Geometry of almost Cliffordian manifolds: classes of subordinated connections

Jaroslav HRDINA*, Petr VAŠÍK
Institute of Mathematics, Faculty of Mechanical Engineering, Brno University of Technology, Brno, Czech Republic

Received: 09.07.2012 ● Accepted: 28.05.2013 ● Published Online: 09.12.2013 ● Printed: 20.01.2014

Abstract: An almost Clifford and an almost Cliffordian manifold is a $G$–structure based on the definition of Clifford algebras. An almost Clifford manifold based on $\mathcal{O} := \mathcal{Cl}(s, t)$ is given by a reduction of the structure group $GL(km, \mathbb{R})$ to $GL(m, \mathcal{O})$, where $k = 2^{s+t}$ and $m \in \mathbb{N}$. An almost Cliffordian manifold is given by a reduction of the structure group to $GL(m, \mathcal{O})GL(1, \mathcal{O})$. We prove that an almost Clifford manifold based on $\mathcal{O}$ is such that there exists a unique subordinated connection, while the case of an almost Cliffordian manifold based on $\mathcal{O}$ is more rich. A class of distinguished connections in this case is described explicitly.

Key words: Clifford algebra, affinor structure, $G$–structure, linear connection, planar curves

1. Introduction

First, let us recall some facts about $G$–structures and their prolongations. There are 2 definitions of $G$–structures. The first reads that a $G$–structure is a principal bundle $P \to M$ with structure group $G$ together with a soldering form $\theta$. The second reads that it is a reduction of the frame bundle $P^1M$ to the Lie group $G$. In the latter case, the soldering form $\theta$ is induced from a canonical soldering form on the frame bundle.

Now let $\mathfrak{g} \subset \wedge^2V$ be the Lie algebra of the Lie group $G$ and let $V$ be a vector space. From the structure theory we know that there is a $G$–invariant complement $\mathcal{D}$ of $\partial(\mathfrak{g} \otimes V^*)$ in $V \otimes \wedge^2V^*$, where $\partial$ is the operator of alternation; see [6]. Let us recall that the torsion of a linear connection lies in the space $V \otimes \wedge^2V^*$.

The almost Clifford and almost Cliffordian structures are $G$–structures based on Clifford algebras. The 2 most important examples are an almost hypercomplex geometry and an almost quaternionic geometry, which are based on Clifford algebra $\mathcal{Cl}(0, 2)$. An important geometric property of almost hypercomplex structures reads that there is no nontrivial $G$–invariant subspace $\mathcal{D}$ in $V \otimes \wedge^2V^*$, because the first prolongation $\mathfrak{g}^{(1)}$ of the Lie algebra $\mathfrak{g}$ vanishes. For almost quaternionic structure, the situation is more complicated, because $\mathfrak{g}^{(1)} = V^*$; see [1]. For these reasons, in the latter case, there exists a distinguished class of linear connections compatible with the structure. Our goal is to describe some of these connections for almost Cliffordian $G$–structures based on Clifford algebras $\mathcal{Cl}(s, t)$ generally.

*Correspondence: hrdina@fme.vutbr.cz

1991 AMS Mathematics Subject Classification: 53C10, 53C15.

179
2. Clifford algebras

The pair \((V, Q)\), where \(V\) is a vector space of dimension \(n\) and \(Q\) is a quadratic form, is called a quadratic vector space. To define Clifford algebras in coordinates, we start by choosing a basis \(e_i, i = 1, \ldots, n\) of \(V\) and by \(I_i, i = 1, \ldots, n\) we denote the image of \(e_i\) under the inclusion \(V \hookrightarrow \text{Cl}(V, Q)\). Then the elements \(I_i\) satisfy the relation

\[I_j I_k + I_k I_j = 2B_{jk} 1,\]

where 1 is the unit in the Clifford algebra and \(B\) is a bilinear form obtained from \(Q\) by polarization. In a quadratic finite dimensional real vector space it is always possible to choose a basis \(e_i\) for which the matrix of the bilinear form \(B\) has the form

\[
\begin{pmatrix}
O_r \\
E_s \\
-E_t
\end{pmatrix}, \quad r + s + t = n,
\]

where \(E_k\) denotes the \(k \times k\) identity matrix and \(O_k\) the \(k \times k\) zero matrix. Let us restrict to the case \(r = 0\), whence \(B\) is nondegenerate. Then \(B\) defines the inner product of signature \((s, t)\) and we call the corresponding Clifford algebra \(\text{Cl}(s, t)\). For example, \(\text{Cl}(0, 2)\) is generated by \(I_1, I_2\), satisfying \(I_1^2 = I_2^2 = -E\) with \(I_1 I_2 = -I_2 I_1\), i.e. \(\text{Cl}(0, 2)\) is isomorphic to \(\mathbb{H}\).

Following the classification of the Clifford algebra, Bott periodicity reads that \(\text{Cl}(0, n) \cong \text{Cl}(0, q) \otimes \mathbb{R}(16p)\), where \(n = 8p + q, \quad q = 0, \ldots, 7\) and \(\mathbb{R}(N)\) denotes the \(N \times N\) matrices with coefficients in \(\mathbb{R}\). To determine explicit matrix representations we use the periodicity conditions

\[
\begin{align*}
\text{Cl}(0, n) & \cong \text{Cl}(n - 2, 0) \otimes \text{Cl}(0, 2), \\
\text{Cl}(n, 0) & \cong \text{Cl}(0, n - 2) \otimes \text{Cl}(2, 0), \\
\text{Cl}(s, t) & \cong \text{Cl}(s - 1, t - 1) \otimes \text{Cl}(1, 1),
\end{align*}
\]

together with the explicit matrix representations of \(\text{Cl}(0, 2), \text{Cl}(2, 0), \text{Cl}(1, 0),\) and \(\text{Cl}(0, 1)\). For example

\[\text{Cl}(3, 0) \cong \text{Cl}(0, 1) \otimes \text{Cl}(2, 0),\]

where the matrix representation of \(\text{Cl}(0, 1)\) on \(\mathbb{R}^{2m}\) is given by the matrices

\[
\begin{pmatrix}
E_m & 0 \\
0 & E_m
\end{pmatrix} \quad \text{and} \quad 
\begin{pmatrix}
0 & E_m \\
-E_m & 0
\end{pmatrix}
\]

and the matrix representation of \(\text{Cl}(2, 0)\) on \(\mathbb{R}^{4m}\) is given by the matrices

\[
E_{4m}, I_1 = \begin{pmatrix}
0 & -E_m & 0 & 0 \\
-E_m & 0 & 0 & 0 \\
0 & 0 & 0 & E_m \\
0 & 0 & E_m & 0
\end{pmatrix}, I_2 = \begin{pmatrix}
0 & 0 & E_m & 0 \\
0 & 0 & 0 & E_m \\
E_m & 0 & 0 & 0 \\
0 & E_m & 0 & 0
\end{pmatrix},
\]

\[
I_3 = I_1 I_2 = \begin{pmatrix}
0 & 0 & 0 & -E_m \\
0 & 0 & -E_m & 0 \\
0 & E_m & 0 & 0 \\
0 & E_m & 0 & 0
\end{pmatrix},
\]

180
where \( E_p \) is an identity matrix \( p \times p \). Now, the matrix representation of \( C(3,0) \) on \( \mathbb{R}^{8m} \) is given by

\[
\begin{pmatrix}
E_{4m} & 0 \\
0 & E_{4m}
\end{pmatrix}, \begin{pmatrix}
I_1 & 0 \\
0 & I_1
\end{pmatrix}, \begin{pmatrix}
I_2 & 0 \\
0 & I_2
\end{pmatrix}, \begin{pmatrix}
I_3 & 0 \\
0 & I_3
\end{pmatrix}, \\
\begin{pmatrix}
0 & E_{4m} \\
-E_{4m} & 0
\end{pmatrix}, \begin{pmatrix}
0 & I_3 \\
-I_1 & 0
\end{pmatrix}, \begin{pmatrix}
0 & I_2 \\
-I_2 & 0
\end{pmatrix}, \begin{pmatrix}
0 & I_1 \\
-I_3 & 0
\end{pmatrix},
\]

for an explicit description see [5].

We now focus on the algebra \( \mathcal{O} := C(s,t) \), i.e. the algebra generated by elements \( I_i, i = 1, \ldots, t \) (called complex units), and elements \( J_j, j = 1, \ldots, s \) (called product units), which are anticommuting, i.e. \( I_j^2 = -E \), \( J_j^2 = E \) and \( K_i K_j = -K_j K_i \), \( i \neq j \), where \( K \in \{I_i, J_j\} \). On the other hand, this algebra is generated by elements \( F_i, i = 1, \ldots, k \) as a vector space. We chose a basis \( F_i, i = 1, \ldots, k \), such that \( F_1 = E, F_i = J_{i-1} \) for \( i = 2, \ldots, t+1 \), \( F_j = J_{j-1} \) for \( j = t+2, \ldots, s+t+1 \) and by all different multiples of \( I_i \) and \( J_j \) of length \( 2, \ldots, s+t \). Let us note that both complex and product units can be found among these multiple generators.

**Lemma 2.1** Let \( F_1, \ldots, F_k \) denote the \( k = 2s+t \) elements of the matrix representation of Clifford algebra \( C(s,t) \) on \( \mathbb{R}^k \). Then there exists a real vector \( X \in \mathbb{R}^k \) such that the dimension of a linear span \( \langle F_i X | i = 1, \ldots, k \rangle \) is equal to \( k \).

**Proof** Let us suppose, without loss of generality, that \( F_1, \ldots, F_k \) are the elements constructed by means of Bott periodicity as above. Then, by induction, we prove that the matrix \( F = \sum_{i=1}^{k} a_i F_i, a_i \in \mathbb{R} \), is a square matrix that has exactly one entry \( a_i \) in each column and each row. For \( C(1,0) \) and \( C(0,1) \), we have

\[
F = \begin{pmatrix}
a_1 & a_2 \\
a_2 & a_1
\end{pmatrix}, F = \begin{pmatrix}
a_1 & a_2 \\
-a_2 & a_1
\end{pmatrix},
\]

respectively. For the rest of the generating cases, \( C(2,0) \), \( C(0,2) \), and \( C(1,1) \), matrix \( F \) can be obtained in a very similar way.

We now restrict to Clifford algebras of type \( C(s,0) \), \( s > 2 \), and show the induction step by means of the periodicity condition

\[
C(s,0) \cong C(0,s-2) \otimes C(2,0).
\]

The rest of the cases according to the Clifford algebra identification above can be proved similarly and we leave it to the reader. Let \( G_i, i = 1, \ldots, l \) denote the \( l \) elements of the matrix representation of Clifford algebra \( C(0,s-2) \) with the required property, i.e. the matrix \( G = \sum_{i=1}^{l} g_i G_i \) is a square matrix with exactly one entry \( g_i \) in each column and each row, i.e.

\[
G := \begin{pmatrix}
g_{\sigma_1(1)} & \cdots & g_{\sigma_1(l)} \\
\vdots & \ddots & \vdots \\
g_{\sigma_l(1)} & \cdots & g_{\sigma_l(l)}
\end{pmatrix},
\]

where \( \sigma_i \) are all permutations of \( \{1, \ldots, l\} \). The matrix of \( C(2,0) \) is

\[
H := \begin{pmatrix}
a_1 & -a_2 & a_3 & -a_4 \\
-a_2 & a_1 & -a_4 & a_3 \\
a_3 & a_4 & a_1 & a_2 \\
-a_4 & a_3 & a_2 & a_1
\end{pmatrix},
\]

181
The matrix for the representation of Clifford algebra $\mathcal{C}l(s,0)$ is then composed as follows:

$$F := \left( g_{\sigma_1(1)}H \quad \cdots \quad g_{\sigma_1(l)}H \right) .$$

Finally, if matrix $G$ has exactly one $g_i$ in each column and each row, matrix $F$ is a square matrix with exactly one $a_{ij}g_i$, where $j = 1, \ldots, 4, i = 1, \ldots, l$ in each column and each row.

Now, let $F = \sum_{i=1}^{k} b_i F_i$ be a $k \times k$ matrix constructed as above and let $e_i$ denote the standard basis of $\mathbb{R}^k$. Then the vector $v_i := F e_i^T$

is the $i$-th column of the matrix $F$ and thus it is composed of $k$ different entries $b_i$. If the dimension of $\langle F_i X | i = 1, \ldots, k \rangle$ is less than $k$, then the vector $v$ has to be zero and thus all $b_i$ have to be zero. $\square$

**Definition 2.2** Let $P^1 M$ be a bundle of linear frames over $M$ (the fiber bundle $P^1 M$ is a principal bundle over $M$ with the structure group $GL(n, \mathbb{R})$). Reduction of the bundle $P^1 M$ to the subgroup $G \subset GL(n, \mathbb{R})$ is called a $G$-structure.

**Definition 2.3** If $M$ is an $km$-dimensional manifold, where $k = 2^{s+t}$ and $m \in \mathbb{N}$, then an almost Clifford manifold is given by a reduction of the structure group $GL(km, \mathbb{R})$ of the principal frame bundle of $M$ to

$$GL(m, O) = \{ A \in GL(km, \mathbb{R}) | AI_i = I_i A , AJ_j = J_j A \},$$

where $O$ is an arbitrary Clifford algebra and $I_i, i = 1, \ldots, t, \quad I_1^2 = -E$ and $J_j, j = 1, \ldots, s, \quad J_j^2 = E$ is the set of anticommuting affinors such that the free associative unitary algebra generated by $\langle I_i, J_j, E \rangle$ is isomorphically equivalent to $O$.

In particular, on the elements of this reduced bundle one can define affinors in the form of $F_1, \ldots, F_k$ globally.

3. $A$-planar curves and morphisms

The concept of planar curves is a generalization of a geodesic on a smooth manifold equipped with certain structure. In [7] the authors proved a set of facts about structures based on 2 different affinors. Following [3, 4], a manifold equipped with an affine connection and a set of affinors $A = \{ F_1, \ldots, F_l \}$ is called an $A$-structure and a curve satisfying $\nabla_x \dot{c} \in \langle F_1(\dot{c}), \ldots, F_l(\dot{c}) \rangle$ is called an $A$-planar curve.

**Definition 3.1** Let $M$ be a smooth manifold such that $\dim(M) = m$. Let $A$ be a smooth $\ell$-dimensional ($\ell < m$) vector subbundle in $T^* M \otimes T M$ such that the identity affinor $E = id_{TM}$ restricted to $T_x M$ belongs to $A_x M \subset T^*_x M \otimes T_x M$ at each point $x \in M$. We say that $M$ is equipped with an $\ell$-dimensional $A$-structure.

It is easy to see that an almost Clifford structure is not an $A$-structure, because the affinors in the form of $F_0, \ldots, F_l \in A$ have to be defined only locally.
**Definition 3.2** The $A$–structure where $A$ is isomorphically equivalent to a Clifford algebra $\mathcal{O}$ is called an almost Cliffordian manifold.

The classical concept of $F$–planar curves defines the $F$–planar curve as the curve $c : \mathbb{R} \to M$ satisfying the condition

$$\nabla_c \dot{c} \in \langle \dot{c}, F(\dot{c}) \rangle,$$

where $F$ is an arbitrary affinor. Clearly, geodesics are $F$–planar curves for all affinors $F$, because $\nabla_c \dot{c} \in \langle \dot{c}, F(\dot{c}) \rangle$.

Now, for any tangent vector $X \in T_x M$ we shall write $A_x(X)$ for the vector subspace

$$A_x(X) = \{ F_i(X) | F_i \in A_x M \} \subset T_x M$$

and call it the $A$–hull of the vector $X$. Similarly, the $A$–hull of a vector field is a subbundle in $TM$ obtained pointwise. For example, the $A$–hull of an almost quaternionic structure is

$$A_x(X) = \{ aX + bI(X) + cJ(X) + dK(X) | a, b, c, d \in \mathbb{R} \}.$$ 

**Definition 3.3** Let $M$ be a smooth manifold equipped with an $A$–structure and a linear connection $\nabla$. A smooth curve $c : \mathbb{R} \to M$ is said to be $A$–planar if

$$\nabla_c \dot{c} \in A(\dot{c}).$$

One can easily check that the class of connections

$$[\nabla]_A = \nabla + \sum_{i=1}^{\dim A} \Upsilon_i \otimes F_i,$$  \hspace{1cm} (1)

where $\Upsilon_i$ are one-forms on $M$, share the same class of $A$–planar curves, but we have to describe them more carefully for Cliffordian manifolds.

**Theorem 3.4** Let $M$ be a smooth manifold equipped with an almost Cliffordian structure, i.e. an $A$–structure, where $A = \text{Cl}(s, t)$, $\dim(M) \geq 2^{s+t+1}$, and let $\nabla$ be a linear connection such that $\nabla A = 0$. The class of connections $[\nabla]$ preserving $A$, sharing the same torsion and $A$–planar curves, is isomorphic to $T^* M$ and the isomorphism has the following form:

$$\Upsilon \mapsto \nabla + \sum_{i=1}^{k} \epsilon_i(\Upsilon \circ F_i) \otimes F_i,$$  \hspace{1cm} (2)

where $\langle F_1, \ldots, F_k \rangle = A$, $k = 2^{s+t}$, as a vector space, $\epsilon_i \in \{ \pm 1 \}$, and $\Upsilon$ is a one-form on $M$.

**Proof** First, let us consider the difference tensor

$$P(X, Y) = \nabla_X(Y) - \nabla_Y(X)$$

and one can see that its value is symmetric in each tangent space because both connections share the same torsion. Since both $\nabla$ and $\nabla$ preserve $F_i$, $i = 1, \ldots, k$, the difference tensor $P$ is Clifford linear in the second
variable. By symmetry it is thus Clifford bilinear and we can proceed by induction. Let \( X = c \) and the deformation \( P(X, X) \) equals to \( \sum_{i=1}^{k} \mathcal{T}_i(X)F_i(X) \) because \( c \) is \( A \)-planar with respect to \( \nabla \) and \( \nabla' \). In this case, we shall verify.

First, for \( s = 1, t = 0 \),

\[
P(X, X) = a(X)X + b(JX)JX,
\]

\[
P(X, X) = J^2P(X, X) = P(JX, JX) = a(JX)JX + b(X)X.
\]

The difference of the first row and the second row implies \( a(X) = b(X) \) and \( a(JX) = b(JX) \) because we can suppose that \( X, JX \) are linearly independent.

For \( s = 0, t = 1 \),

\[
P(X, X) = a(X)X + b(IX)IX,
\]

\[
-P(X, X) = I^2P(X, X) = P(IX, IX) = a(IX)IX - b(X)X.
\]

The sum of the first row and the second row implies \( a(X) = b(X) \) and \( a(IX) = -b(IX) \) because we can suppose that \( X, IX \) are linearly independent.

Let us suppose that the property holds for a Clifford algebra \( Cl(s, t) \), \( k = 2^{s+t} \), i.e.

\[
P(X, X) = \sum_{i=1}^{k} \epsilon_i(\mathcal{Y}(F_i(X)))F_i(X),
\]

where \( \epsilon_i \in \{\pm1\} \).

For \( Cl(s, t+1) \) we have

\[
P(X, X) = \sum_{i=1}^{k} \epsilon_i(\mathcal{Y}(F_i(X)))F_i(X) + \sum_{i=1}^{k} (\xi_i(F_iS(X)))F_iS(X),
\]

and

\[
S^2P(X, X) = \sum_{i=1}^{k} \epsilon_i(\mathcal{Y}(F_i(SX)))F_i(SX) + \sum_{i=1}^{k} (\xi_i(F_i(SX)))F_i(X).
\]

The sum of the first row and the second row implies

\[
\epsilon_i\mathcal{Y}(F_i(X)) = -\xi_i(F_iX) \quad \text{and} \quad \epsilon_i\mathcal{Y}(F_i(SX)) = -\xi_i(F_iSX),
\]

because we can suppose that \( F_iX \) are linearly independent. The case of \( Cl(s+1, t) \) is calculated in the same way.

Now, \( P(X, X) = \sum_{i=1}^{k} \epsilon_i(\mathcal{Y}(F_i(X)))F_i(X) \) and one shall compute

\[
P(X, Y) = \frac{1}{2} \left( \sum_{i=1}^{k} \epsilon_i\mathcal{Y}(F_i(X + Y))F_i(X + Y) - \sum_{i=1}^{k} \epsilon_i\mathcal{Y}(F_i(X))F_i(X) \right)
\]

\[
- \sum_{i=1}^{k} \epsilon_i\mathcal{Y}(F_i(Y))F_i(Y)
\]

by polarization.
Assuming that vectors $F_i(X), F_i(Y), i = 1, \ldots, k$ are linearly independent, we compare the coefficients of $X$ in the expansions of $P(sX, tY) = stP(X, Y)$ as above to get

$$s\Upsilon(sX + tY) - s\Upsilon(sX) = st(\Upsilon(X + Y) - \Upsilon(X)).$$

Dividing by $s$ and putting $t = 1$ and taking the limit $s \to 0$, we conclude that $\Upsilon(X + Y) = \Upsilon(X) + \Upsilon(Y)$.

We have proven that the form $\Upsilon$ is linear in $X$ and

$$(X, Y) \to \sum_{i=1}^{k} \varepsilon_i(\Upsilon(F_i(X)))F_i(Y) + \sum_{i=1}^{k} \varepsilon_i(\Upsilon(F_i(Y)))F_i(X)$$

is a symmetric complex bilinear map that corresponds to $P(X, Y)$ if both arguments coincide; it always agrees with $P$ by polarization and $\nabla$ lies in the projective equivalence class $[\nabla]$. \hfill \Box

4. **$D$-connections**

Let $V = \mathbb{R}^n$, $G \subset GL(V) = GL(n, \mathbb{R})$ be a Lie group with Lie algebra $\mathfrak{g}$ and $M$ be a smooth manifold of dimension $n$.

**Definition 4.1** The first prolongation $\mathfrak{g}^{(1)}$ of $\mathfrak{g}$ is a space of symmetric bilinear mappings $t : V \times V \to V$ such that, for each fixed $v_1 \in V$, the mapping $v \in V \mapsto t(v, v_1) \in V$ is in $\mathfrak{g}$.

**Example 4.2** A complex structure $(M, I), I^2 = -E$, is a $G$-structure where $G = GL(n, \mathbb{C})$ with Lie algebra $\mathfrak{g} = \{ A \in \mathfrak{gl}(2n, \mathbb{R}) | AI = IA \}$. The first prolongation $\mathfrak{g}^{(1)}$ is a space of symmetric bilinear mappings

$$\mathfrak{g}^{(1)} = \{ t : V \times V \to V, t(IX, Y) = It(X, Y), t(Y, X) = t(X, Y) \}.$$ 

On the other hand, a product structure $(M, P), P^2 = E$ is a $G$-structure where $G = GL(n, \mathbb{R}) \oplus GL(n, \mathbb{R})$ with Lie algebra $\mathfrak{g} = \mathfrak{g}(n, \mathbb{R}) \oplus \mathfrak{g}(n, \mathbb{R})$. The first prolongation $\mathfrak{g}^{(1)}$ is a space of symmetric bilinear mappings

$$\mathfrak{g}^{(1)} = \{ t : V_1 \oplus V_2 \to V_1 \oplus V_2, t(V_i, V_i) \in V_i, t(V_2, V_1) = 0 \}.$$ 

**Lemma 4.3** Let $M$ be a $(km)$-dimensional Clifford manifold based on Clifford algebra $\mathcal{O} = Cl(s, t), k = 2^{s+t}, s + t > 1, m \in \mathbb{N}$, i.e. a manifold equipped with $G$-structure, where

$$G = GL(m, \mathcal{O}) = \{ B \in GL(km, \mathbb{R}) | BI_i = I_iB, BJ_j = J_jB \},$$

and $I_i$ and $J_j$ are algebra generators of $\mathcal{O}$. Then the first prolongation $\mathfrak{g}^{(1)}$ of Lie algebra $\mathfrak{g}$ of Lie group $G$ vanishes.

**Proof** Lie algebra $\mathfrak{g}$ of a Lie group $G$ is of the form

$$\mathfrak{g} = \mathfrak{gl}(m, \mathcal{O}) = \{ B \in \mathfrak{gl}(km, \mathbb{R}) | BI_i = I_iB, BJ_j = J_jB \},$$

where $I_i$ and $J_j$ are generators of $\mathcal{O}$, i.e. $K_iK_j = -K_jK_i$ for $K_i, K_j \in \{ I_i, J_j \}, \ i \neq j$. For $t \in \mathfrak{g}^{(1)}$ and $K_i \neq K_j$ we have the equations

$$t(K_iX, K_jX) = K_iK_jt(X, X),$$

185
\[
t(K_iX, K_jX) = t(K_jX, K_iX) = K_jK_i t(X, X) = -K_iK_j t(X, X),
\]
which lead to \( t(X, X) = 0 \). Finally, from polarization,

\[
t(X, Y) = \frac{1}{2} (t(X+Y, X+Y) - t(X, X) - t(Y, Y)) = 0.
\]

Let us shortly note that Example 4.2 covers Clifford manifolds for \( O = Cl(0,1) \) and \( O = Cl(1,0) \). Next, suppose that there is a \( G \)-invariant complement \( D \) to \( \partial(\mathfrak{g} \otimes \mathbb{V}^*) \) in \( \mathbb{V} \otimes \Lambda^2 \mathbb{V}^* \):

\[
\mathbb{V} \otimes \Lambda^2 \mathbb{V}^* = \partial(\mathfrak{g} \otimes \mathbb{V}^*) \oplus D,
\]

where

\[
\partial : \text{Hom}(\mathbb{V}, \mathfrak{g}) = \mathfrak{g} \otimes \mathbb{V}^* \to \mathbb{V} \otimes \Lambda^2 \mathbb{V}^*
\]
is the Spencer operator of alternation.

**Definition 4.4** Let \( \pi : P \to M \) be a \( G \)-structure. A connection \( \omega \) on \( P \) is called a \( D \)-connection if its torsion function

\[
t^\omega : P \to \mathbb{V} \otimes \Lambda^2 \mathbb{V}^* = \partial(\mathfrak{g} \otimes \mathbb{V}^*) \oplus D
\]
has values in \( D \).

**Theorem 4.5** \cite{1}

1. Any \( G \)-structure \( \pi : P \to M \) admits a \( D \)-connection \( \nabla \).
2. Let \( \omega, \bar{\omega} \) be 2 \( D \)-connections. Then the corresponding operators of covariant derivative \( \nabla, \bar{\nabla} \) are related by

\[
\bar{\nabla} = \nabla + S,
\]

where \( S \) is a tensor field such that for any \( x \in M \), \( S_x \) belongs to the first prolongation \( \mathfrak{g}^{(1)} \) of the Lie algebra \( \mathfrak{g} \).

**Definition 4.6** We say that a connected linear Lie group \( G \) with Lie algebra \( \mathfrak{g} \) is of type \( k \) if its \( k \)-th prolongation vanishes, i.e. \( \mathfrak{g}^{(k)} = 0 \) and \( \mathfrak{g}^{(k-1)} \neq 0 \). In this sense, any \( G \)-structure with Lie group \( G \) of type \( k \) is called a \( G \)-structure of type \( k \).

**Theorem 4.7** \cite{1} Let \( \pi : P \to M \) be a \( G \)-structure of type 1 and suppose that there is given a \( G \)-equivariant decomposition

\[
\mathbb{V} \otimes \Lambda^2 \mathbb{V} = \partial(\mathfrak{g} \otimes \mathbb{V}^*) \oplus D.
\]

Then there exists a unique connection, whose torsion tensor (calculated with respect to a coframe \( p \in P \)) has values in \( D \subset \mathbb{V} \otimes \Lambda^2 \mathbb{V}^* \).

**Corollary 4.8** Let \( M \) be a smooth manifold equipped with a \( G \)-structure, where \( G = GL(n, \mathcal{O}) \), \( \mathcal{O} = Cl(s,t) \), \( s + t > 1 \), i.e. an almost Clifford manifold. Then the \( G \)-structure is of type 1 and there exists a unique \( D \)-connection.
5. An almost Cliffordian manifold

One can see that an almost Cliffordian manifold $M$ is given as a $G$–structure provided that there is a reduction of the structure group of the principal frame bundle of $M$ to

$$G := GL(m, \mathcal{O})GL(1, \mathcal{O}) = GL(m, \mathcal{O}) \times Z(GL(1, \mathcal{O})) GL(1, \mathcal{O}),$$

where $Z(G)$ is a center of $G$. The action of $G$ on $T_x M$ looks like

$$QXq, \text{ where } Q \in GL(m, \mathcal{O}), q \in GL(1, \mathcal{O}),$$

where the right action of $GL(1, \mathcal{O})$ is blockwise. In this case the tensor fields in the form $F_1, \ldots, F_k$ can be defined only locally. It is easy to see that the Lie algebra $\mathfrak{gl}(m, \mathcal{O})$ of a Lie group $GL(m, \mathcal{O})$ is of the form

$$\mathfrak{gl}(m, \mathcal{O}) = \{ A \in \mathfrak{gl}(km, \mathbb{R}) | AI_i = I_i A, AJ_j = J_j A \}$$

and the Lie algebra $\mathfrak{g}$ of a Lie group $GL(m, \mathcal{O})GL(1, \mathcal{O})$ is of the form

$$\mathfrak{g} = \mathfrak{gl}(m, \mathcal{O}) \oplus \mathfrak{gl}(1, \mathcal{O}).$$

Let us note that the case of $\text{Cl}(0, 3)$ was studied in a detailed way in [2].

**Remark 5.1** Let $\mathcal{O}$ be the Clifford algebra $\text{Cl}(0, 2)$. For any one-form $\xi$ on $\mathbb{V}$ and any $X, Y \in \mathbb{V}$, the elements of the form

$$S^\xi(X, Y) = -\xi(X)Y - \xi(Y)X + \xi(I_1 X)I_1 Y + \xi(I_1 Y)I_1 X + \xi(I_2 X)I_2 Y$$

$$+ \xi(I_2 Y)I_2 X + \xi(I_1 I_2 X)I_1 I_2 Y + \xi(I_1 I_2 Y)I_1 I_2 X$$

belong to the first prolongation $\mathfrak{g}^{(1)}$ of the Lie algebra $\mathfrak{g}$ of the Lie group $GL(m, \mathcal{O})GL(1, \mathcal{O})$.

**Proof** We fix $X \in \mathbb{V}$ and define $S^\xi_X := S^\xi(X, Y) : \mathbb{V} \to \mathbb{V}$. We have to prove that $S^\xi_X(I_1 Y) = I_i S^\xi_X(Y) + \sum_{i=1}^4 a_i F_i(Y)$, for $i = 1, 2$ and $S^\xi_X(Y) = S^\xi_X(X)$. We compute directly for any $X$ and for $I_1$

$$S^\xi_X(I_1 Y) = -\xi(X)I_1 Y - \xi(I_1 Y)X - \xi(I_1 X)Y - \xi(Y)I_1 X - \xi(I_2 X)I_1 I_2 Y$$

$$- \xi(I_1 I_2 X)I_1 I_2 Y + \xi(I_1 I_2 X)I_2 Y + \xi(I_2 Y)I_1 I_2 X$$

$$= -\xi(X)I_1 Y - \xi(Y)I_1 X - \xi(I_1 Y)I_1 I_2 X$$

$$+ \xi(I_2 Y)I_1 I_2 X + \sum_{l=1}^4 a_l F_l(Y).$$

On the other hand,

$$I_1 S^\xi_X(Y) = -\xi(X)I_1 Y - \xi(Y)I_1 X - \xi(I_1 X)Y - \xi(I_1 Y)X + \xi(I_2 X)I_1 I_2 Y$$

$$+ \xi(I_2 Y)I_1 I_2 X - \xi(I_1 I_2 X)I_2 Y - \xi(I_1 I_2 Y)I_2 X$$

$$= -\xi(Y)I_1 X - \xi(I_1 Y)X + \xi(I_2 Y)I_1 I_2 X$$

$$- \xi(I_1 I_2 Y)I_2 X + \sum_{l=1}^4 a_l F_l(Y)$$

187
and

\[ S_X(I_1 Y) - I_1 S_X(Y) = \]
\[ = -\xi(I_1 Y)X - \xi(Y)I_1 X - \xi(I_1 I_2 Y)I_2 X + \xi(I_2 Y)I_1 I_2 X \]
\[ -(-\xi(Y)I_1 X - \xi(I_1 Y)X + \xi(I_2 Y)I_1 I_2 X - \xi(I_1 I_2 Y)I_2 X) \]
\[ + \sum_{i=1}^{4} a_i F_i(Y) = \sum_{i=1}^{4} a_i F_i(Y). \]

By the same process for \( I_2 \) we obtain

\[ S_X(I_2 Y) - I_2 S_X(Y) = \]
\[ = -\xi(I_2 Y)X + \xi(I_1 I_2 Y)I_1 X - \xi(Y)I_2 X - \xi(I_1 Y)I_1 I_2 X \]
\[ -(-\xi(Y)I_2 X - \xi(I_1 Y)I_1 I_2 X - \xi(I_2 Y)X + \xi(I_1 I_2 Y)I_1 X) \]
\[ + \sum_{i=1}^{4} a_i F_i(Y) = \sum_{i=1}^{4} a_i F_i(Y). \]

Finally, we have to prove the symmetry, but this is obvious.

\[ \square \]

**Lemma 5.2** Let \( Cl(s,t) \) be the Clifford algebra, \( n = s + t \), and let us denote by \( F_i \) the affinors obtained from the generators of \( Cl(s,t) \). Then there exist \( \epsilon_i \in \{ \pm 1 \} \), \( i = 1, \ldots, n \) such that for \( A \in V^* \), the tensor \( S^A \in V \times V \to V \) defined by

\[ S^A(X,Y) = \sum_{i=1}^{n} \epsilon_i A(F_i X)F_i Y, \quad X,Y \in V \]  

(3)

satisfies the identity

\[ S^A(I_j X,Y) - I_j S^A(X,Y) = 0 \]  

(4)

for all algebra generators \( I_j \) of \( Cl(s,t) \).

**Proof** Let us consider the gradation of the Clifford algebra \( Cl(s,t) = Cl^0 \oplus Cl^1 \oplus \ldots \oplus Cl^n \) with respect to the generators of \( Cl(s,t) \). Then we can define gradually: for \( E \in Cl^0 \) we choose \( \epsilon = 1 \). If the identity (4) should be satisfied for the terms in (3), then it must hold that

\[ \epsilon_0 A(I_j X)Y = \epsilon_i A(I_j X)I_j I_j Y \]  

for all \( I_j \),

i.e.

\[ \epsilon_i = \begin{cases} 
1 & \text{for } I_j^2 = 1, \\
-1 & \text{for } I_j^2 = -1. 
\end{cases} \]

For \( F_i \in Cl^w \) the following equality holds:

\[ \epsilon_i A(F_i I_j X)F_i Y = \epsilon_k A(F_k X)I_j F_k Y, \]
and thus \( F_i = I_j F_k \). Note that \( F_k \) can be an element of both \( Cl^{v+1} \) and \( Cl^{v-1} \). WLOG we choose \( I_j \) such that \( F_k \in Cl^{v+1} \). Now 2 possibilities can appear: either

\[
F_i I_j = F_k, \tag{5}
\]

which leads to \( I_j F_k I_j = F_k \) and thus \( \varepsilon_k = \varepsilon_i \), or

\[
F_i I_j = -F_k, \tag{6}
\]

which leads to \( I_j F_k I_j = -F_k \) and thus \( \varepsilon_k = -\varepsilon_i \).

This concludes the definition of \( \varepsilon_i \) such that the identity (4) holds. To prove the consistency, we have to show that the value of \( \varepsilon_k \) does not depend on \( I_j \), i.e. the generators \( I \) such that \( I^2 = 1 \) and \( J \) such that \( J^2 = -1 \), the resulting coefficient \( \varepsilon_k \) obtained after 2 consequent steps of the algorithm with the alternate use of both \( I \) and \( J \), does not depend on the order. Thus let us consider the following cases:

(a) \( F_i = IF_k \), which results in the possibilities \( IF_k I = F_k \), see (5), which leads to \( \varepsilon_k = \varepsilon_i \), or \( IF_k I = -F_k \), see (6), which leads to \( \varepsilon_k = -\varepsilon_i \).

(b) \( F_j = JF_k \), which similarly leads to either \( JF_k J = F_k \) implying \( \varepsilon_k = \varepsilon_j \), or \( JF_k J = -F_k \) implying \( \varepsilon_k = -\varepsilon_j \).

Applying the processes (a) and (b) alternately, we obtain:

\[
JIF_k J = \begin{cases} 
-IF_k & \Rightarrow \varepsilon_i = -\varepsilon_i \\
IF_k & \Rightarrow \varepsilon_i = \varepsilon_i
\end{cases}
\]

for \( F_l = JIF_k \) and

\[
IJJF_k I = \begin{cases} 
-JF_k & \Rightarrow \varepsilon_i = -\varepsilon_j \\
F_k & \Rightarrow \varepsilon_i = \varepsilon_j
\end{cases}
\]

for \( F_l = IJJF_k \). Obviously, the corresponding cases give the same result of \( \varepsilon_k \). \( \square \)

**Theorem 5.3** Let \( \mathcal{O} \) be the Clifford algebra \( Cl(s,t) \). For any one-form \( \xi \) on \( \mathbb{V} \) and any \( X,Y \in \mathbb{V} \), the elements of the form

\[
S^\xi_X(Y) = \sum_{i=1}^{k} \epsilon_i(\xi(F_i X)F_i Y + \xi(F_j Y)F_i X), \quad k = 2^{s+t},
\]

where the coefficients \( \epsilon_i \) depend on the type of \( \mathcal{O} \), belong to the first prolongation \( \mathfrak{g}^{(1)} \) of the Lie algebra \( \mathfrak{g} \) of the Lie group \( GL(m,\mathcal{O})GL(1,\mathcal{O}) \).

**Proof** One can easily see that \( S^\xi \) is symmetric and we have to prove the second condition, i.e. \( S^\xi_X I_i Y - I_i S^\xi_X Y \in \mathcal{O}(Y) \), i.e.

\[
S^\xi_X I_i Y - I_i S^\xi_X Y = \sum_{j=1}^{k} \varepsilon_j \xi(F_j X)F_j Y + \sum_{j=1}^{k} \varepsilon_j \xi(F_j Y)F_j X \\
- \sum_{j=1}^{k} \varepsilon_j I_i \xi(F_j X)F_j Y - \sum_{j=1}^{k} \varepsilon_j I_i \xi(F_j Y)F_j TX.
\]
From Lemma 5.2 we have

\[ S_X^i I_i Y - I_i S_X^i Y = \sum_{j=1}^{k} \epsilon_j \xi (F_j X) F_j Y - \sum_{j=1}^{k} \epsilon_j I_i \xi (F_j Y) F_j TX = \sum_{j=0}^{k} \psi_i F_j Y. \]

\[ \square \]

**Corollary 5.4** Let \( M \) be an almost Cliffordian manifold based on Clifford algebra \( \mathcal{O} = \text{Cl}(s,t) \), where \( \dim(M) \geq 2^{(s+t+1)} \), i.e. a smooth manifold equipped with \( G \)-structure, where \( G = \text{GL}(n, \mathcal{O}) \text{GL}(1, \mathcal{O}) \) or equivalently an \( A \)-structure where \( A = \mathcal{O} \). Then the class of \( \mathcal{D} \)-connections preserving \( A \) and sharing the same \( A \)-planar curves is isomorphic to \((\mathbb{R}^{km})^*\).

**Acknowledgment**

The authors were supported by a grant no. FSI-S-11-3.

**References**

1. Alekseevsky, D. V., Marchiafava, S.: Quaternionic structures on a manifold and subordinated structures. Annali di Mat. Pura a Appl. 171, 205–273 (1996).
2. Burdujan, I.: On almost Cliffordian manifolds. Italian J. Pure Appl. Math. 13, 129–144 (2003).
3. Hrdina, J.: Notes on connections attached to \( A \)-structures. Diff. Geom. Appl. 29, Suppl. 1, 91–97 (2011).
4. Hrdina, J., Slovák, J.: Generalized planar curves and quaternionic geometry. Glob. Anal. Geom. 29, 349–360 (2006).
5. Hrdina, J., Vašík, P.: Generalized geodesics on some almost Cliffordian geometries. Balkan J. Geom. Appl., Vol. 17, 41–48 (2012).
6. Kobayashi, S.: Transformation Groups in Differential Geometry. Springer 1972.
7. Mikeš, J., Sinyukov, N.S.: On quasiplanar mappings of spaces of affine connection. Sov. Math. 27, 63–70 (1983).