Distributed Nash Equilibrium Seeking for Games in Second-Order Systems Without Velocity Measurement

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Abstract—The design of distributed Nash equilibrium-seeking strategies for games in which the involved players are of second-order integrator-type dynamics is investigated in this article. Noticing that velocity signals are usually noisy or not available for feedback control in practical engineering systems, this article supposes that the velocity signals are not accessible for the players. To deal with the absence of velocity measurements, two estimators are designed. The first estimator is established by employing an observer, which has the same order as the players’ dynamics, to estimate the unavailable system states (e.g., the players’ velocities). The second estimator is designed based on a high-pass filter and is motivated by the incentive to reduce the order of the estimator, which in turn saves the computation costs of the seeking algorithms. On the basis of the designed observers/filters, distributed Nash equilibrium-seeking strategies are then established through incorporating them with consensus and gradient algorithms. It is analytically proven that the players’ actions can be regulated to the Nash equilibrium point and their velocities can be regulated to zero by utilizing the proposed velocity-free Nash equilibrium-seeking strategies. A numerical example is provided for the verification of the proposed algorithms.

Index Terms—Filter, Nash equilibrium seeking, second-order dynamics, state observer, without velocity measurement.

I. INTRODUCTION

With the rapid development of Nash equilibrium-seeking algorithms in the past few years, games with second-order integrator-type players have drawn some attention recently. The main reason is that the involved players are of second-order integrator-type dynamics. Many works were reported in part by the National Natural Science Foundation of China (NSFC) under Grant 62173181, Grant 61803202, and Grant 62003282, in part by the Natural Science Foundation of Jiangsu Province under Grant BK20180455, and in part by the Fundamental Research Funds for the Central Universities under Grant 30920032203. Recommended by Associate Editor Andreas A. Malikopoulos. (Corresponding author: Maojiao Ye.)

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control input with full-state feedback, the Nash equilibrium can be stabilized. In [10], we considered games in which the players’ dynamics appear to be heterogeneous in the sense that some players’ dynamics are first-order integrators while the rest are second-order integrators. Based on action and velocity feedback, Nash equilibrium-seeking strategies were proposed for both full information games and partial information games. In [11], games with high-order integrator-type dynamics were concerned and a Nash equilibrium-seeking strategy was proposed by employing adaptive control gains. In [12], a Nash equilibrium-seeking problem for second-order integrator-type games was addressed by designing methods based on projection operators, consensus protocols as well as primal-dual techniques. However, it is worth mentioning that the abovementioned works achieve Nash equilibrium seeking by utilizing full-state feedback, i.e., both the players’ position information and velocity information should be measured to implement the aforementioned methods, which restricts their applications to some extent as practical situations show that it might be challenging or costly to measure the velocities accurately in real time.

It is inadvisable to utilize velocity information as in many practical situations, velocity measurements are usually noisy, which may deteriorate the control performance. Moreover, it is costly and complex to install extra velocity sensors in some engineering systems. Actually, quite a lot of works have been reported to deal with the unavailability of velocity measurements for various control applications. For example, only actuator position measurement units but not velocity measurement devices are included in many commercial robotic systems (e.g., PUMA 560 robot) [13]. To compensate for the limited sensors installed in rigid-link flexible-joint robots, the authors employed a set of filters in the control strategy design to achieve position tracking of the robots [13]. With the development of robots, motion control of mechanical systems without velocity measurement has drawn increasing attention [14]. Moreover, as angular velocities and relative angular velocities are absent, attitude consensus among a group of spacecraft was addressed by introducing some auxiliary dynamics in [15]. Motivated by the fact that ship velocity measurements are usually unavailable, the authors in [16] designed a controller to drive an underactuated ship along a prescribed path without utilizing ship velocities. Furthermore, as it is challenging to obtain velocity signals for electro-hydraulic servomechanisms, an adaptive strategy was proposed for the tracking control of electro-hydraulic servomechanisms based on extended-state-observers and backstepping techniques in [17]. With the lack of velocity feedback, collaborative control (e.g., consensus, formation, to mention just a few) of second-order multiagent systems by utilizing only position information was also reported in quite a few works [18]–[20].

In spirit of relaxing the requirements on velocity measurements, this article considers Nash equilibrium seeking for games in which the players are of second-order integrator-type dynamics without utilizing velocity measurements. In comparison with existing works, the main contributions of the article are summarized as follows.

1) Distributed Nash equilibrium seeking for games with second-order integrator-type players is investigated. Compared with the works in [8]–[12] and [28], velocity measurements are not utilized in the control design, which benefits the applications of games to circumstances in which the players are not equipped...
with any velocity measurement devices or the measured velocities are noisy. Observer-based approaches and filter-based approaches are proposed for velocity estimation, by which distributed Nash equilibrium-seeking strategies are constructed.

2) Stability of the Nash equilibrium under the proposed seeking strategies is analytically explored. It is shown that the players’ actions and velocities can be regulated to the Nash equilibrium and zero, respectively, by employing the Lyapunov stability analysis. 

Notations: This article utilizes \( \mathbb{R} \) to denote the set of real numbers. A matrix whose \((i,j)\)th entry is \( p_{ij} \) is written as \( [p_{ij}] \). Moreover, \( [p_{ij}]_{\text{vec}}(\text{diag}(p_{ij})) \) is a column vector (diagonal matrix) whose \( i \)th (diagonal) entry is \( p_{i} \) and similarly, \( [p_{ij}]_{\text{vec}}(\text{diag}(p_{ij})) \) for \( i,j \in \{1, 2, \ldots, N\} \) defines a column vector (diagonal matrix) whose (diagonal) entries are \( p_{1}, p_{2}, \ldots, p_{N}, p_{N}, \ldots, p_{N} \), respectively. Let \( E_{N} \) be an identity matrix of dimension \( N \times N \). Moreover, \( I_{N} \) and \( 0_{N} \) are \( N \)-dimensional column vectors whose entries are 1 and 0, respectively. We utilize \( \mu_{\min}(\Gamma) \) and \( \mu_{\max}(\Gamma) \) to, respectively, define the minimum and maximum eigenvalues of \( \Gamma \), where \( \Gamma \) is a symmetric real matrix. The minimum and maximum values of real numbers \( p_{1}, p_{2}, \ldots, p_{N} \), where \( r > 1 \) is an integer, are denoted as \( \min\{p_{1}, p_{2}, \ldots, p_{N}\} \) and \( \max\{p_{1}, p_{2}, \ldots, p_{N}\} \), respectively. In addition, \( \otimes \) is the Kronecker product. Unless specified, the constants/parameters defined throughout the article are real numbers.

II. PROBLEM FORMULATION

Problem 1: Consider a game with \( N \) players in which player \( i \)'s action is governed by

\[
\dot{x}_i = v_i, \quad \dot{\bar{u}}_i = u_i \tag{1}
\]

for \( i \in \mathcal{V} \), where \( x_i \in \mathbb{R}, v_i \in \mathbb{R}, \) and \( u_i \in \mathbb{R} \) denote the action, velocity, and control input of the player \( i \), respectively. Moreover, \( \mathcal{V} = \{1, 2, \ldots, N\} \), where \( N > 1 \) is an integer, is the set of players involved in the game. Associate player \( i \) with a cost function \( f_i(x) \), where \( i \in \mathcal{V} \) and \( x = [x_{i1}, x_{i2}, \ldots, x_{iN}]^T \). The objective of this article is to design distributed Nash equilibrium-seeking strategies for the considered game, provided that the players’ velocity measurements are not available.

Let \( \mathbf{x}^* = [\mathbf{x}^*_1, \mathbf{x}^*_2, \ldots, \mathbf{x}^*_i, \ldots, \mathbf{x}^*_N]^T \). Then, the Nash equilibrium \( \mathbf{x}^* = [\mathbf{x}^*_1, \mathbf{x}^*_2, \ldots, \mathbf{x}^*_i, \ldots, \mathbf{x}^*_N]^T \) is achieved if for any initial condition,

\[
\lim_{t \to \infty} \|\mathbf{v}(t) - \mathbf{x}^*\| = 0, \quad \lim_{t \to \infty} \|\mathbf{v}(t)\| = 0 \tag{3}
\]

where \( \mathbf{v} = [v_1, v_2, \ldots, v_N]^T \). Furthermore, if the seeking strategy enables (3) to be satisfied by utilizing only the players’ local information, we say that distributed Nash equilibrium seeking is achieved.

Remark 1: In most of the existing literature, “Nash equilibrium seeking” concerns with the design of action updating strategy that enables the players’ actions to converge to the Nash equilibrium (see, e.g., [4], [7], [9], [11], and many other references). In this article, we further enlarge that the velocities, which are internal states of the players, should converge to zero as well to ensure the stability of the closed-loop system. Different from [8]-[12] and [28] that utilized full-state (including both positions and velocities) feedback in the control law, this article supposes that velocity measurements are not available. Note that the concerned problem is of vital importance as practical experiences have shown that velocity measurements tend to contain noises, which are difficult to be filtered away. Furthermore, many engineering devices (e.g., robots, ships) are not equipped with velocity measurement units and it might be costly to install additional velocity measurement sensors.

Let \( \mathcal{P}(\mathbf{x}) = \left[ \frac{\partial f_i(x)}{\partial x_1}, \frac{\partial f_i(x)}{\partial x_2}, \ldots, \frac{\partial f_i(x)}{\partial x_N} \right] \) and \( H(x) = \left[ \frac{\partial^2 f_i(x)}{\partial x_1 \partial x_2}, \ldots, \frac{\partial^2 f_i(x)}{\partial x_{N-1} \partial x_N} \right] \) for \( i, j \in \mathcal{V} \). The following assumptions will be utilized to develop the main results.

Assumption 1: For each \( i \in \mathcal{V} \), \( f_i(x) \) is twice-continuously differentiable.

Assumption 2: There exists a positive constant \( m \) such that for \( x, y \in \mathbb{R}^N, \)

\[
(x - y)^T (P(x) - P(y)) \geq m||x - y||^2. \tag{4}
\]

Assumption 3: The elements of \( H(x) \) are bounded and there exists a positive constant \( h \) such that \( ||H(x)|| \) is upper bounded by \( h \), i.e., \( \sup_{x \in \mathbb{R}^N} ||H(x)|| = h \).

Remark 2: Assumptions 1-3 are mild for games with second-order integrator-type players in the sense that Assumption 3 can be easily removed with the corresponding results degraded to local/semiglobal ones. By assumption 3, we denote the Lipschitz constant of \( \frac{\partial f_i(x)}{\partial x_j} \) as \( l_i \). Moreover, Assumption 2 is a commonly utilized condition in related existing works (see, e.g., [4], [5], [8]) and is employed to ensure that (1) the Nash equilibrium is unique and satisfies \( P(x^*) = 0_N \); and (2) the Nash equilibrium is globally exponentially stable under the gradient play [1].

In this article, we consider that the players are with partial information about the other players’ actions in the sense that player \( i \) could not directly get \( x_j \) if player \( j \) is not its neighbor. Under this setting, \( \frac{\partial f_i(x)}{\partial x_j} \) is not available for feedback in the control input design as \( x \) is not available for player \( i \). To deal with this situation, we assume that the players are involved in a communication network \( G \), defined as a pair \( (\mathcal{V}, \mathcal{E}) \), where \( \mathcal{V} \) is the node set and \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) is the edge set. The elements of \( \mathcal{E} \) are written as \( (i, j) \), which denotes an edge from node \( i \) to node \( j \). Associate each edge \( (i, j) \) a weight \( a_{ij} \). A directed path is a sequence of nodes \( i_1, i_2, \ldots, i_s \), such that \( (i_1,i_2), (i_2,i_3), \ldots, (i_{s-1},i_s) \in \mathcal{E}, j \in \{1, \ldots, r-1\} \). The directed graph is strongly connected if there is a directed path between any two distinct nodes in the graph. Define \( A = [a_{ij}] \), where \( a_{ij} > 0 \) if \( (i,j) \in \mathcal{E} \) and \( a_{ij} = 0 \), otherwise \( (a_{ij} = 0) \), as the adjacency matrix of the communication graph. Then, \( \mathcal{L} = \mathcal{D} - \mathcal{A} \), where \( \mathcal{D} = \text{diag} \{\sum_{j=1}^{N} a_{ij}\} \) is the Laplacian matrix of \( G \) [26], [27]. In the following, we consider that the following property holds for the communication graph.

Assumption 4: Graph \( G \) is strongly connected.

Lemma 1: Suppose that Assumption 4 holds. Then, for any positive definite diagonal matrix \( k \) of compatible dimension, there exist symmetric positive definite matrices \( \Gamma_1, \Gamma_2 \) such that \( \Gamma_1 k (\mathcal{L} \otimes I_N + A_0) + (\mathcal{L} \otimes I_N + A_0)^T k \Gamma_1 = \Gamma_2 \), where \( A_0 = \text{diag} \{a_{ij}\} \) for \( i, j \in \mathcal{V} \).

Proof: See Appendix A for the proof.

III. MAIN RESULTS

In this section, observer-based seeking strategies and filter-based seeking strategies will be successively established to achieve the goal of the article.

A. Observer-Based Approaches for Distributed Nash Equilibrium Seeking

As the players’ velocities cannot be accessed for feedback in the seeking strategy, it is intuitive that we can design observers to estimate them. Based on this idea, we design the control input of player \( i \) for \( i \in \mathcal{V} \) as

\[
u_i = -k_{i1} \nabla_i f_i(x_i) - k_{i3} \bar{v}_i \tag{5}
\]

where \( \bar{v}_i \) represents player \( i \)'s estimate on its own velocity \( v_i \), \( \nabla_i f_i(x_i) = \frac{\partial f_i(x_i)}{\partial x_i} |_{x=x_i} \), and \( z_i = [z_{i1}, z_{i2}, \ldots, z_{iN}]^T \) is a vector representing player \( i \)'s estimate on \( x \). Moreover, \( \bar{v}_i \) and \( z_{ij} \) are variables updated according to

\[
\dot{\bar{v}}_i = -k_{i2} (\bar{x}_i - x_i) + u_i, \quad \dot{z}_{ij} = -k_{i3} (\bar{x}_i - x_i) + \bar{v}_i \tag{6a}
\]
Assume that Assumptions 1–4 hold and the control is a fixed positive constant. Moreover, $L \otimes \max_{ij} + \rho > 0$ is a vector containing all positive constants defined in (20). (see $\dot{v} + (L \otimes I + A_0)/(L \otimes I + A_0)^{-1})}$]

\begin{equation}
\dot{z}_{ij} = -k_{i4} \left( \sum_{k=1}^{N} a_{ik} (z_{ij} - z_{kj}) + a_{ij} (z_{ij} - x_j) \right)
\tag{6b}
\end{equation}

where $k_{ij}$ for $j \in \{1, 2, 3, 4\}$ are positive constants and for the convenience of stability analysis, define $k_{i4} = \delta k_{i4}$, where $\delta > 0$ is an adjustable parameter and $k_{i4}$ is a fixed positive constant. Moreover, $\bar{x}$ is an auxiliary variable.

Remark 3: It is worth mentioning that in (5) and (6), each player updates its action by utilizing its local information only (e.g., its own information and information from its neighbors). Hence, the method in (5) and (6) is distributed. The main idea of (5) and (6) can be elaborated as follows. As the velocity signal $v_i$ is not accessible, the observer in (6a) is designed to estimate it. As $x$ is a vector containing all players’ actions and it is not directly available for each player, we employ the leader-following consensus algorithm in (6b) for each player to distributively estimate $x$, which is inspired by [1]–[3]. Based on the estimated values, (5) involves a gradient-like term and a velocity-like term for the optimization and stabilization of the players’ objective functions and states, respectively.

In the following, we leave (6a) and focus on the following subsystem, derived by (1), (5), and (6b):

\begin{equation}
\begin{aligned}
\dot{x}_i &= v_i, \quad \dot{v}_i = -k_{i1} \nabla f_i(x_i) - k_{i2} v_i - k_{i1} v_i \\
\dot{z}_{ij} &= -k_{i4} \left( \sum_{k=1}^{N} a_{ik} (z_{ij} - z_{kj}) + a_{ij} (z_{ij} - x_j) \right)
\end{aligned}
\tag{7}
\end{equation}

for $i, j \in \mathcal{V}$, where $\bar{v}_i = v_i - v_i$.

Writing (7) in its vector form, we get that

\begin{equation}
\begin{aligned}
x = \bar{v}, \quad &\dot{\bar{v}} = -k_{i1} [\nabla f_i(x_i)]_{vec} - k_{i2} \bar{v} - k_{i1} \bar{v} \\
\dot{z} &= -\delta k_i (L \otimes I + A_0) (z - 1 \otimes x)
\end{aligned}
\tag{8}
\end{equation}

where $\bar{v} = [\bar{v}]_{vec}$, $k_1 = diag(k_{11})$, $k_4 = diag(k_{44} \otimes I_N)$, and $z = [x_1^T, x_2^T, \ldots, x_N^T]^T$. Note that by Lemma 1, there are symmetric positive definite real matrices $\Gamma_i$ and $\Gamma_2$ such that $\Gamma_i k_i (L \otimes I + A_0) + (L \otimes I + A_0)^T k_i \Gamma_1 = \Gamma_2$.

Define $E = [(v + P(x))^T, (x - x)^T, (z - 1 \otimes x)^T]^T$. Then, regarding $v$ as a virtual control input for (8), it can be proven that (8) is input-to-state stable by appropriately tuning the control gains as given in the following lemma.

**Lemma 2:** Assume that Assumptions 1–4 hold and

\begin{equation}
\begin{aligned}
\min_{\epsilon \in \mathcal{V}} (k_{i1}) &> h + \frac{\sqrt{N} \max_{\epsilon \in \mathcal{V}} \{l_1\} + 1}{2 \epsilon_1} + \sqrt{N} ||\bar{I}_1|| + 1 \\
\delta &> \frac{\sqrt{N} ||\Gamma_1||}{\lambda_{\min}(\Gamma_2)} + \frac{||\bar{I}_1||}{\lambda_{\min}(\Gamma_2)} + \frac{(\max_{\epsilon \in \mathcal{V}} \{l_1\})^2}{2 \lambda_{\min}(\Gamma_2)}
\end{aligned}
\tag{9}
\end{equation}

Proof: See Appendix B.

We are now ready to provide the stability property of the Nash equilibrium under the proposed method in (5)–(6).

**Theorem 1:** Assume that Assumptions 1–4 hold and the control gains $k_{i1}$ for $i \in \mathcal{V}$ and $\delta$ satisfy the tuning rules in (9). Then, the distributed Nash equilibrium seeking is achieved by utilizing (5) and (6).

Proof: See Appendix C.

The strategy developed in (5) and (6) follows [1]–[3] to distributively estimate $x$ by using ideas from leader-following consensus. Alternatively, approaches for distributed estimation of $x$ are available in [22] and [23] by consensus-tracking algorithms. Actually, the idea therein can also be adapted to develop seeking strategies for games considered in this work, and this can be done by simply replacing (6b) with (10c) given later, which results in the seeking strategy as

\begin{equation}
\begin{aligned}
u_i &= -k_{i1} \nabla f_i(z_i) - k_{i1} \bar{v}_i \\
\dot{v}_i &= -k_{i2} (\bar{x}_i - x_i) + v_i \\
\dot{z}_{ij} &= \frac{1}{\sum_{k=1}^{N} a_{ik} + a_{ij}} \left( \sum_{k=1}^{N} a_{ik} (z_{ij} - z_{kj}) \right) + \frac{a_{ij}}{\sum_{k=1}^{N} a_{ik} + a_{ij}} (\bar{v}_i - k_{i4} (z_{ij} - x_j))
\end{aligned}
\tag{10a-c}
\end{equation}

for $i, j \in \mathcal{V}$, where $k_{i1}, k_{i2}, k_{i3}, k_{i4}$, and others variables follow the definitions in (5) and (6).

**Remark 4:** Note that $x$, which is required in [22] and [23], cannot be utilized in the consensus tracking subsystem in (10c) as $x = v$ is considered to be unmeasurable in this article. Hence, $\bar{v}$ is utilized as a replacement, which would result in some observation error for the $x$-estimation subsystem (10c).

The following theorem gives the stability result for (10).

**Theorem 2:** Assume that Assumptions 1–4 hold. Then, the distributed Nash equilibrium seeking is achieved by (10) if

\begin{equation}
\lambda_{\min}(k_{i1}) > h + \frac{1}{2 \epsilon_1} + \frac{h \sqrt{N} \max_{\epsilon \in \mathcal{V}} \{l_1\}}{2 \epsilon_2} \\
\delta > \frac{\sqrt{N} \max_{\epsilon \in \mathcal{V}} \{l_1\}}{2 \epsilon_1} \left( \frac{\max_{\epsilon \in \mathcal{V}} \{l_1\}}{2 \epsilon_2} \right) < m
\tag{11}
\end{equation}

where $\epsilon_1 > 0, \epsilon_2 > 0$ are constants that satisfy $\frac{1}{\epsilon_1} + \frac{h \sqrt{N} \max_{\epsilon \in \mathcal{V}} \{l_1\}}{2 \epsilon_2} < m$.

Proof: See Appendix D.

**Remark 5:** For the observer-based approaches designed in [5], (6), and (10), each player $i$ only needs to tune two parameters, i.e., $k_{i1}$ and $k_{i4}$. To be more specific, each player $i$ should first choose $k_{i1}$ to be sufficiently large and then, for fixed $k_{i1}$, choose $k_{i4}$ to be sufficiently large to obtain the convergence results. Roughly speaking, the tuning of the control gains can be explained as follows. To dominate the action and velocity estimation errors, $k_{i1}$ should not be too small. As the estimated actions are utilized in the optimization part, $k_{i4}$ is required to be sufficiently large to ensure that the action estimation module is faster than the optimization module. Note that $k_{i2}$ and $k_{i3}$ can be chosen as any fixed positive constants, and hence, they do not bring any burden on parameter tuning. Moreover, to further reduce the number of parameters to be tuned for each player, one can adaptively adjust $k_{i1}$ or $k_{i4}$ (see similar ideas in [11] and [25]). If this is the case, each player only needs to tune one parameter. However, it should be mentioned that with adaptive $k_{i1}$ or $k_{i4}$, system complexity is increased to some extent, which indicates that there is a trade-off between the number of parameters to be tuned and system complexity. Note that this article focuses on utilizing velocity measurements, the adaptive parameter designs are not discussed in details to avoid any distraction of the readers’ attention. Interested readers are referred to [11], [25], and the references therein for more insights on adaptive designs.

In this section, the seeking strategy is designed by constructing a state observer given in (6a). It should be noted that the observer is of the same order as the players’ dynamics in (1). An intuitive question is whether it is possible to design reduced-order strategies, which would relax the computation costs, to achieve distributed Nash equilibrium seeking or not. In the following section, we provide another strategy design to answer this question.
B. Filter-Based Distributed Nash Equilibrium-Seeking Strategies

By utilizing ideas from filters, the control input of player $i$ for $i \in \mathcal{V}$ can be designed as

$$u_i = -k_{i1} \nabla_i f_i(z_i) - k_{i1} y_i$$  \hspace{1cm} (12a)

$$y_i = -\dot{x}_i + k_{i2} x_i, \hspace{0.5cm} \dot{\tilde{x}}_i = -k_{i2} \dot{x}_i + k_{i2}^2 x_i$$  \hspace{1cm} (12b)

$$\dot{z}_{ij} = -k_{i3} \left( \sum_{k=1}^{N} a_{ik} (z_{ij} - z_{kj}) + a_{ij} (z_{ij} - x_j) \right)$$  \hspace{1cm} (12c)

where $k_{i3} = \delta \tilde{k}_{i3}$, $\delta$ is a positive constant and $\tilde{k}_{i3}$ is a fixed positive constant. Moreover, $k_{i1, i2}$ are positive constants and $\dot{x}_i$ is an auxiliary variable.

Remark 6: In the control input design (12a), the gradient term is included for the optimization of the players' objective functions. Moreover, $y_i$ serves as an estimate of the velocity of player $i$ and is included to stabilize the system. To provide more insights on how $y_i$ is generated, we can conduct the Laplace transformation for (12b) and get that

$$Y_i(s) = -\tilde{X}_i(s) + k_{i2} X_i(s)$$  \hspace{1cm} (13)

where $s$ is the complex frequency variable and $\tilde{X}_i(s), X_i(s),$ and $Y_i(s)$ are the signals associated with $\dot{x}_i(t), x_i(t),$ and $y_i(t)$ in the complex frequency domain, respectively. Then, by (13),

$$Y_i(s) = \frac{s k_{i2}}{s + k_{i2}} X_i(s) - \frac{1}{s + k_{i2}} \dot{x}_i(0)$$  \hspace{1cm} (14)

where $\frac{1}{s + k_{i2}}$ is a high-pass filter with cut-off frequency $k_{i2}$. This explains the generation of $y_i(t)$ and why we term the method in (12) as a filter-based seeking strategy.

Let $k_1 = \text{diag} \{k_{i1}\}, k_2 = \text{diag} \{k_{i2}\},$ and $k_3 = \text{diag} \{k_{i3} \otimes I_N\}$. Then, by Lemma 1, there are symmetric positive definite matrices $\tilde{\Gamma}_1, \tilde{\Gamma}_2$ that satisfy $\tilde{\Gamma}_1 k_3 (L \otimes I_N + A_0) + (L \otimes I_N + A_0)^T k_3 \tilde{\Gamma}_1 = \tilde{\Gamma}_2$. The following theorem establishes the stability of the Nash equilibrium under (12).

**Theorem 3:** Assume that Assumptions 1–4 hold and $\lambda_{\text{min}}(k_1) > (h \sqrt{N} \max_{i \in \mathcal{V}} \{l_{i1}\} + 1)^2 / (4m) + h$, $\lambda_{\text{min}}(k_2) > \lambda_{\text{max}}(k_1)$

$$\delta > \frac{(\max_{i \in \mathcal{V}} \{k_{i3} l_i\} + \sqrt{\|\tilde{\Gamma}_1\| + \|\tilde{\Gamma}_1\| N \max_{i \in \mathcal{V}} \{l_i\}})^2}{\min \{\lambda_{\text{min}}(A), \lambda_{\text{min}}(k_2) - \lambda_{\text{max}}(k_1)\} \lambda_{\text{min}}(\tilde{\Gamma}_2)}$$  \hspace{1cm} (15)

where $A = \left[ \begin{array}{cc} m & -h \sqrt{N} \max_{i \in \mathcal{V}} \{l_{i1}\} + 1 \\ -h \sqrt{N} \max_{i \in \mathcal{V}} \{l_{i1}\} + 1 & \lambda_{\text{min}}(k_1) - h \end{array} \right]$. Then, the distributed Nash equilibrium seeking is achieved by utilizing the proposed method in (12).

**Proof:** See Appendix E.\hspace{1cm} $\blacksquare$

Similar to Section III-A, by adapting ideas from consensus tracking, the filter-based strategy can also be designed as

$$\dot{z}_i = v_i, \hspace{0.5cm} \dot{v}_i = -k_{i1} \nabla_i f_i(z_i) - k_{i1} y_i$$  \hspace{1cm} (16a)

$$y_i = -\dot{x}_i + k_{i2} x_i, \hspace{0.5cm} \dot{\tilde{x}}_i = -k_{i2} \dot{x}_i + k_{i2}^2 x_i$$  \hspace{1cm} (16b)

$$\dot{z}_{ij} = \sum_{k=1}^{N} a_{ik} (z_{ij} - k_{i3} (z_{ij} - z_{kj})) + a_{ij} (y_j - k_{i3} (z_{ij} - x_j))$$  \hspace{1cm} (16c)

for $i, j \in \mathcal{V}$, where the definitions of $k_{i3}$ for $j \in \{1, 2, 3\}$ and the variables follow those in (12).

Then, the following theorem can be obtained.

**Theorem 4:** Assume that Assumptions 1–4 hold and

$$\lambda_{\text{min}}(k_1) > 1/2 + h + (1 + h \sqrt{N} \max_{i \in \mathcal{V}} \{l_{i1}\}) / (2c_1)$$

$$\lambda_{\text{min}}(k_2) > \max_{i \in \mathcal{V}} \{k_{i3} l_i\} (\|L \otimes I_N + A_0\|^{-1})^2 / 2$$

$$+ \lambda_{\text{max}}(k_1) + \sqrt{\|A_0\|} / 2$$

$$\delta > \frac{(\max_{i \in \mathcal{V}} \{k_{i3} l_i\} (\|L \otimes I_N + A_0\|^{-1}))^2}{2 \lambda_{\text{min}}(k_3)} + \sqrt{\|A_0\| + \max_{i \in \mathcal{V}} \{k_{i3} l_i\} (\|L \otimes I_N + A_0\|^{-1})}$$

where $\epsilon < 1 + h \sqrt{N} \max_{i \in \mathcal{V}} \{l_{i1}\} / (2c_1)$ is a positive constant. Then, the distributed Nash equilibrium seeking is achieved by utilizing (16).

**Proof:** See Appendix F.\hspace{1cm} $\blacksquare$

Remark 7: Note that the proposed methods can be separated into three parts: an optimization and velocity regulation part, a velocity estimation part, and an action estimation part. Hence, the Lyapunov candidate functions should contain an optimization error term, a velocity regulation error term, a velocity estimation error term, and an action estimation error term. Please see the proofs of Theorems 3 and 4 for the details.

Remark 8: Note that the order of the filter subsystem in (12b) is less than that of the observation subsystem in (6a), indicating that filter-based approaches require less computation cost than the observer-based approaches. However, it should be mentioned that each player $i$ needs to tune three parameters ($k_{i1}, k_{i2}, k_{i3}$) for the filter-based algorithm (see Theorems 3 and 4) while the observer-based approaches only require the tuning of two parameters (see Theorems 1 and 2). Note that in practice, each player $i$ can first choose $k_{i1}$ to be sufficiently large and then for fixed $k_{i1}$, choose $k_{i2}$, and $k_{i3}$ to be sufficiently large (such that the action and velocity estimation modules are faster than the optimization module) to ensure the convergence of the proposed filter-based algorithms. Moreover, similar to the observer-based approaches, one can also adaptively adjust $k_{i3}$ to further reduce the number of parameters to be tuned by each player. With adaptive $k_{i3}$, each player $i$ only needs to tune two parameters for the filter-based approaches.

**Remark 9:** In this article, the explicit quantification of the control gains depends on some global information (e.g., the network topology, the players’ objective functions, and the strong monotonicity constant), which is a common characteristic in most of the existing works [25]. It is of great potential that the ideas from adaptive control (see, e.g., [6], [11], [25] and the references therein) can be adapted to achieve fully distributed Nash equilibrium seeking. However, it is still challenging to analytically investigate it as there are multiple correlated control gains in the developed algorithms and we leave it as a future work.

**Remark 10:** Unlike [1]–[3], and [22] that consider first-order players, this article considers that the players are second-order integrators. With the players’ inherent dynamics involved, the Nash equilibrium-seeking algorithm should not only drive the players’ positions to the Nash equilibrium but also steer their velocities to zero. This indicates that the stabilization of the players’ dynamics and optimization of the players’ cost functions should be achieved simultaneously. In particular, the stabilization of the players’ dynamics usually requires the feedback of the players’ velocities, which are difficult to be accurately measured in practice. Hence, this article designs the distributed algorithms without utilizing velocity measurement, which makes the problem more complex. Though in this article, the communication graph is supposed to be fixed, it is of great potential that the developed methods can be adapted to deal with more general communication conditions (see, e.g., [2]). In addition, this article serves as an appetizer to address games in which the players’ internal states are not accessible. Extensions to games with more complicated dynamics in which the internal states are not available for feedback control are also interesting open questions to be further explored.
IV. NUMERICAL EXAMPLE

In this section, the connectivity control game among networked acceleration-actuated mobile sensors considered in [3], [8], and [24] is simulated. More specifically, we consider a game with ten players in which the objective function of player $i$ is given as

$$f_i(x) = x_i^T \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} x_i + [20\ 40]x_i + i + ||x_i - x_{i+1}||^2$$

for $i \in \{1, 2, \ldots, 9\}$ and

$$f_{10}(x) = x_{10}^T \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} x_{10} + [20 \ 40]x_{10} + 10 + ||x_{10} - x_1||^2$$

where $x_i = [x_{i1}, x_{i2}]^T$ denotes the position of sensor $i$. Through direct calculation, it is derived that the unique Nash equilibrium $x^* = [-7.1828, -14.3656, -4.3656, -8.7313, -3.0969, -6.1938, -2.3876, -4.7753, -1.9382, -1.6293, -3.2586, -1.4049, -2.8099, -1.2396, -2.4792, -1.1562, -2.3124, -1.5621, -3.1241]^T$.

In the following, we will simulate the observer-based approach in (5)–(6) and the filter-based approach in (12), successively. In the simulations, the directed communication topology is given in Fig. 1 and all the variables in the proposed methods are initialized at $-5$.

A. The Observer-Based Approach for Distributed Nash Equilibrium Seeking

This section provides simulation results for the method in (5)–(6). In the simulation, $k_{11} = 5, k_{12} = 5, k_{13} = 10$, and $k_{14} = 40$ for all $i \in \{1, 2, \ldots, 10\}$. Generated by (5)–(6), the simulation results are given in Figs. 2 and 3, which plot the players’ positions and velocities, respectively. From the figures, it can be seen that the Nash equilibrium seeking can be achieved by utilizing the observer-based method in (5)–(6) in a distributed fashion. Hence, Theorem 1 is verified.

B. The Filter-Based Approach for Distributed Nash Equilibrium Seeking

This section provides numerical verification for the distributed method in (12). In the numerical study, $k_{11} = 10, k_{12} = 20$, and $k_{14} = 40$. The simulation results generated by (12) are shown in Figs. 4 and 5, which illustrate the players’ positions and velocities, respectively. From the figures, it is clear that the Nash equilibrium seeking is achieved in a distributed fashion by utilizing the method in (12), thus verifying Theorem 3.

Note that in the filter-based approach (12), the players need to generate $\hat{x}_i, i \in \{1, 2, \ldots, 10\}$ for velocity estimation while in the observer-based approach (5)–(6), the players need to generate $\bar{v}_i$ and $\bar{x}_i, i \in \{1, 2, \ldots, 10\}$ for velocity estimation. Hence, in the example, the filter-based approach reduces ten auxiliary variables compared with the observer-based approach.
Players’ velocities generated by (12).

V. CONCLUSION

This article develops Nash equilibrium strategies for games in which the players’ actions are governed by second-order integrator-type dynamics. In particular, the players’ velocities are supposed to be unavailable for feedback control of the players’ positions. Without utilizing velocity measurement, observer-based approaches and filter-based approaches are designed. Through Lyapunov stability analysis, it is theoretically shown that the players’ positions and velocities would be steered to the Nash equilibrium and zero, respectively.

APPENDIX A

A. Proof of Lemma 1

As the communication graph is strongly connected, one can obtain that $k(L \otimes I_N + A_0)$ is a nonsingular $M$-matrix by [26, Lemma 2.4]. Therefore, there exist symmetric positive definite matrices $\Gamma_1$ and $\Gamma_2$ such that $\Gamma_1 k(L \otimes I_N + A_0) + (L \otimes I_N + A_0)^T k \Gamma_1 = \Gamma_2$ by [26, Th. 2.9].

B. Proof of Lemma 2

Let
\[
V = \frac{1}{2} (v + P(x))^T (v + P(x)) + \frac{1}{2} (x - x^*)^T (x - x^*) + (z - 1_N \otimes x)^T \Gamma_1 (z - 1_N \otimes x).
\]

Then, the time derivative of $V$ along (8) is
\[
\dot{V} = (v + P(x))^T (-k_1 P(x) - k_1 v + H(x)v) + (x - x^*)^T v - \delta (z - 1_N \otimes x)^T \Gamma_2 (z - 1_N \otimes x) - 2(z - 1_N \otimes x)^T \Gamma_1 (1_N \otimes v) + (v + P(x))^T k_1 (P(x) + \nabla_i f_i(x_i))_{\text{loc.}}.
\]

Therefore,
\[
\dot{V} \leq -\rho_1 ||v + P(x)||^2 - \rho_2 ||x - x^*||^2 - \rho_3 ||z - 1_N \otimes x||^2 + \frac{||k_1||^2}{2} ||v||^2 + \rho_1 (\max_{i \in V} \{l_i\}) \min \{\Gamma_1\} \min \{\Gamma_2\} \leq \frac{\rho_1}{2} (||v||^2 + ||x - x^*||^2 - \rho_2 ||x - x^*||^2 - \rho_3 ||z - 1_N \otimes x||^2)
\]

where
\[
\rho_1 = \min_{i \in V} \left\{ k_{i1} \right\} - \frac{h \sqrt{N} \max_{i \in V} \{l_i\} + 1}{2 \epsilon_1},
\]
\[
\rho_2 = m - \frac{h \sqrt{N} \max_{i \in V} \{l_i\} + 1}{2 \epsilon_1} - N \max_{i \in V} \{l_i\} \epsilon_2 ||\Gamma_1||
\]
\[
\rho_3 = \delta \min \{\Gamma_2\} - \sqrt{N} ||\Gamma_1|| - \sqrt{N} ||\Gamma_1|| N \max_{i \in V} \{l_i\} ||z - 1_N \otimes x|| - (\max_{i \in V} \{k_{i1}\})^2 / 2
\]

and $\epsilon_1$ and $\epsilon_2$ are positive constants. Choose $\epsilon_1$ and $\epsilon_2$ to be sufficiently small such that $\rho_2 > 0$. Then, for fixed $\epsilon_1$, choose $k_{i1}$ and $\delta$ according to (9). By the above-mentioned tuning rule, we get that $\rho_1 > 0, \rho_2 > 0, \rho_3 > 0$, and
\[
\dot{V} \leq - \frac{\min \{\rho_1, \rho_2, \rho_3\} ||E||^2}{2}
\]

for $||E|| \geq \sqrt{\frac{1}{2} \min \{\rho_1, \rho_2, \rho_3\} ||k_1|| ||v||}$. Hence, by [21, Th. 4.19], there exists a $KL$ function $\beta$ such that $||E(t)|| \leq \beta(||E(0)||, t) + \sqrt{\frac{\max \{\rho_1, \rho_2, \rho_3\} ||k_1|| ||v||}$, thus arriving at the conclusion.

C. Proof of Theorem 1

Defining $\bar{x}_i = \bar{x}_i - x_i$, we get that
\[
\dot{\bar{x}}_i = \dot{\bar{x}}_i - \dot{\bar{x}}_i = -k_{i3} \tilde{x}_i + \bar{v}_i - \bar{v}_i = -k_{i3} \bar{x}_i + \bar{v}_i
\]
\[
\dot{\bar{v}}_i = \dot{\bar{v}}_i - \dot{\bar{v}}_i = -k_{i2} (\bar{x}_i - x_i) = -k_{i2} \bar{x}_i,
\]

Let $\xi_i = [\bar{x}_i, \bar{v}_i]^T$ and define the Lyapunov candidate function as
\[
V_i = \sum_{i=1}^{N} \xi_i^T P_i \xi_i
\]

where $P_i$ is a symmetric positive definite matrix such that
\[
P_i = P_i \left[ \begin{array}{cc} -k_{i3} 1 & -k_{i3} \bar{x}_i \\ -k_{i2} 0 & -k_{i2} 0 \end{array} \right] = -Q_i, \quad Q_i = \text{a symmetric positive definite matrix.}
\]

Note that the existence of $P_i, Q_i$ can be concluded by noticing that $-k_{i3} \bar{x}_i$ is Hurwitz. Then, it can be easily obtained that
\[
\dot{V}_i = -\sum_{i=1}^{N} \lambda_{\min} (Q_i) ||\xi_i||^2
\]

from which it is clear that $\lim_{t \to \infty} ||\xi(t)|| = 0$, where $\xi = [\xi_1^T, \xi_2^T, \ldots, \xi_N^T]^T$.
D. Proof of Theorem 2

Multiplying both sides of (10c) by \(\sum_{k=1}^{N} a_{ik} + a_{ij}\) gives
\[
\sum_{k=1}^{N} \text{vec}(\varepsilon_{ij}) = \sum_{k=1}^{N} a_{ik} \varepsilon_{ij} + a_{ij} \varepsilon_{ij},
\]
whose concatenated vector form is
\[
(\mathcal{L} \otimes I_N + A_0)(z - 1_N \otimes x) = -\delta k_1(\mathcal{L} \otimes I_N + A_0)(z - 1_N \otimes x) + [a_{ij}]_{vec}.
\]
Define \(\Phi = (\mathcal{L} \otimes I_N + A_0)(z - 1_N \otimes x)\). Then, \(\dot{\Phi} = -\delta k_1 \Phi + [a_{ij}]_{vec}\). As (6a) and (10b) are the same and in the proof of Theorem 1, it has been shown that \(\lim_{t \to \infty} ||\varepsilon(t)|| = 0\), indicating that \(\lim_{t \to \infty} ||\varepsilon(t)|| = 0\). Hence, following the proof of Theorem 1, it is clear that it is sufficient to show that
\[
\dot{v} = v, \quad \dot{v} = -k_1 [\nabla f_i(z_i)]_{vec} - k_1 v - k_1 \dot{v}
\]
(25a)
\[
\dot{\Phi} = -\delta k_1 \Phi + [a_{ij}]_{vec}
\]
(25b)
is input-to-state stable by treating \(\dot{v}\) as the virtual control input. To show this, define
\[
V = \frac{1}{2} \Phi^T \Phi + \frac{1}{2} (v + P(x))^T (v + P(x)) + \frac{1}{2} (x - x^*)^T (x - x^*)
\]
Then,
\[
V \leq -\delta \max_{i \in V} (k_1) ||v||^2 + \sqrt{\lambda_{min}(A_0)} ||\Phi||^2 ||\dot{v}||^2 + ||k_1|| (v + P)(x)||v|| + ||x - x^*||^2 + h \lambda_{min}(k_1) ||v + P(x)||^2 + ||x - x^*||^2 + h \lambda_{min}(k_1) ||v + P(x)||^2 + ||x - x^*||^2
\]
As the communication graph is strongly connected, \(\mathcal{L} \otimes I_N + A_0\) is nonsingular by [26, Lemma 2.4]. Recalling the definition of \(\Phi\), it can be obtained that
\[
||z - 1_N \otimes x|| \leq \sqrt{||\mathcal{L} \otimes I_N + A_0||} \leq \sqrt{\lambda_{max}(A_0)} ||\Phi||^2
\]
Hence, by defining \(\rho_1 = \delta \max_{i \in V} (k_1) - \frac{\lambda_{max}(A_0)}{2} \), \(\rho_2 = \frac{m}{2} - \frac{h \lambda_{max}(k_1)}{2} \), \(\rho_3 = \min_{i \in V} (k_1) - \frac{h \lambda_{max}(k_1)}{2} \), choose \(\epsilon_i\) for \(i \in \{1, 2\}\) as positive constants, then,
\[
V \leq -\min_{i \in V} (\rho_1, \rho_2, \rho_3) ||v||^2 + \frac{1}{2} (v + P(x))^T (v + P(x)) + \frac{1}{2} (x - x^*)^T (x - x^*)
\]
(27)
By choosing \(\epsilon_1\) and \(\epsilon_2\) to be sufficiently small, \(\rho_2 > 0\) can be satisfied. Then, for fixed \(\epsilon_1, \epsilon_2\), choose \(k_1\) and \(\delta\) according to (11). By such a tuning rule, it can be obtained that there exists a class \(\mathcal{K}\) function \(\beta\) such that
\[
||E(t)|| \leq \beta(||E(0)||, t) + \frac{h \lambda_{max}(k_1)}{2} ||v||^2
\]
The rest of the argument follows the proof of Theorem 1 and is omitted due to space limitations.

E. Proof of Theorem 3

From (12b), we can obtain that the concatenated vector form of the closed-loop system can be written as
\[
\dot{x} = v, \quad \dot{v} = -k_1 [\nabla f_i(z_i)]_{vec} - k_1 y
\]
(28)
where \(y = [y_1, y_2, \ldots, y_N]^T\). Define \(\dot{y} = y - v\). Then,
\[
\dot{x} = v, \quad \dot{v} = -k_1 [\nabla f_i(z_i)]_{vec} - k_1 v - k_1 \dot{y}
\]
\[
\dot{y} = -k_2 \dot{y} - (k_2 - k_1) [\nabla f_i(z_i)]_{vec} - k_1 v - k_1 \dot{y}
\]
\[
\dot{z} = -\delta k_3 (\mathcal{L} \otimes I_N + A_0)(z - 1_N \otimes x)
\]
(29)
To establish the stability property for (29), one can define the Lyapunov candidate function as
\[
V_1 = V_1 + V_2 + V_3 + V_4, \quad V_1 = \frac{1}{2} (v + P(x))^T (v + P(x))
\]
\[
V_2 = \frac{1}{2} (x - x^*)^T (x - x^*)
\]
\[
V_3 = \frac{1}{2} y^T y
\]
\[
V_4 = (z - 1_N \otimes x)^T \Gamma_1 (z - 1_N \otimes x)
\]
(30)
Then, by similar analysis as that of Lemma 2, we get that
\[
\dot{V}_1 \leq -\lambda_{min}(k_1) ||v + P(x)||^2 - (v + P(x))^T k_1 y
\]
\[
+ \max_{i \in V} (k_1) ||v + P(x)|| ||z - 1_N \otimes x||
\]
\[
+ h ||v + P(x)||^2 + h \lambda_{max}(k_1) ||v + P(x)|| ||x - x^*||
\]
(31)
and
\[
\dot{V}_2 \leq -m ||x - x^*||^2 + ||x - x^*|| ||v + P(x)||
\]
(32)
Moreover,
\[
\dot{V}_3 \leq -\lambda_{min}(k_2) ||y||^2 + \frac{1}{2} \lambda_{max}(k_1) ||v + P(x)|| ||z - 1_N \otimes x||
\]
(33)
Furthermore,
\[
\dot{V}_4 \leq -\min_{i \in V} (\lambda_{min}(k_1)) ||z - 1_N \otimes x||^2
\]
\[
+ 2 h \lambda_{max}(k_1) ||z - 1_N \otimes x|| ||v + P(x)||
\]
\[
+ 2 N \lambda_{min}(k_1) ||z - 1_N \otimes x|| ||x - x^*||
\]
(34)
Hence,
\[
\dot{V} \leq -\lambda_{min}(k_1) - h ||v + P(x)||^2 - m ||x - x^*||^2
\]
\[
- \lambda_{min}(k_2) \lambda_{min}(k_1) ||y||^2 - \delta \lambda_{min}(k_1) ||z - 1_N \otimes x||^2
\]
\[
+ \max_{i \in V} (k_1) ||v + P(x)|| ||z - 1_N \otimes x||
\]
\[
+ \lambda_{min}(k_1) ||y|| ||z - 1_N \otimes x||
\]
(35)
Then, A defined in Theorem 3 is symmetric positive definite by choosing \(\lambda_{min}(k_1) > \frac{h \lambda_{max}(k_1)}{2} ||v||^2\). If this is the case,
\[
V \leq -\min_{i \in V} (\lambda_{min}(k_1)) ||E_i||^2 - (\lambda_{min}(k_2) - \lambda_{max}(k_1)) ||y||^2
\]
\[
- \delta \lambda_{min}(k_1) ||z - 1_N \otimes x||^2 + \max_{i \in V} (k_1) ||y|| ||z - 1_N \otimes x||
\]
\[
+ \max_{i \in V} (k_1) ||z - 1_N \otimes x|| ||x - x^*||
\]
\[
+ 2 N \lambda_{min}(k_1) ||z - 1_N \otimes x|| ||x - x^*||
\]
where \(E_i = (x - x^*)^T (v + P(x))^T\).

Choose \(\lambda_{min}(k_2) > \lambda_{max}(k_1)\). Then,
\[
V \leq -\min_{i \in V} (\lambda_{min}(A), \lambda_{min}(k_1) - \lambda_{max}(k_1)) ||E_i||^2
\]
\[
- \delta \lambda_{min}(k_1) ||z - 1_N \otimes x||^2 + \max_{i \in V} (k_1) ||y|| ||z - 1_N \otimes x||
\]
\[
+ \max_{i \in V} (k_1) ||z - 1_N \otimes x|| ||x - x^*||
\]
\[
+ 2 N \lambda_{min}(k_1) ||z - 1_N \otimes x|| ||x - x^*||
\]

where $E_2 = [E_2^T, \tilde{y}^T]^T$. Hence, by choosing $\delta > \frac{\min(\lambda_{\text{min}}(A), \lambda_{\text{max}}(k_2) - \lambda_{\text{max}}(k_1))]^{1/2}}{\min(\max_{i \in \mathcal{V}}(k_{1i}), 1)}$, we get that

$$V \leq -\lambda_{\text{min}}(A_1)[|E_2|^2]$$

(36)

where $\lambda_{\text{min}}(A_1) > 0$, $E_3 = [E_3^T, \mathbf{1} - (1_N \otimes \mathbf{x})^T]^T$ and $A_1 = [\chi, \delta \lambda_{\text{min}}(\tilde{G}_1)]$, where $\chi = \min(\lambda_{\text{min}}(A_1), \lambda_{\text{max}}(k_2) - \lambda_{\text{max}}(k_1))$, and $\chi_1 = \min(\max_{i \in \mathcal{V}}(k_{1i}), 1) + \sqrt{\lambda_{\text{max}}(k_2)}$. Hence, $||E_3|| \to 0$ as $t \to \infty$. As $\mathbf{x}(t) \to \mathbf{x}^*$ for $t \to \infty$, we obtain that $||P(x)|| \to 0$ as $t \to \infty$ by Assumption 2. Hence, $||\mathbf{v}(t)|| \to 0$ as $t \to \infty$, which further indicates that $||y(t)|| \to 0$ as $t \to \infty$. To this end, the conclusion has been obtained.

**F. Proof of Theorem 4**

Define

$$V = \frac{1}{2}(|v + P(x)|^2 + ||v + P(x)||^2 + \frac{1}{2}y^T\Phi)$$

where $\Phi$ is defined in the proof of Theorem 2. Then, following the proof of Theorem 3,

$$\dot{V} \leq -\rho_1||v + P(x)||^2 - \rho_2||x - x^*||^2 - \rho_3||y||^2 - \rho_4||\Phi||^2$$

(37)

where $\rho_1 = \frac{\rho_{\text{max}}(k_1) - \frac{1}{2}h}{1 + \lambda_{\text{max}}(k_2)}$, $\rho_2 = \frac{h \sqrt{\lambda_{\text{max}}(k_2)}}{\lambda_{\text{max}}(k_2)}$, $\rho_3 = \lambda_{\text{max}}(k_2) - \lambda_{\text{max}}(k_1)$, and $\rho_4 = \frac{\delta \lambda_{\text{min}}(k_2)}{\lambda_{\text{max}}(k_2)} - \frac{\max_{i \in \mathcal{V}}(k_{1i})}{\lambda_{\text{max}}(k_2)}$. Then, for fixed $\epsilon_1$, choose $k_{11}, k_{12}, \dot{\epsilon}$ and $\delta$ according to (4) such that $\rho_1 > 0, \rho_2 > 0, \rho_3 > 0$. By such a tuning rule

$$\dot{V} \leq -\rho_1||v + P(x)||^2 - \rho_2||x - x^*||^2 - \rho_3||y||^2 - \rho_4||\Phi||^2$$

(38)

where $\dot{E} = [(v + P(x))^T, (x - x^*)^T, \tilde{y}^T, \Phi^T]^T$. To this end, the conclusion can be easily drawn.

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