Relation between two measures of entanglement in spin-1/2 and spinless fermion quantum chain systems

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The concepts of concurrence and mode concurrence are the measures of entanglement for spin-1/2 and spinless fermion systems respectively. Based on the Jordan-Wigner transformation, any spin-1/2 system is always associated with a fermion system (called counterpart system). The comparison of concurrence and mode concurrence can be made with the aid of the Marshall’s sign rule for the ground states of spin-1/2 XXZ and spinless fermion chain systems. We observe that there exists an inequality between concurrence and mode concurrence for the ground states of the two corresponding systems. The spin-1/2 XY chain system and its spinless fermion counterpart as a realistic example is discussed to demonstrate the analytical results.

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I. INTRODUCTION

Quantum entanglement is one of the most intriguing features of quantum mechanics for many body systems such as spin, fermion, boson systems etc. It plays a fundamental role in quantum information processing (QIP) schemes and quantum computing, thus is regarded as an important resource for these technologies [1]. It also has been demonstrated that the quantum entanglement can be used to realize quantum teleportation [2, 3, 4, 5] as well as to characterize the quantum critical phenomena in strongly correlated systems [6, 7, 8, 9, 10].

In characterizing the pairwise entanglement in spin-1/2 systems, Wootters proposed a definition of entanglement measure named concurrence [11], which is easy to handle analytically and well accepted by the quantum community. This definition, however, relies on the tensor product structure of the state space with respect to a composite quantum system, which, due to quantum statistics, does not appear obviously for systems of indistinguishable particles, such as fermions and bosons. As a result, many recent efforts are devoted to understand the quantum entanglement of indistinguishable particle systems [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. Especially, a measure of mode entanglement was proposed by Zanardi and Wang in grand canonical ensembles based on the isomorphism between the full Fock space and qubit space [23]. This definition, however, relies on the tensor product structure of the state space with respect to different single particle vector basis or the normal modes, simply called modes, as well as to characterize the quantum critical phenomena in strongly correlated systems [6, 7, 8, 9, 10].

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Although the two measures of entanglement are defined in different quantum systems (spin-1/2 and spinless fermion systems), they are closely related to each other based on the Jordan-Wigner transformation [31]. With this transformation, spin-1/2 operators can be represented by fermion operators. Correspondingly, any spin model has a spinless fermion model as a counterpart, i.e., if we span the Hilbert and Fock space represented by fermion operators. For the BCS model [24], the spinless fermion counterpart is the superfluid state and the spin-1/2 counterpart is the superconductor state. In this way, the two Hamiltonians will share the same matrix form with each other in their corresponding subspaces. Thus the two Hamiltonians have the common vectors as eigen states corresponding to their own basis. In this way, the two systems together with the Jordan-Wigner transformation have provided us a perfect platform to investigate the differences as well as connections between the two different definitions of measure of entanglement so as to lead to some new insights in understanding the nature of spin and fermion systems and also the nature of entanglement [18, 23].

In this paper we will explore the relation between the two measures for quantum chain systems. The key to these observations is the Jordan-Wigner mapping of spins into lattice fermions. We will show that concurrence C...
of a spin-1/2 XXZ chain state is different in magnitude from the mode concurrence (MC) \(23\) of the spinless fermion counterpart state. This difference between the two measures is directly related to, and thus to some extend, is a readout of, the commutation relations of spins and fermions, which are the fundamental features of the two distinct models. Our result is in agreement with the statement in Ref. \(23\) that the entanglement is related to the single-particle basis chosen. Furthermore, with the aid of Marshall’s sign rule \(32, 33\), we reveal that there exists a simple relation between the C and MC for the ground state of the spin-1/2 XXZ chain model and its counterpart, spinless fermion model. The detailed relations depend on different types of pairwise entanglement we concern. We find that for the ground state of a spin-1/2 XXZ chain system and its corresponding ground state of a many-particle spinless fermion system, (i) the C between nearest neighbor (NN) sites in spin-1/2 XXZ chain systems is identical in magnitude to the MC in the corresponding spinless fermion systems, (ii) C is no less than MC between any two non-NN sites. To demonstrate our analytical results, some simple and realistic examples will be discussed in detail at the end of the paper.

II. GENERAL COMPARISON OF THE TWO MEASURES

A. Definitions of C and MC

We first present the definitions of the concurrence and the mode concurrence in spin-1/2 and spinless fermion systems respectively. Consider an N-site spin-1/2 system

\[
H_s = H_s(\{\sigma_i^\alpha\})
\]

where \(\sigma_i^\alpha\) is the Pauli matrix at \(i\)-th site and \(\{\sigma_i^\alpha\} = \{\sigma_i^x, \sigma_i^y, \sigma_i^z; i \in [1, N]\}\). In this paper, we study the Hamiltonian that the \(z\)-component of the total spin is conserved, i.e., \([S^z, H_s] = 0\), where \(S^z = \sum_i S_i^z\) and \(S_i^z = \sigma_i^z/2\) for \(\alpha = x, y, z\). Here and later in the paper we define that \(\sigma_i^\pm = \sigma_i^x \pm i\sigma_i^y\) and \(S_i^\pm = S_i^x \pm iS_i^y\). Then the reduced density matrix of the eigen state \(|\psi_s\rangle\) with respect to two arbitrary sites \(i\) and \(j\) on the basis \(|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\) is given by

\[
\rho_{ij}^{s} = tr_{N-2}(|\psi_s\rangle \langle \psi_s|)
\]

where \(\uparrow (\downarrow)\) denotes the spin up (down) and \(tr_{N-2}\) means tracing over all the variables except the two on the sites \(i, j\). Thus the concurrence \(23\) of the two separated sites \(i, j\) is

\[
C_{i,j} = 2 \max \left\{ 0, |z| - \sqrt{u^+ u^-} \right\},
\]

where, as given in Ref. \(S\),

\[
\begin{align*}
 u^+ &= \frac{1}{4} [1 + (\sigma_i^+ + \sigma_j^- + \sigma_i^- \sigma_j^+)], \\
 u^- &= \frac{1}{4} [1 - (\sigma_i^+ + \sigma_j^- + \sigma_i^- \sigma_j^+)], \\
 z &= \frac{1}{4} \langle \sigma_i^z \sigma_j^z \rangle,
\end{align*}
\]

are the expectations of Pauli matrices.

On the other hand, consider an \(N\)-site spinless fermion system with the Hamiltonian

\[
H_f = H_f(\{a_i^\dagger a_j\}),
\]

where \(a_i^\dagger\) is the fermion operator at the \(i\)-th site, and \(\{a_i^\dagger a_j\} = \{a_i^\dagger a_j \mid i, j \in [1, N]\}\). In this paper, we study the Hamiltonian that the total particle number \(\hat{N} = \sum_i \hat{n}_i = \sum_i a_i^\dagger a_i\) is conserved, i.e., \([\hat{N}, H_f] = 0\). According to Ref. \(23\), the second order reduced density matrix of the state \(|\psi_f\rangle\) with respect to two arbitrary sites \(i\) and \(j\) on the basis \(|11\rangle, |10\rangle, |01\rangle, |00\rangle\) is given as

\[
\rho_{ij}^{f} = tr_{N-2}(|\psi_f\rangle \langle \psi_f|)
\]

where \(|\psi_f\rangle\) is an eigen state of the Hamiltonian \(H_f\). Similarly, \(tr_{N-2}\) means tracing over all the variables except the two on the sites \(i, j\). Similarly, the nonzero elements are determined by the correlation functions of the fermion operators

\[
\begin{align*}
 X^+ &= \langle \hat{n}_i \hat{n}_j \rangle, \\
 X^- &= 1 - \langle \hat{n}_i \rangle - \langle \hat{n}_j \rangle + X^+, \\
 Z &= \langle a_i^\dagger a_j \rangle.
\end{align*}
\]

Correspondingly, the MC between sites \(i\) and \(j\) can be written as \(23\)

\[
(MC)_{i,j} = 2 \max \left\{ 0, |Z| - \sqrt{X^+ X^-} \right\}.
\]

So far we have defined the measures of entanglement both in spin-1/2 and spinless fermion systems respectively. In the following sub section we will discuss the relations between the two measurements.
B. Relations between C and MC for quantum chain systems

As the two definitions of the concurrence correspond to different states and models, we investigate the relation between C and MC based on two typical models, spin-1/2 and spinless fermion chains. First, we concern an N-site XXZ spin chain model, of which the Hamiltonian is given by

\[
H_{s}^{XXZ} = \sum_{j=1}^{N-1} \left[ J_{s} (S_{j}^{+}S_{j+1}^{-} + S_{j}^{-}S_{j+1}^{+}) + J_{s}^{z} S_{j}^{z}S_{j+1}^{z} \right],
\]

where \( J_{s} \) and \( J_{s}^{z} \) are the coupling constants. Obviously, the z-component of the total spin is conserved for this model. The Hilbert space of such a Hamiltonian is spanned by \( 2^{N} \) basis vectors \( |m \rangle_{s} \equiv |m_{1}, m_{2}, ..., m_{N} \rangle_{s} \), where \( m_{1}, m_{2}, ..., m_{N} \in \{ \uparrow, \downarrow \} \) and \( l = 1, 2, 3, ..., N \). Then an arbitrary state of the above Hamiltonian can be generally written as

\[
|\psi_{s} \rangle = \sum_{m} \gamma_{m} |m \rangle_{s} = \sum_{m} \gamma_{m} \otimes_{l=1}^{N} (S_{l}^{+})^{f(m_{l})} |0 \rangle_{s},
\]

where \( \gamma_{m} \) are normalized coefficients, \( |0 \rangle_{s} \equiv |\downarrow \downarrow \cdots \downarrow \rangle_{s} \) represents the saturated ferromagnetic state, \( S_{l}^{+} \) is the raising operator of the \( l \)-th qubit and the function \( f(m_{l}) \) correspond to \( m_{l} = \uparrow, \downarrow \) respectively. We emphasize that \( \otimes_{l=1}^{N} (S_{l}^{+})^{f(m_{l})} \) is arranged in the ascending order of \( l \).

On the other hand, as is well known, there exists a counterpart spinless fermion model for an arbitrary spin-1/2 XXZ model. We now concern the spinless fermion model obtained by employing the Jordan-Wigner transformation \[31\]

\[
\begin{align*}
S_{l}^{+} & = a_{l}^{\dagger} \exp(i\pi \sum_{p=1}^{l-1} \hat{n}_{p}), \\
S_{l}^{-} & = \exp(-i\pi \sum_{p=1}^{l-1} \hat{n}_{p})a_{l}, \\
S_{l}^{z} & = \hat{n}_{l} - \frac{1}{2},
\end{align*}
\]

Based on the transformation, the corresponding spinless fermion Hamiltonian, which is transformed from Eq. \[9\], is given as

\[
H_{f}^{TB} = \sum_{j=1}^{N-1} \left[ J_{j}^{z} \left( \hat{n}_{j} - \frac{1}{2} \right) \left( \hat{n}_{j+1} - \frac{1}{2} \right) \right] + J_{j} (a_{j}^{\dagger}a_{j+1} + \text{H.c.}) \text{.}
\]

This is a typical tight-binding (TB) model with NN interaction \( J_{j}^{z} \) and NN hopping integral \( J_{j} \). The Fock space of the above Hamiltonian is spanned by \( 2^{N} \) basis vectors \( \{|n\rangle_{f} \equiv |n_{1}, ..., n_{t}, ..., n_{N} \rangle_{f} \} \), where \( n = 1, 2, 3, ..., 2^{N} \), \( n_{t} = 1, 0 \) denoting the existence of one or zero particle at the \( l \)-th site respectively and \( l = 1, 2, 3, ..., N \). Accordingly, the corresponding state of the Hamiltonian \[12\], which is transformed from Eq. \[10\], can be expressed explicitly as

\[
|\psi_{f} \rangle = \sum_{n} \gamma_{n} \otimes_{l=1}^{N} |a_{l}^{\dagger} \exp(i\pi \sum_{p=1}^{l-1} \hat{n}_{p})|n \rangle_{f},
\]

where \( |0 \rangle_{f} \) is the vacuum state, i.e., \( a_{l} |0 \rangle_{f} = 0 \).

We now compare the reduced density matrix elements with respect to \( |\psi_{s} \rangle \) and \( |\psi_{f} \rangle \). According to Jordan-Wigner transformation \[11\], we have

\[
\begin{align*}
\hat{n}_{i} & = \frac{1}{4} \left[ 1 + (\sigma_{i}^{x} + \sigma_{i}^{y}) + \sigma_{i}^{z} \right], \\
1 - \hat{n}_{i} & + \hat{n}_{j} = \frac{1}{4} \left[ 1 - (\sigma_{i}^{x} + \sigma_{i}^{y}) + \sigma_{i}^{z} \right], \\
a_{l}^{\dagger}a_{l} & = \frac{1}{4} \left[ \sigma_{l}^{x} + \sigma_{l}^{y} \exp(i\pi \sum_{p=1}^{l-1} \frac{1 + \sigma_{p}^{z}}{2}) \right].
\end{align*}
\]

Therefore, the reduced density matrix elements for the two systems, which are defined in Eqs. \[14\] and \[18\], have the following relations,

\[
u^{\pm} = X^{\pm},
\]

and

\[
\begin{align*}
z & = \frac{1}{4} \langle \psi_{s} | \sigma_{i}^{x} \sigma_{j}^{y} | \psi_{s} \rangle, \\
Z & = \langle \psi_{f} | a_{l}^{\dagger}a_{l} | \psi_{f} \rangle,
= \frac{1}{4} \langle \psi_{s} | \sigma_{l}^{x} \sigma_{j}^{y} \exp(i\pi \sum_{p=1}^{j-1} \frac{1 + \sigma_{p}^{z}}{2}) | \psi_{s} \rangle.
\end{align*}
\]

We notice that \( z \) is different from \( Z \), which is caused by the additional term \( \exp(i\pi \sum_{p=1}^{j-1} \frac{1 + \sigma_{p}^{z}}{2}) \) resulted from the anti-commutation relation. Thus the magnitudes of C and MC for the two corresponding states \( |\psi_{s} \rangle \) and \( |\psi_{f} \rangle \) may differ from each other. However, no specific relations between C and MC can be derived from the above equations. In the following sub section, with the aid of Marshall’s sign rule, we will concern the specific properties of the matrix elements \( z \) and \( Z \) with respect to the corresponding ground states of the spin-1/2 XXZ chain and the corresponding spinless fermion model.

C. Specific relations between C and MC for the ground states

Now we focus on the study of the spin-1/2 XXZ chain system, of which the Hamiltonian is in the form of Eq. \[9\]. When two specific sites \( i \) and \( j \) are concerned, the ground state of the spin-1/2 XXZ chain systems can be explicitly written as
\[ |\psi_{gs}\rangle = |\uparrow\rangle_i |\uparrow\rangle_j \otimes |\psi_1\rangle + |\downarrow\rangle_i |\downarrow\rangle_j \otimes |\psi_2\rangle + \sum_k (x_k |\uparrow\rangle_i |\downarrow\rangle_j + y_k |\downarrow\rangle_i |\uparrow\rangle_j) \otimes |\phi_k\rangle , \]

where
\[
|\psi_1\rangle = \sum_m g_m \otimes \left( S_i^+ \right) f(m) |0\rangle_s ,
\]
\[
|\psi_2\rangle = \sum_n g_n \otimes \left( S_i^+ \right) f(m) |0\rangle_s ,
\]
\[
|\phi_k\rangle = \otimes_{i\neq j} \left( S_i^+ \right) f(m) |0\rangle_s ,
\]
denoting the basis of the rest \(N-2\) sites, and \(x_k, y_k, g_m, g_n\) are normalized coefficients. Here we have used the fact, \([S^z, H_{XXZ}^s] = 0\), which ensures that \(|\psi_{gs}\rangle\) can be written in a single invariant subspace. Then the correlation function \(\langle \sigma_i^x \sigma_j^z \rangle\) can be expressed as
\[
\langle \sigma_i^x \sigma_j^z \rangle = 4 \langle S_i^+ S_j^- \rangle = 4 \sum_k x_k y_k . \tag{19}
\]

Now we will prove an important equation in this paper, i.e., for the spin-1/2 XXZ chain model \([4]\) with arbitrary value of \(J_j\) and \(J_j^z\), it can be verified that for the ground state \(17\),
\[
\sum_k x_k y_k = \sum_k |x_k y_k| . \tag{20}
\]

Before the proof of the above equation, we will first introduce the Marshall’s sign rule \([32]\) for bipartite (i.e., the lattice can be divided into two sublattices, A, B such that all nearest neighbors of a site on one sublattice lie on the other and vice versa) \(XXZ\) systems \([33]\). Obviously, the spin-1/2 \(XXZ\) chain model concerned in the present paper belongs to a bipartite system. In the following, we will present the sign rule of a bipartite \(XXZ\) model in the situation of arbitrary \(J_j^z\) and \(J_j > 0\). The sign rule for arbitrary \(J_j^z\) and \(J_j \leq 0\) is similar to the first situation, and is introduced in detail in Ref. \([33]\).

We rewrite the ground state of the bipartite \(XXZ\) system as
\[
|\psi_{gs}\rangle = \sum_m g_m |m\rangle_s , \tag{21}
\]
where \(g_m\) are normalized coefficients, According to the sign rule \([33]\), in the situation when arbitrary \(J_j^z\) and \(J_j > 0\), the normalized coefficients \(g_m\) can be written in the following form
\[
g_m = (-1)^{\varphi(m)} b_m , \tag{22}
\]
where \(\{b_m\}\) is a positive semi-definite set, and
\[
\varphi(m) = \sum_{l \in A} f(m_l) , \tag{23}
\]
denotes the number of spin ups in sublattice \(A\) for the basis \(|m\rangle_s\).

Applying the above sign rule to the ground state \(17\), it is simply found that the sign of the factor \(x_k y_k\) only depends on the location of \(i\) and \(j\), which is independent of the \(k\). If \(i, j \in A\) or \(i \notin A\) (\(j \in B\)), we have \(x_k y_k \geq 0\) for any \(k\); while if \(i, j\) belong to different sublattices, we have \(x_k y_k \leq 0\) for any \(k\). Then we have proved the Eq. \(20\) for arbitrary \(J_j^z\) and \(J_j > 0\). The situation of arbitrary \(J_j^z\) and \(J_j \leq 0\) will give a similar proof. Therefore, the Eq. \(20\) is valid for arbitrary \(J_j^z\) and \(J_j\), which is crucial for the following discussions.

With the help of sign rules, we now further compare the two measures of entanglement. Based on the identity \([20]\), we have the following conclusions for the correlation functions \(z\) and \(Z\) with respect to the ground state of spin-1/2 \(XXZ\) model and its counterpart many-particle ground state in spinless fermion system.

(i) For NN cases, i.e., \(j = i + 1\), the absolute value of the correlation function \(Z\) is identical to that of the correlation function \(z\),
\[
|Z| = \frac{1}{4} \left| \langle \exp(i \pi \sum_{p=i}^{j} \frac{\sigma_{p}^z + 1}{2} \sigma_{i}^z \sigma_{j}^z) \rangle \right| \tag{24}
\]
\[
= \left| \exp(i \pi \frac{\sigma_{i}^z + 1}{2}) \sum_k x_k y_k \right| \left| \sum_k x_k y_k \right| = |z| ;
\]

(ii) For non-NN cases, i.e., \(j > i + 1\), the absolute values of the corresponding correlation functions have the following relation,
\[
|Z| = \frac{1}{4} \left| \langle \sigma_{i}^+ \sigma_{j}^- \exp(i \pi \sum_{p=i}^{j-1} \frac{\sigma_{p}^z + 1}{2}) \rangle \right| \tag{25}
\]
\[
\leq \sum_k |x_k y_k| = |z| .
\]

Obviously, as we have mentioned in sub section A, the above inequalities are caused by the anti-commutation relations of fermion systems. Since we have \(u^\pm = X^\pm\) as presented in Eq. \(13\), the above inequalities will simply lead to our main observation that there exists a simple and explicit relation between the concurrences of the two counterpart ground states, i.e., for NN cases,
\[
C_{i,j} = 2 \max \left\{ 0, |z| - \sqrt{u^+ u^-} \right\} \tag{26}
\]
\[
= (MC)_{i,j} = 2 \max \left\{ 0, |z| - \sqrt{X^+ X^-} \right\} ,
\]
while for non-NN cases, we have \(C_{i,j} \geq (MC)_{i,j}\). Therefore, by combining the two cases, we get the following
relation for any types of C and MC,

\[ C_{i,j} \geq (MC)_{i,j} \]  

We emphasize that the above inequality is valid in the ground states of spin-1/2 XXZ chain models and their spinless fermion counterparts, where the sign rules hold.

III. FURTHER COMPARISON IN SPECIFIC MODELS

The above discussion in Sec. II. C is based on the ground state of a XXZ chain model, of which the counterpart Hamiltonian \[^{\text{12}}\] in spinless fermion systems corresponds to TB models. The TB models are widely used in modeling metallic, semiconducting, ionic systems \[^{\text{35}}\], and recently optical lattices \[^{\text{36}}\], and are of great interest to both the condensed matter and the quantum information communities. In this section, we further the above discussion and give an example in the XY chain model (30) into its spinless fermion counterpart, which is accordingly the ground state of the counterpart Hamiltonian of Eq. (30). As a result, the correlation functions are given as \( X^+ = 7/18, X^- = 1/9, Z = 1/9 \) for the counterpart ground state, thus we have

\[ (MC)_{1,3} = 2 \max \left\{ 0, |z| - \sqrt{u^+ u^-} \right\} \]
\[ = \frac{1}{9} (4 - \sqrt{14}) \]

On the other hand, we transform the ground state of XY chain model \[^{\text{30}}\] into its spinless fermion counterpart, which is in agreement with Eq. (27). Here the difference between z and Z is apparently resulted from the different commutation relations of the two models.

We remark that our theoretical proof in Sec. II is valid in the above example, which relies on the Marshall’s sign rule. However, we are surprised to notice that the inequality \[^{\text{27}}\] also holds for excited states and even for eigen states of many other models. These situations are beyond our theoretical proof and we will give some examples in the following subsection. Thus the relation (\[^{\text{27}}\]) may provide us a deeper and more general insight into the differences between spin-1/2 and fermion systems.

B. Examples where the sign rule does not apply

We consider the situations where the sign rule is violated. Here we take a TB model with uniform hopping integral as an example, of which the Hamiltonian reads

\[ H^T_B = \sum_{j=1}^{N/(3)} (a_j^+ a_{j+1}^+ + \text{H.c.}) \]

\[ H^X_Y = \sum_{j=1}^{N} (S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+) \]

\[ = 2 \sum_{j=2,3} (S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+) \]

with the ground state

\[ |\psi_{gs}\rangle = \frac{1}{6} (-2 |\uparrow\uparrow\uparrow\uparrow\uparrow\rangle_s + 2 |\uparrow\uparrow\uparrow\uparrow\downarrow\rangle_s + 2 |\uparrow\uparrow\downarrow\uparrow\downarrow\rangle_s + 2 |\uparrow\downarrow\uparrow\uparrow\downarrow\rangle_s + |\downarrow\downarrow\uparrow\uparrow\uparrow\rangle_s + 3 |\uparrow\uparrow\uparrow\downarrow\downarrow\rangle_s - |\uparrow\downarrow\uparrow\downarrow\downarrow\rangle_s) \]

We calculate the concurrence between site 1 and 3, which results in \( u^+ = 7/18, u^- = 1/9, z = 2/9 \), and therefore,

\[ C_{1,3} = 2 \max \left\{ 0, |z| - \sqrt{u^+ u^-} \right\} \]
\[ = \frac{1}{9} (4 - \sqrt{14}) \]

Consequently, we get that \( C_{1,3} > (MC)_{1,3} \), which is in agreement with Eq. (27). Here the difference between z and Z is apparently resulted from the different commutation relations of the two models.

We remark that our theoretical proof in Sec. II is valid in the above example, which relies on the Marshall’s sign rule. However, we are surprised to notice that the inequality \[^{\text{27}}\] also holds for excited states and even for eigen states of many other models. These situations are beyond our theoretical proof and we will give some examples in the following subsection. Thus the relation (\[^{\text{27}}\]) may provide us a deeper and more general insight into the differences between spin-1/2 and fermion systems.
The single-particle eigenstates are $|k\rangle = \sqrt{2/(N+1)} \sum_i \sin(kl) a_i^\dagger |0\rangle$, where $k = n\pi/(N+1)$ and $n = 1, 2, ..., N$, with the corresponding spectrum $\epsilon_k = 2\cos k$. According to the above analysis, an arbitrary two-particle state can be written as

$$|k, k'\rangle = \sum_{i<l'} D(k, k', l, l') a_i^\dagger a_{i'}^\dagger |0\rangle,$$

where

$$D(k, k', l, l') = \frac{2}{N+1} \det \begin{vmatrix} \sin(kl) & \sin(kl') \\ \sin(k' l) & \sin(k' l') \end{vmatrix}.$$  \hspace{1cm} (35)

Assuming $i < j$, we have

$$Z = \sum_k D(k, k', j, l)D(k, k', i, l).$$  \hspace{1cm} (36)

According to the commutation relations, the corresponding correlation function in the XY spin model can be simply written as

$$z = \frac{1}{4} \langle \sigma_i^+ \sigma_j^- \rangle$$

$$= \left( \sum_{i<i} - \sum_{i<j} + \sum_{l>j} \right) D(k, k', j, l)D(k, k', i, l).$$  \hspace{1cm} (37)

Now we consider the sign of three terms in $z$ corresponding to the XY spin model. We take $k = \pi/(N+1)$ and $k' = 2\pi/(N+1)$ as an example. Obviously, the state $|k, k'\rangle$ is an excited state, which does not obey the Marshall's sign rule. A straightforward calculation gives

$$D(k, k', j, l)D(k, k', i, l) > 0,$$

for any $l \neq i, j$. Therefore, from Eqs. 37 and 38, we have

$$|z| > |Z| \implies C_{i,j} > (MC)_{i,j},$$

which shows that our conclusion still holds even the Marshall’s sign rule is violated.

Now we consider another excited state with $k = \pi/(N+1)$ and $k' = N\pi/(N+1)$, which is also beyond the Marshall’s sign rule. For even $j$ or $i$, we have

$$D(k, k', j, l)D(k, k', i, l)$$

$$= D(k, \pi - k, j, l)D(k, \pi - k, i, l)$$

$$= 0.$$  \hspace{1cm} (39)

Although both of the above two examples are excited states, which violate the Marshall’s sign rule, they are still in agreement with the conclusion of Eq. 27. Thus it will be very interesting to give an even more general proof to extend our conclusion for the states that beyond the Marshall’s sign rule.

**IV. SUMMARY AND DISCUSSION**

As a conclusion we have compared the concepts of concurrence for spin-1/2 systems with mode concurrence for spinless fermion systems explicitly. By employing the Jordan-Wigner transformation and the Marshall’s sign rule, we come to our main observations in spin-1/2 XXZ and spinless fermion chain systems that: concurrence and mode concurrence are different from each other for general corresponding states of the two systems and further there exist specific relations between the ground-state concurrence of a spin-1/2 XXZ chain model and the mode concurrence of its counterpart ground state in a many-particle spinless fermion model. (i) The nearest neighbor ground-state concurrence of spin-1/2 XXZ chain models is identical to the nearest neighbor ground-state MC of many-particle spinless fermion systems, i.e., $C_{i,i+1} = (MC)_{i,i+1}$. (ii) For other types of entanglement, concurrences are no less than mode concurrences for any given corresponding ground states, i.e., $C_{i,j} \geq (MC)_{i,j}$ for $j > i + 1$. An example in XY spin chain model and spinless fermion hopping model with practical significance is given to illustrate the simple relation derived from the comparison between concurrence and mode concurrence. The differences between the ground-state entanglement of the two models are closely related to the fundamental features (commutation relations) of the two models and also may indicate some new aspects of the intrinsic distinctions between spin-1/2 XXZ and spinless fermion chain systems. We have also shown some other states and examples that are beyond our analytical proof and we observe that all these results are in agreement with the relation 27. Thus it will be very interesting to further investigate and to generalize the relations between the measures of entanglement in spin-1/2, fermion and even boson systems, as well as the intrinsic properties of the these systems.

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