NONSYMPLECTIC AUTOMORPHISMS OF IRREDUCIBLE SYMPLECTIC MANIFOLDS OF OG$_6$ TYPE

ANNALISA GROSSI

Abstract. We classify nonsymplectic automorphisms of prime order of manifolds of OG$_6$ type. More precisely, we give a classification of the invariant and coinvariant lattices of the second integral cohomology group.

1. Introduction

1.1. Background. Irreducible holomorphic symplectic manifolds are simply connected compact complex Kähler manifolds carrying a nowhere degenerate holomorphic symplectic form $\sigma_X$ which spans $H^{2,0}(X)$.

In dimension 2 irreducible holomorphic symplectic manifolds are K3 surfaces. Fujiki [20] and Beauville [4] found examples in higher dimensions: more precisely the Hilbert scheme of $n$ points on a K3 surface and the generalized Kummer manifold are irreducible holomorphic symplectic manifolds of dimension $2n$. Manifolds which are deformation equivalent to the Hilbert scheme and to the generalized Kummer manifold are called manifolds of K3$[n]$ type and of Kum$[n]$ type respectively.

Mukai [35] discovered a symplectic form on moduli spaces of sheaves with some specific conditions on symplectic surfaces. However he proved that all non-singular irreducible holomorphic symplectic manifolds obtained in this way were a deformation of known examples, and the singular ones admit a resolution of singularities which is irreducible holomorphic symplectic only in two cases discovered by O’Grady: one in dimension 6 [40] and one in dimension 10 [39]. A manifold which is deformation equivalent to the O’Grady’s six dimensional manifold is called a manifold of OG$_6$ type. A manifold deformation equivalent to the O’Grady’s ten dimensional manifold is a manifold of OG$_{10}$ type.

An automorphism of an irreducible holomorphic symplectic manifold is symplectic if it preserves the symplectic form $\sigma_X$. In this paper we study nonsymplectic automorphisms of manifolds of OG$_6$ type.

O’Grady [40] obtains a manifold of OG$_6$ type as a symplectic resolution of the Albanese fiber of a moduli space of stable sheaves on an abelian surface with respect to a specific Mukai vector. Later contributions given first by M. Lehn and Sorger [47] and then by Perego and Rapagnetta [44], allow to conclude that, under specific assumptions on the Mukai vector, the blow up of the fiber of the moduli space along its singular locus always gives a crepant resolution and these crepant resolutions are deformation equivalent, along smooth projective deformations, to the original O’Grady’s example.

1.2. Automorphisms of irreducible holomorphic symplectic manifolds. Automorphisms of irreducible holomorphic symplectic manifolds can be classified studying the induced action on the second integral cohomology group which carries a
lattice structure provided by the Beauville–Bogomolov–Fujiki quadratic form. The
global Torelli theorem for K3 surfaces, due to Piatetski-Shapiro–Shafarevich, allows
to reconstruct automorphisms of a K3 surface $S$ starting from Hodge monodromies
of $H^2(S, \mathbb{Z})$ which preserve the Kähler cone. Huybrechts [24], Markman [27] and
Verbitsky [49] formulated similar results of Torelli type for irreducible holomorphic
symplectic manifolds.

Recently Mongardi–Rapagnetta [30] computed the monodromy group for man-
ifolds of $\text{OG}_6$ type and due to the features of this group, the global Torelli theo-
rem [27, Theorem 1.3] holds in a stronger form for $\text{OG}_6$ type manifolds, namely a
necessary and sufficient condition to have a bimeromorphic map between two man-
ifolds of $\text{OG}_6$ type is to have a Hodge isometry of the second integral cohomology.

Classifying finite groups of automorphisms $G$ of a certain deformation type of
irreducible holomorphic symplectic manifolds can mean one of the following:

1. classifying invariant and coinvariant sublattices of the induced action of $G$
in $H^2(X, \mathbb{Z})$ up to isometry;
2. classifying invariant and coinvariant sublattices up to isometry of $H^2(X, \mathbb{Z})$;
3. classifying the connected components of the moduli space of pairs $(X, G)$.

In general, classification (3) is finer than (2), which in turn is finer than (1).

Until now, much of the literature about automorphisms of irreducible holomor-
phic symplectic manifolds in higher dimension has focused on classification (1) in
the symplectic case and in the nonsymplectic case of prime order.

In the case of manifolds of K3[^2] type the symplectic automorphisms are treated
by Camere [15] and Mongardi [29]; the study of nonsymplectic automorphisms was
started by Beauville [5] and continued by Ohashi–Wandel [42], Boissière–Camere–
Mongardi–Sarti [6], Boissière–Camere–Sarti [9], Camere–G. Kaputska–M. Kaputska–
Mongardi [17]; furthermore Boissière–Nieper-Wiikirchen–Sarti [10] describe the
fixed locus of these automorphisms. Camere–Cattaneo [16] study nonsymplectic
automorphisms of K3[^n] type manifolds, where $n \geq 3$, and Joumaah [26], building
on a work by Ohashi–Wandel [42], gives a criterion to find the classification (3) in
the case of involutions on manifolds of K3[^n] type.

The study of automorphisms of generalized Kummer manifolds was started
by Tari [48] and Mongardi–Tari–Wandel [32] and continued by Boissière–Nieper-
Wiikirchen–Tari [11] and by Brandhorst–Cattaneo [14].

The classification of nonsymplectic automorphisms of O’Grady’s ten dimensional
manifold is started by Brandhorst–Cattaneo [14], and recent progress by Onorati
[43] about the monodromy group and the wall divisors for this deformation class
constitute a starting point for the study of the symplectic case.

1.3. Principal results. We exhibit a complete classification of invariant and co-
invariant sublattices of $H^2(X, \mathbb{Z})$ with respect to a nonsymplectic action of prime
order for manifolds of $\text{OG}_6$ type. In particular we are interested in the image of
the following representation map

$$\nu : \text{Aut}(X) \to O(H^2(X, \mathbb{Z})).$$

We call effective isometries which are in the image of $\nu$. On account of Proposition
3.5 the possible prime orders of a nonsymplectic automorphisms are 2, 3, 5, 7.

To obtain the classification of invariant and coinvariant lattices of $H^2(X, \mathbb{Z})$ we
classify nonsymplectic isometries of prime order of the smallest unimodular lattice
in which $H^2(X, \mathbb{Z})$ embeds, namely $\Lambda = U^{n \oplus}$, hence we prove the following theorem.
Theorem 1.1 (see Theorem 3.16). Let \( \Lambda = U^{\oplus 5} \) and let \( G \subset O(\Lambda) \) be a subgroup of order \( p \). Assume there exists a primitive embedding of \( (2)^{\oplus 2} \) in the invariant lattice of \( G \) in \( \Lambda \). If \( p = 2 \) and the induced action of \( G \) on \( A_X \) is trivial then there are fourteen pairs \( (S_G(\Lambda), T_G(\Lambda)) \) up to isometry. If \( p = 2 \) and the induced action of \( G \) on \( A_X \) is nontrivial then there are twelve pairs \( (S_G(\Lambda), T_G(\Lambda)) \) up to isometry. If \( p = 3 \) then there are three pairs \( (S_G(\Lambda), T_G(\Lambda)) \) up to isometry, and if \( p = 5 \) or \( p = 7 \) there exists a unique pair \( (S_G(\Lambda), T_G(\Lambda)) \) up to isometry. The pairs are listed in Table 1.

Once the classification for \( \Lambda \) is completed we need to determine which isometries of \( \Lambda \) are the extension of isometries of \( H^2(X, \mathbb{Z}) \), i.e. we need to determine in which cases there exists an isometry of the second integral cohomology lattice such that the extension on \( \Lambda \) has the lattices that we have found as invariant and coinvariant sublattices. To do this we use Proposition 4.3.

We compute invariant and coinvariant sublattices of \( H^2(X, \mathbb{Z}) \) up to isometry with respect to a nonsymplectic action of prime order.

Theorem 1.2 (see Theorem 4.13). Let \( X \) be a manifold of OG\( _6 \) type. Let \( G \) be a group of automorphisms where all the nontrivial elements are nonsymplectic of prime order \( p \). If \( p = 2 \) and if the induced action of \( G \) on \( A_X \) is trivial there are twenty-four pairs \((S_G(\Lambda), T_G(\Lambda))\) up to isometry of the invariant and coinvariant sublattices of \( H^2(X, \mathbb{Z}) \). If \( p = 3 \) there are three pairs \((S_G(\Lambda), T_G(\Lambda))\) up to isometry, and if \( p = 5 \) or \( p = 7 \) there exists a unique pair \((S_G(\Lambda), T_G(\Lambda))\) up to isometry. The pairs are listed in Table 3.

Moreover we find a finer classification with respect to the one of Theorem 4.13. We compute invariant and coinvariant sublattices of \( H^2(X, \mathbb{Z}) \) up to isometry of the second integral cohomology lattice with respect to a nonsymplectic action of prime order (see Corollary 4.14).

We obtain the classification of invariant and coinvariant sublattices of \( H^2(X, \mathbb{Z}) \) with respect to an automorphism of an OG\( _6 \) type manifold, hence we can use lattice-theoretic results of [21] to discover which automorphisms are induced and induced at the quotient. We determine when an automorphism is induced by an automorphism of the abelian surface according to the O’Grady’s model, or when an automorphism is induced by an automorphism of the K3\( ^{[3]} \) type manifold according to the Mongardi–Rapagnetta–Saccà’s model [31].

1.4. Contents of the paper. In [32] we give an overview of basic results about irreducible holomorphic symplectic manifolds and we introduce the main tools to approach the study of automorphisms, namely we give basic notions of lattice theory and we recall the properties of the second integral cohomology of an irreducible holomorphic symplectic manifold. Moreover we collect results about monodromy and Kähler cone for the OG\( _6 \) case and we recall Torelli type theorems. There exist two birational models for manifolds of OG\( _6 \) type in literature: the first one due to O’Grady recalled above, and the Mongardi–Rapagnetta–Saccà’s model [31] through which a manifold of OG\( _6 \) type is birational to the quotient of a manifold of K3\( ^{[3]} \) type by a symplectic involution.

To classify nonsymplectic automorphisms of prime order on OG\( _6 \) type manifolds, since their second integral cohomology lattice is not unimodular [45], we consider the following primitive embedding

\[
H^2(X, \mathbb{Z}) \hookrightarrow \Lambda = U^{\oplus 5},
\]
and we extend the induced action on the second integral cohomology on the unimodular lattice $\Lambda$. In $[33]$ we classify nonsymplectic isometries of prime order $p$, and we compute the invariant and coinvariant sublattices which are $p$-elementary sublattices of $\Lambda$. In Proposition $3.4$ we find that, for manifolds of OG$6$ type, a nonsymplectic isometry of prime is effective, hence the strategy is to classify them in order to classify nonsymplectic automorphisms. Then in §4 we look for the isometries of $\Lambda$ that are extensions of isometries of $H^2(X,\mathbb{Z})$ and we classify them finding invariant and coinvariant sublattices of $H^2(X,\mathbb{Z})$.

We recall lattice-theoretic criteria to determine when a manifold of OG$6$ type admits such models $[21]$ and we recall the notions of induced automorphism and automorphism induced at the quotient since we apply them in the classification in $[34]$.

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2. Preliminaries

2.1. Lattices. We gather the required background and we give an overview of lattice theory and of finite quadratic forms, recalling the fundamental definitions and results which we will use throughout this paper. The work $[37]$ due to Nikulin is the most important which we will refer to, but there are also other classical sources, such as $[18]$ and $[25$, Chapther 14$]$. A lattice $L$ is a free $\mathbb{Z}$-module of finite rank together with a non-degenerate symmetric bilinear form $(\cdot,\cdot): L \times L \to \mathbb{Z}$, which we assume to be non-degenerate. A lattice $L$ is called even if $x^2 := (x,x) \in 2\mathbb{Z}$. For any lattice $L$ the discriminant group is the finite group associated to $L$ defined as $A_L := L^*/L$, where $L \mapsto L^* := \text{Hom}_\mathbb{Z}(L,\mathbb{Z})$, $x \mapsto (x,\cdot)$. A lattice is called unimodular if $A_L = \{ \text{id} \}; \ L$ is called $p$-elementary if the discriminant group $A_L$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^\oplus a$ for some $a \in \mathbb{Z}_{>0}$. The length of the discriminant group $A_L$, denoted by $l(A_L)$, is the minimal number of generators of the finite group $A_L$. The divisibility $\text{div}(v)$ of an element $v \in L$ is the positive integer $s$ generator of the ideal $(v,L) = s\mathbb{Z}$. The pairing $(\cdot,\cdot)$ on $L$ induces a $\mathbb{Q}$-valued pairing on $L^*$ and hence a pairing $A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$. If the lattice $\Lambda$ is even, then the $\mathbb{Q}$-valued quadratic form on $L^*$ yields $q_L : A_L \to \mathbb{Q}/2\mathbb{Z}$, which gives back the pairing on $A_L$. The finite group $A_L$ together with $q_L$ is called the discriminant quadratic form on $L$.

A fundamental invariant in the theory of lattices is given by genus. Two lattices, $L$ and $L'$ belong to same genus if $\text{sign}(L) = \text{sign}(L')$ and $L \otimes \mathbb{Z}_p$ and $L' \otimes \mathbb{Z}_p$ are isomorphic (as $\mathbb{Z}_p$-lattices) for all prime integers $p$.

Two lattices $L$ and $L'$ have the same genus if and only if $L \oplus U \cong L' \oplus U$ or equivalently if and only if they have the same signature and discriminant quadratic form.
Theorem 2.1 (Nikulin [37, Corollary 1.9.4]). The genus of an even lattice $L$ is determined by the triple $(l_{++}, l_{--}, q_L)$ where $(l_{++}, l_{--})$ is the signature of the lattice and $q_L$ is its discriminant quadratic form.

Each genus contains only finitely many isomorphism classes of lattices. It is an interesting problem to determine whether there exists an even lattice with given signature and quadratic form, and, if so, whether it is unique, up to isometry.

We recall two lattices that will be useful in §4:

$$H_5 = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}, K_7 = \begin{pmatrix} -4 & 1 \\ 1 & -2 \end{pmatrix}.$$

2.2. Irreducible holomorphic symplectic manifolds. A compact complex Kähler manifold $X$ is an irreducible holomorphic symplectic manifold if

1. $\pi_1(X) = \{\text{id}\}$.
2. $H^0(X, \Omega_X^2) \cong H^{2,0}(X) \cong \mathbb{C} \cdot \sigma_X$, where $\sigma_X$ is the class of a holomorphic 2-form nowhere degenerate as a skew symmetric form on the tangent space.

Remark 2.2. If $X$ is an irreducible holomorphic symplectic manifold then $\dim_{\mathbb{C}}(X)$ is even and $H^2(X, \mathbb{Z})$ is torsion-free due to the Universal coefficient theorem and to the simple connectedness.

Definition 2.3. If $X$ is an irreducible holomorphic symplectic manifold, the Beauville–Bogomolov–Fujiki quadratic form $q_X$ is a canonical integral quadratic form on the free $\mathbb{Z}$-module $H^2(X, \mathbb{Z})$. Its signature is $(3, b_2(X) - 3)$ and it satisfies

$$\int_X \alpha^{2m} = c_X \cdot q_X(\alpha)^m, \forall \alpha \in H^2(X, \mathbb{Z}),$$

where $c_X$, the Fujiki constant, is a positive rational number and $m \colonequals \frac{1}{2} \dim(X)$.

Due to the Beauville–Bogomolov–Fujiki quadratic form for any irreducible holomorphic symplectic manifold $X$ we have that $(H^2(X, \mathbb{Z}), q_X)$ is a lattice. The discriminant group of $H^2(X, \mathbb{Z})$ is called $A_X$. If $X$ is a manifold of OG$_a$ type then $H^2(X, \mathbb{Z}) \cong U \oplus 3 \oplus (-2) \oplus 2$ [25] where $U$ is the unique indefinite unimodular lattice of rank two. In particular $A_X \cong (\mathbb{Z}/2\mathbb{Z})^{2}$.

2.3. Automorphisms of irreducible holomorphic symplectic manifolds. Let $X$ be an irreducible holomorphic symplectic manifold, we denote by $\text{Aut}(X)$ the group of automorphisms of $X$ (biholomorphic maps from $X$ to $X$), and by $\text{Bir}(X)$ the group of birational transformations of $X$ (bimeromorphic maps from $X$ to $X$). Clearly $\text{Aut}(X) \subset \text{Bir}(X)$.

Definition 2.4. If $\varphi \in O(H^2(X, \mathbb{Z}), q_X)$ is an isometry we call $\varphi$ a Hodge operator if the $\mathbb{C}$-linearized action of $\varphi$ is such that $\varphi^*(H^{2,0}(X)) \subseteq H^{2,0}(X)$ and $\varphi^*(H^{1,1}(X)) \subseteq H^{1,1}(X)$.

Definition 2.5. If $\nu$ is the following representation map

$$\nu : \text{Aut}(X) \longrightarrow O(H^2(X, \mathbb{Z}), q_X),$$

$$f \mapsto \nu(f) = (f^*)^{-1}$$

we call $\varphi \in O(H^2(X, \mathbb{Z}), q_X)$ effective if $\varphi \in \text{Im}(\nu)$.

For all compact complex manifolds we have

$$\dim(\text{Aut}(X)) = h^0(TX).$$
If $X$ is an irreducible holomorphic symplectic manifold we have, through the symplectic form defined on the tangent space, an isomorphism $TX \cong \Omega_X$, hence $\dim(\text{Aut}(X)) = h^0(TX) = h^0(\Omega_X) = h^{1,0}(X) = 0$, since $X$ is simply connected. This means that $\text{Aut}(X)$ is a discrete group. Moreover if $X$ is projective then $\text{Bir}(X)$ is finitely generated \cite[Theorem 2]{Mongardi–Wandel}.

**Remark 2.6.** If $G \subset \text{Aut}(X)$ is a finite group and $\varphi \in G$ then the isometry $\varphi^* \in O(H^2(X,\mathbb{Z}), q_X)$ is a Hodge operator.

Hassett–Tschinkel proved that the kernel of the homomorphism $\nu$ in Definition \ref{def:kernel} is invariant under smooth deformations of the manifold $X$ \cite[Theorem 2.1]{Hassett–Tschinkel} and the kernel has been computed for all known deformations type of irreducible holomorphic symplectic manifolds. For manifolds of $\text{OG}_6$ type the kernel of the map is consists of 256 involutions. The following theorem holds.

**Theorem 2.7 (Mongardi–Wandel \cite[Theorem 4.2]{Mongardi–Wandel}).** If $X$ is a manifold of $\text{OG}_6$ type, then $\text{Ker}(\nu) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 8}$.

Given $\varphi \in \text{Aut}(X)$ since $\varphi^*$ is a Hodge isometry it satisfies $\varphi^*(\sigma_X) = \alpha \cdot \sigma_X$ where $\alpha \in \mathbb{C}^*$. If $\varphi$ is of finite order $m$ then $\alpha^m = 1$. An automorphism $\varphi \in \text{Aut}(X)$ is *symplectic* if $\varphi^*(\sigma_X) = \sigma_X$; otherwise $\varphi$ is called *nonsymplectic*; in this case $\varphi^*(\sigma_X) = \alpha \cdot \sigma_X$ where $\alpha \neq 1$, $\alpha \in \mathbb{C}^*$.

An isometry $\varphi \in O(H^2(X,\mathbb{Z}))$ is symplectic if the $\mathbb{C}$-linearized action of $\varphi$ on $H^2(X,\mathbb{C})$ is such that $\varphi(\sigma_X) = \sigma_X$ otherwise it is called nonsymplectic.

**Definition 2.8.** Let $\varphi \in \text{Aut}(X)$ be an automorphism of finite order of an irreducible holomorphic symplectic manifold. The *invariant* lattice of $\varphi$ is

$$T_\varphi(X) = H^2(X,\mathbb{Z})^{\varphi^*} = \{ x \in H^2(X,\mathbb{Z}) | \varphi^*(x) = x \}.$$  

The *co-invariant* lattice of $\varphi$ is

$$S_\varphi(X) = (H^2(X,\mathbb{Z})^{\varphi^*})^\perp \subset H^2(X,\mathbb{Z}).$$

It holds that

$$H^2(X,\mathbb{Z}) \otimes \mathbb{Q} = (T_\varphi(X) \oplus S_\varphi(X)) \otimes \mathbb{Q},$$

Both $T_\varphi(X)$ and $S_\varphi(X)$ are primitive sublattices of the second integral cohomology $H^2(X,\mathbb{Z})$. In fact they can be expressed as kernels of lattice isometries. In particular, if we assume that $m \in \mathbb{N}$ is the order of $\varphi$ then

\begin{equation}
T_\varphi(X) = \text{Ker}(\varphi^* - \text{id}), \quad S_\varphi(X) = \text{Ker}(\text{id} + \varphi^* + \ldots + (\varphi^*)^{m-1})
\end{equation}

2.4. *Néron–Severi and transcendental lattices.* We recall the following definitions of Néron–Severi lattice and transcendental lattice.

**Definition 2.9.** The Néron–Severi lattice is the algebraic $(1, 1)$–part of $H^2(X,\mathbb{C})$, i.e. $\text{NS}(X) := H^2(X,\mathbb{Z}) \cap H^{1,1}(X,\mathbb{C})$. The transcendental lattice $T(X)$ is the orthogonal complement of $\text{NS}(X)$ in the second integral cohomology lattice i.e. $T(X) := \text{NS}(X)^\perp$ and it is the smallest sublattice of $H^2(X,\mathbb{Z})$ such that $H^{2,0}(X) \subset T(X) \otimes_{\mathbb{Z}} \mathbb{C}$.

It holds that

$$H^2(X,\mathbb{Z}) \otimes \mathbb{Q} = (\text{NS}(X) \oplus T(X)) \otimes \mathbb{Q}.$$  

**Remark 2.10.** Since $H^1(X,\mathcal{O}_X) = 0$ the first Chern class $c_1 : \text{Pic}(X) \to H^2(X,\mathbb{Z})$ provides an isomorphism $\text{Pic}(X) \cong \text{NS}(X)$. 

In the following we explain the relative positions of the invariant and of the coinvariant sublattices with respect to $T(X)$ and $\text{NS}(X)$.

**Definition 2.11.** Consider a primitive embedding of lattices
\[ i : L \hookrightarrow L', \]
and consider $\varphi \in O(L)$. The embedding is called $\varphi$-equivariant if there exists $\tilde{\varphi} \in O(L')$ such that $\tilde{\varphi}|_{i(L)} = \varphi$ and $\tilde{\varphi}$ acts as the id on $L'^{\perp'}$.

The following is a version of Lemma 2.12 contained in [28].

**Lemma 2.12.** If $X$ be a manifold of $\text{OG}_6$ type. if $\varphi \in O(H^2(X, \mathbb{Z}))$ is an isometry such that the induced action on $\Lambda = U_{\mathbb{R}}^{\oplus 5}$ is trivial then there exists a $\varphi$-equivariant embedding of $H^2(X, \mathbb{Z})$ in $\Lambda$.

**Proof.** Let $[v_1/2]$ and $[v_2/2]$ be two generators of $A_X$ such that $v_1^2 = -2$ and $v_2^2 = -2$. It holds that $\varphi([v_1/2]) = [v_1/2]$ and $\varphi([v_2/2]) = [v_2/2]$ i.e. $\varphi(v_1) = v_1 + 2v_1$ and $\varphi(v_2) = v_2 + 2v_2$. Consider now a lattice of rank 2 generated by two orthogonal elements $x_1$ and $x_2$ of square 2, its discriminant group is still $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ and it is generated by $[x_1/2]$ and $[x_2/2]$ with discriminant form given by $q(x_1/2) = 1/2$, $q(x_2/2) = 1/2$ and $(x_1, x_2) = 0$. Notice that $H^2 \oplus \mathbb{Z}x_1 \oplus \mathbb{Z}x_2$ has an overlattice isometric to $\Lambda$ which is generated by $H^2$, $x_1 + v_1$ and $x_2 + v_2$. We extend $\varphi$ on $H^2 \oplus x_1 \oplus x_2$ by imposing $\varphi(x_1) = x_1$, $\varphi(x_2) = x_2$ and we obtain an extension $\overline{\varphi}$ of $\varphi$ on $\Lambda$ defined as follows:

\[ \overline{\varphi}(e) = \varphi(e) \quad \forall e \in H^2, \]
\[ \overline{\varphi}(x_1) = x_1, \]
\[ \overline{\varphi}(x_2) = x_2, \]
\[ \overline{\varphi}(\frac{x_1 + v_1}{2}) = \frac{x_1 + \varphi(v_1)}{2}, \]
\[ \overline{\varphi}(\frac{x_2 + v_2}{2}) = \frac{x_2 + \varphi(v_2)}{2}. \]

**Lemma 2.13.** If $X$ is an manifold of $\text{OG}_6$ type and if $\varphi \in O(H^2(X, \mathbb{Z}))$ is an isometry then the induced action on $A_X$ is trivial if and only if the induced action on $A_{\text{SG}(X)}$ is trivial.

**Proof.** We know that the finite index embedding $S_G(X) \oplus T_G(X) \subseteq H^2(X, \mathbb{Z})$ is given by an isotropic subgroup $H$ such that $H \subseteq A_{\text{SG}(X)} \oplus A_{T_G(X)}$ and $A_X = H^\perp/\Lambda$, hence if there exists a non trivial action on an element of $A_X$ it comes from a non trivial action on an element of $A_{\text{SG}(X)}$.

On the other hand, if the action of $G$ is trivial on $A_X$, then we can consider a primitive embedding $H^2(X, \mathbb{Z}) \hookrightarrow \Lambda = U_{\mathbb{R}}^{\oplus 5}$ and extend the action of $G$ trivially on the orthogonal complement. We know that $A_{\text{SG}(\Lambda)} \cong A_{\text{SG}(X)}$ and $A_{T_G(\Lambda)} \cong A_{\text{SG}(\Lambda)}$ since $\Lambda$ is unimodular and the embedding is $G$-equivariant. Obviously the action of $\varphi$ on $A_{T_G(\Lambda)}$ is trivial and so we can conclude that the same holds for $A_{\text{SG}(X)}$. \qed

The following lemma is useful throughout the classification.

**Lemma 2.14** (Mongardi–Tari–Wandel [33] Lemma 1.4). If $L$ is a lattice and $G \subset O(L)$ then the following hold:

- $T_G(L)$ contains $\sum_{g \in G} gv$ for all $v \in L$. 
- $S_G(L)$ contains $v - gv$ for all $v \in L$ and all $g \in G$.
- $R/(T_G(L) \oplus S_G(L))$ is of $|G|$-torsion.

**Definition 2.15.** Let $q$ be a 2-elementary quadratic form on a finite abelian group $G$. We define

\[
\delta(q) = \begin{cases} 
0 & \text{if } q(x) \in \mathbb{Z}/2\mathbb{Z} \text{ for all } x \in G \\
1 & \text{otherwise}
\end{cases}
\]

If $L$ is an even lattice we set $\delta(L) := \delta(q_L)$.

We recall two results that characterize even hyperbolic $p$-elementary lattices, the first one concerns the case $p = 2$, the second one the case $p \neq 2$.

**Theorem 2.16** (Nikulin [38]). An even hyperbolic 2-elementary lattice $L$ of rank $r$ and such that $A_L \cong (\mathbb{Z}/2\mathbb{Z})^a$ is uniquely determined by the invariants $(r, a, \delta)$, where $\delta = \delta(L)$, and exists if and only if the following conditions are satisfied

\[
\begin{cases} 
1 & \text{if } \delta = 0, \text{ then } r \equiv 2 \pmod{4} \\
2 & \text{if } a \equiv 0 \pmod{2}, \text{ then } \delta = 0 \\
3 & \text{if } a \leq 1, \text{ then } r \equiv 2 \pm a \pmod{8} \\
4 & \text{if } a = 2 \text{ and } r \equiv 6 \pmod{8}, \text{ then } \delta = 0 \\
5 & \text{if } \delta = 0 \text{ and } a = r, \text{ then } r \equiv 2 \pmod{4}
\end{cases}
\]

**Theorem 2.17** (Rudakov–Shafarevich [46, Section 1]). An even hyperbolic $p$-elementary lattice of rank $r$ and such that $A_L \cong (\mathbb{Z}/p\mathbb{Z})^a$ with $p \neq 2$ with invariants $(r, a)$ exists if and only if the following conditions are satisfied

\[
\begin{cases} 
1 & \text{if } a \equiv 0 \pmod{2}, \text{ then } r \equiv 2 \pmod{4} \\
2 & \text{if } a \equiv 1 \pmod{2}, \text{ then } p \equiv (-1)^{r/2-1} \pmod{4} \\
3 & \text{if } r \neq 2 \pmod{8}, \text{ then } r > a > 0
\end{cases}
\]

Such a lattice is uniquely determined by the invariants $(r, a)$ if $r \geq 3$.

The following theorem explains how to split a lattice to use the previous results.

**Theorem 2.18** (Nikulin [37 Corollary 1.13.5]). Let $L$ be an even lattice of rank $r$. If $L$ is indefinite, and $r \geq 3 + \ell(A_L)$, then $L \cong U \oplus L_0$ for some lattice $L_0$.

The following lemma allows us to establish some restrictions to have a prime order action on a lattice.

**Lemma 2.19** (Mongardi–Tari–Wandel [33 Lemma 1.8]). Let $L$ be a lattice and $G \subset O(L)$ a subgroup generated by $\varphi$ of prime order $p$. Then

\[
m := \frac{\ell(S_G(L))}{p-1}
\]

is an integer and

\[
\frac{L}{T_G(L) \oplus S_G(L)} \cong (\mathbb{Z}/p\mathbb{Z})^a.
\]
Moreover there are natural embeddings of \( \frac{L}{L(L) \oplus S(L)} \) into the discriminant group \( A_{T_Q(L)} \) and \( A_{S_G(L)} \), and \( a \leq m \).

We need also to recall the following results that will be useful throughout the classification.

**Theorem 2.20** (Nikulin [37 Theorem 1.13.3]). Let \( L \) be an even lattice with signature \((r_+, r_-)\) and discriminant form \( q_L \). If \( L \) is indefinite and \( l(A_L) \leq rk(L) - 2 \), then \( L \) is the only lattice up to isometry with invariants \((r_+, r_-, q_L)\).

**Proposition 2.21** (Boissière–Camere–Mongardi–Sarti [7 Section 4]). Let \( L \) be a lattice with a nontrivial action of order \( p \), with rank \( p - 1 \), and discriminant \( d_L \), then \( \frac{d_L}{p^{m_L}} \) is a square in \( \mathbb{Q} \).

A morphism between two lattices is a linear map that respects the quadratic forms. An injective morphism is called a **primitive embedding** if its cokernel is torsion free.

The following fundamental result proved by Nikulin characterizes primitive embeddings.

**Theorem 2.22** (Nikulin [37 Proposition 1.15.1]). Let \( S \) be an even lattice of signature \((s'(+) s'(-))\) and discriminant form \( q_S \). Primitive embeddings \( i: S \to L \), for \( L \) an even lattice of signature \((l(+) l(-))\) and discriminant form \( q_L \), are determined by quintuples \( \theta_i := (H_S, H_L, \gamma, T, \gamma_T) \) such that:

- \( H_S \) is a subgroup of \( A_S \), \( H_L \) is a subgroup of \( A_L \) and \( \gamma: H_S \to H_L \) is an isomorphism such that \( q_S|H_S \cong q_L|H_L \);
- \( T \) is a lattice of signature \((l(+) - s'(+) s'(-))\) and discriminant form \( q_T \) is the graph of \( \gamma \) and \( \Gamma^\perp \) is its orthogonal complement in \( A_S \oplus A_L \) with respect to the finite bilinear form associated to \( -q_S \oplus q_L \);
- \( \gamma_T \in O(A_T) \).

The lattice \( T \) is isomorphic to the orthogonal complement of \( i(S) \) in \( L \). Moreover, two quintuples \( \theta \) and \( \theta' \) define isomorphic primitive sublattices if and only if \( \overline{\theta}(H_S) = H_S' \) for some \( \mu \in O(S) \) and there exists \( \phi \in O(A_L) \), \( v: T \to T' \) isometries such that \( \gamma' \circ \overline{\gamma} = \varphi \circ \gamma \) and \( \overline{\gamma} \circ \gamma_T = \gamma_T' \circ \overline{\gamma} \), where \( \overline{\gamma} \) and \( \overline{\tau} \) are the induced isometries on the discriminant groups.

Another crucial result due to Nikulin is the following.

**Proposition 2.23** (Nikulin [37 Proposition 1.5.1]). A primitive embedding of an even lattice \( S \) into another even lattice with discriminant form \( q \) and orthogonal complement isomorphic to \( K \), is determined by a pair \( (H, \gamma) \), where \( H \subset A_S \) is a subgroup and \( \gamma: H \to AK \) is a group monomorphism, while \( q_K \circ \gamma = -q_S|H \) and

\[
(q_S \oplus q_K | (\Gamma_{\gamma})^\perp)/\Gamma_{\gamma} \cong q,
\]

where \( \Gamma_{\gamma} \) is the graph of \( \gamma \) in \( A_S \oplus AK \). Two such pairs \( (H, \gamma) \) and \( (H', \gamma') \) determine isomorphic primitive embeddings if and only if \( H = H' \), and the injections \( \gamma \) and \( \gamma' \) are conjugate via some automorphism of \( K \), and they determine primitive sublattices when there exist \( \varphi \in O(S) \) and \( \psi \in O(K) \) such that \( \gamma \circ \overline{\gamma} = \varphi \circ \gamma' \), where \( \overline{\gamma} \) and \( \overline{\psi} \) are the induced isometries on the discriminant groups.

The main results regarding existence and uniqueness of an indefinite lattice are the following.
Theorem 2.24. Let $L$ be an even lattice with discriminant quadratic form $q_L$ and signature $(l_+,l_-)$, with $l_+ \geq 1$ and $l_- \geq 1$. Up to isometry $L$ is the only lattice with invariants $(l_+,l_-,q_L)$ in all of the following cases:

(i) $l_+ + l_- \geq l(A_L) + 2$;
(ii) $l_+ + l_- \geq 3$ and $\text{discr}(L) \leq 127$;
(iii) $l_+ + l_- \geq 3$ and $L$ is $p$-elementary, with $p$ odd;
(iv) $L$ is $2$-elementary.

Proof. The statement combines \cite{[37]} Corollary 1.13.3, \cite{[18]} Chapter 15, Corollary 22, \cite{[33]} Theorem 2.2 and \cite{[19]} Theorem 1.5.2. \hfill $\square$

Theorem 2.25 (Nikulin \cite{[38]}). An even lattice with invariants $(t_+,t_-,q)$ exists if and only if the following conditions are simultaneously satisfied:

1. $t_+ - t_- \equiv \text{sign } q \pmod{8}$;
2. $t_+ \geq 0$, $t_- \geq 0$, $t_+ + t_- \geq l(A_q)$;
3. $(-1)^t |A_q| \equiv \text{discr}(q_p)(\text{mod}(Z_p^2))$ for all odd primes for which $t_+ + t_- = l(A_q)$;
4. $|A_q| = \pm \text{discr}(q_p)(\text{mod}(Z_p^2))$ if $t_+ + t_- = l(A_q)$ and $q_2 \neq q_2^{(2)} + q_2'$.

Finally it is crucial to recall the following result.

Theorem 2.26 (Huybrechts \cite{[23]} Proposition 26.13). Let $X$ be an irreducible holomorphic symplectic manifold. Then $X$ is projective if and only if there exists $l \in \text{NS}(X)$ such that $q_X(l) > 0$.

An equivalent formulation of the previous projectivity criterion is that an irreducible holomorphic symplectic manifold $X$ is projective if and only if the Néron–Severi lattice $\text{NS}(X) \subset H^2(X,\mathbb{Z})$ is hyperbolic.

2.5. O’Grady’s sixfolds. In this section we recall some fundamental results about O’Grady’s sixfolds. O’Grady \cite{[10]} considers $(A,\theta)$ a principally polarized abelian surface and $M_v(A,\theta)$ the moduli space of Gieseker semistable sheaves on $A$ with respect to the $v$-generic stability condition $\theta$ and to a non-primitive Mukai vector $v = (\text{rk}(\mathcal{F}), c_1(\mathcal{F}), \chi(\mathcal{F})) = 10v_0$, $v_0 = (1, 0, -1)$, $v_0^2 = 2$ with respect to the Mukai pairing. Fix $\mathcal{F}_0 \in M_v(A,\theta)$ and consider the map

$$a_v : M_v(A,\theta) \to A \times A^V$$

$$\mathcal{F} \mapsto \left( \sum_{\mathcal{F} \in [\mathcal{F}]} c_2(\mathcal{F}), \det(\mathcal{F}) \otimes \det(\mathcal{F}_0)^{-1} \right).$$

It is an isotrivial fibration. We define $K_v(A,\theta) = a_v^{-1}(0)$. In this case since $v$ is not primitive the moduli space $M_v(A,\theta)$ is singular and admits a symplectic resolution $\pi_v : \tilde{M}_v(A,\theta) \to M_v(A,\theta)$. Then we define $\tilde{a}_v = a_v \circ \pi_v$, and $\tilde{a}_v^{-1}(0) = K_v(A,\theta)$. The map $f_v : \tilde{K}_v(A,\theta) \to K_v(A,\theta)$ is a crepant symplectic resolution and the manifold $K_v(A,\theta)$ is a six dimensional irreducible holomorphic symplectic manifold with second Betti number 8 called O’Grady’s sixfold.

Remark 2.27. If $v$ is primitive and $\theta$ is $v$-generic then $M_v(A,\theta)$ is smooth and also the fiber $K_v(A,\theta)$ is smooth and it is a generalized Kummer manifold.

Later M. Lehn and Soreger \cite{[17]} showed that, under the previous assumption on $w$, the blow up of the fiber of the moduli space along its singular locus always gives a crepant resolution and Perego and Rapagnetta \cite{[14]} proved that these crepant
resolutions are deformation equivalent, along smooth projective deformations, to the original O’Grady example.

2.6. Monodromy and Kähler cone. Let \( X \) be an irreducible holomorphic symplectic manifold whose second cohomology lattice \((H^2(X, \mathbb{Z}), q_X)\) is isometric to a lattice \( L \).

**Definition 2.28.** A marking of \( X \) is a choice of an isometry \( \eta : H^2(X, \mathbb{Z}) \to L \). The pair \((X, \eta)\) is called a marked irreducible holomorphic symplectic manifold. Two marked irreducible holomorphic symplectic manifolds \((X, \eta), (X', \eta')\) are isomorphic if there exists an isomorphism \( f : X \to X' \) such that \( f^* = \eta^{-1} \circ \eta' \).

We can quotient the set of marked irreducible holomorphic symplectic pairs \((X, \eta)\) by the isomorphism relation and we obtain a moduli space.

\[
\mathcal{M}_L := \{(X, \eta) \mid \eta : H^2(X, \mathbb{Z}) \to L \text{ a marking } \} / \cong .
\]

The set \( \mathcal{M}_L \) can be endowed with a structure of compact complex space.

**Definition 2.29.** Let \( X \) and \( Y \) be holomorphic symplectic manifolds. A lattice isometry \( f : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z}) \) is a parallel transport operator if there exists a smooth and proper family \( \pi : \mathcal{X} \to B \) and a continuous path \( \gamma : [0, 1] \to B \), such that \( X \cong \mathcal{X}_{\gamma(0)} \) and \( Y \cong \mathcal{X}_{\gamma(1)} \). Moreover we ask that \( f \) is induced by parallel transport in the local system \( R^2\pi_* \mathbb{Z} \) along \( \gamma \). A parallel transport operator \( f : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z}) \) is called a monodromy operator of \( X \).

The following is a necessary condition for an isometry \( g : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z}) \) to be a parallel transport operator.

**Definition 2.30.** Let \( X \) be an irreducible holomorphic symplectic manifold. The positive cone \( \mathcal{C}_X \) of \( X \) is the connected component of the cone \( \{ \alpha \in H^{1,1}(X, \mathbb{R}) : (\alpha, \alpha) > 0 \} \) which contains the Kähler cone.

Denote by \( \mathcal{C}_X \subset H^2(X, \mathbb{R}) \) the cone \( \{ \alpha \in H^2(X, \mathbb{R}) : (\alpha, \alpha) > 0 \} \). The second cohomology \( H^2(\mathcal{C}_X, \mathbb{Z}) \cong \mathbb{Z} \) comes with a canonical generator, which we call the orientation class of \( \mathcal{C}_X \). Any isometry \( g : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z}) \) induces an isomorphism \( 7 : \mathcal{C}_X \to \mathcal{C}_Y \). The isometry \( g \) is said to be orientation preserving if \( 7 \) is. The subgroup of orientation preserving isometries is denoted by \( \text{O}^+(H^2(X, \mathbb{Z})) \). A parallel transport operator \( g : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z}) \) is orientation preserving (see [27, Section 4]).

We denote by \( \text{Mon}^2(X) \subset \text{O}(H^2(X, \mathbb{Z})) \) the subgroup of monodromy operators, which is of finite index (see [24, Lemma 7.5]). In particular two marked pairs \((X, \eta)\) and \((X', \eta')\) belong to the same connected component of \( \mathcal{M}_L \) if and only if \( \eta^{-1} \circ \eta' \) is a parallel transport operator. As a consequence, the number of connected components of \( \mathcal{M}_L \) is \( \pi_0(\mathcal{M}_L) = [\text{O}(H^2(X, \mathbb{Z})) : \text{Mon}^2(X)] \).

The monodromy group has been studied and completely described for all known irreducible holomorphic symplectic manifolds. If \( X \) is a manifold of \( \text{OG}_6 \) type we have the following result due to Mongardi and Rapagnetta.

**Theorem 2.31** (Mongardi–Rapagnetta [30, Theorem 5.4]). If \( X \) be a manifold of \( \text{OG}_6 \) type then \( \text{Mon}^2(X) = \text{O}^+(H^2(X, \mathbb{Z})) \).
The previous result implies that $\mathcal{M}_L$ has two connected components. If $\mathcal{M}_L^0$ denotes a connected component of the moduli space $\mathcal{M}_L$ and if $(X, \eta) \in \mathcal{M}_L^0$ then $(X, -\eta) \notin \mathcal{M}_L^0$. In fact $-id_{H^2(X, \mathbb{Z})}$ is not a monodromy operator as it does not preserve the orientation of the positive cone.

In the case of K3 surfaces the Kähler cone coincides with the set of real $(1,1)$-classes which have positive intersection with all rational curves on the surface. Boucksom (see [13, Theorem 1.2]) generalizes the result for any irreducible holomorphic symplectic manifold $X$ and we have

$$\mathcal{K}_X = \{ \alpha \in \mathcal{C}_X | \int_C \alpha > 0 \text{ for all rational curves } C \subset X \}.$$  

**Definition 2.32.** Let $X$ be an irreducible holomorphic symplectic manifold. The *birational Kähler cone* is

$$\mathcal{BK}_X = \bigcup_{f: X \to X'} f^* \mathcal{K}_X,$$

where $f: X \to X'$ runs through all birational maps $X \rightarrow X'$ from $X$ to another irreducible holomorphic symplectic manifold $X'$.

**Remark 2.33.** If $G \subset \text{Aut}(X)$ is a finite group of automorphisms of $X$, and $\varphi \in G$, then $\varphi^* \in O(H^2(X, \mathbb{Z}))$ is a Hodge-monodromy operator. This means that $\varphi$ is a Hodge operator and that $\varphi^* \in \text{Mon}^2(X) \subset O(H^2(X, \mathbb{Z}))$.

The monodromy group the Kähler cone and the birational Kähler cone are involved in the study of automorphisms, in particular are useful to study the image of the representation map mentioned in Definition 2.5. The main result that we can use to study effective isometries (Definition 2.5) is the Hodge theoretic Torelli theorem due to Markman and Verbitsky that we recall here for sake of completeness.

**Theorem 2.34** (Markman [27, Theorem 1.3]). If $X_1$ and $X_2$ are irreducible holomorphic symplectic manifolds which are deformation equivalent then:

1. $X_1$ and $X_2$ are bimeromorphic if and only if there exists a parallel transport operator $f: H^2(X, \mathbb{Z}) \to H^2(X_2, \mathbb{Z})$ which is an isometry of integral Hodge structures.
2. Let $f: H^2(X_1, \mathbb{Z}) \to H^2(X_2, \mathbb{Z})$ be parallel transport operator which is an isometry of integral Hodge structures. There exists a birational transformation $\tilde{f}: X_1 \to X_2$ such that $f = \tilde{f}^*$ if and only if $f$ maps a class of the birational Kähler cone of $X_1$ in a class of the birational Kähler cone of $X_2$.
3. Let $f: H^2(X_1, \mathbb{Z}) \to H^2(X_2, \mathbb{Z})$ be parallel transport operator which is an isometry of integral Hodge structures. There exists an isomorphism $\tilde{f}: X_1 \to X_2$ such that $f = \tilde{f}^*$ if and only if $f$ maps some Kähler class on $X_1$ to a Kähler class on $X_2$.

### 2.7. Birational models of O’Grady’s sixfolds

In this paper we refer to the geometrical construction of automorphisms of O’Grady’s sixfolds, and in order to do this we need to recall the birational models of a manifold of OG_6 type. These models are contained in a locus of codimension $\geq 1$ in the moduli space of polarized manifold of OG_6 type. Indeed one model is given by the original construction due to O’Grady ([40]) which we have recalled in Section 2.5 that involves a moduli space of sheaves on an abelian surface, the other one has been more recently introduced by Mongardi, Rapagnetta and Saccà ([31]) and consists in the resolution of singularities of the quotient of a sixfold of $K3^{[3]}$ type by a birational symplectic involution.
In [21] the author gives a lattice-theoretic criterion to identify when a manifold of OG\(_6\) type is birational to an O'Grady's moduli space.

Consider a class \(\sigma \in H^2(X, \mathbb{Z})\) such that \(\sigma^2 = -2\) and \(\text{div}(\sigma) = 2\) and consider an embedding

\[
\sigma : H^2(X, \mathbb{Z}) \cong U^2 \oplus \langle -2 \rangle \hookrightarrow \Lambda_8 = U^4
\]

such that the lattice \(\Lambda_8\) is endowed with the induced Hodge structure, and the embedding is an Hodge embedding such that the orthogonal complement of the image is of (1,1) type. We have the following definition.

**Definition 2.35** (Grossi [21, Definition 3.1]). If \(X\) is a projective manifold of OG\(_6\) type, \(X\) is a *numerical moduli space with respect to \(\sigma\)* if there exists a class \(\sigma \in \text{NS}(X)\) such that \(\sigma^2 = -2\) and \(\text{div}(\sigma) = 2\) and, through the previous embedding, \(\Lambda_8\) contains a copy of the hyperbolic lattice \(U\).

**Proposition 2.36** (Grossi [21, Proposition 3.4]). If \(X\) is a manifold of OG\(_6\) type which is a numerical moduli space, then there exists an abelian surface \(A\) such that \(X\) is birational to \(\tilde{K}_u(A, \theta)\) (see Section 2.5 for a definition of \(\tilde{K}_u(A, \theta)\)).

Moreover in [21] the author summarize a criterion to determine when a manifold of OG\(_6\) type is birational to the quotient of a manifold of \(K3\)\([3]\) type by a birational symplectic involution. The result was implicitly contained in [31].

**Theorem 2.37** (Grossi [21, Theorem 4.3]). If \(X\) is a manifold of OG\(_6\) type and if there exists \(E \in \text{NS}(X)\) such that \(E^2 = -2\) and \(\text{div}(E) = 2\), then there exists a \(K3\) surface \(S\) such that \(X\) is birational to \(Y\). Here \(Y\) is the resolution of singularities of \(S^3/i\) i.e. the blow up of the singular locus of \(S^3/i\), where \(i : S^3 \to S^3\) is a birational symplectic involution and \(S^3\) is the Hilbert scheme of 3 points on \(S\). In this case we say that \(X\) admits a MRS model, after Mongardi–Rapagnetta–Saccà.

2.8. **Induced automorphisms and automorphisms induced at the quotient.** We recall briefly the notions of induced automorphisms and automorphisms induced at the quotient of O'Grady’s sixfolds. For more details we refer to [21].

**Definition 2.38** (Grossi [21, Definition 3.5]). Let \(X\) be a projective manifold of OG\(_6\) type and let \(G \subset \text{Aut}(X)\) a finite group. We say that \(G\) is an induced group of automorphisms if there exists an abelian surface \(A\) with \(G \subset \text{Aut}(A)\), a \(G\)-invariant non-primitive Mukai vector \(v = 2w, v \in H^2(A, \mathbb{Z})\), \(u\), and a \(v\)-generic stability condition \(\theta\) which is \(G\)-invariant, such that \(X\) is birational to \(\tilde{K}_u(A, \theta)\) (see Section 2.5), and the induced action on \(\tilde{K}_u(A, \theta)\) coincides with the given action of \(G\) on \(X\).

If \(X\) is a projective manifold of OG\(_6\) type and \(G \subset \text{Aut}(X)\) is a finite group and the embedding

\[
i : H^2(X, \mathbb{Z}) \hookrightarrow \Lambda = U^5
\]

is primitive, then we can endow \(\Lambda_{10}\) with the unique Hodge structure induced by \(H^2(X, \mathbb{Z})\), in a way such that the orthogonal complement of the embedding is of (1,1)-type.

**Definition 2.39** (Grossi [21, Definition 3.6]). The group \(G \subset \text{Aut}(X)\) is called *numerically induced* if the following requests hold.

1. The group \(G\) induces a trivial action on the discriminant group \(A_X\); the action can be extended to the lattice \(\Lambda_{10}\) and \(S_G(\Lambda_{10}) \cong S_G(X)\).
(2) There exists $\sigma \in \text{NS}(X)$, such that $\sigma^2 = -2$ and $\text{div}(\sigma) = 2$ and $\sigma$ is $G$-invariant, such that $H^2(X, \mathbb{Z}) \hookrightarrow \Lambda_{10}$ is a Hodge embedding such that the $(1,1)$-part of the lattice $T_G(\Lambda_{10})$ contains $U^{\oplus 2}$ as a direct summand.

(3) For all $g \in G$, $\det(g^*) = 1$.

The following result holds.

**Theorem 2.40** (Grossi [21, Theorem 3.9]). Let $X$ be a projective manifold of OG$_6$ type and $G \subset \text{Aut}(X)$ a numerically induced group of automorphisms of $X$. Then $X$ is birational to $\bar{\mathcal{K}}_6(A, \theta)$ (see Section 2.5 for a definition of $\bar{\mathcal{K}}_6(A, \theta)$) and the automorphism is induced.

For the birational model realized as the quotient of an Hilbert cube on a K3 surface by a symplectic involution (see [31] for more details), the natural question is to ask if an automorphism of an O’Grady’s sixfold that admits this birational model lifts to an automorphism of the Hilbert scheme of three points on a K3 surface. I call the automorphisms with such a property automorphisms induced at the quotient.

**Definition 2.41** (Grossi [21, Definition 4.4]). Let $X$ be a manifold of OG$_6$ type, and let $S$ be a K3 surface such that $X \xrightarrow{\psi} S^{[3]}/i$ is a bimeromorphic map. Let $\varphi \in \text{Aut}(X)$ be an automorphism of the O’Grady’s sixfold, $\varphi$ is induced at the quotient if $\varphi$ can be lifted to an automorphism $\varphi \in \text{Aut}(S^{[3]})$ such that the induced action on the quotient coincides with $\varphi$.

\[ \begin{array}{c}
\tau \subset \text{Bl}_Y \to Y' \\
\sigma_1 \circ i \subset Y \to Y \xrightarrow{\psi} \bar{\mathcal{K}} \sim \text{OG}_6 \\
\varphi \circ \bar{\mathcal{K}} \sim \text{OG}_6 \to K \xrightarrow{\text{bir}} Y/i \end{array} \]

In this diagram $Y$ is a manifold of $K3^{[3]}$ type, and the map $Y \to Y$ is a contraction of 256 $\mathbb{P}^3$’s. The involution $i$ is birational on $Y$ and regular on $Y$. The manifold $Y$, which is singular in 256 points, is a 2:1 cover of $K$, which is a singular manifold of OG$_6$ type. $\bar{\mathcal{K}}$ is the resolution of singularities of $K$ and it is birational to $Y/i$.

**Theorem 2.42** (Grossi [21, Theorem 4.15]). If $X$ is a manifold of OG$_6$ type which admits a MRS birational model (see Theorem 2.37), and if $Y$ is the 2:1 cover of $X$ described in the diagram above, and if $\psi$ is the induced map on $Y$, if $\varphi \in \text{Aut}(X)$
is an automorphism of prime order $p$, $p \neq 2$, such that $\text{Sing}(Y) \subset \text{Fix}(\psi)$ and if there exists a class $E \in \text{NS}(X) \cap T_\varphi(X)$ such that $E^2 = -2$ and $\text{div}(E) = 2$, then $\varphi$ is induced at the quotient.

In particular we refer to these notions to identify which of the nonsymplectic isometries classified in the following sections fall into these two classes of induced actions (see [34]).

3. Invariant and coinvariant lattices in $\Lambda$

First of all, we need to recall the following results due to Nikulin and contained in [36].

**Proposition 3.1.** If $X$ is an irreducible holomorphic symplectic manifold and $G \subset \text{Aut}(X)$ is a finite group where all nontrivial elements are purely nonsymplectic automorphisms, then $G$ is cyclic.

**Proposition 3.2.** If $X$ is an irreducible holomorphic symplectic manifold and $G \subset \text{Aut}(X)$ is cyclic group of automorphisms generated by a nonsymplectic element of maximal order $m$, then $\varphi(m)/\text{rk}(T(X))$, where $\varphi$ is the Euler function. In particular $\text{rk}(T(X)) = \varphi(m) \cdot n$ for some $n \in \mathbb{N}$ and this implies that $\varphi(m) \leq b_2(X) - \text{rk}(\text{NS}(X))$.

**Remark 3.3.** If $X$ be an irreducible holomorphic symplectic manifold and $G \subset O(H^2(X, \mathbb{Z}))$ is a finite group where all nontrivial elements are nonsymplectic isometries then in the moduli space of pairs $(X, G)$ the generic element is such that $T_G(X) = \text{NS}(X)$ and $T(X) = S_G(X)$. If the action is symplectic the generic point of the moduli space of the pairs $(X, G)$ is such that $T(X) = T_G(X)$ and $S_G(X) = \text{NS}(X)$. See [41] for more details.

**Proposition 3.4.** If $X$ is an irreducible holomorphic symplectic manifold of OG$_6$ type and if $G \subset \text{Aut}(X)$, $G \cong \mathbb{Z}/p\mathbb{Z}$ is a cyclic group generated by $\varphi$ which is a nonsymplectic automorphism, and $p$ is a prime number then $p \in \{2, 3, 5, 7\}$.

**Proof.** From Proposition 3.2 there exists $n \in \mathbb{Z}$ such that $\text{rk}(T(X)) = \phi(p) \cdot n \leq 7$. Thus $\phi(p) = p - 1 \leq 7$ and this implies that $p \in \{2, 3, 5, 7\}$. □

The main goal of this paper is to investigate the image of the representation map for manifolds of OG$_6$ type

$$\nu : \text{Aut}(X) \to O(H^2(X, \mathbb{Z}), q_X).$$

The following result allows to classify nonsymplectic automorphisms of prime order starting from their action on the second integral cohomology.

**Proposition 3.5.** If $X$ is an irreducible holomorphic symplectic manifold of OG$_6$ type and $\varphi \in O(H^2(X, \mathbb{Z}))$ is a nonsymplectic isometry of prime order $p \in \{3, 5, 7\}$ then $\varphi$ is effective, if $p = 2$ then $\varphi$ is effective up to a sign.

**Proof.** To prove this result we need Theorem 2.34. Since $\varphi$ is a nonsymplectic isometry of prime order we have $\varphi(\sigma_X) = \alpha \sigma_X$ where $\alpha \neq 1$, $\alpha \in \mathbb{C}$ is a $p$-root of unity. Let $G = \langle \varphi \rangle \cong \mathbb{Z}/p\mathbb{Z}$ be the group of isometries generated by $\varphi$. In this setting $T(X) \subset S_G(X)$. Since $T_G(X)$ and $S_G(X)$ are in direct sum and the same holds for $T(X)$ and $\text{NS}(X)$, this implies that $T_G(X) \subset \text{NS}(X)$. From Remark 3.3 it is not restrictive to consider $T_G(X) = \text{NS}(X)$ and consequently $\varphi \in O(H^2(X, \mathbb{Z}))$.
is a Hodge isometry.

We know by [30, Theorem 5.4(1)] that in the \( \text{OG}_6 \) case \( \text{Mon}^2(X) = O^+(H^2(X, \mathbb{Z})) \). Since \( \varphi \) is of prime order \( p \), if \( p \neq 2 \) then \( \varphi \) preserves the orientation of the positive cone i.e. \( \varphi \in \text{Mon}^2(X) \), if \( p = 2 \) then \( \pm \varphi \) preserves the orientation of the positive cone, which means that \( \pm \varphi \in \text{Mon}^2(X) \). In fact, if \( \varphi \), an isometry of prime order \( p \neq 2 \), does not preserve the orientation of the positive cone then \( \varphi^p = \text{id} \) does not preserve the orientation of the positive cone since \( p \) is odd, but this is a contradiction.

The last thing that we need to check is that a Kähler class is sent to a Kähler class. In order to show this we need to show that in this hypothesis, if we consider the generic element of the pair \((X, G)\), the manifold \( X \) is projective. There exists an invariant positive class, the average Kähler class defined as

\[
\omega_G = \sum_{\varphi \in G} \varphi^*(\omega),
\]

where \( \omega \) is a Kähler class. The class \( \omega_G \in T_G(X) \otimes \mathbb{R} \) and this implies that \( \omega_G \in \text{NS}(X) \otimes \mathbb{R} \). Since the symplectic form \( \sigma_X \in T(X) \otimes \mathbb{C} \) and \( \Re(\sigma_X) \) and \( 3(\sigma_X) \) are two positive integral classes in \( T(X) \) hence \( \text{sgn}(T(X)) = (2, *) \). The class \( \omega_G \) is positive, hence \( \text{sgn}(\text{NS}(X)) = (1, *) \). For the Huybrechts’ projectivity criterion, since there is a positive class in \( \text{NS}(X) \), then \( X \) is projective. Since \( \text{NS}(X) = T_G(X) \) then there exists an invariant ample class. In this way we conclude that there exists a Kähler class which is preserved by \( \varphi \) and for this reason \( \varphi \) is effective. \( \square \)

It is convenient here to consider an embedding into a unimodular lattice:

\[ H^2(X, \mathbb{Z}) \hookrightarrow \Lambda = U^{\oplus 5}. \]

We observe that such an embedding is unique up to an isometry of \( \Lambda \). Since \( X \) is a manifold of \( \text{OG}_6 \) type then \( H^2(X, \mathbb{Z}) \cong U^{\oplus 3} \oplus (-2)^{\oplus 2} \); the embedding sends \( U^{\oplus 3} \) identically into the first three copies of \( U \) and sends \( n_1 \) and \( n_2 \) which are the two generators of \((U^{\oplus 3})^\perp \subset H^2(X, \mathbb{Z})\) such that \( n_1^2 = n_2^2 = -2 \), to \( e_4 - f_4 \) and to \( e_5 - f_5 \), respectively, where \( e_4, f_4, e_5, f_5 \) form the usual basis of the last two copies of \( U^{\oplus 2} \).

**Remark 3.6.** The discriminant group of \( X \) is \( \text{Ax}_X \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \). Let \([1, 0]\) and \([0, 1]\) be the two generators of \( \text{Ax}_X \). The induced action of a group of automorphisms on the discriminant group can be of two types: trivial or nontrivial. The nontrivial action is an action which exchanges the generators of the two copies of \( \mathbb{Z}/2\mathbb{Z} \), i.e. \([1, 0]\) and \([0, 1]\). Let \( \varphi \in O(H^2(X, \mathbb{Z})) \) and let \( \tilde{\varphi} \) be the extended action on \( \Lambda \). We know from Lemma 2.12 that if the induced action of \( \varphi \) on \( \text{Ax} \) is trivial then the embedding of \( H^2(X, \mathbb{Z}) \) in \( \Lambda \) is \( \varphi \)-equivariant. In fact, if the action of \( \varphi \in G \) is trivial on \( \text{Ax} \) then \( \tilde{\varphi}(\epsilon_4 + f_4) = \epsilon_4 + f_4 \) and \( \tilde{\varphi}(\epsilon_5 + f_5) = \epsilon_5 + f_5 \), otherwise if the action is not trivial on \( \text{Ax} \) then \( \tilde{\varphi}(\epsilon_4 + f_4) = \epsilon_5 + f_5 \) and \( \tilde{\varphi}(\epsilon_5 + f_5) = \epsilon_4 + f_4 \).

**Remark 3.7.** If the group \( G \) is of prime order \( p \), since \( \Lambda \) is unimodular, then \( T_G(\Lambda) \) and \( S_G(\Lambda) \) are \( p \)-elementary lattices (see Lemma 2.14). This property does not hold in general for \( S_G(X) \) and \( T_G(X) \).

**Proposition 3.8.** Let \( X \) be a manifold of \( \text{OG}_6 \) type and \( G \subset \text{Aut}(X) \) be a group of order \( p \). Consider the primitive embedding of lattices \( i : H^2(X, \mathbb{Z}) \hookrightarrow \Lambda \), and define \( v_1 \) and \( v_2 \) the two vectors of square 2 that are generators of the orthogonal complement of \( i(H^2(X, \mathbb{Z}))^{\perp_{\Lambda}} \cong (2)^{\oplus 2} \).
1. If the induced action of $G$ on $A_X$ is trivial then the embedding in $G$-equivariant and

$$SG(X) \cong SG(\Lambda) \quad TG(X) \cong (v_1 \oplus v_2)^\perp \subset TG(\Lambda).$$

Moreover, it holds that

$$\text{sgn}(SG(\Lambda)) = \text{sgn}(SG(X))$$

$$\text{sgn}(TG(\Lambda)) = \text{sgn}(TG(X)) + (2, 0).$$

2. If the induced action of $G$ on $A_X$ is nontrivial then

$$TG(X) \cong (v_1 + v_2)^\perp \subset TG(\Lambda) \quad SG(X) \cong (v_1 - v_2)^\perp \subset SG(\Lambda).$$

Moreover, it holds that

$$\text{sgn}(SG(\Lambda)) = \text{sgn}(SG(X)) + (1, 0)$$

$$\text{sgn}(TG(\Lambda)) = \text{sgn}(TG(X)) + (1, 0).$$

Proof. If the induced action of $A_X$ is trivial, from Lemma 2.12 we know that the embedding is $G$-equivariant and $SG(X) \cong SG(\Lambda)$. If the induced action of $\varphi \in G$ on $A_X$ is nontrivial then we call $\varphi$ the extension of $\varphi$ on $\Lambda$ and if $\varphi^* \in O(H^2(X, \mathbb{Z}))$ then $\varphi^* \in (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$. Since $G$ exchanges the two generators of $A_X$ thus it exchanges also the two vectors $v_1$ and $v_2$ (see Remark 3.9), then it holds that $\varphi^*(v_1 + v_2) = \varphi^*(v_1) + \varphi^*(v_2) = v_2 + v_1$ and $\varphi^*(v_1 - v_2) = \varphi^*(v_1) - \varphi^*(v_2) = v_2 - v_1 = -(v_1 - v_2)$. Consequently $v_1 + v_2 \in TG(\Lambda)$ and $v_1 - v_2 \in SG(\Lambda)$. □

Remark 3.9. If $G$ is of prime order $p$, then the case $p = 2$ is the only one in which the induced action on $A_X$ can be nontrivial. In fact to have a nontrivial action the order of $G$ has to be a multiple of the order of the automorphism group of the discriminant group $A_X \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$. Since $\text{Aut}(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ this means that the order of $G$ has to be even.

Remark 3.10. Consider the moduli space of pairs $(X, G)$ where $X$ is an irreducible holomorphic symplectic manifold and $G \subset \text{Aut}(X)$. In a generic point of the moduli space, if $G \subset \text{Aut}(X)$ is a group of automorphisms where all the nontrivial elements are nonsymplectic, $T(X) = SG(X)$ and $TG(X) = NS(X)$. By the proof of Proposition 3.3 we have that $\text{sgn}(NS(X)) = (1, *)$ and $\text{sgn}(T(X)) = (2, *)$, hence if the induced action on $A_X$ is trivial we have $\text{sgn}(SG(\Lambda)) = (2, *)$ and $\text{sgn}(TG(\Lambda)) = (3, *)$; if the induced action on $A_X$ is nontrivial we have that $\text{sgn}(SG(\Lambda)) = (3, *)$ and $\text{sgn}(TG(\Lambda)) = (2, *)$.

3.1. $p=2$ - trivial action of $G$ on the discriminant group.

Proposition 3.11. If $G \subset O(\Lambda)$ is a group of order 2 then there exist fourteen pairs of invariant and coinvariant lattices $(TG(\Lambda), SG(\Lambda))$ of $G$ in $\Lambda$ if we assume that $\text{sgn}(SG(\Lambda)) = (2, *)$ and $\text{sgn}(TG(\Lambda)) = (2, *)$ as primitive sublattice.

Proof. Since $G \cong \mathbb{Z}/2\mathbb{Z}$ then $TG(\Lambda)$ and $SG(\Lambda)$ are 2-elementary lattices, i.e. $TG(\Lambda) \cong SG(\Lambda) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$. For each value of $\text{rk}(TG(\Lambda))$ and consequently for each value of $\text{rk}(SG(\Lambda))$ we can determine an upper bound for $a$ using Lemma 2.19. For each possible value of $a$ we apply Theorem 2.16 and Theorem 2.18 for even hyperbolic 2-elementary lattices. We obtain the classification in Table 1. □
3.2. \( p=2 \) - nontrivial action of \( G \) on the discriminant group.

**Proposition 3.12.** If \( \varphi \in O(\Lambda) \) is an isometry of order 2 and \( G = \langle \varphi \rangle \) is the group generated by \( \varphi \) then there exist twelve pairs of invariant and coinvariant lattices \( (T_G(\Lambda), S_G(\Lambda)) \) of \( G \) in \( \Lambda \) if we assume that \( \text{sgn}(S_G(\Lambda)) = (3, \ast) \) and \( T_G(\Lambda) \) admits the lattice \( \langle 2 \rangle^2 \) as primitive sublattice.

**Proof.** This is a direct application of Theorem 2.18 and Theorem 2.17 and we find the classification in Table 1.

3.3. \( p=3 \).

**Proposition 3.13.** If \( \varphi \in O(\Lambda) \) is an isometry of order 3 and \( G = \langle \varphi \rangle \) is the group generated by \( \varphi \) then there exists three pairs of invariant and coinvariant lattices \( (T_G(\Lambda), S_G(\Lambda)) \) of \( G \) in \( \Lambda \) if we assume that \( \text{sgn}(S_G(\Lambda)) = (2, \ast) \) and \( T_G(\Lambda) \) admits the lattice \( \langle 2 \rangle^2 \) as primitive sublattice.

**Proof.** If \( \text{rk}(T_G(\Lambda)) = 4 \) then \( \text{sgn}(T_G(\Lambda)) = (3, 1) \) and \( \text{sgn}(S_G(\Lambda)) = (2, 4) \). By Lemma 2.19 we have that \( a \in \{0, 1, 2, 3\} \). The case \( a = 0 \) does not happen because \( S_G(\Lambda) \) cannot be unimodular by the fact that \( 2 \neq 4 \) (8). The case \( a = 2 \) does not happen since we need that the discriminant of the lattice divided by \( p^{p-2} \) is a square in \( \mathbb{Q} \), as we can find in Proposition 2.21. For this reason \( a \) must be odd.

If \( a = 1 \) then \( S_G(\Lambda) \) is uniquely determined because we can write it as \( U \oplus S' \) by Theorem 2.18 with \( S' \) hyperbolic of rank 4, which is unique by Theorem 2.17. If \( a = 2 \) we can use Theorem 2.18 and Theorem 2.17 to conclude that this case is not allowed. If \( a = 3 \) we can use Theorem 2.17 to conclude that there exists a lattice \( T_G(\Lambda) \) with these properties.

If \( \text{rk}(T_G(\Lambda)) = 6 \) then \( \text{sgn}(T_G(\Lambda)) = (3, 3) \) and \( \text{sgn}(S_G(\Lambda)) = (2, 2) \). By Lemma 2.19 we find that \( a \in \{0, 1, 2\} \), but \( a \) must be odd from Proposition 2.21 then \( a = 1 \).

If \( a = 1 \) we can use Theorem 2.18 and Theorem 2.17 to conclude that this case is not allowed. In fact we split \( S_G(\Lambda) = U \oplus S' \) and \( \text{sgn}(S') = (1, 1) \), hence this is an hyperbolic lattice of rank \( r = 2 \). By Theorem 2.17 we conclude that this case does not occur because \( p = 3 \neq (-1)^{r/2-1} = 1 \) (4).

If \( \text{rk}(T_G(\Lambda)) = 8 \) then \( \text{sgn}(T_G(\Lambda)) = (3, 5) \) and \( \text{sgn}(S_G(\Lambda)) = (2, 0) \). By Lemma 2.19 we find that \( a \in \{0, 1\} \). We cannot have \( a = 0 \) because \( S_G(\Lambda) \) cannot be unimodular by the fact that \( 2 \neq 0 \) (8). If \( a = 1 \) then \( T_G(\Lambda) \) is uniquely determined because we can write it as \( U^{\oplus 2} \oplus T' \) by Theorem 2.18 applied two times. The lattice \( T' \) is hyperbolic of rank 4 and is unique by Theorem 2.17 and equal to \( U \oplus A_2(-1) \). The classification is enumerate in Table 1.

3.4. \( p=5 \).

**Proposition 3.14.** If \( \varphi \in O(\Lambda) \) is an isometry of order 5 and \( G = \langle \varphi \rangle \) is the group generated by \( \varphi \) then there exist only one pair of invariant and coinvariant lattices \( (T_G(\Lambda), S_G(\Lambda)) \) of \( G \) in \( \Lambda \) if we assume that \( \text{sgn}(S_G(\Lambda)) = (2, \ast) \) and \( T_G(\Lambda) \) admits the lattice \( \langle 2 \rangle^2 \) as primitive sublattice.

**Proof.** If \( p = 5 \) then \( \text{rk}(S_G(\Lambda)) = 4 \) because \( \text{rk}(S_G(\Lambda)) = \alpha \cdot 4 \) and \( \text{rk}(S_G(\Lambda)) \leq 7 \). By Lemma 2.19 we have \( a \in \{0, 1\} \). By Proposition 2.21 we know that \( a \) must be odd hence \( a = 1 \). If \( a = 1 \) then we apply Theorem 2.18 to \( S_G(\Lambda) \) and we obtain \( S_G(\Lambda) = U \oplus S' \). The lattice \( S' \) is hyperbolic, 5-elementary of rank \( r = 2 \) and invariants \( (r, a) = (2, 1) \). By Theorem 2.17 we obtain \( S' = H_5 \). In this case \( S_G(\Lambda) = U \oplus H_5 \) and \( T_G(\Lambda) = U^{\oplus 2} \oplus H_5 \) as we can find in Table 1.
3.5. \( p = 7 \).

**Proposition 3.15.** If \( \varphi \in O(\Lambda) \) is an isometry of order 7 and \( G = \langle \varphi \rangle \) is the group generated by \( \varphi \) then there exist fourteen pairs of invariant and coinvariant lattices \((T_G(\Lambda), S_G(\Lambda))\) of \( G \) in \( \Lambda \) if we assume that \( \operatorname{sgn}(S_G(\Lambda)) = (2,*) \) and \( T_G(\Lambda) \) admits the lattice \((2)^{\oplus 2}\) as primitive sublattice.

**Proof.** In this case we have \( \operatorname{sgn}(T_G(\Lambda)) = (3,1) \) and \( \operatorname{sgn}(S_G(\Lambda)) = (2,4) \). By Lemma 2.19 we obtain \( a \in \{0,1\} \). The case \( a = 0 \) is not allowed because \( 2 \neq 4 \) (8). If \( a = 1 \) we can apply Theorem 2.18 to \( S_G(\Lambda) \) and we obtain \( S_G(\Lambda) = U \oplus S' \). The lattice \( S' \) is hyperbolic, 7-elementary of rank \( r = 4 \), signature \((1,3)\) and invariants \( (r, a) = (4, 1) \). By Theorem 2.17 we obtain \( S' = U \oplus K_7 \), hence \( S_G(\Lambda) = U^{\oplus 2} \oplus K_7 \) as we can find in Table 1.

In the classification above we have proved the following theorem.

**Theorem 3.16.** Let \( \Lambda = U^{\oplus 5} \) and let \( G \subseteq O(\Lambda) \) be a subgroup of order \( p \). Assume there exists a primitive embedding of \((2)^{\oplus 2}\) in the invariant lattice of \( G \) in \( \Lambda \). If \( p = 2 \) and the induced action of \( G \) on \( A_X \) is trivial then there are fourteen pairs \((S_G(\Lambda), T_G(\Lambda))\) up to isometry. If \( p = 2 \) and the induced action of \( G \) on \( A_X \) is nontrivial then there are twelve pairs \((S_G(\Lambda), T_G(\Lambda))\) up to isometry. If \( p = 3 \) then there are three pairs of \((S_G(\Lambda), T_G(\Lambda))\) up to isometry, and if \( p = 5 \) or \( p = 7 \) there exists a unique pair \((S_G(\Lambda), T_G(\Lambda))\) up to isometry.

In the following table if \( p = 2 \) in the column \( \delta \) (see Definition 2.15) we indicate whether the quadratic form of the discriminant group of the lattice is integer valued, \( \delta = 0 \), or not \( \delta = 1 \). Moreover \( a = a(S_G(\Lambda)) = a(T_G(\Lambda)) \) since \( \Lambda \) is unimodular.

4. Invariant and coinvariant lattices in \( H^2(X, \mathbb{Z}) \)

4.1. **The classification.** If \( X \) is a manifold of \( OG_6 \) type and \( \varphi \in O(H^2(X, \mathbb{Z})) \) is a nonsymplectic isometry of prime order then by Proposition 3.3 \( \varphi \) is effective (Definition 2.5). If \( G = \langle \varphi \rangle \) is a cyclic group of nonsymplectic isometries generated by \( \varphi \) to classify effective isometries means to classify possible invariant and coinvariant sublattices \((T_G(X), S_G(X))\) of \( H^2(X, \mathbb{Z}) \). There exists three levels of classification.

1. We can classify \( T_G(X) \) and \( S_G(X) \) up to isometry.
2. We can classify \( T_G(X) \) and \( S_G(X) \) up to isometry of \( H^2(X, \mathbb{Z}) \).
3. We can classify \((X, G)\) up to deformation, i.e. we can count the connected components of the moduli space of pairs \((X, G)\).

**Remark 4.1.** The classification in 2. is finer than the classification in 1. and the classification in 3. is finer than the classification in 2. The classification in 2. is equivalent to counting the different primitive embeddings of \( S_G(X) \) in \( H^2(X, \mathbb{Z}) \).

**Remark 4.2.** The third and best level of classification that we can do, concerns to count the different connected components of the moduli space of the pairs \((X, G)\). We can have two manifolds of \( OG_6 \) type, \( X \) and \( Y \), two pairs \((X, G)\) and \((Y, G)\) such that the lattices \((S_G(X), T_G(X))\) and \((S_G(Y), T_G(Y))\) are isometric to each others and such that there exists an isometry of \( H^2(X, \mathbb{Z}) \) such that \( S_G(X) \) is sent to \( S_G(Y) \), but such that there is no way to deform in a continuous way \((X, G)\) to \((Y, G)\) preserving the action of \( G \). This means that \((X, G)\) and \((Y, G)\) are in different connected components of the moduli space of the pairs \((X, G)\). The fact that there could be more than one connected component in the moduli space of
Table 1. Isometries of prime order on $\Lambda$ when $\langle 2 \rangle \otimes 2 \subseteq T_G(\Lambda)$.

| No. | $p$ | $S_G(\Lambda)$ | $T_G(\Lambda)$ | $\text{sgn}(S_G(\Lambda))$ | $a$ | $\delta$ | action on $A_X$ |
|-----|-----|-----------------|----------------|----------------------------|-----|-----|---------------|
| 1   | 2   | $U^{\otimes 2} \oplus \langle -2 \rangle ^{\otimes 3}$ | $\langle 2 \rangle ^{\otimes 3}$ | (2, 5) | 3 | 1 | trivial |
| 2   | 2   | $U \oplus \langle -2 \rangle ^{\otimes 3} \oplus \langle 2 \rangle$ | $\langle 2 \rangle ^{\otimes 3} \oplus \langle -2 \rangle$ | (2, 4) | 4 | 1 | trivial |
| 3   | 2   | $U^{\otimes 2} \oplus \langle -2 \rangle ^{\otimes 2}$ | $U \oplus \langle 2 \rangle ^{\otimes 2}$ | (2, 4) | 2 | 1 | trivial |
| 4   | 2   | $\langle 2 \rangle ^{\otimes 2} \oplus \langle -2 \rangle ^{\otimes 3}$ | $\langle -2 \rangle ^{\otimes 2} \oplus \langle 2 \rangle ^{\otimes 3}$ | (2, 3) | 5 | 1 | trivial |
| 5   | 2   | $U \oplus \langle -2 \rangle ^{\otimes 2} \oplus \langle 2 \rangle$ | $U \oplus \langle 2 \rangle ^{\otimes 2} \oplus \langle -2 \rangle$ | (2, 3) | 3 | 1 | trivial |
| 6   | 2   | $U^{\otimes 2} \oplus \langle -2 \rangle$ | $U^{\otimes 2} \oplus \langle 2 \rangle$ | (2, 3) | 1 | 1 | trivial |
| 7   | 2   | $\langle 2 \rangle ^{\otimes 2} \oplus \langle -2 \rangle ^{\otimes 2}$ | $U \oplus \langle 2 \rangle ^{\otimes 2} \oplus \langle -2 \rangle ^{\otimes 2}$ | (2, 2) | 4 | 1 | trivial |
| 8   | 2   | $U(2)^{\otimes 2}$ | $U \oplus U(2)^{\otimes 2}$ | (2, 2) | 4 | 0 | trivial |
| 9   | 2   | $U \oplus \langle 2 \rangle \oplus \langle -2 \rangle$ | $U^{\otimes 2} \oplus \langle 2 \rangle \oplus \langle -2 \rangle$ | (2, 2) | 2 | 1 | trivial |
| 10  | 2   | $U \oplus \langle 2 \rangle (2)$ | $U^{\otimes 2} \oplus \langle 2 \rangle$ | (2, 2) | 2 | 0 | trivial |
| 11  | 2   | $U^{\otimes 2}$ | $U^{\otimes 3}$ | (2, 2) | 0 | 0 | trivial |
| 12  | 2   | $\langle 2 \rangle ^{\otimes 2} \oplus \langle -2 \rangle ^{\otimes 2} \oplus \langle 2 \rangle$ | $U^{\otimes 2} \oplus \langle -2 \rangle ^{\otimes 2} \oplus \langle 2 \rangle$ | (2, 1) | 3 | 1 | trivial |
| 13  | 2   | $U \oplus \langle 2 \rangle \oplus \langle -2 \rangle$ | $U^{\otimes 2} \oplus \langle -2 \rangle$ | (2, 1) | 1 | 1 | trivial |
| 14  | 2   | $\langle 2 \rangle ^{\otimes 2}$ | $U^{\otimes 3} \oplus \langle -2 \rangle ^{\otimes 2}$ | (2, 0) | 2 | 1 | trivial |

The pairs $(X, G)$ is related to the existence of wall divisors in the invariant lattice $T_G(X) = \text{NS}(X)$.

To obtain the classification of invariant and coinvariant sublattices with respect to a prime order $p$ isometry on $H^2(X, \mathbb{Z})$, we start from a classification of all the possible $p$-elementary sublattices $S'$ of $\Lambda$, and the orthogonal complement $T' = S'^{\perp \Lambda}$. In this setting the pairs $(T', S')$ are candidates to be the invariant and the coinvariant sublattices with respect to the isometry of prime order $p$ that extends from $H^2(X, \mathbb{Z})$ to $\Lambda$. In fact if $p = 2$ we know that all the 2-elementary lattices that we find correspond to an isometry of order two which is the isometry that acts as $-\text{id}$ on $S_G(\Lambda)$. On the other hand, if $p$ is prime, $p \geq 3$ we need the
following proposition to know when an isometry of $\Lambda$ is an extension of an isometry of $H^2(X, \mathbb{Z})$.

**Proposition 4.3.** Let $p \in \{3, 5, 7\}$ and let $(S', T')$ be $p$-elementary lattices such that $S' \hookrightarrow \Lambda = U \oplus 5$ is a primitive embedding and $T' = S'^{\perp \Lambda}$. If $(2)^{\oplus 2} \subseteq T'$ is a primitive sublattice and if there exists a nonsymplectic isometry $\varphi \in O(H^2(X, \mathbb{Z}))$ of prime order $p$ such that $S_{\varphi}(X) = S'$ then $S'$ is the coinvariant sublattice of $H^2(K3, \mathbb{Z})$ with respect to a nonsymplectic isometry of prime order $p$.

**Proof.** Consider $\varphi \in O(H^2(X, \mathbb{Z}))$ an isometry of prime order $p \in \{3, 5, 7\}$. Since $p$ is odd the induced action on $A_X$ is trivial from Remark 3.9 then the embedding of $H^2(X, \mathbb{Z})$ in $\Lambda$ is $\varphi$-equivariant and consequently the action on $H^2(X, \mathbb{Z}) \cap \Lambda \cong (2)^{\oplus 2}$ is trivial. We call $\varphi$ also the isometry of $\Lambda$ that acts as $\varphi$ on $H^2(X, \mathbb{Z})$ and is the identity on the orthogonal complement $(2)^{\oplus 2}$. As a consequence $S_{\varphi}(X) = S_{\varphi}(\Lambda)$. In this way $S_{\varphi}(X)$ is $p$-elementary (see Remark 3.7). Assume that $S' = S_{\varphi}(X)$, and consider the second integral cohomology lattice of a $K3$ surface $H^2(K3, \mathbb{Z}) = U \oplus 3 \oplus E_6(-1)^{\oplus 2}$ which is a unimodular lattice of rank 22. Since $S'$ is the orthogonal complement of $T'$ in $\Lambda$ and $\text{rk}(T') \geq 2$ then $\text{rk}(S') \leq 8$. It is possible to find a primitive embedding $S' = S_{\varphi}(X) \hookrightarrow H^2(K3, \mathbb{Z})$ and moreover since the action is of prime order $p$ it induces a trivial action on $A_X$, hence we can extend the action of $\varphi$ on the $K3$ lattice in a $\varphi$-equivariant way by Lemma 2.13. We call $\widetilde{\varphi} \in O(H^2(K3, \mathbb{Z}))$ the nonsymplectic isometry of order $p$ such that $\widetilde{\varphi}|_{H^2(X, \mathbb{Z})} = \varphi$ and $\widetilde{\varphi}|_{H^2(X, \mathbb{Z}) \cap \Lambda} = \text{id}$ and we have $S_{\varphi}(X) = S_{\varphi}(K3)$. □

In [112] we find a classification of invariant and coinvariant sublattices of $H^2(K3, \mathbb{Z})$ with respect to isometries of prime order $p = 3, 5, 7$. As a consequence using Proposition 4.3 we can determine which $p$-elementary sublattices $S_G(\Lambda)$ classified in Table 1 are the coinvariant sublattices with respect to an extension on $\Lambda$ of an isometry of $H^2(X, \mathbb{Z})$.

More precisely, using Proposition 4.3 and checking the classification of Artebani–Sarti [111] or Artebani–Sarti–Taki [2] we find that $p$-elementary sublattices $S_G(\Lambda)$ for $p = 3, 5, 7$ in Table 1 are coinvariant lattices with respect to an isometry of the same order $p$ on $H^2(K3, \mathbb{Z})$, hence we know that there exist isometries of order 3, 5, 7 on $H^2(X, \mathbb{Z})$ that the coinvariant sublattices are the ones that we find in Table 1 for $p = 3, 5, 7$ respectively.

Consider the following primitive embedding

$$H^2(X, \mathbb{Z}) \hookrightarrow \Lambda := U \oplus 5,$$

and consider $G \subset O(H^2(X, \mathbb{Z}))$ and the induced action of $G$ on $\Lambda$. Lattice-theoretically the embedding is done by choosing $(2)^{\oplus 2} = H^2(X, \mathbb{Z}) \cap \Lambda$.

**Definition 4.4.** Let $X$ be a manifold of OG6 type. We define

$$R = (2)^{\oplus 2} = H^2(X, \mathbb{Z}) \cap \Lambda$$

the orthogonal complement of the second cohomology lattice of $X$ in the smallest unimodular lattice in which we can embed $H^2(X, \mathbb{Z}) \cong U \oplus 3 \oplus (2)^{\oplus 2}$.

In the following we give a list of all configurations of lattices occurring as invariant and coinvariant lattices of nonsymplectic automorphisms of prime order of manifolds of OG6 type. If $G \subset O(H^2(X, \mathbb{Z}))$ and we consider the induced action on $\Lambda$, if $R$ is embedded in the invariant lattice $T_G(\Lambda)$, the induced action on $A_X$ is trivial. In this situation the embedding is $G$-equivariant and $S_G(X) \cong S_G(\Lambda)$,
hence $A_{S_G(X)} \cong A_{S_G(\Lambda)} \cong (\mathbb{Z}/p\mathbb{Z})^{\oplus a}$ for some integer $a = a(S_G(X)) \geq 0$ since $\Lambda$ is unimodular.

We show that if $p \geq 3$ we have that $T_G(X)$ is uniquely determined up to isometry regardless of the embedding of $R$ in $T_G(\Lambda)$. We need to show that the discriminant group of $T_G(X)$, i.e. $A_{T_G(X)}$, is the same for each different embedding of $R$ in $T_G(\Lambda)$. If we have this result we can find an embedding of $R$ in $T_G(\Lambda)$, we compute $R^\perp_{T_G(\Lambda)} = T_G(X)$.

We consider the finite index embedding $R \oplus T_G(X) \subset T_G(\Lambda)$. Here $R$ and $T_G(X)$ are primitive sublattices of $T_G(\Lambda)$. We know that $A_{T_G(\Lambda)} \cong (\mathbb{Z}/p\mathbb{Z})^a$ for some $a$, and $A_R \cong (\mathbb{Z}/2\mathbb{Z})^2$. Using Proposition 4.5 we know that the embedding is given by two subgroups $H_R \subset A_R$ and $H_{T_G(X)} \subset A_{T_G(X)}$, and by an anti-isometry between them, $\gamma : H_R \rightarrow T_G(X)$. We call $H_{T_G(\Lambda)} := \Gamma_\gamma = \{ (a, \gamma(a)), | a \in H_R \}$ and we know that $A_{T_G(\Lambda)} = \Gamma_\gamma / \Gamma_\gamma = (H_{T_G(\Lambda)})^\perp / H_{T_G(\Lambda)}$ where $H_{T_G(\Lambda)} \subset A_R \oplus A_{T_G(X)}$ is an isotropic subgroup.

**Proposition 4.5.** If $p \neq 2$ the following statements hold.

i) The subgroup $H_R = A_R$ and, since $\gamma$ is an anti-isometry, $H_{T_G(\Lambda)} = A_R(-1)$.

ii) The group $A_{T_G(\Lambda)}$ is a subgroup of $A_{T_G(X)} (A_{T_G(\Lambda)} \oplus A_R(-1) \subset A_{T_G(X)}$ and these are orthogonal complements with respect to the induced quadratic form on $A_{T_G(X)}$).

iii) It holds that $A_{T_G(\Lambda)} = A_{T_G(X)} \oplus A_R(-1)$.

**Proof.** i). In this proof we denote $T_G(X)$ by $N$ and $T_G(\Lambda)$ by $T$. Suppose that $H_R \subset A_R$. We would like to find $a \in A_R \setminus H_R$ such that $a \in A_T$. We will show that if $a \in A_T \setminus H_L$ then $a \perp H_L$. This can not happen because $A_T = (\mathbb{Z}/p\mathbb{Z})^a$ and

If we find $a$ such that $b_{A_{A_R}}((a, 0)) = 0 \forall c \in H_R$, we will find $(a, 0) \in \Gamma_\gamma$ such that $(a, 0) \notin \Gamma_\gamma$ i.e. we find $(a, 0) \subset A_T$. The proper subgroups $H_R$ of $A_T \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ are $\{[0, 0], [0, 1], [0, 0], [1, 1], [0, 0] \}$, $\{[1, 1], [0, 0] \}$. We can use the following trick. Let $H_L$ be one of the nontrivial subgroups and let $l$ be the $[0, 0]$ class and $b$ the other one. We have $H_T \subset A_R \oplus A_T$ and $H_T^{-1} \subset A_R \oplus A_N$ and in particular $H_T = \{(a, \gamma(a)), (b, \gamma(b)) \} \cong (\mathbb{Z}/2\mathbb{Z})^2$. Now since $N = R^{-1} \subset T$, we can say that $A_N \cong (\mathbb{Z}/2\mathbb{Z})^b \oplus (\mathbb{Z}/p\mathbb{Z})^a$, and for this reason we obtain $A_T \oplus A_N = (\mathbb{Z}/2\mathbb{Z})^{b+2} \oplus (\mathbb{Z}/p\mathbb{Z})^a$. We have $A_T = H_T^\perp / H_T$ and $q_{A_T \oplus A_N}$ is a non-degenerate quadratic form on $A_T \oplus A_N$, since we are taking the orthogonal of a group of order 2 which is $H_T$, we have $H_T^\perp = (\mathbb{Z}/2\mathbb{Z})^{b+1} \oplus (\mathbb{Z}/p\mathbb{Z})^a$. Moreover $A_T = H_T^\perp / H_T = (\mathbb{Z}/2\mathbb{Z})^b \oplus (\mathbb{Z}/p\mathbb{Z})^a$ and $b$ has to be equal to zero since there are no elements of order 2 in $A_T$. If $b = 0$ $A_N = (\mathbb{Z}/p\mathbb{Z})^a$ which is false because we must have elements of order two in $A_N$.

If we have $H_R = A_R$ as we want.

ii) & iii). By the first step $H_N = A_R(-1) \subset A_N$, $H_N^\perp \subset A_N$. We would like to show that $H_N^\perp = A_T = H_T^\perp / H_T$. We know that $A_R \cong (\mathbb{Z}/2\mathbb{Z})^2$ and this implies
that $H_N = A_R(-1) \cong (\mathbb{Z}/2\mathbb{Z})^2$ which means that $b = 2$.

$H_N = (\mathbb{Z}/2\mathbb{Z})^2 \subseteq A_N = (\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/p\mathbb{Z})^a$ and this implies that $H_N^\perp \cong (\mathbb{Z}/p\mathbb{Z})^a$

since the last orthogonal complement is with respect to the product in $A_N$. Since $|H_N^\perp| = |A_T|$, the idea is to construct an isomorphism $g : H_N^\perp \to H_T^\perp / H_T$ which sends $\alpha \in H_N^\perp$, $\alpha \neq 0$ to $[g(\alpha)] = [(0, \alpha)]$. We need to verify that $g(\alpha) \in H_T^\perp$, in particular $(0, \alpha) \in A_R \oplus A_N$ and $\alpha \in H_N^\perp \subseteq A_N$, so if $(a, b) \in H_T$ we have $q_{A_R \oplus A_N}((0, \alpha), (a, b)) = q_{A_R}(0, a) + q_{A_N}(\alpha, b) = 0$ because $q_{A_R}$ is non-degenerate, $\alpha \in H_N^\perp$ and $b \in H_N^\perp$. This allow us to conclude that $(0, \alpha) \in H_T^\perp$.

In this hypothesis $g$ is well defined. Actually the last thing that we must prove is the injectivity of $g$. We have

$$[g(\alpha)] = g([(\beta)]) \iff [(0, \alpha)] = [(0, \beta)],$$

$$(0, \alpha) = (0, \beta) + (c, \gamma(c)) \Rightarrow (0, \alpha - \beta) = (c, \gamma(c)) \Rightarrow (0, \alpha - \beta) \in H_T.$$

In this setting $\alpha - \beta = \gamma(0) = 0$, since $\gamma$ is an anti-isometry. We can conclude that $\alpha = \beta$ an this forced $g$ to be injective. Actually $g$ is an isomorphism as we wanted to prove. Moreover $H_N^\perp \cong A_T$, $A_R(-1) \cong H_N$, consequently $H_N \cong H_N^\perp \subseteq A_N$, where the orthogonal complement of $H_N$ is in $A_N$. Since $q_{A_N}$ is non-degenerate, $H_N \perp H_N^\perp = A_N$ i.e. $A_N = A_T \oplus A_R(-1)$. \square

4.2. $p=2$, trivial action on the discriminant group. We use the notation of the previous section. Since $p = 2$ we have $A_R \cong (\mathbb{Z}/2\mathbb{Z})^2$ and $A_T \cong (\mathbb{Z}/2\mathbb{Z})^a$. Moreover it holds the following results.

Lemma 4.6. Let $R$ be any lattice and $\varphi \in O(R)$, an involution, then it holds that $S_\varphi(R) = T_{-\varphi}(R)$.

Proof. It holds that $S_\varphi(R) := \text{Ker}(\varphi + 1) = \text{Ker}(-\varphi - 1) = T_{-\varphi}(R).$ \square

Lemma 4.7. Let $T$ be a 2-elementary lattice, and $\varphi \in O(T)$ an involution, such that $\varphi \in O(A_T)$ is trivial, then $T_{\varphi}(T) := T_{-\varphi}$ is 2-elementary.

Proof. Let $T \hookrightarrow \Lambda$ be a primitive embedding of $T$ in the smallest unimodular lattice $\Lambda$, and let $R := T^{\perp}\Lambda$. Let $\tilde{\varphi} := \varphi \oplus -Id_R$ be an extension of $\varphi$ to $\Lambda$. Suppose that $\text{ord}(\varphi) = 2$. Then $-\varphi \in O(\Lambda)$ is of order 2 and we have $T_{-\varphi}(\Lambda) = S_{\tilde{\varphi}}(\Lambda)$ from Lemma 4.6. Moreover $T_{\varphi}(T) = S_{-\varphi}(T) = S_{-\tilde{\varphi}}(\Lambda)$, where the first equal holds from Lemma 4.6 and the second equal holds by construction. Thus $T_{\varphi}(T)$ is a coinvariant sublattice of a unimodular lattice with respect to an action of order 2 and this implies that $T_{\varphi}(T)$ is 2-elementary. \square

Conjecture: The statement of Lemma 4.7 holds also for $p$-elementary lattices.

We can apply the previous result to our case taking $G \subseteq \text{Aut}(X)$, $G \cong \mathbb{Z}/2\mathbb{Z}$ and using $T = T_G(\Lambda)$ and taking into consideration the finite index embedding $T_G(X) \oplus (2)^{\otimes 2} \subseteq T_G(\Lambda)$. We consider the primitive embedding

$$(3) \quad H^2(X, \mathbb{Z}) \hookrightarrow \Lambda,$$

and we extend the action of $G$ on the orthogonal complement of $H^2(X, \mathbb{Z})^{\perp}\Lambda \cong R$ as $-Id_R$. The induced action of $G$ is nontrivial on $R$, but it is trivial on the discriminant group $A_R$ since $[1, 0] = [-1, 0]$ and $[0, 1] = [0, -1]$. Using this setting $T_G(T_G(\Lambda)) = T_G(X)$ and from Lemma 4.7 it is 2-elementary.
In this section we classify the cases in which the embedding in equation \[3\] is $G$-equivariant and with a trivial action on $A_X$. Hence $S_G(X) = S_G(A) = T_G(X)$ has signature $(1,\ast)$ and the latter is obtained as the orthogonal complement of $R \subset T_G(\Lambda)$. The lattice $T_G(X)$, which we denote by $N$, is a 2-elementary lattice, hence we have $A_N \cong (\mathbb{Z}/2\mathbb{Z})^g$, moreover $A_T \cong (\mathbb{Z}/2\mathbb{Z})^a$ and the powers $q$ and $a$ are related, as we will see later. In this case $A_T, A_R, A_N$ are 2-elementary lattices, so we need to use another strategy to obtain all the embeddings of $R$ in $T_G(\Lambda)$ and consequently all the possible lattices $T_G(X)$.

As in the previous section we have $R \oplus T_G(X) \subset T_G(\Lambda)$ which is an embedding of finite index, and $R$ and $T_G(X)$ are primitive sublattices of $T_G(\Lambda)$. This embedding is given by two subgroups $H_R \subset A_R$ and $H_{T_G(X)} \subset T_G(X)$, and by an anti-isometry $\gamma : H_R \rightarrow H_{T_G(X)}$. We call $H_{T_G(\Lambda)} := \Gamma = \{(a,\gamma(a))\,|\,a \in H_R\}$ and we know that $A_{T_G(\Lambda)} = \Gamma^\perp/\Gamma = (T_{T_G(\Lambda)})^\perp/T_{T_G(\Lambda)}$ where $T_{T_G(\Lambda)} \subset A_R \oplus A_{T_G(X)}$ is an isotropic subgroup.

**Proposition 4.8.** In the previous notation, $A_T \cong (\mathbb{Z}/2\mathbb{Z})^a$ and $A_N \cong (\mathbb{Z}/2\mathbb{Z})^\ast$, if the induced action on $A_X$ is trivial then we have one of the following possibilities.

i) If $H_R = A_R$ then $q = a + 2$.

ii) If $H_R = 0$ then $q = a - 2$.

iii) If $H_R \cong \mathbb{Z}/2\mathbb{Z}$ then $q = a$.

**Proof.** To prove i) we refer to Proposition [3] and we obtain $A_N = A_T \oplus A_R(-1)$, which means that $q = a + 2$. To prove point ii) we notice that $H_R = 0$ implies that $\gamma(H_R) = 0$ which means that $H_T = 0$. In this case $H_T^\perp = A_R \oplus A_N$ and $A_T = (H_T)^\perp/H_T = A_R \oplus A_N$, hence $q = a - 2$. For iii) we have $(H_R,\gamma(H_R)) = (0,\gamma(0)), (1,\gamma(1)) \cong \mathbb{Z}/2\mathbb{Z} = H_T$. Moreover $H_T^\perp \subset A_R \oplus A_N \cong (\mathbb{Z}/2\mathbb{Z})^{a+2}$ and for this reason $H_T \cong (\mathbb{Z}/2\mathbb{Z})^{a+1}$. Furthermore $A_T = (H_T)^\perp/H_T = (\mathbb{Z}/2\mathbb{Z})^{a+1}/\mathbb{Z}/2\mathbb{Z} \cong (\mathbb{Z}/2\mathbb{Z})^a$ and we conclude that $q = a$. \[\blacksquare\]

Using these results we can classify possible invariant and coinvariant sublattices of $H^2(X,\mathbb{Z})$ starting from the classification in Table \[1\]. The classification for the second integral cohomology group in the case $p = 2$ is summarized in Table \[3\].

4.3. **p=2, nontrivial action on the discriminant group.** We know that $H^2(X,\mathbb{Z})$ is the orthogonal complement of $R \cong (\mathbb{Z})^{\oplus 2}$ in $\Lambda \cong U^{\oplus 5}$ where $n_1$ and $n_2$ are the two vectors of square 2 in $R$. If the action of $G$ on $A_X \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ is nontrivial we have

$$T_G(X) \cong (n_1 + n_2)^\perp \subset T_G(\Lambda) \quad S_G(X) \cong (n_1 - n_2)^\perp \subset S_G(\Lambda).$$

The lattice $S_G(\Lambda)$ has signature $(3,\ast)$ and $T_G(\Lambda)$ has signature $(2,\ast)$. It follows from the first section that all lattices $S_G(\Lambda)$ and $T_G(\Lambda)$ are 2-elementary, thus their discriminant group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^a$ for some integer $a \geq 0$.

In the following $q$ is the Beauville-Bogomolov quadratic form. By an easy computation we obtain that $q(n_1 + n_2) = 4$ and $q(n_1 - n_2) = 4$. We denote by $(\cdot,\cdot)$ the symmetric bilinear form associated to $q$. Let $x$ and $y$ be two elements in $H^2(X,\mathbb{Z})$. The bilinear form is defined in this way

$$(x, y) := \frac{q(x + y) - q(x) - q(y)}{2}.$$
Another request that we need for \( n_1 \) and \( n_2 \) is that \((n_1 + n_2, n_1 - n_2) = 0\). These two vectors are orthogonal, and \( n_1 + n_2 + n_1 - n_2 = 2n_1 \), i.e. the sum of them is twice a primitive vector. It holds the following lemma.

**Lemma 4.9.** Let \( G \subset O(\Lambda) \) be a finite group of isometries of \( \Lambda \) of order 2, such that the Let \( T_G(\Lambda) \) and \( S_G(\Lambda) \) be respectively the invariant and the coinvariant sublattice of \( \Lambda \) with respect to the action of \( G \). The lattices \( T_G(X) \) and \( S_G(X) \) are the invariant and the coinvariant sublattices with respect to the action of \( G \) on \( H^2(X, \mathbb{Z}) \) such that the induced action of \( G \) on \( Ax \) is nontrivial. The lattices \( T_G(X) \) and \( S_G(X) \) exist if and only if there exist two vectors, \( v_0 = n_1 - n_2 \in S_G(\Lambda) \) and \( v_1 = n_1 + n_2 \in T_G(\Lambda) \), of square 4, orthogonal to each other and such that the sum is twice a primitive vector. In this situation

\[
T_G(X) := (n_1 + n_2)^\perp \subset T_G(\Lambda)
\]

\[
S_G(X) := (n_1 - n_2)^\perp \subset S_G(\Lambda).
\]

**Proof.** For the "if" part we find the vectors \( n_1 \) and \( n_2 \) with the requested properties and we can compute \( S_G(X) \) and \( T_G(X) \). For the "only if" part we need to observe that if we have a nontrivial action of \( G \) on \( Ax \cong \mathbb{Z}/2\mathbb{Z} \), then we know that \( R \) has to be embedded partially in \( S_G(\Lambda) \) and partially in \( T_G(\Lambda) \). Since the two elements \([1,0]\) and \([0,1]\) of are exchanged, for sure the sum of them is preserved and the difference is not preserved. \( \square \)

Starting from the classification in Table 1 we find that in cases 4, 5, 6, 9 it is not possible to find two vectors with the properties of Lemma 4.9.

**Example 4.10.** In case 4, \( S_G(\Lambda) \cong U^\oplus 2 \oplus \langle 2 \rangle \oplus \langle -2 \rangle \) and \( T_G(\Lambda) \cong U \oplus \langle 2 \rangle \oplus \langle -2 \rangle \). Since \( T_G(\Lambda) \oplus S_G(\Lambda) \subset \Lambda \), we can define the two copies of \( U \) in \( S_G(\Lambda) \) generated by \( \{e_1, f_1, e_2, f_2\} \), the vector of square 2 is \( e_3 + f_3 \) in the embedding above and the vector of square \(-2\) is \( e_4 - f_4 \). In the same way we can call \( \{e_5, f_5\} \) the generators of \( U \) in \( T_G(\Lambda) \), the vector of square 2 is \( e_4 + f_4 \) with respect to the choice which we have done before, and the vector of square \(-2\) is \( e_3 - f_3 \). Doing this choice for the basis we can take \( v_0 = e_1 + f_1 + e_3 + f_3 \) or \( v_0 = 2(e_3 + f_3) + e_1 - f_1 + e_2 - f_2 \) and \( v_1 = e_5 + f_5 + e_4 + f_4 \) or \( v_1 = 2(e_4 + f_4) + e_3 - f_3 + e_5 - f_5 \). Since we have these possible choices the sum \( v_0 + v_1 \) is not equal to twice a primitive vector hence we can not find \( S_G(X) \) as a subset of \( S_G(\Lambda) \) that corresponds to a coinvariant sublattice of \( H^2(X, \mathbb{Z}) \) with respect to an isometry of order 2 that induces a nontrivial action on \( Ax \).

**Example 4.11.** In case 2 we can compute for instance \( S_G(\Lambda) \cong U \oplus \langle 2 \rangle^\oplus 2 \oplus \langle -2 \rangle^\oplus 2 \) and \( T_G(\Lambda) \cong \langle 2 \rangle^\oplus 2 \oplus \langle -2 \rangle^\oplus 2 \). Since \( T_G(\Lambda) \oplus S_G(\Lambda) \subset \Lambda \), we can define the copy of \( U \) in \( S_G(\Lambda) \) generated by \( \{e_1, f_1\} \), the two vectors of square 2 are \( e_2 + f_2 \) and \( e_3 + f_3 \) in the embedding above and the two vectors of square \(-2\) are \( e_4 - f_4 \) and \( e_5 - f_5 \). In the same way we can call the vectors of square 2 in \( T_G(\Lambda) e_4 + f_4 \) and \( e_5 + f_5 \) w.r.t. the choice which we have done before, and the vectors of square \(-2\) are \( e_2 - f_2 \) and \( e_3 - f_3 \). We have for sure at least two possible choices for \( v_0 \) and \( v_1 \): the first one is \( v_0 = e_4 - f_4 + e_5 - f_5 + 2(e_1 + f_1) \) and \( v_1 = e_4 + f_4 + e_5 + f_5 \), the second one is \( v_0 = e_4 - f_4 + e_5 - f_5 + 2(e_2 + f_2) \) and \( v_1 = e_4 + f_4 + e_5 + f_5 \). In the first case \( S_G(X) \cong \langle 2 \rangle^\oplus 2 \oplus \langle -2 \rangle^\oplus 2 \oplus \langle 4 \rangle \) and \( T_G(X) \cong \langle -2 \rangle^\oplus 2 \oplus \langle 4 \rangle \). In the second case \( S_G(X) \cong U \oplus \langle 2 \rangle \oplus \langle -2 \rangle \oplus \langle -4 \rangle \) and \( T_G(X) \cong \langle -2 \rangle^\oplus 2 \oplus \langle 4 \rangle \). These two pairs of lattices are not isomorphic.
We notice that the orders of the respective subgroups are co-prime, and for this action on $A_X$ and coinvariant sublattices with respect to isometries that induce a nontrivial action the unique isomorphism between the subgroups that we can have is the one$$ (\Lambda) = 3$$

between a subgroup of $T_G(\Lambda)$ and a subgroup of $A_T G(\Lambda)$ (Theorem 2.22). We notice that the orders of the respective subgroups are co-prime, and for this reason the unique isomorphism between the subgroups that we can have is the one between the trivial subgroups. Consequently we get $A_R \oplus A_T G(\Lambda) \cong A_T G(\Lambda)$, i.e. $(\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/3\mathbb{Z})^2 \cong (\mathbb{Z}/6\mathbb{Z})^2 \oplus \mathbb{Z}/3\mathbb{Z}$, and this implies $R(A_T G(\Lambda)) = 3$. This is a contradiction since the rank of $T_G(\Lambda) = 2$ and for this reason this case is not allowed (see Theorem 2.22).

In another case of Table 1 we have $T_G(\Lambda) = U \oplus A_2$. An embedding consists in taking a vector of square two in $U$ and another orthogonal vector of square two in $A_2$. With this primitive embedding the orthogonal complement is $T_G(\Lambda) = (-2) \oplus (6)$. Now we notice, using Theorem 2.22 that to find a primitive embedding of $R$ in $T_G(\Lambda)$ we need to define an isomorphism between a subgroup of $A_R \cong (\mathbb{Z}/2\mathbb{Z})^2$ and a subgroup of $A_T G(\Lambda) \cong \mathbb{Z}/3\mathbb{Z}$. Since the orders of the subgroups of $A_R$ are 1, 2, 4 and the orders of the subgroups of $A_T G(\Lambda)$ are 1, 3 to have an isomorphism, the only

The same situation happens in cases 7 and 10. There is just one possible choice for $v_0$ and $v_1$ in cases 3, 8, 11, 12 of Table 2. We can find the list of some possible lattices in Table 2.

**Remark 4.12.** The classification in Table 2 is not a complete list of possible invariant and coinvariant sublattices with respect to isometries that induce a nontrivial action on $A_X$. In fact we do not show that the choices of $v_0$ and $v_1$ are unique. The different cases that we find for some invariant and coinvariant lattices of $\Lambda$ depend on the possible embeddings of $v_0$ in $S_G(\Lambda)$ and of $v_1$ in $T_G(\Lambda)$ classified in Table 1.

### Table 2. Some nonsymplectic involutions on $H^2(X, \mathbb{Z})$ with nontrivial action on $A_X$.

| No. | $S_G(X)$ | $T_G(X)$ | $a(S_G(X))$ |
|-----|----------|----------|-------------|
| 1   | $U \oplus (-2) \oplus \langle -4 \rangle$ | $\langle 4 \rangle$ | 3           |
| 2.1 | $(2) \oplus (-2) \oplus \langle -4 \rangle$ | $\langle -2, 2 \rangle \oplus \langle 4 \rangle$ | 6           |
| 2.2 | $U \oplus (-2) \oplus \langle -4 \rangle$ | $\langle -2 \rangle \oplus \langle 4 \rangle$ | 4           |
| 3   | $U \oplus U(2) \oplus \langle -4 \rangle$ | $U(2) \oplus \langle -4 \rangle$ | 4           |
| 7.1 | $(2) \oplus (-2) \oplus \langle -4 \rangle$ | $\langle -2 \rangle \oplus \langle 4 \rangle$ | 5           |
| 7.2 | $(2) \oplus (-2) \oplus \langle -4 \rangle$ | $\langle -2 \rangle \oplus \langle 2 \rangle \oplus \langle -4 \rangle$ | 5           |
| 8   | $U \oplus (-2) \oplus \langle 4 \rangle$ | $\langle 2 \rangle \oplus \langle -2 \rangle \oplus \langle -4 \rangle$ | 3           |
| 10.1| $(2) \oplus (-2) \oplus \langle -4 \rangle$ | $U \oplus \langle -2 \rangle \oplus \langle -4 \rangle$ | 4           |
| 10.2| $U \oplus \langle 4 \rangle$ | $U \oplus \langle -2 \rangle \oplus \langle -4 \rangle$ | 4           |
| 11  | $U \oplus \langle 4 \rangle$ | $U \oplus \langle -2 \rangle \oplus \langle -4 \rangle$ | 2           |
| 12  | $(2) \oplus \langle 4 \rangle$ | $U \oplus \langle -2 \rangle \oplus \langle -4 \rangle$ | 3           |
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possibility is to take the trivial subgroup. In this case \( R^⊥ \cong T_G(X) \subset T_G(\Lambda) \) and we can apply Theorem 2.22 to note that there are no other primitive embeddings, up to isometry, of \( R \) in \( T_G(\Lambda) \) since the unique subgroup of \( A_L \) that we can choose is \( H_R \cong \text{id} \).

In the other case of Table 1 we can embed the two generators of \( R \) in \( U^\oplus 2 \), and we obtain \( T_G(X) = U \oplus (-2)^\oplus 2 \oplus A_2(-1) \). In this case \( R(A_{T_G(X)}) = 2 \) since \( A_{T_G(X)} = (\mathbb{Z}/2\mathbb{Z})^\oplus 2 \oplus 3\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \). We have that the hypothesis of Theorem 2.24 is verified since \( \text{rk}(T_G(X)) \geq 2 \). Also in this case we have uniqueness up to isometry of \( T_G(X) \). The possible cases are classified in Table 3.

4.5. **p=5.** For \( p = 5 \) the action on \( A_X \) is trivial, hence \( S_G(X) = S_G(\Lambda) \) and \( T_G(X) \) has signature \((1,*)) \) and the latter is obtained as the orthogonal of the sublattice \( R \subset T_G(\Lambda) \). As we have seen in Table 1 the only admissible lattice for \( T_G(\Lambda) \) is \( U^\oplus 2 \oplus H_5 \).

As in the second case for \( p = 3 \) we can notice that \( \text{rk}(T_G(X)) = 4 \) and \( R(A_{T_G(X)}) = 2 \), for this reason \( T_G(X) \) is uniquely determined up to isometry of \( T_G(X) \). The possible cases are classified in Table 9.

4.6. **p=7.** As in the second case for \( p = 3 \) we can notice that in Table 1 we have only a possible choice for \( T_G(\Lambda) \). Consequently we have that \( \text{rk}(T_G(X)) = 4 \) and \( R(A_{T_G(X)}) = 2 \), for this reason \( T_G(X) \) is uniquely determined up to isometry of \( T_G(X) \). The possible cases are classified in Table 3.

4.7. **Proofs of theorems.** The following results summarize what we know so far about nonsymplectic automorphisms of prime order of manifolds of OG₆ type, with respect to the 3 levels of classification that we have described in Section 4.1.

**Theorem 4.13.** Let \( X \) be a manifold of OG₆ type. Let \( G \) be a group of automorphisms where all the nontrivial elements are nonsymplectic of prime order \( p \). If \( p = 2 \) and if the induced action of \( G \) on \( A_X \) is trivial there are twenty-four pairs \((S_G(X), T_G(X))\) up to isometry of the invariant and coinvariant sublattices of \( H^2(X,\mathbb{Z}) \). If \( p = 3 \) there are two pairs of \((S_G(X), T_G(X))\) up to isometry, and if \( p = 5 \) or \( p = 7 \) there exists a unique pair \((S_G(X), T_G(X))\) up to isometry.

**Proof.** In Table 1 we obtain the classification (1.) described in Section 4.1. i.e. we compute the pairs \((S_G(X), T_G(X))\) up to isometry.

**Corollary 4.14.** Let \( X \) be a manifold of OG₆ type. Let \( G \) be a group of automorphisms where all the nontrivial elements are nonsymplectic of prime order \( p \). If \( p \in \{3,7\} \) the coinvariant sublattice \( S_G(X) \) (or equivalently the invariant sublattice \( T_G(X) \)) admits a unique embedding in \( H^2(X,\mathbb{Z}) \) up to isometry of \( H^2(X,\mathbb{Z}) \).

**Proof.** For the second level of classification we can consider [3] Theorem 2.9, and using this result we find a unique embedding of \( S_G(X) \) (or equivalently of \( T_G(X) \)) in \( H^2(X,\mathbb{Z}) \), up to isometry of \( H^2(X,\mathbb{Z}) \). This depends on the fact that for the two cases of \( p = 3 \) and in the case of \( p = 7 \), the rank of \( S_G(X) \) or the rank of \( T_G(X) \) is equal to 2 and there are three copies of \( U \) in \( H^2(X,\mathbb{Z}) \).

In Table 3 we indicate \( a = a(S_G(X)) \) and \( \delta = \delta(T_G(X)) \), \( H_L \) is the subgroup of \( A_L \) referring to Proposition 4.8. The last two columns i. and i.q. indicate if an automorphism is induced or induced at the quotient with respect to the criterion recalled in Section 2.8 in Theorem 2.40 and in Theorem 2.42 respectively.
TABLE 3. Nonsymplectic isometries of prime order on $H^2(X, \mathbb{Z})$ with trivial action on $A_X$.

| No. | $p$ | $S_G(X)$ | $T_G(X)$ | $a$ | $\delta$ | $H_L$ | i. | i.q. |
|-----|-----|-----------|-----------|-----|--------|-------|----|-----|
| 1   | 2   | $U^{\oplus 2} \oplus (-2)^{\oplus 3}$ | $(2)$ | 3   | 1      | 0     | no | –   |
| 2.1 | 2   | $U \oplus (-2)^{\oplus 3} \oplus (2)$ | $(2) \oplus (-2)$ | 4   | 1      | 0     | no | –   |
| 2.2 | 2   | $U \oplus (-2)^{\oplus 3} \oplus (2)$ | $U(2)$ | 4   | 0      | 0     | no | –   |
| 3.1 | 2   | $U^{\oplus 2} \oplus (-2)^{\oplus 2}$ | $U$ | 2   | 0      | 0     | no | –   |
| 3.2 | 2   | $U^{\oplus 2} \oplus (-2)^{\oplus 2}$ | $U(2)$ | 2   | 0      | $\mathbb{Z}/2\mathbb{Z}$ | no | –   |
| 3.3 | 2   | $U^{\oplus 2} \oplus (-2)^{\oplus 2}$ | $(2) \oplus (-2)$ | 2   | 1      | $\mathbb{Z}/2\mathbb{Z}$ | no | –   |
| 4   | 2   | $(2)^{\oplus 2} \oplus (-2)^{\oplus 3}$ | $(-2)^{\oplus 2} \oplus (2)$ | 5   | 1      | 0     | no | –   |
| 5.1 | 2   | $U \oplus (-2)^{\oplus 2} \oplus (2)$ | $U \oplus (-2)$ | 3   | 1      | 0     | no | –   |
| 5.2 | 2   | $U \oplus (-2)^{\oplus 2} \oplus (2)$ | $(2) \oplus (-2)^{\oplus 2}$ | 3   | 1      | $\mathbb{Z}/2\mathbb{Z}$ | no | –   |
| 6.1 | 2   | $U^{\oplus 2} \oplus (-2)$ | $(-2)^{\oplus 2} \oplus (2)$ | 1   | 1      | $A_R$ | no | –   |
| 6.2 | 2   | $U^{\oplus 2} \oplus (-2)$ | $U \oplus (-2)$ | 1   | 1      | $\mathbb{Z}/2\mathbb{Z}$ | no | –   |
| 7.1 | 2   | $(2)^{\oplus 2} \oplus (-2)^{\oplus 2}$ | $U \oplus (-2)^{\oplus 2}$ | 4   | 1      | 0     | no | –   |
| 7.2 | 2   | $(2)^{\oplus 2} \oplus (-2)^{\oplus 2}$ | $U \oplus (2) \oplus (-2)^{\oplus 3}$ | 4   | 1      | $\mathbb{Z}/2\mathbb{Z}$ | no | –   |
| 8   | 2   | $(2)^{\oplus 2} \oplus (-2)^{\oplus 2}$ | $(-2)^{\oplus 2} \oplus U(2)$ | 4   | 1      | $\mathbb{Z}/2\mathbb{Z}$ | no | –   |
| 9.1 | 2   | $U \oplus (2) \oplus (2)$ | $(-2)^{\oplus 3} \oplus U(2)$ | 2   | 1      | $A_R$ | yes | –   |
| 9.2 | 2   | $U \oplus (2) \oplus (2)$ | $U \oplus (-2)^{\oplus 2} \oplus U(2)$ | 2   | 1      | $A_R$ | yes | –   |
| 10.1| 2   | $U \oplus U(2)$ | $U \oplus (2) \oplus (-2)^{\oplus 2}$ | 2   | 1      | $A_R$ | yes | –   |
| 10.2| 2   | $U \oplus U(2)$ | $U \oplus (-2)^{\oplus 2} \oplus U(2)$ | 2   | 1      | $A_R$ | yes | –   |
| 11  | 2   | $U^{\oplus 2}$ | $U \oplus (-2)^{\oplus 2}$ | 0   | 1      | $A_R$ | yes | –   |
| 12.1| 2   | $(2)^{\oplus 2} \oplus (-2)$ | $(-2)^{\oplus 3} \oplus (2)$ | 3   | 1      | $A_R$ | no  | –   |
| 12.2| 2   | $(2)^{\oplus 2} \oplus (-2)$ | $U \oplus (-2)^{\oplus 4}$ | 3   | 1      | $\mathbb{Z}/2\mathbb{Z}$ | no | –   |
| 13  | 2   | $U \oplus (2)$ | $U \oplus (-2)^{\oplus 4}$ | 1   | 1      | $A_R$ | no  | –   |
| 14.1| 2   | $(2)^{\oplus 2}$ | $U \oplus (-2)^{\oplus 4}$ | 2   | 1      | $A_R$ | yes | –   |
| 14.2| 2   | $(2)^{\oplus 2}$ | $U \oplus D_4(-1)$ | 2   | 0      | $\mathbb{Z}/2\mathbb{Z}$ | yes | –   |

| No. | $p$ | $S_G(X)$ | $T_G(X)$ | $a$ | $\delta$ | $H_L$ | i. | i.q. |
|-----|-----|-----------|-----------|-----|--------|-------|----|-----|
| 1   | 3   | $U^{\oplus 2} \oplus A_2(-1)$ | $(-2) \oplus (6)$ | 1   | –      | –     | no | yes |
| 2   | 3   | $A_2$ | $A_2 \oplus (2)^{\oplus 2}$ | 1   | –      | –     | yes | yes |
| 1   | 5   | $U \oplus H_5$ | $(-2) \oplus (-10) \oplus U$ | 1   | –      | –     | yes | yes |
| 1   | 7   | $U^{\oplus 2} \oplus K_7$ | $(2) \oplus (14)$ | 1   | –      | –     | yes | no  |

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