Scattering amplitudes and $N$-body post-Minkowskian Hamiltonians in general relativity and beyond

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Abstract: We present a general framework for calculating post-Minkowskian, classical, conservative Hamiltonians for $N$ non-spinning bodies in general relativity from relativistic scattering amplitudes. Novel features for $N > 2$ are described including the subtraction of tree-like iteration contributions and the calculation of non-trivial many-body Fourier transform integrals needed to construct position space potentials. A new approach to calculating these integrals as an expansion in the hierarchical limit is described based on the method of regions. As an explicit example, we present the $O(G^2)$ 3-body momentum space potential in general relativity as well as for charged bodies in Einstein-Maxwell. The result is shown to be in perfect agreement with previous post-Newtonian calculations in general relativity up to $O(G^2\nu^4)$. Furthermore, in appropriate limits the result is shown to agree perfectly with relativistic probe scattering in multi-center extremal black hole backgrounds and with the scattering of slowly-moving extremal black holes in the moduli space approximation.

Keywords: Black Holes, Classical Theories of Gravity, Effective Field Theories, Scattering Amplitudes

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1 Introduction

The problem of predicting the motion of $N$ gravitating, compact bodies is one of the oldest and most basic problems in theoretical physics, and has important applications in astrophysics, cosmology, and gravitational wave science. In Newtonian gravity the equations of motion are simple, pairwise superpositions of 2-body Newtonian forces, but the corresponding solutions are famously chaotic [1] and a complete understanding of periodic solutions for $N \geq 3$ remains an active area of research [2]. In general relativity (GR) the corresponding problem is qualitatively different: even determining the equations of motion themselves is non-trivial since the underlying theory of gravitation is highly non-linear, and there are intrinsic $N$-body forces that cannot be reduced to interactions among pairs of bodies. In the general relativistic $N$-body problem more is different.

Numerical solutions provide a wealth of information for $N$-body dynamics, including strong field effects like black hole mergers [3–5], but can be computationally expensive
and impractical to implement even with only a few bodies. Since finding exact solutions to Einstein’s equations for the motion of $N$ compact bodies appears intractable, it is important to consider a broad range of complementary approaches, including analytic methods. This strategy has been quite successful for the 2-body problem where numerical relativity [6–8], effective one-body [9, 10], and gravitational self-force [11–14] have been applied in concert with perturbative analyses in post-Newtonian (PN) [15–22], post-Minkowskian (PM) [23–31], and non-relativistic general relativity [32–42].

Recently, new techniques based on powerful tools from theoretical high energy physics, such as scattering amplitudes and effective field theory (EFT), have also been applied to the 2-body problem [43–45]. These approaches build on the field theoretic description of gravitons [46–53] by bringing in cutting-edge tools such as the double copy [54–58], on-shell methods [59–63], EFT [32, 42, 64], and advanced multiloop integration [65–74]. The program is motivated in large part by advancing the development of highly accurate waveform models for future gravitational wave detectors [75, 76]; see e.g. [77–80] for recent reviews. Scattering amplitudes have now been applied for deriving a number of state-of-the-art predictions in the PM expansion (fixed order in $G$ and all-orders in velocity), such as the 2-body Hamiltonian at 3PM [81, 82] and 4PM [83, 84], as well as for modeling spin [85–101], tidal corrections [102–107], and radiative effects [108–117]. Much of this recent progress has also been driven by concurrent developments in worldline-based approaches to the calculation of observables in the PM expansion [89, 97, 118–131].

In this paper we extend these amplitudes-based methods to the $N$-body problem, focusing on the calculation of PM Hamiltonians from high-multiplicity relativistic scattering amplitudes. We emphasize novel features that arise for the $N > 2$ case, such as gauge (coordinate) ambiguities, the subtraction of iteration contributions, and non-trivial Fourier transform integrals. As in the 2-body case, deriving classical $N$-body dynamics from scattering amplitudes leads to vast simplifications in the structure of perturbation theory, allowing for both efficient calculation and insight into underlying theoretical structures. These developments offer a path for advancing the state-of-the-art in classical $N$-body dynamics, which can be applied, e.g., for modeling sequential and hierarchical mergers [132–135], and may also enhance our understanding of binaries, and, more broadly, of the theoretical structures that connect scattering amplitudes and classical dynamics.

Previous PN results (an expansion in $v \sim \sqrt{Gm/r} \ll 1$) for the $N$-body problem in GR have been obtained using a variety of purely classical approaches. Important results include the calculation of the leading 1PN $N$-body potential in [15]. The calculation of the 2PN contribution to the 3-body potential was undertaken in [136] and completed in [137]. More recently, by making use of the EFT formalism of [32, 42], these results were extended to include 4-body interactions at 2PN order where the result was given in terms of certain un-evaluated Feynman integrals. The 1PM Hamiltonian was calculated in [138], although at this order there are no genuine 3-body interactions. Using worldline based methods, a formal 2PM effective Lagrangian was calculated in [139] and explicit velocity expanded expressions obtained up to $O(G^2v^4)$. As will be discussed further in section 4.3, the $O(v^2)$, and all orders in $G$, $N$-body potential between extremal (charged) black holes has also been calculated using the moduli space approximation [140]. The main explicit result obtained in
this paper, the 3-body, 2PM, momentum space Hamiltonian for charged bodies in Einstein-Maxwell theory (3.9), is shown to be in perfect agreement with these previous results.

This paper is organized as follows. In section 2 we introduce the general framework for calculating PM (and PN) conservative N-body potentials from N-body scattering amplitudes. The problem of gauge ambiguity is discussed in section 2.4, we introduce a generalization of isotropic gauge to arbitrary reference frames for 2-body dynamics. The systematics of calculating “tree” iteration subtraction contributions is described in section 2.5. The family of non-trivial Feynman integrals relevant for the 3-body potential are described in detail in section 2.6, a new approach to calculating these integrals perturbatively in the hierarchical limit is described based on the method of regions. In section 3 the explicit 3-body, 2PM, momentum space Hamiltonian for charged bodies is given. Further details of the calculation of the subtraction contribution are given in appendix B. Verification of this result is described by comparison with previous PN results in section 4.1, with relativistic probe dynamics on multi-center (extremal) black hole backgrounds in section 4.2 and with the moduli space approximation to extremal black hole interactions in section 4.3. In section 5 important open problems and future directions are described. Finally, the explicit Feynman rules used to calculate the 3-body amplitudes used in section 3 are given in appendix A.

2 General framework

In this section we will describe a generalization of the framework for calculating 2-body conservative PM potentials from scattering amplitudes described in [44] to N-bodies.

2.1 Classical dynamics from scattering amplitudes

We model an N-body system of gravitationally and electromagnetically interacting compact, spinless bodies (e.g. black holes or neutron stars) with masses $m_i$ and electric charges $Q_i$ by $N$ distinguishable scalar fields $\phi_i$:

$$S = \int \mathcal{D}^{d+1}x \sqrt{-g} \left[ -\frac{R}{16\pi G} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{i=1}^{N} \left( |\nabla_\mu \phi_i|^2 - m_i^2 |\phi_i|^2 \right) \right] + S_{GF} + S_{HD}. \quad (2.1)$$

In this action and throughout this paper we will use the $(+,\ldots,-)$ metric convention. Here $S_{GF}$ denotes the gauge fixing terms and $S_{HD}$ denotes higher-derivative interactions encoding (sub-leading) finite-size effects and higher (gravitational and electromagnetic) multipole moments (see e.g. [102, 105]). This effective description is valid in the regime where we have a hierarchy of scales $R \ll r$, where $R$ the typical size of the bodies (for a black hole $R \sim 2Gm$ the Schwarzschild radius) and $r$ is the typical inter-body separation. In the 2-body problem this EFT is appropriate for describing the inspiral phase of a binary black hole merger [32, 42, 44].

We consider the elastic N-to-N process where the incoming scalars have four-momenta $p_i^\mu = (E_i, \vec{p}_i)$, while the outgoing scalars have momenta $p_i^\mu - q_i^\mu$. Conservation of total four-momentum implies $\sum_i q_i^\mu = 0$, while the on-shell condition for external legs implies $p_i \cdot q_i = q_i^2/2$. In the two-body case, it is convenient to work in the center-of-mass (COM) frame where the momentum transfer is purely spatial. This yields results for the Hamiltonian in
isotropic gauge. With $N$ bodies, however, there is no frame choice where all momentum transfers are purely spatial. We will thus work in a generic frame and use the on-shell conditions to fix the energy component of the momentum transfers, $q_0^i$.

In general, scattering amplitudes are manifestly relativistic objects, and so naturally encode dynamics at all orders in velocity. For the $N$-body problem in particular, the non-trivial dynamical input comes from high-multiplicity scattering amplitudes, which can be efficiently calculated using powerful on-shell methods including the double-copy \cite{54–58}, on-shell recursion, and unitarity methods \cite{59–63}.

The scattering amplitude is calculated as a PM expansion in powers of $G$

$$\mathcal{M}_N = \frac{\mathcal{M}^{1\text{PM}}_N}{G} + \frac{\mathcal{M}^{2\text{PM}}_N}{G^2} + \frac{\mathcal{M}^{3\text{PM}}_N}{G^3} + \cdots. \quad (2.2)$$

The classical limit of the full quantum scattering amplitudes $\mathcal{M}_2, \mathcal{M}_3, \ldots, \mathcal{M}_N$ encode conservative $N$-body scattering dynamics. The classical limit follows from having large gravitational charges, $m_i \gg M_{\text{Planck}}$, and large angular momenta in the scattering process, $J \gg \hbar$. From here on we set $\hbar = 1$. The classical limit is implemented by rescaling all graviton momenta $\ell \rightarrow \ell \lambda$ and then expanding in small $\lambda$. For example, $\ell$ could be a loop momentum or one of the momentum transfers $q_i$. These soft graviton momenta can be further separated into potential and radiation subregions, defined by the respective energy-momentum scalings $\ell^\mu \sim (v\lambda, \lambda)$ and $\ell^\mu \sim (v\lambda, v\lambda)$, where $v$ is the typical velocity of the $N$ bodies that defines the PN expansion. We focus here on the conservative dynamics described by potential gravitons.

Order-by-order in PM we subsequently expand in the classical limit

$$\mathcal{M}^{n\text{PM}}_N = \sum_{k \in \mathbb{Z}} \mathcal{M}^{n\text{PM}(k)}_N, \quad \text{where} \quad \mathcal{M}^{n\text{PM}(k)}_N(\{\tilde{p}, \lambda \tilde{q}\}) = \lambda^k \mathcal{M}^{n\text{PM}(k)}_N(\{\tilde{p}, \tilde{q}\}). \quad (2.3)$$

Terms in this series are separated according to the power $k$: \footnote{This scaling can be deduced by dimensional analysis. The $N$-body (quantum) position space potential is a function of the available dimensionful quantities $\{G, h, c, m, p, x\}$ with $[V_N] = ML^2T^{-2}$. The classical part of the potential at $n\text{PM}$, is necessarily homogeneous in re-scaling of the position variables $V_N|_{G^m}\lambda^x = \lambda^{-n(d-2)}V_N|_{G^m}(x)$. Fourier transforming to momentum space, the intrinsic $N$-body potential is multiplied by an overall factor $\delta^{(3)}(\tilde{q}_1 + \cdots + \tilde{q}_N)$, and so $V^{(N)}|_{G^m}\lambda^x = \lambda^{n(d-2)-d(N-1)}V^{(N)}|_{G^m}(\tilde{q})$. The classical part of the amplitude $M_N$ is then defined as the piece generated by the classical momentum space potential in the first Born approximation $M_N \approx -V^{(N)}$.}

- **Super-classical:** $k < n(d-2) - d(N-1)$.
- **Classical:** $k = n(d-2) - d(N-1)$.
- **Quantum:** $k > n(d-2) - d(N-1)$.

Super-classical terms do not contribute to the potential, and must cancel with contributions arising from the iteration of lower-PM-order (and lower-multiplicity) potentials. Quantum terms are discarded as soon as possible since they do not contribute to the classical potential. Note that in (2.3) we are ignoring the overall energy conserving delta function.
which is not homogeneous in the classical scaling \( \vec{q} \to \lambda \vec{q} \). Until a resolution of this constraint is imposed the precise separation between super-classical and classical contributions is ill-defined.

2.2 N-body conservative potentials

After integrating out potential mode gravitons, the resulting conservative dynamics can be described by an effective \( N \)-body Hamiltonian of the form

\[
H(\{\vec{p}, \vec{x}\}) = \sum_{i=1}^{N} \sqrt{\vec{p}_i^2 + m_i^2} + V(\{\vec{p}, \vec{x}\}),
\]

where \( \vec{p}_i \) and \( \vec{x}_i \) denote the 3-momenta and positions of the \( i \)-th particle. Obtaining the potential from matching to a scattering amplitude naturally produces a potential in momentum space, this is related to the position space potential by a Fourier transform which in our conventions will be

\[
V(\{\vec{p}, \vec{x}\}) = \prod_{i=1}^{N} \int \frac{d^d \vec{q}_i}{(2\pi)^d} e^{i \vec{q}_i \cdot \vec{x}_i} V(\{\vec{p}, \vec{q}\}).
\]

To regularize divergences at intermediate stages of calculation, the Fourier transform integrals in this paper will be calculated using dimensional regularization as an expansion around \( d = 3 - 2\epsilon \) dimensions. Overall translation invariance of the position space potential will always produce \( d \) delta functions after Fourier transforming. However, the \( N \)-body potential will also contain terms that depend only on a subset of the bodies, and the Fourier transform of these will produce more singular terms that encode intrinsically \( n \)-body interactions for \( n < N \). For example for \( N = 3 \)

\[
V(\{\vec{p}, \vec{q}\}) = \sum_{(i,j,k) \in S_3} \left[ (2\pi)^{2d} \delta^{(d)}(\vec{q}_i + \vec{q}_j) \delta^{(d)}(\vec{q}_k) \times \frac{1}{2} \times V^{(2)}_{ij}(\vec{p}_i, \vec{p}_j, \vec{q}_i, \vec{q}_j) \right. \\
+ \left. (2\pi)^d \delta^{(d)}(\vec{q}_i + \vec{q}_j + \vec{q}_k) V^{(3)}_{ijk}(\vec{p}_i, \vec{p}_j, \vec{p}_k, \vec{q}_i, \vec{q}_j, \vec{q}_k) \right].
\]

In this formula and subsequently in this paper, the notation \( (i, j, k) \in S_3 \) means that we sum over the \( 3! \) distinct permutations of \( \{1, 2, 3\} \). The factor of \( 1/2 \) multiplying the intrinsic 2-body interaction \( V^{(2)}_{ij} \) in this expression is purely conventional.

Classical scaling for the momentum space potential is defined analogously to (2.3). For \( N \)-bodies in \( d \) spatial dimensions at \( n \)PM, the classical potential (including the delta functions) is homogeneous

\[
V|_{G^n}(\{\vec{p}, \lambda \vec{q}\}) = \lambda^{n(d-2)-dN} V|_{G^n}(\{\vec{p}, \vec{q}\}).
\]

Unlike the scattering amplitude (2.3), there are no super-classical pieces.

\(^2\)Note that here we are making a common abuse of notation using the symbol \( \vec{q}_i \) to denote two strictly different quantities. In (2.2), \( \vec{q}_i \) denotes the momentum transfer in a scattering amplitude, while in (2.5), \( \vec{q}_i \) denotes the Fourier conjugate variable to the position vector \( \vec{x}_i \). These two quantities only coincide when the potential is used to calculate a scattering amplitude using the Born series (2.15).
2.3 Matching calculation

The $N$-body effective potential (2.4) is determined by integrating out potential gravitons through an EFT matching calculation. The same physical observable, the classical part of an $N$-to-$N$ scattering amplitude, is calculated from (2.1) using manifestly relativistic Feynman-Dyson perturbation theory and from (2.4) by (formally) solving the Lippmann-Schwinger equation. Requiring that these two calculations agree defines the effective potential.

In spite of the manifestly relativistic nature of the scattering amplitude $M_N$ in (2.2), the corresponding scattering amplitudes computed from the effective Hamiltonian (2.4) are not manifestly Lorentz invariant since this object depends on a choice of time coordinate. We therefore parametrize the scattering kinematics using 3-momenta. In a general $N$-body scattering process we will denote the incoming 3-momenta by $\vec{p}_i$ and the outgoing 3-momenta by $\vec{p}_i' \equiv \vec{p}_i - \vec{q}_i$, where $\vec{q}_i$ is the 3-momentum transfer. Conservation of 3-momentum corresponds to the constraint $\sum_i \vec{q}_i = 0$. All external particles are assumed to be on-shell, and for a given particle of mass $m_i$ and 3-momentum $\vec{k}_i$ we denote the relativistic energy as $E_i(\vec{k}_i) = \sqrt{|\vec{k}_i|^2 + m_i^2}$. Energy is also conserved in scattering which is given as a non-linear constraint

$$\sum_i E_i(\vec{p}_i) = \sum_i E_i(\vec{p}_i' - \vec{q}_i),$$

which, to leading classical order, takes the form

$$\sum_i \vec{p}_i \cdot \vec{q}_i E_i(\vec{p}_i) = \mathcal{O}(|\vec{q}|^2).$$

Using the momentum space potential we can calculate a scattering amplitude by solving the (relativistic) Lippmann-Schwinger equation [141, 142]

$$T(\{\vec{p}, \vec{q}\}) \equiv -V(\{\vec{p}, \vec{q}\}) + \int_{\{\vec{k}\}} \frac{V(\{\vec{k}, \vec{k} - \vec{p} + \vec{q}\}) T(\{\vec{p} - \vec{k}, \vec{q}\})}{\sum_{j=1}^N [E_j(\vec{p}) - E_j(\vec{k})] + i\epsilon},$$

where

$$\int_{\{\vec{k}\}} \equiv \prod_{i=1}^N \int \frac{d^d\vec{k}_i}{(2\pi)^d},$$

and

$$\langle\{\vec{p} - \vec{q}\}|T|\{\vec{p}\}\rangle \equiv 2\pi\delta\left(\sum_{j=1}^N [E_j(\vec{p}) - E_j(\vec{p} - \vec{q})]\right) T(\{\vec{p}, \vec{q}\}),$$

is the non-trivial part of the S-matrix defined in the convention $S = 1 + iT$. In (2.10) we have introduced the notation $\equiv$ to denote equality on the constraint surface associated with conservation of energy (2.8). Note that for $N > 2$, the $T$-matrix element is not exactly the same as the scattering amplitude since it contains both fully- and partially-connected
contributions; for example for $N = 3$

$$T (\{ \vec{p}, \vec{q} \} ) \overset{i}{=} (2\pi)^{2d} \delta^{(d)} (\vec{q}_3) \delta^{(d)} (\vec{q}_1 + \vec{q}_2) M_2^{(12)} (\vec{p}_1, \vec{p}_2, \vec{q}_1, \vec{q}_2) + (2\pi)^{2d} \delta^{(d)} (\vec{q}_2) \delta^{(d)} (\vec{q}_1 + \vec{q}_3) M_2^{(13)} (\vec{p}_1, \vec{p}_3, \vec{q}_1, \vec{q}_3) + (2\pi)^{2d} \delta^{(d)} (\vec{q}_1) \delta^{(d)} (\vec{q}_2 + \vec{q}_3) M_2^{(23)} (\vec{p}_2, \vec{p}_3, \vec{q}_2, \vec{q}_3) + (2\pi)^d \delta^{(d)} (\vec{q}_1 + \vec{q}_2 + \vec{q}_3) M_3 (\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{q}_1, \vec{q}_2, \vec{q}_3). \quad (2.13)$$

The amplitude $M_N$ differs from the amplitude $M_N$ in (2.2) by a (purely conventional) non-relativistic normalization factor

$$M_N (\{ \vec{p}, \vec{q} \} ) = \prod_{i=1}^{N} \frac{1}{2\sqrt{E_i (\vec{p}_i - \vec{q}_i)}} M_N (\{ \vec{p}, \vec{q} \} ). \quad (2.14)$$

The Lippmann-Schwinger equation (2.10) is formally solved by the Born series

$$T (\{ \vec{p}, \vec{q} \} ) \overset{-1}{=} -V (\{ \vec{p}, \vec{q} \} ) - \int_{\{ \vec{k} \} } \sum_{j=1}^{N} \left[ V (\{ \vec{k}, \vec{k} - \vec{p} + \vec{q} \} ) V (\{ \vec{p}, \vec{p} - \vec{k} \} ) \right] + \ldots \quad (2.15)$$

The connected components of the $T$-matrix can be calculated in a Feynman diagrammatic expansion by treating the expanded potential components as Wilson coefficients in a non-relativistic EFT [44, 64]

$$L_{\text{EFT}} = \sum_i \int_{\vec{k}_i} \left[ i\partial_t - \sqrt{|\vec{k}_i|^2 + m_i^2} \right] \phi_i (\vec{k}_i) - \sum_{ij} \int \frac{d^d \vec{k}_i}{(2\pi)^d} \frac{d^d \vec{k}_j}{(2\pi)^d} (2\pi)^{d} \delta^{(d)} (\vec{k}_i + \vec{k}_j - \vec{k}_i - \vec{k}_j) V^{(2)}_{ij} (\{ \vec{k}, \vec{k} \} ) \times \phi_i^\dagger (\vec{k}_i) \phi_j^\dagger (\vec{k}_j) \phi_i (\vec{k}_i) \phi_j (\vec{k}_j) - \sum_{ijk} \int \frac{d^d \vec{k}_i}{(2\pi)^d} \frac{d^d \vec{k}_j}{(2\pi)^d} \frac{d^d \vec{k}_k}{(2\pi)^d} (2\pi)^d \delta^{(d)} (\vec{k}_i + \vec{k}_j + \vec{k}_k - \vec{k}_i - \vec{k}_j - \vec{k}_k) V^{(3)}_{ijk} (\{ \vec{k}, \vec{k} \} ) \times \phi_i^\dagger (\vec{k}_i) \phi_j^\dagger (\vec{k}_j) \phi_k^\dagger (\vec{k}_k) \phi_i (\vec{k}_i) \phi_j (\vec{k}_j) \phi_k (\vec{k}_k) + (\text{4-body interactions}) + \ldots \quad (2.16)$$

One of the advantages of the EFT is that the tree-like contributions to (2.15), arising for $N > 2$ from the lower-multiplicity contributions to the potential (2.6), become literal tree Feynman diagrams.

The matching procedure for calculating the potential from a scattering amplitudes is a simple rearrangement of the Born series (2.15). For example at 2PM

$$V |_{G^2} (\{ \vec{p}, \vec{q} \} ) \overset{1}{=} -T |_{G^2} (\{ \vec{p}, \vec{q} \} ) - \int_{\{ \vec{k} \} } \frac{V |_G (\{ \vec{k}, \vec{k} - \vec{p} + \vec{q} \} ) V |_G (\{ \vec{p}, \vec{p} - \vec{k} \} )}{\sum_{j=1}^{N} \left[ E_j (\vec{p}) - E_j (\vec{k}) \right] + i\epsilon}. \quad (2.17)$$

The $V$-dependent terms on the right-hand-side are called iteration contributions, and are shown in figure 1. They are discussed in greater detail in section 2.5.
2.4 Gauge ambiguity and conservation of energy

The effective potential between \( N \)-bodies (2.4) depends on the choice of coordinate system and is therefore not a gauge invariant physical quantity. By contrast, scattering amplitudes are gauge invariant. It may seem puzzling, or even inconsistent, that the matching procedure (2.17) seems to relate a gauge invariant quantity to a gauge non-invariant quantity. The resolution is that equations (2.12) and (2.17) hold only on the constraint surface of energy conservation, and therefore do not admit a unique solution for the potential. The correct statement about the gauge invariance of the scattering amplitude is that the \( T \)-matrix element, including the energy conserving delta function, is gauge invariant and unambiguous. The function \( T (\{\vec{p}, \vec{q}\}) \) in (2.12) is defined up to the freedom to add an arbitrary function that vanishes on the support of the energy conserving delta function, and this ambiguity encompasses the coordinate dependence of the potential.

To be more explicit, consider the calculation of the 1PM 2-body potential from the physical scattering amplitude \( M_2 \). In this case, in \( d = 3 \), the matching equation reduces to

\[
V(\vec{p}_1, \vec{p}_2, \vec{q}_1, \vec{q}_2) \overset{!}{=} -\left(2\pi\right)^3 \delta^{(3)}(\vec{q}_1 + \vec{q}_2) M_2(\vec{p}_1, \vec{p}_2, \vec{q}_1, \vec{q}_2),
\]

(2.18)

where on both sides we are truncating at \( O(G|\vec{q}|^{-2}) \). It is straightforward to see that this equation does not uniquely determine the potential, suppose we take any solution and modify it as

\[
V(\vec{p}_1, \vec{p}_2, \vec{q}_1, \vec{q}_2) \rightarrow V(\vec{p}_1, \vec{p}_2, \vec{q}_1, \vec{q}_2) + \left(2\pi\right)^3 \delta^{(3)}(\vec{q}_1 + \vec{q}_2) \times \left(\frac{cG}{|\vec{q}|^4} \left(\frac{\vec{p}_1 \cdot \vec{q}_1}{E_1(\vec{p}_1)} + \frac{\vec{p}_2 \cdot \vec{q}_2}{E_2(\vec{p}_2)}\right)^2\right),
\]

(2.19)

where \( c \) is an arbitrary real number. From (2.9) we see that the second term on the right-hand-side is zero on the energy conserving constraint surface, up to terms that vanish in the classical limit. This expression defines a distinct, but perfectly physical, momentum space potential that is related to the original potential by some change of coordinates in position space. All physical observables calculated using either potential must agree.

For \( N = 2 \) there is a choice of resolution of this ambiguity that dramatically simplifies calculations. Using (2.9) and 3-momentum conservation \( \vec{q}_1 + \vec{q}_2 = 0 \)

\[
\vec{p}_i \cdot \vec{q}_j \overset{!}{=} \frac{E_i(\vec{p}_i) ((\vec{p}_1 + \vec{p}_2) \cdot \vec{q}_i)}{E_1(\vec{p}_1) + E_2(\vec{p}_2)} + O(|\vec{q}|^2).
\]

(2.20)

Order-by-order in the classical expansion we can use this identity to rewrite all \( \vec{p} \cdot \vec{q} \) dot products in terms of \((\vec{p}_1 + \vec{p}_2) \cdot \vec{q}_1\). Making this choice in the definition of \( T (\{\vec{p}, \vec{q}\}) \) uniquely fixes the ambiguity in a way that reduces the total number of independent kinematic invariants that appear in the momentum space potential. This choice is especially useful if we further restrict to the COM frame \( \vec{p}_1 + \vec{p}_2 = 0 \), where this choice is usually called \emph{isotropic gauge}, in which case all \( \vec{p} \cdot \vec{q} \) dot products are absent. As will be discussed further below, the calculation of the iteration contributions to \( N \)-body potentials with \( N > 2 \) requires the 2-body potential in a general frame \( \vec{p}_1 + \vec{p}_2 \neq 0 \). In this case we will refer to this resolution of the ambiguity as \emph{generalized isotropic gauge}, and will be used for the 3-body calculation of the 2PM potential in section 3.
There is an analogous ambiguity present in the definition of $T(\{\vec{p}, \vec{q}\})$ for the scattering of $N > 2$ particles. For $N = 3$, classical energy conservation gives the constraint
\[
\frac{\vec{p}_1 \cdot \vec{q}_1}{E_1(\vec{p}_1)} + \frac{\vec{p}_2 \cdot \vec{q}_2}{E_2(\vec{p}_2)} + \frac{\vec{p}_3 \cdot \vec{q}_3}{E_3(\vec{p}_3)} = \mathcal{O}(|\vec{q}|^2). \tag{2.21}
\]
Together with 3-momentum conservation $\vec{q}_1 + \vec{q}_2 + \vec{q}_3 = 0$, this can be used to reduce the total number of $\vec{p}_i \cdot \vec{q}_j$ invariants appearing from 9 to 5. Unfortunately, any complete fixing of the ambiguity of this kind will produce a 3-body potential that does not have the expected $S_3$ relabelling symmetry. This situation is not improved by restricting to the 3-body center of mass frame where $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$, although the number of invariants is further reduced to 3.

This inherent ambiguity in the potential makes it difficult to compare the result of our calculation to existing results in the literature. One approach following [143] would be to construct an explicit coordinate transformation relating different potentials, we will not attempt this. Another approach is to compare different potentials by calculating gauge invariant quantities. In section 4 we will verify our calculation by comparing gauge invariant classical scattering amplitudes. By generalizing the formalism of KMOC [45] to $N$-bodies, the classical scattering amplitude should determine a range of asymptotic classical observables including the linear $\Delta p^\mu_i$ and angular $\Delta J^\mu\nu_i$ impulses. This will be described in a separate paper.

### 2.5 Matter poles and tree iteration

A qualitatively new feature that appears in the matching calculation for $N > 2$ body interactions is the appearance of iteration diagrams that can be separated into two disconnected pieces by cutting a single matter propagator; examples are shown in figures 1 and 2. Non-local terms in the potential arise from integrating out degrees-of-freedom, such as potential mode gravitons. The matter fields in the EFT are not integrated out, and so we should not expect a singularity in the potential to arise from matter propagators. Whether such singularities are present in a given explicit expression for the potential is sensitive to the chosen resolution of the gauge ambiguity described in section 2.4. We will impose as an additional condition that in a physically acceptable $N$-body momentum space potential, the only singularities should correspond to on-shell (soft) graviton exchange. As we will describe below, it is always possible to construct a potential such that the spurious matter singularities cancel between full-theory and EFT iteration.
Figure 2. Schematic representation of a “tree” iteration subtraction contribution to the effective potential. In a physically acceptable representation of the potential, the spurious matter pole must cancel between full-theory and EFT.

It is useful to write the amplitude (2.2) with the graviton propagator in standard relativistic form but with the matter propagators separated into positive energy (matter) and negative energy (anti-matter) contributions

\[
\frac{1}{p_i^2 - m_i^2} = \frac{1}{2E_i(\vec{p}_i)} \left[ \frac{1}{p_i^0 - E_i(\vec{p}_i)} - \frac{1}{p_i^0 + E_i(\vec{p}_i)} \right]. \tag{2.22}
\]

The negative energy pole appears in time-ordered diagrams describing pair production, a purely quantum mechanical process, and so the classical part of the anti-matter piece does not contain a kinematic singularity. This piece will however, still give a non-singular contribution to the classical potential (excepting graviton poles) and cannot be discarded.

For the matter piece of the full theory amplitude, there are genuine singularities at classical order. These must cancel with corresponding poles arising from denominator factors of the Born series (2.15). The difference of the contributions defines a subtraction contribution to the potential. Diagrammatically, there is a correspondence between full theory and iteration graph topologies with a common matter pole. The general form of the associated subtraction contribution is given in figure 2. It is tempting to imagine that the corresponding diagrams will cancel entirely and so can be simply excluded from the Feynman diagram expansion of the full theory amplitude. Unfortunately this is not the case. The general form of the difference depicted in figure 2 is

\[
V_{\text{subtraction}} \supset N^{(\text{Full})}_L N^{(\text{Full})}_R - N^{(\text{EFT})}_L N^{(\text{EFT})}_R, \tag{2.23}
\]

where the numerator factors \( N \) collectively denote the rest of the diagram on one side of the matter pole. Since both full theory and EFT calculate the same scattering amplitude, on the support of energy conservation the residues of the matter singularities must match, and therefore it must be possible to write

\[
N^{(\text{Full})}_L \equiv N^{(\text{EFT})}_L \left( (p_L - p'_L)^0 - E(\vec{p}_L - \vec{p}'_L) \right) \delta N, \tag{2.24}
\]

where \( \delta N \) is non-singular on the matter pole (though it may be singular elsewhere). We then iterate this procedure on a different spurious matter pole (this is non-trivial only
if there are diagrams with overlapping matter poles), including the subtraction contribution \((2.23)\) defined with \((2.24)\) as part of the EFT amplitude. When all spurious poles have been eliminated we continue the result away from the support of the energy conserving delta function (dropping the ! above the equals sign) to define a potential.

While a potential constructed in this way has only physical singularities, there are further conditions we should impose to define a completely physical result: cancellation of super-classical terms and manifest \(S_N\) symmetry among the bodies. As an explicit example, in appendix B we show how to define the subtraction contribution to the 3-body potential at 2PM such that super-classical contributions cancel between diagrams related by symmetry. A completely systematic approach to engineering such cancellations is not known.

The explicit calculation in appendix B also demonstrates clearly why the naive subtraction \((2.23)\) is not trivially zero. In the PM amplitudes the singularities corresponding to potential graviton exchange are not at the same location, and so cannot cancel. Explicitly for a 3-body 2PM subtraction contribution with \(p_L = p_i + p_j\), \(p'_L = p_i - q_i\) and \(p_R = p_k\), and with the 2-body potential calculated in generalized isotropic gauge, the graviton pole factors in the full theory and EFT are

\[
N_L^{(\text{Full})} \sim \frac{1}{(\frac{p_i \cdot q_k}{E_k})^2 - |q_k|^2}, \quad N_L^{(\text{EFT})} \sim \frac{1}{(\frac{(p_j + p_k) \cdot q_k}{E_j + E_k})^2 - |q_k|^2}.
\]

The misalignment of poles arises from the different energy transfer prescriptions of the 4-momentum transfer in full theory and EFT. This appears to be a generic feature of the subtraction and no choice of 2-body gauge will cause the poles to align. The resulting PM subtraction contribution, after the cancellation of the spurious matter pole, will necessarily have "doubled" graviton poles with factors containing both prescriptions. In the PN expansion the energy transfer component are always subleading to the 3-momentum transfer, so the subtraction contribution will only have a graviton pole at \(\vec{q}_k = 0\) as expected.

For 3-bodies at 3PM and beyond (as well as the more familiar 2-bodies at 2PM), spurious branch cuts are present in both full theory and iteration amplitudes arising from loop integrals with internal matter propagators. The logic in this case is identical to the above: these must match between full theory and EFT iteration and so it must always be possible to define a subtraction contribution without spurious non-analyticities. It is left to future work to carry out such a calculation in detail.

### 2.6 \(N\)-body Fourier transform integrals

The scattering amplitude method for calculating the effective potential described in the previous sections produces a potential in momentum space. For real-time \(N\)-body simulations however we require a potential in position space, and therefore need to calculate a Fourier transform. For the case \(N = 2\) this step of the calculation is essentially trivial, in dimensional regularization

\[
\int \frac{d^d \vec{q}}{(2\pi)^d} \frac{e^{-i\vec{q} \cdot \vec{x}}}{|\vec{q}|^\alpha} = \frac{1}{2^\alpha \pi^{d/2}} \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \frac{1}{|\vec{x}|^{d-\alpha}}.
\]
For 2-body potentials calculated in COM frame (in isotropic gauge) this is all that is needed for the complete PM result. For 2-body potentials in a general frame (in generalized isotropic gauge) the complete, all-orders in momentum, Fourier transform of the 2-body 1PM potential requires the integral

\[
\int \frac{d^d\vec{q}}{(2\pi)^d} \frac{e^{-i\vec{q}\cdot\vec{x}_{12}}}{(\frac{(|\vec{p}_1+\vec{p}_2|\cdot\vec{x}_{12})}{E_1+E_2})^2} = \frac{1}{4\pi r_{12}} \frac{1}{1 - (\frac{r_{12}}{E_1+E_2})^2} \left[ 1 + \left( \frac{(\frac{r_{12}}{E_1+E_2})^2}{1 - (\frac{r_{12}}{E_1+E_2})^2} \right)^{1/2} \right].
\]

(2.27)

For \(N > 2\) the problem of Fourier transforming to position space is highly non-trivial and we are unable to evaluate the PM Fourier transform integrals in closed form. We will therefore proceed by expanding in the non-relativistic limit \(|\vec{p}| \ll m\) before integrating. For \(N = 3\) the type of scalar integrals we require take the form

\[
\int \frac{d^d\vec{q}_1}{(2\pi)^d} \int \frac{d^d\vec{q}_2}{(2\pi)^d} \int \frac{d^d\vec{q}_3}{(2\pi)^d} e^{-i(\vec{q}_1\cdot\vec{x}_1 + \vec{q}_2\cdot\vec{x}_2 + \vec{q}_3\cdot\vec{x}_3)} \left( \frac{2\pi)^d\delta^{(d)}(\vec{q}_1 + \vec{q}_2 + \vec{q}_3)}{\vec{q}_1^{\alpha_1} \vec{q}_2^{\alpha_2} \vec{q}_3^{\alpha_3}} \right).
\]

(2.28)

For graphs with V-type topology, figure 3, the integrand simplifies since for one of the propagators, \(\alpha_i = 0\). The integration over the corresponding \(\vec{q}_i\) is then trivial, and the remaining integrals can be calculated using (2.26). For example

\[
\int \frac{d^d\vec{q}_1}{(2\pi)^d} \int \frac{d^d\vec{q}_2}{(2\pi)^d} \int \frac{d^d\vec{q}_3}{(2\pi)^d} e^{-i(\vec{q}_1\cdot\vec{x}_1 + \vec{q}_2\cdot\vec{x}_2 + \vec{q}_3\cdot\vec{x}_3)} \left( \frac{2\pi)^d\delta^{(d)}(\vec{q}_1 + \vec{q}_2 + \vec{q}_3)}{\vec{q}_1^{\alpha_1} \vec{q}_2^{\alpha_2} \vec{q}_3^{\alpha_3}} \right)
\]

\[
= \frac{1}{2^{\alpha_1+\alpha_2+\alpha_3}} \frac{\Gamma \left( \frac{d-\alpha_1}{2} \right) \Gamma \left( \frac{d-\alpha_2}{2} \right)}{\Gamma \left( \frac{d}{2} \right) \Gamma \left( \frac{d}{2} \right)} \frac{1}{r_{13}^{d-\alpha_1} r_{23}^{d-\alpha_2}}.
\]

(2.29)

where \(r_{ij} = |\vec{x}_i - \vec{x}_j|\). For graphs with Y-type topology all \(\alpha_i \neq 0\) and we need to evaluate these integrals using a different method. The idea is to rewrite the 3-momentum conserving delta function by introducing an integral over an auxiliary spatial point

\[
(2\pi)^d\delta^{(d)}(\vec{q}_1 + \vec{q}_2 + \vec{q}_3) = \int d^d\vec{x}_0 \ e^{i(\vec{q}_1 + \vec{q}_2 + \vec{q}_3)\cdot\vec{x}_0}.
\]

(2.30)
The integral (2.28) then factors into a product for which each factor can be evaluated using (2.26)

\[
\int \frac{d^d \vec{x}_0}{(2\pi)^d} \int \frac{d^d \vec{q}_1}{(2\pi)^d} \int \frac{d^d \vec{q}_2}{(2\pi)^d} \int \frac{d^d \vec{q}_3}{(2\pi)^d} \frac{\epsilon^{i(\vec{q}_1 \cdot \vec{x}_{01} + \vec{q}_2 \cdot \vec{x}_{02} + \vec{q}_3 \cdot \vec{x}_{03})}}{|\vec{q}_1|^{\alpha_1} |\vec{q}_2|^{\alpha_2} |\vec{q}_3|^{\alpha_3}}
\]

\[= \int \frac{d^d \vec{x}_0}{(2\pi)^d} \left( \int \frac{d^d \vec{q}_1}{(2\pi)^d} \frac{\epsilon^{i\vec{q}_1 \cdot \vec{x}_{01}}}{|\vec{q}_1|^{\alpha_1}} \right) \left( \int \frac{d^d \vec{q}_2}{(2\pi)^d} \frac{\epsilon^{i\vec{q}_2 \cdot \vec{x}_{02}}}{|\vec{q}_2|^{\alpha_2}} \right) \left( \int \frac{d^d \vec{q}_3}{(2\pi)^d} \frac{\epsilon^{i\vec{q}_3 \cdot \vec{x}_{03}}}{|\vec{q}_3|^{\alpha_3}} \right) (2.31)
\]

\[= \frac{1}{2^{\alpha_1+\alpha_2+\alpha_3} \pi^{3d/2}} \frac{\Gamma \left( \frac{d-\alpha_1}{2} \right) \Gamma \left( \frac{d-\alpha_2}{2} \right) \Gamma \left( \frac{d-\alpha_3}{2} \right)}{\Gamma \left( \frac{\alpha_1}{2} \right) \Gamma \left( \frac{\alpha_2}{2} \right) \Gamma \left( \frac{\alpha_3}{2} \right)} I_3 \left[ \frac{d-\alpha_1}{2}, \frac{d-\alpha_2}{2}, \frac{d-\alpha_3}{2} \right],
\]

where

\[I_3^\alpha[a_1, a_2, a_3] = \int \frac{d^d \vec{x}_0}{x_{01}^{a_1} x_{02}^{a_2} x_{03}^{a_3}}. \tag{2.32}\]

The non-trivial part of this expression (2.32) is a Euclidean Feynman triangle integral with generically non-integer power propagators. A formal expression for these integrals was obtained using Mellin-Barnes methods, the general result is expressed in terms of the Appell hypergeometric series \( F_4 \) [144]. In the present context, this series expression has a region of convergence [145]

\[r_{12} + r_{13} < r_{23}. \tag{2.33}\]

Interestingly, this inequality is the complement of the region defined by the triangle inequality, and so the formal expression converges nowhere in the physical region. It should be possible to analytically continue the result to the physical region using the methods of [146]. While the general result is very complicated, the integral may simplify for the specific values of \( \alpha_i \) that appear in low-order PM calculations when evaluated in \( d = 3 - 2\epsilon \) up to \( O(\epsilon^0) \). This family of integrals is also well-known to arise in the calculation of CFT 3-point functions in momentum space [147, 148]. In this context they are sometimes referred to as \( \text{triple-K} \) integrals since they can be reduced to a single parameter integral over a product of three Bessel K functions.

At \( O(G^2) \) we need Fourier transforms (2.28) with \( \alpha_i \in 2\mathbb{Z}, \) and therefore the integral (2.32) with \( a_i \in \mathbb{Z} + \frac{1}{2} \). Inspired by [136], the authors of [139] conjectured that for \( a_i \in \mathbb{Z} + \frac{1}{2} \) and \( a_1 + a_2 + a_3 \leq \frac{3}{2} \)

\[I_3^{3-2\epsilon}[a_1, a_2, a_3] = \frac{A}{2\epsilon} + B + C \log(r_{12} + r_{13} + r_{23}) + O(\epsilon), \tag{2.34}\]

where \( A, B \) and \( C \) are homogeneous polynomials in \( r_{ij} \) of degree fixed by dimensional analysis. In [139] an approach to fixing the finitely many unknown coefficients in the Ansatz (2.34) using a Yangian differential equation was proposed. From the triple-K representation this sub-class of integrals can be evaluated directly [148, 150], the results agree with the Yangian approach.

\[\text{The unusual half-integer propagator powers are a consequence of the fact that } d = 3, \text{ is an odd integer. If the number of spatial dimensions was an even integer, then the corresponding Fourier transform integrals would have } \alpha_i \in \mathbb{Z}, \text{ that is they would be ordinary triangle Feynman integrals, see for example [149]. Amusingly, it appears the N-body problem in GR is simpler in odd spacetime dimensions!}\]
Here we provide an alternative approach, with a transparent physical interpretation. We expand the required integral in the hierarchical limit, meaning we calculate the integral (2.32) as a series expansion around the limit

\[ r_{12} \ll r_{13} \sim r_{23}, \quad (2.35) \]

using the method of regions [151]. First we shift \( \vec{x}_0 \) to be centered on body 1 so that the integral takes the form

\[
\int d^d \vec{x}_0 \frac{1}{x_0^{2a_1} |\vec{x}_0 + \vec{x}_{12}|^{2a_2} |\vec{x}_0 + \vec{x}_{13}|^{2a_3}}, \quad (2.36)
\]

We find there are two non-trivial regions that contribute

\[
\text{Region I} : \quad \vec{x}_0 \sim \vec{x}_{12}, \quad \text{Region II} : \quad \vec{x}_0 \sim \vec{x}_{13}. \quad (2.37)
\]

Physically, region I corresponds to the integration point being near the bodies 1 and 2 and region II near body 3. In the method of regions we are instructed to expand the integrand in the small parameter (2.35) in each region, integrate term-by-term and then sum together the result. In region I we make the expansion

\[
|\vec{x}_0 + \vec{x}_{13}|^{-2a_3} = \frac{1}{r_{13}^{2a_3}} \left( 1 + \frac{2\vec{x}_0 \cdot \vec{x}_{13}}{r_{13}^2} + \frac{|\vec{x}_0|^2}{r_{13}^2} \right)^{-a_3}
\]

\[= \frac{1}{r_{13}^{2a_3}} - \frac{2a_3(\vec{x}_0 \cdot \vec{x}_{13})}{r_{13}^{2(a_3+1)}} + \frac{2a_3(a_3+1)(\vec{x}_0 \cdot \vec{x}_{13})^2 - a_3r_{13}^2|\vec{x}_0|^2}{r_{13}^{2(a_3+2)}} + \ldots \quad (2.38)
\]

In region II we make the expansion

\[
|\vec{x}_0 + \vec{x}_{12}|^{-2a_2} = \frac{1}{|\vec{x}_0|^{2a_2}} \left( 1 + \frac{2\vec{x}_0 \cdot \vec{x}_{12}}{|\vec{x}_0|^2} + \frac{r_{12}^2}{|\vec{x}_0|^2} \right)^{-a_2}
\]

\[= \frac{1}{|\vec{x}_0|^{2a_2}} - \frac{2a_2(\vec{x}_0 \cdot \vec{x}_{12})}{|\vec{x}_0|^{2(a_2+1)}} + \frac{2a_2(a_2+1)(\vec{x}_0 \cdot \vec{x}_{12})^2 - a_2r_{12}^2|\vec{x}_0|^2}{|\vec{x}_0|^{2(a_2+2)}} + \ldots \quad (2.39)
\]

Each term in the integral can be evaluated using

\[
\int d^d x_0 \frac{1}{|x_0|^{2a_1} |\vec{x}_0 + \vec{x}|^{2a_2}} = \pi^{d/2} \frac{\Gamma \left( a_1 + a_2 - \frac{d}{2} \right) \Gamma \left( \frac{d}{2} - a_1 \right) \Gamma \left( \frac{d}{2} - a_2 \right)}{\Gamma(a_1) \Gamma(a_2) \Gamma(d - a_1 - a_2)} |\vec{x}|^{d-2a_1-2a_2}, \quad (2.40)
\]

after reducing tensor numerators using

\[
\vec{x}_0 \cdot \vec{x} = \frac{1}{2} \left( |\vec{x}_0 + \vec{x}|^2 - |\vec{x}_0|^2 - |\vec{x}|^2 \right). \quad (2.41)
\]

By matching terms against the series expansion of (2.34) we can fix all unknown coefficients and verify the validity of the ansatz. Unlike the series obtained using Mellin-Barnes methods [144], the hierarchical expansion converges in the physical region and has a straightforward physical interpretation. It may be especially useful at \( \mathcal{O}(G^2) \), where the necessary integrals (with \( a_1 \in \mathbb{Z} \) and \( a_2, a_3 \in \mathbb{Z} + \frac{1}{2} \)) are not known in closed form.
Figure 4. Feynman diagram contributing to 4-body scattering of massive scalars in general relativity at $O(G^3)$.

For higher-multiplicity tree diagrams, similar Fourier transform integrals are required. For $N \geq 4$ we encounter diagrams with internal graviton lines (graviton lines not connected to any matter lines). We introduce a corresponding internal momentum, which is integrated over, and a 3-momentum conserving delta function at each cubic graviton vertex. Using the identity (2.30) on each delta function we can reduce the $N$-body Fourier transform integral to an $N-2$ loop Feynman integral. For example, for the 4-body diagram shown in figure 4, at leading PN order, we require a 2-loop integral

$$
\left(\prod_{j=1}^{4} \int \frac{d^d q_j}{(2\pi)^d} e^{-i\vec{q}_j \cdot \vec{x}_j} \right) \frac{(2\pi)^d \delta^{(d)}(\vec{q}_1 + \vec{q}_2 + \vec{q}_3 + \vec{q}_4)}{[\vec{q}_1]^{\alpha_1} [\vec{q}_2]^{\alpha_2} [\vec{q}_1 + \vec{q}_2]^{\beta} [\vec{q}_3]^{\alpha_3} [\vec{q}_4]^{\alpha_4}}
$$

(2.42)

$$
= \left(\prod_{j=1}^{4} \int \frac{d^d q_j}{(2\pi)^d} e^{-i\vec{q}_j \cdot \vec{x}_j} \right) \frac{(2\pi)^d \delta^{(d)}(\vec{q}_1 + \vec{q}_2 - \vec{k}) (2\pi)^d \delta^{(d)}(\vec{q}_3 + \vec{q}_4 + \vec{k})}{[\vec{q}_1]^{\alpha_1} [\vec{q}_2]^{\alpha_2} [\vec{k}]^{\beta} [\vec{q}_3]^{\alpha_3} [\vec{q}_4]^{\alpha_4}}
$$

$$
= \Gamma \left(\frac{d-d}{2}\right) 2^{d/2} \Gamma \left(\frac{d+2}{2}\right) \left(\prod_{j=1}^{4} \frac{\Gamma \left(\frac{d-\alpha_j}{2}\right)}{\Gamma \left(\frac{d+\alpha_j}{2}\right)}\right)
$$

$$
\times \int \frac{d^d \vec{x}_0 d^d \vec{y}_0}{[\vec{x}_0 - \vec{x}_1]^{d-\alpha_1} [\vec{x}_0 - \vec{x}_2]^{d-\alpha_2} [\vec{x}_0 - \vec{y}_0]^{d-\beta} [\vec{y}_0 - \vec{x}_3]^{d-\alpha_3} [\vec{y}_0 - \vec{x}_4]^{d-\alpha_4}}.
$$

The case $\alpha_i = \beta = 2$ is needed for the 2PN 4-body potential [145], and has never been evaluated in closed form.

3 3-body dynamics in general relativity and Einstein-Maxwell

In this section we apply the general $N$-body framework described above to derive the 3-body potential $V_{ijk}^{(3)}$ in Einstein-Maxwell at $O(G^2)$ or 2PM. The full dynamics of 3-particles at 2PM also requires the 2PM contribution to the 2-body potentials $V_{ij}^{(2)}$, which we do not include here (see e.g. [44] for results albeit in COM frame). We present the results for Einstein-Maxwell (electrically charged scalars). The corresponding result for general relativity can be obtained by simply setting $Q_1 = Q_2 = Q_3 = 0$.

3.1 Full theory amplitudes

To calculate $V_{ijk}^{(3)}$ at 2PM, we need both the 3-to-3 scattering amplitude at 2PM and the 2-to-2 scattering amplitudes at 1PM. We consider scattering of non-spinning (scalar) particles
with masses $m_i$ and charges $Q_i$ for $i = 1, 2, 3$, and derive the necessary amplitudes using standard Feynman diagram methods. At higher-order (both in multiplicity $N$ and $G$), as in the 2-body case, we expect the construction of the relevant classical scattering amplitudes will benefit greatly from using modern methods including the double-copy [54–58], on-shell recursion, and generalized unitarity methods [59–63]. In this case the complexity is comparable and so we have presented the calculation using the more familiar Feynman diagrammatics; the necessary Feynman rules are given in appendix A.

The (relativistically normalized) 2-body amplitude, expanded to classical order is

$$M_{ij}^{(2)} = \frac{16\pi G}{q_i^2 q_j^2} \left[ m_i^2 m_j^2 - 2 (p_i \cdot p_j)^2 + Q_i Q_j (p_i \cdot p_j) \right] + \mathcal{O}(\frac{1}{q_i^2}) \, (3.1)$$

For the 3-body amplitude, we organize the result according to the topology of the Feynman graphs that appear in figure 3 and 5

$$M^{(3)} = \sum_{(i,j,k) \in S_3} \left[ M_{Y,ijk}^{(3)} + M_{V,ijk}^{(3)} + M_{U,ijk}^{(3)} \right], \quad (3.2)$$

where

\begin{align*}
M_{Y,ijk}^{(3)} &= -\frac{256\pi^2 G^2}{q_i^2 q_j^2 q_k^2} \times \left[ (q_i \cdot q_j) \left( 2m_k^2 (p_i \cdot p_j)^2 + 2m_i^2 (p_j \cdot p_k)^2 - 4 (p_i \cdot p_j) (p_i \cdot p_k) (p_j \cdot p_k) - m_i^2 m_j^2 m_k^2 \right) \\
&\quad + (p_k \cdot q_i) (p_k \cdot q_j) \left( m_i^2 m_j^2 - 2 (p_i \cdot p_j)^2 \right) - 2m_k^2 (p_i \cdot p_j) (p_i \cdot q_j) (p_j \cdot q_i) \\
&\quad + (p_j \cdot q_i) (p_k \cdot q_j) \left( 4 (p_i \cdot p_j) (p_i \cdot p_k) - 2m_i^2 (p_j \cdot p_k) \right) \\
&\quad + Q_i Q_j \left( \frac{1}{2} m_k^2 (p_i \cdot q_j) (p_j \cdot q_i) - (q_i \cdot q_j) \left( \frac{1}{2} m_k^2 (p_i \cdot p_j) - (p_i \cdot p_k) (p_j \cdot p_k) \right) \\
&\quad \quad - 2 (p_j \cdot p_k) (p_i \cdot q_j) (p_k \cdot q_i) + (p_i \cdot p_j) (p_k \cdot q_i) (p_k \cdot q_j) \right] + \mathcal{O}(\frac{1}{q_i^2}) \, (3.3)
\end{align*}

\begin{align*}
M_{V,ijk}^{(3)} &= \frac{64\pi^2 G^2}{q_i^2 q_j^2 q_k^2} \left[ 16 (p_i \cdot p_j) (p_i \cdot p_k) (p_j \cdot p_k) - 8m_i^2 (p_j \cdot p_k)^2 \\
&\quad + Q_i Q_j Q_k (p_i \cdot p_k) - 8Q_i Q_j (p_i \cdot p_k) (p_j \cdot p_k) \right] + \mathcal{O}(\frac{1}{q_i^2}) \, (3.4)
\end{align*}

\begin{align*}
M_{U,ijk}^{(3)} &= \frac{128\pi^2 G^2}{q_i^2 q_j^2 q_k^2} \times \left[ 4m_i^2 m_j^2 (p_i \cdot p_j) (p_i \cdot p_j - p_i \cdot q_i) + 4m_i^2 m_j^2 (p_j \cdot p_k) (2p_k \cdot q_i - p_j \cdot q_k + p_j \cdot p_k) \\
&\quad + 8 (p_i \cdot p_j) (p_i \cdot p_k) ((p_j \cdot p_k) (p_j \cdot q_i) + (p_i \cdot p_j) (-2p_k \cdot q_i + p_j \cdot q_k - p_j \cdot p_k)) \\
&\quad - 2m_i^2 m_j^2 m_k^2 \\
&\quad + Q_i Q_j \left[ 2 (p_j \cdot q_i) (p_j \cdot p_k)^2 + m_i^2 m_j^2 (p_j \cdot q_i - 2p_i \cdot p_j) \\
&\quad \quad + (p_i \cdot p_j) (p_j \cdot p_k) (2p_k \cdot q_i - p_j \cdot q_k + p_j \cdot p_k) \right] \\
&\quad \quad - (p_i \cdot p_j) (p_i \cdot p_k) (2p_k \cdot q_i - p_j \cdot q_k + p_j \cdot p_k) \right]
\end{align*}
Figure 5. Additional Feynman diagrams contributing to the 3-body scattering of charged massive scalars in Einstein-Maxwell at $\mathcal{O}(G^2)$. Single wavy lines correspond to photons, and double wavy lines to gravitons.

\begin{align}
+ Q_i Q_k \left[ m_i^2 m_j^2 (-2p_k \cdot q_i + p_j \cdot q_k - 2p_j \cdot p_k) + 2 (p_i \cdot p_j) ((p_i \cdot p_j) (2p_k \cdot q_i - p_j \cdot q_k + 2p_j \cdot p_k) - 2 (p_j \cdot p_k) (p_j \cdot q_i)) \right] \\
+ Q_i Q_j Q_k \left[ (p_j \cdot p_k) (p_j \cdot q_i) + (p_i \cdot p_j) (-2 (p_k \cdot q_i) + p_j \cdot q_k - 2 (p_j \cdot p_k)) \right] \\
+ \mathcal{O}(|\vec{q}|^{-3}).
\end{align}

(3.5)

3.2 Effective potential

Using the general matching framework described in section 2 we can now use the (classical) scattering amplitudes (3.1) and (3.2) to calculate the classical, conservative, momentum space potential for 3 charged bodies in Einstein-Maxwell at 2PM. The general form of the potential is

\[ V(\{\vec{p}, \vec{q}\}) = \sum_{(i,j,k) \in S_3} \left[ (2\pi)^6 \delta^{(3)}(\vec{q_i} + \vec{q_j}) \delta^{(3)}(\vec{q_k}) \times \frac{1}{2} \times V_{ij}^{(2)}(\{\vec{p}, \vec{q}\}) + (2\pi)^3 \delta^{(3)}(\vec{q_i} + \vec{q_j} + \vec{q_k}) \times V_{ijk}^{(3)}(\{\vec{p}, \vec{q}\}) \right]. \]

(3.6)

We begin with the intrinsic 2-body potential $V_{ij}^{(2)}$ at 1PM. As discussed in section 2.4, it is convenient to fix the ambiguity in the definition of this object by working in generalized isotropic gauge. Applying the general matching framework of section 2.3 with the amplitude (3.1) gives the 1PM potential in a general reference frame

\[ V_{ij}^{(2)}(\vec{p}_i, \vec{p}_j, \vec{q}_i, \vec{q}_j) = \frac{4\pi G}{E_i E_j} \left[ E_i E_j (\rho_{ij} - 1) (2E_i E_j (\rho_{ij} - 1) + Q_i Q_j - m_i^2 m_j^2) \right] \]

\[ \frac{E_i}{E_i + E_j} \left[ \frac{E_i \tau_{\mu} + E_j \tau_{\mu}}{E_i + E_j} \right)^2 - |\vec{q_i}|^2} \]

(3.7)
between the positive-energy part of the U-graph matter propagator and the corresponding
full theory U-graph, while the
matter
contribution corresponds to the negative-energy part of the matter propagator in
With this definition the expressions (3.10) are manifestly classical or

\[ V_{ij}^{(3)} = V_{Y,ijk}^{(3)} + V_{V,ijk}^{(3)} + V_{\text{anti-matter},ijk}^{(3)} + V_{\text{subtraction},ijk}^{(3)} \]  

(3.9)

The intrinsic 3-body potential consists of four pieces

\[ V_{Y,ijk}^{(3)} = \frac{1}{8E_iE_jE_k} \frac{256\pi^2G^2}{q_i^2q_j^2q_k^2} \]

\[ \times \left[ (q_i \cdot q_j) \left( 2m_k^2(p_i \cdot p_j)^2 + 2m_i^2(p_j \cdot p_k)^2 - 4(p_i \cdot p_j)(p_i \cdot p_k)(p_j \cdot p_k) - m_i^2m_j^2m_k^2 \right) 
+ (p_k \cdot q_i)(p_k \cdot q_j) \left( m_i^2m_j^2 - 2(p_i \cdot p_j)^2 \right) - 2m_k^2(p_i \cdot p_j)(p_i \cdot q_j)(p_j \cdot q_i) 
+ (p_j \cdot q_i)(p_k \cdot q_j) \left( 4(p_i \cdot p_j)(p_i \cdot p_k) - 2m_i^2(p_j \cdot p_k) \right) 
+ Q_iQ_j \left( \frac{1}{2}m_k^2(p_i \cdot q_j)(p_j \cdot q_i) - (q_i \cdot q_j) \left( \frac{1}{2}m_i^2(p_i \cdot p_j) - (p_i \cdot p_k)(p_j \cdot p_k) \right) 
- 2(p_j \cdot p_k)(p_i \cdot q_j)(p_k \cdot q_i) + (p_i \cdot p_j)(p_k \cdot q_i)(p_k \cdot q_j) \right) \right], \]

\[ V_{V,ijk}^{(3)} = \frac{1}{8E_iE_jE_k} \frac{64\pi^2G^2}{q_i^2q_j^2q_k^2} \left[ 8m_i^2(p_j \cdot p_k)^2 - 16(p_i \cdot p_j)(p_i \cdot p_k)(p_j \cdot p_k) 
- Q_iQ_jQ_k(p_i \cdot p_k) + 8Q_iQ_j(p_i \cdot p_k)(p_j \cdot p_k) \right]. \]  

(3.10)

The 4-vector momentum transfer in these expressions should be understood as a compact shorthand for

\[ q_i^\mu = \left( \frac{\vec{p}_i \cdot \vec{q}_i}{E_i}, \vec{q}_i \right). \]  

(3.11)

With this definition the expressions (3.10) are manifestly classical or \( \mathcal{O} (|q|^{-4}) \). The anti-
matter
contribution corresponds to the negative-energy part of the matter propagator in
the full theory U-graph, while the subtraction contributions correspond to the difference
between the positive-energy part of the U-graph matter propagator and the corresponding
iteration graph. These have a more complicated form

\[
V_{\text{anti-matter,ijk}}^{(3)} = -\frac{1}{2E_j} \frac{N^{(L)}_{ijk} N^{(R)}_{ijk}}{\hat{q}_{ijk} q_{ijk}}
\]

\[
V_{\text{subtraction,ijk}}^{(3)} = -\frac{N^{(L)}_{ijk} \delta N^{(R)}_{ijk} + N^{(R)}_{ijk} \delta N^{(L)}_{ijk}}{\hat{q}_{ijk} q_{ijk}} + \frac{\delta q_{ijk} N^{(L)}_{ijk} N^{(R)}_{ijk}}{\hat{q}_{ijk} q_{ijk} q_{ijk}} + \frac{\delta q_{ijk} N^{(R)}_{ijk} N^{(L)}_{ijk}}{\hat{q}_{ijk} q_{ijk} q_{ijk}}
\]

\[
+ \Delta E_{ijk} \delta q_{ijk} N^{(L)}_{ijk} N^{(R)}_{ijk}
\]

(3.12)

where these expressions depend on a set of universal or theory independent functions

\[
q_{ijk}^{(L)} = \frac{\tau_{ii}^2 - |\vec{q}_i|^2 + \tau_{ii}^3 - |\vec{q}_i|^2 \tau_{ii}}{E_i}
\]

\[
q_{ijk}^{(R)} = \frac{\tau_{kk}^2 - |\vec{q}_k|^2 + \tau_{kk}^3 - |\vec{q}_k|^2 \tau_{kk}}{E_k}
\]

\[
\hat{q}_{ijk}^{(L)} = \frac{(E_j \tau_{ji} + E_i \tau_{ii})^2}{(E_i + E_j)^2} - |\vec{q}_i|^2
\]

\[
\hat{q}_{ijk}^{(R)} = \frac{(E_j \tau_{jk} + E_k \tau_{kk})^2}{(E_j + E_k)^2} - |\vec{q}_k|^2
\]

\[
+ \frac{2(E_j \tau_{jk} + E_k \tau_{kk}) ((E_j + E_k) (\vec{q}_k \cdot \vec{q}_i) - (\tau_{ii} - \tau_{jk} + \tau_{kk}) (E_j \tau_{jk} + E_k \tau_{kk}))}{(E_j + E_k)^3}
\]

\[
\delta q_{ijk}^{(L)} = -\frac{E_j (E_j (\tau_{ji} + \tau_{ii}) + 2E_i \tau_{ii})}{(E_i + E_j)^2}
\]

\[
- E_j |\vec{q}_i|^2 (E_i + E_j) + \tau_{ii} \left( 2E_i E_j + 2E_i^2 + E_j^2 \right) + 2E_i \tau_{ii} (E_i + E_j) \tau_{ji} + E_i E_j \tau_{ji}^2
\]

\[
\delta q_{ijk}^{(R)} = \frac{E_j (E_j (\tau_{jk} + \tau_{kk}) + 2E_k \tau_{kk})}{(E_j + E_k)^2}
\]

\[
+ \frac{1}{2E_k (E_j + E_k)^3} \left[ E_j E_k \left(2(E_j + E_k) (\vec{q}_i \cdot \vec{q}_k) + \tau_{ji} (2\tau_{ii} (E_k - E_j) + (3E_j - E_k) \tau_{jk})\right)
\]

\[
- E_j |\vec{q}_k|^2 (E_j + E_k) + 2E_k^2 \tau_{kk} (2E_k \tau_{ii} + (3E_j - E_k) \tau_{jk})
\]

\[
+ \tau_{kk}^2 \left(3E_j^2 E_k + 4E_j E_k^2 + E_j^3 + 6E_k^2\right)\right]
\]

\[
\Delta E_{ijk} = \frac{1}{2}(-\tau_{ji} + \tau_{ii} + \tau_{jk} - \tau_{kk})
\]

\[
+ \frac{1}{4E_i E_j E_k} \left[ E_k \left(E_i \left(2 (\vec{q}_i \cdot \vec{q}_k) - 2\tau_{ii} \tau_{jk} + \tau_{ji}^2 + \tau_{jk}^2\right) - |\vec{q}_i|^2 (E_i + E_j) + E_j \tau_{ii}^2\right)
\]

\[
+ E_i |\vec{q}_k|^2 (E_j + E_k) - E_i E_j \tau_{kk}^2 - 2E_i E_k \tau_{kk} \tau_{jk}\right],
\]

(3.13)

as well the important non-universal or theory dependent functions

\[
N^{(L)}_{ijk} = \frac{4\pi G}{E_j E_j} \left[ m_i^2 m_j^2 - Q_i Q_j E_i E_j (\rho_{ij} - 1) - 2E_i^2 E_j^2 (\rho_{ij} - 1)^2\right]
\]

\[
+ \frac{2\pi G}{E_i^2 E_j^2} \left[ m_i^2 m_j^2 (E_j \tau_{ii} - E_i \tau_{ji}) + Q_i Q_j E_i E_j (E_i (\rho_{ij} - 1) \tau_{ji} + E_j (\tau_{ji} - \tau_{ii} \rho_{ij}))
\]

\[
- 2E_i^2 E_j^2 (\rho_{ij} - 1) (E_i (\tau_{ii} (\rho_{ij} - 1) - 2\tau_{ji}) - E_i (\rho_{ij} - 1) \tau_{ji})\right]
\]

\[\text{JHEP02}(2023)105\]
\[ N_{ijk}^{(R)} = \frac{4\pi G}{E_j E_k} \left[ m_j^2 m_k^2 - Q_j Q_k E_j E_k (\rho_{jk} - 1) - 2E_j^2 E_k^2 (\rho_{jk} - 1)^2 \right] \\
+ \frac{2\pi G}{E_j^2 E_k^2} \frac{1}{2} \left( \frac{m_j^2 m_k^2}{E_j E_k} \right) (\rho_{jk} - 1) \left( 2\tau_{ii} + \tau_{jk} - 2\tau_{kk} + E_j \tau_{kk} \right) \\
+ \frac{1}{2} Q_j Q_k E_j E_k (E_k (\rho_{jk} - 1) - 2\tau_{kk} (2\tau_{ii} + \tau_{jk} - 2\tau_{kk} + \tau_{jk}) \\
+ E_j (\tau_{jk} - \tau_{kk} \rho_{jk}) \\
+ E_j^2 E_k^2 (\rho_{jk} - 1) (E_k (2\tau_{ii} (\rho_{jk} + 1) - 4\tau_{ii} - \rho_{jk} \tau_{jk} + 2\tau_{kk} (\rho_{jk} - 1) + \tau_{jk}) \\
+ E_j (2\tau_{jk} - \tau_{kk} (\rho_{jk} + 1))) \right) \]

\[ \delta N_{ijk}^{(L)} = \frac{2\pi G}{E_j E_k} E_j [4E_i E_j (\rho_{ij} - 1) + Q_i Q_j] \]

\[ \delta N_{ijk}^{(R)} = -\frac{2\pi G}{E_j E_k} (E_j + 2E_k) [4E_j E_k (\rho_{ij} - 1) + Q_i Q_k] . \quad (3.14) \]

The notation used in these expressions is defined in (3.8). For the sake of notational brevity the expressions (3.12) are written in a form that is not manifestly classical, but does manifest the cancellation of the spurious matter singularity. As discussed in detail in appendix B, the symmetry properties of (3.13) and (3.14) ensure the cancellation of super-classical or \( \mathcal{O} (|q|^{-5}) \) terms after summing over \((i, j, k) \in S_3 \). Quantum or \( \mathcal{O} (|q|^{-3}) \) terms in (3.12) are likewise only included for brevity and should be discarded.

Note that, as discussed in section 2.4, we have fixed the gauge ambiguity in the 3-body potential (3.9) in a convenient but ultimately arbitrary way. This ambiguity does not affect physical, gauge invariant observables calculated from the potential. It is unknown to us if there is a unique “simplest” way to resolve this ambiguity for \( N > 2 \) body dynamics, analogous to generalized isotropic gauge for the 2-body potential.

In the expression (3.9) we have chosen to emphasize the explicit separation of matter and anti-matter contributions and the cancellation of super-classical contributions between diagrams. Alternatively we can derive compact forms of the potential; in general relativity \((Q_1 = Q_2 = Q_3 = 0)\) for example

\[ \dot{V}_{ij}^{(2)} = \frac{4\pi G m_i^2 m_j^2 (2\sigma_{ij} - 1)}{E_i E_j (\omega_{ij}^2 - |q_{ij}|^2)} , \quad (3.15) \]

\[ \dot{V}_{ijk}^{(3)} = \frac{32\pi G^2 m_i^2 m_j^2 m_k^2}{E_i E_j E_k q_{ij}^2 q_{jk}^2} \left[ \frac{q_i^2 (1 - 4\sigma_{ij} \sigma_{jk} \sigma_{ik})}{2} + \frac{(q_i \cdot q_j)^2 (2\sigma_{ij} - 1)}{m_i^2} - \frac{4(p_j \cdot q_i) (p_k \cdot q_i) \sigma_{ij} \sigma_{ik}}{m_j m_k} \right] \]

\[ - \dot{V}_{ij}^{(2)} \dot{V}_{kij}^{(2)} \left[ \frac{1}{2E_j} - \frac{2\omega_{kj}^2}{(\omega_{ij}^2 - |q_{ij}|^2) E_j} \right] + \frac{E_j q_k \cdot q_k (\omega_{ii} + \omega_{ij}) (\omega_{kk} + \omega_{kj})}{2q_i^2 q_j^2 E_i E_j E_k} \]

\[ + \frac{E_j}{E_{jk} (\omega_{kk}^2 - |q_k|^2)} \left( 2\omega_{ii} \omega_{kj} + q_i \cdot q_k \left( 1 - (\omega_{kk} + \omega_{kj})^2 \right) q_k^2 \right) \]

\[ + 16\pi G m_j m_k \dot{V}_{ij}^{(2)} \sigma_{jk} \left[ \frac{1}{E_j (\omega_{kj}^2 - |q_k|^2)} + \frac{q_k \cdot q_i}{E_k m_k} \left( \frac{\omega_{ii} + \omega_{ij}}{q_i^2 E_i} - \frac{\omega_{kk} + \omega_{kj}}{(\omega_{kj}^2 - |q_k|^2) E_{jk}} \right) \right] , \]
where
\[
\sigma_{ij} = \frac{p_i \cdot p_j}{m_i m_j}, \quad \omega_{ij} = \frac{(\vec{p}_i + \vec{p}_j) \cdot \vec{q}_i}{E_i + E_j}, \quad E_{ij} = E_i + E_j, \quad (3.16)
\]
and as before this should be understood as a function of 3-momenta with the 4-momentum transfer defined as in (3.11). The first line in (3.15) is proportional to the sum of full theory amplitudes \( \mathcal{M}_{Y,ijk}^{(3)} + \mathcal{M}_{V,ijk}^{(3)} \), while the remaining terms are the subtraction contributions. The result \( \hat{V}_{ijk}^{(3)} \) in (3.15) is pole-free, manifestly classical, and is physically equivalent to \( V_{ijk}^{(3)} \) in (3.9) for the case of general relativity. Similar results can be derived in Einstein-Maxwell.

Finally, the potential (3.9) is expressed in momentum space. To simulate the real-time dynamics of 3-bodies we need to first Fourier transform to position space. As discussed in section 2.6, the required 3-body PM integrals are unknown in closed form. Expanding order-by-order in small \( \vec{p} \), the integrals needed for (3.9) can be evaluated using either differential equations methods [139], or by the hierarchical expansion using the method of regions as described in section 2.6.

4 Comparison with known results

Given the complicated form of the explicit expression (3.9) for the 3-body potential, it is useful to verify the calculation against known results and alternative methods of calculation. In this section we describe three non-overlapping points of comparison: state-of-the-art post-Newtonian results in general relativity, probe (test mass) dynamics on multi-center black hole backgrounds and slowly moving extremal black hole dynamics via the moduli space approximation. In each case we find exact agreement to the calculated order.

4.1 Post-Newtonian results in general relativity

The momentum space PM expression (3.9) contains \( \mathcal{O}(G^2) \) 3-body interactions at all orders in velocity (momentum). To verify that our formula is correct (in some choice of coordinates) we compare it to state-of-the-art PN potentials, known up to \( \mathcal{O}(G^2 p^4) \) [139]. Since the potential is gauge dependent, we do not expect it to coincide exactly with expressions in the literature. To relate potentials in different gauges we can either construct an explicit change of coordinates [143], or use both potentials to calculate the same gauge invariant observable. In this section we will take the latter approach and calculate gauge invariant scattering amplitudes up to classical order.

The leading 1PN 3-body corrections were calculated long-ago by Einstein, Infeld and Hoffmann [15]. These consist of 3-body interactions at \( \mathcal{O}(G^2 p^4) \) and 2-body interactions at \( \mathcal{O}(G p^2) \). In momentum space the 1PN potential, in the gauge used in [152], is given explicitly by

\[
V_{1PN} = \sum_{(i,j,k) \in S_3} \delta^{(3)}(\vec{q}_i + \vec{q}_j) \delta^{(3)}(\vec{q}_k) \times \frac{4 \pi G m_i m_j}{|\vec{q}_i|^2} \left( 1 + \frac{2|\vec{p}_i|^2}{m_i^2} + \frac{\vec{p}_i \cdot \vec{p}_j}{m_i m_j} - \frac{(\vec{p}_i \cdot \vec{q}_i)(\vec{p}_j \cdot \vec{q}_j)}{m_i m_j |\vec{q}_i|^2} \right) \\
+ \delta^{(3)}(\vec{q}_i + \vec{q}_j + \vec{q}_k) \times \frac{16 \pi^2 G^2 m_i m_j m_k}{|\vec{q}_i|^2 |\vec{q}_j|^2 |\vec{q}_k|^2}. \quad (4.1)
\]
There is also a 2-body $O(G^2p^0)$ contribution at 1PN order, but since this contributes to the 3-body scattering amplitude beginning at $O(G^3)$ we can ignore it for this calculation. Clearly the 2-body part of this potential is not calculated in the generalized isotropic gauge defined in section 2.4. Since a general gauge transformation will mix the 2- and 3-body parts of the potential, we do not expect the 3-body parts of (4.1) to agree exactly with (3.9)

either. To show that these expressions are physically equivalent we use (4.1) together with the Born series (2.15) to calculate a classical scattering amplitude, and compare this with the expected full-theory result (3.9). Since our 2PM potential produces a scattering amplitude in agreement with (3.2) by construction, we conclude that the Einstein-Infeld-Hoffmann potential is related to (3.9) by a coordinate transformation.

The 2PN 3-body corrections were calculated in a series of papers [136, 137]. We will compare with the explicit form given in [139]; transforming the 3-body potential at Hoffmann potential is related to (3.9) by a coordinate transformation.

We will compare with the explicit form given in [139]; transforming the 3-body potential at $O(G^2p^2)$ and 2-body potential at $O(Gp^4)$ given in that paper to momentum space we find

\[
V_{2PN} = \sum_{(i,j,k) \in S_3} \left[ \frac{4\pi G m_i m_j}{|q_i|^2} \frac{4\pi G m_j m_k}{|q_j|^2} \frac{4\pi G m_i m_k}{|q_k|^2} \right] \left( \delta^{(3)}(\bar{q}_i + \bar{q}_j) \delta^{(3)}(\bar{q}_k) \times \frac{4\pi G m_i m_j}{|q_i|^2} \frac{4\pi G m_j m_k}{|q_j|^2} \frac{4\pi G m_i m_k}{|q_k|^2} \right)
\]

\[
\times \left( \frac{4\bar{p}_i \cdot \bar{p}_j}{m_i m_j} - \frac{8\bar{p}_i \cdot \bar{p}_k}{m_i m_k} + \frac{3|\bar{p}_i|^2}{2m_i^2} + \frac{9|\bar{p}_k|^2}{4m_k^2} \right) \left( \frac{4\bar{p}_i \cdot \bar{p}_j}{m_i m_j} - \frac{4|\bar{p}_i|^2}{m_i^2|q_i|^2} \right) \left( \frac{4\bar{p}_i \cdot \bar{p}_j}{m_i m_j} - \frac{3|\bar{p}_i|^2}{m_i m_k|q_i|^2} \right)
\]

\[
+ \delta^{(3)}(\bar{q}_i + \bar{q}_j + \bar{q}_k) \times \frac{32\pi^2 G^2 m_i m_j m_k}{|q_i|^2|q_j|^2|q_k|^2} \left( \frac{4\bar{p}_i \cdot \bar{p}_j}{m_i m_j} - \frac{8(\bar{p}_i \cdot \bar{q}_i)(\bar{p}_i \cdot \bar{q}_j)}{m_i^2} \right)
\]

\[
+ \frac{4(\bar{p}_i \cdot \bar{q}_i)(\bar{p}_i \cdot \bar{q}_j)}{m_i m_j} + \frac{5(\bar{p}_i \cdot \bar{q}_i)(\bar{p}_j \cdot \bar{q}_j)}{m_i m_j} \right).
\]

Again, we are also ignoring a 2-body $O(G^2p^2)$ contribution that does not mix with the 3-body interaction at this PN order. Calculating a gauge invariant scattering amplitude using this expression we find complete agreement with our 2PM expression (3.9).

---

This expression is produced after correcting a small typo in the published version of [139]. In the final line of equation (52) the numerator should read $9(n_{ij} \cdot v_{ij})^2 - 9v_{ij}^2 - 2(n_{ij} \cdot v_{jk})^2 + 2v_{k}^2$. We are very grateful to Florian Loebbert, Jan Plefka, Canxin Shi and Tianheng Wang for communicating with us and confirming this.
Finally, the current state-of-the-art is the explicit 2-body $O(Gp^6)$ and 3-body $O(G^2p^4)$ contributions to the 3PN potential calculated in [139]. The expression for the position space Lagrangian is quite lengthy (around 5000 terms), but with some assistance from a computer algebra system can be converted to a momentum space Hamiltonian and subsequently a scattering amplitude. Once again we find perfect agreement with our 2PM result (3.9).

4.2 Probe scattering on a Majumdar-Papapetrou background

A complementary perturbative approach to gravitational $N$-body dynamics is to consider an expansion in a small mass ratio, but keeping terms to all orders in $G$ and $v$.

Let’s begin by reviewing the setup in the 2-body case. We designate one of the bodies (the background) with mass $m_1$ and the other (the probe) with mass $m_2$, and $m_2/m_1 \ll 1$. To leading order in this small dimensionless parameter, and in the absence of spin and finite-size effects, the dynamics are captured by a probe particle of mass $m_2$ in geodesic motion on a background Schwarzschild metric of mass $m_1$. Explicitly, it is convenient to work with the isotropic gauge form of the metric, in $(+,-,-,-)$ signature, with a COM at $\vec{x}_1$

$$ds^2 = g_{tt}(\vec{x}) \, dt^2 + g_{rr}(\vec{x}) \, d\vec{x} \cdot d\vec{x},$$

where

$$g_{tt}(\vec{x}) = \frac{1 - \frac{Gm_1}{2|\vec{x} - \vec{x}_1|}}{1 + \frac{Gm_1}{2|\vec{x} - \vec{x}_1|}}, \quad g_{rr}(\vec{x}) = -1 - \frac{Gm_1}{2|\vec{x} - \vec{x}_1|}. \quad (4.4)$$

The motion of a point particle of mass $m_2$ on this background is given by extremizing the worldline action

$$S_{\text{probe}} = -m_2 \int ds_2,$$

where $s_2$ is the proper time measured by the probe. Fixing the affine parameter on the worldline to coincide with the time measured by an asymptotic observer, this defines a probe Hamiltonian

$$H_{\text{2-body probe}}^2 = \left[ \left(1 - \frac{Gm_1}{2r_{12}}\right) \left( m_2^2 + \frac{|\vec{p}_2|^2}{1 + \frac{Gm_1}{2r_{12}}} \right) \right]^{1/2}. \quad (4.6)$$

This Hamiltonian describes a relativistic particle of mass $m_2$ interacting with a fixed potential. Order-by-order in $G$ we calculate a classical scattering amplitude using the Born series (2.15), and this should agree with the two-body amplitude calculated using the full PM Hamiltonian, to leading order in the small mass ratio. Up to 2PM (1-loop) for 2-bodies, the probe limit completely fixes the Hamiltonian for arbitrary mass ratio [106].

For the 3-body case there are more possibilities. The simplest is again consider body 1 as a background, and treat bodies 2 and 3 as probes. This means we are considering a mass hierarchy $m_1 \gg m_2 \sim m_3$. The calculation is identical to the above and leads to a Hamiltonian

$$H_{\text{3-body probe}} = \left[ \left(1 - \frac{Gm_1}{2r_{12}}\right) \left( m_2^2 + \frac{|\vec{p}_2|^2}{1 + \frac{Gm_1}{2r_{12}}} \right) \right]^{1/2} + \left[ \left(1 - \frac{Gm_1}{2r_{13}}\right) \left( m_3^2 + \frac{|\vec{p}_3|^2}{1 + \frac{Gm_1}{2r_{13}}} \right) \right]^{1/2}. \quad (4.7)$$
While we can certainly use this to calculate a 3-body classical scattering amplitude and compare with the 2PM result \((3.9)\), such a comparison is not interesting. There are no terms that depend on the coordinates of all 3 bodies, i.e. there are no genuine 3-body interactions. This is not surprising, treating two of the bodies as probes means that they do not interact with each other. It would be very interesting to calculate a physical observable, e.g. a classical impulse, to next-to-leading order in this mass ratio using the self-force formalism \([14]\). We will not pursue this here.

For a non-trivial probe calculation we need to consider a different hierarchy of the form \(m_1 \sim m_2 \gg m_3\). This means we treat body 3 as a probe on a multi-center black hole background of bodies 1 and 2. Unfortunately, in pure GR such solutions are not known analytically.\(^5\) In Einstein-Maxwell theory however, such solutions are possible and are given by the well-known Majumdar-Papapetrou (MP) solution \([155, 156]\)

\[
ds^2 = f(\vec{x})^{-2}dt^2 - f(\vec{x})^2d\vec{x} \cdot d\vec{x}, \quad A = \left(1 - f(\vec{x})^{-1}\right)dt, \tag{4.8}\]

where

\[
f(\vec{x}) = 1 + \frac{Gm_1}{|\vec{x} - \vec{x}_1|} + \frac{Gm_2}{|\vec{x} - \vec{x}_2|}. \tag{4.9}\]

This solution describes two (or more generally \(N\)) extremal, \(Q_i = m_i\), black holes in exact static equilibrium at positions \(\vec{x}_1\) and \(\vec{x}_2\). We also allow the probe to have a charge \(Q_3\) which is assumed to be small compared to \(m_1\) and \(m_2\) in Planck units. This modifies the point particle effective action as

\[
S_{\text{charged probe}} = -m_3 \int ds_3 + Q_3 \int A. \tag{4.10}\]

The result is a Hamiltonian for a charged probe on an MP background

\[
H_{\text{MP}} = \left(1 + \frac{Gm_1}{r_{13}} + \frac{Gm_2}{r_{23}}\right)^{-2} \left[|\vec{p}_3|^2 + m_3^2 \left(1 + \frac{Gm_1}{r_{13}} + \frac{Gm_2}{r_{23}}\right)^2\right]^{1/2}
+ Q_3 \left[1 - \left(1 + \frac{Gm_1}{r_{13}} + \frac{Gm_2}{r_{23}}\right)^{-1}\right]. \tag{4.11}\]

Expanding to 2PM order and Fourier transforming to momentum space, we find the potential

\[
V_{\text{MP}} = \delta^{(3)}(\vec{q}_1 + \vec{q}_3) \delta^{(3)}(\vec{q}_2)
\left[\frac{4\pi Gm_1 (Q_3 E_3 - 2|\vec{p}_3|^2 - m_3^2)}{E_3 |\vec{q}_1|^2}\right]
+ \delta^{(3)}(\vec{q}_2 + \vec{q}_3) \delta^{(3)}(\vec{q}_1)
\left[\frac{4\pi Gm_2 (Q_3 E_3 - 2|\vec{p}_3|^2 - m_3^2)}{E_3 |\vec{q}_2|^2}\right]
+ \delta^{(3)}(\vec{q}_1 + \vec{q}_2 + \vec{q}_3)
\left[\frac{16\pi^2 G^2 m_1 m_2 (6E_3^4 - 2E_3^2 Q_3 - 3E_3^2 m_3^2 - m_3^4)}{E_3^3 |\vec{q}_1|^2 |\vec{q}_2|^2}\right]. \tag{4.12}\]

\(^5\)There do exist formal solutions of the vacuum Einstein equations describing 2 or more collinear Schwarzschild black holes in static equilibrium \([153]\). In these solutions the gravitational attraction is balanced by the (negative) tension of a string-like strut between the black holes at which the metric has a naked conical singularity \([154]\). Due to the pathological nature of this solution, it is not clear if a probe scattering on this background can be related to some limit of the general 3-body potential.
From this potential we calculate a scattering amplitude using the Born series (2.15). Beginning with the 2PM Hamiltonian (3.9), if we take the $Q_i \to m_i$ limit for $i = 1, 2$, expand to leading order in the hierarchy $m_1 \sim m_2 \gg m_3 \sim Q_3$, and then calculate a scattering amplitude, we find exact agreement with the probe calculation at $\mathcal{O}(G^2 p^\infty_0)$.

Interestingly, if we calculate the 2PM amplitude (3.2) for generic $m_i$ and $Q_i$, and then take the limits above we do not find agreement. We will discuss this non-commuting order-of-limits in the next subsection.

4.3 Extremal black hole scattering in the moduli space approximation

Long ago, Manton [157] proposed an interesting approach for describing the scattering of slowly-moving BPS solitons. Consider a field theory model with static $N$-soliton solutions $\phi(x^\mu, \{X\})$, where $\{X\}$ denotes a set of collective coordinates or moduli including the spatial positions of the solitons $\{\vec{x}_i\}$. Manton proposed that up to $\mathcal{O}(v^2)$, but non-perturbatively in all other parameters, the dynamics of the $N$-soliton system is captured by an effective $0 + 1$-dimensional non-linear sigma model on the moduli space with target space metric given by the natural moduli space metric calculated from the overlap of field theory zero modes [157]. This effective model can then be used to calculate physical observables for slowly-moving dynamical solitons, an approach sometimes called the moduli space approximation (MSA). In the original context the solitons considered were BPS magnetic monopole solutions of a Yang-Mills-Higgs model [158, 159]. By an indirect argument, Atiyah and Hitchin obtained the 2-monopole moduli space metric as an exact expression in $g_{YM}$ and used it to analyze monopole-monopole scattering at small-velocity but arbitrary impact parameter [160]. This approach has since been generalized to a variety of physical systems [161, 162], including, importantly for this paper, extremal black holes in an Einstein-Maxwell-Dilaton model [140, 163].

Following Manton’s proposal, Ferrell and Eardley were able to calculate the exact moduli space metric for $N$ extremal Reissner-Nordström black holes in Einstein-Maxwell theory by perturbing around the Majumdar-Papapetrou solution (4.8) [140]. The resulting effective action describes the interactions of $N$ extremal black holes at $\mathcal{O}(G^\infty v^2)$, and is given by

$$S_{\text{MSA}} = \int dt \left[ \sum_{i=1}^{N} \frac{1}{2} m_i |\vec{v}_i|^2 + \frac{3}{8\pi} \int d^3\vec{x}_0 \left( 1 + \sum_{i=1}^{N} \frac{G m_i}{x_{0i}} \right)^2 \sum_{j,k=1}^{N} \frac{G m_j m_k}{x_{0j} x_{0k}} \right. \right. \left. \left. \times \left[ \frac{1}{2} (\vec{x}_{0j} \cdot \vec{x}_{0k}) |\vec{v}_j - \vec{v}_k|^2 - (\vec{x}_{0j} \times \vec{x}_{0k}) \cdot (\vec{v}_j \times \vec{v}_k) \right] \right],$$

(4.13)

Given that this action is exact in $G$ it is surprising that it contains no terms at higher-order than $G^3$. This is a special feature of extremal black hole solutions that can be uplifted to intersecting BPS brane solutions of 11d supergravity [164]. Generic extremal black hole solutions of the Einstein-Maxwell-Dilaton model have an MSA effective action with terms at all orders in $G$ [163].
where \( x_{0i} = |\vec{x}_0 - \vec{x}_i| \). From the effective action we calculate the momentum space potential up to \( \mathcal{O} (G^2 p^2) \),

\[
V_{\text{MSA}} = \sum_{(i,j,k) \in S_3} \left[ \delta^{(3)} (\vec{q}_i + \vec{q}_j) \delta^{(3)} (\vec{q}_k) \left[ \frac{3\pi G m_i m_j}{|\vec{q}_i|^2} \left( \frac{\vec{p}_i}{m_i} - \frac{\vec{p}_j}{m_j} \right)^2 \right] \right] + \delta^{(3)} (\vec{q}_i + \vec{q}_j + \vec{q}_k) \left[ \frac{72\pi^2 G^2 m_i m_j m_k}{|\vec{q}_i|^2 |\vec{q}_j|^2 |\vec{q}_k|^2} \left( \frac{m_k}{m_i m_j} \left( \vec{p}_i \cdot \vec{p}_j \right) - 2 \left( \frac{\vec{p}_i}{m_i} \cdot \frac{\vec{p}_k}{m_k} \right) + \frac{|\vec{p}_k|^2}{m_k} \right) + \frac{48\pi^2 G^2 m_i m_j m_k}{|\vec{q}_i|^2 |\vec{q}_j|^2 |\vec{q}_k|^2} \left( \frac{\vec{p}_i \cdot \vec{q}_j}{m_i m_j} - \frac{\vec{p}_j \cdot \vec{q}_i}{m_i m_j} \right) + \frac{|\vec{p}_j|^2 (\vec{q}_i - \vec{q}_k)}{m_j^2} - \frac{(\vec{p}_j \cdot \vec{q}_k)(\vec{q}_i - \vec{q}_k)}{m_j m_k} \right] \right] + \mathcal{O} (G^3 p^2).
\]

Using this potential we calculate a classical scattering amplitude for three extremal black holes up to \( \mathcal{O} (G^2 p^2) \). We find that this agrees exactly with the scattering amplitude calculated using the 2PM Einstein-Maxwell Hamiltonian (3.9) truncated to this order, in the extremal limit \( Q_i = m_i \). Since (4.13) contains interactions to all orders in \( G \), we can use it to calculate amplitudes and compare with \( N \)-body Hamiltonians obtained from scattering amplitudes at all loop orders. This may be a valuable cross-check for calculations at \( \mathcal{O} (G^3) \) and beyond.

As mentioned in section 4.2 the procedures of taking the large mass limit and calculating an amplitude from a Hamiltonian do not commute. To compare directly with the probe Hamiltonian (4.11) we need to take the large mass limit of the 2PM Hamiltonian first, and then calculate a scattering amplitude treating only the probe as a dynamical degree of freedom. It is possible however to match the calculation taken in the opposite order, where the amplitude is calculated from the general 2PM Hamiltonian (3.9) first, and then the large mass limit is taken. In this case we need to treat the background black holes as dynamical but slowly moving, with interactions given by the MSA potential (4.13). The correct effective Hamiltonian is found to be a hybrid of the probe potential (4.11) and the MSA potential (4.14), which at 2PM and in momentum space is given by

\[
H_{\text{hybrid}} = H_{\text{probe}} + \frac{|\vec{p}_1|^2}{2m_1} + \frac{|\vec{p}_2|^2}{2m_2} + \delta^{(3)} (\vec{q}_1 + \vec{q}_2) \delta^{(3)} (\vec{q}_3) \left[ -\frac{6\pi G m_1 m_2}{|\vec{q}_1|^2} \left( \frac{\vec{p}_1}{m_1} - \frac{\vec{p}_2}{m_2} \right)^2 \right].
\]

The mis-match between the order-of-limits arises because of the iteration graphs shown in figure 1. The matter propagator, at classical order, contains an \( \mathcal{O} \left( |\vec{p}_{1,2}|^{-2} \right) \) kinematic singularity that cancels against the \( \mathcal{O} \left( |\vec{p}_{1,2}|^2 \right) \) MSA two-body interaction (4.15), giving a non-zero contribution in the \( |\vec{p}_{1,2}| \to 0 \) limit. It would be interesting to understand if a similar order-of-limits problem exists in physical observables such as the linear impulse on the probe.
5 Discussion

The main results of this paper are the development of a general approach to calculating classical $N$-body potentials using scattering amplitudes, and the explicit calculation of the intrinsic 3-body potential (3.9) in Einstein-Maxwell at 2PM in momentum space. We have presented a robust set of checks against known results in various limiting cases and find complete agreement. We believe the amplitudes based methods developed in this paper will be a powerful tool for understanding general relativistic $N$-body dynamics. There are many important future directions that should be pursued to further develop this program.

The most obvious direction is to use these methods to calculate conservative potentials at higher powers in $G$. At the next order beyond the calculation presented in this paper, $\mathcal{O}(G^3)$ or 3PM, there are contributions to the general $N$-body potential from 2-body scattering at 2-loops, 3-body scattering at 1-loop and 4-body scattering at tree-level. The 3-body pieces will involve a more complicated iteration subtraction calculation involving EFT 1-loop diagrams with triangle topology as well as non-1PI diagrams. It will be important for such future calculations to develop a systematic approach to cancelling non-physical non-analytic features order-by-order in PM perturbation theory. For instance, in the two-body case reorganizing the structure of iteration terms in the EFT leads to the so-called amplitude-action relation, from which the scattering angle is directly obtained without computing a potential [83]. The generalization of this to $N$-body scattering is an interesting prospect, which we leave to future work. The 4-body pieces at this order are tree-level and should be calculable using a generalization of the procedure described in this paper. In the PN expansion, the static $\mathcal{O}(G^3 \nu^0)$ 3-body potential is known [137] as well as the 4-body potential in an unevaluated form [145].

Possibly the most severe bottleneck for obtaining analytic position space results for $N$-body potentials (in the PN expansion) is the evaluation of the class of Euclidean Feynman integrals described in section 2.6. Without analytic expressions, numerical simulations of real-time $N$-body dynamics may require computationally expensive numerical Fourier transform integrals at each time-step (though this is almost certainly still less computationally expensive than a full numerical relativity simulation [3–5]). For 3-bodies at 3PM, obtaining analytic results would require at least the calculation of (2.32) for the case $a_1 \in \mathbb{Z}$ and $a_2, a_3 \in \mathbb{Z} + \frac{1}{2}$. We are optimistic that with powerful modern Feynman integration technology, especially methods based on differential equations [68, 70, 71, 165, 166], it will be possible to push the state-of-the art for analytic $N$-body potentials. There may be additional advantages or simplifications to such calculations by working directly in coordinate space as described in [167]. Alternatively, in section 2.6 we have argued that it should be possible to make progress by specializing to the hierarchical limit of 3-body dynamics ($r_{12} \ll r_{13} \sim r_{23}$) and demonstrated that the method of regions is a powerful tool in this context for evaluating the integrals (2.32) order-by-order in this hierarchy.

The position space potential is most relevant for the dynamics of bound systems for which the PN expansion is appropriate. For unbound systems (scattering dynamics) the

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7Preliminary investigations for explicit values of the variables $r_{12}$, $r_{13}$ and $r_{23}$ suggest the space of functions in this case involves elliptic functions. We are grateful to Michael Ruf for discussions on this point.
bodies involved may generally have relativistic momenta, and so the PM (velocity re-
summed) expansion is appropriate. In this context the calculation of an analytic position
space potential may be very difficult; even at 2PM the “Y-graph” Fourier transform in-
tegrals are not obviously related to Feynman integrals. Fortunately the construction of a
potential in this context may be unnecessary. By applying the formalism of KMOC [45]
to $N$-body scattering amplitudes it should be possible to calculate gauge invariant phys-
ical conservative observables such as linear and angular impulses, as well as dissipative
observables such as gravitational waveforms, directly from scattering amplitudes. Not only
would this strategy by-pass the difficulty of solving equations of motion for a PM Hamil-
tonian, it also avoids the gauge ambiguities described in section 2.4 by working exclusively
with physical quantities. It is tantalizing to imagine that these unbound $N$-body observ-
ables may be related to bound observables by analytic continuation along the lines of the
boundary-to-bound dictionary introduced in [168], if perhaps only in certain special sym-
metrical configurations. To extract physical observables for bound 3-body systems, such
as a waveform, it may be beneficial to first restrict to the hierarchical limit and match to
a non-relativistic 2-body effective theory as described in [169]. In this context the general
3-body PM Hamiltonian (3.9) may be useful for constructing such an effective theory in
the intermediate case of a bound binary system perturbed by a distant unbound body.

Finally it would be interesting to apply the methods described in this paper to the cal-
culation of spin-dependent $N$-body potentials following the recent successful use of ampli-
tudes based methods for the spinning 2-body problem [86, 170]. To our knowledge, even the
leading-order in PN spin-dependent (intrinsic) 3-body potential has never been calculated.
We leave this and the other interesting open problems described above to future work.

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A Feynman rules for Einstein-Maxwell

At $O(G^2)$ the necessary 3-body scattering amplitudes can be calculated using Feynman
diagrams. For reference the explicit Feynman rules we need for Einstein-Maxwell amplitude
$iM_{MN}$ are given below in Feynman gauge for the photon (single wavy line) and de Donder
gauge for the graviton (double wavy line). Where appropriate, the arrow on the dashed
scalar line indicates both the flow of electric charge and the direction of momentum flow.
Note that the electric charge $Q_i$ is given in Planck units, where $Q_i = m_i$ corresponds to
extremality.
\[ \frac{i}{p^2 - m^2} \]

\[ \mu, \nu \rightarrow \eta_{\mu\nu} \]

\[ \frac{-i\eta_{\mu\nu}}{p^2} \]

\[ \mu, \nu \rightarrow \rho, \sigma \]

\[ \frac{i}{2} \left[ \eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - \eta_{\mu\nu} \eta_{\rho\sigma} \right] \]

\[ \mu \rightarrow \rho \]

\[ -i\sqrt{4\pi G Q} (p_1 + p_2)^\mu \]

\[ \nu \rightarrow \rho \]

\[ 8i\pi G Q^2 \eta_{\mu\nu} \]

\[ \mu, \nu \rightarrow \rho, \sigma \]

\[ i\sqrt{8\pi G} \left[ \eta^{\mu\nu} \left( p_1 \cdot p_2 - m_i^2 \right) - p_1^\mu p_2^\nu - p_1^\nu p_2^\mu \right] \]

\[ \mu, \nu \rightarrow \rho, \sigma \]

\[ 8i\pi G \left[ \left( p_1 \cdot p_2 - m_i^2 \right) \left( \eta^{\mu\nu} \eta^{\sigma\rho} - \eta^{\mu\sigma} \eta^{\nu\rho} - \eta^{\mu\rho} \eta^{\nu\sigma} \right) \right. \]

\[ - \eta^{\mu\nu} (p_1^\rho p_2^\sigma + p_1^\sigma p_2^\rho) + \eta^{\mu\rho} (p_1^\nu p_2^\sigma + p_1^\sigma p_2^\nu) + \eta^{\nu\sigma} (p_1^\rho p_2^\mu + p_1^\mu p_2^\rho) \]

\[ + \eta^{\nu\rho} (p_1^\sigma p_2^\mu + p_1^\mu p_2^\sigma) - \eta^{\rho\sigma} (p_1^\nu p_2^\mu + p_1^\mu p_2^\nu) \]

\[ \left. + \eta^{\rho\nu} (p_1^\sigma p_2^\mu + p_1^\mu p_2^\sigma) - \eta^{\sigma\nu} (p_1^\rho p_2^\mu + p_1^\mu p_2^\rho) \right] \]

\[ \mu, \nu \rightarrow \rho \]

\[ -4i\sqrt{2\pi G Q} \left[ \eta^{\mu\nu} (p_1 + p_2)^\rho - \eta^{\mu\rho} (p_1 + p_2)^\nu - \eta^{\nu\rho} (p_1 + p_2)^\mu \right] \]
\[
\begin{align*}
\mu_1, \nu_1 & = \frac{i}{8\pi G} \left[ p_1^{\mu_1} p_2^{\nu_1} \eta^{\rho \sigma} - p_1^{\rho} p_2^{\sigma} \eta^{\mu \nu} + p_1^{\rho} p_2^{\sigma} \eta^{\mu \nu} - p_1^{\mu_1} p_2^{\nu_1} \eta^{\rho \sigma} \right] \\
\mu_2, \nu_2 & = \frac{i}{8\pi G} \left[ p_1^{\mu_2} p_2^{\nu_2} \eta^{\rho \sigma} - p_1^{\rho} p_2^{\sigma} \eta^{\mu \nu} + p_1^{\rho} p_2^{\sigma} \eta^{\mu \nu} - p_1^{\mu_2} p_2^{\nu_2} \eta^{\rho \sigma} \right] \\
\mu_3, \nu_3 & = \frac{i}{8\pi G} \left[ p_1^{\mu_3} p_2^{\nu_3} \eta^{\rho \sigma} - p_1^{\rho} p_2^{\sigma} \eta^{\mu \nu} + p_1^{\rho} p_2^{\sigma} \eta^{\mu \nu} - p_1^{\mu_3} p_2^{\nu_3} \eta^{\rho \sigma} \right] \\
\end{align*}
\]

The cubic graviton vertex function can be compactly expressed by introducing a set of auxiliary 4-vectors \( z_i^{\mu} \)

\[
V_{hhh}^{(\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3)} (p_1, p_2, p_3) = \frac{1}{24} \frac{\partial^2}{\partial z_1^{\mu_1} \partial z_1^{\nu_1}} \frac{\partial^2}{\partial z_2^{\mu_2} \partial z_2^{\nu_2}} \frac{\partial^2}{\partial z_3^{\mu_3} \partial z_3^{\nu_3}} V_{hhh} (p_1, p_2, p_3, z_1, z_2, z_3),
\]

where

\[
V_{hhh} (p_1, p_2, p_3, z_1, z_2, z_3)
= \sum_{(i,j,k) \in S_3} \left[ 2 \left( p_i \cdot z_k \right) \left( p_j \cdot z_k \right) \left( z_i \cdot z_j \right)^2 - 4 \left( z_i \cdot z_k \right) \left( z_j \cdot z_k \right) \left( p_i \cdot z_j \right) \left( p_k \cdot z_i \right) \right. \\
\left. + \frac{1}{6} \left( p_i \cdot p_j + p_i \cdot p_k + p_j \cdot p_k \right) \left( z_i^2 \left( z_j^2 z_k^2 - 6 (z_j \cdot z_k)^2 \right) + 8 (z_i \cdot z_j) (z_i \cdot z_k) (z_j \cdot z_k) \right) \right].
\]

As in the rest of the paper this notation denotes the sum over the 3! distinct permutations of \( \{1, 2, 3\} \). Note that since this is an off-shell vertex function the vectors \( z_i^{\mu} \) are completely general and do not satisfy any additional identities.

### B Calculation of 3-body subtraction contribution

In this appendix we provide further details of the calculation of the subtraction contribution (3.12) to the 3-body potential in Einstein-Maxwell.

For the calculation of the subtraction contribution (2.23), it is a little simpler (but completely optional) to keep the \( O \left( |q|^{-3} \right) \) and higher terms until the end of the calculation. For 2-body dynamics at \( O \left( G \right) \) the quantum potential in generalized isotropic gauge is

\[
V_{ij}^{(2)} (\vec{p}_i, \vec{p}_j, \vec{q}_i, \vec{q}_j) = \frac{1}{4 \sqrt{E_i (\vec{p}_i) E_j (\vec{p}_j)}} \frac{\partial}{\partial q^i} \frac{\partial}{\partial q^j} \frac{\partial}{\partial q^i} \frac{\partial}{\partial q^j} \left[ \frac{\hat{N}_{ij} (\vec{p}_i, \vec{p}_j, \vec{q}_i)}{(E_i + E_j)^2 - |\vec{q}_i|^2} \right],
\]

where

\[
\hat{N}_{ij} (\vec{p}_i, \vec{p}_j, \vec{q}_i) = 4 \pi G \left[ - \frac{4 E_i E_j (\rho_{ij} - 1) \left( (E_i + E_j)^2 |\vec{q}_i|^2 - (E_i \tau_{ij} + E_j \tau_{ji})^2 + Q_i Q_j (E_i + E_j)^2 \right)}{(E_i + E_j)^2} \\
+ Q_i Q_j \left( \frac{(E_i \tau_{ij} + E_j \tau_{ji})^2}{(E_i + E_j)^2} - |\vec{q}_i|^2 \right) - 8 E_i E_j (\rho_{ij} - 1)^2 + 4 m_i^2 m_j^2 \right].
\]
For the calculation of the U-graph contributions we define the following off-shell numerator function

\[ N_{ij}(p_i, p_j, q_i) = 4\pi G \left[ -8(p_i \cdot p_j)^2 + 4m_i^2 \left( -p_j \cdot q_i - p_j + 2m_j^2 \right) + 4 \left( m_j^2 - p_j^2 \right) (p_i \cdot q_i) + 4p_i^2 \left( p_j \cdot q_i + p_j^2 - m_j^2 \right) + Q_iQ_j \left( -2(p_j \cdot q_i) + 2(p_i \cdot q_i) - q_i^2 \right) + 4(p_i \cdot p_j) \left( 2(p_j \cdot q_i) - 2(p_i \cdot q_i) + q_i^2 + Q_iQ_j \right) \right], \] (B.3)

this is simply the numerator of the off-shell 2-body Feynman diagram that gives the quantum potential (B.1). In terms of these functions, the subtraction contribution (2.23) is calculated by continuing

\[ V_{\text{subtraction,}ijk}^{(3)} = \frac{1}{8\sqrt{E_i(p_i) E_j(p_j) E_k(p_k) E_i(p_i - q_i) E_j(p_j - q_j) E_k(p_k - q_k)}} \times \frac{1}{2E_j(p_j + q_i)} \times \frac{1}{E_i(p_i) + E_j(p_j) - E_i(p_i - q_i) - E_j(p_j - q_j)} \times \left[ \frac{N_{ij}(p_i, p_j, q_i) N_{kj}(p_k, p_j + q_i, q_k)}{(E_i(p_i) - E_i(p_i - q_i))^2 - |q_i|^2} \right] \frac{1}{(E_k(p_k) - E_k(p_k - q_k))^2 - |q_k|^2} \]

\[ - \frac{\tilde{N}_{ij}(p_i, p_j, q_i) \tilde{N}_{kj}(p_k, p_j + q_i, q_k)}{\left( \frac{(p_i + q_i) \cdot q_j}{E_i(p_i) + E_j(p_j)} \right)^2 - |q_i|^2} \frac{1}{\left( \frac{(p_j + q_j) \cdot q_k}{E_i(p_i) + E_j(p_j)} \right)^2 - |q_k|^2} \], \quad (B.4)

away from the constraint surface of energy conservation. Pictorially, this is the contribution from difference of full-theory and EFT graphs of the form depicted in figure 2 with \( p_L = p_i + p_j, p'_L = p_i - q_i \) and \( p_R = p_k \). As discussed in section 2.5 the challenge involved is to manipulate this expression into a form in which the spurious matter pole cancels and that the super-classical \( O(|q|^{-5}) \) part of this expression cancel in the complete subtraction potential (after summing over \( (i, j, k) \in S_3 \)). To engineer such a continuation we will fix the 3-body energy conservation constraint (2.8) differently for different parts of the expression. We will use the notation \( |\tilde{q}_i, \tilde{p}_j, \tilde{q}_k| \) to denote the procedure of first eliminating \( \tilde{q}_j \) using 3-momentum conservation, followed by eliminating \( \tilde{p}_j \cdot \tilde{q}_k \) by solving the energy conservation constraint (2.8). We make the following definitions

\[ N^{(L)}_{ijk} = \frac{N_{ij}(p_i, p_j, q_i)}{4E_i(p_i) E_j(p_j) E_i(p_i - q_i) E_j(p_j - q_j)} \bigg|_{\tilde{q}_j, \tilde{p}_j, \tilde{q}_k} \]

\[ \tilde{N}^{(L)}_{ijk} = \frac{\tilde{N}_{ij}(p_i, p_j, q_i)}{4E_i(p_i) E_j(p_j) E_i(p_i - \tilde{q}_i) E_j(\tilde{p}_j - q_j)} \bigg|_{\tilde{q}_j, \tilde{p}_j, \tilde{q}_k} \]

\[ q^{(L)}_{ijj} = \left[ (E_i(p_i) - E_i(p_i - q_i))^2 - |q_i|^2 \right]_{\tilde{q}_j, \tilde{p}_j, \tilde{q}_k} \]

\[ q^{(L)}_{ijk} = \left[ \left( \frac{(p_i + \tilde{q}_i) \cdot \tilde{q}_j}{E_i(p_i) + E_j(p_j)} \right)^2 - |q_i|^2 \right]_{\tilde{q}_j, \tilde{p}_j, \tilde{q}_k} \]
With these definitions, (B.4) is equivalent to

\[ \Delta E^{(L)}_{ijk} = \left| E_i(p_i) + E_j(p_j) - E_i(p_i - \hat{q}_i) - E_j(p_j + \hat{q}_i) \right|_{\hat{q}_i, \hat{p}_j, \hat{q}_k} \]

\[ \mathcal{N}^{(R)}_{ijk} = \frac{N_{kij}(p_k, p_j + q_i, q_k)}{4\sqrt{E_k(p_k) E_k(p_k - \hat{q}_k) E_j(p_j + \hat{q}_i) E_j(p_j - \hat{q}_j)}} \]

\[ \hat{N}^{(R)}_{ijk} = \frac{\hat{N}_{kij}(\hat{p}_k, \hat{p}_j + \hat{q}_i, \hat{q}_k)}{4\sqrt{E_k(\hat{p}_k) E_k(\hat{p}_k - \hat{q}_k) E_j(\hat{p}_j + \hat{q}_i) E_j(\hat{p}_j - \hat{q}_j)}} \]

\[ q^{(R)}_{ijk} = \left( E_k(p_k) - E_k(p_k - \hat{q}_k) \right)^2 - |\hat{q}_k|^2 \]

\[ \hat{q}^{(R)}_{ijk} = \left( \left( (\hat{p}_j + \hat{q}_i) + \hat{p}_k \right) \hat{q}_k \right)^2 - |\hat{q}_k|^2 \]

\[ \Delta E^{(R)}_{ijk} = \left| E_i(p_i) + E_j(p_j) - E_i(p_i - \hat{q}_i) - E_j(p_j + \hat{q}_i) \right|_{\hat{q}_i, \hat{p}_j, \hat{q}_k}. \] (B.5)

As discussed in section 2.5, consistency requires the hatted and un-hatted quantities differ by an expression proportional to the matter pole. This inspires us to define

\[ \hat{N}^{(L)}_{ijk} = \mathcal{N}^{(L)}_{ijk} + \Delta E^{(L)}_{ijk} \delta \mathcal{N}^{(L)}_{ijk} \]

\[ \hat{N}^{(R)}_{ijk} = \mathcal{N}^{(R)}_{ijk} + \Delta E^{(R)}_{ijk} \delta \mathcal{N}^{(R)}_{ijk} \]

\[ \hat{q}^{(L)}_{ijk} = q^{(L)}_{ijk} + \Delta E^{(L)}_{ijk} \delta q^{(L)}_{ijk} \]

\[ \hat{q}^{(R)}_{ijk} = q^{(R)}_{ijk} + \Delta E^{(R)}_{ijk} \delta q^{(R)}_{ijk}. \] (B.7)

We also note the following

\[ \Delta E^{(L)}_{ijk} \equiv \Delta E^{(R)}_{ijk} \equiv \frac{1}{2} \left[ \Delta E^{(L)}_{ijk} + \Delta E^{(R)}_{ijk} \right]. \] (B.8)

With some elementary algebra we derive the expression

\[ V^{(3)}_{\text{subtraction},ijk} = -\frac{\delta \mathcal{N}^{(L)}_{ijk} \delta \mathcal{N}^{(R)}_{ijk} \Delta E_{ijk}}{q^{(L)}_{ijk} q^{(R)}_{ijk}} - \frac{\mathcal{N}^{(L)}_{ijk} \delta \mathcal{N}^{(R)}_{ijk} + \mathcal{N}^{(R)}_{ijk} \delta \mathcal{N}^{(L)}_{ijk}}{q^{(L)}_{ijk} q^{(R)}_{ijk} \hat{q}^{(L)}_{ijk} \hat{q}^{(R)}_{ijk}} + \frac{\delta q^{(L)}_{ijk} \mathcal{N}^{(L)}_{ijk} \mathcal{N}^{(R)}_{ijk}}{q^{(L)}_{ijk} q^{(R)}_{ijk} \hat{q}^{(L)}_{ijk} \hat{q}^{(R)}_{ijk}} \]

\[ + \frac{\delta q^{(R)}_{ijk} \mathcal{N}^{(R)}_{ijk} \mathcal{N}^{(L)}_{ijk}}{q^{(L)}_{ijk} q^{(R)}_{ijk} \hat{q}^{(L)}_{ijk} \hat{q}^{(R)}_{ijk}} + \frac{\Delta E_{ijk} \delta q^{(L)}_{ijk} \delta q^{(R)}_{ijk} \mathcal{N}^{(L)}_{ijk} \mathcal{N}^{(R)}_{ijk}}{q^{(L)}_{ijk} q^{(R)}_{ijk} \hat{q}^{(L)}_{ijk} \hat{q}^{(R)}_{ijk}}, \] (B.9)

for which the matter pole has cancelled, at the price of introducing some (harmless) doubled graviton poles. Explicit expressions for all of these functions for Einstein-Maxwell theory, expanded in \( q \) to the order relevant for the calculation of the classical potential, are given in (3.13) and (3.14).
What remains is to demonstrate the cancellation of the super-classical contribution. To see this we examine the leading $q$-scaling of each individual object. In general we expect (and confirm from the explicit expressions (3.13) and (3.14))

$$N^{(L/R)} \sim q^0, \quad \delta N^{(L/R)} \sim q^0, \quad q^{(L/R)} \sim q^1, \quad \delta q^{(L/R)} \sim q^1, \quad \Delta E \sim q^1.$$ \hfill (B.10)

By inspection, the first term in (B.9) is purely quantum and so can be dropped giving (3.12), the final three terms contain super-classical contributions. These cancel when we sum this expression with the crossed $(i \leftrightarrow k)$ graph, by virtue of the following symmetry properties

$$N^{(L/R)}_{ijk} = N^{(R/L)}_{kji} + O(q^1)$$
$$q^{(L/R)}_{ijk} = q^{(R/L)}_{kji} + O(q^2)$$
$$\delta q^{(L/R)}_{ijk} = -\delta q^{(R/L)}_{kji} + O(q^2)$$
$$\Delta E_{ijk} = -\Delta E_{kji} + O(q^2).$$ \hfill (B.11)

From this we can see that the final three terms above are odd under $i \leftrightarrow k$ at leading order in $q$, and so the super-classical contributions cancel.

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