Electroweak Precision Observables within a Fourth Generation Model with General Flavour Structure

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Abstract

We calculate the contributions to electroweak precision observables (EWPOs) due to a fourth generation of fermions with the most general (quark-) flavour structure (but assuming Dirac neutrinos and a trivial flavour structure in the lepton sector). We discuss the size of non-oblique contributions arising from Z–quark–anti-quark vertex corrections and the dependence of the EWPOs on all CKM mixing angles involving the fourth generation. We find that the electroweak precision observables are equally sensitive to all three fourth-generation mixing angles and that the corresponding constraints on these angles are competitive with those obtained from flavour physics. For non-trivial $4 \times 4$ flavour structures, the non-oblique contributions lead to relative corrections at the permille level and may well have a noticeable effect in a global fit.

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1 Introduction

With the advent of LHC data the first direct tests for many models of new physics are within reach. Among the conceptionally simplest extensions of the Standard Model (SM3) are those which only add a minimal set of fermions to the SM particle content. This class encompasses both the additional vector-like quarks \([75x718]\) and the fourth-generation scenario (SM4).

The SM4 was fairly popular in the 1980s until electroweak precision observables seemed to rule it out. In the last years models with an additional fourth generation experienced a renaissance as new analyses, e.g. \([75x208]\), somewhat relaxed the electroweak tensions. This realisation also prompted numerous studies of the non-trivial flavour structure of the SM4 \([75x179]\), as well as searches for specific signatures in new physics observables \([75x179]\). Recently, some effort has been directed towards providing an actual fit of the parameters of the model — or, to be precise, of one particular variant which restricts itself to an (almost) decoupled fourth Dirac-like neutrino.\(^1\) This scenario requires merely seven additional parameters and fitting them simultaneously does not seem unrealistic. One of the first attempts in this direction primarily used the electroweak precision observables and restricted itself to only one CKM parameter \([75x114]\); still non-trivial correlations were found, for example, between the Higgs mass and the new mixing angle. More recent studies seek to contain \([75x42]\) or even determine \([75x42]\) the full \(4 \times 4\) CKM matrix. In this case the main challenge is the fact that, if one allows for a generic CKM structure, the flavour and electroweak sector are intertwined and have to be treated simultaneously.

Usually the effects of new physics in the electroweak sector are parametrised by the oblique electroweak parameters \(S, T\) and \(U\), as introduced by Peskin and Takeuchi \([75x38]\). These allow for fairly simple and straightforward estimates of new physics contribution to electroweak observables. However, the validity of this parametrisation relies on certain assumptions about the new physics model, which are, in principle, no longer satisfied in an SM4 with the most general flavour structure.

In this letter we discuss the contributions to electroweak precision observables (EWPOs) due to a fourth generation with general \(4 \times 4\) flavour mixing. For the sake of simplicity we assume Dirac neutrinos and a trivial flavour structure in the lepton sector. Our calculation includes non-oblique contributions, i.e. those which are not captured by the \(S, T\) and \(U\) parameters, and can easily be combined with existing calculations of higher-order QCD and QED corrections within the SM3. We discuss the importance of the non-oblique contributions and the impact of flavour mixing between the fourth and the first three generation in several SM4 scenarios.

In section 2 we briefly review the oblique parameters and their range of applicability. In section 3 we introduce our notations for the SM4 parameters and explain our method

\(^1\)See e.g. \([75x106]\) for a discussion of fourth generation Majorana neutrinos.
for calculating the corrections to the EWPOs. In section 4 we describe our treatment of the Fermi constant $G_F$, which is an observable and not a parameter in our analysis. Our numerical results are presented in section ???. We find that the EWPOs are equally sensitive to all three fourth-generation mixing angles and that the corresponding constraints on these angles are competitive with those obtained from flavour physics. For non-trivial $4 \times 4$ flavour structures, the non-oblique contributions lead to relative corrections of up to one permille for the hadronic $Z$ width $\Gamma_{\text{had}}$ and of several permille for the hadronic $Z \rightarrow bb$ branching ratio $R_b$. A simultaneous fit of the SM4 masses, couplings and CKM matrix should therefore take into account all six SM4 CKM mixing angles and the non-oblique corrections to the EWPOs. We conclude in section ???.

## 2 Oblique Corrections and Electroweak Observables

The constraints imposed on new physics by EWPOs measured at LEP have already been discussed extensively in the literature. In 1992 Peskin and Takeuchi presented a model-independent way of parametrising the new physics contributions to the $Z$ pole observables [?]. Their analysis was based on three assumptions:

1. The electroweak gauge group of the new-physics model is $SU(2)_L \times U(1)_Y$.
2. The new-physics couplings to light fermions (i.e. all SM3 fermions except the top-quark) are negligible.
3. The scale of new physics is much larger than the electroweak scale.

The first assumption forbids the existence of additional gauge bosons coupling directly to leptons. The second assumption guarantees that there are no additional vertex or box-diagrams contributing to the Drell-Yan process. Thus, the only way the new physics contribute to the $Z$ pole observables is through the renormalisation of weak gauge boson wave functions, the electric charge or the Weinberg angle. The third assumption is needed to justify a step in the discussion in [?], where the gauge boson self-energies are expanded to first order around $q^2 = 0$ ($q$ being the momentum flowing through the self-energy graphs). In practice, it is usually sufficient to require that new particles coupling directly to weak gauge bosons are heavier than the $Z$ boson.

In SM extensions that satisfy the criteria above, the new physics contributions to the $Z$ pole observables can be expressed in terms of the oblique electroweak parameters $S$, $T$ and $U$ which were defined in [? ??] and represent different linear combinations of gauge boson self-energies and their derivatives. On the experimental side, the values of $S$, $T$ and $U$ can then be determined from data by performing a global fit of $S$, $T$, $U$ and the SM3 parameters to the $Z$ pole and possibly other low-energy observables. (See [? ??] for a recent analysis of this type.) On the theoretical side one can test to what
extent a given model of new physics agrees with low-energy observables by computing $S$, $T$ and $U$ in this model and comparing the results with the best-fit values.

This method of testing an SM extension against constraints from low-energy experiments is very convenient since it only requires the computation of three quantities. It has been applied to a number of models including the SM4 [? ]. One should, however, keep in mind that the validity of this method depends on the validity of the assumptions listed above. In the SM4 the second assumption is no longer valid if the fourth generation quarks are allowed to mix with the quarks of the first three generations. Hence, the validity of the “oblique method” must be checked explicitly if one attempts to constrain the new mixing angles of the SM4 CKM matrix.

### 3 The $Zq\bar{q}$ Vertex in the SM4

The properties of the $Z$ boson and its couplings to fermions have been measured at LEP 1 with a very high accuracy. Table 1 shows the experimental values and accuracies for a selection of $Z$-pole observables as well as their theoretical predictions within the SM3. The observables are: the partial width for $Z \to$ hadrons ($\Gamma_{\text{had}}$), the hadronic branching fraction for $Z \to b\bar{b}$ ($R_b$), the forward-backward asymmetry for $Z \to b\bar{b}$ ($A_{FB}^b$) and the mass of the $W$ ($M_W$). In the $Z$-pole approximation, the forward-backward asymmetry can be written as $\frac{1}{4}A_eA_b$, where the quantities $A_e$ and $A_b$ only depend on the $Ze^+e^-$ and $Zb\bar{b}$ couplings, respectively. The relative precision of $\Gamma_{\text{had}}$ is approximately 0.1% and $R_b$ is known to an accuracy of 0.3%. The measured value of $A_{FB}^b$ deviates from its SM3 prediction by more than two standard deviations. The discrepancy originates mainly from the factor $A_e$. Oblique corrections due to a fourth generation of fermions affect all $Z$-pole observables, but only observables related to the $Z$–quark–anti-quark vertex are subject to non-oblique corrections; of the observables from table 1, only $\Gamma_{\text{had}}$, $R_b$ and $A_b$ receive non-oblique contributions. Our discussion will therefore mainly focus
Within the SM3 the couplings of $Z$ bosons to quarks have been studied in great detail. Electroweak and QCD corrections to the gauge boson self-energies and the $Z$–quark–anti-quark vertex have been calculated at two-loop order\footnote{The branching fraction $R_c$ and asymmetry factor $A_c$ for the charm quark also receive non-oblique corrections, but these observables are less constraining due to their larger experimental error.} and the results have been implemented in public codes such as TOPAZO\footnote{\cite{TOPAZO}} or ZFITTER\footnote{\cite{ZFITTER}}. Radiative corrections to the partial widths are of the order of 0.1\% (QED) and 4\% (QCD). To match the experimental accuracy of the $Z$-pole observables they must therefore be included in theoretical calculations. In this section we explain how predictions for the $Z$ pole observables within the SM4 can be calculated at the required level of accuracy without the need to re-visit the SM3 calculations.

Before we begin, let us briefly explain our notations for the SM3 and SM4 parameters. For the SM3 CKM matrix we use the standard parametrisation. In this parametrisation the independent parameters are the three mixing angles $\theta_{12}$, $\theta_{13}$ and $\theta_{23}$ and one complex phase $\delta_{13}$. The explicit form of the SM3 CKM matrix in terms of the phase and mixing angles is given in appendix ??.

In the SM4 the CKM matrix is a unitary $4 \times 4$ matrix. After absorbing unphysical complex phases into the definitions of the quark fields, its parametrisation requires only three additional mixing angles $\theta_{14}$, $\theta_{24}$ and $\theta_{34}$ and two additional complex phases $\delta_{14}$ and $\delta_{24}$. The explicit form of the SM4 CKM matrix is also given in appendix ??.

To distinguish the phase $\delta_{13}$ and the mixing angles $\theta_{12}$, $\theta_{13}$ and $\theta_{23}$ of the SM4 CKM matrix from their SM3 counterparts we will use superscripts ‘SM 4’ and ‘SM3’, respectively. The same applies to other parameters like $m_H$ or $M_W$, which exist in both models. We will also use the shorthands $s_{ij}$ and $c_{ij}$ for the sines and cosines of the mixing angles $\theta_{ij}$. Finally, we denote the lepton, neutrino, up and down-type quark of the fourth generation as $\ell_4$, $\nu_4$, $t'$ and $b'$, respectively. Their masses $m_{\ell_4}$, $m_{\nu_4}$, $m_{t'}$ and $m_{b'}$ are independent parameters of the SM4.

Let us now proceed with the discussion of higher order corrections to the $Zq\bar{q}$ vertex. In the limit of vanishing external quark masses $m_q$, the on-shell $Zq\bar{q}$ vertex function only contains two Lorentz structures:

$$\Gamma_{\mu}^q = ie\gamma_{\mu}[F_V^q - F_A^q\gamma_5] .$$

Here and in the following, $q = u, d, s, c, b$ denotes the quark flavour. The form factors $F_V^q$ and $F_A^q$ depend on the quark flavour, the external masses and the parameters of the model under consideration (SM3 or SM4). Following the discussion in \cite{???}, we express
QCD and QED radiative corrections to $F_q^V$ and $F_q^A$ in terms of radiator functions $\mathcal{R}_q^V$ and $\mathcal{R}_q^A$ and write

$$F_q^V = g_q^V\mathcal{R}_q^V , \quad F_q^A = g_q^A\mathcal{R}_q^A . \quad (2)$$

In doing this, we neglect the non-factorisable contributions $\ldots$, whose effect is below the permille level. The effective couplings $g_q^V$ and $g_q^A$ now only contain infrared finite contributions. At leading order $\mathcal{R}_q^V = \mathcal{R}_q^A = 1$ and $g_q^V$ and $g_q^A$ are the tree-level vector and axial couplings of the $Z$ boson.

In this paper we are interested in the difference between predictions for $Z$ pole observables within the SM3 and SM4. For this purpose we denote, for any quantity $X$, the new physics correction by

$$\delta X = X^{SM4} - X^{SM3} , \quad (3)$$

where the superscripts ‘SM4’ and ‘SM3’ indicate that $X$ is evaluated with a given set of SM4 or SM3 parameters, respectively. In principle, the two sets of parameters can be completely unrelated. It is, however, extremely convenient to use the same values of $M_Z$, $M_W$, $m_t$, $\alpha$ and $\alpha_s$ in both sets. In this case, $\delta \mathcal{R}_q^V = \delta \mathcal{R}_q^A = 0$ and the new physics corrections to any $Z$ pole observable can be obtained by only computing the infrared finite quantities $\delta g_q^V$ and $\delta g_q^A$. The form factors $F_q^{SM4}$ and $F_q^{SM4}$ (and thus for the $Z$-pole observables within the SM4) may then be calculated by scaling the corresponding SM3 form factors with the ratios $g_q^{SM4}/g_q^{SM3}$ and $g_q^{SM4}/g_q^{SM3}$, respectively. This way, factorisable QCD and QED corrections are included in $F_q^{SM4}$ and $F_q^{SM4}$ if they were included in the SM3 ‘reference values’ $F_q^{SM3}$ and $F_q^{SM3}$. As we will see below, the ratios $\delta g_q^V/g_q^{(0)}$ and $\delta g_q^A/g_q^{(0)}$ (with $g_q^{(0)}$ and $g_q^{(0)}$ being the tree-level couplings) are typically below 1%. Thus, the approximation

$$F_q^{SM4} \approx F_q^{SM3} \left(1 + \frac{\delta g_q^V}{g_q^{(0)}}\right) , \quad F_q^{SM4} \approx F_q^{SM3} \left(1 + \frac{\delta g_q^A}{g_q^{(0)}}\right) \quad (4)$$

(with $g_q^{(0)}$ and $g_q^{(0)}$ being the tree-level couplings) is generally valid with a relative precision of the order of $10^{-4}$.

The difference between $\mathcal{R}_q^V$ and $\mathcal{R}_q^A$ is of the order of a few percent $\ldots$. Thus, to estimate the size of the new physics contributions to the EWPOs we use the approximation

$$\mathcal{R}_q^V \approx \mathcal{R}_q^A \equiv \mathcal{R}^q \quad (5)$$

and obtain

$$\delta \Gamma_{\mu} = ie\mathcal{R}^q\gamma_{\mu}[\delta g_q^V - \delta g_q^A\gamma_5] . \quad (6)$$

The hadronic $Z$ partial widths and asymmetries are then given by

$$\Gamma(Z \to q\bar{q}) = \alpha M_Z\mathcal{R}^q[(g_q^V)^2 + (g_q^A)^2] , \quad A_q = \frac{g_q^V g_q^A}{(g_q^V)^2 + (g_q^A)^2} \quad (7)$$

3These are independent SM3 input parameters in the on-shell renormalisation scheme $\ldots$, which is the scheme we used in our calculations.
The new physics corrections to these quantities are readily obtained by expanding the effective couplings to first order in $\delta g_V^q$ and $\delta g_A^q$:

$$\frac{\delta \Gamma(Z \to q\bar{q})}{\Gamma^{SM3}(Z \to q\bar{q})} = 2 \frac{g_V^{q(SM3)} \text{Re} \delta g_V^q + g_A^{q(SM3)} \text{Re} \delta g_A^q}{(g_V^{q(SM3)})^2 + (g_A^{q(SM3)})^2} \approx 2 \frac{g_V^{q(0)} \text{Re} \delta g_V^q + g_A^{q(0)} \text{Re} \delta g_A^q}{(g_V^{q(0)})^2 + (g_A^{q(0)})^2}, \quad (8a)$$

$$\frac{\delta \mathcal{A}_q}{\mathcal{A}_q^{SM3}} = \frac{\text{Re} \delta g_V^q}{g_V^{q(SM3)}} + \frac{\text{Re} \delta g_A^q}{g_A^{q(SM3)}} - 2 \frac{g_V^{q(SM3)} \text{Re} \delta g_V^q + g_A^{q(SM3)} \text{Re} \delta g_A^q}{(g_V^{q(SM3)})^2 + (g_A^{q(SM3)})^2} \approx \frac{\text{Re} \delta g_V^q}{g_V^{q(0)}} + \frac{\text{Re} \delta g_A^q}{g_A^{q(0)}} - 2 \frac{g_V^{q(0)} \text{Re} \delta g_V^q + g_A^{q(0)} \text{Re} \delta g_A^q}{(g_V^{q(0)})^2 + (g_A^{q(0)})^2} \quad (8b)$$

Note that, as a result of approximating $\mathcal{R}_V^q \approx \mathcal{R}_A^q$, the radiator functions cancel in the ratios above.

If mixing between the fourth generation quarks and the quarks of the first three generations is neglected, the new physics corrections can be expressed in terms of the oblique electroweak parameters $S$, $T$ and $U$ [? ? ]. In this case, the relations between $\delta g_V^q$, $\delta g_A^q$ and $S$, $T$ and $U$ are

$$\delta g_V^q = \frac{\alpha}{16 c_W s_W} \left[ 2I_3^q S - 4[(c_W^2 - s_W^2)I_3^q + 2s_W^2 Q^q] T - \left( \frac{c_W^2 - s_W^2}{s_W^2} I_3^q + 2Q^q \right) U \right], \quad (9a)$$

$$\delta g_A^q = \frac{\alpha}{16 c_W s_W} \left[ 2S - \frac{c_W^2 - s_W^2}{s_W^2} (4s_W^2 T + U) \right] I_3^q, \quad (9b)$$

where $Q^q$ and $I_3^q$ are the electric charge and weak isospin of the quark $q$ and $s_W$ and $c_W$ are the sine and cosine of the Weinberg angle, defined by $s_W^2 = 1 - M_W^2 / M_Z^2$.

If the fourth generation quarks are allowed to mix with the quarks of the first three generations one also needs to compute the vertex diagrams contributing to $\delta g_V^q$ and $\delta g_A^q$. We used the FeynArts/FormCalc package [? ? ] to compute $\delta g_V^q$ and $\delta g_A^q$ to one-loop order. The renormalisation of the $Zq\bar{q}$ vertex was done in the on-shell scheme [? ]. At the one-loop level only diagrams involving $W$ bosons, charged Goldstone bosons or Higgs bosons contribute to $\delta g_V^q$ and $\delta g_A^q$, as long as $\alpha$, $\alpha_s$, $M_Z$, $M_W$ and $m_t$ are chosen to be the same in the SM3 and SM4. The SM3 parameters and corresponding values for $\Gamma(Z \to q\bar{q})$ and $\mathcal{A}_q$ were taken from [? ]. Specifically, we use

$$1/\alpha(m_Z) = 128.892 \quad , \quad \alpha_s(m_Z) = 0.1185 \quad , \quad M_Z = 91.1875 \text{ GeV} \quad ,$$

$$M_W = 80.384 \text{ GeV} \quad , \quad m_t = 173.2 \text{ GeV} \quad , \quad m_{H^{SM3}} = 90 \text{ GeV} \quad ,$$

$$\Gamma^{SM3}_{\text{had}} = 1.7418 \text{ GeV} \quad , \quad R_b^{SM3} = 0.21578 \quad , \quad A_b^{SM3} = 0.9348 \quad , \quad (10)$$
where
\[
\Gamma_{\text{had}} = \sum_{q=u,d,s,c,b} \Gamma(Z \to q\bar{q}) , \quad R_q = \frac{\Gamma(Z \to q\bar{q})}{\Gamma_{\text{had}}} .
\] (11)

The phase and mixing angles of the SM3 CKM matrix were also taken from [?]:
\[
\theta_{12}^{\text{SM3}} = 0.2273 , \quad \theta_{13}^{\text{SM3}} = 0.003466 , \quad \theta_{23}^{\text{SM3}} = 0.04103 , \quad \delta_{13}^{\text{SM3}} = 1.2020 .
\] (12)

Note that the numerical values for \( \Gamma_{\text{SM3}}^{\text{had}} \) and \( R_{b}^{\text{SM3}} \) are for a fixed “reference” Higgs mass \( m_{H}^{\text{SM3}} = 90 \text{ GeV} \). In the SM4 the Higgs mass is treated as a free parameter.

4 A Note on \( G_F \)

As mentioned above, we use in this work the on-shell renormalisation scheme for the computation of new physics corrections. In this scheme, the quantities \( \alpha(M_Z) \), \( M_Z \) and \( M_W \) are independent parameters. This parametrisation is very convenient for the computation of higher order corrections, but it has its disadvantages if one wants to compare it with experimental data. The Fermi constant \( G_F \), which is determined from the muon lifetime, is a non-trivial function of \( \alpha(M_Z) \), \( M_Z \), \( M_W \) and the other model parameters. Since \( G_F \) is measured very accurately (namely, to a relative precision of \( 10^{-5} \)) it constrains the model to a non-trivial hyper-surface in its parameter space. In other words, one parameter of the model is fixed by the requirement that \( G_F \) assumes its measured value. Typically, one adjusts the value of \( M_W \) to obtain the correct value of \( G_F \).

The relation between \( G_F \) and \( M_W \) is conventionally written as [? ]
\[
G_F = \frac{\pi\alpha}{\sqrt{2}s_W^2 M_W^2 (1-\Delta r)} ,
\] (13)
where \( \Delta r \) encodes higher order corrections and is, in general, a function of all other parameters. New physics, like the existence of a fourth generation of fermions, changes the function \( \Delta r \). Denoting, as before, the new physics correction to \( \Delta r \) as \( \delta \Delta r \) and writing the solutions of (13) in the SM3 and SM4 as \( M_W^{\text{SM3}} \) and \( M_W^{\text{SM4}} \equiv M_W^{\text{SM3}} + \delta M_W \), respectively, we find
\[
\frac{\delta M_W}{M_W^{\text{SM3}}} = \frac{s_W^2}{2(c_W^2 - s_W^2)} \delta \Delta r .
\] (14)

However, since the parameters (10) already satisfy the \( G_F \) constraint we have \( M_W^{\text{SM3}} = M_W \) with \( M_W \) from (10). If the SM4 is to agree with the measured value of the \( W \) mass, the ratio \( \delta M_W / M_W^{\text{SM3}} \) cannot be much larger than one permille. Hence, the shift in \( M_W \) is unimportant for the purpose of computing \( \delta \Delta r \) and the new physics corrections (8) of the hadronic \( Z \) partial widths and asymmetries. We can therefore safely use the value from (10) in these calculations.