The quest for solvable multistate Landau-Zener models

Nikolai A Sinitsyn\textsuperscript{1} and Vladimir Y Chernyak\textsuperscript{2}

\textsuperscript{1} Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545, United States of America
\textsuperscript{2} Department of Chemistry and Department of Mathematics, Wayne State University, 5101 Cass Ave, Detroit, MI 48202, United States of America

E-mail: nsinitsyn@lanl.gov

Received 6 January 2017, revised 15 March 2017
Accepted for publication 21 March 2017
Published 24 May 2017

Abstract
Recently, integrability conditions (ICs) in multistate Landau-Zener (MLZ) theory were proposed [1]. They describe common properties of all known solved systems with linearly time-dependent Hamiltonians. Here we show that ICs enable efficient computer assisted search for new solvable MLZ models that span complexity range from several interacting states to mesoscopic systems with many-body dynamics and combinatorially large phase space. This diversity suggests that nontrivial solvable MLZ models are numerous. In addition, we refine the formulation of ICs and extend the class of solvable systems to models with points of multiple diabatic level crossing.

Keywords: Landau-Zener, quantum integrability, matrix product ansatz, nonadiabatic transitions

(Some figures may appear in colour only in the online journal)

1. Introduction

The future of quantum technology will depend crucially on our ability to manipulate complex quantum systems by time-dependent fields. Achieving sufficient control of driven quantum matter is thus an important challenge, including to mathematical physics. For evolution of more than two interacting states, one of the difficulties here is the scarcity of solvable models that are described by the nonstationary Schrödinger equation with practically interesting explicitly time-dependent Hamiltonians. Most of known solvable driven models are either trivially reducible to independent harmonic oscillators and two-state systems [2–6] or correspond to almost impossible for implementation driving protocols such as multistate shortcuts to adiabaticity [7]. This lack of nontrivial useful results strongly restricts our understanding of
Considerable progress on resolving this problem has been achieved within the MLZ theory, whose goal is to find the scattering $N \times N$ matrix $\hat{S}$ that describes evolution with a linearly time-dependent Hamiltonian \[ H(t) = \hat{A} + \hat{B}t. \] Here, $\Psi$ is the state vector in the space of $N$ states; $\hat{A}$ and $\hat{B}$ are constant Hermitian $N \times N$ matrices. Linear time-dependence of parameters is relatively easy to implement in experiment, so MLZ models find numerous practical applications \[9\].

One can always choose the, so-called, diabatic basis in which the matrix $\hat{B}$ is diagonal, and if any pair of its elements are degenerate then the corresponding off-diagonal element of the matrix $\hat{A}$ is zero. In the diabatic basis, elements of the matrix $\hat{B}$ are called the slopes of diabatic levels, nonzero off-diagonal elements of the matrix $\hat{A}$ are called the coupling constants, and diagonal elements of the Hamiltonian are called diabatic energies.

An element $S_{nn'}$ of the scattering matrix is the amplitude of the state $n$ at $t \to +\infty$, given that at $t \to -\infty$ the system was in the $n'$th diabatic state. In many applications, only the matrix $\hat{P}$, with elements $P_{nn'} \equiv |S_{nn'}|^2$ called transition probabilities, is needed. More precise definition of the scattering matrix and discussion of its general properties can be found in previous reviews \[8, 10\]. Historically, the first study of a model type (1) beyond $N = 2$ was done by Majorana in \[2\].

We will call a particular choice of equation (1) solvable or integrable if one can write a compact analytical expression for elements of the transition probability matrix in terms of the well understood special functions of model parameters. Although the general solution of MLZ problem remains unknown, the number of exactly solved realizations of equation (1) has been growing quickly recently due to the discovery of integrability conditions (ICs) that identify models whose scattering matrices can be found exactly in the form of a product of specific elementary matrices \[1, 11, 12\]. Unfortunately, there is still no known algorithm to derive systems that satisfy ICs. Therefore the search for solvable models has required many trial-and-error efforts, so far.

In this article, we describe tricks that we found useful to alleviate the latter problem. While there is still no algorithm for detailed classification of solvable MLZ models, our goal is to show that there are systematic ways to search for new models that satisfy ICs. For illustrations, we derive and investigate several new solvable cases with different patterns of energy level crossings and different sizes of the phase space. We suggest analytical expressions for elements of the transition probability matrices in these models and provide results of rigorous numerical tests of our solutions.

We would like to stress that ICs in MLZ theory have not been proved mathematically rigorously yet. However, computational algorithms, described in \[12\], can be used to test our predictions with high precision. For example, transition probabilities in models of the type (1) with up to $N = 10$ states can be found numerically with up to three significant digits precision within a few minutes using a standard PC. Numerical algorithms allow 'high throughput' tests of our predictions for a broad range of model parameters and different initial conditions. We considered such a test as confirming the theory only when all tested multiple choices of parameters produced numerical results that were indistinguishable from all our analytical predictions for a given model. We tested usually more than 100 different choices of constant parameters in the nonperturbative regime per model. Thus, although this article is about mathematical questions, it is experimental in spirit. The role of an experimental setup is played here...
by a computer and results of our numerical experiments are obtained in order to provide an input for more mathematically rigorous studies of MLZ theory in the future.

The plan of our article is as follows. In section 2 we summarize basic knowledge about integrability in MLZ theory and suggest refined version of ICs. In section 3, we demonstrate our approach using a simple four-state MLZ Hamiltonian. In section 4, we show how one can generalize this approach to a higher dimensional case using a six-state model as an example, and in section 5 we show how similar intuition can be used to derive a new combinatorially complex but solvable MLZ model. We then summarize our findings in discussion.

2. MLZ models and integrability

In this section, we review basic established facts about integrability in MLZ theory using two previously solved models as examples. We will frequently use this section for references later. The additional goal is to refine definition of ICs.

2.1. Diabatic level diagram

It is convenient to illustrate parameters of any MLZ model in a graph with time-energy axes, as shown in figures 1(a) and (b). Lines of this graph are time-dependent diabatic levels, i.e. diagonal elements of the Hamiltonian in the diabatic basis as functions of time. Small black filled circles mark intersections of levels with nonzero pair-vise couplings. Integers on the left side of diabatic levels mark level indexes. On the right, levels are sometimes marked by analytic expressions for diabatic energies. If intersection of two levels is not specially marked then corresponding coupling between crossing diabatic levels is assumed to be zero, as the coupling between levels 2 and 6 in figure 1(b). If more than two levels intersect at one point, then nonzero couplings between different levels are additionally marked, as between levels 3 and 5 or 5 and 6 in figure 1(b). Direct couplings between parallel levels are always considered zero, such as between levels 2 and 3 in figure 1(a); this is included in the definition of the diabatic basis. Using these rules, we can read off the Hamiltonian from such a diagram. For example, for figure 1(a) the Hamiltonian is a 3 × 3 matrix

$$\hat{H}_{DO} = \begin{pmatrix} t/2 & g_1 & 0 \\ g_1 & e_1 - t/2 & 0 \\ 0 & g_2 & e_2 - t/2 \end{pmatrix}, \quad e_1 < e_2, \quad (2)$$

and the Hamiltonian of the model in figure 1(b) is a 6 × 6 matrix:

$$\hat{H}_{2B} = \begin{pmatrix} 2e_2 - t & 0 & 0 & \sqrt{2}g_2 & 0 & 0 \\ 0 & e_1 + e_2 - t & 0 & g_1 & g_2 & 0 \\ \sqrt{2}g_2 & g_1 & 0 & e_2 & 0 & \sqrt{2}g_2 \\ 0 & g_2 & \sqrt{2}g_1 & 0 & e_1 & \sqrt{2}g_1 \\ 0 & 0 & 0 & \sqrt{2}g_2 & \sqrt{2}g_1 & t \end{pmatrix}, \quad e_1 < e_2. \quad (3)$$

Model (2) is known as the 3-state Demkov–Osherov (DO) model [8, 10, 13]. It is one of the first MLZ models that were solved by methods of complex analysis [13]. Model (3) is also solvable as it is a bosonic extension of the three-state DO model in the sense that it is realized by a quadratic Hamiltonian of three interacting bosonic modes:
\[ H = \frac{1}{2} \left( \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b} - \hat{c}^\dagger \hat{c} \right) + e_1 \hat{b}^\dagger \hat{b} + e_2 \hat{c}^\dagger \hat{c} + \left( g_1 \hat{a}^\dagger \hat{b} + g_2 \hat{a}^\dagger \hat{c} + \text{h.c.} \right), \]  \noindent where \( \hat{a}, \hat{b}, \) and \( \hat{c} \) are bosonic annihilation operators. This Hamiltonian conserves the number of bosons, so if we look at the two-boson sector of this model then in the basis

\[ |1\rangle \equiv \frac{(\hat{c}^\dagger)^2}{\sqrt{2}} |0\rangle, \quad |2\rangle \equiv \hat{b}^\dagger \hat{c}^\dagger |0\rangle, \quad |3\rangle \equiv \frac{(\hat{b}^\dagger)^2}{\sqrt{2}} |0\rangle, \]

\[ |4\rangle \equiv \hat{c}^\dagger \hat{a}^\dagger |0\rangle, \quad |5\rangle \equiv \hat{b}^\dagger \hat{a}^\dagger |0\rangle, \quad |6\rangle \equiv \frac{(\hat{a}^\dagger)^2}{\sqrt{2}} |0\rangle, \]  

where \( |0\rangle \) is the vacuum state, the Hamiltonian (4) has the matrix form (3). On the other hand, if we switch to the Heisenberg picture, then the quadratic Hamiltonian (4) leads to evolution of operators.
\[ i \frac{d}{dt} \begin{pmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{pmatrix} = \hat{H}_{DO} \begin{pmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{pmatrix}, \]  

(6)

where \( \hat{H}_{DO} \) is the same as in (2). Let \( S_{ij}, i, j \in 1, 2, 3 \), be known scattering matrix elements of the DO model. Then we also have relations among operators, such as 

\[ \hat{a}(+\infty) = S_{11}\hat{a}(-\infty) + S_{12}\hat{b}(-\infty) + S_{13}\hat{c}(-\infty) \]

etc. One can use them to connect correlators of an arbitrary number of bosonic operators at \( t = +\infty \) with correlators at \( t = -\infty \), from which one can derive transition probabilities in the model (3). This procedure was designed in [3]. It was applied to solve the model (3) in [14].

2.2. Integrability conditions

The search for new solvable MLZ models by methods of complex analysis and multiparticle generalizations, as described above, has achieved only limited success. Recently, a much bigger class of solvable MLZ systems was discovered with a different approach based on ICs [1, 11, 12].

The simplest way to introduce ICs is by using the diabatic level diagram. Let us say that a path in this diagram is \textit{closed} if it goes along diabatic levels to produce a closed loop such that switching levels along this path is allowed only at level crossings with nonzero couplings. An example of such a path in figure 1(b) is the one that starts at the crossing of levels 3, 5 and 6, then goes along level 6 forward in time, switches to level 4 at crossing with levels 4 and 1, then goes backward in time to level 2; then switches to level 5, and then goes backward in time to return to the original crossing point with levels 3 and 6.

We define the area inside such a closed path as the sum of areas of enclosed plaquettes in the diabatic level diagram counting clockwise and counterclockwise enclosed areas with opposite signs. We also introduce the notion of a \textit{Hamiltonian projected to a subset of levels}. This Hamiltonian is obtained by setting couplings of given levels with all other states to zero and restricting the considered phase space only to the resulting evolution within the given subset of levels. ICs then say that for the full integrability of a MLZ model the following two conditions should be satisfied:

(i) All closed paths of the graph in the diabatic level diagram should enclose zero areas.

(ii) If two or several diabatic levels intersect at one point, and if there is no coupling path that connects two diabatic states within the Hamiltonian projected to this subspace of levels, then there is a corresponding pair of adiabatic energies of the full Hamiltonian that experiences exact crossing at all sufficiently small but finite values of coupling parameters.

For example, the DO model satisfies condition (i) trivially because the graph in figure 1(a) is a simple tree so any closed path on it has zero area naturally. In this model, there are also no diabatic level crossing points without direct coupling between levels, so condition (ii) is also satisfied trivially.

The case of the model in figure 1(b) is less trivial. It is easy to check that the triangle enclosed by levels 2, 4 and 6 is equal to the triangle enclosed by levels 2, 5 and 6, and that there is a single independent closed path in this level diagram that has to enclose one of these triangles clockwise and another one counterclockwise. So, the area of this path is zero, which proves condition (i).

We will not prove validity of (ii) completely analytically because it is generally a complex task while the lack of minigaps at eigenvalue crossing points can be visualized and then tested.
numerically very quickly with an approach that was described in [12]. For example, to make the presence of exact level crossings visually obvious, we diagonalize the Hamiltonian (3) and plot its eigenvalues as function of time in figure 1(c). This figure shows three pairs of adiabatic level crossings that correspond to intersections of diabatic levels in figure 1(b) with zero pairwise couplings, i.e. the pairs (3, 4), (2, 6) and (1, 5), which proves property (ii) for this model.

There are two more pairs of levels without direct couplings, namely (3, 6) and (1, 6) that, however, do not lead to exact crossings of adiabatic levels. This is because these levels cross simultaneously with other levels, respectively 5 and 4, to which they are coupled directly. So, if the Hamiltonian is projected to three such simultaneously crossing levels the phase space does not split into disjoined subspaces. Condition (ii) allows such crossings to happen without emergence of exact adiabatic level crossings.

Finally, we note that our definition of ICs (i)–(ii) is slightly different from the one that was originally suggested in [1]. Condition (i) here is more restrictive than just assumption of equality between dynamic phases of interfering semiclassical trajectories. This would make no difference for all previously found solvable models. However, in appendix A, we discuss an example of a model that breaks the condition (i) but does not break its counterpart in [1]. Our numerical tests showed that, at least by the standard semiclassical ansatz, this model is no longer solvable. So the new version of condition (i) should be preferred. In appendix B we also point that the new definition of the first integrability condition raises interesting analogies with properties of WKB quantized classically integrable multidimensional systems. On the other hand, our definition of condition (ii) is less restrictive than in [1] because now we allow level diagrams with more than two diabatic levels to cross at one point, as it is the case in figure 1(b).

2.3. Solution in the form of the semiclassical ansatz

If ICs are satisfied then one can obtain scattering amplitudes of the model in the form of a product of simple matrices. This form of the solution is called the matrix product ansatz or simply the \textit{semiclassical ansatz}. The name follows from the fact that this ansatz coincides with prediction of the semiclassical approximation in which nonadiabatic processes near different crossing points happen independently of each other. Assuming such real values of all couplings, semiclassical ansatz is generated as follows [1]:

1. We list all level crossing points of the diabatic level diagram in chronological order of their appearance.
2. For each crossing point, we identify the projected Hamiltonian that is obtained by keeping only diabatic levels that cross at this point and assuming that couplings to all other levels are zero.
3. For each projected Hamiltonian, we determine the \textit{projected scattering matrix}, i.e. the scattering matrix of MLZ model described by the projected Hamiltonian. We augment this scattering matrix then to the full phase space of the original model by assuming that other states simply do not experience interactions.
4. The scattering matrix of the full model is then the chronologically ordered product of projected scattering matrices at all encountered crossing points.
5. Final transition probabilities are obtained by taking the absolute value squared of elements of this scattering matrix.

An equivalent formulation of the semiclassical ansatz can be done in terms of the \textit{semiclassical trajectories} [1] that are the paths that go forward in time along the diabatic levels, so that they start and end at, respectively, initially and finally populated levels. Each time such a path goes through a crossing point, one should prescribe a specific amplitude factor that this
point contributes to the full trajectory amplitude as described in [1]. The final amplitude of the transition is then the sum of amplitudes of all semiclassical trajectories connecting the given pair of states.

Here we note that all parameters of scattering matrices at individual crossing points are often not needed to be known. If there are no pairs of semiclassical trajectories that interfere with each other in the full level diagram, then one can skip the step 4) and write the final transition probability matrix as the product of individual transition probability matrices at each crossing point.

It was also found that, in the case when interference of trajectories matters, only trivial phase factors of elements of projected scattering matrices should be included in order to determine final transition probabilities [1]. For example, if only two levels intersect at a given crossing point then the projected Hamiltonian at this crossing is the one of the two-state Landau-Zener model [15] with some coupling $g$ and crossing level slopes $\beta_1$ and $\beta_2$:

$$
\hat{H}_{LZ} = \begin{pmatrix} \beta_1 t & g \\ g & \beta_2 t \end{pmatrix}.
$$

(7)

In all solvable models with a finite number of states, nonzero coupling constants can always be chosen real but not always positive. The simplified scattering matrix that corresponds to the projected two-state Hamiltonian (7) is then known [1]. It depends on the sign of $g$:

$$
\hat{S}_{LZ} = \begin{pmatrix} \sqrt{p} & \pm i \sqrt{1-p} \\ \pm i \sqrt{1-p} & \sqrt{p} \end{pmatrix}, \quad p = e^{-2\pi g^2}/|\beta_1 - \beta_2|,
$$

(8)

where sign ($\pm$) is the same as the sign of the coupling constant $g$. Note that the matrix (8) does not include dynamic coupling-dependent phases of the full scattering matrix of the Landau-Zener model. The reason is that in all exactly solved models such phases factorize in the final result and do not influence final transition probabilities. Therefore, in what follows, we will use only such reduced versions of scattering matrices in our intermediate calculations, and verify validity of this choice at the end numerically.

2.4. Examples of semiclassical ansatz solutions

Let us consider the DO model (2) as an example. Since there is no interference between different paths connecting initial and final states, we can work directly with transition probability matrices that describe each crossing point. Let us define parameters

$$
p_{1,2} = e^{-2\pi g \hat{S}_{LZ}^2}, \quad q_{1,2} = 1 - p_{1,2}.
$$

(9)

There are two crossing points with nonzero couplings. In each case, the Hamiltonians projected to the crossing points are just the two state ones of the form (7), so we can introduce matrices of transition probabilities for evolution with such projected Hamiltonians:

$$
\hat{P}_1 = \begin{pmatrix} p_1 & q_1 & 0 \\ q_1 & p_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{P}_2 = \begin{pmatrix} p_2 & 0 & q_2 \\ 0 & 1 & 0 \\ q_2 & 0 & p_2 \end{pmatrix}.
$$

(10)

The full transition probability matrix of the three-state DO model is then the product of these matrices taken in chronological order:

$$
\hat{P}_{DO} = \hat{P}_2 \hat{P}_1.
$$

(11)
Let us now turn to the model (3). As in any solvable model, its scattering matrix factorizes in chronologically ordered contributions from all diabatic level crossing points with nonzero couplings. There is again no path interference in figure 1(b) of the model (3), so the full transition probability matrix can be directly written as a product of four transition probability matrices describing four crossing points:

\[ \hat{P}_{2B} = \hat{P}_3 \hat{P}_2 \hat{P}_1. \]  

(12)

The matrix \( \hat{P}_1 \) here describes transitions at simultaneous crossing of levels 3, 5, and 6. Projected to the subspace of these three levels Hamiltonian is a \( 3 \times 3 \) matrix. After the shift of the crossing point to \( t = 0 \), this Hamiltonian reads

\[ \hat{H}_1 = \begin{pmatrix} -t & \sqrt{2} g_1 & 0 \\ \sqrt{2} g_1 & 0 & \sqrt{2} g_1 \\ 0 & \sqrt{2} g_1 & t \end{pmatrix}. \]  

(13)

It describes the model of a spin-1 in a linearly time-dependent magnetic field. The transition probability matrix for the model (13) is known since the time of Majorana [2]:

\[ \hat{P}_1 = \begin{pmatrix} p_1^2 & 2p_1 q_1 & q_1^2 \\ 2p_1 q_1 & (1 - 2p_1)^2 & 2p_1 q_1 \\ q_1^2 & 2p_1 q_1 & p_1^2 \end{pmatrix}. \]  

(14)

or in the basis of all six states of the original model

\[ \hat{P}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_1 & 0 & q_1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2p_1 q_1 & 0 & (1 - 2p_1)^2 & 2p_1 q_1 \\ 0 & 0 & q_1^2 & 0 & 2p_1 q_1 & p_1^2 \end{pmatrix}. \]  

(15)

Similarly, \( \hat{P}_2 \) is a unit matrix except the \( 2 \times 2 \) block that corresponds to transitions between levels 2 and 4. Corresponding probabilities are given by the standard two-level Landau-Zener formula:

\[ \hat{P}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & p_2 & 0 & q_2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & q_1 & 0 & p_1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \]  

(16)

Matrices \( \hat{P}_3 \) and \( \hat{P}_4 \) are constructed analogously:

\[ \hat{P}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 & q_2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & q_2 & 0 & 0 & p_2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \hat{P}_4 = \begin{pmatrix} p_2^2 & 0 & 0 & 2p_2 q_2 & 0 & q_2^2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 2p_2 q_2 & 0 & 0 & (1 - 2p_2)^2 & 0 & 2p_2 q_2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ q_2^2 & 0 & 0 & 2p_2 q_2 & 0 & p_2^2 \end{pmatrix}. \]  

(17)
Such an approach to obtain the full transition probability matrix (12) of the model (3) is much more straightforward than dealing with correlators of bosonic operators and relating them to transition probabilities.

3. General 4-state system satisfying integrability conditions

3.1. Distorting previously solved model

In previous section, we showed that if a model satisfies ICs then derivation of its transition probability matrix is straightforward. However, the main challenge usually is to identify systems satisfying these conditions. Here we propose an approach that is based on our new conjecture stating that already solved MLZ models are actually special limits of some more complex solvable models.

For example, we know that solution for spin-1 in a linearly time-dependent field [2] was later found to be a special case of the three-state bow-tie model [16], which in turn was generalized later to the solvable multistate version [17], and eventually to the solvable generalized bow-tie model [18]. So, it is reasonable to expect that other known exact results in MLZ theory have similar generalizations.

The utility of this conjecture is that an already solved system provides the graph of diabatic levels with required topology. Here we say that graphs of diabatic levels, or corresponding models, have the same topology if the graphs have arbitrarily different parameters (couplings, level slopes) but the same set of level crossing points and types of level crossings. The latter means that two conditions are satisfied. First, levels with the same indexes in both graphs cross in the same points. Second, choices of parameters of corresponding Hamiltonians should be such that points leading to exact eigenvalue crossings should be the same in both graphs. One consequence of the latter rule is that, for a pairwise level crossing, if the direct coupling between corresponding two levels is zero in one model it should be also zero in the topologically equivalent model.

We can use the graph of the previously solved MLZ model to design a more complex solvable model by distorting this graph, which means assuming a more general choice of level slopes and coupling parameters while preserving the model’s topology. Generally, assuming arbitrary values of level slopes and couplings would violate ICs, however, for an available graph, we can reimpose such conditions in the form of relations within the new set of parameters. We are going to show that the resulting model is generally richer than the original one in the sense of the larger number of independent parameters. Moreover, it usually contains the original model as a special case.

As the first example, let us consider a simple four-state model that was shown to satisfy ICs and solved in [11]. It has the Hamiltonian

\[
\hat{H}(t) = \begin{pmatrix} -b_1 t + e & 0 & g & -\gamma \\ 0 & b_1 t + e & \gamma & g \\ g & \gamma & -b_2 t & 0 \\ -\gamma & g & 0 & b_2 t \end{pmatrix},
\]

and its level diagram is plotted in figure 2(a). Exact validity of the semiclassical ansatz for this model was rigorously proved in [10].

Let us now deform this graph as shown in figure 2(b) so that the new graph still has two crossings of diabatic levels without direct couplings. All nonzero couplings can now be considered different, which leads to the following Hamiltonian:
with some constants $b_1, b_2, b, e, e_1, e_2, x, y, \gamma$. Here we used the gauge freedom [8] so that slopes of levels 1 and 2 differ only by sign, and we set constant diagonal element of level 4 to zero.

Parameter $e_2$ can be set to zero by time shift $t \rightarrow t + e_2/b_1$ that does not affect transition probabilities for evolution from $-\infty$ to $+\infty$. After this, we can make gauge transformation of diabatic state amplitudes $a_i \rightarrow e^{-ib_1e_2/b_1}a_i$, $i = 1, \ldots, 4$, which results in $e_2 = 0$ at redefined parameters $e \rightarrow e - b_2b_2/e_2/b_1$ and $e_1 \rightarrow e_1 + (b - b_2)e_2/b_1$. We also note that the gauge freedom to choose zero energy point and the arbitrariness of choosing indexes of states allow us to assume that

$$b_1 > b_2, \quad \text{and} \quad b < b_2. \quad (20)$$

Otherwise, we can just rename indexes, followed by proper renaming of constant parameters and obtain the same Hamiltonian at desired relations (20). Therefore, in what follows, we will assume that $e_2 = 0$ with constraints in (20), unless specially stated.

Let $t_{ij}$ be the time moment of diabatic crossing between levels $i$ and $j$. For crossings without direct couplings we have:

$$t_{12} = 0, \quad t_{34} = e_1/(b_2 - b), \quad (21)$$

and for time moments of avoided crossings (i.e. moments of crossing of corresponding diabatic energy levels):

$$t_{13} = e - e_1/b_1 + b, \quad t_{14} = e/b_1 + b_2, \quad t_{23} = -e - e_1/b_1 - b, \quad t_{24} = -e/b_1 - b_2. \quad (22)$$

Let us now consider the condition on that the area inside the closed path in the graph in figure 2(b) is zero. This area is the difference between areas swept over the time axis by two trajectories that connect crossing points at time moments $t_{24}$ and $t_{13}$. One such a trajectory starts at $t_{24}$, then stays on level 2 till time $t_{23}$, and then goes along level 3 till time $t_{13}$. The area that this trajectory sweeps over the time axis is

$$A_1 = \int_{t_{24}}^{t_{23}} (b_1 t + e) \, dt + \int_{t_{13}}^{t_{23}} (bt + e_1) \, dt.$$  

The second trajectory starts at $t_{24}$, then stays on level 4 till time $t_{14}$, and then goes along level 1 till time $t_{13}$. Explicitly, for two trajectories we have then

$$A_1 = \frac{b_1}{2} (t_{23}^2 - t_{24}^2) + \frac{b}{2} (t_{13}^2 - t_{23}^2) + e(t_{23} - t_{24}) + e_1(t_{13} - t_{23}), \quad (23)$$

$$A_2 = \frac{b_2}{2} (t_{14}^2 - t_{24}^2) - \frac{b_1}{2} (t_{13}^2 - t_{14}^2) + e(t_{13} - t_{14}). \quad (24)$$

Setting $A_1 - A_2 = 0$ we find equation that parameters of the model satisfy in order to realize IC (i). In terms of $e_1$ this equation is quadratic with two roots:

$$e_1^{\pm} = e \pm |e| \sqrt{\frac{b_1^2 - b_2^2}{b_1^2 - b_2}}. \quad (25)$$
Equation (25) shows that only values of $b$ in the range $-b_1 < b < b_1$ lead to physical real valued $e^{\pm 1}$.

Our parameters do not satisfy IC (ii) yet. Generally, an analytical proof that some crossing of diabatic levels corresponds to an exact crossing point of adiabatic energy levels is a complex problem. However, there are perturbative conditions that must be satisfied by parameters if there is an exact crossing point. We will derive such constraints analytically and then check numerically whether imposing them is sufficient to satisfy condition (ii).

Let $E_{k}$, $k = 1, \ldots, 4$ be diabatic energies of levels, respectively, at time $t_{12}$. We also define energy gaps

$$E_{13} \equiv E_1 - E_3 = e - e_1, \quad E_{14} \equiv E_1 - E_4 = e.$$ (26)

Similarly, let $E_{k}'$ be energies of levels at time $t_{34}$. We have then

$$E_{32}' \equiv E_3' - E_2' = -e - e_1 (b_1 - b_2) / b_2 - b, \quad E_{31}' \equiv E_3' - E_1' = -e + e_1 (b_1 + b_2) / b_2 - b.$$ (27)

Let us now treat all couplings as small. The necessary condition for the crossing point between levels 1 and 2 to be exact is that coupling between these diabatic states remains zero at 2nd order perturbation in $\gamma$ and $g$. This leads to the constraint

$$-\frac{\gamma x g}{E_{14}} + \frac{\gamma y g}{E_{13}} = 0.$$ Similarily, coupling between levels 3 and 4 should be zero at 2nd order perturbation. This gives the constraint

$$x y = E_{31}' / E_{32}'$$

which leads to

$$x = \pm \sqrt{\frac{E_{31}' E_{14}}{E_{32}' E_{13}}}, \quad y = x E_{13} / E_{14}.$$ (28)
Thus, in order to satisfy ICs, there are three constraints in (25) and (28) on parameters so that only parameter $b$ remains independent among the newly introduced ones. The original model in figure 2(a) is recovered at $b = -b_2$ but other values of $b$ produce models that have not encountered previously in MLZ theory.

Conditions (28) are necessary but not sufficient in order to guarantee the presence of exact crossing points near times $t_{12}$ and $t_{34}$. Therefore, we resort to numerical simulations for additional proof. Figures 3(a) and (b) and 4(a) and (b) show results of explicit numerical calculation of the adiabatic energy levels (i.e. eigenvalues of the Hamiltonian as functions of time $t$) at different values of parameters. It is visually clear that there are, indeed, pairs of exact eigenvalue crossings in each figure when conditions (25) and (28) are satisfied.

Depending on the sign in equation (25) and parameter $e$, one can generate different graph types. Two types correspond to models in which different semiclassical trajectories can interfere, while other two graph types do not feature path interference.

3.2. Phase with path interference

In figures 3(a) and (b), we show adiabatic energies of the Hamiltonian in cases when either (a) $e_1 = e_1^+$ with $e > 0$ or (b) $e_1 = e_1^-$ with $e < 0$. Geometries of these graphs are different by the relative positions of the exact crossing points. The common feature of both graphs is the possibility of more than one semiclassical trajectories that connect one diabatic state at $t = -\infty$ with another diabatic state at $t = +\infty$. In both cases we find

$$x = \pm \sqrt{\frac{b_1 + b_2}{b_1 - b}}, \quad y = \pm \sqrt{\frac{b_1 + b}{b_1 - b_2}},$$

(29)

with $x$ and $y$ having the same sign. Let us introduce

$$p_1 = e^{-2\pi^2/(b_1 - b_2)}, \quad p_2 = e^{-2\pi^2/(b_1 - b)}, \quad q_{1,2} = 1 - p_{1,2}. \quad (30)$$

Semiclassical ansatz calculations with parameters (29) are analogous to calculations in [11] that were done for the original model from figure 2(a). They lead to the following simple form of the transition probability matrix:

$$\hat{P} = \begin{pmatrix}
    p_1 p_2 & 0 & p_2 q_1 & q_2 \\
    0 & p_1 p_2 & q_2 & p_2 q_1 \\
    p_2 q_1 & q_2 & p_1 p_2 & 0 \\
    q_2 & p_2 q_1 & 0 & p_1 p_2
\end{pmatrix}, \quad b_1 > b_2 > b, \quad (31)$$

$$\hat{P} = \begin{pmatrix}
    p_1 p_2 & 0 & q_1 & q_2 p_1 \\
    0 & p_1 p_2 & q_2 p_1 & q_1 \\
    q_1 & q_2 p_1 & p_1 p_2 & 0 \\
    q_2 p_1 & q_1 & 0 & p_1 p_2
\end{pmatrix}, \quad b_1 > b > b_2. \quad (32)$$

Figures 3(c) and (d) compare solution (31) and (32) with results of direct numerical simulations at different values of parameter $b$. The agreement between analytical predictions and numerics is perfect for all values of parameters. This leaves no doubt that semiclassical ansatz prediction here is not an approximation but rather the exact result although, certainly, such an agreement with numerics cannot be accepted as a mathematically rigorous proof. Behavior of parameters $x$ and $y$ is provided in figure 3(e) in order to show that, in our numerical tests, these parameters varied substantially. For generality, we assumed in simulations that $e_2 \neq 0$, so we
Figure 3. Characteristics of the phase with path interference. (a) and (b) Adiabatic energy levels at, respectively, $e > 0$ and $e < 0$ (with, respectively, $e_1 = e_1^-$, and $e_1 = e_1^+$). Parameters in both cases: $b_1 = 1.5$, $b_2 = 0.4$, $g = 0.25$, $\gamma = 0.75g$, $e_2 = 0$, and (a) $b = -0.35$, $e = 1.55$, (b) $b = -0.1$, $e = -1.55$. Avoided crossings are marked by corresponding couplings between diabatic levels. (c) and (d) Test of the validity of equations (31) and (32) for different elements of the transition probability matrix as functions of parameter $b$. Solid curves are theoretical predictions and discrete points are results of numerical simulation of evolution from $t = -2000$ to $t = 2000$ with a discrete time step $\Delta t = 0.0005$. Numerical algorithm is described in [12]. Choice of parameters: $e = 0.3$, $e_2 = 0.1$, $b_1 = 1.5$, $b_2 = 0.4$, $g = 0.25$, $\gamma = 0.75g$. (e) Dependence of coupling asymmetry parameters $x$ and $y$, as well as $e_1$ in the inset, on $b$ for the same other parameters as in (c). (f) Test of independence of transition probabilities of $e_2$. Other parameters are as in (c) with $b = 1.25$. Parameters $e_1$, $x$, $y$ are always tuned to satisfy integrability conditions, including at nonzero $e_2$. 

N A Sinitsyn and V Y Chernyak 

J. Phys. A: Math. Theor. 50 (2017) 255203
Figure 4. Characteristics of the phase without path interference. (a) and (b) Adiabatic energy levels at, respectively, $e > 0$ and $e < 0$ (with, respectively, $e_1 = e_1^+$, and $e_1 = e_1^-$). Parameters in both cases are: $b_1 = 1.5$, $b_2 = 0.4$, $g = 0.25$, $\gamma = 0.75g$, $e_2 = 0$, and (a) $b = -0.35$, $e = 1.55$, (b) $b = -0.1$, $e = -1.55$. Avoided crossings are marked by corresponding coupling between diabatic levels. (c) and (d) Test of the validity of equation (34) for different elements of the transition probability matrix as functions of parameter $b$. Solid curves are theoretical predictions and discrete points are results of numerical simulation. Choice of constant parameters in (c) and (d) are as in, respectively, (a) and (b). (e) Eigenvalues of the Hamiltonian at $b = b_2$ and other parameters are as in (a). (f) Dependence of coupling asymmetry parameters $x$ and $y$, as well as $e_1$ in the inset, on $b$ for the same other parameters as in (a).
also show figure 3(f) confirming that varying $e_2$ does not influence transition probabilities, as long as we adjust $e_1$ properly to satisfy integrability conditions.

Transition probabilities in figures 3(c) and (d) show discontinuous behavior at $b = b_2$. This is not surprising because at $b = b_1$ we have $e_1 = 0$, so diabatic levels 3 and 2 become degenerate not at a single time moment but during all the time. Transition probabilities in MLZ models are known to behave generally discontinuously when parameters pass through values that create such permanent degeneracies [13].

Interestingly, solution (31) and (32) for the model in figure 2(b) has exactly the same form as the solution for the original model in figure 2(a) with only minor generalization that allows free choice of parameter $b$. This hints to possible symmetry that relates arbitrary values of $b$ to the case with $b = -b_2$, which was rigorously proved in [10].

3.3. Phase without path interference

Let us now turn to phases such that (a) $e_1 = e_1^-$ with $e < 0$ or (b) $e_1 = e_1^+$ with $e > 0$ (see equation (25)). Repeating perturbative calculations, we find that in this case parameters $x$ and $y$ have opposite signs:

$$ x = \pm \sqrt{\frac{b_1 + b_2}{b_1 - b}}, \quad y = \mp \sqrt{\frac{b_1 + b}{b_1 - b_2}}, $$

i.e. negative couplings appear in pairs.

In figures 4(a) and (b) we show possible patterns of adiabatic energy levels and demonstrate two exact crossing points. Apparently, there is no path interference in this case. In both cases we find a solution

$$ \tilde{P} = \begin{pmatrix} p_1 & q_1 & q_2 & p_2 \\ q_1 & p_2 & q_1 & p_1 \\ q_2 & p_1 & q_2 & p_2 \\ p_2 & q_2 & p_1 & q_1 \end{pmatrix}. $$

Results of numerical tests of this prediction are shown in figures 4(c) and (d). Again, we find perfect agreement between the semiclassical ansatz and numerical predictions. In figures 4(e) and (f) we show, respectively, adiabatic levels at $b = b_2$ and dependence of parameters $x, y, e_1$ on $b$ that we used to satisfy ICs.

Several properties of solution (34) are to be mentioned. First, up to exchange of positions of some rows and columns, this transition probability matrix is a direct product of $2 \times 2$ matrices

$$ \begin{pmatrix} p_1 & q_1 \\ q_1 & p_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p_2 & q_2 \\ q_2 & p_2 \end{pmatrix}, $$

which are transition probability matrices of two two-state Landau-Zener models. In fact, the special case with $b = -b_2$ corresponds to the solvable model from [3]. This model is, indeed, constructed as a direct product of two two-state Hamiltonians. This fact also suggests that the case with arbitrary $b$ may be related to the direct product of two decoupled two-state Landau-Zener systems.

Second interesting property is the absence of discontinuity at $b = b_2$. This follows from the fact that in this case we have $e_1 = 2e \neq 0$, i.e. there is no permanent degeneracy of two levels. At this point, we have $x = -y = \sqrt{(b_1 + b_2)/(b_1 - b_2)}$, and two levels are parallel to each other, as illustrated in figure 4(e). This case is different from another known solvable model,
called the four-state bow-tie model [17], that has the same geometry of level crossings but different constraints on coupling constants. We find it surprising that the four-state bow-tie model is not a special case of our model. This observation means that, when some of the diabatic levels are parallel to each other, one has more options to satisfy ICs. Determining constraints on parameters in such cases is, however, more difficult because nontrivial constraints emerge then in higher order perturbation theory. We consider such a model in the following section.

4. Distorting Bosonic model

Let us now consider the bosonic model represented by figure 1(b) and look for possibility to distort it in order to generate a new solvable model.

4.1. Model with new couplings only

For simplicity, first, we will not change the slopes of diabatic levels at all and try only to find a different choice of coupling parameters that would satisfy condition (ii). According to the strategy that we described in previous section, we augment the set of parameters to generate the model shown in figure 5(a) with six unknown coupling parameters \( g_1 \) to \( g_6 \). It is easy to check numerically that a random choice of these couplings leads to permanent gaps between all adiabatic energy levels, so this model is generally not integrable.

To narrow the search for proper couplings, we treat coupling constants as small. Diabatic levels 2 and 6 cross at \( t_{cr} = (e_1 + e_2)/2 \). Let \( E_k \), \( k = 1, \ldots, 6 \) be diabatic energies of states at this time moment. We will need

\[ E_2 = E_6 = (e_2 + e_1)/2, \quad E_4 = e_2, \quad E_5 = e_1. \]

At second order in strength of parameters \( g_i \), coupling between levels 2 and 6 is then given by

\[ v_{26} = \frac{g_1 g_6}{E_2 - E_4} + \frac{g_2 g_3}{E_2 - E_5}. \] (35)

In order not to open the minigap between levels 2 and 6, this coupling should be zero. Since \( E_2 - E_4 = -(E_2 - E_5) \), this gives a constraint on couplings:

\[ g_1 g_6 = g_2 g_3. \] (36)

Let us test now the ‘no-coupling’ condition for levels 3 and 4. These diabatic levels cross at \( t_{cr} = 2e_1 - e_2 \), and at this moment

\[ E_6 = 2e_1 - e_2, \quad E_3 = E_4 = e_2, \quad E_5 = e_1, \quad E_2 = 2e_2 - e_1. \]

Generally, a minigap between levels 3 and 4 would open as the third order correction:

\[ v_{34} = \frac{g_6 g_3 g_4}{(E_4 - E_6)(E_4 - E_3)} + \frac{g_1 g_2 g_4}{(E_4 - E_2)(E_4 - E_5)}. \] (37)

By equating \( v_{34} \) to zero we find

\[ g_3 g_6 = 2g_1 g_2. \] (38)

Taking ratio of equations (36) and (38) we find a constraint \( g_3 = \pm \sqrt{2} g_1 \). Similarly, by considering crossing of levels 1 and 5, we find constraint \( g_6 = \pm \sqrt{2} g_2 \).

Such constraints are insufficient to determine values of parameters \( g_4 \) and \( g_5 \). This is expected because when a model has parallel diabatic levels, such levels do not give rise to adiabatic crossing between them, which reduces the number of independent constraints on
coupling parameters. So, for such cases, one should develop higher order perturbative tests for existence of exact level crossings. Here, we will not go for this because the already found constraints are sufficiently helpful. We just recall that presence of exact crossing points usually requires some symmetric choices of couplings. After trying several candidate sets, we found that the choice shown in figure 5(b) does lead to the desired exact eigenvalue crossings, as we prove in figure 5(c). The Hamiltonian of this model reads:

$$
\hat{H} = \begin{pmatrix} 
-t + 2e_2 & 0 & 0 & \gamma & 0 & 0 \\
0 & -t + e_1 + e_2 & 0 & g & g & 0 \\
0 & 0 & -t + 2e_1 & 0 & \gamma & 0 \\
\gamma & g & 0 & e_2 & 0 & \sqrt{2g} \\
0 & g & \gamma & 0 & e_1 & \sqrt{2g} \\
0 & 0 & 0 & \sqrt{2g} & \sqrt{2g} & t
\end{pmatrix},
$$

with arbitrary real parameters $e_1$, $e_2$, $g$, and $\gamma$.

Figure 5. Diabatic level diagram for (a) the Hamiltonian that has the same structure as in equation (3) but with arbitrary choice of coupling constants (b) with specific choice, which is different from the one in equation (3) but satisfying integrability conditions. (c) Three exact crossing points in spectrum of the Hamiltonian that corresponds to figure 5(b) at parameter values $\gamma = 1.123$, $g = 1.5$, $e_1 = 1$, $e_2 = -2$. 
According to section 2.3, as in the original bosonic model, the matrix of transition probabilities factorizes as a product of four transition probability matrices describing all crossing points with nonzero couplings. Consider first the crossing point of diabatic levels 3, 5, and 6. After the time shift that places this crossing point at \( t = 0 \), corresponding projected to the phase space of these levels Hamiltonian becomes a \( 3 \times 3 \) matrix:

\[
\hat{H}_1 = \begin{pmatrix}
-t & \gamma & 0 \\
\gamma & 0 & \sqrt{2g} \\
0 & \sqrt{2g} & t 
\end{pmatrix}.
\] (40)

Transition probabilities for evolution with this Hamiltonian are known because it is a special case of the Hamiltonian of the solvable bow-tie model. Let us define parameters

\[
p_1 = e^{-2\pi^2}, \quad p_2 = e^{-\pi \gamma^2}, \quad q_{1,2} = 1 - p_{1,2}.
\] (41)

The transition probability matrix restricted to the crossing levels is then given by

\[
\hat{P}_1 = \begin{pmatrix}
p_2^2 & q_2(p_1 + p_2) & q_1 q_2 \\
q_2(p_1 + p_2) & (1 - p_1 - p_2)^2 & q_1(p_1 + p_2) \\
q_1 q_2 & q_1(p_1 + p_2) & p_1^2
\end{pmatrix}.
\] (42)

or in the basis of all six states

\[
\hat{P}_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & p_2^2 & 0 & q_2(p_1 + p_2) & q_1 q_2 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & q_2(p_1 + p_2) & 0 & (1 - p_1 - p_2)^2 & q_1(p_1 + p_2) \\
0 & 0 & q_1 q_2 & 0 & q_1(p_1 + p_2) & p_1^2
\end{pmatrix}.
\] (43)

Similarly, \( \hat{P}_2 \) and \( \hat{P}_3 \) are unit matrices except the \( 2 \times 2 \) blocks that describe transitions near crossings of level pairs (2, 4) and (2, 5):

\[
\hat{P}_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & p_1 & 0 & q_1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & q_1 & 0 & p_1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}, \quad \hat{P}_3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & p_1 & 0 & q_1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & q_1 & 0 & p_1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.
\] (44)

and transition probabilities at the crossing of levels 1, 4, and 6 are again obtained from the solution of the three-state bow-tie model:

\[
\hat{P}_4 = \begin{pmatrix}
p_2^2 & 0 & 0 & q_2(p_1 + p_2) & 0 & q_1 q_2 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
q_2(p_1 + p_2) & 0 & (1 - p_1 - p_2)^2 & 0 & q_1(p_1 + p_2) \\
0 & 0 & 0 & 0 & 1 & 0 \\
q_1 q_2 & 0 & q_1(p_1 + p_2) & 0 & p_1^2
\end{pmatrix}.
\] (45)

According to section 2.3, the total transition probability matrix is given by

\[
\hat{P}_\text{fin} = \hat{P}_4 \hat{P}_3 \hat{P}_2 \hat{P}_1.
\]

or explicitly
The effective Hamiltonian near this point is a
probability matrices of each crossing point. Consider first the crossing point of levels 3, 5, and
sical trajectories, so we have to construct semiclassical ansatz from scattering rather than
numerical proof of the existence of three exact level crossings, as required by IC (ii) in
J. Phys. A: Math. Theor. 50 (2017) 255203

4.2. Deformation leading to path interference

Further extensions of the found solution can be searched among models with different level
slopes but the same positions of nonzero couplings and keeping parallel levels to be parallel
after the deformation. As physical meaning of such models is obscure, we will restrict our-
seves to a particularly symmetric case in which we exchange the slope of level 6 with that of
levels 4 and 5, as shown in figure 7(a). We adjust distances between parallel diabatic levels
slopes but the same positions of nonzero couplings and keeping parallel levels to be parallel
Further extensions of the found solution can be searched among models with different level

In figure 6 we test predictions of equation (46) and find that agreement with numerical
simulations is again excellent. Note that one of the signatures of integrability, which can be
tested numerically, is that all transition probabilities are independent of rescaling the distances
between parallel diabatic levels, as we show in figure 6(a). This property of the semiclassical
ansatz is found in all solved models with parallel diabatic levels [12, 13, 18, 19].

\[
P_{in} = \begin{pmatrix}
    p_1^2 & q_1 q_2 (p_1 + p_2) & q_1^2 q_2^2 & p_1 q_2 (p_1 + p_2) & q_1^2 q_2 (p_1 + p_2) & p_1^2 q_1 q_2 \\
    0 & p_1^2 & q_1 q_2 (p_1 + p_2) & p_1 q_2 & q_1^2 (1 - p_1 - p_2)^2 & q_1^2 (p_1 + p_2) \\
    0 & 0 & p_1^2 & q_1 (1 - p_1 - p_2)^2 & q_1^2 (p_1 + p_2) & p_1 (1 - p_1 - p_2)^2 \\
    q_1 (p_1 + p_2) & q_1 (1 - p_1 - p_2)^2 & q_1^2 q_2 (p_1 + p_2) & p_1 (1 - p_1 - p_2)^2 & q_1^2 (p_1 + p_2) \\
    0 & p_1 q_2 & q_1 q_2 (p_1 + p_2) & q_1^2 q_2 & p_1 q_2 (p_1 + p_2) & q_1^2 (p_1 + p_2) \\
    q_1 q_2 & q_1^2 (p_1 + p_2) & q_1 q_2 & q_1 q_2 & q_1 q_2 (p_1 + p_2) & p_1^2 
\end{pmatrix}.
\]

(46)

which is parametrized by three independent parameters \(e, g, \) and \(\gamma\). Figure 7(b) shows
numerical proof of the existence of three exact level crossings, as required by IC (ii) in
this case. The fact that some of the couplings have different signs and that all three exact
crossings happen simultaneously at \(t = 0\) indicates that this model can describe an odd-spin
system [11].

Examining figure 7(a) we find that there is a possibility for interference between semiclassical
trajectories, so we have to construct semiclassical ansatz from scattering rather than probability matrices of each crossing point. Consider first the crossing point of levels 3, 5, and
6. The effective Hamiltonian near this point is a \(3 \times 3\) matrix

\[
\hat{H}_1 = \begin{pmatrix}
    -t + e & 0 & \gamma & 0 & g \\
    0 & -t & 0 & -g & g \\
    0 & 0 & -t - e & 0 & \gamma \\
    \gamma & -g & 0 & t - e & 0 \\
    0 & g & \gamma & 0 & t + e \\
    g & 0 & g & 0 & 0 
\end{pmatrix}.
\]

(47)

This is again the Hamiltonian of the three-state bow-tie model [16] but with different from
equation (40) order of level slopes. Let

\[
X = e^{-\pi \gamma^2/2}, \quad Y = e^{-\pi g^2}.
\]

(49)
The scattering matrix for the Hamiltonian (48) can be obtained by taking square roots of known transition probabilities (given explicitly in [16]), and adding factors \(i\) taken to the power of the number of direct transitions that connect the levels:

\[
\hat{S}_1 = \begin{pmatrix}
XY & i\sqrt{(1 - X)(1 + XY)} & i\sqrt{X(1 - Y)(1 + XY)} \\
i\sqrt{(1 - X)(1 + XY)} & X & -\sqrt{X(1 - X)(1 - Y)} \\
i\sqrt{X(1 - Y)(1 + XY)} & -\sqrt{X(1 - X)(1 - Y)} & 1 - X + XY
\end{pmatrix}. \tag{50}
\]

Note that this scattering matrix is truncated so that we drop its dependence on coupling-dependent phase factors which do not change final transition probabilities in all known exactly solvable models. Lifting \(\hat{S}_1\) to the phase space of all six states, we get

\[
\hat{S}_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & XY & 0 & i\sqrt{(1 - X)(1 + XY)} & i\sqrt{X(1 - Y)(1 + XY)} \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & i\sqrt{(1 - X)(1 + XY)} & 0 & X & -\sqrt{X(1 - X)(1 - Y)} \\
0 & 0 & i\sqrt{X(1 - Y)(1 + XY)} & 0 & -\sqrt{X^2(1 - X)(1 - Y)} & 1 - X + XY
\end{pmatrix}. \tag{51}
\]

Chronologically, the next two crossing points correspond to pairwise transitions between levels \((2, 5)\) and then \((2, 4)\). Corresponding scattering matrices have the form (8) with one subtlety: due to different signs of couplings near those intersections, off-diagonal elements of corresponding scattering matrices have opposite signs.

Figure 6. Test of validity of the transition probability matrix (46) for the model in figure 5(b). Solid curves are analytical predictions and discrete points are results of numerical simulations for evolution from \(t = -2000\) to \(t = 2000\) with a time step \(dt = 0.00005\). Algorithm is explained in [12]. (a) Test of independence of transition probabilities of rescaled distances between parallel levels. Here \(e_1 = -\epsilon, e_2 = 1.3\epsilon\). Evolution starts at level 2. Other parameters: \(\gamma = 0.779, g = 0.38\). (b) Dependence of transition probabilities, starting from level 4, on coupling parameter \(\gamma\). Other parameters: \(e_1 = -0.5, e_2 = 0.2, g = 0.25\).
The last crossing point is between levels 1, 4 and 6, which is again described by the three-state bow-tie model:

\[
\hat{S}_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & Y^{1/2} & 0 & 0 & i\sqrt{1-Y} & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
i\sqrt{1-Y} & 0 & 0 & Y^{1/2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad \hat{S}_3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & Y^{1/2} & 0 & -i\sqrt{1-Y} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & -i\sqrt{1-Y} & 0 & Y^{1/2} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}.
\]

(52)
The final scattering matrix of the model (47) is given by
\[ \hat{S}_4 = \hat{S}_2 \hat{S}_3 \hat{S}_2 \hat{S}_1. \] (54)
Let us introduce parameters
\[ Q_X \equiv 1 - X, \quad Q_Y \equiv 1 - Y, \quad Q_{XY} \equiv 1 + XY. \]
Multiplying scattering matrices in (54) and taking the absolute values squared of all elements of \( \hat{S}' \) we obtain the final matrix of transition probabilities:
\[ \hat{P}' = \begin{pmatrix}
X^2Y^2 & YQ_XQ_YQ_{XY} & YQ_XQ_Y & 0 & XY^2Q_YQ_{XY} \\
0 & Y^2 & YQ_XQ_YQ_{XY} & Q_Y & X^2YQ_Y \\
Q_XQ_Y & X^2YQ_{XY} & 0 & X^2Y & X^2Q_Y \\
0 & Q_Y & YQ_XQ_YQ_{XY} & 0 & X^2Y & XQ_YQ_Y \\
XQ_YQ_X & XQ_YQ_YQ_{XY} & XY^2Q_YQ_{XY} & XYQ_XQ_Y & XQ_YQ_Y & (Q_X + XY)^2
\end{pmatrix} \] (55)
Figures 7(c) and (d) show perfect agreement of this prediction with results of our direct numerical simulations. This model example demonstrates basic nontrivial features of the semiclassical ansatz such as the possibility to describe simultaneous multiple level intersections and interference of semiclassical trajectories that connect complex crossing points.

5. Model of interacting fermions

Now, we turn to the question whether one can use our strategy in order to generate physically meaningful solvable MLZ models with combinatorial complexity. At least one such a model has been already identified previously [12]. It is important, however, to know how common such models are and how different they can be. Here we will present quite a different from [12] model, which describes nonlinear interactions of fermions.

5.1. Six-state sector

The logic that brought us to this model was the following. We started with observation that there are two already known six-state solvable MLZ models with very similar diabatic level diagrams:
\[ \hat{H}_1 = \begin{pmatrix}
e_1 + \beta t & 0 & 0 & -g_2 & -g_3 & 0 \\
e_2 + \beta t & 0 & g_1 & 0 & -g_3 & 0 \\
0 & e_1 + \beta t & 0 & g_1 & 0 & g_2 \\
-g_2 & g_1 & 0 & e_1 + e_2 & 0 & 0 \\
-g_3 & 0 & g_1 & 0 & e_1 + e_3 & 0 \\
0 & -g_3 & g_2 & 0 & 0 & e_2 + e_3
\end{pmatrix}, \quad e_1 < e_2 < e_3, \quad \beta > 0, \] (56)
and
\[
\hat{H}_2 = \begin{pmatrix}
-e_1 + \beta t & 0 & 0 & -g_2 & -g_3 & 0 \\
0 & -e_2 + \beta t & 0 & g_1 & 0 & -g_3 \\
0 & 0 & -e_3 + \beta t & 0 & g_1 & g_2 \\
-g_2 & g_1 & 0 & e_1 + e_2 & 0 & 0 \\
-g_3 & 0 & g_1 & 0 & e_1 + e_3 & 0 \\
0 & -g_3 & g_2 & 0 & 0 & e_2 + e_3
\end{pmatrix}, \quad e_1 < e_2 < e_3, \quad \beta > 0.
\]

(57)

The matrix of transition probabilities in the model (56) was derived in [3] as a two-fermion extension of the DO model. The model (57) was found and solved only recently using ICs [1].

Similarity between Hamiltonians (56) and (57) suggests that they can be special instances of a more general model. The simplest candidate for such a generalization is the following $6 \times 6$ Hamiltonian matrix:
\[
\hat{H}(t) = \begin{pmatrix}
e_1(1 - x) + \beta t & 0 & 0 & -g_2 & -g_3 & 0 \\
0 & e_2(1 - x) + \beta t & 0 & g_1 & 0 & -g_3 \\
0 & 0 & e_3(1 - x) + \beta t & 0 & g_1 & g_2 \\
-g_2 & g_1 & 0 & e_1 + e_2 & 0 & 0 \\
-g_3 & 0 & g_1 & 0 & e_1 + e_3 & 0 \\
0 & -g_3 & g_2 & 0 & 0 & e_2 + e_3
\end{pmatrix},
\]

(58)

where $x$ is a new parameter. At $x = 0$ and $x = 2$ the model (58) transfers into the models (56) and (57), respectively.

It is straightforward to check that IC (i) is always satisfied at $x \neq 1$, however, when $x$ crosses the value $x = 1$ chronological order of diabatic level crossings changes. We tested validity of IC (ii) for both $x < 1$ and $x > 1$ numerically and found that three crossing points of diabatic levels with zero direct coupling between them always resulted in exact crossings of adiabatic energy levels, as illustrated in figures 8(a) and (b).

Since ICs are satisfied, we can apply the semiclassical ansatz to derive transition probability matrices. However, long calculations can be avoided because this analysis is completely analogous to the one in [1] that was used to solve the model (57). Namely, in the case with $x > 1$, topology of the diabatic level graph of the model (58) and chronological order of level crossings is the same as for the graph of the model (57), while coupling constants are also the same. So the transition probability matrix does not depend on $x$, as far as $x > 1$, and solution derived in [1] for the model (57) is equally applicable to any case with $x > 1$. Similarly, at any $x < 1$, topology of the graph of diabatic levels and chronological order of level crossings is the same as in the model (56). Hence, the transition probability matrix that is given by the semiclassical ansatz at any $x < 1$ is the same as the one at $x = 0$, which was found analytically in [3].

Let us introduce parameters:
\[
p_k = e^{-2\pi g_k^0/\beta}, \quad q_k = 1 - p_k, \quad k = 1, 2, 3.
\]

(59)

Final results for the two sectors of the model (58) read:
where $\hat{c}_k$ and $\hat{d}$ are fermionic annihilation operators and $e_k$, $g_{ij}$, $x$ are constants. The matrix form of the Hamiltonian (62) is the same as in equation (58) if we choose $N = 4$ in the two-particle sector and the basis

$$
|1\rangle \equiv \hat{d}^\dagger \hat{c}_1^\dagger |0\rangle, \quad |2\rangle \equiv \hat{d}^\dagger \hat{c}_2^\dagger |0\rangle, \quad |3\rangle \equiv \hat{d}^\dagger \hat{c}_3^\dagger |0\rangle,
$$

$$
|4\rangle \equiv \hat{c}_1^\dagger \hat{c}_2^\dagger |0\rangle, \quad |5\rangle \equiv \hat{c}_1^\dagger \hat{c}_3^\dagger |0\rangle, \quad |6\rangle \equiv \hat{c}_2^\dagger \hat{c}_3^\dagger |0\rangle.
$$

The Hamiltonian (62) conserves the number of fermions, $N_F$, which we can treat as a free integer parameter of the model. We can also allow $N$ to be an arbitrary positive integer. Generally, we can interpret $N$ as the number of different 'quantum dots' in which spinless
fermions of the model reside. There is a single energy level in each quantum dot so, due to the Pauli principle, maximum one fermion can reside in each dot. One of such quantum dots, which is described by the operator $\hat{d}$, is special: fermions can produce transitions between different states only via this dot, which is coupled to all other dots by tunneling amplitudes $g_k$, $k = 1, \ldots, N - 1$, as shown in figure 9(a) for the case $N = 4$. Moreover, we assume that only the energy of the fermion in this dot is time-dependent, e.g. due to a linearly time-dependent
gate voltage. Parameter $x$ describes the nonlinear effect of coupling of the special dot to other dots. This coupling is activated every time the special dot is occupied. Physical meaning of parameter $e_k$ is merely the energy of a fermion when it resides in the quantum dot with index $k \in 1, \ldots, N - 1$.

Does the Hamiltonian (62) satisfy ICs? In this article, we do not provide the definite answer but we do find strong evidence in support of the conjecture that this model is, indeed, solvable. For $N_F > 1$, there are $n = N!/[N_F!(N - N_F)!]$ different states in which one can place $N_F$ fermions among $N$ sites. For example, at $N = 4$ and $N_F = 2$, there are six states available for dynamics, and we already provided the evidence that this model satisfies ICs and its solution coincides with prediction of the semiclassical ansatz.

Let us next consider the case $N_F = 1$, and look at evolution in the basis of Fock states: $|k\rangle \equiv c_k\dagger|0\rangle$, $|N\rangle \equiv d\dagger|0\rangle$, where $|0\rangle$ is the vacuum state of fermions. The Hamiltonian in this basis has the form

$$
\hat{H}^{(N,1)} = \begin{pmatrix}
e_1 & 0 & \ldots & 0 & g_1 \\
0 & e_2 & 0 & \ldots & g_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & e_{N-1} & g_{N-1} \\
g_1 & g_2 & \ldots & g_{N-1} & \beta t 
\end{pmatrix},
$$

(64)

where the upper index in $\hat{H}^{(N,1)}$ means that the Hamiltonian (62) is restricted to one fermion that can be in $N$ different states. The Hamiltonian (64) is precisely the $N$-state DO model, which is exactly solvable [13] and for which ICs are trivially satisfied.

5.3. Ten-state sector

The next in complexity sector of the model (62) has two fermions that can occupy any of five quantum dots, as shown in figure 9(b). In the basis

$$
|1\rangle \equiv \hat{d}\dagger\hat{c}_1\dagger|0\rangle, \quad |2\rangle \equiv \hat{d}\dagger\hat{c}_2\dagger|0\rangle, \quad |3\rangle \equiv \hat{d}\dagger\hat{c}_3\dagger|0\rangle, \quad |4\rangle \equiv \hat{d}\dagger\hat{c}_4\dagger|0\rangle, \quad |5\rangle \equiv \hat{c}_1\dagger\hat{c}_2\dagger|0\rangle, \\
|6\rangle \equiv \hat{c}_1\dagger\hat{c}_3\dagger|0\rangle, \quad |7\rangle \equiv \hat{c}_1\dagger\hat{c}_4\dagger|0\rangle, \quad |8\rangle \equiv \hat{c}_2\dagger\hat{c}_3\dagger|0\rangle, \quad |9\rangle \equiv \hat{c}_2\dagger\hat{c}_4\dagger|0\rangle, \quad |10\rangle \equiv \hat{c}_3\dagger\hat{c}_4\dagger|0\rangle,
$$

(65)

the Hamiltonian (62) is a $10 \times 10$ matrix, which we will not show here explicitly. Diabatic levels of this model are shown in figure 10. The first IC is trivial but tedious to verify, so we will skip this step here. In figure 11(a), we show adiabatic energies of this Hamiltonian that we calculated numerically at $x = 2.5$. One can visually see that there are twelve exact crossing points in this figure, as it is needed to satisfy IC (ii).

Here we note that not all exact crossings in figure 11(a) are robust in the sense that by increasing couplings some of the crossings can annihilate with each other. However, as it was discussed in [12], IC (ii) requires presence of a specific number of exact crossing points only at finite but sufficiently small values of all couplings.

5.3.1. Transition probabilities at $x < 1$. If we restrict attention only to values $x < 1$, topology of the graph of diabatic states is the same as at $x = 0$; hence we can safely set $x = 0$ in order to determine the transition probability matrix. The point $x = 0$ corresponds to the noninteracting case with a quadratic fermionic Hamiltonian. In the Heisenberg’s picture, evolution of operators has then the same form as evolution of amplitudes in the DO model [3]. Consider the DO model with $N$ levels marked so that the level with index $N$ has the positive slope and
levels with indexes $i = 1, \ldots, N - 1$ belong to the band of parallel levels with diabatic energies $e_1 < e_2 < \ldots < e_{N-1}$. Solution of the Heisenberg equation for annihilation operators can then be written in the form

$$\hat{c}_k (+\infty) = \sum_{l=1}^{N} S_{kl} \hat{c}_l (-\infty),$$

where we identify $\hat{d} \equiv \hat{c}_N$. Transition amplitudes in the DO model are known [3, 8, 13]:

$$S_{NN} = \left( \prod_{k=1}^{N-1} p_k \right)^{1/2},$$

$$S_{nN} = i \left( q_n \prod_{k=1}^{n-1} p_k \right)^{1/2},$$

$$S_{Nn} = i \left( q_n \prod_{k=n+1}^{N-1} p_k \right)^{1/2},$$

$$S_{nm}^{(n>m)} = - \left( q_n q_m \prod_{k=m+1}^{n-1} p_k \right)^{1/2},$$

$$S_{nm}^{(n<m)} = 0,$$

Figure 9. (a) Six different states defined in (63) that can be produced by two fermions residing in four different quantum dots. Filled/empty circles correspond to occupied/empty quantum dots; links correspond to allowed transitions between quantum dots when only one of those dots is occupied. (b) Ten different states defined in (65) that can be produced by distributing two fermions among five different quantum dots. State numbering corresponds to our choice of diabatic state indexes.
where we dropped the coupling-dependent phases of amplitudes except the imaginary unit factors that are acquired at every turning to another level. It was discussed in [11] that all other phase factors cancel in semiclassical calculations of transition probabilities in all known exactly solvable models, and we assume this is happening here too. We do not pursue the mathematical proof of this fact because we check the final result numerically.

Consider a sector of such a model with \( M \) fermions. Let indexes \( \gamma_1, \ldots, \gamma_M \) correspond to quantum dots that are initially populated with fermions at \( t \to -\infty \), and let \( \alpha_1, \ldots, \alpha_M \) be indexes of the dots populated with fermions at \( t \to +\infty \) (\( \alpha_1 < \alpha_2 < \ldots < \alpha_M \) etc.). The transition probability between such two states is the same as the product of average populations of final dots:

\[
P_{\gamma_1, \ldots, \gamma_M \to \alpha_1, \ldots, \alpha_M} = \langle \gamma_1, \ldots, \gamma_M | \hat{c}^{\dagger}_{\alpha_1} (+\infty) \hat{c}_{\alpha_1} (+\infty) \ldots \hat{c}^{\dagger}_{\alpha_M} (+\infty) \hat{c}_{\alpha_M} (+\infty) | \gamma_1, \ldots, \gamma_M \rangle, \tag{72}
\]

where

\[
|\gamma_1, \ldots, \gamma_M \rangle \equiv \hat{a}^{\dagger}_{\gamma_1} (-\infty) \ldots \hat{a}^{\dagger}_{\gamma_M} (-\infty) |0\rangle, \tag{73}
\]

and relation between combined indexes, such as \( \gamma_1, \ldots, \gamma_M \), and numbering of diabatic levels is explained in figure 9 for sectors with six and ten interacting states. Simple algebra leads to the final result:

\[
P_{\gamma_1, \ldots, \gamma_M \to \alpha_1, \ldots, \alpha_M} = |\det(\hat{Q})|^2, \tag{74}
\]

where

\[
\hat{Q} = \begin{pmatrix}
S_{\alpha_1 \gamma_1} & S_{\alpha_1 \gamma_2} & \cdots & S_{\alpha_1 \gamma_M} \\
S_{\alpha_2 \gamma_1} & S_{\alpha_2 \gamma_2} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
S_{\alpha_M \gamma_1} & \cdots & \cdots & S_{\alpha_M \gamma_M}
\end{pmatrix}. \tag{75}
\]

Equations (74) and (75) with (67)–(71) provide a simple formal algebraic solution of the problem at \( x = 0 \) and consequently at any \( x < 1 \).
Physically more useful characteristic can be the average population of the site (quantum dot) $\alpha$ irrespectively of occupations of other states. For independent fermions, this characteristic was previously obtained in [3]:

$$P_{\alpha} = (\gamma_1, \ldots, \gamma_M | \hat{c}^\dagger_{\alpha}(+\infty) \hat{c}_{\alpha}(+\infty) | \gamma_1, \ldots, \gamma_M) = \sum_{j=1}^{M} |S_{\alpha\gamma_j}|^2.$$  

(76)

**Figure 11.** (a) Eigenenergies of the Hamiltonian (62) at $N = 5, N_F = 3$ as functions of $t$. For convenience, a term $-\frac{1}{2} \beta t$ was added to the Hamiltonian, where $I$ is a unit matrix; in our ten-state model it merely equally changes slopes of all levels. Parameters: $x = 2.5, \beta = 1$. $g_1 = 0.44, g_2 = 0.36, g_3 = 0.28, g_4 = 0.34, e_1 = 2.5, e_2 = 1.3, e_3 = -0.6, e_4 = -2.7$. Despite large values of couplings, figure shows twelve exact crossing points of energy levels. (b) and (c) Numerical test of semiclassical predictions for transition probabilities as function of the quartic coupling parameter $x$ for initially occupied (b) level-2, (c) level-3. Parameters: $g_1 = 0.4, g_2 = 0.35, g_3 = 0.45, g_4 = 0.5, e_1 = -0.75, e_2 = -0.25, e_3 = 0.4, e_4 = 1, \beta = 1$. 

Physically more useful characteristic can be the average population of the site (quantum dot) $\alpha$ irrespectively of occupations of other states. For independent fermions, this characteristic was previously obtained in [3]:
So, we can only add here that equation (76) is equally valid for the case with quartic interactions at arbitrary \( x < 1 \).

Let us now restrict to the case \( N = 5 \) and \( N_F = 2 \), which is illustrated in figure 10(a). The initially occupied state 3 in figure 10(a) corresponds to initially occupied sites \( \gamma_1 = 3 \) and \( \gamma_2 = 5 \) in figure 9(b). Finally occupied level 2 in figure 10(a), for example, corresponds to finally occupied quantum dots \( \alpha_1 = 2 \) and \( \alpha_2 = 5 \) in figure 9(b).

For initial state 3 (see figure 9(b)) and arbitrary final state \( n = 1, \ldots, 10 \), equation (74) then predicts the following vector of transition probabilities:

\[
P_{3 \rightarrow n}^{<1} = (q_1q_3p_4, p_1q_3q_4q_5p_4, p_1p_2p_4, 0, 0, q_1p_3, q_1q_3q_4, p_1q_2p_3, p_1q_2q_3q_4, p_1p_2q_4) .
\]  

(77)

Similarly, if initial level is 2 then transition probability vector is

\[
P_{2 \rightarrow n}^{<1} = (q_1q_2p_3q_4, p_1p_3p_4, 0, 0, q_1p_2, q_1q_2q_3, q_1q_2p_3q_4, p_1q_3, p_1p_3q_4, 0) .
\]  

(78)

5.3.2. Transition probabilities for \( x > 1 \) in ten-state sector. We do not have a simple formula for \( x > 1 \). Behavior of transition probabilities in this case is more complex because of the possibility of interference between different semiclassical trajectories. For example, there are two paths that connect initial level 2 and final level 10 in figure 10(b), which we mark by red and blue arrows. So, we have to apply the semiclassical ansatz, which is equivalent to deriving all semiclassical paths that connect pairs of initial and final states in the diabatic level diagram and summing amplitudes of all these paths assuming that contributions from all level crossings to each path amplitude factorize.

According to the semiclassical rules, each time a trajectory turns to another level at a crossing point with some pairwise coupling \( g_k \) it receives the factor \( ±i\sqrt{g_k} \), where \( (±) \) is the sign of the coupling. The amplitude of the trajectory marked by red arrows in figure 10(b) is then

\[
S_{\text{red}} = (-i)i(-i)\sqrt{p_1q_3q_4p_3}.
\]

Similarly, the trajectory marked by blue arrows has the amplitude

\[
S_{\text{blue}} = (-i)\sqrt{p_1p_3q_4q_2q_3}.
\]

Comparing them we find that amplitudes of red and blue trajectories have the same absolute magnitude but different signs. Therefore, the total amplitude is \( S_{10,2} = S_{\text{red}} + S_{\text{blue}} = 0 \). Similarly, we can find other transition probabilities starting with initial level 2:

\[
P_{2 \rightarrow n}^{>1} = (0, p_1p_3p_4, p_1q_3q_2p_4, p_1q_2q_4, q_1, 0, 0, p_1q_3p_4, p_1p_3q_4p_2, 0) .
\]

(79)

If initial state is 3 then there is no path interference and transition probability vector is

\[
P_{3 \rightarrow n}^{>1} = (0, 0, p_1p_2p_4, p_1p_2q_4q_3, 0, q_1, 0, p_1q_2, 0, p_1p_2q_4p_3) .
\]

(80)

We compare predictions of equations (77)–(80) with numerical results in figures 11(b) and (c) and again find perfect agreement. Thus, we verified that ICs are valid for, at least, three simplest sectors of the model (62), and that numerically calculated transition probabilities in these sectors coincide with analytical predictions of the semiclassical ansatz within three significant digit precision for all considered parameter values. Complexity of the phase space of the model (62) is growing very quickly with increasing numbers of fermions and quantum dots, which makes further numerical tests problematic but already available results do support our conjecture that the full model (62) is integrable.
6. Discussion

We demonstrated the strategy to search for new solvable MLZ models. The idea is to start with a graph of some already known solvable model and then distort it keeping only topology of the original graph and allowing a broader choice of parameters. ICs can then be converted to a set of algebraic equations on new parameters. Thus, IC (i) of zero area inside any closed path of a graph leads to constraints on slopes of diabatic levels. Treating couplings as small, one can then derive the necessary conditions that such couplings should satisfy in order to agree with IC (ii), which is achieved with the quantum perturbation theory. Sufficiency of these conditions is tested numerically. Following this path, we constructed several new solvable models of different complexity, and we provided detailed numerical check of our predictions for transition probabilities in these models.

Based on our findings, it seems that solvable MLZ models are much more common than it has been previously expected. By creating symmetrized products of already solved systems or populating these systems with some number of noninteracting bosons and fermions, as it is described in [3], we can generate more complex graphs that, in turn, can be distorted, leading to new integrable MLZ models and so on. Moreover, special cases when some of the diabatic levels become parallel produce additional opportunities to distort parameters without breaking ICs. Such opportunities to find new solvable models using distortions can be combined also with previously developed methods to derive systems with smaller phase space sizes from the bigger models. This can be done either by identifying the bigger model as a composite one [10] or by considering limits with level degeneracies and decoupling some of the states in these limits [12, 18].

While the origin of the MLZ integrability remains puzzling, discoveries of new solvable models can have important consequences. For example, we found that transition probability matrices in newly solved models had the same symmetries as in models that had been rigorously solved by other means. Understanding such relations may lead to the rigorous proof of ICs and validity of the semiclassical ansatz. We also find it surprising that, in all promising candidate models, perturbatively derived constraints on coupling constants have lead to truly exact eigenvalue crossings. Formally, perturbatively derived constraints are only necessary conditions that do not have to be sufficient. It seems that, somehow, imposing IC (i) facilitates the emergence of exact crossing points so that a simple perturbative test, which is performed up to some finite order of perturbation theory, provides not only necessary but also sufficient conditions for existence of such points. This observation is one of already many surprisingly fortunate but puzzling properties of MLZ models.

Exact and approximate methods to connect asymptotic behavior of ordinary differential equations are of general interest to mathematics and mathematical physics [20, 21]. For example, the most physically important special functions are solutions of differential equations with well understood asymptotics. Future studies should answer the question whether solvable MLZ models represent a new class of physically important solvable differential equations or such models can be understood in terms of the already well studied special functions, such as the generalized hypergeometric function.

Finally, some of the solutions that have been obtained with ICs are important for practical reasons. For example, our fermionic model in section 5 describes the process of a quantum level interaction with a fermionic bath when this level is driven by a time-dependent gate voltage. Such models have been of interest previously due to the possibility for control of quantum states with minimal dissipation [3, 22]. So, with MLZ solutions, it becomes possible to explore complex explicitly driven quantum mechanical systems analytically and exactly.
We hope that many other physically interesting processes will be eventually understood with such solvable models.

Acknowledgment

This work was carried out under the auspices of the National Nuclear Security Administration of the US Department of Energy at Los Alamos National Laboratory under Contract No. DE-AC52-06NA25396. VYC was supported by the National Science Foundation under Grant No. CHE-1111350. NAS also thanks the support from the LDRD program at LANL.

Appendix A. Counterexample to previous version of the first IC

Our definition of the first integrability condition in section 2.2 is different from the original definition in [1]. Previously, condition (i) requested that if, on the diabatic level diagram, there are more than one semiclassical paths that connect one initial level at \( t = -\infty \) with another level at \( t = +\infty \) then dynamic phases, i.e. integrals of diabatic energies over time, of such trajectories should be the same. The latter property is actually the consequence of IC (i) defined in the present article because the dynamic phase is the area that a trajectory in the level diagram sweeps over the time axis. Equality of such phases means that the path that goes forward along one of these trajectories and then returns to the initial point along the other trajectory encloses zero area in this diagram.

The new definition, however, is more restrictive because it includes the case when different trajectories do not interfere in the physical sense but the graph still has closed loops. We show an example of such a model in figure A1(a). One can verify numerically that IC (ii) is satisfied when

\[
x = \sqrt{(E - e)/(E + e)},
\]

i.e. at such relations among parameters there is an exact crossing point of adiabatic levels near the intersection of levels 1 and 2. As for the first IC (i), it is not satisfied at \( e \neq 0 \) because the graph has a closed path that encloses a finite area. So, semiclassical ansatz should fail.

On the other hand, each semiclassically allowed trajectory in figure A1(a) is unique in the sense that there is no other path that respects causality and connects the same pair of levels, i.e. there are no physically interfering trajectories. So, if definition in [1] is generally true then the semiclassical ansatz should apply.

Figure A1(b) compares prediction of the semiclassical ansatz for this model with results of numerical simulations. Although the semiclassical ansatz gives a reasonable fit to the data, there is noticable disagreement. This makes us to conclude that the more restrictive definition of condition (i) in section 2.2 of the present article should be preferred. So far, we are not aware of any MLZ model that would satisfy new version of ICs (i)–(ii) but would not be solvable by the semiclassical ansatz.

Appendix B. Analogy between first integrability condition and WKB description of classically integrable systems

Here we would like to point that the geometric interpretation of the first integrability condition, presented in section 2.2, may provide a link between the integrable MLZ problems and semiclassical WKB description of classically integrable systems. Consider the graph
$G = (G_0, G_1)$ associated with the diabatic level diagram of a given $n \times n$ MLZ problem. The vertices in $G_0$ of this graph are given by those intersection points for which the coupling between the intersecting diabatic levels is non-zero, and the edges in $G_1$ represent the segments of the straight lines between two consecutive scattering events. Let $X$ be the geometric realization of the graph $G$, i.e. $X$ stands for the graph that is viewed as a set of segments $s_e$, associated with its edges $e \in G_1$, connected to each other at the vertices, rather than just a collection $G = (G_0, G_1)$ of edges and vertices. By definition, every point of $X$ defines diabatic energy $\omega(t)$ of a diabatic state represented by this point. Hence, the geometric space $X$ of our MLZ problem has a natural immersion $f : X \hookrightarrow \mathbb{R}^2$, which is what is shown in the diabatic level diagram.

In the $(t, \omega)$ plane we have a rank 1 form $\alpha = \omega dt$, which, being restricted (strictly speaking pulled-back) to $X$ gives rise to a form $f^* \alpha$ on $X$. The first integrability condition (area rule) is then equivalent to the requirement that for any cycle $C$ in $X$ we have

$$\int_C f^* \alpha = 0, \quad (B.1)$$

which, due to the quasi-one-dimensional nature of $X$ is equivalent to existence of a global function $S : X \rightarrow \mathbb{R}$ with $dS = f^* \alpha$, or, stated differently, $S = \int f^* \alpha$. We will refer to $S$ as the action function.

For 1-dimensional quantum systems, the WKB approximation is based on the action function that can be represented as $S = \int \alpha$, with $\alpha = pdx$. The WKB form $\psi(x) \sim \exp \left( i \hbar^{-1} S(x) \right)$ of an eigenstate may not be generalizable to the multi-dimensional case, however if an $n$-dimensional system is integrable, i.e. it has $n$ mutually commuting integrals $I_j$ of motion, a multi-dimensional counterpart of a WKB wave function can be obtained. Indeed, switching to the classical limit and further fixing the values of all integrals of motion $I_j(p,x) = E_j$, where $E_j$ are some constants for $j = 1, \ldots, n$, we can solve these equations with respect to $p$, resulting in the set of functions $p_j(x)$, which give rise to a 1-form $\alpha = \sum_j p_j dx^j$. It is possible to show

---

**Figure A1.** (a) Diabatic level diagram of the model that breaks condition (i) as defined in this article but satisfies ICs defined in [1]. (b) Numerical test showing that the semiclassical ansatz (solid curves) does not fit results of numerical simulations (discrete points). Parameters: $E = 1, g = 0.5, \gamma = 0.75 g$, and $x$ is given by equation (A.1). Visible agreement for $P_{4\rightarrow 2}$ is not precise either: systematic deviations from the semiclassical ansatz were found in the third significant digit (not shown).

N A Sinitsyn and V Y Chernyak

J. Phys. A: Math. Theor. 50 (2017) 255203
that the integrability condition $\{ I_j, I_k \} = 0$, where $\{ \ldots \}$ stands for Poisson brackets for all $j$ and $k$, is equivalent to the closeness condition $d\alpha = 0$, the latter meaning that one can define a global action function $S = \int \alpha$ that determines the zero-order approximation for a semiclassical WKB function that has exactly the same form as in 1-dimensional case.

Summarizing the analogy: in both cases, namely MLZ problems and quantized integrable systems, the integrability condition, at least in the lowest (principle) order of the semiclassical expansion, boils down to the existence of a globally defined action function $S$. We would like to admit explicitly that at this point we do not see how exactly the above analogy can be applied. It may mean, for example, that integrable MLZ systems describe some ballistic, i.e. large kinetic energy, limits of integrable many-body 1-dimensional quantum systems. Finally, we note that this analogy resembles the property, found at least in some MLZ models, that solvable MLZ Hamiltonians have nontrivial commuting operators that depend polynomially on time [5].

References

[1] Sinitsyn N A 2015 J. Phys. A: Math. Theor. 48 195305
[2] Majorana E 1932 Nuovo Cimento 9 43
[3] Sinitsyn N A 2002 Phys. Rev. B 66 205303
[4] Dziarmaga J 2005 Phys. Rev. Lett. 95 245701
Volkov M V and Ostrovsky V N 2007 Phys. Rev. A 75 022105
[5] Patra A and Yuzbashyan E 2015 J. Phys. A: Math. Theor. 48 245303
[6] Galitski V 2011 Phys. Rev. A 84 012118
[7] Mukherjee V, Montangero S and Fazio R 2016 Phys. Rev. A 93 062108
Deffner S, Jarzynski C and del Campo A 2014 Phys. Rev. X 4 021013
[8] Brundobler S and Elser V 1993 J. Phys. A: Math. Gen. 26 1211
[9] Chen Y-A, Huber S D, Trotzky S, Bloch I and Altman E 2011 Nat. Phys. 7 61
Damski B, Quan H T and Zurek W H 2011 Phys. Rev. A 83 062104
Kiselev M N, Kikoin K and Pennmoe M B 2013 Europhys. Lett. 104 57004
Wubs M, Saito K, Kohler S, Hänggi P and Kayanuma Y 2006 Phys. Rev. Lett. 97 200404
Wernsdorfer W et al 2000 Europhys. Lett. 50 552
[10] Sinitsyn N A, Lin J and Chernyak V Y 2017 Phys. Rev. A 95 012140
[11] Sinitsyn N A 2015 Phys. Rev. B 92 205431
[12] Sinitsyn N A and Li F 2016 Phys. Rev. A 93 063859
[13] Demkov Yu N and Osherov V I 1967 Zh. Exp. Teor. Fiz. 53 1589
Demkov Yu N and Osherov V I 1968 Sov. Phys.—JETP 26 916
Rangelov A A, Piilo J and Vitanov N V 2005 Phys. Rev. A 72 053404
[14] Fai L C, Tchofo M and Jipidi M N 2015 Eur. Phys. J. B 88 181
[15] Nakamura H 2012 Nonadiabatic Transition 2nd edn (Singapore: World Scientific)
Child M S 2010 Molecular Collision Theory (New York: Dover)
Nikitin E E and Umanskii S Y 2011 Theory of Slow Atomic Collisions (Springer Series in Chemical Physics) (Berlin: Springer) (book 30)
[16] Carroll C E and Hioe F T 1986 J. Phys. A: Math. Gen. 19 2061
[17] Ostrovsky V N and Nakamura H 1997 J. Phys. A: Math. Gen. 30 6939
[18] Demkov Y N and Ostrovsky V N 2001 J. Phys. B: At. Mol. Opt. Phys. 34 2419
Demkov Y N and Ostrovsky V N 2000 Phys. Rev. A 61 032705
[19] Sinitsyn N A 2004 J. Phys. A: Math. Gen. 37 10691
[20] Joye A 1997 SIAM J. Math. Anal. 28 669
[21] Aoki T, Kawai T and Takei Y 2002 J. Phys. A.: Math. Gen. 35 2401
[22] Keeling J, Shytov A V and Levitov L S 2008 Phys. Rev. Lett. 101 196404