FOKKER-PLANCK EQUATION FOR DISSIPATIVE 2D EULER EQUATIONS WITH CYLINDRICAL NOISE

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Abstract. After a short review of recent progresses in 2D Euler equations with random initial conditions and noise, some of the recent results are improved by exploiting a priori estimates on the associated infinite dimensional Fokker-Planck equation. The regularity class of solutions investigated here does not allow energy- or enstrophy-type estimates, but only bounds in probability with respect to suitable distributions of the initial conditions. This is a remarkable application of Fokker-Planck equations in infinite dimensions. Among the example of random initial conditions we consider Gibbsian measures based on renormalized kinetic energy.

1. Introduction

This paper is devoted to existence of solutions to the 2-dimensional stochastic Euler equation

\[
\frac{d\omega}{dt} + u \cdot \nabla \omega = -\alpha \omega + \sqrt{2\alpha} \, dW,
\]

where \( u \) is the divergence-less velocity field and \( \omega = \nabla_\perp \cdot u \) the scalar vorticity field, \( \nabla_\perp = (\partial_2, -\partial_1) \). The equation includes a friction term and space-time additive white noise forcing. As a preliminary, we prove an existence result for the associated Fokker-Planck equation: this becomes necessary since solutions to (1.1) under cylindrical white noise forcing exist only in certain distributional spaces where classical energy or enstrophy estimates are not available. Such estimates are thus replaced by probabilistic estimates, taking averages with respect to the solution of the Fokker-Planck equation. We believe this to be a remarkable application of recent techniques developed for Fokker-Planck equations in infinite-dimensional spaces, a topic that has received considerable attention in recent years.

Before we go into technical details it may be convenient to recall the present state of the art on some classes of stochastic partial differential equations (SPDEs) in fluid mechanics and some of the main open questions. The literature on the topic is enormous and we shall discuss only a very small portion of it, neglecting for instance the recent important contributions to compressible models, see for instance [17]. In the more classical incompressible case there are various reviews, like [53, 3, 32, 51]. The three chief open research directions in deterministic incompressible fluid mechanics deal with:

- i) well-posedness;
- ii) inviscid limits;
- iii) turbulence.

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Probability has obvious relations with turbulence, whereas it is less clear how much it can be related to the former two. Problem (ii) has been essentially left untouched by stochastic methods, despite it being quite promising. As for problem (i), a huge effort has been devoted to the attempt at extending and improving the deterministic theory by means of probability and stochastic models. The present work stems from the study of models proposed to describe features of turbulence such as the inverse energy cascade in dimension 2, see [16, 45], and it may fit into the framework of question (i), since it extends the class of function spaces in which Euler equation is solved.

The most important open problems in class (i) consist in well-posedness of basic deterministic equations. Obviously, the outstanding one is well-posedness of 3-dimensional Navier-Stokes equations; we refer to [29] for the statement of the problem in the occasion of the millennium prize declaration. Since all equations discussed in what follows are inviscid, we do not discuss this fundamental problem much further. However, we mention that in sight of striking well-posedness results for stochastic ordinary differential equations (SDEs) with very irregular drift and additive noise such as [60, 50], there exist the general belief that suitably non-degenerate additive noise may regularize several classes of differential equations, providing for instance uniqueness results in cases where the deterministic equation may not have a unique solution. In infinite dimensions there are important examples of such results, see for instance [22, 23, 24]. However, the drift terms in those works is still far from the irregularity and unboundedness of the inertial term of 3D Navier-Stokes equations, and requirements on the noise restrict applications to parabolic equations, involving the Laplacian operator, in dimension $d = 1$. The strategy of those papers consists in solving directly the infinite dimensional Kolmogorov equation associated to the SPDE. In the case of 3D Navier-Stokes equations the corresponding Kolmogorov equation has been solved in [25] but the regularity of solutions is not sufficient to deduce uniqueness results of weak solutions to the stochastic 3D Navier-Stokes equations. That research however was not without interesting consequences; among others, the existence of global in time Markov selections with the Strong-Feller property — a striking continuous dependence on initial conditions — which has no deterministic counterpart in the theory of 3D Navier-Stokes equations, see [40].

In the inviscid, incompressible class, the main open problems concern the 3-dimensional Euler equations: only local results are known, except for special notions of solutions, see [55, 56, 28]. Such equations represent a too difficult task for a first stage understanding of regularization by noise. Let us thus discuss the simpler case of the 2-dimensional Euler equations on the torus $T^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ in vorticity form:

\[
\begin{align*}
\frac{\partial}{\partial t} \omega + u \cdot \nabla \omega &= 0, \\
\text{div } u &= 0, \\
\omega &= \nabla^\perp u.
\end{align*}
\]

When the initial condition $\omega|_{t=0}$ is bounded measurable, a celebrated result of Jukov, see [48], establishes the existence of a unique solution. The result has been extended to additive noise (regular in space) in [11] and to multiplicative transport type stochastic perturbations in [18].
When the regularity of the initial condition $\omega|_{t=0}$ is decreased, say to $L^p(\mathbb{T}^2)$, $p \in (1, \infty)$, global existence can still be proved with arguments based on the formal conservation of the $L^p$-norm of $\omega$; however, uniqueness is open, see [55] for a discussion. It is therefore natural to try stochastic approaches to restore uniqueness below the class $L^\infty$: unfortunately we still do not know whether there exists a noise, either additive or multiplicative, that might do so. This and other closely related open problems originated a considerable amount of research: several attempts have been made to prove that suitable multiplicative transport type noises — a natural choice in inviscid problems due to its conservation properties — regularize first order, transport type PDEs. The case of linear transport equations has been understood quite well, see for instance [30, 57, 32, 39, 7]. The nonlinear case is much more difficult, and only fragmentary results are available: point vortex solutions to the 2D Euler equations are regularized [31]; for dyadic models and their generalizations on trees [6], uniqueness holds thanks to multiplicative noise [4, 13] and a variant of the same technique applied also to a 3D Leray $\alpha$-model [5]; Hamilton-Jacobi equations [41] and scalar conservation laws [42] are also regularized by suitable multiplicative noise, although not of transport type.

The well-posedness problem (i) has another aspect which received attention, both in the seventies and again more recently due to progresses made for nonlinear dispersive equations: extending the existence theory to distributional classes of vorticity fields $\omega$, for the 2D Euler equations (1.2). The motivation is twofold: to understand the limits of PDE theory in terms of roughness of solutions, and to establish a rigorous set-up for investigation of certain explicit invariant measures of Gibbs type (often just Gaussian), which are supported only on such distributional spaces. We recall that invariant measures are of potential interest for turbulence theory, see for instance the classical work [49], thus all explicit examples deserve attention.

The first works in this direction are reviewed in [3]. A basic result is [2], proving existence of stationary solutions of equations (1.2) in the negative order Sobolev space $H^{-1-\delta}(\mathbb{T}^2)$, $\delta > 0$; the time marginal is the 2-dimensional space white noise, also known as Enstrophy measure in this context, or the Energy-Enstrophy Gibbs measure (which we review in section 6). This theory was recently revised by means of an alternative approach based on point vortex approximation [33]. These works, devoted to the deterministic equation (1.2) with random initial conditions, have been generalized to stochastic cases, on one hand to the case of multiplicative transport noise, see in particular [36, 34, 38]; on the other hand to the case of additive space-time white noise and friction [45] (multiplicative noise is formally conservative, while in the case of additive noise a friction is needed to allow stationary solutions). The 2D Euler equations with additive noise, possibly including friction, their corresponding stationary solutions and invariant measures had already been considered before. However, the space regularity of noise is such that solutions are function-valued, not distributions and invariant measures are supported on spaces of functions: we refer for instance to [11, 19, 8, 12, 21, 44, 9, 43, 10], and also to other related results in [52, 53, 32]. Many of those models and results are inspired by the open problem of turbulence (iii); in connection with this question and the previous references we also mention [14, 46, 37].

In the stream of aforementioned works on stochastic 2D Euler equations with damping, but in the specific regime of distributional solutions — corresponding to
the case of cylindrical white noise — investigated by [45], we extend here that result from stationary to non-stationary solutions, with random initial conditions related to a Gaussian invariant measure. As already remarked, this extension is based on preliminary estimates on the associated Fokker-Planck equation, where some technical aspects are inspired by recent works on a different kind of noise, see [36]. Using the method of Galerkin approximation, we shall prove existence of solutions $\rho_t$ to the Fokker-Planck equation with initial data $\rho_0$ which are $L \log L$-integrable with respect to the white noise invariant measure $\mu$ of (1.1). In the case $\alpha > 0$, the relative entropy of these solutions decrease exponentially fast as $t$ grows to $\infty$; this together with an inequality of Kullback [54] implies the convergence to equilibrium of the solutions we constructed. In the case $\rho_0 \in L^2(\mu)$, we also have exponential convergence of $\rho_t$ in $L^2$-norm. These results put forward a difficult question that we will not treat here, namely the search for a notion of uniqueness and ergodicity of the invariant measure $\mu$, and convergence to equilibrium of the non-stationary solutions. Among the non-stationary initial conditions of special physical interest there is the Energy-Enstrophy Gibbs measure associated to the renormalized energy investigated by [46] and previous works, see [3].

This paper is organized as follows. In Section 2 we first recall the definition of the nonlinear term in the weak formulation of the Euler equation and the Fokker-Planck equation, and then state our main results: Theorems 2.4–2.6. The proofs of these results are given in Sections 3–5. In the last section, we discuss the example which takes the Energy-Enstrophy Gibbs measure as the initial condition.

2. Notation and Main Results

Consider the stochastic dissipative Euler equation in vorticity form on the 2-dimensional torus $T^2 = \mathbb{R}^2/(2\pi \mathbb{Z})^2$:

\begin{equation}
\begin{aligned}
\frac{d\omega}{dt} + u \cdot \nabla \omega &= -\alpha \omega dt + \sqrt{2\alpha} dW, \\
\nabla \cdot u &= \omega,
\end{aligned}
\end{equation}

where $\omega$ has zero space average on $T^2$. The latter gauge choice will be assumed throughout this paper, thus all function spaces on $T^2$ are tacitly assumed to have zero averaged elements. We shall henceforth relate the velocity field $u$ to vorticity $\omega$ by the Biot-Savart law inverting the second equation in (2.1):

(2.2) \quad u = u(\omega) = K * \omega, \quad K = -\nabla^\perp G,

with $G$ the (zero averaged) Green function of the Laplacian operator. We recall that $G(x, y) = G(x - y)$ is a translation invariant function, smooth in all points except the origin 0, where it has a logarithmic singularity. As a consequence, $K(x, y) = K(x-y)$ is a translation invariant vector field, exploding at 0 with the speed $|x|^{-1}$.

The forcing term $dW$ is the space-time white noise on $T^2$, so that our model coincides with the one studied in [45], where existence of weak (both in probabilistic and analytic sense) stationary solutions were proved by approximation with a system of Euler point vortices with creation and quenching.

The space-time white noise, a centred delta-correlated Gaussian field, is equivalently understood as the cylindrical Wiener process on $L^2(T^2)$, see [27]. The stationary fixed time marginal considered in [45] is the unique invariant measure of the
linear part of the equation
\begin{equation}
\frac{dZ}{dt} = -\alpha Z \, dt + \sqrt{2\alpha} \, dW,
\end{equation}
that is, the space white noise measure on $\mathbb{T}^2$, often referred to in this context as the \textit{enstrophy measure}. This is due to the fact that it can be realized as the Gaussian measure $\mu$ on the abstract Wiener space $(H^{1-\delta}(\mathbb{T}^2), L^2(\mathbb{T}^2))$ (any $\delta > 0$), where the inner product of the Cameron-Martin space $L^2(\mathbb{T}^2)$ coincides with the quadratic form associated to \textit{enstrophy}, $S(\omega) = \frac{1}{2} \int_{\mathbb{T}^2} \omega^2 \, dx$, which is (formally) a first integral of the 2-dimensional Euler equation.

In what follows, brackets $\langle \cdot, \cdot \rangle$ will denote the $L^2(\mathbb{T}^2)$ inner product or more generally the $L^2(\mathbb{T}^2)$-based duality coupling between distributions and functions on $\mathbb{T}^2$. In order to lighten notation, let us fix $\delta > 0$ and denote $\mathcal{E} = H^{1-\delta}(\mathbb{T}^2)$. Moreover, we denote $\eta$ the isonormal Gaussian process on $L^2(\mathbb{T}^2)$, which is the space white noise on $\mathbb{T}^2$, $\mu$ is its law.

2.1. The Euler nonlinear term in the Gaussian setting. The definition of the nonlinear term in (2.1) when the law of $\omega_\tau$ is $\mu$, or an absolutely continuous measure with respect to $\mu$, is not immediate, and it has been thoroughly discussed in [33] and related works, [35, 26, 45]. We will rely upon the arguments of Subsection 2.5 of [33], which we now review.

Were the stochastic processes $\omega_\tau$ and $W_\tau$ smooth (both in time and space), the differential formulation of (2.1) would be equivalent to the (analytically) weak, integral formulation: for test functions $\phi \in C^\infty(\mathbb{T}^2)$,
\begin{equation}
\langle \omega_\tau - \omega_0, \phi \rangle - \int_0^\tau \langle (K * \omega_s) \omega_s, \nabla \phi \rangle \, ds = -\alpha \int_0^\tau \langle \omega_s, \phi \rangle \, ds + \sqrt{2\alpha} \langle W_\tau, \phi \rangle.
\end{equation}
Since the kernel $K$ is skew-symmetric, we have
\begin{equation}
\langle (K * \omega_s) \omega_s, \nabla \phi \rangle = \langle \omega_s \otimes \omega_s, H_\phi \rangle, \quad H_\phi(x, y) := \frac{\nabla \phi(x) - \nabla \phi(y)}{2} \cdot K(x - y).
\end{equation}
The key fact here is that $H_\phi$ is a bounded symmetric function, smooth outside the diagonal set $\{ (x, x) \in \mathbb{T}^2 \times \mathbb{T}^2 \}$, where it has a jump discontinuity: this is easily seen by means of Taylor expansion. A lengthy but elementary computation in Fourier series reveals that the Sobolev regularity of $H_\phi$ is at best $H^{2-}(\mathbb{T}^2 \times \mathbb{T}^2)$, thus the above symmetrized formulation, dating back to the works of Delort and Schochet, see [58], allows us to give a proper meaning to (2.1) in the case when $\omega_\tau \in H^{-1+}(\mathbb{T}^2)$.

Here comes into play the essential contribution of Gaussian distributions.

\textbf{Proposition 2.1.} Let $\phi \in C^\infty(\mathbb{T}^2)$ and $\omega$ be a random distribution on $\mathbb{T}^2$ with law $\rho \, d\mu$, $\rho \in L^p(\mu)$ for some $p > 1$. For any sequence $(H^n_\phi)_{n \in \mathbb{N}} \subset C^\infty(\mathbb{T}^2 \times \mathbb{T}^2)$ of symmetric functions such that
\begin{equation}
L^2(\mathbb{T}^2 \times \mathbb{T}^2) - \lim_{n \to \infty} H^n_\phi = H_\phi,
\end{equation}
\begin{equation}
\lim_{n \to \infty} \int_{\mathbb{T}^2} H^n_\phi(x, x) \, dx = 0,
\end{equation}
the limit
\begin{equation}
\langle \omega \otimes \omega, H_\phi \rangle := \lim_{n \to \infty} \langle \omega \otimes \omega, H^n_\phi \rangle
\end{equation}
exists in $L^1(\mu)$ and it does not depend on the approximating sequence $H^n_\phi$ among the ones satisfying the above properties. Moreover,

\begin{equation}
(2.8) \quad \mathbb{E}\left[\left|\langle \omega \otimes \omega, H^n_\phi - H_\phi \rangle\right|\right] \leq C_p \|H^n_\phi - H_\phi\|^{1/p'}_{L^p(T^2 \times T^2)} + \int_{T^2} H^n_\phi(x, x) \, dx,
\end{equation}

with $\frac{1}{p} + \frac{1}{p'} = 1$, and for any $q \in [1, \infty)$ it holds

\begin{equation}
(2.9) \quad \mathbb{E}\left[\|\langle \omega \otimes \omega, H_\phi \rangle\|^q\right] \leq C_q \|\rho\|_{L^p(E, \mu)} \|\phi\|^q_{C^2(T^2)},
\end{equation}

with $C_q$ a constant depending only on $q$.

If $\rho_t \in L^\infty([0, T], L^p(E, \mu))$ and $\omega_t$ is a process with trajectories in $C([0, T], E)$ and marginals $\omega_t \sim \rho_t \, d\mu$ (in particular we are assuming $\int_E \rho_t \, d\mu = 1$ for all $t$), the sequence of real processes $\langle \omega_t \otimes \omega_t, H^n_\phi \rangle$ converges in $L^1(E, \mu; L^1([0, T]))$ to a process $\langle \omega_t \otimes \omega_t, H_\phi \rangle$ which does not depend on the approximations $H^n_\phi$ as above.

It is worth noticing that the approximations $H^n_\phi$ as in (2.5) can always be obtained by regularizing the kernel $K$ in the definition of $H_\phi$. The proof of the above Proposition is detailed in [33, Section 2.5]; (2.8) follows easily from the proof of [33, Theorem 14]. We also remark that if $\omega \sim \mu$, the limit (2.7) coincides with the double Wiener-Itô integral of the kernel $H_\phi$ on the Gaussian Hilbert space $(E, \mu)$ (see [45] for a discussion).

With Proposition 2.1 at hand, we are able to give meaning to (2.1) and the associated Fokker-Planck equation when the law of fixed time marginals is (absolutely continuous with respect to) the space white noise measure $\mu$.

2.2. Weak solutions to Euler and Fokker-Planck Equations. On the torus $\mathbb{T}^2 = \mathbb{R}^2 / (2\pi \mathbb{Z})^2$ we consider the normalized Haar measure $dx$ such that $\int_{\mathbb{T}^2} dx = 1$, and the orthonormal Fourier basis $e_k(x) = e^{i k \cdot x}$ of $L^2(\mathbb{T}^2, dx)$. In fact, we will only deal with real-valued objects: Fourier coefficients of opposite modes will henceforth be complex conjugated. Moreover, we will tacitly assume the zero average setting, that is, 0-th Fourier modes of functions and distributions are always null.

Let $\mathcal{F}_{\mathbb{T}}$ be the linear space of cylinder functions of the form

$$\varphi(\omega) = f(\hat{\omega}_{k_1}, \ldots, \hat{\omega}_{k_n}), \quad k_1, \ldots, k_n \in \mathbb{Z}_0^2,$$

with $n \geq 1$ and $f \in C^\infty_b(\mathbb{R}^n)$, the space of bounded functions with bounded derivatives of all orders. The infinitesimal generator associated to (2.3) is $\alpha \mathcal{L}$, with $\mathcal{L}$ the generator of the Ornstein-Uhlenbeck semigroup acting on cylinder functions as

$$\mathcal{L} \varphi(\omega) = \sum_{i=1}^n \partial_i f(\hat{\omega}_{k_1}, \ldots, \hat{\omega}_{k_n}) - \sum_{i=1}^n \partial_i f(\hat{\omega}_{k_1}, \ldots, \hat{\omega}_{k_n}) \hat{\omega}_{k_i}.$$

The infinitesimal generator associated to (2.1) can be written formally as

\begin{equation}
(2.10) \quad \mathcal{A} \varphi(\omega) = B \varphi(\omega) + \alpha \mathcal{L} \varphi(\omega), \quad B \varphi(\omega) = -\langle (K * \omega) \cdot \nabla \omega, D \varphi(\omega) \rangle,
\end{equation}

whose action on cylinder functions $\varphi \in \mathcal{F}_{\mathbb{T}}$ is given in terms of $\mathcal{L}$ defined above and

$$D \varphi(\omega) = \sum_{i=1}^n \partial_i f(\hat{\omega}_{k_1}, \ldots, \hat{\omega}_{k_n}) e_{k_i}.$$
To give a rigorous definition of the Liouville operator $\mathcal{B}$ of Euler equation (1.2), we make use of Proposition 2.1 (see also the discussion in [26]). First, we combine the latter two expressions with (2.4) to obtain

$$
\mathcal{B}\varphi(\omega) = -\sum_{i=1}^{n} \partial_{i} f(\hat{\omega}_{k_{1}}, \ldots, \hat{\omega}_{k_{n}}) \left\langle \left( K * \omega \right) \cdot \nabla \omega, \epsilon_{k_{i}} \right\rangle ,
$$

$$
= \sum_{i=1}^{n} \partial_{i} f(\hat{\omega}_{k_{1}}, \ldots, \hat{\omega}_{k_{n}}) \left\langle \omega \otimes \omega, H_{e_{k_{i}}} \right\rangle ;
$$

hence, by Proposition 2.1, we can define the real random variable $\mathcal{B}\varphi(\eta) \in L^{1}(\mu)$ for all cylinder functions $\varphi \in \mathcal{FC}_{b}$. As already observed above, $\langle \eta \otimes \eta, H_{\phi} \rangle$ is in fact an element of the second Wiener chaos of the Gaussian process $\eta$, since it coincides with the double Itô-Wiener integral. As a consequence, $\mathcal{B}$ is exponentially integrable when acting on cylinder functions:

$$
E[\exp(\varepsilon|\mathcal{B}\varphi(\eta)|)] < \infty \quad \text{for all small } \varepsilon > 0
$$

(see [26, Theorem 8] for an explicit computation).

The singularity of the nonlinear term is such that the operator $\mathcal{B}$, regarded as a vector field acting as a derivation on the whole $E$, does not take values in the Cameron-Martin space $L^{2}(\mathbb{T}^{2})$, or even in $E$, see [2]. Nonetheless, it formally holds $\text{div}_{\mu} \mathcal{B} = 0$, in agreement with the fact that there exist stationary solutions of Euler equation, and thus of (2.1), with invariant measure $\mu$ (see section 6 below). We also notice that $\mathcal{B}$ is skew-symmetric on $\mathcal{FC}_{b}$, as revealed by direct computation.

Let us consider the Fokker-Planck equation associated to (2.1):

$$
\begin{cases}
\partial_{t}\rho = A^{*} \rho - \mathcal{B}\rho + \alpha L\rho, \\
\rho|_{t=0} = \rho_{0}.
\end{cases}
$$

**Definition 2.2.** Given $\rho_{0} \in L^{1}(\mu)$, for any $\alpha \geq 0$, we say that $\rho \in L^{1}_{\text{loc}}(\mathbb{R}_{+}, L^{1}(E, \mu))$ is a weak solution of the Fokker-Planck equation (2.12) if

(a) for any $\varphi \in \mathcal{FC}_{b}$ and $T > 0$,

$$
\int_{0}^{T} \int_{E} |\rho_{t} A\varphi| \, d\mu \, dt < \infty;
$$

(b) for any $f \in C_{c}^{1}(\mathbb{R}_{+})$ and $\varphi \in \mathcal{FC}_{b}$ it holds

$$
f(0) \int_{E} \rho_{0} \varphi \, d\mu + \int_{0}^{\infty} \int_{E} f'(t) \rho_{t} \varphi \, d\mu \, dt + \int_{0}^{\infty} \int_{E} f(t) \rho_{t} A\varphi \, d\mu \, dt = 0.
$$

**Remark 2.3.** Identity (2.13) implies that, in the distributional sense,

$$
\frac{d}{dt} \int_{E} \rho_{t} \varphi \, d\mu = \int_{E} \rho_{t} A\varphi \, d\mu \quad \text{for a.e. } t \in (0, \infty).
$$

Since the right-hand side is locally integrable in $t \in (0, \infty)$, the map $[0, \infty) \ni t \mapsto \int_{E} \rho_{t} \varphi \, d\mu$ is absolutely continuous, thus $\rho_{t}$ is weakly continuous in time. This also gives meaning to the initial condition specification $\rho|_{t=0} = \rho_{0}$. Moreover, taking $\varphi \equiv 1$ yields $\int_{E} \rho_{t} \, d\mu = \int_{E} \rho_{0} \, d\mu$ for all $t > 0$.

Our first aim is to prove existence results of the Fokker-Planck equation with general initial conditions, using a Galerkin type scheme.
Theorem 2.4. Let $\rho_0 \in L \log L(E, \mu; \mathbb{R}_+)$ and $\alpha \geq 0$. Then,

(i) there exists a weak solution $(\rho_t)_{t \in \mathbb{R}_+}$ of the Fokker-Planck equation (2.12) in the sense of Definition 2.2;

(ii) for almost every $t > 0$ it holds

$$\int_E \rho_t \log \rho_t \, d\mu \leq e^{-2\alpha t} \int_E \rho_0 \log \rho_0 \, d\mu + (1 - e^{-2\alpha t}) \|\rho_0\|_{L^1} \log \|\rho_0\|_{L^1}.$$

In particular, if $\rho_0$ is a probability density and $\alpha > 0$, then the relative entropy of the weak solution $\rho_t$ decreases exponentially fast, which in turn implies the convergence to equilibrium of $\rho_t$: for almost every $t > 0$ it holds

$$\|\rho_t - 1\|_{L^1} \leq e^{-\alpha t} \sqrt{2 \int_E \rho_0 \log \rho_0 \, d\mu}.$$

The last assertion is an immediate consequence of the exponential decay of entropy and Kullback’s inequality, see [54, (11)]. Next, we deduce also an existence result for $L^p (p > 1)$ initial densities. We state it explicitly since it will play an important role in building solutions to the stochastic equation (2.1).

Theorem 2.5. Let $\rho_0 \in L^p(E, \mu)$ with $p > 1$ and $\alpha \geq 0$. Then,

(i) there exists a weak solution $\rho \in L^\infty \left( \mathbb{R}_+, L^p(E, \mu) \right)$ to Fokker-Planck equation (2.12) in the sense of Definition 2.2;

(ii) if $p = 2$, then, denoting by $\bar{\rho}_0 = \int_E \rho_0 \, d\mu$, we have, for a.e. $t > 0$,

$$\|\rho_t - \bar{\rho}_0\|_{L^2} \leq e^{-\alpha t} \|\rho_0 - \bar{\rho}_0\|_{L^2}.$$

Our second result is the existence of weak (both in probabilistic and analytical sense) solutions to the Euler equation (2.1) in the setting of Theorem 2.5.

Theorem 2.6. Let $p > 1$, $\alpha \geq 0$, $T > 0$. Assume that $\rho_0 \in L^p(E, \mu; \mathbb{R}_+)$ is a probability density, and let $\rho \in L^\infty \left( [0, T]; L^p(E, \mu) \right)$ be a weak solution obtained in Theorem 2.5 to Fokker-Planck equation (2.12) with initial datum $\rho_0$. There exist a filtered probability space on which a cylindrical Wiener process $W$ on $L^2(T^2)$ and an adapted process $\omega_t$ are defined such that

(i) $\omega \in C([0, T], E)$ with probability one;

(ii) for almost every $t \in [0, T]$, $\omega_t$ has law $\rho_t$ on $\mu$;

(iii) for any $\phi \in C^\infty(T^2)$ and $t \in [0, T]$,

$$\langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \langle \omega_s \otimes \omega_s, H\phi \rangle \, ds - \alpha \int_0^t \langle \omega_s, \phi \rangle \, ds + \sqrt{2\alpha} \langle W_t, \phi \rangle,$$

where the nonlinear term is defined as in Proposition 2.1.

3. The Galerkin approximation and $L \log L$ initial data

Let us define the finite-dimensional projection of $H = L^2(T^2, dx)$ onto the finite set of modes $\Lambda_N = \{ k \in \mathbb{Z}_0^2 : |k|_\infty \leq N \}$,

$$\Pi_N : H \ni f \mapsto \Pi_N f = \sum_{k \in \Lambda_N} \langle f, e_k \rangle_H e_k \in H_N,$$

where we can identify the finite dimensional codomain with

$$H_N = \{ \xi \in \mathbb{C}^{\Lambda_N} : \xi_k = \xi_{-k} \}.$$
(whose dimension is $|\Lambda_N|$). On $H_N$ we consider the Euclidean inner product induced by $\mathbb{C}^{\Lambda_N}$, and the Gaussian measure $\mu_N$ having Fourier coefficients $\hat{\mu}_N(k) = \hat{\mu}_N(-k)$ with the law of independent standard complex Gaussian distributions.

We consider the following Galerkin approximation of (2.1):

\begin{equation}
(3.3) \quad d\Pi_N \omega + \Pi_N((K * \Pi_N \omega) \cdot \nabla \Pi_N \omega) dt = -\alpha \Pi_N \omega \, dt + \sqrt{2\alpha} d\Pi_N W.
\end{equation}

This equation is in fact an SDE in $\omega^N \in H_N$, and it can be rewritten as

\begin{equation}
(3.4) \quad d\omega^N + b_N(\omega^N) \, dt = -\alpha \omega^N \, dt + \sqrt{2\alpha} \, dW^N, \quad W^N = \sum_{k \in \Lambda_N} W^k e_k,
\end{equation}

where the $W^k$’s are independent standard complex Brownian motions such that $\overline{W}^k = W^{-k}$, and the drift is given by

\[ b_N(\xi) = -\sum_{n \in \Lambda_N} e_n \sum_{k \in \Lambda_N} \frac{k^2}{|k|^2} \xi_n \xi_{n-k}, \quad \xi \in H_N, \]

as one can prove by a straightforward computation in Fourier series using that $K(x) = \sum_{k \in \mathbb{Z}^d} \frac{k^2}{|k|^4} e_k(x)$. By means of the above expression, it is easy to check that, for all $\xi \in H_N$,

\begin{equation}
(3.5) \quad \langle b_N(\xi), \xi \rangle_{H_N} = 0, \quad \text{div}_{\mu_N} b_N(\xi) = \text{div} b_N(\xi) - \langle b_N(\xi), \xi \rangle_{H_N} = 0.
\end{equation}

The SDE (3.4) has smooth coefficients, so there exists a unique strong local solution $\omega^N_t$ given an initial datum $\omega^N_0 \in H_N$; the forthcoming estimate shows that it is also global in time.

**Lemma 3.1.** If $\omega^N_t$ is a solution of (3.4), then, for any $t \geq 0$,

\[ \mathbb{E} \left[ |\omega^N_t|_{H_N}^2 \right] \leq |\omega^N_0|_{H_N}^2 e^{-2\alpha t} + |\Lambda_N| (1 - e^{-2\alpha t}). \]

**Proof.** By the Itô formula and (3.5), and omitting all subscripts $H_N$,

\[ d |\omega^N_t|^2 = -2 \langle \omega^N_t, b_N(\omega^N_t) \rangle + \alpha |\omega^N_t|^2 \, dt + 2\sqrt{2\alpha} \, \langle \omega^N_t, dW_t^N \rangle + \alpha \, \langle dW_t^N, dW_t^N \rangle. \]

and therefore

\[ d \left( e^{2\alpha t} |\omega^N_t|^2 \right) = 2\sqrt{2\alpha} e^{2\alpha t} \, \langle \omega^N_t, dW_t^N \rangle + 2\alpha e^{2\alpha t} |\Lambda_N| \, dt. \]

If we define, for $R > 0$, the stopping time

\[ \tau_R = \inf \left\{ t > 0 : |\omega^N_t| \geq R \right\}, \]

then we have

\[ \mathbb{E} \left[ e^{2\alpha (\tau_R \wedge T)} |\omega^N_{\tau_R}|^2 \right] = |\omega^N_0|^2 + 2\sqrt{2\alpha} \mathbb{E} \left[ \int_0^{\tau_R \wedge T} e^{2\alpha s} \, \langle \omega^N_s, dW_s^N \rangle \right] \]

\[ + |\Lambda_N| \mathbb{E} \left( e^{2\alpha (\tau_R \wedge T)} - 1 \right) \]

\[ \leq |\omega^N_0|^2 + |\Lambda_N| \left( e^{2\alpha T} - 1 \right), \]

which concludes the proof if we let $R \uparrow \infty$ by Fatou’s lemma. \qed
3.1. Finite dimensional Fokker-Planck equation. Let $\mathcal{L}_N$ be the Ornstein-Uhlenbeck operator on $H_N$; then $\alpha \mathcal{L}_N$ is the infinitesimal generator of the linear part of (3.4). We can introduce the Galerkin approximation $\mathcal{A}_N$ of $\mathcal{A}$, acting on smooth functions $F \in C_b^2(H_N)$ as

$$
\mathcal{A}_N F(\xi) = -\langle b_N(\xi), \nabla F(\xi) \rangle_{H_N} + \alpha \mathcal{L}_N F(\xi).
$$

We can thus write the Fokker-Planck equation corresponding to (3.4): if the law of $\omega_0^N$ has a smooth probability density $\rho_0^N$ (with respect to $\mu_N$), so does $\omega_t^N$ for any later time, and the density $\rho_t^N$ satisfies

$$
\begin{cases}
\begin{aligned}
\partial_t \rho_t^N &= \mathcal{A}_N^* \rho_t^N, \\
\rho_t^N |_{t=0} &= \rho_0^N.
\end{aligned}
\end{cases}
$$

Remark 3.2. Simple heuristic arguments immediately give rise to an a priori estimate on the entropy of $\rho_t^N$. Indeed, if $\rho_t^N$ is a smooth solution of (3.7), for any $t \geq 0$,

$$
\partial_t \left( \rho_t^N \log \rho_t^N \right) = (1 + \log \rho_t^N) \partial_t \rho_t^N
$$

$$
= (1 + \log \rho_t^N) \langle b_N, \nabla \rho_t^N \rangle_{H_N} + \alpha (1 + \log \rho_t^N) \mathcal{L}_N \rho_t^N.
$$

Integrating on $H_N$ with respect to $\mu_N$ and using (3.5) we get

$$
\int_{H_N} \rho_t^N \log \rho_t^N \, d\mu_N + \alpha \int_0^t \int_{H_N} \frac{1}{\rho_s^N} |\nabla \rho_s^N|^2 \, d\mu_N \, ds = \int_{H_N} \rho_0^N \log \rho_0^N \, d\mu_N.
$$

However, the above computation is somewhat formal, since the drift $b_N$ has quadratic growth. In the following we give a more rigorous proof of the a priori estimate, and at the same time give a meaning to the equation (3.7).

In the remainder of this subsection, we fix $N \in \mathbb{N}$ and assume that the initial condition of (3.7) belongs to

$$
\rho_0^N \in L^\infty(H_N, \mathbb{R}_+).
$$

One can extend the result below to more general initial data, but since the study of (3.7) is only an intermediate step, we do not pursue such generality here. Consider cut-off functions $\chi_n(\xi) = \chi(\xi/n)$, $n \geq 1$, where $\chi \in C_c^\infty(H_N, [0, 1])$ is a radial function (i.e., $\chi(\xi) = \chi(|\xi|_{H_N})$) by a slight abuse of notation) such that $\chi|_{B_N(1)} \equiv 1$ and $\chi|_{B_N(2)} \equiv 0$, $B_N(r)$ being the ball in $H_N$ centered at the origin with radius $r > 0$. Define

$$
b_n^N(\xi) = \chi_n(\xi) b_N(\xi), \quad \xi \in H_N, n \in \mathbb{N};
$$

then $b_n^N$ is a smooth vector field on $H_N$ with compact support for any $n \in \mathbb{N}$. Notice that $b_n^N$ is still divergence-free since by (3.5) and $\nabla \chi_n(\xi) = \chi'(\xi |n|) \frac{\xi}{|\xi|}$ one has

$$
\text{div}_{\mu_N} (b_n^N) = \text{div}_{\mu_N} (\chi_n b_N) = \chi_n \text{div}_{\mu_N} (b_N) - \langle b_N, \nabla \chi_n \rangle_{H_N} = 0.
$$

Now we consider the approximating operators

$$
\mathcal{A}_N^{(n)} F(\xi) = -\langle b_n^N(\xi), \nabla F(\xi) \rangle_{H_N} + \alpha \mathcal{L}_N F(\xi)
$$

and the corresponding Fokker-Planck equations

$$
\begin{cases}
\begin{aligned}
\partial_t \rho_t^{(n)} &= (\mathcal{A}_N^{(n)})^* \rho_t^{(n)}, \\
\rho_t^{(n)} |_{t=0} &= \rho_0^{(n)} = \rho_0^N.
\end{aligned}
\end{cases}
$$
where the initial datum is regularized by means of the Ornstein-Uhlenbeck semigroup $P_t^N = e^{tL_N}$ on $H_N$: for $t \geq 0$ the latter is explicitly given by

\begin{equation}
(3.11) \quad P_t^N \rho_0^N (\xi) = \int_{H_N} \rho_0^N (\eta) \left[ 2\pi (1 - e^{-2t}) \right]^{-|\lambda_N|/2} \exp \left( - \frac{|\eta - e^{-t}\xi|^2}{2(1 - e^{-2t})} \right) d\eta.
\end{equation}

**Lemma 3.3.** For any $n \geq 1$, $\rho_0^{(n)} \in C_b^\infty (H_N, \mathbb{R})$ and

\begin{equation}
(3.12) \quad \int_{H_N} \rho_0^{(n)} \log \rho_0^{(n)} d\mu_N \leq \int_{H_N} \rho_0^N \log \rho_0^N d\mu_N.
\end{equation}

Moreover, the solutions $\rho_t^{(n)}$ of the equations (3.10) satisfy

\begin{equation}
(3.13) \quad \sup_{t \geq 0} \left\| \rho_t^{(n)} \right\|_\infty \leq \left\| \rho_0^N \right\|_\infty,
\end{equation}

\begin{equation}
(3.14) \quad \int_{H_N} \rho_t^{(n)} \log \rho_t^{(n)} d\mu_N \leq e^{-2\alpha t} \int_{H_N} \rho_0^N \log \rho_0^N d\mu_N + \left( 1 - e^{-2\alpha t} \right) \left\| \rho_0^N \right\|_{L^1(\mu_N)} \log \left\| \rho_0^N \right\|_{L^1(\mu_N)} \quad \forall t \geq 0.
\end{equation}

**Proof.** The first assertion follows from (3.8) and (3.11); the estimate (3.12) is a consequence of Jensen’s inequality and the invariance of $\mu_N$ for the semigroup $(P_t^N)_{t \geq 0}$.

Inequality (3.13) follows from (3.8) and the representation

\[ \rho_t^{(n)}(\xi) = E \left[ \rho_0^{(n)}(X_t^{(n)}) \right], \]

where $X_t^{(n)}$ is the solution to the SDE

\[ dX_t^{(n)} = b_N^{(n)} (X_t^{(n)}) dt - \alpha X_t^{(n)} dt + \sqrt{2\alpha} dW_t^N, \quad X_0^{(n)} = \xi. \]

Thanks to (3.9), the arguments in Remark 3.2 are now rigorous and we have

\begin{equation}
(3.15) \quad \frac{d}{dt} \int_{H_N} \rho_t^{(n)} \log \rho_t^{(n)} d\mu_N = -\alpha \int_{H_N} \left\| \nabla \rho_t^{(n)} \right\|_{L^1(\mu_N)}^2 d\mu_N.
\end{equation}

Recall the log-Sobolev inequality on the finite-dimensional Gaussian space $(H_N, \mu_N)$:

\[ \int_{H_N} \varphi^2 \log \left\| \varphi \right\|_{L^2(\mu_N)}^2 d\mu_N \leq 2 \int_{H_N} |\nabla \varphi|^2 d\mu_N, \quad \forall \varphi \in W^{1,2}(H_N, \mu_N). \]

Taking $\varphi = (\rho_t^{(n)})^{1/2}$ yields

\[ \int_{H_N} \rho_t^{(n)} \log \rho_t^{(n)} \left\| \rho_t^{(n)} \right\|_{L^1(\mu_N)} d\mu_N \leq \frac{1}{2} \int_{H_N} \left\| \nabla \rho_t^{(n)} \right\|_{L^1(\mu_N)}^2 d\mu_N. \]

Combining the latter inequality with (3.15) we obtain

\[ \frac{d}{dt} \int_{H_N} \rho_t^{(n)} \log \rho_t^{(n)} d\mu_N \leq -2\alpha \int_{H_N} \rho_t^{(n)} \log \rho_t^{(n)} d\mu_N + 2\alpha \left\| \rho_0^N \right\|_{L^1(\mu_N)} \log \left\| \rho_0^N \right\|_{L^1(\mu_N)}, \]

where we have used the fact that

\[ \left\| \rho_t^{(n)} \right\|_{L^1(\mu_N)} = \left\| \rho_0^{(n)} \right\|_{L^1(\mu_N)} = \left\| \rho_0^N \right\|_{L^1(\mu_N)} \quad \forall t > 0. \]
Integrating in time, we conclude that
\[
\int_{H_N} \rho_t^{(n)} \log \rho_t^{(n)} \, d\mu_N \leq e^{-2\alpha t} \int_{H_N} \rho_0^{(n)} \log \rho_0^{(n)} \, d\mu_N + (1 - e^{-2\alpha t}) \|\rho_0^N\|_{L^1(\mu_N)} \log \|\rho_0^N\|_{L^1(\mu_N)},
\]
which, together with (3.12), leads to the final result. \(\square\)

**Corollary 3.4.** Let \(\rho_0^N \in L^\infty(H_N, \mathbb{R}^+).\) There exists a nonnegative function \(\rho^N \in L^\infty(\mathbb{R}^+, L^\infty(H_N, \mu_N))\) satisfying
\[
\sup_{t \in [0, \infty)} \|\rho_t^N\|_{L^\infty(\mu_N)} \leq \|\rho_0^N\|_{L^\infty(\mu_N)},
\]
\[
\int_{H_N} \rho_t^N \log \rho_t^N \, d\mu_N \leq e^{-2\alpha t} \int_{H_N} \rho_0^N \log \rho_0^N \, d\mu_N + (1 - e^{-2\alpha t}) \|\rho_0^N\|_{L^1(\mu_N)} \log \|\rho_0^N\|_{L^1(\mu_N)}
\]
for almost every \(t > 0\); moreover, for any \(f \in C^1_b(\mathbb{R}^+)\) and \(\psi \in C^\infty_b(\mathbb{R})\),
\[
0 = f(0) \int_{H_N} \psi \rho_0^N \, d\mu_N + \int_0^\infty \int_{H_N} \rho_t^N \left[ f'(t) \psi + f(t) \langle b_N, \nabla \psi \rangle_{H_N} + \alpha f(t) \mathcal{L}_N \psi \right] \, d\mu_N \, dt.
\]
In particular, the above equation shows that \(\rho^N\) satisfies (3.7) in a weak sense.

**Remark 3.5.** It might be possible to give a strong (i.e. pointwise) meaning to (3.7), but the weak sense here, combined with the estimate (3.17), is enough to prove existence of solutions to Fokker-Planck equations in the infinite dimensional case.

**Proof.** Thanks to (3.13), we can find a subsequence \(\{\rho^{(n)}\}_{i \in \mathbb{N}}\) weakly-* converging in \(L^\infty(\mathbb{R}^+, L^\infty(H_N, \mu_N))\) to some \(\rho^N\) satisfying (3.16).

Now we fix any \(T > 0\). We know that \(\rho^{(n)}\) also converges weakly in \(L^1([0, T] \times H_N)\) to \(\rho^N\). The sequence \(\{\rho^{(n)}\}_{i \in \mathbb{N}}\) is contained in the set
\[
\mathcal{S} = \left\{ u \in L^1([0, T] \times H_N) : u_t \geq 0, \int_{H_N} u_t \log u_t \, d\mu_N \leq \Lambda(t) \text{ for all } t \in [0, T] \right\},
\]
where we write \(\Lambda(t)\) for the right hand side of (3.17). The convexity of the function \(s \mapsto s \log s\) implies that \(\mathcal{S}\) is a convex subset of \(L^1([0, T] \times H_N)\). Since the weak closure of \(\mathcal{S}\) coincides with the strong one, there exists a sequence of functions \(u^{(n)} \in \mathcal{S}\) which converge strongly to \(\rho^N\) in \(L^1([0, T] \times H_N)\). Up to a subsequence, \(u^{(n)}\) converge to \(\rho^N\) almost everywhere, thus Fatou's lemma and (3.14) implies that (3.17) holds for a.e. \(t \in (0, T)\). The arbitrariness of \(T > 0\) implies that it holds for a.e. \(t \in (0, \infty)\).

Finally, multiplying both sides of (3.10) (with \(n\) replaced by \(n_i\)) by \(f \in C^1_b(\mathbb{R}^+)\) and \(\psi \in C^\infty_b(\mathbb{R})\), and integrating by parts leads to
\[
0 = f(0) \int_{H_N} \psi \rho_0^{(n)} \, d\mu_N + \int_0^\infty \int_{H_N} \rho_t^{(n)} \left[ f'(t) \psi + f(t) \langle b^{(n)}_N, \nabla \psi \rangle_{H_N} + \alpha f(t) \mathcal{L}_N \psi \right] \, d\mu_N \, dt.
\]
Recall that \( b_N^{(n)} = \chi_n b_N \); it is clear that \( \left< b_N^{(n)}, \nabla \psi \right>_H^N \) converges strongly to \( (b_N, \nabla \psi)_H^N \) in \( L^2(\mu_N) \). By the weak-* convergence of \( \rho_i^{(n)} \), letting \( i \to \infty \) yields (3.18).

\[\square\]

### 3.2. Proof of Theorem 2.4.

We assume that \( \rho_0 \in L \log L(E, \mu; \mathbb{R}_+) \). Define

\[(3.19) \quad \rho_0^N = P_{t/\mu}^N \mathbb{E} [\rho_0 \wedge N | \Pi_N], \quad N \in \mathbb{N},\]

where \( \mathbb{E}[\cdot | \Pi_N] \) is the conditional expectation with respect to the sub-\( \sigma \)-algebra generated by coordinates in \( H_N \). Note that, for any \( f \in L^1(\mu) \) and all \( N \geq 1 \), we can regard \( \mathbb{E}[f | \Pi_N] \) as a function on \( E \). By the invariance of \( \mu_N \) for the Ornstein-Uhlenbeck semigroup \( \{P_t^N\}_{t \geq 0} \) and Jensen’s inequality,

\[
\int_{H_N} \rho_0^N \log \rho_0^N \, d\mu_N \leq \int_{H_N} \mathbb{E} [\rho_0 \wedge N | \Pi_N] \log \mathbb{E} [\rho_0 \wedge N | \Pi_N] \, d\mu_N
= \int_E \mathbb{E} [\rho_0 \wedge N | \Pi_N] \log \mathbb{E} [\rho_0 \wedge N | \Pi_N] \, d\mu.
\]

Using again Jensen’s inequality, for all \( N \in \mathbb{N} \),

\[(3.20) \quad \int_{H_N} \rho_0^N \log \rho_0^N \, d\mu_N \leq \int_E (\rho_0 \wedge N) \log (\rho_0 \wedge N) \, d\mu \leq \int_E \rho_0 \log \rho_0 \, d\mu.
\]

Moreover, it is easy to see that

\[(3.21) \quad \|\rho_0^N\|_{L^1(\mu_N)} \leq \|\rho_0\|_{L^1(\mu)}.
\]

For any \( N \geq 1 \), taking \( \rho_0^N \) as the initial value, by the arguments in the last subsection, we have a nonnegative solution \( \rho^N \) to the finite dimensional Fokker-Planck equation (3.18) which verifies (3.17). We shall regard the solutions as functions on \( E = H^{-1-\delta}(\mathbb{T}^2) \), i.e. \( \rho_t^N(\omega) = \rho_t^N(\Pi_N \omega), (t, \omega) \in \mathbb{R}_+ \times E \). Then, combining (3.17) with (3.20) and (3.21), for a.e. \( t > 0 \),

\[(3.22) \quad \int_E \rho_t^N \log \rho_t^N \, d\mu \leq e^{-2\alpha t} \int_E \rho_0 \log \rho_0 \, d\mu + (1 - e^{-2\alpha t}) \|\rho_0\|_{L^1(\mu)} \log \|\rho_0\|_{L^1(\mu)}.
\]

From this estimate and a diagonal argument, there exist a subsequence \( \{\rho_t^{N_i}\}_{i \geq 1} \) and some function \( \rho : \mathbb{R}_+ \times E \to \mathbb{R}_+ \) such that, for any \( T > 0 \), \( \rho_t^{N_i} \) converges weakly in \( L^1(0, T; L^1(E, \mu)) \) to \( \rho \), and for a.e. \( t > 0 \),

\[
\int_E \rho_t \log \rho_t \, d\mu \leq e^{-2\alpha t} \int_E \rho_0 \log \rho_0 \, d\mu + (1 - e^{-2\alpha t}) \|\rho_0\|_{L^1(\mu)} \log \|\rho_0\|_{L^1(\mu)}.
\]

The proof is similar to that of Corollary 3.4. Moreover, by the duality of Orlicz spaces, one has, for any \( T > 0 \),

\[
\lim_{i \to \infty} \int_0^T \int_E G(t, \omega) \rho_t^{N_i}(\omega) \, d\mu dt = \int_0^T \int_E G(t, \omega) \rho_t(\omega) \, d\mu dt
\]

for any \( G \) such that, for some small \( \varepsilon > 0 \),

\[(3.23) \quad \sup_{t \in [0, T]} \int_E e^{\varepsilon G(t, \omega)} \, d\mu dt < +\infty.
\]
Fix any cylindrical function $\psi$ and $f \in C_c^0(\mathbb{R}_+)$, for $N$ big enough we always have the equation (3.18); replacing $N$ by $N_i$, it can be rewritten as

$$0 = f(0) \int_E \psi_0^N d\mu + \int_0^\infty \int_E \rho_t^N \left[ f'(t)\psi + f(t) \langle b_{N_i}, D\psi \rangle + \alpha f(t) L\psi \right] d\mu dt.$$ 

By the definition (3.19), it is not difficult to show that, for any cylindrical $\psi$,

$$\lim_{i \to \infty} \int_E \psi \rho_0^N d\mu = \int_E \psi \rho_0 d\mu.$$ 

Moreover, the first and the third terms in the second integral also converge to the corresponding limits. The only term that requires our attention is the nonlinear part. We have

$$\left| \int_0^\infty \int_E \rho_t^N f(t) \langle b_{N_i}, D\psi \rangle d\mu dt - \int_0^\infty \int_E \rho_t f(t) \langle B, D\psi \rangle d\mu dt \right| \leq \left| \int_0^\infty \int_E \rho_t^N f(t) \left( \langle b_{N_i}, D\psi \rangle - \langle B, D\psi \rangle \right) d\mu dt \right| + \left| \int_0^\infty \int_E (\rho_t^N - \rho_t) f(t) \langle B, D\psi \rangle d\mu dt \right|.$$ 

By (2.11), $G(t, \omega) := f(t) \langle B, D\psi \rangle$ satisfies (3.23). Thus, the second term on the right hand side tends to 0 as $i \to \infty$. Next, one can prove that $\langle b_{N_i}, D\psi \rangle$ converge strongly in $L^1(E, \mu)$ to $\langle B, D\psi \rangle$ as $i \to \infty$, see for instance [26, Section 3.3.1]. Combining the convergence with the uniform exponential integrability of these quantities, we deduce that the sequence $\langle b_{N_i}, D\psi \rangle$ actually converges to $\langle B, D\psi \rangle$ in the Orlicz norm. Therefore, by (3.22), the first term also vanishes as $i \to \infty$. Thus, we can let $i \to \infty$ in the above equality to get the equation

$$0 = f(0) \int_E \psi \rho_0 d\mu + \int_0^\infty \int_E \rho_t \left[ f'(t)\psi + f(t) \langle B, D\psi \rangle + \alpha f(t) L\psi \right] d\mu dt.$$ 

Therefore, $\rho_t$ solves the Fokker-Planck equation (2.12) for $L \log L$ initial condition. The proof of Theorem 2.4 is complete.

4. $L^p$-initial data

In this section we assume the initial data of the Fokker-Planck equation (2.12) to be integrable of order $p > 1$. In this case, we can follow the arguments in the last section to prove the existence of weak solutions to the Fokker-Planck equations (2.12). Here we only prove new a priori estimates on the Galerkin approximations and the exponential convergence in $L^2(\mu)$ norm in the case $p = 2$.

4.1. A priori estimates for $p > 1$. Assume first $\rho_0^N \in L^\infty(H_N, \mu_N)$ and consider as above the Fokker-Planck equation (3.10):

$$\begin{cases}
\partial_t \rho_t^{(n)}(x) &= (A_N^{(n)})^* \rho_t^{(n)}(x), \\
\rho_t^{(n)}(x)|_{t=0} &= \rho_0^{(n)} = P_{1/n}^N \rho_0^N.
\end{cases}$$ 

Jensen’s inequality implies

$$\int_{H_N} |\rho_t^{(n)}|^p d\mu_N \leq \int_{H_N} |\rho_0^N|^p d\mu_N \quad \text{for all } n \geq 1,$$
and we can extend this bound for all subsequent times.

**Lemma 4.1.** For any \( n \in \mathbb{N} \), it holds that

\[
\int_{H_N} |\rho_t^{(n)}|^p \, d\mu_N \leq \int_{H_N} |\rho_0^{(n)}|^p \, d\mu_N \quad \text{for all } t > 0.
\]

**Proof.** Using equation (3.10),
\[
\partial_t \left[ |\rho_t^{(n)}|^p \right] = p \left( \rho_t^{(n)} \right)^{p-1} \partial_t \rho_t^{(n)}
\]

\[= b_N^{(n)} \cdot \nabla \left[ |\rho_t^{(n)}|^p \right] + p\alpha \left( \rho_t^{(n)} \right)^{p-1} \mathcal{L}_N \rho_t^{(n)}.
\]

Integrating by parts on \( H_N \) with respect to \( \mu_N \) gives us
\[
\frac{d}{dt} \int_{H_N} \left[ (\rho_t^{(n)})^p \right] \, d\mu_N = -p\alpha \int_{H_N} (\rho_t^{(n)})^{p-2} |\nabla \rho_t^{(n)}|^2 \, d\mu_N.
\]

Next, integrating in time between 0 and \( t \) leads to
\[
\int_{H_N} \left[ \rho_t^{(n)} \right]^p \, d\mu_N \leq \int_{H_N} \left[ \rho_0^{(n)} \right]^p \, d\mu_N,
\]

which, together with (4.1), yields the desired estimate. \( \square \)

As a consequence, \( \{\rho^{(n)}\}_{n \geq 1} \) is bounded in \( L^\infty(\mathbb{R}_+, L^p(H_N, \mu_N)) \). Thus we can find a subsequence which converges weakly-* to some limit
\[
\rho^N \in L^\infty(\mathbb{R}_+, L^p(H_N, \mu_N)),
\]

satisfying the estimate
\[
\sup_{t \in \mathbb{R}_+} \int_{H_N} |\rho_t^N|^p \, d\mu_N \leq \int_{H_N} |\rho_0^N|^p \, d\mu_N
\]

and the finite dimensional Fokker-Planck equation
\[
0 = f(0) \int_{H_N} \psi \rho_0^N \, d\mu_N
\]

\[+ \int_0^\infty \int_{H_N} \rho_t^N \left[ f'(t) \psi + f(t) \langle b_N, \nabla \psi \rangle_{H_N} + \alpha f(t) \mathcal{L}_N \psi \right] \, d\mu_N \, dt
\]

for any \( \psi \in C^\infty_c(H_N) \) and \( f \in C^1_c(\mathbb{R}_+) \).

Next, if \( \rho_0 \in L^p(E, \mu) \), we define, for \( N \in \mathbb{N} \),
\[
\rho_0^N = P_{1/N}^N \mathbb{E} \left[ (-N) \vee (\rho_0 \wedge N) \cdot |\Pi_N| \right],
\]

which, by Jensen’s inequality, satisfies
\[
\sup_{N \geq 1} \int_{H_N} |\rho_0^N|^p \, d\mu_N \leq \int_E |\rho_0|^p \, d\mu.
\]

Consider the finite dimensional Fokker-Planck equations (4.3) with initial data \( \rho_0^N \), and regard the solutions \( \rho_t^N \) as functions on \( E \). From estimate (4.2) and inequality (4.5) we deduce
\[
\sup_{N \geq 1} \sup_{t \in \mathbb{R}_+} \int_E |\rho_t^N|^p \, d\mu \leq \int_E |\rho_0|^p \, d\mu.
\]
Hence, we can find a subsequence $\rho^N_n$ converging weakly-* in $L^\infty(\mathbb{R}_+, L^p(E, \mu))$ to some $\rho$, which can be shown to satisfy the Fokker-Planck equation (2.12), thus completing the proof of point (i) of Theorem 2.5. We omit the details.

### 4.2. The case $p = 2$. We want to show the exponential decay of the energy, proving point (ii) of Theorem 2.5. We start again from equation (3.10) with the initial condition $\rho_0^{(n)} = P^N_{1/n} \tilde{\rho}_0^N$, where $\tilde{\rho}_0^N \in L^\infty(H_N)$. It is clear that for all $n \geq 1$,

$$\bar{\rho}_0^{(n)} := \int_{H_N} \rho_0^{(n)} d\mu_N = \int_{H_N} \tilde{\rho}_0^N d\mu_N =: \bar{\rho}_0^N.$$  

**Lemma 4.2.** It holds that

$$\int_{H_N} (\rho_t^{(n)} - \bar{\rho}_0^N)^2 d\mu_N \leq e^{-2\alpha t} \int_{H_N} (\rho_0^N - \bar{\rho}_0^N)^2 d\mu_N \quad \text{for all } t > 0.$$

**Proof.** According to equation (3.10), we have

$$\partial_t \left[ (\rho_t^{(n)} - \bar{\rho}_0^N)^2 \right] = 2(\rho_0^{(n)} - \bar{\rho}_0^N) b_t^{(n)} \cdot \nabla \rho_t^{(n)} + 2\alpha (\rho_t^{(n)} - \bar{\rho}_0^N) \mathcal{L}_N \rho_t^{(n)}.$$

By (3.9), integrating by parts with respect to $\mu_N$ yields

$$\frac{d}{dt} \int (\rho_t^{(n)} - \bar{\rho}_0^N)^2 d\mu_N = -2\alpha \int \left| \nabla \rho_t^{(n)} \right|^2 d\mu_N.$$

Recall that $\mu_N$ satisfies the Poincaré inequality on $H_N$: for any $\varphi \in W^{1,2}(H_N, \mu_N)$,

$$\int (\varphi - \bar{\varphi})^2 d\mu_N \leq \int \left| \nabla \varphi \right|^2 d\mu_N,$$

where $\bar{\varphi} = \int \varphi d\mu_N$. Therefore,

$$\frac{d}{dt} \int (\rho_t^{(n)} - \bar{\rho}_0^N)^2 d\mu_N \leq -2\alpha \int (\rho_t^{(n)} - \bar{\rho}_0^N)^2 d\mu_N,$$

where we used the fact that $\rho_t^{(n)} := \int \rho_t^{(n)} d\mu_N = \bar{\rho}_0^{(n)} = \bar{\rho}_0^N$ for all $t > 0$. As a result,

$$\int (\rho_t^{(n)} - \bar{\rho}_0^N)^2 d\mu_N \leq e^{-2\alpha t} \int (\rho_0^{(n)} - \bar{\rho}_0^N)^2 d\mu_N \quad \text{for all } t > 0.$$

Finally, we complete the proof by noting that

$$\int (\rho_0^{(n)} - \bar{\rho}_0^N)^2 d\mu_N = \int (\rho_0^{(n)} - \bar{\rho}_0^N)^2 d\mu_N - (\rho_0^N)^2 \leq \int (\rho_0^N)^2 d\mu_N - (\rho_0^N)^2 = \int (\rho_0^N - \bar{\rho}_0^N)^2 d\mu_N,$$

where we have used Jensen’s inequality in the second step.

Repeating the arguments below Lemma 4.1, there exists a subsequence $\rho^{(n)}$ converging weakly-* to some $\rho^N \in L^\infty(\mathbb{R}_+, L^2(H_N, \mu_N))$, which is a weak solution to the finite dimensional Fokker-Planck equations (4.3) with the initial datum $\rho_0^N$. Moreover, replacing the set $S$ in the proof of Corollary 3.4 by

$$\tilde{S} = \left\{ u \in L^2([0, T] \times H_N) : \|u_t - \bar{\rho}_0^N\|_{L^2(\mu_N)} \leq e^{-\alpha t} \|\rho_0^N - \bar{\rho}_0^N\|_{L^2(\mu_N)} \quad \forall t \in [0, T] \right\},$$

similar discussions imply that for a.e. $t \in (0, T)$, one has

$$\|\rho_t^N - \bar{\rho}_0^N\|_{L^2(\mu_N)} \leq e^{-\alpha t} \|\rho_0^N - \bar{\rho}_0^N\|_{L^2(\mu_N)}.$$
The arbitrariness of $T > 0$ yields that the above inequality holds for a.e. $t > 0$.

Next, for $\rho_0 \in L^2(E, \mu)$ and $N \in \mathbb{N}$, we define $\rho_N^0$ as in (4.4). We have

$$\bar{\rho}_0^N = \int_{H_N} \rho_0^N d\mu_N = \int_{H_N} \mathbb{E} \left[ (-N) \vee (\rho_0 \wedge N) \right] d\mu_N = \int_E (-N) \vee (\rho_0 \wedge N) d\mu,$$

therefore,

$$\lim_{N \to \infty} \bar{\rho}_0^N = \int_E \rho_0 d\mu = \bar{\rho}_0.$$

This together with (4.5) (taking $p = 2$) implies

$$\limsup_{N \to \infty} \int_{H_N} (\rho_0^N - \bar{\rho}_0^N)^2 d\mu_N \leq \int_E (\rho_0 - \bar{\rho}_0)^2 d\mu.$$  

(4.6)

For any $N \geq 1$, there exists a weak solution $(\rho_t^N)_{t \in \mathbb{R}_+}$ to the equation (4.3) with the initial condition $\rho_0^N$, satisfying

$$\|\rho_t^N - \rho_0^N\|_{L^2(\mu_N)} \leq e^{-\alpha t} \|\rho_0^N - \bar{\rho}_0^N\|_{L^2(\mu_N)}$$

for a.e. $t \in (0, \infty)$.

As usual, we view $\rho_N^N(N \geq 1)$ as functionals on $E$. As in Section 4.1, there is a subsequence $\rho_{N_i}$ converging weakly-* to some $\rho \in L^\infty(\mathbb{R}_+, L^2(E, \mu))$. By (4.7) and (4.8), we can show the exponential decay of the energy of $\rho_t$ for a.e. $t > 0$.

5. Existence of Weak solutions

Thanks to the control on densities we have gained in the last Section, we are now in the position to prove Theorem 2.6. Let us thus take $\rho_0 \in L^p(E, \mu; \mathbb{R}_+)$ for some $p > 1$, satisfying $\rho_0 = \int_E \rho_0 d\mu = 1$. We define $\rho_0^N$ similarly to (4.4):

$$\rho_N^0 = c_N^{-1} P_{1/N}^N \mathbb{E} \left[ (\rho_0 \wedge N) \right],$$

where $c_N$ is the normalizing constant such that $\bar{\rho}_0^N = \int_{H_N} \rho_0^N d\mu_N = 1$. Clearly,

$$\lim_{N \to \infty} c_N = 1.$$

Let $\rho_t^N$ be the solution of the finite dimensional Fokker-Planck equations (4.3) with initial data $\rho_0^N$. Combining the above fact with (4.6), we see that

$$\sup_{N \geq 1} \sup_{t \in [0, T]} \|\rho_t^N\|_{L^p(\mu)} \leq c_0 \|\rho_0\|_{L^p(\mu)}.$$  

(5.1)

Consider the solution $\omega_t^N$ of the SDEs (3.4), for which the initial values $\omega_0^N$ is distributed as $\rho_0^N \mu_N$; then $\rho_t^N$ is the probability density function (with respect to $\mu_N$) of $\omega_t^N$. In this part we regard $\omega_t^N$ and $\rho_t^N$ as objects defined on $E = H^{-1-\delta/2}$, i.e. $\omega_t^N(\omega) = \omega_t^N(\Pi_N \omega)$, $\rho_t^N(\omega) = \rho_t^N(\Pi_N \omega)$. We want to show that the laws $Q^N$ of $\omega_t^N$ on $C([0, T], E)$ are tight. To this end we will use the compactness criterion proved in [59, Corollary 9, p. 90]. The arguments here follow those of [34, Section 3].

Take $\delta \in (0, 1)$, $\kappa > 5$ (this choice is due to estimates below) and consider the spaces

$$X = H^{-1-\delta/2}(\mathbb{T}^2), \quad B = H^{-1-\delta}(\mathbb{T}^2), \quad Y = H^{-\kappa}(\mathbb{T}^2).$$

Then $X \subset B \subset Y$ with compact embeddings and we also have, for a suitable constant $C > 0$ and for

$$\theta = \frac{\delta/2}{\kappa - 1 - \delta/2},$$

(5.2)
the interpolation inequality
\[ \|\omega\|_B \leq C\|\omega\|^{1-\theta}_X \|\omega\|_{\theta}^\theta, \quad \omega \in X. \]
These are the preliminary assumptions of [59, Corollary 9, p. 90]. We consider here a particular case:
\[ S = L^{p_0}(0,T; X) \cap W^{1/3,4}(0,T; Y), \]
where for \(0 < \alpha < 1\) and \(p \geq 1\),
\[ W^{\alpha,p}(0,T; Y) = \left\{ f : f \in L^p(0,T; Y) \text{ and } \int_0^T \int_0^T \frac{\|f(t) - f(s)\|^p_Y}{|t-s|^{\alpha p+1}} \, dt \, ds < \infty \right\}. \]

**Lemma 5.1.** Let \(\delta \in (0,1)\) and \(\kappa > 5\) be given. If
\[ p_0 > \frac{12(\kappa - 1 - 3\delta/2)}{\delta}, \]
then \(S\) is compactly embedded into \(C([0,T], H^{-1-\delta}(T^2))\).

**Proof.** Recall that \(\theta\) is defined in (5.3). In our case, we have \(s_0 = 0, r_0 = p_0\) and \(s_1 = 1/3, r_1 = 4\). Hence \(s_\theta = (1 - \theta)s_0 + \theta s_1 = \theta/3\) and
\[ \frac{1}{r_\theta} = \frac{1 - \theta}{r_0} + \frac{\theta}{r_1} = \frac{1 - \theta}{p_0} + \frac{\theta}{4}. \]
It is clear that for \(p_0\) given above, it holds \(s_\theta > 1/r_\theta\), thus the desired result follows from the second assertion of [59, Corollary 9]. \(\square\)

For \(N \geq 1\), let \(Q^N\) be the law of \(\omega^N\) on \(\mathcal{X} := C([0,T], H^{-1-\delta}(T^2))\). We want to prove that the family \(\{Q^N\}_{N \geq 1}\) is tight in \(\mathcal{X}\). The next result follows from the definition of the topology in \(\mathcal{X}\).

**Lemma 5.2.** The family \(\{Q^N\}_{N \geq 1}\) is tight in \(\mathcal{X}\) if and only if it is tight in the space \(C([0,T], H^{-1-\delta}(T^2))\) for any \(\delta > 0\).

In view of the above two lemmas, it is sufficient to prove that \(\{Q^N\}_{N \geq 1}\) is bounded in probability in \(W^{1/3,4}(0,T; H^{-\kappa}(T^2))\) and in each \(L^{p_0}(0,T; H^{-1-\delta}(T^2))\) for any \(p_0 > 0\) and \(\delta > 0\).

We show first that the family \(\{Q^N\}_{N \geq 1}\) is bounded in probability on the space \(L^{p_0}(0,T; H^{-1-\delta}(T^2))\). Let us recall that, for any \(q > 1\) and \(\delta > 0\), there exists \(C_{q,\delta} > 0\) such that
\[ \int \|\omega\|_{H^{-1-\delta}}^q \, d\mu \leq C_{q,\delta}. \]
We have
\[
\mathbb{E} \left[ \int_0^T \|\omega_t^N\|_{H^{-1-\delta}}^{p_0} \, dt \right] = \int_0^T \mathbb{E} \left[ \|\omega_t^N\|_{H^{-1-\delta}}^{p_0} \right] \, dt
\leq \int_0^T \int \|\omega\|_{H^{-1-\delta}}^{p_0} \rho_t^N(\omega) \, d\mu \, dt
\leq \int_0^T \left[ \int \|\omega\|_{H^{-1-\delta}}^{pq} \, d\mu \right]^{1/q} \left[ \int (\rho_t^N(\omega))^p \, d\mu \right]^{1/p} \, dt
\leq C_{p_0,q,\delta} T \sup_{t \in [0,T]} \|\rho_t^N\|_{L^p(\mu)} \leq C_{p_0,q,\delta} T \|\rho_0\|_{L^p(\mu)}.
\]
where \( q \) is the conjugate number of \( p \) and we have used the above estimate and (5.2) in the last two steps. By Chebyshev’s inequality, the family \( \{ Q^N \} \) is bounded in probability in \( L^{p_0}(0, T; H^{-1-\delta}(\mathbb{T}^2)) \).

Next, we prove boundedness in probability of \( \{ Q^N \} \) in \( W^{1/3,4}(0, T; H^{-\kappa}(\mathbb{T}^2)) \) where \( \kappa > 5 \). Again by Chebyshev’s inequality, it suffices to show that

\[
\sup_{N \geq 1} \mathbb{E} \left[ \int_0^T \| \omega^n_s - \omega^N_s \|_{H^{-\kappa}}^4 \, dt + \int_0^T \int_0^T \frac{\| \omega^n_t - \omega^N_s \|_{H^{-\kappa}}^4}{|t-s|^{7/3}} \, dt \, ds \right] < \infty.
\]

In view of (5.4), we see that it is sufficient to establish a uniform estimate on the expectation \( \mathbb{E} \| \omega^n_t - \omega^N_s \|_{H^{-\kappa}}^4 \). We write \( \langle \cdot, \cdot \rangle \) for the inner product in \( L^2(\mathbb{T}^2) \).

**Lemma 5.3.** There exists \( C > 0 \) depending on \( \alpha, \delta \) and \( \| \rho_0 \|_{L^p(\mu)} \) such that for any \( k \in \Lambda_N \), we have

\[
\mathbb{E} \left[ \langle \omega^n_t - \omega^N_s, e_k \rangle^4 \right] \leq C(t - s)^2 (|k|^8 + 1).
\]

**Proof.** By equation (3.4),

\[
\langle \omega^n_t, e_k \rangle = \langle \omega^n_0, e_k \rangle + \int_0^t \langle \omega^n_s, u(\omega^n_s) \cdot \nabla e_k \rangle \, ds \nonumber
\]

\[
- \alpha \int_0^t \langle \omega^n_s, e_k \rangle \, ds + \sqrt{2\alpha} \int_0^t \langle dW^N_s, e_k \rangle \nonumber
\]

\[
= \langle \omega^n_0, e_k \rangle + \int_0^t \langle \omega^n_s \otimes \omega^n_s, H_{e_k} \rangle \, ds - \alpha \int_0^t \langle \omega^n_s, e_k \rangle \, ds + \sqrt{2\alpha} W^k_t. \tag{5.5}
\]

Therefore, for \( 0 \leq s < t \leq T \),

\[
\langle \omega^n_t - \omega^N_s, e_k \rangle = \int_s^t \langle \omega^n_r \otimes \omega^n_r, H_{e_k} \rangle \, dr - \alpha \int_s^t \langle \omega^n_r, e_k \rangle \, dr + \sqrt{2\alpha}(W^k_t - W^k_s).
\]

First, we control by Hölder’s inequality:

\[
\mathbb{E} \left[ \left( \int_s^t \langle \omega^n_r \otimes \omega^n_r, H_{e_k} \rangle \, dr \right)^4 \right] \nonumber
\]

\[
\leq (t - s)^3 \mathbb{E} \left[ \int_s^t \langle \omega^n_r \otimes \omega^n_r, H_{e_k} \rangle^4 \, dr \right] \nonumber
\]

\[
= (t - s)^3 \int_s^t \int_s^t \langle \omega \otimes \omega, H_{e_k} \rangle^4 \, d\mu \, dr \nonumber
\]

\[
\leq (t - s)^3 \int_s^t \left[ \int_s^t \langle \omega \otimes \omega, H_{e_k} \rangle^4 \, d\mu \right]^{1/q} \left[ \int_t \langle \rho_t \rangle^p \, d\mu \right]^{1/p} \, dr. \nonumber
\]

By (2.9) and the uniform density estimate (5.2),

\[
\mathbb{E} \left[ \left( \int_s^t \langle \omega^n_r \otimes \omega^n_r, H_{e_k} \rangle \, dr \right)^4 \right] \leq C_q \| e_k \|^4_{L^2(\mathbb{T}^2)} (t - s)^4 \sup_{t \in [0, T]} \| \rho^N_t \|_{L^p(\mu)} \nonumber
\]

\[
\leq C_q (t - s)^4 |k|^8 \| \rho_0 \|_{L^p(\mu)}.
\]

Similarly,
\[
\mathbb{E} \left[ \left( \int_s^t \langle \omega_r^N, e_k \rangle \, dr \right)^4 \right] \leq (t-s)^3 \mathbb{E} \int_s^t \langle \omega_r^N, e_k \rangle^4 \, dr
\]
(5.7)
\[
= (t-s)^3 \int_s^t \int \langle \omega, e_k \rangle^4 \rho_\kappa^N \, d\mu \, dr
\]
\[
\leq C_9 (t-s)^4 \| \rho_0 \|_{L^p(\mu)}.
\]
Finally,
\[
\mathbb{E} \left[ (W_t^I - W_s^k)^4 \right] \leq C (t-s)^2.
\]
Combining this estimate with (5.5)-(5.7) yields the result. \qed

As a result of Lemma 5.3, by Cauchy’s inequality,
\[
\mathbb{E}(\| \omega_t^N - \omega_s^N \|^4_{H^{1/2}}) = \mathbb{E} \left[ \left( \sum_{k \in \mathbb{Z}_0^2} |k|^{-2\kappa} \langle \omega_t^N - \omega_s^N, e_k \rangle^2 \right)^2 \right]
\]
\[
\leq \left( \sum_{k \in \mathbb{Z}_0^2} |k|^{-2\kappa} \right) \sum_{k \in \mathbb{Z}_0^2} |k|^{-2\kappa} \mathbb{E} \left[ \langle \omega_t^N - \omega_s^N, e_k \rangle^4 \right]
\]
\[
\leq \tilde{C} (t-s)^2 \sum_{k \in \mathbb{Z}_0^2} |k|^{-2\kappa}|k|^8 \leq \tilde{C} (t-s)^2,
\]
since $2\kappa - 8 > 2$ due to the choice of $\kappa$. Consequently,
\[
\mathbb{E} \left[ \int_0^T \int_0^T \frac{\| \omega_t^N - \omega_s^N \|^4_{H^{1/2}}}{|t-s|^{7/3}} \, dt \, ds \right] \leq \tilde{C} \int_0^T \int_0^T \frac{|t-s|^2}{|t-s|^{7/3}} \, dt \, ds < \infty.
\]
The proof of the boundedness in probability of $\{Q^N\}_{N \geq 1}$ in $W^{1/3,4}(0,T; H^{-\kappa}(\mathbb{T}^2))$ is complete.

To summarize, we have shown that the family $\{Q^N\}_{N \geq 1}$ of laws of $\{\omega^N\}_{N \geq 1}$ is tight on $\mathcal{X} = C([0,T], E)$. Since we are dealing with the SDEs (3.4), it is necessary to consider the laws of $\omega^N$ together with the law $W$ on $\mathcal{Y} = C([0,T], \mathbb{R}^{Z_0^2})$ of the family of Brownian motions $W := \{W_k\}_{k \in \mathbb{Z}_0^2}$. For any $N \in \mathbb{N}$, we denote $Q^N \otimes W$ the joint law (not the product measure) of $(\omega^N, W)$ on $\mathcal{X} \times \mathcal{Y} = C([0,T], E) \times C([0,T], \mathbb{R}^{Z_0^2})$.

Then, it is easy to see that the family $\{Q^N \otimes W\}_{N \geq 1}$ of joint laws is tight on $\mathcal{X} \times \mathcal{Y}$, cf. the arguments above [34, Lemma 3.4]. Thus, by Prohorov’s theorem (see [15, Theorem 5.1, p. 59]), we can find a subsequence $\{Q^N_i \otimes W\}_{i \geq 1}$ which converge weakly to some $Q \otimes W$, a probability measure on $\mathcal{X} \times \mathcal{Y}$. Next, the Skorokhod theorem (see [15, Theorem 6.7, p. 70]) implies that there exist a probability space $(\hat{\Theta}, \hat{F}, \hat{P})$, a sequence of processes $\{\hat{\omega}^N_i, \hat{W}^N_i\}_{i \in \mathbb{N}}$ and a limit process $(\tilde{\omega}, \tilde{W})$ defined on this probability space such that, for all $i \in \mathbb{N}$, the law of $(\tilde{\omega}^N_i, \hat{W}^N_i)$ is $Q^N_i \otimes W$, and $\hat{P}$-a.s., $(\hat{\omega}^N_i, \hat{W}^N_i)$ converge in $\mathcal{X} \times \mathcal{Y}$ to $(\tilde{\omega}, \tilde{W})$ as $i \to \infty$. Note that $\tilde{W}^N_i$ and $\tilde{W}$ are families of Brownian motions indexed by $\mathbb{Z}_0^2$.

We need one last result before proving the existence of solutions to (2.1).
Lemma 5.4. For a.e. \( t \in [0, T] \), the law of \( \tilde{\omega}_t \) on \( E \) has a density \( \rho_t \) with respect to \( \mu \), where \( \rho_t \) is a weak solution to the Fokker–Planck equation (2.12).

Proof. Fix any \( F \in C_b(E, \mathbb{R}) \) and \( f \in C([0, T]) \). By the \( \tilde{\mathbb{P}} \)-a.s. convergence of \( \tilde{\omega}^{N_i} \) to \( \tilde{\omega} \) in \( \mathcal{X} = C([0, T], E) \), we have

\[
\mathbb{E}_{\tilde{\mathbb{P}}} \int_0^T f(t)F(\tilde{\omega}_t)dt = \lim_{i \to \infty} \mathbb{E}_{\tilde{\mathbb{P}}} \int_0^T f(t)F(\tilde{\omega}_t^{N_i})dt = \lim_{i \to \infty} \mathbb{E}_{\tilde{\mathbb{P}}} \int_0^T f(t)F(\omega_t^{N_i})dt = \lim_{i \to \infty} \int_0^T f(t)\int_E F(\omega)\rho_t^{N_i}(\omega) d\mu(\omega)dt.
\]

The densities \( \rho^{N_i}(i \in \mathbb{N}) \) satisfy the estimates (5.2), thus, taking a further subsequence if necessary, we can assume that \( \rho^{N_i} \) converge weakly to some limit \( \rho \), which by the first half of Theorem 2.5, is a weak solution of the Fokker–Planck equation (2.12). Next, we have

\[
\int_0^T f(t)\mathbb{E}_{\tilde{\mathbb{P}}} F(\tilde{\omega}_t) dt = \int_0^T f(t)\int_E F(\omega)\rho_t(\omega) d\mu(\omega)dt.
\]

The arbitrariness of \( f \in C([0, T]) \) implies that, for a.e. \( t \in [0, T] \),

\[
\mathbb{E}_{\tilde{\mathbb{P}}} F(\tilde{\omega}_t) = \int_E F(\omega)\rho_t(\omega) d\mu(\omega).
\]

We can take a countable dense subset \( C \subset C_b(E, \mathbb{R}) \) of functionals \( F \) such that, for a.e. \( t \in [0, T] \), the above equality holds for all \( F \in C \). Thus the law of \( \tilde{\omega}_t \) is \( \rho_t \).

Up to now, we have indeed obtained the assertions (i) and (ii) of Theorem 2.6. Finally, we can prove the existence of weak solutions to the stochastic Euler equation (2.1).

Proof of Theorem 2.6(iii). Recall that \( \omega^{N_i} \) solves the finite dimensional equation (3.4) with \( N_i \) in place of \( N \), and \( (\tilde{\omega}^{N_i}, \tilde{W}^{N_i}) \) has the same law as \( (\omega^{N_i}, W) \), where we write \( W \) for the family of Brownian motions \( \{W^k\}_{k \in \mathbb{Z}_0^2} \), similarly for \( \tilde{W}^{N_i} \). Therefore, for any \( \phi \in C^\infty(\mathbb{T}^2) \),

\[
\langle \tilde{\omega}^{N_i}_t, \phi \rangle = \langle \omega_0^{N_i}, \phi \rangle + \int_0^t \langle \tilde{\omega}^{N_i}_s, (K * \tilde{\omega}^{N_i}_s) \cdot \nabla \phi_{N_i} \rangle ds - \alpha \int_0^t \langle \tilde{\omega}^{N_i}_s, \phi \rangle ds + \sqrt{2\alpha} \langle \tilde{W}^{N_i}_t, \phi \rangle,
\]

where \( \phi_{N_i} = \Pi_{N_i} \phi = \sum_{k \in \Lambda_{N_i}} \langle \phi, e_k \rangle e_k \). In this equation, we write \( \langle \cdot, \cdot \rangle \) for the inner product in \( L^2(\mathbb{T}^2) \), which will also be used for the pairing between the distributions \( C^\infty(\mathbb{T}^2)' \) and smooth functions \( C^\infty(\mathbb{T}^2) \).

By the \( \tilde{\mathbb{P}} \)-a.s. convergence of \( (\tilde{\omega}^{N_i}, \tilde{W}^{N_i}) \) to \( (\tilde{\omega}, \tilde{W}) \) in \( \mathcal{X} \times \mathcal{Y} \) as \( i \to \infty \), it is clear that all the terms, except the nonlinear part, converge in \( L^1(\tilde{\Theta}, \tilde{\mathbb{P}}, C([0, T], \mathbb{R})) \) to the corresponding one in the limit. Next,

\[
\int_0^t \langle \tilde{\omega}^{N_i}_s, (K * \tilde{\omega}^{N_i}_s) \cdot \nabla \phi_{N_i} \rangle ds = \int_0^t \langle \tilde{\omega}^{N_i}_s \otimes \tilde{\omega}^{N_i}_s, H_{\phi_{N_i}} \rangle ds.
\]
We have

\[
E_\tilde{\rho} \left[ 1 \land \sup_{0 \leq s \leq T} \left| \int_0^t \langle \tilde{\omega}_s^{N_i} \otimes \tilde{\omega}_s^{N_i}, H_{\phi N_i} \rangle ds - \int_0^t \langle \tilde{\omega}_s \otimes \tilde{\omega}_s, H_{\phi} \rangle ds \right| \right] \\
\leq E_\tilde{\rho} \left[ 1 \land \int_0^T \left| \langle \tilde{\omega}_s^{N_i} \otimes \tilde{\omega}_s^{N_i}, H_{\phi N_i} \rangle - \langle \tilde{\omega}_s \otimes \tilde{\omega}_s, H_{\phi} \rangle \right| ds \right] \\
\leq E_\tilde{\rho} \left[ 1 \land \int_0^T \left| \langle \tilde{\omega}_s^{N_i} \otimes \tilde{\omega}_s^{N_i}, H_{\phi N_i} - H_{\phi} \rangle \right| ds \right] \\
+ E_\tilde{\rho} \left[ 1 \land \int_0^T \left| \langle \tilde{\omega}_s^{N_i} \otimes \tilde{\omega}_s^{N_i} - \tilde{\omega}_s \otimes \tilde{\omega}_s, H_{\phi} \rangle \right| ds \right].
\]

We denote the two terms on the right hand side by \( I_1^{N_i} \) and \( I_2^{N_i} \), respectively. By the definition of \( H_{\phi} \), we have \( H_{\phi N_i} - H_{\phi} = H_{\phi N_i} - \phi \). Therefore, by (2.9),

\[
I_1^{N_i} \leq E_\tilde{\rho} \left[ 1 \land \int_0^T \left| \langle \tilde{\omega}_s^{N_i} \otimes \tilde{\omega}_s^{N_i}, H_{\phi N_i} - H_{\phi} \rangle \right| ds \right] \\
\leq CT \left\| \phi_{N_i} - \phi \right\|_{C^2(T^2)} \sup_{0 \leq s \leq T} \left\| \rho_s^{N_i} \right\|_{L^p(\mu)} \leq C' T \left\| \rho_0 \right\|_{L^p(\mu)} \| \phi_{N_i} - \phi \|_{C^2(T^2)},
\]

where the last step follows from (5.2). Since \( \phi \in C^\infty(\mathbb{T}^2) \), the Fourier series \( \phi_{N_i} = \Pi_{N_i} \phi \) converge to \( \phi \) in \( C^\infty(\mathbb{T}^2) \). Thus we deduce

(5.8) \[ \lim_{i \to \infty} I_1^{N_i} = 0. \]

Next, let \( H_{\phi}^{n_i} \in C^\infty(\mathbb{T}^2 \times \mathbb{T}^2) \) be an approximating sequence of \( H_{\phi} \) as in (2.5) and (2.6). By the triangle inequality,

\[
I_2^{N_i} \leq E_\tilde{\rho} \left[ 1 \land \int_0^T \left| \langle \tilde{\omega}_s^{N_i} \otimes \tilde{\omega}_s^{N_i}, H_{\phi}^{n_i} - H_{\phi} \rangle \right| ds \right] \\
+ E_\tilde{\rho} \left[ 1 \land \int_0^T \left| \langle \tilde{\omega}_s \otimes \tilde{\omega}_s, H_{\phi}^{n_i} - H_{\phi} \rangle \right| ds \right] \\
+ E_\tilde{\rho} \left[ 1 \land \int_0^T \left| \langle \tilde{\omega}_s^{N_i} \otimes \tilde{\omega}_s^{N_i} - \tilde{\omega}_s \otimes \tilde{\omega}_s, H_{\phi}^{n_i} \rangle \right| ds \right] \\
=: J_{1,n}^{N_i} + J_{2,n} + J_{3,n}^{N_i}.
\]

Recall that, by Lemma 5.4, \( \tilde{\omega}_s \) has the density \( \rho_s \) for a.e. \( s \in (0, T) \) and the estimate below holds:

\[
\sup_{0 \leq s \leq T} \left\| \rho_s \right\|_{L^p(\mu)} \leq \liminf_{i \to \infty} \sup_{0 \leq s \leq T} \left\| \rho_s^{N_i} \right\|_{L^p(\mu)} \leq c_0 \left\| \rho_0 \right\|_{L^p(\mu)}.
\]

Therefore, by (2.8),

\[
J_{2,n} \leq E_\tilde{\rho} \int_0^T \left| \langle \tilde{\omega}_s \otimes \tilde{\omega}_s, H_{\phi}^{n_i} - H_{\phi} \rangle \right| ds \\
\leq T \left[ C_p \| H_{\phi}^{n_i} - H_{\phi} \|_{L^2(\mathbb{T}^2 \times \mathbb{T}^2)} + \int_{\mathbb{T}^2} H_{\phi}^{n_i}(x, x) \, dx \right]
\]
which tends to 0 as \( n \to \infty \). Next, thanks to the uniform estimates (5.2) on the densities \( \rho_s^{N_i} \) of \( \tilde{\omega}_s^{N_i} \), the same arguments as above yield
\[
\lim_{n \to \infty} J_{1,n}^{N_i} = 0 \quad \text{uniformly in } i \in \mathbb{N}.
\]
Finally, fix any \( n \in \mathbb{N} \); \( \tilde{\omega}_s^{N_i} \) converge in \( C([0,T],E) \) to \( \tilde{\omega} \) as \( i \to \infty \), thus
\[
\lim_{i \to \infty} \int_0^T \left| \langle \tilde{\omega}_s^{N_i} \otimes \tilde{\omega}_s^{N_i} - \tilde{\omega}_s \otimes \tilde{\omega}_s, H^n_\phi \rangle \right| \, ds = 0.
\]
As a result, for any fixed \( n \), the dominated convergence theorem implies
\[
\lim_{i \to \infty} J_{3,n}^{N_i} = 0.
\]
Therefore, first letting \( i \to \infty \) and then \( n \to \infty \) in (5.9), we obtain
\[
\lim_{i \to \infty} J_{2,n}^{N_i} = 0.
\]
Combining this limit with (5.8) we finish the proof. \( \square \)

6. Gibbsian Energy-Enstrophy Measures

We conclude our study with a relevant example of an absolutely continuous measure with respect to the white noise measure \( \mu \) from which to start the stochastic dynamics we have discussed so far.

The enstrophy measure \( \mu \) is formally represented by
\[
\mu(\omega) = \frac{1}{Z} e^{-S(\omega)} d\omega,
\]
where \( S(\omega) = \frac{1}{2} \int_{\mathbb{T}^2} \omega^2 \, dx \) is the enstrophy of \( \omega \). As already mentioned in section 2, \( \mu \) is interpreted as the law of the centred, zero averaged (recall that all our function spaces are subject to the zero space average condition), Gaussian random field \( \eta \) with identity covariance kernel, or equivalently the \( L^2(\mathbb{T}^2) \) inner product as covariance quadratic form.

Besides enstrophy, 2-dimensional Euler equation preserves energy,
\[
E(\omega) = \frac{1}{2} \int_{\mathbb{T}^2} |u|^2 \, dx = \frac{1}{2} \int_{\mathbb{T}^2} \omega \Delta^{-1} \omega \, dx,
\]
where the second expression is readily obtained from the first one recalling that \( u = \nabla \perp \Delta^{-1} \omega \) and integrating by parts. With this in mind, it is natural to consider another candidate invariant measure for Euler equation, the energy-enstrophy measure
\[
\mu_\beta(\omega) = \frac{1}{Z_\beta} e^{-\beta E(\omega) - S(\omega)} d\omega,
\]
with \( \beta \in \mathbb{R} \) a real parameter. The measure \( \mu_\beta \) is rigorously defined as the law of the centred, zero averaged, Gaussian random field \( \eta_\beta \) on \( \mathbb{T}^2 \) with covariance
\[
\forall f, g \in L^2(\mathbb{T}^2), \quad \mathbb{E} \left[ \langle \eta_\beta, f \rangle \langle \eta_\beta, g \rangle \right] = \langle f, Q_\beta g \rangle, \quad Q_\beta = (1 + \beta (-\Delta)^{-1})^{-1},
\]
whenever \( Q_\beta \) is well-defined as a positive definite operator, that is for \( \beta > -1 \). Equivalently, \( \eta_\beta \) is a centred Gaussian stochastic process indexed by \( L^2(\mathbb{T}^2) \) with
the specified covariance. Since the embedding of $Q^{1/2}_s L^2(T^2)$ into $H^s(T^2)$ is Hilbert-Schmidt for all $s < -1$, $\eta_\beta$ can be identified with a random distribution taking values in the latter spaces (see [27]).

The Gaussian random distributions we just introduced are best understood in terms of Fourier series: we can write

$$\eta_\beta = \sum_{k \in \mathbb{Z}_0^2} \hat{\eta}_{\beta,k} e_k,$$

where $\hat{\eta}_{\beta,k} = \langle \eta_\beta, e_k \rangle \sim N_c \left( 0, \frac{|k|^2}{\beta + |k|^2} \right)$

are independent $\mathbb{C}$-valued Gaussian variables, and the Fourier expansion thus converges in $L^2(H^s(T^2),\mu_\beta)$ for $s < -1$. The measure $\mu_\beta$ is also characterized by its Fourier transform (characteristic function) on $H^s(T^2)$: for any $f \in H^{-s}(T^2)$,

$$\int e^{i\langle \omega, f \rangle} d\mu_\beta(\omega) = \exp \left( -\frac{1}{2} \sum_{k \in \mathbb{Z}_0^2} \frac{4\pi^2 |k|^2 |\beta|}{\beta + 4\pi^2 |k|^2} |\hat{f}_k|^2 \right).$$

Let us give an equivalent definition of $\mu_\beta$: in sight of (6.1), energy can be written in terms of Fourier components as

$$2E(\omega) = -\langle \omega, \Delta^{-1}\omega \rangle = \sum_{k \in \mathbb{Z}_0^2} |\hat{\omega}_k|^2 |k|^2.$$

This expression does not make sense as a random variable if $\omega = \eta$, since in that case $\hat{\eta}_k$’s are i.i.d. Gaussian variables, and the series diverges almost surely. However, one can define a renormalized energy by means of Wick ordering:

$$2: E(\eta) = \lim_{K \to \infty} \sum_{|k| \leq K} \frac{\hat{\eta}_k \hat{\eta}_k^*}{|k|^2} = \lim_{K \to \infty} \sum_{|k| \leq K} \left( \frac{|\hat{\eta}_k|^2}{|k|^2} - \int \frac{|\hat{\eta}_k|^2}{|k|^2} d\mu(\eta) \right),$$

where the limit holds in $L^2(\mu)$ (see [1]), and it defines an element of the second Wiener chaos $H^2(\mu)$. As a consequence, $2E$ can be expressed as a double Itô-Wiener stochastic integral with respect to the white noise $\eta$, the kernel being naturally Green’s function $G$:

$$2: E(\eta) = I^2(G,\eta),$$

where the right-hand side is the double Itô-Wiener integral of $G(x,y)$ with respect to the white noise $\eta$ (that is, on the Gaussian space $(E,\mu)$), as defined in [47]. The proof of the forthcoming Proposition is detailed in [45], and has an analogue in infinite product representations of energy-entrstrophy measures in [2].

**Proposition 6.1.** The probability measure on $E$ defined by

$$d\tilde{\mu}_\beta = \frac{1}{Z_\beta} e^{-\beta 2: E(\omega)} d\mu(\omega), \quad Z_\beta = \int e^{-\beta 2: E(\omega)} d\mu(\omega),$$

is well-posed. It coincides with the energy-entrstrophy measure, $\tilde{\mu}_\beta = \mu_\beta$.

Intuition suggests that the renormalized energy is invariant for Euler’s equation, and we can express this fact rigorously by means of the above discussion. The idea is to exhibit a solution of the Fokker-Planck equation (2.12) — in the case where friction and forcing are absent, $\alpha = 0$ — such that $\rho_t \equiv 2E$. In fact, since no uniqueness results are available, this is the best notion of invariance we can
produce, and as we see below it is a consequence of the infinitesimal invariance already observed in the literature.

**Proposition 6.2.** For any cylinder function \( \phi \in \mathcal{FC}_b \) and \( \beta > -1 \) it holds

\[
E [ :E (\eta) B \phi (\eta) ] = E \left[ \frac{1}{Z_\beta} e^{-\beta :E (\eta)} B \phi (\eta) \right] = 0.
\]

As a consequence, for \( \alpha = 0 \), there exist constant solutions of (2.12) (in the sense specified in section 2) such that \( \rho_t \equiv :E \) or \( \frac{1}{Z_\beta} e^{-\beta :E} \). Moreover, there exists a weak solution of (1.1) (again in the sense of section 2) whose fixed time marginals are constant in time, and coincide with \( \frac{1}{Z_\beta} e^{-\beta :E} \).

**Proof.** The fact that \( E [ :E (\eta) B \phi (\eta) ] = 0 \) is detailed in [20, Theorem 3.1], and infinitesimal invariance of Gibbs density can be obtained by a completely analogous computation. By means of (6.6), one can straightforwardly check that the constant densities \( \rho_t \equiv :E \) or \( \frac{1}{Z_\beta} e^{-\beta :E} \) solve the Fokker-Planck equation (2.12) for \( \alpha = 0 \) in the sense of Definition 2.2. In order to apply Theorem 2.6 and deduce existence of a stationary solution to Euler equation, we are only left to verify suitable integrability conditions.

Since \( :E \) belongs to the second Wiener chaos of \( \mu \), it has finite moments of all orders, as well as exponential moments: we already mentioned that \( e^{-\beta :E} \) is integrable as soon as \( \beta > -1 \). This threshold can be deduced from the explicit Gaussian expression (6.4) and the standard result [47, Theorem 6.1].

Thanks to the integrability properties of \( :E \) and \( \frac{1}{Z_\beta} e^{-\beta :E} \), we just recalled, Theorem 2.5 and Theorem 2.6 provide existence of solutions to the stochastic Euler equation (1.1) and the associated Fokker-Planck equation with initial data \( \mu_\beta \) also for \( \alpha > 0 \).

However, \( :E \) is not invariant for the Ornstein-Uhlenbeck generator \( \mathcal{L} \), as one can verify with an elementary computation in Fourier series in the same fashion of the above ones. The resulting flow is thus not stationary. When \( \beta > - \frac{1}{2} \), by the decay estimate in Theorem 2.5 for the case \( p = 2 \) we know that the solutions we have built converge for large time to the space white noise. Since uniqueness results are not available, we cannot rule out existence of “anomalous” solutions with a different behavior. As already remarked in the Introduction, just like uniqueness of weak solutions, convergence to equilibrium in this setting remains a fascinating open problem.

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