An Algorithm Enumerating All Infinite Repetitions in a D0L-system

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Abstract. We describe a simple algorithm which, for a given D0L-system, returns all words \(v\) such that \(v^k\) is a factor of the language of the system for all \(k\). This algorithm can be used to decide whether a D0L-system is repetitive.

Keywords: D0L-system; periodicity; repetition

1 Introduction

Repetitions in words were studied already by Thue [15] in 1906. Since then this topic has been addressed by many authors from many points of view. This work deals with repetitions in languages of D0L-systems. In particular, we focus on the case when the language of a D0L-system contains an arbitrarily long repetition as a factor. The other case, when these repetitions are bounded, has been recently profoundly studied by Krieger [8,9].

Ehrenfeucht and Rozenberg proved in [3] that an infinite repetition can appear in the language of a D0L-system as a factor only if there is a word \(v\) such that \(v^k\) is a factor of an element of the language for all \(k \in \mathbb{N}\); formally, they proved that if a D0L-system is repetitive (i.e., there is an element of the language that contains a \(k\)-power as a factor for all positive \(k\)), then it is strongly repetitive (i.e., there exists a non-empty word \(v\) such that \(v^k\) appears in the language of the system as a factor for all positive \(k\)). Moreover, it follows from the work of Mignosi and Séébold in [12] that this is the only source of infinite repetitions since for any D0L-system there is a constant \(M\) such that \(v^M\) being in the language as a factor implies that \(v^k\) is in the language as a factor for all \(k \in \mathbb{N}\). Hence, in order to describe all infinite repetitions occurring in the language of a D0L-system as factors we can limit ourselves to all finite words \(v\) having arbitrarily large repetitions in some element of the language.

The first problem we encounter is deciding whether the language of a given D0L-system contains factors with arbitrarily large repetitions or not. Decidability of this problem was proved for the first time in [3] and then, using different strategy, also in [12]. But the algorithms are quite complicated and their complexity is unknown. Another algorithm working in polynomial time is given in [7] by Kobayashi and Otto. Their approach uses the notion of quasi-repetitive elements:
v is a quasi-repetitive element for a morphism \( \varphi \) if there exist positive integers \( n \) and \( p \) with \( p \neq 0 \) such that \( \varphi^n(v) = v_1^p \) where \( v_1 \) is a conjugate of \( v \). They proved that repetitiveness is equivalent to the existence of quasi-repetitive factors.

Here we slightly broaden this idea as we prove that a non-pushy D0L-system (pushy systems are known to be repetitive) is repetitive if and only if there is a letter \( a \) and an integer \( \ell \) such that the fixed point of \( \varphi^\ell \) starting in \( a \) is purely periodic. For given \( \ell \) and \( a \), this is very easy to decide employing the algorithm introduced by Lando in [10]. Further, using the notion of simplification [2] and some technical results from [3] to handle pushy systems, we assemble an algorithm enumerating all arbitrarily long repetitions that appear in some element of the studied language.

The decidability of a more general problem whether \( \varphi^\ell \) has an eventually periodic fixed point has been shown in [13] and [5]. Recently, Honkala in [6] gave a simple algorithm to decide the problem. However, the algorithm needs to compute a power of the morphism \( \varphi \) that is greater than \( k! \) where \( k \) is the cardinality of the alphabet.

## 2 Definitions and basic notions

An alphabet \( \mathcal{A} \) is a finite set of letters. We denote by \( \mathcal{A}^* \) the free monoid on \( \mathcal{A} \) and by \( \mathcal{A}^+ \) the set of all non-empty words. The empty word is denoted \( \varepsilon \). A subset of \( \mathcal{A}^* \) is called a language and its elements are words. Let \( v = v_0 \cdots v_{n-1} \) with \( v_i \in \mathcal{A} \) for \( 0 \leq i < n \). The length of \( v \) is \( n \) and is denoted by \(|v|\). We denote by \( \text{first}(v) \) the first letter of the word \( v \in \mathcal{A}^+ \), i.e., here \( \text{first}(v) = v_0 \). By repeating the word \( v \) \( k \)-times with \( k \in \mathbb{N} \) we get the \( k \)-power of \( v \) denoted by \( v^k = vv \cdots v \). Any infinite sequence of letters \( u = u_0 u_1 \cdots \) is called an infinite word over \( \mathcal{A} \). A word \( v \in \mathcal{A}^* \) is a factor of a finite or infinite word \( u \) if there exist words \( x \) and \( y \) such that \( u = xvy \); if \( x \) is empty (resp. \( y \) is empty), \( v \) is a prefix (resp. suffix) of \( u \). A word \( w \) is a conjugate of \( v \in \mathcal{A}^* \) if \( w = yx \) and \( v = xy \) for some \( x, y \in \mathcal{A}^* \); the set of all conjugates of \( v \) is denoted by \([v]\). A word \( v \) is primitive if \( v = z^k \) implies \( k = 1 \). The shortest word \( x \) such that \( v = x^k \), \( k \in \mathbb{N}^+ \), is the primitive root of \( v \). An infinite word \( u \) is eventually periodic if it is of the form \( u = xyyyy \cdots = xy^\omega \); it is (purely) periodic if \( x \) is empty and aperiodic if it is not eventually periodic.

Given two alphabets \( \mathcal{A} \) and \( \mathcal{B} \), any homomorphism \( \varphi \) from \( \mathcal{A}^* \) to \( \mathcal{B}^* \) is called a morphism. An infinite word \( u \) is a periodic point of a morphism \( \varphi : \mathcal{A}^* \rightarrow \mathcal{A}^* \) if \( \varphi^\ell(u) = u \) for some \( \ell \geq 1 \); periodic point starting with the letter \( a \) such that \( \varphi^\ell(a) = av \), for some \( v \in \mathcal{A}^+ \), is the infinite word \( \varphi^\ell(a) = \lim_{k \rightarrow +\infty} \varphi^{k\ell}(a) \). A non-empty word \( w \) is mortal with respect to a morphism \( \varphi \) if \( \varphi^k(w) = \varepsilon \) for some \( k \), otherwise it is immortal. A morphism \( \varphi \) over \( \mathcal{A} \) is non-erasing if \( \varphi(a) \) is non-empty for all \( a \in \mathcal{A} \).

The triplet \( G = (\mathcal{A}, \varphi, w) \), where \( \mathcal{A} \) is an alphabet, \( \varphi \) a morphism on \( \mathcal{A}^* \) and \( w \in \mathcal{A}^+ \) is a D0L-system. The language of \( G \) is the set \( L(G) = \{ \varphi^k(w) \mid k \in \mathbb{N} \} \). The system \( G \) is reduced if every letter of \( \mathcal{A} \) occurs in some element of \( L(G) \). In what follows, we will naturally suppose that we have a reduced system since if
a system is not reduced, then we may consider a subset of the alphabet and a restriction of the morphism to get a reduced system with the same properties. If the set \( L(G) \) is finite, then \( G \) is finite. A letter \( a \in \mathcal{A} \) is bounded if the D0L-system \((\mathcal{A}, \varphi, a)\) is finite, otherwise \( a \) is unbounded. The set of all bounded letters is denoted by \( \mathcal{A}_0 \). The D0L-system \( G \) is non-erasing (resp. injective) if the morphism \( \varphi \) is non-erasing (resp. injective). We denote by \( S(L(G)) \) the set of all factors of the elements of the set \( L(G) \).

If for any \( k \in \mathbb{N}^+ \) there is a word \( v \) such that \( v^k \in S(L(G)) \), then \( G \) is repetitive; if there is a word \( v \) such that \( v^k \in S(L(G)) \) for all \( k \in \mathbb{N}^+ \), then \( G \) is strongly repetitive. By [3], all repetitive D0L-systems are strongly repetitive. An important class of repetitive D0L-systems are pushy D0L-systems: \( G \) is pushy if \( S(L(G)) \) contains infinitely many words over \( \mathcal{A}_0 \).

### 3 Infinite periodic factors

As explained above, to describe all infinite repetitions appearing as factors in the language of a D0L-system \( G \) it suffices to study words \( v \) such that \( v^k \in S(L(G)) \) for all \( k \in \mathbb{N} \). Therefore we introduce the following notions.

**Definition 1.** Given a D0L-system \( G \), we say that \( v^\omega \) is an infinite periodic factor of \( G \) if \( v \) is a non-empty word and \( v^k \in S(L(G)) \) for all positive integers \( k \).

Let \( v \) be non-empty. We say that infinite periodic factors \( v^\omega \) and \( u^\omega \) are equivalent if the primitive root of \( u \) is a conjugate of the primitive root of \( v \). We denote the equivalence class containing \( v^\omega \) by \([v]^\omega\).

#### 3.1 Simplification

In what follows, some of the proofs are based on the assumption that the respective D0L-system is injective. This does not mean any loss of generality of our results since any non-injective D0L-system can be mapped to an injective one that has the same structure of infinite periodic factors. This mapping is given by a simplification of morphisms [2].

**Definition 2.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be two finite alphabets and let \( f : \mathcal{A}^* \to \mathcal{A}^* \) and \( g : \mathcal{B}^* \to \mathcal{B}^* \) be morphisms. We say that \( f \) and \( g \) are twined if there exist morphisms \( h : \mathcal{A}^* \to \mathcal{B}^* \) and \( k : \mathcal{B}^* \to \mathcal{A}^* \) satisfying \( k \circ h = f \) and \( h \circ k = g \). If \( \#\mathcal{B} < \#\mathcal{A} \) and \( f \) and \( g \) are twined, then \( g \) is a simplification (with respect to \( (h, k) \)) of \( f \).

If a D0L-system has no simplification, then it is called elementary. It is known [2] that elementary D0L-systems are injective and thus non-erasing.

Moreover, it is easy to see that every non-injective morphism has a simplification that is injective. A simple algorithm to find an injective simplification of a morphism works as follows. If the morphism is erasing, one can find a non-erasing simplification using Proposition 3.6 in [7]. A simplification of a non-erasing morphism is related to the defect theorem and one can use the algorithm described in [4].
Example 1. The morphism $f$ determined by $a \mapsto ac, b \mapsto bade, c \mapsto acab$ and $d \mapsto adc$ is not injective as $f(ab) = f(cd)$. Therefore, there must exist a simplification; let $h$ be a morphism given by $a \mapsto x, b \mapsto yz, c \mapsto xy, d \mapsto z$ and $k$ a morphism given by $x \mapsto ac, y \mapsto b, z \mapsto adc$. Since $f = k \circ h$, $f$ is twined with the morphism $g = h \circ k$, which is determined by $x \mapsto xxy, y \mapsto yz, z \mapsto zxyz$ and defined over the alphabet $\{x, y, z\}$. One can easily check that $g$ has no simplification and hence it is elementary, and thus injective.

Let $G = (A, f, w)$ and let $g : B^* \rightarrow B^*$ be an injective simplification of $f$ with respect to $(h, k)$. We say that the D0L-system $(B, g, h(w))$ is an injective simplification of $G$ (with respect to $(h, k)$). The role of injective simplifications in the study of repetitions is given by the following lemma.

Lemma 1 (Kobayashi, Otto [7]). Let $G$ be a D0L-system and $G'$ its injective simplification with respect to $(h, k)$. Then $v^\omega$ is an infinite periodic factor of $G$ if and only if $(h(v))^\omega$ is an infinite periodic factor of $G'$.

With this lemma in hand, we can consider only injective D0L-systems in the sequel.

3.2 Graph of infinite periodic factors

If $v^\omega$ is an infinite periodic factor of $G = (A, \varphi, w)$, then $(\varphi(v))^\omega$ is an infinite periodic factor, too. This gives us a structure that can be captured as a graph.

Definition 3. Let $G = (A, \varphi, w)$ be a D0L-system. The graph of infinite periodic factors of $G$, denoted $P_G$, is a directed graph with loops allowed and defined as follows:

1. the set of vertices of $P_G$ is the set

$$V(P_G) = \{[v]^\omega \mid v^\omega \text{ is an infinite periodic factor of } S(L(G))\};$$

2. there is a directed edge from $[v]^\omega$ to $[z]^\omega$ if and only if $\varphi(v^\omega) \in [z]^\omega$.

Obviously, the outdegree of any vertex of $P_G$ is equal to one.

Lemma 2. If $G = (A, \varphi, w)$ is an injective D0L-system, then any vertex $[v]^\omega \in P_G$ has indegree at least 1.

Proof. Let $[v]^\omega \in V(P_G)$. Since $v^\omega$ is an infinite factor of $G$, there exists an infinite word $u$ such that $su^\omega = \varphi(u)$ and the word $s$ is a non-empty suffix of $v$ such that $|s| < |v|$.  

To obtain a contradiction suppose that $u$ is not eventually periodic, i.e., is aperiodic. By the pigeonhole principle there exist words $v_1$ and $v_2$ such that $v = v_1v_2$ and there exist infinitely many prefixes $u$ of $u$ such that $\varphi(u) = sv^k v_1$ for some integer $k$.

Let $(u_1, u_2, u_3)$ be a subsequence of the sequence of all such prefixes ordered by increasing length. Denote $u_1x = u_2$ and $u_2y = u_3$. The situation is demonstrated
in Figure 1. It is known (see, e.g., Proposition 1.3.2 in [11]) that for any \( w_1, w_2 \in A^* \) the equality \( w_1 w_2 = w_2 w_1 \) implies that there exists a word \( z \) such that \( w_1 = z^m \) and \( w_2 = z^n \) for some integers \( m \) and \( n \). Therefore, since \( u \) is aperiodic, we can choose the prefixes \( u_1, u_2 \) and \( u_3 \) such that \( xy \neq yx \). We have \( \varphi(x) = v_2 v^k v_1 \) and \( \varphi(y) = v_2 v^\ell v_1 \) for some integers \( k \) and \( \ell \). This implies \( \varphi(xy) = \varphi(yx) \) which is a contradiction.

Thus, \( u \) is an infinite periodic factor of \( G \). It follows that there exists a primitive word \( u \) such that \( u = zu^\omega \), \( z \in A^* \), and there is a directed edge in \( \mathcal{P}_G \) from \([u]^\omega \) to \([u]^\omega \).

**Corollary 1.** If \( G = (A, \varphi, w) \) is an injective D0L-system, then its graph of infinite periodic factors \( \mathcal{P}_G \) is 1-regular. In other words, \( \mathcal{P}_G \) consists of disjoint cycles.

So, each vertex of \( \mathcal{P}_G \) is a vertex of a cycle. In what follows, we will distinguish two types of cycles:

**Lemma 3.** Let \( G = (A, \varphi, w) \) be an injective D0L-system. If \([v]^\omega \) and \([w]^\omega \) are two vertices of the same cycle in \( \mathcal{P}_G \), then \( v \) consists of bounded letters if and only if \( w \) consists of bounded letters.

A proof follows from the obvious fact that if \( v \in A_0^+ \), then \( \varphi^k(v) \in A_0^+ \) for all \( k \in \mathbb{N}^+ \). Similarly, if \( v \) contains an unbounded letter, then \( \varphi^k(v) \) must contain an unbounded letter as well for all \( k \in \mathbb{N}^+ \). So, in \( \mathcal{P}_G \) we have two types of cycles: over bounded letters only or containing some unbounded letters. We treat these two cases separately.

### 3.3 Bounded letters

Obviously, if \( v^\omega \) is an infinite periodic factor of a D0L-system \( G \) with \( v \) containing bounded letters only, then \( G \) is pushy by definition. In [3] it is proved that it is decidable whether a D0L-system is pushy or not. In particular, the authors proved that a D0L-system is pushy if and only if it satisfies the *edge condition:*
there exist \( a \in A, k \in \mathbb{N}^+, v \in A^* \) and \( u \in A_0^+ \) such that \( \varphi^k(a) = vau \) or \( \varphi^k(a) = uav \). Here we slightly broaden the idea and show how to describe all factors over \( A_0 \).

**Definition 4.** Let \( G = (A, \varphi, w) \) be a D0L-system. The graph of unbounded letters of \( G \) to the right, denoted \( \text{UR}_G \), is the labeled directed graph defined as follows:

(i) the set of vertices is \( V(\text{UR}_G) = A \setminus A_0 \),
(ii) there is a directed edge from \( a \) to \( b \) with label \( u \in A^* \) if there exists \( v \in A^* \) such that \( \varphi(k)(a) = vbu \).

The graph of unbounded letters of \( G \) to the left \( \text{UL}_G \) is defined analogously: the only difference is that the roles of \( v \) and \( u \) in the definition of a directed edge are switched.

Clearly, the edge condition is satisfied if and only if one of the graphs \( \text{UL}_G \) and \( \text{UR}_G \) contains a cycle including an edge with an immortal label. Therefore, we have this:

**Proposition 1.** A D0L-system \( G \) is pushy if and only if one of the graphs \( \text{UL}_G \) and \( \text{UR}_G \) contains a cycle including an edge with an immortal label.

Since the outdegree of any vertex of the graphs \( \text{UL}_G \) and \( \text{UR}_G \) is 1, the graphs consist of components each containing exactly one cycle. Assume that \( a \) is a vertex of a cycle of \( \text{UR}_G \) that contains at least one edge with a non-empty immortal label, i.e., there exist \( k \in \mathbb{N}^+, u_1, \ldots, u_k \in A^*_0 \) and \( a_1, \ldots, a_{k-1} \in A \setminus A_0 \) so that

\[
a \xrightarrow{u_1} a_1 \xrightarrow{u_2} \cdots \xrightarrow{u_k} a_{k-1} \xrightarrow{u_k} a
\]

is a cycle of \( \text{UR}_G \) with \( u_j \) being immortal for some \( j \) such that \( 1 \leq j \leq k \). Denote \( u = u_k \varphi(u_{k-1}) \cdots \varphi^{-1}(u_1) \), then for all \( \ell \in \mathbb{N}^+ \) the word \( \varphi^{\ell k}(a) \) has a suffix

\[
u \varphi^k(u) \varphi^{2k}(u) \cdots \varphi^{(\ell-1)k}(u) \in A^+_0.
\]

**Lemma 4.** Let \( G = (A, \varphi, w) \) be a D0L-system. If \( u \in A_0^+ \) is immortal and \( k \in \mathbb{N}^+ \), then the infinite word

\[
u = u \varphi^k(u) \varphi^{2k}(u) \varphi^{3k}(u) \ldots
\]

is eventually periodic.

**Proof.** Clearly, the sequence \( (\varphi^j(u))_{j=0}^{+\infty} \) is eventually periodic. Let \( s \) and \( t \) be numbers such that \( \varphi^s(u) = \varphi^{s+t}(u) \) with \( t > 0 \). Define \( \ell_0 \) and \( \ell_1 \) so that \( \ell_0 k \geq s \) and \( (\ell_1 - \ell_0)k \) is a multiple of \( t \). We get that

\[
u = u \varphi^k(u) \cdots \varphi^{\ell_0 k}(u) \left( \varphi^{(\ell_0+1)k}(u) \varphi^{(\ell_0+2)k}(u) \cdots \varphi^{\ell_1 k}(u) \right) \omega.
\]

\( \square \)
The situation is analogous for the graph \( UL_G \). Since the cycles of the graphs \( UR_G \) and \( UL_G \) are clearly the only sources of factors over \( A_0 \) of arbitrary length, we get the following theorem. This result has been already obtained in [1], Proposition 4.7.62, where one can find a distinct proof.

**Theorem 1.** If \( G \) is a pushy D0L-system, then there exist \( L \in \mathbb{N} \) and a finite set \( U \) of words from \( A_0^+ \) such that any factor from \( S(L(G)) \cap A_0^+ \) is of one of the following three forms:

(i) \( w_1 \),

(ii) \( w_1u_{k_1}w_2 \),

(iii) \( w_1u_{k_1}w_{k_2}w_3 \),

where \( u_1, u_2 \in U \), \( |w_j| < L \) for all \( j \in \{1, 2, 3\} \), and \( k_1, k_2 \in \mathbb{N}^+ \).

**Example 2.** Consider the D0L-system \( G = (A, \varphi, 0) \) with \( A = \{0, 1, 2\} \) and \( \varphi \) determined by \( 0 \mapsto 012, 1 \mapsto 2 \) and \( 2 \mapsto 1 \). There are two bounded letters, \( A_0 = \{1, 2\} \), and one unbounded letter. The graphs \( UL_G \) and \( UR_G \) both contain one loop on the only vertex 0 labeled with the empty word and 12, respectively. Therefore, \( G \) is pushy and the corresponding infinite periodic factor over bounded letters is a suffix of 

\[ 12 \varphi(12) \varphi^2(12) \varphi^3(12) \ldots, \]

namely \( (1221)^\omega \). In fact, this infinite periodic factor is a suffix of the eventually periodic fixed point of \( \varphi \) starting in 0.

The D0L-system \( H = (B, \psi, 0) \) with \( B = \{0, 1, 2, 3\} \) and \( \psi \) determined by \( 0 \mapsto 0123, 1 \mapsto 2, 2 \mapsto 1 \) and \( 3 \mapsto 123 \) is also pushy. The graph \( UL_H \) contains two loops: one on the vertex 0 labeled with the empty word and one on the vertex 3 labeled with 12; the labels in \( UR_H \) are all equal to the empty word. Hence, there is an infinite periodic factor over bounded letters which is an infinite prefix of the left-infinite word

\[ \ldots \varphi^3(12) \varphi^2(12) \varphi(12)12. \]

In this case, it is the word itself, namely \( (2112)^\omega \). In this case, the fixed point of \( \psi \) is not eventually periodic, but still all prefixes of \( (2112)^\omega \) are its factors.

### 3.4 Unbounded letters

Now we address the other case: infinite periodic factors containing an unbounded letter.

**Theorem 2.** Let \( G = (A, \varphi, w) \) be an injective repetitive D0L-system. If \( [v]^{\omega} \) is a vertex of \( P_G \) such that \( v \) contains an unbounded letter, then there exist \( b \in A \) and \( \ell \in \mathbb{N}, 1 \leq \ell \leq \#A \), such that \( (\varphi^\ell)^{\omega}(b) = z^{\omega} \) for some \( z \in [v] \).

**Proof.** Clearly, we can assume that \( v \) is primitive. If \( [v]^{\omega} \) is a vertex of a cycle of length \( m \) in the graph \( P_G \), then \( \varphi^m([v]^{\omega}) \in [v]^{\omega} \). Since \( v \) contains an unbounded letter, we get \( |\varphi^m(v)| > |v| \) and so there exist a nonnegative integer \( k_1 \), a suffix \( s_1 \) of \( v \) and a prefix \( p_1 \) of \( v \) such that \( \varphi^m(v) = s_1v^{k_1}p_1 \). In fact, \( \varphi^m(v) \) is a factor
of \( v^\omega \) for all \( t \), thus there exist a sequence of integers \((k_t)_{t \geq 1}\), sequence of suffixes of \( v \) \((s_t)_{t \geq 1}\) and prefixes of \( v \) \((p_t)_{t \geq 1}\) such that \( \varphi^{k_t}(v) = s_tv^{k_t}p_t \). Clearly it must hold that \( p_ts_t = v \) for all \( t \) (since \( \varphi^{k_t}(vv) \) is also a factor of \( v^\omega \)).

Obviously, the sequence \((s_t)_{t \geq 1}\) is eventually periodic (and so is \((p_t)_{t \geq 1}\)). Therefore, there exist integers \( t_1 < t_2 \) such that \( s_{t_1} = s_{t_2} \). It follows that 

\[ \varphi^{m(t_2 - t_1)}(s_{t_1}p_{t_1}) = (s_{t_1}p_{t_1})^k \] 

for some \( k > 1 \).

Put \( z = s_{t_1}p_{t_1} \), \( t = m(t_2 - t_1) \) and \( b = \text{first}(z) \). Clearly, \( \varphi^t(b) \) is a prefix of \( z^\omega \) for all \( k \). If \( b \) is bounded, we are done. Assume \( b \) is unbounded, then there exists \( f \) such that \( u = \varphi^f(b) = \varphi^{2f}(b) \). Hence, \( u \) is a prefix of \( z = uz' \) with \( |z'| > 0 \) and we get \( \varphi^t(z'uv) = (z'uv)^k \) for some \( k > 1 \). Since \( z \) must contain at least one unbounded letter and \( u \) consists of bounded letters only, we can repeat this until the first letter of \( z' \) is unbounded.

Assume now that \( (\varphi^t)^\omega(b) = z^\omega \). It remains to prove that \( t \) can be taken less than or equal to \( \#A \). Put \( \ell_{\min} = \min\{j \in \mathbb{N} : \text{first}(\varphi^j(b)) = b\} \). Obviously such \( \ell_{\min} \) exists and is at most \( \#A \). Moreover, \( t \) must be a multiple of \( \ell_{\min} \) and hence we can put \( t = \ell_{\min} \) and still have \( (\varphi^t)^\omega(b) = z^\omega \).

By Theorems 1 and 2 and by Lemma 1 we get the following claim.

**Corollary 2.** Any repetitive D0L-system contains a finite number of primitive words \( v \) such that \( v^\omega \) is an infinite periodic factor.

### 4 Algorithm

Given an injective D0L-system \( G = (A, \varphi, w) \), all infinite periodic factors over bounded letters can be obtained from the cycles in graphs \( UL_G \) and \( UR_G \). It remains unclear how to find infinite periodic factors containing an unbounded letter. Theorem 2 says that equivalence classes \( [v]^\omega \) that contains those factors are in one-to-one correspondence with periodic periodic points of the morphism \( \varphi \). Consider the following graph of first letters over the set of vertices equal to \( A \): there is a directed edge from \( a \) to \( b \) if \( b = \text{first}(\varphi(a)) \). It follows that \( (\varphi^t)^\omega(a) \) is an infinite periodic point of \( \varphi \) if and only if \( a \in A \setminus A_0 \) is a vertex of a cycle of a length that divides \( \ell \). Therefore, we have only finitely many candidates \( a \in A \) and \( \ell \in \mathbb{N}^+ \) for which we need to verify whether \( (\varphi^t)^\omega(a) \) is a periodic infinite word. In fact, it suffices to check only one letter from each cycle of the graph of first letters. Finally, to verify whether the word \( (\varphi^t)^\omega(a) \) is periodic we can use the algorithm described in 10. This algorithm is very effective, in short it works as follows:

1. Find the least \( k \leq \#A \) such that \( (\varphi^t)^k(a) \) contains at least two occurrences of one unbounded letter.
2. If \( (\varphi^t)^k(a) \) does not contain two occurrences of \( a \), then \( (\varphi^t)^\omega(a) \) is not periodic. Otherwise denote by \( v \) the longest prefix of \( (\varphi^t)^k(a) \) containing \( a \) only as the first letter.

Note that “periodic periodic” is not a typing error: it is a periodic point of a morphism, which happens to be periodic, i.e., of the form \( w^\omega \).
Input: Alphabet $\mathcal{A}$, morphism $\varphi: \mathcal{A}^* \rightarrow \mathcal{A}^*$.

Output: List of all primitive words $v$ such that $v^k$ can be generated by $\varphi$ for all $k$.

$\varphi \leftarrow$ injective simplification of $\varphi$ with respect to $(h,k)$;
calculate the list of bounded letters $\mathcal{A}_0$;
construct the graphs UL and UR;
OutputList $\leftarrow$ empty list;

foreach cycle $c$ in UL do
  if $c$ contains an edge with non-empty immortal label then
    $u \leftarrow u_h \varphi(u_{h-1}) \cdots \varphi^{k-1}(u_1)$ where $u_1, \ldots, u_h$ are the labels of edges in the cycle $c$;
    find the least $s$ and $t$ such that $t > 0$ and $\varphi^t(u) = \varphi^{s+t}(u)$;
    $l_0 \leftarrow \lfloor s/k \rfloor \cdot k$;
    $l_1 \leftarrow l_0 + \text{LeastCommonMultiple}(t,k)/k$;
    append $\text{PrimitiveRoot} (\varphi^{(0+1)k}(u) \varphi^{(0+2)k}(u) \cdots \varphi^{lk}(u))$ to OutputList;
  end
end

foreach cycle $c$ in UR do
  // analogous procedure as for cycles in UL
end

foreach letter $a$ in $\mathcal{A} \setminus \mathcal{A}_0$ do
  find the least $l$ such that $\text{FirstLetter}(\varphi^l(a)) = a$ with $l \leq \#\mathcal{A}$;
  if $l$ exists then
    find the least $s$ such that $\varphi^s(a)$ contains at least two occurrences of one unbounded letter;
    if $\varphi^s(a)$ contains at least two occurrences of $a$ then
      $v \leftarrow$ the longest prefix of $\varphi^s(a)$ containing only one occurrence of $a$;
      if $\varphi^s(v) = v^m$ for some integer $m \geq 2$ then
        append $v$ to OutputList;
      end
    end
  end
end

OutputList $\leftarrow \{\text{PrimitiveRoot}(k(w)): w \in \text{OutputList}\}$;
add conjugates to OutputList;
return OutputList;

Algorithm 1: Pseudocode for the main algorithm.

3. Now, $(\varphi^h)^\omega(a)$ is periodic if and only if $\varphi^h(v) = v^m$ for some integer $m \geq 2$.

This, together with the algorithm to construct an injective simplification, gives us an effective algorithm that decides whether a given DOL-system is repetitive and, moreover, returns the list of all infinite periodic factors.

The described algorithm is given in pseudocode in Algorithm 1.
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