ASPECTS OF SOLITONS IN AFFINE INTEGRABLE HIERARCHIES

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ABSTRACT

We consider a very large class of hierarchies of zero-curvature equations constructed from affine Kac-Moody algebras $\hat{G}$. We argue that one of the basic ingredients for the appearance of soliton solutions in such theories is the existence of “vacuum solutions” corresponding to Lax operators lying in some abelian (up to central term) subalgebra of $\hat{G}$. Using the dressing transformation procedure we construct the solutions in the orbit of those vacuum solutions, and conjecture that the soliton solutions correspond to some special points in those orbits. The generalized tau-function for those hierarchies are defined for integrable highest weight representations of $\hat{G}$, and it applies for any level of the representation and it is independent of its realization. We illustrate our methods with the recently proposed non abelian Toda models coupled to matter fields. A very special class of such theories possess a $U(1)$ Noether charge that, under a suitable gauge fixing of the conformal symmetry, is proportional to a topological charge. That leads to a mechanism that confines the matter fields inside the solitons.

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1 Introduction

The study of soliton solutions of non-linear differential equations has been developed considerably in the last decades using (apparently) quite diverse methods. In spite of the great variety of types of equations considered, some basic features seem to be common to a large class of them. Attempts to unify such various aspects of basic classical Soliton Theory is clearly very important and may lead to new insights into the role of solitons in Physics and Mathematics.

Among the several methods of constructing solutions for non-linear differential equations we have the Hirota method [1], the dressing transformation procedure [2, 3, 4, 5, 6], Backlund transformations [7, 8], the inverse scattering method [9], the Leznov-Saveliev algebraic construction [10] and the tau-function approach [11, 3]. Each one of these methods have their own advantages, and the choice of one or the other depends on the particular problem and model one wants to address. However, soliton solutions are special and the theories presenting them must possess some common structures. By solitons here we mean a solution localized in space that travels without dispersion, and that keeps its form when scattered by other soliton, suffering just a shift in its position with respect to the one it would have if not for the scattering. In fact, in some theories like the Toda models which possess two dimensional Lorentz invariance, the solitons can have a particle interpretation and there are indications for the existence of a duality between the one-soliton solutions and the fundamental particles. In four dimensions, a similar duality, generalizing the electromagnetic duality, seems to exist between monopoles and gauge particles [12].

Pratically all theories in one and two space time dimensions, presenting soliton solutions, have a representation in terms of a zero curvature condition or Lax-Zakharov-Shabat equation. In addition, the corresponding Lax operators lie in some infinite dimensional Lie algebra. In fact, we can say that basically almost all known soliton equations are related to Kac-Moody algebras [13, 14].

In this paper we try to unify some aspects concerning soliton solutions. We argue that a basic ingredient for the appearance of soliton solutions is that there must exist one or several solutions, which we call “vacuum solutions”, such that the Lax operators, when evaluated on them, should lie in some abelian subalgebra (up to central term) of the Kac-Moody algebra associated...
to the model. Such subalgebra can be written as an algebra of oscillators $b_i$

$$[b_i, b_j] = i \beta_i C \delta_{i+j,0}$$  \hspace{1cm} (1)$$

with $\beta_i$ being some complex numbers and $C$ the central element of the Kac-Moody algebra $\mathcal{G}$. In several cases, but not in all, (1) constitutes a Heisenberg subalgebra of $\mathcal{G}$. In addition, we argue that the components of the Lax operators in the direction of the $b_i$'s should be constant for the vacuum solutions.

Using the dressing transformation method one can then construct, out of a given vacuum solution, an orbit of solutions parametrized by a constant group element $\rho$ of the Kac-Moody group. We conjecture that the soliton solutions of those theories lie on such orbits, and that they correspond to points of the orbits where $\rho$ is the product of exponentials of eigenvectors of the constant elements defined by the Lax operators evaluated on the corresponding vacuum solution. Such observations provide not only a powerful and elegant method of constructing soliton solutions, but also allow us to connect and generalize several results known in the literature. The so called solitonic specialization of the Leznov-Saveliev solution proposed in the context of Toda type models, can then be connected to the dressing method applied to such vacuum solutions. In addition, we believe that in several cases there is a connection with the Backlund transformation method.

We can also connect that observation with tau-functions. We define tau-functions as states in integrable highest weight representations of the Kac-Moody algebra, lying in the orbit, under the action of the group elements performing the dressing transformations, of the highest weight state. Our definition is independent of the level of the representation and also on the way it is realised, constituting a generalization of the previous definitions of tau-functions for level one vertex operator representations. The connection with the Hirota method is then made by realizing the Hirota’s tau-function as projections of the tau-function on some suitable states of representation. The truncation of the Hirota’s expansion is then understood in terms of the nilpotency of some operators in those integrable representations.

We illustrate our methods with the recently proposed non abelian Toda models coupled to matter fields. In fact, we discuss a very special class of such theories where the solitons play a crucial role. These models possess
a $U(1)$ Noether charge that, under a suitable gauge fixing of the conformal symmetry, is proportional to a topological charge. That leads to a mechanism that confines the matter fields inside the solitons.

The paper contains the material presented by the authors in their talks at the “International Workshop on Selected Topics of Theoretical and Modern Mathematical Physics - SIMI/96”, held in Tbilisi, Georgia (September/96), and summarizes results of references [22, 21, 23]. It is organized as follows. In section 2 we introduce the type of hierarchies of soliton equations we shall consider, and discuss their vacuum solutions. In section 3 we define the dressing transformations and in section 4 we construct the soliton solutions. Section 5 introduces the tau-functions and discuss how they connect to previous definitions and to the Hirota’s tau functions. The non abelian Toda models are introduced in section 6, their soliton solutions constructed in section 7, and their properties discussed in sections 8 and 9. In section 10 we present in great detail, the example of the model associated to the principal gradation of $sl(2)^{(1)}$.

\section{Hierarchies and vacuum solutions}

Non-linear integrable hierarchies of equations are most conveniently discussed by associating them with a system of first-order differential equations

$$\mathcal{L}_N \Psi = 0 ,$$

where $\mathcal{L}_N$ are Lax operators of the form

$$\mathcal{L}_N \equiv \frac{\partial}{\partial t_N} - A_N,$$

and the variables $t_N$ are the various “times” of the hierarchy. Then, the equivalent zero-curvature formulation is obtained through the integrability conditions of the associated linear problem

$$[\mathcal{L}_N , \mathcal{L}_M] = 0 .$$

An equivalent way to express the relation between the solutions of the zero-curvature equations and of the associated linear problem is

$$A_N = \frac{\partial \Psi}{\partial t_N} \Psi^{-1} .$$
The class of integrable hierarchies of zero-curvature equations that will be studied here is constructed from graded Kac-Moody algebras in the following way. Consider a complex affine Kac-Moody algebra $\hat{\mathcal{G}} = \hat{G} + C \mathcal{D}$, associated to a simple finite Lie algebra $G$ of rank $r$, and an integer gradation of its derived algebra $\hat{\mathcal{G}}$ labelled by a vector $s = (s_0, s_1, \ldots, s_r)$ of $r+1$ non-negative co-prime integers such that

$$\hat{\mathcal{G}} = \bigoplus_{i \in \mathbb{Z}} \hat{\mathcal{G}}_i(s) \quad \text{and} \quad [\hat{\mathcal{G}}_i(s), \hat{\mathcal{G}}_j(s)] \subseteq \hat{\mathcal{G}}_{i+j}(s). \quad (6)$$

According to [24], integral gradations of $\hat{\mathcal{G}}$ are labelled by a set of co-prime integers $s = (s_0, s_1, \ldots, s_r)$, and the grading operators are given by

$$Q_s \equiv H_s + N_s D - \frac{1}{2N_s} \text{Tr}(H_s)^2 C. \quad (7)$$

Here

$$H_s \equiv \sum_{a=1}^{r} s_a \lambda^\psi_a \cdot H^0, \quad N_s \equiv \sum_{i=0}^{r} s_i m^\psi_i, \quad \psi = \sum_{a=1}^{r} m^\psi_a \alpha_a, \quad m^\psi_0 = 1; \quad (8)$$

$H^0$ is an element of the Cartan subalgebra of $\mathcal{G}$; $\alpha_a, a = 1, 2, \ldots, r$, are its simple roots; $\psi$ is its maximal root; $m^\psi_a$ the integers in expansion $\psi = \sum_{a=1}^{r} m^\psi_a \alpha_a$; and $\lambda^\psi_a$ are the fundamental co–weights satisfying the relation $\alpha_a \cdot \lambda^\psi_b = \delta_{ab}$.

We have in mind basically two types of integrable systems. The first one corresponds to the Generalized Drinfel’d-Sokolov Hierarchies considered in [14], and [25], which are generalizations of the KdV type hierarchies studied in [13]. In particular, and using the parlance of the original references, we will be interested in the generalized mKdV hierarchies, whose construction can be summarised as follows (see [14] and, especially, [25] for details). Given an integer gradation $s$ of $\hat{\mathcal{G}}$ and a semisimple constant element $E_l$ of grade $l$ with respect to $s$, one defines the Lax operator

$$L \equiv \partial_x + E_l + A, \quad (9)$$

where the components of $A$ are the fields of the hierarchy. In [25], it was shown that the component of $A$ along the central term of $\hat{\mathcal{G}}$ should not be considered as an actual degree of freedom of the hierarchy. This is the
reason why these hierarchies can be equivalently formulated both in terms of affine Kac-Moody algebras or of the corresponding loop algebras. They are functions of \( x \) and of the other times of the hierarchy taking values in the subspaces of \( \hat{G} \) with grades ranging from 0 to \( l - 1 \). For each element in the centre of \( \text{Ker}(\text{ad} E_l) \) with positive \( s \)-grade \( N \), one constructs a local functional of those fields, \( B_N \), whose components take values in the subspaces \( \hat{G}_0(s), \ldots, \hat{G}_N(s) \). Then, \( B_N \) defines the flow equation
\[
\frac{\partial L}{\partial t_N} = [B_N, L],
\] (10)
and the resulting Lax operators \( \mathcal{L}_N = \partial/\partial t_N - B_N \) commute among themselves \([14]\).

The second type of integrable systems corresponds to the non-abelian affine Toda theories \([26, 23, 21, 27]\), and a very general class of these models will be described in section 6.

An important common feature of all those hierarchies is that they possess trivial solutions which will be called “vacuum solutions”. These particular solutions are singled out by the condition that the Lax operators evaluated on them lie on some abelian subalgebra of \( \hat{G} \), up to central terms. Then, the dressing transformation method can be used to generate an orbit of solutions out of each “vacuum”. Moreover, it is generally conjectured that multi-soliton solutions lie in the resulting orbits. As a bonus, the fact that we only consider the particular subset of solutions connected with a generic vacuum allows one to perform the calculations in a very general way and, consequently, our results apply to a much broader class of hierarchies.

For a given choice of the Kac-Moody algebra \( \hat{G} \) and the gradation \( s \), let us consider Lax operators of the form \([8]\) where the potentials can be decomposed as
\[
A_N = \sum_{i=\text{N}_-}^{\text{N}_+} A_{N,i}, \quad \text{where} \quad A_{N,i} \in \hat{G}_i(s)
\] (11)
\( N_- \) and \( N_+ \) are non-positive and non-negative integers, respectively, and the times \( t_N \) are labelled by (positive or negative) integer numbers. The particular form of these potentials will be constrained only by the condition that the corresponding hierarchy admits vacuum solutions where they take
the form

\[ A_N^{(\text{vac})} = \sum_{i=\text{vac}}^{N} c_i b_i + f_N(t) c \equiv \varepsilon_N + f_N(t) C. \]  

(12)

In this equation, \( C \) is the central element of \( \hat{G} \), and \( b_i \in \hat{G}_i(s) \) are the generators of a subalgebra \( \hat{s} \) of \( \hat{G} \) defined by

\[ \hat{s} = \{ b_i \in \hat{G}_i(s), i \in E \subset \mathbb{Z} \mid [b_i, b_j] = i \beta_i C \delta_{i+j,0} \}. \]  

(13)

where \( \beta_i \) are arbitrary (vanishing or non-vanishing) complex numbers such that \( \beta_{-i} = \beta_i \), and \( E \) is some set of integers numbers. Moreover, \( c_i^N \) are also arbitrary numbers, and \( f_N(t) \) are \( C \)-functions of the times \( t_N \) that satisfy the equations

\[ \frac{\partial f_N(t)}{\partial t_M} - \frac{\partial f_M(t)}{\partial t_N} = \sum_i i \beta_i c_M^i c_N^i. \]  

(14)

These vacuum potentials correspond to the solution of the associated linear problem given by the group element \( \hat{G} \)

\[ \Psi^{(\text{vac})} = \exp \left( \sum_N \varepsilon_N t_N + \gamma(t) C \right) \]  

(15)

where the numeric function \( \gamma(t) \) is a solution of the equations

\[ \frac{\partial \gamma(t)}{\partial t_N} = f_N(t) + \frac{1}{2} \sum_{M,i} i \beta_i c_M^i c_N^i t_M. \]  

(16)

3 Dressing Transformations

In terms of the associated linear problem, one can define an important set of transformations called “dressing transformations”, which take known solutions of the hierarchy to new solutions. Regarding the structure of the integrable hierarchies, these transformations have a deep meaning and, in fact, the group of dressing transformations can be viewed as the classical precursor of the quantum group symmetries \[ \hat{G}_\text{class}. \] Denote by \( \hat{G}_-(s) \), \( \hat{G}_+(s) \), and \( \hat{G}_0(s) \) the subgroups of the Kac-Moody group \( \hat{G} \) formed by exponentiating the subalgebras \( \hat{G}_{<0}(s) \equiv \bigoplus_{i<0} \hat{G}_i(s) \), \( \hat{G}_{>0}(s) \equiv \bigoplus_{i>0} \hat{G}_i(s) \), and \( \hat{G}_0(s) \), respectively. According to Wilson \[ [16, 17] \], the dressing transformations can
be described in the following way. Consider a solution $\Psi$ of the linear problem (2), and let 
\[ \rho = \rho_0 \rho_+ \]
be a constant element in the “big cell” of $\hat{G}$, i.e., in the subset $\hat{G}_-(s) \hat{G}_0(s) \hat{G}_+(s)$ of $\hat{G}$, such that
\[ \Psi \rho \Psi^{-1} = (\Psi \rho \Psi^{-1})_{<0} (\Psi \rho \Psi^{-1})_0 (\Psi \rho \Psi^{-1})_{>0} . \] (17)

Notice that these conditions are equivalent to say that both $\rho$ and $\Psi \rho \Psi^{-1}$ admit a generalized Gauss decomposition with respect to the gradation $s$.

Define
\[ \Psi^\rho = \Theta^{(0)}(\rho \Psi^{-1})_{<0} \Psi \rho \]
\[ \Theta^{(0)}_+ (\rho \Psi^{-1})_0 \Psi \] (18)

where
\[ \Theta^{(0)}_+ (\rho \Psi^{-1})_0 = (\Psi \rho \Psi^{-1})_{>0} . \] (19)

Then, $\Psi^\rho$ is another solution of the linear problem. In order to prove it, introduce the notation $g_\pm \equiv (\Psi \rho \Psi^{-1})_\pm$ and $\partial_N \equiv \partial / \partial t_N$, and consider
\[ \partial_N \Psi^\rho \Psi^{-1} = \partial_N \Theta^{(0)}_+ \Theta^{(0)}_- \rho - \Theta^{(0)}_+ \rho_+ \partial_N \rho_+ g_+ \Theta^{(0)}_- \rho_+ \]
\[ + \Theta^{(0)}_+ g_+ (\rho_+ \partial_N \rho_+ g_+ \Theta^{(0)}_- \rho_+ \)
\[ = \partial_N \Theta^{(0)}_- \Theta^{(0)}_+ + \Theta^{(0)}_- \partial_N g_+ g_+ \Theta^{(0)}_- \rho_+ \]
\[ + \Theta^{(0)}_+ g_+ (\partial_N \rho_+ g_+ \Theta^{(0)}_- \rho_+) \] (20)

Then, the first identity implies that $\partial_N \Psi^\rho \Psi^{-1} \in \bigoplus_{i \leq N_+} \hat{G}_i(s)$, and the second that $\partial_N \Psi^\rho \Psi^{-1} \in \bigoplus_{i \geq N_-} \hat{G}_i(s)$. Consequently
\[ A^\rho_N = \frac{\partial \Psi^\rho}{\partial t_N} (\Psi^\rho)^{-1} \in \bigoplus_{i = N_-}^{N_+} \hat{G}_i(s) , \] (21)

and, taking into account (11), it is a solution of the hierarchy of zero-curvature equations. If the fields of the hierarchy are such that $A_{N,i}$ does not span the whole subspace $\hat{G}_i(s)$ then we have to impose further constraints on the group elements performing the dressing transformation.

For any $\rho$ lying in the big cell of $\hat{G}$, the transformation
\[ D_\rho : \Psi \mapsto \Psi^\rho , \quad A_N \mapsto A^\rho_N , \] (22)

is called a dressing transformation.
4 Solitons out of vacuum solutions

We now consider the orbit of the vacuum solution (15) under the group of dressing transformations. For any element \( \rho \) of the big cell of \( \hat{G} \), let us define

\[
\Theta_+^\geq = \left(\Psi^{(\text{vac})} \rho \Psi^{(\text{vac})}^{-1}\right)_{>0}, \quad \Theta_-^\leq = \left(\Psi^{(\text{vac})} \rho \Psi^{(\text{vac})}^{-1}\right)_{<0}.
\]

(23)

and

\[
\Theta_+ = \Theta_+^{(0)} \Theta_+^\geq, \quad \Theta_- = \Theta_-^{(0)} \Theta_-^\leq; \tag{24}
\]

where \( \Theta_+^{(0)} \) and \( \Theta_-^{(0)} \) are the same as in (19), but with \( \Psi \) replaced by \( \Psi^{(\text{vac})} \) defined in (15).

Now, the orbit of the vacuum solution (15) can be easily constructed using Eqs. (18) and (21). Under the dressing transformation generated by \( \rho \),

\[
\Psi^{(\text{vac})} \mapsto \Psi = \Theta_- \Psi^{(\text{vac})} \rho = \Theta_+ \Psi^{(\text{vac})},
\]

(25)

or, equivalently, \( A_N^{(\text{vac})} \) becomes

\[
A_N^\rho - f_N(t) c = \Theta_- \varepsilon_N \Theta_-^{-1} + \partial_N \Theta_- \Theta_-^{-1} \in \bigoplus_{i \leq N_+} \hat{G}_i(s)
\]

\[
= \Theta_+ \varepsilon_N \Theta_+^{-1} + \partial_N \Theta_+ \Theta_+^{-1} \in \bigoplus_{i \geq N_-} \hat{G}_i(s),
\]

(26)

Eqs. (24), (23) and (26) summarize the outcome of the dressing transformation method, which, starting with some vacuum solution (12), associates a solution of the zero-curvature equations (4) to each constant element \( \rho \) in the big cell of \( \hat{G} \). The construction of this solution involves two steps. First, the Eqs. (26) can be understood as a local change of variables between the components of the potential \( A_N \) and some components of the group elements \( \Theta_+ \) and \( \Theta_- \).

The second step consists in obtaining the value of the required components of \( \Theta_+ \) and \( \Theta_- \) from Eqs. (24) and (23). This is usually done by considering matrix elements of the form

\[
\langle \mu | \Theta_-^{-1} \Theta_+ | \mu' \rangle = \langle \mu | e^{\sum_{N} \varepsilon N i N} \rho e^{-\sum_{N} \varepsilon N i N} | \mu' \rangle,
\]

(27)

where \( | \mu \rangle \) and \( | \mu' \rangle \) are vectors in a given representation of \( \hat{G} \). The appropriate set of vectors is specified by the condition that all the required
components of $\Theta_-$ and $\Theta_+$ can be expressed in terms of the resulting matrix elements. It will be shown below that the required matrix elements, considered as functions of the group element $\rho$, constitute the generalization of the Hirota’s tau-functions for these hierarchies. Moreover, Eq. (27) is the analogue of the, so called, solitonic specialization of the Leznov-Saveliev solution proposed in [18, 19, 20, 23, 21] for the affine (abelian and non-abelian) Toda theories.

Consider now the common eigenvectors of the adjoint action of the $\varepsilon_N$’s that specify the vacuum solution (12). Then, the important class of multi-soliton solutions is conjectured to correspond to group elements $\rho$ which are the product of exponentials of eigenvectors

$$\rho = e^{F_1} e^{F_2} \ldots e^{F_n}, \quad [\varepsilon_N, F_k] = \omega^{(k)}_N F_k, \quad k = 1, 2, \ldots n.$$  (28)

In this case, the dependence of the solution upon the times $t_N$ can be made quite explicit

$$\langle \mu | \Theta^-_1 \Theta_+ | \mu' \rangle = \langle \mu | \prod_{k=1}^{n} \exp(e^{\sum N \omega^{(k)}_N t_N F_k}) | \mu' \rangle.$$  (29)

We emphasize that not all solutions of the type (29) are soliton solutions, but we conjecture that the soliton and multi-soliton solutions are among them. The conjecture that multi-soliton solutions are associated with group elements of the form (28) naturally follows from the well known properties of the multi-soliton solutions of affine Toda equations and of hierarchies of the KdV type, and, in the sine-Gordon theory, it has been explicitly checked in Ref. [6]. Actually, in all these cases, the multi-soliton solutions are obtained in terms of representations of the “vertex operator” type where the corresponding eigenvectors are nilpotent. Then, for each eigenvector $F_k$ there exists a positive integer number $m_k$ such that $(F_k)^m \neq 0$ only if $m \leq m_k$.

This remarkable property simplifies the form of (29) because it implies that $e^{F_k} = 1 + F_k + \ldots + (F_k)^{m_k}/m_k!$, which provides a group-theoretical justification of Hirota’s method.

An interesting feature of the dressing transformations method is the possibility of relating the solutions of different integrable equations. Consider two different integrable hierarchies whose vacuum solutions are compatible, in the sense that the corresponding vacuum Lax operators commute. Then, one can consider the original integrable equations as the restriction of a
larger hierarchy of equations. Consequently, the solutions obtained through the group of dressing transformations can also be understood in terms of the solutions of the larger hierarchy, which implies certain relations among them. (see section 4 of [23] for more details).

5 The tau-functions

According to the discussion in the previous section, the orbits generated by the group of dressing transformations acting on some vacuum provide solutions of certain integrable hierarchies of equations. Making contact with the method of Hirota, the generalized “tau-functions” that will be defined in this section constitute a new set of variables to describe those solutions. One of the characteristic properties of these variables is that they substantially simplify the task of constructing multi-soliton solutions [23]. The group-theoretical interpretation of this property has already been pointed out in the previous section. Tau-functions are given by certain matrix elements in an appropriate representation of the Kac-Moody Group $\hat{G}$. Moreover, the tau-functions corresponding to the multi-soliton solutions are expected to involve nilpotent elements of $\hat{G}$, which is the origin of their remarkable simple form.

The tau-function formulation of the Generalized Drinfel’d-Sokolov Hierarchies of [14] has already been worked out in [23], which, in fact, has largely inspired our approach. However, there are two important differences between our results and those of [23]. Firstly, our approach applies to the affine Toda equations too, and, secondly, it does not rely upon the use of (level-one) vertex operator representations.

At this point, it is worth recalling that the solutions constructed in sections 3 and 4 are completely representation-independent. In contrast, our definition of tau-functions makes use of a special class of representations of the Kac-Moody algebra $\hat{G}$ called “integrable highest-weight” representations. The reason why these representations are called “integrable” is the following. For an infinite-dimensional representation, it is generally not possible to go from a representation of the algebra $\hat{G}$ to a representation of the corresponding group $\hat{G}$ via the exponential map $x \mapsto e^x$. However, the construction does work if, for instance, the formal power series terminates at a certain power of $x$, or if the representation space admits a basis of eigenvalues of $x$. These conditions, applied to the Chevalley generators of $\hat{G}$, single out this
special type of representations.

The generalized tau-functions will be sets of matrix elements of the form indicated on the right-hand-side of (27), considered as functions of the group element $\rho$. They are characterized by the condition that they allow one to parameterize all the components of $\Theta_+$ and $\Theta_-$ required to specify the solutions (26) of the zero-curvature equations (4). As we have discussed before, the tau-functions corresponding to the multi-soliton solutions are expected to have a very simple form. However, in contrast with the original method of Hirota, we cannot ensure in general that the equations of the hierarchy become simpler in terms of this new set of variables.

First, let us discuss the generalized Hirota tau-functions associated with the components of $B$. In equation (27), these components can be isolated by considering the vectors $|\mu_0\rangle$ of an integrable highest-weight representation $L(\tilde{s})$ of $\tilde{G}$ which are annihilated by all the elements in $\hat{G}_{>0}(s)$, i.e., $T |\mu_0\rangle = 0$ and $\langle \mu_0 | T' = 0$ for all $T \in \hat{G}_{>0}(s)$ and $T' \in \hat{G}_{<0}(s)$, respectively. Then, the corresponding tau-functions are defined as

$$
\tau_{\mu_0,\mu'_0}(t) = \langle \mu'_0 | \Psi^{(\text{vac})} \rho \Psi^{(\text{vac})^{-1}} | \mu_0 \rangle \\
= \langle \mu'_0 | e^{\sum_N \varepsilon N t_N} \rho e^{-\sum_N \varepsilon N t_N} | \mu_0 \rangle,
$$

(30)

and, in terms of them, equation (27) becomes just

$$
\langle \mu'_0 | B^{-1} | \mu_0 \rangle = \tau_{\mu_0,\mu'_0}(t).
$$

(31)

where we have denoted

$$
B^{-1} \equiv \left( \Psi^{(\text{vac})} \rho \Psi^{(\text{vac})^{-1}} \right)_0
$$

(32)

By construction, $\hat{G}_0(s)$ always contains the central element $C$ of the Kac-Moody algebra, but it is always possible to split the contribution of the corresponding field in (31). Let $s_q \neq 0$ and consider the subalgebra $\hat{G}^{(q)}$ of $\hat{G}$ generated by the $e_i^\pm$ with $i = 0,\ldots, r$ but $i \neq q$, which is a semisimple finite Lie algebra of rank $r$ ($\hat{G}^{(q)}$ is always simple if $q = 0$). Then, $\hat{G}_0(s) = (\hat{G}_0(s) \cap \hat{G}^{(q)}) \oplus C C$ and, correspondingly, $B$ can be split as $B = b \exp(\nu c)$.

Since the resulting relations between tau-functions and components of the $A_N$’s will be considered as generic changes of variables, we will not generally indicate the intrinsic dependence of the tau-functions on the group element $\rho$. 

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Here, $\nu$ is the field along $C$, and $b$ is a function taking values in the semisimple finite Lie group $G_0^{(q)}$ whose Lie algebra is $\hat{G}_0^{(s)} \cap \hat{G}^{(q)}$. Since $\hat{K} = \sum_{i=0}^r k_i\hat{s}_i$ is the level of the representation $L(\hat{s})$, Eq. (31) is equivalent to
\[
\langle \mu'_0 | B^{-1} | \mu_0 \rangle = e^{-\nu \hat{K}} \langle \mu'_0 | b^{-1} | \mu_0 \rangle = \tau_{\mu_0,\mu'_0}(t) .
\] (33)
Moreover, it is always possible to introduce a tau-function for the field $\nu$. Let us consider the highest-weight vector $|v_q\rangle$ of the fundamental representation $L(q)$, which is obviously annihilated by all the elements in $\hat{G}^{(q)}$. Therefore,
\[
\langle v_q | B^{-1} | v_q \rangle = e^{-\nu k_q^v} = \tau_{v_q,v_q}(t) \equiv \tau_q^{(0)}(t) ,
\] (34)
which leads to
\[
\langle \mu'_0 | b^{-1} | \mu_0 \rangle = \frac{\tau_{\mu_0,\mu'_0}(t)}{(\tau_q^{(0)}(t))^{k_q/k_q^v}} \quad \text{and} \quad \nu = - \ln \frac{\tau_q^{(0)}(t)}{k_q^v} .
\] (35)

Finally, recall that the vectors $|\mu_0\rangle$ form a representation of the semisimple Lie group $G_0^{(q)}$. Therefore, if $L(s)$ is chosen such that this representation is faithful, Eq. (33) allows one to obtain all the components of $b$ in terms of the generalized tau-functions $\tau_{\mu_0,\mu'_0}$ and $\tau_q^{(0)}$. Notice that, in this case, the definition of generalized tau-functions coincide exactly with the quantities involved in the solitonic specialization of the Leznov-Saveliev solution proposed in [20].

Let us now discuss the generalized tau-functions associated with the components of $\Theta^{-1}_\leq$. Consider the gradation $s$ of $\hat{G}$ involved in the definition of the integrable hierarchy. For each $s_i \neq 0$, let us consider the highest-weight vector of the fundamental representation $L(i)$ and define the (right) tau-function vector
\[
|\tau_i^R(t)\rangle = \Psi^{(vac)}_i h \Psi^{(vac)-1}_i |v_i\rangle = e^{\sum_N \varepsilon N t_N} h e^{-\sum_N \varepsilon N t_N} |v_i\rangle .
\] (36)
Notice that $|\tau_i^R(t)\rangle$ is a vector in the representation $L(i)$. Therefore, it has infinite components, and it will be shown soon that the role of the Hirota tau-functions will be played by a finite subset of them. Taking into account that $|v_i\rangle$ is annihilated by all the elements in $g_{>0}(s)$, equation (27) implies
\[
\Theta^{-1}_\leq B^{-1} |v_i\rangle = |\tau_i^R(t)\rangle , \quad i = 0, \ldots, r \quad \text{and} \quad s_i \neq 0 .
\] (37)
The definition (36) is inspired by the tau-function approach of \[11, 25, 23\]. However, in \[25\], and \[23\], the authors consider a unique tau-function |τs⟩ ∈ L(s). In fact, one could equally consider different tau-functions |τs′⟩ associated with any integrable representation L(s′) such that s′i ̸= 0 if, and only if, si ̸= 0. Since the highest weight state of L(s) is obtained by the tensor product of the |vi⟩’s, as

\[ |v_s⟩ = \bigotimes_{i=0}^{r} \{ |v_i⟩ \otimes^{s_i} \} . \]  

(38)

one sees that all these choices lead to the same results, but the one presented here is the most economical.

Since, for any integrable representation, the grading operator (7) can be diagonalized acting on L(s), these tau-functions vectors can be decomposed as

\[ |τ^R_i(t)⟩ = \sum_{-j \in \mathbb{Z}} |τ^{R(-j)}_i(t)⟩, \quad Q_i |τ^{R(-j)}_i(t)⟩ = -j |τ^{R(-j)}_i(t)⟩, \]  

(39)

where we have used that Θ− ∈ ˆG<0(s) and B ∈ ˆG0(s), and Qi indicates the derivation corresponding to the grading operator (7) with sj = δji. Moreover, the highest-weight vector is an eigenvector of the subalgebra ˆG0(s) and, consequently, of B. Therefore,

\[ |τ^{R(0)}_i(t)⟩ = B^{-1} |v_i⟩ = τ^{(0)}_i(t) |v_i⟩ , \]  

(40)

where, τ^{(0)}_i(t) is a C-function, not a vector of L(i), whose definition is

\[ τ^{(0)}_i(t) = ⟨v_i | e^\sum_{N<d} e^{Nt} e^{-\sum_{N<d} ε_{NT}} |v_i⟩ \equiv τ_{t_i,v_i}(t) \]  

(41)

(compare with Eq. (33)). Therefore, Eq. (37) becomes

\[ Θ^{-1} |v_i⟩ = \frac{1}{τ^{(0)}_i(t)} |τ^R_i(t)⟩ , \]  

(42)

\[To compare with (31), notice that |µ0⟩ = |v_i⟩ forms a one-dimensional representation of ˆG0(s) and, consequently, τvis,µ0′(t) vanishes unless |µ0′⟩ = |v_i⟩. Therefore, for non-abelian G0, the required tau-functions τvis,µ0′(t) have to involve the fundamental integrable representations L(j) corresponding to sj = 0, in contrast with |τ^R_i(t)⟩ (see Eq. (33)).\]  

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which is the generalization of the Eq. (5.1) of [25] for general integrable highest-weight representations of \( \hat{G} \). Eq. (42) allows one to express all the components of \( \Theta_- \) in terms of the components of \( | \tau_i^R(0) \rangle \) for all \( i = 0, \ldots, r \) with \( s_i \neq 0 \) (for instance, by using the positive definite Hermitian form of \( L(i) \)). However, it is obvious that only a finite subset of them enter in the definition of the potentials \( A_N \) through Eq. (26).

In exactly the same way, one can introduce another set of “left” tau-function vectors through

\[
\langle \tau_i^L(t) | = \langle v_i | \Psi(\text{vac}) \ h \Psi(\text{vac})^{-1},
\]

which leads to

\[
\langle v_i | \Theta_+ = \langle \tau_i^L(t) | \frac{1}{\tau_i^0(t)},
\]

and allows one to express all the components of \( \Theta_+ \) in terms of the components of \( \langle \tau_i^L(t) | \) for all \( i = 0, \ldots, r \) with \( s_i \neq 0 \).

Summarising, the generalized Hirota tau-functions of these hierarchies consist of the subset of functions \( \tau_{\mu_0,\mu'_0} \) and of components of \( | \tau_i^R \rangle \) and \( \langle \tau_i^L | \) required to parameterize all the components of the potentials \( A_N \) in Eq. (26). Then, for the multi-soliton solutions corresponding to the group element \( \rho \) specified in (28), their truncated power series expansion follows from the possible nilpotency of the eigenvectors \( F_k \) in these representations. For instance, if \( n = 1 \) in (28) and \( F_1^m | \mu_0 \rangle = F_1^m | v_i \rangle = 0 \) unless \( m \leq m_1 \), then

\[
\tau_{\mu_0,\mu'_0}(t) = \tau_{\mu_0,\mu'_0}^0 + \tau_{\mu_0,\mu'_0}^1 + \ldots + \tau_{\mu_0,\mu'_0}^{m_1},
\]

\[
| \tau_i^R(t) \rangle = \sum_{k=0}^{m_1} \frac{1}{k!} e^k \sum_{N} w_N t^N \langle \mu_0' | F_1^k | \mu_0 \rangle,
\]

and

\[
| \tau_i^L(t) \rangle = \sum_{k=0}^{m_1} \frac{1}{k!} e^k \sum_{N} w_N t^N F_1^k | v_i \rangle.
\]

6 The example of the Non Abelian Toda Models

Consider an untwisted affine Kac-Moody algebra \( \hat{G} \) endowed with an integral gradation \( \hat{G} = \oplus_{n \in \mathbb{Z}} \hat{G}_n \) (see (3),(13)). By an affine Kac-Moody algebra we
mean a loop algebra corresponding to a finite dimensional simple Lie algebra \( \mathcal{G} \) of rank \( r \), extended by the center \( C \) and the derivation \( D \).

Let \( \mathcal{M} \) be a two dimensional manifold with local coordinates \( x_+ \) and \( x_- \); \( \hat{\mathcal{G}} \) be an affine Kac-Moody algebra corresponding to a finite dimensional complex simple Lie algebra \( \mathcal{G} \) with the Lie group \( G \); \( \mathcal{A} \) be a flat connection in the trivial holomorphic principal fibre bundle \( \mathcal{M} \times \hat{\mathcal{G}} \rightarrow \mathcal{M} \). Specify the connection in such a way that its \((1,0)\)-component takes values in the subspaces \( \bigoplus_{n=0}^{l} \hat{\mathcal{G}}_n \), and \((0,1)\)-component takes values in \( \bigoplus_{n=0}^{l} \hat{\mathcal{G}}_{-n} \), with \( l \) being a fixed positive integer. In other words, up to a relevant gauge transformation, these components, satisfying the zero curvature condition

\[
\partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0, \quad (46)
\]

are of the form

\[
A_+ = -B F^+ B^{-1}, \quad A_- = -\partial_- B B^{-1} + F^- . \quad (47)
\]

Here \( B \) is a mapping from \( \mathcal{M} \) to the Lie group \( \hat{\mathcal{G}}_0 \) with the Lie algebra \( \hat{\mathcal{G}}_0 \); \( F^\pm \) are mappings to \( \bigoplus_{l=1}^{l} \hat{\mathcal{G}}_{\pm l} \) of the form

\[
F^+ = E_1 + \sum_{m=1}^{l-1} F^+_m, \quad F^- = E_{-l} + \sum_{m=1}^{l-1} F^-_m, \quad (48)
\]

with \( E_{\pm l} \) being some fixed elements of \( \hat{\mathcal{G}}_{\pm l} \); and \( F^+_m, 1 \leq m \leq l-1 \), take values in \( \hat{\mathcal{G}}_{\pm m} \).

Substituting the gauge potentials (47) into (46), one gets the equations of motion

\[
\partial_+ \left( \partial_- B B^{-1} \right) = [E_{-l}, B E_l B^{-1}] + \sum_{n=1}^{l-1} [F^-_n, B F^+_n B^{-1}], \quad (49)
\]

\[
\partial_- F^+_m = [E_l, B^{-1} F^-_{l-m} B] + \sum_{n=1}^{l-m-1} [F^+_n, B^{-1} F^-_n B] , \quad (50)
\]

\[
\partial_+ F^-_m = -[E_{-l}, B F^+_l B^{-1}] - \sum_{n=1}^{l-m-1} [F^-_{n+m}, B F^+_n B^{-1}] . \quad (51)
\]

Since \( Q_s \), defined in (7), and \( C \) are in \( \hat{\mathcal{G}}_0 \), we parametrise \( B \) as

\[
B = b e^{\eta Q_s} e^{\nu C} , \quad (52)
\]
where $b$ is a mapping to $G_0$, the subgroup of $\mathcal{G}_0$ generated by all elements of $\mathcal{G}_0$ other than $Q_s$ and $C$. The fields $\eta$ and $\nu$ correspond to the extension of the loop algebra, and, as we will show below, are responsible for making the system conformally invariant \cite{28, 29}. Clearly, the order of the three factors in (52) is irrelevant, since they commute. In addition, we will use a special basis for the generators of $\hat{G}_0$ such that they are all orthogonal to $Q_s$ and $C$. From (7) one observes that the generators of $\hat{G}_0$ are, besides $C$ and $Q_s$, the elements $H_a^0$, $a = 1, 2, \ldots, r$, of the Cartan subalgebra, and step operators $E_{\pm \alpha}^0$ and $E_{\pm \beta}^1$, such that $\sum_{a=1}^r s_a \lambda_a^\nu \cdot \alpha = 0$, and $\sum_{a=1}^r s_a \lambda_a^\nu \cdot \beta = N_s$. There can be no step operators $E_n^\gamma$, with $|n| > 1$, as explained in appendix C of ref. \cite{23}. Therefore, shifting the Cartan elements as

$$\tilde{H}_a^0 = H_a^0 - \frac{1}{N_s} \text{Tr} \left( H_s H_a^0 \right) C = H_a^0 - \frac{2}{\alpha_a^2 N_s} C,$$

(53)

one gets

$$\text{Tr} \left( C^2 \right) = \text{Tr} \left( C \tilde{H}_a^0 \right) = \text{Tr} \left( Q_s^2 \right) = \text{Tr} \left( Q_s \tilde{H}_a^0 \right) = 0, \quad \text{Tr} \left( Q_s C \right) = N_s,$$

$$\text{Tr} \left( \tilde{H}_a^0 \tilde{H}_b^0 \right) = \text{Tr} \left( H_a^0 H_b^0 \right) = 4\alpha_a \cdot \alpha_b / \alpha_a^2 \alpha_b^2 \equiv \eta_{ab},$$

(54)

for all $a, b = 1 \ldots, r$.

Here we have used $H_a^0 = 2\alpha_a \cdot H^0 / \alpha_a^2$, $\text{Tr} \left( x \cdot H^0 y \cdot H^0 \right) = x \cdot y$, and $\text{Tr} \left( C D \right) = 1$. For more detail of such a special basis, see appendix C of ref. \cite{23}.

Substituting (54) into the equations of motion (49)–(51), one has

$$\partial_+ \left( \partial_- b^{-1} \right) + \partial_+ \partial_- \nu C = e^{\nu \eta} \left[ E_{-l}, b E_l b^{-1} \right] + \sum_{n=1}^{l-1} e^{\nu \eta} \left[ F_n^{-}, b F_n^{+} b^{-1} \right],$$

(55)

$$\partial_- F_m^+ = e^{(l-m)\eta} \left[ E_l, b^{-1} F_{l-m} b \right] + \sum_{n=1}^{l-m-1} e^{\nu \eta} \left[ F_n^{+}, b^{-1} F_n^{-} b \right],$$

(56)

$$\partial_+ F_m^- = -e^{(l-m)\eta} \left[ E_{-l}, b F_{l-m} b^{-1} \right] - \sum_{n=1}^{l-m-1} e^{\nu \eta} \left[ F_n^{-}, b F_n^{+} b^{-1} \right],$$

(57)

$$\partial_+ \partial_- \eta Q_s = 0,$$

(58)

where the last equation is a consequence of the fact that $D$, and hence $Q_s$, can not be obtained as the Lie bracket of any two elements of $\hat{G}$.

The structure of the vacuum of the system (53)–(58) is rather complicated. We will discuss some aspects of it below. However, there is a simple
condition that guarantees the existence of static (vacuum) solutions. If the elements $E_{\pm l}$ satisfy the relation

$$[E_l, E_{-l}] = \beta C,$$

where $\beta = \frac{1}{N_s} \text{Tr}(E_l E_{-l}),$ (59)

then

$$b = 1, \quad F^\pm_m = 0, \quad \eta = 0, \quad \nu = -\beta x_+ x_-,$$

(60)
is a (vacuum) solution of (55)–(58).

Another possibility for vacuum solutions arises when $E_{\pm l}, \quad l > 1,$ belong to a Heisenberg subalgebra of $\hat{G},$ see [24, 15],

$$[E_M, E_N] = \text{Tr}(E_M E_{-M}) M \delta_{M+N,0} C,$$

(61)

where $M, N$ belong to some (infinite) subset $\mathbb{Z}_E$ of the integer numbers $\mathbb{Z}.$ In such cases one has that

$$b = 1, \quad \eta = 0, \quad F^\pm_M = c^\pm_M E_{\pm M}, \quad F^\pm_m = 0, \quad m \notin \mathbb{Z}_E, \quad \nu = -\Omega x_+ x_-,$$

(62)
is a solution of (55)–(58) with $c^\pm_M$ being constants, and

$$\Omega \equiv \beta + \sum_{M=1}^{l-1} \text{Tr}(E_M E_{-M}) M c^+_M c^-_M.$$

(63)

Obviously, the system (55)–(58) may have many more vacuum solutions besides (60) and (62). However, the condition (59) guarantees the existence of at least one vacuum solution. Such a fact, as we will see below, favors the existence of soliton solutions.

The models introduced above are completely characterised by the data

$$\{\hat{G}, Q_s, l, E_{\pm l}\};$$

and we have a quite large class of systems with physical properties crucially depending on a choice of those data.

Equations (55) – (58) are invariant under the conformal transformation

$$x_+ \to f(x_+), \quad x_- \to g(x_-),$$

(64)

with $f$ and $g$ being analytic functions; and with the fields transforming as

$$b(x_+, x_-) \to \tilde{b}(\tilde{x}_+, \tilde{x}_-) = b(x_+, x_-),$$

(65)

$$e^{-\nu(x_+, x_-)} \to e^{-\nu(\tilde{x}_+, \tilde{x}_-)} = (f')^\delta (g')^\delta e^{-\nu(x_+, x_-)},$$

(66)

$$e^{-\eta(x_+, x_-)} \to e^{-\eta(\tilde{x}_+, \tilde{x}_-)} = (f')^{1/l} (g')^{1/l} e^{-\eta(x_+, x_-)},$$

(67)

$$F^+_m(x_+, x_-) \to \tilde{F}^+_m(\tilde{x}_+, \tilde{x}_-) = (f')^{-1+m/l} F^+_m(x_+, x_-),$$

(68)

$$F^-_m(x_+, x_-) \to \tilde{F}^-_m(\tilde{x}_+, \tilde{x}_-) = (g')^{-1+m/l} F^-_m(x_+, x_-),$$

(69)
where the conformal weight $\delta$, associated to $e^{-\nu}$, is arbitrary.

Notice that the Lorentz transformation $x_\pm \rightarrow \lambda^{\pm 1}x_\pm$ is obtained from (14) by taking $f(x_+) = x_+/\lambda$ and $g(x_-) = \lambda x_-$. Equations (55)–(58) are also invariant under the transformation $x_\pm \rightarrow \lambda^{\pm 1}x_\pm$.

Equations (53)–(58) are also invariant under the transformation $s^b(x_+, x_-) \rightarrow h_L(x_-) h^R(x_+)$, (70)

$$F^+_m(x_+, x_-) \rightarrow h_R^{-1}(x_+) F^+_m(x_+, x_-) h_R(x_+),$$

(71)

$$F^-_m(x_+, x_-) \rightarrow h_L(x_-) F^-_m(x_+, x_-) h^{-1}_L(x_-),$$

(72)

where $h_L(x_-)$ and $h_R(x_+)$ are elements of subgroups $\mathcal{H}_L^0$ and $\mathcal{H}_R^0$ of $G_0$, respectively, satisfying the conditions

$$h_R(x_+) E_l h^{-1}_R(x_+) = E_l, \quad h^{-1}_L(x_-) E_{-l} h_L(x_-) = E_{-l}. \quad (73)$$

The left and right gauge transformations commute, and so the gauge group is $\mathcal{H}_L^0 \otimes \mathcal{H}_R^0$. Whenever $\mathcal{H}_L^0$ and $\mathcal{H}_R^0$ have a set of common generators, we get an important subgroup of the gauge group, namely $\mathcal{H}_D \equiv \mathcal{H}_L^0 \cap \mathcal{H}_R^0$. These are global gauge transformations, where the fields are transformed under conjugation ($h_L = h_R^{-1} \equiv h_D = \text{const}$),

$$b \rightarrow h_D b h_D^{-1}, \quad F^\pm_m \rightarrow h_D F^\pm_m h_D^{-1}, \quad (74)$$

and $E_{\pm l} = h_D E_{\pm l} h_D^{-1}$. We discuss the relevance of these transformations below.

## 7 Soliton Solutions

We now perform the dressing transformation, discussed in sections 3 and 4, by taking as an initial configuration a vacuum solution of (55)–(58). As we have said, the model under consideration may have several types of vacuum solutions. However, here we will deal with the solutions of type (60) or (62).

For the vacuum solutions (52), the gauge potentials (17) become

$$A_+^{(0)} = -\mathcal{E}_+, \quad A_-^{(0)} = \mathcal{E}_- + \Omega x_+ C,$$

(75)

with $\mathcal{E}_\pm$ given by

$$\mathcal{E}_\pm \equiv E_{\pm l} + \sum_{N=1}^{l-1} c_N^\pm E_{\pm N}, \quad \text{and so} \quad [\mathcal{E}_+, \mathcal{E}_-] = \Omega C; \quad (76)$$

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where $c^\pm_N$ and $\Omega$ were introduced in (62) and (63), respectively.

They can be written as

$$A^{(0)}_\pm = -\partial_\pm \Psi^{(\text{vac})} \Psi^{(\text{vac})}_\pm^{-1}, \quad \text{with} \quad \Psi^{(\text{vac})} = e^{x_+ E_+} e^{-x_- E_-}. \quad (77)$$

The gauge potentials for the vacuum solution (60) are obtained from (75) by taking $c^\pm_n = 0$. In fact, they are connected by the gauge transformation

$$A^{(0)}_\pm = \tilde{\Psi}^{(\text{vac})} \bigg|_{c^\pm_n = 0} \left( \tilde{\Psi}^{(\text{vac})} \right)^{-1} - \partial_\pm \tilde{\Psi}^{(\text{vac})} \left( \tilde{\Psi}^{(\text{vac})} \right)^{-1}, \quad (78)$$

with

$$\tilde{\Psi}^{(\text{vac})} = \exp[x_+ (E_+ - E_t)] \exp[-x_- (E_- - E_{-t})]. \quad (79)$$

However, in general, the vacuum solutions (60) and (62) may not be connected by any dressing transformation, and, in such a case, the existence of two elements of form (24), is not always possible. Consequently, one can have soliton solutions lying on different orbits under the dressing transformations.

In order to perform the dressing procedure, we take (75) as initial gauge potentials. Then, we obtain, under the dressing procedure, the solutions on the orbit of vacuum (62), and for $c^\pm_n = 0$ those on the orbit of the vacuum (10). From the structure of the dressing transformations and from the fact that the grading operator (7) is never the result of any commutation, since it contains $D$, it follows that the dressing transformations do not excite the field $\eta$. Therefore, from (47), (52), (75) and (26) we get

$$b E_t b^{-1} + \sum_{m=1}^{l-1} b F^+_m b^{-1} = \Theta_\pm \left( E_t + \sum_{n=1}^{l-1} c^+_n E_n + \Theta^{-1}_- \partial_+ \Theta_+ \right) \Theta_\pm^{-1}; \quad (80)$$

$$- \partial_- \nu b^{-1} - (\partial_- \nu + \Omega x_+) C \quad + \quad E_{-t} + \sum_{m=1}^{l-1} F^-_m = \Theta_\pm \left( E_{-t} + \sum_{m=1}^{l-1} c^-_m E_{-m} - \Theta^{-1}_+ \partial_- \Theta_+ \right) \Theta_\pm^{-1}; \quad (81)$$

Note that in the above relations, the fields $b$, $\nu$ and $F^\pm_m$ stand for the solutions on the orbit of the vacuum solution (32). The procedure to construct the
solution requires to split the above equations into the eigensubspaces of the grading operator \([\hat G]\). It is convenient to write
\[
\Theta^> = \exp \left( \sum_{s>0} t^{(s)} \right), \quad \Theta^< = \exp \left( \sum_{s>0} t^{(-s)} \right), \quad \text{where } t^{(\pm s)} \in \hat G_{\pm s}. \quad (82)
\]
The mappings \(t^{(\pm s)}\), for each choice of \(\rho\), are determined from (23) with \(\Psi\) being \(\Psi^{(\text{vac})}\) given in (77). Then, the components of (80) and (82) in each eigensubspace, give an equation connecting the fields with \(t^{(\pm s)}\). Thus the solutions for the fields \(b, \nu\) and \(F^\pm_m\) are determined from \(t^{(\pm s)}\). Such a procedure is rather cumbersome, but fortunately, one needs to know very few \(t^{(\pm s)}\)'s to get the solution. For instance, taking relations (80) and (82) for \(\Theta^>\) with grade components 0 and \(-l\) \((l \text{ and } 0)\), one gets
\[
\Theta_+^{(0)} = h^{-1}_L (x_-), \quad \Theta^-_{(0)} = b e^{(\nu + \Omega x_+) C} h_R (x_+), \quad (83)
\]
with \(h_L (x_-)\) and \(h_R (x_+)\) defined in (73).

From (17), (24), (77) and (83) it follows that
\[
\Theta_-^{-1} \left( h_L (x_-) b e^{(\nu + \Omega x_+) C} h_R (x_+) \right)^{-1} \Theta^+_+ = e^{x_+ \xi_+} e^{-x_- \xi_-} \rho e^{x_- \xi_-} e^{-x_+ \xi_+}. \quad (84)
\]

The space–time dependence of the r.h.s. of the above relation is given explicitly. One can extract the solutions out of (84) by taking the expectation value of its both sides between suitable states of a given representation of \(\hat G\), in a similar way to that one explained in section 4.

The solitons solutions are obtained from (84) by choosing the fixed group element \(\rho\), characterising the dressing transformation, as the exponential of an eigenvector of \(\mathcal{E}_\pm\), i.e.
\[
\rho = e^V. \quad (85)
\]
That is the solitonic specialization discussed in section 4. Indeed, if \(V\) satisfies the relations
\[
[\mathcal{E}_\pm, V] = \omega_\pm V, \quad (86)
\]
then (84) reads as
\[
\exp \left( e^{x_+ \omega_+ - x_- \omega_-} V \right) \equiv \exp \left( e^{\gamma (x-v t)} V \right), \quad (87)
\]
with \(\gamma = \omega_+ + \omega_-\), and \(v = (\omega_- - \omega_+) / (\omega_+ + \omega_-)\), since \(x_\pm = t \pm x\).
Therefore, for each eigenvector $V$, expression (87) corresponds to a solution that travels with a constant velocity $v$ without dispersion. Depending upon the properties of $V$, as we will see below in the examples, such solutions correspond to one–soliton solutions.

The multi–soliton solutions are obtained by taking $\rho$ to be the product of several one–soliton $\rho$’s, i.e.,

$$\rho = e^{V_1} e^{V_2} e^{V_3} \ldots e^{V_N},$$

(88)

with each $V_i$ satisfying $[E_\pm, V_i] = \omega_\pm V_i$.

Notice that, under the global gauge transformations (74), the gauge potentials (47) are transformed as $A_\pm \rightarrow h_D A_\pm h_D^{-1}$. Therefore, since the potentials are pure gauge, $A_\mu = -\partial_\mu TT^{-1}$, one has $T \rightarrow h_D T$, and consequently (23) implies $\Theta_\pm \rightarrow h_D \Theta_\pm h_D^{-1}$ and $\Theta_\pm' \rightarrow h_D \Theta_\pm' h_D^{-1}$. Hence, with solution (84) corresponding to a fixed element $\rho$, a solution, obtained from that by a global gauge transformation (74), is given by (84) with the replacement

$$\rho \rightarrow h_D \rho h_D^{-1},$$

(89)

if the condition $h_D E_\pm h_D^{-1} = E_\pm$ is satisfied. For the solutions on the orbit of the vacuum (60), that is indeed true, since $E_\pm = E_{\pm l}$; see (76). For the solitonic case, one then obtains for each eigenvector $V$ of $E_\pm$, an orbit of equivalent one–soliton (or multi–soliton) solutions generated by $h_D V h_D^{-1}$.

## 8 Masses of fundamental particles and solitons

As we have seen above, the system under consideration is conformally invariant. Therefore, since we do not have a continuum mass spectrum, its fundamental particles have to be massless. However, such a symmetry can be spontaneously broken by choosing a particular constant solution for the field $\eta$, say $\eta = \eta_0$. The resulting theory is then massive. Representing the mapping $B$ as $B \equiv \exp T$, and considering only the linear field approximation, i.e., the free part of the equations of motion (49)–(51), one gets

$$\partial_+ \partial_- T = -v_\eta [E_{-l}, [E_l, T]],$$

(90)

$$\partial_+ \partial_- F^+_m = -v_\eta [E_{-l}, [E_l, F^+_m]],$$

(91)

$$\partial_+ \partial_- F^-_m = -v_\eta [E_{-l}, [E_l, F^-_m]],$$

(92)
where \( v_\eta = e^{i \eta \phi} \).

Therefore, the masses of fundamental particles in such a theory are given by the eigenvalues of the operator \([E_{-l}, [E_l, *]]\) in the subspaces \( \mathcal{G}_n, n = 0, \pm 1, \pm 2, \ldots \pm (l - 1) \), i.e.,

\[
[E_{-l}, [E_l, X]] = \lambda X. \tag{93}
\]

Since \( \partial_+ \partial_- = \frac{1}{4} (\partial_+^2 - \partial_-^2) \), we obtain the masses from the Klein–Gordon type equations (90) – (92) as

\[
m^2 = 4 \lambda v_\eta. \tag{94}
\]

That result constitute a generalization of the arguments used in the abelian and non abelian affine Toda models [30, 23]. Of course, we are interested in those cases where the eigenvalues of the operator \([E_{-l}, [E_l, *]]\) are real and positive on the subspaces under consideration. That will be, in fact, one of the conditions we use to select the data \( \{ \mathcal{G}, Q, l, E_{\pm l} \} \) for defining physical models through (47).

Notice that the field \( e^{i \eta \phi} \) plays the role of a Higgs field, since it not only spontaneously breaks the conformal symmetry, but also because its vacuum expectation value sets the mass scale of the theory. We have here the same mechanism as in non abelian affine Toda theories [31, 23].

Let us explain now, following the reasonings of [31] and [23], that the masses of solitons are also generated by the spontaneous breakdown of the conformal symmetry.

The energy momentum tensor of such theories is of the form (see [21] for more details)

\[
L_{\mu\nu} = \Theta_{\mu\nu} + S_{\mu\nu} \tag{95}
\]

where \( S_{\mu\nu} \) is the improvement term

\[
S_{\mu\nu} \equiv -\frac{k}{l} \text{Tr} \left( Q_s \left( \partial_\mu \left( B^{-1} \partial_\nu B \right) - g_{\mu\rho} \partial_\rho \left( B^{-1} \partial^\rho B \right) \right) \right) = -\frac{k N_s}{l} \left( \partial_\mu \partial_\nu - g_{\mu\nu} \partial^2 \right) \nu, \tag{96}
\]

Due to the fact we are dealing with a conformally invariant theory, \( L_{\mu\nu} \) satisfies

\[
\partial_- L_{++} = 0, \quad \partial_+ L_{--} = 0, \quad L_{+-} = L_{-+} = 0. \tag{97}
\]
Even though it is not traceless, $\Theta_{\mu\nu}$ is symmetric and conserved,

$$\partial^\mu \Theta_{\mu\nu} = 0,$$  \hspace{1cm} (98)

The energy of classical solutions are given by the space integral of the $(0,0)$ component of energy–momentum tensor $L_{\mu\nu}$. In the Lorentz frame where the classical soliton solution is static, the energy should be interpreted as the mass of the soliton. However, since the theory is conformally invariant, it has no mass scale, and the soliton mass should vanish. When the conformal symmetry is spontaneously broken by choosing a particular constant solution for the field $\eta$, we obtain a massive theory. Construct the energy–momentum tensor of such a theory as follows. Clearly, the tensor $\Theta_{\mu\nu}$, introduced in (95) and evaluated at any classical solution, satisfies (98). Therefore, the tensor defined by

$$\Theta_{\mu\nu}^{\text{broken}} \equiv \Theta_{\mu\nu} |_{\eta=\text{constant}},$$  \hspace{1cm} (99)

is symmetric and conserved,

$$\partial^\mu \Theta_{\mu\nu}^{\text{broken}} = 0,$$  \hspace{1cm} (100)

since $\eta = \text{constant}$ is a solution of the equations of motion. Then, let the energy in the massive theory be proportional to the space integral of $\Theta_{00}^{\text{broken}}$. Using (96) and (95), we obtain the soliton mass in the form

$$\frac{M}{\sqrt{1 - v^2}} \equiv - \int_{-\infty}^{\infty} dx \Theta_{00}^{\text{broken}} + E_{\text{vac.}} = - \frac{k N_s}{l} \partial_x (\nu + \Omega x_+ x_-) \big|_{-\infty}^{\infty},$$  \hspace{1cm} (101)

because the integral of $L_{00}^{\text{red.}}$ vanishes by the above arguments. Here $v$ is the soliton velocity in the units of the speed of the light. Notice that we have subtracted the energy $E_{\text{vac.}}$ of the vacuum solution which is, in fact, divergent. Of course, the vacuum solution is not unique, and it is not clear which one provides the absolute minimum of the energy. We will use the following prescription for the soliton mass formula. For the soliton solutions lying, under the dressing transformations, on the orbit of the vacuum solution (62), we take $\Omega$ in (101) to be that one given in (63). However, for those soliton solutions lying on the orbit of the vacuum (60), we take $\Omega$ in (101) to be equal to the parameter $\beta$ introduced in (59). Such a prescription guarantees the finiteness of the soliton masses.
The soliton masses are determined solely by the behaviour at \(x = \pm \infty\) of the space derivative of the field \(\nu\). That is quite a remarkable fact. In addition, as we now explain, it is very easy to obtain such a behaviour in the general case from the solitonic solutions (87).

Consider an integral gradation of \(\hat G\), with \(s_i' = \frac{\psi^2}{\alpha^2} s_i\), \(\alpha_0 = -\psi\), and \(s_i\) labeling the gradation that defines the model (55)–(58). Consider the integrable highest weight representation with highest weight state \(|\lambda_{s'}\rangle = \bigotimes_{i=0}^r |\hat{\lambda}_i\rangle^{\otimes s_i'}, (102)\), where \(|\hat{\lambda}_i\rangle\) are the highest weight states of the fundamental representations of \(\hat G\), and \(\hat{\lambda}_i\) are the corresponding fundamental weights of \(\hat G\).

Then it is possible to show [21] that taking the expectation value of both sides of (84) in such state, one gets (with the gauge choice \(h_L(x-) = h_R(x+) = 1\))

\[
e^{-\left(\nu + \Omega x + x\right) N_s \frac{\psi^2}{2}} = \langle \lambda_{s'} | e^{x_+ \xi_+} e^{-x_- \xi_-} \rho e^{x_- \xi_-} e^{-x_+ \xi_+} | \lambda_{s'} \rangle. (103)
\]

Now, choosing \(\rho\) to be the exponential of an eigenvector of \(\xi_\pm\),

\[
[\xi_\pm, V] = \omega_\pm V, (104)
\]

we obtain a soliton solution

\[
e^{-\left(\nu + \Omega x + x\right) N_s \frac{\psi^2}{2}} = \langle \lambda_{s'} | e^{\Gamma V} | \lambda_{s'} \rangle (105)
\]

with \(\Gamma = \omega_+ x_+ - \omega_- x_- \equiv \gamma (x - vt)\).

Suppose \(V\) is an operator in such a representation for which there is a positive integer \(N'_V\), such that

\[
\langle \lambda_{s'} | V^n | \lambda_{s'} \rangle = 0 \quad \text{for} \quad n > N'_V. (106)
\]

Then the soliton mass is easily obtained from (101), where for \(\gamma > 0 \ (\gamma < 0)\) only the upper (lower) limit \(x = \infty \ (x = -\infty)\) contributes in the integral\(^4\),

\[
M = \frac{2 \ k \sqrt{N'_V}}{l} \gamma \sqrt{1 - v^2} = \frac{2 \ k \ N'_V}{l} \sqrt{\omega_+ \omega_-}. (107)
\]

\(^4\)We point out that the soliton mass formula (107) could be equally obtained by defining the mass through the momentum formula, instead through the energy like in (101), as \(\frac{M}{\sqrt{1-v^2}} = \int dx \Theta_{01}^{\text{broken}}\). In this case, we do not have to subtract the vacuum momentum, since it vanishes.
Notice that we must have $\omega_+ \omega_- > 0$ in order to have the soliton velocity $v = (\omega_- - \omega_+)/(\omega_- + \omega_+)$, not exceeding the light velocity ($c = 1$).

The soliton mass formula \( (107) \) has some remarkable properties. One of them concerns the relation particle–soliton in the theory, indicating some sort of duality similar to the electromagnetic duality of some four dimensional gauge theories possessing the Bogomolny (monopole) limit \[12\]. As we have seen, the soliton solutions are created by the eigenvectors \( V \) of \( E_{\pm} \). From \( (104) \) one has \([E_+, [E_-, V]] = \omega_+ \omega_- V\). Expanding \( V \) over the eigenvectors of the grading operator \( Q_s \) as \( V = \sum n V(n) \), one observes that \([E_+, [E_-, V(n)] = \omega_+ \omega_- V(n)\). Therefore, if some \( V(n) \in \hat{G}_m, n = 0, \pm 1, \pm 2, \ldots \pm (l - 1)\), does not vanish, it implies that \( V(n) \) must be one of the eigenvectors \( X \) in \( \{ \mathcal{X} \} \). Then we associate a soliton with a fundamental particle. In addition, we have \( \lambda \equiv \omega_+ \omega_- \), and, consequently, from \( (14) \) and \( (107) \), the masses of the corresponding soliton and fundamental particle are determined by the same eigenvalue. In fact, we have from \( (14) \) and \( (107) \), with \( v_\eta = 1 \), that

\[
M_{\text{sol}} = \frac{2}{\psi^2} \frac{k}{l} N_\lambda^m \text{m}^\text{part}.
\] (108)

Of course, in the expansion of \( V \), we may have more than one non vanishing \( V(n) \), with \( n = 0, \pm 1, \pm 2, \ldots \pm (l - 1) \). Then we would associate a one–soliton solution to more than one fundamental particle. The counting of one–soliton solutions has to be better analysed in each particular case. We discuss this issue in section \[14\].

9 The matter fields

It is clear from \( (65)-(69) \), that the massive fields associated with non vanishing grade (namely \( F_{m}^{\pm} \)), are chiral fields with non vanishing spins, in contrast with the Toda type fields. In fact, we show that the free equations for such fields take the form of the massive Dirac equation, as could be expected from general covariance arguments.

Consider the subspace \( \hat{G}_m \) for \( 0 < m < l \). Let \( \hat{G}^{(F)}_m \) be the subspace of \( \hat{G}_m \), generated by the eigenvectors of \([E_{\pm}, [E_\ell, \cdot]]\) with non zero eigenvalues, i.e.,

\[
\hat{G}^{(F)}_m \equiv \{ T^{(m)} \in \hat{G}_m | \lambda^{(m)} \neq 0 \},
\] (109)
where $\lambda^{(m)}$ is defined as
\[
[E_{-l}, [E_l, T^{(m)}]] = [E_l, [E_{-l}, T^{(m)}]] = \lambda^{(m)} T^{(m)}. \tag{110}
\]
Decompose the subspace $\hat{G}_m$, as a vector space, into the sum
\[
\hat{G}_m = \hat{G}_m^{(F)} + \hat{G}_m^{(K)}, \tag{111}
\]
where $\hat{G}_m^{(K)}$ is the complement of $\hat{G}_m^{(F)}$ in $\hat{G}_m$.

It is possible to show (see [21] for details) that the subspaces $\hat{G}_{-l+m}^{(F)}$ and $\hat{G}_m^{(F)}$ are isomorphic. The mapping is given by the action of $E_{-l}$ on $\hat{G}_m^{(F)}$, or equivalently by the action of $E_l$ on $\hat{G}_{-l+m}^{(F)}$. Therefore, we can put in one-to-one correspondence the fields in $\hat{G}_{-l+m}^{(F)}$ and $\hat{G}_m^{(F)}$. Then, one can show that each pair of such fields constitute a Dirac spinor under the two dimensional Lorentz group, and their equations of motion can indeed be written in the form of (obviously not free) Dirac equations. Consequently, we interpret the massive fields associated to generators of non zero grading as matter fields.

10 An example of a special class of models

There is a class of models possessing a $U(1)$ Noether current, which, under some circumstances, is proportional to a topological current. That occurs for those models where the grade $l$ of the operator $E_l$, introduced in (112), is equal to the integer $N_s$ defined in (7). In addition, it is necessary that the operators $E_{\pm N_s}$ satisfy the condition
\[
z E_{-N_s} = \mu z^{-1} E_{N_s} \in \text{center of } \hat{G}_0 \tag{112}
\]
where, $\mu$ is some constant independent of $z$, and $z$ is a complex variable used to realize the generators of the affine Kac-Moody algebra $\hat{G}$, in terms of those of the finite simple Lie algebra $G$ as
\[
H_a^n \equiv z^n H_a, \quad E_a^n \equiv z^n E_a, \quad D \equiv \frac{d}{dz} \tag{113}
\]
That means that the “projections” of $E_{\pm N_s}$ onto $\hat{G}_0$, are parallel and lie in the center of $\hat{G}_0$. When those condition are satisfied it is possible to gauge fix the...
conformal symmetry, such that a special $U(1)$ Noether charge is proportional to a topological charge.

In this section we discuss an example where that happens. It corresponds to the principal gradation of $sl(2)^{(1)}$ with $l = 2$. Let us denote by $H^n$, $E^n_\pm$, $D$ and $C$ the Chevalley basis generators of the $sl(2)^{(1)}$. The commutation relations are

\[
[H^m, H^n] = 2mC \delta_{m+n,0}, \quad [E^n_+, E^n_-] = H^{m+n} + mC \delta_{m+n,0},
\]

\[
[H^m, E^n_\pm] = \pm 2E^m_\pm, \quad [D, T^m] = mT^m, \quad T^m \equiv H^m, E^m_\pm;
\]

all other commutation relations are trivial. The grading operator for the principal gradation ($s = (1, 1)$) is $Q \equiv \frac{1}{2}H^0 + 2D$. Then the eigensubspaces are $\hat{G}_0 = \{H^0, C, Q\}$, $\hat{G}_{2n+1} = \{E^n_+, E^n_{n+1}\}$, with $n \in \mathbb{Z}$, and $\hat{G}_{2n} = \{H^n\}$, with $n \in \{\mathbb{Z} - 0\}$.

The mapping $B$ is parametrised as

\[
B = e^{\varphi H^0} e^{\bar{\nu} C} e^{\eta Q} = e^{\varphi \bar{H}^0} e^{\nu C} e^{\eta Q},
\]

where $\bar{H}^0 = H^0 - \frac{1}{2}C$ is the Cartan generator in the special basis introduced in (53), and so $\bar{\nu} = \nu - \frac{1}{2}\varphi$.

In the case $l = 2$, we choose

\[
E_2 \equiv m H^1, \quad E_{-2} \equiv m H^{-1},
\]

where $m$ is a constant. We then have

\[
[E_2, [E_{-2}, E^n_\pm]] = 4m^2 E^n_\pm.
\]

Therefore, each of the subspaces $\hat{G}_{\pm}$ has two generators with the same eigenvalue $4m^2$. Following section 9 we write

\[
F_1^+ = 2\sqrt{im} \left( \psi_R E^0_+ + \bar{\psi}_R E^1_+ \right), \quad F_1^- = 2\sqrt{im} \left( \psi_L E^{-1}_+ - \bar{\psi}_L E^0_- \right),
\]

and introduce the Dirac fields

\[
\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}; \quad \bar{\psi} = \begin{pmatrix} \bar{\psi}_R \\ \bar{\psi}_L \end{pmatrix}
\]

From (14) we obtain the masses of the particles,

\[
m_\varphi = m_{\bar{\nu}} = m_\eta = 0; \quad m_\psi = 4m;
\]
The equations of motion derived from (55)–(58), are
\[
\partial^2 \varphi = -4m_\psi \overline{\gamma}_5 e^{\eta + 2 \varphi} \gamma_5 \psi, \quad (122)
\]
\[
\partial^2 \tilde{\nu} = -2m_\psi \overline{\psi}(1 - \gamma_5) e^{\eta + 2 \varphi} \gamma_5 \psi - \frac{1}{2} m_\psi^2 e^{2\eta}, \quad (123)
\]
\[
\partial^2 \eta = 0, \quad (124)
\]
\[
i \gamma^\mu \partial_\mu \psi = m_\psi e^{\eta + 2 \varphi} \gamma_5 \psi, \quad (125)
\]
\[
i \gamma^\mu \partial_\mu \tilde{\psi} = m_\psi e^{\eta - 2 \varphi} \gamma_5 \tilde{\psi}, \quad (126)
\]
where the gamma matrices are defined as
\[
\gamma_0 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \gamma_0 \gamma_1, \quad \text{and} \quad \bar{\psi} \equiv \tilde{\psi}^T \gamma_0. \quad (127)
\]
and \( \gamma_5 = \gamma_0 \gamma_1 \), and \( \bar{\psi} \equiv \tilde{\psi}^T \gamma_0 \). Recall that \( \partial^2 = \partial^2_t - \partial^2_x \), \( x_\pm = t \pm x \). The corresponding Lagrangian has the form
\[
\frac{1}{k} \mathcal{L} = \frac{1}{4} \partial_\mu \varphi \partial^\mu \varphi + \frac{1}{4} \partial_\mu \varphi \partial^\mu \eta + \frac{1}{2} \partial_\mu \tilde{\nu} \partial^\mu \eta - \frac{1}{8} m_\psi^2 e^{2\eta} + i \bar{\psi} \gamma^\mu \partial_\mu \psi - m_\psi \bar{\psi} e^{\eta + 2 \varphi} \gamma_5 \psi. \quad (128)
\]
It is real (for \( \eta = \text{real constant} \)) if \( \bar{\psi} \) is the complex conjugate of \( \psi \), and if \( \varphi \) is pure imaginary. This will be true for the soliton solution as we shall see below.

Notice that such model is invariant under the transformations
\[
x_+ \leftrightarrow x_-; \quad \psi_R \leftrightarrow e^{i\theta} \psi_L; \quad \tilde{\psi}_R \leftrightarrow -e^{i\theta} \psi_L; \quad \varphi \leftrightarrow \varphi; \quad \eta \leftrightarrow \eta; \quad \nu \leftrightarrow \nu \quad (129)
\]
where \( \epsilon = \pm 1 \). It should be interpreted as the product CP of charge conjugation times parity. Parity alone is clearly violated.

The generator \( H^0 \in \hat{G}_0 \) commutes with \( E_{\pm 2} \), and, therefore, the gauge symmetry (70)–(72) of the model is \( U(1)_L \otimes U(1)_R \),
\[
h_L(x_-) = e^{\xi(x_-) H^0}, \quad h_R(x_+) = e^{\xi(x_+) H^0}. \quad (130)
\]
Since the generators of \( U(1)_L \) and \( U(1)_R \) are the same, we have the global gauge transformations (44) generated by \( h_D \equiv h_L = h_R^{-1} \equiv e^{i\theta H^0/2} \) (\( \theta = \text{const.} \)). The fields are transformed as
\[
\psi \rightarrow e^{i\theta} \psi \quad \tilde{\psi} \rightarrow e^{-i\theta} \tilde{\psi} \quad \varphi \rightarrow \varphi; \quad \tilde{\nu} \rightarrow \tilde{\nu}; \quad \eta \rightarrow \eta; \quad (131)
\]
and the corresponding Noether current is

\[ J^\mu = \bar{\psi} \gamma^\mu \psi, \quad \partial_\mu J^\mu = 0. \] (132)

The fields \( \psi \) and \( \tilde{\psi} \) have charges 1 and \(-1\), respectively; and \( \varphi, \tilde{\nu} \) and \( \eta \) have charge zero.

Let us next see how the general arguments given above concerning Noether and topological charges apply here. The topological current and charges are

\[ j^\mu = \frac{1}{2\pi i} \epsilon^{\mu\nu} \partial_\nu \varphi, \quad Q_{\text{topol.}} \equiv \int dx j^0, \] (133)

Indeed, the Lagrangian (128) has infinitely many degenerate vacua due to the invariance under \( \varphi \rightarrow \varphi + i\pi \). Making use of the field equations, one easily verifies that

\[ \partial_\mu \left[ i\bar{\psi} \gamma_5 \gamma^\mu \psi + \frac{1}{2} \partial^\mu \varphi \right] = 0 \] (134)

Combining this equation with the conservation of the vector current \( \bar{\psi} \gamma^\mu \psi \), one deduces that there exist two charges defined by

\[ J = -i\bar{\psi}_R \psi_R + \frac{1}{2} \partial_+ \varphi, \quad \bar{J} = i\bar{\psi}_L \psi_L + \frac{1}{2} \partial_- \varphi \] which satisfy \( \partial_- J = 0, \partial_+ \bar{J} = 0 \). We now make a “gauging fixing” of the conformal symmetry by choosing \( J = \bar{J} = 0 \). We call it a “gauging fixing”, because any values of \( J \) and \( \bar{J} \) can be transformed to zero by a conformal transformation. This gives, altogether,

\[ \frac{1}{2\pi i} \epsilon^{\mu\nu} \partial_\nu \varphi = \frac{1}{\pi} \bar{\psi} \gamma^\mu \psi, \] (135)

so that the topological and Noether currents are proportional. As discussed at the beginning of this section, that is a consequence of the fact that \( E_{\pm2} \) satisfies (112).

Let us turn to the Noether charge which here is simply the fermion number. It should be defined such that it satisfies the Poisson bracket relation

\[ i \{ \psi, Q_{\text{Noether}} \}_{\text{P.B.}} = \psi \] (136)

Since the coupling constant \( k \) was taken as an overall factor, this is satisfied by

\[ Q_{\text{Noether}} = k \int dx \bar{\psi} \gamma^0 \psi \] (137)
so that

\[ Q_{\text{topol.}} = \frac{1}{k\pi} Q_{\text{Noether}} \quad (138) \]

As argued in general, this means that \( k \) should only take discrete values as expected, since our actions are related with the one of WZNW.

Let us now construct the soliton solutions. The operators \( E_{\pm 2} \) given in (117), lie in the homogeneous Heisenberg subalgebra generated by \( H^n \), with the commutation relations (114). Such a subalgebra has no generators of grade \( \pm 1 \) for the principal gradation. Therefore, the model under consideration has no vacuum solutions of type (62). Then, from (76), we get

\[ E_{\pm} = E_{\pm 2} = m H^{\pm 1}. \quad (139) \]

We perform the dressing transformation starting from the vacuum solution (60), namely

\[ \varphi = \eta = \psi = \bar{\psi} = 0, \quad \tilde{\nu} = -\frac{1}{8} m^2 \psi x_+ x_- \equiv \nu_0. \quad (140) \]

Now, let \( | \hat{\lambda}_0 \rangle \) and \( | \hat{\lambda}_1 \rangle \) be the highest weight states of two fundamental representations of the affine Kac–Moody algebra \( sl(2)^{(1)} \), respectively the scalar and spinor ones. Then, from (84) with \( \eta = 0 \), we obtain the solutions on the orbit of the vacuum (140),

\[
\begin{align*}
e^{-\varphi} &= \frac{\langle \hat{\lambda}_1 | G | \hat{\lambda}_1 \rangle}{\langle \hat{\lambda}_0 | G | \hat{\lambda}_0 \rangle}, & e^{-(\tilde{\nu}-\nu_0)} &= \langle \hat{\lambda}_0 | G | \hat{\lambda}_0 \rangle, \\
\psi_R &= \sqrt{\frac{m}{i}} \frac{\langle \hat{\lambda}_0 | E_+^0 G | \hat{\lambda}_0 \rangle}{\langle \hat{\lambda}_0 | G | \hat{\lambda}_0 \rangle}, & \bar{\psi}_R &= -\sqrt{\frac{m}{i}} \frac{\langle \hat{\lambda}_1 | G E_+^0 | \hat{\lambda}_1 \rangle}{\langle \hat{\lambda}_1 | G | \hat{\lambda}_1 \rangle}, \\
\psi_L &= -\sqrt{\frac{m}{i}} \frac{\langle \hat{\lambda}_1 | G E_0^0 | \hat{\lambda}_1 \rangle}{\langle \hat{\lambda}_1 | G | \hat{\lambda}_1 \rangle}, & \bar{\psi}_L &= -\sqrt{\frac{m}{i}} \frac{\langle \hat{\lambda}_0 | G E_+^{-1} | \hat{\lambda}_0 \rangle}{\langle \hat{\lambda}_0 | G | \hat{\lambda}_0 \rangle}.
\end{align*}
\]

where

\[ G \equiv e^{x_+ \mathcal{E}_+} e^{-x_+ \mathcal{E}_-} \rho e^{x_- \mathcal{E}_-} e^{-x_- \mathcal{E}_+}. \quad (142) \]

In order to get the soliton solutions, we choose the fixed mapping \( \rho \) to be an exponentiation of an eigenvector of \( \mathcal{E}_\pm \) (solitonic specialization); namely, \( \rho = e^V \), with \( [\mathcal{E}_\pm, V] = \omega_\pm V \). Therefore,

\[ G = \exp \left(e^\Gamma V\right) \quad \text{with} \quad \Gamma = \omega_+ x_+ - \omega_- x_- \equiv \gamma (x-v t). \quad (143) \]
In this case the eigenvectors of $\mathcal{E}_\pm$ are

$$V_\pm(z) = \sum_{n \in \mathbb{Z}} z^{-n} E^n_\pm. \quad (144)$$

Indeed,

$$[\mathcal{E}_+, V_\pm(z)] = \pm 2m z V_\pm(z) \equiv \omega^\pm_+ V_\pm(z), \quad (145)$$

$$[\mathcal{E}_-, V_\pm(z)] = \pm 2m/z V_\pm(z) \equiv \omega^\pm_- V_\pm(z). \quad (146)$$

The solution, associated with $V_+(z)$, is

$$\nu = \nu_0, \quad \varphi = \tilde{\psi} = 0, \quad \psi = \sqrt{\frac{m}{i}} e^\Gamma \begin{pmatrix} z \\ -1 \end{pmatrix}; \quad (147)$$

while those, associated with $V_-(z)$, is given by

$$\nu = \nu_0, \quad \varphi = \psi = 0, \quad \tilde{\psi} = -\sqrt{\frac{m}{i}} e^{-\Gamma} \begin{pmatrix} 1 \\ 1/z \end{pmatrix}, \quad (148)$$

where

$$\Gamma = 2m(zx_+ - \frac{1}{z}x_-) \equiv \gamma (x - vt) \quad (149)$$

The masses of these solutions are obtained from (107). Here the relevant state $| \lambda_{\nu'} \rangle$ in (106) is

$$| \lambda_{\nu'} \rangle = | \hat{\lambda}_0 \rangle \otimes | \hat{\lambda}_1 \rangle. \quad (150)$$

Using level one vertex operators, one can verify that

$$\langle \hat{\lambda}_i | (V_\pm(z))^n \mid \hat{\lambda}_i \rangle = 0, \quad \text{for } n \geq 1 \text{ and } i = 0, 1. \quad (151)$$

Therefore, $N'_V = 0$ in (106), and from (107) one gets that the masses of the solutions (147) and (148) vanish. Such solutions correspond to the objects which travel with velocities $v = \pm (1 - z^2)/(1 + z^2)$; and keeping $z^2 > 0$, one has $|v| < 1$. Therefore, these solutions cannot be interpreted as solitons (particles), since they would correspond to massless particles traveling with velocity smaller that light velocity. We should interpret them as vacuum configurations, since they have the same energy as vacuum (140).

The true soliton solutions of the system are constructed as follows. Notice that $V_+(z)$ and $V_-(z)$ have the same eigenvalues. Therefore, any linear
combination of them, leads to solutions traveling with a constant velocity without dispersion. So, we let
\[ V(a_{\pm}, z) \equiv \sqrt{i} (a_+ V_+(z) + a_- V_-(z)); \]  
(152)

\[ [\mathcal{E}_+, V(a_{\pm}, z)] = 2mz V(a_{\pm}, z), \qquad [\mathcal{E}_-, V(a_{\pm}, z)] = \frac{2m}{z} V(a_{\pm}, z), \]  
(153)

and so \( \omega_+ = 2mz \) and \( \omega_- = \frac{2m}{z} \). The particular factor \( \sqrt{i} \) is chosen such that the reality condition will be obeyed with \( a_- = a_+^* \). Again, using level one vertex operators, one can verify that\(^5\)

\[ \langle \lambda_{s'} \mid V(a_{\pm}, z)^n \mid \lambda_{s'} \rangle = 0 \quad \text{for } n > 4. \]  
(154)

Therefore, \( N'_V = 4 \) in (106), and from (107) with \( \psi^2 = 2 \), and \( \psi \) being the highest root of \( \text{sl}(2) \), one gets that the mass of such solutions is

\[ M = 8k m = 2k m_{\psi}, \]  
(155)

where \( k \) is the coupling constant appearing in the Lagrangian (128). The solutions generated by (152), have two parameters, namely \( a_{\pm} \). One parameter is always present, because one can scale an eigenvector of \( \mathcal{E}_\pm \) without changing the width \( \gamma \) and velocity \( v \) of the soliton, obtained from the eigenvectors \( \omega_{\pm} \); see (87). However, in this case, the second parameter comes from a symmetry. As we have pointed out in (89), associated to the fixed element \( \rho = e^{V(a_{\pm}, z)} \), we have an orbit of equivalent solutions due to the global transformations (131),

\[ V(a_{\pm}, z) \to \sqrt{i} \left( a_+ e^{i\theta} V_+(z) + a_- e^{-i\theta} V_-(z) \right). \]  
(156)

The explicit form of the solutions generated by (152), is obtained using (141),

\[ \varphi = \log \left( \frac{1 + i\sigma e^{2\Gamma}}{1 - i\sigma e^{2\Gamma}} \right), \]  
(157)

\[ \tilde{\nu} = -\log \left( 1 + i\sigma e^{2\Gamma} \right) - \frac{1}{8} m_{\psi}^2 x_+ x_-, \]  
(158)

\[ \eta = 0; \]  
(159)

\(^5\)Notice that the truncation occurs for powers greater than 4, and not 2, because \( \langle \lambda_{s'} \rangle \) lies in the tensor product representation, see (150).
\[ \psi = a_+ \sqrt{m} e^{\Gamma} \left( \frac{z}{1 + i\sigma e^{2\Gamma}} \right), \quad \bar{\psi} = a_- \sqrt{m} e^{\Gamma} \left( \frac{z}{1 - i\sigma e^{2\Gamma}} \right); \quad (160) \]

where \( \Gamma \) is given in (149), and \( \sigma = a_+ a_- z/4 \). Keeping \( m \) and \( z \) real, we have the mass \( M \) of the soliton, from (155), real and positive, and also the parameters \( \gamma \) and \( v \) (149) are real. The reality condition is obeyed if \( a_- = a_+^* \), as anticipated. At this point, it is useful to re-express the expressions just given in terms of the physical parameters of the soliton. Using equations (127) and (121) and one deduces that

\[ \gamma = \frac{m_\psi}{\sqrt{1 - v^2}}, \quad z = \frac{\sqrt{(1 - v)/(1 + v)}}. \quad (161) \]

Moreover, since \( a_\pm \) are complex conjugate, we may write

\[ a_\pm = e^{\pm i\theta} 2 \sqrt{\frac{\sigma}{z}}. \quad (162) \]

The dependence upon space-time appears only through \( \sqrt{\sigma} \exp(\Gamma) \). We will write

\[ \sqrt{\sigma} e^{\Gamma} = \exp((\gamma (x - x_0 - vt)) \quad (163) \]

where \( x_0 \) is the position of the soliton at time zero. Then we have

\[ \varphi = 2i \arctan \left( \exp \left( \frac{2m_\psi (x - x_0 - vt)}{\sqrt{1 - v^2}} \right) \right), \quad (164) \]

which is the sine–Gordon soliton. The Dirac fields are given by

\[ \psi = e^{i\theta} \sqrt{m_\psi} e^{m_\psi (x - x_0 - vt)/\sqrt{1 - v^2}} \left( \frac{1 + i\sigma}{1 + i\sigma e^{2\Gamma}} \right)^{1/4} \frac{1}{1 + i\sigma e^{2\Gamma} (x - x_0 - vt)/\sqrt{1 - v^2}} \left( \frac{1 + i\sigma}{1 - i\sigma e^{2\Gamma} (x - x_0 - vt)/\sqrt{1 - v^2}} \right)^{1/4} \quad (165) \]

and \( \bar{\psi} \) is the complex conjugate of \( \psi \). Thus the only parameters are the soliton mass and velocity, together with the angle \( \theta \) which reflects the global invariance (131). Notice that the sign of the topological charge can be reversed by reversing the sign of \( z \). Therefore, the solutions (164)–(165) contain the sine–Gordon soliton and anti–soliton.

\[ ^6 \text{by convention, we choose } \sigma \text{ to be positive} \]
Finally, we come to the very important feature of the present model already mentioned above in general, namely it is clear from the explicit expressions Eqs. 165 that $\psi$ vanishes exponentially when $x - x_0 \to \pm \infty$, so that the Dirac field is confined inside the soliton. That this must be true is of course a general consequence of Eq. 135 which may be verified directly on the explicit solution. This phenomenon has been much studied for electron phonon systems. Models of a similar type describe the electron self-localization in quasi-one-dimensional dielectrics (for recent reviews see [32], [33]). At low temperature these systems go over to dielectric states characterized by charge density waves which can be constructed on the basis of the Peierls model. The continuous limits are described by Lagrangians similar to Eq. 128. Discussing this important issue is beyond the scope of the present article, so we will not dwell upon it here. Let us simply recall that the typical example of the polyacteline molecule was much discussed in connection with fermion number fractionization [34]. Clearly, on the other hand one may regard our soliton solution a sort of one dimensional bag model for QCD. In this connection let us note that, if we introduce the two-by-two matrix $U = \exp(\eta + 2\varphi \gamma_5)$, we may rewrite the Lagrangian Eq. 128 as

$$L = \frac{1}{16} \left\{ \text{tr} \left[ U^{-1} \partial_\mu U \frac{1 + \gamma_5}{2} U^{-1} \partial^\mu U \right] - \frac{1}{2} \text{tr} \left[ U^{-1} \partial_\mu U \right] \text{tr} \left[ U^{-1} \partial^\mu U \right] \right\}$$

$$+ i \bar{\psi} \gamma_\mu \partial_\mu \psi - \bar{\psi} U \psi - \frac{m^2}{8} \text{det}(U),$$

(166)

which is similar to a two-dimensional version of the low energy effective action for QCD (see e.g. [33]).

11 Conclusions

From the common description based on zero curvature equations, a unified treatment has been given of the procedures to obtain solutions of nonlinear equations for a large class of integrable hierarchies including the generalized KdV and non-abelian Toda coupled to matter. Interesting examples of them have been discussed in detail, taking advantage of the simplifications achieved, and the same procedure can be equally well applied to more complex equations. The role of the tau functions has been clarified, as specific matrix elements in special integrable highest weight representations (for any
level). The results are a further step towards establishing the conjecture that all multisoliton solutions lie in the orbit of the vacuum, generated by dressing transformations, which are now clearly related to the tau functions and solitonic specialization. These results should be very useful for the classification of integrable theories in two dimensions and for the generalization to the quantum case and to higher dimensions.

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