Distributional Representation of Longitudinal Data: Visualization, Regression and Prediction

Álvaro Gajardo¹, Xiongtao Dai*,² and Hans-Georg Müller†

¹Department of Statistics, University of California, Davis, Davis, CA 95616 U.S.A.
²Department of Statistics, Iowa State University, Ames, IA 50011, U.S.A.

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Abstract

We develop a representation of Gaussian distributed sparsely sampled longitudinal data whereby the data for each subject are mapped to a multivariate Gaussian distribution; this map is entirely data-driven. The proposed method utilizes functional principal component analysis and is nonparametric, assuming no prior knowledge of the covariance or mean structure of the longitudinal data. This approach naturally connects with a deeper investigation of the behavior of the functional principal component scores obtained for longitudinal data, as the number of observations per subject increases from sparse to dense. We show how this is reflected in the shrinkage of the distribution of the conditional scores given noisy longitudinal observations towards a point mass located at the true but unobservable FPCs. Mapping each subject’s sparse observations to the corresponding conditional score distribution leads to useful visualizations and representations of sparse longitudinal data. Asymptotic rates of convergence as sample size increases are obtained for the 2-Wasserstein metric between the true and estimated conditional score distributions, both for a $K$-truncated functional principal component representation as well as for the case when $K = K(n)$ diverges with sample size $n \to \infty$. We apply these ideas to construct predictive distributions aimed at predicting outcomes given sparse longitudinal data.

Key words and phrases: Functional Data Analysis, Functional Principal Components, Wasserstein Metric, Sparse Design, Sparse-to-Dense, Predictive Distribution.

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1 Introduction

The study of data that consist of samples of random trajectories, also known as Functional Data Analysis (FDA), has found a wide range of applications (Horvath and Kokoszka 2012; Wang et al. 2016). For longitudinal studies, commonly conducted in the social and biomedical sciences, functional principal component analysis (FPCA), a core technique of FDA, was shown to play a fundamental role. A major reason for this is that FPCA can be naturally adjusted to account for the commonly observed sparsity of the available observations per subject. These observations are inherently correlated and are often available at only a few irregular times and are possibly contaminated with measurement error; FPCA thus provides an important bridge between functional and longitudinal data, especially in the Gaussian case, where errors and random trajectories have joint Gaussian distributions (Yao et al. 2005a).

FPCA characterizes the main modes of variation in the underlying trajectories (Kleffe 1973; Castro et al. 1986) and allows to achieve dimension reduction of the inherently infinite-dimensional functional objects by considering just a few principal components that explain most of the variation. When subjects are recorded densely over time, one can consistently recover the entire random trajectories from the available data, simply by employing, for example, Riemann sums to recover the integrals that correspond to projections of the trajectories on the eigenfunctions of the auto-covariance operator of the underlying stochastic process. These integrals then correspond to the unobserved FPCs, and their approximation improves as the number of observations per subject increases (Müller 2005). However, functional data sparsely observed over time with noisy measurements, which are the quintessential observations in longitudinal studies, pose a major challenge, as here the number of observations per subject is constrained and can be as low as 2 or 3, so that simple Riemann sums become rather infeasible.

To address the interface between functional data analysis and sparse longitudinal studies, Yao et al. (2005a) introduced the Principal Analysis through Conditional Expectation (PACE) approach which recovers the underlying trajectories by targeting the best prediction of the FPCs conditional on the observations, a quantity that can be consistently estimated based on consistent estimates of population quantities such as mean and covariance functions. These estimates are nonparametric and are obtained by pooling all sparse observations across subjects.
together and thus borrowing strength from all subjects in the estimation step, and no prior knowledge of the structure of either covariance or mean functions is needed beyond some basic smoothness properties. Even though the predicted FPCs obtained in this way are unbiased, they have non-vanishing variance and are thus not consistent. This often produces a bias when considering subsequent modeling such as nonlinear functional regression. Predicting a scalar response \( Y \) from sparsely observed predictor processes \( X(t) \) is also a major challenge even for the simplest case of a functional linear model (Petrovich et al. 2018), due to the inconsistency of the predictors that is a consequence of the sparsity of the longitudinal observations.

Functional regression models have proved useful to model the relationship between a scalar or functional response and functional predictors \( X(t) \), where \( t \) ranges over a compact interval \( T \) (Ramsay and Silverman 2005; Shi and Choi 2011; Wang et al. 2016). A simple model is the Functional Linear Regression Model (FLM) (Yao et al. 2005b; Kneip et al. 2016; Chiou et al. 2016), modeling the relationship between a scalar response \( Y \) and a functional predictor \( X(t) \) as

\[
E[Y|X] = \mu_Y + \int_T \beta(t)X^c(t)dt,
\]

where \( \mu_Y = EY \) and \( X^c(t) = X(t) - EX(t) \) is the centered process. This is a direct extension of the standard linear regression model to the functional case, but is more complex as the predictors lie in an infinite-dimensional space, usually assumed to be the Hilbert space \( L^2(T) \). Analogous to the estimation of FPCs, the inner product involving the slope function \( \beta \) cannot be feasibly approximated by Riemann sums in the situation where the functional predictor \( X \) is only observed over a few time points and possibly contaminated with measurement errors.

To obtain a consistent estimate of the slope function \( \beta \) in the FLM with sparse observations, one can use the fact that the linear model structure allows to express the slope in terms of the cross covariance and covariance functions of the predictor process \( X \) and the response, which are quantities that can be consistently estimated under mild assumptions (Yao et al. 2005b). Alternative multiple imputation methods based on conditioning on both the predictor observations and the response \( Y \) have also been explored (Petrovich et al. 2018), and these also rely on cross-covariance estimation.

Our main contributions are as follows. First, we provide a map from sparse and irregularly sampled data to a multivariate Gaussian distribution that provides a representation of the sparse
functional/longitudinal data. We show how this map and the ensuing distributional representation of longitudinal data at the subject level can be harnessed for a novel visualization of sparsely sampled functional/longitudinal data. Moreover, the features of the distributional representation provide useful uncertainty quantification about the random trajectory of a subject.

Second, we present a deeper investigation of the behavior of the functional principal component scores as the number of observations per subject becomes larger, and how this is reflected in the shrinkage of the conditional score distribution given the data towards a point mass located at the true but unobserved FPCs. This study complements previous work on the changing behavior of mean and eigenfunction estimates when transitioning from sparse to dense data (Li and Hsing 2010; Zhang and Wang 2016, 2018); this previous work emphasizes population properties of functional data, while our current study aims at individual level FPCs and trajectories, along the lines of Müller (2005) and Dai et al. (2018). The latter work targets the Karhunen-Loève representation of the derivative process $X'(t)$ and provides convergence results for the predicted FPCs associated with $X'(t)$ when transitioning from sparse to dense designs under Gaussianity assumptions throughout. In this paper we develop a novel distributional approach. For some of our key results, no distributional assumptions are imposed on the random trajectories $X$. For Gaussian processes, we quantify the shrinkage of the joint distribution of conditional FPCs to a point mass that is located at the true but unobserved FPCs.

Third, in the Gaussian setting, we derive theoretical results for the $L^2$-Wasserstein convergence towards the true $K$-truncated conditional Gaussian process distribution given the sparse data, and show that under mild conditions the infinite-dimensional Gaussian process counterparts can also be consistently estimated provided that the number of truncated principal components $K = K(n)$ diverges as the sample size $n$ increases.

Fourth, in the context of predicting the scalar response $Y$ in a functional linear regression model, it is a problem of key interest how to handle the prediction problem for the response when the predictor for a subject is uncertain due to sparse functional/longitudinal measurements that do not specify the predictor trajectory beyond its conditional distribution in the Gaussian setting. We advocate here to focus on the construction of predictive distributions, which specify the distribution of the response given the information available for a subject. We show that these predictive distributions can be consistently estimated in the Wasserstein and Kolmogorov
metric, and we introduce a Wasserstein measure of discrepancy to assess the predictability of
the response by the predictive distribution. We then proceed to show that this measure is
well interpretable and can be consistently recovered under mild assumptions. Our simulations
support the utility of the Wasserstein discrepancy measure under different sparsity designs and
noise levels for both the functional predictor and response in finite sample situations.

The paper is structured as follows. Preliminary results are in Section 2, where we establish
the convergence of the best predicted FPCs towards the true unobserved FPCs when transition-
ing from sparse to dense data. Crucially, this study does not require distributional assumptions.
The concept of representing sparse functional/longitudinal data by galaxies, i.e., multivariate
Gaussian distributions, for Gaussian functional data and our main results are the theme of
Section 3, followed by an analysis of the shrinkage of the conditional FPC score distributions
towards the point mass located at the true scores in Section 3.1. We then demonstrate in Sec-
tion 3.2 how the galaxy representations lead to practically useful graphical representations
and visualization of sparsely sampled functional and longitudinal data. Asymptotic results pertaining
to the $L^2$-Wasserstein convergence of the estimated functional galaxy objects towards their
true counterparts are in Section 3.3, followed by a study of prediction of scalar responses $Y$
in a Functional Linear Model (1) when predictors are sparsely observed in Section 4. Here we
introduce galaxies as predictive distributions and assess the predictability of the response by
the predictive distribution through a Wasserstein discrepancy measure and present asymptotic
results for the consistent estimation of the predictive distributions and the Wasserstein discrep-
ancy. This is followed by data illustrations for the proposed representations and visualizations
in Section 5. Finally, simulation results to demonstrate the finite sample performance of the
methods are reported in Section 6. Proofs and auxiliary results as well as additional materials
for the data applications can be found in the Supplement.

2 Transition from Sparse to Dense Sampling

We assume that for each individual $i = 1, \ldots, n$ there is an underlying unobserved function
$X_i(t)$, where the functions $X_i$ are i.i.d. realizations of a stochastic process $X(t)$, $t \in T$, and $T$ is
a closed and bounded interval on the real line. Without loss of generality we assume $T = [0, 1]$. 
Sparsely sampled and error-contaminated observations $X_{ij} = X_i(T_{ij}) + \epsilon_{ij}, \ j = 1, \ldots, n_i$, are obtained at random times $T_{ij} \in \mathcal{T}$ that are distributed according to a continuous distribution $F_T$. The measurement errors $\epsilon_{ij}$ are assumed to be i.i.d. Gaussian with mean 0 and variance $\sigma^2$, and independent of the underlying process $X_i(\cdot)$. Throughout, our analysis is conditional on the random number of observations per subject $n_i$ (Zhang and Wang 2016).

We denote the auto-covariance function of the process $X$ by

$$\Gamma(s, t) = \text{cov}(X(s), X(t)) = \sum_{k=1}^{\infty} \lambda_k \phi_k(s) \phi_k(t), \quad s, t \in \mathcal{T},$$

where $\lambda_1 > \lambda_2 > \cdots \geq 0$ are the ordered eigenvalues which satisfy $\sum_{k=1}^{\infty} \lambda_k < \infty$, and $\phi_k$, $k \geq 1$, are the orthonormal eigenfunctions associated with the Hilbert-Schmidt operator $\Xi(g) = \int_{\mathcal{T}} \Gamma(\cdot, t) g(t) dt$, $g \in L^2(\mathcal{T})$. We denote by $\mu(t) = E(X_i(t))$ the mean function, $X_i^c(t) = X_i(t) - \mu(t)$ the centered process, and $\xi_{ik} = \int_{\mathcal{T}} X_i^c(t) \phi_k(t) dt$ the $k$th functional principal component score (FPC), $k = 1, 2, \ldots$. The FPCs satisfy $E(\xi_{ik}) = 0$, $E(\xi_{ik}^2) = \lambda_k$ and $E(\xi_{ik} \xi_{il}) = 0$ for $k, l = 1, 2, \ldots$ with $l \neq k$. The individual trajectories can then be represented through the Karhunen-Loève decomposition

$$X_i(t) = \mu(t) + \sum_{k=1}^{\infty} \xi_{ik} \phi_k(t),$$

where in practice it is often useful to consider a truncated expansion using the first $K > 0$ components that explain most of the variation, for example through the fraction of variance explained (FVE) criterion (Yao et al. 2005a). Denote by $\mathbf{T}_i = (T_{i1}, \ldots, T_{in_i})^T$ the time observations for the $i$th subject.

Writing $\mathbf{X}_i = (\tilde{X}_{i1}, \ldots, \tilde{X}_{in_i})^T$ and conditional on $\mathbf{T}_i$, it follows that $\text{cov}(\tilde{X}_{ij}, \xi_{ik} | \mathbf{T}_i) = \lambda_k \phi_k(T_{ij}), \ j = 1, \ldots, n_i$ and $k = 1, \ldots, K$. Define

$$\Phi_{ik} = \begin{pmatrix} \phi_1(T_{i1}) & \cdots & \phi_K(T_{i1}) \\ \vdots & \ddots & \vdots \\ \phi_1(T_{in_i}) & \cdots & \phi_K(T_{in_i}) \end{pmatrix},$$

$$\mu_i = E(\mathbf{X}_i | \mathbf{T}_i) = (\mu(T_{i1}), \ldots, \mu(T_{in_i}))^T$$

and the $n_i \times n_i$ conditional covariance matrix $\Sigma_i = \text{cov}(\mathbf{X}_i | \mathbf{T}_i)$, for which the $(j, l)$ entry is given by $\sigma^2 \delta_{jl} + \Gamma(T_{ij}, T_{il})$, where $\delta_{jl} = 1$ if $j = l$ and 0 otherwise. To predict the FPCs $\xi_{ik} = (\xi_{i1}, \xi_{i2}, \ldots, \xi_{iK})^T$, we utilize best linear unbiased predictors (BLUP) (Rice and Wu 2001) of $\xi_{ik}$ given $\mathbf{X}_i$ and $\mathbf{T}_i$, which are $\hat{\xi}_{ik} = \Lambda_K \Phi_{ik}^T \Sigma_i^{-1} (\mathbf{X}_i - \mu_i)$, without any distributional assumptions on the process $X$ (Dai et al. 2018).
We now show that as the number of observations for an individual increases as the functional sampling gets denser, the predicted FPCs $\tilde{\xi}_{ik}$ converge to the true but unobserved FPCs $\xi_{ik}$, so that the true trajectory can be consistently recovered, which is not the case for the sparse situation, where only best predictors can be consistently recovered (Yao et al. 2005a). Our analysis requires the following regularity assumptions.

(S1) The process $X(t)$ is continuously differentiable a.s. for $t \in T$.

(S2) $\partial \Gamma(s,t)/\partial s$ exists and is continuous, where $s,t \in T$.

(S3) $\{T_{ij} : i = 1, \ldots, n, j = 1, \ldots, n_i\}$ are i.i.d. copies of a random variable $T$ defined on $T$, and $n_i$ are regarded as fixed. The density $f(\cdot)$ of $T$ is bounded below, $\min_{t \in T} f(t) \geq m_f > 0$.

Assumption (S2) is a requirement for the smoothness of the covariance function, while (S3) is a standard assumption (Zhang and Wang 2016) to ensure there no sampling gaps, and (S3) allows to control the gaps between the order statistics of the time points, as needed for the Riemann sum approximation when transitioning from sparse to dense sampling (see Lemma S2).

**Theorem 1.** Suppose that (S1)-(S3) hold and the number of observations $n_i$ for the $i$th subject satisfies $n_i = m \to \infty$, $i = 1, \ldots, n$. Then, for any fixed $K \geq 1$ and $k = 1, \ldots, K$, we have

$$|\tilde{\xi}_{ik} - \xi_{ik}| = O_p(m^{-1/2}).$$

This is the same rate of convergence as derived previously in Dai et al. (2018) for the FPCs of the derivative process $X'(t)$ under Gaussian assumptions. This previous analysis utilized convergence results for nonparametric posterior distributions (Shen 2002) that are tied to the Gaussian assumption, whereas here we develop a novel direct approach that does not require distributional assumptions on $X$. The result in Theorem 1 is based on the population values for mean and covariance functions as well as the measurement error variance. However, these quantities are unknown and must be estimated from the available data. Taking this estimation step into account motivates the following results. We consider two different settings where either the subjects are observed on dense designs, with $n_i = m \to \infty$, or on sparse designs, with $n_i \leq N_0 < \infty$ for a fixed number $N_0 < \infty$, reflecting few and irregularly timed observations per subject. To simplify notations, we will use the following abbreviations for rates of convergence.
for the mean and covariance of the underlying stochastic process $X$,

\[ a_{n1} = h^2_{\mu} + \frac{\log(n)}{nh_{\mu}}, \quad b_{n1} = h^2_G + \frac{\log(n)}{nh^2_G}, \]
\[ a_{n2} = h^2_{\mu} + \left(1 + \frac{1}{m h_{\mu}}\right) \frac{\log(n)}{n}, \quad b_{n2} = h^2_G + \left(1 + \frac{1}{m h_G}\right) \frac{\log(n)}{n}. \]

Quantities $a_n$ and $b_n$ will be used in the following in dependence on the design setting as follows: For sparse designs, $a_n = a_{n1}$ and $b_n = b_{n1}$, while for dense designs, $a_n = a_{n2}$ and $b_n = b_{n2}$.

The estimation of population quantities is carried out in a similar fashion as in Yao et al. (2005a). Theorem 2 below shows that the estimated FPCs for a new independent subject $i^*$ that is not part of the training data sample ($i = 1, \ldots, n$), but for which measurements are available over a dense but possibly irregular grid, converge to the true FPCs, irrespective of whether the subjects in the training set are observed under sparse or dense designs. Specifically, given an independent realization $X^*$ of the Gaussian process $X$, and independent of $X_1, \ldots, X_n$, we observe the measurements of the process $X^*$ made at times $T_j^*$ with added noise, $X^* = (X^*(T_1^*) + \epsilon_1^*, \ldots, X^*(T_{m^*}^*) + \epsilon_{m^*}^*)$, where the errors $\epsilon_j^*$ are Gaussian with mean zero and variance $\sigma^2$, and independent of all other random quantities. In the following, we consider the Karhunen-Loève decomposition $X^*(t) = \mu(t) + \sum_{k=1}^{\infty} \xi_k \phi_k(t)$ and the FPC score estimates \( \hat{\xi}_k = \hat{\lambda}_k \hat{\phi}_k(T^*)^T \hat{\Sigma}^{*-1}(X^* - \hat{\mu}^*) \), where $\hat{\mu}^* = (\hat{\mu}(T_1^*), \ldots, \hat{\mu}(T_{m^*}^*))^T$, $\hat{\phi}_k(T^*) = (\hat{\phi}_k(T_1^*), \ldots, \hat{\phi}_k(T_{m^*}^*))^T$, $T^* = (T_1^*, \ldots, T_{m^*}^*)^T$, and $\hat{\Sigma}^{*-1}$ is analogous to $\Sigma_i^{-1}$ but replacing the $T_{ij}$ with $T_j^*$ and the population quantities by their estimated counterparts. Here we require the following additional standard assumption.

(B1) The eigenvalues $\lambda_1 > \lambda_2 > \cdots > 0$ are all distinct.

**Theorem 2.** Suppose that assumptions (S1), (B1) and (A1)–(A8) in the Appendix are satisfied. Consider either a sparse design setting when $n_i \leq N_0 < \infty$ or a dense design when $n_i = m \to \infty$, $i = 1, \ldots, n$. Set $a_n = a_{n1}$ and $b_n = b_{n1}$ for the sparse case, and $a_n = a_{n2}$ and $b_n = b_{n2}$ for the dense case. For a new independent subject $i^*$ and $k \geq 1$, if $m^*(a_n + b_n) = o(1)$ as $n \to \infty$, where $m^* = m^*(n) \to \infty$,

\[ |\hat{\xi}_k - \xi_k^*| = O_p(m^{-1/2}) + a_n + b_n. \]

When the underlying functional data is Gaussian, the infinite-dimensional random trajectory $X$ can be characterized by its mean function and covariance integral operator (Hsing and Eubank
This introduces a rich geometry and structure to represent and visualize longitudinal data; we refer to these visualizations as galaxy objects.

3 Galaxies for Gaussian Processes

3.1 Galaxy Objects and the Transition from Sparse to Dense Sampling

In the important case when \(X(t), t \in T\), is a Gaussian process, we have \(\xi_{iK} = (\xi_{i1}, \xi_{i2}, \ldots, \xi_{iK})^T \sim N(0, \Lambda_K)\), where \(\Lambda_K = \text{diag}(\lambda_1, \ldots, \lambda_K)\) and \(K\) is a positive integer that corresponds to a truncation parameter. Conditional on \(T_i\), it follows that \(\xi_{iK}\) and \(X_i\) are jointly normal

\[
\begin{pmatrix}
X_i \\
\xi_{iK}
\end{pmatrix}
\sim N
\left(
\begin{pmatrix}
\mu_i \\
0
\end{pmatrix},
\begin{pmatrix}
\Sigma_i & \Phi_i \Lambda_K \\
\Lambda_K \Phi_i^T & \Lambda_K
\end{pmatrix}
\right),
\]

and by a property of multivariate normal distributions (see for example Mardia et al. (1979)),

\[
\xi_{iK} | X_i, T_i \sim N_K(\tilde{\xi}_{iK}, \Sigma_{iK}),
\]

where \(\tilde{\xi}_{iK} = E(\xi_{iK} | X_i, T_i) = \Lambda_K \Phi_i^T \Sigma_i^{-1} (X_i - \mu_i)\) is the best linear unbiased predictor (BLUP) of \(\xi_{iK}\) given \(X_i\) and \(T_i\), and \(\Sigma_{iK} = \Lambda_K - \Lambda_K \Phi_i \Phi_i^T \Sigma_i^{-1} \Phi_i \Lambda_K\) is the conditional variance. We note that the relation in (4) was previously exploited, for example in Yao et al. (2005a), to construct simultaneous confidence bands for estimated trajectories; compare also Wang and Shi (2014). We will refer to the conditional distribution in (4) as a \(K\)-truncated galaxy since it has elliptical contours and is a distributional representation for the subject’s truncated true but unobserved scores \(\xi_{iK}\) (see Figure 4), and provides a key tool for visualizing the available information for a given subject.

Theorem 1 implies that the center of the \(K\)-truncated galaxy converges to the true FPCs \(\xi_{iK}\) in the transition from sparse to dense functional data. We now show that entire \(K\)-truncated galaxy shrinks to a point mass located at its true \(K\)-truncated FPCs. Recall that \(\Sigma_{iK}\) is the conditional covariance as in (4) and for a matrix \(A \in \mathbb{R}^{p \times q}\) denote by \(\|A\|_{op,2} = \sup_{\|v\|_2 = 1} \|Av\|_2\) the 2-matrix norm, where \(\|\cdot\|_2\) is the Euclidean norm in \(\mathbb{R}^p, p, q > 0\). For the following, we require Gaussianity.

(S4) The process \(X(t), t \in T\), is Gaussian.
Theorem 3. Suppose that (S1)-(S4) hold and the number of observations for the \(i\)th subject diverges, namely \(n_i = m \to \infty, i = 1, \ldots, n\). Then, for any fixed \(K \geq 1\) and \(k = 1, \ldots, K\),

\[
\|\Sigma_{iK}\|_{op,2} = O_p(m^{-1}).
\] (5)

Theorem 1 and Theorem 3 show that the \(K\)-truncated galaxy of a given subject shrinks to the true \(K\)-truncated FPCs \(\xi_{iK}\) at a root-\(m\) rate as the number of observations per subject diverges. The size of a \(K\)-truncated galaxy implicitly reflects the number of observations for that subject. Consider now an independent densely measured subject \(i^*\) as in Section 2.

Theorem 4. Suppose that (S1), (S4), (B1) and (A1)–(A8) in the Appendix hold. Let \(K > 0\) and consider either a sparse design setting when \(n_i \leq N_0 < \infty\) or a dense design when \(n_i = m \to \infty, i = 1, \ldots, n\). Set \(a_n = a_{n1}\) and \(b_n = b_{n1}\) for the sparse case, and \(a_n = a_{n2}\) and \(b_n = b_{n2}\) for the dense design. For a new independent subject \(i^*\), if \(m^*(a_n + b_n) = o(1)\) as \(n \to \infty\), where \(m^* = m^*(n) \to \infty\),

\[
\|\hat{\Sigma}_{iK}^* - \Sigma_{iK}^*\|_{op,2} = O_p(m^*-1 + a_n + b_n).
\]

The following theoretical framework is a direct consequence of the theory of square integrable Gaussian processes. For the separable real Hilbert space \(\mathcal{H} = L^2(\mathcal{T})\) with inner product \(\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L^2(\mathcal{T})}\), a probability measure \(\nu\) defined over the Borel sets \(\mathcal{B}(\mathcal{H})\) is Gaussian if for any \(h \in \mathcal{H}^*\), where \(\mathcal{H}^*\) denotes the dual space consisting of continuous and linear functionals on \(\mathcal{H}\), \(\mu \circ h\) is a Gaussian measure on \(\mathbb{R}\) (Gelbrich 1990). Such measures \(\nu\) are characterized by their mean \(m_\nu \in \mathcal{H}\) and covariance operator \(\Xi_\nu : \mathcal{H} \to \mathcal{H}\) (Kuo 1975), defined through

\[
\langle m_\nu, a \rangle = \int_\mathcal{H} \langle x, a \rangle \nu(dx), \quad a \in \mathcal{H},
\]

\[
\langle \Xi_\nu(a), b \rangle = \int_\mathcal{H} \langle x - m_\nu, a \rangle \langle x - m_\nu, b \rangle \nu(dx), \quad a, b \in \mathcal{H}.
\]

Denote the Gaussian measure \(\nu\) by \(\mathcal{G}(m_\nu, \Xi_\nu)\) in what follows. The \(K\)-truncated conditional distribution of the centered process \(X_i^c(\cdot)\) given \(X_i\) is defined as

\[
\mathcal{G}_{iK} = (\text{The conditional distribution of } \xi_{iK}^T \Phi_K | X_i, T_i) = \mathcal{G}(\tilde{\mu}_{iK}, \tilde{\Sigma}_{iK}),
\] (6)

where \(\tilde{\mu}_{iK} = \xi_{iK}^T \Phi_K, \Phi_K = (\phi_1, \ldots, \phi_K)^T\) are the first \(K\) eigenfunctions, and \(\tilde{\Sigma}_{iK} : L^2(\mathcal{T}) \to L^2(\mathcal{T})\) is the integral operator associated with \(\Gamma_{iK}(s, t) := \sum_{1 \leq j, l \leq K} \{\Sigma_{iK}\}_{jl} \phi_j(s) \phi_l(t), \) with \([A]_{ij}\)
denoting the \((i, j)\)th entry of a matrix \(A\). This object is the functional (and finite-dimensional) counterpart of the \(K\)-truncated galaxy in (4) as it involves the first \(K\) eigenfunctions \(\Phi_K\). We thus refer to \(G_{iK}\) as the \(K\)-truncated representation of the \(i\)th subject’s (unobserved) trajectory. Since functional data is inherently infinite-dimensional, we may regard the truncated representations as a form of dimension reduction, where the \(K\)-truncated representation \(G_{iK}\) targets the true infinite-dimensional representation, also referred to as true galaxy,

\[
G_i = (\text{The conditional distribution of } (X - \mu) \mid X_i, T_i) = G(\hat{\mu}_i, \Xi_i),
\]

where \(\hat{\mu}_i = \Gamma(\cdot, T_i)\Sigma_i^{-1}(X_i - \mu_i), \ t \in \mathcal{T}\) and \(\Xi_i\) is the integral operator associated with the covariance function \(\Gamma_i(s, t) = \Gamma(s, t) - \Gamma(s, T_i)\Sigma_i^{-1}\Gamma(T_i, t), s, t \in \mathcal{T}\), taking the convention that \(\Gamma(s, T_i)\) and \(\Gamma(T_i, t)\) are row and column vectors containing the evaluations of \(\Gamma\), respectively.

Next we study in which way the truncated galaxies provide an approximation to the true galaxies as the truncation point \(K\) increases.

The estimated versions to each of the previous objects are obtained by replacing population quantities by their estimates, where estimated conditional distributions are then

\[
\hat{F}(\xi_i | X_i, T_i) = N_K(\hat{\xi}_{iK}, \hat{\Sigma}_{iK}),
\]

with \(\hat{\xi}_{iK} = \hat{\Lambda}_K \hat{\Phi}_{iK}^T \hat{\Sigma}_i^{-1}(X_i - \hat{\mu}_i)\) and \(\hat{\Sigma}_{iK} = \hat{\Lambda}_K - \hat{\Lambda}_K \hat{\Phi}_{iK}^T \hat{\Sigma}_i^{-1} \hat{\Phi}_{iK} \hat{\Lambda}_K\), leading to the following estimate for the truncated galaxy \(G_{iK}\),

\[
\hat{G}_{iK} = G(\hat{\mu}_{iK}, \hat{\Xi}_{iK}).
\]

Here \(\hat{\mu}_{iK} = \hat{\xi}_{iK}^T \hat{\Phi}_K\), and \(\hat{\Xi}_{iK}\) is the integral operator associated with the covariance function \(\hat{\Gamma}_{iK}(s, t) := \sum_{1 \leq j, l \leq K} [\hat{\Sigma}_{iK}]_{jl} \hat{\phi}_j(s) \hat{\phi}_l(t)\). The infinite-dimensional version is

\[
\hat{G}_i = G(\hat{\mu}_i, \hat{\Xi}_i),
\]

where \(\hat{\mu}_i(t) = \hat{\Gamma}(t, T_i) \hat{\Sigma}_i^{-1}(X_i - \hat{\mu}_i), \ t \in \mathcal{T}\) and \(\hat{\Xi}_i\) is the integral operator associated with the covariance function \(\hat{\Gamma}_i(s, t) := \hat{\Gamma}(s, t) - \hat{\Gamma}(s, T_i) \hat{\Sigma}_i^{-1} \hat{\Gamma}(T_i, t), s, t \in \mathcal{T}\).

### 3.2 Visualization of Sparsely Sampled Gaussian Functional Data Through Galaxy Plots

Once a truncation point \(K\) has been chosen, for example by requiring that \(K\) components explain a fraction \((1 - \alpha)\) of the total variance by the FVE criterion, by utilizing (4), the data available
for any subject can be mapped to a uniquely determined $K$-truncated galaxy; a common choice is $\alpha = 0.05$. A $K$-truncated galaxy is suited for visualization by choosing an ellipsoidal contour of a pre-specified level, especially when $K$ is relatively small such as $K = 2$ or 3. While for larger dimensions the visualizations can be based on lower-dimensional projections of the ellipsoids, we note that in many sparse settings it is common to recover not more than the first two eigenfunctions. Accordingly, in the following we represent Gaussian conditional distributions using only the first two FPCs, i.e., restricting the visualization to 2-truncated galaxies.

This leads to a galaxy plot, depicting $(1 - \gamma) \times 100\%$ high probability elliptical or ellipsoid contours with default chosen as $\gamma = 0.05$; the plot also includes the conditional mean in (4) at the center. The plot’s elliptical contour and center characterize the 2-dimensional Gaussian distribution. Figure 1 indicates the dependency of the 2-truncated galaxies on the number and time locations of the measurements, and the galaxies are seen to shrink as the number of observations available for a subject increases.

For another illustration of galaxy plots we utilize the body mass index (BMI) data available from the Baltimore Longitudinal Study of Aging (BLSA, Shock et al. 1984). Here we consider the first 30 individuals who are at least 50 years old with more than 10 observations over their lifetime and employ the \textit{fdapace} \textit{R} package (Gajardo et al. 2021) to construct the 2-truncated galaxies. For these data the first $K = 2$ components explain 96\% of the total variation. Figure 2 illustrates the 2-truncated galaxies, each representing a subject in the study; the number of observations ranges between 11 and 19, with a median of 14 measurements. Smaller galaxies correspond to subjects with more measurements. Larger galaxies indicate subjects with a higher uncertainty in their true but unobserved underlying trajectory due to fewer or more noisy measurements. Additional illustrations of galaxies can be found in Section 5.

### 3.3 Wasserstein Distance Between Conditional Distributions

When only sparse observations are available, one cannot use the usual $L^2$ distance to characterize the dissimilarity between individuals. A previous proposal by Peng and Müller (2008) is to use the conditional $L^2$ distance between two subjects curves, which is defined as

$$\tilde{D}(i, j) = \{E(D^2(i, j) | X_i, T_i, X_j, T_j)\}^{\frac{1}{2}}, \quad 1 \leq i, j \leq n,$$
Figure 1: Galaxy plot of simulated sparsely observed Gaussian processes over $\mathcal{T} = [0, 10]$ and eigenpairs and mean function as considered in Yao et al. (2005a). We generate random trajectories where subjects are sampled from four groups with equal probability and each group varies in their time locations and number of observations. For the first and second group $n_i \in \{1, \ldots, 5\}$ uniformly and their time locations have a triangular distribution over $\mathcal{T}$ with modes at 0.5 and 9.5, respectively, so that their time observations lie mostly towards the left and right boundaries. The third group has $n_i \in \{1, \ldots, 5\}$ uniformly with triangular time distribution over $\mathcal{T}$ with mode at 5. For the fourth group $n_i \in \{15, \ldots, 20\}$ uniformly with uniformly distributed time locations, corresponding to a relatively dense design. The left panel displays some 2-truncated galaxies for the first two groups (in dark orange and orchid, resp.) and the right panel for the third and fourth groups (in blue and red, resp.).

where $D(i, j)$ is the $L^2$ distance between trajectories $X_i(\cdot)$ and $X_j(\cdot)$. Equivalently, $\tilde{D}(i, j)^2 = E(\sum_{k=1}^{\infty} (\xi_{ik} - \xi_{jk})^2|X_i, T_i, X_j, T_j)$, which in turn can be cast in terms of the conditional mean and variance of the subject FPCs that can be estimated consistently (Yao et al. 2005a) under Gaussian assumptions. This conditional $L^2$ distance approach does however not capture the full extent of the uncertainty that is inherent in the conditional distributions. To remedy this, we introduce here a new similarity measure that corresponds to the $L^2$-Wasserstein distance between the (conditional) Gaussian distributions of two subjects.

For $p \geq 1$, the $L^p$-Wasserstein distance between two measures $\nu$ and $\tau$ is defined as

$$W_p(\nu, \tau) = \left\{ \inf_{A \sim \nu, B \sim \tau} E(\|A - B\|^p) \right\}^{\frac{1}{p}},$$

where the norm $\|\cdot\|$ is either the Euclidean norm for measures supported on $\mathbb{R}^d$, $d \geq 1$, or $L^2$-norm for measures on $L^2$ space, and the infimum is taken over all pairs of random variables $A$ and $B$. 
with marginal distribution $\nu$ and $\tau$, respectively (Villani 2003). The $L^2$-Wasserstein distance between two Gaussian measures $G(m_{\mu_1}, \Xi_{\mu_1})$ and $G(m_{\mu_2}, \Xi_{\mu_2})$ over the infinite-dimensional Hilbert space $L^2(T)$ has a particularly simple form (Gelbrich 1990), given by

$$W_2^2(G(m_{\mu_1}, \Xi_{\mu_1}), G(m_{\mu_2}, \Xi_{\mu_2})) = \|m_{\mu_1} - m_{\mu_2}\|_{L^2}^2 + \text{tr}(\Xi_{\mu_1} + \Xi_{\mu_2} - 2(\Xi_{\mu_1}^{1/2} \Xi_{\mu_2} \Xi_{\mu_1}^{1/2})^{1/2}),$$  \hspace{1cm} (12)

where for a positive, self-adjoint and compact operator $R : L^2(T) \to L^2(T)$, the square root operator $R^{1/2}$ is defined through its spectral decomposition (Hsing and Eubank 2015).

We then employ the $L^2$-Wasserstein distance on the space of $K$-truncated representations $\mathcal{G}_{iK}$, defined conditionally on both the measurements $X_i$ and time observations $T_i$. Since the $\mathcal{G}_{iK}$ depend on unknown population quantities, we obtain estimates $\hat{\mathcal{G}}_{iK}$ by plugging in the sample quantities as in (9). We show below that the estimates $\hat{\mathcal{G}}_{iK}$ are consistent for $\mathcal{G}_{iK}$ in the $L^2$-Wasserstein distance under mild regularity conditions on the eigengaps of the conditional covariances $\Sigma_{iK}$. The eigengap of $\Sigma_{iK}$ is defined as $\min_{j=1,...,K-1}[\tilde{\lambda}_j(\Sigma_{iK}) - \tilde{\lambda}_{j+1}(\Sigma_{iK})]$, where $\tilde{\lambda}_1(\Sigma_{iK}) \geq \cdots \geq \tilde{\lambda}_K(\Sigma_{iK})$ are the ordered eigenvalues of $\Sigma_{iK}$. These condition holds a.s. under weak assumptions on the eigenfunctions as shown in Lemma S1 in the Supplement.

**Theorem 5.** Suppose that (S4), (B1), and (A1)–(A8) in the Appendix hold and consider a sparse design setting, where $n_i \leq N_0 < \infty$ for all $i = 1, \ldots, n$, setting $a_n = a_{n1}$ and $b_n = b_{n1}$. If
for a given $K \geq 1$ the eigengap of $\Sigma_{iK}$ is positive, then

$$W_2^2(\hat{\mathcal{G}}_{iK}, \mathcal{G}_{iK}) = O_p(a_n + b_n), \quad \text{as} \quad n \to \infty.$$ 

Our next result states that the $L^2$-Wasserstein distance between two estimated $K$-truncated representations can be recovered consistently.

**Theorem 6.** Suppose that (S4), (B1), (A1)–(A8) in the Appendix hold and consider a sparse design setting, where $n_i \leq N_0 < \infty$ for all $i = 1, \ldots, n$, setting $a_n = a_{n1}$ and $b_n = b_{n1}$. For any two distinct subjects $i$ and $j$, where $i, j \in \{1, \ldots, n\}$, and any given $K \geq 1$, if the eigengaps of $\Sigma_{iK}, \Sigma_{jK}$ and $\Sigma_{iK}^{-1/2} \Sigma_{jK} \Sigma_{iK}^{-1/2}$ are positive, then

$$W_2^2(\hat{\mathcal{G}}_{iK}, \hat{\mathcal{G}}_{jK}) - W_2^2(\mathcal{G}_{iK}, \mathcal{G}_{jK}) = O_p(a_n + b_n), \quad \text{as} \quad n \to \infty.$$ 

The following assumptions are needed to obtain the rate of convergence of the $L^2$-Wasserstein distance between the true infinite-dimensional representation and its estimate.

(B2) Let $c_n = \max(a_n, b_n) \to 0$ as $n \to \infty$, where $a_n$ and $b_n$ are defined in (3).

(B3) The number of components $K = K(n)$ satisfies $\sum_{k=1}^{K} \lambda_k^{-1/2} \delta_k^{-1} = o(c_n^{-1})$, where $\delta_k = \min(\lambda_{k-1} - \lambda_k, \lambda_k - \lambda_{k+1})$ is the $k^{th}$ eigengap, $k = 1, 2, \ldots$, taking the convention that $\lambda_0 = \infty$.

**Theorem 7.** Suppose that (S4), (B1), (A1)–(A8) in the Appendix hold and consider a sparse design setting, where $n_i \leq N_0 < \infty$ for all $i = 1, \ldots, n$, setting $a_n = a_{n1}$ and $b_n = b_{n1}$. Under (B2)–(B3), for any given subject,

$$W_2^2(\hat{\mathcal{G}}_i, \mathcal{G}_i) = o(1) \quad \text{a.s.}$$

**Theorem 8.** Suppose that (S4), (B1), (A1)–(A8) in the Appendix hold and consider a sparse design setting, where $n_i \leq N_0 < \infty$ for all $i = 1, \ldots, n$, setting $a_n = a_{n1}$ and $b_n = b_{n1}$. Under (B2) and if the number of components $K = K(n) \to \infty$ as $n \to \infty$, then for any given subject

$$W_2(\hat{\mathcal{G}}_{iK}, \hat{\mathcal{G}}_i) = o(1) \quad \text{a.s.}$$

As an immediate consequence of these results, if the number of components $K = K(n)$ increases in $n$, the estimated galaxies are consistent for their targets.
Corollary 1. Under the conditions of Theorem 7

\[ W_2(\hat{G}_{ik}, G_i) = o(1) \quad \text{a.s.}, \]

for any fixed subject \( i \geq 1 \).

4 Prediction in Functional Linear Models

In this section we explore the concept of galaxies, i.e., conditional FPC distributions given the available data for a subject, in the framework of Functional Linear Models (FLM) as defined in (1) (Yao et al. 2005b). Suppose one has an infinite-dimensional Gaussian predictor process \( X(t), t \in T \), with Karhunen-Loève decomposition \( X(t) = \mu(t) + \sum_{j=1}^{\infty} \xi_j \phi_j(t) \), and a Euclidean response \( Y \in \mathbb{R} \), which are related through a linear regression model

\[ E(Y|X) = \beta_0 + \sum_{j=1}^{\infty} \xi_j \beta_j =: \eta, \tag{13} \]

where \( \beta_0 = E(Y) \) is the intercept and \( \eta \) is the linear predictor. The response is generated by adding Gaussian noise \( Y = \beta_0 + \sum_{j=1}^{\infty} \xi_j \beta_j + \epsilon_Y \), where \( \epsilon_Y \sim N(0, \sigma_Y^2) \) is independent of all other random quantities.

There has been substantial work on estimating the slope function \( \beta \) based on a sample of functional predictors \( X_1, \ldots, X_n \) which are paired to their corresponding response values \( Y_1, \ldots, Y_n \), such that \( (X_i, Y_i) \overset{i.i.d.}{\sim} (X,Y), \ i = 1, \ldots, n \), where \( X \) and \( Y \) are related through (13) (Cai and Hall 2006). A second important issue is prediction of the response \( Y \) (Hall and Horowitz 2007). If however the predictor curves are observed on sparse and irregular times and contaminated with measurement error, the FPCs cannot be consistently recovered (Yao et al. 2005a). For estimating the slope function \( \beta \), it is well known that this difficulty can be overcome by exploiting the fact that \( \beta \) can be expressed in terms of covariances and cross-covariances in the linear model, which are quantities that can be consistently estimated (Yao et al. 2005b), and thus \( \beta \) can be consistently recovered even in sparse settings.
Predicting the scalar response $Y$ based on a sparsely observed predictor process $X$ is a different story. This has remained a challenging task as the FPCs of the trajectory $X$ cannot be recovered consistently. Indeed, for a new subject whose predictor process $X$ is sparsely observed at irregular times and moreover contaminated with measurement error, even knowledge of the true $\beta$ does not make it possible to predict $Y$ since the term $\int_{T} \beta(s) X(s) ds$ in the linear predictor $\eta$ cannot be consistently recovered. One might try to circumvent this basic problem by instead targeting the expected value of the response conditional on the sparse and noise measurements of the predictor process $X$ (Yao et al. 2005b).

Here we propose an alternative approach for which a consistent estimation method is available. The idea is to focus on the predictive distribution of the linear predictor $\eta$ given the data. The target is thus moved from constructing a point prediction to that of constructing a distribution for the expected response given the data available for the subject. Note that we do not aim at a distribution for the observed response $Y$ as it also contains the additional error $\epsilon_Y$ that is independent of all other random quantities and thus inherently unpredictable. To construct the distribution for the predictable part of the response $Y$, we consider $\eta_K = \beta_0 + \beta_K^T \xi_K$, the truncated real-valued predictor employing the first $K$ principal components, where $K$ can be chosen by the FVE criterion and $\beta_K = (\beta_1, \ldots, \beta_K)^T$ are the (truncated) slope coefficients. Thus $\eta = \eta_K + R_K$, where $R_K = \sum_{j \geq K+1} \xi_j \beta_j$ corresponds to the linear predictor part that remains unexplained when truncating at $K$ components but decreases asymptotically as $E(R_K) = 0$ and $\text{Var}(R_K) = \sum_{j \geq K+1} \lambda_j \beta_j^2 = o(1)$ as $K$ increases, where the latter rate can be controlled by suitable assumptions (Hall and Horowitz 2007). For responses $Y = \eta_K + R_K + \epsilon_Y$, a natural predictive distribution $P_{iK}$ of the predictable part of the response $\eta_{iK}$ is thus the conditional distribution of $\eta_{iK}$ given the data $X_i$ and $T_i$.

Since $X$ is a Gaussian process, we have $P_{iK} \overset{d}{=} N(\beta_0 + \beta_K^T \tilde{\xi}_{iK}, \beta_K^T \Sigma_{iK} \beta_K)$, which corresponds to a projection of the $K$-truncated galaxy into the real line and utilizes all data available for a subject. To quantify the performance of the predictive distribution $P_{iK}$ we employ the 2-Wasserstein distance between two probability measures $\nu_1, \nu_2$,

$$W_2^2(\nu_1, \nu_2) = \int_0^1 (Q_1(p) - Q_2(p))^2 dp,$$

where $Q_j(p) = \inf\{s \in \mathbb{R}: F_j(s) \geq p\}$, $p \in (0, 1)$, is the (generalized) quantile function corre-
sponding to \( \nu_j, j = 1, 2 \) (Villani 2003). Since the true predictive distribution is unknown and the predictable part of the response \( Y \) also is not observable, as a practical tool we utilize a Wasserstein discrepancy \( D_{nK} \), which is the average Wasserstein distance between \( \mathcal{P}_{iK} \) and the point mass measure \( \delta_{Y_i} \) located at \( Y_i \). Formally,

\[
D_{nK} := n^{-1} \sum_{i=1}^{n} W_2^2(\delta_{Y_i}, \mathcal{P}_{iK}) = n^{-1} \sum_{i=1}^{n} (Y_i - \tilde{\eta}_{iK})^2 + n^{-1} \sum_{i=1}^{n} \beta_K^T \Sigma_i \beta_K, \tag{15}
\]

where \( \tilde{\eta}_{iK} = E(\eta_{iK} | X_i, T_i) = \beta_0 + \beta_K^T \hat{\xi}_{iK} \) is the best prediction of the (truncated) linear predictor, or equivalently the center of \( \mathcal{P}_{iK} \). Note that (15) follows from (14) and similar ideas as in Amari and Matsuda (2021) when computing the Wasserstein distance between the predictive distribution and the actual observed response.

Assume that the number of observations \( n_i = m_0 < N_0 \) is common across subjects, so that the \( \Sigma_{iK} \) form an i.i.d. sequence of random positive definite matrices. It can be shown (see the proof of Theorem 10) that \( D_{nK} \) converges to the population level Wasserstein discrepancy measure, which is given by

\[
D_K = 2\beta_K^T E(\Sigma_{1K}) \beta_K + \sigma_Y^2 + \sum_{k \geq K+1} \lambda_k \beta_k^2 - 2\beta_K^T E[\Lambda_K \Phi_{1K}^T \Sigma_1^{-1} \sum_{k \geq K+1} \phi_k(T_1) \lambda_k \beta_k]. \tag{16}
\]

The first term in (16) encapsulates both the number of observations and the time locations, where increased values of \( m_0 \) are related to a shrinkage of \( \beta_K^T E(\Sigma_{1K}) \beta_K \) (see Theorem 3) and thus lower discrepancy values, i.e. increased predictability. Similarly, increased predictor and response noise levels encapsulated in \( \sigma^2 \) and \( \sigma_Y^2 \), respectively, are related to higher discrepancy, i.e. worse predictability. The last two terms come from the unexplained linear predictor part \( \mathcal{R}_K \) due to the truncation at \( K \) components; this term can be shrunk by increasing \( K = K(n) \).

Consider an example with eigenbasis \( \phi_k(t) = \sin(k \pi t) / \sqrt{2}, t \in T \). If the Fourier coefficients \( \beta_k \) and eigenvalues \( \lambda_k \) exhibit polynomial decay \(|\beta_k| = O(k^{-\alpha_1})\) and \( \lambda_k = O(k^{-\alpha_2}), \alpha_1, \alpha_2 > 1 \), then by the Cauchy-Schwarz inequality we have \( \sum_{k \geq K+1} \lambda_k \beta_k^2 = O(K^{2-2\alpha_1-\alpha_2}) \) and similarly \( \beta_K^T E[\Lambda_K \Phi_{1K}^T \Sigma_1^{-1} \sum_{k \geq K+1} \phi_k(T_1) \lambda_k \beta_k] = O(K^{1-\alpha_1-\alpha_2}) \) with \( K^{1-\alpha_1-\alpha_2} \leq K^{-1} \), where we use that \( \|\beta_K\|_2 \leq \|\beta\|_{L^2} \) and the uniform bound on the remaining quantities (see for example the proof of Lemma S5). In practice, the predictive distribution \( \mathcal{P}_{iK} \) and therefore also \( D_{nK} \) are unknown as they depend on unknown population quantities that need to be estimated. For this, we introduce \( \hat{\mathcal{P}}_{iK} \) and \( \hat{D}_{nK} \) by replacing population quantities by their estimated counterparts, where the intercept \( \beta_0 \) and slope coefficients \( \beta_K \) are replaced by estimates introduced below.
Let $C(t) = \text{Cov}(X(t), Y) = \sum_{k=1}^{\infty} E(Y \xi_k) \phi_k(t)$ be the cross-covariance function between the process $X$ and response $Y$ and $\sigma_k = \int_T C(t) \phi_k(t) dt = E(Y \xi_k)$, $k = 1, 2, \ldots$. We estimate $C(t)$ using a local linear smoother on the raw covariances $C_i(T_{ij}) = (\hat{X}_{ij} - \hat{\mu}(T_{ij}))Y_i$ (Yao et al. 2005b), leading to estimates $\hat{C}(t) = \hat{\beta}^X_0$, where

$$\hat{\beta}^X_0, \hat{\beta}^X_1 = \arg\min_{\beta^X_0, \beta^X_1 \in \mathbb{R}} \sum_{i, j} n_i w_i K_h(T_{ij} - t)(C_i(T_{ij}) - \beta^X_0 - \beta^X_1(t - T_{ij}))^2,$$

and $w_i = (\sum_{i=1}^{n_i} n_i)^{-1}$. Since $\sigma_k = \lambda_k \beta_k$, under the following common regularity condition,

$$(B4) \|\beta\|_{L^2}^2 = \sum_{m=1}^{\infty} \sigma_m^2 / \lambda_m^2 < \infty,$$

it holds that $\beta(t) = \sum_{m=1}^{\infty} \frac{\sigma_m}{\lambda_m} \phi_m(t)$, $t \in T$, where the equality is understood in the $L^2$ sense. This motivates to estimate $\beta$ by

$$\hat{\beta}_M(t) = \sum_{m=1}^{M} \frac{\hat{\sigma}_m}{\hat{\lambda}_m} \phi_m(t), \quad t \in T,$$

where $\hat{\sigma}_k = \int_T \hat{C}(t) \hat{\phi}_k(t) dt$ is an estimate of $\sigma_k$ and $M = M(n)$ is a positive integer sequence that diverges as $n \to \infty$. The intercept $\beta_0 = E(Y)$ is estimated by $\hat{\beta}_0 = n^{-1} \sum_{i=1}^{n} Y_i$. Convergence of $\hat{\beta}_M$ towards $\beta_M$ is inherently tied to the eigengaps of the process $X$ (Cai and Hall 2006; Müller and Yao 2010).

With estimates $\hat{\beta}_M$ of $\beta$ in hand, we can readily construct the predictive distributions $\hat{P}_{iK}$. For the following, we assume for simplicity that the optimal asymptotic tuning parameters are used for estimating the mean, covariance and cross-covariance, $h_\mu \asymp (\log(n)/n)^{1/5}$, $h_G \asymp (\log(n)/n)^{1/6}$ (Dai et al. 2018) and $h \asymp n^{-1/3}$ in the sparse design case that we consider here; $\beta_n \asymp \gamma_n$ means that $c_1 \beta_n \leq \gamma_n \leq c_2 \beta_n$ for some constants $c_1, c_2 > 0$. In particular, this implies $c_n \asymp (\log(n)/n)^{1/3}$. Defining sequences $v_M = \sum_{m=1}^{M} \delta^{-1} \lambda^{-1}_m$, $\tau_M = \sum_{m=1}^{M} \lambda^{-1}_m$ and a remainder term $\Theta_M = \lVert \sum_{m \geq M+1} \frac{\sigma_m}{\lambda_m} \phi_m \rVert_{L^2}$, we note that $M = M(n)$ should not grow too fast with sample size $n$, which we formalize in the following assumption,

$$(B5) \text{The integer sequence } M = M(n) \to \infty \text{ as } n \to \infty \text{ is such that } \sum_{m=1}^{M} \lambda^{-1/2} \delta^{-1} = O(\rho^{-1})$$

for some $\rho \in (1/3, 1)$,

with an additional regularity assumption to obtain uniform convergence,

$$(C1) \text{There exist a scalar } \kappa_0 > 0 \text{ such that } \lambda_{\min}(\Sigma_{iK}) \geq \kappa_0 \text{ almost surely, for all } i \geq 1.$$
(C1) is a mild assumption, as $\Sigma_{iK}$ corresponds to the conditional variance of $\xi_{iK} - \tilde{\xi}_{iK}$ given $T_i$, which is positive definite and cannot shrink to zero in the sparse case due to the constraint on the number of observations per subject $n_i \leq N_0 < \infty$.

Our next result demonstrates that $\hat{P}_{iK}$ is consistent for $P_{iK}$ in the 2-Wasserstein metric, which implies the $L^1$ convergence between the cumulative distribution functions associated with the empirical and true predictive distributions (Villani 2003), convergence in the Kolmogorov metric between the corresponding distribution functions and in the $L^2$ metric between the corresponding (predictive) densities. Let $F_{iK}$ denote the cumulative distribution function corresponding to $P_{iK} \overset{d}{=} N(\beta_0 + \beta_K^T \tilde{\xi}_{iK}, \beta_K^T \Sigma_{iK} \beta_K)$ and $\hat{F}_{iK}$ the corresponding cdf obtained by replacing $\tilde{\xi}_{iK}$ and $\Sigma_{iK}$ by $\hat{\xi}_{iK}$ and $\hat{\Sigma}_{iK}$, respectively, and $\beta_0$ and $\beta_K$ by then above estimates. Denote the estimated and true predictive densities by $\hat{f}_i(t) = d\hat{F}_i(t)/dt$ and $f_i(t) = dF_i(t)/dt$. For a function $g: T \to \mathbb{R}$, let $\|g\|_{L^2(\mathbb{R})} = (\int_{\mathbb{R}} g^2(s)ds)^{1/2}$ denote its $L^2$ norm over $\mathbb{R}$.

**Theorem 9.** Suppose that (S4), (B1)-(B2), (B4)-(B5), (C1), (A1)–(A8) in the Appendix hold and consider a sparse design with $n_i \leq N_0 < \infty$. For a fixed $K \geq 1$, setting $a_n = a_{n1}$ and $b_n = b_{n1}$,

$$W_2(\hat{P}_{iK}, P_{iK}) = O_p(\alpha_n),$$

$$\sup_{t \in \mathbb{R}} |\hat{F}_{iK}(t) - F_{iK}(t)| = O_p(\alpha_n),$$

$$\|\hat{f}_{iK} - f_{iK}\|_{L^2(\mathbb{R})} = O_p(\alpha_n),$$

where $\alpha_n = c_n v_M + c_n^\rho \tau_M^{1/2} + \Theta_M$ and the $O_p(\alpha_n)$ terms are uniform in $i$.

Under the conditions of Theorem 9, $\alpha_n \to 0$ is a consequence of $\tau_M \leq v_M = O(c_n^{\rho - 1})$, which implies $\alpha_n \leq O(c_n^{(\beta \rho - 1)/2} + \Theta_M)$ as shown in the proof of Lemma S15. There is a natural trade-off between how fast $M$ can grow and the rate of convergence for the estimates of the population quantities. In effect, a larger $M$ entails a lower remainder term $\Theta_M$ but affects the rate at which $\beta$ is recovered through $\hat{\beta}_M$, which involves $M$ components, and vice versa. Since the former term is connected to the decay of the covariance terms $\sigma_m/\lambda_m$, the optimal growth rate of $M(n)$ is inherently tied to the decay rate of $\sigma_m$, $\lambda_m$ and the eigengaps $\delta_m$.

As an example, we consider the special case where $X$ is a Brownian motion, for which the $\lambda_m$ and $\phi_m$ are known (Hsing and Eubank 2015). Lemma S19 in the Supplement shows that if $M = M(n) \asymp (\log(n)/n)^{\rho - 1/15}$, then $M$ satisfies (B5) with $\sum_{m=1}^M \lambda_m^{-1/2} \delta_m^{-1} \asymp c_n^{\rho - 1}$. Moreover,
if the decay of $\sigma_m$ is such that $\sigma_m^2 \leq Cm^{-(6+\delta)}$ for some constant $C > 0$ and $\delta > 0$, then (B4) is satisfied, the remainder $\Theta_M = O\left( M^{-\delta/2} \right)$ and the rate $\alpha_n$ satisfies the following conditions:

If $\rho \leq (3 + \delta)/(13 + \delta)$, then $\alpha_n = O((\log(n)/n)^{(13\rho-3)/30})$ while if $\rho > (3 + \delta)/(13 + \delta)$ it holds that $\alpha_n = O((\log(n)/n)^{\delta(1-\rho)/30})$. The optimal rate is achieved when $\rho = (3 + \delta)/(13 + \delta)$ and leads to $\alpha_n = O((\log(n)/n)^{\delta})$, where $q = (\delta/(13 + \delta))/3$. A sufficiently large $\delta$ implies that $q$ is closer to $1/3$ so that the rate $\alpha_n$ approaches $c_n = (\log(n)/n)^{1/3}$, which is the rate at which population quantities such as the covariance function $\Gamma$ are recovered.

Regarding the Wasserstein discrepancy $D_{nK}$, Theorem 10 demonstrates that the proposed predictability measure and the response measurement error variance $\sigma_Y^2$ can be consistently estimated in the sparse case. We consider the special case when the number of observations $n_i = m_0 < N_0$ is common across subjects, and show that the estimated Wasserstein discrepancy measure $\hat{D}_n$ converges to the population target $D_K$, which is inherently related to the predictability of the response by the $K$-truncated galaxy objects.

**Theorem 10.** Suppose that (S4), (B1)-(B2), (B4)-(B5), (C1), (A1)-(A8) in the Appendix hold and consider a sparse design with $n_i = m_0 \leq N_0 < \infty$, setting $a_n = a_{n1}$ and $b_n = b_{n1}$. For $K \geq 1$,

$$\hat{D}_{nK} = D_K + O_p(\alpha_n), \quad \alpha_n = c_nv_M + c^2_n\tau_M^{1/2} + \Theta_M,$$

and furthermore

$$n^{-1}\sum_{i=1}^{n}(Y_i - \bar{Y}_n)^2 - \sum_{m=1}^{M}\hat{\lambda}_j\hat{\beta}_j^2 = \sigma_Y^2 + O_p(\alpha_n) + \sum_{m \geq M+1}\lambda_m^2 + \Theta_M,$$

with $\bar{Y}_n = n^{-1}\sum_{i=1}^{n}Y_i$. (22)

Given that the predictive distribution can be consistently recovered, one can utilize it to perform an interval prediction of the response $Y_i$, for example, using the quantiles of the estimated predictive distribution $\hat{P}_i$ which are consistent to their true counterparts. Specifically, for a new subject $i^{**}$, denoting by $q_i^{**}(\alpha)$ the $\alpha \times 100\%$ quantile of $\hat{P}_i^{**}$, an interval estimate of $Y_{i^{**}}$ can be obtained through $[q_i^{**}(\alpha_0), q_i^{**}(\alpha_1)]$, where $\alpha_0, \alpha_1 \in (0, 1)$ and satisfy $\alpha_0 + \alpha_1 = 1$. Thus, due to the shrinkage of the galaxies as the number of observations increases as in Theorem 1 and Theorem 3, the interval prediction shrinks and reflects the uncertainty in the estimate which is encapsulated in the number of observations and noise level.

Table 1 in Section 6 summarizes simulation results for the Wasserstein discrepancy $\hat{D}_{nK}$ under different sparsity levels and measurement error noise levels in the predictor and response.
One finds that $\hat{D}_{nK}$ decreases monotonically as the design becomes denser while keeping the noise level $\sigma$ and $\sigma_Y$ fixed. Increasing $\sigma$ or $\sigma_Y$ is seen to entail larger values of $\hat{D}_{nK}$, which means decreasing predictability of the response.

5 Data Illustrations

In this section we further showcase the concept of galaxies for longitudinal data. The left panel of Figure 4 demonstrates a galaxy plot for the Brown Kiwi study in the North Island of New Zealand (Jones et al. 2009), which is a longitudinal study of the weight patterns of wild and captive kiwis. We employed the fdapace package (Gajardo et al. 2021) separately for male and female kiwis. More than 98% of the variation is explained by the first $K = 2$ components. The first eigenfunction corresponds to the overall weight over the observation period while the second eigenfunction is a contrast in weight between early and later stages in time (see Figure 3). More positive values of the first FPC score $\xi_1$ are related to an overall higher weight of the kiwi throughout the observation period. For males, a positive $\xi_2$ is related to an accelerated weight growth when the kiwi age is around 1 year old and with a strong weight decrease three years later. For females, the relationship is similar as a positive $\xi_2$ is related to weight gain a few months after birth and with a weight decrease around age 2.8 and 5.5.

The left panel of Figure 4 shows the galaxy plot for a few kiwis that differ on their number of observations and time locations, as well as upon being either wild or captive and their gender. Here wild kiwis are colored in blue while captive in red. A larger confidence ellipsoid indicates higher uncertainty in the underlying growth curve. Figure S7 in the Supplement shows a boxplot for the time locations while Figure S8 presents the number of observations for these kiwis. It is clearly seen that the 2-truncated galaxies are affected by the number and time locations of the observations. For example, the male kiwis labeled 13 and 14 are observed densely at more than 38 different time points, so that their ellipsoids are smaller than other males while their orientation strongly differ. The latter is due to the kiwi labeled 13 being observed mostly around ages 1.3 to 2.7 years old while the kiwi labeled 14 is almost only observed very young below 1 years old. Figure 5 shows the galaxy plot for all the wild and captive kiwis, indicating males and females. Female kiwis are found to have a higher variability compared to males.
Figure 3: First two eigenfunctions for the male (left panel) and female (right panel) kiwi data. For male kiwis, the first $K = 2$ eigenfunctions explain 98.5% of the variation while 99.8% is explained for females. Here the first eigenfunction (in black) corresponds to an overall weight over the entire time window while the second eigenfunction (in blue) is a contrast in weight between early and later stages.

Figure 4: Galaxy plot of the sparsely observed weights for the Brown Kiwi study (left panel, red for captive kiwis and blue for wild while ‘F’ for females and ‘M’ for males), where a few kiwis are displayed, and densely observed height growth trajectories (right panel, red for girls and blue for boys). Here $(1 - \alpha)100\% = 95\%$ is used for the contours.

The right panel of Figure 4 presents the galaxy plot for the Berkeley growth data (Tuddenham
and Snyder 1954), where each individual has 31 regularly spaced observations. Girls and boys are shown in red and blue, respectively. Since the individuals are observed on the same grid, the covariance matrix in (4) is the same for all individuals and thus the shape and orientation of the 2-truncated galaxies remain unaltered. Most of the galaxies are well separated from each other except for some individuals with their first FPC around zero, meaning that the individual trajectories are mostly distinct.

The galaxy concept arises naturally when constructing confidence bundles for individual sparsely observed functional data. This can be achieved by mapping each point \( \xi_\alpha \) lying on the \((1 - \alpha) \times 100\% \) elliptical contour corresponding to the \( K \)-truncated galaxy (4) into the function \( \mu(t) + \xi_\alpha^T \Phi_K(t) \), and then mapping all such points obtaining a confidence band for the sparsely observed individual trajectory; see also Yao et al. (2005a). We showcase this construction for the body mass index (BMI) data in the Baltimore Longitudinal Study of Aging (BLSA, Shock et al. 1984) using a sample of subjects as explained in Section 3.2. Figure 6 shows the BMI trajectory with its 95\% confidence band for a randomly selected individual in the BLSA dataset. The conditional mean function is shown in black, and we select four possible trajectories for this subject (shown in blue) by mapping the boundary points lying along the main and minor axis of the elliptical contour corresponding to the 2- truncated galaxy. Functions between the conditional mean and one of the curves in the boundary of the shaded region are regarded to be within the confidence band.
Figure 5: Galaxy plot for wild (blue) and captive (red) kiwis, where the left and right panels display male and female kiwis, respectively. The first $K = 2$ components explain at least 98% of the variation for both males and females.

Figure 6: Confidence bundle for one subject’s BMI trajectory in the BLSA dataset. The conditional mean function is shown in black, and four possible trajectories lying on the 95% confidence contour of the 2-truncated galaxy are displayed in blue, indicating the typical shapes of the most extreme trajectories within the confidence contour. Functions lying between the conditional mean and one of the curves in the boundary of the shaded region are regarded to be within the confidence band.
6 Simulations

We consider a finite-dimensional Gaussian process $X(t)$, $t \in \mathcal{T} = [0, 10]$, using $K = 4$ principal components, where the population quantities are given by $\phi_1(t) = -\cos(\pi t/10)/\sqrt{5}$, $\phi_k(t) = \sin((2k-3)\pi t/10)/\sqrt{5}$, $k = 2, \ldots, K$, $\mu(t) = t/2$, $\lambda_k = 4/(1 + k)^2$, $k = 1, \ldots, K$. For the functional linear model, the intercept and slope coefficients are given by $\beta_0 = 0.5$, $\beta_1 = 1$, $\beta_2 = -1$, $\beta_3 = 0.5$ and $\beta_4 = -0.5$. We investigate different noise levels in the predictor process $X$ and response $Y$ as well as different sparse settings, where we select a random number of time points $n_i = m_0$ for the $i$th subject, $i = 1, \ldots, n$. Here $m_0 = 2$ reflects a very sparse design, $m_0 = 8$ a medium sparse and $m_0 = 20$ a dense case. Then, given the number $n_i$, we select the time points at random and without replacement from an equispaced grid of 100 points over $\mathcal{T}$. We perform 2,000 simulations, where the methods were implemented in Julia, interfacing with R and the fdapace package.

Table 1 presents the results for the Wasserstein discrepancy $\hat{D}_{nK}$ under different sparsity designs and noise levels in both the functional predictor and scalar response $Y$. As explained before, the discrepancy $\hat{D}_{nK}$ clearly reflects the improvements in predictability for lower noise levels and under increasingly denser designs and increases monotonically in both $\sigma$ and $\sigma_Y$.

| Measurement Error Noise level | Sparsity setting |
|------------------------------|------------------|
| Predictor Response           | Very Sparse      | Medium Sparse   | Dense           |
| $\sigma$                     | $\sigma_Y$       | $n = 500$       | $n = 2000$      | $n = 500$       | $n = 2000$      | $n = 500$       | $n = 2000$      |
| 0.5                          | 0.5              | 3.008           | 2.645           | 1.492           | 1.477           | 0.863           | 0.853           |
|                              | 1.0              | 3.863           | 3.421           | 2.255           | 2.237           | 1.612           | 1.606           |
| 1.0                          | 0.5              | 3.639           | 3.449           | 2.540           | 2.418           | 1.729           | 1.715           |

Table 1: Simulation results for the Wasserstein discrepancy $\hat{D}_{nK}$, which measures the predictability of the response $Y_i$ by the predictive distribution $\mathcal{P}_{iK}$. The true regression parameters are $\beta_0 = 0.5$ and $\beta_K = (1, -1, 0.5, -0.5)^T$. Different sparsity levels are investigated, where very sparse corresponds to $n_i = 2$ observations per subject, for medium sparse $n_i = 8$ and for dense design $n_i = 20$. Different measurement error levels for the predictor and response are explored. Here 2,000 simulations are performed. The values displayed are the average values of $\hat{D}_{nK}$ across simulations.

As an additional measure of performance for $\mathcal{P}_{iK}$, we computed the estimated Wasserstein distance between $\hat{F}_{iK}(\beta_0 + \int_{\mathcal{T}} \beta(s)(X_i(s) - \mu(s))ds)$ and a uniform distribution on $(0, 1)$. This is of interest as $F_{1K}(\eta_{1K}), \ldots, F_{nK}(\eta_{nK})$ constitute an i.i.d. sample from a uniform random variable.
U in (0, 1). A conditioning argument gives $P(F_{iK}(\eta_{iK}) \leq p) = E(P(F_{iK}(\eta_{iK}) \leq p|X_i, T_i)) = E(P(\eta_{iK} \leq F_{iK}^{-1}(p)|X_i, T_i)) = p, p \in (0, 1)$. Thus, if we denote by $F_K(\eta_K)$ a generic probability transformation of the linear response $\eta_K$ through the cdf corresponding to $\eta_K|X, T$, then one should expect the random variable $F_K(\eta_K)$ to be close to a uniform distribution over $(0, 1)$, where we utilize the Wasserstein distance to measure the discrepancy between these distributions,

$$W_2^2(F_K(\eta_K), U) = \int_0^1 (Q_K(p) - p)^2 dp,$$

(23)

where $Q_K$ is the quantile function of the random variable $F_K(\eta_K)$. Since $F_{1K}(\eta_{1K}), \ldots, F_{nK}(\eta_{nK})$ are i.i.d. with $F_K(\eta_K)$, we may estimate $Q_K$ by the empirical quantile of the $F_{iK}(\eta_{iK})$.

Defining $Z_i$ to be the $i$th order statistic of the $F_{iK}(\eta_{iK}), j = 1, \ldots, n$, a natural estimate $U_W$ of $W_2^2(F_K(\eta_K), U)$ in (23) is given by (Amari and Matsuda 2021)

$$U_W = \sum_{i=1}^n \frac{z_i^2}{n} - z_i \left( \frac{i^2}{n^2} - \frac{(i - 1)^2}{n^2} \right) + \frac{1}{3} \left( \frac{i^3}{n^3} - \frac{(i - 1)^3}{n^3} \right),$$

and we define $\hat{U}_W$ analogously after replacing population quantities by their estimated versions.

Table 2 shows the simulations results, where it is clearly seen that as $n$ increases, the distance $\hat{U}_W$ diminishes which reflects better performance of the predictive distributions $P_{iK}$. Higher noise levels are reflected in worse performance as it becomes harder to estimate population quantities with the same sample size. Similarly, denser designs have a lower average value of $\hat{U}_W$ as more observations are available.

| Measuremet Error Noise level | Sparsity setting | Very Sparse | Medium Sparse | Dense |
|-----------------------------|-----------------|-------------|---------------|-------|
| Predictor | Response | $n = 500$ | $n = 2000$ | $n = 500$ | $n = 2000$ | $n = 500$ | $n = 2000$ |
| $\sigma$ | $\sigma_Y$ | 0.00174 | 0.00062 | 0.00085 | 0.00046 | 0.00076 | 0.00037 |
| 0.5 | 0.5 | 0.00218 | 0.00075 | 0.00122 | 0.00058 | 0.00125 | 0.00052 |
| 1.0 | 0.5 | 0.00295 | 0.00154 | 0.00105 | 0.00044 | 0.00082 | 0.00045 |

Table 2: Simulation results for the Wasserstein measure against a uniform distribution $\hat{U}_W$ defined through (23). The simulation settings are the same as the ones shown in Table 1 and 2,000 simulations are performed. The values displayed are the average of $\hat{U}_W$ across simulations.

7 Appendix: Assumptions

We assume the following regularity conditions (A1)–(A8) which are utilized in Dai et al. (2018) and are presented here for completeness. Define $w_i = (\sum_{i=1}^{n} n_i)^{-1}$ and $v_i = (\sum_{i=1}^{n} n_i(n_i - 1))^{-1}$. 

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(A1) $K(\cdot)$ is a symmetric probability density function on $[-1, 1]$ and is Lipschitz continuous: There exists $0 < L < \infty$ such that $|K(u) - K(v)| \leq L|u - v|$ for any $u, v \in [0,1]$.

(A2) $\{T_{ij} : i = 1, \ldots, n, j = 1, \ldots, n_i\}$ are i.i.d. copies of a random variable $T$ defined on $\mathcal{T}$, and $n_i$ are regarded as fixed. The density $f(\cdot)$ of $T$ is bounded below and above,

$$0 < m_f \leq \min_{t \in \mathcal{T}} f(t) \leq \max_{t \in \mathcal{T}} f(t) \leq M_f < \infty.$$  

Furthermore $f^{(2)}$, the second derivative of $f(\cdot)$, is bounded.

(A3) $X, \epsilon, \text{ and } T$ are independent.

(A4) $\mu^{(3)}(t)$ and $\partial^4 \Gamma(s, t)/\partial^{4-p}$ exist and are bounded on $\mathcal{T}$ and $\mathcal{T} \times \mathcal{T}$, respectively, for $p = 0, \ldots, 4$.

(A5) $h_\mu \to 0$ and $\log(n) \sum_{i=1}^n n_i w_i^2 / h_\mu \to 0$.

(A6) For some $\alpha > 2$, $E(\sup_{t \in \mathcal{T}} |X(t) - \mu(t)|^\alpha) < \infty$, $E(|\epsilon|^\alpha) < \infty$, and

$$n \left[ \sum_{i=1}^n n_i w_i^2 h_\mu + \sum_{i=1}^n n_i(n_i - 1) w_i^2 h_\mu^2 \right] \left[ \frac{\log(n)}{n} \right]^{2/\alpha - 1} \to \infty.$$

(A7) $h_G \to 0$, $\log(n) \sum_{i=1}^n n_i(n_i - 1) v_i^2 / h_G^2 \to 0$.

(A8) For some $\beta > 2$, $E(\sup_{t \in \mathcal{T}} |X(t) - \mu(t)|^{2\beta}) < \infty$, $E(|\epsilon|^{2\beta}) < \infty$, and

$$n \left[ \sum_{i=1}^n n_i(n_i - 1) v_i^2 h_G^2 + \sum_{i=1}^n n_i(n_i - 2)(n_i - 3) v_i^2 h_G^3 \right] \left[ \frac{\log(n)}{n} \right]^{2/\beta - 1} \to \infty.$$

We remark that assumption (A2) implies (S3), where the latter is used in Theorem 1. Also we note that assumption (A4) implies the weaker assumption (S2).

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Supplement

For notational simplicity, for a function $g_1 : T \rightarrow \mathbb{R}$ and a vector $z = (z_1, \ldots, z_p)^T \in \mathbb{R}^p$, $p > 0$, denote by $g_1(z) = (g_1(z_1), \ldots, g_1(z_p))^T$ the application of $g_1$ to $z$ entry-wise. Similarly, for a function $g_2 : T \times T \rightarrow \mathbb{R}$ and a second vector $r = (r_1, \ldots, r_q)^T \in \mathbb{R}^q$, $q > 0$, denote by $g_2(z, r^T)$ the $p \times q$ matrix, for which the $(l, k)$ element is given by $g_2(z_l, r_k)$, where $1 \leq l \leq p$ and $1 \leq k \leq q$.

S.1 Auxiliary Results and Proofs of Main Results in Section 3

In this section we provide auxiliary lemmas which will be used to derive the main results in section 3. For the next lemma, we say that a process $X$ is explained by its first $K$ principal components if $X(t) = \mu(t) + \sum_{k=1}^{K} \xi_k \phi_k(t)$ and thus it is of finite dimension $K$. In what follows, for a matrix $A \in \mathbb{R}^{p \times q}$, $p, q > 0$, denote by $\|A\|_{op,2} = \sup_{\|x\|_2=1} \|Ax\|_2$ its matrix operator norm with respect to the Euclidean norm $\|\cdot\|_2$.

Lemma S1. Suppose that the process $X$ is finite dimensional and explained by its first $K = 2$ principal components. If $\phi_1$ and $\phi_2$ are bijective and differentiable in a finite partition of $T$, then $\Sigma_{iK}$ has a positive eigengap almost surely.

Proof of Lemma S1. Recalling that $\Sigma_{iK} = \Lambda_{iK} - \Lambda_{iK} \Phi_{iK}^T \Sigma_{i}^{-1} \Phi_{iK} \Lambda_{iK}$ and since $K = 2$, it follows that the characteristic polynomial of $\Sigma_{iK}$ is given by $p(\lambda) = \lambda^2 - \text{tr}(\Sigma_{iK}) \lambda + \det(\Sigma_{iK})$, and thus the eigengap is equal to $\sqrt{\Delta_p}$, where $\Delta_p$ is the discriminant of the quadratic polynomial $p$.

It is easy to show that

$$\Delta_p = (\lambda_1 - \lambda_2 + \lambda_1^2 \phi_{12}^T \Sigma_{i}^{-1} \phi_{12}^{-1} \phi_{12} - \lambda_1^2 \phi_{11}^T \Sigma_{i}^{-1} \phi_{11}^{-1} \phi_{11})^2 + 4\lambda_1^2 \lambda_2^2 (\phi_{11}^T \Sigma_{i}^{-1} \phi_{11})^2,$$

so that it suffices to check that $\phi_{11}^T \Sigma_{i}^{-1} \phi_{12}$ is not identically zero almost surely. Let $B = \sigma^2 I_{n_i} + \lambda_1 \phi_{11}^T \phi_{11}$, where $I_{n_i}$ denotes the $n_i \times n_i$ identity matrix, and denote by $\|\cdot\|_2$ the Euclidean norm in $\mathbb{R}^{n_i}$. By the Sherman-Morrison formula, it follows that $B^{-1} = \sigma^{-2} \left( I_{n_i} - \frac{\lambda_1 \phi_{11} \phi_{11}^T}{\sigma^2 + \lambda_1 \|\phi_{11}\|_2^2} \right)$, and a second application of the formula leads to

$$\Sigma_{i}^{-1} = B^{-1} - \frac{B^{-1} \lambda_2 \phi_{12} \phi_{12}^T B^{-1}}{1 + \lambda_2 \phi_{12}^T B^{-1} \phi_{12}}.$$

Thus

$$\phi_{11}^T \Sigma_{i}^{-1} \phi_{12} = \frac{\phi_{11}^T B^{-1} \phi_{12}}{1 + \lambda_2 \phi_{12}^T B^{-1} \phi_{12}},$$

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where $\phi_{ij}^T B^{-1} \phi_{i2} = \frac{\phi_{ij}^T \phi_{i2}}{\sigma^2 + \lambda_1 \|\phi_{i1}\|_2^2}$ and $\phi_{ij}^T B^{-1} \phi_{i2} > 0$ a.s. since the eigenvalues of $B$ are bounded below by $\sigma^2$. The conclusion then follows if we can show that $\phi_{ij}^T \phi_{i2} \neq 0$ almost surely. Note that $\phi_{ij}^T \phi_{i2} = \sum_{j=1}^n \phi_1(T_{ij})\phi_2(T_{ij})$ and the $T_{ij}$ are i.i.d. with a continuous distribution supported on $\mathcal{T}$. Thus, the distribution of $\phi_{ij}^T \phi_{i2}$ corresponds to the $n$-fold convolution of the continuous distribution associated with $\phi_1(T_{i1})\phi_2(T_{i1})$, which is a continuous probability measure, and hence $\phi_{ij}^T \phi_{i2} \neq 0$ holds almost surely. \hfill $\square$

**Lemma S2.** Let $T_1, \ldots, T_m$ be i.i.d. with density function $f(t)$, $t \in \mathcal{T} = [0, 1]$ and let $T_{(1)}, \ldots, T_{(m)}$ be the order statistics. Let $w_l := T_{(l)} - T_{(l-1)}$, $l = 1, \ldots, m$, where $T_{(0)} := 0$, be the spacing between the order statistics. Suppose that there exists $c_0 > 0$ such that $f(t) \geq c_0$ for all $t \in \mathcal{T}$. Then

$$E(w_l^2) = O(m^{-2}), \quad l = 1, \ldots, m,$$

and

$$E\left(1 - T_{(m)}\right)^2 = O(m^{-2}).$$

**Proof of Lemma S2.** One can replace $T_l$ with i.i.d. copies $Q(U_l)$, $l = 1, \ldots, m$, where the $U_l \overset{i.i.d.}{\sim} U(0, 1)$ and $Q$ is the quantile function corresponding to $f$. Since $f$ is strictly positive, then $T_{(l)} = Q(U_{(l)})$, $l = 1, \ldots, m$, and moreover, from a Taylor expansion of $Q(\cdot)$, we have

$$E\left(w_l^2\right) = E[Q'(\eta)(U_{(l)} - U_{(l-1)})]^2 \leq c_0^{-1} E[U_{(l)} - U_{(l-1)}]^2,$$

where $\eta_l$ is between $U_{(l-1)}$ and $U_{(l)}$, and the last inequality follows from the fact that $Q'(t) = 1/f(Q(t)) \leq c_0^{-1}$. The first result follows since $E[U_{(l)} - U_{(l-1)}]^2 = 2/(m^2 + 3m + 2) = O(m^{-2})$. Similarly, by expanding $Q(U_m)$ around $Q(1) = 1$, the second result follows. \hfill $\square$

**Proof of Theorem 1.** Fix $i \in \{1, \ldots, n\}$ and $k \in \mathbb{N}$, and recall that

$$\tilde{\xi}_{ik} = \lambda_k \phi_{ik}^T \Sigma_i^{-1}(X_i - \mu_i), \quad (S.24)$$

where $\phi_{ik} = (\phi_k(T_{i1}), \ldots, \phi_k(T_{im}))^T$. Define $W = \text{diag}(w_l)$, where $w_l$ are quadrature weights chosen according to the left endpoint rule, i.e. $w_l = T_{il} - \max_{j:T_{ij} < T_{il}} T_{ij}$ for $l = 1, \ldots, m$, and we set $\max_{j:T_{ij} < T_{il}} T_{ij} = 0$ whenever $\{j : T_{ij} < T_{il}\} = \emptyset$. Let $g_m$ be the size of the maximal gap between $\{0, T_{i1}, \ldots, T_{im}, 1\}$ for $\mathcal{T} = [0, 1]$ and consider the quadrature approximation errors

$$e_k = \int_{\mathcal{T}} \Gamma(T_i, t)\phi_k(t) dt - \Sigma_i W \phi_{ik},$$

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where $\Gamma(T_i, t) = (\Gamma(T_{i1}, t), \ldots, \Gamma(T_{im}, t))^T$. Here note that since $\Sigma_i = \sigma^2 I_m + \Gamma(T_i, T_i^T)$, where $\Gamma(T_i, T_i^T)$ corresponds to the matrix with elements $[\Gamma(T_i, T_i^T)]_{jl} = \Gamma(T_{ij}, T_{il})$, $j, l \in \{1, \ldots, m\}$, we have $\Sigma_i W \phi_{ik} = \sigma^2 W \phi_{ik} + \Gamma(T_i, T_i^T) W \phi_{ik}$ where the second term in the previous expression corresponds to the numerical quadrature approximation to $\int_T \Gamma(T_i, t) \phi_k(t) dt$ and the first term will be shown to be diminishable as $m \to \infty$.

Next, from the quadrature approximation error for integrating a continuously differentiable function $g$ over $[0, 1]$ under the left-endpoint rule and denoting $T_i^{(m)} := \max_{1 \leq j \leq m} T_{ij}$ we have

$$
\left| \int_0^1 g(t) dt - \sum_{l=1}^m g(T_{il}) w_l \right| \leq \frac{\sup_{t \in \mathcal{T}} |g'(t)|}{2} \left( \sum_{l=1}^m w_l^2 + (1 - T_i^{(m)})^2 \right) + |(1 - T_i^{(m)}) g(1)| \quad (S.25)
$$

$$
= O_p(m^{-1}), \quad (S.26)
$$

where (S.26) follows from Lemma S2. Denoting by $\|\cdot\|_2$ the Euclidean norm in $\mathbb{R}^m$, we have

$$
\|e_k\|_2 \leq \left\| \int_T \Gamma(T_i, t) \phi_k(t) dt - \Gamma(T_i, T_i^T) W \phi_{ik} \right\|_2 + \|\sigma^2 W \phi_{ik}\|_2 = O_p(m^{-1/2}), \quad (S.27)
$$

which follows by noting that the integration error rates for all entries in $e_k$ are uniform due to (S2) and (S.25), and that

$$
\|W \phi_{ik}\|_2^2 \leq \sum_{l=1}^m w_l^2 \sup_{t \in \mathcal{T}} \phi_k(t)^2 = O_p(m^{-1}). \quad (S.28)
$$

Next, since $\lambda_k \phi_{ik} = \Sigma_i W \phi_{ik} + e_k$, we have

$$
\lambda_k \phi_{ik}^T \Sigma_i^{-1}(X_i - \mu_i) = \phi_{ik}^T W(X_i - \mu_i) + e_k^T \Sigma_i^{-1}(X_i - \mu_i)
$$

$$
= \phi_{ik}^T W(Y_i - \mu_i) + \phi_{ik}^T W e_i + e_k^T \Sigma_i^{-1}(X_i - \mu_i), \quad (S.29)
$$

where $Y_i = (X_{i1}, \ldots, X_{im})^T$ and $e_i = (\epsilon_{i1}, \ldots, \epsilon_{im})^T$. Let $g_k(t) = \phi_k(t)(X_i(t) - \mu(t))$. Then, from (S2) and since the process $X_i(t)$ is assumed continuously differentiable almost surely, we have $g_k(t)$ is continuously differentiable a.s. over the compact set $\mathcal{T} = [0, 1]$ so that $\sup_{t \in \mathcal{T}} |g_k(t)| = O_p(1)$. Thus, using (S.25) and the fact that $\int_0^1 \phi_k(t)(X_i(t) - \mu(t)) dt = \xi_{ik}$, we obtain

$$
\xi_{ik} - \sum_{l=1}^m \phi_k(T_{il})(X_i(T_{il}) - \mu(T_{il})) w_l = O_p(m^{-1}),
$$

whence

$$
\phi_{ik}^T W(Y_i - \mu_i) = \xi_{ik} + O_p(m^{-1}). \quad (S.30)
$$
Next, note that by conditioning and using the independence between $\epsilon_i$ and $T_i$, we have

$$E(\phi_{ik}^T W \epsilon_i)^2 = E[E(\phi_{ik}^T W \epsilon_i^T W \phi_{ik} | T_i)] = E[\phi_{ik}^T W E(\epsilon_i^T W \phi_{ik})] = \sigma^2 E(\|W \phi_{ik}\|_2^2).$$

Hence, from (S.28) it follows that $E(\phi_{ik}^T W \epsilon_i)^2 = O(m^{-1})$ and thus

$$\phi_{ik}^T W \epsilon_i = O_p(m^{-1/2}). \tag{S.31}$$

We now show that $Z_m := e_k^T \Sigma_i^{-1}(X_i - \mu_i) = O_p(m^{-1/2})$. Note that for any $M > 0$

$$P \left( \sqrt{m} |Z_m| > M |T_i| \right) \leq \frac{m}{M^2} \|e_k\|_2^2 \|\Sigma_i^{-1/2}\|_{op,2} \leq \frac{m}{M^2 \sigma^2} \|e_k\|_2^2, \tag{S.32}$$

where the last inequality follows since $\|\Sigma_i^{-1/2}\|_{op,2} \leq \sigma^{-1}$. Next, from (S.27) we have $m \|e_k\|_2^2 = O_p(1)$ and thus for any $\epsilon > 0$ there exist $M_0 = M_0(\epsilon) > 0$ and $m_0 = m_0(\epsilon) \in \mathbb{N}^+$ such that

$$P \left( m \|e_k\|_2^2 > M_0 \right) \leq \epsilon, \quad \forall m \geq m_0. \tag{S.33}$$

Hence, by choosing $M = M_\epsilon := \sqrt{M_0/(\epsilon \sigma^2)}$ and defining $u_{im} := P \left( \sqrt{m} |Z_m| > M |T_i| \right)$,

$$P \left( \sqrt{m} |Z_m| > M \right) = E(u_{im}) = E[u_{im}1_{u_{im} \leq \epsilon} + u_{im}1_{u_{im} > \epsilon}] \leq \epsilon + P(u_{im} > \epsilon), \tag{S.34}$$

where the last inequality follows since $u_{im} \leq 1$. Next, (S.32) and (S.33) imply $P(u_{im} > \epsilon) \leq \epsilon$ for $m \geq m_0$, whence

$$P \left( \sqrt{m} |e_k^T \Sigma_i^{-1}(X_i - \mu_i)| > M_\epsilon \right) \leq 2\epsilon, \quad \forall m \geq m_0, \tag{S.35}$$

which shows that $e_k^T \Sigma_i^{-1}(X_i - \mu_i) = O_p(m^{-1/2})$. The proof of (2) follows by combining (S.29), (S.30), (S.31) and (S.35).

**Lemma S3.** Suppose that the same assumptions as in Theorem 2 hold. Let $k \geq 1$ and define

$$\hat{e}_k^* = \int_T \hat{\Gamma}(T^*, t) \phi_k(t) dt - \Sigma^* W^* \phi_k^*, \text{ where } \phi_k^* = \hat{\phi}_k(T^*) \text{ and the quadrature weight matrix } W^* \text{ is defined analogously as } W \text{ in the proof of Theorem 1 but using the time points } T^*. \text{ Then}

$$\|\hat{e}_k^*\|_2 = O_p(m^{*-1/2}) \quad \text{as} \quad n \to \infty. \tag{S.36}$$

**Proof of Lemma S3.** Similar to the proof of Theorem 1, we have

$$\|\hat{e}_k^*\|_2 \leq \left\| \int_T \hat{\Gamma}(T^*, t) \phi_k(t) dt - \hat{\Gamma}(T^*, T^* T) W^* \phi_k^* \right\|_2 + \left\| \sigma^2 W^* \phi_k^* \right\|_2, \tag{S.37}$$
where the \((i, j)\) element of \(\hat{\Gamma}(\mathbf{T}^*, \mathbf{T}^{*T})\) is given by \(\hat{\Gamma}(T_i^*, T_j^*)\), \(1 \leq i, j \leq m^*\). Note that
\[
\left\| \sigma^2 \mathbf{W}^* \hat{\phi}_k^* \right\|_2 \leq (|\sigma^2| + \sigma^2) \left( \sum_{l=1}^{m} w_l^2 \sup_{t \in \mathcal{T}} |\hat{\phi}_k(t)|^2 \right)^{1/2} \\
\leq (|\sigma^2| + \sigma^2) \left( \sum_{l=1}^{m} w_l^2 \sup_{t \in \mathcal{T}} \{|\hat{\phi}_k(t)| - \phi_k(t)\}^2 \right)^{1/2} \\
= O_p(m^{-1/2}),
\]  
(S.38)
where the last equality holds since \(\sup_{t \in \mathcal{T}} |\hat{\phi}_k(t) - \phi_k(t)| = o(1)\) and \(|\sigma^2| = o(1)\) a.s.\(n \to \infty\), which follows from Proposition 1 in Dai et al. (2018). Next
\[
\int_{\mathcal{T}} \hat{\Gamma}(\mathbf{T}^*, t)\hat{\phi}_k(t)dt - \hat{\Gamma}(\mathbf{T}^*, \mathbf{T}^{*T}) \mathbf{W}^* \hat{\phi}_k^* = \int_{\mathcal{T}} \hat{\Gamma}(\mathbf{T}^*, t)\hat{\phi}_k(t)dt - \int_{\mathcal{T}} \Gamma(\mathbf{T}^*, t)\phi_k(t)dt \\
+ \int_{\mathcal{T}} \Gamma(\mathbf{T}^*, t)\phi_k(t)dt - \Gamma(\mathbf{T}^*, \mathbf{T}^{*T}) \mathbf{W}^* \phi_k^* \\
+ \Gamma(\mathbf{T}^*, \mathbf{T}^{*T}) \mathbf{W}^* \phi_k^* - \hat{\Gamma}(\mathbf{T}^*, \mathbf{T}^{*T}) \mathbf{W}^* \hat{\phi}_k^*,
\]  
(S.39)
where \(\hat{\phi}_k^* = \phi_k(\mathbf{T}^*)\). Hence, it suffices to study the rate of each of the differences in (S.39). First
\[
\int_{\mathcal{T}} \hat{\Gamma}(\mathbf{T}^*, t)\hat{\phi}_k(t)dt - \Gamma(\mathbf{T}^*, t)\phi_k(t)dt = \int_{\mathcal{T}} (\hat{\Gamma}(\mathbf{T}^*, t) - \Gamma(\mathbf{T}^*, t))\hat{\phi}_k(t)dt + \int_{\mathcal{T}} \Gamma(\mathbf{T}^*, t)(\hat{\phi}_k(t) - \phi_k(t))dt,
\]
where, for \(j = 1, \ldots, m^*\), and by using the orthonormality of the \(\hat{\phi}_k\),
\[
\left| \int_{\mathcal{T}} (\hat{\Gamma}(T_j^*, t) - \Gamma(T_j^*, t))\hat{\phi}_k(t)dt \right| \leq \left( \int_{\mathcal{T}} (\hat{\Gamma}(T_j^*, t) - \Gamma(T_j^*, t))^2 dt \right)^{1/2} \leq \sup_{s, t \in \mathcal{T}} |\hat{\Gamma}(s, t) - \Gamma(s, t)|,
\]
and
\[
\left| \int_{\mathcal{T}} \Gamma(T_j^*, t)(\hat{\phi}_k(t) - \phi_k(t))dt \right| \leq \sup_{s, t \in \mathcal{T}} \sup_{t \in \mathcal{T}} |\Gamma(s, t)|\sup_{t \in \mathcal{T}} |\hat{\phi}_k(t) - \phi_k(t)|,
\]
where we use that \(|\mathcal{T}| = 1\). Thus, since \(\sup_{s, t \in \mathcal{T}} |\hat{\Gamma}(s, t) - \Gamma(s, t)|\) and \(\sup_{t \in \mathcal{T}} |\hat{\phi}_k(t) - \phi_k(t)|\) are \(O(a_n + b_n)\) a.s.\(n \to \infty\), which follows from Proposition 1 in Dai et al. (2018), and using the fact that \(\Gamma(s, t)\) is continuous over the compact set \(\mathcal{T}^2\), leads to
\[
\left\| \int_{\mathcal{T}} \hat{\Gamma}(\mathbf{T}^*, t)\hat{\phi}_k(t)dt - \Gamma(\mathbf{T}^*, t)\phi_k(t)dt \right\|_2 = O \left( \sqrt{m^* (a_n + b_n)} \right) \quad \text{a.s.} \quad (S.40)
\]
as \(n \to \infty\). Second, from the proof of Theorem 1 we have
\[
\left\| \int_{\mathcal{T}} \Gamma(\mathbf{T}^*, t)\hat{\phi}_k(t)dt - \Gamma(\mathbf{T}^*, \mathbf{T}^{*T}) \mathbf{W}^* \hat{\phi}_k^* \right\|_2 = O_p(m^{-1/2}). \quad (S.41)
\]
Third

\[
\Gamma(T^*, T^{*T})W^*\phi_k^* - \hat{\Gamma}(T^*, T^{*T})W^*\hat{\phi}_k^* = (\Gamma(T^*, T^{*T}) - \hat{\Gamma}(T^*, T^{*T}))W^*\phi_k^* + \hat{\Gamma}(T^*, T^{*T})(W^*\phi_k^* - W^*\hat{\phi}_k^*).
\]

(S.42)

Next

\[
\left\| (\Gamma(T^*, T^{*T}) - \hat{\Gamma}(T^*, T^{*T}))W^*\phi_k^* \right\|_2 \leq \left\| \Gamma(T^*, T^{*T}) - \hat{\Gamma}(T^*, T^{*T}) \right\|_{op, 2} \left\| W^*\phi_k^* \right\|_2 \\
= \left\| \Gamma(T^*, T^{*T}) - \hat{\Gamma}(T^*, T^{*T}) \right\|_{op, 2} O_p(m^{-1/2}),
\]

where the last equality follows similarly as (S.28). Since \( \|A\|_{op, 2} \leq \|A\|_F \), where \( \|A\|_F \) denotes the Frobenius norm of a squared matrix \( A \), and

\[
\left\| \Gamma(T^*, T^{*T}) - \hat{\Gamma}(T^*, T^{*T}) \right\|_F^2 = \sum_{j=1}^{m^*} \sum_{l=1}^{m^*} \left( \Gamma(T_j^*, T_l^{*T}) - \hat{\Gamma}(T_j^*, T_l^{*T}) \right)^2 \leq m^2 \sup_{s,t \in \mathcal{T}} |\hat{\Gamma}(s,t) - \Gamma(s,t)|^2 \\
= O(m^2(a_n + b_n)^2),
\]

a.s. as \( n \to \infty \), it follows that

\[
\left\| (\Gamma(T^*, T^{*T}) - \hat{\Gamma}(T^*, T^{*T}))W^*\phi_k^* \right\|_2 = O_p(\sqrt{m^*}(a_n + b_n)).
\]

(S.43)

Next, as \( n \to \infty \) we have

\[
\left\| \hat{\Gamma}(T^*, T^{*T})(W^*\phi_k^* - W^*\hat{\phi}_k^*) \right\|_2 = \left( \left\| \hat{\Gamma}(T^*, T^{*T}) - \Gamma(T^*, T^{*T}) \right\|_{op, 2} + \left\| \Gamma(T^*, T^{*T}) \right\|_{op, 2} \right) \left\| W^*\phi_k^* - W^*\hat{\phi}_k^* \right\|_2 \\
\leq (O(m^*(a_n + b_n)) + O(m^*)) \left( \sum_{l=1}^{m^*} w_l^2 \sup_{t \in \mathcal{T}} |\hat{\phi}_k(t) - \phi_k(t)|^2 \right)^{1/2} \text{ a.s.} \\
= O(m^*(a_n + b_n) + m^*)O_p(m^{*-1/2}(a_n + b_n)) = O_p(\sqrt{m^*}(a_n + b_n)).
\]

(S.44)

Hence, combining (S.42), (S.43) and (S.44) we obtain

\[
\left\| \Gamma(T^*, T^{*T})W^*\phi_k^* - \hat{\Gamma}(T^*, T^{*T})W^*\hat{\phi}_k^* \right\|_2 = O_p(m^{*-1/2}(a_n + b_n)),
\]

(S.45)

and then using (S.40), (S.41) and (S.45) along with (S.39) leads to

\[
\left\| \int_{\mathcal{T}} \hat{\Gamma}(T^*, t)\hat{\phi}_k(t)dt - \hat{\Gamma}(T^*, T^{*T})W^*\hat{\phi}_k^* \right\|_2 = O_p(m^{*-1/2}(a_n + b_n)) + O_p(m^{*-1/2}).
\]

(S.46)

The condition \( m^*(a_n+b_n) = o(1) \) implies \( \left\| \int_{\mathcal{T}} \hat{\Gamma}(T^*, t)\hat{\phi}_k(t)dt - \hat{\Gamma}(T^*, T^{*T})W^*\hat{\phi}_k^* \right\|_2 = O_p(m^{*-1/2}) \),

and the result then follows by using (S.37) and (S.38).
Lemma S4. Suppose that the same assumptions as in Theorem 2 hold. Let \( k \geq 1 \) and \( \hat{e}_k^* \) be defined as in Lemma S3. Then

\[
Z_{m^*,n} := \hat{e}_k^T \hat{\Sigma}^{*-1} (X^* - \hat{\mu}^*) = O_p(m^{*-1/2}), \quad (S.47)
\]

as \( n \to \infty \).

Proof of Lemma S4. For any \( M > 0 \) and by independence of the new subject’s observations from the estimated population quantities, we have

\[
P \left( \sqrt{m^*} |Z_{m^*,n} - \mu_{m^*,n}| > M|T^*, \hat{\Gamma}, \hat{\phi}_k, \hat{\sigma}, \hat{\mu} \right) \leq \frac{m^*}{M^2} \hat{e}_k^T \hat{\Sigma}^{*-1} \Sigma^* \hat{\Sigma}^{*-1} \hat{e}_k^* \quad (S.48)
\]

where \( \mu_{m^*,n} := \hat{e}_k^T \hat{\Sigma}^{*-1} (\mu^* - \hat{\mu}^*) \) and

\[
\hat{e}_k^* \Sigma^{*-1} \Sigma^* \hat{e}_k^* \leq \hat{e}_k^T (\Sigma^{*-1} - \Sigma^* \Sigma^{*-1}) \Sigma^* \hat{e}_k + 2 \hat{e}_k^T (\Sigma^{*-1} - \Sigma^* \Sigma^{*-1}) \hat{e}_k + \hat{e}_k^T \Sigma^{*-1} \hat{e}_k^* \\
\leq O(m^*(a_n + b_n)^2) \frac{\| \hat{e}_k^* \|^2}{2} + O(m^*(a_n + b_n)) \| \hat{e}_k^* \|^2 + \| \hat{e}_k^* \|^2 \\
\leq c \| \hat{e}_k^* \|^2 \quad \text{a.s. for large enough } n.
\]

Here \( c \) is a positive universal constant. The second inequality follows since \( \| \Sigma^{*-1} \|_{op,2} \leq \sigma^{-2} \)

and the fact that \( \| \Sigma^{*-1} - \Sigma^* \Sigma^{*-1} \|_{op,2} = O(m^*(a_n + b_n)) \) a.s., which was shown in the proof of Theorem 2 in Dai et al. (2018), and the last inequality follows from the assumption that \( m^*(a_n + b_n) = o(1) \) as \( n \to \infty \). Thus, for large enough \( n \) we have

\[
P \left( \sqrt{m^*} |Z_{m^*,n} - \mu_{m^*,n}| > M|T^*, \hat{\Gamma}, \hat{\phi}_k, \hat{\sigma}, \hat{\mu} \right) \leq \frac{m^*}{M^2} \| \hat{e}_k^* \|^2 \quad \text{a.s.} \quad (S.49)
\]

where \( m^* = m^*(n) \). Next, from (S.36) in Lemma S3 we have \( m^* \| \hat{e}_k^* \|^2 = O_p(1) \) for large enough \( n \), and hence it follows that for any \( \epsilon > 0 \) there exist \( M_0 = M_0(\epsilon) > 0 \) and \( m_0 = m_0(\epsilon) \in \mathbb{N}^+ \) such that

\[
P \left( m^* \| \hat{e}_k^* \|^2 > M_0 \right) \leq \epsilon, \quad \forall m^* \geq m_0. \quad (S.50)
\]

The previous inequality is satisfied for large enough \( n \) since \( m^* = m^*(n) \to \infty \) as \( n \to \infty \). Hence, choosing \( M = M_\epsilon := \sqrt{cM_0/\epsilon} \) and defining \( u_{m^*,n} := P \left( \sqrt{m^*} |Z_{m^*,n} - \mu_{m^*,n}| > M|T^*, \hat{\Gamma}, \hat{\phi}_k, \hat{\sigma}, \hat{\mu} \right) \)

we obtain

\[
P \left( \sqrt{m^*} |Z_{m^*,n} - \mu_{m^*,n}| > M_\epsilon \right) = E[u_{m^*,n} 1_{u_{m^*,n} \leq \epsilon} + u_{m^*,n} 1_{u_{m^*,n} > \epsilon}] \leq \epsilon + P(u_{m^*,n} > \epsilon), \quad (S.51)
\]

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where the last inequality follows since \( u_{m^*n} \leq 1 \). Next, \((S.49)\) and \((S.50)\) imply \( P(u_{m^*n} > \epsilon) \leq \epsilon \) for \( m^* \geq m_0 \), whence

\[
P \left( \sqrt{m^*} \left| \hat{\mathbf{e}}_k^T \hat{\Sigma}^{*-1}\mathbf{X}^* - \hat{\mu}^* \right| - \mu_{m^*,n} \right) > M_e \right) \leq 2\epsilon \quad \forall m^* \geq m_0. \tag{S.52}
\]

Similar arguments show that \( \mu_{m^*,n} = \hat{\mathbf{e}}_k^T \hat{\Sigma}^{*-1}\hat{\mu}^* = O_p(m^{*-1/2}) \). The result follows. \( \square \)

**Proof of Theorem 2.** Let \( \hat{\mathbf{e}}_k^* = \int_T \hat{\Gamma}(T^*, t) \hat{\phi}_k(t) dt - \hat{\Sigma}^* \hat{\mathbf{W}}^* \hat{\phi}_k^* \), where the quadrature weight matrix \( \mathbf{W}^* \) is defined analogously as \( \mathbf{W} \) in the proof of Theorem 1 but using the time points \( T^* \), and \( \hat{\phi}_k = \hat{\phi}_k(T^*) \). Write \( \hat{\mu}^* = \mu(T^*) \), \( \phi_k^* = \phi_k(T^*) \), \( \mathbf{e}_k^* = \int_T \Gamma(T^*, t) \phi_k(t) dt - \Sigma^* \mathbf{W}^* \phi_k^* \) and \( \hat{\xi}_k^* = \lambda_k \phi_k(T^*) \hat{\Sigma}^{*-1}(\mathbf{X}^* - \mu^*) \). Then

\[
|\hat{\xi}_k^* - \hat{\xi}_k^*| = \left| \phi_k^* \mathbf{W}^*(\mathbf{X}^* - \hat{\mu}^*) + \mathbf{e}_k^* \hat{\Sigma}^{*-1}(\mathbf{X}^* - \mu^*) \right| - \left| \phi_k^* \mathbf{W}^*(\mathbf{X}^* - \mu^*) + \mathbf{e}_k^* \hat{\Sigma}^{*-1}(\mathbf{X}^* - \mu^*) \right| + O_p(m^{*-1/2})
\]

\[
= |(\phi_k^* - \phi_k^*) \mathbf{W}^*(\mu^* - \mu^*) + (\phi_k^* - \phi_k^*) \mathbf{W}^*(\mu^* - \mu^*) + \phi_k^* \mathbf{W}^*(\mu^* - \mu^*)| + O_p(m^{*-1/2})
\]

\[
\leq \left| \mathbf{W}^*(\phi_k^* - \phi_k^*) \right|_2 \left| \mu^* - \hat{\mu}^* \right|_2 + \left| \mathbf{W}^*(\phi_k^* - \phi_k^*) \right|_2 \left| \mathbf{X}^* - \mu^* \right|_2 + \left| \mathbf{W}^* \phi_k^* \right|_2 \left| \hat{\mu}^* - \mu^* \right|_2 + O_p(m^{*-1/2}),
\]

where the second equality is due to \((S.47)\) in Lemma S4 and the fact that \( \mathbf{e}_k^T \hat{\Sigma}^{*-1}(\mathbf{X}^* - \mu^*) = O_p(m^{*-1/2}) \), which was shown in the proof of Theorem 1. Term-by-term estimates are

\[
\left| \hat{\mu}^* - \mu^* \right|_2 = O(m^{1/2}a_n) \quad \text{a.s.}, \tag{S.54}
\]

\[
\left| \mathbf{W}^*(\phi_k^* - \phi_k^*) \right|_2 = O(m^{1/2}(a_n + b_n)) \quad \text{a.s.}, \tag{S.55}
\]

\[
\left| \mathbf{X}^* - \mu^* \right|_2 = O_p(m^{1/2}), \tag{S.56}
\]

\[
\left| \mathbf{W}^* \phi_k^* \right|_2 = O_p(m^{*-1/2}), \tag{S.57}
\]

where the rates \( a_n \) and \( b_n \) in \((S.54)\) and \((S.55)\) are by Proposition 1 in Dai et al. (2018), see also Zhang and Wang (2016), \((S.56)\) is by the Weak Law of Large Numbers, and \((S.57)\) was shown in the proof of Theorem 1. Combining \((S.53)\)–\((S.57)\) and (3) completes the proof. \( \square \)

**Proof of Theorem 3.** Recalling that \( \Sigma_{iK} = \Lambda_K - \Lambda_K \hat{\Phi}_{iK}^T \Sigma_{iK}^{-1} \hat{\Phi}_{iK} \Lambda_K \) we have

\[
\| \Sigma_{iK} \|_{op,2} \leq \text{trace}(\Sigma_{iK}) = \sum_{k=1}^K (\lambda_k - \lambda_k \phi_{ik}^T \Sigma_{iK}^{-1} \lambda_k \phi_{ik}). \tag{S.58}
\]
Moreover, since $\lambda_k \phi_{ik} = e_k + \Sigma_i W \phi_{ik}$ it follows that

$$
\lambda_k \phi_{ik} \Sigma_i^{-1} \lambda_k \phi_{ik} = e_k^T \Sigma_i^{-1} e_k + 2 e_k^T W \phi_{ik} + \phi_{ik}^T W \Sigma_i W \phi_{ik}.
$$

(S.59)

Next, from the proof of Theorem 1, since in (S.27) it was shown that $\|\lambda_k \phi_{ik} - \Gamma(T_i, T_i^T) W \phi_{ik}\|_2 = O_p(m^{-1/2})$ and using (S.28),

$$
\phi_{ik}^T W \Sigma_i W \phi_{ik} = \sigma^2 \phi_{ik}^T W W \phi_{ik} + \phi_{ik}^T W \Gamma(T_i, T_i^T) W \phi_{ik} = O_p(m^{-1}) + \phi_{ik}^T W \left( \lambda_k \phi_{ik} - O_p(m^{-1/2}) \right) = \lambda_k \phi_{ik}^T W \phi_{ik} + O_p(m^{-1}),
$$

where $\lambda_k \phi_{ik}^T W \phi_{ik} = \lambda_k + O_p(m^{-1})$. This follows from the quadrature approximation error (S.26), observing $\int_0^1 \phi_k^2(t) dt = 1$, and implies

$$
\phi_{ik}^T W \Sigma_i W \phi_{ik} = \lambda_k + O_p(m^{-1}).
$$

(S.60)

Then (5) follows by combining (S.58), (S.59), (S.60), (S.27), (S.28), and the fact that $\|\Sigma_i^{-1}\|_{op,2} \leq \sigma^{-2}$.  

Proof of Theorem 4. Recall that $\hat{\mu}^* = \hat{\mu}(T^*)$, $T^* = (T_1^*, \ldots, T_m^*)^T$, $\hat{\Sigma}^* = \hat{\Sigma}^*(X^* - \hat{\mu}^*), \hat{\Phi}_K^*$ is analogous to $\hat{\Phi}_i K$ while replacing the $T_{ij}$ with $T_j^*$, and similarly for quantities such as $\Phi_K^*, \Sigma^*-1$, and $\Sigma^*$. Note that

$$
\Sigma_K^* - \hat{\Sigma}_K^* = \hat{\Lambda}_K + \hat{\Phi}_K^* \Sigma^*-1 \hat{\Phi}_K^* \hat{\Lambda}_K - \Lambda_K \Phi_K^* \Sigma^*-1 \Phi_K^T \Lambda_K.
$$

(S.61)

where $\|\Lambda_K - \hat{\Lambda}_K\|_{op,2} = O_p(a_n + b_n)$ follows from Proposition 1 in Dai et al. (2018) and $\|\Lambda_K - \hat{\Lambda}_K\|_{op,2} \leq \sqrt{K} \max_{1 \leq k \leq K} |\lambda_k - \hat{\lambda}_k|$. Since $\hat{\lambda}_k \hat{\phi}_k = \int_T \hat{\Gamma}(T^*, t) \hat{\phi}_k(t) dt$ and writing $\hat{\Sigma}^* = \int_T \hat{\Gamma}(T^*, t) \hat{\phi}_k(t) dt - \hat{\Sigma}^* \hat{\Phi}_k^T \hat{\phi}_k^*$, we have that the $(j, l)$ entry of $\hat{\Lambda}_K \hat{\Phi}_K^* \hat{\Sigma}^*-1 \hat{\Phi}_K^T \hat{\Lambda}_K$ is given by

$$
[\hat{\Lambda}_K \hat{\Phi}_K^* \hat{\Sigma}^*-1 \hat{\Phi}_K^T \hat{\Lambda}_K]_{j,l} = (\hat{e}_j^T \hat{\Sigma}^*-1 \phi_j^T W^*) (\hat{e}_l^T + \hat{\Sigma}^* \hat{W}^* \hat{\phi}_l^*),
$$

(S.62)

where $1 \leq j, l \leq K$. Denote by $\hat{\Gamma}(T^*, T^*)$ the matrix whose $(i, j)$ element is $\hat{\Gamma}(T_i^*, T_j^*)$, $1 \leq i, j \leq m^*$, and similarly define $\hat{\Gamma}(T^*, T^* T)$. Also note that $\hat{\Sigma}^* = \hat{\sigma}^2 I_{m^*} + \hat{\Gamma}(T^*, T^* T)$, where $I_{m^*} \in \mathbb{R}^{m^* \times m^*}$ is the identity matrix. From the proof of Theorem 1, Lemma S3 and Lemma S2 along with the condition $m^*(a_n + b_n) = o(1)$ as $n \to \infty$, it follows that
\[ \| \hat{\Gamma}(\mathbf{T}^*, \mathbf{T}^s) - \Gamma(\mathbf{T}^*, \mathbf{T}^s) \|_2 = O_p(m^s(a_n + b_n)), \| \mathbf{W}^* (\hat{\phi}_p^* - \phi_p^*) \|_2 = O_p(m^{s-1/2}(a_n + b_n)), \]
\[ p = j, l, \| \hat{\Gamma}(\mathbf{T}^*, \mathbf{T}^s) \|_{op, 2} = O(m^s), \| \Sigma^* \|_{op, 2} = O(m^s), \| \hat{\Sigma}^* \|_{op, 2} = O_p(m^s), \| \mathbf{W}^* \phi_p^* \|_2 = O_p(m^{s-1/2}), p = j, l, \| \hat{\Sigma}^* - \Sigma^* \|_{op, 2} = O_p(m^s(a_n + b_n)), \| \hat{\Sigma}^* - \Sigma^* \|_{op, 2} = O_p(m^s(a_n + b_n)), \| \mathbf{W}^* \|_2 = O_p(m^{s-1/2}), \| \mathbf{e}_p^* \|_2 = O_p(m^{s-1/2}) \text{ and } \| \mathbf{e}_p^* - \mathbf{e}_p^* \|_2 = O_p(m^{1/2}(a_n + b_n)), p = j, l. \]

These bounds imply
\[ \hat{\phi}_j^T \mathbf{W}^* \hat{\Sigma}^* \mathbf{W}^* \hat{\phi}_l^* - \phi_j^T \mathbf{W}^* \Sigma^* \mathbf{W}^* \phi_l^* = O_p(a_n + b_n), \]
\[ \hat{e}_j^T \hat{\Sigma}^* \hat{e}_l^* - e_j^T \Sigma^* e_l^* = O_p(m^{s-1} + a_n + b_n), \]
\[ \hat{e}_j^T \mathbf{W}^* \hat{\phi}_l^* - e_j^T \mathbf{W}^* \phi_l^* = O_p(a_n + b_n), \]
\[ \hat{\phi}_j^T \mathbf{W}^* \hat{e}_l^* - \phi_j^T \mathbf{W}^* e_l^* = O_p(a_n + b_n), \]

which combined with (S.62) leads to
\[ [\hat{\Lambda}_K \hat{\Phi}_K^s \hat{\Sigma}^s \hat{\Phi}_K^s \hat{\Lambda}_K]_{j,l} - [\Lambda_K \Phi_K^s \Sigma^s \Phi_K^s \Lambda_K]_{j,l} = O_p(m^{s-1} + a_n + b_n). \]

Hence \[ \| \hat{\Lambda}_K \hat{\Phi}_K^s \hat{\Sigma}^s \hat{\Phi}_K^s \hat{\Lambda}_K - \Lambda_K \Phi_K^s \Sigma^s \Phi_K^s \Lambda_K \|_F = O_p(m^{s-1} + a_n + b_n) \] and the result follows from (S.61).

In what follows, for a function \( g_2 : \mathcal{T} \times \mathcal{T} \to \mathbb{R} \), denote by \( \| g_2 \|_\infty := \sup_{s,t \in \mathcal{T}} |g_2(s,t)| \) its supremum norm over \( \mathcal{T}^2 \).

**Lemma S5.** Suppose that (A1)–(A8) in the Appendix hold and let \( K \geq 1 \). Under the sparse design setting of Theorem 6, it holds that
\[ \| \Gamma_{iK} - \hat{\Gamma}_{iK} \|_\infty = O(a_n + b_n), \]
a.s. as \( n \to \infty \).

**Proof of Lemma S5.** First consider a sparse design setting. Note that
\[ \| \Gamma_{iK} - \hat{\Gamma}_{iK} \|_\infty \leq \sum_{1 \leq j, l \leq K} \sup_{s,t \in \mathcal{T}} |[\Sigma_{iK}]_{j,l} \phi_j(s) \phi_l(t) - [\hat{\Sigma}_{iK}]_{j,l} \hat{\phi}_j(s) \hat{\phi}_l(t)| \]
\[ \leq \sum_{1 \leq j, l \leq K} \sup_{s,t \in \mathcal{T}} |\phi_j(s) \phi_l(t) - \hat{\phi}_j(s) \hat{\phi}_l(t)| \| \Sigma_{iK} \|_{1,l} + \| \hat{\Sigma}_{iK} \|_{1,l} \| \phi_j(s) \phi_l(t) \|, \]
\[ (S.64) \]
where by the triangle inequality

\[
\sup_{s,t \in T} |\hat{\phi}_j(s)\hat{\phi}_t(t)| \leq \sup_{s,t \in T} |\hat{\phi}_j(s)\hat{\phi}_t(t) - \hat{\phi}_j(s)\phi_t(t)| + \|\phi_j\|_\infty \|\phi_t\|_\infty,
\]

\[
\sup_{s,t \in T} |\hat{\phi}_j(s)\phi_t(t) - \hat{\phi}_j(s)\phi_t(t)| \leq \|\phi_t\|_\infty \|\phi_j - \hat{\phi}_j\|_\infty + \|\phi_j - \hat{\phi}_j\|_\infty \|\phi_t - \hat{\phi}_t\|_\infty + \|\phi_j\|_\infty \|\phi_t - \hat{\phi}_t\|_\infty.
\]

From Proposition 1 in Dai et al. (2018) we have \(\|\phi_k - \hat{\phi}_k\|_\infty = O(a_n + b_n)\) a.s. as \(n \to \infty\), whence the above bounds lead to

\[
\sup_{s,t \in T} |\hat{\phi}_j(s)\phi_t(t) - \hat{\phi}_j(s)\phi_t(t)| = O(1) \quad \text{a.s.}
\]

(S.66)

as \(n \to \infty\). We first show the rate for \([\Sigma_{iK}]_{jl} - [\hat{\Sigma}_{iK}]_{jl}\) which is obtained if we can control the Frobenius norm of \(\Sigma_{iK} - \hat{\Sigma}_{iK}\). Note

\[
\Sigma_{iK} - \hat{\Sigma}_{iK} = (\Lambda_K - \hat{\Lambda}_K) + \hat{\Lambda}_K \hat{\Phi}_{iK}^T \hat{\Sigma}_i^{-1} \hat{\Phi}_{iK} \hat{\Lambda}_K - \Lambda_K \Phi_{iK}^T \Sigma_i^{-1} \Phi_{iK} \Lambda_K
\]

\[
= (\Lambda_K - \hat{\Lambda}_K) + (\hat{\Lambda}_K \hat{\Phi}_{iK}^T - \Lambda_K \Phi_{iK}^T) \Sigma_i^{-1} \hat{\Phi}_{iK} \hat{\Lambda}_K + \Lambda_K \Phi_{iK}^T (\hat{\Sigma}_i^{-1} \hat{\Phi}_{iK} \hat{\Lambda}_K - \Sigma_i^{-1} \Phi_{iK} \Lambda_K).
\]

(S.67)

Next, denoting \(C_i := (\hat{\Sigma}_i^{-1} \hat{\Phi}_{iK} \hat{\Lambda}_K - \Sigma_i^{-1} \Phi_{iK} \Lambda_K)\), we have

\[
C_i = (\Sigma_i^{-1} - \Sigma_i^{-1}) (\hat{\Phi}_{iK} \hat{\Lambda}_K - \Phi_{iK} \Lambda_K) + \Sigma_i^{-1} (\hat{\Phi}_{iK} \hat{\Lambda}_K - \Phi_{iK} \Lambda_K) + (\hat{\Sigma}_i^{-1} - \Sigma_i^{-1}) \Phi_{iK} \Lambda_K,
\]

(S.68)

where

\[
\hat{\Phi}_{iK} \hat{\Lambda}_K - \Phi_{iK} \Lambda_K = (\hat{\Phi}_{iK} - \Phi_{iK})(\hat{\Lambda}_K - \Lambda_K) + \Phi_{iK} (\hat{\Lambda}_K - \Lambda_K) + (\hat{\Phi}_{iK} - \Phi_{iK}) \Lambda_K.
\]

(S.69)

Note that \(\|\hat{\Phi}_{iK} - \Phi_{iK}\|_F \leq \sqrt{N_0 K} \max_{1 \leq k \leq K} \|\hat{\phi}_k - \phi_k\|_\infty = O(\sqrt{N_0}(a_n + b_n))\) a.s. as \(n \to \infty\).

Next, using perturbation results Bosq (2000), Proposition 1 in Dai et al. (2018) and the Cauchy Schwarz inequality, it follows that \(|\hat{\lambda}_k - \lambda_k| \leq \|\hat{\Gamma} - \Gamma\|_\infty = O(a_n + b_n)\) a.s. as \(n \to \infty\). Thus \(\|\hat{\Lambda}_K - \Lambda_K\|_F \leq \sqrt{K} \max_{1 \leq k \leq K} \|\hat{\lambda}_k - \lambda_k\|_\infty = O(a_n + b_n)\) a.s. as \(n \to \infty\). Furthermore, from the proof of Theorem 2 in Dai et al. (2018) we have \(\|\hat{\Sigma}_i^{-1} - \Sigma_i^{-1}\|_{op,2} = O(N_0(a_n + b_n))\) a.s. which implies \(\|\hat{\Sigma}_i^{-1} - \Sigma_i^{-1}\|_F \leq \sqrt{N_0} \|\hat{\Sigma}_i^{-1} - \Sigma_i^{-1}\|_{op,2} = O(N_0^{3/2}(a_n + b_n))\) a.s. as \(n \to \infty\). Thus, from (S.68) and (S.69), \(\|\Sigma_i^{-1}\|_{op,2} \leq \sigma^{-2}\) and \(\|\hat{\Phi}_{iK}\|_F \leq \sqrt{N_0 K} \max_{1 \leq k \leq K} \|\hat{\phi}_k\|_\infty\), it follows that \(\|\hat{\Phi}_{iK} \hat{\Lambda}_K - \Phi_{iK} \Lambda_K\|_F = O(\sqrt{N_0}(a_n + b_n))\) and \(\|C_i\|_F = O(N_0^2(a_n + b_n))\) a.s. as \(n \to \infty\).
Thus \( \lambda \) which is equivalent to the linear system 
\[
q > K
\]
be positive. Next, from (S.71) and the fact that
\[
\phi
\]
Next, taking inner product with respect to
\[
\Xi
\]
and for an operator
\[
F
\]
and thus \( e \) are positive eigengaps.

Proof of Lemma S6. Note that \( \Sigma_{iK} \) is positive definite since it corresponds to the variance of the conditional Gaussian distribution of \( \xi_{iK} \) given \( X_i \) and \( T_i \). Since \( \Xi_{iK} \) is a non-negative self adjoint operator of rank \( K \), by the spectral theorem we have \( \Xi_{iK} = \sum_{k=1}^{K} \lambda_{ik} e_{ik} \otimes e_{ik} \), where \( (\lambda_{ik}, e_{ik})_{k=1}^{K} \) are the eigenpairs of the operator, so that \( \Xi_{iK}^{1/2} = \sum_{k=1}^{K} \sqrt{\lambda_{ik}} e_{ik} \otimes e_{ik} \). Note that since \( \Xi_{iK} e_{ik} = \lambda_{ik} e_{ik} \), it follows by construction of \( \Xi_{iK} \) that
\[
\Xi_{iK} e_{ik} = \sum_{1 \leq l, p \leq K} [\Sigma_{iK}]_{lp} \phi_l \phi_p^\ast L^2 = \lambda_{ik} e_{ik}.
\] (S.71)
Next, taking inner product with respect to \( \phi_q, q = 1, \ldots, K \), implies
\[
\sum_{1 \leq p \leq K} [\Sigma_{iK}]_{qp} \phi_p^\ast e_{ik} L^2 = \lambda_{ik} \phi_q \phi_p^\ast L^2,
\]
which is equivalent to the linear system
\[
\Sigma_{iK} (\langle \phi_1, e_{ik} \rangle L^2, \ldots, \langle \phi_K, e_{ik} \rangle L^2)^T = \lambda_{ik} (\langle \phi_1, e_{ik} \rangle L^2, \ldots, \langle \phi_K, e_{ik} \rangle L^2)^T.
\]
Thus \( \lambda_{ik} \) is an eigenvalue of \( \Sigma_{iK} \) with eigenvector \( (\langle \phi_1, e_{ik} \rangle L^2, \ldots, \langle \phi_K, e_{ik} \rangle L^2)^T \) so that it must be positive. Next, from (S.71) and the fact that \( \lambda_{ik} > 0 \), it follows that \( \langle \phi_q, e_{ik} \rangle L^2 = 0 \) for any \( q > K \) and thus \( e_{ik} \in \text{span}\{\phi_1, \ldots, \phi_K\} \). Therefore \( e_{ik} = \sum_{j=1}^{K} \langle \phi_j, e_{ik} \rangle L^2 \phi_j \). Suppose that the
eigenvalue $\lambda_{ik}$ of $\Xi_{iK}$ has multiplicity greater than one, so that we can find $1 \leq k' \neq k \leq K$ such that the eigenfunction $e_{ik'}$ has eigenvalue $\lambda_{ik}$. Denoting by $\zeta_l = ((\phi_1, e_{il})_{L^2}, \ldots, (\phi_K, e_{il})_{L^2})^T$, $l = k, k'$, the orthogonality of $e_{ik}$ and $e_{ik'}$ implies that $\zeta_k$ and $\zeta_{k'}$ are orthogonal as elements of $\mathbb{R}^K$. This contradicts the fact that $\Sigma_{iK}$ is positive definite since the arguments above show that $\zeta_k$ and $\zeta_{k'}$ are orthogonal eigenvectors with the same eigenvalue. Finally, if $\lambda_{i1} > \cdots > \lambda_{iK} > 0$ are the corresponding ordered eigenvalues of $\Xi_{iK}$, then $\lambda_{i1}^2 > \cdots > \lambda_{iK}^2 > 0$ are the ordered ones for $\Xi_{iK}^2$. Thus, for $1 \leq l < p \leq K$, we obtain

$$\lambda_{il}^2 - \lambda_{ip}^2 = (\lambda_{il} - \lambda_{ip})(\lambda_{il} + \lambda_{ip}) > 2\lambda_{iK}(\lambda_{il} - \lambda_{ip}) > 0,$$

and the result follows.

**Lemma S7.** Suppose that $A$ is a strictly positive operator on $L^2$, of finite rank $K$ and with positive eigengaps. Let $A_n$ be a sequence of positive operators on $L^2$ of finite rank $K$ and such that $\|A_n - A\|_{op} = O(\alpha_n)$ a.s. for some sequence $\alpha_n \to 0$ as $n \to \infty$. Then

$$\|A_n^{\frac{1}{2}} - A^{\frac{1}{2}}\|_{op} = O(\alpha_n),$$

a.s. as $n \to \infty$.

**Proof of Lemma S7.** Let $x \in L^2$ with $\|x\|_{L^2} = 1$ and $(\lambda_{Ak}, e_{Ak})_{k=1}^K$ be the eigenpairs of $A$ and $(\lambda_{nk}, e_{nk})_{k=1}^K$ the eigenpairs of $A_n$. Define the projections $x_{nk} = \langle x, e_{nk} \rangle_{L^2}$ and $x_{Ak} = \langle x, e_{Ak} \rangle_{L^2}$. Then

$$\|(A_n^{\frac{1}{2}} - A^{\frac{1}{2}})x\|_{L^2} = \left\| \sum_{k=1}^K \sqrt{\lambda_{nk}} e_{nk} x_{nk} - \sqrt{\lambda_{Ak}} e_{Ak} x_{Ak} \right\|_{L^2}$$

$$\leq \sum_{k=1}^K |\sqrt{\lambda_{nk}} - \sqrt{\lambda_{Ak}}| \|x_{nk}\| + \sum_{k=1}^K \sqrt{\lambda_{Ak}} \|e_{nk} x_{nk} - e_{Ak} x_{Ak}\|_{L^2}$$

$$\leq \sum_{k=1}^K \frac{\|A_n - A\|_{op}}{\sqrt{\lambda_{Ak}}} \|x_{nk}\| + \sum_{k=1}^K \sqrt{\lambda_{Ak}} \|e_{nk} x_{nk} - e_{Ak} x_{Ak}\|_{L^2},$$

where $|x_{nk}| \leq |x_{nk} - x_{Ak}| + |x_{Ak}| \leq \|e_{nk} - e_{Ak}\|_{L^2} + 1$ and

$$\|e_{nk} x_{nk} - e_{Ak} x_{Ak}\|_{L^2} \leq \|e_{nk} - e_{Ak}\|_{L^2}(\|e_{nk} - e_{Ak}\|_{L^2} + 1) + \|e_{nk} - e_{Ak}\|_{L^2}. $$

Next, since the eigenvalues of the operator $A$ have multiplicity one and $\|A - A_n\|_{op} = O(\alpha_n)$ a.s. as $n \to \infty$, from Bosq (2000) we have $\|e_{nk} - e_{Ak}\|_{L^2} = O(\alpha_n)$ a.s. as $n \to \infty$, where the eigenfunction $e_{nk}$ is chosen such that $\langle e_{nk}, e_{Ak} \rangle_{L^2} \geq 0$. The result follows.
Proof of Theorem 5. First note that $\Xi_{iK}$ and $\hat{\Xi}_{iK}$ are positive, self adjoint and finite rank operators. The same properties are then shared by the operator $\hat{\Xi}_{iK}^{\frac{1}{2}}\Xi_{iK}\hat{\Xi}_{iK}^{\frac{1}{2}}$. From the closed form expression of the $L^2$ Wasserstein distance between Gaussian measures on a finite dimensional Hilbert space (12), we have

$$W_2^2(\hat{G}_{iK}, G_{iK}) = \|\hat{\mu}_{iK} - \hat{\mu}_{iK}\|^2 + \text{tr}(\hat{\Xi}_{iK} + \Xi_{iK} - 2(\hat{\Xi}_{iK}\Xi_{iK}\hat{\Xi}_{iK})^{\frac{1}{2}}).$$ (S.72)

Similar arguments as in the proof of Theorem 2 in Dai et al. (2018) show that $|\hat{\xi}_{ik} - \xi_{ik}| = O(a_n + b_n)\|X_i - \mu_i\|_2 = O(a_n + b_n)O_p(1) = O_p(a_n + b_n)$ and $\|\hat{\phi}_k - \phi_k\|_{L^2} \leq \|\hat{\phi}_k - \phi_k\|_{\infty} = O(a_n + b_n)$ a.s. as $n \to \infty$, $k = 1, \ldots, K$. By the orthonormality of the $\hat{\phi}_k$, we then have

$$\|\hat{\mu}_{iK} - \hat{\mu}_{iK}\|_{L^2} = \|\hat{\xi}_{iK}^T\hat{\Phi}_K - \hat{\xi}_{iK}\Phi_K\|_{L^2} = \|\hat{\xi}_{iK} - \hat{\xi}_{iK}\|_2 \sum_{k=1}^K (|\hat{\xi}_{iK} - \hat{\xi}_{ik}| + |\hat{\xi}_{ik}|) \|\hat{\phi}_k - \phi_k\|_{L^2} = O_p(a_n + b_n),$$

where we use the fact that $\hat{\xi}_{ik} = O_p(1)$, $k = 1, \ldots, K$. This shows that

$$\|\hat{\mu}_{iK} - \hat{\mu}_{iK}\|_{L^2} = O_p(a_n + b_n).$$

Next, writing $\Delta_{iK}^{\frac{1}{2}} = \hat{\Xi}_{iK}^{\frac{1}{2}} - \Xi_{iK}^{\frac{1}{2}}$, we have

$$\|\Xi_{iK}^{\frac{1}{2}}\Xi_{iK}\hat{\Xi}_{iK}^{\frac{1}{2}} - \Xi_{iK}^{\frac{1}{2}}\|_{op} = \|\Delta_{iK}^{\frac{1}{2}}\Xi_{iK}\Delta_{iK}^{\frac{1}{2}} + \Delta_{iK}^{\frac{1}{2}}\Xi_{iK}\Xi_{iK}^{\frac{1}{2}} + \Xi_{iK}\Xi_{iK}^{\frac{1}{2}}\|_{op} \leq \|\Delta_{iK}^{\frac{1}{2}}\|_{op}\|\Xi_{iK}\|_{op} + \|\Delta_{iK}^{\frac{1}{2}}\|_{op}\|\Xi_{iK}\|_{op} + \|\Xi_{iK}\|_{op}\|\Xi_{iK}\|_{op}\|\Delta_{iK}^{\frac{1}{2}}\|_{op} \leq O(\|\Delta_{iK}^{\frac{1}{2}}\|_{op}^2 + \|\Delta_{iK}^{\frac{1}{2}}\|_{op}),$$ (S.73)

where the last inequality follows since $\|\Xi_{iK}^{\frac{1}{2}}\|_{op} \leq K\sqrt{\|\Xi_{iK}\|_{op}}$ and $\|\Xi_{iK}\|_{op} \leq K^2\|X_{iK}\|_F = O(1)$, which is due to $\max_{k=1,\ldots,K} \|\phi_k\|_{\infty} < \infty$ and $\|\Sigma^{-1}\|_{op,2} \leq \sigma^{-2}$. Applying the Cauchy-Schwarz inequality and (S.63) in Lemma S5,

$$\|\Xi_{iK} - \hat{\Xi}_{iK}\|_{op} = \max_{\|x\|_{L^2} = 1} \|(\Xi_{iK} - \hat{\Xi}_{iK})x\|_{L^2} \leq \|\Gamma_{iK} - \hat{\Gamma}_{iK}\|_{\infty} = O(a_n + b_n) \text{ a.s.,}$$

as $n \to \infty$. Thus, from Lemma S6 and Lemma S7, it follows that $\Delta_{iK}^{\frac{1}{2}} = O(a_n + b_n)$ a.s. as $n \to \infty$, which combined with (S.73) leads to

$$\|\Xi_{iK}^{\frac{1}{2}}\Xi_{iK}\hat{\Xi}_{iK}^{\frac{1}{2}} - \Xi_{iK}^{\frac{1}{2}}\|_{op} = O(a_n + b_n) \text{ a.s.}$$
as \( n \to \infty \). Then, by again using Lemma S6 and Lemma S7, and since \( \Xi_{iK} - \left(\hat{\Xi}_{iK}^\frac{1}{2}\Xi_{iK}\hat{\Xi}_{iK}^\frac{1}{2}\right) \) is of rank at most 2\( K \), it follows that

\[
\text{tr}(\Xi_{iK} - \left(\hat{\Xi}_{iK}^\frac{1}{2}\Xi_{iK}\hat{\Xi}_{iK}^\frac{1}{2}\right)) \leq 2K\|\Xi_{iK}^\frac{1}{2} - \left(\hat{\Xi}_{iK}^\frac{1}{2}\Xi_{iK}\hat{\Xi}_{iK}^\frac{1}{2}\right)\|_{op} = O(a_n + b_n) \quad \text{a.s.,}
\]

as \( n \to \infty \). This implies

\[
\text{tr}(\tilde{\Xi}_{iK} + \Xi_{iK} - 2(\hat{\Xi}_{iK}^\frac{1}{2}\Xi_{iK}\hat{\Xi}_{iK}^\frac{1}{2})) = \text{tr}(\tilde{\Xi}_{iK} - \Xi_{iK} + 2(\Xi_{iK} - \left(\hat{\Xi}_{iK}^\frac{1}{2}\Xi_{iK}\hat{\Xi}_{iK}^\frac{1}{2}\right))) \\
\leq 2K\|\tilde{\Xi}_{iK} - \Xi_{iK}\|_{op} + 2\text{tr}(\Xi_{iK} - \left(\hat{\Xi}_{iK}^\frac{1}{2}\Xi_{iK}\hat{\Xi}_{iK}^\frac{1}{2}\right)) \\
= O(a_n + b_n) \quad \text{a.s.,}
\]

as \( n \to \infty \), which completes the proof. \( \square \)

**Lemma S8.** Suppose that the eigengaps of \( \Sigma_{iK}^\frac{1}{2}\Sigma_{jK}\Sigma_{iK}^\frac{1}{2} \) are positive. Then the eigenvalues of the operator \( \Xi_{iK}^\frac{1}{2}\Xi_{jK}\Xi_{iK}^\frac{1}{2} \) are exactly those of \( \Sigma_{iK}^\frac{1}{2}\Sigma_{jK}\Sigma_{iK}^\frac{1}{2} \).

**Proof of Lemma S8.** Let \( x \in L^2 \) and define the vector of truncated Fourier coefficients \( \tilde{x}_K = (\langle x, \phi_1 \rangle_{L^2}, \ldots, \langle x, \phi_K \rangle_{L^2})^T \). Recall that \( \Phi_{iK}(t) = (\phi_1(t), \ldots, \phi_K(t))^T \). Then

\[
\Xi_{iK}(x) = \sum_{1 \leq l, p \leq K} [\Sigma_{iK}]_{lp}\phi_l(x, \phi_p)_{L^2} = \Phi_{iK}^T \Sigma_{iK}^\frac{1}{2}\Phi_{iK} \tilde{x}_K,
\]

and similarly for the operator \( \Xi_{jK} \). Let \( \Sigma_{iK} = P_i\Lambda_i P_i^T \) be the spectral decomposition. We first establish \( \Xi_{iK}^\frac{1}{2}(x) = \Phi_{iK}^T \Sigma_{iK}^\frac{1}{2}P_i^T \tilde{x}_K = \Phi_{iK}^T P_i\Lambda_i^\frac{1}{2}P_i^T \tilde{x}_K \). This follows since the square root operator of the self adjoint and positive operator \( \Xi_{iK} \) is unique and

\[
\Xi_{iK}^\frac{1}{2}\Xi_{iK}^\frac{1}{2}(x) = \Xi_{iK}^\frac{1}{2}(\Phi_{iK}^T P_i\Lambda_i^\frac{1}{2}P_i^T \tilde{x}_K) = \Phi_{iK}^T P_i\Lambda_i^\frac{1}{2}P_i^T \tilde{x}_K = \Xi_{iK}(x),
\]

where the second equality follows since \( z_q := (\Phi_{iK}^T P_i\Lambda_i^\frac{1}{2}P_i^T \tilde{x}_K, \phi_q)_{L^2} \) corresponds to the qth element of the vector \( P_i\Lambda_i^\frac{1}{2}P_i^T \tilde{x}_K, q = 1, \ldots, K \), so that \( (z_1, \ldots, z_K)^T = P_i\Lambda_i^\frac{1}{2}P_i^T \tilde{x}_K \) and then the vector of truncated Fourier coefficients of \( \Phi_{iK}^T P_i\Lambda_i^\frac{1}{2}P_i^T \tilde{x}_K \) is exactly \( P_i\Lambda_i^\frac{1}{2}P_i^T \tilde{x}_K \). Then

\[
\Xi_{iK}^\frac{1}{2}\Xi_{jK}^\frac{1}{2}(x) = \Xi_{iK}^\frac{1}{2}(\Phi_{iK}^T \Sigma_{iK}^\frac{1}{2}\Phi_{iK} \tilde{x}_K) = \Xi_{iK}^\frac{1}{2}(\Phi_{iK}^T \Sigma_{jK}\Sigma_{iK}^\frac{1}{2}\Phi_{iK} \tilde{x}_K) = \Phi_{iK}^T \Sigma_{iK}^\frac{1}{2}\Sigma_{jK}\Sigma_{iK}^\frac{1}{2}\Phi_{iK} \tilde{x}_K.
\]

Similar arguments as those in the proof of Lemma S6 complete the proof. \( \square \)
Proof of Theorem 6. Let $K \geq 1$ be fixed. From the closed form expression of the $L^2$-Wasserstein distance between Gaussian measures on a finite-dimensional Hilbert space (12), we have

$$|W^2_2(\hat{G}_{iK}, \hat{G}_{jK}) - W^2_2(\hat{G}_{iK}, G_{jK})| \leq |\Delta_\mu| + |\Delta_\Sigma|,$$  \hfill (S.74)

where $\Delta_\mu := \|\hat{\mu}_i - \mu_j\|^2_2 - \|\hat{\mu}_i - \mu_j\|^2_2$ and $\Delta_\Sigma := \text{tr}(\hat{\Sigma}_{iK} + \hat{\Sigma}_{jK} - 2(\hat{\Sigma}_{iK} \hat{\Sigma}_{jK} \hat{\Sigma}_{iK})^{1/2}) - \text{tr}(\Sigma_{iK} + \Sigma_{jK} - 2(\Sigma_{iK} \Sigma_{jK} \Sigma_{iK})^{1/2})$. Thus

$$\Delta_\Sigma = \text{tr}(\hat{\Sigma}_{iK} - \Sigma_{iK}) + \text{tr}(\hat{\Sigma}_{jK} - \Sigma_{jK}) + 2\text{tr}(\Sigma_{iK} \Sigma_{jK} \Sigma_{iK})^{1/2} - (\Sigma_{iK} \Sigma_{jK} \Sigma_{iK})^{1/2})$$  \hfill (S.75)

Defining $\Delta_{iKp} := \Sigma_{iK} - \hat{\Sigma}_{iK}$ and similarly for $\Delta_{jKp}, p \in \{1/2, 1\}$, we have

$$(\Sigma_{iK} \Sigma_{jK} \Sigma_{iK})^{1/2} - (\Sigma_{iK} \Sigma_{jK} \Sigma_{iK})^{1/2} = \Delta_{iK1} \Sigma_{iK} \Sigma_{jK} \Sigma_{iK} + \Delta_{iK1} A + \Sigma_{iK} A,$$

where $A := \Delta_{iK1} \Sigma_{iK} \Sigma_{jK} \Sigma_{iK} + \Sigma_{iK} \Sigma_{jK} \Delta_{iK1}$. By the arguments outlined in the proof of Theorem 5, we have $\|\Delta_{qKp}\|_{op} = O(a_n + b_n)$ a.s. as $n \to \infty$, $\|\Sigma_{iK}^{1/2}\|_{op} = O(1)$ and $\|\Sigma_{qK}\|_{op} = O(1)$, where $q = i, j$ and $p = 1, 1/2$. This shows that

$$\|((\Sigma_{iK}^{1/2} \Sigma_{jK} \Sigma_{iK})^{1/2} - (\Sigma_{iK}^{1/2} \Sigma_{jK} \Sigma_{iK})^{1/2})\|_{op} = O(a_n + b_n) \text{ a.s.,}$$  \hfill (S.76)

as $n \to \infty$. Next, since $\|\hat{\Sigma}_{qK}\|_{op} \leq \|\Sigma_{qK} - \hat{\Sigma}_{qK}\|_{op} + \|\Sigma_{qK}\|_{op} = O(1)$ a.s. as $n \to \infty$, $q = i, j$, and using that $\Sigma_{iK}^{1/2} \Sigma_{jK} \Sigma_{iK}^{1/2}$ is a strictly positive and finite rank operator with positive eigengaps, which follows from Lemma S8, similar arguments as in the proof of Theorem 5 along with (S.76) imply $\text{tr}(\Sigma_{iK}^{1/2} \Sigma_{jK} \Sigma_{iK}^{1/2} - (\Sigma_{iK}^{1/2} \Sigma_{jK} \Sigma_{iK}^{1/2})^{1/2}) = O(a_n + b_n)$ a.s. as $n \to \infty$. Then (S.75) leads to

$$\Delta_\Sigma = O(a_n + b_n) \text{ a.s.,}$$  \hfill (S.77)

as $n \to \infty$. For the mean part, similar arguments as in the proof of Theorem 5 show that

$$\|\hat{\mu}_{qK} - \tilde{\mu}_{qK}\|_{L^2} = O_p(a_n + b_n), q = i, j, \text{ as } n \to \infty,$$

implying

$$\Delta_\mu = \|\hat{\mu}_i - \mu_j\|^2_2 - \|\hat{\mu}_i - \mu_j\|^2_2$$

$$\leq \|\hat{\mu}_i - \mu_j\|^2_2 + \|\hat{\mu}_j - \mu_j\|^2_2 + 2\|\hat{\mu}_i - \mu_j\|_{L^2} \|\hat{\mu}_j - \mu_j\|_{L^2}$$

$$+ 2(\|\hat{\mu}_i - \mu_j\|^2_2 + \|\hat{\mu}_j - \mu_j\|^2_2)\|\hat{\mu}_i - \mu_j\|_{L^2}$$

$$= O_p(a_n + b_n),$$

where for the last equality one uses the fact that $\|\hat{\mu}_i - \mu_j\|_{L^2} = O_p(1)$. Then

$$\|\hat{\mu}_i - \mu_j\|^2_2 \leq \sum_{k=1}^K |\xi_{ik} - \tilde{\xi}_{jk}| \leq \sum_{k=1}^K (|\xi_{ik}| + |\tilde{\xi}_{jk}|) = O_p(1),$$

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Δμ = O_p(\text{coefficients}). \quad (S.78)

The result follows by combining (S.74), (S.77) and (S.78).

**Lemma S9.** Under the assumptions of Theorem 7, it holds that

\[
\sum_{j=1}^{K} \frac{\lambda_j}{\delta_j^2} = O \left( \sum_{j=1}^{K} \frac{1}{\lambda_j \delta_j^2} \right) \quad (S.79)
\]

as \( n \to \infty \).

**Proof of Lemma S9.** Since \( \lambda_j \to 0 \) as \( j \to \infty \), there exists \( J^* \geq 1 \) such that \( \lambda_j \geq 1 \) for \( j \leq J^* \) and \( \lambda_j < 1 \) whenever \( j > J^* \). Note that

\[
\sum_{j=1}^{K} \frac{\lambda_j}{\delta_j^2} \leq \sum_{j=1}^{J^*} \frac{\lambda_j}{\delta_j^2} + \sum_{j=J^*+1}^{K} \left( \lambda_j - \frac{1}{\lambda_j} \right) \frac{1}{\delta_j^2}
\]

whence it suffices to show that the third term in (S.80) diverges to \(-\infty\) as \( n \to \infty \). For this,

\[
\sum_{j=J^*+1}^{K} \left( \lambda_j - \frac{1}{\lambda_j} \right) \frac{1}{\delta_j^2} \leq \lambda_{J^*+1}^2 - 1 \sum_{j=J^*+1}^{K} \frac{1}{\lambda_j \delta_j^2} - \sum_{j=J^*+1}^{K} \frac{1}{\lambda_j \delta_j^2} = \sum_{j=J^*+1}^{K} \frac{1}{\lambda_j \delta_j^2} (\lambda_{J^*+1}^2 - 1).
\]

The result follows from the fact that \( \lambda_{J^*+1}^2 - 1 < 0 \) and since \( \sum_{j=1}^{K} \lambda_j^{-1/2} \delta_j^{-1} \to \infty \) as \( n \to \infty \) implies \( \sum_{j=J^*+1}^{K} \lambda_j^{-1} \delta_j^{-2} \to \infty \) as \( n \to \infty \).

**Proof of Theorem 7.** Recall that the eigenpairs of (the integral operator associated with) \( \hat{\Gamma} \) are \((\hat{\lambda}_k, \hat{\phi}_k)\), and those of \( \Gamma \) are \((\lambda_k, \phi_k), k \geq 1\). Let \( \epsilon_1, \ldots, \epsilon_n \) and \( Z_k, k \geq 1\), be a sequence of independent standard normal random variables, which are independent of all other random quantities; also let \( \epsilon = [\epsilon_1, \ldots, \epsilon_n]^T \). Let

\[
Y^*(t) = \hat{\mu}(t) + \sum_{k=1}^{\infty} \sqrt{\lambda_k} Z_k \hat{\phi}_k(t), \quad Y^*_t = Y^*(T_t) + \sigma \epsilon, \quad Y^{*\epsilon}(t) = Y^*(t) - \hat{\mu}(t), \quad (S.81)
\]

\[
Z^*(t) = \mu(t) + \sum_{k=1}^{\infty} \sqrt{\lambda_k} Z_k \phi_k(t), \quad Z^*_t = Z^*(T_t) + \sigma \epsilon, \quad Z^{*\epsilon}(t) = Z^*(t) - \mu(t), \quad (S.82)
\]

where the convergence of the infinite sums can be interpreted in terms of convergence of random elements in \( L^2(\mathcal{T}) \) or in mean square pointwise at each time \( t \in \mathcal{T} \). We take the convention
that all expected values are taken conditional on the observations \((X_i, T_i), i = 1, \ldots, n\). Then, given \((X_i, T_i)_{i=1}^n\), it holds that \(\hat{\mu}_i + Y^* - E(Y^* | Y_i^*)\) share the same distribution as \(\hat{G}_i\), and so do \(\hat{\mu}_i + Z^* - E(Z^* | Z_i^*)\) and \(\hat{G}_i\). By (11) and defining the auxiliary quantities \(\Delta_{\mu_i} = \hat{\mu}_i - \mu_i\), \(\Delta_1^* = Y^{*c} - Z^{*c}\) and \(\Delta_2^* = E(Y^{*c} | Y_i^*) - E(Z^{*c} | Z_i^*)\), it follows that

\[
W_2^2(\hat{G}_i, G_i) \leq E \| (\hat{\mu}_i + Y^* - E(Y^* | Y_i^*)) - (\hat{\mu}_i + Z^* - E(Z^* | Z_i^*)) \|_2^2 = E \| \Delta_{\mu_i} + \Delta_1^* - \Delta_2^* \|_2^2
\]

\[
\leq E \| \Delta_{\mu_i} \|_2^2 + E \| \Delta_1^* \|_2^2 + E \| \Delta_2^* \|_2^2 + 2E(\| \Delta_{\mu_i} \|_2 \| \Delta_1^* \|_2 + \| \Delta_{\mu_i} \|_2 \| \Delta_2^* \|_2 + \| \Delta_1^* \|_2 \| \Delta_2^* \|_2)
\]

(S.83)

Next

\[
E \| \Delta_1^* \|_2^2 = E \left| \sum_{k=1}^{\infty} Z_k (\sqrt{\lambda_k} \hat{\phi}_k - \lambda_k \phi_k) \right|^2_{L^2}
\]

\[
= \sum_{k=1}^{\infty} \left\| \sqrt{\lambda_k} \hat{\phi}_k - \lambda_k \phi_k \right\|^2_{L^2}
\]

\[
\leq \sum_{k=1}^{K} \left\| \sqrt{\lambda_k} \hat{\phi}_k - \lambda_k \phi_k \right\|^2_{L^2} + \sum_{k=K+1}^{\infty} \left( \lambda_k + \lambda_k - 2 \sqrt{\lambda_k \lambda_k} \langle \hat{\phi}_k, \phi_k \rangle \right)
\]

\[
\leq \sum_{k=1}^{K} \left\| \sqrt{\lambda_k} \hat{\phi}_k - \lambda_k \phi_k \right\|^2_{L^2} + \sum_{k=K+1}^{\infty} \left( \lambda_k + \lambda_k \right),
\]

where we use the fact that \(\hat{\phi}_k\) is chosen such that \(\langle \hat{\phi}_k, \phi_k \rangle \geq 0\). Also note that by the Cauchy-Schwarz inequality and from Proposition 1 in Dai et al. (2018), we have \(\| \hat{\Xi} - \Xi \|_{op} = O(a_n + b_n)\) a.s.. Here \(K = K(n)\) is chosen according to

\[
K = \sup \left\{ J \geq 1 : \sum_{k=1}^{J} \frac{1}{\lambda_k^{1/2} \delta_k} \leq c_n^{-1/2} \right\},
\]

(S.85)

which implies \(\sum_{k=1}^{K} \| \hat{\Xi} - \Xi \|_{op} \lambda_k^{-1/2} \delta_k^{-1} = O(\sqrt{c_n})\) and \(K \to \infty\) as \(n \to \infty\). Using auxiliary results (Bosq 2000) we have

\[
\left| \sqrt{\lambda_k} - \sqrt{\lambda_k} \right| = \left| \frac{\hat{\lambda}_k - \lambda_k}{\sqrt{\hat{\lambda}_k} + \sqrt{\lambda_k}} \right| \leq \left| \frac{\hat{\lambda}_k - \lambda_k}{\sqrt{\lambda_k}} \right| \leq \frac{\| \hat{\Xi} - \Xi \|_{op}}{\sqrt{\lambda_k}},
\]

\[
\| \hat{\phi}_k - \phi_k \|_{L^2} \leq \frac{2\sqrt{2}}{\delta_k} \| \hat{\Xi} - \Xi \|_{op},
\]

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and thus
\[ \left\| \sqrt{\hat{\lambda}_k} \hat{\phi}_k - \sqrt{\lambda_k} \phi_k \right\|_{L^2}^2 \leq \left( \left\| \sqrt{\hat{\lambda}_k} - \sqrt{\lambda_k} \right\|_{L^2} + \left\| \sqrt{\lambda_k} (\hat{\phi}_k - \phi_k) \right\|_{L^2} + \left\| \sqrt{\hat{\lambda}_k} (\hat{\phi}_k - \phi_k) \right\|_{L^2} \right)^2 \]
\[ \leq 2 \sqrt{2} \frac{\left\| \hat{\Xi} - \Xi \right\|_{op}^2}{\sqrt{\lambda_k} \delta_k} + 2 \sqrt{2} \frac{\sqrt{\lambda_k}}{\delta_k} \left\| \hat{\Xi} - \Xi \right\|_{op} + \left\| \hat{\Xi} - \Xi \right\|_{op} \]
\[ = O \left( \frac{\left\| \hat{\Xi} - \Xi \right\|_{op}^2}{\sqrt{\lambda_k} \delta_k} + \frac{\sqrt{\lambda_k}}{\delta_k} \left\| \hat{\Xi} - \Xi \right\|_{op}^2 \right), \quad (S.86) \]

where the last equality follows since \( \delta_k \leq \lambda_k \). This implies
\[ \left\| \sqrt{\hat{\lambda}_k} \hat{\phi}_k - \sqrt{\lambda_k} \phi_k \right\|_{L^2}^2 = O \left( \frac{\left\| \hat{\Xi} - \Xi \right\|_{op}^2}{\lambda_k \delta_k^2} + \frac{\lambda_k}{\delta_k^2} \left\| \hat{\Xi} - \Xi \right\|_{op}^2 \right) \]
\[ = O \left( \frac{\left\| \hat{\Xi} - \Xi \right\|_{op}^2}{\lambda_k \delta_k^2} + \frac{\lambda_k}{\delta_k^2} \left\| \hat{\Xi} - \Xi \right\|_{op}^2 \right), \quad (S.86) \]
a.s., where the second equality is a consequence of \( \delta_k \leq \lambda_k \) and \( \left\| \hat{\Xi} - \Xi \right\|_{op} = o(1) \) a.s. as \( n \to \infty \).

From (S.85), (S.86) and (S.79) in Lemma S9 it follows that
\[ \sum_{k=1}^K \left\| \sqrt{\hat{\lambda}_k} \hat{\phi}_k - \sqrt{\lambda_k} \phi_k \right\|_{L^2}^2 = O \left( \left\| \hat{\Xi} - \Xi \right\|_{op}^2 \sum_{k=1}^K \frac{1}{\lambda_k \delta_k^2} \right) \]
\[ = O \left( \left\| \hat{\Xi} - \Xi \right\|_{op}^2 \left( \sum_{k=1}^K \frac{1}{\lambda_k \delta_k} \right)^2 \right) = O(c_n) = o(1) \quad \text{a.s.} \quad (S.87) \]

and we have also
\[ \sum_{k=K+1}^\infty (\hat{\lambda}_k + \lambda_k) = o(1) \quad \text{a.s.,} \quad (S.88) \]
since \( \sum_{k=1}^\infty \lambda_k < \infty \) and \( \sum_{k=K+1}^\infty \hat{\lambda}_k = o(1) \) a.s., as in the proof of Theorem 5. Combine (S.84), (S.87), and (S.88) to obtain
\[ E \left\| \Delta_i^* \right\|_{L^2}^2 = E \left\| Y^{*c} - Z^{*c} \right\|_{L^2}^2 = o(1) \quad \text{a.s.} \quad (S.89) \]

Writing
\[ \hat{A}_1 = \hat{\Gamma}(\cdot, T_i), \quad \hat{A}_2 = \hat{\Sigma}_i^{-1}, \quad \hat{A}_3 = Y_i^* - \hat{\mu}_i, \quad (S.90) \]
\[ A_1 = \Gamma(\cdot, T_i), \quad A_2 = \Sigma_i^{-1}, \quad A_3 = Z_i^* - \mu_i, \quad (S.91) \]
by (B2) and similar arguments to the proof of Theorem 2 in Dai et al. (2018),

\[ \|\hat{A}_1 - A_1\|_{L^2} = o(1) \quad \text{a.s.,} \quad \tag{S.92} \]
\[ \|\hat{A}_2 - A_2\|_{\text{op},2} = o(1) \quad \text{a.s..} \quad \tag{S.93} \]

Similar to the derivation of (S.89), for any \( t \in \mathcal{T} \),

\[
E[Y_\ast^c(t) - Z_\ast^c(t)]^2 = E[\sum_{k=1}^{\infty} Z_k (\hat{\lambda}_k \hat{\phi}_k(t) - \lambda_k \phi_k(t))]^2 \\
= \sum_{k=1}^{\infty} (\sqrt{\hat{\lambda}_k \hat{\phi}_k(t) - \sqrt{\lambda_k \phi_k(t)})^2 \\
= K \sum_{k=1}^{\infty} (\sqrt{\hat{\lambda}_k \hat{\phi}_k(t) - \sqrt{\lambda_k \phi_k(t)})^2 + \sum_{k=K+1}^{\infty} (\hat{\lambda}_k \hat{\phi}_k^2(t) + \lambda_k \phi_k^2(t)) \\
- 2 \sum_{k=K+1}^{\infty} \sqrt{\hat{\lambda}_k \lambda_k \hat{\phi}_k(t) \phi_k(t)}. \tag{S.94} \]

Note that \( \sum_{k=K+1}^{\infty} \hat{\lambda}_k \hat{\phi}_k^2(t) = o(1) \) a.s. as \( n \to \infty \). This is because for any fixed \( K_0 \geq 1 \), we have \( K = K(n) > K_0 \) for large enough \( n \) and

\[
\sum_{k=K+1}^{\infty} \hat{\lambda}_k \hat{\phi}_k^2(t) = \sum_{k \geq 1} (\hat{\lambda}_k \hat{\phi}_k^2(t) - \lambda_k \phi_k^2(t)) + \sum_{k=K_0}^{\infty} (\lambda_k \phi_k^2(t) - \hat{\lambda}_k \hat{\phi}_k^2(t)) + \sum_{k \geq K_0+1} \lambda_k \phi_k^2(t) \\
= \hat{\Gamma}(t, t) - \Gamma(t, t) + \sum_{k \leq K_0} (\lambda_k \phi_k^2(t) - \hat{\lambda}_k \hat{\phi}_k^2(t)) + \sum_{k \geq K_0+1} \lambda_k \phi_k^2(t),
\]

where \( |\hat{\Gamma}(t, t) - \Gamma(t, t)| \leq \sup_{s \in \mathcal{T}} |\hat{\Gamma}(s, s) - \Gamma(s, s)| = o(1) \) a.s. and it is easy to show that \( \sum_{k \leq K_0} (\lambda_k \phi_k^2(t) - \hat{\lambda}_k \hat{\phi}_k^2(t)) = o(1) \) a.s. using that \( |\hat{\phi}(t) - \phi(t)| = o(1) \) and \( |\hat{\lambda}_k - \lambda_k| = o(1) \) a.s. as \( n \to \infty \), which, along with the uniform convergence of \( \hat{\Gamma} \) towards \( \Gamma \) over \( \mathcal{T} \), follows from Proposition 1 in Dai et al. (2018). This shows that \( \limsup_{n \to \infty} \sum_{k=K+1}^{\infty} \hat{\lambda}_k \hat{\phi}_k^2(t) \leq \sum_{k \geq K_0+1} \lambda_k \phi_k^2(t) \) for any \( K_0 > 0 \), and then taking \( K_0 \to \infty \) and using the fact that \( \sum_{k \geq 1} \lambda_k \phi_k^2(t) = \Gamma(t, t) < \infty \) leads to \( \sum_{k=K+1}^{\infty} \hat{\lambda}_k \hat{\phi}_k^2(t) = o(1) \) a.s. as \( n \to \infty \).

Note that by (S.85), the Cauchy-Schwarz inequality and Mercer’s theorem,

\[
\sum_{k=K+1}^{\infty} \sqrt{\hat{\lambda}_k \lambda_k \hat{\phi}_k(t) \phi_k(t)} \leq \left( \sum_{k=K+1}^{\infty} \hat{\lambda}_k \hat{\phi}_k^2(t) \right)^{1/2} \left( \sum_{k=K+1}^{\infty} \lambda_k \phi_k^2(t) \right)^{1/2} = o(1), \tag{S.95} \]

as \( n \to \infty \). Next, since \( \sum_{k=1}^{K} \|\hat{\Xi} - \Xi\|_{\text{op}, \lambda_k^{-1/2} \delta_k^{-1}} = o(1) \) a.s. and \( \lambda_K = o(1) \) as \( n \to \infty \), for large enough \( n \) it holds that \( \|\hat{\Xi} - \Xi\|_{\text{op}, \lambda_K^{-1/2} \delta_K^{-1}} \leq \sum_{k=1}^{K} \|\Delta\|_{\text{op}, \lambda_k^{-1/2} \delta_k^{-1}} \leq 1/2 \) and \( \lambda_K < 1 \), so
that $\|\hat{\Xi} - \Xi\|_{op} \leq \lambda_{K}^{1/2}\delta_{k} \leq \delta_{k}/2 \leq \lambda_{K}/2$ a.s. Thus, for large enough $n$ and $k \leq K$ we have

$|\hat{\lambda}_{k} - \lambda_{k}| \leq \|\hat{\Xi} - \Xi\|_{op} \leq \lambda_{k}/2$ a.s., leading to

$$
\left| \frac{1}{\sqrt{\lambda_{k}}} - \frac{1}{\sqrt{\hat{\lambda}_{k}}} \right| = \left| \frac{\hat{\lambda}_{k} - \lambda_{k}}{\sqrt{\lambda_{k}\hat{\lambda}_{k}(\sqrt{\lambda_{k}} + \sqrt{\hat{\lambda}_{k}})}} \right| \leq \frac{\sqrt{2}\|\hat{\Xi} - \Xi\|_{op}}{\lambda_{k}^{3/2}}.
$$

(S.96)

Similar to the proof of Theorem 2 in Yao et al. (2005a) we have

$$
|\hat{\lambda}_{k}\hat{\phi}_{k}(t) - \lambda_{k}\phi_{k}(t)| \leq O \left( c_{n} + \frac{\|\hat{\Xi} - \Xi\|_{op}}{\delta_{k}} \right)
$$

(S.97)

where the upper bound is uniform over $t \in T$. Hence

$$
\left| \sqrt{\hat{\lambda}_{k}}\hat{\phi}_{k}(t) - \sqrt{\lambda_{k}}\phi_{k}(t) \right| = \left| \frac{1}{\sqrt{\lambda_{k}}}\hat{\lambda}_{k}\hat{\phi}_{k}(t) - \frac{1}{\sqrt{\hat{\lambda}_{k}}}\lambda_{k}\phi_{k}(t) \right|
$$

$$
\leq \left| \left( \frac{1}{\sqrt{\lambda_{k}}} - \frac{1}{\sqrt{\hat{\lambda}_{k}}} \right) (\hat{\lambda}_{k}\hat{\phi}_{k}(t) - \lambda_{k}\phi_{k}(t)) \right| + \left| \left( \frac{1}{\sqrt{\lambda_{k}}} - \frac{1}{\sqrt{\hat{\lambda}_{k}}} \right) \lambda_{k}\phi_{k}(t) \right|
$$

$$
+ \left| \frac{1}{\sqrt{\lambda_{k}}} (\hat{\lambda}_{k}\hat{\phi}_{k}(t) - \lambda_{k}\phi_{k}(t)) \right|
$$

$$
\leq O \left( \frac{\|\hat{\Xi} - \Xi\|_{op}}{\lambda_{k}^{3/2}} \left( c_{n} + \frac{\|\hat{\Xi} - \Xi\|_{op}}{\delta_{k}} \right) \right) + \frac{\sqrt{2}\|\hat{\Xi} - \Xi\|_{op}}{\lambda_{k}} \sqrt{\lambda_{k}}|\phi_{k}(t)|
$$

$$
+ O \left( \frac{c_{n}}{\sqrt{\lambda_{k}}} + \frac{\|\hat{\Xi} - \Xi\|_{op}}{\sqrt{\lambda_{k}\delta_{k}}} \right).
$$

(S.98)

With

$$
\sum_{k=1}^{K} \frac{1}{\lambda_{k}^{3/2}\delta_{k}} = O \left( \frac{1}{\lambda_{K}\delta_{K}} \right)
$$

and (S.79) this leads to

$$
\sum_{k=1}^{K} \left| \sqrt{\hat{\lambda}_{k}}\hat{\phi}_{k}(t) - \sqrt{\lambda_{k}}\phi_{k}(t) \right| \leq O \left( \sum_{k=1}^{K} \frac{c_{n}}{\delta_{k}\sqrt{\lambda_{k}}} \right) + O \left( \|\hat{\Xi} - \Xi\|_{op}^{2} \left( \sum_{k=1}^{K} \frac{1}{\lambda_{k}\delta_{k}} \right) \right)
$$

$$
+ O \left( \|\hat{\Xi} - \Xi\|_{op}^{2} \sum_{k=1}^{K} \frac{1}{\delta_{k}\sqrt{\lambda_{k}}} \sqrt{\sum_{k=1}^{K} \lambda_{k}\phi_{k}^{2}(t)} \right)
$$

$$
+ O \left( \frac{c_{n} + \|\hat{\Xi} - \Xi\|_{op}^{2}}{\sqrt{\lambda_{K}\delta_{K}}} \right)
$$

$$
= o(1),
$$

(S.99)
where the last equality holds uniformly over $t \in \mathcal{T}$ due to (S.85) and the fact that $\sum_{k=1}^{K} \lambda_k \phi_k^2(t) \leq \sum_{k=1}^{\infty} \lambda_k \phi_k^2(t) = \Gamma(t, t)$ which is uniformly bounded owing to the continuity of $\Gamma$ over the compact set $\mathcal{T}$. Thus

$$
\sum_{k=1}^{K} \left| \sqrt{\lambda_k} \hat{\phi}_k(t) - \sqrt{\lambda_k} \phi_k(t) \right|^2 \leq \left( \sum_{k=1}^{K} \left| \sqrt{\lambda_k} \hat{\phi}_k(t) - \sqrt{\lambda_k} \phi_k(t) \right| \right)^2 = o(1), \quad (S.100)
$$
a.s. as $n \to \infty$. Next, (S.94), (S.95) and (S.100) imply

$$
E[Y^{*c}(t) - Z^{*c}(t)]^2 = o(1) + \sum_{k=K+1}^{\infty} \hat{\lambda}_k \hat{\phi}_k^2(t) + \lambda_k \phi_k^2(t) = o(1) \quad \text{a.s.,}
$$
uniformly over $t \in \mathcal{T}$ due to Mercer’s theorem and since it was already shown that $\sum_{k=K+1}^{\infty} \hat{\lambda}_k \hat{\phi}_k^2(t) = o(1)$ a.s. as $n \to \infty$. Thus

$$
\left\| \hat{A}_3 - A_3 \right\|_2 = o(1) \quad \text{a.s.,}
$$

$$
\| \Delta_2^2 \|_2^2 = \| E(Y^{*c} | Y_t^*) - E(Z^{*c} | Z_t^*) \|_2 = \left\| \hat{A}_1 \hat{A}_2 \hat{A}_3 - A_1 A_2 A_3 \right\|_2^2
\leq \sum_{p=1}^{3} \left\| \hat{A}_p - A_p \right\| \prod_{q \neq p} \| A_q \| + \sum_{p=1}^{3} \| A_p \| \prod_{q \neq p} \left\| \hat{A}_q - A_q \right\| + \sum_{q=1}^{3} \| \hat{A}_q - A_q \|
= o(1) \quad \text{a.s.,}
$$

where the norms in the second equation are as in in (S.92), (S.93), and (S.102). By dominated convergence,

$$
E \| \Delta_2^2 \|_2^2 = E \| E(Y^{*c} | Y_t^*) - E(Z^{*c} | Z_t^*) \|_2^2 = o(1) \quad \text{a.s.} \quad (S.103)
$$

Next

$$
\| \Delta_{\mu_i} \|_2 = \| \hat{\mu}_i - \mu_i \|_2 \leq \| (\hat{\Gamma}(\cdot, T_i) \Sigma_i^{-1} - \Gamma(\cdot, T_i) \Sigma_i^{-1})(X_i - \hat{\mu}_i) \|_2 + \| \Gamma(\cdot, T_i) \Sigma_i^{-1}(\mu_i - \hat{\mu}_i) \|_2,
$$

where we used that $\| \Sigma_i^{-1} \|_{op, 2} \leq \sigma^{-2}$ and $\sup_{t \in \mathcal{T}} |\mu(t) - \hat{\mu}(t)| = o(1)$ a.s., as per Proposition 1 in Dai et al. (2018), imply $\| \Gamma(\cdot, T_i) \Sigma_i^{-1}(\mu_i - \hat{\mu}_i) \|_2 = o(1)$ as $n \to \infty$. Note

$$
\| (\hat{\Gamma}(\cdot, T_i) \Sigma_i^{-1} - \Gamma(\cdot, T_i) \Sigma_i^{-1})(X_i - \hat{\mu}_i) \|_2 \leq \| \hat{\Gamma}(\cdot, T_i) (\Sigma_i^{-1} - \Sigma_i^{-1})(X_i - \hat{\mu}_i) \|_2 \leq o(1) \| X_i - \hat{\mu}_i \|_2
\leq o(1) (\| X_i - \mu_i \|_2 + \| \mu_i - \hat{\mu}_i \|_2)
= o(1) \quad \text{a.s.}
$$
where the second inequality follows from \( \sup_{s,t \in \mathcal{T}} |\hat{\Gamma}(s, t) - \Gamma(s, t)| = o(1) \), \( \sup_{s \in \mathcal{T}} |\mu(s) - \hat{\mu}(s)| = o(1) \), and \( \|\hat{\Sigma}_i^{-1} - \Sigma_i^{-1}\|_{\text{op,2}} = o(1) \) a.s., which is due to Proposition 1, arguments in the proof of Theorem 2 in Dai et al. (2018), and \( \|\Gamma\|_\infty < \infty \). Thus, combining with (S.104) and since the expectations are taken conditionally on the observations \((X_i, T_i)\), \(i = 1, \ldots, n\), it follows that

\[
E \|\Delta_{\mu_i}\|_2^2 = \|\Delta_{\mu_i}\|_2^2 = o(1) \quad \text{a.s.} \tag{S.105}
\]

Combining (S.83), (S.89), (S.103) and (S.105) completes the proof. \( \square \)

**Proof of Theorem 8.** Recall that the eigenpairs of the integral operator associated with \( \hat{\Gamma} \) are \((\hat{\lambda}_k, \hat{\phi}_k)\), and those of \( \Gamma \) are \((\lambda_k, \phi_k)\), \(k \geq 1\). Let \(\epsilon_1, \ldots, \epsilon_n\) and \(Z_k\), \(k \geq 1\), be a sequence of independent standard normal random variables, which are independent of all other random quantities; and also let \(\epsilon = [\epsilon_1, \ldots, \epsilon_n]^T\). Let

\[
Z^*(t) = \hat{\mu}(t) + \sum_{k=1}^{\infty} \sqrt{\hat{\lambda}_k} Z_k \hat{\phi}_k(t), \quad Z_i^* = Z^*(T_i) + \sigma \epsilon, \quad Z^{sc}(t) = Z^*(t) - \hat{\mu}(t),
\]

\[
Y^*(t) = \hat{\mu}(t) + \sum_{k=1}^{K} \sqrt{\hat{\lambda}_k} Z_k \hat{\phi}_k(t), \quad Y^{sc}(t) = Y^*(t) - \hat{\mu}(t),
\]

and recall that \(\hat{\mu}_{iK} = \xi_{iK}^T \hat{\Phi}_K\) and \(\hat{\mu}_i(\cdot) = \hat{\Gamma}(\cdot, T_i) \Sigma_i^{-1}(X_i - \hat{\mu}_i)\). Take the convention that all expected values are taken conditional on the observations \((X_i, T_i)_{i=1}^n\). Then, given \((X_i, T_i)_{i=1}^n\), it holds that \(\hat{\mu}_{iK} + Y^* - E(Y^*|Z_i^*)\) share the same distribution as \(\hat{G}_{iK}\), and so do \(\hat{\mu}_i + Z^* - E(Z^*|Z_i^*)\) and \(\hat{G}_i\). By (11) and defining the auxiliary quantities \(\Delta_{\mu_i} = \hat{\mu}_{iK} - \hat{\mu}_i\), \(\Delta_1^* = Y^{sc} - Z^{sc}\) and \(\Delta_2^* = E(Y^{sc}|Z_i^*) - E(Z^{sc}|Z_i^*)\), it follows that

\[
W_2^2(\hat{G}_{iK}, G_i) \leq \|E(\hat{\mu}_{iK} + Y^* - E(Y^*|Y_i^*)) - (\hat{\mu}_i + Z^* - E(Z^*|Z_i^*))\|_2^2 = E \|\Delta_{\mu_i} + \Delta_1^* - \Delta_2^*\|_2^2 \leq E \|\Delta_{\mu_i}\|_2^2 + E \|\Delta_1^*\|_2^2 + 2E(\|\Delta_{\mu_i}\|_2 \|\Delta_1^*\|_2 + \|\Delta_{\mu_i}\|_2 \|\Delta_2^*\|_2 + \|\Delta_1^*\|_2 \|\Delta_2^*\|_2),
\]

\[
E \|\Delta_1^*\|_2^2 = E \left\| \sum_{k=K+1}^{\infty} \sqrt{\hat{\lambda}_k} Z_k \hat{\phi}_k \right\|_2^2 = \sum_{k=K+1}^{\infty} \hat{\lambda}_k.
\]

Note that for any \(K_0 \geq 1\) and using that \(\hat{\lambda}_k \geq 0\) almost surely, \(\sum_{k=K+1}^{\infty} \hat{\lambda}_k \leq \sum_{k=K_0+1}^{\infty} \hat{\lambda}_k\) a.s. for large enough \(n\) due to the fact that \(K = K(n)\) diverges as \(n \to \infty\). Also

\[
\sum_{k \geq K_0+1} \hat{\lambda}_k = \sum_{k \geq 1} \hat{\lambda}_k - \sum_{k \leq K_0} \hat{\lambda}_k = \sum_{k \geq 1} \hat{\lambda}_k - \lambda_k + \sum_{k \leq K_0} \lambda_k - \hat{\lambda}_k + \sum_{k \geq K_0+1} \lambda_k = \sum_{k \geq K_0+1} \lambda_k + o(1),
\]

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a.s. as \( n \to \infty \), where the last equality follows since \( \sum_{k \geq 1} \hat{\lambda}_k - \lambda_k = \int_{\mathcal{T}} \hat{\Gamma}(t) - \Gamma(t) \, dt = O(a_n + b_n) \) and \( \sum_{k \geq K_0} \lambda_k - \hat{\lambda}_k = O(a_n + b_n) \) a.s. as \( n \to \infty \). Hence

\[
\sum_{k \geq K+1} \hat{\lambda}_k \leq \sum_{k \geq K_0+1} \hat{\lambda}_k - \lambda_k + \sum_{k \geq K_0+1} \lambda_k = o(1) + \sum_{k \geq K_0+1} \lambda_k \quad \text{a.s.,}
\]
as \( n \to \infty \). This shows that \( \limsup_{n \to \infty} \sum_{k \geq K+1} \hat{\lambda}_k \leq \sum_{k \geq K_0+1} \lambda_k \) a.s. for any \( K_0 \geq 1 \), and therefore \( \sum_{k \geq K+1} \hat{\lambda}_k = o(1) \) a.s. as \( n \to \infty \). Thus

\[
E \| \Delta_1^* \|_{L_2}^2 = o(1) \quad \text{a.s.,}
\]
as \( n \to \infty \). Writing \( a_K(t) = \sum_{k \geq K+1} \hat{\lambda}_k \hat{\phi}_k(t) \), we have

\[
\| \Delta_2^* \|_{L_2}^2 = \| E(Y^*|Z_i^*) - E(Z^*|Z_i^*) \|_{L_2}^2 = \left\| \sum_{k \geq K+1} \hat{\lambda}_k \hat{\phi}_k (T_{ii}) \hat{\phi}_k(t) \right\|_{L_2}^2 = \left\| a_K^T(\cdot) \Sigma_i^{-1} (Z_i^* - \hat{\mu}_i) \hat{\phi}_k(t) \right\|_{L_2}^2 = \int_0^1 \| a_K(t) \Sigma_i^{-1} (Z_i^* - \hat{\mu}_i) \|_{L_2}^2 dt \leq \int_0^1 \| a_K(t) \|_{L_2}^2 \| \Sigma_i^{-1} (Z_i^* - \hat{\mu}_i) \|_{L_2}^2 dt,
\]
where the last inequality follows from \( \sum_{k \geq K+1} \hat{\lambda}_k \hat{\phi}_k^2(s) = o(1) \) a.s., uniformly over \( s \in \mathcal{T} \), which was shown in the proof of Theorem 7, and the \( o(1) \) term in the upper bound is uniform in \( t \). Combining with (S.108) and utilizing the orthonormality of the \( \hat{\phi}_k \),

\[
\| \Delta_2^* \|_{L_2}^2 \leq \| \Sigma_i^{-1} (Z_i^* - \hat{\mu}(T_i)) \|_{L_2}^2 \sum_{l=1}^{n_i} \left( \sum_{k \geq K+1} \hat{\lambda}_k \hat{\phi}_k^2(T_{iil}) \right) \leq \sum_{l=1}^{n_i} \left( \sum_{k \geq K+1} \hat{\lambda}_k \hat{\phi}_k^2(T_{iil}) \right) \leq o(1) \sum_{l=1}^{n_i} \sum_{k \geq K+1} \hat{\lambda}_k \hat{\phi}_k^2(t) \quad \text{a.s.,}
\]
where the last equality follows from the fact that \( \sum_{k \geq K+1} \hat{\lambda}_k = o(1) \) a.s. as \( n \to \infty \). This implies

\[
E \| \Delta_2^* \|_{L_2}^2 \leq o(1) E \| \Sigma_i^{-1} (Z_i^* - \hat{\mu}(T_i)) \|_{L_2}^2 \leq o(1) \text{tr}(\Sigma_i^{-1}) = o(1) \quad \text{a.s.,}
\]
(S.109)
as $n \to \infty$, where the last equality follows from the fact that $\left\| \Sigma^{-1}_i \right\|_{op, 2} \leq \sigma^2 + o(1)$ almost surely, which follows from the proof of Theorem 2 in Dai et al. (2018). Next

$$E \| \Delta \mu_i \|^2_{L^2} = \| \hat{\mu}_i K - \tilde{\mu}_i \|^2_{L^2} = \left\| \hat{\xi}^T_{iK} \hat{\Phi}_K - \hat{\Gamma}(\cdot, T_i) \hat{\Sigma}^{-1}_i (X_i - \hat{\mu}_i) \right\|^2_{L^2}$$

$$= \left\| \left( \hat{\Phi}_K \hat{\lambda}_K \hat{\Phi}_K^T - \hat{\Gamma}(\cdot, T_i) \right) \hat{\Sigma}^{-1}_i (X_i - \hat{\mu}_i) \right\|^2_{L^2}$$

$$= \left\| \sum_{k \geq K+1} \hat{\lambda}_k \hat{\varphi}_k \hat{\Phi}^T_{ik} \hat{\Sigma}^{-1}_i (X_i - \hat{\mu}_i) \right\|^2_{L^2}$$

$$= \left\| \hat{a}^T_K (\cdot) \hat{\Sigma}^{-1}_i (X_i - \hat{\mu}_i) \right\|^2_{L^2} \leq o(1) \left\| \hat{\Sigma}^{-1}_i (X_i - \hat{\mu}_i) \right\|^2_{L^2},$$

where the last inequality follows by similar arguments as before. Combining this with $\| X_i - \tilde{\mu}_i \|_{2} \leq \| X_i - \mu_i \|_{2} + \| \mu_i - \tilde{\mu}_i \|_{2} \leq \| X_i - \mu_i \|_{2} + o(1)$, which was shown in the proof of Theorem 7, implies

$$E \| \Delta \mu_i \|^2_{L^2} \leq o(1) \left\| X_i - \tilde{\mu}_i \right\|^2_{L^2} = o(1) \quad \text{a.s.,} \quad \text{(S.110)}$$

as $n \to \infty$. The result then follows by combining (S.106), (S.107), (S.109) and (S.110).

\[ \square \]

S.2 Proof of Main Results for Prediction in Functional Linear Models

We first provide some auxiliary lemmas that will be used in the proof of the main results in section 4. Here we derive a slightly more general result without using optimal bandwidths. Recall that $w_i := (\sum_{l=1}^{n} n_l)^{-1}$, $v_M = \sum_{m=1}^{M} \frac{1}{\delta_m}$ and $C(t) = E((X(t) - \mu(t))Y) = \int_{T} \beta(s) \Gamma(t, s) ds$, $t \in T$. In what follows, for a function $h : T \in \mathbb{R}$, let $\| h \|_{L^2} = (\int_{T} h^2(t) dt)^{1/2}$ be its $L^2$ norm over $T$. For a vector $z \in \mathbb{R}^k$ and a matrix $A \in \mathbb{R}^{p \times k}$, $k, p > 0$, denote by $\| z \|_2$ its Euclidean norm and $\| A \|_{op} = \sup_{z \in \mathbb{R}^k : \| z \|_2 = 1} \| Az \|_2$ the matrix operator norm.

**Lemma S10.** Suppose that (S4), (B1)-(B2), (A1)–(A8) in the Appendix hold and consider a sparse design with $n_i \leq N_0 < \infty$, setting $a_n = a_{n1}$ and $b_n = b_{n1}$. Then

$$n^{-1} \sum_{i=1}^{n} \| \hat{\xi}_{iK} - \tilde{\xi}_{iK} \|^2_{L^2} = O_p((a_n + b_n)^2), \quad \text{(S.111)}$$

and

$$n^{-1} \sum_{i=1}^{n} \| \hat{\xi}_{iK} \|^2_{L^2} = O_p(1). \quad \text{(S.112)}$$
Proof of Lemma S10. From arguments in the proof of Theorem 2 in Dai et al. (2018) and noting that the constant \( c \) that appears in Lemma A.3 in Facer and Müller (2003) can be taken as a universal constant \( c = 2 \),

\[
\|\hat{\xi}_{iK} - \hat{\xi}_{iK}\|_2^2 \leq O((a_n + b_n)^2)\|\mathbf{X}_i - \hat{\mu}_i\|_2^2 + O(a_n^2 + O(a_n(a_n + b_n)))\|\mathbf{X}_i - \hat{\mu}_i\|_2 \quad \text{a.s., (S.113)}
\]

where the \( O((a_n + b_n)^2) \), \( O(a_n^2) \) and \( O(a_n(a_n + b_n)) \) terms are uniform in \( i \). Let \( \mathbf{U}_i = (X_i(T_{i1}), \ldots, X_i(T_{im}))^T \) be the true but unobserved values of the trajectory for the \( i \)th subject at the time points \( T_{ij} \), so that by construction \( \mathbf{X}_i = \mathbf{U}_i + \mathbf{e}_i \). Then

\[
n^{-1} \sum_{i=1}^n \|\mathbf{X}_i - \hat{\mu}_i\|_2 = n^{-1} \sum_{i=1}^n \|\mathbf{U}_i + \mathbf{e}_i - \hat{\mu}_i\|_2 \\
\quad \leq n^{-1} \sum_{i=1}^n \|\mathbf{U}_i - \mu_i\|_2 + n^{-1} \sum_{i=1}^n \|\mathbf{e}_i\|_2 + n^{-1} \sum_{i=1}^n \|\mu_i - \hat{\mu}_i\|_2, \quad \text{(S.114)}
\]

where from Proposition 1 in Dai et al. (2018) we have \( n^{-1} \sum_{i=1}^n \|\mu_i - \hat{\mu}_i\|_2 = O(a_n) \) almost surely. Since \( n_i \leq N_0 \) in the sparse case, it is easy to show that \( n^{-1} \sum_{i=1}^n \|\mathbf{e}_i\|_2 = O_p(1) \) and by Jensen’s inequality

\[
E \left( n^{-1} \sum_{i=1}^n \|\mathbf{U}_i - \mu_i\|_2 \right) \leq n^{-1} \sum_{i=1}^n \left( \sum_{j=1}^{n_i} E(X_i(T_{ij}) - \mu(T_{ij}))^2 \right)^{1/2} \\
\quad = n^{-1} \sum_{i=1}^n \left( \sum_{j=1}^{n_i} E(\Gamma(T_{ij}, T_{ij})) \right)^{1/2} \leq (\|\Gamma\|_{\infty} N_0)^{1/2} = O(1),
\]

where the first equality follow by conditioning on \( T_{ij} \). This shows that \( n^{-1} \sum_{i=1}^n \|\mathbf{U}_i - \mu_i\|_2 = O_p(1) \). Combining with (S.114) leads to

\[
n^{-1} \sum_{i=1}^n \|\mathbf{X}_i - \hat{\mu}_i\|_2 = O_p(1). \quad \text{(S.115)}
\]

Next, by the triangle inequality

\[
\|\mathbf{X}_i - \hat{\mu}_i\|_2 \leq \|\mathbf{U}_i - \mu_i\|_2 + \|\mathbf{e}_i\|_2 + \|\mu_i - \hat{\mu}_i\|_2 \\
\quad + 2\|\mathbf{U}_i - \mu_i\|_2 \|\mathbf{e}_i\|_2 + 2\|\mathbf{U}_i - \mu_i\|_2 \|\mu_i - \hat{\mu}_i\|_2 + 2\|\mathbf{e}_i\|_2 \|\mu_i - \hat{\mu}_i\|_2,
\]

where \( \|\mu_i - \hat{\mu}_i\|_2 \leq \sqrt{N_0} \sup_{t \in T} (\mu(t) - \hat{\mu}(t))^2 = O(a_n) \) a.s. and uniformly over \( i \). This along with the independence of \( \mathbf{e}_i \) and \( \mathbf{U}_i \), conditionally on \( T_i \), and using similar arguments as before, leads to \( E\|\mathbf{X}_i - \hat{\mu}_i\|_2 = O(1) \) uniformly over \( i \). Thus

\[
n^{-1} \sum_{i=1}^n \|\mathbf{X}_i - \hat{\mu}_i\|_2^2 = O_p(1). \quad \text{(S.116)}
\]
Combining (S.113), (S.115) and (S.116) leads to the first result in (S.111). Next, note that
\[ E(\tilde{\xi}_{ik}^T \Phi_{ik} \Sigma_i^{-1})^2 \leq E(\|\Lambda_K \Phi_{ik}^T \Sigma_i^{-1}\|_{op,2}^4) \leq O(1), \]
where the O(1) term is uniform in \( i \) and the last inequality follows from \( \|\Lambda_K\|_{op,2} \leq \lambda_1 K \), \( \|\Phi_{ik}\|_{op,2} \leq N_0 \sum_{j=1}^K \phi_j \|\Sigma_i^{-1}\|_{op,2} \leq \sigma^{-2} \), \( E(\|X_i - \mu_i\|_2^4 | T_i) \leq O(1) \) uniformly over \( i \), where the latter is a consequence of the Gaussian process assumption on \( X_i(\cdot) \) and \( ||\Gamma||_\infty < \infty \). Thus, \( E(\|\tilde{\xi}_{ik}\|_2^2) = O(1) \) uniformly in \( i \) which implies \( E(n^{-1} \sum_{i=1}^n \|\tilde{\xi}_{ik}\|_2^2) = O(1) \) and the second result in (S.112).

**Lemma S11.** Suppose that (S4), (B1)-(B2), (B4)-(B5), (A1)–(A8) in the Appendix hold and consider a sparse design with \( n_i \leq N_0 < \infty \), setting \( a_n = a_n1 \) and \( b_n = b_{n1} \). Let \( \tilde{Z}_i(t) := \sum_{j=1}^{n_i} w_i K_h(T_{ij} - t) \left( \frac{T_{ij} - t}{h} \right)^r (U_{ij} Y_i - C(t)) \), where \( U_{ij} = X(T_{ij}) - \mu(T_{ij}) \). Then
\[ E[\tilde{Z}_i^2(t)] = O((n^2 h)^{-1}), \]
where the \( O((n^2 h)^{-1}) \) term is uniform in \( i \) and \( t \).

**Proof.** Observe
\[
E[\tilde{Z}_i^2(t)] = E \left( \sum_{j=1}^{n_i} w_i^2 K_h^2(T_{ij} - t) \left( \frac{T_{ij} - t}{h} \right)^{2r} (U_{ij} Y_i - C(t))^2 \right) \\
+ E \left( \sum_{j=1}^{n_i} \sum_{l \neq j} w_i^2 K_h(T_{ij} - t) K_h(T_{il} - t) \left( \frac{T_{ij} - t}{h} \right)^r \left( \frac{T_{il} - t}{h} \right)^r (U_{ij} Y_i - C(t))(U_{il} Y_i - C(t)) \right)
\]
and note that for any \( t_1, t_2 \in \mathcal{T} \), with \( \mu_Y = E(Y) \),
\[
E(U(t_1)U(t_2)Y^2) = E(U(t_1)U(t_2)[\mu_Y + \int_{\mathcal{T}} \beta(s)U(s)ds + \epsilon_Y]^2) \\
= (\mu_Y^2 + \sigma_Y^2) \Gamma(t_1, t_2) + 2 \int_{\mathcal{T}} \mu_Y \beta(s)E(U(t_1)U(t_2)U(s))ds \\
+ \int_{\mathcal{T}} \int_{\mathcal{T}} \beta(s_1)\beta(s_2)E(U(t_1)U(t_2)U(s_1)U(s_2))ds_1ds_2 \\
= O(1),
\]
where the \( O(1) \) term is uniform over \( t_1 \) and \( t_2 \), which follows from \( ||\Gamma||_\infty < \infty \) and \( U(t) \sim N(0, \Gamma(t, t)) \), owing to (S4). This implies that \( E((U_{ij} Y_i - C(t))^2 | T_{ij}) \) is uniformly bounded
above, and by a conditioning argument it follows that
\[ E \left( \sum_{j=1}^{n_i} w_i^2 K_h^2(T_{ij} - t) \left( \frac{T_{ij} - t}{h} \right)^{2r} (U_{ij} Y_i - C(t))^2 \right) \leq O(1) E \left( \sum_{j=1}^{n_i} w_i^2 K_h^2(T_{ij} - t) \left( \frac{T_{ij} - t}{h} \right)^{2r} \right) \]
\[ = O((n^2 h)^{-1}), \]
where the last equality is due to \( w_i \leq n^{-1} \). Denote by \( R_{ijr,h}(t) = w_i K_h(T_{ij} - t) \left( \frac{T_{ij} - t}{h} \right)^r \), \( q = j, l \).
Since \( E((U_{ij} Y_i - C(t))(U_{il} Y_i - C(t))|T_{ij}, T_{il}) = O(1) \) uniformly in \( i \) and \( t \), similar arguments as before show that
\[ E \left( \sum_{j=1}^{n_i} \sum_{l \neq j} R_{ijr,h}(t) R_{ilr,h}(t)(U_{ij} Y_i - C(t))(U_{il} Y_i - C(t)) \right) \leq O(1) \sum_{j=1}^{n_i} \sum_{l \neq j} E[R_{ijr,h}(t)] E[R_{ilr,h}(t)] \]
\[ = O(n^{-2}), \]
whence the result follows. \( \square \)

**Lemma S12.** Suppose that (S4), (B1)-(B2), (B4)-(B5), (A1)–(A8) in the Appendix hold and consider a sparse design with \( n_i \leq N_0 < \infty \), setting \( a_n = a_{n1} \) and \( b_n = b_{n1} \). For \( r = 0, 1 \) we have
\[ \| \sum_{i=1}^{n} \sum_{j=1}^{n_i} w_i K_h(T_{ij} - \cdot) \left( \frac{T_{ij} - \cdot}{h} \right)^{r} \epsilon_{ij} Y_i \|_{L^2} = O_p((nh)^{-1/2}), \]
\[ \text{(S.117)} \]
and
\[ \| \sum_{i=1}^{n} \sum_{j=1}^{n_i} w_i K_h(T_{ij} - \cdot) \left( \frac{T_{ij} - \cdot}{h} \right)^{r} (U_{ij} Y_i - C(\cdot)) \|_{L^2} = O_p \left( \left( \frac{1}{nh} + h^2 \right)^{1/2} \right), \]
\[ \text{(S.118)} \]
where \( U_{ij} = X(T_{ij}) - \mu(T_{ij}) \).

**Proof.** Define \( Z_i(t) := \sum_{j=1}^{n_i} w_i K_h(T_{ij} - t) \left( \frac{T_{ij} - t}{h} \right)^r \epsilon_{ij} Y_i \). Note that the \( Z_i \) are independent and by independence of the \( \epsilon_{ij} \) along with a conditioning argument, \( E(Z_i(t)) = 0 \) and
\[ E(\| \sum_{i=1}^{n} Z_i \|_{L^2}^2) = n \int_{\tau} E(Z_i(t))^2 dt, \]
\[ E(Z_i^2(t)) = E \left( \sum_{j=1}^{n_i} \sum_{l=1}^{n_i} w_i^2 K_h^2(T_{ij} - t) \left( \frac{T_{ij} - t}{h} \right)^{2r} \epsilon_{ij} K_h(T_{il} - t) \left( \frac{T_{il} - t}{h} \right)^{r} \epsilon_{il} Y_i^2 \right) \]
\[ = \sum_{j=1}^{n_i} E \left( w_i^2 K_h^2(T_{ij} - t) \left( \frac{T_{ij} - t}{h} \right)^{2r} \epsilon_{ij}^2 Y_i^2 \right) \]
\[ = E(Y^2) \sigma^2 \sum_{j=1}^{n_i} E \left( w_i^2 K_h^2(T_{ij} - t) \left( \frac{T_{ij} - t}{h} \right)^{2r} \right) = O((n^2 h)^{-1}), \]
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where the $O(h^{-1})$ is uniform in $i$ and $t$. Thus $E(\|\sum_{i=1}^{n} Z_i\|_{L^2}^2) = O((nh)^{-1})$ and the first result in (S.117) follows. Next, defining $\tilde{Z}_i(t) := \sum_{j=1}^{n_i} w_i h(T_{ij} - t) \left(\frac{T_{ij} - t}{h}\right)^r (U_{ij} Y_i - C(t))$, we have

$$E(\|\sum_{i=1}^{n} \tilde{Z}_i\|_{L^2}^2) = \sum_{i=1}^{n} \int_T E[\tilde{Z}_i^2(t)] dt + \sum_{i=1}^{n} \sum_{k \neq i} \int_T E(\tilde{Z}_i(t))E(\tilde{Z}_k(t)). \quad \text{(S.119)}$$

By a conditioning argument, it follows that

$$|E(\tilde{Z}_i(t))| = \left| \sum_{j=1}^{n_i} w_i E \left( K_h(T_{ij} - t) \left(\frac{T_{ij} - t}{h}\right)^r (C(T_{ij}) - C(t)) \right) \right| \leq \sum_{j=1}^{n_i} w_i \int_{-t/h}^{(1-t)/h} |u'| K(u)|C(t + uh) - C(t)| f(t + uh) du \sup_{s \in [-1,1]} |C'(s)| \|f\|_{\infty} h \int_{-t/h}^{(1-t)/h} |u'| K(u) du \leq O(n^{-1}h),$$

where the $O(n^{-1}h)$ is uniform in $i$ and $t$. This implies $|\sum_{i=1}^{n} \sum_{k \neq i} \int_T E(\tilde{Z}_i(t))E(\tilde{Z}_k(t))| = O(h^2)$. Combining with (S.119) and Lemma S11, the second result in (S.118) follows.

**Lemma S13.** Suppose that (S4), (B1)-(B2), (B4)-(B5), (A1)–(A8) in the Appendix hold and consider a sparse design with $n_i \leq N_0 < \infty$, setting $a_n = a_{n1}$ and $b_n = b_{n1}$. For $r = 0, 1$,

$$\left\| \sum_{i=1}^{n} \sum_{j=1}^{n_i} w_i K_h(T_{ij} - t) \left(\frac{T_{ij} - t}{h}\right)^r (\mu(T_{ij}) - \hat{\mu}(T_{ij})) Y_i \right\|_{L^2} = O_p(a_n).$$

**Proof.** Setting $Z_i := \sum_{j=1}^{n_i} w_i K_h(T_{ij} - t) \left(\frac{T_{ij} - t}{h}\right)^r (\mu(T_{ij}) - \hat{\mu}(T_{ij})) Y_i$, note that

$$E\left(\left\| \sum_{i=1}^{n} Z_i \right\|_{L^2}^2\right) = \int_T \sum_{i=1}^{n} E[Z_i^2(t)] dt + \int_T \sum_{i=1}^{n} \sum_{k \neq i} E[Z_i(t)Z_k(t)]. \quad \text{(S.120)}$$

Since $|Z_i(t)| \leq \|\mu - \hat{\mu}\|_{\infty} \sum_{j=1}^{n_i} w_i K_h(T_{ij} - t) \left(\frac{|T_{ij} - t|}{h}\right)^r |Y_i|$, it follows that

$$E[Z_i^2(t)] \leq E\left[\|\mu - \hat{\mu}\|_{\infty}^2 \sum_{j=1}^{n_i} \sum_{l=1}^{n_i} w_i^2 Y_i^2 K_h(T_{ij} - t) K_h(T_{il} - t) \left(\frac{|T_{ij} - t|}{h}\right)^r \left(\frac{|T_{il} - t|}{h}\right)^r \right] \leq O(a_n^2) \left\{ \sum_{j=1}^{n_i} w_i^2 E(Y^2) E\left[ K_h^2(T_{ij} - t) \left(\frac{T_{ij} - t}{h}\right)^{2r} \right] \right. \left. + \sum_{j=1}^{n_i} \sum_{l \neq j} w_i^2 E(Y^2) E\left[ K_h(T_{ij} - t) \left(\frac{|T_{ij} - t|}{h}\right)^r \right] E\left[ K_h(T_{il} - t) \left(\frac{|T_{il} - t|}{h}\right)^r \right] \right\} \leq O(a_n^2)[O(n^{-2}h^{-1}) + O(n^{-2})] = O(a_n^2 n^{-2}h^{-1}), \quad \text{(S.121)}$$
where the first inequality follows from Proposition 1 in Dai et al. (2018) and the term \( O(a_n^2 n^{-2} h^{-1}) \) is uniform in \( i \) and \( t \). Similarly, for \( k \neq i \) and setting \( h_{qdr}(t) := \left( \frac{T_{qd} - t}{h} \right)^r \), \( q = i, k \) and \( d = j, l \), we have

\[
E(|Z_i(t)Z_k(t)|) \leq E\left[ \sum_{j=1}^{n_i} \sum_{l=1}^{n_k} w_i K_h(T_{ij} - t) h_{ijr}(t) \mu(T_{ij}) - \mu(T_{ij}) Y_i w_k K_h(T_{kl} - t) h_{klr}(t) \mu(T_{kl}) - \mu(T_{kl}) Y_k \right]
\]

\[
\leq O(a_n^2) \sum_{j=1}^{n_i} \sum_{l=1}^{n_k} w_i w_k E[K_h(T_{ij} - t) h_{ijr}(t)] E[K_h(T_{kl} - t) h_{klr}(t)] [E(Y)]^2
\]

\[
= O(a_n^2 n^{-2}),
\]

where the \( O(a_n^2 n^{-2}) \) term is uniform in \( i, k \) and \( t \). Combining this with (S.120) and (S.121) leads to the result.

\[\square\]

**Lemma S14.** Suppose that (S4), (B1)-(B2), (B4)-(B5), (A1)–(A8) in the Appendix hold and consider a sparse design with \( n_i \leq N_0 < \infty \), setting \( a_n = a_{n1} \) and \( b_n = b_{n1} \). Then

\[
\| \hat{C} - C \|_{L^2} = O_p \left( \frac{1}{nh^2} + h^2 \right)^{1/2} + a_n.
\]

**Proof.** Proceeding similarly to the proof of Theorem 3.1 in Zhang and Wang (2016), using (17),

\[
\hat{C}(t) = \frac{S_2(t) \tilde{R}_0(t) - S_1(t) \tilde{R}_1(t)}{S_0(t) S_2(t) - S_1^2(t)},
\]

where

\[
S_r(t) = \sum_{i=1}^{n} \sum_{j=1}^{n_i} w_i K_h(T_{ij} - t) \left( \frac{T_{ij} - t}{h} \right)^r,
\]

\[
\tilde{R}_r(t) = \sum_{i=1}^{n} \sum_{j=1}^{n_i} w_i K_h(T_{ij} - t) \left( \frac{T_{ij} - t}{h} \right)^r C_i(T_{ij}),
\]

and \( r = 0, 1, 2 \). Then

\[
\hat{C}(t) - C(t) = \frac{(\tilde{R}_0(t) - C(t) S_0(t)) S_2(t) - (\tilde{R}_1(t) - C(t) S_1(t)) S_1(t)}{S_0(t) S_2(t) - S_1^2(t)}.
\]
Since \( C_i(T_{ij}) = (\hat{X}_{ij} - \hat{\mu}(T_{ij}))Y_i = (U_{ij} + \epsilon_{ij})Y_i + (\mu(T_{ij}) - \hat{\mu}(T_{ij}))Y_i \), where \( U_{ij} = X(T_{ij}) - \mu(T_{ij}) \),

\[
\| \hat{R}_0(t) - C(t)S_0(t) \|_{L^2} \leq \| \sum_{i=1}^{n} \sum_{j=1}^{n_i} w_i K_h(T_{ij} - t)(U_{ij}Y_i - C(t)) \|_{L^2} + \| \sum_{i=1}^{n} \sum_{j=1}^{n_i} w_i K_h(T_{ij} - t)\epsilon_{ij}Y_i \|_{L^2}
\]

\[
= O_p \left( \left( \frac{1}{nh} + h^2 \right)^{1/2} \right) + O_p(nh^{-1/2}) + O_p(a_n)
\]

\[
= O_p \left( \left( \frac{1}{nh} + h^2 \right)^{1/2} \right) + O_p(a_n),
\]

where the last equality follows from Lemma S12 and Lemma S13. Similarly

\[
\| \hat{R}_1(t) - C(t)S_1(t) \|_{L^2} \leq \| \sum_{i=1}^{n} \sum_{j=1}^{n_i} w_i K_h(T_{ij} - t) \left( \frac{T_{ij} - t}{h} \right)(U_{ij}Y_i - C(t)) \|_{L^2}
\]

\[
= O_p \left( \left( \frac{1}{nh} + h^2 \right)^{1/2} \right) + O_p(a_n).
\]

These along with (S.122) and similar arguments as in the proof of Theorem 4.1 in Zhang and Wang (2016) show that \( S_0(t)S_2(t) - S_2^2(t) \) is positive and bounded away from 0 with probability tending to 1 and \( \sup_{t \in T} |S_r(t)| = O_p(1), r = 1, 2 \). The result then follows.

**Lemma S15.** Suppose that (S4), (B1)-(B2), (B4)-(B5), (A1)–(A8) in the Appendix hold and consider a sparse design with \( n_i \leq N_0 < \infty \), setting \( a_n = a_{n1} \) and \( b_n = b_{n1} \). Then, setting \( \tau_M = \sum_{m=1}^{M} \frac{1}{\lambda_m} \), for large enough \( n \), the following relations hold almost surely,
\[
\sum_{m=1}^{M} \frac{\hat{\sigma}_m - \sigma_m}{\lambda_m} = \tau_M \| \hat{C} - C \|_{L^2} + \tau_M^{1/2} O(c_n^p), \tag{S.123}
\]

\[
\sum_{m=1}^{M} \frac{\hat{\sigma}_m - \sigma_m}{\lambda_m} \left| \frac{\hat{\lambda}_m - \lambda_m}{\lambda_m} \right| \leq O(c_n^{2\rho}) + \| \hat{C} - C \|_{L^2} \tau_M^{1/2} O(c_n^p), \tag{S.124}
\]

\[
\sum_{m=1}^{M} \frac{\sigma_m}{\lambda_m} \left| \frac{\hat{\lambda}_m - \lambda_m}{\lambda_m} \right| \leq O(c_n) \tau_M, \tag{S.125}
\]

\[
\sum_{m=1}^{M} \frac{\hat{\sigma}_m - \sigma_m}{\lambda_m} \| \hat{\phi}_m - \phi_m \|_{L^2} \leq O(c_n^{2\rho}) + O(c_n^p) (\| \hat{C} - C \|_{L^2} + c_n) \tau_M^{1/2}, \tag{S.126}
\]

\[
\sum_{m=1}^{M} \frac{\sigma_m}{\lambda_m} \| \hat{\phi}_m - \phi_m \|_{L^2} \leq O(c_n) v_M, \tag{S.127}
\]

**Proof.** First note

\[
\sum_{m=1}^{M} \frac{1}{\delta_m} \leq \left( \sum_{m=1}^{M} \frac{1}{\lambda_m^2} \right)^{1/2} \left( \sum_{m=1}^{M} \lambda_m \right)^{1/2} \leq \left( \sum_{m=1}^{M} \frac{1}{\sqrt{\lambda_m} \delta_m} \right) \left( \sum_{m=1}^{\infty} \lambda_m \right)^{1/2} = O(c_n^{1\rho}) - 1,
\]

implying \( c_n v_M = O(c_n^p) = o(1) \) as \( n \to \infty \). From the proof of Theorem 7, \( \| \hat{\Xi} - \Xi \|_{\text{op}} = O(a_n + b_n) \) a.s. Note that from the orthonormality of the \( \phi_k \) and using perturbation results (Bosq 2000), we have \( \| \hat{\phi}_k - \phi_k \|_{L^2} \leq 2\sqrt{2} \| \hat{\Xi} - \Xi \|_{\text{op}} / \delta_k \), \( k \geq 1 \), so that for any \( m \geq 1 \)

\[
|\hat{\sigma}_m - \sigma_m| = |\langle \hat{C}, \hat{\phi}_m \rangle_{L^2} - \langle C, \phi_m \rangle_{L^2}| \leq 2\sqrt{2} \| \hat{C} - C \|_{L^2} \frac{\| \hat{\Xi} - \Xi \|_{\text{op}}}{\delta_m} + \| \hat{C} - C \|_{L^2} + 2\sqrt{2} \| C \|_{L^2} \frac{\| \hat{\Xi} - \Xi \|_{\text{op}}}{\delta_m}, \tag{S.128}
\]

and from \( \delta_m \leq \lambda_m \),

\[
\sum_{m=1}^{M} \frac{\| \hat{\Xi} - \Xi \|_{\text{op}}}{\lambda_m \delta_m} \leq \tau_M^{1/2} \sum_{m=1}^{M} \frac{\| \hat{\Xi} - \Xi \|_{\text{op}}}{\sqrt{\lambda_m} \delta_m} = \tau_M^{1/2} O(c_n^p) \quad \text{a.s.} \tag{S.129}
\]

Thus

\[
\sum_{m=1}^{M} \frac{|\hat{\sigma}_m - \sigma_m|}{\lambda_m} \leq \tau_M^{1/2} O(c_n^{1\rho}) \| \hat{C} - C \|_{L^2} + \tau_M \| \hat{C} - C \|_{L^2} + \tau_M^{1/2} O(c_n^{1\rho}) \quad \text{a.s.}
\]

\[
= \tau_M \| \hat{C} - C \|_{L^2} + \tau_M^{1/2} O(c_n^{1\rho}),
\]

which shows the first result in (S.123). Next, since \( M = M(n) \) is such that \( \sum_{m=1}^{M} \frac{1}{\sqrt{\lambda_m} \delta_m} = O(c_n^{1\rho}) = o(1) \) as \( n \to \infty \), analogous arguments as in the proof of Theorem 7 show that
there exists \( n_0 \geq 1 \) such that for all \( n \geq n_0 \) it holds that \( \|\hat{\Xi} - \Xi\|_{op} \leq \lambda_M/2 \) a.s.. Then \( |\hat{\lambda}_m - \lambda_m| \leq \|\hat{\Xi} - \Xi\|_{op} \) implies \( |\hat{\lambda}_m| \geq \lambda_m/2 \) a.s. for large enough \( n \). With (S.128), (S.129),

\[
\sum_{m=1}^{M} |\sigma_m| \frac{|\hat{\lambda}_m - \lambda_m|}{|\lambda_m| \lambda_m} \leq 2 \sum_{m=1}^{M} |\sigma_m| \frac{\|\hat{\Xi} - \Xi\|_{op}}{\lambda_m^2}
\leq 4\sqrt{2}\|\hat{C} - C\|_{L^2} \sum_{m=1}^{M} \frac{\|\hat{\Xi} - \Xi\|_{op}^2}{\lambda_m^2 \delta_m} + 2\|\hat{C} - C\|_{L^2} \sum_{m=1}^{M} \frac{\|\hat{\Xi} - \Xi\|_{op}^2}{\lambda_m^2}
+ 4\sqrt{2}\|C\|_{L^2} \sum_{m=1}^{M} \frac{\|\hat{\Xi} - \Xi\|_{op}^2}{\lambda_m^2 \delta_m}
\leq \|\hat{C} - C\|_{L^2} O(c_n^{2\rho}) + \|\hat{C} - C\|_{L^2} \tau_M^{1/2} O(c_n^2) + O(c_n^{2\rho}) \quad \text{a.s.}
= O(c_n^{2\rho}) + \|\hat{C} - C\|_{L^2} \tau_M^{1/2} O(c_n^2) \quad \text{a.s.,}
\]

for large enough \( n \), implying the second result in (S.124). Similarly, for large enough \( n \) and a.s.

\[
\sum_{m=1}^{M} |\sigma_m| \frac{|\hat{\lambda}_m - \lambda_m|}{|\lambda_m| \lambda_m} \leq 2 \sum_{m=1}^{M} |\sigma_m| \frac{\|\hat{\lambda}_m - \lambda_m|}{\lambda_m^2} \leq O(c_n) \left( \sum_{m=1}^{M} \frac{\sigma_m^2}{\lambda_m^2} \right)^{1/2} \tau_M = O(c_n) \tau_M,
\]

where the last equality is due to \( \sum_{m=1}^{\infty} \sigma_m^2 / \lambda_m^2 < \infty \). This shows the third result in (S.125).

Next,

\[
\sum_{m=1}^{M} \frac{|\hat{\sigma}_m|}{\lambda_m} \frac{2|\hat{\lambda}_m - \lambda_m|}{|\lambda_m| \lambda_m} \|\hat{\phi}_m - \phi_m\|_{L^2} \leq 2 \sum_{m=1}^{M} \frac{|\hat{\sigma}_m - \sigma_m|}{\lambda_m} \|\hat{\phi}_m - \phi_m\|_{L^2} + \sum_{m=1}^{M} \frac{|\sigma_m|}{\lambda_m} \|\hat{\lambda}_m - \lambda_m|}{|\lambda_m| \lambda_m} \|\hat{\phi}_m - \phi_m\|_{L^2}
+ \sum_{m=1}^{M} \frac{|\hat{\sigma}_m - \sigma_m|}{\lambda_m} \|\hat{\lambda}_m - \lambda_m|}{|\lambda_m| \lambda_m} \|\hat{\phi}_m - \phi_m\|_{L^2}.
\]

From (S.128), (S.129) and using that \( \|\hat{\phi}_m - \phi_m\|_{L^2} \leq 2\sqrt{2}\|\hat{\Xi} - \Xi\|_{op}/\delta_m \), we obtain

\[
\sum_{m=1}^{M} \frac{|\hat{\sigma}_m - \sigma_m|}{\lambda_m} \|\hat{\phi}_m - \phi_m\|_{L^2} \leq 8\|\hat{\Xi} - \Xi\|_{L^2} \sum_{m=1}^{M} \frac{|\hat{\sigma}_m - \sigma_m|}{\lambda_m} \|\hat{\Xi} - \Xi\|_{op}^2 + 2\sqrt{2}\|\hat{C} - C\|_{L^2} \sum_{m=1}^{M} \frac{\|\hat{\Xi} - \Xi\|_{op}^2}{\lambda_m \delta_m}
\leq \|\hat{C} - C\|_{L^2} O(c_n^{2\rho}) + \|\hat{C} - C\|_{L^2} \tau_M^{1/2} O(c_n^2) + O(c_n^{2\rho}) \quad \text{a.s.}
= O(c_n^{2\rho}) + \|\hat{C} - C\|_{L^2} \tau_M^{1/2} O(c_n^2).
\]
Next, for large enough $n$, 
\[
\sum_{m=1}^{M} \frac{|\sigma_m| |\hat{\lambda}_m - \lambda_m|}{|\hat{\lambda}_m| \lambda_m} \|\hat{\phi}_m - \phi_m\|_{L^2} \leq 4\sqrt{2} \sum_{m=1}^{M} |\sigma_m| \|\hat{\Xi} - \Xi\|_{op}^2 \lambda_m^2 \delta_m \quad \text{a.s.}
\leq \left( \sum_{m=1}^{M} \frac{\sigma_m^2}{\lambda_m^2} \right)^{1/2} O(c_n^{1+\rho}) \tau_M^{1/2} \quad \text{a.s.}
= O(c_n^{1+\rho}) \tau_M^{1/2}.
\]  
(S.132)

Similarly, from (S.128) we obtain 
\[
\sum_{m=1}^{M} \frac{|\hat{\sigma}_m - \sigma_m||\hat{\lambda}_m - \lambda_m|}{|\hat{\lambda}_m| \lambda_m} \|\hat{\phi}_m - \phi_m\|_{L^2} \leq 4\sqrt{2} \sum_{m=1}^{M} |\sigma_m| \|\hat{\Xi} - \Xi\|_{op}^2 \lambda_m^2 \delta_m \leq 16 \|\hat{C} - C\|_{L^2} \sum_{m=1}^{M} \frac{\|\hat{\Xi} - \Xi\|_{op}^3}{\lambda_m^2 \delta_m^2} + 4\sqrt{2} \|\hat{C} - C\|_{L^2} \sum_{m=1}^{M} \frac{\|\hat{\Xi} - \Xi\|_{op}^2}{\lambda_m^2 \delta_m} + 16 \|C\|_{L^2} \sum_{m=1}^{M} \frac{\|\hat{\Xi} - \Xi\|_{op}^3}{\lambda_m^2 \delta_m^2} \leq O(c_n^{1+2\rho}) \tau_M \|\hat{C} - C\|_{L^2} + O(c_n^{2\rho}) \|\hat{C} - C\|_{L^2} + O(c_n^{1+2\rho}) \tau_M \quad \text{a.s.}
= O(c_n^{1+2\rho}) \tau_M + O(c_n^{2\rho}) \|\hat{C} - C\|_{L^2} \quad \text{a.s.}
\]  
(S.133)

Combining (S.130), (S.131), (S.132) and (S.133) with the fact that $c_n \tau_M \leq c_n v_M = o(1)$ as $n \to \infty$, which was already shown, leads to the fourth result in (S.126). Finally 
\[
\sum_{m=1}^{M} \frac{|\sigma_m|}{\lambda_m} \|\hat{\phi}_m - \phi_m\|_{L^2} \leq 2\sqrt{2} \sum_{m=1}^{M} \frac{|\sigma_m|}{\lambda_m^2 \delta_m} \|\hat{\Xi} - \Xi\|_{op} \leq \left( \sum_{m=1}^{M} \frac{\sigma_m^2}{\lambda_m^2} \right)^{1/2} \|\hat{\Xi} - \Xi\|_{op} v_M = O(c_n) v_M,
\]
a.s., which shows the last result in (S.127).

The next lemma provides the $L^2$ convergence of the empirical estimate $\hat{\beta}_M$ towards $\beta$, which is required to construct the estimated predictive distribution $\hat{P}_{iK}$. Recall that 
\[
\hat{\beta}_M(t) := \sum_{m=1}^{M} \frac{\hat{\sigma}_m}{\hat{\lambda}_m} \hat{\phi}_m(t), \quad t \in \mathcal{T},
\]
\[
\Theta_M = \left\| \sum_{m=1}^{M} \frac{\sigma_m}{\lambda_m} \phi_m \right\|_{L^2} \quad \text{and} \quad \tau_M = \sum_{m=1}^{M} \lambda_m^{-1}.
\]

**Lemma S16.** Suppose that (S4), (B1)-(B2), (B4)-(B5), (A1)–(A8) in the Appendix hold and consider a sparse design with $n_i \leq N_0 < \infty$, setting $a_n = a_{n1}$ and $b_n = b_{n1}$. Let $K \geq 1$. Then 
\[
\|\hat{\beta}_M - \beta\|_{L^2} = O_p(r_n),
\]  
(S.134)
and

$$\int_T \hat{\beta}_M(t) \hat{\phi}_k(t) dt = \int_T \beta(t) \phi_k(t) dt + O_p(r_n),$$  

(S.135)

where \( r_n = c_n v_M + c_n^{\rho \frac{1}{2}} + \tau_M \left[ \left( \frac{1}{\bar{m}^2} + h^2 \right)^{1/2} + a_n \right] + \Theta_M \) and \( k = 1, \ldots, K \).

Proof. Observe

$$\|\hat{\beta}_M - \beta\|_{L^2} \leq \sum_{m=1}^{M} \left\| \frac{\hat{\sigma}_m}{\hat{\lambda}_m} - \frac{\sigma_m}{\lambda_m} \phi_m \right\|_{L^2} + \sum_{m=M+1}^{\infty} \left\| \frac{\sigma_m}{\hat{\lambda}_m} \phi_m \right\|_{L^2},$$  

(S.136)

and

$$\sum_{m=1}^{M} \left\| \frac{\hat{\sigma}_m}{\hat{\lambda}_m} - \frac{\sigma_m}{\lambda_m} \phi_m \right\|_{L^2} \leq \sum_{m=1}^{M} \left| \frac{\hat{\sigma}_m - \sigma_m}{\hat{\lambda}_m} \phi_m \right|_{L^2} + \sum_{m=1}^{M} \left| \frac{\sigma_m}{\hat{\lambda}_m} \phi_m \right|_{L^2} \leq \sum_{m=1}^{M} \left| \frac{\hat{\sigma}_m - \sigma_m}{\hat{\lambda}_m} \phi_m \right|_{L^2} + \sum_{m=M+1}^{\infty} \left| \frac{\sigma_m}{\hat{\lambda}_m} \phi_m \right|_{L^2},$$  

(S.137)

By the triangle inequality and Lemma S15, we have that for large enough \( n \)

$$\sum_{m=1}^{M} \left| \frac{\hat{\sigma}_m - \sigma_m}{\hat{\lambda}_m} \phi_m \right|_{L^2} \leq \sum_{m=1}^{M} \left| \frac{\hat{\sigma}_m - \sigma_m}{\hat{\lambda}_m} \phi_m \right|_{L^2} + \sum_{m=1}^{M} \left| \frac{\sigma_m}{\hat{\lambda}_m} \phi_m \right|_{L^2} \leq O(c_n^{\rho}) \tau_M^{1/2} + O(c_n) \tau_M \quad \text{a.s.}$$

where the second equality is due to \( c_n \tau_M = c_n^{\rho \frac{1}{2}} c_n^{1-\rho \frac{1}{2}} \tau_M^{1/2} = o(1) c_n^{\rho \frac{1}{2}} \tau_M^{1/2} \), and

$$\sum_{m=1}^{M} \left| \frac{\hat{\sigma}_m - \sigma_m}{\hat{\lambda}_m} \phi_m \right|_{L^2} \leq O(c_n^{2\rho}) + O(c_n) \| \hat{C} - C \|_{L^2} \tau_M^{1/2} + O(c_n) v_M.$$  

With (S.136), (S.137) and the fact that \( v_M = O(c_n^{\rho - 1}) \) as \( n \to \infty \), which was shown in the proof of Lemma S15, we arrive at

$$\|\hat{\beta}_M - \beta\|_{L^2} \leq O(c_n) v_M + O(c_n) \tau_M^{1/2} + \tau_M \| \hat{C} - C \|_{L^2} + \sum_{m=M+1}^{\infty} \left\| \frac{\sigma_m}{\hat{\lambda}_m} \phi_m \right\|_{L^2}$$

and the result in (S.134) follows from Lemma S14. Finally, recalling that \( \hat{\beta}_k = \int_T \hat{\beta}_M(t) \hat{\phi}_k(t) dt \) and \( \beta_k = \int_T \beta(t) \phi_k(t) dt \), we have

$$|\hat{\beta}_k - \beta_k| = \left| \int_T [\hat{\beta}_M(t) \hat{\phi}_k(t) - \beta(t) \phi_k(t)] dt \right| \leq \|\hat{\beta}_M - \beta\|_{L^2} \|\hat{\phi}_k - \phi_k\|_{L^2} + \|\hat{\beta}_M - \beta\|_{L^2} \|\hat{\phi}_k - \phi_k\|_{L^2} = O_p(r_n + a_n + b_n) = O_p(r_n),$$

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where the second equality is due to the fact that \( \| \hat{\phi}_k - \phi_k \|_{L^2} \leq O(a_n + b_n) \) a.s., which follows from the proof of Lemma S15. This shows the second result in (S.135).

We remark that when choosing the optimal bandwidth \( h \asymp n^{-1/3} \), then the rate \( \tau_M[(nh)^{-1} + h^2]^{1/2} + a_n \) is faster than \( c_n v_M \) and thus the rate \( r_n \) is equivalent to \( a_n \) defined as in Theorem 9.

Recall that \( \mathcal{P}_{iK} \) corresponds to the true predictive distribution \( \eta_{iK}|X_i, T_i \), or equivalently \( N(\beta_0 + \beta_K^T \xi_{iK}, \beta_K^T \Sigma_{iK} \beta_K) \), while \( \hat{\mathcal{P}}_{iK} \overset{d}{=} N(\hat{\beta}_0 + \hat{\beta}_K^T \hat{\xi}_{iK}, \hat{\beta}_K^T \hat{\Sigma}_{iK} \hat{\beta}_K) \) corresponds to an intermediate target, replacing population quantities by their estimated counterparts but keeping the true intercept and slope coefficients \( \beta_0 \) and \( \beta_K \). Also \( \hat{\mathcal{P}}_{iK} \) corresponds to the estimated predictive distribution, i.e. \( \hat{\mathcal{P}}_{iK} \overset{d}{=} N(\hat{\beta}_0 + \hat{\beta}_K^T \hat{\xi}_{iK}, \hat{\beta}_K^T \hat{\Sigma}_{iK} \hat{\beta}_K) \). Finally, recall that \( F_{iK}(t), \tilde{F}_{iK}(t) \) and \( \hat{F}_{iK} \) are the distribution functions associated with \( \mathcal{P}_{iK}, \hat{\mathcal{P}}_{iK} \) and \( \mathcal{P}_{iK} \), respectively.

**Proof of Theorem 9.** Recall that for a normal random variable \( Z_1 \sim N(\kappa_1, \kappa_2^2) \) and \( t \in (0, 1) \) it holds that \( Q_1(t) = \kappa_2 q(t) + \kappa_1 \), where \( Q_1(\cdot) \) and \( q(\cdot) \) are the quantile functions corresponding to \( Z_1 \) and a standard normal random variate, respectively. Note that since \( |\lambda_{\min}(\hat{\Sigma}_{iK}) - \lambda_{\min}(\Sigma_{iK})| \leq \| \Sigma_{iK} - \hat{\Sigma}_{iK} \|_{op,2} = o_p(1) \), where the \( o_p(1) \) term is uniform in \( i \) (see the proof of Lemma S5), and \( \lambda_{\min}(\Sigma_{iK}) \geq \kappa_0 \) a.s., we have

\[
P\left( \| \Sigma_{iK} - \hat{\Sigma}_{iK} \|_{op,2} \leq \kappa_0/2 \right) = P\left( \kappa_0 - \| \hat{\Sigma}_{iK} - \Sigma_{iK} \|_{op,2} \geq \kappa_0/2 \right)
\leq P\left( \lambda_{\min}(\Sigma_{iK}) - \| \Sigma_{iK} - \hat{\Sigma}_{iK} \|_{op,2} \geq \kappa_0/2 \right)
\leq P\left( \lambda_{\min}(\Sigma_{iK}) \geq \kappa_0/2 \right),
\]

which implies \( \lambda_{\min}(\hat{\Sigma}_{iK}) \geq \kappa_0/2 \) with probability tending to 1. For the remainder of the proof we work on this event. From the closed form expression for the 2-Wasserstein distance between one-dimensional distributions with finite second moments,

\[
W_2^2(\tilde{\mathcal{P}}_{iK}, \mathcal{P}_{iK}) = \int_0^1 \left( (\beta_K^T \Sigma_{iK} \beta_K)^{1/2} - (\beta_K^T \Sigma_{iK} \beta_K)^{1/2} q(t) + \beta_K^T (\hat{\xi}_{iK} - \hat{\xi}_{iK}) \right)^2 dt
= \left( (\beta_K^T \Sigma_{iK} \beta_K)^{1/2} - (\beta_K^T \Sigma_{iK} \beta_K)^{1/2} \right)^2 \int_0^1 q^2(t) dt + (\beta_K^T (\hat{\xi}_{iK} - \hat{\xi}_{iK}))^2
+ 2[(\beta_K^T \Sigma_{iK} \beta_K)^{1/2} - (\beta_K^T \Sigma_{iK} \beta_K)^{1/2}] (\beta_K^T (\hat{\xi}_{iK} - \hat{\xi}_{iK})) \int_0^1 q(t) dt
\leq \frac{(\beta_K^T (\Sigma_{iK} - \Sigma_{iK} \beta_K))^2}{\beta_K^T \Sigma_{iK} \beta_K} \int_0^1 q^2(t) dt + (\beta_K^T (\hat{\xi}_{iK} - \hat{\xi}_{iK}))^2,
\] (S.138)
where the last inequality follows from the fact that $\int_0^1 q(t)dt = E(Z) = 0$, where $Z \sim N(0, 1)$, and using the inequality $(\sqrt{x} - \sqrt{y})^2 \leq (x - y)^2 / y$ which is valid for any scalars $x \geq 0$ and $y > 0$. Since $\int_0^1 q^2(t)dt = E(Z^2) < \infty$, it then suffices to control the terms $\beta_K^T (\hat{\Sigma}_iK - \Sigma_iK)\beta_K$ and $(\beta_K^T (\hat{\xi}_iK - \xi_iK))^2$. From the proof of Lemma S5, we have $\|\Sigma_iK - \hat{\Sigma}_iK\|_F = O(a_n + b_n)$ a.s. as $n \to \infty$, where the $O(a_n + b_n)$ term is uniform over $i$, and from the proof of Theorem 5 we have $|\xi_{ik} - \hat{\xi}_{ik}| = O_p(a_n + b_n)$, $k = 1, \ldots, K$. Thus, $(\beta_K^T (\xi_iK - \hat{\xi}_iK))^2 \leq \|\beta_K\|^2 \|\xi_iK - \hat{\xi}_iK\|^2 = O_p((a_n + b_n)^2)$ and properties of the operator norm show that $|\beta_K^T (\hat{\Sigma}_iK - \Sigma_iK)\beta_K| \leq \|\beta_K\|^2 \|\hat{\Sigma}_iK - \Sigma_iK\|_F = O(a_n + b_n)$ a.s. as $n \to \infty$. This along with (S.138) leads to

$$W_2(\hat{P}_{ik}, P_{ik}) = O_p(a_n + b_n).$$

(S.139)

Similar arguments show that

$$W_2^2(\hat{P}_{ik}, \tilde{P}_{ik}) \leq \frac{(\beta_K^T \hat{\Sigma}_iK \beta_K - \beta_K^T \Sigma_iK \beta_K)^2}{\beta_K^T \Sigma_iK \beta_K} \int_0^1 q^2(t)dt + ((\hat{\beta}_K - \beta_K)^T \hat{\xi}_iK + \hat{\beta}_0 - \beta_0)^2,$$

(S.140)

and

$$|\beta_K^T \hat{\Sigma}_iK \beta_K - \beta_K^T \Sigma_iK \beta_K| = |(\beta_K - \beta_K)^T \Sigma_iK \beta_K + \beta_K^T \Sigma_iK (\beta_K - \beta_K)|$$

$$\leq \left\| \beta_K - \beta_K \right\|^2_2 \left\| \Sigma_iK - \Sigma_iK \right\|_{op, 2} + \left\| \hat{\beta}_K - \beta_K \right\|^2_2 \left\| \hat{\Sigma}_iK - \Sigma_iK \right\|_{op, 2} + \left\| \hat{\beta}_K - \beta_K \right\|^2_2 \left\| \hat{\Sigma}_iK \right\|_{op, 2} \left\| \beta_K \right\|^2_2$$

$$= O_p(a_n),$$

(S.141)

where the first inequality follows from properties of the operator norm and the last equality is due to Lemma S16 along with the fact that $h \asymp n^{-1/3}$ implies that the rate $\tau_M \left[ \left( \frac{1}{m^2} + h^2 \right)^{1/2} + a_n \right]$ is faster than $c_n v_M$, $\left\| \hat{\Sigma}_iK - \Sigma_iK \right\|_F = O(a_n + b_n)$ a.s. as $n \to \infty$ and that $\left\| \Sigma_iK \right\|_{op, 2}$ is uniformly bounded in $i$ in the sparse case. Since $|\lambda_{\min}(\hat{\Sigma}_iK) - \lambda_{\min}(\Sigma_iK)| \leq \left\| \hat{\Sigma}_iK - \Sigma_iK \right\|_{op, 2}$, we have $\beta_K^T \hat{\Sigma}_iK \beta_K \geq \beta_K^T \beta_K \lambda_{\min}(\Sigma_iK) \geq \beta_K^T \beta_K (\lambda_{\min}(\Sigma_iK) - \left\| \hat{\Sigma}_iK - \Sigma_iK \right\|_{op, 2})^1 \{\lambda_{\min}(\Sigma_iK) \geq \|\hat{\Sigma}_iK - \Sigma_iK\|_{op, 2}\}$. Thus, using that $\left\| \Sigma_iK - \hat{\Sigma}_iK \right\|_{op, 2} = o_p(1)$, where the $o_p(1)$ term is uniform in $i$, $\lambda_{\min}(\Sigma_iK) \geq \kappa_0$ a.s., and writing

$$p_0 = P \left[ \frac{1}{\beta_K^T \Sigma_iK \beta_K} \leq \frac{2}{\beta_K^T \beta_K \lambda_{\min}(\Sigma_iK)} \right] \quad \text{and} \quad \lambda_{\min}(\hat{\Sigma}_iK) \geq \kappa_0 / 2,$$
it follows that
\[
p_0 \geq P[\beta_K^T \beta_K \lambda_{\min}(\Sigma_i K) \leq 2 \beta_K^T \beta_K \lambda_{\min}(\hat{\Sigma}_i K) \text{ and } \lambda_{\min}(\hat{\Sigma}_i K) \geq \kappa_0/2] \\
\geq P[\beta_K^T \beta_K \lambda_{\min}(\Sigma_i K) \leq 2 \beta_K^T \beta_K (\lambda_{\min}(\Sigma_i K) - \|\hat{\Sigma}_i K - \Sigma_i K\|_{\text{op}, 2}) \text{ and } \lambda_{\min}(\hat{\Sigma}_i K) \geq \kappa_0/2] \\
\geq P[\|\hat{\Sigma}_i K - \Sigma_i K\|_{\text{op}, 2} > \kappa_0/2] - P[\lambda_{\min}(\hat{\Sigma}_i K) < \kappa_0/2].
\]

This implies \( p_0 \to 1 \) as \( n \to \infty \) and hence the event \((\beta_K^T \hat{\Sigma}_i K \beta_K)^{-1} \leq 2(\beta_K^T \beta_K \lambda_{\min}(\Sigma_i K))^{-1} \) with \( \lambda_{\min}(\hat{\Sigma}_i K) \geq \kappa_0/2 \) occurs with probability tending to 1. It then suffices to work on this event in what follows. Combining with (S.140), (S.141) and
\[
\|\beta_K - \beta_K\|^2 \xi_{iK} + \beta_0 - \beta_0 \leq \|\beta_K - \beta_K\|^2 (\|\xi_{iK} - \hat{\xi}_{iK}\|^2 + \|\hat{\xi}_{iK}\|^2) + |\hat{\beta}_0 - \beta_0| = O_p(\alpha_n),
\]
which follows from Lemma S16 and the facts that \( \hat{\beta}_0 - \beta_0 = \bar{Y}_n - E(Y) = O_p(n^{-1/2}), \)
\[
\|\hat{\xi}_{iK} - \xi_{iK}\|^2 = O_p(a_n + b_n) \text{ and } \|\hat{\xi}_{iK}\|^2 = O_p(1) \text{ hold uniformly in } i,
\]
then leads to
\[
W_2(\hat{P}_{iK}, \bar{P}_{iK}) = O_p(\alpha_n). \tag{S.142}
\]

The result in (18) then follows from (S.139) and (S.142).

Next, denote by \( \phi \) and \( \Phi \) the density and cdf of a standard normal random variable, and define the quantities \( \bar{u}_{in} = \beta_0 + \beta_K^T \hat{\xi}_{iK}, \bar{\sigma}_{in} = (\beta_K^T \hat{\Sigma}_i K \beta_K)^{1/2}, u_i = \beta_0 + \beta_K^T \hat{\xi}_{iK}, \sigma_i = (\beta_K^T \Sigma_i K \beta_K)^{1/2} \) and \( \Delta_{in}(t) = (t - u_i)/\sigma_i \) and \( (t - \bar{u}_{in})/\bar{\sigma}_{in}, t \in \mathbb{R}. \) Then
\[
\sup_{t \in \mathbb{R}} |\bar{F}_{iK}(t) - F_{iK}(t)| = \sup_{t \in \mathbb{R}} \left| \phi \left( \frac{t - \bar{u}_{in}}{\bar{\sigma}_{in}} \right) - \Phi \left( \frac{t - u_i}{\sigma_i} \right) \right| \leq \sup_{t \in \mathbb{R}} |\phi(\varepsilon_s)\Delta_{in}(t)|, \tag{S.143}
\]
where the second equality follows by a Taylor expansion and \( \varepsilon_s \) is between \((t - \bar{u}_{in})/\bar{\sigma}_{in} \) and \((t - \mu_i)/\sigma_i \). Defining \( r_{in}(t) = (t - \bar{u}_{in})/\bar{\sigma}_{in}, r_i(t) = (t - u_i)/\sigma_i \) and setting \( I_{in} = [\min \{u_i, \bar{u}_{in}\}, \max \{u_i, \bar{u}_{in}\}], \)
\[
|\phi(\varepsilon_s)\Delta_{in}(t)| \leq \phi(0)|\Delta_{in}(t)|\mathbf{1}_{\{t \in I_{in}\}} + \phi(\min \{|r_{in}(t)|, |r_i(t)|\})|\Delta_{in}(t)|\mathbf{1}_{\{t \notin I_{in}\}} \\
\leq \phi(0)|\Delta_{in}(t)|\mathbf{1}_{\{t \in I_{in}\}} + [\phi(r_{in}(t)) + \phi(r_i(t))]|\Delta_{in}(t)|. \tag{S.144}
\]

Since \( \bar{u}_{in} - u_i = O_p(a_n + b_n), |\bar{\sigma}_{in} - \sigma_i| \leq |\bar{\sigma}_{in}^2 - \sigma_i^2|/\sigma_i = O_p(a_n + b_n), |\bar{\sigma}_{in}^{-1} - \sigma_i^{-1}| \leq |\bar{\sigma}_{in} - \sigma_i| \leq \bar{\sigma}_{in} - \sigma_i \)
\[
\sigma_i/(\tilde{\sigma}_{in}\sigma_i) \leq |\hat{\sigma}_{in} - \sigma_i|/\sqrt{2}(\beta_{\tilde{K}}^T \beta_{\tilde{K}} \lambda_{\min}(\Sigma_{iK}))^{-1/2}\sigma_i^{-1}
\]
and \(\lambda_{\min}(\Sigma_{iK}) \geq \kappa_0\) a.s., it follows that
\[
|\Delta_{in}(t)| = |(t - u_i)\sigma_i - (t - \tilde{u}_{in})|/\sigma_i
\]
\[
\leq \frac{1}{\sigma_i}|\tilde{u}_{in} - u_i| + |t - u_i| \left| \frac{1}{\tilde{\sigma}_{in}} - \frac{1}{\sigma_i} \right| + |\tilde{u}_{in} - u_i| \left| \frac{1}{\tilde{\sigma}_{in}} - \frac{1}{\sigma_i} \right|
\]
\[
= O_p(a_n + b_n) + O_p(a_n + b_n)|t - u_i|,
\] (S.145)
where both \(O_p(a_n + b_n)\) terms are uniform in \(t\). This implies
\[
\sup_{t \in \mathbb{R}} |\Delta_{in}(t)|1_{\{t \in I_{in}\}} \leq O_p(a_n + b_n) + O_p(a_n + b_n)|\tilde{u}_{in} - u_i| = O_p(a_n + b_n).
\] (S.146)
Since \(\|\Sigma_{iK}\|_{op}\) is uniformly bounded above in the sparse case, it is easy to show that \(\varphi(r_i(t))|t - u_i| \leq O(1)\), where the \(O(1)\) term is uniform in both \(t\) and \(i\). This combined with (S.145) leads to
\[
\sup_{t \in \mathbb{R}} \varphi(r_i(t))|\Delta_{in}(t)| = O_p(a_n + b_n).
\] (S.147)
Next, from (S.145) we have
\[
\sup_{t \in \mathbb{R}} \varphi(r_{in}(t))|\Delta_{in}(t)| \leq O_p(a_n + b_n) + O_p(a_n + b_n) \sup_{t \in \mathbb{R}} \varphi(r_{in}(t))|t - u_i|,
\]
and the result then follows from (S.143), (S.144), (S.146) and (S.147) if we can show that \(\varphi(r_{in}(t))|t - u_i| = O_p(1)\) uniformly in \(t\). It is easy to see that
\[
\varphi(r_{in}(t))|t - u_i| \leq \varphi(r_{in}(t_1^*))(t_1^* - u_i)1_{\{t_1^* \leq u_i\}} + \varphi(r_{in}(t_2^*))(u_i - t_2^*)1_{\{t_2^* \leq u_i\}}
\]
\[
\leq \varphi(0)(t_1^* - t_2^*),
\]
where \(t_1^* = (u_i + \tilde{u}_{in} + \sqrt{(u_i - \tilde{u}_{in})^2 + 4\tilde{\sigma}^2_{in}})/2\) and \(t_2^* = (u_i + \tilde{u}_{in} - \sqrt{(u_i - \tilde{u}_{in})^2 + 4\tilde{\sigma}^2_{in}})/2\). Since \(\tilde{\sigma}_{in}\) is uniformly upper bounded in the sparse setting and \(\tilde{u}_{in} - u_i = O_p(a_n + b_n)\), we obtain
\[
\sup_{t \in \mathbb{R}} \varphi(r_{in}(t))|t - u_i| \leq \varphi(0)\sqrt{(u_i - \tilde{u}_{in})^2 + 4\tilde{\sigma}^2_{in}} = O_p(1).
\]
Therefore
\[
\sup_{t \in \mathbb{R}} |\tilde{F}_{iK}(t) - F_{iK}(t)| = O_p(a_n + b_n),
\] (S.148)
so that it then remains to control the term \( \sup_{t \in \mathbb{R}} |\hat{F}_{iK}(t) - \tilde{F}_{iK}(t)| \). For this purpose, define auxiliary quantities \( \hat{u}_{in} = \hat{\beta}_0 + \hat{\beta}_K^T \hat{\xi}_{IK} \), \( \hat{\sigma}_{in} = (\hat{\beta}_K^T \hat{\Sigma}_{iK} \hat{\beta}_K)^{1/2} \) and \( \hat{\Delta}_{in}(t) = (t - \hat{u}_{in})/\hat{\sigma}_{in} - (t - \tilde{u}_{in})/\tilde{\sigma}_{in}, \) \( t \in \mathbb{R} \). From Lemma S16 it follows that \( \hat{u}_{in} - \tilde{u}_{in} = \hat{\beta}_0 - \tilde{\beta}_0 + (\hat{\beta}_K - \beta_K)^T (\hat{\xi}_{IK} - \xi_{IK}) + (\hat{\beta}_K - \beta_K)^T \xi_{IK} = O_p(\alpha_n), |\hat{\sigma}_{in} - \tilde{\sigma}_{in}| \leq |\hat{\sigma}_{in}^2 - \tilde{\sigma}_{in}^2|/\hat{\sigma}_{in} = O_p(\alpha_n) \), which is due to (S.141) and since \( \tilde{\sigma}_{in}^{-1} \leq \sqrt{2}(\beta_K^T \beta_K \lambda_{\min}(\Sigma_{iK}))^{-1/2} \). Also, from (S.141) and using \( \lambda_{\min}(\Sigma_{iK}) \geq \kappa_0 \) a.s. we have \( |\hat{\sigma}_{in} - \tilde{\sigma}_{in}| \leq |\hat{\sigma}_{in}^2 - \tilde{\sigma}_{in}^2|/\hat{\sigma}_{in} \leq |\hat{\sigma}_{in}^2 - \tilde{\sigma}_{in}^2|/\sqrt{2}(\beta_K^T \beta_K \lambda_{\min}(\Sigma_{iK}))^{-1/2} = o_p(1) \) and then \( |\hat{\sigma}_{in} - \sigma_i| \leq |\hat{\sigma}_{in} - \tilde{\sigma}_{in}| + |\tilde{\sigma}_{in} - \sigma_i| = o_p(1) \). This along with the fact that \( \hat{\sigma}_{in} \geq \|\beta_K\|_2 \kappa_0/2 \geq \|\beta_K\|_2 \kappa_0/4 \) holds with probability tending to 1 implies \( \hat{\sigma}_{in}^{-1} \leq 2\sigma_i^{-1} \) with probability tending to 1 as \( n \to \infty \). Combining this with \( \lambda_{\min}(\Sigma_{iK}) \geq \kappa_0 \) a.s. then leads to

\[
\left| \frac{1}{\hat{\sigma}_{in}} - \frac{1}{\tilde{\sigma}_{in}} \right| = O_p(\alpha_n),
\]

where the bound is uniform in \( i \), and similarly as in (S.145) we obtain

\[
|\hat{\Delta}_{in}(t)| \leq |t - \hat{u}_{in}| \left| \frac{1}{\hat{\sigma}_{in}} - \frac{1}{\tilde{\sigma}_{in}} \right| + |\hat{u}_{in} - \tilde{u}_{in}| \left| \frac{1}{\hat{\sigma}_{in}} - \frac{1}{\tilde{\sigma}_{in}} \right| + |\hat{u}_{in} - \tilde{u}_{in}| \frac{1}{\tilde{\sigma}_{in}} \leq O_p(\alpha_n) + O_p(\alpha_n)|t - \hat{u}_{in}|.
\]

Next

\[
\varphi(r_{in}(t))|t - \hat{u}_{in}| \leq \varphi(1) \sqrt{\beta_K^T \Sigma_{iK} \beta_K} \varphi(1) (\beta_K^T \beta_K)^{1/2} \left( \|\Sigma_{iK} - \Sigma_{IK}\|_{op,2} + \|\Sigma_{IK}\|_{op,2} \right)^{1/2} = O_p(1),
\]

where the \( O_p(1) \) term is uniform in both \( t \) and \( i \). This combined with (S.149) shows that

\[
\sup_{t \in \mathbb{R}} \varphi(r_{in}(t))|\hat{\Delta}_{in}(t)| = O_p(\alpha_n).
\]

Setting \( \hat{r}_{in}(t) = (t - \hat{u}_{in})/\hat{\sigma}_{in} \), similar arguments as before lead to

\[
\sup_{t \in \mathbb{R}} \varphi(\hat{r}_{in}(t))|t - \hat{u}_{in}| \leq \varphi(0) \sqrt{(\hat{u}_{in} - \tilde{u}_{in})^2 + 4\hat{\sigma}_{in}^2} = O_p(1),
\]

where the last equality is due to \( |\hat{u}_{in} - \tilde{u}_{in}| = O_p(\alpha_n) \) and \( \hat{\sigma}_{in}^2 \leq \beta_K^T \beta_K \left( \|\Sigma_{iK} - \Sigma_{IK}\|_{op,2} + \|\Sigma_{IK}\|_{op,2} \right) = O_p(1) \). With (S.149) this implies

\[
\sup_{t \in \mathbb{R}} \varphi(\hat{r}_{in}(t))|\hat{\Delta}_{in}(t)| = O_p(\alpha_n).
\]
Setting $\hat{I}_{in} = [\min\{\hat{u}_{in}, \tilde{u}_{in}\}, \max\{\hat{u}_{in}, \tilde{u}_{in}\}]$, then similar arguments as the ones outlined in (S.143) and (S.144) shows that
\[
\sup_{t \in \mathbb{R}} |\hat{F}_{iK}(t) - \tilde{F}_{iK}(t)| \leq \varphi(0) |\hat{\Delta}_{in}(t)| 1_{\{t \in \hat{I}_{in}\}} + |\varphi(\hat{r}_{in}(t)) + \varphi(\tilde{r}_{in}(t))||\hat{\Delta}_{in}(t)|. \tag{S.152}
\]
This together with \(\sup_{t \in \mathbb{R}} |\hat{\Delta}_{in}(t)| 1_{\{t \in \hat{I}_{in}\}} \leq O_p(\alpha_n) + O_p(\alpha_n)|\hat{u}_{in} - \tilde{u}_{in}| = O_p(\alpha_n)\), where the latter follows from (S.149), as well as (S.150) and (S.151) then leads to
\[
\sup_{t \in \mathbb{R}} |\hat{F}_{iK}(t) - \tilde{F}_{iK}(t)| = O_p(\alpha_n). \tag{S.153}
\]
The result in (19) then follows from (S.148), (S.153) and the triangle inequality.

For the next result in (20), similarly as before we first start by showing that \(\|\hat{f}_{ik} - f_{ik}\|_{L^2(\mathbb{R})} = O_p(a_n + b_n)\), where \(\hat{f}_i(t) := \hat{F}'_i(t) = \varphi((t - \tilde{u}_i)/\tilde{\sigma}_i)/\tilde{\sigma}_i\). Since \(f_{i}(t) = F'_i(t) = \varphi((t - u_i)/\sigma_i)/\sigma_i\), we have
\[
\begin{align*}
\left\| \frac{1}{\tilde{\sigma}_i} \varphi \left( \frac{\cdot - \tilde{u}_i}{\tilde{\sigma}_i} \right) - \frac{1}{\sigma_i} \varphi \left( \frac{\cdot - u_i}{\sigma_i} \right) \right\|_{L^2(\mathbb{R})} & \leq \frac{1}{\tilde{\sigma}_i} \left\| \varphi \left( \frac{\cdot - \tilde{u}_i}{\tilde{\sigma}_i} \right) - \varphi \left( \frac{\cdot - u_i}{\sigma_i} \right) \right\|_{L^2(\mathbb{R})} \\
& \quad + \left| \frac{1}{\tilde{\sigma}_i} - \frac{1}{\sigma_i} \right| \left\| \varphi \left( \frac{\cdot - u_i}{\sigma_i} \right) \right\|_{L^2(\mathbb{R})}. \tag{S.154}
\end{align*}
\]
Thus, since \(\|\varphi \left( \frac{\cdot - u_i}{\sigma_i} \right)\|_{L^2(\mathbb{R})} = O(\sigma_i^{1/2})\) and \(|\tilde{\sigma}_i^{-1} - \sigma_i^{-1}| = O_p(a_n + b_n)\), we obtain
\[
\left| \frac{1}{\tilde{\sigma}_i} - \frac{1}{\sigma_i} \right| \left\| \varphi \left( \frac{\cdot - u_i}{\sigma_i} \right) \right\|_{L^2(\mathbb{R})} = O_p(a_n + b_n). \tag{S.155}
\]
Next, using the relation \(\varphi'(t) = -t\varphi(t)\) and a Taylor expansion, it follows that
\[
\left\| \varphi \left( \frac{\cdot - \tilde{u}_i}{\tilde{\sigma}_i} \right) - \varphi \left( \frac{\cdot - u_i}{\sigma_i} \right) \right\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} (\varphi'(\varepsilon_t))^2 \Delta_{in}^2(t) dt = \int_{\mathbb{R}} \varepsilon_t^2 \varphi^2(\varepsilon_t) \Delta_{in}^2(t) dt,
\]
where \(\varepsilon_t\) is between \(r_{in}(t)\) and \(r_i(t)\). Hence, from (S.154) and (S.155) it suffices to show that
\[
\int_{\mathbb{R}} \varepsilon_t^2 \varphi^2(\varepsilon_t) \Delta_{in}^2(t) dt = O_p((a_n + b_n)^2). \quad \text{Indeed, from the fact that } |\varepsilon_t| \leq |r_{in}(t)| + |r_i(t)|, \quad \text{sup}_{t \in I_{in}} |r_{in}(t)| = O_p(a_n + b_n), \quad \text{sup}_{t \in I_{in}} |r_i(t)| = O_p(a_n + b_n) \text{ and } \varphi(\varepsilon_t) 1_{\{t \in I_{in}\}} \leq \varphi(r_{in}(t)) + \varphi(r_i(t)), \quad \text{we have}
\]
\[
\begin{align*}
\int_{\mathbb{R}} \varepsilon_t^2 \varphi^2(\varepsilon_t) \Delta_{in}^2(t) dt &= \int_{I_{in}} \varepsilon_t^2 \varphi^2(\varepsilon_t) \Delta_{in}^2(t) dt + \int_{r_{in}^c} \varepsilon_t^2 \varphi^2(\varepsilon_t) \Delta_{in}^2(t) dt \\
& \leq \varphi^2(0) O_p((a_n + b_n)^5) + \int_{I_{in}} [\varphi(r_{in}(t)) + \varphi(r_i(t))]^2 (r_{in}(t) + r_i(t))^2 \Delta_{in}^2(t) dt \\
& \leq O_p((a_n + b_n)^5) + \int_{\mathbb{R}} [\varphi(r_{in}(t)) + \varphi(r_i(t))]^2 (r_{in}(t) + r_i(t))^2 \Delta_{in}^2(t) dt,
\end{align*}
\tag{S.156}
\]
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where the first inequality follows from (S.146) and the relation $|r_{in}(t)||r_i(t)|{1}_{\{t \in T_n\}} = r_{in}(t)r_i(t){1}_{\{t \in T_n\}}$.

Next, using that $\int_{\mathbb{R}} \varphi^2(s)|s|^p ds < \infty$, $p \in \mathbb{N}$, we obtain the following facts:

\[
\int_{\mathbb{R}} \varphi^2(r_{in}(t))r_{in}^2(t)\Delta_{in}^2(t)dt \leq \sigma_i^{-2}O_p((a_n + b_n)^2),
\]

\[
\int_{\mathbb{R}} \varphi^2(r_{in}(t))r_i^2(t)\Delta_{in}^2(t)dt \leq \sigma_i^{-4}O_p((a_n + b_n)^2),
\]

\[
\int_{\mathbb{R}} \varphi^2(r_{in}(t))r_{in}(t)r_i(t)\Delta_{in}^2(t)dt \leq \sigma_i^{-3}O_p((a_n + b_n)^2),
\]

\[
\int_{\mathbb{R}} \varphi^2(r_i(t))r_i(t)r_i(t)\Delta_{in}^2(t)dt \leq (\beta_K^T \beta_K \lambda_{\min}(\Sigma_{iK}))^{-3/2}O_p((a_n + b_n)^2),
\]

\[
\int_{\mathbb{R}} \varphi^2(r_i(t))r_i^2(t)\Delta_{in}^2(t)dt \leq (\beta_K^T \beta_K \lambda_{\min}(\Sigma_{iK}))^{-1}O_p((a_n + b_n)^2),
\]

\[
\int_{\mathbb{R}} \varphi^2(r_i(t))r_i^2(t)\Delta_{in}^2(t)dt \leq (\beta_K^T \beta_K \lambda_{\min}(\Sigma_{iK}))^{-2}O_p((a_n + b_n)^2),
\]

\[
\int_{\mathbb{R}} \varphi(r_{in}(t))\varphi(r_{i}(t))r_{in}(t)r_{in}(t)\Delta_{in}^2(t)dt \leq \sigma_i^{-3}O_p((a_n + b_n)^2).
\]

These facts along with (S.156) imply $\int_{\mathbb{R}} \varepsilon_i^2 \varphi^2(\varepsilon_i)\Delta_{in}^2(t)dt \leq O_p((a_n + b_n)^5) + O_p((a_n + b_n)^2) = O_p((a_n + b_n)^2)$ and

\[
\left\| \hat{f}_{iK} - f_{iK} \right\|_{L^2(\mathbb{R})} = O_p(a_n + b_n).
\]

Similar arguments imply $\|\hat{f}_{iK} - \hat{f}_{iK}\|_{L^2(\mathbb{R})} = O_p(a_n)$ and the result in (20).

Finally, from condition (C1) we have $\lambda_{\min}(\Sigma_{iK}) \geq \kappa_0$ and also $\sigma_i^2 = (\beta_K^T \Sigma_{iK} \beta_K) \geq \beta_K^T \beta_K \lambda_{\min}(\Sigma_{iK}) \geq \beta_K^T \beta_K \kappa_0$ a.s., which implies $\sigma_i^{-1} = O(1)$ and $\lambda_{\min}(\Sigma_{iK})^{-1} = O(1)$ a.s., where the $O(1)$ terms are uniform in $i$. Since $\left\| \Sigma_{iK} - \hat{\Sigma}_{iK} \right\|_F = O(a_n + b_n)$ a.s. as $n \to \infty$, where the $O(a_n + b_n)$ term is uniform over $i$, and $\left\| \hat{\xi}_{iK} - \xi_{iK} \right\|_2 = O_p(a_n + b_n)$, where the $O_p(a_n + b_n)$ term is also uniform over $i$, it can be easily checked from the previous arguments that the rates of convergence in (18), (19) and (20) are uniform in $i$.

The following auxiliary lemmas will be used in the proof of Theorem 10.

**Lemma S17.** Suppose that (S4), (B1)-(B2), (B4)-(B5), (A1)–(A8) in the Appendix hold and consider a sparse design with $n_s \leq N_0 < \infty$, setting $a_n = a_{n1}$ and $b_n = b_{n1}$. Then

\[
n^{-1} \sum_{i=1}^n (\hat{\eta}_{iK} - \eta_{iK})\epsilon_{iY} = O_p(a_n),
\]

where $\hat{\eta}_{iK} = \hat{\beta}_0 + \hat{\beta}_K^T \hat{\xi}_{iK}$, and $(\hat{\beta}_0, \hat{\beta}_K^T)^T$ are the estimates in the functional linear model as in Theorem 9.
Proof of Lemma S17. By the Cauchy-Schwarz inequality

$$\left| n^{-1} \sum_{i=1}^{n} (\tilde{\eta}_{iK} - \hat{\eta}_{iK}) \epsilon_{iY} \right| \leq \left( n^{-1} \sum_{i=1}^{n} (\tilde{\eta}_{iK} - \hat{\eta}_{iK})^2 \right)^{1/2} \left( n^{-1} \sum_{i=1}^{n} \epsilon_{iY}^2 \right)^{1/2}, \tag{S.157}$$

where \( n^{-1} \sum_{i=1}^{n} \epsilon_{iY}^2 \)^{1/2} = O_p(1), whence \( |\tilde{\eta}_{iK} - \hat{\eta}_{iK}| \leq |\beta_0 - \hat{\beta}_0| + \|\hat{\beta}_K - \beta_K\|_2 \|\hat{\xi}_{iK}\|_2 + \|\hat{\beta}_K\|_2 \|\hat{\xi}_{iK} - \hat{\xi}_{iK}\|_2 \), and then

$$\begin{align*}
(\tilde{\eta}_{iK} - \hat{\eta}_{iK})^2 & \leq (\beta_0 - \hat{\beta}_0)^2 + \|\hat{\beta}_K - \beta_K\|_2^2 \|\hat{\xi}_{iK}\|_2^2 + \|\hat{\beta}_K\|_2^2 \|\hat{\xi}_{iK} - \hat{\xi}_{iK}\|_2^2 \\
& \quad + 2|\beta_0 - \hat{\beta}_0| \|\hat{\beta}_K - \beta_K\|_2 \|\hat{\xi}_{iK}\|_2 \|2|\beta_0 - \hat{\beta}_0| \|\hat{\beta}_K\|_2 \|\hat{\xi}_{iK} - \hat{\xi}_{iK}\|_2 \\
& \quad + 2\|\hat{\beta}_K - \beta_K\|_2 \|\hat{\xi}_{iK}\|_2 \|\hat{\beta}_K\|_2 \|\hat{\xi}_{iK} - \hat{\xi}_{iK}\|_2.
\end{align*}$$

From Lemma S16 we have \(|\beta_0 - \hat{\beta}_0| = O_p(n^{-1/2})\) and \(\|\hat{\beta}_K - \beta_K\|_2 = O_p(\alpha_n)\), which combined with Lemma S10 and the Cauchy-Schwarz inequality leads to

$$n^{-1} \sum_{i=1}^{n} (\tilde{\eta}_{iK} - \hat{\eta}_{iK})^2 = O_p((\alpha_n)^2). \tag{S.158}$$

The result then follows from (S.157) and (S.158).

Lemma S18. Under the conditions of Theorem 10, it holds that

$$n^{-1} \sum_{i=1}^{n} (\eta_{iK} - \tilde{\eta}_{iK})^2 - \beta_K^T E(\Sigma_{1K}) \beta_K = O_p(n^{-1/2}).$$

Proof of Lemma S18. Since \( \xi_{iK} - \tilde{\xi}_{iK} | T_i \sim N(0, \Sigma_{iK}) \), by conditioning on \( T_i \),

$$E(\eta_{iK} - \tilde{\eta}_{iK})^2 = E \left( E \left[ \left( \beta_K^T (\xi_{iK} - \tilde{\xi}_{iK}) \right)^2 \mid T_i \right] \right) = \beta_K^T E(\Sigma_{1K}) \beta_K,$$

where the last equality is due to the fact that \( n_i = m_0 \) implies that \( \Sigma_{iK} \) are a sequence of i.i.d. random positive definite matrices. Similarly, since \( \eta_{iK} - \tilde{\eta}_{iK} | T_i \sim N(0, \beta_K^T \Sigma_{iK} \beta_K) \) we have \( E((\eta_{iK} - \tilde{\eta}_{iK})^4 | T_i) = 3(\beta_K^T \Sigma_{iK} \beta_K)^2 \) and thus

$$\begin{align*}
\Var((\eta_{iK} - \tilde{\eta}_{iK})^2) &= E(\Var((\eta_{iK} - \tilde{\eta}_{iK})^2 | T_i)) + \Var(\beta_K^T \Sigma_{iK} \beta_K) \\
&= 2E((\beta_K^T \Sigma_{iK} \beta_K)^2) + \Var(\beta_K^T \Sigma_{iK} \beta_K) \\
&= O(1),
\end{align*}$$

where the \( O(1) \) term is uniform in \( i \) since \( \|\Sigma_{iK}\|_{op} \) is uniformly bounded in the sparse case. Since the \( \eta_{iK} - \tilde{\eta}_{iK} \) are independent, the result then follows from the Central Limit Theorem. \( \square \)
Proof of Theorem 10. Recall that \( \eta_{ik} := \beta_0 + \beta_k^T \xi_{ik} \) is the \( K \)-truncated linear predictor for the \( i \)th subject and \( \hat{\eta}_{ik} := \beta_0 + \beta_k^T \hat{\xi}_{ik} \) its best prediction. Also, recall that \( P_{ik} \) corresponds to the predictive distribution of \( \eta_{ik} \) given \( X_i \) and \( T_i \), and \( \hat{P}_{ik} \) is the corresponding estimate. Writing \( Y_i = \beta_0 + \beta_k^T \xi_{ik} + \sum_{k>\text{K}+1} \beta_k \xi_{ik} + \epsilon_i Y = \eta_{ik} + R_{ik} + \epsilon_i Y \), where \( R_{ik} = \sum_{k>\text{K}+1} \beta_k \xi_{ik} \), the estimated Wasserstein discrepancy is given by \( \hat{\beta}_{ik} = \sum_{i=1}^n \epsilon_i Y = \sum_{i=1}^n \beta_k (Y_i - \hat{\eta}_{ik}) \), where

\[
\begin{align*}
&n^{-1} \sum_{i=1}^n W^2_2(\delta_{Y_i}, \hat{P}_{ik}) = n^{-1} \sum_{i=1}^n (Y_i - \hat{\eta}_{ik})^2 + n^{-1} \sum_{i=1}^n \beta_k^T \Sigma_{ik} \beta_k \\
&= n^{-1} \sum_{i=1}^n (\eta_{ik} - \hat{\eta}_{ik})^2 + n^{-1} \sum_{i=1}^n \epsilon_i^2 Y + n^{-1} \sum_{i=1}^n \beta_k^2 + 2n^{-1} \sum_{i=1}^n (\eta_{ik} - \hat{\eta}_{ik}) \epsilon_i Y \\
&+ 2n^{-1} \sum_{i=1}^n (\eta_{ik} - \hat{\eta}_{ik}) R_{ik} + n^{-1} \sum_{i=1}^n R_{ik} \epsilon_i Y + n^{-1} \sum_{i=1}^n \beta_k^T \Sigma_{ik} \beta_k.
\end{align*}
\]  
(S.159)

Since \( n_i = m_0 < N_0 \), by the central limit theorem,

\[
n^{-1} \sum_{i=1}^n (\eta_{ik} - \hat{\eta}_{ik}) R_{ik} = -\beta_k^T E \left( A_K \Phi_{ik}^T \Sigma^{-1} \sum_{k>\text{K}+1} \phi_k(T_i) \lambda_k \beta_k \right) + O_p(n^{-1/2}),
\]

and

\[
n^{-1} \sum_{i=1}^n R_{ik}^2 = \sum_{k>\text{K}+1} \beta_k^2 \lambda_k + O_p(n^{-1/2}).
\]  
(S.160)

Combining this with \( n^{-1} \sum_{i=1}^n (\eta_{ik} - \hat{\eta}_{ik})^2 = O_p(\alpha_n^2) \), as shown in the proof of Lemma S17,

\[
n^{-1} \sum_{i=1}^n (\eta_{ik} - \hat{\eta}_{ik}) R_{ik} = n^{-1} \sum_{i=1}^n (\eta_{ik} - \hat{\eta}_{ik}) R_{ik} + n^{-1} \sum_{i=1}^n (\hat{\eta}_{ik} - \hat{\eta}_{ik}) R_{ik} \\
= -\beta_k^T E \left( A_K \Phi_{ik}^T \Sigma^{-1} \sum_{k>\text{K}+1} \phi_k(T_i) \lambda_k \beta_k \right) + O_p(\alpha_n).
\]  
(S.161)

Next

\[
n^{-1} \sum_{i=1}^n (\eta_{ik} - \hat{\eta}_{ik}) \epsilon_i Y = n^{-1} \sum_{i=1}^n (\eta_{ik} - \hat{\eta}_{ik}) \epsilon_i Y + n^{-1} \sum_{i=1}^n (\hat{\eta}_{ik} - \hat{\eta}_{ik}) \epsilon_i Y \\
= O_p(n^{-1/2}) + O_p(\alpha_n) = O_p(\alpha_n),
\]  
(S.162)

where the last equality follows from Lemma S17 and since \( n^{-1} \sum_{i=1}^n (\eta_{ik} - \hat{\eta}_{ik}) \epsilon_i Y = O_p(n^{-1/2}) \), which is due to the Central Limit Theorem. Similarly, from Lemma S18 we have

\[
n^{-1} \sum_{i=1}^n (\eta_{ik} - \hat{\eta}_{ik})^2 = \beta_k^T E(\Sigma_{1k}) \beta_k + O_p(n^{-1/2}),
\]  
(S.163)

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and

\[ n^{-1} \sum_{i=1}^{n} (\eta_{iK} - \tilde{\eta}_{iK})^2 - n^{-1} \sum_{i=1}^{n} (\eta_{iK} - \hat{\eta}_{iK})^2 = n^{-1} \sum_{i=1}^{n} (\eta_{iK} - \hat{\eta}_{iK})^2 \]

\[ + 2n^{-1} \sum_{i=1}^{n} (\eta_{iK} - \tilde{\eta}_{iK})(\tilde{\eta}_{iK} - \hat{\eta}_{iK}) = O_p(\alpha_n), \tag{S.164} \]

where the last equality follows from the fact that \( n^{-1} \sum_{i=1}^{n} (\tilde{\eta}_{iK} - \hat{\eta}_{iK})^2 = O_p(\alpha_n^2) \), (S.163) and the Cauchy-Schwarz inequality. Combining (S.163) and (S.164) leads to

\[ n^{-1} \sum_{i=1}^{n} (\eta_{iK} - \hat{\eta}_{iK})^2 = \beta_K^T E(\Sigma_{1K}) \beta_K + O_p(\alpha_n). \tag{S.165} \]

Next, note that

\[ |\hat{\beta}_K^T \hat{\Sigma}_{iK} \hat{\beta}_K - \beta_K^T \Sigma_{iK} \beta_K| = |\hat{\beta}_K^T (\hat{\Sigma}_{iK} - \Sigma_{iK}) \hat{\beta}_K + (\hat{\beta}_K - \beta_K)^T \Sigma_{iK} \hat{\beta}_K + \beta_K^T \Sigma_{iK} (\beta_K - \beta_K)| \]

\[ \leq \|\hat{\beta}_K\|_2 \|\Sigma_{iK} - \Sigma_{iK}\|_{op,2} + \|\hat{\beta}_K - \beta_K\|_2 \|\Sigma_{iK}\|_{op,2}(\|\hat{\beta}_K\|_2 + \|\beta_K\|_2). \]

From the proof of Theorem 9, we have \( \|\Sigma_{iK} - \hat{\Sigma}_{iK}\|_F = O(a_n + b_n) \) a.s. as \( n \to \infty \), where the \( O(a_n + b_n) \) term is uniform in \( i \). Since \( \|\Sigma_{iK}\|_F = O(1) \) uniformly over \( i \),

\[ n^{-1} \sum_{i=1}^{n} (\hat{\beta}_K^T \hat{\Sigma}_{iK} \hat{\beta}_K - \beta_K^T \Sigma_{iK} \beta_K) \leq n^{-1} \sum_{i=1}^{n} |\hat{\beta}_K^T \hat{\Sigma}_{iK} \hat{\beta}_K - \beta_K^T \Sigma_{iK} \beta_K| \]

\[ \leq \|\hat{\beta}_K - \beta_K\|_2 (\|\hat{\beta}_K\|_2 + \|\beta_K\|_2) n^{-1} \sum_{i=1}^{n} \|\Sigma_{iK}\|_F \]

\[ + \|\beta_K\|_2^2 n^{-1} \sum_{i=1}^{n} \|\hat{\Sigma}_{iK} - \Sigma_{iK}\|_F \]

\[ \leq \|\hat{\beta}_K - \beta_K\|_2 (\|\hat{\beta}_K\|_2 + \|\beta_K\|_2) O(1) + \|\beta_K\|_2^2 O(a_n + b_n) \text{ a.s.}, \]

as \( n \to \infty \). From Lemma S16, we have \( \|\hat{\beta}_K - \beta_K\|_2 = O_p(\alpha_n) \), which combined with \( \|\hat{\beta}_K\|_2 \leq \|\hat{\beta}_K - \beta_K\|_2 + \|\beta_K\|_2 = O_p(1) \) leads to

\[ n^{-1} \sum_{i=1}^{n} \hat{\beta}_K^T \hat{\Sigma}_{iK} \hat{\beta}_K - n^{-1} \sum_{i=1}^{n} \beta_K^T \Sigma_{iK} \beta_K = O_p(\alpha_n). \]

This along with an application of the Central Limit Theorem shows that

\[ n^{-1} \sum_{i=1}^{n} \hat{\beta}_K^T \hat{\Sigma}_{iK} \hat{\beta}_K = \beta_K^T E(\Sigma_{1K}) \beta_K + O_p(\alpha_n). \tag{S.166} \]
Finally, it is easy to show that \( n^{-1} \sum_{i=1}^{n} R_i \epsilon_i Y = O_p(n^{-1/2}) \) and \( n^{-1} \sum_{i=1}^{n} \epsilon_i Y = \sigma_Y^2 + O_p(n^{-1/2}) \), applying the CLT. Combining with (S.160), (S.161), (S.162), (S.165) and (S.166),

\[
\hat{D}_{nK} = 2\beta_K^T E(\Sigma_{1K}) \beta_K + \sigma_Y^2 + \sum_{k \geq K+1} \beta_k^2 \lambda_k - 2\beta_K^T E \left( \Lambda_K \Phi_{1K}^T \Sigma_1^{-1} \sum_{k \geq K+1} \phi_k(T_1) \lambda_k \beta_k \right) + O_p(\alpha_n),
\]

implying the first result in (21). Similar arguments show that the Wasserstein distance using true population quantities \( D_{nK} \) is such that

\[
D_{nK} = n^{-1} \sum_{i=1}^{n} W_2^2(\delta_{Y_i}, \mathcal{P}_{iK}) = n^{-1} \sum_{i=1}^{n} (Y_i - \tilde{\eta}_{iK})^2 + n^{-1} \sum_{i=1}^{n} \beta_K^T \Sigma_{iK} \beta_K
\]

\[
= D_K + O_p(n^{-1/2}),
\]

where \( D_K = 2\beta_K^T E(\Sigma_{1K}) \beta_K + \sigma_Y^2 + \sum_{k \geq K+1} \beta_k^2 \lambda_k - 2\beta_K^T E \left( \Lambda_K \Phi_{1K}^T \Sigma_1^{-1} \sum_{k \geq K+1} \phi_k(T_1) \lambda_k \beta_k \right) \).

Next, since \( Y = \mu_Y + \int_T \beta(t) U(t) + \epsilon_Y \), where \( \mu_Y = E(Y) \) and \( U(t) = X(t) - \mu(t) \), we have \( E(Y^2) = \mu_Y^2 + \sigma_Y^2 + E(\langle \beta, U \rangle_{L^2}^2) \), where \( \langle \cdot, \cdot \rangle_{L^2} \) is the \( L^2(T) \) inner product. From (S4) it follows that \( E(\langle \beta, U \rangle_{L^2}^2) = \sum_{j=1}^\infty \beta_j^2 \lambda_j \) as the FPCs are independent in the Gaussian case. Then

\[
n^{-1} \sum_{i=1}^{n} (Y_i - \tilde{Y}_n)^2 = \text{Var}(Y) + O_p(n^{-1/2}) = \sigma_Y^2 + \sum_{j=1}^\infty \lambda_j \beta_j^2 + O_p(n^{-1/2}). \tag{S.167}
\]

Also, \( |\hat{\beta}_j| \leq \|\hat{\beta}_M\|_{L^2} \) and \( |\beta_j| \leq \|\beta\|_{L^2} \). With perturbation results as used in the proof of Lemma S15 this leads to

\[
\left| \sum_{m=1}^{M} \hat{\lambda}_m \hat{\beta}_m^2 - \lambda_m \beta_m^2 \right| \leq \sum_{m=1}^{M} |\hat{\lambda}_m - \lambda_m| \|\hat{\beta}_m^2 - \beta_m^2\| + \sum_{m=1}^{M} |\hat{\lambda}_m - \lambda_m| \beta_m^2 + \sum_{m=1}^{M} \lambda_m |\hat{\beta}_m^2 - \beta_m^2| \\
\leq \|\hat{\Xi} - \Xi\|_{\text{op}} (\|\hat{\beta}_M\|_{L^2} + \|\beta\|_{L^2}) \sum_{m=1}^{M} |\hat{\beta}_m - \beta_m| + \|\hat{\Xi} - \Xi\|_{\text{op}} \sum_{m=1}^{M} \beta_m^2 \\
+ (\|\hat{\beta}_M\|_{L^2} + \|\beta\|_{L^2}) \sum_{m=1}^{M} \lambda_m |\hat{\beta}_m - \beta_m|. \tag{S.168}
\]

Next, from the proof of Lemma S16 and since \( \sum_{j=1}^{\infty} \lambda_j < \infty \), we have

\[
\sum_{m=1}^{M} \lambda_m |\hat{\beta}_m - \beta_m| \leq \|\hat{\beta}_M - \beta\|_{L^2} \sum_{m=1}^{M} \lambda_m \|\hat{\phi}_m - \phi_m\|_{L^2} + \|\hat{\beta}_M - \beta\|_{L^2} \left( \sum_{m=1}^{M} \lambda_m \right) \\
+ \|\beta\|_{L^2} \sum_{m=1}^{M} \lambda_m \|\hat{\phi}_m - \phi_m\|_{L^2} \\
\leq \left( \sum_{j=1}^{\infty} \lambda_j \right) \|\hat{\beta}_M - \beta\|_{L^2} + 2\sqrt{2}\|\hat{\Xi} - \Xi\|_{\text{op}} (\|\hat{\beta}_M - \beta\|_{L^2} + \|\beta\|_{L^2}) \left( \sum_{m=1}^{M} \lambda_m \right) \right) \text{ a.s.} \\
\leq O_p(\alpha_n) + O_p(1)O(c_n^p) = O_p(\alpha_n), \tag{S.169}
\]
where the last inequality follows from Lemma S9 and Lemma S16. Similarly
\[
\sum_{m=1}^{M} |\hat{\beta}_m - \beta_m| \leq 2\sqrt{2}\|\hat{\Xi} - \Xi\|_\text{op} \left( \|\hat{\beta}_M - \beta\|_{L^2} + \|\beta\|_{L^2} \right) \left( \sum_{m=1}^{M} \frac{1}{\delta_m} \right) + \|\hat{\beta}_M - \beta\|_{L^2} M \quad \text{a.s.}
\]
\[
\leq O(c_n)O(c_n^{-1}) \left( \|\hat{\beta}_M - \beta\|_{L^2} + \|\beta\|_{L^2} \right) + \|\hat{\beta}_M - \beta\|_{L^2} O(c_n^{-1}) \quad \text{a.s.}
\]
\[
\leq O_p(c_n) + O_p(c_n^{-1} \alpha_n),
\]
(S.170)

where the second and third inequalities follow from Lemma S16 and using that \(\sum_{m=1}^{M} \frac{1}{\delta_m} \sim O(c_n^{-1})\), which was shown in the proof of Lemma S15, along with the fact that \(M \sim O(c_n^{-1})\), which is due to the condition \(\sum_{m=1}^{M} \frac{1}{\sqrt{\lambda_m \delta_m}} = O(c_n^{-1})\) and \(0 < \delta_m < \lambda_m \leq \lambda_1\). Combining (S.168), (S.169) and (S.170) leads to
\[
\left| \sum_{m=1}^{M} \hat{\lambda}_m \hat{\beta}_m^2 - \sum_{m=1}^{M} \lambda_m \beta_m^2 \right| = O_p(\alpha_n).
\]

This implies
\[
\left| \sum_{m=1}^{M} \hat{\lambda}_m \hat{\beta}_m^2 - \sum_{m=1}^{\infty} \lambda_m \beta_m^2 \right| \leq O_p(\alpha_n) + \sum_{m=\infty}^{M+1} \lambda_m \beta_m^2,
\]
and the result in (22) follows from (S.167).

\[
\square
\]

**S.3 Additional Results for Section 4**

Consider the Brownian motion as an example of a Gaussian process for which \(\lambda_m = 4/(\pi^2(2m-1)^2)\) and \(\phi_m(t) = \sqrt{2} \sin((2m-1)\pi t/2)\) (Hsing and Eubank 2015). Adopting the optimal bandwidth choices as discussed in section 4 leads to \(c_n \asymp (\log(n)/n)^{1/3}\).

**Lemma S19.** For the Brownian motion, if \(M = M(n)\) satisfies (B5) with \(\sum_{m=1}^{M} \frac{1}{\sqrt{\lambda_m \delta_m}} \asymp c_n^{-1}\), then
\[
M(n) \asymp \left( \frac{\log(n)}{n} \right)^{(\rho-1)/15},
\]
(S.171)
\[
\tau_M \asymp \left( \frac{\log(n)}{n} \right)^{(\rho-1)/5},
\]
(S.172)
\[
\upsilon_M \asymp \left( \frac{\log(n)}{n} \right)^{4(\rho-1)/15}.
\]
(S.173)

Moreover, if \(\sigma_m^2 \leq C m^{-(6+\delta)}\) for some constant \(C > 0\) and \(\delta > 0\), then (B4) is satisfied, \(\Theta_M = O(M^{-\delta/2})\) and the rate \(\alpha_n\) in Theorem 9 satisfies the following conditions: If \(\rho \leq\)
(3 + δ)/(13 + δ), then \( \alpha_n = O((\log(n)/n)^{(13\rho-3)/30}) \) while if \( \rho > (3 + \delta)/(13 + \delta) \) it holds that \( \alpha_n = O((\log(n)/n)^{\delta(1-\rho)/30}) \). The optimal rate is achieved when \( \rho = (3 + \delta)/(13 + \delta) \) and leads to \( \alpha_n = O((\log(n)/n)^9) \), where \( q = (\delta/(13 + \delta))/3 \).

**Proof.** For any \( m \geq 1 \)

\[
\lambda_m - \lambda_{m+1} = \frac{32}{\pi^2} \frac{m}{(2m - 1)^2(2m + 1)^2},
\]

which is decreasing as \( 1 \leq m \to \infty \) and thus the eigengaps are given by

\[
\delta_m = \frac{32}{\pi^2} \frac{m}{(2m - 1)^2(2m + 1)^2}, \quad m \geq 1.
\]

Since the harmonic sum \( H(M) = \sum_{m=1}^{M} 1/m \) satisfies \( H(M) \leq 1 + \log(M) \) and \( M = M(n) \to \infty \) as \( n \to \infty \), we obtain

\[
\sum_{m=1}^{M} \frac{1}{\sqrt{\lambda_m \delta_m}} = \frac{\pi^3}{64} \sum_{m=1}^{M} \frac{(2m - 1)^3(2m + 1)^2}{m} \asymp M(n)^5.
\]

Taking \( \sum_{m=1}^{M} \frac{1}{\sqrt{\lambda_m \delta_m}} \asymp c_{\rho} n^{-\rho} \) shows the first result in (S.171). Next, simple calculations show that

\[
\tau_M = \sum_{m=1}^{M} \frac{1}{\lambda_m} = \sum_{m=1}^{M} \frac{\pi^2(2m - 1)^2}{4} \asymp M(n)^3,
\]

and

\[
u_M = \sum_{m=1}^{M} \frac{1}{\delta_m} = \frac{\pi^2}{32} \sum_{m=1}^{M} \frac{(2m - 1)^2(2m + 1)^2}{m} \asymp M(n)^4.
\]

The results in (S.172) and (S.173) then follow. If \( \sigma_m^2 \leq Cm^{-(6+\delta)} \) for some \( C, \delta > 0 \), then \( \sum_{m=1}^{\infty} \sigma_m^2/\lambda_m^2 \leq O(1) \sum_{m=1}^{\infty} m^{-(2+\delta)} < \infty \) and condition (B4) is satisfied. Next, from the orthonormality of the \( \phi_m \)

\[
\Theta_M = \left\| \sum_{m \geq M+1} \frac{\sigma_m}{\lambda_m} \phi_m \right\|_{L^2} \leq \sum_{m \geq M+1} \left| \frac{\sigma_m}{\lambda_m} \right| \leq O(1) \sum_{m \geq M+1} m^{-(1+\delta/2)} \leq O(1) \int_{M}^{\infty} s^{-(1+\delta/2)} ds \]

\[
= O \left( \frac{1}{\delta M^{\delta/2}} \right),
\]

which implies \( \Theta_M = (\log(n)/n)^{\delta(1-\rho)/30} \). Also note that \( c_n \nu_M \asymp (\log(n)/n)^{(1+4\rho)/15} \) and \( c_{\rho} \tau_M^{1/2} \asymp (\log(n)/n)^{(13\rho-3)/30} \). This implies

\[
\alpha_n = c_n \nu_M + c_{\rho} \tau_M^{1/2} + \Theta_M \leq O\left((\log(n)/n)^{(13\rho-3)/30} + (\log(n)/n)^{\delta(1-\rho)/30}\right).
\]
Thus, if $\rho \leq (3 + \delta)/(13 + \delta)$, then $\alpha_n = O((\log(n)/n)^{(13\rho - 3)/30})$. Similarly, if $\rho > (3 + \delta)/(13 + \delta)$, then $\alpha_n = O((\log(n)/n)^{\delta(1-\rho)/30})$. The optimal rate is achieved when $\rho = (3 + \delta)/(13 + \delta)$ and leads to $\alpha_n = O((\log(n)/n)^{q})$, where $q = (\delta/(13 + \delta))/3$.

S.4 Additional Plots for the Kiwi Data

In this section we provide additional results for the brown kiwi dataset. Figure S7 shows the boxplots for the time point observations corresponding to each kiwi’s 2-truncated galaxy that appear in Figure 4 while Figure S8 presents the total number of counts for each of the kiwis.

Figure S7: Boxplot for the time locations in days of the male and female kiwis shown in Figure 4. These kiwis differ in their time points locations and spread as well as their total number of observations.
Figure S8: Total number of observations for each of the kiwis shown in Figure 4. The kiwis labeled 1, 13 and 14 as in Figure 4 have a large number of observations compared to the rest, where the counts can be as low as 5 per kiwi.