Non-local orders, entanglement entropy, and quantum fidelity are investigated in an infinite-size bond-alternating Ising chain with the Dzyaloshinskii-Moriya interaction by employing the infinite matrix product state representation with the infinite time evolving block decimation method. Directly computing two distinct types of finite string correlations for very large lattice distances, in contrast to an extrapolated extreme value for finite size chains, reveals two topologically ordered phases. As the bond alternation varies, a topological quantum phase transition with continuously variable critical exponents along the phase boundary occurs between the two Haldane phases for the Dzyaloshinskii-Moriya interaction stronger than the Ising interaction, while a topological quantum crossover between them happens through an intermediate antiferromagnetic phase demonstrated with the quantum fidelity for the Dzyaloshinskii-Moriya interaction weaker than the Ising interaction. The critical exponents of the order parameters and the central charges from the entanglement entropy quantify the universality classes of the phase transition points. Anisotropic Heisenberg types of spin chains with bond alternations are finally discussed to share the same criticality.

**Introduction.**—Topologically ordered states [1] beyond the Landau paradigm of spontaneous symmetry breaking [2] have been studied intensively and extensively in condensed matter systems. Moreover, such robust states against troublesome decoherence are of rapidly growing interest in the field of quantum information processing and computation [3, 4]. Between them, we observe (i) a direct phase transition shown to enable to realize the topologically distinct symmetry phase for the topological quantum crossover (TQC) as well as the phase transitions.

**Model and string order parameters.**—Let us consider the model Hamiltonian

\[ H = \sum_{j=\infty}^{\infty} \left( 1 + (-1)^j \delta \right) \left( D \cdot S_j \times S_{j+1} + J S_j^x S_{j+1}^x \right), \]  

where \( S_j^x (\alpha = x, y, z) \) are the spin-1/2 operators at lattice site \( i \), \( J > 0 \) and \( D \) denote the Ising and the DM interactions, respectively. The bond-alternation parameter ranges as \(-1 \leq \delta \leq 1\). We choose \( D = D_2 \) with \( D > 0 \). For clarity, we study mainly the system in Eq. (1). However, adding a term \( \sum_j (1 + (-1)^j \delta) (S_j^x S_{j+1}^y + S_j^y S_{j+1}^x) \) in Eq. (1) does not change the physics of criticality, will be discussed later.

Based on the bond alternation, one can define two string order parameters \([22, 23]\) as

\[ O_{\text{str,even}}^\alpha = \lim_{|i-j|\to\infty} \left\langle -4 \left( S_{2i}^x \exp \left[ \frac{i\pi}{2} \sum_{k=2i+1}^{2j-2} S_k^x \right] \right)^{S_{2j-1}^x} \right\rangle, \]  

\[ O_{\text{str,odd}}^\alpha = \lim_{|i-j|\to\infty} \left\langle -4 \left( S_{2i+1}^x \exp \left[ \frac{i\pi}{2} \sum_{k=2i+2}^{2j-1} S_k^x \right] \right)^{S_{2j}^x} \right\rangle, \]

where \( \alpha = x, y, \) and \( z \). Our iMPS groundstate wavefunction allows us to directly calculate the defined string orders \([16]\).

Figure 1(a) and 1(b) show, respectively, the even and the odd string order parameters \( O_{\text{str,even/odd}}^\alpha \) in \( \delta \sim (D/J) \) plane. In fact, \( O_{\text{str,even}}(\delta) = O_{\text{str,odd}}(-\delta) \) for a given \( D/J \). As the bond alternation \( \delta \) varies, the even (odd) string order parameter is finite for \( \delta > \delta_c^+ (\delta < \delta_c^-) \), where the phase boundary functions are obtained as

\[ \delta_c^\pm (J, D) = \mp \delta_c \Theta(J - D) \]

with \( \delta_c = (D - J)^2 / (AD + J)^2 \), the numerical fitting constant \( A = 5/4 \), and the unit step function \( \Theta(x) \) representing equal to
similar hidden symmetry breaking may occur for each phase. The gapful even (odd) Haldane phase to the other Haldane phase as the bond alternation varies. (i) For \(\delta < \delta_b \) as function of \(|\delta - \delta_c|\) for various DM interactions \(D = J, 1.8J, 4J, \) and \(9J\), corresponding to the numerical critical exponents \(\beta = 1/12, 1/8, 1/5.688, \) and \(1/5\), respectively. (d) \(\beta\) as a function of \(D/J\) in the \(O_{str}\) phase. Here, note that \(O_{str} = O_{str}(-\delta)\). In (a) and (b), the red lines are the phase boundaries. The truncation dimension is \(\chi = 32\).

For \(x < 0\) and \(1 \geq x \geq 0\). Other components of the string order parameters are zero, i.e., \(O_{str}^{xy} = 0\). Thus, the finite even (odd) string order parameter characterizes a topologically ordered phase, i.e., the gapful even (odd) Haldane phase \([12, 13, 24]\). Similarly to the spin-1 Heisenberg chain understood by the hidden \(Z_2 \times Z_2\) breaking symmetry \([25]\), a similar hidden symmetry breaking may occur for each phase.

In addition, the phase boundaries of topological characterizations in Eq. (3) expose two ways changing from one Haldane phase to the other Haldane phase as the bond alternation varies. (i) For \(J \leq D\), in the even (odd) string order parameter is finite for \(\delta > 0\) (\(\delta < 0\)) with \(\delta_c^x = 0\), which implies that a TQPT occurs at \(\delta = 0\). (ii) For \(J > D\), i.e., \(\delta_c^x = \mp \delta_c\), the even and odd string order parameters have a finite value for \(\delta > -\delta_c\) and \(\delta < \delta_c\), respectively. The system is then in the odd (even) Haldane phase for \(\delta < -\delta_c\) (\(\delta > \delta_c\)). However, the order parameters coexist for \(-\delta_c < \delta < \delta_c\). This implies that a TQC occurs from the even Haldane phase to the odd Haldane phase or vice versa in the range of the bond alternation, \(-\delta_c < \delta < \delta_c\). In some sense, our system with such topological phase changes resembles an anisotropic antiferromagnet, possessing a magnetic crossover with two continuous phase transitions at phase boundaries, such as GdAlO_3, where anisotropies favors spin alignment along particular lattice directions, breaks an \(O_{str}\) symmetry, and give rise to a multicritical point, particularly, a tetracritical point \([24, 28]\).

**Topological quantum Gaussian transition.** In order to obtain the critical exponents of the string order parameters, in Fig. (1c), we plot the string order parameter \(O_{str}^{xy}\) as a function of \(|\delta - \delta_c^x|\) for various values of \(D/J\). Note that the string order parameters scale as \(O_{str}^{xy} \propto |\delta - \delta_c^x|^{\beta}\) and the critical exponents \(\beta(D/J)\) depend on the chosen values of \(D/J\).

FIG. 1: (Color online) (a) Even and (b) odd string order parameters, \(O_{str}^{xy}\), in the \(\delta - (D/J)\) plane. (c) \(O_{str}^{xy}\) as a function of \(|\delta - \delta_c|\) for various DM interactions \(D = J, 1.8J, 4J, \) and \(9J\), corresponding to the numerical critical exponents \(\beta = 1/12, 1/8, 1/5.688, \) and \(1/5\), respectively. (d) \(\beta\) as a function of \(D/J\) in the \(O_{str}\) phase. Here, note that \(O_{str} = O_{str}(-\delta)\). In (a) and (b), the red lines are the phase boundaries. The truncation dimension is \(\chi = 32\).

FIG. 2: (Color online) (a) Von Neumann entropies \(S_{odd}\) in the \(\delta - (D/J)\) plane with \(\chi = 32\). Here, \(S_{odd} = S_{str}(-\delta)\). (b) Divergence of the correlation lengths \(\xi\) as a function of the truncation dimension \(\chi\) at the critical points. (c) Divergence of von Neumann entropies as a function of \(\chi\) at the critical points in (b). The C’s are given in the text.
scaling law in one-dimensional lattice systems in the thermodynamic limit and, at critical points, are related to a universal factor, i.e., a central charge of associated conformal field theory \cite{[17][18]}. Let us consider quantum entanglements between two half-infinite subsystems. Our system is partitioned into the left and the right half-infinite subsystems denoted by \( L \) and \( R \). The von Neumann entropy between \( L \) and \( R \) is defined as \( S = -\text{Tr} \rho_L \log_2 \rho_L = -\text{Tr} \rho_R \log_2 \rho_R \) in terms of the reduced density matrix of subsystems \( \rho_L \) or \( \rho_R \). In the iMPS representation, the von Neumann entropy can be expressed in terms of the Schmidt coefficients, \( \lambda \), as \( S = -\sum_{\lambda \in \mathcal{L}} \lambda^2 \log_2 \lambda^2 \). Due to the bond alternation, there are two types of Schmidt coefficient matrices that describe two possible ways of the partitions, i.e., one is on the even sites, the other is on the odd sites.

In Fig. 2(a), the von Neumann entropy \( S_{\text{odd}} \) is plotted in \( \delta-\langle D/J \rangle \) plane. The singular behaviors (peaks) of the entropy indicate the phase transition along the phase boundaries \( \delta = \delta_c \). Note that the entropy peak at \( \delta = 0 \) for \( J \leq D \) is split into the two peaks at \( \delta = \pm \delta_c \) for \( J > D \). Since the two von Neumann entropies \( S_{\text{even/odd}} \) depending on the even- or the odd-site partitions satisfy \( S_{\text{odd}}(\delta) = S_{\text{even}}(-\delta) \), the \( S_{\text{odd}} \) has the same singular behaviors along the phase boundaries \( \delta = \delta_c \). In Fig. 2(b), the correlation lengths \( \xi(\delta) \) are plotted as a function of the truncation dimension \( \chi \) for the critical points. As the truncation dimension increases, the correlation lengths \( \xi \) scale to diverge as \( \xi(\chi) = \xi_0 \chi^z \), which means the scale invariance of the system in the thermodynamic limit \( \chi \rightarrow \infty \), with the numerical finite-entanglement scaling exponents \( \chi \), (i) \( \xi_0 = 0.073 \) and \( k = 2.031 \) at \( C_1(\delta, D/J) = (0.2651, 0.3) \), (ii) \( \xi_0 = 0.0727 \) and \( k = 1.5126 \) at \( C_2 = (0.0085, 0.8) \), (iii) \( \xi_0 = 0.217 \) and \( k = 1.323 \) at \( C_3 = (0, 1) \), and (iv) \( \xi_0 = 0.298 \) and \( k = 1.376 \) at \( C_4 = (0, 4) \). In Fig. 2(c), we show the logarithmic scaling of the von Neumann entropy \( S(\chi) \) for the critical points. From the \( \kappa \)'s in Fig. 2(b), the linear fittings \( S(\chi) = S_0 + (\kappa/6) \log_2 \chi \) yield (i) the central charge \( c \approx 0.505 \) with \( S_0 = 0.0089 \) at \( C_1 \), (ii) \( c \approx 0.4979 \) with \( S_0 = 0.5491 \) at \( C_2 \), (iii) \( c \approx 1.002 \) with \( S_0 = 0.3816 \) at \( C_3 \), and (iv) \( c \approx 1.003 \) with \( S_0 = 0.521 \) at \( C_4 \). Consequently, the characteristic entanglement properties confirm that for \( J \leq D \), the TQPT between the even- and the odd-Haldane phases at the critical points \( \delta = 0 \) is a Gaussian transition which is characterized by the central charge \( c = 1 \) and the occurrence of a phase transition between two gapful phases with the continuous variable critical exponent of the string order parameters. For \( J > D \), the central charge \( c = 1/2 \) at the phase boundaries \( \delta = \pm \delta_c \) also confirms that an Ising type of phase transitions occurs along the boundaries of the TQC.

**Quantum fidelity per site and intermediate antiferromagnetic phase for topological quantum crossover.** Although the TQPT occurring without any explicit symmetry breaking has been understood by the string order parameters, our system can undergo a symmetry breaking because the Hamiltonian in Eq. 1 is invariant under the unitary transformation \( U = \prod U_{2j} \otimes U_{2j+1} \) with \( U_{2j} = \sigma^x \) and \( U_{2j+1} = \sigma^y \), i.e., \( UHU^{-1} = H \), and then possesses a \( Z_2 \) symmetry generated by the transformation \( U \). If the system undergoes explicitly a spontaneous breaking of the \( Z_2 \) symmetry in the interaction parameter space, it has a \( Z_2 \) broken-symmetry phase with a doubly degenerate groundstate. Thus, in order to clarify whether the \( Z_2 \) symmetry breaking occurs, let us consider a quantum fidelity that allows us to determine groundstate degeneracy in one-dimensional infinite quantum lattice systems \cite{[19][20]}. We employ the FLS \( d(|\psi_n\rangle, |\psi\rangle) \) in Ref. \cite{[20]} as \( \ln d(|\psi_n\rangle, |\psi\rangle) \equiv \lim_{\lambda \rightarrow -\infty} (1/\lambda) \ln F(|\psi_n\rangle, |\psi\rangle) \), where the quantum fidelity is \( F(|\psi_n\rangle, |\psi\rangle) = |\langle \psi_n | \psi \rangle| \), \( L \) is the system size, \( |\psi_n\rangle \) is an iMPS groundstate calculated with the randomly chosen \( n \)-th initial state for given parameters, and \( |\psi\rangle \) is an arbitrary chosen reference state. If \( F \) has \( N \) projection values onto the reference state, the system has \( N \) degenerate groundstates. By using many random initial states for given parameters, we detect a degenerate groundstate. In Fig. 3 the FLS \( d(|\psi_n\rangle, |\psi\rangle) \) shows that the system has a doubly generate groundstate for \( -\delta_c < \delta < \delta_c \) and \( J > D \) [Fig. 3(a)] and a single groundstate for \( J \leq D \) [Fig. 3(b)]. Note that the bifurcation points \cite{[19][20]} at \( \delta = \pm \delta_c \) correspond to the phase boundaries [Fig. 3(a)] and the singular behavior \cite{[21][32]} at \( \delta = 0 \) in the derivative of the FLS over the bold alternation \( \delta \) indicates the phase transition point [the inset of Fig. 3(b)]. As a result, the doubly degenerate groundstate implies that the topological quantum crossover region, i.e., \( -\delta_c < \delta < \delta_c \) for \( J > D \), is a \( Z_2 \) broken-symmetry phase.

For the \( Z_2 \) broken-symmetry phase, actually, there are the two groundstates \( |\psi_1\rangle, U|\psi_2\rangle \) that satisfy \( H|\psi_1\rangle = E_g|\psi_1\rangle \) or \( HU|\psi_2\rangle = E_gU|\psi_2\rangle \) with the groundstate energy \( E_g \), and
they are not equal, i.e., \(|\psi_j\rangle \neq U|\psi_j\rangle\). One can then denote \(|\psi_1\rangle = |\psi_j\rangle\) and \(|\psi_2\rangle = U|\psi_j\rangle\). Due to the transformations of the spin operators as \(US_j^zU^\dagger = -S_j^z\) and \(US_j^zS_j^zU^\dagger = S_j^zS_j^z\), the local magnetizations and the spin-spin correlations from the two groundstate wavefunctions might have the relations 

\[
\langle S_j^z|\psi_1\rangle = -\langle S_j^z|\psi_2\rangle\quad \text{and} \quad \langle S_j^zS_k^z|\psi_1\rangle = \langle S_j^zS_k^z|\psi_2\rangle,
\]

respectively. Further, for \(J > D\), the nearest spin-spin correlation is antiferromagnetic, i.e., \(\langle \phi|S_j^zS_{j+1}^z|\psi\rangle < 0\). In Fig. 3(c), then, we plot the staggered magnetization \(M_z = \langle (S_j^z - S_{j+1}^z)/2\rangle\) as a function of \(\delta\). The two groundstates give a finite staggered magnetization for \(-\delta_1 < \delta < \delta_1\) and \(M_z = \langle (S_j^z - S_{j+1}^z)/2\rangle\) for \(\delta_1 < \delta < \delta_2\). The string order parameters calculated from the two degenerate groundstates are the same each other, i.e., \(\langle \phi_1|O_{\text{string,even/odd}}|\psi_1\rangle = \langle \phi_2|O_{\text{string,even/odd}}|\psi_2\rangle\), as it should. In addition, Fig. 3(d) shows that the staggered magnetization scales as \(M_z \propto |\delta - \delta|^\beta\) with the critical exponent \(\beta = 1/8\). Consequently, the TQC region is characterized by the local order, i.e., the staggered magnetization. The valance bond solid picture may state that a nonlocal string order for a Haldane phase captures so-called ‘dilute’ antiferromagnetic phase. Hence, the TQC between the two distinct ‘dilute’ antiferromagnetic phases, i.e., the even- and the odd-Haldane phases, occurs via the intermediate antiferromagnetic state as the bond alternation \(\delta\) varies for \(J > D\).

**Dimer and chiral orders.**— Due to the symmetry of the Hamiltonian in Eq. (1), the system can have the other local orders that have nothing to do with the \(Z_2\) symmetry breaking. To show this point explicitly, in Fig. 4(a) and (b), the odd dimer and the odd chiral orders are plotted in \(\delta = (D/J)\) plane. Here, the dimer and the chiral orders are defined as \(O_D^{\text{even}} = \langle S_j^z \cdot S_{j+1}^z - S_{j+1}^z \cdot S_j^z \rangle\), \(O_D^{\text{odd}} = \langle S_j^z \cdot S_{j+1}^z - S_{j+1}^z \cdot S_j^z \rangle\), \(O_C^{\text{even}} = \langle S_j^z \cdot S_{j+1}^z \rangle\), and \(O_C^{\text{odd}} = \langle S_j^z \cdot S_{j+1}^z \rangle\). They satisfy the relations \(O_D^{\text{even}}(\delta) = O_D^{\text{even}}(-\delta)\) with \(O_D(0) = 0\) and \(O_C^{\text{even}}(\delta) = O_C^{\text{even}}(-\delta)\) with \(O_C(0) = 0\). As they should be, \(\langle \phi_1|O_D^{\text{even/odd}}|\psi_1\rangle = \langle \psi_2|O_D^{\text{even/odd}}|\psi_2\rangle\) and \(\langle \phi_1|O_C^{\text{even/odd}}|\psi_1\rangle = \langle \psi_2|O_C^{\text{even/odd}}|\psi_2\rangle\). Both the dimer and the chiral orders are finite in the whole parameter range, which implies that they cannot distinguish the two Haldane phases. Hence, only the two string order parameters distinguish the two Haldane phases.

**Relations to other spin models.**— Actually, our results can be shared to understand other spin-1/2 lattice models with bond alternation. Using the non-local transformation \(\tilde{H} = \exp[-i \sum_j \alpha_j S_j^z]H \exp[i \sum_j \alpha_j S_j^z]\), the anisotropic Heisenberg chain with the DM interaction \(H' = \sum_j (1 + (-1)^j \delta)(\mathbf{D} \cdot S_j \times S_{j+1} + \Delta (S_j^z S_{j+1}^z + S_{j+1}^z S_j^z) + J S_j^z S_{j+1}^z)\) can be mapped to a spin-1/2 XXZ chain with the bond alternation, \(\tilde{H}' = \sum_j (1 + (-1)^j \delta)(\mathbf{D} \cdot S_j \times S_{j+1} + \Delta (S_j^z S_{j+1}^z + S_{j+1}^z S_j^z) + J S_j^z S_{j+1}^z)\), with \(\alpha_j - \alpha_{j+1} = \tan^{-1}(D/\Delta)\) or a form of Eq. (1). \(H' = \sum_j (1 + (-1)^j \delta)(\mathbf{D} \cdot S_j \times S_{j+1} + \Delta (S_j^z S_{j+1}^z + S_{j+1}^z S_j^z) + J S_j^z S_{j+1}^z)\), with \(\alpha_j - \alpha_{j+1} = \pi/2\). Hence, the bond alternation can lead the even and the odd Haldane phases in the systems described by the Hamiltonian \(H'\) with the DM interaction. Furthermore, according to Kohmoto, den Nijs, and Kadanoff [31], the transformed spin-1/2 XXZ chain with the bond alternation can be mapped to the one dimensional quantum Ashkin-Teller model, \(H_{AT} = -\sum_{j} (\sigma_j^x \tau_{j+1}^+ + \tau_j^+ \sigma_{j+1}^x) + \lambda \sigma_j^y \tau_j^z - \eta \sum (\sigma_j^x + \tau_j^+ + \lambda \sigma_j^y \tau_j^z)\) with \(J = J' / \sqrt{2D^2 + 2\Delta^2}\) and \(\delta = (\eta - 1)/(\eta + 1)\), where \(\sigma_j^x\) and \(\tau_j^z\) are Pauli matrices. This Ashkin-Teller model possesses a \(Z_2\times Z_2\) symmetry because it is invariant under the unitary transformation \(\sigma_j^x \rightarrow -\sigma_j^x\) and \(\tau_j^z \rightarrow -\tau_j^z\). Then, for \(\delta = 0\), our topological quantum Gaussian critical line corresponds to the \(\delta = 0\) line for \((1 + (-1)^j \delta)(\mathbf{D} \cdot S_j \times S_{j+1} + \Delta (S_j^z S_{j+1}^z + S_{j+1}^z S_j^z) + J S_j^z S_{j+1}^z)\) with \(\alpha_j - \alpha_{j+1} = \pi/2\). The continuously varying critical exponents in our topological quantum Gaussian transition line [Fig. 4(b)] agrees well with the exact critical exponent function for the Gaussian critical line of the Ashkin-Teller model. At the point \(\delta = 0\) for \(\lambda = (D/J) = 1\), our model and the spin-1/2 XXZ chain are \(U(1)\) symmetric, whereas the Ashkin-Teller model is \(Z_2 \times Z_2\) symmetric. The universality class of the point \(\delta = 0\) for \(D/J = 1\) is of the Berezinskii-Kosterlitz-Thouless type.

**Summary.**— We have established that the Ising chain with the DM interaction is in the even- or the odd Haldane phases induced by the bond alternation. For \(D \geq J\), the direct transition between the topologically ordered states belongs to the Gaussian type transition. For \(D < J\), the indirect phase transition is undergone through the intermediate antiferromagnetic state that is distinguished by the Ising type of quantum phase transitions from the topologically ordered states. In addition, the quantum entanglement was shown to detect the TQPTs as well as to classify the universality classes. The FLS was shown to be a useful tool to detect the degenerate groundstates indicating a spontaneous broken-symmetry phase as well as the TQPTs. Those results have been obtained from the iMPS numerical calculation. We have finally discussed that the same critical phenomena can be seen in various anisotropic Heisenberg types of spin chain models with the bond alternation.

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