Generalized Tensor Function via the Tensor Singular Value Decomposition based on the T-Product

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Abstract

In this paper, we present the definition of generalized tensor function according to the tensor singular value decomposition (T-SVD) via the tensor T-product. Also, we introduce the compact singular value decomposition (T-CSVD) of tensors via the T-product, from which the projection operators and Moore Penrose inverse of tensors are also obtained. We also establish the Cauchy integral formula for tensors by using the partial isometry tensors and applied it into the solution of tensor equations. Then we establish the generalized tensor power and the Taylor expansion of tensors. Explicit generalized tensor functions are also listed. We define the tensor bilinear and sesquilinear forms and proposed theorems on structures preserved by generalized tensor functions. For complex tensors, we established an isomorphism between complex tensors and real tensors. In the last part of our paper, we find that the block circulant operator established an isomorphism between tensors and matrices. This isomorphism is used to prove the F-stochastic structure is invariant under generalized tensor functions. The concept of invariant tensor cones is also raised.

Keywords. T-product, T-SVD, T-CSVD, generalized tensor function, pseudo-inverse, Cauchy integral formula, tensor bilinear form, Jordan algebra, Lie algebra, block tensor multiplication, complex-to-real isomorphism, tensor-to-matrix isomorphism.

AMS Subject Classifications. 15A48, 15A69, 65F10, 65H10, 65N22.

1 Introduction

Matrix functions have wide applications in many fields. They emerge as exponential integrators in differential equations. For square matrices, people usually define the matrix function by using its Jordan canonical form [15] [18]. Unfortunately, this kind of method could not be extended to rectangular matrices. In 1972, Hawkins and Ben-Israel [17] (or [5, Chapter 6]) first introduced the generalized matrix functions by using the singular value decomposition and compact singular value decomposition for rectangular matrices. It has been recognized that the generalized matrix functions have great uses in data science, matrix optimization problems, Hamiltonian dynamical systems and etc. Recently, Benzi and Huang considered the structural properties which are preserved by generalized matrix functions [1, 3, 6]. Noferini [41] provided the formula for the Fréchet derivative of a generalized matrix function.

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As high-dimension analogues of matrices, a tensor means a hyperdimensional matrix, they are extensions of matrices. The difference is that a matrix entry $a_{ij}$ has two indices $i$ and $j$, while a tensor entry $a_{i_1 \ldots i_m}$ has $m$ indices $i_1, \ldots, i_m$. We call $m$ the order of tensor, if the tensor has $m$ lower indices. Let $\mathbb{C}$ be the complex field and $\mathbb{R}$ be the real field. For a positive integer $N$, let $[N] = \{1, 2, \cdots, N\}$. We say a tensor is a real tensor if all its entries are in $\mathbb{R}$ and a tensor is a complex tensor if all its entries are in $\mathbb{C}$.

Recently, the tensor T-product has been established and proved to be a useful tool in many areas, such as image processing [27, 28, 40, 44, 46, 56], computer vision [4, 16, 49, 51], signal processing [9, 33, 36, 45], data completion and denoising [11, 24, 25, 34, 36, 38, 39, 42, 47, 50, 52, 53, 54, 55], since the tensor T-product change problems to block circulant matrices which could be block diagonalizable by the fast Fourier transformation [8, 14]. Because of the importance of tensor T-product, in 2018, Lund [37] gave the definition for tensor functions via the T-product of third-order F-square tensors which means all the front slices of a tensor is square matrices.

In this paper, we generalize the tensor T-function from F-square third order tensors to rectangular tensors $A \in \mathbb{R}^{m \times n \times p}$ by using tensor singular value decomposition (T-SVD) and tensor compact singular value decomposition (T-CSVD). Kilmer [29] gives the tensor singular value decomposition in 2011 (See Fig. 1), which gives a new tensor representation and compression idea based on the tensor T-product method especially for third order tensors. The tensor singular value decomposition of tensor $A \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ is given by [16, 28, 29]

$$A = U \ast S \ast V^H,$$

where $U \in \mathbb{C}^{n_1 \times n_1 \times n_3}$ and $V \in \mathbb{C}^{n_2 \times n_2 \times n_3}$ are orthogonal tensors and $S \in \mathbb{C}^{n_1 \times n_2 \times n_3}$ is a F-diagonal tensor respectively. The entries in $S$ are called the singular tubes of $A$. By using this kind of decomposition, the definition of general tensor functions can be raised.

This paper is organized as follows. We make some review of the definition of tensor T-product and some algebraic structure of third order tensors via this kind of product in Preliminaries. Then we recall the definition of T-function given by Lund and some of its properties. In the main part of our paper, we introduce the definition of tensor
singular value decomposition and the generalized matrix functions. Then we extend the
generalized matrix functions to tensors. Properties are also given in the following part.
In order to illustrate the generalized tensor functions explicitly, we present the definition
of tensor compact singular value decomposition and tensor rank. Orthogonal projection
tensors and Cauchy integral formula of tensor functions are also provided. As a special
case of generalized tensor functions, the expression of the Moore-Penrose inverse and the
resolvent of a tensor are also introduced, which have the applications to give the solution
of the tensor equation
\[ A * X * B = D. \]

We give the definition of tensor power by using orthogonal projection. Taylor expansion
of some explicit tensor function are also listed. It should be noticed that, for simplicity
of illustration, we only propose results for third order tensors. Results for \( n \)-th order
tensors can also be deduced by our methods, see [35]. By establishing the block tensor
multiplication via T-product, we summarize many kinds of special tensor structures which
are preserved by generalized tensor functions, which is mainly in multiplicative group
\( G \), Lie algebra \( L \) and Jordan algebra \( J \). Centrohermitian structure and block circulant
structure are also considered. Since isomorphism relations are very important relation in
algebra, we further establish the complex-to real isomorphism which commutes with the
generalized tensor function, so complex tensor functions can be isomorphically changed
to real tensor functions. In the last part of our paper, we find that the block circulant
operator ‘bcirc’ establishes an isomorphism between matrices and tensors, which means
the generalized functions of tensors commutes with the operator ‘bcirc’. Now we can say
the generalized matrix function becomes a special case of the generalized tensor function.
As a beautiful application of this theorem, we defined the F-stochastic tensor structure
and proved that this kind of tensor structure is invariant under the generalized tensor
functions.

2 Preliminaries

2.1 Notation and index

A new concept is proposed for multiplying third-order tensors, based on viewing a tensor
as a stack of frontal slices. Suppose two tensors \( A \in \mathbb{R}^{m \times n \times p} \) and \( B \in \mathbb{R}^{n \times s \times p} \) and denote
their frontal faces respectively as \( A^{(k)} \in \mathbb{R}^{m \times n} \) and \( B^{(k)} \in \mathbb{R}^{n \times s}, k = 1, 2, \ldots, p \). We also
define the operations bcirc, unfold and fold as [16, 28, 29],

\[
\text{bcirc}(A) := \begin{bmatrix}
A^{(1)} & A^{(p)} & A^{(p-1)} & \cdots & A^{(2)} \\
A^{(2)} & A^{(1)} & A^{(p)} & \cdots & A^{(3)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A^{(p)} & A^{(p-1)} & \cdots & A^{(2)} & A^{(1)}
\end{bmatrix},
\text{unfold}(A) := \begin{bmatrix}
A^{(1)} \\
A^{(2)} \\
\vdots \\
A^{(p)}
\end{bmatrix},
\]

and \( \text{fold(unfold}(A)) := A \). We can also define the corresponding inverse operation
bcirc\(^{-1} : \mathbb{R}^{mp \times np} \rightarrow \mathbb{R}^{m \times n \times p} \) such that bcirc\(^{-1}(\text{bcirc}(A)) = A \).

2.2 The tensor T-Product

The following definitions and properties are introduced in [16, 28, 29].
Definition 1. (T-product) Let $A \in \mathbb{R}^{m \times n \times p}$ and $B \in \mathbb{R}^{n \times s \times p}$ be two real tensors. Then the T-product $A \ast B$ is a $m \times s \times p$ real tensor defined by

$$A \ast B := \text{fold}(\text{bcirc}(A)\text{unfold}(B)).$$

We can also introduce definitions of transpose, identity and orthogonal of tensors as follows.

Definition 2. (Transpose and conjugate transpose) If $A$ is a third order tensor of size $m \times n \times p$, then the transpose $A^\top$ is obtained by transposing each of the frontal slices and then reversing the order of transposed frontal slices $2$ through $n$. The conjugate transpose $A^H$ is obtained by conjugate transposing each of the frontal slices and then reversing the order of transposed frontal slices $2$ through $n$.

Definition 3. (Identity tensor) The $n \times n \times p$ identity tensor $I_{nnp}$ is the tensor whose first frontal slice is the $n \times n$ identity matrix, and whose other frontal slices are all zeros.

It is easy to check $A \ast I_{nnp} = A$ for $A \in \mathbb{R}^{m \times n \times p}$.

Definition 4. (Orthogonal and unitary tensor) An $n \times n \times p$ real-valued tensor $P$ is orthogonal if $P^\top \ast P = P \ast P^\top = I$. An $n \times n \times p$ complex-valued tensor $Q$ is unitary if $Q^H \ast Q = Q \ast Q^H = I$.

For a frontal square tensor $A$ of size $n \times n \times p$, it has inverse tensor $B(= A^{-1})$, provided that

$$A \ast B = I_{nnp} \quad \text{and} \quad B \ast A = I_{nnp}.$$

It should be noticed that invertible third order tensors of size $n \times n \times p$ forms a group, since the invertibility of tensor $A$ is equivalent to the invertibility of the matrix $\text{bcirc}(A)$, and the set of invertible matrices forms a group. Also, the orthogonal tensors via the tensor t-product also forms a group, since $\text{bcirc}(Q)$ is an orthogonal matrix.

2.3 Tensor T-Function

First, we make some recall for the functions of square matrices based on the Jordan canonical form \[15, 18\].

Let $A \in \mathbb{C}^{n \times n}$ be a matrix with spectrum $\lambda(A) := \{\lambda_j\}_{j=1}^n$, where $N \leq n$ and $\lambda_j$ are distinct. Each $m \times m$ Jordan block $J_m(\lambda)$ of an eigenvalue $\lambda$ has the form

$$J_m(\lambda) = \begin{bmatrix}
\lambda & 1 \\
& \ddots & \ddots \\
& & \ddots & 1 \\
& & & \lambda
\end{bmatrix} \in \mathbb{C}^{m \times m}.
$$

Suppose that $A$ has the Jordan canonical form

$$A = XJX^{-1} = X\text{diag}(J_{m_1}(\lambda_{j_1}), \cdots, J_{m_p}(\lambda_{j_p}))X,$$

with $p$ blocks of sizes $m_i$ such that $\sum_{i=1}^p m_i = n$, and the eigenvalues $\{\lambda_{j_k}\}_{k=1}^p \in \text{spec}(A)$.

Definition 5. (Matrix function) Suppose $A \in \mathbb{C}^{n \times n}$ has the Jordan canonical form and the matrix function is defined as

$$f(A) := Xf(J)X^{-1},$$
where \( f(J) := \text{diag}(f(J_{n_1}(\lambda_{jk})), \ldots, f(J_{m_1}(\lambda_{jk}))) \), and

\[
f(J_{m_1}(\lambda_{jk})) := \begin{bmatrix}
f(\lambda_{jk}) & f'(\lambda_{jk}) & \frac{f''(\lambda_{jk})}{2!} & \cdots & \frac{f^{(n_{jk}-1)}(\lambda_{jk})}{(n_{jk}-1)!}
f(\lambda_{jk}) & f'(\lambda_{jk}) & \cdots & \cdots & \frac{f''(\lambda_{jk})}{2!}
\vdots & \ddots & \ddots & \ddots & \vdots
\vdots & \ddots & \ddots & \ddots & \frac{f''(\lambda_{jk})}{2!}
0 & \cdots & \cdots & \cdots & f(\lambda_{jk})
0 & \cdots & \cdots & \cdots & f(\lambda_{jk})
\end{bmatrix} \in \mathbb{C}^{m_i \times m_i}.
\]

There are various matrix function properties throughout the theorems of matrix analysis. Here we make review of these properties and the proofs could be found in the excellent monograph [18].

**Lemma 1.** Suppose \( A \) is a complex matrix of size \( n \times n \) and \( f \) is a function defined on the spectrum of \( A \). Then we have

1. \( f(A)A = Af(A) \),
2. \( f(A^H) = f(A)^H \),
3. \( f(XAX^{-1}) = Xf(A)X^{-1} \),
4. \( f(\lambda) \in \text{spec}(f(A)) \) for all \( \lambda \in \text{spec}(A) \).

By using the concept of T-product, the matrix function is generalized to tensors of size \( n \times n \times p \). Suppose we have tensors \( A \in \mathbb{C}^{n \times n \times p} \) and \( B \in \mathbb{C}^{n \times n \times p} \), then the tensor T-function of \( A \) is defined by [37]

\[
f(A) \ast B := \text{fold}(f(\text{bcirc}(A)) \cdot \text{unfold}(B)),
\]

or equivalently

\[
f(A) := \text{fold}(f(\text{bcirc}(A))\hat{E}_{1 \times n \times n}^{np \times n}),
\]

here \( \hat{E}_{1 \times n \times n}^{np \times n} = \hat{e}_k^p \otimes I_{n \times n} \), where \( \hat{e}_k^p \in \mathbb{C}^p \) is the vector of all zeros except for the \( k \)-th entry and \( I_{n \times n} \) is the identity matrix.

There is another way to express \( \hat{E}_{1 \times n \times n}^{np \times n} \):

\[
\hat{E}_{1 \times n \times n}^{np \times n} = \begin{bmatrix}
I_{n \times n} & 0 \\
0 & \vdots \\
\vdots & 0 \\
0 & 0
\end{bmatrix} \cdot 1 \otimes I_{n \times n} = \text{unfold}(I_{n \times n \times p}).
\]

Note that \( f \) on the right-hand side of the equation is merely the matrix function defined above, so the T-function is well-defined.

From this definition, we could see that for a tensor \( A \in \mathbb{C}^{n \times n \times p} \) and \( \text{bcirc}(A) \) is a block circulant matrix of size \( np \times np \). The frontal faces of \( A \) are the block entries of \( A\hat{E}_{1 \times n \times n}^{np \times n} \), then \( A = \text{fold}(AE_{1 \times n \times n}^{np \times n}) \).

In order to get further properties of generalized tensor functions, we make some review of the results on block circulant matrices and the tensor T-product.

**Lemma 2.** [8] Suppose \( A, B \in \mathbb{C}^{np \times np} \) are block circulant matrices with \( n \times n \) blocks. Let \( \{\alpha_j\}_{j=1}^k \) be scalars. Then \( A^T, A^H, \alpha_1A + \alpha_2B, AB, q(A) = \sum_{j=1}^k \alpha_jA^j \) and \( A^{-1} \) are also block circulant.
Lemma 3. Suppose we have tensors $A \in \mathbb{C}^{n \times n \times p}$ and $B \in \mathbb{C}^{n \times s \times p}$. Then

1. $\text{unfold}(A) = \text{bcirc}(A)\overline{E_1}^{np \times n}$,
2. $\text{bcirc}(\text{fold}(\text{bcirc}(A)\overline{E_1}^{np \times n})) = \text{bcirc}(A)$,
3. $\text{bcirc}(A \ast B) = \text{bcirc}(A)\text{bcirc}(B)$,
4. $\text{bcirc}(A^j) = \text{bcirc}(A^j)$, for all $j = 0, 1, \cdots$,
5. $(A \ast B)^H = B^H \ast A^H$.
6. $\text{bcirc}(A^\top) = (\text{bcirc}(A))^\top$, $\text{bcirc}(A^H) = (\text{bcirc}(A))^H$.

3 Main Results

3.1 Generalized tensor function

Since the tensor T-product and tensor T-function could only be defined for F-square tensors, i.e., tensors of size $n \times n \times p$. In this section, we generalize the concept of tensor T-functions to tensors of size $n \times m \times p$, that is F-rectangular tensors. In order to do this, we first introduced the T-SVD decomposition of a f-rectangular tensor.

Lemma 4. Let $A$ be an $n_1 \times n_2 \times n_3$ real-valued tensor. Then $A$ can be factored as

$$A = U \ast S \ast V^\top,$$

where $U$, $V$ are unitary $n_1 \times n_1 \times n_3$ and $n_2 \times n_2 \times n_3$ tensor respectively, and $S$ is a $n_1 \times n_2 \times n_3$ F-diagonal tensor.

In matrix theories, the generalize matrix function of a $m \times n$ matrix has been introduced by using the matrix singular value decomposition (SVD) and the Moore-Penrose inverse of the matrix.

Let $A \in \mathbb{C}^{m \times n}$ and the singular value decomposition be

$$A = U \Sigma V^H.$$

Let $r$ be the rank of $A$. Consider the matrices $U_r$ and $V_r$ formed with the first $r$ columns of $U$ and $V$, and let $\Sigma_r$ be the leading $r \times r$ principal submatrix of $\Sigma$ whose diagonal entries are $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$. Then we have the compact SVD,

$$A = U_r \Sigma_r V_r^H.$$

Hawkins and Ben-Israel (or [5, Chapter 6]) present the spectral theory of rectangular matrices by the SVD. Let $f : \mathbb{R} \to \mathbb{R}$ be a scalar function such that $f(\sigma_i)$ is defined for all $i = 1, 2, \cdots, r$.

Define the generalized matrix function induced by $f$ as

$$f^\ast(A) = U_r f(\Sigma_r) V_r^H,$$

where

$$f(\Sigma_r) = \begin{bmatrix} f(\sigma_1) & \cdots & f(\sigma_r) \\ \vdots & \ddots & \vdots \\ f(\sigma_r) & \cdots & f(\sigma_1) \end{bmatrix}.$$

The induced function $f^\ast(A)$ reduces to the standard matrix function $f(A)$ whenever $A$ is Hermitian positive definite, or when $A$ is Hermitian positive semi-definite and $f$ satisfies $f(0) = 0$. 
Lemma 5. Let $A \in \mathbb{C}^{m \times n}$ be a matrix of rank $r$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a scalar function, and let $f^\circ : \mathbb{C}^{m \times n} \rightarrow \mathbb{C}^{m \times n}$ be the induced generalized matrix function. Then
\begin{enumerate}
  
  \item $[f^\circ(A)]^H = f^\circ(A^H)$,
  
  \item Let $X \in \mathbb{C}^{m \times m}$ and $Y \in \mathbb{C}^{n \times n}$ be two unitary matrices, then $f^\circ(XAY) = X[f^\circ(A)]Y$,
  
  \item If $A = A_1 \oplus A_2 \oplus \cdots \oplus A_k$, then $f^\circ(A) = f^\circ(A_1) \oplus f^\circ(A_2) \oplus \cdots \oplus f^\circ(A_k)$, here $\oplus$ is the direct sum of matrices.
  
  \item $f^\circ(A) = f((\sqrt{AA^H})(\sqrt{AA^H})^\dagger A) = A(\sqrt{AH})^\dagger f(\sqrt{AH}A)$, where $M^\dagger$ is the Moore-Penrose inverse of $M$ \cite{2}, and $\sqrt{AH}A$ is the square root of $A^H A$ \cite{3} Chapter 6.
\end{enumerate}

Also we introduce these simple results without proof.

Corollary 1. Let $A \in \mathbb{C}^{m \times n}$ be a matrix of rank $r$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a scalar function, and let $f^\circ : \mathbb{C}^{m \times n} \rightarrow \mathbb{C}^{m \times n}$ be the induced generalized matrix function. Then
\begin{enumerate}
  
  \item $[f^\circ(A)]^H = f^\circ(A^H)$,
  
  \item $f^\circ(A) = f^\circ(\bar{A})$.
\end{enumerate}

Now, we use Lemma 5 to establish the generalized tensor function for F-rectangular tensors of size $n_1 \times n_2 \times n_3$.

Theorem 1. (Generalized tensor function) Suppose $A \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ is a third order tensor, $A$ has the $T$-SVD decomposition

\[ A = U \ast S \ast V^H, \]

where $U, V$ are unitary $n_1 \times n_1 \times n_3$ and $n_2 \times n_2 \times n_3$ tensor respectively, and $S$ is a $n_1 \times n_2 \times n_3$ $F$-diagonal tensor, which can be factorized as follows:

\[ \text{bcirc}(U) = (F_{n_3} \otimes I_{n_1}) \begin{bmatrix} U_1 & U_2 & \cdots & U_{n_3} \\ \Sigma_1 & & & \\ & \Sigma_2 & & \\ & & \ddots & \\ & & & \Sigma_{n_3} \end{bmatrix} (F_{n_3}^H \otimes I_{n_1}), \quad U_i \in \mathbb{C}^{n_1 \times n_1}, \ i = 1, 2, \cdots, n_3. \] \hfill (1)

\[ \text{bcirc}(S) = (F_{n_3} \otimes I_{n_1}) \begin{bmatrix} V_1^H \\ \Sigma_1 \\ \Sigma_2 \\ \vdots \\ \Sigma_{n_3} \end{bmatrix} (F_{n_3}^H \otimes I_{n_1}), \quad \Sigma_i \in \mathbb{R}^{n_1 \times n_2}, \ i = 1, 2, \cdots, n_3. \] \hfill (2)

\[ \text{bcirc}(V^T) = (F_{n_3} \otimes I_{n_2}) \begin{bmatrix} V_1^H \\ \vdots \\ V_{n_3}^H \end{bmatrix} (F_{n_3}^H \otimes I_{n_2}), \quad V_i^H \in \mathbb{C}^{n_2 \times n_2}, \ i = 1, 2, \cdots, n_3, \] \hfill (3)

where $F_{n_3}$ is the discrete Fourier matrix of size $n_3 \times n_3$, which is defined as \cite{3}

\[
F_{n \times n} = \frac{1}{\sqrt{n}} \begin{bmatrix}
1 & 1 & 1 & 1 & \cdots & 1 \\
1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\
1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\
1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)}
\end{bmatrix},
\]
where \( \omega = e^{-2\pi i/n} \) is the a primitive \( n \)th root of unity in which \( i^2 = -1 \). \( F_{n_3}^H \) is the conjugate transpose of \( F_{n_3} \).

If \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a scalar function, the induced function \( f^\diamond : \mathbb{R}^{n_1 \times n_2 \times n_3} \rightarrow \mathbb{R}^{n_1 \times n_2 \times n_3} \) could be defined as:

\[
f^\diamond(A) = U \ast \hat{f}(S) \ast V^H,
\]

where \( \hat{f}(S) \) is given by

\[
\hat{f}(S) = \text{bcirc}^{-1} \left( \begin{pmatrix} \hat{f}(\Sigma_1) & \hat{f}(\Sigma_2) & \cdots & \hat{f}(\Sigma_{n_3}) \end{pmatrix} \right),
\]

where

\[
\hat{f}(\Sigma_i) = f(\sqrt{\Sigma_i} \Sigma_i^H)(\sqrt{\Sigma_i} \Sigma_i^H)^\dagger \Sigma_i = \Sigma_i(\sqrt{\Sigma_i^H \Sigma_i})^\dagger f(\sqrt{\Sigma_i^H \Sigma_i}), \quad i = 1, 2, \ldots, n_3.
\]

**Proof.** Since tensor \( A \) has the T-SVD decomposition [28, 29]

\[
A = U \ast S \ast V^H,
\]

taking “bcirc” on both sides of the equation

\[
\text{bcirc}(A) = \text{bcirc}(U \ast S \ast V^H)
\]

\[
= \text{bcirc}(U) \cdot \text{bcirc}(S) \cdot \text{bcirc}(V^H),
\]

where \( \text{bcirc}(U) \) and \( \text{bcirc}(V^H) \) are unitary matrices of order \( n_1 n_3 \times n_1 n_3 \) and \( n_2 n_3 \times n_2 n_3 \) respectively, and

\[
\text{bcirc}(S) = (F_{n_3} \otimes I_{n_1}) \begin{bmatrix} \Sigma_1 & \Sigma_2 & \cdots & \Sigma_{n_3} \end{bmatrix} (F_{n_3}^H \otimes I_{n_2}),
\]

is a block circulant matrix in \( \mathbb{C}^{n_1 n_3 \times n_2 n_3} \). By Lemma 1, the induced function on \( S \) could be defined by

\[
\hat{f}(S) = \text{bcirc}^{-1} \left( \begin{pmatrix} \hat{f}(\Sigma_1) & \hat{f}(\Sigma_2) & \cdots & \hat{f}(\Sigma_{n_3}) \end{pmatrix} \right),
\]

then we define

\[
\text{bcirc}(f^\diamond(A)) = \text{bcirc}(U) \cdot \text{bcirc}(\hat{f}(S)) \cdot \text{bcirc}(V^H).
\]

Taking ‘bcirc’ on both sides of the equation, it turns out

\[
f^\diamond(A) = U \ast \hat{f}(S) \ast V^H.
\]
Remark 1. An important observation is that the scalar function $f$ could be assumed to be an odd function, since $f$ is only defined for non-negative numbers, we can complete it to an odd function by setting $f(0) = 0$ and $f(x) = -f(-x)$ for all $x < 0$. 

Corollary 2. Suppose $A \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ is a third order tensor. Let $f : \mathbb{R} \to \mathbb{R}$ be a scalar function and let $f^{\hat{\circ}} : \mathbb{R}^{n_1 \times n_2 \times n_3} \to \mathbb{R}^{n_1 \times n_2 \times n_3}$ be the corresponding generalized function of third order tensors. Then

1. $[f^{\hat{\circ}}(A)]^\top = f^{\hat{\circ}}(A^\top)$.

2. Let $P \in \mathbb{R}^{n_1 \times n_1 \times n_3}$ and $Q \in \mathbb{R}^{n_2 \times n_2 \times n_3}$ be two orthogonal tensors, then $f^{\hat{\circ}}(P \ast A \ast Q) = P \ast f^{\hat{\circ}}(A) \ast Q$.

Proof. (1) From Theorem 1, if $A$ has the T-SVD decomposition $A = U \ast S \ast V^H$, taking the transpose on both sides, then it turns to be $A^\top = V \ast S \ast U^\top$.

It follows that $f^{\hat{\circ}}(A^\top) = V \ast \hat{f}(S) \ast U^\top = [U \ast \hat{f}(S) \ast V^H]^\top = [f^{\hat{\circ}}(A)]^\top$.

(2) The results follows from the fact that the unitary tensors form a group under the multiplication ‘$\ast$’. Define $B = P \ast A \ast Q$, then we obtain

$$f^{\hat{\circ}}(B) = f^{\hat{\circ}}(P \ast U \ast S \ast V^H \ast Q) = (P \ast U) \hat{f}(S)(Q^\top \ast V)^H = P \ast U \ast \hat{f}(S) \ast V^H \ast Q = P \ast f^{\hat{\circ}}(A) \ast Q.$$ 

Corollary 3. (Composite functions) Suppose $A \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ is a third order tensor and it has T-SVD decomposition $A = U \ast S \ast V^H$. Let $\{\sigma_1, \sigma_2, \cdots, \sigma_r\}$ be the singular values set of $bcirc(S)$. Assume $h : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are two scalar functions and $g(h(\sigma_i))$ exists for all $i$.

Let $g^{\hat{\circ}} : \mathbb{R}^{n_1 \times n_2 \times n_3} \to \mathbb{R}^{n_1 \times n_2 \times n_3}$ and $h^{\hat{\circ}} : \mathbb{R}^{n_1 \times n_2 \times n_3} \to \mathbb{R}^{n_1 \times n_2 \times n_3}$ be the induced generalized tensor functions. Moreover, let $f : \mathbb{R} \to \mathbb{R}$ be the composite function $f = g \circ h$. Then the induced tensor function $f^{\hat{\circ}} : \mathbb{R}^{n_1 \times n_2 \times n_3} \to \mathbb{R}^{n_1 \times n_2 \times n_3}$ satisfies

$$f^{\hat{\circ}}(A) = g^{\hat{\circ}}(h^{\hat{\circ}}(A)).$$

Proof. Let $B = h^{\hat{\circ}}(A) = U \ast \hat{f}(\Sigma) \ast V^H = U \ast \Theta \ast V^H$. Let $P \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ be a permutation tensor such that $\Theta = P \ast \Theta \ast P^\top$ has diagonal entries ordered in non-increasing order. Then it follows that tensor $B$ is given by

$$B = \tilde{U} \ast \tilde{\Theta} \ast \tilde{V}^H,$$
where \( \tilde{U} = U \ast P \) and \( \tilde{V} = V \ast P \). It follows that
\[
g^\wedge(h^\wedge(A)) = g^\wedge(B)
\]
\[
= \tilde{U} \ast \hat{g}(\hat{\Theta}) \ast \tilde{V}^H
\]
\[
= U \ast P \ast \hat{g}(\hat{\Theta}) \ast P^T \ast V^H
\]
\[
= U \ast \hat{g}(P \ast \hat{\Theta} \ast P^T) \ast V^H
\]
\[
= U \ast \hat{g}(\Theta) \ast V^H
\]
\[
= U \ast \hat{f}(\Sigma) \ast V^H
\]
\[
= f^\wedge(A).
\]

The next corollaries describe the relationship between the generalized tensor functions and the tensor T-functions.

**Corollary 4.** Suppose \( A \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) is a third order tensor, and let \( f : \mathbb{R} \to \mathbb{R} \) be a scalar function, and we also use \( f \) to denote the tensor T-function. Let \( f^\wedge : \mathbb{R}^{n_1 \times n_2 \times n_3} \to \mathbb{R}^{n_1 \times n_2 \times n_3} \) be the induced generalized tensor function. Then
\[
f^\wedge(A) = f(\sqrt{A \ast A}^T) \ast (\sqrt{A \ast A}^T)^\dagger \ast A = A \ast (\sqrt{A^T \ast A})^\dagger \ast f(\sqrt{A^T \ast A}).
\]

*Proof.* This identity is the consequence of the fact that the generalized tensor function is equal to the tensor T-function when the tensor is a F-square tensor. \( \square \)

**Corollary 5.** Suppose \( A \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) is a third order tensor, and let \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) be two scalar functions, such that \( f^\wedge(A) \) and \( g(A \ast A^T) \) are defined. Then we have
\[
g(A \ast A^T) \ast f^\wedge(A) = f^\wedge(A) \ast g(A^T \ast A).
\]

*Proof.* From \( A = U \ast S \ast V^H \), it follows \( A \ast A^T = U \ast (S \ast S^H) \ast U^H \); thus we have
\[
g(A \ast A^T) \ast f^\wedge(A) = U \ast g(S \ast S^H) \ast U^H \ast U \ast \hat{f}(S) \ast V^H
\]
\[
= U \ast \hat{g}(S \ast S^H) \ast \hat{f}(S) \ast V^H
\]
\[
= U \ast \hat{f}(S) \ast U \ast \hat{g}(S^H \ast S) \ast V^H
\]
\[
= U \ast \hat{f}(S) \ast U \ast g(S \ast S^H) \ast U \ast \hat{g}(S^H \ast S) \ast V^H
\]
\[
= f^\wedge(A) \ast g(A^T \ast A).
\]

\( \square \)

### 3.2 Tensor T-CSVD and orthogonal projection

Since the compact singular value decomposition (CSVD) plays an important role in the Moore-Penrose inverse theorem and the least squares problem. In this subsection, we extend the T-SVD theorem to tensor T-CSVD via the T-product of tensors. We first introduce some concept of tensors analogue to the matrix analysis.
Definition 6. Let $\mathcal{A}$ be an $n_1 \times n_2 \times n_3$ real-valued tensor.

1. The range space of $\mathcal{A}$, $\mathcal{R}(\mathcal{A}) := \text{Ran}(\text{bcirc}(\mathcal{A}))$,
2. The null space of $\mathcal{A}$, $\mathcal{N}(\mathcal{A}) := \text{Null}(\text{bcirc}(\mathcal{A}))$,
3. The tensor norm $\|\mathcal{A}\| := \|\text{bcirc}(\mathcal{A})\|$,
4. The Moore-Penrose inverse $\mathcal{A}^\dagger = \text{bcirc}^{-1}((\text{bcirc}(\mathcal{A}))^\dagger)$.

Now we introduce the tensor T-CSVD via the T-product according to the T-SVD of tensors.

Suppose $\mathcal{A}$ be an $n_1 \times n_2 \times n_3$ real-valued tensor. Then we have the tensor T-SVD of $\mathcal{A}$

$$\mathcal{A} = \mathcal{U} \ast \mathcal{S} \ast \mathcal{V}^H,$$

where $\mathcal{U}$, $\mathcal{V}$ are unitary $n_1 \times n_1 \times n_3$ and $n_2 \times n_2 \times n_3$ tensor respectively, and $\mathcal{S}$ is a $n_1 \times n_2 \times n_3$ F-diagonal tensor, which can be factorized as Equations (1), (2), and (3).

Suppose $\Sigma_i$ has rank $r_i$, denote $U_i = (x_1^{(i)}, x_2^{(i)}, \ldots, x_{n_1}^{(i)})$ and $V_i = (y_1^{(i)}, y_2^{(i)}, \ldots, y_{n_2}^{(i)})$, $i = 1, 2, \ldots, n_3$ and $r = \max_{i=1}^{n_3}\{r_i\}$ is usually called the tubal-rank of $\mathcal{A}$ via the T-product [29]. We introduce the T-CSVD by ignoring the “0” singular values and the corresponding tubes. That is

$$(\Sigma_i)_r = \text{diag}(c_1^{(i)}, c_2^{(i)}, \ldots, c_r^{(i)}) \in \mathbb{R}^{r \times r},$$

$$(U_i)_r = (x_1^{(i)}, x_2^{(i)}, \ldots, x_r^{(i)}) \in \mathbb{C}^{n_1 \times r},$$

$$(V_i)_r = (y_1^{(i)}, y_2^{(i)}, \ldots, y_r^{(i)}) \in \mathbb{C}^{n_2 \times r},$$

then it turns out to be

$$\text{bcirc}(\mathcal{A}) = (F_{n_3} \otimes I_{n_1}) \begin{bmatrix} (U_1)_r & (U_2)_r & \cdots & (U_{n_3})_r \end{bmatrix} \begin{bmatrix} (\Sigma_1)_r \\ (\Sigma_2)_r \\ \vdots \\ (\Sigma_{n_3})_r \end{bmatrix} \begin{bmatrix} (V_1)_r^H & (V_2)_r^H & \cdots & (V_{n_3})_r^H \end{bmatrix} \times (F_{n_3} \otimes I_{n_2})$$

$$= \text{bcirc}(\mathcal{U}(r)) \text{bcirc}(\mathcal{S}(r)) \text{bcirc}(\mathcal{V}(r)^H),$$

where $\mathcal{U}(r) \in \mathbb{C}^{n_1 \times r \times n_3}$, $\mathcal{S}(r) \in \mathbb{C}^{r \times r \times n_3}$, $\mathcal{V}(r) \in \mathbb{C}^{r \times n_2 \times n_3}$, that is to say

$$\mathcal{A} = \mathcal{U}(r) \ast \mathcal{S}(r) \ast \mathcal{V}(r)^H. \quad (6)$$

Remark 2. In matrix analysis, if a matrix $A$ has CSVD

$$A = U_r S_r V_r^H,$$

where $S_r = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$, we have $\sigma_i \neq 0$, for all $i \in [r]$, but in T-CSVD for tensors, we may have $c_j^{(i)} = 0$ for some $i$ and $j$, since we choose $r = \max_i\{r_i\}$.
Corollary 6. Suppose $A$ is an $n_1 \times n_2 \times n_3$ real-valued tensor. Then we have the tensor $T$-SVD of $A$ (see Fig. 2) is

$$A = U * S * V^H,$$

the $T$-CSVD of $A$ is

$$A = U^{(r)} * S^{(r)} * V^{H(r)},$$

and the Moore-Penrose inverse of tensor $A$ is given by

$$A^\dagger = V^{(r)} * S^{\dagger (r)} * U^{H(r)}.$$

Proof. It is obvious by using the identity

$$bcirc(A^\dagger) = (bcirc(A))^\dagger,$$

and the $(bcirc(A))^\dagger$ is defined by matrix Moore-Penrose inverse \cite{5}.

It is easy to establish the following results of projection.

Corollary 7. Suppose $A$ is an $n_1 \times n_2 \times n_3$ real-valued tensor and the $T$-CSVD of $A$ is

$$A = U^{(r)} * S^{(r)} * V^{H(r)},$$

then we have

1. $bcirc(U^{(r)} * U^{H(r)}) = P_{R(A)} = bcirc(A * A^\dagger), U^{H(r)} * U^{(r)} = I^r$,
2. $bcirc(V^{(r)} * V^{H(r)}) = P_{R(A^\dagger)} = bcirc(A^\dagger * A), V^{H(r)} * V^{(r)} = I^r$,
3. The tensor $E := U^{(r)} * V^{H(r)}$ is a real partial isometry tensor satisfying $bcirc(E * E^\top) = P_{R(A)}$ and $bcirc(E^\top * E) = P_{R(A^\dagger)}$.

By using the Tensor T-CSVD, we can also derive the generalized tensor function of tensors. For simplicity, we give the following results without the proof.

Theorem 2. (Compact tensor singular value decomposition) Suppose $A$ be an $n_1 \times n_2 \times n_3$ real-valued tensor and the $T$-CSVD of $A$ is

$$A = U^{(r)} * S^{(r)} * V^{H(r)},$$
If \( f : \mathbb{R} \to \mathbb{R} \) is a scalar function, the induced function \( f^\circ : \mathbb{R}^{n_1 \times n_2 \times n_3} \to \mathbb{R}^{n_1 \times n_2 \times n_3} \) could be defined as:

\[
f^\circ(A) = U(r) \ast \hat{f}(S(r)) \ast V^H(r),
\]

here \( \hat{f}(S(r)) \) is defined as above.

**Remark 3.** From the above illustration, since \( S(r) \) is a special tensor, so we can find that \( \hat{f}(S(r)) \) is \( f^\circ(S(r)) \), that is to say, \( f^\circ(A) = U(r) \ast f^\circ(S(r)) \ast V^H(r) \).

By using the projection tensor \( E \) of the T-CSVD of a tensor, we can obtain the following results.

**Corollary 8.** If \( f, g, h : \mathbb{R} \to \mathbb{R} \) are scalar functions, and \( f^\circ, g^\circ, h^\circ : \mathbb{R}^{n_1 \times n_2 \times n_3} \to \mathbb{R}^{n_1 \times n_2 \times n_3} \) are the corresponding induced generalized tensor functions. Suppose \( A \) is an \( n_1 \times n_2 \times n_3 \) real-valued tensor and the T-CSVD of \( A \) is

\[
A = U(r) \ast S(r) \ast V^H(r),
\]

the partial isometry tensor \( E = U(r) \ast V^H(r) \) is a real tensor, and we have

1. If \( f(z) = k \), then \( f^\circ(A) = kE \),
2. If \( f(z) = z \), then \( f^\circ(A) = A \),
3. If \( f(z) = g(z) + h(z) \), then \( f^\circ(A) = g^\circ(A) + h^\circ(A) \),
4. If \( f(z) = g(z)h(z) \), then \( f^\circ(A) = g^\circ(A) \ast E^\ast \ast h^\circ(A) \).

**Proof.** It follows easily from \( E^\ast = E^\dagger \). \( \Box \)

For illustration of the following part of our paper, we need to introduce \( r \times n_3 \) partial isometries tensors of \( A \), which need to satisfy the equations:

\[
E_j^{(i) \ast}E_l^{(k)} = 0, E_j^{(i)} \ast E_l^{(k)} = 0, \text{ for } i \neq k \text{ or } j \neq l.
\]

\[
E_j^{(i) \ast}E^\ast = A \ast E^\ast \ast E_j^{(i)}.
\]

By the definition of \( E \), we have

\[
\text{bcirc}(E) = (F_{n_3} \otimes I_{n_1}) \begin{bmatrix} (U_1)^r & (U_2)^r & \cdots & (U_{n_3})^r \end{bmatrix} \begin{bmatrix} (V_1)^H_r & (V_2)^H_r & \cdots & (V_{n_3})^H_r \end{bmatrix} (F_{n_3}^H \otimes I_{n_2}),
\]

then we define the complex tensor \( E_j^{(i)} \) by

\[
\text{bcirc}(E_j^{(i)}) = (F_{n_3}^H \otimes I_{n_1})(u_j^{(i)}v_j^{(i)H})(F_{n_3} \otimes I_{n_2}), i = 1, 2, \cdots, n_3, j = 1, 2, \cdots, r.
\]

and it is easy to find

\[
E = \sum_{i,j} E_j^{(i)},
\]

which means the projection tensor \( E \) is the sum of the partial isometry tensors \( E_j^{(i)} \), i.e.,

\[
E_j^{(i)} = E_j^{(i)H}.
\]

Partial isometries in Hilbert spaces were studied by von Neumann and Halmos.

It should be notice that although \( u_j^{(i)} \) and \( v_j^{(i)H} \) are complex vectors, the isometry matrix \( \text{bcirc}(E) \) is proved to be a real bi-circulant matrix (without proof), and this is the main reason why \( f^\circ \) maps a real tensor to a real tensor, not a complex tensor.
3.3 T-Eigenvalue and Cauchy integral theorem of tensors

Suppose \( A \) is an \( n_1 \times n_2 \times n_3 \) real-valued tensor and the T-CSVD of \( A \) is

\[
A = U_{(r)} \ast S_{(r)} \ast V_{(r)}^{H},
\]

the set of T-eigenvalues of \( A \) is defined to be the set \( \text{spec}(\text{bcirc}(S_{(r)} \ast S_{(r)}^{H})) \), which is equal to the set of all the positive eigenvalues of the matrix \( \text{bcirc}(A \ast A^{H}) \).

Denote the T-eigenvalue set \( \text{spec}(\text{bcirc}(S_{(r)} \ast S_{(r)}^{H})) = \{ |c_{ij}^{(i)}|^2, 1 \leq i \leq n_3, 1 \leq j \leq r \} \), where \( r = \sum_{i=1}^{n_3} r_i \). We call the non-zero elements of \( c_{ij}^{(i)} \) the T-singular values of \( A \). It can be noticed that

\[
A = \sum_{i,j} c_{ij}^{(i)} E_{ij}^{(i)}.
\]

We use the set \( \{ \gamma_k^{(l)} \} \) to be the set of distinct \( c_{ij}^{(i)} \)'s.

In the following theorems we need our function \( f \) to satisfy the following conditions: suppose \( f : \mathbb{C} \rightarrow \mathbb{C} \) is a scalar function, \( \Gamma_k^{(l)} \) is a simple closed positively oriented contour such that

1. If there is some \( c_{ij}^{(i)} = 0 \), then we need \( f \) is analytic on and inside \( \Gamma_k^{(l)} \) and \( f(0) = 0 \).

2. \( \gamma_k^{(l)} \) is inside \( \Gamma_k^{(l)} \) but no other \( \gamma_{k'}^{(l)} \) is on or inside \( \Gamma_k^{(l)} \), and the tensor \( \bar{E}_{k}^{(l)} \) is defined by

\[
\bar{E}_{k}^{(l)} = \sum_{c_{ij}^{(i)} = \gamma_k^{(l)}} c_{ij}^{(i)} E_{ij}^{(i)}.
\]

**Theorem 3.** Suppose \( f : \mathbb{C} \rightarrow \mathbb{C} \) is a scalar function and \( \Gamma_k^{(l)} \) is a contour satisfying the above conditions respectively.

1. The relationship between \( \bar{E}_{k}^{(l)} \) and \( \Gamma_k^{(l)} \) is

\[
\bar{E}_{k}^{(l)} = E \ast \left( \frac{1}{2\pi i} \int_{\Gamma_k^{(l)}} (zE - A)^{\dagger} dz \right) \ast E.
\]  

(7)

2. The tensor function \( f^{\circ} : \mathbb{C}^{n_1 \times n_2 \times n_3} \rightarrow \mathbb{C}^{n_1 \times n_2 \times n_3} \) induced by \( f : \mathbb{C} \rightarrow \mathbb{C} \) is

\[
f^{\circ}(A) = E \ast \left( \frac{1}{2\pi i} \int_{\Gamma} f(z)(zE - A)^{\dagger} dz \right) \ast E,
\]  

(8)

where \( \Gamma = \bigcup_{k,l} \Gamma_k^{(l)} \).

In particular, if all \( c_{ij}^{(i)} \neq 0 \) for the tensor \( A \), then

\[
A^\dagger = \frac{1}{2\pi i} \int_{\Gamma} \frac{(zE - A)^{\dagger}}{z} dz.
\]  

(9)

**Proof.** (1) From the T-CSVD of \( A \), we have

\[
zE - A = U_{(r)} \ast (zI - S_{(r)}) \ast V_{(r)}^{H},
\]

taking Moore-Penrose inverse on both sides of the equation, it comes to

\[
(zE - A)^{\dagger} = V_{(r)} \ast (zI - S_{(r)})^{-1} \ast U_{(r)}^{H}
\]
Let $\mathcal{W}_k^{(l)}$ be the right-hand side of the equality need to be prove,

$$
\mathcal{W}_k^{(l)} = \mathcal{E} \ast \left( \frac{1}{2\pi i} \int_{\Gamma_k^{(l)}} (z\mathcal{E} - \mathcal{A})^\dagger dz \right) \ast \mathcal{E}
$$

$$
= \mathcal{U}_{(r)} \ast \mathcal{V}_{(r)}^H \ast \left( \frac{1}{2\pi i} \int_{\Gamma_k^{(l)}} (z\mathcal{I} - \mathcal{S}_{(r)})^{-1} \ast \mathcal{U}_{(r)}^H dz \right) \ast \mathcal{U}_{(r)} \ast \mathcal{V}_{(r)}^H
$$

$$
= \mathcal{U}_{(r)} \ast \left( \frac{1}{2\pi i} \int_{\Gamma_k^{(l)}} (z\mathcal{I} - \mathcal{S}_{(r)})^{-1}dz \right) \ast \mathcal{V}_{(r)}^H,
$$

by taking $\text{bcirc}$ and block fast Fourier transformation on both sides of the above equation we have

$$
\text{bcirc}(\mathcal{W}_k^{(l)}) = \text{bcirc}(\mathcal{U}_{(r)} \ast \left( \frac{1}{2\pi i} \int_{\Gamma_k^{(l)}} (z\mathcal{I} - \mathcal{S}_{(r)})^{-1}dz \right) \ast \mathcal{V}_{(r)}^H)
$$

$$
= \text{bcirc}(\mathcal{U}_{(r)}) \text{bcirc}(\frac{1}{2\pi i} \int_{\Gamma_k^{(l)}} (z\mathcal{I} - \mathcal{S}_{(r)})^{-1}dz) \text{bcirc}(\mathcal{V}_{(r)}^H)
$$

$$
= \text{bcirc}(\mathcal{U}_{(r)}) \text{bcirc}(\frac{1}{2\pi i} \int_{\Gamma_k^{(l)}} (z - c_{k'}^{(l)})^{-1}dz) \text{bcirc}(\mathcal{V}_{(r)}^H)
$$

$$
= \text{bcirc}(\mathcal{U}_{(r)}) \text{diag}(w_{k'}^{(l)'}) \text{bcirc}(\mathcal{V}_{(r)}^H),
$$

where

$$
w_{k'}^{(l)'} = \frac{1}{2\pi i} \int_{\Gamma_k^{(l)}} (z - c_{k'}^{(l)})^{-1}dz = \begin{cases} 0, & \text{if } c_{k'}^{(l)'} \notin \Gamma_k^{(l)} \\ 1, & \text{if } c_{k'}^{(l)'} \in \Gamma_k^{(l)}, \end{cases}
$$

by the definition of $\Gamma = \bigcup_{k,l} \Gamma_k^{(l)}$ and Cauchy integral formula,

$$
\text{bcirc}(\mathcal{W}_k^{(l)}) = \text{bcirc}\left( \sum_{c_{k'}^{(l)'} = \gamma_k^{(l)}} (u_{j}^{(l)} v_{j}^{(l)H}) \right) = \text{bcirc}(\bar{\mathcal{E}}_k^{(l)}).
$$

(2) Similarly,

$$
\text{bcirc}(\mathcal{E} \ast \left( \frac{1}{2\pi i} \int_{\Gamma} f(z)(z\mathcal{E} - \mathcal{A})^\dagger dz \right) \ast \mathcal{E})
$$

$$
= \text{bcirc}(\mathcal{U}_{(r)}) \text{bcirc}\left( \text{diag}\left( \frac{1}{2\pi i} \int_{\Gamma_k^{(l)}} f(z)(z - c_{k'}^{(l)})^{-1} dz \right) \right) \text{bcirc}(\mathcal{V}_{(r)}^H)
$$

$$
= \sum_{i,j} f(c_{k'}^{(l)}) \text{bcirc}(\mathcal{E}_k^{(l)})
$$

$$
= \text{bcirc}(\hat{f}(\mathcal{A})).
$$

The final step is because of the assumption, if there is some $c_{k'}^{(l)'} = 0$, then we have $f(0) = 0$. \qed

Since the resolvent of matrices and tensors plays an important role in the matrix equations such as $AXB = D$, so we expand the definition of resolvent to tensors via the tensor T-product. The tensor resolvent satisfies the following important property:
Theorem 4. The generalized tensor resolvent of $\mathcal{A}$ is the function $\hat{\mathcal{R}}(z, \mathcal{A})$ given by

$$\hat{\mathcal{R}}(z, \mathcal{A}) = (z\mathcal{E} - \mathcal{A})^\dagger, \quad (10)$$

then

$$\hat{\mathcal{R}}(\lambda, \mathcal{A}) - \hat{\mathcal{R}}(\mu, \mathcal{A}) = (\mu - \lambda)\hat{\mathcal{R}}(\lambda, \mathcal{A}) \ast \hat{\mathcal{R}}(\mu, \mathcal{A}) \quad (11)$$

for any $\lambda, \mu \neq c_j^{(i)}$.

Proof. From the above proof, we have

$$(z\mathcal{E} - \mathcal{A})^\dagger = \sum_{i,j} \frac{1}{z - c_j^{(i)}} \mathcal{E}_j^{(i)H}.$$ 

Then the left-hand side comes to

$$\hat{\mathcal{R}}(\lambda, \mathcal{A}) - \hat{\mathcal{R}}(\mu, \mathcal{A}) = \sum_{i,j} \left( \frac{1}{\lambda - c_j^{(i)}} - \frac{1}{\mu - c_j^{(i)}} \right) \mathcal{E}_j^{(i)H}$$

$$= \sum_{i,j} \left( \frac{\mu - \lambda}{(\lambda - c_j^{(i)})(\mu - c_j^{(i)})} \right) \mathcal{E}_j^{(i)H}$$

$$= (\mu - \lambda) \left( \sum_{i,j} \frac{1}{\lambda - c_j^{(i)}} \mathcal{E}_j^{(i)H} \right) \ast \mathcal{E} \ast \left( \sum_{k,l} \frac{1}{\lambda - c_k^{(k)}} \mathcal{E}_l^{(k)H} \right)$$

$$= (\mu - \lambda) \hat{\mathcal{R}}(\lambda, \mathcal{A}) \ast \hat{\mathcal{R}}(\mu, \mathcal{A}).$$

We illustrate now the application of the above concepts to the solution of the tensor equation:

$$\mathcal{A} \ast \mathcal{X} \ast \mathcal{B} = \mathcal{D}.$$ 

Here the tensors $\mathcal{A} \in \mathbb{R}^{m \times n \times p}$, $\mathcal{B} \in \mathbb{R}^{k \times l \times p}$ and $\mathcal{D} \in \mathbb{R}^{m \times l \times p}$ and the respective partial isometries are given by

$$\mathcal{A} = \sum_{i,j=1}^{n_3, r^A} c_j^{(i)A} \mathcal{E}_j^{(i)A}, \quad \mathcal{E}^A = \sum_{i,j=1}^{n_3, r^A} \mathcal{E}_j^{(i)A},$$

$$\mathcal{B} = \sum_{i,j=1}^{n_3, r^B} c_j^{(i)B} \mathcal{E}_j^{(i)B}, \quad \mathcal{E}^B = \sum_{i,j=1}^{n_3, r^B} \mathcal{E}_j^{(i)B}.$$ 

Theorem 5. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$ be as above, and let $\Gamma_1$ and $\Gamma_2$ be contours surrounding $c(\mathcal{A}) = \{c_j^{(i)A}, i = 1, 2, \ldots, n_3, j = 1, 2, \ldots, r^A\}$ and $c(\mathcal{B}) = \{c_j^{(i)B}, i = 1, 2, \ldots, n_3, j = 1, 2, \ldots, r^B\}$, respectively. Then the above tensor equation has the following solution:

$$\mathcal{X} = -\frac{1}{4\pi^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{\hat{\mathcal{R}}(\lambda, \mathcal{A}) \ast \mathcal{D} \ast \hat{\mathcal{R}}(\mu, \mathcal{B})}{\lambda \mu} \, d\mu d\lambda. \quad (12)$$

Proof. It follows from Theorem 3 that

$$\mathcal{A} = \mathcal{E}^A \ast \left( \frac{1}{2\pi i} \int_{\Gamma_1} \lambda \hat{\mathcal{R}}(\lambda, \mathcal{A}) \, d\lambda \right) \ast \mathcal{E}^A,$$
\[ B = \mathcal{E}^B \ast \left( \frac{1}{2\pi i} \int_{\Gamma_2} \mu \hat{\mathcal{R}}(\mu, B) \, d\mu \right) \ast \mathcal{E}^B. \]

Therefore,

\[
\mathcal{A} \ast \mathcal{X} \ast \mathcal{B} = \mathcal{A} \ast \left( \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\hat{\mathcal{R}}(\lambda, \mathcal{A})}{\lambda} \, d\lambda \right) \ast \mathcal{D} \ast \left( \frac{1}{2\pi i} \int_{\Gamma_2} \frac{\hat{\mathcal{R}}(\mu, \mathcal{A})}{\mu} \, d\mu \right) \ast \mathcal{E}^B
\]

\[
= \mathcal{E}^A \ast \left( \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\hat{\mathcal{R}}(\lambda, \mathcal{A})}{\lambda} \, d\lambda \right) \ast \mathcal{D} \ast \left( \frac{1}{2\pi i} \int_{\Gamma_2} \frac{\hat{\mathcal{R}}(\mu, \mathcal{B})}{\mu} \, d\mu \right) \ast \mathcal{E}^B
\]

\[
= \mathcal{E}^A \ast \mathcal{E}^{A\top} \ast \mathcal{D} \ast \mathcal{E}^{B\top} \ast \mathcal{E}^B
\]

\[
= \mathcal{P}_{\mathcal{R}(\mathcal{A})} \ast \mathcal{D} \ast \mathcal{P}_{\mathcal{R}(\mathcal{B})}
\]

\[
= \mathcal{A} \ast \mathcal{A}^\dagger \ast \mathcal{D} \ast \mathcal{B}^\dagger \ast \mathcal{B}
\]

\[
= \mathcal{D}.
\]

At the end of this subsection, we consider the least squares problem via the T-product,

\[
\min_x \| \mathcal{A} \ast \mathcal{X} - \mathcal{B} \|_F
\]

the least squares solution to this problem is \( X_{LS} = \mathcal{A}^\dagger \ast \mathcal{B} \).

### 3.4 Generalized tensor power

**Definition 7.** (Generalized tensor power) Suppose \( \mathcal{A} \) is an \( n_1 \times n_2 \times n_3 \) real-valued tensor and the T-CSVD of \( \mathcal{A} \) is

\[
\mathcal{A} = \mathcal{U}(r) \ast \mathcal{S}(r) \ast \mathcal{V}^H(r).
\]

The generalized power \( \mathcal{A}^{(k)} \) of \( \mathcal{A} \) is defined as:

\[
\mathcal{A}^{(k)} = \mathcal{A}^{(k-1)} \ast \mathcal{E}^\top \ast \mathcal{A}, \quad k \geq 1,
\]

where

\[
\mathcal{A}^{(0)} = \mathcal{E} = \mathcal{U}(r) \ast \mathcal{V}^H(r).
\]

From Definition 7, it is easy to come to the following results.

**Corollary 9.** Suppose \( \mathcal{A} \) is an \( n_1 \times n_2 \times n_3 \) real-valued tensor, then

\[
(\mathcal{A})^{2k+1} = (\mathcal{A} \ast \mathcal{A}^\top)^k \ast \mathcal{A},
\]

\[
(\mathcal{A})^{2k} = (\mathcal{A} \ast \mathcal{A}^\top)^k \ast \mathcal{E}.
\]

Now, the Taylor expansion of a function can be extended to the generalized tensor function.

**Theorem 6.** Suppose \( \mathcal{A} \) is an \( n_1 \times n_2 \times n_3 \) real-valued tensor, \( f : \mathbb{C} \to \mathbb{C} \) is a scalar function given by,

\[
f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k
\]

for

\[
|z - z_0| < R.
\]
Then the generalized tensor function $f^\otimes : \mathbb{C}^{n_1 \times n_2 \times n_3} \to \mathbb{C}^{n_1 \times n_2 \times n_3}$ is given by

$$f^\otimes(A) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(A - z_0E)^k,$$

for

$$|z_0 - c_j^{(i)}| < R, \ i = 1, 2, \ldots, n_3, \ j = 1, 2, \ldots, r_i.$$

**Proof.** By induction, it is easy to verify the equation

$$(A - z_0E)^k = U_{(r)} \ast (S_{(r)} - z_0I)^k \ast V_{(r)}^H, \ k = 0, 1, \ldots,$$

For $n = 0, 1, \ldots$ we define

$$f_n^\otimes(A) = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}(A - z_0E)^k \ast U_{(r)} \ast \left( \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}(S_{(r)} - z_0I)^k \right) \ast V_{(r)}^H,$$

so that

$$\|f^\otimes(A) - f_n^\otimes(A)\| \leq \|U_{(r)}\| \left\| \left( \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}(S_{(r)} - z_0I)^k \right) \right\| \|V_{(r)}^H\|,$$

by Definition 6 of tensor norm via the T-SVD, we obtain

$$\|f^\otimes(A) - f_n^\otimes(A)\| \to 0, \ (n \to 0).$$

**Example 1.** Suppose $A$ is an $n_1 \times n_2 \times n_3$ real-valued tensor, $f : \mathbb{C} \to \mathbb{C}$ is a scalar function given by the Laurent expansion,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}z^k.$$

Define the functions $f_1(z)$ and $f_2(z)$ as follows:

$$f_1(z) = \sum_{k=0}^{\infty} \frac{f^{(2k)}(0)}{(2k)!}z^{2k}, \ f_2(z) = \sum_{k=0}^{\infty} \frac{f^{(2k+1)}(0)}{(2k+1)!}z^{2k+1},$$

So it comes to

$$f^\otimes(A) = f_1^\otimes(A \ast A^\top) \ast E + f_2^\otimes(A \ast A^\top) \ast A.$$

We have the following Maclaurin formulae for explicit functions,

$$\exp(z) = \sum_{k=0}^{\infty} \frac{1}{k!}z^k, \ \ln(1 + z) = \sum_{k=0}^{\infty} \frac{1}{2k + 1}z^{2k+1} - \sum_{k=0}^{\infty} \frac{1}{2k}z^{2k},$$

$$\sin(z) = (-1)^k \sum_{k=0}^{\infty} \frac{1}{(2k + 1)!}z^{2k+1}, \ \cos(z) = (-1)^k \sum_{k=0}^{\infty} \frac{1}{(2k)!}z^{2k}.$$
Then generalized tensor function comes to

\[
\exp^\Diamond (A) = \sum_{k=0}^{\infty} \frac{1}{2k!} (A \ast A^\top)^k \ast \mathcal{E} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (A \ast A^\top)^k \ast A,
\]

\[
\ln^\Diamond (I + A) = \sum_{k=0}^{\infty} \frac{1}{2k+1} (A \ast A^\top)^k \ast A - \sum_{k=0}^{\infty} \frac{1}{2k} (A \ast A^\top)^k \ast \mathcal{E},
\]

\[
\sin^\Diamond (A) = (-1)^k \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (A \ast A^\top)^k \ast A,
\]

\[
\cos^\Diamond (A) = (-1)^k \sum_{k=0}^{\infty} \frac{1}{(2k)!} (A \ast A^\top)^k \ast \mathcal{E}.
\]

Remark 4. Since \( \exp(0) = 1 \neq 0 \) and \( \cos(0) = 1 \neq 0 \), so the above generalized tensor function \( \exp^\Diamond (A) \) and \( \cos^\Diamond (A) \) hold only when \( c^{(i)}_{j} \neq 0 \) for all \( i = 1, 2, \ldots, n_3 \), \( j = 1, 2, \ldots, r \).

3.5 Structures preserved by generalized tensor functions

In matrix cases, knowledge of the structural structures of generalized matrix function \( f(A) \) can lead to more accurate and efficient algorithms such as Toeplitz structures, triangular structures, circulant structures and so on \[12, 22, 23\], which will lead to significant savings when computing it. Similar savings may be expected to generalized tensor functions. So in this subsection, we indicate to propose some structures preserved by generalized tensor functions via the tensor T-product.

First, we need the following lemma which is the similar case of Corollary 2 for complex tensors.

**Lemma 6.** Suppose \( A \in \mathbb{C}^{n_1 \times n_2 \times n_3} \) is a third order tensor. Let \( f : \mathbb{C} \to \mathbb{C} \) be a scalar function and let \( f^\Diamond : \mathbb{C}^{n_1 \times n_2 \times n_3} \to \mathbb{C}^{n_1 \times n_2 \times n_3} \) be the corresponding generalized function of third order tensors. Then

1. \( [f^\Diamond (A)]^H = f^\Diamond (A^H) \).
2. Let \( P \in \mathbb{C}^{n_1 \times n_1 \times n_3} \) and \( Q \in \mathbb{C}^{n_2 \times n_2 \times n_3} \) be two unitary tensors, then \( f^\Diamond (P \ast A \ast Q) = P \ast f^\Diamond (A) \ast Q \).

We also need the concept of block tensor which is corresponded to the tensor T-product.

**Definition 8.** (Block tensor via T-product) Suppose \( A \in \mathbb{C}^{n_1 \times m_1 \times p} \), \( B \in \mathbb{C}^{n_1 \times m_2 \times p} \), \( C \in \mathbb{C}^{n_2 \times m_1 \times p} \) and \( D \in \mathbb{C}^{n_2 \times m_2 \times p} \) are four tensors. The block tensor

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \in \mathbb{C}^{(n_1 + n_2) \times (m_1 + m_2) \times p}
\]

is the block tensor given by compositing the frontal slices of the four tensors.

By the definition of tensor T-product, matrix block and block matrix multiplication, we give the following result.
Theorem 7. (Tensor block multiplication via T-product) Suppose \( A_1 \in \mathbb{C}^{n_1 \times m_1 \times p}, B_1 \in \mathbb{C}^{m_1 \times m_2 \times p}, C_1 \in \mathbb{C}^{m_2 \times m_1 \times p}, D_1 \in \mathbb{C}^{m_1 \times m_2 \times p}, A_2 \in \mathbb{C}^{m_1 \times r_1 \times p}, B_2 \in \mathbb{C}^{m_1 \times r_2 \times p}, C_2 \in \mathbb{C}^{m_2 \times r_1 \times p} \) and \( D_2 \in \mathbb{C}^{m_2 \times r_2 \times p} \) are complex tensors, then we have

\[
\begin{bmatrix}
A_1 & B_1 \\
C_1 & D_1
\end{bmatrix} \ast \begin{bmatrix}
A_2 & B_2 \\
C_2 & D_2
\end{bmatrix} = \begin{bmatrix}
A_1 \ast A_2 + B_1 \ast C_2 & A_1 \ast B_2 + B_1 \ast D_2 \\
C_1 \ast A_2 + D_1 \ast C_2 & C_1 \ast D_2 + D_1 \ast D_2
\end{bmatrix}. \tag{13}
\]

Proof. For the simplicity of illustration, we only choose \( p = 2 \) and prove the simple case,

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \ast \begin{bmatrix}
\mathcal{E} \\
\mathcal{F}
\end{bmatrix} = \begin{bmatrix}
A \ast \mathcal{E} + B \ast \mathcal{F} \\
C \ast \mathcal{E} + D \ast \mathcal{F}
\end{bmatrix}.
\]

LHS = \( \text{unfold} \left( \text{bcirc} \left( \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \right) \right) \text{unfold} \left( \begin{bmatrix}
\mathcal{E} \\
\mathcal{F}
\end{bmatrix} \right) \)

\[
= \text{unfold} \left( \begin{bmatrix}
A^{(1)} & B^{(1)} \\
C^{(1)} & D^{(1)}
\end{bmatrix} \begin{bmatrix}
A^{(2)} & B^{(2)} \\
C^{(2)} & D^{(2)}
\end{bmatrix} \begin{bmatrix}
E^{(1)} \\
F^{(1)}
\end{bmatrix} \right)
\]

\[
= \text{unfold} \left( \begin{bmatrix}
A^{(1)} E^{(1)} + B^{(1)} F^{(1)} + A^{(2)} E^{(2)} + B^{(2)} F^{(2)} \\
C^{(1)} E^{(1)} + D^{(1)} F^{(1)} + C^{(2)} E^{(2)} + D^{(2)} F^{(2)}
\end{bmatrix} \right).
\]

RHS = \( \text{unfold} \left( \text{bcirc}(\mathcal{A}) \text{unfold}(\mathcal{E}) \right) + \text{unfold} \left( \text{bcirc}(\mathcal{B}) \text{unfold}(\mathcal{F}) \right) \)

\[
= \begin{bmatrix}
\text{unfold} \left( \text{bcirc}(\mathcal{A}) \text{unfold}(\mathcal{E}) \right) + \text{unfold} \left( \text{bcirc}(\mathcal{B}) \text{unfold}(\mathcal{F}) \right)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\text{unfold} \left( \text{bcirc}(\mathcal{A}) \text{unfold}(\mathcal{E}) \right) + \text{bcirc}(\mathcal{B}) \text{unfold}(\mathcal{F})
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\text{unfold} \left( \begin{bmatrix}
A^{(1)} \\
C^{(1)}
\end{bmatrix} \begin{bmatrix}
E^{(1)} \\
E^{(1)}
\end{bmatrix} + \begin{bmatrix}
A^{(2)} \\
C^{(2)}
\end{bmatrix} \begin{bmatrix}
E^{(2)} \\
E^{(2)}
\end{bmatrix} + \begin{bmatrix}
B^{(1)} \\
B^{(2)}
\end{bmatrix} \begin{bmatrix}
F^{(1)} \\
F^{(1)}
\end{bmatrix}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\text{unfold} \left( \begin{bmatrix}
A^{(1)} E^{(1)} + B^{(1)} F^{(1)} + A^{(2)} E^{(2)} + B^{(2)} F^{(2)} \\
C^{(1)} E^{(1)} + D^{(1)} F^{(1)} + C^{(2)} E^{(2)} + D^{(2)} F^{(2)}
\end{bmatrix} \right).
\]

\[
= \text{LHS}.
\]

Remark 5. Theorem 7 is of great importance since it shows that our definition of block tensor confirms the definition of the tensor T-product, which will have great impact on obtaining the results of tensor T-decomposition theorems.

Remark 6. It should be noticed that, in the proof, the following equations do not hold:

\[
\text{bcirc} \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \neq \begin{bmatrix}
\text{bcirc}(\mathcal{A}) & \text{bcirc}(\mathcal{B}) \\
\text{bcirc}(\mathcal{C}) & \text{bcirc}(\mathcal{D})
\end{bmatrix}, \text{unfold} \begin{bmatrix}
\mathcal{E} \\
\mathcal{F}
\end{bmatrix} \neq \begin{bmatrix}
\text{unfold}(\mathcal{E}) \\
\text{unfold}(\mathcal{F})
\end{bmatrix}.
\]

Remark 7. For non-F-square tensors, if we set

\[
\mathcal{B} = \begin{bmatrix}
0 & A^H \\
A & 0
\end{bmatrix}
\]

\[
20
\]
then $\mathcal{B}$ is a $F$-square tensor. We have that for any real-valued odd function $f$, the induced generalized tensor function satisfies

$$f^\diamond(\mathcal{B}) = \begin{bmatrix} 0 & f^\diamond(\mathcal{A}) \\ f^\diamond(\mathcal{A})^H & 0 \end{bmatrix}.$$  

By using the concept of tensor block and tensor multiplication, now we can define the bilinear forms of tensors, which will lead to further results of invariance of generalized tensor functions and several classes of tensors whose properties are preserved by generalized tensor functions. For matrix cases, there is already a lot of papers on this problem.

We emphasize that since the generalized tensor function $f^\diamond(\mathcal{A})$ cannot be represented as a polynomial of the tensor $\mathcal{A}$, so the associative tensor T-product algebra may not be closed under our definition of generalized tensor functions. Not only the tensor classes already given in Table 1 and Table 2, several other classes of tensors are also introduced. First, we introduced the definition of tensor bilinear form as follows.

**Definition 9.** (Bilinear form of tensors) Let $\mathbb{K}^{1 \times 1 \times p} = \mathbb{R}^{1 \times 1 \times p}$ or $\mathbb{C}^{1 \times 1 \times p}$, a scalar product of $\mathbb{K}^{n \times n \times p}$ is a bilinear or sesquilinear form $\langle \cdot, \cdot \rangle_T$ defined by any nonsingular tensor $T \in \mathbb{K}^{n \times n \times p}$ for $x, y \in \mathbb{K}^{n \times n \times p}$

$$\langle x, y \rangle_T = \begin{cases} x^T * T * y, \text{ for real or complex bilinear forms,} \\ x^H * T * y, \text{ for sesquilinear forms.} \end{cases}$$  \hspace{1cm} (14)

The adjoint of $\mathcal{A}$ with respect to the scalar product $\langle \cdot, \cdot \rangle_T$ denoted by $\mathcal{A}^*$ is uniquely defined by the property $\langle A^* x, y \rangle_T = \langle x, A^* y \rangle_T$ for all $x, y \in \mathbb{K}^{n \times n \times p}$.

Associated with $\langle \cdot, \cdot \rangle_T$ is an automorphism group $\mathbb{G}$, a Lie algebra $\mathbb{L}$, and a Jordan algebra $\mathbb{J}$, which are the subsets of $\mathbb{K}^{n \times n \times p}$ defined by

$$\mathbb{G} := \{ G : \langle G * x, G * y \rangle_T = \langle x, y \rangle_T, \forall x, y \in \mathbb{K}^{n \times n \times p} \} = \{ G : G^* = G^{-1} \},$$

$$\mathbb{L} := \{ L : \langle L * x, y \rangle_T = -\langle x, L * y \rangle_T, \forall x, y \in \mathbb{K}^{n \times n \times p} \} = \{ L : L^* = -L \},$$

$$\mathbb{J} := \{ S : \langle S * x, y \rangle_T = \langle x, S * y \rangle_T, \forall x, y \in \mathbb{K}^{n \times n \times p} \} = \{ S : S^* = S \}.$$  Here, $\mathbb{G}$ is a multiplicative group, while $\mathbb{L}$ and $\mathbb{J}$ are linear subspaces of $\mathbb{K}^{n \times n \times p}$.

Similar to matrix cases [6], we present several important structures for tensors. These tensors forms Lie and Jordan tensors algebras over real and complex fields and can be named in terms of symmetry, anti-symmetry and so on, with respect to tensor bilinear and sesquilinear form. We first introduce some special notations of tensors:

1. The reverse tensor $\mathcal{R}_n \in \mathbb{R}^{n \times n \times p}$ is the tensor whose first frontal slice is

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ & & \cdot \\ & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

and other frontal slices are all zeros.

2. The skew Hamiltonian tensor

$$\mathcal{J} = \begin{bmatrix} 0 & I_{nmp} \\ -I_{nmp} & 0 \end{bmatrix} \in \mathbb{R}^{2n \times 2n \times p}.$$  

3. The pseudo-symmetric tensor

$$\Sigma_{a,b} = \begin{bmatrix} I_{aap} & 0 \\ 0 & -I_{bpb} \end{bmatrix} \in \mathbb{R}^{n \times n \times p},$$
with \( a + b = n \).

(4) The adjoint tensor

\[
\mathcal{A}^* = \begin{cases} 
\mathcal{T}^{-1} \ast \mathcal{A}^\top \ast \mathcal{T}, & \text{for bilinear forms,} \\
\mathcal{T}^{-1} \ast \mathcal{A}^H \ast \mathcal{T}, & \text{for sesquilinear forms,}
\end{cases}
\]

(15)

where \( \mathcal{T} \) is one of the tensors defining the above bilinear or sesquilinear forms.

We find the generalized tensor function preserves some tensor structures as the following theorem.

**Theorem 8.** Let \( \mathcal{T} \) be one of the classes in Table 1. If \( \mathcal{A} \in \mathcal{T} \) and \( f^\circ (\mathcal{A}) \) is well defined, then \( f^\circ (\mathcal{A}) \in \mathcal{T} \).

**Proof.** It is obvious that \( \mathcal{R}_n, \mathcal{J} \) and \( \Sigma_{a,b} \) are unitary tensors via the T-product. By Lemma 6 we have

\[
f^\circ (\mathcal{A}^*) = \begin{cases} 
\mathcal{T}^{-1} \ast f^\circ (\mathcal{A})^\top \ast \mathcal{T} = f^\circ (\mathcal{A})^*, & \text{for bilinear forms,} \\
\mathcal{T}^{-1} \ast f^\circ (\mathcal{A})^H \ast \mathcal{T} = f^\circ (\mathcal{A})^*, & \text{for sesquilinear forms.}
\end{cases}
\]

Hence, for Jordan algebra \( \mathcal{J} \) we have \( f^\circ (\mathcal{A})^* = f^\circ (\mathcal{A}^*) = f^\circ (\mathcal{A}) \), for Lie algebra \( \mathcal{L} \) we have \( f^\circ (\mathcal{A})^* = f^\circ (\mathcal{A}^*) = f^\circ (-\mathcal{A}) = -f^\circ (\mathcal{A}) \) because of Remark 1, which means we can set \( f \) to be an odd function. \( \square \)

**Table 1:** Structured tensors associated with certain bilinear and sesquilinear forms

| Space | \( \mathcal{T} \) | Jordan Algebra \( \mathcal{J} = \{ S : S^* = S \} \) | Lie algebra \( \mathcal{L} = \{ \mathcal{E} : \mathcal{E}^* = -\mathcal{E} \} \) |
|-------|------------------|-------------------|-------------------|
| \( \mathbb{C}^{n \times 1 \times p} \) | \( \mathcal{I} \) | Symmetrics | Skew-symmetrics |
| \( \mathbb{C}^{n \times 1 \times p} \) | \( \Sigma_{a,b} \) | Complex symmetric tensors | Complex skew-symmetric tensors |
| \( \mathbb{C}^{n \times 1 \times p} \) | \( \mathcal{R}_n \) | Pseudo-symmetric tensors | Pseudo-skew-symmetric tensors |
| \( \mathbb{C}^{n \times 1 \times p} \) | \( \mathcal{J} \) | Complex symmetric tensors | Complex skew-symmetric tensors |
| \( \mathbb{C}^{n \times 1 \times p} \) | \( \Sigma_{a,b} \) | Complex pseudo-symmetric tensors | Complex pseudo-skew-symmetric tensors |
| \( \mathbb{C}^{n \times 1 \times p} \) | \( \mathcal{R}_n \) | Persymmetric tensors | Perskew-symmetric tensors |
| \( \mathbb{C}^{n \times 1 \times p} \) | \( \mathcal{J} \) | Complex skew-symmetric tensors | Complex \( \mathcal{J} \)-symmetric tensors |

| Space | \( \mathbb{C}^{n \times 1 \times p} \) | \( \mathcal{I} \) | Hermitians | Skew-Hermitians |
|-------|------------------|-------------------|-------------------|
| \( \mathbb{C}^{n \times 1 \times p} \) | \( \Sigma_{a,b} \) | Pseudo-Hermitians | Pseudo-skew-Hermitians |
| \( \mathbb{C}^{n \times 1 \times p} \) | \( \mathcal{R}_n \) | Perhermitians | Skew-perhermitians |
| \( \mathbb{C}^{n \times 1 \times p} \) | \( \mathcal{J} \) | \( \mathcal{J} \)-skew-Hermitians | \( \mathcal{J} \)-Hermitians |

There are other kind of tensor classes that are preserved by arbitrary generalized tensor functions.

**Definition 10.** (Centrohermitian tensor) \( \mathcal{A} \in \mathbb{C}^{n \times m \times p} \) is centrohermitian (skew-centrohermitian) if \( \mathcal{R}_m \ast \mathcal{A} \ast \mathcal{R}_n = \overline{\mathcal{A}} \) (respectively, \( \mathcal{R}_m \ast \mathcal{A} \ast \mathcal{R}_n = -\overline{\mathcal{A}} \)).

**Theorem 9.** Suppose \( \mathcal{A} \in \mathbb{C}^{n \times m \times p} \) is a tensor. Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be a scalar function and let \( f^\circ : \mathbb{C}^{n_1 \times n_2 \times n_3} \rightarrow \mathbb{C}^{n_1 \times n_2 \times n_3} \) be the corresponding generalized function of third order tensors which is assumed to be well defined at \( \mathcal{A} \).

1. If \( \mathcal{A} \) is centrohermitian (skew-centrohermitian), then \( f^\circ (\mathcal{A}) \) is also centrohermitian (skew-centrohermitian).
2. If \( m = n \) and \( \mathcal{A}^H \ast \mathcal{A} = \mathcal{A} \ast \mathcal{A}^H \) (which is called normal), then \( f^\circ (\mathcal{A}) \) is also normal.
3. If \( m = n \) and \( \mathcal{A} \) is F-circulant then \( f^\circ (\mathcal{A}) \) is also F-circulant.
4. If \( \mathcal{A} \) is a F-block-circulant tensor with F-circulant blocks, then \( f^\circ (\mathcal{A}) \) is also a F-block-circulant tensor with F-circulant blocks.
Proof. (1) If $\mathcal{A}$ is centrohermitian, then we have $\mathcal{R}_m \ast \mathcal{A} \ast \mathcal{R}_n = \mathcal{A}$. Since $\mathcal{R}_n$ and $\mathcal{R}_n$ are unitary tensors, so by Lemma 6, we have $\mathcal{R}_m \ast f^\Diamond(\mathcal{A}) \ast \mathcal{R}_n = f^\Diamond(\mathcal{R}_m \ast \mathcal{A} \ast \mathcal{R}_n) = f^\Diamond(\mathcal{A}) = f^\Diamond(\overline{\mathcal{A}})$.

If $\mathcal{A}$ is skew-centrohermitian, then we have $\mathcal{R}_m \ast \mathcal{A} \ast \mathcal{R}_n = -\overline{\mathcal{A}}$. Since $\mathcal{R}_n$ and $\mathcal{R}_n$ are unitary tensors, so by Lemma 6, we have $\mathcal{R}_m \ast f^\Diamond(\mathcal{A}) \ast \mathcal{R}_n = f^\Diamond(\mathcal{R}_m \ast \mathcal{A} \ast \mathcal{R}_n) = f^\Diamond(-\overline{\mathcal{A}}) = -f^\Diamond(\mathcal{A})$.

(2) Suppose $\mathcal{A} \in \mathbb{C}^{m \times n \times p}$ is normal, that is $\mathcal{A} \ast \mathcal{A}^H = \mathcal{A}^H \ast \mathcal{A}$. So we take the bcirc operator on both sides of the equation, notice that $\text{bcirc}(\mathcal{A}^H) = (\text{bcirc}(\mathcal{A}))^H$, we get

$$\text{bcirc}(\mathcal{A}) = (F_p \otimes I_m) \begin{bmatrix} D_1 & D_2 & \cdots & D_p \end{bmatrix} (F_p^H \otimes I_n).$$

We get $D_1, D_2, \cdots, D_p$ are normal matrices, by the definition of generalized tensor function, we get the result.

(3) and (4) hold because of the same reason of (2) by taking ‘bcirc’ operator on the tensor $\mathcal{A}$.

For some structured tensors, their structures may not be preserved by every generalized tensor function $f^\Diamond$, but if we put some restrictions on the origin scalar function $f$, we can get $f^\Diamond(\mathcal{A})$ holds the same structure with $\mathcal{A}$. The following theorems are good examples.

Theorem 10. Let $\mathbb{G}$ be one of the tensor groups in Table 2. If $\mathcal{A} \in \mathbb{G}$, $f : \mathbb{R} \to \mathbb{R}$ is defined for $x > 0$ and satisfies

$$f(x)f\left(\frac{1}{x}\right) = 1 \quad \text{and} \quad f(0) = 0,$$

then $f^\Diamond(\mathcal{A}) \in \mathbb{G}$.

Proof. Since $\mathcal{R}_n$, $\mathcal{J}$ and $\Sigma_{a,b}$ are unitary, by Lemma 6 we have

$$f^\Diamond(\mathcal{A}^*) = \begin{cases} \mathcal{T}^{-1} \ast f^\Diamond(\mathcal{A})^\top \ast \mathcal{T} = f^\Diamond(\mathcal{A})^*, & \text{for bilinear forms}, \\ \mathcal{T}^{-1} \ast f^\Diamond(\mathcal{A})^H \ast \mathcal{T} = f^\Diamond(\mathcal{A})^*, & \text{for sesquilinear forms}. \end{cases}$$

Hence, for $\mathcal{A}$ in each of the above tensor automorphism groups $\mathbb{G} = \{ \mathcal{G} : \mathcal{G}^* = \mathcal{G}^{-1} \}$, we have

$$f^\Diamond(\mathcal{A})^* = f^\Diamond(\mathcal{A}^*) = f^\Diamond(\mathcal{A}^{-1}) = \mathcal{V}_R \ast f^\Diamond(\mathcal{S}^{-1}) \ast \mathcal{U}_R^H = \mathcal{V}_R \ast f^\Diamond(\mathcal{S})^{-1} \ast \mathcal{U}_R^H = f^\Diamond(\mathcal{A})^{-1}.$$

Remark 8. It could be noticed that for any unitary tensor $\mathcal{A}$, we have $f^\Diamond(\mathcal{A}) = f(1)\mathcal{A}$, therefore $f^\Diamond(\mathcal{A}) = \mathcal{A}$ for any function $f$ satisfying $f(1) = 1$. Hence, some generalized tensor functions may be trivial functions for unitary tensors. A complete characterization of all (meromorphic) functions satisfying the condition that $f(x)f\left(\frac{1}{x}\right) = 1$ can be found in [18].

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Table 2: Structured tensors associated with certain bilinear and sesquilinear forms

| Space (\(n \times 1 \times p\)) | \(T\) | Automorphism Group \(G = \{G : G^{*} = G^{\dagger}\}\) |
|-------------------------------|-----|----------------------------------|
| Real orthogonals | Real orthogonals | \(\mathbb{R}^{n \times 1 \times p}\) |
| Complex orthogonals | Complex orthogonals | \(\mathbb{C}^{n \times 1 \times p}\) |
| Pseudo-orthogonals | Pseudo-orthogonals | \(\mathbb{R}^{n \times 1 \times p}\) |
| Complex pseudo-orthogonals | Complex pseudo-orthogonals | \(\mathbb{C}^{n \times 1 \times p}\) |
| Real perplectics | Real perplectics | \(\mathbb{R}^{2n \times 1 \times p}\) |
| Complex symplectics | Complex symplectics | \(\mathbb{C}^{2n \times 1 \times p}\) |

**Theorem 11.** Let \(A \in \mathbb{R}^{m \times n \times p}\) be an nonnegative tensor and \(f\) be the odd part of an analytic function which has the Laurent expansion for the form \(f(z) = \sum_{k=0}^{\infty} c_{2k+1} z^{2k+1}\) with \(c_{2k+1} \geq 0\), assumed to be convergent for \(|z| < R\) with \(R = \|A\|_{2}\). Then \(f^{\dagger}(A)\) is well-defined, and \(f^{\dagger}(A)\) is also a nonnegative tensor.

**Proof.** Without loss of generality, we assume \(f\) is an odd function which has the Laurent expansion

\[
f(z) = \sum_{k=0}^{\infty} c_{2k+1} z^{2k+1}
\]

with coefficients \(c_{2k+1} \geq 0\). The condition on the radius of convergence of the Laurent series guarantees that \(f^{\dagger}(A)\) is well-defined. Since \(A\) has the T-CSVD

\[
A = U_{(r)} * \Sigma_{(r)} * V_{(r)}^{H},
\]

then \((A * A^{\top}) A = U_{(r)} * \Sigma_{(r)}^{2k+1} * V_{(r)}^{H}\). It turns out to be that

\[
f^{\dagger}(A) = U_{(r)} * \left(\sum_{k=0}^{\infty} c_{2k+1} \Sigma_{(r)}^{2k+1}\right) * V_{(r)}^{H} \geq 0.
\]

**Definition 11.** (Permutation tensor) A tensor \(P \in \mathbb{R}^{n \times n \times p}\) is called a permutation tensor if its first frontal slice is a permutation matrix and other frontal slices are all zeros.

It is obvious that permutation tensors are all orthogonal tensors, i.e

\[
P * P^{\top} = P^{\top} * P = \mathcal{I}.
\]

The next theorem shows that the generalized tensor function preserves zero slices in certain positions of tensors.

**Theorem 12.** Let \(A \in \mathbb{C}^{n \times n \times p}\) be a complex tensor and \(f^{\dagger}(A)\) be well-defined.

1. If the \(i\)th lateral (horizontal) slice of \(A\) consists of all zeros, then the \(i\)-th lateral (horizontal) slice of \(f^{\dagger}(A)\) consists of all zeros.

2. If there exist permutation tensors \(P \in \mathbb{R}^{n \times n \times p}\) and \(Q \in \mathbb{R}^{n \times n \times p}\) such that \(P * A * Q\) is \(F\)-block diagonal, then \(f^{\dagger}(A)\) is also a \(F\)-block diagonal tensor.
Figure 3: (a) Frontal, (b) horizontal, and (c) lateral slices of a 3rd order tensor. The lateral slices are also referred to as oriented matrices. (d) A lateral slice as a vector of tube fibers.

**Proof.** (1) Without loss of generality, we may assume the last lateral slice of \( A \) is 0, since for any permutation tensor \( P \) we have

\[
f^\circ (A \ast P) = f^\circ (A) \ast P.
\]

We write \( A = \begin{bmatrix} \hat{A} & 0 \end{bmatrix} \) and assume that \( \hat{A} \) has T-SVD decomposition

\[
\hat{A} = \hat{U} \ast \hat{\Sigma} \ast \hat{V}^H.
\]

It follows this equation that \( A \) has the T-SVD decomposition via block tensor multiplication

\[
A = \begin{bmatrix} \hat{A} & 0 \end{bmatrix} = \hat{U} \ast \begin{bmatrix} \hat{\Sigma} & 0 \end{bmatrix} \ast \begin{bmatrix} \hat{V} & 0 \\ 0 & 1 \end{bmatrix} = U \ast \Sigma \ast V^H,
\]

where the ‘1’ in the tensor block is a tube tensor whose first element is 1 and the other element are all zeros.

We assume that \( A \) has tubal rank \( r \), we have the T-CSVD

\[
f^\circ (A) = U_{(r)} \ast f^\circ (\Sigma_{(r)}) \ast V_{(r)}^H.
\]

We can find the last lateral slice of \( V_{(r)} \) consists of all zeros, so it comes to the conclusion that the last lateral slice of \( f^\circ (A) \) is also zero.

For the similar reason, if \( A \) has zero horizontal slices, then we can use the same method for \( A^H \). By using the conclusion that \( f^\circ (A)^H = f^\circ (A^H) \), we can get the corresponding result.

(2) By using the tensor block multiplication via T-block, we can easily get the result. \( \square \)

### 3.6 Complex-to-real isomorphism

In this section, we show that the generalized tensor functions are well-behaved with respect to the canonical isomorphism between the algebra of \( n \times n \times p \) complex tensors and the subalgebra of the algebra of the real \( 2n \times 2n \times p \) tensors consisting of all block tensors of the form:

\[
\begin{bmatrix}
B & -C \\
C & B
\end{bmatrix},
\]

where \( B \) and \( C \) are tensors in \( \mathbb{R}^{n \times n \times p} \).
Theorem 13. Let \( \mathcal{A} = \mathcal{B} + i \mathcal{C} \in \mathbb{C}^{n \times n \times p} \), where \( \mathcal{B} \) and \( \mathcal{C} \) are real tensors. Let \( f : \mathbb{R} \to \mathbb{R} \) be a scalar function satisfies \( f(0) = 0 \). \( f^\diamond : \mathbb{C}^{n \times n \times p} \to \mathbb{C}^{n \times n \times p} \) is the induced generalized tensor function. Let \( \Phi : \mathbb{C}^{n \times n \times p} \to \mathbb{R}^{2n \times 2n \times p} \) be the mapping:

\[
\Phi(\mathcal{A}) = \begin{bmatrix} \mathcal{B} & -\mathcal{C} \\ \mathcal{C} & \mathcal{B} \end{bmatrix}.
\]

We also denote by \( f^\diamond \) the generalized tensor function from \( \mathbb{R}^{2n \times 2n \times p} \) to \( \mathbb{R}^{2n \times 2n \times p} \) induced by \( f \). Then \( f^\diamond(\Phi(\mathcal{A})) \) is well defined and \( f^\diamond \) commutes with \( \Phi \):

\[
f^\diamond(\Phi(\mathcal{A})) = \Phi(f^\diamond(\mathcal{A})).
\]

That is to say, we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{C}^{n \times n \times p} (\ast, +, \mathcal{I}_n, 0) & \xrightarrow{f^\diamond} & \mathbb{C}^{n \times n \times p} (\ast, +, \mathcal{I}_n, 0) \\
\downarrow \Phi & & \downarrow \Phi \\
\mathbb{R}^{2n \times 2n \times p} (\ast, +, \mathcal{I}_{2n}, 0) & \xrightarrow{f^\diamond} & \mathbb{R}^{2n \times 2n \times p} (\ast, +, \mathcal{I}_{2n}, 0)
\end{array}
\]

Proof. By using the T-SVD of tensors, we have

\[
\mathcal{A} = \mathcal{B} + i \mathcal{C} = \mathcal{U} * \Sigma * \mathcal{V}^H
\]

\[
= (\mathcal{U}_1 + i \mathcal{U}_2) * \Sigma * (\mathcal{V}_1 + i \mathcal{V}_2)^H
\]

\[
= (\mathcal{U}_1 + i \mathcal{U}_2) * \Sigma * (\mathcal{V}_1^T - i \mathcal{V}_2^T)
\]

\[
= \mathcal{U}_1 * \Sigma * \mathcal{V}_1^T + \mathcal{U}_2 * \Sigma * \mathcal{V}_2^T + i (\mathcal{U}_2 * \Sigma * \mathcal{V}_1^T - \mathcal{U}_1 * \Sigma * \mathcal{V}_2^T),
\]

and

\[
f^\diamond(\mathcal{A}) = \mathcal{U}_1 * f^\diamond(\Sigma) * \mathcal{V}_1^T + \mathcal{U}_2 * f^\diamond(\Sigma) * \mathcal{V}_2^T + i (\mathcal{U}_2 * f^\diamond(\Sigma) * \mathcal{V}_1^T - \mathcal{U}_1 * f^\diamond(\Sigma) * \mathcal{V}_2^T).
\]

Hence, by the above block tensor multiplication theorem, we obtain

\[
\Phi(f^\diamond(\mathcal{A})) = \begin{bmatrix}
\mathcal{U}_1 * f^\diamond(\Sigma) * \mathcal{V}_1^T + \mathcal{U}_2 * f^\diamond(\Sigma) * \mathcal{V}_2^T & -\mathcal{U}_2 * f^\diamond(\Sigma) * \mathcal{V}_1^T + \mathcal{U}_1 * f^\diamond(\Sigma) * \mathcal{V}_2^T \\
\mathcal{U}_2 * f^\diamond(\Sigma) * \mathcal{V}_1^T - \mathcal{U}_1 * f^\diamond(\Sigma) * \mathcal{V}_2^T & \mathcal{U}_1 * f^\diamond(\Sigma) * \mathcal{V}_1^T + \mathcal{U}_2 * f^\diamond(\Sigma) * \mathcal{V}_2^T
\end{bmatrix}.
\]

Applying tensor singular value decomposition to \( \Phi(\mathcal{A}) = \begin{bmatrix} \mathcal{B} & -\mathcal{C} \\ \mathcal{C} & \mathcal{B} \end{bmatrix} \), we have the decomposition:

\[
\begin{bmatrix} \mathcal{B} & -\mathcal{C} \\ \mathcal{C} & \mathcal{B} \end{bmatrix} = \begin{bmatrix} \mathcal{U}_1 & -\mathcal{U}_2 \\ \mathcal{U}_2 & \mathcal{U}_1 \end{bmatrix} * \begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix} * \begin{bmatrix} \mathcal{V}_1 & -\mathcal{V}_2 \\ \mathcal{V}_2 & \mathcal{V}_1 \end{bmatrix}^T.
\]

Since \( \mathcal{U} \) is a unitary tensor, we obtain

\[
(\mathcal{U}_1 + i \mathcal{U}_2) * (\mathcal{U}_1^T - i \mathcal{U}_2^T) = \mathcal{I},
\]

and

\[
(\mathcal{U}_1^T - i \mathcal{U}_2^T) * (\mathcal{U}_1 + i \mathcal{U}_2) = \mathcal{I}.
\]

Therefore, \( \mathcal{U}_1 * \mathcal{U}_1^T + \mathcal{U}_2 * \mathcal{U}_2^T = \mathcal{I} \) and \( \mathcal{U}_1 * \mathcal{U}_2^T = \mathcal{U}_2 * \mathcal{U}_1^T \).

Thus \( \begin{bmatrix} \mathcal{U}_1 & -\mathcal{U}_2 \\ \mathcal{U}_2 & \mathcal{U}_1 \end{bmatrix} \) is an orthogonal tensor. Similarly, \( \begin{bmatrix} \mathcal{V}_1 & -\mathcal{V}_2 \\ \mathcal{V}_2 & \mathcal{V}_1 \end{bmatrix} \) is also an orthogonal tensor.
So, on the other hand, it comes to
\[
f^\circ(\Phi(A)) = \begin{bmatrix}
U_1 & -U_2 \\
U_2 & U_1
\end{bmatrix} \ast \begin{bmatrix}
f^\circ(\Sigma) & 0 \\
0 & f^\circ(\Sigma)
\end{bmatrix} \ast \begin{bmatrix}
V_1^T & V_2^T \\
-V_2^T & V_1^T
\end{bmatrix}
\]
\[
= \begin{bmatrix}
U_1 \ast f^\circ(\Sigma) \ast V_1^T + U_2 \ast f^\circ(\Sigma) \ast V_2^T - U_2 \ast f^\circ(\Sigma) \ast V_1^T + U_1 \ast f^\circ(\Sigma) \ast V_2^T \\
U_2 \ast f^\circ(\Sigma) \ast V_1^T - U_1 \ast f^\circ(\Sigma) \ast V_2^T
\end{bmatrix}.
\]
Therefore,
\[
f^\circ(\Phi(A)) = \Phi(f^\circ(A)).
\]

This theorem gives us a transformation between the function \(f^\circ : \mathbb{C}^{n \times n \times p} \to \mathbb{C}^{n \times n \times p}\) and \(f^\circ : \mathbb{R}^{2n \times 2n \times p} \to \mathbb{R}^{2n \times 2n \times p}\) induced by the same scalar function \(f : \mathbb{C} \to \mathbb{C}\). This theorem will be very useful to avoid complex number computations of generalized tensor functions. Since the map \(\Phi\) is invertible, we also have the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{C}^{n \times n \times p} (\ast, +, I_n, 0) & \xrightarrow{f^\circ} & \mathbb{C}^{n \times n \times p} (\ast, +, I_n, 0) \\
\Phi \downarrow & & \Phi^{-1} \uparrow \\
\mathbb{R}^{2n \times 2n \times p} (\ast, +, I_{2n}, 0) & \xrightarrow{f^\circ} & \mathbb{R}^{2n \times 2n \times p} (\ast, +, I_{2n}, 0)
\end{array}
\]

### 3.7 Tensor to matrix isomorphism

It should be noticed that if we have a tensor \(A \in \mathbb{C}^{m \times n \times p}\), when \(p = 1\), our definition of generalized tensor function degenerate to the generalized matrix function. We also denote the generalized matrix function to be \(f^\circ\).

When \(p \neq 1\), we also want to establish some isomorphism structures between matrices and tensors which might be useful to transpose generalized tensor function problems to generalized matrix function problems. We have the following theorem which shows the bcirc operator on tensors is an isomorphism between the tensor space \(\mathbb{C}^{m \times n \times p}\) and the matrix space \(\mathbb{C}^{mp \times np}\).

**Theorem 14.** Let \(A \in \mathbb{C}^{m \times n \times p}\) be a complex tensor and \(f : \mathbb{C} \to \mathbb{C}\) be a scalar function. Denote \(f^\circ\) to be both the induced generalized tensor function and the generalized matrix function. ‘bcirc’ is the tensor block circulant operator. Then \(f^\circ\) commutes with bcirc:

\[
f^\circ(\text{bcirc}(A)) = \text{bcirc}(f^\circ(A)). \tag{17}
\]

That is to say, we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{C}^{n \times m \times p} (\ast, +, I_n, 0) & \xrightarrow{f^\circ} & \mathbb{C}^{n \times m \times p} (\ast, +, I_n, 0) \\
\downarrow \text{bcirc} & & \downarrow \text{bcirc} \\
\mathbb{C}^{np \times mp} (\ast, +, I_{np}, 0) & \xrightarrow{f^\circ} & \mathbb{C}^{np \times mp} (\ast, +, I_{np}, 0)
\end{array}
\]
Proof.

\[ f^\circ (\text{bcirc}(A)) = f^\circ \begin{bmatrix} A^{(1)} & A^{(p)} & A^{(p-1)} & \cdots & A^{(2)} \\ A^{(2)} & A^{(1)} & A^{(p)} & \cdots & A^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A^{(p)} & A^{(p-1)} & \cdots & \cdots & A^{(2)} \\ A^{(2)} & A^{(1)} & \cdots & \cdots & A^{(3)} \end{bmatrix} \]

\[ = f^\circ \begin{bmatrix} D_1 & & & \cdots & \frac{F_p \otimes I_n}{D_{p+1}} \end{bmatrix} \begin{bmatrix} f^\circ(D_1) \\ f^\circ(D_2) \\ \vdots \\ f^\circ(D_p) \end{bmatrix} \]

On the other hand,

\[ \text{bcirc}(f^\circ(A)) = \text{bcirc}^{-1} \begin{bmatrix} f^\circ(D_1) \\ f^\circ(D_2) \\ \vdots \\ f^\circ(D_p) \end{bmatrix} \]

So we have \( f^\circ (\text{bcirc}(A)) = \text{bcirc}(f^\circ(A)). \)

**Remark 9.** Theorem 14 also shows that the generalized matrix functions defined by Ben-Israel [5] is the degenerate case of our generalized tensor functions. The characteristics and properties of generalized tensor functions also hold for generalized matrix functions.

Since the ‘bcirc’ operator is an invertible operator, the following commutative diagram holds:

\[ \mathbb{C}^{n \times m \times p} (\cdot, +, I_n, 0) \xrightarrow{f^\circ} \mathbb{C}^{n \times m \times p} (\cdot, +, I_n, 0) \]

\[ \xrightarrow{\text{bcirc}} \mathbb{C}^{np \times mp} (\cdot, +, I_{np}, 0) \xrightarrow{f^\circ} \mathbb{C}^{np \times mp} (\cdot, +, I_{np}, 0) \]

The above diagram shows that we can transpose the generalized tensor functions to generalized matrix functions and some algorithms in matrices cases maybe useful upon the transposed matrix problems.
In probability theories, a stochastic matrix is a square matrix with all rows and columns summing to 1. They are very useful to describe the transitions of a Markov chain. The stochastic matrix was first developed by Andrey Markov at the beginning of the 20th century and it is found varieties of usage throughout quite a lot of scientific fields, such as probability theories, statistics, finance and linear algebra, as well as computer science and population genetics and so on [2]. In this paper, we generalize the concept of doubly stochastic matrices to third order tensors as follows:

**Definition 12.** (Doubly F-stochastic tensor) A tensor $A \in \mathbb{R}^{n \times n \times p}$ is called doubly F-stochastic if and only if
\[ A \ast e = A^\top \ast e = e, \] where $e \in \mathbb{R}^{n \times 1 \times p}$ is a tensor whose elements are all 1.

This definition is to say, if a tensor $A \in \mathbb{R}^{n \times n \times p}$ is F-doubly stochastic, the sum of all the elements of its horizontal slices and lateral slices are all 1 (See Fig. 3).

We can also define the right stochastic tensor (left stochastic tensor) with each lateral (horizontal) slice summing to 1.

As a beautiful application and illustration of how to use the above isomorphism theorem, we have introduced the concept of doubly F-stochastic tensor and we will prove this set is invariant under the generalized tensor function. In order to prove this, we need the following lemma.

**Lemma 7.** [6] If $A \in \mathbb{R}^{n \times n}$ is doubly stochastic, $f$ satisfies the same assumptions as in Theorem 11 and $f(1) = 1$, then $f^\circ(A)$ is also doubly stochastic.

**Theorem 15.** If $A \in \mathbb{R}^{n \times n \times p}$ is F-doubly stochastic, $f$ satisfies the same assumptions as above and $f(1) = 1$, then $f^\circ(A)$ is also F-doubly stochastic.

**Proof.** Since we have
\[ A \ast e = A^\top \ast e = e, \] that is equivalent to the equation
\[ \text{bcirc}(A)\text{unfold}(e) = \text{bcirc}(A)^\top\text{unfold}(e) = \text{unfold}(e). \] So we have bcirc($A$) is a stochastic matrix. By the previous lemma, we have $f^\circ(\text{bcirc}(A))$ is also a stochastic matrix. Since we have the matrix tensor isomorphism, it comes to bcirc($f^\circ(A)$) is a stochastic matrix. That is to say,
\[ \text{bcirc}(f^\circ(A))\text{unfold}(e) = \text{bcirc}(f^\circ(A))^\top\text{unfold}(e) = \text{unfold}(e) \] which is equivalent to
\[ f^\circ(A) \ast e = f^\circ(A)^\top \ast e = e. \] That is to say $f^\circ(A)$ is also a F-doubly stochastic tensor.

### 3.8 Invariant tensor cones

In the previous subsections, we proposed structures of tensors which is preserved under generalized tensor functions. In this subsection, we will talk about tensor cones which is another type of invariant tensor structure.

Let $U \in \mathbb{C}^{m \times m \times p}$ and $V \in \mathbb{C}^{n \times n \times p}$ be two fixed unitary tensors. Denote the set $\mathcal{S}_{U,V}$ be the set of $m \times n \times p$ complex tensors of the form:
\[ A = U \ast \mathcal{S} \ast V^H, \]
where

$$(F_p \otimes I_m) \begin{bmatrix}
(\Sigma_1)_r \\
(\Sigma_2)_r \\
\vdots \\
(\Sigma_p)_r
\end{bmatrix} (F^H_p \otimes I_n),$$

$$(\Sigma_i)_r = \text{diag}(c^{(i)}_1, c^{(i)}_2, \ldots, c^{(i)}_r, 0, \ldots, 0) \in \mathbb{R}^{m \times n},$$

here $r$ is the tubal rank of the tensor $\mathcal{A}$. The singular values $c^{(i)}_j$ satisfy $c^{(i)}_1 \geq c^{(i)}_2 \geq \cdots \geq c^{(i)}_r \geq 0$. Then the set $\mathcal{S}_{U,V}$ is a closed convex cone, i.e. the tensor cone is a closed set under the Euclidean topology. It’s interior is the set of all tensors $\mathcal{A} \in \mathcal{S}_{U,V}$ whose tubal-rank rank$_t(\mathcal{A}) = r$.

For $f : \mathbb{R} \rightarrow \mathbb{R}$ is any nonnegative non-increasing function for $x > 0$, then because of the definition of generalized tensor function, $\mathcal{S}_{U,V}$ is invariant under $f^\circ$. Further, if $f(x) > 0$ for $x > 0$, the induced function $f^\circ$ maps the interior of the tensor cone $\mathcal{S}_{U,V}$ to itself.

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