Majority Edge-Colorings of Graphs

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Abstract

We propose the notion of a majority $k$-edge-coloring of a graph $G$, which is an edge-coloring of $G$ with $k$ colors such that, for every vertex $u$ of $G$, at most half the edges of $G$ incident with $u$ have the same color. We show the best possible results that every graph of minimum degree at least 2 has a majority 4-edge-coloring, and that every graph of minimum degree at least 4 has a majority 3-edge-coloring. Furthermore, we discuss a natural variation of majority edge-colorings and some related open problems.

Mathematics Subject Classifications: 05C15

1 Introduction

Motivated by similar notions considered for vertex-colorings, we propose and study majority edge-colorings of graphs: For a (finite, simple, and undirected) graph $G$, an edge-coloring $c : E(G) \rightarrow [k]$ is a majority $k$-edge-coloring if, for every vertex $u$ of $G$ and every color $\alpha$ in $[k]$, at most half the edges incident with $u$ have the color $\alpha$. 

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Before we present our results, we discuss some related research. Lovász [9] showed that every graph $G$ has a 2-vertex-coloring such that, for every vertex $u$ of $G$, at most half the neighbors of $u$ have the same color as $u$. For infinite graphs, this leads to the Unfriendly Partition Conjecture [2]. Kreutzer, Oum, Seymour, van der Zypen, and Wood [8] showed that every digraph $D$ has a 4-vertex-coloring such that, for every vertex $u$ of $D$, at most half the out-neighbors of $u$ have the same color as $u$, and they conjecture that 3 colors suffice. Anholcer, Bosek, and Grytczuk [4] studied a choosability version for digraphs. It follows from a result of Wood [13] that every digraph $D$ has a 4-arc-coloring such that, for every vertex $u$ of $D$, at most half the arcs leaving $u$ have the same color. Further related research concerns defective or frugal edge-colorings [1, 3, 7], where maximum degree conditions are imposed on the subgraphs formed by edges having the same color.

Our first result is that 2 colors almost suffice for a majority edge-coloring.

**Theorem 1.** Let $G$ be a connected graph.

(i) If $G$ has an even number of edges or $G$ contains vertices of odd degree, then $G$ has a 2-edge-coloring such that, for every vertex $u$ of $G$, at most $\left\lfloor \frac{d_G(u)}{2} \right\rfloor$ of the edges incident with $u$ have the same color.

(ii) If $G$ has an odd number of edges, all vertices of $G$ have even degree, and $u_G$ is any vertex of $G$, then $G$ has a 2-edge-coloring such that, for every vertex $u$ of $G$ distinct from $u_G$, exactly $\frac{d_G(u)}{2}$ of the edges incident with $u$ have the same color, and exactly $\frac{d_G(u)}{2} + 1$ of the edges incident with $u_G$ have the same color.

Using Vizing’s bound [12] on the chromatic index leads to our second result.

**Theorem 2.** Every graph of minimum degree at least 2 has a majority 4-edge-coloring.

Clearly, a graph containing a vertex of degree 1 does not have a majority edge-coloring, which motivates the minimum degree condition in Theorem 2. Furthermore, since graphs of minimum degree at least 2, maximum degree 3, and chromatic index 4 have no majority 3-edge-coloring, the number of colors in Theorem 2 is best possible under this minimum degree condition. In fact, if a graph $G$ of minimum degree at least 2 has an induced subgraph $H$ such that $H$ is a graph of maximum degree 3 and chromatic index 4 such that all vertices of $H$ have degree 2 or 3 in $G$, then $G$ has no majority 3-edge-coloring. We conjecture that all graphs for which 4 colors are needed contain an induced subgraph of maximum degree 3 and chromatic index 4.

Our third result supports this conjecture.

**Theorem 3.** Every graph of minimum degree at least 4 has a majority 3-edge-coloring.

Since a graph containing a vertex of odd degree at least 3 does not have a majority 2-edge-coloring, the number of colors in Theorem 3 is best possible under the minimum degree condition in that result. In Section 2 we prove our results, and in a conclusion we discuss a variation of majority edge-colorings.
2 Proofs

Theorem 1 is a consequence of Euler’s Theorem \[6\].

**Proof of Theorem 1.**

(i) Let the multigraph \(G'\) arise from \(G\) by adding the edges of a perfect matching \(M\) on the possibly empty set of vertices of odd degree. Clearly, the multigraph \(G'\) is connected and every vertex has even degree in \(G'\). Let \(e_0e_1\cdots e_{m-1}\) be an Euler tour of \(G'\), where, provided that \(M\) is not empty, we may assume that \(e_{m-1} \in M\). Setting \(c(e_i) = (i \mod 2) + 1\) for every index \(i\) such that \(e_i\) belongs to \(G\), yields the desired 2-edge-coloring of \(G\).

(ii) Let \(e_0e_1\cdots e_{m-1}\) be an Euler tour of \(G\) such that \(e_0\) is incident with \(u_G\). Now, setting \(c(e_i) = (i \mod 2) + 1\) for every index \(i\), yields the desired 2-edge-coloring of \(G\).

Theorem 2 is a consequence of Vizing’s Theorem \[12\].

**Proof of Theorem 2.** Let \(G\) be a graph of minimum degree at least 2. If \(u\) is a vertex of degree \(d\), and \(d = d_1 + \cdots + d_k\) is a partition of \(d\) into positive integers \(d_i\), then the graph \(H\) arises from \(G\) by splitting \(u\) into vertices of degrees \(d_1, \ldots, d_k\) if there is a partition \(N_G(u) = N_1 \cup \cdots \cup N_k\) of \(N_G(u)\) with \(|N_i| = d_i\) for \(i \in [k]\), \(V(H) = (V(G) \setminus \{u\}) \cup \{u_1, \ldots, u_k\}\) for \(u_1, \ldots, u_k \not\in V(G)\), and \(E(H) = E(G - u) \cup \bigcup_{i \in [k]} \{u_iv : v \in N_i\}\). See Figure 1 for an illustration.

![Figure 1: Splitting a vertex \(u\) of degree 7 into vertices of degrees 2, 2, and 3.](image)

Now, let \(G^*\) arise from \(G\) by splitting every vertex of degree \(d > 3\) into vertices of degrees

- \(3, \ldots, 3\), if \(d \equiv 0 \mod 3\),
- \(2, 2, 3, \ldots, 3\), if \(d \equiv 1 \mod 3\), and
- \(2, 3, \ldots, 3\), if \(d \equiv 2 \mod 3\).

Note that there is a natural bijection between the edges of \(G\) and those of \(G^*\). By Vizing’s Theorem \[12\], the graph \(G^*\) has a proper 4-edge-coloring, which yields a majority 4-edge-coloring of \(G\). In fact, we obtain an edge-coloring of \(G\) such that, for every vertex of degree \(d\) at least 4, at most \((d + 2)/3\) of the incident edges have the same color.
We proceed to the proof of Theorem 3.

Proof of Theorem 3. Let $G$ be a graph of minimum degree $\delta$ at least 4. Let $V(G) = D \cup A \cup C$ be the Gallai-Edmonds decomposition of $G$, that is, $D$ is the set of all vertices of $G$ that are missed by some maximum matching, $A$ is the set of all vertices of $G$ outside of $D$ that have a neighbor in $D$, and $C$ contains the remaining vertices, cf. [10].

Let $D'$ be the set of isolated vertices in $G[D]$.

Claim 4. It is possible to select, for every vertex $u$ in $D'$, exactly one edge incident with $u$ in such a way that every vertex $v$ in $A$ is incident with at most $\left\lfloor \frac{d_G(v)}{2} \right\rfloor$ of the selected edges.

Proof of Claim 4. Let $H_0$ be the bipartite subgraph of $G$ with partite sets $D'$ and $A$ whose edges are exactly all edges of $G$ between $D'$ and $A$. Let $H$ arise from $H_0$ by replacing each vertex $v$ in $A$ by $\left\lfloor \frac{d_G(v)}{2} \right\rfloor$ copies having the same neighbors in $D'$ as $v$. Clearly, the desired statement follows if $H$ has a matching saturating all vertices in $D'$. Suppose, for a contradiction, that such a matching does not exist. By Hall’s Theorem [5], there is a subset $S$ of $D'$ with $|S| > |N_H(S)|$. By the definition of $D'$ and the construction of $H$, we have $|N_H(S)| = \sum_{v \in N_G(S)} \left\lfloor \frac{d_G(v)}{2} \right\rfloor$. Let $m$ denote the number of edges of $G$ between $S$ and $N_G(S)$. Since every vertex in $D'$ has all its neighbors in $A$, we have $m \geq \delta|S|$. Furthermore, $m \leq \sum_{v \in N_G(S)} d_G(v)$. Combining these estimates, we obtain

$$\sum_{v \in N_G(S)} \delta \left\lfloor \frac{d_G(v)}{2} \right\rfloor = \delta |N_H(S)| < \delta |S| \leq m \leq \sum_{v \in N_G(S)} d_G(v). \quad (1)$$

For integers $\delta$ and $d$ with $3 \leq \delta \leq d$, it is easy to verify that $\delta \left\lfloor \frac{d}{2} \right\rfloor \geq d$, which yields a contradiction to (1). This completes the proof of Claim 4.

The properties of the Gallai-Edmonds decomposition imply that $G[C]$ has a perfect matching $M_C$, that there is a matching $M_A$ using edges between $A$ and $D$ that connects each vertex from $A$ to a distinct component of $G[D]$, and that every component of $G[D]$ is factor-critical; recall that a graph $H$ is factor-critical if $H - u$ has a perfect matching for every vertex $u$ of $H$.

We now construct a subset $E_1$ of the edge set $E(G)$ of $G$ as follows, starting with the empty set:

- We add to $E_1$ all $|D'|$ selected edges as in Claim 4.
- We add $M_C$ to $E_1$.
- For every vertex $v$ from $A$ that is not incident with a selected edge, we add to $E_1$ the unique edge from $M_A$ incident with $v$. Let $M'_A$ be the subset of $M_A$ added to $E_1$. 

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• For every component $K$ of $G'[D]$ of order at least 3 such that some vertex $x$ of $K$ is incident with an edge from $M'_A$, we add to $E_1$ a perfect matching of $K - x$.

• For every component $K$ of $G'[D]$ of order at least 3 such that no vertex of $K$ is incident with an edge from $M'_A$, we add to $E_1$ a perfect matching of $K - x$ for some vertex $x$ of $K$ as well as one further edge of $K$ incident with $x$.

Up to some small modifications explained below, this completes the description of $E_1$.

By construction, the spanning subgraph $G_1$ of $G$ with edge set $E_1$ satisfies

$$1 \leq d_{G_1}(u) \leq \left\lfloor \frac{d_G(u)}{2} \right\rfloor$$

for every vertex $u$ of $G$. (2)

Let $G_2$ be the spanning subgraph of $G$ with edge set $E(G) \setminus E_1$.

For every component $K$ of $G_2$ such that all vertices of $K$ have even degree in $G_2$, $K$ has an odd number of edges, and all vertices from $V(K)$ have degree 1 in $G_1$, we select any edge $e_K$ from $K$ and move it from $G_2$ to $G_1$. Note that $K - e_K$ contains exactly two vertices of odd degree, and, hence, is still connected. Furthermore, since $G$ has minimum degree at least 4, it follows that (2) still holds after each such modification. Having performed these modifications for each such component of $G_2$, every component $K$ of (the modified) $G_2$ now

• either contains at least one vertex of odd degree in $K$,

• or all vertices of $K$ have even degrees in $K$, and the number of edges of $K$ is even,

• or all vertices of $K$ have even degrees in $K$, the number of edges of $K$ is odd, and $K$ contains a vertex $u_K$ such that the degree of $u_K$ in $G_1$ is at least 2.

The components of $G_2$ as in the final point are called type 2 components, and the remaining components of $G_2$ are called type 1 components.

We are now in a position to describe a majority 3-edge-coloring $c : E(G) \rightarrow [3]$.

• For all edges $e$ of $G_1$, let $c(e) = 3$.

• For every component $K$ of $G_2$ that is of type 1, let $c : E(K) \rightarrow [2]$ be as in Theorem 1(i) (applied to $K$ as $G$).

• For every component $K$ of $G_2$ that is of type 2, let $c : E(K) \rightarrow [2]$ be as in Theorem 1(ii) (applied to $K$ and $u_K$ as $G$ and $u_G$).

It is now easy to verify that $c$ is a majority 3-edge-coloring of $G$, which completes the proof. ~\[\square\]
3 Conclusion

The most natural question motivated by our results is which graphs of minimum degree at least 2 do not have a majority 3-edge-coloring.

As a variation of majority edge-colorings, we propose the study of \(\alpha\)-majority edge-colorings for \(\alpha \in (0, 1)\), where at most an \(\alpha\)-fraction of the edges incident with each vertex are allowed to have the same color. If \(k\) is a positive integer at least 2, then every positive integer at least \(k(k-1)\) can be written as a non-negative integral linear combination of \(k\) and \(k+1\). Using this fact, a straightforward adaptation of the proof of Theorem 2 yields the following statement: If a graph \(G\) has minimum degree at least \(k(k-1)\), then \(G\) has a \(1\)-majority \((k+2)\)-edge-coloring. A probabilistic argument implies that, for a sufficiently large minimum degree, one color less suffices.

Theorem 5. For every integer \(k\) at least 2, there is a positive integer \(\delta_k\) such that every graph of minimum degree at least \(\delta_k\) has a \(1\)-majority \((k+1)\)-edge-coloring.

Proof. Let \(G\) be a graph of minimum degree \(\delta\) at least \(\delta_k\), where we specify \(\delta_k\) later. Let \(c: E(G) \rightarrow [k+1]\) be a random \((k+1)\)-edge-coloring, where we choose the color of each edge uniformly and independently at random. For every vertex \(u\) of \(G\), we consider the bad event \(A_u\) that more than \(1\)-fraction of the edges incident with \(u\) have the same color.

For every vertex \(u\) of \(G\), the event \(A_u\) is determined only by the colors of the edges incident with \(u\), which are chosen uniformly and independently at random. Therefore, the event \(A_u\) is mutually independent of all events \(A_v\) with \(v \in V(G) \setminus \{u\} \cup N_G(u)\). In order to complete the proof, we use the weighted Lovász Local Lemma, cf. [11], which states that with positive probability none of the bad events \(A_u\) occurs provided that there is a positive integer \(t_u\) for every vertex \(u\) of \(G\) and there is some real \(p\) with \(0 \leq p \leq \frac{1}{3}\) such that

- \(\mathbb{P}[A_u] \leq p^{t_u}\) for every vertex \(u\) of \(G\) and
- \(\sum_{v \in N_G(u)} (2p)^{t_v} \leq \frac{t_u}{2}\) for every vertex \(u\) of \(G\).

Let \(p = (k+1)e^{-\frac{\delta}{\max(k,k+1)}}\) and, for every vertex \(u\) of \(G\), let \(t_u = \left\lceil \frac{d_G(u)}{\delta} \right\rceil\). Note that \(d_G(u) \geq \delta\) implies that \(t_u\) is a positive integer, and that \(2t_u = 2\left\lceil \frac{d_G(u)}{\delta} \right\rceil \geq \frac{d_G(u)}{\delta}\).
Choosing $\delta_k$ sufficiently large, we may ensure that $p \leq \frac{1}{4}$, and, hence, $\mathbb{P}[A_u] \leq p^{\frac{d_{G(u)}}{\delta}} \leq p^{t_u}$. Furthermore, we obtain

$$\sum_{v \in N_G(u)} (2p)^{t_v} \leq 2pd_G(u) \leq 4p\delta t_u = \left(4(k+1)e^{-\frac{\delta}{3k(k+1)}}\right)t_u,$$

which is at most $t_u/2$ for $\delta_k$ sufficiently large. Altogether, choosing $\delta_k$ sufficiently large, the weighted Lovász Local Lemma implies that with positive probability none of the bad events $A_u$ occurs, which implies the existence of a $\frac{1}{k}$-majority $(k+1)$-edge-coloring and completes the proof.

The estimates in the above proof allow to show that $\delta_k$ can be chosen to be $O(k^3 \log k)$. Our Theorem 3 implies that 4 is the smallest possible value for $\delta_2$.

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