Asymptotic properties of the solutions to the Vlasov-Maxwell system in the exterior of a light cone

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Abstract

This paper is concerned with the asymptotic behavior of small data solutions to the three-dimensional Vlasov-Maxwell system in the exterior of a light cone. The plasma does not have to be neutral and no compact support assumptions are required on the data. In particular, the initial decay in the velocity variable of the particle density is optimal and we only require an $L^2$ bound on the electromagnetic field with no additional weight. We use vector field methods to derive improved decay estimates in null directions for the electromagnetic field, the particle density and their derivatives. In contrast with [2], where we studied the behavior of the solutions in the whole spacetime, the initial data have less decay and we do not need to modify the commutation vector fields of the relativistic transport operator. To control the solutions under these assumptions, we crucially use the strong decay satisfied by the particle density in the exterior of the light cone, null properties of the Vlasov equation and certain hierarchies in the energy norms.

Contents

1 Introduction
  1.1 Small data results for the VM system
  1.2 Previous works on Vlasov systems using vector field methods
  1.3 Statement of the main result
  1.4 Main ingredients of the proof
  1.5 Structure of the paper
  1.6 Acknowledgements

2 Preliminaries
  2.1 Basic notations
  2.2 A null foliation
  2.3 The commutation vector fields
  2.4 The null components of the velocity vector and the weights preserved by $T$
  2.5 The null decomposition of the electromagnetic field

3 Energy and pointwise decay estimates
  3.1 Estimates for velocity averages
  3.2 Estimates for the electromagnetic field

4 Null properties of the Vlasov equation

5 Bootstrap assumptions and strategy of the proof

6 Improvement of the energy bound on the particle density
  6.1 Proof of inequality
  6.2 Proof of Proposition 6.1
    6.2.1 If $|\gamma| \leq N - 2$
    6.2.2 When $|\gamma| \geq N - 1$
  6.3 The remaining energy norm
  6.4 $L^2$ estimates on velocity averages

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1 Introduction

In this article, we study the asymptotic properties of small data solutions of the Vlasov-Maxwell (VM) system in the exterior of a light cone \( V_b := \{ (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3 / |x| > t - b \} \), where, say, \( b \leq -1 \). More precisely, our main goal is to derive sharp decay estimates. The system, which is of particular importance in plasma physics, is given for one species of particles by

\[
v^0 \partial_t f + v^i \partial_i f + ev^j F_{\mu j} \partial_{\nu} f = 0, \tag{1}
\]

\[
\nabla^\mu F_{\mu \nu} = J(f)_{\nu} := \int_{v \in \mathbb{R}^3} \frac{v^\nu}{v^0} f dv, \tag{2}
\]

\[
\nabla^\mu F^*_{\mu \nu} = 0, \tag{3}
\]

where

- \( v^0 = \sqrt{m^2 + |v|^2}, m > 0 \) is the mass of the particles and \( e \neq 0 \) their charge. For the remaining of this paper, we take \( m = e = 1 \) and we denote \( \sqrt{1 + |v|^2} \) by \( v^0 \).
- The function \( f(t, x, v) \) is the particle density, the 2-form \( F(t, x) \) is the electromagnetic field and \( *F(t, x) \) is its Hodge dual.

1.1 Small data results for the VM system

The study of the small data solutions of the VM system has been initiated in [9] by Glassey-Strauss. Under a compact support assumption in space and in velocity on the initial data, they proved global existence and obtained the optimal decay rate on \( \int_0^t \int_{|v| > \epsilon} f dv \). The compact support assumption in \( v \) is replaced by Schaeffer in [12] by a polynomial decay but the data still have to be compactly supported in space. Moreover, the optimal decay rate on \( \int_0^t \int_{|v| > \epsilon} f dv \) is not obtained by this method. None of these results contain estimates on the derivatives of \( \int_0^t \int_{|v| > \epsilon} f dv \) nor on the higher order derivatives of the electromagnetic field.

In [1], we removed all compact support assumptions for the dimensions \( d \geq 4 \). For this, we used vector field methods, developed in [4] for the electromagnetic field and [8] for relativistic transport equations. We then obtained almost optimal decay on the solutions of the system and their derivatives and we described precisely the behavior each null component of the electromagnetic field. We recently extended these results to the 3d case and we also relaxed the assumptions on the initial data, allowing in particular the presence of a non-zero total charge. A better understanding of the null structure of the VM system as well as the use of modified vector fields\(^3\) were the key for dealing with the slower decay rates of the solutions. We splitted the electromagnetic field into two parts. The chargeless one on which we could then propagate a weighted \( L^2 \) norm and the pure charge part, given by an explicit formula, which decays as \( \epsilon r^{-2} \) despite of its infinite energy.

We also investigate the case where the particles are massless (i.e. \( m = 0 \)). First in [1] for the high dimensions, where we proved that similar results to the massive case hold provided that the velocity support of the the particle density is bounded away from 0. These extra hypothesis appears to be necessary since we also proved in [11] that the VM system do not admit a local classical solution for certain smooth initial data which do not vanish for small velocities. Secondly, in our recent work [3], we proved sharp asymptotics on the small data solutions and their derivatives to the massless VM system in 3d. Contrary to the massive case, the proof does not require the use of modified vector fields but still necessitates a strong understanding of the null properties of the system.

\(^1\)We choose to lighten the notations by considering only one species since the presence of other ones does not complicate the analysis.

\(^2\)We will, throughout this article, use the Einstein summation convention so that \( v^i \partial_i f = \sum_{i=1}^{\dim} v^i \partial_i f \). A sum on latin letters starts from 1 whereas a sum on greek letters starts from 0.

\(^3\)Modified Vector fields, which depend on the solution itself, were already used by [1] (respectively [7]) in the context of the Vlasov-Nordström (respectively the Einstein-Vlasov) system. They are built over the commutation vector fields of the relativistic transport operator \( v^\mu \partial_\mu \) in order to compensate the worst source terms of the commuted Vlasov equation.
In this article, we study the asymptotic properties of the solutions to the VM system in the exterior of a light cone under a smallness assumption but weaker decay near infinity. We obtain in particular almost optimal pointwise decay estimates on the velocity average of the Vlasov field as well as its derivatives. The hypotheses on the particle density in the variable \( v \) are optimal in the sense that we merely suppose \( f \) and its derivatives to be initially integrable in \( v \), which is a necessary condition for the source term of the Maxwell equations \([2]\) to be well defined. As \( f \) strongly decay in the domain studied, our proof merely requires the boundedness of the \( L^2 \) norm of the electromagnetic field. This has to be compared with our proof in \([2]\], where we study the same problem in the whole spacetime, which crucially relies on the propagation of a weighted energy norm of \( F \). Another remarkable point, still related to the good behavior of \( f \) in the region \( V_b \), concerns the commutation vector fields used to study the Vlasov equation. Contrary to \([2]\), we do not need to modify the commutation vector fields of the relativistic transport operator \( \dd^\mu \partial_\mu \) in order to compensate the worst source terms of the commuted Vlasov equations and then close the energy estimates. This leads in particular to a much simpler proof.

Finally, let us mention the recent result \([14]\) of Wang concerning the small data solutions of the massive 3d VM system. Using both vector field methods and Fourier analysis, he proved optimal pointwise decay estimates on \( \int_0^1 f dv \) and its derivatives under strong polynomial decay hypotheses in \((x,v)\) on \( f(t=0) \). In particular, the initial data does not have to be compactly supported.

1.2 Previous works on Vlasov systems using vector field methods

Results on the asymptotic behavior of solutions of several Vlasov systems were recently derived using vector field methods. Let us mention the pioneer work \([8]\) of Fajman-Joudioux-Smulevici on the Vlasov-Norström system (see also \([7]\)) as well as the results of \([13]\) on the Vlasov-Poisson system. The two different proofs, obtained independently by \([6]\) and \([10]\), of the stability of the Minkowski spacetime as a solution to the Einstein-Vlasov system constitute a culmination of these vector field methods.

1.3 Statement of the main result

In order to present our main theorem, we call initial data set for the VM system any ordered pair \((f_0,F_0)\) where \( f_0 : \mathbb{R}^3_+ \times \mathbb{R}^3 \to \mathbb{R} \) and \( F_0 \) are both sufficiently regular and satisfy the constraint equations

\[
\nabla^i (F_0)_{i0} = - \int_{v \in \mathbb{R}^3} f_0 dv \quad \text{and} \quad \nabla^i (F_0)_{i0} = 0.
\]

We refer to Section \([2]\) for the notations not yet defined.

**Theorem 1.1.** Let \( N \geq 8, b \leq -1, 0 < \eta < \frac{1}{72N}, \epsilon > 0, (f_0,F_0) \) an initial data set for the Vlasov-Maxwell equations \([11,3]\) satisfying\(^4\)

\[
\sum_{|\beta|+|\gamma|\leq N+3} \int_{|x|\geq b} \int_{v \in \mathbb{R}^3} (1+|x|)^{\frac{N+4+|\beta|}{2}} (1+|v|)^{|\gamma|} \left| \partial_\gamma^\beta \partial_\nu^\sigma f_0 \right| dv dx + \sum_{|\gamma|\leq N+2} \int_{|x|\geq b} (1+|x|)^{2|\gamma|} \left| \nabla_\gamma F_0 \right|^2 dx \leq \epsilon
\]

and \((f,F)\) be the unique classical solution of the system which satisfies \( f(t=0) = f_0 \) and \( F(t=0) = F_0 \). Then, there exists \( C > 0 \) and \( \epsilon_0 > 0 \), depending only on \( N \) and \( \eta \), such that, if \( 0 \leq \epsilon \leq \epsilon_0 \), \((f,F)\) is well defined in \( V_b = \{ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 / r > t - b \} \) and verifies the following estimates.

- **Energy bound for the electromagnetic field** \( F \): \( \forall t \in \mathbb{R}_+ \),

\[
\sum_{0 \leq k \leq N} \sum_{Z^\gamma \in \mathbb{R}^k} \int_{|x|\geq t-b} \left| \mathcal{L}^\gamma_{Z^\gamma} (F) \right|^2 dx \leq C \epsilon,
\]

- **Pointwise decay estimates** for the null components of \( \mathcal{L}^\gamma_{Z^\gamma} (F) \): \( \forall |\gamma| \leq N-2, (t,x) \in V_b \),

\[
|\alpha (\mathcal{L}^\gamma_{Z^\gamma} (F))|(t,x) \lesssim \frac{\sqrt{\epsilon}}{\tau_+^\gamma}, \quad |\rho (\mathcal{L}^\gamma_{Z^\gamma} (F))|(t,x) + |\alpha (\mathcal{L}^\gamma_{Z^\gamma} (F))|(t,x) + |\sigma (\mathcal{L}^\gamma_{Z^\gamma} (F))|(t,x) \lesssim \frac{\sqrt{\epsilon}}{\tau_+^\gamma}.
\]

\(^4\) We could save three powers of \( x \) in the condition on the initial norm of \( f_0 \) with easy but cumbersome modifications of our proof (mostly in Section \([2.2,3]\) and Proposition \([3]\)). Note also that following the strategy used in Subsection 17.2 of \([2]\) to derive \( L^2 \) estimates on the Vlasov field, we could avoid any hypotheses on the derivatives of order \( N+1 \) and \( N+2 \) of \( F_0 \).
• Energy bound for the particle density: \( \forall \, t \in \mathbb{R}_+ , \)
\[
\sum_{0 \leq k \leq N} \sum_{\beta \in \mathbb{N}^3} \int_{|x| \geq 2^{-t}} \int_{v \in \mathbb{R}^3} |\hat{Z}^\beta f| \, dv \, dx \leq C e(1 + t)^{(N+1)\eta}.
\]

• Pointwise decay estimates for the velocity averages of \( \hat{Z}^\beta f \): \( \forall \, |\beta| \leq N - 3 , \,(t,x) \in V_b , \)
\[
\int_{v \in \mathbb{R}^3} |\hat{Z}^\beta f| \, dv \lesssim \frac{\epsilon}{\tau^2(1+(N+1)\eta)_{\tau-}} \quad \text{and} \quad \forall \, a \in \left[0, \frac{9}{2}\right] , \quad \int_{v \in \mathbb{R}^3} |\hat{Z}^\beta f| \, dv \lesssim \frac{\epsilon}{(1 + a)_{\tau-}}.
\]

Remark 1.2. Note that we can study the solutions to the Vlasov-Maxwell equations in the exterior of a light cone, without any information on their behavior in the remaining part of the Minkowski space, by finite speed of propagation. Every inextendible past causal curves of such a region intersect the hypersurface \( t = 0 \) once and only once, i.e. the region is globally hyperbolic.

Remark 1.3. By a time translation, one can prove a similar result for \( b \in \mathbb{R} \) (\( \epsilon_0 \) would then also depends on \( b \)).

Remark 1.4. Assuming more decay on the electromagnetic field at \( t = 0 \), one could propagate a stronger energy norm as in [2] or [3]. We then could assume less decay in \( x \) on \( f_0 \) and improve the decay rate of the null components of the electromagnetic field. Note however that if the total electromagnetic charge
\[
Q(F)(t) := \lim_{r \to +\infty} \int_{S_{t,r}} \frac{x^i}{r} F_{0i} \, dS_{t,r} = - \lim_{r \to +\infty} \int_{S_{t,r}} \rho(F) \, dS_{t,r} = \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} f \, dx \, dv,
\]
which is a conserved quantity in \( t \), is non zero, we cannot obtain a better decay rate than \( r^{-2} \) on \( \rho(F) \) and assume that \( \int_{\mathbb{R}^3} |\rho(F)| \, dx \) is initially finite. We point out that our hypotheses on the electromagnetic field are compatible with the presence of a non zero total charge.

Remark 1.5. The results of [6], [9] and [7] are obtained using a hyperboloidal foliation and then require compactly supported initial data in space. These compact restrictions on the data could be removed by adapting the method used in this article to the Vlasov-Nordström and the Einstein-Vlasov systems.

Theorem 1.1 immediately implies the following result, concerning solutions arising from large data.

Corollary 1.6. Let \( N \geq 8 \) and \((f_0,F_0)\) an initial data set for the Vlasov-Maxwell equations (1)-(5) satisfying
\[
\sum_{|\beta|+|\gamma| \leq N+3} \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} (1 + |x|) \frac{N+4+|\gamma|}{2} (1 + |v|)^{|\gamma|} |\partial^2_x \partial^\gamma_v f_0| \, dv \, dx + \sum_{|\gamma| \leq N} \int_{x \in \mathbb{R}^3} (1 + |x|)^2 |\nabla \partial^\gamma F_0|^2 \, dx < +\infty
\]
and \((f,F)\) be the unique local classical solution to the system which satisfies \( f(t=0) = f_0 \) and \( F(t=0) = F_0 \). Then, there exists \( b \leq -1 \) such that \((f,F)\) is well defined in \( V_b \) and verifies similar estimates as those presented in Theorem 1.1.

Proof. One only has to notice that there exists \( b \leq -1 \) such that \((f_0,F_0)\) satisfies the hypotheses of Theorem 1.1.

Global existence in the whole Minkowski spacetime for classical solutions to the VM system with large data still remains an open problem. For the weak solutions, the problem was solved in [5] and revisited in [11].

1.4 Main ingredients of the proof

The proof of the main result of this paper is based on vector field methods and then essentially relies on bounding sufficiently well the spacetime integrals of the source terms of the commuted equations. In the exterior of the light cone \( V_0 \), the solutions to the Vlasov equation behave better than in the interior region. One can already see that with the following estimate (see Lemma 2.9), for \( g \) a solution to the free transport equation \( v^\beta \partial_\mu g = 0, \)
\[
\forall \, |x| \geq t , \quad \int_v |g(t,x,v)| \, dv \lesssim \sum_{|\beta| \leq 3} \left\| (v^0)^{2k} (1 + r)^{|\beta|+k+q} \partial^2_v g \right\|_{L^1_v}(t=0) \quad \frac{1}{(1 + t + r)^{2+k} (1 + |t-r|)^{1+q}}.
\]
Contrary to [2], where we study solutions to the VM system in the whole Minkowski spacetime, this strong decay should allow us in principle to avoid the use of modified vector fields. This also allows us to assume less decay on the electromagnetic field and to avoid any difficulty due to the presence of a non zero total charge.
However, as we start with optimal decay in \( v \), we cannot fully use (4). In particular, no extra decay in the \( t + r \) direction can be obtained in that way. Moreover, since the initial data are not compactly supported in \( v \), a problem arises from large velocities, for which \( v^0 \sim |v| \), so that the characteristics of the transport equation ultimately approach those of the Maxwell equations. The consequence is that, in a product such as \( L_{Z^\gamma} (F) \tilde{Z}^\beta f \), one cannot, in view of support considerations, transform a \( |t - r| \) decay in a \( t + r \) one anymore. To circumvent this difficulty, we take advantage of the null structure of the non linealities such as

\[
v^\mu L_{Z^\gamma} (F) \mu^i \partial_{\nu} \tilde{Z}^\beta f, \tag{5}
\]

where \( Z \) is a Killing vector field and \( \tilde{Z} \) its complete lift. The problem is that, for \( g \) solution to \( v^\mu \partial_\mu g = 0 \), \( \partial_\nu g \) essentially behaves as \( (1 + t + r) \partial_{t,x} g \) and the electromagnetic field, as a solution of a wave equation, only decay with a rate of \( (1 + t + r)^{-1} \) in the \( t + r \) direction. However, from [4] (respectively [2]), we know that certain null components of the Maxwell field (respectively the velocity vector \( v \)) are expected to behave better than others. As we propagate a weaker energy norm on \( F \) than [2], the null components \( \rho \) and \( \sigma \) do not decay faster than \( \alpha \) but still have a better behavior. Indeed, they allow us to take advantage of the \( t - r \) decay as they permit us to estimate spacetime integrals by using a null foliation. For the velocity vector, the component \( v^\mu \) allows us to integrate according to a null foliation and provides, as the angular components, an extra decay in \( t + r \) at the cost of weights preserved by the flow of \( v^\mu \partial_\mu \) (see Lemma 2.9). Finally, the radial component of \( (0, \partial_\nu \tilde{Z}^\beta f, \partial_r \tilde{Z}^\beta f, \partial_\nu \tilde{Z}^\beta f) \) costs a power of \( t - r \) instead of \( t + r \). The null structure of \( \tilde{Z} \) is fully depicted in Lemma 2.9 and we can observe that each term contains either the better null component \( \alpha \) of the electromagnetic field, the better null component \( v^\mu \) of the velocity vector or at least two good components.

Finally, the weak decay assumptions on the electromagnetic field force us to consider several hierarchies in the energy norms of the Vlasov field in order to close the energy estimates. Let us illustrate how appears such a hierarchy by an example.

- One of the worst source term of the transport equation satisfied by \( \tilde{Z} f \), where \( \tilde{Z} \) is the complete lift of the Killing vector field \( Z \), is bounded by \( (1 + t + r) \frac{v^\mu}{v^0} |L_{Z} (F)||\partial_{t,x} f| \).
- As \( |L_{Z} (F)| \) merely decay as \( (1 + t + r)^{-1} (1 + |t - r|)^{-\frac{4}{3}} \), we obtain an (almost) integrable decay rate through the utilization of the inequality \( 1 + |t - r| \lesssim z \), where \( z \) is a combination of weights preserved by \( v^\mu \partial_\mu \), so that
  \[
  (1 + t + r) \frac{v^\mu}{v^0} |L_{Z} (F)||\partial_{t,x} f| \lesssim \frac{1}{1 + |t - r|} \frac{v^\mu}{v^0} \sqrt{z} |\partial_{t,x} f|.
  \]
- Thus, we schematically have \( \| \tilde{Z} f \|_{L^1_{t,v}} (t) \lesssim \| \sqrt{z} |\partial_{t,x} f| \|_{L^1_{t,v}} (t) \log (3 + t) \). This leads us to consider energy norms controlling quantities such as \( \| z^{\frac{2 + \beta_P}{2 + \beta_P - 1}} \tilde{Z}^\beta f \|_{L^1_{t,v}} \), where \( \beta_P \) is the number of homogeneous vector fields composing \( \tilde{Z}^\beta \).

1.5 Structure of the paper

Section 2 contains most of the notations used in this article. The vector fields used in this paper and the commuted equations are presented in Subsection 2.3. In Subsection 2.4 fundamental properties of the null components of the velocity vector are proved. The energy norms used to study the Vlasov-Maxwell system are introduced in Section 3. During this section, we also prove approximate conservation laws as well as Klainerman-Sobolev type inequalities in order to control these norms and derive pointwise decay estimates from them. Section 4 is devoted to the study of the null structure of the commuted Vlasov equations. In Section 5, we set up the bootstrap assumptions, present their immediate consequences and describe the strategy of the proof of our main result. Sections 6 (respectively 7) concerns the improvement of the energy bounds on the particle density (respectively the electromagnetic field). Finally, we prove in Section 8 \( L^2 \) estimates for the velocity averages of the higher order derivatives of the Vlasov field.

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\[5\text{Note that this property of } v^\mu \text{ is specific to the exterior of the light cone. In the whole spacetime, the extra decay is merely } \frac{1 + r}{1 + t + r}.\]

\[6\text{Actually, because of other source terms, we will consider slightly different energy norms.}\]
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2 Preliminaries

2.1 Basic notations

In this article we work on the 3 + 1 dimensional Minkowski spacetime ($\mathbb{R}^{3+1}$, $\tilde{\eta}$) and we will use two sets of coordinates. The Cartesian coordinates ($x^0 = t, x^1, x^2, x^3$) and null coordinates $(u, u_1, u_2)$, where

$$u = t + r, \quad u = t - r$$

and $(u_1, u_2)$ are spherical variables, which are spherical coordinates on the spheres $(t, r) = constant$. Apart from $r = 0$ and the usual degeneration of spherical coordinates, these coordinates are defined globally on the Minkowski space. We will also use the following classical weights,

$$\tau_+ := \sqrt{1 + u^2} \quad \text{and} \quad \tau_- := \sqrt{1 + u^2}.$$

**Remark 2.1.** *In this paper, we exclusively work in regions where $1 + t \leq \tau_+(t, x) \lesssim |x|$. We denote by $\nabla$ the intrinsic covariant differentiation on the spheres $(t, r) = constant$ and by $(e_1, e_2)$ an orthonormal basis on them.* Capital Roman indices such as $A$ or $B$ always correspond to spherical variables. The null derivatives are defined by

$$L = \partial_t + \partial_r \quad \text{and} \quad L = \partial_t - \partial_r, \quad \text{so that} \quad L(u) = 2, \quad L(u) = 0, \quad L(u) = 0 \quad \text{and} \quad L(u) = 2.$$ 

The velocity vector $(v^\mu)_{0 \leq \mu \leq 3}$ is parametrized by $(v^t)_{1 \leq \mu \leq 3}$ and $v^0 = \sqrt{1 + |v|^2}$ since we normalize the mass of the particles to $m = 1$. Let $T$ be the operator defined by

$$T : f \mapsto v^\mu \partial_\mu f,$$

for all sufficiently regular function $f : [0, T] \times \mathbb{R}^3 \times \mathbb{R}$. We will raise and lower indices using the metric $\tilde{\eta}$. For instance, $v_0 = v^\mu \tilde{\eta}_{00} = -v^0$ and $x^1 = x_\mu \tilde{\eta}^{\mu 1} = x_1$. Finally, we will use the notation $Q \lesssim R$ for an inequality of the form $Q \leq CR$, where $C > 0$ is a constant independent of the solutions but which could depend on $N$, the maximal number of derivatives, or on fixed parameters ($\delta$ and $\eta$).

2.2 A null foliation

We start by presenting various subsets of the Minkowski space which depends on $t \in \mathbb{R}_+, \ r \in \mathbb{R}_+, \ u \in \mathbb{R}$ or $b \in \mathbb{R}$. Let $\Sigma_{t,r}$, $\Sigma_{t}^{\mu}$, $C_u(t)$ and $V_u(t)$, be the sets defined as

$$S_{t,r} := \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / (s, |y|) = (t, r)\}, \quad \Sigma_{t}^{\mu} := \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / s = t, \ |y| > s - b\}, \quad C_u(t) := \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / s < t, \ s - |y| = u\}, \quad V_u(t) := \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / s < t, \ s - |y| < u\}.$$

The volume form on $C_u(t)$ is given by $dC_u(t) = \sqrt{2}^{-1} r^2 dudS^2$, where $dS^2$ is the standard metric on the 2 dimensional unit sphere.
The following lemma illustrates that we can foliate \(V_b(T)\) by \((\mathcal{S}_s)_{0 \leq s < T}\) or \((\mathcal{C}_a(T))_{a < b}\) and will be used several times during this article.

**Lemma 2.2.** Let \(T > 0, b \in \mathbb{R}\) and \(g \in L^1(V_b(T))\). Then
\[
\int_{V_b(T)} gdV_b(T) = \int_0^T \int_{\mathcal{S}_s} gdxds = \int_{u=-\infty}^b \int_{\mathcal{C}_a(T)} gdC_a(T) \frac{du}{\sqrt{2}}.
\]
We will use the second foliation in order to take advantage of decay in the \(t-r\) direction as \(\|\tau_1^{-1}\|_{L^1(C_u(t))} = \tau_1^{-1}\) whereas \(\|\tau_1^{-1}\|_{L^1(\mathcal{S}_s)} \geq (1 + b^2)^{1/2}\).

### 2.3 The commutation vector fields

The aim of this subsection is to introduce the commutation vector fields for the Maxwell equations, those for the relativistic transport operator and certain of their basic properties. Let \(\mathbb{P}\) be the generators of the Poincaré algebra, i.e. the set containing
- the translations \(\partial_\mu := \partial_{x^\mu}, \ 0 \leq \mu \leq 3\).
- the rotations \(\Omega_{ij} = x^i \partial_j - x^j \partial_i, \ 1 \leq i < j \leq 3\).
- the Lorentz boosts \(\Omega_{0k} = t \partial_k + x^k \partial_t, \ 1 \leq k \leq 3\).

Let also \(\mathbb{O} := \{\Omega_{12}, \Omega_{13}, \Omega_{23}\}\) be the set of the rotational vector fields and \(\mathbb{K} := \mathbb{P} \cup \{S\}\), where \(S = x^\mu \partial_\mu\) is the scaling vector field. We will use the vector fields of \(\mathbb{K}\) for commuting the Maxwell equations. To commute the operator \(T = v^\mu \partial_\mu\), we will rather use the complete lifts of the vector fields of \(\mathbb{P}\).

**Definition 2.3.** Let \(\Gamma\) be a vector field of the form \(\Gamma^\beta \partial_\beta\). Then, the complete lift \(\hat{\Gamma}\) of \(\Gamma\) is defined by
\[
\hat{\Gamma} = \Gamma^\beta \partial_\beta + v^i \frac{\partial \Gamma^i}{\partial x^j} \partial_\nu.
\]
Consequently, for all \(\mu \in [0,3], 1 \leq i < j \leq 3\) and \(k \in [1,3]\),
\[
\partial_\mu = \partial_\mu, \quad \hat{\Omega}_{ij} = x^i \partial_j - x^j \partial_i + v^i \partial_\nu - v^j \partial_\nu \quad \text{and} \quad \hat{\Omega}_{0k} = t \partial_k + x^k \partial_t + v^k \partial_\nu.
\]
Since \([T,\hat{Z}] = 0\) for all \(Z \in \mathbb{P}\) and \([T,S] = T\), we consider, as \(\mathbb{K}\), the following set
\[
\hat{\mathbb{P}}_0 := \{\hat{Z} / Z \in \mathbb{P}\} \cup \{S\}.
\]
For simplicity, we denote by \(\hat{Z}\) an arbitrary vector field of \(\hat{\mathbb{P}}_0\), even if \(S\) is not a complete lift. Note that the vectorial space engendered by each of these sets is an algebra. More precisely, if \(L\) is either \(\mathbb{K}, \mathbb{P}\) or \(\mathbb{O}\), then for all \((Z_1, Z_2) \in L^2, (Z_1, Z_2)\) is a linear combinations of vector fields of \(L\). We also consider an ordering on each of the sets \(\mathbb{O}, \mathbb{P}, \mathbb{K}\) and \(\hat{\mathbb{P}}_0\), such that, if \(\mathbb{P} = \{Z'i / 1 \leq i \leq |\mathbb{P}|\}\), then \(\mathbb{K} = \{Z'i / 1 \leq i \leq |\mathbb{K}|\}\), with \(Z'|\mathbb{K}| = S\), and
\[
\hat{\mathbb{P}}_0 = \{\hat{Z}' / 1 \leq i \leq |\hat{\mathbb{P}}_0|\}, \quad \text{with} \quad \{\hat{Z}'\}_{1 \leq i \leq |\mathbb{P}|} = \{\hat{Z}'\}_{1 \leq i \leq |\mathbb{P}|} \quad \text{and} \quad |\hat{Z}'|_0 = S.
\]
If \(L\) denotes \(\mathbb{O}, \mathbb{P}, \mathbb{K}\) or \(\hat{\mathbb{P}}_0\), and \(\beta \in \{1, ..., |L|\}^q\), with \(q \in \mathbb{N}^+\), we will denote the differential operator \(\Gamma^{\beta_1}...\Gamma^{\beta_q} \in L^{[\beta]}\) by \(\Gamma^\beta\). For a vector field \(X\), we denote by \(L_X\) the Lie derivative with respect to \(X\) and if \(Z^\gamma \in \mathbb{K}^q\), we will write \(L_{Z^\gamma}\) for \(L_{Z^\gamma_1}...L_{Z^\gamma_q}\). We denote moreover the number of translations composing \(\Gamma^\beta\) by \(\beta_T\) and the number of homogeneous vector fields by \(\beta_p\), so that \(\beta = \beta_T + \beta_p\).

Let us recall, by the following classical result, that the derivatives tangential to the cone behave better than others.

**Lemma 2.4.** The following relations hold,
\[
(t-r)\hat{L} = S - \frac{x^i}{r} \Omega_{0i}, \quad (t+r)L = S + \frac{x^i}{r} \Omega_{0i} \quad \text{and} \quad v e_A = \sum_{1 \leq i < j \leq 3} C_{ij}^A \Omega_{ij},
\]
where the \(C_{ij}^A\) are uniformly bounded and depend only on spherical variables. Similarly, we have
\[
(t-r)\hat{\partial}_i = \frac{t}{t+r} S - \frac{x^i}{t+r} \Omega_{0i}, \quad \text{and} \quad (t-r)\partial_i = \frac{t}{t+r} \Omega_{0i} - \frac{x^i}{t+r} S - \frac{x^j}{t+r} \Omega_{ij}.
\]
We introduce now the notation $\nabla_v g := (0, \partial_{v^0} g, \partial_{v^1} g, \partial_{v^2} g)$, so that (11) can be rewritten

$$T_F(f) := v^\mu \partial_\mu f + F(v, \nabla_v f) = 0.$$ 

In order to commute the Vlasov-Maxwell system, we recall the following result (see Lemma 2.8 of [8] for a proof) where the Kronecker symbol is extended to vector fields, i.e. $\delta_{X, Y} = 1$ if $X = Y$ and $\delta_{X, Y} = 0$ otherwise.

**Lemma 2.5.** Let $G$ be a 2-form and $g$ a function both sufficiently regular. Then, for all $\hat{Z} \in \hat{P}_0$,

$$\hat{Z} (G(v, \nabla_v g)) = L_Z(G(v, \nabla_v g)) + G(v, \nabla_v \hat{Z} g) - 2\delta_{\hat{Z}, S}G(v, \nabla_v g).$$

If $G$ and $g$ satisfy $\nabla^\mu G_{\mu\nu} = J(g)_{\nu}$ and $\nabla^\mu G_{\mu\nu} = 0$, then

$$\forall \ Z, G, \hat{Z} \in \mathbb{K}, \quad \nabla^\mu L_Z(G)_{\mu\nu} = J(\hat{Z} g)\nu + 3\delta_{\hat{Z}, S}J(g)\nu \quad \text{and} \quad \nabla^\mu* L_Z(G)_{\mu\nu} = 0.$$

We then deduce the form of the source terms of the commuted Vlasov-Maxwell equations.

**Proposition 2.6.** Let $(f, F)$ be a sufficiently regular solution to the VM system (11)-(3) and $Z^\kappa \in \mathbb{K}^{[\kappa]}$. There exists integers $n^\kappa_{\gamma, \beta}$ and $m^\kappa_{\xi}$ such that

$$[T_F, \hat{Z}^\kappa](f) = T_F(\hat{Z}^\kappa f) = \sum_{|\gamma| + |\beta| \leq |\kappa|} n^\kappa_{\gamma, \beta} \nabla^\gamma L_{\gamma}^\beta(F)(v, \nabla_v \hat{Z}^\beta(f)), $$

$$\nabla^\mu L_{Z^\kappa}(F)_{\mu\nu} = \sum_{|\xi| \leq |\kappa|} m^\kappa_{\xi} J(\hat{Z}^\xi f)_{\nu}, $$

$$\nabla^\mu* L_{Z^\kappa}(F)_{\mu\nu} = 0.$$

Moreover, the number of homogeneous vector fields $\beta_\nu$ of $\hat{Z}^\beta$ satisfies the following condition.

- Either $\beta_\nu < \kappa_\nu$
- or $\beta_\nu = \kappa_\nu$ and $\gamma_\nu \geq 1$.

Note that the structure of the non-linearity $F(v, \nabla_v f)$ as well as the one of $J(f)$ is preserved by commutation, which reflects the null properties of the system. This is crucial for us since, as mentioned earlier, if the source terms of the Vlasov equation (respectively the Maxwell equations) behaved as $v^0|f||\partial_v f|$ (respectively $\int_v |f| dv$), we would not be able to close the energy estimates for the Vlasov field (respectively the electromagnetic field).

**Remark 2.7.** Let us explain why we count the number of the homogeneous vector fields in the source terms of the Vlasov equation. As $\partial_v f \sim \tau_x \partial_{v_x} f + \hat{Z} f$, the decay rate of the solutions will not be strong enough for us to close the energy estimates without using a hierarchy on the derivatives of $f$. If $\gamma_\nu \geq 1$, Lemma 2.4 will give us an extra decay in the $u$ direction. Otherwise, the worst source terms to control in order to bound $\hat{Z}^\kappa f||_{L^1_v}$ will only involve $\hat{Z}^\beta f$, with $\beta_\nu < \kappa_\nu$.

### 2.4 The null components of the velocity vector and the weights preserved by $T$

We denote by $(v^L, v^\perp, v^e_1, v^e_2)$ the null components of the velocity vector $v$, so that

$$v = v^L + v^\perp, \quad v^L = \frac{v^0 + v^r}{2} \quad \text{and} \quad v^\perp = \frac{v^0 - v^r}{2}.$$ 

If there is no ambiguity, we will write $v^A$ for $v^e_\nu$. Let $k_1$ and $z$ be defined as

$$k_1 := \left\{ \frac{v^\mu}{v^0} \mid 0 \leq \mu \leq 3 \right\} \cup \left\{ \frac{z_{\mu\nu}}{\mu \neq \nu} \right\}, \quad \text{where} \quad z_{\mu\nu} := x^\mu \frac{v^\nu}{v^0} - x^\nu \frac{v^\mu}{v^0}, \quad \text{and} \quad z^2 := \sum_{w \in k_1} w^2.$$

Because of regularity issues, we will rather work with $z$ than with the elements of $k_1$. Two fundamental properties of these weights is that they are preserved by the flow of $T$ and by the action of $	ilde{P}_0$.

**Lemma 2.8.** For all $\hat{Z} \in \hat{P}_0$ and $a \in \mathbb{R}_+$, we have

$$T(z) = 0 \quad \text{and} \quad |\hat{Z}(z^a)| \lesssim a z^a.$$
Proof. Let \( w \in k_1 \). By straightforward computations, one can prove that
\[
T(w) = 0 \quad \text{and} \quad \tilde{Z}(v^0 w) \in v^0 k_1 \cup \{0\}, \quad \text{so that} \quad |\tilde{Z}(w)| \lesssim \sum_{w_0 \in k_1} |w_0|.
\]
Indeed, considering for instance \( tv^1 - x^1 v^0, x^1 v^2 - x^2 v^1, \tilde{\Omega}_{12} \) and \( S \), we have
\[
\tilde{\Omega}_{12}(tv^1 - x^1 v^0) = -tv^2 - x^2 v^0, \quad \tilde{\Omega}_{12}(x^1 v^2 - x^2 v^1) = 0,
\]
\[
S(tv^1 - x^1 v^0) = tv^1 - x^1 v^0 \quad \text{and} \quad S(x^1 v^2 - x^2 v^1) = x^1 v^2 - x^2 v^1.
\]
Then,
\[
T(z) = \sum_{w \in k_1} \frac{w}{z} T(w) = 0 \quad \text{and} \quad |\tilde{Z}(z^a)| = |az^{a-1} \sum_{w \in k_1} \frac{w}{z} \tilde{Z}(w)| \lesssim az^{a-1} \sum_{w_0 \in k_1} |w_0| \lesssim az^a.
\]
\[\square\]

Recall that if \( k_0 := k_1 \cup \{ x^\nu v_\nu \} \), then \( \tau_{-} v^L + \tau_{+} v^L \lesssim \sum_{w \in k_0} |w| \). Unfortunately, the weight \( x^\nu v_\nu \) is not preserved by \( T \) so we will not be able to take advantage of this inequality during this paper. In the following lemma, which reflects the good behavior of the components \( v^2 \) and \( v^A \) of the velocity vector, we prove a similar inequality specific to the exterior of the lightcone and adapted to the study of massive particles.

Lemma 2.9. We have, for all \( |x| \geq t \),
\[
1 \leq 4v^0 v^L, \quad |v^A| \lesssim \sqrt{v^0 v^L} \quad \text{and} \quad \tau_{+} - (1 + r) \frac{v^L}{v^0} + (1 + r) \frac{|v^A|}{v^0} \lesssim z.
\]

Proof. Note first that \( 4v^2 v^L v^L \geq r^2 + \sum_{k<l} |v^0 z_{kl}|^2 \). Indeed, as we study massive particles, we have \( v^0 = \sqrt{1 + |v|^2} \), so that
\[
4v^2 v^L v^L = (r v^0)^2 - (x^i v_i)^2 = r^2 + \sum_{i=1}^3 (r^2 - |x^i|^2 |v_i|^2) - 2 \sum_{1 \leq k < l \leq 3} x^k x^l v_k v_l,
\]
\[
\sum_{1 \leq k < l \leq 3} |v^0 z_{kl}|^2 = \sum_{1 \leq k < l \leq 3} |x^k|^2 |v_l|^2 + |x^l|^2 |v_k|^2 - 2 \sum_{1 \leq k < l \leq n} x^k x^l v_k v_l.
\]
The first inequality then comes from \( v^L \leq r^0 \). The second one and \( (1 + r) \frac{v^L}{v^0} \lesssim z \) then ensue from \( r v_A = v^0 C_{ij} z_{ij} \), where \( C_{ij}^0, C_{ij}^A \) are bounded functions depending only on the spherical variables such as \( r e_A = C_{ij}^0 \tilde{\Omega}_{ij} \). The last part of the third inequality is specific to the exterior of the light cone. Recall that \( x^i - t \frac{v^i}{v^0} \in k_1 \). Then, \( \tau_{-} \lesssim z \) follows from \( 1 \leq z \) and
\[
(r - t) \leq r - |\frac{v}{v^0}| \leq |x - t \frac{v}{v^0}| \leq \sum_{i=1}^3 |x^i - t \frac{v^i}{v^0}| = \sum_{i=1}^3 |z_{0i}| \leq z.
\]
Finally, remark first that \( v^L \leq v^0 \), which treats the case \( |x| \leq 1 \). If \( |x| \geq \max(t, 1) \), note that
\[
2r v^i = r v^0 - r \frac{x^i}{r} v_i = r v^0 + (t - r) \frac{x^i}{r} v_i - x^i \left( \frac{v^i v^0}{v^0} - x_i \right) - r v^0 = (t - r) v_i \frac{x^i}{r} - v^0 \frac{x^i}{r} z_{0i}
\]
and use (6).
\[\square\]

2.5 The null decomposition of the electromagnetic field

In order to capture its geometric properties, the electromagnetic field will be represented all along this paper by a 2-form. Let \( G \) be a 2-form defined on \( [0, T] \times \mathbb{R}_3^\nu \). Its Hodge dual \( 'G \) is the 2-form given by
\[
'G_{\mu \nu} = \frac{1}{2} C^{\lambda \sigma} e_{\lambda \sigma \mu \nu},
\]
\[\text{Note however that } x^\mu v_\mu \text{ is preserved by the massless relativistic transport operator } |v| \partial_t + v^i \partial_i.\]
where \( \varepsilon \) is the Levi-Civita symbol, and its energy-momentum tensor is

\[
T[G]_{\mu\nu} := G_{\mu\sigma} G_{\nu}^{\beta} - \frac{1}{4} \eta_{\mu\nu} G G^{\rho\sigma}. 
\]

Note that \( T[G]_{\mu\nu} \) is symmetric, i.e. \( T[G]_{\mu\nu} = T[G]_{\nu\mu} \). The null decomposition of \( G \), \((\alpha(G), \mathbf{a}(G), \rho(G), \sigma(G))\), introduced by [1], is defined by

\[
\alpha_A(G) = G_{A\mu}, \quad \mathbf{a}_A(G) = G_{A\mu}, \quad \rho(G) = \frac{1}{2} G_{\mu\nu}^{\mu} \quad \text{and} \quad \sigma(G) = G_{e_1 e_2},
\]

so that the null components of \( T[G] \) are then given by

\[
T[G]_{\mu\nu} = |\alpha(G)|^2, \quad T[G]_{\mu\nu}^{\mu\nu} = |\mathbf{a}(G)|^2 \quad \text{and} \quad T[G]_{\mu\nu}^{\mu\nu} = |\rho(G)|^2 + |\sigma(G)|^2. \quad (7)
\]

For a proof of the following classical results, we refer to [1] or to [2] (Subsection 2.3 and Lemma D.3).

**Lemma 2.10.** Let \( G \) be a 2-form and \( J \) be a 1-form both sufficiently regular and such that

\[
\nabla^\mu G_{\mu
u} = J_\nu, \quad \nabla^\mu G^ {\mu\nu} = 0.
\]

Then, \( \nabla^\mu T[G]_{\mu\nu} = G_{\nu\lambda} J^\lambda \) and, denoting by \((\alpha, \mathbf{a}, \rho, \sigma)\) the null decomposition of \( G \),

\[
\nabla_L \alpha_A - \frac{\alpha_A}{r} + \nabla_{\mathbf{e}_A} \rho + \varepsilon_B A \nabla_{\mathbf{e}_B} \sigma = J_A.
\]

### 3 Energy and pointwise decay estimates

We recall here classical energy estimates for both the Vlasov field and the electromagnetic field and how obtain pointwise decay estimates from them. For all this section, we define \( T > 0 \) and \( b \leq -1 \). The energies defined below are adapted to the study of the Vlasov-Maxwell system in the exterior of the light cone \( u \geq b \).

#### 3.1 Estimates for velocity averages

For the Vlasov field, we will use the following approximate conservation law.

**Proposition 3.1.** Let \( H : V_b(T) \times \mathbb{R}^3 \rightarrow \mathbb{R} \) and \( g_0 : \Sigma^b_0 \times \mathbb{R}^3 \rightarrow \mathbb{R} \) be two sufficiently regular functions and \( F \) a sufficiently regular 2-form. Then, \( g_0 \), the unique classical solution of

\[
T_F(g) &= H \\
g(0, \ldots) &= g_0,
\]

satisfies, for all \( t \in [0, T] \), the following estimates,

\[
\left\| \int_{v \in \mathbb{R}^3} |g| dv \right\|_{L^1}\left(\Sigma^{(b)}_t\right) + \sup_{u < b} \left\| \int_{v \in \mathbb{R}^3} \frac{d}{dv} |g| dv \right\|_{L^1}\left(C_u(t)\right)} \leq 2 \left\| \int_{v \in \mathbb{R}^3} |g_0| dv \right\|_{L^1}\left(\Sigma^{(b)}_0\right) + 2 \int_0^t \int_{v \in \mathbb{R}^3} \left| H \right| \frac{dv}{v^b} dx ds.
\]

**Proof.** As \( T(|g|) = \frac{d}{dv} H - \frac{d}{dv} F(v, \nabla_v g) \) and since \( F \) is a 2-form, integration by parts in \( v \) gives us

\[
\partial_v \int_{v} \frac{v^u}{v^b} dv = \int_{v} \left( \frac{d}{dv} \frac{H}{v^b} - \frac{d}{dv} F \right) \left( \frac{v^u}{v^b} \right) dv = \int_{v} \left( \frac{g}{|g|} \frac{H}{v^b} - \frac{v^u v^i}{(v^b)^3} F_{ji}\right) dv = \int_{v} \frac{g}{|g|} \frac{H}{v^b} dv.
\]

Apply now the divergence theorem to \( \int_{v} \frac{v^u}{v^b} dv \) in the region \( C_u(t) \), for \( u < b \), in order to get

\[
\int_{\Sigma^b_t} \int_{v} |g| dv dx + \sqrt{2} \int_{C_u(t)} \int_{v} \frac{d}{dv} |g| dv dx \left( C_u(t) \right) = \int_{\Sigma^b_0} \int_{v} |g| dv dx + \int_0^t \int_{\Sigma^b_v} \int_{v} \frac{g}{|g|} \frac{H}{v^b} dv dx ds.
\]

We then deduce that

\[
\int_{\Sigma^b_t} \int_{v} |g| dv dx = \sup_{u < b} \int_{\Sigma^b_t} \int_{v} |g| dv dx \leq \int_{\Sigma^b_0} \int_{v} |g| dv dx + \int_0^t \int_{\Sigma^b_v} \int_{v} \frac{g}{|g|} \frac{H}{v^b} dv dx ds,
\]

which allows us to conclude the proof.
In view of Remark 2.7 and the previous proposition, we then define hierarchised energy norms. For $(Q, \lambda) \in \mathbb{N} \times [0, \frac{1}{2}]$ and $q \in [Q, +\infty[$, let
\[
E_b[g](t) := \left\| \int_{v \in \mathbb{R}^3} |g| dv \right\|_{L^1(\mathbb{S}_t^1)} + \sup_{u < b} \left\| \int_{v \in \mathbb{R}^3} \frac{L}{v^\theta} |g| dv \right\|_{L^1(C_u(t))},
\]
(8)

\[
E_{Q, \lambda}^\beta[f](t) := \sum_{0 \leq k \leq Q} \sum_{Z \in \mathbb{Z}^3 \cap [0, b]^3} (1 + t)^{-\beta r/\lambda} E_b[|\sqrt{z}^{-\beta} f(t)dv].
\]
(9)

**Remark 3.2.** As $z \geq 1$, we have $E[|\sqrt{Z}^\beta f(t)dv] \leq (1 + t)^{-\beta r/\lambda} E_{Q, \lambda}^\beta[f](t)$ for all $0 \leq a \leq q - (1 - 2\lambda)\beta r$.

The remaining of this subsection is devoted to the proof of a Klainerman-Sobolev type inequality. The constants hidden by $\lesssim$ will here depend on $a$. We start with a commutator property between the vector fields of $\mathbb{K}$ and the averaging in $v$.

**Lemma 3.3.** Let $v : V_b(T) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be a sufficiently regular function and $a \in \mathbb{R}_+$. We have, almost everywhere,
\[
\forall Z \in \mathbb{K}, \quad Z \left( \int_{v \in \mathbb{R}^3} z^a (|f| dv) \right) \lesssim \int_{v \in \mathbb{R}^3} z^a |f| dv + \int_{v \in \mathbb{R}^3} z^a |:\hat{Z} f| dv.
\]
Proof. Consider for instance the case where $Z = \Omega_{01} = t\partial_t + x^1 \partial_x$. We have, almost everywhere,
\[
\left| Z \left( \int_{v \in \mathbb{R}^3} |z^a f| dv \right) \right| = \left| \int_{v \in \mathbb{R}^3} \hat{\Omega}_{01} (|z^a f|) dv - \int_{v \in \mathbb{R}^3} v^0 \partial_v (|z^a f|) dv \right|
\leq \int_{v \in \mathbb{R}^3} \hat{\Omega}_{01} (z^a f) dv + \int_{v \in \mathbb{R}^3} v^0 |z^a f| dv.
\]
It then remains to use $|\hat{\Omega}_{01} (z^a)| \lesssim az^a$.

Before presenting the Klainerman-Sobolev inequality used in this article, we prove the following estimate.

**Lemma 3.4.** Let $g : S^2 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be a sufficiently regular function and $a \in \mathbb{R}_+$. Then,
\[
\forall \omega \in S^2, \quad \int_{v \in \mathbb{R}^3} z^a |g(\omega, v)| dv \lesssim \sum_{0 \leq k \leq 2} \sum_{\Omega^\beta \in \Omega^k} \left\| \int_{v \in \mathbb{R}^3} z^a \hat{\Omega}^\beta g(\omega, v)| dv \right\|_{L^1(S^2)}.
\]
Proof. Let $\omega \in S^2$ and $(\theta, \varphi)$ a local coordinate map in a neighborhood of $w$. By the symmetry of the sphere we can suppose that $\theta$ and $\varphi$ take their values in an interval of a size independent of $\omega$. Using a one dimensional Sobolev inequality, that $|\partial_\theta u| \lesssim \sum_{\Omega^r \in \Omega^0} \Omega^r u$ and Lemma 3.3 we have,
\[
\int_{\omega} z^a |g(\omega, w, v)| dv \lesssim \int_{\theta} \int_{\varphi} z^a |g(\omega, \theta, \varphi, v)| dv + \int_{v} z^a |g(\omega, \theta, \varphi, v)| dv d\theta
\lesssim \int_{\theta} \int_{\varphi} z^a |g(\omega, \theta, \varphi, v)| dv d\theta + \sum_{\Omega^r \in \Omega^0} \int_{\theta} \int_{\varphi} z^a |\hat{\Omega}^\beta g(\omega, \theta, \varphi, v)| dv d\theta.
\]

Doing the same for the second spherical variable of $\int_{\omega} z^a |\hat{\Omega}^\beta g(\omega, \theta, \varphi, v)| dv$, we obtain the result.

**Proposition 3.5.** Let $v : V_b(T) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be a sufficiently regular function and $a \in \mathbb{R}_+$. Then,
\[
\forall (t, x) \in V_b(T), \quad \int_{v \in \mathbb{R}^3} z^a |f(t, x, v)| dv \lesssim \frac{1}{\tau_+ \tau} \sum_{0 \leq k \leq 3} \sum_{Z \in \mathbb{Z}^3 \cap [0, b]^3} \int_{|y| \geq |x|} \int_{v \in \mathbb{R}^3} z^a |\hat{Z}^\beta f| dv dx.
\]
Proof. Let $(t, x) = (t, |x| \in V_b(T)$. One has, using successively Lemmas 2.4 and 3.3
\[
|x|^2 \tau_+ \int_{v} z^a |f|(t, |x|, v) dv = -|x|^2 \int_{r=|x|}^{+\infty} \partial_r \left( \int_{v \in \mathbb{R}^3} z^a |f|(t, r, x, v) dv \right) dr
\leq |x|^2 \int_{|x|}^{+\infty} \int_{r=|x|}^{+\infty} \int_{v \in \mathbb{R}^3} z^a |f|(t, r, x, v) dv + Z \left( \int_{v} z^a |f|(t, r, x, v) dv \right) dr
\lesssim \int_{|x|}^{+\infty} \int_{v} z^a |f|(t, r, x, v) dv dr + \sum_{|x| \in \mathbb{Z}^3 \cap [0, b]^3} \int_{r=|x|}^{+\infty} \int_{v \in \mathbb{R}^3} z^a |\hat{Z}^\beta f|(t, r, x, v) dv dr.
\]

It then remains to apply the previous lemma and to recall that $\tau_+ \lesssim r$ in $V_b(T)$.
We can improve the decay rate in the $u$ direction if we pay the price in terms of weights in $v^0$ and $z$. More precisely, by Lemma 2.9 we have $|v^0|^{-a} \lesssim |v^0z|^a$, so that

$$\forall (t, x) \in V_b(T), \quad \int_{s \in \mathbb{R}^3} |f(t, x, v)| \frac{dv}{(v^0)^{2a}} \lesssim \frac{1}{\tau^{2+a}} \sum_{0 \leq k \leq 3} z^a \int_{v \in \mathbb{R}^3} \left| \tilde{\nabla}^b f \right| dv \left\|_{L^1(\Sigma^t)} \right.) .$$

### 3.2 Estimates for the electromagnetic field

In this subsection, we introduce first the energy norm used in this paper to study the electromagnetic field and, secondly, we derive pointwise decay estimates from it through Klainerman-Sobolev inequalities. We consider, for the remaining of this section, $G$ a sufficiently regular 2-form defined on $V_b(T)$ and we denote by $(\alpha, \Omega, \rho, \sigma)$ its null decomposition. We suppose that $G$ satisfies

$$\nabla^\mu G_{\mu\nu} = J, \quad \nabla^{\mu\nu} G_{\mu\nu} = 0,$$

with $J$ a sufficiently regular 1-form defined on $V_b(T)$.

**Definition 3.6.** Let $N \in \mathbb{N}$. We define, for $t \in [0, T[$,

$$E^b[G](t) := \int_{\Sigma^t} (|\alpha|^2 + |\Omega|^2 + 2|\rho|^2 + 2|\sigma|^2) \, dx + \sup_{u < b} \int_{C_u(t)} \left( |\alpha|^2 + |\rho|^2 + |\sigma|^2 \right) dC_u(t),$$

$$E^b_N[G](t) := \sum_{0 \leq k \leq N} \sum_{\lambda \in \mathbb{R}^k} E^b[\mathcal{L}_{2^k}(G)](t).$$

**Proposition 3.7.** We have, for all $t \in [0, T[$,

$$E^b[G](t) \leq 2E^b[G](0) + 8 \int_0^t \int_{\Sigma^u} |G_{\rho0} J^\mu| \, dx \, ds.$$

**Proof.** Recall from Lemma 2.10 that $\nabla^{\mu\nu} T[G]_{\mu\nu} = G_{0\nu} J^\nu$. Hence, applying the divergence theorem in $V_u(t)$, for $u < b$, we get

$$\int_{\Sigma^t} T[G]_{00} \, dx + \frac{1}{\sqrt{2}} \int_{C_u(t)} T[G]_{00} dC_u(t) = \int_{\Sigma^t} T[G]_{00} \, dx - \int_0^t \int_{\Sigma^u} G_{0\nu} J^\nu \, dx \, ds. \tag{10}$$

We then obtain

$$\sup_{u < b} \int_{C_u(t)} T[G]_{00} dC_u(t) \leq \int_{\Sigma^t} |T[G]_{00}| \, dx + \int_0^t \int_{\Sigma^u} |G_{0\nu} J^\nu| \, dx \, ds,$n

$$\int_{\Sigma^t} T[G]_{00} \, dx = \sup_{u < b} \int_{\Sigma^u} T[G]_{00} \, dx \leq \int_{\Sigma^t} |T[G]_{00}| \, dx + \int_0^t \int_{\Sigma^u} |G_{0\nu} J^\nu| \, dx \, ds.$$

It then remains to add the previous two inequalities and to notice, using $\left(1\right)$, that

$$4T[G]_{00} = |\alpha|^2 + |\Omega|^2 + 2|\rho|^2 + 2|\sigma|^2 \quad \text{and} \quad 2T[G]_{00} = |\alpha|^2 + |\rho|^2 + |\sigma|^2.$$

□

In order to prove pointwise decay estimates on $G$, we will use the following three Lemmas. The first one, which is proved in Appendix D of [3], extends the results of Lemma 2.4 for the null components of a 2-form.

**Lemma 3.8.** We have, denoting by $\zeta$ any of the null component $\alpha$, $\Omega$, $\rho$, or $\sigma$,

$$\tau_- |\nabla_{\mathcal{L}} \zeta(G)| + \tau_+ |\nabla_L \zeta(G)| \lesssim \sum_{|\gamma| \leq 1} |\zeta(\mathcal{L}_{2^k}(G))|, \quad (1 + r) |\nabla \zeta(G)| \lesssim |\zeta(G)| + \sum_{\Omega \in \mathcal{D}} |\zeta(\mathcal{L}_{0}(G))|$$

and, on $V_b(T)$, $\forall \mu \in [0, 3]$, $\tau_- |\nabla_{\partial_\mu} \zeta(G)| \leq \tau_- |\nabla_L \zeta(G)| + \tau_+ |\nabla_{\mathcal{L}} \zeta(G)| + \tau_- |\nabla \zeta(G)| \lesssim \sum_{|\gamma| \leq 1} |\zeta(\mathcal{L}_{2^k}(G))|.$

The following result, also proved in Appendix D of [3], presents commutation properties between $\mathcal{L}_\Omega$, $\nabla_{\partial_\mu}$, $\nabla_L$ or $\nabla_{\mathcal{L}}$ and the null decomposition of $G$. 

12
Lemma 3.9. Let $\Omega \in \mathbb{R}$. Then, denoting by $\zeta$ any of the null component $\alpha, \alpha, \rho$ or $\sigma$,

$$[L_\Omega, \nabla_\alpha]G = 0, \quad L_\Omega(\zeta(G)) = \zeta(L_\Omega(G)) \quad \text{and} \quad \nabla_\alpha(\zeta(G)) = \zeta(\nabla_\alpha(G)).$$

Similar results hold for $L_\Omega$ and $\nabla_\alpha, \nabla_L$ or $\nabla_L$. For instance, $\nabla_L(\zeta(G)) = \zeta(\nabla_L(G))$.

We now recall the Sobolev inequalities which will be used to prove the pointwise decay estimates on the null components of the electromagnetic field. For this, we introduce $|U(y)|_{3,k}^3 := \sum_{|\beta| \leq k} |L_\Omega U|^2$, where $\Omega^\beta \in \mathbb{R}^{[\beta]}$.

Lemma 3.10. Let $U$ be a sufficiently regular tensor field defined on $\mathbb{R}$. Then,

$$\forall x \neq 0, \quad |U(x)| \lesssim \frac{1}{|x|} \left( \int_{|y| \geq |x|} |U(y)|_{3,2}^2 + |y|^2 |\nabla_\alpha U(y)|_{3,1}^2 dy \right)^{\frac{1}{2}}.$$

If $t \in \mathbb{R}_+$ and $|x| \geq t - b$, we have

$$\forall x \neq 0, \quad |U(x)| \lesssim \frac{1}{|x|} \left( \int_{|y| \geq t-b} |U(y)|_{3,2}^2 + \tau^2 |\nabla_\alpha U(y)|_{3,1}^2 dy \right)^{\frac{1}{2}}.$$

\textbf{Proof.} The first inequality is proved in Lemma 2.3 of [4] and the second one can be proved similarly as inequality (ii) of Lemma 2.3 of [4].

We now prove the pointwise decay estimates used in this article.

Proposition 3.11. For all $(t, x) \in V_0(T)$, we have

$$|\rho(t, x) + |\sigma(t, x) + |\alpha(t, x)| \lesssim \frac{\sqrt{\|E_2^{-1}G\|_2}}{\tau + \tau_+} \quad \text{and} \quad |\alpha(t, x)| \lesssim \frac{\sqrt{E_2^{-1}G(t)}}{\tau} + \sum_{|\beta| \leq 1} \|rL_{\Omega^\beta}(J)A\|_{L^2(\mathbb{R})}.$$
4 Null properties of the Vlasov equation

In order to take advantage of the null structure of the commuted transport equation, we will expand quantities such as $L_{Z'}(F)(v, \nabla_v g)$, with $g$ a regular function, in null coordinates. We will then use the following lemma.

**Lemma 4.1.** Let $G$ be a sufficiently regular 2-form, $(\alpha, \rho, \sigma, \sigma)$ its null components and $g$ a sufficiently regular function. Then,

$$|G(v, \nabla_v g)| \lesssim \left( |\rho| + \frac{|v^A|}{v^0} |\alpha| \right) \left( \tau_- |\nabla_{t,x} g| + \sum_{\mathcal{Z} \in \mathcal{P}_0} \hat{Z}_g \right) + \left( |\alpha| + \frac{|v^A|}{v^0} |\sigma| + \frac{v^L}{v^0} |\sigma| \right) \left( \tau_+ |\nabla_{t,x} g| + \sum_{\mathcal{Z} \in \mathcal{P}_0} \hat{Z}_g \right).$$

**Proof.** Expanding $G(v, \nabla_v g)$ with null components, we get

$$G(v, \nabla_v g) = 2 \rho \left( v^L (\nabla_v g)^L_1 - v^L (\nabla_v g)^L_2 \right) + v^B \varepsilon_{BA} \sigma (\nabla_v g)^A - v^L \alpha_A (\nabla_v g)^A + v^A \alpha_A (\nabla_v g)^L.$$

We bound the angular components of $\nabla_v g$ using $v^0 \partial_{x^i} = \hat{\Omega}_{0i} - t \partial_i - x^i \partial_t$. The radial component $(\nabla_v g)^r = 2 (\nabla_v g)^r = -2 (\nabla_v g)^L$ has a better behavior since

$$v^0 (\nabla_v g)^r = \frac{x^i}{r} v^0 \partial_{x^i} g = \frac{x^i}{r} \hat{\Omega}_{0i} g - Sg + (t - r)Lg.$$  

Let us explain how this result reflects the null structure of the Vlasov equation. For this, we use the notation $Q \prec R$ if $R$ is expected to behave better than $Q$, so that

- $\alpha \prec \rho \sim \sigma \prec \alpha$,
- $v^L \prec v^A \prec v^L$.

Note now that each term given by the previous lemma contains either two good factors, $\alpha$ or $v^L$.

5 Bootstrap assumptions and strategy of the proof

Let $N \geq 8$, $b \leq -1$ and $(\delta, \eta) \in \mathbb{R}_+^2$ be two constants such that $0 < 5\delta < \eta < \frac{1}{16N}$. From now, we drop the dependence in $b$ of all the quantities defined previously (such as the energy norms $\mathcal{E}_0$ and $\mathcal{E}_0^\Lambda$ defined in [3] and $\mathcal{E}_N^b$ or $\mathcal{E}_N^b(N)$). Let $(f_0, F_0)$ be an initial data set satisfying the assumptions of Theorem 1.1. Then, by a local well-posedness argument, there exists a unique maximal solution to the Vlasov-Maxwell system defined in $V_b(T^*)$, with $T^* \in \mathbb{R}_+ \cup \{+\infty\}$. Let $T \in ]0, T^*[$ be the largest time such that $\mathcal{E}_N^b$ for all $t \in [0, T]$,

$$\mathcal{E}_N^{N+13,\delta} \lesssim 4 \epsilon (1 + t)^\delta,$$

$$\mathcal{E}_N^{N+9,\eta} \lesssim 4 \epsilon (1 + t)^\eta,$$

$$\mathcal{E}_N[F](t) \lesssim 4 \epsilon,$$

$$\sum_{|\beta| \leq N-1} \left\| \int v^A \hat{Z}_\beta f dv \right\|_{L^2(\mathbb{R}^3_v)} \lesssim \sqrt{\epsilon}.$$  

We consider the last bootstrap assumption in order to simplify the proof. The remainder of this paper is devoted to the improvement of these inequalities which will prove that $T = T^*$ and then $T^* = +\infty$, implying Theorem 1.1. Let us expose the immediate consequences of the bootstrap assumptions. Using the Klainerman-Sobolev inequality of Proposition 3.5 and (15) (respectively (16)), one has

$$\forall (t, x) \in V_b(T), \ |\beta| \leq N - 5, \ \int v \sqrt{2}^{N+10-(1-2\delta)\beta_p} \hat{Z}_\beta f \ dv \lesssim \epsilon \frac{(1 + t)^{(\beta_p+4)\delta}}{\tau_+^{\tau_-}},$$  

$$\forall (t, x) \in V_b(T), \ |\beta| \leq N - 3, \ \int v \sqrt{2}^{N+6-(1-2\eta)\beta_p} \hat{Z}_\beta f \ dv \lesssim \epsilon \frac{(1 + t)^{(\beta_p+4)\eta}}{\tau_+^{\tau_-}}.$$  

Note that $T > 0$ by continuity. Remark also that, considering if necessary $\epsilon_1 = C_1 \epsilon$, with $C_1$ a constant depending only on $N$, we can suppose without loss of generality that the energy norms are initially smaller than $\epsilon$. We refer to Appendix B of [1] for the details of the computations for similar energy norms.
By Proposition 5.11 and 17, we obtain that, for all \((t, x) \in V_b(T)\) and \(|\gamma| \leq N - 2\),
\[
|\alpha (L_{Z\gamma}(F))| (t, x) \lesssim \frac{\sqrt{\xi}}{\tau_+^{\frac{1}{2}}}.
\]

\[
|\rho (L_{Z\gamma}(F))| (t, x) + |\sigma (L_{Z\gamma}(F))| (t, x) + |\omega (L_{Z\gamma}(F))| (t, x) \lesssim \frac{\sqrt{\xi}}{\tau_+^{\frac{1}{2}}}.
\]

(21)

The proof is organized as follows:

- We start by improving the bootstrap assumptions 15 and 19 by several applications of the approximate conservation law of Proposition 5.1. Exploiting the null structure of the non-linearity \(L_{Z\gamma}(F)(v, \nabla_v \hat{Z}_\beta f)\) is then fundamental in order to bound the spacetime integrals arising from the energy estimates.

- Then, we improve the bound on the energy norm of the electromagnetic field 17. For this, we use the energy estimate of Proposition 5.7 and make crucial use of the null structure of the source terms of the Maxwell equations.

- The last step consists in proving an estimate on \(\| \int_{v} |\hat{Z}_\beta f| dv \|_{L^2(\mathcal{I}^0)}\) for \(|\beta| \geq N - 2\). We then rewrite all Vlasov equations as an inhomogeneous system of transport equations. We deal with the homogeneous part by taking advantage of the smallness assumption on the \(N + 3\) derivatives of \(f\) at \(t = 0\) as well as the pointwise decay estimates 21. We will decompose the inhomogeneous part as a product \(K Y\) where \(K^2 Y \in L^1 L^1(\Sigma^0)\) and \(\int_y Y |dv| = 0\), a decaying function.

6 Improvement of the energy bound on the particle density

The aim of this section is to prove that, for \(\epsilon\) small enough, \(E_{N-2}^{N+9,\eta}[f] \leq 3\epsilon(1 + t)^\eta\) for all \(t \in [0, T]\) (we will sketch the improvement of the estimate on \(E_{N-2}^{N+13,\beta}[f]\) as it is very similar and simpler). For this, recall that

\[
E_{N-2}^{N+9,\eta}[f](0) \leq \epsilon
\]

and let us prove that

\[
\forall |\kappa| \leq N, \quad \forall t \in [0, T], \quad E[\sqrt{z}^{N+9-(1-2\eta)\kappa P} \hat{Z}_\kappa f](t) - 2E[\sqrt{z}^{N+9-(1-2\eta)\kappa P} \hat{Z}_\kappa f](0) \lesssim \epsilon^2 (1 + t)^{\kappa P + \eta}.
\]

We then fix \(|\kappa| \leq N\) and denote \(\frac{1}{2}(N + 9 - (1 - 2\eta)\kappa P)\) by \(a\). Note, by Lemma 2.8, that

\[
T_P(z^a \hat{Z}_\kappa f) = F(v, \nabla_v z^a) \hat{Z}_\kappa f + z^a T_P(\hat{Z}_\kappa f).
\]

(22)

Thus, in view of the energy estimate of Proposition 5.1 and commutation formula of Proposition 2.4, it suffices to prove that

\[
\int_0^t \int_{\Sigma^0} \int_{v} z^a F(v, \nabla_v z) \hat{Z}_\kappa f |dv| v^0 dx ds \lesssim \epsilon^2 (1 + t)^{\kappa P + 1 + \eta}
\]

and that the following proposition holds, where \([\gamma] := \max(0, 1 - \gamma_T)\).

**Proposition 6.1.** Let \(\gamma\) and \(\beta\) be such that \(|\gamma| + |\beta| \leq |\kappa|, \ |\beta| \leq |\kappa| - 1\) and \(\beta P + |\gamma| \leq \kappa P\). Then,

\[
\int_0^t \int_{\Sigma^0} \int_{v} |z^a L_{Z\gamma}(F)(v, \nabla_v \hat{Z}_\beta f)| |dv| v^0 dx ds \lesssim \epsilon^2 (1 + t)^{\kappa P + 1 + \eta}.
\]

The remaining of the section is divided in four parts. The first two ones are devoted to the proof of 23 and Proposition 6.1. Then, we explain briefly how to improve the bound on \(E_{N-2}^{N+13,\beta}[f]\). Finally, we prove an \(L^2\) estimate on \(\int_v z|\hat{Z}_\beta f| dv\) which will be useful for Section 7.

6.1 Proof of inequality 23

Note first that we have \(|\nabla_{t,z} z| \leq 1\) and, using Lemma 2.8, \(|\hat{Z} z| \lesssim z\). Applying Lemma 4.1 with \((G, g) = (F, z)\), we can then observe that it suffices to prove that

\[
I_1 := \int_0^t \int_{\Sigma^0} \int_{v} (|\tau_+ + z| |\rho(F)| + (\tau_+ + z)|\alpha(F)|) |z^{a-1} \hat{Z}_\kappa f| |dv| v^0 dx ds \lesssim \epsilon^2 (1 + t)^{\kappa P + 1 + \eta}
\]

and

\[
I_2 := \int_0^t \int_{\Sigma^0} \int_{v} (\frac{|v A|}{v^0} |\sigma(F)| + \frac{|v L| + |v A|}{v^0} |\alpha(F)|) |z^{a-1} \hat{Z}_\kappa f| |dv| v^0 dx ds \lesssim \epsilon^2 (1 + t)^{\kappa P + 1 + \eta}.
\]
Recall, from Lemma 2.9, the inequalities $1 \lesssim \sqrt{v_0 v_L}$, $1 \lesssim \tau_+^{-1} z$ and $v_L + |u_A| \lesssim \tau_+^{-1} v_0 z$, so that

$$1 \lesssim \sqrt{v_0 v_L} \lesssim \frac{v_0 z}{\sqrt{\tau_+ \tau_-}}$$

and

$$|v_A| + v_L \lesssim \frac{z}{\tau_+}.$$

Hence, according to (21), it comes

$$(\tau_- + 1|\rho(F)|(s, x) + (\tau_+ + 1|\alpha(F)|(s, x) + (\tau_+ + 1) \begin{pmatrix} \frac{|u_A|}{v_0} \sigma(F)(s, x) + \frac{v_L + |u_A|}{v_0} |\alpha(F)(s, x) \end{pmatrix} \lesssim \sqrt{\tau_+^{-1} z}.$$

Consequently, using the bootstrap assumption (16),

$$I_1 + I_2 \lesssim \int_0^t \int_\Sigma \frac{\sqrt{\tau_+}}{v} \int \big|z^a \hat{\nabla}^2 f\big| dv dx ds \lesssim \sqrt{\tau} \int_0^t \frac{E[z^a \hat{\nabla}^2 f](s)}{1 + s} ds$$

$$\lesssim \sqrt{\tau} \int_0^t \frac{(1 + s)^{\kappa_p} E_N^{9,9}[f](s)}{1 + s} ds \lesssim \sqrt{\tau} \int_0^t \frac{(1 + s)^{\kappa_p + 1)}{1 + s} ds \lesssim \sqrt{\tau}(1 + t)^{(\kappa_p + 1)}.$$

### 6.2 Proof of Proposition 6.1

Let $\gamma$ and $\beta$ satisfying $|\beta| + |\gamma| \leq |\kappa|$, $|\beta| \leq |\kappa| - 1$ and $\beta_p + |\gamma| \leq \kappa_p$. Using Lemma 4.1, we need to bound by $\epsilon(1 + t)^{(\kappa_p + 1)}$, for all $\hat{\Gamma} \in \mathbb{P}_0$, the following integrals,

$$I_{\hat{\Gamma}} := \int_0^t \int_\Sigma \big|\hat{\nabla} f\big| dv dx ds,$$

$$I_{\beta, \alpha} := \int_0^t \int_\Sigma (\tau_- |\beta(LZ, (F))| + \tau_+ |\alpha(LZ, (F))) \int_v \big|z^a \hat{\nabla}^2 f\big| dv dx ds,$$

$$I_{\beta, \alpha} := \int_0^t \int_\Sigma (\sigma(LZ, (F)) + |\alpha(LZ, (F))) \int_v \big|z^a \hat{\nabla}^2 f\big| dv dx ds.$$

In order to close the energy estimates, we will have to pay attention to the hierarchy discussed in Remark 2.7. For the remaining of this subsection, we fix $\hat{\Gamma} \in \mathbb{P}_0$ and we denote by $(\alpha, \rho, \sigma)$ the null decomposition of $LZ, (F)$. The proof is divided in two parts. First, we treat the case where the electromagnetic field can be estimated pointwise ($|\gamma| \leq N - 2$). Otherwise we necessarily have $|\beta| \leq 1$ and we can use the estimate (19) on the Vlasov field.

#### 6.2.1 If $|\gamma| \leq N - 2$

Suppose first that $\beta_p < \kappa_p$, which implies $a + \frac{1}{2} - \eta \leq \frac{1}{2}(N - (1 - 2\eta)\beta_p)$. The bootstrap assumption (16) then gives

$$E[z^a \hat{\nabla}^2 f](t) \lesssim \epsilon(1 + t)^{(\kappa_p + 1)}.$$
In order to lighten the notations, we denote \( |z^{a+\frac{1}{2}-\eta}\nabla_{t,x}\tilde{Z}^\beta f| \) by \( g \). We then have
\[
I_{\rho,a} + I_{\sigma,a} \lesssim \int_0^t \int_0^{\frac{1}{1+s}a} | \int_0^{1-s} \int_0^a |g|dvdxds + \int_0^t \int_0^{\frac{1}{1+s}a} \int_0^a |g|dvdxds.
\]
To deal with the second integral, we split \( V_0(t) \) as follows,
\[
V_0(t) = \{(s,x) \in V_0(t) / s - |x| \leq -t\} \cup \{(s,x) \in V_0(t) / -t \leq s - |x| \leq b\} := V_1 \cup V_2.
\]
Note that if \( s \leq t \), then \( s - |x| \leq -t \) implies \( |x| \geq 2s \) so that \( \tau_+ \lesssim \tau_- \) on \( V_1 \). Consequently, using \( V_1 \subset V_0(t) \), \( \mathbb{E}[g](t) \lesssim \epsilon(1+t)^{(\beta_P+1)\eta} \) (see (24) and Lemma 2.2), it comes
\[
I_{\rho,a} + I_{\sigma,a} \lesssim \int_0^t \int_0^{\frac{1}{1+s}a} \int_0^a |g|dvdxds + \int_0^t \int_0^{\frac{1}{1+s}a} \int_0^a |g|dvdxds.
\]
We suppose now that \( \beta_P = \kappa_P \), so that \( \tau_T \geq 1 \). Since, for \( Z \in \mathbb{K} \) and \( 0 \leq \mu \leq 3 \), \( [Z, \partial_u] \) is either equal to 0 or \( \pm \partial_v \) for \( \nu \in [0,3] \), we can assume that \( Z^\gamma = \partial Z^{\gamma_0} \) with \( |\gamma_0| = |\gamma| - 1 \). Note also that (24) does not hold in that case. The bootstrap assumption (10) merely gives us
\[
\mathbb{E}[z^{a+\frac{1}{2}+\eta}\tilde{Z}^\beta f](t) \lesssim \epsilon(1+t)^{(\beta_P+1)\eta}
\]
and
\[
\mathbb{E}[z^{a}\nabla_{t,x}\tilde{Z}^\beta f](t) \lesssim \epsilon(1+t)^{(\beta_P+1)\eta}.
\]
Applying Lemma 3.8 and using again \( 1 \lesssim \sqrt{v_0 v_{\mu}} \leq \sqrt{v_0 v_{\mu}} \) as well as (21), we have
\[
\tau_+ |\rho(LZ^\gamma(F))| + \tau_+ |\alpha(LZ^\gamma(F))| \lesssim \sum_{|\xi| \leq |\xi_0|} |\rho(LZ^\gamma(F))| + \sqrt{v_0 v_{\mu}} \lesssim \epsilon(1+t)^{(\beta_P+1)\eta}.
\]
We then have, using Lemma 2.2 and (20) and \( \beta_P = \kappa_P \),
\[
I_{\rho,a} + I_{\sigma,a} \lesssim \int_0^t \int_0^{\frac{1}{1+s}a} \int_0^a |z^{a}\nabla_{t,x}\tilde{Z}^\beta f|dvdxds + \int_0^t \int_0^{\frac{1}{1+s}a} \int_0^a |z^{a}\nabla_{t,x}\tilde{Z}^\beta f|dvdxds.
\]
Finally, as \( z^{a} \leq 2\tau_+^{\frac{1}{2}+\gamma_0}z^{a+\frac{1}{2}+\eta} \), we have by (20) and (24),
\[
I_\Gamma \lesssim \int_0^t \int_0^{\frac{1}{1+s}a} \int_0^a |z^{a+\frac{1}{2}+\eta}\Gamma^{\tilde{Z}^\beta f}|dvdxds + \int_0^t \int_0^{\frac{1}{1+s}a} \int_0^a |z^{a+\frac{1}{2}+\eta}\Gamma^{\tilde{Z}^\beta f}|dvdxds.
\]

To deal with the remaining integral, let us introduce, for all $u < b$ and $i \in \mathbb{N}$, the following truncated cone
\[ C^i_u(t) := \{(s, x) \in C_u(t) / 2^i - 1 \leq s \leq T_{i+1}(t)\}, \quad \text{where} \quad T_{i+1}(t) = \min(t, 2^{i+1} - 1). \quad (27) \]

Notice that $\|T_+^{-\eta}\|_{L^\infty(C^i_u(t))} \leq C2^{-i\eta}$, with $C > 0$ a constant independent of $i \in \mathbb{N}$, and
\[
\int_{C^i_u(t)} \int_u^b \|z^{a - \frac{1}{2} + \eta} \hat{\Gamma} \hat{Z}^\beta f\| dudC^i_u(t) \leq \mathbb{E}[\|z^{a - \frac{1}{2} + \eta} \hat{\Gamma} \hat{Z}^\beta f\|(T_{i+1}(t)) \leq 4\epsilon(1 + T_{i+1}(t))^{(\beta P + 2)\eta} \leq 8\epsilon 2^{(\beta P + 2)\eta}.
\]

Consequently, as $V_k(t)$ can be foliated by $(C^i_u(t))_{u < b, i \leq \log_2(1+t)}$,
\[
\int_0^t \int_{\Sigma_u} \frac{\sqrt{t}}{\tau_+^{\frac{\beta}{4}}} \int_u^b \|z^{a - \frac{1}{2} + \eta} \hat{\Gamma} \hat{Z}^\beta f\| dv ds \lesssim \log_{(1+t)} \int_0^t \int_{\Sigma_u} \frac{\sqrt{t}}{\tau_+^{\frac{\beta}{4}}} \int_u^b \|z^{a - \frac{1}{2} + \eta} \hat{\Gamma} \hat{Z}^\beta f\| dv ds \lesssim \log_{(1+t)} \mathbb{E}[\|z^{a - \frac{1}{2} + \eta} \hat{\Gamma} \hat{Z}^\beta f\|(T_{i+1}(t)) \leq 4\epsilon(1 + T_{i+1}(t))^{(\beta P + 2)\eta} \lesssim \epsilon^{\frac{\beta}{4}} \log_{(1+t)} \sum_{i=0}^\infty 2^{(\beta P + 1)\eta} \lesssim \epsilon^{\frac{\beta}{4}} (1 + t)^{(\beta P + 1)\eta}
\]

and we then deduce that $I_{\Gamma} \lesssim \epsilon^{\frac{\beta}{4}} (1 + t)^{(\beta P + 1)\eta}$.

### 6.2.2 When $|\gamma| \geq N - 1$

In that case, we have $|\beta| \leq 1$. Using first the inequality $|v^A|^{\frac{1}{2}} + |\Delta_u|^{\frac{1}{2}} \lesssim \tau_+^{-\frac{\beta}{2}} z^{\frac{\beta}{2}}$, coming from Lemma 2.9, as well as the Cauchy-Schwarz inequality in $(t, x)$ and secondly the bootstrap assumption \[\ref{bootstrap}\] we get
\[
I_{\sigma, \alpha} \lesssim \left( \int_0^t \|L z^{\gamma}(F)\|^2_{L^2(\Sigma_u)} ds \int_0^t \left( 1 + s \right)^{\frac{\beta}{4}} \tau_+^{\frac{\beta}{4}} \int_u^b \|z^{a + \frac{1}{2} + \eta} \nabla_{t,x} \hat{Z}^\beta f\| dv \right)^2 dx ds \lesssim \left( \int_0^t \left( 1 + s \right)^{\frac{\beta}{4}} \tau_+^{\frac{\beta}{4}} \int_u^b \|z^{a + \frac{1}{2} + \eta} \nabla_{t,x} \hat{Z}^\beta f\| dv \right)^2 dx ds \lesssim \epsilon^{\frac{\beta}{4}} \log_{(1+t)} \sum_{i=0}^\infty 2^{(\beta P + 1)\eta} \lesssim \epsilon^{\frac{\beta}{4}} (1 + t)^{(\beta P + 1)\eta}.
\]

Using this time Lemma 2.2, the inequality $1 \lesssim \tau_+^{-\frac{\beta}{2}} z^{\frac{\beta}{2}}$ and the Cauchy-Schwarz inequality in $(u, \omega)$, it comes
\[
I_{\rho, \alpha} \lesssim \int_{u=\infty}\int_0^b ||\rho + \alpha||_{L^2(C_u(t))} \left( \int_{u=\infty}^{b} \frac{1}{\tau_+^{\frac{\beta}{4}}} \left( 1 + s \right)^{\frac{\beta}{4}} \tau_+^{\frac{\beta}{4}} \int_u^b \|z^{a + \frac{1}{2} + \eta} \nabla_{t,x} \hat{Z}^\beta f\| dv \right)^2 dx ds \lesssim \epsilon^{\frac{\beta}{4}} \log_{(1+t)} \sum_{i=0}^\infty 2^{(\beta P + 1)\eta} \lesssim \epsilon^{\frac{\beta}{4}} (1 + t)^{(\beta P + 1)\eta}.
\]

Now, by the Cauchy-Schwarz inequality in $v$ and $1 \lesssim v^0 v^\frac{1}{2}$, we have
\[
\left( \int_{u=\infty}^{b} \frac{1}{\tau_+^{\frac{\beta}{4}}} \left( 1 + s \right)^{\frac{\beta}{4}} \tau_+^{\frac{\beta}{4}} \int_u^b \|z^{a + \frac{1}{2} + \eta} \nabla_{t,x} \hat{Z}^\beta f\| dv \right)^2 dx ds \lesssim \epsilon^{\frac{\beta}{4}} \log_{(1+t)} \sum_{i=0}^\infty 2^{(\beta P + 1)\eta} \lesssim \epsilon^{\frac{\beta}{4}} (1 + t)^{(\beta P + 1)\eta}.
\]

As $\beta P \leq \kappa P$, we have $a \leq \frac{1}{2} (N + 9 - (1 - 2\eta)\beta P) \leq \frac{1}{2} (N + 10 - (1 - 2\delta)\beta P)$. The pointwise decay estimate \[\ref{pointwise}\] and the bootstrap assumption \[\ref{bootstrap}\] then gives us
\[
\left( \int_{u=\infty}^{b} \frac{1}{\tau_+^{\frac{\beta}{4}}} \left( 1 + s \right)^{\frac{\beta}{4}} \tau_+^{\frac{\beta}{4}} \int_u^b \|z^{a + \frac{1}{2} + \eta} \nabla_{t,x} \hat{Z}^\beta f\| dv \right)^2 dx ds \lesssim \epsilon(1 + t)^{\beta P + 4\delta}(1 + t)^{\beta P} \epsilon^{\beta P} \lesssim \epsilon^2 (1 + t)^7\delta. 
\]

Combining the last inequality with \[\ref{I_{sigma, alpha}}\] and \[\ref{I_{rho, alpha}}\], we finally deduce that $I_{\sigma, \alpha} + I_{\rho, \alpha} \lesssim \epsilon^{\frac{\beta}{4}} (1 + t)^\eta$ since $7\delta \leq 2\eta$. Finally, one can obtain that $I_\Gamma \lesssim \epsilon^{\frac{\beta}{4}} (1 + t)^\eta$ by simpler considerations, which concludes the proof of Proposition 6.1.
6.3 The remaining energy norm

For the improvement of $E_{N-2}^{N+13,\delta}[f] \leq 4\epsilon(1+t)^{\delta}$ we have, in view of (22) as well as Propositions 3.1 and 2.6 to prove similar estimates than (23) and those of Proposition 6.1. More precisely, $E_{N-2}^{N+13,\delta}[f] \leq 3\epsilon(1+t)^{\delta}$ on $[0,T]$ ensues, for $\epsilon$ small enough, from the following proposition.

Proposition 6.2. Let $|\alpha| \leq N-2$, $\gamma$ and $\beta$ be such that $|\gamma| + |\beta| \leq |\alpha|$, $|\beta| \leq |\alpha| - 1$ and $\beta + |\gamma| \leq \kappa_p$. Then,

$$
\int_0^t \int_{\Sigma_s} \int
\left| \frac{\alpha + 13}{(1 - 2\delta)} - \eta \right| \mathcal{P}_\mathcal{L}_Z(F)(v, \nabla_z f) \frac{dv}{v^6} \, dx \, ds \\
\int_0^t \int_{\Sigma_s} \int\left| \frac{\alpha + 13}{(1 - 2\delta)} - \eta \right| \mathcal{P}_\mathcal{L}_Z(F)(v, \nabla_z f) \frac{dv}{v^6} \, dx \, ds \\
\leq \epsilon^2(1 + t)^{\kappa_p+1}.
$$

Proof. One only has to follow Subsections 6.1 and, as $|\gamma| \leq N-2$, 6.2.1 and to use the bootstrap assumption (15) instead of (10).

6.4 $L^2$ estimates on velocity averages

The following result improves the bootstrap assumption (18) if $\epsilon$ is small enough and will allow us to improve our estimate on $E_N[F]$.

Proposition 6.3. We have, for all $t \in [0, T]$,

$$
\sum_{|\beta| \leq N} \left\| \int v^\alpha v^0 \hat{Z}^\beta f \, dv \right\|_{L^2(\Sigma_t)} \lesssim \sum_{|\beta| \leq N} \left\| \int v \hat{Z}^\beta f \, dv \right\|_{L^2(\Sigma_t)} \lesssim \frac{\epsilon}{(1 + t)^{\delta}}.
$$

Proof. The first inequality ensues from $r|v^\alpha| \lesssim v^0 z$ (see Lemma 2.9). For the second one, we start by considering $|\beta| \leq N-3$. Using successively the Cauchy-Schwarz inequality in $v$, the pointwise decay estimate (20) and the bootstrap assumption (10), we get

$$
\left\| \int v \hat{Z}^\beta f \, dv \right\|_{L^2(\Sigma_t)}^2 \lesssim \left\| \int v \hat{Z}^\beta f \, dv \int v z^2 \hat{Z}^\beta f \, dv \right\|_{L^1(\Sigma_t)} \lesssim \left\| \int v \hat{Z}^\beta f \, dv \right\|_{L^\infty(\Sigma_t)} \left\| \int v z^2 \hat{Z}^\beta f \, dv \right\|_{L^1(\Sigma_t)} \lesssim \frac{\epsilon^2}{(1 + t)^{2 - (2\beta_p + 5)\eta}} \lesssim \frac{\epsilon^2}{(1 + t)^{\frac{5}{2}}}. 
$$

The cases $N - 2 \leq |\beta| \leq N$ are the purpose of Section 8.

7 The energy bound on the electromagnetic field

According to the energy estimate of Proposition 5.7 and the commutator formula of Proposition 2.6 and since $E_N[F](0) \leq \epsilon$, we would obtain $E_N[F] \leq 3\epsilon$ on $[0, T]$ for $\epsilon$ small enough if we could prove

$$
\sum_{|\gamma| \leq N} \sum_{|\beta| \leq N} \int_0^t \int_{\Sigma_s} \left| \mathcal{L}_Z^{-1}(F)(v) v^\alpha v^0 \hat{Z}^\beta f \right| \, dx \, ds \lesssim \epsilon^2.
$$

We then fix $|\beta| \leq N$, $|\gamma| \leq N$ and we denote by $(\alpha, \omega, \rho, \sigma)$ the null decomposition of $\mathcal{L}_Z^{-1}(F)$. Expanding $\mathcal{L}_Z^{-1}(F)(v) v^\alpha v^0 \hat{Z}^\beta f$ in null coordinates, we can observe that it suffices to prove that

$$
I_\rho := \int_0^t \int_{\Sigma_s} |\rho| \int v \hat{Z}^\beta f \, dv \, dx \, ds \lesssim \epsilon^2 \\
I_{\alpha, \omega} := \int_0^t \int_{\Sigma_s} (|\alpha| + |\omega|) \int v |v^\alpha v^\omega \hat{Z}^\beta f | \, dv \, dx \, ds \lesssim \epsilon^2.
$$

Using successively the Cauchy-Schwarz inequality in $(s, x)$, the bootstrap assumption (17), the inequality $\tau_+ |v^\alpha| \lesssim v^0 z$ which comes from Lemma 2.9 and Proposition 6.3 we have

$$
|I_{\alpha, \omega}|^2 \lesssim \int_0^t \left| \frac{|\alpha| + |\omega|}{(1 + s)^2} \right| ds \int_0^t (1 + s)^2 \left\| \int v |v^\alpha v^\omega \hat{Z}^\beta f | \, dv \right\|_{L^2(\Sigma_s)}^2 \, ds \lesssim \int_0^t \frac{E_N[F](s)}{(1 + s)^2} ds \int_0^t \left| \frac{v}{v^0} \hat{Z}^\beta f \right| \, dv \, dx \, ds \lesssim \int_0^t \frac{\epsilon}{(1 + s)^2} ds \int_0^t \frac{\epsilon^2}{(1 + s)^{\xi_2}} ds \lesssim \epsilon^3.
$$

19
Similarly, using \( \tau_- \lesssim z \) instead of \( \tau_+ |v^A| \lesssim v^0 z \) (see also Lemma 2.9 and Lemma 2.2) it comes
\[
|I_\rho|^2 \lesssim \int_0^t \int_{\mathbb{R}^3_v} |\rho|^2 \tau_+^2 dx ds \int_0^t \int_{\mathbb{R}^3_v} \tau_- \int v \hat{Z}^\beta f \left| dv \right|^2 dx ds
\]
\[
\lesssim \int_{u=-\infty}^b \int_{\mathcal{C}_{u}(t)} \frac{|\rho|^2}{\tau_+^2} dC_u(t) du \int_0^t \int_{\mathbb{R}^3_v} \int v \left| \hat{Z}^\beta f \right| dv \left| dv \right|^2 dx ds
\]
\[
\lesssim \int_{u=-\infty}^b \mathcal{E}_N[F]^4(t) du \int_0^t \left| \int v \hat{Z}^\beta f \right| dv \left| dv \right|^2 dx ds \lesssim \int_{u=-\infty}^0 \frac{c}{\tau_+^2} du \int_0^{+\infty} \frac{c^2}{(1+s)^2} ds \lesssim c^3.
\]
This concludes the improvement of the bootstrap assumption (1.7).

8 \( L^2 \) estimates for the higher order derivatives of the Vlasov field

In this last section, we complete the proof of Proposition 6.3. For this purpose, we slightly modify the strategy used in Section 4.5.7 of [8] in order to keep more of the null structure of the system. The first step of the proof consists in rewriting all transport equations as a hierarchised system. Let \( I \) and \( J \) be the following two ordered sets,
\[
I := \{ \beta \text{ multi-index} / N-3 \leq |\beta| \leq N \} = \{ \beta^1, ..., \beta^I \},
\]
\[
J := \{ \xi \text{ multi-index} / |\xi| \leq N-3 \} = \{ \xi^1, ..., \xi^J \}.
\]

Remark 8.1. Even if it only remains us to estimate \( \| \int v \hat{Z}^\beta f dv \|_{L^2(\mathbb{R}^3_v)} \) for all \( |\beta| \geq N-2 \), we included the multi-indices of length \( N-3 \) in \( I \). It will allow us to conserve the null structure of the Vlasov equations.

We also consider \( I^k := \{ \beta \in I / |\beta| = k \} \), for \( N-3 \leq k \leq N \), and two vector valued fields \( R \) and \( W \) of respective length \( |I| \) and \( |J| \) such that
\[
R_i = \hat{Z}^\beta f \quad \text{and} \quad W_i = \hat{Z}^\xi f.
\]

For simplicity, we will sometimes abusively write \( i \in I^k \) instead of \( \beta^i \in I^k \). We denote by \( \mathcal{V} \) the module over the ring \( C^0(\mathbb{V}_0(T) \times \mathbb{R}^3_v) \) engendered by \( \{ \partial_i \} \leq 1 \leq 3 \) and we recall that \( |\gamma| := \text{max}(0,1-\gamma_T) \). Let us now rewrite the Vlasov equations satisfied by the components of \( R \).

Lemma 8.2. There exists two matrices valued functions \( A : \mathbb{V}_0(T) \times \mathbb{R}^3_v \rightarrow \mathcal{M}_{|I|,|J|}(\mathcal{V}) \) and \( B : \mathbb{V}_0(T) \times \mathbb{R}^3_v \rightarrow \mathcal{M}_{|I|,|J|}(\mathcal{V}) \) such that
\[
\mathbf{T}_F(R) + AR = BW.
\]

These two matrices are such that \( \mathbf{T}_F(R_i) \), for \( 1 \leq i \leq |I| \), is a linear combination of
\[
L_{Z^\gamma}(F)(v, \nabla_v R_j), \quad \text{with} \quad |\gamma| < |\beta^i|, \quad |\gamma| + |\beta^j| \leq |\beta^j| \quad \text{and} \quad \beta^i + |\gamma| \leq \beta^j, \quad \text{and} \quad \beta^i + |\gamma| \leq \beta^j.
\]

Remark 8.3. Note that if \( \beta^i \in I^{\gamma-3} \), then \( A_i^q = 0 \) for all \( 1 \leq q \leq |I| \). If \( p \geq 1 \) and \( \beta^i \in I^{N-3+p} \), we have \( |\gamma| \leq p \). The condition \( \beta^i + |\gamma| \leq \beta^j \) expresses that either \( \hat{Z}^\gamma \) is composed by a translation, and will then give an extra decay in the \( u \) direction, or that the number of homogeneous vector fields composing \( R_j \), \( \beta^j \), is strictly lower than \( \beta^i \).

Proof. One only has to apply the commutation formula of Proposition 2.6 to \( \hat{Z}^\beta f \) and to replace each quantity such as \( \hat{Z}^\beta f \), for \( |\gamma| \neq N-3 \), by the corresponding component of \( R \) or \( W \). If \( |\gamma| = N-3 \), we replace it by the corresponding component of \( R \).

In order to establish an \( L^2 \) estimate on the velocity average of \( R \), we split it in \( R := H + G \), where
\[
\begin{align*}
\left\{ \begin{array}{ll}
\mathbf{T}_F(H) + AH = 0, & H(0,\ldots) = R(0,\ldots), \\
\mathbf{T}_F(G) + AG = BW, & G(0,\ldots) = 0
\end{array} \right.
\end{align*}
\]
and then prove \( L^2 \) estimates on \( \int v H dv \) and \( \int v G dv \). For the homogeneous part \( H \), we will commute the transport equation and take advantage of the decaying properties of the matrix \( A \) in order to obtain boundedness on a certain \( L^1 \) norm as for \( f \) in Section 8. The \( L^2 \) estimate will then follow from a Klainerman-Sobolev inequality and the bound obtained on \( E[H] \). The inhomogeneous part will be schematically decomposed as \( G = KW \), with \( K \) a matrix such that \( E[|K|^2W(t)] \leq c(1+t)^{\chi} \). The expected decay rate on \( \| \int v G dv \|_{L^2} \) will then be obtained using the pointwise decay estimates satisfied by the components of \( W \). Note that contrary to [8], we keep the \( v \) derivatives in the matrices \( A \) and \( B \). This allows us to crucially exploit the good behavior of \( (\nabla v g)^v \) (see [14]) but it forces us to put the derivatives of order \( N-3 \) in both \( R \) and \( W \).
8.1 The homogeneous system

With the aim of obtaining an $L^\infty$ estimate on $\int_v |H|dv$, we will have to commute at least three times the transport equation satisfied by each component of $H$. However, in order to control $\|\tilde{Z}^\beta H_i\|_{L^1_v}$, where $|\beta| = 3$, $\beta^i \in I^k$ and $k \geq N - 2$, a bound on the $L^1$ norm of $\tilde{Z}^\alpha H_j$, with $|\alpha| = 4$ and $j \in I^{k-1}$, is required. This leads us to introduce the following energy norm

$$E_H(t) := \sum_{k=0}^3 \sum_{|\beta| \leq 3+k} \sum_{i \in I^{N-k}} (1+t)^{-|\beta P + (\beta^i_P) P|} E \left[ \sqrt{z}^{N+9-(1-2\eta)(|\beta P + (\beta^i_P) P|)} \tilde{Z}^\beta H_i(t) \right].$$

We have the following commutation formula.

**Lemma 8.4.** Let $0 \leq k \leq 3$, $|\beta| \leq 3 + k$ and $i \in I^{N-k}$. Then, $T_F(\tilde{Z}^\beta H_i)$ can be written as a linear combination of terms such as

$$L_{Z^\alpha}(F) \left( v, \nabla_v \tilde{Z}^\alpha H_j \right),$$

with $|\alpha| \leq 6 \leq N - 2$, $|\alpha| \leq |\beta|$, $|\beta^i| \leq |\beta|$, $|\beta P + (\beta^i_P) P| < |\beta| + |\beta^i|$, $\beta^i_P + \kappa_P < \beta_P + \beta_P$ or $\beta^i_P + \kappa_P = \beta_P + \beta_P$ and $\gamma_T \geq 1$.

**Proof.** According to Proposition 2.6, the source terms coming from $[T_F, \tilde{Z}^\beta](H_i)$ are such as those described in this lemma, with $j = i$. The other ones arise from $\tilde{Z}^\beta (T_F(H_i))$ and the result follows from Lemma 8.2 and $|\beta|$ applications of Lemma 2.6.

Hence, as $R(0,\ldots) = H(0,\ldots)$, there exists $C_0 > 0$ such that $E_H(0) \leq C_0 \epsilon$. Following the proof of (23) and Proposition 6.1 (for the cases where $|\alpha| \leq 6 - 2$), one can prove, if $\epsilon$ small enough, that $E_H(t) \leq 3C_0 \epsilon (1+t)^\eta$ for all $t \in [0,T]$. By Proposition 2.6, we then obtain, for $0 \leq k \leq 3$,

$$\forall (t,x) \in V_k(T), \quad 1 \leq j \leq |I^{N-k}|, \quad |\beta| \leq k, \quad \int_v \sqrt{z}^{N+6-(1-2\eta)(\beta_P - \beta^i_P)} \tilde{Z}^\beta H_j|dv \lesssim \epsilon (1+t)^{\eta+4}. \quad (30)$$

8.2 The inhomogenous system

The purpose of this subsection is to prove an $L^2$ estimate on $\int_v |G|dv$. We cannot proceed by commuting $T_F(G) + AG = BW$ since $B$ contains top order derivatives of $F$ and we then follow the strategy exposed earlier in this section. For this, in order to prove $L^1$ estimates on quantities related to $G$, we need

- to rewrite the $v$ derivatives hidden in the matrix $A$.
- to ensure that the (transformed) matrix $A$ decay sufficiently fast. We will then modify each component $G_i$ of $G$ by $z^{a_i}G_i$, with a well chosen exponent $a_i$, in order to take advantage of similar hierarchies than those used in Section 6.

We start by introducing some notations and proving certain preparatory results.

**Definition 8.5.** Let $L$ be the vector valued field of length $|I|$ such that, for $i \in [1,|I|]$

$$L_i = \sqrt{z}^{N-(1-2\eta)\beta^i_P} G_i.$$

For $\tilde{Z} \in \mathbb{P}_0$ and $i \in I \setminus I^N$, we define $i_{\tilde{Z}}$ such that

$$R_{i_2} = \tilde{Z} \tilde{Z}^\beta f.$$ 

We will transform the $v$ derivatives by several applications of the following result.

**Lemma 8.6.** Let $\tilde{Z} \in \mathbb{P}_0$ and $i \in I \setminus I^N$. Then, as $R = H + G$ and $\tilde{Z} R_i = \tilde{Z} \tilde{Z}^\beta f = R_{i_2}$,

$$\tilde{Z} G_i = G_{i_2} + H_{i_2} - \tilde{Z} H_i.$$ 

The aim of the next lemma is to describe in details the transport equation satisfied by $L$. 

21
Lemma 8.7. There exists a, b ≥ 1, a vector valued field Y of length p and three matrices valued functions \( \overline{A} : V_b(T) \times \mathbb{R}_b \rightarrow \mathbb{M}_{1,1}(\mathbb{R}) \), \( \overline{B} : V_b(T) \times \mathbb{R}_b \rightarrow \mathbb{M}_{1,1}(\mathbb{R}) \) and \( \overline{D} : V_b(T) \times \mathbb{R}_b \rightarrow \mathbb{M}_{p}(\mathbb{R}) \) such that

\[
T_F(L) + \overline{A}L = \overline{B}Y, \quad T_F(Y) = \overline{D}Y \quad \text{and} \quad \forall (t, x) \in V_b(T), \quad \int_v z^2 |Y|(t, x, v)dv \leq \epsilon \frac{(1 + t)^{N+4}}{\tau_+ \tau_-}.
\]

The matrices \( \overline{A} \) and \( \overline{B} \) are such that \( T_F(L_i) \) can be bounded, for \( 1 \leq i \leq |I| \), by a linear combination of the following terms,

- \( \left( |\rho(L_{Z,1}(F))| + |\sigma(L_{Z,1}(F))| + |\alpha(L_{Z,1}(F))| + \frac{\tau^+}{\tau_-} |\alpha(L_{Z,1}(F))| \right) |Y| \), with \( |\gamma| \leq N \).
- \( \sqrt{\epsilon} \frac{v^0}{\tau_+ \tau_-} |L_k|, \quad \sqrt{\epsilon} \frac{v^0}{\tau_+ \tau_-} |L_q|, \quad \sqrt{\epsilon} \frac{v^0}{\tau_+ \tau_-} |L_j|, \quad \sqrt{\epsilon} \frac{v^0}{\tau_+ \tau_-} |L_k|, \quad \sqrt{\epsilon} \frac{v^0}{\tau_+ \tau_-} |L_q|, \quad \sqrt{\epsilon} \frac{v^0}{\tau_+ \tau_-} |L_j|, \)
  where \( k, q, j \in [1, |I|] \), \( \beta^k_p < \beta^q_p \), \( \beta^q_p = \beta^p_p \) and \( \beta^p_p = \beta^i_p + 1 \).

In order to describe the components of the matrix \( \overline{D} \), we use the quantity \( |j| \) which will be defined during the proof for all \( j \in [1, p] \). \( \overline{D} \) is such that \( T_F(Y_i) \) can be bounded, for \( 1 \leq i \leq p \), by a linear combination of the following terms,

- \( \sqrt{\epsilon} \frac{v^0}{\tau_+ \tau_-} |Y_k|, \quad \sqrt{\epsilon} \frac{v^0}{\tau_+ \tau_-} |Y_q|, \quad \sqrt{\epsilon} \frac{v^0}{\tau_+ \tau_-} |Y_j|, \quad \sqrt{\epsilon} \frac{v^0}{\tau_+ \tau_-} |Y_k|, \quad \sqrt{\epsilon} \frac{v^0}{\tau_+ \tau_-} |Y_q|, \quad \sqrt{\epsilon} \frac{v^0}{\tau_+ \tau_-} |Y_j|, \)
  where \( k, q, j \in [1, p] \), \( |k| < |i| \), \( |q| = |i| \) and \( |j| = |i| + 1 \).

Proof. The key element of the proof will be to rewrite all terms of the form \( \partial_{\alpha}G_j \) appearing in the equation \( T_F(G) + AG = BW \) with the formula \( v^0 \partial_{\alpha} = \tilde{O}_{\alpha k} - \partial_{\alpha} t_d k \). As \( j \in I \setminus I \) by Lemma 8.2 we will express \( \tilde{O}G_j \), by Lemma 8.3, as a combination of \( (G_j)_{\alpha \in I}, (H_j)_{\alpha \in I} \) and \( \tilde{Z}H_j \). This suggests us to take for \( Y \) the vector valued field of length \( p \) composed by the following components \( Y_i \), where \( i \in [1, p] \),

- \( Y_i = \sqrt{z^{N+2-|1-2\alpha|+\beta^i_p}} \tilde{Z}^\beta \), with \( |\beta| \leq N - 3 \). We then define \( [i] := \beta^i_p \).
- \( Y_i = \sqrt{z^{N+2-|1-2\alpha|+\beta^i_p}} \tilde{Z}^\beta H_j \), with \( |\beta| + |\beta^i_p| \leq N \). In that case, we define \( [i] := \beta^i_p + \beta^i_p \).

In view of \( (20) \) and \( (50) \), \( \int_v z^2 |Y|dv \) satisfies the expected pointwise decay estimate. The construction of the matrix \( \overline{D} \) is similar to the one of \( \overline{A} \) detailed below and then sketched. To obtain it, apply Lemmas 2.6 and 8.4 and then make similar operations as those made in the proof of \( (23) \) and Proposition 6.1 (for the cases where \( |\gamma| \leq N - 2 \)). We now turn on the construction of \( \overline{A} \) and \( \overline{B} \). Fix \( i \in [1, |I|] \) and note that

\[
T_F(L_i) = T_F(\sqrt{z^{N-\beta^i_p}} G_i) = \sqrt{z^{N-\beta^i_p}} F(v, \nabla_v z)G_i + \sqrt{z^{N-\beta^i_p}} T_F(G_i).
\]

Following the computations of Subsection 6.1, we have

\[
\left| F(v, \nabla_v z)\sqrt{z^{N-\beta^i_p}} G_i \right| \lesssim \sqrt{\epsilon} \frac{v^0}{\tau_+ \tau_-} \sqrt{z^{N-\beta^i_p}} |G_i| \lesssim \sqrt{\epsilon} \frac{v^0}{\tau_+ \tau_-} |L_i|.
\]

By Lemma 8.2, \( \sqrt{z^{N-\beta^i_p}} T_F(G_i) \) can be written as a linear combination of the following terms.

- Those coming from \( BW \),

\[
\sqrt{z^{N-\beta^i_p}} L_{Z_{1}}(F)(v, \nabla_v W_j), \quad \text{with} \quad |\xi_j| \leq N - 4, \quad |\gamma| \leq N \text{ and } \xi_j \leq \beta^i_p.
\]

Let \( |\gamma| \leq N \) and \( (\alpha, \beta, \rho, \sigma) \) be the null decomposition of \( L_{Z_{1}}(F) \). Using Lemma 4.1 we can bound \( (31) \) by terms of the form

\[
|L_{Z_{1}}(F)| \sqrt{z^{N-\beta^i_p}} \left| \tilde{G} \tilde{Z}^{\xi_j} f \right|, \quad \left( \tau_- |\rho| + \tau_+ |\alpha| + \tau_+ v^0 v^0 |\sigma| + |\alpha| \right) \sqrt{z^{N-\beta^i_p}} \left| \nabla_{t, x} \tilde{G} \tilde{Z}^{\xi_j} f \right|
\]

where \( \tilde{F} \in \tilde{F}_0 \). As \( \xi_j \leq \beta^i_p \), there exists \( \kappa_1 \) and \( \kappa_2 \) satisfying

\[
|\tilde{G} \tilde{Z}^{\xi_j} f| = \tilde{Z}^{\kappa_1} f, \quad |\kappa_1| \leq N - 3, \quad \kappa_1 \leq \beta^i_p + 1 \quad \text{and} \quad \nabla_{t, x} \tilde{G} \tilde{Z}^{\xi_j} f = \tilde{Z}^{\kappa_2} f, \quad |\kappa_2| \leq N - 3, \quad \kappa_2 \leq \beta^i_p.
\]
Consequently, there exists \((j, q) \in [1, p]\) such that
\[
\sqrt{z}^{-N-(1-2n)\beta_p} \left| \hat{\mathbf{V}}^k f \right| \leq \sqrt{z}^{-N+2-(1-2n)\kappa_0} \left| \hat{\mathbf{V}}^k f \right| = |Y_j| \quad \text{and} \quad \sqrt{z}^{-N+2-(1-2n)\beta_p} \left| \hat{\mathbf{W}}^k f \right| \leq |Y_q|. \quad (33)
\]

Then, combine (32) with (43) and use the inequality \(\tau_- v^0 + \tau_+ v^A + \tau_+ v L \lesssim z\) of Lemma 2.9 in order to obtain terms involving \(Y\) of the expected form.

- Those coming from \(A W\),
\[
\sqrt{z}^{-N-(1-2n)\beta_p} L_{Z^+} \left( v, \nabla_v G_j \right), \quad \text{with} \quad |\beta| < |\beta^i|, \quad |\gamma| \leq 3 \quad \text{and} \quad \beta_p + |\gamma| \leq \beta_p^i.
\]

In order to rewrite \(v\) derivatives of \(\hat{L}\), note that Lemma 8.9 gives, using \(v^0 \partial_{\hat{v}^i} = \hat{\nabla}_k - x^k \partial_k - t \partial_k\) and (141),
\[
v^0 \partial_{\hat{v}^i} G_j = \mathcal{G}_{j00k} - \mathcal{G}_{j00k} - t \mathcal{G}_{j00k} - \mathcal{G}_{j00k} - \partial_{\hat{v}^i} H_j + x^k \partial_k H_j + t \partial_k H_j,
\]
\[
v^0 \partial_{\hat{v}^i} G_j = \mathcal{G}_{j00k} + x^k \partial_k H_j - \mathcal{G}_{j00k} - \partial_{\hat{v}^i} H_j + S H_j + (t - r) \left( \mathcal{G}_{j00k} + \partial_{\hat{v}^i} H_j - \partial_{\hat{v}^i} H_j + \mathcal{G}_{j00k} - \partial_{\hat{v}^i} H_j \right).
\]

We then have schematically
\[
v^0 \left| \left( \nabla_v G_j \right)^T \right| \lesssim \left| G_j \right| + \tau_- \left| G_j \right| + \left| H_j \right| + \left| \hat{\mathcal{Z}} H_j \right| + \tau_- \left| H_j \right| + \left| \hat{\mathcal{Z}} H_j \right|, \quad (34)
\]
\[
v^0 \left| \left( \nabla_v G_j \right)^A \right| \lesssim \left| G_j \right| + \tau_- \left| G_j \right| + \left| H_j \right| + \left| \hat{\mathcal{Z}} H_j \right| + \tau_- \left| H_j \right| + \left| \hat{\mathcal{Z}} H_j \right|. \quad (35)
\]

Denote by \((\alpha, \beta, \rho, \sigma)\) the null decomposition of \(L_{Z^+} \left( v, \nabla_v G_j \right)\) in null components using formula (133). As the computations are similar to those made in Subsection 6.2, we only bound certain terms given by (133). Consider for instance \(\sqrt{z}^{-N-(1-2n)\beta_p} \epsilon BA v^B \sigma \left( \nabla_v G_j \right)^A\) and use (34) in order to bound it. As \(\beta_p^j \leq \beta_p\), we have \(\beta_p^\rho \leq \beta_p\), as well as \(\beta_p \leq \beta_p^i + 1\) and the terms related to \(H\) can be estimated, using \(\tau_+ |v^B| \lesssim v^0 z\), as follows
\[
\tau_+ |\sigma| \left| \frac{|B|}{v^0} \right| \sqrt{z}^{-N-(1-2n)\beta_p} \left( |H_j| + |\partial H_j| \right) + |\sigma| \sqrt{z}^{-N-(1-2n)\beta_p} \left( |H_j| + |\hat{\mathcal{Z}} H_j| \right) \lesssim |\sigma| |Y|.
\]

For the ones associated to \(G\), suppose first that \(\beta_p^j < \beta_p\). By Lemma 2.9, we have \(|v^B| \lesssim v^0 v L\) as well as \(\tau_-^{\frac{1}{4} - \eta} \lesssim z^{\frac{1}{4} - \eta}\). Hence, using also \(|\sigma| \lesssim \sqrt{\tau_+^{1-\tau_-^{\frac{1}{2}}} \cdot \tau_-^{\frac{1}{2}}} \) which comes from (21), we get
\[
\tau_+ |\sigma| \left| \frac{|B|}{v^0} \right| \sqrt{z}^{-N-(1-2n)\beta_p} \left| G_{j0} \right| \lesssim \sqrt{\frac{\tau_+}{\tau_-^{\frac{1}{2}}}} \left| |L_{j0}| \right| \lesssim \sqrt{\frac{\tau_+}{\tau_-^{\frac{1}{2}}}} \left| L_{j0} \right|, \quad \beta_p^\rho \leq \beta_p,
\]
\[
|\sigma| \left| \frac{|B|}{v^0} \right| \sqrt{z}^{-N-(1-2n)\beta_p} \left| G_{j0} \right| \lesssim \sqrt{\frac{\tau_+}{\tau_-^{\frac{1}{2}}}} \left| |L_{j0}| \right| \lesssim \sqrt{\frac{\tau_+}{\tau_-^{\frac{1}{2}}}} \left| L_{j0} \right|, \quad \beta_p^\rho \leq \beta_p^j.
\]

If \(\beta_p^j = \beta_p\), then \(|\gamma| = 0\) and \(\tau_- \geq 1\) so that \(|\sigma| \lesssim \sqrt{\tau_+^{1-\tau_-^{\frac{1}{2}}} \cdot \tau_-^{\frac{1}{2}}} \) by Lemma 3.8. We then have
\[
\tau_+ |\sigma| \left| \frac{|B|}{v^0} \right| \sqrt{z}^{-N-(1-2n)\beta_p} \left| G_{j0} \right| \lesssim \sqrt{\tau_+^{1-\tau_-^{\frac{1}{2}}} \cdot \tau_-^{\frac{1}{2}}} \left| |L_{j0}| \right| \lesssim \sqrt{\tau_+^{1-\tau_-^{\frac{1}{2}}} \cdot \tau_-^{\frac{1}{2}}} \left| L_{j0} \right|, \quad \beta_p^\rho = \beta_p,
\]
\[
|\sigma| \left| \frac{|B|}{v^0} \right| \sqrt{z}^{-N-(1-2n)\beta_p} \left| G_{j0} \right| \lesssim \sqrt{\tau_+^{1-\tau_-^{\frac{1}{2}}} \cdot \tau_-^{\frac{1}{2}}} \left| |L_{j0}| \right| \lesssim \sqrt{\tau_+^{1-\tau_-^{\frac{1}{2}}} \cdot \tau_-^{\frac{1}{2}}} \left| L_{j0} \right|, \quad \beta_p^\rho = \beta_p^j + 1.
\]

We now treat the terms involving \(\rho\), which can be estimated by \(|v^0 \rho \left( \nabla_v G_j \right)^T|\), and we use (34) to bound them. As \(\beta_p^j \leq \beta_p^j\) and \(\tau_- \lesssim z\), the terms related to \(H\) can be estimated as follows
\[
\tau_- |\rho| \sqrt{z}^{-N-(1-2n)\beta_p} \left( |H_j| + |\partial H_j| \right) + |\rho| \sqrt{z}^{-N-(1-2n)\beta_p} \left( |H_j| + |\hat{\mathcal{Z}} H_j| \right) \lesssim |\rho| |Y|.
\]

For the ones associated to \(G\), start again by assuming \(\beta_p^j \leq \beta_p\). As \(\tau_-^{\frac{1}{4} - \eta} \lesssim z^{\frac{1}{4} - \eta}\) and \(|\rho| \lesssim \sqrt{\tau_+^{1-\tau_-^{\frac{1}{2}}} \cdot \tau_-^{\frac{1}{2}}} \) by (21), we get
\[
\tau_- |\rho| \sqrt{z}^{-N-(1-2n)\beta_p} \left| G_{j0} \right| \lesssim \sqrt{\frac{\tau_+}{\tau_-^{\frac{1}{2}}}} \left| L_{j0} \right| \lesssim \sqrt{\frac{\tau_+}{\tau_-^{\frac{1}{2}}}} \left| L_{j0} \right|, \quad \beta_p^\rho < \beta_p,
\]
\[
|\rho| \sqrt{z}^{-N-(1-2n)\beta_p} \left| G_{j0} \right| \lesssim \sqrt{\frac{\tau_+}{\tau_-^{\frac{1}{2}}}} \left| L_{j0} \right| \lesssim \sqrt{\frac{\tau_+}{\tau_-^{\frac{1}{2}}}} \left| L_{j0} \right|, \quad \beta_p^\rho \leq \beta_p^j.
\]
In order to apply Proposition 3.1, remark that terms of the integral (they are all described in Lemma 8.7). Fix for instance that

$$\int \hat{\beta} = |Z| \int A$$

$$\tau \beta_{\beta} g_{\beta} \int G_{\beta} \lesssim \sqrt{v^0_{\beta}} \int G_{\beta}$$

$$\beta_{\beta} g_{\beta} \int G_{\beta} \lesssim \sqrt{v^0_{\beta}} \int G_{\beta}$$

$$\beta_{\beta} \int G_{\beta}$$

$$\beta_{\beta} = \beta_{\beta} + 1.$$ As the other terms can be handled similarly, this concludes the construction of $G$ and $\hat{G}$. □

Let $K$ be the solution of $\mathcal{T}_F(K)+\overline{\mathcal{A}}K+\overline{\mathcal{D}} = \overline{\mathcal{B}}$ satisfying $K(0,.) = 0$. Note that $KY$ satisfies $\mathcal{T}_F(KY)+\overline{\mathcal{A}}KY = \overline{\mathcal{B}}Y$ and initially vanishes, so that $KY = L$. The goal now is to prove a sufficiently good estimate on the energy norm

$$\mathcal{E}(t) := \sum_{i=0}^|I| \sum_{j=0}^P \sum_{q=0}^p (1 + \tau^{-\eta})^{-2|\beta|} \mathcal{E}\left[|K|^2 |Y\rangle(t)\right].$$

In order to apply Proposition 3.1 we remark that

$$\mathcal{T}_F \left(|K|^2 |Y\rangle(t)\right) = |K|^2 |\mathcal{D}_Q |Y\rangle - 2 \left(\mathcal{A}_k |K^2 + \mathcal{D}_k |K^2 |Y\rangle \right) = 2 |K|^2 |Y\rangle - 2 \mathcal{D}_k |K^2 |Y\rangle.$$ (36)

**Proposition 8.8.** If $\epsilon$ is small enough, we have $\mathcal{E}_G(t) \lesssim \epsilon(1 + t)^{4N\eta}$ for all $t \in [0, T]$.

**Proof.** We use again the continuity method. Let $T_0 \in [0, T]$ be the smallest time such that $\mathcal{E}_G(t) \lesssim \epsilon(1 + t)^{4N\eta}$ for all $t \in [0, T_0]$. Fix $i \in [1, |I|]$ and $(j, q) \in [1, |I|^2]$. According to the energy estimate of Proposition 3.1 and (36), we would improve the bootstrap assumption, for $\epsilon$ small enough, if we could prove that

$$\int_0^t \int_{Q} \int_{Q} |K|^2 |\mathcal{D}_Q |Y\rangle - 2 \left(\mathcal{A}_k |K^2 + \mathcal{D}_k |K^2 |Y\rangle \right) dx ds \lesssim \epsilon^2 (1 + t)^{2(\eta - \frac{2}{3} + \frac{N\eta}{2})},$$

(37)

$$\mathcal{I}_Q := \int_0^t \int_{Q} \int_{Q} |K|^2 |\mathcal{D}_Q |Y\rangle dx ds \lesssim \epsilon^2 (1 + t)^{2(\eta - \frac{2}{3} + \frac{N\eta}{2})}.$$ (38)

Let us start with (37). Since the computations are similar to those of Subsection 6.2.1 we only study certain terms of the integral (they are all described in Lemma 8.7). Fix $1 \leq r \leq p$ as well as $1 \leq k \leq |I|$ and suppose for instance that

$$|A_k| \lesssim \sqrt{v^0_{\beta}} \quad \beta_{\beta} = \beta_{\beta} \quad |D_k| \lesssim \sqrt{v^0_{\beta}} \quad |J| < |r| \quad \text{and} \quad |D_k| \lesssim \sqrt{v^0_{\beta}} \quad |r| = |q| + 1.$$ Without any summation in $r$ and by the Cauchy-Schwarz inequality in $(u, \omega, v)$,

$$\int_0^t \int_{Q} \int_{Q} |K|^2 |\mathcal{D}_Q |Y\rangle dx ds \lesssim \int_0^t \int_{Q} \int_{Q} |K|^2 |\mathcal{D}_Q |Y\rangle dx ds \lesssim \epsilon^2 (1 + t)^{2(\eta - \frac{2}{3} + \frac{N\eta}{2})}.$$ Without any summation in $r$ and by the Cauchy-Schwarz inequality in $(x, v)$ as well as $-|r| + 1 \leq -|J|$, 

$$\int_0^t \int_{Q} \int_{Q} |K|^2 |\mathcal{D}_Q |Y\rangle dx ds \lesssim \epsilon^2 (1 + t)^{2(\eta - \frac{2}{3} + \frac{N\eta}{2})}.$$ Without any summation in $r$ and by the Cauchy-Schwarz inequality in $(x, v)$ as well as $-|r| + 1 \leq -|J|$,
Recall now from (27) the definition of $C_u(t)$ and $T_{i+1}(t)$. Without any summation in $r$ and since $[r] = [q] + 1$, we have

$$\int_0^t \int_{\Sigma_s} \left| K_i^j \right|^2 |s| T_{i+r} Y_r \, dv \, ds \lesssim \log_2(1+t) \sum_{i=0}^{t \log_2(1+t)} \int_{u=-\infty}^{b} \int_{C_u(t)} \frac{\sqrt{t}}{\tau_+ \tau_-} \int_{v} \frac{du}{v} \left| K_i^j \right|^2 |Y_r| dv dC_u(t) \, du$$

By the bootstrap assumption (17) and dropping the dependence of $\alpha$, $\rho$ and $\sigma$ in $\mathcal{L}_Z(F)$, we have

$$\forall s \in [0, T], \forall u < b, \quad ||\rho| + |\sigma| + |\alpha||L^2(\Sigma_s^1) + \int_{C_u(s)} |\alpha|^2 dC_u(s) \leq \mathcal{E}_N[F](s) \leq \epsilon.$$  

Consequently, using the Cauchy-Schwarz inequality in $v$, $\int_v |Y_r| dv \lesssim \epsilon (1 + s)^{2N} \frac{\sqrt{t}}{\tau_+ \tau_-} \tau_0$ and $1 \lesssim \tau_0^{-1}$,

$$I_{\Sigma_s} \lesssim \int_0^t \int_{\Sigma_s} \left( |\rho| + |\sigma| + |\alpha| + \frac{\tau_+}{\tau_-} |\alpha| \right) \int_{v} |Y_r| dv \int_v \left| K_i^j \right|^2 |Y_q| \frac{dv}{v^\theta} \frac{1}{2} dx ds$$

By the Cauchy-Schwarz inequality in $x$ and $\mathbb{E}_G(s) \lesssim \epsilon (1 + t)^{4N\eta}$, we get

$$\int_0^t \int_{\Sigma_s} \left( |\rho| + |\sigma| + |\alpha| \right) \frac{\sqrt{t}}{\tau_+ \tau_-} \int_{v} \frac{dv}{v^\theta} \left| K_i^j \right|^2 |Y_q| \frac{1}{2} dx ds \lesssim \int_0^t \frac{\sqrt{t}}{1 + s} \mathcal{E}_N[F](s) \mathbb{E} \left[ \left| K_i^j \right|^2 |Y_q| \right] (s) \frac{ds}{1 + s}$$

Using the null foliation $(C_u(t))_{u < b}$ of $V_b(t)$ and the Cauchy-Schwarz inequality in $(u, \omega)$, it comes

$$\int_0^t \int_{\Sigma_s} |\alpha| \frac{\sqrt{t}}{\tau_+ \tau_-} \int_{v} \frac{dv}{v^\theta} \left| K_i^j \right|^2 |Y_q| \frac{1}{2} dx ds \lesssim \int_{-\infty}^b \frac{\sqrt{t}}{\tau_+ \tau_-} \int_{C_u(t)} |\alpha|^2 dC_u(t) \int_{\Sigma_s} \frac{dv}{v^\theta} \left| K_i^j \right|^2 |Y_q| dv dC_u(t) \, du$$

Combining (39), (40) and (41), we finally obtain, since $[j] \leq N$,

$$I_{\Sigma_s} \lesssim \epsilon^2 (1 + t)^{\frac{3}{2}(4N-[j]+[\beta]+\frac{1}{2}[q])} \lesssim \epsilon^2 (1 + t)^{\frac{3}{2}(4N-2[j]+2[\beta]+[q])}.$$  

This concludes the improvement of the bootstrap assumption and then the proof. \qed
8.3 End of the proof of Proposition 6.3

Let $i \in I$. Using the Cauchy-Schwarz inequality in $v$, $\mathbb{E}_H(t) \lesssim (1 + t)^{\eta}$ and the pointwise decay estimates \cite{30}, we have

$$
\left\| \int_v z|H_i| dv \right\|_{L^2(\mathcal{S}_i^t)} \lesssim \left\| \int_v \left| H_i \right| dv \right\|_{L^\infty(\mathcal{S}_i^t)} \left\| \int_v z^2 |H_i| dv \right\|_{L^1(\mathcal{S}_i^t)}
\lesssim \left\| \frac{\epsilon}{\tau_+^{-(N+4)\eta}} \right\|_{L^\infty(\mathcal{S}_i^t)} (1 + t)^{\beta + \eta} \mathbb{E}_H(t) \lesssim \frac{\epsilon^2}{(1 + t)^{2-(2N+5)\eta}} \lesssim \frac{\epsilon^2}{(1 + t)^{\frac{1}{2}}}
$$

As $L_i = K_i^j Y_j$, the Cauchy-Schwarz inequality in $v$, $\mathbb{E}_G(t) \lesssim (1 + t)^{4N\eta}$ and $\int_v z^2 |Y| dv \lesssim \epsilon \tau_+^{-2+(N+4)\eta}$ gives

$$
\left\| \int_v z|L_i| dv \right\|_{L^2(\mathcal{S}_i^t)} \lesssim \left\| \int_v z^2 |Y| dv \int_v \left| K_i^j \right|^2 |Y_j| dv \right\|_{L^1(\mathcal{S}_i^t)} \lesssim \left\| \int_v z^2 |Y| dv \right\|_{L^\infty(\mathcal{S}_i^t)} \left\| \int_v \left| K_i^j \right|^2 |Y_j| dv \right\|_{L^1(\mathcal{S}_i^t)}
\lesssim \left\| \frac{\epsilon}{\tau_+^{-(N+4)\eta}} \right\|_{L^\infty(\mathcal{S}_i^t)} (1 + t)^{\eta (-2|j| + 2|\beta| + |j|)} \mathbb{E}_L(t) \lesssim \frac{\epsilon^2}{(1 + t)^{2-8N\eta}} \lesssim \frac{\epsilon^2}{(1 + t)^{\frac{1}{2}}}
$$

To conclude the proof of Proposition 6.3 notice that for all $N - 2 \leq |\beta| \leq N$, there exists $i \in I$ verifying $\hat{Z}^\beta f = H_i + G_i$ and that $|G_i| \leq |L_i|$.

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