On weak solutions to a fractional Hardy-Hénon equation:  
Part II: Existence

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Abstract
This paper and [29] treat the existence and nonexistence of stable weak solutions to a fractional Hardy-Hénon equation  
\((-\Delta)^s u = |x|^\ell |u|^{p-1} u\) in \(\mathbb{R}^N\), where 0 < \(s < 1\), \(\ell > -2s\), \(p > 1\), \(N \geq 1\) and \(N > 2s\). In this paper, when \(p\) is critical or supercritical in the sense of the Joseph-Lundgren, we prove the existence of a family of positive radial stable solutions, which satisfies the separation property. We also show the multiple existence of the Joseph-Lundgren critical exponent for some \(\ell \in (0, \infty)\) and \(s \in (0, 1)\), and this property does not hold in the case \(s = 1\).

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1 Introduction

This paper is a continuation of [29], and we consider the existence of stable solutions for a fractional Hardy-Hénon equation
\((-\Delta)^s u = |x|^\ell |u|^{p-1} u\) in \(\mathbb{R}^N\).
(1.1)

Throughout this paper, we always assume the following condition on \(s, \ell, p, N\):
\[0 < s < 1, \quad \ell > -2s, \quad p > 1, \quad N \geq 1, \quad N > 2s.\]  
(1.2)

Here \((-\Delta)^s\) is the fractional Laplacian, which is defined for any \(\varphi \in C^\infty_c(\mathbb{R}^N)\) by
\((-\Delta)^s \varphi(x) := C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} \, dy = C_{N,s} \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} \, dy\)
for \( x \in \mathbb{R}^N \), where P.V. stands for the Cauchy principal value integral and
\[
C_{N,s} := 2^{2s} s(1-s) \pi^{-\frac{N}{2}} \Gamma\left(\frac{N}{2} + s\right) / \Gamma\left(2 - s\right)
\]
with the gamma function \( \Gamma \). For the notion of solutions and stability, see Definitions 1.1 and 1.3 below.

1.1 The case \( s = 1 \)

When \( s = 1 \), we regard \((-\Delta)^s\) in (1.1) as the usual Laplacian \(-\Delta\) and (1.1) becomes
\[
- \Delta u = |x|^{\ell} |u|^{p-1} u \quad \text{in } \mathbb{R}^N.
\]
(1.3)

Equation (1.3) is called the Lane–Emden equation (\( \ell = 0 \)), the Hénon equation (\( \ell \geq 0 \)) or the Hardy–Hénon equation (\( \ell > -2 \)) and there are a lot of works for (1.3) in which the existence/nonexistence of solutions and their properties were studied. For instance, we refer to [18,41] (\( \ell = 0 \)), [9,35,43,44] (\( \ell > -2 \)) and references therein. There are also works for (1.3) on manifolds [2–4,27,28,36] and for (1.3) with the higher order operators [12,21,24,25,33,45].

For (1.3) with \( \ell = 0 \), among other things, in the seminal work [18], Farina proved the nonexistence of nontrivial stable solutions for \( p \in (1,c(N)) \) and the existence of positive radial stable solutions for \( p \in [c(N),\infty) \), where \( c(N) \) is the Joseph–Lundgren exponent and defined by
\[
c(N) := \infty \quad \text{if } N \leq 10, \quad c(N) := \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} \quad \text{if } N \geq 11.
\]
This result is extended to the case \( \ell > -2 \) by [9,44] and in this case, the threshold \( c(N) \) is replaced by
\[
p_+(N, \ell) := \begin{cases} \frac{(N-2)^2 - 2(\ell+2)(\ell+N) + 2(\ell+2)^2(\ell+2N-2)}{(N-2)(N-4\ell-10)} & \text{if } N > 10 + 4\ell, \\ \infty & \text{if } 2 \leq N \leq 10 + 4\ell. \end{cases}
\]
(1.4)

1.2 The case \( 0 < s < 1 \)

Next, we turn to (1.1), which is a fractional counterpart of (1.3). For the nonexistence of stable solutions to (1.1), in [29], the authors of this paper addressed this issue under the condition that \( p \) is subcritical in the sense of Joseph–Lundgren (see Definition 1.2 below). For references on the nonexistence of stable solutions, we refer to [29] and references therein. In this paper, we focus on the existence of stable solutions.

To state known results of (1.1), we first remark that due to [7], the fractional Laplacian \((-\Delta)^s u\) can be expressed as the limit \(-\lim_{\ell \searrow 0} t^{1-2s} \partial_t U(x,t)\), where \( U(x,t) \) is a solution of some elliptic equation on \( \mathbb{R}^{N+1}_+ \) with \( U(x,0) = u(x) \). Exploiting this idea, we may rewrite (1.1) as the equation in \( \mathbb{R}^{N+1}_+ \) with the nonlinear boundary condition:
\[
\begin{cases}
- \text{div} \left( t^{1-2s} \nabla U \right) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\
- \lim_{t \searrow 0} t^{1-2s} \partial_t U(x,t) = \kappa_s |x|^{\ell} |U(x,0)|^{p-1} U(x,0) & \text{for } x \in \mathbb{R}^N,
\end{cases}
\]
(1.5)

where \( \kappa_s := 2^{1-2s} \Gamma(1-s)/\Gamma(s) \). In particular, when \( s = 1/2 \), (1.5) becomes
\[
- \Delta U = 0 \quad \text{in } \mathbb{R}^{N+1}_+, \quad - \lim_{t \searrow 0} t^{1-2s} \partial_t U(x,t) = \kappa_s |x|^{\ell} |U(x,0)|^{p-1} U(x,0) \quad \text{for } x \in \mathbb{R}^N.
\]
(1.6)
We first state results for \( s = 1/2 \) and \( \ell = 0 \), namely, (1.6). In [8], Chipot, Chlebík, Fila and Shafrir proved the existence of positive solutions for \( p \geq (N + 1)/(N - 1) = p_S(N,0) \), where \( p_S(N,\ell) \) is defined in (1.7). On the other hand, Quittner and Reichel [40] showed the existence of the singular solution. Moreover, in [26], under the assumption that \( p \) is either critical or supercritical in the sense of the Joseph–Lundgren, Harada showed the existence of a family \((u_\alpha)_{\alpha > 0}\) of solutions of (1.1) with the separation property \( u_{\alpha_1} < u_{\alpha_2} \) for \( \alpha_1 < \alpha_2 \) and \( u_\alpha \to u_\infty \) as \( \alpha \to \infty \), where \( u_\infty \) is a singular solution of (1.1). In addition, though it is not explicitly stated in [26], the family \((u_\alpha)_{\alpha > 0}\) becomes stable. We also refer to [11] for the notion of the Joseph–Lundgren exponent when \( s = 1/2 \) and \( \ell = 0 \).

In the case \( s \in (0,1) \) with \( \ell = 0 \), the existence of a family \((u_\alpha)_{\alpha > 0}\) of positive radial stable solutions of (1.1) was proved in [13]. Moreover, we remark that the existence of singular solutions was not dealt. We say that

\[ u \text{ becomes stable.} \]

Remark 1.1. In [29], we treat solutions in \( H^s_{\text{loc}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} dx) \), where

\[ L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} dx) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \mid \int_{\mathbb{R}^N} (1 + |x|)^{-N-2s} |u(x)| \, dx < \infty \right\}. \]

However, since we deal with singular solutions of (1.1) in this paper, we slightly change the notion of solutions as follows:

**Definition 1.1.** We say that \( u \) is a *solution of (1.1)* if \( u \in H^s_{\text{loc}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} dx) \) and \( u \) satisfies

\[ |x|^{\ell}|u|^{p-1} u \in L^{\frac{2N}{N+2s}}_{\text{loc}}(\mathbb{R}^N), \quad \langle u, \varphi \rangle_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |x|^{\ell}|u|^{p-1} u \varphi \, dx \quad \text{for each } \varphi \in C_c^\infty(\mathbb{R}^N). \]

**Remark 1.1.**

1. For later use, we require the stronger condition \( |x|^{\ell}|u|^{p-1} u \in L^{\frac{2N}{N+2s}}_{\text{loc}}(\mathbb{R}^N) \) than \( |x|^{\ell}|u|^{p-1} u \in L^{\frac{2N}{N+2s}}_{\text{loc}}(\mathbb{R}^N) \) in Definition 1.1.
Since \( \ell > -2s \) and \( N > 2s \) hold by (1.2), we see that if \( u \in L^\infty_{\text{loc}}(\mathbb{R}^N) \), then \( |x|^\ell |u|^{p-1} u \in L^{\frac{2N}{N+2\ell}}_{\text{loc}}(\mathbb{R}^N) \). We remark that in Theorem 1.1, we find positive solutions of (1.1) belonging to \( H^s_{\text{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N) \).

Next, following [11,13,26], we define subcritical, critical and supercritical in the sense of Joseph–Lundgren. For this purpose, we set

\[
p_S(N, \ell) := \frac{N + 2s + 2\ell}{N - 2s}, \quad \lambda(\alpha) := 2^{2s} \Gamma\left(\frac{N + 2s + 2\alpha}{4}\right) \Gamma\left(\frac{N + 2s - 2\alpha}{4}\right) \Gamma\left(\frac{N - 2s + 2\alpha}{4}\right) \Gamma\left(\frac{N - 2s - 2\alpha}{4}\right).
\]

(1.7)

**Definition 1.2.** We say that \( p \) is **subcritical in the sense of the Joseph–Lundgren** (for short, we write **JL-subcritical**) if

\[
either 1 < p \leq p_S(N, \ell) \quad \text{or} \quad p > p_S(N, \ell), \quad p\lambda\left(\frac{N - 2s}{2} - 2s + \ell \right) > \lambda(0).
\]

On the other hand, \( p \) is said to be **supercritical in the sense of the Joseph–Lundgren** (for short, **JL-supercritical**) provided

\[
p > p_S(N, \ell), \quad p\lambda\left(\frac{N - 2s}{2} - 2s + \ell \right) < \lambda(0).
\]

We also call \( p \) **critical in the sense of the Joseph–Lundgren** (for short, **JL-critical**) when

\[
p > p_S(N, \ell), \quad p\lambda\left(\frac{N - 2s}{2} - 2s + \ell \right) = \lambda(0).
\]

We remark that Definition 1.2 coincides with those in [11,13,26] when \( \ell = 0 \).

Finally, we introduce the notation of stability of solutions of (1.1):

**Definition 1.3.** Let \( u \) be a solution of (1.1). We say that \( u \) is **stable** if \( u \) satisfies

\[
p \int_{\mathbb{R}^N} |x|^\ell |u|^{p-1} \varphi^2 \, dx \leq \|\varphi\|^2_{H^s(\mathbb{R}^N)} \quad \text{for every} \quad \varphi \in C_c^\infty(\mathbb{R}^N),
\]

where

\[
\|\varphi\|^2_{H^s(\mathbb{R}^N)} := \int_{\mathbb{R}^N} (2\pi |\xi|^2)^s |\mathcal{F}\varphi(\xi)|^2 \, d\xi = \frac{C_{N,s}}{2} \|\varphi\|^2_{H^s(\mathbb{R}^N)},
\]

\[
\mathcal{F}\varphi(\xi) := \int_{\mathbb{R}^N} e^{-2\pi i x \cdot \xi} \varphi(x) \, dx, \quad \|\varphi\|^2_{H^s(\mathbb{R}^N)} := \int_{\mathbb{R}^N \times \mathbb{R}^N} (\varphi(x) - \varphi(y))^2 \, dx \, dy.
\]

Now we are in position to state our main results in this paper:

**Theorem 1.1.** Suppose (1.2) and put

\[
\theta_0 := \frac{2s + \ell}{p - 1}.
\]

(1.8)

Assume \( p \) is JL-critical or JL-supercritical, namely

\[
p > p_S(N, \ell), \quad p\lambda\left(\frac{N - 2s}{2} - \theta_0 \right) \leq \lambda(0).
\]

(1.9)

Then the following hold:
(i) There exists a singular stable solution \( u_S(x) := A_0|x|^{-\theta_0} \in H_\text{loc}^s(\mathbb{R}^N) \) of \( (1.1) \) with \( u_S \notin H^s(\mathbb{R}^N) \), where

\[
A_0 := \lambda \left( \frac{N - 2s}{2} - \theta_0 \right)^{\frac{1}{p-1}}.
\]

(ii) There exists a family \( (u_\alpha)_{\alpha > 0} \subset H_\text{loc}^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N) \) of solutions of \( (1.1) \) such that

(a) \( u_\alpha \) is positive, radial, stable and \( u_\alpha(x) < u_S(x) \) for each \( x \in \mathbb{R}^N \setminus \{0\} \),
(b) for each \( \alpha > 0 \) and \( x \in \mathbb{R}^N \), \( u_\alpha(0) = \alpha \) and \( u_\alpha(x) = \alpha u_1(\alpha^{1/\theta_0}x) \),
(c) for every \( \alpha < \beta \) and \( x \in \mathbb{R}^N \), \( u_\alpha(x) < u_\beta(x) \),
(d) if \( u \in H_\text{loc}^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) is a positive radial stable solution of \( (1.1) \) satisfying \( u(x) \to 0 \) as \( |x| \to \infty \), then \( u = u_\alpha \) with \( u(0) = \alpha \),
(e) As \( \alpha \to \infty \), \( u_\alpha(x) \to u_S(x) \) for each \( x \in \mathbb{R}^N \setminus \{0\} \) and \( u_\alpha(x) \to u_S(x) \) in \( H_\text{loc}^s(\mathbb{R}^N) \).

Next we look at the range of \( p \) in which \( p \) is JL-subcritical, JL-critical and JL-supercritical for the case \( \ell \leq 0 \):

**Theorem 1.2.** Suppose \( (1.2) \). Then the following hold:

(i) Assume \(-2s \leq \ell \leq 0\). Then there exists a unique \( p_{JL} \in (p_S(N, \ell), \infty] \) such that each \( p \in (1, p_{JL}) \) is JL-subcritical. In addition, if \( p_{JL} < \infty \), then \( p = p_{JL} \) (resp. \( p > p_{JL} \)) is JL-critical (resp. JL-supercritical);

(ii) Assume \( \ell = 0 \).

(a) When \( N = 1 \) with \( 0 < s < 1/2 \) and \( 2 \leq N \leq 7 \) with \( 0 < s < 1 \), \( p_{JL} = \infty \) holds.
(b) When \( N = 8, 9 \), there exists a unique \( s_N \in (0, 1) \) such that \( p_{JL} < \infty \) if \( 0 < s < s_N \) and \( p_{JL} = \infty \) if \( s_N \leq s < 1 \).
(c) When \( N \geq 10 \) with \( 0 < s < 1 \), \( p_{JL} < \infty \) always holds;

(iii) Assume \(-2 \leq \ell < 0\).

(a) When \( N = 1 \), there exists an \( s_{1, \ell} \in (-\ell/2, 1/2) \) such that \( p_{JL} < \infty \) if \(-\ell/2 < s < s_{1, \ell} \) and \( p_{JL} = \infty \) if \( s_{1, \ell} < s < 1/2 \).
(b) When \( 2 \leq N < 10 + 4\ell \), there exist \( s_{N, \ell}, \tilde{s}_{N, \ell} \in (-\ell/2, 1) \) with \( s_{N, \ell} < \tilde{s}_{N, \ell} \), \( s_{N, \ell} \leq \tilde{s}_{N, \ell} \) such that \( p_{JL} < \infty \) if \(-\ell/2 < s < s_{N, \ell} \) and \( p_{JL} = \infty \) if \( \tilde{s}_{N, \ell} < s \leq 1 \). Moreover, \( s_{N, \ell} = \tilde{s}_{N, \ell} \) holds when \( 2 \leq N \leq 6 \).
(c) When \( N \geq 10 + 4\ell, p_{JL} < \infty \) holds for \(-\ell/2 < s < 1 \).

Theorem 1.2 is also useful for \( \ell > 0 \). To see this, we introduce the following condition on \( N \) and \( s \):

(A) \( N = 1 \) with \( 0 < s < 1/2 \), or \( 2 \leq N \leq 7 \) with \( 0 < s < 1 \), or \( N = 8, 9 \) with \( s_N \leq s < 1 \).

By Theorem 1.2 (ii), \( p_{JL} = \infty \) and each \( p \in (1, \infty) \) is JL-subcritical under (A) and \( \ell = 0 \). As a corollary of Theorem 1.2, we obtain the following result in which we also compare \( s_N \) and \( s_{N, \ell} \) for the case \( \ell < 0 \) and \( N = 8, 9 \):

**Corollary 1.1.** Suppose \( (1.2) \), and let \( s_N \) and \( s_{N, \ell} \) be the numbers in Theorem 1.2.

(i) Assume \( (A) \) and \( \ell > 0 \). Then every \( p \in (1, \infty) \) is JL-subcritical.
(ii) Assume $-2s < \ell < 0$ and $N = 8,9$ with $0 < s < s_N$ and $N < 10 + 4\ell$. Then $p_{JL} < \infty$.
Therefore, $s_N \leq s_{N,\ell}$ holds.

Finally, we focus on the case $0 < s < 1$ and $\ell > 0$. Under a certain condition, we show that
there are at least two JL-critical exponents (Theorem 1.3 (ii)), which never occurs in the case $s = 1$.

**Theorem 1.3.** Let $N \geq 8$. Then there exist $s_N \in (0,1)$, $(\ell_{1,s})_{0<s<s_N}$ and $(\ell_{2,s})_{0<s<s_N}$ with

- if $s \in (0, s_N)$ and $\ell \in (0, \ell_{1,s})$, then there exist $p_i = p_i(N, s, \ell)$ ($i = 1,2$) such that
  - (a) $p_{S}(N,\ell) < p_1 \leq p_2 < \infty$,
  - (b) every $p \in (1,p_1)$ is JL-subcritical,
  - (c) each $p \in (p_2,\infty)$ is JL-supercritical,
  - (d) $p_1$ and $p_2$ are JL-critical,

and

- if $s \in (0, s_N)$ and $\ell \in (\ell_{1,s},\ell_{2,s})$, then there exist $p_i = p_i(N, s, \ell)$ ($i = 1,2,3,4$) such that
  - (a) $p_{S}(N,\ell) < p_1 \leq p_2 < p_3 \leq p_4 < \infty$,
  - (b) every $p \in (1,p_1) \cup (p_4,\infty)$ is JL-subcritical,
  - (c) each $p \in (p_2,p_3)$ is JL-supercritical,
  - (d) $p_1,p_2,p_3$ and $p_4$ are JL-critical.

In addition, there exist $(\ell_{3,s})_{0<s<1}$ when $N \geq 10$ and $(\ell_{3,s})_{0<s<s_N}$ when $N = 8,9$ with $\ell_{3,s} = \ell_{3}(N, s)$ such that if $\ell \geq \ell_{3,s}$, then each $p > 1$ is JL-subcritical.

**Remark 1.2.** We can also show that for any given $(\ell, s) \in (-2,\infty) \times (0,1)$ with $-2s < \ell$, there
exist $N_{s}(\ell, s) \in \mathbb{N}$ and $p_{s}(\ell, s) > 1$ such that if $N \geq N_{s}(\ell, s)$, then every $p \in (p_{s}(\ell, s),\infty)$ is
JL-supercritical. See Lemma 5.5 and Remark 5.1.

### 1.4 Comments

We first compare our results with the literature and give some remarks on the results. About the
existence of the family of positive radial stable solutions, Theorem 1.1 is an extension of [13, 26].
In [26], the case $s = 1/2$ and $\ell = 0$ was considered and a similar result to Theorem 1.1 was shown.
On the other hand, in [13], the case $0 < s < 1$ and $\ell = 0$ was considered, however, the properties (c),
(d) and (e) in Theorem 1.1 were not proved (see [13, section 7]). Therefore, Theorem 1.1 (c)–(e) are
new even in the case $s \neq 1/2$ and $\ell = 0$, and we obtain a counterpart of [26] for the case $0 < s < 1$
and $-2s < \ell$. Here we point out that for the case $s = 1$, the separation property of positive radial
stable solutions was shown in [35, 43]. In addition, when $\ell = 0$, the separation property is useful in
the study of the corresponding parabolic problem. For instance, see [22, 23, 38, 39]. We hope that
our results in this paper might be useful in the study of parabolic problem corresponding to (1.1).

For Theorem 1.2, in [26, section 4], when $2 \leq N \leq 5$, $\ell = 0$ and $s = 1/2$, he proved $p_{JL} = \infty$.
Therefore, Theorem 1.2 (i) and (ii) generalize this result even in the case $s = 1/2$ and $\ell = 0$.
Moreover, we may also treat the case $s \in (0,1)$ and $\ell \in (-2s,0)$ in Theorem 1.2. From Theorem
1.2, we see that the case $s \in (0,1)$ and $-2s < \ell < 0$ is similar to the case $s = 1$ in the sense that
the range of $p$ for being JL-subcritical is bounded by the range of $p$ for being JL-supercritical
and there is at most one JL-critical exponent.
On the number $10 + 4\ell$ in Theorem 1.2, we recall that this is the threshold of $N$ for $p_{+}(N, \ell) = \infty$ when $s = 1$ (see [9] and (1.4)). In Section 5 (Lemma 5.4 and Remark 5.1), we observe that as $s \nearrow 1$ and $p \to \infty$, Definition 1.2 coincides with $N < 10 + 4\ell$, $N = 10 + 4\ell$ and $N > 10 + 4\ell$, respectively. Hence, we recover the case $s = 1$ from Definition 1.2. This seems not observed in the literature.

About Theorem 1.3, we find the situation which is completely different from the case $s = 1$. In Theorem 1.3 (ii), we show that there are at least two JL-critical exponents. To the best of the authors’ knowledge, this is the first result for the multiple existence of JL-critical exponents. We also remark that the range of $p$ for being JL-supercritical is bounded from above and below by the range of $p$ for being JL-subcritical. This structure is different from the case $s = 1$, too. We emphasize that this structure and the multiple existence of JL-critical exponents are subtle since these phenomena disappear when $\ell > 0$ is sufficiently large due to the last statement of Theorem 1.3.

Next, we comment on the proofs of Theorems 1.1–1.3. For Theorems 1.2 and 1.3, we first introduce the change of variables and show the concavity of some function related to the inequalities in Definition 1.2 for the case $\ell \leq 0$. We remark that this change of variables also enables us to treat the case $p = \infty$ and to find a function whose positivity (resp. negativity) is equivalent to being JL-subcritical (reps. JL-supercritical) at $p = \infty$. Through the analysis at $p = \infty$, the critical number $10 + 4\ell$ appears as $s \nearrow 1$ (see Lemma 5.4).

For Theorem 1.1, we first show the existence of the family of positive radial stable solutions satisfying Theorem 1.1 (a) and (b). To this end, as in [13], we use an idea in [6] (see also [10]). More precisely, using the singular solution $u_S$ and an auxiliary function in Proposition 3.3, we construct a bounded supersolution to the equation on $B_1$ (see (3.1) and (3.3)). Then we find a family of minimal radial solutions of (3.3) and show the existence of bounded positive stable solutions of (1.1) via the family of minimal solutions. We remark that in [13, 26], they considered the equation on $\mathbb{R}^{N+1}$, namely, (1.5). On the other hand, we mainly study the equation on $B_1 \subset \mathbb{R}^N$. Since we study the case $\ell < 0$, weak solutions and unbounded supersolutions on $B_1$, (3.1) or (3.3) seems simple to treat. Therefore, the details are different from [13, 26].

After the existence of the family of solutions, we show the separation property (Proposition 4.1) for positive radial stable solutions of (1.1), which is a key to prove Theorem 1.1 (c)–(e). Here we argue in the spirit of [26, section 7]. As mentioned in the above, [26] treated the case $s = 1/2$ with $\ell = 0$ and the extension problem (1.6) was considered. Since $U$ in (1.6) is harmonic in $\mathbb{R}_+^{N+1}$, he could use the analyticity of solutions to (1.6) on $\mathbb{R}^{N+1}_+$ as well as the properties of the spherical harmonic functions for proving the separation property. On the other hand, we consider (1.5) and clearly, the analyticity of solutions to (1.5) on $\mathbb{R}^{N+1}_+$ cannot be used. Thus, we need to modify the argument in [26, section 7]. Though our proof is lengthy compared to [26, section 7], we stress that our arguments are mostly elementary.

To end this introduction, we list some open problems related to our results:

(i) Let $u$ be a radial stable solution of (1.1). Then does $u \geq 0$ (resp. $u \leq 0$) in $\mathbb{R}^N$ hold?

(ii) Let $u$ be a positive radial solution with $\lim_{r \to 0} u(r) = \infty$. Then does $u = u_S$ hold?

When $s = 1$ and $\ell = 0$, (i) was proved in [18, Theorem 5]. On the other hand, when $s = 1$, (ii) was addressed in [9, Theorem 2.4]. In both proofs, the ODE techniques were used. Thus, we cannot apply the arguments to the case $s \in (0, 1)$ directly and need another idea. For (ii), we remark that we show $u = u_S$ under additional assumptions in Corollary 4.2. It is not clear that we may remove these additional assumptions. Finally, as mentioned above, when $s = 1$, the assertions in Theorem 1.1 are useful in the study of the corresponding parabolic problem. Therefore, it is interesting to study the parabolic problem corresponding to (1.1).
This paper is organized as follows. In Section 2, we study the existence of the singular solution and its property, and prove Theorem 1.1 (i). In Section 3, we construct the family of positive radial stable solutions. Section 4 is devoted to the proof of the separation property (Proposition 4.1) and its consequences. After that, we complete the proof of Theorem 1.1. In Section 5, we prove Theorem 1.2, Corollary 1.1 and Theorem 1.3. Appendices contain proofs of some technical lemmata.

2 Proof of Theorem 1.1 (i)

In this section, we prove Theorem 1.1 (i).

Proposition 2.1. Suppose (1.2). Let \( u_S(x) := A_0|x|^{-\theta_0} \), where \( \theta_0 \) and \( A_0 \) are as in (1.8) and (1.10), respectively.

(i) Let \( p > (N + \ell)/(N - 2s) \). Then \( u_S \) is a singular solution of (1.1) in the sense of \( \mathcal{S}^* \), where \( \mathcal{S} \) denotes the Schwartz space.

(ii) Let \( p > p_S(N, \ell) \). Then \( u_S \in H^s_{\text{loc}}(\mathbb{R}^N) \), \( u_S \not\in \dot{H}^s(\mathbb{R}^N) \) and \( u_S \) is a solution of (1.1) in the sense of Definition 1.1.

Proof. (i) For \( 0 < \alpha < N \), we set

\[
\gamma(\alpha) := \frac{\pi^N 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{N - \alpha}{2}\right)}.
\]

It is known that (see, e.g., [42, Chapter V])

\[
\mathcal{F}\left[|x|^{-\alpha}\right](\xi) = \gamma(N - \alpha)(2\pi|\xi|)^{-(N - \alpha)} \quad \text{in } (\mathcal{S})^*.
\]

(2.1)

Notice that for \( \varphi \in \mathcal{S} \), \( (-\Delta)^s \varphi = (-\Delta)^s \mathcal{F}\varphi \) and

\[
(-\Delta)^s \varphi = \mathcal{F}^{-1}\left((2\pi|\xi|)^{2s} \mathcal{F}\varphi\right) = \mathcal{F}\left((2\pi|\xi|)^{2s} \mathcal{F}^{-1}\varphi\right).
\]

(2.2)

Thus, it suffices to show

\[
(2\pi|\xi|)^{2s} \mathcal{F} u_S(\xi) = \mathcal{F}\left(|x|^{\ell} u_S^p(x)\right)(\xi) \quad \text{in } (\mathcal{S})^*.
\]

Since it follows from (1.2) and \( p > (N + \ell)/(N - 2s) \) that \( 0 < \theta_0 < N \), by (2.1) with \( \alpha = \theta_0 \) we observe that

\[
(2\pi|\xi|)^{2s} \mathcal{F} u_S(\xi) = A_0 \gamma (N - \theta_0) (2\pi|\xi|)^{-(N - \theta_0) + 2s} = A_0 \gamma (N - \theta_0) (2\pi|\xi|)^{-(N - \frac{2sp + \ell}{p - 1})}.
\]

(2.3)

On the other hand, since

\[
|x|^{\ell} u_S^p(x) = A_0^p|x|^{\ell - p\theta_0} = A_0^p|x|^{-\frac{\ell + 2sp}{p - 1}},
\]

and it follows from \( p > (N + \ell)/(N - 2s) \) that \( (2sp + \ell)/(p - 1) < N \), by (2.1) with \( \alpha = (2sp + \ell)/(p - 1) \) we have

\[
\mathcal{F}\left(|x|^{\ell} u_S^p\right)(\xi) = A_0^p \gamma \left(N - \frac{\ell + 2sp}{p - 1}\right)(2\pi|\xi|)^{-(N - \frac{\ell + 2sp}{p - 1})}.
\]

(2.4)

Thus, by (1.1), (2.3) and (2.4), (2.2) holds provided

\[
A_0^{p - 1} = \frac{\gamma (N - \theta_0)}{\gamma \left(N - \frac{\ell + 2sp}{p - 1}\right)}.
\]
Recalling the definitions of $\gamma$ and $\lambda$, we have
\[
\frac{\gamma (N - \theta_0)}{\gamma (N - \frac{\ell + 2sp}{p-1})} = 2^{2s} \frac{\Gamma \left( \frac{1}{2} (N - \theta_0) \right) \Gamma \left( \frac{2sp + \ell}{2(p-1)} \right)}{\Gamma \left( \frac{1}{2} (N - \frac{2sp + \ell}{p-1}) \right) \Gamma \left( \frac{\theta_0}{2} \right)} = \lambda \left( \frac{N - 2s - \theta_0}{2} \right) = A_0^{p-1},
\]
\[\text{hence, (2.2) and (i) hold.}\]

(ii) We first remark that $p > p_S(N, \ell)$ yields
\[
\frac{\ell + 2sp}{p-1} < N, \quad |x|^\ell u_S^p = A_0^{p} |x|^{-\frac{\ell + 2sp}{p-1}} \in L^2_{loc}(\mathbb{R}^N), \quad u_S \in L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s}).
\]

Next, we prove $u_S \in H^s_{loc}(\mathbb{R}^N)$. Let $\varphi \in \mathcal{S}$. We shall show $(1 + |\xi|^2)^s |\mathcal{F} [\varphi u_S](\xi)|^2 \in L^1(\mathbb{R}^N)$. Since $\mathcal{F} u_S(\xi) = C|\xi|^{-N+\theta_0}$ for some $C > 0$ and $\theta_0 < N$, we have $\mathcal{F} [\varphi u_S] = (\mathcal{F} \varphi \ast \mathcal{F} u_S) \in L^\infty(\mathbb{R}^N)$. Moreover, it follows from (1.2) and $p > p_S(N, \ell)$ that $\theta_0 < (N - 2s)/2$ and
\[
-N < -N + \theta_0 < -\frac{N}{2} - s.
\]

Then it is easily seen that there exists a $C_0 > 0$ such that
\[
|(\mathcal{F} \varphi \ast \mathcal{F} u_S)(\xi)| \leq C_0 (1 + |\xi|)^{-N+\theta_0} \quad \text{for all } \xi \in \mathbb{R}^N.
\]

By (2.6) one sees that
\[
2s + 2 (-N + \theta_0) < -N, \quad (1 + |\xi|^2)^s |\mathcal{F} [\varphi u_S](\xi)|^2 \in L^1(\mathbb{R}^N).
\]

This implies $\varphi u_S \in H^s(\mathbb{R}^N)$, and we obtain $u_S \in H^s_{loc}(\mathbb{R}^N)$.

Next, we prove $u_S \notin H^s(\mathbb{R}^N)$. Since $\mathcal{F} u_S(\xi) = C|\xi|^{-N+\theta_0}$ for some $C > 0$, by considering $|\xi|^{2s} |\mathcal{F} u_S(\xi)|^2$ and looking at the index, we get
\[
2s - 2N + 2\theta_0 < 2s - 2N + (N - 2s) = -N, \quad |\xi|^{2s} |\mathcal{F} u_S(\xi)|^2 \notin L^1(B_1).
\]

Thus, $u_S \notin H^s(\mathbb{R}^N)$.

Finally, by [29, Lemma 2.1], $u_S \in H^s_{loc}(\mathbb{R}^N)$ and (2.5), we observe that for any $\varphi \in C_0^\infty(\mathbb{R}^N)$,
\[
\int_{\mathbb{R}^N} |x|^\ell u_S^p \varphi \, dx = \langle (-\Delta)^s u_S, \varphi \rangle_{\mathcal{F} \ast \mathcal{F}} = \int_{\mathbb{R}^N} u_S (-\Delta)^s \varphi \, dx = \langle u_S, \varphi \rangle_{\dot{H}^s(\mathbb{R}^N)}.
\]

Thus, $u_S$ is a solution of (1.1) in the sense of Definition 1.1 and we complete the proof. \hfill\Box

\textbf{Proof of Theorem 1.1 (i).} By Proposition 2.1 it is enough to show that $u_S$ is stable, that is,
\[
\| \varphi \|^2_{\dot{H}^s(\mathbb{R}^N)} - p \int_{\mathbb{R}^N} |x|^\ell u_S^{p-1} \varphi^2 \, dx \geq 0 \quad \text{for every } \varphi \in C_0^\infty(\mathbb{R}^N).
\]

To this end, we first recall the Hardy type inequality (see, e.g., [20, 30, 46])
\[
\lambda(0) \int_{\mathbb{R}^N} |x|^{-2s} \varphi^2 \, dx \leq \int_{\mathbb{R}^N} |\xi|^{2s} |F [\varphi](\xi)|^2 \, d\xi,
\]
where
\[
F [\varphi](\xi) := (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-ix \cdot \xi} \varphi(x) \, dx.
\]
By $F[\varphi](\xi) = (2\pi)^{-N/2} \mathcal{F} \varphi (\xi/(2\pi))$ and
\[ \int_{\mathbb{R}^N} |\xi|^{2s} |F[\varphi](\xi)|^2 \, d\xi = \int_{\mathbb{R}^N} (2\pi |\xi|)^{2s} |\mathcal{F} \varphi(\xi)|^2 \, d\xi = \|\varphi\|_{H^s(\mathbb{R}^N)}^2, \] (2.9)
it follows from (2.8) and (2.9) that
\[ \lambda(0) \int_{\mathbb{R}^N} |x|^{-2s} \varphi^2 \, dx \leq \|\varphi\|_{H^s(\mathbb{R}^N)}^2 \text{ for each } \varphi \in C_0^\infty(\mathbb{R}^N). \] (2.10)

On the other hand, since
\[ |x|^s u^{p-1}_S(x) = A_0^{p-1} |x|^{-2s} = \lambda \left( \frac{N - 2s}{2} - \theta_0 \right) |x|^{-2s}, \]
we observe from (1.9) and (2.10) that
\[ \|\varphi\|_{H^s(\mathbb{R}^N)}^2 - p \int_{\mathbb{R}^N} |x|^s u^{p-1}_S \varphi^2 \, dx = \|\varphi\|_{H^s(\mathbb{R}^N)}^2 - p \lambda \left( \frac{N - 2s}{2} - \theta_0 \right) \int_{\mathbb{R}^N} |x|^{-2s} \varphi^2 \, dx \geq \|\varphi\|_{H^s(\mathbb{R}^N)}^2 - \lambda(0) \int_{\mathbb{R}^N} |x|^{-2s} \varphi^2 \, dx \geq 0. \]

Hence, (2.7) holds and we complete the proof. \(\square\)

3 Construction of a family to stable radial solutions

In this section, we construct stable solutions of (1.1). Throughout this section, we assume (1.9).
Let $\mu \in (0, 1)$ and $u_S$ be a singular solution of (1.1) in Theorem 1.1 (i). We first consider the following equation:
\[ (-\Delta)^s u = |x|^s |u|^{p-1} u \text{ in } B_1, \quad u = \mu \eta_\mu u_S \text{ in } \mathbb{R}^N \setminus B_1, \] (3.1)
where $\eta_\mu \in C_c^\infty(\mathbb{R}^N)$ is a cut-off function satisfying
\[ \eta_\mu(x) := \eta_0 ((1 - \mu)x), \quad \eta_0(x) = \eta_0(|x|), \quad 0 \leq \eta_0(r) \leq 1 \text{ in } [0, \infty), \]
\[ \eta_0(r) \equiv 1 \text{ in } [0, 1], \quad \eta_0(r) \equiv 0 \text{ in } [2, \infty), \quad \eta'_0(r) \leq 0 \text{ in } [0, \infty). \] (3.2)

Remark that $\eta_\mu u_S \in H^s(\mathbb{R}^N)$ and $\eta_\mu(x) \to 1$ as $\mu \nearrow 1$.

In what follows, we only treat the existence of positive solutions. Therefore, we replace $|u|^{p-1} u$ in (3.1) by $u^+_p$ and consider
\[ (-\Delta)^s u = |x|^s u^+_p \text{ in } B_1, \quad u = \mu \eta_\mu u_S \text{ in } \mathbb{R}^N \setminus B_1. \] (3.3)

First we give the definition of solutions, supersolutions and subsolutions of (3.3). Remark that $L^\infty(\mathbb{R}^N) \subset L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} \, dx)$.

**Definition 3.1.** We say that $u$ is a solution of (3.3) if $u$ satisfies
\[ u \in H^s_{\text{loc}}(\mathbb{R}^N), \quad u = \mu \eta_\mu u_S \text{ in } \mathbb{R}^N \setminus B_1, \quad |x|^s u^+_p \in L^{2N/(2N-1)}(B_1), \]
\[ \langle u, \varphi \rangle_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |x|^s u^+_p \varphi \, dx \text{ for every } \varphi \in C_c^\infty(B_1). \]
Similarly, \( u \) is said to be a **supersolution of (3.3)** (resp. **subsolution**) if \( u \) satisfies

\[
\begin{align*}
&u \in H^s_{\text{loc}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} \, dx), \quad |x|^\ell u^p_+ \in L^{\frac{2N}{N+s}}(B_1), \\
&u \geq \mu \eta_{\mu} u_S \quad \text{in } \mathbb{R}^N \setminus B_1 \quad \text{(resp. } u \leq \mu \eta_{\mu} u_S \text{ in } \mathbb{R}^N \setminus B_1), \\
&\langle u, \varphi \rangle_{H^s(\mathbb{R}^N)} \geq \int_{\mathbb{R}^N} |x|^\ell u^p_+ \varphi \, dx \quad \text{for all } \varphi \in C^\infty_c(B_1) \text{ with } \varphi \geq 0 \\
&\left(\text{resp. } \langle u, \varphi \rangle_{H^s(\mathbb{R}^N)} \leq \int_{\mathbb{R}^N} |x|^\ell u^p_+ \varphi \, dx \quad \text{for all } \varphi \in C^\infty_c(B_1) \text{ with } \varphi \geq 0\right).
\end{align*}
\]

**Remark 3.1.**

(i) Replacing \( u^p_+ \) by \(|u|^{p-1}u\) in Definition 3.1, we may define the notation of solutions, supersolutions and subsolutions of (3.1).

(ii) Since \( p > p_S(N, \ell) \) implies \(|x|^\ell u^p_+ \in L^{\frac{2N}{N+s}}(B_1)\), it is easily seen that \( u_S \) is a supersolution of (3.3) for all \( \mu \in (0, 1] \).

(iii) For solutions \( u \) of (3.1), we require \( u \) to satisfy \( u = \mu \eta_{\mu} u_S \) in \( \mathbb{R}^N \setminus B_1 \). Hence, \( u \in H^s(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} \, dx) \).

(iv) Since \( H^s(\mathbb{R}^N) \subset L^{2s}(\mathbb{R}^N) \) with \( 2s := (2N)/(N-2s) \), \( L^{2s}(B_1) = (L^{\frac{2N}{N+s}}(B_1))^s \) and we require the integrability condition \(|x|^\ell u^p_+ \in L^{\frac{2N}{N+s}}(B_1)\), in Definition 3.1, we may replace \( C^\infty_c(B_1) \) by \( \mathcal{H}_0^s(B_1) \), where

\[
\mathcal{H}_0^s(B_1) := \left\{ u \in H^s(\mathbb{R}^N) \mid u \equiv 0 \quad \text{in } \mathbb{R}^N \setminus B_1 \right\}.
\]

We remark that the following may be checked:

\[
\mathcal{H}_0^s(B_1) = C^\infty_c(B_1)^{\|\cdot\|_{H^s(\mathbb{R}^N)}}.
\]

In order to prove the existence of solutions of (3.1), we shall find a bounded supersolution of (3.3) for \( 0 < \mu < 1 \). To this end, we need some preparations.

**Proposition 3.1.** **Suppose that** \( u, v \in H^s_{\text{loc}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} \, dx) \) **satisfy**

\[
\begin{align*}
&u \leq v \quad \text{in } \mathbb{R}^N \setminus B_1, \quad \langle u, \varphi \rangle_{H^s(\mathbb{R}^N)} \leq \langle v, \varphi \rangle_{H^s(\mathbb{R}^N)} \quad \text{for all } \varphi \in \mathcal{H}_0^s(B_1) \text{ with } \varphi \geq 0 \text{ in } B_1. \quad (3.4)
\end{align*}
\]

**Then** \( u \leq v \) **in** \( \mathbb{R}^N \). **In particular, if** \( v \) **is any supersolution of (3.3), then** \( v \geq 0 \) **in** \( \mathbb{R}^N \).

**Remark 3.2.** **By** [29, Lemma 2.1] **and the density argument,** \( \langle u, \varphi \rangle_{H^s(\mathbb{R}^N)} \in \mathbb{R} \) **for all** \( u \in H^s_{\text{loc}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} \, dx) \) **and** \( \varphi \in \mathcal{H}_0^s(B_1) \).

**Proof of Proposition 3.1.** **Set** \( w := u - v \). **Remark that** \((a_+ - b_+)^2 \leq (a_+ - b_-)(a - b) \leq (a - b)^2 \) **for all** \( a, b \in \mathbb{R} \). **Thus,** \( w_+ \in \mathcal{H}_0^s(B_1) \) **holds by the assumption and**

\[
0 \geq \langle w, w_+ \rangle_{H^s(\mathbb{R}^N)} \geq \|w_+\|^2_{H^s(\mathbb{R}^N)}. \quad (3.5)
\]

**Hence,** \( w_+ \equiv 0 \) **in** \( \mathbb{R}^N \) **and** \( u \leq v \) **in** \( \mathbb{R}^N \) **holds.**

Next, **let** \( u \equiv 0 \). **If** \( v \) **is any supersolution of (3.3), then for every** \( \varphi \in \mathcal{H}_0^s(B_1) \) **with** \( \varphi \geq 0 \) **in** \( B_1 \),

\[
\langle v, \varphi \rangle_{H^s(\mathbb{R}^N)} \geq \int_{\mathbb{R}^N} |x|^\ell v^p_+ \varphi \, dx \geq 0 = \langle u, \varphi \rangle_{H^s(\mathbb{R}^N)}, \quad v \geq \mu \eta_{\mu} u_S \geq 0 = u \quad \text{in } \mathbb{R}^N \setminus B_1.
\]

**Since** \( u \) **and** \( v \) **satisfy** (3.4), **we have** \( 0 = u(x) \leq v(x) \) **in** \( \mathbb{R}^N \). \( \square \)
Next we recall the strong maximum principle for the fractional Laplacian.

**Proposition 3.2.** ([31, Theorem 1.1] and [37, Theorem 4.1]) Let \( \Omega \subset \mathbb{R}^N \) be a domain, \( c \in L^\infty(\Omega) \) and \( u \in \mathcal{H}^s(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, (1 + |x|)^{-N-2s} \, dx) \) satisfy

\[
    u \geq 0 \quad \text{in} \quad \mathbb{R}^N, \quad \langle u, \varphi \rangle_{\mathcal{H}^s(\mathbb{R}^N)} \geq \int_\Omega c(x) u \varphi \, dx \quad \text{for any} \quad \varphi \in C^\infty_c(\Omega) \quad \text{with} \quad \varphi \geq 0.
\]

Then either \( u \equiv 0 \) in \( \Omega \) or else \( \inf_{K} u > 0 \) for each compact \( K \subset \Omega \). In particular, if \( u \neq 0 \), \( u \geq 0 \) in \( \mathbb{R}^N \) and \( \langle u, \varphi \rangle_{\mathcal{H}^s(\mathbb{R}^N)} \geq 0 \) for every \( \varphi \in C^\infty_c(\Omega) \) with \( \varphi \geq 0 \), then \( u > 0 \) a.e. in \( \Omega \).

Next, to construct a bounded supersolution of (3.3), we follow the scheme in [6] and show

**Proposition 3.3.** For each \( \mu \in (0, 1) \), there exists \( \Phi_\mu : [0, \infty) \to [0, \infty) \) satisfying the following:

(i) \( \Phi_\mu \in W^{2, \infty}((0, \infty)) \);

(ii) \( \Phi_\mu'(u) \) is nonincreasing on \([0, \infty)\) and \( \Phi_\mu'(u) > 0 \) in \([0, \infty)\);

(iii) \( \Phi_\mu(u) = u \) if \( u \in [0, M_0] \), where \( M_0 := u_S(1) \);

(iv) For each \( u \in [0, \infty) \), \( \Phi_\mu'(u)u^p \geq \mu^{p-1}\Phi_\mu(u) \).

**Proof.** Fix \( \varphi_0 \in C^1([0, M_0 + 1]) \) so that \( \varphi_0(u) = 1 \) on \([0, M_0]\), \( \varphi_0 \) is nonincreasing in \([M_0, M_0 + 1]\) and \( \varphi_0(M_0 + 1) = 0 \). Remark that due to \( \mu \in (0, 1) \) and \( p > 1 \),

\[
    \varphi_0(u)u^p > \mu^{p-1}u^p = \mu^{p-1}\left(\int_0^u \varphi_0(t) \, dt\right)^p \quad \text{for all} \quad u \in (0, M_0].
\]

We set

\[
    \zeta_\mu := \sup\left\{ u \in [M_0, M_0 + 1] \mid \varphi_0(t)t^p > \mu^{p-1}\left(\int_0^t \varphi_0(\tau) \, d\tau\right)^p \quad \text{for each} \quad t \in [M_0, u]\right\}.
\]

By \( \varphi_0(M_0 + 1) = 0 \) and the monotonicity of \( \varphi_0 \), we see that

\[
    M_0 < \zeta_\mu < M_0 + 1, \quad \varphi_0(\zeta_\mu)c_\mu^p = \mu^{p-1}\left(\int_0^{\zeta_\mu} \varphi_0(\tau) \, d\tau\right)^p,
\]

\[
    \varphi_0(u)u^p \geq \mu^{p-1}\left(\int_0^u \varphi_0(t) \, dt\right)^p \quad \text{for any} \quad u \in [0, \zeta_\mu], \quad \varphi_0(\zeta_\mu) < 1. \quad (3.6)
\]

Now we define \( \Phi_\mu(u) \) by

\[
    \Phi_\mu(u) := \begin{cases} 
    \int_0^u \varphi_0(t) \, dt & \text{if} \quad 0 \leq u \leq \zeta_\mu, \\
    \Psi_\mu(u) & \text{if} \quad \zeta_\mu < u,
    \end{cases} \quad (3.7)
\]

where \( \Psi_\mu(u) \) is a unique solution of

\[
    \Psi_\mu'(u) = \mu^{p-1}\Psi_\mu^p(u)u^{-p} \quad \text{in} \quad (\zeta_\mu, \infty), \quad \Psi_\mu(\zeta_\mu) = \int_0^{\zeta_\mu} \varphi_0(\tau) \, d\tau. \quad (3.8)
\]

Rewriting (3.8) as

\[
    \frac{d}{du}\Psi_\mu^{1-p}(u) = \mu^{p-1}\frac{d}{du}u^{1-p} \quad \text{in} \quad (\zeta_\mu, \infty),
\]

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we see that
\[
\Psi_\mu(u) = \left\{ \Psi_\mu^{1-p}(\zeta_\mu) + \mu^{p-1} (u^{1-p} - \zeta_\mu^{1-p}) \right\}^{\frac{1}{p-1}}.
\] (3.9)
Moreover, it follows from (3.6) and (3.8) that
\[
\Psi_\mu'(\zeta_\mu) = \mu^{p-1} \Psi_\mu^{p}(\zeta_\mu) \zeta_\mu^{-p} = \varphi_0(\zeta_\mu) < 1.
\] (3.10)

We shall prove \( \Psi_\mu \in W^{2,\infty}((\zeta_\mu, \infty)) \). To this end, thanks to (3.9), it suffices to show
\[
\zeta_\mu^{p-1} > \mu^{p-1} \Psi_\mu^{p-1}(\zeta_\mu).
\] (3.11)
Noting the monotonicity of \( \varphi_0 \) and \( \varphi_0(t) \leq 1 \) in \([0, M_0 + 1]\) with \( \varphi_0(\zeta_\mu) < 1 \), we get
\[
\Psi_\mu(\zeta_\mu) = \int_0^{\zeta_\mu} \varphi_0(t) \, dt < \zeta_\mu.
\] (3.12)
Since \( \mu \in (0, 1) \), (3.11) is verified. Thus, \( \Phi_\mu \in W^{2,\infty}((0, \infty)) \), and (i) holds.

For (ii), by the choice of \( \varphi_0 \) and (3.7), it is enough to show \( \Psi_\mu''(u) \leq 0 \) on \((\zeta_\mu, \infty)\). Noting (3.10) and (3.12), we have \( \Psi_\mu(u) \leq u \) in \([\zeta_\mu, \zeta_\mu + L] \) for some \( L_0 > 0 \). By (3.8), if the inequality \( \Psi_\mu(u) \leq u \) in \([\zeta_\mu, \zeta_\mu + L] \) holds for some \( L > 0 \), then \( \Psi_\mu(u) \leq u^{p-1} < 1 \) in \([\zeta_\mu, \zeta_\mu + L] \) and \( \Psi_\mu(u) \leq u \) in \([\zeta_\mu, \zeta_\mu + L] \) for some \( L_1 > L \). Hence, repeating this argument and noting \( \Psi_\mu(u) \leq u \) in \([\zeta_\mu, \zeta_\mu + L_0] \), we observe \( \Psi_\mu(u) \leq u \) for all \( u \in [\zeta_\mu, \infty) \). This together with (3.8) implies that for \( \zeta_\mu < u \),
\[
\Psi_\mu''(u) = p\mu^{p-1}\Psi_\mu^{p-1}(u)\Psi_\mu'(u)u^{-p} - p\mu^{p-1}\Psi_\mu^{p}(u)u^{-p-1}
\]
\[
= p\mu^{2(p-1)}u^{-2p}\Psi_\mu^{p-1}(u) - p\mu^{p-1}\Psi_\mu'(u)u^{-p-1}
\]
\[
= p\mu^{p-1}u^{-2p}\Psi_\mu(u) \left\{ \mu^{p-1}\Psi_\mu'(u) - u^{-p-1} \right\} \leq 0.
\]
Hence, \( \Phi_\mu''(u) \leq 0 \) in \((\zeta_\mu, \infty)\).

(iii) and (iv) follow from (3.6), (3.8) and the definition of \( \Phi_\mu \) and \( \varphi_0 \), and we complete the proof.

**Remark 3.3.** From the proof of Proposition 3.3, we observe that \( \Phi_\mu(u) \leq u \) and \( 0 < \Phi_\mu'(u) \leq 1 \) for any \( u \in [0, \infty) \).

**Proposition 3.4.** For \( 0 < \mu \leq 1 \), set
\[
\overline{S}_\mu := \{ u \mid u \text{ is a supersolution of (3.3) } \}.
\]
If \( u \in \overline{S}_1 \), then for any \( \mu \in (0, 1) \), \( \Phi_\mu(\mu u) \in \overline{S}_\mu \cap L^\infty(\mathbb{R}^N) \).

**Proof.** Let \( u \in \overline{S}_1 \). Then, since it follows from Proposition 3.1 that \( u \geq 0 \) in \( \mathbb{R}^N \), we see that \( v(x) := \Phi_\mu(\mu u(x)) \) is well-defined. By \( u \in H^{s}_{\text{loc}}(\mathbb{R}^N) \) and \( \Phi_\mu \in W^{2,\infty}((0, \infty)) \) due to Proposition 3.3, we find \( v \in H^{s}_{\text{loc}}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N) \) and \( |x|^t v \in L^{\frac{2N}{N+2s}}(B_1) \).

In what follows, we shall show \( v \in \overline{S}_\mu \). Set
\[
\varphi_1(x) := \varphi_1(k^{-1}x), \quad \varphi_1 \in C_c^\infty(\mathbb{R}^N), \quad \varphi_1 \equiv 1 \text{ in } B_1, \quad \varphi_1 \equiv 0 \text{ in } \mathbb{R}^N \setminus B_2.
\]
For each \( \psi \in C_c^\infty(B_1) \), thanks to \( u \in H^{s}_{\text{loc}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} \, dx) \), by [29, Lemma 2.1] we have
\[
\left| \langle u, \psi \rangle_{H^s(\mathbb{R}^N)} - \langle \varphi_k u, \psi \rangle_{H^s(\mathbb{R}^N)} \right|
\]
\[
\leq C \left( \| u - \varphi_k u \|_{L^2(B_2)} \| \psi \|_{H^s(B_2)} + \| u - \varphi_k u \|_{L^2(B_2)} \| \psi \|_{L^2(B_2)} \right.
\]
\[
\left. + \| u - \varphi_k u \|_{L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} \, dx)} \| \psi \|_{L^1(B_2)} \right),
\] (3.13)
where $C > 0$ depends only on $N$ and $s$. Therefore, it is easily seen that
\[
\langle \varphi_k u, \psi \rangle_{H^s(\mathbb{R}^N)} \to \langle u, \psi \rangle_{H^s(\mathbb{R}^N)} \quad \text{as } k \to \infty.
\]

Next, let $(\rho_\varepsilon)_\varepsilon$ be a mollifier and put $u_{\varepsilon,k} := \rho_\varepsilon * (\varphi_k u)$. Then $\|u_{\varepsilon,k} - \varphi_k u\|_{H^s(\mathbb{R}^N)} \to 0$ as $\varepsilon \to 0$. Now we define $v_{\varepsilon,k}$ and $v_k$ by
\[
v_{\varepsilon,k}(x) := \Phi_\mu (\mu u_{\varepsilon,k}(x)), \quad v_k(x) := \Phi_\mu (\mu \varphi_k(x)u(x)).
\]
Since $\Phi_\mu \in W^{2,\infty}((0,\infty))$ and $\varphi_k u \in H^s(\mathbb{R}^N)$, we see that
\[
\begin{align*}
v_{\varepsilon,k} & \in W^{2,\infty}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \quad \text{supp } v_{\varepsilon,k} \text{ is compact}, \quad v_k \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \\
v_{\varepsilon,k} & \to v_k \quad \text{weakly in } H^s(\mathbb{R}^N) \quad \text{as } \varepsilon \to 0.
\end{align*}
\]
(3.14)

Moreover, by Proposition 3.3 (ii) and Lemma B.1, it may be verified that
\[
(-\Delta)^s v_{\varepsilon,k}(x) = (-\Delta)^s \Phi_\mu (\mu u_{\varepsilon,k}(x)) \geq \Phi'_\mu (\mu u_{\varepsilon,k}(x)) (-\Delta)^s (\mu u_{\varepsilon,k})(x) \quad \text{for each } x \in \mathbb{R}^N.
\]
Thus, for $\psi \in C^\infty_c(B_1)$ with $\psi \geq 0$, we obtain
\[
\langle v_{\varepsilon,k}, \psi \rangle_{H^s(\mathbb{R}^N)} \geq \langle \mu u_{\varepsilon,k}, \Phi'_\mu(\mu u_{\varepsilon,k})\psi \rangle_{H^s(\mathbb{R}^N)}.
\]
(3.15)

Due to $\Phi'_\mu(t) \in W^{1,\infty}((0,\infty))$, we observe that $\langle \Phi'_\mu(\mu u_{\varepsilon,k})\psi \rangle_\varepsilon$ is bounded in $H^s(\mathbb{R}^N)$ and it is easily seen that
\[
\Phi'_\mu(\mu u_{\varepsilon,k})\psi \to \Phi'_\mu(\mu \varphi_k u)\psi \quad \text{weakly in } H^s(\mathbb{R}^N) \quad \text{as } \varepsilon \to 0.
\]

By letting $\varepsilon \to 0$ in (3.15), it follows from (3.14) and $\|u_{\varepsilon,k} - \varphi_k u\|_{H^s(\mathbb{R}^N)} \to 0$ that
\[
\langle v_k, \psi \rangle_{H^s(\mathbb{R}^N)} \geq \langle \mu \varphi_k u, \Phi'_\mu(\mu \varphi_k u)\psi \rangle_{H^s(\mathbb{R}^N)}.
\]
(3.16)

Next, noting for $k \geq 2$, $\varphi_k u \equiv u$ on $B_2$ and $\Phi'_\mu(\mu \varphi_k u)\psi \equiv \Phi'_\mu(\mu u)\psi$ on $\mathbb{R}^N$, $|\varphi_k u| \leq |u|$ and $u \in L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s}dx)$, as in (3.13), we may verify that, for any $\psi \in C^\infty_c(B_1)$ with $\psi \geq 0$,
\[
\langle v_k, \psi \rangle_{H^s(\mathbb{R}^N)} \to \langle v, \psi \rangle_{H^s(\mathbb{R}^N)}, \quad \langle \mu \varphi_k u, \Phi'_\mu(\mu \varphi_k u)\psi \rangle_{H^s(\mathbb{R}^N)} \to \langle \mu u, \Phi'_\mu(\mu u)\psi \rangle_{H^s(\mathbb{R}^N)}.
\]

Sending $k \to \infty$ in (3.16) and noting $u \in \overline{\mathcal{S}_1}$ and $\Phi'_\mu(\mu u)\psi \in \mathcal{H}_0^s(B_1)$, we see from Proposition 3.3 (iv) that
\[
\langle v, \psi \rangle_{H^s(\mathbb{R}^N)} \geq \langle \mu u, \Phi'_\mu(\mu u)\psi \rangle_{H^s(\mathbb{R}^N)} \geq \mu \int_{B_1} |x|^s \Phi'_\mu(\mu u)\psi \, dx \geq \mu \int_{B_1} |x|^s \Phi'_\mu(\mu u)\psi \, dx = \int_{B_1} |x|^s \Phi'_\mu(\mu u)\psi \, dx.
\]

Finally, since $u \in \overline{\mathcal{S}_1}$, we have $u_S(x) \leq u(x)$ for all $x \in \mathbb{R}^N \setminus B_1(0)$. If $\mu u(x) \geq M_0$ holds at $x \in \mathbb{R}^N \setminus B_1$, then $\mu u_S(x) \leq M_0 \leq \mu u(x)$, and Proposition 3.3 (ii) and (iii) yield $v(x) = \Phi_\mu(\mu u(x)) \geq \Phi_\mu(\mu u_S(x)) = M_0$. On the other hand, if $\mu u(x) \leq M_0$ holds at $x \in \mathbb{R}^N \setminus B_1$, then $v(x) = \Phi_\mu(\mu u(x)) = \mu u(x) \geq M_0$. Thus, $v \in \overline{\mathcal{S}_\mu}$. \hfill \Box

Remark 3.4. Since $u_S \in \overline{\mathcal{S}_1}$, it follows that $\Phi_\mu(\mu u_S) \in \overline{\mathcal{S}_\mu} \cap L^\infty(\mathbb{R}^N)$ for every $\mu \in (0,1)$.

Now, we shall construct a positive solution of (3.3) (and (3.1)).

Proposition 3.5. For each $\mu \in (0,1)$, there exists a positive minimal solution $u_\mu \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ of (3.3). Furthermore, $u_\mu$ is radial and $u_\mu \leq u$ in $\mathbb{R}^N$ holds for any $u \in \overline{\mathcal{S}_\mu}$. 

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Proof. Let \( w_1 \in H^s(\mathbb{R}^N) \) be a solution of
\[
(-\Delta)^s w_1 = 0 \quad \text{in} \quad B_1, \quad w_1 = \mu \eta \mu u_S \quad \text{in} \quad \mathbb{R}^N \setminus B_1
\] (3.17)
in the sense that
\[
\langle w_1, \varphi \rangle_{H^s(\mathbb{R}^N)} = 0 \quad \text{for all} \quad \varphi \in C_c^\infty(B_1), \quad w_1 = \mu \eta \mu u_S \quad \text{in} \quad \mathbb{R}^N \setminus B_1.
\]
To see that (3.17) has a solution, choose \( \tilde{u}_S \in C^\infty(\mathbb{R}^N) \) satisfying
\[
\tilde{u}_S = u_S \quad \text{in} \quad \mathbb{R}^N \setminus B_1, \quad \tilde{u}_S \geq 0 \quad \text{in} \quad B_1.
\]
Define \( \tilde{u}_{S,\mu} \) by
\[
\tilde{u}_{S,\mu} := \mu \eta \mu \tilde{u}_S \in C_c^\infty(\mathbb{R}^N).
\] (3.18)
Then the equation
\[
(-\Delta)^s \tilde{w}_1 = (-\Delta)^s (\tilde{u}_{S,\mu}) \quad \text{in} \quad B_1, \quad \tilde{w}_1 \in H_0^s(B_1)
\]
has a unique solution \( \tilde{w}_1 \), where we use an equivalent norm \( \| \cdot \|_{H^s(\mathbb{R}^N)} \) on \( H_0^s(B_1) \) and \( \tilde{w}_1 \) satisfies
\[
\langle \tilde{w}_1, \varphi \rangle_{H^s(\mathbb{R}^N)} = \langle \tilde{u}_{S,\mu}, \varphi \rangle_{H^s(\mathbb{R}^N)} \quad \text{for all} \quad \varphi \in H_0^s(B_1).
\]
Hence, \( w_1 := \tilde{u}_{S,\mu} - \tilde{w}_1 \in H^s(\mathbb{R}^N) \) is a solution of (3.17):
\[
\langle w_1, \varphi \rangle_{H^s(\mathbb{R}^N)} = 0 \quad \text{for all} \quad \varphi \in H_0^s(B_1), \quad w_1 = \tilde{u}_{S,\mu} = \mu \eta \mu u_S \quad \text{in} \quad \mathbb{R}^N \setminus B_1.
\]
Notice that for any \( u \in \mathfrak{F}_\mu \), we have
\[
\langle w_1, \varphi \rangle_{H^s(\mathbb{R}^N)} = 0 \leq \int_{B_1} |x|^s |u|^p \varphi \, dx \leq \langle u, \varphi \rangle_{H^s(\mathbb{R}^N)} \quad \text{for all} \quad \varphi \in H^s_0(B_1) \text{ with} \quad \varphi \geq 0.
\]
Hence, by Proposition 3.1, \( 0 \leq w_1 \leq u \) in \( \mathbb{R}^N \) for all \( u \in \mathfrak{F}_\mu \). In particular, \( w_1 \leq \Phi_{\mu}(\mu u_S) \) in \( \mathbb{R}^N \) since \( \Phi_{\mu}(\mu u_S) \in \mathfrak{F}_\mu \) due to Remark 3.4, hence, \( w_1 \in L^\infty(\mathbb{R}^N) \). In addition, by (3.17) and Proposition 3.2, we have \( w_1 > 0 \) in \( B_1 \).

Next, by \( w_1 \in L^\infty(\mathbb{R}^N) \), let \( w_2 \in H^s(\mathbb{R}^N) \) be a solution of
\[
(-\Delta)^s w_2 = |x|^s |u|^p \geq 0 = (-\Delta)^s w_1 \quad \text{in} \quad B_1, \quad w_2 = \mu \eta \mu u_S \quad \text{in} \quad \mathbb{R}^N \setminus B_1.
\]
Since \( |x|^s |u|^p \leq |x|^s |u|^p \) in \( B_1 \) for each \( u \in \mathfrak{F}_\mu \), as in the above, we get
\[
w_1 \leq w_2 \leq u \quad \text{in} \quad \mathbb{R}^N \quad \text{for any} \quad u \in \mathfrak{F}_\mu, \quad w_2 \leq \Phi_{\mu}(\mu u_S), \quad w_2 \in L^\infty(\mathbb{R}^N).
\]
We repeat this procedure and let \( w_{n+1} \in H^s(\mathbb{R}^N) \) be a solution of
\[
(-\Delta)^s w_{n+1} = |x|^s |u|^p \quad \text{in} \quad B_1, \quad w_{n+1} = \mu \eta \mu u_S \quad \text{in} \quad \mathbb{R}^N \setminus B_1.
\] (3.19)
Then \( w_n \leq w_{n+1} \leq u \) in \( \mathbb{R}^N \) for each \( u \in \mathfrak{F}_\mu, w_{n+1} \leq \Phi_{\mu}(\mu u_S) \) in \( \mathbb{R}^N \) and \( w_{n+1} \in L^\infty(\mathbb{R}^N) \).

Next, we show that \( (w_n)_n \) is bounded in \( H^s(\mathbb{R}^N) \). To see this, since \( w_{n+1} - \tilde{u}_{S,\mu} \in H_0^s(B_1) \), it follows that
\[
\int_{B_1} |x|^s |w|^p_n (w_{n+1} - \tilde{u}_{S,\mu}) \, dx = \langle w_{n+1}, w_{n+1} - \tilde{u}_{S,\mu} \rangle_{H^s(\mathbb{R}^N)}
\]
\[
\geq \| w_{n+1} \|^2_{H^s(\mathbb{R}^N)} - \| w_{n+1} \|_{H^s(\mathbb{R}^N)} \| \tilde{u}_{S,\mu} \|_{H^s(\mathbb{R}^N)}.
\] (3.20)
Since \( w_n \leq \Phi_\mu(\mu u_S) \) in \( \mathbb{R}^N \) for each \( n \geq 1 \) and \( \Phi_\mu(\mu u_S) \in L^\infty(\mathbb{R}^N) \), from (3.20), it follows that \( (\|w_n\|_{H^s(\mathbb{R}^N)})_n \) is bounded. The boundedness of \( (\|w_n\|_{L^2(\mathbb{R}^N)})_n \) follows from \( w_n \leq \Phi_\mu(\mu u_S) \) in \( \mathbb{R}^N \) and \( w_n = \mu \eta_\mu u_S \) in \( \mathbb{R}^N \setminus B_1 \).

Owing to the monotonicity of \( w_n \), we set \( u_\mu(x) := \lim_{n \to \infty} w_n(x) \) and remark that \( w_n \rightharpoonup u_\mu \) weakly in \( H^s(\mathbb{R}^N) \). Then it is easily seen from (3.19) that
\[
\langle u_\mu, \varphi \rangle_{H^s(\mathbb{R}^N)} = \int_{B_1} |x|^\ell u_\mu^p \varphi \, dx \quad \text{for any } \varphi \in \mathcal{H}_0^s(B_1(0)).
\]

Since \( \mathcal{S}_\mu \) contains all nontrivial solutions of (3.3) by definition, we observe that \( u_\mu \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) is a positive minimal solution of (3.3).

Finally, noting that \( u_\mu(Rx) \) is also a solution of (3.3) for all \( R \in O(N) \) due to the radial symmetry of \( \eta_\mu u_S \), we have \( u_\mu(x) \leq u_\mu(Rx) \) for any \( R \in O(N) \), which implies \( u_\mu(R^{-1}x) \leq u_\mu(x) \leq u_\mu(Rx) \) for any \( R \in O(N) \). Thus \( u_\mu(x) = u_\mu(Rx) \) for every \( R \in O(N) \) and \( u_\mu \) is radial.

In the following, using \( u_\mu \) in Proposition 3.5 and the stability of \( u_S \), we shall construct stable solutions of (1.1). To this end, we show the separation property and the convergence result of \( (u_\mu)_{0<\mu<1} \).

**Proposition 3.6.** For each \( \mu \in (0,1) \), let \( u_\mu \) be a positive minimal solution of (3.3) in Proposition 3.5. Then the following hold:

(i) Let \( 0 < \mu_1 < \mu_2 < 1 \). Then \( u_{\mu_1} \leq u_{\mu_2} \) in \( \mathbb{R}^N \);

(ii) As \( \mu \nearrow 1 \), \( u_\mu \rightharpoonup u_S \) weakly in \( H^s_{\text{loc}}(\mathbb{R}^N) \).

**Proof.** We first prove the separation property. Let \( 0 < \mu_1 < \mu_2 < 1 \). Then it follows from (3.2) that \( \mu_1 \eta_{\mu_1}(x) \leq \mu_2 \eta_{\mu_2}(x) \) for all \( x \in \mathbb{R}^N \). Hence, by (3.3), \( u_{\mu_1} \leq u_{\mu_2} \) in \( \mathbb{R}^N \setminus B_1 \). Therefore, since \( (u_{\mu_1} - u_{\mu_2})_+ \in \mathcal{H}_0^s(B_1) \), similarly to (3.5), it holds that
\[
\|(u_{\mu_1} - u_{\mu_2})_+\|^2_{H^s(\mathbb{R}^N)} \leq \langle u_{\mu_1} - u_{\mu_2}, (u_{\mu_1} - u_{\mu_2})_+ \rangle_{H^s(\mathbb{R}^N)}
= \int_{B_1} |x|^\ell (u_{\mu_1}^p - u_{\mu_2}^p) (u_{\mu_1} - u_{\mu_2})_+ \, dx.
\]

If \( (u_{\mu_1} - u_{\mu_2})_+ \neq 0 \), then we have
\[
\int_{B_1} |x|^\ell (u_{\mu_1}^p - u_{\mu_2}^p) (u_{\mu_1} - u_{\mu_2})_+ \, dx = \int_{B_1} |x|^\ell p \int_0^1 (\theta u_{\mu_1} + (1 - \theta) u_{\mu_2})^{p-1} \, d\theta (u_{\mu_1} - u_{\mu_2})_+^2 \, dx
\leq p \int_{B_1} |x|^\ell u_{\mu_1}^{p-1} (u_{\mu_1} - u_{\mu_2})_+^2 \, dx.
\]

Since \( (u_{\mu_1} - u_{\mu_2})_+ \in \mathcal{H}_0^s(B_1) \) and \( C^\infty_c(B_1) \) is dense in \( \mathcal{H}_0^s(B_1) \), we may find a \( w \in C^\infty_c(B_1) \) so that
\[
\|w\|^2_{H^s(\mathbb{R}^N)} < p \int_{B_1} |x|^\ell u_{\mu_1}^{p-1} w^2 \, dx.
\]

On the other hand, since \( u_\mu \leq \Phi_\mu(\mu u_S) \leq \mu u_S \) with \( \mu \in (0,1) \) and \( u_S \) is stable, it follows that
\[
\|\varphi\|^2_{H^s(\mathbb{R}^N)} - p \int_{\mathbb{R}^N} |x|^\ell u_\mu^{p-1} \varphi^2 \, dx \geq 0 \quad \text{for all } \varphi \in C^\infty_c(B_1),
\]
which contradicts (3.21) and \( u_{\mu_1} \leq u_{\mu_2} \) in \( \mathbb{R}^N \).

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Next we prove the convergence property. To this end, we first show that \((u_\mu)_{\mu<1/2}\) is bounded in \(H^s_{\text{loc}}(\mathbb{R}^N)\). By (3.2) and (3.18) it is easily seen that
\[
\tilde{u}_{S,\mu}\varphi \to \tilde{u}_S\varphi \quad \text{in} \quad C^2(\mathbb{R}^N) \quad \text{for any} \quad \varphi \in C_c^\infty(\mathbb{R}^N) \quad \text{as} \quad \mu \to 1.
\] (3.22)

Now choose \(\varphi_k \in C_c^\infty(\mathbb{R}^N)\) with \(0 \leq \varphi_k \leq 1\), \(\varphi_k(x) \equiv 1\) on \(B_k\) and \(\varphi_k \equiv 0\) on \(\mathbb{R}^N \setminus B_{2k}\). Since \(\varphi_k(u_\mu - \tilde{u}_{S,\mu}) \in H^s_0(B_1)\) holds with \(u_\mu \leq \Phi_{\mu}(u_S) \leq \mu u_S\) and \(\tilde{u}_S \geq 0\) in \(\mathbb{R}^N\), we find that
\[
\langle u_\mu, \varphi_k(u_\mu - \tilde{u}_{S,\mu}) \rangle_{H^s(\mathbb{R}^N)} = \int_{B_1} |x|^f u_\mu^p \varphi_k(u_\mu - \tilde{u}_{S,\mu}) \, dx \leq \int_{B_1} |x|^f u_\mu^{p+1} \, dx < \infty.
\] (3.23)

Since \(k \geq 1\) and \((1 - \varphi_k)u_\mu = (1 - \varphi_k)\tilde{u}_{S,\mu}\), we remark that
\[
\langle u_\mu, \varphi_k(u_\mu - \tilde{u}_{S,\mu}) \rangle_{H^s(\mathbb{R}^N)} = \langle \varphi_k u_\mu + (1 - \varphi_k)u_\mu, \varphi_k u_\mu - \varphi_k \tilde{u}_{S,\mu} \rangle_{H^s(\mathbb{R}^N)}
\]
\[
= \|\varphi_k u_\mu\|^2_{H^s(\mathbb{R}^N)} - \langle \varphi_k u_\mu, \varphi_k \tilde{u}_{S,\mu} \rangle_{H^s(\mathbb{R}^N)} + \langle (1 - \varphi_k) \tilde{u}_{S,\mu}, \varphi_k u_\mu \rangle_{H^s(\mathbb{R}^N)} - \langle (1 - \varphi_k) \tilde{u}_{S,\mu}, \varphi_k \tilde{u}_{S,\mu} \rangle_{H^s(\mathbb{R}^N)}.
\] (3.24)

As in (3.13), it follows from [29, Lemma 2.1] that
\[
\|\langle (1 - \varphi_k) \tilde{u}_{S,\mu}, \varphi_k u_\mu \rangle \|
\leq C \left( \|\varphi_k u_\mu\|_{H^s(\mathbb{R}^N)}^2 + \|\varphi_k \tilde{u}_{S,\mu}\|_{H^s(\mathbb{R}^N)}^2 + \|\varphi_k u_\mu\|_{L^2(B_{4k})} + \|\varphi_k \tilde{u}_{S,\mu}\|_{L^2(B_{4k})} + \|\varphi_k u_\mu\|_{L^1(B_{4k})} + \|\varphi_k \tilde{u}_{S,\mu}\|_{L^1(B_{4k})} \right)
\]

Remark that a similar estimate holds for \(\langle (1 - \varphi_k) \tilde{u}_{S,\mu}, \varphi_k \tilde{u}_{S,\mu} \rangle_{H^s(\mathbb{R}^N)}\), \((1 - \varphi_k) \tilde{u}_{S,\mu}\) is bounded in \(H^s(B_{4k}) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} \, dx)\), and \((\varphi_k \tilde{u}_{S,\mu})_{\mu<1/2}\) is bounded in \(H^s(\mathbb{R}^N)\). Since \(\|\varphi_k u_\mu\|_{L^2(\mathbb{R}^N)} \leq C \|\varphi_k u_\mu\|_{H^s(\mathbb{R}^N)}\) and supp \(\varphi_k \subset B_{2k}\), it follows from (3.22), (3.23), (3.24) and \(u_\mu \leq u_S\) that for any \(\mu \in (1/2, 1)\),
\[
\int_{B_1} |x|^f \tilde{u}^{p+1} \, dx \geq \|\varphi_k u_\mu\|_{H^s(\mathbb{R}^N)}^2 - \|\varphi_k u_\mu\|_{H^s(\mathbb{R}^N)}^2 - \|\varphi_k \tilde{u}_{S,\mu}\|_{H^s(\mathbb{R}^N)}^2
\]
\[
- C_k \left( \|\varphi_k u_\mu\|_{H^s(\mathbb{R}^N)}^2 + \|\varphi_k u_\mu\|_{L^2(B_{4k})}^2 + \|\varphi_k u_\mu\|_{L^1(B_{4k})}^2 + 1 \right)
\]
\[
\geq \|\varphi_k u_\mu\|_{H^s(\mathbb{R}^N)}^2 - C_k \left( \|\varphi_k u_\mu\|_{H^s(\mathbb{R}^N)}^2 + \|\varphi_k u_\mu\|_{L^2(\mathbb{R}^N)}^2 + 1 \right)
\]
\[
\geq \|\varphi_k u_\mu\|_{H^s(\mathbb{R}^N)}^2 - C_k \left( \|\varphi_k u_\mu\|_{H^s(\mathbb{R}^N)}^2 + 1 \right).
\]

Thus, \((\|\varphi_k u_\mu\|_{H^s(\mathbb{R}^N)})_{1/2<\mu<1}\) is bounded for each \(k\).

Let \(u_1(x) := \lim_{\mu \to 1} u_\mu(x) \leq u_S(x)\). Then, by Proposition 2.1 (ii), \(u_1 \in L^2_{\text{loc}}(\mathbb{R}^N)\). Furthermore, from the boundedness of \((\|\varphi_k u_\mu\|_{H^s(\mathbb{R}^N)})_{1/2<\mu<1}\), it follows that \(\varphi_k u_1 \in H^s(\mathbb{R}^N)\) and
\[
\varphi_k u_\mu \to \varphi_k u_1 \quad \text{weakly in} \quad H^s(\mathbb{R}^N) \quad \text{as} \quad \mu \to 1.
\] (3.25)

for each \(k\). Next, we shall show that for each \(\varphi \in C_c^\infty(B_1)\),
\[
\int_{B_1} |x|^f \tilde{u}^p \varphi \, dx = \lim_{\mu \to 1} \langle u_\mu, \varphi \rangle_{H^s(\mathbb{R}^N)} = \langle u_1, \varphi \rangle_{H^s(\mathbb{R}^N)}.
\] (3.26)

To this end, notice that \(u_\mu \leq u_S\) on \(\mathbb{R}^N\) and \((1 - \varphi_k) (u_\mu - u_1) = 0\) on \(B_2\) for \(k \geq 2\). In particular, \(u_\mu \to u_1\) in \(L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} \, dx)\). Thus, as in (3.13), we infer from [29, Lemma 2.1], (3.25) and (1 - \varphi_k)u_\mu \equiv 0 \equiv (1 - \varphi_k)u_1 on \(B_2\) that for each \(k \geq 2\), as \(\mu \to 1\),
\[
\langle \varphi_k u_\mu, \varphi \rangle_{H^s(\mathbb{R}^N)} \to \langle \varphi_k u_1, \varphi \rangle_{H^s(\mathbb{R}^N)}, \quad \langle (1 - \varphi_k) u_\mu, \varphi \rangle_{H^s(\mathbb{R}^N)} \to \langle (1 - \varphi_k) u_1, \varphi \rangle_{H^s(\mathbb{R}^N)}.
\]
Therefore,

\[
\int_{B_1} |x|^p u_1^p \varphi \, dx = \lim_{\mu \nearrow 1} \int_{B_1} |x|^p u_\mu^p \varphi \, dx = \lim_{\mu \nearrow 1} \langle u_\mu, \varphi \rangle_{H^s(\mathbb{R}^N)} = \langle \varphi_k u_\mu + (1 - \varphi_k) u_\mu, \varphi \rangle_{H^s(\mathbb{R}^N)} = \langle u_\mu, \varphi \rangle_{H^s(\mathbb{R}^N)},
\]

and \((3.26)\) holds.

Since \(|x|^p u_0^p \in L^{2N/(N+2s)}(B_1)\), \(u_1\) is a solution of \((3.1)\) with \(\mu = 1\) and \(u_1 \leq u_S\). Noting \(u_S - u_1 \in H_0^s(B_1)\), we have

\[
\|u_S - u_1\|_{H^s(\mathbb{R}^N)}^2 = \langle u_S - u_1, u_S - u_1 \rangle_{H^s(\mathbb{R}^N)} = \int_{B_1} |x|^p (u_S - u_1)^2 \, dx.
\]

If \(u_S - u_1 \neq 0\), then

\[
\frac{\|u_S - u_1\|_{H^s(\mathbb{R}^N)}^2}{\|u_1\|_{H^s(\mathbb{R}^N)}^2} = \int_{B_1} |x|^p \int_0^1 (\theta u_S + (1 - \theta) u_1)^p - 1 \, d\theta \, (u_S - u_1)^2 \, dx
\]

which contradicts \((2.7)\). Thus, \(u_S \equiv u_1\). \(\Box\)

**Remark 3.5.** Since \(u_\mu \in L^\infty(\mathbb{R}^N)\), we may prove \(u_\mu \in C(B_1)\) (see [29, Proposition 2.1]).

Now, we shall construct a stable solution of \((1.1)\).

**Proposition 3.7.** There exists a positive radial stable solution \(u\) of

\[
(-\Delta)^s u = |x|^p u^p \quad \text{in } \mathbb{R}^N, \quad \|u\|_{L^\infty(\mathbb{R}^N)} = 1, \quad 0 < u \leq u_S \quad \text{in } \mathbb{R}^N.
\]

**Proof.** For each \(\mu \in (0, 1)\), let \(u_\mu\) be as in Proposition 3.5. Let \((\mu_j)_j \subset (0, 1)\) with \(\mu_j \searrow 1\) as \(j \to \infty\), and set

\[
\tilde{u}_j(x) := \frac{1}{m_j} u_{\mu_j} \left( m_j^{-1/\theta_0} x \right), \quad m_j := \|u_{\mu_j}\|_{L^\infty(\mathbb{R}^N)},
\]

where \(\theta_0\) is as in \((1.8)\). By Propositions 3.5 and 3.6, Remark 3.5 and \(u_\mu \leq u_S\) in \(B_1\), we have \(m_j \to \infty\) as \(j \to \infty\) and

\[
\langle \tilde{u}_j, \varphi \rangle_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |x|^p \tilde{u}_j^p \varphi \, dx \quad \text{for all } \varphi \in C_c^\infty(B_{m_j^{-1/\theta_0}}),
\]

\[
\tilde{u}_j(x) \leq \frac{1}{m_j} u_S \left( m_j^{-1/\theta_0} x \right) = u_S(x) \quad \text{for each } x \in \mathbb{R}^N \setminus \{0\}, \quad \|\tilde{u}_j\|_{L^\infty(\mathbb{R}^N)} = 1.
\]

Next, we claim that \((\varphi \tilde{u}_j)_j\) is bounded in \(H^s(\mathbb{R}^N)\) for any \(\varphi \in C_c^\infty(\mathbb{R}^N)\). To this end, we set \(\tilde{U}_j(x, t) := (P_s(\cdot, t + \tilde{u}_j)(x) \text{ where } P_s(x, t) := p_{N+1, t}^{2s} (|x|^2 + t^2)^{-\frac{N+2s}{2}}. \quad \text{As in [29, Lemmata 2.1 and 2.2], from } \tilde{u}_j \in H_{loc}^s(\mathbb{R}_+^N) \cap L^\infty(\mathbb{R}_+^N), \text{ we have}

\[
\lim_{t \to 0} - \int_{\mathbb{R}^N} t^{1-2s} \partial_t \tilde{U}_j(x, t) \psi(x, t) \, dx = \kappa_s \langle \tilde{u}_j, \psi(\cdot, 0) \rangle_{H^s(\mathbb{R}^N)} \quad \text{for each } \psi \in C_c^\infty(\mathbb{R}_+^{N+1}).
\]
In particular, if \( \psi(x,t) \in C_c^\infty(\mathbb{R}^{N+1}_+ \setminus \{0\}) \) with supp \( \psi(\cdot,0) \subset B_{m_j^{1/\theta_0}} \), then we may choose \( \psi^2(\cdot,0)\tilde{u}_j \) as a test function in (3.28). Thus,

\[
\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla \tilde{U}_j \cdot \nabla \left( \psi^2 \tilde{U}_j \right) \, dX = \kappa_8 \langle \tilde{u}_j, \psi^2(\cdot,0)\tilde{u}_j \rangle_{H^s(\mathbb{R}^N)} = \kappa_8 \int_{B_{m_j^{1/\theta_0}}} |x|^{\ell_{p+1}^B} \psi^2(x,0) \, dx. \tag{3.29}
\]

Furthermore, it follows from \( \|P_s(\cdot,t)\|_{L^1(\mathbb{R}^N)} = 1 \) for all \( t > 0 \) and (3.28) that

\[\|\tilde{U}_j\|_{L^\infty(\mathbb{R}^{N+1}_+)} \leq \sup_{t > 0} \|P_s(\cdot,t)\|_{L^1(\mathbb{R}^N)} \|\tilde{u}_j\|_{L^\infty(\mathbb{R}^N)} = 1.\]

Since

\[
\nabla \tilde{U}_j \cdot \nabla \left( \psi^2 \tilde{U}_j \right) \\
\geq \left| \nabla \tilde{U}_j \right|^2 \psi^2 - 2 |\tilde{U}_j| |\nabla \psi| \\
\geq \frac{1}{2} \left| \nabla \tilde{U}_j \right|^2 \psi^2 - 2 \tilde{U}_j^2 |\nabla \psi|^2 \\
= \frac{1}{4} \left| \nabla \tilde{U}_j \right|^2 \psi^2 - \frac{1}{4} \left( \left| \nabla \tilde{U}_j \right|^2 - 2 \psi \tilde{U}_j \nabla \psi \cdot \nabla \tilde{U}_j - \tilde{U}_j^2 |\nabla \psi|^2 \right) - 2 \tilde{U}_j^2 |\nabla \psi|^2 \geq \frac{1}{4} \left| \nabla \tilde{U}_j \right|^2 - 3 \tilde{U}_j^2 |\nabla \psi|^2,
\]

we see from (3.29) that \( (\psi \tilde{U}_j)_j \) is bounded in \( H^1(\mathbb{R}^{N+1}_+, t^{1-2s} \, dX) \). By this boundedness, the trace theorem \( H^1(\mathbb{R}^{N+1}_+, t^{1-2s} \, dX) \subset H^s(\mathbb{R}^N) \) and \( \tilde{U}_j(x,0) = \tilde{u}_j(x) \), for any \( \varphi \in C_c^\infty(\mathbb{R}^N) \), \( (\varphi \tilde{u}_j)_j \) is bounded in \( H^s(\mathbb{R}^N) \).

Now let \( \tilde{u}_j \to \tilde{u}_0 \) weakly in \( H^s(B_R) \) as \( j \to \infty \) for any \( R > 0 \). By (3.29), \( \|\tilde{u}_j\|_{L^\infty(\mathbb{R}^N)} = 1 \), \( |x|^{\ell} \in L^q_{\text{loc}}(\mathbb{R}^N) \) for some \( q < \frac{N}{2s} \) and [29, Proposition 2.1] (see [32]), we observe that \( \tilde{u}_j \to \tilde{u}_0 \) in \( C^\beta_{\text{loc}}(\mathbb{R}^N) \) for some \( \beta \in (0,1) \). Then \( \|\tilde{u}_0\|_{L^\infty(\mathbb{R}^N)} = 1 \) holds due to \( \tilde{u}_j \leq u_S \) in \( \mathbb{R}^N \setminus \{0\} \). Moreover, for \( \varphi \in C_c^\infty(\mathbb{R}^N) \), as (3.26), it follows that

\[
\langle \tilde{u}_0, \varphi \rangle_{H^s(\mathbb{R}^N)} = \lim_{j \to \infty} \langle \tilde{u}_j, \varphi \rangle_{H^s(\mathbb{R}^N)} = \lim_{j \to \infty} \int_{\mathbb{R}^N} |x|^{\ell} \tilde{u}_j^{p+} \varphi \, dx = \int_{\mathbb{R}^N} |x|^{\ell} \tilde{u}_0^{p+} \varphi \, dx.
\]

Hence, \( \tilde{u}_0 \) is a solution of (3.27). In addition, from

\[
\langle \tilde{u}_0, \varphi \rangle_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |x|^{\ell} \tilde{u}_0^{p+} \varphi \, dx \geq 0 = \int_{\mathbb{R}^N} 0 \varphi \, dx \quad \text{for each } \varphi \in C_c^\infty(\mathbb{R}^N) \text{ with } \varphi \geq 0
\]

and Proposition 3.2, we observe that \( \tilde{u}_0 > 0 \) in \( B_R \) for each \( R > 0 \), hence, \( \tilde{u}_0 > 0 \) in \( \mathbb{R}^N \). The stability follows from \( \tilde{u}_0 \leq u_S \) in \( \mathbb{R}^N \) and (2.7).

As a corollary of Proposition 3.7, we have

**Corollary 3.1.** Let \( \tilde{u}_0 \) be a bounded radial stable solution of (3.27) in Proposition 3.7. For \( m > 0 \), set

\[
u_{m,0}(x) := m \tilde{u}_0 \left( m^{1/\theta_0} x \right), \tag{3.31}
\]

where \( \theta_0 \) is as in (1.8). Then \( u_{m,0} \) satisfies

\[
(-\Delta)^s u_{m,0} = |x|^{\ell} u_{m,0}^p \quad \text{in } \mathbb{R}^N, \quad \|u_{m,0}\|_{L^\infty(\mathbb{R}^N)} = m, \quad 0 < u_{m,0} < u_S \quad \text{in } \mathbb{R}^N \tag{3.32}
\]

and \( u_{m,0} \) is stable.
Proof. It is easy to check that \( u_{m,0} \) satisfies \((-\Delta)^s u_{m,0} = |x|^s u_{m,0}^p \) in \( \mathbb{R}^N \) and \( \| u_{m,0} \|_{L^\infty(\mathbb{R}^N)} = m \). In addition, by \( u_{m,0}(x) = m\tilde{u}_0(m^{1/\theta_0}x) \leq mu_S(m^{1/\theta_0}x) = u_S(x) \) and (2.7), we infer that \( u_{m,0} \) is stable. Finally, from \( u_{m,0} \leq u_S \) in \( \mathbb{R}^N \) and \( \langle u_S - u_{m,0}, \varphi \rangle_{H^s(\mathbb{R}^N)} \geq 0 \) for each \( \varphi \in C^\infty_c(\mathbb{R}^N) \) with \( \varphi \geq 0 \), Proposition 3.2 yields \( u_{m,0} < u_S \) in \( \mathbb{R}^N \).

**Remark 3.6.** By the regularity result it holds that \( \tilde{u}_0 \in C(\mathbb{R}^N) \). Furthermore, by (3.31) and (3.32) we see that \( \| u_{m,0} \|_{L^\infty(\mathbb{R}^N)} \tilde{u}_0(0) = m\tilde{u}_0(0) = u_{m,0}(0) \). These together with \( \tilde{u}_0 > 0 \) in \( \mathbb{R}^N \) imply that \( u_{m,1,0}(0) < u_{m,2,0}(0) \) is equivalent to \( m_1 < m_2 \) and instead of \( (u_{m,0})_{m>0} \), we may write \( (u_\alpha)_{\alpha>0} \) where \( \alpha = u_\alpha(0) = m\tilde{u}_0(0) \).

### 4 Separation property and proof of Theorem 1.1 (ii)

In this section, we aim to prove the comparison of positive radial stable solutions of (1.1) and study the properties of \( (u_{m,0})_{m>0} \) in Corollary 3.1.

#### 4.1 Separation property

The aim of this subsection is to prove the following comparison result which is an extension of [26, Proposition 7.1]:

**Proposition 4.1.** Let \( u_1, u_2 \in H^s_{\text{loc}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1+|x|)^{-N-2s}dx) \) be positive radial stable solutions of

\[ (-\Delta)^s u = |x|^s u^p \quad \text{in } \mathbb{R}^N. \]

In addition, suppose that

\[
\begin{align*}
  u_1 &\in L^\infty(\mathbb{R}^N), \\
  u_2 &\in L^\infty(\mathbb{R}^N \setminus \{0\}), \\
  u_2 &\leq Cu_S \quad \text{in } B_1, \\
  u_1(x), u_2(x) &\to 0 \text{ as } |x| \to \infty, \\
  u_1(0) &< \lim_{|x| \to 0} u_2(x),
\end{align*}
\]

for some \( C > 0 \). Then \( u_1 < u_2 \) in \( \mathbb{R}^N \).

**Remark 4.1.**

(i) In Proposition 4.1, the case \( u_2(x) \to \infty \) as \( |x| \to 0 \) may occur, however, by \( u_2 \leq Cu_S \) in \( B_1 \) and (4.2), we have \( |x|^s u_2^p \in L^{\frac{N}{2(N+s)}}(\mathbb{R}^N) \).

(ii) From \( u_2 \leq Cu_S \) in \( B_1 \), \( u_2 \in L^\infty(\mathbb{R}^N \setminus \{0\}) \) and the Hardy type inequality (2.8), we see that the functional

\[ \mathcal{H}_{B_R}^s(u_2) \ni w \mapsto \int_{\mathbb{R}^N} |x|^s u_2^{p-1} w^2 dx \in \mathbb{R} \]

is bounded for each \( R > 0 \), where \( \mathcal{H}_{B_R}^s(u_2) := \{ w \in H^s(\mathbb{R}^N) \mid w \equiv 0 \text{ on } \mathbb{R}^N \setminus B_R \} \). Since we assume that \( u_2 \) is stable, it follows that

\[ p \int_{\mathbb{R}^N} |x|^s u_2^{p-1} w^2 dx \leq \| w \|_{H^s(\mathbb{R}^N)}^2 \]

for all \( w \in H^s(\mathbb{R}^N) \), which have compact support in \( \mathbb{R}^N \).

(iii) From (4.2) and the fact that \( u_1 \) and \( u_2 \) are solutions of (4.1), we have \( u_1 \in C(\mathbb{R}^N) \) and \( u_2 \in C(\mathbb{R}^N \setminus \{0\}) \) as in Remark 3.5. If \( \lim_{|x| \to 0} u_2(x) < \infty \), then \( u_2 \in C(\mathbb{R}^N) \).

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Moreover, let \( u \in H^s_{loc}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s}dx) \) be a positive solution of (4.1). Since we require \( u \) to satisfy \( |x|^fu^p \in L^{\frac{2N}{N+2}}(B_1) \), [29, Lemma 2.2] still holds for \( U(x,t) := (P_s(\cdot,t)*u)(x) \), that is, \( U \in H^1_{loc}(\mathbb{R}^n_+), t^{1-2s}dX \) satisfies

\[
\int_{\mathbb{R}^N_+} t^{1-2s} \nabla U \cdot \nabla \psi \, dX = \kappa_s \langle u, \psi(\cdot,0) \rangle_{\dot{H}^s(\mathbb{R}^N)} = \kappa_s \int_{\mathbb{R}^N} |x|^f u^p \psi(x,0) \, dx
\]

for every \( \psi \in C^1_c(\mathbb{R}^N) \).

We first remark that if \( u_1 \leq u_2 \) in \( \mathbb{R}^N \) holds, then for each \( \varphi \in C_c^\infty(\mathbb{R}^N) \) with \( \varphi \geq 0 \),

\[
\langle u_2 - u_1, \varphi \rangle_{\dot{H}^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |x|^f (u_2^p - u_1^p) \varphi \, dx \geq 0 = \int_{\mathbb{R}^N} 0 \varphi \, dx.
\]

Since \( u_i \in H^s_{loc}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s}dx) \) by (4.2), Proposition 3.2 implies that \( u_2 - u_1 > 0 \) holds in \( \mathbb{R}^N \) if \( u_2 \in L^\infty(B_1) \) and in \( \mathbb{R}^N \setminus \{0\} \) if \( u_2(x) \to 0 \) as \( |x| \to \infty \). By (4.2), in both cases, we have \( u_1 < u_2 \) in \( \mathbb{R}^N \) and Proposition 4.1 holds. Therefore, in what follows, it suffices to prove

\[
u_1 \leq u_2 \quad \text{in} \quad \mathbb{R}^N.
\]

The argument below is inspired by [26, Section 7].

To show (4.3), we set

\[
[u_1 - u_2 > 0] := \{ \, x \in \mathbb{R}^N \mid u_1(x) - u_2(x) > 0 \}.
\]

We will prove Proposition 4.1 indirectly and hereafter, suppose

\[
[u_1 - u_2 > 0] \neq \emptyset. \tag{4.4}
\]

We aim to derive a contradiction.

**Lemma 4.1.** Under (4.4), both of \([u_1 - u_2 > 0]\) and \([u_2 - u_1 > 0]\) are unbounded and have infinitely many components.

**Proof.** Notice that both of \([u_1 - u_2 > 0]\) and \([u_2 - u_1 > 0]\) are nonempty open sets in \( \mathbb{R}^N \) due to Remark 4.1 (iii), (4.2) and (4.4). If \([u_1 - u_2 > 0]\) is bounded, then

\[
v(x) := (u_1(x) - u_2(x))^+ \in H^s(\mathbb{R}^N), \quad \text{supp} \, v \, \text{is compact,} \quad v \neq 0.
\]

Thus, due to Remark 3.1 (iv), we can take \( v \) as a test function, and

\[
\langle u_1 - u_2, v \rangle_{\dot{H}^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |x|^f (u_1^p - u_2^p) v \, dx.
\]

Since \((a_+ - b_+)^2 \leq (a - b)(a_+ - b_+)\) holds, we have

\[
\|v\|_{\dot{H}^s(\mathbb{R}^N)}^2 \leq \langle u_1 - u_2, v \rangle_{\dot{H}^s(\mathbb{R}^N)}.
\]

Moreover, from \( v \neq 0 \) and

\[
\int_{\mathbb{R}^N} |x|^f (u_1^p - u_2^p) v \, dx = \int_{\mathbb{R}^N} |x|^f \int_0^1 (\theta u_1 + (1 - \theta)u_2)^p - 1 \, d\theta v^2 \, dx
\]

\[
< p \int_{\mathbb{R}^N} |x|^f u_1^{p-1} v^2 \, dx,
\]

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it follows that
\[ \|v\|^2_{H^s(\mathbb{R}^N)} < p \int_{\mathbb{R}^N} |x|^\alpha u^p \, dx, \]
and this contradicts the stability of \( u_1 \).

On the other hand, when \([u_2 - u_1 > 0]\) is bounded, we exploit
\[ w(x) := (u_2(x) - u_1(x))_+ \]
and test this function to \((-\Delta)^s(u_2 - u_1) = |x|^\alpha(u^p_2 - u^p_1)\). Then we have a contradiction.

Finally, suppose that either \([u_1 - u_2 > 0]\) or \([u_2 - u_1 > 0]\) has only finitely many components. Here we first assume that \([u_1 - u_2 > 0]\) admits only finitely many components \( I_1, \ldots, I_k \). We know that \([u_1 - u_2 > 0]\) is unbounded and each \( I_j \) is either ball or annulus due to the radial symmetry of \( u_1 \) and \( u_2 \). Therefore, one of \( I_1, \ldots, I_k \) should be unbounded, say \( I_1 \). Since \( I_1 \) is unbounded and either ball or annulus, we may find an \( r_0 > 0 \) such that \( \mathbb{R}^N \setminus B_{r_0} \subset I_1 \). This means \( u_1(x) - u_2(x) > 0 \) for all \( |x| \geq r_0 \) and \([u_2 - u_1 > 0]\) becomes bounded. However, this is a contradiction. In a similar way, we may prove that the case \([u_2 - u_1 > 0]\) has only finitely many components never occurs.

\[ \square \]

**Remark 4.2.** Let \( I_{k,+} \) (resp. \( I_{k,-} \)) be a component of \([u_2 - u_1 > 0]\) (resp. \([u_1 - u_2 > 0]\)) . Thanks to (4.2), we may suppose \( 0 \in I_{1,+} \). Furthermore, from the above argument, we may verify that each \( I_{k,\pm} \) is bounded.

Let \( u_i \) be as in Proposition 4.1. For \( u_i \), set
\[ U_i(X) := (P_s(\cdot, t) * u_i)(x). \] (4.5)
Then \( U_i(X) \to 0 \) as \(|X| \to \infty\) thanks to (4.2) and Lemma B.2. Moreover, noting Remark 4.1, we may prove
\begin{align*}
U_1 &\in H^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\}, t^{-\frac{2s}{s-1}} \, dX) \cap C(\mathbb{R}^N \setminus \{0\}) \cap C^\infty(\mathbb{R}^N), \\
U_2 &\in H^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\}, t^{-\frac{2s}{s-1}} \, dX) \cap C(\mathbb{R}^N \setminus \{0\}) \cap C^\infty(\mathbb{R}^N). \quad (4.6)
\end{align*}
and for every \( \varphi \in C^1(\mathbb{R}^{N+1}_+) \),
\[ \int_{\mathbb{R}^{N+1}_+} t^{-2s} \nabla U_i \cdot \nabla \varphi \, dx = \kappa_s \langle u_i, \varphi(\cdot, 0) \rangle_{H^s(\mathbb{R}^N)} = \kappa_s \int_{\mathbb{R}^N} |x|^\alpha u^p \varphi(x, 0) \, dx. \] (4.7)
For \( U_i \), we also use the following notation:
\[ V(X) := U_2(X) - U_1(X) \in H^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\}, t^{-\frac{2s}{s-1}} \, dX) \cap C(\mathbb{R}^N \setminus \{0\}) \cap C^\infty(\mathbb{R}^N), \]
\[ [V > a] := \left\{ X \in \mathbb{R}^{N+1}_+ \mid V(X) > a \right\}, \quad [V < a] := \left\{ X \in \mathbb{R}^{N+1}_+ \mid V(X) < a \right\}. \] (4.8)
Then, by (4.2), (4.4), (4.5), (4.6) and (4.8) we see that \( [V > 0] \neq \emptyset \) and \( [V < 0] \neq \emptyset \). When \( u_2 \in L^\infty(\mathbb{R}^N) \), \( V \in C(\overline{\mathbb{R}^{N+1}_+}) \) holds due to Remark 4.1. In addition, by virtue of Lemma B.2,
\[ \lim_{|X| \to 0} V(X) = \infty \quad \text{when } u_2(x) \to \infty \text{ as } |x| \to 0. \] (4.9)
Moreover, for each component \( \mathcal{O}_+ \subset \mathbb{R}^{N+1}_+ \) of \([V > 0] \) and \( \mathcal{O}_- \subset \mathbb{R}^{N+1}_+ \) of \([V < 0] \), define
\[ V_{\mathcal{O}_+}(X) := V(X)\chi_{\mathcal{O}_+}(X), \quad V_{\mathcal{O}_-}(X) := V(X)\chi_{\mathcal{O}_-}(X), \] (4.10)
where \( \overline{\mathcal{O}_\pm} \subset \mathbb{R}^{N+1}_+ \) is the closure of \( \mathcal{O}_\pm \) in \( \mathbb{R}^{N+1}_+ \) and \( \chi_A \) is the characteristic function of \( A \).
Lemma 4.2. For each any component $O_\pm \subset \mathbb{R}_+^{N+1}$ of $[V > 0]$ and $[V < 0]$, let $V_{O_\pm}$ be as in (4.10). Then $V_{O_\pm} \in C(\mathbb{R}_+^{N+1} \setminus \{0\}) \cap H^1_{\text{loc}}(\mathbb{R}_+^{N+1}, t^{1-2s}dX)$, and $V_{O_\pm}$ satisfy

$$\int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla V_{O_\pm}|^2 \psi dX$$

$$= - \int_{\mathbb{R}_+^{N+1}} t^{1-2s} \nabla \left( \frac{V^2}{2} \right) \cdot \nabla \psi dX + \kappa_s \int_{\mathbb{R}^N} |x|^t (u_2^p - u_1^p) V_{O_\pm}(x,0) \psi(x,0) dx$$

(4.11)

for each $\psi \in C^\infty_c(\mathbb{R}_+^{N+1})$. Furthermore, if $u_2 \in L^\infty(\mathbb{R}^N)$, then $V_{O_\pm} \in C(\mathbb{R}_+^{N+1})$.

Proof. We only treat $V_{O_+}$ and a similar argument works for $V_{O_-}$. We first show $V_{O_+} \in C(\mathbb{R}_+^{N+1} \setminus \{0\})$. Since $V = 0$ on $\partial O_+ \cap \mathbb{R}_+^{N+1}$ and $V \in C(\mathbb{R}_+^{N+1} \setminus \{0\})$, it suffices to show the continuity on $(\mathbb{R}_+^{N} \setminus \{0\}) \times \{0\}$. Let $x \in \mathbb{R}_+^{N} \setminus \{0\}$. If $V(x,0) = 0$, then the continuity of $V_{O_+}$ at $(x,0)$ follows from the continuity of $V(x,t)$. If $V(x,0) \neq 0$, then since $V \in C(\mathbb{R}_+^{N+1} \setminus \{0\})$, for some $r_0 > 0$,

$$V(y,t) \neq 0 \text{ for every } (y,t) \in B_{r_0}^+(x,0) := \{(x,t) \in \mathbb{R}_+^{N+1} : |x-y| + t \leq r_0 \}.$$ 

Thus, if $V(x,0) < 0$, then $V_{O_+}(y,t) = 0$ for every $(y,t) \in B_{r_0}^+(x,0)$ and $V_{O_+}$ is continuous at $(x,0)$. If $V(x,0) > 0$, then $B_{r_0}(x,0) \cap \mathbb{R}_+^{N+1} \subset \tilde{O}$ where $\tilde{O}$ is some component of $[V > 0]$. When $O_+ \neq \tilde{O}$, $V_{O_+}(y,t) = 0$ for all $(y,t) \in B_{r_0}^+(x,0)$ and $V_{O_+}$ is continuous at $(x,0)$. On the other hand, when $O_+ = \tilde{O}$, we have $V_{O_+}(y,t) = V(y,t)$ for all $(y,t) \in B_{r_0}^+(x,0)$. Hence, $V_{O_+}$ is continuous on $\mathbb{R}_+^{N+1} \setminus \{0\}$. In addition, when $u_2 \in L^\infty(\mathbb{R}^N)$, we have $V \in C(\mathbb{R}_+^{N+1})$ and from the above argument, it is easily seen that $V_{O_+} \in C(\mathbb{R}_+^{N+1})$.

Next, we prove $V_{O_+} \in H^1_{\text{loc}}(\mathbb{R}_+^{N+1}, t^{1-2s}dX)$. We argue as in [5, Theorem 9.17]. We choose $\zeta_0 \in C^\infty(\mathbb{R})$ such that

$$\zeta_0(\tau) = 0 \text{ if } \tau \leq 1, \quad 0 \leq \zeta_0(\tau) \leq \tau \text{ for every } \tau \in [0, \infty), \quad \zeta_0(\tau) = \tau \text{ if } \tau \geq 2, \quad (4.12)$$

and define

$$W_n(X) := \frac{1}{n} \zeta_0(n V_{O_+}(X)) \psi(X) \text{ where } \psi \in C^\infty_c(\mathbb{R}_+^{N+1}).$$

Remark that

$$\text{supp} W_n \text{ is compact in } \mathbb{R}_+^{N+1}, \quad V_{O_+} = 0 \text{ on } \partial O_+ \cap \mathbb{R}_+^{N+1}, \quad V(x,0) = u_2(x) - u_1(x) > 0 \text{ if } V_{O_+}(x,0) > 0.$$ 

Thus, by (4.2) and (4.9), we may find a $\delta_0 > 0$ such that $B_{\delta_0}^+(0,0) \cap \partial O_+ = \emptyset$. By $V_{O_+} \in C(\mathbb{R}_+^{N+1} \setminus \{0\})$ and $\psi \in C^\infty_c(\mathbb{R}_+^{N+1})$, for each $n \geq 1$, we may find an $\varepsilon_n > 0$ such that

$$W_n(X) = 0 \text{ if } X \in \mathbb{R}_+^{N+1} \text{ and dist}(X, \partial O_+ \cap \mathbb{R}_+^{N+1}) < \varepsilon_n.$$ 

From this and the fact $V_{O_+} \equiv V$ in $[V_{O_+} > 0] \cap \mathbb{R}_+^{N+1}$, it follows that

$$W_n \in C^\infty(\mathbb{R}_+^{N+1}) \cap H^1(\mathbb{R}_+^{N+1}, t^{1-2s}dX).$$ 

By $V_{O_+}(X) > 0$ for every $X \in O_+$ and $V_{O_+} = 0$ on $\partial O_+ \cap \mathbb{R}_+^{N+1}$, as $n \to \infty$, we have

$$\zeta_0'(n V_{O_+}(X)) \to 1 \text{ for all } X \in O_+, \quad \zeta_0'(n V_{O_+}(X)) = 0 \text{ for all } X \in \mathbb{R}_+^{N+1} \setminus O_+, \quad \frac{1}{n} \zeta_0(n V_{O_+}(X)) \to V_{O_+}(X) \text{ for each } X \in \mathbb{R}_+^{N+1} \cap O_+.$$

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Since
\[ \nabla W_n = \psi \zeta_0(nV_{O_+}) \nabla V + \frac{1}{n} \zeta_0(nV_{O_+}) \nabla \psi, \quad \| \zeta_0 \|_{L^\infty(\mathbb{R})} < \infty, \]
and
\[ \left| \frac{1}{n} \zeta_0(nV_{O_+}(X)) \right| \leq |V_{O_+}(X)| \quad \text{for all } X \in \mathbb{R}_+^{N+1}, \]
the dominated convergence theorem implies
\[ \int_{\mathbb{R}_+^{N+1}} t^{1-2s} |W_n - V_{O_+} \psi|^2 dX + \int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla W_n - \psi \chi_{O_+} \nabla V - V_{O_+} \nabla \psi|^2 dX \rightarrow 0. \]
Thus, for every \( \psi \in C_c^{\infty}(\mathbb{R}_+^{N+1}) \),
\[ W_n \rightarrow V_{O_+} \psi \quad \text{in } H^1(\mathbb{R}_+^{N+1}, t^{1-2s} dX), \quad V_{O_+} \psi \in H^1(\mathbb{R}_+^{N+1}, t^{1-2s} dX), \]
\[ \nabla (V_{O_+} \psi) = \psi \chi_{O_+} \nabla V + V_{O_+} \nabla \psi. \]
Hence, \( V_{O_+} \in H^{1}_{loc}(\mathbb{R}_+^{N+1}, t^{1-2s} dX) \).

Now we prove (4.11). Since \( C_c^{\infty}(\mathbb{R}_+^{N+1}) \) is dense in \( H^1(\mathbb{R}_+^{N+1}, t^{1-2s} dX) \), in view of Remark 4.1 and \( W_n(x, 0) \in H^s(\mathbb{R}^N) \) thanks to the trace theorem, (4.7) yields
\[ \int_{\mathbb{R}_+^{N+1}} t^{1-2s} \nabla V \cdot \nabla W_n dX = \kappa_x \int_{\mathbb{R}^N} |x|^\ell (u_2^p - u_1^p) W_n(x, 0) dx. \]
Observe that
\[ \int_{\mathbb{R}_+^{N+1}} t^{1-2s} \nabla V \cdot \nabla W_n dX \]
\[ = \int_{O_+} t^{1-2s} \nabla V \cdot \nabla \left( \frac{1}{n} \zeta_0(nV_{O_+}) \psi \right) dX \]
\[ = \int_{O_+} t^{1-2s} |\nabla V_{O_+}|^2 \zeta_0(nV_{O_+}) \psi dX + \int_{O_+} t^{1-2s} \frac{1}{n} \zeta_0(nV_{O_+}) \nabla V_{O_+} \cdot \nabla \psi dX. \]
The dominated convergence theorem yields
\[ \lim_{n \rightarrow \infty} \int_{\mathbb{R}_+^{N+1}} t^{1-2s} \nabla V \cdot \nabla W_n dX = \int_{O_+} t^{1-2s} |\nabla V_{O_+}|^2 \psi dX + \int_{O_+} t^{1-2s} V_{O_+} \nabla V_{O_+} \cdot \nabla \psi dX \]
\[ = \int_{O_+} t^{1-2s} |\nabla V_{O_+}|^2 \psi dX + \int_{O_+} t^{1-2s} \nabla \left( \frac{V_{O_+}^2}{2} \right) \cdot \nabla \psi dX. \]
On the other hand, since \( W_n \rightarrow V_{O_+} \psi \) strongly in \( H^1(\mathbb{R}_+^{N+1}, t^{1-2s} dX) \), by the trace theorem and \( V_{O_+} \in C(\mathbb{R}_+^{N+1} \setminus \{0\}) \), we see \( W_n(x, 0) \rightarrow (V_{O_+}, \psi)(x, 0) \) strongly in \( H^s(\mathbb{R}^N) \). Therefore, thanks to
\[ |x|^\ell u_2^p \in L^{2N+2\ell}_N(B_1) \quad \text{and} \quad u_i \in L^\infty_{loc}(\mathbb{R}^N \setminus \{0\}), \]
we obtain
\[ \lim_{n \rightarrow \infty} \kappa_s \int_{\mathbb{R}^N} |x|^\ell (u_2^p - u_1^p) W_n(x, 0) dx = \kappa_s \int_{\mathbb{R}^N} |x|^\ell (u_2^p - u_1^p) V_{O_+}(x, 0) \psi(x, 0) dx. \]
Thus, (4.11) holds. \( \square \)

**Lemma 4.3.** All components \( O_+ \subset \mathbb{R}_+^{N+1} \) of \( [V > 0] \) and \( [V < 0] \) are unbounded.
Proof. As in Lemma 4.2, we only deal with \( \mathcal{O}_+ \) since the case \( \mathcal{O}_- \) can be shown similarly. We argue by contradiction and suppose that \( \mathcal{O}_+ \) is bounded. Then \( V_{\mathcal{O}_+} \) (resp. \( V_{\mathcal{O}_+}(\cdot, 0) \)) in (4.10) has compact support in \( \mathbb{R}^{N+1}_+ \) (resp. \( \mathbb{R}^N_+ \)), and \( V_{\mathcal{O}_+} \in H^1(\mathbb{R}^{N+1}_+, t^{1-2s}dX) \) holds in view of Lemma 4.2. In addition, we may take \( \psi \equiv 1 \) in (4.11) due to the compactness of \( \text{supp} V_{\mathcal{O}_+} \) and \( \text{supp} V_{\mathcal{O}_+}(\cdot, 0) \).

Now we distinguish two cases: (i) \( V_{\mathcal{O}_+}(\cdot, 0) \equiv 0 \), (ii) \( V_{\mathcal{O}_+}(\cdot, 0) \not\equiv 0 \). When (i) happens, then putting \( \psi \equiv 1 \) in (4.11), we obtain
\[
\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla V_{\mathcal{O}_+}|^2 dX = \kappa_s \int_{\mathbb{R}^N} |x|^\ell(u_2^p - u_1^p) V_{\mathcal{O}_+}(x, 0) \, dx = 0.
\]
Thus, \( V_{\mathcal{O}_+} \equiv 0 \) in \( \mathbb{R}^{N+1}_+ \), however, this contradicts \( V_{\mathcal{O}_+} > 0 \) in \( \mathcal{O}_+ \).

Next, we consider case (ii). In this case, let \( x_0 \in \mathbb{R}^N \) satisfy \( V_{\mathcal{O}_+}(x_0, 0) > 0 \). Then \( 0 < V_{\mathcal{O}_+}(x_0, 0) = V(x_0, 0) = u_2(x_0) - u_1(x_0) \) and \( B_r^+(x_0, 0) \subset \mathcal{O}_+ \) holds for some \( r > 0 \) and \( V_{\mathcal{O}_+}(x, 0) > 0 \) if \( |x - x_0| < r \). Let \( \psi \equiv 1 \) in (4.11) to get
\[
\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla V_{\mathcal{O}_+}|^2 dX = \kappa_s \int_{\mathbb{R}^N} |x|^\ell(u_2^p - u_1^p) V_{\mathcal{O}_+}(x, 0) \, dx.
\]
Moreover, by the trace theorem, \( V_{\mathcal{O}_+}(x, 0) \in H^s(\mathbb{R}^N) \) and put \( W_{\mathcal{O}_+}(x, t) := P_s(\cdot, t) * V_{\mathcal{O}_+}(\cdot, 0) \). By [29, (2.8)] and the density of \( C_0^\infty(\mathbb{R}^N) \) in \( H^s(\mathbb{R}^N) \) and \( C_0^\infty(\mathbb{R}^{N+1}_0) \) in \( H^1(\mathbb{R}^{N+1}_+, t^{1-2s}dX) \), respectively, we have
\[
\kappa_s\|V_{\mathcal{O}_+}(\cdot, 0)\|_{H^s(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla W_{\mathcal{O}_+}|^2 dX \leq \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla V_{\mathcal{O}_+}|^2 dX,
\]
which gives
\[
\|V_{\mathcal{O}_+}(\cdot, 0)\|_{H^s(\mathbb{R}^N)}^2 \leq \int_{\mathbb{R}^N} |x|^\ell(u_2^p - u_1^p) V_{\mathcal{O}_+}(x, 0) \, dx
\]
\[
= \int_{\mathbb{R}^N} |x|^\ell p \int_0^1 (\theta u_2 + (1 - \theta)u_1)^{p-1} d\theta (u_2 - u_1) V_{\mathcal{O}_+}(x, 0) \, dx.
\]
From the fact that \( V_{\mathcal{O}_+}(x, 0) > 0 \) yields \( V_{\mathcal{O}_+}(x) = u_2(x) - u_1(x) > 0 \), we infer that
\[
\|V_{\mathcal{O}_+}(\cdot, 0)\|_{H^s(\mathbb{R}^N)}^2 < p \int_{\mathbb{R}^N} |x|^\ell u_2^{p-1} V_{\mathcal{O}_+}^2 (x, 0) \, dx. \tag{4.13}
\]
Since \( V_{\mathcal{O}_+}(\cdot, 0) \in H^s(\mathbb{R}^N) \) and \( \text{supp} V_{\mathcal{O}_+}(\cdot, 0) \) is compact in \( \mathbb{R}^N \), (4.13) contradicts the stability of \( u_2 \) (see Remark 4.2). Hence, we conclude that \( \mathcal{O}_+ \) is unbounded. \( \square \)

Lemma 4.4. For \( k \in \mathbb{N} \), let \( I_{k,-} \) be a component of \([u_1 - u_2] > 0\) in Remark 4.2 and let \( \mathcal{O}_{k,-} \subset \mathbb{R}^{N+1}_+ \) be a component of \([V < 0]\) satisfying \( B_r^+(x, 0) \cap \mathbb{R}^{N+1}_+ \subset \mathcal{O}_{k,-} \) holds for each \( x \in I_{k,-} \) and some \( r_x > 0 \). For each \( \mathcal{O}_{k,-} \), define \( \mathcal{O}_{k,-} \) same as in (4.10). Then \( \text{supp} V_{\mathcal{O}_{k,-}}(\cdot, 0) \) is compact in \( \mathbb{R}^N \).

Remark 4.3. For each \( I_{k,\pm} \), we can choose the components \( \mathcal{O}_{k,\pm} \subset \mathbb{R}^{N+1}_+ \) of \([V > 0]\) and \([V < 0]\) in same way as above. In fact, if \( I_{k,\pm} = A_{ak,\pm, bk,\pm} = \{ x \in \mathbb{R}^N \mid a_{k,\pm} < |x| < b_{k,\pm} \} \), then we may choose a small \( \varepsilon_k > 0 \) so that
\[
V(x, t) = U_2(x, t) - U_1(x, t) > 0 \quad \text{if} \quad a_{k,\pm} + \varepsilon_k \leq |x| \leq b_{k,\pm} - \varepsilon_k, \quad 0 \leq t \leq \varepsilon_k,
\]
\[
V(x, t) = U_2(x, t) - U_1(x, t) < 0 \quad \text{if} \quad a_{k,\pm} + \varepsilon_k \leq |x| \leq b_{k,\pm} - \varepsilon_k, \quad 0 \leq t \leq \varepsilon_k.
\]
Proof of Lemma 4.4. Suppose that supp$V_{\mathcal{O}_{\kappa,-}}(\cdot, 0)$ is unbounded. By the assumption and $V_{\mathcal{O}_{\kappa,-}} \in C(\overline{\mathbb{R}^{N+1}_+ \setminus \{0\}})$ thanks to Lemma 4.2, we may take $(x_n)_n$ and $(r_n)_n$ so that

$$|x_n| < |x_{n+1}|, \quad |x_n| \to \infty \quad \text{as } n \to \infty, \quad V_{\mathcal{O}_{\kappa,-}} < 0 \quad \text{in } B_{r_n}(x_n, 0).$$

Hence, $B_{r_n}(x_n, 0) \cap \mathbb{R}^{N+1}_+ \subset \mathcal{O}_{\kappa,-}$ for each $n \geq 1$.

Next, we claim $V(x, 0) \leq 0$ for all $|x| \geq |x_1|$. If this is true, then the component $|V(x, 0) > 0| = |u_2 - u_1| > 0$ is bounded and this contradicts Lemma 4.1, and we conclude that $V_{\mathcal{O}_{\kappa,-}}(\cdot, 0)$ has compact support.

We show $V(x, 0) \leq 0$ for all $|x| \geq |x_1|$. Suppose that there exists a $y_0 \in \mathbb{R}^N$ such that $|y_0| > |x_1|$ and $V(y_0, 0) > 0$. Select $s_0 > 0$ and $n_0 \in \mathbb{N}$ so that $|x_{n_0}| < |y_0| < |x_{n_0+1}|$ and $V > 0$ in $B_{s_0}^+(y_0, 0)$. Since $B_{r_{n_0}}^+(x_{n_0}, 0) \cap \mathbb{R}^{N+1}_+ \subset \mathcal{O}_{\kappa,-}, B_{r_{n_0+1}}^+(x_{n_0+1}, 0) \cap \mathbb{R}^{N+1}_+ \subset \mathcal{O}_{\kappa,-}$ and $\mathcal{O}_{\kappa,-}$ is open and connected in $\mathbb{R}^{N+1}$, there exists a $\gamma_0(\tau) \in C([0, 1], \overline{\mathbb{R}^{N+1}})$ such that

$$\gamma_0(0) = (x_{n_0}, 0), \quad \gamma_0(1) = (x_{n_0+1}, 0), \quad \gamma_0(\tau) \in \mathcal{O}_{\kappa,-} \quad \text{for each } \tau \in (0, 1).$$

Let $\mathcal{O}_+ \subset \mathbb{R}^{N+1}_+$ be a component of $[V > 0]$ satisfying $B_{s_0}^+(y_0, 0) \cap \mathbb{R}^{N+1}_+ \subset \mathcal{O}_+$. Due to the existence of $\gamma_0$ and the radial symmetry of $V$, $\mathcal{O}_+$ becomes bounded (see Appendix A for the details). However, this contradicts Lemma 4.3. Thus, $V(x, 0) \leq 0$ for all $|x| \geq |x_1|$ and we complete the proof. \qed

Lemma 4.5. For each component $\mathcal{O}_- \subset \mathbb{R}^{N+1}_+$ of $[V < 0]$, let $V_{\mathcal{O}_-}$ be as in (4.10). If $\psi \in C^2_c(\overline{\mathbb{R}^{N+1}_+})$ satisfies

$$t^{1-2s} \partial_t \psi(x,t) \to 0 \quad \text{as } t \to 0,$$

then

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla V_{\mathcal{O}_-}|^2 \psi \, dX = \frac{1}{2} \int_{\mathcal{O}_-} V_{\mathcal{O}_-}^2 \text{div} \left( t^{1-2s} \nabla \psi \right) \, dX + \kappa_s \int_{\mathbb{R}^N} |x|^s (u_2 - u_1) V_{\mathcal{O}_-}(x, 0) \psi(x, 0) \, dx.$$ (4.14)

Proof. From (4.11), it suffices to prove

$$-\frac{1}{2} \int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla \left( V_{\mathcal{O}_-}^2 \right) \cdot \nabla \psi \, dX = \frac{1}{2} \int_{\mathcal{O}_-} V_{\mathcal{O}_-}^2 \text{div} \left( t^{1-2s} \nabla \psi \right) \, dX.$$ (4.15)

To this end, we use $\zeta_0$ in (4.12). We note that for each $X \in \mathcal{O}_-$, as $n \to \infty$,

$$\nabla \left( \frac{1}{n} \zeta_0 \left( nV_{\mathcal{O}_-}(X) \right) \right)^2 \cdot \nabla \psi(X) = 2 \frac{1}{n} \zeta_0 \left( nV_{\mathcal{O}_-}(X) \right) \zeta'_0 \left( nV_{\mathcal{O}_-}(X) \right) V_{\mathcal{O}_-}(X) \cdot \nabla \psi(X) \to 2V_{\mathcal{O}_-}(X) \nabla V_{\mathcal{O}_-}(X) \cdot \nabla \psi(X).$$

Let $R > 0$ satisfy supp$\psi \subset (-R, R)^N \times [0, R)$. Since $V_{\mathcal{O}_-} \in C(\overline{\mathbb{R}^{N+1}_+})$ due to (4.9) and Lemma 4.2,
by \((4.14)\), \(\zeta_0(nV_{\mathcal{O}_-}) \in C^\infty(\mathbb{R}_+^{N+1})\) and the dominated convergence theorem,

\[
-\frac{1}{2} \int_{\mathbb{R}_+^{N+1}} t^{1-2s} \nabla \left( V_{\mathcal{O}_-}^2 \right) \cdot \nabla \psi \, dX = -\frac{1}{2} \lim_{n \to \infty} \int_{\mathcal{O}_-} t^{1-2s} \nabla \left\{ \frac{1}{n} \zeta_0 \left( nV_{\mathcal{O}_-} \right) \right\}^2 \cdot \nabla \psi \, dX
\]

\[
= -\frac{1}{2} \lim_{n \to \infty} \int_{[-R,R]^{N} \times [0,R]} t^{1-2s} \nabla \left\{ \frac{1}{n} \zeta_0 \left( nV_{\mathcal{O}_-} \right) \right\}^2 \cdot \nabla \psi \, dX
\]

\[
= \frac{1}{2} \lim_{n \to \infty} \int_{[-R,R]^{N} \times [0,R]} \left\{ \frac{1}{n} \zeta_0 \left( nV_{\mathcal{O}_-} \right) \right\}^2 \text{div} \left( t^{1-2s} \nabla \psi \right) \, dX
\]

\[
= \frac{1}{2} \int_{\mathcal{O}_-} V_{\mathcal{O}_-}^2 \text{div} \left( t^{1-2s} \nabla \psi \right) \, dX.
\]

Therefore, under \((4.14)\), we get \((4.15)\) through \((4.11)\).

\[\square\]

**Lemma 4.6.** For \(k \in \mathbb{N}\), let \(\mathcal{O}_{k,-}\) and \(V_{\mathcal{O}_{k,-}}\) be as in Lemma 4.4. Then

\[
|V_{\mathcal{O}_{k,-}}(x,t)| \leq C_0 \left( |x|^2 + t^2 \right)^{-\frac{N-2s}{2}} \quad \text{for all } (x,t) \in \mathbb{R}_+^{N+1}.
\]

\[(4.16)\]

Furthermore, \(\nabla V_{\mathcal{O}_{k,-}} \in L^2(\mathbb{R}_+^{N+1}, t^{1-2s} dX)\).

**Proof.** We first prove \((4.16)\). Following [7], we set

\[
\Gamma(x,t) := \left( |x|^2 + t^2 \right)^{-\frac{N-2s}{2}}.
\]

Then \(-\text{div}(t^{1-2s} \nabla \Gamma) = 0 \) in \(\mathbb{R}_+^{N+1}\). Since \(V_{\mathcal{O}_{k,-}}(\cdot,0)\) has compact support in \(\mathbb{R}^N\) by Lemma 4.4 and \(V_{\mathcal{O}_{k,-}}(X) = 0\) for \(|X| \ll 1\) due to \((4.9)\), we may find a \(C_0 > 0\) such that

\[
0 \leq -V_{\mathcal{O}_{k,-}}(x,0) < C_0 \Gamma(x,0) \quad \text{for all } x \in \mathbb{R}^N.
\]

Notice also that \(-\text{div}(t^{1-2s} \nabla V) = 0 \) in \(\mathbb{R}_+^{N+1}\) due to \((4.5)\) and \((4.8)\). This together with \(V_{\mathcal{O}_{k,-}} = V\) in \(\mathcal{O}_{k,-}\) yields

\[
-\Delta_{t,x} \left( C_0 \Gamma + V_{\mathcal{O}_{k,-}} \right) - \frac{1}{t} \frac{2s}{t} \partial_t \left( C_0 \Gamma + V_{\mathcal{O}_{k,-}} \right) = 0 \quad \text{in } \mathcal{O}_{k,-}, \quad V_{\mathcal{O}_{k,-}} = 0 \quad \text{on } \partial \mathcal{O}_{k,-} \cap \mathbb{R}_+^{N+1}.
\]

Since \(U_r(X) \to 0\) as \(|X| \to \infty\), we have \(V_{\mathcal{O}_{k,-}}(X) \to 0\) as \(|X| \to \infty\) and the strong maximum principle asserts that \(C_0 \Gamma + V_{\mathcal{O}_{k,-}} \geq 0\) in \(\mathcal{O}_{k,-}\). Otherwise, since \(C_0 \Gamma + V_{\mathcal{O}_{k,-}} \geq 0\) on \(\partial \mathcal{O}_{k,-} \cup \mathbb{R}^N \times \{0\}\), we may find a negative global minimum \(X_0 \in \mathcal{O}_{k,-}\) of \(C_0 \Gamma + V_{\mathcal{O}_{k,-}}\). Then the strong maximum principle yields \(C_0 \Gamma + V_{\mathcal{O}_{k,-}} \equiv (C_0 \Gamma + V_{\mathcal{O}_{k,-}})(X_0) < 0\) in \(B_r^+(X_0)\) as long as \(B_r^+(X_0) \subset \mathcal{O}_{k,-}\). Thus, enlarging \(r > 0\) and noting \(V_{\mathcal{O}_{k,-}}(X) = 0\) for \(|X| \ll 1\), we have \(B_r^+(X_0) \cap \partial \mathcal{O}_{k,-} \neq \emptyset\) or \(\overline{B_r^+(X_0)} \cap \mathbb{R}^N \times \{0\} \neq \emptyset\) and this leads a contradiction. Therefore, \(C_0 \Gamma + V_{\mathcal{O}_{k,-}} \geq 0\) in \(\mathcal{O}_{k,-}\). By \(V_{\mathcal{O}_{k,-}} \equiv 0\) in \(\mathbb{R}_+^{N+1} \setminus \mathcal{O}_{k,-}\), \((4.16)\) holds.

Next we prove \(\nabla V_{\mathcal{O}_{k,-}} \in L^2(\mathbb{R}_+^{N+1}, t^{1-2s} dX)\). In \((4.15)\), we compute the term:

\[
\int_{\mathcal{O}_{k,-}} V_{\mathcal{O}_{k,-}}^2 \text{div} \left( t^{1-2s} \nabla \psi \right) \, dX.
\]

For \(R > 0\), we consider

\[
\psi(x,t) = \psi_R(x,t) := \psi_0(R^{-1}|x|)\psi_0(R^{-1}t),
\]
where \( \psi_0 \in C_c^\infty(\mathbb{R}) \) with \( \psi_0(\tau) = 1 \) for \( |\tau| \leq 1 \) and \( \psi_0(\tau) = 0 \) for \( |\tau| \geq 2 \). Then (4.16) is satisfied and

\[
div \left( t^{1-2s} \nabla \psi \right) = t^{1-2s} \Delta_{t,x} \psi + (1 - 2s)t^{-2s} \partial_t \psi
\]

By (4.16), we observe that

\[
\int R^{-2}t^{1-2s} \psi'_0'(R^{-1}t) \psi_0(R^{-1}x) + t^{1-2s} \psi'_0'(R^{-1}t) \left[ R^{-2} \psi''_0(R^{-1}x) + R^{-1} \frac{N-1}{|x|} \psi'_0(R^{-1}x) \right] + (1 - 2s)t^{-2s} R^{-1} \psi_0(R^{-1}x) \psi'_0(R^{-1}t).
\]

By (4.16), we observe that

\[
\begin{align*}
\int_{O_{k,-}} V_{O_{k,-}}^2 & \left| t^{1-2s} \psi'_0'(R^{-1}t) \psi_0(R^{-1}x) \right| dX \\
& \leq CR^{-2} \int_R^{2R} dt \int_{|x| \leq 2R} \frac{t^{1-2s}}{(|x|^2 + t^2)^{N-2s}} dx \\
& = CR^{-2} \int_R^{2R} t^{1-2s} dt \int_{|y| \leq 2R/t} t^{-2N+4s} (1 + |y|^2)^{-(N-2s)} t^N dy \\
& \leq CR^{-2} \int_R^{2R} t^{-N+2s+1} dt \int_{|y| \leq 2} (1 + |y|^2)^{-(N-2s)} dy \\
& \leq CR^{-N+2s}.
\end{align*}
\]

In a similar way,

\[
\begin{align*}
\int_{O_{k,-}} V_{O_{k,-}}^2 & \left| t^{-2s} R^{-1} \psi_0(R^{-1}x) \psi'_0(R^{-1}t) \right| dX \\
& \leq CR^{-1} \int_R^{2R} dt \int_{|x| \leq 2R} \frac{t^{-2s}}{(|x|^2 + t^2)^{N-2s}} dx, \quad \text{for } CR^{-1} \int_R^{2R} t^{-N+2s} dt \leq CR^{-N+2s}
\end{align*}
\]

and

\[
\begin{align*}
\int_{O_{k,-}} V_{O_{k,-}}^2 & t^{1-2s} \psi'_0'(R^{-1}t) \left[ R^{-2} \psi''_0(R^{-1}x) + R^{-1} \frac{N-1}{|x|} \psi'_0(R^{-1}x) \right] dX \\
& \leq C \int_0^{2R} dt \int_{R \leq |x| \leq 2R} \left( |x|^2 + t^2 \right)^{-(N-2s)} t^{1-2s} R^{-2} dx \\
& \leq C \int_0^{2R} \left( R^2 + t^2 \right)^{-N+2s} t^{1-2s} R^{2-N} dt \\
& = C \int_0^2 R^{-2N+4s} (1 + \tau^2)^{-N+2s} \tau^{1-2s} R^{N-2} R d\tau = CR^{-N+2s}.
\end{align*}
\]

Thus,

\[
\limsup_{R \to \infty} \int_{O_{k,-}} V_{O_{k,-}}^2 \left| \operatorname{div} \left( t^{1-2s} \nabla \psi_R \right) \right| dX = 0.
\]

Noting that \( V_{O_{k,-}}(x,0) \) has compact support by Lemma 4.4, we also have

\[
\limsup_{R \to \infty} \int_{\mathbb{R}^N} |x|^\ell \left( u_2^p - u_1^p \right) V_{O_{k,-}}(x,0) \psi_R(x,0) dx = \int_{\mathbb{R}^N} |x|^\ell \left( u_2^p - u_1^p \right) V_{O_{k,-}}(x,0) dx < \infty.
\]
Hence, (4.15), \( u_1 \in L^\infty(\mathbb{R}^N) \) and the fact that \( V_{\mathcal{O}_{k,-}}(x,0) < 0 \) implies \( u_2(x) - u_1(x) < 0 \) give

\[
\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla V_{\mathcal{O}_{k,-}}|^2 \, dX = \kappa_s \int_{\mathbb{R}^N} |x|^s (u_1^p - u_2^p) V_{\mathcal{O}_{k,-}}(x,0) \, dx \\
= \kappa_s \int_{\mathbb{R}^N} |x|^s (u_1^p - u_2^p) (-V_{\mathcal{O}_{k,-}}(x,0)) \, dx \\
< \kappa_s p \int_{\mathbb{R}^N} |x|^s u_1^{p-1} V_{\mathcal{O}_{k,-}}^2(x,0) \, dx < \infty.
\]

Therefore, \( \nabla V_{\mathcal{O}_{k,-}} \in L^2(\mathbb{R}^{N+1}_+, t^{1-2s} dX) \).

Now, we shall complete the proof of Proposition 4.1.

**Proof of Proposition 4.1.** Recalling Lemmata 4.2 and 4.4, we have \( V_{\mathcal{O}_{k,-}}(\cdot, 0) \in H^s(\mathbb{R}^N) \) and

\[
\kappa_s \| V_{\mathcal{O}_{k,-}}(\cdot, 0) \|^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla (P_s(\cdot, t) * V_{\mathcal{O}_{k,-}}(\cdot, 0)))(x) |^2 \, dX.
\]

Notice that the last assertion follows from [29, (2.7)].

In the following, we claim

\[
\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla (P_s(\cdot, t) * V_{\mathcal{O}_{k,-}}(\cdot, 0)) |^2 \, dX \leq \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla V_{\mathcal{O}_{k,-}}|^2 \, dX.
\]

(4.20)

If this is true, then we get a contradiction for the stability of \( u_1 \) from (4.19), (4.20) and the compactness of \( \text{supp}\, V_{\mathcal{O}_{k,-}}(\cdot, 0) \):

\[
\| V_{\mathcal{O}_{k,-}}(\cdot, 0) \|^2_{H^s(\mathbb{R}^N)} < p \int_{\mathbb{R}^N} |x|^s u_1^{p-1} V_{\mathcal{O}_{k,-}}^2(x,0) \, dx.
\]

Hence, (4.4) does not hold and we complete the proof of Proposition 4.1.

We show (4.20). Set \( W(x,t) := (P_s(\cdot, t) * V_{\mathcal{O}_{k,-}}(\cdot, 0)))(x) \). Notice that

\[- \text{div}(t^{1-2s} \nabla W) = 0 \quad \text{in} \ \mathbb{R}^{N+1}_+, \quad W(x,0) = V_{\mathcal{O}_{k,-}}(x,0) \leq 0, \quad \nabla W \in L^2(\mathbb{R}^{N+1}_+, t^{1-2s} dX).
\]

By Lemma B.2 and the fact that \( V_{\mathcal{O}_{k,-}}(x,0) \) has compact support, \( W(X) \to 0 \) as \( |X| \to \infty \). Since \( V_{\mathcal{O}_{k,-}}(x,0) = 0 \) for \( |x| \ll 1 \) by (4.2) and \( V_{\mathcal{O}_{k,-}}(\cdot,0) \in C(\mathbb{R}^N) \), as in Lemma 4.6, one may check that

\[
0 \leq -W(X) \leq C_0(|x|^2 + t^2)^{-\frac{N-2s}{2}} \quad \text{for all} \ X \in \mathbb{R}^{N+1}_+.
\]

(4.21)

Moreover, for \( \psi_R(x,t) = \psi_0(R^{-1} t) \psi_0(R^{-1} |x|) \) with \( R \gg 1 \) and \( \psi_0 \) as in the proof of Lemma 4.6, we obtain \( \psi_R(x,0)V_{\mathcal{O}_{k,-}}(x,0) = V_{\mathcal{O}_{k,-}}(x,0) \). By \( \psi_R(W - V_{\mathcal{O}_{k,-}}) \in H^1(\mathbb{R}^{N+1}_+, t^{1-2s} dX) \) with \( \psi_R(x,0)(W(x,0) - V_{\mathcal{O}_{k,-}}(x,0)) \equiv 0 \), we infer from [29, Lemma 2.2] that

\[
\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla W \cdot \nabla ( \psi_R (W - V_{\mathcal{O}_{k,-}})) \, dX = \kappa_s \langle V_{\mathcal{O}_{k,-}}, \psi_R(\cdot,0)(W(\cdot,0) - V_{\mathcal{O}_{k,-}}(\cdot,0)) \rangle_{H^s(\mathbb{R}^N)} = 0.
\]

(4.22)

Thus, if

\[
\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla W \cdot \nabla \psi_R (W - V_{\mathcal{O}_{k,-}}) \, dX \to 0 \quad \text{as} \ R \to \infty,
\]

(4.23)
then the facts \( \nabla V_{O_{k,-}}, \nabla W \in L^2(\mathbb{R}_+^{N+1}, t^{1-2s}dX) \), (4.22) and (4.23) imply

\[
\int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla W|^2 dX = \int_{\mathbb{R}_+^{N+1}} t^{1-2s} \nabla W \cdot \nabla V_{O_{k,-}} dX \\
\leq \left[ \int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla W|^2 dX \right]^{1/2} \left[ \int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla V_{O_{k,-}}|^2 dX \right]^{1/2} < \infty
\]

and (4.20) holds.

To prove (4.23), we see that

\[
\int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla W : (\nabla \psi_R) (W - V_{O_{k,-}})| dX \\
\leq \left[ \int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla W|^2 dX \right]^{1/2} \left[ \int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla \psi_R|^2 (V_{O_{k,-}}^2 + W^2) dX \right]^{1/2}.
\]

By Lemma 4.6 and (4.21), as in (4.17) and (4.18), we may prove

\[
\int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla \psi_R|^2 (V_{O_{k,-}}^2 + W^2) dX \\
\leq CR^{-2} \int_R^{2R} dt \int_{|x| \leq 2R} t^{1-2s} \left( |x|^2 + t^2 \right)^{N-2s} dx + CR^{-2} \int_0^{2R} dt \int_{R \leq |x| \leq 2R} t^{1-2s} \left( |x|^2 + t^2 \right)^{N-2s} dx \\
\leq CR^{-N+2s} \rightarrow 0 \quad (R \rightarrow \infty).
\]

Thus, (4.23) holds and we complete the proof of Proposition 4.1.

\[\square\]

### 4.2 Some properties of stable solutions

In this subsection, we investigate some properties of positive radial stable solutions. We first obtain the separation property of \((u_{m,0})_{m>0}\) as a corollary of Corollary 3.1 and Proposition 4.1:

**Corollary 4.1.** For \((u_{m,0})_{m>0}\) in Corollary 3.1, if \(m_1 < m_2\), then \(0 < u_{m_1,0} < u_{m_2,0}\) in \(\mathbb{R}^N\).

**Proof.** Notice that \(\bar{u}_0 \in C(\mathbb{R}^N)\), \(\bar{u}_0(0) > 0\), \(m \bar{u}_0(0) = u_{m,0}(0)\) and \(m = \|u_{m,0}\|_{L^\infty(\mathbb{R}^N)}\). Applying Proposition 4.1 for \(u_{m_1,0}\) and \(u_{m_2,0}\) with \(m_1 < m_2\), we have \(u_{m_1,0} < u_{m_2,0}\) in \(\mathbb{R}^N\). \(\square\)

Remark that by Corollary 4.1, the limit \(\lim_{m \to \infty} u_{m,0}(x) =: u_{\infty,0}(x)\) exists for any \(x \in \mathbb{R}^N\).

**Proposition 4.2.** The sequence \((\varphi u_{m,0})_{m}\) is bounded in \(H^s(\mathbb{R}^N)\) for each \(\varphi \in C_c^\infty(\mathbb{R}^N)\). Furthermore, \(u_{\infty,0} = u_S\) and \(u_{m,0} \to u_S\) strongly in \(H^s_{\text{loc}}(\mathbb{R}^N)\) as \(m \to \infty\).

**Proof.** Set \(U_m(x,t) := (P_s(\cdot, t) \ast u_{m,0})(x)\) and let \(\psi \in C_c^\infty(\overline{\mathbb{R}_+^{N+1}})\). By the trace theorem, it suffices to show that \((\psi U_m)_m\) is bounded in \(H^1(\mathbb{R}_+^{N+1}, t^{1-2s}dX)\). From \(0 \leq u_{m,0} \leq u_S\), we remark that

\[0 \leq U_m(X) \leq U_S(X) = (P_s(\cdot, t) \ast u_S)(x).
\]

By \(U_S \in H^1_{\text{loc}}(\overline{\mathbb{R}_+^{N+1}}, t^{1-2s}dX)\) due to [29, Lemma 2.1], we observe that \((U_m)_m\) is bounded in \(L^2_{\text{loc}}(\overline{\mathbb{R}_+^{N+1}}, t^{1-2s}dX)\).

Next, since \(U_m\) satisfies

\[
\int_{\mathbb{R}_+^{N+1}} t^{1-2s} \nabla U_m \cdot \nabla \varphi dX = \kappa_s \langle u_{m,0}, \varphi(\cdot, 0) \rangle_{H^s(\mathbb{R}^N)} = \kappa_s \int_{\mathbb{R}^N} |x|^2 u_{m,0}^p \varphi(x, 0) dx
\]

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for all \( \varphi \in H^1(\mathbb{R}_+^{N+1}, t^{1-2s}dX) \) with compact support in \( \mathbb{R}_+^{N+1} \), we may test \( \psi^2 U_m \) and get

\[
\int_{\mathbb{R}_+^{N+1}} t^{1-2s} \nabla U_m \cdot \nabla (\psi^2 U_m) \, dX = \kappa_s \int_{\mathbb{R}^N} |x| t^{p+1} \psi^2(x, 0) \, dx \leq \kappa_s \int_{\mathbb{R}^N} |x| t^{p+1} \psi^2(x, 0) \, dx.
\]

Since \( |x| t^{p+1} \in L^1_{\text{loc}}(\mathbb{R}^N) \) due to \( p > p_S(N, \ell) \), applying the same argument as in (3.30), we may verify that \( (\nabla(\psi U_m))_m \) is bounded in \( L^2(\mathbb{R}_+^{N+1}, t^{1-2s}dX) \).

Since \( u_m(x) \not\rightarrow u_{\infty,0}(x) \) as \( m \rightarrow \infty \), it holds that \( u_{m,0} \rightarrow u_{\infty,0} \) weakly in \( H^s_{\text{loc}}(\mathbb{R}^N) \) as \( m \rightarrow \infty \).

By \( u_{m,0}(x) \leq u_{m,0}(x) \) for \( m_1 < m_2 \) and \( u_{m,0}(x) \leq u_S(x) \), one sees that

\[
u_{\infty,0}(x) \leq u_S(x), \quad \int_{\mathbb{R}^N} |x| t^{p+1} u_{m,0}(x) \varphi(x) \, dx \rightarrow \int_{\mathbb{R}^N} |x| t^{p+1} u_{\infty,0}(x) \varphi(x) \, dx \quad \text{for all} \ \varphi \in C^\infty_c(\mathbb{R}^N).
\]

Furthermore, from \( u_{m,0}(x) \leq u_{\infty,0}(x) \leq u_S(x) \) and \( u_{m,0} \rightarrow u_{\infty,0} \) weakly in \( H^s_{\text{loc}}(\mathbb{R}^N) \), as in (3.26), it follows that \( 0 < u_{\infty,0} \) and

\[
\langle u_{m,0}, \varphi \rangle_{H^s(\mathbb{R}^N)} \rightarrow \langle u_{\infty,0}, \varphi \rangle_{H^s(\mathbb{R}^N)} \quad \text{for each} \ \varphi \in C^\infty_c(\mathbb{R}^N).
\]

Hence, \( (-\Delta)^s u_{\infty,0} = |x| t^{p+1} u_{\infty,0} \) in \( \mathbb{R}^N \).

From the definition of \( u_{m,0} \) in Corollary 3.1, it follows that for every \( m > 0 \),

\[
mu_{\infty,0}(m^{1/\theta_0} x) = \lim_{k \to \infty} mu_{k,0}(m^{1/\theta_0} x) = \lim_{k \to \infty} mk\tilde{u}_0(m^{1/\theta_0} k^{1/\theta_0} x) = \lim_{t \to \infty} t\tilde{u}_0(t^{1/\theta_0} x) = \lim_{t \to \infty} u_t(x) = u_{\infty,0}(x).
\]

For any \( |x| > 0 \), choose \( m = |x|^{-\theta_0} \). Then, since \( u_{\infty,0} \) is radially symmetric, we obtain

\[
u_{\infty,0}(x) = |x|^{-\theta_0} u_{\infty,0}(|x|^{-\theta_0})^{1/\theta_0} x = |x|^{-\theta_0} u_{\infty,0}(1) = C u_S(x).
\]

Recalling that \( u_{\infty,0} \) satisfies \( (-\Delta)^s u_{\infty,0} = |x| t^{p+1} u_{\infty,0} \) in \( \mathbb{R}^N \), we have \( C = 1 \) and \( u_{\infty,0} = u_S \).

Since \( u_S \in L^2_{\text{loc}}(\mathbb{R}^N) \) and \( u_{m,0}(x) \not\rightarrow u_{\infty,0}(x) = u_S(x) \) for \( x \in \mathbb{R}^N \) as \( m \to \infty \), we have \( u_{m,0} \to u_S \) strongly in \( L^2_{\text{loc}}(\mathbb{R}^N) \). What remains to prove is

\[
\lim_{m \to \infty} \int_{B_R \times B_R} \frac{(v_m(x) - v_m(y))^2}{|x-y|^{N+2s}} dxdy = 0 \quad \text{for each} \ R > 0,
\]

where \( v_m(x) := u_S(x) - u_{m,0}(x) \).

Let \( R > 0 \) be given and choose a radial cut-off function \( \varphi_R \in C^\infty_c(\mathbb{R}^N) \) so that

\[
\varphi_R \equiv 1 \quad \text{on} \ B_R, \quad \varphi_R \equiv 0 \quad \text{on} \ B_R^c.
\]

Since \( \varphi_R v_m \in H^s(\mathbb{R}^N) \) with compact support, we obtain

\[
\langle v_m, \varphi_R v_m \rangle_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |x|^{\ell} (u^p_S - u^p_{m,0}) \varphi_R v_m \, dx.
\]

By \( v_m(x) \to 0 \) for each \( x \in \mathbb{R}^N \setminus \{0\} \) and the dominated convergence theorem,

\[
\lim_{m \to \infty} \int_{\mathbb{R}^N} |x|^{\ell} (u^p_S - u^p_{m,0}) \varphi_R v_m \, dx = 0.
\]
To compute the left-hand side of (4.26), we decompose \( \mathbb{R}^N \) into \( \mathbb{R}^N = B_R \cup A_{R,2R} \cup B_{2R}^c \), where \( A_{R,2R} := \{ x \in \mathbb{R}^N \mid R \leq |x| < 2R \} \). It follows from (4.25) that

\[
\langle v_m, \varphi_R v_m \rangle_{H^s(\mathbb{R}^N)} = \int_{B_R \times B_R} \frac{(v_m(x) - v_m(y))^2}{|x-y|^{N+2s}} \, dx \, dy + \int_{A_{R,2R} \times A_{R,2R}} \frac{(v_m(x) - v_m(y))((\varphi_R v_m)(x) - (\varphi_R v_m)(y))}{|x-y|^{N+2s}} \, dx \, dy + 2 \left( \int_{B_R \times B_{2R}} + \int_{B_{2R} \times B_{2R}} + \int_{A_{R,2R} \times B_{2R}} \right) \frac{(v_m(x) - v_m(y))((\varphi_R v_m)(x) - (\varphi_R v_m)(y))}{|x-y|^{N+2s}} \, dx \, dy \]

\[
= \int_{B_R \times B_R} \frac{(v_m(x) - v_m(y))^2}{|x-y|^{N+2s}} \, dx \, dy + I_{m,1} + 2(I_{m,2} + I_{m,3} + I_{m,4}).
\]

(4.28)

We shall prove \( I_{m,j} \to 0 \) as \( m \to \infty \) for \( j = 1, 2, 3, 4 \). To this end, by [17, Proposition 3.2 and Lemma 3.3] ( [29, Proposition 2.1]), we remark that \( (v_m)_{m \geq 1} \) is bounded in \( C^{1,\beta}_\text{loc}(B_{R/2}^c) \) for some \( \beta > 0 \). From \( v_m(x) \to 0 \) for any \( x \in \mathbb{R}^N \setminus \{0\} \), we obtain

\[
v_m \to 0 \quad \text{in} \quad C^{1}_\text{loc} \left( B_{R/2}^c \right).
\]

(4.29)

Since \( v_m \) and \( \varphi_R \) are radially symmetric, for each \( x \in A_{R/2,2R} \) and \( y \in A_{R/2,2R} \), we have

\[
|v_m(x) - v_m(y)| = |v_m(|x|e_1) - v_m(|y|e_1)| \leq \|v_m\|_{C^1(A_{R/2,2R})} |x| - |y|, \quad (4.30)
\]

and

\[
|\varphi_R(x)v_m(x) - \varphi_R(y)v_m(y)| \leq \|\varphi_R v_m\|_{C^1(A_{R/2,2R})} |x - y|. \quad (4.31)
\]

Using (4.30) and (4.31), we obtain

\[
|I_{m,1}| \leq \|v_m\|_{C^1(A_{R/2,2R})} \|\varphi_R v_m\|_{C^1(A_{R/2,2R})} \int_{A_{R,2R} \times A_{R,2R}} |x-y|^{-N+2-2s} \, dx \, dy = o(1). \quad (4.32)
\]

Next, by \( v_m(x) \to 0 \) and \( |v_m(x)| \leq u_S(x) \) for each \( x \in \mathbb{R}^N \setminus \{0\} \), (4.29), (4.30) and the dominated convergence theorem, we obtain

\[
|I_{m,2}| \leq \int_{B_R \times A_{R,2R}} \frac{|v_m(x) - v_m(y)|((\varphi_R v_m)(x) - (\varphi_R v_m)(y))}{|x-y|^{N+2s}} \, dx \, dy
\]

\[
= \left( \int_{B_{R/2} \times A_{R,2R}} + \int_{A_{R/2,2R} \times A_{R,2R}} \right) \frac{|v_m(x) - v_m(y)|((\varphi_R v_m)(x) - (\varphi_R v_m)(y))}{|x-y|^{N+2s}} \, dx \, dy
\]

\[
\leq \left( \frac{R}{2} \right)^{N+2s} \int_{B_{R/2} \times A_{R,2R}} |v_m(x) - v_m(y)|((\varphi_R v_m)(x) - (\varphi_R v_m)(y)) \, dx \, dy
\]

\[
+ \|v_m\|_{C^1(A_{R/2,2R})} \|\varphi_R v_m\|_{C^1(A_{R/2,2R})} \int_{A_{R/2,2R} \times A_{R,2R}} |x-y|^{2-N+2s} \, dx \, dy
\]

\[
= o(1).
\]
For $I_{m,3}$, the fact $\|v_m\|_{L^\infty(B_{2R}^c)} \leq \|u_S\|_{L^\infty(B_{2R}^c)} \leq C_R$ and the dominated convergence theorem yield

$$|I_{m,3}| \leq \int_{B_R \times B_{2R}^c} \frac{|v_m(x) - v_m(y)| v_m(x)}{|x - y|^{N+2s}} \, dx \, dy \leq \int_{B_R} \left( \|v_m\|_{L^\infty(B_{2R}^c)} v_m(x) \int_{B_{2R}^c} |x - y|^{-N-2s} \, dy \right) dx \tag{4.34}$$

$$\leq C_R \int_{B_R} \{v_m(x) + 1\} v_m(x) \, dx = o(1).$$

For $I_{m,4}$, we have

$$|I_{m,4}| \leq \int_{A_{2R} \times B_{2R}^c} \frac{|v_m(x) - v_m(y)| |(\varphi_R v_m)(x) - (\varphi_R v_m)(y)|}{|x - y|^{N+2s}} \, dx \, dy$$

$$= \left( \int_{A_{2R} \times A_{2R} \setminus B_{2R}^c} + \int_{A_{2R} \times B_{2R}^c} \right) \frac{|v_m(x) - v_m(y)| |(\varphi_R v_m)(x) - (\varphi_R v_m)(y)|}{|x - y|^{N+2s}} \, dx \, dy.$$

We remark that (4.30) and (4.32) hold for $(i)$ Corollary 4.2. (ii) Let $u$ be a positive radial stable solution of (1.1) and $u(x) \to 0$ as $|x| \to \infty$.

(a) If $u \in L^\infty(\mathbb{R}^N)$, then $u(x) = u_{\tilde{m},0}(x)$ holds for $\tilde{m} = \|u\|_{L^\infty(\mathbb{R}^N)}$.

(b) If $u \in L^\infty_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$, $u \leq C u_S$ in $B_1$ for some $C > 0$ and $u(x) \to \infty$ as $|x| \to 0$, then $u_S \leq u$ in $\mathbb{R}^N$.

Thus, $I_{m,4} = o(1). \tag{4.35}$

From (4.26)–(4.28) and (4.32)–(4.35), it follows that

$$\int_{B_R \times B_R} \frac{(v_m(x) - v_m(y))^2}{|x - y|^{N+2s}} \, dx \, dy = o(1)$$

and (4.24) holds. Thus, we complete the proof.

Finally, as a corollary of Proposition 4.1, we obtain the following:

**Corollary 4.2.** (i) Let $u$ be a positive radial stable solution of (1.1) and $u(x) \to 0$ as $|x| \to \infty$.

(a) If $u \in L^\infty(\mathbb{R}^N)$, then $u(x) = u_{\tilde{m},0}(x)$ holds for $\tilde{m} = \|u\|_{L^\infty(\mathbb{R}^N)}$.

(b) If $u \in L^\infty_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$, $u \leq C u_S$ in $B_1$ for some $C > 0$ and $u(x) \to \infty$ as $|x| \to 0$, then $u_S \leq u$ in $\mathbb{R}^N$.

(ii) If $u(x) \in H^s_{\text{loc}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ is a positive radial solution of (1.1) with $u \leq u_S$ in $\mathbb{R}^N$, then either $u = u_{\tilde{m},0}$ with $\tilde{m} = \|u\|_{L^\infty(\mathbb{R}^N)}$ or else $u = u_S$. 

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By Definition 1.2, we only consider the case \( u \) obtained in Corollary 3.1. By Remark 3.6, this family may be rewritten as (5.1). Thus, Theorem 1.1 (ii) holds.

5.1 Preliminaries

In this section, we prove Theorems 1.2 and 1.3, and Corollary 1.1. Throughout this section, we always assume (1.2).

5 Proof of Theorems 1.2 and 1.3, and Corollary 1.1

In this section, we prove Theorems 1.2 and 1.3, and Corollary 1.1. Throughout this section, we always assume (1.2).

5.1 Preliminaries

We first recall the definition of \( \lambda(\alpha) \):

\[
\lambda(\alpha) := 2^\alpha \frac{\Gamma\left(\frac{N+2s}{4} + \frac{\alpha}{2}\right) \Gamma\left(\frac{N+2s}{4} - \frac{\alpha}{2}\right)}{\Gamma\left(\frac{N-2s}{4} - \frac{\alpha}{2}\right) \Gamma\left(\frac{N-2s}{4} + \frac{\alpha}{2}\right)}.
\]

By Definition 1.2, we only consider the case

\[
p_S(N, \ell) = \frac{N + 2s + 2\ell}{N - 2s} < p < \infty.
\] (5.1)

Under (5.1), \( p \) is JL-subcritical (resp. JL-supercritical) if and only if

\[
p \lambda(\beta_{\ell,N,p,s}) > \lambda(0) \quad \text{(resp. } p \lambda(\beta_{\ell,N,p,s}) < \lambda(0)\text{)}, \quad \beta_{\ell,N,p,s} := \frac{N - 2s}{2} - \frac{2s + \ell}{p - 1}.
\] (5.2)

Next, we change the variable from \( p \) to \( x \) as follows. Noting that \( \beta_{\ell,N,p,s} \) is strictly increasing in \( p \in (p_S(N, \ell), \infty) \) and

\[
\lim_{p \to p_S(N, \ell)} \beta_{\ell,N,p,s} = 0, \quad \lim_{p \to \infty} \beta_{\ell,N,p,s} = \frac{N - 2s}{2},
\]
we set
\[ A_{N,s} := \frac{N - 2s}{4}, \quad x := \frac{\beta_{N,p,s}}{2} \in (0, A_{N,s}). \]
Then \( p \) is expressed as
\[ p = \frac{N + 2s + 2\ell - 4x}{N - 2s - 4x}. \]
It is also convenient to use \( A_{N,s} \) instead of \( N \) and notice that
\[ \frac{N + 2s}{4} = A_{N,s} + s, \quad N - 2s - 4x = 4A_{N,s} - 4x = 4(A_{N,s} - x), \]
\[ N + 2s + 2\ell - 4x = 4(A_{N,s} - x) + 4s + 2\ell. \]
Using \( x \), we shall study the validity of the following inequality for \( x \in (0, A_{N,s}) \) instead of (5.2):
\[ g_{\ell}(x) > \frac{(\Gamma(A_{N,s} + s))^2}{(\Gamma(A_{N,s}))^2} =: \tilde{M}_{N,s} \quad \text{(resp. } g_{\ell}(x) < \tilde{M}_{N,s}), \]
where
\[ g_{\ell}(x) := \frac{A_{N,s} - x + s + \frac{\ell}{2}}{A_{N,s} - x} \cdot \frac{\Gamma(A_{N,s} + s - x)\Gamma(A_{N,s} + s + x)}{\Gamma(A_{N,s} - x)\Gamma(A_{N,s} + x)} \]
\[ = \left( A_{N,s} - x + s + \frac{\ell}{2} \right) \frac{\Gamma(A_{N,s} + s - x)\Gamma(A_{N,s} + s + x)}{\Gamma(A_{N,s} + 1 - x)\Gamma(A_{N,s} + x)} \in C([0, A_{N,s}]). \] (5.4)
Remark that (5.3) is equivalent to the case where \( p \) is JL-subcritical (resp. JL-supercritical). In addition, \( x = 0 \) (resp. \( x = A_{N,s} \)) corresponds to \( p = p_S(N, \ell) \) (resp. \( p = \infty \)). It is also easily seen that
\[ g_0(x) = \frac{\Gamma(A_{N,s} + 1 + s - x)\Gamma(A_{N,s} + s + x)}{\Gamma(A_{N,s} + 1 - x)\Gamma(A_{N,s} + x)} \] (5.5)
and
\[ g_{\ell}(x) = \frac{A_{N,s} - x + s + \frac{\ell}{2}}{A_{N,s} - x + s} g_0(x). \] (5.6)

We first prove that \( p \) is JL-subcritical if \( p \) is close to \( p_S(N, \ell) \).

**Lemma 5.1.** There exists a \( p_0 = p_0(\ell, N, p, s) \in (p_S(N, \ell), \infty) \) such that each \( p \in (p_S(N, \ell), p_0) \) is JL-subcritical.

**Proof.** It is easily seen from (1.2) that
\[ \lim_{x \downarrow 0} \frac{A_{N,s} - x + s + \frac{\ell}{2}}{A_{N,s} - x} = \frac{A_{N,s} + s + \frac{\ell}{2}}{A_{N,s}} = p_S(N, \ell) > 1. \]
Furthermore,
\[ \lim_{x \downarrow 0} \frac{\Gamma(A_{N,s} + s - x)\Gamma(A_{N,s} + s + x)}{\Gamma(A_{N,s} - x)\Gamma(A_{N,s} + x)} = \frac{\left(\Gamma(A_{N,s} + s)\right)^2}{(\Gamma(A_{N,s}))^2}. \]
From these two facts with (5.4), we may find an \( x_0 = x_0(\ell, N, p, s) \in (0, A_{N,s}) \) such that (5.3) holds for any \( x \in (0, x_0) \). From the change of variables, we see that Lemma 5.1 holds. \( \square \)
Let $-2s < \ell \leq 0$ and set
\[ h_\ell(x) := \log g_\ell(x) = \log \left( A_{N,s} + s + \frac{\ell}{2} - x \right) + \log \Gamma(A_{N,s} + s - x) - \log \Gamma(A_{N,s} + 1 - x) \]
\[ + \log \Gamma(A_{N,s} + s + x) - \log \Gamma(A_{N,s} + x). \] (5.7)

Then we see that (5.3) is equivalent to
\[ h_\ell(x) > 2 \left[ \log \Gamma(A_{N,s} + s) - \log \Gamma(A_{N,s}) \right] \] (resp. \[ h_\ell(x) < 2 \left[ \log \Gamma(A_{N,s} + s) - \log \Gamma(A_{N,s}) \right] \]. (5.8) (5.9)

We next show that $h_\ell(x)$ is strictly concave in $[0, A_{N,s}]$. This is helpful to see the range of $x$, where (5.8) or (5.9), namely (5.3), holds.

**Lemma 5.2.** Let $-2s < \ell \leq 0$ and $h_\ell$ be as in (5.7).

(i) The function $h_\ell$ is strictly concave in $[0, A_{N,s}]$.

(ii) Suppose $\ell = 0$.

(a) If $A_{N,s} > 1/2$, then $h_0'(x) > 0$ for each $x \in (0, 1/2)$, $h_0'(1/2) = 0$ and $h_0'(x) < 0$ for each $x \in (1/2, A_{N,s})$.

(b) If $A_{N,s} \leq 1/2$, then $h_0'(x) > 0$ for all $x \in (0, A_{N,s})$.

**Proof.** (i) Denote by $\psi(z)$ the digamma function, that is
\[ \psi(z) := \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \] (5.10)

By (5.7) and (5.10) it holds that
\[ h_\ell'(x) = -\frac{1}{A_{N,s} + s + \ell/2 - x} - \psi(A_{N,s} + s - x) + \psi(A_{N,s} + 1 - x) \]
\[ + \psi(A_{N,s} + s + x) - \psi(A_{N,s} + x) \]
and
\[ h_\ell''(x) = -\frac{1}{(A_{N,s} + s + \ell/2 - x)^2} + \psi'(A_{N,s} + s - x) - \psi'(A_{N,s} + 1 - x) \]
\[ + \psi'(A_{N,s} + s + x) - \psi'(A_{N,s} + x). \] (5.11)

For $\psi'(z)$, we have the following expression (see, e.g., [1,34])
\[ \psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z + n)^2}. \] (5.12)

Thus, for $x \in (0, A_{N,s})$, it follows that
\[ \psi'(A_{N,s} + s + x) - \psi'(A_{N,s} + x) = \sum_{n=0}^{\infty} \left[ \frac{1}{(A_{N,s} + s + x + n)^2} - \frac{1}{(A_{N,s} + x + n)^2} \right] < 0. \]
For the first three terms in (5.11), since $-2s < \ell \leq 0$, we observe that

$$
- \frac{1}{(A_{N,s} + s + \ell/2 - x)^2} + \psi'(A_{N,s} + s - x) - \psi'(A_{N,s} + 1 - x)
$$

$$
= - \frac{1}{(A_{N,s} + s + \ell/2 - x)^2} + \sum_{n=0}^{\infty} \frac{1}{(A_{N,s} + s - x + n)^2} - \sum_{n=0}^{\infty} \frac{1}{(A_{N,s} + 1 - x + n)^2}
$$

$$
= - \frac{1}{(A_{N,s} + s + \ell/2 - x)^2} + \sum_{n=0}^{\infty} \left\{ \frac{1}{(A_{N,s} + s - x + 1 + n)^2} - \frac{1}{(A_{N,s} + 1 - x + n)^2} \right\}
$$

$$
< 0.
$$

Hence, $h_\ell''(x) < 0$ for each $x \in (0, A_{N,s})$ when $-2s < \ell \leq 0$ and $h_\ell$ is strictly concave.

(ii) We assume $\ell = 0$ and compute $h_0'(x)$. To this end, we use the following expression for $\psi(z)$ (see, e.g., [1]):

$$
\psi(z) = -\gamma + \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+z} \right), \quad \gamma := \lim_{n \to \infty} \left( -\log n + \sum_{k=1}^{n} \frac{1}{k} \right) \in (0, 1).
$$

(5.13)

Remark that when $\ell = 0$, by (5.5), we have

$$
h_0'(x) = -\psi(A_{N,s} + s + 1 - x) + \psi(A_{N,s} + 1 - x) + \psi(A_{N,s} + s + x) - \psi(A_{N,s} + x).
$$

(5.14)

Since

$$
- \psi(A_{N,s} + 1 + s - x) + \psi(A_{N,s} + 1 - x)
$$

$$
= \sum_{n=0}^{\infty} \left( \frac{1}{n + A_{N,s} + 1 + s - x} - \frac{1}{n + A_{N,s} + 1 - x} \right)
$$

$$
= \sum_{n=0}^{\infty} \frac{-s}{(n + A_{N,s} + 1 + s - x)(n + A_{N,s} + 1 - x)},
$$

and

$$
\psi(A_{N,s} + s + x) - \psi(A_{N,s} + x)
$$

$$
= \sum_{n=0}^{\infty} \left( -\frac{1}{n + A_{N,s} + s + x} + \frac{1}{n + A_{N,s} + x} \right)
$$

$$
= \sum_{n=0}^{\infty} \frac{s}{(n + A_{N,s} + s + x)(n + A_{N,s} + x)},
$$

it follows from (5.14) that

$$
h_0'(x) = \sum_{n=0}^{\infty} \frac{(n + A_{N,s} + 1 + s - x)(n + A_{N,s} + 1 - x) - (n + A_{N,s} + s + x)(n + A_{N,s} + x)}{(n + A_{N,s} + 1 + s - x)(n + A_{N,s} + 1 - x)(n + A_{N,s} + s + x)(n + A_{N,s} + x)}.
$$

(5.15)

Then we compute the numerator in (5.15), that is

$$
(n + A_{N,s} + 1 + s - x)(n + A_{N,s} + 1 - x) - (n + A_{N,s} + s + x)(n + A_{N,s} + x)
$$

$$
= (-2x + 1)(2n + 2A_{N,s} + 1 + s).
$$

(5.16)

Hence, from (5.15) and (5.16), (a) and (b) clearly hold.
5.2 Analysis at \( x = A_{N,s} \)

Let \( h_\ell \) be as in (5.7). Then, by Lemmata 5.1 and 5.2, when \(-2s < \ell \leq 0\), in order to determine the range of \( x \) (namely, \( p \)), where (5.3) holds, it suffices to see the validity of (5.8) or (5.9) at \( x = A_{N,s} \).

In what follows, we analyze the cases \( N = 1 \) and \( N \geq 2 \) separately since the range of \( s \) is different in these two cases.

Suppose \( N = 1 \). In this case, we have \( A_{1,s} = 1/4 - s/2 \) and (5.8) and (5.9) with \( x = A_{1,s} \) become

\[
\log \left( s + \frac{\ell}{2} \right) + \log \Gamma(s) + \log \Gamma \left( \frac{1}{2} \right) - \log \Gamma \left( \frac{1}{2} - s \right) - 2 \log \Gamma \left( \frac{1}{4} + \frac{s}{2} \right) + 2 \log \Gamma \left( \frac{1}{4} - \frac{s}{2} \right) > 0, 
\]

\[
\log \left( s + \frac{\ell}{2} \right) + \log \Gamma(s) + \log \Gamma \left( \frac{1}{2} \right) - \log \Gamma \left( \frac{1}{2} - s \right) - 2 \log \Gamma \left( \frac{1}{4} + \frac{s}{2} \right) + 2 \log \Gamma \left( \frac{1}{4} - \frac{s}{2} \right) < 0. 
\]

(5.17)

(5.18)

For \((\ell, s)\), notice that \(-2s < \ell \leq 0\) and \(0 < s < 1/2\) are equivalent to \(-1 < \ell \leq 0\) and \(-\ell/2 < s < 1/2\). To analyze (5.17), we first fix an \( \ell \in (-1, 0] \) and vary \( s \in (-\ell/2, 1/2) \):

**Lemma 5.3.** Assume \( N = 1 \), fix an \( \ell \in (-1, 0] \) and put \( s_\ell := -\ell/2 \). Then there exists an \( \bar{s}_\ell \in [s_\ell, 1/2) \) such that (5.17) holds if and only if \( s \in (\bar{s}_\ell, 1/2) \). Moreover, when \( \ell = 0 \), \( \bar{s}_\ell = 0 = s_\ell \), and when \(-1 < \ell < 0 \), \( \bar{s}_\ell > s_\ell \) and (5.18) holds for \( s \in (s_\ell, \bar{s}_\ell) \).

**Proof.** For \( s \in (s_\ell, 1/2) \), let us consider

\[ H_\ell(s) := \log \left( s + \frac{\ell}{2} \right) + \log \Gamma(s) + \log \Gamma \left( \frac{1}{2} \right) - \log \Gamma \left( \frac{1}{2} - s \right) - 2 \log \Gamma \left( \frac{1}{4} + \frac{s}{2} \right) + 2 \log \Gamma \left( \frac{1}{4} - \frac{s}{2} \right). \]

Note that when \( \ell = 0 \), we have

\[ s_\ell = 0, \quad \log \left( s + \frac{\ell}{2} \right) + \log \Gamma(s) = \log s \Gamma(s) = \log \Gamma(s + 1), \quad H_0(0) = 0. \]

(5.19)

On the other hand, when \( \ell < 0 \), we get \( H_\ell(s) \to -\infty \) as \( s \searrow s_\ell \).

Next, we compute \( H'_\ell(s) \). It follows from (5.13) that for \( s \in (s_\ell, 1/2) \)

\[
H'_\ell(s) = \frac{1}{s + \frac{\ell}{2}} + \psi(s) + \psi \left( \frac{1}{2} - s \right) - \psi \left( \frac{1}{4} + \frac{s}{2} \right) - \psi \left( \frac{1}{4} - \frac{s}{2} \right) 
= \frac{1}{s + \frac{\ell}{2}} + \sum_{n=0}^{\infty} \left[ -\frac{1}{n + s} - \frac{1}{n + \frac{1}{2} - s} + \frac{1}{n + \frac{1}{4} + \frac{s}{2}} + \frac{1}{n + \frac{1}{4} - \frac{s}{2}} \right] 
= \frac{1}{s + \frac{\ell}{2}} - \frac{1}{s} + \sum_{n=0}^{\infty} \left[ -\frac{1}{n + 1 + s} - \frac{1}{n + \frac{1}{2} - s} + \frac{1}{n + \frac{1}{4} + \frac{s}{2}} + \frac{1}{n + \frac{1}{4} - \frac{s}{2}} \right]. 
\]

Since \(-1 < \ell \leq 0\), we see that

\[ \frac{1}{s + \frac{\ell}{2}} - \frac{1}{s} \geq 0. \]

In addition, one sees that

\[ \frac{1}{n + \frac{1}{4} + \frac{s}{2}} - \frac{1}{n + 1 + s} > 0, \quad \frac{1}{n + \frac{1}{4} - \frac{s}{2}} - \frac{1}{n + \frac{1}{2} - s} = \frac{1}{n + \frac{1}{4} - \frac{s}{2}} - \frac{1}{n + 2 \left( \frac{1}{4} - \frac{s}{2} \right)} > 0. \]
Therefore,
\[ H'_\ell(s) > 0 \quad \text{for every } s \in \left( s_\ell, \frac{1}{2} \right). \quad (5.20) \]

In particular, by (5.19), when \( \ell = 0 \), we obtain \( H_\ell(s) > 0 \) for each \( s \in (0, 1/2) \) and (5.17) holds for each \( s \in (0, 1/2) \).

To show (5.17) in the case \( \ell < 0 \), we observe the behavior of \( H_\ell(s) \) as \( s \nearrow 1/2 \). For this purpose, we use the following expansion:
\[ \log \Gamma(z) = -\log z + O(1) \quad \text{as } z \searrow 0. \quad (5.21) \]

As \( s \nearrow 1/2 \), we obtain
\[
- \log \Gamma \left( \frac{1}{2} - s \right) = \log \left( \frac{1}{2} - s \right) + O(1), \\
2 \log \Gamma \left( \frac{1}{4} - \frac{s}{2} \right) = -2 \log \left( \frac{1}{2} \left( \frac{1}{2} - s \right) \right) + O(1) = 2 \log 2 - 2 \log \left( \frac{1}{2} - s \right) + O(1)
\]
and
\[ H_\ell(s) = - \log \left( \frac{1}{2} - s \right) + O(1) \to \infty \quad \text{as } s \nearrow \frac{1}{2}. \]

From these, we infer that if \( \ell < 0 \), then by (5.20) and \( H_\ell(s) \to -\infty \) as \( s \searrow s_\ell \), there exists a unique \( \tilde{s}_\ell \in (s_\ell, 1/2) \) such that \( H_\ell(\tilde{s}_\ell) = 0 \) and \( H_\ell(s) > 0 \) in \( (\tilde{s}_\ell, 1/2) \). This completes the proof. \( \Box \)

Next, we turn to the case \( N \geq 2 \). As in the above, we consider (5.8) at \( x = A_{N,s} \) and set
\[
H_\ell(s,N) := \log \left( s + \frac{s_N}{2} \right) + \log \Gamma(s) + \log \Gamma \left( \frac{N}{2} \right) - \log \Gamma \left( \frac{N - 2s}{2} \right) \\
- 2 \log \Gamma \left( \frac{N + 2s}{4} \right) + 2 \log \Gamma \left( \frac{N - 2s}{4} \right).
\]

We shall observe when the inequality \( H_\ell(s,N) > 0 \) holds, which is equivalent to (5.8) at \( x = A_{N,s} \). Remark that in Lemmata 5.4 and 5.5, we also treat not only \( \ell \in (-2,0] \), but also \( \ell > 0 \).

**Lemma 5.4.** Suppose \( N \geq 2 \) and \( \ell \in (-\infty, \infty) \). Then the following hold:

(i) Assume \( \ell \in (-2,0] \) and set \( s_\ell := -\ell/2 \). If \( \ell = 0 \), then \( H_\ell(s,N) \to 0 \) as \( s \searrow s_\ell \). On the other hand, if \( -2 < \ell < 0 \), then \( H_\ell(s,N) \to -\infty \) as \( s \searrow s_\ell \);

(ii) When \( 2 \leq N \leq 6 \) and \( -2 < \ell \leq 0 \), \( \frac{\partial H_\ell}{\partial s}(s,N) > 0 \) for each \( s \in (s_\ell, 1) \);

(iii) When \( N \geq 6 \) and \( \ell \geq 0 \), \( H_\ell(\cdot,N) \) is strictly convex in \((0,1)\);

(iv) When \( N = 7 \) and \( \ell = 0 \), \( \lim_{s\searrow 0} \frac{\partial H_\ell}{\partial s}(s,7) > 0 \). When \( N \geq 8 \) and \( \ell = 0 \), there exists a \( t_N \in (0,1] \) such that \( \frac{\partial H_\ell}{\partial s}(s,N) < 0 \) for \( s \in (0,t_N) \) and \( \frac{\partial H_\ell}{\partial s}(s,N) > 0 \) if \( t_N < 1 \) and \( t_N < s < 1 \);

(v) For each \( \ell \in (-\infty, \infty) \), we have
\[
\lim_{s \searrow 1} H_\ell(s,N) = H_\ell(1,N) \begin{cases} 
> 0 & \text{if } 2 \leq N < 10 + 4\ell, \\
= 0 & \text{if } N = 10 + 4\ell, \\
< 0 & \text{if } 10 + 4\ell < N.
\end{cases}
\]
Remark 5.1. When $s = 1$, it is known that each $p \in (1, \infty)$ is JL-subcritical if and only if $N \leq 10 + 4\ell$ (see (1.4) and [9]). Since $x = A_{N,s}$ corresponds to $p = \infty$, by Lemma 5.4 (v), we see that the case $s = 1$ and $p = \infty$ appear in the limit $s \nearrow 1$.

Proof of Lemma 5.4. (i) When $\ell = 0$, similarly to (5.19), we have $H_0(\cdot, N) \in C([0, 1])$ and $H_0(0, N) = 0$. On the other hand, when $-2 < \ell < 0$, $\lim_{s \nearrow \ell} H_\ell(s, N) = -\infty$ due to the term $\log(s + \ell/2)$ and $s_\ell > 0$.

(ii) Recalling (5.13), we compute $\partial H_\ell/\partial s(s, N)$:

$$\frac{\partial H_\ell}{\partial s}(s, N) = \frac{1}{s + \frac{\ell}{2}} + \psi(s) + \psi\left(\frac{N - 2s}{2}\right) - \psi\left(\frac{N + 2s}{4}\right) - \psi\left(\frac{N - 2s}{4}\right)$$

$$= \frac{1}{s + \frac{\ell}{2}} + \sum_{n=0}^{\infty} \left[ \frac{1}{n + s} - \frac{1}{n + \frac{N - 2s}{2}} + \frac{1}{n + \frac{N + 2s}{4}} + \frac{1}{n + \frac{N - 2s}{4}} \right]$$

$$= \frac{1}{s + \frac{\ell}{2}} - \frac{1}{s} + \sum_{n=0}^{\infty} \left[ \frac{1}{n + 1 + s} - \frac{1}{n + \frac{N - 2s}{2}} + \frac{1}{n + \frac{N + 2s}{4}} + \frac{1}{n + \frac{N - 2s}{4}} \right].$$

(5.22)

Since $-2 < \ell \leq 0$, we have

$$\frac{1}{s + \frac{\ell}{2}} - \frac{1}{s} \geq 0 \quad \text{for} \quad s \in (s_\ell, 1).$$

On the other hand, we have

$$= \frac{1}{s + \frac{\ell}{2}} - \frac{1}{s} = \frac{1 + \frac{\ell}{2} - \frac{N}{4}}{N - 2s} \frac{N - 2s}{4} \frac{N + 2s}{N - 2s} \frac{N + 2s}{4} \frac{N + 2s}{N - 2s} \frac{N - 2s}{4} \frac{N - 2s}{4} \frac{N - 2s}{4}$$

$$= \left\{ n + \frac{N + 2s}{4} \right\} \left\{ n + 1 + s \right\} \left\{ n + \frac{N - 2s}{4} \right\} \left\{ n + \frac{N - 2s}{4} \right\}$$

$$= \frac{N - 2s}{2} \frac{N - 2s}{4} \frac{N + 2s}{4} \frac{N + 2s}{4} \frac{N + 2s}{4} \frac{N + 2s}{4} \frac{N + 2s}{4} \frac{N + 2s}{4}$$

From this expression, we infer that for any $\ell \in (-2, 0]$ and $2 \leq N \leq 6$, $\partial H_\ell/\partial s(s, N) > 0$ for every $s \in (s_\ell, 1)$.

(iii) Assume $N \geq 6$ and $\ell \geq 0$. Then it follows from (5.12) and (5.22) that

$$\frac{\partial^2 H_\ell}{\partial s^2}(s, N)$$

$$= -\frac{1}{(s + \frac{\ell}{2})^2} + \psi'(s) - \psi'\left(\frac{N - 2s}{2}\right) - \frac{1}{2} \psi'\left(\frac{N + 2s}{4}\right) + \frac{1}{2} \psi'\left(\frac{N - 2s}{4}\right)$$

$$= -\frac{1}{(s + \frac{\ell}{2})^2} + \sum_{n=0}^{\infty} \left[ \frac{1}{(n + s)^2} - \frac{1}{(n + \frac{N - 2s}{2})^2} + \frac{1}{2} \left\{ \frac{1}{(n + \frac{N - 2s}{2})^2} - \frac{1}{(n + \frac{N + 2s}{4})^2} \right\} \right]$$

$$= \frac{1}{s^2} - \frac{1}{(s + \frac{\ell}{2})^2}$$

$$+ \sum_{n=0}^{\infty} \left[ \frac{1}{(n + 1 + s)^2} - \frac{1}{(n + \frac{N - 2s}{2})^2} + \frac{1}{2} \left\{ \frac{1}{(n + \frac{N - 2s}{2})^2} - \frac{1}{(n + \frac{N + 2s}{4})^2} \right\} \right].$$

Since $\ell \geq 0$ and $N \geq 6$, we see that

$$\frac{1}{s^2} - \frac{1}{(s + \frac{\ell}{2})^2} \geq 0, \quad 1 + s < \frac{N - 2s}{2}.$$
Thus, \( \frac{\partial^2 H_\ell}{\partial s^2}(s, N) > 0 \) holds for each \( s \in (0, 1) \) and \( H_\ell(\cdot, N) \) is strictly convex in \((0, 1)\).

(iv) Assume \( N = 7 \) and \( \ell = 0 \). Then by (5.13) and (5.22),
\[
\lim_{s \downarrow 0} \frac{\partial H_0}{\partial s}(s, 7) = \sum_{n=0}^{\infty} \left[ -\frac{1}{n+1} - \frac{1}{n+\frac{1}{2}} + \frac{2}{n+\frac{3}{4}} \right] = \psi(1) + \psi \left( \frac{7}{2} \right) - 2\psi \left( \frac{7}{4} \right).
\]

It follows from [34, (1.3.3)–(1.3.5)] that
\[
\psi \left( \frac{7}{2} \right) = \psi \left( \frac{5}{2} \right) + \frac{2}{5} = \psi \left( \frac{3}{2} \right) + \frac{2}{3} + \frac{2}{5} = \psi \left( \frac{1}{2} \right) + 2 + \frac{2}{3} + \frac{2}{5}
\]
and
\[
-2\psi \left( \frac{7}{4} \right) = -2 \left[ \psi \left( \frac{3}{4} \right) + \frac{4}{3} \right] = -\frac{8}{3} - 2\psi \left( \frac{1}{2} \right) - \pi + 2 \log 2.
\]

From \( \psi(1) = -\gamma \) and \( \psi(1/2) = -2 \log 2 - \gamma \), we infer that
\[
\lim_{s \downarrow 0} \frac{\partial H_0}{\partial s}(s, 7) = \psi(1) + \psi \left( \frac{1}{2} \right) + \frac{2}{5} - 2\psi \left( \frac{1}{2} \right) - \pi + 2 \log 2 = \frac{2}{5} + 4 \log 2 - \pi > 0.
\]

Hence, the first assertion holds.

To prove the second assertion, we first suppose \( N = 8 \) and \( \ell = 0 \). Then by (5.22), we have
\[
\lim_{s \downarrow 0} \frac{\partial H_0}{\partial s}(s, 8) = \sum_{n=0}^{\infty} \left[ -\frac{1}{n+1} - \frac{1}{n+\frac{1}{2}} + \frac{2}{n+\frac{3}{4}} \right] = \left[ -1 - \frac{1}{2} + 2 \left( \frac{1}{2} + \frac{1}{3} \right) \right] = -\frac{1}{6} < 0.
\]

Hence, there exists an \( \tilde{s}_0 \in (0, 1) \) such that
\[
\frac{\partial H_0}{\partial s}(s, 8) < 0 \quad \text{for all } s \in (0, \tilde{s}_0).
\]

On the other hand, for \( H_0(s, N) \), we regard \( N \) as a real variable in \([8, \infty)\) and differentiate \( \frac{\partial H_0}{\partial s} \) with respect to \( N \) to obtain
\[
\frac{\partial^2 H_0}{\partial N \partial s}(s, N) = \frac{1}{2} \psi' \left( \frac{N - 2s}{2} \right) - \frac{1}{4} \psi' \left( \frac{N + 2s}{4} \right) - \frac{1}{4} \psi' \left( \frac{N - 2s}{4} \right)
\]
\[
= \frac{1}{4} \sum_{n=0}^{\infty} \left[ \frac{2}{(n + N - 2s)^2} - \frac{1}{(n + N + 2s)^2} - \frac{1}{(n + N - 2s)^2} \right].
\]

Since
\[
\frac{N - 2s}{2} > \frac{N + 2s}{4} > \frac{N - 2s}{4} \quad \text{for every } (s, N) \in (0, 1) \times [8, \infty),
\]
we get
\[
\frac{\partial^2 H_0}{\partial N \partial s}(s, N) < 0 \quad \text{for each } (s, N) \in (0, 1) \times [8, \infty).
\]

Combining this with (5.23), we obtain
\[
\frac{\partial H_0}{\partial s}(s, N) < 0 \quad \text{for any } (s, N) \in (0, \tilde{s}_0) \times [8, \infty).
\]

Set
\[
t_N := \sup \left\{ t \in (0, 1) \left| \frac{\partial H_0}{\partial s}(t, N) < 0 \right. \right\}.
\]
Recalling (iii), we have \( \frac{\partial H_\ell}{\partial s}(s, N) < 0 \) for every \( s \in (0, t_N) \) and \( \frac{\partial H_\ell}{\partial s}(s, N) > 0 \) if \( t_N < 1 \) and \( t_N < s < 1 \).

(v) Suppose \( -2 < \ell < \infty \). We first remark that when \( N = 2 \), by (5.21),

\[
\lim_{s \nearrow 1} H_\ell(s, 2) = \log \left( 1 + \frac{\ell}{2} \right) + \lim_{s \nearrow 1} \left[ -\log \Gamma(1 - s) + 2 \log \Gamma \left( \frac{1}{2} - \frac{s}{2} \right) \right]
\]

\[
= \log \left( 1 + \frac{\ell}{2} \right) + \lim_{s \nearrow 1} \left[ (1 - s) - \log \frac{(1 - s)^2}{4} + O(1) \right] = \infty.
\]

Hence, we consider the case \( N \geq 3 \). We notice that

\[
\log \Gamma \left( \frac{N}{2} \right) - \log \Gamma \left( \frac{N}{2} - 1 \right) = \log \frac{\Gamma(N/2)}{\Gamma(N/2 - 1)} = \log \left( \frac{N}{2} - 1 \right)
\]

and that

\[
2 \log \Gamma \left( \frac{N}{4} - \frac{1}{2} \right) - 2 \log \Gamma \left( \frac{N}{4} + \frac{1}{2} \right) = 2 \log \frac{\Gamma(N/4 - 1/2)}{\Gamma(N/4 + 1/2)} = 2 \log \frac{1}{\frac{N}{4} - \frac{1}{2}}
\]

\[
= -2 \log \left( \frac{N}{4} - \frac{1}{2} \right).
\]

Therefore, we obtain

\[
\lim_{s \nearrow 1} H_\ell(s, N) = H_\ell(1, N) = \log \left[ \left( 1 + \frac{\ell}{2} \right) \left( \frac{N}{2} - 1 \right) \left( \frac{N}{4} - \frac{1}{2} \right)^{-2} \right]
\]

\[
= \log \left[ \left( 1 + \frac{\ell}{2} \right) 4 \left( \frac{N}{2} - 1 \right)^{-1} \right].
\]

Since

\[
4 \left( 1 + \frac{\ell}{2} \right) \left( \frac{N}{2} - 1 \right)^{-1} \geq 1 \iff 8 + 4\ell \geq N - 2 \iff 10 + 4\ell \geq N,
\]

we see that (v) holds. \( \square \)

**Lemma 5.5.** Let \( (\ell, s) \in (-2, \infty) \times (0, 1) \) with \( -2s < \ell \). Then there exists an \( N_* = N_*(s, \ell) \geq 2 \) such that \( H_\ell(s, N) < 0 \) if \( N \geq N_* \).

**Proof.** As in the proof of Lemma 5.4 (iv), we regard \( N \) as a real variable. Let \( (\ell, s) \in (-2, \infty) \times (0, 1) \) with \( \ell > -2s \). For \( N \geq 2 \), it follows from (5.13) that

\[
\frac{\partial H_\ell}{\partial N}(s, N) = \frac{1}{2} \psi \left( \frac{N}{2} \right) - \frac{1}{2} \psi \left( \frac{N - 2s}{2} \right) - \frac{1}{2} \psi \left( \frac{N + 2s}{4} \right) + \frac{1}{2} \psi \left( \frac{N - 2s}{4} \right)
\]

\[
= \frac{1}{2} \sum_{n=0}^{\infty} \left[ \frac{1}{n + \frac{N}{2}} + \frac{1}{n + \frac{N - 2s}{2}} + \frac{1}{n + \frac{N + 2s}{4}} - \frac{1}{n + \frac{N - 2s}{4}} \right]
\]

\[
= \frac{1}{2} \sum_{n=0}^{\infty} \left[ \frac{s}{(n + \frac{N}{2})(n + \frac{N - 2s}{2})} - \frac{s}{(n + \frac{N + 2s}{4})(n + \frac{N - 2s}{4})} \right]
\]

\[
= \frac{s}{32} \sum_{n=0}^{\infty} \frac{-3N^2 - 8Ns + 4s^2 - (8N - 16s)n}{n + \frac{N}{2}(n + \frac{N - 2s}{2})(n + \frac{N + 2s}{4})(n + \frac{N - 2s}{4})}.
\]
For \( N \in [2, \infty) \), we have \( 3N^2 - 8Ns + 4s^2 > 0 \) and \( 8N - 16s > 0 \), which yields
\[
\frac{\partial H_\ell}{\partial N}(s, N) < 0 \quad \text{for each } \ell \in (-2, \infty), \ s \in \left( \max \left\{ 0, -\frac{\ell}{2} \right\}, 1 \right) \quad \text{and } N \in [2, \infty). \tag{5.24}
\]

By (5.24), to prove Lemma 5.5, it is enough to observe the asymptotic behavior of \( H_\ell(s, N) \) as \( N \to \infty \). To this end, we shall use the following: \( \text{see [34, p.10–p.11]} \)
\[
\log \Gamma(z) = z \log z - z - \frac{1}{2} \log z + O(1) \quad \text{as } z \to \infty.
\]

Since \( \log(1 + t) = t + O(t^2) \) as \( t \searrow 0 \), we have
\[
\log \Gamma\left( \frac{N^2}{2} \right) - \log \Gamma\left( \frac{N-2s}{2} \right) = \frac{N}{2} \log \frac{N}{2} - \frac{N-2s}{2} \log \frac{N-2s}{2} + O(1)
\]
\[
= \frac{N}{2} \log \frac{N}{N-2s} + s \log \frac{N-2s}{2} + O(1)
\]
\[
= s \log(N - 2s) + O(1)
\]
and
\[
2 \log \Gamma\left( \frac{N-2s}{4} \right) - 2 \log \Gamma\left( \frac{N+2s}{4} \right)
\]
\[
= 2 \left[ \frac{N-2s}{4} \log \frac{N-2s}{4} - \frac{N+2s}{4} \log \frac{N+2s}{4} \right] + O(1)
\]
\[
= -2s \log(N + 2s) + O(1),
\]
which gives
\[
H_\ell(s, N) = s \log(N - 2s) - 2s \log(N + 2s) + O(1) \to -\infty \quad \text{as } N \to \infty.
\]

From this fact and (5.24), we observe that Lemma 5.5 holds.

**Remark 5.2.** By the change of variables and Lemma 5.5, for any given \( (\ell, s) \in (-2, \infty) \times (0,1) \) with \( \ell > -2s \), if \( p \) and \( N \) are sufficiently large, then \( p \) is JL-supercritical.

### 5.3 Proof of Theorem 1.2, Corollary 1.1 and Theorem 1.3

We prove Theorem 1.2.

**Proof of Theorem 1.2.** (i) Let \( 0 < s < \min\{1, N/2\} \), \(-2s < \ell \leq 0\). Remark that the assertion in (i) is equivalent to the following claim: either (5.8) holds for all \( x \in [0, A_{N,s}] \) or else there exists an \( x_{JL} \in (0, A_{N,s}) \) such that \( h_\ell(x_{JL}) = 2 \log \Gamma(A_{N,s} + s) - \log \Gamma(A_{N,s}) \) and (5.8) (resp. (5.9)) holds in \((0,x_{JL})\) (resp. in \((x_{JL}, A_{N,s})\)). By Lemmata 5.1 and 5.2, it is easily seen that this claim follows and (i) holds.

(ii) Assume \( \ell = 0 \). When \( N = 1 \) and \( 0 < s < 1/2 \), Lemma 5.3 gives
\[
h_0(A_{N,s}) > 2 \left[ \log \Gamma(A_{N,s} + s) - \log \Gamma(A_{N,s}) \right],
\]

hence, Lemmata 5.1 and 5.2 yield
\[
h_0(x) > 2 \left[ \log \Gamma(A_{N,s} + s) - \log \Gamma(A_{N,s}) \right] \quad \text{for all } x \in [0, A_{N,s}].
\]
Therefore, when \( \ell = 0, N = 1 \) and \( s \in (0,1/2) \), any \( p \in (1, \infty) \) is JL-subcritical.
When $2 \leq N \leq 7$, from Lemma 5.4 (i)–(iv), it follows that

$$\frac{\partial H_0}{\partial s}(s, N) > 0 \quad \text{for each } s \in (0, 1), \quad \lim_{s \to 0} H_0(s, N) = 0.$$ 

Thus, $H_0(s, N) > 0$ holds for every $2 \leq N \leq 7$ and $s \in (0, 1)$. Since $H_0(s, N) > 0$ is equivalent to $h_0(A_{N,s}) > 2[\log \Gamma(A_{N,s} + s) - \log \Gamma(A_{N,s})]$, the assertion for $2 \leq N \leq 7$ holds.

Suppose $N = 8, 9$. In this case, by Lemma 5.4 (i), (iii), (iv) and (v), there exists a unique $s_N \in (0, 1)$ such that $H_0(s, N) < 0 = H_0(s, N) < H_0(t, N)$ hold for $0 < s < s_N < t < 1$. By Lemma 5.2 and the fact

$$H_0(s, N) = h_0(A_{N,s}) - 2[\log \Gamma(A_{N,s} + s) - \log \Gamma(A_{N,s})],$$

$$h_0(0) - 2[\log \Gamma(A_{N,s} + s) - \log \Gamma(A_{N,s})] > 0$$

when $0 < s < s_N$, there exists a unique $x_{JL} \in (0, A_{N,s})$ such that

$$h_0(x_{JL}) = 2[\log \Gamma(A_{N,s} + s) - \log \Gamma(A_{N,s})].$$

(5.25)

On the other hand, for every $s \in [s_N, 1)$, we get $h_0(x) > 2[\log \Gamma(A_{N,s} + s) - \log \Gamma(A_{N,s})]$ in $[0, A_{N,s})$. Therefore, the assertion for $N = 8, 9$ holds.

Finally, when $N \geq 10$, Lemma 5.4 (i), (iii), (iv) and (v) yield $H_0(s, N) < 0$ for all $s \in (0, 1)$. Hence, by Lemma 5.2, we may find a unique $x_{JL} \in (0, A_{N,s})$ satisfying (5.25). Thus, the assertion for $N \geq 10$ holds.

(iii) According to Lemmata 5.1–5.3, we may prove the assertion for $N = 1$ as in the case (ii). Therefore, we omit the details for $N = 1$.

Suppose $2 \leq N < 10 + 4\ell$. From Lemma 5.4 (i) and (v), we find $x_{N,\ell}, \tilde{s}_{N,\ell} \in (-\ell/2, 1)$ with $s_{N,\ell} \leq \tilde{s}_{N,\ell}$ such that $H_\ell(s, N) < 0$ for $s \in (-\ell/2, s_{N,\ell})$ and $H_\ell(s, N) > 0$ for $s \in (\tilde{s}_{N,\ell}, 1)$. In particular, when $2 \leq N \leq 6$, Lemma 5.4 (ii) yields $s_{N,\ell} = \tilde{s}_{N,\ell}$. Then, by the same argument as in the case $N = 8, 9$ of (ii), the assertion for $2 \leq N < 10 + 4\ell$ holds.

When $N \geq 10 + 4\ell$, applying Lemma 5.4 (ii) and (v), we derive $H_\ell(s, N) < 0$ for $s \in (-\ell/2, 2)$ and $3 \leq N \leq 6$. Then, it follows from (5.24) that $H_\ell(s, N) < 0$ for $s \in (-\ell/2, 2)$ and $N \geq 3$. Hence, by the same method as in the case $N \geq 10$ of (ii), the assertion for $N \geq 10 + 4\ell$ holds.

Next we show Corollary 1.1:

**Proof of Corollary 1.1.** We only treat $p > p_{S}(N, \ell)$.

(i) In this case, by Theorem 1.2 (ii) and (5.3), we know that $g_\ell(x) > \tilde{M}_{N,s}$ for all $x \in [0, A_{N,s})$. From $p > p_{S}(N, \ell)$, we observe that $g_\ell(x) > g_\ell(0) > \tilde{M}_{N,s}$ for any $x \in [0, A_{N,s})$. Hence, each $p > p_{S}(N, \ell)$ is JL-subcritical.

(ii) Since $-2s < \ell < 0$, (5.6) gives $g_\ell(x) < g_\ell(0)$ for each $x \in [0, A_{N,s})$. By $s \in (0, s_N)$ and Theorem 1.2, there exists a unique $x_0 \in (0, A_{N,s})$ such that $g_\ell(x_0) = \tilde{M}_{N,s}$. Thus, $g_\ell(x_0) < \tilde{M}_{N,s}$ and Theorem 1.2 yield $p_{JL} < \infty$. This fact implies that we may choose $s_{N,\ell}$ in Theorem 1.2 (ii) to satisfy $s_{N,\ell} \geq s_N$.

Finally, we prove Theorem 1.3:

**Proof of Theorem 1.3.** We remark that due to $N \geq 8$ and $0 < s < 1$, $A_{N,s} > 1/2$ holds. Moreover, $g_0(A_{N,s}) < \tilde{M}_{N,s}$ holds provided $N = 8, 9$ with $0 < s < s_N$ or $10 \leq N$ with $0 < s < 1$. We shall find an $\tilde{s}_N \in (0, 1)$ with $\tilde{s}_N \leq s_N$ when $N = 8, 9$ such that if $s \in (0, \tilde{s}_N)$, then there exists an $x_1 = x_1(N, s) \in (1/2, A_{N,s})$ such that

$$g_0(x_1) < \tilde{M}_{N,s}, \quad \ell_{1,s} := \frac{2s}{M_{0,s}} \left(\tilde{M}_{N,s} - M_{0,s}\right), \quad \ell_{2,s} := \frac{2}{M_{1,s}} \left(A_{N,s} - x_1 + s\right) \left(\tilde{M}_{N,s} - M_{1,s}\right)$$

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satisfy Theorem 1.3 (i) and (ii) with \( \ell_{1,s} < \ell_{2,s} \)
where
\[
M_{0,s} := g_0(A_{N,s}), \quad M_{1,s} := g_0(x_1).
\]

We remark that (5.6) gives
\[
\widetilde{M}_{N,s} \geq g_\ell(x_1) = A_{N,s} - x_1 + s + \frac{\ell}{2} M_{1,s} \iff \frac{\ell}{2} M_{1,s} \leq (A_{N,s} - x_1 + s) (\widetilde{M}_{N,s} - M_{1,s}) \iff \ell \leq \ell_{2,s}
\]
and
\[
\widetilde{M}_{N,s} \geq g_\ell(A_{N,s}) = \frac{s + \frac{\ell}{2}}{s} M_{0,s} \iff \frac{\ell}{2} M_{0,s} \leq s (\widetilde{M}_{N,s} - M_{0,s}) \iff \ell \leq \ell_{1,s}.
\]

Assuming \( \ell_{1,s} < \ell_{2,s} \) for a while, we check Theorem 1.3 (i) and (ii). Indeed, if \( s \in (0, \bar{s}_N) \) and \( \ell \in (0, \bar{\ell}_1) \), (5.27) and Lemma 5.1 yield \( g_\ell(A_{N,s}) < \widetilde{M}_{N,s} \) and \( g_\ell(x) > \widetilde{M}_{N,s} \) for sufficiently small \( x > 0 \). Therefore, Theorem 1.3 (i) holds by virtue of (5.3). On the other hand, if \( s \in (0, \bar{s}_N) \) and \( \ell \in (\ell_{1,s}, \ell_{2,s}) \), then (5.26) and (5.27) give \( g_\ell(x_1) < \widetilde{M}_{N,s} < g_\ell(A_{N,s}) \). By Lemma 5.1 and (5.3), Theorem 1.3 (ii) holds.

Now let us show the existence of \( \bar{s}_N \) and \( \ell_{1,s} < \ell_{2,s} \) for all \( s \in (0, \bar{s}_N) \). Set
\[
L(x) := \frac{2}{g_0(x)} (A_{N,s} - x + s) (\widetilde{M}_{N,s} - g_0(x)).
\]
Then \( L(A_{N,s}) = \ell_{1,s} \) and \( L(x_1) = \ell_{2,s} \). We shall prove \( L'(A_{N,s}) < 0 \) for each \( s \in (0, \bar{s}_N) \). By
\[
L'(A_{N,s}) = -\frac{2s}{M_{0,s}^2} (\widetilde{M}_{N,s} - M_{0,s}) g_0'(A_{N,s}) - \frac{2}{M_{0,s}^2} (\widetilde{M}_{N,s} - M_{0,s}) - \frac{2s}{M_{0,s}} g_0'(A_{N,s})
= \frac{2}{M_{0,s}^2} \left[ -M_{0,s} (\widetilde{M}_{N,s} - M_{0,s}) - s \widetilde{M}_{N,s} g_0'(A_{N,s}) \right].
\]
From (5.3) and (5.5), we have \( \widetilde{M}_{N,s} \to 1, M_{0,s} = g_0(A_{N,s}) \to 1 \) as \( s \to 0 \). Therefore,
\[
\lim_{s \searrow 0} \frac{L'(A_{N,s})}{s} = -2 \left[ \frac{\partial \widetilde{M}_{N,s}}{\partial s} \bigg|_{s=0} - \frac{\partial M_{0,s}}{\partial s} \bigg|_{s=0} + \lim_{s \searrow 0} g_0'(A_{N,s}) \right]. \tag{5.28}
\]

It follows from (5.3) that
\[
\frac{\partial \widetilde{M}_{N,s}}{\partial s} \bigg|_{s=0} = \widetilde{M}_{N,s} \frac{\partial}{\partial s} \log \widetilde{M}_{N,s} \bigg|_{s=0} = 2 \psi \left( \frac{N}{4} \right).
\]

On the other hand, by (5.14), \( h_0'(A_{N,s}) \to 0 \) as \( s \searrow 0 \). Since \( g_0'(A_{N,s}) = g_0(A_{N,s}) h_0'(A_{N,s}) \), we see that
\[
\lim_{s \searrow 0} g_0'(A_{N,s}) = 0.
\]

Moreover,
\[
\frac{\partial M_{0,s}}{\partial s} \bigg|_{s=0} = \frac{\Gamma(1+s) \Gamma(2A_{N,s}+s)}{\Gamma(2A_{N,s})} \bigg|_{s=0} = \psi(1) + \psi \left( \frac{N}{2} \right).
\]
Therefore, (5.28) yields
\[
\lim_{s \searrow 0} \frac{L'(A_{N,s})}{s} = -2 \left[ 2 \psi \left( \frac{N}{4} \right) - \psi(1) - \psi \left( \frac{N}{2} \right) \right].
\]
Since (5.12) gives
\[
\frac{d}{dN} \left[ 2\psi\left(\frac{N}{4}\right) - \psi(1) - \psi\left(\frac{N}{2}\right) \right] = \frac{1}{2} \left[ \psi'\left(\frac{N}{4}\right) - \psi'\left(\frac{N}{2}\right) \right] > 0,
\]
by \( N \geq 8 \), we observe that
\[
\lim_{s \searrow 0} \frac{L'(A_{N,s})}{s} \leq -2 \left[ 2\psi(2) - \psi(1) - \psi(4) \right] = -\frac{1}{3} < 0.
\]
Hence, there exists an \( \tilde{s}_N > 0 \) such that \( L'(A_{N,s}) < 0 \) for any \( s \in (0, \tilde{s}_N) \) and \( \ell_{1,s} < \ell_{2,s} \) follows if we choose \( x_1 \) close to \( A_{N,s} \).

Finally, we prove the existence of \( (\ell_{3,s}) \). By the assumption \( N = 8, 9 \) with \( 0 < s < s_N \) or \( N \geq 10, (5.3) \), Lemma 5.2 (ii) and Theorem 1.2 (ii), there exists a unique \( \tilde{x}_{N,s} \in (0, A_{N,s}) \) such that \( g_0(\tilde{x}_{N,s}) = \tilde{M}_{N,s} \). For any \( \ell > 0 \), (5.6) and Lemma 5.2 (ii) yield
\[
\min_{\tilde{x}_{N,s} \leq x \leq A_{N,s}} g_\ell(x) > \left[ \min_{\tilde{x}_{N,s} \leq x \leq A_{N,s}} \frac{A_{N,s} - x + \frac{\ell}{2}}{A_{N,s} - \tilde{x}_{N,s} + s} \right] \left[ \min_{\tilde{x}_{N,s} \leq x \leq A_{N,s}} g_0(x) \right]
= \frac{A_{N,s} - \tilde{x}_{N,s} + s + \frac{\ell}{2}}{A_{N,s} - \tilde{x}_{N,s} + s} M_{0,s}.
\]
We set
\[
\ell_{3,s} := \frac{2}{M_{0,s}} \left( \frac{A_{N,s} - \tilde{x}_{N,s} + s}{A_{N,s} - \tilde{x}_{N,s} + s} \right) (\tilde{M}_{N,s} - M_{0,s}).
\]
Then it is easily seen that
\[
\frac{A_{N,s} - \tilde{x}_{N,s} + s + \frac{\ell}{2}}{A_{N,s} - \tilde{x}_{N,s} + s} M_{0,s} \geq \tilde{M}_{N,s} \iff \frac{\ell}{2} M_{0,s} \geq (A_{N,s} - \tilde{x}_{N,s} + s) (\tilde{M}_{N,s} - M_{0,s}) \iff \ell \geq \ell_{3,s}.
\]
Since \( g_\ell(x) > g_0(x) \) for any \( x \in [0, \tilde{x}_{N,s}] \) and \( \ell > 0 \), if \( \ell \geq \ell_{3,s} \), then
\[
\min_{0 \leq x \leq A_{N,s}} g_\ell(x) > \tilde{M}_{N,s}.
\]
This completes the proof.

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A Appendix A

In this Appendix, we provide the details of the proof that \( \mathcal{O}_+ \) is bounded in the proof of Lemma 4.4. We recall that there exists a \( \gamma_0 \in C([0, 1], \mathbb{R}^{N+1}) \) such that
\[
\gamma_0(0) = (x_{n_0}, 0), \quad \gamma_0(1) = (x_{n_0+1}, 0), \quad \gamma_0(\tau) \in \mathcal{O}_{k, -} \quad \text{for each} \, \tau \in (0, 1),
\]
where $V(X) = U_2(X) - U_1(X)$, $V(x_{n_0}, 0), V(x_{n_0+1}, 0) < 0 < V(y_0, 0)$ and $|x_{n_0}| < |y_0| < |x_{n_0+1}|$. In what follows, by $V(x, t) = V(|x|, t)$, set $\gamma_1(\tau) := (|y_0, x(\tau)|, y_0, t(\tau))$ where $y_0(\tau) = (y_0, x(\tau), y_0, t(\tau)) \in \mathbb{R}^N \times [0, \infty)$. Then,

$$
\gamma_1 \in C([0, 1], [0, \infty)^2), \quad \gamma_1(0) = (|x_{n_0}|, 0), \quad \gamma_1(1) = (|x_{n_0+1}|, 0), \quad V(\gamma_1(\tau)) < 0 \quad \text{for each } \tau \in [0, 1].
$$

To proceed, we choose an $R_0 > 0$ so that

$$
\max_{\tau \in [0,1]} \{ \gamma_1,|x| (\tau) + \gamma_1,|x| (\tau) \} < R_0, \quad (A.1)
$$

where $\gamma_1(\tau) = (\gamma_1,|x| (\tau), \gamma_1,|x| (\tau))$. Now we prove the following result:

**Lemma A.1.** If $\gamma_2 = (\gamma_2,|x|, \gamma_2,|t|) \in C([0, 1], [0, \infty)^2)$ satisfies

$$
\gamma_2(0) = (|y_0|, 0), \quad \gamma_2,|t| (\tau) > 0 \quad \text{for each } 0 < \tau \leq 1, \quad R_0 < \gamma_2,|x| (1) + \gamma_2,|t| (1), \quad (A.2)
$$

then there exist $0 < \tau_1, \tau_2 < 1$ such that $\gamma_1(\tau_1) = \gamma_2(\tau_2)$.

As a consequence of Lemma A.1, the component $O_+ \subset \mathbb{R}^N_{+1}$ of $[V < 0]$ satisfying $(y_0, 0) \in \overline{O_+}$ becomes bounded. In fact, if $O_+$ is unbounded, then from the fact that $O_+$ is open and connected, we may find $\gamma_3 \in C([0, 1], \mathbb{R}^N_{+1})$ such that

$$
\gamma_3(0) = (y_0, 0), \quad \gamma_3,|x| (\tau) \in O_+ \quad \text{for each } 0 < \tau \leq 1, \quad 2R_0 < |\gamma_3(1)|.
$$

However, by setting $\gamma_2(\tau) := (|\gamma_3,|x| (\tau), |\gamma_3,|x| (\tau))$, $\gamma_2$ satisfies (A.2) and there exist $0 < \tau_1, \tau_2 < 1$ so that $\gamma_1(\tau_1) = \gamma_2(\tau_2) \in O_+ \cap O_k, \tau$. However, this is a contradiction since $V < 0$ on $O_k, \tau$ and $V > 0$ on $O_+$. Therefore, $O_+$ must be bounded.

What remains to prove is Lemma A.1:

**Proof of Lemma A.1.** From (A.1), we observe that $\gamma_1$ does not intersect with $\gamma_2$ in the region

$$
\Omega := \{ (z, t) \in [0, \infty)^2 \mid z + t > R_0 \}.
$$

Since $\Omega$ is path-connected, without loss of generality, we may suppose that $\gamma_2(1) = (R_0, R_0)$.

Next, consider a function $\eta : [0, 1]^2 \to \mathbb{R}^2$ defined by

$$
\eta(\tau_1, \tau_2) = \left( \begin{array}{c} \eta_1(\tau_1, \tau_2) \\ \eta_2(\tau_1, \tau_2) \end{array} \right) := \gamma_2(\tau_2) - \gamma_1(\tau_1) = \left( \begin{array}{c} \gamma_2,|x| (\tau_2) - \gamma_1,|x| (\tau_1) \\ \gamma_2,|t| (\tau_2) - \gamma_1,|t| (\tau_1) \end{array} \right).
$$

To prove Lemma A.1, it is enough to find $(\tau_1, \tau_2) \in (0, 1)^2$ so that $\eta(\tau_1, \tau_2) = (0, 0)$. To this end, we shall show

$$
\text{deg} (\eta, [0, 1]^2, (0, 0)) \neq 0, \quad (A.3)
$$

To see (A.3), we will exploit the homotopy invariance of degree and observe $\eta$ on $\partial [0, 1]^2$. When $(\tau_1, \tau_2) = (0, \tau_2)$ with $0 < \tau_2 \leq 1$, recalling $\gamma_2,|t| (\tau) > 0$ for $0 < \tau \leq 1$ and $\gamma_1,|t| (0) = 0$, we have

$$
\eta_2(0, \tau_2) = \gamma_2,|t| (\tau_2) > 0 \quad \text{for every } 0 < \tau_2 \leq 1.
$$

(A.4)

Similarly, when $(\tau_1, \tau_2) = (\tau_1, 0)$ with $0 < \tau_1 < 1$, it follows that

$$
\eta_2(\tau_1, 0) = -\gamma_1,|t| (\tau_1) < 0 \quad \text{for every } 0 < \tau_1 < 1.
$$

(A.5)
On the other hand, since \( \gamma_{1,t}(1) = 0 \) and \( \gamma_{2,t}(1) = R_0 \), it follows that

\[
\begin{align*}
\eta_2(1, \tau) &= \gamma_{2,t}(\tau) > 0 \quad \text{for every } 0 < \tau \leq 1, \\
\eta_2(\tau, 1) &= R_0 - \gamma_{1,t}(\tau) > 0 \quad \text{for every } 0 \leq \tau \leq 1.
\end{align*}
\] (A.6)

Next, by \( \eta(0, 0) = (|y_0| - |x_n|, 0), \eta(1, 0) = (|y_0| - |x_{n0+1}|, 0) \) and the continuity of \( \gamma_i \), we may find \( 0 < \varepsilon_0 < 1/4 \) satisfying \( 0 \leq \tau_1, \tau_2 \leq \varepsilon_0 \Rightarrow \eta_1(\tau_1, \tau_2) > 0, \)
\( 0 \leq 1 - \tau_1, \tau_2 \leq \varepsilon_0 \Rightarrow \eta_1(\tau_1, \tau_2) < 0. \) (A.7)

With this choice of \( \varepsilon_0 \), we define \( F \) by

\[
F(\tau_1, \tau_2) = \left( \frac{F_1(\tau_1, \tau_2)}{F_2(\tau_1, \tau_2)} \right) := \nabla \left[ -\left( \tau_1 - \frac{1}{2} \right)^2 + \left( \tau_2 - \frac{\varepsilon_0}{2} \right)^2 \right] = \left( -2\tau_1 + 1, 2\tau_2 - \varepsilon_0 \right).
\]

Then it follows that

\[
\begin{align*}
0 \leq \tau_1 \leq \varepsilon_0 & \Rightarrow F_1(\tau_1, \tau_2) = 1 - 2\tau_1 > 0, \\
1 - \varepsilon_0 \leq \tau_1 \leq 1 & \Rightarrow F_1(\tau_1, \tau_2) = 1 - 2\tau_1 < 0, \\
\varepsilon_0 \leq \tau_2 \leq 1 & \Rightarrow F_2(0, \tau_2) = F_2(1, \tau_2) = 2\tau_2 - \varepsilon_0 > 0, \\
0 \leq \tau_1 \leq 1 & \Rightarrow F_2(\tau_1, 0) = -\varepsilon_0 < 0, \\
0 \leq \tau_1 \leq 1 & \Rightarrow F_2(\tau_1, 1) = 2 - \varepsilon_0 > 0.
\end{align*}
\] (A.8)

Finally, we define

\[
H_\theta(\tau_1, \tau_2) := \theta \eta(\tau_1, \tau_2) + (1 - \theta) F(\tau_1, \tau_2).
\]

By (A.4)–(A.8), it is easily seen that \( H_\theta(\tau_1, \tau_2) \neq (0, 0) \) for all \( (\tau_1, \tau_2) \in \partial[0,1]^2 \) and \( \theta \in [0,1] \).

Hence, the homotopy invariance of degree implies

\[
\deg \left( \eta, [0,1]^2, (0,0) \right) = \deg \left( F, [0,1]^2, (0,0) \right) = -1.
\]

Thus, (A.3) holds and we complete the proof. \( \square \)

\section*{B Appendix B}

Here we prove some technical lemmata.

**Lemma B.1.** Let \( \varphi \in W^{2,\infty}((0, \infty)) \) with \( \varphi(0) = 0 \) and \( u \in C^\infty_c(\mathbb{R}^N) \) with \( u \geq 0 \) in \( \mathbb{R}^N \). Assume that \( \varphi'(t) > 0 \) and \( \varphi' \) is nonincreasing in \( (0, \infty) \). Then

\[
(-\Delta)^s (\varphi(u))(x) \geq \varphi'(u(x))(\Delta)^s u(x) \quad \text{for each } x \in \mathbb{R}^N. \tag{B.1}
\]

**Proof.** By \( \varphi(u) \in W^{2,\infty}(\mathbb{R}^N) \) and the fact \( \text{supp} \varphi(u) \) is compact, we have

\[
(-\Delta)^s (\varphi(u))(x) = C_{N,s} \lim_{\varepsilon \to 0} \int_{|y-x| \geq \varepsilon} \frac{\varphi(u(x)) - \varphi(u(y))}{|x - y|^{N+2s}} dy.
\]

Since

\[
\varphi(u(y)) - \varphi(u(x)) = \int_0^1 \varphi'(u(x) + \theta(u(y) - u(x))) d\theta (u(y) - u(x)),
\]

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we observe that

\[
(-\Delta)^s (\varphi(u))(x) - \varphi'(u(x)) (-\Delta)^s u(x)
\]

\[
= C_{N,s} \lim_{\varepsilon \to 0} \int_{|y-x| \geq \varepsilon} \frac{dy}{|x-y|^{N+2s}} \times \left[ \int_0^1 \varphi'(u(x) + \theta (u(y) - u(x))) d\theta (u(x) - u(y)) - \varphi'(u(x))(u(x) - u(y)) \right] \tag{B.2}
\]

\[
= C_{N,s} \lim_{\varepsilon \to 0} \int_{|y-x| \geq \varepsilon} \frac{dy}{|x-y|^{N+2s}} \int_0^1 (u(x) - u(y)) \times \{ \varphi'(u(x) + \theta (u(y) - u(x))) - \varphi'(u(x)) \} d\theta.
\]

Since \( \varphi'(t) \) is nonincreasing in \( t \), we see that for each \( x,y \in \mathbb{R}^N \) and \( \theta \in (0,1) \),

\[
(u(x) - u(y)) \left\{ \varphi'(u(x) + \theta (u(y) - u(x))) - \varphi'(u(x)) \right\} \geq 0.
\]

Using this inequality and (B.2), we infer that (B.1) holds. \( \square \)

**Lemma B.2.**

(i) Let \( u \in H^s_{\text{loc}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} dx) \) satisfy \( u(x) \to 0 \) as \( |x| \to \infty \). Then, \( U(X) := (P_s(\cdot, t) * u)(x) \to 0 \) as \( |X| \to \infty \).

(ii) If \( u \in H^s_{\text{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N \setminus \{ 0 \}) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} dx) \) satisfies \( u(x) \to 0 \) as \( |x| \to \infty \) and \( u(x) \to 0 \) as \( |x| \to 0 \), then \( U(X) \to \infty \) as \( |X| \to 0 \).

**Proof.** (i) From the assumption, for any given \( \varepsilon > 0 \), there exists an \( R_\varepsilon > 0 \) so that \( |u(x)| \leq \varepsilon \) for \( |x| \geq R_\varepsilon \). Write

\[
u(x) = \chi_{B_{R_\varepsilon}}(x)u(x) + (1 - \chi_{B_{R_\varepsilon}}(x))u(x) =: u_{\varepsilon,1}(x) + u_{\varepsilon,2}(x), \quad U_{\varepsilon,i}(X) := (P_s(\cdot, t) * u_{\varepsilon,i})(x).
\]

By \( \|u_{\varepsilon,2}\|_{L^\infty(\mathbb{R}^N)} \leq \varepsilon \), we see that

\[
\sup_{t > 0} \|U_{\varepsilon,2}(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq \sup_{t > 0} \|P_s(\cdot, t)\|_{L^1(\mathbb{R}^N)} \|u_{\varepsilon,2}\|_{L^\infty(\mathbb{R}^N)} \leq \varepsilon.
\]

On the other hand, we see from the compactness of \( \text{supp} \ u_{\varepsilon,1} \) that \( u_{\varepsilon,1} \in L^1(\mathbb{R}^N) \) and

\[
|U_{\varepsilon,1}(X)| \leq \int_{|y| \leq R_\varepsilon} \frac{p_{N,s}t^{2s}}{|x-y|^{N+2s}} |u_{\varepsilon,1}(y)| dy \leq C \int_{|y| \leq R_\varepsilon} t^{2s} |y|^{N-2s} |u_{\varepsilon,1}(y)| dy = C \|u_{\varepsilon,1}\|_{L^1(\mathbb{R}^N)} t^{-N}. \tag{B.3}
\]

Hence, there exists a \( t_\varepsilon > 0 \) such that

\[
\sup_{t \geq t_\varepsilon, x \in \mathbb{R}^N} |U_{\varepsilon,1}(X)| \leq \varepsilon.
\]

Finally, notice that \( |x-y| \geq |x|/2 \) for all \( |x| \geq 2R_\varepsilon \) and \( |y| \leq R_\varepsilon \). Thus, in (B.3), if \( |x| \geq 2R_\varepsilon \), then

\[
|U_{\varepsilon,1}(X)| \leq \int_{|y| \leq R_\varepsilon} p_{N,s}t^{2s} |x|^{-N-2s} |u_{\varepsilon,1}(y)| dy = C_{N,s} t^{2s} \|u_{\varepsilon,1}\|_{L^1(\mathbb{R}^N)} |x|^{-N-2s}.
\]
Therefore, we may find an $r_\varepsilon > 0$ such that
\[
\sup_{0 < t \leq t_\varepsilon, |x| \geq r_\varepsilon} |U_{\varepsilon,1}(X)| \leq \varepsilon.
\]
Now we conclude that $|X| \geq t_\varepsilon + r_\varepsilon$ implies $|U(X)| \leq 2\varepsilon$.

(ii) For each $L > 0$, set
\[
\begin{align*}
u_L(x) &:= \min\{u(x), L\} \in H^s_{\text{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s}dx), \\
U_L(X) &:= (P_s(\cdot, t) * u_L)(x) \in C(\mathbb{R}_+^{N+1}).
\end{align*}
\]
Since $u_L(x) \leq u(x)$ in $\mathbb{R}^N$, we have $U_L(X) \leq U(X)$ for each $L > 0$. From the assumption $u(x) \to \infty$ as $|x| \to 0$ and $U_L(x,0) = u_L(x)$, there exists an $r_L > 0$ such that
\[
|y| + t \leq r_L \Rightarrow \frac{L}{2} \leq U_L(y,t) \leq U(y,t).
\]
Since $L > 0$ is arbitrary, $U(X) \to \infty$ as $|X| \to 0$. \hfill \Box

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