On a growth model for complex networks capable of producing power-law out-degree distributions with wide range exponents

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The out-degree distribution is one of the most reported topological properties to characterize real complex networks. This property describes the probability that a node in the network has a particular number of outgoing links. It has been found that in many real complex networks the out-degree has a behavior similar to a power-law distribution, therefore some network growth models have been proposed to approximate this behavior. This paper introduces a new growth model that allows to produce out-degree distributions that decay as a power-law with an exponent in the range from 1 to $2$.

Among the topological properties of real complex networks (CN), one of the most studied is the out-degree distribution. This property describes the probability that a node in the network has a particular number of outgoing links. It has been found that in many real complex networks the out-degree has a behavior similar to a power-law distribution, that is, $P(k)\propto k^{-\gamma}$. In order to approximate this type of out-degree distribution, some growth models for CN have been proposed. For example, Dorogovtsev et.al. and Bollobás et.al. have each developed a model capable of producing out-degree distributions that decay as a power-law with exponent $\gamma=2+\frac{n_0+n+B}{m}$ and $\gamma=1+\frac{1+\delta_{out}(x+\beta)}{\beta+\gamma}$, respectively. Hence in both models the $\gamma$ exponent is greater than 2. Esquivel et.al. proposed a model that produces out-degree distributions that decay as a power-law where the $\gamma$ exponent value is in the range between 0 and 1.

The previous models are not able to produce out-degree distributions with $\gamma$ exponents in the range between 1 and 2. However, there are real CN where the $\gamma$ exponent value is within this interval. For example, the social network of Flickr users, the Any Beat network, the online social network Epinions and the network of flights between airports of the world (OpenFlights) where the $\gamma$ exponent for the out-degree distribution of these CN is close to 1.74, 1.71, 1.69 and 1.74 respectively.

This paper introduces a new model for growth of directed CN that allows to obtain out-degree distributions that decay as a power-law with exponents in the range $1<\gamma<2$. That is, the proposed model is able to generate all exponent values found in documented real CN.
**Proposed model**

In this model, the growth of the network is performed by adding nodes one at a time. At the beginning of the network, only node \( n_0 \) is present in the network and its out-degree is 0. Then, the out-degree of any new node \( n_{\text{new}} \) added to this network is determined as follows:

- With probability \( N^{-\alpha} \), \( n_{\text{new}} \) randomly selects an out-degree uniformly distributed from 0 to \( N - 1 \). That is, \( n_{\text{new}} \) may have out-degree 0, 1, 2, \ldots, \( N - 1 \). It is important to notice that it is possible that \( n_{\text{new}} \) has an out-degree of the order of \( N - 1 \). In this situation \( n_{\text{new}} \) would connect to all the other nodes in the network. This scenario may not be realistic for large values of \( N \), because in many real networks, the maximum degree for a node is much smaller than the total number of nodes \( N \). However, the probability \( N^{-\alpha} \) decreases when \( N \) increases for \( \alpha > 0 \).

- With complementary probability \( 1 - N^{-\alpha} \), \( n_{\text{new}} \) copies the out-degree of a randomly selected node from the network. It is important to notice that as the number of \( Q_s \) nodes with out-degree \( s \) increases, the probability that \( n_{\text{new}} \) has out-degree \( s \) also increases to \( \frac{Q_s}{N-1} \).

It is possible to employ the continuum method\(^\text{12} \) to obtain the analytical solution for the proposed model. This method is implemented using the following differential equation:

\[
\frac{dQ_s(N)}{dN} = N^{-\alpha} + (1 - N^{-\alpha}) \frac{Q_s(N)}{N-1} \quad (1)
\]

The previous equation describes the variation of the number of nodes with out-degree \( s \) with respect to the total number \( N \) of nodes in the network. The term \( g_s \) describes the situation that a new node randomly selects an out-degree value and the term \( g_s \), the situation that a new node copies this value from a randomly selected node in the network.

Eq. 1 may be written in the standard form for a linear differential equation:

\[
\frac{dQ_s(N)}{dN} + \frac{N^{-\alpha} - 1}{N-1} Q_s(N) = \frac{N^{-\alpha}}{N}. \quad (2)
\]

From Eq. 2, it is possible to deduce the integrating factor \( I(N) = e^{\int \frac{N^{-\alpha} - 1}{N-1} dN} \). Solving for \( I(N) \) produces non-elementary functions, which complicate the solution of Eq. 2. In order to obtain an integrating factor in terms of elementary functions, it is best to simplify Eq. 2 as follows:

\[
\frac{dQ_s(N)}{dN} + \frac{N^{-\alpha} - 1}{N-1} Q_s(N) = \frac{N^{-\alpha}}{N}. \quad (3)
\]

This simplification has little implications for large values of \( N \), because \( N \gg 1 \), as \( N \gg 1 \). This allows to employ the following integrating factor: \( I(N) = e^{\int \frac{N^{-\alpha} - 1}{N-1} dN} = e^{\frac{N^{-\alpha}}{N}} \). Multiplying Eq. 3 by \( I(N) \) produces:

\[
\frac{N^{-\alpha}}{N} Q_s(N) = \int \frac{N^{-\alpha} - 1}{N-1} e^{\frac{N^{-\alpha}}{N}} \, dN. \quad (4)
\]

Solving for \( Q_s(N) \):

\[
\frac{N^{-\alpha}}{N} Q_s(N) = \frac{e^{\frac{N^{-\alpha}}{N}} - 1}{N^{-\alpha} - 1} dN. \quad (5)
\]

\[
Q_s(N) = 1 + \frac{Ne^{\frac{N^{-\alpha}}{N}}\Gamma\left(\frac{1}{\alpha}, \frac{N^{-\alpha}}{N}\right)}{2^{1 - \frac{1}{\alpha}}} + kNe^{\frac{N^{-\alpha}}{N}}, \quad (6)
\]

where \( k \) is a constant and \( \Gamma(\cdot) \) is the incomplete Gamma function. In order to obtain the out-degree distribution \( Q_s(N) \), it is necessary to solve Eq. 6 for \( s = 1, 2 \), and so on as follows:

- for \( Q_1(N) \), consider the initial condition

\[
Q_1(2) = \frac{2^{-\alpha}}{2};
\]

this initial condition is due to the fact that, at the beginning the network only has one node, \( n_0 \), with no outgoing links \((N = 1)\). When the next node, \( n_1 \), is added \((N = 2)\), the probability that node \( n_1 \) has out-degree \( s = 1 \) is \( \frac{2^{-\alpha}}{2} \).

- Then, solving Eq. 6 for the initial condition \( Q_1(2) = \frac{2^{-\alpha}}{2} \) produces:

\[
Q_1(N) = 1 + \frac{Ne^{\frac{N^{-\alpha}}{N}}\Gamma\left(\frac{1}{\alpha}, \frac{N^{-\alpha}}{N}\right)}{2^{1 - \frac{1}{\alpha}}} + \left[2^{-(x+1)} - 1 - \frac{2e^{\frac{N^{-\alpha}}{N}}\Gamma\left(\frac{1}{\alpha}, \frac{N^{-\alpha}}{N}\right)}{2^{1 - \frac{1}{\alpha}}} \right] \frac{Ne^{\frac{N^{-\alpha}}{N}}}{2}, \quad (7)
\]

- for \( Q_2(N) \), consider the initial condition

\[
Q_2(3) = \frac{3^{-\alpha}}{3};
\]

this initial condition is due to the fact that, before adding node \( n_2 \) only \( n_0 \) and \( n_1 \) exist in the network \((N = 2)\) and both have \( s < 2 \), therefore \( Q_2(2) = 0 \). When \( n_2 \) is added \((N = 3)\), the probability that node \( n_2 \) has out-degree \( s = 2 \) is \( \frac{3^{-\alpha}}{3} \).

- Then, solving Eq. 6 with the initial condition \( Q_2(3) = \frac{3^{-\alpha}}{3} \), one obtains:

\[
Q_2(N) = 1 + \frac{Ne^{\frac{N^{-\alpha}}{N}}\Gamma\left(\frac{1}{\alpha}, \frac{N^{-\alpha}}{N}\right)}{2^{1 - \frac{1}{\alpha}}} + \left[3^{-(x+1)} - 1 - 3e^{\frac{N^{-\alpha}}{N}}\Gamma\left(\frac{1}{\alpha}, \frac{N^{-\alpha}}{N}\right) + \frac{3e^{\frac{N^{-\alpha}}{N}}\Gamma\left(\frac{1}{\alpha}, \frac{N^{-\alpha}}{N}\right)}{2^{1 - \frac{1}{\alpha}}} \right] \frac{Ne^{\frac{N^{-\alpha}}{N}}}{3} \quad (8)
\]

From the results in Eqs. 7 and 8, it is possible to deduce that:

\[
Q_s(N) = 1 + \frac{Ne^{\frac{N^{-\alpha}}{N}}\Gamma\left(\frac{1}{\alpha}, \frac{N^{-\alpha}}{N}\right)}{2^{1 - \frac{1}{\alpha}}} + \left[(x+1)^{-(x+1)} - 1 - \frac{(x+1)e^{\frac{N^{-\alpha}}{N}}\Gamma\left(\frac{1}{\alpha}, \frac{N^{-\alpha}}{N}\right)}{2^{1 - \frac{1}{\alpha}}} \right] \frac{Ne^{\frac{N^{-\alpha}}{N}}}{(x+1)} \quad (9)
\]

Normalizing Eq. 9, yields:

\[
P_s(N) = \frac{1 + \sqrt{x+1}e^{\frac{N^{-\alpha}}{N}} + \left[(x+1)^{-(x+1)} - 1 - \frac{(x+1)e^{\frac{N^{-\alpha}}{N}}\Gamma\left(\frac{1}{\alpha}, \frac{N^{-\alpha}}{N}\right)}{2^{1 - \frac{1}{\alpha}}} \right] \frac{Ne^{\frac{N^{-\alpha}}{N}}}{(x+1)}}{N} \quad (10)
\]

Eq. 10 describes the out-degree distribution \( P_s(N) \) obtained with the proposed model for \( 1 < s < N \). It can also be noted that, as \( s \to N \), Eq. 10 predicts that \( P_s(N) \approx \frac{1}{N^{1+\frac{1}{\alpha}}} \). That is, \( P_s(N) \) decays to 0 rapidly as \( s \to N \) and \( N \gg 1 \), therefore the power-law behavior exhibits a cutoff (Figure 1a).

In order to obtain the scaling exponent of the out-degree distribution, terms \( \Gamma(\cdot) \) into Eq. 10 are simplified using:

\[
\gamma(a,x) = \Gamma(a) - \gamma(a,x),
\]

where \( \gamma(a,x) \) and \( \Gamma(a,x) \) are the lower and upper incomplete Gamma functions, respectively. By the following asymptotic property:
Using Eq. 11 it is possible rewrite the $\Gamma(\cdot)$ terms of Eq. 10 as follows:

$$\Gamma\left(\frac{1}{x} \frac{N - s}{a}\right) \to \Gamma\left(\frac{1}{x}\right) \frac{a^{-1} - 1}{N} \text{ for } N > 1,$$

and

$$\Gamma\left(\frac{1}{x} \frac{(s + 1)^{-2}}{a}\right) \to \Gamma\left(\frac{1}{x}\right) \frac{a^{-1} - 1}{s + 1} \text{ for } s > 1.$$

Substituting Eqs. 12 and 13 into Eq. 10 and considering that $s + 1 = s$ as $s \gg 1$, Eq. 10 can be expressed as:

$$P_s(N) \approx \frac{1}{N} + \frac{\Gamma\left(\frac{1}{x}\right) - \frac{a^{-1} - 1}{x} e^{\frac{a^{-1}}{x}}}{s^{a^{-1}} - 1 - \frac{\Gamma\left(\frac{1}{x}\right) - \frac{a^{-1} - 1}{x}}{s^{rac{a^{-1}}{x}}} e^{\frac{a^{-1}}{x}}},$$

and

$$P_s(N) \approx \frac{1}{N} + \frac{\Gamma\left(\frac{1}{x}\right) - \frac{a^{-1} - 1}{x} e^{\frac{a^{-1}}{x}}}{s^{a^{-1}} - 1 - \frac{\Gamma\left(\frac{1}{x}\right) - \frac{a^{-1} - 1}{x}}{s^{rac{a^{-1}}{x}}} e^{\frac{a^{-1}}{x}}},$$

$$P_s(N) \approx \frac{1}{N} + \frac{\Gamma\left(\frac{1}{x}\right) - \frac{a^{-1} - 1}{x} e^{\frac{a^{-1}}{x}}}{s^{a^{-1}} - 1 - \frac{\Gamma\left(\frac{1}{x}\right) - \frac{a^{-1} - 1}{x}}{s^{rac{a^{-1}}{x}}} e^{\frac{a^{-1}}{x}}},$$

Using the two first terms of the series expansion of $e^{-\frac{a^{-1}}{x}} \approx 1 - \frac{s^{a^{-1}}}{x}$ in Eq. 15 and simplifying

$$P_s(N) \approx \frac{1}{N} + \frac{\Gamma\left(\frac{1}{x}\right) - \frac{a^{-1} - 1}{x} e^{\frac{a^{-1}}{x}}}{s^{a^{-1}} - 1 - \frac{\Gamma\left(\frac{1}{x}\right) - \frac{a^{-1} - 1}{x}}{s^{rac{a^{-1}}{x}}} e^{\frac{a^{-1}}{x}}},$$

$$P_s(N) \approx \frac{1}{N} + \frac{\Gamma\left(\frac{1}{x}\right) - \frac{a^{-1} - 1}{x} e^{\frac{a^{-1}}{x}}}{s^{a^{-1}} - 1 - \frac{\Gamma\left(\frac{1}{x}\right) - \frac{a^{-1} - 1}{x}}{s^{rac{a^{-1}}{x}}} e^{\frac{a^{-1}}{x}}},$$

$$P_s(N) \approx \frac{1}{N} + \frac{\Gamma\left(\frac{1}{x}\right) - \frac{a^{-1} - 1}{x} e^{\frac{a^{-1}}{x}}}{s^{a^{-1}} - 1 - \frac{\Gamma\left(\frac{1}{x}\right) - \frac{a^{-1} - 1}{x}}{s^{rac{a^{-1}}{x}}} e^{\frac{a^{-1}}{x}}},$$

for $s \gg 1, \frac{1}{s^{rac{a^{-1}}{x}} + \frac{1}{x}} \approx \frac{1}{\frac{1}{s^{rac{a^{-1}}{x}}}}$, thus it is possible to rewrite Eq. 16 as:

$$P_s(N) \approx \frac{1}{N} + \frac{\Gamma\left(\frac{1}{x}\right) - \frac{a^{-1} - 1}{x} e^{\frac{a^{-1}}{x}}}{s^{a^{-1}} - 1 - \frac{\Gamma\left(\frac{1}{x}\right) - \frac{a^{-1} - 1}{x}}{s^{rac{a^{-1}}{x}}} e^{\frac{a^{-1}}{x}}}.$$
Conclusions

Local processes participate in the growth and evolution of real CN which, in turn, shape the out-degree of its nodes. The model proposed here incorporates two local processes: a random out-degree selection and a copy of an out-degree for the nodes added to the network. This model is able to produce out-degree distributions that decay as a power-law with the $\gamma$ exponent in the range from 1 to $\infty$. That is, the proposed model reproduces all exponent values found in distributions of documented real complex networks.

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Author contributions

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Additional information

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Figure 2 | (a) Out-degree distribution of the Flickr social network. (b) Comparison of the out-degree distribution produced by the proposed model (Eq. 10) with $\alpha = 0.74$ and $N = 2, 302, 925$ and the actual out-degree distribution of the Flickr social network.