A multi-linear geometric estimate

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Erdős and Szemerédi made the following significant conjecture in additive number theory: If $A$ is a finite set of integers with $|A| = n$ then either $A + A$ or $A \cdot A$ must have size at least $C_\epsilon n^{2-\epsilon}$ for any $\epsilon > 0$.

The sum-project conjecture naturally leads one to the consideration of the size of sets of the form $A \cdot A + A \cdot A$.

Hart and Iosevich previously showed that if $E \subset \mathbb{F}_q^d$ with $|E| > q^{d+1/2}$ then $\mathbb{F}_q^* \subset \varpi(E^2)$ where $\varpi$ is any non-degenerate bilinear form.

This estimate can be used to show that if $A \subset \mathbb{F}_q^*$ is sufficiently large then

$$F_q^* \subset dA^2 = A \cdot A + \cdots + A \cdot A.$$ 

We generalize this estimate to the case that $\varpi$ is a multi-linear form.
Talk outline

- Preliminaries on forms
- The main result
- Applications when $n = 3$ and $d = 2$
- Why $n = 3$ and $d = 2$?
- Proof sketch
Preliminaries on forms

- Given two $\mathbb{F}_q$-vector spaces $V$ and $W$ let $\boxslash: V \times W \to V \otimes W$ denote the canonical map taking $(v, w)$ to $v \otimes w$.
- Note that when $E$ is a subspace of $V$ both $E^{\boxtimes n}$ and $E^\otimes n$ are defined and are in general distinct.
- An $n$-linear form is a linear transformation $\varpi: V^\otimes n \to \mathbb{F}$ where $V$ is an $\mathbb{F}$-vector space.
- For example, the usual dot product over $\mathbb{F}_q^d$ is a bilinear ($n = 2$) form.
Preliminaries on forms

Definition (Space of multi-linear forms)

Given a vector space $V$ over a field $\mathbb{F}$ and some $n \in \mathbb{N}$ we denote by $\text{Form}(V, n)$ the dual $\mathbb{F}$-vector space to $V^\otimes n$. That is, $\text{Form}(V, n) := \text{Hom}(V^\otimes n, \mathbb{F})$. 
Preliminaries on forms

Definition (Level set)

Given an \( \mathbb{F} \)-vector space \( V \), a form \( \varpi \in \text{Form}(V, n) \), \( E \subset V \), \( t \in \mathbb{F} \) we define the \( t \)-level set of \( \varpi \) (with respect to \( E \)) to be

\[
L_t := \left\{ (z, w) \in E^{\otimes (n-1)} \times E \mid \varpi(z, w) = t \right\}
\]

and we define \( \nu(t) := |L_t| \).
Preliminaries on forms

**Definition (Evaluation map)**

Given a vector space $V$, some $n \in \mathbb{N}$, some $k \in [n]$, and subspaces $A \leq V \otimes (n-1)$ and $B \leq V$ the $k^{th}$ evaluation map on $(A, B)$ is

$$\text{eval}_{k, A, B}: \text{Form}(V, n) \otimes B \rightarrow \text{Hom}(A, \mathbb{F})$$

is given by

$$(\text{eval}_{k, A, B}(\varpi \otimes y))(x_1, \ldots, x_{n-1}) := \varpi(x_1, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_{n-1}).$$
Preliminaries on forms

Definition \(((A, B)\text{-non-degenerate form})\)

Given a form \(\varpi \in \text{Form}(V, n)\) and subspaces \(A \leq V^{\otimes (n-1)}\) and \(B \leq V\) we say that \(\varpi\) is \((A, B)\text{-non-degenerate} \) in the \(k^{\text{th}}\) coordinate when \(\text{Ker}(\text{eval}_{k, A, B}^{\varpi}) = 0\).
Preliminaries on forms

- Let $V = \mathbb{F}_q^d$, $A = V^{\otimes(n-1)}$, $B = V$, $n = 3$, and

  $\varpi(x, y, z) = x_1y_1z_1 + x_2y_2z_2 + \cdots + x_dy_dz_d$.

  It is not difficult to see that this form is $(A, B)$-non-degenerate.

- If we keep $B$ the same, change $A$ to $W^{\otimes(n-1)}$, where

  $W = \left\{ x \in \mathbb{F}_q^d \mid x_1 = 0 \right\}$,

  and use the same form as above, we get an $(A, B)$-degenerate form.
Preliminaries on forms

Definition (Projective index)

Given $E \subset \mathbb{F}^d_q$ we say that $E$ has projective index $\alpha$ when

$$\left| \left\{ (a, w) \in \left( \mathbb{F}^*_q \setminus \{1\} \right) \times \mathbb{F}^d_q \mid w, aw \in E \right\} \right| \geq \alpha \cdot (q - 2)|E|$$
The main result

Theorem (A., Iosevich (2021))

Suppose that \( \varpi \in \text{Form}(q, d, n) \) for some \( n \geq 2 \), that \( E \subset \mathbb{F}_q^d \), and that \( E \) has projective index \( \alpha \). If there exists an \( r \)-dimensional subspace \( A \) of \( (\mathbb{F}_q^d)^{\otimes (n-1)} \) and a subspace \( B \) of \( \mathbb{F}_q^d \) such that

1. \( E^{\otimes (n-1)} \subset A \),
2. \( E \subset B \),
3. \( \varpi \) is \( (A, B) \)-non-degenerate, and
4. \( |E| > q^{\frac{r+n-1}{n}} \left( 1 - \alpha \left( 1 - \frac{2}{q} \right) \right)^{\frac{1}{n}} \)

then \( \mathbb{F}_q^* \subset \varpi(E^n) \). This bound is sharp.
Applications when $n = 3$ and $d = 2$

**Definition (Omphalos)**

We say that a set $E \subset \mathbb{F}_q^2$ is a $(q, k, \ell)$-omphalos when

$$E = \bigcup_{h \in H} E_h$$

where $H$ is a set of $k$ distinct lines through the origin in $\mathbb{F}_q^2$ and each $E_h$ consists of exactly $\ell$ nonzero points from $h$. 
Applications when $n = 3$ and $d = 2$

Proposition

Suppose that $E$ is a $(q, k, \ell)$-omphalos and that $\varpi$ is a non-degenerate ternary form on $\mathbb{F}_q^2$. If

$$k^3 \ell^3 > q^6 - (\ell - 1)q^5$$

then $\varpi(E^3) \supset \mathbb{F}_q^*$. 
Applications when $n = 3$ and $d = 2$

- Let $\Gamma$ be a subgroup of $\mathbb{F}_q^*$ of order $\frac{q-1}{s}$ and take $H \subset \mathbb{F}_q^*$ where $|H| = r$ and if $h_1, h_2 \in H$ with $h_1 \neq h_2$ then $h_1 \Gamma \neq h_2 \Gamma$. That is, let $H$ consist of representatives of $r$ distinct cosets of $\Gamma$.

- Define

$$E := \{ x(1, y) \in \mathbb{F}_q^2 \mid x, y \in H\Gamma \}.$$  

- Note that $\Gamma$ is a $(q, k, \ell)$-omphalos where

$$k = \ell = \frac{r(q-1)}{s}.$$
Applications when $n = 3$ and $d = 2$

- Taking $E$ from the previous example we have that if

$$(q - 1)^6 r^6 + (q - 1)s^5 q^5 r - s^6 (q^6 + q^5) > 0$$

and $\varpi$ is a non-degenerate ternary form on $\mathbb{F}_q^2$ then $\mathbb{F}_q^* \subset \varpi(E^3)$. 
Applications when \( n = 3 \) and \( d = 2 \)

Consider the set \( E \subset \mathbb{F}_q^2 \) from the previous example where \( q = 160001, s = 20, \) and \( r = 16 \) and take \( \varpi \) to be the ternary dot product.

Since

\[
(q - 1)^6 r^6 + (q - 1)s^5 q^5 r - s^6 (q^6 + q^5) > 0
\]

in this case and \( \varpi \) is non-degenerate we have that every nonzero member of \( \mathbb{F}_q^* \) may be written as

\[
\varpi(h_1 \gamma_1(1, h_2 \gamma_2), h_3 \gamma_3(1, h_4 \gamma_4), h_5 \gamma_5(1, h_6 \gamma_6))
\]

where the \( h_i \) are from a fixed set \( H \) consisting of \( r = 16 \) coset representatives of the subgroup \( \Gamma \) of \( \mathbb{F}_q^* \) of order \( \frac{q-1}{s} = 8000 \) and the \( \gamma_i \) are members of \( \Gamma \).
Applications when $n = 3$ and $d = 2$

- Each member of $\mathbb{F}_q^*$ is of the form

$$h_1 h_3 h_5 \psi_1 (1 + h_2 h_4 h_6 \psi_2)$$

where $\psi_1$ and $\psi_2$ are 20th powers in $\mathbb{F}_q$ and the $h_i$ belong to $H$.

- Moreover,

$$A \cdot A \cdot A + A \cdot A \cdot A \cdot A \cdot A \cdot A \supset \mathbb{F}_q^*$$

when $A = H\Gamma$. 
Why $n = 3$ and $d = 2$?

When $\dim(\text{Span}(E)) = \ell$ we need

$$\ell^{n-1} - n\ell + n < 2 - \log_q(q - \alpha(q - 2)).$$
Proof sketch

Theorem (A., Iosevich (2021))

Suppose that $\varpi \in \text{Form}(q, d, n)$ for some $n \geq 2$, that $E \subset \mathbb{F}_q^d$, and that $E$ has projective index $\alpha$. If there exists an $r$-dimensional subspace $A$ of $(\mathbb{F}_q^d)^{\otimes (n-1)}$ and a subspace $B$ of $\mathbb{F}_q^d$ such that

1. $E^{\otimes (n-1)} \subset A$,
2. $E \subset B$,
3. $\varpi$ is $(A, B)$-non-degenerate, and
4. $|E| > q^{\frac{r+n-1}{n}} \left(1 - \alpha \left(1 - \frac{2}{q}\right)\right)^{\frac{1}{n}}$

then $\mathbb{F}_q^* \subset \varpi(E^n)$. This bound is sharp.
Proof sketch

Write

\[
\nu(t) = \sum_{z \in E^{\boxtimes(n-1)}} q^{-1} \sum_{w \in E} \sum_{s \in \mathbb{F}_q} \chi(s(\omega(z, w) - t)).
\]

Thus,

\[
\nu(t) = q^{-1} \left| E^{\boxtimes(n-1)} \right| |E| + R
\]

where

\[
R := \sum_{z \in E^{\boxtimes(n-1)}} q^{-1} \sum_{w \in E} \sum_{s \in \mathbb{F}_q^*} \chi(s(\omega(z, w) - t)).
\]
Proof sketch

- View $R$ as a sum in $z$ and apply Cauchy-Schwarz.
- We find that $R^2 \leq U + V$ where

$$U = \left| E^{(n-1)} \right| q^{-2} \sum_{s,s' \in \mathbb{F}_q^*} \chi(t(s' - s)) E(w) E(w')$$

and

$$V = \left| E^{(n-1)} \right| q^{-2} \sum_{s,s' \in \mathbb{F}_q^*} \sum_{z \in A} \chi(\varpi(z, sw - s' w'))$$

- The $(A, B)$-nondegeneracy of $\varpi$ and orthogonality of $\chi$ gives $V = 0$. 
Proof sketch

- We have $R^2 \leq U = C + D$ where

$$C := \left| E^\otimes(n-1) \right| q^{r-2} \sum_{s, s' \in \mathbb{F}_q^*} \chi(t(s' - s)) E(w) E(w')$$

and

$$D := \left| E^\otimes(n-1) \right| q^{r-2} \sum_{s, s' \in \mathbb{F}_q^*} \chi(t(s' - s)) E(w) E(w').$$

- Without using the projective index $\alpha$ we can just note that $C < 0$, but in general this is not enough.
Proof sketch

Since

\[ C \leq - \left| E^{(n-1)} \right| |E| r^{-1} \alpha \left( 1 - \frac{2}{q} \right) \]

and

\[ D = \left| E^{(n-1)} \right| |E| r^{-1} \]

we have \( \nu(t) > 0 \) and the result follows.
References

Derrick Hart and Alex Iosevich. “Sums and products in finite fields: an integral geometric viewpoint”. In: *Radon transforms, geometry, and wavelets*. Vol. 464. Contemp. Math. Providence, RI: Amer. Math. Soc., 2008, pp. 129–135