DYNAMICAL BOREL–CANTELLI LEMMA FOR
RECURRENCE UNDER LIPSCHITZ TWISTS

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Abstract. In the study of some dynamical systems the limsup set of a sequence of measurable sets is often of interest. The shrinking targets and recurrence are two of the most commonly studied problems that concern limsup sets. However, the zero-one laws for the shrinking targets and recurrence are usually treated separately and proved differently. In this paper, we introduce a generalized definition that can specialize into the shrinking targets and recurrence; our approach gives a unified proof of the zero-one laws for the two problems.

1. Introduction

Throughout the paper, let \((X,d)\) be a separable and compact metric space, and let \((X,\mu,T)\) be a probability measure preserving system. One of the most fundamental results in ergodic theory is the Poincaré Recurrence Theorem, see e.g. [EW, Theorem 2.11], which asserts that almost all points in measurable dynamical systems return close to themselves under a measure-preserving map; namely, that

\[
\mu(R_T) = 1, \tag{1.1}
\]

where \(R_T\) is the set of recurrence for \(T\):

\[
R_T := \{x \in X : \liminf_{n \to \infty} d(T^n x, x) = 0\}.
\]

One of the first results concerning the speed of recurrence is due to Borisnitzan in [B]. Namely, assume that the \(\alpha\)-dimensional Hausdorff measure of \(X\) is \(\sigma\)-finite for some \(\alpha > 0\). Then

\[
\liminf_{n \to \infty} n^{1/\alpha} d(T^n x, x) = 0
\]

for \(\mu\)-almost every \(x \in X\). In other words, for a function \(\psi : \mathbb{N} \to \mathbb{R}^+\) let us define the following set:

\[
R_T(\psi) := \{x \in X : d(T^n x, x) < \psi(n) \text{ for infinitely many } n \in \mathbb{N}\}.
\]

Then the Poincaré Recurrence Theorem says that the set

\[
R_T = \bigcap_{\varepsilon > 0} R_T(\varepsilon 1_X)
\]
has full measure, and, with the notation $\psi_s(x) := x^{-s}$, Boshernitzan’s result says that $R_T(\varepsilon \psi_{1/\alpha})$ has full measure for any $\varepsilon > 0$.

It is a natural problem to find necessary and sufficient conditions on $\psi$ to guarantee that the set $R_T(\psi)$ has measure zero or one. In fact, under some additional assumptions one expects this condition to be the convergence/divergence of the sum of measures of the sets

$$A_T(n, \psi) := \{ x \in X : d(T^n x, x) < \psi(n) \}.$$  

Equivalently, letting $B(x, r)$ stand for the open ball in $X$ centered in $x$ of radius $r$, we can write $R_T^\psi(\psi) = \limsup A_T^\psi(n, \psi)$, where

$$A_T^\psi(n, \psi) := \{ x \in X : d(T^n x, y) < \psi(n) \} = T^{-n} B(y, \psi(n)),$$

and $B(y, r)$ stands for the open ball centered at $y \in X$ and radius $r > 0$. Clearly

$$\mu(R_T^\psi) = 1 \text{ for any } y \in \text{supp } \mu \text{ if } T \text{ is ergodic;}$$

furthermore, there have been plenty of results in the literature giving 0–1 laws for $\mu(R_T^\psi(\psi))$. In fact, one can often use mixing properties of $T$ to conclude that $\mu(R_T^\psi(\psi))$ is equal to zero/one if and only if the series

$$\sum_{n=1}^{\infty} \mu(A_T^\psi(n, \psi)) = \sum_{n=1}^{\infty} \mu(B(y, \psi(n)))$$

converges/diverges. See [Ph, CK, KM, FMP, HNPV] and many other references.

The goal of the current paper is to study a property unifying these two settings, and to prove a zero–one law applying to both. Namely, for a Borel measurable function $f : X \to X$ define $R_T^f$, the set of $f$-twisted recurrent points for $T$, by

$$R_T^f := \{ x \in X : \liminf_{n \to \infty} d(T^n x, f(x)) = 0 \}.$$

The two previous settings correspond to $f$ being the identity and constant functions respectively. We will show in the next section that $\mu(R_T^f) = 1$ for
any measurable $f$ if $T$ is ergodic and $\mu$ has full support. Furthermore, one can study the rate of twisted recurrence as follows: for $\psi : \mathbb{N} \to \mathbb{R}^+$ define

$$R_T^f(\psi) := \left\{ x \in X \Big| d(T^n x, f(x)) < \psi(n) \right\},$$

so that $R_T^f = \bigcap_{\varepsilon > 0} R_T^f(\varepsilon 1_X)$. In general the rate of twisted recurrence can be arbitrary slow, see §2 for examples. The main goal of the paper is to prove, under assumptions similar to those of [HLSW], a zero–one law for the sets $R_T^f(\psi)$ for a large class of functions $f$.

To state the main result of the paper, we need to adapt and modify the settings and assumptions from [HLSW]. Throughout the paper we write $a \lesssim b$ if $a \leq Cb$ for some constant $C > 0$, and $a \asymp b$ if $a \lesssim b$ and $b \lesssim a$.

Our main assumption is that there exist at most countably many pairwise disjoint open subsets $X_i$, $i \in I$, of $X$ such that $T|_{X_i}$ is continuous and injective for each $i$, and $\mu(X \setminus \bigcup_i X_i) = 0$. Those will be called cylinders of order 1. Then for any $m \in \mathbb{N}$ one can define

$$F_m := \left\{ X_{i_1} \cap T^{-1} X_{i_2} \cap \cdots \cap T^{-(m-1)} X_{i_m} : i_1, \ldots, i_m \in I \right\}$$

(1.6)

to be the collection of cylinders of order $m$. Note that for $J \in F_m$ and $x, y \in J$, the points $T^n x$ and $T^n y$ are in the same partition set $X_i$ for $0 \leq n < m$, and hence $T, \ldots, T^m$ are injective on $J$. Also, since $T$ is continuous, each cylinder in $F_m$ is open.

Now let us list our assumptions on the measure $\mu$. The first one is Ahlfors regularity of dimension $\delta > 0$; namely, that there exist positive real numbers $\eta_1, \eta_2, r_0$ such that

$$\eta_1 r^\delta \leq \mu(B(x, r)) \leq \eta_2 r^\delta \text{ for any ball } B(x, r) \subset X \text{ with } 0 < r < r_0. \quad (1.7)$$

As a consequence, since $\mu$ was assumed to be a probability measure, the space $X$ has finite diameter.

Next, we assume that $(X, \mu, T)$ is uniformly mixing (a property introduced in [FMP]), that is: there exist a summable sequence of positive real numbers $(a_n)_{n \in \mathbb{N}}$ such such that

$$|\mu(E \cap T^{-n} F) - \mu(E)\mu(F)| \leq a_n \mu(F) \quad \text{for any balls } E, F \subset X \text{ and for all } n \geq 1. \quad (1.8)$$

Note that it was proved in [FMP] that under the aforementioned mixing assumption, for any $y \in X$ and any $\psi$ the set $R_T^y(\psi)$ is null (resp., conull) if the series

$$\sum_{n=1}^\infty \mu(B(y, \psi(n))) \asymp \sum_{n=1}^\infty \psi^\delta(n)$$

(1.7) converges (resp., diverges). However, in order to similarly treat the sets $R_T^f(\psi)$ for more general functions $f$ we will require some more information
on the expanding properties of $T$. We define

$$K_J := \inf_{x, y \in J, x \neq y} \frac{d(T^m x, T^m y)}{d(x, y)},$$

and impose the following additional assumptions:

- **Bounded distortion:** There exists a constant $K_1 > 0$ such that
  $$K^{-1}_1 \leq \frac{d(T^m x, T^m y) / d(x, y)}{d(T^m x, T^m z) / d(x, z)} \leq K_1$$
  for all $m \in \mathbb{N}$ and $x, y, z \in J \in F_m$ with $x \neq y$ and $x \neq z$. \hfill (1.9)

- **Expanding properties:**
  $$\inf_{J \in F_m} K_J \to \infty \text{ as } m \to \infty$$
  \hfill (1.10)

  and
  $$\sup_{m \in \mathbb{N}} \sum_{J \in F_m} K^{-\delta}_J < \infty.$$ \hfill (1.11)

- **Conformality:** There exists a constant $K_2 \geq 1$ such that
  $$B(T^m x, K^{-1}_2 K_J r) \subset T^m B(x, r) \subset B(T^m x, K_2 K_J r)$$
  for any $m \in \mathbb{N}$ and any ball $B(x, r) \subset J \in F_m$. \hfill (1.12)

**Remark 1.1.** We note that conditions (1.7)–(1.12) are essentially equivalent to Conditions I–V from [HLSW]. Namely:

- (1.7) is a slightly weaker version of [HLSW, Condition I].
- (1.8) replaces [HLSW, Condition II] where the rate of mixing was assumed to be exponential.
- As for (1.9)–(1.12), in [HLSW] the standing assumption was that the restriction of $T$ to $X_i$ for every $i$ is differentiable and expanding, namely it was assumed that
  $$\|D x T^{-1}\|^{-1} > 1 \text{ for any } x \in \cup_i X_i.$$ \hfill (1.13)

  The role of (1.9) was played there by [HLSW, Condition III] stated as follows: there exists a constant $K_1 > 0$ such that
  $$K^{-1}_1 \leq \frac{d(T^m x, T^m y)}{d(x, y)\|D x T^m\|} \leq K_1 \forall m \in \mathbb{N} \text{ and } \forall x, y \in J \in F_m \text{ with } x \neq y.$$ \hfill (1.14)

- Similarly, the constant $K_J$ for $J \in F_m$ was defined in [HLSW] by $K_J := \inf_{x \in J} \|D x T^m\|$, and the role of (1.10) was played by $\inf_{J \in F_m} K_J > 1$ for some $m \in \mathbb{N}$, which, in view of (1.13), is easily seen to be equivalent to $\inf_{J \in F_m} K_J \to \infty$ as $m \to \infty$. Conditions IV and V of [HLSW] are identical to (1.11) and (1.12) respectively.

Let us now specify the class of functions $f$ which we can treat by our technique. Say that $f : X \to X$ is Lipschitz if

$$\sup_{x, y \in X, x \neq y} \frac{d(f(x), f(y))}{d(x, y)} < \infty,$$
and that \( f \) is piecewise Lipschitz if there exist at most countably many measurable subsets \( Y_i \) of \( X \) and Lipschitz functions \( f_i : X \to X \), \( i \in I \), such that \( \mu(X \setminus \bigcup_i Y_i) = 0 \) and \( f|_{Y_i} = f_i \) for each \( i \). An example: when \( X = [0, 1] \), the function \( f(x) = \sqrt{x} \) is piecewise Lipschitz but not Lipschitz.

Now we are ready to state our main theorem.

**Theorem 1.2.** Assume that \((X, \mu, T)\) satisfies conditions (1.7)–(1.12). Then for any function \( \psi : \mathbb{N} \to \mathbb{R}_+ \) with \( \lim_{n \to \infty} \psi(n) = 0 \) and any piecewise Lipschitz function \( f : X \to X \), the set \( R^f_T(\psi) \) is null (resp., conull) if the series \( \sum_{n=1}^{\infty} \psi^n(n) \) converges (resp., diverges).

Note that the twisted recurrence set-up was recently considered in [LVW] for the special case \( X = [0, 1] \) and \( T : x \mapsto \beta x \mod 1 \), where \( \beta > 1 \), establishing the conclusion of Theorem 1.2 in that case. We also remark that the paper [DFL] suggests an even more general set-up: there the authors consider a uniformly Lipschitz function \( \Phi : X \times X \to \mathbb{R} \) and under certain assumptions recover zero–one laws for sets of the form
\[
\{ x \in X \mid \psi_1(n) \leq \Phi(x, T^n x) \leq \psi_2(n) \text{ for infinitely many } n \in \mathbb{N} \}.
\]
Our set-up corresponds to \( \psi_1 = 0, \psi_2 = \psi \) and \( \Phi(x, y) = d(f(x), y) \). It would be interesting to see if the methods of our paper can be applied to the generalized setting of [DFL].

The structure of the paper is as follows. In §2 we discuss several basic properties of \( f \)-twisted recurrence sets and some examples of such sets. In §3 we study quasi-independence properties of the sequence of measurable sets whose limsup set is given by (1.5). In §4 we complete the proof of Theorem 1.2. The final section contains several applications of our main theorem.

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2. More about \( f \)-twisted recurrence

We start with several elementary observations concerning sets of \( f \)-twisted recurrence.

**Lemma 2.1.** Let \( \psi : \mathbb{N} \to \mathbb{R}_+ \) be an arbitrary function, and let \( f : X \to X \) be such that there exist at most countably many measurable subsets \( Y_i \) of \( X \) and functions \( f_i : X \to X \), \( i \in I \), such that \( \mu(X \setminus \bigcup_i Y_i) = 0 \),
\[
f|_{Y_i} = f_i \text{ and } \mu(R^f_i(\psi)) = 1 \text{ for each } i \in I.
\]
Then \( \mu(R^f_T(\psi)) = 1 \).

**Proof.** Indeed, it follows from (1.5) and (2.1) that
\[
\mu(R^f_T(\psi) \cap Y_i) = \mu(R^f_i(\psi) \cap Y_i) = \mu(Y_i)
\]
for each \( i \in I \).

Let us say that a function is \textit{simple} if it takes at most countably many values.

**Corollary 2.2.** Suppose \( T \) is ergodic and \( \text{supp} \mu \) is dense in \( X \). Then \( \mu(R_T^f) = 1 \) for any simple function \( f : X \to X \).

**Proof.** Immediate from Lemma 2.1 and (1.4).

**Lemma 2.3.** Let \( (f_n)_{n \in \mathbb{N}} \) be a sequence of functions \( X \to X \) such that \( \mu(R_T^{f_n}) = 1 \) for each \( n \). Suppose that \( f_n \to f \) uniformly on a set of full measure. Then \( \mu(R_T^f) = 1 \).

**Proof.** Since \( \bigcap_n R_T^{f_n} \) has full measure, for almost every \( x \in X \) and each \( n \in \mathbb{N} \) one has

\[
\liminf_{k \to \infty} d(T^k x, f_n(x)) = 0.
\]

Fix \( \varepsilon > 0 \); then there exists \( N \) so that for all \( n > N \), \( d(f_n(x), f(x)) < \frac{\varepsilon}{2} \) for almost every \( x \in X \); on the other hand, for almost every \( x \in X \) such that \( d(f_n(x), f(x)) < \frac{\varepsilon}{2} \), \( d(T^k x, f_n(x)) < \frac{\varepsilon}{2} \) for infinitely many \( k \). This implies \( d(T^k x, f(x)) < \varepsilon \) for infinitely many \( k \). Since \( \varepsilon \) is chosen arbitrarily, we have \( \liminf_{k \to \infty} d(T^k x, f(x)) = 0 \).

**Corollary 2.4.** Suppose that \( T \) is ergodic and \( \text{supp} \mu \) is dense in \( X \). Then \( \mu(R_T^f) = 1 \) for any Borel-measurable \( f : X \to X \).

**Proof.** Let \( \{x_n\}_{n=1}^\infty \) be a dense subset of \( X \). Let \( \varepsilon > 0 \) and \( f : X \to X \) be a Borel-measurable function. Then \( \{B(x_n, \varepsilon)\}_{n=1}^\infty \) covers \( X \). Define

\[
g_\varepsilon(x) = x_n \text{ where } n = \inf_m \{m : f(x) \in B(x_m, \varepsilon)\}
\]

Then \( g_\varepsilon \) is simple and \( \|g_\varepsilon - f\|_\infty \leq \varepsilon \). Since \( \varepsilon \) is chosen arbitrarily, \( f \) is a uniform limit of simple functions. By Corollary 2.2 and Lemma 2.3, \( x \) is \( f \)-recurrent for almost every \( x \in X \).

Next, let us observe that the properties of sets \( R_T^f(\psi) \) could be strikingly different from the conclusion of Theorem 1.2 if the assumptions of that theorem are not imposed. Let us start with the simplest possible non-trivial \(^1 \) example of an ergodic dynamical system: an irrational circle rotation \( X = \mathbb{R}/\mathbb{Z}, \mu = \text{Lebesgue measure}, T_\alpha(x) = x + \alpha \mod \mathbb{Z} \) where \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \).

Then the condition defining the recurrence set

\[
R_{T_\alpha}(\psi) = \{x : |n\alpha - m| < \psi(n) \text{ for infinitely many } n \in \mathbb{N} \text{ and some } m \in \mathbb{Z}\}
\]

is independent on \( x \); hence \( R_{T_\alpha}(\psi) \) is either \( X \) or \( \emptyset \), and this dichotomy is different for different \( \alpha \). More precisely, Dirichlet’s Theorem implies that \( R_{T_{\alpha_1}}(\psi_1) = X \) for any \( \alpha \), and the same is true for \( \psi_1 \) replaced with \( \frac{1}{\sqrt{3}} \psi_1 \), but

\footnote{For us ergodic self-maps \( T \) of finite sets \( X \) will be trivial: indeed, since those are transitive, it easily follows that \( R_T^f(\psi) = X \) for any \( f \) and any positive \( \psi \).}
not with \( c \psi_1 \) for \( c < \frac{1}{\sqrt{5}} \). In particular, \( \alpha \) is badly approximable if and only if \( R_{T_{\alpha}}(c \psi_1) = \emptyset \) for some \( c > 0 \). On the other hand, the theory of continued fractions shows that for any positive non-increasing \( \psi \) (decaying arbitrarily fast) there exists \( \alpha \) such that \( R_{T_{\alpha}}(\psi) \) contains 0 (and hence coincides with \( \mathbb{R}/\mathbb{Z} \)).

Likewise, studying targets shrinking to \( y \in X \) for the above system reduces to inhomogeneous Diophantine approximation:

\[
R_{T_{\alpha}}^y(\psi) = \{ x : \text{dist}(n \alpha, y - x) < \psi(n) \text{ for infinitely many } n \in \mathbb{N} \}
\]

According to Minkowski’s theorem [C, Chapter III, Theorem II], for any irrational \( \alpha \) and any \( y \in \mathbb{R}/\mathbb{Z} \), the complement of \( R_{T_{\alpha}}^y(\frac{1}{4} \psi_1) \) is at most countable. A precise zero-one law for sets \( R_{T_{\alpha}}^y(\psi) \) again depends on the Diophantine properties of \( \alpha \). For example, it is a theorem of Kurzweil [Ku] that \( \alpha \) is badly approximable if and only if the following statement holds: for any non-increasing \( \psi \), the set \( R_{T_{\alpha}}^y(\psi) \) is null/conull if \( \sum_{k=1}^{\infty} \psi(k) \) converges/diverges. However, well approximable \( \alpha \) come with their own convergence/divergence condition on \( \psi \) guaranteeing that \( R_{T_{\alpha}}^y(\psi) \) is null or conull; see [FK] for the most general statement.

Clearly the set-up of \( f \)-twisted recurrence can be similarly and straightforwardly restated in a Diophantine approximation language:

\[
R_{T_{\alpha}}^f(\psi) = \{ x : \text{dist}(n \alpha, f(x) - x) < \psi(n) \text{ for infinitely many } n \in \mathbb{N} \}.
\]

Thus if \( f(x) = x + \beta \mod \mathbb{Z} \) for a fixed \( \beta \), then \( R_{T_{\alpha}}(\psi) \) is either \( X \) or \( \emptyset \); alternatively, if the pushforward of Lebesgue measure by the map \( x \mapsto f(x) - x \) is absolutely continuous with respect to Lebesgue, then the zero/one law for the sets \( R_{T_{\alpha}}^f(\psi) \) depends on the Diophantine properties of \( \alpha \) as described in [FK].

The situation is even trickier if one considers irrational rotations of higher-dimensional tori. Namely, if we let \( X = \mathbb{R}^d/\mathbb{Z}^d \) and \( \mu \) is Lebesgue measure, then it is shown in [GP] that for any (arbitrarily slowly decaying) non-increasing function \( \psi \) with \( \lim_{t \to \infty} \psi(t) = 0 \) there exists an ergodic translation \( T_{\alpha} : x \mapsto x + \alpha \mod \mathbb{Z}^d \) such that \( \mu(R_{T_{\alpha}}^y(\psi)) = 0 \) for any \( y \in X \). Moreover, by suitably reparametrizing the aforementioned example one can construct a smooth mixing transformation on the three dimensional torus with the same property. Thus some conditions on the speed of mixing is crucial for a zero-one law as in Theorem 1.2.

### 3. A QUASI-INDEPENDENCE ESTIMATE

In the next two sections we prove Theorem 1.2, thereby assuming that \( (X, \mu, T) \) satisfies conditions (1.7)-(1.12), and fixing \( \psi : \mathbb{N} \to \mathbb{R}^+ \) with \( \lim_{n \to \infty} \psi(n) = 0 \). Similarly to (1.2) and (1.3), for an arbitrary \( f : X \to X \) let us define

\[
A_n = A_f^T(n, \psi) := \{ x \in X : d(T^n x, f(x)) < \psi(n) \}.
\]
Clearly $R^f_\psi = \limsup A_n$.

Unlike the shrinking target case, corresponding to constant functions $f$, the sets $A_n$ cannot be expressed in the form $T^{-n}B_n$ for some balls $B_n$. Our strategy is to consider the intersection of $A_n$ with $f^{-1}B(x_0, r)$, where $x_0 \in X$ and $r > 0$, and approximate this intersection by the preimages of some balls under $T$.

**Lemma 3.1.** For any $x_0 \in X$, any $r > 0$ and any subset $E$ of $f^{-1}B(x_0, r)$,
\[
E \cap A_n \subseteq E \cap T^{-n}B(x_0, \psi(n) + r)
\]  
Furthermore, if $r \geq \psi(n)$, then
\[
E \cap T^{-n}B(x_0, \psi(n) - r) \subseteq E \cap A_n
\]

**Proof.** Fix a point $x \in E \cap A_n$. Then
\[
d(f(x), x_0) < r \text{ and } d(T^n x, f(x)) < \psi(n),
\]
which implies that
\[
d(T^n x, x_0) < d(T^n x, f(x)) + d(f(x), x_0) < \psi(n) + r.
\]
Hence $E \cap A_n \subseteq E \cap T^{-n}B(x_0, \psi(n) + r)$.

On the other hand, fix $x \in E \cap T^{-n}B(x_0, \psi(n) - r)$. Then $d(f(x), x_0) < r$ and $d(T^n x, x_0) < \psi(n) - r$. Hence
\[
d(T^n x, f(x)) \leq d(T^n x, x_0) + d(x_0, f(x)) < \psi(n),
\]
thus $E \cap T^{-n}B(x_0, \psi(n) - r) \subseteq E \cap A_n$. \qed

Choose $n_0 \in \mathbb{N}$ such that $5\psi(n) < r_0$ for all $n > n_0$, where $r_0$ is as in (1.7); the next several statements in this section will be proved for $n > n_0$.

**Lemma 3.2.** Let $0 \leq \varepsilon \leq \frac{1}{2}$ and $B = B(x_0, \varepsilon \psi(n))$ for some $x_0 \in X$ and $n > n_0$. Then for any open ball $E$ contained in $f^{-1}B$,
\[
C_1 \mu(E) \psi^\delta(n) - C_2 \mu(E) \psi^\delta(n) \leq \mu(E \cap A_n) \leq C_3 \mu(E) \psi^\delta(n) + C_4 a_n \psi^\delta(n),
\]
where $C_1 = \eta_1(1 - \varepsilon)^\delta, C_2 = \eta_2(1 - \varepsilon)^\delta, C_3 = \eta_2(1 + \varepsilon)^\delta, C_4 = \eta_2(1 + \varepsilon)^\delta$, with $\eta_1, \eta_2$ as in (1.7) and $(a_n)_{n \in \mathbb{N}}$ as in (1.8).

**Proof.** Let $r = \varepsilon \psi(n)$, and let $E$ be an open ball contained in $f^{-1}B(x_0, r)$. Combining (3.2) with (1.8), we get
\[
\mu(E \cap A_n) \geq \mu\left\{(E \cap T^{-n}B(x_0, \psi(n) - r)\right\}
\]
\[
\geq \mu(E) \mu(T^{-n}B(x_0, \psi(n) - r)) - a_n \mu(E \cap T^{-n}B(x_0, \psi(n) - r))
\]
\[
\geq \mu(E) \mu\left\{T^{-n}B(x_0, (1 - \varepsilon)\psi(n))\right\} - a_n \mu\left\{T^{-n}B(x_0, (1 - \varepsilon)\psi(n))\right\}
\]
\[
= \mu(E) \mu\left\{B(x_0, (1 - \varepsilon)\psi(n))\right\} - a_n \mu\left\{B(x_0, (1 - \varepsilon)\psi(n))\right\}
\]
\[
\geq \mu(E) \eta_1(1 - \varepsilon)^\delta \psi^\delta(n) - \eta_2 a_n (1 - \varepsilon)^\delta \psi^\delta(n)
\]

(1.7)
and

\[
\mu(E \cap A_n) \leq \mu \left(E \cap T^{-n}B(x_0, \psi(n) + r)\right) \\
\leq \mu(E) \mu(T^{-n}B(x_0, \psi(n) + r)) + a_n \mu(E \cap T^{-n}B(x_0, \psi(n)) + r) \\
\leq \mu(E) \mu \left(T^{-n}B(x_0, (1 + \varepsilon)\psi(n))\right) + a_n \mu \left(T^{-n}B(x_0, (1 + \varepsilon)\psi(n))\right) \\
\leq \mu(E) \mu \left(B(x_0, (1 + \varepsilon)\psi(n))\right) + a_n \mu \left(B(x_0, (1 + \varepsilon)\psi(n))\right) \\
\leq \mu(E) \eta_2 (1 + \varepsilon)^\delta \psi^\delta(n) + \eta_2 a_n (1 + \varepsilon)^\delta \psi^\delta(n).
\]

(1.7)

□

To prove Theorem 1.2, in view of Lemma 2.1 it is enough to assume that \( f \) is Lipschitz. Thus for the rest of the paper we let \( f : X \rightarrow X \) be a \( p \)-Lipschitz function for some \( p > 0 \).

The next lemma estimates the measure of the sets \( A_n \).

**Lemma 3.3.** For \( n > n_0 \),

\[
\eta_2^{-1} \eta_1 C_1 5^{-\delta} \psi^\delta(n) - \eta_2^{-1} 2^\delta C_2 a_n p^\delta \leq \mu(A_n) \\
\leq \eta_1^{-1} \eta_2 C_3 2^{-\delta} \psi^\delta(n) + \eta_1^{-1} 10^\delta C_4 a_n p^\delta,
\]

where \( C_1, C_2, C_3, C_4 \) are as in Lemma 3.2.

**Proof.** Take \( x \in X \) and \( y \in f^{-1}\{x\} \). For \( \varepsilon > 0 \), take \( z \in B \left(y, \frac{\varepsilon \psi(n)}{p}\right) \).

Then by the \( p \)-Lipschitz condition, \( d(x, f(z)) \leq p d(y, z) < \varepsilon \psi(n) \). Thus

\[
B(y, \varepsilon \psi(n)/p) \subset f^{-1}B(x, \varepsilon \psi(n)).
\]

We have an open covering

\[
\{ B(y, \varepsilon \psi(n)/p) : y \in X \}
\]

with each \( B(y, \varepsilon \psi(n)/p) \subset f^{-1}B(x, \varepsilon \psi(n)) \) for some \( x \in X \).

By Vitali’s covering theorem (5r-covering lemma), we can find countably many disjoint balls \( \{ B(y_j, \varepsilon \psi(n)/p) \}_{j \in \mathcal{J}} \) such that

\[
X \subset \bigcup_{j \in \mathcal{J}} B(y_j, 5\varepsilon \psi(n)/p).
\]

(3.4)

By the disjointness of \( \{ B(y_j, \varepsilon \psi(n)/p) \}_{j \in \mathcal{J}} \), we have

\[
\sum_{j \in \mathcal{J}} \eta_1 (\varepsilon \psi(n)/p)^\delta \leq \sum_{j \in \mathcal{J}} \mu \left(B(y_j, \varepsilon \psi(n)/p)\right) \leq \mu(X) = 1.
\]

Hence \( |\mathcal{J}| \leq \eta_1^{-1} \left(\frac{\varepsilon \psi(n)}{p}\right)^\delta \). On the other hand, by (3.4) we have

\[
\sum_{j \in \mathcal{J}} \eta_2 (5\varepsilon \psi(n)/p)^\delta \geq \sum_{j \in \mathcal{J}} \mu \left(B(y_j, 5\varepsilon \psi(n)/p)\right) \geq \mu(X) = 1;
\]
hence $|J| \geq \eta_2^{-1} \left( \frac{p}{5\varepsilon \psi(n)} \right)^{\delta}$.

By Lemma 3.2 and taking $\varepsilon = \frac{1}{10}$, since for each $j$ we have $B(y_j, \varepsilon \psi(n)/p) \subset f^{-1}B(x_j, \varepsilon \psi(n))$ and $B(y_j, 5\varepsilon \psi(n)/p) \subset f^{-1}B(x_j, 5\varepsilon \psi(n))$, it follows that

$$\mu(A_n) \leq \sum_{j \in J} \mu \left( B(y_j, \varepsilon \psi(n)/p) \cap A_n \right)$$

$$\leq |J| \left[ C_3 \mu \left( B(y_j, 5\varepsilon \psi(n)/p) \right) \psi^\delta(n) + C_4 a_n \psi^\delta(n) \right]$$

$$\leq \eta_1^{-1} \left( \frac{p}{\varepsilon \psi(n)} \right)^{\delta} \left[ C_3 (5\varepsilon \psi(n)/p)^{\delta} \eta_2 \psi^\delta(n) + C_4 a_n \psi^\delta(n) \right]$$

$$\leq \eta_1^{-1} \eta_2 C_3 2^{-\delta} \psi^\delta(n) + \eta_1^{-1} 10^\delta C_4 a_n p^\delta$$

and

$$\mu(A_n) \geq \sum_{j \in J} \mu \left( B(y_j, \varepsilon \psi(n)/p) \cap A_n \right)$$

$$\geq |J| \left[ C_1 \mu \left( B(y_j, \varepsilon \psi(n)/p) \cap A_n \right) \psi^\delta(n) - C_2 a_n \psi^\delta(n) \right]$$

$$\geq \eta_1^{-1} \left( \frac{p}{5\varepsilon \psi(n)} \right)^{\delta} \left[ C_1 (\varepsilon \psi(n)/p)^{\delta} \eta_1 \psi^\delta(n) - C_2 a_n \psi^\delta(n) \right]$$

$$\geq \eta_1^{-1} \eta_2 C_1 5^{-\delta} \psi^\delta(n) - \eta_2^{-1} 2^\delta C_2 a_n p^\delta.$$

\[ \square \]

**Proposition 3.4.**

$$\sum_{n=1}^{\infty} \psi^\delta(n) = \infty \iff \sum_{n=1}^{\infty} \mu(A_n) = \infty \quad (3.5)$$

**Proof.** By Lemma 3.3 we know that

$$\eta_2^{-1} \eta_1 C_1 5^{-\delta} \sum_{n>n_0} \psi^\delta(n) - \eta_2^{-1} 2^\delta C_2 p^\delta \sum_{n>n_0} a_n$$

$$\leq \sum_{n>n_0} \mu(A_n)$$

$$\leq \eta_1^{-1} \eta_2 C_3 2^{-\delta} \sum_{n>n_0} \psi^\delta(n) + \eta_1^{-1} 10^\delta C_4 a_n p^\delta \sum_{n=1}^{\infty} a_n.$$

Since $\{a_n\}$ is summable, (3.5) holds. \[ \square \]

**Remark 3.5.** Note that Proposition 3.4 and the Borel–Cantelli Lemma immediately imply the convergence case of Theorem 1.2: if $\sum_{n=1}^{\infty} \psi(n) < \infty$, then $\mu(B^T(\psi)) = \mu(\limsup_n A_n) = 0$.

Now let us make use of assumptions (1.9)–(1.12). The following two lemmas were stated and used in [HLSW]. Some of the following statements only hold for large enough indices, which we specify here. Let $m_0 \geq n_0$ be
such that \( K_J > \max \left( \frac{K_1 \text{diam}(X)}{r_0}, 2p \right) \) for all \( m > m_0 \) and \( J \in \mathcal{F}_m \) (which is possible in view of (1.10)).

**Lemma 3.6.** For \( m \in \mathbb{N} \), \( J \) a cylinder in \( \mathcal{F}_m \) and for any open set \( U \) contained in \( J \), \( \mu(T^m U) \asymp K_J^4 \mu(U) \).

**Proof.** By (1.7) and (1.12), we know that for all open balls \( B \subset J \) with radius smaller than \( r_0 \), it holds that \( \mu(T^m B) \asymp K_J^4 \mu(B) \). Let \( U \subset J \) be an open subset. Consider the cover

\[ S = \{ B(x, r) : x \in U, B(x, 5r) \subset U, r < r_0 \}. \]

By Vitali's covering theorem, \( S \) has a countable sub-collection \( B \) of disjoint balls so that

\[ \bigcup_{B(x, r) \in B} B(x, r) \subset U \subset \bigcup_{B(x, r) \in B} B(x, 5r). \]

Since \( T^m \) is injective on \( J \),

\[ \bigcup_{B(x, r) \in B} T^m B(x, r) \subset T^m U \subset \bigcup_{B(x, r) \in B} T^m B(x, 5r). \]

Hence

\[ K_J^4 \sum_{B(x, r) \in B} \mu(B) \asymp \mu(T^m U). \]

On the other hand, \( \mu(U) \asymp \sum_{B(x, r) \in B} \mu(B) \), and the lemma is proved. \( \square \)

**Lemma 3.7.** Let \( m > m_0 \), and let \( J \) be a cylinder in \( \mathcal{F}_m \). Then \( \text{diam}(J) \lesssim K_J^{-1} \) and \( \mu(J) \lesssim K_J^{-\delta} \).

**Proof.** Let \( x, y \in J \). By (1.9) and the definition of \( K_J \),

\[ d(x, y) \leq \frac{K_1 d(T^m x, T^m y)}{K_J}, \]

hence

\[ d(x, y) \leq K_1 \text{diam}(X) K_J^{-1}, \]

which implies the bound on \( \text{diam}(J) \). Also, by the assumption \( m > m_0 \) we have \( K_1 \text{diam}(X) K_J^{-1} < r_0 \), hence

\[ J \subset B\left(x_0, K_1 \text{diam}(X) K_J^{-1}\right) \]

for some \( x_0 \in X \). Then by (1.7), \( \mu(J) \lesssim K_J^{-\delta} \). \( \square \)

**Lemma 3.8.** For \( m > m_0 \) and for every cylinder \( J \in \mathcal{F}_m \) there exists a ball of radius

\[ r = \frac{2\psi(m)}{K_J - p}, \quad (3.6) \]

say \( B(z, r) \), such that

\[ J \cap A_m \subset B(z, r) \cap J \quad (3.7) \]
Proposition 3.9. For

Proof. Choose any \( x, z \in J \cap A_m \). Since \( J \in \mathcal{F}_m \), in view of (1.9) we have

\[
d(T^m x, T^m z) K^{-1}_J \geq d(x, z).
\]

On the other hand,

\[
d(T^m x, T^m z) \leq d(T^m x, f(x)) + d(f(x), f(z)) + d(T^m z, f(z)) \lesssim 2\psi(m) + pd(x, z)
\]

Then \( K_J d(x, z) < 2\psi(m) + pd(x, z) \), i.e. \( d(x, z) < \frac{2\psi(m)}{K_J - p} \).

For any \( m \in \mathbb{N} \) and \( J \in \mathcal{F}_m \) define

\[ J^* := B(z, r) \cap J, \]

where \( r \) and \( z \) are defined in (3.6) and (3.7).

We now come to the main result of the section.

Proposition 3.9. For \( n > m > m_0 \),

\[ \mu(A_m \cap A_n) \lesssim \psi^\delta(m) \psi^\delta(n) + a_{n-m} \psi^\delta(n) + a_n \psi^\delta(m). \]

Proof. Write

\[ A_m = \bigsqcup_{J \in \mathcal{F}_m} J \cap A_m \subset \bigsqcup_{J \in \mathcal{F}_m} J^*. \]

Now let us estimate \( \mu(J^* \cap A_m) \) for any fixed \( J \in \mathcal{F}_m \).

Case (i): \( pr = \frac{2\psi(m)}{K_J - p} \leq \psi(n) \).

Note that for all \( x \in B(z, r) \), \( d(f(x), f(z)) < pd(x, z) < pr \), therefore \( B(z, r) \subset f^{-1}(B(f(z), pr)) \). Thus \( J^* \) is a subset of \( f^{-1}(B(f(z), pr)) \), and we can apply Lemma 3.1 to \( J^* \) and obtain

\[
J^* \cap A_m \subset J^* \cap T^{-n} B(f(z), \psi(n) + pr) \subset J^* \cap T^{-n} B(f(z), 2\psi(n)). \tag{3.8}
\]

Then apply Lemma 3.6 to \( J^* \cap T^{-n} B(f(z), 2\psi(n)) \), getting

\[
\mu\left(J^* \cap T^{-n} B(f(z), 2\psi(n))\right) \lesssim K_J^{-\delta} \left(T^m J^* \cap T^{-(n-m)} B(f(z), 2\psi(n))\right) \lesssim K_J^{-\delta} \mu\left(T^m B(z, r) \cap T^{-(n-m)} B(f(z), 2\psi(n))\right).
\]

By the conformality assumption (1.12), for \( m \) large enough such that \( \inf_{J \in \mathcal{F}_m} K_J > 2p \), we have

\[
T^m B(z, r) = T^m B\left(z, \frac{2\psi(m)}{K_J - p}\right) \subset B\left(T^m z, K^2_{2} \frac{2\psi(m)}{K_J - p}\right) \subset B(T^m z, K_2 4\psi(m)),
\]

where \( K_2 \) is defined in (1.12). Thus

\[
\mu(J^* \cap A_n) \lesssim K_J^{-\delta} \mu\left(B(T^m z, 4K_2 \psi(m)) \cap T^{-(n-m)} B(f(z), 2\psi(n))\right).
\]
By the mixing property (1.8),
\[
\mu(J^* \cap A_n) \lesssim K_J^{-\delta} \mu(T^m B(T^m z, 4K_2 \psi(m)) \cap T^{-(n-m)} B(f(z), 2\psi(n)))
\]
\[
\lesssim K_J^{-\delta} \left[ \mu\left( B(T^m z, 4K_2 \psi(m)) \right) \mu\left( B(z, 2\psi(n)) \right) + Ca_{n-m} \mu\left( B(z, 2\psi(n)) \right) \right]
\]
\[
\lesssim K_J^{-\delta} \left[ \psi^\delta(m) \psi^\delta(n) + a_{n-m} \psi^\delta(n) \right]
\]

Hence, since \( \sum_{J \in F_m} K_J^{-\delta} \) is uniformly bounded for all \( m \) in view of (1.11), we conclude that
\[
\sum_{J \in F_m, r \leq \psi(n)} \mu(J^* \cap A_n) \lesssim \psi^\delta(m) \psi^\delta(n) + a_{n-m} \psi^\delta(n).
\]

\underline{Case (ii): } \( pr = \frac{2 \psi(m)}{K_J - p} > \psi(n) \).

We replace the ball \( B(f(z), pr) \) by a collection of balls of radius \( \psi(n) \).

Choose a maximal \( \frac{3}{2} \psi(n) \)-separated set in \( B(f(z), pr) \), denoted by \( \{z_i\}_{1 \leq i \leq N_{m,n}} \).

Then
\[
B(f(z), pr) \subset \bigcup_{i=1}^{N_{m,n}} B(z_i, \psi(n)) \subset B(f(z), 2pr).
\]

Since \( \mu \) is Ahlfors regular and \( m > n_0 \),
\[
N_{m,n} \asymp \left( \frac{p \psi(m)}{(K_J - p) \psi(n)} \right)^\delta.
\]

Since \( B(z_i, \psi(n)) \subset f^{-1}B(f(z_i), p\psi(n)) \), we can apply Lemma 3.1 to each ball \( B(z_i, \psi(n)) \) with \( 1 \leq i \leq N_{m,n} \) and obtain
\[
\mu(B(z_i, \psi(n)) \cap A_n)
\]
\[
\leq \mu\left( B(z_i, \psi(n)) \cap T^{-n} B(z_i, (1+p)\psi(n)) \right)
\]
\[
\leq \mu\left( B(z_i, \psi(n)) \right) \mu\left( B(z_i, (1+p)\psi(n)) \right) + O(a_n) \mu\left( B(z_i, (1+p)\psi(n)) \right)
\]
\[
\leq [\psi^\delta(n) + O(a_n)] \psi^\delta(n)
\]

Now summing over \( 1 \leq i \leq N_{m,n} \), we have
\[
\mu(J^* \cap A_n) \leq \sum_{i=1}^{N_{m,n}} \mu\left( B(z_i, \psi(n)) \cap A_n \right)
\]
\[
\lesssim \left( \frac{p \psi(m)}{K_J - p} \right)^\delta \left[ \psi^\delta(n) + a_n \right]
\]
Then
\[
\sum_{J \in F_{m}, r > \psi(n)} \mu(J^* \cap A_n) \lesssim \sum_{J \in F_{m}, r = \psi(n)} \left( \frac{p \psi(m)}{K_m - p} \right)^\delta \left[ \psi^\delta(n) + a_n \right] \\
\lesssim \psi^\delta(m) \psi^\delta(n) + a_n \psi^\delta(n).
\]

(1.11)

Conclusion:
\[
\mu(A_m \cap A_n) = \sum_{J \in F_{m}, r \leq \psi(n)} \mu(J^* \cap A_n) + \sum_{J \in F_{m}, r > \psi(n)} \mu(J^* \cap A_n) \\
\lesssim \psi^\delta(m) \psi^\delta(n) + a_n \psi^\delta(n) + a_n \psi^\delta(m).
\]

\[\square\]

4. Full measure of \( R^f_T(\psi) \)

We start with the following abstract lemma.

**Lemma 4.1.** Let \( \mu \) be a probability measure on \( X \). Let \((E_k)\) be a sequence of measurable subsets of \( X \), and let \((a_k),(b_k)\) be sequences of positive real numbers such that \( \sum_{k=1}^\infty a_k < \infty \) and \( \sum_{k=1}^\infty b_k = \infty \). Assume that for some positive constants \( s_1, s_2, s_3 \) it holds that
\[
s_1(b_n - a_n) \leq \mu(E_n) \leq s_3(b_n + a_n) \quad \text{for all } n \in \mathbb{N},
\]
\[
\mu(E_m \cap E_n) \leq s_2(b_n b_m + a_n - m a_n b_m) \quad \text{for all } m < n.
\]
Then \( \mu(\limsup_k E_k) > 0 \).

**Proof.** The proof of this lemma uses ideas introduced in §2.3.1 of [HLSW]. Define \( f_n = \sum_{k=1}^n \chi_{E_k} \). In particular, since \( \sum_{k=1}^\infty b_k = \infty \), if
\[
f_n(x) \geq \frac{1}{2} \sum_{k=1}^n b_k \text{ infinitely often},
\]
then \( x \in E_k \) infinite often, i.e.,
\[
\limsup_n \left\{ x : f_n(x) \geq \frac{1}{2} \sum_{k=1}^n b_k \right\} \subset \limsup_k E_k
\]

So it suffices to show that \( \limsup_n \{ f_n \geq \frac{1}{2} \sum_{k=1}^n b_k \} \) has positive measure. To show this, we use the Paley–Zygmund inequality, which states
\[
P(Z > \theta \mathbb{E}[Z]) \geq (1 - \theta)^2 \frac{\mathbb{E}[Z]^2}{\mathbb{E}[Z^2]}
\]
for all random variables \( Z \) with finite variance, and for all real numbers \( 0 \leq \theta \leq 1 \). We will show that \( \frac{\int f_n \mu^p}{\int f_n^p \mu} \) is greater than a positive constant independent of \( n \).
Let us denote $\sum_{k=1}^{\infty} a_k = S$. On the one hand,

$$\int f_n \, d\mu = \sum_{k=1}^{n} \mu(E_k) \geq s_1 \sum_{k=1}^{n} (b_k - a_k) \geq s_1 \sum_{k=1}^{n} b_k - s_1 S$$

$$\geq \frac{s_1}{2} \sum_{k=1}^{n} b_k \quad \text{when } n \text{ is sufficiently large, because } \sum_{k=1}^{\infty} b_k = \infty.$$

On the other hand,

$$\int f_n^2 \, d\mu = \sum_{k=1}^{n} \mu(E_k) + 2 \sum_{k < \ell \leq n} \mu(E_k \cap E_\ell)$$

$$\leq s_3 \sum_{k=1}^{n} (b_k + a_k) + s_2 \sum_{k=1}^{n} \sum_{\ell=k}^{n} (b_\ell b_k + a_{\ell-k} b_\ell + a_\ell b_k)$$

$$\leq s_3 \sum_{k=1}^{n} b_k + s_3 S + s_2 \sum_{k=1}^{n} \sum_{\ell=k}^{n} b_\ell b_k + s_2 S \sum_{\ell=1}^{n} b_\ell + s_2 S \sum_{k=1}^{n} b_k$$

$$\leq (s_3 + s_2 (1 + 2S)) \left( \sum_{k=1}^{n} b_k \right)^2,$$

when $n$ is sufficiently large.

Hence

$$\mu(\limsup \{E_k \}) \geq \mu \left( \limsup \left\{ f_n \geq \frac{1}{2} \sum_{k=1}^{n} b_k \right\} \right)$$

$$\geq \limsup_n \left( 1 - \frac{1}{2} \right)^2 \cdot \frac{\left( \int f_n \, d\mu \right)^2}{\int f_n^2 \, d\mu} \geq \frac{1}{4} \frac{s_1^2}{2 \left( s_3 + s_2 (1 + 2S) \right)} > 0.$$

**Corollary 4.2.** Assume that $(X, \mu, T)$ satisfies conditions (1.7)-(1.12), and let $\psi : \mathbb{N} \to \mathbb{R}_+$ be such that $\lim_{n \to \infty} \psi(n) = 0$ and $\sum_{n=1}^{\infty} \psi^\delta(n) = \infty$. Then $\mu(R_T^f(\psi)) > 0$ for any Lipschitz function $f : X \to X$.

**Proof.** Take $E_k = A_k$. By Lemma 3.3, there exist positive constants $s_1$ and $s_3$ so that $s_1 (\psi^\delta(n) - a_n) \leq \mu(A_n) \leq s_3 (\psi^\delta(n) + a_n)$. By Proposition 3.9 there exists a positive constant $s_2$ so that

$$\mu(A_m \cap A_n) \leq s_2 (\psi^\delta(m) \psi^\delta(n) + a_n - m \psi^\delta(n) + a_n \psi^\delta(m))$$

for $m < n$. Take $b_n = \psi^\delta(n)$. Hence the above lemma implies that $\mu(R_T^f(\psi)) = \mu(\limsup_n A_n) > 0$. \qed

We now show $R_T^f(\psi)$ has full measure. The following proof is adapted from [HLSW].

**Lemma 4.3.** Let $(X, \mu, T)$, $\psi : \mathbb{N} \to \mathbb{R}_+$ and $f : X \to X$ be as in Corollary 4.2. Then $\mu(R_T^f(\psi)) = 1$. 


Proof. Consider the set
\[ R'(\psi) := \left\{ x \in X : \liminf_{n \to \infty} \psi^{-\delta}(n)d(T^n x, f(x)) < \infty \right\} \]
Take a point \( x \in R'(\psi) \cap f^{-1}X_i \). By definition, there exist \( c(x) > 0 \) and \( \{n_k\}_k \subset \mathbb{N} \) so that
\[ \psi^{-\delta}(n_k)d(T^{n_k} x, f(x)) < c(x) \forall k \geq 1 \]
Let \( s(x) \) be a positive real number so that \( B((f(x), s(x)) \subset X_i \). Since \( \psi(n) \to 0 \), take \( N \in \mathbb{N} \) such that for all \( k \geq N \),
\[ c(x)\psi(n_k) < s(x) \]
Then \( d(T^{n_k} x, f(x)) < c(x)\psi(n_k) < s(x) \), so \( T^{n_k} x \in X_i \) for all \( k \geq N \). Then for all \( k \geq N \),
\[ d(T^{n_k}(Tx), T(f(x))) = d(T(T^{n_k} x), T(f(x))) \leq d(T^{n_k} x, f(x)) < c(x)\psi(n_k) \]
Hence \( R'(\psi) \cap (\bigcup_i f^{-1}X_i) \subset T^{-1}R'(\psi) \), so \( \mu(R'(\psi) \setminus T^{-1}R'(\psi)) = 0 \). But \( R^{-1}_T(\psi) \subset R' \), so \( \mu(R'(\psi)) > 0 \), and by the ergodicity of \( T \), \( \mu(R'(\psi)) = 1 \).

Now we show that \( \mu(R^{-1}_T(\psi)) = 1 \). Take a sequence of positive numbers \( \{\ell(n) : n \geq 1\} \) such that
\[ \sum_{n=1}^{\infty} \left( \frac{\psi(n)}{\ell(n)} \right)^\delta = \infty \quad \text{and} \quad \lim_{n \to \infty} \ell(n) = \infty. \]
Consider \( \tilde{\psi}(n) = \psi(n)^\delta/\ell(n)^\delta \); then \( R'(\tilde{\psi}) \) has full measure, i.e. for \( \mu \)-almost every \( x \in X \),
\[ \liminf_{n \to \infty} \tilde{\psi}(n)d(T^n x, f(x)) < \infty. \]
By Egorov’s theorem, for any \( \varepsilon > 0 \) there exists \( M > 0 \) such that the set
\[ R_M = \left\{ x \in X : \frac{\ell(n)^\delta}{\psi(n)^\delta}d(T^n x, f(x)) < M \text{ for infinitely many } n \in \mathbb{N} \right\} \]
is of measure at least \( 1 - \varepsilon \). Then \( R_M \subset R_T^{-1}(\tilde{\psi}) \), by letting \( \ell(n) \to \infty \). Since \( \varepsilon \) is arbitrary, it implies \( \mu(R_T^{-1}(\tilde{\psi})) = 1 \).

5. Applications

Here we list several examples of dynamical systems to which our theorem applies. The first two come from the paper [HLSW]:

- \( X = [0,1], T : x \mapsto \beta x \mod 1 \), where \( \beta > 1 \), and \( \mu \) is the \( T \)-invariant probability measure absolutely continuous with respect to Lebesgue measure, namely (see [R])
\[ \mu(E) = \begin{cases} \text{Leb}(E) & \text{if } \beta \text{ is an integer,} \\
\frac{1}{\sum_{k=0}^{\infty} \frac{1}{\beta^k}} \left( \sum_{k=0}^{\infty} \frac{\text{Leb}(E \cap [0,(\beta^k)])}{\beta^k} \right) & \text{if } \beta \text{ is not an integer}, \end{cases} \]
where \( \{x\} \) denotes the fractional part of \( x \);

\[ X = [0, 1], \quad T : x \mapsto \frac{1}{2} \mod 1, \quad \text{and} \quad \mu \text{ is the Gauss measure given by} \]
\[
d\mu = \frac{dx}{(\log 2)(1+x)}.
\]

Sections 3.1–3.2 of [HLSW] together with Remark 1.1 show that in both cases the assumptions of Theorem 1.2 are satisfied. In fact, in both cases uniform mixing with exponential rate was first exhibited in [Ph], together with a quantitative shrinking target property of these systems.

Our last example deals with self-similar sets. Let
\[ \Theta := \{\theta_i(x) : \mathbb{R} \to \mathbb{R}\}_{i=1}^L \]
be a system of similarities with
\[
|\theta_i(x) - \theta_j(y)| = r_i|x - y| \quad \text{for all} \quad x, y \in \mathbb{R}, \quad i = 1, \ldots, L,
\]
where \( 0 < r_i < 1 \) for all \( i \). Then by [H, Theorem 3.1.(3)] there exists a unique nonempty compact set \( X \subset \mathbb{R} \), called the attractor of the system, such that
\[
X = \bigcup_{i=1}^L \theta_i(X).
\]
Furthermore, we assume that \( \Theta \) satisfies the open set condition: that is, there exists a non-empty bounded open set \( U \subset \mathbb{R} \) such that
\[
\bigcup_{i=1}^L \theta_i(U) \subset U \quad \text{and} \quad \theta_i(U) \cap \theta_j(U) = \emptyset \quad \text{for all} \quad i \neq j.
\]
Then it is known that the Hausdorff dimension of \( X \) is equal to the unique solution \( \delta \in [0, 1] \) of the equation \( \sum_{i=1}^L r_i^\delta = 1 \) (see [Fa, Theorem 9.3]). Furthermore, the normalized restriction \( \mu \) of the \( \delta \)-dimensional Hausdorff measure to \( X \) is positive, finite and satisfies
\[
\mu = \sum_{i=1}^L r_i^\delta \cdot (\theta_i)_* \mu. \tag{5.1}
\]
(For a proof, see [H, Theorem 4.4.(1)].)

To define the corresponding expanding map and construct the cylinders, we consider the following lemma from [Sc, Gr]:

**Lemma 5.1** ([Sc], Theorem 2.2; [Gr], Lemma 3.3). Let \( \{\theta_i : \mathbb{R}^n \to \mathbb{R}^n\}_{i=1}^L \) be a system of similarities satisfying the open set condition, \( X \) its attractor, and \( \mu \) the self-similar measure given by (5.1). Then there exists a nonempty compact set \( A \) with

(i) \( \theta_i(A) \subset A \) for all \( i = 1, \ldots, L \);

(ii) \( \theta_i(\text{Int}(A)) \cap \theta_j(\text{Int}(A)) = \emptyset \) for each \( i \neq j \);

(iii) \( \mu(\text{int}(A) \cap X) = 1 \).
We remark that parts (i) and (ii) are stated in [Sc, Theorem 2.2], and part (iii) follows from the proof of [Gr, Lemma 3.3], where it is shown that \( \mu((A \setminus \text{int}(A)) = 0 \) and \( X = \text{supp} \mu \subset A \).

Now define

\[
X_i := \theta_i((\text{int}(A) \cap X).
\]  

(5.2)

Each \( X_i \) is open in \( X \) because \( \theta_i \) is an open map. The disjointness of \( X_i \) and \( X_j \) for \( i \neq j \) follows from Lemma 5.1(ii). Finally, one can write

\[
\mu \left( \bigcup_{i=1}^{L} X_i \right) = \sum_{i=1}^{L} \mu(\theta_i((\text{int}(A) \cap X)) = \sum_{i=1}^{L} r_i^\delta \mu(\text{int}(A) \cap X)
\]

(by Lemma 5.1(iii))

\[
= \sum_{i=1}^{L} r_i^\delta = 1 = \mu(X).
\]

Hence one can define the map \( T : X \rightarrow X \) \( \mu \)-almost everywhere by setting \( T|_{X_i} = \theta_i^{-1}|_{X_i} \). It follows from (5.1) that \( (X, \mu, T) \) is a measure-preserving system. Clearly \( T|_{X_i} \) is continuous and injective for every \( i \). Therefore the collection \( \{X_i\}_{i=1}^{L} \) satisfies our assumption for being cylinders of order 1.

For \( i = (i_1, \ldots, i_m) \in \{1, \ldots, L\}^m \) let us define

\[
\theta_i := \theta_{i_1} \circ \cdots \circ \theta_{i_m} \quad \text{and} \quad r_1 := \prod_{k=1}^{m} r_{i_k}.
\]

Using the definition (1.6) of cylinders of order \( m \) together with (5.2), it is easy to see that the set \( F_m \) of cylinders of order \( m \) is precisely

\[
\{ X_i := \theta_i((\text{int}(A) \cap X) : i \in \{1, \ldots, L\}^m \}
\]

and the restriction of \( T^m \) onto \( X_i \in \mathcal{F}_m \) coincides with \( \theta_i^{-1} \). This, in particular, implies that

\[
\mu(X_i) = r_1^\delta
\]  

(5.3)

and

\[
K_J = \frac{|T^m x - T^m y|}{|x - y|} = r_1^{-1} \quad \text{for all } x, y \in J = X_i \in \mathcal{F}_m.
\]  

(5.4)

Denote

\[
r_{\max} := \max_{i=1, \ldots, L} r_i.
\]

Let us now verify assumptions (1.7)–(1.12) of Theorem 1.2.

(1.7): By [H, Theorem 5.3(1)(i)],

\[
\gamma_1 \leq \liminf_{r \to 0} \frac{\mu(B(x, r))}{r^\delta} \leq \limsup_{r \to 0} \frac{\mu(B(x, r))}{r^\delta} \leq \gamma_2
\]

for some \( 0 < \gamma_1 < \gamma_2 < \infty \) and all \( x \in X \). Clearly it implies that \( \mu \) is Ahlfors regular of dimension \( \delta \).
(1.8): (T is uniformly mixing) Let $E$ be a non-empty open ball in $X$, let $F$ be a measurable set in $X$, and let $m \in \mathbb{N}$. Note that for all cylinders $J = X_i \in \mathcal{F}_m$, where $i \in \{1, \ldots, L\}^m$ one can write

$$
\mu(J \cap T^{-m} F) = r_i^\delta \mu(T^m J \cap F) = r_i^\delta \mu(\text{int}(A) \cap F) = \mu(J) \mu(F).
$$

Note that since $E$ is an interval, we can (up to a set of measure zero) write $E$ as a disjoint union of cylinders of order $m$ and at most two balls contained in cylinders of order $m$; i.e.,

$$
E = \left( \bigcup_{J \in \mathcal{F}_m, J \subset E} J \right) \cup E_1 \cup E_2
$$

where the unions above are disjoint, and $E_1, E_2$ are contained in some cylinders $J_1, J_2 \in \mathcal{F}_m$ respectively, hence have measure not greater than $r_{\max}^m$. Then

$$
\left| \mu(E \cap T^{-m} F) - \mu(E) \mu(F) \right|
= \left| \sum_{J \in \mathcal{F}_m, J \subset E} (J \cap T^{-m} F) - \sum_{J \in \mathcal{F}_m, J \subset E} \mu(J) \mu(F) + \mu(E_1 \cap T^{-m} F) - \mu(E_1) \mu(F)
+ \mu(E_2 \cap T^{-m} F) - \mu(E_2) \mu(F) \right|
= \mu(E_1 \cap T^{-m} F) - \mu(E_1) \mu(F) + \mu(E_2 \cap T^{-m} F) - \mu(E_2) \mu(F)
\leq \mu(J_1 \cap T^{-m} F) + \mu(J_1) \mu(F) + \mu(J_2 \cap T^{-m} F) + \mu(J_2) \mu(F)
= 4r_{\max}^m \mu(F),
$$

and (1.8) follows with $a_n = 4r_{\max}^n$.

(1.9): Follows from (5.4) with $K_1 = 1$.

(1.10): Again from (5.4), for all $m \in \mathbb{N}$ we have

$$
\inf_{J \in \mathcal{F}_m} K_J = \inf_{i \in \{1, \ldots, L\}^m} r_i^{-1} = r_{\max}^{-m},
$$

which goes to $\infty$ as $m \to \infty$.

(1.11): In view of (5.4), for all $m \in \mathbb{N}$ one can write

$$
\sum_{J \in \mathcal{F}_m} K_J^{-\delta} \sum_{i \in \{1, \ldots, L\}^m} r_i^\delta = (r_1^\delta + \cdots + r_m^\delta)^m = 1.
$$

(1.12): Also follows from (5.4) with $K_2 = 1$.

Thus, by Theorem 1.2, for any function $\psi : \mathbb{N} \to \mathbb{R}^+$ and any Lipschitz function $f : X \to X$, the shifted recurrence set $R_f^T(\psi)$ satisfies

- $\mu(R_f^T(\psi)) = 0$ if $\sum_{n=1}^{\infty} \psi^\delta(n) < \infty$;
- $\mu(R_f^T(\psi)) = 1$ if $\sum_{n=1}^{\infty} \psi^\delta(n) = \infty$. 

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