Non-typical points for $\beta$-shifts

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February 4, 2022

Abstract

We study sets of nontypical points under the map $f_\beta \mapsto \beta x \mod 1$, for non-integer $\beta$ and extend our results from [2] in several directions. In particular we prove that sets of points whose forward orbit avoid certain Cantor sets, and set of points for which ergodic averages diverge, have large intersection properties. We remove the technical condition $\beta > 1.541$ found in [2].

Acknowledgements Both authors were supported by EC FP6 Marie Curie ToK programme CODY. Part of the paper was written when the authors were visiting institut Mittag-Leffler in Djursholm. The authors are grateful for the hospitality of the institute.

Mathematics Subject Classification 2010: 37E05, 37C45, 11J83.

1 $\beta$-shifts

Let $[x]$ denote the integer part of the real number $x$, and let $\lfloor x \rfloor$ denote the largest integer strictly smaller than $x$. Let $\beta > 1$. For any $x \in [0, 1)$ we associate the sequence $d(x, \beta) = (d(x, \beta)_n)_{n=0}^\infty \in \{0, 1, \ldots, \lfloor \beta \rfloor\}^N$ defined by

$$d(x, \beta)_n := \lfloor \beta f_n \beta(x) \rfloor,$$

where $f_\beta(x) = \beta x \mod 1$. The closure, with respect to the product topology, of the set $\{ d(x, \beta) : x \in [0, 1) \}$ is denoted by $S_\beta$ and it is called the $\beta$-shift. We will denote the set of all finite words occurring in $S_\beta$ by $S_\beta^*$. The sets $S_\beta$ and $S_\beta^*$ are invariant under the left-shift $\sigma: (i_n)_{n=0}^\infty \mapsto (i_{n+1})_{n=0}^\infty$ and the map $d(\cdot, \beta): x \mapsto d(x, \beta)$ satisfies the equality $\sigma^n(d(x, \beta)) = d(f^n_\beta(x), \beta)$. If we order $S_\beta$ with the lexicographical ordering then the map $d(\cdot, \beta)$ is one-to-one and monotone increasing. Let $d_-(1, \beta)$ be the limit in the product topology of $d(x, \beta)$ as $x$ approaches 1 from below. Then the subshift $S_\beta$ satisfies

$$S_\beta = \{ (j_k)_{k=0}^\infty : \sigma^n(j_k)_{k=0}^\infty \leq d_-(1, \beta) \ \forall n \}.$$  

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Note that \( d_{\{1, \beta\}} = d(1, \beta) \) if and only if \( d(1, \beta) \) contains infinitely many non-zero digits.

Parry proved in [5] that the map \( \beta \mapsto d(1, \beta) \) is monotone increasing and injective. For a sequence \((j_k)_{k=0}^\infty\) there is a \( \beta > 1 \) such that \((j_k)_{k=0}^\infty = d(1, \beta)\) if and only if \( \sigma^n((j_k)_{k=0}^\infty) < (j_k)_{k=0}^\infty \) for every \( n > 0 \). The number \( \beta \) is then the unique positive solution of the equation
\[
1 = \sum_{k=0}^\infty \frac{d_k(1, \beta)}{x^{k+1}}.
\]

One observes that the fact that the map \( \beta \mapsto d(1, \beta) \) is monotone increasing and injective together with (1) imply that \( S_{\beta_1} \subseteq S_{\beta_2} \) holds if and only if \( \beta_1 \leq \beta_2 \).

If \( x \in [0,1] \) then
\[
x = \sum_{k=0}^\infty \frac{d_k(x, \beta)}{\beta^{k+1}}.
\]

This formula can be seen as an expansion of \( x \) in the non-integer base \( \beta \), and thereby generalises the ordinary expansion in integer bases.

We let \( \pi_{\beta} \) be the map \( \pi_{\beta} : S_\beta \to [0,1) \) defined by
\[
\pi_{\beta} : (i_k)_{k=0}^\infty \mapsto \sum_{k=0}^\infty \frac{i_k}{\beta^{k+1}}.
\]

Hence, \( \pi_{\beta}(d(x, \beta)) = x \) holds for any \( x \in [0,1) \) and \( \beta > 1 \).

We define cylinder sets as
\[
[i_0 \cdots i_{n-1}] := \{ (j_k)_{k=0}^\infty \in S_\beta : i_k = j_k, \ 0 \leq k < n \},
\]
and say that \( n \) is the generation of the cylinder \([i_0 \cdots i_{n-1}].\) We will also call the half-open interval \( \pi_{\beta}([i_0 \cdots i_{n-1}) \) a cylinder of generation \( n \). The set \([i_0 \cdots i_{n-2}]\) will be called the parent cylinder of \([i_0 \cdots i_{n-1}].\)

Note that if \( d(1, \beta) \) has only finitely many non-zero digits, then \( S_\beta \) is a subshift of finite type, so there is a constant \( C > 0 \) such that
\[
C\beta^{-n} \leq |\pi_{\beta}([i_0 \cdots i_{n-1}])| \leq \beta^{-n}.
\]

2  Transversality and large intersection classes

In [1], Falconer defined \( G^s, 0 < s \leq n \), to be the class of \( G_\delta \) sets \( F \) in \( \mathbb{R}^n \) such that \( \dim_H(\cap_{i=1}^\infty f_i(F)) \geq s \) for all sequences of similarity transformations \((f_i)_{i=1}^\infty\). He characterised \( G^s \) in several equivalent ways and proved among other things that countable intersections of sets in \( G^s \) are also in \( G^s \).

In [2], the following approximation theorem was proven, where \( G^s \) are restrictions of Falconer’s classes to the unit interval.

**Theorem 1.** Let \( \beta \in (1.541, 2) \) and let \((\beta_n)_{n=1}^\infty\) be any sequence with \( \beta_n \in (1.541, \beta) \) for all \( n \), such that \( \beta_n \to \beta \) as \( n \to \infty \). Assume that \( E \subseteq S_\beta \) and \( \pi_{\beta_n}(E \cap S_\beta) \) is in the class \( G^s \) for all \( n \). If \( F \) is a \( G_\delta \setminus E \) such that \( F \supset \pi_{\beta}(E) \), then \( F \) is also in the class \( G^s \).
When expanding a number $x$ in base $\beta > 1$ as $d(x, \beta) = (x_k)_{k=0}^{\infty}$, one can consider how often a given word $y_1 \ldots y_m$ occurs. If the expression

$$\# \left\{ i \in \{0, \ldots, n-1\} : x_i \ldots x_{i+m-1} = y_1 \ldots y_m \right\}$$

converges as $n \to \infty$, it gives an asymptotic frequency of the occurrence of the word $y_1 \ldots y_m$ in the expansion of $x$ to the base $\beta$. Theorem 1 was used in 2 to prove the following.

**Proposition 1.** For any sequence of bases $(\beta_n)_{n=1}^{\infty}$, such that $\beta_n \in (1.541, 2)$ for all $n$, the set of points for which the frequency of any finite word does not converge in the expansion to any of these bases, has Hausdorff dimension 1.

The reason for the condition $\beta \in (1.541, 2)$ in Theorem 1 and Proposition 1 is that we needed some estimates on the map

$$\sum_{k=0}^{\infty} \frac{a_k - b_k}{\beta_k^2} \mapsto \sum_{k=0}^{\infty} \frac{a_k - b_k}{\beta_k^2}, \quad (a_1, a_2 \ldots), (b_1, b_2 \ldots) \in S_{\beta_1},$$

(3)

when $\beta_1 < \beta_2$, provided by the following transversality lemma by Solomyak 7.

**Lemma 1.** Let $x_0 < 0.649$. There exists a constant $\delta > 0$ such that if $x \in [0, x_0]$ then

$$|g(x)| < \delta \implies g'(x) < -\delta$$

holds for any function of the form

$$g(x) = 1 + \sum_{k=1}^{\infty} a_k x^k, \text{ where } a_k \in \{-1, 0, 1\}.$$  (4)

The condition $x_0 < 0.649$ in Lemma 1 introduces the condition $\beta > 1/0.649$ or for simplicity $\beta > 1.541$. But, when studying the map 3, the coefficients in the power series 3 will not be free to take values in $\{-1, 0, 1\}$; they will be the difference of two sequences from $S_\beta$. This allows us to remove the condition $x_0 < 0.649$.

**Lemma 2.** Let $\beta > 1$. There exists a constant $\delta > 0$ such that if $x \in [0, 1/\beta]$ then

$$|g(x)| < \delta \implies g'(x) < -\delta$$

holds for any function of the form

$$g(x) = 1 + \sum_{k=1}^{\infty} (a_k - b_k) x^k, \text{ where } (a_1, a_2 \ldots), (b_1, b_2 \ldots) \in S_\beta.$$  (4)

**Proof.** Assume that no such $\delta$ exists. Then there is a sequence $g_i$ of power series and a sequence of numbers $x_n \in [0, 1/\beta]$, such that $\lim_{n \to \infty} g_n(x_n) = 0$ and $\lim \inf_{n \to \infty} g_n(x_n) \geq 0$.

We can take a subsequence such that $g_n$ converges termwise to a series $g(x) = 1 + \sum_{k=1}^{\infty} (a_k - b_k) x^k$, with $(a_1, a_2 \ldots), (b_1, b_2 \ldots) \in S_\beta$, and $x_n$ converges to some number $x_0$. Clearly, $g(x_0) = 0$ and $g'(x_0) \geq 0$, so $x_0 \neq 0$. 


Let $\beta_0 = 1/x_0 \geq \beta$. Then $(a_1, a_2, \ldots), (b_1, b_2, \ldots) \in S_{\beta_0}$ and $g(x_0) = 0$ implies that

$$\pi_{\beta_0}(a_1, a_2, \ldots) - \pi_{\beta_0}(b_1, b_2, \ldots) = \sum_{k=1}^{\infty} \frac{a_k}{\beta_0^k} - \sum_{k=1}^{\infty} \frac{b_k}{\beta_0^k} = -1.$$ 

Since both sums are in $[0, 1]$, we conclude that

$$\sum_{k=1}^{\infty} \frac{a_k}{\beta_0^k} = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{b_k}{\beta_0^k} = 1.$$ 

We must therefore have $(a_1, a_2, \ldots) = (0, 0, \ldots)$. This implies that $g'(x) < 0$ for all $x \in (0, 1/\beta]$, contradicting the fact that $g'(x_0) \geq 0$.

Replacing Lemma 1 by Lemma 2 in the proofs of [2], we immediately get the following improved versions of Theorem 1 and Proposition 1. Note that allowing $\beta > 1$ instead of $\beta \in (1, 2)$ only affects notation slightly by adding new symbols to the shift space $S_\beta$. The proofs in [2] go through almost verbatim. Also the result from [3], which is used in [2] to prove Proposition 1, is easily extended from $\beta \in (1, 2)$ to $\beta > 1$.

**Theorem 2.** Let $\beta > 1$ and let $(\beta_n)_{n=1}^{\infty}$ be any sequence with $\beta_n < \beta$ for all $n$, such that $\beta_n \to \beta$ as $n \to \infty$. Assume that $E \subset S_\beta$ and $\pi_{\beta_n}(E \cap S_{\beta_n})$ is in the class $G^\alpha$ for all $n$. If $F$ is a $G_\delta$ set such that $F \supset \pi_\beta(E)$, then $F$ is also in the class $G^\alpha$.

**Proposition 2.** For any sequence of bases $(\beta_n)_{n=1}^{\infty}$, such that $\beta_n > 1$ for all $n$, the set of points for which the frequency of any finite word does not converge in the expansion to any of these bases, has Hausdorff dimension 1.

### 3 Schmidt games and avoiding Cantor sets

In [6], Schmidt introduced a set-theoretic game which can be seen as a metric version of the Banach–Mazur game (see for example [4]). We present here a modified version of Schmidt’s game that was used in [2].

Consider the unit interval $[0, 1]$ with the usual metric and a set $E \subset [0, 1]$. Two players, Black and White, play the game in $[0, 1]$ with two parameters $0 < \alpha, \gamma < 1$ according to the following rules:

- In the initial step Black chooses a closed interval $B_0 \subset [0, 1]$.
- Then the following step is repeated. At step $k$ White chooses a closed interval $W_k \subset B_k$ such that $|W_k| \geq \alpha |B_k|$. Then Black chooses a closed interval $B_{k+1} \subset W_k$ such that $|B_{k+1}| \geq \gamma |W_k|$.

We say that $E$ is $(\alpha, \gamma)$-winning if there is a strategy that White can use to make sure that $\bigcap_k W_k \subset E$, and $\alpha$-winning if this holds for all $\gamma$. As was shown in [2], the following proposition easily follows from the methods in [4].

**Proposition 3.**

- If $E$ is $\alpha$-winning for $\alpha = \alpha_0$, then $E$ is $\alpha$-winning for all $\alpha \leq \alpha_0$. 

Let \( \alpha \) is \( \beta \) will depend on \( \alpha \). See Remark 2 at the end of the paper for an estimate of \( \alpha \).

Proposition 4. For any \( S \alpha \) is \( \beta \) will depend on \( \alpha \) which the sets are bounded away from \( x \).

Proposition 5. Let \( \beta > 1 \) and let \( \Sigma_A \subset S_\beta \) be a subshift of finite type, such that there is a finite word \( i_1 \ldots i_n \) from \( S_\beta \setminus \Sigma_A \). Then there exist \( \alpha > 0 \) such that

\[
G_{f_\beta}(\pi_\beta(\Sigma_A)) = \left\{ y \in [0,1] : \pi_\beta(\Sigma_A) \cap \bigcup_{n=1}^\infty f_\beta^n(y) = \emptyset \right\}
\]

is \( \alpha \)-winning.

A quick look at Proposition 3 gives us the following corollary.

Corollary 1. Let \( N \in \mathbb{N}, \beta_1, \ldots, \beta_N > 1 \) and for each \( 1 \leq n \leq N \), let \( \Sigma_{A_n} \subset S_{\beta_n} \) such that there is a finite word \( i_1 \ldots i_{k_n} \) from \( S_{\beta_n} \setminus \Sigma_{A_n} \). The the set

\[
\bigcap_{n=1}^N G_{f_{\beta_n}}(\pi_\beta(\Sigma_{A_n}))
\]

has Hausdorff dimension 1.

The reason that \( N \) in Corollary 14 must be finite is that \( \alpha_0 \) from Proposition 3 will depend on \( \beta \) and \( \Sigma_A \). When taking intersections we need a uniform \( \alpha \) for which the sets are \( \alpha \)-winning, to be able to say anything about the intersection. See Remark 4 at the end of the paper for an estimate of \( \alpha \).

Before giving the proof of Proposition 5 we note that if \( S_\beta \) is a subshift of finite type, then Proposition 5 is easy. Indeed, by 2 we have good control over the size of each cylinder. So, it is not hard to see that there is an \( \alpha_0 \) such that each time White plays he can introduce the word \( i_1 \ldots i_n \). Again by (2) this implies that the word \( i_1 \ldots i_n \) occurs regularly in \( \{y\} = \cap_k W_k \), and this means that \( \bigcup_{n=1}^\infty f_\beta^n(y) \) is bounded away from \( \pi_\beta(\Sigma_A) \). Hence Proposition 5 need only be proved in the case when \( S_\beta \) is not of finite type.

The case when \( S_\beta \) is not of finite type is much more difficult, since we have no uniform lower bound on the size of cylinders, such as 2. The key step in proving Proposition 3 was the following theorem from 2. It will be used in the proof of Proposition 5.

Theorem 3. Let \( \beta \in (1,2) \) and let \( (\beta_n)_{n=1}^\infty \) be any sequence with \( \beta_n \in (1,\beta) \) for all \( n \) such that \( \beta_n \to \beta \) as \( n \to \infty \). Let also \( E \subset S_\beta \) and \( \alpha \in (0,1) \). If \( \pi_{\beta_n}(E \cap S_{\beta_n}) \) is \( \alpha \)-winning for \( \alpha = \alpha_0 \) for all \( n \), then \( \pi_\beta(E) \) is \( \alpha \)-winning for any \( \alpha \leq \min\left\{ \frac{1}{16}, \frac{\alpha_0}{2} \right\} \).
Remark 1. The condition $\beta \in (1, 2)$ in Theorem $\ref{thm:main}$ comes from the fact that in $\ref{thm:main}$, we chose to work with $\beta < 2$ to simplify notation. It is not difficult to extend the proof of Theorem $\ref{thm:main}$ to hold for all $\beta > 1$.

The only place in which $\beta \in (1, 2)$ was used is in what is called “An auxiliary strategy”. There we use the fact that in any cylinder $\pi_\beta([i_0 \ldots i_n])$, the player White needs at most a factor 2 to make sure that the game continues in $\pi_\beta([i_0 \ldots i_n])$, thereby avoiding the cylinder $\pi_\beta([i_0 \ldots i_n,1])$ which may have bad properties. If $\beta > 2$, a factor 2 is still enough for White to avoid the cylinder $\pi_\beta([i_0 \ldots i_n,|\beta|])$ which may have bad properties. The factor 2 is not enough for White to choose any other cylinder $\pi_\beta([i_0 \ldots i_nk])$ in one move, but after a couple of moves, the game is already played in such a small set that at most two of these cylinders remain, so White can pick at least one of them. That is all what is needed for the strategy to work.

Proposition $\ref{prop:main}$ follows from Theorem $\ref{thm:main}$ and Remark $\ref{rem:main}$ once we have proven the following proposition.

**Proposition 6.** Let $\beta > 1$ such that $S_\beta$ is not of finite type and let $\Sigma_A \subset S_\beta$ be a subshift of finite type. Then there exist $\alpha > 0$ and $\beta_0 < \beta$ such that

$$G_{\beta'}(\pi_\beta'(\Sigma_A)) = \{y \in [0,1) : \pi_\beta'(\Sigma_A) \cap \bigcup_{n=0}^{\infty} f_{\beta'}^n(y) = \emptyset\}$$

is $\alpha$-winning for any $\beta' \in [\beta_0, \beta]$ such that $S_{\beta'}$ is of finite type.

To prove Proposition $\ref{prop:main}$ we need some lemmata.

**Lemma 3.** Let $\beta > 1$ and let $i_1 \ldots i_n$ be a finite word in $S_\beta^*$ such that we have $i_1 \ldots i_n j_1 \ldots j_m \in S_\beta^*$ for all finite words $j_1 \ldots j_m \in S_\beta^*$. Then $|\pi([i_1 \ldots i_n])| = \beta^{-n}$ and also $|\pi([i_1 \ldots i_n j_1 \ldots j_m])| = \beta^{-n}|\pi([j_1 \ldots j_m])|$ for all finite words $j_1 \ldots j_m \in S_\beta^*$.

**Proof.** It is clear that $\sigma^n([i_1 \ldots i_n]) = S_\beta$, so $f_{\beta}^n(\pi_\beta([i_1 \ldots i_n])) = (0,1)$, where $f_{\beta}$ is just the scaling $x \mapsto \beta^nx$ on $\pi_\beta([i_1 \ldots i_n])$. Thus, $\pi_\beta([i_1 \ldots i_n])$ is just a smaller copy of $[0,1)$.

**Lemma 4.** Let $\beta > 1$, $M \in \mathbb{N}$ and $k \in \mathbb{N}$ be such that $(d(1,\beta)n)_{n=0}^M0^k1 \in S_\beta^*$. If $\beta_0 \in (1, \beta)$ is such that $(d(1,\beta)n)_{n=0}^M0^k1 \in S_{\beta_0}^*$, then for all $i_1 \ldots i_n \in S_{\beta_0}$ such that $M = \max\{m : i_{n-m} \ldots i_n = (d(1,\beta)n)_{n=0}^m\}$, it holds that

$$|\pi_\beta([i_1 \ldots i_n])| \geq \beta^{-(n+k+1)}, \text{ for all } \beta' \in [\beta_0, \beta].$$

**Proof.** Let $\beta' \in [\beta_0, \beta]$. From $\ref{lem:main}$ and the maximality of $M$ we conclude that $i_1 \ldots i_{n-M} j_1 \ldots j_m \in S_{\beta'}^*$ for all $j_1 \ldots j_m \in S_{\beta'}^*$. From Lemma $\ref{lem:main}$ we then get $|\pi_\beta([i_1 \ldots i_n])| \geq |\pi_\beta([i_1 \ldots i_n 0^{k+1}])| = \beta^{-(n+k+1)}$.

**Lemma 5.** Let $\beta > 1$ and $M \in \mathbb{N}$. There exist $\epsilon > 0$ and $\beta_0 < \beta$ such that for any $\beta' \in [\beta_0, \beta]$ and for any interval $I \subset [0,1]$, there exists a cylinder $\pi_{\beta'}([i_0 \ldots i_n])$ such that $\max\{m : i_{m-n} \ldots i_n = (d(1,\beta)n)_{n=0}^m\} \geq M$ for which $|\pi_{\beta'}([i_0 \ldots i_n]) \cap I| > \epsilon |I|$. Moreover, if $S_{\beta'}$ is of finite type, then $|\pi_{\beta'}([i_0 \ldots i_n]) \cap I| > \delta_{\beta'}|\pi_{\beta'}([i_0 \ldots i_n])|$, where $\delta_{\beta'} > 0$ is independent of $I$. 

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Proof. Let $\beta' \in [\beta_0, \beta]$ as in Lemma 4 and let $I \subset [0, 1]$ be an interval. Note that all cylinders in this proof will be with respect to $S_{\beta'}$. Let $n$ be the smallest generation for which there is a cylinder contained in $I$. Let $\pi_{\beta'}([i_0 \ldots i_{n-1}])$ be one of these generation $n$ cylinders in $I$. By the minimality of $n$ we know that the parent cylinder, $\pi_{\beta'}([i_0 \ldots i_{n-2}])$ covers at least one endpoint of $I$. If $\pi_{\beta'}([i_0 \ldots i_{n-2}])$ does not cover $I$, let $m$ be the smallest generation for which there is a cylinder contained in $I \setminus \pi_{\beta'}([i_0 \ldots i_{n-2}])$. Let $\pi_{\beta'}([j_0 \ldots j_{m-1}])$ be one of these generation $m$ cylinders. By the minimality of $m$ we know that the parent cylinder, $\pi_{\beta'}([j_0 \ldots j_{m-2}])$ covers the other endpoint of $I$.

Together, $\pi_{\beta'}([i_0 \ldots i_{n-2}])$ and $\pi_{\beta'}([j_0 \ldots j_{m-2}])$ cover $I$. Indeed, if not, then there is a smallest generation $l$ for which there is a cylinder $\pi_{\beta'}([k_0 \ldots k_{l-1}])$ between $\pi_{\beta'}([i_0 \ldots i_{n-2}])$ and $\pi_{\beta'}([j_0 \ldots j_{m-2}])$. Consider its parent cylinder $\pi_{\beta'}([k_0 \ldots k_{l-2}])$. If $\pi_{\beta'}([k_0 \ldots k_{l-2}])$ would intersect one of $\pi_{\beta'}([i_0 \ldots i_{n-2}])$ and $\pi_{\beta'}([j_0 \ldots j_{m-2}])$, then it would have to contain it. But this is impossible since the minimality of $n$ and $m$ implies $l \leq n, m$. Thus, $\pi_{\beta'}([k_0 \ldots k_{l-2}])$ is also between $\pi_{\beta'}([i_0 \ldots i_{n-2}])$ and $\pi_{\beta'}([j_0 \ldots j_{m-2}])$, which contradicts the minimality of $l$.

Consider the one of $\pi_{\beta'}([i_0 \ldots i_{n-2}])$ and $\pi_{\beta'}([j_0 \ldots j_{m-2}])$ that covers at least half of $I$. Let us assume it is $\pi_{\beta'}([i_0 \ldots i_{n-2}])$ but it makes no difference for the argument.

If $\max\{m : i_{n-m-2} \ldots i_{n-2} = (d(1, \beta)_n)_{m=0}^n \} \geq M$, then we can choose the set $\pi_{\beta'}([i_0 \ldots i_{n-2}]) \cap I$ as long as $\epsilon \leq 1/2$ and we are done with the first claim. The second claim, that $|\pi_{\beta'}([i_0 \ldots i_{n-2}]) \cap I| > \sigma_{\beta'}|\pi_{\beta'}([i_0 \ldots i_{n-2}])|$ follows from the fact that $|\pi_{\beta'}([i_0 \ldots i_{n-2}])| \leq I$ and (2), since $S_{\beta'}$ is of finite type.

Assume instead that $\max\{m : i_{n-m-2} \ldots i_{n-2} = (d(1, \beta)_n)_{m=0}^n \} = N < M$. Then $i_0 \ldots i_{n-2}(d(1, \beta)_n)_{M-N}^M \in S_{\beta'}^*$ by (1). By Lemma 5, there is a $k$ only depends on $\beta$ and $M$ such that

$$|\pi_{\beta'}([i_0 \ldots i_{n-2}(d(1, \beta)_n)_{M-N}^M])| \geq \beta^{-(n+M+k)} \geq \beta^{-(M+k+1)}|\pi_{\beta'}([i_0 \ldots i_{n-2}])|.$$ 

We conclude that if $\epsilon \leq \beta^{-(M+k+1)/2}$, then we can choose the cylinder $\pi_{\beta'}([i_0 \ldots i_{n-2}(d(1, \beta)_n)_{M-N}^M]) \subset I$. This ensures the truth of both claims and we are done.

We are now ready to prove Proposition 6.

Proof of Proposition 6. Note that since $\Sigma_A$ is of finite type while $S_{\beta}$ is not, there is an $M > 1$ such that $(d(1, \beta)_n)_{n=0}^M$ is not allowed in $\Sigma_A$. For this $M$ choose $\epsilon$ and $\beta_0$ as in Lemma 5. Let $\beta' \in [\beta_0, \beta]$ such that $S_{\beta'}$ is of finite type, let $\gamma > 0$ and $\alpha = \epsilon/2$.

Assume that Black has chosen his first interval $B_0$. We will construct a strategy that White can use to make sure that $\cap_k W_k = \{x\} \subset G_{\beta'}(\Sigma_A)$, or equivalently that $f_{\beta'}^* (x)_{n=0}^\infty$ is bounded away from $\pi_{\beta'}(\Sigma_A)$.

Each time Black has chosen an interval $B_k$, Lemma 5 ensures that White can choose $W_k \subset \pi_{\beta'}([i_1 \ldots i_n]) \cap B_k$, where $\max\{m : i_{n-m} \ldots i_n = (d(1, \beta)_n)_{m=0}^n \} \geq M$ and $|W_k| \leq \sigma(\beta')|\pi_{\beta'}([i_1 \ldots i_n])|$. Since $\beta' < \beta$, there is an $N$ such that $(d(1, \beta)_n)_{n=0}^N \notin S_{\beta'}$, so $M \leq \max\{m : i_{n-m} \ldots i_n = (d(1, \beta)_n)_{m=0}^N \} \leq N$.

If White plays like this, it ensures that the sequence $d(x, \beta')$ contains the word $(d(1, \beta)_n)_{n=0}^M$ regularly. Thus, $f_{\beta'}^* (x)_{n=0}^\infty$ is always in a cylinder outside $\pi_{\beta'}(\Sigma_A)$. If $(f_{\beta'}^* (x))_{n=0}^\infty$ would be bounded away from the endpoints of these cylinders, then $(f_{\beta'}^* (x))_{n=0}^\infty$ would be bounded away from $\pi_{\beta'}(\Sigma_A)$. But $\alpha = \epsilon/2,$
so there is a factor 2 left after placing $W_k$ in $\pi_{\beta'}[i_1 \ldots i_n] \cap B_k$. White can place $W_k$ in the middle of $\pi_{\beta'}[i_1 \ldots i_n] \cap B_k$, thereby avoiding the endpoints.

We conclude that $G_{\beta'}(\Sigma_A)$ is $\alpha$-winning and we are done.

**Remark 2.** The $\alpha$ in Proposition 6 can be extracted quite easily from the proofs. Let $M$ be such that $(d_k(1, \beta))_{k=1}^M$ is not at word in $\Sigma_A$. Take $k$ such that $(d_j(1, \beta))_{j=0}^M 0^k 1 < d(1, \beta)$. Then $\alpha = \beta^{-(M+k+1)}/4$ is small enough. It follows that in Proposition 6 $\alpha = \beta^{-(M+k+1)}/16$ is small enough. Note that these values for $\alpha$ are not optimal, but they make it possible to extend Corollary 4 to infinite intersections, for some cases.

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NON-TYPICAL POINTS FOR $\beta$-SHIFTS

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Abstract. We study sets of nontypical points under the map $f_\beta \mapsto \beta x \mod 1$, for non-integer $\beta$ and extend our results from [3] in several directions. In particular we prove that sets of points whose forward orbit avoid certain Cantor sets, and set of points for which ergodic averages diverge, have large intersection properties. We observe that the technical condition $\beta > 1.541$ found in [3] can be removed.

1. $\beta$-shifts

Let $[x]$ denote the integer part of the real number $x$, and let $\lfloor x \rfloor$ denote the largest integer strictly smaller than $x$. Let $\beta > 1$. For any $x \in [0, 1)$ we associate the sequence $d(x, \beta) = (d(x, \beta)_n)_{n=0}^\infty \in \{0, 1, \ldots, \lfloor \beta \rfloor \}^\mathbb{N}$ defined by $d(x, \beta)_n := \lfloor \beta f^n_\beta(x) \rfloor$, where $f_\beta(x) = \beta x \mod 1$. The closure, with respect to the product topology, of the set

\[ \{ d(x, \beta) : x \in [0, 1) \} \]

is denoted by $S_\beta$ and it is called the $\beta$-shift. We will denote the set of all finite words occuring in $S_\beta$ by $S_\beta^\ast$. The sets $S_\beta$ and $S_\beta^\ast$ are invariant under the left-shift $\sigma : (i_n)_{n=0}^\infty \mapsto (i_{n+1})_{n=0}^\infty$ and the map $d(-, \beta) : x \mapsto d(x, \beta)$ satisfies the equality $\sigma^n(d(x, \beta)) = d(f^n_\beta(x), \beta)$. If we order $S_\beta$ with the lexicographical ordering then the map $d(-, \beta)$ is one-to-one and monotone increasing. Let $d_-(1, \beta)$ be the limit in the product topology of $d(x, \beta)$ as $x$ approaches 1 from below. Then the subshift $S_\beta$ satisfies

\[ S_\beta = \{ (j_k)_{k=0}^\infty : \sigma^n(j_k)_{k=0}^\infty \leq d_-(1, \beta) \forall n \}. \]

Note that $d_-(1, \beta) = d(1, \beta)$ if and only if $d(1, \beta)$ contains infinitely many non-zero digits.

Parry proved in [7] that the map $\beta \mapsto d(1, \beta)$ is monotone increasing and injective. For a sequence $(j_k)_{k=0}^\infty$ there is a $\beta > 1$ such that $(j_k)_{k=0}^\infty = d(1, \beta)$ if and only if $\sigma^n((j_k)_{k=0}^\infty) < (j_k)_{k=0}^\infty$ for every $n > 0$. The number $\beta$ is then the unique positive solution of the equation

\[ 1 = \sum_{k=0}^\infty \frac{d_k(1, \beta)}{x_{k+1}}. \]

Date: February 4, 2022.

2000 Mathematics Subject Classification. 37E05, 37C45, 11J83.

Both authors were supported by EC FP6 Marie Curie ToK programme CODY. Part of the paper was written when the authors were visiting institut Mittag-Leffler in Djursholm in 2010. The authors are grateful for the hospitality of the institute.
One observes that the fact that the map $\beta \mapsto d(1, \beta)$ is monotone increasing and injective together with (1) imply that $S_{\beta_1} \subseteq S_{\beta_2}$ holds if and only if $\beta_1 \leq \beta_2$.

If $x \in [0, 1]$ then

$$x = \sum_{k=0}^{\infty} d_k(x, \beta) \beta^k.$$  

This formula can be seen as an expansion of $x$ in the non-integer base $\beta$, and thereby generalises the ordinary expansion in integer bases.

We let $\pi_{\beta}$ be the map $\pi_{\beta}: S_\beta \to [0, 1)$ defined by

$$\pi_{\beta}: (i_k)_{k=0}^{\infty} \mapsto \sum_{k=0}^{\infty} \frac{i_k}{\beta^{k+1}}.$$  

Hence, $\pi_{\beta}(d(x, \beta)) = x$ holds for any $x \in [0, 1)$ and $\beta > 1$.

We define cylinder sets as

$$[i_0 \cdots i_{n-1}] := \{ (j_k)_{k=0}^{\infty} \in S_{\beta} : i_k = j_k, \ 0 \leq k < n \},$$

and say that $n$ is the generation of the cylinder $[i_0 \cdots i_{n-1}]$. We will also call the half-open interval $\pi_{\beta}([i_0 \cdots i_{n-1}])$ a cylinder of generation $n$. The set $[i_0 \cdots i_{n-2}]$ will be called the parent cylinder of $[i_0 \cdots i_{n-1}]$.

Note that if $d(1, \beta)$ has only finitely many non-zero digits, then $S_{\beta}$ is a subshift of finite type, so there is a constant $C > 0$ such that

$$C\beta^{-n} \leq |\pi_{\beta}([i_0 \cdots i_{n-1}])| \leq \beta^{-n}.$$  

2. Transversality and large intersection classes

In [2], Falconer defined $G^s$, $0 < s \leq n$, to be the class of $G_\delta$ sets $F$ in $\mathbb{R}^n$ such that $\dim_H(\cap_{s=1}^{\infty} f_s(F)) \geq s$ for all sequences of similarity transformations $(f_i)_{i=1}^{\infty}$. He characterised $G^s$ in several equivalent ways and proved among other things that countable intersections of sets in $G^s$ are also in $G^s$.

In [3], the following approximation theorem was proven, where $G^s$ are restrictions of Falconer’s classes to the unit interval.

**Theorem 1.** Let $\beta \in (1.541, 2)$ and let $(\beta_n)_{n=1}^{\infty}$ be any sequence with $\beta_n \in (1.541, \beta)$ for all $n$, such that $\beta_n \to \beta$ as $n \to \infty$. Assume that $E \subset S_{\beta}$ and $\pi_{\beta_n}(E \cap S_{\beta_n})$ is in the class $G^s$ for all $n$. If $F$ is a $G_\delta$ set such that $F \supset \pi_{\beta}(E)$, then $F$ is also in the class $G^s$.

When expanding a number $x$ in base $\beta > 1$ as $d(x, \beta) = (x_k)_{k=0}^{\infty}$, one can consider how often a given word $y_1 \cdots y_m$ occurs. If the expression

$$\frac{\#\{i \in \{0, \ldots, n - 1\} : x_i \cdots x_{i+m-1} = y_1 \cdots y_m\}}{n}$$

converges as $n \to \infty$, it gives an asymptotic frequency of the occurrence of the word $y_1 \cdots y_m$ in the expansion of $x$ to the base $\beta$. Theorem 1 was used in [3] to prove the following.

**Proposition 1.** For any sequence of bases $(\beta_n)_{n=1}^{\infty}$, such that $\beta_n \in (1.541, 2)$ for all $n$, the set of points for which the frequency of any finite word does not converge in the expansion to any of these bases, has Hausdorff dimension 1.
The reason for the condition $\beta \in (1.541, 2)$ in Theorem 1 and Proposition 1 is that in [3], we needed some estimates on the map

\[
\sum_{k=1}^{\infty} \frac{a_k - b_k}{\beta_k^1} \mapsto \sum_{k=1}^{\infty} \frac{a_k - b_k}{\beta_k^2}, \quad (a_1, a_2 \ldots), (b_1, b_2 \ldots) \in S_{\beta_1},
\]

when $\beta_1 < \beta_2$, provided by the following transversality lemma by Solomyak [9].

**Lemma 1.** Let $x_0 < 0.649$. There exists a constant $\delta > 0$ such that if $x \in [0, x_0]$ then

\[|g(x)| < \delta \implies g'(x) < -\delta\]

holds for any function of the form

\[
g(x) = 1 + \sum_{k=1}^{\infty} a_k x^k, \text{ where } a_k \in \{-1, 0, 1\}.
\]

The condition $x_0 < 0.649$ in Lemma 1 introduces the condition $\beta > 1/0.649$ or for simplicity $\beta > 1.541$. But, when studying the map defined in [3], the coefficients in the power series (1) will not be free to take values in $\{-1, 0, 1\}$, they will be the difference of two sequences from $S_\beta$. This allows us to remove the condition $x_0 < 0.649$, which is done by using Lemma 2 below instead of Lemma 1.

**Lemma 2.** Let $\beta > 1$. There exists a constant $\delta > 0$ such that if $x \in [0, 1/\beta]$ then

\[|g(x)| < \delta \implies g'(x) < -\delta\]

holds for any function of the form

\[
g(x) = 1 + \sum_{k=1}^{\infty} (a_k - b_k) x^k, \text{ where } (a_1, a_2 \ldots), (b_1, b_2 \ldots) \in S_\beta.
\]

This lemma was stated and proved in [3], were it was used for other purposes. We refer to [5] for the proof, were in fact, the lemma was proved with the condition $x \in [0, 1/\beta]$ replaced by the weaker condition $x \in [0, 1/\beta + \varepsilon]$, where $\varepsilon$ is a small positive constant. In this note we will however only need the weaker form stated above.

Replacing Lemma 1 by Lemma 2 in the proofs of [3], we immediately get the following improved versions of Theorem 1 and Proposition 1. Note that allowing $\beta > 1$ instead of $\beta \in (1, 2)$ only affects notation slightly by adding new symbols to the shift space $S_\beta$. The proofs in [3] go through almost verbatim. Also the result from [4], which is used in [3] to prove Proposition 1, is easily extended from $\beta \in (1, 2)$ to $\beta > 1$.

**Theorem 2.** Let $\beta > 1$ and let $(\beta_n)_{n=1}^{\infty}$ be any sequence with $\beta_n < \beta$ for all $n$, such that $\beta_n \to \beta$ as $n \to \infty$. Assume that $E \subset S_\beta$ and $\pi_{\beta_n}(E \cap S_{\beta_n})$ is in the class $G^* \forall n$. If $F$ is a $G_\delta$ set such that $F \supset \pi_{\beta}(E)$, then $F$ is also in the class $G^*$.  

**Proposition 2.** For any sequence of bases $(\beta_n)_{n=1}^{\infty}$, such that $\beta_n > 1$ for all $n$, the set of points for which the frequency of any finite word does not converge in the expansion to any of these bases, has Hausdorff dimension 1.
3. SCHMIDT GAMES AND AVOIDING CANTOR SETS

In [8], Schmidt introduced a set-theoretic game which can be seen as a metric version of the Banach–Mazur game (see for example [6]). We present here a modified version of Schmidt’s game that was used in [3].

Consider the unit interval \([0, 1]\) with the usual metric and a set \(E \subset [0, 1]\). Two players, Black and White, play the game in \([0, 1]\) with two parameters \(0 < \alpha, \gamma < 1\) according to the following rules:

In the initial step Black chooses a closed interval \(B_0 \subset [0, 1]\).

Then the following step is repeated. At step \(k\), White chooses a closed interval \(W_k \subset B_k\) such that \(|W_k| \geq \alpha |B_k|\). Then Black chooses a closed interval \(B_{k+1} \subset W_k\) such that \(|B_{k+1}| \geq \gamma |W_k|\).

We say that \(E\) is \((\alpha, \gamma)\)-winning if there is a strategy that White can use to make sure that \(\bigcap_k W_k \subset E\), and \(\alpha\)-winning if this holds for all \(\gamma\). As was shown in [3], the following proposition easily follows from the methods in [8].

Proposition 3.

1. If \(E\) is \(\alpha\)-winning for \(\alpha = \alpha_0\), then \(E\) is \(\alpha\)-winning for all \(\alpha \leq \alpha_0\).
2. If \(E_i\) is \(\alpha\)-winning for \(i = 1, 2, 3, \ldots\), then \(\bigcap_{i=1}^{\infty} E_i\) is also \(\alpha\)-winning.
3. If \(E\) is \(\alpha\)-winning, then the Hausdorff dimension of \(E\) is 1.

In [3], the following proposition was proven.

Proposition 4. For any \(\beta \in (1, 2)\) and any \(x \in [0, 1]\),

\[
G_\beta(x) = \left\{ y \in [0, 1] : x \notin \bigcup_{n=0}^{\infty} f_\beta^n(y) \right\},
\]

is \(\alpha\)-winning for any \(\alpha \leq 1/16\).

The set \(G_\beta(x)\) consists of points for which the forward orbit under \(f_\beta\) is bounded away from \(x\). One can also think of \(G_\beta(x)\) as the union over all \(\delta > 0\), of sets of points with orbits not falling into a hole of radius \(\delta\) around \(x\).

Let us at this point compare our result with a result by Dolgopyat [1]. Dolgopyat proved that if \(E\) is a set of Hausdorff dimension strictly smaller than 1, and \(f\) is a piecewise expanding map on an interval, then the set of points, for which the orbit under \(f\) avoids the set \(E\), has full Hausdorff dimension. Dolgopyat’s result is stronger in the sense that the result holds for a much larger class of maps. However, Dolgopyat’s result does not give any intersection property, and in this sense our result is stronger, since we can treat countably many different maps at the same time, whereas Dolgopyat’s result only gives results for one fixed map.

Here, we will extend our results in the spirit of Dolgopyat, and instead of considering only orbits avoiding a point, we consider orbits avoiding a more general set \(E\). In doing so, we will prove that the set of points that avoid a set \(E\) is \(\alpha\)-winning, and so get a stronger statement than only full Hausdorff dimension, which is the result of Dolgopyat.

\footnote{We are grateful to a referee for pointing out this paper to us.}
However, we will need to impose some restrictions on the set \( E \). More precisely, we will prove the following proposition which shows that we can avoid entire Cantor sets instead of just single points. We consider sets of the form

\[
G_{f_\beta}(\pi_\beta(\Sigma_A)) = \left\{ y \in [0,1) : \pi_\beta(\Sigma_A) \cap \bigcup_{n=0}^\infty f_\beta^n(y) = \emptyset \right\},
\]

where \( \Sigma_A \) denotes a subshift of finite type. We then have that \( \pi_\beta(\Sigma_A) \) is a Cantor set in \([0,1]\). Hence the set \( G_{f_\beta}(\pi_\beta(\Sigma_A)) \) is the set of points with forward orbit bounded away from the Cantor set \( \pi_\beta(\Sigma_A) \). One can also think of the set \( G_{f_\beta}(\pi_\beta(\Sigma_A)) \) as the union over all \( \delta > 0 \), of sets of points with orbits not falling into a hole consisting of a \( \delta \)-neighbourhood of \( \pi_\beta(\Sigma_A) \).

**Proposition 5.** Let \( \beta > 1 \) and let \( \Sigma_A \subset S_\beta \) be a subshift of finite type, such that there is a finite word \( i_0 \ldots i_n \) from \( S_\beta \setminus \Sigma_A \). Then there exist \( \alpha > 0 \) such that

\[
G_{f_\beta}(\pi_\beta(\Sigma_A)) = \left\{ y \in [0,1) : \pi_\beta(\Sigma_A) \cap \bigcup_{n=0}^\infty f_\beta^n(y) = \emptyset \right\}
\]

is \( \alpha \)-winning.

A quick look at Proposition 3 gives us the following corollary.

**Corollary 1.** Let \( N \in \mathbb{N} \), \( \beta_1, \ldots, \beta_N > 1 \) and for each \( 1 \leq n \leq N \), let \( \Sigma_{A_n} \subset S_{\beta_n} \) such that there is a finite word \( i_1 \ldots i_{k_n} \) from \( S_{\beta_n} \setminus \Sigma_{A_n} \). The set

\[
\bigcap_{n=1}^N G_{f_{\beta_n}}(\pi_\beta(\Sigma_{A_n}))
\]

has Hausdorff dimension 1.

The reason why \( N \) in Corollary 1 must be finite is that \( \alpha_0 \) from Proposition 5 will depend on \( \beta \) and \( \Sigma_A \). When taking intersections we need a uniform \( \alpha \) for which the sets are \( \alpha \)-winning, to be able to say anything about the intersection. See Remark 2 at the end of the paper for an estimate of \( \alpha \). If we have uniform estimates on the \( \alpha \), then we can take countable intersections in Corollary 1.

Before giving the proof of Proposition 5 we note that if \( S_\beta \) is a subshift of finite type, then Proposition 5 is easy to prove. Indeed, then there is a constant \( C > 0 \) such that

\[
(5) \quad C \leq \frac{|\pi_\beta([i_0 \ldots i_k])|}{\beta^{k+1}} \leq 1
\]

holds for all cylinders \([i_0 \ldots i_k]\). Using (5), it is not hard to see that there is an \( \alpha_0 > 0 \) such that each time White plays she can introduce the word \( i_0 \ldots i_n \), that is missing in \( \Sigma_A \). By (5) this implies that the word \( i_0 \ldots i_n \) occurs regularly in \( \{y\} = \cap_k W_k \), and this means that \( \bigcup_{n=0}^\infty f_\beta^n(y) \) is bounded away from \( \pi_\beta(\Sigma_A) \). Hence Proposition 5 need only be proved in the case when \( S_\beta \) is not of finite type.

The case when \( S_\beta \) is not of finite type is much more difficult, since we have no uniform lower bound on the size of cylinders, such as (2). The key
step in proving Proposition 4 was the following theorem from [3]. It will be used in the proof of Proposition 5.

Theorem 3. Let $\beta \in (1, 2)$ and let $(\beta_n)_{n=1}^{\infty}$ be any sequence with $\beta_n \in (1, \beta)$ for all $n$ such that $\beta_n \to \beta$ as $n \to \infty$. Let also $E \subset S_\beta$ and $\alpha \in (0, 1)$. If $\pi_{\beta_n}(E \cap S_{\beta_n})$ is $\alpha$-winning for $\alpha = \alpha_0$ for all $n$, then $\pi_\beta(E)$ is $\alpha$-winning for any $\alpha \leq \min\{\frac{1}{16}, \frac{\alpha_0}{2}\}$.

Remark 1. The condition $\beta \in (1, 2)$ in Theorem 3 comes from the fact that in [3], we chose to work with $\beta > 2$ to simplify notation. It is not difficult to extend the proof of Theorem 3 to hold for all $\beta > 1$.

The only place in which $\beta \in (1, 2)$ was used is in what is called “An auxiliary strategy”. There we use the fact that in any cylinder $\pi_\beta([i_0 \ldots i_n])$, the player White needs at most a factor 2 to make sure that the game continues in $\pi_\beta([i_0 \ldots i_n[0])$, thereby avoiding the cylinder $\pi_\beta([i_0 \ldots i_n[1])$ which may have bad properties. If $\beta > 2$, a factor 2 is still enough for White to avoid the cylinder $\pi_\beta([i_0 \ldots i_n[\beta])$ which may have bad properties. The factor 2 is not enough for White to choose any other cylinder $\pi_\beta([i_0 \ldots i_nk])$ in one move, but after a couple of moves, the game is already played in such a small set that at most two of these cylinders remain, so White can pick at least one of them. That is all what is needed for the strategy to work.

Proposition 5 follows from Theorem 3 and Remark 1 once we have proven the following proposition.

Proposition 6. Let $\beta > 1$ such that $S_\beta$ is not of finite type and let $\Sigma_A \subset S_\beta$ be a subshift of finite type. Then there exist $\alpha > 0$ and $\beta_0 < \beta$ such that

$$G_{\beta'}(\pi_{\beta'}(\Sigma_A)) = \{ y \in [0, 1) : \pi_{\beta'}(\Sigma_A) \cap \bigcup_{n=0}^{\infty} f_{\beta'}^n(y) = \emptyset \}$$

is $\alpha$-winning for any $\beta' \in [\beta_0, \beta]$ such that $S_{\beta'}$ is of finite type.

To prove Proposition 6 we need some lemmata.

Lemma 3. Let $\beta > 1$ and let $i_0 \ldots i_n$ be a finite word in $S_\beta^*$ such that we have $i_0 \ldots i_nj_0 \ldots j_m \in S_\beta^*$ for all finite words $j_0 \ldots j_m \in S_\beta^*$. Then we have $|\pi([i_0 \ldots i_n])| = \beta^{-n-1}$ and

$$|\pi_\beta([i_0 \ldots i_nj_0 \ldots j_m])| = \beta^{-n-1}|\pi_\beta([j_0 \ldots j_m])|$$

for all finite words $j_0 \ldots j_m \in S_\beta^*$.

Proof. It is clear that $\sigma^{n+1}([i_0 \ldots i_n]) = S_\beta$, so $f_{\beta}^{n+1}(\pi_\beta([i_1 \ldots i_n])) = [0, 1)$, where $f_{\beta}^{n+1}$ is just a scaling with factor $\beta^{n+1}$ on $\pi_\beta([i_0 \ldots i_n])$. Thus, $\pi_\beta([i_0 \ldots i_n])$ is just a smaller copy of $[0, 1)$.

Lemma 4. Let $\beta > 1$, $M \in \mathbb{N}$ and $k \in \mathbb{N}$ be such that $(d(1, \beta)_n)_{n=0}^{M}0^k1 \in S_\beta^*$. If $\beta_0 \in (1, \beta)$ is such that $(d(1, \beta)_n)_{n=0}^{M}0^k1 \in S_{\beta_0}^*$, then for all $i_0 \ldots i_n \in S_{\beta_0}^*$ such that $M = \max\{ m : i_{n-m} \ldots i_n = (d(1, \beta)_n)_{m=0}^{m} \}$, it holds that

$$|\pi_{\beta'}([i_0 \ldots i_n])| \geq \beta^{-(n+k+2)}$$

for all $\beta' \in [\beta_0, \beta]$.
Proof. Let $\beta' \in [\beta_0, \beta]$. From (1) and the maximality of $M$ we conclude that $i_0 \ldots i_{n-M} j_0 \ldots j_m \in S_{\beta'}^n$ for all $j_0 \ldots j_m \in S_{\beta'}^M$. From Lemma 2 we then get $|\pi_\beta([i_0 \ldots i_n])| \geq |\pi_\beta([i_0 \ldots i_n, 0^{k+1}])| = \beta^{-(n+k+2)}$.

Lemma 5. Let $\beta > 1$ and $M \in \mathbb{N}$. There exist $\epsilon > 0$ and $\beta_0 < \beta$ such that for any $\beta' \in [\beta_0, \beta]$ and for any interval $I \subset [0, 1]$, there exists a cylinder $\pi_{\beta'}([i_0 \ldots i_n])$ such that $\max\{m : i_{n-m} \ldots i_n = (d(1, \beta)n)_{m=0}^m \} \geq M$ for which $|\pi_{\beta'}([i_0 \ldots i_n]) \cap I| > \epsilon|I|$. Moreover, if $S_{\beta'}$ is of finite type, then $|\pi_{\beta'}([i_0 \ldots i_n]) \cap I| > \sigma_{\beta'}|\pi_{\beta'}([i_0 \ldots i_n])|$, where $\sigma_{\beta'} > 0$ is independent of $I$.

Proof. Let $\beta' \in [\beta_0, \beta]$ as in Lemma 4 and let $I \subset [0, 1]$ be an interval. Note that all cylinders in this proof will be with respect to $S_{\beta'}$. Let $n$ be the smallest generation for which there is a cylinder contained in $I$. Let $\pi_{\beta'}([i_0 \ldots i_{n-1}])$ be one of these generation $n$ cylinders in $I$. By the minimality of $n$ we know that the parent cylinder, $\pi_{\beta'}([i_0 \ldots i_{n-2}])$ covers at least one endpoint of $I$. If $\pi_{\beta'}([i_0 \ldots i_{n-2}])$ does not cover $I$, let $m$ be the smallest generation for which there is a cylinder contained in $I \setminus \pi_{\beta'}([i_0 \ldots i_{n-2}])$. Let $\pi_{\beta'}([j_0 \ldots j_{m-1}])$ be one of these generation $m$ cylinders. By the minimality of $m$ we know that the parent cylinder, $\pi_{\beta'}([j_0 \ldots j_{m-2}])$ covers the other endpoint of $I$.

Together, the cylinders $\pi_{\beta'}([i_0 \ldots i_{n-2}])$ and $\pi_{\beta'}([j_0 \ldots j_{m-2}])$ cover $I$. Indeed, if not, then there is a smallest generation $l$ for which there is a cylinder $\pi_{\beta'}([k_0 \ldots k_{l-1}])$ between $\pi_{\beta'}([i_0 \ldots i_{n-2}])$ and $\pi_{\beta'}([j_0 \ldots j_{m-2}])$. Consider its parent cylinder $\pi_{\beta'}([k_0 \ldots k_{l-2}])$. If $\pi_{\beta'}([k_0 \ldots k_{l-2}])$ would intersect one of $\pi_{\beta'}([i_0 \ldots i_{n-2}])$ and $\pi_{\beta'}([j_0 \ldots j_{m-2}])$, then it would have to contain it. But this is impossible since the minimality of $n$ and $m$ implies $l \geq n, m$. Thus, $\pi_{\beta'}([k_0 \ldots k_{l-2}])$ is also between $\pi_{\beta'}([i_0 \ldots i_{n-2}])$ and $\pi_{\beta'}([j_0 \ldots j_{m-2}])$, which contradicts the minimality of $l$.

Consider the one of $\pi_{\beta'}([i_0 \ldots i_{n-2}])$ and $\pi_{\beta'}([j_0 \ldots j_{m-2}])$ that covers at least half of $I$. Let us assume it is $\pi_{\beta'}([i_0 \ldots i_{n-2}])$ but it makes no difference for the argument.

If $\max\{m : i_{n-m-2} \ldots i_{n-2} = (d(1, \beta)n)_{m=0}^m \} \geq M$, then we can choose the set $\pi_{\beta'}([i_0 \ldots i_{n-2}]) \cap I$ as long as $\epsilon \leq 1/2$ and we are done with the first claim. The second claim, that $|\pi_{\beta'}([i_0 \ldots i_{n-2}]) \cap I| > \sigma_{\beta'}|\pi_{\beta'}([i_0 \ldots i_{n-2}])|$ follows from the fact that $|\pi_{\beta'}([i_0 \ldots i_{n-1}])| < I$ and (2), since $S_{\beta'}$ is of finite type.

Assume instead that $\max\{m : i_{n-m-2} \ldots i_{n-2} = (d(1, \beta)n)_{m=0}^m \} < M$. Then $i_0 \ldots i_{n-2}(d(1, \beta)n)_{M-N-1}^M \in S_{\beta'}^M$ by (1). By Lemma 4 there is a $k$ that only depends on $\beta$ and $M$ such that

$$|\pi_{\beta'}([i_0 \ldots i_{n-2}(d(1, \beta)n)_{M-N-1}^M])| \geq \beta^{-(n+M+k+1)}$$

$$\geq \beta^{-(M+k+2)}|\pi_{\beta'}([i_0 \ldots i_{n-2}])|.$$

Since $|\pi_{\beta'}([i_0 \ldots i_{n-2}])| \geq |I|/2$, we conclude that if $\epsilon \leq \beta^{-(M+k+2)}/2$, then we can choose the cylinder $\pi_{\beta'}([i_0 \ldots i_{n-2}(d(1, \beta)n)_{M-N-1}^M]) \subset I$. This ensures the truth of both claims and we are done.

Proof of Proposition 6. Note that since $\Sigma_A$ is of finite type while $S_\beta$ is not, there is an $M > 1$ such that $(d(1, \beta)n)_{M=0}^M$ is not allowed in $\Sigma_A$. For this $M$
choose \( \epsilon \) and \( \beta_0 \) as in Lemma \([5]\). Let \( \beta' \in [\beta_0, \beta] \) such that \( S_{\beta'} \) is of finite type, and let \( \alpha = \epsilon/2 \).

Assume that Black has chosen his first interval \( B_0 \). We will construct a strategy that White can use to make sure that \( \cap_k W_k = \{ x \} \subset G_{\beta'}(\Sigma_A) \), or equivalently that \( (f_{\beta'}^n(x))_{n=0}^\infty \) is bounded away from \( \pi_{\beta'}(\Sigma_A) \).

Each time Black has chosen an interval \( B_k \), Lemma \([5]\) ensures that White can choose \( W_k \subset \pi_{\beta'}[i_0 \ldots i_n] \cap B_k \), where

\[
\max\{ m : i_{n-m} \ldots i_n = (d(1, \beta)n(m=0) \}\} \geq M
\]

and \( |W_k| \geq \sigma(\beta')|\pi_{\beta'}[i_0 \ldots i_n]| \). Since \( \beta' < \beta \), there is an \( N \) such that \( (d(1, \beta)n)_{n=0}^N \not\in S_{\beta'} \). It implies that for the cylinders \( \pi_{\beta'}([i_0 \ldots i_n]) \) that occur here, the numbers \( \max\{ m : i_{n-m} \ldots i_n = (d(1, \beta)n=0) \} \) will be bounded by \( N \).

If White plays like this, it ensures that the sequence \( d(x, \beta') \) contains the word \( (d(1, \beta)n)_{n=0}^M \) regularly. Thus, \( f_{\beta'}^n(x) \) is always in a cylinder outside \( \pi_{\beta'}(\Sigma_A) \). If \( f_{\beta'}^n(x) \) would be bounded away from the endpoints of these cylinders, then \( (f_{\beta'}^n(x))_{n=0}^\infty \) would be bounded away from \( \pi_{\beta'}(\Sigma_A) \). But \( \alpha = \epsilon/2 \), so there is a factor 2 left after placing \( W_k \) in \( \pi_{\beta'}[i_0 \ldots i_n] \cap B_k \). White can place \( W_k \) in the middle of \( \pi_{\beta'}[i_0 \ldots i_n] \cap B_k \), thereby avoiding the endpoints.

We conclude that \( G_{\beta'}(\Sigma_A) \) is \( \alpha \)-winning and we are done. \( \square \)

Remark 2. The \( \alpha \) in Proposition \([5]\) can be extracted quite easily from the proofs. Let \( M \) be such that \( (d_k(1, \beta))_{k=0}^M \) is not at word in \( \Sigma_A \). Take \( k \) such that \( (d_j(1, \beta))_{j=0}^k 1 < d(1, \beta) \). Then \( \alpha = \beta^{-(M+k+1)}/4 \) is small enough. It follows that in Proposition \([5]\) \( \alpha = \beta^{-(M+k+1)}/16 \) is small enough. Note that these values for \( \alpha \) are not optimal, but they make it possible to extend Corollary \([1]\) to countable intersections, for some cases.

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