On the definition of the $\Delta$ mass and width

D. Djukanovic,1 J. Gegelia,1,2 and S. Scherer1

1 Institut für Kernphysik, Johannes Gutenberg-Universität, D-55099 Mainz, Germany
2 High Energy Physics Institute, Tbilisi State University, Tbilisi, Georgia

(Dated: February 1, 2008)

In the framework of effective field theory we show that, at two-loop order, the mass and width of the $\Delta$ resonance defined via the (relativistic) Breit-Wigner parametrization both depend on the choice of field variables. In contrast, the complex-valued position of the pole of the propagator is independent of this choice.

PACS numbers: 14.20.Gk 12.39.Fe, 12.39.Gd

The problem of defining masses of unstable particles has a long history. A popular definition corresponding to a (relativistic) Breit-Wigner formula makes use of the zero of the real part of the inverse propagator to identify the mass. The field-redefinition dependence of such a definition was shown in Refs. [1, 2] in the scalar sector of the Standard Model. Another important example is the definition of the Z-boson mass. The gauge-parameter dependence of the Breit-Wigner mass starting at two-loop order was shown in Refs. [3, 4, 5, 6, 7]. In contrast, the complex-valued position of the pole of the propagator is independent of this choice.

We consider the interaction terms of the form

$$\mathcal{L}_{\text{int}} = g \partial^\nu \pi \bar{\psi}^\mu (\delta_{\mu\nu} - z \gamma_\mu \gamma_\nu) \Psi + \text{H.c.} + \cdots,$$

where the ellipsis refers to an infinite number of interaction terms which are present in the EFT. These terms also include all counter-terms which take care of divergences appearing in our calculations. The consistency of the interaction terms with the constraints of the spin-3/2 system fixes the value of the parameter $z$ to $-1$ for $A = -1$ [16, 21]. Throughout this paper we use dimensional regularization. Although our results are renormalization scheme independent, for simplicity we use the minimal subtraction scheme [22]. It is implemented by subtracting the divergent parts of one- and two-loop diagrams using the standard procedure [23].

Let us consider the field transformation

$$\bar{\psi}^\mu \rightarrow \bar{\psi}^\mu + \xi \partial^\mu \pi \bar{\Psi}, \quad \psi^\nu \rightarrow \psi^\nu + \xi \partial^\nu \pi \bar{\Psi},$$

where $\xi$ is an arbitrary real parameter. When inserted into the Lagrangians of Eqs. (1) and (2), the field redefinition generates additional interaction terms. The terms linear in $\xi$ are given by

$$\mathcal{L}_{\text{add int}} = \xi \partial^\mu \pi \bar{\Psi} A_{\mu\nu} \psi^\nu + \xi \partial^\nu \pi \bar{\psi}^\mu A_{\mu\nu} \Psi.$$

Note that the contribution generated from the expression explicitly shown in Eq. (2) vanishes identically. Because of the equivalence theorem physical quantities cannot depend on the field redefinition parameter $\xi$. Below we demonstrate that the complex-valued position of the pole of the $\Delta$ propagator does not depend on $\xi$. In contrast, the mass and width defined via the real and imaginary parts of the inverse propagator depend on $\xi$ at two-loop order.

The dressed propagator of the $\Delta$ in $n$ space-time dimensions can be written as [16, 22]

$$-i \left[ g^{\mu\nu} \frac{g^{\rho\nu}}{n-1} \frac{p^\rho \gamma^\nu - \gamma^\rho p^\nu}{(n-1)m_\Delta} - \frac{(n-2)p^\rho p^\nu}{(n-1)m_\Delta} \right].$$
where we parameterized the self-energy of the $\Delta$ as
\[
\Sigma_1(p^2)g^{\mu\nu} + \Sigma_2(p^2)\gamma^\mu\gamma^\nu + \Sigma_3(p^2)p^\mu\gamma^\nu + \Sigma_4(p^2)\gamma^\mu p^\nu
+ \Sigma_5(p^2)\gamma^\mu p^\nu + \Sigma_6(p^2)\gamma^\mu\gamma^\nu
+ \Sigma_7(p^2)\gamma^\mu p^\nu + \Sigma_8(p^2)\gamma^\mu\gamma^\nu + \Sigma_9(p^2)\gamma^\mu p^\nu + \Sigma_{10}(p^2)p^\mu p^\nu.
\]

The complex pole $z$ of the $\Delta$ propagator is obtained by solving the equation
\[
z - m_\Delta - \Sigma_1(z^2) - z \Sigma_6(z^2) = 0.
\]
The pole mass is defined as the real part of $z$.

On the other hand, the mass $m_R$ and width $\Gamma$ of the $\Delta$ resonance are often determined from the real and imaginary parts of the inverse propagator (corresponding to the Breit-Wigner parametrization), i.e.,
\[
m_R - m_\Delta - \text{Re} \Sigma_1(m_R^2) - m_R \text{Re} \Sigma_6(m_R^2) = 0,
\]
\[
\Gamma = -2 \text{Im} \Sigma_1(m_R^2) - 2 m_R \text{Im} \Sigma_6(m_R^2).
\]

Below we calculate the $\Delta$ mass using both definitions and analyze their $\xi$ dependence to first order.

The $\Delta$ self-energy at one loop-order is given by the diagram in Fig. 1(a). The corresponding results for $\Sigma_1$ and $\Sigma_6$ read
\[
\Sigma_1^{(a)} = -g^2 m_N I_1 - 2 \xi g \left( p^2 - m_\Delta m_N \right) I_1 + p^2 J_1,
\]
\[
\Sigma_6^{(a)} = -g^2 (I_1 + J_1) + 2 \xi g \left[ m_\Delta J_1 + (m_\Delta - m_N) I_1 \right],
\]
where $I_1$, $J_1$ are defined through the one-loop integrals
\[
I^{\alpha\beta}, I^{\alpha\beta\gamma} = \int \frac{i d^nk}{(2\pi)^n} \frac{k^\alpha k^\beta, k^\alpha k^\beta k^\gamma}{[k^2 + i0^+]} [(p + k)^2 - m_N^2 + i0^+],
\]
which we parameterize as
\[
I^{\alpha\beta} = I_1 g^{\alpha\beta} + I_2 p^\alpha p^\beta,
\]
\[
I^{\alpha\beta\gamma} = J_1 \left( g^{\alpha\beta} p^\gamma + g^{\alpha\gamma} p^\beta + g^{\beta\gamma} p^\alpha + J_2 p^\alpha p^\beta p^\gamma \right).
\]

The two-loop contributions to the $\Delta$ self-energy are given in Fig. 1(b) - (d). We are interested in terms linear in $\xi$. Calculating diagram (b) and (c) we find that they give vanishing contributions. The result of diagram (d), linear in $\xi$, can be reduced to the form
\[
\Sigma_1^{(d)} = 2 g^3 \xi \left[ m_N^2 I_1^2 + p^2 (I_1 + J_1)^2 \right],
\]
\[
\Sigma_6^{(d)} = 4 g^3 \xi m_N I_1 (I_1 + J_1).
\]

Note that the vanishing of the contributions of diagrams (b) and (c) as well as the simple expression of Eq. (10) have to be attributed to the special choice of the field transformation of Eq. (3).

To find the pole of the propagator we insert its loop expansion
\[
z = m_\Delta + \delta_1 z + \delta_2 z + \cdots
\]
in Eq. (7) and solve the resulting equation order by order. Using Eq. (9) we obtain for the one-loop result
\[
\delta z_1 = -g^2 \left[ m_\Delta \tilde{J}_1 + (m_\Delta + m_N) \tilde{I}_1 \right],
\]
\[
\tilde{J}_1 = J_1|_{p^2=m_\Delta^2}, \quad \tilde{I}_1 = I_1|_{p^2=m_\Delta^2}.
\]
The contribution to the two-loop expression $\delta z_2$, linear in $\xi$, generated by the one-loop diagram reads
\[
\delta z_2^{\xi=1L} = 2 g^3 \xi \left[ m_\Delta \tilde{J}_1 + (m_\Delta + m_N) \tilde{I}_1 \right] \left( 1 - \frac{1}{\xi} \right).
\]

These two contributions exactly cancel each other leading to the $\xi$-independent pole of the propagator.

We perform the same analysis inserting the loop expansion $m_R$.
\[
m_R = m_\Delta + \delta_1 m + \delta_2 m + \cdots
\]
in Eq. (8). For $\delta m_1$ we obtain
\[
\delta m_1 = -g^2 \left[ m_\Delta \text{Re} \tilde{J}_1 + (m_\Delta + m_N) \text{Re} \tilde{I}_1 \right].
\]
The contribution to $\delta m_2$ generated by the one-loop diagram reads
\[
\delta m_2^{\xi=1L} = 2 g^3 \xi \left[ m_\Delta \text{Re} \tilde{J}_1 + (m_\Delta + m_N) \text{Re} \tilde{I}_1 \right] \left( 1 - \frac{1}{\xi} \right).
\]

For an unstable $\Delta$ resonance $\tilde{I}_1$ and $\tilde{J}_1$ have imaginary parts and therefore Eqs. (17) and (18) do not cancel each other, thus leading to a $\xi$-dependent mass $m_R$. An analogous result holds for the width $\Gamma$ obtained from Eq. (3).

To conclude, we addressed the issue of defining the mass and width of the $\Delta$ resonance in the framework...
of a low-energy EFT of QCD. In general, the scattering amplitude of a resonant channel can be presented as a sum of the resonant contribution expressed in terms of the dressed propagator of the resonance and the background contribution. The resonant contribution defines the Breit-Wigner parameters through the real and imaginary parts of the inverse (dressed) propagator. The resonant part and the background separately depend on the chosen field variables, while the sum is of course independent of this choice. We have performed a particular field transformation with an arbitrary parameter $\xi$ in the effective Lagrangian. In a two-loop calculation we have demonstrated that the mass and width of the $\Delta$ resonance determined from the real and imaginary parts of the inverse propagator depend on the choice of field variables. On the other hand, the complex pole of the full propagator does not depend on the field transformation parameter $\xi$.

Note that according to general theorems \cite{26, 27, 28, 29} it is expected that in quantum field theories the $S$-matrix is independent of field redefinitions (change of variables). The pole of the $\Delta$ propagator corresponds to the pole of the $S$-matrix. The Laurent-series expansion of the $S$-matrix around this pole has to be independent of the field redefinition term-by-term. Therefore the pole of the $\Delta$ propagator is expected to be independent of the field redefinition.

The conclusions from this work are not restricted to the considered toy model or EFT in general. Rather, our results are a manifestation of the general feature that the (relativistic) Breit-Wigner parametrization leads to model- and process-dependent masses and widths of resonances. Although in some cases (like the $\Delta$ resonance) the background has small numerical effect on the Breit-Wigner parameters — a pole-zero cancellation — the mass and width should be considered preferable as these are free of conceptual ambiguities. This agrees with and supports the recent results of Ref. \cite{30}.

We would like to thank L. Tiator for useful discussions. D.D. and J.G. acknowledge the support of the Deutsche Forschungsgemeinschaft (SFB 443).

\begin{thebibliography}{99}
\bibitem{1} S. Willenbrock and G. Valencia, Phys. Lett. B \textbf{247}, 341 (1990).
\bibitem{2} G. Valencia and S. Willenbrock, Phys. Rev. D \textbf{42}, 853 (1990).
\bibitem{3} A. Sirlin, Phys. Rev. Lett. \textbf{67}, 2127 (1991).
\bibitem{4} A. Sirlin, Phys. Lett. B \textbf{267}, 240 (1991).
\bibitem{5} S. Willenbrock and G. Valencia, Phys. Lett. B \textbf{259}, 373 (1991).
\bibitem{6} J. Gegelia, G. Japaridze, A. Tkabladze, A. Khelashvili and K. Turashvili, in Quarks ’92, Proceedings of the International Seminar on Quarks, Zelenigord, Russia, 1992, edited by D. Yu. Grigoriev, V. A. Matveev, V. A. Rubakov, and P. G. Tinyakov (World Scientific, River Edge, N. J., 1993), [ArXiv:hep-ph/9910527].
\bibitem{7} P. Gambino and P. A. Grassi, Phys. Rev. D \textbf{62}, 076002 (2000).
\bibitem{8} G. ’t Hooft and M. J. G. Veltman, Nucl. Phys. B\textbf{50}, 318 (1972).
\bibitem{9} B. W. Lee and J. Zinn-Justin, Phys. Rev. D \textbf{5}, 3121 (1972).
\bibitem{10} D. Gross, in Methods in Field Theory, edited by R. Balian and J. Zinn-Justin (North-Holland, Amsterdam, 1976).
\bibitem{11} S. Weinberg, Physica \textbf{A96}, 327 (1979).
\bibitem{12} J. Gasser and H. Leutwyler, Annals Phys. \textbf{158}, 142 (1984).
\bibitem{13} J. Gasser, M. E. Sainio, and A. Švarc, Nucl. Phys. B\textbf{307}, 779 (1988).
\bibitem{14} S. Scherer, Adv. Nucl. Phys. \textbf{27}, 277 (2003); S. Scherer and M. R. Schindler, [ArXiv:hep-ph/0505265].
\bibitem{15} T. R. Hemmert, B. R. Holstein, and J. Kambor, J. Phys. G \textbf{24}, 1831 (1998).
\bibitem{16} C. Hacker, N. Wies, J. Gegelia, and S. Scherer, Phys. Rev. C \textbf{72}, 055203 (2005).
\bibitem{17} V. Pascalutsa, M. Vanderhaeghen, and S. N. Yang, Phys. Rept. \textbf{437}, 125 (2007).
\bibitem{18} G. Höhler, Against Breit-Wigner parameters — a pole-zero cancellation, in C. Caso et al. [Particle Data Group], Eur. Phys. J. C \textbf{3}, 624 (1998).
\bibitem{19} W. Rarita and J. S. Schwinger, Phys. Rev. \textbf{60}, 61 (1941).
\bibitem{20} N. Wies, J. Gegelia, and S. Scherer, Phys. Rev. D \textbf{73}, 094012 (2006).
\bibitem{21} L. M. Nath, B. Etemadi and J. D. Kimel, Phys. Rev. D \textbf{3}, 2153 (1971).
\bibitem{22} For the purposes of this work we could have also worked in terms of bare parameters. In this case no counter term diagrams occur and the loop integrals do not need to be subtracted.
\bibitem{23} J. C. Collins, Renormalization (Cambridge University Press, Cambridge, UK, 1984).
\bibitem{24} The formulas of Ref. \cite{16} are given in 4 dimensions but can be easily generalized for $n$ dimensions.
\bibitem{25} The loop integrals $I_3$ and $J_3$ are made finite by applying the minimal subtraction scheme, i.e., by subtracting the parts proportional to $1/(n – 4)$.
\bibitem{26} J. Chisholm, Nucl. Phys. \textbf{26}, 469 (1961).
\bibitem{27} S. Kamefuchi, L. O’Raifeartaigh and A. Salam, Nucl. Phys. \textbf{28}, 529 (1961).
\bibitem{28} S. R. Coleman, J. Wess and B. Zumino, Phys. Rev. \textbf{177}, 2239 (1969).
\bibitem{29} S. Weinberg, The Quantum theory of fields. Vol. 1: Foundations (Cambridge University Press, Cambridge, UK, 1995).
\bibitem{30} A. R. Bohm and Y. Sato, Phys. Rev. D \textbf{71}, 085018 (2005).
\end{thebibliography}