CONVEX BODIES ASSOCIATED TO TENSOR NORMS

MAITE FERNÁNDEZ-UNZUETA and LUISA F. HIGUERAS-MONTAÑO

Abstract. We determine when a convex body in $\mathbb{R}^d$ is the closed unit ball of a reasonable crossnorm on $\mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_l}$, $d = d_1 \cdots d_l$. We call these convex bodies “tensorial bodies”. We prove that, among them, the only ellipsoids are the closed unit balls of Hilbert tensor products of Euclidean spaces. It is also proved that linear isomorphisms on $\mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_l}$ preserving decomposable vectors map tensorial bodies into tensorial bodies. This leads us to define a Banach-Mazur type distance between them, and to prove that there exists a Banach-Mazur type compactum of tensorial bodies.

1. Introduction

Tensor products of finite dimensional spaces play a fundamental role in a wide range of problems in applications. They arise, among others, in quantum computing [15], in theoretical computer science [10], and in the use of tensor decompositions to extract and explain properties from data arrays (see [18] and the references therein). This fact has motivated the current research into their geometric, topologic and algebraic properties, as can be seen in [6, 12, 14, 19, 29].

On the other hand, there is a well developed theory of norms defined on tensor products of Banach spaces. This theory was established by A. Grothendieck [13]. It has had a great impact in the Geometry of Banach spaces, as can be traced in [7, 8, 9, 23, 25, 28]. Indeed, its impact extends even beyond Mathematical Analysis. By way of example, we refer to the survey [17] where applications of Grothendieck’s theorem (usually called Grothendieck’s inequality) to the design of polynomial time algorithms for computing approximate solutions of NP problems are detailed. In the other direction, we refer to [5] where results from theoretical computer science are used to prove that for some indices $p_1, p_2, p_3$, the space $\ell_{p_1} \hat{\otimes} \pi \ell_{p_2} \hat{\otimes} \pi \ell_{p_3}$ fails to have non trivial cotype. The interested reader can consult [1, 24, 27] for further information about tensor products of Banach spaces and its applications.

In the case of finite dimensions, the Minkowski functional enables the use of convex geometry to study finite dimensional Banach spaces (also known as Minkowski spaces) and vice versa. With it, a bijection between norms and 0-symmetric convex bodies in $\mathbb{R}^d$ is established. This result was originally due to H. Minkowski [22], and nowadays is a standard result (see [20] Remark 1.7.7 for a modern statement). Thus, in the context of tensors of finite dimensional spaces, a natural question to ask is if it is possible to determine the convex bodies that are the unit balls of tensor normed spaces, as well as 0-symmetric convex bodies are the unit balls of normed spaces.

2010 Mathematics Subject Classification. 46M05, 52A21, 46N10, 15A69.

Key words and phrases. Convex body, Tensor norm, Minkowski space, Banach-Mazur distance, Tensor product of convex sets, Linear mappings on tensor spaces.
The main result of this paper, Theorem 3.2, provides an affirmative answer to this question.

This work, as well as [3], lies between the theory of tensor norms and convex geometry. In [3], G. Aubrun and S. Szarek establish connections between tensor norms on finite dimensions and convex geometry to estimate the volume of the set of separable mixed quantum states.

We now briefly expose our results. Bringing together the theory of tensor norms and convex geometry, we immediately obtain that the convex bodies \( Q \subset \mathbb{R}^d \) that are the unit ball of a reasonable crossnorm defined on \( \mathbb{R}^d = \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_l}, d = d_1 \cdots d_l, \) are those \( Q \subset \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_l} \) for which there exist norms \( \| \cdot \|_i \) on \( \mathbb{R}^{d_i} \) such that

\[
B_{\otimes^i}(\mathbb{R}^{d_i},\|\cdot\|_i) \subseteq Q \subseteq B_{\otimes^i}(\mathbb{R}^{d_i},\|\cdot\|_i),
\]

where \( B \) denotes the closed unit ball of the projective and the injective tensor norms. In Proposition 3.1, we prove that (1.1) is equivalent to say that

\[
Q_1 \otimes_{\pi} \cdots \otimes_{\pi} Q_l \subseteq Q \subseteq Q_1 \otimes_{\pi} \cdots \otimes_{\pi} Q_l,
\]

where for each \( i, Q_i \subset \mathbb{R}^{d_i} \) is the closed unit ball of \( (\mathbb{R}^{d_i},\|\cdot\|_i) \), and \( \otimes_{\pi}, \otimes_{\pi} \) are the projective and the injective tensor products of 0-symmetric convex bodies, defined by G. Aubrun and S. Szarek in [3].

Our main result (Theorem 3.2) lies much deeper than Proposition 3.1. There, we establish the conditions on \( Q \subset \mathbb{R}^d = \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_l} \) that guarantee the existence of the convex sets \( Q_i \subset \mathbb{R}^{d_i} \) in (1.2) and give an explicit description of them.

Proposition 3.1 and Theorem 3.2 allow us to introduce “tensorial bodies”: a 0-symmetric convex body \( Q \subset \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_l} \) is a tensorial body if there exist 0-symmetric convex bodies \( Q_i \subset \mathbb{R}^{d_i}, i = 1, \ldots, l \) such that (1.2) holds (see Definition 3.2).

Corollary 3.4 shows that the reasonable crossnorms on \( \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_d} \) are the image of the tensorial bodies in \( \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_d} \) under the bijection given by the Minkowski functional. With it, we can go further with the study of this class of convex sets. We prove that the polar set of a tensorial body is a tensorial body and prove the stability of tensorial bodies by multiplying for positive scalars (Proposition 3.5). We also show that the convex bodies \( Q_i \) in (1.2) are essentially unique (see Proposition 3.6).

In Theorem 3.12, we prove that the subgroup of linear isomorphisms on \( \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_d} \) preserving decomposable vectors also preserve tensorial bodies. We denote this group by \( GL_\otimes (\otimes_{i=1}^l \mathbb{R}^{d_i}) \). By means of \( GL_\otimes (\otimes_{i=1}^l \mathbb{R}^{d_i}) \), we start a geometric study of the set of tensorial bodies, defining the following distance:

\[
\delta_{BM}^B (P, Q) := \inf \left\{ \lambda \geq 1 : Q \subseteq T P \subseteq \lambda Q \text{ for } T \in GL_\otimes \left( \otimes_{i=1}^l \mathbb{R}^{d_i} \right) \right\},
\]

where \( P, Q \subset \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_l} \) are tensorial bodies. We call \( \delta_{BM}^B \) the tensorial Banach-Mazur distance. We use it to show that there is a Banach-Mazur type compactum of tensorial bodies in \( \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_l} \) (Theorem 3.13).

Finally, we apply the ideas developed through the paper to prove that the only ellipsoids that are also tensorial bodies in \( \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_l} \) are the unit balls of the Hilbert tensor product of Euclidean spaces (see Theorem 3.2 and Corollary 3.3).
The paper is organized as follows: in Subsection 1.1, we introduce the notation and basic results that we will use throughout the paper. In Section 2, we recall the main properties of the projective and the injective tensor product of 0-symmetric convex bodies. In Section 3, we prove Theorem 3.2 and establish the fundamental properties of tensorial bodies. There, we exhibit examples of tensorial bodies and show that not every 0-symmetric convex body is of this type. In Subsection 3.2, we establish the relation between $GL_\otimes \left( \otimes_{i=1}^l \mathbb{R}^{d_i} \right)$ and the set of tensorial bodies, and settle the fundamental properties of $\delta_{\otimes}^{BM}$. We finish this section by giving upper bounds for $\delta_{\otimes}^{BM}$ (Corollary 3.15). In Section 4, we characterize the ellipsoids in the class of tensorial bodies (Theorem 4.2 and Corollary 4.3).

We like to point out that Theorems 3.2 and 3.12 remain true in $\mathbb{C}^d = \otimes_{i=1}^l \mathbb{C}^{d_i}$, $d = d_1 \cdots d_l$, when circled convex bodies (i.e. a convex body $Q \subset \mathbb{C}^d$ s.t. $e_i Q = Q$) are considered. As a consequence, it is possible to provide the corresponding notion of “tensorial body in $\otimes_{i=1}^l \mathbb{C}^{d_i}$” as well as the definition of the tensorial Banach-Mazur distance. Here, for the sake of transparency we will concentrate in the case of 0-symmetric convex bodies in real spaces.

1.1. Preliminaries. Throughout this paper, $X$, $Y$ or $X_i$ will denote Banach spaces. The closed unit ball of $X$ will be denoted by $B_X$ and its dual space by $X^*$. We write $\mathcal{L}(X,Y)$ to denote the space of bounded linear operators from $X$ to $Y$.

Let $V_i$, $i = 1, \ldots, l$ be vector spaces over the same field $\mathbb{R}$ or $\mathbb{C}$. By $\otimes_{i=1}^l V_i$ we denote its tensor product, and by $\otimes$ we denote the canonical multilinear map:

$$\otimes : V_1 \times \cdots \times V_l \rightarrow \otimes_{i=1}^l V_i$$

$$(x^1, \ldots, x^l) \rightarrow x^1 \otimes \cdots \otimes x^l.$$ 

In the case of Banach spaces, a norm $\alpha (\cdot)$ on the tensor product $\otimes_{i=1}^l X_i$ is called a \textbf{reasonable crossnorm} if

1. $\alpha (x^1 \otimes \cdots \otimes x^l) \leq \|x^1\| \cdots \|x^l\|$ for every $x^i \in X_i$ with $i = 1, \ldots, l$.

2. If $x_i^* \in X_i^*$ for $i = 1, \ldots, l$ then $x_i^1 \otimes \cdots \otimes x_i^l \in (\otimes_{i=1}^l X_i, \alpha)^*$ and $\|x_i^1 \otimes \cdots \otimes x_i^l\| \leq \|x_i^1\| \cdots \|x_i^l\|$.

If $\alpha (\cdot)$ is a reasonable crossnorm on $\otimes_{i=1}^l X_i$, $\otimes_{i=1}^l X_i$ will denote the normed space $(\otimes_{i=1}^l X_i, \alpha)$, and $X_1 \hat{\otimes} \cdots \hat{\otimes} X_l$ its completion.

For each $u \in \otimes_{i=1}^l X_i$, the projective norm $\pi$ and the injective norm $\epsilon$ are defined by:

$$\pi (u) := \inf \left\{ \sum_{i=1}^n \|x_i^1\| \cdots \|x_i^l\| : u = \sum_{i=1}^n x_i^1 \otimes \cdots \otimes x_i^l \right\}$$

and

$$\epsilon (u) := \sup \{ \|x_i^* \otimes \cdots \otimes x_i^* (u)\| : x_i^* \in B_{X_i^*}, \text{ for } i = 1, \ldots, l \}.$$ 

Both the projective and the injective norm are reasonable crossnorms on $\otimes_{i=1}^l X_i$. Indeed, these norms provide the next fundamental characterization of reasonable crossnorms:

A norm $\alpha (\cdot)$ on $\otimes_{i=1}^l X_i$ is a reasonable crossnorm if and only if

$$\epsilon (u) \leq \alpha (u) \leq \pi (u) \text{ for every } u \in \otimes_{i=1}^l X_i.$$
The proof of this equivalence in the case of two normed spaces can be consulted in [25, Proposition 6.3]. For a deeper discussion about tensor norms we also refer to [7].

1.1.1. Convex bodies in Euclidean spaces. Let $\mathbb{E}$ be a real Euclidean space with scalar product $\langle \cdot, \cdot \rangle_\mathbb{E}$ and Euclidean ball $B_\mathbb{E}$. A subset $P \subset \mathbb{E}$ is called a convex body if $P$ is a compact convex set with nonempty interior. Every convex body $P \subset \mathbb{E}$ for which $P = -P$ is called a 0-symmetric (or centrally symmetric) convex body. The set of 0-symmetric convex bodies in $\mathbb{E}$ is denoted by $\mathcal{B}(\mathbb{E})$ (resp. $\mathcal{B}(d)$ if $\mathbb{E} = \mathbb{R}^d$).

If $C$ is a nonempty subset of $\mathbb{E}$, then its polar set is defined by

$$C^\circ := \{ y \in \mathbb{E} : \sup_{x \in C} |\langle x, y \rangle_\mathbb{E}| \leq 1 \}.$$ 

The Minkowski functional (or gauge function) of $P \in \mathcal{B}(\mathbb{E})$ is defined as

$$g_P(x) := \inf \left\{ \lambda > 0 : \frac{1}{\lambda} x \in P \right\} \text{ for } x \in \mathbb{E}.$$ 

A fundamental result concerning 0-symmetric convex bodies is the bijection between norms defined on $\mathbb{E}$ and 0-symmetric convex bodies in $\mathbb{E}$. This result, originally due to H. Minkowski [22], will be used throughout the paper without making an explicit reference. We will use it in the following form:

**Theorem 1.1.** Let $\mathbb{E}$ be a Euclidean space. If $A \in \mathcal{B}(\mathbb{E})$, then

$$\|x\|_A := g_A(x) \text{ for } x \in \mathbb{E}$$

defines a norm $\|\cdot\|_A$ on $\mathbb{E}$ for which $A$ is the closed unit ball. Furthermore, for every $x \in \mathbb{E}$ we have

$$\|x\|_{A^\circ} = \|\langle \cdot, x \rangle : (\mathbb{E}, \|\cdot\|_A) \to \mathbb{R}\|.$$ 

This statement as well as the theory of convex bodies and convex geometry that will be used in this paper, can be found in [26].

2. The projective and injective tensor products of 0-symmetric convex bodies

To introduce the projective and the injective tensor products of 0-symmetric convex bodies, it is convenient to first recall two well known facts about tensor products of Banach spaces. The first one is that

$$B_{X_1 \hat{\otimes} \cdots \hat{\otimes} X_l} = \text{conv} \{ x^1 \otimes \cdots \otimes x^l : x^i \in B_{X_i} \}.$$ 

The second one is the duality between the injective and projective tensor product of Banach spaces given by the canonical isometry:

$$X_1 \otimes_{\varepsilon} \cdots \otimes_{\varepsilon} X_l \mapsto (X_1^* \otimes_{\pi} \cdots \otimes_{\pi} X_l^*)^*,$$

which on finite dimensions is an isometric isomorphism (see [4] pp. 27, 46, respectively).

Let $Q_i \subset \mathbb{R}^{d_i}$, $i = 1, \ldots, l$ be 0-symmetric convex bodies with associated Minkowski functionals $g_{Q_i}$, $i = 1, \ldots, l$. By (2.1), $\text{conv} \{ x^1 \otimes \cdots \otimes x^l : x^i \in Q_i \}$ is the closed unit ball of the projective norm on $\hat{\otimes}_{i=1}^l (\mathbb{R}^{d_i}, g_{Q_i})$. This fact provides a natural way to define the projective tensor product of 0-symmetric convex bodies: the projective
tensor product of $Q_1, \ldots, Q_l$ is the 0-symmetric convex body in $\otimes_{i=1}^l \mathbb{R}^{d_i}$ defined by:

$$Q_1 \otimes_\pi \cdots \otimes_\pi Q_l := \text{conv} \left\{ x^1 \otimes \cdots \otimes x^l \in \otimes_{i=1}^l \mathbb{R}^{d_i} : x^i \in Q_i, i = 1, \ldots, l \right\}.$$  

This definition was introduced by G. Aubrun and S. Szarek in [3]. There, the projective tensor product of more general classes of convex sets is considered.

Since $\text{conv} \left\{ x^1 \otimes \cdots \otimes x^l \in \otimes_{i=1}^l \mathbb{R}^{d_i} : x^i \in Q_i \right\}$ is compact (see Proposition 2.4), it coincides with its closure. Then,

$$Q_1 \otimes_\pi \cdots \otimes_\pi Q_l = B_{\otimes_{i=1}^l (\mathbb{R}^{d_i}, g_{Q_i})}.$$  

The duality between the injective and the projective tensor norms given in (2.2) gives rise to a notion of injective tensor product of 0-symmetric convex bodies. To be precise, we first fix the scalar products that will be used throughout the paper.

Given $d \in \mathbb{N}$, we will denote by $\langle \cdot, \cdot \rangle$ the standard scalar product on $\mathbb{R}^d$, and by $\| \cdot \|_2$, $B_2^d$ its associated norm and Euclidean ball respectively.

The scalar product on $\otimes_{i=1}^l \mathbb{R}^{d_i}$ will be the one associated to the Hilbert tensor product $\otimes_{H, i=1}^l \mathbb{R}^{d_i}$, that is, $\langle \cdot, \cdot \rangle_H$ will be the bilinear form determined by the relation

$$\left\langle x^1 \otimes \cdots \otimes x^l, y^1 \otimes \cdots \otimes y^l \right\rangle_H := \prod_{i=1}^l \langle x^i, y^i \rangle$$

(see [16] Section 2.5]). The closed unit ball of $\otimes_{H, i=1}^l \mathbb{R}^{d_i}$ will be denoted by $B_{2}^{d_1, \ldots, d_l}$, and its norm by $\| \cdot \|_H$. In this way, given a 0-symmetric convex body $Q \subset \otimes_{i=1}^l \mathbb{R}^{d_i}$, its polar set acquires the form $Q^\circ = \{ z \in \otimes_{i=1}^l \mathbb{R}^{d_i} : \sup_{u \in Q} |\langle u, z \rangle_H| \leq 1 \}$.

Now, if $Q_i \subset \mathbb{R}^{d_i}$, $i = 1, \ldots, l$ are 0-symmetric convex bodies, the injective tensor product of $Q_1, \ldots, Q_l$ is the 0-symmetric convex body in $\otimes_{i=1}^l \mathbb{R}^{d_i}$ defined as follows:

$$Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l := (Q_1^\circ \otimes_\pi \cdots \otimes_\pi Q_l^\circ)^\circ.$$  

This definition appeared for the first time in the remarkable monograph [3] Subsection 4.1.4] published in 2017. Later we will use this identity written in the following equivalent ways:

**Proposition 2.1.** Let $Q_i \subset \mathbb{R}^{d_i}$, $i = 1, \ldots, l$ be 0-symmetric convex bodies. Then,

1. $(Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l)^\circ = Q_1^\circ \otimes_\pi \cdots \otimes_\pi Q_l^\circ$.
2. $(Q_1 \otimes_\pi \cdots \otimes_\pi Q_l)^\circ = Q_1^\circ \otimes_\epsilon \cdots \otimes_\epsilon Q_l^\circ$.

Due to the duality between the projective and the injective tensor norms (2.2), along with (2.3), we have that

$$Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l = B_{\otimes_{i=1}^l (\mathbb{R}^{d_i}, g_{Q_i})}.$$  

**2.1. The unit balls of $\ell_1^{d_i}$ and $\ell_\infty^{d_i}$.** Proposition 2.2 below, together with (2.2) and (2.3) show that the convex bodies $B_1^{d_i}$, $B_\infty^{d_i}$, $d = d_1 \cdots d_l$, are the closed unit balls of $\otimes_{\pi, i=1}^l \ell_1^{d_i}$ and $\otimes_{\epsilon, i=1}^l \ell_\infty^{d_i}$, respectively.

In effect, let $B_p^{d_i}$ be the closed unit ball of $\ell_p^{d_i}$, $d \in \mathbb{N}$ and $1 \leq p \leq \infty$. For each $i = 1, \ldots, l$, let $\{ e_{d_i}^j \}_{j=1,\ldots,d_i}$ be the standard basis of $\mathbb{R}^{d_i}$. Then, the set of vectors
\{e_{d_1} \otimes \cdots \otimes e_{d_l}\} is an orthonormal basis in $\otimes_{i=1}^l \mathbb{R}^{d_i}$, and it can be identified with the standard basis of $\mathbb{R}^d$, $d = d_1 \cdots d_l$. Consequently, for each $1 \leq p \leq \infty$, the sets

$$B_{p}^{d_1, \ldots, d_l} := \left\{ z \in \otimes_{i=1}^l \mathbb{R}^{d_i} : \sum_{j_1, \ldots, j_l} \left| \langle z, e_{d_{j_1}} \otimes \cdots \otimes e_{d_{j_l}} \rangle_H \right|^p \leq 1 \right\}$$

for $p \neq \infty$ and

$$B_{\infty}^{d_1, \ldots, d_l} := \left\{ z \in \otimes_{i=1}^l \mathbb{R}^{d_i} : \max_{j_1, \ldots, j_l} \left| \langle z, e_{d_{j_1}} \otimes \cdots \otimes e_{d_{j_l}} \rangle_H \right| \leq 1 \right\}$$

are naturally identified with the closed unit balls of $\ell_p^d$. Thus, $B_p^d = B_{p}^{d_1, \ldots, d_l}$ for $1 \leq p \leq \infty$.

**Proposition 2.2.** Let $d \in \mathbb{N}$. For every factorization of $d$ in natural numbers $d = d_1 \cdots d_l$,

$$B_{1}^{d} = B_{1}^{d_1} \otimes \cdots \otimes B_{1}^{d_l} \quad \text{and} \quad B_{\infty}^{d} = B_{\infty}^{d_1} \otimes \cdots \otimes B_{\infty}^{d_l}.$$

The previous proposition is a well known result, see for instance [25, Exercise 2.6] or [4, pp. 83].

In Subsection 3.1, we will treat the case $1 < p < \infty$. We will see that $B_p^d$ is the closed unit ball associated to a reasonable crossnorm on $\otimes_{i=1}^l \ell^d_{p,i}$. In this case it is not the projective nor the injective tensor norm on $\otimes_{i=1}^l \ell^d_{p,i}$.

We finish this section stating without proof two results that will be used throughout the paper. Proposition 2.2 is a well known result (for a proof see [4, Proposition 4.2]). Proposition 2.3 is a direct consequence of the continuity of the canonical multilinear map $\otimes : \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_l} \to \otimes_{i=1}^l \mathbb{R}^{d_i}$.

**Proposition 2.3.** The set of decomposable vectors $\{x_1 \otimes \cdots \otimes x_l \in \otimes_{i=1}^l \mathbb{R}^{d_i} : x_i \in \mathbb{R}^{d_i}\}$ is a closed subset of $\otimes_{i=1}^l \mathbb{R}^{d_i}$.

**Proposition 2.4.** If $A_i \subseteq \mathbb{R}^{d_i}$, $i = 1, \ldots, l$ are compact sets then $\otimes (A_1, \ldots, A_l) := \{x_1 \otimes \cdots \otimes x_l \in \otimes_{i=1}^l \mathbb{R}^{d_i} : x_i \in A_i\}$ is a compact subset of $\otimes_{i=1}^l \mathbb{R}^{d_i}$.

### 3. Tensorial Bodies

In this section we characterize the convex bodies in $\otimes_{i=1}^l \mathbb{R}^{d_i}$ that are the closed unit balls of reasonable crossnorms. They will be called tensorial bodies (Definition 3.3). A main tool to study them is the group of linear isomorphisms that preserve decomposable vectors. With it, we will introduce a Banach-Mazur type distance between tensorial bodies, and prove that there is a Banach-Mazur type compactum associated to them (see Subsection 3.2).

Recall that we have already fixed the scalar product $\langle \cdot, \cdot \rangle_H$ on $\otimes_{i=1}^l \mathbb{R}^{d_i}$ and that $g_Q$ denotes the Minkowski functional of a 0-symmetric convex body $Q$. With them, we have:

**Proposition 3.1.** Let $Q \subseteq \otimes_{i=1}^l \mathbb{R}^{d_i}$ and let $Q_i \subseteq \mathbb{R}^{d_i}$, $i = 1, \ldots, l$ be 0-symmetric convex bodies. Then, $g_Q(\cdot)$ is a reasonable crossnorm on $\otimes_{i=1}^l \mathbb{R}^{d_i}$ if and only if

$$Q_1 \otimes \cdots \otimes Q_l \subseteq Q \subseteq Q_1 \otimes \cdots \otimes Q_l.$$
In this case, for every decomposable vector \( x^1 \otimes \cdots \otimes x^l \in \otimes_{i=1}^l \mathbb{R}^{d_i} \) we have:

\[
\begin{align*}
g_Q \left( x^1 \otimes \cdots \otimes x^l \right) &= g_{Q_1} \left( x^1 \right) \cdots g_{Q_l} \left( x^l \right), \\
g_{Q_1} \cdots g_{Q_l} \left( x^1 \otimes \cdots \otimes x^l \right) &= g_{Q_1} \left( x^1 \right) \cdots g_{Q_l} \left( x^l \right).
\end{align*}
\]

Proof. Let \( Q_i \subset \mathbb{R}^{d_i}, i = 1, \ldots, l \), be 0-symmetric convex bodies. Then, \([2,3]\) and \((2.3)\) tell us that \( Q_1 \otimes \cdots \otimes Q_l \) is the closed unit ball of \( \otimes_{i=1}^l \left( \mathbb{R}^{d_i}, g_{Q_i} \right) \), and \( Q_1 \otimes \cdots \otimes Q_l \) is the closed unit ball of \( \otimes_{i=1}^l \left( \mathbb{R}^{d_i}, g_{Q_i} \right) \). Now, the proof of the first part follows from the characterization of a reasonable crossnorm \([1,3]\). The second part follows using the two properties that define being a reasonable crossnorm.

This proposition can be understood as the definition of a reasonable crossnorm written in terms of convex bodies. It determines when a 0-symmetric convex body in \( \otimes_{i=1}^l \mathbb{R}^{d_i} \) is the unit ball of a reasonable crossnorm when the norms on each \( \mathbb{R}^{d_i} \) are fixed \((g_{Q_i})\). Our next result goes further: it determines when a 0-symmetric convex body in \( \otimes_{i=1}^l \mathbb{R}^{d_i} \) is the unit ball of a reasonable crossnorm, with respect to some norms (not determined a priori) on the spaces \( \mathbb{R}^{d_i} \).

For every non-zero decomposable vector \( a \in \otimes_{i=1}^l \mathbb{R}^{d_i} \) and every 0-symmetric convex body \( Q \subset \otimes_{i=1}^l \mathbb{R}^{d_i} \). If \( a = a^1 \otimes \cdots \otimes a^l \), consider the 0-symmetric convex bodies in \( \mathbb{R}^{d_i}, i = 1, \ldots, l \), defined as:

\[
Q^1 a^1 \cdots a^l := \left\{ x^i \in \mathbb{R}^{d_i} : a^1 \otimes \cdots \otimes a^{i-1} \otimes x^i \otimes a^{i+1} \otimes \cdots \otimes a^l \in Q \right\}.
\]

Theorem 3.2. Let \( Q \subset \otimes_{i=1}^l \mathbb{R}^{d_i} \) be a 0-symmetric convex body. Then, there exist norms \( \| \cdot \|_i \) on \( \mathbb{R}^{d_i}, i = 1, \ldots, l \), such that \( Q \) is the closed unit ball of a reasonable crossnorm on \( \otimes_{i=1}^l \left( \mathbb{R}^{d_i}, \| \cdot \|_i \right) \) if and only if for an arbitrary decomposable vector \( a^1 \otimes \cdots \otimes a^l \in \partial Q \) it holds:

\[
Q^1 a^1 \cdots a^l \otimes \cdots \otimes Q^l a^1 \cdots a^l \subseteq Q \subseteq Q^1 a^1 \cdots a^l \otimes \cdots \otimes Q^l a^1 \cdots a^l.
\]

In such a situation, \( Q^1 a^1 \cdots a^l = \| a^i \|_i B_{\left( \mathbb{R}^{d_i}, \| \cdot \|_i \right)} \).

Proof. Suppose that \( Q \) is the closed unit ball of a reasonable crossnorm \( \alpha(\cdot) \) on \( \otimes_{i=1}^l \left( \mathbb{R}^{d_i}, \| \cdot \|_i \right) \).

Clearly \( g_Q (\cdot) = \alpha (\cdot) \) and for each \( x^1 \otimes \cdots \otimes x^l \in \otimes_{i=1}^l \mathbb{R}^{d_i} \) we have:

\[
\begin{align*}
g_Q \left( x^1 \otimes \cdots \otimes x^l \right) &= \| x^1 \|_1 \cdots \| x^l \|_l \text{ and } \left\langle \cdot, x^1 \otimes \cdots \otimes x^l \right\rangle_H = \left\| \left\langle \cdot, x^1 \right\rangle \cdots \left\langle \cdot, x^l \right\rangle \right\|,
\end{align*}
\]
where \( \left\langle \cdot, x^i \right\rangle \) is a linear functional on \( \left( \mathbb{R}^{d_i}, \| \cdot \|_i \right) \), \( i = 1, \ldots, l \).

Now, if we fix an arbitrary \( a^1 \otimes \cdots \otimes a^l \in \partial Q \), then \( g_Q \left( a^1 \otimes \cdots \otimes a^l \right) = \| a^1 \|_1 \cdots \| a^l \|_l = 1 \), and

\[
g_Q \left( a^1 \otimes \cdots \otimes a^{i-1} \otimes x^i \otimes a^{i+1} \otimes \cdots \otimes a^l \right) = \| a^1 \|_1 \cdots \| a^{i-1} \|_{i-1} \| x^i \|_i \| a^{i+1} \|_{i+1} \cdots \| a^l \|_l = \frac{1}{\| a^i \|_i} \| x^i \|_i.
\]

Thus, from the definition of \( Q^1 a^1 \cdots a^l \), we obtain \( g_{Q^i a^1 \cdots a^l} (x^i) = \frac{1}{\| a^i \|_i} \| x^i \|_i \) for \( i = 1, \ldots, l \) and \( Q^1 a^1 \cdots a^l = \| a^i \|_i B_{\left( \mathbb{R}^{d_i}, \| \cdot \|_i \right)} \). Since the latter is equivalent to \( g_{Q^i a^1 \cdots a^l} \circ (x^i) = \frac{1}{\| a^i \|_i} \| x^i \|_i \)}
bodies \( Q \) are tensorial bodies. Indeed, if \( \lambda Q \) is a tensorial body with respect to \( \otimes \), then from \((3.4)\) and the previous equalities we have:

\[
gQ \left( x^1 \otimes \cdots \otimes x^l \right) = g_{Q_1^{a_1}, \ldots, a_l} \left( x^1 \right) \cdots g_{Q_l^{a_1}, \ldots, a_l} \left( x^l \right)
\]

and

\[
gQ^o \left( x^1 \otimes \cdots \otimes x^l \right) = \left\| \left\langle \cdot, x^1 \otimes \cdots \otimes x^l \right\rangle \right\| H = \left\| \left\langle \cdot, x^1 \right\rangle \right\| \cdots \left\| \left\langle \cdot, x^l \right\rangle \right\|
\]

\[
= g_{\left( Q_1^{a_1}, \ldots, a_l \right)}^o \left( x^1 \right) \cdots g_{\left( Q_l^{a_1}, \ldots, a_l \right)}^o \left( x^l \right).
\]

Therefore, by Proposition 3.1, \((3.3)\) holds.

To prove the converse, suppose that \( Q \) satisfies \((3.3)\) for \( a_1 \otimes \cdots \otimes a_l \in \partial Q \), then from Proposition 3.1 we conclude that \( gQ \) is a reasonable crossnorm on \( \otimes_{i=1}^l \left( \mathbb{R}^{d_i}, g_{Q_i^{a_1}, \ldots, a_l} \right) \).

This completes the proof.

Now, we introduce the formal notion of a tensorial body:

**Definition 3.3.** A 0-symmetric convex body \( Q \subset \otimes_{i=1}^l \mathbb{R}^{d_i} \) is called a **tensorial body** in \( \otimes_{i=1}^l \mathbb{R}^{d_i} \) if there exist 0-symmetric convex bodies \( Q_i \subset \mathbb{R}^{d_i}, i = 1, \ldots, l \) such that

\[
Q_1 \otimes_{\pi} \cdots \otimes_{\pi} Q_l \subset Q \subset Q_1 \otimes_{\pi} \cdots \otimes_{\pi} Q_l.
\]

If \( Q \) satisfies the inclusions in Definition 3.3 we will say that \( Q \) is a **tensorial body with respect to** \( Q_1, \ldots, Q_l \). The set of tensorial bodies in \( \otimes_{i=1}^l \mathbb{R}^{d_i} \) is denoted by \( \mathcal{B}_{Q_1, \ldots, Q_l} \left( \otimes_{i=1}^l \mathbb{R}^{d_i} \right) \). The set of tensorial bodies with respect to \( Q_1, \ldots, Q_l \) is denoted by \( \mathcal{B}_{Q_1, \ldots, Q_l} \left( \otimes_{i=1}^l \mathbb{R}^{d_i} \right) \).

In the next corollary, we summarize the relation between tensorial bodies in \( \otimes_{i=1}^l \mathbb{R}^{d_i} \) and reasonable crossnorms. We omit its proof, since it follows directly from Proposition 3.1 and Theorem 3.2.

**Corollary 3.4.** Let \( Q \subset \otimes_{i=1}^l \mathbb{R}^{d_i} \) be a 0-symmetric convex body. The following are equivalent:

1. \( Q \) is a tensorial body in \( \otimes_{i=1}^l \mathbb{R}^{d_i} \).
2. \( Q \) satisfies \((3.3)\) for any \( a_1 \otimes \cdots \otimes a_l \in \partial Q \).
3. There exist norms \( \| \cdot \| \) on \( \mathbb{R}^{d_i}, i = 1, \ldots, l \), such that \( gQ \) is a reasonable crossnorm on \( \otimes_{i=1}^l \left( \mathbb{R}^{d_i}, \| \cdot \| \right) \).

In this case, \( g_{Q_i^{a_1}, \ldots, a_l}^o (\cdot) \triangleq \frac{1}{\| a_i \|} \| \cdot \|_i \) for \( i = 1, \ldots, l \).

**Proposition 3.5.** Let \( Q \subset \otimes_{i=1}^l \mathbb{R}^{d_i} \) be a tensorial body. Then \( Q^o = \mathcal{B}_{Q, \ldots, Q} \left( \otimes_{i=1}^l \mathbb{R}^{d_i} \right) \) are tensorial bodies. Indeed, if \( Q \in \mathcal{B}_{Q_1, \ldots, Q_l} \left( \otimes_{i=1}^l \mathbb{R}^{d_i} \right) \) for some 0-symmetric convex bodies \( Q_i \subset \mathbb{R}^{d_i}, i = 1, \ldots, l \), then

1. \( Q^o \in \mathcal{B}_{Q_1^o, \ldots, Q_l^o} \left( \otimes_{i=1}^l \mathbb{R}^{d_i} \right) \).
2. \( \lambda Q \in \mathcal{B}_{Q_1, \ldots, (\lambda Q_k), \ldots, Q_l} \left( \otimes_{i=1}^l \mathbb{R}^{d_i} \right) \).

**Proof.** (1). If \( Q \subset \otimes_{i=1}^l \mathbb{R}^{d_i} \) is a tensorial body with respect to \( Q_1, \ldots, Q_l \) then

\[
Q_1 \otimes_{\pi} \cdots \otimes_{\pi} Q_l \subset Q \subset Q_1 \otimes_{\pi} \cdots \otimes_{\pi} Q_l.
\]

Thus, \( (Q_1 \otimes_{\pi} \cdots \otimes_{\pi} Q_l)^o \subset Q^o \subset (Q_1 \otimes_{\pi} \cdots \otimes_{\pi} Q_l)^o \). By Proposition 2.1 this implies that \( Q_1^o \otimes_{\pi} \cdots \otimes_{\pi} Q_l^o \subset Q^o \subset Q_1^o \otimes_{\pi} \cdots \otimes_{\pi} Q_l^o \) which is equivalent to \( Q^o \in \mathcal{B}_{Q_1, \ldots, Q_l} \left( \otimes_{i=1}^l \mathbb{R}^{d_i} \right) \).
(2) We assume, w.l.o.g., that \( k = 1 \). To prove this part, it is enough to observe that, by definition, for each real number \( \lambda > 0 \), we have \( \lambda(Q_1 \otimes \cdots \otimes Q_l) = (\lambda Q_1) \otimes \cdots \otimes (\lambda Q_l) \) and
\[
\lambda Q_1 \otimes \cdots \otimes \lambda Q_l = \lambda (Q_1^\circ \otimes \cdots \otimes Q_l^\circ) = ((\lambda Q_1) \otimes \cdots \otimes (\lambda Q_l))^\circ.
\]
From this, it follows that \( \lambda Q \in B_{(\lambda Q_1), \ldots, Q_l} \left( \bigotimes_{i=1}^l \mathbb{R}^{d_i} \right) \), if \( Q \in B_{Q_1, \ldots, Q_l} \left( \bigotimes_{i=1}^l \mathbb{R}^{d_i} \right) \).  

A tensorial body in \( \bigotimes_{i=1}^l \mathbb{R}^{d_i} \) is a tensorial body with respect to an essentially unique \( l \)-tuple of convex bodies. More precisely:

**Proposition 3.6.** Let \( P_i, Q_i \subset \mathbb{R}^{d_i}, \ i = 1, \ldots, l \) be 0-symmetric convex bodies. If
\[
Q \in B_{P_1, \ldots, P_l} \left( \bigotimes_{i=1}^l \mathbb{R}^{d_i} \right) \cap B_{Q_1, \ldots, Q_l} \left( \bigotimes_{i=1}^l \mathbb{R}^{d_i} \right),
\]
then there exist real numbers \( \lambda_i > 0, \ i = 1, \ldots, l \) such that \( \lambda_1 \cdots \lambda_l = 1 \) and \( P_i = \lambda_i Q_i \) for \( i = 1, \ldots, l \).

**Proof.** Let \( g_Q, g_{Q_i} \) and \( g_{P_i} \) be the Minkowski functionals associated to \( Q, Q_i \) and \( P_i \) respectively. If \( Q \) is a tensorial body with respect to \( P_i, \ i = 1, \ldots, l \), and with respect to \( Q_i, \ i = 1, \ldots, l \), then Proposition 3.1 implies that:
\[
g_{P_i} (x^1) \cdots g_{P_i} (x^l) = g_Q (x^1 \otimes \cdots \otimes x^l) = g_{Q_1} (x^1) \cdots g_{Q_l} (x^l).
\]
Therefore, if we fix \( a^1 \otimes \cdots \otimes a^l \in \partial Q \), then
\[
g_{Q_1} (a^1) \cdots g_{Q_l} (a^l) = g_{P_1} (a^1) \cdots g_{P_l} (a^l) = 1.
\]
Analogously, for \( a^1 \otimes \cdots \otimes a^{i-1} \otimes x^i \otimes a^{i+1} \otimes \cdots \otimes a^l, \ i = 1, \ldots, l \), we have:
\[
g_{Q_1} (a^1) \cdots g_{Q_{i-1}} (a^{i-1}) g_{Q_i} (x^i) g_{Q_{i+1}} (a^{i+1}) \cdots g_{Q_l} (a^l) = g_{P_1} (a^1) \cdots g_{P_{i-1}} (a^{i-1}) g_{P_i} (x^i) g_{P_{i+1}} (a^{i+1}) \cdots g_{P_l} (a^l).
\]
Now, if we multiply both sides of the above equation by \( g_{Q_i} (a^i) g_{P_i} (a^i) \), we obtain that \( g_{P_i} (a^i) g_Q (x^i) = g_{Q_i} (a^i) g_{P_i} (x^i) \), which is equivalent to \( g_{P_i} (x^i) = \frac{g_{Q_i}(a^i)}{g_{P_i}(a^i)} g_Q (x^i) \). Thus, if \( \lambda_i := \frac{g_{Q_i}(a^i)}{g_{P_i}(a^i)}, \ i = 1, \ldots, l \) then we have proved that \( \lambda_1 \cdots \lambda_l = 1 \) and \( P_i = \lambda_i Q_i \), as required.

In order to simplify the arguments, we will choose the convex bodies defined (3.2) in a specific way: for every 0-symmetric convex body \( Q \subset \bigotimes_{i=1}^l \mathbb{R}^{d_i}, \ Q^i \) will denote the convex bodies generated by \( e_1^{d_1} \otimes \cdots \otimes (\lambda e_1^{d_1}) \), \( \lambda = \frac{1}{\lambda_Q (e_1^{d_1} \otimes \cdots \otimes e_1^{d_1})} \). That is,
\[
Q^i := Q_{e_1^{d_1}, \ldots, \lambda e_1^{d_1}} \text{ for } i = 1, \ldots, l.
\]

**Proposition 3.7.** If \( Q_n, n \in \mathbb{N}, \) are tensorial bodies in \( \bigotimes_{i=1}^l \mathbb{R}^{d_i} \) such that \( g_{Q_n} \) converges uniformly on compact sets to \( g_Q \), for some 0-symmetric convex body \( Q \), then \( Q \) is a tensorial body in \( \bigotimes_{i=1}^l \mathbb{R}^{d_i} \). In this case, \( g_{Q_n} \) and \( g_{(Q_n)^\circ} \) converge uniformly on compact sets to \( g_Q^\circ \) and \( g_{(Q)^\circ} \), respectively.
Proof. Since $Q_n$, $n \in \mathbb{N}$, are tensorial bodies, then (2) of Corollary 3.4 implies that $Q_n^1 \otimes \cdots \otimes Q_n^l \subseteq Q_n \subseteq Q_n^1 \otimes \cdots \otimes Q_n^l$, for $n \in \mathbb{N}$.

Suppose that we already proved the uniform convergence (on compact sets) of $gQ_n$ to $gQ$. From this, it follows that $g(Q_n^i)^o$ converges uniformly on compact sets to $g(Q)^o$. Thus, we get:

$$gQ \left( x^1 \otimes \cdots \otimes x^l \right) = \lim_{n \to \infty} gQ_n \left( x^1 \otimes \cdots \otimes x^l \right) = \lim_{n \to \infty} gQ_n^1 \left( x^1 \right) \cdots gQ_n^l \left( x^l \right) = gQ^1 \left( x^1 \right) \cdots gQ^l \left( x^l \right).$$

Similarly, since the uniform convergence of $gQ_n$ to $gQ$ implies the convergence $gQ_n$ to $gQ^o$, we have $gQ^o \left( x^1 \otimes \cdots \otimes x^l \right) = g(Q_n^1)^o \left( x^1 \right) \cdots g(Q_n^l)^o \left( x^l \right)$. Therefore, from Proposition 3.4, $Q$ is a tensorial body w.r.t. $Q^i$, $i = 1, \ldots, l$.

Now, we turn to prove that for each $i = 1, \ldots, l$, $gQ_n^i$ converges pointwise to $gQ^i$. Then, by [26, Theorem 1.8.12], we know that this implies the uniform convergence.

From the convergence of $gQ_n$ to $gQ$, and the definition of $Q^i, Q_n^i$, it follows directly that $gQ_n^i$ converges pointwise to $gQ^i$. For the case $i = 1, \ldots, l - 1$, it is enough to observe that

$$gQ_n^i \left( x^i \right) = \frac{1}{gQ^i \left( e_1^i \otimes \cdots \otimes e_l^i \right)} gQ_n \left( e_1^i \otimes \cdots \otimes e_{i-1}^i \otimes x^i \otimes e_{i+1}^i \otimes \cdots \otimes e_l^i \right),$$

for $x^i \in \mathbb{R}^d$. Thus, from the definition of $Q^i$ and the convergence of $gQ_n$ to $gQ$, we know that $gQ_n^i$ converges pointwise $gQ^i$. \hfill \square

3.1. Examples of tensorial bodies.

3.1.1. The trivial case. Every 0-symmetric convex body $Q \subset \mathbb{R} \otimes \mathbb{R}^d$ is a tensorial body in $\mathbb{R} \otimes \mathbb{R}^d$.

**Proposition 3.8.** Let $Q \subset \mathbb{R} \otimes \mathbb{R}^d$ be a 0-symmetric convex body. Then $Q = [-1, 1] \otimes \pi Q$ where $[-1, 1] = \{ \lambda \in \mathbb{R} : -1 \leq \lambda \leq 1 \}$ and $\pi Q := \{ x \in \mathbb{R}^d : 1 \otimes x \in Q \}$.

**Proof.** Let $u \in \mathbb{R} \otimes \mathbb{R}^d$, then $u = \sum_{i=1}^N \lambda_i \otimes x_i = 1 \otimes \left( \sum_{i=1}^N \lambda_i x_i \right)$. Thus, $u \in Q$ if and only if $u = 1 \otimes \tilde{u}$ with $\tilde{u} = \sum_{i=1}^N \lambda_i x_i \in \pi Q$. Therefore, from the definition of $[-1, 1] \otimes \pi Q$ we obtain the desired result. \hfill \square

The proposition above and [26, Proposition 3.8] show that every 0-symmetric convex body $Q$ in $\mathbb{R} \otimes \mathbb{R}^d$ is the closed unit ball associated to the projective tensor norm on $\mathbb{R} \otimes \left( \mathbb{R}^d, g_Q \right)$. It is also worth to notice that on $\mathbb{R} \otimes \mathbb{R}^d$, the projective and the injective tensor product of 0-symmetric convex bodies are equal.

3.1.2. The closed unit balls of $\ell_p^d$.

**Proposition 3.9.** Let $d \in \mathbb{N}$. For every factorization of $d$ in natural numbers $d = d_1 \cdots d_l$ and for every $1 \leq p \leq \infty$, $B_p^d = B_p^{d_1, \ldots, d_l}$ is a tensorial body in $\otimes_{i=1}^l \mathbb{R}^{d_i}$. It holds that

$$B_p^{d_1, \ldots, d_l} \subseteq B_p^d = B_p^{d_1, \ldots, d_l} \subseteq B_p^{d_1} \otimes \cdots \otimes B_p^{d_l}, \ 1 \leq p \leq \infty.$$  

In the cases where $1 < p < \infty$, $d_i \geq 2$, $i = 1, \ldots, l$, $B_p^{d_1, \ldots, d_l}$ is not the projective nor the injective tensor product of $B_p^{d_i}$, $i = 1, \ldots, l$. 


Proof. We will use the notation fixed in Example 2.1. The cases $p = 1, \infty$ were already proved in Proposition 2.2. We will give the proof for $1 < p < \infty$.

Let $x^i \in \mathbb{R}^{d_i}$, $i = 1, \ldots, l$ then:

\[
g_{B_{p_1}^{d_1}, \ldots, d_l} (x^1 \otimes \cdots \otimes x^l) = \left( \sum_{j_1, \ldots, j_l} \left| \langle x^1 \otimes \cdots \otimes x^l, e_{i_{j_1}}^{d_1} \otimes \cdots \otimes e_{i_{j_l}}^{d_l} \rangle_{H} \right|^p \right)^{\frac{1}{p}}
\]

Thus, from the last equality for $p^*$ and the relation \( (B_{p_1}^{d_1}, \ldots, d_l)^{\circ} = B_{p^*}^{d_1}, \ldots, d_l \) we have that \( g_{B_{p_1}^{d_1}, \ldots, d_l} \) is a tensorial body w.r.t. \( B_{p^*}^{d_1}, \ldots, d_l \).

To prove the other statement, first observe that if \( d_1 = 1 \), $i = 1, \ldots, l$ then \( B_{p_1}^{d_1}, \ldots, d_l = B_1^{d_1} \otimes \cdots \otimes B_{p_i}^{d_i} = B_1^{d_1} \otimes \cdots \otimes B_{p_i}^{d_i} \). To avoid this case, we assume that each \( d_i \geq 2 \).

Let \( E \subset \otimes_{i=1}^{l} l_{p_i}^{d_i} \) be the vector space generated by \( \{ e_1^{d_1} \otimes \cdots \otimes e_1^{d_l}, e_2^{d_1} \otimes \cdots \otimes e_2^{d_l} \} \). Then, from [2, Theorem 1.3], it follows that \( E \) is isometrically isomorphic to \( l_\infty^2 \) for \( \frac{2}{p} = \min \{ 1, \frac{1}{p} \} \). Hence, for each \( u = a_1 e_1^{d_1} \otimes \cdots \otimes e_1^{d_l} + a_2 e_2^{d_1} \otimes \cdots \otimes e_2^{d_l} \in E \) we have \( \pi (u) = (|a_1|^p + |a_2|^p)^{\frac{1}{p}} \). Therefore, \( B_{p_1}^{d_1}, \ldots, d_l \neq B_{p_1}^{d_1} \otimes \cdots \otimes B_{p_i}^{d_i} \).

Now, suppose that \( B_{p_1}^{d_1}, \ldots, d_l = B_{p_1}^{d_1} \otimes \cdots \otimes B_{p_i}^{d_i} \) for some \( 1 < p < \infty \). Then, from Proposition 3.1, \( B_{p_1}^{d_1}, \ldots, d_l = B_{p_1}^{d_1} \otimes \cdots \otimes B_{p_i}^{d_i} \). Since the latter equality is not possible, we must have \( B_{p_1}^{d_1}, \ldots, d_l \neq B_{p_1}^{d_1} \otimes \cdots \otimes B_{p_i}^{d_i} \) for all \( 1 < p < \infty \).

Proposition 3.9 together with Corollary 3.3 imply that \( B_{p_1}^{d_1}, \ldots, d_l \) is not a tensorial body in \( \mathbb{R}^{m} \otimes \mathbb{R}^{n} \).

Let \( m, n \in \mathbb{N}, m, n \geq 2 \). Let

\[
\mathcal{E} = \left\{ z = \sum_{i,j=1}^{m,n} z_{ij} e_i^m \otimes e_j^n : \left| z_{11} \right|^2 + \left| z_{nn} \right|^2 + \sum_{(i,j) \neq (1,1), (m,n)} \left| z_{ij} \right|^2 \leq 1 \right\}
\]

Then \( \mathcal{E} \subset B(\mathbb{R}^{m} \otimes \mathbb{R}^{n}) \setminus B_{\infty}(\mathbb{R}^{m} \otimes \mathbb{R}^{n}) \). To verify this, consider the convex bodies generated by \( e_1^m \otimes \sqrt{3} e_1^n \) and \( e_m^m \otimes \sqrt{2} e_n^n \), according to the relation (3.2):

\[
\mathcal{E}_{1}^{e_1^m, \sqrt{3} e_1^n} = \left\{ x \in \mathbb{R}^{m} : x \otimes \sqrt{3} e_1^n \in \mathcal{E} \right\}, \quad \mathcal{E}_{1}^{e_m^m, \sqrt{2} e_n^n} = \left\{ x \in \mathbb{R}^{m} : x \otimes \sqrt{2} e_n^n \in \mathcal{E} \right\}.
\]

We will proceed by contradiction. Suppose that \( \mathcal{E} \) is a tensorial body in \( \mathbb{R}^{m} \otimes \mathbb{R}^{n} \), then Corollary 3.3 and Proposition 3.6 imply that there exists \( \lambda_1 > 0 \) such that \( \mathcal{E}_{1}^{e_1^m, \sqrt{3} e_1^n} = \lambda_1 \mathcal{E}_{1}^{e_m^m, \sqrt{2} e_n^n} \). However, for every \( x = (x_1, \ldots, x_m) \in \mathbb{R}^{m} \) we have:

\[
g_{\mathcal{E}_{1}^{e_1^m, \sqrt{3} e_1^n}} (x) = g_{\mathcal{E}} (x \otimes \sqrt{3} e_1^n) = \sqrt{\left| x_1 \right|^2 + 3 \sum_{i=2}^{m} \left| x_i \right|^2},
\]

\[
g_{\mathcal{E}_{1}^{e_m^m, \sqrt{2} e_n^n}} (x) = g_{\mathcal{E}} (x \otimes \sqrt{2} e_n^n) = \sqrt{\left| x_m \right|^2 + 2 \sum_{i=1}^{m-1} \left| x_i \right|^2}.
\]
Thus $E_1^{e_1} \vee \mathbb{R}^n \neq \lambda E_1^{e_1} \vee \mathbb{R}^n$ for all $\lambda > 0$. This is a contradiction, hence $Q$ is not a tensorial body in $\mathbb{R}^m \otimes \mathbb{R}^n$.

Analogous examples $E$ so that $E \in B(\otimes_{i=1}^l \mathbb{R}^{d_i}) \setminus B(\otimes_{i=1}^l \mathbb{R}^{d_i})$, can be constructed in $\otimes_{i=1}^l \mathbb{R}^{d_i}$ when $l \geq 2$.

**Remark 3.10.** The previous example, in contrast with Example 3.1.1 (the trivial case), shows that if $d = mn$ is not a trivial factorization of $d$ (i.e. if $m \neq 1$ or $n \neq 1$) then $B(\mathbb{R}^m \otimes \mathbb{R}^n) \not\subseteq B(\mathbb{R}^m \otimes \mathbb{R}^n)$. As a consequence being a tensorial body depends on the tensor decomposition defined on $\mathbb{R}^d$.

### 3.2. Linear isomorphisms preserving tensorial bodies

A linear map $T : \otimes_{i=1}^l \mathbb{R}^{d_i} \to \otimes_{i=1}^l \mathbb{R}^{d_i}$ preserves decomposable vectors if $T (x^1 \otimes \cdots \otimes x^l)$ is a decomposable vector for every $x^i \in \mathbb{R}^{d_i}$, $i = 1, \ldots, l$. By $GL_\otimes(\otimes_{i=1}^l \mathbb{R}^{d_i})$ we denote the set of linear isomorphisms $T : \otimes_{i=1}^l \mathbb{R}^{d_i} \to \otimes_{i=1}^l \mathbb{R}^{d_i}$ preserving decomposable vectors. To shorten notation we usually write $GL_\otimes$.

Linear mappings preserving decomposable vectors have been deeply studied. For an account on this topic as well as for the fundamentals about it, we refer the reader to [20, 21, 30, 31]. In [20, Corollary 2.14], it is proved that if $d_i \geq 2$, $i = 1, \ldots, l$, then for each element $T \in GL_\otimes(\otimes_{i=1}^l \mathbb{R}^{d_i})$ there exists a permutation $\sigma$ on $\{1, \ldots, l\}$ and linear isomorphisms $T_i : \mathbb{R}^{d_i(\sigma)} \to \mathbb{R}^{d_i}$, $i = 1, \ldots, l$ such that for every $x^1 \otimes \cdots \otimes x^l \in \otimes_{i=1}^l \mathbb{R}^{d_i}$,

$$
T \left( x^1 \otimes \cdots \otimes x^l \right) = T_1 \left( x^{\sigma(1)} \right) \otimes \cdots \otimes T_l \left( x^{\sigma(l)} \right).
$$

Using this characterization and the fact that the set of decomposable vectors is closed in $\otimes_{H, i=1}^l \mathbb{R}^{d_i}$ (see Proposition 3.3), we can easily obtain the next result.

**Proposition 3.11.** Let $d_i \geq 2$, $i = 1, \ldots, l$ be natural numbers. Then $GL_\otimes(\otimes_{i=1}^l \mathbb{R}^{d_i})$ is a closed subgroup of $GL \left( \otimes_{H, i=1}^l \mathbb{R}^{d_i} \right)$.

**Theorem 3.12.** ($GL_\otimes$ preserves tensorial bodies). Assume $d_i \geq 2$ for $i = 1, \ldots, l$. If $T \in GL_\otimes(\otimes_{i=1}^l \mathbb{R}^{d_i})$ and $Q \in B_\otimes(\otimes_{i=1}^l \mathbb{R}^{d_i})$, then $TQ \in B_\otimes(\otimes_{i=1}^l \mathbb{R}^{d_i})$.

**Proof.** Suppose that $Q$ is a tensorial body in $\otimes_{i=1}^l \mathbb{R}^{d_i}$, then (3.1) holds for suitable 0-symmetric convex bodies $Q_i \subset \mathbb{R}^{d_i}$, $i = 1, \ldots, l$.

On the other hand, let $T$ be an element in $GL_\otimes(\otimes_{i=1}^l \mathbb{R}^{d_i})$ and let $T_i$, $i = 1, \ldots, l$, be as in (3.3). Then, by the definition of $\otimes_\pi$, we have:

$$
T \left( Q_1 \otimes_\pi \cdots \otimes_\pi Q_l \right) = T_1 Q_{\sigma(1)} \otimes_\pi \cdots \otimes_\pi T_l Q_{\sigma(l)}.
$$

Similarly,

$$
T \left( Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l \right) = \left( \left( T^\epsilon \right)^{-1} \left( Q_1 \otimes_\pi \cdots \otimes_\pi Q_l \right) \right)^{\otimes_\epsilon} = \left( T_1^{\epsilon} \right)^{-1} Q_{\sigma(1)}^{\otimes_\pi} \otimes_\pi \left( T_l^{\epsilon} \right)^{-1} Q_{\sigma(l)}^{\otimes_\pi} = T_1^{\epsilon} Q_{\sigma(1)} \otimes_\epsilon \cdots \otimes_\epsilon T_l^{\epsilon} Q_{\sigma(l)}.
$$

Therefore, $T_1 Q_{\sigma(1)} \otimes_\pi \cdots \otimes_\pi T_l Q_{\sigma(l)} \subseteq TQ \subseteq T_1 Q_{\sigma(1)} \otimes_\epsilon \cdots \otimes_\epsilon T_l Q_{\sigma(l)}$. This proves that $TQ$ is a tensorial body in $\otimes_{i=1}^l \mathbb{R}^{d_i}$.

$\square$
A Banach-Mazur type distance. From now on, we will assume that each space $\mathbb{R}^{d_i}$, $i = 1, \ldots, l$ has dimension $d_i \geq 2$. Using Theorem 3.12, we are able to define a distance $\delta_{BM}^{\otimes}$ between tensorial bodies in $\otimes_{i=1}^l \mathbb{R}^{d_i}$, which is the analogue, for tensorial bodies, of the Banach-Mazur distance.

Recall that the Banach-Mazur distance between isomorphic Banach spaces $X$ and $Y$ is defined as:

$$\delta^{BM}(X, Y) := \inf \{ \| T \| \| T^{-1} \| : T \in \mathcal{L}(X, Y) \text{ and } T^{-1} \in \mathcal{L}(Y, X) \}.$$ 

Between 0-symmetric convex bodies in a Euclidean space $\mathbb{E}$, it is defined as:

$$\delta^{BM}(P, Q) := \inf \{ \lambda \geq 1 : T : \mathbb{E} \to \mathbb{E} \text{ is a bijective linear map and } Q \subseteq TP \subseteq \lambda Q \}. $$

A complete exposition of the Banach-Mazur distance and its properties can be found in [28].

Let $P, Q$ be tensorial bodies in $\otimes_{i=1}^l \mathbb{R}^{d_i}$. We define the tensorial Banach-Mazur distance $\delta^{BM}_{\otimes}(P, Q)$ as follows:

$$(3.7) \quad \delta^{BM}_{\otimes}(P, Q) := \inf \{ \lambda \geq 1 : Q \subseteq TP \subseteq \lambda Q, \text{ for } T \in GL_{\otimes} \left( \otimes_{i=1}^l \mathbb{R}^{d_i} \right) \}.$$ 

It is well defined, since for every $P, Q \in B_{\otimes} \left( \otimes_{i=1}^l \mathbb{R}^{d_i} \right)$ there exist real numbers $r_1, r_2 > 0$ such that $Q \subseteq r_1 P \subseteq r_2 Q$. It holds that

$$(3.8) \quad \delta^{BM}(P, Q) \leq \delta^{BM}_{\otimes}(P, Q) \text{ for } P, Q \in B_{\otimes} \left( \otimes_{i=1}^l \mathbb{R}^{d_i} \right).$$

Using Proposition 3.11 and Theorem 3.12, it can be directly proved that for each pair $P, Q \subset \otimes_{i=1}^l \mathbb{R}^{d_i}$ of tensorial bodies, the infimum in (3.7) attains its value at some $\lambda > 0$ and some $T \in GL_{\otimes} \left( \otimes_{i=1}^l \mathbb{R}^{d_i} \right)$. Indeed, it is possible to define the following equivalence relation:

**For every $P, Q \in B_{\otimes} \left( \otimes_{i=1}^l \mathbb{R}^{d_i} \right)$, $P \sim Q$ if and only if $\delta^{BM}_{\otimes}(P, Q) = 1$.**

We denote $BM_{\otimes} \left( \otimes_{i=1}^l \mathbb{R}^{d_i} \right)$ the set of equivalence classes of tensorial bodies determined by this relation. Elementary arguments show that $\log\delta^{BM}_{\otimes}$ is a metric on this set. Moreover, this metric gives rise to a Banach-Mazur type compactum of tensorial bodies:

**Theorem 3.13. (The compactum of tensorial bodies)** $BM_{\otimes} \left( \otimes_{i=1}^l \mathbb{R}^{d_i} \right), \log\delta^{BM}_{\otimes}$ is a compact metric space.

**Proof.** The proof is essentially a standard argument of compactness. Given a tensorial body $P \subset \otimes_{i=1}^l \mathbb{R}^{d_i}$, $[P]$ denotes its associate equivalence class.

Let $\{[P_n]\}$ be a sequence in $BM_{\otimes} \left( \otimes_{i=1}^l \mathbb{R}^{d_i} \right)$ and let $P_n^l$ be the convex sets introduced in Proposition 3.7. By Corollary 3.1, $P_n^1 \otimes_\pi \cdots \otimes_\pi P_n^l \subseteq P_n \subseteq P_n^1 \otimes_\epsilon \cdots \otimes_\epsilon P_n^l$. Thus, from [11], Proposition 2.4, it follows that

$$(3.9) \quad P_n^1 \otimes_\pi \cdots \otimes_\pi P_n^l \subseteq P_n \subseteq \frac{d}{dl}P_n^1 \otimes_\pi \cdots \otimes_\pi P_n^l,$$

for every $n \in \mathbb{N}$. On the other hand, from a general well known fact, for every $P_i^l$, $i = 1, \ldots, l$ there exists a linear isomorphism $T_{i,n} : \mathbb{R}^{d_i} \to \mathbb{R}^{d_i}$, such that $B_1^{d_i} \subseteq T_{i,n}P_n^i \subseteq d_iB_1^{d_i}$. Hence, applying $T_{1,n} \otimes \cdots \otimes T_{l,n}$ to (3.9) together with (3.6), we have

$$B_1^{d_1} \otimes_\pi \cdots \otimes_\pi B_1^{d_l} \subseteq (T_{1,n} \otimes \cdots \otimes T_{l,n})P_n \subseteq \frac{d^l}{dl}B_1^{d_1} \otimes_\pi \cdots \otimes_\pi B_1^{d_l}.$$
Now, for each \( n \in \mathbb{N} \) denote by \( Q_n \) the tensorial body \((T_{1,n} \otimes \cdots \otimes T_{l,n}) P_n\). By the Arzela-Ascoli theorem, there is a subsequence \( \{g_{Q_{n_k}}\} \) converging uniformly (on compact sets of \( \otimes_{i=1}^l \mathbb{R}^{d_i} \)) to \( g_Q \) for some 0-symmetric convex body \( Q \). Hence, by Proposition [3.7], \( Q \) is a tensorial body in \( \otimes_{i=1}^l \mathbb{R}^{d_i} \).

What is left is to show that \([P_{n_k}]\) converges to \([Q]\). To prove this, notice that the uniform convergence of \( g_{Q_{n_k}} \) to \( g_Q \) implies that the identity map \( I_k : \left( \otimes_{i=1}^l \mathbb{R}^{d_i}, g_{Q_{n_k}} \right) \rightarrow \left( \otimes_{i=1}^l \mathbb{R}^{d_i}, g_Q \right) \) is such that \( \lim_{k \rightarrow \infty} \| I_k \| \| I_k^{-1} \| = 1 \). Thus, \( \lim_{k \rightarrow \infty} \delta_{BM} (Q_{n_k}, Q) = 1 \). Since \( \delta_{BM} (P_{n_k}, Q) = \delta_{BM} (Q_{n_k}, Q) \), we conclude that \([P_{n_k}]\) converges to \([Q]\) as required.

We finish this section by giving some upper bounds for the tensorial Banach-Mazur distance \( \delta_{BM} \).

**Proposition 3.14.** Let \( P_i, Q_i \subset \mathbb{R}^{d_i}, i = 1, \ldots, l \) be 0-symmetric convex bodies. Then,

1. \( \delta_{BM} (P_1 \otimes \cdots \otimes P_l, Q_1 \otimes \cdots \otimes Q_l) \leq \delta_{BM} (P_1, Q_1) \cdots \delta_{BM} (P_l, Q_l) \).
2. \( \delta_{BM} (P_1 \otimes Q_1 \otimes \cdots \otimes Q_l) \leq \delta_{BM} (P_1, Q_1) \cdots \delta_{BM} (P_l, Q_l) \).

**Proof.** We give the proof only for the projective tensor product of 0-symmetric convex bodies. The proof for \( \otimes \) is analogous. First, we will show that for each \( i \in \{1, \ldots, l\} \) the following inequality holds:

\[
\delta_{BM} (P_1 \otimes \cdots \otimes P_i \otimes \cdots \otimes Q_i \otimes \cdots \otimes P_l, P_1 \otimes \cdots \otimes P_i \otimes \cdots \otimes Q_i \otimes \cdots \otimes P_l) \leq \delta_{BM} (P_1, Q_1) \cdot \delta_{BM} (P_i, Q_i) \cdot \delta_{BM} (P_l, Q_l).
\]

Let \( \lambda \geq \delta_{BM} (P_i, Q_i) \). Then, \( Q_i \subseteq T_i (P_i) \subseteq \lambda Q_i \) for some linear isomorphism \( T_i : \mathbb{R}^{d_i} \rightarrow \mathbb{R}^{d_i} \). By [14] we have \( P_1 \otimes \cdots \otimes T_i P_i \otimes \cdots \otimes P_l = I_{\mathbb{R}^{d_1}} \otimes \cdots \otimes T_i \otimes \cdots \otimes I_{\mathbb{R}^{d_l}} \) \( P_1 \otimes \cdots \otimes P_i \otimes \cdots \otimes P_l \). Therefore, if \( S = I_{\mathbb{R}^{d_1}} \otimes \cdots \otimes T_i \otimes \cdots \otimes I_{\mathbb{R}^{d_l}} \), then

\[
P_1 \otimes \cdots \otimes Q_i \otimes \cdots \otimes P_l \subseteq S (P_1 \otimes \cdots \otimes P_i \otimes \cdots \otimes Q_i \otimes P_i)
\subseteq P_1 \otimes \cdots \otimes Q_i \otimes \cdots \otimes P_l
= \lambda (P_1 \otimes \cdots \otimes Q_i \otimes \cdots \otimes P_l).
\]

From this, it follows (3.10).

To prove (1), observe that from the multiplicative triangle inequality of \( \delta_{BM} \) and

\[
\prod_{i=1}^l \delta_{BM} (Q_{i-1} \otimes \cdots \otimes Q_i, P_{i-1} \otimes \cdots \otimes P_i, Q_i \otimes \cdots \otimes P_i) \leq \delta_{BM} (P_1, Q_1) \cdots \delta_{BM} (P_l, Q_l).
\]

Using the previous proposition and [11] Proposition 2.4], we obtain the following upper bound for the tensorial Banach-Mazur distance.

**Corollary 3.15.** For every pair of tensorial bodies \( P, Q \subset \otimes_{i=1}^l \mathbb{R}^{d_i} \) we have:

1. \( \delta_{BM} (P, Q) \leq (d_1 \cdots d_{l-1})^2 \left( \prod_{i=1}^l \delta_{BM} (P^i, Q^i) \right). \)
2. \( \delta_{BM} (P, Q) \leq (d_1 \cdots d_{l-1})^2 (d_1 \cdots d_l). \)
4. Tensorial Ellipsoids

In this section we give a complete description of the ellipsoids in $\otimes_{i=1}^l \mathbb{R}^{d_i}$ which are also tensorial bodies (Corollary 4.3). To this end, we first introduce some definitions.

Recall that an ellipsoid $E \subset V$ in a vector space of dimension $d$ is defined as the image of the Euclidean ball $B_d^2$ by a linear isomorphism $T : \mathbb{R}^d \rightarrow V$.

In the case of ellipsoids in $\otimes_{i=1}^l \mathbb{R}^{d_i}$, alternatively, we will say that $E \subset \otimes_{i=1}^l \mathbb{R}^{d_i}$ is an ellipsoid if $E = T \left( B_{d_1}^2 \cdot \cdot \cdot B_{d_l}^2 \right)$ for some linear isomorphism $T : \otimes_{i=1}^l \mathbb{R}^{d_i} \rightarrow \otimes_{i=1}^l \mathbb{R}^{d_i}$, providing we have identified $B_{d_i}^2 = B_{d_i}^2 \otimes \cdot \cdot \cdot \otimes B_{d_i}^2$ (see Subsection 2.1).

**Definition 4.1.** An ellipsoid $E \subset \otimes_{i=1}^l \mathbb{R}^{d_i}$ is called a tensorial ellipsoid in $\otimes_{i=1}^l \mathbb{R}^{d_i}$ if $E$ is also a tensorial body in $\otimes_{i=1}^l \mathbb{R}^{d_i}$.

The set of tensorial ellipsoids in $\otimes_{i=1}^l \mathbb{R}^{d_i}$ will be denoted by $\mathcal{E}_\otimes \left( \otimes_{i=1}^l \mathbb{R}^{d_i} \right)$.

If $E_i = T_i \left( B_{d_i}^2 \right)$, $i = 1, \ldots, l$ are ellipsoids in $\mathbb{R}^{d_i}$, $i = 1, \ldots, l$ respectively, then the Hilbertian tensor product of $E_1, \ldots, E_l$, introduced in [3], is defined as

$$E_1 \otimes_2 \cdot \cdot \cdot E_l := T_1 \otimes \cdot \cdot \cdot \otimes T_l \left( B_{d_1}^2 \cdot \cdot \cdot B_{d_l}^2 \right).$$

It can be directly proved that $E_1 \otimes_2 \cdot \cdot \cdot E_l$ is the closed unit ball of the Hilbert tensor product $\otimes_{i=1}^l \left( \mathbb{R}^{d_i}, g_{E_i} \right)$. Thus, Hilbertian tensor products of ellipsoids are the first examples of tensorial ellipsoids. In particular for the Euclidean ball we have:

$$B_{d_1}^2 \cdot \cdot \cdot B_{d_l}^2 = B_{2}^2 \otimes_2 \cdot \cdot \cdot \otimes_2 B_{2}^2 \in \mathcal{E}_\otimes \left( \otimes_{i=1}^l \mathbb{R}^{d_i} \right).$$

Actually, in Theorem 4.2 we prove that $B_{d_1}^2 \cdot \cdot \cdot B_{d_l}^2$ is the only ellipsoid between $B_{2}^2 \otimes_\pi \cdot \cdot \cdot \otimes_\pi B_{2}^2$ and $B_{2}^2 \otimes_\epsilon \cdot \cdot \cdot \otimes_\epsilon B_{2}^2$. From this, we obtain that the only tensorial ellipsoids are the Hilbertian tensor product of ellipsoids (Corollary 4.3).

**Theorem 4.2.** Let $E \subset \otimes_{i=1}^l \mathbb{R}^{d_i}$ be an ellipsoid such that

$$B_{d_1}^2 \otimes_\pi \cdots \otimes_\pi B_{d_l}^2 \subset E \subset B_{2}^2 \otimes_\pi \cdots \otimes_\pi B_{2}^2,$$

then $E = B_{d_1}^2 \cdot \cdot \cdot B_{d_l}^2$.

We will give the proof of the theorem at the end of the section. Before, we will prove Corollary 4.3 and several related results.

**Corollary 4.3.** If $E$ is a tensorial ellipsoid in $\otimes_{i=1}^l \mathbb{R}^{d_i}$, then there exist linear isomorphisms $T_i : \mathbb{R}^{d_i} \rightarrow \mathbb{R}^{d_i}$ for $i = 1, \ldots, l$ such that

$$E = T_1 \otimes \cdot \cdot \cdot \otimes T_l \left( B_{d_1}^2 \cdot \cdot \cdot B_{d_l}^2 \right) = T_1 \left( B_{d_1}^2 \right) \otimes_2 \cdot \cdot \cdot \otimes_2 T_l \left( B_{d_l}^2 \right).$$

**Proof.** Assume that $E$ belongs to $\mathcal{E}_\otimes \left( \otimes_{i=1}^l \mathbb{R}^{d_i} \right)$. Then there exist 0-symmetric convex bodies $A_i \subset \mathbb{R}^{d_i}$, $i = 1, \ldots, l$ such that

$$A_1 \otimes_\pi \cdots \otimes_\pi A_l \subset E \subset A_1 \otimes_\epsilon \cdots \otimes_\epsilon A_l.$$

Since $E$ is an ellipsoid we must have that all $A_i$, $i = 1, \ldots, l$ are ellipsoids. Thus, there exist linear isomorphisms $T_i : \mathbb{R}^{d_i} \rightarrow \mathbb{R}^{d_i}$, $i = 1, \ldots, l$ with $A_i = T_i \left( B_{d_i}^2 \right)$. From
this and Theorem 3.12 we obtain:

\[
T_1 \left( B_2^{d_1} \right) \otimes \ldots \otimes \pi \left( B_2^{d_1} \right) \subset \mathcal{E} \subset T_1 \left( B_2^{d_1} \right) \otimes \ldots \otimes \pi \left( T_1^{-1} \otimes \ldots \otimes T_1^{-1} \right) \mathcal{E} \subset B_2^{d_1} \otimes \ldots \otimes B_2^{d_1}.
\]

Therefore, Theorem 4.2 implies that \((T_1^{-1} \otimes \ldots \otimes T_1^{-1}) \mathcal{E} = B_2^{d_1 \ldots d_l} \). Thus, \(\mathcal{E} = T_1 \left( B_2^{d_1} \right) \otimes \ldots \otimes T_1 \left( B_2^{d_1 \ldots d_l} \right) \) or equivalently \(\mathcal{E} = T_1 \left( B_2^{d_1} \right) \otimes 2 \ldots \otimes 2 T_1 \left( B_2^{d_1} \right) \). \(\square\)

Every ellipsoid \(\mathcal{E} = T \left( B_2^{d_1 \ldots d_l} \right) \subset \otimes \mathbb{R}^{d_l} \) is the closed unit ball associated to the scalar product \(\langle \cdot, \cdot \rangle_E := \langle T^{-1} (\cdot), T^{-1} (\cdot) \rangle_H \). In view of this, the following proposition describes the relation between \(\langle \cdot, \cdot \rangle_E \) and \(\langle \cdot, \cdot \rangle_H \) on decomposable vectors, when \(\mathcal{E} \) is a tensorial ellipsoid in \(\mathbb{R}^m \otimes \mathbb{R}^n \).

**Proposition 4.4.** Let \(m, n\) be natural numbers. If \(\mathcal{E} = T \left( B_2^{m,n} \right) \subset \mathbb{R}^m \otimes \mathbb{R}^n\) is an ellipsoid, then

\[
\langle x, z \rangle \langle y, w \rangle = \frac{\langle L (x \otimes y), L (z \otimes w) \rangle_H + \langle L (x \otimes w), L (z \otimes y) \rangle_H}{2}
\]

for each \(x, z \in \mathbb{R}^m\) and \(y, w \in \mathbb{R}^n\).

**Proof.** Recall that if \(\mathcal{E} = T \left( B_2^{m,n} \right) \subset \mathbb{R}^m \otimes \mathbb{R}^n\) is an ellipsoid then \(\mathcal{E}^0 = (T^t)^{-1} \left( B_2^{m,n} \right) \).

Assume that (4.3) holds for \(L = T^{-1}, T^t\) the following relations hold:

\[
\langle x, z \rangle \langle y, w \rangle = \frac{\langle L (x \otimes y), L (z \otimes w) \rangle_H + \langle L (x \otimes w), L (z \otimes y) \rangle_H}{2}
\]

for each \(x, z \in \mathbb{R}^m\) and \(y, w \in \mathbb{R}^n\).

From Proposition 3.1, we have \(g_E (x, y) = \|x\|_2 \|y\|_2\) and \(g_{E^0} (x, y) = \|x\|_2 \|y\|_2\). Thus, from Proposition 4.1 we get that (4.2) holds.

Now, the polarization formula applied to \(\langle T^{-1} (x \otimes y), T^{-1} (x \otimes w) \rangle_H \) and the latter equality imply:

\[
\langle T^{-1} (x \otimes y), T^{-1} (x \otimes w) \rangle_H = \|x\|_2^2 \langle y, w \rangle.
\]

From the polarization formula and (4.4), we have

\[
\langle x, z \rangle \langle y, w \rangle = \frac{\|x + z\|^2_2 - \|x - z\|^2_2}{4} \langle y, w \rangle.
\]

Thus, using (4.4) in the last equality, we get that (4.3) holds for \(L = T^{-1}\). To finish the proof, observe that \(\mathcal{E}^0\) also satisfies (4.2), see Proposition 3.1. Hence, (4.3) holds for \(T^t\).

**Lemma 4.5.** Let \(\mathcal{E}\) be a tensorial ellipsoid in \(\otimes \mathbb{R}^{d_l}\). For every \(z^l \in \partial B_2^{d_l}\), let

\[
i_{z^l} : \mathbb{R}^{d_1} \otimes \ldots \otimes \mathbb{R}^{d_l-1} \to \mathbb{R}^{d_1} \otimes \ldots \otimes \mathbb{R}^{d_l-1} \otimes \mathbb{R}^{d_l}
\]

\[
x^1 \otimes \ldots \otimes x^{l-1} \to x^1 \otimes \ldots \otimes x^{l-1} \otimes z^l,
\]

and \(\mathcal{E}_{z^l} := i_{z^l}^{-1} (\mathcal{E})\). Then, if

\[
B_2^{d_1} \otimes \ldots \otimes B_2^{d_l} \subset \mathcal{E} \subset B_2^{d_1} \otimes \ldots \otimes B_2^{d_l}
\]

for each
one has

\[ B_{2}^{d_{1}} \otimes \cdots \otimes B_{2}^{d_{l-1}} \subset \mathcal{E}_{\varepsilon} \subset B_{2}^{d_{1}} \otimes \cdots \otimes B_{2}^{d_{l-1}}. \]

**Proof.** Let \( \langle \cdot, \cdot \rangle_{\mathcal{E}} \) be the scalar product associated to \( \mathcal{E} \). From the definition of \( \mathcal{E}_{\varepsilon} \), we know that it is an ellipsoid. By \( \langle \cdot, \cdot \rangle_{\varepsilon} \), \( g_{\mathcal{E}_{\varepsilon}}(\cdot) \) we denote the scalar product and the Minkowski functional determined by \( \mathcal{E}_{\varepsilon} \). Thus, for every \( u \in \mathbb{R}^{d_{1}} \otimes \cdots \otimes \mathbb{R}^{d_{l-1}} \), we have \( g_{\varepsilon}(u) = g_{\varepsilon}(i_{\varepsilon}(u)) \). Since \( \mathcal{E} \) is a tensorial ellipsoid, from Proposition 3.1

\[ g_{\varepsilon}(x^{1} \otimes \cdots \otimes x^{l-1}) = \|x^{1}\|_{2} \cdots \|x^{l-1}\|_{2}. \]

We also have:

\[ g_{\varepsilon_{\varepsilon}}(x^{1} \otimes \cdots \otimes x^{l-1}) = \sup_{g_{\varepsilon_{\varepsilon}}(a) \leq 1} \left\langle a, x^{1} \otimes \cdots \otimes x^{l-1} \right\rangle_{H} \]

\[ = \sup_{g_{\varepsilon}(i_{\varepsilon}(a)) \leq 1} \left\langle i_{\varepsilon}(a), x^{1} \otimes \cdots \otimes x^{l-1} \otimes z \right\rangle_{H} \]

\[ \leq g_{\varepsilon}(x^{1} \otimes \cdots \otimes x^{l-1} \otimes z) \]

\[ = \|x^{1}\|_{2} \cdots \|x^{l-1}\|_{2}. \]

Therefore, from Proposition 3.1 we know \( \mathcal{E}_{\varepsilon} \) is a tensorial body w.r.t. \( B_{2}^{d_{i}}, i = 1, \ldots, l. \)

**Proof.** (of Theorem 4.2) The proof will be divided into two parts. First, we will prove the theorem for tensorial ellipsoids in \( \mathbb{R}^{m} \otimes \mathbb{R}^{n} \). Then, for the general case we will use induction on \( l \), the number of factors on the tensor product \( \otimes_{i=1}^{l} \mathbb{R}^{d_{i}} \).

**Step 1.** Suppose \( \mathcal{E} \subset \mathbb{R}^{m} \otimes \mathbb{R}^{n} \) is an ellipsoid such that \( B_{2}^{m} \otimes B_{2}^{n} \subset \mathcal{E} \subset B_{2}^{m} \otimes \varepsilon B_{2}^{n} \).

If \( \mathcal{E} = T(\mathbb{R}^{m} \otimes \mathbb{R}^{n}) \) for some linear isomorphism on \( \mathbb{R}^{m} \otimes \mathbb{R}^{n} \), then from Proposition 4.3 \( \|x\|_{\mathcal{E}_{\varepsilon}} \|y\|_{\mathcal{E}_{\varepsilon}} \) holds for \( T^{1}, T^{l} \). Thus, for \( x, z \in \mathbb{R}^{m} \) and \( y, w \in \mathbb{R}^{n} \) and \( S = TT^{t} \) we have:

\[ \langle x, z \rangle \langle y, w \rangle = \frac{\langle S^{-1}(x \otimes y), z \otimes w \rangle_{H} + \langle S^{-1}(x \otimes w), z \otimes y \rangle_{H}}{2} \]

\[ \langle x, z \rangle \langle y, w \rangle = \frac{\langle S(x \otimes y), z \otimes w \rangle_{H} + \langle S(x \otimes w), z \otimes y \rangle_{H}}{2}. \]

On the other hand, for the canonical basis \( \{e_{i}\}_{i=1}^{d} \subset \mathbb{R}^{d} \), \( \{e_{i}\}_{i=1}^{m} \subset \mathbb{R}^{m} \) and \( \{e_{j}\}_{j=1}^{n} \subset \mathbb{R}^{n} \) let \( \Phi_{(m,n)} : \mathbb{R}^{m} \otimes \mathbb{R}^{n} \rightarrow \mathbb{R}^{d} \) be the bijective map such that \( \Phi_{(m,n)}(e_{i} \otimes e_{j}) = e_{(i-1)n+j} \) for \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \). Clearly, \( \Phi_{(m,n)} \) preserves \( \langle \cdot, \cdot \rangle_{H} \) and the standard scalar product \( \langle \cdot, \cdot \rangle \) on \( \mathbb{R}^{d} \). Hence if \( \tilde{S} := \Phi_{(m,n)} S \Phi_{(m,n)}^{-1} \) we have:

\[ \langle e_{(i-1)n+j}, e_{(k-1)n+l} \rangle = \frac{\langle \tilde{S}^{-1} e_{(i-1)n+j}, e_{(k-1)n+l} \rangle + \langle \tilde{S}^{-1} e_{(i-1)n+l}, e_{(k-1)n+j} \rangle}{2}, \]

\[ \langle e_{(i-1)n+j}, e_{(k-1)n+l} \rangle = \frac{\langle \tilde{S} e_{(i-1)n+j}, e_{(k-1)n+l} \rangle + \langle \tilde{S} e_{(i-1)n+l}, e_{(k-1)n+j} \rangle}{2}. \]
for $1 \leq i, k \leq m$ and $1 \leq j, l \leq n$. Therefore, for $W = \tilde{S}, \tilde{S}^{-1}$:

$$\langle We_{(i-1)n+j}, e_{(k-1)n+l} \rangle = \begin{cases} 1 & \text{if } k = i, l = j \\ 0 & \text{if } k = i, l \neq j \\ 0 & \text{if } k \neq i, l = j \\ -\langle We_{(i-1)n+l}, e_{(k-1)n+j} \rangle & \text{if } k \neq i, l \neq j \end{cases}$$

(4.5)

Hence, the positive definite matrices associated to $\tilde{S}, \tilde{S}^{-1}$ can be written using the matrices: $A_{ki} := \langle \tilde{S}e_{(i-1)n+j}, e_{(k-1)n+l} \rangle_{i,j}$ and $B_{ki} := \langle \tilde{S}^{-1}e_{(i-1)n+j}, e_{(k-1)n+l} \rangle_{i,j}$.

That is, $\tilde{S} = (A_{ki})_{k,i}$ and $\tilde{S}^{-1} = (B_{ki})_{k,i}$ for $1 \leq k, i \leq m$. Clearly, $A_{ki}, B_{ki} \in M_{n,n}(\mathbb{R})$ for all $1 \leq k, i \leq m$. Moreover, from (4.3), it follows that $A_{ki}$ and $B_{ki}, k \neq i$, are antisymmetric matrices and $A_{kk} = B_{kk} = I_n$, $k = i$ ($I_n$ is the identity matrix). From this and the symmetry of $\tilde{S}, \tilde{S}^{-1}$, we know that $A_{ik} = -A_{ki}$. Thus, $\tilde{S}, \tilde{S}^{-1}$ satisfy (4.4) (see Lemma 4.5 at the end of this section) and $\tilde{S} = I_d$. The latter implies that the linear isomorphism $T$ is such that $TT^t = I_d$, so it is an orthogonal map on $\otimes_{t=1}^l \mathbb{R}_{d_i}$ and $E = B_{m,n}^d$. This finishes the first part of the proof.

Step 2. As we mentioned at the beginning of the proof, this case will be proved by induction on the number $l$ of factors on the tensor product. To simplify the notation, in this part of the proof we use the symbol $||\cdot||_Q$ to denote the Minkowski functional associated to a 0-symmetric convex body $Q$.

The case $l = 2$ was already proved. Now we assume that the result holds for $l - 1$. This means that for every tensorial ellipsoid $E \subset \otimes_{i=1}^{l-1} \mathbb{R}_{d_i}$ such that $B_{21}^{d_1} \otimes \cdots \otimes B_{21}^{d_{l-1}} \subset E \subset B_{2}^{d_1} \otimes \cdots \otimes B_{2}^{d_{l-1}}$, we have $E = B_{1}^{d_1,\ldots,d_{l-1}}$

Let $E$ be a tensorial ellipsoid in $\otimes_{i=1}^l \mathbb{R}_{d_i}$ satisfying (4.1), and let $||\cdot||_E$ be its Minkowski functional. By Lemma 4.3 for every $z \in \partial B_{2}^{d_1}$ we have $B_{2}^{d_1} \otimes \cdots \otimes B_{2}^{d_{l-1}} \subset E_z \subset B_{2}^{d_1} \otimes \cdots \otimes B_{2}^{d_{l-1}}$. Applying the induction hypothesis we obtain $E_z = B_{2}^{d_1,\ldots,d_{l-1}}$. Therefore, for every $\sum_{i=1}^N x_i^{1} \otimes \cdots \otimes x_i^{l-1} \in \otimes_{i=1}^{l-1} \mathbb{R}_{d_i}$ we have

$$\left\| \sum_{i=1}^N x_i^{1} \otimes \cdots \otimes x_i^{l-1} \otimes \varepsilon \right\|_E = \left\| \sum_{i=1}^N x_i^{1} \otimes \cdots \otimes x_i^{l-1} \right\|_{E_z} = \left\| \sum_{i=1}^N x_i^{1} \otimes \cdots \otimes x_i^{l-1} \right\|_H.$$ 

Since, by Proposition 4.3, $E_0$ also satisfies (4.1), we also have

$$\left\| \sum_{i=1}^N x_i^{1} \otimes \cdots \otimes x_i^{l-1} \otimes \varepsilon \right\|_{E_0} = \left\| \sum_{i=1}^N x_i^{1} \otimes \cdots \otimes x_i^{l-1} \right\|_H.$$ 

Now, consider the canonical isomorphism

$$\psi : (\mathbb{R}_{d_1} \otimes \cdots \otimes \mathbb{R}_{d_{1-1}}) \otimes \mathbb{R}_{d_i} \rightarrow \mathbb{R}_{d_1} \otimes \cdots \otimes \mathbb{R}_{d_{1-1}} \otimes \mathbb{R}_{d_i}$$

$$x_1^{1} \otimes \cdots \otimes x_i^{l-1} \otimes \varepsilon \rightarrow x_1^{i} \otimes \cdots \otimes x_i^{l-1} \otimes \varepsilon,$$

and denote by $\tilde{E}$ the ellipsoid in $(\mathbb{R}_{d_1} \otimes \cdots \otimes \mathbb{R}_{d_{1-1}}) \otimes \mathbb{R}_{d_i}$ determined by this isomorphism and $E$. Then, for each non-zero $x_i \in \mathbb{R}_{d_i}$, and $u = \sum_{i=1}^N x_i^{1} \otimes \cdots \otimes x_i^{l-1} \in \cdots \otimes \mathbb{R}_{d_{1-1}} \otimes \mathbb{R}_{d_i}$.
Thus, by Proposition 3.1, \( A \) is a tensorial ellipsoid in \( (\mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_{l-1}}) \otimes \mathbb{R}^{d_l} \), and \( B_2^{d_1,\ldots,d_{l-1}} \otimes \tau B_2^{d_l} \subset \tilde{E} \subset B_2^{d_1,\ldots,d_{l-1}} \otimes \tau B_2^{d_l} \).

Now, let \( d = d_1 \cdots d_{l-1} \). Then, by the identification \( B_2^{d_1,\ldots,d_{l-1}} = B_2^d \) (see Subsection 2.1) and the case of two factors proved in Step 1., we know that \( \tilde{E} = B_2^{d_{k+1}} \). Finally, since \( E = \psi \left( \tilde{E} \right) \) and \( \psi \) is an orthogonal map, we have that \( E = B_2^{d_1,\ldots,d_l} \) which is the desired result. \( \square \)

The next lemma is used in the proof of Theorem 4.2. For a given \( n \in \mathbb{N} \), \( I_n \in M_{n \times n} (\mathbb{R}) \) will denote the identity matrix of dimension \( n \).

**Lemma 4.6.** Let \( m, n \in \mathbb{N} \) and \( d = mn \). If \( S \in M_{d \times d} (\mathbb{R}) \) is a positive definite matrix and there exist antisymmetric matrices \( A_{ki}, B_{ki} \in M_{n,n} (\mathbb{R}) \), \( k = 1, \ldots, m-1, i = k+1, \ldots, m \) such that

\[
(4.6) \quad S = \begin{bmatrix}
I_n & A_{12} & \cdots & A_{1,m}
\end{bmatrix}, \quad S^{-1} = \begin{bmatrix}
I_n & B_{12} & \cdots & B_{1,m}
\end{bmatrix} - \begin{bmatrix}
B_{12} & I_n & \cdots & B_{2,m}
\end{bmatrix} - \begin{bmatrix}
B_{1,m} & B_{2,m} & \cdots & I_n
\end{bmatrix}
\]

Then \( S = I_d \).

**Proof.** Let \( n \geq 1 \) be fixed. We will prove the result by induction on \( m \).

Step 1. If \( m = 1 \), then \( d = n \). By definition of \( S \), \( S = I_d \) and the result is proved.

Assume now that \( m = 2 \). In this case,

\[
S = \begin{bmatrix}
I_n & A_{12}
\end{bmatrix} \quad \text{and} \quad S^{-1} = \begin{bmatrix}
I_n & B_{12}
\end{bmatrix} - \begin{bmatrix}
B_{12} & I_n
\end{bmatrix}
\]

Then

\[
SS^{-1} = \begin{bmatrix}
I_n - A_{12}B_{12} & A_{12} + B_{12}
\end{bmatrix} - \begin{bmatrix}
I_n - A_{12}B_{12} & I_n - A_{12}B_{12}
\end{bmatrix} = \begin{bmatrix}
I_n & 0
\end{bmatrix}.
\]

Thus, \( B_{12} = -A_{12} \) and \( A_{12}^2 = 0 \). Since \( A_{12} \) is antisymmetric, the latter equality implies that \( A_{12}^2A_{12} = 0 \) so \( A_{12} = 0 \) which completes the proof.
Step 2. Assume the result is valid for \( m-1 \). By \( E, F, G, H \) we denote the following matrices:

\[
E := \begin{bmatrix}
I_n & A_{12} & \ldots & A_{1,m-1} \\
-A_{12} & I_n & \ldots & A_{2,m-1} \\
\vdots & \vdots & \ddots & \vdots \\
-A_{1,m-1} & -A_{2,m-1} & \ldots & I_n
\end{bmatrix}_{n(m-1),n(m-1)}, \quad F := \begin{bmatrix}
A_{1,m} \\
A_{2,m} \\
\vdots \\
A_{m-1,m}
\end{bmatrix}_{n(m-1),n(n-1)},
\]

\[
G := \begin{bmatrix}
I_n & B_{12} & \ldots & B_{1,m-1} \\
-B_{12} & I_n & \ldots & B_{2,m-1} \\
\vdots & \vdots & \ddots & \vdots \\
-B_{1,m-1} & -B_{2,m-1} & \ldots & I_n
\end{bmatrix}_{n(m-1),n(m-1)}, \quad H := \begin{bmatrix}
B_{1,m} \\
B_{2,m} \\
\vdots \\
B_{m-1,m}
\end{bmatrix}_{n(m-1),n(n-1)}.
\]

Clearly, since \( S \) is a positive definite matrix then \( S^{-1}, E, G \) are positive definite matrices. Also,

\[
S = \begin{bmatrix}
E & F \\
F^t & I_n
\end{bmatrix}, \quad S^{-1} = \begin{bmatrix}
G & H \\
H^t & I_n
\end{bmatrix}
\]

and

\[
SS^{-1} = \begin{bmatrix}
EG + FH^t & EH + FI_n \\
F^tG + I_nH^t & F^tH + I_n
\end{bmatrix} = \begin{bmatrix}
I_{n(m-1)} & 0_{n(n-1)} \\
0_{n(n-1)} & I_{n^{-1}}
\end{bmatrix}.
\]

Therefore, \( F^tH + I_n = 0_{n,n} \) and \( F^tH = 0_{n,n} \). Since we also have \( F^tG + I_nH^t = 0_{n,n(m-1)} \) then \( H^t = -F^tG \). This yields to

\[
\begin{equation}
H = -GF.
\end{equation}
\]

From the previous equations we get \( 0 = F^tH = -F^tGF \) and

\[
\begin{equation}
F^tGF = 0_{n,n}.
\end{equation}
\]

Now, if we write \( F_i, i = 1, 2, \ldots, n \) for the columns of \( F \), then from (4.8) we have

\[
\begin{equation}
F_i^tGF_i = 0.
\end{equation}
\]

Since \( G \) are positive definite matrix, from (4.9) we know that each \( F_i = 0, i = 1, 2, \ldots, n \) and \( F = 0 \). This and (4.7) imply \( H = 0 \).

Finally, we are in position to apply our inductive hypothesis to \( E \) and \( E^{-1} = G \). Then, \( E = I_{n(m-1)} \) which implies \( S = I_d \).

\[\square\]

\section*{References}

[1] A. Acín, N. Gisin, and B. Toner. Grothendieck's constant and local models for noisy entangled quantum states. \textit{Phys. Rev. A}, 73:062105, 2006.

[2] A. Arias and J. Farmer. On the structure of tensor products of \( l_p \)-spaces. \textit{Pacific Journal of Mathematics}, 175(1):13–37, 1996.

[3] G. Aubrun and S. Szarek. Tensor products of convex sets and the volume of separable states on n qudits. \textit{Physical Review A}, 73(2):022109, 2006.

[4] G. Aubrun and S. Szarek. Alice and Bob Meet Banach: The Interface of Asymptotic Geometric Analysis and Quantum Information Theory, volume 223. American Mathematical Soc., 2017.

[5] J. Briët, A. Naor, and O. Regev. Locally decodable codes and the failure of cotype for projective tensor products. \textit{Electron. Res. Announc. Math. Sci.}, 19:120–130, 2012.

[6] V. de Silva and L-H. Lim. Tensor rank and the ill-posedness of the best low-rank approximation problem. \textit{SIAM Journal on Matrix Analysis and Applications}, 30(3):1084–1127, 2008.

[7] K. Defant and K. Floret. \textit{Tensor Norms and Operator Ideals}, volume 176. North Holland Mathematics Studies, 1st Edition edition, 1992.
[8] J. Diestel, A. Grothendieck, J. Fourie, and J. Swart. *The metric theory of tensor products: Grothendieck’s résumé revisited*. American Mathematical Soc., 2008.

[9] J. Diestel, H. Jarchow, and A. Tonge. *Absolutely summing operators*, volume 43 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1995.

[10] K. Efremenko. 3-query locally decodable codes of subexponential length. In *STOC’09—Proceedings of the 2009 ACM International Symposium on Theory of Computing*, pages 39–44. 2009.

[11] M. Fernández-Unzueta. The Segre cone of Banach spaces and multilinear mappings. arXiv:1804.10641.

[12] O. Giladi, J. Prochno, C. Schütt, N. Tomczak Jaegermann, and E. Werner. On the Geometry of Projective Tensor Products. *J. Funct. Anal.*, 273(2):471–495, 2017.

[13] A. Grothendieck. *Résumé de la théorie métrique des produits tensoriels topologiques*. Soc. de Matemática de São Paulo, 1956.

[14] W. Hackbusch. Numerical tensor calculus. *Acta numerica*, 23:651–742, 2014.

[15] K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein. Volume of the set of separable states. *Phys. Rev. A*, 58:883–892, 1998.

[16] R. Kadison and J. Ringrose. *Fundamentals of the theory of operator algebras*. Academic Press, 1983.

[17] S. Khot and A. Naor. Grothendieck-type inequalities in combinatorial optimization. *Comm. Pure Appl. Math.*, 65(7):992–1035, 2012.

[18] T. Kolda and B. Bader. Tensor decompositions and applications. *SIAM review*, 51(3):455–500, 2009.

[19] J. Landsberg. *Tensors: geometry and applications*. Graduate Studies in Mathematics, 128, 2011.

[20] M-H. Lim. Additive preservers of non-zero decomposable tensors. *Linear Algebra and Its Applications*, 428:239–253, 2008.

[21] M. Marcus and B. Moyls. Transformation on tensor product spaces. *Pacific Journal of Mathematics*, 9(4):1215–1221, 1959.

[22] H. Minkowski. *Geometrie der Zahlen*. 1910. Teubner, Leipzig, 1927.

[23] G. Pisier. Grothendieck’s theorem, past and present. *Bull. Amer. Math. Soc. (N.S.)*, 49(2):237–323, 2012.

[24] I. Pitowsky. New Bell inequalities for the singlet state: going beyond the Grothendieck bound. *J. Math. Phys.*, 49(1):012101, 11, 2008.

[25] R. Ryan. *Introduction to tensor products of Banach spaces*. Springer monographs in mathematics, 2002.

[26] R. Schneider. Convex bodies: the Brunn-Minkowski theory, volume 44 of Encyclopedia of Mathematics and Applications, 1993.

[27] S. Szarek, E. Werner, and K. Życzkowski. How often is a random quantum state k-entangled? *Journal of Physics A: Mathematical and Theoretical*, 44(4):045303, 15, 2011.

[28] N. Tomczak-Jaegermann. *Banach-Mazur distances and finite-dimensional operator ideals*, volume 38. Longman Sc & Tech, 1989.

[29] N. Vannieuwenhoven, J. Nicaise, R. Vandervel, and K. Meerbergen. On generic nonexistence of the schmidt–eckart–young decomposition for complex tensors. *SIAM Journal on Matrix Analysis and Applications*, 35(3):886–903, 2014.

[30] R. Westwick. Transformation on tensor spaces. *Pacific Journal of Mathematics*, 23(3), 1967.

[31] R. Westwick. Transformation on tensor spaces II. *Linear and Multilinear Algebra*, 40(1):81–92, 1995.

Centro de Investigación en Matemáticas (CIMAT), A.P. 402 Guanajuato, Gto., México

E-mail address: maite@cimat.mx, fher@cimat.mx