The transition matroid of a 4-regular graph: an introduction

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Abstract

Given a 4-regular graph $F$, we introduce a binary matroid $M_\tau(F)$ on the set of transitions of $F$. Parametrized versions of the Tutte polynomial of $M_\tau(F)$ yield several well-known graph and knot polynomials, including the Martin polynomial, the homflypt polynomial, the Kauffman polynomial and the Bollobás-Riordan polynomial.

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1 Introduction

A graph is determined by two finite sets, one set containing vertices and the other containing edges. Each edge is incident on one or two vertices; an edge incident on only one vertex is a loop. We think of an edge as consisting of two distinct half-edges, each of which is incident on precisely one vertex. In this paper we are especially interested in 4-regular graphs, i.e., graphs in which each vertex has precisely four incident half-edges. The special theory of 4-regular graphs was initiated by Kotzig and although his definitions and results have been generalized and modified over the years, most of the basic ideas of the theory appear in his seminal paper [50].

Matroids were introduced by Whitney [81], and there are several standard texts about them [36, 61, 77, 78, 79, 80]. In this paper we will only encounter binary matroids. If $M$ is a $GF(2)$-matrix with columns indexed by the elements of a set $S$, then the binary matroid represented by $M$ is given by defining the rank of each subset $A \subseteq S$ to be equal to the dimension of the $GF(2)$-space spanned by the corresponding columns of $M$. Matroids can be defined in many other ways. In particular, the minimal nonempty subsets of $S$ that correspond to linearly dependent sets of columns of $M$ are the circuits of the
Figure 1: A 4-regular graph with one vertex has four distinct circuits.

matroid represented by \( M \). We will not refer to matroid circuits often, to avoid confusion with the following definition.

A circuit in a graph is a sequence \( v_1, h_1, h'_1, v_2, h_2, \ldots, h_k, h'_k = h'_0, v_{k+1} = v_1 \) such that for each \( i \in \{1, \ldots, k\} \), \( h_i \) and \( h'_i \) are half-edges of a single edge and \( h'_{i-1} \) and \( h_i \) are both incident on \( v_i \). The half-edges that appear in a circuit must be pairwise distinct, but vertices may be repeated. Two circuits are considered to be the same if they differ only by a combination of cyclic permutations \( (1, \ldots, k) \mapsto (i, \ldots, k, 1, \ldots, i-1) \) and reversals \( (1, \ldots, k) \mapsto (k, \ldots, 1) \).

Notice that these definitions seem to be essentially non-matroidal: circuits may be nested, and distinct circuits may involve precisely the same vertices and half-edges, in different orders. For instance if a graph has one vertex \( v \) and two edges \( e_1 = \{ f, f' \} \) and \( e_2 = \{ h, h' \} \) (both loops), then it has four different circuits: \( v, f, f', v; f, f', v, h, h', v, v; f, f', v, h, h', v \) and \( v, h, h', v \). See Fig. 1 where these circuits are indicated from left to right, using the convention that when a circuit traverses a vertex, the dash style (dashed or undashed) is maintained.

A circuit \( v_1, h_1, h'_1, v_2, \ldots, h_k, h'_k = h'_0, v_{k+1} = v_1 \) in a 4-regular graph is specified by the triples \( h'_{i-1}, v_i, h_i \) where \( h'_{i-1} \) and \( h_i \) are distinct half-edges incident on \( v_i \). We call such a triple a single transition. Kotzig called these triples “transitions” [50], but we adopt the convention used by other authors (including Ellis-Monaghan and Sarmiento [31], Jaeger [40] and Las Vergnas [52, 53]) that a transition consists of two disjoint single transitions at the same vertex.

A circuit partition (or Eulerian partition or \( \xi \)-decomposition) of a 4-regular graph \( F \) is a partition of \( E(F) \) into edge-disjoint circuits. These partitions were mentioned by Kotzig [50], and since then it has become clear that they are of fundamental significance in the theory of 4-regular graphs. Expanding on earlier work of Martin [51], Las Vergnas [53] introduced the generating function that records the sizes of circuit partitions of \( F \), and also the generating functions that record the sizes of directed circuit partitions of directed versions of \( F \); he called these generating functions the Martin polynomials of \( F \). A circuit
partition of $F$ is determined by choosing one of the three transitions at each vertex, and Jaeger [40] used this fact in defining his transition polynomial, a form of the Martin polynomial that incorporates transition labels. A labeled form of the Martin polynomial was independently discovered by Kauffman, who used it in his bracket polynomial definition of the Jones polynomial of a knot or link [43, 47].

For plane graphs, there is an indirect connection between Martin polynomials and graphic matroids, introduced by Martin [57] and further elucidated by Las Vergnas [51] and Jaeger [40]. (The corresponding result for the Kauffman bracket is due to Thistlethwaite [68].) The complementary regions of a 4-regular graph $F$ imbedded in the plane can be colored checkerboard fashion, yielding a pair of dual graphs with $F$ as medial; the cycle matroid of either of the two dual graphs yields the Martin polynomial of a directed version of $F$. This indirect connection has been extended to several formulas, each time weakening the connection with matroids: Jaeger extended it to include information from the undirected Martin polynomial [41], Las Vergnas extended it to medial graphs in the projective plane and the torus [52], and Ellis-Monaghan and Moffatt extended it to include medials in surfaces of all genera [28, 29].

The purpose of the present paper is to introduce a more general connection between matroids and Martin polynomials, which holds for all 4-regular graphs and does not require surface geometry. Let $F$ be a 4-regular graph, and let $T(F)$ be the set of transitions of $F$; we call this the canonical partition of $T(F)$. Let $M_\tau(F)$ be the set of transversals of the canonical partition, i.e., subsets of $T(F)$ that contain exactly one transition for each vertex of $F$. Then there is a bijection

$$\tau : \{\text{circuit partitions of } F\} \rightarrow T(F)$$

that assigns to a circuit partition $P$ the set of transitions involved in $P$.

**Theorem 1.** Let $F$ be a 4-regular graph with $c(F)$ connected components and $n$ vertices. Then there is a rank-$n$ matroid $M_\tau(F)$ defined on $\Sigma(F)$ such that for each circuit partition $P$ of $F$, the rank of $\tau(P)$ in $M_\tau(F)$ is given by

$$r(\tau(P)) = n + c(F) - |P|.$$ 

Here $|P|$ denotes the size of $P$, i.e., the number of circuits included in $P$.

We call $M_\tau(F)$ the transition matroid of $F$, and we call the equation displayed in Theorem 1 the circuit-nullity formula. (The name may seem more appropriate after the formula is rewritten as $|P| - c(F) = n - r(\tau(P))$: the number of “extra” circuits in $P$ equals the nullity of $\tau(P)$.) The formula has been rediscovered in one form or another many times over the years [3, 6, 19, 21, 38, 43, 49, 54, 56, 80, 85, 60, 67, 71, 74, 83]. What is surprising about Theorem 1 is not the circuit-nullity formula, but the fact that the formula can be extended to give well-defined ranks for arbitrary subsets of $\Sigma(F)$, including subsets not associated with circuit partitions in $F$ because they contain two transitions at some vertex, or none.
We define $M_\tau(F)$ in Section 2. The definition is not concise enough to summarize conveniently here, but we might mention that in the terminology of [75], $M_\tau(F)$ is the isotropic matroid of the interlacement graph of any Euler system of $F$. Notice the word “any” in the preceding sentence: a typical 4-regular graph has many different Euler systems, with nonisomorphic interlacement graphs; but they all give rise to the same transition matroid.

In Section 3 we observe that the transition matroid of $F$ is closely related to the $\Delta$-matroids and isotropic system associated to $F$ by Bouchet [11, 12, 15, 16].

**Theorem 2** These statements about 4-regular graphs $F$ and $F'$ are equivalent:

1. The transition matroid of $F$ is isomorphic to the transition matroid of $F'$.
2. The isotropic system of $F$ is isomorphic to the isotropic system of $F'$.
3. Any $\Delta$-matroid of $F$ is isomorphic to a $\Delta$-matroid of $F'$.
4. All $\Delta$-matroids of $F$ are isomorphic to $\Delta$-matroids of $F'$.
5. Any interlacement graph of $F$ is isomorphic to an interlacement graph of $F'$.
6. All interlacement graphs of $F$ are isomorphic to interlacement graphs of $F'$.

Notice again the word “any”: a typical 4-regular graph has many distinct interlacement graphs and many distinct $\Delta$-matroids, but only one transition matroid and only one isotropic system.

**Theorem 2** tells us that if an operation on 4-regular graphs preserves interlacement graphs, then it also preserves transition matroids. In Section 4 we discuss a theorem of Ghier [24], a version of which was subsequently discovered independently by Chmutov and Lando [25]. Ghier’s theorem implies that there are two fundamental types of interlacement-preserving operations, associated with edge cuts of size two or four. Borrowing some terminology from knot theory, we call operations of the first type connected sums; their inverses are separations. (See Fig. 2) Operations of the second type are balanced mutations. (See Fig. 3) The formal definitions follow.

**Definition 3** Suppose $F_1$ and $F_2$ are two separate 4-regular graphs. Let $e_1 = \{h_1, h'_1\} \in E(F_1)$ and $e_2 = \{h_2, h'_2\} \in E(F_2)$. Then a connected sum of $F_1$ and
F_2 with respect to e_1 and e_2 is a graph obtained from F_1 ⋃ F_2 by replacing e_1 and e_2 with new edges \{h_1, h_2\} and \{h'_1, h'_2\}.

Notice that there are two distinct connected sums of F_1 and F_2 with respect to e_1 and e_2, which involve matching different pairs of half-edges. Moreover there are many different connected sums of F_1 and F_2, with different choices of e_1 and e_2. In contrast, the inverse operation is uniquely defined.

**Definition 4** Suppose two edges \(e_1 = \{h_1, h'_1\}\) and \(e_2 = \{h_2, h'_2\}\) constitute an edge cut of \(F\). Then the separation of \(F\) with respect to \(e_1\) and \(e_2\) is the graph \(F'\) with \(c(F') = c(F) + 1\) that is obtained from \(F\) by replacing \(e_1\) and \(e_2\) with new edges \(\{h_1, h_2\}\) and \(\{h'_1, h'_2\}\).

The second type of interlacement-preserving operation is more complicated.

**Definition 5** Suppose \(F\) has nonempty subgraphs \(F_1\) and \(F_2\) such that \(V(F) = V(F_1) \cup V(F_2), V(F_1) \cap V(F_2) = \emptyset\) and precisely four edges of \(F\) connect \(F_1\) to \(F_2\). Group these four edges into two pairs, and reassemble the four half-edges from each pair into two new edges, each of which connects \(F_1\) to \(F_2\). The resulting graph \(F'\) is obtained from \(F\) by balanced mutation.

The reader familiar with knot theory should find balanced mutation rather strange, in at least three ways. 1. Knot theorists use diagrams in the plane, but we are discussing abstract graphs, not plane graphs. In particular, the apparent edge-crossings in Fig. 3 are mere artifacts of the figure; they do not reflect any significant information about the graphs involved in the operation. 2. Similarly, knot-theoretic mutation is usually depicted by rotating \(F_2\) through an angle of \(\pi\), rather than repositioning half-edges. This rotation implicitly requires that new edges and old edges correspond in pairs. As our discussion is abstract, we mention this requirement explicitly. 3. In knot theory it is not required that the four edges shown in the diagram be distinct; two might be parts of an arc that simply passes through the region of the plane denoted \(F_1\) or \(F_2\), without encountering any crossing. This third issue is not significant, though, because such trivial knot-theoretic mutations can be accomplished with separations and connected sums.

Ghier’s theorem [34] implies that Definitions 3, 4 and 5 characterize the 4-regular graphs with isomorphic transition matroids.
Theorem 6 Let $F$ and $F'$ be 4-regular graphs. Then $M_\tau(F) \cong M_\tau(F')$ if and only if $F$ can be obtained from $F'$ using connected sums, separations and balanced mutations.

In particular, Theorem 6 tells us that if $F$ is a connected 4-regular graph that admits no separation or balanced mutation, then $F$ is determined by its transition matroid:

Corollary 7 Let $F$ be a connected 4-regular graph in which every edge cut with fewer than six edges consists of the edges incident on one vertex. Then a 4-regular graph $F'$ is isomorphic to $F$ if and only if their transition matroids are isomorphic.

Theorem 6 is the analogue for transition matroids of Whitney’s famous characterization of graphs with isomorphic cycle matroids (see Chapter 5 of [61]): connected sums are analogous to vertex identifications in disjoint graphs, separations are analogous to cutpoint separations, and balanced mutations are analogous to Whitney twists. Corollary 7 is the analogue of a special case of Whitney’s theorem: 3-connected simple graphs are determined by their cycle matroids.

These analogies may be explained using an idea that appeared implicitly in Jaeger’s work [38] and was later discussed in detail by Bouchet [12]. A circuit partition $P$ in a 4-regular graph $F$ has an associated touch-graph $Tch(P)$, which has a vertex for each circuit of $P$ and an edge for each vertex of $F$; the edge of $Tch(P)$ corresponding to $v$ is incident on the vertex or vertices of $Tch(P)$ corresponding to circuit(s) of $P$ that pass through $v$. In Section 5 we observe that connected sums, separations and balanced mutations of 4-regular graphs induce vertex identifications, cutpoint separations and Whitney twists in touch-graphs. We also show that Theorem 6 implies the formula of Las Vergnas and Martin for plane graphs that was mentioned above.

A parametrized Tutte polynomial of a matroid $M$ on a ground set $S$ is a sum of the form

$$f(M) = \sum_{A \subseteq S} \left( \prod_{a \in A} \alpha(a) \right) \left( \prod_{s \in S-A} \beta(s) \right) (x-1)^{r(S)-r(A)}(y-1)^{|A|-r(A)}$$

where $r$ is the rank function of $M$ and $\alpha$, $\beta$ are weight functions mapping $S$ into some commutative ring. (Other types of parametrized Tutte polynomials appear in the literature [7, 32, 82], but they will not enter into our discussion.) Different choices of $\alpha$ and $\beta$ produce parametrized Tutte polynomials with different amounts of information, but if we are free to choose $\alpha$ and $\beta$ as we like, we can produce a parametrized Tutte polynomial in which the products of parameters identify the contributions of individual subsets. Such a polynomial determines the rank of each subset of $S$, so it determines the matroid $M$. Consequently, an equivalent form of Theorem 6 is the following.

Theorem 8 Let $F$ and $F'$ be 4-regular graphs. Then $F$ can be obtained from $F'$ using connected sums, separations and balanced mutations if and only if for
all weight functions on $\mathcal{I}(F)$, there are corresponding weight functions on $\mathcal{I}(F')$ such that the resulting parametrized Tutte polynomials of $M_\tau(F)$ and $M_\tau(F')$ are the same.

Theorem 9 provides a direct connection between Martin polynomials and parametrized Tutte polynomials of transition matroids.

Theorem 9. If $F$ is a 4-regular graph then $c(F)$ and the parametrized Tutte polynomial of $M_\tau(F)$ determine the Martin polynomial of $F$ and the directed Martin polynomials of all balanced orientations of $F$.

Proof. First, we define a parametrization for which every nonzero parameter product in the parametrized Tutte polynomial $f(M_\tau(F))$ corresponds to a circuit partition. Begin with a ring $R$ of polynomials in $6|V(F)| + 2$ indeterminates, $x$ and $y$ in addition to $\alpha(\tau)$ and $\beta(\tau)$ for each transition $\tau \in \mathcal{I}(F)$. Let $I$ be the ideal of $R$ generated by all products $\alpha(\tau_1)\alpha(\tau_2)$, $\alpha(\tau_1)\alpha(\tau_3)$, $\alpha(\tau_2)\alpha(\tau_3)$ and $\beta(\tau_1)\beta(\tau_2)\beta(\tau_3)$ such that $\tau_1, \tau_2, \tau_3$ are the three transitions corresponding to a single vertex. Consider the parametrized Tutte polynomial $f(M_\tau(F))$ in the quotient ring $R/I$. Then the nonzero terms of $f(M_\tau(F))$ correspond to subsets $A \subset \mathcal{I}(F)$ that include precisely one transition for each vertex of $F$. Each such subset $A$ has $|A| = n = r(\mathcal{I}(F))$, so Theorem 9 tells us that

$$f(M_\tau(F)) = \sum_P \left( \prod_{\tau \in P} \alpha(\tau) \right) \left( \prod_{\tau \not\in P} \beta(\tau) \right) (x - 1)^{|P| - c(F)} (y - 1)^{|P| - c(F)}$$

with a term for each circuit partition $P$ of $F$.

Note that if $\pi : R \to R/I$ is the canonical map onto the quotient, then $\pi$ is injective on the additive subgroup of $R$ generated by monomials that include no more than one of the $\alpha(\tau)$ corresponding to any vertex and no more than two of the $\beta(\tau)$ corresponding to any vertex. The version of the Martin polynomial used by Las Vergnas [23], $\sum_P (\zeta - 1)^{|P| - 1}$, may be obtained from the inverse image $\pi^{-1}f(M_\tau(F))$ by setting $\alpha(\tau) = 1$ and $\beta(\tau) = 1$ for every transition $\tau$, setting $x = \zeta$ and $y = 2$, and multiplying by $(\zeta - 1)^{c(F) - 1}$. To obtain a directed Martin polynomial, enlarge $I$ by including $\alpha(\tau)$ for every direction-violating transition $\tau$.

Theorem 9 is surprising because it has been assumed that for non-planar 4-regular graphs, Martin polynomials are not connected with matroids. (As noted above, the circuit theory of 4-regular graphs seems to be fundamentally non-matroidal, because distinct circuits can be built from the same vertices and half-edges.) The theorem also implies that several other graph polynomials associated with circuit partitions in 4-regular graphs can be recovered directly from the Tutte polynomial of $M_\tau(F)$: like the Martin polynomials, these polynomials were previously assumed to be essentially non-matroidal.

Theorem 10. If $F$ is a 4-regular graph then $c(F)$ and the parametrized Tutte polynomial of $M_\tau(F)$ determine the interlace polynomials of all circle graphs associated with $F$ [1, 2, 3, 4, 25, 73], the transition polynomial of $F$ [27, 40].
and the homflypt [53, 63] and Kauffman [40] polynomials of all knots and links with diagrams whose underlying 4-regular graphs are isomorphic to $F$.

Another kind of structure connected with 4-regular graphs is a ribbon graph. These objects may be defined in several different ways, and in Section 6 we focus on one particular definition, which associates a ribbon graph with a pair of compatible circuit partitions in a 4-regular graph, i.e., two partitions that do not involve the same transition at any vertex. Using this definition it is not hard to deduce the following.

**Theorem 11** If $F$ is a 4-regular graph then $c(F)$ and the parametrized Tutte polynomial of $M_\tau(F)$ determine all weighted Bollobás-Riordan and topological Tutte polynomials of ribbon graphs whose medial graphs are isomorphic to $F$.

We hope that Theorems 9–11 will provide both a unifying theme for the many polynomials mentioned and useful applications of the theory of parametrized Tutte polynomials. The representation of ribbon graphs using compatible circuit partitions also yields an embedding-free expression of the theory of twisted duality due to Chmutov [22], Ellis-Monaghan and Moffatt [28]: two ribbon graphs are twisted duals if and only if they correspond to compatible circuit partitions of the same 4-regular graph.

In Section 8 we relate these ideas to a characterization of planar 4-regular graphs that appears in discussions of the famous Gauss crossing problem (see for instance [26, 35, 64]): a 4-regular graph is planar if and only if it has an Euler system whose interlacement graph is bipartite.

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## 2 Defining the transition matroid

Recall that an Euler circuit of a 4-regular graph $F$ is a circuit that includes every half-edge of $F$. If $F$ is disconnected it cannot have an Euler circuit, of course, but every 4-regular graph $F$ has an Euler system, i.e., a set containing one Euler circuit for each connected component of $F$. **Kotzig’s theorem** [50] states that given one Euler system of $F$, all the others are obtained by applying sequences of $\kappa$-transformations: given an Euler system $C$ and a vertex $v$, the $\kappa$-transform $C \ast v$ is the Euler system obtained from $C$ by reversing one of the two $v$-to-$v$ walks within the circuit of $C$ incident on $v$. See Fig. 3.

Let $C$ be an Euler system of a 4-regular graph $F$. The interlacement graph $\mathcal{I}(C)$ is a simple graph with the same vertices as $F$. Two vertices $v \neq w \in V(F)$ are adjacent in $\mathcal{I}(C)$ if and only if they appear in the order $v...w...v...w$ on one of the circuits of $C$; if this happens then $v$ and $w$ are interlaced with respect to $C$. Interlacement graphs were discussed by Bouchet [10] and Read and Rosenstiehl
[64], who observed that the relationship between $\mathcal{I}(C)$ and $\mathcal{I}(C*v)$ is described by simple local complementation at $v$: if $v \neq x \neq y \neq v$ and $x, y$ are both neighbors of $v$ in $\mathcal{I}(C)$ then they are adjacent in $\mathcal{I}(C*v)$ if and only if they are not adjacent in $\mathcal{I}(C)$.

Recall that a transition at a vertex $v$ is a partition of the four half-edges incident at a vertex $v$ into two pairs, and $\mathcal{T}(F)$ denotes the set of all transitions in $F$. For each Euler system $C$ of $F$ and each $v \in V(F)$ we may label the transitions at $v$ according to their relationships with $C$, following a notational scheme from [71, 72, 73, 74].

**Definition 12** The transition that is used by the circuit of $C$ incident at $v$ is labeled $\phi_C(v)$, the other transition that is consistent with an orientation of this circuit is labeled $\chi_C(v)$, and the transition that is not consistent with an orientation of the incident circuit of $C$ is labeled $\psi_C(v)$.

It is easy to see that transition labels with respect to $C$ are not changed when different orientations of the circuits of $C$ are used. However, a given transition may have different labels with respect to different Euler systems. The differences among these labels are not arbitrary:

**Proposition 13** If $C$ is an Euler system of $F$ then the labels of elements of $\mathcal{T}(F)$ with respect to $C$ determine the labels of elements of $\mathcal{T}(F)$ with respect to all other Euler systems of $F$.

**Proof.** As illustrated in Fig. 5 a $\kappa$-transformation changes transition labels in the following way: $\psi_{C*v}(v) = \phi_C(v)$ and $\phi_{C*v}(v) = \psi_C(v)$; if $w$ is a neighbor of $v$ in $\mathcal{I}(C)$ then $\psi_{C*v}(w) = \chi_C(w)$ and $\chi_{C*v}(w) = \psi_C(w)$; and all other transitions of $F$ have the same $\phi, \chi, \psi$ labels with respect to $C$ and $C*v$. The proposition now follows from Kotzig’s theorem, as every other Euler system of $F$ can be obtained from $C$ through $\kappa$-transformations. ■

Here is a simple analogy: a vector space has many bases, and each particular basis gives rise to a coordinatization of the vector space. The coordinatizations corresponding to different bases are not independent of each other; once we know one coordinatization, all the others are determined through multiplication by nonsingular matrices. This simple analogy is more apt than it may appear at first glance, by the way; see [72, 74] for details.
Definition 14 Let $F$ be a 4-regular graph with an Euler system $C$, let $I$ denote the identity matrix, and let $\mathcal{A}(I(C))$ denote the adjacency matrix of $I(C)$. Then

$$M(C) = (I \big| \mathcal{A}(I(C)) \big| I + \mathcal{A}(I(C))).$$

The rows of this matrix are indexed using vertices of $F$ as in $\mathcal{A}(I(C))$, and the columns are indexed using transitions of $F$ as follows: the column of $I$ corresponding to $v$ is indexed by $\phi_C(v)$, the column of $\mathcal{A}(I(C))$ corresponding to $v$ is indexed by $\chi_C(v)$, and the column of $I + \mathcal{A}(I(C))$ corresponding to $v$ is indexed by $\psi_C(v)$.

Different Euler systems of $F$ will certainly give rise to different matrices, but these matrices are tied together in a special way.

Proposition 15 If $C$ and $C'$ are Euler systems of $F$ then $M(C)$ and $M(C')$ represent the same matroid on $\mathcal{T}(F)$.

Proof. Clearly the matroid $M_r(F)$ is not affected if elementary row operations are applied to $M(C)$; also, permuting the columns of $M(C)$ does not affect the matroid so long as the same permutation is applied to the column indices.

According to Kotzig’s theorem, it suffices to prove the proposition when $C' = C \ast v$. Each of $M(C)$, $M(C \ast v)$ consists of three square submatrices, which we refer to as their $\phi$, $\chi$ and $\psi$ parts. The columns of each part are indexed by
We can obtain \( M(C \ast v) \) from \( M(C) \) using row and column operations as follows. First, add the \( v \) row of \( M(C) \) to every other row corresponding to a neighbor of \( v \). Second, interchange the \( v \) columns of the \( \phi \) and \( \psi \) parts of the resulting matrix. Third, for each neighbor \( w \) of \( v \), interchange the \( w \) columns of the \( \chi \) and \( \psi \) parts of the resulting matrix.

Notice that the column interchanges of the preceding paragraph correspond precisely to the label changes induced by the \( \kappa \)-transformation \( C \mapsto C \ast v \), as discussed in the proof of Proposition 13. \( \phi \) and \( \psi \) are interchanged at \( v \), and \( \chi \) and \( \psi \) are interchanged at each neighbor \( w \) of \( v \). Consequently, \( M(C) \) and \( M(C \ast v) \) represent the same matroid on \( \Sigma(F) \).

Proposition 15 allows us to make the following definition:

**Definition 16** If \( F \) is a 4-regular graph with an Euler system \( C \), then the transition matroid \( M_\tau(F) \) is the binary matroid on \( \Sigma(F) \) represented by the matrix \( M(C) \).

It is not hard to verify that \( M_\tau(F) \) satisfies the requirements of Theorem 1. That is, \( M_\tau(F) \) is a rank-\( n \) matroid defined on \( \Sigma(F) \) such that for each circuit partition \( P \) of \( F \), the rank of \( \tau(P) \) in \( M_\tau(F) \) is given by \( r(\tau(P)) = n + c(F) - |P| \). For if \( C \) is any Euler system of \( F \) then the \( I \) submatrix of \( M(C) \) guarantees that the rank of \( M(C) \) is \( n \), no matter what \( A(\mathcal{I}(C)) \) is. Also, if \( P \) is a circuit partition of \( F \) then the rank of \( \tau(P) \) in \( M_\tau(F) \) is the \( GF(2) \)-rank of the submatrix of \( M(C) \) consisting of the columns corresponding to elements of \( \tau(P) \).

As mentioned in the introduction, the fact that this rank equals \( n - |P| + c(F) \) has been discussed by many researchers. (The reader who has not seen the circuit-nullity formula before may find the account in [74] convenient, as the notation is close to ours.) This completes the proof of Theorem 1.

We briefly discuss some basic properties of transition matroids. Notice first that the three columns of \( M(C) \) corresponding to a vertex \( v \) sum to 0, and the \( \phi_C(v) \) and \( \psi_C(v) \) columns are never 0. We conclude that the cell \( \{\phi_C(v), \chi_C(v), \psi_C(v)\} \) of the canonical partition contains either a 3-cycle of \( M_\tau(F) \), or a loop and a pair of non-loop parallel elements. Cells of the latter sort correspond to isolated vertices in interlacement graphs of \( F \), i.e., cut vertices and looped vertices of \( F \).

As the rank of \( M_\tau(F) \) is \( n \), the circuit-nullity formula \( r(\tau(P)) = n - |P| + c(F) \) indicates that a circuit partition \( P \) is an Euler system of \( F \) if and only if \( \tau(P) \) is a basis of \( M_\tau(F) \). Here are two important features of the relationship between Euler systems and matroid bases.

1. A fundamental property of matroids is this: the bases of a matroid are all interconnected by basis exchanges (single-element swaps between bases). This property holds in \( M_\tau(F) \), of course, but it does not mean that Euler systems are interconnected by single-vertex transition changes. In general \( M_\tau(F) \) will have many bases that do not correspond to Euler systems because they do not include precisely one transition for each vertex. For comparison we should regard \( \kappa \)-transformations as specially modified basis exchanges, and we should regard Kotzig’s theorem as asserting that those bases of \( M_\tau(F) \) that correspond
to Euler systems are distributed densely enough that they form a connected set under these special exchanges.

2. A basis of a matroid is a subset of the matroid’s ground set, so it is a simple matter to compare two bases $B$ and $B'$; when they disagree about an element it must be that the element is included in one of $B, B'$ and excluded from the other. For $M_r(F)$ this simple analysis is correct if we focus our attention on $\Sigma(F)$, but the fact there are three different transitions at each vertex complicates matters if we focus our attention on $V(F)$ instead of $\Sigma(F)$. There are five different ways that two Euler systems $C, C'$ might disagree about a vertex $v$, corresponding to the five nontrivial permutations of the set $\{\phi, \chi, \psi\}$.

A transition matroid is a $3n$-element matroid of rank $n$, so deleting or contracting a single element cannot yield another transition matroid. However, if the triple corresponding to a vertex $v$ is removed by contracting one transition at $v$ and deleting the other two, then the resulting matroid is isomorphic to the transition matroid of the 4-regular graph $F'$ obtained from $F$ by detachment (or splitting) along the contracted transition. (As indicated in Fig. 6, the detachment is constructed by first removing $v$, and then connecting the loose half-edges paired by the contracted transition. If $v$ is looped in $F$, this construction may produce one or two arcs with no incident vertex; any such arc is simply discarded.) This property is easy to verify if we first choose an Euler system $C$ in which the contracted transition is $\phi_C(v)$. Then the only nonzero entry of the column of $M(C)$ corresponding to the contracted transition is a 1 in the $v$ row, so the definition of matroid contraction tells us that the matroid $(M_r(F)/\phi_C(v)) - \chi_C(v) - \psi_C(v)$ is represented by the submatrix of $M(C)$ obtained by removing the row corresponding to $v$ and removing all three columns corresponding to $v$. The resulting matrix is clearly the same as $M(C')$, where $C'$ is the Euler system of $F'$ obtained in the natural way from $C$.

Many binary matroids occur as submatroids of transition matroids of 4-regular graphs. Jaeger [39] proved that every cographic matroid is represented by a symmetric matrix whose off-diagonal entries equal those of the interlacement matrix of some 4-regular graph; consequently, the submatroids of transition matroids include all cographic matroids. (See the end of Section 6 for more discussion of this point.) In addition, many non-cographic matroids occur as submatroids of transition matroids. For example, consider the 4-regular graph $F$ of Fig. 7. Every pair of vertices is interlaced with respect to the indicated
Figure 7: The indicated Euler circuit has $\mathcal{I}(C) \cong K_3$.

Euler circuit, so $M_\tau(F)$ is represented by the matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1
\end{pmatrix}.
$$

The first seven columns represent a well-known non-cographic submatroid of $M_\tau(F)$, the Fano matroid.

Before proceeding, we should mention that it is likely that some form of the transition matroid has been noticed before. See [18], where Bouchet comments that “the main applications – for instance, to eulerian graphs – involve multimatroids that can be sheltered by matroids.” (In Bouchet’s terminology, the rank function of the multimatroid associated with a 4-regular graph $F$ is the restriction of the rank function of $M_\tau(F)$ to the subtransversals of the canonical partition of $\mathcal{I}(F)$; he defined this restriction using the circuit-nullity formula. $M_\tau(F)$ shelters the multimatroid because its rank function extends the multimatroid’s rank function to arbitrary subsets of $\mathcal{I}(F)$.) As far as we know, though, Bouchet’s published work includes neither any explicit description of his sheltering matroid nor any results about it.

The reason we mention “some form of the transition matroid” is that there are at least three other ways to define $M_\tau(F)$.

1. The rank function of $M_\tau(F)$ may be described in two stages: First, use the circuit-nullity formula to determine $r(\tau(P))$ as $n + c(F) - |P|$ for each circuit partition $P$. Second, describe how the ranks of subsets of the form $\tau(P)$ determine the ranks of all other subsets of $\mathcal{I}(F)$. Bouchet detailed the first stage in [18], but did not discuss the second stage. It is possible to complete this description without explicitly mentioning the matrix $M(C)$; the crucial fact is that each cell of the canonical partition has rank one or two. We leave the details as an exercise for the interested reader.

2. Similarly, the bases of the matroid $M_\tau(F)$ may be described in two stages: First, observe that $\tau(C)$ is a basis for each Euler system $C$. Second, describe the rest of the bases. Bouchet noted the connection between Euler systems and submatrices of $A(\mathcal{I}(C))$ in [16], but did not discuss the other bases of $M_\tau(F)$.

3. The line graph $L(F)$ is a 6-regular graph that has a vertex for each edge of $F$ and an edge for each pair of distinct half-edges incident at the same vertex. (With this convention, a loop in $F$ gives rise to a loop in $L(F)$.) That
is, there is an edge of $L(F)$ for each single transition of $F$. Like any graph, $L(F)$ has a cycle matroid $M(L(F))$ consisting of vectors in the $GF(2)$-vector space $GF(2)^{V(L(F))}$; the vector corresponding to a non-loop edge of $L(F)$ has nonzero coordinates corresponding to the two vertices incident on that edge, and the vector corresponding to a loop is 0. If $Z$ is the subspace of $GF(2)^{V(L(F))}$ spanned by $\{z_1 - z_2 \mid z_1$ and $z_2$ are the vectors corresponding to two single transitions that constitute a transition of $F\}$ then $M_T(F)$ is the matroid on $T(F)$ in which each transition is represented by the image of either of its constituent single transitions in the quotient space $GF(2)^{V(L(F))}/Z$.

3 Proof of Theorem 2

In this section we provide a brief outline of arguments verifying equivalences among the conditions of Theorem 2. The substance of the arguments appears in the references.

Recall that Theorem 2 asserts equivalences among the following statements about 4-regular graphs $F$ and $F'$:

1. The transition matroid of $F$ is isomorphic to the transition matroid of $F'$.
2. The isotropic system of $F$ is isomorphic to the isotropic system of $F'$.
3. Any $\Delta$-matroid of $F$ is isomorphic to a $\Delta$-matroid of $F'$.
4. All $\Delta$-matroids of $F$ are isomorphic to $\Delta$-matroids of $F'$.
5. Any interlacement graph of $F$ is isomorphic to an interlacement graph of $F'$.
6. All interlacement graphs of $F$ are isomorphic to interlacement graphs of $F'$.

The implication 6 $\Rightarrow$ 5 is obvious, and the implication 5 $\Rightarrow$ 1 follows immediately from Definition 10. The implication 1 $\Rightarrow$ 6 follows from two results. The first of these results is a theorem of the author [75], which tells us that if $G$ and $G'$ are simple graphs with adjacency matrices $A$ and $A'$, then the matrices

$$(I \mid A \mid I + A) \text{ and } (I \mid A' \mid I + A')$$

represent isomorphic binary matroids if and only if $G$ and $G'$ are equivalent under simple local complementation. The second of these results is the fundamental relationship between $\kappa$-transformations and local complementation, which tells us that any sequence of simple local complementations on interlacement graphs is induced by a corresponding sequence of $\kappa$-transformations on Euler systems. This implies that if $F$ and $F'$ are 4-regular graphs with Euler systems $C$ and $C'$, then $I(C)$ and $I(C')$ are equivalent under simple local complementation if and only if $F'$ has an Euler system $C''$ such that $I(C)$ and $I(C'')$ are isomorphic.
The equivalence of statements 2, 5 and 6 of Theorem 2 is readily extracted from Bouchet’s discussion of graphic isotropic systems in Section 6 of [15], although he does not give quite the same statements. Similarly, the equivalence of statements 3, 4, 5 and 6 may be extracted from Section 5 of [16]. We do not repeat the details of these discussions, but we might summarize them by saying that as the isotropic system of \( F \) is determined by the sizes of the circuit partitions of \( F \), and the \( \Delta \)-matroids associated with \( F \) are determined by the Euler systems of \( F \), these structures are tied to the transition matroid \( M_\tau(F) \) by the circuit-nullity formula. A more thorough account of the connections tying isotropic matroids to \( \Delta \)-matroids and isotropic systems is given in [75].

In the balance of the paper we focus on transition matroids and interlacement graphs.

4 Ghier’s theorem

Ghier [34] characterized the double occurrence words with connected, isomorphic interlacement graphs, and several years later a similar characterization of chord diagrams with isomorphic interlacement graphs was independently discovered by Chmutov and Lando [23]. In this section we deduce Theorem 6 from Ghier’s theorem.

4.1 Ghier’s theorem for double occurrence words

A double occurrence word in letters \( v_1, ..., v_n \) is a word of length \( 2n \) in which every \( v_i \) occurs twice, and the interlacement graph of a double occurrence word is the simple graph with vertices \( v_1, ..., v_n \) in which \( v_i \) and \( v_j \) are adjacent if and only if they are interlaced, i.e., they appear in the order \( v_i ... v_j ... v_i \) or \( v_j ... v_i ... v_j \).

**Definition 17** An equivalence between two double occurrence words is a finite sequence of cyclic permutations \( x_1 ... x_{2n} \mapsto x_i ... x_{2n}x_1 ... x_{i-1} \) and reversals \( x_1 ... x_{2n} \mapsto x_{2n} ... x_1 \).

**Definition 18** [34] Let \( \{v_1, ..., v_n\} \) be the union of two disjoint subsets, \( V' \) and \( V'' \). Let \( W \) be a double occurrence word in \( v_1, ..., v_n \) of the form \( W'_1W'_2W''_1W''_2 \), where \( W'_1, W'_2 \) involve only letters in \( V' \) and \( W''_1, W''_2 \) involve only letters in \( V'' \). Then the corresponding turnarounds of \( W \) are the double occurrence words \( W'_1\overrightarrow{W''_2}W'_2\overrightarrow{W''_1}W''_2, \overrightarrow{W'_1W''_2}W'_2W''_1 \) and \( \overrightarrow{W'_1W''_2}W'_2\overrightarrow{W''_1}W''_2 \), where the arrow indicates reversal of a subword.

In Ghier’s definition each of \( V', V'' \) must have at least two elements, but there is no harm in ignoring this stipulation because the effect is simply to include some equivalences among the turnarounds. Ghier observed that the first type of turnaround suffices, up to equivalence: a turnaround of the second type is a composition of cyclic permutations with a turnaround of the first type, and a
Theorem 19 [34] Two double occurrence words have isomorphic, connected interlacement graphs if and only if some finite sequence of equivalences and turnarounds transforms one word into the other.

In order to consider double occurrence words with disconnected interlacement graphs, let us define a concatenation of two disjoint double occurrence words \( w_1 \ldots w_{2m} \) and \( x_1 \ldots x_{2n} \) in the obvious way: it is \( w_1 \ldots w_{2m}x_1 \ldots x_{2n} \) or \( x_1 \ldots x_{2n}w_1 \ldots w_{2m} \). (Here “disjoint” means that no letter occurs in both words.) Clearly then the interlacement graph of a concatenation is the disjoint union of the interlacement graphs of the concatenated words. The converse holds too:

Lemma 20 The interlacement graph of a double occurrence word is disconnected if and only if the word is equivalent to a concatenation of two nonempty subwords.

Proof. Suppose \( x_1 \ldots x_{2n} \) is a double occurrence word in \( v_1, \ldots, v_n \), with a disconnected interlacement graph. Using a cyclic permutation if necessary, we may presume that \( x_1 \) and \( x_{2n} \) lie in different connected components of the interlacement graph. Let \( i \) be the greatest index such that \( x_i \) lies in the same connected component of the interlacement graph as \( x_1 \); necessarily then \( x_i \) is the second appearance of the corresponding letter.

We claim that there is no \( j > i \) such that \( x_j = x_{j'} \) with \( j' < i \). Suppose instead that there is such a \( j \). As \( j > i \), \( x_j \) is not in the connected component of the interlacement graph that contains \( x_1 \) and \( x_i \), so \( x_j \) is not interlaced with either; hence the double occurrence word must be of the form \( x_1Ax_1Bx_2Cx_1Dx_1Ex_2F \) for some double occurrence word \( ABCDEF \) in the remaining letters. Choose a path from \( x_1 \) to \( x_i \) in the interlacement graph; say it is \( x_1, x_{k_1}, x_{k_2}, \ldots, x_{k_{n-1}}, x_i \). As \( x_{k_1} \) is interlaced with \( x_1, x_{k_1} \) must appear precisely once in \( A \). As \( x_{k_1} \) is not interlaced with \( x_j \), its other appearance must be in \( B \) or \( F \); hence \( x_{k_1} \) appears twice in \( Fx_1Ax_1B \). As \( x_{k_3} \) is interlaced with \( x_{k_1}, x_{k_3} \) must appear at least once in \( Fx_1Ax_1B \). As \( x_{k_3} \) is not interlaced with \( x_j \), its other appearance cannot be in \( Cx_1Dx_1E \), so it must be in \( Fx_1Ax_1B \); hence \( x_{k_3} \) appears twice in \( Fx_1Ax_1B \). Similarly, \( x_{k_3} \) must appear at least once in \( Fx_1Ax_1B \) (as it is interlaced with \( x_{k_3} \)) and cannot appear precisely once in \( Fx_1Ax_1B \) (as it is not interlaced with \( x_j \)), so \( x_{k_3} \) also appears twice in \( Fx_1Ax_1B \). The same reasoning holds for \( x_{k_4} \),
Eventually we conclude that $x_{k-1}$ appears twice in $Fx_1Ax_1B$; but this contradicts the fact that $x_{k-1}$ is interlaced with $x_i$.

The contradiction verifies the claim. It follows that $x_1\ldots x_{2n}$ is the concatenation of $x_1\ldots x_i$ and $x_{i+1}\ldots x_{2n}$. ■

**Definition 21** A disjoint family of double occurrence words is a finite set $\{W_1, \ldots, W_k\}$ of double occurrence words, no two of which share any letter. The interlacement graph of $\{W_1, \ldots, W_k\}$ is the union of the interlacement graphs of $W_1, \ldots, W_k$.

In particular, if $W$ is a double occurrence word then $\{W\}$ is a singleton disjoint family.

**Definition 22** An equivalence between two disjoint families of double occurrence words is some sequence of the following operations: cyclically permuting a word, reversing a word, replacing two disjoint words with their concatenation, and separating a concatenation of two disjoint words into the two separate words.

If $\{W_1, \ldots, W_k\}$ is a disjoint family of double occurrence words whose interlacement graph has $c$ connected components then $1 \leq k \leq c$, and $k = c$ if and only if each $W_i$ has a connected interlacement graph. Considering Lemma 20 we see that $\{W_1, \ldots, W_k\}$ is equivalent to disjoint families of all cardinalities in the set $\{1, \ldots, c\}$.

**Definition 23** A turnaround of a disjoint family of double occurrence words is a disjoint family obtained by applying a turnaround to some element of the original disjoint family.

Here is the general form of Ghier’s theorem.

**Corollary 24** Two disjoint families of double occurrence words have isomorphic interlacement graphs if and only if one disjoint family can be obtained from the other by some finite sequence of equivalences and turnarounds.

**Proof.** If necessary, replace each disjoint family by an equivalent one in which every double occurrence word has a connected interlacement graph. Then apply the original form of Ghier’s theorem to the individual words. ■

### 4.2 Ghier’s theorem for 4-regular graphs

An Euler system $C$ in a 4-regular graph $F$ with $c(F)$ connected components gives rise to many equivalent disjoint families of double occurrence words $W(C) = \{W_1(C), \ldots, W_{c(F)}(C)\}$. For $1 \leq i \leq c(F)$ the double occurrence word $W_i(C)$ is obtained by reading off the vertices visited by the $i$th element of $C$, in order. Clearly $W(C)$ is defined only up to cyclic permutations and reversals of its elements, but in any case the interlacement graph of $W(C)$ is $I(C)$. Corollary 24 implies the following.
Corollary 25 Let $F$ and $F'$ be 4-regular graphs with Euler systems $C$ and $C'$. Then $\mathcal{I}(C) \cong \mathcal{I}(C')$ if and only if $W(C)$ and $W(C')$ differ only by equivalences and turnarounds.

On the other hand, a disjoint family $W = \{W_1, ..., W_k\}$ of double occurrence words in the letters $v_1, ..., v_n$ gives rise to an Euler system $C(W)$ in a 4-regular graph $F(W)$ with $V(F(W)) = \{v_1, ..., v_n\}$. $F(W)$ has $k$ connected components, and the vertices of the $i$th connected component are the letters $v_j$ that appear in $W_i$. If $W_i = x_1 ... x_{2p}$ then for $1 \leq q < 2p$ there is an edge of this connected component from $x_q$ to $x_{q+1}$, and there is also an edge from $x_{2p}$ to $x_1$. $C(W)$ includes the Euler circuits obtained directly from $W_1, ..., W_k$. Clearly $W$ is equivalent to $W(C(W))$, and $C(W(C)) = C$ for any Euler system $C'$ in a 4-regular graph.

A double occurrence word is oriented in a natural way: we read the word from beginning to end. In order to deduce Theorem 6 of the introduction from Corollary 25, then, we must understand the balanced mutations of Definition 5 as operations on directed graphs. Recall that a direction of an edge of a graph is determined by designating one of the half-edges as initial, and the other as terminal. A balanced orientation of a 4-regular graph is a set of edge-directions with the property that at each vertex, there are two initial half-edges and two terminal half-edges. It is easy to see that every 4-regular graph $F$ has balanced orientations: just walk along Euler circuits to determine edge-directions. If $F$ has subgraphs $F_1$ and $F_2$ as in Definition 5 then a balanced orientation of $F$ must include two edges directed from $F_1$ to $F_2$, and two edges directed from $F_2$ to $F_1$. There are three different pairings of these four edges; two of them pair oppositely directed edges, and the third pairs like-directed edges. No matter which edge-pairing is used, balanced orientations of $F$ and $F'$ correspond to each other directly. This correspondence is illustrated in Fig. 8, where the notation

![Figure 8: Orientation-reversing and orientation-preserving balanced mutations.](image-url)
Figure 9: Orientation-reversing balanced mutations suffice.

Figure 10: A turnaround of $W$ yields a balanced mutation of $F(W)$.

$\overset{\sim}{F}_2$ is used to indicate that edge-directions within $F_2$ have been reversed.

Fig. 9 depicts two types of balanced mutations but as indicated in Fig. 9, the first type actually suffices: a balanced mutation involving pairs of like-directed edges can be performed by combining two balanced mutations involving pairs of oppositely-directed edges. In Fig. 9 the edge-pairing used in the second of these orientation-reversing mutations is indicated by dashes.

**Theorem 26** Let $F$ and $F'$ be 4-regular graphs. Then $F$ and $F'$ have Euler systems $C$ and $C'$ with $\mathcal{I}(C) \cong \mathcal{I}(C')$ if and only if $F$ and $F'$ differ only by connected sums, separations and balanced mutations.

**Proof.** If $F$ and $F'$ have Euler systems $C$ and $C'$ with $\mathcal{I}(C) \cong \mathcal{I}(C')$ then Corollary 25 tells us that $W(C')$ can be obtained from $W(C)$ by equivalences and turnarounds. Applying cyclic permutations and reversals to elements of $W$ has no effect on $F(W)$ or $C(W)$; a concatenation has the effect of replacing two connected components of $F(W)$ with a connected sum; separating a concatenation has the effect of separating a connected sum; and a turnaround of $W$ yields a balanced mutation of $F(W)$. See Fig. 10 for a schematic illustration of the turnaround $W_1W_2'W_2W_1' \rightarrow W_1\overset{\sim}{W}_1W_2'\overset{\sim}{W}_2$.

For the converse, suppose $F'$ can be obtained from $F$ by a single connected sum, separation or balanced mutation. Let $C$ be any Euler system of $F$. If $F'$ is obtained from $F$ with a single connected sum operation then clearly $F'$ has an Euler system $C'$ such that $W(C')$ is obtained from $W(C)$ by concatenating two words. If $F'$ is obtained from $F$ by separating a connected sum then clearly $F'$ has an Euler system $C'$ such that $W(C')$ is obtained from $W(C)$ by first (if
necessary) cyclically permuting one word to yield a concatenation of two disjoint
subwords, and then replacing the concatenation with the two disjoint words.

Suppose now that $F'$ is obtained from $F$ by a balanced mutation. As noted
above, we may presume that $F$ and $F'$ are given with balanced orientations
that agree within $F_1$ and disagree within $F_2$. Let $C$ be a directed Euler system
of $F$. Then the four edges connecting $F_1$ to $F_2$ appear in either two circuits
or one circuit of $C$, according to whether or not they lie in different connected
components of $F$. Fig. 11 indicates schematically that in either case, we can
construct a directed Euler system $C'$ of $F'$ by using the new edges to con-
nect the walks within $F_1$ and $F_2$ that constitute Euler circuits in $C$. (The
Euler circuits of other connected components are the same in $C'$ and $C$.) In
the first case, $W(C')$ can be obtained from $W(C)$ using concatenations and
separations. In the second case, $W(C')$ is obtained from $W(C)$ by a trans-
formation of the form $W_1 W_2 X_1 X_2 \mapsto W_1 \overrightarrow{X_2 X_1} \overrightarrow{W_2 W_1}$, with $W_1$ and $X_1$ the
indicated walks within $F_1$; this transformation is the composition of a reversal
$W_1 W_2 X_1 X_2 \mapsto \overrightarrow{X_2 X_1} \overrightarrow{W_2 W_1}$, a turnaround $\overrightarrow{X_2 X_1} \overrightarrow{W_2 W_1} \mapsto \overrightarrow{X_2 X_1} \overrightarrow{W_2 W_1}$, and a
cyclic permutation $\overrightarrow{X_2 X_1} \overrightarrow{W_2 W_1} \mapsto W_1 \overrightarrow{X_2 X_1} \overrightarrow{W_2}$.

We conclude that if $F'$ can be obtained from $F$ by connected sums, separa-
tions and balanced mutations then it has an Euler system $C'$ such that $W(C')$
is obtained from $W(C)$ by equivalences and turnarounds. Corollary 25 then
implies that $I(C) \cong I(C')$. □

Theorem 26 allows us to deduce Theorem 6 from Theorem 2.

Before proceeding, we should emphasize that the correspondence between
Euler systems illustrated in Fig. 11 is peculiar to balanced mutations. As
indicated in Fig. 12 graphs that are not quite balanced mutations do not
exhibit the same correspondence. They also do not have isomorphic transition
matroids, in general.
5 Touch-graphs and a formula of Las Vergnas and Martin

Suppose $P$ is a circuit partition in a 4-regular graph $F$. As mentioned in the introduction, Bouchet [12] discussed a way to construct a touch-graph $Tch(P)$ from $P$, with a vertex for each circuit of $P$ and an edge for each vertex of $F$; the edge corresponding to $v$ is incident on the vertex or vertices corresponding to circuit(s) of $P$ incident at $v$. (This construction appeared implicitly in earlier work of Jaeger [38].)

**Theorem 27** Let $F$ and $F'$ be 4-regular graphs with isomorphic transition matroids. Then some isomorphism $f : M_\tau(F) \to M_\tau(F')$ has the following properties:

1. There is a bijection $g : V(F) \to V(F')$ that is compatible with $f$, in the sense that for each $v \in V(F)$, the images under $f$ of the three transitions at $v$ are the three transitions of $F'$ at $g(v)$.

2. If $P$ and $P'$ are circuit partitions with $\tau(P') = f(\tau(P))$, then $g$ defines an isomorphism between the cycle matroids of $Tch(P)$ and $Tch(P')$.

**Proof.** By Theorem [5] it suffices to consider the possibility that $F'$ is obtained from $F$ by a single connected sum, separation or balanced mutation. Then $F$ and $F'$ have the same vertices and the same half-edges, though the half-edges displayed in Fig. 2 or 3 (whichever figure is appropriate) are combined into edges in different ways. As transitions are pairings of half-edges incident at the same vertex, $F$ and $F'$ also have the same transitions. If we use $g$ and $f$ to denote the identity maps of $V(F) = V(F')$ and $\mathcal{E}(F) = \mathcal{E}(F')$, respectively, then these identity maps certainly satisfy assertion 1.
If $C$ is an Euler system of $F$ then as discussed in the proof of Theorem \[23\] $F'$ has an Euler system $C'$ such that $W(C)$ and $W(C')$ differ only by equivalences and turnarounds. Then the interlacement graphs $I(C)$ and $I(C')$ are the same, so the identity map $f$ defines an isomorphism of transition matroids.

It remains to verify assertion 2. Each circuit partition $P$ of $F$ corresponds to a circuit partition $P'$ of $F'$, with $\tau(P') = f(\tau(P))$. The edges of $Tch(P)$ and $Tch(P')$ correspond to elements of $V(F) = V(F')$, so the identity map $g$ is a bijection between the edge sets of $Tch(P)$ and $Tch(P')$. If $F'$ is obtained from $F$ by a connected sum of $F_1$ and $F_2$, then $P'$ is obtained by “splicing” two circuits of $P$, one contained in $F_1$ and the other contained in $F_2$. Clearly then $Tch(P')$ is obtained from $Tch(P)$ by identifying two vertices from different connected components, so $Tch(P)$ and $Tch(P')$ have the same cycle matroid. If $F'$ is obtained from $F$ by a separation, the situation is reversed.

Suppose now that $F'$ is obtained from $F$ by a balanced mutation which is not a composition of connected sums and separations. Then the four edges connecting $F_1$ to $F_2$ lie in the same connected component of $F$. Let $P$ and $P'$ be circuit partitions of $F$ and $F'$, respectively, with $\tau(P') = f(\tau(P))$. As the only difference between $F$ and $F'$ is that different half-edges constitute the four edges connecting $F_1$ to $F_2$, the only differences between $P$ and $P'$ involve the circuit(s) containing the four edges that connect $F_1$ to $F_2$. In particular, the circuits of $P$ that are contained in $F_1$ or $F_2$ are the same as the circuits of $P'$ that are contained in $F_1$ or $F_2$.

We claim that there is a bijection $f_P : P \to P'$ that is the identity map for circuits contained in $F_1$ or $F_2$. If the claim is false then we may suppose without loss of generality that the four edges connecting $F_1$ to $F_2$ appear in one circuit of $P$, but in two distinct circuits of $P'$. We use induction on $|P|$ to prove that this is impossible. If $|P| = c(F)$ then $P$ is an Euler system of $F$. As illustrated in Fig. 11 it follows that $P'$ is an Euler system of $F'$, and all four edges connecting $F_1$ to $F_2$ in $F'$ lie in one connected component; but then they all appear in one circuit of $P'$, a contradiction. Proceeding inductively, suppose $|P| > c(F)$ and the claim holds for all circuit partitions of size smaller than $|P|$. As the four edges connecting $F_1$ to $F_2$ lie on a single circuit of $P$, $|P| > c(F)$ implies that there is a circuit $\gamma_1 \in P$ that is contained in either $F_1$ or $F_2$, and is not an Euler circuit of a connected component of $F$. Consequently there is a vertex $v$ where $\gamma_1$ and some other circuit $\gamma_2 \in P$ are both incident. As noted earlier, $P$ and $P'$ include the same circuits contained in $F_1$ or $F_2$, so $\gamma_1 \in P'$ and $P'$ also includes a circuit $\gamma_2 \neq \gamma_1$ that is incident at $v$. Let $\overline{P}$ be a circuit partition of $F$ obtained by changing the transition of $P$ at $v$, and let $\overline{P}'$ be the corresponding circuit partition of $F'$. As indicated in Fig. 12 \[|P| = |P'|-1\]; consequently the inductive hypothesis tells us that all four edges connecting $F_1$ to $F_2$ in $F'$ lie on a single circuit $\overline{\gamma}$ of $\overline{P}'$. By hypothesis $P'$ does not contain any such circuit, so $\overline{\gamma} \notin \overline{P}'$, i.e., $\overline{\gamma}$ is the circuit obtained by unifying $\gamma_1$ and $\gamma_2$. As $\gamma_1$ is contained in $F_1$ or $F_2$, $\gamma_2$ must contain all four edges connecting $F_1$ to $F_2$ in $F'$. But again, by hypothesis $P'$ does not contain any such circuit. This contradiction establishes the claim.
As \( P = V(Tch(P)) \) and \( P' = V(Tch(P')) \), the claim tells us that \( f_P \) is a bijection between the vertex sets of \( Tch(P) \) and \( Tch(P') \). The relationship between the two touch-graphs falls into one of the following cases. Case 1: The four edges connecting \( F_1 \) to \( F_2 \) in \( F \) all appear in one circuit of \( P \). In this case the four edges connecting \( F_1 \) to \( F_2 \) in \( F' \) all appear in one circuit of \( P' \), and \( f_P \) defines an isomorphism \( Tch(P) \cong Tch(P') \). Case 2: The four edges connecting \( F_1 \) to \( F_2 \) in \( F \) appear in two different circuits \( \gamma_1 \neq \gamma_2 \in P \), and the four edges connecting \( F_1 \) to \( F_2 \) in \( F' \) appear in two different circuits \( f_P(\gamma_1) \neq f_P(\gamma_2) \in P' \). In this case \( f_P(\gamma_1) \) and \( f_P(\gamma_2) \) are the circuits of \( F' \) obtained by “splicing” the portions of \( \gamma_1 \) and \( \gamma_2 \) within \( F_1 \) and \( F_2 \), using the four edges of \( F' \) that connect \( F_1 \) and \( F_2 \). Interchanging \( f_P(\gamma_1) \) and \( f_P(\gamma_2) \) if necessary, we may presume that \( f_P \) has been defined in such a way that \( f_P(\gamma_1) \) contains the \( F_1 \) portion of \( \gamma_1 \). Subcase 2a: The two portions of \( \gamma_1 \) are spliced to form \( f_P(\gamma_1) \) and the two portions of \( \gamma_2 \) are spliced to form \( f_P(\gamma_2) \). In this case, \( f_P \) defines an isomorphism \( Tch(P) \cong Tch(P') \). Subcase 2b: The \( F_1 \) portion of \( \gamma_1 \) and the \( F_2 \) portion of \( \gamma_2 \) are spliced to form \( f_P(\gamma_1) \), and \( f_P(\gamma_2) \) is formed by splicing the \( F_1 \) portion of \( \gamma_2 \) and the \( F_2 \) portion of \( \gamma_1 \). In this case the bijection \( f_P : V(Tch(P)) \rightarrow V(Tch(P')) \) preserves all adjacencies, except that each edge in \( Tch(P) \) between \( \gamma_1 \) (respectively, \( \gamma_2 \)) and a circuit \( \gamma \) contained in \( F_2 \) is replaced in \( Tch(P') \) with an edge between \( f_P(\gamma_2) \) (respectively, \( f_P(\gamma_1) \)) and \( \gamma \). If no circuit of \( P \) is contained in \( F_2 \) then this edge-swapping is immaterial and \( f_P \) defines an isomorphism \( Tch(P) \cong Tch(P') \). If no circuit of \( P \) is contained in \( F_1 \) then this edge-swapping tells us that an isomorphism \( Tch(P) \cong Tch(P') \) is given by the bijection obtained from \( f_P \) by interchanging the values of \( f_P(\gamma_1) \) and \( f_P(\gamma_2) \). If both \( F_1 \) and \( F_2 \) contain circuits of \( P \) then \( \{\gamma_1, \gamma_2\} \) is a vertex cut of \( Tch(P) \), which separates the circuits contained in \( F_1 \) from those contained in \( F_2 \). The edge-swapping is then a Whitney twist with respect to \( \{\gamma_1, \gamma_2\} \).

In every case \( Tch(P) \) and \( Tch(P') \) have isomorphic cycle matroids, and the matroid isomorphisms respect the identification of touch-graph edges with elements of \( V(F) = V(F') \). Consequently assertion 2 holds.

Examples of Cases 2a and 2b appear in Figs. 14 and 15, respectively. Transitions are indicated in the figures by plain, dashed and bold line styles; when a circuit traverses a vertex the style is maintained, but sometimes it is necessary to change the style in the middle of an edge. To indicate the correspondence between circuits and touch-graph vertices, some circuits and vertices are numbered; also, every circuit with dashed transitions is represented by a dashed vertex.
Figure 14: Corresponding circuit partitions in balanced mutations, whose touchgraphs are the same.

Figure 15: Corresponding circuit partitions in balanced mutations, whose touchgraphs are related by a Whitney twist.
If $F$ is a 4-regular graph, then Theorem 27 tells us indirectly that $M_{\tau}(F)$ determines the cycle matroids of all the touch-graphs of circuit partitions in $F$. There is a more direct way to see this property, using matroid duality. Here are three ways to describe the dual $M^*$ of a binary matroid $M$ on a set $S$:

1. The bases of $M^*$ are the complements of the bases of $M$.

2. The rank functions $r$ of $M$ and $r^*$ of $M^*$ are related by the fact that for every $A \subseteq S$, the nullity of $A$ in $M^*$ equals the corank of $S - A$ in $M$:

   $$|A| - r^*(A) = r(S) - r(S - A).$$

3. The subspace of $GF(2)^S$ spanned by the rows of a matrix representing $M$ is the orthogonal complement of the subspace spanned by the rows of a matrix representing $M^*$.

Let $P$ be a circuit partition of $F$, and recall that $\tau(P)$ is the subset of $\mathcal{T}(F)$ consisting of the transitions involved in $P$; let $M_{\tau}(P)$ denote the matroid obtained by restricting $M_{\tau}(F)$ to $\tau(P)$. Notice that both $M_{\tau}(P)$ and the cycle matroid of $Tch(P)$ have elements corresponding to the vertices of $F$. It turns out that these matroids are closely connected:

**Theorem 28** The cycle matroid of the graph $Tch(P)$ and the dual of $M_{\tau}(P)$ define the same matroid on $V(F)$.

We do not provide a proof of Theorem 28 because several equivalent results have already appeared in the literature. An algebraic form of Theorem 28 is this:

**Theorem 29** For any Euler system $C$ of $F$, let $M(C,P)$ be the submatrix of $M(C)$ consisting of columns corresponding to elements of $\tau(P)$. Then the orthogonal complement of the row space of $M(C,P)$ is the subspace of $GF(2)^{V(F)}$ spanned by the elementary cocycles of $Tch(P)$.

Here the elementary cocycle corresponding to a vertex of a graph $G$ is the subset of $E(G)$ consisting of the non-loop edges incident at that vertex. Jaeger [38] verified Theorem 29 in case $C$ and $P$ are compatible, i.e., $P$ does not involve the $\phi_C(v)$ transition at any vertex. The general case of Theorem 29 is proven in [74]. A detailed discussion of Theorem 28 is given in [20]: like Jaeger’s original version the argument of [20] requires that $C$ and $P$ be compatible, but Proposition 15 renders the requirement moot. Yet another equivalent version of Theorem 28 appears in Bouchet’s work on isotropic systems; see Section 6 of [12].

Theorem 28 tells us that the duals of the cycle matroids of the touch-graphs of all the different circuit partitions of $F$ are interwoven in the single matroid $M_{\tau}(F)$. This property might well remind the reader of Bouchet’s isotropic systems [12, 15] or the cycle family graphs of Ellis-Monaghan and Moffatt [28, 29, 30]; indeed, transition matroids are implicit in both these sets of ideas.
Theorem 28 also provides a new proof of a famous formula introduced by Martin [57] and explained and extended by Las Vergnas [51], which ties the Martin polynomial of a 4-regular plane graph to the diagonal Tutte polynomial of either of its associated “checkerboard” graphs. The formula is given in Corollary 32 below.

Theorem 30 Suppose \( F \) is a 4-regular graph with \( n \) vertices. Let \( P_1 \) and \( P_2 \) be compatible circuit partitions in \( F \), and for \( v \in V(F) \) let \( \tau_1(v), \tau_2(v) \) be the transitions involved in \( P_1 \) and \( P_2 \), respectively. If \( B_1 \) is any basis of \( M_\tau(P_1) \) then

\[
B_1 \cup \{ \tau_2(v) \mid \tau_1(v) \notin B_1 \}
\]

is a basis of \( M_\tau(F) \). It follows that

\[
r(\tau(P_1) \cup \tau(P_2)) = n \leq r(\tau(P_1)) + r(\tau(P_2)).
\]

Proof. Let \( \{v_1, ..., v_p\} = \{v \in V(F) \mid \tau_1(v) \notin B_1 \} \). For \( 0 \leq i \leq p \) let \( P'_i \) be the circuit partition of \( F \) such that

\[
\tau(P'_i) = \{ \tau_1(v) \mid v \notin \{v_1, ..., v_i\} \} \cup \{ \tau_2(v_1), ..., \tau_2(v_i) \}.
\]

We claim that \( |P'_i| = |P_1| - i \) for each \( i \); as \( P'_0 = P_1 \), the claim is true for \( i = 0 \). Suppose \( i \geq 1 \) and the claim holds for \( P'_0, ..., P'_{i-1} \). The three circuit partitions that involve the same transitions as \( P'_{i-1} \) at all vertices other than \( v_i \) are pictured in Fig. 13. Suppose \( P'_{i-1} \) is not the partition pictured in the middle. Let \( \tau(v_i) \) be the transition of the circuit partition pictured in the middle, and let \( P'' \) be the circuit partition with \( \tau(P'') = (\tau(P'_{i-1}) - \{\tau_1(v_i)\}) \cup \{\tau(v_i)\} \). Then \( |P''| = |P'_{i-1}| + 1 \), so the circuit-nullity formula tells us that \( r(\tau(P'')) = r(\tau(P'_{i-1})) - 1 \). Consequently, adjoining \( \tau_1(v_i) \) to \( \tau(P'') \) raises the rank. This is impossible, though, because \( B_1 \subseteq \tau(P'') \) and \( B_1 \) spans \( \tau(P_1) \). We conclude that \( P'_{i-1} \) is the partition pictured in the middle of Fig. 13 so \( P'_i \) is not; hence \( |P'_i| = |P'_{i-1}| - 1 \), and the claim holds for \( P'_i \).

The circuit-nullity formula now tells us that \( r(\tau(P'_i)) = i + r(\tau(P_1)) \) for each \( i \). In particular, \( r(\tau(P'_p)) = p + r(\tau(P_1)) = n - |B_1| + r(\tau(P_1)) \). As \( B_1 \) is a basis of \( M_\tau(P_1) \), \( |B_1| = r(\tau(P_1)) \) and hence \( r(\tau(P'_p)) = n \). As \( \tau(P'_p) \subseteq \tau(P_1) \cup \tau(P_2) \), we conclude that \( r(\tau(P_1) \cup \tau(P_2)) = n \).

The proof is completed by recalling that \( r(A_1 \cup A_2) \leq r(A_1) + r(A_2) \) for any two subsets of any matroid.

If a subset \( A \subseteq V(F) \) has the property that \( \tau_2(A) = \{ \tau_2(v) \mid v \in A \} \) is a circuit in \( M_\tau(P_2) \), then the contrapositive of Theorem 30 implies that \( \{ \tau_1(v) \mid v \notin A \} \) does not contain any basis of \( M_\tau(P_1) \). Consequently \( \tau_1(A) = \{ \tau_1(v) \mid v \in A \} \) is not contained in any basis of \( M_\tau(P_1)^* \); that is, \( \tau_1(A) \) contains a circuit of \( M_\tau(P_1)^* \). This property – if \( \tau_2(A) \) is a circuit in \( M_\tau(P_2) \) then \( \tau_1(A) \) contains a circuit of \( M_\tau(P_1)^* \) – is expressed by saying that the identity map of \( V(F) \) defines a weak map from \( M_\tau(P_2) \) to \( M_\tau(P_1)^* \). The weak map property is sufficient for our present purposes, but it is worth noting that in fact, the identity map of \( V(F) \) defines a strong map from \( M_\tau(P_2) \) to \( M_\tau(P_1)^* \). That is, if \( \tau_2(A) \)
is a circuit in $M_\tau(P_2)$ then $\tau_1(A)$ is a disjoint union of circuits of $M_\tau(P_1)^\tau$.
(Edmonds [27] proved an equivalent result, in which $Tch(P_1)$ and $Tch(P_2)$ are geometrically dual embedded graphs on a surface; the fact that Edmonds’ result yields a strong map was mentioned by Las Vergnas [52].) To see why this is true, consider that the circuit-nullity formula tells us that a circuit of $F$ yields a strong map was mentioned by Las Vergnas [52].

To see why this is true, consider that the circuit-nullity formula tells us that a circuit of $M_\tau(P_2)$ corresponds to a minimal circuit of $F$ involving only transitions from $P_2$; say the circuit is $v_1, h_1, h'_1, v_2, \ldots, h_{\ell-1}, h'_{\ell-1} = h'_0, v_\ell = v_1$. As $P_1$ and $P_2$ are compatible, none of the single transitions $h_{i-1}'$, $v_i$, $h_i$ appears in $P_1$. Hence each of them corresponds to an edge of $Tch(P_1)$, and the circuit corresponds to a closed walk in $Tch(P_1)$. A closed walk is a union of edge-disjoint minimal circuits, of course, and the minimal circuits of $Tch(P_1)$ correspond to circuits of its cycle matroid. Finally, Theorem [28] tells us that the identity map of $V(F)$ defines an isomorphism between the cycle matroid of $Tch(P_1)$ and $M_\tau(P_1)^\tau$.

**Corollary 31** In the situation of Theorem [30] these conditions are equivalent:

1. $r(\tau(P_1)) + r(\tau(P_2)) = n$.
2. Each of $M_\tau(P_1), M_\tau(P_2)$ is isomorphic to the dual of the other.
3. $M_\tau(P_1)$ and $M_\tau(P_2)$ define dual matroids on $V(F)$.
4. The matroid obtained by restricting $M_\tau(F)$ to $\tau(P_1) \cup \tau(P_2)$ is the direct sum $M_\tau(P_1) \oplus M_\tau(P_2)$.

**Proof.** As the rank of a direct sum is simply the total of the ranks of the summands, the implication 4 $\Rightarrow$ 1 follows directly from the fact that $n = r(\tau(P_1)) + r(\tau(P_2))$. For the opposite implication, suppose $r(\tau(P_1)) + r(\tau(P_2)) = n$. If there were any nontrivial intersection between the linear span of the columns of $M(C)$ corresponding to $\tau(P_1)$ and the linear span of the columns corresponding to $\tau(P_2)$, we would have $r(\tau(P_1)) + r(\tau(P_2)) > r(\tau(P_1) \cup \tau(P_2))$; $r(\tau(P_1)) + r(\tau(P_2)) = n$, so there is no such nontrivial intersection. We conclude that 1 $\Rightarrow$ 4.

The implication 3 $\Rightarrow$ 2 is obvious, and 2 $\Rightarrow$ 1 follows immediately from any of the descriptions of dual matroids mentioned at the beginning of the section.

It remains to verify the implication 1 $\Rightarrow$ 3. Suppose $r(\tau(P_1)) + r(\tau(P_2)) = n$, and let $B_1 = \{ \tau_1(a) \mid a \in A \}$ and $B_2 = \{ \tau_2(v) \mid v \notin A \}$. If $B_1$ is a basis of $M_\tau(P_1)$, then according to Theorem [30] $B_1 \cup B_2$ is an independent set of $M_\tau(F)$, so $B_2$ is an independent set of $M_\tau(P_2)$. As $|B_2| = n - |B_1|$ is the rank of $M_\tau(P_2)$, $B_2$ must be a basis of $M_\tau(P_2)$. Similarly, if $B_2$ is a basis of $M_\tau(P_2)$ then $B_1$ is a basis of $M_\tau(P_1)$.

Notice that Theorem [28] implies that if the equivalent conditions of Corollary [31] hold then $Tch(P_1)$ and $Tch(P_2)$ are dual graphs, and hence must be planar. It turns out that in this situation $F$ must be planar too; see Section 8 for details.

**Corollary 32** [27, 57] Suppose the equivalent conditions of Corollary [31] hold, and let $r_\tau$ be the rank function of the cycle matroid of $Tch(P_1)$. Let $A \subseteq V(F)$ be any subset, and let $P_A$ be the circuit partition of $F$ with $\tau(P_A) = \{ \tau_1(a) \mid a \in A \} \cup \{ \tau_2(v) \mid v \in V(F) - A \}$. Then $|P_A| - c(F) = r_\tau(V(F)) + |A| - 2r_T(A)$. 

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Proof. The circuit-nullity formula tells us that $|P_A| - c(F) = n - r(\tau(P_A))$, where $r$ is the rank function of $M_\tau(F)$. As $\tau(P_A) \subseteq \tau(P_1) \cup \tau(P_2)$, part 4 of Corollary 31 tells us that

$$r(\tau(P_A)) = r(\tau(P_A) \cap \tau(P_1)) + r(\tau(P_A) \cap \tau(P_2)).$$

Part 3 of Corollary 31 tells us that the corank of $\tau(P_A) \cap \tau(P_1)$ in $M_\tau(P_1)$ is the same as the nullity of $\tau(P_A) \cap \tau(P_2)$ in $M_\tau(P_2)$, i.e.,

$$r(\tau(P_1)) - r(\tau(P_A) \cap \tau(P_1)) = n - |A| - r(\tau(P_A) \cap \tau(P_2)).$$

It follows that

$$|P_A| - c(F) = n - r(\tau(P_A)) = n - r(\tau(P_A) \cap \tau(P_1)) - r(\tau(P_A) \cap \tau(P_2)) = r(\tau(P_1)) - r(\tau(P_A) \cap \tau(P_1)) + |A| - r(\tau(P_A) \cap \tau(P_1)).$$

This is the sum of the corank and the nullity of $\tau(P_A) \cap \tau(P_1)$ in $M_\tau(P_1)$, so it equals the sum of the nullity and the corank of $\tau(P_A) \cap \tau(P_1)$ in the dual matroid of $M_\tau(P_1)$. The formula of the statement follows, because Theorem 28 tells us that the cycle matroid of $Tch(P_1)$ and the dual matroid of $M_\tau(P_1)$ define the same matroid on $V(F)$. ■

6 4-regular graphs and ribbon graphs

In this section we provide a brief exposition of some ideas from topological graph theory. For thorough discussions we refer to the literature [8, 9, 22, 28, 29, 30, 37, 59].

A ribbon graph is a graph $G$ given with additional information regarding the edges and vertices of $G$. Each vertex $v \in V(G)$ is given with a prescribed order of the half-edges incident at $v$; the collection of these orders is called a rotation system on $G$. Also, each edge $e \in E(G)$ is labeled 1 or $-1$. Two ribbon graphs are equivalent if there is a graph isomorphism between them, such that the rotation systems and $\pm 1$ labels are related by a sequence of operations of these two types: (a) cyclically permute the half-edge order at a vertex, leaving the $\pm 1$ labels unchanged, and (b) reverse the half-edge order at a vertex, reversing all the $\pm 1$ labels of non-loop edges incident at that vertex (n.b. loops retain their $\pm 1$ labels).

The definition of equivalence is motivated by thinking of a ribbon graph $G$ as a blueprint for constructing a surface with boundary $S(G)$. The first step of the construction is to replace each vertex with a disk, whose boundary carries a preferred orientation. The second step is to choose, for each half-edge of $G$, two points on the boundary of the disk representing the vertex incident on that half-edge; the two points that represent a half-edge should be close together, and separated from those representing any other half-edge. Moreover, these pairs of points should be chosen so that as one walks around the boundary circle in the preferred direction, one encounters the pairs in the prescribed order of the
corresponding half-edges. The third step is to replace each edge with a narrow band, whose two ends correspond to the two half-edges of the edge. Each end of the band is connected to the disk corresponding to the vertex incident on that half-edge, by identifying the end with the short arc on the disk’s boundary bounded by the points corresponding to that half-edge. If the edge is labeled 1 then the band-ends are attached so that the band’s boundary may be oriented consistently with the preferred orientation(s) of the incident disk(s); if the edge is labeled $-1$ then the band-ends are attached so that the band’s boundary cannot be oriented consistently with the preferred orientation(s). See Fig. 16, where bands representing edges labeled 1 are on the left, and bands representing edges labeled $-1$ are on the right.

The purpose of this section is to present another description of equivalence classes of ribbon graphs, using circuit partitions in 4-regular graphs. This description appears at least implicitly in writings of Bouchet [17], Edmonds [27] and Tutte [76], but it does not seem to have been used in more recent work.

Suppose a ribbon graph $G$ is given, with disks $D_1, \ldots, D_k$ corresponding to the vertices of $G$ and bands $B_1, \ldots, B_n$ corresponding to the edges of $G$. The medial graph $F = F(G)$ is a 4-regular graph constructed from $G$ as follows. First, $V(F) = E(G)$. Second, the four half-edges of $F$ incident on a vertex $v_i$ correspond to the four corners of the band $B_i$. Finally, the edges of $F$ are obtained by pairing together the half-edges that correspond to consecutive points on a disk boundary, but do not correspond to the same band-end. See Fig. 17 for an example.

There are two natural circuit partitions in the medial graph, which we denote $\delta$ and $\varepsilon$; like any circuit partitions, they are determined by choosing the appropriate transitions at each vertex. The $\delta$ transition at a vertex $v_i$ of $F(G)$ pairs together the half-edges corresponding to the same end of the band $B_i$. We use the letter $\delta$ because the circuits of this partition correspond to the disks $D_1, \ldots, D_k$. The $\varepsilon$ transition at $v_i$ pairs together the half-edges corresponding to the same edge of the band $B_i$. The circuits of the $\varepsilon$ partition correspond to the boundary curves of $S(G)$.

The $\delta$ and $\varepsilon$ circuit partitions of the example of Fig. 17 are indicated in Fig. 18. Circuits are indicated using the convention that when a circuit is followed through a vertex, the style (dashed or plain) is maintained; however it is sometimes necessary to change the style in the middle of an edge, to make sure that the transitions are indicated unambiguously.

It is a simple matter to reverse the construction. Suppose a 4-regular graph $F$ is given with two compatible circuit partitions, i.e., two circuit partitions that do not involve the same transition at any vertex. Label the two compatible circuit partitions $\delta$ and $\varepsilon$, and let $\{\Delta_1, \ldots, \Delta_6\}$ be the circuit partition of $F$ determined by the $\delta$ transitions. Let $V(F) = \{v_1, \ldots, v_n\}$ and for each $j \in \{1, \ldots, k\}$, let $\Delta_j$ be $v_j, h_j, h'_j, v_j$, $h_j, h'_j, v_j$, $\ldots$, $h_j(\ell_j - 1)$, $v_j = v_j$. Let $D_1, \ldots, D_k$ be pairwise disjoint disks and for each $j \in \{1, \ldots, k\}$, let $w_{j1}, w'_{j1}, w_{j2}, w'_{j2}, \ldots$, $w_{j(\ell_j - 1)}$, $w'_{j(\ell_j - 1)}$ be $2(\ell_j - 1)$ distinct, consecutive points on the boundary circle of $D_j$. Also, let $w_{j\ell_j} = w_{j1}$. Associate a band $B_i$ with each vertex $v_i$
Figure 16: Disks and bands representing edges labeled 1 and $-1$, respectively.

Figure 17: At the top, a ribbon graph $G$. In the middle, the disks and bands of $S(G)$. At the bottom, the medial graph $F(G)$. 
Figure 18: The $\delta$ and $\varepsilon$ circuit partitions of the example of Fig. 17.

of $F$; if $v_i = v_{ab} = v_{cd}$ then one end of $B_i$ is the arc from $w'_{a(b-1)}$ to $w_{ab}$ on $D_a$, and the other end of $B_i$ is the arc from $w'_{c(d-1)}$ to $w_{cd}$ on $D_c$. The $\delta$ and $\varepsilon$ circuit partitions are compatible, so the $\varepsilon$ transition at $v_i$ pairs $h'_{a(b-1)}$ with one of $h'_{c(d-1)}$, $h_{cd}$, and pairs $h_{ab}$ with the other one of $h'_{c(d-1)}$, $h_{cd}$. One edge of the band $B_i$ should connect $w'_{a(b-1)}$ to one of $w'_{c(d-1)}$, $w_{cd}$ and the other edge of $B_i$ should connect $w_{ab}$ to the other one of $w'_{c(d-1)}$, $w_{cd}$, as dictated by the $\varepsilon$ transition.

There are several reasons that representing a ribbon graph $G$ using circuit partitions in the medial graph $F(G)$ is of value.  

1. Clearly if $G$ and $G'$ are ribbon graphs with the same medial, then they differ only in their choices of which of the three transitions at each vertex are designated $\delta$ and $\varepsilon$. (No particular designation is used for the third transition.) This observation provides a new explanation of Chmutov’s theory of partial duality [22] and Ellis-Monaghan’s and Moffatt’s more general twisted duality [28, 30]. Namely: partial duals are obtained by interchanging the ($\delta, \varepsilon$) designations at some vertices, and twisted duals are obtained by permuting the ($\delta, \varepsilon$, other) designations at some vertices. A distinctive feature of these duality theories is their reliance on surface geometry; this reliance can be problematic because twisted duals of ribbon graphs are not naturally embedded in the same surface. They are, however, represented by different pairs of circuit partitions
2. In particular, the compatible circuit partition representation helps one to understand a theorem of Ellis-Monaghan and Moffatt [28], that two ribbon graphs are twisted duals if and only if they have the same medial graph. Their proof seems to require embedding the ribbon graphs, constructing embedded medials, and then “disembedding” the medials. The compatible circuit partition representation gives an alternative way to think about this theorem.

3. As we discuss in the next section, topological properties of the surface \( S(G) \) are represented by combinatorial properties of the medial graph \( F(G) \) and transition matroid \( M_\tau(F(G)) \).

Before closing this section, we recall a result of Jaeger [39] that was mentioned in Section 2: every cographic matroid is represented by a square \( GF(2) \)-matrix whose off-diagonal entries agree with some interlacement matrix \( A(I(C)) \).

The discussion above contains the following outline of a proof of Jaeger’s theorem. If \( G \) is a graph then we may consider it as a ribbon graph, with any rotation system. \( G \) is then isomorphic to the touch-graph of the circuit partition of \( F(G) \) given by the \( \delta \) transitions. As discussed in Section 5, the dual of the cycle matroid of \( G \) is then isomorphic to the submatroid of \( M_\tau(F(G)) \) consisting of the various elements \( \delta(v), v \in V(F(G)) \). This outline for a proof of Jaeger’s theorem is detailed in [20], without any explicit reference to topological graph theory. J. A. Ellis-Monaghan noticed it, and was kind enough to suggest that the first part of the argument might be modified to involve ribbon graphs. The discussion above verifies her insight.

7 Topological Tutte polynomials

Several authors have studied polynomials associated with a ribbon graph \( G \), which combine combinatorial information regarding \( G \) and topological information regarding \( S(G) \). Two such polynomials were introduced in the 1970s by Las Vergnas [51] and Penrose [62], but broader interest in such polynomials seems to have been relatively quiet until the work of Bollobás and Riordan [8, 9]. (In contrast, the intervening decades were a time of very intense research regarding knot polynomials.) In particular, the discussion of duality in [9] has stimulated several interesting developments, including the geometrically inspired partial duality of Chmutov [22] and twisted duality of Ellis-Monaghan and Moffatt [28, 30]. We do not attempt to provide a detailed account of these topics here. Instead we content ourselves with the obvious observation that in the representation of ribbon graphs using compatible circuit partitions discussed in the previous section, the (geometric) dual ribbon graph of Bollobás and Riordan [9] is represented by interchanging the \( \delta \) and \( \epsilon \) transitions at all vertices of \( F(G) \), the dual with respect to an edge of Chmutov [22] is represented by interchanging the \( \delta \) and \( \epsilon \) transitions at a single vertex of \( F(G) \), and the half-twist of an edge of Ellis-Monaghan and Moffatt [28] is represented by interchanging the \( \epsilon \) transition at a vertex with the non-\( \delta \), non-\( \epsilon \) transition.

As discussed in the introduction, Theorem [1] implies that any polynomial
which has a description as a vertex-weighted Martin polynomial of a 4-regular graph \( F \) can be derived from a parametrized Tutte polynomial of \( M_4(F) \). Theorems 9 and 10 follow directly, because the polynomials mentioned in these theorems have such descriptions. For instance, the homflypt polynomial \([33, 63]\) has a circuit partition model due to Jaeger [42], which has been extended to the Kauffman polynomial [46] by Kauffman [48].

At first glance, topological Tutte polynomials like those of Bollobás and Riordan [8, 9] may seem to fall into a different category, because their definitions do not involve circuit partitions. Instead, the topological Tutte polynomials of a ribbon graph \( G \) are defined using combinatorial properties of \( G \) and topological properties of \( S(G) \). It turns out, though, that almost all of this information is available in the transition matroid of \( F(G) \).

**Proposition 33** Let \( G \) be a ribbon graph. For each \( v \in V(F(G)) \), let \( \delta(v) \) and \( \varepsilon(v) \) denote the \( \delta \) and \( \varepsilon \) transitions at \( v \).

1. \( G \), the surface \( S(G) \) and the medial \( F(G) \) all have the same number of connected components.

2. The boundary curves of \( S(G) \) correspond to the circuits in the \( \varepsilon \) partition.

3. The cycle matroid of \( G \) is the dual matroid of \( M_4(\delta) \).

4. For each subset \( A \subseteq V(F(G)) \), let \( P_A \) be the circuit partition of \( F(G) \) that involves \( \delta \) transitions at vertices in \( A \), and \( \varepsilon \) transitions at vertices not in \( A \). Then \( S(G) \) is orientable if and only if \( |P_A| \neq |P_{A-\{a\}}| \) \( \forall A \subseteq V(F(G)) \) \( \forall a \in A \).

**Proof.** For part 1, note that two vertices are connected by a walk in \( G \) if and only if the corresponding disks are connected by a sequence of disks and bands in \( S(G) \), and such a sequence of disks and bands corresponds to a walk in \( F(G) \). The walk in \( F(G) \) may be considerably longer than the original walk in \( G \), of course, because vertices of \( G \) (which may be traversed in a single step) are replaced with segments of the \( \delta \) circuits of \( F(G) \).

Part 2 is clear, as the circuits of the \( \varepsilon \) partition are defined to follow the boundary of \( S(G) \).

Part 3 is discussed in Section 5.

For part 4, consider that \( S(G) \) is orientable if and only if it is possible to coherently orient the boundaries of all the disks and bands involved in its construction. Clearly this holds if and only if it is possible to choose edge-directions in \( F(G) \) that are respected by all the \( \delta \) and \( \varepsilon \) transitions. If it is possible to choose such edge-directions, then it must be that \( |P_A| \neq |P_{A-\{a\}}| \) for every choice of \( A \subseteq V(F(G)) \) and \( a \in A \). See Fig. 19 where the circuit partition pictured on the left is one of \( P_A, P_{A-\{a\}} \) and the circuit partition pictured in the middle is the other of \( P_A, P_{A-\{a\}} \); clearly the partition pictured in the middle includes one more circuit.

For the converse, recall Kotzig’s observation that \( F \) must have an Euler system \( C \) that involves only \( \delta \) and \( \varepsilon \) transitions [54]. The recursive construction
of such an Euler system is simple. Begin with any circuit partition $P_0$ that involves only $\delta$ and $\varepsilon$ transitions. If $P_0$ is not an Euler system, there is a vertex $v$ at which two distinct circuits of $P_0$ are incident. The transition of $P_0$ at $v$ is either $\delta(v)$ or $\varepsilon(v)$; let $P_1$ be the circuit partition that involves the same transition as $P_0$ at every other vertex, and involves the different element of $\{\delta(v), \varepsilon(v)\}$. Then $|P_1| = |P_0| - 1$, and $P_1$ inherits the property that it involves only $\delta$ and $\varepsilon$ transitions. Repeat this process until an Euler system $C$ is obtained.

As $C$ involves only $\delta$ and $\varepsilon$ transitions, $C = P_B$ for some $B \subseteq V(F(G))$. Choose orientations for the circuits in $C$, and use these orientations to direct all the edges of $F(G)$. If $b \in B$ and $\varepsilon(b)$ is inconsistent with these edge-directions then Fig. 19 indicates that $|P_B| = |P_{B-\{b\}}|$. If $v \notin B$ and $\delta(v)$ is inconsistent with these edge-directions then Fig. 19 indicates that $|P_B| = |P_{B\cup\{v\}}|$. Consequently if $|P_A| \neq |P_{A-\{a\}}|$ $\forall A \subseteq V(F(G)) \forall a \in A$ then the edge-directions must be respected by all the $\delta$ and $\varepsilon$ transitions.

As the circuit-nullity formula ties the number of circuits in a circuit partition $P$ to the rank of $\tau(P)$ in the matroid $M_\tau(F(G))$, it follows that $M_\tau(F(G))$ contains enough information to determine the cycle matroid of $G$, the number of boundary curves in $S(G)$, and the orientability of $S(G)$. The matroid cannot detect the number of connected components, of course, because it is invariant under connected sums and separations. Theorem 11 follows, for the various topological Tutte polynomials are all defined by subset expansions in which the contribution of a subset $X \subseteq E(G)$ is determined by the cycle matroid of $G$, the cycle matroid of the geometric dual, and topological characteristics of the surface $S(G[X])$, where $G[X]$ is the graph obtained from $G$ by removing edges not in $X$. When we remove an edge from $G$ the effect on the medial $F(G)$ is simply to perform the detachment corresponding to the $\delta$ transition at the corresponding vertex. As discussed in Section 2, the transition matroid of the detached graph is a minor of $M_\tau(F(G))$, so all information about that matroid is present in $M_\tau(F(G))$.

We should note that all the information about $G$ and $S(G)$ discussed in Proposition 33 is contained in the submatroid of $M_\tau(F(G))$ consisting of the $\delta$ and $\varepsilon$ transitions. The Euler systems of $F(G)$ involving only $\delta$ and $\varepsilon$ transitions are determined by this submatroid, and these Euler systems in turn determine a $\Delta$-matroid. The corresponding $\Delta$-matroid version of Proposition 33 is due to Bouchet [17], and Proposition 33 could be deduced from the $\Delta$-matroid version simply by observing that the matroid $M_\tau(F(G))$ contains enough information.

Figure 19: Three circuit partitions that differ at only one vertex. The one on the right is inconsistent with the indicated edge-directions.
to determine all the $\Delta$-matroids associated with ribbon graphs with $F(G)$ as medial. This observation is a special case of the fact that all binary $\Delta$-matroids are determined by isotropic matroids of graphs; see [75] for details.

8 Planar 4-regular graphs

If a 4-regular graph $F$ is imbedded in the plane, then the complementary regions can be colored black or white, in such a way that regions that share an edge of $F$ have different colors. (This observation dates back to the founding of topology [55].) Let $P_B$ and $P_W$ be the circuit partitions of $F$ that give the boundaries of the black and white regions, respectively. Euler’s formula tells us that

$$2c(F) = |V(F)| - |E(F)| + |P_B| + |P_W| = -n + |P_B| + |P_W|,$$

and the circuit-nullity formula tells us that

$$|P_B| - c(F) = n - r(\tau(P_B)) \text{ and } |P_W| - c(F) = n - r(\tau(P_W)).$$

Combining these formulas we conclude that

$$r(\tau(P_B)) + r(\tau(P_W)) = 2n + 2c(F) - |P_B| - |P_W| = n.$$ 

That is, $P_B$ and $P_W$ satisfy the equivalent conditions of Corollary 31.

Conversely, suppose $F$ is a 4-regular graph with circuit partitions $P_1$ and $P_2$, which satisfy the equivalent conditions of Corollary 31. Let $G$ be the ribbon graph constructed from $F$ using $P_1$ and $P_2$ as the $\delta$ and $\varepsilon$ circuit partitions (respectively), as in Section 6. $S(G)$ is a surface with boundary constructed from $|P_1|$ disks and $n$ bands, which has $|P_2|$ boundary curves. Consequently, if $S'(G)$ is the closed surface obtained from $S(G)$ by attaching a disk to each boundary curve, then the Euler characteristic of $S'(G)$ is $|P_1| - n + |P_2|$. The circuit-nullity formula tells us that

$$|P_1| - n + |P_2| = c(F) + n - r(\tau(P_1)) - n + c(F) + n - r(\tau(P_2))$$

so since $r(\tau(P_1)) + r(\tau(P_2)) = n$, the Euler characteristic of $S'(G)$ is $2c(F)$. As $c(F)$ is the number of connected components of $S'(G)$, we conclude that each connected component of $S'(G)$ has Euler characteristic 2; that is, each connected component is a sphere. $F$ is imbedded in $S'(G)$, so $F$ is planar.

We have proven the following.

Proposition 34 A 4-regular graph $F$ is planar if and only if it has a pair of compatible circuit partitions that satisfy the equivalent conditions of Corollary 31.

The intent of this proposition is not to provide a new criterion for planarity, but merely to indicate that $M_\tau(F)$ incorporates information connected to familiar planarity criteria. For instance, the reader will certainly not be surprised

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that Corollary 31 indicates a connection between planarity and duality. It takes only a little longer to see that Corollary 31 is also connected to the following planarity criterion, which is part of several solutions of the Gauss crossing problem that have appeared in the literature [26, 33, 63].

**Corollary 35** Let $F$ be a 4-regular graph. Then $F$ is planar if and only if it has an Euler system whose interlacement graph is bipartite.

**Proof.** Suppose $F$ is planar, and let $P_1$ and $P_2$ be compatible circuit partitions of $F$ that satisfy the equivalent conditions of Corollary 31. For each $v \in V(F)$, let $\tau_1(v)$ and $\tau_2(v)$ be the transitions involved in $P_1$ and $P_2$, respectively. Let $B_1$ be a basis of $M_{\tau}(P_1)$, let $V_1 = \{ v \in V(F) \mid \tau_1(v) \in B_1 \}$, and let $V_2 = V(F) - V_1$. According to condition 3 of Corollary 31, $B_2 = \{ \tau_2(v) \mid v \in V_2 \}$ is a basis of $M_{\tau}(P_2)$. Condition 4 of Corollary 31 then tells us that $B_1 \cup B_2$ is a basis of $M_{\tau}(F)$.

Let $C$ be the Euler system of $F$ with $\tau(C) = B_1 \cup B_2$. Let

$$A(\mathcal{I}(C)) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where the rows of $A_{ij}$ correspond to vertices from $V_i$ and the columns of $A_{ij}$ correspond to vertices from $V_j$. By Definition 16

$$r(M_{\tau}(P_1)) = r \begin{pmatrix} I_1 & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

where $I_1$ is a $|V_1| \times |V_1|$ identity matrix and $A_{22}$ is obtained from $A_{22}$ by placing a 1 at each diagonal entry corresponding to a vertex $v \in V_2$ where $\tau_1(v) = \psi_C(v)$. As $r(M_{\tau}(P_1)) = |B_1| = |V_1|$, $A_{22}$ cannot have any nonzero entry. It follows that no two vertices of $V_2$ are neighbors in $\mathcal{I}(C)$. Similarly,

$$r(M_{\tau}(P_2)) = r \begin{pmatrix} A_{11} & 0 \\ A_{21} & I_2 \end{pmatrix},$$

where $I_2$ is a $|V_2| \times |V_2|$ identity matrix and $A_{11}$ agrees with $A_{11}$ off the diagonal.

As $r(M_{\tau}(P_2)) = |V_2|$, $A_{11}$ cannot have any nonzero entry. It follows that no two vertices of $V_1$ are neighbors in $\mathcal{I}(C)$, so $\mathcal{I}(C)$ is bipartite.

Suppose conversely that $F$ has an Euler system $C$ whose interlacement graph is bipartite. Let $V(F) = V_1 \cup V_2$, so that $V_1 \cap V_2 = \emptyset$ and every edge of $\mathcal{I}(C)$ connects a vertex from $V_1$ to a vertex from $V_2$. Then

$$A(\mathcal{I}(C)) = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix},$$

where $A_{12}$ and $A_{21}$ are transposes. For $i \in \{1, 2\}$ let $P_i$ be the circuit partition of $F$ with

$$\tau(P_i) = \{ \phi_C(v) \mid v \in V_i \} \cup \{ \chi_C(v) \mid v \notin V_i \}. $$
Then Definition 16 implies that 
\[ r(M_\tau(P_1)) = r \begin{pmatrix} I_1 & A_{12} \\ 0 & 0 \end{pmatrix} \] and 
\[ r(M_\tau(P_2)) = r \begin{pmatrix} 0 & 0 \\ A_{21} & I_2 \end{pmatrix}, \]
where \( I_1 \) and \( I_2 \) are identity matrices. It follows that 
\[ r(M_\tau(P_1)) + r(M_\tau(P_2)) = n, \]
so \( P_1 \) and \( P_2 \) satisfy the equivalent conditions of Corollary 31.

9 Conclusion

The transition matroids of 4-regular graphs provide unified descriptions of several objects of graph theory and knot theory. One virtue of this unified context is to allow the application to these objects of matroid techniques, which have been well studied. Algorithmic and combinatorial properties of parametrized Tutte polynomials of matroids \([7, 32, 69, 70, 82]\) provide activities descriptions, complexity results and substitution techniques that apply directly to all the different polynomials mentioned in the introduction. Much remains to be done, though, to provide details of these applications, and to relate them to results already in the literature. For example, the circuit-nullity formula tells us that some bases of transition matroids correspond to Euler systems, so it seems natural to guess that matroidal basis activities can be used to explain the significance for the Bollobás-Riordan polynomial of the quasi-trees introduced by Champanerkar, Kofman and Stoltzfus \([21]\). Similarly, it seems natural to guess that formulas for parametrized Tutte polynomials of series/parallel extensions of matroids can be used to explain the tangle substitution formula of Jin and Zhang \([44]\) for the homflypt polynomial. Guessing that there are such explanations is not the same as actually providing them, though.

10 Dedication

This paper is dedicated to the memory of Michel Las Vergnas, in appreciation of the beauty and importance of his many contributions to the theory of graphs and matroids. In particular, his papers on circuit partitions and embedded graphs from the 1970s and 1980s \([51, 52, 53]\) have proven to be crucial to the development of the theory of 4-regular graphs, and they inspire the results presented here.

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