**Radial quantum deformation for Schrodinger equation on Coulomb potential by using Hypergeometric method**

**A Suparmi*, D A Dianawati† and C Cari†**

†Physics Departement, Graduate Program, SebelasMaret University
Jl. Ir. Sutami 36A KeninganJebres Surakarta 57126, INDONESIA

*E-mail: soeparmi@staff.uns.ac.id

**Abstract.** The Hypergeometric method was used to obtain the solution of Schrodinger equation for radial quantum deformation on Coulomb potential. The Schrodinger equation was reduced into the general form of Hypergeometric function with variable and parameter substitutions. As a result, the energy was calculated from the energy equation and wave function was visualized by Matlab R2013a software. The decrease of energy values causes the increase of quantum number and quantum deformation parameters, while the wave functions have the depth of deep amplitude by the increase of quantum number and quantum number parameters.

**1. Introduction**

In quantum mechanics, the Schrodinger equation has an important role to present electron [1,2]. The Schrodinger equation was used to describe about the motion of electron to proton in Coulomb potential that the potential energy system was the bound energy of electron to nucleus [1,3].

The behavior of particle in Schrodinger equation was influenced by some potentials, especially shape invariance potential, such as Coulomb, Three-dimension Harmonic Oscillator, Rosen Morse, Manning Rosen, Pöschl-Teller [3,4], Eckart, Hulthen[2] and Kratzer potentials [5]. In classical mechanics and quantum mechanics, Coulomb potential was used to solve the relativistic and non-relativistic equations [6]. The quantum deformation has many applications in nuclear and high energy physics, quantum of hall effect, black hole and cosmic string [7].

The solution of Schrodinger was obtained by using some methods. The method to solve the Schrodinger equation for shape invariance potentials were factorization, Nikiforov-Uvarov [8,9], and Hypergeometric methods [3]. The Hypergeometric method was used to obtain the solution of Schrodinger equation into the Hypergeometric function by variable and parameter substitution [10-12].

This research will obtain and analyzed the energy and wave function as the solution of Schrodinger equation with radial quantum deformation for Coulomb potential by using Hypergeometric method that will be calculated and visualized by Matlab R2013a software. This paper is organized as follows. Section 2 is experimental that contain Schrodinger equation, q-deformed of quadratic radial momentum, Coulomb potential and Hypergeometric method. Section 3 is result and discussion, and the last is conclusion in Section 4.
2. Experimental

2.1. Schrödinger Equation
The Schrödinger equation could be expressed by [2,3,13]

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + V(r)\right)\psi_{nlm}(r,\theta,\phi) = E\psi_{nlm}(r,\theta,\phi)$$

with \(m, \nabla^2, V(r)\) and \(\psi\) are mass of particle, spherical coordinate, spherical symmetric potential and wave function, respectively.

The Schrödinger equation in the radial part was written as [14,15]

$$\left(-\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr}\right) + \frac{l(l+1)}{2mr^2} V(r)\right)G_n(r) = EG_n(r)$$

The radial part of Schrödinger equation with centrifugal term approximate as

$$\frac{d^2 G(r)}{dr^2} + \frac{2m}{\hbar^2} \left(E - V(r) - \frac{l(l+1)\hbar^2}{2mr^2}\right)G(r) = 0$$

with \(V(r)\) is the central potential.

2.2. QDeformed of Quadratic Radial Momentum
The q-deformation equation was written as [16,17]

$$\frac{d^2}{dr^2} = D_x^q$$

with

$$D_x^q = \left(1 + qx^2\right) \frac{d}{dx}$$

By setting new variable as

$$r = \frac{x}{\sqrt{q}} \rightarrow \frac{dx}{dr} = \sqrt{q}$$

By using variable substitution equations (6) into (5), the equation (4) becomes

$$D_x^q = \frac{d^2}{dx^2} = q(1 + x^2)^2 \frac{d^2}{dx^2} + 2qx(1 + x^2) \frac{d}{dx}$$

Equation (7) was the quadratic of radial momentum quantum deformation equation.

2.3. Coulomb Potential
The behavior of electron in Hydrogen atoms was confluence by Coulomb potential [4]. The symmetric potential as Coulomb potential was expressed by [3,6,18]

$$V(r) = -\frac{Ze^2}{r}$$

By inserting equations (6) into (8), the Coulomb potential was rewritten by

$$V(x) = -\frac{Ze^2}{x\sqrt{q}}$$

By inserting equations (9) and (7) into (3), the Schrödinger equation for Coulomb potential with q-deformed of quadratic radial momentum as
Equation (10) will be changed into the Hypergeometric function equation with variable substitution to obtain the solution of Schrodinger equation.

2.4. Hypergeometric Method
The differential of Hypergeometric equation could be expressed by [4,19]
\[
s(1-s)\frac{\partial^2 g}{\partial s^2} + \left[c-(a+b+1)s\right]\frac{\partial g}{\partial s} - abg = 0
\]
(11)
The simplest solution for equation (11) with regular singular point in \(z = 0\) as series form
\[
g = s^e \sum_{n=-\infty}^{\infty} a_n s^n
\]
(12)
By inserting equations (12) into (11), we have the solution of wave function for Hypergeometric function that was written as [10,11]
\[
2F_1(a,b;c;s) = g_1(s) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} s^n
\]
(13)
By setting \(a = -n\) or \(b = -n\), the solution of equation (13) becomes the solution of polynomial series rank \(n\) to obtain the energy system.

3. Results and Discussions
By using variable substitution to obtain the solution of equation (10) as
\[
x = \cot y \rightarrow dx = \cos e c^2 y dy
\]
(14)
By inserting equations (14) into (10), the equation (10) becomes
\[
\frac{d^2 G(y)}{dy^2} + l(l+1)\tan^2 y G(y) + \left[\frac{2mE}{qh^2} + \frac{2mZe^2}{\sqrt{q}h^2}\tan y\right] G(y) = 0
\]
(15)
By setting \(\tan^2 y = \sec^2 y - 1\), the equation (15) becomes
\[
\frac{d^2 G(y)}{dy^2} + l(l+1)\cos^2 y G(y) + \frac{2mZe^2}{\sqrt{q}h^2}\tan y - \left[l(l+1) - \frac{2mE}{qh^2}\right] G(y) = 0
\]
(16)
with
\[
l(l+1) = \nu(\nu + 1) \rightarrow l = \nu
\]
(17)
\[
\frac{2mZe^2}{\sqrt{q}h^2} = 2q' \rightarrow q' = \frac{mZe^2}{\sqrt{q}h^2}
\]
(18)
\[
E' = \left[l(l+1) - \frac{2mE}{qh^2}\right]
\]
(19)
Equation (16) becomes
\[
\frac{d^2 G(y)}{dy^2} + \left[\frac{\nu(\nu + 1)}{\cos^2 y} + 2q'\tan y\right] G(y) - E'G(y) = 0
\]
(20)
The equation (20) was the general form of differential Schrodinger equation with Rosen Morse II potential [20] that would be solved by Hypergeometric method.

The equation (20) would be reduced into the differential of Hypergeometric function with the variable approximate as
\[ \tan \psi = i(1-2s) \rightarrow \frac{ds}{dy} = 2is(1-s) \]  

By inserting equations (21) into (20), equation (20) becomes

\[ s(1-s)\frac{d^2G(s)}{ds^2} + (1-2s)\frac{dG(s)}{ds} + \left[ -\nu(\nu+1) + \frac{(E'-2q'i)}{4s} + \frac{(E'+2q'i)}{4(1-s)} \right] G(s) = 0 \]  

with

\[ \alpha^2 = \frac{(E'-2q'i)}{4} \]  

\[ \beta^2 = \frac{(E'+2q'i)}{4} \]  

From the equation (12), the wave function in equation (22) becomes

\[ G = s^\alpha (1-s)^\beta g(s) \]  

By inserting equations (25) into (22), the form of equation (22) could be rewritten as

\[ s(1-s)g'' + \left((2\alpha+1) - (2\alpha + 2\beta + 2)s\right)g' + \left(-\nu(\nu+1) + (\alpha + \beta)(\alpha + \beta + 1)\right)g = 0 \]  

By comparing equation (26) with (11), we have parameters

\[ a = \alpha + \beta - \nu \]  

\[ b = \alpha + \beta + \nu + 1 \]  

\[ c = 2\alpha + 1 \]  

The energy equation would be obtained from equation (27) with the condition \( a = -n \) that would be written as

\[ E' = (\nu - n)^2 - \frac{q'^2}{(\nu - n)^2} \]  

By inserting equations (18) into (30), we have

\[ E = \frac{q\hbar^2}{2m} \left( l(l+1) - (\nu - n)^2 + \frac{q'^2}{(\nu - n)^2} \right) \]  

with \( \nu \) and \( q' \) parameters in equation (17) and (18).

The equation (30) was the energy equation for Schrödinger equation on Coulomb potential with \( q \)-deformed of quadratic radial momentum by Hypergeometric method that the result was shown in Table 1.

**Table 1. The Energy for Coulomb Potential with Q-Deformed of Quadratic Radial Momentum**

| \( n \) | \( q \) | \( E_n \) | \( n \) | \( q \) | \( E_n \) | \( n \) | \( q \) | \( E_n \) |
|---|---|---|---|---|---|---|---|---|
| 0.01 | -0.005 | -0.005 | 0.01 | -0.020 | -0.019 | 0.01 | -0.045 | -0.044 |
| 0.04 | -0.020 | -0.020 | 0.04 | -0.080 | -0.079 | 0.04 | -0.180 | -0.179 |
| 0.07 | -0.035 | -0.035 | 0.07 | -0.140 | -0.139 | 0.07 | -0.315 | -0.314 |
| 1 | 0.1 | -0.050 | 2 | 0.1 | -0.200 | 3 | 0.1 | -0.450 |
| 0.4 | -0.200 | -0.200 | 0.4 | -0.800 | -0.799 | 0.4 | -1.800 | -1.799 |
| 0.7 | -0.350 | -0.350 | 0.7 | -1.400 | -1.399 | 0.7 | -3.150 | -3.149 |
| 1 | -0.500 | -0.500 | 1 | -2.000 | -1.999 | 1 | -4.500 | -4.499 |
Table 1 shows the energy for Schrodinger equation on Coulomb potential with quantum deformation and quantum number variations. The energy result was negative that indicate the bound energy on each atom’s shell.

Overall, the decrease of energy values causes the increase of quantum number and quantum deformation parameters. The increase of quantum number indicates particle has the probability to avoid from nucleus when the bound energy value decrease. Besides that, the quantum deformation indicates the gap energy value. The increase of quantum deformation parameter causes the decrease of energy value and the the increase of gap energy.

Furthermore, the wave function for Schrodinger equation could be obtained by inserting parameters in equations (21), (23), (24), (27-29) into (25), the equation (25) becomes

\[
G_n = \frac{1+i \tan \gamma}{2} \left( \frac{1-i \tan \gamma}{2} \right)^4 C'(-1)^n (2\alpha + 1) \frac{1+i \tan \gamma}{2}
\]

The equation (32) was the wave function equation for Schrodinger equation on Coulomb potential with q-deformed of quadratic radial momentum. The result of wave function of radial part equation was shown in Table 2 with quantum number variation. Then, the wave function was visualized by Matlab R2013a software that was shown in Figure 1, Figure 2 and Figure 3.

**Table 2.** The Wave Function for Schrodinger Equation on Coulomb potential with Q-Deformed of Quadratic Radial Momentum

| \( n \) | \( G_n(y) \) |
|---|---|
| 0 | \( F_0(x) = \frac{1+i \tan \gamma}{2} \left( \frac{1-i \tan \gamma}{2} \right)^4 \)
| 1 | \( F_1(x) = -\frac{1+i \tan \gamma}{2} \left( \frac{1-i \tan \gamma}{2} \right)^4 \times C'(2\alpha + 1) \left( 2\alpha + 2 \right) \left( 1+i \tan \gamma \right) \)
| 2 | \( F_2(x) = \frac{1+i \tan \gamma}{2} \left( \frac{1-i \tan \gamma}{2} \right)^4 \times C'(2\alpha + 1)(2\alpha + 2) \left( 2\alpha + 2 \right) \left( 1+i \tan \gamma \right)^2 \)
Figure 1. The wave function for Coulomb potential in \( q = 0.01 \)

Figure 2. The wave function for Coulomb potential in \( q = 0.1 \)
Figure 3. The wave function for Coulomb potential in $q = 1$

The wave function in Figure 1, Figure 2 and Figure 3 show about the vibration motion of particle that influence by Coulomb potential. The wave function moves from the high to low amplitude and back from low to high amplitude. It means that the motion of wave function was periodic and the motion of wave function as radius function. Besides that, the increase of quantum number and quantum deformation parameters causes the decrease of the range of wave function and the increase of the depth of amplitude. For example, on $n = 1$ has the depth of amplitude $0.2 \, f \, m$ at $q = 0.01$, $0.8 \, f \, m$ at $q = 0.1$, and $1 \, f \, m$ at $q = 1$. Then, on $n = 2$ has the depth of amplitude $1 \, f \, m$ at $q = 0.01$, $0.8 \, f \, m$ at $q = 0.1$, and $1.5 \, f \, m$ at $q = 1$. Meanwhile, on $n = 1$ and $n = 2$ have the range of wave function for one wavelength $3.1 \, f \, m$ at $q = 0.01$, $3.13 \, f \, m$ at $q = 0.1$, and $3.15 \, f \, m$ at $q = 1$.

4. Conclusions
The radial quantum deformation for Schrodinger equation on Coulomb potential could be solved by Hypergeometric method. The result were analytical energy and wave function, then calculated and visualized by Matlab R2013a software. The decrease of bound energy values causes the increase of quantum number and quantum deformation parameters, while the wave functions have the depth of deep amplitude by the increase of quantum number and quantum number parameters.

5. References
[1] A.Suparmi 2013 Mekanika Kuantum I:FMIPA UNS ISBN 978-602-99344-1-0
[2] F.Taskin and G.Kocak 2010 Chin. Phys. B 19 9 090314
[3] D.A.Dianawati, A.Suparmi, C.Cari 2018 AIP Phys. Conf. Ser.020071
[4] A.Suparmi 2011 Mekanika Kuantum II:FMIPA UNS ISBN 978-602-99344-2-7
[5] S.M.Ikhdair and R.Sever 2008 Formerly Central European Journal of Physics 6 3 697-703
[6] Dong, Shi-Hai 2011 Wave Equations in Higher Dimensions: New York SpingerDordecht Heidelberg London
[7] H.Sobhani, W.S.Chung, H.Hassanabadi 2017 Indian J Phys 92 4 529-536
[8] B.I. Ita, A.I. Ikeuba, A.N. Ikot 2013 *Commun. Theor. Phys* 61 149-152
[9] C. Cari and A. Suparmi 2013 *J. Phys. Conf. Ser.* 423 012031
[10] I.L. Elviyanti, A. Suparmi, C. Cari, D.A. Nugraha, B.N. Pratiwi 2017 *J. Phys. Conf. Ser.* 909
[11] D.A. Dianawati, A. Suparmi, C. Cari, D. Anggraini, U. Ulfa, D.C. Fitri, P. Mega 2018 *AIP Phys. Conf. Ser.* 020164
[12] A. Suparmi, D.A. Dianawati, C. Cari 2019 *J. Phys. Conf. Ser.* 1153
[13] N. Ahmed, S.Z. Alamri, M. Rassem 2018 *NRIAG Journal of Astronomy and Geophysics* 7 1-3
[14] C. Berkdemir, A. Berkdemir, J. Han 2005 *Chem. Phys. Let* 417 326-329
[15] B.I. Ita and P. Ekuri 2010 *Ecl. Quim, Sao Paulo* 35 3 103-107
[16] H. Hassanabadi, W.S. Chung, S. Zare, S.B. Bhardwaj 2017 *Hindawi Advances in High Energy Physics* 2017 1730834
[17] H. Sobhani, W.S. Chung, H. Hassanabadi 2017 *Hindawi Advances in High Energy Physics* 2017 9530874
[18] A.D. Anita, E.E. Ituen, H.P. Obong, C.N. Isonguyo 2015 *International Journal of Recent Advances in Physics* 4 1 55-65
[19] S. Mubeen, M. Naz, A. Rehman, G. Rahman 2014 *Hindawi Journal of Applied Mathematics* 2014 128787
[20] C. Cari 2013 *MekanikaKuantum*: UNS Press ISBN 978-979-498-830-5

Acknowledgement
This research was partly supported by Research grant of Pascasarjana UNS 516/UN27.21/PP/2019.