Differences between the trajectory representation and Copenhagen regarding the past and present in quantum theory

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We examine certain pasts and presents in the classically forbidden region. We show that for a given past the trajectory representation does not permit some presents while Copenhagen predicts a finite probability for these presents to exist. This suggests another gedanken experiment to invalidate either Copenhagen or the trajectory representation.

I. INTRODUCTION

While Copenhagen postulates that the wave function $\psi$ contains an exhaustive knowledge of quantum phenomenon, the trajectory representation (TR) in contrast considers $\psi$ to be incomplete because an eigenfunction $\psi$ with energy $E$ may have microstates. Copenhagen makes its predictions based upon $\psi$ being complete. Copenhagen for given past conditions predicts only the probabilities of later events. If Copenhagen’s probability density, $\psi^\dagger \psi$, is finite for some spatial point for a particular present, then that present is permitted. We define $\{\text{Copenhagen}\}$ to be the domain of the presents allowed by Copenhagen. On the other hand, TR is a causal representation. Each microstate has its own trajectory. The present for a given past is microstate dependent. In turn, we define $\{\text{TR}\}$ to be the domain of the presents allowed by TR. The question we investigate here is whether the family of trajectories of the microstates of a given wave function, $\psi$, for a given past covers the spatial domain allowed by Copenhagen at a later time. In other words, is $\{\text{TR}\} \cup \{\text{Copenhagen}\} = \{\text{TR}\}$ always true? It turns out that sometimes it isn’t true. We show where and how for a given initial (past) position the family of trajectories of TR does not cover the domain of present positions allowed by Copenhagen.

Other differences between Copenhagen and TR have previously been presented. Recently, Copenhagen and TR have been shown to predict differences regarding perturbing impulses and regarding the overdetermination of the motion by a redundant set of constants of the motion. For a perturbing impulse, TR, which is a causal theory couched in a Hamilton-Jacobi formulation, has the perturbing impulse act on a particle at its instantaneous position while Copenhagen, as a probability theory, mandates that even impulses must be averaged over Copenhagen’s probability density $\psi^\dagger \psi$. We note that for adiabatic perturbations, the trajectory representation and Copenhagen are in agreement. With regard to the overdetermination of the motion, the generator of the motion and the subsequent trajectory can be described by a finite number of constants of the motion. Any overdetermination of the motion by a super-sufficient number of constants of the motion would still have only a prescribed number of independent constants of the motion while all surplus constants of the motion would be redundant as they could be described in terms of the independent constants of the motion. Meanwhile, Copenhagen regards measuring these surplus constants of the motion as independent measurements. This investigation herein needs only a sufficient set of constants of the motion to determine the microstate to present differences between Copenhagen and TR. We note for completeness that position and momentum are insufficient by themselves to specify motion regardless of the Heisenberg uncertainty principle.

In Section 2, we show that, for a given past, TR does not generate trajectories that cover the Copenhagen-allowed domain of present positions when we specify an initial (past) position in the clas-
physically forbidden region inside a semi-infinite step barrier. In Section 3, we show that for a square well TR does generate trajectories that cover the Copenhagen-allowed domain of present positions. We discuss our results in Section 4. This investigation borrows the computational results from prior TR investigations that had examined other aspects of quantum theory. The interested reader who may be uninitiated in TR will find an introduction to TR in “Extended Version of ‘The Philosophy of the Trajectory Representation of Quantum Mechanics'”, quant-ph/0009070.

II. SEMI-INFINITE STEP BARRIER

Let us first investigate a case where the family of trajectories of TR does not cover the domain of present positions allowed by Copenhagen, {Copenhagen}. We consider a particle has sub-barrier energy \( E \). We also assume a barrier given by

\[
V = \begin{cases} 
0, & x < 0 \\
U > E, & x \geq 0 
\end{cases}
\]

For this potential, the wave function, \( \psi \), is finite over the finite \((r, t)\)-manifold. We investigate whether a particle initially posited in a classically forbidden region inside the barrier. This \( t_D \) is the time for a particle to complete its round trip from the wall of the step barrier at \( x = 0 \) out to its turning point at \( x = \infty \) and then its return back to \( x = 0 \). This dwell time, \( t_D \), has been studied for TR and is given by

\[
t_D = 2 \frac{(ab - c^2/4)^{1/2}}{a ± c(\kappa/k) + b(\kappa/k)^2} \frac{m}{\hbar k} \tag{1}
\]

where \( k = (2mE)^{1/2}/\hbar \) and \( \kappa = [2m(U - E)]^{1/2}/\hbar \). The coefficients \( a, b, c \) are additional constants of the motion that specify the microstate. One of these coefficients may be eliminated by noting that TR scales the Wronskian for the pair of independent solutions of the associated Schrödinger equation to be equal to \( \{2m/[\hbar^2(ab - c^2/4)]\}^{1/2} \). Two additional constants of the motion are needed because the quantum stationary Hamilton-Jacobi equation in one dimension,

\[
\frac{W_x^2 + V - E}{2m} = \frac{\hbar^2}{4m} \left( \frac{W_{xx}}{W_x} - \frac{3}{2} \left( \frac{W_{xx}}{W_x} \right)^2 \right)
\]

classical HJE

quantum effects \( \Rightarrow \) third-order ODE

is a third-order nonlinear differential equation while the classical stationary Hamilton-Jacobi equation is only first-order. By Eq. (1), the dwell time is a function of the microstate coefficients \( (a, b, c) \).

Let us now briefly digress on coefficients. While the particular values for the coefficients \( (a, b, c) \) are dependent upon which set of independent solutions of the Schrödinger equation is chosen consistent with the normalization of the Wronskian, the observable results are independent of the particular choice. This is analogous to vector analysis where the results of vector operations remain independent of the choice of coordinate systems.

Let us now compare our \( t_D \) calculated by TR to the barrier dwell times calculated by other workers not using TR. We consider a monochromatic incident particle \( (x < 0) \). As shown elsewhere\(^6\) the particle is monochromatic iff \( a = b \) and \( c = 0 \). Then \( t_D \) reduces to\(^6\)

\[
t_D = 2m/(\hbar \kappa) = \hbar/[E(U - E)]^{1/2}
\]

which is consistent\(^6\) with the dwell times for barriers presented by Hartman\(^9\) and Fletcher\(^10\).

We also note that \( t_D \) is inversely proportional to \( \kappa \) or \( (U - E)^{1/2} \). Particle velocity inside the barrier increases as \( (U - E)^{1/2} \) increases consistent with Barton\(^11\). This seems to be counterintuitive and to support superluminal velocities. As the trajectory transverses an infinite distance between the interface at \( x = 0 \) and the turning point at \( x = \infty \) in a finite duration of time, the velocity along the trajectory in this non-relativistic opus must become infinite for at least an infinitesimal duration\(^6\). This is another manifestation that TR is a nonlocal theory. Others
have made much ado about this aspect of nonlocality in tunnelling.

We now take another brief divergence. For completeness, we note that the superluminal velocity of TR implies the particle spends less time in these regions. Copenhagen alleges that this “reduce amount of time in a region” just manifests a reduced probability density in this very region. Here Copenhagen has tacitly assumed that quantum systems are ergodic where distributions over time converge to probability densities. In higher dimensions, the infinite velocity that occurs at the turning point at \( \infty \) implies that the trajectory for oblique incidence has a cusp at this turning point at \( \infty \). Furthermore, the trajectory for oblique incidence has a cusp at this turning point at \( \infty \).

The maximum dwell time, \( t_{\text{MD}} \), for a particle in the step barrier occurs at

\[
a = (1 + c^2/4)^{1/2} \kappa/k = 2^{1/2} \kappa/k,
b = (1 + c^2/4)a^{-1} = 2^{1/2} k/\kappa,
c = 2 - \epsilon \text{ where } 0 < \epsilon \ll 1.
\]

Then the dwell time has a least upper bound given by

\[
t_{\text{MD}} \leq \frac{1 + (\kappa/k)^2}{2^{1/2} - 1} \frac{m}{\hbar \kappa^2}
\]

Hence, if the particle is in the barrier at a particular time, \( t_0 \), then, after the time \( t_0 + t_{\text{MD}} \) at the most, the particle must have been reflected from the barrier and can no longer be in the barrier. This differs with Copenhagen, which postulates that the particle could be found again in the barrier forever. And we conclude that

\[
\{\text{TR}\}_{\text{SB}} \cup \{\text{Copenhagen}\}_{\text{SB}} \neq \{\text{TR}\}_{\text{SB}}
\]

for the particles reflected from a semi-infinite step barrier (the subscript “SB” denotes step barrier).

### III. SQUARE WELL

Let us now investigate a case where the family of trajectories of TR does cover the \((r,t)\)-domain allowed by Copenhagen. We assume a square well given by

\[
V = \begin{cases} U, & |x| \geq q \\ 0, & |x| < q. \end{cases}
\]

The trajectories for the bound states, \( E < U \), for this square well have already been studied. The period of libration, \( t_L \), for the trajectory for microstate \((a,b,c)\) has been determined to be

\[
t_L = 4[1 + (\kappa/k)^2] \frac{m(q + \kappa^{-1})}{\hbar k} \times \frac{(ab - c^2/4)^{1/2}(a + b(\kappa/k)^2)}{a^2 + (2ab - c^2)(\kappa/k)^2 + b^2(\kappa/k)^2}. \tag{2}
\]

Note that Eq. 2 is valid for symmetric and antisymmetric microstates of bound states.

The maximum time, \( t_{\text{ML}} \), for libration is given at

\[
a = (1 + c^2/4)^{1/2} \kappa/k = 2^{1/2} \kappa/k,
b = (1 + c^2/4)a^{-1} = 2^{1/2} k/\kappa,
|c| = 2 - \epsilon \text{ where } 0 < \epsilon \ll 1
\]

where again the Wronskian has been normalized so that \((ab - c^2/4)^{-1/2} = 1\). We note that here we need the magnitude of coefficient \( c \) for \( t_{\text{ML}} \) rather than just its value as we did for \( t_{\text{MD}} \). This is because we must consider reflection from both step barriers of the square well with interfaces at \( x = \pm q \). From
Eq. (2), the maximum time for libration has a least upper bound given by
\[ t_{ML} < 2^{3/2}[1 - (\kappa/k)^2] \frac{m(q + \kappa^{-1})}{\hbar \kappa}. \]

On the other hand, the minimum time for libration has a greatest lower bound given by zero. Either coefficient, \(a\) or \(b\), in Eq. (2) can grow without finite bound inducing \(t_L\) to become zero since
\[ t_L \propto \left[ \frac{(ab - c^2/4)(a + b(\kappa/k)^2)}{a^2 + (2ab - c^2)(\kappa/k)^2 + b^2(\kappa/k)^4} \right] \]
while still maintaining \((ab - c^2/4) = 1\). Therefore, there always exists for a bound-state particle a trajectory with an appropriate \(t_L\) to connect any past position with any present. Additional libration periods may be added consistently to a corresponding sub-orbital duration to cover \(\Delta t > t_{ML}\). We find that
\[
\{\text{TR}\}_{SW} \cup \{\text{Copenhagen}\}_{SW} = \{\text{TR}\}_{sw}
\]
for bound state particles in a square well (the subscript “SW” denotes square well). For bound states of a square well, Copenhagen’s independence of the present from the past corresponds to the spanning of the \((r,t)\)-manifold by the trajectories for a family of microstates.

For completeness, we note that the converse of the above is not true for excited states of the square well since
\[
\{\text{TR}\}_{SW} \cup \{\text{Copenhagen}\}_{SW} \neq \{\text{Copenhagen}\}_{SW}
\]
for excited states. The reason is that the isolated zeros in the classically allowed region of the excited wave function are accessible in TR. Nevertheless, we note for completeness that every neighborhood of these isolated zeros of the excited wave function contains points that are accessible to the corresponding \(\{\text{TR}\}_{SW}\). The isolated zeros of excited states are limit points of the corresponding \(\{\text{TR}\}_{SW} \cup \{\text{Copenhagen}\}_{SW}\).

**IV. DISCUSSION**

The semi-infinite step barrier is a counterexample where some presents are excluded by a single past. Our dwell times for the particle in the step barrier are shown herein to be consistent with the findings of other workers. In contrast, overdetermination of the constants of the motion corresponds to multiple pasts (i.e., \(t_0 < t_{00} < t_{000} \cdots\) where a sufficient number of multiple pasts may overdetermine a microstate.

Still the step barrier only presents a gedanken experiment to distinguish TR from Copenhagen because an ideal semi-infinite step potential is corrupted by reality.

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1. E. R. Floyd, Phys. Rev. D 26, 1339 (1982); 34, 3246 (1986); Found. Phys. Lett. 9, 489 (1996), quant-ph/9707051.
2. A. E. Faraggi and M. Matone, Int. J. Mod. Phys. A 15, 1869 (2000), hep-th/9809127.
3. R. Carroll, Can. J. Phys. 77, 319 (1999), quant-ph/9903081.
4. D. Bohm, *Quantum Theory* (Prentice Hall, 1951, Englewood Cliffs) pp. 26–29.
5. E. R. Floyd, Int. J. Mod. Phys. A 14, 1111 (1999), quant-ph/9708026.
6. E. R. Floyd, Found. Phys. Lett. 13 235 (2000), quant-ph/9708007.
7. R. Carroll, *Quantum Theory, Deformation and Integrability* (Elsevier, 2000, Amsterdam) pp. 50–56.
8. E. R. Floyd, Int. J. Mod. Phys. A 15, 1363 (2000), quant-ph/9907092.
9. T. E. Hartman, J. App. Phys. 33, 3247 (1962).
10. J. R. Fletcher, J. Phys. C 18, L55 (1985).
11. G. Barton, Ann. Phys. (N.Y.) 166, 322 (1986).
12. R. Y. Chiao, P. G. Kwiat and A. M. Steinberg, Sc. Am. 269(2), 52 (Aug. 1993).