One particle self correlations for a separable representation of the singlet spin state beyond standard Quantum Mechanics

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Abstract
A new pure quantum state, isotropic in spin variables, is defined in an extended spin phase space beyond standard quantum mechanics. It allows to represent the entangled singlet state in separable form. The statistical correlations between Alice and Bob measurements become self correlations between hidden spin values for each particle, together with perfect anti correlation between spin values on the pair. Alice determines through measurement on her particle the value of spin in some direction. Spin in another direction is inferred from Bob measurement on the companion. Bell’s inequalities are violated because of the wave like behaviour of quantum systems. In full analogy with the two slit experiment, interference terms between spin field components appear determining the contextual character of quantum distributions of probability.

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1 After Bell

Violations of Bell’s type inequalities in spatially separated measurements\cite{1} have been empirically tested beyond any reasonable doubt\cite{2,3,4,5}. All relevant loopholes have been satisfactorily closed, and the predicted quantum correlations confirmed. It is time to look for physical evidence of the non local influence between measurement events. However, known interactions are mediated by physical systems, either particles following time–like or light–like paths or distributed fields evolving relativistically. Space–like curves as paths of particles are discarded because they have a frame dependent time orientation, according to relativity. The value of a field at a space time event depends on its values along a spatial sheet inside the past light cone; in other words, it commutes with values of the field at spatially separated events. Even in case of time–like separation between measurements it is unlikely the existence of a mediating system connecting them without observable decay for increasing distance and with other systems in between that do not shield its propagation.

We can, alternatively, go beyond the standard formulation of Quantum Mechanics (SQM) and develop an explicitly local, separable description of entangled states for composites. The celebrated EPR paper\cite{6} about incompleteness of SQM, the quantum potential in Bohm mechanics\cite{7}, the analysis of Renninger\cite{8} of the wave particle duality in an interferometer, inconsistency between SQM and the action reaction principle\cite{9,10}, among other considerations\cite{11,12,13}, are enough arguments to explore the possibility of a formulation of Quantum Mechanics in extended phase spaces\cite{14}(EQM).

States of quantum systems could be described by some \([x, \Phi]\), \(x\) commuting and non commuting variables of a corpuscular subsystem and \(\Phi\) an accompanying de Broglie\cite{15} (or pre–quantum, sub quantum\cite{16,17}) field. In the double field \(\Phi–\Psi\) model, the distribution of amplitude \(\Psi = R \exp(i\theta)\) is a statistical representation of an ensemble of composite systems, \(R^2\) distribution of probability for variables \(x\) of the particle and \(\Delta\theta\) relative phases between field components.

No go theorems\cite{18,19,20,21,22} are dead end paths for the pursuit of an extended phase space in EQM. The existence of global, non contextual distributions of probability \(P(x)\) in spaces of (so called) hidden variables for the particle and whose marginals match the quantum distributions are mathematically forbidden. But these distributions would ignore the accompanying field and its interaction with the particle. Obviously, models that do not fulfil some hypothesis of these no go theorems are not ruled out\cite{14}. The empirical fact is that entanglement appears exclusively after local in-
teraction in the past between the correlated systems, or it is conditional to some intermediate interaction if both systems of interest have not been in contact in the past.

In this letter, a local description for the quantum correlations in the singlet spin state is formulated in the framework of an extended phase space for spin variables. In section II, quantum distributions of amplitude, not classical distributions of probability, describe all standard pure quantum states, each one eigenstate of some spin operator, and a new pure quantum state isotropic in spin, which has no counterpart in SQM. It is this ingredient of the formalism, the use, as in SQM, of quantum amplitudes in the extended space instead of classical probabilities, which allows to overcome the thesis of no go theorems. The calculation of marginal amplitudes through projection over the standard phase spaces, followed by Born rule, reproduces the results of SQM, that is, the correlations between a previously known (the eigenvalue) and a measured value of spin. Correlations between hidden values of spin in two arbitrary directions can be consistently computed in the isotropic state. In the singlet state of a composite, section III, two values of spin can be determined for each particle. One is obtained through direct measurement and the other is inferred from the perfect anti correlation with the measured companion. The state of the composite is separable, each particle is in the new isotropic spin state, with its associated individual self correlations. This formalism can be relevant for the ontological interpretation of Quantum Mechanics, the ensemble character of pure quantum states.

2 The isotropic spin state

If locality is assumed for joint and spatially separated measurements on the singlet spin state, perfect anti correlation between outputs for any common, freely and independently chosen by Alice and Bob, direction \( \mathbf{n} \) of measurement implies that a complete representation of the physical state is characterised by all values of spin. If, to avoid mathematical complications, we consider a finite set of directions \( \{ \mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_N \} \), values \( (s_1^\alpha, s_2^\alpha, \ldots, s_N^\alpha)_\nu \) and \( (s_1^\beta, s_2^\beta, \ldots, s_N^\beta)_\nu \) for each pair of jointly generated particles \( (\alpha(\nu), \beta(\nu)) \), \( \nu \in \{1, 2, \ldots\} \), are fixed from the generation event, fulfilling \( s_1^\alpha + s_2^\beta = 0 \). Three independent values as \( s_x, s_y, s_z \) do not determine the other variables of spin, e.g. \( s_\theta \) for the magnitude (operator) \( S_\theta = \cos(\theta)S_x + \sin(\theta)S_y \). The functional relations between non commuting operators are not fulfilled by their eigenvalues, \( s_\theta \neq \cos(\theta)s_x + \sin(\theta)s_y \). In SQM, the dimension of the
phase space is lower than in Classical Hamiltonian Mechanics, e.g. position and momentum variables \{ (q, p) \} are restricted to \{ q \} (resp. \{ p \}) in the position (momentum) representation. The phase space of EQM has higher dimensions than its classical counterpart, according to the infinite degrees of freedom of the accompanying field.

Let us consider the extended spin phase space \( \mathcal{P}h = \{(s_1, \ldots, s_N)|s_k = \pm\}, |\mathcal{P}h| = 2^N \), associated to an elementary spin 1/2 particle. Bell’s inequalities state that for \( N > 2 \) there are not global, non contextual distributions of probability on \( \mathcal{P}h \), describing a classical statistical ensemble from which the quantum probabilities for the singlet could be obtained.

\[
P_{QM}(s_1^\alpha, s_2^\beta) = \frac{1}{4}(1 - s_1^\alpha s_2^\beta \mathbf{n}_1 \cdot \mathbf{n}_2) = \\
= \frac{1}{4}(1 + s_1^\alpha s_2^\alpha \mathbf{n}_1 \cdot \mathbf{n}_2) = P_{EQM}(s_1^\alpha, s_2^\alpha)
\]
can not be reproduced by a global distribution of probability \( P_{Cl}(s_1^\alpha, \ldots, s_N^\alpha) \) through marginals \( \sum_{l \neq 1, 2} \sum_{s_l} P_{Cl}(s_1^\alpha, \ldots, s_N^\alpha) \). The existence of a classical probabilistic mixture \( P_{Cl} \) of physical states with hidden variables, representing an ensemble quantum state, is a “natural” hypothesis systematically considered in the literature of no go theorems. However, it is not unavoidable, and interference phenomena as in the paradigmatic two slit experiment point to the need of other mathematical tools. An alternative algorithm must be applied in \( \mathcal{P}h \), able to reproduce the quantum distributions for a statistical sample of measurements over the same pure/ensemble quantum state. Let us apply the “quantum way”, a distribution of amplitude of probability \( Z(s_1, \ldots, s_N), Z : \mathcal{P}h \rightarrow K \) (in the spin phase space, \( K \) will be the set of imaginary quaternions). We can mimic the paradigmatic two slit experiment and obtain marginals for the distribution of amplitude \( Z \)

\[
Z(s_j) = \sum_{l \neq j} \left( \sum_{s_l} Z(s_1, \ldots, s_N) \right)
\]
Applying now Born rule, we get the probabilities

\[
P(s_j) = \frac{|Z(s_j)|^2}{|Z(+j)|^2 + |Z(-j)|^2},
\]
where there will appear generically interference terms in the squared sum of amplitudes. Compare it with
\[ \Psi(x_0, y_0) = \Psi_L(x_0, y_0) + \Psi_R(x_0, y_0) \]

\[ P(x_0, y_0) = \frac{|\Psi(x_0, y_0)|^2}{\sum_{(x,y)} |\Psi(x, y)|^2}, \]

where \((x, y)\) are the position variables at the final screen of the two slit experiment, and \(\Psi(x_0, y_0) = \Psi_L(x_0, y_0) + \Psi_R(x_0, y_0)\) is the marginal amplitude, sum of left and right slit field components. Interference terms in \(|\Psi(x_0, y_0)|^2\) are here responsible of the diffraction pattern. Similarly, interference terms in \(|Z(s_j)|^2\) are responsible of the contextual character of quantum distributions of probability, i.e., its dependence on the field components allowed by the physical context.

Formal distributions of probability for correlated values of spin in two arbitrary directions \(n_j\) and \(n_k\) can be similarly determined, although in different, alternative ways. One of these values remains necessarily counterfactual because of the incompatibility of joint measurements; measurement of \(s_j\) unavoidably perturbs the previous value of \(s_k\). From the marginals

\[ Z(s_j, s_k) = \sum_{l \neq j,k} \left( \sum_{s_l} Z(s_1, \ldots, s_N) \right) \]

we could formally define the joint, unobservable distribution

\[ \Pi(s_j, s_k) = \frac{|Z(s_j, s_k)|^2}{\sum_{s_j', s_k'} |Z(s_j', s_k')|^2}; \]

and the same definition can be generalized to \(\Pi(s_j, s_k, s_l)\), etc. Generically, \(P(s_j) \neq \Pi(s_j, +k) + \Pi(s_j, -k)\) because of the interference terms when Born rule is applied to a sum of amplitudes

\[ |Z(s_j, +k) + Z(s_j, -k)|^2 = |Z(s_j, +k)|^2 + |Z(s_j, -k)|^2 + \]

\[ + \left( Z^*(s_j, +k)Z(s_j, -k) + Z^*(s_j, -k)Z(s_j, +k)b \right)_{\text{interf}} \]

We can interpret \(P(s_j)\) and \(\Pi(s_j, s_k)\) as corresponding to incompatible physical contexts, as in the two slit experiment. Alternatively, we can also define conditional probabilities
\[ \Pi(s_k|s_j) = \frac{|Z(s_j, s_k)|^2}{\sum_{s'_k} |Z(s_j, s'_k)|^2} \]

from which

\[ \Pi(s_j; s_k) = P(s_j)\Pi(s_k|s_j) \]

and similarly for \( \Pi(s_k; s_j) \). Now, \( P(s_j) = \Pi(s_j; +k) + \Pi(s_j; -k) \) but generically \( \Pi(s_j; s_k) \neq \Pi(s_j, s_k) \neq \Pi(s_k; s_j) \). When values \( s_j \) and \( s_k \) are jointly observable, as in the singlet state, it must happen that \( \Pi(s_j; s_k) = \Pi(s_j, s_k) = \Pi(s_k; s_j) \), matching the observed \( P(s_j, s_k) \). This will happen if the physical contexts associated to \( P(s_j) \) and \( P(s_j, s_k) \) are compatible.

Let us consider the quaternion

\[ N[n] = (n \cdot i)I + (n \cdot j)J + (n \cdot k)K, \]

with null real part, associated to a unit vector \( n \). Each spin state \((s_1, \ldots, s_N)\) will have a fixed associated amplitude \( Z \), sum of elementary amplitudes \( s_j N_j \equiv s_j N[n_j] \), in analogy with the elementary amplitudes \( e^{iS[\text{path}]/\hbar} \) for virtual paths in the path integral formalism, \( Z(s_1, \ldots, s_N) \equiv \sum_j s_j N_j \). The physical context determines which virtual spin states are considered, in the same way that different physical configurations determine the virtual paths to be taken into account, e.g. in the two slit experiment. The SQM state \(|+1\rangle\), spin up in direction \( n_1 \), can be prepared using a Stern–Gerlach apparatus that splits the incoming trajectory into up and down spin output paths. The up path does not have down spin field components, so that in the extended formalism the ensemble state \(|+1\rangle\) has associated distribution of amplitude \( Z_{+1}(s_1, s_2, \ldots, s_N) \) where \( Z_{+1}(-1, \ldots) \equiv 0 \). The marginals become \( Z_{+1}(-1) = 0 \), \( Z_{+1}(+1) = 2^{N-1}N_1 \), \( Z_{+1}(s_2) = 2^{N-2}(N_1 + s_2 N_2) \). When \( s_2 \) is measured \( \pm j \) terms interfere for \( j \geq 3 \). These marginal amplitudes determine the associated observable probabilities, \( P_{+1}(-1) = 0 \), \( P_{+1}(+1) = 1 \), as well as \( P_{+1}(s_2) = (1 + s_2 n_1 \cdot n_2)/2 \), where the relations

\[ N^* = -N \quad N^2 = -1 \quad N_1^* N_2 = n_1 \cdot n_2 = N[n_1 \times n_2] \]

have been used. The SQM distributions are reproduced.

When the context allows both spin up and down field components in all directions the corresponding state \( S_0 \) becomes isotropic, with distribution of amplitude \( Z_0 \) containing all components in \( \mathcal{P}h \), \( Z_0 \equiv Z \), and distributions
of probability \( P_0(s_j) = 1/2 \) for all \( j \). This quantum state has no counterpart in the Hilbert space of SQM, where every vector of state is up eigenstate for the spin operator in some direction. A classical mixture like

\[
\rho = \frac{1}{2}|+1><+1| + \frac{1}{2}|-1><-1|
\]

reproduces the isotropic distribution too, but it has different ontological content; \( \rho \) represents two sub-ensembles of pure states, \( |-1> \) and \( |+1> \), each one lacking the other field components, while \( S_0 \) contains all of them which can interfere. \( S_0 \) and \( \rho \) are associated to different physical contexts.

Formal distributions of probability for two or more values of spin are obtained through marginal amplitudes and Born rule,

\[
\Pi_0(s_1, s_2) = \frac{(1 + s_1 s_2 \mathbf{n}_1 \cdot \mathbf{n}_2)}{4},
\]

proportional to the (not normalized) squared marginal amplitude

\[
|2^{N-2} (s_1 \mathbf{N}_1 + s_2 \mathbf{N}_2)|^2
\]

as well as \( \Pi_0(s_1, s_2, s_3) = \frac{1}{24} (3 + 2 s_1 s_2 \mathbf{n}_1 \cdot \mathbf{n}_2 + 2 s_1 s_3 \mathbf{n}_1 \cdot \mathbf{n}_3 + 2 s_2 s_3 \mathbf{n}_2 \cdot \mathbf{n}_3) \)

proportional to

\[
|2^{N-3} (s_1 \mathbf{N}_1 + s_2 \mathbf{N}_2 + s_3 \mathbf{N}_3)|^2
\]

\( \Pi_0(s_1, s_2) \) is not observable, but it is consistently defined: \( \Pi_0(s_1, +2) + \Pi_0(s_1, -2) = P_0(s_1) \), and \( \Pi_0(s_1; s_2) = \Pi_0(s_1, s_2) \), so that we can consider a “classical” distribution \( P_0(s_1, s_2) \). On the other hand, a \( P(s_1, s_2, s_3) \) is not consistently defined,

\[
\Pi_0(s_1, s_2, +3) + \Pi_0(s_1, s_2, -3) \neq P_0(s_1, s_2)
\]

Notice the analogy with the two slit experiment

\[
P(x, y, L) + P(x, y, R) \neq P(x, y)
\]
3 The singlet state

Two particles $\alpha$ and $\beta$ are jointly generated in the singlet spin state

$$|S_{\text{singlet}}> = |+\alpha> \otimes |-\beta> - |-\alpha> \otimes |+\beta>$$

Each particle (marginal) density is isotropic in spin variables, $P(s^\alpha_j) = 1/2$ for all directions $j$; no individual pure quantum state of $\alpha$ in the two-dimensional Hilbert spin space of SQM can represent it. In the usual interpretation of SQM, with one to one correspondence between physical and pure quantum states, a separable description of the singlet is not possible. On the other hand, in the extended phase space where pure quantum states, distributions of quaternion amplitudes, represent ensembles of physical states, the $\alpha-\beta$ correlation applies to jointly generated pairs $(\alpha(\nu), \beta(\nu))$ and not to isotropic spin states $S^\alpha_0$ and $S^\beta_0$, which describe statistical ensembles for each particle separately. It is obvious that there is not correlation between pairs of outputs for Alice and Bob measurements over particles $\alpha(\nu)$ and $\beta(\nu)$ belonging to different pairs $\nu \neq \nu'$. Correlations apply to jointly generated particles $s^\alpha_j(\nu) + s^\beta_j(\nu) = 0$. The singlet state is expressed in separable form

$$S_{\text{singlet}} = S^\alpha_0 \otimes_{\text{corr}} S^\beta_0$$

if $\otimes_{\text{corr}}$ is understood as the perfect anti correlation between jointly generated pairs. Each particle, if we ignore the companion, is in the pure state $S^\alpha_0$ of EQM.

Equivalently, a distribution $Z_{\text{singlet}}$ can be defined on the subset $\mathcal{P}h_{\text{corr}} \subset \mathcal{P}h_\alpha \times \mathcal{P}h_\beta$ defined by the correlation equations, or

$$Z_{\text{singlet}}((s^\alpha_1, \ldots, s^\alpha_N), (s^\beta_1, \ldots, s^\beta_N)) \equiv 0$$

outside $\mathcal{P}h_{\text{corr}}$ ($s^\alpha_j + s^\beta_j \neq 0$ for some $j$) and

$$Z_{\text{singlet}}((s^\alpha_1, \ldots, s^\alpha_N), (-s^\alpha_1, \ldots, -s^\alpha_N)) \equiv Z_0(s^\alpha_1, \ldots, s^\alpha_N) = -Z_0(s^\beta_1, \ldots, s^\beta_N)$$

The $\alpha$ ($\beta$) marginal of $Z_{\text{singlet}}$, when projecting from $\mathcal{P}h_{\text{corr}}$ onto $\mathcal{P}h_\alpha$ ($\mathcal{P}h_\beta$), becomes trivially (there is only one non vanishing term in the fibre of the projection) the isotropic $S^\alpha_0$ ($S^\beta_0$), i.e., they are pure quantum states and not mixtures as the marginals of the density $\rho_{\text{singlet}} = |S_{\text{singlet}}> < S_{\text{singlet}}|$. The formal distribution $\Pi_0(s^\alpha_j(\nu), s^\alpha_k(\nu))$ is observable. Recall it is consistent with $\Pi_0(s^\alpha_j(\nu), s^\beta_k(\nu))$ and $\Pi_0(s^\beta_j(\nu), s^\alpha_k(\nu))$, defining a classical distribution. The
second value of spin is inferred from the output of Bob measurement over $\beta(\nu)$, without perturbing the state of $\alpha(\nu)$. It means we observe (infer) the value of $s_{\alpha(\nu)}^n$ previous to measurement of $s_{\beta(\nu)}^n$. Physical splitting into $\pm_k$ spin field components of $\beta(\nu)$, at Bob’s apparatus, does not perturb Alice’s $\alpha(\nu)$ particle. Splitting of the $\alpha(\nu) \pm_j$ spin field components maintains on each branch both $\pm_k$ (and other $\pm_l$) spin components of the total spin field, which interfere. Both $n_j$ and $n_k$ are freely and independently chosen by Alice and Bob. The correlations in each individual isotropic spin state $S_0$ are an inner property of each particle separately. The predicted distributions of probability obtained from $S_0$ through marginal amplitudes and Born rule, observable because of perfect anti correlation, match the SQM predictions for the entangled singlet state.

When considering a third direction, interference in 

$$|Z_0(s_1, s_2, +3) + Z_0(s_1, s_2, -3)|^2$$

does not vanish. A global classical distribution of probability $P_{Cl}(s_1, \ldots)$ does not exist, according to Bell’s inequalities. As in the two slit experiment, field components of hidden, not measured magnitudes are superposed and interfere. The only relevant distinction between both physical processes is that $x$ and $y$ position coordinates at the final screen commute and can be jointly measured on an individual particle, while $s_1$ and $s_2$ do not commute and one of them can only be inferred from measurement on the correlated companion. Counterfactual values are widespread in Physics, and our degree of confidence in them is linked to our confidence (empirically grounded) in the applied theory, in this case Quantum Mechanics. The property of consistency depends on the quantum state, here the isotropic spin $S_0$. We could calculate in an orthodox $Z_{+1}$ state formal joint or conditional correlations between $s_2$ and $s_3$, but $\Pi_{+1}(s_2; s_3) \neq \Pi_{+1}(s_2, s_3) \neq \Pi_{+1}(s_3; s_2)$ are incompatible. $s_2$ and $s_3$ variables are not jointly observable.

The contextual character of the quantum distributions is already present in the paradigmatic two slit experiment. Two non vanishing probabilities $P_R(x, y)$ and $P_L(x, y)$, applied each to the physical context with one slit open and the other closed, do not add to the distribution $P(x, y)$ in the third context with both slits open. Wave superposition and interference, a typical phenomenon for distributed fields, is behind this contextual behaviour of the quantum probabilities, and suggest to interpret elementary particles as composites made of a corpuscular system and a distributed, relativistically (locally, causally) evolving field. The same phenomenon, applied to spin field components, is found in the isotropic spin state, which is a pure quantum
state in EQM without counterpart in SQM. Other entangled composites in SQM could also find a local, separable and contextual representation through new states in adequate extended phase space of EQM.

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