Differentiation Operator in the Beurling Space of Ultradifferentiable Functions of Normal Type on an Interval

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Abstract—In this paper we study closed subspaces of ultradifferentiable functions which are invariant under the differentiation operator. We propose a version of spectral synthesis which takes into account the presence of non-trivial differentiation invariant subspaces containing no exponential monomials. As an application, we describe the sets of solutions of finite and infinite systems of "local" homogeneous convolution equations.

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1. INTRODUCTION

Let \( \omega : [0; \infty) \to [0; \infty) \) be non-decreasing continuous function satisfying the conditions
the function \( \varphi(t) := \omega(e^t) \) is convex on \( [0; \infty) \),
\( \omega(x) = o(x) \), and \( \ln x = o(\omega(x)) \) as \( x \to \infty \),
\( \omega \) is an “almost subadditive” weight:
\[
\forall \sigma > 1 \exists C > 0 \colon \omega(x + y) \leq \sigma(\omega(x) + \omega(y)) + C, \quad \forall x, y \geq 0,
\]
\( \omega \) is non-quasianalytic weight:
\[
\int_1^\infty \frac{\omega(x)}{x^2} \, dx < \infty.
\]

Let \( \{c_k\} \), \( k = 1, 2, \ldots \), be such that \( 0 < c_k \nearrow a \), \( 0 < a \leq +\infty \). Then, \( \{-c_k; c_k\} \) is the increasing sequence of closed intervals exhausting \( (-a; a) \). Given \( f \in C^\infty(-a; a) \), \( q \in (0; 1) \) and \( k \in \mathbb{N} \), we set
\[
||f||_{\omega,q,k} = \sup_{j \in \mathbb{N}_0} \sup_{|x| \leq c_k} \left| \frac{f^{(j)}(x)}{e^{q\varphi^*(j/q)}} \right|,
\]
where \( \varphi^*(s) = \sup(t - \varphi(t)) \).

The space of ultradifferentiable functions (UDF) of normal type on \( (-a; a) \) is defined as
\[
\mathcal{U}_a = \{ f \in C^\infty(-a; a) : ||f||_{\omega,q,k} < \infty \forall q \in (0; 1), \forall k = 1, 2, \ldots \}.
\]

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The system of semi-norms $|| \cdot ||_{\omega,q,k}$ defines a locally convex topology in $U_a$. Equipped with this topology, $U_a$ becomes a locally convex space of $(M^*)$-type.

For an arbitrary interval $(a; b) \subseteq \mathbb{R}$, we denote by $U(a; b)$ the space of UDF with shifted argument

$$f \in U(a; b) \iff f(x + (a + b)/2) \in U((b-a)/2).$$

Let $D = \frac{d}{dx}$ be the differentiation operator acting in $U_a$, $W \subset U_a$ be a closed differentiation invariant subspace, that is $D(W) \subset W$. Shortly, $W$ is said to be $D$-invariant subspace.

It is well-known that root elements of the differentiation operator are exponential monomials $t^k e^{-\lambda t}$, $k \in \mathbb{Z}_+$, $\lambda \in \mathbb{C}$. We denote by $\text{Exp}(W)$ the set of all exponential monomials contained in $W$.

Given an ultradistribution $S \in U'_a$, the formula

$$g(y) = T_S(f)(y) = T_S(f(x + y))$$

defines a convolution operator $T_S : U_a \rightarrow U(c + a; a - d)$, where $[c; d] = \text{ch supp } S$ is convex hall of the support of $S$. The kernel of $T_S$ provides a classical example of $D$-invariant subspace.

Let $S \in U'_a$. It is known that if its Fourier–Laplace transform $F(S)$ is a divisor of $F(U'_a)$ then every function $f \in \text{ker } T_S$ can be represented as series of exponential polynomials contained in $\text{ker } T_S$. The series converges with grouping in $U_a$. This is the result due to D.A. Abanina [1]. In the next sections, we recall the explicit definitions of Fourier–Laplace transform $F$, multiplier and divisor of the space $F(U'_a)$.

At the moment, we only need to mention that if $F(S)$ is a divisor of $F(U'_a)$ then $\text{supp } S = \{0\}$ (see [1]).

On the other hand, for an ultradistribution $S$ with an arbitrary compact support in $(-\alpha; a)$, we do not know any results even on the approximation of functions in $W := \text{ker } T_S$ by the elements of $\text{span Exp}(W_S)$.

In this paper, we are going to consider more general situation. Given $A$, $B \subseteq \mathbb{R}$, denote by $A \div B$ their geometric difference consisting of all $x \in \mathbb{R}$ such that $x + B \subseteq A$.

Let $S \in U'_a$, and $I \subseteq (-\alpha; a)$ be a relatively closed interval such that $\bar{I} := I \div \text{ch supp } S \neq \emptyset$. We define “local” convolution operator $T_{S,I} : U_a \rightarrow U(\bar{I})$ by setting

$$g = T_{S,I}(f), \quad g(y) = S(f(x + y)), \quad y \in \bar{I}.$$  

Clearly, the kernel $W_{S,I} := \text{ker } T_{S,I}$ is a closed $D$-invariant subspace in $U_a$. It is also not difficult to see that $W_{S,I}$ contains all functions $f \in U_a$ vanishing on $I$. So, we face a problem because any non-zero function $f \in U_a$ vanishing on $I$ cannot belong to the closure of $\text{span Exp}(W_{S,I})$ in $U_a$. This is the consequence of two facts:

1) for any interval $X \subseteq \mathbb{R}$, there are continuous embeddings

$$U(X) \subset C^\infty(X) \subset C(X);$$

2) for any relatively closed interval $X \subseteq (-\alpha; a)$ and mean periodic function $f$ on $X$, its mean periodic continuation to $(-\alpha; a)$ is unique (see [2, 3]).

Denote by $U_{\text{min}}(\mathbb{R})$ the space of UDF of minimal type on $\mathbb{R}$. In [4], the authors consider a subspace $W \subset U_{\text{min}}(\mathbb{R})$ which is similar to $W_{S,I}$. They prove that any function in $W$ can be locally expanded in a series (with grouping) of exponential polynomials contained in $W$. This series converges in a small interval under the condition that $F(S)$ is a divisor of the space $F(U'_{\text{min}}(\mathbb{R}))$, where $S$ is the ultradistribution defining $W$.

In fact, the above questions are particular cases of the following spectral synthesis problem: how to reconstruct $D$-invariant subspace $W \subset U_a$ knowing $\text{Exp}(W)$? Generally speaking, it is not equivalent to establish the relation $W = \text{span Exp}(W)$.

Given relatively closed interval $I \subseteq (-\alpha; a)$, we set

$$W_I = \{ f \in U_a : f = 0 \text{ on } I \}.$$  

If $I \neq (-\alpha; a)$ then $W_I$ is non-trivial $D$-invariant subspace containing no exponential monomials. Following [5], we call $W_I$ residual subspace. We will show that any $D$-invariant subspace $W \subset U_a$ has residual interval $I_W$ (see Proposition 3 below). It is defined by the following requirements: $I_W$ is relatively closed in $(-\alpha; a)$, $W_{I_W} \subset W$, and $W_I \setminus W \neq \emptyset$ for any relatively closed interval $I \subseteq I_W$.  

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Now, taking into account the presence of residual subspaces, we formulate spectral synthesis problem for the differentiation operator $D$ in $\mathcal{U}_a$ in a similar way as it was done for $D$ in $C^\infty(a;b)$ (see [5–8]).

Let $W \subset \mathcal{U}_a$ be a $D$-invariant subspace with residual interval $I_W$ and supplies of exponential monomials $\text{Exp}(W)$. We say that $W$ admits weak spectral synthesis if

$$W = \text{span}(\text{Exp}(W)) + W_{I_W}. \quad (4)$$

$D$-invariant subspace $W$ admits spectral synthesis if

$$W = \text{span}(\text{Exp}(W)). \quad (5)$$

Clearly, in this case, $I_W = (-a;a)$. Spectral synthesis problem: given $D$-invariant subspace $W \subset \mathcal{U}_a$, to obtain the conditions under which the relation (4) holds?

In [6, 8], we modified classical dual scheme to solve the similar problem in $C^\infty(a;b)$. This scheme goes back to L. Ehrenpreis and Krasichkov-Ternovskii [9]. There, we also partially used some results on spectral analysis and synthesis for the differentiation operator in $C^\infty(-a;a)$ due to A. Aleman and B. Koremblum [5]. These authors got their results by different methods and approaches. However, all of them may be obtained by using our dual scheme.

In this paper, we study the spectral synthesis problem for the differentiation operator in $\mathcal{U}_a$. The results on the general theory of UDF and ultradistributions due to A.V. Abanin (see [10, 11]) make it possible to use the dual approach.

First, we obtain the solution of the spectral synthesis problem in general situation. Then, from the obtained results, we derive assertions on weak spectral synthesis for $D$-invariant subspaces of one of the following type. Let $I \subset (-a;a)$ be an arbitrary fixed respectively closed interval. We consider

1) $W = \ker T_{S,I},$ where $T_{S,I} : \mathcal{U}_a \to \mathcal{U}(\overline{I}),$ $S \in \mathcal{U}_a'$ is an arbitrary ultradistribution.

2) $W = \bigcap_{j=1}^m \ker T_{S_j,I},$ where $S_j \in \mathcal{U}_a', \text{supp } S_j = \{0\}, j = 1, \ldots, m,$ and $S_j \ast S_{j+1}, j = 1, \ldots, m - 1.$

3) $W = \bigcap_{\alpha} \ker T_{S_\alpha,I},$ where $\{S_\alpha\} \subset \mathcal{U}_a'$ and for any $\alpha \mathcal{F}(S_\alpha)$ is a multiplier of $\mathcal{F}(\mathcal{U}_a').$

We stick to the following plan.

In Section 2, we introduce a topological module of entire functions $\mathcal{P}_a,$ describe duality between $D$-invariant subspaces in $\mathcal{U}_a$ and closed submodules in $\mathcal{P}_a,$ establish equivalence of the spectral synthesis problem in $\mathcal{U}_a$ and the dual problem of local description in $\mathcal{P}_a.$ In Section 3, we explore characteristics and properties of closed submodule $\mathcal{J} \subset \mathcal{P}_a$ we will need. We also solve the local description problem there. Section 4 is devoted to the assertions on the spectral synthesis for $D$-invariant subspaces in $\mathcal{U}_a$ including (intersections of) kernels of “local” convolution operators.

2. DUALITY OF $D$-INVARIANT SUBSPACES AND WEIGHTED SUBMODULES OF ENTIRE FUNCTIONS

In this section, we establish duality between $D$-invariant subspaces of UDF and submodules of entire functions. It leads to the equivalence of the spectral synthesis problem for $D$-invariant subspaces and the local description problem for submodules. We apply duality scheme mentioned in Introduction. The classical version of this scheme deals with $D$-invariant subspaces of functions which are holomorphic in a convex domain (see [9]). We have modified the scheme to study $D$-invariant subspaces in $C^\infty(a;b)$ (see [6, 8, 12]). There, we have also faced some difficulties because of presence of non-trivial residual subspaces in $C^\infty(a;b)$.

According the general theory of UDF developed by A. V. Abanin ([10, 11]), all polynomials and exponentials $e^{i\lambda t}, \lambda \in \mathbb{C},$ belong to $\mathcal{U}_a'$, and this space forms a ring with respect the operation of multiplication of functions. The space of ultradifferentiable test functions contains cut-off functions and partition of unity. The strong dual space $\mathcal{U}_a'$ is a space of all ultradistributions with compact supports in $(-a;a).$ Both spaces, $\mathcal{U}_a$ and $\mathcal{U}_a'$ are reflexive because of first of them is of $(M^*)$-type, and the second is of $(LM^*)$-type. It implies the following assertion.
**Proposition 1** (General duality principle). Between the set of all closed subspaces \( \{W\} \) of the space \( U_a \) and the set of all closed dual spaces \( \{V\} \) of the strong dual space \( U_a' \) there is one-to-one correspondence defined by the rule \( W \leftrightarrow V \iff V = W^0, \) where \( W^0 = \{S \in U_a' : S(f) = 0 \ \forall f \in W\} \).

The subspace \( W^0 \) is called annihilator subspace for \( W \).

Let \( c_k \) be the same as in (3), \( 0 < r_k \not\to 1. \) We set
\[
P_k = \left\{ \varphi \in H(\mathbb{C}) : \|\varphi\|_k = \sup_{z \in \mathbb{C}} \frac{|\varphi(z)|}{e^{\rho_k \omega(|z|)} + c_k |\operatorname{Im} z|} < +\infty \right\}, \quad k = 1, 2, \ldots
\]

It is easy to check that \( P_k \) is a Banach space, \( k = 1, 2, \ldots \).

Denote by \( \mathcal{F} \) the Fourier–Laplace acting in \( U_a'; \) \( \mathcal{F}(S)(z) = S(e^{-it}z), \) \( z \in \mathbb{C}. \) It is known that \( \mathcal{F} \) is a linear topological isomorphism between \( U_a' \) and \( \mathcal{P}_a := \lim \operatorname{ind} P_k \) (see [10, 11]).

Because of (2), any \( \varphi \in \mathcal{P}_a \) is of the Cartwright class of entire functions. In particular, it is an entire function of completely regular growth with respect to the order 1, and the indicator diagram of \( \varphi \) equals \([-ih_\varphi(-\pi/2); ih_\varphi(\pi/2)]\), where \( h_\varphi \) is the indicator function of \( \varphi \).

By Fourier–Laplace transform, the generalized differentiation operator \( D' : U_a' \to U_a' \) transforms to the operator of multiplication by \((-iz)\). The last one acts continuously in \( \mathcal{P}_a. \) It means that \( \mathcal{P}_a \) is a topological module over the ring \( \mathbb{C}[z]; \) and if \( W \subset U_a \) is a \( D \)-invariant subspace then \( \mathcal{J} = \mathcal{F}(W^0) \) is a closed submodule in \( \mathcal{P}_a. \)

In our further considerations, we will always deal with closed submodules \( \mathcal{P}_a \) and will omit the word “closed” talking about them. Given \( D \)-invariant subspace \( W, \) we denote by \( \Lambda_W \) the sequence of multiple points \( \{(\lambda_k; m_k)\} \) such that
\[
\operatorname{Exp}(W) = \{t^j e^{-i\lambda_k t}, \ j = 0, 1, \ldots, m_k - 1\}_{k=1}^\infty.
\]

Denote by \( \mathcal{Z}_\varphi \) zero set of function \( \varphi \in \mathcal{P}_a, \) and by \( \mathcal{Z}_\mathcal{J} \) zero set of a submodule \( \mathcal{J} \subset \mathcal{P}_a, \) that is \( \mathcal{Z}_\mathcal{J} = \bigcap_{\varphi \in \mathcal{J}} \mathcal{Z}_\varphi. \) In the other words, \( (\mu_k; n_k) \in \mathcal{Z}_\mathcal{J} \) if and only if any function \( \psi \in \mathcal{J} \) vanishes at \( \mu_k \) with a multiplicity at least \( n_k, \) and there exists \( \varphi \in \mathcal{J} \) vanishing at \( \mu_k \) with the multiplicity \( n_k. \)

By Proposition 1, taking into account the isomorphism \( \mathcal{P}_a = \mathcal{F}(U_a') \) and the relation between differentiation in \( U_a \) and multiplication by \((-iz)\) in \( \mathcal{P}_a, \) we get

**Proposition 2** (Special duality principle). Between the set of all \( D \)-invariant subspaces \( \{W\} \) and the set of all submodules \( \{\mathcal{J}\} \) there is one-to-one correspondence defined by the rule
\[
W \leftrightarrow \mathcal{J} \iff \mathcal{J} = \mathcal{F}(W^0);
\]
in addition, \( \Lambda_W = \mathcal{Z}_\mathcal{J}. \)

The submodule \( \mathcal{J} = \mathcal{F}(W^0) \) is said to be an annihilator submodule for \( W. \)

Given submodule \( \mathcal{J} \subset \mathcal{P}_a, \) we define its indicator segment \( [c_\mathcal{J}; d_\mathcal{J}] \subset \mathbb{R} \) setting \( c_\mathcal{J} = -\sup_{\varphi \in \mathcal{J}} h_\varphi(-\pi/2), d_\mathcal{J} = \sup_{\varphi \in \mathcal{J}} h_\varphi(\pi/2). \)

**Proposition 3.** For any \( D \)-invariant subspace \( W \subset U_a, \) there exists its residual interval \( I_W \) equal to \([c_\mathcal{J}; d_\mathcal{J}] \cap (-a; a), \) where \( \mathcal{J} = \mathcal{F}(W^0). \)

**Proof.** Set \( I_0 = (-a; a) \cap [c_\mathcal{J}; d_\mathcal{J}]. \) It is known that the shift operators
\[
f \mapsto f(\cdot + y), \quad (f \mapsto f(\cdot - y)), \quad y > 0,
\]
act continuously in \( U(-a; +\infty) \) and in \( U(-\infty; a), \) respectively. For any \( f \in W_{I_0}, \) we have the representation
\[
f = f_+ + f_-, \quad f_-, f_+ \in W_{I_-}, W_{I_+} \subset U(-\infty; a), W_{I_-} \subset U(-a; +\infty). \]

Also, for any \( S \in F^{-1}(\mathcal{J}), \)
\[
\operatorname{supp} g(\cdot - y) \cap \operatorname{supp} S = \emptyset, \quad g \in W_{I_-}, \quad y > 0,
\]

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From the above we derive that
\[ S(f) = S(f_- + f_+) = \lim_{y \to 0^+} (S(f_-(x-y)) + S(f_+(x+y))) = 0 \]
for any \( S \in \mathcal{F}^{-1}(\mathcal{J}) \). The duality principle (Proposition 1) implies that \( W_{I_0} \subset W \).

Let an interval \( I' \subset I_0 \) be relatively closed in \((-a; a)\). From the definitions of \( c_{\mathcal{J}} \) and \( d_{\mathcal{J}} \) and general theory of UDF and ultradistributions, we get that for any \( c' \in (c_{\mathcal{J}}; d_{\mathcal{J}}) \setminus I' \), there exist \( S \in \mathcal{F}^{-1}(\mathcal{J}) \), \( f \in \mathcal{U}_a \) and \( \delta > 0 \) satisfying

\[ S(f) \neq 0, \quad \text{supp } f \subset (c' - \delta; c' + \delta) \subset (c_{\mathcal{J}}; d_{\mathcal{J}}) \setminus I'. \]

By the duality principle, we conclude that \( f \notin W \). However, \( f \in W_I \). It means that \( I_0 \) is the minimal one among all relatively closed in \((-a; a)\) intervals with the property \( W_I \subset W \). That is, \( I_W = I_0 \).

We are going to show that the spectral synthesis problem for a \( D \)-invariant subspace \( W \subset \mathcal{U}_a \) is equivalent to the problem of reconstructing of submodule \( \mathcal{J} = \mathcal{F}(W^0) \) under the assumption that its zero set and indicator segment are known.

Notice that \( D \)-invariant subspace \( W \) admitting weak spectral synthesis (4), is minimal among all \( D \)-invariant subspaces \( W \subset \mathcal{U}_a \) such that

\[ I_W = I_W, \quad \exp(W) = \exp(W). \]

By Proposition 2, we see that the annihilator submodule \( \mathcal{J} = \mathcal{F}(W^0) \) is maximal among all submodules \( \mathcal{J} \subset \mathcal{P}_a \) with \( \mathcal{Z}_J = \mathcal{Z}_\mathcal{J} \) and \( [c_{\mathcal{J}}; d_{\mathcal{J}}] = [c_{\mathcal{J}}; d_{\mathcal{J}}] \).

A submodule \( \mathcal{J} \subset \mathcal{P}_a \) is said to be weakly localisable if it has the described maximality property. Such submodule contains all functions \( \psi \in \mathcal{P}_a \) with indicator diagrams \( [ic_{\psi}; id_{\psi}] \subset [ic_{\mathcal{J}}; id_{\mathcal{J}}] \) vanishing on \( \mathcal{Z}_\mathcal{J} \).

We have proved the following assertion (compare with [8, Proposition 2]).

**Proposition 4.** \( D \)-invariant subspace \( W \subset \mathcal{U}_a \) admits weak spectral synthesis if and only if its annihilator submodule \( \mathcal{J} \subset \mathcal{P}_a \) is weakly localisable.

**Remark 1.** In particular, if \( I_W = (-a; a) \) then Proposition 4 establishes classical duality between the spectral synthesis problem and the problem of local description of submodules. In this case, the submodule \( \mathcal{J} \) is called localisable or ample (see [6, 8, 9, 13]).

3. LOCAL DESCRIPTION OF SUBMODULES IN \( \mathcal{P}_a \)

3.1. Weak Localisability of Stable Submodules

Let \( \mathcal{P} \) be a locally convex space of entire functions. It is called \( b \)-stable if, for any bounded subset \( \mathcal{B} \subset \mathcal{P} \), the set of all entire functions of the form

\[ \psi = \frac{\varphi}{z - \lambda}, \quad \lambda \in \mathbb{C}, \quad \varphi \in \mathcal{B}, \]

is a bounded subset of \( \mathcal{P} \).

As we have noticed above, \( \mathcal{P}_a \) is a locally convex space of \((LN^*)\)-type. It is well-known that a subset \( \mathcal{B} \subset \mathcal{P}_a \) is bounded if and only if \( \mathcal{B} \) is a bounded subset in some \( \mathcal{P}_k \) (see [14, Theorem 2]). Taking into account this fact and the description of the topology in \( \mathcal{P}_a \), we can easily derive that \( \mathcal{P}_a \) is a bornological and \( b \)-stable space. It means that we may apply abstract methods developed in [13, 15] to study submodules in \( \mathcal{P}_a \).

A submodule \( \mathcal{J} \subset \mathcal{P}_a \) is generated by functions \( \varphi_1, \ldots, \varphi_m \) \((m \text{-generated})\) if it is the closure of the set \( \{p_1\varphi_1 + \cdots + p_m\varphi_m\} \), where \( p_1, \ldots, p_m \) are polynomials.

A submodule \( \mathcal{J} \subset \mathcal{P}_a \) generated by one function \( \varphi \in \mathcal{P}_a \), is said to be principal. We denote it by \( \mathcal{J}_\varphi \).

Given a set \( \mathcal{S} = \{\varphi_\alpha\} \subset \mathcal{P}_a \), a submodule \( \mathcal{J} \subset \mathcal{P}_a \) generated by \( \mathcal{S} \) equals the closure in \( \mathcal{P}_a \) of the set of all finite sums \( p_1\varphi_{\alpha_1} + \cdots + p_k\varphi_{\alpha_k} \), where \( \varphi_{\alpha_j} \in \{\varphi_\alpha\}, \quad p_j \in \mathbb{C}[z] \). That is, \( \mathcal{J} \) is minimal among all submodules containing \( \mathcal{S} \).
Given a submodule $J \subset P_a$, let $\lambda \in \mathbb{C}$ be a zero of $J$ of multiplicity $n_\lambda \geq 0$, that is any function $\psi \in J$ vanishes at $\lambda$ with the multiplicity at least $n_\lambda$ and there exists $\psi_\lambda \in J$ for which $\lambda$ is a zero of multiplicity equaled to $n_\lambda$. A submodule $J$ is stable at the point $\lambda$ if the following implication holds: if $\varphi \in J$, and $\lambda$ is a zero of $\varphi$ of the multiplicity $n > n_\lambda$ then $\frac{\varphi}{x^\lambda} \in J$.

A submodule $J$ is stable if it is stable at any point $\lambda \in \mathbb{C}$.

From the results of [15, §4] we derive that

1) stability of a submodule $J$ at one point $\lambda_0 \in \mathbb{C}$ implies its stability at any point $\lambda \in \mathbb{C}$;

2) principal submodules in $P_a$ are stable.

It is easy to see that any weakly localisable submodule $J \subset P$ is stable. However,

a) there are stable submodules in $P_a$ which are not weakly localisable (see remarks at the end of this subsection);

b) there are unstable submodules in $J$ (Proposition 7 below).

We are going to obtain a condition guaranteeing weak localisability of stable submodule. In [13], Krasichkov-Ternovski studied abstract spaces of holomorphic vector functions. For our purpose, we cite his results formulating them for entire scalar functions.

Let $P$ be a locally convex space of entire functions and a topological module over the ring $\mathbb{C}[z]$. A closed submodule $J \subset P$ is said to be $b$-saturated with respect to a function $\psi \in P$ if there exists a bounded subset $B \subset P$ such that the following implication holds: $\rho$ is an entire function and

$$|\rho(z)\psi(z)| \leq |\varphi(z)| + |\psi(z)|, \quad \forall z \in \mathbb{C}, \quad \forall \varphi \in B \bigcap J,$$

implies $\rho = \text{const}$.

Denote by $P$ a bornological and $b$-stable space of entire functions.

**Bornological version of the individual theorem** [13]. Let $J$ be a stable closed submodule in $P$ and a function $\psi \in P$ satisfy the condition $Z_\psi \subset Z_\psi$. Then, $\psi \in J$ if and only if $J$ is $b$-saturated with respect to $\psi$.

Given $\varphi \in P_a$, we denote by $J(\varphi)$ the submodule consisting of all functions $\psi \in P_a$ such that $\psi = \rho \varphi$, where $\rho$ is an entire function of minimal type with respect to the order 1. It means that the submodule $J(\varphi)$ contains all functions $\psi \in P_a$ which zero sets satisfy $Z_\varphi \subset Z_\psi$ and which indicator diagrams equal the indicator diagram of $\varphi$. Clearly, $J(\varphi)$ is a weakly localisable submodule; in particular, $J_a \subset J(\varphi)$.

Now, we formulate and prove Proposition 5 and Theorem 1. They are very similar to Lemma 1 and Theorem 1 in [8]. There we considered the Schwartz module $P_a = F((C^\infty(-a; a))^\prime)$. Proofs of both assertions are also similar to ones in [8]. Nevertheless, for the sake of completeness, we state them here too.

**Proposition 5.** Let $J \subset P_a$ be a stable submodule. Assume that $\varphi \in P_a$ satisfies the conditions $Z_\varphi \subset Z_\varphi$ and $[-ih_\varphi(-\pi/2); i\varphi(\pi/2)] \subset (c_J; d_J)$. Then, $J(\varphi) \subset J$.

**Proof.** Consider an arbitrary function $\psi \in J(\varphi)$. By the inequalities $c_J < c_\varphi$, $d_J > d_\varphi$ and the definitions of $c_J$ and $d_J$, we get that $c_J \leq c_\varphi < d_\varphi < d_J$ for some $\varphi_1, \varphi_2 \in J$.

Set $\varphi_B = \varphi_1 + \varphi_2$. This is a function of completely regular growth with respect to the order 1. Notice that the indicator diagram of $\psi \in J(\varphi)$ equals the closed interval $[c_\varphi; d_\varphi]$. Hence, it is compactly contained in the indicator diagram of $\varphi_B$. It follows that

$$\frac{\psi(z)}{\varphi_B(z)} \to 0, \quad z = re^{i\theta}$$

as $r \to \infty$ staying outside of some set of zero respective measure. The relation (6) holds uniformly with respect to $\theta \in \{\pi/2 - \theta < \delta\} \cup \{-\pi/2 + \theta < \delta\}$, where $\delta > 0$ is a sufficiently small fixed number.

Show that the submodule $J$ is $b$-saturated with respect to the function $\psi$. We set $B = \{\varphi_B\}$ and consider an entire function $\rho$ satisfying

$$|\rho(z)\varphi_B(z)| \leq |\psi(z)| + |\varphi_B(z)|, \quad z \in \mathbb{C}.$$
By the maximum modulus principle, from (6) we get that $\rho$ is bounded on the imaginary axis. It follows that $\rho = \text{const}$, and the stable submodule $J$ is $b$-saturated with respect to $\psi$. By the bornological version of the individual theorem, we conclude that $\psi \in J$.

**Theorem 1.** A stable submodule $J \subset P_a$ is weakly localisable if and only if it contains a function $\varphi$ with the property $J(\varphi) \subset J$.

**Proof.** Clearly, we only need to prove the sufficient part.

1) First, consider the situation, when $J(\varphi) \subset J$, and the indicator diagram of $\varphi$ equals $i[c_J; d_J]$ (Unlike in the Schwartz module [8, Theorem 1], the case of $c_J = d_J$ is non-trivial in $P_a$).

Let $\psi \in P_a$ be such that $Z_\psi \supset Z_J$, and $i[c_\psi; d_\psi] \subset i[c_J; d_J]$. To make it sure that $\psi \in J$, it is sufficient to check that $J$ is $b$-saturated with respect to $\psi$.

Set

$$B = \{ \varphi \in H(\mathbb{C}) : |\varphi(z)| \leq |\psi(z)| + |\varphi(z)|, \ z \in \mathbb{C} \}.$$  

By the topological properties of $P_a$, $B$ is bounded. Consider an entire function $\rho$ satisfying

$$|\rho(z)\varphi(z)| \leq |\varphi(z)| + |\psi(z)|, \ \forall \ z \in \mathbb{C}, \ \varphi \in B \cap J.$$  

In particular, we have

$$|\rho(z)\varphi(z)| \leq |\varphi(z)| + |\psi(z)|, \ \forall \ z \in \mathbb{C}.$$  

It follows that $\rho$ is of minimal type with respect to the order $1$ and $\rho \varphi \in J(\varphi)$. Taking into account the definition of $B$ and the relation $J(\varphi) \subset J$, we obtain that $\rho \varphi \in B \cap J$. It means that (8) holds with $\varphi = \rho \varphi$. Hence,

$$|\rho^2(z)\varphi(z)| \leq |\rho(z)\varphi(z)| + |\psi(z)| \leq 2(|\varphi(z)| + |\psi(z)|), \ z \in \mathbb{C}.$$  

This estimate implies that $\frac{\rho^2}{2} \varphi \in B \cap J$. Now, setting $\varphi = \frac{\rho^2}{2} \varphi$ in (8), we obtain

$$\left| \frac{\rho^2(z)}{2} \varphi(z) \right| \leq 2(|\varphi(z)| + |\psi(z)|), \ z \in \mathbb{C},$$

and $\frac{\rho^2}{2} \varphi \in B \cap J$. Continuing to argue by the same way, we prove that

$$\left| \frac{\rho^n(z)}{2^n - 1} \varphi(z) \right| \leq |\varphi(z)| + |\psi(z)|, \ z \in \mathbb{C},$$

holds for all $n \in \mathbb{N}$. Hence, $\rho = \text{const}$, and $J$ is $b$-saturated with respect to $\psi$.

2) Now, suppose that there exists $\varphi \in P_a$ satisfying

$$J(\varphi) \subset J, \ [c_\varphi; d_\varphi] \subset [c_J; d_J] \subset (a; b).$$

It means that at least one of the expressions, $\delta_1 = c_\varphi - c_J$ or $\delta_2 = d_J - d_\varphi$, is positive. Consider in details the case when both, $\delta_1$ and $\delta_2$, are positive.

By Proposition 5, we have

$$J(e^{\delta_1 z} \varphi) \subset J, \ J(e^{-\delta_2 z} \varphi) \subset J$$

for all $\delta' \in [0; \delta_1]$ and $\delta'' \in [0; \delta_2]$. In particular,

$$e^{\delta_1 z} \varphi, e^{-\delta_2 z} \varphi \in J, \ \delta' \in [0; \delta_1], \ \delta'' \in [0; \delta_2].$$  

Set $\Phi = (e^{\delta_1 z} + e^{-\delta_2 z}) \varphi$, The relations

$$\lim_{\delta' \to \delta_1} e^{\delta' z} \varphi = e^{\delta_1 z} \varphi, \ \lim_{\delta'' \to \delta_2} e^{-\delta'' z} \varphi = e^{-\delta_2 z} \varphi,$$

hold with respect to the topology of $P_a$. Together with (9), they lead to the inclusion $\Phi \in J$.

Further, any function $\Psi \in J(\Phi)$ is of the form

$$\Psi = \rho \Phi = \rho(e^{\delta_1 z} + e^{-\delta_2 z}) \varphi,$$

where $\rho$ is an entire function of minimal type with respect to the order $1$. 

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Again by (10), we obtain the required relation \( \Psi \). By this relation and Proposition 5, we derive that

\[
\rho \varphi \in J, \quad \Psi_{\delta'} = e^{i\delta'z} \rho \varphi \in J, \quad \forall \delta' \in (0; \delta_1), \quad \Psi_{\delta''} = e^{-i\delta''z} \rho \varphi \in J, \quad \forall \delta'' \in (0; \delta_2).
\]

The relation

\[
\Psi = \lim (\Psi_{\delta'} + \Psi_{\delta''}) \quad \text{as} \quad \delta' \to \delta_1, \quad \delta'' \to \delta_2,
\]

leads to the inclusion \( \Psi \in J \). Because of this is true for an arbitrary function \( \Psi \in J(\Phi) \), we conclude that \( J(\Phi) \subset J \).

The indicator diagram of the function \( \Phi \) equals the indicator segment of \( J \). It means that we are in the conditions of the first part of the proof.

3) It remains to consider the case when \( c_J = a \) or (and) \( d_J = b \).

Let \( \Psi \in P_a \) and \([c_\Psi; d_\Psi] \subset [c_J; d_J], Z_\Psi \supset Z_J \). Prove that \( \Psi \in J \). For this purpose, we choose and fix a closed interval \([c'; d']\) such that

\[
[c'; d'] \subset (a; b), \quad [c_\Psi; d_\Psi] \subset [c'; d'], \quad [c_\varphi; d_\varphi] \subset [c'; d'].
\] (10)

Denote by \( J' \) a weakly localisable submodule with the indicator segment equaled to \([c'; d']\) and zero set \( Z_{J'} = Z_J \). It is easy to see that \( J' = J \cap J' \) is the closed stable submodule with the indicator segment \([c'; d']\) and zero set \( Z_J = Z_J \).

From (10), it follows that \( J(\varphi) \subset J' \). Applying first two parts of the proof, we conclude that \( J = J' \).

Again by (10), we obtain the required relation \( \Psi \in J \subset J \).

**Corollary 1.** Let \( J \subset P_a \) be a stable submodule. If there exists a function \( \varphi_0 \in J_0 \) such that the principal submodule generated by \( \varphi_0 \) is weakly localisable then \( J \) is also weakly localisable.

**Proof.** Clearly, we have \( J(\varphi_0) = J_{\varphi_0} \subset J \). By Theorem 1, \( J \) is weakly localisable.

**Corollary 2.** If the indicator segment of stable submodule \( J \subset P_a \) is non-compact in \((-a; a)\) then \( J \) is weakly localisable. In particular, a stable submodule \( J \subset P_a \) is localisable if and only if \( c_J = -a, \ d_J = a \).

Recall that an entire function \( \psi_0 \) is a multiplier of the space \( P_a \) if the correspondence \( \varphi \mapsto \varphi \psi_0 \) defines a continuous map of \( P_a \) into itself.

According [20, Proposition 1], the set \( M_a \) of all multipliers of \( P_a \) has the description

\[
M_a = \left\{ \psi \in H(\mathbb{C}) : \forall \varepsilon > 0 \sup_{z \in \mathbb{C}} \frac{|\psi(z)|}{e^{\varepsilon \omega(z) + \varepsilon |\text{Im } z|}} < \infty \right\}.
\] (11)
A divisor of $\mathcal{P}_a$ is a function $\psi_0 \in \mathcal{M}_a$ such that the implication

$$\psi \in \mathcal{P}_a, \quad \frac{\psi}{\psi_0} \in H(\mathbb{C}) \implies \frac{\psi}{\psi_0} \in \mathcal{P}_a$$

holds.

**Remark 2.** Because of Theorem 2, Corollary 1 is worth considering only for non-trivial submodule $\mathcal{J}$ satisfying the relation

$$d_\mathcal{J} - c_\mathcal{J} = 2\pi D_{BM}(\mathcal{Z}_\mathcal{J}). \quad (12)$$

Principal submodules generated by slowly decreasing functions (see [19]) or by divisors of the space $\mathcal{P}_a$ (see [20]) satisfy (12), and they are weakly localisable. It may be proved by the scheme we applied in [21] together with the results on weighted polynomial approximation obtained in [22, VI.H.2].

It is known that there exist stable submodules satisfying (12) in the Schwartz module $\mathcal{P}_a$, which are not weakly localisable (see [7, 8, 19]). We guess that the same is true in $\mathcal{P}_a$. We are going to study in details the critical case (12) for submodules in $\mathcal{P}_a$ later.

### 3.2. Stability

Theorems 1 and 2 and their corollaries motivate to study conditions of stability for submodules in $\mathcal{P}_a$. As we have noticed at the beginning of the previous subsection, any principal submodule is stable. However, for 2-generated submodule, it may fail.

**Proposition 7.** Let $\mathcal{J}$ be a submodule generated in $\mathcal{P}_a$ by functions $\varphi_1, \varphi_2 \in \mathcal{P}_a$, and $i[c_1; d_1], i[c_2; d_2]$ be their indicator diagrams.

If $[c_1; d_1] \subset (-a; 0), [c_2; d_2] \subset (0; a)$ then the submodule $\mathcal{J}$ is not stable.

**Proof.** By Proposition 2, there exists a unique $D$-invariant subspace $W \subset U_a$ such that $\mathcal{F}(W^0) = \mathcal{J}$. Its residual interval equals $[c_1; d_2]$ (Proposition 3).

Given a function $f \in U_a$, by the duality principles (Propositions 1, 2), we see that

$$f \in W \iff S_j(f^{(k)}) = 0, \quad k = 0, 1, 2, \ldots, \quad j = 1, 2,$$

where $S_j = \mathcal{F}^{-1}(\varphi_j)$ $j = 1, 2$. In particular, for an arbitrary $\varepsilon > 0$ if $f \in U_a$ vanishes on the set $[c_1 - \varepsilon; d_1 + \varepsilon] \cup [c_2 - \varepsilon; d_2 + \varepsilon]$ then $f \in W$.

Notice that $\mathcal{Z}_\mathcal{J} = \mathcal{Z}_{\varphi_1} \cap \mathcal{Z}_{\varphi_2}$, and $2\pi D_{BM}(\mathcal{Z}_\mathcal{J}) < d_2 - c_1$. Hence, if $\mathcal{J}$ is a stable submodule then it is weakly localisable (by Proposition 4 and Theorem 2), that is

$$W = W_{I_W} + \text{span}(\text{Exp}(W)).$$

The set of exponents of the system $\text{Exp}(W)$ equals $\mathcal{Z}_\mathcal{J}$. This system is incomplete in $C[c_j; d_j], j = 1, 2$. Taking into account these two facts, consider a function $g \in W$ vanishing on $[c_1; d_1] \cup [c_2; d_2]$, but $g \neq 0$ on $I_W$. By the above, $g$ belongs to the closure of $\text{span}(\text{Exp}(W))$ in each of the spaces $C[c_1; d_1], C[c_2; d_2], C[c_1; d_2]$. It follows that $g = 0$ $I_W = [c_1; d_2]$ according to the result on uniqueness of mean periodic continuation ([2, §1], [3, §9]).

It leads to the contradiction. Hence, the submodule $\mathcal{J}$ is not stable.

Below, we consider the situations, when finitely generated submodule is necessarily stable.

**Theorem 3.** Let $\varphi_1, \varphi_2 \in \mathcal{P}_a$ be functions of minimal type with respect to the order 1, and $\varphi_1 \varphi_2 \in \mathcal{P}_a$. Then, they generate stable submodule $\mathcal{J} \subset \mathcal{P}_a$.

**Proof.** Without loss of generality (wlog), we may assume that $\varphi_1(0) = \varphi_2(0) = 1$.

It is known that $\mathcal{J}$ is stable if and only if the identically zero function is contained in the closure in $\mathcal{P}_a$ of the set $\{p\varphi_1 - q\varphi_2\}$, where $p, q \in \mathbb{C}[z]$, and $p(0) = q(0) = 1$ ([15, Proposition 4.9]).

Because of the condition $\varphi_1 \varphi_2 \in \mathcal{P}_a$, there exist $r_1, r_2 \in (0; 1)$ such that $r_1 + r_2 < 1$ and for any $\varepsilon > 0$

$$\sup_{z \in \mathbb{C}} \frac{\varphi_j(z)}{e^r\omega(|z|) + \varepsilon|\text{Im } z|} < +\infty, \quad j = 1, 2.$$

Fix an arbitrary $\delta > 0$ satisfying $r_1 + r_2 + 2\delta < 1$. Taking into account the initial properties of $\omega$, we apply the theorem on weighted polynomial approximation on the real line for the restrictions of entire
functions of minimal type [22, VI.H.2]. By this result, we find two sequences of polynomials, \( \{p_k\}, \{q_k\} \), such that \( p_k(0) = 1, q_k(0) = 1, k = 1, 2, \ldots \), and

\[
\sup_{x \in \mathbb{R}} \frac{|\varphi_1(x) - p_k(x)|}{e^{(\gamma + \delta)\omega(|x|)}} \to 0, \quad k \to \infty, \tag{13}
\]

\[
\sup_{x \in \mathbb{R}} \frac{|\varphi_2(x) - q_k(x)|}{e^{(\gamma + \delta)\omega(|x|)}} \to 0, \quad k \to \infty. \tag{14}
\]

Now, by standard argument including the application of Phragmén–Lindelöf principle (see [22, VI.H.2]) and the criterion of convergency for a countable sequence in locally convex space of \((LN^*)\)-type (see [14]), we derive that

\[
q_k \varphi_1 - p_k \varphi_2 \to 0, \quad k \to \infty,
\]

with respect to the topology of \( \mathcal{P}_a \). By the above cited criterion of stability, we conclude that \( \mathcal{J} \) is stable submodule.

**Corollary 3.** Let \( \varphi_1, \ldots, \varphi_m \in \mathcal{P}_a \) be functions of minimal type with respect to the order 1 and

\[
\varphi_j \varphi_{j+1} \in \mathcal{P}_a, \quad j = 1, \ldots, m - 1.
\]

Then,

1) the submodule \( \mathcal{J} \) generated by \( \varphi_1, \ldots, \varphi_m \) in \( \mathcal{P}_a \) is stable;

2) if, in addition, there is a divisor of \( \mathcal{P}_a \) among functions \( \varphi_j, j = 1, \ldots, m \), then the submodule \( \mathcal{J} \) is weakly localisable.

**Proof.** 1) Wlog,

\[
\varphi_j(0) = 1, \quad j = 1, 2, \ldots, m.
\]

From (15) and Theorem 3, it follows that 2-generated submodules

\[
\mathcal{J}_{\varphi_j, \varphi_{j+1}} = \{p \varphi_j + q \varphi_{j+1}\}, \quad p, q \in \mathbb{C}[z], \quad j = 1, \ldots, m - 1,
\]

are stable. Applying the necessary part of the stability criterion ([15, Proposition 4.9]) for these submodules, we find generalized sequences of polynomials \( p_{a, j}^{(1)}, q_{a, j}^{(1)} \) such that

\[
p_{a, j}^{(1)}(0) = q_{a, j}^{(1)}(0) = 1, \quad p_{a, j}^{(1)} \varphi_j - q_{a, j}^{(1)} \varphi_{j+1} \to 0, \quad \alpha \nearrow
\]

By the sufficient part of the same criterion, we see that \( \mathcal{J} \) is stable if for any complex numbers \( c_j, j = 1, \ldots, m \), satisfying \( c_1 + \cdots + c_m = 0 \), zero function belongs to the closure of the set \( \{s_{1, \varphi_1} + \cdots + s_{m, \varphi_m}\} \), where

\[
s_j \in \mathbb{C}[z], \quad s_j(0) = c_j, \quad j = 1, \ldots, m.
\]

Set

\[
s_{1, \alpha} = c_1 p_{a, 1}^{(1)}, \quad s_{2, \alpha} = (c_1 + c_2)p_{a, 2}^{(2)} - c_1 q_{a, 1}^{(1)},
\]

\[
\begin{align*}
s_{3, \alpha} &= (c_1 + c_2 + c_3)p_{a, 3}^{(3)} - (c_1 + c_2)q_{a, 2}^{(2)}, \\
\vdots \\
s_{m-1, \alpha} &= (c_1 + \cdots + c_{m-1})p_{a, m-1}^{(m-1)} - (c_1 + \cdots + c_{m-2})q_{a, m-2}^{(m-2)}, \\
s_{m, \alpha} &= c_m q_{a, m}^{(m-1)}.
\end{align*}
\]

It is easy to check that \( s_{j, \alpha}(0) = c_j, \forall \alpha, j = 1, \ldots, m \), and \( s_{1, \alpha} \varphi_1 + \cdots + s_{m, \alpha} \varphi_m \to 0 \) with respect to the topology of \( \mathcal{P}_a \). It follows that \( \mathcal{J} \) is a stable submodule.

2) Assume that \( \varphi_1 \) is a divisor of \( \mathcal{P}_a \). According to Remark 2, the principal submodule \( \mathcal{J}_{\varphi_1} \) is weakly localisable, that is \( \mathcal{J}_{\varphi_1} = \mathcal{J}(\varphi_1) \subset \mathcal{J} \). By the first assertion of the corollary, \( \mathcal{J} \) is a stable submodule. Applying Corollary 1, we conclude that \( \mathcal{J} \) is a weakly localisable submodule. \( \square \)
Corollary 4. Let \( \{ \varphi_\alpha \} \subset M_\alpha \). Then,
1) \( \{ \varphi_\alpha \} \) generates a stable submodule \( J \) in \( P_\alpha \);
2) if, in addition, \( \{ \varphi_\alpha \} \) contains a divisor of \( P_\alpha \) then \( J \) is weakly localisable.

Proof. 1) Wlog, we assume that \( \varphi_\alpha (0) = 1 \) for any \( \alpha \). Let \( \psi, \varphi \in J \) and \( \psi(0) = 0 \).

Every function in \( J \) is of minimal type with respect to the order 1. Taking into account the description of \( M_\alpha \) cited above and Theorem 3, we conclude that the functions \( \psi \) and \( \varphi_\alpha \) generate stable submodule \( J_{\psi,\varphi_\alpha} \). It leads to the relations \( \frac{\psi}{\varphi} \in J_{\psi,\varphi_\alpha} \subset J \).

2) The second assertion is proved by the similar way as it was done to justify the second part of Corollary 3.

4. APPLICATIONS TO D-INARIANT SUBSPACES

In this section, we apply the results of the previous sections to study weak spectral synthesis for the differentiation operator in \( U_\alpha \).

By Propositions 3, 4, Corollary 1 and Theorem 2, we obtain

Theorem 4. Let \( W \subset U_\alpha \) be D-invariant subspace with residual interval \( I_W \) of length \( d \), and \( \Lambda \) be a sequence of exponents of \( \text{Exp}(W) \). Assume that the annihilator submodule \( J = \text{F}(W^0) \) is stable, and one of the conditions a) \( 2\pi D_{BM}(\Lambda) < d \) or b) there exists \( \varphi_0 \in J \) generating weakly localisable principal submodule \( J_{\varphi_0} \) hold. Then, \( W \) admits weak spectral synthesis.

Corollary 5. Let \( S \in U_\alpha^* \), and \( I \subset (-\alpha; \alpha) \) is a relatively closed interval such that \( I \div \text{ch supp} S \neq \emptyset \). Then, \( W = \ker T_{S,1} \) is D-invariant subspace admitting weak spectral synthesis with \( I_W = I \); and the sequence of exponents of \( \text{Exp}(W) \) equals \( Z_\varphi \), where \( \varphi = \text{F}(S) \).

Proof. By standard methods, one can easily check that \( W = \ker T_{S,1} \) is a D-invariant subspace. Applying the duality principles (Propositions 1, 2), we justify that

\[ J = \text{F}(W^0) = \text{span} \{ e^{-itp} : t \in I \div \text{ch supp} S, p \in \mathbb{C}[z] \}, \]

is the annihilator submodule of \( W \). It follows that the indicator segment of \( J \) is \( I \), and its zero set equals \( Z_\varphi \). Together with Propositions 2, 3, these facts imply that \( I_W = I \), and the sequence of exponents of \( \text{Exp}(W) \) is \( Z_\varphi \).

Remembering that \( I \div \text{ch supp} S \neq \emptyset \), we apply Beurling–Malliavin theorem on radius of completeness and Paley–Wiener–Schwartz theorem for ultradistributions. It gives us the inequality \( 2\pi D_{BM}(Z_{J_2}) < d \), where \( d \) is the length of \( I \). Now, by Theorem 4, to justify that \( W \) admits weak spectral synthesis it is sufficient to check that \( J \) is a stable submodule.

Wlog, we assume that \( \varphi(0) = 1 \) and prove that \( J \) is stable at the origin. Together with [15, Proposition 4.2] it will lead to the required assertion.

Consider \( \Psi \in J \), \( \Psi(0) = 0 \). By (17), we have \( \Psi = \lim_{\alpha \to \infty} \Psi_\alpha \), where

\[ \Psi_\alpha = (a_1, \alpha e^{-it_1, \alpha z}p_1, \alpha + \cdots + a_n, \alpha e^{-it_n, \alpha z}p_n, \alpha) \varphi, \]

\[ a_j, \alpha \in \mathbb{C}, t_j, \alpha \in I \div \text{ch supp} S, p_j, \alpha \in \mathbb{C}[z], j = 1, \ldots, n. \]

Notice that

\[ \Psi_\alpha(0) = (a_1, \alpha p_1, \alpha(0) + \cdots + a_n, \alpha p_n, \alpha(0)) \to 0. \]

(18)

Setting

\[ \Phi_\alpha = \sum_{j=1}^{n} a_j, \alpha e^{-it_j, \alpha z}(p_j, \alpha - p_j, \alpha(0)) \varphi + \sum_{j=1}^{n} a_j, \alpha p_j, \alpha(0)(e^{-it_j, \alpha z} - 1) \varphi, \]

we get the representation \( \Psi_\alpha = \Phi_\alpha + \Psi_\alpha(0) \varphi \). Clearly, \( \Phi_\alpha(0) = 0 \) and

\[ \Phi_\alpha z = \sum_{j=1}^{n} a_j, \alpha e^{-it_j, \alpha z}q_j, \alpha \varphi + \sum_{j=1}^{n} a_j, \alpha p_j, \alpha(0) e^{-it_j, \alpha z} \frac{1}{z} \varphi, \]

where
where \( q_{j,\alpha} = \frac{p_{j,\alpha} - p_{j,\alpha}(0)}{z} \) are polynomials.

Further, by the results of [22, VI.E.1], we have
\[
e^{-it_{j,\alpha}z} - 1 \in \text{span} \left\{ e^{-it\varphi} : t \in I, \varphi \in \text{ch supp } S \right\},
\]
j = 1, \ldots, n, \alpha, \forall \alpha, \text{ where the closure is taken with respect to the topology of } \mathcal{P}_\alpha. \text{ Hence, } \frac{\varphi_\alpha}{z} \in \mathcal{J} \text{ for every } \alpha. \text{ Because of (18), we have } \Psi = \lim \frac{\varphi_\alpha}{z}.

From the topological properties of \( \mathcal{P}_\alpha \) it is not difficult to derive that \( \frac{\Psi}{z} = \lim \frac{\varphi_\alpha}{z} \). It follows that \( \frac{\Psi}{z} \in \mathcal{J} \), and \( \mathcal{J} \) is a stable submodule. \( \square \)

**Corollary 6.** Let \( I \subset (-\alpha; \alpha) \) be a relatively closed interval,
\[
S_j \in \mathcal{U}_\alpha', \quad \text{supp } S_j = \{0\}, \quad j = 1, \ldots, m,
\]
and \( S_j * S_{j+1} \in \mathcal{U}_\alpha, j = 1, \ldots, m - 1 \). Then, \( W = \bigcap_{j=1}^m \ker T_{S_j,1} \) admits weak spectral synthesis, \( I_W = I \), and the sequence of exponents of \( \text{Exp}(W) \) is \( \bigcap_{j=1}^m Z_{\varphi_j} \), where \( \varphi_j = \mathcal{F}(S_j) \).

**Proof.** Wlog, we assume that \( 0 \in I \) and \( \varphi_j(0) = 1, j = 1, \ldots, m \).

By standard methods including the duality principles (Propositions 1, 2), we derive that \( W \) is a \( D \)-invariant subspace in \( \mathcal{U}_\alpha \) and
\[
\mathcal{J} = \mathcal{F}(W^0) = \text{span} \left\{ e^{-it_{j,\alpha}z}p_{j,\alpha} \varphi_j : t_j \in I, p_{j,\alpha} \in \mathbb{C}[z], j = 1, \ldots, m \right\}
\]
is its annihilator submodule. From (20), it follows that \( \mathcal{T} \) is the indicator segment of \( \mathcal{J} \), and \( \mathcal{Z}_{\mathcal{J}} = \bigcap_{j=1}^m Z_{\varphi_j} \). It implies that \( I_W = I \), and the sequence of exponents of \( \text{Exp}(W) \) equals \( \mathcal{Z}_{\mathcal{J}} = \bigcap_{j=1}^m Z_{\varphi_j} \).

By Beurling–Malliavin theorem on radius of completeness, Paley–Wiener–Schwartz theorem for ultradistributions and the condition (19) we obtain that \( 2\pi \text{BM}(\mathcal{Z}_{\mathcal{J}}) < d \), where \( d \) is the length of \( I \).

Because of Theorem 4 and [15, Proposition 4.2], the only thing we need to do is to check that \( \mathcal{J} \) is stable at the origin.

Let \( \Psi \in \mathcal{J} \) and \( \Psi(0) = 0 \). From (20) it follows that \( \Psi = \lim \Psi_\alpha \), where
\[
\Psi_\alpha = \sum_{j=1}^m a_{j,\alpha} e^{-it_{j,\alpha}z}(p_{j,\alpha} - p_{j,\alpha}(0)) \varphi_j + \sum_{j=1}^m a_{j,\alpha} p_{j,\alpha}(0)(e^{-it_{j,\alpha}z} - 1) \varphi_j + \sum_{j=1}^m a_{j,\alpha} p_{j,\alpha}(0)(\varphi_j - \varphi_1) + \psi_\alpha.
\]

Notice that
\[
\psi_\alpha = \left( \sum_{j=1}^m a_{j,\alpha} p_{j,\alpha}(0) \right) \varphi_1 \to 0, \quad \alpha \nearrow,
\]
because \( \sum_{j=1}^m a_{j,\alpha} p_{j,\alpha}(0) = \Psi_\alpha(0) \to 0 \). It implies that \( \Psi = \lim \Phi_\alpha \), \( \alpha \nearrow \), where \( \Phi_\alpha = \Psi_\alpha - \psi_\alpha \).

Further, \( \Phi_\alpha(0) = 0 \) and
\[
\Phi_\alpha \frac{z}{z} = \sum_{j=1}^m a_{j,\alpha} e^{-it_{j,\alpha}z} q_{j,\alpha} \varphi_j + \sum_{j=1}^m a_{j,\alpha} p_{j,\alpha}(0) \frac{e^{-it_{j,\alpha}z} - 1}{z} \varphi_j + \sum_{j=1}^m a_{j,\alpha} p_{j,\alpha}(0) \frac{\varphi_j - \varphi_1}{z} = \Sigma_1 + \Sigma_2 + \Sigma_3,
\]
where \( q_{j,\alpha} = (p_{j,\alpha} - p_{j,\alpha}(0))z^{-1} \) are polynomials. Clearly, \( \Sigma_1 \in \mathcal{J} \).
Let $J_0$ be a submodule generated by $\varphi_1, \ldots, \varphi_m$. Because of Corollary 3, $J_0$ is stable. It follows that $\overline{\varphi_j \varphi_k} \in J_0$, $j = 1, \ldots, m$. Hence, $\Sigma_3 \in J_0 \subset J$.

Applying the results of [22, VI.E.1], we approximate every function

$$e^{-itjz} - 1 \overline{\varphi_j}, \quad j = 1, \ldots, m,$$

by the elements of span $\{e^{-itz}\varphi, \ t \in I\}$, and get $\Sigma_2 \in J$. Finally,

$$\frac{\Phi_{\alpha}}{z} \in J, \quad \text{and} \quad \frac{\Psi}{z} = \lim \frac{\Phi_{\alpha}}{z} \in J.$$

That is, $J$ is stable at the origin. \hfill \Box

**Corollary 7.** Let $I \subset (-a; a)$ be a relatively closed interval, and $\{S_\alpha\} \subset U'_I$ be such that $\varphi_\alpha = F(S_\alpha) \in \mathcal{M}_a \forall \alpha$. Then, $W = \bigcap_\alpha \ker T_{S_\alpha}$ is a weakly synthesable subspace, $I_W = I$, and the sequence of exponents of the system $\text{Exp}(W)$ equals $\bigcap_\alpha Z_{\varphi_\alpha}$.

Corollary 7 is justified by the same scheme as Corollary 6 was done (using Corollary 4 instead of Corollary 3).

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