Coxeter’s Frieze Patterns Arising from Dyck Paths

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Abstract
Frieze patterns are defined by objects of a category of Dyck paths, to do that, it is introduced the notion of diamond of Dynkin type $A_n$. Such diamonds constitute a tool to build integral frieze patterns.

Keywords
Diamond \cdot Dyck path \cdot Dyck paths category \cdot Frieze pattern \cdot Seed vector \cdot Triangulation

Mathematics Subject Classification 16G20 \cdot 16G30 \cdot 16G60

1 Introduction

Frieze patterns (as shown below) were introduced by Coxeter in the early 70s [9]. According to Propp [14], they arose from Coxeter’s study of metric properties of

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polytopes, and served as useful scaffolding for various sorts of metric data.

\[
\begin{array}{ccccccc}
\ldots & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 1 & 1 & 1 & 1 & 1 \\
\ldots & m_{-1,-1} & m_{00} & m_{11} & m_{22} & \ldots \\
m_{-2,-1} & m_{-1,0} & m_{0,1} & m_{12} & m_{23} & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
1 & 1 & 1 & 1 & 1 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & \ldots 
\end{array}
\]

Such patterns are defined as grids of numbers bounded from above by an infinite row of 0s followed by a row of 1s and such that every four adjacent numbers of the following form

\[
\begin{bmatrix}
  b \\
  a & c \\
  d 
\end{bmatrix}
\]

satisfy the arithmetic (or frieze) rule

\[ac - bd = 1.\] (1)

The third line (i.e., the first nontrivial row) can be chosen in an arbitrary way and then complete using the rule (1), which was named the modular equation by Coxeter [9].

A frieze is called closed, if it is also bounded from below by a line of 1s (followed by a line of 0s). A frieze is called integral if it consists of positive integers. The sequence of integers in the first non-trivial row is called quiddity sequence. This sequence completely determines the frieze pattern. Each frieze pattern is also periodic since it is invariant under glide reflection. The order of the frieze pattern is defined to be the number of rows minus one. It follows that each frieze pattern of order \( n \) is \( n \)-periodic [3]. Conway and Coxeter classified completely the frieze patterns whose entries are positive integers, and show that these frieze patterns constitute a manifestation of the Catalan numbers [7,8] giving a bijection between positive integer frieze patterns and triangulations of regular polygons with labeled vertices.

Frieze patterns appear independently in the 70s in the context of quiver representations, in such a case, the local arithmetic rule is an additive analogue of Coxeter’s unimodular rule. The generalization of the Coxeter’s unimodular rule on Auslander-Reiten quivers was found by Caldero and Chapoton [5]. Assem, Reutenauer and Smith introduced also a generalization of friezes by associating frieze patterns to Cartan matrices [1].

According to Morier-Genoud [13] there are mainly three approaches for the study of friezes:
1. A representation theoretical and categorical approach, in deep connection with the theory of cluster algebras, where entries in the friezes are rational functions.
2. A geometrical approach, in connection with moduli spaces of points in projective space and Grassmannians, where entries in the friezes are more often real or complex numbers.
3. A combinatorial approach, focusing on friezes with positive integer entries.

Many authors have studied friezes from the different points of view finding connections with different branches of mathematics [13]. For instance, Baur et al. [3] studied mutation of friezes proving how mutation of a cluster affects the associated frieze. On the other hand, Fontaine and Plamondon [11] presented a formula for the number of friezes of type $B_n$, $C_n$, $D_n$, and $G_2$. They conjectured that the number of friezes of type $E_6$, $E_7$, $E_8$, and $F_4$ is 868, 4400, 26592 and 112, respectively. In this way, the number of friezes can be defined as a Dynkin function in the sense of Ringel [15].

In this paper, frieze patterns are interpreted as objects of a novel category of Dyck paths introduced recently by Cañadas and Rios [6].

The following is a list of our main results, all of them dealing with integral closed friezes.
1. It is introduced the notion of diamond of Dynkin type $A_n$, and some of its properties are proved (see, Propositions 5, 6 and Theorem 8). Such diamonds are used as a tool to build frieze patterns.
2. It is proved that there is a bijective correspondence between the set of all vectors associated to positive integral diamonds of Dynkin type $A_n$ and triangulations of a polygon with $n + 3$ vertices. This result is a consequence of a bijection between such triangulations and Dyck paths of length $2(n + 1)$ (see Lemma 16 and Theorem 17).
3. It is proved that if $C(A_0, t) = \{A_i\}_{0 \leq i \leq p-1}$ is the minimal $p$-cycle generated by a diamond $A_0$ of Dynkin type $A_n$. Then $C(A_0, t)$ is in surjective correspondence with a direct sum of $p$ indecomposable objects of a Dyck paths category (see Theorem 21).

The paper is distributed as follows. In Sect. 2, we recall main definitions and notation to be used throughout the paper [2,6,12,17].

2 Preliminaries

In this section, we recall main definitions and notation to be used throughout the paper [2,6,12,17].

2.1 Cluster algebras from quivers

For quivers, cluster algebras are defined as follows:

Fix an integer $n \geq 1$. In this case, a seed $(Q, u)$ consists of a finite quiver $Q$ without loops or 2-cycles with vertex set $\{1, \ldots, n\}$, whereas $u$ is a free-generating set $\{u_1, \ldots, u_n\}$ of the field $\mathbb{Q}(x_1, \ldots, x_n)$.
Let \((Q, u)\) be a seed and \(k\) a vertex of \(Q\). The mutation \(\mu_k(Q, u)\) of \((Q, u)\) at \(k\) is the seed \((Q', u')\), where:

(a) \(Q'\) is obtained from \(Q\) as follows;

1. reverse all arrows incident with \(k\),
2. for all vertices \(i \neq j\) distinct from \(k\), modify the number of arrows between \(i\) and \(j\), in such a way that a system of arrows of the form \((i \xrightarrow{r} j, i \xrightarrow{s} k, k \xrightarrow{t} j)\) is transformed into the system \((i \xleftarrow{r} j, k \xrightarrow{s} i, j \xleftarrow{t} k)\). And the system \((i \xrightarrow{r} j, j \xrightarrow{s} k, k \xrightarrow{t} i)\) is transformed into the system \((i \xleftarrow{r} j, k \xrightarrow{s} i, j \xleftarrow{t} k)\). Where, \(r, s\) and \(t\) are non-negative integers,

\(\text{an arrow} \ i \xrightarrow{l} j\), with \(l \geq 0\) means that \(l\) arrows go from \(i\) to \(j\) and an arrow \(i \xleftarrow{l} j\), with \(l \leq 0\) means that \(-l\) arrows go from \(j\) to \(i\).

(b) \(u'\) is obtained form \(u\) by replacing the element \(u_k\) with

\[
u_k = \frac{1}{u_k} \prod_{\text{arrows } i \rightarrow k} u_i + \prod_{\text{arrows } k \rightarrow j} u_j.\tag{2}\]

If there are no arrows from \(i\) with target \(k\), the product is taken over the empty set and equals 1. It is not hard to see that \(\mu_k(\mu_k(Q, u)) = (Q, u)\). In this case the matrix mutation \(B'\) has the form

\[
b'_{ij} = \begin{cases} 
b_{ij}, & \text{if } i = k \text{ or } j = k, \\
b_{ij} + sgn(b_{ik})[b_{ik}b_{kj}]_+, & \text{else}, \end{cases}
\]

where \([x]_+ = \max(x, 0)\). Thus, if \(Q\) is a finite quiver without loops or 2-cycles with vertex set \(\{1, \ldots, n\}\), the following interpretations take place:

1. the clusters with respect to \(Q\) are the sets \(u\) appearing in seeds, \((Q, u)\) obtained from a initial seed \((Q, x)\) by iterated mutation,
2. the cluster variables for \(Q\) are the elements of all clusters,
3. the cluster algebra \(\mathcal{A}(Q)\) is the \(\mathbb{Q}\)-subalgebra of the field \(\mathbb{Q}(x_1, \ldots, x_n)\) generated by all the cluster variables.

As example, the cluster variables associated to the quiver \(Q = 1 \rightarrow 2\) are:

\[
\left\{ x_1, x_2, \frac{1 + x_2}{x_1}, \frac{1 + x_1 + x_2}{x_1x_2}, \frac{1 + x_1}{x_2} \right\}.
\]

Regarding cluster algebras arising from quivers, we recall that, Fomin and Zelevinsky [10] proved that any cluster algebra \(\mathcal{A}(Q)\) of finite type has a finite set of cluster variables and the following result.

**Theorem 1** The cluster algebra \(\mathcal{A}(Q)\) is of finite type if and only if \(Q\) is mutation-equivalent to an orientation of a simply-laced Dynkin diagram, \(\mathbb{A}_n, n \geq 1, \mathbb{D}_n, n \geq 4, \mathbb{E}_6, \mathbb{E}_7\) and \(\mathbb{E}_8\).
2.2 Friezes

An alternative way to define friezes is to say that they are ring homomorphisms from a cluster algebra to the ring of integers such that all cluster variables are sent to positive integers [11]. Let \( Q \) be a quiver without loops and 2-cycles and let \( \mathcal{A}(Q) \) be the corresponding cluster algebra with trivial coefficients [12], then:

(i) A frieze of type \( Q \) is a ring homomorphism \( F : \mathcal{A}(Q) \to R \) from the cluster algebra to an integral domain \( R \). The frieze is called integral if \( R = \mathbb{Z} \).

(ii) An integral frieze is said to be positive if every cluster variable in \( \mathcal{A}(Q) \) is mapped by \( F \) to a positive integer.

Let \( x = (x_1, \ldots, x_n) \) be a cluster of \( \mathcal{A}(Q) \), then:

(iii) A vector \( (a_1, \ldots, a_n) \in R^n \) is called a frieze vector relative to \( x \) if the frieze \( F \) defined by the assignment \( F(x_i) = a_i \) has values in \( R \), \( F \) is said to be unitary if there exists a cluster \( x \) such that \( F(x) \) is a unit in \( R \), for all \( x \in x \). If the frieze \( F \) is unitary we say that the frieze vector \( (a_1, \ldots, a_n) \) is unitary.

(iv) A vector \( (a_1, \ldots, a_n) \in \mathbb{Z}_{>0}^n \) is called a positive frieze vector relative to \( x \) if the frieze \( F \) defined by \( F(x_i) = a_i \) is positive integral.

The following is an example of a frieze pattern. Hereinafter, frieze patterns are assumed to be integral closed friezes.

\[
\begin{array}{cccccccc}
... & 0 & 0 & 0 & 0 & 0 & 0 & ...
\end{array}
\]
\[
\begin{array}{cccccccc}
... & 1 & 1 & 1 & 1 & 1 & ...
\end{array}
\]
\[
\begin{array}{cccccccc}
... & 2 & 3 & 1 & 2 & 3 & ...
\end{array}
\]
\[
\begin{array}{cccccccc}
... & 5 & 2 & 1 & 5 & ...
\end{array}
\]
\[
\begin{array}{cccccccc}
... & 2 & 3 & 1 & 2 & 3 & ...
\end{array}
\]
\[
\begin{array}{cccccccc}
... & 1 & 1 & 1 & 1 & ...
\end{array}
\]
\[
\begin{array}{cccccccc}
... & 0 & 0 & 0 & 0 & 0 & ...
\end{array}
\]

2.3 Dyck paths categories

In this section we recall the definition and main properties of the category of Dyck paths as Cañadas and Rios describe in [6].

A Dyck path is a lattice path in \( \mathbb{Z}^2 \) from \((0,0)\) to \((n,n)\) with steps \((1,0)\) and \((0,1)\) such that the path never passes below the line \(y = x\). The number of Dyck paths of length \(2n\) is equal to \(C_n = \frac{1}{n+1}\binom{2n}{n}\), the \(n\)th Catalan number [17].

The set of Dyck words is the set of words \(w\) in the free monoid \(X^* = \{U, D\}^*\) satisfying the following two conditions [2]:

- for any left factor \(u\) of \(w\) (i.e., \(w = uv\) for some suitable word \(v\)), \(|u|_U \geq |u|_D\),
- \(|w|_U = |w|_D\),
where $|w|_a$ is the number of occurrences of the letter $a \in X = \{U, D\}$ in the word $w$.

Henceforth, Dyck words defined as before are used to denote Dyck paths.

Let $\mathcal{D}_{2n}$ be the set of all Dyck paths of length $2n$, let $UWD = Uw_1 \ldots w_{n-1}D$ be a Dyck path in $\mathcal{D}_{2n}$ with $A = \{UD, DU,UU,DD\}$ being the set of all possible choices for each $w_i \in W$, $1 \leq i \leq n - 1$.

The support of $UWD$ (denoted by $\operatorname{Supp} UWD \subseteq \{1, 2, \ldots, n - 1\} = n - 1$) is a set of indices (of the $w_i$s) such that

$$\operatorname{Supp} UWD = \{q \in n - 1 \mid w_q = UD \text{ or } w_q = UU, 1 \leq q \leq n - 1\}.$$  

A map $f : A \rightarrow A$ such that for any $w \in A$, it holds that $f(w) = f(ab) = w^{-1} = ba$, $a, b \in \{U, D\}$ is said to be a shift. An unitary shift is a map $f_i : \mathcal{D}_{2n} \rightarrow \mathcal{D}_{2n}$ such that

$$f_i(Uw_1 \ldots w_{i-1}w_iw_{i+1} \ldots w_{n-1}D) = Uw_1 \ldots w_{i-1}f(w_i)w_{i+1} \ldots w_{n-1}D.$$  

We will denote a unitary shift by a vector of maps from $\mathcal{D}_{2n}$ to itself of the form $(1, \ldots, 1_{i-1}, f_i, 1_{i+1}, \ldots, 1_{n-1})$, where $1_k$ is the identity map associated to the $i$th coordinate.

An elementary shift is a composition of unitary shifts. A shift path of length $m$ $UWD \rightarrow UW_1D \rightarrow \cdots \rightarrow UW_mD \rightarrow UVD$ from $UWD$ to $UV$ is a composition of $m$ elementary shifts. The set of all Dyck paths in a shift path between $UWD$ and $UV$ will be denoted by $J(W, V)$. For notation, we introduce the identity shift as the elementary shift $(1, \ldots, 1_{n-1})$.

Suppose that a relation $R : \mathcal{D}_{2n} \rightarrow \mathcal{D}_{2n}$ is defined by applying successive elementary shifts to a given Dyck path. Then $R$ is said to be an irreversible relation over $\mathcal{D}_{2n}$ if and only if elementary shifts transforming Dyck paths (from one to the other) are not reversible. In other words, if an elementary shift $F = f_{p_1} \circ \cdots \circ f_{p_q}$ transforms a Dyck path $UWD$ into a Dyck path $UV$ then there is not an elementary shift $F' = f_{p_1} \circ \cdots \circ f_{p_q}$ transforming $UV$ into $UWD$, for some $p, q \in \mathbb{Z}^+$. If there exist two paths $G \circ F$ and $G' \circ F'$ of irreversible relations (of length 2) transforming a Dyck path $UWD$ into the Dyck path $UV$ over $R$ in the following form:

$$\begin{array}{ccc}
UWD & \xrightarrow{F} & UWD' \\
\downarrow{F'} & & \downarrow{G'} \\
UWD'' & \xrightarrow{G} & UVD,
\end{array}$$

with $W' \neq W''$. Then $G \circ F$ is said to be related with $G' \circ F'$ (denoted $G \circ F \sim_R G' \circ F'$) whenever $G' = F$ and $G = F'$.

As for the case of diagonals [4], Cañadas and Rios [6] defined a $\mathbb{F}$-linear additive category $(\mathcal{D}_{2n}, R)$ based on Dyck paths, in this case, objects are $\mathbb{F}$-linear combinations of Dyck paths in $\mathcal{D}_{2n}$ with space of morphisms from a Dyck path $UWD$ to a Dyck
path \( UVD \) over \( \mathbb{F} \) associated to \( R \) being the vector space

\[
\text{Hom}(\mathbb{D}_n, R)(UWD, UVD) = \langle \{g \mid g \text{ is a shift path associated to } R\}\rangle/\langle \sim_R \rangle.
\]

\( \text{Hom}(\mathbb{D}_n, R)(UWD, UVD) \neq 0 \) if and only if there are shift paths transforming \( UWD \) into \( UVD \) and

\[
\bigcap_{UW_i D \in J(W, V)} \text{Supp } UW_i D \neq \emptyset,
\]

for each shift path, with \( UWD \) and \( UVD \) in \( \mathbb{D}_n \).

Figure 1 shows the elementary shifts over \( (\mathbb{D}_6, R) \) associated to an irreversible relation \( R \) defined over the set of all Dyck paths of length 6. And such that,

\[
R(UWD) = \begin{cases} f_1(UWD), & \text{if } w_1 = UD, \\ f_2(UWD), & \text{if } w_2 = UD. \end{cases}
\]

If \( n = \{1, 2, \ldots, n\} \) is an \( n \)-point chain then \( \mathcal{C}_{(1,n)} \) stands for all \textit{admissible subchains} \( \mathcal{C} \) of \( n \) with \( \min \mathcal{C} = 1 \) and \( \max \mathcal{C} = n \). Given an admissible subchain \( \mathcal{C} = \{j_1, \ldots, j_m, i_1, \ldots, i_l\} \) then two Dyck paths \( D \) and \( D' \) of length \( 2n \) are said to be related by a relation of type \( R^{i_1 \ldots i_l}_{j_1 \ldots j_m} \) if there is an elementary shift associated to the points of the subchain transforming one into the other [6].

We let \( \mathcal{C}_{2n} \) denote the subcategory of \( (\mathbb{D}_2n, R^{i_1 \ldots i_l}_{j_1 \ldots j_m}) \) whose objects are \( \mathbb{F} \)-linear combinations of Dyck paths with exactly \( n - 1 \) peaks and related by a relation of type \( R^{i_1 \ldots i_l}_{j_1 \ldots j_m} \).

The following results describe the structure of the category of Dyck paths [6]. We recall that a point \( x \) where a Dyck path \( UWD \) changes from the north to the east is said to be a \textit{peak} of the path.

In the following theorem the symbol \( Q \) is used to denote a quiver of type \( A_{n-1} \) \( (n \geq 2) \) for which \( \{i_1, \ldots, i_l\} \) and \( \{j_1, \ldots, j_m\} \) are the corresponding sets of sinks and sources. \( \text{rep } Q \) denotes the corresponding category of representations.

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Theorem 2 [6, Theorem 15] There is a categorical equivalence between categories $\mathcal{C}_{2n}$ and $\text{rep } Q$.

Corollary 3 [6, Corollary 16] There exists a bijection between the set of representatives of indecomposable representations $\text{Ind } Q$ of $\text{rep } Q$ and the set of Dyck paths of length $2n$ with exactly $n - 1$ peaks.

Corollary 4 [6, Corollary 17] The category $\mathcal{C}_{2n}$ is an abelian category.

3 Relationships between Friezes and Dyck paths

In this section, we present the main results of the paper. In particular, it is introduced the notion of integral diamond of Dynkin type $A_n$, which are integer arrays used to build frieze patterns associated to triangulations of an $(n + 3)$-polygon.
### 3.1 $\mathbb{A}_n$-diamonds

Let $R$ be an integral domain then a diamond of Dynkin type $\mathbb{A}_n$ or $\mathbb{A}_n$-diamond is an array $A = (a_{i,j})$ with entries in $R$, such entries satisfy conditions (D1) and (D2) associated to arrays with the following shape:

\[
\begin{array}{ccccccc}
& & a_{2,0} & & & & \\
& a_{1,1} & & a_{2,1} & & & \\
& & a_{1,2} & & & & \\
& & & & \ddots & & \\
& & & & & a_{2,n-1} & \\
& & & & & & a_{1,n} \\
& & & & & & & a_{2,n} \\
& & & & & & & a_{1,n+1} \\
\end{array}
\]

(D1) $a_{2,0} = a_{1,n+1} = 1$,
(D2) $a_{1,j}a_{2,j} - a_{2,j-1}a_{1,j+1} = 1$ for $1 \leq j \leq n$,

where, $1$ is the identity element of $R$.

If $R = \mathbb{Z}$ then $A$ is said to be a positive integral diamond of Dynkin type $\mathbb{A}_n$, if it also satisfies the following condition (D3)

(D3) $a_{1,1} = a$ (or $a_{1,1} = a + ma$), $a_{2,1} = a + ma$ (or $a_{2,1} = a$) and $a_{1,2} = a^2 + am_a - 1$, with $1 \leq a \leq \lfloor \frac{n+2}{2} \rfloor$, $1 \leq m_1 \leq n$ and $0 \leq m_a \leq n + 2(1-a)$ if $a > 1$.

Henceforth, if it is not mentioned explicitly, $\mathbb{A}_n$-diamonds are assumed to be positive integral diamonds of Dynkin type $\mathbb{A}_n$.

Two $\mathbb{A}_n$-diamonds $A$ and $B$ constitute a coupling, denoted $A \parallel B$ if and only if $a_{2,j} = b_{1,j}$ for $1 \leq j \leq n$.

A set $\{A^t\}_{t \geq 0}$ is an $\mathbb{A}_n$-sequence of couplings of $\mathbb{A}_n$ if and only if $A^r \parallel A^{r+1}$ for $r \geq 0$ ($X^0 = X$ for any $\mathbb{A}_n$-diamond $X$). An $\mathbb{A}_n$-sequence $\{A^t\}_{t \geq 0}$ of couplings is a $p$-cycle if there exists $p \in \mathbb{N}$ such that $A^t = A^{t+p}$. If the $\mathbb{A}_n$-sequence of couplings $S_t = \{A^t\}_{t \geq 0}$ constitute a frieze pattern $\mathcal{F}$ then we will say that $S_t$ generates $\mathcal{F}$. In this case, we point out that Lemma 7 proves that $A^0 = A$ generates $S_t$, meaning that $S_t$ can be obtained from a sequence of couplings starting with $A$.

For example, let $R = \mathbb{Z}$, the sets $\{A^t\}_{t \geq 0}$ and $\{B^t\}_{t \geq 0}$ ($A^0$ and $B^0$ as shown below) are $\mathbb{A}_1$-sequences which are 2-cycles with $A^{2k} = B^{2k+1} = A$, $A^{2k+1} = B^{2k} = B$ and $k \geq 0$.

\[
A = \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}, \quad B = \begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}
\]

(3)

In general, it can be written an $\mathbb{A}_n$-sequence $\{A^t\}_{t \geq 0}$ as an $\mathbb{A}_n$-array $C_{At} = (c_{i,j})$ such that $c_{t+1,j} = a_{1,j}^t$, and $c_{t+1,0} = c_{t+1,n+1} = 1$, for $t \geq 0$. For the previous
example,

\[
C_{A'} = \begin{pmatrix} 1 & 1 & 1 & 1 & \ldots \\ 1 & 2 & 1 & 2 & \ldots \\ 1 & 1 & 1 & 1 & \ldots \\ \end{pmatrix}, \quad C_{B'} = \begin{pmatrix} 1 & 1 & 1 & 1 & \ldots \\ 2 & 1 & 2 & 1 & \ldots \\ 1 & 1 & 1 & 1 & \ldots \\ \end{pmatrix}
\]

\(C_{A'}\) and \(C_{B'}\) are \(A_1\)-arrays associated to \(\{A'_t\}_{t \geq 0}\) and \(\{B'_t\}_{t \geq 0}\), respectively.

If the \(A_n\)-sequence of couplings is finite of length \(m\), then it can be associated to an infinity \(A_n\)-array, \(C^m_{A'} = (c^m_{i,j})\) such that

\[
c^m_{(s_0+1)+km,j} = a^s_{1,j}, \quad c^m_{(s_0+1)+km,0} = c^m_{(s_0+1)+km,n+1} = 1,
\]

for \(k \in \mathbb{Z}\). For any \(A_n\)-sequence of couplings \(\{A'_t\}_{t \geq 0}\), it is possible to choose an \(A_n\)-subsequence \(\{B^z_t\}_{t \geq 0}\) with \(B^z = A^{x+z}\) for some \(x \geq t\). In particular, if \(\{A'_t\}_{t \geq 0}\) is a \(p\)-cycle then the subsequence \(\{B^{s_0}_t\}_{0 \leq s_0 \leq p-1}\) such that \(B^{s_0} = A^t\) is called the minimal \(p\)-cycle of \(\{A'_t\}_{t \geq 0}\).

The following results give the main properties of diamonds of Dynkin type \(A_n\).

**Proposition 5** Let \(\{A'_t\}_{t \geq 0}\) be a \(p\)-cycle and let \(B = \{B^{s_0}_t\}_{0 \leq s_0 \leq p-1}\) be its minimal \(p\)-cycle. Then, the array \(C^p_B\) is a frieze pattern of order \(n+3\). In particular, \(p\) divides \(n+3\).

**Proof** Let \(C^p_B = (c^p_{ij})\) be the infinity \(A_n\)-array associated to \(\{B^{s_0}_t\}_{0 \leq s_0 \leq p-1}\), identity (4) implies that

\[
c^p_{(s_0+1)+kp,j} = a^{s_0}_{1,j}, \quad c^p_{(s_0+1)+kp,0} = c^p_{(s_0+1)+kp,n+1} = 1,
\]

for \(k \in \mathbb{Z}\), since that \(\{A'_t\}_{t \geq 0}\) is a \(p\)-cycle, then \(C^p_B\) is a frieze pattern. \(\square\)

**Proposition 6** Let \(\{A'_t\}_{t \geq 0}\) be a \(p\)-cycle of length \(2p\), then the subsequences \(\{B^{s_i}_t\}\) with \(0 \leq s_i \leq p-1\) generate the same frieze pattern of order \(n+3\), for \(0 \leq i \leq p-1\), and \(B^{s_i} = A^{i+s_i}\).

**Proof** Let \(\{A'_t\}_{t \geq 0}\) be a \(p\)-cycle of length \(2p\). And \(C^p_A = (c^p_{ij})\) and \(C^p_B = (c^p_{ij})\) two infinity arrays in subsequences \(A = \{B^{s_i}_t\}_{0 \leq s_i \leq p-1}\) and \(B = \{B^{s_i}_{t'}\}_{0 \leq s_i \leq p-1}\) of \(\{A'_t\}_{t \geq 0}\) for \(0 \leq i < i' \leq p-1\). Then, the following identities (5) hold by applying the translation \(s_{i'} = s_i - |i' - i|\),

\[
c^p_{s_i+1+kp,j} = a^{s_i+1+i'}_{1,j} = a^{s_i-i+i'}_{1,j} = a^{s_i+i}_{1,j} = c^p_{s_i+1+kp,j}.
\]

We are done. \(\square\)

**Lemma 7** Let \(\{A'_t\}_{t \geq 0}\) be a sequence of couplings. Then \(\{A'_t\}_{t \geq 0}\) is generated by \(A^0\). In particular, \(A^0\) generates a \(p\)-cycle for some \(p > 0\).
Proof Let \( \{A^t\}_{t \geq 0} \) be a sequence, then

\[
a_{2,j}^x = \frac{1 + (a_{2,j}^{x-1}) (a_{2,j+1}^{x-1})}{a_{2,j}^{x-1}},
\]

for \( 1 \leq j \leq n \), and \( x \geq t \), \( a_{2,j}^x \) can be written by using the set \( \{a_{2,j}^0\}_{1 \leq j \leq n} \) for \( x > 0 \).

In particular, the set \( \{a_{2,1}^0, \ldots, a_{2,n}^0\} \) is a seed of the cluster algebra associated to the linearly oriented quiver of type \( \mathbb{A}_n \). Since the cluster variables are finite in the case \( \mathbb{A}_n \), then there is \( p = n + 3 \) (in some cases, it is not minimal) such that \( A^0 = A^p \).

\[ \square \]

Theorem 8 Let \( A \) be a diamond of Dynkin type \( \mathbb{A}_n \) then \( A \) generates a frieze pattern.

**Proof** It is a direct consequence of Lemma 7, and Proposition 5.

For instance, diamonds \( A \) and \( B \) given in (3) generate the following frieze pattern.

\[
\ldots 1 1 1 1 1 \ldots \\
\ldots 1 2 1 2 \ldots \\
\ldots 1 1 1 1 1 \ldots 
\]

### 3.2 Seed vectors

In this section, we give an algorithm to build a family of positive integral frieze vectors associated to the linearly oriented quiver of type \( \mathbb{A}_n \). These vectors allow to find out a connection between the positive integral diamonds of Dynkin type \( \mathbb{A}_n \), triangulations, and Dyck paths.

Let \( A \) be a diamond of Dynkin type \( \mathbb{A}_n \) then we can write its first column as a vector with the form \( v_A = (a_1, \ldots, a_n) \), where \( a_j = a_{1,j} \). In such a case, we say that \( v_A \) is associated to \( A \) and that \( v_A \) generates \( A \).

**Proposition 9** If \( v = (a_1, \ldots, a_n) \) is a vector associated to a positive integral diamond of Dynkin type \( \mathbb{A}_n \) with \( a_n = 1 \), then the vector \( v' = (a_1, \ldots, a_i, a_i + a_{i+1}, a_{i+1}, \ldots, a_{n-1}) \) is also associated to a positive integral diamond of Dynkin type \( \mathbb{A}_n \), for \( 1 \leq i < n \).

**Proof** Let \( v_A = (a_1, \ldots, a_n) \) be a vector associated to a positive integral diamond \( A = (a_{j,m}) \) of Dynkin type \( \mathbb{A}_n \), then we take the vector \( v_{A+i} = (a_1, \ldots, a_i, a_i + a_{i+1}, a_{i+1}, \ldots, a_{n-1}) \) and the array \( A + i \) of the following form:

\[
b_{1,m} = \begin{cases} 
a_{1,m}, & \text{if } m \leq i, \\
a_{1,i} + a_{1,i+1}, & \text{if } m = i + 1, \\
a_{1,m-1}, & \text{if } m > i + 1, 
\end{cases}
\]
and

\[ b_{2,m} = \begin{cases} 
    a_{2,m}, & \text{if } m \leq i - 1, \\
    a_{2,i-1} + a_{2,i}, & \text{if } m = i, \\
    a_{2,m-1}, & \text{if } m \geq i + 1,
\end{cases} \]

then \( b_{1,m}b_{2,m} - b_{2,m-1}b_{2,m+1} = 1 \), for \( 1 \leq m \leq n \) and \( 1 \leq i < n \). Therefore \( A + i \) is a positive integral diamond of Dynkin type \( \mathbb{A}_n \).

\[ \square \]

**Proposition 10** For each vector \( v_{n,z} = (a_1, \ldots, a_n) \) with

\[ a_i = \begin{cases} 
    z + 1 - i, & \text{if } i < z, \\
    1, & \text{if } i \geq z.
\end{cases} \]  \hspace{1cm} (6)

there is associated a unique positive integral diamond of Dynkin type \( \mathbb{A}_n \), for \( z \in \{1, \ldots, n+1\} \).

**Proof** Let \( v_{n,z} \) be a vector and let \( z \) be a natural number between 1 and \( n+1 \), we define a positive integral diamond \( A \) with \( a_{1,i} = a_i \) and \( a_{2,i} = b_i \) where

\[ b_i = \begin{cases} 
    1, & \text{if } i < z, \\
    i + 2 - z, & \text{if } i \geq z,
\end{cases} \]  \hspace{1cm} (7)

then \( a_{1,i}a_{2,i} - a_{2,i-1}a_{2,i+1} = 1 \) for \( 1 \leq i \leq n \).

\[ \square \]

**Remark 11** \( v_{n,z} \) is called a *seed vector*. The vector \( v^{n,z} = (b_1, \ldots, b_n) \) defines a positive integral diamond \( B \) of Dynkin type \( \mathbb{A}_n \) such that \( b_{2,i} \) satisfies the following identity

\[ b_{2,i} = \begin{cases} 
    i - 1, & \text{if } i < z - 1, \\
    (b_{1,i} + 1)z - 1, & \text{if } z - 2 < i < n, \\
    z, & \text{if } i = n.
\end{cases} \]  \hspace{1cm} (8)

and \( b_i = b_{1,i} \) is defined as in (7).

**Proposition 12** The positive integral diamonds \( A \) and \( B \) of Dynkin type \( \mathbb{A}_n \) generated by vectors \( v_{n,z} \) and \( v^{n,z} \) respectively constitute a coupling.

**Proof** It is a direct consequence of Proposition 10 and Lemma 7.

\[ \square \]

The number of ways of applying recursively Proposition 9 to a vector \( w_A = (a_1, \ldots, a_{z-1}, 1, \ldots, 1) \in \mathbb{N}^n \) is given by the next identity (denoted by \( f_{n,z} \)),

\[ f_{n,z} = \begin{cases} 
    \sum_{i=1}^{n} f_{n-1,i}, & \text{if } z > 1, \\
    \sum_{i=1}^{n} f_{n-1,i}, & \text{if } z = 1.
\end{cases} \]
where it is included the trivial move $w_{A+0} = w_A$, for $n > 1$, and any $z \in \{1, \ldots, n+1\}$. Actually, these numbers can be arranged as follows:

\[
\begin{align*}
&f_{1,2} \quad f_{1,1} \\
&f_{2,3} \quad f_{2,2} \quad f_{2,1} \\
&f_{3,4} \quad f_{3,3} \quad f_{3,2} \quad f_{3,1} \\
&f_{4,5} \quad f_{4,4} \quad f_{4,3} \quad f_{4,2} \quad f_{4,1} \\
\end{align*}
\]

for any of the vectors $w_A$. Since the first choices are $v_{1,1} = (1)$ and $v_{1,2} = (2)$, then $f_{1,1} = 1$ and $f_{1,2} = 1$. The previous triangle (8) appears in the On-Line Encyclopedia of Integer Sequences (OEIS) as A009766 (Catalan triangle [16]). In particular, we generate all positive integral diamonds of Dynkin type $\mathbb{A}_n$ via the seed vectors $v_{n,z}$. For example, for $n = 3$, all vectors that generate positive integral diamonds of Dynkin type $\mathbb{A}_3$ are:

\[
\begin{align*}
&(1, 1, 1) \quad (1, 1, 2) \quad (1, 2, 1) \quad (1, 2, 3) \quad (1, 3, 2) \quad (2, 1, 1) \quad (2, 1, 2) \\
&(2, 3, 1) \quad (2, 3, 4) \quad (2, 5, 3) \quad (3, 2, 1) \quad (3, 2, 3) \quad (3, 5, 2) \quad (4, 3, 2)
\end{align*}
\]

Let $G = UD \ldots UD \ldots$ be a Dyck path of length $2n$ and let $m_i$ be the number of $Us$ before the occurrence of the $i$th $D$ in $G$ then $G$ can be written as a vector $v_G = (v_1, \ldots, v_{n-1})$ where $v_i = m_i - i + 1$. As an example, consider the Dyck path $G$ shown in Fig. 3, which has associated the vector $v_G = (5, 4, 3, 5, 4, 3, 2)$.

In what follows, it is defined a map $T_i$ between vectors associated to positive integral diamonds of Dynkin type $\mathbb{A}_n$ and Dyck paths by using a relation over the coordinates of a vector $u = (a_1, \ldots, a_m)$. $T_i$ is defined in such a way that, $T_i : \mathbb{N}^m \to \mathbb{N}$ and:

- If $a_i - a_l > 0$ for some $l \in \{1, \ldots, i\}$, then we select the maximum of such indexes $\max \{l\}$ and write $r_1 = a_i - a_l$. In the same way, it is chosen $\max \{l\}$ such that $r_1 - a_l > 0$ and write $r_2 = r_1 - a_l$, this process ends if $\{l \mid r_l - a_l > 0\} = \emptyset$. Thus, if $u \in \mathbb{N}^m$ then $T_i(u) = r_i + t$, for some $t$.
- If $a_i - a_l \leq 0$ for all $l \in \{1, \ldots, i\}$, then $T_i(u) = a_i$.
For instance, for the vector \( u = (14, 52, 4, 23, 9, 2) \), it holds that \( T_1(u) = 14, T_2(u) = 13, T_3(u) = 4, T_4(u) = 8, T_5(u) = 3, \) and \( T_6(u) = 2 \).

**Proposition 13** Let \( v_{n,z} \) be a seed vector, then \((T_1(v_{n,z}), \ldots, T_n(v_{n,z}))\) defines a Dyck path of length 2(n + 1).

**Proof** For any \( z \in \{1, \ldots, n + 1\} \), \( T_i(v_{n,z}) = a_i \) with \( a_i \) given by identity (6), there is a word \( G_{v_{n,z}} = u_1 \ldots u_{2(n+1)} \) in the free monoid \( \{U, D\}^* \) such that

\[
G_{v_{n,z}} = U \underbrace{U \ldots U}_{z-1} D \underbrace{D \ldots D}_{z-1} U D U D \ldots U D U D,
\]

for any left factor \( u_s \) in \( G_{v_{n,z}} \) of length \( s \in \{1, \ldots, 2(n+1)\} \), \( 0 \leq |u_s|_U - |u_s|_D \leq z-1 \), therefore \( G_{v_{n,z}} \in \mathcal{D}_{2(n+1)} \). \( \square \)

**Proposition 14** Let \( v_A = (a_1, \ldots, a_n) \) be a vector associated to a positive integral diamond \( A \) of Dynkin type \( \tilde{A}_n \) with \( a_n = 1 \), such that \((T_1(v_A), \ldots , T_n(v_A))\) defines a Dyck path in \( \mathcal{D}_{2(n+1)} \). Then \((T_1(v_{A+i}), \ldots , T_n(v_{A+i}))\) also defines a Dyck path in \( \mathcal{D}_{2(n+1)} \).

**Proof** Let \( v_A = (a_1, \ldots, a_n) \) be a vector associated to a positive integral diamond \( A \) with \( a_n = 1 \), then there exists a Dyck path \( G_{v_A} \in \mathcal{D}_{2(n+1)} \) such that any left factor \( u_s \) of length \( s \) satisfies \( |u_s|_U \geq |u_s|_D \) for \( 1 \leq s \leq 2(n+1) \). If \( v_{A+i} \) is a vector associated to the positive integral diamond \( A + i \) with

\[
T_m(v_{A+i}) = \begin{cases} T_m(v_A), & \text{if } 1 \leq m \leq i, \\ T_m(v_A) + 1, & \text{if } m = i + 1, \\ T_{m-1}(v_A), & \text{if } m \geq i + 1, \end{cases}
\]

then there is a word \( G_{A+i} = w'_1, \ldots, w'_{2(n+1)} \) in \( \{U, D\}^* \). Now, we choose an index \( m_1 \) of the \( i \)th \( D \) in \( G_{A+i} \) such that for any left factor \( u'_s \) in \( G_{A+i} \) the following identities hold:

\[
|u'_s|_U = \begin{cases} |u_s|_U, & \text{if } 1 \leq s \leq m_1, \\ |u_{m_1}|_U + 1, & \text{if } s = m_1 + 1, \\ |u_{s-2}|_U + 1, & \text{if } s \geq m_1 + 2, \end{cases}
\]

and

\[
|u'_s|_D = \begin{cases} |u_s|_D, & \text{if } 1 \leq s \leq m_1, \\ |u_{m_1}|_D, & \text{if } s = m_1 + 1, \\ |u_{s-2}|_U + 1, & \text{if } s \geq m_1 + 2, \end{cases}
\]

thus, the following options for \( U_s \) and \( D_s \) take place:

- If \( 1 \leq s \leq m_1 \), \( |u'_s|_U = |u_s|_U \geq |u_s|_D = |u'_s|_D \).
• If \( s = m_1 + 1 \), \( |u^\prime_{m_1+1}|_U = |u_{m_1}|_U + 1 > |u_{m_1}|_D = |u^\prime_{m_1+1}|_D \).
• If \( m_1 + 2 \leq s \leq 2(n+1) \), \( |u^\prime_s|_U = |u_{s-2}|_U + 1 \geq |u_{s-2}|_D + 1 = |u^\prime_s|_D \).

Therefore, \( G_{A+i} \in \mathcal{D}_{2(n+1)} \).

**Lemma 15** There is a bijective correspondence between the set of all vectors associated to positive integral diamonds of Dynkin type \( \mathbb{A}_n \) and the set of all Dyck paths of length \( 2(n+1) \).

**Proof** Let \( \mathcal{D}_{\mathbb{A}_n} \) be the set of all vectors associated to positive integral diamonds of Dynkin type \( \mathbb{A}_n \) and let \( \mathcal{D}_{2(n+1)} \) be the set of all Dyck paths of length \( 2(n+1) \) then we define a map \( f : \mathcal{D}_{\mathbb{A}_n} \to \mathcal{D}_{2(n+1)} \) with \( f(u_A) = (T_1(u_A), \ldots, T_n(u_A)) \), Propositions 13 and 14 allow us to establish that \( f \) is well defined.

In order to prove that the map \( f \) is injective, suppose that \( u_A \neq v_B \), and that \( l \) is the minimum index for which \( u_{l} \neq v_{l} \). Thus, if \( l = 1 \) then \( T_1(u_A) \neq T_1(v_B) \). If \( l > 1 \), \( u_l = m(u_{l-1}) + a \) and \( v_l = m'(u_{l-1}) + a \) with \( m \neq m' \) is a consequence of Proposition 9 then \( r_{u_{l-1}} \neq r_{v_{l-1}} \), therefore \( T_l(u_A) \neq T_l(v_B) \).

An alternative way of writing a Dyck path \( G \in \mathcal{D}_{2(n+1)} \) can be defined by using a vector \( \lambda_G = (\lambda_1, \ldots, \lambda_n) \) where \( \lambda_i \) is the number of \( Ds \) before the occurrence of the \( (n+2-i) \)th \( U \) in \( G \). For example the Dyck path illustrated in Figure 3 has associated the following vector \( \lambda_G = (4, 4, 4, 3, 0, 0, 0, 0) \).

Let \( \lambda \) be a vector associated to a Dyck path of length \( 2(n+1) \) then a triangulation of an \((n+3)\)-polygon can be realized by \( \lambda \) as follows:

• Fix a labeling for the vertices of polygon \( K_0^{n+3} = (v_0^{n+3}, \ldots, v_{n+2}^{n+3}) \) with \( v_i^{n+3} = i \), for \( 0 \leq i \leq n + 2 \).
• For \( \lambda_i \), we draw a diagonal \( l_i^{\lambda_i} \) between \( \lambda_i \) and \( \lambda_i + 2 \). Afterwards, we label the last polygon with \( n + 3 - i \) vertices \( K_i^{n+3-i} = (v_0^{n+3-i}, \ldots, v_{n+2-i}^{n+3-i}) \), and

\[
 v_j^{n+3-i} = \begin{cases} 
 v_j^{n+3-(i-1)}, & \text{if } j \leq \lambda_i, \\
 v_j^{n+3-(i-1)} - 1, & \text{if } j > \lambda_i,
\end{cases}
\]

for \( i = 1, \ldots, n \).

The previous algorithm suggests that if \( l_i^{\lambda_i} \) is a diagonal then it does not cross the diagonals \( l_1^{\lambda_1}, \ldots, l_{i-1}^{\lambda_{i-1}} \) for \( 1 \leq i \leq n \). Figure 4 illustrates the construction of a triangulation associated to the vector \( \lambda_G = (2, 2, 1) \) of the Dyck path \( G = UDU DU UDD \).
If we fix a labeling $K$ over all vertices of a polygon with $n+3$ vertices, a triangulation $T$ is written as a sequence $T = (l_1^\lambda, \ldots, l_n^\lambda)$, where $v_i$ belongs to the set of vertices.

**Lemma 16** There is a bijective correspondence between the set of all triangulations of a polygon with $n+3$ vertices and the set of all Dyck paths of length $2(n+1)$.

**Proof** Let $T_n$ be the set of all triangulations of a polygon with $n+3$ vertices then we can define a map $g : D_{2(n+1)} \rightarrow T_n$ with $g(\lambda) = T_\lambda$. In order to prove that $g$ is one to one, we fix a labeling $K$ and suppose that $g(\lambda_G) = g(\sigma_G')$, then $(l_1^\lambda, \ldots, l_n^\lambda) = (l_1^\sigma, \ldots, l_n^\sigma)$, provided that $l_j^\lambda = l_j^\sigma$. Since by definition there are diagonals connecting vertices $\lambda_j$ with $(\lambda_j+2)$ and $\sigma_j$ with $(\sigma_j+2)$, therefore $\lambda_j = \sigma_j$ for $j = 1, \ldots, n$. We are done. \(\square\)

The next theorem presents the main result regarding relationships between positive integral diamonds of Dynkin type $A_n$ and triangulations of an $(n+3)$-polygon.

**Theorem 17** There is a bijective correspondence between the set of all vectors associated to positive integral diamonds of Dynkin type $A_n$, and triangulations of a polygon with $n+3$ vertices.

**Proof** Fix a labeling $K$ of a polygon with $n+3$ vertices. Then the map $F : H_n \rightarrow T_n$ defined by the formula

$$F(v_A) = (g \circ f)(v_A)$$

is a bijection (see Lemmas 15 and 16). \(\square\)

Figure 5 presents an example of the bijective correspondence between a positive integral diamond of Dynkin type $A_4$, a Dyck path of length 10, and a triangulation of a polygon with 7 vertices.

### 3.3 Frieze patterns and Dyck paths

In this section, we describe an algebraic interpretation of frieze patterns as a direct sum of indecomposable objects of Dyck paths categories.

**Lemma 18** Vectors $v_{n,z}$ and $v^{n,z}$ realize the same triangulation except for one anti-clockwise rotation.

**Proof** Let $v_{n,z}$ and $v^{n,z}$ be frieze vectors, fixed a labeling $K_1$ in an $(n+3)$-polygon, then

$$f(v_{n,z})$$

$\square$ Springer
the following identities hold by applying map $F$ (see (9)) as follows:

$$F(v_{n,z}) = \left( \underbrace{n, \ldots, z}_{1}, \underbrace{0, \ldots, 0}_{n-z+1} \right),$$

$$F(v^{n,z}) = \left( \underbrace{n-1, \ldots, z-1}_{1}, \underbrace{z-1, \ldots, 1}_{n-z+1} \right).$$
Proof Let \( 1 \leq i < j \leq n \), such that \( \sigma(1, \ldots, n) = (\sigma(1), \ldots, \sigma(n)) \) and \( \sigma \) realizes the same triangulation except for one anti-clockwise rotation then there exists a permutation \( \sigma \in S_n \) such that \( \sigma(1), \ldots, \sigma(n) \) \( F(\sigma(v)) = F(v) \) in \( K_2 \). In general, if \( v \) and \( w \) realize the same triangulation except for one anti-clockwise rotation, then there exists a permutation \( \sigma' \in S_n \) such that \( \sigma'(F(v)) = F(w) \) in \( K_2 \).

Note that, there exists a permutation
\[
\sigma = \left( \begin{array}{cccccccc}
1 & 2 & \ldots & n-z-1 & n-z & n-z+1 & n-z+2 & \ldots & n-1 & n \\
1 & 2 & \ldots & n-z-1 & n-z & n-1 & \ldots & n-z+2 & n-z+1
\end{array} \right)
\]
in \( S_n \) that describes a bijection between the coordinates of the vector \( F(v_{n,z}) = (u_1, \ldots, u_n) \) and \( F(v^{n,z}) = (u'_1, \ldots, u'_n) \) such that \( \sigma(F(v_{n,z})) = (u_{\sigma(1)}, \ldots, u_{\sigma(n)}) = (u'_1, \ldots, u'_n) = F(v^{n,z}) \) in \( K_2 \).

Lemma 19 Let \( A \) and \( B \) be a coupling of positive integral diamonds of Dynkin type \( \mathbb{A}_n \), and \( v_A = (a_1, \ldots, a_z, \ldots, a_n) \) a corresponding associated vector with \( a_1 = 1 \) for \( 1 \leq z \leq n \). If \( v_A \) and \( v_B \) realize the same triangulation except for one anti-clockwise rotation. Then vectors
\[
v_{A+i} = (a_1, \ldots, a_{i-1}, a_{i-1} + a_i, a_{i+1}, \ldots, a_{n-1}) \quad \text{and} \quad v_{B+i-1} = (b_1, \ldots, b_{i-2}, b_{i-2} + b_{i-1}, b_{i-1}, \ldots, b_{n-1}),
\]
realize the same triangulation except for one anti-clockwise rotation for \( z-1 \leq i \leq n \), and \( i \geq 2 \).

Proof Let \( v_A \) and \( v_B \) be associated vectors to the \( A_n \)-diamonds \( A \) and \( B \), respectively. Since \( v_A \) and \( v_B \) realize the same triangulation except for one anti-clockwise rotation then there exists a permutation \( \sigma \in S_n \) such that \( \sigma(F(v_A)) = F(v_B) \) in \( K_2 \). The following options arise from the map \( f \), such that:

1. If \( i > z \geq 1 \), then
\[
f(v_A) = (\ldots, \frac{1}{i-1}, \frac{1}{i}, \ldots), \quad f(v_B) = (\ldots, \frac{d}{i-2}, 2, \frac{2}{i}, \ldots). \quad \text{(see Fig. 6)}
\]

2. If \( d = 1 \), then
\[
F(v_A) = (\ldots, \frac{i}{n-i}, \frac{i-1}{n+1-i}, \ldots), \quad F(v_B) = (\ldots, \frac{i-1}{n-i}, \frac{i-2}{n+1-i}, \ldots).
\]
and \( \sigma \) satisfies the expression,

\[
\sigma(r) = \begin{cases} 
  r, & \text{if } r \leq n + 1 - i, \\
  m, & \text{otherwise},
\end{cases}
\]

for some \( m > n + 1 - i \). Applying \( F \) to \( v_{A+i} \) and \( v_{B+i-1} \), it holds that

\[
F(v_{A+i}) = (\ldots, i - 1, i - 1, \ldots), \\
F(v_{B+i-1}) = (\ldots, i - 2, i - 2, \ldots),
\]

then there exists \( \sigma' \in S_n \) such that \( \sigma' = \sigma \) and \( \sigma'(F(v_{A+i})) = F(v_{B+i-1}) \) in \( K_2 \) (see Fig. 6).

(1.2) The case for \( d = 2 \) is the same as the previous case.

(1.3) If \( d = 3 \), then \( A \) and \( B \) do not realize the same triangulation.

Note that, if \( z = 1 \), this case satisfies the condition (1.1) and (1.2) without \( d \).

(2) If \( i = z \geq 2 \), then

\[
f(v_A) = (\ldots, 2, 1, \ldots), \\
f(v_B) = (\ldots, b, a, 2, \ldots),
\]

(see Fig. 7).

(2.1) If \( a = 1 \) and \( b = 1 \), then

\[
F(v_A) = (\ldots, i, \ldots), \\
F(v_B) = (\ldots, i - 1, i - 1, \ldots),
\]

and \( \sigma_1 \) is defined by the following cases:

\[
\sigma_1(r) = \begin{cases} 
  r, & \text{if } r \leq n - i, \\
  n + 1 - i, & \text{if } r = n, \\
  m, & \text{otherwise},
\end{cases}
\]

for some \( m > n + 1 - i \). Applying \( F \), we obtain

\[
F(v_{A+i}) = (\ldots, i - 1, \ldots), \\
F(v_{B+i-1}) = (\ldots, i - 2, \ldots),
\]

then there exists \( \sigma_1' \in S_n \) satisfying the following cases:

\[
\sigma_1'(r) = \begin{cases} 
  n - i, & \text{if } r = n, \\
  n + 1 - i, & \text{if } r = n - i, \\
  \sigma_1(r), & \text{otherwise},
\end{cases}
\]

therefore \( \sigma_1'(F(v_{A+i})) = F(v_{B+i-1}) \) in \( K_2 \) (see Fig. 8).

(2.2) If \( a = 1 \) and \( b = 2 \), then conditions defined in the case (2.1) hold.

(2.3) For \( a = 2 \) and \( b = 1 \) or \( b = 2 \), we have only contradictions.
(2.4) If \( a = 2 \) and \( b = 3 \), \( F(v_B) = (\ldots, i_{i-1}^1, \ldots) \) and \( \sigma_2 = \sigma \). Applying \( F \) to \( v_B+i-1 \), it holds that \( F(v_B+i-1) = (\ldots, i_{i-2}, \ldots) \) then there exits \( \sigma'_2 \in S_n \) such that \( \sigma'_2 = \sigma \) and
\[
\sigma'_2(F(v_{A+i})) = F(v_B+i-1) \text{ in } K_2 \text{ (see Fig. 8)}.
\]

(2.5) Case (2.3) holds for \( a = 3 \) and \( b = 1, 2, 3 \).

Note that, if \( z = 2 \) then conditions for \( a = 1, 2 \) without \( b \) hold.

(3) If \( i = z - 1 \geq 3 \), then
\[
f(v_A) = (\ldots, 2_i, 1_{i+1}, \ldots),
f(v_B) = (\ldots, b_i, a_i, 2_{i+1}, \ldots) \text{ (see Fig. 9)}.
\]

(3.1) If \( a = 1 \) and \( b = 1 \), then
\[
F(v_A) = (\ldots, i_{i+1}, \ldots),
\]
\[
F(v_B) = (\ldots, i_{i-1}, i_n, i_{n-i}, \ldots), \quad \text{and} \quad
\sigma_3(r) = \begin{cases} 
  r, & \text{if } r \leq n - i - 1, \\
  n - i, & \text{if } r = n, \\
  n + 1 - i, & \text{if } r = n - 1, \\
  m, & \text{otherwise},
\end{cases}
\]

for some \( m > n + 1 - i \). Provided that
\[
F(v_{A+i}) = (\ldots, i_{i-1}, \ldots), \quad \text{and}
\]
\[
F(v_{B+i-1}) = (\ldots, i_{i-2}, \ldots),
\]
then there exits \( \sigma'_3 \in S_n \) such that
\[
\sigma'_3(r) = \begin{cases} 
  n - i - 1, & \text{if } r = n, \\
  n - i, & \text{if } r = n - 1, \\
  n + 1 - i, & \text{if } r = n - i - 1, \\
  \sigma_3(r), & \text{otherwise},
\end{cases}
\]

then \( \sigma'_3(F(v_{A+i})) = F(v_{B+i-1}) \) in \( K_2 \) (see Fig. 10).

(3.2) If \( a = 1 \) and \( b = 2 \). It holds that,
\[
F(v_B) = (\ldots, i_{i-1}, i_n, \ldots),
\]
whereas, \( \sigma_4 \) is given by the identities
\[
\sigma_4(r) = \begin{cases} 
  r, & \text{if } r \leq n - i - 1, \\
  n - i, & \text{if } r = n, \\
  m, & \text{otherwise},
\end{cases}
\]

for some \( m > n - i \). Applying \( F \) to \( v_{B+i-1} \), we get
\[
F(v_{B+i-1}) = (\ldots, i_{i-1}, i_{i-2}, \ldots),
\]
then there exists \( \sigma'_4 \) with
\[
\sigma'_4(r) = \begin{cases} 
  n - i - 1, & \text{if } r \leq n, \\
  n - i, & \text{if } r = n - i - 1, \\
  \sigma_4(r), & \text{otherwise},
\end{cases}
\]

therefore \( \sigma'_4(F(v_{A+i})) = F(v_{B+i-1}) \) in \( K_2 \) (see Fig. 10).
Fig. 7 Dyck paths associated to vectors $v_A$ and $v_B$ for $i = z$

Fig. 8 Dyck paths associated to vectors $v_{A+i}$ and $v_{B+i-1}$ for $i = z$

Fig. 9 Dyck paths associated to vectors $v_A$ and $v_B$ for $i - 1 = z$

(3.3) If $a = 2$ and $b = 3$, $F(v_B) = (\ldots, \underbrace{i}_{n-i-1}, \ldots)$, provided that

$$\sigma_5(r) = \begin{cases} r, & \text{if } r \leq n - i - 1, \\ m, & \text{otherwise}, \end{cases}$$

for some $m > n - i$. In this case, $F(v_{B+i-1}) = (\ldots, \underbrace{i - 2}_{n-i-1}, \ldots)$, and there is $\sigma_5' = \sigma_5$ such that $\sigma_5'(F(v_{A+i})) = F(v_{B+i-1})$ in $K_2$ (see Fig. 10).

Same arguments are used for the remaining cases (see item (2) of this proof).

\[\square\]

**Proposition 20** Two positive integral diamonds of Dynkin type $\Delta_n$ are in the same minimal $p$-cycle if their triangulations are in the same mutation class.
The following result gives a way to build frieze patterns.

**Theorem 21** Let $A^0$ be a positive integral diamond of Dynkin type $A_n$ and let $\{A^t\}_{0 \leq t \leq p-1}$ be the minimal $p$-cycle generated by $A^0$. Then:

(i) $A^0$ and $F(v_A^0)$ generate the same frieze pattern (see (9)).

(ii) $\{A^t\}_{0 \leq t \leq p-1}$ is in surjective correspondence with a direct sum of $p$ indecomposable objects of a Dyck paths category.

**Proof** Let $\mathcal{D}_{A_n}$ be the set of all vectors associated to positive integral diamonds of Dynkin type $A_n$, $A^0$ a positive integral diamond of Dynkin type $A_n$, and $\{A^t\}_{0 \leq t \leq p-1}$ the minimal $p$-cycle generated by $A^0$.

(i) Let $K$ be a labeling of an $(n+3)$-polygon, Theorem 17 implies that

$$F(v_A^0) = \tilde{g}((a_0^{11}, T_2(v_A^0), \ldots, T_n(v_A^0))) = \tilde{g}(\lambda_1, \ldots, \lambda_{n+1-a_0^{11}}, 0, \ldots, 0) = (l_1^{n+1-a_0^{11}}, l_0^{n-a_0^{11}}, \ldots, l_0^n),$$

then, there are $a_0^{11} - 1$ diagonals from the vertex 0 to other vertices, i.e., there are $a_0^{11}$ triangles incident with vertex 0. Proposition 20 allows us to establish that $a_i^{11}$ is the number of triangles incident with the vertex $i$, for $1 \leq i \leq n+3$, $i = pm$ and $1 \leq m \leq p \mid (n+3)$. Therefore $A^0$ and $F(v_A^0)$ generate the same frieze pattern.

(ii) Let $(\mathcal{D}_{2(n+1)}, R)$ be any Dyck paths category, we take objects of $(\mathcal{D}_{2(n+1)}, R)$ defined by the following identity

$$\mathcal{O}(\mathcal{D}_{2n}, R) = \bigoplus_{G_i \in \mathcal{D}_{2n}} G_i \quad \text{where} \quad g(\lambda_{G_i}) \text{ and } g(\lambda_{G_j}) \text{ are in the same mutation class},$$

the map $\varphi : \mathcal{D}_{A_n} \rightarrow \mathcal{O}(\mathcal{D}_{2n}, R)$, such that

$$\varphi(v_A^0) = f(v_A^0) \oplus \cdots \oplus f(v_A^{p-1}),$$

with $\{A^t\}_{0 \leq t \leq p-1}$ is surjective as a consequence of Theorem 17 and Proposition 20.

□

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As an example of the use of Theorem 21, we choose the object $D$ of a Dyck paths category $(\mathcal{D}_{2(n+1)}, \mathcal{R})$ shown in Fig. 11 which has associated the following frieze pattern where red numbers is a positive integral diamond of Dynkin type $A_n$.

\[
\begin{array}{cccccccc}
... & 1 & 1 & 1 & 1 & 1 & ... \\
... & 1 & 3 & 2 & 1 & 3 & 2 & ... \\
... & 2 & 5 & 1 & 2 & 5 & ... \\
... & 1 & 3 & 2 & 1 & 3 & 2 & ... \\
... & 1 & 1 & 1 & 1 & 1 & ... \\
\end{array}
\]

References

1. Assem, I., Reutenauer, C., Smith, D.: Friezes. Adv. Math. 225, 3134–3165 (2010)
2. Barcucci, E., Del Lungo, A., Fezzi, S., Pinzani, R.: Nondecreasing Dyck paths and q-Fibonacci numbers. Discrete Math. 170, 211–217 (1997)
3. Baur, K., Faber, E., Gratz, S., Serhiyenko, K., Todorov, G.: Mutation of friezes. Bull. Sci. Math. 142, 1–48 (2018)
4. Caldero, P., Chapoton, F., Schiffler, R.: Quivers with relations arising from clusters ($A_n$ case). Trans. Am. Math. Soc. 358(3), 1347–1364 (2006)
5. Caldero, P., Chapoton, F.: Cluster algebras as Hall algebras of quiver representations. Comment. Math. Helv. 81(3), 595–616 (2006)
6. Cañadas, A.M., Rios, G.B.: Dyck paths categories and its relationships with cluster algebras. arXiv:2102.02974 (2021) Preprint
7. Conway, J.H., Coxeter, H.S.M.: Triangulated polygons and frieze patterns. Math. Gaz. 57, 87–94 (1973)
8. Conway, J.H., Coxeter, H.S.M.: Triangulated polygons and frieze patterns. Math. Gaz. 57, 175–183 (1973)
9. Coxeter, H.S.M.: Frieze patterns. Acta Arith. 18, 297–310 (1971)
10. Fomin, S., Zelevinsky, A.: Cluster algebra. II: Finite type classification. Invent. Math. 154(1), 63–121 (2003)
11. Fontaine, B., Plamondon, P.-G.: Counting friezes in type $\mathbb{D}_n$. J. Algebr. Combin. 44(2), 433–445 (2016)
12. Gunawan, E., Schiffler, R.: Frieze vectors and unitary friezes. J. Combin. 11(4), 681–703 (2020)
13. Morier-Genoud, S.: Coxeter’s frieze patterns at the crossroads of algebra, geometry and combinatorics. Bull. Lond. Math. Soc 47(6), 895–938 (2015)
14. Propp, J.: The combinatorics of frieze patterns and Markoff numbers. Integers 20, 1–38 (2020)
15. Ringel, C.M.: Catalan combinatorics of the hereditary Artin algebras. In: Developments in Representation Theory. Contemporary Mathematics, 673, pp. 51–177, Providence, RI, AMS (2016)
16. Sloane, N.J.A.: On-Line Encyclopedia of Integer Sequences The OEIS Foundation. http://oeis.org/A009766
17. Stanley, R.P.: Enumerative Combinatorics, vol. 2. Cambridge University Press, Cambridge (1999)

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