Quillen model categories without equalisers or coequalisers

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Abstract
Quillen defined a model category to be a category with finite limits and colimits carrying a certain extra structure. In this paper, we show that only finite products and coproducts (in addition to the certain extra structure alluded to above) are really necessary to construct the homotopy category. This leads to the interesting observation that the homotopy category construction could feasibly be iterated.

1 Introduction
This paper concerns the definition of Quillen model category and the most basic fact which follows from said definition. In particular, we show that this same result can be obtained with slightly weaker hypotheses; i.e., that the definition of Quillen model category can be weakened “at no extra cost”.

The bone of contention is the number—or, more accurately, class—of limits and colimits which the category is required to possess. We recall that a category $K$ has arbitrary (finite) limits if and only if it has equalisers and arbitrary (finite) products, [6, p.113]. Dually, $K$ has arbitrary (finite) colimits if and only if it has coequalisers and arbitrary (finite) coproducts.

The original motivation for this result lay in the author’s attempt to apply the theory of Quillen model structures to categories arising in the study of linear logic, [2, 3]; such categories frequently possess only products and coproducts. Understanding that such a motivation may strike a topologist as (frankly) obscure, we conclude the article by discussing some potential applications of this result within the more traditional demesne of topology and geometry.

2 Background
The most basic fact about Quillen model categories, alluded to above, is this: given a Quillen model category $\mathcal{K}$, one can define a category-theoretic congruence [6, p.52] $\sim$ on a certain full

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subcategory of $\mathcal{K}$, denoted $\mathcal{K}_{cf}$, such that the resultant quotient category $\mathcal{K}_{cf}/\sim$ is equivalent to the category of fractions obtained by inverting the weak equivalences in $\mathcal{K}$.

We will not repeat the definition of Quillen model structure—which can be found, for example, in [5, 1.1.3]. But we will quickly review the definition(s) of the homotopy relation, $\sim$. For the remainder of the paper, $\mathcal{K}$ will denote a category with finite products and finite coproducts equipped with a Quillen model structure.

**Definitions 2.1**

Two arrows $x \xrightarrow{\alpha} y$ in $\mathcal{K}$ are called

1. *left-homotopic*, denoted $\alpha \sim_\ell \beta$, if there exists a diagram

$$
\begin{array}{ccc}
\n & x + x \xrightarrow{[\alpha, \beta]} y \\
\downarrow \nabla & & \downarrow \omega \\
\n & x \xleftarrow{\sigma} c \\
\downarrow \mu & & \downarrow \omega \\
\n & c \xrightarrow{\omega} y
\end{array}
$$

with $\mu$ a cofibration and $\sigma$ a weak equivalence.

2. *right-homotopic*, denoted $\alpha \sim_\ell \beta$, if there exists a diagram

$$
\begin{array}{ccc}
\n & p \xleftarrow{\kappa} y \\
\downarrow \varpi & & \downarrow \Delta \\
\n & x \xrightarrow{(\beta, \gamma)} \times y \\
\downarrow \psi & & \downarrow \Delta \\
\n & \times y \xrightarrow{\psi} y
\end{array}
$$

with $\varpi$ a fibration and $\kappa$ a weak equivalence.

The following statements are well-known, and their proofs do not use the existence of limits and colimits other than products and coproducts. [Indeed, the first two are almost tautologous. But the third is quite interesting, as we shall see below.]

**Lemmata 2.2**

1. The relation $\sim_\ell$ is reflexive, symmetric and satisfies left-congruity—i.e., $\alpha \sim_\ell \beta \Rightarrow \theta \alpha \sim_\ell \theta \beta$, whenever this makes sense.

2. The relation $\sim_\ell$ is reflexive, symmetric and satisfies right-congruity—i.e., $\alpha \sim_\ell \beta \Rightarrow \alpha \theta \sim_\ell \beta \theta$, whenever this makes sense.

3. The restriction of $\sim_\ell$ to $\mathcal{K}_{cf}$ coincides with that of $\sim_\ell$.  

2
What remains to show is that the common restriction of $\sim^L$ and $\sim^R$ to $\mathcal{K}_{cf}$, henceforth denoted $\sim$, is transitive.

It is worth noting at this stage that the usual proof of the transitivity of $\sim$ (as found in Lemma 4), requires the existence of pushouts but not of pullbacks. Of course, there is a dual proof of the same fact which requires pullbacks but not pushouts. It would seem quite odd if the existence of pushouts-or-pullbacks were necessary as well as sufficient.

3 Transitivity of $\sim$

The following lemma, although stated in somewhat more general terms, essentially amounts to the transitivity of $\sim$. Its proof is, in fact, an adaptation of the usual proof of that $\sim^L$ and $\sim^R$ coincide on $\mathcal{K}_{cf}$.

Lemma 3.1

Let $\alpha, \beta$ and $\gamma$ be parallel arrows $x \rightarrow y$, and suppose $\alpha \sim^L \beta \sim^R \gamma$. Then $y$ fibrant implies $\alpha \sim^L \gamma$ dually, $x$ cofibrant implies $\alpha \sim^R \gamma$.

Proof

Suppose that $y$ is fibrant and that the relations $\alpha \sim^L \beta \sim^R \gamma$ are witnessed as follows:

\[
\begin{array}{cccccc}
\downarrow & & & & & \\
\sigma & \alpha & \beta & \gamma & & \\
& x & & & & y
\end{array}
\]

with $\mu$ a cofibration, $\omega$ a fibration, and $\sigma, \kappa$ weak equivalences.

Let $\varpi_0, \varpi_1$ denote the two components of $\varpi$ so that $\varpi = (\varpi_0, \varpi_1)$. By a standard argument, $y$ fibrant implies that $y \times y \xrightarrow{\pi_0} y$ and $y \times y \xrightarrow{\pi_1} y$ are fibrations; hence also $\varpi_0 = \pi_0 \varpi$ and $\varpi_1 = \pi_1 \varpi$. Moreover, $\varpi_0 \kappa = \text{id}_y = \varpi_1 \kappa$, so by two-out-of-three, $\varpi_0, \varpi_1$ are also weak equivalences.

Now we can factor $[\alpha, \gamma]$ through $\varpi_1$ as follows:

\[
[\alpha, \gamma] = [\varpi_1 \kappa \alpha, \varpi_1 \psi] = \varpi_1 [\kappa \alpha, \psi]
\]

and moreover,

\[
\varpi_0 [\kappa \alpha, \psi] = [\varpi_0 \kappa \alpha, \varpi_0 \psi] = [\alpha, \beta] = \omega \mu.
\]

1In this, my notation is slightly non-standard. More often, one writes $\psi \sim \omega$ to mean that both $\psi \sim^L \omega$ and $\psi \sim^R \omega$ hold.
Hence, we have a diagram

\[
\begin{array}{ccc}
  x + x & \xrightarrow{[\alpha, \gamma]} & y \\
  \downarrow \mu & & \downarrow p \\
  c & \xrightarrow{\omega} & y
\end{array}
\]

with \( \mu \) a cofibration and \( \varpi_0 \) a trivial fibration. Therefore, we can find a diagonal lift

\[
\begin{array}{ccc}
  x + x & \xrightarrow{[\alpha, \gamma]} & y \\
  \downarrow \mu & & \downarrow p \\
  c & \xrightarrow{\omega} & y
\end{array}
\]

and so the composite \( \varpi_1 \delta \) witnesses \( \alpha \sim \gamma \).

Q.E.D.

**Theorem 3.2**

The relation \( \sim \) is a congruence on \( K_{cf} \); moreover, the quotient category \( K_{cf}/\sim \) is equivalent to \( K[\mathcal{W}^{-1}] \).

**Proof**

We have already established the first statement via a series of lemmata; the second is proven as in [5].

Q.E.D.

### 4 Iterated Homotopy?

It is well known that, for an arbitrary Quillen model category \( \mathcal{K} \), the homotopy category \( \text{Ho}[\mathcal{K}] \) does not have arbitrary limits and colimits. It does, however, inherit discrete limits and colimits from \( \mathcal{K} \). Thus the result of this article shows that, if we can find a Quillen model structure on \( \mathcal{H} = \text{Ho}[\mathcal{K}] \), then there is no obstruction to forming a further homotopy category, \( \text{Ho}[\mathcal{H}] = \text{Ho}[\text{Ho}[\mathcal{K}]] \).

This observation could be utilised in two opposite ways: one might try to find Quillen model structures on already known homotopy categories—this might prove a useful way of studying individual homotopy invariants; or, perhaps more interestingly, one might attempt
to ‘subdivide’ ordinary homotopy into smaller, and hopefully more tractable, steps. Let us illustrate the latter idea with an example.

Consider the concept of thin homotopy which arises in differential geometry, [1]. For our present purposes, let us define a smooth space to be a finite disjoint union of finite-dimensional differential manifolds, and a smooth map to be a map between smooth spaces whose restriction to each connected component of the domain is smooth. Then the category of smooth spaces and smooth maps, $\mathcal{M}$, has finite products and coproducts.

Questions 4.1

1. Does there exist a Quillen model structure on $\mathcal{M}$ resulting in a suitable thin homotopy category, $\text{Ho}[\mathcal{M}]$?

2. If so, does there exist a Quillen model structure on $\mathcal{H} = \text{Ho}[\mathcal{M}]$ such that $\text{Ho}[\mathcal{H}]$ ($= \text{Ho}[\text{Ho}[\mathcal{M}]]$) is equivalent to the usual homotopy category of smooth spaces?

Note that, while one could replace $\mathcal{M}$ by an even larger category—say diffeological spaces [2], or Frölicher spaces [3]—in order to avoid using the result of this article with respect to the first question posed above, there is no guarantee that one could similarly avoid the result of this article with respect to the second.

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