Instability of interfaces in the antiferromagnetic XXZ chain at zero temperature

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Abstract

For the antiferromagnetic, highly anisotropic XZ and XXZ quantum spin chains, we impose periodic boundary conditions on chains with an odd number of sites to force an interface (or kink) into the chain. We prove that the energy of the interface depends on the momentum of the state. This shows that at zero temperature the interface in such chains is not stable. This is in contrast to the ferromagnetic XXZ chain for which the existence of localized interface ground states has been proven for any amount of anisotropy in the Ising-like regime.
1 Introduction

Interfaces or domain walls in classical spin systems have been the subject of mathematical study for several decades. Dobrushin proved [12] that in the three-dimensional Ising model at low temperatures, under suitable (Dobrushin) boundary conditions, there is a stable interface orthogonal to the 001-direction. These boundary conditions hence yield a non-translation invariant Gibbs state at low temperatures. However, Gallavotti proved [14] that the two-dimensional model shows a very different behavior; thermal fluctuations destabilize the interface and the corresponding Gibbs state is translation invariant.

Interfaces in quantum-mechanical systems can exhibit a much richer and more complex behavior than their classical counterparts. A review of some of this behavior may be found in [24]. For example, quantum fluctuations may lift a classical degeneracy and, in doing so, stabilize an interface (against thermal fluctuations) that is unstable in the corresponding classical system. Such a stabilization is an example of the phenomenon of ground state selection [16]. It is expected to occur for the 111–(or diagonal) interface in the three-dimensional ferromagnetic, anisotropic XXZ model [see e.g. [5, 6]], and has been proved to occur for the 111-interface in the three-dimensional Falicov-Kimball model [11]. These models can be viewed as quantum perturbations of the classical Ising model. In contrast to these quantum–mechanical models, the diagonal interface in the three-dimensional classical Ising model is expected to be unstable at non–zero temperatures. This is due to the massive degeneracy of the zero-temperature configurations compatible with the boundary conditions which favor such an interface [see [17]].

Another interesting feature of interfaces in quantum-mechanical systems is the diverse nature of the low–lying excitations above the interface ground states for different models and for different orientations of the interface. For example, there are gapless excitations above the conjectured diagonal interface states in the spin-1/2 ferromagnetic, anisotropic, XXZ model. These excitations were described in the two-dimensional case by Koma and Nachtergaele [6, 20, 21], and proved to exist in all dimensions greater than one by Matsui [23]. In contrast, it is expected that there is a gap in the spectrum above a ground state that describes an interface perpendicular to a coordinate direction.

For quantum-mechanical systems, the stability of an interface is a nontrivial question even in the ground state, since quantum fluctuations can destabilize the interface at zero temperature. In this case quantum fluctuations play a role analogous to that of thermal fluctuations in classical systems. In one dimension we expect interface states to be unstable for generic Hamiltonians. However, there are notable exceptions, e.g. the anisotropic ferromagnetic XXZ chain. In addition to its two ferromagnetically ordered, translation invariant ground states, this model has ground states corresponding to an interface between two domains of opposite magnetization. The stability of this interface was proved independently by Alcaraz, Salinas and Wreszinski [1] and Gottstein and Werner [15]. This stability is a direct consequence of the conservation of the total $z$-component of the spin. There are no terms in the Hamiltonian that can simply move the interface across one lattice spacing. To conserve the spin, one must at the same time create a new excitation in the chain, thus raising the energy of the state.
More precisely, it was proved in \[1, 15\] that, under suitable boundary conditions, there exists a family of interface ground states which describe a localized domain wall. The localization length depends on the anisotropy of the model and diverges in the limit of the isotropic model. Alternative proofs of the stability of this interface were given in \[4\], by using the path integral representation of interface states, and in \[3\], by employing the principle of exponential localization \[13\]. The above results show that in the spin-1/2 ferromagnetic, anisotropic XXZ model, an arbitrarily small amount of anisotropy is sufficient to stabilize the interface against quantum fluctuations.

Quantum perturbations do not always have the drastic effect of either stabilizing an unstable classical interface or destabilizing an interface at zero temperature. There exist quantum lattice models which are quantum perturbations of suitable classical systems such that an interface in the classical system remains essentially unchanged under the quantum perturbations. For example, if we add a quantum perturbation to the three dimensional Ising model, then the so-called Dobrushin condition induces a stable interface in the system, in the sense that there is a low temperature non–translation invariant Gibbs state describing an asymptotically horizontal interface. This was proved in a more general setting by Borgs, Chayes, and Fröhlich \[8\] for systems in dimensions \(d \geq 3\), by using a quantum version of the Pirogov Sinai theory \[7, 9\]. One expects that adding a quantum perturbation to the two-dimensional Ising model at low temperatures will not stabilize the 10-interface in this model but we are not aware of any proof of this.

In this paper we consider the stability of the interface states in the anisotropic, antiferromagnetic(AF) XXZ and XZ models at zero temperature. We prove that in these models the interface is not stable in one dimension. We study the question of stability by analyzing the dispersion relation for the energy of the interface, i.e., its energy as a function of its momentum. For the AF models we can force an interface into the system by imposing periodic boundary conditions on a chain with an odd number of sites. We can study the energy of the interface by comparing the energies for chains with an even and odd number of sites. The AF Hamiltonians that we consider are invariant both under lattice translations and global spin flips. The combined symmetry of translating by one lattice spacing and then performing a global spin flip, which we denote by \(\tilde{T}\), is a useful symmetry for studying the interfaces since it leaves the Néel states invariant. We refer to the eigenvalue of this symmetry operator as a “generalized momentum.” We study the difference between the lowest energy of an eigenstate with generalized momentum \(k\) for a chain with an odd number of sites and that with an even number of sites. We take this difference to be the definition of the dispersion relation for the interface. If the interface is stable, then there should be an eigenstate \(|\Psi\rangle\) of the Hamiltonian (for a chain with an odd number of sites) which has some localized structure. So the states \(\tilde{T}^l|\Psi\rangle\) should be linearly independent. By taking linear combinations of these states,

\[
|\Phi_k\rangle = \sum_l e^{ikl}\tilde{T}^l|\Psi\rangle,
\]

we can form eigenstates of the Hamiltonian with different generalized momenta. Since \(\tilde{T}\)
commutes with the Hamiltonian, these states all have the same energy. Thus the dispersion relation is independent of the generalized momentum if there is a stable interface. We prove that in the infinite volume limit the dispersion relation for the AF chain depends on the generalized momentum, and so the chain does not admit ground states that correspond to a stable interface. In contrast, for the anisotropic, ferromagnetic XXZ chain, we prove that the dispersion relation is “flat” (i.e., $k$–independent) in the infinite length limit. This provides another approach to studying the stability of the interface in this model at zero temperature to complement the approaches of [1, 15, 4, 3].

The XZ chain is exactly solvable, and Araki and Matsui used this to prove the absence of non-translationally invariant infinite volume ground states [2]. This shows the interface is unstable in this model since infinite volume ground states containing an interface would be non-translationally invariant. The XXZ model is also exactly solvable, so one might be able to use this solvability to study the dispersion relation we study. We emphasize, however, that in our approach we do not use the exact solvability of either of these models. The techniques that we use to study the interface are based on a novel approach to the analysis of ground states of quantum spin systems, introduced by Kirkwood and Thomas [18]. They considered spin–1/2 models, but their approach was applied to some higher spin models by Matsui [22]. Their method originally required a Perron-Frobenius condition on the Hamiltonian. We removed this condition and simplified the proof of convergence of the expansion in [10]. Although we restrict our attention to the XZ and XXZ models in this paper, we expect the methods and results to be applicable to a much broader class of models.

The paper is organized as follows: To keep the paper self–contained, we first give a summary of our version of the Kirkwood–Thomas approach (as developed in [10]) by using it to study the ground state of the AF anisotropic XZ Hamiltonian. This is done in Section 2 for a $d$–dimensional lattice under periodic boundary conditions. The results of this section, for the case $d = 1$, are used later in our analysis of interface states in the AF anisotropic XZ chain. If the number of sites $N$ in such a chain is even then the ground state does not have an interface. However, if $N$ is odd then the periodic boundary conditions force an interface in the chain. The latter situation is studied in Section 3. We prove that the dispersion relation for the energy of the interface depends non–trivially on the generalized momentum $k$ even in the limit $N \to \infty$. This allows us to conclude that the ground state of the AF anisotropic XZ chain does not have a stable interface. In Section 4 we prove a similar result for the AF anisotropic XXZ chain. In contrast, in Section 5, we prove that for the corresponding ferromagnetic model the energy of the interface does not depend on $k$ in the limit $N \to \infty$.

## 2 XZ ground state : the Kirkwood–Thomas approach

We consider the following antiferromagnetic Hamiltonian defined on a finite lattice $\Lambda \subset \mathbb{Z}^d$

$$\widetilde{H} = \sum_{\langle ij \rangle \subseteq \Lambda} \sigma_i^z \sigma_j^z + \epsilon \sum_{\langle ij \rangle \subseteq \Lambda} \sigma_i^x \sigma_j^x,$$  \hspace{1cm} (2)
where the sums are over all nearest neighbor pairs (denoted by $\langle ij \rangle$) in $\Lambda$. We impose periodic boundary conditions and assume that $\Lambda$ has an even number of sites in each coordinate direction. The Hamiltonian $\tilde{H}$ acts on the Hilbert space $\mathcal{H}_\Lambda = (\mathbb{C}^2)^{\otimes|\Lambda|}$, where $|\Lambda|$ denotes the number of sites in the lattice $\Lambda$. The Hamiltonian and most of the quantities that follow depend on the volume $\Lambda$. However, for notational simplicity, we often suppress this explicit dependence. The above Hamiltonian commutes with the global spin flip operator given by

$$\tilde{P} = \prod_{i \in \Lambda} \sigma_z^i. \quad (3)$$

The above form of the Hamiltonian seems natural for perturbation theory in $\epsilon$ since the $\epsilon = 0$ Hamiltonian is diagonal. However, following Kirkwood and Thomas, we study a unitarily equivalent Hamiltonian obtained by a rotation about the $Y$-axis in spin space caused by the operator

$$R = \exp \left( i \frac{\pi}{4} \sum_{j \in \Lambda} \sigma_y^j \right). \quad (4)$$

Hence,

$$R \sigma_i^x R^{-1} = \sigma_i^z; \quad R \sigma_i^z R^{-1} = -\sigma_i^x,$$

and therefore

$$R \tilde{H} R^{-1} = \sum_{\langle ij \rangle \subset \Lambda} \sigma_i^x \sigma_j^x + \epsilon \sum_{\langle ij \rangle \subset \Lambda} \sigma_i^z \sigma_j^z. \quad (5)$$

The global spin flip operator transforms into

$$R \tilde{P} R^{-1} = \prod_{i \in \Lambda} \sigma_z^i. \quad (6)$$

Finally, we perform a unitary transformation to change the $\epsilon = 0$ Hamiltonian from antiferromagnetic to ferromagnetic. Define

$$U = \prod_{j \in \Lambda \text{ even}} \sigma_z^j \quad (7)$$

where $j \text{ odd}$ means that the sum of the components of $j$ is odd. Since $\Lambda$ has an even number of sites in each coordinate direction, the transformed Hamiltonian, $H$, is given by

$$H = U \tilde{H} R^{-1} U^{-1} = -\sum_{\langle ij \rangle \subset \Lambda} \sigma_i^x \sigma_j^x + \epsilon \sum_{\langle ij \rangle \subset \Lambda} \sigma_i^z \sigma_j^z. \quad (8)$$

Since $[H, P] = 0$, the state space of the Hamiltonian $H$ can be decomposed into two subspaces corresponding to the eigenvalues $+1$ and $-1$ of $P$. We refer to these two subspaces as the even and odd sectors respectively. The transformed global spin flip operator, $R \tilde{P} R^{-1}$, remains unchanged under the action of the unitary operator $U$:

$$P = U \tilde{P} R^{-1} U^{-1} = \prod_{i \in \Lambda} \sigma_i^z. \quad (9)$$
We emphasize that eq.(8) is not true if Λ has an odd number of sites in any lattice direction. This fact plays a key role in our study of interfaces in the one dimensional case [see e.g. Section 3].

Let us introduce some definitions and notations. A classical spin configuration on the lattice is defined to be an assignment of a +1 or a −1 to each site in the lattice. Hence, for each \( i \in \Lambda \), \( \sigma_i = \pm 1 \). We will abbreviate the classical spin configuration \( \{\sigma_i\}_{i \in \Lambda} \) by \( \sigma \). For each such \( \sigma \) we let \( |\sigma\rangle \) be the state in the Hilbert space, \( \mathcal{H}_\Lambda \), which is the tensor product of a spin–up state at each site with \( \sigma_i = +1 \) and a spin–down state at each site with \( \sigma_i = -1 \). Thus \( |\sigma\rangle \) is an eigenstate of all the \( \sigma_z^i \) with \( \sigma_z^i |\sigma\rangle = \sigma_i |\sigma\rangle \). The states \( |\sigma\rangle \) form a complete orthonormal basis of \( \mathcal{H}_\Lambda \). Any state \( |\Psi\rangle \) can be written in terms of this basis:

\[
|\Psi\rangle = \sum_{\sigma} \psi(\sigma) |\sigma\rangle
\]  

(10)

where \( \psi(\sigma) \) is a complex-valued function on the spin configurations \( \sigma \). For a single site, the vectors \((|+1\rangle + |-1\rangle)\) and \((|+1\rangle - |-1\rangle)\) are the eigenstates of \( \sigma_x^i \) with eigenvalues +1 and −1, respectively. Thus the (unnormalized) ground states of the Hamiltonian, \( H \), \([8]\) for \( \epsilon = 0 \) are given by (10) with \( \psi(\sigma) = 1 \) and \( \psi(\sigma) = \prod_{i \in \Lambda} \sigma_i \). We define

\[
\sigma(X) = \prod_{i \in X} \sigma_i
\]  

(11)

and use the convention that \( \sigma(\emptyset) = 1 \). Note that \( \sigma(\Lambda) \) is equal to +1(−1) in the even (odd) sector.

In the Kirkwood–Thomas method one expands the ground state with respect to the basis \( \{ |\sigma\rangle \} \), as in eq. (10), and writes \( \psi(\sigma) \) in the form

\[
\psi(\sigma) = \exp\left[-\frac{1}{2} \sum_X g(X) \sigma(X)\right]
\]  

(12)

for some real \( g(X) \). As in [10], we justify the above exponential form of \( \psi(\sigma) \) by a two–step procedure: First, we consider (12) to be an ansatz and prove that it satisfies the Schrödinger equation. This ensures that there is an eigenstate of the form (12). Next we give an argument to show that this eigenstate must in fact be the ground state.

Consider the Schrödinger equation

\[
H \Psi = E_0 \Psi
\]  

(13)

The operator \( \sigma_i^z \sigma_j^z \) is diagonal in the chosen basis, so

\[
\sigma_i^z \sigma_j^z \sum_{\sigma} \psi(\sigma) |\sigma\rangle = \sum_{\sigma} \sigma_i \sigma_j \psi(\sigma) |\sigma\rangle.
\]  

(14)

The operator \( \sigma_i^z \sigma_j^z \) just flips the spins at sites \( i \) and \( j \), i.e., \( \sigma_i^z \sigma_j^z |\sigma\rangle = |\sigma^{(ij)}\rangle \), where \( \sigma^{(ij)} \) is the spin configuration \( \sigma \) but with \( \sigma_i \) replaced by \( -\sigma_i \) and \( \sigma_j \) replaced by \( -\sigma_j \). Hence

\[
\sigma_i^z \sigma_j^z \sum_{\sigma} \psi(\sigma) |\sigma\rangle = \sum_{\sigma} \psi(\sigma) |\sigma^{(ij)}\rangle = \sum_{\sigma} \psi(\sigma^{(ij)}) |\sigma\rangle.
\]  

(15)
The last equality follows by a change of variables in the sum.

We now see that if we use (10) in the Schrödinger equation (13) and pick out the coefficient of $|\sigma\rangle$, then for each spin configuration $\sigma$ we have

$$-\sum_{\langle ij \rangle} \psi(\sigma^{(ij)}) + \epsilon \sum_{\langle ij \rangle} \sigma_i \sigma_j \psi(\sigma) = E_0 \psi(\sigma).$$

(16)

Henceforth, the condition $\langle ij \rangle \subset \Lambda$ will be implicit in all our sums on $\langle ij \rangle$. Dividing both sides of (16) by $\psi(\sigma)$ we have

$$-\sum_{\langle ij \rangle} \frac{\psi(\sigma^{(ij)})}{\psi(\sigma)} + \epsilon \sum_{\langle ij \rangle} \sigma_i \sigma_j = E_0.$$  

(17)

Now $\sigma^{(ij)}(X)$ is $\sigma(X)$ when both of $i$ and $j$ are in $X$, and when both of them are not in $X$. If exactly one of $i$ and $j$ is in $X$, then $\sigma^{(ij)}(X)$ is $-\sigma^{(ij)}(X)$. We will let $\partial X$ denote the set of nearest neighbor bonds which connect a site in $X$ with a site not in $X$. (Henceforth, we will always use the word bond to denote a nearest neighbor bond.) Then the condition that exactly one of $i$ and $j$ belongs to $X$ may be written as $\langle ij \rangle \in \partial X$. We will often abbreviate this condition as $X : \langle ij \rangle$. Thus

$$\psi(\sigma^{(ij)}) = \exp\left[-\frac{1}{2} \sum_X g(X \sigma(X) + \sum_{X : \langle ij \rangle} g(X) \sigma(X) \right]$$

(18)

and so the Schrödinger equation is now

$$-\sum_{\langle ij \rangle} \exp\left[ \sum_{X : \langle ij \rangle} g(X \sigma(X) \right] + \epsilon \sum_{\langle ij \rangle} \sigma_i \sigma_j = E_0.$$  

(19)

As in [10], we refer to this equation as the Kirkwood-Thomas equation.

We expand the exponential in a power series. The contribution from the linear term may be rewritten as

$$\sum_{\langle ij \rangle} \sum_{X : \langle ij \rangle} g(X \sigma(X) = \sum_X |\partial X| g(X \sigma(X)$$

(20)

where $|\partial X|$ is the number of bonds in $\partial X$, i.e., the number of bonds that connect a site in $X$ with a site not in $X$. Hence the Kirkwood Thomas equation becomes

$$\sum_X |\partial X| g(X \sigma(X) + E_0 + d|\Lambda| = -\sum_{\langle ij \rangle} \frac{1}{n} \sum_{X_1, X_2, \ldots, X_n : \langle ij \rangle} \prod_{k=1}^n g(X_k \sigma(X_k) + \epsilon \sum_{\langle ij \rangle} \sigma_i \sigma_j.$$  

(21)

Here $d|\Lambda|$ is the number of bonds in the lattice.

Since $\sigma^2_i = 1$, $\sigma(X) \sigma(Y) = \sigma(X \triangle Y)$ where the symmetric difference $X \triangle Y$ of $X$ and $Y$ is defined by $X \triangle Y = X \cup Y \setminus (X \cap Y)$. Thus $\prod_{k=1}^n \sigma(X_k) = \sigma(X_1 \triangle \cdots \triangle X_n)$. If we equate the
coefficient of $\sigma(X)$ on both sides of eq. (21), we obtain, for $X \neq \emptyset$,

$$g(X) = \frac{1}{|\partial X|} \left[ -\sum_{\langle ij \rangle} \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{X_1 \Delta \cdots \Delta X_n = X} g(X_1)g(X_2) \cdots g(X_n) + \epsilon 1_{nn}(X) \right].$$

(22)

where $1_{nn}(X)$ is 1 if $X$ consists of two nearest neighbor sites and is 0 otherwise.

If $X = \Lambda$, then $\partial X = \emptyset$. So the coefficient of $g(\Lambda)$ on the LHS of equation (21) is zero. This looks like a fatal problem since the RHS of the equation will contain a multiple of $\sigma(\Lambda)$. We solve this problem by exploiting the decomposition of the state space into even and odd sectors (as in [18]). We look for eigenstates of the form

$$|\Psi_e\rangle = \sum_{\sigma: \text{even}} \psi(\sigma)|\sigma\rangle$$

(23)

and

$$|\Psi_o\rangle = \sum_{\sigma: \text{odd}} \psi(\sigma)|\sigma\rangle$$

(24)

where the sums are only over configurations $\sigma$ for which the number of sites $i$ with $\sigma_i = -1$ is even or odd, respectively. (Equivalently, $\sigma(\Lambda) := \prod_{i \in \Lambda} \sigma_i = +1$, or $-1$.) The Schrödinger equation is still equivalent to (19), but now to find an eigenstate in the even (respectively, odd) sector, this equation need only hold for $\sigma$ with $\prod_{i \in \Lambda} \sigma_i = +1$ (respectively, $-1$). Thus the terms on the RHS of (21) which contain $\sigma(\Lambda)$ may be included in the equation for $X = \emptyset$. So for $X = \emptyset$, we obtain the equation

$$E_{\pm} + d |\Lambda| = -\sum_{\langle ij \rangle} \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{X_1 \Delta \cdots \Delta X_n = \emptyset} g(X_1)g(X_2) \cdots g(X_n) + \epsilon 1_{nn}(X)$$

$$\pm \sum_{\langle ij \rangle} \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{X_1 \Delta \cdots \Delta X_n = \Lambda} g(X_1)g(X_2) \cdots g(X_n).$$

(25)

Here and henceforth, the upper (lower) sign corresponds to the even (odd) sector. We have replaced $E_0$ by $E_{\pm}$ since the eigenvectors in the even and odd sectors have different eigenvalues. We will see later that the difference between the two eigenvalues is exponentially small in the number of sites in the lattice $\Lambda$. Note that eq. (19) for the two sectors can be combined into the single equation

$$-\sum_{Y:j} \exp(\sum Y g(Y)\sigma(Y)) + \epsilon \sum_{j=1}^{N} \sigma_j \sigma_{j+1} = \frac{E_{+} + E_{-}}{2} + \frac{E_{+} - E_{-}}{2} \sigma(|\Lambda|).$$

(26)

We let $g$ denote the collection of coefficients $\{g(X) : X \subset \Lambda, X \neq \emptyset, X \neq \Lambda\}$, and think of eq.(22) as a fixed point equation, $g = F(g)$. We define a norm by

$$||g|| = \sum_{X \neq \emptyset} |g(X)||\partial X| (|\epsilon| M)^{-w(X)},$$

(27)
where $b$ is a nearest neighbor bond and $w(X)$ is defined as follows: We consider two bonds to be “connected” if they share an endpoint or if the distance between them is 1. We consider a set of bonds to be “connected” if we can get from one bond in the set to any other bond in the set by going through a sequence of connected bonds in the set. Then $w(X)$ is the cardinality of the smallest set of bonds which contains $X$ and is “connected.” Note that the symmetries of the lattice imply that the norm $||g||$ does not depend on the choice of $b$.

**Theorem 1** There exists a constant $M > 0$ which depends only on the number of dimensions of the lattice, such that if $|\epsilon| M \leq 1$, then the fixed point equation (22) has a solution $g$, and $||g|| \leq \delta$ for some constant $\delta$ which depends only on the lattice.

**Proof:** We will prove that $F$ is a contraction on a small ball about the origin, and that it maps this ball back into itself. The contraction mapping theorem will then imply that $F$ has a fixed point in this ball. For the sake of concreteness, we prove it is a contraction with constant $1/2$, but there is nothing special about the choice of $1/2$.

Define

$$\delta = \frac{4(2d - 1)}{M}$$

We will show that

$$||F(g) - F(g')|| \leq \frac{1}{2}||g - g'|| \quad \text{for} \quad ||g||, ||g'|| \leq \delta,$$

and

$$||F(g)|| \leq \delta \quad \text{for} \quad ||g|| \leq \delta.$$  

The proof of (29) proceeds as follows: Fix a bond $b$ to use in the definition of $||F(g) - F(g')||$. Then

$$||F(g) - F(g')|| \leq \sum_{(ij)} \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{X_1, \ldots, X_n:(ij), b \in \partial \Delta} |g(X_1) \cdots g(X_n) - g'(X_1) \cdots g'(X_n)| (|\epsilon| M)^{-w(\Delta)},$$

where $\Delta = X_1 \Delta \cdots \Delta X_n$. If $b \in \partial \Delta$, then $b$ is in at least one $\partial X_k$. Using the symmetry under permutations of the $X_k$, we can take $b \in \partial X_1$ at the cost of a factor of $n$. We claim that if $\langle ij \rangle \in \partial X_k$ for $k = 1, 2, \ldots, n$, then

$$w(X_1 \Delta \cdots \Delta X_n) \leq \sum_{k=1}^{n} w(X_k).$$

To prove the claim, for $k = 1, 2, \ldots, n$, let $C_k$ be sets of bonds such that $X_k \subset C_k$, $|C_k| = w(X_k)$ and $C_k$ is connected in the sense used to define $w(X_k)$ [see discussion after (27)]. Define $C = \cup_{k=1}^{n} C_k$. Since $X_k$ contains exactly one of the sites $i$ and $j$, $C_k$ contains at least one of
the sites \( i \) and \( j \). Since \( C_1, \ldots, C_n \) are connected this implies that \( C \) is connected. Clearly, \( X_1 \triangle \cdots \triangle X_n \subset C \). So

\[
w(X_1 \triangle \cdots \triangle X_n) \leq |C| \leq \sum_{k=1}^{n} |C_k| = \sum_{k=1}^{n} w(X_k),
\]

which proves the claim (32).

Using

\[
| \prod_{k=1}^{n} g(X_k) - \prod_{k=1}^{n} g'(X_k) | \leq \sum_{k=1}^{n-1} \prod_{i=1}^{k} |g(X_i)| |g(X_k) - g'(X_k)| \prod_{i=k+1}^{n} |g(X_i)|
\]

we have

\[
||F(g) - F(g')|| \leq \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \sum_{X_1 \triangle \cdots \triangle X_n} \sum_{(ij) \in \partial X_1} \sum_{X_2, \ldots, X_n,(ij)} \sum_{k=1}^{n-1} \prod_{i=1}^{k} |g(X_i)| (|\epsilon| M)^{-w(X_i)} \prod_{i=k+1}^{n} |g'(X_i)| (|\epsilon| M)^{-w(X_i)}
\]

\[
\leq \sum_{n=2}^{\infty} \frac{1}{(n-1)!} ||g - g'|| \sum_{k=1}^{n} ||g||^{k-1} ||g'||^{n-k}
\]

\[
\leq K ||g - g'||,
\]

where

\[
K = \sum_{n=2}^{\infty} \frac{n}{(n-1)!} \delta^{n-1} = e^\delta - 1 + \delta e^\delta,
\]

and we have used the fact that both \( ||g|| \) and \( ||g'|| \) are bounded by \( \delta \). By choosing \( \delta \) to be sufficiently small we obtain \( K \leq 1/2 \).

To prove (30), we use (29) with \( g' = 0 \). From (22) it follows that

\[
||F(0)|| \leq \sum_{X} \epsilon 1_{\text{nn}}(X) (|\epsilon| M)^{-w(X)}
\]

\[
\leq 2(2d - 1)\epsilon (|\epsilon| M)^{-1} \leq 2(2d - 1) M^{-1} = \frac{\delta}{2}.
\]

Hence,

\[
||F(g)|| \leq ||F(g) - F(0)|| + ||F(0)|| \leq \frac{1}{2} ||g|| + \frac{1}{2} \delta \leq \delta
\]
Eq. (25) may be used to study the difference between the ground state energies in the odd and even sectors. It is straightforward to show that

$$|E_- - E_+| \leq c(|\epsilon| M)^{w(\Lambda)} d|\Lambda|,$$

(39)

where the constant $c$ depends on $||g||$. Since $w(\Lambda) = |\Lambda|/2$, the difference between these two eigenvalues is exponentially small in the number of sites in the lattice.

We conclude this section by showing that the eigenstates we have constructed in the even and odd sectors are indeed the lowest eigenstates in these sectors. The argument is similar to that in [10], but some small modifications are needed to take account of the decomposition into even and odd sectors. We know our eigenstates are the lowest in their sectors when $\epsilon = 0$. Since we have a finite lattice, our eigenvalue problem is finite dimensional. So in each sector, our eigenstate will remain the lowest eigenstate provided its eigenvalue does not cross another eigenvalue associated with that sector, i.e., provided the eigensubspace in the sector associated with our eigenvalue continues to be one-dimensional. Hence, if we show that there exists an $\epsilon_0 > 0$ such that our eigenfunction is non–degenerate for all $\epsilon$ with $|\epsilon| < \epsilon_0$, then it would follow that our eigenfunction is the ground state for all such $\epsilon$.

Suppose that there is a value of $\epsilon$ for which there is another eigenvector $|\Psi'_e\rangle$ with the same eigenvalue as $|\Psi_e\rangle$. (The argument in the case of the odd sector is identical.) Define $\psi'(\sigma)$ for even $\sigma$ by

$$|\Psi'_e\rangle = \sum_{\sigma: even} \psi'(\sigma)|\sigma\rangle,$$

(40)

and let $\psi'(\sigma) = 0$ for odd $\sigma$. Now consider $\psi(\sigma) + \alpha \psi'(\sigma)$ where $\alpha$ is a small real number and $\psi(\sigma)$ is defined through (23). As $\alpha \to 0$, this converges to $\psi(\sigma)$ for each $\sigma$. There are only finitely many values of $\sigma$, so for small enough $\alpha$, this function is always positive (since $\psi(\sigma) > 0 \forall \sigma$). So it can be written as $\exp[-\frac{1}{2} \sum_X g_\alpha(X)\sigma(X)]$. Moreover, as $\alpha \to 0$, $g_\alpha(\sigma) \to g(\sigma)$ for each $\sigma$, and by construction $g_\alpha$ satisfies the fixed point equation. So for sufficiently small $\alpha$, $g_\alpha$ is a solution of the fixed point equation which is inside the ball in which we know the fixed point equation has a unique solution. This contradiction completes the argument.

### 3 Interfaces in the Antiferromagnetic XZ chain

In this section we consider the model of the previous section in one dimension. So $\Lambda = \{1, \cdots, N\}$, and

$$\tilde{H} = \sum_{j=1}^{N} \sigma_j^z \sigma_{j+1}^z + \epsilon \sum_{j=1}^{N} \sigma_j^z \sigma_{j+1}^x.$$

(41)

The indices should be taken to be periodic, e.g., $\sigma_{N+1}^z$ means $\sigma_1^z$. When $N$ is even, we have as before

$$H = UR\tilde{H}R^{-1}U^{-1} = -\sum_{j=1}^{N} \sigma_j^z \sigma_{j+1}^x + \epsilon \sum_{j=1}^{N} \sigma_j^z \sigma_{j+1}^z,$$

(42)

11
and the ground state may be constructed as in the previous section. If $N$ is odd, then the periodic boundary conditions force an interface into the antiferromagnetic chain. In this case we have

$$H = URHR^{-1}U^{-1} = -\sum_{j=1}^{N} J_j \sigma_j^x \sigma_{j+1}^x + \epsilon \sum_{j=1}^{N} \sigma_j^z \sigma_{j+1}^z,$$

(43)

where the coupling $J_j$ is +1 except when $j = N$, in which case it is −1. So the $\epsilon = 0$ Hamiltonian has ferromagnetic couplings for all the bonds except the bond between the sites 1 and $N$.

As before the Hamiltonian $\tilde{H}$ commutes with the global spin flip operator $\tilde{P}$ [(3)]. It also commutes with the translation operator $T$ defined by

$$T \sigma_i^\alpha T^{-1} = \sigma_{i+1}^\alpha, \quad \alpha = x, y, z.$$

(44)

When $\epsilon = 0$ and $N$ is even, the ground states of the Hamiltonian $\tilde{H}$ [(41)] are the two Néel states. These states are not invariant under translation. However, if we translate and then perform the global spin flip, the Néel states remain unchanged. So if we define

$$\tilde{T} = \tilde{P}T$$

(45)

then $\tilde{T}$ commutes with $\tilde{H}$ and leaves the Néel states invariant. This combined symmetry of the Hamiltonian will be the most useful one in our study of interface states, since its action on an interface is to simply translate the interface by one site. Let

$$T = URT\tilde{R}R^{-1}U^{-1}$$

(46)

be this combined symmetry after our unitary transformations. Simple calculations show that when $N$ is even, $T$ is equal to the pure translation operator $T$. However, for odd values of $N$ we find that

$$T = \sigma_1^z T.$$

(47)

In words, $T$ translates by one lattice spacing and rotates the spin at the site $i = 1$. We can refer to it as a generalized translation operator. Throughout this section we will assume $N$ to be odd.

Since $H$ and $T$ commute, we choose the eigenfunctions of $H$ to be eigenfunctions of $T$ as well. So they can be labeled by an index $k$, where $k$ can be regarded as the generalized “momentum”, i.e.,

$$T \psi_k(\sigma) = e^{-ik}\psi_k(\sigma).$$

(48)

It is important to note that $T^N$ is not the identity operator. In fact,

$$T^N = P = \prod_{i=1}^{N} \sigma_i^z,$$

(49)

the transformed global spin flip operator [(9)] of Section 2. The state space may again be decomposed into two subspaces corresponding to the eigenvalues +1 and −1 of $P = T^N$, which
we refer to as the even and odd sectors respectively. We see that \( T^2N = 1 \), and so the possible values of \( k \) are \( k = \pi j/N \) with \( j = 0, 1, 2, \ldots, 2N - 1 \). An eigenstate of \( T \) with eigenvalue \( e^{-ik} \) will be in the even sector if \( e^{-ikN} = 1 \) and in the odd sector if \( e^{-ikN} = -1 \).

Almost every quantity depends on \( N \), the number of sites. We usually suppress this dependence, but in the statement of the following theorem we make it explicit. As we saw in the last section, for even \( N \), the lowest eigenvalues in the even and odd sectors, which we now denote by \( E^N_+ \) and \( E^N_- \), respectively, are slightly different. The expansion of the previous section shows that with our periodic boundary conditions, they are both equal, up to a correction that is exponentially small in \( N \), to \( N \) times a constant \( \epsilon_0 \), the infinite volume ground state energy per site. We define \( E^N_0(k) \) to be \( E^N_+ \) if \( k \) is in the even sector and \( E^N_- \) if \( k \) is in the odd sector. So

\[
E^N_0(k) = \frac{E^N_+ + E^N_-}{2} + \frac{E^N_+ - E^N_-}{2}e^{-ikN}.
\]  

(50)

For odd \( N \) we let \( E^N_1(k) \) denote the lowest eigenvalue in the subspace of generalized momentum \( k \) for the Hamiltonian of this section. The difference \( E^N_1(k) - E^N_0(k) \) with \( N \) even is equal to \( \epsilon_0 \) plus the energy of an interface with momentum \( k \). Our goal is to study this quantity in the infinite \( N \) limit. If there is a localized interface, then this difference would be independent of \( k \), as explained in the Introduction.

The quantities \( E^N_1(k) \) and \( E^N_0(k) \) are only defined for a finite set of values of \( k \), and the two functions are defined on different sets of values. To make sense of this difference, we extend the definitions of these two functions to all \( k \). The Fourier coefficients \( e^N_{0,s} \) are defined by

\[
E^N_0(k) = \sum_{s=-N+1}^{N} e^N_{0,s} e^{iks}.
\]  

(51)

The RHS of this equation is defined for all \( k \), so we can take it to be the definition of the LHS for all \( k \). We extend the definition of \( E^N_1(k) \) to all \( k \) in the same way. It is useful to define \( e^N_{0,s} \) and \( e^{N+1}_{1,s} \) for all \( s \) by making them periodic function of \( s \) with periods \( 2N \) and \( 2(N + 1) \). Then we can rewrite our Fourier series so that they are centered around \( s = 0 \), e.g.,

\[
E^N_0(k) = \sum_{s=-N+1}^{N} e^N_{0,s} e^{iks}.
\]  

(52)

This form is better suited for taking the \( N \to \infty \) limit.

**Theorem 2** There exists an \( \epsilon_0 > 0 \) such that for all \( |\epsilon| < \epsilon_0 \) the following is true: For \( s \in \mathbb{Z} \) there are coefficients \( \epsilon_s \) such that for all \( k \)

\[
\lim_{N \to \infty} \left( E^N_1(k) - E^N_0(k) \right) = \sum_s \epsilon_s e^{iks}.
\]  

(53)

Moreover, there is a constant \( c \) such that

\[
|\epsilon_s| \leq (c|\epsilon|)^{|s|/2}.
\]  

(54)
where the notation $\lceil l \rceil$ denotes the smallest integer which is not smaller than $l$. We have

$$\varepsilon_2 = \varepsilon_{-2} = \epsilon + O(\epsilon^2).$$

(55)

So the dispersion relation (53) is not a constant function of $k$.

The remainder of this section is devoted to the proof of this theorem. In the last section we assumed that $N$ was even. It is only for even $N$ that the periodic boundary conditions for the original Hamiltonian (2) lead to the Hamiltonian (8), and hence to the Kirkwood-Thomas equation (19). However, eq. (19) is defined for all $N$ and the proof of the existence of a solution works for odd $N$ as well. Hence, to prove the theorem we can consider the difference $\left(E_1^N(k) - E_0^N(k)\right)$ with $N$ odd. Throughout the proof we will work with this quantity and suppress the superscript $N$.

We start by studying what the eigenfunctions of $T$ look like. For $k = \pi j/N$ with $j = 0, 1, 2, \ldots, 2N - 1$ we define

$$\phi_{X,k}(\sigma) = \sum_{l=1}^{2N} e^{ikl} \sigma_1\sigma_2\cdots\sigma_l \sigma(X + l).$$

(56)

Indices should be taken to be periodic, i.e., $\sigma_{N+i} = \sigma_i$ for $i = 1, 2, \ldots, N$. However, for $l > N$ one should not interpret $\sigma_1\sigma_2\cdots\sigma_l$ as $\sigma_1\sigma_2\cdots\sigma_{l-N}$. Since $\sigma_i^2 = 1$, it is $\sigma_{l-N+1}\cdots\sigma_N$. Note that $\sigma_1\sigma_2\cdots\sigma_l \sigma(X + l) = T^l \sigma(X)$, so we can write the above as

$$\phi_{X,k}(\sigma) = \sum_{l=1}^{2N} e^{ikl} T^l \sigma(X),$$

(57)

from which it is clear that $\phi_{X,k}(\sigma)$ is an eigenfunction of $T$ with eigenvalue $e^{-ik}$.

These functions span the subspace of generalized momentum $k$, but they are not linearly independent. For some choices of $X$ and $k$, $\phi_{X,k}(\sigma)$ will be zero. We define the action of $T$ on a set of sites by $\sigma(T^l X) = T^l \sigma(X)$. More explicitly, we have $TX = \{1\} \Delta(X + 1)$. Then

$$\phi_{T^l X,k}(\sigma) = \sum_{l=1}^{2N} e^{ikl} T^l \sigma(T^l X) = \sum_{l=1}^{2N} e^{ikl} T^{l-t} \sigma(X) = \sum_{l=1}^{2N} e^{ik(l-t)} T^l \sigma(X) = e^{-ikt} \phi_{X,k}(\sigma).$$

(58)

Hence, if two subsets of the lattice are related by a generalized translation then the corresponding functions are the same up to a multiplicative constant. If we define two sets $X$ and $Y$ to be equivalent if $X = T^n Y$ for some $n$, then we can partition the subsets of $\Lambda$ into equivalence classes. Pick one set from each equivalence class and let $\mathcal{X}$ be the resulting collection of subsets.
of Λ. The φ_X,k will still span the subspace of generalized momentum k if we only consider X ∈ X.

As we remarked before, the proof of the previous section that the Kirkwood–Thomas eq. (19) has a solution works for odd N just as for even N. We let Ω(σ) be the solution,

\[ \Omega(σ) = \exp \left[ -\frac{1}{2} \sum_Y g(Y)σ(Y) \right]. \]  

(59)

This is the ground state of the Hamiltonian in (42) for odd N, or equivalently of the Hamiltonian in (43) with all the J_j = +1. Ω(σ) is translationally invariant, so if ψ_k(σ) has generalized momentum k, then ψ_k(σ)/Ω(σ) does too. Now suppose that for each k we have an eigenstate ψ_k(σ) with momentum k. Then ψ_k(σ) can be written in the form

\[ ψ_k(σ) = \Omega(σ) \sum_{X \in X} c(X, k)φ_X,k(σ) = Ω(σ) \sum_{l=1}^{2N} e^{ikl}σ_1σ_2⋯σ_l \sum_{X \in X} c(X, k)σ(X + l) \]  

(60)

for some coefficients c(X, k), which depend on k. Let us rewrite the expression for ψ_k(σ) in a manner that makes the k–dependence more explicit: For each X we can write c(X, k) as a Fourier series

\[ c(X, k) = \sum_{n=1}^{2N} e^{-ikn}e(X, n). \]  

(61)

The coefficients c(X, k) are functions of k = πj/N with j = 0, 1, 2, ⋅⋅⋅, 2N − 1, and hence the sum on the RHS of (61) is over 2N values (rather than just N). Using (58) we have

\[ ψ_k(σ) = \Omega(σ) \sum_{X \in X} \sum_{n} e(X, n) φ_{T^nX,k} = Ω(σ) \sum_{X} e(X) φ_{X,k}, \]  

(62)

where the coefficients e(X) are defined by the equations

\[ e(X) = \sum_{Y \in X, n:T^nY=X} e(Y, n). \]  

(63)

The wavefunction ψ_k(σ) can now be written in the form

\[ ψ_k(σ) = \Omega(σ) \sum_{l=1}^{2N} e^{ikl}σ_1σ_2⋯σ_l \sum_{X} e(X)σ(X + l). \]  

(64)

Note that the k–dependence is now entirely contained in the factor e^{ikl}.

We will abbreviate ⟨j, j + 1⟩ ∈ ∂X by j : X or X : j. Recall that σ_j^xσ_{j+1}^xσ(X) = −σ(X) if j : X and it equals σ(X) otherwise. It easily follows that

\[ J_jσ_j^xσ_{j+1}^xσ_1σ_2⋯σ_l = s(j, l)σ_1σ_2⋯σ_l, \]  

(65)
where

\[ s(j, l) = \begin{cases} +1 & \text{if } j \neq l \mod N \\ -1 & \text{if } j = l \mod N. \end{cases} \] (66)

Thus

\[
(H \psi_k)(\sigma) = \Omega(\sigma) \sum_{l=1}^{2N} e^{ikl} \sigma_1 \sigma_2 \cdots \sigma_l \\
\left[ -\sum_{j=1}^N \exp\left[\sum_{Y:j} g(Y)\sigma(Y)\right] s(j, l) \left( \sum_X e(X)\sigma(X + l) - 2 \sum_{X:j-l} e(X)\sigma(X + l) \right) \\
+ \epsilon \sum_{j=1}^N \sigma_j \sigma_{j+1} \sum_X e(X)\sigma(X + l) \right].
\] (67)

The above must equal \(E_1(k)\psi_k(\sigma)\). Canceling the common factor of \(\Omega(\sigma)\), the Schrödinger equation for the Hamiltonian \(H\) [[43]] becomes

\[
\sum_{l=1}^{2N} e^{ikl} \sigma_1 \sigma_2 \cdots \sigma_l \left[ -\sum_{j=1}^N \exp\left[\sum_{Y:j} g(Y)\sigma(Y)\right] s(j, l) \left( \sum_X e(X)\sigma(X + l) - 2 \sum_{X:j-l} e(X)\sigma(X + l) \right) \\
+ \epsilon \sum_{j=1}^N \sigma_j \sigma_{j+1} \sum_X e(X)\sigma(X + l) \right] = 0
\] (68)

If eq.(68) was of the form

\[
\sum_{l=1}^{2N} e^{ikl} f(l, \sigma) = 0
\] (69)

then we would have been able to conclude that \(f(l, \sigma) = 0\) for all \(l\). However, even though eq.(68) resembles (69), the two equations are not quite identical in form. This is because \(E_1(k)\) depends on \(k\). To cast (68) in the form (69), we write \(E_1(k)\) as a Fourier series in \(k\). When \(\epsilon = 0\), \(E_1(k) - E_0(k) = 2\). So we write it as

\[
E_1(k) = E_0(k) + 2 + \sum_{s=1}^{2N} e_s e^{-iks}
\] (70)

Using the definition of \(E_0(k)\) [eq. (50)],

\[
\sum_{l=1}^{2N} e^{ikl} \sigma_1 \cdots \sigma_l E_1(k) \sum_X e(X)\sigma(X + l) = \left( 2 + \frac{E_+ + E_-}{2} \right) \sum_{l=1}^{2N} e^{ikl} \sigma_1 \cdots \sigma_l \sum_X e(X)\sigma(X + l) \\
+ \sum_{l,s=1}^{2N} e^{ikl} \sigma_1 \cdots \sigma_{l+s} e_s \sum_X e(X)\sigma(X + s + l) \\
+ \frac{E_+ - E_-}{2} \sum_{l=1}^{2N} e^{ikl} \sigma_1 \cdots \sigma_l \sum_X e(X)\sigma(X + l)
\] (71)
where we have made a change of variables \( l \to l + s \). In the expression \( \sigma_1 \cdots \sigma_{l+s} \) the index \( l + s \) can be as large as \( 4N \). For \( i = 1, 2, \ldots, N \), we interpret \( \sigma_{i+N}, \sigma_{i+2N} \) and \( \sigma_{i+3N} \) to all be \( \sigma_i \). By making a change of variables \( l \to l + N \), and using \( \sigma_{i+1} \cdots \sigma_{i+N} = \sigma(\Delta) \) and \( \sigma(X + l + N) = \sigma(X + l) \), we rewrite the last term on the RHS of (71) as follows:

\[
\frac{E_+ - E_-}{2} e^{-ikN} \sum_{l=1}^{2N} e^{ikl} \sigma_1 \cdots \sigma_l \sum_X e(X) \sigma(X + l) = \frac{E_+ - E_-}{2} \sigma(\Lambda) \sum_{l=1}^{2N} e^{ikl} \sigma_1 \cdots \sigma_l \sum_X e(X) \sigma(X + l).
\]

(72)

If we use (71) in (68) the resulting equation is of the form (69). Hence, after canceling a common factor of \( \sigma_1 \sigma_2 \cdots \sigma_l \), we conclude that

\[
- \sum_{j=1}^{N} \exp [\sum_{Y:j} g(Y) \sigma(Y)] s(j, l) \left( \sum_X e(X) \sigma(X + l) - 2 \sum_{X:j-l} e(X) \sigma(X + l) \right)
+ e^{N} \sum_{j=1}^{N} \sigma_j \sigma_{j+1} \sum_X e(X) \sigma(X + l) - (2 + \frac{E_+ + E_-}{2}) \sum_X e(X) \sigma(X + l)
- \sum_{s=1}^{2N} \sigma_{l+1} \cdots \sigma_{l+s} e_s \sum_X e(X) \sigma(X + s + l) - \frac{E_+ - E_-}{2} \sigma(\Lambda) \sum_X e(X) \sigma(X + l) = 0.
\]

(73)

Recall that the coefficients \( g(Y) \) satisfy eq.(26):

\[
- \sum_{j=1}^{N} \exp(\sum_{Y:j} g(Y) \sigma(Y)) + \epsilon \sum_{j=1}^{N} \sigma_j \sigma_{j+1} = \frac{E_+ + E_-}{2} + \frac{E_+ - E_-}{2} \sigma(\Lambda).
\]

(74)

Multiplying this equation by \( \sum_X e(X) \sigma(X + l) \) and subtracting the result from (73)

\[
\sum_{j=1}^{N} \exp(\sum_{Y:j} g(Y) \sigma(Y)) \left[ (1 - s(j, l)) \sum_X e(X) \sigma(X + l) + 2s(j, l) \sum_{X:j-l} e(X) \sigma(X + l) \right]
- 2 \sum_X e(X) \sigma(X + l) - \sum_{s=1}^{2N} \sigma_{l+1} \cdots \sigma_{l+s} e_s \sum_X e(X) \sigma(X + s + l) = 0.
\]

(75)

Defining \( h(Y) \) by

\[
\exp(\sum_{Y:N} g(Y) \sigma(Y)) = 1 + \sum_Y h(Y) \sigma(Y)
\]

(76)

we have

\[
h(Y) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{Y_1, \ldots, Y_n: \Delta = Y} g(Y_1) \cdots g(Y_n).
\]

(77)
Using the translation invariance of the $g(Y)$
\[
\exp(\sum_{Y:j} g(Y)\sigma(Y)) = \exp(\sum_{Y:N} g(Y + j)\sigma(Y + j)) = \exp(\sum_{Y:j} g(Y)\sigma(Y + j)) = 1 + \sum_{Y} h(Y)\sigma(Y + j). \tag{78}
\]
Inserting (78) in (75) we have
\[
\sum_{j=1}^{N} \left[ (1 - s(j, l)) \sum_{X} e(X)\sigma(X + l) + 2s(j, l) \sum_{X;j-l} e(X)\sigma(X + l) \right] \\
+ \sum_{j=1}^{N} h(Y)\sigma(Y + j) \left[ (1 - s(j, N)) \sum_{X} e(X)\sigma(X + l) + 2s(j, N) \sum_{X;j-l} e(X)\sigma(X + l) \right] \\
- 2 \sum_{X} e(X)\sigma(X + l) - \sum_{s=1}^{2N} \sigma_{l+1} \cdots \sigma_{l+s} e_s \sum_{X} e(X)\sigma(X + s + l) = 0. \tag{79}
\]
Eq. (79) must hold for all $l$ and $\sigma$. The equations for different values of $l$ are in fact identical. To see this we make a change of variables $j \to j + l$ in the sums over $j$. Note that $s(j + l, l) = s(j, N)$. The resulting equation must hold for all configurations $\sigma$. Hence, we can also replace $\sigma$ by the configuration obtained by translating $\sigma$ by $l$ sites so that $\sigma(X + l)$ becomes $\sigma(X)$. The result of these two changes of variables is that, for each value of $l$, eq. (79) reduces to the following equation, which is the $l = N$ case of eq. (79):
\[
\sum_{j=1}^{N} \left[ (1 - s(j, N)) \sum_{X} e(X)\sigma(X) + 2s(j, N) \sum_{X;j} e(X)\sigma(X) \right] \\
+ \sum_{j=1}^{N} h(Y)\sigma(Y) \left[ (1 - s(j, N)) \sum_{X} e(X)\sigma(X) + 2s(j, N) \sum_{X;j} e(X)\sigma(X) \right] \\
- 2 \sum_{X} e(X)\sigma(X) - \sum_{s=1}^{2N} \sigma_{1+1} \cdots \sigma_{l+s} e_s \sum_{X} e(X)\sigma(X + s) = 0. \tag{80}
\]
Note that
\[
\sum_{j} s(j, N) \sum_{X;j} e(X)\sigma(X) = \sum_{X} e(X)\sigma(X) \sum_{j:j} s(j, N) \\
= \sum_{X} n(X) e(X)\sigma(X), \tag{81}
\]
where we have defined
\[
n(X) := \sum_{j:X} s(j, N), \tag{82}
\]
and the sum is over \( j \) such that \( \langle j, j+1 \rangle \in \partial X \). Note that \( n(X) \) is either zero or an even integer. Moreover,

\[
1 - s(j, N) = \begin{cases} 
2 & \text{if } j = N \\
0 & \text{if } j \neq N.
\end{cases}
\]

(83)

Hence, eq. (80) can be written as

\[
2 \sum_{X} n(X) e(X)\sigma(X) + 2 \sum_{Y} h(Y)\sigma(Y) \sum_{X} e(X)\sigma(X) + 2 \sum_{Y} \sum_{X} \sum_{j: X} h(Y)\sigma(Y + j) s(j, N) e(X)\sigma(X) - \sum_{s=1}^{2N} \sigma_1 \cdots \sigma_s e_s \sum_{X} e(X)\sigma(X + s) = 0.
\]

(84)

Recall that \( j : X \) means that exactly one of the sites \( j \) and \( j+1 \) is in \( X \). Define \( j :: X \) as follows: If \( j \neq N \), \( j :: X \) means the same as \( j : X \). However, \( N :: X \) means either both of the sites \( N \) and 1 are in \( X \) or both are not. This is a natural definition since the sites \( j \) for which \( j :: X \) are precisely the sites for which there is an interface between the sites \( j \) and \( j+1 \). With this definition,

\[
2 \sum_{Y} h(Y)\sigma(Y) \sum_{X} e(X)\sigma(X) + 2 \sum_{Y} \sum_{X} \sum_{j: X} h(Y)\sigma(Y + j) s(j, N) e(X)\sigma(X) = 2 \sum_{Y} \sum_{X} \sum_{j: X} h(Y)\sigma(Y + j) e(X)\sigma(X).
\]

(85)

Since

\[
\sigma_1 \cdots \sigma_s \sigma(X + s) = \sigma(T^s X),
\]

the last term in (84) can be written as

\[
\sum_{s=1}^{2N} e_s \sum_{X} e(X)\sigma(T^s X) = \sum_{s=1}^{2N} e_s \sum_{X} e(T^{-s} X)\sigma(X),
\]

(87)

where the equality follows by a change of variables in the sum. (Since \( T^{2N} = 1 \), \( T^{-s} = T^{2N-s} \).) Thus (84) holds for all configurations \( \sigma \) if and only if for all \( X \),

\[
2n(X)e(X) + 2 \sum_{Y,Z,j::Z} h(Y)e(Z)1((Y + j)\Delta Z = X) = \sum_{s=1}^{2N} e_s e(T^{-s} X).
\]

(88)

The integer \( n(X) \) is zero for sets of the form \( X = \{1, 2, \ldots, s\} \) and \( X = \{s+1, s+2, \ldots, N\} \). These are the sets \( T^m \emptyset \) where \( m = 0, 1, \ldots, 2N-1 \). Let us assume that \( e(X) = 0 \) for all \( X \)
for which \( n(X) = 0 \), except for \( X = \emptyset \) for which it is equal to unity. This is essentially a normalization condition. (A priori there is no reason that a solution with these properties must exist, but we will show that it does.) With this assumption, if \( X = T^m\emptyset \) then

\[
\sum_{s=1}^{2N} e_s e(T^{-s}X) = e_m. \tag{89}
\]

Thus eq. (88) gives

\[
e_m = 2 \sum_{Y,Z,j : Z} h(Y)e(Z)1((Y + j)\Delta Z = T^m\emptyset). \tag{90}
\]

For \( X \) for which \( n(X) \neq 0 \), we obtain the relation

\[
e(X) = \frac{1}{2n(X)} \left[ -2 \sum_{Y,Z,j : Z} h(Y)e(Z)1((Y + j)\Delta Z = X) + \sum_{s=1}^{2N} e_s e(T^{-s}X) \right]. \tag{91}
\]

We will show that these equations [(90) and (91)] have a solution by writing them as a fixed point equation. Consider the set of variables

\[
e := \{ e(X) : n(X) \neq 0 \} \cup \{ e_s : s = 1, 2, \ldots, 2N \}. \tag{92}
\]

Equations (90) and (91) form a fixed point equation for \( e \)

\[
F(e) = e. \tag{93}
\]

Let us introduce the norm

\[
||e|| := \sum_{l=1}^{2N} |e_l|(e|M)^{-w_N(T^l\emptyset)} + 2 \sum_{n(X) \neq 0} |e(X)|n(X)(e|M)^{-w_N(X)}, \tag{94}
\]

where \( w_N(X) \) is the number of bonds in the smallest set of bonds which contains \( X \) and intersects the bond \( \langle N, 1 \rangle \) and which is connected in the sense used in the previous section to define \( w(X) \) [see the discussion after (27)]. The factor of 2 in the norm is included merely for later convenience.

We prove that the fixed point equation for \( e \) has a solution by using the contraction mapping theorem as we did in the previous section. We must show that there is a \( \delta' > 0 \) such that

\[
||F(e) - F(\tilde{e})|| \leq \frac{1}{2}||e - \tilde{e}|| \quad \text{for} \quad ||e||, ||\tilde{e}|| \leq \delta'; \tag{95}
\]

\[
||F(e)|| \leq \delta' \quad \text{for} \quad ||e|| \leq \delta'. \tag{96}
\]
To verify (95), we use (90) and (91) to see that

\[ ||F(e) - F(\tilde{e})|| \leq 2 \sum_{Y} \sum_{Z : j \in Z} |h(Y)| |e(Z) - \tilde{e}(Z)||(|e|M)^{-w_N(Y + j)\Delta Z} \]

+ \sum_{s} \sum_{X : n(X) \neq 0} |e_s e(T^{-s}X) - \tilde{e}_s \tilde{e}(T^{-s}X)||(|e|M)^{-w_N(X)}. \tag{97} \]

To continue we need the following two inequalities.

\[ w_N((Y + j)\Delta Z) \leq w_N(Y) + w_N(Z), \quad \text{for } j :: Z \tag{98} \]

\[ w_N(X) \leq w_N(T^{-s}X) + w_N(T^s\emptyset) \tag{99} \]

The inequality (99) can equivalently be written as

\[ w_N(T^sX) \leq w_N(X) + w_N(T^s\emptyset). \tag{100} \]

In the following proofs of these inequalities, “a connected set of bonds” will always mean connected in the sense used to define \( w_N(X) \). To prove (98), let \( A \) and \( B \) be connected sets of bonds which contain \( Y \) and \( Z \) respectively, both of which intersect the bond \( \langle N, 1 \rangle \), and such that \( w_N(Y) = |A| \), \( w_N(Z) = |B| \). We consider the cases of \( j = N \) and \( j \neq N \) separately. First let \( j = N \). Then \( A \cup B \) is a connected set of bonds which contains \( (Y + j)\Delta Z = Y\Delta Z \) and intersects the bond \( \langle N, 1 \rangle \). So

\[ w_N((Y + j)\Delta Z) \leq |A \cup B| \leq |A| + |B| = w_N(Y) + w_N(Z). \tag{101} \]

Now suppose \( j \neq N \). Then \( j :: Z \) means that either \( j \) or \( j + 1 \) is in \( Z \) and so is in \( B \). Since \( \langle N, 1 \rangle \) intersects \( A \), the set \( A + j \) contains at least one of the sites \( j \) and \( j + 1 \). Thus \( (A + j) \cup B \) is a connected set of bonds. It contains \( (Y + j)\Delta Z \) and intersects the bond \( \langle N, 1 \rangle \). So

\[ w_N((Y + j)\Delta Z) \leq |(A + j) \cup B| \leq |A + j| + |B| = w_N(Y) + w_N(Z). \tag{102} \]

This proves (98). The inequality (100) is a special case of (98). To see this, note that

\[ T^sX = (X + s) \Delta T^s\emptyset, \tag{103} \]

so if we take \( Y = X \), \( Z = T^s\emptyset \) and \( j = s \), then (98) becomes (100). (It is easy to check that \( s :: T^s\emptyset \) for all \( s \).)

We will also need the relation,

\[ |\{j : j :: Z\}| = n(Z) + 1. \tag{104} \]

Recalling the definition of \( n(Z) \) [(82)], and of \( s(j, N) \) [(66)],

\[ 1 + n(Z) = 1 + \sum_{j : Z} s(j, N) = 1 - 1(N : Z) + \sum_{j : Z, j \neq N} 1, \tag{105} \]
where $1(\cdot)$ denotes an indicator function. Now $j : Z$ and $j : Z$ are equivalent if $j \neq N$. Moreover, $N : Z$ holds if and only if $N : Z$ does not hold. So $(1 - 1(N : Z)) = 1(N : Z)$. This proves (104).

Using (98) and (104), the first term in (97) is

\[
\leq 2 \sum_{Y} \sum_{Z : j \neq Z} |h(Y)|(|\epsilon| M)^{-w_{N}(Y)} |e(Z) - \bar{e}(Z)||(|\epsilon| M)^{-w_{N}(Z)}
\]

\[
= 2 \sum_{Y} \sum_{Z} [n(Z) + 1]|h(Y)||(|\epsilon| M)^{-w_{N}(Y)} |e(Z) - \bar{e}(Z)||(|\epsilon| M)^{-w_{N}(Z)}
\]

(106)

If $n(Z) = 0$ then either both of $e(Z)$ and $\bar{e}(Z)$ are 0, or both are 1. So $|e(Z) - \bar{e}(Z)| = 0$ when $n(Z) = 0$. Thus we can bound $(n(Z) + 1)$ by $2n(Z)$ on the RHS of (106). Hence,

\[
\text{RHS of (106)} \leq 2 \sum_{Y} |h(Y)||(|\epsilon| M)^{-w_{N}(Y)}||e - \bar{e}|.
\]

(107)

Using (99) and the triangle inequality in the form

\[
|e_s e(T^{-s} X) - \bar{e}_s \bar{e}(T^{-s} X)| \leq |e_s| |e(T^{-s} X) - \bar{e}(T^{-s} X)| + |e_s - \bar{e}_s| |\bar{e}(T^{-s} X)|,
\]

(108)

the second term in (97) is bounded by

\[
\sum_{s} \sum_{X : n(X) \neq 0} |e_s||(|\epsilon| M)^{-w_{N}(T^{s}\emptyset)} |e(T^{-s} X) - \bar{e}(T^{-s} X)||(|\epsilon| M)^{-w_{N}(T^{-s} X)}
\]

\[
+ \sum_{s} \sum_{X : n(X) \neq 0} |e_s - \bar{e}_s||(|\epsilon| M)^{-w_{N}(T^{s}\emptyset)} |\bar{e}(T^{-s} X)||(|\epsilon| M)^{-w_{N}(T^{-s} X)}.
\]

(109)

since $||e||$ and $||\bar{e}||$ are no greater than $\delta'$.

Using the above inequalities (107) and (109), we have

\[
||F(e) - F(\bar{e})|| \leq [2 \sum_{Y} |h(Y)||(|\epsilon| M)^{-w_{N}(Y)} + \delta'] ||e - \bar{e}|.
\]

(110)

It is easily shown that

\[
w_{N}(X_1 \triangle \cdots \triangle X_n) \leq \sum_{k=1}^{n} w_{N}(X_k).
\]

(111)

So using (77)

\[
\sum_{Y} |h(Y)||(|\epsilon| M)^{-w_{N}(Y)} \leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{Y_1:N,\ldots,Y_n:N} \prod_{k=1}^{n} (||\epsilon| M)^{-w_{N}(Y_k)} |g(Y_k)|.
\]

(112)

The constraint $Y_k : N$ implies that $Y_k$ intersects the bond $<N,1>$, and so $w_{N}(Y_k) = w(Y_k)$. Hence,

\[
\text{RHS of (112)} = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{Y_1:N,\ldots,Y_n:N} \prod_{k=1}^{n} (||\epsilon| M)^{-w(Y_k)} |g(Y_k)| = e^{|g|} - 1 \leq e^{\delta} - 1.
\]

(113)

22
The last inequality follows from Theorem 1. So

\[ ||F(e) - F(\tilde{e})|| \leq K ||e - \tilde{e}||. \]  

(114)

where

\[ K = 2(e^\delta - 1) + \delta'. \]  

(115)

If \( \delta \) and \( \delta' \) are small enough, then \( K \leq 1/2 \).

To prove (96), we use (90) and (91) to compute \( F(0) \). Note that \( e = 0 \) means that \( e_s = 0 \) for all \( 1 \leq s \leq 2N \), and \( e(X) = 0 \) for all \( X \) except \( X = \emptyset \). We always have \( e(\emptyset) = 1 \). Letting \( \tilde{e} \) denote \( F(0) \), we have

\[ \tilde{e}_m = 2h(T^m\emptyset), \]  

(116)

and for \( X \) with \( n(X) \neq 0 \)

\[ \tilde{e}(X) = \frac{1}{2n(X)}[-2h(X)]. \]  

(117)

Thus

\[ ||F(0)|| \leq 2 \sum_Y |h(Y)|(|\epsilon| M)^{-w_N(Y)} \leq 2(e^\delta - 1). \]  

(118)

If we decrease \( \delta \), then \( K \) decreases. Hence, we can assume \( \delta \) to be small enough so that \( 2(e^\delta - 1) < \delta'/2 \). So

\[ ||F(e)|| \leq ||F(e) - F(0)|| + ||F(0)|| \leq \frac{1}{2} ||e|| + 2(e^\delta - 1) \leq \delta' \]  

(119)

since \( ||e|| \leq \delta' \).

This finishes the proof that the fixed point equation has a solution and thus completes the construction of eigenstates of \( H \) with generalized momentum \( k \). When \( \epsilon = 0 \) these states are the lowest eigenstates in the subspaces of generalized momentum \( k \) for \( k \neq 0 \), and the next to lowest for \( k = 0 \). The same sort of argument that was used in Section 2 proves that this is true for all \( \epsilon \) such that \( |\epsilon| < \epsilon_0 \), for some \( \epsilon_0 > 0 \). We refer the reader to section 3 of [10] for a completely analogous argument.

We now consider the convergence of the \( N \to \infty \) limit. We start by asking how the volume \( \Lambda \) enters the ground state fixed point equation (22). The sets \( X_i \) in this equation are subsets of \( \Lambda \) and the definition of nearest neighbor for the term \( 1_{nn}(X) \) depends on \( \Lambda \). The solution \( g \) of eq. (22) will depend on \( \Lambda \), and so we denote it by \( g_{\Lambda} \). However, we can consider this equation for the infinite lattice \( \mathbb{Z}^d \). This means that the sets can be any finite subset of \( \mathbb{Z}^d \), and nearest neighbor is defined in the usual way for \( \mathbb{Z}^d \). The proof of the ground state section shows that this infinite volume fixed point equation has a solution, which we denote by \( g_{\infty} \). One can prove that \( g_{\Lambda} \) converges to \( g_{\infty} \) in an appropriate sense by showing \( g_{\Lambda} \) is an approximate solution of the fixed point equation that defines \( g_{\infty} \). We refer the reader to [10] for details.

The fixed point equations, (90) and (91), of this section can also be defined for the infinite lattice \( \mathbb{Z}^d \), and the fixed point argument of this section proves it has a solution. This solution
includes the Fourier coefficients \( \varepsilon_s \), so in this way the coefficients \( \varepsilon_s \) of Theorem 2 are defined. The convergence of \( E_N^{N+1}(k) - E_N(k) \) to \( \sum_s \varepsilon_s e^{ik_s} \) can be proved by the methods of [10] as well.

The last step in the proof is to show that \( e_2 \) and \( e_{-2} = e_{2N-2} \) are not zero in the infinite length limit. We start with (22) to compute \( g \) to first order in \( \varepsilon \). At first order in \( \varepsilon \) the only nonzero coefficients \( g(X) \) are for sets \( X \) which consist of a pair of adjacent sites. In this case \( g(X) = \varepsilon/2 + O(\varepsilon^2) \). By (77), the only \( Y \) for which \( h(Y) \) is nonzero at first order in \( \varepsilon \) is a set of nearest neighbor sites satisfying \( N : Y \). There are two such sets, \( \{1, 2\} \) and \( \{N - 1, N\} \). They have \( h(Y) = \varepsilon/2 + O(\varepsilon^2) \). Now consider eq.(90). \( h(Y) \) is always at least first order in \( \varepsilon \), but there is one \( Z \) for which \( e(Z) \) is zeroth order in \( \varepsilon \), namely, \( e(\emptyset) = 1 \). For this \( Z \) the only \( j \) satisfying \( j :: Z \) is \( j = N \). Thus the first order contribution to \( e_m \) is of the form

\[
2 \sum_Y h(Y) 1(Y = T^m \emptyset). \tag{120}
\]

The sets \( \{1, 2\} \) and \( \{N - 1, N\} \) are of the form \( T^m \emptyset \) for \( m = 2 \) and \( m = 2N - 2 \), respectively. Thus

\[
e_2 = e_{2N-2} = \varepsilon + O(\varepsilon^2). \tag{121}
\]

This proves (55) of Theorem 2.

4 Antiferromagnetic XXZ Chain

In this section we study the antiferromagnetic XXZ model whose Hamiltonian on the 1-dimensional lattice \( \Lambda = \{1, 2, \ldots, N\} \) is

\[
\tilde{H} = \sum_{j=1}^{N} \sigma_j^z \sigma_{j+1}^z + \varepsilon \sum_{j=1}^{N} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) \tag{122}
\]

Using \( \sigma^y = i\sigma^x \sigma^z \) we have

\[
\tilde{H} = \sum_{j=1}^{N} \sigma_j^z \sigma_{j+1}^z + \varepsilon \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x (1 - \sigma_j^x \sigma_{j+1}^x) \tag{123}
\]

As before consider a unitary operator that causes a rotation about the \( Y \)-axis in spin space:

\[
R := \exp \left( \frac{i\pi}{4} \sum_{j \in \Lambda} \sigma_j^y \right)
\]

so that

\[
R\tilde{H}R^{-1} = \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x + \varepsilon \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x (1 - \sigma_j^x \sigma_{j+1}^x) \tag{124}
\]
For the antiferromagnet we proceed as in the previous section and use the unitary transformation $U$ [eq. (7)]:

$$H = UR\tilde{H}R^{-1}U^{-1} = -\sum_{j=1}^{N} J_{j} \sigma_{j}^{x} \sigma_{j+1}^{x} + \epsilon \sum_{j=1}^{N} \sigma_{j}^{z} \sigma_{j+1}^{z} (1 + J_{j} \sigma_{j}^{x} \sigma_{j+1}^{x})$$ (125)

where $J_{j} = 1$ for $j \neq N$ and $J_{N}$ is 1 when $N$ is even and $-1$ when $N$ is odd. $\tilde{H}$ is translation invariant and commutes with the global spin flip operator $\tilde{P}$ [(3)]. So $H$ commutes with $T$ [(46)], as it did in the previous section.

When $N$ is even (and so $J_{j} = +1 \forall j$), the ground state wave function

$$\Omega(\sigma) = \exp \left[ -\frac{1}{2} \sum_{Y} g(Y) \sigma(Y) \right]$$ (126)

must satisfy

$$-\sum_{j} \exp \left[ \sum_{X:j} g(X) \sigma(X) \right] + \epsilon \sum_{j} \sigma_{j} \sigma_{j+1} \left[ 1 + \exp \left[ \sum_{X:j} g(X) \sigma(X) \right] \right] = \frac{E_{+} + E_{-}}{2} + \frac{E_{+} - E_{-}}{2} \sigma(\Lambda)$$ (127)

where $X : j$ means $(j, j+1) \in X$. Theorem 1 of Section 2 holds for this model. We omit the proof since it is analogous to the proof in Section 2. Since the dimension $d = 1$, we choose $\delta = 4M^{-1}$ as given by (28).

To study interfaces in this model we take $N$ to be odd. So $J_{N} = -1$. We recall that in this case, $T = \sigma_{1}^{z} T$ [(47)]. As before we look for a solution of the form

$$\psi_{k}(\sigma) = \Omega(\sigma) \sum_{l=1}^{2N} e^{ikl} \sigma_{1} \sigma_{2} \cdots \sigma_{l} \sum_{X} e(X) \sigma(X + l).$$ (128)

The Schrödinger equation $(H \psi_{k})(\sigma) = E_{1}(k) \psi_{k}(\sigma)$, becomes (after canceling a common factor of $\Omega(\sigma)$)

$$\sum_{l=1}^{2N} e^{ikl} \sigma_{1} \sigma_{2} \cdots \sigma_{l} \left[ -\sum_{j=1}^{N} \exp \left[ \sum_{Y:j} g(Y) \sigma(Y) \right] s(j, l) \left( \sum_{X} e(X) \sigma(X + l) - 2 \sum_{X:j-l} e(X) \sigma(X + l) \right) \right] + \epsilon \sum_{j=1}^{N} \sigma_{j} \sigma_{j+1} \sum_{X} e(X) \sigma(X + l) + \epsilon \sum_{j=1}^{N} \sigma_{j} \sigma_{j+1} \exp \left[ \sum_{Y:j} g(Y) \sigma(Y) \right] s(j, l) \left( \sum_{X} e(X) \sigma(X + l) - 2 \sum_{X:j-l} e(X) \sigma(X + l) \right) - E_{1}(k) \sum_{X} e(X) \sigma(X + l) = 0.$$ (129)
This is the analog of (68). We now proceed by analogy with the derivation of (90) and (91) from (68). This leads to the equation:

\[
2 \sum_X n(X) e(X) \sigma(X) - 2\epsilon \sigma_N \sigma_1 \sum_X e(X) \sigma(X) \\
-2\epsilon \sum_{j=1}^{N} \sigma_j \sigma_{j+1} s(j, N) \sum_X e(X) \sigma(X) - \sum_{s=1}^{2N} \sigma_1 \cdots \sigma_s \epsilon_s \sum_X e(X) \sigma(X + s) \\
+2 \sum_Y h(Y) \sigma(Y) \sum_X e(X) \sigma(X) + 2 \sum_{j=1}^{N} \sum_Y h(Y) \sigma(Y + j) s(j, N) \sum_X e(X) \sigma(X) \\
+\epsilon \sum_{j=1}^{N} \sigma_j \sigma_{j+1} \sum_Y h(Y) \sigma(Y + j) (s(j, N) - 1) \sum_X e(X) \sigma(X) \\
-2\epsilon \sum_{j=1}^{N} \sigma_j \sigma_{j+1} \sum_Y h(Y) \sigma(Y + j) s(j, N) \sum_X e(X) \sigma(X) \\
= 0
\]

(130)

Eq. (130) yields the following equations which are the analogs of (90) and (91). For \( X \) for which \( n(X) \neq 0 \) we have

\[
e(X) = \frac{1}{2n(X)} \left[ 2\epsilon \sum_{j:Z} \sum_Z e(Z) 1(Z \triangle \{j, j + 1\} = X) \right.
\]

\[
+ \sum_{s=1}^{2N} \epsilon_s e(T^{-s}X) - 2\sum_{j:Z} \sum_Y h(Y) e(Z) 1((Y + j) \triangle Z = X) \\
+2\epsilon \sum_{j:Z} \sum_Y h(Y) e(Z) 1(Z \triangle (Y + j) \triangle \{j, j + 1\} = X) \left. \right]
\]

(131)

Recall that \( j : Z \) means that exactly one of \( j \) and \( j + 1 \) is in \( Z \) if \( j \neq Z \), and \( N : Z \) means that either both of \( N \) and \( 1 \) are in \( Z \) or neither of them is. For \( X = T^m \emptyset \) we have

\[
e_m = 2\epsilon \sum_{j:Z} \sum_Z e(Z) 1(Z \triangle \{j, j + 1\} = T^m \emptyset) \\
+2\epsilon \sum_{j:Z} \sum_Y h(Y) e(Z) 1(Z \triangle (Y + j) \triangle \{j, j + 1\} = T^m \emptyset) \\
-2\sum_{j:Z} \sum_Y h(Y) e(Z) 1((Y + j) \triangle Z = T^m \emptyset).
\]

(132)

Recall that \( n(X) = 0 \) if and only if \( X \) is of the form \( T^m \emptyset \) for some integer \( m \). As in the previous section, we assume \( e(\emptyset) = 1 \) and \( e(T^m(\emptyset)) = 0 \) for \( m \neq 0 \).
We let \( e \) denote the same collection of variables as in the previous section and continue to use the norm \((94)\). Equations \((131)\) and \((132)\) form a fixed point equation which can be written as \( F(e) = e \). (Of course, the function \( F \) is different from the \( F \) of the previous section.) We prove there is a solution to the fixed point equation by proving \((95)\) and \((96)\).

To prove \((95)\) we use \((131)\) and \((132)\) to see that
\[
||F(e) - F(\tilde{e})|| \leq 2|\epsilon| \sum_{Z} \sum_{j \in Z} |e(Z) - \tilde{e}(Z)||(|\epsilon|M)^{-w_N(Z \Delta \{j,j+1\})}
\]
\[+ \sum_{s} \sum_{X,n(X) \neq 0} |e_s e(T^{-s}X) - \tilde{e}_s \tilde{e}(T^{-s}X)||(|\epsilon|M)^{-w_N(X)}
\]
\[+ 2|\epsilon| \sum_{Y} |h(Y)| \sum_{Z} \sum_{j \in Z} |e(Z) - \tilde{e}(Z)||(|\epsilon|M)^{-w_N(Z \Delta (Y+j) \Delta \{j,j+1\})}
\]
\[+ 2 \sum_{Y} |h(Y)| \sum_{Z} \sum_{j \in Z} |e(Z) - \tilde{e}(Z)||(|\epsilon|M)^{-w_N(Z \Delta (Y+j))}
\]
\[=: (a1) + (a2) + (a3) + (a4). \tag{133}
\]
We proved the following inequalities \([98\) and \((99)\)] in the previous section
\[
w_N((Y + j) \Delta Z) \leq w_N(Y) + w_N(Z), \quad \text{for} \quad j \in Z \tag{134}
\]
\[
w_N(X) \leq w_N(T^{-s}X) + w_N(T^{s}\emptyset). \tag{135}
\]
In addition, we need the following two inequalities.
\[
w_N((Y + j) \Delta Z \Delta \{j,j+1\}) \leq w_N(Y) + w_N(Z) + 1, \quad \text{for} \quad j \in Z \tag{136}
\]
\[
w_N(Z \Delta \{j,j+1\}) \leq w_N(Z) + 1. \tag{137}
\]
Inequality \((136)\) can be proved with two applications of \((134)\) as follows.
\[
w_N((Y + j) \Delta Z \Delta \{j,j+1\}) = w_N(\{(Y \Delta \{N,1\}) + j\} \Delta Z)
\]
\[\leq w_N(Y \Delta \{N,1\}) + w_N(Z) \leq w_N(Y) + 1 + w_N(Z) \tag{138}
\]
Similarly, inequality \((137)\) follows from \((134)\).

Using inequality \((137)\) and eq.\((104)\) we obtain
\[
(a1) \leq 2|\epsilon| \sum_{Z} |e(Z) - \tilde{e}(Z)||(|\epsilon|M)^{-w_N(Z)} \sum_{j \in Z} 1,
\]
\[= 2M^{-1} \sum_{n(Z) \neq 0} (n(Z) + 1)|e(Z) - \tilde{e}(Z)||(|\epsilon|M)^{-w_N(Z)} . \tag{139}
\]
We have added the constraint \(n(Z) \neq 0\) on the sum because \(|e(Z) - \tilde{e}(Z)| = 0\) for \(n(Z) = 0\). Hence we can bound \((n(Z) + 1)\) in the above sum by \(2n(Z)\). This yields
\[
(a1) \leq 2M^{-1}|e - \tilde{e}|. \tag{140}
\]
Using the triangle inequality,

\[ |e_s e(T^{-s}X) - \bar{e}_s e(T^{-s}X)| \leq |e_s| |e(T^{-s}X) - \bar{e}(T^{-s}X)| + |e_s - \bar{e}_s| |\bar{e}(T^{-s}X)|, \]

and (135) we get

\[ (a2) \leq ||e|| \frac{1}{2} |e - \bar{e}| + ||e - \bar{e}|| \frac{1}{2} |\bar{e}|. \] (141)

Using (136) and (104) we get

\[ (a3) \leq 2|\epsilon| (|\epsilon|M)^{-1} \sum_Y |h(Y)| (|\epsilon|M)^{-w_N(Y)} \sum_Z (n(Z) + 1)|e(Z) - \bar{e}(Z)|(\epsilon|M)^{-w_N(Z)} \]

\[ \leq 2M^{-1} \left( \sum_Y |h(Y)| (|\epsilon|M)^{-w_N(Y)} \right) ||e - \bar{e}||. \]

\[ \leq 2M^{-1}(\epsilon^\delta - 1) ||e - \bar{e}||, \] (142)

where we have used (112) - (113).

Similarly we get

\[ (a4) \leq 2 \sum_Y |h(Y)| (|\epsilon|M)^{-w_N(Y)} \sum_Z (n(Z) + 1)|e(Z) - \bar{e}(Z)|(\epsilon|M)^{-w_N(Z)} \]

\[ \leq 2 \left( \sum_Y |h(Y)| (|\epsilon|M)^{-w_N(Y)} \right) ||e - \bar{e}||. \]

\[ \leq 2(\epsilon^\delta - 1) ||e - \bar{e}||. \] (143)

From (140), (141), (142) and (143) we obtain

\[ ||F(e) - F(\bar{e})|| \leq K ||e - \bar{e}||, \] (144)

where

\[ K = \delta' + (1 + 2M^{-1})(\epsilon^\delta - 1) + 2M^{-1} \]

\[ = \delta' + (1 + \frac{\delta}{2})(\epsilon^\delta - 1) + \frac{\delta}{2}, \] (145)

since we have chosen \( \delta = 4M^{-1} \). Hence, if \( \delta' \) and \( \delta \) are small enough then \( K \leq 1/2 \). To prove (96) we use (132) and (131) to compute \( F(0) \). Note that \( e = 0 \) means that \( e_s = 0 \) for all \( s \in \Lambda \), and \( e(X) = 0 \) for all \( X \) except \( X = \emptyset \). We always have \( e(\emptyset) = 1 \). Letting \( \bar{e} \) denote \( F(0) \), we have

\[ \bar{e}_m = 2\epsilon \sum_Y h(Y)1(Y \triangle \{N, 1\} = T^m\emptyset) - \sum_Y h(Y)1(Y = T^m\emptyset). \] (146)

For \( X \) with \( n(X) \neq 0 \)

\[ \bar{e}(X) = \frac{1}{2n(X)} \left[ 2\epsilon 1(X = \{N, 1\}) - 2h(X) + 2eh(X \triangle \{N, 1\}) \right]. \] (147)
Thus
\[ ||F(0)|| \leq 2M^{-1} + 2(e^\delta - 1) + 2M^{-1}(e^\delta - 1). \] (148)

If we decrease \( \delta \), then \( K \) decreases. So we can assume that \( \delta \) is small enough that \( 2(e^\delta - 1) + \delta e^\delta/2 < \delta'/2 \). So
\[ ||F(\epsilon)|| \leq ||F(\epsilon) - F(0)|| + ||F(0)|| \leq \frac{1}{2}\delta' + \frac{\delta}{2} + 2(e^\delta - 1) + \frac{\delta}{2}(e^\delta - 1) \leq \delta' \] (149)
since \( ||e|| \leq \delta' \).

This finishes the proof that the fixed point equation (93) has a solution and thus completes the construction of eigenstates of \( H \) with generalized momentum \( k \). When \( \epsilon = 0 \) these states are the lowest eigenstates in the subspaces of generalized momentum \( k \) for \( k \neq 0 \), and the next to lowest for \( k = 0 \). The same argument that we used in Section 2 proves that this is true for small \( \epsilon \). As in Section 3, we can explicitly compute the lowest order term in the dispersion relation for the interface and see that it is not zero. So the dispersion relation depends on \( k \), indicating that the ground state does not correspond to a stable interface.

5 Ferromagnetic XXZ Chain

In this section we will prove that the ground state of the ferromagnetic chain has a stable interface at zero temperature by showing that, for \( s \neq 0 \), the Fourier coefficients \( e_s^N \) for the dispersion relation vanish in the limit \( N \to \infty \). Thus, in the infinite length limit the dispersion relation is flat, i.e., independent of the generalized momentum \( k \). As discussed in the Introduction, the zero–temperature stability of the interface for the ferromagnet has been proven before. The point of this section is to show that this result can also be obtained by our methods. We will construct the wave function for ground states with an interface in them just as we did for the antiferromagnet. However, we will use very different weights in the norm. The weight for the terms \( e_s^N \) will be exponentially large in \( N \) for \( s \neq 0 \). So the existence of a fixed point in this norm will prove that \( e_s^N \) goes to zero exponentially fast as \( N \) goes to infinity. The weights we use for the norm are based on considerations of how many applications of terms in the Hamiltonian it takes to get between various states. So we begin by studying the action of the Hamiltonian.

A ferromagnetic XXZ chain of \( N \) sites is governed by the Hamiltonian
\[ \tilde{H} = - \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x - \epsilon \sum_{j=1}^{N} \sigma_j^z \sigma_{j+1}^z (1 - \sigma_j^z \sigma_{j+1}^z) \] (150)
(which is the ferromagnetic analog of (123)). However, unlike the antiferromagnetic case, we cannot force an interface into such a chain by considering \( N \) to be odd and imposing periodic boundary conditions. So instead, to induce an interface we change the coupling between the
sites $N$ and 1 as follows: We write the Hamiltonian in the form

$$\tilde{H} = -\sum_{j=1}^{N} J_j \sigma_j^z \sigma_{j+1}^z - \epsilon \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x (1 - J_j \sigma_j^z \sigma_{j+1}^z).$$ (151)

If $J_j = 1$ for all $j$ then (151) reduces to (150). Such a Hamiltonian has two translation-invariant ground states – with all spins up and all spins down, respectively. However, the choice $J_N = -1$ and $J_j = 1$ for all $j \neq N$, induces an interface into the chain by causing at least one bond in the chain to be frustrated. Moreover, this particular choice of coupling yields a unitarily equivalent Hamiltonian $H$ [(152) below] which commutes with the generalized translation operator $T$ [(47)]. Hence, it allows us to exploit this symmetry to study the interface states, as in the case of the antiferromagnetic chain.

As before, we take $R$ to be the rotation operator defined by (4). Hence,

$$H := R\tilde{H}R^{-1} = -\sum_{j=1}^{N} J_j \sigma_j^z \sigma_{j+1}^z - \epsilon \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x (1 - J_j \sigma_j^z \sigma_{j+1}^z).$$ (152)

The original Hamiltonian $\tilde{H}$ [(151)] is not translation invariant when $J_N = -1$. Nonetheless, our choice of boundary conditions for the original Hamiltonian is such that the transformed Hamiltonian $H$ [(152)] commutes with $T$, as is easily checked. (Note that for the ferromagnetic chain we do not use the unitary transformation $U$.)

Recall that $\mathcal{H}_\Lambda = (\mathbb{C}^2)^{\otimes |\Lambda|}$ is the Hilbert space of the lattice. In (152) the indices should be taken to be periodic e.g., $\sigma_{N+1}^x$ means $\sigma_1^x$. We can write the Hamiltonian as

$$H = H_0 + H_1,$$ (153)

where

$$H_0 := -\sum_{j=1}^{N} J_j \sigma_j^z \sigma_{j+1}^z,$$ (154)

and

$$H_1 := -\epsilon \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x (1 - J_j \sigma_j^z \sigma_{j+1}^z).$$ (155)

For any $X \subset \Lambda$ let $|X\rangle \in \mathcal{H}_\Lambda$ be given by

$$|X\rangle = \sum_\sigma \sigma(X)|\sigma\rangle,$$ (156)

where $\sigma(X) = \prod_{j \in X} \sigma_j$. Hence,

$$\sigma_i^x |X\rangle = \sum_\sigma \sigma(X)|\sigma^{(i)}\rangle,$$ (157)
where $\sigma^{(i)}$ is the spin configuration $\sigma$ but with $\sigma_i$ replaced by $-\sigma_i$. By making a change of variables in the sum we obtain

$$\sigma_i^\tau |X\rangle = \sum_{\sigma} \sigma^{(i)}(X)|\sigma\rangle,$$

(158)

where

$$\sigma^{(i)}(X) = -\sigma(X) \quad \text{if} \quad i \in X, \quad = \sigma(X) \quad \text{if} \quad i \notin X. \quad (159)$$

Hence,

$$\sigma_i^\tau |X\rangle = - |X\rangle \quad \text{for} \quad i \in X, \quad = |X\rangle \quad \text{for} \quad i \notin X. \quad (160)$$

Note that the states $|X\rangle$ are eigenstates of $H_0 \ [(154)]$.

The ground states of the original Hamiltonian $\tilde{H} \ [(151)]$ for the choice $J_N = -1$ and $\epsilon = 0$ corresponds to a configuration consisting of a string of up–spins ($+$) next to a string of down–spins ($-$). We refer to such ground states of $\tilde{H}$ as its $\epsilon = 0$ interface states. However, the configuration corresponding to the $\epsilon = 0$ interface states of the unitarily equivalent Hamiltonian $H \ [(152)]$ (i.e., ground states of $H_0 \ [(154)]$) cannot be visualized as clearly. The unitary transformation $R$ obscures the picture. So, to describe $|X\rangle$, it is useful to think about $R^{-1}|X\rangle$.

For example the state $R^{-1}|\emptyset\rangle$ corresponds to the configuration

$$++\cdots+,$$

where the labels of the sites increase from 1 to $N$ from left to right. It has a single interface between the nearest neighbor sites $N$ and 1. We say that there is an interface between two nearest neighbor sites $j$ and $j + 1$ if the nearest neighbor bond $\langle j, j+1 \rangle$ is frustrated, i.e., if the spins are antiparallel for $j \neq N$ and parallel for $j = N$. Note that for $X = \Lambda$, $R^{-1}|X\rangle$ is the configuration with all $-$’s, this being another configuration with an interface between the sites $N$ and 1. If $X = \{1, 2, \cdots, j\}$, then $R^{-1}|X\rangle$ looks like

$$++\cdots+--\cdots--,$$

where the last $+$ occurs at the site $j$; for $X = \{j, j+1, \cdots, N\}$, $R^{-1}|X\rangle$ looks like

$$--\cdots+-++\cdots++,$$

where the first $+$ occurs at the site $j$.

Let $\mathcal{I}(X)$ denote the set of sites for which the configuration corresponding to the state $R^{-1}|X\rangle$ has interfaces between each site $i$ in this set and its nearest neighbor $i + 1$. For $j \neq N$,
\[ j \in \mathcal{I}(X) \] if and only if exactly one of \( j \) and \( j + 1 \) is in \( X \), and for \( j = N, j \in \mathcal{I}(X) \) if and only if both \( N \) and \( 1 \) are either in \( X \) or outside it. Then
\[
H_1|X\rangle = -2\epsilon \sum_{j \in \mathcal{I}(X)} |X\triangle\{j, j + 1\}\rangle. \tag{161}
\]

Let \( X, Y \subset \Lambda \) and consider the states \( |X\rangle \) and \( |Y\rangle \). If \( |X\rangle \) and \( |Y\rangle \) are both even (or both odd), then after repeated applications of the Hamiltonian on the state \( |X\rangle \) we can obtain a state which has a nonzero overlap with \( |Y\rangle \). This is, however, not possible if one of \( |X\rangle \) and \( |Y\rangle \) is even and the other is odd. For \( |X\rangle \) and \( |Y\rangle \) both even (or odd) we define \( \alpha(X \rightarrow Y) \) to be the minimum number of applications of the Hamiltonian necessary to get from \( |X\rangle \) to a state which has a nonzero overlap with \( |Y\rangle \). We denote such a transition by the symbol \( X \rightarrow Y \). Hence, \( \alpha(X \rightarrow Y) \) is equal to the smallest integer \( n \) for which
\[
\langle Y|H^n_1|X\rangle \neq 0 \tag{162}
\]

Equivalently, we consider all sequences \( X_0, X_1, X_2, \ldots, X_n \) such that \( X_0 = X, X_n = Y \), and for each \( k \) there is a \( j : X_{k-1} \) so that \( X_k = X_{k-1} \triangle \{j, j + 1\} \). Then \( \alpha(X \rightarrow Y) \) is the smallest \( n \) for all such sequences. In addition, we define \( \alpha(X) := \alpha(X \rightarrow \emptyset) \). It is clear that \( \alpha(X) \) is infinite for \( |X\rangle \) odd. Since \( T^s \) is a unitary operator which commutes with \( H_1 \), we have
\[
\langle T^sY|H^n_1|T^sX\rangle = \langle Y|T^{-s}H^n_1T^s|X\rangle = \langle Y|H^n_1|X\rangle \tag{163}
\]
The above equation implies that
\[
\alpha(T^sX \rightarrow T^sY) = \alpha(X \rightarrow Y). \tag{164}
\]

We start with the analog of eq.(130) for the case of Hamiltonian \( H \) [(152)] (which is unitarily equivalent to the ferromagnetic Hamiltonian \( \widetilde{H} \) (151)). The change of the Hamiltonian \( \widetilde{H} \) from the antiferromagnetic (123) to the ferromagnetic (151) case (and hence the corresponding change of \( H \) from (125) to (152)) changes some of the signs in eq.(130). Moreover, since \( h(X) = 0 \) for the Hamiltonian given by (152), many of the terms in this equation reduce to zero. Taking into account these changes, we obtain the following equation:
\[
2 \sum_X n(X)e(X)\sigma(X) - 2\epsilon\sigma_N\sigma_1 \sum_X e(X)\sigma(X) \\
-2\epsilon \sum_{j=1}^N \sigma_j\sigma_{j+1} s(j, N) \sum_X e(X)\sigma(X) - \sum_{s=1}^{2N} \sigma_1 \cdots \sigma_s e_s \sum_X e(X)\sigma(X + s) = 0. \tag{165}
\]

Using eqs. (66) and (86), and picking out the coefficient of \( \sigma(X) \) we have
\[
2n(X)e(X) - 2\epsilon e(X\triangle\{N, 1\}) + 2\epsilon e(X\triangle\{N, 1\}) \mathbf{1}(N : X) \\
-2\epsilon \sum_{j=1}^{N-1} e(X\triangle\{j, j + 1\}) \mathbf{1}(j : X) - \sum_{s=1}^{2N} e_s e(T^{-s}X) = 0, \tag{166}
\]

32
Since $1 - 1(N : X) = 1(N :: X) \equiv 1(N \in \mathcal{I}(X))$, and for $j \neq N$, $1(j : X) = 1(j \in \mathcal{I}(X))$, the above equation can be written as

$$2n(X) e(X) - 2 \epsilon \sum_{j \in \mathcal{I}(X)} e(X \triangle \{j, j + 1\}) - \sum_{s=1}^{2N} e_s e(T^{-s}X) = 0,$$

(167)

which we can write as

$$2n(X) e(X) - 2 \epsilon \sum_{j=1}^{N} \sum_{Z \in \mathcal{I}(Z) \triangle j} e(Z)1(Z \triangle \{j, j + 1\} = X) - \sum_{s=1}^{2N} e_s e(T^{-s}X) = 0,$$

(168)

since $j \in \mathcal{I}(X)$ implies that $j \in \mathcal{I}(Z)$ for $X = Z \triangle \{j, j + 1\}$.

For $X$ such that $n(X) \neq 0$ we rewrite this as

$$e(X) = \frac{1}{2n(X)} \left[ +2 \epsilon \sum_{j=1}^{N} \sum_{Z \in \mathcal{I}(Z) \triangle j} e(Z)1(Z \triangle \{j, j + 1\} = X) + \sum_{s=1}^{2N} e_s e(T^{-s}X) \right].$$

(169)

Recall that $n(X) = 0$ if and only if $X$ is of the form $T^m(\emptyset)$ for some $m$. We assume that $e(X) = 0$ for all $X \subset \Lambda$ for which $n(X) = 0$, except for $X = \emptyset$ for which we assume that

$$e(\emptyset) = 1.$$

(170)

Hence, for $X = T^m(\emptyset)$, (166) becomes

$$e_m = -2 \epsilon \sum_{j \in \mathcal{I}(X)} e(X \triangle \{j, j + 1\}).$$

(171)

When $X = T^m(\emptyset)$ the set $\mathcal{I}(X)$ contains only one site and we find that

$$e_m = -2 \epsilon e(X_m) \quad \text{for} \quad m \leq N,$$

(172)

where $X_m = \{1, 2, \ldots m\} \triangle \{m, m + 1\}$, and

$$e_{m+N} = -2 \epsilon e(\Lambda \setminus X_m) = -2 \epsilon e(T^N X_m) \quad \text{for} \quad m \leq N.$$

(173)

Note that $n(X_m) \neq 0$.

Consider the set of variables

$$e := \{e(X) : n(X) \neq 0\} \cup \{e_s : s = 1, 2, \ldots, 2N\}$$

(174)

Equations (169),(172) and (173) form a fixed point equation for $e$:

$$F(e) = e$$

(175)
Let us introduce the norm

$$
||e|| := 2^N \sum_{m=1}^{2N} |e_m|(|e|M)^{-\beta_m} + 2 \sum_{n(X) \neq 0} |e(X)|n(X)(|e|M)^{-\alpha(X)},
$$

(176)

where $M$ is a positive constant and $\beta_m = \alpha(T^m \emptyset \to \emptyset)$. Recall that $\alpha(X \to Y)$ is the least number of applications of the Hamiltonian it takes to get from $|X\rangle$ to a state which has a nonzero overlap with $|Y\rangle$. For $m$ odd, repeated applications of the Hamiltonian to $|T^m \emptyset\rangle$ can never produce a state with a nonzero overlap with $|\emptyset\rangle$. So $\beta_m$ is taken to be infinite for odd values of $m$. The factor of 2 in the second term on the RHS of (176) is included merely for convenience.

For $m \leq N$,

$$
\beta_m = \alpha(T^m \emptyset),
$$

$$
\beta_{m+N} = \alpha(T^{m+N} \emptyset) \equiv \alpha(A \setminus T^m \emptyset).
$$

(177)

**Theorem 3** There exists a constant $M > 0$ such that if $||e|M \leq 1$, then the fixed point equation (175) has a solution $e$, and $||e|| \leq c$ for some constant $c$ which depends only on $M$. Furthermore,

$$
\sum_{s=-N+1, s \neq 0}^{N} |e_s| \leq c(||e|M)^{N-1}
$$

(178)

So in the infinite length limit, the dispersion relation for an interface is independent of the generalized momentum $k$.

**Proof:** It is not hard to show that $\beta_2 = \beta_{N-2} = N - 1$, and $\beta_s$ for other nonzero $s$ is even larger. So (178) will follow from the existence of a fixed point in the norm (176). As before we prove the existence of a fixed point by proving

$$
||F(e) - F(\bar{e})|| \leq \frac{1}{2} ||e - \bar{e}|| \quad \text{for} \quad ||e||, ||\bar{e}|| \leq \delta;
$$

(179)

$$
||F(e)|| \leq \delta \quad \text{for} \quad ||e|| \leq \delta.
$$

(180)

with

$$
\delta = \frac{4}{M}
$$

(181)

From (172) and (173) we get (using the definition of $\beta_m$)

$$
\sum_{m=1}^{2N} |e_m - \bar{e}_m|(|e|M)^{-\beta_m} \leq \sum_{m=1}^{N} |e_m - \bar{e}_m|(|e|M)^{-\beta_m} + \sum_{m=1}^{N} |e_{m+N} - \bar{e}_{m+N}|(|e|M)^{-\beta_{m+N}}
$$
This is because, for $m \leq N$,
\[ \beta_m = \alpha(T^m\emptyset) = \alpha(X_m) + 1, \]
and
\[ \beta_{m+N} = \alpha(T^{N+m}\emptyset) = \alpha(\Lambda \setminus X_m) + 1, \]
Hence,
\[ \text{RHS of (182)} \leq 2|\epsilon| (|\epsilon|\mathcal{M})^{-1} \sum_{n(Y) \neq 0} |e(Y) - \tilde{c}(Y)|(|\epsilon|\mathcal{M})^{-Z} \]
\[ \leq M^{-1} ||e - \tilde{c}||. \]  
(183)

Further, from (169) we get
\[ 2 \sum_{n(X) \neq 0} n(X)|e(X) - \tilde{c}(X)|(|\epsilon|\mathcal{M})^{-\alpha(X)} \]
\[ \leq 2|\epsilon| \sum_{n(X) \neq 0} \sum_{j=1}^{N} \sum_{Z \in \mathcal{I}(Z) \cap j} |e(Z) - \tilde{c}(Z)|(|\epsilon|\mathcal{M})^{-\alpha(Z \triangle \{j, j+1\})} 1(Z \triangle \{j, j+1\} = X) \]
\[ + \sum_{n(X) \neq 0} |e_{\ast}e(T^{-s}X) - \tilde{e}_{\ast}\tilde{c}(T^{-s}X)|(|\epsilon|\mathcal{M})^{-\alpha(X)}. \]
\[ =: (a) + (b). \]  
(184)

We claim that
\[ \alpha(Z \triangle \{j, j+1\}) \leq \alpha(Z) + 1 \quad \text{for} \quad j \in \mathcal{I}(Z). \]  
(185)
To see this note that if $j \in \mathcal{I}(Z)$, then $j \in \mathcal{I}(Z \triangle \{j, j+1\})$. So a single application of the Hamiltonian can cause the transition $Z \triangle \{j, j+1\} \rightarrow Z$. Using (185) we get
\[ (a) \leq 2|\epsilon| (|\epsilon|\mathcal{M})^{-1} \sum_{n(Z) \neq 0} |e(Z) - \tilde{c}(Z)|(|\epsilon|\mathcal{M})^{-\alpha(Z)} \sum_{j \in \hat{Z}(Z)} 1 \]
\[ \leq 2M^{-1} \sum_{n(Z) \neq 0} |e(Z) - \tilde{c}(Z)|(|\epsilon|\mathcal{M})^{-\alpha(Z)}|\delta Z| \]
\[ \leq 2M^{-1} \sum_{n(Z) \neq 0} |e(Z) - \tilde{c}(Z)|(|\epsilon|\mathcal{M})^{-\alpha(Z)}(n(Z) + 2) \]
\[ \leq 3M^{-1} ||e - \tilde{c}||, \]  
(186)
where we have used the inequality

$$|\delta Z| \leq n(Z) + 2.$$  

Moreover, using the triangle inequality we get

$$(b) \leq \sum_{s=1}^{2N} \sum_{X \neq 0} \sum_{n(X) \neq 0} |e_s| |e(T^{-s}X) - \tilde{e}(T^{-s}X)| \left( |\epsilon| \|M\|^{-\alpha(X)} + |e_s - \tilde{e}_s| \|\tilde{e}(T^{-s}X)\| |\epsilon| \|M\|^{-\alpha(X)} \right).$$  

(187)

Let $Y = T^{-s}X$. Hence, $X = T^sY$. Since $n(X) \neq 0$ in the above sum, we must have $Y \neq \emptyset$ and $n(Y) \neq \emptyset$. We claim that

$$\alpha(T^sY) \leq \alpha(Y) + \beta.$$  

(188)

If we have a sequence of applications of the Hamiltonian that causes the transition $T^sY \rightarrow T^s\emptyset$ and another sequence that causes the transition $T^s\emptyset \rightarrow \emptyset$, then together they give a sequence which results in $T^sY \rightarrow \emptyset$. Thus

$$\alpha(T^sY) = \alpha(T^sY \rightarrow \emptyset) \leq \alpha(T^sY \rightarrow T^s\emptyset) + \alpha(T^s\emptyset \rightarrow \emptyset) = \alpha(Y) + \alpha(T^s\emptyset) = \alpha(Y) + \beta.$$  

(189)

where we have used (164). From (187) and (188) it follows that

$$(b) \leq \sum_{s=1}^{2N} |e_s| \left( |\epsilon| \|M\|^{-\alpha(Y)} \right) + \sum_{s=1}^{2N} |e_s - \tilde{e}_s| \left( |\epsilon| \|M\|^{-\alpha(Y)} \right)$$

$$+ \sum_{s=1}^{2N} |e_s - \tilde{e}_s| \left( |\epsilon| \|M\|^{-\alpha(Y)} \right)$$

$$\leq \frac{1}{2} |\epsilon| \|e - \tilde{e}| + \frac{1}{2} |\tilde{e}| \|e - \tilde{e}|.$$  

(190)

(191)

From (186) and (191) it follows that

$$\text{RHS of (184)} \leq ||e - \tilde{e}|| \left[ \frac{1}{2} ||e|| + \frac{1}{2} ||\tilde{e}|| + 4M^{-1} \right] \leq K||e - \tilde{e}||$$  

(192)

where we have used $||e|| \leq \delta$, $||\tilde{e}|| \leq \delta$, and defined

$$K = \delta + 4M^{-1} = 2\delta.$$  

(193)

If $\delta \leq 1/4$, then $K \leq 1/2$.

To prove (180), we use (169), (172) and (173) to compute $F(0)$. Note that $e = 0$ means that $e_s = 0$ for all $s \in \Lambda$, and $e(X) = 0$ for all $X$ except $X = \emptyset$. We always have $e(\emptyset) = 1$. Letting $\tilde{e}$ denote $F(0)$, we have

$$\tilde{e}_m = 0 \quad \text{for all} \quad m.$$  

(194)
For $X = \{N, 1\}$ we have
\[
\tilde{e}(X) = \frac{1}{2n(X)}[2\epsilon]
\] (195)
and $\tilde{e}(X) = 0$ for all other $X$ for which $n(X) \neq 0$. Thus
\[
||F(0)|| \leq 2|\epsilon|(|\epsilon|M)^{-\alpha(\{N,1\})} = 2|\epsilon|(|\epsilon|M)^{-1} = 2M^{-1},
\] (196)
since $\alpha(\{N,1\}) = 1$. Hence, for $||\epsilon|| \leq \delta$, where $\delta = 4M^{-1}$
\[
||F(\epsilon)|| \leq ||F(\epsilon) - F(0)|| + ||F(0)|| \leq \frac{1}{2}||\epsilon|| + 2M^{-1} \leq \delta
\] (197)
This finishes the proof that the fixed point equation has a solution and so completes the proof of Theorem 3.

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