The number of chains of subgroups of a finite elementary abelian $p$-group

Marius Tărnăuceanu

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Abstract
In this short note we give a formula for the number of chains of subgroups of a finite elementary abelian $p$-group. This completes our previous work [5].

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1 Introduction

Let $G$ be a group. A chain of subgroups of $G$ is a set of subgroups of $G$ totally ordered by set inclusion. A chain of subgroups of $G$ is called rooted (more exactly $G$-rooted) if it contains $G$. Otherwise, it is called unrooted. Notice that there is a bijection between the set of $G$-rooted chains of subgroups of $G$ and the set of distinct fuzzy subgroups of $G$ (see e.g. [5]), which is used to solve many computational problems in fuzzy group theory.

The starting point for our discussion is given by the paper [5], where a formula for the number of rooted chains of subgroups of a finite cyclic group is obtained. This leads in [3] to precise expression of the well-known central Delannoy numbers in an arbitrary dimension and has been simplified in [2]. Some steps in order to determine the number of rooted chains of subgroups of a finite elementary abelian $p$-group are also made in [5]. Moreover, this counting problem has been naturally extended to non-abelian groups in other
works, such as [1, 4]. The purpose of the current note is to improve the results of [5], by indicating an explicit formula for the number of rooted chains of subgroups of a finite elementary abelian $p$-group.

Given a finite group $G$, we will denote by $C(G)$, $D(G)$ and $F(G)$ the collection of all chains of subgroups of $G$, of unrooted chains of subgroups of $G$ and of $G$-rooted chains of subgroups of $G$, respectively. Put $C(G) = |C(G)|$, $D(G) = |D(G)|$ and $F(G) = |F(G)|$. The connections between these numbers have been established in [2], namely:

**Theorem 1.** Let $G$ be a finite group. Then $$F(G) = D(G)+1 \quad \text{and} \quad C(G) = F(G) + D(G) = 2F(G) - 1.$$ 

In the following let $p$ be a prime, $n$ be a positive integer and $\mathbb{Z}_p^n$ be an elementary abelian $p$-group of rank $n$ (that is, a direct product of $n$ copies of $\mathbb{Z}_p$). First of all, we recall a well-known group theoretical result that gives the number $a_{n,p}(k)$ of subgroups of order $p^k$ in $\mathbb{Z}_p^n$, $k = 0, 1, ..., n$.

**Theorem 2.** For every $k = 0, 1, ..., n$, we have $$a_{n,p}(k) = \frac{(p^n - 1) \cdots (p - 1)}{(p^k - 1) \cdots (p - 1)(p^{n-k} - 1) \cdots (p - 1)}.$$ 

Our main result is the following.

**Theorem 3.** The number of rooted chains of subgroups of the elementary abelian $p$-group $\mathbb{Z}_p^n$ is $$F(\mathbb{Z}_p^n) = 2 + 2f(n) \sum_{k=1}^{n-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n-1} \frac{1}{f(n-i_k)f(i_k-i_{k-1}) \cdots f(i_2-i_1)f(i_1)},$$ where $f: \mathbb{N} \to \mathbb{N}$ is the function defined by $f(0) = 1$ and $f(r) = \prod_{s=1}^{r}(p^s - 1)$ for all $r \in \mathbb{N}^*$. 

Obviously, explicit formulas for $C(\mathbb{Z}_p^n)$ and $D(\mathbb{Z}_p^n)$ also follow from Theorems 1 and 2. By using a computer algebra program, we are now able to calculate the first terms of the chain $f_n = F(\mathbb{Z}_p^n)$, $n \in \mathbb{N}$, namely:
- $f_0 = 1$;
- $f_1 = 2$;
- $f_2 = 2p + 4$;
- $f_3 = 2p^3 + 8p^2 + 8p + 8$;
- $f_4 = 2p^6 + 12p^5 + 24p^4 + 36p^3 + 36p^2 + 24p + 16$.

Finally, we remark that the above $f_3$ is in fact the number $a_{3,p}$ obtained by a direct computation in Corollary 10 of [5].

2 Proof of Theorem 3

We observe first that every rooted chain of subgroups of $\mathbb{Z}_p^n$ are of one of the following types:

(1) \[ G_1 \subset G_2 \subset ... \subset G_m = \mathbb{Z}_p^n \] with $G_1 \neq 1$

and

(2) \[ 1 \subset G_2 \subset ... \subset G_m = \mathbb{Z}_p^n. \]

It is clear that the numbers of chains of types (1) and (2) are equal. So

(3) \[ f_n = 2x_n, \]

where $x_n$ denotes the number of chains of type (2). On the other hand, such a chain is obtained by adding $\mathbb{Z}_p^n$ to the chain

\[ 1 \subset G_2 \subset ... \subset G_{m-1}, \]

where $G_{m-1}$ runs over all subgroups of $\mathbb{Z}_p^n$. Moreover, $G_{m-1}$ is also an elementary abelian $p$-group, say $G_{m-1} \cong \mathbb{Z}_p^k$ with $0 \leq k \leq n$. These show that the chain $x_n$, $n \in \mathbb{N}$, satisfies the following recurrence relation

(4) \[ x_n = \sum_{k=0}^{n-1} a_{n,p}(k)x_k, \]
which is more facile than the recurrence relation founded by applying the Inclusion-Exclusion Principle in Theorem 9 of [5].

Next we prove that the solution of (4) is given by

\( x_n = 1 + \sum_{k=1}^{n-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n-1} a_{n,p}(i_k) a_{i_{k-1},p}(i_{k-1}) \cdots a_{i_2,p}(i_1) \).

We will proceed by induction on \( n \). Clearly, (5) is trivial for \( n = 1 \). Assume that it holds for all \( k < n \). One obtains

\[
x_n = \sum_{k=0}^{n-1} a_{n,p}(k)x_k = 1 + \sum_{k=1}^{n-1} a_{n,p}(k)x_k =
\]

\[
= 1 + \sum_{k=1}^{n-1} a_{n,p}(k) \left( 1 + \sum_{r=1}^{k-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq k-1} a_{k,p}(i_r) a_{i_{r-1},p}(i_{r-1}) \cdots a_{i_2,p}(i_1) \right) =
\]

\[
= 1 + \sum_{k=1}^{n-1} a_{n,p}(k) + \sum_{k=1}^{n-1} a_{n,p}(k) \sum_{r=1}^{k-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq k-1} a_{k,p}(i_r) a_{i_{r-1},p}(i_{r-1}) \cdots a_{i_2,p}(i_1) =
\]

\[
= 1 + \sum_{k=1}^{n-1} a_{n,p}(k) + \sum_{k=1}^{n-1} a_{n,p}(k) \sum_{r=1}^{n-2} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq k-1} a_{k,p}(i_r) a_{i_{r-1},p}(i_{r-1}) \cdots a_{i_2,p}(i_1) =
\]

\[
= 1 + \sum_{k=1}^{n-1} a_{n,p}(k) + \sum_{k=1}^{n-1} a_{n,p}(k) \sum_{r=2}^{n-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_{r-1} \leq k-1} a_{k,p}(i_r) a_{i_{r-1},p}(i_{r-1}) \cdots a_{i_2,p}(i_1) =
\]

\[
= 1 + \sum_{1 \leq i_1 \leq n-1} a_{n,p}(i_1) + \sum_{r=2}^{n-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_{r-1} \leq k-1} a_{k,p}(i_r) a_{i_{r-1},p}(i_{r-1}) \cdots a_{i_2,p}(i_1) =
\]

\[
= 1 + \sum_{1 \leq i_1 \leq n-1} a_{n,p}(i_1) + \sum_{r=2}^{n-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_{r-1} \leq n-1} a_{n,p}(i_r) a_{i_{r-1},p}(i_{r-1}) \cdots a_{i_2,p}(i_1) =
\]

\[
= 1 + \sum_{r=1}^{n-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n-1} a_{n,p}(i_r) a_{i_{r-1},p}(i_{r-1}) \cdots a_{i_2,p}(i_1),
\]

as desired.
Since by Theorem 2
\[
a_{n,p}(k) = \frac{(p^n - 1) \cdots (p - 1)}{(p^k - 1) \cdots (p - 1)(p^{n-k} - 1) \cdots (p - 1)} = \frac{f(n)}{f(k) f(n - k)}, \forall 0 \leq k \leq n,
\]
the equalities (3) and (5) imply that
\[
f_n = 2 + 2f(n) \sum_{k=1}^{n-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n-1} \frac{1}{f(n-i_k)f(i_k-i_{k-1}) \cdots f(i_2-i_1)f(i_1)},
\]
completing the proof. □

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Marius Tărnăuceanu
Faculty of Mathematics
“Al.I. Cuza” University
Iaşi, Romania
e-mail: tarnauc@uaic.ro