K-theory of regular compactification bundles

V. Uma

Department of Mathematics, Indian Institute of Technology-Madras, Chennai-600036, India

Correspondence
V. Uma, Department of Mathematics, Indian Institute of Technology-Madras, Chennai-600036, India.
Email: vuma@iitm.ac.in

Abstract
Let $G$ be a split connected reductive algebraic group. Let $E \rightarrow B$ be a $G \times G$-torsor over a smooth base scheme $B$ and $X$ be a regular compactification of $G$. We describe the Grothendieck ring of the associated fibre bundle $E(X) := E \times_{G \times G} X$, as an algebra over the Grothendieck ring of a canonical toric bundle over a flag bundle over $B$. These are relative versions of the corresponding results on the Grothendieck ring of $X$ in the case when $B$ is a point, and generalize the classical results on the Grothendieck rings of projective bundles, toric bundles and flag bundles.

KEYWORDS
flag bundles, K-theory, regular compactification bundles, toric bundles

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1 | INTRODUCTION

In this article we shall consider algebraic groups and varieties to be defined over an algebraically closed field $F$. By an algebraic group we mean an affine, smooth group scheme of finite type over $F$. By a scheme we mean a separated scheme of finite type over $F$. By a variety we mean an integral scheme of finite type over $F$. All schemes we consider are assumed to be noetherian. By a torsor we shall always mean locally isotrivial in the sense of [26].

Let $G$ denote a connected reductive algebraic group. Let $C$ be the center of $G$ and let $G_{ad} := G/C$ be the corresponding semisimple adjoint group.

A normal complete variety $X$ is called an equivariant compactification of $G$ if $X$ contains $G$ as an open subvariety and the action of $G \times G$ on $G$ by left and right multiplication extends to $X$. We say that $X$ is a regular compactification of $G$ if $X$ is an equivariant compactification of $G$ which is regular as a $G \times G$-variety ([8, Section 2.1]). Smooth complete toric varieties are regular compactifications of the torus. For the adjoint group $G_{ad}$, the wonderful compactification $\overline{G_{ad}}$ constructed by De Concini and Procesi in [13] is the unique regular compactification of $G_{ad}$ with a unique closed $G_{ad} \times G_{ad}$-orbit.

Let $E \rightarrow B$ be a $G \times G$-torsor over a base scheme $B$. Let $X$ be a projective regular compactification of a connected reductive algebraic group $G$. Let $E(X) := E \times_{(G \times G)} X$ denote the associated bundle with fibre $X$ and base $B$. Since $E$ is the total space of a $G \times G$-torsor over $B$, it is a $G \times G$-scheme. Further, the space $E(X)$ also gets the structure of a scheme (see [16, Proposition 23]). Moreover, since $X$ is a smooth $G$-scheme, when the base scheme $B$ is smooth, it can be seen that $E(X)$ is a smooth scheme (see below for details).

The main aim of this article is to describe the Grothendieck ring of algebraic vector bundles on $E(X)$ as an algebra over the Grothendieck ring of algebraic vector bundles on a smooth base scheme $B$. This is with a view to generalize and is motivated by the corresponding classical results on projective bundles [3], toric bundles [29], and flag bundles [18, 24]. This is also a relative version of the results in [32, 33] on K-theory of regular compactifications when $B = pt$.

We now fix some notations before we give an overview of the main results.
For a $G$-scheme $X$, let $K_0^G(X)$ denote the Grothendieck ring of $G$-equivariant vector bundles on $X$ and $K_0^G(Y)$ denote the Grothendieck group of $G$-equivariant coherent sheaves on $Y$. Let $R(G)$ denote the Grothendieck ring of representations of $G$ over the field $F$. Identifying $R(G)$ with $K_0^G(pt)$, the pullback by the projection $X \to pt$ gives a canonical map $R(G) \to K_0^G(X)$. This gives $K_0^G(X)$ the structure of an $R(G)$-algebra. Also $K_0^G(X)$ is a module over the ring $K_0^G(X)$ and is hence an $R(G)$-module. Moreover, when $X$ is smooth $K_0^G(X) \cong K_0^G(X)$.

In Section 1 we prove some preliminary results on the $G$-equivariant K-theory of schemes having equivariant cellular structure, assuming that $\pi_1(G)$ is torsion free. Let $X$ be any $G$-cellular scheme, that is a $G$-scheme with plying by $G$-stable affine cells (see Definition 2.1).

We prove an equivariant Kunneth theorem (see Theorem 2.2) where we show that we have an isomorphism $K^G_0(Y) \otimes_{R(G)} K^G_0(X) \cong K^G_0(Y \times X)$ of $K^G_0(Y)$-modules. Moreover, when $X$ and $Y$ are smooth then the Kunneth map is an isomorphism of $K^G_0(Y)$-algebras.

Let $p : \mathcal{E} \to B$ denote a $G$-torsor over a base scheme $B$ and $X$ be a $G$-scheme. Then by [16, Proposition 23], $\mathcal{E}$ is a $G$-scheme and the associated bundle $\mathcal{E}(X) := \mathcal{E} \times_X G$ is a scheme. We have the projection $\pi : \mathcal{E}(X) \to B$ defined as $\pi[e, x] = p(e)$ for $[e, x] \in \mathcal{E}(X)$. Furthermore, when $B$ and $X$ are smooth, $\mathcal{E}(X)$ is a smooth scheme. (Since $G$ is smooth, the quotient map $\mathcal{E} \to B$ is a smooth morphism. Thus when the base scheme $B$ is smooth, by the local triviality of the $G$-torsor $\mathcal{E} \to B$, it follows that the total space $\mathcal{E}$ is smooth. When $X$ is a smooth $G$-scheme, $\mathcal{E} \times X$ is smooth. Thus the quotient $\mathcal{E}(X)$ of $\mathcal{E} \times X$ by the free action of $G$ is a smooth scheme.)

For a $G$-cellular scheme (resp. a $G$-scheme with a $T$-cellular structure, where $T$ denotes a maximal torus of $G$ acting on $X$ by restriction), in Corollary 2.7 (resp. Corollary 2.8) we derive the structure of the Grothendieck group $K_0(\mathcal{E}(X))$ of coherent sheaves on $\mathcal{E}(X)$, as a module over the Grothendieck ring $K^G_0$ of algebraic vector bundles on $B$. We further note that Corollary 2.8 holds for any projective $G$-scheme $X$ having finitely many $T$-fixed points.

In particular, when $X$ is smooth and the base scheme $B$ is smooth we get the structure of $K_0(\mathcal{E}(X)) = K^G_0(\mathcal{E}(X))$ as a $K^G_0(B)$-algebra. More precisely, using the Kunneth formula and the fact that $K^G_0(\mathcal{E}) = K^G_0(\mathcal{E}/G) = K^G_0(B)$, we show that $K^G_0(\mathcal{E}(X))$ is isomorphic to a canonical extension of scalars of the $R(G)$-algebra $K_0^G(X)$, to the ring $K_0^G(B)$.

Let $B$ be a smooth scheme and let $p : \mathcal{E} \to B$ be a $T$-torsor and $X$ be a smooth projective $T$-variety with finitely many $T$-fixed points and invariant curves. Let $X_T$ denote the set of $T$-fixed points in $X$. If $|X_T| = m$, we consider $\prod_{i=1}^m K^G_0(B)$ which is a ring under pointwise addition and multiplication. It further gets a canonical $K^G_0(B)$-algebra structure through the diagonal inclusion. The ring $K^G_0(\mathcal{E}(X))$ also has a $K^G_0(B)$-algebra structure induced by pull back of vector bundles under the projection $\pi$. We next prove Theorem 2.10 which is a relative version of the localization theorem for $K^G_0(X)$ (see [34], [32, Theorem 1.3]). Here we show that the restriction to the $T$-fixed points of the fiber induces a canonical inclusion

$$K^G_0(\mathcal{E}(X)) \simeq \prod_{i=1}^m K^G_0(B)$$

of $K(B)$-algebras. We further show that the image of $K^G_0(\mathcal{E}(X))$ in $\prod_{i=1}^m K^G_0(B)$ is precisely the intersection of the images of the Grothendieck rings of $\mathbb{P}^1$-bundles associated to $\mathcal{E} \to B$, corresponding to the projective lines joining two distinct $T$-fixed points.

In Section 2 we prove our main results. Let $G$ be a connected reductive algebraic group and let $\bar{G}$ be a factorial cover of $G$ (see 2.35). In particular, $\pi_1(\bar{G})$ torsion free.

Let $p : \mathcal{E} \to B$ be a $\bar{G} \times \bar{G}$-torsor over a smooth base scheme $B$. We let $\tilde{X}$ be a projective regular compactification of $G$. Let $\mathcal{E}(X) := \mathcal{E} \times_{\bar{G} \times \bar{G}} X$ where $\bar{G}$ acts on $X$ via its canonical projection to $G$.

Let $T$ denote a maximal torus of $G$ and $B$ a Borel subgroup containing $T$. Let $W$ denote the Weyl group of $(G, T)$. In Theorem 3.3, using Theorem 2.2 and [32, Corollary 2.3] we describe the Grothendieck ring of $\mathcal{E}(X)$ as $\text{diag}(W)$-invariants of the Grothendieck ring of a toric bundle, with fibre the toric variety $\overline{T} \subseteq \overline{G} = X$, and base another bundle over $B$ with fibre $G/B \times G/B$. We note that here the $\text{diag}(W)$-action on the Grothendieck ring of the toric bundle is induced from its canonical action on $\overline{T}$ (see [8, Proposition A1, A2]). This is the relative version of [32, Proposition 2.15].

In Theorem 3.4, we use Theorem 2.2, Theorem 2.10, [32, Corollary 2.2] and [33, Theorem 2.4], to further describe the multiplicative structure of $K^G_0(\mathcal{E}(X))$, as an algebra over the Grothendieck ring of a toric bundle with fibre the toric variety $\overline{T}^+$, and base a flag bundle. The toric variety $\overline{T}^+$ is associated to a smooth fan in the lattice of one parameter subgroups of $T$, supported on the positive Weyl chamber (see [8, Proposition A1, A2]). This is the relative version of [33, Theorem 3.1]. Finally, in Section 3 we retrieve the known results on the Grothendieck ring of toric bundles and flag bundles.
2 | PRELIMINARIES ON EQUIVARIANT K-THEORY

Throughout this section we shall assume that $G$ is a connected reductive algebraic group. In Subsections 1.1 and 1.2 we shall assume in addition that $\pi_1(G)$ is torsion free. This is with a view to apply the results in [23] which require this hypothesis.

2.1 | Kunneth formula for equivariant K-theory

For $G$-schemes $X$ and $Y$, the map

$$\boxtimes : K^G_0(Y) \boxtimes_{\mathbb{Z}} K^G_0(X) \to K^G_0(Y \times X)$$

(2.1)

induced by the external tensor product of $G$-equivariant coherent sheaves is defined by

$$(G, G') \mapsto G \boxtimes G' := p_Y^* (G) \boxtimes_{o_{Y \times X}} p_X^* (G')$$

(2.2)

where $p_X$ and $p_Y$ are the projections from $Y \times X$ to $X$ and $Y$ respectively. Moreover, since $p_X^*$ and $p_Y^*$ are $R(G)$-module maps, the elements of the form $1 \boxtimes a - a \boxtimes 1$ for $a \in R(G)$ map to 0 under $\boxtimes$. This induces a map of $R(G)$-modules

$$\varphi : K^G_0(Y) \boxtimes_{R(G)} K^G_0(X) \to K^G_0(Y \times X).$$

(2.3)

**Definition 2.1.** A $G$-cellular scheme is a $G$-scheme $X$ equipped with a $G$-stable algebraic cell decomposition. In other words there is a filtration

$$X = X_1 \supsetneq X_2 \supsetneq \cdots \supsetneq X_m \supsetneq X_{m+1} = \emptyset$$

where each $X_i$ is a closed $G$-stable subscheme of $X$ and $X_1 \setminus X_{i+1} = U_i$ is $G$-equivariantly isomorphic to the affine space $\mathbb{A}^{K_i}$ equipped with a linear action of $G$, for $1 \leq i \leq m$.

For $Y$ any $G$-scheme and for $X_i$ and $U_i$ for $1 \leq i \leq m$, as in Definition 2.1, we have the following maps of $R(G)$-modules

$$\varphi_i : K^G_0(Y) \boxtimes_{R(G)} K^G_0(X_i) \to K^G_0(Y \times X_i)$$

(2.4)

and

$$\psi_i : K^G_0(Y) \boxtimes_{R(G)} K^G_0(U_i) \to K^G_0(Y \times U_i).$$

(2.5)

We now show that the $R(G)$-modules on either side of (2.3) have a canonical structure of $K^G_0(Y)$-modules. This is induced from the $K^G_0(Y)$-module structure on $K^G_0(Y)$.

For every $l \geq 0$, we have $p_Y^* : K^G_0(Y) \to K^G_0(Y \times X)$ induced by the pull back of $G$-equivariant coherent sheaves under $p_Y$. In particular, $p_Y^* : K^G_0(Y) \to K^G_0(Y \times X)$ maps $K^G_0(Y)$ to $K^G_0(Y \times X)$ via pull back of $G$-equivariant vector bundles under $p_Y$, giving $K^G_0(Y \times X)$ the structure of a $K^G_0(Y)$-algebra.

Furthermore, since $K^G_l(Y \times X)$ is a $K^G_0(Y \times X)$-module (induced by tensor product of $G$-equivariant vector bundles with $G$-equivariant coherent sheaves see [12, p. 247–248], [25]), it follows that $K^G_l(Y \times X)$ gets the structure of a $K^G_0(Y)$-module for $l \geq 0$. Furthermore, $p_Y^*$ is a morphism of $K^G_0(Y)$-modules.

On the other hand we note that $K^G_0(Y) \boxtimes_{R(G)} K^G_0(X)$ has a canonical structure of $K^G_0(Y) \boxtimes_{R(G)} R(G) \cong K^G_0(Y)$-module.
We now claim that $\varphi$ preserves the $K^0_G(Y)$-module structure. Note that for a $G$-equivariant vector bundle $E$ on $Y$, $p_Y^*(E)$ is a $G$-equivariant vector bundle on $Y \times X$ and $p_Y^*(E \otimes^G F) = p_Y^*(E) \otimes_{O_X} p_Y^*(F)$ for every $G$-equivariant coherent sheaf $F$ on $Y$. The claim follows from the $K^0_G(Y)$-module structure on $K^0_G(Y) \otimes_{R(G)} K^0_G(X)$ and $K^0_G(Y \times X)$.

We recall below the Thom isomorphism theorem in higher $G$-equivariant $K$-theory (see [25, 31] or [12, Theorem 5.4.17]).

**Theorem** (Thom isomorphism). Let $\pi^t : E \to X$ be a $G$-equivariant affine bundle which is a torsor under the vector group scheme associated with a $G$-equivariant vector bundle over a $G$-scheme $X$. For any $j \geq 0$ the morphism $\pi^{t*} : K^j_G(X) \to K^j_G(E)$ is an isomorphism.

**Theorem 2.2.** (Equivariant Kunneth formula) Let $X$ be a $G$-cellular scheme and let $Y$ be any $G$-scheme. Then the canonical map $\varphi$ defined in (2.3) is an isomorphism of $K^0_G(Y)$-modules. Moreover, when $X$ and $Y$ are smooth then $\varphi$ is an isomorphism of $K^0_G(Y)$-algebras.

**Proof.** Let $\alpha : X_{i+1} \hookrightarrow X_i$ denote the embedding of the closed subscheme and $\beta : U_i \to X_i$ denote the immersion of the open subscheme. For a $G$-scheme $Y$ consider the equivariant cellular fibration

$$Y \times X_1 \supseteq Y \times X_2 \supseteq \cdots \supseteq Y \times X_m = Y$$

over $Y$. Since $id_Y \times \alpha$ and $id_Y \times \beta$ are maps of $G$-schemes, these induce morphisms in higher $G$-equivariant $K$-theory ([25, 31]) $(id_Y \times \alpha)_* : K^0_G(Y \times X_{i+1}) \to K^0_G(Y \times X_i)$ and $(id_Y \times \beta)^* : K^0_G(Y \times U_i) \to K^0_G(Y \times U_i)$ for each $l \geq 0$. We have the following long exact sequence in higher $G$-equivariant $K$-theory

$$K^0_G(Y \times X_i) \xrightarrow{(id_Y \times \beta)^*} K^0_G(Y \times U_i) \xrightarrow{\partial} K^0_G(Y \times X_{i+1}) \xrightarrow{(id_Y \times \alpha)_*} K^0_G(Y \times X_i) \xrightarrow{(id_Y \times \beta)^*} K^0_G(Y \times U_i) \to 0$$

(2.6)

(see [12, Lemma 5.5.1]). Moreover, we have the pull back maps from $K^0_G(Y) \to K^0_G(Y \times X_i)$ and the following commutative triangle for every $l \geq 0$.

$$K^0_G(Y \times X_i) \xrightarrow{(id_Y \times \beta)^*} K^0_G(Y \times U_i) \cong K^0_G(Y) \xleftarrow{\partial} K^0_G(Y \times X_i) \xrightarrow{(id_Y \times \alpha)_*} K^0_G(Y \times X_i) \xrightarrow{(id_Y \times \beta)^*} K^0_G(Y \times U_i) \to 0$$

Note that the isomorphism in the above diagram is the Thom isomorphism for the $G$-equivariant affine bundle $Y \times U_i \to Y$. It follows that the maps $(id_Y \times \beta)^* : K^0_G(Y \times X_i) \to K^0_G(Y \times U_i)$ are surjective. This in turn implies from (2.6) that the connecting homomorphism $\partial$ is trivial and $(id_Y \times \alpha)_*$ is injective. Thus we have the following short exact sequence of $K^0_G(Y)$-modules

$$0 \to K^0_G(Y \times X_{i+1}) \xrightarrow{(id_Y \times \alpha)_*} K^0_G(Y \times X_i) \xrightarrow{(id_Y \times \beta)^*} K^0_G(Y \times U_i) \to 0.$$

(2.7)

When $Y = pt$ (2.7) reduces to the following short exact sequence of $R(G)$-modules

$$0 \to K^0_G(X_{i+1}) \xrightarrow{\alpha_*} K^0_G(X) \xrightarrow{\beta^*} K^0_G(U) \to 0.$$

(2.8)

We now claim that the map (2.4) is an isomorphism for each $1 \leq i \leq m$. When $i = 1$, this will imply that (2.3) is an isomorphism. We prove this by downward induction on $i$. This is trivially true for $i = m$, since in this case $X_m = \emptyset$. Consider the commutative diagram of $R(G)$-modules

$$\begin{array}{ccc}
K^0_G(Y) \otimes_{R(G)} K^0_G(X_{i+1}) & \xrightarrow{id_Y \otimes \alpha_*} & K^0_G(Y) \otimes_{R(G)} K^0_G(X_i) \\
\varphi_{i+1} \downarrow & & \downarrow \varphi_i \\
K^0_G(Y \times X_{i+1}) & \xrightarrow{(id_Y \times \alpha)_*} & K^0_G(Y \times X_i) \\
\end{array}$$

$$0 \to K^0_G(Y \times X_{i+1}) \xrightarrow{(id_Y \times \alpha)_*} K^0_G(Y \times X_i) \xrightarrow{(id_Y \times \beta)^*} K^0_G(Y \times U_i) \to 0$$

(2.9)
where the bottom row is a part of (2.7) when \( l = 0 \) and the top horizontal row is obtained from taking tensor product of the exact sequence (2.8) when \( l = 0 \) with the \( R(G) \)-module \( K^G_0(Y) \) on the left.

Furthermore, the map \( \psi_i \) is an isomorphism of \( R(G) \)-modules for every \( 1 \leq i \leq m \). This can be seen from the following commuting diagram:

\[
\begin{array}{ccc}
K^G_0(Y) \otimes_{R(G)} K^G_0(U_i) & \xrightarrow{=^*} & K^G_0(Y) \otimes_{R(G)} K^G_0(pt) \\
\psi_i \downarrow & & \downarrow \cong \\
K^G_0(Y \times U_i) & \xrightarrow{=^*} & K^G_0(Y)
\end{array}
\]  

(2.10)

where the top and the bottom isomorphisms follow from the Thom isomorphisms for the \( G \)-equivariant affine bundles \( U_i \rightarrow pt \) and \( Y \times U_i \rightarrow Y \) respectively and the second vertical arrow is the canonical isomorphism.

Therefore if we assume that \( \varphi_{i+1} \) is surjective, it follows by diagram chase that \( \varphi_i \) is surjective. Note that the bottom horizontal row of (2.9) is left exact. Hence if we assume that \( \varphi_{i+1} \) is injective, it follows again by diagram chase that \( \varphi_i \) is injective.

\[\square\]

**Definition 2.3.** Let \( S \) be a \( G \)-scheme. By a relative \( G \)-cellular scheme we mean a \( G \)-scheme \( X \) over \( S \) equipped with a \( G \)-stable relative algebraic cell decomposition. In other words there is a filtration

\[ X = X_1 \supseteq X_2 \supseteq \cdots \supseteq X_m \supseteq X_{m+1} = \emptyset \]

where each \( X_i \) is a closed \( G \)-stable subscheme of \( X \) over \( S \) and \( X_i \setminus X_{i+1} = U_i \) is isomorphic to a \( G \)-equivariant affine bundle which is a torsor under the vector group scheme associated to a \( G \)-equivariant vector bundle over \( S \) of rank \( k_i \), for \( 1 \leq i \leq m \). Equivalently \( X \rightarrow S \) is a \( G \)-equivariant cellular fibration in the sense of [12].

For \( X \) and \( Y \) schemes over \( S \), we have a map of \( K^G_0(S) \)-modules

\[ \varphi^S : K^G_0(Y) \otimes_{K^G_0(S)} K^G_0(X) \rightarrow K^G_0(Y \times_S X) \]

induced again by external tensor product of \( G \)-equivariant coherent sheaves on \( Y \) and \( X \).

We state below the relative version of Theorem 2.2.

**Theorem 2.4.** Let \( X \) be a relative \( G \)-cellular scheme over \( S \) and let \( Y \) be any \( G \)-scheme over \( S \). The map \( \varphi^S \) is an isomorphism of \( K^G_0(Y) \)-modules. Moreover, if \( X \) and \( Y \) are smooth then \( \varphi^S \) is an isomorphism of \( K^G_0(Y) \)-algebras.

**Proof.** The proof follows verbatim as that of Theorem 2.2 by replacing everywhere \( Y \times X \) by the fibre product \( Y \times_S X \) and \( R(G) \) by the \( R(G) \)-algebra \( K^G_0(S) \). \[\square\]

**Remark 2.5.** Since a \( G \)-cellular scheme is in particular \( T \)-linear the proof of Theorem 2.2 also follows directly from that of [2, Proposition 6.4] by using the fact that the \( T \)-equivariant Kunneth map is a faithfully flat extension of the \( G \)-equivariant map, as in [19, proof of Theorem A.3]. (Also see Corollary 2.8 below where this argument is being used.) However the filtration argument we use in the proof of Theorem 2.2 is better adapted to the setting of \( G \)-cellular and \( T \)-cellular schemes, and allows an easy proof of Theorem 2.4. Indeed the proof of Theorem 2.4 follows verbatim from that of [2, Proposition 6.4] by replacing everywhere \( Y \times X \) by \( Y \times_S X \). But it is good to see it explicitly as above.

2.1.1 Applications of the Kunneth formula

Let \( X \) be a \( T \)-cellular scheme. In other words we have a stratification

\[ X = X_1 \supseteq X_2 \supseteq \cdots \supseteq X_m \supseteq X_{m+1} = \emptyset \]  

(2.11)

by \( T \)-stable closed subschemes such that \( F^{k_i} \cong U_i = X_i \setminus X_{i+1} \) is a \( T \)-representation.
Let \( X := G \times_B X \), where the \( B \) action on \( X \) is through the canonical projection \( B \to T \). Then \( X \) gets a \( G \)-scheme structure via the natural action of \( G \) on the left. We further have a \( G \)-equivariant stratification \( X = X_1 \supseteq X_2 \supseteq \cdots \supseteq X_m \) where \( X_i := G \times_B X_i \) for \( 1 \leq i \leq m \). Further, \( G \times_B U_i = X_i \setminus X_{i+1} \) is a \( G \)-equivariant vector bundle over \( G/B \). Thus \( X \) has the structure of a relative \( G \)-cellular scheme over \( G/B \).

**Proposition 2.6.** Let \( Y \) be any \( G \)-scheme. Then the canonical map
\[
\varphi : K^G_0(Y) \bigotimes_{R(G)} K^G_0(X) \to K^G_0(Y \times X)
\]
defined as in (2.3) is an isomorphism of \( K^G_0(Y) \)-modules. Moreover, when \( X \) and \( Y \) are smooth then \( \varphi \) is an isomorphism of \( K^G_0(Y) \)-algebras.

**Proof.** Since \( X \) is a relative \( G \)-cellular scheme over \( S = G/B \) by Theorem 2.4 we have the isomorphism
\[
K^G_0(Y_S) \otimes_{K^0_0(S)} K^G_0(X) \cong K^G_0(Y_S \times_S X)
\]
(2.12)
where \( Y_S = Y \times S \) is the base change of \( Y \) to \( S \). Since \( Y_S \times_S X = (Y \times S) \times_S X = Y \times X \), the right hand side of (2.12) is isomorphic to \( K^G_0(Y \times X) \). By [23, proof of Proposition 4.1, p.30] we have the following isomorphism
\[
K^G_0(Y_S) \cong K^G_0(Y) \otimes_{R(G)} K^G_0(S)
\]
(2.13)
since \( K^G_0(S) = R(B) = R(T) \). Furthermore, we have
\[
K^G_0(Y) \otimes_{R(G)} K^G_0(X) \cong K^G_0(Y) \otimes_{R(G)} K^G_0(S) \otimes_{K^0_0(S)} K^G_0(X).
\]
(2.14)
Note that the right hand side of (2.14) follows by Theorem 2.4 applied to \( X = X \) and \( Y = S \). Now, by (2.13) and (2.14) we get that the left hand side of (2.12) is isomorphic to \( K^G_0(Y) \otimes_{R(G)} K^G_0(X) \). The proposition now follows from (2.12). (Note here that \( K^G_0(S) = K^G_0(S) \) since \( S \) is smooth.) \( \square \)

Let \( B \) be a smooth base scheme and \( p : \mathcal{E} \to B \) a \( G \)-torsor. Let \( \mathcal{E}(X) := \mathcal{E} \times_G X \) denote the associated bundle with fibre a \( G \)-cellular scheme \( X \) and projection \( \pi : \mathcal{E}(X) \to B \). We recall that \( \mathcal{E} \) is a smooth \( G \)-scheme and \( \mathcal{E}(X) \) is a scheme. Further, \( K_0(\mathcal{E}(X)) \) becomes a \( K^0(B) \)-module via pull back of classes of vector bundles under \( \pi^* \). Furthermore, we note that \( K^0(B) \) is an \( R(G) \)-algebra via the map which takes the isomorphism class of any \( G \)-representation \( V \) to the class in \( K^0(B) \) of the associated vector bundle \( \mathcal{E} \times_G V \).

**Corollary 2.7.** We have the following isomorphism of \( K^0(B) \)-modules:
\[
K^0(B) \bigotimes_{R(G)} K^0_0(X) \cong K_0(\mathcal{E}(X))
\]
where the left hand side has a canonical \( K^0(B) \)-module structure by extension of scalars to the \( R(G) \)-algebra \( K^0(B) \). The above isomorphism is an isomorphism of \( K^0(B) \)-algebras if \( X \) is a smooth \( G \)-cellular scheme.

**Proof.** Note that \( X \) satisfies the hypothesis of Theorem 2.2. Further, since \( G \) acts freely on \( \mathcal{E} \) as well as on \( \mathcal{E} \times X \) diagonally, we have the isomorphisms
\[
K^0(B) = K^0(\mathcal{E}/G) = K^0_0(\mathcal{E})
\]
and
\[
K_0(\mathcal{E}(X)) = K_0((\mathcal{E} \times X)/G) = K^0_0(\mathcal{E} \times X)
\]
Moreover, the $R(G)$-module structure on $K^0_G(E)$ is via the pull back under the structure morphism $E \to \text{Spec}(F)$. Thus the class of a $G$-representation $V$ pulls back to the class of the trivial bundle $E \times V$ with the diagonal action of $G$ in $K^0_G(E)$. This further maps to the class in $K^0(B)$ of the vector bundle $E \times_G V$ over $E/G = B$. Note that since $B$ and hence $E$ are smooth schemes $K^0_G(E) = K^0(B) = K^0_G(E)$. Also if in addition $X$ is a smooth $G$-scheme then $E \times X$ and hence $E(X)$ are smooth schemes. The proof now follows readily from Theorem 2.2. □

In the following corollary we show that the assertion of Corollary 2.7 holds under a weaker assumption that the $G$-scheme $X$ is $T$-cellular and not necessarily $G$-cellular. This is always true if for instance we assume that $X$ is smooth, projective and has only finitely many $T$-fixed points (see [4–6] or [9, Section 3.1, 3.2]).

**Corollary 2.8.** Let $X$ be a $G$-scheme with a $T$-cellular structure. We have the following isomorphism of $K^0(G)$-modules

$$K^0_G(B) \bigotimes_{R(G)} K^0_G(X) \cong K^0_G(E(X)).$$

The above isomorphism is an isomorphism of $K^0(G)$-algebras if $X$ is a smooth $T$-cellular $G$-scheme.

**Proof.** Since $X$ is $T$-cellular we can apply Theorem 2.2 for the action of $T$, taking $Y = E$. It follows that we have an isomorphism of $R(T)$-modules

$$K^T_0(E) \bigotimes_{R(T)} K^T_0(X) \cong K^T_0(E \times X). \tag{2.15}$$

By [23, Proposition 2.10] the isomorphism (2.15) can be rewritten as

$$K^G_0(E \times G/B) \bigotimes_{R(T)} K^G_0(X \times G/B) \cong K^G_0(E \times X \times G/B). \tag{2.16}$$

Now, for a $G$-scheme $Y$ we have the following canonical isomorphism of $R(G)$-modules

$$R(G) \bigotimes_{R(G)} K^G_0(Y) \cong K^G_0(Y \times G/B)$$

(see [23, proof of Proposition 4.1, p. 30]). It follows that (2.16) can be rewritten as

$$\left[ R(T) \bigotimes_{R(G)} K^G_0(E) \right] \bigotimes_{R(G)} \left[ R(T) \bigotimes_{R(G)} K^G_0(X) \right] \cong R(T) \bigotimes_{R(G)} K^G_0(E \times X). \tag{2.17}$$

Further, the left hand side of (2.17) is isomorphic to $R(T) \bigotimes_{R(G)} \left[ K^G_0(E) \bigotimes_{R(G)} K^G_0(X) \right]$. It follows that the canonical map

$$K^G_0(E) \bigotimes_{R(G)} K^G_0(X) \to K^G_0(E \times X) \tag{2.18}$$

becomes an isomorphism after tensoring with $R(T)$ which is a free $R(G)$-module of rank $|W|$ (see [23, Proposition 1.22]) and hence a faithfully flat extension. (Also see [19, proof of Theorem A3].) Therefore (2.18) must be an isomorphism. The corollary follows by observing that since $B$ is smooth, we have $K^G_0(E) = K^0_0(B) = K^0(B)$.

□

### 2.2 Relative Localization theorem over $B = E/T$

Throughout this subsection we let $E \to B = E/T$ denote a $T$-torsor and not a $G$-torsor.

Let $X$ be a smooth projective variety on which the torus $T$ acts with finitely many $T$-fixed points.
We show below that \(X\) admits plus and minus Bialynicki Birula cell decomposition which are both filtrable (see [4–6] and [9, Section 3.1, 3.2]). (The author is grateful to Prof. M. Brion for the following explanation.)

Since \(X\) is a smooth projective variety with \(T\)-action having only finitely many fixed points \(\{x_1, \ldots, x_m\}\), by a theorem of Sumihiro (see [30]) there exists a \(T\)-equivariant embedding of \(X\) in a projective space \(\mathbb{P}(V)\) where \(V\) is a finite dimensional \(T\)-module. We can write \(V = \bigoplus_{\chi \in X^*(T)} V_{\chi}\) which is a decomposition of \(V\) into \(T\)-eigenspaces. There are finitely many characters \(\chi's\) (say \(\chi_1, \ldots, \chi_m\)) such that \(V_{\chi_i}\) is non-zero. Thus we may find a one-parameter subgroup \(\lambda\) of \(T\) such that the pairings \(n_j := \langle \chi_j, \lambda \rangle\) are pairwise distinct for \(i = 1, \ldots, m\). We further reorder the indices such that \(n_1 < \cdots < n_m\).

We get an action of the multiplicative group \(\mathbb{G}_m\) on \(\mathbb{P}(V)\) via \(\lambda\). The fixed points of this action are the projective subspaces \(\mathbb{P}(V_{\chi_i})\) which are same as the fixed points for the action of \(T\). Thus \(\lambda\) is a *generic* one-parameter subgroup of \(T\).

Moreover, for a fixed point \(x_i\) of \(X\) we can define the plus cell
\[
X_+(x_i, \lambda) := \left\{ x \in X \mid \lim_{t \to 0} \lambda(t) \cdot x = x_i \right\}
\]
and the minus cell
\[
X_-(x, \lambda) := \left\{ x \in X \mid \lim_{t \to \infty} \lambda(t) \cdot x = x_i \right\}.
\]

Thus for the fixed point \(x_i = [v]\) for \(v \in V_{\chi_i}\) it can be seen by direct computation that \(X_+(x_i, \lambda)\) is the intersection of \(X\) with the image in \(\mathbb{P}(V)\) of \(v + \bigoplus_{j \geq i} V_{\chi_j}\). Similarly \(X_-(x_i, \lambda)\) is the intersection of \(X\) with the image in \(\mathbb{P}(V)\) of \(v + \bigoplus_{j < i} V_{\chi_j}\).

Thus we see that
\[
\overline{X_+(x_i, \lambda)} \subseteq \bigcup_{j \geq i} X_+(x_j, \lambda) \quad (2.19)
\]
and
\[
\overline{X_-(x_i, \lambda)} \subseteq \bigcup_{j \leq i} X_-(x_j, \lambda) \quad (2.20)
\]

In other words the plus and minus Bialynicki Birula cell decompositions of \(X\) are filtrable.

Moreover, it can be seen that the image in \(\mathbb{P}(V)\) of \(v + \bigoplus_{j \geq i} V_{\chi_j}\) and \(v + \bigoplus_{j < i} V_{\chi_j}\), intersect transversally in \(\mathbb{P}(V)\) at the \(T\)-fixed point \([v]\). It follows that \(X_+(x_i, \lambda)\) and \(X_-(x_i, \lambda)\) intersect transversally in \(X\) at \(x_i\) for \(1 \leq i \leq m\). Alternately this also follows from the decomposition \(T_{x_i}X = (T_{x_i}X)_- \oplus (T_{x_i}X)_0 \oplus (T_{x_i}X)_+\) of the tangent space of \(X\) at \(x_i\) into weight subspaces with negative, zero and positive weights under the \(\mathbb{G}_m\) action via \(\lambda\), and from the fact that that \((T_{x_i}X)_0\) is zero dimensional, \((T_{x_i}X)_- = T_{x_i}X_- (x_i, \lambda)\) and \((T_{x_i}X)_+ = T_{x_i}X_+(x_i, \lambda)\).

Let \(U_i := X_+(\lambda, x_i)\) and let \(U_i' := X_-(\lambda, x_i)\). Further, let \(V_i := \overline{U_i}\) and \(V_i' := \overline{U_i'}\). Then (2.19) and (2.20) can be rewritten as
\[
V_i \subseteq \bigcup_{j \geq i} U_j \quad (2.21)
\]
and
\[
V_i' \subseteq \bigcup_{j \leq i} U'_j \quad (2.22)
\]
Furthermore, \((U_i)^T = \{ x_i \}, U_i \cong \mathbb{A}^{k_i}, (U_i')^T = \{ x_i \} \) and \( U_i' \cong \mathbb{A}^{n-k_i} \) for every \( 1 \leq i \leq m \).

Then it follows from (2.21) and (2.22) that \( V_i \) and \( V_i' \) are closed \( T \)-invariant subvarieties of \( X \) of dimensions \( k_i \) and \( n - k_i \) respectively. Moreover,

\[
V_i \text{ and } V_i' \text{ intersect transversally along } x_i. \tag{2.23}
\]

It further follows from (2.21) and (2.22) that if \( x_k \in V_i \) then \( k \geq i \) and if \( x_k \in V_j' \) then \( k \leq j \). Thus if the intersection of \( V_i \) with \( V_j' \) is nonempty then it must contain a \( T \)-fixed point being a complete \( T \)-variety. Thus it follows that if

\[
V_i \cap V_j' \neq \emptyset \Rightarrow i \leq j. \tag{2.24}
\]

When \( X \) is a smooth projective toric variety then \( V_i \) and \( V_i' \) are described explicitly in \([17, \text{pages 102–104}]\). When \( X \) is the flag variety then \( V_i \) and \( V_i' \) are the Schubert and the opposite Schubert varieties respectively (see for example \([10, \text{Section 1.2}]\)).

Further let \( X_i = \bigcup_{j=1}^m U_j \) and \( X_i' = \bigcup_{j=1}^i U_j' \) so that

\[
X_i = X \supset X_2 \supset \cdots \supset X_m = \{ x_m \}
\]

and

\[
X_i' = X \supset X_{i-1} \supset \cdots \supset X_1 \]

are the filtrations of \( X \) by closed subvarieties given by the plus and minus Bialynicki-Birula cell decompositions respectively.

We note that \( \mathcal{E} \times X \rightarrow \mathcal{E} \) is a \( T \)-equivariant cellular fibration

\[
\mathcal{E} \times X = \mathcal{E} \times X_1 \supseteq \mathcal{E} \times X_2 \supseteq \cdots \supseteq \mathcal{E} \times X_m = \mathcal{E} \times \{ x_m \}
\]

Further, \( \mathcal{E} \times U_i = \mathcal{E} \times X_i \setminus \mathcal{E} \times X_{i+1} \) is isomorphic to a trivial \( T \)-equivariant vector bundle of rank \( k_i \) over \( \mathcal{E} \times x_i \). There is a dual \( T \)-equivariant cellular fibration structure on \( \mathcal{E} \times X \rightarrow \mathcal{E} \)

\[
\mathcal{E} \times X = \mathcal{E} \times X'_1 \supseteq \mathcal{E} \times X'_{i-1} \supseteq \cdots \supseteq \mathcal{E} \times X'_1 = \mathcal{E} \times \{ x_1 \}
\]

where \( \mathcal{E} \times U'_i = \mathcal{E} \times X'_i \setminus \mathcal{E} \times X'_{i-1} \) is isomorphic to a trivial \( T \)-equivariant vector bundle over \( \mathcal{E} \times x_i \) of rank \( n - k_i \).

For \( 1 \leq i \leq m \), consider \( \mathcal{E}(V_i) := \mathcal{E} \times x_i \subseteq \mathcal{E}(X) \) and \( \mathcal{E}(V'_i) := \mathcal{E} \times x_i \subseteq \mathcal{E}(X) \) which are closed subvarieties of \( \mathcal{E}(X) \) of codimensions \( n - k_i \) and \( k_i \) respectively. The following can be derived respectively from (2.23) and (2.24)

\[
\mathcal{E}(V_i) \text{ and } \mathcal{E}(V'_i) \text{ intersect transversally along } \mathcal{E}(x_i). \tag{2.25}
\]

\[
\mathcal{E}(V_i) \cap \mathcal{E}(V'_i) = \mathcal{E}(V_i \cap V'_i) \neq \emptyset \Rightarrow i \leq j. \tag{2.26}
\]

Notation 1. Since \( X, B \) and hence \( \mathcal{E}(X) \) are smooth, throughout this section and the remaining part of the paper, henceforth we shall let \( K_T(X) := K_0(X) = K_0^T(X), K(X) := K_0(X) = K^0(X), K_\mathcal{E}(x) := K_0^T(\mathcal{E}) = K^0_\mathcal{E}(\mathcal{E}), K_\mathcal{T}(\mathcal{E} \times X) := K_0^T(\mathcal{E} \times X) = K^0_\mathcal{E}(\mathcal{E} \times X), K(B) := K_0(B) = K^0(B) \) and \( \mathcal{K}(\mathcal{E}(X)) := K_0(\mathcal{E}(X)) = K^0(\mathcal{E}(X)) \).

We shall fix a base point \( b_0 \in B \) and identify \( X \) with the fibre \( \pi^{-1}(b_0) \subseteq \mathcal{E}(X) \). We then have the restriction homomorphism \( K(\mathcal{E}(X)) \rightarrow K(X) \). We note here that the classes \( \mathcal{O}_{\mathcal{E}(V_i)} \in K(\mathcal{E}(X)) \) restrict to the classes \( \mathcal{O}_{V_i} \) for \( 1 \leq i \leq m \) which form a basis of \( K(X) \) as a free \( \mathbb{Z} \)-module. The following is a version of Leray–Hirsch theorem in this setting. (See \([29, \text{p. 153}, \text{proof of Theorem 1.2 (iv)}]\) for the case when \( X \) is a smooth projective toric variety.)

**Proposition 2.9.** The classes \( \mathcal{O}_{\mathcal{E}(V_i)} \in K(\mathcal{E}(X)) \) for \( 1 \leq i \leq m \) form a basis of \( K(\mathcal{E}(X)) \) as a free \( K(B) \)-module.
Proof. The classes of structure sheaves of the cell closures \([\mathcal{O}_V]_T\) form a basis of \(K_T(X)\) as an \(R(T)\)-module by \([12, \text{Lemma 5.5.1(b)}]\). Now, under the isomorphism \(K(\mathcal{E}(X)) \cong K_T(\mathcal{E} \times X) \cong K(B) \otimes_{R(T)} K_T(X)\) given by Corollary 2.7 the classes \([\mathcal{O}_{E \times V}]_T\) correspond to \([\mathcal{O}_{E_0 \times V}]_T \in K_T(\mathcal{E} \times X)\) and hence to the class \(1 \otimes [\mathcal{O}_V]_T\) in \(K(B) \otimes_{R(T)} K_T(X)\) for \(1 \leq i \leq m\). Since \(1 \otimes [\mathcal{O}_V]_T\) for \(1 \leq i \leq m\) is a \((K(B))\)-module basis of \((K(B) \otimes_{R(T)} K_T(X)\) the proposition follows.

In the rest of this section we shall assume that \(X\) is a smooth projective variety with an action of the torus \(T\) having finitely many \(T\)-fixed points as well as finitely many \(T\)-invariant curves.

Let \(\mathcal{E} \times T \xrightarrow{\iota} \mathcal{E} \times X\) denote the inclusion of \(T\)-subschemas. In this section we prove a precise form of localization theorem for the \(K\)-ring of the space \(\mathcal{E} \times X\) which generalizes \([33, \text{Theorem 1.3}]\) to the relative setting.

Let \(C_{ij}\) denote the \(T\)-invariant irreducible curve in \(X\) joining the \(T\)-fixed points \(x_i\) and \(x_j\). Choose an isomorphism of \(C_{ij}\) with \(\mathbb{P}^1\) that sends \(x_i\) to 0 and \(x_j\) to \(\infty\) without loss of generality. Further, let \(T\) act on \(C_{ij} \setminus \{x_j\} \cong \mathbb{P}^1 \setminus \\{\infty\}\) via a character \(\chi_{ij}\). Let \(C\) denote the finite collection of invariant curves in \(X\).

Let \(\mathcal{Y}\) denote the subring of \(\prod_{k=1}^m R(T)\) consisting of \((y_k)\) such that \((1 - e^{-\chi_{ij}})\) divides \(y_i - y_j\) for each \(C_{ij} \in C\). Clearly \(\mathcal{Y}\) is an \((R(T))\)-subalgebra of \(\prod_{k=1}^m R(T)\) where \(R(T) \subseteq \prod_{k=1}^m R(T)\) is embedded diagonally.

Let \(\mathcal{Y}_{ij}\) denote the subring of \(\prod_{k=1}^m R(T)\) consisting of \((y_k)\) satisfying the condition that \((1 - e^{-\chi_{ij}})\) divides \(y_i - y_j\) corresponding to \(C_{ij}\) and \(y_k \in R(T)\) is arbitrary for \(k \neq i, j\). Again \(\mathcal{Y}_{ij}\) is an \((R(T))\)-subalgebra of \(\prod_{k=1}^m R(T)\) under the diagonal embedding. Further, by definition

\[
\mathcal{Y} = \bigcap_{C_{ij} \in C} \mathcal{Y}_{ij}. \tag{2.27}
\]

Recall that \(K(B) = K_T(\mathcal{E})\) has a canonical \((R(T))\)-algebra structure via the map that sends the class \([V]\) of a \(T\)-representation to the class \([E \times T]_T\) of the associated vector bundle on \(B\). In particular, \(e^X \in R(T)\) maps to the class \([L_X]\) of the associated line bundle \(L_X := E \times R(T) C_X\).

For \(1 \leq i \leq m\), we have canonical sections \(s_i : B \rightarrow E \times T x_i \subseteq E \times T X\) defined by \(s_i(b) = [e, x_i]\), where \(e \in p^{-1}(b)\). Moreover, \(s_i\) and \(s_j\) can be identified with the sections at 0 and \(\infty\) of the \(\mathbb{P}^1\)-bundle \(E \times T C_{ij}\) on \(B\).

We have

\[
K_T(\mathcal{E} \times T) = K_T\left(\bigcup_{j=1}^m E \times x_j\right) \cong \prod_{j=1}^m K_T(\mathcal{E} \times x_j). \tag{2.28}
\]

Since \(s_k\) maps \(B\) isomorphically onto \(E(x_k) := E \times T x_k\) with inverse \(\pi_k = \pi |_{E(x_k)}\) we have \(K_T(\mathcal{E} \times x_k) = K(E(x_k)) \cong K(B)\) for \(1 \leq k \leq m\). Thus \(s_k^* : K(E(X)) \rightarrow K(B)\) can be identified with the composition of \(\iota^*\) with the projection onto the direct factor \(K_T(E \times x_k) \subseteq K_T(E \times T)\).

Further, \(\iota^*\) can be identified with

\[
(s_k^*) : K_T(\mathcal{E} \times X) \rightarrow \prod_{k=1}^m K(B). \nonumber
\]

The ring \(K(\mathcal{E}(X))\) gets the structure of \((K(B))\)-algebra via pull back by \(\pi^*\), \((K(B))\)-algebra structure on \(\prod_{k=1}^m K(B)\) is via the diagonal inclusion and \(\iota^*\) is a morphism of \((K(B))\)-algebras.

Let \(\mathcal{Y}' := K_T(\mathcal{E}) \otimes_{R(T)} \mathcal{Y}\) and \(\mathcal{Y}_{ij}' := K_T(\mathcal{E}) \otimes_{R(T)} \mathcal{Y}_{ij}\) denote respectively the extension of scalars of the \((R(T))\)-algebras \(\mathcal{Y}\) and \(\mathcal{Y}_{ij}\) to \(K_T(\mathcal{E}) \cong K(B)\). We can identify \(\mathcal{Y}'\) with

\[
\left\{(y'_{ij} \in \prod_{k=1}^m K(B) \mid 1 - \left[ L_{X_{ij}}^\vee\right] \text{ divides } y'_i - y'_j \forall C_{ij} \in C\right\}. \nonumber
\]

Also \(\mathcal{Y}_{ij}'\) can be identified with

\[
\left\{(y'_k \in \prod_{k=1}^m K(B) \mid 1 - \left[ L_{X_{ij}}^\vee\right] \text{ divides } y'_i - y'_j \text{ corresponding to } C_{ij} \in C \text{ and } y'_k \text{ is arbitrary for } k \neq i,j\right\}. \nonumber
\]
In particular, \( \mathcal{Y}' \) and \( \mathcal{Y}_{ij}' \) are \( K(B) \)-subalgebras of \( \prod_{k=1}^{m} K(B) \) under the diagonal embedding. Furthermore, we note that

\[
\mathcal{Y}' = \bigcap_{C_{ij} \in C} \mathcal{Y}_{ij}'.
\]

(2.29)

**Theorem 2.10.** Let \( E \to B \) be a \( T \)-torsor. Then the restriction map

\[
t^*: K_T(\mathcal{E} \times X) \to K_T(\mathcal{E} \times X^T) \cong \prod_{i=1}^{m} K(B)
\]

is injective and the image is isomorphic to the \( K(B) \) subalgebra \( \mathcal{Y}' \).

**Proof.** We first prove the injectivity of \( t^* \). By Proposition 2.9, any element of \( K(\mathcal{E}(X)) \) is uniquely expressible as a free \( K(B) \) module as follows:

\[
\sum_{i=1}^{m} \pi^*(b_i) \cdot [\mathcal{O}_{\mathcal{E}(V_i)}]
\]

where \( b_i \in K(B) \) for \( 1 \leq i \leq m \). Let

\[
t^*\left( \sum_{i=1}^{m} \pi^*(b_i) \cdot [\mathcal{O}_{\mathcal{E}(V_i)}] \right) = 0.
\]

Let \( k \geq 1 \) be the least so that \( b_k \neq 0 \). Since \( t^* \) is a \( K(B) \)-algebra homomorphism this implies that

\[
t^*\left( [\mathcal{O}_{\mathcal{E}(V'_k)}] \right) \cdot \sum_{i=k}^{m} \pi^*(b_i) \cdot [\mathcal{O}_{\mathcal{E}(V'_i)}] = 0.
\]

(2.30)

By (2.26), in \( K(\mathcal{E}(X)) \) we have

\[
[\mathcal{O}_{\mathcal{E}(V'_i)}] \cdot [\mathcal{O}_{\mathcal{E}(V'_j)}] = 0
\]

(2.31)

whenever \( i > k \). Thus from (2.25) and (2.31), we get

\[
[\mathcal{O}_{\mathcal{E}(V'_k)}] \cdot \sum_{i=k}^{m} \pi^*(b_i) \cdot [\mathcal{O}_{\mathcal{E}(V'_i)}] = \pi^*(b_k) \cdot [\mathcal{O}_{\mathcal{E}(x_k)}].
\]

Thus (2.30) implies that

\[
t^*\left( \pi^*(b_k) \cdot [\mathcal{O}_{\mathcal{E}(x_k)}] \right) = 0.
\]

(2.32)

Furthermore, \( t^* \) is a \( K(B) \)-algebra homomorphism where the \( K(B) \)-algebra structure on \( \prod_{i=1}^{m} K(B) \) is via the diagonal embedding. Thus (2.32) implies that

\[
b_k \cdot t^*\left( [\mathcal{O}_{\mathcal{E}(x_k)}] \right) = 0.
\]

(2.33)

Since \( t^* = (s_k^*) \), (2.33) in particular implies that

\[
b_k \cdot s_k^*\left( [\mathcal{O}_{\mathcal{E}(x_k)}] \right) = 0.
\]

(2.34)

Now,

\[
s_k^*\left( [\mathcal{O}_{\mathcal{E}(x_k)}] \right) = [\mathcal{O}_B]
\]
since \( s_k \) is an isomorphism of varieties from \( B \) onto \( \mathcal{E}(x_k) \) whose inverse is \( \pi_k = \pi |_{\mathcal{E}(x_k)} \). Thus from (2.34) we get \( b_k \cdot [\mathcal{O}_B] = 0 \) in \( K(B) \). This implies that \( b_k = 0 \) which is a contradiction to our assumption. Thus we conclude that \( t' \) is injective.

Now, by Theorem 2.2 we have \( K_T(\mathcal{E}) \otimes_{R(T)} K_T(X) \cong K_T(\mathcal{E} \times X) \). Moreover, \( t = \text{id}_\mathcal{E} \times t' : (\mathcal{E} \times X)^T = \mathcal{E} \times X \hookrightarrow \mathcal{E} \times X \) where \( t' \) denotes the inclusion \( X^T \hookrightarrow X \). Thus \( t'' \) can be identified with \( \text{id}_{K(B)} \otimes t'' \).

More explicitly, by (2.28) it follows that

\[
K_T(\mathcal{E} \times X^T) \cong \prod_{j=1}^{m} K_T(\mathcal{E} \times x_j) \cong \prod_{j=1}^{m} K_T(\mathcal{E}) \otimes_{R(T)} K_T(x_j) \cong K_T(\mathcal{E}) \otimes_{R(T)} \left( \prod_{j=1}^{m} K_T(x_j) \right) \cong K_T(\mathcal{E}) \otimes_{R(T)} K_T(X^T).
\]

By [32, Theorem 1.3], the image of the restriction map \( t'' : K_T(X) \hookrightarrow K_T(X^T) \cong \prod_{i=1}^{m} R(T) \) is identified with the \( R(T) \)-subalgebra \( \mathcal{Y} \). It follows that the image of \( t'' : K_T(\mathcal{E}) \otimes_{R(T)} K_T(X) \hookrightarrow K_T(\mathcal{E}) \otimes_{R(T)} K_T(X^T) \) can be identified with \( \mathcal{Y}' \). Hence the theorem.

We have the following geometric interpretation of Theorem 2.10.

**Corollary 2.11.** We have a canonical embedding of \( K(B) \)-algebras

\[
K(\mathcal{E}(X)) \hookrightarrow \prod_{i=1}^{m} K(B).
\]

Furthermore, the image of \( t'' \) is the intersection of the images of

\[
K(\mathcal{E}(C_{ij})) \hookrightarrow K(\mathcal{E} \times_T x_i) \times K(\mathcal{E} \times_T x_j) \hookrightarrow K(\mathcal{E} \times_T X^T) \cong \prod_{i=1}^{m} K(B).
\]

**Proof.** Recall that we can identify \( K_T(\mathcal{E} \times X) \) with \( K(\mathcal{E}(X)) = (\mathcal{E} \times X)/T \) and \( K_T(\mathcal{E} \times x_i) = K(\mathcal{E} \times x_i) \) canonically with the ring \( K(B \times \mathcal{E}/T) \) for every \( 1 \leq i \leq m \). Furthermore, Theorem 2.10 applied for the principal \( T \)-bundle \( \mathcal{E} \rightarrow B \) and the smooth projective \( T \)-variety \( C_{ij} = \mathbb{P}^1 \) implies that \( K_T(\mathcal{E} \times C_{ij}) = K(\mathcal{E}(C_{ij})) \) embeds in \( K(\mathcal{E} \times_T x_i) \times K(\mathcal{E} \times_T x_j) \hookrightarrow K_T(\mathcal{E}) \otimes_{R(T)} K_T(X^T) \) and its image is isomorphic to the subring \( \mathcal{Y}'_{ij} \). The proof now follows readily from (2.27) and Theorem 2.10.

2.3 Some further notations

In this subsection and the remaining part of the paper we shall assume that \( G \) is an arbitrary reductive algebraic group and drop the additional assumption of Subsections 1.1 and 1.2 that \( \pi_1(G) \) is torsion free.

Let \( W \) denote the Weyl group and \( \Phi \) denote the root system of \( (G, T) \). We further have the subset \( \Phi^+ \) of positive roots fixing \( B \supseteq T \), and its subset \( \Delta = \{ \alpha_1, \ldots, \alpha_r \} \) of simple roots where \( r \) is the semisimple rank of \( G \). For \( \alpha \in \Delta \) we denote by \( s_\alpha \) the corresponding simple reflection. For any subset \( I \subseteq \Delta \), let \( W_I \) denote the subgroup of \( W \) generated by all \( s_\alpha \) for \( \alpha \in I \). At the extremes we have \( W_\emptyset = \{ 1 \} \) and \( W_\Delta = W \).

Let \( \Lambda := X^+(T) \). Then \( R(T) \) (the representation ring of the torus \( T \)) is isomorphic to the group algebra \( \mathbb{Z}[\Lambda] \). Let \( e_\lambda \) denote the element of \( \mathbb{Z}[\Lambda] = R(T) \) corresponding to a weight \( \lambda \in \Lambda \). Then \( \{ e_\lambda \}_{\lambda \in \Lambda} \) is a basis of the \( \mathbb{Z} \) module \( \mathbb{Z}[\Lambda] \). Further, since \( W \) acts on \( X^+(T) \), on \( \mathbb{Z}[\Lambda] \) we have the following natural action of \( W \) given by \( \omega(\lambda) = \omega e_\lambda \) for each \( \omega \in W \) and \( \lambda \in \Lambda \). Recall that we can identify \( R(G) \) with \( R(T)^W \) via restriction to \( T \), where \( R(T)^W \) denotes the subring of \( R(T) \) invariant under the action of \( W \) (see [23, Example 1.19]).

Recall from [21, Corollary 3.7] that there exists an exact sequence:

\[
1 \rightarrow \mathcal{Z} \rightarrow \tilde{G} := \tilde{C} \times G^{ss} \overset{\pi}{\rightarrow} G \rightarrow 1 \tag{2.35}
\]

where \( \mathcal{Z} \) is a finite central subgroup, \( \tilde{C} \) is a torus and \( G^{ss} \) is semisimple and simply-connected. The condition that \( G^{ss} \) is simply connected implies that \( \tilde{G} \) is factorial (see [23]).
Now, from (2.35) it follows that $\bar{B} := \pi^{-1}(B)$ and $\bar{T} := \pi^{-1}(T)$ are respectively a Borel subgroup and a maximal torus of $\tilde{G}$. Further, by restricting the map $\pi$ to $\bar{T}$ we get the following exact sequence:

$$1 \to \mathcal{Z} \to \tilde{T} \to T \to 1. \quad (2.36)$$

Let $\bar{W}$ and $\bar{\Phi}$ denote respectively the Weyl group and the root system of $(\tilde{G}, \bar{T})$. Then by (2.35), it also follows that $\bar{W} = W$ and $\bar{\Phi} = \Phi$. Further we have

$$R(\tilde{G}) = R(\bar{C}) \bigotimes R(G^{ss}) \quad (2.37)$$

and

$$R(\tilde{T}) \cong R(\bar{C}) \bigotimes R(T^{ss}) \quad (2.38)$$

where $T^{ss}$ is the maximal torus $\bar{T} \cap G^{ss}$ of $G^{ss}$.

Recall we can identify $R(\tilde{G})$ with $R(\bar{T})^W$ via restriction to $\bar{T}$, and further $R(\bar{T})^W$ is a free $R(\bar{G})$-$\Phi$ module of rank $|W|$ (see [28, Theorem 2.2]). Moreover, since $G^{ss}$ is semi-simple and simply connected, $R(G^{ss}) \cong \mathbb{Z}[x_1, \ldots, x_r]$ is a polynomial ring on the fundamental representations ([23, Example 1.20]). Hence $R(\tilde{G}) = R(\bar{C}) \bigotimes R(G^{ss})$ is the tensor product of a polynomial ring and a Laurent polynomial ring, and hence a regular ring of dimension $r + \text{dim}(\bar{C}) = \text{rank}(G)$ where $r$ is the rank of $G^{ss}$.

For $X$ any $G$-scheme, we shall consider the $\bar{T}$ and $\tilde{G}$-equivariant $K$-theory of $X$ where we take the natural actions of $\bar{T}$ and $\tilde{G}$ on $X$ through the canonical surjections to $T$ and $G$ respectively.

We consider $\mathbb{Z}$ as an $R(\bar{G})$-$\Phi$ module by the augmentation map $\varepsilon : R(\bar{G}) \to \mathbb{Z}$ which maps any $\bar{G}$-representation $V$ to $\text{dim}(V)$. Moreover, we have the natural restriction homomorphisms $K_{\bar{G}}(X) \to K_{\bar{T}}(X)$ and $K_{\tilde{G}}(X) \to K(X)$ where $K(X)$ denotes the ordinary Grothendieck ring of algebraic vector bundles on $X$. We then have the following isomorphisms (see [23, Proposition 4.1 and Theorem 4.2]).

$$R(\bar{T}) \bigotimes_{R(\bar{G})} K_{\bar{G}}(X) \cong K_{\bar{T}}(X), \quad (2.39)$$

$$K_{\bar{G}}(X) \cong K_{\bar{T}}(X)^W, \quad (2.40)$$

$$\mathbb{Z} \bigotimes_{R(\bar{G})} K_{\bar{G}}(X) \cong K(X). \quad (2.41)$$

Let $R(\bar{T})^{W_I}$ denote the invariant subring of the ring $R(\bar{T})$ under the action of the subgroup $W_I$ of $W$ for every $I \subset \Delta$. Thus in particular we have, $R(\bar{T})^{W} = R(\bar{G})$ and $R(\bar{T})^{\{1\}} = R(\bar{T})$. Further, for every $I \subset \Delta$, $R(\bar{T})^{W_I}$ is a free module over $R(\bar{G}) = R(\bar{T})^W$ of rank $|W/W_I|$ (see [28, Theorem 2.2]). Indeed, [28, Theorem 2.2] which we apply here holds for $R(T^{ss})$.

However, since $W$ acts trivially on the central torus $\bar{C}$ and hence trivially on $R(\bar{C})$ we have

$$R(\bar{T})^{W_I} = R(\bar{C}) \bigotimes R(T^{ss})^{W_I} \quad (2.42)$$

for every $I \subseteq \Delta$, and hence we obtain the analogous statement for $R(\bar{T})$.

Let $W^I$ denote the set of minimal length coset representatives of the parabolic subgroup $W_I$ for every $I \subset \Delta$. Then

$$W^I := \{ w \in W \mid l(wv) = l(w) + l(v) \ \forall \ v \in W_I \} = \{ w \in W \mid w(\Phi^+_I) \subset \Phi^+ \}$$

where $\Phi_I$ is the root system associated to $W_I$, where $I$ is the set of simple roots. Recall (see [20, p. 19]) that we also have:

$$W^I = \{ w \in W \mid l(uss) > l(w) \ \text{for all} \ s \in I \},$$
Note that \( J \subseteq I \) implies that \( W^{\Delta \setminus J} \subseteq W^{\Delta \setminus I} \). Let

\[
C^I := W^{\Delta \setminus I} \setminus \left( \bigcup_{J \subseteq I} W^{\Delta \setminus J} \right). \tag{2.43}
\]

Let \( \alpha_1, \ldots, \alpha_r \) be an ordering of the set \( \Delta \) of simple roots and \( \omega_1, \ldots, \omega_r \) denote respectively the corresponding fundamental weights for the root system of \( (G^{ss}, T^{ss}) \). Since \( G^{ss} \) is simply connected, the fundamental weights form a basis for \( X^* (T^{ss}) \) and hence for every \( \lambda \in X^* (T^{ss}) \), \( e^\lambda \in R (T^{ss}) \) is a Laurent monomial in the elements \( e^{\omega_i} : 1 \leq i \leq r \).

In [28, Theorem 2.2] Steinberg has defined a basis \( \{ f^J_v : v \in W^J \} \) of \( R (T^{ss})^W \) as an \( R (T^{ss})^W \)-module. We recall here this definition: For \( v \in W^I \) let

\[
p_v := \prod_{\alpha_i < 0, v^{-1} \alpha_i < 0} e^{\omega_i} \in R(\tilde{T}). \tag{2.44}
\]

Then

\[
f^J_v := \sum_{x \in W^J (v) \setminus W^I} x^{-1} v^{-1} p_v \tag{2.45}
\]

where \( W_J (v) \) denotes the stabilizer of \( v^{-1} p_v \) in \( W_J \).

We shall also denote by \( \{ f^J_v : v \in W^J \} \) the corresponding basis of \( R (\tilde{T})^W_J \) as an \( R (\tilde{T})^W \)-module where it is understood that

\[
f^J_v := \bigotimes_{v \in C^I} f^J_v \in R (\tilde{G}) \bigotimes R (T^{ss})^W_J. \tag{2.46}
\]

**Notation 2.** Whenever \( v \in C^I \) we denote \( f^I_{v \setminus J} \) simply by \( f_v \). We can drop the superscript in the notation without any ambiguity since \( \{ C^I : I \subseteq \Delta \} \) are disjoint. Therefore with the modified notation [32, Lemma 1.10] implies that: \( \{ f_v : v \in W^{\Delta \setminus J} = \bigcup_{J \subseteq \Delta} C^J \} \) form an \( R (\tilde{T})^W \)-basis for \( R (\tilde{T})^W_{\Delta \setminus J} \) for every \( I \subseteq \Delta \). Further, let

\[
R (\tilde{T})_I := \bigoplus_{v \in C^I} R (\tilde{T})^W \cdot f_v. \tag{2.47}
\]

In \( R (T) \) let

\[
f_v \cdot f_{v'} = \sum_{J \subseteq (I \cup I')} \sum_{w \in C^J} a^w_{v,v'} \cdot f_w \tag{2.48}
\]

for certain elements \( a^w_{v,v'} \in R (\tilde{G}) = R (\tilde{T})^W \forall v \in C^I, v' \in C^{I'} \) and \( w \in C^J, J \subseteq (I \cup I') \).

**Remark 2.12.** In the next section, for a regular compactification \( X \) of \( G \) we shall consider the action of \( \tilde{G} \times \tilde{G} \) on \( X \) via its canonical surjection to \( G \times G \) and further consider \( K_{\tilde{G} \times \tilde{G}}(X) \) instead of \( K_{G \times G} (X) \). This is in order to apply the results in Subsections 1.1 and 1.2, since \( \pi_1 (\tilde{G}) \) is torsion free. Moreover, this also enables us to use the Steinberg basis defined in Notation 2 and its structure constants (2.48) in the description of the multiplicative structure of \( K_{\tilde{G} \times \tilde{G}} (X) \).

### 3 K-THEORY OF BUNDLES WITH FIBRE REGULAR COMPACTIFICATIONS OF G

In this section \( X \) denotes a projective regular compactification of \( G \).

Let \( \overline{T} \) denote the closure of \( T \) in \( X \). It is known that for the left action of \( T \) (i.e. for the action of \( T \times \{ 1 \} \)), \( \overline{T} \) is a smooth projective toric variety. (see [8]). Moreover, \( X^{T \times T} \) is contained in the union \( X_c \) of all closed \( G \times G \)-orbits in \( X \); moreover all such orbits are isomorphic to \( G / B^- \times G / B \).

Let \( F \) be the fan associated to \( \overline{T} \) in \( X_c (T) \otimes \mathbb{R} \). Since \( \overline{T} \) is complete, \( F \) is a subdivision of \( X_c (T) \otimes \mathbb{R} \). There is a canonical action of \( \text{diag}(W) := \{ (w_1, w_2) \in W \times W \mid w_1 = w_2 \} \) on \( \overline{T} \) induced from the conjugation action on \( T \) which corresponds...
to an action of $W$ on $F$. By [8, Proposition A2], it follows that $F = WP_+$ where $P_+$ is the subdivision of the positive Weyl chamber formed by the cones in $F$ contained in this chamber. Therefore $F$ is a smooth subdivision of the fan associated to the Weyl chambers, and $W$ acts on $F$ by reflection about the Weyl chambers. Let $\tilde{T}^+$ denote the toric variety associated to the fan $P_+$.

Now, since $X$ is projective, the canonical morphism $f : X \longrightarrow G_{ad}$ is projective. Also, $\tilde{T}^+$ is the inverse image of $\mathbb{A}^r$ under $f$ and the restriction $g : \tilde{T}^+ \longrightarrow \mathbb{A}^r$ of $f$ is a projective morphism of toric varieties. This implies that $\tilde{T}^+$ is a semi-projective $T$-toric variety.

Let $\mathbb{A}^r(\ell)$ denote the set of maximal cones of $\mathbb{A}^r$. Then we know that $\mathbb{A}^r(\ell)$ parameterizes the closed $G \times G$-orbits in $X$. Hence $X^{T \times T}$ is parametrized by $F_+(l) \times W \times W$ (see [8, Proposition A1 and A2]).

Recall by [34, Theorem 2] and [32, Theorem 2.1], that $K_{\tilde{T} \times \tilde{T}}(X)$ embeds into $K_{\tilde{T} \times \tilde{T}}(X_{\tilde{T}})$, the latter being a product of copies of the ring $K_{\tilde{T} \times \tilde{T}}(G/B^- \times G/B)$.

Let $\mathcal{Y}$ denote

$$(f_{\sigma, u, v}) \in \prod_{\sigma \in F_+(l)} \prod_{u, v \in W} K_{\tilde{T} \times \tilde{T}}(X) = K_{\tilde{T} \times \tilde{T}}(X^{T \times T})$$

satisfying the congruences:

(i) $f_{\sigma, u, v} \equiv f_{\sigma, u, v} \pmod{1 - e^{-u(\alpha)} \otimes e^{-v(\alpha)}}$ whenever $\alpha \in \Delta$ and the cone $\sigma \in F_+(l)$ has a facet orthogonal to $\alpha$, and that

(ii) $f_{\sigma, u, v} \equiv f_{\sigma', u, v} \pmod{1 - e^{-\chi}}$ whenever $\chi \in X^*(T)$ and the cones $\sigma$ and $\sigma' \in F_+(l)$ have a common facet orthogonal to $\chi$.

(In (ii), $\chi$ is viewed as a character of $T \times T$ which is trivial on $\text{diag}(T)$ and hence is a character of $T$.)

Then $\mathcal{Y}$ is an $R(\tilde{T}) \otimes R(\tilde{T})$-subalgebra of $K_{\tilde{T} \times \tilde{T}}(X^{T \times T})$ (see [32]).

In this section we consider $\mathcal{E} \longrightarrow B$ as a $\widetilde{G} \times \widetilde{G}$-torsor over a scheme $B$. We shall consider the associated fibre bundle $\mathcal{E} \times \widetilde{G} \times \widetilde{G} X$ with fibre the regular compactification $X$ of $G$ in view of Remark 2.12.

The following proposition is the relative version of [32, Theorem 2.1].

**Theorem 3.1.** Let $X$ be a projective regular compactification of $G$ and let $\mathcal{E} \longrightarrow B$ be a $\widetilde{G} \times \widetilde{G}$-torsor. The map

$$\prod_{\sigma \in F_+(l)} t_{\sigma} : K_{\tilde{T} \times \tilde{T}}(\mathcal{E} \times X) \longrightarrow \prod_{\sigma \in F_+(l)} K_{\tilde{T} \times \tilde{T}}(\mathcal{E} \times G/B^- \times G/B)$$

is injective and its image is $K(\mathcal{E} / \tilde{T} \times \tilde{T}) \otimes R(\tilde{T}) \otimes R(\tilde{T}) \mathcal{Y}$.

**Proof.** Consider the $\tilde{T} \times \tilde{T}$-torsor $\mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{E} \times \mathcal{E} / \tilde{T} \times \tilde{T}$. Consider the action of $\tilde{T} \times \tilde{T}$ on $X$ by restriction. By Theorem 2.10 we have

$$t : K_{\tilde{T} \times \tilde{T}}(\mathcal{E} \times X) \longrightarrow K_{\tilde{T} \times \tilde{T}}(\mathcal{E} \times X^T)$$

is injective. Further, by applying Theorem 2.2 on either side of 3.2 we get

$$K_{\tilde{T} \times \tilde{T}}(\mathcal{E} \times X) \cong K(\mathcal{E} / \tilde{T} \times \tilde{T}) \otimes K_{\tilde{T} \times \tilde{T}}(X) \longrightarrow K(\mathcal{E} / \tilde{T} \times \tilde{T}) \otimes K_{\tilde{T} \times \tilde{T}}(X^T) \cong K_{\tilde{T} \times \tilde{T}}(\mathcal{E} \times X^T)$$

(3.3)

can be identified with the map $id_{K(\mathcal{E} / \tilde{T} \times \tilde{T})} \otimes t'$, where $t'$ is the map $K_{\tilde{T} \times \tilde{T}}(X) \longrightarrow K_{\tilde{T} \times \tilde{T}}(X^T)$ induced by restriction to the $T$-fixed points. By [32, Theorem 2.1] the image of $t'$ lies in

$$\prod_{\sigma \in F_+(l)} K_{\tilde{T} \times \tilde{T}}(G/B^- \times G/B)$$
and can be identified with \( \mathcal{Y} \). Thus it follows that the image of \( \iota \) lies in

\[
K(\mathcal{E}/T \times \overline{T}) \bigotimes_{R(T)} \prod_{\sigma \in \mathcal{P}_+(l)} K_{T \times T}(G/B^- \times G/B)
\]

and can be identified with \( K(\mathcal{E}/T \times \overline{T}) \bigotimes_{R(T)} \prod_{\sigma \in \mathcal{P}_+(l)} K_{T \times T}(G/B^- \times G/B) \). \( \Box \)

Let \( \mathcal{Z} \) consist in all families \( (f_\sigma)_{\sigma \in \mathcal{P}_+(l)} \) of elements of \( R(T \times 1) \bigotimes R(\mathfrak{diag}(\overline{T})) \) such that

(i) \((1, x_\alpha) f_\sigma(u, v) \equiv f_\sigma(u, v) (\mod (1 - e^{-\alpha(u)}))\) whenever \( \alpha \in \Delta \) and the cone \( \sigma \in \mathcal{P}_+(l) \) has a facet orthogonal to \( \alpha \), and that

(ii) \( f_\sigma \equiv f_{\sigma'} (\mod (1 - e^{-\chi}))\) whenever \( \chi \in X^*(T) \) and the cones \( \sigma \) and \( \sigma' \in \mathcal{P}_+(l) \) have a common facet orthogonal to \( \chi \).

In particular, \( \mathcal{Z} \) is \( R(G) \bigotimes R(G) \)-subalgebra of \( \prod_{\sigma \in \mathcal{P}_+(l)} R(T \times 1) \bigotimes R(\overline{T}) \).

The following is the relative version of [32, Corollary 2.2].

**Proposition 3.2.**

(i) We have a canonical inclusion

\[
K(\mathcal{E}(X)) \hookrightarrow \prod_{\sigma \in \mathcal{P}_+(l)} K(\mathcal{E}/B^- \times \overline{B}).
\]

Here \( K(\mathcal{E}/B^- \times \overline{B}) \) is the K-ring of the bundle \( \mathcal{E}(G/B^- \times G/B) \) over \( B \) with fibre \( G/B^- \times G/B \).

(ii) The image of \( K(\mathcal{E}(X)) \) in the above inclusion is identified with \( K(\mathcal{E} \times \overline{T}^+) \) which is the K-ring of a toric bundle with fibre \( \overline{T}^+ \) over \( E/B^- \times B = E(G/B^- \times G/B) \).

(iii) The ring \( K(\mathcal{E}(X)) \) is further isomorphic to \( K(B) \bigotimes_{R(\mathfrak{g}) \bigotimes R(\mathfrak{g})} \mathcal{Z} \).

**Proof.**

(i) By taking \( W \times W \)-invariants on either side of (3.1) in Proposition 3.1 we get the inclusion

\[
[K_{T \times T}(\mathcal{E} \times X)]^{W \times W} \hookrightarrow \prod_{\sigma \in \mathcal{P}_+(l)} [K_{T \times T}(\mathcal{E} \times G/B^- \times G/B)]^{W \times W}.
\]

Now, by applying [32, Theorem 1.8] or [23, Proposition 4.1] on either side of (3.5) we get:

\[
K_{\mathfrak{g} \times \mathfrak{g}}(\mathcal{E} \times X) \hookrightarrow \prod_{\sigma \in \mathcal{P}_+(l)} K_{\mathfrak{g} \times \mathfrak{g}}(\mathcal{E} \times G/B^- \times G/B).
\]

This is further equivalent to

\[
K(\mathcal{E} \times \mathfrak{g} \times \mathfrak{g} \times X) \hookrightarrow \prod_{\sigma \in \mathcal{P}_+(l)} [K(\mathcal{E} \times \mathfrak{g} \times \mathfrak{g}) \mathfrak{g}/B^- \times \mathfrak{g}/B = K(\mathcal{E}/B^- \times B)]
\]

and (3.4) follows.

(ii) Recall that we have a split exact sequence

\[
1 \longrightarrow \mathfrak{diag} \overline{T} \longrightarrow T \times \overline{T} \longrightarrow \overline{T} \longrightarrow 1
\]
where the second map is given by \((t_1, t_2) \mapsto t_1 \cdot t_2^{-1}\) and the splitting given by \(t \mapsto (t, 1)\). Thus we get canonical isomorphism

\[
R(\text{diag } \overline{T}) \otimes R(\overline{T} \times 1) \cong R(\overline{T} \times \overline{T}).
\]

(3.8)

Using the change of variables coming from (3.8), \([33, \text{Proposition 2.1}]\) implies that the image of \(K_{\tilde{G} \times \tilde{G}}(X)\) in \(\prod_{\sigma \in P_+(l)} [K_{\tilde{G} \times \tilde{G}}(G/B^- \times G/B) = R(\overline{T}) \otimes R(\overline{T})]\) can be identified with \(K_{\overline{T}}(\overline{T}^+) \otimes R(\overline{T})\). Note that Corollary 2.8 implies

\[
K_{\tilde{G} \times \tilde{G}}(\mathcal{E} \times X) \cong K_{\tilde{G} \times \tilde{G}}(\mathcal{E}) \otimes_{R(\tilde{G}) \otimes R(\tilde{G})} K_{\tilde{G} \times \tilde{G}}(X)
\]

(3.9)

and

\[
K_{\tilde{G} \times \tilde{G}}(\mathcal{E} \times G/B^- \times G/B) \cong K_{\tilde{G} \times \tilde{G}}(\mathcal{E}) \otimes_{R(\tilde{G}) \otimes R(\tilde{G})} K_{\tilde{G} \times \tilde{G}}(\tilde{G}/B^- \times \tilde{G}/B).
\]

(3.10)

Thus under the inclusion (3.6) the image of

\[
K_{\tilde{G} \times \tilde{G}}(\mathcal{E}) \otimes_{R(\tilde{G}) \otimes R(\tilde{G})} K_{\tilde{G} \times \tilde{G}}(X)
\]

in

\[
K_{\tilde{G} \times \tilde{G}}(\mathcal{E}) \otimes_{R(\tilde{G}) \otimes R(\tilde{G})} \prod_{\sigma \in P_+(l)} R(\overline{T}) \otimes R(\overline{T})
\]

can be identified with

\[
K_{\tilde{G} \times \tilde{G}}(\mathcal{E}) \otimes_{R(\tilde{G}) \otimes R(\tilde{G})} K_{T}(\overline{T}^+) \otimes R(\overline{T}).
\]

(3.11)

By Theorem 2.2, (3.11) can further be identified with

\[
K\left(\mathcal{E} \times_{\tilde{G} \times \tilde{G}} \left(\tilde{G} \times \tilde{G} \times_{\tilde{B}^- \times \tilde{B}} \overline{T}^+ \times pt\right)\right) = K\left(\mathcal{E} \times_{\tilde{B}^- \times \overline{T}} \overline{T}^+\right),
\]

where \(\tilde{B}^- \times \tilde{B}\) acts on \(\overline{T}^+\) via the canonical projection to \(\overline{T} \times 1\).

(iii) Since \(Z \cong K_{\tilde{G} \times \tilde{G}}(X)\) by \([32, \text{Proposition 2.5}]\) and \(K(B) \cong K_{\tilde{G} \times \tilde{G}}(\mathcal{E})\), the claim readily follows from (3.9). \(\square\)

### 3.1 First description of \(K(\mathcal{E}(X))\)

Recall that

\[
\mathcal{E}/(\tilde{B}^- \times \tilde{B}) = \mathcal{E} \times_{\tilde{G} \times \tilde{G}} (\tilde{G} \times \tilde{G})/(\tilde{B}^- \times \tilde{B})
\]

is a bundle with fibre \(\tilde{G}/\tilde{B}^- \times \tilde{G}/\tilde{B}\) over \(B\). Thus \(K(\mathcal{E}(\tilde{G}/\tilde{B}^- \times \tilde{G}/\tilde{B}))\) gets a \(R(\tilde{B}^-) \times R(\tilde{B})\)-module structure by sending a representation \(\mathcal{V} \otimes \mathcal{W}\) of \(\tilde{B}^- \times \tilde{B}\) to the associated vector bundle \(\mathcal{E} \times_{\tilde{B}^- \times \tilde{B}} (\mathcal{V} \otimes \mathcal{W})\) on \(\mathcal{E}/(\tilde{B}^- \times \tilde{B})\). Since \(\tilde{G} \times \tilde{G} \times_{\tilde{B}^- \times \tilde{B}} \mathcal{V} \otimes \mathcal{W}\) is a \(\tilde{G} \times \tilde{G}\)-linearized vector bundle on the space \(\tilde{G}/\tilde{B}^- \times \tilde{G}/\tilde{B}\). This is also the associated bundle \(\mathcal{E} \times_{\tilde{G} \times \tilde{G}} (\tilde{G} \times \tilde{G} \times_{\tilde{B}^- \times \tilde{B}} \mathcal{V} \otimes \mathcal{W})\). Let \(K_{\tilde{B}^- \times \overline{T}}(\overline{T})\) denote the \(\tilde{B}^- \times \tilde{B}\)-equivariant K-ring of \(\overline{T}\) where we take the natural action of \(\tilde{B}^- \times \tilde{B}\) on \(\overline{T}\) via the canonical projection to \(T \times T\).

We now prove the first main theorem of this section.
Theorem 3.3. Let $\mathcal{E} \rightarrow B$ be a $\tilde{G} \times \tilde{G}$-torsor. Consider the associated bundle $\mathcal{E}(X) := \mathcal{E} \times_{\tilde{G} \times \tilde{G}} X$ with fibre the regular compactification $X$ of $G$ over $B$. Here again the action on $\tilde{G} \times \tilde{G}$ on $X$ is via the natural projection to $G \times G$. The ring $K(\mathcal{E}(X))$ is isomorphic to the ring

$$K\left(\mathcal{E} \times_{\tilde{B} \times \tilde{B}} \tilde{T}\right)^{\text{diag}(W)} = \left[K(\mathcal{E}(\tilde{G} \times \tilde{G})/(\tilde{B}^{-} \times \tilde{B})) \otimes_{R(\tilde{B}^{-}) \otimes R(\tilde{B})} K_{\tilde{B}^{-} \times \tilde{B}}(\tilde{T})\right]^{\text{diag}(W)} \quad (3.12)$$

as a $K(B)$-module.

Proof. By Corollary 2.8 we have

$$K(\mathcal{E}(X)) \cong K(B) \otimes_{R(\tilde{G} \times \tilde{G})} K_{\tilde{G} \times \tilde{G}}(X). \quad (3.13)$$

By [32, Corollary 2.3] (3.13) implies

$$K(\mathcal{E}(X)) \cong K(B) \otimes_{R(\tilde{G} \times \tilde{G})} K_{\tilde{T}}(\tilde{T})^{\text{diag}(W)}. \quad (3.14)$$

By [23, Corollary 2.15] this can further be rewritten as

$$K(\mathcal{E}(X)) \cong K(B) \otimes_{R(\tilde{G} \times \tilde{G})} K_{\tilde{B}^{-} \times \tilde{B}}(\tilde{T})^{\text{diag}(W)}. \quad (3.15)$$

and

$$K(\mathcal{E}(X)) \cong K(B) \otimes_{R(\tilde{G} \times \tilde{G})} K_{\tilde{G} \times \tilde{G}}(\tilde{G} \times \tilde{G} \times_{\tilde{B}^{-} \times \tilde{B}} \tilde{T})^{\text{diag}(W)}. \quad (3.16)$$

Since $\text{diag}(W)$ acts trivially on $B$ and hence on $K(B)$, by Proposition 2.6 it follows that the right hand side of (3.16) is isomorphic to $K\left(\mathcal{E} \times_{\tilde{G} \times \tilde{B}} (\tilde{G} \times \tilde{G} \times_{\tilde{B}^{-} \times \tilde{B}} \tilde{T})\right)^{\text{diag}(W)}$. This reduces to $K\left(\mathcal{E} \times_{\tilde{B}^{-} \times \tilde{B}} \tilde{T}\right)^{\text{diag}(W)}$. Now, (3.12) follows by applying Corollary 2.7 to the $\tilde{B}^{-} \times \tilde{B}$-torsor $\mathcal{E} \rightarrow \mathcal{E} / (\tilde{B}^{-} \times \tilde{B})$ and the associated $\tilde{T}$-bundle. \qed

3.2 Second description of $K(\mathcal{E}(X))$

We first set up some notations necessary to state the main theorem.

Consider the ring

$$\mathcal{K} := K\left(\mathcal{E} \times_{\tilde{B} \times \tilde{G}} \tilde{T}^{+}\right) \quad (3.17)$$

where $\tilde{B} \times \tilde{G}$ acts on $\tilde{T}^{+}$ via the canonical projection $\tilde{B} \times \tilde{G} \rightarrow \tilde{T} \times 1$. The ring (3.17) is the $K$-ring of a $\tilde{T}^{+}$-bundle over the flag bundle $\mathcal{E} / \tilde{B} \times \tilde{G} = \mathcal{E} \times_{\tilde{G} \times \tilde{G}} (\tilde{G} / \tilde{B} \times \text{pt})$ over $B$. Since $\tilde{T}^{+}$ is a semi-projective toric variety, by [33, Theorem 4.1] the ring $\mathcal{K}$ gets a $K(\mathcal{E} / \tilde{G} \times \tilde{B})$-algebra structure.

Let $\overline{f}_v := 1 \otimes (1 \otimes f_v) \in K(B) \otimes_{R(\tilde{G})} R(\tilde{G}) \otimes R(\tilde{T}) = K(\mathcal{E} / \tilde{G} \times \tilde{B}) \quad (3.18)$
where \( f_v \in \mathcal{R}(\overline{T}) = K_{\overline{G}}(G/B) \) is as in Notation 2. Note that \( \mathcal{E}/\overline{G} \times \overline{B} = \mathcal{E} \times_{\overline{G} \times \overline{G}} (pt \times \overline{G}/\overline{B}) \) is a flag bundle over \( B = \mathcal{E}/\overline{G} \times \overline{G} \).

Let \( \lambda : = 1 \bigotimes (\mu \bigotimes 1) \in K(B) \bigotimes \mathcal{R}(\overline{T}) \bigotimes \mathcal{R}(\overline{G}) = K(\mathcal{E}/\overline{B} \times \overline{G}) \quad (3.19) \)

where \( \mu : = \prod_{\alpha \in \Delta} (1 - e^{-\alpha}) \in \mathcal{R}(\overline{T}) \) for \( \Delta \subset \Delta \).

Let \( c_{w,v,v'} = 1 \bigotimes (1 \bigotimes a_{w,v,v'}) \in K(B) \bigotimes \mathcal{R}(\overline{T}) \bigotimes \mathcal{R}(\overline{G}) = K(B) \quad (3.20) \)

where \( a_{w,v,v'} \in \mathcal{R}(\overline{G}) \) as in (2.48).

Let

\[
\mathcal{K}' : = K\left( \mathcal{E} \times_{\overline{G} \times \overline{G}} \overline{T}^+ \right). \quad (3.21)
\]

Here \( \overline{B} \times \overline{B} \)-acts on \( \overline{T} \times 1 \). Then \( \mathcal{K}' \) is the \( K \)-ring of a \( \overline{T}^+ \)-bundle over the bundle \( \mathcal{E}/\overline{B}^{-} \times B \) having fibre \( G/B^{-} \times G/B \) over \( B \). Again since \( \overline{T}^+ \) is a semi-projective toric variety, by [33, Theorem 4.1] the ring \( \mathcal{K}' \) gets a \( K(\mathcal{E}/\overline{B} \times \overline{B}) \)-algebra structure.

Further, we note that \( \mathcal{E}/\overline{B} \times \overline{B} \) is a flag bundle over \( \mathcal{E}/\overline{B} \times \overline{G} \) with fibre the flag variety \( pt \times \overline{G}/\overline{B} \). Moreover, \( \mathcal{E} \times_{\overline{B} \times \overline{B}} \overline{T}^+ \) is the pull back of \( \mathcal{E} \times_{\overline{B} \times \overline{G}} \overline{T}^+ \) to \( \mathcal{E}/\overline{B} \times \overline{B} \). Thus the canonical inclusion

\[
K(\mathcal{E}/\overline{B} \times \overline{G}) \hookrightarrow K(\mathcal{E}/\overline{B} \times \overline{B}) \quad (3.22)
\]

is the restriction of \( \mathcal{K} \hookrightarrow \mathcal{K}' \).

Moreover, \( f_v, \lambda_i \) and \( c_{w,v,v'} \) lie in \( K(\mathcal{E}/\overline{B} \times \overline{B}) \) via pull back from \( K(\mathcal{E}/\overline{G} \times \overline{B}), K(\mathcal{E}/\overline{B} \times \overline{G}) \) and \( K(B) \) respectively.

We now prove the second main theorem of this section. This is the relative version of [32, Theorem 3.8].

**Theorem 3.4.**

1. We have the following isomorphism as submodules of \( \mathcal{K}' \):

\[
K(\mathcal{E}(X)) \cong \bigoplus_{v \in \Delta} \mathcal{K} \cdot f_v. \quad (3.23)
\]

In particular, the ring \( K(\mathcal{E}(X)) \) gets a canonical structure of a \( \mathcal{K} \)-module of rank \( |W| \).

2. Furthermore, (3.23) is an isomorphism of \( \mathcal{K} \)-algebras where any two basis elements \( f_v \) and \( f_{v'} \) multiply in \( \mathcal{K}' \) as follows

\[
f_v \cdot f_{v'} = \sum_{J \subseteq I \cup I'} \sum_{w \in C_J} (\lambda_{J \cup I'} \cdot \lambda_{(I \cup I') \setminus J}) \cdot c_{w,v,v'} \cdot f_w. \quad (3.24)
\]

**Proof.**

1. Note that the isomorphism in [33, Theorem 2.2 (i)] is as \( \mathcal{R}(\overline{G}) \bigotimes \mathcal{R}(\overline{G}) \)-algebras. Thus by base changing to the \( \mathcal{R}(\overline{G}) \bigotimes \mathcal{R}(\overline{G}) \)-module \( K(B) \) on either side we get the following isomorphism of \( K(B) \)-algebras

\[
K(B) \bigotimes \mathcal{R}(\overline{G}) \bigotimes \mathcal{R}(\overline{G}) \cong \bigoplus_{I \subseteq \Delta} K(B) \bigotimes \mathcal{R}(\overline{G}) \bigotimes \mathcal{R}(\overline{G}) \quad (3.25)
\]
By Corollary 2.7 this can be rewritten as

\[ K(\mathcal{E}(X)) \cong \bigoplus_{I \subseteq \Delta} \bigoplus_{\nu \in C^I} K(B) \bigotimes_{R(\tilde{G}) \otimes R(\tilde{G})} K(T^+ \nu) \bigotimes_{R(\tilde{G})} R(\tilde{G}) \cdot f_\nu. \]

Now, \( R(\tilde{G}) \otimes R(\tilde{G}) \) acts on \( K(T^+ \nu) \) and \( R(\tilde{G}) \cdot f_\nu \) via the first and second projections respectively. Thus (3.17) and (3.18) together imply (3.23). Note that \( \mathcal{K} \cdot f_\nu \) is a \( \mathcal{K} \)-submodule of \( \mathcal{K}' \) for every \( \nu \in C^I \) and \( I \subseteq \Delta \). Furthermore, the direct sum decomposition (3.23) gives \( K(\mathcal{E}(X)) \) a structure of a free \( \mathcal{K} \)-module of rank \( |W| \). Also by Proposition 3.2 (ii), (3.23) is an equality of \( \mathcal{K} \)-submodules of \( \mathcal{K}' \).

2. We observe that

\[
\overline{f_\nu} \cdot \overline{f_{\nu'}} = \left[ 1 \bigotimes \left( 1 \bigotimes f_{\nu} \right) \right] \cdot \left[ 1 \bigotimes \left( 1 \bigotimes f_{\nu'} \right) \right] = 1 \bigotimes \left[ \left( 1 \bigotimes f_{\nu} \right) \cdot \left( 1 \bigotimes f_{\nu'} \right) \right].
\]

Now, [33, Theorem 2.2 (ii)] implies that

\[
\overline{f_\nu} \cdot \overline{f_{\nu'}} = 1 \bigotimes \sum_{J \subseteq (I \cup I')} \sum_{w \in C^J} \left( \mu_{I \cap I'} \cdot \mu_{I \cup I' \setminus J} \bigotimes a_{\nu, \nu'}^{w} \right) \cdot \left( 1 \bigotimes f_{w} \right).
\]

This can further be written as

\[
1 \bigotimes \sum_{J \subseteq (I \cup I')} \sum_{w \in C^J} \left( \mu_{I \cap I'} \cdot \mu_{I \cup I' \setminus J} \bigotimes 1 \right) \cdot \left( 1 \bigotimes a_{\nu, \nu'}^{w} \right) \cdot \left( 1 \bigotimes f_{w} \right).
\]

The equality (3.24) now follows by applying (3.18), (3.19) and (3.20) successively. \( \square \)

**Example 3.5.** Let \( X_0 \) denote the wonderful compactification of \( PGL(n, \mathbb{C}) \). Now, regarding \( \mathbb{P}^n(\mathbb{C}) \) as \( PGL(n + 1, \mathbb{C})/Q \) where \( Q \) denotes the maximal parabolic in \( PGL(n + 1, \mathbb{C}) \) which fixes the one dimensional subspace of \( \mathbb{C}^{n+1} \) generated by the coordinate vector \( e_1 \) we can consider the principal \( Q \times Q \)-bundle

\[
PGL(n + 1, \mathbb{C}) \times PGL(n + 1, \mathbb{C}) \rightarrow \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C}) \cong PGL(n + 1, \mathbb{C}) \times PGL(n + 1, \mathbb{C})/Q \times Q.
\]

The Levi subgroup \( L_Q \) of \( Q \) is identified with \( GL(n, \mathbb{C}) \) and the adjoint group \( L_Q/C(L_Q) \) is \( PGL(n, \mathbb{C}) \). We can construct the associated bundle

\[
X_1 := (PGL(n + 1, \mathbb{C}) \times PGL(n + 1, \mathbb{C})) \times_{Q \times Q} X_0
\]

where \( Q \times Q \) acts on \( X_0 \) via its projection to \( L_Q/C(L_Q) \times L_Q/C(L_Q) \). Since \( Q/L_Q = Q_u \) is the unipotent radical of \( Q \), we have

\[
K(X_1) \cong K(PGL(n + 1, \mathbb{C}) \times PGL(n + 1, \mathbb{C}) \times_{Q \times Q} X_0)
\]

and

\[
K(PGL(n + 1, \mathbb{C}) \times PGL(n + 1, \mathbb{C})/Q \times Q) \cong K(PGL(n + 1, \mathbb{C}) \times PGL(n + 1, \mathbb{C})/L_Q \times L_Q) \cong K(\mathcal{P}^n(\mathbb{C}) \times \mathcal{P}^n(\mathbb{C}))
\]

(see [12, 5.2.18]). Thus by Theorem 3.3 we have \( K(X_1) \cong K(\mathcal{P}^n(\mathbb{C}) \times \mathcal{P}^n(\mathbb{C})) \bigotimes_{R(L_Q) \otimes R(L_Q)} K_{L_Q \times L_Q}(X_0) \). Here \( R(L_Q \times L_Q) \cong R(T \times T)^{S_n \times S_n} \) (see Section 1.3 above) and \( K_{L_Q \times L_Q}(X_0) \cong K_{T \times T}(Y_0)^{S_n} \) where \( T \) the maximal torus of \( GL(n, \mathbb{C}) \) consisting of the diagonal matrices acts canonically on the toric variety \( Y_0 \) associated to the Weyl chambers of \( A_{n-1} \)-type. Thus we have

\[
K(X_1) = K(\mathcal{P}^n(\mathbb{C}) \times \mathcal{P}^n(\mathbb{C})) \bigotimes_{R(T \times T)^{S_n \times S_n}} K_{T \times T}(Y_0)^{S_n}.
\]
In this section we retrieve known results on $K$-theory of toric bundles [29, Theorem 1.2(iv)] and flag bundles.

Let $X$ be a smooth $T$-cellular toric variety associated to a fan $\Sigma$ in the lattice $N = \mathbb{Z}^n$. Let $\Sigma(1) = \{ \rho_1, \ldots, \rho_d \}$ denote the edges and $v_1, \ldots, v_d$ primitive lattice points along the edges. Let $M = \text{Hom}(N, \mathbb{Z})$ be the dual lattice.

Let $E \xrightarrow{\pi} B$ be a principal $T$-bundle. Let $\mathcal{E}(X)$ denote the associated toric bundle $E \times_T X$.

**Proposition 4.1.** Then $K(\mathcal{E}(X))$ has the following presentation as $K(B)$-algebra:

$$K(B)[x_1, \ldots, x_d]/I$$

where $I$ is the ideal generated by the following two types of relations:

$$x_{i_1} \cdots x_{i_k} | \langle v_{i_1}, \ldots, v_{i_k} \rangle \notin \Sigma,$$ \hspace{1cm} (4.1)

$$\prod_{l \mid (u, v_l) \geq 0} (1 - x_l)^{(u, v_l)} - [L_u] \prod_{l \mid (u, v_l) \leq 0} (1 - x_l)^{- \langle u, v_l \rangle} \forall u \in M.$$ \hspace{1cm} (4.2)

**Proof.** Since $X$ is $T$-cellular it satisfies the hypothesis of the Theorem 2.2 and Corollary 2.7. Hence by Corollary 2.7, $K(\mathcal{E}(X)) = K(B) \otimes_{R(T)} K_T(X)$ where the extension of scalars to $K(B)$ is obtained by sending $e^u \in R(T)$ to $[L_u]$ for every $u \in M = \text{Hom}(T, \mathbb{C}^*)$. Now the theorem follows readily from the presentation of the ring $K_T(X)$ as an $R(T)$-algebra described in [34, Theorem 6.4].

Let $E \xrightarrow{\pi} B$ be a principal $T$-bundle and let $\mathcal{E}(G/B)$ denote the associated flag bundle $E \times_T G/B$.

**Proposition 4.2.** The ring $K(\mathcal{E}(G/B))$ has the following presentation

$$\frac{K(B) \otimes R(T)}{I}$$

(4.3)

where $I$ is the ideal generated by the relations

$$[\mathcal{E} \times_G \mathcal{V}] \otimes 1 - 1 \otimes [\mathcal{V}] \text{ for every } [\mathcal{V}] \in R(G) = R(T)^W.$$

**Proof.** Now $G/B$ is a projective variety with a $T$-cellular structure given by the Bruhat decomposition. Therefore by Corollary 2.8 we have $K(\mathcal{E}(G/B)) = K(B) \otimes_{R(G)} K_G(G/B)$. The theorem now follows by using the fact that $K_G(G/B) = R(T)$ and the $R(G)$ algebra structure on $R(T)$ and $K(B)$ (see [24, (2)]).

We further derive the following description of $K(\mathcal{E}(G/B))$.

**Proposition 4.3.** The ring $K(\mathcal{E}(G/B))$ is isomorphic to the subring of $K(B)^W$ consisting of tuples $(f_w)_{w \in W}$ satisfying the condition that $f_w - f_{s_w \cdot w}$ is divisible by $1 - [L_{s_w \cdot w}]$ for every $w \in W$ and $\alpha \in \Phi^+$.

**Proof.** Recall that the $T$-fixed point set in $G/B$ can be identified with $W$. Thus from the description of the $T$-invariant curves in $G/B$ (see [11] or [9, Section 6.5]) and from [32, Theorem 1.3] it follows that $K_T(G/B)$ can be described as the $R(T)$-subalgebra of $R(T)^W$ consisting of tuples $(f_w)$ such that $f_w - f_{s_w \cdot w}$ is divisible by $1 - e^{-\alpha}$ for $w \in W$ and $\alpha \in \Phi^+$. The proposition now follows readily from Theorem 2.10.

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