THE ASYMPTOTIC BEHAVIOR OF THE REIDEMEISTER TORSION FOR SEIFERT MANIFOLDS AND PSL$_2(\mathbb{R})$-REPRESENTATIONS OF FUCHSIAN GROUPS

YOSHIKAZU YAMAGUCHI

ABSTRACT. We show that a PSL$_2(\mathbb{R})$-representation of a Fuchsian group induces the asymptotics of the Reidemeister torsion for the Seifert manifold corresponding to the euler class of the PSL$_2(\mathbb{R})$-representation. We also show that the limit of leading coefficient of the Reidemeister torsion is determined by the euler class of a PSL$_2(\mathbb{R})$-representation of a Fuchsian group. In particular, the leading coefficient of the Reidemeister torsion for the unit tangent bundle over a two–orbifold converges to $-\chi \log 2$ where $\chi$ is the Euler characteristic of the two–orbifold. We also give a relation between $\mathbb{Z}_2$-extensions for PSL$_2(\mathbb{R})$-representations of a Fuchsian group and the asymptotics of the Reidemeister torsion.

1. Introduction

The previous works [Yam13, Yam] of the author have investigated the asymptotic behavior of the Reidemeister torsion for a Seifert manifold with a sequence of SL$_2(N)(\mathbb{C})$-representation of the fundamental group. Here the sequence of SL$_2(N)(\mathbb{C})$-representations starts with an SL$_2(\mathbb{C})$-representation and the remaining representations are given by the composition with the irreducible $2N$-dimensional representations of SL$_2(\mathbb{C})$. A sequence of SL$_2(N)(\mathbb{C})$-representations defines a sequence of the Reidemeister torsion of a Seifert manifold. We can consider the asymptotic behavior of the sequence given by the Reidemeister torsions. Under the constraint that SL$_2(N)(\mathbb{C})$-representations send a regular fiber to $-I_{2N}$, the observation in [Yam] revealed the growth order and the convergence of the leading coefficient of the Reidemeister torsion for a Seifert manifold.

In this paper, we observe when we have a natural situation for a Seifert manifold and SL$_2(\mathbb{R})$-representations satisfying our constraint. We will deal with closed Seifert manifolds whose base orbifolds have the negative Euler characteristics. Namely, they admit the $\mathbb{H}^2 \times \mathbb{R}$ or SL$_2(\mathbb{R})$-geometry. We can regard the fundamental group of the base orbifold as a cocompact Fuchsian group $\Gamma$ which can be embedded in PSL$_2(\mathbb{R})$. The fundamental group of a Seifert manifold is a central extension of the Fuchsian group $\Gamma$ by $\mathbb{Z}$. The universal cover $\widetilde{\text{PSL}}_2(\mathbb{R})$ is also a central extension of PSL$_2(\mathbb{R})$ by $\mathbb{Z}$. If we choose a homomorphism $\tilde{\rho}$ from $\Gamma$ to PSL$_2(\mathbb{R})$ for a Seifert manifold $M$, then we have a lift $\tilde{\rho}$ from $\pi_1(M)$ to PSL$_2(\mathbb{R})$. This is due to the work [JN85] by M. Jankins and W. Neumann (we review this in Section 2.2). Since PSL$_2(\mathbb{R})$ is also the universal cover of SL$_2(\mathbb{R})$, we have the SL$_2(\mathbb{R})$-representation $\rho$ of $\pi_1(M)$ given by the composition of $\tilde{\rho}$ with the projection from PSL$_2(\mathbb{R})$ onto SL$_2(\mathbb{R})$. The SL$_2(\mathbb{R})$-representation $\rho$ induces the SL$_2(N)(\mathbb{R})$-representations which define the acyclic chain complexes of the Seifert manifold $M$. We can observe the

2010 Mathematics Subject Classification. Primary: 57M27, 57M05, Secondary: 57M50.

Key words and phrases. Seifert fibered spaces; geometric structure; Reidemeister torsion; asymptotic behaviors.
asymptotics of the Reidemeister torsion starting with a $\text{PSL}_2(\mathbb{R})$-representation of a Fuchsian group.

We refer the case of a Seifert homology sphere and $\text{SU}(2)$-representations to the previous work [Yam], in which the maximum and minimum of the limits have been observed in detail. The limit of the leading coefficient is determined by the images of exceptional fibers by a given $\text{SU}(2)$-representation.

First, we will show that the limit of leading coefficient in the asymptotic behavior of the Reidemeister torsions is determined by a representation of a Fuchsian group. Here we start with a $\text{PSL}_2(\mathbb{R})$-representation of a cocompact Fuchsian group $\Gamma = \Gamma(g; \alpha_1, \ldots, \alpha_n)$ (for the notation, see Section 2.2). If we have the following diagram:

$$
0 \to \mathbb{Z} \to \pi_1(M(\frac{a}{b}, \frac{a}{b_1}, \ldots, \frac{a}{b_n})) \to \Gamma \to 1
$$

then we have the $\text{SL}_2(\mathbb{R})$-representation $\rho$ of $\pi_1(M(\frac{a}{b}, \frac{a}{b_1}, \ldots, \frac{a}{b_n}))$ which induces a sequence of the Reidemeister torsion $\text{Tor}(M; \rho_{2N})$ (for the details on this sequence, we refer to Theorem 2.1 and Lemma 3.3). The sequence of $\log|\text{Tor}(M(\frac{a}{b}, \frac{a}{b_1}, \ldots, \frac{a}{b_n}); \rho_{2N})|$ has the growth order of $2N$ and the following leading coefficient.

**Theorem** (Theorem 3.5). For the induced $\text{SL}_2(\mathbb{R})$-representations $\rho_{2N}$, the limit of the leading coefficient of $\log|\text{Tor}(M(\frac{a}{b}, \frac{a}{b_1}, \ldots, \frac{a}{b_n}); \rho_{2N})|$ is expressed as

$$
\lim_{N \to \infty} \frac{\log|\text{Tor}(M(\frac{a}{b}, \frac{a}{b_1}, \ldots, \frac{a}{b_n}); \rho_{2N})|}{2N} = -(2 - 2g - \sum_{j=1}^{n} \frac{\lambda_j - 1}{\lambda_j}) \log 2.
$$

Here $\lambda_j$ is the order of $\tilde{\rho}(q_j)$ where $q_j$ is the homotopy class of a loop around the $j$-th branched point on the two–orbifold.

**Remark.** The existence of a lift $\tilde{\rho}$ is determined by the euler class $e(\tilde{\rho})$ in $H^2(\Gamma; \mathbb{Z})$. The explicit value of $\lambda_j$ is given in Theorem 3.5. The limit of the leading coefficient is also determined by the euler class $e(\tilde{\rho})$.

The unit tangent bundle over a two–orbifold $\mathbb{H}^2/\Gamma$ is also a Seifert manifold with the index $((1, 2g - 2), (\alpha_1, \alpha_1 - 1), \ldots, (\alpha_n, \alpha_n - 1))$. We can think of the unit tangent bundle $T^1(\mathbb{H}^2/\Gamma)$ as $\text{PSL}_2(\mathbb{R})/\Gamma$. For a Seifert manifold $\text{PSL}_2(\mathbb{R})/\Gamma$, we can take a $\text{PSL}_2(\mathbb{R})$-representation $\rho$ of $\pi_1(\text{PSL}_2(\mathbb{R})/\Gamma)$ as a lift of an embedding from $\Gamma$ into $\text{PSL}_2(\mathbb{R})$.

**Corollary** (Corollary 3.7). Suppose that an $\text{SL}_2(\mathbb{R})$-representation $\rho$ of $\pi_1(\text{PSL}_2(\mathbb{R})/\Gamma)$ is the composition of $\tilde{\rho}$ with the projection from $\text{PSL}_2(\mathbb{R})$ onto $\text{SL}_2(\mathbb{R})$. Then we have the following limit of the leading coefficient of $\log|\text{Tor}(\text{PSL}_2(\mathbb{R})/\Gamma; \rho_{2N})|$:

$$
\lim_{N \to \infty} \frac{\log|\text{Tor}(\text{PSL}_2(\mathbb{R})/\Gamma; \rho_{2N})|}{2N} = -(2 - 2g - \sum_{j=1}^{n} \frac{\alpha_j - 1}{\alpha_j}) \log 2
$$

$$
= -\chi \log 2
$$

where $\chi$ is the Euler characteristic of the base orbifold $\mathbb{H}^2/\Gamma$. 

We also investigate which $\mathrm{PSL}_2(\mathbb{R})$-representation $\tilde{\rho}$ of $\Gamma$ induces $\mathrm{SL}_2(\mathbb{R})$-representation of $\pi_1(M(\frac{1}{p_1}, \frac{a_1}{p_1}, \ldots, \frac{a_n}{p_n}))$. We show a sufficient condition for $\tilde{\rho}$ to be lifted to an $\mathrm{SL}_2(\mathbb{R})$-representation for a given Seifert manifold $M(\frac{1}{p_1}, \frac{a_1}{p_1}, \ldots, \frac{a_n}{p_n})$ in the following diagram:

$$
\begin{array}{c}
0 \to \mathbb{Z} \to \pi_1(M(\frac{1}{p_1}, \frac{a_1}{p_1}, \ldots, \frac{a_n}{p_n})) \to \Gamma \to 1 \\
\downarrow \rho \quad \downarrow \tilde{\rho} \\
0 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R}) \to 1.
\end{array}
$$

Our sufficient condition is given in terms of the euler class of $\tilde{\rho}$.

**Theorem** (Theorem 4.3). A $\mathrm{PSL}_2(\mathbb{R})$-representation $\tilde{\rho}$ of $\Gamma$ induces an $\mathrm{SL}_2(\mathbb{R})$-one of $\pi_1(M(\frac{1}{p_1}, \frac{a_1}{p_1}, \ldots, \frac{a_n}{p_n}))$ such that $\rho(h) = -I$ if the euler class $e(\tilde{\rho})$ satisfies the criteria of Theorem 2.2 and gives the equivalent class $[b_0 + \beta_1 x_1 + \cdots + \beta_n x_n] \in \Ext(\Gamma; \mathbb{Z}/2\mathbb{Z})$.

Moreover the limit of the leading coefficient for $\mathrm{Tor}(M(\frac{1}{p_1}, \frac{a_1}{p_1}, \ldots, \frac{a_n}{p_n}); \rho_{2N})$ is expressed in Theorem 4.6. We will compute explicit examples for $\mathrm{SL}_2(\mathbb{R})$-representations of Brieskorn manifolds by using Theorem 4.3.

**Organization.** We review the previous result on the asymptotic behavior of the Reidemeister torsion in Section 2.1. Section 2.2 gives a brief review the work on $\mathrm{PSL}_2(\mathbb{R})$-representations of a cocompact Fuchsian group by Jankins and Neumann [JN85]. In Section 3 we observe the asymptotic behavior of the Reidemeister torsion for a Seifert manifold and the $\mathrm{SL}_2(\mathbb{R})$-representation induced by a $\mathrm{PSL}_2(\mathbb{R})$-one. We deal with $\mathrm{SL}_2(\mathbb{R})$-representations for a Seifert manifold, which are given by $\mathrm{PSL}_2(\mathbb{R})$-representations of a Fuchsian group with different euler classes in Section 4. The last Section 5 shows explicit examples of $\mathrm{SL}_2(\mathbb{R})$-representations induced by $\mathrm{PSL}_2(\mathbb{R})$-ones for Brieskorn manifolds.

2. Preliminaries

For a Seifert index $(g; (1, b), (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$, we follow the convention of Jankins and Neumann [JN85]. This notation differs from that of [Yam] in sign of $b$ and $\beta_1, \ldots, \beta_n$.

2.1. Asymptotic behavior of the Reidemeister torsion. Let $M$ be a Seifert manifold of the index $(g; (1, b), (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$. The fundamental group of $M$ is expressed as

$$
\pi_1(M) = \langle a_1, b_1, \ldots, a_n, b_n, q_1, \ldots, q_n, h \mid h \text{ is central}, q_j^{a_j} = h^{b_j}, q_1 \cdots q_n \prod_{j=1}^{g} [a_i, b_i] = h^{-b} \rangle.
$$

We use the symbol $\rho$ for an $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(M)$. We denote by $\rho_{2N}$ the composition of $\rho$ with the irreducible $2N$-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$ and by $\mathrm{Tor}(M; \rho_{2N})$ the Reidemeister torsion of $M$ and $\rho_{2N}$. The asymptotic behavior of the Reidemeister torsion is expressed as follows.

**Theorem 2.1** (Theorem 4.5 in [Yam]). If $\pi_1(M)$ has an $\mathrm{SL}_2(\mathbb{C})$-representation $\rho$ sending the homotopy class $h$ of a regular fiber to $-I$, the asymptotic behavior of the sequence given by the Reidemeister torsion $\mathrm{Tor}(M; \rho_{2N})$ is expressed as

$$
\lim_{N \to \infty} \frac{\log |\mathrm{Tor}(M; \rho_{2N})|}{(2N)^2} = 0,
$$

$$
\lim_{N \to \infty} \frac{\log |\mathrm{Tor}(M; \rho_{2N})|}{2N} = -(2 - 2g - \sum_{j=1}^{n} \lambda_j - \frac{1}{\lambda_j}) \log 2.
$$
The right hand side of the Euler class is determined by the orders of the $\text{SL}_2(\mathbb{C})$-matrices for the exceptional fibers. When we denote by $\ell_j$ the homotopy class of $j$-th exceptional fiber, each $\lambda_j$ is half the order of $\rho(\ell_j)$.

2.2. $\text{PSL}_2(\mathbb{R})$-representations of Fuchsian groups and the Euler classes. We use the symbol $\Gamma = \Gamma(g; \alpha_1, \ldots, \alpha_n)$ for a cocompact Fuchsian group of genus $g$ with branch indices $\alpha_1, \ldots, \alpha_n$. The Fuchsian group $\Gamma$ has the following presentation:

$$\Gamma = \langle a_1, b_1, \ldots, a_g, b_g, q_1, \ldots, q_n \mid q_j^{a_j} = 1, q_1 \cdots q_n, [a_i, b_i] = 1 \rangle.$$  

In [JN85], Jankins and Neumann determined the set of components in $\text{Hom}(\Gamma, \text{PSL}_2(\mathbb{R}))$ by an Euler class:

$$e : \text{Hom}(\Gamma, \text{PSL}_2(\mathbb{R})) \to H^2(\Gamma; \mathbb{Z}) = ab(x_0, \ldots, x_n \mid a_i x_i = x_0, i = 1, \ldots, n)$$

The Euler class $e$ is defined as follows. Taking the pull-back central extension for $f \in \text{Hom}(\Gamma, \text{PSL}_2(\mathbb{R}))$ from

$$\begin{array}{c}
\Gamma \\
\downarrow f \\
0 \to \mathbb{Z} \to \text{PSL}_2(\mathbb{R}) \to \text{PSL}_2(\mathbb{R}) \to 1,
\end{array}$$

we have the following commutative diagram:

$$\begin{array}{c}
0 \to \mathbb{Z} \to \Gamma \to \text{PSL}_2(\mathbb{R}) \to \text{PSL}_2(\mathbb{R}) \to 1
\end{array} \quad \begin{array}{c}
\downarrow \| \\
\downarrow \tilde{f} \\
\downarrow f
\end{array}$$

The central extension $\tilde{\Gamma}$ has a presentation:

$$\langle a_1, b_1, \ldots, a_g, b_g, q_1, \ldots, q_n, h \mid h \text{ is central, } q_j^{a_j} = h^\beta, q_1 \cdots q_n, [a_i, b_i] = h^{-b} \rangle$$

with $0 \leq \beta_i < \alpha_i$. By the presentation of $\tilde{\Gamma}$, we define $e(f)$ to be $b x_0 + \beta_1 x_1 + \cdots + \beta_n x_n$.

**Theorem 2.2** (Theorem 1 in [JN85]). Suppose that $x \in H^2(\Gamma; \mathbb{Z})$ satisfies $x = b x_0 + \beta_1 x_1 + \cdots + \beta_n x_n$ ($0 \leq \beta_i < \alpha_i$). There exists some $f$ such that $e(f) = x$ if and only if the following holds:

(i) If $g > 0$ then $2 - 2g - n \leq b \leq 2g - 2$;

(ii) If $g = 0$ then either

(a) $2 - n \leq b \leq -2$ or;

(b) $b = -1$ and $\sum_{j=1}^n (\beta_j / \alpha_j) \leq 1$ or;

(c) $b = 1 - n$ and $\sum_{j=1}^n (\beta_j / \alpha_j) \geq n - 1$.

**Remark 2.3.** The above definition of the Euler class $e$ is the alternative one used in the proof of [JN85] Theorem 1]. The Euler class $e$ is defined as $e(f) = f^*(c)$ where $f^* : H^2(\text{PSL}_2(\mathbb{R}); \mathbb{Z}) \to H^2(\text{PSL}_2(\mathbb{R}); \mathbb{Z}) = H^2(\Gamma; \mathbb{Z})$ is the induced homomorphism by $f$ and $c$ is a generator of $H^2(\text{PSL}_2(\mathbb{R}); \mathbb{Z}) \cong \mathbb{Z}$.

Jankins and Neumann also proved the following theorem on the components of the $\text{PSL}_2(\mathbb{R})$-representation space of a Fuchsian group.

**Theorem 2.4** (Theorem 2 in [JN85]). Let $\Gamma$ be a cocompact Fuchsian group. The fibers of $e$ are the components of $\text{Hom}(\Gamma, \text{PSL}_2(\mathbb{R}))$. 
3. Asymptotics of the Reidemeister torsion via \( \widetilde{\text{PSL}}_2(\mathbb{R}) \)-representations

In the observation of the asymptotic behavior for the Reidemeister torsion, we require that an \( \text{SL}_2(\mathbb{C}) \)-representation \( \rho \) sends the central element \( h \) to \(-I\). Since \( \pi_1(M) \) is a central extension of \( \Gamma \), this is equivalent to that we have the following commutative diagram:

\[
\begin{array}{c}
0 \to \mathbb{Z} \to \pi_1(M) \to \Gamma \to 1 \\
\downarrow \quad \downarrow \rho \quad \downarrow \tilde{\rho} \\
0 \to \{\pm I\} \to \text{SL}_2(\mathbb{C}) \to \text{PSL}_2(\mathbb{C}) \to 1.
\end{array}
\]

The limit in Theorem \( \ref{thm:main} \) is determined by the induced \( \text{PSL}_2(\mathbb{C}) \)-representation \( \tilde{\rho} \) of \( \Gamma \).

**Proposition 3.1.** The integer \( \lambda_j \) in the equation \( \ref{eq:lambda} \) coincides with the order of \( \tilde{\rho}(q_j) \).

**Proof.** The order of \( \rho(\ell_j) \) is equal to \( 2\lambda_j \). This means that \( \lambda_j \) is the minimum of natural numbers such that \( \rho(\ell_j)^4 = -I \). On the other hand, the order of \( \tilde{\rho}(\ell_j) \) in \( \text{PSL}_2(\mathbb{C}) \) is the minimum of natural numbers such that \( \rho(\ell_j) = \pm I \). Hence the order of \( \tilde{\rho}(\ell_j) \) is equal to \( \lambda_j \). \( \square \)

We can rewrite the statement of Theorem \( \ref{thm:main} \) in terms of \( \tilde{\rho} \).

**Corollary 3.2.** If a Seifert manifold \( M \) has an \( \text{SL}_2(\mathbb{C}) \)-representation \( \rho \) such that \( \rho(h) = -I \), then we have

\[
\lim_{N \to \infty} \frac{\log |\text{Tor}(M; \rho_{2N})|}{2N} = -\left(2 - 2g - \sum_{j=1}^{n} \frac{\lambda_j - 1}{\lambda_j}\right) \log 2.
\]

where \( \lambda_j \) is the order of \( \tilde{\rho}(q_j) \) in \( \text{PSL}_2(\mathbb{C}) \).

It is natural to try to start with a \( \text{PSL}_2(\mathbb{C}) \)-representation of \( \Gamma \). Here and subsequently, we focus our attention on \( \text{PSL}_2(\mathbb{R}) \)-representations of a Fuchsian group and the induced \( \text{SL}_2(\mathbb{R}) \)-representation of \( \pi_1(M) \).

Given a Fuchsian group \( \Gamma = \Gamma(g; \alpha_1, \ldots, \alpha_n) \) and a \( \text{PSL}_2(\mathbb{R}) \)-representation \( \tilde{\rho} \), we have a Seifert manifold \( M \) and a \( \text{PSL}_2(\mathbb{R}) \)-representation \( \rho \) of \( \pi_1(M) \), induced from the diagram:

\[
\begin{array}{c}
0 \to \mathbb{Z} = \langle h \rangle \to \pi_1(M) \to \Gamma \to 1 \\
\downarrow \tilde{\rho} \quad \downarrow \rho \\
0 \to \mathbb{Z} \to \text{PSL}_2(\mathbb{R}) \to \text{PSL}_2(\mathbb{R}) \to 1.
\end{array}
\]

The Seifert index \( (g; (1, b), (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)) \) of \( M \) is given by the euler class \( e(f) \).

We also have the \( \text{SL}_2(\mathbb{R}) \)-representation \( \rho \) of \( \pi_1(M) \) defined by the composition with the projection \( \text{PSL}_2(\mathbb{R}) \to \text{SL}_2(\mathbb{R}) \). Here we identify the universal cover \( \widetilde{\text{SL}}_2(\mathbb{R}) \) with \( \text{PSL}_2(\mathbb{R}) \).

**Lemma 3.3.** Suppose that \((b; \beta_1, \ldots, \beta_n)\) satisfies the criteria of Theorem \( \ref{thm:main} \). Then there exist \( \text{PSL}_2(\mathbb{R}) \)-representations \( \tilde{\rho} \) of \( \pi_1(M(\frac{1}{b}; \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_n}{\beta_n})) \) such that \( \tilde{\rho}(h) = \text{sh}(1) \).

**Remark 3.4.** Here for any \( \gamma \in \mathbb{R} \) we write \( \text{sh}(\gamma) \) for the shift by \( \gamma \), that is the self-homeomorphism of \( \mathbb{R} \), \( r \to r + \gamma \). We can consider \( \text{PSL}_2(\mathbb{R}) \) as a subgroup of the group of homeomorphisms \( f: \mathbb{R} \to \mathbb{R} \) which are lifts of homeomorphisms of the circle. The shift \( \text{sh}(\gamma) \) is an element in \( \text{PSL}_2(\mathbb{R}) \), which projects to \( \begin{pmatrix} \cos(2\pi\gamma) & -\sin(2\pi\gamma) \\ \sin(2\pi\gamma) & \cos(2\pi\gamma) \end{pmatrix} \) in \( \text{PSL}_2(\mathbb{R}) \) by the projection. The center of \( \text{PSL}_2(\mathbb{R}) \) is \( \{\text{sh}(k) | k \in \mathbb{Z}\} = \mathbb{Z} \).

**Proof.** This is a consequence of Theorems \( \ref{thm:main} \) and \( \ref{thm:main} \). \( \square \)
Theorem 3.5. Suppose that \((b; \beta_1, \ldots, \beta_n)\) satisfies the criterion of Theorem 2.2. If \(\bar{\rho}\) is a \(\text{PSL}_2(\mathbb{R})\)-representation of \(\Gamma\) in \(e^{-1}(b\alpha_0 + \beta_1 x_1 + \cdots + \beta_n x_n)\) and \(\rho\) is the induced \(\text{SL}_2(\mathbb{R})\)-representation of \(M(\frac{1}{\rho}, \frac{\alpha_1}{\rho}, \ldots, \frac{\alpha_n}{\rho})\) by \(\bar{\rho}\), then \(\lambda_j\) in the asymptotic behavior is expressed as

\[
\lambda_j = \frac{\alpha_j}{(\alpha_j, \beta_j)}
\]

where \((\alpha_j, \beta_j)\) is the greatest common divisor of \(\alpha_j\) and \(\beta_j\).

If all \((\alpha_j, \beta_j)\) are equal to 1, then we have

\[
\lim_{N \to \infty} \frac{\log |\text{Tor}(M(\frac{1}{\rho}, \frac{\alpha_1}{\rho}, \ldots, \frac{\alpha_n}{\rho}); \rho_{2N})|}{2N} = -(2 - 2g - \sum_{j=1}^{n} \frac{\alpha_j - 1}{\alpha_j}) \log 2 = -\chi \log 2
\]

where \(\chi\) is the Euler characteristic of the base orbifold.

Remark 3.6. If \(M\) is the unit tangent bundle over a two-orbifold \(\mathbb{H}^2/\Gamma\) whose fundamental group is embedded as a Fuchsian group \(\Gamma\) in \(\text{PSL}_2(\mathbb{R})\), i.e., \(M = T^1\mathbb{H}^2/\Gamma(= \text{PSL}_2(\mathbb{R})/\Gamma)\), then \(M\) is the Seifert manifold with the index of \((g; (2g - 2), (\alpha_1, \alpha_1 - 1), \ldots, (\alpha_n, \alpha_n - 1))\) and there exists a lift \(\bar{\rho}\) of an embedding \(\rho\) of \(\Gamma\) into \(\text{PSL}_2(\mathbb{R})\) as follows:

\[
0 \to \mathbb{Z} \to \pi_1(M) \to \Gamma \to 1
\]

\[
\bar{\rho} \downarrow \bar{\rho} \downarrow \rho
\]

\[
0 \to \mathbb{Z} \to \text{PSL}_2(\mathbb{R}) \to \text{PSL}_2(\mathbb{R}) \to 1.
\]

We can regard this \(\bar{\rho}\) as an embedding of \(\pi_1(M)\) into \(\text{Isom}^+(\text{PSL}_2(\mathbb{R}))\) since \(\text{PSL}_2(\mathbb{R})\) is a subgroup of \(\text{Isom}^+(\text{PSL}_2(\mathbb{R}))\). For more details, we refer to [Sco83].

Corollary 3.7. Suppose that \(M\) is the quotient of \(\text{PSL}_2(\mathbb{R})\) by a Fuchsian group \(\Gamma\) and \(\bar{\rho}\) is an embedding of \(\pi_1(M)\) into \(\text{PSL}_2(\mathbb{R})\) (in \(\text{Isom}^+(\text{PSL}_2(\mathbb{R}))\)). For the \(\text{SL}_2(\mathbb{R})\)-representation \(\rho\) induced by \(\bar{\rho}\), the asymptotic behavior of the Reidemeister torsion is expressed as

\[
\lim_{N \to \infty} \frac{\log |\text{Tor}(M(\frac{1}{\rho}, \frac{\alpha_1}{\rho}, \ldots, \frac{\alpha_n}{\rho}); \rho_{2N})|}{2N} = -\chi \log 2
\]

Proof. It follows from Euclidean algorithm that \((\alpha_i, \alpha_i - 1) = 1\). Together with Theorem 3.5, we obtain the limit in our claim. \(\square\)

Remark 3.8. The Seifert manifold \(\text{PSL}_2(\mathbb{R})/\Gamma\) is also regarded as \(\text{PSL}_2(\mathbb{R})/\rho^{-1}(\Gamma)\) where \(\rho\) is the projection from \(\text{PSL}_2(\mathbb{R})\) onto \(\text{PSL}_2(\mathbb{R})\).

The next lemma was shown in the proof of [IN85, Theorem 1].

Lemma 3.9. Let \(\bar{\rho}\) be a \(\text{PSL}_2(\mathbb{R})\)-representation of \(\Gamma\) in \(e^{-1}(b\alpha_0 + \beta_1 x_1 + \cdots + \beta_n x_n)\). The induced \(\text{PSL}_2(\mathbb{R})\)-representation \(\rho\) of \(M(\frac{1}{\rho}, \frac{\alpha_1}{\rho}, \ldots, \frac{\alpha_n}{\rho})\) satisfies that every \(\bar{\rho}(q_j)\) is conjugate to \(\text{sh}(\beta_j/\alpha_j)\) in \(\text{PSL}_2(\mathbb{R})\) for \(j = 1, \ldots, n\).

We enclose this section with the proof of Theorem 3.5.

Proof of Theorem 3.5. Our \(\text{SL}_2(\mathbb{R})\)-representation \(\rho\) of \(\pi_1(M(\frac{1}{\rho}, \frac{\alpha_1}{\rho}, \ldots, \frac{\alpha_n}{\rho}))\) is given by the \(\text{PSL}_2(\mathbb{R})\)-representation \(\bar{\rho}\) as follows:

\[
\rho : \pi_1(M(\frac{1}{\rho}, \frac{\alpha_1}{\rho}, \ldots, \frac{\alpha_n}{\rho})) \to \text{PSL}_2(\mathbb{R}) \to \text{SL}_2(\mathbb{R}).
\]

By Lemmas 3.4 and 3.9 we can see that

\[
\rho(q_j) \sim \left( \frac{\cos \beta_j}{\alpha_j}, \frac{-\sin \beta_j}{\alpha_j} \right),
\]

\[
\sin \beta_j/\alpha_j, \cos \beta_j/\alpha_j.
\]
Hence the order of $\tilde{\rho}(g)$ in $\text{PSL}_2(\mathbb{R})$ is equal to $\alpha_j/(\alpha_j,\beta_j)$. Together with Proposition 3.1 we can obtain that $\lambda_j = \alpha_j/(\alpha_j,\beta_j)$.

4. $\mathbb{Z}_2$-extension of a Fuchsian group and $\text{SL}_2(\mathbb{R})$-representation

Let $\Gamma$ be a Fuchsian group of genus $g$ with branch indices $\alpha_1,\ldots,\alpha_n$ and $M$ be a Seifert manifold with the index $(g; (1, b), (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$. Theorem 2.2 shows that when the integers $(b, \beta_1, \ldots, \beta_n)$ corresponds to some euler class $e(\tilde{\rho})$ in $H^2(\Gamma; \mathbb{Z})$, we have a $\text{PSL}_2(\mathbb{R})$-representation $\tilde{\rho}$ of $\pi_1(M)$ induced by $\tilde{\rho}$. Then $\tilde{\rho}$ also induces an $\text{SL}_2(\mathbb{R})$-representation $\rho$ such that $\rho(h) = -I$.

We can also find $\text{SL}_2(\mathbb{R})$-representations of $\pi_1(M)$ induced by other $(b', \beta'_1, \ldots, \beta'_n)$ in $H^2(\Gamma; \mathbb{Z})$. We classify $(b', \beta'_1, \ldots, \beta'_n)$ which gives an $\text{SL}_2(\mathbb{R})$-representation of $\pi_1(M)$.

4.1. $\mathbb{Z}_2$-extension and $\text{SL}_2(\mathbb{R})$-representation. Every irreducible $\text{SL}_2(\mathbb{R})$-representation $\rho$ of $\pi_1(M)$ factors through $\pi_1(M)/\langle h \rangle$ since $\rho$ sends the central element $h$ into the center of $\text{SL}_2(\mathbb{R})$. When we regard $\Gamma$ as $\pi_1(M)/\langle h \rangle$, we have the following central extension of $\Gamma$:

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \pi_1(M)/\langle h \rangle \rightarrow \Gamma \rightarrow 1.$$ 

Taking the pull–back central extension from

$$\begin{array}{c}
\Gamma \\
\downarrow \tilde{\rho} \\
0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \text{SL}_2(\mathbb{R}) \rightarrow \text{PSL}_2(\mathbb{R}) \rightarrow 1,
\end{array}$$

we have the following diagram:

$$\begin{array}{c}
0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \hat{\Gamma} \rightarrow \Gamma \rightarrow 1 \\
\| \\
\downarrow \rho \\
0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \text{SL}_2(\mathbb{R}) \rightarrow \text{PSL}_2(\mathbb{R}) \rightarrow 1.
\end{array}$$

Here $\hat{\Gamma}$ is isomorphic to $\pi_1(M)/\langle h \rangle$ for some $M$.

**Lemma 4.1.** The group of central extensions of $\Gamma$ by $\mathbb{Z}/2\mathbb{Z}$ is expressed as

$$\text{Ext}(\Gamma; \mathbb{Z}/2\mathbb{Z}) \cong \text{Ext}(\Gamma; \mathbb{Z})/2\text{Ext}(\Gamma; \mathbb{Z}).$$

**Remark 4.2.** Lemma 4.1 is a consequence of the remark following the proof of [NJ81, Theorem 10.4].

**Proof.** In proving the surjectivity of the following homomorphism:

$$H^2(\Gamma; \mathbb{Z}) = \text{Ext}(\Gamma; \mathbb{Z}/2\mathbb{Z}) \rightarrow \text{Ext}(\Gamma; \mathbb{Z}/2\mathbb{Z}) \rightarrow ab(x_0, \ldots, x_n | \alpha_i x_i = x_0, i = 1, \ldots, n),$$

we define a function

$$\nu : ab(x_0, \ldots, x_n | \alpha_i x_i = x_0, i = 1, \ldots, n) \rightarrow S(\mathbb{Z}) = \{\text{subgroups of } \mathbb{Z}\}$$

by $[s_0, \ldots, s_n] \mapsto \ker(\mathbb{Z} \rightarrow \pi(s_0, \ldots, s_n))$. Here $\pi(s_0, \ldots, s_n)$ is a central extension of $\Gamma$ given by $\pi_1(M(g; (1, s_0), (\alpha_1, s_1), \ldots, (\alpha_n, s_n)))$. The function $\nu$ gives a central extension of $\Gamma$ by

$$1 \rightarrow \mathbb{Z}/\nu([s_0, \ldots, s_n]) \rightarrow \pi(s_0, \ldots, s_n) \rightarrow \Gamma \rightarrow 1$$

for each $[s_0, \ldots, s_n]$. It is shown in [NJ81, the proof of Theorem 10.4] that

$$\text{Ext}(\Gamma; \mathbb{Z}) \cong \nu^{-1}(0), \quad \nu^{-1}(0) = ab(x_0, \ldots, x_n | \alpha_i x_i = x_0, i = 1, \ldots, n).$$

In the case of $\text{Ext}(\Gamma; \mathbb{Z}/2\mathbb{Z})$, we also define a function

$$\nu' : ab(x_0, \ldots, x_n | \alpha_i x_i = x_0, i = 1, \ldots, n) \rightarrow S(\mathbb{Z}) = \{\text{subgroups of } \mathbb{Z}\}$$

by $[s_0, \ldots, s_n] \mapsto \ker(\mathbb{Z} \rightarrow \pi(s_0, \ldots, s_n))$. When we regard $\Gamma$ as $\pi_1(M)/\langle h \rangle$, we have the following central extension of $\Gamma$:

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \pi_1(M)/\langle h \rangle \rightarrow \Gamma \rightarrow 1.$$ 

Taking the pull–back central extension from

$$\begin{array}{c}
\Gamma \\
\downarrow \tilde{\rho} \\
0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \text{SL}_2(\mathbb{R}) \rightarrow \text{PSL}_2(\mathbb{R}) \rightarrow 1,
\end{array}$$

we have the following diagram:

$$\begin{array}{c}
0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \hat{\Gamma} \rightarrow \Gamma \rightarrow 1 \\
\| \\
\downarrow \rho \\
0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \text{SL}_2(\mathbb{R}) \rightarrow \text{PSL}_2(\mathbb{R}) \rightarrow 1.
\end{array}$$

Here $\hat{\Gamma}$ is isomorphic to $\pi_1(M)/\langle h \rangle$ for some $M$.
by \([s_0, \ldots, s_n] \mapsto \ker(\mathbb{Z} \to \pi(s_0, \ldots, s_n)/\langle h^2 \rangle)\). One can see that

\[
\text{Ext}(\Gamma; \mathbb{Z}/2\mathbb{Z}) \cong \nu^{-1}(2\mathbb{Z})/2ab(x_0, \ldots, x_n | \alpha_i x_i = x_0, i = 1, \ldots, n)
\]

since the submodule \(2ab(x_0, \ldots, x_n | \alpha_i x_i = x_0, i = 1, \ldots, n)\) corresponds to the trivial extension. We can see that each element of \(\nu^{-1}(\{0\})\) is contained in \(\nu^{-1}(2\mathbb{Z})\) by the following diagram:

\[
\begin{array}{ccc}
1 & \to & \mathbb{Z} \\
\downarrow & & \downarrow \\
\mathbb{Z}/2\mathbb{Z} & \to & \pi(s_0, \ldots, s_n)/\langle h^2 \rangle.
\end{array}
\]

Hence it follows from \(\nu^{-1}(\{0\}) \subset \nu^{-1}(2\mathbb{Z}) \subset ab(x_0, \ldots, x_n | \alpha_i x_i = x_0, i = 1, \ldots, n)\) and \(\text{Proposition 4.5}\) that \(\nu^{-1}(2\mathbb{Z}) = ab(x_0, \ldots, x_n | \alpha_i x_i = x_0, i = 1, \ldots, n) \cong \text{Ext}(\Gamma; \mathbb{Z})\). Therefore we have the isomorphism between \(\text{Ext}(\Gamma; \mathbb{Z}/2\mathbb{Z})\) and \(\text{Ext}(\Gamma; \mathbb{Z})/2\text{Ext}(\Gamma; \mathbb{Z})\). □

**Theorem 4.3.** Suppose that both of \((b, \beta_1, \ldots, \beta_n)\) and \((b', \beta'_1, \ldots, \beta'_n)\) give the same class in \(\text{Ext}(\Gamma; \mathbb{Z}/2\mathbb{Z})\). If \((b', \beta'_1, \ldots, \beta'_n)\) gives an euler class as in Theorem 2.2 then the central extension \([(b', \beta'_1, \ldots, \beta'_n)]\) in \(\text{Ext}(\Gamma; \mathbb{Z})\) induces an \(\text{SL}_2(\mathbb{R})\)-representation \(\rho'\) of \(\pi_1(M)\) such that \(\rho'(h) = -I\).

**Proof.** It follows from the assumption that we have an isomorphism \(\varphi\) from \(\pi_1(M)/\langle h^2 \rangle\) to \(\pi_1(M')/\langle h^2 \rangle\) as

\[
\begin{array}{ccc}
0 & \to & \mathbb{Z}/2\mathbb{Z} \\
\downarrow & & \downarrow \varphi \\
0 & \to & \pi_1(M)/\langle h^2 \rangle \\
\end{array}
\]

where \(M\) and \(M'\) are the Seifert manifolds with the indices \((g; 1, b), (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\) and \((g; 1, b'), (\alpha_1, \beta'_1), \ldots, (\alpha_n, \beta'_n)\). Since \(b'x_0 + \beta'_1 x_1 + \cdots + \beta'_n x_n\) is an euler class in \(H^2(\Gamma; \mathbb{Z}) = \text{Ext}(\Gamma; \mathbb{Z})\), there exists a \(\text{PSL}_2(\mathbb{R})\)-representation \(\bar{\rho}\) of \(\pi_1(M')\). This \(\text{PSL}_2(\mathbb{R})\)-representation \(\bar{\rho}\) gives an \(\text{SL}_2(\mathbb{R})\)-representation \(\rho'\) such that \(\rho'(h') = -I\). Taking the pull–back of the homomorphism from \(\pi_1(M')/\langle h^2 \rangle\) to \(\text{SL}_2(\mathbb{R})\) by \(\varphi\), we obtain a homomorphism from \(\pi_1(M)/\langle h^2 \rangle\) to \(\text{SL}_2(\mathbb{R})\). The composition with the projection from \(\pi_1(M)\) gives an \(\text{SL}_2(\mathbb{R})\)-representation of \(\pi_1(M)\) sending \(h\) to \(-I\). □

**Remark 4.4.** Although \((b, \beta_1, \ldots, \beta_n)\) is contained in the equivalent class of \((b', \beta'_1, \ldots, \beta'_n)\) in \(\text{Ext}(\Gamma; \mathbb{Z}/2\mathbb{Z})\), the induced representation \(\rho\) by \((b, \beta_1, \ldots, \beta_n)\) is not necessarily conjugate to \(\rho'\) induced by \((b', \beta'_1, \ldots, \beta'_n)\). We can see an example in Section 5.

**Proposition 4.5.** Suppose that \(\bar{\rho}\) and \(\bar{\rho}'\) are \(\text{PSL}_2(\mathbb{R})\)-representations of \(\Gamma\) satisfying that

\[
[e(\bar{\rho})] = [e(\bar{\rho}')]
\]

in \(\text{Ext}(\Gamma; \mathbb{Z}/2\mathbb{Z})\) such that \(\bar{\rho}\) is an euler class in \(H^2(\Gamma; \mathbb{Z}) = \text{Ext}(\Gamma; \mathbb{Z})\). Then the \(\text{SL}_2(\mathbb{R})\)-representation \(\rho\) of \(\pi_1(M(\frac{1}{b}, \frac{1}{\beta_1}, \ldots, \frac{1}{\beta_n}))\) is not conjugate to \(\rho'\).

**Proof.** If \(\rho\) were conjugate to \(\rho'\), then \(\bar{\rho}\) would also be conjugate to \(\bar{\rho}'\). Since the euler class is invariant under conjugation (we refer to the proof of \(\text{[JN85]}\), Theorem 1), the euler class \(e(\bar{\rho})\) must coincide with \(e(\bar{\rho}')\). This is a contradiction to the assumption that \(e(\bar{\rho}) \neq e(\bar{\rho}')\). □

**Theorem 4.6.** Suppose that \(M\) denotes a Seifert manifold \(M(\frac{1}{b}, \frac{1}{\beta_1}, \ldots, \frac{1}{\beta_n})\) and integers \((b', \beta'_1, \ldots, \beta'_n)\) gives an euler class satisfying the criteria of Jankins and Neumann in Theorem 2.2 then

If \((b, \beta_1, \ldots, \beta_n)\) and \((b', \beta'_1, \ldots, \beta'_n)\) give the same class in \(\text{Ext}(\Gamma; \mathbb{Z}/2\mathbb{Z})\), then
there exists an $\text{SL}_2(\mathbb{R})$-representation $\rho'$ of $\pi_1(M)$ such that the asymptotic behavior of $\text{Tor}(M; \rho'_{2N})$ is expressed as

$$\lim_{N \to \infty} \frac{\log |\text{Tor}(M; \rho'_{2N})|}{2N} = \left(2 - 2g - \sum_{j=1}^{n} \frac{\lambda_j}{\lambda'_j} - 1\right) \log 2$$

where $\lambda'_j = \alpha_j/(\alpha_j, \beta'_j)$.

5. Examples

5.1. $\text{SU}(1, 1)$-representations of a Brieskorn manifold. It is known that $\text{SL}_2(\mathbb{R})$ is conjugate to $\text{SU}(1, 1) = \left\{ \begin{pmatrix} \xi & \eta \\ \bar{\eta} & \bar{\xi} \end{pmatrix} \mid |\xi|^2 - |\eta|^2 = 1 \right\}$ by $\left\{ \begin{pmatrix} 1 & \sqrt{-1} \\ 1 & \sqrt{-1} \end{pmatrix} \right\}$. We can consider $\text{SU}(1, 1)$-representations instead of $\text{SL}_2(\mathbb{R})$.

Suppose that $M$ is a Seifert homology sphere with three exceptional fibers. The genus of the base orbifold must be zero. The Seifert index is given by

$(0; (1, b), (a_1, \beta_1), (a_2, \beta_2), (a_3, \beta_3))$.

We can express $\pi_1(M)$ as

$$\pi_1(M) = \langle q_1, q_2, q_3, h \mid h: \text{central}, q_1^{a_j} = h^j (j = 1, 2, 3), q_1q_2q_3 = h^b \rangle.$$ 

We write $\begin{pmatrix} \xi_j & \eta_j \\ \bar{\eta}_j & \bar{\xi}_j \end{pmatrix}$ for $\rho(q_j)$ and $a_j + bj \sqrt{-1}$ for each $\xi_j$.

We will compute irreducible $\text{SU}(1, 1)$-representations of $\pi_1(M)$ such that $\rho(h) = -I$ up to conjugation.

Definition 5.1. For an irreducible $\text{SU}(1, 1)$-representation of $\pi_1(M)$, we have the triple $(k_1, k_2, k_3)$ of natural numbers such that

- $\text{tr} \rho(q_j) = 2 \cos(k_j \pi / \alpha_j)$;
- $0 < k_j < \alpha_j$;
- $k_j \equiv \beta_j \mod 2$.

Lemma 5.2. Let $\rho$ be an irreducible $\text{SU}(1, 1)$-representation of $\pi_1(M)$. The conjugacy class of $\rho$ is determined by the pair of the triple $(k_1, k_2, k_3)$ and the sign of $b_1$.

Proof. By the relations of $\pi_1(M)$, the representation of $\rho$ is determined by $\xi_1, \eta_1, \xi_2$ and $\eta_2$. We can assume that $\rho(q_1)$ is diagonal i.e., $\eta_1 = 0$, and $\eta_2$ is a positive real number. The second assumption is realized by the conjugation of a diagonal matrix. Then $a_1$ equals to $\cos(k_1 \pi / \alpha_1)$ and $b_1 = \pm \sin(k_1 \pi / \alpha_1)$. Since $\text{tr} \rho(q_2)$ equals to $2a_2$, we also have that $a_2 = \cos(k_2 \pi / \alpha_2)$. By the relation that $q_1q_2 = h^{-b}q_3^{-1}$, we have that

$$\text{tr} \rho(q_3) = \text{tr} \rho(q_3)^{-1} = (-1)^b 2(a_1a_2 - b_1b_2).$$

Together with $\text{tr} \rho(q_3) = 2 \cos(k_3 \pi / \alpha_3)$, we can see that $b_2$ is determined by $(k_1, k_2, k_3)$ and the sign of $b_2$. Last $\eta_2$ is given by the equality that $a_2^2 + b_2^2 - \eta_2^2 = 1$. □

Remark 5.3. The diagonal matrix $\begin{pmatrix} e^{\sqrt{-1} \theta} & 0 \\ 0 & e^{-\sqrt{-1} \theta} \end{pmatrix}$ is not conjugate to $\begin{pmatrix} e^{-\sqrt{-1} \theta} & 0 \\ 0 & e^{\sqrt{-1} \theta} \end{pmatrix}$ in $\text{SU}(1, 1)$.

Proposition 5.4. Suppose that an $\text{SU}(1, 1)$-representation $\rho$ of $\pi_1(M)$ is irreducible and satisfies that $\rho(h) = -I$. 

If $b$ is even, then the corresponding triple $(k_1, k_2, k_3)$ satisfies the either inequality:

$$0 < \frac{k_1}{a_3} \leq \left| \frac{k_1}{a_1} - \frac{k_2}{a_2} \right| \quad \text{or} \quad 1 - \left| \frac{k_1 + k_2}{a_2} - 1 \right| \leq \frac{k_3}{a_3} < 1. \tag{6}$$

If $b$ is odd, then the corresponding triple $(k_1, k_2, k_3)$ satisfies the either inequality:

$$0 < \frac{k_1}{a_3} \leq \left| \frac{k_1}{a_1} + \frac{k_2}{a_2} - 1 \right| \quad \text{or} \quad 1 - \left| \frac{k_1}{a_1} - \frac{k_2}{a_2} \right| \leq \frac{k_3}{a_3} < 1. \tag{7}$$

Proof. We prove the case that $b$ is even. In this case, we have the equality that $\rho(q_1) = \rho(q_2)^{-1} \rho(q_3)^{-1}$. We can assume $\rho(q_1)$ is diagonal, i.e.,

$$\rho(q_1) = \begin{pmatrix} \xi_1 & 0 \\ 0 & \bar{\xi}_1 \end{pmatrix}$$

without loss of generality. Hence the trace $\text{tr} \rho(q_3) = 2 \cos(k_3 \pi/a_3)$ equals to $\xi_1 \xi_2 + \bar{\xi}_1 \bar{\xi}_2 = 2(a_1 a_2 - b_1 b_2)$. We have that

$$a_1 a_2 - b_1 b_2 = \cos \left( \frac{k_3 \pi}{a_3} \right). \tag{8}$$

It follows from the assumption that $a_1 = \cos(k_1 \pi/a_1)$, $b_1 = \pm \sin(k_1 \pi/a_1)$. We also have the inequality that $b_1^2 \geq \sin^2(k_2 \pi/a_2)$ from $|\xi_2|^2 = 1 + |\eta|^2 \geq 1$, $|\xi_2|^2 = a_2^2 + b_2^2$ and $a_2 = \cos(k_2 \pi/a_2)$. Together with Eq. (8), we obtain the following constrains:

$$\cos \left( k_1 \pi/a_1 \right) \cos \left( k_2 \pi/a_2 \right) \mp \sin \left( k_1 \pi/a_1 \right) b_2 \geq \cos \left( k_3 \pi/a_3 \right).$$

$$b_2 \geq \sin^2 \left( k_2 \pi/a_2 \right).$$

We can derive our inequality (6) from the above constrains. Similarly we can also derive the inequality (7) in the case that $b$ is odd.

Remark 5.5. One can find the similar inequality for SU(2)-representations in [FS90, Sav99 §14.5].

5.2. Brieskorn manifold of type $(2, 3, 7)$. Let $M$ be the Seifert manifold of the index $(\gamma; (1, b), (a_1, b_1), (a_2, b_2), (a_3, b_3)) = (0; (1, -1), (2, 1), (3, 1), (7, 1))$. Then the fundamental group $\pi_1(M)$ is expressed as

$$\pi_1(M) = \langle q_1, q_2, q_3, h | h: \text{central}, q_1^2 = h, q_1^3 = h, q_3^7 = h, q_1 q_2 q_3 = h^{-1} \rangle.$$

Lemma 5.6. There are two conjugacy classes of irreducible SU(1, 1)-representations $\rho$ of $\pi_1(M)$ such that $\rho(h) = -I$.

Proof. We have the triple $(k_1, k_2, k_3)$ of natural numbers for the SU(1, 1)-representation $\rho$, as in Definition 5.1. This triple satisfies the constrain (7). Therefore we have only one triple $(k_1, k_2, k_3) = (1, 1, 1)$. Since we have two possibility of the sign of $b_1$, we can conclude that there are the two conjugacy classes of irreducible SU(1, 1)-representations of $\pi_1(M)$.

We see the correspondence between the conjugacy classes and the euler classes of PSL$_2(\mathbb{R})$-representations of $\Gamma$. From Theorem 4.3 it is enough to find equivalent classes $[\{b', \beta_1', \ldots, \beta_3'\}]$ in $\text{Ext}(\Gamma; \mathbb{Z}/2\mathbb{Z})$, which satisfy the criteria of Theorem 2.2. Since the genus $g$ in the Seifert index is equal to 0 and the number $n$ of exceptional fibers is equal to 3, we consider the cases (ib) and (ic) in Theorem 2.2.

(ub) We suppose that $b' = -1$. If we find $(\beta_1', \beta_2', \beta_3')$ such that $\beta_1'/2 + \beta_2'/3 + \beta_3'/7 \leq 1$, then we have only solution $(1, 1, 1)$. Since these integers $(b', \beta_1', \beta_2', \beta_3')$ coincides with $(b, \beta_1, \beta_2, \beta_3) = (-1, 1, 1, 1)$ in the Seifert index of $M$, this solution gives an PSL$_2(\mathbb{R})$-representation of $\pi_1(M)$.
We suppose that $b' = -2$. If we find $(\beta'_1, \beta'_2, \beta'_3)$ such that $\beta'_1/2 + \beta'_2/3 + \beta'_3/7 \geq n-1 = 2$, then we have only solution $(1, 2, 6)$. This solution induces an $\text{PSL}_2(\mathbb{R})$-representation of $\pi_1(M')$ where $M'$ is the Seifert manifold with the index of $(g; (1, b), (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_n, \beta_n)) = (0; (1, -2), (2, 1), (3, 2), (7, 6))$.

Remark 5.7. The Seifert manifold $M'$ in \textbf{lit} is obtained by the homeomorphism by reversing the orientation of a fiber in $M$. This is due to that the Seifert index of $M'$ is obtained from $M$ as follows:

$$(0; (1, -1), (2, 1), (3, 1), (7, 1)) \xrightarrow{\text{ori, rev}} (0; (1, 1), (2, -1), (3, -1), (7, -1))$$

$$= (0; (1, 1 - 3), (2, -1 + 2), (3, -1 + 3), (7, -1 + 7))$$

$$= (0; (1, -2), (2, 1), (3, 2), (7, 6)).$$

We refer to \[NJ81\] Theorem 1.5 for the above operations of Seifert index.

Lemma 5.8. The isomorphism from $\pi_1(M)$ to $\pi_1(M')$ induced by the homeomorphism in Remark\textbf{5.7} is given by the following correspondence:

$$h \mapsto h^{-1}, \quad q_j \mapsto q_j' h^{-1}.$$ 

Proof. The homeomorphism by reversing the orientation of a fiber induces the correspondence sending $h$ to $h^{-1}$. The change of Seifert index from $(0; (1, 1), (2, -1), (3, -1), (7, -1))$ to $(0; (1, -2), (2, 1), (3, 2), (7, 6))$ corresponds to changing the generators $q_j$ to $q_j'$. If we write $q_j'$ for the new generators, then we have the presentation of $\pi_1(M')$.

Proposition 5.9. Let $\rho$ and $\rho'$ be irreducible $\text{SU}(1, 1)$-representations of $\pi_1(M)$ induced by $(b, \beta_1, \beta_2, \beta_3) = (-1, 1, 1, 1)$ and $(b', \beta'_1, \beta'_2, \beta'_3) = (-2, 1, 1, 1)$. The conjugacy class of $\rho$ is different from that of $\rho'$.

Proof. The euler class of $e(\rho')$ is equal to $-e(\bar{\rho})$. It follows from Proposition \[4.5\] that $\rho$ is not conjugate to $\rho'$.

Corollary 5.10. The $\text{SU}(1, 1)$-representations $\rho$ and $\rho'$ give all representatives in the conjugacy classes of irreducible $\text{SU}(1, 1)$-representations sending $h$ to $-I$.

Therefore we have the isomorphism $\varphi$ from $\pi_1(M)$ to $\pi_1(M')$ induced by the orientation reversing homeomorphism. The composition of $\varphi$ gives an $\text{PSL}_2(\mathbb{R})$-representation of $\pi_1(M)$. These two $\text{PSL}_2(\mathbb{R})$-representations give the representatives of different conjugacy classes in the set of $\text{SL}_2(\mathbb{R})$-representations of $\pi_1(M)$.

This observation can be extend to a general Seifert manifold.

Theorem 5.11. Suppose that $M$ is a Seifert manifold and $M'$ is the orientation reversed manifold $-M$ along fibers. If there exists a $\text{PSL}_2(\mathbb{R})$-representation $\bar{\rho}$ of $\pi_1(M)$, then we also have a $\text{PSL}_2(\mathbb{R})$-representation $\bar{\rho}'$ of $\pi_1(M')$ such that

- $\bar{\rho}'$ induces an $\text{SL}_2(\mathbb{R})$-representation $\rho'$ of $\pi_1(M)$ which arises the following asymptotic behavior of the Reidemeister torsion:

$$\lim_{N \to \infty} \log |\text{Tor}(M; \frac{1}{\rho}_{\mathbb{R}}, \ldots, \frac{\alpha_n}{\mathbb{R}}; \rho_{2N})| / 2N = \lim_{N \to \infty} \log |\text{Tor}(M; \frac{1}{\rho}_{\mathbb{R}}, \ldots, \frac{\alpha_n}{\mathbb{R}}; \rho_{2N})| / 2N$$

where $\rho$ is the $\text{SL}_2(\mathbb{R})$-representation induced by $\bar{\rho}$;
- $e(\bar{\rho}') = -e(\bar{\rho})$ where $\bar{\rho}$ and $\bar{\rho}'$ are the induced $\text{PSL}_2(\mathbb{R})$-representation of $\Gamma$. 

\[\square\]
Proof. Let $\varphi$ denote the orientation reversing homeomorphism from $M$ to $M'$. Then we can choose the composition $\rho \circ \varphi^{-1}$ as $\rho'$. By the construction, we can see the equality on the asymptotic behaviors of the Reidemeister torsions. We also have the equality on the euler classes of $\text{PSL}_2(\mathbb{R})$-representations of $\Gamma$. □

Acknowledgment

The author wishes to express his thanks to Professor Makoto Sakuma for drawing the author’s attention to $\text{PSL}_2(\mathbb{R})$-representations of a Fuchsian group and the induced asymptotic behavior of the Reidemeister torsion. This research was supported by by JSPS KAKENHI Grant Number 26800030.

References

[FS90] S. Fintushel and R. Stern, *Instanton homology of Seifert fibered homology three spheres*, Proc. London Math. Soc. 61 (1990), 109–137.

[JN85] M. Jankins and W. Neumann, *Homomorphisms of fuchsian groups to $\text{PSL}(2, \mathbb{R})$*, Comment. Math. Helv. 60 (1985), 480–495.

[NJ81] W. Neumann and M. Jankins, *Seifert manifolds*, Lecture notes, Brandeis Univ., 1981.

[Sav99] N. Saveliev, *Lectures on the topology of 3-manifolds*, de Gruyter Textbook, Walter de Gruyter & Co., Berlin, 1999.

[Sco83] P. Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. 15 (1983), 401–487.

[Yam] Y. Yamaguchi, *A surgery formula for the asymptotics of the higher dimensional reidemeister torsion and seifert fibered spaces*, arXiv:1210.8040.

[Yam13] ______. *Higher even dimensional Reidemeister torsion for torus knot exteriors*, Math. Proc. Cambridge Philos. Soc. 155 (2013), 297–305.