Super-Penrose process due to collisions inside ergosphere

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If two particles collide inside the ergosphere, the energy in the centre of mass frame can be made unbound provided at least one of particles has a large negative angular momentum (A. A. Grib and Yu. V. Pavlov, Europhys. Lett. 101, 20004 (2013)). We show that the same condition can give rise to unbounded Killing energy of debris at infinity, i.e. super-Penrose process. Proximity of the point of collision to the black hole horizon is not required.

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I. INTRODUCTION

Investigation of high energy collisions in the black hole background now attracts much attention. It was stimulated by the observation that collision of two particles moving towards a black hole can produce an indefinitely large energy $E_{c.m.}$ in the centre of mass frame [1]. The same happens if particles move in opposite directions [2] - [4]. In this context, there are two different issues, connected with obtaining (i) large energies in the centre of mass $E_{c.m.}$ and (ii) large Killing energies $E$ of debris at infinity. It turned out that there are serious restrictions on $E$ even in spite of large $E_{c.m.}$ since strong gravitational redshift almost compensates the excess of energy [5] - [7]. Nonetheless, there exist scenarios, in which $E$ is also significantly amplified or even unbounded (hence, extraction of energy from a black hole is big) [8] - [10]. Such cases are called the super-Penrose process in [9], and we stick
to this terminology. However, the super-Penrose process near black holes has its own severe restrictions \[11\], \[12\].

Up to now, all discussion in literature concerning the super-Penrose process in the black hole background applied to collisions near the event horizon only. In the present paper we show that there exists an alternative mechanism in which large $E$ and $E_{c.m.}$ are compatible with each other. It is based on the Grib-Pavlov mechanism of collision. It was shown in \[13\] that if two particles collide inside the ergosphere of the Kerr metric and at least one of particles has the large negative angular momentum $L$, the resulting $E_{c.m.}$ is also large. Later on, it was shown in \[14\] that this is a universal property of ergoregions of generic axially symmetric rotating black holes. In both aforementioned papers, only the properties of $E_{c.m.}$ were considered. Below, we will see that for such a type of collision the super-Penrose mechanism is possible. It is worth stressing that, although in the scenarios under discussion $L$ for initial particles is supposed to be large, their Killing energies $E$ are finite. The similar combination (large $L$ and modest $E$) occurs also in some other scenarios of high energy collisions - say, in the vicinity of magnetized black holes \[15\] - \[17\].

II. BASIC FORMULAS

Let us consider the metric

$$ds^2 = -N^2 dt^2 + g_\phi(d\phi - \omega dt)^2 + \frac{dr^2}{A} + g_\theta d\theta^2. \quad (1)$$

We assume that all metric coefficient do not depend on $t$ and $\phi$. This gives rise to the conservation of the energy $E = -mu_0$ and angular momentum $L = mu_\phi$. Here, $m$ is the particle’s mass, $u^\mu = \frac{dx^\mu}{dt}$ is the four-velocity, $\tau$ is the proper time. In what follows, we restrict ourselves by motion in the equatorial plane $\theta = \frac{\pi}{2}$. For such a motion, one can always redefine the radial coordinate to achieve $N^2 = A$. Then, equations of motion read

$$m \frac{dt}{d\tau} = \frac{X}{N^2}, \quad (2)$$
$$X = E - \omega L, \quad (3)$$
$$m \frac{d\phi}{d\tau} = \frac{L}{g_\phi} + \frac{\omega X}{N^2}, \quad (4)$$
$$m \frac{dr}{d\tau} = \sigma Z, \quad Z = \sqrt{X^2 - N^2 \left( \frac{L^2}{g_\phi} + m^2 \right)}. \quad (5)$$
Here, \( \sigma = \pm 1 \) depending on the direction of motion.

We assume the forward-in time condition \( \frac{d\tau}{d\xi} > 0 \), whence (for \( N \neq 0 \))

\[
X > 0. \tag{6}
\]

If two particles 1 and 2 collide to produce particles 3 and 4, the conservation of energy and angular momentum gives us

\[
E_1 + E_2 = E_3 + E_4, \tag{7}
\]
\[
L_1 + L_2 = L_3 + L_4. \tag{8}
\]

The conservation of the radial momentum reads

\[
\sigma_1 Z_1 + \sigma_2 Z_2 = \sigma_3 Z_3 + \sigma_4 Z_4. \tag{9}
\]

It is implied that masses of all particles are fixed. Say, one can take \( m_3 = m_1, m_4 = m_2 \) for the elastic collision or \( m_3 = m_4 = 0 \) for annihilation of two initial particles into gamma quanta. The quantities \( E_1, E_2, L_1 \) and \( L_2 \) are fixed. We can also fix, say, \( L_4 \). Then, three equations (7) - (9) determine three unknowns \( E_3, E_4, L_3 \).

### III. SCENARIOS OF COLLISION

We assume that particle 3 moves outward right after collision and escapes, so \( \sigma_3 = +1 \). By assumption, particle 2 has a large negative angular momentum \( L_2 = -|L_2| \). In general, eq. (9) is quite cumbersome algebraically. As our goal is just to demonstrate the existence of the super-Penrose process, we will make several simplifications. We assume that

\[
L_1 = L_2 b, L_4 = a L_2, \tag{10}
\]

where \( b \) and \( a \) are numbers. Then, the conservation of the angular momentum entails that

\[
L_3 = L_2 (1 + b - a). \tag{11}
\]

In general, this still leads to rather bulky algebraic expressions in (9). We restrict ourselves by the case \( a = 1 + b \), so \( L_3 = 0 \). This is quite sufficient for our purpose - to demonstrate the existence of the super-Penrose process. We are interested in the scenario in which \( E_3 \) is large and has the order \( |L_2| \), so we put

\[
E_3 = |L_2| y + O(1), \tag{12}
\]
Correspondingly,

\[ E_4 = -|L_2|y + O(1), \]  

so collision must occur inside the ergosphere where negative energies are allowed. Then,

\[ X_1 \approx \omega b|L_2|, \]  
\[ X_2 \approx \omega |L_2|, \]  
\[ X_3 \approx |L_2|y, \]  
\[ X_4 \approx |L_2|[(\omega (1 + b) - y]. \]

Here, condition (6) for particle 1 requires \( b > 0 \). It is satisfied automatically for particle 2. It is also satisfied for particle 3, provided (13) holds true. For particle 4 it gives us

\[ \omega (1 + b) - y > 0. \]  

By substitution into (9), we have in the leading order in \( L_2 \) the equation

\[ \Omega_c(\sigma_2 + \sigma_1 b) - y = \sigma_4 \sqrt{[(y - \omega_c(1 + b)]^2 + (1 + b)^2(\Omega_c^2 - \omega_c^2)}, \]  

where subscript ”c” means that the corresponding quantity is taken in the point of collision,

\[ \Omega = \sqrt{\frac{g_{00}}{g_\phi}}, g_{00} = -N^2 + g_\phi \omega^2. \]  

As inside the ergosphere \( g_{00} > 0 \), the quantity \( \Omega \) is real. This is again the point where the properties of the ergoregion come into play.

If eq. (20) has a positive root \( y \) and for this root the forward-in time condition (6) is satisfied, the super-Penrose process does occur.

Taking the square of (20), one can find that

\[ b\Omega_c^2(\varepsilon - 1) + y[(1 + b)\omega_c - \Omega_c(\sigma_2 + b\sigma_1)] = 0, \]

where

\[ \varepsilon = \sigma_1 \sigma_2. \]  

As we are interested in the existence of the root \( y \neq 0 \), we must take

\[ \varepsilon = -1. \]
Then,
\[ y = \frac{2b\Omega_c^2}{(b + 1)\omega_c + \Omega_c\sigma_2(b - 1)}, \]  
(25)
\[ \frac{X_4}{L_2} = \omega_c(1 + b) - y = \frac{V}{(b + 1)\omega_c + \Omega_c\sigma_2(b - 1)}, \]  
(26)
where
\[ V = \omega_c^2(1 + b)^2 + \omega_c\Omega_c\sigma_2(b^2 - 1) - 2b\Omega_c^2. \]  
(27)
Taking into account that according to (21), \( \omega > \Omega \), it is seen that for any \( b > 0 \), the denominator in (26) is positive. For the numerator we have the condition \( V > 0 \). We can rewrite (27) as
\[ V = 2b(\omega_c^2 - \Omega_c^2) + \omega_c[\omega_c(1 + b^2) + \sigma_2\Omega_c(b^2 - 1)]. \]  
(28)
As \( \omega > \Omega \), it is clear from (28) that indeed \( V > 0 \) for all values of parameters.

One should be careful about the sign of \( \sigma_4 \) to avoid fake roots after taking the square. This sign must coincide with that of the left hand side of (20). It is straightforward to check that
\[ \text{sign} \sigma_4 = \text{sign}[\omega_c\sigma_2(1 - b^2) - \Omega_c(1 + b^2)]. \]  
(29)
For example, for \( \sigma_2 = -1 \) and \( b < 1 \), we must take \( \sigma_4 = -1 \). However, if, say, \( \sigma_2 = -1 \), \( b > 1 \) and \( \omega > \Omega_c\frac{b^2 + 1}{b^2 - 1} \), we have \( \sigma_4 = +1 \).

After collision, particle 3 moves away from a black hole. In doing so, there are no turning points for it. Indeed, for this particle \( L_3 = 0 \) and
\[ Z_3^2 = E_3^2 - \frac{N^2}{g_\phi}m_3^2, \]  
(30)
where we took into account the mass term which was discarded before in (5) as small correction. Here, \( E_3^2 = O(L_3^2) \) is large, the second term is finite, so indeed \( Z > 0 \). As far as particle 4 is concerned, it falls into a black hole, if \( \sigma_4 = -1 \). If \( \sigma_4 = +1 \), it moves from a black hole to the turning point and bounces back. Particle 4 cannot escape to the asymptotically flat infinity since its energy is negative.

Thus we succeed in the sense that the unbounded energy \( E_3 \) is obtained. For this purpose, it is necessary in our scenario with \( L_3 = 0 \) that particles 1 and 2 move in the opposite directions before collisions according to (23), (24).
IV. ENERGY IN THE CENTRE OF MASS

To evaluate the energy of the centre of mass $E_{c.m.}$, one can use the known formula (see, e.g. eq. 19 of [14] in which one should put $\theta = \text{const}$). It is more convenient to apply it to the pair of particles 3 and 4 than to the original ones 1 and 2 since now $L_3 = 0$. Then, it follows from the aforementioned formula that

$$E_{c.m.}^2 = \frac{X_3 X_4 - \sigma_4 Z_3 Z_4}{N^2}. \quad (31)$$

Taking into account (5), (17) and (18) we obtain in the main approximation

$$E_{c.m.}^2 = \frac{L_2^2 \mu}{N^2}. \quad (32)$$

$$\mu = y\{[\omega c(1 + b) - y] - \sigma_4 \sqrt{[\omega c(1 + b) - y]^2 - (1 + \sigma_4 \Omega_c^2)} \}. \quad (33)$$

Obviously, $\mu > 0$ for any sign of $\sigma_4$. When $L_2^2 \to \infty$, the energy $E_{c.m.} \to \infty$ as well.

V. COLLISIONS NEAR THE BOUNDARY OF ERGOSPHERE

In the previous section, we mainly concentrated on the case when an escaping particle 3 has the angular momentum $L_3 = 0$. We found that the scenario with unbounded $E_3$ are possible, provided $|L_2|$ is large enough. One can ask, whether this value is singled out and what changes if $L_3 \neq 0$. Although, as is said above, formulas become in general cumbersome, there is a situation when analysis can be carried out analytically in a rather simple form. This is the case when collisions occur in the vicinity of the boundary of the ergoregion (see below).

As before, we assume that all angular momenta are proportional to $L_2$. However, now both the coefficients $a$ and $b$ introduced in the beginning of Section III are free parameters. Then, instead of eqs. (17) and (18) we have

$$X_3 \approx |L_2| [y + \omega(1 + b - a)]. \quad (34)$$

$$X_4 \approx |L_2| [\omega a - y]. \quad (35)$$

Equations (15) and (16) are still valid. Using the expression (5) and taking into account (10), (11) we can write the conservation of radial momentum (9) in main approximation in
the form
\[(\sigma_1 b + \sigma_2) \Omega = \sqrt{y^2 + 2y\omega(1 + b - a) + \Omega^2(1 + b - a)^2} + \sigma_4\sqrt{y^2 - 2y\omega a + a^2\Omega^2}.\] (36)

Here, we put \(\sigma_3 = +1\) for the escaping particle, as before. All quantities in (36) are taken in the point of collision. When \(a = 1 + b\), we return to (20).

It is sufficient to find at least one scenario with unbounded \(E_3\). Let us choose
\[\sigma_1 = 1, \sigma_2 = -1, b > 1, 0 < a < 1 + b.\] (37)

By definition, \(g_{00} = 0\) on the boundary of the ergoregion, so \(\Omega = 0\) according to (21). Let collision occur inside the ergoregion but very close to its boundary. Then, \(\Omega \to 0\). We expect the existence of the solution of (36) in the form
\[y\omega = \Omega^2 x,\] (38)
where \(x = O(1)\). Although \(\Omega\) is small, we imply that \(\Omega^2 |L_2|\) is still large enough to have \(E_3\) large according to (12). Now we can neglect in (36) terms \(y^2\) inside the radicals and obtain the equation
\[F(b) = b - 1 = f(x) \equiv \sqrt{(1 + b - a)(2x + (1 + b - a))} - \sqrt{-2x + a + a^2},\] (39)
where we chose \(\sigma_4 = -1\). Then, \(x \leq x_{\text{max}} = \frac{a}{2}\) to guarantee that the expression inside the second radical is nonnegative.

We want to show that the positive solution of this equation with \(x = O(1)\) does exist. It is seen from (39) that \(f(0) = 1 + b - 2a\). Thus \(f(0) < F\). Meanwhile, the function \(f(x)\) is monotonically increasing. Therefore, if we achieve \(f\left(\frac{a}{2}\right) > F\), it will mean that in some intermediate point \(0 < x < x_{\text{max}}\) the curve \(f(x)\) intersects the line of constant \(F\), so the solution exists. This condition is rendered as
\[b - 1 < \sqrt{(1 + b - a)(1 + b)},\] (40)
whence
\[b > \frac{a}{4 - a}.\] (41)
This is quite compatible with (37), so the solution does exist. For instance, we can take \(b = 3, a = 2\). Then, eq. (39) has the form
\[1 = \sqrt{x + 1} - \sqrt{1 - x}\] (42)
that has a solution \(x = \frac{\sqrt{3}}{2}\).
VI. SUMMARY AND CONCLUSIONS

Thus we showed that there exist scenarios in which particles collide inside the ergoregion in such a way that not only (i) their energy in the centre of mass diverges, but also (ii) the Killing energy of one of particles escaping to infinity is unbounded. This suggests a more easy way of extracting energy since now (i) there is no problem with the redshift and time delay [18], [19], (ii) there is no problem with fine-tuning typical of high energy collisions near the horizon [1]. The restrictions of the super-Penrose process indicated in [11], [12] are also irrelevant now. We analyzed in detail two situations: (a) escaping particle 3 has \( L_3 = 0 \) and (b) collision occurs very closely to the boundary of the ergoregion. Meanwhile, it is clear from derivation that collisions with unbounded \( E_3 \) can occur everywhere inside the ergosphere.

A separate interesting question that remained outside the scope of the present work is the conditions under which particles with unbounded negative \( L \) can occur inside the ergosphere. It was pointed out in [13] that such values can be obtained as a result of preceding collision. However, more thorough inspection showed that the situation is not so simple since there are obstacles against such a scenario in that collisions of particles with finite \( E \) and \( L \) cannot give rise to indefinitely large negative \( L \) (see Sec. VI of [21] for details). Therefore, other mechanisms should be relevant here to achieve large negative \( L \) (thermal fluctuations, variable or chaotic electromagnetic fields, etc.). They are model-dependent and need separate treatment. Meanwhile, the results of our work have general model-independent character irrespective of the way the initial state is prepared.

One can hope that the observation made in the present work can be of use for investigation of the role of collisional Penrose process in astrophysics [20].

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