On Almost Uniform Continuity of Borel Functions on Polish Metric Spaces

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Abstract

We show that, on any given finite Borel measure space with the ambient space being a Polish metric space, every Borel real-valued function is almost a bounded, uniformly continuous function in the sense that for every \( \varepsilon > 0 \) there is some bounded, uniformly continuous function such that the set of points at which they would not agree has measure < \( \varepsilon \). In particular, this result complements the known result of almost uniform continuity of Borel real-valued functions on a finite Radon measure space whose ambient space is a locally compact metric space. As direct applications in connection with some common modes of convergence, under our assumptions it holds that i) for every Borel real-valued function there is some sequence of bounded, uniformly continuous functions converging in measure to it, and ii) for every bounded, Borel real-valued function there is some sequence of bounded, uniformly continuous functions converging in \( L^p \) to it.

Keywords: almost uniform continuity; Borel functions; convergence; extension theorems; finite Borel measures; Lusin’s theorem; Lusin topology; Polish metric spaces

MSC 2020: 30L99; 60A10; 26A15; 28A99

1 Introduction

Let \( \Omega \) be a metric space; let \( M \) be a finite Borel measure over \( \Omega \). It follows from a well-known version of Lusin’s Theorem (e.g. Theorem 2.24 in Rudin [4]) that, if \( \Omega \) is locally compact, if \( M \) is Radon, and if \( f : \Omega \to \mathbb{R} \) is Borel-measurable, then for every \( \varepsilon > 0 \) there is some bounded, uniformly continuous function \( \Omega \to \mathbb{R} \) such that the set of points at which they possibly disagree has measure < \( \varepsilon \). For our purposes, we refer to a Borel function \( \Omega \to \mathbb{R} \) satisfying the conclusion of the above proposition as \((M-)almost uniformly continuous\), where \( \Omega \) is simply a metric space and \( M \), with respect to which the meaning of “almost” is clearly assigned, is simply a finite Borel measure over \( \Omega \). Thus boundedness is also a requirement of almost uniform continuity.

Now, depending on the purposes, local compactness is not always a helping property. For instance, there are metric spaces that are interesting and important in analysis but that are not locally compact; it can be shown that the space \( \mathbb{R}^\mathbb{N} \) of real sequences, equipped with the product metric (in terms of summation) of the (equivalent) Euclidean metric \((x, y) \mapsto 1 \wedge |x - y|\) of \( \mathbb{R} \), is not locally compact, and that the classical Wiener space \( C([0,1], \mathbb{R}) \) is not locally compact with respect to the uniform metric;

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no closed ball in either space is compact. On the other hand, these spaces are indeed complete and separable with respect to the respective metrics, i.e. they are Polish metric spaces. Given the importance of Polish metric spaces, in particular of compact metric spaces, in analysis (and geometry), it would be desirable to have an almost-uniform-continuity result, serving Polish metric spaces and requiring no local compactness, for Borel real-valued functions, such that the assumption on the underlying finite measure is hopefully mild.

It turns out that it suffices for the underlying (finite) measure to be Borel. In this short article, we obtain the desired result that every Borel real-valued function on a Polish metric space is almost uniformly continuous with respect to a pre-specified finite Borel measure. Indeed, given an arbitrary metric space and a finite Borel measure over it, one can already assert that every Borel real-valued function is almost a continuous function in the following sense: It follows from a proposition in Federer [2] (Section 2.3.6) that every Borel real-valued function on the metric space has the property that for every $\varepsilon > 0$ there is some continuous function such that the set of points where they possibly disagree has measure $< \varepsilon$. The proposition is obtained directly from the Federer’s version of Lusin’s theorem and the Tietze’s extension theorem. However, without further assumptions, the approximating continuous functions need not be bounded or uniformly continuous.

The main message of the proof of our main result is, rather than claiming a “significant” advancement in the classical topics, that one might as well obtain the readily applicable result — almost uniform continuity of Borel real-valued functions on a Polish metric space taken as a finite Borel measure space — simply with new twists of known facts. Some interesting by-products of the main result, under the same assumptions, are i) every Borel real-valued function is the (essential) convergence-in-measure limit of bounded, uniformly continuous functions, and ii) every bounded, Borel real-valued function is the (essential) $L^p$-limit of bounded, uniformly continuous functions. After introducing the necessary preliminaries, we proceed to the proofs.

2 Preliminaries

By a Borel measure over a metric space $\Omega$ we mean a measure defined on the Borel sigma-algebra $\mathcal{B}_\Omega$ of $\Omega$ generated by the topology of $\Omega$ induced by the given metric of $\Omega$.

We will stick to the standard measure-theoretic definitions of outer regularity and inner regularity of measures. For our purposes, it would be convenient to introduce another kind of regularity associated with measures. If $M$ is a finite Borel measure over a metric space $\Omega$, the measure $M$ is called co-outer regular if and only if $M(B) = \sup\{M(F) \mid F \subset B \text{ is closed}\}$ for every $B \in \mathcal{B}_\Omega$. The terminology reflects the elementary fact that a closed set is the complement of some open set. Since $\sup\{M(K) \mid K \subset B \text{ is compact}\} \leq \sup\{M(F) \mid F \subset B \text{ is closed}\}$ whenever $B \in \mathcal{B}_\Omega$, the inner regularity (resp. co-outer regularity) of a finite Borel measure need not imply co-outer regularity (resp. inner regularity).

1Although the Federer’s proof is for outer measures in the “usual” sense, it happens to apply to Borel measures in the “usual” sense. A measure in the geometric-measure-theoretic sense is precisely an outer measure in the “usual” sense.

2In the literature of probability theory, co-outer regularity is sometimes also termed inner regularity, and is associated with Borel probability measures over a metric space. Since our arguments will involve both inner regularity in the standard measure-theoretic sense and co-outer regularity, and since every Borel probability measure over a metric space is a finite Borel measure, we choose to assign a new name to the property.
The topology of a metric space always refers to the topology induced by the given metric. By a *Polish metric space* we mean a metric space that is complete and separable with respect to the given metric. If $\Omega$ is a metric space, and if $\mathcal{M}$ is a finite Borel measure over $\Omega$, we will denote by $L^0(\mathcal{M})$ the collection of all Borel functions $\Omega \to \mathbb{R}$, by $L^0_b(\mathcal{M})$ the collection of all bounded, Borel functions $\Omega \to \mathbb{R}$, by $C(\Omega)$ the collection of all continuous functions $\Omega \to \mathbb{R}$, by $C_u(\Omega)$ the collection of all uniformly continuous functions $\Omega \to \mathbb{R}$, and by $C_{b,u}(\Omega)$ the collection of all bounded, uniformly continuous functions $\Omega \to \mathbb{R}$.

Throughout, we will in general write a set $\{x \in \Omega \mid P(x) \text{ holds}\}$ obtained by specification simply as $\{P\}$ whenever no confusion is possible. Thus, if $f, g : \Omega \to \mathbb{R}$, then $\{x \in \Omega \mid f(x) \neq g(x)\} = \{f \neq g\}$. Moreover, when written in juxtaposition with a measure, the set $\{f \neq g\}$ will also be written simply as $(f \neq g)$. For example, we have $\mathcal{M}(\{f \neq g\}) = \mathcal{M}(f \neq g)$. This notation is common in probability theory.

If $A_1, A_2$ are subsets of a topological space, then $A_1$ is said to be relatively dense in $A_2$ if and only if the closure of $A_1$ includes $A_2$. If $A_2$ coincides with the given ambient space, then the relative denseness of $A_1$ in $A_2$ is simply the denseness of $A_1$ in $A_2$ in the usual sense.

We will argue in terms of the language of topology, which may be more conceptually "compact". If $\Omega$ is a metric space, and if $\mathcal{M}$ is a finite Borel measure over $\Omega$, let

$$V(f, \varepsilon) := \{g \in L^0(\mathcal{M}) \mid \mathcal{M}(f \neq g) < \varepsilon\}$$

for every $f \in L^0(\mathcal{M})$ and every $\varepsilon > 0$. We have $f \in V(f, \varepsilon)$ for every $f \in L^0(\mathcal{M})$ and every $\varepsilon > 0$; moreover, the triangle inequality ensures that the intersection of any two $V(f, \varepsilon)$ is some union of $V(f, \varepsilon)$. Topologize $L^0(\mathcal{M})$ in terms of the topology generated by $\{V(f, \varepsilon)\}_{f, \varepsilon}$. Considering the conclusion of the aforementioned Rudin’s version of Lusin’s theorem, we will refer to the topology of $L^0(\mathcal{M})$ thus obtained as a Lusin topology (so $\{V(f, \varepsilon)\}_{f, \varepsilon}$ may naturally be called a Lusin basis), for ease of reference. Accordingly, the topological properties of sets such as closedness with respect to the Lusin topology of $L^0(\mathcal{M})$ will be referred to in terms of the modifier “Lusin”; for instance, a closed subset of $L^0(\mathcal{M})$ with respect to the corresponding Lusin topology will also be said to be Lusin-closed. Now the almost uniform continuity of elements of $L^0(\mathcal{M})$ may be translated as follows: An element $f$ of $L^0(\mathcal{M})$ is almost uniformly continuous if and only if $f$ lies in the Lusin-closure of $C_{b,u}(\Omega)$. Moreover, the almost uniform continuity of elements of $L^0(\mathcal{M})$, which is the conclusion of our main result, may now be stated neatly as: the space $C_{b,u}(\Omega)$ is Lusin-dense in $L^0(\mathcal{M})$.

### 3 Results

We should like to prove our main result:

**Theorem 1.** If $\Omega$ is a Polish metric space, and if $\mathcal{M}$ is a finite Borel measure over $\Omega$, then $C_{b,u}(\Omega)$ is Lusin-dense in $L^0(\mathcal{M})$.

**Proof.** We have $C_{b,u}(\Omega) \subset L^0(\mathcal{M})$; so the Lusin-closure of $C_{b,u}(\Omega)$ is included in $L^0(\mathcal{M})$. It then suffices to show that every element of $L^0(\mathcal{M})$ lies in the Lusin-closure of $C_{b,u}(\Omega)$.

Let $f \in L^0(\mathcal{M})$; let $\varepsilon > 0$. Since every finite Borel measure over a metric space is both outer regular and co-outer regular, which is known and may be obtained neatly from an immediate, apparent

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3When we say “Rudin’s version” or “Federer’s version”, we merely make a nominal distinction, which facilitates the communication.
generalization of the simple proof of Theorem 1.1. in Billingsley [1], adapting the proof of the Federer’s version of Lusin’s theorem for arbitrary metric spaces (i.e. Section 2.3.5, Federer [2]) in the apparent way ensures the existence of some closed subset $F_\varepsilon$ of $\Omega$ such that $\mathcal{M}(F_\varepsilon) < \varepsilon/2$ and $f|_{F_\varepsilon}$ is continuous.

On the other hand, we claim that $\mathcal{M}$ is in fact also inner regular. Indeed, for $\mathcal{M}$ to be inner regular it is sufficient for $\mathcal{M}$ to be inner regular at $\Omega$, i.e. for it to hold that $\mathcal{M}(\Omega) = \sup\{\mathcal{M}(K) \mid K \subset \Omega \text{ is compact}\}$. To see this, fix any $B \in \mathscr{B}_\Omega$. Then, since $\mathcal{M}$ is co-outer regular, for every $\delta > 0$ there is some closed $F \subset B$ such that $\mathcal{M}(F) > \mathcal{M}(B) - \delta/2$; moreover, there is some compact $K \subset \Omega$ such that $\mathcal{M}(K) > \mathcal{M}(\Omega) - \delta/2$. Then $K \cap F$ is compact and included in $B$, and

$$
\mathcal{M}(K \cap F) = \mathcal{M}(F) - \mathcal{M}(F \setminus K) \\
> \mathcal{M}(B) - \delta/2 - \mathcal{M}(K^c) \\
> \mathcal{M}(B) - \delta;
$$

the inner regularity of $\mathcal{M}$ follows. But $\mathcal{M}$ is indeed inner regular at $\Omega$; this follows from a direct apparent application of the simple proof of Theorem 1.3 in Billingsley [1]. We have proved the claim of the inner regularity of $\mathcal{M}$.

Now there is some compact $K \subset F_\varepsilon$ such that $\mathcal{M}(F_\varepsilon \setminus K) < \varepsilon/2$, and so $f|_K$ is bounded and uniformly continuous. Then the McShane’s extension theorem (Corollary 2, McShane [3]) asserts the existence of some $g \in C_{b,u}(\Omega)$ such that $g|_K = f|_K$ (and $g$ preserves the bounds). Since $\{f \neq g\} \subset K^c$, we have

$$
\mathcal{M}(f \neq g) \leq \mathcal{M}(K^c) \\
\leq \mathcal{M}(F_\varepsilon \setminus K) + \mathcal{M}(F_\varepsilon^c) \\
< \varepsilon;
$$

so $f$ lies in the Lusin-closure of $C_{b,u}(\Omega)$. □

**Remark.** As some branches of probability theory admitting extensive literature such as weak convergence theory (e.g. Billingsley [1]) or optimal transport (e.g. Villani [5]) serve as a natural, significant context directly deeply connected with Polish metric spaces taken as a finite Borel measure space, we would stress that the applicability of Theorem 1 covers the Borel probability spaces whose ambient space is a Polish metric space, although this remark is technically apparent. □

There is an interesting application of Theorem 1 for convergence in measure:

**Theorem 2.** If $\Omega$ is a Polish metric space, and if $\mathcal{M}$ is a finite Borel measure over $\Omega$, then $C_{b,u}(\Omega)$ is dense in $L^0(\mathcal{M})$ with respect to the convergence-in-measure topology of $L^0(\mathcal{M})$.

**Proof.** In accordance with the topological flavor, we topologize $L^0(\mathcal{M})$ in terms of the topology generated by the subsets

$$
\{g \in L^0(\mathcal{M}) \mid \inf\{r \in [0, +\infty] \mid \mathcal{M}(|f - g| \geq r) < \varepsilon\} < \varepsilon\}
$$

where $f \in L^0(\mathcal{M})$ and $\varepsilon > 0$. It is well-known (and readily seen) that a sequence in $L^0(\mathcal{M})$ converges in measure in $L^0(\mathcal{M})$ if and only if it converges with respect to this topology in $L^0(\mathcal{M})$. Call the topology the convergence-in-measure topology of $L^0(\mathcal{M})$.

Let $f \in L^0(\mathcal{M})$. If $\varepsilon > 0$, then there is by Theorem 1 some $g \in C_{b,u}(\Omega)$ such that $\mathcal{M}(|f - g| > 0) < \varepsilon/2$, and so $\mathcal{M}(|f - g| \geq \varepsilon/2) \leq \mathcal{M}(|f - g| > 0) < \varepsilon/2$. This implies that $g$ lies in the basic (convergence-in-measure-)neighborhood of $f$ with radius $\varepsilon$. □
We have, as a side observation potentially of interest, more information on the relations between the two topologies of \( L^0(\mathbb{M}) \):

**Proposition 1.** If \( \Omega \) is a metric space, and if \( \mathbb{M} \) is a finite Borel measure over \( \Omega \), then every element of the convergence-in-measure topology of \( L^0(\mathbb{M}) \) is some union of elements of the Lusin topology of \( L^0(\mathbb{M}) \).

**Proof.** For convenience, denote by \( d_c \) the (pseudo-)metric defining a basic open set for the convergence-in-measure topology.

Let \( f \in L^0(\mathbb{M}) \); let \( \varepsilon > 0 \); let \( d_c(f, g) < \varepsilon \). If \( h \) is contained in the basic open set \( V(g, \varepsilon - d_c(f, g)/2) \) of the Lusin topology, then \( d_c(h, g) \leq \varepsilon - d_c(f, g)/2 \); and so

\[
d_c(h, f) \leq d_c(h, g) + d_c(g, f) < \varepsilon.
\]

Theorem 1 may also be applied to obtain an interesting result regarding \( L^p \)-convergence:

**Theorem 3.** If \( \Omega \) is a Polish metric space, and if \( \mathbb{M} \) is a finite Borel measure over \( \Omega \), then \( C_{b,u}(\Omega) \) is relatively \( L^p \)-dense in \( L^0_b(\mathbb{M}) \) for every \( 1 \leq p < +\infty \).

**Proof.** Let \( 1 \leq p < +\infty \). Since \( \mathbb{M} \) is by assumption finite, we have \( C_{b,u}(\Omega) \subset L^0_b(\mathbb{M}) \subset L^p(\mathbb{M}) \). The proof is complete if the \( L^p \)-closure of \( C_{b,u}(\Omega) \) includes \( L^0_b(\mathbb{M}) \).

Let \( f \in L^0_b(\mathbb{M}) \); let \( T > 0 \) be a bound of \( f \). We have \( f \in L^0(\mathbb{M}) \) by definition, and, given any \( \varepsilon > 0 \), there is by Theorem 1 some \( g \in C_{b,u}(\Omega) \) with \( T \) being a bound such that

\[
M(|f - g| > 0) < \left( \frac{\varepsilon}{T} \right)^p = \eta.
\]

Then Minkowski inequality implies

\[
\left( \int_{\Omega} |f - g|^p \, d\mathbb{M} \right)^{1/p} = \left( \int_{|f - g| > 0} |f - g|^p \, d\mathbb{M} \right)^{1/p} < T \eta^{1/p} = \varepsilon.
\]

Applying the proof ideas of Theorems 2, 3 and using the Rudin’s version of Lusin’s theorem together give

**Proposition 2.** If \( \Omega \) is a locally compact metric space, and if \( \mathbb{M} \) is a finite Radon measure over \( \Omega \), then i) \( C_{b,u}(\Omega) \) is dense in \( L^0(\mathbb{M}) \) with respect to the convergence-in-measure topology of \( L^0(\mathbb{M}) \), and ii) \( C_{b,u}(\Omega) \) is relatively \( L^p \)-dense in \( L^0_b(\mathbb{M}) \) for every \( 1 \leq p < +\infty \).

**Remark.** Under the assumptions of Proposition 1, Proposition 1 contains, as far as almost uniform continuity is concerned, information in addition to the corollary to Theorem 2.24, i.e. the Rudin’s version of Lusin’s theorem, in Rudin [4]; in some directions, Proposition 1 contains more information.

As a compact metric space is both locally compact and Polish, a generic corollary certainly follows:

**Corollary 1.** If \( \Omega \) is a compact metric space, and if \( \mathbb{M} \) is a finite Borel measure over \( \Omega \), then i) \( C_{b,u}(\Omega) \) is Lusin-dense in \( L^0(\mathbb{M}) \), ii) \( C_{b,u}(\Omega) \) is dense in \( L^0(\mathbb{M}) \) with respect to the convergence-in-measure topology of \( L^0(\mathbb{M}) \), and iii) \( C_{b,u}(\Omega) \) is relatively \( L^p \)-dense in \( L^0_b(\mathbb{M}) \) for every \( 1 \leq p < +\infty \).
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