NON-ABELIAN DUALITY, PARAFERMIONS
AND SUPERSYMMETRY

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ABSTRACT

Non–Abelian duality in relation to supersymmetry is examined. When the action of the isometry group on the complex structures is non–trivial, extended supersymmetry is realized non–locally after duality, using path ordered Wilson lines. Prototype examples considered in detail are, hyper–Kahler metrics with SO(3) isometry and supersymmetric WZW models. For the latter, the natural objects in the non–local realizations of supersymmetry arising after duality are the classical non–Abelian parafermions. The canonical equivalence of WZW models and their non–Abelian duals with respect to a vector subgroup is also established.

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1 Introduction

Target space duality (T–duality) interpolates between effective field theories corresponding to backgrounds with different spacetime and even topological properties. Since strings propagating in T–dual backgrounds are equivalent and since the validity of the corresponding effective field theories is limited, we may use T–duality as a way of probing truly stringy phenomena. The latter have to be taken into account in order to resolve paradoxes that appear in attempts to describe various phenomena solely in terms of the local effective field theories. Taking one step further this way of reasoning, we may view some long standing problems in fundamental Physics, for instance in black holes Physics, as nothing but paradoxes of the effectively field theory description, which will cease to exist once string theoretical effects are properly taken into account. Though this is a speculation at the moment, it provides the main motivation for this work.

The best ground to test these ideas is in the relation between duality and supersymmetry, in the presence of rotational isometries. In these cases non–local world–sheet effects have to be taken into account in order for supersymmetry and Abelian T–duality to reconcile. Various aspects in this interplay between supersymmetry and Abelian T–duality have been considered in the literature. This paper is a natural continuation of these works for the cases where the duality group is a non–Abelian one (for earlier work see and references therein).

The plan of the paper is as follows: In section 2 we consider 2–dimensional bosonic σ–models that are invariant under the action of a non–Abelian group $G$ on the left. We briefly review the canonical transformation that generates the dual σ–model in a way suitable for transformations of other geometrical objects. Then we extend it to models with $N = 1$ world–sheet supersymmetry. For cases that admit $N = 2$ or $N = 4$ extended supersymmetry we derive the transformation rules of the corresponding complex structures. We show that, when these belong to non–trivial representations of a rotational subgroup of the duality group $G$, non–local world–sheet effects manifested with Wilson lines, are necessary to restore extended supersymmetry at the string theoretical level. Nevertheless, this appears to be lost after non–Abelian duality from a local effective field theory point of view. As examples, 4–dimensional Hyper–Kahler metrics with $SO(3)$ isometry are considered in detail. The Eguchi–Hanson, Taub–NUT and Atiyah–Hitchin metrics are famous examples among them. Explicit expressions for the three complex structures are given in general, which could be useful for other independent applications.

In section 3 we consider the dual of a WZW model for a group $G$ with respect to the vector action of a non–Abelian subgroup $H$. In such cases extended supersymmetry is always realized non–locally after duality. We show how these realizations become natural using classical non–Abelian parafermions of the $G/H$ coset conformal field theory. We also establish the, so far lacking, canonical equivalence of these models. As an example we consider the dual, with respect to $SU(2)$, of the WZW model based on $SU(2) \otimes U(1)$.

In section 4 we present our conclusions, and discuss feature directions of this work.
We consider classical string propagation in 2 Left invariant models of the models of section 3 following Dirac’s method for constrained Hamiltonian systems.

2 Left invariant models

We consider classical string propagation in d-dimensional backgrounds that are invariant under the left action\(^1\) of a group \(G\) with dimension \(\dim(G) \leq d\). We may split the target space variables as \(X^M = \{X^\mu, X^i\}\), where \(X^\mu, \mu = 1, 2, \ldots, \dim(G)\) parametrize a group element in \(G\) and \(X^i, i = 1, 2, \ldots, d - \dim(G)\) are some internal coordinates which are inert under the group action. It will be convenient to think of them as parametrizing a group locally isomorphic to \(U(1)^{d-\dim(G)}\). We also introduce a set of representation matrices \(\{t^A\} = \{t^\mu, t^i\}\), with \(a = 1, 2, \ldots, \dim(G)\) and \(i = 1, 2, \ldots d - \dim(G)\), which we normalize to unity. The components of the left and right invariant Maurer–Cartan 1–forms are defined as

\[
L^A_M = -i Tr(t^A \hat{g}^{-1} \partial_M \hat{g}) , \quad R^A_M = -i Tr(t^A \partial_M \hat{g} \hat{g}^{-1}) = C^{AB}(\hat{g}) L^B_M ,
\]

where \(\hat{g} = g^{it,X}Y\), with \(g \in G\) and \(C^{AB}(\hat{g}) = Tr(t^A \hat{g} t^B \hat{g}^{-1})\). When we specialize to the internal space, \(L^i_j = R^i_j = \delta^i_j\), \(C^{ij} = \delta^{ij}\) and the corresponding structure constants are zero. The inverses of (2.1) will be denoted by \(L^M_A\) and \(R^M_A\) respectively.

The most general Lagrangian density which is manifestly invariant under the transformation \(g \to \Lambda g\), for some constant matrix \(\Lambda \in G\), is given by

\[
\mathcal{L} = E^+_A E^+_B L^A_M L^B_N \partial_+ X^M \partial_- X^N ,
\]

where the couplings \(E^+_{MN}\) can only depend on the \(X^i\)’s and thus are also invariant under the action of the group \(G\). For later use we also introduce \(E^-_{AB} = E^+_BA\). An equivalent expression to (2.2) is

\[
\mathcal{L} = E^+_{ij} \partial_+ X^i \partial_- X^j + E^+_a L^a_\mu \partial_+ X^\mu \partial_- X^\nu + E^+_a L^a_\mu \partial_+ X^\mu \partial_- X^i + E^+_a L^a_\mu \partial_+ X^i \partial_- X^\mu .
\]

The natural time coordinate on the world–sheet is \(\tau = \sigma^+ + \sigma^-\), while \(\sigma = \sigma^+ - \sigma^-\) denotes the corresponding spatial variable. The Poisson bracket of the variable \(X^\mu\) and its conjugate momentum \(P_\mu\) is \(\{X^\mu(\sigma), P_\nu(\sigma')\} = \delta^\mu_\nu \delta(\sigma - \sigma')\). Since the only dependence of (2.3) on the variables \(X^\mu\) is via the combinations \(L^a_\mu \partial_\sigma X^\mu\), it is convenient to know the Poisson brackets of \(L^a_\mu \partial_\sigma X^\mu\) and \(L^a_\mu P_\nu\). After a simple computation we find

\[
\{ \partial_\sigma X^\mu L^a_\mu(\sigma), L^b_\nu P_\nu(\sigma') \} = f^{ab}_{bc} L^c_\mu \partial_\sigma X^\mu \delta(\sigma - \sigma') + \delta^a_b \delta_\sigma \delta(\sigma - \sigma') ,
\]

\[
\{ L^a_\mu P_\nu(\sigma), L^b_\nu P_\nu(\sigma') \} = f^{ab}_{bc} L^c_\mu P_\delta(\sigma - \sigma') .
\]

\(^1\)In the language of Poisson–Lie T–duality [13] we concentrate on cases of semi–Abelian doubles, where the coalgebra is Abelian, or in other words to traditional non–Abelian duality. The reason is that there are no known non–trivial examples of Poisson–Lie T–duality where supersymmetry enters also into the game. Nevertheless, we comment on section 4 on how Poisson–Lie T–duality may be used as a manifest supersymmetry restoration technique, in a string theoretical context.
At this point we perform the transformation \((X^\mu, P_\nu) \to (\tilde{X}^\mu, \tilde{P}_\nu)\) defined as \[ L^a_\mu \partial_\sigma X^\mu = \tilde{P}^a, \quad L^{a\mu} P_\mu = \partial_\sigma \tilde{X}^a - f^{ab}_c \tilde{X}^c, \tag{2.5} \]
where \(f^{ab} \equiv f^{abc} \tilde{X}^c\). One can show that it preserves the Poisson brackets \((2.4)\) and hence it is a canonical one. The \(X^i\)’s remain unaffected by this transformation, so that \(\tilde{X}^i = X^i\). It is then a straightforward procedure to find the Lagrangian density to the dual to \((2.3)\) \(\sigma\)-model by applying the usual rules of canonical transformations in the Hamiltonian formalism. Here we only quote the final result:

\[ \tilde{\mathcal{L}} = E_{ij} \partial_+ X^i \partial_- X^j + (\partial_+ \tilde{X}^a - E_{ai} \partial_+ X^i) (M^{-1})_{ab} (\partial_- \tilde{X}^b + E_{bj} \partial_- X^j), \tag{2.6} \]

with

\[ M_{-}^{ab} = E_{ab}^+ + f_{ab}. \tag{2.7} \]

In addition conformal invariance requires the shift of the dilaton \([1]\) by \(\ln \det(M_-)\). The action \((2.7)\) was obtained in \([8]\) in the traditional approach to non–Abelian duality, where one adds to \((2.2)\) a Lagrange multiplier term and introduces non–dynamical gauge fields which are then integrated out using their classical equations of motion.

The transformation \((2.3)\) was first applied to Principal Chiral Models (PCMs); with \(G = SU(2)\) in \([13]\) and for general group in \([10]\). In PCMs there is no internal space and \(E_{ab}^+ = \delta_{ab}\). Hence, after non–Abelian duality with respect to the left action of the group there are still conserved currents associated with the right action of the group which generate symmetries in the dual model. It is tempting to attribute the success of the canonical transformation \((2.3)\) to the existence of such conserved (local) currents. However, this is not true since \((2.6)\), which has generically no conserved (local) currents, correctly follows from \((2.5)\). Instead, what is common in the models \((2.2)\) is the fact that the group action is entirely from the left. As a consequence, in the traditional approach with gauge fields, we can completely fix a unitary gauge as \(g = 1\), by appropriately choosing the \(X^\mu\)’s. In some sense \((2.3)\) is a straightforward generalization of the corresponding transformation for Abelian isometry groups \([22]\). We will see in section 3 that when the action of the isometry group is not entirely on the left or on the right, the analog of the canonical transformation \((2.5)\) is radically different.

It is important to know how the world–sheet derivatives of the target space variables \(\partial_+ X^M\) transform under the canonical transformation. It is quite straightforward, and in fact easier than applying the canonical transformation in all of its glory, to show that \((2.3)\) and the fact that the canonical transformation preserves the Lorentz invariance of the 2–dimensional \(\sigma\)-model action \((2.2)\), imply the dual model \((2.6)\) as well as\[ L^A M_\pm \partial_\pm X^M = Q^A_{\pm B} \partial_\pm \tilde{X}^B, \tag{2.8} \]
where the matrix \(Q_{\pm}\) is defined as

\[ (Q_{\pm})^A_B = \begin{pmatrix} \pm (M_{\pm}^{-1})^{ab} & -(M_{\pm}^{-1})^{ac} E_{jc}^\pm \\ 0 & \delta_{ij} \end{pmatrix}, \tag{2.9} \]

\(^2\) Details on the application of this method, for the case of Abelian T–duality, can be found in \([23]\).
with $M^a_{\pm} \equiv M^{ba}$. Of course this transformation acts trivially on the internal variables $X^i$, as it should. Notice that $Q_\pm$ only depends on the dual model variables $\tilde{X}^M = (\tilde{X}^a, X^i)$. Also $A^\pm_\pm \equiv t_a Q^a_\pm B \partial_\pm \tilde{X}^B$ can be identified with the on shell values of the gauge fields introduced in the traditional approach to non–Abelian duality. For later convenience, the inverse matrix to (2.9) is also given:

$$
(Q^\pm_\pm)^A_B = \begin{pmatrix} \pm(M^a_{\pm})^{ab} & \pm(E^i_{ja}) \\
0 & \delta^i_j \end{pmatrix},
$$

In terms of these matrices the metric corresponding to (2.6) can be written as

$$
\tilde{G}_{AB} = Q^C_\pm A Q^D_\pm B G_{CD}, \quad G_{CD} \equiv \frac{1}{2}(E^+_CD + E^-_{CD}),
$$

where both expressions corresponding to the plus and the minus signs give the same result for $\tilde{G}_{AB}$, as they should.

Let us consider the transformation (2.8) for the plus sign. It amounts to a non–local redefinition of the target space variables $X^\mu$ in the group element $g \in G$ associated with the isometry

$$
g = Pe^i f^{\sigma^+} A^\pm_\pm,
$$

where $P$ stands for path ordering of the exponential. The integration is carried out for fixed $\sigma^-$ and connects a base point with $\sigma^+$. Since the equations of motions for (2.6) imply the vanishing of the field strength associated with $A_\pm$, the expression (2.12) for $g$ can be replaced by a similar one using $A_-$ and integration carried out for fixed $\sigma^+$. The dual background (2.6) is a local function of $\tilde{X}^M$ due to the fact that in the original background (2.2) all group dependence was via the left–invariants $L^a_{\mu}$. However, other geometrical objects are not bound to have such a dependence. In these cases they become non–local in the dual picture. We will shortly encounter examples of that kind.

N=1 world–sheet supersymmetry: Any background can be made $N = 1$ supersymmetric [24]. Thus, it is expected that a manifestly supersymmetric version of the non-abelian duality transformation exists. Indeed, this was found, in the traditional formalism, for the general model (2.3) in [14]. In terms of a canonical transformation there is work for the supersymmetric version of the non–linear Chiral Model on $O(4)$ and its dual [25]. Since supersymmetry dictates the form of any transformation compatible with it once the bosonic part is known, it is straightforward to find the canonical transformation for the general supersymmetric model by applying the following procedure: First, one obtains the supersymmetric version of (2.8) by simply replacing the bosonic fields and world–sheet derivatives by their respective superfields and world–sheet superderivatives

$$
L_M^A(Z) D_\pm Z^M = Q^A_\pm B(\tilde{Z}) D_\pm \tilde{Z}^B,
$$

where $Z^M = X^M - i\theta_+ \Psi^M_+ + i\theta_- \Psi^M_+ - i\theta_+ \theta_- F^M$ is a generic $N = 1$ superfield, with a similar expression for $\tilde{Z}^M$, and $D_\pm = \mp i\partial_\pm \mp \theta_\pm \partial_\pm$. We have denoted by $\Psi^M_\pm$ the world–sheet fermions and by $\theta^\pm$ the two Grassman variables. The highest component
of the superfield \( F^M \) is eliminated by using its equations of motion. It finally assumes the form \( F^M = i(\Omega^\pm)^M_{\Lambda} \Psi^N \Psi^\Lambda_\pm \). Next we expand both sides of (2.13) and read the corresponding transformation rules of the components. We find that the transformation of the bosonic part is given by (2.3) with the right hand side modified by a quadratic term in the world–sheet fermions \( \Psi^M_\pm \) of similar chirality as the world–sheet derivative,

\[
L^A_M \partial^\pm X^M = Q^A_{\pm B} \partial^\pm \tilde{X}^B - i \partial_B Q^A_{\pm C} \tilde{\Psi}^B \tilde{\Psi}^C_\pm + \frac{i}{2} f^A_{BCD} Q^B_{\pm E} Q^C_{\pm F} \tilde{\Psi}^D \tilde{\Psi}^E_\pm .
\]

We note that bosons in the dual model are composites of bosons and fermions of the original model. This boson–fermion synphysis is a common characteristic of duals to supersymmetric \( \sigma \)-models and was first observed in [20] for the non–Abelian dual of the supersymmetric extension of the Chiral Model on \( O(4) \) and for Abelian duality in [1]. Accordingly the redefinition of the group element \( g \in G \), though similar to (2.12), will also involve the quadratic in the fermions terms that appear in the right hand side of (2.14). Nevertheless, since they can always be generated from the bosonic first term we will refrain, in the rest of the paper, from writing them explicitly. The transformation of \( L^A_M \Psi^M_\pm \) is similar to (2.8),

\[
L^A_M \Psi^M_\pm = Q^A_{\pm B} \tilde{\Psi}^B_\pm .
\]

The on shell expression for the highest component of the superfield can be used to find the transformation of the generalized connection,

\[
(\Omega^\pm)^A_{\ BC} = (Q^{-1}_{\pm} L)^A_M (L^{-1} Q^N_\pm)_{\ BC} \left( (\Omega^\pm)^M_{\ NA} - \partial_N L^D_M L^M_D \right)
\]  
+ \partial_B (Q^N_\pm)^D_C (Q^{-1}_{\pm})^A_D .
\]

When the group \( G \) is Abelian the transformations of the world–sheet fermions and of the connections reduce to the corresponding ones in [1][1]. Consider now a field \( V^M_\pm \) that transforms under non-abelian duality similarly to (2.8). Namely,

\[
L^A_M V^M_\pm = Q^A_{\pm B} V^B_\pm .
\]

We will call such a field a \((1_\pm, 0)\) tensor, since its transformation under duality resembles that of a vector field under diffeomorphisms. In general, a \((n_+, n_-; m_+, m_-)\) tensor will have \( n_\pm \) upper and \( m_\pm \) lower indices of the indicated chirality. It is a straightforward computation to prove, using (2.16), (2.18), that

\[
\tilde{D}^A_\pm \tilde{V}^B_\pm = (L^{-1} Q^N_\pm)^M_A (Q^{-1}_{\pm} L)^B_N D^\pm_M V^N .
\]

Hence, the covariant derivative of a \((1_\pm, 0)\) tensor is a \((1_\pm, 1_\mp)\) tensor. More generally the covariant derivative \( D^+_M \) on a tensor of type \((n_+, 0; m_+, 0)\) will transform it into a tensor of type \((n_+, 1_\mp; m_+, 1_\pm)\). When the group \( G \) is Abelian the transformations of the world–sheet fermions and of the connections reduce to the corresponding ones in [1][1]. Consider now a field \( V^M_\pm \) that transforms under non-abelian duality similarly to (2.8). Namely,

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\((n_+, 0; m_+, 1_-)\) type–tensor. Similarly, the covariant derivative \(D_{\tilde{M}}^{-}\) on a \((0, n_-; 0, m_-)\) type–tensor will transform it into a \((0, n_-; 1_+, m_-)\) type one. The fact that the action of covariant derivatives on tensors of the type we have indicated, preserves their tensor character, is not a trivial statement. Any other combination of covariant derivatives on these or more general tensors produces objects that transform anomalously under duality. For instance, the generalized curvature \(R_{\tilde{M}NKA}^+\), though a tensor under diffeomorphisms, it is not one under duality \([14, 4, 6]\). This is ultimately connected to the non–local nature of the duality transformation when the latter is viewed merely as a redefinition of the target space variables (cf. (2.12)).

Extended world–sheet supersymmetry: Conventionally, extended \(N = 2\) supersymmetry \([27, 28, 29]\) requires that the background is such that an (almost) complex (hermitian) structure \(F^{\pm}_{MN}\) in each sector, associated to the right and left-handed fermions, exists. Similarly, \(N = 4\) extended supersymmetry \([28, 29, 30]\) requires that, in each sector, there exist three complex structures \((F^{\pm}_{I})_{MN}, I = 1, 2, 3\). The complex structures are covariantly constant, with respect to the generalized connections, they are represented by antisymmetric matrices and in the case of \(N = 4\) they obey the \(SU(2)\) Clifford algebra. If in addition they are integrable they also satisfy the Nijenhuis conditions, though these are not necessary for the existence of extended supersymmetry \([31]\). The above requirements put severe restrictions on the backgrounds that admit a solution. For instance in the absence of torsion the metric should be Kahler for \(N = 2\) and hyper–Kahler for \(N = 4\) \([28]\).

In order to determine the fate of extended supersymmetry under non–Abelian duality it is useful to assign the complex structures to representations of the isometry group \(G\). The simplest cases to consider are those with complex structures belonging to the singlet representation, thus remaining invariant under the group action on the left. The most general form of such complex structures is

\[
F^{\pm}_{MN} = F^{\pm}_{AB} L_A^M L_B^N, \tag{2.20}
\]

where \(F^{\pm}_{AB}\) is an antisymmetric matrix independent of the \(X^\mu\)’s which obeys \((F^2)^{AB} = -\delta^{AB}\). Its functional dependence on the internal space variables \(X^i\) is determined by demanding that (2.20) is covariantly constant. In order to find how (2.20) transforms under non–Abelian duality we consider, similarly to the case of Abelian duality \([4]\), the 2–form \(F^{\pm} = F^{\pm}_{MN} dX^M \wedge dX^N\) and its transformation properties induced by (2.8). The result is

\[
\tilde{F}^{\pm}_{AB} = Q_{\pm A}^C Q_{\pm B}^D F^{\pm}_{CD}. \tag{2.21}
\]

Hence, \(F^{\pm}_{AB}\) transforms as a \((0, 2_{\pm})\) tensor under duality. Then, it follows that \(\tilde{D}^{\pm}_{A} F^{\pm}_{BC} = 0\). Similarly, one verifies that all properties of the original complex structure are properties of its duals as well. In the case of \(N = 4\) with three complex structures that are singlets, each one of them is of the form (2.21), with the corresponding \((F^{\pm}_{I})_{AB}, I = 1, 2, 3\) obeying the \(SU(2)\) Clifford algebra. They transform as in (2.21) under duality and they similarly define a locally realized \(N = 4\) in the dual model.
Consider now cases where the complex structures transform in a non-trivial representation of the duality group $G$. This is impossible if we only have $N = 2$ extended supersymmetry since there should be at least two complex structures to form a non-trivial representation. On the other hand, it is well known that this implies the existence of a third one and thus we are led to consider the case of $N = 4$ extended supersymmetry.

If the duality group is $SO(3) \simeq SU(2)$, with structure constants $f_{IJK} = \sqrt{2} \varepsilon_{IJK}$ in our normalization, then this implies that the Lie–derivative acts as $\mathcal{L}_{R} F_{J}^{\pm} = f_{IJK} F_{K}^{\pm}$. Thus the complex structures $F_{I}^{\pm}$, $I = 1, 2, 3$ transform in the triplet representation. For bigger groups the same transformation is valid if we restrict to an appropriate rotational $SO(3)$ subgroup of $G$. Let us introduce a singlet under the group $G$ matrix $(\Phi_{I}^{\pm})_{AB}$, which satisfies the same properties as the matrix $(F_{I}^{\pm})_{AB}$. The form of the triplet complex structures is then

\[
(F_{I}^{\pm})_{MN} = C^{IJ}(g)(\Phi_{J}^{\pm})_{MN} , \quad (\Phi_{I}^{\pm})_{MN} = (\Phi_{J}^{\pm})_{AB} L_{M}^{A} L_{N}^{B} ,
\]

where $I, J = 1, 2, 3$, but note that, as always $A = 1, 2, \ldots \text{dim}(G), \ldots, d$. In order to prove this, it is enough to notice that $\mathcal{L}_{R} C_{JK} = f_{JL} C_{LK}$ and $\mathcal{L}_{R} \Phi_{I}^{\pm} = 0$. Consider now the effects of non–Abelian duality on complex structures of the form (2.22). The singlet factor $(\Phi_{I}^{\pm})_{MN}$ remains local and transforms similarly to (2.19). However, the matrix $C_{IJ}$ involves the group element $g \in G$ explicitly, which then will be given by the path ordered Wilson line (2.12). Hence, in the dual model

\[
(\tilde{F}_{I}^{\pm})_{AB} = C^{IJ}(g)(\tilde{\Phi}_{J}^{\pm})_{AB} , \quad (\tilde{\Phi}_{I}^{\pm})_{AB} = Q^{C}_{\pm A} Q^{D}_{\pm B} (\tilde{\Phi}_{J}^{\pm})_{CD} .
\]

The complex structure as a whole is non–local precisely due to the attached Wilson line. The question is whether or not it can still be used to define an extended supersymmetry. The non–local complex structure (2.23) still satisfies the $SU(2)$ Clifford algebra, but it is no longer covariantly constant. This is similar to the case of Abelian duality, as it was first found in [3] and further elaborated in [4]. Instead, they have to satisfy the general conditions for existence of non–local complex structures [6]

\[
\tilde{D}_{A}^{\pm} (\tilde{F}_{I}^{\pm})_{BC} \partial_{\pi} \tilde{X}^{A} + \tilde{\partial}_{\bar{\pi}} (\tilde{F}_{I}^{\pm})_{BC} = 0 ,
\]

where the tilded world–sheet derivative acts only on the non–local part of the complex structure. Using (2.23), we find that (2.24) implies the following equation for $\tilde{\Phi}_{I}^{\pm}$

\[
C_{IJ} \tilde{D}_{A}^{\pm} (\tilde{\Phi}_{J}^{\pm})_{BC} + C_{IJ} f^{E} f^{D} Q_{\pm A}^{E} (\tilde{\Phi}_{J}^{\pm})_{BC} = 0 .
\]

Then the transformations (2.23), (2.19) imply

\[
C_{IJ} D_{M}^{\pm} (\Phi_{J}^{\pm})_{NA} + C_{IJ} f^{A} f^{B} L_{M}^{B} (\Phi_{J}^{\pm})_{NA} = 0 ,
\]

which is nothing but the covariantly constancy equation for the local complex structure (2.22) rewritten as an equation for $(\Phi_{I}^{\pm})_{MN}$. Thus, we have proved that the original local $N = 4$ breaks down to a local $N = 1$, whereas the part corresponding to the extended supersymmetry gets realized non–locally. Nevertheless, in a string setting $N = 4$ remains a genuine supersymmetry.
Hyper–Kahler metrics with SO(3) isometry

In order to fully illustrate the previous general discussion it will be enough to focus on the special class of 4–dim hyper–Kahler metrics with $SO(3)$ symmetry. An additional reason is that hyper–Kahler geometry is an interesting subject by itself, especially in connection with the theory of gravitational instantons, supersymmetric models and supergravity, and various moduli problems in monopole physics, string theory and elsewhere. The line element of 4–dim hyper–Kahler metrics with $SO(3)$ symmetry, in the Bianchi IX formalism is given by

$$ds^2 = f^2(t)dt^2 + a_1^2(t)\sigma_1^2 + a_2^2(t)\sigma_2^2 + a_3^2(t)\sigma_3^2 .$$ (2.27)

Here, $\sigma_i$, $i = 1, 2, 3$ are the left–invariant 1–forms of $SO(3)$ \[ In the parametrization of the group element in terms of Euler angles, $g = e^{i\phi\sigma_3}e^{i\theta\sigma_2}e^{i\psi\sigma_3}$, they assume the form

$$\sigma_1 = \frac{1}{2}(\sin \theta \cos \psi d\phi - \sin \psi d\theta) ,$$

$$\sigma_2 = \frac{1}{2}(\sin \theta \sin \psi d\phi + \cos \psi d\theta) ,$$

$$\sigma_3 = \frac{1}{2}(d\psi + \cos \theta d\phi) .$$ (2.28)

The coordinate $t$ of the metric can always be chosen so that

$$f(t) = \frac{1}{2}a_1a_2a_3,$$ (2.29)

using a suitable reparametrization. It was established some time ago [32] that the second–order differential equations that provide the self–duality condition for the class of metrics (2.27) in the parametrization (2.29), can be integrated once to yield the following first–order system in $t$

$$\frac{a_i'}{a_i} = \frac{1}{2}a_i^2 - a_i^2 - 2\frac{\lambda_i}{a_i} , \quad i = 1, 2, 3 ,$$ (2.30)

where the three parameters $\lambda_i$ remain undetermined for the moment. The derivatives (denoted by prime) are taken with respect to $t$. We essentially have two distinct categories of solutions to (2.30), depending on the values of the parameters $\lambda_1, \lambda_2, \lambda_3$. The first is described by $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and the second by $\lambda_1 = \lambda_2 = \lambda_3 = 1$. The Eguchi–Hanson metric belongs to the first category and the Taub–NUT and the Atiyah–Hitchin metrics to the second. These three cases provide the only non–trivial hyper–Kahler 4–metrics with $SO(3)$ isometry that are complete and non–singular [33].

Complex structures: It is known (see, for instance, [34]) that the complex structures for the Eguchi–Hanson metric are singlets under the the $SO(3)$ action whereas those

\[ Since the internal space parametrized by the variable $t$ is 1-dimensional, it will not be confusing to use instead of upper case letters $I, J, K$, lower case ones $i, j, k$. Also in order to comply with standard notation in the literature and avoid factors of $\sqrt{2}$ we will use $\sigma_i = \frac{1}{\sqrt{2}}L^i$ for the left invariant Maurer–Cartan forms. Then also $f_{ijk} = \sqrt{2}\epsilon_{ijk}$.\]
for the Taub–NUT and the Atiyah–Hitchin transform as a triplet. Moreover, for the Eguchi–Hanson and the Taub–NUT metrics explicit expressions are known [33, 34]. For the Atiyah–Hitchin metric the complex structures are only known in the Toda–frame formulation of the metric [35], which was found using the fact that $\partial/\partial\phi$ is a manifest Killing vector field of (2.27). Recently also, one of the complex structures of the Atiyah–Hitchin metric, in the parametrization (2.27), appeared in [36]. However, the result for the general metric (2.27) is not known, so that we will proceed with its derivation.

We will prove that any hyper–Kahler metric that is $SO(3)$–invariant with line element given by (2.27) and (2.30), has three complex structures given by

$$F_i = \begin{cases} K_i & \text{if } \lambda_1 = \lambda_2 = \lambda_3 = 0 \\ C_{ij}K_j & \text{if } \lambda_1 = \lambda_2 = \lambda_3 = 1 \end{cases},$$

(2.31)

where $K_i$ is given by

$$K_i = 2e_0 \wedge e_i + \epsilon_{ijk}e_j \wedge e_k,$$

(2.32)

with the tetrads defined as $e_0 = fdt$ and $e_i = a_i\sigma_i$. In accordance with (2.20), (2.22) the $F_i$’s for $\lambda_i = 0$ are singlets of $SO(3)$ whereas for $\lambda_i = 1$ transform in the triplet representation. In order to prove (2.31) let us first note that clearly the $K_i$’s obey the quaternionic algebra. Since $C_{ik}C_{jk} = \delta_{ij}$ it is easy to verify that the $F_i$’s, in general, obey the same algebra as well. Then, it remains to prove that $D_\mu(F_i)_{\nu\rho} = 0$. Since the torsion is zero, it suffices to show that $F_i$ is a closed 2–form and that the associated Nijenhuis tensor vanishes. A short computation using (2.30) to substitute for derivatives with respect to $t$, gives

$$dK_i = -4f\epsilon_{ijk}\lambda_ja_kdt \wedge \sigma_j \wedge \sigma_k.$$

(2.33)

Thus, in the cases where $\lambda_i = 0$, we find that indeed $F_i = K_i$ are closed forms. Next using the property $dC_{ij} = 2C_{im}\epsilon_{mjk}\sigma_k$ we compute that

$$d(CK)_i = -4fC_{ij}\epsilon_{jmk}(\lambda_m - 1)a_kdt \wedge \sigma_m \wedge \sigma_k$$

$$+ 2C_{ij}\epsilon_{jmn}\sigma_l \wedge \sigma_n \wedge \sigma_n.$$

(2.34)

It can be easily seen that the second line in the above equation vanishes identically. Hence also for the cases $\lambda_i = 1$, $F_i = C_{ij}K_j$ are closed forms. Verifying the vanishing of the Nijenhuis tensor is a bit harder task, but nevertheless straightforward, and will not yield any details.

The dual $\sigma$-model: Non–Abelian duality on (2.27) with respect to the $SO(3)$ isometry group corresponds to a canonical transformation which for the world–sheet derivatives

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5 Any hyper–Kahler metric with a rotational Killing symmetry can be formulated in the Toda–frame [37], in which case the explicit expressions for the complex structures are known in general [3].

6 This implies that there exist an atlas such that one of the $F_i$’s is constant. The integrability of the quaternionic structure which would have implied that an atlas existed such that all three $F_i$’s were constant, requires much stronger conditions to be satisfied [38]. Nevertheless, for the existence of $N = 4$ supersymmetry this integrability is not needed.
assumes the form (cf. (2.8))

\[ \sigma_i^\pm = \pm 2e^{-\Phi} \left( \frac{4f^2}{a_i^2} \partial_\pm \chi^i \mp \chi^i \chi \cdot \partial_\pm \chi \pm \epsilon_{ijk} \chi_k a_k^2 \partial_\pm \chi^j \right) , \]  

(2.35)

where \( \sigma_i^\pm \) are the (1, 0) and (0, 1) components of the decomposition of the 1–forms (2.28) on the world–sheet and the \( \chi^i \)'s represent the three variables dual to the Euler angles. The dual to the background (2.27) can be obtained by specializing (2.4) in this case. The explicit form for the fields is [12]

\[ d\tilde{s}^2 = f^2 dt^2 + e^{-\tilde{\Phi}} (\chi_i \chi_j + \delta_{ij} \frac{4f^2}{a_i^2}) d\chi_i d\chi_j , \]

\[ \tilde{B}_{ij} = -e^{-\tilde{\Phi}} \epsilon_{ijk} a_k^2 , \]

\[ e^{\tilde{\Phi}} = 4(4f^2 + a_i^2 \chi_i^2) . \]  

(2.36)

The dual complex structures: The dual to the 2–form (2.32) can be obtained from (2.23) or by directly transforming it using (2.35). The result is

\[ \tilde{K}_i^\pm = e^{-\Phi} \left( \pm 4f dt \wedge \left( \frac{4f^2}{a_i^2} d\chi^i + \chi^i \chi \cdot d\chi \pm \epsilon_{ijk} \chi_k a_k^2 d\chi^j \right) \right. \]

\[ \left. \pm \frac{4f}{a_i} \chi \cdot d\chi \wedge d\chi^i + a_i^2 \epsilon_{ijk} a_j a_k d\chi^j \wedge d\chi^k \right) . \]  

(2.37)

For the cases where the original hyper–Kahler metric corresponds to the choice \( \lambda_i = 0 \) in (2.30) these are in fact the three complex structures for the dual background (2.36), which has locally realized \( N = 4 \) supersymmetry. It can be shown that the (anti)self–duality conditions of the dilaton–axion field are solved and therefore we have found that (2.36) is a new class of axionic–instantons which are related to hyper–Kahler metrics (2.27) via non–Abelian duality. Though not obvious, it can be shown that the metric in (2.36) is conformally flat (for the case where (2.27) is the Eguchi–Hanson metric this was observed in [12]), and the conformal factor \( e^{-\Phi} \) satisfies the Laplace equation adapted to the flat metric. This is in agreement with a theorem proved in [39] for 4-dim backgrounds with \( N = 4 \) world–sheet supersymmetry and non–vanishing torsion. The particular form of the coordinate change needed to explicitly demonstrate this is complicated and not very illuminating. Here we mention the result for the non–Abelian dual to 4-dimensional flat space which corresponds to the choice \( a_1 = a_2 = a_3 = (-t)^{-1/2} \) in (2.27). We found that the dual metric can be written in terms of Cartesian coordinates \( x_i \), as \( d\tilde{s}^2 = e^{-\tilde{\Phi}} dx_i dx_i \), where \( e^{\tilde{\Phi}} = 2r \sqrt{r^2 + x_4^2} \), with \( r^2 = x_4 x_4 \).

For the cases where the original hyper–Kahler metric corresponds to the choice \( \lambda_i = 1 \) in (2.30) the dual background has non–locally realized \( N = 4 \) world–sheet supersymmetry. The complex structures are \( \tilde{K}_i^\pm = C_{ij}(g) \tilde{K}_j^\pm \), with \( C_{ij}(g) \) being non–local functionals of the dual space variables according to (2.35).
3 Duals of WZW Models

We would like to make contact with exact conformal field theoretical results. The hyper–Kahler metrics and their non–Abelian duals we have examined are not appropriate for such an investigation since their description in terms of exact conformal field theories is, at present, unknown. The best examples to consider in this respect are non–Abelian duals of WZW models, since, as it turns out, the non–local realizations of supersymmetry that arise after duality can be naturally expressed in terms of non-abelian parafermions.

The WZW model action, to be denoted by $I_{\text{wzw}}(g)$, for a group element $g \in G$ corresponds to a background with metric and torsion given by

$$G_{MN} = L^A_M L^A_N = R^A_M R^A_N, \quad H_{MNA} = f_{ABC} L^A_M L^B_N L^C_A = f_{ABC} R^A_M R^B_N R^C_A. \quad (3.1)$$

A WZW model for a general group can be made $N = 1$ supersymmetric on the world–sheet [40]. If the group is an even dimensional one the supersymmetry is promoted to an $N = 2$ [41]. Moreover, WZW models based on quaternionic groups have actually $N = 4$ [41]. The general form of the complex structures is very similar to (2.20),

$$F^+_M = F^+_A L^A_M L^B_N, \quad F^-_M = F^-_A R^A_M R^B_N, \quad (3.2)$$

where the constant matrices $F^\pm_{AB}$ are Lie algebra complex structures [41]. The covariant constancy of $F^\pm_{MN}$ follows trivially from the fact that $D^+_M L^A_N = D^-_M R^A_N = 0$, which are valid for any WZW model. It is obvious that $F^+$ ($F^-$) is invariant under the left (right) group action. Thus, under the vector action of a non–Abelian subgroup $H$ of $G$, i.e. $g \to \Lambda^{-1} g \Lambda$, none of the $F^+$, $F^-$ is invariant.

The analog of the canonical transformation (2.5) or (2.8) for the non–Abelian dual of a WZW model with respect to its vector subgroup $H$ will be presented in the next subsection. Here, we proceed traditionally by starting with the usual gauged WZW action [42, 43] plus a Langrange multiplier term,

$$S = I_{\text{wzw}}(g) + \frac{k}{\pi} \int \text{Tr} \left( A_+ \partial_- gg^{-1} - g^{-1} \partial_+ g A_- + A_+ g A_- g^{-1} - A_+ A_- \right) + i \text{Tr} \left( v F^+ - \right), \quad (3.3)$$

where $A_\pm$ are gauge fields in the Lie algebra of a subgroup $H$ of $G$ with corresponding field strength $F^+ = \partial_+ A_- - \partial_- A_+ - [A_+, A_-]$ and $v$ are some Lie algebra variables in $H$ that play the role of Lagrange multipliers. We also split indices as $A = (a, \alpha)$, where $a \in H$ and $\alpha \in G/H$. Variation of (3.3) with respect to all fields gives the classical equations of motion

$$\delta A_+ : \quad D_+ gg^{-1}|_H + iD_- v = 0, \quad (3.4)$$
$$\delta A_- : \quad g^{-1} D_+ g|_H + iD_+ v = 0, \quad (3.5)$$
$$\delta g : \quad D_+ (D_- gg^{-1}) + F^+_+ = 0, \quad (3.6)$$
$$\delta v : \quad F^+_- = 0. \quad (3.7)$$

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To find the dual $\sigma$-model a unitary gauge should be chosen. This is done by fixing $\dim(H)$ variables among the total number of $\dim(G) + \dim(H)$ ones, thus remaining with a total of $\dim(G)$ variables, which we will denote by $X^M$. If $H \neq G$ then generically there is no isotropy subgroup and we can gauge fix all $\dim(H)$ variables in the group element $g \in G$. If $H = G$ then the non-trivial isotropy subgroup corresponding to the Cartan subalgebra of $G$ cannot be gauge fixed away. In such cases we gauge fix $\dim(G) - \text{rank}(G)$ parameters in $g$ and the remaining rank($G$) ones among the Lagrange multipliers $v^a$. Then we eliminate the gauge fields using their classical equations of motion (3.4), (3.5),

\begin{align}
A_+^a &= +i(C^T - I - f)^{-1}_{ab}(L^b_{\mu} \partial_+ X^\mu + \partial_+ v^b) \equiv A_{+M}^a \partial_+ X^M , \\
A_-^a &= -i(C - I + f)^{-1}_{ab}(R^b_{\mu} \partial_- X^\mu + \partial_- v^b) \equiv A_{-M}^a \partial_- X^M .
\end{align}

(3.8)

Finally, the dual $\sigma$-model is given by [8, 11]

\[ S = I_{wzw}(g) - \frac{k}{\pi} \int (L^a_{\mu} \partial_+ X^\mu + \partial_+ v^a)(C - I + f)^{-1}_{ab}(R^b_{\mu} \partial_- X^\mu + \partial_- v^b) . \]

(3.9)

A dilaton $\Phi = \ln \det (C - I + f)$ is also induced in order to preserve conformal invariance at 1–loop [1].

As in the previous section it will be convenient to have an explicit expression for the generalized connections of the dual model (3.9). For this we utilize the classical string equations for the dual action (3.3), $D_+(D_- gg^{-1}) = 0$, which follow from (3.6) after we use (3.7). In these equations the gauged fields entering the covariant derivatives should be replaced by their on shell values (3.8). We define

\[ \text{Tr}(t^A g^{-1} D_+ g) = i \mathcal{L}^A_M \partial_+ X^M , \quad \text{Tr}(t^A D_- gg^{-1}) = i \mathcal{R}^A_M \partial_- X^M . \]

(3.10)

Under gauge transformations $\mathcal{L}^A_M$ and $\mathcal{R}^A_M$ are left and right invariant respectively. Then it is easy to cast the classical equations of motion into the standard form for any 2–dimensional $\sigma$–model

\[ \partial_+ \partial_- X^M + (\Omega^-)^M_{N\Lambda} \partial_+ X^N \partial_- X^\Lambda = 0 , \]

(3.11)

from which we read off the generalized connection of the dual model

\[ (\Omega^-)^M_{N\Lambda} = \mathcal{L}^M_A \partial_A \mathcal{L}^A_N + if^A_{BC} \mathcal{L}^M_A \mathcal{L}^B_N \mathcal{L}^C_L A_L^- . \]

(3.12)

It is convenient to define the following gauge invariant elements, in the Lie algebra of $G$

\[ \Psi_+ = -ih_-^{-1} g^{-1} D_+ gh_+ , \quad \Psi_- = -ih_+^{-1} D_- gg^{-1} h_+ , \]

(3.13)

where the group elements $h_\pm \in H$ are given by path ordered exponentials similar to (2.12)

\[ h_+^{-1} = Pe^{-\int g^+ A_+} , \quad h_-^{-1} = Pe^{-\int g^- A_-} , \]

(3.14)

with the gauge fields $A_\pm$ determined by (3.8). They obey $A_\pm = \partial_\pm h_\mp h_\pm^{-1}$. Using the classical equations of motion (3.4)–(3.7), it can be shown that $\Psi_+$ and $\Psi_-$ are chiral

\[ \partial_- \Psi_+ = 0 , \quad \partial_+ \Psi_- = 0 . \]

(3.15)
We will also denote $\Psi^A = \Psi^A_{\pm M} \partial_{\mp} X^M$, where
\[
\Psi^A_{+M} = C^{BA}(h_-) L^B_M, \quad \Psi^A_{-M} = C^{BA}(h_+) R^B_M. \tag{3.16}
\]
Because they have Wilson lines attached to them, $\Psi_{\pm}$ are non–local. Since, the action we started with (3.3) contains the standard gauged WZW action corresponding to the coset $G/H$, it is expected that $\Psi_{\pm}$ will be related to the classical non–Abelian parafermions \[44, 45\]. The precise relationship will be uncovered in the next subsection.

We are now in the position to examine the fate of world–sheet supersymmetry under non–Abelian duality. We will show that the dual action (3.9) has non–locally realized extended supersymmetry with complex structures, corresponding to (3.2), given by
\[
\tilde{F}^+_{MN} = F^+_{AB} \Psi^A_{+M} \Psi^B_{+N}, \quad \tilde{F}^-_{MN} = F^-_{AB} \Psi^A_{-M} \Psi^B_{-N}. \tag{3.17}
\]
It is obvious that the dual complex structures (3.17) obey all properties of their counterparts (3.2) except that they are not covariantly constant. Being non–local they should satisfy instead, the equation \[6\]
\[
\tilde{D}^\pm_M (\tilde{F}^\pm)_{NA} \partial_{\mp} \tilde{X}^M + \tilde{\partial}_{\mp} (\tilde{F}^\pm)_{NA} = 0, \tag{3.18}
\]
where the tilded derivative acts only on the non–local part of the complex structures contained in $h_{\pm}$, which are given by the path ordered exponentials (3.14). For this it is enough to prove that
\[
\tilde{D}^\pm_M \Psi^A_{\pm M} \partial_{\mp} X^M + \tilde{\partial}_{\mp} \Psi^A_{\pm} = 0, \tag{3.19}
\]
where, similarly to (3.18), the tilded world–sheet derivative acts only on the non–local part of $\Psi^A_{\pm}$. This becomes a straightforward computation after we use the expression for the generalized connection of the dual model (3.12).

Thus, we have shown that as long as $H$ is non–Abelian, T–duality breaks all local extended supersymmetries which are then realized non–locally with complex structures given by (3.17). Our treatment is equally applicable to the cases where $H$ is an Abelian subgroup of $G$. However, in such cases T–duality preserves one extended supersymmetry. In order to see that let us recall \[11\] that for any even dimensional WZW model the non–vanishing elements of the matrix $F^\pm_{AB}$ in the Cartan basis are $F^\pm_{\alpha\bar{\alpha}} = i$ and $F^\pm_{ij}$, where $i, j$ here are labels in the Cartan subalgebra of $G$ and $\alpha$ ($\bar{\alpha}$) is a positive (negative) root label. Since the group $H$ is Abelian we have $C_{ij}(h_{\pm}) = \delta_{ij}$. Using the fact that $C^\beta_{\alpha}(h_{\pm}) C^\delta_{\gamma}(h_{\pm}) = \delta_{\beta\gamma}$ and \[3.16\], we find that the complex structures (3.17) are local functions of the target space variables and assume the form
\[
\tilde{F}^+_{MN} = i L^\alpha_M L^\bar{\alpha}_N + F^+_{ij} L^i_M L^j_N, \quad \tilde{F}^-_{MN} = i R^\alpha_M R^\bar{\alpha}_N + F^-_{ij} R^i_M R^j_N. \tag{3.20}
\]
We conclude that, if $H$ is Abelian T–duality preserves the local $N = 2$ of the even dimensional supersymmetric WZW models. However, this is not the case for the two additional complex structures present in WZW models based on quaternionic groups,
which actually have $N = 4$ extended supersymmetry. These cannot be written in a form similar to (3.20) and remain genuinely non–local. More details for the case of the WZW model based on $SU(2) \otimes U(1)$ can be found in [3, 35, 46] and for a general quaternionic group in [3].

Non–Abelian parafermions

We will now find the precise relation of $\Psi_\pm$ to the non–Abelian classical parafermions of the coset theory $G/H$ [45]. Moreover, we will show that their Poisson brackets obey the same algebra as the currents of the original WZW model. This provides the, so far lacking, canonical equivalence between a WZW model for $G$ and its dual with respect to a vector subgroup $H$ as it is given by (3.9). In retrospect the emergence of parafermions is not a surprise since the non–Abelian duals of WZW models are related to gauged WZW models, as it was shown in [11] and [12].

Since we are interesting in the computation of Poisson brackets, our treatment here will be completely classical. Hence, the non–trivial Jacobians arising from changing variables inside the functional path integral [43] will be ignored. Let us define the gauge invariant analogs of $g, h, v$

\[ f = h_-^{-1}gh_-, \quad h = h_-^{-1}h_- \in H, \quad \tilde{v} = h_-^{-1}vh_- \in H. \]  

(3.21)

and introduce a group element $\lambda \in H$ such that the $i\partial_- \tilde{v} = -\partial_- \lambda \lambda^{-1}$. With these definitions the gauge field strength $F_{\pm-} = h_- \partial_- (h_-^{-1} \partial_+ h) h_-^{-1}$. Then with the help of the Polyakov–Wiegman formula the action (3.3) assumes the form

\[ S = I_{wzw}(hf) - I_{wzw}(h\lambda) + I_{wzw}(\lambda). \]  

(3.22)

The form of $\Psi_-$, defined in (3.13), in terms of gauge invariant quantities is $\Psi_- = -ih\partial_- ff^{-1}h^{-1}$. The latter expression contains $h_+$ whose definition (3.14) involves a timelike integral, when we regard $\sigma^+$ as “time”. This makes the computation of the corresponding Poisson brackets very difficult to perform. Thus, as in [14, 15], we make use of the equation of motion $F_{\pm-} = 0$ to replace $h_+$ by $h_-$ in the definition of $\Psi_-$ in (3.13) or equivalently to consider Poisson brackets of $\Psi = \frac{ik}{\pi} \partial_- ff^{-1}$, 

\[ \Psi = \frac{ik}{\pi} \partial_- ff^{-1}, \]  

(3.23)

where for notational convenience we have modified the normalization factor and have dropped the minus sign as a subscript. We should point out that the on shell condition $\partial_+ \Psi = 0$ is still obeyed. The computation of the Poisson brackets using directly the action (3.22) will be done systematically in the appendix using Dirac’s canonical approach to

\[ \text{From now on we concentrate on one chiral sector only. We will use } x \text{ or } y \text{ to denote the world–sheet coordinate } \sigma^-. \]
constrained systems. Here we follow a shortcut which enables us to make direct contact with the parafermions. We rewrite the action (3.22) by shifting $h \rightarrow h\lambda^{-1}$, as [12]

$$S = I_{wzw}(h\lambda^{-1}f) - I_{wzw}(h) + I_{wzw}(\lambda) .$$  \hspace{1cm} (3.24)

The first two terms correspond to the gauged WZW action for the coset $G/H$ and the third to an additional WZW action. Parafermions are introduced, similarly to [44, 45], by defining

$$\Psi_{G/H}^\alpha = i k \pi \partial_-(\lambda^{-1}f) j \lambda f^{-1} \lambda ,$$  \hspace{1cm} (3.25)

where the superscript emphasizes that they are valued in the coset $G/H$. Their Poisson brackets have been computed in [44, 45]

$$\{\Psi_{G/H}^\alpha (x), \Psi_{G/H}^\beta (y)\} = -k \pi \delta_\alpha\beta \delta'(x-y) - f_{\alpha\beta\gamma} \Psi_{G/H}^\gamma (y) \delta(x-y)$$

$$- \pi \frac{2k}{k} f_{\alpha\beta\gamma} \epsilon(x-y) \Psi_{G/H}^\gamma (x) \Psi_{G/H}^\beta (y) ,$$  \hspace{1cm} (3.26)

where the antisymmetric step function $\epsilon(x-y)$ equals $+1(-1)$ if $x > y$ ($x < y$). The last term in (3.26) is responsible for their non-trivial monodromy properties and unusual statistics. The currents corresponding to the WZW model action $I_{wzw}(\lambda)$ in (3.24) are defined as

$$J = i k \pi \partial_- \lambda \lambda^{-1} = k \pi \partial_- \tilde{v} ,$$  \hspace{1cm} (3.27)

with $\partial_+ J = 0$ on shell. Using the basic Poisson bracket for a WZW model [47]

$$\{TTr(t^a \lambda^{-1}\delta\lambda)(x), TTr(t^b \lambda^{-1}\delta\lambda)(y)\} = - \frac{2k}{k} \pi \epsilon(x-y) \delta^{ab} ,$$  \hspace{1cm} (3.28)

and the variation under infinitesimal transformations

$$\delta J_a = i k \pi C_{ab}(\lambda) TTr(t^b \partial_-(\lambda^{-1}\delta\lambda)) ,$$  \hspace{1cm} (3.29)

one proves that the following current algebra is obeyed [47]

$$\{J_a (x) , J_b (y)\} = - \frac{k}{\pi} \delta_{ab} \delta'(x-y) - f_{abc} J_c (y) \delta(x-y) .$$  \hspace{1cm} (3.30)

In addition due to the “decoupling” in (3.24) we have $\{\Psi_{G/H}^\alpha , J_b\} = 0$. In order to compute the Poisson brackets of (3.23) we first note that $\Psi_a = J_a$, due to (3.4). Hence, the bracket $\{\Psi_a, \Psi_b\}$ is the same as (3.30). On the other hand $\Psi_a = C_{ab}(\lambda) \Psi_{G/H}^b$. To determine $\{\Psi_a, \Psi_b\}$ and $\{\Psi_a, J_b\}$ we need the variation

$$\delta \Psi_a = C_{ab}(\lambda) \delta \Psi_{G/H}^b + i TTr(t^b \lambda^{-1}\delta\lambda) f_{b\gamma\delta} C_{a\delta}(\lambda) \Psi_{G/H}^\gamma .$$  \hspace{1cm} (3.31)

Then, using (3.26), (3.28) and (3.31) we find

$$\{\Psi_a (x) , \Psi_b (y)\} = - \frac{k}{\pi} \delta_{ab} \delta'(x-y) - \left( f_{\alpha\beta\gamma} \Psi_{\gamma}(y) + f_{\alpha\beta c} J_c (y) \right) \delta(x-y) ,$$  \hspace{1cm} (3.32)
and
\[ \{J_a(x), \Psi_\beta(y)\} = -f_{a\beta\gamma} \Psi_\gamma(y) \delta(x - y) . \] (3.33)

Thus the closed algebra obeyed by \( \Psi_A = \{J_a, \Psi_\alpha\} \) is given by (3.30), (3.32) and (3.33), which is the current algebra for \( G \). We emphasize the fact that, even though the \( \Psi_\alpha \)'s are related to the coset parafermions \( \Psi^{G/H}_\alpha \)'s, they are not parafermions themselves since in their Poisson bracket (3.32) there is no term similar to the third term in (3.26). The reason, is precisely the “dressing” provided by the extra fields (Lagrange multipliers). This is equivalent to the well known realizations of current algebras in conformal field theory using parafermions. Hence, we have shown a canonical equivalence between a WZW model for a general group \( G \) and its dual with respect to a vector subgroup \( H \) in the sense that the algebras obeyed by the natural (equivalently, symmetry generating) objects in the two models are the same.

Non–Abelian dual to \( SU(2) \otimes U(1) \)

The corresponding WZW action is given by
\[ S = \frac{k}{4\pi} \int \partial_+ \phi \partial_\phi + \partial_+ \theta \partial_\theta + \partial_+ \psi \partial_\psi + 2 \cos \theta \partial_+ \phi \partial_\psi + \partial_+ \rho \partial_\rho . \] (3.34)

This is the most elementary non–trivial model with \( N = 4 \) world–sheet supersymmetry. The three complex structures in the right sector are given by
\[ F_i^+ = 2d\rho \land \sigma_i - \epsilon_{ijk} \sigma_j \land \sigma_k , \] (3.35)
where the left invariant Maurer–Cartan forms of \( SU(2) \), defined in (2.28), have been used. The complex structures for the left sector can be similarly written down
\[ F_i^- = 2d\rho \land \bar{\sigma}_i - \epsilon_{ijk} \bar{\sigma}_j \land \bar{\sigma}_k , \] (3.36)
where \( \bar{\sigma}_i \) are the right invariant Maurer–Cartan forms of \( SU(2) \). Their explicit expressions can be obtained from (2.28) by letting \( (\phi, \theta, \psi) \rightarrow (-\psi, -\theta, -\phi) \), up to an overall minus sign. We can readily see that (3.35),(3.36) are of the general form (3.2).

Under \( SU(2) \) transformations the variable \( \rho \) is inert. The non–Abelian dual of (3.34) with respect to a vector \( SU(2) \) was found in [8], and we will not repeat all the steps of the derivation here. We only mention that a proper unitary gauge choice is \( \phi = \psi = 0 \) among the variables of the \( SU(2) \) group element and \( v_3 = 0 \) among the Lagrange multipliers. The latter choice becomes necessary because, according to our discussion after (3.7), there is a non–trivial isotropy group in this case. After we make the shift \( v_2 \rightarrow v_2 - \theta \) the classical solutions for the gauge fields \( A_\pm = \frac{i}{2} \vec{A}_\pm \cdot \vec{\sigma}_{Pauli} \) are
\[ \vec{A}_\pm = \frac{\mp 1}{2v_1^2 \sin^2 \frac{\theta}{2}} \left( v_1^2 \partial_\pm v_1 + v_1 (\sin \theta - \theta + v_2) \partial_\pm v_2, \ v_1 (\sin \theta - \theta + v_2) \partial_\pm v_1 \right) + \left( 4 \sin^4 \frac{\theta}{2} + (\sin \theta - \theta + v_2)^2 \right) \partial_\pm v_2, \pm 2v_1 \sin^2 \frac{\theta}{2} \partial_\pm v_2 \right) , \] (3.37)
and the background fields of the dual model are found to be

\[ ds^2 = d\rho^2 + d\theta^2 + \frac{1}{v_1^2 \sin^2 \frac{\theta}{2}} \left( 4 \sin^4 \frac{\theta}{2} dv_2^2 + (v_1 dv_1 + (v_2 - \theta + \sin \theta) dv_2)^2 \right), \]

\[ \Phi = \ln(v_1^2 \sin^2 \frac{\theta}{2}), \]

with zero antisymmetric tensor. Note that, even though the torsion vanishes, the Ricci tensor is not zero due to the presence of a non–trivial dilaton. This means that the manifold is not hyper–Kahler, as the latter property implies Ricci flatness [28].

The reason for this apparent paradox is of course the fact that the original local \( N = 4 \) world–sheet supersymmetry is realized in the dual model (3.38) non–locally, except for the \( N = 1 \) part. In the right sector the expressions for the non–local complex structures are given by

\[ \tilde{F}_i^+ = C_{ji}(h_-) \left( 2d\rho \wedge \mathcal{L}_j - \epsilon_{jkl} \mathcal{L}_k \wedge \mathcal{L}_l \right), \]

where \( \mathcal{L}_i = \mathcal{L}_i^\mu dX^\mu, X^\mu = \{ \theta, v_1, v_2 \} \) and

\[ (\mathcal{L}_i^\mu) = \begin{pmatrix} 0 & -1 & -\frac{v_2 - \theta}{v_1} \\ 1 & 0 & 0 \\ 0 & -\cot \frac{\theta}{2} & -\frac{2 + \cot \frac{\theta}{2} (v_2 - \theta)}{v_1} \end{pmatrix}. \]

In the left sector the non–local complex structures are

\[ \tilde{F}_i^- = C_{ji}(h_+) \left( 2d\rho \wedge \mathcal{R}_j - \epsilon_{jkl} \mathcal{R}_k \wedge \mathcal{R}_l \right), \]

where \( \mathcal{R}_i = \mathcal{R}_i^\mu dX^\mu \) and

\[ (\mathcal{R}_i^\mu) = \begin{pmatrix} 0 & -1 & -\frac{v_2 - \theta}{v_1} \\ 1 & 0 & 0 \\ 0 & \cot \frac{\theta}{2} & \frac{2 + \cot \frac{\theta}{2} (v_2 - \theta)}{v_1} \end{pmatrix}. \]

The group elements \( h_{\pm} \in SU(2) \) are given by the path ordered Wilson lines (3.14), with gauge fields (3.37).

4 Discussion and concluding remarks

In this paper we examined the behavior of supersymmetry under non–Abelian T–duality. We considered models that are invariant under the left action of a general semi–simple group. We gave the general form of the corresponding \( \sigma \)–models as well as of the complex structures, in cases that admit extended world–sheet supersymmetry, and found their transformation rules under non–Abelian duality by utilizing a canonical transformation. Although, as a general rule, \( N = 1 \) world–sheet supersymmetry is preserved under duality, whenever the action of the group on the complex structures is non–trivial, extended
supersymmetry seems to be incompatible with non–Abelian duality. However, this is only an artifact of the description in terms of an effective field theory, since non–local world–sheet effects restore supersymmetry at the string level. As examples, $SO(3)$–invariant hyper–Kahler metrics which include the Eguchi–Hanson, the Taub–NUT and the Atiyah–Hitchin metrics were considered in detail. Explicit expressions for the three complex structures were given which should be useful in moduli problems in monopole physics. We have also considered WZW models and their non–Abelian duals with respect to the vector action of a subgroup. The canonical equivalence of these models was shown by explicitly demonstrating that the algebra obeyed by the Poisson brackets of chiral currents of the WZW model is preserved under the non–Abelian duality transformation. The effect of non–Abelian duality is that the currents are represented in terms coset parafermions. The latter are non–local and have non–trivial braiding properties due to Wilson lines attached to them. We believe that this type of canonical equivalence is not restricted to just WZW models and their duals but to other models with vector action of the isometry group.

Non–Abelian duality destroys manifest target space supersymmetry as well, in the sense that the standard Killing spinor equations do not have a solution. In fact, the breaking of manifest target space supersymmetry occurs hand–in–hand with the breaking of local $N = 4$ extended world–sheet supersymmetry. This is attributed to the relation between Killing spinors and complex structures [18], using $F_{\mu\nu} = \bar{\xi} \Gamma_{\mu\nu} \xi$. The situation is similar to the case of Abelian duality [2, 14, 4, 6] with the difference that the non–local Killing spinors arising after duality do not define a local $N = 2$ world–sheet supersymmetry using the above relation between Killing spinors and complex structures. The lowest order effective field theory is not enough at all to understand the fate of target space supersymmetry under non–Abelian duality, since one has to generate the whole supersymmetry algebra and not just its truncated part corresponding to the Killing spinor equations. In the realization of the supersymmetry algebra after non–Abelian duality massive string modes play a crucial role and a complete truncation to only the massless modes is inconsistent. This becomes apparent by making contact with the work of Scherk and Schwarz [50] on coordinate dependent compactifications. The arguments are similar to the case of Abelian duality and were presented in [46].

This investigation is part of a program whose goal is to use non–trivial stringy effects occurring in duality symmetries, in physical situations that seem paradoxical in the effective field theory approach. In particular, we would like to view T–duality as a mechanism of restoring various symmetries, such as supersymmetry, in a manifest way. An example of how this works is based on the background corresponding to $SU(2)_k/U(1) \otimes SL(2, \mathbb{R})_k/U(1)$. This has $N = 4$ world–sheet supersymmetry which however, is not manifest and is realized using parafermions [71]. An appropriate Abelian duality transformation leads to an axionic instanton background with manifest $N = 4$ and target space supersymmetry restored [6]. An equivalent model where target space supersymmetry was restored by making a moduli parameter dynamical was considered in [6]. In order to advance these ideas and use non–Abelian duality as the symmetry restora-
tion mechanism one has to relax the condition that an isometry group exists at all, since in any case this is being destroyed by non–Abelian duality. The notion of non–Abelian duality in the absence of isometries is now well defined and under the name Poisson–Lie T–duality [13] and the closely related quasi–axial–vector duality which was initiated in [17], explicitly constructed in [18] whereas its relation to the Poisson-Lie T–duality was investigated in [19]. The idea is to search in various backgrounds of interest in black hole Physics or cosmology, for “non–commutative conservation laws” that generalize [15] the usual conservation laws. The hope is that in the dual description various properties, which were hidden, become manifest and possibly resolve certain paradoxes with field theoretical origin. We hope to report work in this direction in the future.

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A Derivation of Poisson brackets

In this appendix we derive the Poisson brackets of section 3 in a more systematic way. Because of the gauging procedure it turns out that we are dealing with constrained Hamiltonian systems. A consistent way of implementing the constraints was provided by Dirac (see, for instance, [52]). For our purposes the relevant part of his analysis is that given a set of second class constraints \( \{ \varphi_a \} \) one first computes the matrix generated by their Poisson brackets

\[
D_{ab} = \{ \varphi_a , \varphi_b \} . \tag{A.1}
\]

In this and in similar computations we are free to use the constraints only after calculating their Poisson brackets. When \( D_{ab} \) is invertible one simply postulates that the usual Poisson brackets are replaced by the so called Dirac brackets defined as

\[
\{ A, B \}_D = \{ A, B \} - \{ A, \varphi_a \} D_{ab}^{-1} \{ \varphi_b , B \} , \tag{A.2}
\]

for any two phase space variables \( A \) and \( B \). Then the constraints can be strongly set to zero since they have vanishing Dirac brackets among themselves and with anything else.

As a very elementary application of this method consider an arbitrary action that is first order in time derivatives

\[
S = \int dt A_a (X) \dot{X}^a . \tag{A.3}
\]

The conjugate momentum to \( X^a \) is given by \( P_a = A_a \) and therefore we cannot solve for the velocity \( \dot{X}^a \) in terms of the momentum \( P_a \). Hence we impose the constraint

\[
\varphi_a = P_a - A_a \approx 0 \tag{A.4}
\]
and follow Dirac’s procedure. Using the basic Poisson bracket \( \{ X^a, P_b \} = \delta^a_b \) we find that the matrix (A.1) is given by

\[
D_{ab} \equiv M_{ab} = \frac{\partial A_b}{\partial X^a} - \frac{\partial A_a}{\partial X^b}.
\]

This Dirac bracket coincides with the Poisson bracket postulated in [47] for the action (A.3). In practice we read off the matrix \( M_{ab} \) by simply considering the variation of (A.3)

\[
\delta S = \int dt M_{ab} \delta X^a \dot{X}^b.
\]

In the rest of this appendix as well as in the bulk of the paper we will call the Dirac bracket (A.5) simply a Poisson bracket in order to comply with standard terminology in the literature.

The models we encountered in section 3 belong to the general type (A.3) where \( \sigma^+ \) is considered as the time variable, whereas \( \sigma^- \) is treated as a continuous index. In that respect our treatment differs from the one in [53] where \( \tau = \sigma^+ + \sigma^- \) was taken as the time variable and computation of brackets of parafermions was not considered.

Gauged WZW Models: The purpose is to reproduce (3.26) in a straightforward way compared to that in [44, 45] and mainly to be able to compare with the analogous derivation of (3.30), (3.32), (3.33) which will follow.

Using the definitions (3.21) we can write the gauged WZW action as

\[ S = I_{wzw}(hf) - I_{wzw}(h). \]

A general variation of the action gives

\[
\delta S = \frac{k}{\pi} \int h^{-1} \delta h \left( f \partial_-(f^{-1} \partial_+ f) f^{-1} + f \partial_-(f^{-1} h^{-1} \partial_+ h) f^{-1} - \partial_-(h^{-1} \partial_+ h) \right) + f^{-1} \delta f \left( \partial_-(f^{-1} \partial_+ f) + \partial_-(f^{-1} h^{-1} \partial_+ h) \right).
\]

Using for notational convenience the definition

\[
Z_I = \left( \text{Tr}(t^a h^{-1} \delta h), \text{Tr}(t^A f^{-1} \delta f) \right),
\]

we compute the basic Poisson brackets (cf. footnote 7)

\[
\{ Z_I(x), Z_J(y) \} = -\frac{\pi}{2k} M^{-1}_{IJ}(x, y) \epsilon(x - y),
\]

where the matrix \( M(x, y) \) is defined as

\[
M(x, y) = \begin{pmatrix}
C_{ab}(f(x)f^{-1}(y)) - \delta_{ab} & C_{aB}(f(x)) \\
C_{bA}(f(y)) & \delta_{AB}
\end{pmatrix}.
\]
Inverting the above matrix and explicitly writing out (A.11) we obtain
\[
\{ \text{Tr}(t^a f^{-1} \delta f)(x), \text{Tr}(t^b f^{-1} \delta f)(y) \} = \frac{\pi}{2k} \epsilon(x-y) \left( C_{cA}(f(y)) C_{cB}(f(x)) - \delta_{AB} \right),
\]
\[
\{ \text{Tr}(t^a h^{-1} \delta h)(x), \text{Tr}(t^b h^{-1} \delta h)(y) \} = \frac{\pi}{2k} \epsilon(x-y) \delta_{ab},
\]
\[
\{ \text{Tr}(t^a h^{-1} \delta h)(x), \text{Tr}(t^b f^{-1} \delta f)(y) \} = -\frac{\pi}{2k} \epsilon(x-y) C_{AB}(f(x)).
\]
(A.12)

We would like to compute Poisson brackets of the gauged invariant quantities \( \Psi = \frac{i k}{\pi} \partial_- f f^{-1} \), obeying \( \partial_+ \Psi = 0 \) on shell, and \( H = -\frac{i k}{\pi} \partial_- hh^{-1} \). Using the variations
\[
\delta \Psi_A = \frac{i k}{\pi} C_{AB}(f) \text{Tr}(t^B \partial_-(f^{-1} \delta f)),
\]
\[
\delta H_a = -\frac{i k}{\pi} C_{ab}(h) \text{Tr}(t^B \partial_-(h^{-1} \delta h)),
\]
and (A.12) we obtain
\[
\{ \Psi_a(x), \Psi_b(y) \} = -\frac{k}{\pi} \delta_{ab} \delta'(x-y) - \left( f_{a\beta\gamma} \Psi_\gamma(y) + f_{a\beta c} \Psi_c(y) \right) \delta(x-y)
\]
\[-\frac{\pi}{2k} f_{c\alpha\gamma} f_{c\beta\delta} \Psi_\gamma(x) \Psi_\delta(y) \epsilon(x-y),
\]
(A.14)

\[
\{ \Psi_a(x), \Psi_b(y) \} = f_{abc} \Psi_c(y) \delta(x-y) - \frac{\pi}{2k} f_{cad} f_{cbe} \Psi_d(x) \Psi_e(y) \epsilon(x-y),
\]
(A.15)

\[
\{ \Psi_a(x), \Psi_b(y) \} = -\frac{\pi}{2k} f_{cad} f_{c\beta\gamma} \Psi_d(x) \Psi_\gamma(y) \epsilon(x-y),
\]
(A.16)

and
\[
\{ H_a(x), H_b(y) \} = \frac{k}{\pi} \delta_{ab} \delta'(x-y) - f_{abc} H_c(y) \delta(x-y),
\]
(A.17)

\[
\{ H_a(x), \Psi_b(y) \} = \frac{k}{\pi} C_{ab}(h(x)) \delta'(x-y),
\]
(A.18)

\[
\{ H_a(x), \Psi_b(y) \} = 0.
\]
(A.19)

The form of the action (A.7) suggests that the equation of motion corresponding to the gauge field \( A_+ \) has to be imposed as a constraint, i.e. \( \varphi_1^a(x) = \Psi^a(x) \approx 0 \). Then, the on shell condition \( F_{++} = 0 \) implies the constraint \( \varphi_2^a = H^a \approx 0 \) (or \( h \approx 1 \)). However, due to (A.17) and (A.18) we observe that these cannot be imposed strongly. They are second class constraints and the matrix (A.1), in the basis \( \varphi_a(x) = \{ \varphi_1^a(x), \varphi_2^a(x) \} \), is given by
\[
D_{ab}(x, y) = \frac{k}{\pi} \left( \begin{array}{cc} 0 & \delta_{ab} \\ \delta_{ab} & \delta_{ab} \end{array} \right) \delta'(x-y),
\]
(A.20)

whereas its inverse is
\[
D^{-1}_{ab}(x, y) = \frac{\pi}{2k} \left( \begin{array}{cc} -\delta_{ab} & \delta_{ab} \\ \delta_{ab} & 0 \end{array} \right) \epsilon(x-y).
\]
(A.21)
Then using (A.2) we can compute the Dirac brackets of the $\Psi_\alpha$’s. It turns out that, 
$\{\Psi_\alpha, \Psi_\beta\}_D \approx \{\Psi_\alpha, \Psi_\beta\}$, hence obtaining the result (3.26).

**Non–abelian duals of WZW Models:** In this case the starting point is the action (3.22). Its general variation is given by

$$
\delta S = \frac{k}{\pi} \int h^{-1} \delta h \left( f_\partial \left( f^{-1} f_+ \right) f^{-1} + f_\partial \left( f^{-1} h^{-1} h f \right) f^{-1} - \lambda \partial_\lambda \left( \lambda^{-1} h^{-1} \partial_\lambda h \right) \lambda^{-1} \right.
$$

$$
- \lambda \partial_\lambda \left( \lambda^{-1} \partial_\lambda \right) \lambda^{-1} + f_\delta \left( \partial_\lambda \left( f^{-1} f_+ \right) + \partial_\lambda \left( f^{-1} h^{-1} h f \right) \right)
$$

$$
- \lambda^{-1} \delta \lambda \partial_\lambda \left( \lambda^{-1} h^{-1} \partial_\lambda h \right).
$$

(A.22)

Similarly to (A.9) we define

$$
Z_I = \left( Tr(t^a h^{-1} \delta h), Tr(t^a \lambda^{-1} \lambda), Tr(t^A f^{-1} f_\delta) \right).
$$

(A.23)

These obey (A.10) with the matrix $M(x, y)$ now defined as

$$
M(x, y) = \begin{pmatrix}
C_{ab}(f(x)f^{-1}(y)) - C_{ab}(\lambda(x)\lambda^{-1}(y)) & -C_{ab}(\lambda(x)) & C_{ab}(f(x)) \\
-C_{ba}(\lambda(y)) & 0 & 0 \\
C_{bA}(f(y)) & 0 & \delta_{AB}
\end{pmatrix}.
$$

(A.24)

Inverting this matrix we find that the non–zero basic Poisson brackets are given by

$$\{Tr(t^A f^{-1} f_\delta(x), Tr(t^B f^{-1} f_\delta(y))\} = -\frac{\pi}{2k} \delta_{AB} \epsilon(x - y),$$

$$\{Tr(t^a \lambda^{-1} \lambda(x), Tr(t^b \lambda^{-1} \lambda)(y)\} = -\frac{\pi}{2k} \delta_{ab} \epsilon(x - y),$$

$$\{Tr(t^a \lambda^{-1} \lambda(x), Tr(t^b f^{-1} f_\delta(y))\} = -\frac{\pi}{2k} C_{ab}(\lambda^{-1}(x)f(x)) \epsilon(x - y),$$

$$\{Tr(t^a h^{-1} \delta h(x), Tr(t^b \lambda^{-1} \lambda)(y)\} = \frac{\pi}{2k} C_{ab}(\lambda(y)) \epsilon(x - y).$$

(A.25)

Using them and the variations (A.13),(3.29) we calculate the Poisson brackets

$$\{\Psi_A(x), \Psi_B(y)\} = -\frac{k}{\pi} \delta_{AB} \delta'(x - y) - f_{ABC} \Psi_C(y) \delta(x - y),$$

(A.26)

$$\{J_a(x), J_b(y)\} = -\frac{k}{\pi} \delta_{ab} \delta'(x - y) - f_{abc} J_c(y) \delta(x - y),$$

(A.27)

$$\{J_a(x), \Psi_b(y)\} = -\frac{k}{\pi} \delta_{ab} \delta'(x - y) - f_{abc} J_c(y) \delta(x - y),$$

(A.28)

$$\{J_a(x), \Psi_\beta(y)\} = 0,$$

(A.29)

and

$$\{H_a(x), J_b(y)\} = -\frac{k}{\pi} C_{ab}(h(x)) \delta'(x - y) - f_{abc} C_{ad}(h(y)) J_c(y) \delta(x - y),$$

(A.30)

$$\{H_a(x), H_b(y)\} = \{H_a(x), \Psi_A(y)\} = 0 .$$

(A.31)
As in the case of gauged WZW models we have to impose the equation of motion corresponding to $A_+$ as a constraint, i.e., $\varphi_1^a(x) = \Psi^a(x) - J^a(x) \approx 0$, as well as $\varphi_2^a(x) = H^a(x) \approx 0$ corresponding to $F_+ = 0$. Since they cannot imposed strongly we again follow Dirac’s procedure. We first compute the matrix (A.1)

$$D_{ab}(x, y) = \frac{k}{\pi} \begin{pmatrix} 0 & C_{ab}(\lambda(x)\lambda^{-1}(y)) \\ C_{ba}(\lambda(x)\lambda^{-1}(y)) & 0 \end{pmatrix} \delta'(x - y),$$

and its inverse

$$D^{-1}_{ab}(x, y) = \frac{\pi}{2k} \begin{pmatrix} 0 & C_{ab}(\lambda(y)\lambda^{-1}(x)) \\ C_{ba}(\lambda(y)\lambda^{-1}(x)) & 0 \end{pmatrix} \epsilon(x - y).$$

Then using (A.2) we obtain that the Dirac bracket $\{\Psi_A, \Psi_B\}_D$ coincides with the corresponding Poisson bracket (A.26). As a consistency check the Dirac brackets of the $J_a$’s should coincide with the Dirac brackets of the $\Psi_a$’s because the constraint $\varphi_1^a$ is imposed strongly. This can be verified using (A.2) and the explicit form of the matrix $D^{-1}_{ab}$ in (A.33). We note that this is not the case for the corresponding Poisson brackets as one can see from (A.28), (A.29).

Finally, let us mention that the conclusion we have reached about WZW models and their non–Abelian duals, would have of course been the same even if we had worked, within Dirac’s general framework, with the action (3.24) instead of (3.22).
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