STABILIZATION OF THE WITT GROUP

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Abstract.
In this Note, using an idea due to Thomason [8], we define a “homology theory” on the category of rings which satisfies excision, exactness, homotopy (in the algebraic sense) and periodicity of order 4. For regular noetherian rings, we find Balmer’s higher Witt groups. For more general rings, this homology is isomorphic to the KT-theory of Hornbostel [3], inspired by the work of Williams [9]. For real or complex C*-algebras, we recover - up to 2 torsion - topological K-theory.

1. Let A be a ring with an antiinvolution a → ȧ and let ε be an element of the center of A such that εε = 1. We assume also that 2 is invertible in the ring. There are now well known definitions of the higher hermitian K-group (denoted by εLn(A), as in [5]) and the higher Witt group εWn(A) : this is the cokernel of the map induced by the hyperbolic functor

K_n(A) → εL_n(A)

where the K_n(A) denote the Quillen K-group (which is defined for all values of n ∈ Z).

One of the fundamental results of higher Witt theory is the periodicity isomorphism (where Z' = Z[1/2], cf.[4])

εWn(A) ⊗ Z' ≅ εW_{n-2}(A) ⊗ Z'

It is induced by the cup-product with a genuine element u_2 ∈ εL_{-2}(Z'). By analogy with algebraic topology, we shall call u_2 the Bott element in Witt theory. This element is explicitly described in the following way. We consider the 2 x 2 matrix (with the involution defined by ȧ = ȧ^-1 and ť = ť^-1 and where we put λ = ȧ = 1/2).

M = \begin{pmatrix}
    b(t+t^{-1}-2) & a + (1-a)t \\
    -(1-d+dt^{-1}) & -c
\end{pmatrix}

where \begin{pmatrix} a & b \\ c & d \end{pmatrix} = u p_0 u^{-1}

with p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} and u = \begin{pmatrix} \lambda \overline{\lambda} + \lambda \\ \lambda \overline{\lambda}(z - 1) \\ \lambda z + \overline{\lambda} \\ z - 1 \end{pmatrix}

This 2 x 2 matrix represents an element of εL_{0}(Z'[t, t^{-1}, z, z^{-1}]) whose image in εL_{-2}(Z') ≅ Z ⊕ Z/2 is a free generator (cf. [5] for the details).
2. The higher Witt groups $\varepsilon W_n(A)$ do not have all the nice formal properties one should expect. For instance, a cartesian square of rings with antiinvolutions (where the vertical maps are surjective)

\[
\begin{array}{c}
A \\ \downarrow \\
A_2
\end{array} \rightarrow \begin{array}{c}
A_1 \\ \downarrow \\
A'
\end{array}
\]

does not induce in general a long Mayer-Vietoris exact sequence of Witt groups

\[
\varepsilon W_{n+1}(A') \rightarrow \varepsilon W_n(A) \rightarrow \varepsilon W_n(A_1) \oplus \varepsilon W_n(A_2) \rightarrow \varepsilon W_n(A') \rightarrow \varepsilon W_{n-2}(A_2)
\]

As a counterexample for $n = 0$ coming from topology, one might take the ring of complex continuous functions on finite CW-complexes, provided with the trivial involution (otherwise this will imply - with topological notations - that the obvious cokernel from KU to KO is a cohomology theory and therefore a direct factor in KO). Note however that by tensoring with $\mathbb{Z}'$, we restore the Mayer-Vietoris axiom as a consequence of the periodicity theorem and the direct splitting of hermitian K-theory shown in [4] p. 253.

Following an idea due to Thomason [8], one may overcome this difficulty by stabilizing the higher Witt groups. More precisely, we define a new theory $\varepsilon W_n(A)$ as the limit of the inductive system

\[
\varepsilon W_n(A) \rightarrow \varepsilon W_{n-2}(A) \rightarrow \varepsilon W_{n-4}(A) \rightarrow ...
\]

where the arrows are induced by the cup-product with the Bott element $u_2$ mentioned above. As a matter of fact, the periodicity map $\varepsilon W_n(A) \rightarrow \varepsilon W_{n-2}(A)$ can be factored as

\[
\varepsilon W_n(A) \rightarrow \varepsilon L_{n-2}(A) \rightarrow \varepsilon W_{n-2}(A)
\]

Therefore $\varepsilon W_n(A)$ is also the limit of the inductive system

\[
\varepsilon L_n(A) \rightarrow \varepsilon L_{n-2}(A) \rightarrow \varepsilon L_{n-4}(A) \rightarrow ...
\]

3. **THEOREM.** This new theory $\varepsilon W_n(A)$ satisfies the following properties

a) **Homotopy invariance.** The polynomial extension $A \rightarrow A[t]$, where $t = t$, induces an isomorphism

\[
\varepsilon W_n(A) \cong \varepsilon W_n(A[t])
\]

b) **Exactness and excision.** From a cartesian square as above (with $\psi$ surjective)
one deduces an isomorphism of the associated relative groups

\[ \epsilon \mathcal{U}_n(\phi) \cong \epsilon \mathcal{U}_n(\psi) \]

and therefore a Mayer-Vietoris exact sequence (for all \( n \in \mathbb{Z} \))

\[ \epsilon \mathcal{U}_n(A) \longrightarrow \epsilon \mathcal{U}_n(A') \longrightarrow \epsilon \mathcal{U}_n(A_1) \oplus \epsilon \mathcal{U}_n(A_2) \longrightarrow \epsilon \mathcal{U}_n(A') \longrightarrow \]

c) **Periodicity.** The cup-product with the Bott element induces the isomorphisms

\[ \epsilon \mathcal{U}_n(A) \cong \epsilon \mathcal{U}_{n-2}(A) \cong \epsilon \mathcal{U}_{n-4}(A) \]

d) **Normalization.** Let us assume now that \( A \) is a regular noetherian ring. Then the natural map

\[ \epsilon \mathcal{W}_0(A) \longrightarrow \epsilon \mathcal{W}_0(A) \]

is an isomorphism and the group \( \epsilon \mathcal{W}_1(A) \) is isomorphic to the cokernel of the map defined in [5]

\[ k_0(A) \longrightarrow \epsilon \mathcal{W}_1(A) \]

Moreover, the groups \( \epsilon \mathcal{W}_n(A) \) coincide with the higher Witt groups of Balmer [1].

**Proof:** Periodicity is imposed by the definition (as in Thomason’s theory). Homotopy invariance is a consequence of the same property for the Witt groups. Since the \( L_n \)-groups satisfy the excision and exactness properties for \( n < 0 \) (cf. [6] for instance), this is also true of the theory \( \epsilon \mathcal{U}_n : \) as we have noticed before, \( \epsilon \mathcal{U}_n(A) \) is also the limit of the inductive system

\[ \epsilon L_n(A) \longrightarrow \epsilon L_{n-2}(A) \longrightarrow \epsilon L_{n-4}(A) \longrightarrow \]

If \( A \) is regular noetherian, the \( K \)-theory groups \( K_n(A) \) are 0 for \( n < 0 \). Therefore, according to the 12 term exact sequence proved in [5], we have an exact sequence

\[ 0 = k_{-1}(A) \longrightarrow \epsilon W_0(A) \longrightarrow \epsilon W_{-2}(A) \longrightarrow k'_{-1}(A) = 0 \]

with an obvious isomorphism \( \epsilon W_{-2}(A) \cong \epsilon W_{-2}(A) \) since again \( K_n(A) = 0 \) for \( n = -1 \) and -2. With the same argument, we prove that \( \epsilon W_n(A) \cong \epsilon W_{n-2}(A) \) for all \( n \leq 0 \).

In the same spirit, we have an exact sequence

\[ k_0(A) \longrightarrow \epsilon W_1(A) \longrightarrow \epsilon \mathcal{W}_1(A) \longrightarrow 0 \]

The first map is the following. An element of \( k_0(A) \) is the class of a module \( E \) which is isomorphic to its dual. Its image in \( \epsilon W_1(A) \) is associated to the automorphism of the hyperbolic module \( E \oplus E^* \cong E \oplus E \) defined by the matrix
Finally, the isomorphism with the higher Witt groups defined by Balmer will follow from next theorem.

4. THEOREM. The homology $\varepsilon W_\ast(A)$ is isomorphic to the KT$_\ast$-theory of Hornbostel (cf. [3] § 5).

Proof. This KT-theory is the direct limit of the system

$$
\varepsilon L_n(A) \longrightarrow \varepsilon L_n(U_A) \longrightarrow \ldots \longrightarrow \varepsilon L_n(U_A^r) \longrightarrow
$$

where $U_A$ is the ring defined in [5], p. 263 and $U_A^r$ the $r$-iteration of the “U-construction”. All the arrows above are $L_\ast$-module maps as defined in [4] p. 233 and [5] p. 276. This implies that the homomorphism

$$
\varepsilon L_n(A) \longrightarrow \varepsilon L_n(U_A^r)
$$

is the cup-product with a well defined element $w_r$ in $1L_0(U_Z^r)$ (this is probably related to the question 6.6 raised by Hornbostel in his paper [3]). On the other hand, as a consequence of the fundamental theorem of hermitian K-theory (cf. [5] p. 264), we have an isomorphism of $L_\ast$-modules between $\varepsilon L_n(U_A^r)$ and $\varepsilon L_n(U_A^{r+4})$ (as noticed also by Williams [9]). Therefore, the previous direct limit is simply the limit of the system

$$
\varepsilon L_n(A) \longrightarrow \varepsilon L_n(A) \longrightarrow \ldots
$$

where the arrows are defined by the cup-product with a specific element $w$ in $1L_\ast(U_Z^r)$. On the other hand, we know that if we apply this construction to the ring $A = Z'$ and $\varepsilon = 1$, we find an isomorphism between $1W_0(Z')$ and $1L_4(Z') = 1W_4(Z')$ (because the ring $Z'$ is regular). As a matter of fact, we find a chain of isomorphisms

$$
1W_0(Z') \cong 1L_0(U_Z^r) \cong 1L_0(U_Z^2) \cong 1L_0(U_Z^3) \cong 1L_0(U_Z^4) \cong 1L_4(Z')
$$

I claim that $w$, the image of 1 by this chain of isomorphisms, is $(u_2)^2$ up to a unipotent element. This is exactly the well known computation of the classical Witt ring of $Z'$ which is $Z \oplus Z/2$ generated by the classes of the following elements in the Grothendieck Witt group: $<x^2>$ and $<\bar{x}^2>-<2x^2>$.

If $A$ is regular noetherian and $\varepsilon = 1$, Hornbostel has proved moreover in [3] that KT$_n(A)$ is isomorphic to the n-Witt group defined by Balmer, which proves the last part of theorem 3.

5. Remark. For simplicity’s sake, we have just considered hermitian K-theory groups. One could have taken as well homotopy colimits of the corresponding classifying
spaces, using for instance the machinery developed in [4] § 1, greatly generalized by Schlichting.

6. THEOREM. The theory $\e L_0(A)$ is invariant under nilpotent extensions. In other words, let $I$ be a nilpotent ideal of $A$, stable under the antiinvolution. Then the quotient map $A \longrightarrow A/I$ induces an isomorphism

$$\e L_0(A) \cong \e L_0(A/I)$$

Proof. For $n < 0$, $\e L_n(A)$ is $\e L_0$ of the $(-n)^{th}$ suspension $S^{-n}A$. Therefore, it suffices to show that the induced map $\e L_0(A) \longrightarrow \e L_0(A/I)$ is an isomorphism. To prove surjectivity, we remark that every $A/I$-hermitian module $M$ is the image of a self-adjoint projection operator $Q$ in some hyperbolic module $H(B^n)$, where $B = A/I$ (cf. [6]). We write $Q = (J - 1)/2$ where $J$ is an involution. We lift $J$ to an operator on $H(A^n)$ which we call $R$. Then $R^* = R + \eta$, where $\eta$ is a matrix with coefficients in $I$. By replacing $R$ by $S = R + \eta/2$, we see that we may assume $R$ to be self-adjoint. Now $R^2 = 1 + \gamma$ where $\gamma$ is a matrix with coefficients in $I$ and the power expansion $U = (1 + \gamma)^{-1/2}$ is convergent since $I$ is nilpotent. Therefore, the product $RU$ is a self-adjoint involution which is a lift of $J$. This proves the surjectivity of the map $\e L_0(A) \longrightarrow \e L_0(A/I)$ (take the image of $1 - RU$).

For the proof of injectivity, the argument is quite similar. Let $E_1$ et $E_2$ be two hermitian modules over the ring $A$ which become isomorphic over the ring $A/I$. This means (up to stabilization) that $E_1$ and $E_2$ are associated to self-adjoint involutions $J_1$ and $J_2$ which are conjugate mod $I$. In other words, we have $J_2 = \alpha J_1 (\alpha)^{-1}$ where $\alpha$ is a unitary matrix mod. $I$. This invertible matrix can be lifted to an invertible matrix $\beta$ in $H(A^n)$ and we put

$$\gamma = \beta (\beta^*)^{-1/2}$$

(polar decomposition of matrices). This is a unitary matrix which is a lift of $\alpha$. Therefore, by replacing $J_1$ by $\gamma J_1 \gamma^{-1}$, we may assume that $J_1 = J_2$ mod $I$.

We now consider the matrix $\delta = (1 + J_1 J_2)/2$. Since $\delta = 1$ mod $I$, $\delta$ is an invertible matrix such that $\delta J_1 = J_2, \delta$. We consider again the polar decomposition of $\delta$, i.e. we replace $\delta$ by $\delta' = \delta (\delta^*)^{-1/2}$. Since $\delta, \delta^*$ commutes with $J_1$, we also have $\delta'. J_1 = J_2, \delta'$. This shows that $\delta'$ is a unitary matrix which conjugates $J_1$ and $J_2$ and completes the proof of injectivity.

7. Remark. Using [2] and most of the above properties of the theory $\e L_n$, Schlichting was able to prove cdh descent for the theory $\e L_*$, extended to the category of (commutative) schemes of finite type over a field of characteristic 0 [7].

8. When $A$ is a Banach algebra, we may consider the topological analogs of the

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1 More precisely, $E_1$ (resp. $E_2$) is the image of $(J_1 - 1)/2$ (resp. $(J_2 - 1)/2$).
previous definitions. In that case, the group $\varepsilon W_n^{\text{top}}(A)$ is simply isomorphic to $\varepsilon W_n^{\text{top}}(A) \otimes \mathbb{Z}'$. We prove this fact by looking at the image of $(u_2)^4$ in $1W_{-8}^{\text{top}}(R) \cong \mathbb{Z}$. According to [5] we find 8 times the generator. Therefore, by taking the inductive limit we localize with respect to the multiplicative system $(2^r)$; the natural map $\varepsilon W_n^{\text{top}}(A) \rightarrow \varepsilon W_n^{\text{top}}(A)$ coincides with this localization. One should also notice that $\varepsilon W_n^{\text{top}}(A) \otimes \mathbb{Z}'$ is isomorphic to $\varepsilon W_n(A) \otimes \mathbb{Z}'$, as a consequence of the periodicity theorem and the well-known computation of $\varepsilon W_0$ and $\varepsilon W_1$. Finally, if $A$ is a $C^*$-algebra, it is well known that $1W_n^{\text{top}}(A)$ is isomorphic to the topological K-theory of $A$.

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