Eventually fixed points of endomorphisms of virtually free groups

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Abstract

We consider the subgroup of points of finite orbit through the action of an endomorphism of a finitely generated virtually free group, with particular emphasis on the subgroup of eventually fixed points, EvFix(ϕ): points whose orbit contains a fixed point. We provide an algorithm to compute the subgroup of fixed points of an endomorphism of a finitely generated virtually free group and prove that finite orbits have cardinality bounded by a computable constant, which allows us to solve several algorithmic problems: deciding if ϕ is a finite order element of End(G), if ϕ is aperiodic, if EvFix(ϕ) is finitely generated and, in the free group case, whether EvFix(ϕ) is a normal subgroup of $F_n$ or not. We also present a bound for the rank of EvFix(ϕ) in case it is finitely generated.

1 Introduction

The study of fixed subgroups of endomorphisms of groups started with the (independent) work of Gersten [5] and Cooper [4], using respectively graph-theoretic and topological approaches. They proved that the subgroup of fixed points Fix(ϕ) of some fixed automorphism ϕ of $F_n$ is always finitely generated, and Cooper succeeded on classifying from the dynamical viewpoint the fixed points of the continuous extension of ϕ to the boundary of $F_n$. Bestvina and Handel subsequently developed the theory of train tracks to prove that Fix(ϕ) has rank at most n in [2]. This was shown to hold for general endomorphisms by Imrich and Turner by reducing to problem to the automorphism case in [8]. The problem of computing a basis for Fix(ϕ) had a tribulated history and was finally settled by Bogopolski and Maslakova in 2016 in [3] for automorphisms and by Mutanguha [13] for general endomorphisms of a free group. This line of research extended early to wider classes of groups. For instance, Paulin proved in 1989 that the subgroup of fixed points of an automorphism of a hyperbolic group is finitely generated [15]. Fixed points were also studied for right-angled Artin groups [16] and lamplighter groups [11].

In this paper, we extend the algorithmic result of [13] proving that Fix(ϕ) is computable if ϕ is an endomorphism of a finitely generated virtually free group.

Theorem 4.2. Let G be a finitely generated virtually free group and ϕ ∈ End(G). Then Fix(ϕ) is computable.

In [14], Myasnikov and Shpilrain study finite orbits of elements of a free group under the action of an automorphism proving that, in a free group $F_n$, there is an orbit of cardinality k if and only if there is an element of order k in Aut($F_n$). Moreover, the authors prove that this result does not hold for general endomorphisms, by providing an example of an endomorphism of $F_3$ for which there is a point whose orbit has 5 elements.

In this paper, we study finite orbits of elements under the action of an endomorphism of a finitely generated virtually free group. The orbit of an element $x ∈ F_n$ through an
endomorphism \( \varphi \) is finite if and only if it intersects the subgroup of periodic points, \( \text{Per}(\varphi) \), of \( \varphi \). The set of such points forms a subgroup of \( F_n \) and so do the points whose orbit intersects the fixed subgroup \( \text{Fix}(\varphi) \). We call these subgroups \( \text{EvPer}(\varphi) \) and \( \text{EvFix}(\varphi) \), respectively. It is easy to see that \( \text{EvFix}(\varphi) \) coincides with \( \text{Fix}(\varphi) \) if (and only if) \( \varphi \) is injective. For this reason, we will mainly focus on noninjective endomorphisms. Given an endomorphism \( \varphi \in \text{End}(F_n) \), we will find conditions for \( \text{EvFix}(\varphi) \) to be a normal subgroup of \( F_n \). Also, despite the fact that the result in [14] cannot be generalized to endomorphisms, we prove that, replacing finite orbits by periodic orbits, the result holds for endomorphisms, i.e., there is an endomorphism of \( F_n \) with a periodic orbit of cardinality \( k \) if and only if there is an element of order \( k \) in \( \text{Aut}(F_n) \). Moreover, we prove that, for endomorphisms of finitely generated virtually free groups, finite orbits have bounded cardinality.

**Corollary 5.6** Let \( G \) be a finitely generated virtually free group and \( \varphi \in \text{End}(G) \). There is a computable constant \( k \) that

\[
\max\{|\text{Orb}_\varphi(x)| \mid x \in \text{EvPer}(\varphi)\} \leq k.
\]

This allows us to solve some algorithmic questions: we can decide if \( \varphi \) is a finite order element of \( \text{End}(G) \), if \( \varphi \) is aperiodic or not and, in case \( G \) is free, whether \( \text{EvFix}(\varphi) \) is a normal subgroup of \( F_n \) or not.

Unlike the case of \( \text{Fix}(\varphi) \) and \( \text{Per}(\varphi) \), the subgroups \( \text{EvFix}(\varphi) \) and \( \text{EvPer}(\varphi) \) are not necessarily finitely generated. However, we prove that we can always decide if that is the case, by proving the following result, which might have independent interest. Also, if \( \text{EvFix}(\varphi) \) is finitely generated, then a set of generators can be effectively computed.

**Proposition 5.12** Let \( G \) be a finitely generated virtually free group having a free subgroup \( F \) of finite index, \( H \leq G \) and \( \varphi \in \text{End}(G) \) be an endomorphism. If \( \ker(\varphi) \) is finite, then \( H \varphi^{-1} \) is finitely generated. If not, the following are equivalent:

1. \( H \varphi^{-1} \) is finitely generated
2. \( H \varphi^{-1} \cap F \) is a finite index subgroup of \( F \)
3. \( H \varphi^{-1} \cap F \) is a finite index subgroup of \( G \)
4. \( H \varphi^{-1} \) is a finite index subgroup of \( G \)
5. \( H \cap G \varphi \) is a finite index subgroup of \( G \varphi \)
6. \( H \cap F \varphi \) is a finite index subgroup of \( F \varphi \)

Finally, we provide an upper bound for the rank of \( \text{EvFix}(\varphi) \) for endomorphisms of \( G \). However, we do not know if the bound is sharp.

2 General properties

Let \( G \) be a group and \( \varphi \in \text{End}(G) \). A point \( x \in G \) is said to be a fixed point if \( x\varphi = x \). The set of all fixed points forms a subgroup which we denote by \( \text{Fix}(\varphi) \). A point \( x \in G \) is said to be a periodic point if there is some \( m \in \mathbb{N} \) such that \( x\varphi^m = x \). The set of all periodic points forms a subgroup which we denote by \( \text{Per}(\varphi) \). Obviously, we have that

\[
\text{Per}(\varphi) = \bigcup_{k=1}^{\infty} \text{Fix}(\varphi^k).
\]

Given \( x \in G \), the orbit of \( x \) through \( \varphi \) is defined by

\[
\text{Orb}_\varphi(x) = \{x\varphi^k \mid k \in \mathbb{N}\}.
\]
A point $x$ is said to be eventually periodic if its orbit is finite, i.e., there is some $m \in \mathbb{N}$ such that $x\varphi^m \in \text{Per}(\varphi)$ and similarly, $x$ is said to be eventually fixed if there is some $m \in \mathbb{N}$ such that $x\varphi^m \in \text{Fix}(\varphi)$ or, equivalently, such that $x\varphi^m = x\varphi^{m+1}$. In this case, for every $k \geq m$, we have that $x\varphi^k = x\varphi^m$. We denote by $\text{EvPer}(\varphi)$ (resp. $\text{EvFix}(\varphi)$) the set of all eventually periodic (resp. fixed) points of $\varphi$. It is clear from the definitions that

$$\text{EvPer}(\varphi) = \bigcup_{k=1}^{\infty} (\text{Per}(\varphi))\varphi^{-k} \quad \text{and} \quad \text{EvFix}(\varphi) = \bigcup_{k=1}^{\infty} (\text{Fix}(\varphi))\varphi^{-k}.$$

**Proposition 2.1.** Let $\varphi \in \text{End}(G)$. Then $\text{EvPer}(\varphi)$ and $\text{EvFix}(\varphi)$ are groups.

**Proof.** Let $x_1, x_2 \in \text{EvPer}(\varphi)$. Then, there are $m_1, m_2 \in \mathbb{N}$ such that $x_1\varphi^{k_1}$ and $x_1\varphi^{k_2}$ are periodic points for all $k_1 > m_1$ and $k_2 > m_2$. So, taking $M = \max\{m_1, m_2\}$, we have that $(x_1 x_2)\varphi^M$ is periodic. Also, if there is some $m \in \mathbb{N}$ such that $x_1\varphi^m \in \text{Per}(\varphi)$, then $x_1^{-1}\varphi^m \in \text{Per}(\varphi)$.

Similarly, let $x_1, x_2 \in \text{EvFix}(\varphi)$. Then, there are $m_i \in \mathbb{N}$ such that $x_i\varphi^{m_i} = x_i\varphi^{m_i+1}$, for $i = 1, 2$. Then, putting $M = \max\{m_1, m_2, m_3\}$, we have that $(x y)\varphi^{M+1} = x\varphi^{M+1} y\varphi^{M+1} = x\varphi^{m_1} y\varphi^{m_2} = x\varphi^M y\varphi^M$. Also, we have that $(x_1^{-1}\varphi^{m_1+1} = (x_1\varphi^{m_1+1})^{-1} = (x_1\varphi^m)^{-1} = x_1^{-1}\varphi^m$.

Now, we present some natural properties of $\text{EvFix}(\varphi)$.

**Lemma 2.2.** Let $\varphi \in \text{End}(G)$. Then

(i) $\bigcup_{k=1}^{\infty} \text{Ker}(\varphi^k) \subseteq \text{EvFix}(\varphi)$

(ii) $\text{Fix}(\varphi) \subseteq \text{EvFix}(\varphi)$

(iii) $\text{EvFix}(\varphi) \cap \text{Per}(\varphi) = \text{Fix}(\varphi)$

(iv) $\text{EvFix}(\varphi) = \text{Fix}(\varphi) \iff \varphi$ is a monomorphism.

**Proof.** (i), (ii) and (iii) are obvious by definition. If $\varphi$ is a monomorphism, then $(\text{Fix}(\varphi))\varphi^{-k} = \text{Fix}(\varphi)$ for every $k \in \mathbb{N}$, and so $\text{EvFix}(\varphi) = \text{Fix}(\varphi)$. If there exists $1 \neq x \in \text{Ker}(\varphi)$, then $x \in \text{EvFix}(\varphi) \setminus \text{Fix}(\varphi)$. 

In case $G$ is a finitely generated virtually free group, it is well known that $\text{Fix}(\varphi)$ is finitely generated. However, that might not be the case of $\text{EvFix}(\varphi)$, even in the free group case.

**Example 2.3.** Let $\varphi : F_2 \to F_2$ be defined by $a \mapsto aba$ and $b \mapsto 1$. Then for $x \in F_2$, we have that $x\varphi = (aba)^{\lambda_a(x)}$, where $\lambda_a : F_2 \to \mathbb{Z}$ is the endomorphism defined by $a \mapsto 1$ and $b \mapsto 0$. So, $\text{Fix}(\varphi)$ is trivial and $\text{Ker}(\varphi) = \{w \mid \lambda_a(w) = 0\}$. Also $(\text{Ker}(\varphi))\varphi^{-1} = \text{Ker}(\varphi)$. So, $\text{EvFix}(\varphi) = \text{Ker}(\varphi)$, which is not finitely generated.

**Example 2.4.** Let $\varphi : F_2 \to F_2$ be defined by $a \mapsto bab^{-1}$ and $b \mapsto 1$. Then for $x \in F_2$, we have that $x\varphi = bab^{\lambda_b(x)}b^{-1}$. So, $\text{Fix}(\varphi) = \{bab^{k}b^{-1} \mid k \in \mathbb{Z}\}$ and $\text{EvFix}(\varphi) = (\text{Fix}(\varphi))\varphi^{-1} = F_2$, which is finitely generated.

We will see later in the paper that we can decide whether $\text{EvFix}(\varphi)$ and $\text{EvPer}(\varphi)$ are finitely generated for endomorphisms of finitely generated virtually free groups.

## 3 Normality in free groups

The purpose of this section is to describe the cases where, for an endomorphism $\varphi \in \text{End}(F_n)$, we have that $\text{EvFix}(\varphi) \subseteq F_n$. We start with a technical lemma.

**Lemma 3.1.** Let $u \in F_n \setminus \{1\}$ be a nontrivial non proper power. If there exists $w \in F_n$ and $p, q \in \mathbb{Z}$ such that $w^{-1} u^p w = u^q$, then $p = q$ and $w \in \langle u \rangle$. 


Proof. Let \( u \in F_n \setminus \{1\} \). If there exist \( w \in F_n \) and \( p, q \in \mathbb{Z} \) such that \( w^{-1}u^pw = u^q \), then the cyclic reduced cores of \( u^p \) and \( u^q \) are equivalent under a cyclic permutation of their letters. This implies in particular that \( p = q \) and \( w \) commutes with \( u^p \).

We are now able to describe the cases where \( \text{EvFix}(\varphi) \) is a normal subgroup of \( F_n \).

**Proposition 3.2.** Let \( \varphi \in \text{End}(F_n) \). Then one of the following holds:

1. \( \text{EvFix}(\varphi) = F_n \)
2. \( \text{EvFix}(\varphi) = \bigcup_{k=1}^{\infty} \ker(\varphi^k) \)
3. \( \text{EvFix}(\varphi) \) is not a normal subgroup of \( F_n \).

Proof. Suppose that \( \text{EvFix}(\varphi) \neq F_n \), \( \text{EvFix}(\varphi) \neq \bigcup_{k=1}^{\infty} \ker(\varphi^k) \) and that \( \text{EvFix}(\varphi) \leq F_n \) and take \( g \in F_n \setminus \text{EvFix}(\varphi) \). Since \( \text{EvFix}(\varphi) \) is normal, then for every \( x \in \text{EvFix}(\varphi) \setminus \bigcup_{k=1}^{\infty} \ker(\varphi^k) \) we have that \( gxg^{-1} \in \text{EvFix}(\varphi) \), and \( g^{-1}xg \in \text{EvFix}(\varphi) \), which means that there are some \( n, m, p \in \mathbb{N} \) for which \( (gxg^{-1})^{\varphi^m+1} = (gxg^{-1})^{\varphi^n} \) and \( (g^{-1}xg)^{\varphi^m+1} = (g^{-1}xg)^{\varphi^n} \). Also, there is some \( p \in \mathbb{N} \) such that \( x^{\varphi^p} = x^{\varphi^p+1} \). So, letting \( M = \max\{n, m, p\} \), it follows that

\[
(gxg^{-1})^{\varphi^M+1} = (gxg^{-1})^{\varphi^M},
\]

(1)
and

\[
(g^{-1}xg)^{\varphi^M+1} = (g^{-1}xg)^{\varphi^M}
\]

(2)

We can rewrite (1) as

\[
x^{\varphi^{M+1}}g^{-1}x^{\varphi^{M+1}}g^{\varphi^M} = g^{-1}x^{\varphi^{M+1}}g^{\varphi^M}x^{\varphi^M},
\]

and so \( g^{-1}x^{\varphi^{M+1}}g^{\varphi^M} \) and \( x^{\varphi^M} \) commute. Similarly, we can rewrite (2) as

\[
g^{\varphi^M}g^{-1}x^{\varphi^{M+1}}g^{\varphi^M}g^{-1}x^{\varphi^{M+1}}g^{\varphi^M} = x^{\varphi^{M+1}}g^{\varphi^M}g^{-1}x^{\varphi^{M+1}}g^{\varphi^M},
\]

(3)

Since \( g \notin \text{EvFix}(\varphi) \), then \( g^{\varphi^M}g^{-1}x^{\varphi^{M+1}}g^{\varphi^M} \neq 1 \), and since \( x \notin \bigcup_{k=1}^{\infty} \ker(\varphi^k) \), then \( x^{\varphi^M} \neq 1 \). We then have that \( g^{\varphi^M}g^{-1}x^{\varphi^{M+1}}g^{\varphi^M}g^{-1}x^{\varphi^{M+1}}g^{\varphi^M} \) and \( x^{\varphi^M} \) are powers of the same primitive word \( u \in F_n \). So, put

\[
g^{-1}x^{\varphi^{M+1}}g^{\varphi^M} = u^q
\]

(4)
and

\[
g^{\varphi^M}g^{-1}x^{\varphi^{M+1}}g^{\varphi^M} = u^k
\]

(5)
and

\[
x^{\varphi^M} = u^r.
\]

(6)

From (4) we get that

\[
g^{-1}x^{\varphi^{M+1}}g^{\varphi^M} = u^q
\]

(4)
and

\[
g^{\varphi^M}g^{-1}x^{\varphi^{M+1}}g^{\varphi^M} = u^k
\]

(5)
and

\[
x^{\varphi^M} = u^r.
\]

(6)

From (4) we get that

\[
g^{-1}x^{\varphi^{M+1}}g^{\varphi^M}g^{-1}x^{\varphi^{M+1}}g^{\varphi^M} = u^{2q}
\]

(4)
and

\[
g^{\varphi^M}g^{-1}x^{\varphi^{M+1}}g^{\varphi^M}g^{-1}x^{\varphi^{M+1}}g^{\varphi^M} = u^{2k}
\]

(5)

and

\[
x^{\varphi^M} = u^r.
\]

(6)

From (4) we get that

\[
g^{-1}x^{\varphi^{M+1}}g^{\varphi^M}g^{-1}x^{\varphi^{M+1}}g^{\varphi^M} = u^{2q}
\]

Applying (3.3), we obtain that \( g^{-1}x^{\varphi^{M+1}}u^{k}g^{\varphi^M} = u^{2q} \), and so that

\[
g^{\varphi^M} = u^{k}g^{\varphi^M}u^{-2q}.
\]
But from (4), we know that
\[ g_{\varphi}^{M+1} = g_{\varphi}^{M}u^{-q}. \]

Hence \( g_{\varphi}^{M}u^{-q} = u^{k}g_{\varphi}^{M}u^{-2q}, \) and \( g_{\varphi}^{M}u^{q}g^{-1}_{\varphi} = u^{k}. \) From Lemma 3.1 we know that \( k = q \) and \( g_{\varphi}^{M} \) is a power of \( u. \)

Since \( x_{\varphi}^{M} \neq 1, \) then \( r \neq 0. \) From (3) and (6), we know that \( (u_{\varphi})^{r} = u^{r} = u^{r} \) and so \( u \in \text{Fix}(\varphi) \) and \( g_{\varphi}^{M} \) is fixed, which contradicts the assumption that \( g \notin \text{EvFix}(\varphi). \)

We now present some remarks on the previous Proposition.

**Remark 3.3.** Condition 2. in Proposition 3.2 is equivalent to \( \text{Fix}(\varphi) \) being trivial. Indeed, if there is some nontrivial element \( x \in \text{Fix}(\varphi), \) then \( x_{\varphi}^{k} = x \neq 1, \) for every \( k \in \mathbb{N}. \) If \( \text{Fix}(\varphi) = 1, \) then 2. holds by definition.

**Remark 3.4.** If \( \text{EvFix}(\varphi) \) is finitely generated, then there must be a bound on the size of the orbits of eventually fixed points, but the converse is false by Example 2.3. Indeed, suppose that \( \text{EvFix}(\varphi) = \langle w_{1}, \ldots, w_{k} \rangle. \) Then
\[ M = \max\{|\text{Orb}_{\varphi}(w_{i})| \mid i \in [k]\} \]
is a bound on the size of the finite orbits. We will see later that such a bound always exists if \( \varphi \) is an endomorphism of a finitely generated virtually free group.

**Remark 3.5.** Conditions 1. and 2. are not mutually exclusive. However, if \( F_{n} = \bigcup_{k=1}^{\infty} \ker(\varphi^{k}), \) then every point is eventually sent to 1 and, by the observation above, orbit sizes must be bounded. So, this happens if and only if \( \varphi \) is a vanishing endomorphism, i.e., if there is some \( r \) such that \( \varphi^{r} \) maps every element to 1.

## 4 An algorithm to compute the fixed subgroup of an endomorphism of a finitely generated virtually free group

The purpose of this section is to provide an algorithm to compute the fixed subgroup of an endomorphism of a finitely generated virtually free group. When we take a finitely generated virtually free group as input, we assume that we are given a decomposition as a disjoint union
\[ G = Fb_{1} \cup Fb_{2} \cup \cdots \cup Fb_{m}, \quad (7) \]
where \( F = F_{A} \leq G \) is a finitely generated free group and a presentation of the form \( \langle A, b_{1}, \ldots, b_{m} \mid R \rangle, \) where the relations in \( R \) are of the form \( b_{i}a = u_{ia}b_{i} \) and \( b_{ibj} = v_{ij}b_{r_{ij}}, \) with \( u_{ia}, v_{ij} \in F_{A} \) and \( r_{ij} \in [m], \) \( i, j = 1, \ldots, m, \ a \in A. \)

We start by presenting a technical lemma, which is simply an adaptation of [7] Lemma 2.2 with the additional condition of the subgroups being normal. The proof follows in the exact same way as theirs, noting that the preimage of a normal subgroup by an endomorphism is still a normal subgroup.

A subgroup \( H \) of a group \( G \) is **fully invariant** if \( \varphi(H) \leq H \) for every endomorphism \( \varphi \) of \( G. \)

**Lemma 4.1.** Let \( G \) be a group, \( n \) be a natural number and \( N \) be the intersection of all normal subgroups of \( G \) of index \( \leq n. \) Then \( N \) is fully invariant, and if \( G \) is finitely generated, then \( N \) has finite index in \( G. \)

**Theorem 4.2.** Let \( G \) be a finitely generated virtually free group and \( \varphi \in \text{End}(G). \) Then \( \text{Fix}(\varphi) \) is computable.
Proof. Take a decomposition as in \((7)\). By Lemma \([13]\) the intersection \(F\) of all normal subgroups of \(G\) of index at most \(m\) is a fully invariant finite index subgroup, i.e. \([G : F'] < \infty\) and \(F'\varphi \subseteq F'\) for all endomorphisms \(\varphi \in \text{End}(G)\). Also, since \([G : F] = m\) and \(F \leq G\), then \(F' \leq F\), and so \(F'\) is free. We will now prove that \(F'\) is computable. We start by enumerating all finite groups of cardinality at most \(m\). For each such group \(K = \{k_1, \ldots, k_s\}\) we enumerate all homomorphisms from \(G\) to \(K\) by defining images of the generators and checking all the relations. For each homomorphism \(\theta : G \to K\), we have that \([G : \text{Ker}(\theta)] = |\text{Im}(\theta)| \leq |K| \leq m\). In fact, all normal subgroups of \(G\) of index at most \(m\) are of this form. We compute generators for the kernel of each \(\theta\), which is possible since we can test membership in \(\ker(\theta)\), which is a finite index subgroup. We can also find \(a_2, \ldots, a_s \in G\) such that

\[
G = \ker(\theta) \cup \ker(\theta)a_2 \cup \cdots \cup \ker(\theta)a_s,
\]

taking \(a_i\) such that \(a_i\theta = k_i\).

Hence, we can compute \(F'\), since it is a finite intersection of computable subgroups, and a decomposition of \(G\) as a disjoint union

\[
G = F'b_1' \cup Fb_2' \cup \cdots \cup Fb_m'.
\]

Now, take \(\psi = \varphi|_{F'}\). Since \(F'\) is fully invariant, then \(\psi \in \text{End}(F')\). Now, for \(u \in F'\), put \(X_u = \{x \in F' \mid x\psi = xu\}\). We claim that \(X_u\) is computable. Since \([G : F'] < \infty\), then \(F'\) is finitely generated and so it has a finite basis \(X\). Consider a new letter \(c\) not belonging \(X\), let \(F'' = F' \ast \langle c \rangle\) and \(\psi' \in \text{End}(F'')\) defined by mapping the letters \(x \in F''\) to \(x\psi\) and \(c\) to \(u^{-1}c\). By \([13]\), we can compute a basis for \(\text{Fix}(\psi')\). It is easy to see that \(X_u,c = \text{Fix}(\psi') \cap F'c\). Indeed, if \(x \in X_u\), then

\[
(x)c\psi' = (x\psi')(c\psi') = (x\psi)u^{-1}c = xu^{-1}c = xc
\]

and if \(x \in \text{Fix}(\psi') \cap F'c\), then there is \(y \in F'\) such that \(x = yc\) and

\[
yc = (yc)\psi' = (y\psi)u^{-1}c,
\]

which means that \(y\psi = yu\) and so \(y \in X_u\). Therefore, for all \(u \in F'\), \(X_u,c\) (and so \(X_u\)) is computable. We claim that, for \(i \in [m']\),

\[
\text{Fix}(\varphi) \cap F'b_i' = \begin{cases} \emptyset & \text{if } b_i'((b_i')^{-1}\varphi) \notin F' \\ X_{b_i'((b_i')^{-1}\varphi)}b_i' & \text{if } b_i'((b_i')^{-1}\varphi) \in F' \end{cases}
\]

Suppose that \(b_i'((b_i')^{-1}\varphi) \in F'\). Let \(x \in X_{b_i'((b_i')^{-1}\varphi)}\). Then, \(x\varphi = x\varphi b_i'((b_i')^{-1}\varphi)\), and so \((x\varphi)(b_i') \varphi = xb_i'\varphi\). Thus, \(xb_i' \in \text{Fix}(\varphi) \cap F'b_i'\). Now, let \(x \in \text{Fix}(\varphi) \cap F'b_i'\). Then, \(x(b_i')^{-1} \in F'\)

\[
(x(b_i')^{-1})\varphi = x\varphi b_i'((b_i')^{-1}\varphi) = x((b_i')^{-1}\varphi) = x(b_i')^{-1}b_i'((b_i')^{-1}\varphi)
\]

and so \(x(b_i')^{-1} \in X_{b_i'((b_i')^{-1}\varphi)}\).

If \(\text{Fix}(\varphi) \cap F'b_i' \neq \emptyset\), then there is some \(x \in F'\) such that \((x\varphi)(b_i') \varphi = (xb_i') \varphi = xb_i'\varphi\) and so \(x\varphi = xb_i'((b_i')^{-1}\varphi)\). Since \(F'\) is fully invariant, then \(xb_i'((b_i')^{-1}\varphi) \in F'\), which yields that \(b_i'((b_i')^{-1}\varphi) \in F'\).

Clearly,

\[
\text{Fix}(\varphi) = \bigcup_{i=1}^m \text{Fix}(\varphi) \cap F'b_i',
\]

and so \(\text{Fix}(\varphi)\) is computable.

\(\square\)

Remark 4.3. We remark that, since \(\text{Fix}(\psi) = \text{Fix}(\varphi) \cap F'\), then \(\text{Fix}(\varphi) : \text{Fix}(\psi) \leq [G : F']\), and so \(\text{Fix}(\varphi)\) has a generating with at most \(\text{rank}(\text{Fix}(\psi)) + [G : F']\) elements. By \([8]\), it follows that \(\text{rank}(\text{Fix}(\varphi)) \leq \text{rank}(F') + [G : F']\).
5 Finite Orbits

Given a finite orbit $\text{Orb}_\varphi(x)$, we say that $\text{Orb}_\varphi(x) \cap \text{Per}(\varphi)$ is the **periodic part of the orbit** and $\text{Orb}_\varphi(x) \setminus \text{Per}(\varphi)$ is the **straight part of the orbit**.

![Figure 1: A finite orbit](image)

In Figure 1, the straight part of the orbit corresponds to $\{x, x\varphi, \ldots, x\varphi^{r-1}\}$ and the periodic part of the orbit corresponds to $\{x\varphi^k \mid k \geq r\} = \{x\varphi^r, \ldots, x\varphi^{r+p-1}\}$, where $p$ is the period of $x\varphi^r$.

In [14], the authors show that, for an automorphism of a free group $F_n$, there is an orbit of cardinality $k$ if and only if there is an element of order $k$ in $\text{Aut}(F_n)$. Moreover, the authors prove that this result does not hold for general endomorphisms, by providing an example of an endomorphism of $F_3$ for which there is a point whose orbit has 5 elements.

However, using a standard argument, we present a similar result for periodic parts of orbits of general endomorphisms of $F_n$.

**Lemma 5.1.** Let $\varphi \in \text{End}(F_n)$. There is a periodic point of period $k$ for some $\varphi \in \text{End}(F_n)$ if and only if there is an element of order $k$ in $\text{Aut}(F_n)$.

**Proof.** Let $x \in \text{Per}(\varphi)$ be a periodic point of period $k$. Consider the stable image of $\varphi$,

$$S = \bigcap_{s \geq 1} F_n \varphi^s.$$

It is well known that $S$ is a free group of rank at most $n$ and that $\varphi|_S$ is an automorphism (see [8]). Also, it is obvious that $\text{Per}(\varphi) \subseteq S$, and so $x$ is a point of $S$ with a finite orbit of cardinality $k$. Therefore, by [13] Theorem 1.1, there is an element of order $k$ in $\text{Aut}(S)$. Since $\text{rank}(S) = r \leq n$, then there is an automorphism of $F_n$ of order $k$, which can be defined by applying the automorphism induced by $\varphi|_S$ to the first $r$ letters and the identity in the remaining letters.

Conversely, if there is an element of order $k$ in $\text{Aut}(F_n)$, then there is an orbit of cardinality $k$ for some automorphism of $F_n$. Since finite orbits of automorphisms are periodic, the result follows.

**Corollary 5.2.** There is a computable constant $k$ that bounds the size of the periodic parts of every orbit $\text{Orb}_\varphi(x)$, when $\varphi$ runs through $\text{End}(F_n)$ and $x$ runs through $F_n$.

**Proof.** By [12] and [9], $\text{Aut}(F_n)$ has an element of order $m = p_1^{\alpha_1} \cdots p_s^{\alpha_s} \in \mathbb{N}$, where $p_i$'s are different primes, if and only if $\sum_{i=1}^s (p_i^{\alpha_i} - p_i^{\alpha_i-1}) \leq n$. We have that $\sum_{i=1}^s (p_i^{\alpha_i} - p_i^{\alpha_i-1}) = \sum_{i=1}^s (p_i - 1) p_i^{\alpha_i-1}$ and so, if a natural number $m \in \mathbb{N}$ is the order of some automorphism of $F_n$, then it only admits in its factorization primes $p$ such that $p - 1 \leq n$ and each of them can have exponent at most $\log_p(n) + 1$. There are finitely many integers in those conditions, and so, $m$ must be bounded above by some constant $k$ that depends only on $n$.

We now prove that, given an endomorphism of a finitely generated virtually free group, we can bound the size of periodic parts of finite orbits by a computable constant. This constant
depends on the endomorphism unlike the one obtained in Corollary 5.2 for endomorphisms of free groups.

**Proposition 5.3.** Let $G$ be a finitely generated virtually free group and $\varphi \in \text{End}(G)$. Then, there is a computable constant $k$ such that the infinite ascending chain

$$\text{Fix}(\varphi) \subseteq \text{Fix}(\varphi^2) \subseteq \text{Fix}(\varphi^3) \subseteq \cdots$$

stabilizes after $k$ steps. Equivalently, if $x \in \text{EvPer}(\varphi)$, then the periodic part of the orbit of $x$ has cardinality at most $k$.

**Proof.** Proceeding as in the proof of Theorem 4.2, we compute a decomposition

$$F = Fb_1 \cup \cdots \cup Fb_m,$$

where $F$ is a fully invariant free subgroup of $G$. We want to compute $k$ such that the ascending chain $C$ defined by

$$\text{Fix}(\varphi) \subseteq \text{Fix}(\varphi^2) \subseteq \text{Fix}(\varphi^3) \subseteq \cdots$$

stabilizes after at most $k$ steps. For $i \in [m]$, consider the chains $C_i$ given by

$$\text{Fix}(\varphi) \cap Fb_i \subseteq \text{Fix}(\varphi^2) \cap Fb_i \subseteq \text{Fix}(\varphi^3) \cap Fb_i \subseteq \cdots$$

Since, for all $j \in \mathbb{N}$, we have that

$$\text{Fix}(\varphi^j) = \bigcup_{i \in [m]} (\text{Fix}(\varphi^j) \cap Fb_i),$$

it follows that $C$ stabilizes after $n$ steps if and only if all chains $C_i$ stabilize after at most $n$ steps.

We will prove that, for all $i \in [m]$, we can compute a constant $k_i$ such that the chain $C_i$ stabilizes after $k_i$ steps and so, taking $k = \max\{k_i \mid i \in [m]\}$ suffices.

Let $i \in [m]$. Since $F$ is fully invariant, we have that, for all $k \in \mathbb{N}$, $(Fb_i)^{\varphi^k} \subseteq F(b_i)^{\varphi^k}$. Hence, the mapping $\theta : G/F \to G/F$ defined by $Fb_i \mapsto F(b_i^\varphi)$ is a well-defined endomorphism. Since $G/F$ is finite, we can compute the orbit $\text{Orb}_\theta(Fb_i)$ of $Fb_i$ through $\theta$. In particular, we can check if $Fb_i$ is periodic. If it is not, then, $\text{Fix}(\varphi^k) \cap Fb_i = \emptyset$, for all $k \in \mathbb{N}$. Indeed, if there were some $k \in \mathbb{N}$, $x \in F$ such that $(xb_i)^{\varphi^k} = xb_i$, then, since $x\varphi^k \in F$, we have

$$(Fb_i)^{\theta^k} = F(b_i^{\varphi^k}) = F((xb_i)^{\varphi^k}) = F(xb_i) = Fb_i,$$

If $Fb_i$ is periodic, then let $p$ be its period and take $z \in F$ such that $b_i^{\varphi^p} = zb_i$. Clearly, if $j \in \mathbb{N}$ is such that $\text{Fix}(\varphi^j) \cap Fb_i \neq \emptyset$, then $p$ divides $j$. Also, let $C$ be the bound given by Corollary 5.2 for $n = \text{rank}(F) + 1$. Let $c$ be a letter not belonging to the alphabet of $F$ and $\psi : F \ast \langle c \rangle \to F \ast \langle c \rangle$ be defined by mapping the letters of the alphabet of $F$ through $\varphi^p$ and $c$ to $zc$. Notice that, for all $j \in \mathbb{N}$,

$$c^{\psi^j} = \left( \prod_{s=0}^{j-1} z^{\varphi((j-1-s)p)} \right) c \quad \text{and} \quad b_i^{\varphi^p} = \prod_{s=0}^{j-1} z^{\varphi((j-1-s)p)b_i}.$$

We claim that, for $x \in F$ and $q \in \mathbb{N}$,

$$(xc)^{\psi^q} = xc \iff (xb_i)^{\varphi^q} = xb_i.$$

Indeed, let $x \in F$ and $q \in \mathbb{N}$ be such that

$$xc = (xc)^{\psi^q} = x\varphi^{qp}c^{\psi^q} = x\varphi^{qp} \left( \prod_{s=0}^{q-1} z^{\varphi((q-1-s)p)} \right) c.$$
Then,

$$(xb_i)\varphi^{bp} = x\varphi^{bp} \left( \prod_{s=0}^{q-1} z\varphi^{q-1-s})^p \right) b_i = xb_i.$$ 

The converse is analogous.

Since, by Corollary 5.2, the periods by the action of $\psi$ are bounded above by $C$, then the periods of points in $Fb_i$ by the action of $\varphi$ are bounded above by $Cp$, which is computable since both $C$ and $p$ are. Hence, the chain $C_i$ stabilizes after at most $Cp$ steps. □

**Corollary 5.4.** Let $G$ be a finitely generated virtually free group and $\varphi \in \text{End}(G)$. There is a computable constant $k \in \mathbb{N}$ such that $\text{EvPer}(\varphi) = \text{EvFix}(\varphi^k)$.

**Proof.** Let $k$ be the constant given by Proposition 5.3. It is obvious that $\text{EvFix}(\varphi^k) \subseteq \text{EvPer}(\varphi)$. Now, let $x \in \text{EvPer}(\varphi)$. We have that there is some $s \in \mathbb{N}$ such that $x\varphi^s \in \text{Per}(\varphi)$. By Proposition 5.3, the period of $x\varphi^s$ is bounded above by $k$, and so it divides $k!$. Take $n \in \mathbb{N}$ such that $nk! > s$. This way, we have that $x\varphi^nk!$ belongs to the periodic part of the orbit of $x$. Thus, $x\varphi^nk! \varphi^k = x\varphi^k$ and so, $x \in \text{EvFix}(\varphi^k)$. □

Now we show that, for a fixed endomorphism, we can also bound the size of the straight part of the orbits by a computable constant, which, in combination with Proposition 5.3 gives us a way of computing an upper bound on the cardinality of finite orbits.

**Proposition 5.5.** Let $G$ be a finitely generated virtually free group and $\varphi \in \text{End}(G)$. Then, there is a computable constant $k$ that bounds the size of the straight part of every finite orbit.

**Proof.** Compute a decomposition

$$G = Fb_1 \cup \cdots \cup Fb_m,$$

where $F$ is a fully invariant free subgroup of $G$ and write $\psi = \varphi|_F$. We can assume that $b_1 = 1$. For all $j \in \mathbb{N}$, consider the surjective mappings $\varphi_j : \text{Im}(\varphi^j) \to \text{Im}(\varphi^{j+1})$ and $\psi_j : \text{Im}(\psi^j) \to \text{Im}(\psi^{j+1})$ given by restricting $\varphi$. It suffices to prove that for some computable $k$, we have that $\varphi_k$ is injective and this implies that the straight part of a finite orbit must contain at most $k$ elements. Indeed, suppose that there is some $x \in \text{EvPer}(\varphi)$ such that the straight part of $\text{Orb}_\varphi(x)$ has $r > k$ elements. Put $y = x\varphi^r$ and let $\pi$ be the period of $y$. Clearly, $y \in \text{Im}(\varphi^r) \subseteq \text{Im}(\varphi^k)$. Then $y = x\varphi^r \varphi_k$ and $y = y\varphi^r \varphi_k$. But $y\varphi^r \varphi_k = x\varphi^r \varphi_k$ since $x\varphi^r \varphi_k$ belongs to the straight part of the orbit and $y\varphi^r \varphi_k$ belongs to the periodic part. This contradicts the injectivity of $\varphi_k$.

So, it remains to prove that $\text{Im}(\varphi^k) \simeq \text{Im}(\varphi^{k+1})$ for some computable $k$, which, by Hopfianity of $\text{Im}(\varphi^k)$ implies that $\varphi_k$ is injective. We have that, for all $i \in \mathbb{N}$,

$$G\varphi^i = F\varphi^i(b_1 \varphi^i) \cup \cdots \cup F\varphi^i(b_m \varphi^i),$$

and so $F\varphi^i$ is a finite index subgroup of $G\varphi^i$ and $[G\varphi^i : F\varphi^i] \leq [G : F] = m$. Also, $0 \leq \text{rank}(\text{Im}(\psi^i)) \leq \text{rank}(\text{Im}(\psi^i))$, for every $i \in \mathbb{N}$.

Now, we describe the algorithm to compute $k$. Start by computing the smallest positive integer $j_1 \in \mathbb{N}$ such that $\text{rank}(\text{Im}(\psi^{j_1+1})) = \text{rank}(\text{Im}(\psi^{j_1}))$. Clearly, $j_1$ is computable: we have generators for $\text{Im}(\psi^i)$ for every $i \in \mathbb{N}$ and so we can compute its rank by computing the graph rank of its Stallings automaton. If $\text{rank}(\text{Im}(\psi^{j_1})) = 0$, then $\psi$ is a vanishing endomorphism and so $\text{Im}(\psi^{j_1})$ is finite. In that case, the orbits of elements in $\text{Im}(\psi^{j_1})$ must be finite, since $\text{Im}(\varphi^k) \subseteq \text{Im}(\varphi^{j_1})$, for $k > j_1$, and so after at most $|\text{Im}(\varphi^{j_1})|$ iterations, we must reach a periodic point. We can compute the entire orbit of all the elements in $\text{Im}(\psi^{j_1})$, put $M$ to be the cardinality of the largest orbit, $k = M + j_1$ and we are done. So, suppose that $\text{Im}(\psi^{j_1})$ is nontrivial.
Since free groups are Hopfian, then a free group is not isomorphic to any of its proper quotients. Thus, \( \psi_{j_1} \) must be injective. If \( \text{Im}(\varphi^{j_1}) \simeq \text{Im}(\varphi^{j_1+1}) \), then, we are done. If not, by Hopfianity, \( \varphi_{j_1} \) is not injective, and so there are some \( f \in F \) and \( i \in [m] \) such that \( (fb_i)\varphi^{j_1}(f) = 1 \). Since \( \psi_{j_1} \) is injective, then \( i \neq 1 \). So, there is some \( i \in \{2, \ldots, m\} \) such that \( (fb_i)\varphi^{j_1+1} = 1 \) and so \( b_i\varphi^{j_1+1} \in F\varphi^{j_1+1} \), thus

\[
[G\varphi^{j_1+1} : F\varphi^{j_1+1}] \leq m - 1
\]

and for all \( i \geq j_1 \), \( [G\varphi^i : F\varphi^i] \leq [G\varphi^{j_1+1} : F\varphi^{j_1+1}] \leq m - 1 \).

Now, we compute \( j_2 \), the second least positive integer such that \( \text{rank}(\text{Im}(\psi^{j_2+1})) = \text{rank}(\text{Im}(\psi^{j_2})) \) and proceed as above. After \( m \) steps, we have either found \( k \) or we have that \( G\varphi^{j_m} = F\varphi^{j_m} \simeq F\varphi^{j_m+1} = G\varphi^{j_m+1}, \) and we are done.

Combining Proposition 5.3 and Proposition 5.5 we obtain the following corollary.

**Corollary 5.6.** Let \( G \) be a finitely generated virtually free group and \( \varphi \in \text{End}(G) \). There is a computable constant \( k \) that

\[
\max\{|\text{Orb}_x(x)| \mid x \in \text{EvPer}(\varphi)\} \leq k.
\]

Given an endomorphism \( \varphi \in \text{End}(G) \), we denote by \( C_\varphi \) the computable constant that bounds the size of all finite orbits through \( \varphi \).

**Corollary 5.7.** Let \( \varphi \in \text{End}(F_n) \). It is decidable whether \( \text{EvFix}(\varphi) \) is a normal subgroup of \( F_n \) or not.

**Proof.** To decide if condition 1. in Proposition 3.2 holds, we check if the generators of the free group \( F_n \) are eventually fixed by computing the first \( C_\varphi \) elements of their orbits. By Remark 3.3, condition 2. is equivalent to \( \text{Fix}(\varphi) \) being trivial which is known to be decidable (in fact, by [13], we can find a basis for \( \text{Fix}(\varphi) \)).

**Corollary 5.8.** Let \( G \) be a finitely generated virtually free group and \( \varphi \in \text{End}(G) \) be defined by the image of the generators. It is decidable whether \( \varphi \) is a finite order element of \( \text{End}(G) \) or not. We can also decide if \( \varphi \) is aperiodic or not.

**Proof.** The endomorphism \( \varphi \) has finite order if there are \( p, q \in \mathbb{N} \) such that for every \( x \in G \), \( x\varphi^p = x\varphi^q \). Since the size of finite orbits is bounded above by a computable constant, then we can check if the letters have finite orbits. If there is some letter \( a \) with infinite orbit, then \( a\varphi^p \neq a\varphi^q \), for \( p, q \in \mathbb{N} \) with \( p \neq q \). If every letter is eventually periodic, then let \( p \) be the maximum length of the straight parts of the orbits of the letters, so that \( a\varphi^p \) is a periodic point for every letter \( a \) and let \( m \) be the least common multiple between the length of the periodic parts. Then \( a\varphi^p = a\varphi^{p+m} \) for every letter and so \( \varphi^p = \varphi^{p+m} \).

Aperiodicity is similar. We have that there is an \( m \in \mathbb{N} \) such that \( \varphi^m = \varphi^{m+1} \) if and only if for each letter \( a \) there is a \( p \) such that \( a\varphi^p = a\varphi^{p+1} \) and that is decidable simply by computing the orbits of the letters.

**Corollary 5.9.** Let \( G \) be a finitely generated virtually free group and \( \varphi \in \text{End}(G) \). The infinite ascending chain \( \text{Ker}(\varphi) \subseteq \text{Ker}(\varphi^2) \subseteq \ldots \) stabilizes and \( \bigcup_{k=1}^{\infty} \text{Ker}(\varphi^k) = \text{Ker}(\varphi^{C_\varphi}). \)

**Proof.** For \( k > C_\varphi \), if \( x\varphi^k = 1 \), then \( x\varphi^{C_\varphi} = 1 \), since \( |\text{Orb}_x(x)| \leq C_\varphi \).

**Corollary 5.10.** Let \( G \) be a finitely generated virtually free group and \( \varphi \in \text{End}(G) \). Then, \( \text{EvFix}(\varphi) = (\text{Fix}(\varphi))\varphi^{-C_\varphi} = \text{Ker}(\varphi^{C_\varphi}) \cup \text{Fix}(\varphi) \) and \( \text{Fix}(\varphi) \simeq \text{EvFix}(\varphi)/\text{Ker}(\varphi^{C_\varphi}). \)

**Proof.** We have that \( \text{EvFix}(\varphi) \) is the subgroup of points that get mapped to \( \text{Fix}(\varphi) \) by \( \varphi^{C_\varphi} \). Also, \( \text{Ker}(\varphi^{C_\varphi}) \subseteq \text{EvFix}(\varphi) \) and for every element \( x \in \text{EvFix}(\varphi) \), there is some \( y \in \text{Fix}(\varphi) \),
such that \( x\varphi^{C}\varphi = y = y\varphi^{C}\varphi \). So there must be some \( z \in \text{Ker}(\varphi^{C}\varphi) \) such that \( x = yz \). Thus, \( \text{EvFix}(\varphi) = \text{Ker}(\varphi^{C}\varphi) \vee \text{Fix}(\varphi) \).

Letting \( \psi \) denote the restriction of \( \varphi^{C}\varphi \) to \( \text{EvFix}(\varphi) \), we have that

\[
\text{Fix}(\varphi) = \text{Im}(\psi) \simeq \text{EvFix}(\varphi)/\text{Ker}(\psi) = \text{EvFix}(\varphi)/\text{Ker}(\varphi^{C}\varphi).
\]

We now present a result which, despite being easy and in the author’s opinion, of independent interest, doesn’t seem to appear in the literature. But first, we present a technical lemma.

**Lemma 5.11.** Let \( G \) be a group and \( H, K \leq G \). Then for all \( x, y \in G \), \( Hx \cap Ky \) is either empty or a coset of \( H \cap K \).

**Proof.** Suppose that \( Hx \cap Ky \neq \emptyset \) and let \( z \in Hx \cap Ky \). Then, \( Hz = Hx \) and \( Kz = Ky \). We will prove that \((H \cap K)z = Hx \cap Ky \). Let \( w \in (H \cap K)z \). Then, there are \( h \in H \), \( k \in K \) such that \( w = hz = kz \) and so \( h = k \in H \cap K \). Now, let \( v \in Hx \cap Ky \). Then \( Hv = Hx = Hz \) and \( Kv = Ky = Kz \) and so \( vz^{-1} \in H \cap K \).  

**Proposition 5.12.** Let \( G \) be a finitely generated virtually free group having a free subgroup \( F \) of finite index, \( H \leq G \) and \( \varphi \in \text{End}(G) \) be an endomorphism. If \( \text{Ker}(\varphi) \) is finite, then \( H\varphi^{-1} \) is finitely generated. If not, the following are equivalent:

1. \( H\varphi^{-1} \) is finitely generated
2. \( H\varphi^{-1} \cap F \) is a finite index subgroup of \( F \)
3. \( H\varphi^{-1} \cap F \) is a finite index subgroup of \( G \)
4. \( H\varphi^{-1} \) is a finite index subgroup of \( G \)
5. \( H \cap G\varphi \) is a finite index subgroup of \( G\varphi \)
6. \( H \cap F\varphi \) is a finite index subgroup of \( F\varphi \)

**Proof.** It is easy to see that if \( \text{Ker}(\varphi) \) is finite, then \( H\varphi^{-1} \) is finitely generated, since it is generated by the preimages of the generators of \( H \) together with the kernel.

So, assume that \( \text{Ker}(\varphi) \) is infinite. It is obvious that \( 2 \Rightarrow 3 \) and that \( 3 \Rightarrow 4 \). It is also well known that \( 4 \Rightarrow 1 \). We will prove that \( 1 \Rightarrow 2 \), \( 4 \Leftrightarrow 5 \). and \( 5 \Leftrightarrow 6 \). and that suffices.

We start by proving that \( 1 \Rightarrow 2 \). Suppose that \( H\varphi^{-1} \) is finitely generated. Since virtually free groups are Howson (free groups are Howson [3] and it is easy to see that the Howson property is preserved by taking finite extensions), then \( H\varphi^{-1} \cap F \) is also finitely generated. Then, by Marshall Hall’s Theorem, there is some finite index subgroup \( H' \) of \( F \) such that \( H\varphi^{-1} \cap F \) is a free factor of \( H' \). Take \( H'' \) such that

\[
H' = (H\varphi^{-1} \cap F) * H''.
\]

Since \( H' \) is a finite index subgroup of \( F \), it is finitely generated. Let \( A = \{a_1, \ldots, a_n\} \) be a basis of \( H' \) such that \( H\varphi^{-1} \cap F = \langle a_1, \ldots, a_k \rangle \) and \( H'' = \langle a_{k+1}, \ldots, a_n \rangle \). If \( H'' \) is trivial, then \( H\varphi^{-1} \cap F = H' \) is a finite index subgroup of \( F \) and we are done. Suppose then that \( H'' \) is nontrivial. Obviously, since \( \varphi \) is noninjective, \( \{1\} \neq \text{Ker}(\varphi) \subseteq H\varphi^{-1} \). Thus,

\[
(\text{Ker}(\varphi) \cap F) \subseteq (H\varphi^{-1} \cap F).
\]

Moreover, \( \text{Ker}(\varphi) \cap F \) is not trivial. Indeed, the fact that the kernel is infinite implies that \( \varphi|F \) is noninjective. Let \( 1 \neq x \in H'' \) and \( 1 \neq y \in \text{Ker}(\varphi) \cap F \). Then

\[
x y z^{-1} \in (\text{Ker}(\varphi) \cap F) \subseteq (H\varphi^{-1} \cap F),
\]
which is absurd, since the letters in \( x \) don’t belong to the basis set of \( (H\varphi^{-1} \cap F) \).

Now, we prove that 4. \( \Rightarrow \) 5. Suppose that \( H\varphi^{-1} \) is a finite index subgroup of \( G \). Then there are \( b_i \in G, i \in \{0, \ldots, k\} \), such that

\[
G = b_0(H\varphi^{-1}) \cup \cdots \cup b_k(H\varphi^{-1}).
\]

Clearly,

\[
G\varphi = (b_0\varphi)(H \cap G\varphi) \cup \cdots \cup (b_k\varphi)(H \cap G\varphi)
\]

and so \( H \cap G\varphi \) is a finite index subgroup of \( G\varphi \).

Similarly, if \( H \cap G\varphi \) is a finite index subgroup of \( G\varphi \), then there are \( b_i \in G, i \in \{0, \ldots, k\} \), such that

\[
G\varphi = (b_0\varphi)(H \cap G\varphi) \cup \cdots \cup (b_k\varphi)(H \cap G\varphi).
\]

Hence,

\[
G = b_0(H\varphi^{-1}) \cup \cdots \cup b_k(H\varphi^{-1}),
\]

because, given \( x \in G \), we have that there are some \( y \in G \) and \( i \in \{0, \ldots, k\} \) such that \( y\varphi x = (b_i)(y\varphi) \) and so \( x = b_iyk \) for some \( k \in \text{Ker}(\varphi) \leq H\varphi^{-1} \). So, \( yk \in H\varphi^{-1} \).

Hence, 5. \( \Rightarrow \) 4.

Finally, we prove that 5. \( \Leftrightarrow \) 6. Assume that \( H \cap G\varphi \) is a finite index subgroup of \( G\varphi \). Then, there are \( m \in \mathbb{N} \) and elements \( b_i \in G\varphi \) such that

\[
G\varphi = (H \cap G\varphi)b_1 \cup \cdots \cup (H \cap G\varphi)b_m.
\]

Thus,

\[
F\varphi = G\varphi \cap F\varphi = ((H \cap G\varphi)b_1 \cap F\varphi) \cup \cdots \cup ((H \cap G\varphi)b_m \cap F\varphi).
\]

By Lemma 5.11 \( H \cap F\varphi = H \cap G\varphi \cap F\varphi \) is a finite index subgroup of \( F\varphi \) of index at most \( m \). Conversely, since \( F\varphi \) is a finite index subgroup of \( G\varphi \), then, if \( H \cap F\varphi \) is a finite index subgroup of \( F\varphi \), it must also be a finite index subgroup of \( G\varphi \), by transitivity. Since \( H \cap G\varphi \geq H \cap F\varphi \), then \( H \cap G\varphi \) is also a finite index subgroup of \( G\varphi \).

Notice that noninjective endomorphisms of free groups satisfy the hypothesis of Proposition 5.12. Also, the following corollary follows directly from the proof of Proposition 5.12.

**Corollary 5.13.** The index of the subgroups in conditions 4. and 5. of Proposition 5.12 must coincide.

**Corollary 5.14.** Let \( G \) be a finitely generated virtually free group and \( \varphi \in \text{End}(G) \). It is decidable whether \( \text{EvFix}(\varphi) \) (resp. \( \text{EvPer}(\varphi) \)) is finitely generated and, in case the answer is affirmative, a set of generators can be effectively computed.

**Proof.** Compute a decomposition

\[
G = Fb_1 \cup \cdots \cup Fb_m,
\]

where \( F \) is a fully invariant free subgroup of \( G \) and write \( \psi = \varphi|_F \). We can assume that \( b_1 = 1 \).

Since \( F \) is fully invariant, we have that

\[
G\varphi^{C\varphi} = F\varphi^{C\varphi}(b_1\varphi^{C\varphi}) \cup \cdots \cup F\varphi^{C\varphi}(b_m\varphi^{C\varphi}). \tag{9}
\]

Now, it is easy to see that \( \varphi^{C\varphi} \) has finite kernel if and only if \( \psi^{C\varphi} \) is injective, which is decidable, since, by Hopfianity, \( \psi^{C\varphi} \) is injective if and only if \( \text{rank}(F) = \text{rank}(F\psi^{C\varphi}) \). If \( \varphi^{C\varphi} \) has finite kernel, then it must be computable. Indeed, if, for \( x \in F \), \( (xb_i)^{C\varphi} = 1 \), then \( x\varphi^{C\varphi} = x\psi^{C\varphi} = b_i^{-1}\varphi^{C\varphi} \), and so for all \( i \in [m] \), we check if \( h_i^{-1}\varphi^{C \varphi} \in \text{Im}(\psi^{C\varphi}) \), and if it is, we compute \( x \in F \) such that \( x\psi^{C\varphi} = b_i^{-1}\varphi^{C\varphi} \). We then have that \( \text{EvFix}(\varphi) = \)...
(Fix(\(\varphi\)))^{\varphi^{-C_{\varphi}}} \text{ is finitely generated and, since Fix(\(\varphi\)) is computable by Theorem 4.2 and Ker(\(\varphi\)) is computable, a set of generators for EvFix(\(\varphi\)) can be computed. So, suppose that the kernel of \(\varphi^{C_{\varphi}}\) is infinite.}

We know that EvFix(\(\varphi\)) = (Fix(\(\varphi\)))^{\varphi^{-C_{\varphi}}} and so, by Proposition 5.12, it is finitely generated if and only if Fix(\(\varphi\)) \cap F^{\varphi^{C_{\varphi}}} is a finite index subgroup of F^{\varphi^{C_{\varphi}}}.

Using Theorem 4.2, we can compute a basis for Fix(\(\varphi\)), and so compute a set of generators for Fix(\(\varphi\)) \cap F^{\varphi^{C_{\varphi}}}.

Notice that, since \(F\) is fully invariant then \(F^{\varphi^{C_{\varphi}}}\) (and so Fix(\(\varphi\)) \cap F^{\varphi^{C_{\varphi}}} is a subgroup of \(F\). Then, we can decide if Fix(\(\varphi\)) \cap F^{\varphi^{C_{\varphi}}} has finite index on \(F^{\varphi^{C_{\varphi}}}\) and compute right coset representatives \(b'_i \in F^{\varphi^{C_{\varphi}}}\).

\[
F^{\varphi^{C_{\varphi}}} = (\text{Fix}(\varphi) \cap F^{\varphi^{C_{\varphi}}}) \ b'_1 \cup \cdots \cup (\text{Fix}(\varphi) \cap F^{\varphi^{C_{\varphi}}}) \ b'_k. \tag{10}
\]

Combining (9) and (10), we have that

\[
G^{\varphi^{C_{\varphi}}} = \bigcup_{i=1}^{m} \bigcup_{j=1}^{k}(\text{Fix}(\varphi) \cap F^{\varphi^{C_{\varphi}}}) \ b'_j (b_i^{\varphi^{C_{\varphi}}}),
\]

and so

\[
G^{\varphi^{C_{\varphi}}} = \bigcup_{i=1}^{m} \bigcup_{j=1}^{k} \text{Fix}(\varphi) \ b'_j (b_i^{\varphi^{C_{\varphi}}}).
\]

By testing membership in Fix(\(\varphi\)) \(\cap G^{\varphi^{C_{\varphi}}}\), we can check whether any two cosets coincide and refine the decomposition to obtain a proper subdecomposition where all cosets are distinct of the form (eventually relabeling the coset representatives)

\[
G^{\varphi^{C_{\varphi}}} = \bigcup_{i=1}^{m'} \bigcup_{j=1}^{k'} (\text{Fix}(\varphi) \cap G^{\varphi^{C_{\varphi}}}) \ b'_j (b_i^{\varphi^{C_{\varphi}}}) = \bigcup_{i=1}^{m'} \bigcup_{j=1}^{k'} (\text{Fix}(\varphi)) b'_j (b_i^{\varphi^{C_{\varphi}}}),
\]

where \(k' \leq k\) and \(m' \leq m\).

For all \((i, j) \in [m'] \times [k']\), we can compute \(b_{i,j} \in G\) such that \(b_{i,j}^{\varphi^{C_{\varphi}}} = b'_j (b_i^{\varphi^{C_{\varphi}}})\), and so, by the proof of Proposition 5.12

\[
G = \bigcup_{i=1}^{m'} \bigcup_{j=1}^{k'} (\text{Fix}(\varphi) \varphi^{-C_{\varphi}}) b_{i,j}.
\]

Having the coset representative elements \(b_{i,j}\) and being able to check membership in Fix(\(\varphi\)) \(\varphi^{-C_{\varphi}}\), we can compute a set of generators for Fix(\(\varphi\)) \(\varphi^{-C_{\varphi}} = \text{EvFix}(\varphi)\). By Corollary 5.3, it is clear that the result also holds for EvPer(\(\varphi\)).

Finally, we will prove that in the cases where EvFix(\(\varphi\)) is finitely generated, we can bound its rank. The rank of a finitely generated virtually free group \(G\) is defined as the minimal cardinality of a set of generators of \(G\).

**Proposition 5.15.** Let \(G\) be a finitely generated virtually free group, \(\varphi \in \text{End}(G)\) and \(F\) be a fully invariant free subgroup of \(G\). If EvFix(\(\varphi\)) is finitely generated, then rank(EvFix(\(\varphi\))) \leq \left[ G : F \right] + \text{max}\{\text{rank}(F), \text{rank}(F)^2 - 3\text{rank}(F) + 3\}.

Proof.

If \(\varphi\) is injective, then rank(EvFix(\(\varphi\))) = rank(Fix(\(\varphi\))) \leq \left[ G : F \right] + \left[ G : F \right], \text{ by Remark 4.3. So, assume that \(\varphi\) is not injective and put } \psi = \varphi|_F. \text{ We have that EvFix(\(\psi\)) = EvFix(\(\varphi\)) \cap F \text{ and } C_{\psi} \leq C_{\varphi}. \text{ Thus, by [1] Proposition 2.1,}

\[
[\text{EvFix}(\varphi) : \text{EvFix}(\psi)] \leq \left[ G : F \right].
\]

Hence, EvFix(\(\psi\)) is finitely generated and rank(EvFix(\(\varphi\))) \leq \text{rank}(\text{EvFix}(\psi)) + \left[ G : F \right].
If $\psi$ is injective, then $\text{rank}(\text{EvFix}(\psi)) = \text{rank}(\text{Fix}(\psi)) \leq \text{rank}(F)$. So, assume that $\psi$ is noninjective.

If $\text{rank}(F\varphi^{C_\psi}) = 0$, then $\psi$ is a vanishing endomorphism, and so $\text{EvFix}(\psi) = F$ and, in this case, we have that $\text{rank}(\text{EvFix}(\varphi)) \leq \text{rank}(F) + [G : F]$.

If $\text{rank}(F\varphi^{C_\psi}) = 1$, then $F\varphi^{C_\psi}$ is abelian, thus $\text{Fix}(\psi) \leq F\varphi^{C_\psi}$, and so

$$\text{EvFix}(\psi) = \text{Fix}(\psi)^{-C_\psi} \leq F.$$

By Proposition 3.2, we must have that either $\text{EvFix}(\psi) = F$, in which case we are done, or $\text{EvFix}(\psi) = \bigcup_{k=1}^{\infty} \text{Ker}(\psi^k) = \text{Ker}(\psi^{C_\psi})$. But we know that $\text{EvFix}(\psi)$ is a finite index subgroup of $F$, because $\text{EvFix}(\psi) = (\text{Fix}(\psi))^{\psi^{-C_\psi}}$ is finitely generated (see condition 4. of Proposition 5.12 with $G = F$). This means that $\text{Im}(\psi^{C_\psi})$ is finite, which implies that $\psi^{C_\psi}$ is trivial. Hence, in this case $\psi$ is a vanishing endomorphism, and so $\text{EvFix}(\psi) = F$. In this case, we have that $\text{rank}(\text{EvFix}(\varphi)) \leq \text{rank}(F) + [G : F]$.

Now, suppose that $\text{rank}(F\varphi^{C_\psi}) > 1$. Since $\text{EvFix}(\psi)$ is finitely generated, then, by condition 5. of Proposition 5.12 (with $G = F$), $\text{Fix}(\psi)$ must be a finite index subgroup of $F\psi^{C_\psi}$. By [10] Proposition 3.9 and [17] Corollary 2, we have that

$$[F\varphi^{C_\psi} : \text{Fix}(\psi)] = \frac{\text{rank}(\text{Fix}(\psi)) - 1}{\text{rank}(F) - 1} \leq \text{rank}(F) - 2.$$

By Corollary 5.14 we have that $[F : \text{EvFix}(\psi)] = [F\varphi^{C_\psi} : \text{Fix}(\psi)] \leq \text{rank}(F) - 2$. But now, using [10] Proposition 3.9 again, we get that

$$\frac{\text{rank}(\text{EvFix}(\psi)) - 1}{\text{rank}(F) - 1} = [F : \text{EvFix}(\psi)] \leq \text{rank}(F) - 2,$$

and so $\text{rank}(\text{EvFix}(\psi)) \leq \text{rank}(F)^2 - 3\text{rank}(F) + 3$ and

$$\text{rank}(\text{EvFix}(\varphi)) \leq \text{rank}(F)^2 - 3\text{rank}(F) + 3 + [G : F].$$

6 Further work

The main question left open by this work concerns the generalization of these results for other classes of groups.

**Problem 6.1.** Let $G$ be a (torsion-free) hyperbolic group and $\varphi \in \text{End}(G)$. Is it decidable whether $\text{EvFix}(\varphi)$ (resp. $\text{EvPer}(\varphi)$) is finitely generated and, in case the answer is affirmative, can a set of generators be effectively computed?

Another potentially interesting problem could be checking the existence of a better bound to the rank of $\text{EvFix}(\varphi)$ than the one provided by Proposition 5.15.

**Problem 6.2.** Is the bound given by Proposition 5.15 sharp?

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