Monodromy and Tangential Center Problems *

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Abstract
We consider families of Abelian integrals arising from perturbations of planar Hamiltonian systems. The tangential center focus problem asks for the conditions under which these integrals vanish identically. The problem is closely related to the monodromy problem, which asks when the monodromy of a vanishing cycle generates the whole homology of the level curves of the Hamiltonian. We solve both these questions for the case when the Hamiltonian is hyperelliptic. As a side-product, we solve the corresponding problems for the “0-dimensional Abelian integrals” defined by Gavrilov and Movasati.

1 Introduction

The weak Hilbert 16th problem, as posed by Arnold [1], asks:

**Problem 1.1 (Weak Hilbert 16th problem)** Let $F \in \mathbb{C}[x,y]$ and $\omega = P(x,y)dx + Q(x,y)dy$, with $P,Q \in \mathbb{C}[x,y]$ and consider the system

$$dF + \varepsilon \omega = 0. \tag{1.1}$$

Bound the number of real limit cycles in the system (1.1) for small values of $\varepsilon$.

The problem leads to the study of the zeros of the Abelian integral

$$I(t) = \int_{\delta(t)} \omega, \tag{1.2}$$

where $\delta(t)$ is a family of cycles lying in $F^{-1}(t)$. Provided this integral does not vanish identically, limit cycles of (1.1) correspond to zeros of $I(t)$ for generic values of $t$.

That is, to first order we are led to solve the following simpler problem.

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Problem 1.2 (Tangential Hilbert 16th problem) Bound the number of zeros of the Abelian integral (1.2) in terms of the degrees of $F$ and $\omega$.

When, the Abelian integral vanishes identically, it provides no information about the limit cycles of (1.1), and higher order perturbation theory must be used. It is therefore of interest to understand under what conditions this can happen.

The classical center focus problem asks for a characterization of centers of planar polynomial vector fields. The problem of when an Abelian integral vanishes identically along a vanishing cycle, can be seen as a tangential version of this problem.

Problem 1.3 (Tangential center focus problem) Characterize the conditions under which the Abelian integral $I(t)$ of (1.2) vanishes identically along a vanishing cycle $\delta(t)$ associated to a Morse singular point $p$ of $F$.

If such a $\delta(t)$ exists, we say that (1.1) has a tangential center at $p$.

Problem 1.3 was solved by Il’yashenko for generic $F$ by proving that, for generic $F$, the monodromy acts transitively on the first homology group of the generic fiber. The vanishing of $I(t)$ therefore implies the vanishing of the Abelian integral along all cycles in $H_1(F^{-1}(t))$. This in turn implies that the form $\omega$ is relatively exact.

In fact, the condition that the vanishing of an Abelian integral (1.2) along a family of cycles $\delta(t)$ implies the relative exactness of $\omega$ is called “condition (*)” by Françoise. Under condition (*), Françoise [6] (see also [11]) gives an algorithm for calculating higher order terms of the displacement function.

By the results of Bonnet and Dimca [2] (and, in a more restricted setting, Gavrilov [7] and Il’yashenko [10]), if we assume the vanishing of the Abelian integrals on all cycles, then $P(F)\omega$ must be relatively exact, for some polynomial $P$, whose roots correspond to some exceptional fibers. Condition (*) therefore follows automatically (after possible multiplication of $\omega$ by a factor $P(F)$) if we can show that under the action of the monodromy, the cycle $\delta(t)$ generates the whole of the homology of the generic fiber of $F$ over $\mathbb{Q}$.

This leads to a natural problem:

Problem 1.4 (Monodromy problem) Under what conditions on $F$ is the $\mathbb{Q}$-subspace of $H_1(F^{-1}(t), \mathbb{Q})$ generated by the images of a vanishing cycle of a Morse point under monodromy, equal to the whole of $H_1(F^{-1}(t), \mathbb{Q})$?

The principal motivation for this paper was to solve these last two problems in the case when $F(x,y) = y^2 + f(x)$ (the hyperelliptic case). In more detail, we prove the following two theorems.

Theorem 1.5 The system (1.1), with $F = y^2 + f(x)$, has a tangential center with associated vanishing cycle $\delta(t)$, if and only if (i) or (ii) is verified:

(i) the form $\omega$ is relatively exact i.e. $\omega = A\, df + dB$, with $A, B \in \mathbb{C}[x,y].$

(ii) $f$ is decomposable i.e. $f = g \circ h$, and $\omega = \tilde{\omega} + \pi^* \eta$, where $\tilde{\omega}$ is relatively exact, and $\pi_* \delta(t)$ is homotopic to zero in $y^2 + g(z) = t$, where $\pi(x,y) = (h(x), y) = (z, y)$.

Theorem 1.6 Let $F = y^2 + f(x)$, with associated vanishing cycle $\delta(t)$ at a Morse point, then one of the following must hold.
(i) the monodromy of $\delta(t)$ generates the homology $H_1(F^{-1}(t), \mathbb{Q})$.

(ii) $f$ is decomposable i.e. $f = g \circ h$, and $\pi_* \delta(t)$ is homotopic to zero in $y^2 + g(z) = t$, where $\pi(x, y) = (h(x), y) = (z, y)$.

To prove the above theorems we first reduce them to analogous 0-dimensional problems which we consider next.

We define a 0-dimensional Abelian integral following Gavrilov and Movasati [8].

Let $f \in \mathbb{C}[x]$ be a polynomial and $\delta(t) \in H_0(f^{-1}(t))$ a 0-cycle: that is, $\delta(t) = \sum n_i x_i(t) \in f^{-1}(t)$, $n_i \in \mathbb{C}$, with $\sum n_i = 0$ and let $\omega \in \mathbb{C}[x]$ be a polynomial (0-form). A 0-dimensional Abelian integral is given by a function

$$I_0(t) = \int_{\delta(t)} \omega := \sum n_i \omega(x_i(t)).$$ \hfill (1.3)

A cycle of the form $\delta(t) = x_i(t) - x_j(t)$, with $f(x_i(t)) = f(x_j(t)) = t$ is called a simple cycle.

We characterize the vanishing of 0-dimensional Abelian integrals along simple cycles (the 0-dimensional tangential center focus problem) and the conditions under which a simple cycle generates the whole of the reduced homology $H_0(f^{-1}(t))$ of the generic fiber (the 0-dimensional monodromy problem).

**Theorem 1.7** Let $f, \omega \in \mathbb{C}[x]$, $\delta(t) = x_i(t) - x_j(t)$ be a simple cycle in the generic fiber of $f$. The Abelian integral $I(t) = \int_{\delta(t)} \omega$ vanishes identically if and only if there exists a polynomial $h$ with $\deg(h) > 1$ such that $f = g \circ h$ and $\omega = \eta \circ h$, for some polynomials $g$ and $\eta$, and $\delta(t) = \tilde{\delta}(h(t))$ for some simple cycle $\tilde{\delta}$ of $g$.

**Theorem 1.8** Let $\delta(t) = x_i(t) - x_j(t)$ be a simple cycle in the generic fiber of $f$. Then either

(i) The cycle $\delta(t)$ generates the reduced homology $H_0(f^{-1}(t))$.

(ii) $f$ decomposes as $f = g \circ h$, ($\deg(h) > 1$), and $\delta(t) = \tilde{\delta}(h(t))$ for some simple cycle $\tilde{\delta}$ of $g$.

The principal tools in the proof of these theorems is Lüroth’s theorem on field extensions and the Burnside-Schur theorem on group actions with a regular cyclic subgroup. We recall both these theorems in Section 2 below.

**Remark 1.9** If a cycle $\delta(t)$ is not simple, then the theorems above do not hold. A counter-example is provided if $f(x) = T_p(x)$, a Chebyshev polynomial of prime degree. We examine this case in detail in the final section.

Similarly, the polynomial $F(x, y) = y^2 + T_p(x)$ gives a counter-example to a generalization of Theorems 1.5 and 1.6.

### 2 Preliminaries

We recall some definitions from group theory.

**Definition 2.1**
1. Let $G$ be a group acting on a finite set $S$. We say that the action is **imprimitive** if there exists a non-trivial decomposition of $S$, $S = \bigcup S_i$, such that for each element of $g$ and each $i$, $g$ sends $S_i$ into $S_j$ for some $j$. The action is called **primitive** if it is not imprimitive.

2. An action is **transitive** if given any pair of elements of $S$, $s_1$ and $s_2$, there is an element $g \in G$ which sends $s_1$ to $s_2$.

3. An action is **2-transitive** if given any two pairs of elements of $S$, $(s_1, s_2)$ and $(s_3, s_4)$, there is an element $g \in G$ which sends $s_1$ to $s_3$ and $s_2$ to $s_4$.

4. An action is **regular** if given two elements $s_1$ and $s_2$ of $S$ there is a unique element $g$ of $G$ which sends $s_1$ to $s_2$.

5. Given $s \in S$, we denote the group of all elements of $G$ which fix $s$ (the **stabilizer** of $s$) by $G_s$.

The following theorem is classical, but we state it here for convenience.

**Theorem 2.2 (Lüroth)** Let $k(t)$ be a transcendental extension of a field $k$. Any subfield $K \subset k(t)$, such that $k \subsetneq K$, is of the form $K = k(r)$ for some $r \in k(t)$.

**Proposition 2.3** Let $G$ be a group acting transitively on a finite set $S$. The action of $G$ on $S$ is imprimitive if and only if for some element $s$ of $S$ there is a subgroup $H$ of $G$ such that

$$G_s \subsetneq H \subsetneq G,$$  \hspace{1cm} (2.1)

where $G_s$ is the subgroup of $G$ of all elements which leave $s$ fixed.

**Proof** Suppose that the action of $G$ on $S$ is imprimitive, and let $S_0$ be the subset which contains $s$ in the decomposition of $S$. We let $H$ be the subset of $G$ consisting of all elements which fix $S_0$. Since $S_0$ is non-trivial it must have more than one element but be strictly contained in $S$. From the transitivity of $G$, $H$ must be therefore strictly larger than $G_s$, but smaller than $G$.

Conversely, if (2.1) holds, we can consider the orbit of $s$ under the action of $H$: call this $S_1$. This cannot be the whole of $S$, or else $H$ would be the same as $G$ (since $H$ already contains $G_s$). However, it must contain more elements than just $s$. Now consider the action of $G$ on $S_1$. If $s' \in g_1(S_1) \cap g_2(S_1)$ then there exist some $h_1, h_2 \in H$ such that $g_1h_1(s) = s' = g_2h_2(s)$. Thus $h_2^{-1}g_2^{-1}g_1h_1 \in G_s$, and hence $g_2^{-1}g_1 \in H$ and $g_1(S_1) = g_2(S_1)$. Therefore the images of $S$ under $G$ give a partition of $S$ on which $G$ acts imprimitively.

Recall that the affine group $\text{Aff}(\mathbb{Z}_p)$ is the group of all affine transformations of $\mathbb{Z}_p$ to itself. That is, it is the group of all maps from $\mathbb{Z}_p$ to itself of the form $x \mapsto ax + b$ for $a, b \in \mathbb{Z}_p$ with multiplication given by composition. Note that every element of $\text{Aff}(\mathbb{Z}_p)$ fixes at most one element of $\mathbb{Z}_p$. We will use this fact in the proof of Theorem 3.8.

**Theorem 2.4 (Burnside-Schur)** Every primitive finite permutation group containing a regular cyclic subgroup is either 2-transitive or permutationally isomorphic to a subgroup of the affine group $\text{Aff}(\mathbb{Z}_p)$, where $p$ is a prime.

**Proof** See [3] or [4].
3 Monodromy groups of polynomials

Let \( f(x) \) be a polynomial of degree \( n > 0 \), and consider the solutions, \( x_i(t) \), of the equation \( f(x) = t \). Let \( \Sigma \) be the set of critical points \( t \in \mathbb{C} \) for which \( f(x) = t \) and \( f'(x) = 0 \) have a common solution. Clearly there are at most \( n(n-1) \) of these points. As \( t \) takes values in \( \mathbb{C} \setminus \Sigma \) the functions \( x_i(t) \) are well-defined. The group \( G = \pi_1(\mathbb{C} \setminus \Sigma) \) acts on the \( x_i(t) \). The action is always transitive (Proposition 3.2).

**Definition 3.1** Let \( G \) be as above, then the action of \( G \) on the set of \( x_i \) is called the *monodromy* group of the polynomial \( f \), denoted \( \text{Mon}(f) \).

**Proposition 3.2** Let \( f \) be a polynomial over \( \mathbb{C} \) of degree \( n \), then its monodromy group, \( \text{Mon}(f) \), is transitive and has a cyclic subgroup of degree \( n \) which acts regularly on the roots of \( f \).

**Proof** The first statement follows from the second. When \( t \) is large the \( x_i \) can be expanded as

\[
x_i = \omega^t i^{1/n} + O(t^{(1/n)-1}),
\]

where \( \omega \) is an \( n \)-th root of unity. Thus, taking a sufficiently large loop in \( \mathbb{C} \setminus \Sigma \), we obtain an element of \( G \) which is an \( n \)-cycle. This element generates a cyclic subgroup of \( G \) which acts regularly on the roots of \( f(x) = t \).

Elements of the monodromy group clearly lie in the Galois group of \( f(x) - t = 0 \) over \( \mathbb{C}(t) \). The following fundamental theorem [5] states that all elements of the Galois group can be generated in this way.

**Theorem 3.3** The monodromy group of \( f \), \( \text{Mon}(f) \), is isomorphic to the Galois group of \( f(x) - t \) considered as a polynomial over \( \mathbb{C}(t) \).

**Definition 3.4** We say that a polynomial \( f(x) \) is *decomposable* if and only if there exist two polynomials \( g \) and \( h \), both of degree greater than one, such that \( f(x) = g(h(x)) \).

**Lemma 3.5** Suppose that \( f(x) \) is a polynomial over \( \mathbb{C} \) which can be expressed as \( g(h(x)) \) for \( g \) and \( h \) rational functions of degree greater than one over \( \mathbb{C} \), then there is a decomposition \( f(x) = \tilde{g}(\tilde{h}(x)) \), where \( \tilde{g} \) and \( \tilde{h} \) can be chosen to be polynomials over \( \mathbb{C} \).

**Proof** Let \( h(x) = r(x)/s(x) \), where \( r \) and \( s \) are polynomials over \( \mathbb{C} \). Without loss of generality, if \( m \) is a Möbius transformation, we can rewrite the decomposition of \( f \) as \( f = \tilde{g} \circ \tilde{h} \) with \( \tilde{g} = g \circ m\) and \( \tilde{h} = m \circ h \). In this way, we can assume that \( \tilde{h} = r/\tilde{s} \), with \( \deg(\tilde{s}) < \deg(r) \), and both \( r \) and \( s \) monic. Now,

\[
\tilde{g}(\tilde{h}(x)) = \frac{\prod_{i=1}^{q} \alpha_i \tilde{r}(x) + \beta_i \tilde{s}(x)}{\prod_{i=1}^{q} \gamma_i \tilde{r}(x) + \delta_i \tilde{s}(x)},
\]

for some constants \( \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C} \). If \( \alpha_i \tilde{r} + \beta_i \tilde{s} \) shares a common factor with \( \gamma_i \tilde{r} + \delta_i \tilde{s} \), these two polynomials must be the same up to a constant multiple, whence we can assume that the fraction above allows no further cancelations. Since \( \tilde{g} \circ \tilde{h} \) is a polynomial, \( \prod \gamma_i \tilde{r} + \delta_i \tilde{s} \) must be a constant, and hence the denominator has no dependence on \( r \), and \( \tilde{s} \) must be a constant (and therefore \( \tilde{s} = 1 \)). The result follows directly.

**Proposition 3.6** Let \( f \) be a polynomial as above and let \( G = \text{Mon}(f) \) be its monodromy group. Then
(i) The action of $G$ is imprimitive if and only if the polynomial $f$ is decomposable.

(ii) The action is 2-transitive if and only if the divided differences polynomial

$$\Delta(x, y) = (f(x) - f(y))/(x - y)$$

is irreducible.

\textbf{Proof}

(i) Let $t \in \mathbb{C} \setminus \Sigma$, let $s$ be a root of $f(x) = t$, and $G_s$ the stabilizer of $s$. From Proposition 2.3, we have

$$G_s \subset H \subset G.$$  \hfill (3.1)

The splitting field of $f(x) - t$ over $\mathbb{C}(t)$ is just $\mathbb{C}(x_1(t), \ldots, x_n(t))$. Under the Galois correspondence, we have

$$\mathbb{C}(x_k(t)) \supseteq K \supseteq \mathbb{C}(t),$$  \hfill (3.2)

where $K$ is the fixed field of $H$, and $x_k(t)$ is the root of $f(x) = t$ corresponding to $s$. From Lüroth’s theorem, we must have $K = \mathbb{C}(r(x_k))$, for some rational function $r$ over $\mathbb{C}$. Then (3.2) implies that $t = s(r(x_k))$ for some rational function $s$. Thus $f(x) = s(r(x))$, and Lemma 3.5 shows that $s$ and $r$ can in fact be chosen to be polynomials. Conversely, given a decomposition $f(x) = s(r(x))$, we take $K = \mathbb{C}(r(x))$ and obtain (3.1) from (3.2) via the Galois correspondence.

(ii) Let $y = x_1$ be a root of $f(x) - t = 0$. Then, for any other root $z$ of $f(x) = t$, we must have

$$f(z) - f(y) = 0 = (z - y)R(z, y),$$

for some polynomial $R(x, y)$, which must therefore contain the minimal polynomial for $z$ over $\mathbb{C}(y, t) = \mathbb{C}(y)$. Clearly, $G$ is 2-transitive if and only if there is an automorphism of $\mathbb{C}(x_1, \ldots, x_n)$ which fixes $y$, and sends $z$ to any of the roots $x_2$ to $x_n$. In turn, this can happen if and only if the polynomial $R$ is irreducible.

\textbf{Definition 3.7} The unique polynomial $T_n(x)$ which satisfies $T_n(\cos(\theta)) = \cos(n\theta)$ is called the \textit{Chebyshev} polynomial of degree $n$. Equivalently $T_n((z + z^{-1})/2) = (z^n + z^{-n})/2$.

From the definition, the Chebyshev polynomial $T_n$ has $n - 1$ distinct turning points when $T_n = \pm 1$. Conversely, it can be shown that any polynomial $T(x)$ with just two critical values and with all turning points distinct must be equivalent to $T_n(x)$ for some $n$ after pre- and post- composition with suitable linear functions.

We would like to thank Peter Müller for bringing the following result to our attention. We give a proof for completeness.

\textbf{Theorem 3.8} Let $f(x)$ be a polynomial of degree $n$ and $G = \text{Mon}(f)$, then one of the following holds.

(i) The action of $G$ on the $x_i$ is 2-transitive

(ii) The action of $G$ on the $x_i$ is imprimitive

(iii) $f$ is equivalent to a Chebyshev polynomial $T_p$ where $p$ is prime.
Remark 3.9 In particular, the question of whether $f$ is a composite polynomial or not, can be solved very simply by considering whether or not the divided differences polynomial factorizes or not, having excluded the two exceptional cases above. “Equivalence” refers to pre- and post-composition by linear functions.

Proof From Proposition [3.2], we can apply the Burnside-Schur Theorem to show that the group must be 2-transitive, imprimitive, or a subgroup of $\text{Aff}(\mathbb{Z}_p)$. In the latter case we note that $n = p$, and every element of $\text{Aff}(\mathbb{Z}_p)$ fixes at most one element of $\mathbb{Z}_p$. This means that for every critical value of $f$ there is at most one $x_i$ that remains fixed as we turn around this value.

Now, suppose $f$ has $r$ distinct critical values, $t_1, \ldots, t_r$, and $f$ has $r_i$ distinct turning points associated to the critical value $t_i$. Let the multiplicities of the roots of $f'$ at these turning points be $m_{i,1}, \ldots, m_{i,r_i}$. Since a root of multiplicity $m_{i,j}$ gives a cycle of order $m_{i,j}$, then for all $i$ we must have

$$n - 1 \leq \sum_{j=1}^{r_i} m_{i,j} \leq n,$$  \hspace{1cm} (3.3)

since at most one of the $x_i$ remains fixed when turning around each critical value. Summing these equations over $i$ we obtain

$$r(n - 1) \leq \sum_{i=1}^{r} \sum_{j=1}^{r_i} m_{i,j} \leq rn.$$  \hspace{1cm} (3.4)

But the number of turning points of $f$ counted with multiplicity is just the sum of the $m_{i,j}$, and hence

$$r(n - 1) \leq (n - 1) + \sum_{i=1}^{r} r_i \leq rn.$$  \hspace{1cm} (3.5)

Since the sum of the $r_i$ is at most $n - 1$ we must have $r \leq 2$.

If $r = 1$, then [3.5] shows that $r_1 = 1$, and therefore $f(x)$ must have a root of multiplicity $n$. This is just Case (iv), noting that $n$ is prime.

If $r = 2$ we need $n - 1 \leq r_1 + r_2 \leq n + 1$. But since $r_i$ can be no more than $n/2$ this means that both $r_i$ lie between $(n - 1)/2$ and $n/2$. This implies that every turning point must have multiplicity 1 and the polynomial must be Chebyshev with $n$ prime.

4 Proof of the 0-dimensional theorems

Having dealt with the preliminaries, the proof of the 0-dimensional theorems are straightforward.

4.1 Proof of Theorem [1.7]

Let $\delta(t) = x_i(t) - x_j(t)$ be a simple cycle, and let $\mathcal{F} \subset \mathbb{C}(x)$ denote the field of all rational functions $R \in \mathbb{C}(x)$ for which $R(x_i(t)) = R(x_j(t))$. Clearly $\mathbb{C} \subset \mathcal{F}$, and from the hypothesis of the theorem, $f$ and $\omega$ both lie in $\mathcal{F}$, so it contains at least one non-trivial element. However, $x$ does not lie in $\mathcal{F}$, so we have

$$\mathbb{C} \subsetneq \mathcal{F} \subsetneq \mathbb{C}(x).$$

By Lüroth’s theorem, there exists a non-trivial rational function $h(x)$, necessarily of degree greater than one, such that $\mathcal{F}$ is generated by $h(x)$. In particular, $f$ and $\omega$ are rational functions of $h(x)$.
However, by Lemma 3.5 this implies that after a Möbius transformation, the generator \( h(x) \) can be taken to be a polynomial, and \( f \) and \( \omega \) are polynomials of \( h(x) \). Since \( h(x) \) lies in \( \mathcal{F} \), \( h(x_i) = h(x_j) \) and we are done.

### 4.2 Proof of Theorem 1.8

Let \( \delta(t) = x_i(t) - x_j(t) \) be a simple cycle, and let \( G = \text{Mon}(f) \) be the monodromy group of \( f \). Consider the graph with vertices \( x_1, \ldots, x_n \) and whose edges consist of all pairs \( \{x_r, x_s\} \) for which there exists a \( \sigma \in G \) such that \( \{\sigma(x_r), \sigma(x_s)\} = \{x_r, x_s\} \). Every vertex lies on at least one edge, since \( G \) is transitive.

If two roots \( x_r \) and \( x_s \) lie in a connected component of the graph, then it is clear that we can obtain \( x_r - x_s \) as a sum of terms of the form \( \pm(\sigma_k(x_r) - \sigma_k(x_s)) \). Thus, if the monodromy of the cycle \( \delta(t) \) does not generate the whole of \( H_0(f^{-1}, \mathbb{Z}) \), then there must be more than one connected component of the graph. Let \( S \) be the connected component of the graph which contains \( x_i \) and \( x_j \).

Each element of \( G \) gives an automorphism of the graph in a natural way. Take \( H \) to be a subgroup of \( G \) which sends \( S \) to itself. Clearly \( H \) contains \( G_{x_i} \) and also some element, \( \sigma_{ij} \), which sends \( x_i \) to \( x_j \). However, if the graph is not connected, \( H \) is strictly smaller than \( G \). Thus, from the proof of Proposition 3.6, \( f(x) \) is decomposable with \( f(x) = g(h(x)) \), where \( h(x) \) generates the fixed field of \( H \). Finally, \( h(x_j) = \sigma_{ij}(h(x_i)) = h(x_i) \), since \( \sigma_{ij} \) lies in \( H \).

### 5 The tangential center focus problem in the hyperelliptic case

**Proposition 5.1** Let \( \omega \) be a polynomial 1-form, and \( F(x, y) = y^2 + f(x) \) a polynomial. Then, there exists polynomials \( A, B \in \mathbb{C}[x, y] \) and \( g \in \mathbb{C}[x] \), such that

\[
\omega = AdF + dB + yg \, dx.
\]

**Proof** First, it is clear that we can write \( \omega = dB'(x, y) + A'(x, y) \, dx \) for an appropriate choice of polynomials \( A', B' \in \mathbb{C}[x, y] \). Then, using inductively the identity

\[
a(x)y^{n+2} \, dx = \frac{(n+2)}{2} A(x)f'(x)y^n \, dx + d(A(x)y^{n+2}) - \frac{(n+2)}{2} A(x)y^n \, dF,
\]

where \( A(x) \) is a primitive of \( a(x) \), we obtain the result.

We now prove Theorem 1.5 from Proposition 5.1 we need only consider the case \( \omega = yk(x) \, dx \).

Without loss of generality, we can assume that the tangential center is at the origin, and that we have scaled \( x \) so that \( f(x) = x^2 + O(x^3) \). We define an analytic function \( X \) to be the unique solution of the equation

\[
X^2 = f(x), \quad X = x + O(x^2).
\]

With respect to the coordinates \((X, y)\), the vanishing cycles can be reparameterized to give the circles \( X^2 + y^2 = t \). Furthermore, our Abelian integral (1.2) becomes

\[
\int_{\delta(t)} yk(x) \, dx = \int_{X^2 + y^2 = t} ym(X)X \, dX,
\]

where

\[
m(X(x)) = \frac{2k(x)}{f'(x)}.
\]
Clearly this integral vanishes for small values of $X$ if and only if $k(0) = 0$, and $m(X)$ is even in $X$. That is,

$$m(X(x)) = \phi(X(x)^2) = \phi(f(x)),$$

for some analytic function $\phi$. Thus,

$$2k(x) = f'(x)\phi(f(x)).$$

Taking $\Phi$ to be a primitive of $\phi$ with $\Phi(0) = 0$, and $K$ to be a primitive of $2k$ with $K(0) = 0$, we obtain

$$K(x) = \Phi(f(x)).$$

Now, this means that $K(x)$ vanishes with respect to the cycle defined by $f(x) = X^2$, and by the proof of Theorem 1.7 in the previous section, we must have both $K$ and $f$ to be composites of a common polynomial $h(x)$: $K = r \circ h$ and $f = g \circ h$.

Finally, taking $\pi(x,y) = (h(x),y) = (z,y)$, we find

$$\omega = yk(x)dx = yr'(h(x))h'(x)dx = \pi^*(yr'(z)dz).$$

This concludes the theorem once we note that the vanishing cycle is pushed forward to a cycle homotopic to zero in the $(z,y)$ coordinates. This is true as they lie on a family of parabolas $X + y^2 = t$.

### 6 The monodromy problem in the hyperelliptic case

We consider the level curves of the Hamiltonian $H = y^2 - f(x) = t$ as a two sheeted covering of the complex plane $\mathbb{C}$ given by projection onto the $x$-axis. The sheets ramify at the roots of $f(x) = t$. Taking $\Sigma$ to be the set of critical points as above, we let $t$ vary in $\mathbb{C} \setminus \Sigma$, and follow the effect on the homology group $H_1(F^{-1}(t),\mathbb{Z})$. We wish to relate this group to the monodromy group of the polynomial $f(x)$. As $x$ tends to infinity along the positive real axis, we can distinguish the two sheets as “upper” and “lower” depending on whether $y = \pm x^{n/2}$. We let $\tau$ denote the deck transformation which takes $y$ to $-y$ fixing $x$.

Let $H^i_1(F^{-1}(t),\mathbb{Z})$ represent the homology with closed support of $F^{-1}(t)$ over $\mathbb{Z}$. This can be obtained from $H_1(F^{-1}(t),\mathbb{Z})$ by adding unbounded closed curves. Let $x_i(t)$ be the roots of $f(x) = t$. Generically, the $x_i$ will having distinct imaginary parts, and so any closed path in $\mathbb{C} \setminus \Sigma$ can be deformed so that only two of the $x_i$’s have the same imaginary part at the same time. In other words, we can decompose every element of $\text{Mon}(f)$ as a number of swaps of $x_i$’s with neighboring real values.

Suppose that the $x_i$ are initially numbered in order of decreasing imaginary part for a value of $t$ close to zero. We let $L_i$ represent the path from infinity (from the direction of the positive real axis) on the upper sheet, turning around $x_i$ in the positive direction and returning to infinity on the lower sheet. Clearly $\tau(L_i) + L_i$ is homotopic to zero, and so the $L_i$ generate $H^i_1(F^{-1}(t),\mathbb{Z})$. Furthermore, the elements $L_i - L_{i+1}$ generate $H_1(F^{-1}(t),\mathbb{Z})$.

The effect of a swap of $x_i$ and $x_{i+1}$ is to take $L_{i+1}$ to $L_i$ and $L_i$ to $2L_i - L_{i+1}$. This is a little too complex to analyze in general, except for very specific systems. Instead we shall work for the moment over $\mathbb{Z}_2$. That is, we consider the images of the $L_i$ in $H^i_1(F^{-1}(t),\mathbb{Z}_2)$ and $H^i_1(F^{-1}(t),\mathbb{Z}_2)$.

Working modulo 2 means that a swap of $x_i$ and $x_{i+1}$ takes $L_{i+1}$ to $L_i$ and $L_i$ to $L_{i+1}$. That is, the action of $\text{Mon}(f)$ on the $L_i$ (mod 2) is exactly the same as the action on the $x_i$. 

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We now apply the results of Theorem 3.8 in order to prove Theorem 1.6. According to Theorem 3.8 we only need to consider four cases.

We shall show below that the Cases (i) and (ii) of Theorem 3.8 correspond to Cases (i) and (ii) of Theorem 1.6. Case (iii) can be dealt with by adapting the proof for case (i), and in Case (iv) the Hamiltonian does not have a Morse point, and hence there are no tangential centers.

Case (i)/(iii) If the monodromy group of $f$ is 2-transitive then we can find a transformation which takes any two $x_i$’s to any other two. Since, working modulo two, the action on the loops $L_i$ is the same as the action on the $x_i$, we can find an element of the monodromy group which takes $L_i - L_{i+1}$ to $L_j - L_{j+1}$ modulo 2 for all $i$ and $j$.

Now, the vanishing cycle $\delta(t)$ occurs at the coalescence of two of these $x_i$’s and so must correspond to one of the $L_k - L_{k+1}$ for some $k$. Thus, there exist paths $\ell_i$ in $\mathbb{C} \setminus \Sigma$ such that

$$\sigma(\ell_i) \delta(t) = L_i - L_{i+1} \pmod{2},$$

for all $i$.

Now let $N = 2\lfloor (n-1)/2 \rfloor$. Then $L_i - L_{i+1}$ form a basis of $H_1(F^{-1}(t), \mathbb{Z})$. From the discussion above, we have

$$
\begin{pmatrix}
\sigma(\ell_1) \delta(t) \\
\sigma(\ell_2) \delta(t) \\
\vdots \\
\sigma(\ell_N) \delta(t)
\end{pmatrix} = A
\begin{pmatrix}
L_1 - L_2 \\
L_2 - L_3 \\
\vdots \\
L_N - L_{N+1}
\end{pmatrix},
$$

where the matrix $A$ reduces to the identity matrix if we reduce modulo 2. In particular, $A$ is invertible, and we can express the basis of $H_1(F^{-1}(t), \mathbb{Z})$ as sums of the $\sigma(\ell_i) \delta(t)$ with coefficients in $\mathbb{Q}$. That is, $\delta(t)$ generates $H_1(F^{-1}(t), \mathbb{Q})$. This gives us Case (i) of Theorem 1.6.

Note that in Case (iii), the monodromy group is not 2-transitive. However it is still possible to generate each of the $L_i - L_j$ over $\mathbb{Z}_2$ as a sum of $\sigma(\ell_k) \delta(t)$. This follows directly from the $\mathbb{Z}_p$ action of Proposition 3.2 on the roots, and hence on the $L_i$ over $\mathbb{Z}_2$. The proof then proceeds as above.
Case (ii) In this case, we can assume that Mon$(f)$ is imprimitive (but not 2-transitive) on the roots of $f(x) = t$. From Theorem 1.3 the function $f(x)$ decomposes as $f(x) = g(h(x))$ with $h(x_i) = h(x_j)$. This gives us Case (ii) of Theorem 1.6 once we note that the vanishing cycle for two irreducible sub-representations. One is the trivial one, and one the space result ([9], p281) that the permutation representation over $v_1$ is pushed forward to a cycle homotopic to zero via $h$.

This completes the proof of Theorem 1.6.

7 Generalized monodromy, tangential problems and Chebyshev polynomials

In this final section we would like to consider the possibility of generalizing the tangential center focus problem or monodromy problem to the case where the cycle $\delta(t)$ lies in $H_1(F^{-1}(t), \mathbb{C})$. We will show that the Chebyshev polynomials give counterexamples to both Theorems 1.5 and 1.6 in this case. That is, in the Chebyshev case there are non-trivial subspaces of $H_1$ and it is either equivalent to either $x^p$ or the Chebyshev polynomial $T_p$ for some prime $p$.

Conjecture 7.1 If there exists a non-trivial subspace of $H_1(F^{-1}(t), \mathbb{C})$ which is invariant under the monodromy, then either the polynomial $f$ decomposes as $f = g \circ h$, or $f$ is equivalent to either $x^p$ or the Chebyshev polynomial $T_p$ for some prime $p$.

For completeness, it would be also interesting to investigate, in analogy with Theorems 1.5 and 1.6, whether any cycles $\delta(t)$ which lie in the invariant subspace of $H_1(F^{-1}(t), \mathbb{C})$ and any 1-form $\omega$ for which $\int_{\delta(t)} \omega \equiv 0$ also must factor through $h$ if there is a decomposition. We do not consider these questions here.

In the 0-dimensional case, Conjecture 7.1 is in fact a theorem. From Propositions 3.6 and 3.8 if the monodromy is not imprimitive (and hence the polynomial $f$ decomposes), it is either equivalent to $x^p$ or $T_p(x)$ for some prime $p$, or the monodromy group is 2-transitive. In the latter case, it is a classical result ([9], p281) that the permutation representation over $\mathbb{C}$ of a 2-transitive group decomposes into two irreducible sub-representations. One is the trivial one, and one the space $\{\sum k_i x_i | k_i \in \mathbb{C}, \sum k_i = 0\}$, which is just our space $H_0(f^{-1}(t), \mathbb{C})$.

Theorem 7.2 Let $f(x) = T_p(x)$ be the Chebyshev polynomial for some prime $p > 2$, and let $F = y^2 + f(x)$.

(i) The space $H_0(f^{-1}(t), \mathbb{C})$ splits into $p - 1$ invariant subspaces $W_{x^k/p}$, $k = 1, \ldots, (p - 1)/2$. Furthermore, if $\delta(t) \in W_{x^{k+1}/p}$, $k = 2, \ldots, (p - 1)/2$, then there exists a 0-form $\omega$ such that $\int_{\delta(t)} \omega \equiv 0$, but $\omega$ is not decomposable.

(ii) The space $H_0(F^{-1}(t), \mathbb{C})$ splits into $p - 1$ invariant subspaces $V_{x^k/p}$, $k = 1, \ldots, (p - 1)/2$. Furthermore, if $\delta(t) \in V_{x^{k+1}/p}$, $k = 2, \ldots, (p - 1)/2$, then there exist a 1-form $\omega$ such that $\int_{\delta(t)} \omega \equiv 0$, but $\omega$ is not relatively exact.

Remark 7.3 It would be sufficient to consider homology groups with coefficients in $\mathbb{Q}(w)$ for some $p$-th root of unity $w$. 

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The rest of this section is devoted to proving Theorem 7.2. We assume that \( n \) is odd throughout.

Recall that \( T_n(x) \) is defined by

\[
T_n(x) = T_n(\cos(\theta)) = \cos(n\theta), \quad \text{where} \quad x = \cos \theta \tag{7.1}
\]

Clearly \( T_n \) has degree \( n \), and hence \( T_n(x) \) is not decomposable for \( p \) prime.

We prove the theorem by pulling back to \( \theta \) coordinates. Let \( \tilde{F} : \mathbb{C}^2 \to \mathbb{C} \) be the function

\[
\tilde{F}(\theta, y) = y^2 + \cos(n\theta)
\]

and let \( X_F \) be the Hamiltonian vector field associated to \( \tilde{F} \). The vector field \( X_F \) has infinitely many singular points \( p_\ell = \left( \frac{2\ell\pi}{n}, 0 \right), \ell \in \mathbb{Z} \). These points are saddles \( s_{2k} = \left( \frac{2k\pi}{n}, 0 \right) \) for \( \ell = 2k \) even and centers \( c_{2k+1} = \left( \frac{2k\pi}{n} + \frac{2\ell\pi}{n}, 0 \right) \), for \( \ell = 2k + 1 \) odd.

For \( t \in (-1, 1) \), let \( \tilde{C}_{2k+1}, k \in \mathbb{Z} \), be the cycle turning once in the positive direction around the center \( c_{2k+1} \). All cycles \( \tilde{C}_{2k+1} \) vanish for \( t = -1 \). Similarly, let \( \tilde{S}_{2k} \) be the complex cycle vanishing at the saddle \( s_{2k} \), for \( t = 1 \). The orientation is chosen by the condition:

\[
(\tilde{C}_{2i-1}, \tilde{S}_{2i}) = (\tilde{S}_{2i}, \tilde{C}_{2i+1}) = 1.
\]

We denote \( \tilde{P}_t \) the cycle \( \tilde{S}_t \), for \( \ell \) even or \( \tilde{C}_t \), for \( \ell \) odd. The complex fiber \( \tilde{F}^{-1}(t) \) can be represented as a two-sheeted Riemann surface \( y = \sqrt{t - \cos(n\theta)} \), with a countable number of cuts. The homology group of a fiber \( H_1(\tilde{F}^{-1}(t), \mathbb{Z}) \) for \( t \in \mathbb{C} \setminus \{-1, 1\} \), is the free abelian group on the set of cycles \( U_{i=\mathbb{Z}} \{ C_i, S_i \} \).

The flow of the gradient vector field of \( \tilde{F} \) allows us to define a compact support fibration on \( \mathbb{C}^2 \setminus \{ \tilde{F}^{-1}(-1) \cup \tilde{F}^{-1}(1) \} \). That is, for any \( t_0 \in \mathbb{C} \setminus \{-1, 1\} \), and any compact \( K \) in \( F^{-1}(t_0) \), there exists a neighborhood \( U \) of \( t_0 \in \mathbb{C} \setminus \{-1, 1\} \) and an embedding \( \Phi : U \times K \to \mathbb{C}^2 \setminus \{-1, 1\} \), such that \( \Phi(t_0, p) = p \) and \( F \circ \Phi(t, p) = t \), for any \( t \in U \). Moreover, the trivialization \( \Phi \) is well defined up to an isotopy which is identity on \( K \) and preserves the fibers.

The existence of the compact support fibration enables the definition of the monodromy acting on \( H_1(\tilde{F}^{-1}(t), \mathbb{Z}) \). In fact, by the Picard-Lefschetz formula, it follows:

\[
\begin{align*}
\mathcal{M}_1(\tilde{C}_{2i+1}) &= \tilde{C}_{2i+1} + \tilde{S}_{2i} - \tilde{S}_{2i+2}, & \mathcal{M}_1(\tilde{S}_{2i}) &= \tilde{S}_{2i}, \\
\mathcal{M}_{-1}(\tilde{C}_{2i+1}) &= \tilde{C}_{2i+1}, & \mathcal{M}_{-1}(\tilde{S}_{2i}) &= \tilde{S}_{2i} + \tilde{C}_{2i-1} - \tilde{C}_{2i+1}.
\end{align*} \tag{7.2}
\]

Consider the mapping \( \cos : \mathbb{C} \to \mathbb{C} \) and denote \( \Pi = \cos \times \text{Id} \), then

\[
(\theta, y) \in \mathbb{C}^2 \underset{\tilde{F}}{\mapsto} (x, y) \in \mathbb{C} \setminus \{ -1, 1 \} \times \mathbb{C} \underset{\mathbb{C} \setminus \{ -1, 1 \}}{\mapsto} \mathbb{C} \setminus \{ -1, 1 \}
\]

Let

\[
P_t = \Pi_*(\tilde{P}_t), \quad S_{2t} = \Pi_*(\tilde{S}_{2t}), \quad C_{2t+1} = \Pi_*(\tilde{C}_{2t+1}).
\]

The map \( \cos : \mathbb{C} \setminus \pi\mathbb{Z} \to \mathbb{C} \setminus \{ -1, 1 \} \) is a covering with covering group \( G = \mathbb{Z}_2 \times \mathbb{Z}_2 = D_\infty \) generated by two transformations of order 2: \( a(\theta) = -\theta \) and \( b(\theta) = 2\pi - \theta \). The composition \( b \circ a \) is the translation \( \theta \mapsto 2\pi + \theta \), which we denote \( T \). We take \( a \) and \( T \) as the generators of \( D_\infty \), with \( Ta = aT^{-1} \).

The map \( \Pi : (\mathbb{C} \setminus \pi\mathbb{Z}) \times \mathbb{C} \to (\mathbb{C} \setminus \{ -1, 1 \}) \times \mathbb{C} \) is a covering with the same covering group \( G = D_\infty \) generated by the two transformations \( a \times \text{id} \) and \( T \times \text{id} \).
The action of the group $G = D_\infty$ on the cycles $\tilde{P}_t$ (i.e. $\tilde{C}_{2k+1}$ or $\tilde{S}_{2k}$) is given by

$$T \times \text{id}(\tilde{P}_t) = \tilde{P}_{t+2\ell},$$

$$a \times \text{id}(\tilde{P}_t) = -\tilde{P}_{-\ell},$$

We let $H^1_\ell(\tilde{F}^{-1}(t), \mathbb{C})$ be the homology with closed support of $\tilde{F}^{-1}(t)$ with complex coefficients. An element of $H^1_\ell(\tilde{F}^{-1}(t), \mathbb{C})$ is of the form $C = \sum_{t \in \mathbb{Z}} z_t \tilde{P}_t$.

We define the action of the covering group $G$ on $H^1_\ell(\tilde{F}^{-1}(t), \mathbb{C})$ as follows:

$$g(C) = g(\sum_{t \in \mathbb{Z}} z_t P_t) = \sum_{t \in \mathbb{Z}} z_t g(P_t).$$

Let $H^G_\ell(\tilde{F}^{-1}(t), \mathbb{C})$ be the subspace of $H^1_\ell(\tilde{F}^{-1}(t), \mathbb{C})$ consisting of elements of $H_1(\tilde{F}^{-1}(t), \mathbb{C})$ invariant under the action of the group $G$. The monodromy operators $\tilde{M}_\sigma, \sigma = \pm 1$ extend naturally to $H_1(\tilde{F}^{-1}(t), \mathbb{C})$.

The space $H^G_\ell(\tilde{F}^{-1}(t), \mathbb{C})$ is the $\mathbb{C}$-vector space generated by $\sum_{g \in G} g(P_t), \ell \in \mathbb{Z}$, and the extended monodromy $\tilde{M}_\sigma$ preserves it.

Let $\Pi' : H_1(F^{-1}(t), \mathbb{C}) \to H^G_\ell(\tilde{F}^{-1}(t), \mathbb{C})$ be the pullback via the map $\Pi$, then $\Pi'$ gives an isomorphism of $H_1(F^{-1}(t), \mathbb{C})$ onto $H^G_\ell(\tilde{F}^{-1}(t), \mathbb{C})$ with inverse $\Pi'' = (\Pi')^{-1}$ from $H^G_\ell(\tilde{F}^{-1}(t), \mathbb{C})$ to $H_1(F^{-1}(t), \mathbb{C})$.

Let $M_\sigma$ and $\tilde{M}_\sigma, \sigma = \pm 1$, be the monodromies corresponding to turning around $\sigma = \pm 1$, as given in (7.3). Then the following diagram is commutative

$$
\begin{array}{ccc}
H^G_\ell(\tilde{F}^{-1}(t), \mathbb{C}) & \xrightarrow{\Pi} & H_1(F^{-1}(t), \mathbb{C}) \\
\tilde{M}_\sigma \downarrow & & \downarrow M_\sigma \\
H^G_\ell(\tilde{F}^{-1}(t), \mathbb{C}) & \xrightarrow{\Pi'} & H_1(F^{-1}(t), \mathbb{C})
\end{array}
$$

Let $w$ be an $n$-th rooth of unity. The vectors $\sum w^\ell \tilde{S}_{2\ell}$ and $\sum w^\ell \tilde{C}_{2\ell+1}$, are clearly invariant by the translation $T$, but not by $a$. Taking

$$\tilde{S}_w = (1 + a) \sum w^\ell \tilde{S}_{2\ell}, \quad \tilde{C}_w = (1 + a) \sum w^\ell \tilde{C}_{2\ell+1},$$

we therefore obtain elements of $H^G_\ell(\tilde{F}^{-1}(t), \mathbb{C})$.

We let $S_w$ and $C_w$ in $H_1(F^{-1}(t), \mathbb{C})$ represent the images of $\tilde{S}_w$ and $\tilde{C}_w$ under $\Pi'$. That is

$$S_w = \Pi' \tilde{S}_w = \sum_{\ell=0}^{n-1} (w^{\ell} - w^{-\ell}) S_{2\ell},$$

$$C_w = \Pi' \tilde{C}_w = \sum_{\ell=0}^{n-1} (w^{\ell} - w^{-\ell-1}) C_{2\ell+1}.$$

By direct substitution from (7.2) we can calculate the variation, $\var_{\ell_0}, t_0 = \pm 1$, on $H^G_\ell(\tilde{F}^{-1}(t), \mathbb{C})$ around $t_0 = \pm 1$. Due to (7.3), these calculations push forward to $H_1(F^{-1}, \mathbb{C})$ via $\Pi'$, to obtain

$$\var_1(C_w) = (1 + w^{-1}) S_w,$$

$$\var_{-1}(S_w) = 0,$$

$$\var_1(S_w) = (1 + w) C_w.$$

We denote $\tilde{V}_w = \text{Span}(\tilde{C}_w, \tilde{S}_w) \subset H^G_\ell(\tilde{F}^{-1}(t), \mathbb{C})$ and $V_w = \Pi' \tilde{V}_w \subset H_1(F^{-1}(t), \mathbb{C})$. The spaces $V_w \subset H_1(F^{-1}(t), \mathbb{C})$ are invariant under the action of the monodromy group $\mathbb{M}$ of the fibration given by $F$. Moreover, for $n$ odd,

$$H_1(F^{-1}(t), \mathbb{C}) = \bigoplus_{w^{\ell} = 1, \text{Im}(w) > 0} V_w.$$
Proposition 7.4 Let $\delta = S_w$ or $\delta = C_w$ be the family of cycles in $H_1(F^{-1}(t), \mathbb{C})$, given by (7.3), for $w = e^{2\pi i k}$, $k = 2,\ldots,(n-1)/2$ and let $\omega = ydx$. Then
\[
\int_{\delta} \omega \equiv 0,
\]
but the form $\omega$ is not relatively exact.

Proof Let $w = e^{2\pi i k}$, $k = 2,\ldots,(p-1)/2$, where $\xi = e^{2\pi i k}$.

Consider first the case $\delta = S_w$. Let $I = \int_{S_w} y \cos \theta d\theta$. We calculate $I_{2\ell} = \int_{S_{\xi}(\xi^{2\ell})} y \sin \theta d\theta$. We make a change of coordinates $\theta \mapsto \theta + \frac{2\pi i}{n}$. This gives $I_{2\ell} = -\cos \frac{2\pi i}{n} \int_{S_{\xi}} y \sin \theta d\theta - \sin \frac{2\pi i}{n} \int_{S_{\xi}} \cos \theta d\theta$.

The first integral vanishes, giving
\[
I_{2\ell} = -\frac{2\pi}{n} \frac{1}{i} I = -\frac{i \xi^\ell - i \xi^{-\ell}}{2i}.
\]

This gives
\[
\int_{S_{\xi}} y dx = \sum_{\ell=0}^{n-1} (w^\ell - w^{-\ell}) \frac{1}{i} I_{2\ell} = -\frac{1}{2i} \sum_{\ell=0}^{n-1} (\xi^{\ell+k} - \xi^{-\ell-k}) (\xi^{\ell+1/2} - \xi^{-\ell-1/2}) \frac{1}{i} I_{2\ell} = 0.
\]

The last equality holds as each of the four sums which appear vanishes. Consider now the case $\delta = C_w$. Denote $T_{-\pi/n}(\tilde{C}_1)$ the transport of the translation of the cycle $\tilde{C}_1$ by $-\pi/n$, thus giving a cycle centered at the origin. Let $J = \int_{T_{-\pi/n}(\tilde{C}_1)} y \cos \theta d\theta$. We calculate $J_{2\ell+1} = \int_{T_{-\pi/n}(\tilde{C}_1)} y \sin \theta d\theta$. We make the change of coordinates $\theta \mapsto \theta + \frac{(2\ell+1)\pi}{n}$. This gives $J_{2\ell+1} = -\cos \frac{(2\ell+1)\pi}{n} \int_{T_{-\pi/n}(\tilde{C}_1)} y \sin \theta d\theta - \sin \frac{(2\ell+1)\pi}{n} \int_{T_{-\pi/n}(\tilde{C}_1)} y \cos \theta d\theta = -\sin \frac{(2\ell+1)\pi}{n} J$. That is
\[
J_{2\ell+1} = -\sin \frac{(2\ell+1)\pi}{n} J = -\frac{i \xi^{\ell+1/2} - i \xi^{-\ell-1/2}}{2} J.
\]

\[
\int_{C_w} y dx = \sum_{\ell=0}^{n-1} (w^\ell - w^{-\ell-1}) J_{2\ell+1} = -\frac{1}{2i} \sum_{\ell=0}^{n-1} (\xi^{\ell+k} - \xi^{-\ell-k}) (\xi^{\ell+1/2} - \xi^{-\ell-1/2}) = 0.
\]

similarly to (7.7).

Note that it is obvious that the form $\omega = ydx$ is not relatively exact since for instance $\int_{C_1} y dx \neq 0$ is the non-zero area bounded by $C_1$. $lacktriangle$

This completes the proof of part (ii) of the Theorem 7.2. We now prove the statement in part (i).

Let $\theta_{\ell}^\pm = \pm \frac{\arccos}{n}$, $\theta_{\ell}^\pm = \theta_{0}^\pm + \frac{2\pi}{n}$, $x_{\ell}^\pm = \cos(\theta_{\ell}^\pm)$. Note that $x_{\ell}^\pm = \cos(\theta_{0}^\pm + \frac{2\pi}{n}) = \cos(\theta_{0}^\pm \cos \frac{2\pi}{n} - \sin \theta_{0}^\pm \sin \frac{2\pi}{n})$. Hence,
\[
x_{\ell}^\pm - x_{\ell}^- = -2 \sin \theta_{0}^\pm \sin \frac{2\pi}{n} = i \sin \theta_{0}^\pm (\xi^\ell - \xi^{-\ell}).
\]

Let
\[
\delta_{2\ell}(c) = x_{\ell}^+(c) - x_{\ell}^-(c), \quad \ell = 0,\ldots,n-1,
\]
be the families of simple cycles of the Chebyshev polynomial $T_n$, and let
\[
\delta_{w}(t) = \sum_{\ell=0}^{n-1} (w^\ell - w^{-\ell-1}) \delta_{2\ell} \in H_0(f^{-1}(t), \mathbb{C})
\]

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where \( w = e^{\frac{2k\pi}{n}} \), \( k = 1, \ldots, (n-1)/2 \). Taking \( W_w \) as the subspace of \( H_0(f^{-1}(t), \mathbb{C}) \) spanned by \( \delta_w \), it is clear that \( W_w \) is invariant under the monodromy, and

\[
H_0(f^{-1}(t), \mathbb{C}) = \bigoplus_{\sigma = 1, f_m(w) > 0} W_w.
\]

**Proposition 7.5** The 0-dimensional Abelian integral \( I = \int_{\delta} \omega \) vanishes identically, for the cycle \( \delta = \delta_w \), with \( w = e^{\frac{2k\pi}{n}} \), \( k = 2, \ldots, (n-1)/2 \) and \( \omega(x) = x \), but the 0-form \( \omega \) is not relatively exact.

**Proof** The proof is similar to the proof of the previous theorem. In fact it is simpler. The simple cycles \( \delta_2^\ell \) entering in the definition of the cycle \( \delta \) corresponds to the ramification points around which the cycle \( S_2^\ell \) turns. We have

\[
\int_{\delta} \omega = \sum_{\ell=0}^{n-1} (w^\ell - w^{-\ell})(x^+_{\ell} - x^-_{\ell}) = i \sin \theta_0 \sum_{\ell=0}^{n-1} (\xi^k \ell - \xi^{-k} \ell)(\xi \ell - \xi^{-\ell}) = 0.
\]

On the other hand \( \int_{\delta} \omega = x^+_{\ell} - x^-_{\ell} = 2\arccos \frac{t}{n} \neq 0 \), so \( \omega \) is not relatively exact. \( \square \)

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