Probabilistic Measures and Algorithms Arising from the Macdonald Symmetric Functions

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Abstract

The Macdonald symmetric functions are used to define measures on the set of all partitions of all integers. Probabilistic algorithms are given for growing partitions according to these measures. The case of Hall-Littlewood polynomials is related to the finite classical groups, and the corresponding algorithms simplify. The case of Schur functions leads to a $q$-analog of Plancherel measure, and a conditioned version of the corresponding algorithms yields generalizations of the hook walk of combinatorics.

1 Introduction

The Macdonald symmetric functions are the most general class of symmetric functions known at present. Various specializations give the Schur functions, the Hall-Littlewood polynomials, Jack’s symmetric functions, and zonal polynomials. A good account of symmetric function theory is Macdonald’s book [21].

The present work considers probabilistic aspects of Macdonald’s symmetric functions. Our initial motivation came from the theory of random matrices. Recently there has been much interest in studying what a random element of a finite general linear group $GL(n,q)$ “looks like” [5], [6], [7], [12], [15], [24], [25]. Many properties of a matrix (e.g. its characteristic polynomial, its order, the dimension of its fixed space) are functions only of its conjugacy class. Thus a logical step in understanding a random matrix is to first understand the conjugacy class of a random matrix. Recall that the conjugacy classes of $GL(n,q)$ correspond to the rational canonical form of a matrix (this is a generalization of Jordan canonical form which works over non-algebraically closed fields—see Chapter 6 of Herstein [16]). Rational canonical form admits the following combinatorial description. To each monic irreducible polynomial $\phi$ over $F_q$, a field of size $q$, associate a partition (perhaps the trivial partition) $\lambda_\phi$ of some non-negative integer $|\lambda_\phi|$. Let $m_\phi$ be the degree of $\phi$. Then the data $\lambda_\phi$ represents a conjugacy class when:

1. $|\lambda_\phi| = 0$
2. $\sum_\phi |\lambda_\phi|m_\phi = n$

Fulman [3] defines a measure on the set of all partitions $\lambda$ of all integers as follows. Fix $u$ such that $0 < u < 1$. Then pick the size of a general linear group with probability of size $n$ equal to $(1-u)n^n$. Next pick $\alpha$ uniformly in $GL(n,q)$ and take the partition $\lambda_\phi(\alpha)$ corresponding to $\phi$ in the rational canonical form of $\alpha$.

Theorem 3 of Section 7 proves that these group theoretic measures on partitions can be defined in terms of the Hall-Littlewood symmetric functions. This, together with the identities in Section 3, led us to a general definition of measures which works for the Macdonald symmetric functions. The measures defined from the Macdonald symmetric functions can be grown probabilistically (Sections 5, 6, 8). One remarkable feature of these algorithms is that they blend nicely with the algebraic structure of symmetric functions. For instance, the algorithms can be divided into...
steps corresponding to each variable \( x_i \). The Pieri formula of algebraic geometry also makes an appearance as a probabilistic transition rule.

This paper is structured as follows. Section 2 collects notation which will be used freely in following sections. Section 3 reviews identities satisfied by the Macdonald symmetric functions. Section 4 uses the Macdonald symmetric functions to define measures on the set of all partitions of all integers. Section 5 gives probabilistic algorithms for growing partitions according to the measures of Section 4. Section 6 shows that the algorithms of Section 5 simplify for the case of Hall-Littlewood polynomials. Section 7 considers further specializations of the measures coming from the Hall-Littlewood polynomials, explaining the connection with the finite classical groups. Section 8 develops the “Young Tableau Algorithm”, a simplification which works only in the case relevant to the general linear groups. Section 9 develops a formula for the specialized Hall-Littlewood measures in terms of weights on the Young lattice; this extends to the unitary groups as well. Section 10 specializes the measures of Section 8 to the Schur functions, leading to a \( q \)-analog of Plancherel measure. Section 11 explains how the algorithm of Section 5 is related to Kerov’s \( q \)-generalization of the hook walk of combinatorics. Section 12 gives suggestions for future research.

Most of the results of this paper are taken from Fulman’s Ph.D. thesis [9] done under the guidance of Persi Diaconis. The purpose of this paper is to emphasize symmetric function theory and combinatorics with a minimum of group theory. A companion paper to this one is Fulman [10], which applies the results of Sections 8 and 9 to prove group theoretic results about the general linear and unitary groups.

2 Notation

We begin by reviewing some standard notation about partitions, as on pages 2-5 of Macdonald [21]. Let \( \lambda \) be a partition of some non-negative integer \( |\lambda| \) into parts \( \lambda_1 \geq \lambda_2 \geq \cdots \). Let \( m_i(\lambda) \) be the number of parts of \( \lambda \) of size \( i \), and let \( \lambda' \) be the partition dual to \( \lambda \) in the sense that \( \lambda'_i = m_i(\lambda) + m_{i+1}(\lambda) + \cdots \). Let \( n(\lambda) \) be the quantity \( \sum_{i \geq 1} (i-1)\lambda_i = \sum_i \binom{\lambda_i}{2} \).

It is also useful to define the diagram associated to \( \lambda \) as the set of points \((i, j)\) \( \in \mathbb{Z}^2 \) such that \( 1 \leq j \leq \lambda_i \). We use the convention that the row index \( i \) increases as one goes downward and the column index \( j \) increases as one goes across. So the diagram of the partition \((5441)\) is:

\[
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \\
\cdot \\
\end{array}
\]

It is sometimes useful to think of these dots as boxes. Given a square \( s \) in the diagram of a partition \( \lambda \), let \( l'_\lambda(s), l_\lambda(s), a_\lambda(s), a'_\lambda(s) \) be the number of squares in the diagram of \( \lambda \) to the north, south, east, and west of \( s \) respectively. The subscript \( \lambda \) will sometimes be omitted if the partition \( \lambda \) is clear from context. So the diagram

\[
\begin{array}{cccc}
\cdot & \cdot & \cdot & \\
\cdot & s & \cdot & \cdot \\
\cdot & \cdot & \cdot & \\
\cdot \\
\end{array}
\]

has \( l'(s) = l(s) = a'(s) = 1 \) and \( a(s) = 2 \).

A skew-diagram is the set theoretic difference \( \lambda - \mu \) of two diagrams \( \lambda \) and \( \mu \), where the diagram of \( \lambda \) contains the diagram of \( \mu \). A horizontal strip is a skew-diagram with at most one square in each column. For instance the following diagram is a horizontal strip:
Letting $f(u)$ be a polynomial in the variable $u$, the notation $[u^n]f(u)$ means the coefficient of $u^n$ in $f(u)$. The following notation is less widely known, and is taken from Chapter 6 of Macdonald [21].

1. Given a partition $\lambda$ and a square $s$, set $b_\lambda(s) = 1$ if $s \not\in \lambda$. Otherwise set:

$$b_\lambda(s) = \frac{1 - q^{a_\lambda(s)}t_{\lambda}(s)+1}{1 - q^{a_\lambda(s)+1}t_{\lambda}(s)}$$

Let $b_\lambda(q, t) = \prod_{s \in \lambda} b_\lambda(s)$.

2. Define

$$\phi_{\lambda/\mu}(q, t) = \prod_{s \in C_{\lambda/\mu}} \frac{b_\lambda(s)}{b_\mu(s)}$$

where $C_{\lambda/\mu}$ is the union of the columns intersecting $\lambda - \mu$.

3. The skew Macdonald polynomials (in one variable) are defined as:

$$P_{\lambda/\mu}(x; q, t) = \frac{b_\mu(q, t)}{b_\lambda(q, t)} \phi_{\lambda/\mu}(q, t)x^{\lambda|-\mu|}$$

if $\lambda - \mu$ is a horizontal strip, and 0 otherwise.

4. Let $(x, q)_\infty$ denote $\prod_{i=1}^\infty (1 - qx^{i-1})$. Then define $\prod(x, y; q, t)$ by:

$$\prod(x, y; q, t) = \prod_{i,j=1}^\infty \frac{(tx_iy_j, q)_\infty}{(x_iy_j, q)_\infty}$$

Also define $g_n(y; q, t)$ as the coefficient of $x^n$ in $\prod_j \frac{(tx_jy, q)_\infty}{(xy, q)_\infty}$.

### 3 Properties of the Macdonald Symmetric Functions

The Macdonald symmetric functions $P_\lambda(x; q, t)$ are a two-parameter family of symmetric functions. A precise definition is in Chapter 6 of Macdonald [21]. The Macdonald symmetric functions have five properties which we shall need. It is convenient to name them (the Pieri Formula is already named).

1. **Measure Identity** [21], page 324:

$$\sum_\lambda P_\lambda(x; q, t)P_\lambda(y; q, t)b_\lambda(q, t) = \prod(x, y; q, t)$$
2. **Factorization Theorem** [21], page 310:

\[
\prod(x, y; q, t) = \prod_{n \geq 1} e^{\frac{1}{n} \frac{1 - t^n}{1 - q^n} p_n(x)p_n(y)}
\]

3. **Principal Specialization Formula** [21], page 337:

\[
P_\lambda(1, t, \cdots, t^{N-1}; q, t) = t^{n(\lambda)} \prod_{s \in \lambda} \frac{1 - q^{a'(s)}t^{N - l'(s)}}{1 - q^{a(s)}t^{l(s) + 1}}
\]

4. **Skew Expansion** [21], pages 343-7:

\[
P_\lambda(x_1, \cdots, x_N; q, t) = \sum_\mu P_\mu(x_1, \cdots, x_{N-1}; q, t)P_{\lambda/\mu}(x_N; q, t)
\]

5. **Pieri Formula** [21], page 340:

\[
P_\mu(y; q, t)g_r(y; q, t) = \sum_{\lambda: |\lambda - \mu| = r} \phi_{\lambda/\mu}(q, t)P_\lambda(y; q, t)
\]

It is worth remarking that the Pieri Formula has its history in algebraic geometry, as a rule for multiplying classes of Schubert varieties in the cohomology ring of Grassmanians.

4 **Defining Measures** \(P_{x,y,q,t}\) from the Macdonald Symmetric Functions

In this section the Macdonald symmetric functions are used to define families of probability measures on the set of all partitions of all numbers. It is assumed throughout this paper that \(x, y, q, t\) satisfy the following conditions:

1. \(0 \leq t, q < 1\)
2. \(x_i, y_i \geq 0\)
3. \(\sum_{i,j} \frac{x_i y_i}{1 - x_i y_j} < \infty\)

The following formula defines a probability measure \(P_{x,y,q,t}\) on the set of all partitions of all numbers:

\[
P_{x,y,q,t}(\lambda) = \frac{P_\lambda(x; q, t)P_\lambda(y; q, t)b_\lambda(q, t)}{\prod(x; y; q, t)}
\]

**Lemma 1** \(P_{x,y,q,t}\) is a measure.
Proof: By the Measure Identity and the fact that there are countably many partitions, it suffices to check that $0 \leq P_{x,y,q,t}(\lambda) < \infty$ for all $\lambda$. For this it is sufficient to show (again by the Measure Identity) that $P_\lambda(x; q, t), b_\lambda(q, t) \geq 0$ for all $\lambda$ and that $0 \leq \prod(x, y; q, t) < \infty$.

Condition 1 implies that $b_\lambda(q, t) \geq 0$ for all $\lambda$. We claim that $x_i \geq 0$ implies that $P_\lambda(x; q, t) \geq 0$. To see this, note that when $P_\lambda(x; q, t)$ is expanded in monomials in the $x$ variables, all coefficients are non-negative. For any particular monomial, this follows by repeated use of the Skew Expansion.

By the Factorization Theorem, showing that $0 \leq \prod(x, y; q, t) < \infty$ is equivalent to showing that:

$$0 \leq \sum_{n \geq 1} \frac{1}{n} \frac{1 - t^n}{1 - q^n} p_n(x)p_n(y) < \infty$$

Conditions 1 and 2 imply that this expression is non-negative. To see that it is finite, use Condition 3 as follows:

$$\sum_{n \geq 1} \frac{1}{n} \frac{1 - t^n}{1 - q^n} p_n(x)p_n(y) \leq \frac{1}{1 - q} \sum_{n \geq 1} p_n(x)p_n(y)$$

$$= \frac{1}{1 - q} \sum_{i,j \geq 1} x_i y_j$$

$$< \infty$$

Define truncated measures $P_{x,y,q,t}^N(\lambda)$ to be 0 if $\lambda$ has more than $N$ parts, and otherwise:

$$P_{x,y,q,t}^N(\lambda) = \frac{P_\lambda(x_1, \cdots, x_N, 0, \cdots; q, t)P_\lambda(y; q, t)b_\lambda(q, t)}{\prod(x_1, \cdots, x_N, 0, \cdots, y; q, t)}$$

Let $P^0_0(x, y, q, t) = 1$ on the empty partition and 0 elsewhere. Arguing as in Lemma 1 shows that the $P_{x,y,q,t}^N$ are probability measures. It is also clear that $\lim_{N \to \infty} P_{x,y,q,t}^N = P_{x,y,q,t}$. There are other possible definitions of $P_{x,y,q,t}^N$ which converge to $P_{x,y,q,t}$ in the $N \to \infty$ limit (for instance one can truncate both the $x$ and $y$ variables). These deserve further investigation.

5 A Probabilistic Algorithm for Picking From $P_{x,y,q,t}$

This section gives a stochastic method for picking from $P_{x,y,q,t}$ under conditions 1-3 of Section 4.

Algorithm for Picking from $P_{x,y,q,t}$

Step 0 Start with $\lambda$ the empty partition and $N$ (which we call the interval number) equal to 1.

Step 1 Pick an integer $n_N$ so that $n_N = k$ with probability $\prod_j \frac{(x_N y_j q)}{(x_N y_j q)} \frac{1}{\prod_j ((x_N y_j q) \frac{1}{\prod_j (x_N y_j q))}} g_k(y; q, t)x_N^k$. (These probabilities sum to 1 by the definition of $g_k$).

Step 2 Let $\Lambda$ be a partition containing $\lambda$ such that the difference $\Lambda - \lambda$ is a horizontal strip of size $n_N$. There are at most a finite number of such $\Lambda$. Change $\lambda$ to $\Lambda$ with probability:

$$\frac{\phi_{\Lambda/\lambda}(q, t)}{g_{n_N}(y; q, t)} \frac{P_\Lambda(y; q, t)}{P_\lambda(y; q, t)}$$

(These probabilities sum to 1 by the Pieri Formula). Then set $N = N + 1$ and go to Step 1.
Lemma 2 will show that this algorithm terminates with probability 1.

As an example of the algorithm, suppose we are at Step 1 with $N = 3$ and the partition $\lambda$: $$\ldots$$

We then pick $n_3$ according to the rule in Step 1. Suppose that $n_3 = 2$. We thus add a horizontal strip of size 2 to $\lambda$, giving $\Lambda$ equal to one of the following four partitions with probability given by the rule in Step 2:

$$\ldots$$

We then set $N = 4$ and return to Step 1.

**Lemma 2** The algorithm terminates with probability 1.

**Proof:** Recall the Borel-Cantelli lemmas of probability theory, which say that if $A_N$ are events with probability $P(A_N)$ and $\sum_N P(A_N) < \infty$, then with probability 1 only finitely many $A_N$ occur. Let $A_N$ be the event that at least one box is added to the partition during interval $N$. To prove the lemma it is sufficient to show that only finitely many $A_N$ occur.

The Factorization Theorem implies that $g_0 = 1$. Again using the Factorization Theorem and the fact that $1 - e^{-x} \leq x$ for $x \geq 0$ shows that:

$$\sum_{N \geq 1} P(A_N) = \sum_{N \geq 1} [1 - \prod_j (\frac{xN y_j; q; \infty)}{t xN y_j; q; \infty) g_0)]$$

$$= \sum_{N \geq 1} [1 - e^{- \sum_{n \geq 1} \frac{1}{n} - \frac{t^n}{1 - q^n} (xN)p_n(y)}]$$

$$\leq \sum_{N \geq 1} \sum_{n \geq 1} \frac{1}{n} \frac{1 - t^n}{1 - q^n} (xN)p_n(y)$$

$$= \sum_{n \geq 1} \frac{1}{n} \frac{1 - t^n}{1 - q^n} p_n(x)p_n(y)$$
Theorem 1 is the main result of this section. Since \( q \) and \( t \) are fixed, the notation in the proof of Theorem 1 will be abbreviated somewhat by omitting the explicit dependence on these variables.

**Theorem 1**  The chance that the algorithm yields the partition \( \lambda \) at the end of interval \( N \) is \( P_{x,y,q,t}^N(\lambda) \). Consequently, the algorithm for picking from \( P_{x,y,q,t}^0 \) works.

**Proof:**  Since the algorithm proceeds by adding horizontal strips, it is clear that the partition produced at the end of interval \( N \) has at most \( N \) parts.

The base case \( N = 0 \) is clear since the algorithm starts with the empty partition and \( P_{x,y,q,t}^0 \) is 1 on the empty partition and 0 elsewhere.

For the induction step, the Skew Expansion gives:

\[
P_{x,y,q,t}^N(\Lambda) = \prod_{i=1}^{N-1} \left( \sum_{\text{horizontal strip } \lambda} \prod_{j=1}^{N-1} \left( \frac{x_i y_j}{t x_i y_j} \right) P_\lambda(x_1, \ldots, x_N) P_\lambda(y) b_\lambda \right) \prod_{j=1}^N \left( \frac{x_N y_j}{t x_N y_j} \right) g_{|\lambda|-|\lambda|} \phi_{\lambda/\lambda}(y) P_\lambda(y)
\]

Probabilistically, this equality says that the chance that the algorithm gives \( \Lambda \) at the end of interval \( N \) is equal to the sum over all \( \lambda \) such that \( \Lambda/\lambda \) is a horizontal strip of the chance that the algorithm gives \( \lambda \) at the end of interval \( N - 1 \) and that \( \lambda \) then grows to \( \Lambda \) in interval \( N \). This proves the theorem. \( \square \)

As a corollary of the above algorithm, one obtains a probability generating function with the size of the partition \( \lambda \).
Corollary 1 The distribution of the size of a partition $\lambda$ chosen from $P_{x,y,q,t}$ has as its probability generating function in the variable $z$:

$$\frac{\prod (xz, y; q, t)}{\prod (x, y; q, t)}$$

**Proof:** By the way the algorithm works, the growth of $\lambda$ during different intervals is independent. So it suffices to show that the chance $\lambda$ grows by $k$ in interval $N$ is:

$$\prod_{j}(xz, y^{j}; q, t) \prod_{j}(x, y^{j}; q, t)$$

This is clear from Step 1 of the algorithm and the definition of $g_{k}$. □

This section closes by noting that in the case $y_{i} = t^{i-1}$, there is a nice expression for $g_{n}$. For this and future use, recall the following lemma of Stong [24].

**Lemma 3** For $|q| > 1$ and $0 < u < 1$,

1. $\prod_{r=1}^{\infty} \left( 1 - \frac{1}{q^{r}} \right) = \sum_{n=0}^{\infty} \frac{u^{n}q^{(n)}}{(q^{n}-1)\cdots(q-1)}$

2. $\prod_{r=1}^{\infty} \left( 1 - \frac{u}{q^{r}} \right) = \sum_{n=0}^{\infty} \frac{(-u)^{n}q^{n}(q^{n}q^{(n)}-1)}{(q^{n}-1)\cdots(q-1)}$

**Corollary 2** If $0 < t, q < 1$, then $g_{n}(t^{i-1}; q, t) = \frac{1}{(1-q^{n})\cdots(1-q)}$

**Proof:** By Lemma 3.

$$g_{n}(t^{i-1}; q, t) = [u^{n}] \prod_{i=1}^{\infty} \left( 1 - \frac{uq^{i}}{q^{i}-1} \right)$$

$$= \frac{1}{q^{n}} [u^{n}] \prod_{i=1}^{\infty} \left( 1 - \frac{u}{q^{i}} \right)$$

$$= \frac{1}{q^{n}} \frac{1}{q^{(n)}(q^{(n)}-1)\cdots(q^{n}-1)}$$

$$= \frac{1}{(1-q^{n})\cdots(1-q)}$$

□

6 Hall-Littlewood Polynomials: Simplified Algorithms

In this section the measure $P_{x,y,q,t}$ is studied under the specialization $y^{i} = t^{i-1}, q = 0$. As one motivation for these choices, note that setting $q = 0$ in the Macdonald polynomials gives the Hall-Littlewood polynomials. The further specialization $x_{i} = ut^{i}$ will be considered in Sections 8 - 9. This further specialization is the case relevant to the finite classical groups. Nevertheless, this section will show that the probabilistic algorithm of Section 5 simplifies without having to assume that $x_{i} = ut^{i}$.

Supposing that $0 < t, x_{i} < 1, \sum x_{i} < 1$, we give a simplified algorithm which allows one to grow the partition $\lambda$ by adding 1 box at a time. Using the Borel-Cantelli lemmas it is straightforward to check that this algorithm always halts.
Simplified Algorithm for Picking from \( P_{x,t-1,0,t} \)

**Step 0** Start with \( \lambda \) the empty partition and \( N = 1 \). Also start with a collection of coins indexed by the natural numbers such that coin \( i \) has probability \( x_i \) of heads and probability \( 1 - x_i \) of tails.

**Step 1** Flip coin \( N \).

**Step 2a** If coin \( N \) comes up tails, leave \( \lambda \) unchanged, set \( N = N + 1 \) and go to Step 1.

**Step 2b** If coin \( N \) comes up heads, let \( j \) be the number of the last column of \( \lambda \) whose size was increased during a toss of coin \( N \) (on the first toss of coin \( N \) which comes up heads, set \( j = 0 \)). Pick an integer \( S > j \) according to the rule that \( S = j + 1 \) with probability \( t_{\lambda}^{j+1} \) and \( S = s > j + 1 \) with probability \( t_{\lambda}^s - t_{\lambda}^{j+1} \) otherwise. Then increase the size of column \( S \) of \( \lambda \) by 1 and go to Step 1.

For example, suppose we are at Step 1 with \( \lambda \) equal to the following partition:

```
. . .
```

Suppose also that \( N = 4 \) and that coin 4 had already come up heads once, at which time we added to column 1, giving \( \lambda \). Now we flip coin 4 again and get heads, going to Step 2b. We have that \( j = 1 \). Thus we add a dot to column 1 with probability 0, to column 2 with probability \( t^2 \), to column 3 with probability \( t - t^2 \), to column 4 with probability 0, and to column 5 with probability \( 1 - t \). We then return to Step 1.

Note that the dots added during the tosses of a given coin form a horizontal strip.

Theorem 2 shows that the simplified algorithm works.

**Theorem 2** The simplified algorithm for picking from \( P_{x,t-1,0,t} \) refines the general algorithm.

**Proof:** Let interval \( N \) denote the time between the first and last tosses of coin \( N \). To prove the theorem, it will be shown that the two algorithms add horizontal strips in the same way during interval \( N \).

For this observe that the size of the strips added in interval \( N \) is the same for the two algorithms. Since \( q = 0 \) the integer \( n_N \) in Step 1 of the general algorithm is equal to \( k \) with probability \( (1 - x_N)x_N^k \). This is equal to the chance of \( k \) heads of coin \( N \) in the simplified algorithm.

Given that a strip of size \( k \) is added during interval \( N \), the general algorithm then increases \( \lambda \) to \( \Lambda \) with probability:

\[
\frac{\phi_{\Lambda/\lambda}(0,t)}{g_k(t^{-1};0,t)} \frac{P_\Lambda(1,t,t^2,\ldots;0,t)}{P_\lambda(1,t,t^2,\ldots;0,t)}
\]

This probability can be simplified. Lemma 2 shows that \( g_k(t^{-1};0,t) = 1 \). The definition of \( \phi_{\Lambda/\lambda}(0,t) \) and the Principal Specialization Formula show that the probability can be rewritten as:

\[
\left( \prod_{s \in C_{\Lambda/\lambda}} \frac{b_\lambda(s)}{b_\lambda(s)} \right)^{t^n(\lambda)} \frac{\prod_{s \in \Lambda} \frac{1}{1 - 0^{n_{\Lambda}(s)}p_{\Lambda}(s)+t}}{\prod_{s \in \lambda} \frac{1}{1 - 0^{n_\lambda(s)}p_\lambda(s)+t}}
\]

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where $0^0 = 1$. Let $A$ be the set of column numbers $a > 1$ such that $\Lambda - \lambda$ intersects column $a$ but not column $a - 1$. Let $A'$ be the set of column numbers $a$ such that either $a = 1$ or $a > 1$ and $\Lambda - \lambda$ intersects both columns $a$ and $a - 1$. Most of the terms in the above expression cancel, giving:

$$
\frac{t^n(\Lambda)}{t^n(\lambda)} \prod_{a \in A} (1 - t^{\lambda'_{a-1} - \lambda'_a}) = \prod_{a \in A'} t^{\lambda'_a} \prod_{a \in A} (t^{\lambda'_a} - t^{\lambda'_{a-1}})
$$

It is easily seen that the simplified algorithm can go from $\lambda$ to $\Lambda$ in exactly 1 way, and this also happens with probability equal to:

$$
\prod_{a \in A'} t^{\lambda'_a} \prod_{a \in A} (t^{\lambda'_a} - t^{\lambda'_{a-1}})
$$

\[\square\]

### 7 Hall-Littlewood Polynomials: Relation with the Finite Classical Groups

This section explains the relation of measures defined from the Hall-Littlewood polynomials with the finite classical groups. The case of the general linear groups will be worked out in detail. Analogous results will then be described for the other classical groups.

Recall that the conjugacy classes of $GL(n, q)$ correspond to the possible rational canonical forms of a matrix. Rational canonical form is a generalization of Jordan canonical form which works over non-algebraically closed fields. See Chapter 6 of Herstein [16] for a clear discussion of canonical forms of matrices. Rational canonical form admits the following combinatorial description. To each monic irreducible polynomial $\phi$ over $F_q$, a field of size $q$, associate a partition (perhaps the trivial partition) $\lambda_\phi$ of some non-negative integer $|\lambda_\phi|$. Let $m_\phi$ be the degree of $\phi$. Then the data $\lambda_\phi$ represents a conjugacy class when:

1. $|\lambda_\phi| = 0$
2. $\sum_{\phi} |\lambda_\phi| m_\phi = n$

Given an element $\alpha$ in $GL(n, q)$, let $\lambda_\phi(\alpha)$ be the partition associated to $\phi$ in the rational canonical form of $\alpha$. For example, the identity matrix has $\lambda_{z-1}$ equal to $(1^n)$ and an elementary matrix with $a \neq 0$ in the $(1, 2)$ position, ones on the diagonal and zeros elsewhere has $\lambda_{z-1}$ equal to $(2, 1^{n-2})$.

Following Fulman [9], one can now define a random partition $\lambda_\phi$ as follows. Fix $u$ such that $0 < u < 1$. Then pick the size of a general linear group with probability of size $n$ equal to $(1-u)n^u$. Next pick $\alpha$ uniformly in $GL(n, q)$ and take the partition $\lambda_\phi(\alpha)$ corresponding to $\phi$ in the rational canonical form of $\alpha$.

Theorem 3 is the main result of this section. It shows that the random partitions $\lambda_\phi$ are independent for different $\phi$ and relates their distributions to measures defined using the Hall-Littlewood polynomials.

**Theorem 3** The random partitions $\lambda_\phi$ (defined on the union of all the groups $GL$) are independent with distribution $P_{\frac{m_\phi}{q^{m_\phi}}, \frac{1}{q^{(i-1)m_\phi}}, 0, \frac{1}{q^{0}}}$.
Proof: Kung [20] proved that the conjugacy class of $GL(n,q)$ corresponding to the data $\lambda_\phi$ has size:

$$\frac{|GL(n,q)|}{\prod_\phi \prod_i \prod_{k=1}^{m_i(\lambda_\phi)} (q^{m_\phi d_i(\lambda_\phi)} - q^{m_\phi (d_i(\lambda_\phi) - k)})}$$

where for any partition $\lambda$,

$$d_i(\lambda) = m_1(\lambda)1 + m_2(\lambda)2 + \cdots + m_{i-1}(\lambda)(i - 1) + (m_i(\lambda) + m_{i+1}(\lambda) + \cdots + m_j(\lambda))i.$$  

Next, observe that:

$$\prod_i \prod_{k=1}^{m_i(\lambda_\phi)} (q^{m_\phi d_i(\lambda_\phi)} - q^{m_\phi (d_i(\lambda_\phi) - k)}) = q^{m_\phi \sum [d_i m_i(\lambda_\phi) - (m_i(\lambda_\phi))^2]} \prod_i \prod_{k=1}^{m_i(\lambda_\phi)} (1 - q^{k m_\phi})$$

$$= q^{m_\phi \sum [i m_i(\lambda_\phi) + 2 \sum_{h<i} h m_h(\lambda_\phi)]} m_i(\lambda_\phi) \prod_{k=1}^{m_i(\lambda_\phi)} (1 - q^{k m_\phi})$$

$$= q^{m_\phi |\lambda_\phi| + 2n(\lambda_\phi)} \prod_{k=1}^{m_i(\lambda_\phi)} (1 - q^{k m_\phi})$$

$$= q^{m_\phi |\lambda_\phi| + 2n(\lambda_\phi)} \prod_{s \in \lambda_\phi} (1 - q^{(l(s) + 1)m_\phi})$$

$$= q^{m_\phi n(\lambda_\phi)} P_\lambda(\frac{1}{\lambda}, \frac{1}{m_\phi}, \cdot \cdot \cdot ; 0, \frac{1}{m_\phi})$$

The second equality follows from the identity $d_i(\lambda) = \sum_{h<i} h m_h(\lambda) + i m_i(\lambda) + \sum_{i<k} i m_k(\lambda)$. The third equality follows from the identity $\chi'_i = m_i(\lambda) + m_{i+1}(\lambda) + \cdots$. The fourth equality follows from the Principal Specialization Formula of Section 3.

Now define a “cycle index” for $GL$ as in Stong [24],

$$Z_{GL(n,q)} = \frac{1}{|GL(n,q)|} \sum_{\alpha \in GL(n,q)} \prod x_{\phi, \lambda}(\alpha)$$

The observation that Kung’s conjugacy class size formula factors in $\phi$ leads to the equation:

$$1 + \sum_{n=1}^{\infty} Z_{GL(n,q)} u^n = \prod_{\lambda} x_{\phi, \lambda} P_\lambda(\frac{u}{q m_\phi}, \frac{u}{q m_\phi}, \cdot \cdot \cdot ; 0, \frac{1}{q m_\phi})$$

The definition of the measure $P_\lambda(\frac{u}{q m_\phi}, \frac{u}{q m_\phi}, \cdot \cdot \cdot ; 0, \frac{1}{q m_\phi})$ gives that:

$$P_\lambda(\frac{u}{q m_\phi}, \frac{u}{q m_\phi}, \cdot \cdot \cdot ; 0, \frac{1}{q m_\phi}) = \prod_{r=1}^{\infty} (1 - u^{m_\phi}) P_\lambda(\frac{u}{q m_\phi}, \frac{u}{q m_\phi}, \cdot \cdot \cdot ; 0, \frac{1}{q m_\phi})$$
Therefore,

\[ 1 + \sum_{n=1}^{\infty} Z_{\text{GL}(n,q)} u^n = \prod_{\phi \neq z} \sum_{\lambda} x_{\phi,\lambda} P_{\frac{m_\phi}{q^1 \phi}, \frac{1}{q^{(1-1)m_\phi}}, 0, \frac{1}{q^m_\phi}}(\lambda) \]

Setting all \( x_{\phi,\lambda} \) to 1 in this equation gives:

\[ \frac{1}{1 - u} = \prod_{\phi \neq z} \prod_{r=1}^{\infty} \left( 1 - \frac{u}{q^m_\phi} \right) \]

Combining these last two equations proves that:

\[ (1 - u)[1 + \sum_{n=1}^{\infty} Z_{\text{GL}(n,q)} u^n] = \prod_{\phi \neq z} \sum_{\lambda} x_{\phi,\lambda} P_{\frac{m_\phi}{q^1 \phi}, \frac{1}{q^{(1-1)m_\phi}}, 0, \frac{1}{q^m_\phi}}(\lambda) \]

The statement of the theorem is exactly a probabilistic interpretation of this last equation.

Theorem 3 leads to a corollary which is useful for studying the \( n \to \infty \) asymptotics of random matrix theory.

**Lemma 4** If \( f(1) < \infty \) and \( f \) has a Taylor series around 0, then:

\[ \lim_{n \to \infty} \left[ u^n \frac{f(u)}{1 - u} \right] = f(1) \]

**Proof:** Write the Taylor expansion \( f(u) = \sum_{n=0}^{\infty} a_n u^n \). Then observe that \( [u^n] \frac{f(u)}{1 - u} = \sum_{i=0}^{n} a_i \).

**Corollary 3** The \( n \to \infty \) limit of the random variables \( \lambda_{\phi} \) with the uniform distribution on \( \text{GL}(n,q) \) is \( P_{\frac{1}{q^m_\phi}, \frac{1}{q^{(1-1)m_\phi}}, 0, \frac{1}{q^m_\phi}} \).

**Proof:** Apply Lemma 4 to the final equation in the proof of Theorem 3.

It is worth remarking that one possible motivation for a result like Theorem 3 comes from the theory of the symmetric groups, in particular the “Polya Cycle Index”. Let \( a_i(\pi) \) be the number of \( i \)-cycles of a permutation \( \pi \). Using the fact that there are \( n! \prod_{i} a_i^i(\pi) \) elements of \( S_n \) with \( a_i \) \( i \)-cycles, one proves that:

\[ \sum_{n=0}^{\infty} \frac{(1 - u)u^n}{n!} \sum_{\pi \in S_n} \prod_{i} a_i^i(\pi) = \prod_{m=1}^{\infty} e^{x_m(n)(x_m-1)} \]

This last equation has the following probabilistic interpretation. Fix \( u \) such that \( 0 < u < 1 \) and pick the size of the symmetric group with chance of size \( n \) equal to \( (1 - u)u^n \). Next choose \( \pi \) uniformly in that \( S_n \). Then the random variables \( a_i(\pi) \) are independent Poisson \( \frac{u^i}{i!} \). Combining the above equation with Lemma 4 shows that for any \( i < \infty \), the joint distribution of \( (a_1(\pi), \ldots, a_i(\pi)) \) converges to independent (Poisson(1), \ldots, Poisson(\frac{1}{i!})) as \( n \to \infty \). The Poisson distribution, naturally arising in the symmetric groups, is of fundamental mathematical importance; it is reasonable to expect the distributions \( P_{\frac{m_\phi}{q^1 \phi}, \frac{1}{q^{(1-1)m_\phi}}, 0, \frac{1}{q^m_\phi}} \), naturally arising in the general linear groups, to be of equal importance.

Let us now consider briefly analogs of Theorem 3 for the other finite classical groups (proofs appear in Fulman [9]).
1. **Unitary Groups** The unitary group $U(n,q)$ can be defined as the subgroup of $GL(n,q^2)$ preserving a non-degenerate skew-linear form, for instance $<\vec{x},\vec{y}> = \sum_{i=1}^{n} x_i y_i^q$.

Wall [28] found that the conjugacy classes of the unitary group $U(n,q)$ correspond to the following combinatorial data. As was the case with $GL(n,q^2)$, Wall [28] proved that the conjugacy classes of the unitary group correspond to the following combinatorial description analogous to rational canonical form for the general linear groups. Given a polynomial $\phi$ with coefficients in $F_q[z]$ and non-vanishing constant term, define a polynomial $\tilde{\phi}$ by:

$$\tilde{\phi} = z^{m_{\phi}} \phi\left(\frac{1}{z}\right)$$

where $\phi^q$ raises each coefficient of $\phi$ to the $q$th power. Writing this out, a polynomial $\phi(z) = z^{m_{\phi}} + \alpha z + \alpha_0$ with $\alpha_0 \neq 0$ is sent to $\tilde{\phi}(z) = z^{m_{\phi}} + \left(\frac{\alpha}{\alpha_0}\right)^q z^{m_{\phi} - 1} + \cdots + \left(\frac{\alpha_{m_{\phi} - 1}}{\alpha_0}\right)^q z + \left(\frac{1}{\alpha_0}\right)^q$. Fulman [9] proves that all irreducible polynomials such that $\phi = \tilde{\phi}$ have odd degree.

Wall [28] proves that the conjugacy classes of the unitary group correspond to the following combinatorial data. As was the case with $GL(n,q^2)$, an element $\alpha \in U(n,q)$ associates to each monic, non-constant, irreducible polynomial $\phi$ over $F_q[z]$ a partition $\lambda_{\phi}$ of some non-negative integer $|\lambda_{\phi}|$ by means of rational canonical form. The restrictions necessary for the data $\lambda_{\phi}$ to represent a conjugacy class are:

(a) $|\lambda_z| = 0$
(b) $\lambda_{\phi} = \tilde{\lambda}_{\phi}$
(c) $\sum_{\phi} |\lambda_{\phi}| m_{\phi} = n$

Random partitions $\lambda_{\phi}$ can be defined exactly as in the case of the general linear groups. Fix $u$ such that $0 < u < 1$. Pick the size with probability of size $n$ equal to $(1 - u)u^n$. Next pick $\alpha$ uniformly in $U(n,q)$ and take the partition $\lambda_{\phi}(\alpha)$ corresponding to $\phi$ in Wall’s description of the conjugacy class of $\alpha$ in $U(n,q)$.

Fulman [9] uses Wall’s conjugacy class size formula and the fact that all polynomials invariant under $\phi$ have odd degree to prove the following analog of Theorem 3.

**Theorem 4** If $\phi = \tilde{\phi}$, then $\lambda_{\phi}$ has distribution $P_{\frac{(-u)^m_{\phi}}{q^{m_{\phi}}} \frac{1}{(-q)^{m_{\phi}}} \frac{1}{q^{m_{\phi}}} \frac{1}{(-q)^{m_{\phi}}}}$. If $\phi \neq \tilde{\phi}$, then $\lambda_{\phi} = \tilde{\lambda}_{\phi}$ have distribution $P_{\frac{2m_{\phi}}{q^{2m_{\phi}}} \frac{1}{q^{2(m_{\phi})}} \frac{1}{q^{2m_{\phi}}} \frac{1}{q^{2m_{\phi}}} \frac{1}{q^{2m_{\phi}}} \frac{1}{q^{2m_{\phi}}}}$. These random partitions are independent and as with $GL$, the case $u = 1$ corresponds to the $n \to \infty$ limit.

2. **Symplectic Groups** Assume for simplicity that the characteristic of $F_q$ is not equal to 2. The symplectic group $Sp(2n,q)$ can be defined as the subgroup of $GL(2n,q)$ preserving a non-degenerate alternating form on $F_q$, for instance $<\vec{x},\vec{y}> = \sum_{i=1}^{n} (x_{2i-1} y_{2i} - x_{2i} y_{2i-1})$.

Given a polynomial $\phi$ with coefficients in $F_q$ and non-vanishing constant term, define a polynomial $\tilde{\phi}$ by:

$$\tilde{\phi} = z^{m_{\phi}} \phi\left(\frac{1}{z}\right)$$
where \( \phi^q \) raises each coefficient of \( \phi \) to the \( q \)th power. Explicitly, a polynomial \( \phi(z) = z^m \phi + \alpha_m \phi^{-1} + \cdots + \alpha_1 z + \alpha_0 \) with \( \alpha_0 \neq 0 \) is sent to \( \bar{\phi}(z) = z^m \phi + \left( \frac{\alpha_m}{\alpha_0} \right)^q z + \left( \frac{\alpha_1}{\alpha_0} \right)^q. \) (The notation \( \bar{\phi} \) breaks from Wall \footnote{Wall [28], in which \( \tilde{\phi} \) was used, but these maps are different. Namely \( \tilde{\phi} \) is defined on polynomials with coefficients in \( F_q \), but \( \bar{\phi} \) is defined on polynomials with coefficients in \( F_{q^2} \)). Fulman \footnote{Fulman [9] showed that all irreducible polynomials such that \( \phi = \bar{\phi} \) have even degree, except for the polynomials \( z \pm 1 \).} Wall [28] proved that a conjugacy class of \( \text{Sp}(2n, q) \) corresponds to the following data. To each monic, non-constant, irreducible polynomial \( \phi \neq z \pm 1 \) associate a partition \( \lambda_\phi \) of some non-negative integer |\( \lambda_\phi \)|. To \( \phi \) equal to \( z - 1 \) or \( z + 1 \) associate a symplectic signed partition \( \lambda_{\pm}^\phi \), by which is meant a partition of some natural number \( |\lambda_{\pm}^\phi| \) such that the odd parts have even multiplicity, together with a choice of sign for the set of parts of size \( i \) for each even \( i > 0 \).

Example of a Symplectic Signed Partition

\[
\begin{array}{cccccccccc}
+ & . & . & . & . & . & . & . & . & . \\
& & \ddots & & & & & & & \\
& & & & . & . & . & . & . & . \\
& & & & & & \ddots & & & \\
& & & & & & & & & . \\
& & & & & & & & & - \\
\end{array}
\]

Here the + corresponds to the parts of size 4 and the − corresponds to the parts of size 2. This data represents a conjugacy class of \( \text{Sp}(2n, q) \) if and only if:

\begin{enumerate}
  \item |\( \lambda_z \)| = 0
  \item \( \lambda_\phi = \lambda_{\tilde{\phi}} \)
  \item \( \sum_{\phi = z \pm 1} |\lambda_{\phi}^\pm| + \sum_{\phi \neq z \pm 1} |\lambda_\phi| m_\phi = 2n \)
\end{enumerate}

The symplectic groups can be used to define measures on partitions \( \lambda_\phi \) and symplectic signed partitions \( \lambda_{\pm}^{z \pm 1} \) as follows. Fix \( u \) so that \( 0 < u < 1 \) and pick the dimension with probability of dimension \( 2n \) equal to \( (1 - u^2)u^{2n} \). Then pick \( \alpha \) uniformly in \( \text{Sp}(2n, q) \) and let \( \lambda_\phi \) and \( \lambda_{\pm}^{z \pm 1} \) be the data corresponding to the conjugacy class of \( \alpha \).

Fulman \footnote{Fulman [9] uses Wall’s conjugacy class size formula and the fact that all polynomials other than \( z \pm 1 \) which are invariant under \( \bar{\phi} \) have even degree to prove the following result.} uses Wall’s conjugacy class size formula and the fact that all polynomials other than \( z \pm 1 \) which are invariant under \( \bar{\phi} \) have even degree to prove the following result.

**Theorem 5** If \( \phi = \tilde{\phi} \) and \( \phi \neq z \pm 1 \), then \( \lambda_\phi \) has distribution
\[
P \cdot \frac{(-u)^m_\phi}{(-q)^{m_\phi}} \cdot \frac{1}{(-q)^{n(1)_m_\phi}} \cdot \frac{1}{(-q)^{m_\phi}}.
\]
If \( \phi \neq \tilde{\phi} \), then \( \lambda_\phi = \lambda_{\tilde{\phi}} \) have distribution
\[
P \cdot \frac{2^{m_\phi}}{q^{2m_\phi}} \cdot \frac{1}{(-q)^{2(n-1)_m_\phi}} \cdot \frac{1}{(-q)^{2m_\phi}}.
\]
These random partitions are independent and as with \( GL \), the case \( u = 1 \) corresponds to the \( n \rightarrow \infty \) limit.

The distribution of the symplectic signed partitions \( \lambda_{z \pm 1}^\phi \) is more elusive (see Fulman \footnote{Fulman [9] for some results.} for some results.
3. **Orthogonal Groups** For simplicity assume that the characteristic of $F_q$ is not equal to 2. The orthogonal groups can be defined as subgroups of $GL(n,q)$ preserving a non-degenerate symmetric bilinear form. For $n = 2l + 1$ odd, there are two such forms up to isomorphism, with inner product matrices $A$ and $\delta A$, where $\delta$ is a non-square in $F_q$ and $A$ is equal to:

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & l & I_l \\
0 & I_l & 0
\end{pmatrix}
$$

Denote the corresponding orthogonal groups by $O^+(2l+1, q)$ and $O^-(2l+1, q)$. This distinction will be useful, even though these groups are isomorphic.

For $n = 2l$ even, there are again two non-degenerate symmetric bilinear forms up to isomorphism with inner product matrices:

$$
\begin{pmatrix}
0 & I_l \\
I_l & 0
\end{pmatrix}
\quad \quad \quad
\begin{pmatrix}
0_{l-1} & I_{l-1} & 0 & 0 \\
I_{l-1} & 0_{l-1} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -\delta
\end{pmatrix}
$$

where $\delta$ is a non-square in $F_q$. Denote the corresponding orthogonal groups by $O^+(2l, q)$ and $O^-(2l, q)$.

To describe the conjugacy classes of the finite orthogonal groups, it is necessary to use the notion of the Witt type of a non-degenerate quadratic form, as in Chapter 9 of Bourbaki. Call a non-degenerate form $N$ null if the vector space $V$ on which it acts can be written as a direct sum of 2 totally isotropic subspaces (a totally isotropic space is one on which the inner product vanishes identically). Define two non-degenerate quadratic forms $Q'$ and $Q$ to be equivalent if $Q'$ is isomorphic to the direct sum of $Q$ and a null $N$. The Witt type of $Q$ is the equivalence class of $Q$ under this equivalence relation. There are 4 Witt types over $F_q$, which Wall denotes by $0, 1, \delta, \omega$, corresponding to the forms $0, x^2, \delta x^2, x^2 - \delta y^2$ where $\delta$ is a fixed non-square of $F_q$. These 4 Witt types form a ring, but only the additive structure is relevant here. The sum of two Witt types with representatives $Q_1, Q_2$ on $V_1, V_2$ is the equivalence class of $Q_1 + Q_2$ on $V_1 + V_2$. It is easy to see that the four orthogonal groups $O^+(2n+1, q), O^-(2n+1, q), O^+(2n, q), O^-(2n, q)$ arise from forms $Q$ of Witt types $1, \delta, 0, \omega$ respectively.

Consider the following combinatorial data. To each monic, non-constant, irreducible polynomial $\phi \neq z \pm 1$ associate a partition $\lambda_\phi$ of some non-negative integer $|\lambda_\phi|$. To $\phi$ equal to $z - 1$ or $z + 1$ associate an orthogonal signed partition $\lambda_\phi^\pm$, by which is meant a partition of some natural number $|\lambda_\phi^\pm|$ such that all even parts have even multiplicity, and all odd $i > 0$ have a choice of sign. For $\phi = z - 1$ or $\phi = z + 1$ and odd $i > 0$, we denote by $\Theta_i(\lambda_\phi^\pm)$ the Witt type of the orthogonal group on a vector space of dimension $m_i(\lambda_\phi^\pm)$ and sign the choice of sign for $i$.
Example of an Orthogonal Signed Partition

\[
\begin{array}{ccccccc}
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
\end{array}
\]

Here the $-$ corresponds to the part of size 3 and the $+$ corresponds to the parts of size 1.

The following theorem, though not stated there, is implicit in the discussion on pages 38-40 of Wall [28]. The polynomial $\tilde{\phi}$ is defined as for the symplectic groups.

**Theorem 6** The data $\lambda_{z-1}^\pm, \lambda_{z+1}^\pm, \lambda_\phi$ represents a conjugacy class of some orthogonal group if:

(a) $|\lambda_z| = 0$
(b) $\lambda_\phi = \lambda_{\tilde{\phi}}$
(c) $\sum_{\phi = z \pm 1} |\lambda_\phi^\pm| + \sum_{\phi \neq z \pm 1} |\lambda_\phi|m_\phi = n$

In this case, the data represents the conjugacy class of exactly one orthogonal group $O(n,q)$, with sign determined by the condition that the group arises as the stabilizer of a form of Witt type:

\[
\sum_{\phi = z \pm 1 \ i \ \text{odd}} \Theta_i(\lambda_\phi^\pm) + \sum_{\phi \neq z \pm 1 \ i \geq 1} \sum_{m_\phi} i m_i(\lambda_\phi)\omega
\]

The definition of measures on partitions for the tower $O(n,q)$ differs from that of the other groups. For $0 < u < 1$, pick an integer $n$ with the probability of $n = 0$ equal to $\frac{1-u}{1+u}$ and probability of $n \geq 1$ equal to $\frac{1-u}{1+u}2u^n$. If $n \geq 1$, choose $O^+(n,q)$ or $O^-(n,q)$ with probability $\frac{1}{2}$ and then choose within that group uniformly. This defines random orthogonal signed partitions $\lambda_{z-1}^\pm, \lambda_{z+1}^\pm$ and random partitions $\lambda_\phi$ for $\phi \neq z \pm 1$. If $\phi \neq z \pm 1$, the random partitions $\lambda_\phi$ have the same distribution as for the symplectic groups. The orthogonal signed partitions are again elusive.

8 Hall-Littlewood Polynomials: The Young Tableau Algorithm

This section, as the previous, studies the measures $P_{x,y,q,t}$ with $q = 0, y_i = t^{i-1}$, and $x = ut^i$. We also set $t = \frac{1}{q}$ where $q$, different from the $q$ above, is the size of a finite field. Section 7 showed that this is the case relevant to the finite classical groups. As will emerge, the algorithm in this section is quite different from the simplified algorithm of Section 6 which works by adding horizontal strips.

Recall that a standard Young tableau $T$ of size $n$ is a partition of $n$ with each box containing one of $\{1, \ldots, n\}$ such that each of $\{1, \ldots, n\}$ appears exactly once and the numbers increase in each row and column of $T$. For instance,
is a standard Young tableau. We call the algorithm in this section the Young Tableau Algorithm because numbering the boxes in the order in which they are created gives a standard Young tableau. It is assumed that $0 < u < 1$ and $q > 1$.

The Young Tableau Algorithm

**Step 0** Start with $N = 1$ and $\lambda$ the empty partition. Also start with a collection of coins indexed by the natural numbers, such that coin $i$ has probability $\frac{u}{q^i}$ of heads and probability $1 - \frac{u}{q^i}$ of tails.

**Step 1** Flip coin $N$.

**Step 2a** If coin $N$ comes up tails, leave $\lambda$ unchanged, set $N = N + 1$ and go to Step 1.

**Step 2b** If coin $N$ comes up heads, choose an integer $S > 0$ according to the following rule. Set $S = 1$ with probability $\frac{q^N - \lambda_1'}{q^N - 1}$. Set $S = s > 1$ with probability $\frac{q^N - \lambda_s' - q^N - \lambda_{s-1}'}{q^N - 1}$. Then increase the size of column $s$ of $\lambda$ by 1 and go to Step 1.

Note that as with the previous algorithms, this algorithm halts by the Borel-Cantelli lemmas.

Let us now look at the same example as in Section 6, so as to see that the Young Tableau Algorithm is quite different from the simplified algorithm for the Hall-Littlewood polynomials.

So suppose we are at Step 1 with $\lambda$ equal to the following partition:

```
. .
. .
```

Suppose also that $N = 4$ and that coin 4 had already come up heads once, at which time we added to column 1, giving $\lambda$. Now we flip coin 4 again and get heads, going to Step 2b. We add to column 1 with probability $\frac{q^4 - 1}{q^4 - 1}$, to column 2 with probability $\frac{q^3 - q^2}{q^4 - 1}$, to column 3 with probability $\frac{q^2 - q}{q^4 - 1}$, to column 4 with probability 0, and to column 5 with probability $\frac{q^4 - q^3}{q^4 - 1}$. We then return to Step 1.

Note that there is a non-0 probability of adding to column 1, and that the dots added during the toss of a given coin need not form a horizontal strip. This contrasts sharply with the algorithm in Section 6.

We use the notation that $(x)_N = (1 - x)(1 - \frac{x}{q})\cdots(1 - \frac{x}{q^{N-1}})$. Recall from Section 8 that $m_i(\lambda)$ is the number of parts of $\lambda$ of size $i$, that $n(\lambda) = \sum i(i - 1)\lambda_i$, that $a(s)$ is the number of squares in $\lambda$ to the east of $s$, and that $l(s)$ is the number of squares in $\lambda$ to the south of $s$. Lemma 5 gives a formula for the truncated measure $P^N_{\frac{1}{q^N}, \frac{1}{q^N}, 0, \frac{1}{q}}$ in terms of this notation.

**Lemma 5** $P^N_{\frac{1}{q^N}, \frac{1}{q^N}, 0, \frac{1}{q}}(\lambda) = 0$ if $\lambda$ has more than $N$ parts. Otherwise:

$$P^N_{\frac{1}{q^N}, \frac{1}{q^N}, 0, \frac{1}{q}}(\lambda) = \frac{u^{\lambda_1}(\frac{u}{q})N(\frac{1}{q})N}{(\frac{1}{q})N - \lambda_1'} \prod_{i \geq 1} \frac{1}{q^{(\lambda_i')^2} \frac{1}{q^{(\lambda_i')^2}} m_i(\lambda)}$$
Proof: The first statement is clear from the definition of the measure $P^N_{\frac{1}{q^2}, \frac{1}{q^{1-r}}, 0, \frac{1}{q}}$ in Section 4. The second equality can be deduced from the definition of $P^N_{\frac{1}{q^2}, \frac{1}{q^{1-r}}, 0, \frac{1}{q}}$ and the Principal Specialization Formula as follows:

$$P^N_{\frac{1}{q^2}, \frac{1}{q^{1-r}}, 0, \frac{1}{q}}(\lambda) = \frac{P_{\lambda}(\frac{u}{q}, \ldots, \frac{u}{q}, 0, \ldots; 0, \frac{1}{q}) P_{\lambda}(\frac{1}{q^{1-r}}; 0, \frac{1}{q}) b_{\lambda}(0, t)}{\prod_{i=1}^n \frac{(1 - \frac{u}{q^2}) P_{\lambda}(\frac{u}{q}, \ldots, \frac{u}{q}, 0, \ldots; 0, \frac{1}{q}) P_{\lambda}(\frac{1}{q^{1-r}}; 0, \frac{1}{q}) b_{\lambda}(0, t)}{q_n(\lambda)}}$$

$$= \frac{\prod_{i=1}^N (1 - \frac{u}{q^2}) P_{\lambda}(\frac{u}{q}, \ldots, \frac{u}{q}, 0, \ldots; 0, \frac{1}{q}) P_{\lambda}(\frac{1}{q^{1-r}}; 0, \frac{1}{q}) b_{\lambda}(0, t)}{\prod_{i=1}^{N - \lambda_1'} (1 - \frac{1}{q^2}) q_n(\lambda)}$$

$$= \frac{\prod_{i=1}^{N - \lambda_1'} (1 - \frac{1}{q^2}) q_n(\lambda)}{q^{n(\lambda)}}$$

$$= \frac{\prod_{i=1}^{N - \lambda_1'} (1 - \frac{1}{q^2}) q_n(\lambda)}{q^{n(\lambda)}}$$

$$= \frac{\prod_{i=1}^{N - \lambda_1'} (1 - \frac{1}{q^2}) q_n(\lambda)}{q^{n(\lambda)}}$$

$$= \frac{\prod_{i=1}^{N - \lambda_1'} (1 - \frac{1}{q^2}) q_n(\lambda)}{q^{n(\lambda)}}$$

$$= \frac{\prod_{i=1}^{N - \lambda_1'} (1 - \frac{1}{q^2}) q_n(\lambda)}{q^{n(\lambda)}}$$

where the last equality uses the fact that $n(\lambda) = \sum_i (\lambda_i^')$. □

**Theorem 7** The chance that the Young Tableau algorithm yields $\lambda$ at the end of interval $N$ is $P^N_{\frac{1}{q^2}, \frac{1}{q^{1-r}}, 0, \frac{1}{q}}(\lambda)$.

Proof: The theorem is clear if $N < \lambda_1'$ for then $P^N_{\frac{1}{q^2}, \frac{1}{q^{1-r}}, 0, \frac{1}{q}}(\lambda) = 0$, and Step 2b does not permit the number of parts of the partition to exceed the number of the coin being tossed at any stage in the algorithm.

For the case $N \geq \lambda_1'$, use induction on $|\lambda| + N$. The base case is that $\lambda$ is the empty partition. This means that coins 1, 2, ⋅⋅⋅, $N$ all came up tails on their first tosses, which occurs with probability $(\frac{u}{q})^N$. So the base case checks.

Let $s_1 \leq s_2 \leq \cdots \leq s_k$ be the columns of $\lambda$ with the property that changing $\lambda$ by decreasing the size of one of these columns by 1 gives a partition $\lambda^*$. It then suffices to check that the claimed formula for $P^N_{\frac{1}{q^2}, \frac{1}{q^{1-r}}, 0, \frac{1}{q}}(\lambda)$ satisfies the equation:

$$P^N_{\frac{1}{q^2}, \frac{1}{q^{1-r}}, 0, \frac{1}{q}}(\lambda) = (1 - \frac{u}{q^2}) P^{N-1}_{\frac{1}{q^2}, \frac{1}{q^{1-r}}, 0, \frac{1}{q}}(\lambda) + \frac{u}{q^N} q^{N - \lambda_1'} - 1 \frac{P^N_{\frac{1}{q^2}, \frac{1}{q^{1-r}}, 0, \frac{1}{q}}(\lambda^*)}{q^N - 1}$$

$$+ \sum_{s_i > 1} \frac{u}{q^N} q^{N - \lambda_1' + 1} - q^{N - \lambda_1'} - 1 \frac{P^N_{\frac{1}{q^2}, \frac{1}{q^{1-r}}, 0, \frac{1}{q}}(\lambda^*)}{q^N - 1}$$

This equation is based on the following logic. Suppose that when coin $N$ came up tails, the algorithm gave the partition $\lambda$. If coin $N$ came up tails on its first toss, then we must have had $\lambda$ when coin $N - 1$ came up tails. Otherwise, for each $s_i$ we add the probability that “The algorithm
gave the partition $\lambda^{s_i}$ on the penultimate toss of coin $N$ and the partition $\lambda$ on the last toss of coin $N''$. It is not hard to see that this probability is equal to the probability of getting $\lambda^{s_i}$ on the final toss of coin $N$, multiplied by the chance of a heads on coin $N$ which then gives the partition $\lambda$ from $\lambda^{s_i}$.

We divide both sides of this equation by $P_N^{\frac{u}{q^{N-1}}, \frac{1}{q}}(\lambda)$ and show that the terms on the righthand side sum to 1. First consider the terms with $s_i > 1$. Induction gives that:

$$\sum_{s_i > 1} \frac{u}{q^{N-1}} \frac{q^{N-\lambda_i^{s_i}+1} - q^{N-\lambda_i^{s_i-1}}}{q^{N-1}} P_N^{\frac{u}{q^{N-1}}, \frac{1}{q}}(\lambda^{s_i}) \frac{1}{q}$$

$$= \sum_{s_i > 1} \frac{q^{-\lambda_i^{s_i}+1} - q^{-\lambda_i^{s_i-1}}}{q^{N-1}} q^{(\lambda_i^{s_i})_2} \left( \frac{1}{q}\right)_{\lambda_i^{s_i-1} - \lambda_i^{s_i}} \left( \frac{1}{q}\right)_{\lambda_i^{s_i} - \lambda_i^{s_i+1}}$$

$$= \sum_{s_i > 1} \frac{q^{-\lambda_i^{s_i}+1} - q^{-\lambda_i^{s_i-1}}}{q^{N-1}} q^{2\lambda_i^{s_i}-1} \left(1 - \frac{1}{q^{\lambda_i^{s_i}}}ight)$$

$$= \sum_{s_i > 1} \frac{q^{\lambda_i^{s_i}} - q^{\lambda_i^{s_i+1}}}{q^{N-1}}$$

$$= \frac{q^{\lambda_2} - 1}{q^{N-1}}$$

Next consider the term coming from $P_N^{\frac{u}{q^{N-1}}, \frac{1}{q}}(\lambda^{s_1})$. If $N = \lambda_1$, then $\lambda_1 > N - 1$, so by what we have proven $P_N^{\frac{u}{q^{N-1}}, \frac{1}{q}}(\lambda^{s_1})$ is 0. Otherwise,

$$\left(1 - \frac{u}{q^N}\right) P_N^{\frac{u}{q^{N-1}}, \frac{1}{q}}(\lambda) = \left(1 - \frac{u}{q^N}\right) \left( \frac{u}{q} \right)_{N-1} \left( \frac{1}{q} \right)_{N-\lambda_i^{s_1}}$$

$$= \left(1 - \frac{1}{q^{N-\lambda_i^{s_1}}}ight)$$

$$= \frac{q^N - q^{\lambda_i^{s_1}}}{q^{N-1}}$$

So this term always contributes $\frac{q^{N-\lambda_2}}{q^{N-1}}$.

Finally, consider the term coming from $P_N^{\frac{u}{q^{N-1}}, \frac{1}{q}}(\lambda^1)$. This vanishes if $\lambda_1 = \lambda_2$ since then $\lambda^1$ is not a partition. Otherwise,

$$\frac{u}{q^N} \frac{q^{N-\lambda_1^{s_1}+1} - 1}{q^{N-1}} P_N^{\frac{u}{q^{N-1}}, \frac{1}{q}}(\lambda^{s_1}) \frac{1}{q} = \frac{q^{-\lambda_i^{s_1}+1} - q^{-N}}{q^{N-1}} \left( \frac{1}{q}\right)_{N-\lambda_i^{s_1} + 1} \left( \frac{1}{q}\right)_{\lambda_i^{s_1} - \lambda_i^{s_1+1}}$$

$$= \frac{q^{-\lambda_i^{s_1}+1} - q^{-N}}{q^{N-1}} \left(1 - \frac{1}{q^{\lambda_i^{s_1} - \lambda_i^{s_1+1}}}ight)$$

$$= \frac{q^{\lambda_i^{s_1} - \lambda_i^{s_1+1}}}{q^{N-1}}$$

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\[ q^N - 1 \]

So in all cases this term contributes \( \frac{q_1^{\lambda_1} - q_2^{\lambda_2}}{q^N - 1} \).

Adding up the three terms completes the proof. \( \square \)

As an example of Lemma 5 and Theorem 7, suppose that \( N = 4 \) and \( \lambda \) is the partition:

\[
\begin{array}{ccc}
& & 1 \\
& 2 & 4 \\
1 & 3 & 5 & 6 \\
8 & 9
\end{array}
\]

Then the chance that the Young Tableau Algorithm gives the partition \( \lambda \) when coin 4 comes up tails is:

\[
\frac{u^4(1 - \frac{u}{q})(1 - \frac{u}{q^2})(1 - \frac{u}{q^3})(1 - \frac{1}{q^4})}{q^{10}(1 - \frac{1}{q})^2}
\]

9 Hall-Littlewood Polynomials: Weights on the Young Lattice

In this section \( T \) denotes a standard Young tableau and \( \lambda \) denotes the partition corresponding to \( T \). Let \( |T| \) be the size of \( T \). As explained in Section 8, the Young Tableau algorithm constructs a standard Young tableau, and thus defines a measure on the set of all standard Young tableaux.

Let \( P_{q, q^{-1}, 0, \frac{1}{q}}(T) \) be the chance that the Young Tableau algorithm of Section 8 outputs \( T \), and let \( P_{q, q^{-1}, 0, \frac{1}{q}}(T) \) be the chance that it outputs \( T \) when coin \( N \) comes up tails.

We also introduce the following notation. Let \( T_{i,j} \) be the entry in the \((i, j)\) position of \( T \) (recall that \( i \) is the row number and \( j \) the column number). For \( j \geq 2 \), let \( A(i,j) \) be the number of entries \((i', j-1)\) such that \( T_{i', j-1} < T_{i,j} \). Let \( B(i,j) \) be the number of entries \((i', 1)\) such that \( T_{i', 1} < T_{i,j} \). For instance the tableau:

\[
\begin{array}{ccc}
1 & 3 & 5 & 6 \\
2 & 4 & 7 \\
8 & 9
\end{array}
\]

has \( T_{1,3} = 5 \). Also \( A_{(1,3)} = 2 \) because there are 2 entries in column \( 3 - 1 = 2 \) which are less than 5 (namely 3 and 4). Finally, \( B_{(1,3)} = 2 \) because there are 2 entries in column 1 which are less than 5 (namely 1 and 2).

There is a simple formula for \( P_{q, q^{-1}, 0, \frac{1}{q}}(T) \) in terms of this data.

**Theorem 8** \( P_{q, q^{-1}, 0, \frac{1}{q}}(T) = 0 \) if \( T \) has greater than \( N \) parts. Otherwise:

\[
P_{q, q^{-1}, 0, \frac{1}{q}}(T) = \frac{u^{|T|}}{|GL(\lambda', q)|} \prod_{r=1}^{N}(1 - \frac{u}{q^r})(1 - \frac{1}{q^r}) \prod_{(i,j) \in \lambda, j \geq 2} q^{1-i} - q^{-A(i,j)} \frac{q^{B(i,j)} - 1}{q^{B(i,j)}}
\]

**Proof:** The case where \( T \) has more than \( N \) parts is proven as in Theorem 5.
The case \( \lambda_1 \leq N \) is proven by induction on \(|T| + N\). If \(|T| + N = 1\), then \(T\) is the empty tableau and \(N = 1\). This means that coin 1 in the Tableau algorithm came up tails on the first toss, which happens with probability \(1 - \frac{u}{q}\). So the base case checks.

For the induction step, there are two cases. The first case is that the largest entry in \(T\) occurs in column \(s > 1\). Removing the largest entry from \(T\) gives a tableau \(T_s\). We have the equation:

\[
P_{\frac{u}{q^t}, \frac{1}{q^{t-t}}, 0, \frac{1}{q}}^{N} (T) = (1 - \frac{u}{q^N})P_{\frac{u}{q^t}, \frac{1}{q^{t-t}}, 0, \frac{1}{q}}^{N-1} (T) + \frac{u}{q^N} \frac{q^{N-\lambda_s+1} - q^{N-\lambda_{s-1}}}{q^N - 1} P_{\frac{u}{q^t}, \frac{1}{q^{t-t}}, 0, \frac{1}{q}}^{N} (T^s)
\]

The two terms in this equation correspond to whether or not \(T\) was completed at time \(N\). We divide both sides of the equation by \(P_{\frac{u}{q^t}, \frac{1}{q^{t-t}}, 0, \frac{1}{q}}^{N} (T)\), substitute in the conjectured formula, and show that it satisfies this recurrence. The two terms on the right hand side then give:

\[
\frac{q^N - q^{\lambda_1}}{q^N - 1} + \frac{q^{-\lambda_s+1} - q^{-\lambda_{s-1}}}{q^N - 1} \frac{1}{q^{\lambda_1+1} - q^{\lambda_{s-1}}} = 1
\]

The other case is that the largest entry of \(T\) occurs in column 1. We then have the equation:

\[
P_{\frac{u}{q^t}, \frac{1}{q^{t-t}}, 0, \frac{1}{q}}^{N} (T) = (1 - \frac{u}{q^N})P_{\frac{u}{q^t}, \frac{1}{q^{t-t}}, 0, \frac{1}{q}}^{N-1} (T) + \frac{u}{q^N} \frac{q^{N-\lambda_1+1} - 1}{q^N - 1} P_{\frac{u}{q^t}, \frac{1}{q^{t-t}}, 0, \frac{1}{q}}^{N} (T^1)
\]

As in the previous case, we divide both sides of the equation by \(P_{\frac{u}{q^t}, \frac{1}{q^{t-t}}, 0, \frac{1}{q}}^{N} (T)\), substitute in the conjectured formula, and show that it satisfies this recurrence. The two terms on the right hand side then give:

\[
\frac{q^N - q^{\lambda_1}}{q^N - 1} + \frac{1}{q^N} \frac{q^{N-\lambda_s+1} - 1}{q^N - 1} \frac{1}{(1 - \frac{1}{q^{\lambda_1+1}})} \frac{|GL(\lambda_1, q)|}{|GL(\lambda_1 - 1, q)|} = 1
\]

This completes the induction, and the proof of the theorem. \(\Box\)

For instance, Theorem 8 says that if

\[
S = \frac{u^4(1 - \frac{u}{q})(1 - \frac{u}{q^2})(1 - \frac{u}{q^3})(1 - \frac{u}{q^4})(1 - \frac{1}{q})(1 - \frac{1}{q^2})(1 - \frac{1}{q^3})}{|GL(3, q)|}
\]

then the chances that the Young Tableau Algorithm gives the following tableaux:

\[
\begin{align*}
1 & \quad 2 \\
3 & \\
4 & \\
1 & \quad 3 \\
2 & \\
4 & \\
1 & \quad 4 \\
2 & \\
3 & 
\end{align*}
\]
when coin 4 comes up tails are \( \frac{S}{q} \), \( \frac{S}{q^2} \), and \( \frac{S}{q^3} \) respectively. Note that the sum of these probabilities is:

\[
\frac{u^4(1 - \frac{u}{q})(1 - \frac{u}{q^2})(1 - \frac{u}{q^3})(1 - \frac{1}{q^4})(1 - \frac{1}{q^5})}{q^{10}(1 - \frac{1}{q})^2}
\]

As must be the case and as was proved at the end of Section 8, this quantity is also equal to the chance that the Young Tableau Algorithm gives the partition:

. . .

An interesting object in combinatorics is the Young lattice. The elements of this lattice are all partitions of all numbers. An edge is drawn between partitions \( \lambda \) and \( \Lambda \) if \( \Lambda \) is obtained from \( \lambda \) by adding one box. Note that a standard Young tableau \( T \) of shape \( \lambda \) is equivalent to a path in the Young lattice from the empty partition to \( \lambda \). This equivalence is given by growing the partition \( \lambda \) by adding boxes in the order 1, \( \cdots \), \( n \) in the positions determined by \( T \). For instance the tableau:

\[
\begin{array}{ccc}
1 & 3 & 4 \\
2 & & \\
\end{array}
\]

corresponds to the path:

\[
\emptyset \rightarrow \ldots \rightarrow \ldots \rightarrow \ldots \rightarrow \ldots
\]

The measure \( P_{u^4, \frac{1}{q^3} - \frac{1}{q^3}}(T) \) on standard Young tableaux has the following description in terms of weights on the Young lattice.

**Corollary 4** Put weights \( m_{\lambda, \Lambda} \) on the Young lattice according to the rules:

1. \( m_{\lambda, \Lambda} = \frac{u}{q^{\lambda_1}(q^{\lambda_1+1} - 1)} \) if \( \Lambda \) is obtained from \( \lambda \) by adding a box to column 1

2. \( m_{\lambda, \Lambda} = \frac{u(q^{-\lambda_s} - q^{-\lambda_s-1})}{q^{\lambda_1 - 1}} \) if \( \Lambda \) is obtained from \( \lambda \) by adding a box to column \( s > 1 \)

Then the chance that the Tableau algorithm produces \( T \) is equal to:

\[
\prod_{r=1}^{\infty} (1 - \frac{u}{q^r}) \prod_{i=0}^{[T]-1} m_{\gamma_i, \gamma_{i+1}}
\]

where the \( \gamma_i \) are the partitions in the path along the Young lattice which corresponds to the tableau \( T \).

**Proof:** This follows by letting \( N \to \infty \) in Theorem 3 and the fact that \( T \) corresponds to a unique path in the Young lattice. \( \square \)

The following remarks may be of interest.
1. Note that the total weight out of the empty partition is \( \frac{u}{q-1} \) and that the total weight out of any other partition \( \lambda \) is:

\[
\frac{u}{q^{\lambda_1}(q^{\lambda_1+1} - 1)} + \sum_{i \geq 2} \frac{u(q^{-\lambda_1'} - q^{-\lambda_1'-1})}{q^{\lambda_i} - 1} = \frac{u}{q^{\lambda_1}(q^{\lambda_1+1} - 1)} + \frac{u}{q^{\lambda_1'}} = \frac{uq}{q^{\lambda_1+1} - 1} < 1
\]

Since the sum of the weights out of a partition \( \lambda \) to a larger partition \( \Lambda \) is less than 1, the weights can also be viewed as transition probabilities, provided that one allows for halting.

2. Note that the Young Tableau Algorithm of Section 8 for growing \( \lambda \phi \) according to the group theoretic measures of Section 8 does not carry over to unitary case if \( \phi = \tilde{\phi} \), since then some of the probabilities involved would be negative. The description in terms of weights on the Young lattice in Corollary 4, however, does extend to the unitary groups. The weight formula should be altered as follows. In the case \( \phi = \tilde{\phi} \) one replaces the variables \((u, q)\) by \((-u, -q)\), and in the case \( \phi \neq \tilde{\phi} \) one replaces the variables \((u, q)\) by \((u^2, q^2)\).

3. Some applications of the results of this and the preceding section toward proving group theoretic theorems can be found in the companion paper by Fulman [10].

10 Schur Functions: A \( q \)-analog of the Plancherel Measure of the Symmetric Group

To begin, let us recall the definition of the Plancherel measure of the symmetric group. This is a measure on the partitions \( \lambda \) of size \( n \). Letting \( h(s) = a(s) + l(s) + 1 \) be the hook-length of \( s \in \lambda \), the Plancherel measure assigns to \( \lambda \) the probability \( \prod_{s \in \lambda} \frac{1}{h(s)^2} \). Kerov and Vershik [19], [26], [27] have studied Plancherel measure extensively. The connection with the representation theory of the symmetric group is that the irreducible representations of \( S_n \) can be parameterized by partitions \( \lambda \) of \( n \) such that the representation corresponding to \( \lambda \) has dimension \( \prod_{s \in \lambda} \frac{1}{h(s)} \) (see pages 53-96 of Sagan [23]).

Plancherel measure has another description. Robinson and Schensted found a bijection from the symmetric group to the set of pairs \((P, Q)\) of standard Young tableau of the same shape (see pages 97-101 of Sagan [23] for details). Call the shape associated to \( \pi \) under the Robinson-Schensted correspondence \( \lambda(\pi) \). Then \( \lambda(\pi) \) has Plancherel measure if \( \pi \) is chosen uniformly from the symmetric group. This follows from the fact that the dimension of the irreducible representation of \( S_n \) corresponding to \( \lambda \) is the number of standard tableaux of shape \( \lambda \).

Let us now see how the measures \( P_{x,y,q,t} \) of Section 4 lead to a \( q \)-analog of Plancherel measure. This section studies the specialization \( x_i = t^i, y_i = t^{i-1}, q = t \). We then set \( t = \frac{1}{q} \), where this \( q \) is the size of a finite field. Lemma 3 gives a formula for the measure \( P^N_{x,y,q,t} \). We use the notation that \( (x)_N = (1-x)(1-\frac{x}{q}) \cdots (1-\frac{x}{q^{N-1}}) \). Let \( c(s) = a'(s) - l'(s) \) denote the content of \( s \in \lambda \) (here \( l'(s), l(s), a(s) \), and \( a'(s) \) are the number of squares in \( \lambda \) to the north, south, east, and west of \( s \) respectively).
Lemma 6

\[ P_N^{1/q^t \cdot \frac{1}{q^{t-1}} \cdot \frac{1}{q}} \left( \lambda \right) = \left[ \prod_{r=1}^{N} \prod_{t=0}^{\infty} \left( 1 - \frac{1}{q^{t+\tau}} \right) \right] \frac{1}{q^{2n(\lambda) + |\lambda|}} \prod_{s \in \lambda} \frac{1 - \frac{1}{q^{N+c(s)}}}{1 - \frac{1}{q^{h(s)}}}^{2} \]

Proof: This can be deduced from the definition of the measure \( P_N^{1/q^t \cdot \frac{1}{q^{t-1}} \cdot \frac{1}{q}} \) and the Principal Specialization Formula as follows:

\[
P_N^{1/q^t \cdot \frac{1}{q^{t-1}} \cdot \frac{1}{q}} = \frac{P_\lambda\left(\frac{1}{q}, \ldots, \frac{1}{q} \cdot 0, \ldots, \frac{1}{q} \cdot 1, \frac{1}{q} \cdot \frac{1}{q} \right)}{\prod_{r=1}^{N} \prod_{t=0}^{\infty} \left( 1 - \frac{1}{q^{t+\tau}} \right)} P_{\lambda}(\frac{1}{q}, \ldots, \frac{1}{q}, 0, \ldots, \frac{1}{q}, \frac{1}{q}) \\
= \frac{\prod_{s \in \lambda} \left( 1 - \frac{1}{q^{n(s)}} \right) P_{\lambda}(\frac{1}{q}, \ldots, \frac{1}{q}, 0, \ldots, \frac{1}{q}, \frac{1}{q})}{\prod_{s \in \lambda} \left( 1 - \frac{1}{q^{n(s)}} \right) \frac{1 - \frac{1}{q^{N+c(s)}}}{1 - \frac{1}{q^{h(s)}}}^{2}} \\
= \frac{\prod_{s \in \lambda} \left( 1 - \frac{1}{q^{n(s)}} \right) P_{\lambda}(\frac{1}{q}, \ldots, \frac{1}{q}, 0, \ldots, \frac{1}{q}, \frac{1}{q})}{\prod_{s \in \lambda} \left( 1 - \frac{1}{q^{n(s)}} \right) \frac{1 - \frac{1}{q^{N+c(s)}}}{1 - \frac{1}{q^{h(s)}}}^{2}} \\
\]

Renormalizing the measure \( P_N^{1/q^t \cdot \frac{1}{q^{t-1}} \cdot \frac{1}{q}} \) to live on partitions of size \( n \) will give a \( q \)-analog of the Plancherel measure. To this end, we introduce polynomials \( J_n(q) \). First define \( J_\lambda(q) \) by:

\[ J_\lambda(q) = \frac{q^{(|\lambda|)^2 - |\lambda| - 2n(\lambda)}([\lambda]! \frac{1}{q^{|\lambda|}})^2}{\prod_{s \in \lambda} \left( 1 - \frac{1}{q^{n(s)}} \right)^2} \]

The measure \( P_N^{1/q^t \cdot \frac{1}{q^{t-1}} \cdot \frac{1}{q}} \) can then be written as:

\[ P_N^{1/q^t \cdot \frac{1}{q^{t-1}} \cdot \frac{1}{q}}(\lambda) = \left[ \prod_{r=1}^{\infty} \prod_{t=0}^{\infty} \left( 1 - \frac{1}{q^{t+\tau}} \right) \right] J_\lambda(q) \frac{J_\lambda(q)}{J_n(q)} \]

It is not clear that the \( J_\lambda(q) \) are polynomials in \( q \), but this will turn out to be true. Define \( J_n(q) = \sum_{\lambda:|\lambda|=n} J_\lambda(q) \) and \( J_0(q) = 1 \). Proposition B which follows immediately from the definitions in this section, explains why one might be interested in the polynomials \( J_\lambda(q) \) and \( J_n(q) \).

Proposition 1 Under the measure \( P_N^{1/q^t \cdot \frac{1}{q^{t-1}} \cdot \frac{1}{q}} \), the conditional probability of \( \lambda \) given that \( |\lambda| = n \) is equal to \( \frac{J_\lambda(q)}{J_n(q)} \).

Lemma 6 is an easy exercise from page 11 of Macdonald [21] and will be useful.

Lemma 7

\[ \sum_{s \in \lambda} h(s) = n(\lambda) + n(\lambda') + |\lambda| \]

It is possible to relate the polynomials \( J_\lambda(q) \) to the Kostka-Foulkes polynomials \( K_\lambda(q) \) (sometimes denoted \( K_{\lambda(1^n)}(q) \)). The Kostka-Foulkes polynomials are defined as:

\[ K_\lambda(q) = \frac{q^{n(\lambda)}[|\lambda|]!}{\prod_{s \in \lambda} |h(s)|} \]
where \([n] = 1+q+\cdots+q^{i-1}\), the \(q\)-analog of the number \(n\). One can also check from Chapter 4 of Macdonald \([21]\) that \(K_{\lambda'}(q)\) is the degree of the unipotent representation of \(GL(n, q)\) corresponding to the partition \(\lambda'\).

Proposition 3 connects the \(J_{\lambda}(q)\) to the Kostka-Foulkes polynomials.

**Proposition 2** \(J_{\lambda}(q) = [K_{\lambda'}(q)]^2\)

**Proof:** Using Lemma 7, observe that:

\[
J_{\lambda}(q) = \frac{q^{|\lambda|^2 - |\lambda| - 2n(\lambda)} [\frac{1}{q^{|\lambda|}}]^2}{\prod_{s \in \lambda} (1 - \frac{1}{q^{h(s)}})^2} = q^{|\lambda|^2 - |\lambda| - 2n(\lambda)} \prod_{s \in \lambda} (q^{h(s)} - 1)^2 \prod_{i=1}^{[|\lambda|]} (q^i - 1)^2 q^{|\lambda|^2 + |\lambda|} = q^{2n(\lambda')} \prod_{s \in \lambda'} [h(s)]^2 = K_{\lambda'}(q)^2
\]

\(\square\)

Theorem 9 gives some properties of the \(J_n(q)\). By the remark before Proposition 2, \(J_n(q)\) is the sum of the squares of the degrees of the irreducible unipotent representations of \(GL(n, q)\). Recall that \([u^n]f(u)\) means the coefficient of \(u^n\) in \(f(u)\).

**Theorem 9**

1. \(J_n(q)\) is a symmetric polynomial of degree \(2\binom{n}{2}\) which has non-negative integer coefficients and satisfies \(J_n(1) = n!\).

2. \(\frac{J_n(q)}{q^{n^2}(1-\frac{1}{q^2})^2(1-\frac{1}{q^n})^2} = [u^n] \prod_{r=1}^{\infty} \prod_{s=-\infty}^{\infty} \frac{1}{1 - q^{r+s+1}}\)

**Proof:** Proposition 2 shows that \(J_n(q)\) is a polynomial with non-negative integer coefficients. Note by Lemma 7 that:

\[
\text{deg}(J_{\lambda}) = 2\text{deg}(K_{\lambda'}) = 2[n(\lambda') + \left(\frac{|\lambda| + 1}{2}\right) - \sum_{s \in \lambda'} h(s)] = 2 \left(\frac{|\lambda|}{2}\right) - 2n(\lambda)
\]

Thus \(J_{\lambda}\) has degree \(2\binom{|\lambda|}{2}\) for \(\lambda = (|\lambda|)\) and smaller degree for all other \(\lambda\). So \(J_n(q)\) has degree \(2\binom{n}{2}\). Symmetry means that \(J_n(q) = q^{2\binom{n}{2}} J_n(\frac{1}{q})\). In fact \(J_{\lambda}(q) + J_{\lambda'}(q)\) satisfies this property, by Lemma 7.

To see that \(J_n(1) = n!\), observe that:
\[ J_n(1) = \sum_{\lambda \vdash n} [K_{\lambda'}(1)]^2 \]
\[ = \sum_{\lambda \vdash n} [K_{\lambda}(1)]^2 \]
\[ = \sum_{\lambda \vdash n} \left[ \frac{n!}{\prod_{s \in \lambda} h(x)} \right]^2 \]
\[ = n! \]

For the second part of the theorem, it is useful to consider the measure \( P_{\frac{u}{q^1}, \frac{1}{q^1}, \frac{1}{q^1}, \frac{1}{q^1}} \). Arguing as in Lemma 3 shows that:

\[ P_{\frac{u}{q^1}, \frac{1}{q^1}, \frac{1}{q^1}, \frac{1}{q^1}}(\lambda) = \prod_{r=1}^{\infty} \prod_{t=0}^{\infty} \left( 1 - \frac{u}{q^{r+t}} \right) \frac{u^{[n]} J_\lambda(q)}{q^{[\lambda]}(1 - \frac{1}{q})^2 \cdots (1 - \frac{1}{q^n})^2} \]

The fact that this is a measure means that:

\[ \sum_{n=1}^{\infty} q^n J_n(q) \frac{u^n}{(1 - \frac{1}{q}) \cdots (1 - \frac{1}{q^n})^2} = \frac{1}{\prod_{r=0}^{\infty} \prod_{s=1}^{\infty} \left( 1 - \frac{u}{q^{r+s}} \right)} \]

Taking coefficients of \( u^n \) on both sides proves the second part. \( \square \)

Corollary 5 of Theorem 9 shows that conditioning the measure \( P_{\frac{u}{q^1}, \frac{1}{q^1}, \frac{1}{q^1}, \frac{1}{q^1}} \) on \( |\lambda| = n \) gives a \( q \)-analog of the Plancherel measure on partitions of size \( n \).

**Corollary 5**  The conditional probability of \( \lambda \) given that \( |\lambda| = n \) under the measure \( P_{\frac{u}{q^1}, \frac{1}{q^1}, \frac{1}{q^1}, \frac{1}{q^1}} \) reduces to the Plancherel measure of the symmetric group when one sets \( q = 1 \).

**Proof:** Proposition 1 shows that the conditional probability is \( \frac{J_{\lambda}(q)}{J_n(q)} \). The result follows from the definition of \( J_\lambda(q) \), and the fact that \( J_n(1) = n! \), which is part of the first statement of Theorem 9. \( \square \)

The following observations show that this \( q \)-analog of Plancherel measure has properties similar to the Plancherel measure of the symmetric group.

1. By Proposition 3 and the remark before it, our \( q \)-analog of Plancherel measure assigns a probability to \( \lambda \) which is proportional to the square of the degree of the unipotent representation of \( GL(n, q) \) parameterized by \( \lambda' \), the transpose partition. This is in direct analogy with the Plancherel measure of the symmetric group, which assigns a probability to \( \lambda \) which is proportional to the square of the degree of the irreducible representation of \( S_n \) parameterized by \( \lambda' \).

2. The description of the Plancherel measure of the symmetric group in terms of the Robinson-Schensted correspondence carries over to the above \( q \)-analog of Plancherel measure. To state this precisely recall that the major index of a permutation \( \pi \in S_n \) is defined by:

\[ maj(\pi) = \sum_{\substack{i + 1 \leq i \leq n - 1 \\pi(i) > \pi(i+1)}} i \]
Theorem 10 Choose $\pi \in S_n$ with probability proportional to $q^{\text{maj}(\pi) + \text{maj}(\pi^{-1})}$. Then $\lambda(\pi)'$, the transpose of the partition associated to $\pi$ through the Robinson-Schensted correspondence, has the $q$-analog of Plancherel measure defined in Corollary 3.

Proof: Define the major index of a standard Young tableau as the sum of the entries $i$ such that $i + 1$ is in a row below that of $i$. Reasoning similar to that of page 243 of Macdonald [21] shows that

$$K_{\lambda'}(q) = \sum_{T \in \text{SYT}(\lambda')} q^{\text{maj}(T)}$$

where the sum is over all standard Young tableaux of shape $\lambda'$.

From the way the Robinson-Schensted correspondence works (pages 97-101 of Sagan [23]), one sees that if $\pi$ corresponds to the pair $(P, Q)$, then $\text{maj}(\pi) = \text{maj}(Q)$. It is also known (Theorem 3.86 of Sagan [23]) that if $\pi$ corresponds to the pair $(P, Q)$, then $\pi^{-1}$ corresponds to the pair $(Q, P)$.

Proposition 2 and the Robinson-Schensted correspondence thus give that:

$$J_{\lambda}(q) = \left[ \sum_{T \in \text{SYT}(\lambda')} q^{\text{maj}(T)} \right]^2$$

$$= \sum_{(P, Q) \in \{\text{SYT}(\lambda') \times \text{SYT}(\lambda')\}} q^{\text{maj}(P)} q^{\text{maj}(Q)}$$

$$= \sum_{\pi \in S_n : \lambda(\pi) = \lambda'} q^{\text{maj}(\pi) + \text{maj}(\pi^{-1})}$$

\[\square\]

11 Schur Functions: A Comparison with Kerov’s $q$-analogs of Plancherel Measure and the Hook Walk

Kerov [18] has a $q$-analog of Plancherel measure which comes from the Schur functions. His $q$-analog of Plancherel measure is defined implicitly by means of a probabilistic algorithm called the $q$ hook walk. This walk starts with the empty partition, and adds a box at a time. The partition $\lambda$ grows to $\Lambda$ (here $|\Lambda| = |\lambda| + 1$) with probability:

$$q^n(\Lambda) \prod_{s \in \lambda} [h(s)] q^n(\lambda) \prod_{s \in \Lambda} [h(s)]$$

It can now be seen that Kerov’s $q$-analog of Plancherel measure is different from the $q$-analog introduced in Section 10, because the partition

$$\left( \frac{q^{\ell}}{q^{\ell+1}} \right)$$

has mass $\frac{1}{q^\ell}$ under Kerov’s $q$-analog of Plancherel measure and mass $\frac{q^{\ell}}{q^{\ell+1}}$ under our $q$-analog of Plancherel measure.

Proposition 3 relates Kerov’s $q$ hook walk to the algorithm of Section 3.
Proposition 3  Suppose that $n N$ is equal to 1 for all $N$ in Step 1 of the algorithm of Section 5 for picking from $P_{\frac{1}{q^t}, \frac{1}{q^t}, \frac{1}{q^t}, \frac{1}{q}}$. The growth process on partitions this defines is exactly Kerov's $q$ hook walk.

Proof: Step 2 in the algorithm of Section 5 changes $\lambda$ to $\Lambda$ with probability:

$$\frac{\phi_{\Lambda/\lambda}(\frac{1}{q^t}, \frac{1}{q}) P_{\lambda}(\frac{1}{q^t}, \frac{1}{q})}{g_1(\frac{1}{q^t})} \frac{P_{\Lambda}(\frac{1}{q^t}, \frac{1}{q})}{P_{\lambda}(\frac{1}{q^t}, \frac{1}{q})}$$

The definition of $\phi_{\Lambda/\lambda}$ shows that $\phi_{\Lambda/\lambda}(\frac{1}{q}, \frac{1}{q}) = 1$. Corollary 3 shows that $g_1 = \frac{1}{1-q^t}$. The Principal Specialization Formula shows that $P_{\lambda}(\frac{1}{q^t}, \frac{1}{q})$ is equal to $\frac{1}{q^{n(\lambda)}} \prod_{s \in \lambda} \frac{1}{1-q^t s}$. Combining these facts proves that:

$$\frac{\phi_{\Lambda/\lambda}(\frac{1}{q^t}, \frac{1}{q}) P_{\lambda}(\frac{1}{q^t}, \frac{1}{q})}{g_1(\frac{1}{q^t})} \frac{P_{\Lambda}(\frac{1}{q^t}, \frac{1}{q})}{P_{\lambda}(\frac{1}{q^t}, \frac{1}{q})} = \frac{q^{n(\Lambda)}}{q^{n(\lambda)}} \prod_{s \in \Lambda} [h(s)]$$

as desired. □

12 Suggestions for Future Research

This section suggests some possibilities for future research.

1. Read group theoretic information off of the probabilistic algorithms of Sections 3 and 4. As was shown in Section 3, these algorithms are related to the finite classical groups. Some group theoretic applications of these algorithms are given in the companion paper [10].

2. Develop probabilistic algorithms for picking from the measures $\lambda_{\pm 1}^{\pm 1}$ for the symplectic and orthogonal groups. These will be more complicated than the algorithms for the general linear and unitary groups, since there are size restrictions on the partitions (for instance in the symplectic groups $|\lambda_{\pm 1}^{\pm 1}|$ is always even). Presumably one adds $1*2$ or $2*1$ tiles according to some rules.

3. Persi Diaconis suggested the problem of implementing this paper’s algorithms in a computer program. The Young Tableau Algorithm, for instance, involves flipping infinitely many coins. How can this practical obstacale be overcome?

4. Study the shapes of partitions under the measures $P_{x,y,q,t}(\lambda)$ for various specializations of the variables $x, y, q, t$. For instance find generating functions for various functionals of the partitions such as the number of parts, largest part, number of 1’s, etc. (A generating function for the size was found as Corollary 1 of Section 5). It should also be possible to extend work of Vershik [26], [27] which shows that random partitions under measures such as the Plancherel measure have an asymptotic limit shape.

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References

[1] Andrews, G., The theory of partitions. Encyclopedia of Mathematics and its Applications, Vol. 2. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976.

[2] Arratia, R. and Tavare, S., The cycle structure of random permutations. Ann. Probab. 20 (1992), no. 3, 1567-1591.

[3] Bourbaki, N., Formes sesquilineaires et formes quadratiques (Elements de Mathematique I, livre II), Hermann (Paris), 1959.

[4] Carter, R., Simple groups of lie type. John Wiley and Sons, 1972.

[5] Celler, F., Leedham-Green, C., Murray, S., Niemeyer, A., and O'Brien, E.A., Generating random elements of a finite group. Communications in algebra, 23 (13), (1995), 4931-4948.

[6] Diaconis, P. and Shahshahani, M., On the eigenvalues of random matrices, J. Appl. Prob. 31 (1994), 49-61.

[7] Fine, N.J. and Herstein, I. N., The probability that a matrix is nilpotent, Illinois J. Math. 2 (1958), 499-504.

[8] Fristedt, B., The structure of random partitions of large integers. Trans. Amer. Math. Soc. 337 (1993), no. 2, 703-735.

[9] Fulman, J., Probability in the classical groups over finite fields: symmetric functions, stochastic algorithms and cycle indices, PhD Thesis, Harvard University, 1997.

[10] Fulman, J., A probabilistic approach toward the finite general linear and unitary groups, preprint.

[11] Gerstenhaber, M., On the number of nilpotent matrices with coefficients in a finite field, Illinois J. Math. 5 (1961), 330-333.

[12] Goh, W. and Schmutz, E., A central limit theorem on $GL_n(F_q)$, Preprint. Department of Math. Drexel University.

[13] Greene, C., Nijenhuis, A. and Wilf, H., A probabilistic proof of a formula for the number of Young tableaux of a given shape. Adv. in Math 31 (1979), no. 1, 104-109.

[14] Greene, C., Nijenhuis, A. and Wilf, H., Another probabilistic method in the theory of Young tableaux. J. Combin. Theory Series A, 37 (1984), 127-135.

[15] Hansen, J. and Schmutz, E., How random is the characteristic polynomial of a random matrix? Math. Proc. Cambridge Philos. Soc. 114 (1993), no. 3, 507-515.

[16] Herstein, I.N., Topics in algebra. Second edition. Xerox College Publishing, Lexington, Mass.-Toronto, Ont., 1975.
[17] Kerov, S.V., The boundary of Young lattice and random Young tableaux. Formal power series and algebraic combinatorics. DIMACS Ser. Discrete Math. Theoret. Comput. Sci. 24, pg. 133-158.

[18] Kerov, S.V., A $q$-analog of the hook walk algorithm for random Young tableaux. Journal of Algebraic Combinatorics 2 (1993), 383-396.

[19] Kerov, S.V. and Vershik, A.M., Asymptotic behavior of the Plancherel measure of the symmetric group and the limit form of Young tableaux. Dokl. Akad. Nauk SSSR 233 (1977), no. 6, 1024-1027.

[20] Kung, J., The cycle structure of a linear transformation over a finite field, Linear Algebra Appl. 36 (1981), 141-155.

[21] Macdonald, I.G., Symmetric functions and Hall polynomials, Second Edition. Claredon Press, Oxford. 1995.

[22] Mehta, M.L., Random matrices. Academic Press, San Diego. (1991).

[23] Sagan, B., The symmetric group: representations, combinatorial algorithms, and symmetric functions. Wadsworth and Brooks/Cole 1991.

[24] Stong, R., Some asymptotic results on finite vector spaces, Advances in Applied Mathematics 9, 167-199 (1988).

[25] Stong, R., The average order of a matrix. Journal of Combinatorial Theory, Series A. Vol. 64, No. 2, November 1993.

[26] Vershik, A.M., Asymptotic combinatorics and algebraic analysis. Proceedings of the International Congress of Mathematicians, Zurich 1994, 1384-1394.

[27] Vershik, A.M., Statistical mechanics of combinatorial partitions, and their limit shapes. Functional Analysis and its Applications, Vol. 30, No. 2, 1996, pg. 90-105.

[28] Wall, G.E., On conjugacy classes in the unitary, symplectic, and orthogonal groups, Journal of the Australian Mathematical Society 3 (1963), 1-63.