Heisenberg’s and Hardy’s Uncertainty Principles in Real Clifford Algebras

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Abstract. Recently, many surveys are devoted to study the Clifford Fourier transform. Dealing with the real Clifford Fourier transform introduced by Hitzer [10], we establish analogues of the classical Heisenberg’s inequality and Hardy’s theorem in the real Clifford algebra $\text{Cl}(p, q)$.

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1. Introduction

According to the relevance of the Clifford Fourier transform in applied mathematics, many works focus on the study of Fourier transform in the framework of Clifford algebras. In [1], the Fourier transform is extended to a multivector valued function-distributions with compact support. Then several definitions of Fourier transform in Clifford algebras are given such as Quaternion Fourier transform (QFT) [6, 8] and Clifford Fourier transform (CFT) [3, 4, 5]. We refer the reader to [2] for more details of Fourier transforms over Clifford algebras.

Regarding the fundamental property of the complex imaginary unit $j \in \mathbb{C}$, i.e. $j^2 = -1$, researchers concentrate on finding Clifford numbers satisfying this property (square roots minus one in Clifford algebras) [7, 9]. Replacing $j$ by a square root of $-1$ allows to define a generalized Fourier transform in Clifford algebras [5, 10]. Our aim in the present paper is to obtain uncertainty principle for the Clifford Fourier transform stated by E. Hitzer in [10].

The paper is structured as follows. In section 2, we recall necessary background knowledge of Clifford algebras and some results that will be useful in the sequel. In section 3, we adopt the real Clifford Fourier transform of [10] and we review its properties. In section 4, we provide Heisenberg’s inequality and Hardy’s theorem for the Clifford Fourier transform.
2. Preliminaries

The Clifford geometric algebra $\mathcal{C}l(p,q)$ is defined as a non-commutative algebra generated by the basis \( \{ e_1, e_2, \ldots, e_n \} \), with \( n = p + q \), satisfying the multiplication rule:

\[
e_k e_l + e_l e_k = 2 \epsilon_k \delta_{k,l}, \quad k, l = 1, \ldots, n,
\]

with \( \epsilon_k = +1 \) for \( k = 1, \ldots, p \) and \( \epsilon_k = -1 \) for \( k = p+1, \ldots, n \). Here \( \delta_{k,l} \) denotes the Kronecker symbol, i.e., \( \delta_{k,l} = 1 \) for \( k = l \) and \( \delta_{k,l} = 0 \) for \( k \neq l \).

This algebra can be decomposed as

\[
\mathcal{C}l(p,q) = \bigoplus_{k=0}^{n} \mathcal{C}l_k(p,q),
\]

with \( \mathcal{C}l_k(p,q) \) the space of \( k \)-vectors given by

\[
\mathcal{C}l_k(p,q) := \text{span}\{ e_{i_1} \ldots e_{i_k} | i_1 < \ldots < i_k \}.
\]

Hence \( \{ e_A = e_{i_1} e_{i_2} \ldots e_{i_k} | A \subseteq \{1, \ldots, n\}, 1 \leq i_1 < \ldots < i_k \leq n \} \) with \( e_0 = 1 \), presents a \( 2^n \)-dimensional graded basis of \( \mathcal{C}l(p,q) \).

Every element \( \alpha \) of \( \mathcal{C}l(p,q) \) (Clifford number) can be written as

\[
\alpha = \sum_A \alpha_A e_A = \langle \alpha \rangle_0 + \langle \alpha \rangle_1 + \cdots + \langle \alpha \rangle_n,
\]

with \( \alpha_A \) real numbers. Here \( \langle \cdot \rangle_k \) is the grade \( k \)-part of \( \alpha \). As example, \( \langle \cdot \rangle_0 \) denotes the scalar part, \( \langle \cdot \rangle_1 \) is the vector part and \( \langle \cdot \rangle_2 \) is the bivetor part.

A vector \( x \in \mathbb{R}^{p,q} \) can be identified by

\[
x = \sum_{l=1}^{n} x_l e^l = \sum_{l=1}^{n} x^l e_l, \quad \text{with} \quad e^l := e_l e_l.
\]

The scalar product and outer product for a \( k \)-vector \( A_k \in \mathcal{C}l(p,q) \) and an \( s \)-vector \( B_s \in \mathcal{C}l(p,q) \) are defined, respectively, by

\[
A_k \ast B_s = \langle A_k B_s \rangle_0, \quad A_k \wedge B_s = \langle A_k B_s \rangle_{k+s}.
\]

In \( \mathcal{C}l(p,q) \), the complex and the quaternion conjugations are replaced by the principle reverse which is introduced for all \( \alpha \in \mathcal{C}l(p,q) \) by

\[
\tilde{\alpha} = \sum_{k=0}^{n} (-1)^{k(k-1)/2} \langle \alpha \rangle_k,
\]

where \( \overline{e_A} = e_{i_1} e_{i_2} \ldots e_{i_k} e_0, \) \( 1 \leq i_1 < i_2 < \ldots < i_k \leq n \). In the case of \( \mathbb{C} \otimes \mathcal{C}l(p,q) \), we add the rule \( \tilde{j} = -j \) with \( j \in \mathbb{C} \) is the complex imaginary unit.

Note that \( \tilde{e_A} \ast e_B = \delta_{A,B} \), for all \( 1 \leq A, B \leq 2^n \). In particular, we have \( e^l \ast e_k = e^l \cdot e_k = \delta_{k,l} \).

This leads to

\[
M \ast \tilde{N} = \sum_A M_A N_A, \quad \forall M, N \in \mathcal{C}l(p,q).
\]
The modulus of a multivector $M \in Cl(p, q)$ is given by
\[ |M|^2 = M \ast \widetilde{M} = \sum_A M_A^2, \] (2.9)
from which we can find Cauchy-Schwartz’s inequality for multivectors:
\[ |M \ast \widetilde{N}| \leq |M||N|. \] (2.10)
In the sequel, we will consider functions defined on $\mathbb{R}^{p,q}$ and taking values in $Cl(p, q)$. These functions can be expressed as follows
\[ f(x) = \sum_A f_A(x) e_A, \] (2.11)
where $f_A$ are real-valued functions.

**Definition 2.1.** Let $f, g : \mathbb{R}^{p,q} \rightarrow Cl(p, q)$. Then inner product of $f$ and $g$ is defined by
\[ (f, g) = \int_{\mathbb{R}^{p,q}} f(x) g(x) dx = \sum_{A,B} e_A e_B \int_{\mathbb{R}^{p,q}} f_A(x) g_B(x) dx, \] (2.12)
with symmetric scalar part
\[ <f, g> = \int_{\mathbb{R}^{p,q}} f(x) \ast \widetilde{g(x)} dx = \sum_A \int_{\mathbb{R}^{p,q}} f_A(x) g_A(x) dx, \] (2.13)
which induces the following $L^2(\mathbb{R}^{p,q}, Cl(p, q))$-norm
\[ ||f||^2 := <f, f> = \int_{\mathbb{R}^{p,q}} |f(x)|^2 dx = \sum_A \int_{\mathbb{R}^{p,q}} f_A^2(x) dx. \] (2.14)
Then $L^2(\mathbb{R}^{p,q}, Cl(p, q))$ is introduced as
\[ L^2(\mathbb{R}^{p,q}, Cl(p, q)) = \{ f : \mathbb{R}^{p,q} \rightarrow Cl(p, q) \mid ||f|| < \infty \}. \] (2.15)

The vector derivative $\nabla$ of a function $f$ is defined by
\[ \nabla = \sum_{l=1}^n e^l \partial_l \quad \text{with} \quad \partial_l = \frac{\partial}{\partial x_l}, \quad 1 \leq l \leq n. \] (2.16)
For an arbitrary vector $a \in \mathbb{R}^{p,q}$, the vector differential in the $a$-direction is given by
\[ a \cdot \nabla f(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon a) - f(x)}{\epsilon}. \] (2.17)

**Proposition 2.2 (Integration of parts).** \[5 \] Proposition 3.9]
\[ \int_{\mathbb{R}^n} f(x) [a \cdot \nabla g(x)] dx = \left[ \int_{\mathbb{R}^{n-1}} f(x) g(x) dx \right]_{a \cdot x = -\infty}^{a \cdot x = \infty} - \int_{\mathbb{R}^n} [a \cdot \nabla f(x)] g(x) dx. \]

Recently, an interest is given to study square roots minus one in the real Clifford algebra, i.e. Clifford numbers $i \in Cl(p, q)$ such that $i^2 = -1$, since it generalizes the complex imaginary unit. Note that every multivector with respect to any square root can split into commuting and anticommuting parts.
Lemma 2.3. [9] Let $A \in \text{Cl}(p,q)$. Then with respect to a square root $i \in \text{Cl}(p,q)$ of $-1$, $A$ has a unique decomposition

$$A = A_{+i} + A_{-i},$$

with

$$A_{+i} = \frac{1}{2} (A + i^{-1}Ai) \quad \text{and} \quad A_{-i} = \frac{1}{2} (A - i^{-1}Ai).$$

Moreover one has

$$A_{+i}i = iA_{+i} \quad \text{and} \quad A_{-i}i = -iA_{-i}.$$ 

3. Clifford-Fourier transform

In this section, we recall the real Clifford Fourier transform and its properties. For more details, we refer the reader to [10]. Furthermore, we add some results related to the Clifford Fourier transform and its kernel.

Definition 3.1. Let $f \in L^1(\mathbb{R}^{p,q}, \text{Cl}(p,q))$ and $i \in \text{Cl}(p,q)$ be a square root of $-1$. The Clifford-Fourier transform (CFT), with respect to $i$, is defined by

$$\mathcal{F}^i \{f\}(w) = \int_{\mathbb{R}^{p,q}} f(x)e^{-iu(x,w)}d^n x, \quad (3.1)$$

where $d^n x = dx_1 \cdots dx_n$, $x, w \in \mathbb{R}^{p,q}$ and $u : \mathbb{R}^{p,q} \times \mathbb{R}^{p,q} \to \mathbb{R}$.

Due to the non-commutativity of $\text{Cl}(p,q)$, we have right and left linearity for the Fourier transform defined above:

$$\mathcal{F}^i \{\alpha h_1 + \beta h_2\}(w) = \alpha \mathcal{F}^i \{h_1\}(w) + \beta \mathcal{F}^i \{h_2\}(w), \quad (3.2)$$

$$\mathcal{F}^i \{h_1 \alpha + h_2 \beta\}(w) = \mathcal{F}^i \{h_1\}(w)\alpha_{+i} + \mathcal{F}^{-i} \{h_1\}(w)\alpha_{-i}$$  

$$+ \mathcal{F}^i \{h_2\}(w)\beta_{+i} + \mathcal{F}^{-i} \{h_2\}(w)\beta_{-i}, \quad (3.3)$$

for all $h_1, h_2 \in L^1(\mathbb{R}^{p,q}, \text{Cl}(p,q))$ and $\alpha, \beta \in \text{Cl}(p,q)$.

In the rest of this work, we will assume that

$$u(x, w) = x \ast \tilde{w} = \sum_{l=1}^{n} x^l w^l = \sum_{l=1}^{n} x_l \tilde{w}_l \quad (3.4)$$

and $i = \sum_{A} i_A e_A$ be a square root $-1$ satisfying

$$\tilde{i} = -i. \quad (3.5)$$

The Clifford Fourier transform admits the following properties:

i) Inversion formula: for $h, \mathcal{F}^i \{h\} \in L^1(\mathbb{R}^{p,q}, \text{Cl}(p,q))$, we have

$$h(x) = \mathcal{F}^{-i} \{-1\} \{\mathcal{F}^i \{h\}\}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{p,q}} \mathcal{F}^i \{h\}(w)e^{iu(x,w)}d^n w, \quad (3.6)$$

where $d^n w = dw_1 \cdots dw_n$ and $x, w \in \mathbb{R}^{p,q}$.

ii) Plancherel identity: for $h_1, h_2 \in L^2(\mathbb{R}^{p,q}, \text{Cl}(p,q))$,

$$< h_1, h_2 > = \frac{1}{(2\pi)^n} < \mathcal{F}^i \{h_1\}, \mathcal{F}^i \{h_2\} >. \quad (3.7)$$
iii) Parseval identity: for $h \in L^2(\mathbb{R}^{p,q}, Cl(p,q))$, we have
\[ ||h|| = \frac{1}{(2\pi)^n} ||\mathcal{F}^i\{h\}||. \] (3.8)

4i) Scaling property: for $a \in \mathbb{R} \setminus \{0\}$,
\[ \mathcal{F}^i\{h_d\}(w) = \frac{1}{|a|^n} \mathcal{F}^i\{h\}(\frac{1}{a}w), \] (3.9)
with $h_d(x) := h(ax), x \in \mathbb{R}^{p,q}$.

**Proposition 3.2.** The Gaussian function $f(x) = e^{-\frac{|x|^2}{4}}$ on $\mathbb{R}^n$ defines an eigenfunction for the Clifford Fourier transform:
\[ \mathcal{F}^i\{f\}(w) = (2\pi)^{\frac{n}{2}} f(w). \] (3.10)

**Proposition 3.3.** Let $t > 0$. Then
\[ \mathcal{F}^i\{e^{-t|x|^2}\}(w) = (2\pi)^{\frac{n}{2}} \frac{1}{(2t)^{\frac{n}{2}}} e^{-\frac{|w|^2}{4t}}. \] (3.11)

**Proof.** An application of (3.9) and Proposition 3.2 yields the desired result. \qed

**Proposition 3.4.**
\[ \mathcal{F}^i\{a \cdot \nabla f\}(w) = a \cdot w \mathcal{F}^i\{f\}(w)i \] (3.12)

**Proof.** Following Corollary 3.13 with substituting $i_n$ by $i$, we can show that
\[ a \cdot \nabla e^{iu(x,w)} = a \cdot we^{iu(x,w)}. \] (3.13)
Thus we have
\[
\begin{align*}
    a \cdot \nabla f(x) &= a \cdot \nabla \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{p,q}} \mathcal{F}^i\{f\}(w)e^{iu(x,w)}d^m w \\
    &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{p,q}} \mathcal{F}^i\{f\}(w) \left( a \cdot \nabla e^{iu(x,w)} \right) d^m w \\
    &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{p,q}} \mathcal{F}^i\{f\}(w) \left( a \cdot we^{iu(x,w)} \right) d^m w \\
    &= \mathcal{F}^{-1}_{\mathcal{F}^i\{a \cdot \nabla f\}(w)}(a \cdot w \mathcal{F}^i\{f\}(w)i).
\end{align*}
\]
Using inversion formula, we get
\[ \mathcal{F}^i\{a \cdot \nabla f\}(w) = a \cdot w \mathcal{F}^i\{f\}(w)i. \] \qed

**Theorem 3.5.** Let $z = a + jb \in \mathbb{C} \otimes \mathbb{R}^{p,q}$ with $a,b \in \mathbb{R}^{p,q}$ and $j \in \mathbb{C}$ is the complex imaginary unit. Put $u(x,z) = x \ast \bar{z}, x \in \mathbb{R}^{p,q}$. Then we have
\[ |e^{-iu(x,z)}| \leq (1 + |i|^2)^\frac{1}{2} e^{|x||b|}. \] (3.14)
Proof. Observe that
\[ u(x, z) = x \ast z = x \ast \tilde{a} + \tilde{j}(x \ast \tilde{b}) \]
\[ = x \ast \tilde{a} - j(x \ast \tilde{b}) \]
\[ = u(x, a) - j(u(x, b)). \]

So we compute
\[ ju(x, z) = u(x, b) + j(u(x, a)). \quad (3.15) \]

We should notice that
\[ e^{-iu(x, z)} = \cos(u(x, z)) - i\sin(u(x, z)), \]
where
\[ \cos(u(x, z)) = \frac{e^{ju(x, z)} + e^{-ju(x, z)}}{2} \]
and
\[ \sin(u(x, z)) = \frac{e^{ju(x, z)} - e^{-ju(x, z)}}{2j}. \]

By (3.15) it follows that
\[ |\cos(u(x, z))| \leq \frac{|e^{ju(x, z)}| + |e^{-ju(x, z)}|}{2} \]
\[ \leq \frac{|e^{u(x, b)}||e^{ju(x, a)}| + |e^{-u(x, b)}||e^{-ju(x, a)}|}{2} \]
\[ \leq \frac{|e^{u(x, b)}| + |e^{-u(x, b)}|}{2}. \]

Similarly we obtain
\[ |\sin(u(x, z))| \leq \frac{|e^{u(x, b)}| + |e^{-u(x, b)}|}{2}. \]

See that for all \( \gamma \in \mathbb{R} \),
\[ \frac{e^{-\gamma} + e^{\gamma}}{2} \leq e^{\gamma}. \]

So we deduce that
\[ |\cos(u(x, z))| \leq e^{\|u(x, b)\|} \]
and
\[ |\sin(u(x, z))| \leq e^{\|u(x, b)\|}. \]

Note that
\[ |e^{-iu(x, z)}|^2 = |\cos(u(x, z))|^2 + \sum_A |i_A|^2 |\sin(u(x, z))|^2. \]

Thus
\[ |e^{-iu(x, z)}| \leq (1 + |i|^2)^{\frac{1}{2}} e^{\|u(x, b)\|}. \]

Using (2.10), we get
\[ |u(x, b)| = |x \ast b| \leq |x||b|. \]

This leads to
\[ |e^{-iu(x, z)}| \leq (1 + |i|^2)^{\frac{1}{2}} e^{\|x||b\|}. \] \[ \square \]
4. Uncertainty Principles

In this section, we study uncertainty principles for the Clifford Fourier transform stated in section 3. Actually, we establish Heisenberg’s inequality and Hardy’s theorem in the setting of the real Clifford algebras $Cl(p, q)$.

4.1. Heisenberg’s Inequality

We remind that Heisenberg’s inequality was given for $n = 3$ in [11] and for $n = 2, 3(\text{mod}4)$ in [5]. We will prove Heisenberg’s inequality for the real Clifford Fourier transform in $Cl(p, q)$ with a similar way.

**Theorem 4.1 (Directional Uncertainty Principle).** Let $f \in L^2(\mathbb{R}^{p,q}, Cl(p, q))$. Assume $F = \int_{\mathbb{R}^{p,q}} |f(x)|^2 d^n x$. Then

$$\int_{\mathbb{R}^n} (a \cdot x)^2 |f(x)|^2 d^n x \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (b \cdot w)^2 |F^i\{f\}(w)|^2 d^n w \geq (a \cdot \tilde{b})^2 \frac{1}{4} F^2,$$

(4.1)

with $a, b$ arbitrary constant vectors.

**Proof.** Using (2.14), (3.5) and Proposition 3.4, we obtain

$$\int_{\mathbb{R}^n} (b \cdot w)^2 |F^i\{f\}(w)|^2 d^n w = \langle b \cdot w F^i\{f\}(w), b \cdot w F^i\{f\}(w) \rangle > \langle F^i\{b \cdot \nabla f\}(w), F^i\{b \cdot \nabla f\}(w) \rangle > \int_{\mathbb{R}^n} |F^i\{b \cdot \nabla f\}(w)|^2 d^n w.$$

An application of Parseval identity (see (3.8)) leads to

$$\int_{\mathbb{R}^n} (b \cdot w)^2 |F^i\{f\}(w)|^2 d^n w = (2\pi)^n \int_{\mathbb{R}^n} |b \cdot \nabla f(w)|^2 d^n w. \quad (4.2)$$

See that for $\phi, \psi : \mathbb{R}^n \to \mathbb{C}$,

$$\int_{\mathbb{R}^n} |\phi(x)|^2 d^n x \int_{\mathbb{R}^n} |\psi(x)|^2 d^n x \geq \left( \int_{\mathbb{R}^n} \phi(x) \overline{\psi(x)} d^n x \right)^2.$$

Thus it follows that

$$I := \int_{\mathbb{R}^n} (a \cdot x)^2 |f(x)|^2 d^n x \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (b \cdot w)^2 |F^i\{f\}(w)|^2 d^n w$$

$$= \int_{\mathbb{R}^n} (a \cdot x)^2 |f(x)|^2 d^n x \int_{\mathbb{R}^n} |b \cdot \nabla f(x)|^2 d^n w$$

$$\geq \left( \int_{\mathbb{R}^n} (a \cdot x) |f(x)| |b \cdot \nabla f(x)| d^n x \right)^2.$$

By (2.10) we get

$$I \geq \left( \int_{\mathbb{R}^n} (a \cdot x) \tilde{f(x)} \ast b \cdot \nabla f(x) d^n x \right)^2$$

$$\geq \left( \int_{\mathbb{R}^n} (a \cdot x) \left( \tilde{f(x)} \ast b \cdot \nabla f(x) \right) d^n x \right)^2.$$
Since $2f(x) \ast b \cdot \nabla f(x) = b \cdot \nabla |f(x)|^2$, then

$$I \geq \frac{1}{4} \left( \int_{\mathbb{R}^n} (a \cdot x) (b \cdot \nabla |f(x)|^2) d^n x \right)^2.$$  

Proposition 2.2 yields

$$I \geq \frac{1}{4} \left( \left[ \int_{\mathbb{R}^n-1} (a \cdot x) |f(x)|^2 d^{n-1} x \right]_{b \cdot x = \infty}^{b \cdot x = -\infty} - \int_{\mathbb{R}^n} |b \cdot \nabla (a \cdot x)| |f(x)|^2 d^n x \right)^2 \geq \frac{1}{4} \left( 0 - \int_{\mathbb{R}^n} a \cdot \tilde{b} |f(x)|^2 d^n x \right)^2 = \frac{1}{4} (a \cdot \tilde{b})^2 F^2.$$

\[ \square \]

**Corollary 4.2 (Uncertainty Principle).** We have

$$\int_{\mathbb{R}^n} (a \cdot x)^2 |f(x)|^2 d^n x \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (a \cdot w)^2 |\mathcal{F}^i \{ f \}(w)|^2 d^n w \geq \frac{1}{4} F^2,$$  

(4.3)

with equality when $f(x) = C_0 e^{-k|x|^2}$, $C_0 \in Cl(p, q)$ is an arbitrary constant multivector and $0 < k \in \mathbb{R}$.

**Proof.** Put $b = \pm a$ and $|a|^2 = 1$. The desired inequality is a straightforward consequence of Theorem 4.1. Now we move to prove the equality. For $f = C_0 e^{-k|x|^2}$ with $C_0 \in Cl(p, q)$, observe that

$$a \cdot \nabla f = -2ka \cdot xf.$$

Combining this result with (4.2), we obtain

$$J := \int_{\mathbb{R}^n} (a \cdot x)^2 |f(x)|^2 d^n x \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (a \cdot w)^2 |\mathcal{F}^i \{ f \}(w)|^2 d^n w \geq \frac{1}{4} F^2.$$  

By the fact that $2f(x) \ast a \cdot \nabla f(x) = a \cdot \nabla |f(x)|^2$ and Proposition 2.2, we get

$$J = \frac{1}{4} \left( \int_{\mathbb{R}^n} a \cdot xa \cdot \nabla |f(x)|^2 d^n x \right)^2 \geq \frac{1}{4} \left( \int_{\mathbb{R}^n} a \cdot \nabla a \cdot x |f(x)|^2 d^n x \right)^2 = \frac{1}{4} |a|^2 F^2.$$  

Since $|a|^2 = 1$, we conclude the proof. \[ \square \]
Theorem 4.3. For $a \cdot \tilde{b} = 0$, we find
\[
\int_{\mathbb{R}^n} (a \cdot x)^2 |f(x)|^2 d^n x \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (b \cdot w)^2 |\mathcal{F}^i \{f\}(w)|^2 d^n w \geq 0. \tag{4.4}
\]

Proof. See that for $a \cdot \tilde{b} = 0$, the right side of Theorem 4.1's inequality is 0. \hfill \square

Theorem 4.4. One has
\[
\int_{\mathbb{R}^n} |x|^2 |f(x)|^2 d^n x \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |w|^2 |\mathcal{F}^i \{f\}(w)|^2 d^n w \geq n \frac{1}{4} F^2. \tag{4.5}
\]

Proof. Note that $|x|^2 = \sum_{k=1}^{n} (e_k \cdot x)^2$ and $|w|^2 = \sum_{k=1}^{n} (e_k \cdot w)^2$. We compute
\[
K := \int_{\mathbb{R}^n} |x|^2 |f(x)|^2 d^n x \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |w|^2 |\mathcal{F}^i \{f\}(w)|^2 d^n w
\]
\[
= \sum_{k,l=1}^{n} (e_k \cdot x)^2 |f(x)|^2 d^n x \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (e_l \cdot w)^2 |\mathcal{F}^i \{f\}(w)|^2 d^n w
\]
\[
= \sum_{k=1}^{n} (e_k \cdot x)^2 |f(x)|^2 d^n x \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (e_k \cdot w)^2 |\mathcal{F}^i \{f\}(w)|^2 d^n w
\]
\[
+ \sum_{k \neq l}^{n} (e_k \cdot x)^2 |f(x)|^2 d^n x \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (e_l \cdot w)^2 |\mathcal{F}^i \{f\}(w)|^2 d^n w.
\]

From Theorem 4.1 and Theorem 4.3 we obtain
\[
K \geq \sum_{k=1}^{n} (e_k \cdot x)^2 |f(x)|^2 d^n x \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (e_k \cdot w)^2 |\mathcal{F}^i \{f\}(w)|^2 d^n w
\]
\[
\geq \sum_{k=1}^{n} (e_k \cdot \tilde{e}_k)^2 \frac{1}{4} F^2,
\]
which completes the proof. \hfill \square

4.2. Hardy’s Theorem

Theorem 4.5. Let $p, q$ be positive constants. Suppose $f$ is a measurable function on $\mathbb{R}^{p,q}$ satisfying the following estimates:
\[
|f(x)| \leq C e^{-p|x|^2}, \quad x \in \mathbb{R}^{p,q} \tag{4.6}
\]
\[
|\mathcal{F}^i \{f\}(y)| \leq C e^{-q|y|^2}, \quad y \in \mathbb{R}^{p,q}, \tag{4.7}
\]
where $C$ is a positive constant. Then:

1. If $pq > \frac{1}{4}$, then $f = 0$.
2. If $pq = \frac{1}{4}$, then $f(x) = A e^{-p|x|^2}$, with $A$ a Clifford constant.
3. If $pq < \frac{1}{4}$, then there exist many such functions.
Proof. According to (3.9), we can assume that $p = q$. For $pq < \frac{1}{4}$, by Proposition 3.3 the functions $f(x) = Ce^{-t|x|^2}$ for $t \in [p, \frac{1}{4q}]$ and for some Clifford constant $C$ satisfies (4.6) and (4.7). Since (1) is deduced from (2), it is sufficient to prove (2). Hence we will show that if $p = q = \frac{1}{2}$, then $f(x) = Ae^{-\frac{|x|^2}{2}}$ with $A$ a constant.

Using Theorem 3.5 and (4.6), we obtain for all $\lambda \in \mathbb{C} \otimes \mathbb{R}^{p,q}$

$$|F^i\{f\}(\lambda)| \leq \int_{\mathbb{R}^{p,q}} |f(x)| e^{-i\langle u(x,\lambda) \rangle} |d^n x|$$

$$\leq \left(1 + |i|^2\right)^{\frac{1}{2}} \int_{\mathbb{R}^{p,q}} |f(x)| e^{\frac{1}{2}|\text{Im}(\lambda)|} |d^n x|$$

$$\leq C \left(1 + |i|^2\right)^{\frac{1}{2}} \int_{\mathbb{R}^{p,q}} e^{-\frac{|x|^2}{2}} e^{\frac{1}{2}|\text{Im}(\lambda)|} |d^n x|$$

$$\leq C' e^{\frac{|\lambda|^2}{4}},$$

with $C'$ is a positive constant.

Furthermore we get $F^i\{f\}$ is an entire function satisfying (4.8) and (4.7). Thus [12, Lemma.2.1] implies that

$$F^i\{f\}(x) = Ae^{-\frac{|x|^2}{2}}, \quad \text{for some constant } A.$$

By (3.6) and Proposition 3.2 it follows that $f(x) = Ae^{-\frac{|x|^2}{2}}$.

□

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