Abstract

The leading short-time behaviour of the Yang Mills Schrödinger functional is obtained within a local expansion in the fields.
1 Introduction

In the Schrödinger representation the vacuum of a quantum field theory is a functional of the eigenvalue, $\phi(x)$, of the field on a fixed, spacelike, quantisation surface with co-ordinate $x$. In perturbation theory its logarithm, $W[\phi]$, is a sum of connected Feynman diagrams, and is, in general, a non-local functional. If $\phi$ varies very slowly on a length scale of the inverse of the lightest mass in the theory, then $W[\phi]$ reduces to a sum of local functionals, that is to say, integrals of products of $\phi$ and a finite number of its derivatives at the same spatial point. As it stands this observation appears to have little physical interest since we need more rapidly varying fields to probe the internal structure of particles which occurs on shorter scales. However, when $W$ is evaluated for a field that is obtained from $\phi$ by a scale transformation in which $x \to \lambda x$ then it is analytic in $\lambda$. This may be used to reconstruct $W[\phi]$ for arbitrary $\phi$ from the local expansion \[1\]. Furthermore, the coefficients of the expansion are determined by the Schrödinger equation which may be constructed so as to act directly on this expansion, again by exploiting analyticity \[2\]. For theories that are classically massive this local expansion appears already within the framework of standard semi-classical perturbation theory, but for Yang-Mills theory, which is classically massless, the leading order contribution to $W$ does not reduce to a local expansion for slowly varying fields. Nonetheless quantum effects generate a non-zero mass-gap, and so the full expression for $W$ does have such an expansion, as has been seen in Monte-Carlo simulations of lattice gauge theory \[3\] \[4\] \[5\]. In principle this will be determined by solving the Schrödinger equation, but in practice the construction of this equation to a sufficient order to generate reliable results is some way off. It would have been useful to study this local expansion using standard semi-classical techniques. As we cannot we will study instead the so-called Schrödinger functional, which is just the matrix element of the Euclidean time-evolution operator for time $\tau$, between eigenstates of the gauge-field $A$.

$$\Phi_{\tau}[A, A'] = \langle A | e^{-H\tau/\hbar} | A' \rangle.$$  \tag{1}$$

(Here $H$ is the Hamiltonian $H[A, E] = -\frac{1}{2g^2} \int d^3x \ tr(B^2 + E^2)$, where $E = -\dot{A}$, $B = \nabla \wedge A + A \wedge A$, $A = A^A T^A$, $tr(T^A T^B) = -\delta^{AB}$, $[T^A, T^B] = f^{ABC} T^C$ and we work in the Weyl gauge, $A_0 = 0$.)

Again the logarithm of this functional, $W_{\tau}[A, A']$ is a sum of connected diagrams, and is non-local, but having introduced the length-scale $\tau$ into the problem results in a local expansion for fields that vary slowly, even within the semi-classical expansion. In other words we can solve for the short-time behaviour using both a local expansion, and semi-classical perturbation theory. This enables us to compare the efficacy of solving the problem in two different ways, i.e. by solving the Hamilton-Jacobi equation for the derivative expansion and by evaluating the leading order contribution to the functional integral. Furthermore the result of the semi-classical calculation is useful as it will still be a good leading order approximation to the full expression for times that are small in comparison to the inverse of the lightest glueball mass. The Schrödinger functional has been studied in lattice QCD \[6\] as a functional integral over a space-time with boundaries given by the quantisation surfaces at times 0 and $\tau$. The divergences have been studied at one-loop, where it was found that they could be cancelled by quark counter-terms on
the space-time boundary [7, 8].

We begin by considering the differential equation approach to \( W_\tau[A, A'] \). Now \( \Phi_\tau \) satisfies the Schrödinger equation with initial condition

\[
- \hbar \frac{\partial}{\partial \tau} \Phi_\tau[A, A'] = H \Phi_\tau[A, A'], \quad \lim_{\tau \to 0} \Phi_\tau[A, A'] = \delta[A - A'] \tag{2}
\]

In the Schrödinger representation the Yang-Mills electric field is represented by \( \tilde{E}_C^\mu(x) = i\hbar g^2 \frac{\delta}{\delta \bar{A}_C^\mu(x)} \), so that

\[
H = \left( -\frac{1}{2} \hbar g^2 \Delta + g^{-2}B \right), \quad \Delta \equiv \int d^3x \frac{\delta}{\delta \bar{A}_C^\mu(x)} \cdot \frac{\delta}{\delta \bar{A}_C^\mu(x)}
\]

and

\[
B = -\frac{1}{2} \int d^3x \ trB^2. \tag{3}
\]

The Schrödinger equation (1) must be regularized as the kinetic term \( \Delta \) contains two functional derivatives acting at the same point of space, however this will not affect the leading order calculation, since if we set \( W_\tau[A, A'] = w[A, A']/(\hbar g^2) \) the Schrödinger equation reads:

\[
\frac{1}{2} \hbar g^2 \Delta w - \frac{1}{2} \left( \int d^3x \ \frac{\delta w}{\delta \bar{A}_C^\mu(x)} \cdot \frac{\delta w}{\delta \bar{A}_C^\mu(x)} \right) + B + \frac{\partial w}{\partial t} = 0. \tag{4}
\]

Neglecting the \( O(\hbar g^2) \) term leads to the Hamilton-Jacobi equation which we now solve in a derivative expansion subject to the conditions that \( w[A, A'] \) be real and invariant under simultaneous time-independent gauge transformations of \( A \) and \( A' \) and that \( w[A, A'] = w[A', A] \). If we order the local expansion of \( w \) according to the mass dimension of local functionals, then the first gauge invariant term is \( \int d^3x \ trA^2 \), where \( A = A - A' \). Since the only length-scale in this classical problem is \( \tau \) this enters \( w \) multiplied by \( 1/\tau \). This term will enable us to fit the initial condition since as \( \tau \) becomes small \( \exp -\tau^{-1} \int d^3x \ trA^2 \sim \delta[A - A'] \). Next we must include dimension four fields, for example \( \int d^3x \ trB^2 \) which is needed to cancel a similar term in the Hamiltonian. The Hamilton-Jacobi equation generates cross-terms from these two functionals of the form

\[
\left( \frac{\delta}{\delta \bar{A}^R_\mu(x)} \right) \int d^3x' A_\nu(x') A_\nu(x') \left( \frac{\delta}{\delta \bar{A}^R_\mu(x)} \right) \left( \frac{1}{2} \int d^3x' B_\rho^A(x')B_\rho^A(x') \right)
\]

\[
(2 \ A_\mu | x) (D_{\mu \rho}^R B_\rho^A | x) \tag{5}
\]

where

\[
D_{\mu \rho}^R \equiv \epsilon_{\mu \rho \sigma} D_\sigma^R, \quad D_\sigma^R = (\partial_\sigma \delta^R + f^R S A_\sigma). \tag{6}
\]

This is a gauge-invariant dimension four field that should in turn be included in the expansion of \( w \), as too should all further terms generated by repeated applications of
\[ \int d^3x \, A \cdot \delta/\delta A. \] Thus we arrive at an ansatz for the the local expansion including terms up to dimension four

\[
W[A, A'] = \int d^3x \, tr \{ A^2 |_{x} a/\tau + B^2 |_{x} b\tau + A \cdot DB |_{x} c\tau \\
+ \int d^3y \, A(x) \cdot \Lambda(x, y) \cdot A(y) \, d\tau + DA \cdot [A, A] |_{x} e\tau \\
+ [A, A] \cdot [A, A] |_{x} f\tau
\]

where

\[
\Lambda_{\mu\nu}^{AB}(x, y) = \frac{\delta}{\delta A_{\mu}^{A}(x)} \frac{\delta}{\delta A_{\nu}^{B}(y)} \frac{1}{2} \int d^3x' B_{\rho}^{R} B_{\rho}^{R} \mid_{x'}
\]

\[
= (D_{\mu\rho}^{AR} D_{\rho\nu}^{RB} + f_{AB}^{RR} \epsilon_{\mu\nu\rho} B_{\rho}^{R}) \mid_{x} \delta^3(x - y)
\]

\[
= (D_{\mu\rho}^{AR} D_{\rho\nu}^{RB} + f_{AB}^{RR}) \mid_{x} \delta^3(x - y)
\]

The constants \( b, c, d, e \) are related by imposing \( w[A, A'] = w[A', A] \), which, after some algebra implies that \( c = -b \) and \( e = \frac{1}{2} b - 3 d \). Finally, if we substitute (8) into the Hamilton-Jacobi equation the remaining coefficients are determined leading to our final result for \( W \):

\[
W_{\tau}[A, A'] = \frac{1}{2\bar{h}g^2} \int d^3x \, tr \{ -\frac{A^2}{\tau} \mid_{x} - B^2 \mid_{x} \tau + A \cdot DB \mid_{x} \tau \\
- \int d^3y A(x) \cdot \Lambda(x, y) \cdot A(y) \, \frac{\tau}{3} + DA \cdot [A, A] \mid_{x} \frac{\tau}{2} \\
- [A, A] \cdot [A, A] \mid_{x} \frac{\tau}{10} \}
\]

We will now compute this expression using a different method, so as to gauge the efficiency of the above approach. The functional integral representation of \( W_{\tau}[A_{\text{out}}, A_{\text{in}}] \) leads to a saddle-point approximation in which this is given to leading order by minus the Euclidean action

\[
\frac{1}{2\bar{h}g^2} \int d^3x \, dt \, (E^A \cdot E^A + B^A \cdot B^A) \mid_{x}
\]

where, in the temporal gauge,

\[
E = -\dot{A} \quad \text{and} \quad B = \nabla \wedge A + A \wedge A.
\]

The fields \( A \) are required to be on-shell, i.e. they satisfy the Euler Lagrange equations

\[
\ddot{A} = -D \wedge B, \quad D \wedge B = \nabla \wedge B + A \wedge B + B \wedge A
\]

subject to the boundary conditions that at \( A(x, 0) = A_{\text{in}} \) and \( A(x, \tau) = A_{\text{out}} \). If we expand the field as a power series in \( \tau \): \( A(x, \tau) = \sum_{n=0}^{\infty} A_n(x) \, \tau^n \) and substitute into the equations of motion and boundary conditions we eventually obtain
\[ A_0 = A_{\text{in}} \]
\[ A_1 = \frac{A}{\tau} - A_2 \tau - A_3 \tau^2 - A_4 \tau^3 - A_5 \tau^4 - O(\tau^2), \]
\[ A_2 = -\frac{1}{2} D_0 \wedge B_0, \]
\[ A_3 = -\frac{1}{6\tau} \left\{ D_0 \wedge (D_0 \wedge \mathcal{A}) + \mathcal{A} \wedge B_0 + B_0 \wedge \mathcal{A} \right\}, \]
\[ A_4 = -\frac{1}{12\tau^2} \left\{ D_0 \wedge (\mathcal{A} \wedge \mathcal{A}) + \mathcal{A} \wedge (D_0 \wedge \mathcal{A}) + (D_0 \wedge \mathcal{A}) \wedge \mathcal{A} \right\}, \]
\[ A_5 = -\frac{1}{20\tau^3} \left\{ \mathcal{A} \wedge (\mathcal{A} \wedge \mathcal{A}) + (\mathcal{A} \wedge \mathcal{A}) \wedge \mathcal{A} \right\}. \] (14)

where we neglected terms of \( O(L^{-n}) \) \( n > 4 \) and \( D_0 \wedge = \nabla \wedge +A_0 \wedge + \wedge A_0 \). The subscript 0 denotes that the gauge potential has been taken as \( A_0 \), and \( \mathcal{A} = A_{\text{out}} - A_{\text{in}} \).

Substituting this into the Euclidean action and some considerable algebra eventually leads to the same expression as previously for \( W_\tau[A, A'] \).

To conclude, we have computed the local expansion of the leading order term in the semi-classical expansion of the Schrödinger functional for Yang-Mills theory using two different approaches, the first by substituting our local expansion directly into the Hamilton-Jacobi equation, the second by computing the on-shell Euclidean action. It turned out that the former approach was far more efficient. The result describes the leading short-time behaviour. As a final check on our results note that when the initial and final gauge-fields are identified \( A = 0 \) and our expression reduces, as it should, to \( -\tau \int d^3xB^2/(2g^2) \), i.e. minus the Euclidean action evaluated for a time-independent potential.

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