FIRST-KIND BOUNDARY INTEGRAL EQUATIONS FOR THE
DIRAC OPERATOR IN 3D LIPSCHITZ DOMAINS∗
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Abstract. We develop novel first-kind boundary integral equations for Euclidean Dirac operators in 3D Lipschitz domains. They comprise square-integrable potentials and involve only weakly singular kernels. Generalized Gårding inequalities are derived and we establish that the obtained boundary integral operators are Fredholm of index zero. Their finite dimensional nullspaces are characterized and we show that their dimensions are equal to the number of topological invariants of the domain’s boundary, in other words, to the sum of its Betti numbers. This is explained by the fundamental discovery that the associated bilinear forms agree with those induced by the 2D Dirac operators for surface de Rham Hilbert complexes whose underlying inner-products are the non-local inner products defined through the classical single-layer boundary integral operators for the Laplacian. Decay conditions for well-posedness in natural energy spaces of the Dirac system in unbounded exterior domains are also presented.

Key words. Dirac, Hodge–Dirac, potential representation, representation formula, jump relations, first-kind boundary integral equations, coercive boundary integral equations

AMS subject classifications. 31A10, 45A05, 45E05, 45P05, 35F15, 34L40, 35Q61

1. Introduction. We develop first-kind boundary integral equations for the Hodge-Dirac operator in 3-dimensional Euclidean space

\begin{equation}
D := d + \delta : \mathbf{H}(d, \Omega^-) \cap \mathbf{H}(\delta, \Omega^+) \rightarrow L^2(\Omega^+)^8,
\end{equation}

involving the exterior derivative and codifferential

\begin{equation}
d := \begin{pmatrix}
\nabla & 0 \times 3 & 0 \times 3 & 0 \\
0 & 0 \times 3 & 0 \times 3 & 0 \\
0 & \text{curl} & 0 \times 3 & 0 \\
0 & 0 \times \text{div} & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
\delta := \begin{pmatrix}
0 & -\text{div} & 0^\top & 0 \\
0 & 0 \times 3 & \text{curl} & 0 \\
0 & 0 \times 3 & 0 \times 3 & -\nabla \\
0 & 0 \times \text{div} & 0^\top & 0
\end{pmatrix}.
\end{equation}

We are concerned with the partial differential equations

\begin{equation}
\begin{aligned}
-\text{div} \, U_1 &= F_0, \\
\nabla U_0 + \text{curl} \, U_2 &= F_1, \\
-\nabla U_3 + \text{curl} \, U_1 &= F_2, \\
\text{div} \, U_2 &= F_3.
\end{aligned}
\end{equation}

We will consider both interior and exterior boundary value problems, and assume that (1.3) is either posed on a bounded domain \( \Omega^- \) having a Lipschitz boundary \( \Gamma := \partial \Omega^- \), or on the unbounded complement \( \Omega^+ := \mathbb{R}^3 \setminus \overline{\Omega^-} \). In the latter case, suitable decay conditions at infinity will be needed. Throughout, \( \Omega \in \{ \Omega^-, \Omega^+ \} \).

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1.1. Related work. Current work discussing Dirac operators from the point of view of Hodge theory offers solutions to boundary value problems for (1.3) and related eigenvalue problems based on domain variational formulations [13,26]. The operator matrix in (1.1) appears under a change of variables in the works of M. Taskinen, S. Vänskä and P. Ylää-Oijala [41–43] as R. Picard’s extended Maxwell operator. It was originally assembled by R. Picard by combining the first-order Maxwell operator with the principal part of the equations of linear acoustics [25, 34, 35]. In [41–43], Helmholtz-like boundary value problems for Picard’s operator are studied with a focus on second-kind boundary integral equations.

Eigenvalue problems related to acoustic and electromagnetic scattering, that is transmission problems for the so-called perturbed Dirac operator, have also guided the study of second-kind boundary integral equations in the literature of harmonic and hypercomplex analysis. Important contributions were made in that direction by E. Marmolejo-Olea, I. Mitrea, M. Mitrea, Q. Shi [28], A. Axelsson, A. Rosén and J. Helsing [4,20,36]. There, the Dirac operator enters larger systems of equations that encompass or correspond to Maxwell’s equations [20,28]. An extensive body of work, created by these authors together with R. Grognard and J. Hogan [5], S. Keith [6], A. McIntosh and S. Monniaux [30, 31], is devoted to the harmonic analysis of Dirac operators in $L_p$ spaces [7,29].

1.2. Our contributions. In this work, we derive novel first-kind boundary integral equations for the Dirac equation $D\vec{U} = 0$ with suitable boundary and decay conditions. Two boundary integral operators are obtained and shown to satisfy generalized Gårding inequalities, making them Fredholm of index 0. Their finite dimensional nullspaces are characterized in Section 7, where we show that their dimension equals the number of topological invariants of the boundary—counted as the sum of its Betti numbers. Indeed, the integral representations of their associated bilinear forms turn out to be related to the variational formulations of the surface Dirac operators introduced in Section 8. Recognizing these surface operators will simultaneously reveal how the boundary integral operators introduced in Section 5, which are related to two different sets of boundary conditions, arise as “rotated” versions of one another. The exterior representation formula of Lemma 4.15 and the condition at infinity identified in (4.66) eventually lead, together with the coercivity results of Section 6, to well-posedness of Euclidean Dirac exterior boundary value problems in natural energy spaces in the complement of the finite dimensional nullspaces.

The new integral formulas display desirable properties: the surface potentials are square-integrable and the kernels of the bilinear forms associated with the boundary integral operators are merely weakly singular, i.e. they are bounded by $|x - y|^{-\alpha}$, $\alpha < 2$, cf. [24, Sec. 2.4]. Nevertheless, we want to emphasize that the main result is the discovery that they relate to the Hodge–Dirac operators of surface de Rham Hilbert complexes equipped with the non-local inner products defined as the bilinear forms associated with the classical single-layer potential for the Laplacian. As a consequence, we already know a lot about these first-kind boundary integral operators for the Dirac operator. Moreover, this relationship suggests that they are related to the first-kind boundary integral operators for the Hodge–Laplacian.

For the sake of readability, we adopt the framework of classical vector analysis rather than exterior calculus. It is in this framework that the structural relationship between the following development and the standard theory for second-order elliptic operators seemed most explicit.

In summary, our main contributions are:
We derive representation formulas for the Dirac equation posed on domains having a Lipschitz boundary by following the approach pioneered by M. Costabel [16]. The novelty here is to follow and extend the elegant strategy used in [14]—there used to find a representation formula for Hodge–Laplace and Helmholtz operators—that leads to potentials having simple explicit expressions. By adapting the arguments in the now classical monographs by W. McLean [32, Chap. 7] and A. Sauter and C. Schwab [37, Chap. 3], we also establish an exterior representation formula. We will observe that the development of this theory is possible due to the strong structural similarity between integration by parts for the first-order Dirac operator and Green’s second formula for second-order elliptic operators.

A sneak peek at the potentials presented in (4.39) and (4.42) will already convince the reader that the approach we have adopted leads to simple formulas for the square-integrable potentials involved in the representation formula. Some terms are recognizable from [14, 15], while others occur in well-known theory for elliptic second-order operators. The simplicity that comes with the calculation procedure provided by Lemma 4.5 allows for a straightforward analysis of their mapping and jump properties.

Given the previous items, it is not surprising that decay conditions at infinity for exterior boundary value problems posed on the unbounded domain Ω⁺ can be easily established by adapting the approach for second-order elliptic operators presented in [32, Chap. 7].

The crux of our calculations are the formulas (5.12) and (5.13) for the bilinear forms associated with the obtained weakly-singular first-kind boundary integral operators. We provide generalized Gårding inequalities for the two operators and characterize their null-spaces.

Our main discovery is presented in Section 8, where we expose the relationship between these boundary integral operators and surface Dirac operators in an Hilbert complex framework.

2. Function spaces and traces. As usual, \( L^2(Ω) \) and \( \mathbf{L}^2(Ω) \) denote the Hilbert spaces of complex square-integrable scalar and vector-valued functions defined over \( Ω \). We denote their inner products using round brackets, e.g. \( ⟨·,·⟩ \). The spaces \( H^1(Ω) \) and \( H^1(Ω) \) refer to the corresponding Sobolev spaces. The notation \( C_0^∞(Ω) \) is used for smooth functions. The subscript in \( C_0^∞(Ω) \) further specifies that these smooth functions have compact support in \( Ω \). \( C_0^∞(Ω) \) is defined as the space of uniformly continuous functions over \( Ω \) that have uniformly continuous derivatives of all order. A subscript is used to identify spaces of locally integrable functions/vector fields, e.g. \( U ∈ L^2_{\text{loc}}(Ω) \) if and only if \( φU \) is square-integrable for all \( φ ∈ C_0^∞(\mathbb{R}^3) \). We denote with an asterisk the spaces of functions with zero mean, e.g. \( H_3^1(Ω) \).

In general, given an operator \( L \) acting on square-integrable fields in the sense of distributions, we equip

\[
H(L,Ω) := \{U ∈ (L^2(Ω))^\bullet | LU ∈ (L^2(Ω))^{\dagger}\}
\]

with the natural graph norm, where \( \bullet = 8 \) or \( 3 \) and \( \dagger = 8, 3 \) or \( 1 \). Important specimens
are
\[ H(\text{div}, \Omega) := \left\{ U \in \left( L^2(\Omega) \right)^3 \mid \text{div} U \in L^2(\Omega) \right\}, \]
\[ H(\text{curl}, \Omega) := \left\{ U \in \left( L^2(\Omega) \right)^3 \mid \text{curl} U \in \left( L^2(\Omega) \right)^3 \right\}. \]

Of course, in all of the above definitions, \( \Omega \) can be replaced by \( \mathbb{R}^3 \), or any other domain. We understand restrictions in the sense of distributions when working with domains having disconnected components. For example, in line with the above notation we mean in particular
\[ H \left( D, \mathbb{R}^3 \backslash \Gamma \right) := H(D, \Omega) \times H \left( D, \mathbb{R}^3 \backslash \Omega \right) \subset \left( L^2(\mathbb{R}^3) \right)^8. \]

We use a prime superscript to denote dual spaces, for instance \( C_0^\infty(\Omega)' \) is the space of distributions in \( \Omega \). Angular brackets indicate duality pairings, e.g. \( \langle \cdot, \cdot \rangle_{\Gamma} \). The former will be used for domain-based quantities in \( \Omega \), while the latter will pair spaces on \( \Gamma \).

Trace-related theory for Lipschitz domains can be found in [8, 9, 11] and [19, 32], where it is established that the traces
\[ \gamma W := W|_{\Gamma}, \] \[ \forall W \in C^\infty(\Omega), \]
\[ \gamma_n W := \gamma W \cdot n, \] \[ \forall W \in C^\infty(\Omega), \]
\[ \gamma_\tau W := \gamma W \times n, \] \[ \forall W \in C^\infty(\Omega), \]
\[ \gamma_t W := n \times (\gamma_\tau W), \] \[ \forall W \in C^\infty(\Omega), \]
extend to continuous and surjective linear operators
\[ \gamma : H^1(\Omega) \to H^{1/2}(\Gamma), \] \[ \text{[22, Thm. 4.2.1]} \]
\[ \gamma_n : H(\text{div}, \Omega) \to H^{-1/2}(\Gamma), \] \[ \text{[19, Thm. 2.5, Cor. 2.8]} \]
\[ \gamma_\tau : H(\text{curl}, \Omega) \to H^{-1/2}(\text{div}, \Gamma), \] \[ \text{[11, Thm. 4.1]} \]
\[ \gamma_t : H(\text{curl}, \Omega) \to H^{-1/2}(\text{curl}, \Gamma), \] \[ \text{[11, Thm. 4.1]} \]
with nullspaces
\[ H^1_0(\Omega) := \overline{C_0^\infty(\Omega)}^{H^1(\Omega)} = \ker \gamma, \] \[ \text{[32, Thm 3.40]} \]
\[ H_0(\text{div}, \Omega) := \overline{C_0^\infty(\Omega)}^{H(\text{div}, \Omega)} = \ker \gamma_n, \] \[ \text{[33, Thm. 3.25]} \]
\[ H_0(\text{curl}, \Omega) := \overline{C_0^\infty(\Omega)}^{H(\text{curl}, \Omega)} = \ker \gamma_\tau = \ker \gamma_t. \] \[ \text{[33, Thm. 3.33]} \]

Here, \( n \in L^\infty(\Gamma) \) is the essentially bounded unit normal vector field on \( \Gamma \) directed toward the exterior of \( \Omega^- \). Detailed definitions can be found in [8, 9, 11] together with a study of the involved surface differential operators. Short practical summaries are also provided in [12, 14, 23, 38].

Similarly as for the Hodge–Laplace operator [14, 15, 38, 39], a theory of boundary value problems for the Hodge–Dirac problem in three dimensions entails partitioning our collection of traces into two “dual” pairs. Accordingly, we assemble the traces into
\[ \gamma_T \left( \vec{U} \right) := \begin{pmatrix} \gamma (U_0) \\ \gamma_t (U_1) \\ \gamma_n (U_2) \end{pmatrix} \] and \[ \gamma_R \left( \vec{U} \right) := \begin{pmatrix} \gamma_n (U_1) \\ \gamma_\tau (U_2) \\ \gamma (U_3) \end{pmatrix}. \]
In this sense, we can identify with $\Omega$, e.g. (2.6) will be tagged with a minus subscript (only when required to avoid confusion), (2.12) are dual to each other with respect to the $L^2$ duality pairing (c.f. [14, Lem. 5.6]). In this sense, we can identify

\begin{equation}
\mathcal{H}_\Gamma':= \mathcal{H}_\Gamma \quad \text{and} \quad \mathcal{H}_\Gamma'' = \mathcal{H}_\Gamma.
\end{equation}

Naturally, the traces can also be taken from the exterior domain. The extensions (2.6) will be tagged with a minus subscript (only when required to avoid confusion), e.g. $\gamma^-$, to distinguish them from the extensions obtained from (2.5) by replacing $\Omega$ with $\Omega^+ := \mathbb{R}^3 \setminus \overline{\Omega}$, which we will label with a plus superscript, e.g. $\gamma^+$.

**Lemma 2.1.** (See [14, Lem. 6.4]). *The linear mappings*

\begin{equation}
\gamma_{\Gamma}^\pm: \mathcal{H}_{\text{loc}}(D, \Omega^+) \to \mathcal{H}_\Gamma, \quad \gamma_{\text{R}}^\pm: \mathcal{H}_{\text{loc}}(D, \Omega^+ \setminus \Gamma) \to \mathcal{H}_\Gamma,
\end{equation}

defined by (2.10) are continuous and surjective. There exist continuous lifting maps $\mathcal{E}_{\Gamma}: \mathcal{H}_\Gamma \to \mathcal{H}_{\text{loc}}(D, \mathbb{R}^3 \setminus \Gamma)$ and $\mathcal{E}_\Gamma: \mathcal{H}_\Gamma \to \mathcal{H}_{\text{loc}}(D, \mathbb{R}^3 \setminus \Gamma)$ such that $\gamma_{\Gamma} \circ \mathcal{E}_{\Gamma} = \text{Id}$ and $\gamma_{\text{R}} \circ \mathcal{E}_\Gamma = \text{Id}$.

**Lemma 2.2.** (See [14, Lem. 6.4]). *The surface divergence extends to a continuous surjection* $\text{div}_\Gamma: \mathcal{H}^{-1/2}(\text{div}_\Gamma, \Gamma) \to H^{-1/2}_\Gamma(\Gamma)$, *while* $\text{curl}_\Gamma: H^{1/2}_\Gamma(\Gamma) \to H^{-1/2}(\text{div}_\Gamma, \Gamma)$ *is a bounded injection with closed range such that* $\text{curl}_\Gamma \xi = \nabla_\Gamma \xi \times n$ *for all* $\xi \in H^{1/2}(\Gamma)$. *These operators satisfy* $\text{div}_\Gamma \circ \text{curl}_\Gamma = 0$.

**Lemma 2.3.** For all $\mathbf{U} \in \mathbf{H}(d, \Omega^+)$ and $\mathbf{V} \in \mathbf{H}(\delta, \Omega^+)$,

\begin{equation}
\int_{\Omega^+} d\mathbf{U} \cdot \delta \mathbf{V} \, dx = \int_{\Omega^+} \mathbf{U} \cdot \delta \mathbf{V} \, dx \pm \langle \gamma_{T} \mathbf{U}, \gamma_{R} \mathbf{V} \rangle_\Gamma.
\end{equation}

*Proof.* We integrate by parts using Green’s identities to obtain

\begin{align*}
\int_{\Omega^+} d\mathbf{U} \cdot \mathbf{V} \, dx &= \int_{\Omega^+} \nabla U_0 \cdot V_1 \, dx + \int_{\Omega^+} \nabla U_1 \cdot V_2 \, dx + \int_{\Omega^+} \nabla U_2 \cdot V_3 \, dx \\
&= -\int_{\Omega^+} U_0 \, (\text{div} \mathbf{V}_1) \, dx + \int_{\Omega^+} U_1 \cdot \text{curl} \mathbf{V}_2 \, dx - \int_{\Omega^+} U_2 \cdot \nabla V_3 \, dx \\
&\quad + \langle \gamma_{U_0}, \gamma_{n} V_1 \rangle_{\Gamma} + \langle \gamma_{U_1}, \gamma_{\tau} V_2 \rangle_{\Gamma} + \langle \gamma_{n} U_2, \gamma_{\tau} V_3 \rangle_{\Gamma} \\
&= \int_{\Omega^+} \mathbf{U} \cdot \delta \mathbf{V} \, dx + \langle \gamma_{T} \mathbf{U}, \gamma_{R} \mathbf{V} \rangle_{\Gamma}.
\end{align*}

\hfill \Box

**Corollary 2.4.** (Green’s formula for Dirac operator). *For all* $\mathbf{U}, \mathbf{V} \in \mathbf{H}(D, \Omega^+)$, *we have*

\begin{equation}
\int_{\Omega^+} D\mathbf{U} \cdot \delta \mathbf{V} \, dx = \int_{\Omega^+} \mathbf{U} \cdot D\mathbf{V} \, dx \pm \langle \gamma_{T} \mathbf{U}, \gamma_{R} \mathbf{V} \rangle_{\Gamma} \mp \langle \gamma_{T} \mathbf{V}, \gamma_{R} \mathbf{U} \rangle_{\Gamma}.
\end{equation}

*Remark 2.5.* It is remarkable that despite the fact that $D$ is a first-order operator, (2.15) nevertheless resembles Green’s classical second formula for the Laplacian. This
induces profound structural similarities between the representation formula, potentials and boundary integral equations for the Dirac operator established in the next sections and the already well-known theory for second-order elliptic operators. As emphasized in [39], a formula such as (2.15) paves the way for harnessing powerful established techniques.

We will indicate with curly brackets the average \( \{ \gamma \} := \frac{1}{2} (\gamma^- + \gamma^+) \) of a trace and with square brackets its jump \([\gamma] := \gamma^- - \gamma^+\) over the interface \(\Gamma\).

**Warning.** Notice the sign in the jump \([\gamma] = \gamma^- - \gamma^+\), which is often taken to be the opposite in the literature!

### 3. Boundary value problems.

In light of Lemma 2.1 and the duality in (2.12), the integration by parts formula (2.15) points towards two types of boundary conditions. Consider the boundary value problems of finding \(\vec{U} \in H(D; \Omega)\) satisfying

\[
\begin{aligned}
\{ \vec{D} \vec{U} \} &= \vec{0}, & \text{in } \Omega, \\
\{ \gamma_T \vec{U} \} &= \vec{b}, & \text{on } \Gamma,
\end{aligned}
\]

or

\[
\begin{aligned}
\{ \vec{D} \vec{U} \} &= \vec{0}, & \text{in } \Omega, \\
\{ \gamma_R \vec{U} \} &= \vec{a}, & \text{on } \Gamma,
\end{aligned}
\]

For \(\Omega = \Omega^+\), also impose the decay condition that \(\vec{U}(x) \to 0\) uniformly as \(x \to \infty\), cf. Lemma 4.16. In the following sections, development related to problem (T) will be colored in blue, while red will be used for (R).

When \(\Omega\) is bounded, the self-adjoint Dirac operator behind (R) is

\[
D_R^\Omega = d + d^*,
\]

where \(d : L^2(\Omega)^8 \to L^2(\Omega)^8\) is the closed densely defined Fredholm-nilpotent linear operator associated with the \(L^2\) de Rham cochain complex [1,26]

\[
H^1(\Omega) \xrightarrow{-\nabla} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega),
\]

cf. [1, Chap. 3-4], [26, Sec. 2]. The Hilbert space adjoint \(d^*\) is the nilpotent operator associated with the dual chain complex [1, Sec. 4.3, Thm. 6.5]

\[
L^2_+ (\Omega) \xrightarrow{-\text{div}} H_0(\text{div}, \Omega) \xrightarrow{\text{curl}} H_0(\text{curl}, \Omega) \xrightarrow{-\nabla} H^1_0(\Omega).
\]

The mapping properties of \(D_R\) and its domain are detailed in Figure 1.

Similarly, the self-adjoint operator

\[
D_T^\Omega := \delta + \delta^*
\]

behind (T) arises from the dual perspective, where we view the codifferential operator \(\delta : L^2(\Omega)^8 \to L^2(\Omega)^8\) as the nilpotent operator associated with the Hilbert chain complex

\[
L^2(\Omega) \xrightarrow{-\text{div}} H(\text{div}, \Omega) \xrightarrow{\text{curl}} H(\text{curl}, \Omega) \xrightarrow{-\nabla} H^1(\Omega).
\]
The adjoint $\delta^*$ is spawned by the chain complex

$$H^1_0(\Omega) \rightarrow H_0(\text{curl}, \Omega) \rightarrow H_0(\text{div}, \Omega) \rightarrow L_2^*(\Omega).$$

See Figure 2 for the explicit mapping properties of $D^T$ and its domain of definition.

So unlike second-order operators, the Hodge–Dirac operator admits two distinct fundamental symmetric bilinear forms

$$\mathcal{A}_d(\vec{U}, \vec{V}) = \int_\Omega \delta \vec{U} \cdot \vec{V} + \vec{U} \cdot \delta \vec{V} \, dx, \quad \vec{U}, \vec{V} \in H(\delta, \Omega),$$

$$\mathcal{A}_d(\vec{U}, \vec{V}) = \int_\Omega d \vec{U} \cdot \vec{V} + \vec{U} \cdot d \vec{V} \, dx, \quad \vec{U}, \vec{V} \in H(d, \Omega),$$

that rest on an equal footing. They readily appear upon integrating by parts with Lemma 2.3 and they are involved in the first-order analogs of Green’s identities

$$\int_{\Omega^+} D \vec{U} \cdot \vec{V} = \mathcal{A}_d(\vec{U}, \vec{V}) \pm \langle \gamma_T \vec{U}, \gamma_R \vec{V} \rangle_\Gamma,$$

$$\int_{\Omega^+} D \vec{U} \cdot \vec{V} = \mathcal{A}_d(\vec{U}, \vec{V}) \pm \langle \gamma_T \vec{V}, \gamma_R \vec{U} \rangle_\Gamma,$$

which hold for all $\vec{U}, \vec{V} \in H(D, \Omega)$.
These identities lead to the variational problems:

\[
\text{(VT)} \quad \vec{U} \in H(\delta, \Omega) : \quad A_\delta(\vec{U}, \vec{V}) = -\langle \vec{b}, \gamma R \vec{V} \rangle_\Gamma, \quad \forall \vec{V} \in H(\delta, \Omega),
\]

and

\[
\text{(VR)} \quad \vec{U} \in H(d, \Omega) : \quad A_d(\vec{U}, \vec{V}) = \langle \vec{a}, \gamma T \vec{V} \rangle_\Gamma, \quad \forall \vec{V} \in H(d, \Omega).
\]

3.1. Compatibility conditions. Either from Green’s second formula for the Dirac operator (2.15) or the variational problems themselves, we see that the boundary values \( \vec{b} \in H_T \) and \( \vec{a} \in H_R \) must fulfill compatibility conditions. For the problems to admit solutions, we require that

\[
\text{(CCT)} \quad \langle \vec{b}, \gamma R \vec{V} \rangle_\Gamma = 0, \quad \forall \vec{V} \in \mathcal{H}_T,
\]

and

\[
\text{(CCR)} \quad \langle \vec{a}, \gamma T \vec{V} \rangle_\Gamma = 0, \quad \forall \vec{V} \in \mathcal{H}_R,
\]

where

\[
\mathcal{H}_T(\Omega) := \{ \vec{V} \in H(D, \Omega) : \nabla \vec{V} = 0, \gamma T \vec{V} = \vec{0} \}
\]

and

\[
\mathcal{H}_R(\Omega) := \{ \vec{V} \in H(D, \Omega) : \nabla \vec{V} = 0, \gamma R \vec{V} = \vec{0} \}
\]

are spaces of harmonic vector-fields. We refer to [1–3] and [26] for explanations on how these spaces exactly correspond to the nullspaces of the Hodge-Laplacian with natural and essential boundary conditions.

The fact that there are two distinct bilinear forms in the expressions (VT) and (VR) is one of the appealing use of the dual perspective involving the codifferential \( \delta \). It points to the symmetry presented in Remark 5.1 below, and it highlights the necessity of imposing compatibility conditions on the data. For example, we could alternatively formulate (T) as the variational problem

\[
\text{(3.10)} \quad \vec{U} \in H(d, \Omega) \text{ with } \quad \gamma T \vec{U} = \vec{b} : \quad A_d(\vec{U}, \vec{V}) = 0, \quad \forall \vec{V} \in H_0(d, \Omega),
\]

where \( H_0(d, \Omega) = H_0^1(\Omega) \times H_0(\text{curl}, \Omega) \times H_0(\text{div}, \Omega) \times L^2(\Omega) \). But according to (2.15) the condition (CCT) must remain, and it now appears less obviously so when the type of boundary condition is essential. Anyway, in a formulation such as (3.10), one proceeds with a lifting of the boundary data and is left with the solvability of the problem

\[
\text{(3.11)} \quad \vec{U}_0 \in H_0(d, \Omega) : \quad A_d(\vec{U}_0, \vec{V}) = -A_d(\delta T \vec{b}, \vec{V}), \quad \forall \vec{V} \in H_0(d, \Omega).
\]

So the question of compatibility cannot be avoided: integrating by parts with the right-hand side evaluated at a nullspace element in \( \mathcal{H}_T \) using (3.8b) leads to (CCT). We discuss in greater details the reason why the two boundary conditions can be formulated both as natural and essential in Remark 5.1.
3.2. Well-posedness. Since the bilinear form $\mathcal{A}_δ$ is associated with the self-adjoint operator $D_T$ obtained from the chain complex (3.6) and $\mathcal{A}_d$ to the self-adjoint operator $D_R$ spawned by the cochain complex (3.2), they fit the framework of [26, Sec. 2]. The abstract inf-sup inequality supplied in [26, Thm. 6] applies to both bilinear forms and leads to well-posedness of the mixed variational problems:

\[
\mathcal{A}_δ(\vec{U}, \vec{V}) + \left( \vec{P}, \vec{V} \right)_\Omega = -\left( \vec{b}, \gamma \vec{V} \right)_\Gamma \quad \forall \vec{V} \in H(\delta, \Omega^-),
\]

\[
\left( \vec{U}, \vec{W} \right)_\Omega = 0 \quad \forall \vec{W} \in \ker D_T,
\]

and

\[
\mathcal{A}_d(\vec{U}, \vec{V}) + \left( \vec{Q}, \vec{V} \right)_\Omega = \left( \vec{a}, \gamma \vec{V} \right)_\Gamma \quad \forall \vec{V} \in H(d, \Omega^-),
\]

\[
\left( \vec{U}, \vec{W} \right)_\Omega = 0 \quad \forall \vec{W} \in \ker D_R,
\]

for unknown pairs $(\vec{U}, \vec{P}) \in H(\delta, \Omega^-) \times \ker D_T$ and $(\vec{U}, \vec{Q}) \in H(d, \Omega^-) \times \ker D_R$.

Consistency of the right-hand side in (VT) exactly corresponds to requiring that (CCT) holds for the given data $\vec{b} \in \mathcal{H}_\Gamma$, while (CCR) similarly guarantees consistency of the right-hand side in (VR). We conclude that if the compatibility conditions are satisfied, solutions to (VT) and (VR) in $\Omega^-$ are unique up to contributions of harmonic vector-fields in $\ker D_T$ and $\ker D_R$. Moreover, they continuously depend on the boundary data.

4. Representation formulas. We derive interior and exterior representation formulas for solutions of the Dirac equation. It is expressed through known boundary potentials, whose jump properties across $\Gamma$ are elaborated.

4.1. Fundamental solution. Convolution of a vector field $\vec{U} : \mathbb{R}^3 \to \mathbb{R}^8$ by a matrix-valued function $K : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^{8 \times 8}$ possibly having a singularity at the $0 \in \mathbb{R}^3$ is defined, if the limit exists, as the Cauchy principal value

\[
(K * \vec{U})(x) := \lim_{\epsilon \to 0} \int_{\mathbb{R}^3 \setminus B_\epsilon(0)} K(x - y)\vec{U}(y) \, dy \in \mathbb{R}^8,
\]

where $B_\epsilon(0) \subset \mathbb{R}^3$ is a ball of radius $\epsilon$ centered at the origin.

Let $G : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ be given by $G(z) := (4\pi |z|)^{-1}$, and set

\[
G(z) := G(z) I_8 \in \mathbb{R}^{8 \times 8}, \quad z \neq 0,
\]

where $I_8$ is the identity matrix on $\mathbb{R}^8$. Then, define $\Phi : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^{8 \times 8}$ by applying the Dirac operator to the columns of $G$ as

\[
\Phi(z) := \begin{pmatrix}
0 & -(\nabla G)^T(z) & 0^T & 0 \\
(\nabla G)(z) & 0_{3 \times 3} & A_{3 \times 3}(z) & 0 \\
0 & A_{3 \times 3}(z) & 0_{3 \times 3} & -(\nabla G)^T(z) \\
0 & 0^T & (\nabla G)^T(z) & 0
\end{pmatrix} \in \mathbb{R}^{8 \times 8}, \quad z \neq 0,
\]

where the anti-symmetric blocks

\[
A_{3 \times 3}(z) := \begin{pmatrix}
0 & -(\partial_3 G)(z) & (\partial_2 G)(z) \\
(\partial_3 G)(z) & 0 & -(\partial_1 G)(z) \\
-(\partial_2 G)(z) & (\partial_1 G)(z) & 0
\end{pmatrix} \in \mathbb{R}^{3 \times 3}, \quad z \neq 0,
\]

are associated with the curl operator.
Lemma 4.1. For $z \neq 0$,
\begin{equation}
\Phi (-z) = -\Phi (z) \quad \text{and} \quad \Phi (z) \vec{U} \cdot \vec{V} = -\vec{U} \cdot \Phi (z) \vec{V}
\end{equation}
for all $\vec{U}, \vec{V} \in \mathbb{R}^8$.

Proof. Let $s : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the sign flip operation $s(z) = -z$. For the fist identity,
we simply rely on the fact that $G(x) = G(|x|)$ to verify that for any $\vec{U} \in \mathbb{R}^8$,
\begin{equation}
\Phi (-z) \vec{U} = D(G(\vec{U}))(s(z)) = -D_x(G(s(x)) \vec{U})|_{x=z} = -D_x(G(x) \vec{U})|_{x=z} = -\Phi (z) \vec{U}.
\end{equation}
The second identity is clear by definition.

This lemma allows to extend the domain of the Newton-type potential $N$:
\begin{equation}
N : C_0^\infty (\mathbb{R}^3)^8 \rightarrow C^\infty (\mathbb{R}^3)^8
\end{equation}
\begin{equation}
\vec{U} \mapsto \Phi * \vec{U}
\end{equation}
to distributions.

Lemma 4.2. For all $\vec{U}, \vec{V} \in C_0^\infty (\mathbb{R}^3)^8$,
\begin{equation}
(N \vec{U}, \vec{V}) = (\vec{U}, N \vec{V}).
\end{equation}

Proof. Using Lemma 4.1, we can change the order of integration using Fubini’s theorem and evaluate
\begin{equation}
(N \vec{U}, \vec{V}) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi (x - y) \vec{U}(y) \cdot \vec{V}(x) \, dx \, dy
\end{equation}
\begin{equation}
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \vec{U}(y) \cdot \Phi (y - x) \vec{V}(x) \, dx \, dy
\end{equation}
\begin{equation}
= \int_{\mathbb{R}^3} \vec{U}(y) \cdot \int_{\mathbb{R}^3} \Phi (y - x) \vec{V}(x) \, dx \, dy
\end{equation}
\begin{equation}
= (\vec{U}, N \vec{V}).
\end{equation}

Remark 4.3. Lemma 4.2 reflects the fact that the Dirac operator is symmetric as an unbounded operator on $(L^2(\mathbb{R}^3))^8$.

The extension
\begin{equation}
N : (C^\infty (\mathbb{R}^3)^8)' \rightarrow (C_0^\infty (\mathbb{R}^3)^8)'
\end{equation}
is obtained as in [37, Sec. 3.1.1] via dual mapping by defining the action of the distribution $N \vec{U} \in (C_0^\infty (\mathbb{R}^3)^8)'$ on $\vec{V} \in C_0^\infty (\mathbb{R}^3)^8$ as
\begin{equation}
(N \vec{U}, \vec{V}) := \langle \vec{U}, N \vec{V} \rangle.
\end{equation}

Proposition 4.4 (Fundamental solution). For all compactly supported distributions $\vec{U} \in (C^\infty (\mathbb{R}^3)^8)'$,
\begin{equation}
N D \vec{U} = \vec{U} = D N \vec{U}
\end{equation}
holds in $(C_0^\infty (\mathbb{R}^3)^8)'$. 
Proof. We first show that for $\tilde{U} \in (C^\infty(\mathbb{R}^3)^8)'$,

\begin{equation}
\langle N D\tilde{U}, \tilde{V} \rangle = \langle \tilde{U}, \tilde{V} \rangle
\end{equation}

for all $\tilde{V} \in C^\infty_0(\mathbb{R}^3)^8$.

The argument is inspired by the proof of [18, Thm.1]. Let $e_i \in \mathbb{R}^3$ be the vector with 1 at the i-th entry and zeros elsewhere, $i = 1, 2, 3$. Since

\begin{equation}
N\tilde{V} = \int_{\mathbb{R}^3} \Phi(x - y) \tilde{V}(y) \, dy = \int_{\mathbb{R}^3} \Phi(y) \tilde{V}(x - y) \, dy,
\end{equation}

we have

\begin{equation}
\frac{N\tilde{V}(x + he_i) - N\tilde{V}(x)}{h} = \int_{\mathbb{R}^3} \Phi(y) \frac{\tilde{V}(x + he_i - y) - \tilde{V}(x - y)}{h} \, dy.
\end{equation}

Hence,

\begin{equation}
D_x N\tilde{V}(x) = \int_{\mathbb{R}^3} \Phi(y) D\tilde{V}(x - y) \, dy,
\end{equation}

because the assumption that $\tilde{V}$ is smooth and compactly supported guarantees that

\begin{equation}
\frac{\tilde{V}(x + he_i - y) - \tilde{V}(x - y)}{h} \to \frac{\partial}{\partial x_i} \tilde{V}(x - y)
\end{equation}

uniformly for $h \to 0$. The main idea is to isolate $\Phi$’s singularity at the origin by splitting the right hand side of (4.17) into two integrals as

\begin{equation}
D_x N\tilde{V}(x) = \int_{B_\epsilon(0)} \Phi(y) D\tilde{V}(x - y) \, dy + \int_{\mathbb{R}^3 \setminus B_\epsilon(0)} \Phi(y) D\tilde{V}(x - y) \, dy
\end{equation}

whose limits as $\epsilon \to 0$ we can control.

The main difficulty is that we cannot readily mimic the standard proof commonly given for the Poisson equation, because the integration by parts formula supplied for the product of two vectors by (2.15) is not applicable to the matrix–vector multiplication involved in the integrands of (4.19). The analysis of

\begin{equation}
\Phi(y) D\tilde{V}(x - y) = \begin{pmatrix}
-\nabla G(y) \cdot \nabla V_0(x - y) - \nabla G(y) \cdot \text{curl} V_1(x - y) \\
-\text{div} V_1(x - y) \nabla G(y) - \nabla G(y) \times \nabla V_3(x - y) + \nabla G(y) \times \text{curl} V_2(x - y) \\
-\nabla G(y) \times \nabla V_1(x - y) + \nabla G(y) \times \text{curl} V_2(x - y) - \text{div} V_2(x - y) \nabla G(y) \\
-\nabla G(y) \cdot \nabla V_3(x - y) + \nabla G(y) \cdot \text{curl} V_2(x - y)
\end{pmatrix}
\end{equation}

is carried out component-wise.

There are five different types of terms whose limit need to be investigated. Let $V \in (C^\infty_0(\mathbb{R}^3))^3$ and $\tilde{V} \in C^\infty_0(\mathbb{R}^3)$ be arbitrary fields. To ease the reading, we write $V_{\epsilon}(x) := V(x - y)$ and $\tilde{V}_{\epsilon}(x) := \tilde{V}(x - y)$. We denote by $n$ the unit normal vector field pointing towards the interior of $B_\epsilon(0)$.
Similarly, integrating by parts yields

\[
\int_{\mathbb{R}^3 \setminus \{0\}} \nabla G(y) \cdot \nabla V(x - y) dy = -\int_{\partial B_r(0)} \nabla G(y) \cdot \mathbf{n}_r(y) V(x - y) d\sigma(y)
\]

\[
= \frac{1}{4\pi} \int_{\partial B_r(0)} \frac{V(x - y)}{|y|^3} \left(-\mathbf{y} \cdot \frac{\mathbf{y}}{|\mathbf{y}|}\right) d\sigma(y) = -\frac{1}{4\pi \epsilon^2} \int_{\partial B_r(0)} V(x - y) d\sigma(y)
\]

\[
= -\int_{\partial B_r(0)} V(y) d\sigma(y) \xrightarrow{\epsilon \to 0} -V(x)
\]

and

\[
\int_{\mathbb{R}^3 \setminus \{0\}} \nabla G(y) \cdot \nabla V(x - y) dy
\]

\[
= -\int_{\partial B_r(0)} \left(\nabla G(y) \times \mathbf{n}_r(y)\right) \cdot V(x - y) d\sigma(y)
\]

\[
= -\frac{1}{4\pi \epsilon^2} \int_{\partial B_r(0)} (\mathbf{y} \times \mathbf{y}) \cdot V(x - y) d\sigma(y) = 0.
\]

Similarly, integrating by parts component-wise yields

\[
\int_{\mathbb{R}^3 \setminus \{0\}} \nabla G(y) \times \nabla V_x(y) dy
\]

\[
= \int_{\mathbb{R}^3 \setminus \{0\}} \left(\frac{\partial_2 G(y) \partial_1 V_x(y) - \partial_3 G(y) \partial_2 V_x(y)}{\partial_1 G(y) \partial_2 V_x(y) - \partial_2 G(y) \partial_1 V_x(y)}\right) dy
\]

\[
= \int_{\mathbb{R}^3 \setminus \{0\}} \left(\frac{2G(y) \partial_2 \partial_3 V_x(y) - G(y) \partial_3 \partial_2 V_x(y)}{G(y) \partial_2 \partial_1 V_x(y) - G(y) \partial_1 \partial_2 V_x(y)}\right) dy
\]

\[
+ \int_{\partial B_r(0)} \left(-\left(\mathbf{n}_r\right)_2(y) G(y) \partial_3 V_x(y) + \left(\mathbf{n}_r\right)_3(y) G(y) \partial_2 V_x(y)\right) dy
\]

Since \( V \) is smooth everywhere in \( \mathbb{R}^3 \), partial derivatives commute and the volume integral vanishes, leading to

\[
\int_{\mathbb{R}^3 \setminus \{0\}} \nabla G(y) \times \nabla V_x(y) dy = -\int_{\partial B_r(0)} G(y) \mathbf{n}_r(y) \times \nabla V_x(y) d\sigma(y).
\]

This integral vanishes under the limit \( \epsilon \to 0 \), because

\[
\sup_{x \in \mathbb{R}^3} \left| \int_{\partial B_r(0)} G(y) \mathbf{n}_r(y) \times \nabla V(x - y) d\sigma(y) \right|
\]

\[
\leq \| \nabla V \|_\infty \int_{\partial B_r(0)} |G(y)| d\sigma(y) = O(\epsilon).
\]

Moving on to the next term, one eventually obtains from similar calculations that

\[
\int_{\mathbb{R}^3 \setminus \{0\}} \nabla G(y) \times \nabla V_x(y) dy = \int_{\mathbb{R}^3 \setminus \{0\}} G(y) \nabla V_x(y) dy
\]

\[
+ \int_{\partial B_r(0)} G(y) \left(\nabla V_x(y) \times \mathbf{n}_r(y)\right) d\sigma(y).
\]
Since \( \| \text{curl} \, V \|_\infty < \infty \), the boundary integral on the right hand side vanishes under the limit by repeating the argument of \( (4.25) \). Finally, commuting partial derivatives after integrating by parts also yields

\[
(4.27) \quad \int_{\mathbb{R}^3 \backslash B_i(0)} \nabla G(x - y) \cdot \nabla V \, dy = \int_{\mathbb{R}^3 \backslash B_i(0)} G(y) \nabla \text{div} V(x - y) - \int_{\partial B_i(0)} G(y) \text{div} V(x - y) \, n_\gamma(y) \, d\sigma(y)
\]

Putting the two previous calculations together, we find that

\[
(4.28) \quad \lim_{\epsilon \to 0} \int_{\mathbb{R}^3 \backslash B_i(0)} \nabla G(y) \times \text{curl} \, V(x - y) - \text{div} \, V(x - y) \nabla G(y) \, dy
\]

\[
= -\lim_{\epsilon \to 0} \int_{\mathbb{R}^3 \backslash B_i(0)} G(y) \Delta V(x - y) \, dy = V(x),
\]

where we recognized the vector (Hodge-) Laplace operator \(-\Delta \equiv \text{curl} \, \text{curl} - \nabla \text{div} \). We have found that \( J_\epsilon \to \widehat{V}(x) \) as \( \epsilon \to 0 \). Meanwhile,

\[
(4.29) \quad \| J_\epsilon \|_{\infty} \leq \| D\nabla \Phi \|_{\infty} \int_{B_i(0)} \| \Phi \|_{\infty} \, dy = \Theta \left( \int_{B_i(0)} \| \nabla G \|_\infty \, dy \right) = \Theta(\epsilon).
\]

The calculations for \( \widehat{U} = D \, N \, \Phi \) follow similarly starting from \( (4.17) \).

In light of Proposition 4.4, we say that the kernel \( \Phi \) of \( N \) is a fundamental solution for the Dirac operator.

4.2. Surface potentials. Adopting the perspective on first-kind boundary integral operators from [16, 32, 37] and [14]—in the later works for the study of second-order elliptic operators—for the first-order Dirac operator, we define the surface potentials

\[
(4.30) \quad \mathcal{L}_T(\vec{a}) := N(\gamma_T \vec{a}), \quad \forall \, \vec{a} = (a_0, a_1, a_2) \in \mathcal{H}_R,
\]

\[
(4.31) \quad \mathcal{L}_R(\vec{b}) := -N(\gamma_R \vec{b}), \quad \forall \, \vec{b} = (b_0, b_1, b_2) \in \mathcal{H}_T,
\]

where the mappings \( \gamma_T : \mathcal{H}_R = \mathcal{H}_T' \to \mathcal{H}_{loc}(D, \mathbb{R}^3 \backslash \overline{\Omega})' \) and \( \gamma_R : \mathcal{H}_T = \mathcal{H}_R' \to \mathcal{H}_{loc}(D, \mathbb{R}^3 \backslash \overline{\Omega})' \) are adjoint to the trace operators \( \gamma_T \) and \( \gamma_R \) defined in \( (2.10) \).

It will be convenient to denote by \( \Phi_x \) the map \( y \mapsto \Phi(x - y) \). Let \( \vec{E}_j \in \mathbb{R}^8 \) denote the constant vector with 1 at the \( j \)-th entry and zeros elsewhere, \( j = 1, \ldots, 8 \). Similarly for \( \vec{E}_k \in \mathbb{R}^3, \, k = 1, 2, 3 \).

Adapting the calculations found in [14, Sec. 4.2], we will establish integral representation formulas for these potentials by splitting the pairings into their components.

**Lemma 4.5.** Given \( \vec{a} \in \mathcal{H}_R \) and \( \vec{b} \in \mathcal{H}_T \), it holds for \( x \in \Omega \setminus \Gamma \) that

\[
(4.32a) \quad \mathcal{L}_T(\vec{a})(x) \cdot \vec{E}_j = \langle \vec{a}, \gamma_T(\Phi_x \vec{E}_j) \rangle_{\Gamma},
\]

\[
(4.32b) \quad \mathcal{L}_R(\vec{b})(x) \cdot \vec{E}_j = \langle \vec{b}, \gamma_R(\Phi_x \vec{E}_j) \rangle_{\Gamma}.
\]
Proof. Let \( V \in C^0_0(\mathbb{R}^3) \) and suppose that \( \tilde{a} \) is the trace of a smooth 8-dimensional vector-field. Using Fubini’s theorem and the fact that \( \Phi \) is smooth away from the origin,

\[
\begin{align*}
(4.33) & \quad \langle N(\gamma \tilde{a}), V E_j \rangle_{\mathbb{R}^3} = \langle \tilde{a}, \gamma T N(V E_j) \rangle_T \\
(4.34) & \quad = \int_T \tilde{a}(y) \cdot \gamma T \int_{\mathbb{R}^3} \Phi(y - x) V(x) E_j(x) \, dx \, d\sigma(y) \\
(4.35) & \quad = -\int_{\mathbb{R}^3} V(x) \left( \int_T \tilde{a}(y) \cdot \gamma T \Phi(x - y) E_j \, d\sigma(y) \right) \, dx,
\end{align*}
\]

where the sign was obtained in (*) thanks to Lemma 4.1. The integrals on the right-hand side of (4.35) can be extended to duality pairings by a standard density argument exploiting Lemma 2.1.

Similar calculations can be carried out for \( L_R \).

In particular,

\[
(4.36) \quad \Phi_x(y) E_1 = \begin{pmatrix} 0 \\ \nabla G(x - y) \\ 0 \end{pmatrix}, \quad \Phi_x(y) E_8 = \begin{pmatrix} 0 \\ 0 \\ -\nabla G(x - y) \end{pmatrix},
\]

\[
(4.37) \quad \Phi_x(y) E_i = \begin{pmatrix} -\partial_{z_{\mu(i)}} G(z) \\ 0 \\ \nabla G(z) \times E_{\mu(i)} \end{pmatrix} \bigg|_{z = x - y}, \quad \Phi_x(y) E_k = \begin{pmatrix} 0 \\ \nabla G(z) \times E_{\nu(k)} \\ 0 \end{pmatrix} \bigg|_{z = x - y},
\]

for \( i = 2, 3, 4, k = 5, 6, 7, \mu(i) = i - 1 \) and \( \nu(k) = k - 4 \).

Therefore, we can evaluate

\[
(4.38a) \quad L_T(\tilde{a})(x) \cdot E_1 = -\int_T a_1(y) \cdot \nabla G(x - y) \, d\sigma(y)
\]

\[
(4.38b) \quad L_T(\tilde{a})(x) \cdot E_i = \int_T a_0(y) \partial_{\mu(i)} G(x - y) \, d\sigma(y)
- \int_T a_2(y) \left( \nabla G(x - y) \times E_{\mu(i)} \right) \cdot n(y) \, d\sigma(y)
+ \partial_{\mu(i)} \int_T a_0(y) G(x - y) \, d\sigma(y)
+ E_{\mu(i)} \cdot \int_T a_2(y) \nabla G(x - y) \times n(y) \, d\sigma(y)
\]

\[
(4.38c) \quad L_T(\tilde{a})(x) \cdot E_k = -\int_T a_1(y) \cdot \left( \nabla G(x - y) \times E_{\nu(k)} \right) \, d\sigma(y)
= E_{\nu(k)} \cdot \int_T \nabla_y G(x - y) \times a_1(y) \, d\sigma(y)
\]

\[
(4.38d) \quad L_T(\tilde{a})(x) \cdot E_8 = \int_T a_2(y) \nabla_y G_x(y) \cdot n(y) \, d\sigma(y),
\]
where we have used the fact that $a_l \in H^{-1/2}(\text{div}, \Gamma)$ was “tangential” to safely drop the trace $\gamma_\nu$ everywhere. Similarly as in the proof of Lemma 4.5, all these integrals should be understood as duality pairings and the following explicit representations do not only hold in the sense of distributions, but also pointwise on $\mathbb{R}^3 \setminus \Gamma$.

We collect the above entries to obtain

\begin{equation}
\mathcal{L}_\Gamma (\vec{a}) = \begin{pmatrix}
- \text{div } \Psi (a_1) \\
\nabla \psi (a_0) + \text{curl } \Upsilon (a_2) \\
\text{curl } \Psi (a_1) \\
\text{div } \Upsilon (a_2)
\end{pmatrix}, \quad \text{pointwise on } \mathbb{R}^3 \setminus \Gamma,
\end{equation}

where we respectively recognize in

\begin{align}
\psi(q)(x) &:= \int_\Gamma q(y)G(x - y) \, d\sigma(y), \quad x \in \mathbb{R}^3 \setminus \Gamma, \\
\Psi(p)(x) &:= \int_\Gamma p(y)G(x - y) \, d\sigma(y), \quad x \in \mathbb{R}^3 \setminus \Gamma, \\
\Upsilon(q)(x) &:= \int_\Gamma q(y)G(x - y) \, n(y) \, d\sigma(y), \quad x \in \mathbb{R}^3 \setminus \Gamma,
\end{align}

the well-known single layer, vector single layer and normal vector single layer potentials. They notably enter (4.39) in the expression for the classical double layer potential $\text{div } \Upsilon (q)$ and for the Maxwell double layer potential $\text{curl } \Psi (p)$ as they arise in acoustic and electromagnetic scattering respectively.

Similarly, for $i = 2, 3, 4$ and $k = 5, 6, 7$,

\begin{align}
\mathcal{L}_\Gamma (\vec{b})(x) \cdot \vec{E}_1 &= \int_\Gamma b_0(y) \, \nabla G (x - y) \cdot n(y) \, d\sigma(y) \\
\mathcal{L}_\Gamma (\vec{b})(x) \cdot \vec{E}_i &= \int_\Gamma b_1(y) \cdot \left( \nabla G (x - y) \times E_{\mu(i)} \right) \times n(y) \, d\sigma(y) \\
&= \int_\Gamma \left( \nabla G (x - y) \times E_{\mu(i)} \right) \cdot n(y) \times b_1(y) \, d\sigma(y) \\
&= E_{\mu(i)} \cdot \int_\Gamma (n(y) \times b_1(y)) \times \nabla G(x - y) \, d\sigma(y) \\
\mathcal{L}_\Gamma (\vec{b})(x) \cdot \vec{E}_k &= \int_\Gamma b_0(y) \left( \nabla G(x - y) \times E_{\nu(k)} \right) \cdot n(y) \, d\sigma(y) \\
&\quad + \int_\Gamma \delta_2(y) \, \partial_j G(x - y) \, d\sigma(y) \\
&= E_{\nu(k)} \cdot \int_\Gamma b_0(y) \, n(y) \times \nabla G(x - y) \, d\sigma(y) \\
&\quad + \int_\Gamma \delta_2(y) \, \partial_j G(x - y) \, d\sigma(y) \\
\mathcal{L}_\Gamma (\vec{b})(x) \cdot \vec{E}_8 &= - \int_\Gamma b_1(y) \cdot \nabla G(x - y) \times n(y) \, d\sigma(y) \\
&= - \int_\Gamma \nabla G(x - y) \cdot n(y) \times b_1(y) \, d\sigma(y)
\end{align}
so that we have

\[
L_R \begin{bmatrix} \text{div} \Psi (b_0) \\ \text{curl} \Psi (b_1 \times n) \\ - \text{curl} \Phi (b_0) + \nabla \psi (b_2) \\ \text{div} \Phi (b_1 \times n) \end{bmatrix}, \text{ pointwise on } \mathbb{R}^3 \setminus \Gamma.
\]

\[\textbf{4.3. Mapping properties of the surface potentials.} \text{ Fortunately, we already know a lot about each potential entering } (4.39) \text{ and } (4.42).
\]

**Lemma 4.6.** The potentials \(L_T : H_T \rightarrow H(D, \mathbb{R}^3 \setminus \Gamma)\) and \(L_R : H_T \rightarrow H(D, \mathbb{R}^3 \setminus \Gamma)\) explicitly given by (4.39) and (4.42) are continuous.

**Proof.** Recall that if \(b_1 \in H^{-1/2}(\text{curl}, \Gamma)\), then \(n \times b_1 \in H^{-1/2}(\text{div}, \Gamma)\). So the proof simply boils down to extracting from the discussion of Section 5 in [14] the mapping properties

\[
(4.43a) \quad \nabla \psi : H^{-1/2}(\Gamma) \rightarrow H_{\text{loc}}(\text{curl}^2, \mathbb{R}^3 \setminus \Gamma) \cap H_{\text{loc}}(\nabla \text{div}, \mathbb{R}^3 \setminus \Gamma),
\]

\[
(4.43b) \quad \text{div} \Psi : H^{1/2}(\Delta, \mathbb{R}^3 \setminus \Gamma),
\]

\[
(4.43c) \quad \text{curl} \, \Psi : H^{-1/2}(\text{div}, \Gamma) \rightarrow H_{\text{loc}}(\text{curl}, \mathbb{R}^3 \setminus \Gamma),
\]

\[
(4.43d) \quad \text{div} \Psi : H^{-1/2}(\text{div}, \Gamma) \rightarrow H_{\text{loc}}(\nabla \text{div}, \mathbb{R}^3 \setminus \Gamma),
\]

\[
(4.43e) \quad \text{curl} \, \Psi : H^{-1/2}(\text{div}, \Gamma) \rightarrow H_{\text{loc}}(\text{curl}, \mathbb{R}^3 \setminus \Gamma).
\]

Since \(\text{div} \circ \text{curl} \equiv 0\), we have in particular

\[
(4.44a) \quad \text{curl} \, \Psi : H^{-1/2}(\text{div}, \Gamma) \rightarrow H_{\text{loc}}(\text{curl}, \mathbb{R}^3 \setminus \Gamma) \cap H_{\text{loc}}(\text{div}, \mathbb{R}^3 \setminus \Gamma),
\]

\[
(4.44b) \quad \text{curl} \, \Phi : H^{-1/2}(\text{div}, \Gamma) \rightarrow H_{\text{loc}}(\text{curl}, \mathbb{R}^3 \setminus \Gamma) \cap H_{\text{loc}}(\text{div}, \mathbb{R}^3 \setminus \Gamma).
\]

Now, for \(z \neq 0\), the kernels of the two surface potentials decay as

\[
\| \nabla G(z) \| \lesssim \| z \|^{-2},
\]

thus are not only square-integrable locally, but in fact belong to \((L^2(\mathbb{R}^3 \setminus \Gamma))^8\). \(\Box\)

The next lemma shows that the surface potentials solve the homogeneous Dirac equation.

**Lemma 4.7.** For all \(\tilde{b} \in \mathcal{H}_T\) and \(\tilde{a} \in \mathcal{H}_R\), it holds on \(\mathbb{R}^3 \setminus \Gamma\) that

\[
(4.45a) \quad D L_R (\tilde{b}) \equiv \tilde{0},
\]

\[
(4.45b) \quad D L_T (\tilde{a}) \equiv \tilde{0}.
\]

**Proof.** The well-known vector and scalar potentials of (4.40) are harmonic. Hence,
since \( \text{div} \circ \text{curl} \equiv 0 \) and \( \text{curl} \circ \nabla \equiv 0 \), we directly evaluate

\[
\mathbb{D}L_T(\vec{a}) = \begin{pmatrix}
-\Delta \psi(a_0) \\
-\nabla \psi(a_0) + \nabla \Psi(a_1) \\
curl \nabla \psi(a_0) + curl \Psi(a_1) \\
\text{div} \Psi(a_1) \\
\text{div} \psi(a_1) + curl \Psi(a_1) \\
0
\end{pmatrix} = 0.
\]

A similar calculation holds for \( \mathbb{D}L_R(\vec{b}) \).

**Remark 4.8.** Lemma 4.7 was proved using the explicit representations (4.39) and (4.42). The technique revealed some structure behind the two boundary potentials. However, notice that adapting the argument found in the proof of [37, Thm. 3.1.6], the desired result could also be obtained by observing that

\[
\gamma_T^0 : \mathcal{H}_R \to (H_{\text{loc}}(D, \mathbb{R}^3 \setminus \Gamma))' \subset (C^\infty(\mathbb{R}^3 \setminus \Gamma)^8)',
\]

together with Proposition 4.4, guarantees the equality \( \mathbb{D}L_T \vec{a} = \gamma_T^0 \vec{a} \) as continuous linear functionals on \( C_0^\infty(\mathbb{R}^3 \setminus \Gamma) \).

**Remark 4.9.** It is a nice and unusual property for the potentials to belong to \( (L^2(\Omega^+))^8 \) as opposed to being only locally square-integrable. We see from Lemma 4.5 that this is a consequence of two ingredients: the stronger singularity of the Dirac fundamental solution, combined with the absence of differential operators acting on the relevant traces.

**Lemma 4.10 (Jump relations).** For all \( \vec{a} \in \mathcal{H}_R \) and \( \vec{b} \in \mathcal{H}_T \),

\[
\begin{align*}
[\gamma_T] L_T(\vec{a}) &= 0, \\
[\gamma_R] L_T(\vec{a}) &= \text{Id}, \\
[\gamma_T] L_R(\vec{b}) &= \text{Id}, \\
[\gamma_R] L_R(\vec{b}) &= 0.
\end{align*}
\]

**Proof.** For the most part, the following jump relations can be inferred from known theory. We carefully evaluate

\[
\begin{align*}
[\gamma_T] L_T(\vec{a}) &= \begin{pmatrix}
-\gamma_1 \nabla \psi(a_0) + \gamma_3 \text{curl} \Psi(a_1) \\
\gamma_1 \nabla \psi(a_0) + \gamma_3 \text{curl} \Psi(a_1) \\
\gamma_2 \text{curl} \Psi(a_1) \\
\gamma_2 \nabla \Psi(a_1) \\
\gamma_2 \nabla \Psi(a_1) \\
0
\end{pmatrix} = \begin{pmatrix}0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\
[\gamma_R] L_T(\vec{a}) &= \begin{pmatrix}
-\gamma_1 \text{div} \Psi(a_1) \\
\gamma_1 \nabla \psi(a_0) + \gamma_3 \text{curl} \Psi(a_1) \\
\gamma_2 \text{curl} \Psi(a_1) \\
\gamma_2 \nabla \Psi(a_1) \\
\gamma_2 \nabla \Psi(a_1) \\
\gamma_0 \Psi(a_0)
\end{pmatrix} = \begin{pmatrix}a_0 \\ a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\
[\gamma_T] L_R(\vec{b}) &= \begin{pmatrix}
\gamma_0 \text{div} \Psi(b_0) \\
\gamma_1 \nabla \psi(b_0) + \gamma_2 \text{curl} \Psi(b_1 \times n) \\
\gamma_2 \text{curl} \Psi(b_1 \times n) \\
\gamma_1 \nabla \Psi(b_0) + \gamma_2 \nabla \psi(b_2) \\
\gamma_1 \nabla \Psi(b_0) + \gamma_2 \nabla \psi(b_2) \\
\gamma_0 \Psi(b_0)
\end{pmatrix} = \begin{pmatrix}b_0 \\ b_1 \\ b_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\
[\gamma_R] L_R(\vec{b}) &= \begin{pmatrix}
-\gamma_1 \text{div} \Psi(b_0) \\
\gamma_1 \nabla \psi(b_0) + \gamma_3 \text{curl} \Psi(b_1 \times n) \\
\gamma_2 \text{curl} \Psi(b_1 \times n) \\
\gamma_1 \nabla \Psi(b_0) + \gamma_3 \nabla \psi(b_2) \\
\gamma_1 \nabla \Psi(b_0) + \gamma_3 \nabla \psi(b_2) \\
\gamma_0 \Psi(b_0)
\end{pmatrix} = \begin{pmatrix}0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\end{align*}
\]

The individual terms appearing in the above calculations can be found in [14, Sec. 5] and [21, Sec. 4], possibly up to tangential rotation by 90°. Some terms slightly
differ. In both (4.50b) and (4.50c), we are particularly concerned with the normal jump of \( \text{curl} \, \Upsilon \) across \( \Gamma \). Fortunately, we know that the restriction of \( \Upsilon \) to \( H^{1/2}(\Gamma) \) is a continuous map with codomain \( H_{\text{loc}}(\text{curl}^2, \Omega) \). Its image is therefore regular enough for the identity

\[
[\gamma_n] \text{curl} \, \Upsilon (q) = \text{div}_\Gamma ([\gamma_n] \, \Upsilon (q)) = 0
\]

to hold for all \( q \in H^{1/2}(\Gamma) \) [12, Eq. 8].

**Remark 4.11.** The formal structure of these jump relations is the same as that of the jump identities for the potentials associated with other operators such as

- scalar second-order strongly elliptic operators [32, 37],
- second-order Maxwell wave operators [10, 12],
- Hodge–Laplace and Hodge–Helmholtz operators [14, 15].

### 4.4. Representation by surface potentials

Following McLean in [32, Chap. 7], we mimic the approach introduced by Costabel and Dauge [16, 17]. Corollary 2.4 plays the role of Green’s second identity. We begin with the case where a solution

\[
\gamma_n \Upsilon, \gamma_T \Upsilon \text{ are continuous across } \Gamma.
\]

**Proposition 4.12 (Interior representation formula).** If \( \hat{\Upsilon} \in H(D, \mathbb{R}^3 \setminus \Gamma) \) is compactly supported and \( \hat{\Phi} \in (L^2(\mathbb{R}^3))^8 \) is such that \( \hat{\Phi}|_\Gamma := (D \Phi)|_\Omega \) and \( \Phi|_{\Omega^+} := (D \Phi)|_{\Omega^+} \). Then

\[
(\Phi \ast \hat{\Phi})(x) = \Phi \ast \hat{\Phi}(x) + \mathcal{L}_T \left[ [\gamma_R \hat{\Upsilon}, [\gamma_T \hat{\Upsilon}] + \mathcal{L}_R \left[ [\gamma_R \hat{\Upsilon}, [\gamma_T \hat{\Upsilon}])
\right. \right.,
\]

**Proof.** According to (2.15),

\[
\begin{align*}
(D \hat{\Upsilon}, \hat{V})_{\mathbb{R}^3} &= \int_\Omega \hat{U} \cdot \hat{V} \, d\mathbf{x} + \int_{\mathbb{R}^3 \setminus \Gamma} \hat{U} \cdot D \hat{V} \, d\mathbf{x} \\
&= \int_\Omega \hat{F} \cdot \hat{V} \, d\mathbf{x} + \int_{\mathbb{R}^3 \setminus \Gamma} \hat{U} \cdot D \hat{V} \, d\mathbf{x} \\
&= \int_{\mathbb{R}^3} \hat{F} \cdot \hat{V} \, d\mathbf{x} + \int_{\mathbb{R}^3 \setminus \Gamma} \hat{U} \cdot D \hat{V} \, d\mathbf{x} + \int_{\Gamma} \hat{U} \cdot \hat{V} \, d\mathbf{S}.
\end{align*}
\]

for all \( \hat{V} \in (C_0^\infty(\mathbb{R}^3))^8 \). The regularity assumptions on \( \hat{U} \) guarantee that the traces are well-defined. We have used the fact that \( \hat{V} \) is smooth across the boundary to obtain the last equality, because smoothness guarantees that \( \gamma_T^{-1} \hat{V} = \gamma_T^+ \hat{V} \) and \( \gamma_R^{-1} \hat{V} = \gamma_R^+ \hat{V} \). Therefore, in the sense of distributions, we have

\[
D \hat{U} = F + \left( \gamma_T^{-1} \hat{U} \right) - \left( \gamma_R^{-1} \hat{U} \right).
\]

Since \( \hat{U} \) is assumed to have compact support, it can interpreted as a continuous linear functional on \( C_c^\infty(\mathbb{R}^3)^8 \) and convolution with \( \Phi \) using Proposition 4.4 shows that the identity is valid when interpreted in the sense distributions. Lemma 4.6 confirms that the equality holds in \( (L^2(\mathbb{R}^3))^8 \).
In the following, we will work over the domains defined as the interior $B_\rho$ and exterior $B^+_\rho$ of an open ball of radius $\rho$. Therefore, we must introduce the traces $\gamma^o_T$ and $\gamma^0_T$ that extend the operators defined in (2.5) where $\Gamma$ is replaced by the boundary $\partial B_\rho$ of the open ball. The surface potentials $\mathcal{L}^o_\rho$ and $\mathcal{L}^0_\rho$ are defined accordingly with respect to these trace mappings. Similarly, a dagger $\dagger$ will refer to any given Lipschitz domain $\Omega_1 \subset \mathbb{R}^3$. The following development parallels that of [32, Sec. 7].

**Lemma 4.13.** For $\vec{U} \in (C^\infty_0 (\Omega^{+})^8)'$ such that $\mathbf{D} \vec{U}$ has compact support in $\Omega^+$, there exists a unique vector field $\mathbf{M} \vec{U} \in (C^\infty (\mathbb{R}^3))^8$ such that

$$
\mathbf{M} \vec{U} (x) = \mathcal{L}^0_\rho \left( \gamma^0_T \vec{U} \right) (x) + \mathcal{L}^o_\rho \left( \gamma^o_T \vec{U} \right) (x)
$$

for all $x$ inside any bounded Lipschitz domain $\Omega_1$ such that

$$
\overline{\Omega} \cup \text{supp} (\mathbf{D} \vec{U}) \subseteq \Omega_1.
$$

**Remark 4.14.** It is key in the statement of Lemma 4.13 that the vector field $\mathbf{M} \vec{U}$ is independent of $\Omega_1$.

**Proof.** Under the above hypotheses, $\vec{U}$ is harmonic in $\Omega^+ \setminus \text{supp}(\mathbf{D} \vec{U})$, because $\mathbf{D} \vec{U} = \vec{0}$ implies that $\Delta \vec{U} = \mathbf{D}^2 \vec{U} = \vec{0}$. Standard elliptic regularity theory [32, Thm. 6.4] further tells us that $\vec{U}$ is a regular distribution whose components are smooth in that domain. Therefore, we can define $\mathbf{M} \vec{U}$ in $B_{\rho_1}$ as in the right hand side of (4.54) by

$$
\mathbf{M} \vec{U} (x) := \mathcal{L}^0_\rho \left( \gamma^0_T \vec{U} \right) (x) + \mathcal{L}^o_\rho \left( \gamma^o_T \vec{U} \right) (x),
$$

where the radius $\rho_1$ is large enough that $\overline{\Omega} \cup \text{supp}(\mathbf{D} \vec{U}) \subseteq B_{\rho_1}$.

Applying (2.15) inside $B_{\rho_1} \setminus \overline{B}_{\rho_1}$ with $\rho_1 < \rho_2$ eventually shows that this definition is independent of the radius. Indeed, for any $x \in B_{\rho_1}$, $\Phi^i$ is a smooth matrix in $\mathbb{R}^3 \setminus \overline{B}_{\rho_1}$, and, thus, $\text{supp}(\mathbf{D} \vec{U}) \subseteq B_{\rho_1}$ guarantees for $i = 1, ..., 8$ that

$$
0 = \int_{B_{\rho_2} \setminus \overline{B}_{\rho_1}} \Phi (x - y) \mathbf{D} \vec{U} (y) \, dy \cdot \vec{E}_i = \int_{B_{\rho_2} \setminus \overline{B}_{\rho_1}} \Phi^i (x - y) \mathbf{D} \vec{U} (y) \, dy
$$

$$
(\ast) \quad = - \int_{B_{\rho_2} \setminus \overline{B}_{\rho_1}} \Phi^i (x - y) \cdot \mathbf{D} \vec{U} (y) \, dy = - \int_{B_{\rho_2} \setminus \overline{B}_{\rho_1}} \mathbf{D} \Phi^i (x - y) \cdot \vec{U} (y) \, dy
$$

$$
- \langle \gamma^o_T \vec{U}, \gamma^o_T \Phi^i (x - \cdot) \rangle_\Gamma - \langle \gamma^o_T \Phi^i (x - \cdot), \gamma^0_T \vec{U} \rangle_\Gamma
$$

$$
+ \langle \gamma^o_T \vec{U}, \gamma^0_T \Phi^i (x - \cdot) \rangle_\Gamma + \langle \gamma^o_T \Phi^i (x - \cdot), \gamma^0_T \vec{U} \rangle_\Gamma,
$$

where $\Phi^i$ corresponds to the $i$-th row of $\Phi$, $\Phi^i : j$ to its $j$-th column, and Lemma 4.1 was used to obtain $\ast$.

On the one hand, for $x \neq y$,

$$
\mathbf{D} \Phi^i (x - y) \cdot \vec{U} (y) = \mathbf{D} \Phi^i (x - y) \cdot \vec{E}_i
$$

$$
= \mathbf{D} \mathbf{D} \Phi^i (x - y) \cdot (\mathbf{G} (x - y) \vec{U} (y)) \cdot \vec{E}_i = (-\Delta_x \mathbf{G} (x - y)) \vec{U} (y) \cdot \vec{E}_i = 0.
$$

On the other hand,

$$
\langle \gamma^o_T \vec{U}, \gamma^o_T \Phi^i (x - \cdot) \rangle = - \langle \gamma^o_T \vec{U}, \gamma^o_T \Phi^i (x) \rangle = - \mathcal{L}^o_\rho \left( \gamma^o_T \vec{U} \right) (x) \cdot \vec{E}_i
$$

$$
- \langle \gamma^o_T \vec{U}, \gamma^o_T \Phi^i (x - \cdot) \rangle_\Gamma = - \langle \gamma^o_T \vec{U}, \gamma^o_T \Phi^i (x) \rangle_\Gamma = - \mathcal{L}^o_\rho \left( \gamma^o_T \vec{U} \right) (x) \cdot \vec{E}_j.
$$
by Lemma 4.5, and similarly for the remaining boundary terms. These two pieces of information together prove the validity of the independence claim.

In fact, the same argument can be repeated in $B_{\rho} \setminus \Omega_\bigtriangleup$ to confirm that (4.54) holds independently of the chosen Lipschitz domain satisfying the hypotheses.

Smoothness of $M \vec{U}$ is inherited from the smoothness of the integrands.

**Lemma 4.15.** Let $\vec{F} \in (L^2(\Omega^+))^8$ be compactly supported and suppose that $\vec{U} \in (C_0^\infty(\Omega^+))^8$ satisfies $D \vec{U} = \vec{F}$ on $\Omega^+$. If the restriction of $\vec{U}$ to $\Omega^+ \cap B_{\rho}$ belongs to $H(\Omega \cup \Gamma \in B_{\rho}$ and $\text{supp} \vec{F} \subseteq B_{\rho}$, then

$$\vec{U} = \Phi * \vec{F} - \mathcal{L}_T \gamma_R^+ \vec{U} - \mathcal{L}_R \gamma_T^+ \vec{U} + M \vec{U}$$

holds in $H(\Omega, \Omega^+)$. 

**Proof.** Upon applying Proposition 4.12 to the distribution

$$\vec{U}_0 := \begin{cases} \vec{0}, & \text{in } \Omega, \\ \vec{U}, & \text{in } \Omega^+ \cap B_{\rho}, \\ \vec{0}, & \text{in } \mathbb{R}^3 \setminus B_{\rho}, \end{cases}$$

that is compactly supported and belongs to $H_{\text{loc}}(\mathbb{D}, \mathbb{R}^3 \setminus (\Gamma \cup \partial B_{\rho}))$, we obtain

$$\vec{U}_0 = \Phi * \vec{F} - \mathcal{L}_T \gamma_R^+ \vec{U} - \mathcal{L}_R \gamma_T^+ \vec{U} + \mathcal{L}_R^\rho \gamma_R^+ \vec{U}$$

as a functional on $(C_0^\infty(\mathbb{R}^3))^8$. Since $B_{\rho}$ satisfies the hypotheses imposed on $\Omega_\bigtriangleup$ in the statement of Lemma 4.13, we recognize that

$$\mathcal{L}_T^\rho \gamma_R^+ \vec{U}(x) + \mathcal{L}_R^\rho \gamma_T^+ \vec{U}(x) = M \vec{U}(x)$$

for all $x \in B_{\rho}$. Hence,

$$\vec{U} = \Phi * \vec{F} - \mathcal{L}_T \gamma_R^+ \vec{U} - \mathcal{L}_R \gamma_T^+ \vec{U} + M \vec{U}$$

in $\Omega^+ \cap B_{\rho}$.

As in Lemma 4.13, it follows from $\text{supp} \vec{F} \subseteq B_{\rho}$ that $\vec{U}$ is harmonic in $\mathbb{R}^3 \setminus B_{\rho}$, and thus smooth everywhere outside the ball $B_{\rho}$ by well-known elliptic regularity theory [32, Thm. 6.4]. Hence, the hypothesis that $\vec{U} \in H(\mathbb{D}, \Omega^+ \cap B_{\rho})$ for at least one ball $B_{\rho}$ satisfying the hypotheses in fact guarantees that it belongs to that space independently of the radius satisfying those same requirements. Therefore, (4.15) holds in the whole of $\Omega^+$. Based on Lemma 4.13, the mapping properties of the potentials established in Lemma 4.6 and Proposition 4.4, we conclude that the equality (4.64) holds in fact not only in $H_{\text{loc}}(\mathbb{D}, \Omega^+)$, but in $H(\mathbb{D}, \Omega^+)$—which is the desired result.

**Lemma 4.16.** Under the hypotheses of Lemma 4.15,

$$M \vec{U} = \vec{0}$$

if and only if

$$\|\vec{U}(z)\| \to 0 \text{ uniformly as } z \to \infty.$$
The operator form of the interior and exterior Calderón projectors defined on \( \vec{U} \) by taking the traces \( \gamma \vec{U} \) that appear in the following inequalities are smooth boundary fields.

Recall that for \( z \neq 0 \),

\[
\| \nabla G(z) \| \lesssim \| z \|^{-2}. \tag{4.67}
\]

Therefore, it is easily seen from (4.39) and (4.42) that if \( \rho_2 > \rho_1 \),

\[
\left\| \mathcal{L}_t \left( \gamma^2 \vec{U} \right)(x) \right\| \lesssim \rho_2^{-2} \left\| \int_{\partial B_{\rho_2}} \gamma \vec{U}(y) \, d\sigma(y) \right\| \lesssim \max_{y \in \partial B_{\rho_2}} \| \vec{U}(y) \|
\]

for all \( x \in B_{\rho_1} \), \( \bullet = T \) or \( R \). Notice that the left hand side of (4.68) is well-defined, because as in Lemma 4.13, Lemma 4.7 and \( D^2 = -\Delta \) guarantee that away from the boundary \( \partial B_{\rho_2} \) the potentials are smooth harmonic vector fields. No differential operator appears in the definition of the trace mappings \( \gamma_R \) and \( \gamma_T \). The independence of \( M \vec{U} \) from its domain of definition thus directly yields one implication of the lemma upon taking \( \rho_2 \to \infty \).

The converse follows from the exterior representation formula (4.60) with \( M \vec{U} = \vec{\Phi} \) and an analysis exploiting (4.67) that leads to an inequality similar to (4.68). However, this time the potentials are computed as integrals (duality pairings) on the fixed boundary \( \Gamma \) and an inverse square decay is inherited from the decay of the fundamental solution.

\[ \square \]

**Proposition 4.17** (Exterior representation formula). If \( \vec{U} \in H_{0}\text{loc}(D, \Omega^+) \) is such that \( \vec{U}(z) \to 0 \) as \( z \to \infty \) and \( \vec{F} := DU \) is compactly supported. Then

\[
\vec{U}(x) = \Phi * \vec{F}(x) - \mathcal{L}_T \gamma^+_R \vec{U}(x) - \mathcal{L}^+_R \gamma_T \vec{U}(x), \quad x \in \Omega^+. \tag{4.69}
\]

### 5. Boundary integral equations.

Boundary integral equations are obtained by taking the traces \( \gamma_R \) and \( \gamma_T \) on both sides of the representation formulas (4.51) and (4.69). The operator form of the interior and exterior Calderón projectors defined on \( \mathcal{H}_R \times \mathcal{H}_T \), which we denote \( P^- \) and \( P^+ \) respectively, enter the Calderón identities

\[
\begin{pmatrix}
\{ \gamma_R \} \mathcal{L}_T + \frac{1}{2} \text{Id} \\
\{ \gamma_T \} \mathcal{L}_T
\end{pmatrix}
\begin{pmatrix}
\{ \gamma_R \} \mathcal{L}_R \\
\{ \gamma_T \} \mathcal{L}_R + \frac{1}{2} \text{Id}
\end{pmatrix}
\begin{pmatrix}
\gamma^-_R (U) \\
\gamma^-_T (U)
\end{pmatrix} = \begin{pmatrix}
\gamma^-_R (U) \\
\gamma^-_T (U)
\end{pmatrix}, \tag{5.1}
\]

\[
\begin{pmatrix}
-\{ \gamma_R \} \mathcal{L}_T - \frac{1}{2} \text{Id} \\
-\{ \gamma_T \} \mathcal{L}_T
\end{pmatrix}
\begin{pmatrix}
-\{ \gamma_R \} \mathcal{L}_R \\
-\{ \gamma_T \} \mathcal{L}_R - \frac{1}{2} \text{Id}
\end{pmatrix}
\begin{pmatrix}
\gamma^+_R (U) \\
\gamma^+_T (U)
\end{pmatrix} = \begin{pmatrix}
\gamma^+_R (U) \\
\gamma^+_T (U)
\end{pmatrix}. \tag{5.2}
\]

For example, extend a solution \( \vec{U} \in H(D, \Omega) \) of the homogeneous Dirac equation in \( \Omega^- \) to the whole of \( \mathbb{R}^3 \) by zero. Using Proposition 4.12,

\[
\vec{U}(x) = \mathcal{L}_T \gamma^-_R \vec{U}(x) + \mathcal{L}^-_R \gamma_T \vec{U}(x), \quad x \in \mathbb{R}^3 \setminus \Gamma. \tag{5.3}
\]

Then, applying \( \gamma^-_R \) on both sides of the equation yields

\[
\gamma^-_R \vec{U}(x) = \gamma^-_R \mathcal{L}_T \gamma^-_R \vec{U}(x) + \gamma^-_R \mathcal{L}^-_R \gamma_T \vec{U}(x), \quad x \in \Gamma. \tag{5.4}
\]
It is a simple calculation to verify that the jump identities of Lemma 4.10 implies

\begin{align}
\{\gamma_T\} L_T(\tilde{\alpha}) &= \gamma_T L_T(\tilde{\alpha}), & \{\gamma_R\} L_T(\tilde{\alpha}) &= \gamma_R L_T(\tilde{\alpha}) - \frac{1}{2} \tilde{\alpha}, \\
\{\gamma_T\} L_R(\tilde{\beta}) &= \gamma_T L_R(\tilde{\beta}) - \frac{1}{2} \tilde{\beta}, & \{\gamma_R\} L_R(\tilde{\beta}) &= \gamma_R L_R(\tilde{\beta}).
\end{align}

Substituting the interior traces for the averages using these relations leads to the top row of (5.1). The other identities are obtained similarly.

A classical argument, cf. [40, lem. 6.18], shows that \( P^- \) and \( P^+ \) are indeed projectors, i.e. \((P^\mp)^2 = P^\mp \). The proof, which for the homogeneous Dirac equation is essentially based on Lemma 4.7, also shows as a byproduct, cf. [44, Thm. 3.7], that the images of \( P^- \) and \( P^+ \) are spaces of valid interior and exterior Cauchy data, respectively. In fact, as observed in [12, Sec. 5], we have \( P^- + P^+ = \text{Id} \). So the range of \( P^- \) coincides with the nullspace of \( P^+ \) and vice versa. Therefore, we find the important property that \((\tilde{\alpha}, \tilde{\beta}) \in \mathcal{H}_R \times \mathcal{H}_T \) is valid interior or exterior Cauchy data if and only if it lies in the nullspace of \( P^+ \) or \( P^- \), respectively.

The two direct boundary integral equations of the first-kind related to (R) and (T) then read as follows. Given \( \gamma_R \tilde{U} = \tilde{\alpha} \in \mathcal{H}_R \), the task is to determine the unknown \( \tilde{\beta} = \gamma_T \tilde{U} \in \mathcal{H}_T \) by solving

\[(BR) \quad \gamma_R L_R(\tilde{\beta}) = \frac{1}{2} \tilde{\alpha} - \{\gamma_R\} L_T(\tilde{\alpha}).\]

If \( \tilde{\beta} \in \mathcal{H}_T \) is known instead, then we solve

\[(BT) \quad \gamma_T L_T(\tilde{\alpha}) = \frac{1}{2} \tilde{\beta} - \{\gamma_T\} L_R(\tilde{\beta})\]

for the unknown \( \tilde{\alpha} \in \mathcal{H}_R \).

**Remark 5.1 (Duality and symmetry).** Let us revisit the boundary value problems of Section 3. We wish to highlight that (T) and (R) are really the same problem in hiding. For example, we can always relabel the components of an unknown vector-field \( \tilde{U} \in \mathbf{H}(D, \Omega) \) to

\begin{align}
V_0 &:= U_3, & V_1 &:= -U_2 & V_2 &:= -U_1 & V_3 &:= V_0,
\end{align}

and set

\begin{align}
a_0 &:= -b_2 & a_1 &:= n \times b_1 & a_3 &= b_0.
\end{align}

This turns a problem (T) for \( \tilde{U} \) into a problem (R) for \( \tilde{V} \in \mathbf{H}(D, \Omega) \).

Since both a solution \( \tilde{U} \) of (T) and a solution \( \tilde{V} \) of (R) can be written using the representation formula (4.51), we expect (5.8) to define an isomorphism \( \Xi : \mathcal{H}_T \to \mathcal{H}_R \) that also turns one of the boundary integral equation into the other. And indeed, one can verify that

\[\{\gamma_R\} L_T \left( \Xi \tilde{b} \right) = \Xi \gamma_T L_R \left( \tilde{b} \right) \quad \text{and} \quad \{\gamma_T\} L_T \left( \Xi \tilde{b} \right) = \Xi \{\gamma_R\} L_R \left( \tilde{b} \right).\]

Hence, (BT) can be equivalently formulated as a problem (BR) with unknown \( \Xi^{-1} \tilde{\alpha} \) and given data \( \Xi \tilde{b} \).
Let us take a closer look at the bilinear forms naturally associated with the continuous first-kind boundary integral operators

\begin{align}
\gamma_T \mathcal{L}_T : \mathcal{H}_R &\to \mathcal{H}_T, \\
\gamma_R \mathcal{L}_R : \mathcal{H}_T &\to \mathcal{H}_R,
\end{align}

that map trace spaces to their dual spaces.

Let \( \mathbf{a} \) and \( \mathbf{c} \) be trial and test boundary vector fields lying in \( \mathcal{H}_R \), and similarly for \( \mathbf{b} \) and \( \mathbf{d} \) in \( \mathcal{H}_T \). Catching up with the calculations of subsection 4.2, we want to derive convenient integral formulas for

\[ \langle \mathbf{c}, \gamma_T \mathcal{L}_T (\mathbf{a}) \rangle = -\langle c_0, \gamma \text{div} \Psi(a_1) \rangle + \langle c_1, \gamma_\ell \nabla \psi(a_0) \rangle + \langle c_1, \gamma_n \text{curl} \Psi(a_2) \rangle + \langle c_2, \gamma_n \text{curl} \Psi(a_1) \rangle \]

and

\[ \langle \mathbf{d}, \gamma_R \mathcal{L}_R (\mathbf{b}) \rangle = \langle d_0, \gamma_n \text{curl} \Psi(b_1 \times n) \rangle - \langle d_1, \gamma_\tau \text{curl} \Psi(b_0) \rangle + \langle d_1, \gamma_\tau \nabla \psi(b_2) \rangle + \langle d_2, \gamma \text{div} \Psi(b_1 \times n) \rangle. \]

In the course of our derivation, we will often rely implicitly on the fact that \( a_1 \) and \( b_1 \) are tangential vector fields.

Using the fact that \( \text{div} \Psi(a_1) = \psi(\text{div}_\Gamma a_1) \) and \( \text{div} \Psi(b_1 \times n) = \psi(\text{curl}_\Gamma b_1) \) \cite{[27, Lem. 2.3]}, we immediately find that

\[ (c_0, \gamma \text{div} \Psi(a_1)) \Gamma = \int_{\Gamma} \int G_x(y) c_0(x) \text{div}_\Gamma a_1(y) \, d\sigma(y) \, d\sigma(x) \]

and

\[ \langle d_2, \gamma \text{div} \Psi(b_1 \times n) \rangle \Gamma = \int_{\Gamma} \int G_x(y) d_2(x) \text{curl}_\Gamma b_1(y) \, d\sigma(y) \, d\sigma(x). \]

We know from \cite{[14, Sec. 6.4]} that

\[ \langle d_1, \gamma_\tau \text{curl} \Psi(b_0) \rangle \]

\[ = -\int_{\Gamma} \int G_x(y) (n(x) \times d_1(x)) \cdot (n(y) \times \nabla_\Gamma b_0(y)) \, d\sigma(y) \, d\sigma(x) \]

\[ = \int_{\Gamma} \int G_x(y) (n(x) \times d_1(x)) \cdot \text{curl}_\Gamma b_0(y) \, d\sigma(y) \, d\sigma(x). \]

Adapting the arguments, we also obtain

\[ \langle c_1, \gamma_\ell \text{curl} \Psi(a_2) \rangle = \langle c_1 \times n, \gamma_\tau \text{curl} \Psi(a_0) \rangle \]

\[ = \int_{\Gamma} \int G_x(y) (n(x) \times (c_1(x) \times n(x))) \cdot \text{curl}_\Gamma a_2(y) \, d\sigma(y) \, d\sigma(x) \]

\[ = \int_{\Gamma} \int G_x(y) c_1(x) \cdot \text{curl}_\Gamma a_2(y) \, d\sigma(y) \, d\sigma(x). \]

Again, from \cite{[14, Sec. 6.4]}, we can similarly extract

\[ \langle c_2, \gamma_n \text{curl} \Psi(a_1) \rangle = -\int_{\Gamma} \int G_x(y) a_1(y) \cdot (n(x) \times \nabla_\Gamma c_2(x)) \, d\sigma(y) \, d\sigma(x) \]

\[ = \int_{\Gamma} \int G_x(y) a_1(y) \cdot \text{curl}_\Gamma c_2(x) \, d\sigma(y) \, d\sigma(x). \]
and
\[
(d_0 \cdot \gamma_n \text{curl}\Psi(b_1 \times n))_\Gamma = - \int_\Gamma \int_\Gamma G_x(y) \left( n(y) \times b_1(y) \right) \cdot \text{curl}_\Gamma d_0(x) \, d\sigma(y) \, d\sigma(x)
\]

Finally, it follows almost directly by definition that
\[
\langle e_1, \gamma_1 \nabla \psi(a_0) \rangle_\Gamma = - \int_\Gamma \int_\Gamma G_x(y) a_0(y) \text{div}_\Gamma e_1(x) \, d\sigma(y) \, d\sigma(x),
\]
and
\[
\langle d_1, \gamma_\tau \nabla \psi(b_2) \rangle_\Gamma = \int_\Gamma \int_\Gamma G_x(y) b_2(y) \text{curl}_\Gamma d_1(x) \, d\sigma(y) \, d\sigma(x).
\]

Putting everything together yields the symmetric bilinear forms

\[
\langle \tilde{e}, \gamma_\tau \mathcal{L}_\Gamma (\tilde{a}) \rangle = - \int_\Gamma \int_\Gamma G(x - y) c_0(x) \text{div}_\Gamma a_1(y) \, d\sigma(x) \, d\sigma(y) - \int_\Gamma \int_\Gamma G(x - y) a_0(y) \text{div}_\Gamma c_1(x) \, d\sigma(y) \, d\sigma(x) + \int_\Gamma \int_\Gamma G(x - y) c_1(x) \cdot \text{curl}_\Gamma a_2(y) \, d\sigma(y) \, d\sigma(x) + \int_\Gamma \int_\Gamma G(x - y) a_1(y) \cdot \text{curl}_\Gamma c_2(x) \, d\sigma(y) \, d\sigma(x),
\]

\[
\langle \tilde{d}, \gamma_R \mathcal{L}_R (\tilde{b}) \rangle = - \int_\Gamma \int_\Gamma G(x - y) \left( n(y) \times b_1(y) \right) \cdot \text{curl}_\Gamma d_0(x) \, d\sigma(y) \, d\sigma(x) - \int_\Gamma \int_\Gamma G(x - y) \left( n(x) \times d_1(x) \right) \cdot \text{curl}_\Gamma b_0(y) \, d\sigma(y) \, d\sigma(x) + \int_\Gamma \int_\Gamma G(x - y) b_2(y) \text{curl}_\Gamma d_1(x) \, d\sigma(y) \, d\sigma(x) + \int_\Gamma \int_\Gamma G(x - y) d_2(x) \text{curl}_\Gamma b_1(y) \, d\sigma(y) \, d\sigma(x).
\]

The above integrals must be understood as duality pairings.

Remark 5.2. Let us highlight here, as we have announced in the introduction, that in the sense of [24, Chap. 2.5], these double integrals feature only weakly singular kernels!

The non-local inner products

\[
(u, v)_{-1/2} := \int_\Gamma \int_\Gamma G_x(y) u(x) v(y) \, d\sigma(x) \, d\sigma(y),
\]

\[
(u, v)_{-1/2, T} := \int_\Gamma \int_\Gamma G_x(y) u(x) \cdot v(y) \, d\sigma(x) \, d\sigma(y),
\]

\[
(u, v)_{-1/2, R} := \int_\Gamma \int_\Gamma G_x(y) \left( n(x) \times u(x) \right) \cdot \left( n(y) \times v(y) \right) \, d\sigma(x) \, d\sigma(y).
\]
respectively defined over $H^{-1/2}(\Gamma)$, $H_T^{-1/2}(\Gamma) := (H_T^{1/2}(\Gamma))'$ and $H_R^{-1/2}(\Gamma) := (H_R^{1/2}(\Gamma))'$, where

\begin{align}
H_T^{1/2}(\Gamma) := \gamma_T(H^1(\Omega)) \quad \text{and} \quad H_R^{1/2}(\Gamma) := \gamma_R(H^1(\Omega)),
\end{align}

are positive definite Hermitian forms, and induce equivalent norms on the trace spaces \cite[Sec. 4.1]{10}. In the following, we will concern ourselves with the coercivity and geometric structure of the bilinear forms

\begin{align}
B_T((\tilde{a}, \tilde{c}), (\tilde{b}, \tilde{d})) := \langle \gamma_T L_T(\tilde{a}), \tilde{c} \rangle &= \langle -\text{div}_\Gamma a_1, c_0 \rangle_{-1/2} + \langle a_0, -\text{div}_\Gamma c_1 \rangle_{-1/2} \\
&\quad + \langle \text{curl}_\Gamma a_2, c_1 \rangle_{-1/2,T} + \langle a_1, \text{curl}_\Gamma c_2 \rangle_{-1/2,T},
\end{align}

and

\begin{align}
B_R((\tilde{b}, \tilde{d}), (\tilde{b}, \tilde{d})) := \langle \gamma_R L_R(\tilde{b}), \tilde{d} \rangle &= \langle b_1, \nabla_\Gamma d_0 \rangle_{-1/2,R} + \langle \nabla_\Gamma b_0, d_1 \rangle_{-1/2,R} \\
&\quad + \langle \delta_2, \text{curl}_\Gamma d_1 \rangle_{-1/2} + \langle \text{curl}_\Gamma b_1, \delta_2 \rangle_{-1/2}.
\end{align}

**6. T-coercivity.** Based on the space decomposition introduced by the next lemma, we design isomorphisms $\mathcal{H}_R \rightarrow \mathcal{H}_R$ and $\mathcal{H}_T \rightarrow \mathcal{H}_T$ that are instrumental for obtaining the desired generalized Gårding inequalities for $B_T$ and $B_R$.

**Lemma 6.1 (See \cite[Sec. 7]{21} and \cite[Lem. 2]{12}).** There exists a continuous projection $Z^T : H^{-1/2}(\text{div}_\Gamma, \Gamma) \rightarrow H_R^{1/2}(\Gamma)$ with

\begin{align}
\text{ker}(Z^T) &= \text{ker}(\text{div}_\Gamma) \cap H^{-1/2}(\text{div}_\Gamma, \Gamma)
\end{align}

and satisfying

\begin{align}
\text{div}_\Gamma(Z^T(v)) &= \text{div}_\Gamma(v).
\end{align}

The closed subspaces $X(\text{div}_\Gamma, \Gamma) := Z^T(H^{-1/2}(\text{div}_\Gamma, \Gamma))$ and $N(\text{div}_\Gamma, \Gamma) := \text{ker}(\text{div}_\Gamma) \cap H^{-1/2}(\text{div}_\Gamma, \Gamma)$ provide a stable direct regular decomposition

\begin{align}
H^{-1/2}(\text{div}_\Gamma, \Gamma) &= X(\text{div}_\Gamma, \Gamma) \oplus N(\text{div}_\Gamma, \Gamma).
\end{align}

Hence, it follows from \eqref{6.2} that

\begin{align}
v \mapsto \|\text{div}_\Gamma(v)\|_{-1/2} + \|(\text{Id} - Z^T)v\|_{-1/2}
\end{align}

also defines an equivalent norm in $H^{-1/2}(\text{div}_\Gamma, \Gamma)$.

Note that since, by Rellich’s embedding theorem, $H_R^{1/2}(\Gamma)$ compactly embeds in the space $L^2_t(\Gamma) := \{u \in L^2(\Gamma) \mid u \cdot n \equiv 0\}$ of square-integrable tangential vector-fields, this is also the case for $X(\text{div}_\Gamma, \Gamma)$.
From Lemma 2.2, \( \text{div}_\Gamma : X(\text{div}_\Gamma, \Gamma) \to H^{1/2}_\ast(\Gamma) \) is a continuous bijection, thus the bounded inverse theorem guarantees the existence of a continuous inverse \((\text{div}_\Gamma)^\dagger : H^{1/2}_\ast(\Gamma) \to X(\text{div}_\Gamma, \Gamma)\) such that

\[
(\text{div}_\Gamma)^\dagger \circ \text{div}_\Gamma = \text{Id}|_{X(\text{div}_\Gamma, \Gamma)}, \quad \text{div}_\Gamma \circ (\text{div}_\Gamma)^\dagger = \text{Id}|_{H^{1/2}_\ast(\Gamma)}.
\]

The existence of an operator \( \text{curl}^\dagger : N(\text{div}_\Gamma, \Gamma) \to H^{1/2}_\ast(\Gamma) \) satisfying \( \text{curl}^\dagger \circ \text{div}_\Gamma = \text{Id} \) and \( \text{curl}_\Gamma \circ \text{curl}^\dagger = (H^{-1/2}(\text{div}_\Gamma, \Gamma)-\text{orthogonal projection onto (surface) divergence-free vector-fields}) \) also follows by Lemma 2.2.

In the following, we will denote by \( \text{curl}_\Gamma \) both the projection \( H^{1/2}(\Gamma) \to H^{1/2}_\ast(\Gamma) \) onto mean zero functions and the projection \( H^{-1/2}(\Gamma) \to H^{-1/2}_\ast(\Gamma) \) onto the space of annihilators of the characteristic function.

**Lemma 6.2.** The bounded linear operator

\[
\Xi : H^{1/2}_\ast(\Gamma) \times H^{-1/2}(\text{div}_\Gamma, \Gamma) \times H^{1/2}(\Gamma) \to H^{-1/2}_\ast(\Gamma) \times H^{-1/2}(\text{div}_\Gamma, \Gamma) \times H^{1/2}(\Gamma)
\]

defined by

\[
\Xi \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -\text{div}_\Gamma a_1 \\ -(\text{div}_\Gamma)^\dagger (Q_\ast a_0) + \text{curl}_\Gamma (Q_\ast a_2) \\ (\text{curl}_\Gamma)^\dagger ((\text{Id} - Z^\Gamma) a_1) \end{pmatrix}
\]

is a continuous involution. In particular, \( \Xi \) is an isomorphism of Banach spaces.

**Proof.** We directly evaluate

\[
\Xi^2 \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \Xi \begin{pmatrix} -\text{div}_\Gamma a_1 \\ -(\text{div}_\Gamma)^\dagger (Q_\ast a_0) + \text{curl}_\Gamma (Q_\ast a_2) \\ (\text{curl}_\Gamma)^\dagger ((\text{Id} - Z^\Gamma) a_1) \end{pmatrix}
\]

\[
= \begin{pmatrix} \text{div}_\Gamma((\text{div}_\Gamma)^\dagger (Q_\ast a_0)) - \text{div}_\Gamma(\text{curl}_\Gamma (Q_\ast a_2)) \\ -(\text{div}_\Gamma)^\dagger (Q_\ast (\text{div}_\Gamma a_1)) + \text{curl}_\Gamma (Q_\ast (\text{curl}_\Gamma)^\dagger ((\text{Id} - Z^\Gamma) a_1)) \\ (\text{curl}_\Gamma)^\dagger ((\text{Id} - Z^\Gamma)((\text{div}_\Gamma)^\dagger (Q_\ast a_0))) + (\text{curl}_\Gamma)^\dagger ((\text{Id} - Z^\Gamma)(\text{curl}_\Gamma (Q_\ast a_2))) \end{pmatrix}
\]

\[
= \begin{pmatrix} Z^\Gamma a_1 + (\text{Id} - Z^\Gamma) a_1 \\ Q_\ast a_0 \\ Q_\ast a_2 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}. \quad \square
\]

**Proposition 6.3.** There exists a constant \( C > 0 \) and a compact bilinear form \( C : H_R \times H_R \to \mathbb{R} \) such that

\[
(6.5) \quad \|\Xi \tilde{a}, \gamma_T \mathcal{L}_T (\tilde{a})\|_X + C (\tilde{a}, \tilde{a}) \geq C \|\tilde{a}\|^2_{H_R} \quad \forall \tilde{a} \in H_R.
\]

**Proof.** The operator \( \text{curl}_\Gamma : H^1(\Gamma) \to H^{-1/2}(\text{div}_\Gamma) \) is a continuous injection with closed range, it is thus bounded below. Since the mean operator has finite rank, it is
compact. Moreover, \((\text{div}_\Gamma)^\dagger \left( H^{-1/2}_s (\Gamma) \right) \subset H^{1/2}_R (\Gamma)\) is compactly embedded in \(L^2_R (\Gamma)\). Hence, the proof ultimately follows from

\[
\{ \Xi, \gamma_\Gamma L_\Gamma (\tilde{a}) \}_x = (\text{div}_\Gamma \, a_1, \text{div}_\Gamma \, a_1)_{-1/2} + (a_2, \, Q_\gamma a_2)_{-1/2} + \left( (\text{div}_\Gamma)^\dagger \, Q_\gamma a_2, \, \text{curl}_\Gamma a_0 \right)_{-1/2} + \left( \text{curl}_\Gamma \, Q_\gamma a_0, \, \text{curl}_\Gamma a_0 \right)_{-1/2} + \left( a_1, \, (\text{Id} - Z_\Gamma) \, a_1 \right)_{-1/2}
\]

and the opening observations of this section.

Since \(\text{curl}_\Gamma (d) = \text{div}_\Gamma (n \times d)\) for all \(d \in H^{-1/2}(\text{curl}_\Gamma, \Gamma)\), tinkering with the signs and introducing rotations in the definition of \(\Xi\) easily leads to an analogous generalized Gårding inequality for \(\gamma_\Gamma L_\Gamma\).

**Corollary 6.4.** The boundary integral operators \(\gamma_\Gamma L_\Gamma : \mathcal{H}_R \to \mathcal{H}_T\) and \(\gamma_\Gamma L_R : \mathcal{H}_T \to \mathcal{H}_R\) are Fredholm of index 0.

### 7. Kernels.

We conclude from Corollary 6.4 that the nullspaces of \(\gamma_\Gamma L_\Gamma\) and \(\gamma_\Gamma L_R\) are finite dimensional. In this section, we proceed similarly as in [14, Sec. 7.1] and [15, Sec. 3] to characterize them explicitly.

Suppose that \(\tilde{a} \in \mathcal{H}_R\) is such that \(\gamma_\Gamma L_\Gamma (\tilde{a}) = 0\).

- Since \(\text{div}_\Gamma \, a_1 \in H^{-1/2}(\Gamma)\), we can test the bilinear form of Equation (5.12) with \(c_1 = 0 = c_2\) and \(c_2 = 0\) to find that \(\text{div}_\Gamma \, a_1 = 0\).
- Testing with \(c_0 = 0 \quad \text{and} \quad c_1 = 0\) shows that \((a_1, \, \text{curl}_\Gamma \, v)_{-1/2} = 0 \quad \forall \, v \in H^{1/2}(\Gamma)\).
- Because \(\text{div}_\Gamma \circ \text{curl}_\Gamma = 0\), we can choose \(c_2 = 0\), \(c_0 = 0\) and \(c_1 = \text{curl}_\Gamma a_2\) to conclude that \(\text{curl}_\Gamma a_2 = 0\).
- We are left with \((a_0, \, \text{div}_\Gamma \, v)_{-1/2} = 0 \quad \forall \, v \in H^{1/2}(\text{div}_\Gamma, \Gamma)\).

In \(H^{1/2}(\Gamma)\), \(\ker (\text{curl}_\Gamma) = \ker (\nabla_\Gamma \psi)\) is the space of functions \(C(\Gamma)\) that are constant over connected components of \(\Gamma\). Defining \(\Psi_\Gamma := \gamma_\Gamma \Psi\), we have found that

\[
\ker (\gamma_\Gamma L_\Gamma) = \left\{ \tilde{a} \in \mathcal{H}_R \mid a_0 \in C(\Gamma), \, \text{curl}_\Gamma \Psi_\Gamma (a_1) = 0, \, \text{div}_\Gamma a_1 = 0, \, \nabla_\Gamma \psi (a_0) = 0 \right\}.
\]

Now, suppose that \(\tilde{b} \in \mathcal{H}_T\) is such that \(\gamma_\Gamma L_R (\tilde{b}) = 0\).

- As \(\text{curl}_\Gamma \, (b_1) \in H^{-1/2}(\Gamma)\), we may test Equation (5.13) with \(d_2 = \text{curl}_\Gamma \, b_1\), \(d_1 = 0\) and \(d_0 = 0\) to find that \(\text{curl}_\Gamma \, b_1 = 0\).
- Testing with \(d_2 = 0 \quad \text{and} \quad d_1 = 0\) we find that \((n \times b_1, \, \text{curl}_\Gamma \, v)_{-1/2} = 0 \quad \forall \, v \in H^{1/2}(\Gamma)\).
- Since \(\text{curl}_\Gamma \circ \nabla_\Gamma = 0\), we can choose \(d_0 = 0\), \(d_2 = 0\) and \(d_1 = \nabla_\Gamma b_0\) to conclude that \(\text{curl}_\Gamma b_0 = 0\).
- Finally, it follows that \((\delta_2, \, \text{curl}_\Gamma \, v)_{-1/2} = 0 \quad \forall \, v \in H^{-1/2}(\text{curl}_\Gamma, \Gamma)\).

Notice that since \(\nabla_\Gamma (v)\) is tangential for all \(v \in H^{1/2}(\Gamma)\),

\[
(n \times b_1, \, \text{curl}_\Gamma \, v)_{-1/2} = (n \times b_1, \, \nabla_\Gamma v \times n)_{-1/2} = (n \times \Psi \, (n \times b_1), \, \nabla_\Gamma v)
\]
for all $v \in H^{1/2}(\Gamma)$. Therefore, we let $\Psi_{\tau}(\cdot) := -\gamma_{\tau}\Psi(n \times \cdot)$ and conclude that

\[
\text{ker} (\gamma_R L_R) = \left\{ \vec{b} \in \mathcal{H}_T \mid b_0 \in C(\Gamma), \text{curl}_\Gamma b_1 = 0, \text{div}_\Gamma \Psi_{\tau} (b_1) = 0, \text{curl}_\Gamma \psi \left( b_0' \right) = 0 \right\}.
\]

Equation (7.1) and Equation (7.2) together with the mapping properties of the scalar and vector single layer potentials allow us to determine as in [14, Sec. 7.2] and [15, Lem. 2, Lem. 6] that the dimension of these nullspaces relate to the Betti numbers of $\Gamma$.

**Proposition 7.1.** The dimensions of $\text{ker} (\gamma_T L_T)$ and $\text{ker} (\gamma_R L_R)$ are finite and equal to the sum of the Betti numbers $\beta_0(\Gamma) + \beta_1(\Gamma) + \beta_2(\Gamma)$.

**Remark 7.2.** The zeroth Betti number $\beta_0(\Gamma)$ indicates the number of connected components of $\Gamma$. The first Betti number $\beta_1(\Gamma)$ amounts to the number of equivalence classes of non-bounding cycles in $\Gamma$. For the second Betti number, it holds that $\beta_2(\Gamma) = \beta_2(\Omega^+) + \beta_2(\Omega^-)$, which sums the number of holes in $\Omega^+$ and $\Omega^-$, respectively.

**8. Surface Dirac operators.** In this section, we reveal the geometric structure behind the formulas of the bilinear forms $B_R$ and $B_T$ established in Section 5. They turn out to be associated with the 2D surface Dirac operators induced by the chain and cochain Hilbert complexes

\[
H^{-1/2}(\Gamma) \xrightarrow{\nabla_F} H_{T}^{-1/2}(\Gamma) \xrightarrow{\text{curl}_F} H^{-1/2}(\Gamma)
\]

and

\[
H^{-1/2}(\Gamma) \xrightarrow{-\text{div}_\Gamma} H_{R}^{-1/2}(\Gamma) \xrightarrow{\text{curl}_\Gamma} H^{-1/2}(\Gamma),
\]

equipped with the non-local inner products (5.14), (5.15) and (5.16). Their associated domain complexes

\[
H^{1/2}(\Gamma) \xrightarrow{\nabla_F} H^{-1/2}(\text{curl}_F, \Gamma) \xrightarrow{\text{curl}_F} H^{-1/2}(\Gamma)
\]

and

\[
H^{-1/2}(\Gamma) \xrightarrow{-\text{div}_\Gamma} H^{-1/2}(\text{div}_F, \Gamma) \xrightarrow{\text{curl}_F} H^{1/2}(\Gamma),
\]

are equipped with the natural graph inner products.

**Remark 8.1.** Notice that (8.3) and (8.4) are dual to each other with respect to the duality pairing on the boundary introduced in Section 2.

The Hilbert space adjoint $d_T^*$ and $\delta_T^*$ of the nilpotent operators

\[
d_T^* : \mathcal{H}_T \rightarrow \mathcal{H},
\]

\[
\delta_T^* : \mathcal{H}_R \rightarrow \mathcal{H}_R,
\]
represented by the block operator matrices
\[
d_{\Gamma} := \begin{pmatrix} 0 & 0^T & 0 \\ \nabla_{\Gamma} & 0_{3\times3} & 0 \\ 0 & \text{curl}_{\Gamma} & 0 \end{pmatrix} \quad \text{and} \quad \delta_{\Gamma} := \begin{pmatrix} 0 & -\text{div}_{\Gamma} & 0 \\ 0 & 0_{3\times3} & \text{curl}_{\Gamma} \\ 0 & 0^T & 0 \end{pmatrix}
\]
are non-local operators.

In terms of variational formulations, the bilinear forms associated with the surface Dirac operators
\[
D_{\Omega}^{\Gamma} := d_{\Gamma} + d_{\Gamma}^* \\
D_{\Omega}^T := \delta_{\Gamma} + \delta_{\Gamma}^*
\]
are precisely \( B_{\mathcal{R}} \) and \( B_{\mathcal{T}} \) defined in (5.19) and (5.18), previously associated to the boundary integral operators \( \gamma_{\mathcal{R}}\mathcal{L}_{\mathcal{R}} \) and \( \gamma_{\mathcal{T}}\mathcal{L}_{\mathcal{T}} \):

\[
\left( D_{\mathcal{R}}^{\Gamma} \vec{b}, \vec{d} \right)_{\mathcal{H}_{\mathcal{R}}} = \left( d_{\Gamma} \vec{b}, \vec{d} \right)_{\mathcal{H}_{\mathcal{R}}} + \left( \vec{b}, d_{\Gamma} \vec{d} \right)_{\mathcal{H}_{\mathcal{R}}} \\
= (\nabla_{\Gamma} b_0, d_1)_{-1/2, \mathcal{R}} + (\text{curl}_{\Gamma} b_1, d_2)_{-1/2} \\
+ (b_1, \nabla_{\Gamma} d_0)_{-1/2, \mathcal{R}} + (b_2, \text{curl}_{\Gamma} d_1)_{-1/2}
\]

and similarly

\[
\left( D_{\mathcal{T}}^{\Gamma} \vec{a}, \vec{c} \right)_{\mathcal{H}_{\mathcal{T}}} = \left( \delta_{\Gamma} \vec{a}, \vec{c} \right)_{\mathcal{H}_{\mathcal{T}}} + (\vec{a}, \delta_{\Gamma} \vec{c})_{\mathcal{H}_{\mathcal{T}}} \\
= (-\text{div}_{\Gamma} a_1, c_0)_{-1/2} + (a_0, -\text{div}_{\Gamma} c_1)_{-1/2} \\
+ (\text{curl}_{\Gamma} a_2, c_1)_{-1/2, \mathcal{T}} + (a_1, \text{curl}_{\Gamma} c_2)_{-1/2, \mathcal{T}}
\]

First-kind boundary integral operators spawned by the (volume) Dirac operators in 3D Euclidean space thus coincide with (surface) Dirac operators on 2D boundaries: boundary value problems related to \( D^{\Omega}_{\mathcal{R}} = d + d^* \) in \( \Omega \) can be formulated as problems for \( D_{\mathcal{R}}^{\Gamma} = d_{\Gamma} + d_{\Gamma}^* \) in \( \Gamma \), and similarly problems for \( D^{\Omega}_{\mathcal{T}} = \delta + \delta^* \) in \( \Omega \) correspond to problems for \( D_{\mathcal{T}}^{\Gamma} = \delta_{\Gamma} + \delta_{\Gamma}^* \) in \( \Gamma \).

This explains why the dimension of the nullspaces of first-kind boundary integral operators is the sum of the dimensions of the standard spaces of surface harmonic scalar and vector fields.

9. Solvability. Thanks to the duality between the trace spaces, \( (B_{\mathcal{T}}) \) and \( (B_{\mathcal{R}}) \) can be reformulated into the variational problems:

\[\text{(BVT)} \quad \vec{a} \in \mathcal{H}_{\mathcal{R}}: \quad B_{\mathcal{T}}(\vec{a}, \vec{c}) = \ell_{\mathcal{T}}(\vec{c}), \quad \forall \vec{c} \in \mathcal{H}_{\mathcal{R}},\]

and

\[\text{(BVR)} \quad \vec{b} \in \mathcal{H}_{\mathcal{T}}: \quad B_{\mathcal{R}}(\vec{b}, \vec{d}) = \ell_{\mathcal{R}}(\vec{d}), \quad \forall \vec{d} \in \mathcal{H}_{\mathcal{T}},\]
with right-hand side functionals

(9.1) \[ \ell_T(\vec{c}) = \langle \frac{1}{2} \vec{b} - \{\gamma_T\} L_R(\vec{b}), \vec{c} \rangle_\Gamma \]

and

(9.2) \[ \ell_R(\vec{d}) = \langle \frac{1}{2} \vec{a} - \{\gamma_R\} L_T(\vec{a}), \vec{d} \rangle_\Gamma. \]

As explained in Remark 5.1, it is sufficient when it comes to well-posedness to restrict our considerations to only one of the two boundary integral equations stated in Section 5. The following result makes explicit the condition under which a solution to \((BV^R)\) exists.

**Proposition 9.1.** If the boundary data \(\vec{a} \in \mathcal{H}_R\) satisfies the compatibility condition \((\text{CCR})\), then the right-hand side \(\ell \in \mathcal{H}_T^\prime\) of \((BV^R)\) is consistent in the sense that

(9.3) \[ \ell_R(\vec{d}) = 0, \quad \forall \vec{d} \in \ker B_T. \]

**Proof.** Following the strategy found in the proofs of \([15, \text{Lem. 4}]\) and \([15, \text{Lem. 8}]\), we use (4.39) to directly evaluate

\[
\ell_R(\vec{d}) = \langle \frac{1}{2} \vec{a} - \{\gamma_R\} L_T(\vec{a}), \vec{d} \rangle_\Gamma
\]

\[
= \langle \frac{1}{2} \vec{a}, \vec{d} \rangle_\Gamma - \langle \{\gamma_n\} \text{curl}(a_2), d_0 \rangle_\Gamma + \langle K'(a_0), d_0 \rangle_\Gamma
\]

\[
+ \langle a_1, \text{C}(b_1) \rangle_\Gamma - \langle K(a_2), \vec{d}_2 \rangle_\Gamma
\]

\[
= \langle (\frac{1}{2} \text{Id} - K') a_0, d_0 \rangle_\Gamma + \langle \{\gamma_n\} \text{curl}(a_2), d_0 \rangle_\Gamma
\]

\[
+ \langle a_1, (\frac{1}{2} \text{Id} + \text{C}) b_1 \rangle_\Gamma + \langle (\frac{1}{2} \text{Id} + K) a_2, \vec{d}_2 \rangle_\Gamma,
\]

where we recognize the “Maxwell double layer boundary integral operator” \(\text{C}\), and the double layer boundary integral operator \(K\) for the Laplacian.

Locally constant functions are trivially harmonic. They can thus be written using the classical representation formula for the scalar Laplacian in which the Neumann trace vanishes to yield \(d_0 = \gamma (\frac{1}{2} \text{Id} - K)d_0\). Since \(K\) is dual to \(K'\), the first term on the right-hand side vanishes because of the compatibility condition \((\text{CCR})\).

The second term also evaluates to zero. On the one hand, \(\ker \text{curl}_\Gamma = \ker \nabla_\Gamma\). On the other hand, \(\gamma_n \text{curl} = \text{curl}_\Gamma \gamma_t\) in \(H(\text{curl}_\Gamma, \Omega)\), and \(\text{curl}_\Gamma\) is dual to \(\text{curl}_\Gamma\).

The third and fourth terms are shown to vanish in \([15, \text{Lem.4}]\) and \([15, \text{Lem.3}]\) with similar arguments.

In the framework of Section 8, a standard result is the Poincaré inequality: \(\exists C > 0\), only depending on \(\Gamma\), such that \([1, 26]\)

(9.4) \[ \|\vec{b}\|_{\mathcal{H}_R} \leq C \|d_\Gamma \vec{b}\|_{\mathcal{H}_R}, \quad \forall \vec{b} \in \mathcal{R}, \]

where \(\mathcal{R} := (\ker d_\Gamma)^\perp \cap \text{dom}(d_\Gamma)\) and orthogonality is taken in the non-local inner products introduced in Section 5. From the complex (8.3),

(9.5) \[ \text{dom}(d_\Gamma) = H^{1/2}(\Gamma) \times H^{-1/2}(\text{curl}_\Gamma, \Gamma) \times H^{-1/2}(\Gamma), \]
and thus

\[ K = K_0 \times K_1 \times K_2 \in H \]

with

\[ K_0 := \ker \nabla, \quad K_1 := \ker \text{curl} \cap \left( \nabla H^\frac{1}{2} (\Gamma) \right), \quad K_2 := \left( \text{curl}^{-1} H^{-\frac{1}{2}} (\text{curl}, \Gamma) \right). \]

It is routine to verify from (7.2) that \( K = \ker B_R \). Hence, due to the inf-sup inequality supplied in [26, Thm. 2.4], the problem of finding \( \vec{b} = H_T \) and \( \vec{p} \in \alpha \) such that

\[
B_R \left( \vec{b}, \vec{d} \right) + \left\langle \vec{p}, \vec{d} \right\rangle_T = \ell_R(\vec{d}) \quad \forall \vec{d} \in H_T,
\]

\[
\left\langle \vec{b}, \vec{g} \right\rangle_T = 0 \quad \forall \vec{g} \in \ker B_R
\]

is well-posed.

Similarly, the problem of solving

\[
B_T \left( \vec{a}, \vec{e} \right) + \left\langle \vec{q}, \vec{e} \right\rangle_T = \ell_T(\vec{e}), \quad \forall \vec{e} \in H_T,
\]

\[
\left\langle \vec{a}, \vec{g} \right\rangle_T = 0, \quad \forall \vec{g} \in \ker B_T
\]

for the unknown pair \( (\vec{a}, \vec{q}) \in H_R \times \ker B_T \) is well-posed.

**Theorem 9.2.** The mixed variational problems (MBVR) has a unique solution \( \vec{b} = H_T \) such that \( \vec{b} \perp \ker B_R \). Moreover,

\[
\left\| \vec{b} \right\|_{-1/2} + \left\| \vec{p} \right\|_{-1/2} \lesssim \left\| \frac{1}{2} \vec{a} - \{ \gamma_R \} \mathcal{L}_T(\vec{a}) \right\|_{\mathcal{H}_R},
\]

where the constant depends only on the constant in the Poincaré inequality (9.4). If \( \vec{a} \) satisfies (CCR), then this result extends to the variational problem (BVR) and (9.7) holds with \( \vec{p} = 0 \).

Similarly, the mixed variational problems (MBVT) has a unique solution \( \vec{a} = H_R \) such that \( \vec{a} \perp \ker B_T \). Moreover,

\[
\left\| \vec{a} \right\|_{-1/2} + \left\| \vec{q} \right\|_{-1/2} \lesssim \left\| \frac{1}{2} \vec{b} - \{ \gamma_T \} \mathcal{L}_R(\vec{b}) \right\|_{\mathcal{H}_T},
\]

where the constant depends only on the constant in the Poincaré inequality for \( \delta_T \). If \( \vec{b} \) satisfies (CCT), then this result extends to the variational problem (BVT) and (9.8) holds with \( \vec{q} = 0 \).

**10. Conclusion.** First-kind boundary integral equations are appealing to the numerical analysis community because they lead to variational problems posed in natural “energy” trace spaces that are generally well-suited for Galerkin discretization. Therefore, on the one hand, the new equations pave the way for development of new Galerkin boundary element methods. On the other hand, our results simultaneously open a new perspective towards the recent developments in boundary integral equations for Hodge-Laplace problems. As it stands, the rich theories of Hilbert complexes and nilpotent operators not only support our observations with the help of already established abstract inf-sup conditions, but in fact also supply the framework and analysis tools needed to relate the studied non-standard surface Dirac operators to the mixed variational formulations associated with the first-kind boundary integral
operators for the Hodge-Laplacian. In fact, this insight already led us to observe that the variational formulation [15, Eq. 25] is associated with the Laplace-Beltrami of the Hilbert complex (8.1). We note that [15, Eq. 34] also appears to be related to higher-order differential forms on surfaces. The significant observation that our integral operators arise as “non-standard” surface Dirac operators associated to trace Hilbert complexes suggests a new analysis of Hodge-Dirac and Hodge-Laplace related first-kind boundary integral equations which has yet to be explored.

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