COMPACTIFICATION OF SL(2)

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ABSTRACT. We discuss ‘hd-compactifications’ of SL(2, K) for K = C or R. These are compact manifolds with boundary on which both the Schwartz and the Harish-Chandra Schwartz spaces are shown to be relatively standard spaces of conormal functions relative to the boundary. Closure under convolution and other module properties are shown to follow from the structure of appropriate generalized product spaces and the functorial properties of conormal functions and smooth maps between manifolds with corners. It is anticipated that a similar approach applies to general real reductive Lie groups, with the additional complications for SL(n, K) being essentially combinatorial.

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INTRODUCTION

In this note we discuss real compactifications of the groups SL(2, R) and SL(2, C) and associated spaces. These are special cases of the ‘hd-compactification’ which will be described elsewhere for SL(n, K) and GL(n, K). We conjecture that such a compactification exists, and in an appropriate sense is unique, for any real reductive Lie group. We view the hd-compactification as the real analogue of the ‘wonderful compactification’ of de Concini and Procesi, to which it is closely related. Similar compactifications have been considered elsewhere, in particular by Mazzeo and Vasy [6], especially for homogeneous spaces.

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We hope that the approach to the subject of analysis on groups presented here may have more substantial consequences, in this note we restrict attention to relatively well-know results approached with these less familiar techniques, which have their origins in scattering theory and geometric analysis of non-compact and singular spaces. Perhaps most relevant is the study of edge vector fields and the corresponding geometric constructions. In fact we encounter here not only the edge calculus of Mazzeo, [5], which has its origins in the study of hyperbolic space in [7], but also the b-calculus [9].

**Definition 1.** By an hd-compactification of a Lie group $G$ we mean a compactification, in principle to a compact manifold with corners, $G[1]$, with three properties:

- Inversion extends to a diffeomorphism of $G[1]$.
- The right-invariant vector fields span the ‘iterated edge’ vector fields associated to an iterated boundary fibration structure of $G[1]$.
- Together the left- and right-invariant vector fields span the Lie algebra, $\mathfrak{v}(G[1])$ of tangent vector fields.

As is indicated below, these properties imply that the Schwartz space is identified with the space of smooth functions on the compactification, vanishing to infinite order at the boundary. More significantly Harish-Chandra’s Schwartz space, denoted here HC($G$), is identified with the space of conormal functions with respect to the boundary which are log-rapidly decaying relative to a fixed power weight determined by the Haar measure of the group (and encoded by Harish-Chandra in the decay properties of a spherical function). This corresponds to the smallest power of a boundary defining function (in general a product of powers of defining functions) not in $L^2$. For the convenience of the reader an appendix on conormal functions is included.

For SL(2, $\mathbb{R}$) the hd-compactification is a solid 3-torus, i.e. is diffeomorphic to the product of the circle, SO(2), and a closed 2-disc which can be identified with the closure of the positive symmetric $2 \times 2$ matrices of trace 1. This in turn is a radial compactification of the space of positive matrices of determinant one. In general, for SL($n, \mathbb{C}$) or SL($n, \mathbb{R}$), it is necessary to desingularize the stratified space given as the closure of the positive hermitian or symmetric matrices of trace 1, hence the designation ‘hd’. This hd-compactification is shown to be closely related to (and at least for SL($n, \mathbb{C}$) derivable from) the wonderful compactification of de Concini and Procesi, [2]. As pointed out to us by Eckhard Meinrenken, the compactification of SL(2, $\mathbb{R}$) can be obtained as the closure of the image of the radial projection into the sphere in the $2 \times 2$ matrices, i.e. as the closure in the sphere of the matrices with positive determinant.

One of the fundamental properties of the Harish-Chandra Schwartz space is that it is closed under convolution. A geometric proof of this is given here by defining an associated compactification, $G[2]$, of $G^2$ with the property, amongst others, that multiplication $(g, h) \mapsto gh^{-1}$ extends to a smooth map $G[2] \rightarrow G[1]$. Closure of HC($G$) under convolution then follows from push-forward/pull-back properties of conormal functions under b-fibrations, of which this map is an example. The space $G[2]$ is the ‘double’ space for the (in general iterated) edge structure.

We give a second (larger) compactification of SL(2, $\mathbb{R}$) relative to a parabolic subgroup and use it to recover the result that the Harish-Chandra functions on the quotient by the associated unipotent group form a convolution module over the
Harish-Chandra space of the group. Use of this space also serves to show that the spherical function, $\Phi$, is log-smooth.

As an elementary illustration of our approach, consider the 1-dimensional multiplicative group, which appears below as the positive diagonal subgroup of SL(2). We radially compactify $GL_+ = \mathbb{R}_+$ to a closed interval, for instance using the diffeomorphism to the interior

$$GL_+ \ni \tau \mapsto \frac{2}{\pi} \arctan \tau \in [0, 1] = GL_+[1].$$

Thus $\frac{1}{\tau}$ is a defining function near the top boundary, and $\tau$ itself defines the lower boundary. Rather trivially, this is the unique hd-compacification, with the Lie algebra, spanned by $\tau \partial_{\tau}$ generating the b-vector fields. The Harish-Chandra space is

$$HC(GL_+) = (\log \rho)^\infty \mathcal{A}(GL_+[1])$$

is the space of conormal functions (having stable regularity under application of b-differential operators) which decay faster at both boundaries than any inverse power of the logarithm of a defining function.

As noted above, the twisted product $\chi : GL_+^2 \ni (\sigma, \tau) \mapsto \sigma / \tau$ extends smoothly to a b-fibration

$$\chi : GL_+^2 = [GL_+^2 : \{(0, 0), (1, 1)\}] \longrightarrow GL_+^1.$$  

In this case the double space, defined by blowing up the corner, is the usual product space for the b-calculus. The two stretched projections $\pi_R, \pi_L : GL_+^2 \longrightarrow GL_+^1$ are also b-fibrations. The convolution product is captured by the diagram and formula

$$f_1 * f_2 = (\pi_L)_* (\chi^* f_1 \pi_R f_2 dg_R);$$

which carries over to the general case; see also Figure.

In §1 the notion of hd-compactification is discussed and the compactification of SL(2) is described. In §2 the Harish-Chandra and Schwartz spaces are identified and in §3 they are shown to be closed under convolution. The compactification $G[1; N]$ corresponding to the action of unipotent subgroup is presented in §4 and used in §5 to show that the space HC$(G/N)$ is a module over HC$(G)$. Following Crisp and Higson, §6, the relevance of these constructions for induced representations is recalled in §9 and some properties of the intertwining operators for a parabolic group and its opposite are given in §7.

The results concerning the Harish-Chandra Schwartz space given here are largely in reply to questions raised by Nigel Higson; we hope to answer more of these in due course. The authors also thank Roman Bezrukavnikov and Eckhard Meinrenken for helpful discussions.
1. Compactification

By a compactification of a non-compact manifold without boundary $M$ we mean a compact manifold, necessarily with boundary and generally with corners, $M[1]$, and a diffeomorphism

$$M \xrightarrow{i} M[1]$$

onto the interior of $M[1]$. The ‘1’ here corresponds to a compactification of $M$, $M[2]$ to a compactification of $M^2$, etc. Often there is more than one compactification of interest for a given manifold. In that case we add a distinguishing modifier, e.g., $M[2; b]$.

Two compactifications are equivalent if there is a diffeomorphism between them intertwining the injection diffeomorphisms

$$M[1, i] \xrightarrow{i} M \xleftarrow{i'} M[1, i']. \xrightarrow{\pi_R} GL_+[1] \xleftarrow{\pi_L} GL_+[2].$$

Compactifications often arise through the resolution of a singular space and result in manifolds with corners with iterated boundary fibrations. As in [2] we call these simply ‘iterated spaces’. Such an iterated space is a compact manifold with corners with compatible fibrations at each of its boundary hypersurfaces. Since these do not arise in any significant way in the present discussion of $SL(2, \mathbb{R})$ and $SL(2, \mathbb{C})$ (denoted collectively $SL(2)$) we do not recall the notion – but it is needed for
n > 2. For SL(2) the compactifications of the groups are manifolds with boundary and correspondingly the iterated structure is an edge structure. This corresponds to a fibration of the boundary, i.e. it is the total space of a smooth fibre bundle with compact fibres

\[ F \to \partial M[1] \to B. \]

The edge vector fields associated to this structure are

\[ \mathcal{V}_e(M[1]) = \{ V \in \mathcal{V}_b(M[1]); V \text{ is tangent to the fibres } (1.1) \}. \]

Here \( \mathcal{V}_b(M[1]) \) is the Lie algebra of all smooth vector fields tangent to the boundary (or, for a manifold with corners, tangent to all boundary faces).

**Definition 2.** By an hd-compactification of a Lie group we mean a compactification \( G \hookrightarrow G[1] \) in the sense discussed above, with the additional properties:

(a) Inversion extends to a diffeomorphism of \( G[1] \),

(b) The right-invariant vector fields on \( G \) lift to be smooth on \( G[1] \), tangent to all boundary faces, and to span (over \( C^\infty(G[1]) \)) the Lie algebra of iterated edge vector fields – those vector fields tangent to an appropriate iterated structure on \( \partial G[1] \),

(c) The span of left and right invariant vector fields on \( G[1] \) is the Lie algebra of all vector fields tangent to the boundary, \( \mathcal{V}_b(G[1]) \).

Observe that (a) and (b) imply that the left invariant vector fields span the iterated structure which is the image of the one in (b) under inversion. Thus (c) is by way of a transversality condition for these two fibrations. Similarly it follows from (b) that the left and right actions of \( G \) on itself extend to actions on \( G[1] \).

**Conjecture 1.** Any real reductive Lie group has a unique hd-compactification up to equivalence.

Although not discussed here the construction of \( G[1] \) and \( G[2] \) below can be extended to higher products giving a ‘generalized product’ \( G[^*] \) which is a simplicial space with additional functorial properties.

To compactify SL\((n, \mathbb{K})\) we start from the right polar decomposition

\[ G = KA, \ g = ka, \ a = (g^* g)^{1/2}. \]

Thus \( K \) is the maximal compact subgroup. For \( n = 2, \ K = SO(2) \subset SU(2) \) in the real and complex cases and and \( A \) is the space of positive definite Hermitian \( 2 \times 2 \) matrices of determinant one in the complex case and the real subspace for SL\((2, \mathbb{R})\).

Let \( B \) be the corresponding space, of Hermitian or symmetric matrices respectively, that are positive definite and of trace 1. Since \( a \) in \( (1.3) \) has determinant equal to 1 the map

\[ A \ni a \mapsto b = (\text{Tr}(a))^{-1} a \in B, \ \text{Tr}(a) = (\det(b))^{-\frac{1}{2}}, \]

is a diffeomorphism.

Let \( B[1] \) be the closure of \( B \) in the \( 2 \times 2 \) matrices - the space of non-negative Hermitian or real symmetric matrices of trace 1. In the general case, \( n > 2 \), the closure is not smooth and \( B[1] \) is defined as a resolution of the resulting stratified space. Here however,
Lemma 1. $B[1]$ is a closed ball of dimension 3 for $\text{SL}(2, \mathbb{C})$ and dimension 2 for $\text{SL}(2, \mathbb{R})$.

Proof. The elements of $B[1] \setminus B$ are non-negative Hermitian or real-symmetric $2 \times 2$ matrices of rank 1 and trace 1. Thus 1 is an eigenvalue. In a small neighborhood of $\partial B[1]$ in $B[1]$ there is necessarily an eigenvalue close to one and another close to zero, $1 - s$ and $s$, respectively. Since the multiplicity of the eigenvalues is constant the eigenspaces are smooth and the eigen-decomposition allows the neighbourhood to be identified with, in the complex case,

$$
\mathbb{P} \times [0, \varepsilon) \mapsto (1 - s)q(\xi) + sq^\perp(\xi), \quad \xi \in \mathbb{P}.
$$

Here $q$ is orthogonal projection onto the line in $\mathbb{C}^2$ determined by $\xi \in \mathbb{P} = S^2$. In the real case the eigenvectors are necessarily real so the identification becomes

$$
S \times [0, \varepsilon) \mapsto (1 - s)q(\theta) + sq^\perp(\theta).
$$

Now $q(\theta)$ is projection onto $\cos \theta e_1 + \sin \theta e_2, \theta \in [0, \pi)$. Thus it follows that $B[1]$ is a compact ball, or disk, with $s$, defined near the boundary, as a boundary defining function.

In fact it is generally more convenient to take a slightly different representation of a neighbourhood of the boundary of $B[1]$. Since the eigenvalue near 1 is smooth near the boundary we may divide by it and consider instead the space of Hermitian/symmetric $2 \times 2$ non-negative matrices with 1 as an eigenvalue and with the other eigenvalue suitably small. This gives the representation

$$
\beta = q(\theta) + t q^\perp(\theta), \quad t \in [0, \varepsilon).
$$

Then the corresponding element of $A$ is

$$
a = t^{-1/2} \beta = t^{-1/2}q(\theta) + t^{1/2}q^\perp(\theta).
$$

Proposition 1. For $G = \text{SL}(2, \mathbb{R})$ and $\text{SL}(2, \mathbb{C})$

\begin{equation}
G[1] = K \times B[1]
\end{equation}

is an hd-compactification in which the fibration of the boundary corresponding to the right-invariant vector fields is given by the restriction of the product decomposition \((1.4)\) to $\partial G[1] = K \times \partial B[1]$.

Proof. First note that the adjoint action of $K$ on $A$ extends to a smooth action on $B[1]$. Indeed under conjugation $g \mapsto kgk^{-1}$ both the determinant and the trace are invariant so this action projects to $B$ to the conjugation action there and so extends smoothly to the closure $B[1]$. On the boundary $S^2$, respectively $S$, the action of $\text{SU}(2)$ projects to the rotation action of $\text{SO}(3) = \text{SU}(2)/\mathbb{Z}_2$, respectively $\text{SO}(2)$ projects to the rotation action of $\text{SO}(2)/\mathbb{Z}_2$. It follows that this action is smooth (and transitive) on the boundary of $B[1]$, so the compactification obtained by taking the opposite polar decomposition is equivalent to \((1.4)\).

We next compute the span of the right-invariant vector fields, of course this is locally all vector fields on $G$, so we are only interested in the behaviour near the boundary. If $g = ka$ and $u \in g$ is an element of the Lie algebra then

\begin{equation}
\exp(su)ka = k(s)a(s)
\end{equation}

is the integral curve of a general right-invariant vector field near $g$. If

\begin{equation}
g = t + a
\end{equation}
and \( u \in \mathfrak{k} \) then \( \exp(su) \in K \) and it follows that the right-invariant vector fields on \( K \) lift smoothly to \( G[1] = K \times B[1] \) and, by transitivity, span all vector fields on \( K \).

Thus it suffices to consider (1.5) for \( u \in \mathfrak{a} \). Then the polar decomposition gives

\[
a(s)^2 = g^*g = a \exp(2su)a, \quad a = a(0).
\]

Mapping \( A \) into \( B \) gives the curve \( b(s) = (\Tr(a(s)))^{-1}a(s) = \det(b(s))^{1/2}a(s) \) defined by

\[
b(s)^2 = \det(b(s))a \exp(2su)a.
\]

If \( a \in A \) approaches the boundary of \( G[1] \) along the curve of diagonal matrices,

\[
a = \begin{pmatrix} t^{-\frac{1}{2}} & 0 \\ 0 & t^{\frac{1}{2}} \end{pmatrix} \quad \text{as } t \downarrow 0
\]

then

\[
u = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \implies a(s) = \begin{pmatrix} t^{-\frac{1}{2}}e^{s} & 0 \\ 0 & t^{\frac{1}{2}}e^{-s} \end{pmatrix}, \quad \beta(s) = \begin{pmatrix} 1 & 0 \\ 0 & te^{-2s} \end{pmatrix}.
\]

Similarly

\[
u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \implies a(s)^2 = \begin{pmatrix} t^{-1}\cosh(2s) & \sinh(2s) \\ \sinh(2s) & t \cosh(2s) \end{pmatrix}.
\]

The large eigenvalue \( \lambda \) of \( a(s)^2 \) satisfies

\[
(\cosh(2s) - t\lambda)(t^2\cosh(2s) - t\lambda) = t^2\sinh^2(2s) \implies \\
\lambda = t^{-1}\cosh(2s)(1 + t^2s^2F(t^2,s^2))
\]

where the implicit function theorem shows that \( F \) is smooth near 0 and \( F(0,0) \neq 0 \). The corresponding eigenspace is spanned by

\[
e_1 - tsL(t^2, s^2)e_2
\]

with \( L \) smooth and \( L(0)F(0) = 1 \).

From (1.10) it follows that the corresponding right-invariant vector field, projected to \( B[1] \), is \(-2t\partial_t\). Similarly for (1.11) the vector field vanishes with \( t \) but with a coefficient which is a non-vanishing vector field on \( \partial B[1] \). Taking into account the conjugation action discussed above, this identifies the span of the right-invariant vector fields with the edge vector fields for \( K \times \partial B[1] \rightarrow \partial B[1] \).

The inverse of \( kuau^{-1} \), where \( a \) is positive and diagonal, is \( ua^{-1}u^{-1}k^{-1} \). Inversion of diagonal matrices in \( A \) clearly extends smoothly to \( B[1] \) so it follows that inversion on \( G \) extends smoothly to \( G[1] \).

It is noted above that the radial vector field on \( B \) is, near the boundary, in the span of the right-invariant vector fields. Since the conjugation action is in the span of the left and right vector fields and acts transitively on \( \partial B[1] \) it follows that all tangent vector fields on \( G[1] \) are in the smooth span of the left- and right-invariant vector fields. Thus \( G[1] \) is an hd-compactification. \( \square \)

**Remark 1.** An equivalent compactification of \( A \) can be obtained by projecting to the trace-free Hermitian matrices

\[
A \ni a \mapsto a - \frac{1}{2} \Tr(a) \Id \in T.
\]
However it corresponds to the quadratic compactification, rather than the usual radial compactification, in which \((\text{Tr}(\alpha^2))^{-1}\) is introduced as a defining function near infinity. This is well-defined for a linear space (i.e. linear transformations lift to be smooth) but not for an affine space.

**Remark 2.** The stabilizer of a fibre \(K \times \{q\}, q \in \partial B[1]\), of the right fibration of the boundary under the right action of \(G\) is the parabolic subgroup

\[(1.15)\quad P(q^\perp) = \{g \in G; gq^\perp = cq^\perp\}\]

as is discussed further below. Thus the compactification amounts to the simultaneous addition of the homogeneous spaces \(G/P\) for all parabolic subgroups of \(G\).

Geometrically, the resolution \(B[1]\) in the case of \(\text{SL}(n)\) has a strong iterative property. Namely the boundary hypersurfaces are labelled by a ‘depth’ index which can be taken to be the corank of a limiting matrix. The corresponding boundary hypersurface of \(B[1]\) is a bundle over the Grassmannian corresponding to the limiting rank, with fibre the product of two versions of \(B[1]\), one for the point in the Grassmannian and another for its orthogonal.

There is a close relationship between the hd-compactification and the wonderful compactification of de Concini and Procesi.

**Proposition 2.** The adjoint group \(\text{SL}(n, \mathbb{C})/\mathbb{Z}_n\) has the same positive part as \(\text{SL}(n, \mathbb{C})\) and the closure of the image of \(A\) in the wonderful compactification of \(\text{SL}(n, \mathbb{C})/\mathbb{Z}_n\) is diffeomorphic to \(B[1]\).

This is quite elementary for \(n = 2\). For \(\text{SL}(n, \mathbb{C})/\mathbb{Z}_n\) the hd-compactification is given by the real blow-up of the divisors in the wonderful compactification – similar constructions occur in [10] for the Deligne-Mumford compactification of the Riemann moduli space.

## 2. Schwartz spaces

It follows from Proposition 1.4 that for \(\text{SL}(2)\) Haar measure is an edge density on \(G[1]\) – if \(t\) is a boundary defining function then

\[(2.1)\quad dg = t^{-1-2\kappa}\nu = t^{-2\kappa}\nu_b \text{ near } \partial G[1]\]

where \(\nu\) is a smooth, strictly positive, measure and \(\nu_b\) is a b-measure, so near the boundary is of the form

\[(2.2)\quad \nu_b = \frac{dt}{t} \nu_\partial\]

with \(\nu_\partial\) a positive smooth measure on the boundary. The weight \(2\kappa\) is the codimension of the fibres over the boundary, i.e.

\[(2.3)\quad \kappa = \begin{cases} \frac{1}{2} & \text{for } \text{SL}(2, \mathbb{R}) \\ 1 & \text{for } \text{SL}(2, \mathbb{C}). \end{cases}\]

Thus if \(L^2_\rho(G)\) and \(L^2_\rho(G[1])\) are the \(L^2\) spaces, computed relative to Haar measure and a b-measure respectively, then

\[(2.4)\quad L^2_\rho(G) = \rho^\kappa L^2_\rho(G[1])\]

where \(\rho\) is any boundary defining function.
One indication of the relevance of the hd-compactification is that the extended Schwartz and Harish-Chandra spaces are readily characterized in terms of $G[1]$. The definition and some of the properties of the spaces of conormal functions bounded with respect to a weight are recalled in the appendix.

The space of bounded conormal functions is defined by

\[
 u \in \mathcal{A}(X) \iff \sup |Du| < \infty \text{ for all } D \in \text{Diff}^*_b(X)
\]

where $\text{Diff}^*_b(X)$ is the enveloping algebra of $\mathcal{V}_b(X)$. As follows directly, $\mathcal{A}(X)$ is a Fréchet algebra containing the space, $C^\infty(X)$, of functions smooth up to the boundary. If $w < 0$ is a weight in the sense of (A.8) then

\[
 u \in w\mathcal{A}(X) \iff u/w \in \mathcal{A}(X) \iff \sup |(Du)/w| < \infty \text{ for all } D \in \text{Diff}^*_b(X).
\]

The most obvious weights on a compact manifold with corners are the products of real powers of defining functions for the various boundary hypersurfaces. Here logarithmic weights are also important so, always choosing a boundary defining function with $\rho < 1$, we define

\[
 \text{ilog } \rho = \frac{1}{\log \varrho} \in \mathcal{A}(X).
\]

Indeed, if $V \in \mathcal{V}_b(X)$ is a vector field tangent to the boundary then $(V \rho)/\rho \in C^\infty(X)$ and

\[
 (2.5) \quad V \text{ilog } \rho = (\text{ilog } \rho)^2 \frac{V \rho}{\rho}
\]

so it follows that ilog $\rho$ is a weight, vanishing at the boundary. We use the formal notation of $w^\infty w'$ for a weight $w$, required to vanish at the boundary and a second weight $w'$, to denote the intersections of the weighted spaces

\[
 (2.6) \quad w^\infty w' \mathcal{A}(X) = \bigcap_{m \in \mathbb{R}} w^m w' \mathcal{A}(X);
\]

these are again Fréchet spaces, now with $C^\infty_c(X \setminus \partial X)$ a dense subspace.

**Proposition 3.** For an hd-compactification of a semisimple Lie group the Schwartz space is $\dot{C}^\infty(G[1])$, the space of smooth functions vanishing to infinite order at all boundary faces, and the Harish-Chandra (Schwartz) space is

\[
 (2.7) \quad \text{HC}(G) = (\text{ilog } \rho)^\infty \rho^\kappa A(G[1]),
\]

the space of conormal functions with ‘log-rapid vanishing’ at the boundary relative to the weighted space $\rho^\kappa \mathcal{A}(G[1])$.

**Proof.** Here we consider only $G = \text{SL}(2, K)$ but in fact the proof persists for $\text{SL}(n, K)$.

The definition given by Knapp, [4], and Wallach, [12], amounts to the condition

\[
 (2.8) \quad u \in \text{HC}(G) \iff \|g\|^{p} \Xi D_1 D_2 u \in L^\infty(G), \quad \text{for all } p, D_1, D_2
\]

where $D_1$ and $D_2$ are in the left and right enveloping algebras. The weight $\|g\| \sim 1/\text{ilog } \rho$ and the spherical function $\Xi$ is ‘almost in $L_2^+$ – in this case

\[
 ct^\kappa \leq \Xi \leq C t^\kappa \log 1/t \text{ near } \partial G[1], \quad c, C > 0.
\]

For $\text{SL}(2, \mathbb{R})$, this follows from the fact that $\kappa$ is a double root of the indicial polynomial of the radial part of the Laplacian on $A$ shifted corresponding to
the bottom of the continuous spectrum but can also be extracted from results in Varadarajan’s book [11]. A direct proof by push-forward is given below in Lemma 5. Then (2.5) is equivalent to (2.7). Note that we have used the consequence of the \( h_d \)-compactification conditions that

\[
D_1 D_2 \in \text{Diff}_b^*(G[1])
\]

and these products of left and right invariant operators span the \( b \)-differential operators. □

If one thinks in terms of standard harmonic analysis then (2.7) is equivalent to a statement on the Mellin transform near the boundary. Namely the Mellin transform – the Fourier transform in terms of \( \log t \) – is, in an appropriate normalization, holomorphic in the dual half-space \( \text{Im } s > \kappa \) and uniformly a Schwartz function of \( \text{Re } s \) up to, and on, the limiting line with values in \( C^\infty(\partial G[1]) \). Thus, in terms of the variable \( x = i \log t \), these are smooth functions in the usual sense, vanishing rapidly as \( x \downarrow 0 \) but with a factor of \( \exp(-\kappa/x) \).

3. Convolution

The left action of \( G \) on \( G \) is given by integration of the image of the Lie algebra and since these vector fields extend smoothly to \( G[1] \), where they are complete, the left action extends smoothly. Similarly for the right action:

\[
(3.1) \quad G \times G[1] \to G[1], \ G[1] \times G \to G[1].
\]

However the product itself does not extend to a smooth map from \( G[1]^2 \).

To resolve this issue we consider an appropriate compactification of \( G^2 \) obtained by blow-up from \( G[1]^2 \). For \( \text{SL}(2) \) we take \( G[2] = G[2, R] \) to be the edge compactification of \( G[1]^2 \) with respect to the right fibration. More explicitly,

\[
(3.2) \quad G[2] = G[2, R] = K^2 \times B[2, 0], \ B[2, 0] = [B[2, b]; \beta_b^{-1}(\partial \text{Diag})].
\]

Here \( B[2, b] \) is the ‘\( b \)-resolution’ of \( B[1]^2 \), obtained by blowing up the corner:

\[
(3.3) \quad B[2, b] = [B[1]^2; (\partial B[1])^2], \ \beta_b : B[2, b] \to B[1]^2
\]

being the blow-down map. The subsequent blow up in (3.2) is of the preimage of the diagonal in the boundary. In terms of the product with \( K^2 \) on the left, the second blow-up corresponds to the preimage of the fibre diagonal of the boundary. Neither blow-up affects the interior which remains \( K^2 \times B^2 \). The inclusion of \( G^2 \) is through the ‘right’ product inclusion in \( (K \times B[1]) \times (K \times B[1]) \).

Note that the first blow-up is not really necessary to resolve the edge structure of the manifold. However it seems that this larger resolution (the blow-ups can be performed in either order) is the more appropriate one here.

**Proposition 4.** The twisted product map

\[
(3.4) \quad \chi : G \times G \ni (g, h) \mapsto gh^{-1} \in G
\]
and the two projections lift to \( b \)-fibrations

\[
\begin{array}{ccc}
G[1] & \xrightarrow{\pi_L} & G[2] \\
\downarrow & & \downarrow \\
G[1] & \xrightarrow{\pi_R} & G[1]
\end{array}
\]

Although there is a corresponding, but different, left compactification, \( G[2, L] \), of \( G^2 \) we will denote this right compactification by \( G[2] \).

**Proof.** We give a computational proof, although this follows more abstractly from the properties of the hd-compactification. As noted above, the product map extends to \( G \times G[1] \) and \( G[1] \times G \) and, since the blow-ups in \((3.3)\) and \((3.2)\) are in the factors of \( B[1] \) and the conjugation action of \( K \) on \( B[1] \) is smooth, it suffices to consider the behaviour of the product of two elements of \( B[1] \) near the boundary. The diagonal adjoint action of \( K \) on the factors of \( B[1] \) preserves both centres of blow-up, so also extends smoothly to \( G[2] \). Thus it suffices to consider the product where one factor is diagonal

\[
(t_1^{-\frac{1}{2}} q(\xi) + t_1^{\frac{1}{2}} q^\perp(\xi)) 
\begin{pmatrix}
t_2^{-\frac{1}{2}} & 0 \\
0 & t_2^{-\frac{1}{2}}
\end{pmatrix};
\]

here the second factor has been inverted as in \((3.4)\) and, the center of blow-up being in the corner, we may suppose that both \( t_1 \) and \( t_2 \) are close to zero.

The polar part of the product in \((3.6)\) is readily computed

\[
a(t_1, t_2)^2 = \begin{pmatrix}
t_2^{-\frac{1}{2}} & 0 \\
0 & t_2^{-\frac{1}{2}}
\end{pmatrix} (t_1^{-1} q(\xi) + t_1^{\frac{1}{2}} q^\perp(\xi)) 
\begin{pmatrix}
t_2^{-\frac{1}{2}} & 0 \\
0 & t_2^{-\frac{1}{2}}
\end{pmatrix}
\]

\[
= (t_1 t_2)^{-1} \left( q(e_2) (q(\xi) + t_1^{\frac{1}{2}} q^\perp(\xi)) q(e_2) + t_2 q(e_2) (q(\xi) + t_2^{\frac{1}{2}} q^\perp(\xi)) q(e_1) \right.
\]

\[
+ t_2 q(e_1) (q(\xi) + t_1^{\frac{1}{2}} q^\perp(\xi)) q(e_2) + t_2 q(e_1) (q(\xi) + t_2^{\frac{1}{2}} q^\perp(\xi)) q(e_1) \bigg),
\]

written as a sum of the four terms corresponding to the basis \( e_1, e_2, \) so each has rank at most one.

If \( t_1 \downarrow 0 \) and \( t_2 \downarrow 0 \) but \( \xi \) is bounded away from \( e_1 \) then the first, most singular, term is non-zero and there is necessarily an eigenvalue which is a positive smooth multiple of \((t_1 t_2)^{-1}\); the other eigenvalue is its inverse. Thus the trace of the square-root must be of the form \( \alpha^{-\frac{1}{2}} (t_1 t_2)^{-\frac{1}{2}} \) with \( \alpha > 0 \) and smooth. It follows that the corresponding rescaled matrix of trace one satisfies

\[
b(t_1, t_2) = \alpha q(e_2) q(\xi) q(e_2) + t_1 E_1 + t_2 E_2.
\]

It is therefore a smooth curve in \( B \) approaching the boundary. A similar computation shows that the factor in \( K \) in the polar decomposition of \((3.6)\) is also smooth down to \( t_1 = t_2 = 0 \).
This actually proves smoothness of the product without the first blow-up in (3.2), so it certainly remains smooth after this blow-up but away from the preimage of \( \xi = e_2 \).

The blow-up of \( t_1 = t_2 = 0 \) sets \( t_i = \tau_i s \) where the \( \tau_i \) and \( s \) are defining functions for the resulting three boundary faces, moreover \( \tau_1 + \tau_2 > 0 \) since the two ‘old’ boundary hypersurfaces no longer intersect. Then (3.7) becomes

\[
(3.9) \quad a(t_1, t_2)^2 = s^{-2}(\tau_1 \tau_2)^{-1} \left( q(e_2)q(\xi)q(e_2) + s\tau_2 (q(e_2)q(\xi)q(e_1) + q(e_1)q(\xi)q(e_2)) + s^2 F \right)
\]

with \( F \) smooth.

The first term is a multiple \( R_2^2(\xi)q(e_2) \) of the projection onto \( e_2 \) with coefficient which is the square of a defining function for \( \xi = e_1 \) and the coefficient of \( s \) vanishes at \( \xi = e_1 \). The second blow-up, in (3.2), is the introduction of polar coordinates in the sense that a defining function for the new front face is \( x^2 = R^2 + s^2 \). Then \( s = \sigma x \) where \( \sigma \) is a defining function for the lift of \( s = 0 \) and \( R^2 = x^2 r^2 \) where \( r^2 \) is smooth, non-negative, and vanishes precisely at the lift of the diagonal \( \xi = e_2 \). Thus (3.9) becomes

\[
(3.10) \quad a(t_1, t_2)^2 = x^{-2}(\tau_1 \tau_2)^{-1} \left( r^2 q(e_2) + \sigma e + xf \right)
\]

where all terms are smooth and \( e \) is linearly independent of \( q(e_2) \) and does not vanish at \( r = 0 \).

It follows that

\[
(3.11) \quad a(t_1, t_2) = x^{-1}(\tau_1 \tau_2)^{-\frac{1}{2}} b(x, \sigma, \tau_1, \tau_2)
\]

projects to a smooth family in \( B[1] \).

Again a similar analysis shows the smoothness of the compact factor, so the product does extend to a smooth map. A boundary defining function for the left factor \( G[1] \) lifts to the product of boundary defining functions for the three boundary faces excepting the remaining boundary hypersurface which projects onto the boundary of the right factor of \( G[1] \). The \( b \)-submersion condition follows from analysis of invariant vector fields; hence the map is \( b \)-fibration.

That the two projections lift to be \( b \)-fibrations is standard for the edge stretched product for any boundary fibration. \( \square \)

Convolution on \( \mathcal{C}_c^\infty(G) \) is given by the standard formula

\[
(f_1, f_2) \mapsto f_1 \ast f_2(g) = \int_G f_1(gh^{-1})f_2(h) \, dh.
\]

This can be interpreted geometrically as

\[
f_1 \ast f_2 = (\pi_L)_*(\chi^* f_1 \cdot \pi^*_R f_2 \cdot dh)
\]

where \( \chi(g, h) = gh^{-1} \). Since it is most natural to push forward densities we multiply by Haar measure on \( G \) and write the convolution formula as

\[
(3.12) \quad f_1 \ast f_2 dg = (\pi_L)_*(\chi^* f_1 \cdot \pi^*_R f_2 \cdot dg dh)
\]

A basic result due to Harish-Chandra which follows from this geometric setup is:-
**Proposition 5.** Convolutions extend by density from $C^\infty_c(G) \subseteq \text{HC}(G)$ to

\begin{equation}
\text{HC}(G) \times \text{HC}(G) \to \text{HC}(G)
\end{equation}

The proof, below, depends on an examination of the functions and measures in (3.12). The notion of a b-fibration and some of the properties of such maps are briefly recalled in the appendix. Note that for a b-fibration all hypersurfaces in the domain are of one of two types, either ‘fixed’ – those which are mapped onto the image space – or ‘non-fixed’ if mapped into (and then necessarily onto) a boundary hypersurface.

**Lemma 2.** If $f : X \to Y$ is a b-fibration between compact manifolds with corners then pull-back defines a continuous map

\begin{equation}
f^* : (\log \rho)^\infty \mathcal{A}(Y) \to (\log \rho')^{\kappa \mathcal{A}(X)}
\end{equation}

where $\rho$ is a total boundary defining function on $Y$, $\rho'$ is a collective boundary defining function for the non-fixed hypersurfaces in $X$ and $\kappa$ is the formal weight denoting smoothness up to the fixed hypersurfaces. Similarly

\begin{equation}
f_* : (\log \rho_H)^{\infty} w_{H'} \mathcal{A}(X; \Omega_0) \to (\log \rho'_H)^{\kappa} \mathcal{A}(Y; \Omega_0)
\end{equation}

provided $w_{H'}$ is an integrable weight at the fixed hypersurfaces.

**Proof.** Under pull-back with respect to a b-map conormal functions with weight $w$ lift to be conormal with weight $f^* w$ at the non-fixed hypersurfaces and with smoothness up to the fixed hypersurfaces. This gives (3.14) since the pull-back of the logarithmic weight $\log \rho$ at a hypersurface $H'$ in the base satisfies

\begin{equation}
c \prod_{f(H) = H'} (\log \rho_H) \leq f^* \log \rho_H \leq C \prod_{f(H) = H'} (\log \rho_H)^{1/p(H')}, \quad c, \quad C > 0,
\end{equation}

where $p(H')$ is the multiplicity of $f$ at $H'$, the maximum number of hypersurfaces in the preimage of $H'$ with non-empty mutual intersection.

Under push-forward, there is in general a fixed ‘logarithmic growth’ of order $p(H')$ in essentially the same sense. That is

\begin{equation}
f_* : \prod_{H' \in \mathcal{M}_1(Y)} f^*(\log \rho_{H'})^{k} w_{H'} \mathcal{A}(X; \Omega_0) \to \prod_{H' \in \mathcal{M}_1(Y)} (\log \rho_{H'})^{k-p(H')} \mathcal{A}(Y; \Omega_0)).
\end{equation}

Using (3.16) again in both domain and range, (3.15) follows. □

Note that if $X$ is any compact manifold with corners the space of log-rapid decaying conormal functions can be defined in two ways, since

\begin{equation}
(\prod_{H \in \mathcal{M}_1(X)} \log \rho_H) w_{H' \mathcal{A}(X)} = (\log \rho_H)^{\kappa \mathcal{A}(X)}, \quad \rho = \prod_{H \in \mathcal{M}_1(X)} \rho_H.
\end{equation}

**Proof of Proposition 5.** The first step is to analyze the boundary behaviour of the product $dgdh$ of the Haar measures on $G[2]$. On $G[1]^2$ it is an edge density

\begin{equation}
dgdh = l_1^{-2\kappa} l_2^{-2\kappa} \nu_{b}
\end{equation}

where $\nu_{b}$ is a positive b-density. Under blow up of a boundary face, in this case the corner, a positive b-density lifts to a positive b-density so

\begin{equation}
dgdh = \tau_1^{-2\kappa} \tau_2^{-2\kappa} s^{-4\kappa} \nu_{b}, \quad \text{on } [K^2 \times B[2, b])
\end{equation}
with \( \tau_1, \tau_2 \) and \( s \) defining functions for the three boundary hypersurfaces. The second blow up is of a \( p \)-submanifold of \( s = 0 \) of codimension \( 2\kappa \), i.e. the dimension of \( \partial B[1] \). It follows that

\[
dg dh = \tau_1^{-2\kappa} \tau_2^{-2\kappa} s^{-4\kappa} x^{-2\kappa} \nu_b \quad \text{on } G[2].
\]

This is essentially the formula for Lebesgue measure in polar coordinates.

Next consider the pull-back \( \pi^*_R f_2 \) of an element of \( \text{HC}(G) \) to \( G[2] \) under the b-fibration \( \pi_R \). Lemma 2 shows that for the pull-back of bounded conormal functions on a compact manifold with boundary, log-rapid decrease is reflected in log-rapid decay at all non-fixed hypersurfaces and smoothness up to the ‘fixed’ hypersurfaces. Since

\[
\pi^*_Rf_2 \in (\text{ilog} \, \tau_2)^\infty (\text{ilog} \, s)^\infty (\text{ilog} \, x)^\infty (\tau_2 s x)^\kappa \quad \text{for } \tau_1 = 0. \quad \text{A}(G[2]), \quad f_2 \in \text{HC}(G[1]).
\]

Now essentially the same analysis shows that

\[
\chi^*_f_1 \in (\text{ilog} \, \tau_1)^\infty (\text{ilog} \, \tau_2)^\infty (\text{ilog} \, s)^\infty (\tau_1 s^2 \tau_2)^\kappa \quad \text{A}(G[2]), \quad f_1 \in \text{HC}(G[1]).
\]

It follows that the product

\[
(\chi^*_f_1)(\pi^*_R f_2) \in (\text{ilog} \, \tau_1 \text{ilog} \, \tau_2 \text{ilog} \, s \text{ilog} \, x)^\infty (\tau_1 s^3 \tau_2^2 x)^\kappa \quad \text{A}(G[2]); \quad \Rightarrow
\]

\[
(\chi^*_f_1)(\pi^*_R f_2) \text{dg} \text{dh} \in (\text{ilog} \, \tau_1 \text{ilog} \, \tau_2 \text{ilog} \, s \text{ilog} \, x)^\infty (\tau_1 s^2 x)^{-\kappa} \quad \text{A}(G[2]; \Omega_b).
\]

The absence of any power weight in \( \tau_2 \), with the log-rapid decay, means that the density (3.25) is fibre-integrable for \( \pi_L \), so again using Lemma 2 the push-forward of the product is well-defined and in the space \( \text{HC}(G[1])d\nu \), which is Harish-Chandra’s result, (3.13).

4. IWASAWA DECOMPOSITION

As noted above, the base of the fibration given by the product decomposition

\[
\partial G[1] = K \times \partial B[1]
\]

can be identified with the space of parabolic subgroups of \( \text{SL}(2) \). Each parabolic is conjugate to the upper triangular subgroup \( P_+ \subset \text{SL}(2) \), so \( \partial B[1] \) is identified as the 1-dimensional real, respectively complex, projective space. The boundary of \( B[1] \), realized as the rank one positive matrices, is given by the corresponding projections. In this identification, \( P_+ \) is identified with the line \( [e_1] \), or the corresponding projection, \( q(e_1) \).

Consider the Iwasawa decomposition of \( \text{SL}(2) \)

\[
G = KQN_+.
\]

The quotient \( G/N_+ \) can therefore be identified with \( KQ \) where \( Q \) is the subgroup of positive definite diagonal matrices. This leads to the induced compactification given by the closure \( Q[1] \) of \( Q \) in \( B[1] \)

\[
(G/N_+)[1] = K \times Q[1] \subseteq G[1].
\]

Thus, for \( \text{SL}(2) \), \( Q[1] \subset B[1] \) is the same closed interval for both complex and real cases.

The relationship between the base of the boundary fibration (4.1) and the parabolic subgroups can be seen more geometrically.
Lemma 3. The closure in $G[1]$ of each orbit for the right action of $N$ on $G$ contains two points of $\partial G[1] = K \times \partial B[1]$ of the form $\pm kq(N)$ for a fixed point $q_1(N) \in \partial B[1]$ and some $k \in K$.

Proof. Since the parabolic subgroups are all conjugate, it suffices to consider the right action of $N_+$. The action of $K$ on the left commutes with composition with $n(x) \in N_+$ on the right, so it suffices to consider the orbit starting at a point of $A$. In fact from the Iwasawa decomposition it is enough to consider initial points in $Q$. Consider the polar decomposition of the corresponding curve

$$g(x; \tau) = \begin{pmatrix} \tau & 0 & 1 & -x \\ 0 & \tau & 1 & 0 \\ 1 & -s & c \\ 0 & c & b & d \end{pmatrix} = k(x; \tau)a(x; \tau).$$

Since $a$ is positive definite and

$$a(x; \tau)^2 = \begin{pmatrix} \tau & -x\tau \\ -x\tau & x^2\tau + \tau^{-1} \end{pmatrix},$$

it follows that

$$a(x; \tau) = \frac{1}{(\tau + x^2\tau + \tau^{-1} + 2)^{\frac{1}{2}}} \begin{pmatrix} \tau + 1 & -x\tau \\ -x\tau & x^2\tau + \tau^{-1} + 1 \end{pmatrix}.$$

Dividing by the trace it follows that the image curve is smooth in $B[1]$ in terms of the radially compactified variable $x/(1 + x^2)^{\frac{1}{2}}$ and

$$a(x; \tau) \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

at both end-points. Similarly

$$c = \frac{\tau + 1}{((\tau + 1)^2 + (x\tau)^2)^{\frac{1}{2}}}, \quad s = \frac{-x\tau}{((\tau + 1)^2 + (x\tau)^2)^{\frac{1}{2}}},$$

which are smooth functions of $1/|x|$ as $x \rightarrow \infty$ and

$$k(\tau; x) \rightarrow \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

as $x \rightarrow \mp\infty$.

Notice that the limit point of the right $N$ orbits projected to $B$

$$q_1(N_+) = e_2$$

is the opposite end-point of $Q[1]$ to that coming from the limiting projection above, which will be denoted $q_0(N_+) = e_1$.

We define a second compactification of $G$ associated to a choice of parabolic by two levels of blow-up, associated to $q_1(N)$, from $G[1]$

Definition 3. The compactification of $G$ relative to $N$, acting on the right, is

$$G[1; N] = K \times B[1; N], \quad B[1; N] = [[B[1]; \{q(N)\}]; \partial ff].$$

Here $\partial ff$ is the codimension two corner which is the boundary of the front face introduced in the first blow-up, so consisting of two points for SL(2, $\mathbb{R}$) and a circle for SL(2, $\mathbb{C}$).

Thus, $G[1; N]$ has three bounding hypersurfaces, an ‘old’ one, numbered ‘0’ corresponding to the original boundary, and the two front faces numbered ‘1’ and ‘2’ from the two blow-ups. So the ‘2’ face separates the ‘0’ and ‘1’ faces.
**Figure 2. Resolving the unipotent flow**

**Proposition 6.** The quotient map $G \rightarrow G/N$ extends to a fibration

\[
\begin{array}{ccc}
G & \xrightarrow{\pi_N} & G/N \\
\downarrow & & \downarrow \\
N[1] & \xrightarrow{\pi_N} & G[1; N] \\
\downarrow & & \downarrow \\
G[1] & \xrightarrow{\beta} & G/N[1]
\end{array}
\]

where $N[1]$ is the radial compactification of $N$ as a Euclidean space. Defining functions $\rho_i$, $i = 0, 1$ for the boundaries of $G/N[1]$ and $\rho$ for the boundary of $G[1]$ pull back in terms of defining functions for the boundaries of $G[1; N]$ as

\[
\begin{align*}
\pi_N^* \rho_0 &= \tilde{\rho}_0, \\
\pi_N^* \rho_1 &= \tilde{\rho}_1 \tilde{\rho}_2^2, \\
\beta^* \rho &= \tilde{\rho}_0 \tilde{\rho}_1 \tilde{\rho}_2^2
\end{align*}
\]

and the generating vector field for the right action of $N$ lifts to $G[1; N]$ to be of the form

\[
V(N) = \tilde{\rho}_1 \tilde{\rho}_2^2 W, \quad 0 \neq W \text{ smooth}, \quad W \tilde{\rho}_2 \neq 0 \text{ at } \tilde{\rho}_2 = 0
\]

where $W$ is tangent to the boundary surfaces $\{\tilde{\rho}_0 = 0\}, \{\tilde{\rho}_1 = 0\}$. The inclusion makes $G/N[1]$ into a p-submanifold transversal to the fibration such that $\tilde{\rho}_0$ and $\tilde{\rho}_1$ restrict to boundary defining functions.

So this blow-up ‘resolves’ the right action of $N$ in the sense that the closures of the the orbits in $G$ become the orbits of the smooth vector field $W$ and are precisely the fibres of $\pi_N$. Note however that while the action of $N$ does extend smoothly to $G[1; N]$ the points on the two front faces are all fixed points for the action. So although this gives a meaning to the ‘quotient’ formula

\[
G[1; N]/N[1] = G/N[1]
\]

this is not strictly correct in terms of orbit spaces.

**Proof.** The conjugation action by $K$ allows the discussion to be reduced to the case of $N_+ \subset G = \text{SL}(2, \mathbb{R})$ We proceed to compute the form of the generating vector field for the right action of $N_+$, which we know to be smooth on $G[1]$. In fact we have already seen that $V(N)$ is tangent to the boundary and non-vanishing except at $K \times \{q_1(N)\}$.

A neighborhood of $q_1(N) \in B[1]$ is smoothly parameterized by the matrices

\[
\beta(t, s) = \begin{pmatrix} t + s^2 & s \\ s & 1 \end{pmatrix}, \quad t \geq 0, \quad |s| < \delta.
\]
Applying $N_+$ on the right and taking the polar decomposition gives
\[
\beta \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} = KA,
\]
where
\[
K = T \begin{pmatrix} t + s^2 - xs + 1 & -x(t + s^2) \\ x(t + s^2) & t + s^2 - xs + 1 \end{pmatrix},
\]
\[
A = T \begin{pmatrix} (t + s^2)^2 + s^2 + t & -x(t + s^2)^2 + s(t + s^2 - sx + 1) \\ -x(t + s^2)^2 + s(t + s^2 - sx + 1) & (-x(t + s^2) + s^2 + (-sx + 1)^2 + t) \end{pmatrix},
\]
\[
T = T(x, s, t) = ((x(t + s^2))^2 + (t + s^2 - xs + 1))^{\frac{1}{2}}.
\]

Differentiating at $x = 0$ gives the generating vector field
\[
(4.13) \ V(N) = (s^2 + t + 1)^{-1} (2st(2s^2 + 2t + 1)\partial_t + (s^4 + s^2 - t^2)\partial_s - (s^2 + t)\partial_\theta).
\]
In particular all coefficients vanish at $q_1(N)$. The blow up of the boundary point introduces a boundary face which is an interval. A neighbourhood of this boundary hypersurface of $[B[1]; q_1(N_+)]$ is covered by the three coordinate systems
\[
(4.14) \ \frac{s}{t}, t \text{ over } |s| < 2t, \ \frac{t}{|s|}, |s| \text{ over } t < 2|s|
\]
where the second pair of coordinate systems cover the endpoints and the first actually covers the interior. From (4.20) and (4.25) it follows that along the integral curves
\[
(4.15) \ c_\tau s^2 < |t| < G_\tau s^2, \ \tau > 0.
\]
So after this first blow-up these curves approach one of the end-points of the front face.

The second blow-up replaces each of these end-points by an interval with a neighbourhood covered by two coordinate systems. Restricting attention to the end-point in $t < 4s$ these are given by
\[
(4.16) \ \eta = \frac{t}{s^2}, s \text{ over } t < 2s^2, \ \sigma = \frac{s^2}{t}, \frac{t}{s} \text{ over } 2t > s^2.
\]
In particular near points of the interior of the interval, either system is admissible. It follows from (4.15) that each integral curve of $N$ starting at a finite point, $\tau > 0$, hits the boundary in the interior of this second front face at a unique point. Indeed along the curves defined by $b(x, \tau)$
\[
(4.17) \ \eta \rightarrow \tau^{-1}, s = |x|^{-1} + O(|x|^{-3})
\]
and $\eta$ and $s$ are smooth functions of $x^{-1}$ and $0 < \tau < \infty$.

From (4.13), in polar coordinates at $t = 0, s = 0, S = s/t$,
\[
V = (s^2 + t + 1)^{-1} t((2s^2 + 2t + 1)(S(t\partial_t - S\partial_s)
+ (S^4t^2 + S^2 - 1)\partial_S - (1 + S^2)\partial_\theta)) = tW
\]
where $W|_{t=0} \neq 0$. Making the second blow-up introduces $\eta = t/s^2$ and $s' = s$ in $t << s^2$ and $\sigma = t^2/s, T = t/s$ in $t >> s^2$ in terms of which
\[
(4.18) \ 2t\partial_t + s\partial_s = s'\partial_{s'}, t\partial_t = \eta\partial_\eta \Rightarrow V = (s')^2W, Ws' \neq 0.
\]
Similar analysis in the other regions gives the stated form of $W$. 

It is convenient to have a ‘universal’ version of this resolution. If we consider the space $G[1] \times K$ as a bundle over $K$ then in each fibre we may consider the action of the parabolic group parameterized by $k' \in K$ corresponding to the unipotent group

$$N_{k'} = k'N_+(k')^{-1}.$$ 

The total space of the fibre bundle in which the action of this group in each fibre is resolved, as above, is a compact manifold with corners

$$\langle 4.19 \rangle \quad (G \times K)[1; N_+] = (G[1] \times K; \partial_H)$$

where the p-submanifold $q_1(N_+)$ is the graph of $K \ni k' \mapsto q_1(k') \in \partial G[1]$ and the second blow-up is of its boundary. The ‘universal’ quotient map corresponding to the collective fibration by the $N_+$ is

$$\langle 4.20 \rangle \quad (G \times K)[1; N_+] \longrightarrow (G \times K)/N_+[1]$$

where the central space is the fibre bundle over $K$ with fibre at $k'$ the compactified space $(K \times k'Q(k')^{-1})[1] \subset G[1]$ of $k'$-conjugates of diagonal matrices. This fibrewise action gives a diffeomorphism to the product bundle over the last factor

$$\langle 4.21 \rangle \quad (G \times K)/N_+[1] \longrightarrow K \times Q[1] \times K$$

extending

$$G \times K/N_+ \ni (k, q', k') \longmapsto (kk', (k')^{-1}q'k', k') \in K \times Q \times K.$$ 

The transversality of this action of $K$ on $G \times K$ means that the projection back is a b-fibration

$$\langle 4.22 \rangle \quad \pi_L : (G \times K)[1; N_+] \longrightarrow G[1].$$

Following standard prescriptions

**Lemma 4.** The Harish-Chandra space of $G/N$ is identified with the conormal space

$$\langle 4.23 \rangle \quad HC(G/N) = (i \log \rho)^{\infty} \rho_0^\infty \rho_1^{-\infty} A(G[N[1]])$$

where $\rho = \rho_0 \rho_1$ is a total boundary defining function and $\rho_1$ defines the end corresponding to $q(N)$, boundaries respectively.

**Proof.** The invariant vector field on $Q$ is the radial vector field $\tau \partial_\tau$ so the span with the generating vector field(s) from $K$ gives b-regularity with respect to the weight $\delta^{-\frac{1}{2}}$ plus log-rapid decay. 

Having established that $G[1; N]$ fibres over $G/N[1]$, consider the pull-back via the total blow-down map, $\beta : G[1; N] \longrightarrow G[1]$, of $HC(G[1])$ to $G[1; N]$. In fact

$$\langle 4.24 \rangle \quad \beta^* HC(G) \subset (\log \rho)^{\infty} (\rho_0 \rho_1 \rho_2)^{-\infty} A(G[1; N]).$$

where the weight follows from $\langle 4.10 \rangle$. This is not an equality of spaces since the pulled back functions have more regularity at the front faces.

The properties of the compactification $(G \times K)[1; N_+]$ also allow us to analyse the boundary behaviour of Harish-Chandra’s spherical function for $SL(2, K)$. The spherical function is defined in terms of the pseudocharacter given by the eigenvalue quotient

$$\langle 4.25 \rangle \quad \delta : Q \ni \begin{pmatrix} \tau^{-\frac{1}{2}} & 0 \\ 0 & \tau^\frac{1}{2} \end{pmatrix} \longrightarrow \tau^{-1} \in [0, \infty)$$
extended to $G$ through the Iwasawa decomposition
\begin{equation}
\delta : G \to Q \to (0, \infty).
\end{equation}
Then the spherical function (associated to $N$) is
\begin{equation}
\Phi(g) = \frac{1}{\text{Vol}(K)} \int_K \delta^{-\frac{1}{2}}(gk^{-1})dk.
\end{equation}
Notice that the invariant measure on $G/N_+ = KQ$ is $\delta^{-1}dkda$.

**Lemma 5.** The spherical function for $\text{SL}(2)$ with respect to $N$ is polyhomogenous conormal, positive, bi-invariant for the action of $K$ and takes the form near the boundary
\begin{equation}
\Phi(g) = -bt^\mu \log t + at^\mu, \ a, b \text{ smooth, } b|_{\rho_G[1]} > 0.
\end{equation}
In particular, as a weight function, $1/\Phi$, with rapid logarithmic decay added, gives the same space as $\rho^\mu$ on $G[1]$.

**Proof.** Consider the real case for $N_+$. Since $\delta$ is pulled-pack from $Q$ using the Iwasawa decomposition with respect to $N$ the integrand in $\Phi$ as a function on $G \times K$ is, for each $k \in K$, the pull-back of the corresponding function on the $k'$-diagonal matrices under the Iwasawa decomposition for $N_k = kNk^{-1}$. These actions are resolved on $(G \times K)[1; N_*]$. Thus the spherical function satisfies
\begin{equation}
\delta^{-1}(gk^{-1}) = \tilde{\rho}^{-1}_0 \tilde{\rho}_1 \text{ on } (G \times K)[1; N_*]
\end{equation}
is the product of the inverse of a defining functions for the ‘old’ boundary and a defining function for the first front face.

Thus $\Phi$ is the push-forward to $G[1]$ of $\tilde{\rho}^{-1}_0 \tilde{\rho}_1$ with respect to the fibre density $dk$ under the b-fibration. To compute the form of $\Phi$, choose a positive b-density $\nu_G$ on $G[1]$. Then $\nu_G dk$ is a positive b-density on $G[1] \times K$ and lifts after the first blow-up to be of the form $\rho_1 \nu_b$. Since the second blow-up is of a corner, which is a boundary hypersurface of the first front face, the lift to $(G \times K)[1; N_*]$ is of the form $\rho_1 \rho_2 \nu_b$. Thus the spherical function satisfies
\begin{equation}
\Phi_{\nu_G} = (\pi_G)_* \tilde{\rho}^{-1}_0 \tilde{\rho}_1 \nu_b \implies \Phi = at^{\frac{\mu}{2}} - bt^\mu \log t, \ a \text{ smooth}, b > 0 \text{ near } \partial G[1].
\end{equation}
Here the positivity follows from the positivity of the integrand and the coefficient of the logarithm corresponds precisely to the integral over the corner.

The argument for the spherical function on $\text{SL}(2, \mathbb{C})$ with respect to the upper triangular Borel subgroup is very similar.

**Lemma 6.** Averaging over a unipotent subgroup gives a continuous linear map
\begin{equation}
\int_N dn : \text{HC}(G) \to \text{HC}(G/N).
\end{equation}

**Proof.** This follows from the properties of pull-back and push-forward for the space $(G \times K)[1; N_*]$. Following the lifting property it suffices to show that along the central row in push-forward gives
\begin{equation}
\int_N dn : (\log \rho)^\infty (\rho_0 \rho_1 \rho_2)^\kappa \mathcal{A}(G[1]; N) \to (\log \rho)^\infty \rho_0^\kappa \rho_1^{-\kappa} \mathcal{A}(G/N[1]).
\end{equation}
It follows from Proposition that $dn = \rho_1^{-2\kappa} \rho_2^{-2\kappa} d\bar{n}_b$, where $d\bar{n}_b$ is a non-vanishing b-measure on the closed interval $N[1]$. This is transversal to the boundary
Thus the decay at $\rho_2 = 0$ in (4.24) is indeed sufficient to give integrability across the fixed hypersurface $\{\rho_2 = 0\}$ and the extra factor of $\rho_1^{-2\kappa}$ gives the change of weight in (4.32).

It is immediate that the map (4.32) is surjective but in fact (4.31) is also surjective although this is not so elementary.

5. HC$(G/N)$ as a module

For a unipotent subgroup and
\[
\pi_{\infty}(G), \ u \in \pi_{\infty}(G/N),
\]
\[f * u(h) = \int_G f(g)u(g^{-1}h)dg = \int_G f(hg^{-1})u(g)dg \in \pi_{\infty}(G/N)
\]
defines a continuous linear map
\[
(5.2) \quad f * : \pi_{\infty}(G/N) \to \pi_{\infty}(G/N).
\]
For the upper triangular case, fixing $h = kq$, $(k,q) \in KQ$,
\[
f * u(k,q) = \int_{K \times Q} S(f)(k,q,k',q')u(k',q')dk'dq'
\]
where for fixed $k$ and $q$
\[
(5.3) \quad S(f)(k,q,k',q') = (\pi_N)_{(f(kq'q^{-1})(u')^{-1})}^*(k')^{-1}u(k',q')\delta(q')^{-1}dn = \tilde{S}(f)(k\tilde{q}(k')^{-1}),
\]
\[n' = q'(q')^{-1}, \quad \tilde{q} = q(q')^{-1}
\]
and
\[dn' = \delta(q')dn \Rightarrow \tilde{S}(f)(k,\tilde{q},k') = (\pi_N)_{(f(kq^{-1})q(k')^{-1})}^*\text{ on } KQ \times KQ
\]
where $\pi_N$ is the push-forward map in Lemma 6.

This corresponds to the resolution given by the space $(G \times K)[1; N_s]$. Thus it follows as in Lemma 6 that
\[
(5.4) \quad \tilde{S} : HC(G) \to (\log \rho)^{\infty} \rho_1^{-\kappa} \rho_0^\kappa A(K \times Q[1] \times K)
\]
where we have used the push-forward theorem for the b-fibration (4.21).

**Proposition 7.** The map (5.2) extends by continuity to a bilinear map
\[
(5.5) \quad HC(G) \times HC(G/N) \to HC(G/N)
\]
making $HC(G/N)$ a module over $HC(G)$ extending the product (5.2); the action of $HC(G)$ is through a family of $\mathbb{R}^+$-invariant b-smoothing operators on $G/N[1]$.  

**Proof.** From (5.1) it follows that the kernel map (5.3) has image in the corresponding space of conormal densities
\[
(5.6) \quad S : HC(G) \to (\log \rho')^{\infty} \rho_{0L}^\kappa \rho_0 \rho_1^\kappa \rho_{1L}^\kappa \iota''A(G/N[2; b])dk'dq'
\]
where $\rho'$ is a collective defining function for the ‘old boundaries’ and similarly $\iota''$ corresponds to smoothness up to the two front faces. Now the mapping property
(5.3) is a direct consequence of the action diagram for the b-calculus

\[ (\log \rho)^{\infty} \rho_1^{\kappa} \rho_1^{\kappa} \rho_0^{\kappa} \rho_0^{\kappa} A(G/N[2]; b) dk' dq' \]

where now \( \rho \) is a total boundary defining function. This pushes forward into HC(G/N) giving (5.5).

The natural action of the diagonal group \( Q \) on \( C_\infty^c(G/N) \) includes the pseudocharacter

\[ C_\infty^c(G/N) \times Q \ni (u, \lambda) \mapsto \delta^{-\frac{1}{2}}(\lambda) u(k \mu \lambda^{-1}) \]

where \( G/N \) is identified with \( KQ \). The convolution action of \( v \in C_\infty^c(Q) \) is therefore

\[ u \ast v(k \mu) = \int u(k \mu \lambda^{-1}) v(\lambda) \delta(\lambda^{-\frac{1}{2}}) d\lambda. \]

**Lemma 7.** The product (5.9) extends by continuity to a jointly continuous bilinear map

\[ HC(G/N) \times HC(Q) \longrightarrow HC(G/N). \]

**Proof.** For the radial compactification of the group \( \mathbb{R}_+ \) the Harish-Chandra space is

\[ HC(Q) = (i \log \rho)^{\infty} A(Q[1]). \]

The composition can be realized in terms of the diagram of b-fibrations

\[ \begin{array}{ccc}
G/N[1] & \xrightarrow{\pi_R} & G/N[1] \\
\pi_L & \downarrow & \downarrow \\
G/N[2] & \xrightarrow{\pi_L} & G/N[1] \\
\chi & \downarrow & \\
Q[2] & & \\
\end{array} \]

where \( Q[2] = Q[2, b] \) and the lower map is the lift of the product \( (q, q') \mapsto q(q')^{-1}. \)

6. **Parabolic induction**

By definition, a tempered representation of a reductive group \( G \) is a smooth representation in a Fréchet space \( \mathbb{V} \) so a smooth map

\[ (6.1) \quad \pi : G \longrightarrow \text{Hom}(\mathbb{V}), \quad \pi(gh) = \pi(g) \pi(h) \]

which has the regularity property that the convolution integral

\[ (6.2) \quad C_\infty^c(G) \ni f \longrightarrow \int_G \phi(gh^{-1}) \pi(h) v \]
extends by continuity to a jointly continuous bilinear map
\[
\tilde{\pi} : HC(G) \times V \longrightarrow V
\]
which is a module over convolution
\[
(6.3) \quad \tilde{\pi}(f \ast g, v) = \tilde{\pi}(f, \tilde{\pi}(g))
\]
and is surjective
\[
(6.5) \quad \tilde{\pi}(HC(G), V) = V.
\]
In fact this last property is a consequence of the others. Conversely, (6.3), (6.4) and (6.5) (apparently) imply (6.1) exists so that (6.3) is recovered from (6.2).

Now, we wish to consider the functor of parabolic induction – construction of representations of \(G = SL(2, \mathbb{R})\) from representation of \(L = D \cup -D\) using the upper triangular parabolic \(LN, N = N_+\). To do so consider the action of \(L\) on the right on \(G/N_+ = KD\). This gives rise to a diagram of maps
\[
(6.6) \quad \\
\begin{array}{cccc}
G/N & \longrightarrow & G/N \times L & \longrightarrow \\
\uparrow & & \downarrow & \\
L & & C & \longrightarrow \\
\end{array}
\]
where (this may be a bad choice of normalization) the top map is the left projection, the lower left map is product map, \((kd, l) \mapsto kd\), and the lower right map is projection and inversion, \((kd, l) \mapsto l^{-1}\).

Now, if we define the compactification of the product to be the b-stretched product
\[
(6.7) \quad (G/N \times L)[1] = K \times [D[1] \times L[1], \{q(e_1), q(e_1)\}, \{q(e_2), q(e_2)\}]
\]
obtained by blowing up four of the eight corners, where \(L[1] = D[1] \cup -D[1]\), then:-

**Lemma 8.** The diagram of fibrations (6.6) extends to a diagram of b-fibrations
\[
(6.8) \quad \\
\begin{array}{cccc}
G/N[1] & \longrightarrow & (G/N \times L)[1] & \longrightarrow \\
\uparrow & & \downarrow & \\
L[1] & & C & \longrightarrow \\
\end{array}
\]

**Proof.** Basically this is the stretched product for the multiplicative group \(L\) and ultimately \(D\). \(\Box\)

The Harish-Chandra space of \(L\) is
\[
(6.9) \quad HC(L) = (i \log \rho)\infty A(L[1]).
\]
That is, the bounded conormal functions with log-rapid decay, without weight.
As above, we can deduce a product from (6.8) as a bilinear map
\[ \phi \circ \psi = L_*(C^*\phi \cdot R^*(\delta^{1/2}\psi)dl). \]
Here \( \delta : L = \begin{pmatrix} I & 0 \\ 0 & l^{-1} \end{pmatrix} \to l^2 \). So \( \delta(l^{-1}) = \delta(l)^{-1} \).

**Proof.** The compactified space is really two copies of the product \( K \times D[2] \) where \( D[2] \) is the b-resolution of \( D^2 \), so with the two diagonal corners blown up. I believe \( dl \) is the b-differential (so confusingly \( dl/l \)). If I have not messed up here, the factor \( \delta^{1/2} \) shifts the weighting on the unweighted space \( HC(L) \) so that it looks like the restriction of \( HC(G/N) \) to \( L[1] \), as a p-submanifold of \( G/N[1] \). As a result this should be like the action of \( HC(G) \) on \( HC(G/N) \) below. \( \square \)

For any Fréchet space \( V \) there is no problem in defining \( HC(G;V), \ HC(G/N;V) \) and so on, just as the subspace of \( C^\infty \) maps into \( V \) which satisfy the same estimates as \( HC \) but for each of the seminorms on \( V \). Now, the induced representation corresponding to \( \pi \) acts on a Fréchet space which is a closed subspace of \( HC(G/N;V) \). Namely
\[ HC_\pi(G/N;V) = \{ u \in HC(G/N;V); u(kdl) = \delta(l)^{-1/2} \pi(l)^{-1} u(kdl) \}. \]

This is supposed to carry an induced tempered representation of \( G \). Clearly \( HC(G;N) \) itself has a left action of \( G \) and this leaves \( HC_\pi(G/N;V) \) invariant.

The first claim is that the condition in (6.11) can be expressed in terms of the maps in (6.8).

If \( u \in HC(G/N;V) \), \( \tilde{\pi} \) is the bilinear map from \( \pi \) and \( \psi \in HC(L) \) then \( L^*u : G/N \times L \to V \) with corresponding boundary behaviour on \( (G/N \times L)[1] \). Then consider the composite map corresponding to (6.11)
\[ \delta(l)^{1/2} \psi(l)\pi(l)u(kdl) : G/N \times L \to V \]
Then the condition (6.11) should imply, and actually reduce to
\[ \int \delta(l)^{1/2} \psi(l)\pi(l)u(kdl) = \tilde{\pi}(\psi,u(kd)) \]
and this in turn is written more compactly as
\[ L_*(R^*(\delta^{1/2}\psi\pi)C^*u \cdot dl) = \tilde{\pi}(\psi,u) \text{ in } HC(G/N;V). \]

Of course in C+H this is written in terms of quotients of completed tensor products.

### 7. Intertwining

If \( N_- \) is the opposite, the transpose, of \( N_+ = N \) then
\[ G/N_- \cong KA \]
so the two space are naturally identified. However under this identification the corresponding Harish-Chandra spaces \( HC(G/N_\pm) \) are not identified. Rather
\[ HC(G/N_\pm) = (i\log \rho)^\infty \rho_\pm^{\mu}A(KA) \]
\[ \implies HC(G/N_+) = \delta_+ HC(G/N_-), \ \delta_+ = \rho_0^{-2\mu} \rho_1^{2\mu} = \rho_1^{-2\mu} \rho_0^{2\mu} = \delta^{-1}. \]
The ‘limiting element’ for \( N_+ \), \( q(e_2) \), is replaced for \( N_+ \) by
\[
q(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]
and since this is antipodal we may simultaneously perform the resolutions for both \( N_+ \) and for \( N_- \) obtaining

\[
G[1; N_\pm] = K \times B[1; N_+, N_-],
\]
\[
B[1; N_+, N_-] = [[B[1]; \{q(e_2)\}; \{q(e_1)\}] ; \partial_{2^+} \partial_{2^-}]
\]
since the centres of blow-up are disjoint – see Figure 3.

This space can be used to analyze the well-known intertwining operators \( J_\pm \) which using (7.1) can be seen as integral transforms

\[
J_\pm : C_\infty^c(KA) \rightarrow C_\infty^c(KA), \quad J_\pm(u) = (\pi_{N_\pm})_*((\pi_{N_\pm}^* u) \cdot dn_\pm).
\]

**Proposition 8.** The fibrations \( \pi_\pm : G[1; N_\pm] \rightarrow G/N_\pm[1] \) lift to b-fibrations

\[
\xymatrix{ G[1; N_\pm] & G/N_-[1] \ar[l] \ar[r] & G/N_+[1] }
\]
and the intertwining operators \( J_\pm \) in (7.4) extend to continuous linear operators

\[
J_\pm : HC(G/N_\pm) \rightarrow \delta_{\pm}^\frac{1}{2} C_\infty^c(G/N_\pm) + HC(G/N_\pm)
\]
where the non-trivial leading term is given explicitly as an integral

\[
K_\pm : HC(G/N_\pm) \rightarrow C_\infty^c(K), \quad K_\pm f = \delta_{0_\pm} \frac{1}{2} J_\pm f \bigg|_{\rho_{0_\pm} = 0},
\]

\[
K_\pm(f)(\theta) = \int_A \delta_{\pm}^\frac{1}{2} (f(a, \theta + \pi/2) - f(a, \theta - \pi/2)) \, da.
\]
Remark 3. The computations of Crisp and Higson [1] show that $\mathcal{J}_\pm$ have continuous right inverses $I_\pm$

(7.7) \hspace{1cm} HC(G/N_\pm) \xrightarrow{I_\pm} \text{Nul}(K_\pm), \text{ } \mathcal{J}_\pm I_\pm = \text{Id}.

Proof. The first step is to analyse the pull-back of $HC(G/N_\pm)$ to $G[1, N_\pm]$. This factors through the pull-back to $G[1; N_\pm]$ where the projection to $G/N_\pm[1]$ is a fibration, so

(7.8) \hspace{1cm} \pi_+^* HC(G/N_\pm) \subset \nu_2(\text{ilog } \rho_0^+)\infty(\text{ilog } \rho_1^+)\infty \rho_0^{-\mu} \rho_1^\mu \mathcal{A}(G[1; N_\pm]).

The extra blow-ups in the passage from $G[1, N_\pm]$ to $G[1, N_\pm]$ occur at $q(e_1)$, in the interior of the face defined by $\rho_0^+$, and at the boundary of the resulting front face. It follows directly that

(7.9) \hspace{1cm} \pi_+^* HC(G/N_\pm) \subset \nu_2(\text{ilog } \rho_0^+)\infty(\text{ilog } \rho_1^+)\infty(\text{ilog } \rho_2^-)\infty(\text{ilog } \rho_1^-)\infty \rho_0^\mu \rho_2^\mu \rho_1^- \rho_1^\mu \mathcal{A}(G[1; N_\pm]).

where now $\rho_0$ defines the ‘old boundary’ outside the two blow-ups. As usual, this is not an equality.

Push-forward is relative to the generating vector field $V_-$ for the action of $N_-$. From Proposition 6 this is smooth on $G[1, N_-]$ and of the form $\rho_1 - \rho_2^2 W_-$ with $W_\rho_2 \neq 0$ at the boundary but $W_-$ tangent to the other boundaries and in particular non-zero at $q(e_2)$. Lifted to $G[1; N_+, N_-]$ this becomes singular and of the form

(7.10) \hspace{1cm} V_- = \rho_1^{-1} \rho_2^{-1} \rho_1 - \rho_2^2 W_-, \text{ } W_\in V_0(G[1; N_\pm])

where now $\tilde{W}_-$ is a non-vanishing smooth b-vector field which spans the null space of the b-differential of the stretched projection to $G/N_-[1]$. Overall then $(J_+ f)\nu_0$, for $f \in HC(G/N_+)$ is the image of some some b-density

\begin{multline*}
\nu_2(\text{ilog } \rho_0^+)\infty(\text{ilog } \rho_0^+)\infty(\text{ilog } \rho_1^-)\infty \rho_0^\mu \rho_2^\mu \rho_1^- \rho_1^\mu \mathcal{A}(G/N_[-1])\nu_0, \\
g \in \mathcal{A}(G[1; N_+, N_-]).
\end{multline*}

under pushforward with respect to $\pi_-$. This is indeed integrable across the fixed boundary $\{\rho_2 = 0\}$ (because of the rapid log-decay) and by the push-forward theorem therefore lies in

(7.11) \hspace{1cm} (\text{ilog } \rho_0^-)\infty(\text{ilog } \rho_1^-)\infty \rho_0^{-\mu} \rho_1^- \mathcal{A}(G/N_-[1])\nu_0,

so (7.7) holds.

Continuity also follows from this argument. \hfill \Box

**Appendix: Conormal functions**

Since the spaces of log-rapid decay conormal functions are not well-known we recall here, without proofs, some of the properties of conormal functions to put these in context.

We start by recalling the case of a compact manifold with boundary, $X$. If $\mathcal{V}(X)$ is the Lie algebra of all smooth vector fields – meaning smooth up to the boundary – then using the action on extendible distributions (so just in the interior) smooth functions are characterized by

(A.1) \hspace{1cm} \mathcal{C}\infty(X) = \{u \in L^\infty(X); \text{Diff}^*(X) u \subset L^\infty(X)\}.
Here \( \text{Diff}^*(X) \) is the enveloping algebra of \( \mathcal{V}(X) \), the space of linear differential operators with coefficients smooth on \( X \). The spaces of order at most \( k \) are finitely spanned over \( C^\infty(X) \) and the Fréchet topology on \( C^\infty(X) \) is given by the corresponding \( L^\infty \) norms in \( (A.1) \).

The conormal functions (with respect to \( L^\infty \), these could also very properly be called ‘symbols’) are defined by direct analogy with \( (A.1) \) but replacing \( \mathcal{V}(X) \) by its (more intrinsic) sub-algebra

\[
(A.2) \quad \mathcal{V}_b(X) = \{ V \in \mathcal{V}(X); V \text{ is tangent to } \partial X \}.
\]

The tangency condition can be restated in terms of a smooth boundary defining function \( \rho \in C^\infty(X) \), \( \{ \rho > 0 \} = X \setminus \partial X \), \( d\rho \neq 0 \) on \( \partial X \). Namely if \( V \in \mathcal{V}(X) \) then \( V \in \mathcal{V}_b(X) \) if and only if \( (V\rho)/\rho \in C^\infty(X) \). Then \( \text{Diff}^*_b(X) \subset \text{Diff}^*(X) \) is the corresponding enveloping algebra and we define the space of conormal functions by

\[
(A.3) \quad \mathcal{A}(X) = \{ u \in L^\infty(X); \text{Diff}^*_b(X) u \subset L^\infty(X) \}.
\]

This is a Fréchet space with the seminorms defined in the same manner and

\[
(A.4) \quad C^\infty(X) \subset \mathcal{A}(X)
\]

with the inclusion continuous.

We can recover this smooth subspace by considering a ‘radial vector field’. This is an element \( R \in \mathcal{V}_b(X) \), usually taken to be real, with the normalizing condition that

\[
(A.5) \quad R\rho = \rho + a\rho^2, \ a \in C^\infty(X).
\]

In local coordinates in which \( \rho = x \) then \( R = x\partial_x + xT \) where \( T \in \mathcal{V}_b(X) \) locally, and local radial vector fields can be patched to give a global radial vector field. Having chosen the radial vector field consider the ‘test operators’

\[
(A.6) \quad T(R, k) = R(R - 1) \ldots (R - k) \in \text{Diff}^*_b(X).
\]

The smooth subspace is characterized by the ‘Taylor series’ conditions

\[
(A.7) \quad u \in \mathcal{A}(X), \ T(R, k)u \in \rho^k L^\infty(X) \ \forall \ k \implies u \in C^\infty(X).
\]

As well as the ‘bounded conormal functions’ defined by \( (A.3) \) we need weighted versions of such spaces. By a weight \( 0 < \alpha \in C^\infty(X \setminus \partial X) \) (defined only on the interior of \( X \)) we mean functions with the iterative property

\[
(A.8) \quad P\alpha \in \alpha L^\infty(X) \iff (P\alpha)/\alpha \in L^\infty(X) \ \forall \ P \in \text{Diff}^*_b(X).
\]

The most obvious example is a defining function \( \rho \in C^\infty(X) \). Two weights are equivalent if they are bounded relative to each other

\[
(A.9) \quad \frac{1}{c} \alpha_1 \leq \alpha_2 \leq c\alpha_1, \ c > 0
\]

and only the behaviour near the boundary is significant. The weighted spaces discussed below only depend on the equivalence class of the weight and any weight is equivalent to one which is a function of a radial variable, reducing to the one-dimensional case. The only examples which arise here are powers \( x^t \) and \( -\log x \).

The product of two weights is also a weight. If \( \alpha \) is a weight then for any \( t \in \mathbb{R} \), \( \alpha^t \) is a weight. Significantly in the present setting if \( \inf \alpha > 0 \) then \( \log \alpha \) is also a weight.

For any weight the corresponding weighted conormal space is defined by

\[
(A.10) \quad \alpha \mathcal{A}(X) = \{ u : X \setminus \partial X \to \mathbb{C}; (Pu)/\alpha \in L^\infty(X) \ \forall \ P \in \text{Diff}^*_b(X) \}.
\]
In particular, $\alpha \in \alpha \mathcal{A}(X)$ and as the notation implicitly indicates
\[(A.11) \quad u \in \alpha \mathcal{A}(X) \iff u/\alpha \in \mathcal{A}(X)\]
as a consequence of the estimates (A.8). That is, multiplication by $\alpha$ is an isomorphism of $\mathcal{A}(X)$ onto $\alpha \mathcal{A}(X)$. For two weights
\[(A.12) \quad \alpha \leq C\beta \quad \iff \quad \alpha \mathcal{A}(X) \subset \beta \mathcal{A}(X)\]
If $\alpha$ is a bounded weight and $\beta$ is a weight is convenient to consider $\alpha^\infty \beta$ as a formal weight in the sense that
\[(A.13) \quad \alpha^\infty \beta \mathcal{A}(X) = \bigcap_k \alpha^k \beta \mathcal{A}(X)\]
These are again Fréchet spaces and if $\alpha$ vanishes at the boundary
\[(A.14) \quad \lim_{\epsilon \downarrow 0} \sup_{\rho < \epsilon} \alpha = 0 \quad \text{then} \quad C^\infty_c(\mathcal{X} \setminus \partial \mathcal{X}) \quad \text{is dense in} \quad \alpha^\infty \beta \mathcal{A}(X)\]
The Harish-Chandra space in the case of $\text{SL}(2, \mathbb{K})$ is $(\text{ilog} \rho)^\infty \rho^\kappa \mathcal{A}(\mathcal{G}[1])$ where for $\rho < 1$,
\[\text{ilog} \rho = \frac{1}{\log \frac{1}{\rho}}\]
is a boundary defining function so in particular this density statement applies.
For $\text{SL}(n, \mathbb{K})$ and even for $\text{SL}(2, \mathbb{K})$ when we consider $G[2]$ and related compactifications, we need to consider conormal functions on compact manifolds with corners. Recall that such a manifold, still denoted $\mathcal{X}$, is locally modelled on $[0, \infty)^n$ instead of $\mathbb{R}^n$ and we impose the additional requirement that boundary hypersurfaces - the closures of the components of the subsets of points at which the local model is $[0, \infty) \times \mathbb{R}^{n-1}$ - are embedded. This is equivalent to requiring that each such boundary hypersurface $H$ has a boundary defining function $\rho_H \in C^\infty(\mathcal{X})$ in the sense completely analogous to the boundary case
\[(A.15) \quad \{\rho_H > 0\} = \mathcal{X} \setminus H, \quad d\rho_H \neq 0 \quad \text{at} \quad H.\]
It follows that each of the boundary hypersurfaces has a neighbourhood in $\mathcal{X}$ diffeomorphic to $H \times [0, \epsilon)_{\rho}$. This allows all the statements above to be generalized rather directly. Namely $\mathcal{V}_h(X)$ is the Lie algebra of smooth vector fields tangent to all boundary hypersurfaces (and hence to all boundary faces). The definition of the bounded-conormal space and weights is then formally the same as (A.3), (A.8) and (A.10). There are intermediate Lie algebras between $\mathcal{V}_h(X)$ and $\mathcal{V}(X)$, in particular if $H$ is a hypersurface then
\[(A.16) \quad \mathcal{V}_H(X) = \{V \in \mathcal{V}(X); V \rho_H \in \rho_H C^\infty(X)\}\]
consists of the vector fields which are tangent to $H$. A weight at $H$ is then defined by the condition
\[(A.17) \quad 0 < \alpha \in C^\infty(\mathcal{X} \setminus H), \quad \text{Diff}^*_H \alpha \subset \alpha L^\infty(X)\]
which implies that $\alpha$ is a weight on $\mathcal{X}$ but is also smooth, so trivial as a weight, up to hypersurfaces other than $H$. Then if $\mathcal{M}_1(X)$ is the set of boundary hypersurfaces and $\alpha_H$ is a weight for each $H \in \mathcal{M}_1(X)$ then taking $\alpha_\ast$ to be the products of these weights there are corresponding conormal spaces
\[(A.18) \quad \alpha_\ast \mathcal{A}(X) = \{u \in L^\infty(X); u/\Pi_{H \in \mathcal{M}_1(X)} \alpha_H \in \mathcal{A}(X)\}.\]
The properties listed above carry over in a rather direct way and in particular formal weighted spaces, corresponding to \( \alpha, \beta \) at any combination of hypersurfaces, are defined if the weight \( \alpha \) vanish at \( H \) in the sense corresponding to (A.14).

We also use hybrid \( C^\infty \)-conormal spaces; that these make good sense is a consequence of the local product decomposition near a boundary hypersurface. We define another formal weight at each boundary hypersurfaces, \( \iota_H \). If \( \alpha \) is a collection of weights one of which is \( \iota_H \), then let \( \hat{\alpha} \) be the weights where \( \iota_H \) is replaced by \( \rho \). This allows us to define

(A.19) \[
\hat{\alpha}_*(X) = \{ u \in \hat{\alpha}(0)A(X); T(R_H, k)u \in \hat{\alpha}(k)A(X) \ \forall \ k \}.
\]

In a local product decomposition this corresponds to smoothness in the normal variable with values in the conormal space for \( H \) where the formal smooth ‘weight’ is deleted.

One can take the formal smooth weight at any collection of hypersurfaces and and in particular if one takes this weight at all boundary hypersurfaces then one recovers \( C^\infty(X) \).

The conormal spaces have interpolation properties corresponding to multiplicative properties of the weights. For instance if \( w_1(H) \) and \( w_2(H') \) are vanishing weights at different, but possibly intersecting, hypersurfaces then

(A.20) \[
w_1A(X) \cap w_2A(X) \subset w_1^{1/2}w_2^{1/2}A(X).
\]

From this point onwards we will only consider the special weights given by the defining functions \( \rho_H \) themselves, the formal smoothing weight and weights related to \( \text{ilog} \rho_H \).

As remarked above these conormal spaces are analogues of \( C^\infty(X) \) on a compact manifold without boundary. In the case of compact manifolds with corners many of the standard functorial results carry over to the smooth spaces. In particular if \( f : M \rightarrow N \) is a smooth map between compact manifolds with corners then

(A.21) \[
f^* : C^\infty(N) \rightarrow C^\infty(M).
\]

For push-forward a stronger condition is needed, that \( f \) be a submersion,

(A.22) \[
\begin{align*}
&f_* : T_pM \rightarrow T_{f(p)}N \text{ surjective } \forall \ p \in M \implies \\
&f_* : C^\infty(M; \Omega) \rightarrow C^\infty(N; \Omega)
\end{align*}
\]

where it is only natural to push forward densities.

General smooth maps are not particularly natural in the context of manifolds with corners – in general there need be little relationship to the boundary. So for instance under pull-back, (A.21), vanishing of \( u \in C^\infty(N) \) at a boundary hypersurface does not have direct implications for the vanishing of \( f^*u \) at boundary hypersurfaces.

It is more natural to work in the category of b-maps – and these are indeed the maps that are typically encountered. Here we only consider interior b-maps (meaning the image meets the interior) but drop the qualifier. A b-map is a smooth map \( f : M \rightarrow N \) with the additional property that the defining functions pull back appropriately

(A.23) \[
f^* \rho_H = a_H : \prod_{H \in \mathcal{M}_1(N)} \rho_H^{\mu(H, H')}, \ \forall \ H' \in \mathcal{M}_1(N), \ 0 < a_H \in C^\infty(M).
\]

The powers \( \mu(H, H') \) are necessarily non-negative integers but can all vanish.
For such a b-map an analogue of (A.24) holds for the conormal spaces. Namely if \( w \) is a weight on \( N \) then \( f^\# w \) is the weight on \( M \) given by \( f^* w \) with the addition of the formal smoothing weights at all hypersurfaces \( H \in \mathcal{M}_1(M) \) for which \( \mu(H, H') = 0 \) for all \( H' \), then
\[
(A.24) \quad f^* : \alpha \mathcal{A}(N) \longrightarrow (f^\# \alpha) \mathcal{A}(M).
\]
For power weights this corresponds to composition in the indices
\[
(A.25) \quad f^* : \rho^\kappa \mathcal{A}(N) \longrightarrow \rho^\kappa \mathcal{A}(M), \quad \kappa(H) = \sum_{H' \in \mathcal{M}_1(N)} \mu(H, H') \kappa'(H').
\]
In general the index \( \mu(H_i, H') \) can be non-zero for more than one \( H_i \in \mathcal{M}_1(M) \) and the same \( H' \in \mathcal{M}_1(N) \). If this does not happen, so for each \( H \in \mathcal{M}_1(M) \) there is at most one \( H' \) such that \( \mu(H, H') = 0 \), the b-map is said to be b-normal – this corresponds to the absence of boundary hypersurfaces in \( M \) which are mapped into corners of codimension two (or higher) in \( N \).

For the logarithmic weights the pull-back \( f^* \log \rho_{H'} \) is not a product of weights. However it is bounded between such products:
\[
(A.26) \quad \frac{1}{c} \prod_{H_i \in \mathcal{M}_1(M); \mu(H_i, H') \neq 0} (\log(\rho_{H_i}))^{1/p} \leq f^* \log \rho_H \leq c \prod_{H_i \in \mathcal{M}_1(M); \mu(H_i, H') \neq 0} (\log(\rho_{H_i})).
\]
Here \( p \) is the number of hypersurfaces in the preimage of \( H' \) but can be improved to the maximal number of mutually intersecting hypersurfaces in the preimage.

For push-forward it is necessary to make stronger assumptions on \( f \), but weaker than the assumption of a fibration as is needed for (A.22). Namely it suffices to take \( f \) to be a b-fibration. This corresponds to the three conditions that \( f \) be a b-map, that further it satisfies the b-normal condition, and finally that the b-differential be surjective. This latter condition can be stated infinitesimally or globally as the condition that every element \( V \in \mathcal{V}_b(N) \) is \( f \)-related to an element \( W \in \mathcal{V}_b(M) \),
\[
WF^* u = f^*(V u) \quad \forall u \in \mathcal{C}^\infty(N).
\]
For a b-fibration there is an analogue of (A.22) under an integrability assumption on the domain. Consider the ‘fixed’ hypersurfaces on \( M \), those which are not mapped by \( f \) into the boundary of \( N \). These are precisely the hypersurfaces such that \( \mu(H, *) = 0 \). Then a suitable ‘integral’ weight is
\[
(A.27) \quad I_f = \prod_{H \in \mathcal{M}_1(M); \mu(H, *) = 0} (\log \rho_H)^2
\]
where any power greater than one suffices. We also define a weight on \( N \) corresponding to the number, \( p(H') \), of boundary hypersurfaces in \( M \) mapped into \( H \). As in (A.26) this can be refined to the maximal number of mutually intersecting hypersurfaces in the preimage of \( H' \). Then
\[
(A.28) \quad J_f = \prod_{H' \in \mathcal{M}_1(N)} (\log \rho_{H'})^{1-p(H')}
\]
Then for any weight \( \alpha' \) on \( N \),
\[
(A.29) \quad f_* : (f^* \omega) I_f \mathcal{A}(M; \Omega_b) \longrightarrow w J_f \mathcal{A}(N; \Omega_b)).
\]
Thus there is in general ‘logarithmic growth’ of the push-forward. Note that the case of a fibration corresponds to $p(H') = 1$ and hence no such factors appear. In [S] the existence of expansions for push-forward of an integrable function with expansions is discussed – there may indeed be additional logarithmic terms.

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