UNIFORM MEASURES ON THE
ARBITRARY COMPACT METRIC SPACES,
with applications.

E.Ostrovsky\textsuperscript{a}, L.Sirota\textsuperscript{b}

\textsuperscript{a} Corresponding Author. Department of Mathematics and computer science, Bar-Ilan University, 84105, Ramat Gan, Israel.

E - mail: galo@list.ru; eugostrovsky@list.ru

\textsuperscript{b} Department of Mathematics and computer science. Bar-Ilan University, 84105, Ramat Gan, Israel.

E - mail: sirota3@bezeqint.net

Abstract.

We introduce and investigate in this short report the new notion of uniform measure (distribution) on the arbitrary compact metric space.

We consider also some possible applications of these measures in the theory of imbedding theorems and in the theory of random processes (fields), in particular, in the so-called majorizing (and minorizing) measures method, belonging to X.Fernique and M.Talagrand.

These considerations based on the L.Arnold and P.Imkeller generalization of the classical A.M.Garsia-E.Rodemich-H.Jr.Rumsey inequality and X.Fernique-M.Talagrand estimation for random fields.

Key words and phrases: Compact metric space, distance, ball, majorizing and minorizing measures, Prokhorov's theorem, radii, uniform measure (distribution), weak homogeneity, quasi-homogeneous random field, exponential estimates, Riesz - Lebesgue and Grand Lebesgue spaces, metric entropy, upper and lower estimates, spherical packing, weak convergence of measures, distribution of maximum of random field, tail of distribution, Arnold-Imkeller, Garsia-Rodemich-Rumsey and Fernique-Talagrand inequalities.

2000 Mathematics Subject Classification. Primary 37B30, 33K55; Secondary 34A34, 65M20, 42B25.

1 Notations. Definitions. Statement of problem.

Let \((X, d)\) be compact metric spaces relative a distance (or possible semi-distance) \(d = d(x_1, x_2), x_1, x_2 \in X\). The semi-distance \(d\) differs on the distance only in that
Let also \( \mu(\cdot) \) be Borelian probability: \( \mu(X) = 1 \) measure on \( X \); we will consider in this article only probability Borelian measures. The set of all such a measures will be denoted by \( M\mu = M\mu(X) \).

We denote as usually \( D = \text{diam}(X) = \max(d(x_1,x_2), x_1,x_2 \in X) \) the diameter \( X \) relative the distance \( d; \)

\[
B(x,r) = B_d(x,r) = \{y : y \in X, d(x,y) \leq r\}
\]

the closed \( d - \) ball with center \( x \) and radii \( r; r \in [0,D]. \)

Let us define for any measure \( \mu \in M\mu(X) \) and for introduced distance \( d = d(\cdot,\cdot) \) the following two important (for us) functions:

\[
h_-(\mu; r) = h_-(d,\mu; r) = \inf_{x \in X} \mu(B(x,r)), \quad \tag{1.0a}
\]

\[
h_+(\mu; r) = h_+(d,\mu; r) = \sup_{x \in X} \mu(B(x,r)), \quad r \in [0,D]. \quad \tag{1.0b}
\]

This functions play a very important role in the theory of random processes (fields), see e.g. [30], chapter 3, p. 95 - 99; [33]; another applications, in the theory of majorizing measures, will be discuss below.

We define by \( N(T,\epsilon) = N(T,d,\epsilon) \), \( \epsilon > 0 \) as usually for the (compact) metric subset \( (T,d) \), \( T \subset X \) the minimal number of closed balls with radii \( \epsilon : B(x_j,\epsilon) \) which cover the set \( T : \)

\[
T \subset \bigcup_{j=1}^{N(T)} B(x_j,\epsilon); \quad N(\epsilon) := N(X,d,\epsilon) \to \min.
\]  

Recall that the metric space \((X,d)\) is compact iff it is closed, bounded and \( \forall \epsilon > 0 \Rightarrow N(X,d,\epsilon) < \infty \) (Hausdorff’s theorem).

The (natural) logarithm of \( N(X,d,\epsilon) \)

\[
H(X,d,\epsilon) = \log N(X,d,\epsilon)
\]

is called entropy of the set \( X \) relative the distance \( d(\cdot,\cdot) \) and also widely used in the entropy approach to the investigation of continuity of random processes and fields and theory of limit theorems for random processes, see [10], [11], [12], [30] etc.

We denote also by \( N(x;\delta,\epsilon) = N(d,x;\delta,\epsilon) \) the \( \epsilon - \) entropy of \( \delta - \) ball with center \( x : \)

\[
N(x;\delta,\epsilon) = N(d,x;\delta,\epsilon) = N(B(x,\delta),d,\epsilon);
\]

and correspondingly

\[
N_-(\delta,\epsilon) = \inf_{x \in X} N(d,x;\delta,\epsilon), \quad N_+(\delta,\epsilon) = \sup_{x \in X} N(B(x,\delta),d,\epsilon).
\]

The last notions was introduced by Dmitrovsky [9] and was used also in the theory of random fields.

It is clear that
\[ \epsilon \geq \delta \Rightarrow N(x; \delta, \epsilon) = 1. \]

The similar concept to the entropy forms the so-called capacity \( M = M(X, d, \epsilon) \) of the compact metric space \( X \) relative to distance \( d \) at the point \( \epsilon; \ \epsilon > 0 \). Namely, \( M(X, d, \epsilon) \) is maximal number of disjoint balls \( B(y, \epsilon) \) belonging to the set \( X \); and we denote for simplicity

\[
N(\epsilon) = N(X, d, \epsilon), \quad M(\epsilon) = M(X, d, \epsilon).
\]

It is known \([23]\) that

\[
N(X, d, 2\epsilon) \leq M(X, d, \epsilon) \leq N(X, d, \epsilon).
\]

Analogously may be defined the values \( M(x; \delta, \epsilon) = M(d, x; \delta, \epsilon) \) the \( \epsilon \) – capacity of \( \delta \) – ball with center \( x \):

\[
M(x; \delta, \epsilon) = M(d, x; \delta, \epsilon) = M(B(x, \delta), d, \epsilon)
\]

and correspondingly

\[
M_-(\delta, \epsilon) = \inf_{x \in X} M(d, x; \delta, \epsilon), \quad M_+(\delta, \epsilon) = \sup_{x \in X} M(B(x, \delta), d, \epsilon).
\]

Our purpose is to estimate both the functions \( h_\pm(\mu; r) \) through the entropy \( H(X, d, \epsilon), \epsilon \in (0, D) \) and construction an optimal in natural sense measure, so-called uniform measure, on arbitrary compact metric space.

Let \( \epsilon \in (0, D) \) and let \( S(\epsilon) = \{x_k(\epsilon), \ k = 1, 2, \ldots, N(X, d, \epsilon) = N(\epsilon)\} \) be arbitrary \( d \) – optimal \( \epsilon \) – net. This net may be in general case not unique, for example in the case when \( X \) is unit circle on the plane \( R^2 \) equipped with ordinary distance the turn any optimal \( S(\epsilon) \) net on an arbitrary angle remains its optimality.

We can and will conclude by virtue of Egorov’s theorem that in the set \( X \) there is a point \( x_0 \) (the ”center” of space) for which

\[
\sup_{x \in X} d(x, x_0) \leq D/2.
\]

The value \( D/2 \) may be called radii of the set (space) \( X \) : radii \( (X) := D/2 \). Evidently, under our agreement

\[
\forall \epsilon \geq D/2 \Rightarrow H(X, d, \epsilon) = 0.
\]

Let \( F = \{\nu_\epsilon(A)\}, \ \epsilon \in (0, D) \) be a family of discrete Borelian probability measures uniformly distributed on the set \( S(\epsilon) \):

\[
\nu_\epsilon(A) = \text{card}\{x_k(\epsilon) \in S(\epsilon), \ x_k(\epsilon) \in A\}/N(\epsilon).
\]  

The family \( F \) satisfies all the conditions of famous Prokhorov’s theorem \([39]\), namely, it is weakly compact. Therefore, there exists a sequence \( \nu_{\epsilon_m}(\cdot) \), \( \epsilon_m > 0, \lim_{m \to \infty} \epsilon_m = 0 \) which weakly converges:
where \( \nu \) is also Borelian probability measure on the set \( X \).

Recall that the equality (1.3) denotes that for arbitrary continuous bounded function \( f : X \to R \)

\[
\lim_{m \to \infty} \int_X f(x) \nu_{\epsilon_m}(dx) = \int_X f(x) \nu(dx).
\]

**Definition of the uniform measure (distribution).**

The arbitrary Borelian probability measure \( \nu \) which is weak limit of some sequence \( \nu_{\epsilon_m} \) as \( \epsilon_m \to 0^+ \) is said to be uniform measure, or equally uniform distribution, defined on the Borelian sets of the space \( X \).

We really proved the existence of uniform measure (or measures) on the arbitrary metric compact. We investigate further its properties and consider some applications.

It is easily to verify that if \( X \) is compact metrizable topological group, then the uniform measure in our sense is unique and coincides with the classical normed Haar’s measure.

### 2 Upper estimates. Main result.

**Definition 2.1.** The metric space \( (X,d) \) is said to be weakly homogeneous (w.h.), if there is a constant \( C_- \); \( 0 < C_- \leq 1 \) for which

\[
\forall \epsilon : 0 < \epsilon \leq r \Rightarrow \frac{N_-(r,\epsilon)}{N(\epsilon)} \geq \frac{C_-}{N(r)}.
\]

(2.1)

For example, if \( X \) convex bounded non-trivial domain in the space \( R^l, \ l = 1,2, \ldots \)
with Euclidean distance \( |x - y| \) satisfying the cone property, and

\[
C_1 |x - y|^\alpha \leq d(x,y) \leq C_2 |x - y|^\alpha, \ \alpha = \text{const} \in (0,1],
\]

\[
C_1 , C_2 = \text{const}, 0 < C_1 \leq C_2 < \infty,
\]

where \( |x| \) is ordinary Euclidean norm (or any other equivalent to Euclidean norm), then the condition (2.1) is satisfied. Moreover,

\[
N(r,\epsilon) \asymp N_-(r,\epsilon) \asymp \left[ \frac{r}{\epsilon} \right]^{1/\alpha}.
\]

**Theorem 2.1.** Let the space \( (X,d) \) be weakly homogeneous. Then for all the uniform normed Borelian measure \( \mu \).
\[ h_-(\mu; r) \geq \frac{C}{N(r)}, \quad 0 < r < \infty, \quad C = \text{const} \in (0, \infty). \]  

**Proof.** Let \( \nu_\epsilon \) be discrete measure defined by the equality (1.2). Then

\[ h_-(\nu_\epsilon, r) = \frac{N_-(r, \epsilon)}{N(\epsilon)}, \quad \epsilon \leq r. \]

It follows from the direct definition of the weak homogeneity that

\[ h_-(\nu_\epsilon, r) \geq \frac{C}{N(r)}, \quad \epsilon \leq r. \]  

(2.3)

Let \( \mu \) be arbitrary uniform measure on the \( X \) and let \( \epsilon_m, m = 1, 2, \ldots \) be any sequence of positive numbers, \( \lim_{m \to \infty} \epsilon_m = 0 \) such that

\[ \mu = (w) \lim_{m \to \infty} \nu_{\epsilon_m}. \]

We deduce passing to the limit as \( \epsilon_m \to 0+ \) in the inequality (2.3)

\[ h_-(\mu, r) \geq \frac{C}{N(r)}, \quad r > 0; \]  

(2.4)

the case \( r \geq D/2 \) is trivial.

This completes the proof of theorem 2.1.

### 3 Lower estimates.

**Proposition 3.1.** Let \( \nu(\cdot) \) be additive non-negative probability function defined on the Borelian sets of the space \( X \). Then

\[ \frac{1}{h_+(\nu; 2\epsilon)} \leq M(\epsilon) \leq \frac{1}{h_-(\nu; \epsilon)}. \]  

(3.1)

**Proof** is alike to one in the coding theory, in the proof of spherical packing boundaries. Indeed, let the set \( \{x_i\}, \; i = 1, 2, \ldots, M(\epsilon) \) be such that

\[ X \supset \bigcup_{i=1}^{M(\epsilon)} B(x_i, \epsilon), \; B(x_i, \epsilon) \cap B(x_j, \epsilon) = \emptyset, \; i \neq j; \]

then

\[ 1 = \nu(X) \geq \sum_{i=1}^{M(\epsilon)} \nu(B(x_i, \epsilon)) \geq M(\epsilon) h_-(\nu; \epsilon). \]

On the other hands, as long as the set \( \{x_i\}, \; i = 1, 2, \ldots, M(\epsilon) \) is one of the best,

\[ X \subset \bigcap_{i=1}^{M(\epsilon)} B(x_i, 2\epsilon), \]

following
4 Majorizing and minorizing measures.

Some applications in the theory of imbedding theorems.

Let $(Y, \rho)$ be another separable metric spaces, $m$ be arbitrary distribution on the Borelian sets $X$, $f : X \to Y$ be (measurable) function. Let also $\Phi(z), z \in \mathbb{R}_+ = [0, \infty)$ be continuous Young-Orlicz function, i.e. strictly increasing function such that

$$\Phi(z) = 0 \iff z = 0; \lim_{z \to \infty} \Phi(z) = \infty.$$  

We denote as usually

$$\Phi^{-1}(w) = \sup\{z, z \geq 0, \Phi(z) \leq w\}, w \geq 0$$

the inverse function to the function $\Phi$.

Let us introduce the Orlicz space $L(\Phi) = L(\Phi; m \times m, X \otimes X)$ on the set $X \otimes X$ equipped with the Young - Orlicz function $\Phi$.

We assume henceforth that for all the values $x_1, x_2 \in X, x_1 \neq x_2$ (the case $x_1 = x_2$ is trivial) the value $\rho(f(x_1), f(x_2))$ belongs to the space $L(\Phi)$.

Note that for the existence of such a function $\Phi(\cdot)$ is necessary and sufficient only the integrability of the distance $\rho(f(x_1), f(x_2))$ over the product measure $m \times m$:

$$\int_X \int_X \rho(f(x_1), f(x_2)) m(dx_1) m(dx_2) < \infty,$$

see [25], chapter 2, section 8; where is described a method for building of this function $\Phi(\cdot)$.

Another natural way, which may gives an optimal up to linear scaling the Orlicz function $\Phi = \Phi(z)$. Introduce the following functions:

$$\psi(p) := \left[ \int_X \int_X \rho^p(f(x_1), f(x_2)) m(dx_1) m(dx_2) \right]^{1/p}, p \geq 1;$$

$$\phi(p) := \left[ \frac{p}{\psi(p)} \right]^{-1},$$

$$\phi^*(y) := \sup_{p \geq 1} \{p|y| - \phi(p)\}$$

the Young - Fenchel, or Legendre transform for the function $\phi(\cdot)$.

The function $\Phi = \Phi(z)$ may be constructively defined under some natural conditions as follows:
\[
\Phi(z) := e^{\phi(z)} - 1,
\]
see [24], [30], chapter 1,2.

Under this assumption the distance \( d = d(x_1, x_2) \) may be constructively defined by the formula:

\[
d_\Phi(x_1, x_2) := ||\rho(f(x_1), f(x_2))||L(\Phi). \tag{4.1}
\]

Since the function \( \Phi = \Phi(z) \) is presumed to be continuous and strictly increasing, \( \lim_{z \to \infty} \Phi(z) = \infty \), it follows from the relation (4.1) that \( V(d_\Phi) \leq 1 \), where by definition for arbitrary distance function \( d = d(\cdot, \cdot) \)

\[
V = V(d) := \int_X \int_X \Phi \left[ \frac{\rho(f(x_1), f(x_2))}{d(x_1, x_2)} \right] m(dx_1) m(dx_2). \tag{4.2}
\]

Let us define also the following important distance function on the set \( X \):

\[
w(x_1, x_2) = w(x_1, x_2; V) = w(x_1, x_2; V, m, \Phi) = w(x_1, x_2; V, m, \Phi, d) \overset{\text{def}}{=} 6 \int_0^{d(x_1, x_2)} \left\{ \Phi^{-1} \left[ \frac{4V}{m^2(B(r, x_1))} \right] + \Phi^{-1} \left[ \frac{4V}{m^2(B(r, x_2))} \right] \right\} dr, \tag{4.3}
\]

where \( m(\cdot) \) is probabilistic Borelian measure on the set \( X \) and \( V = V(d) \).

The triangle inequality and other properties of the distance function \( w = w(x_1, x_2) \) are proved in [26].

**Definition 4.1.** (See [26]). The measure \( m \) is said to be minorizing measure relative the distance \( d = d(x_1, x_2) \), if for each values \( x_1, x_2 \in X \) \( V(d) < \infty \) and moreover \( w(x_1, x_2; V(d)) < \infty \).

We will denote the set of all minorizing measures on the metric set \( (X, d) \) by \( \mathcal{M} = \mathcal{M}(X) \).

Evidently, if the function \( w(x_1, x_2) \) is bounded, then the minorizing measure \( m \) is majorizing. Inverse proposition is not true, see [26], [1].

**Remark 4.1.** If the measure \( m \) is minorizing, then

\[
w(x_n, x; V(d)) \to 0 \iff d(x_n, x) \to 0, \quad n \to \infty.
\]

Therefore, the continuity of a function relative the distance \( d \) is equivalent to the continuity of this function relative the distance \( w \).

**Remark 4.2.** If

\[
\sup_{x_1, x_2 \in X} w(x_1, x_2; V(d)) < \infty,
\]

then the measure \( m \) is called majorizing measure. This classical definition with theory explanation and applications basically in the investigation of local structure...
The following important inequality belongs to L.Arnold and P.Imkeller [1], [20]; see also [22], [2].

**Theorem of L.Arnold and P.Imkeller.** Let the measure $m$ be minorizing. Then there exists a modification of the function $f$ on the set of zero measure, which we denote also by $f$, for which

$$\rho(f(x_1), f(x_2)) \leq w(x_1, x_2; V, m, \Phi, d). \quad (4.4)$$

As a consequence: this function $f$ is $d-$ continuous and moreover $w-$ Lipshitz continuous with unit constant.

The inequality (4.4) of L.Arnold and P.Imkeller is significant generalization of celebrated Garsia - Rodemich - Rumsey inequality, see [16], with at the same applications as mentioned before [19], [31], [32], [33], [40].

The inequality (4.4) may be also interpreted as an imbedding theorems of the Sobolev’s fractional order space into the space of continuous functions.

**Open question:** how to define the majorizing (minorizing) measure (measures)?

Some attempts have been made in the works [5], [10], [13], [27], [42], [44], [46], [47] etc.

We intend to offer as the capacity of majorizing measure the (arbitrary) uniform distribution on the compact metric space introduced before.

By our opinion, this approach is somewhat easier and more convenient than the above.

**Remark 4.3.** The inequality of L.Arnold and P.Imkeller (4.4) is closely related with the theory of fractional order Sobolev’s - rearrangement invariant spaces, see [2], [16], [19], [29], [32], [40].

**Remark 4.4.** In the previous articles [26], [6] was imposed on the function $\Phi(\cdot)$ the following $\Delta^2$ condition:

$$\Phi(x)\Phi(y) \leq \Phi(K(x + y)), \exists K = \text{const} \in (1, \infty), \ x, y \geq 0$$

or equally

$$\sup_{x,y>0} \left[ \frac{\Phi^{-1}(xy)}{\Phi^{-1}(x) + \Phi^{-1}(y)} \right] < \infty. \quad (4.5)$$

We do not suppose this condition. For instance, we can consider the function of a view $\Phi(z) = |z|^p$, which does not satisfy (4.5).

**Proposition 4.1.** We deduce using the direct definition of the variable $h_-(m,r) : w(x_1, x_2; V) \leq w(x_1, x_2; V)$, where $w = \overline{w}(x_1, x_2, V, d, m) =$
\[ w(x_1, x_2, V) \overset{\text{def}}{=} 12 \int_0^{d(x_1, x_2)} \Phi^{-1} \left[ \frac{4V}{h^2(m, r)} \right] \, dr. \quad (4.6) \]

**Proposition 4.2.** We conclude using as the capacity of the measure \( m \) described below measure \( \nu_\epsilon \):

\[
6^{-1} w(x_1, x_2, d) \leq \int_\epsilon^{d(x_1, x_2)} \Phi^{-1} \left[ \frac{4V}{N^2(\epsilon)} N^2(x_2, r, \epsilon) \right] \, dr + \int_\epsilon^{d(x_1, x_2)} \Phi^{-1} \left[ \frac{4V}{N^2(\epsilon)} N^2(x_1, r, \epsilon) \right] \, dr, \quad \epsilon \in (0, D). \quad (4.7)
\]

As a consequence: \( 6^{-1} w(x_1, x_2, d) \leq \)

\[
\inf_{\epsilon \in (0, D)} \left\{ \int_\epsilon^{d(x_1, x_2)} \left[ \Phi^{-1} \left( \frac{4V}{N^2(\epsilon)} N^2(x_2, r, \epsilon) \right) + \Phi^{-1} \left( \frac{4V}{N^2(\epsilon)} N^2(x_1, r, \epsilon) \right) \right] \, dr \right\}.
\]

or \( 6^{-1} w(x_1, x_2, d) \leq \)

\[
\lim_{\epsilon \to 0^+} \left\{ \int_\epsilon^{d(x_1, x_2)} \left[ \Phi^{-1} \left( \frac{4V}{N^2(\epsilon)} N^2(x_2, r, \epsilon) \right) + \Phi^{-1} \left( \frac{4V}{N^2(\epsilon)} N^2(x_1, r, \epsilon) \right) \right] \, dr \right\}.
\]

**Proposition 4.3.** Suppose in addition that the (compact) metric space \((X, d)\) is weak homogeneous. If we choose as a capacity of the measure \( m \) arbitrary uniform distribution on the set \( X \), then

\[
w(x_1, x_2, V) \leq 12 \int_0^{d(x_1, x_2)} \Phi^{-1} \left[ \frac{4VN^2(r)}{C^2} \right] \, dr. \quad (4.9)
\]

\section{Application to the problem of boundedness and continuity of random fields.}

General Orlicz and majorizing measures method approach.

Let \( \xi = \xi(x), \ x \in X \) be separable numerical stochastic continuous (i.e. continuous in probability relative the distance \( d \)) random field (r.f.), defined aside the probability space on the introduced before set \( X \), not necessary to be Gaussian.

The correspondent set of all elementary events, probability and expectation we will denote as ordinary correspondingly by \( \Omega, P, E \), and the probabilistic Lebesgue-Riesz \( L_p \) norm of a random variable (r.v) \( \eta \) we will denote as follows:

\[
|\eta|_p \overset{\text{def}}{=} \left[ E|\eta|^p \right]^{1/p}.
\]
We find in this section some sufficient condition for boundedness of $\xi(x)$ (a.e.).

Recall that the first publication about fractional Sobolev’s inequalities [16] was devoted in particular to the such a problem; see also articles [19], [32], [40].

Let $\Phi = \Phi(u)$ be again the Young-Orlicz function. We impose together with M.Ledoux and M.Talagrand, [27], chapter 11 the following conditions on the function $\Phi(\cdot)$:

$$\Phi^{-1}(xy) \leq C \left( \Phi^{-1}(x) + \Phi^{-1}(y) \right), \quad \exists C = \text{const} < \infty$$

$$\int_0^1 \Phi^{-1}(x^{-1}) \, dx < \infty.$$  (5.0)

We will denote the Orlicz norm by means of the function $\Phi$ of a r.v. $\kappa$ defined on our probabilistic space $(\Omega, P)$ as $|||\kappa|||_{L(\Omega, \Phi)}$ or for simplicity $|||\kappa|||_{\Phi}$.

We introduce the so-called natural (new) distance $d_{\Phi}(x_1, x_2)$ on the set $X$ as follows:

$$d_{\Phi} = d_{\Phi}(x_1, x_2) := |||\rho(\xi(x_1), \xi(x_2))|||_{L(\Omega, \Phi)}, \quad x_1, x_2 \in X.$$  (5.1)

**Definition 5.1., see [27], chapter 11.**

The Borelian probabilistic measure $m(\cdot)$ on the Borelian sets $X$ equipped by the distance $d_{\Phi} = d_{\Phi}(\cdot, \cdot)$ is called majorizing, iff

$$\gamma_m(X, d, \Phi) := \sup_{x \in X} \int_D \Phi^{-1} \left( \frac{1}{m(B_{d_{\Phi}}(x, r))} \right) \, dr < \infty.$$  (5.2)

**Theorem 5.1., see [27], section 11.2.** Let $m(\cdot)$ be the probabilistic majorizing measure on the set $X$ relative the distance $d_{\Phi}(\cdot, \cdot)$. Then

$$\sup_{y \in X} \left\{ E \sup_{x \in X} [\xi(x) - \xi(y)] \right\} \leq K(\Phi) \gamma_m(X, d_{\Phi}, \Phi).$$  (5.3)

As a consequence: the random field $\xi = \xi(x), \quad x \in X$ is bounded with probability one.

**Remark 5.1.** M.Ledoux and M.Talagrand proved in addition that if

$$\lim_{\delta \to 0^+} \sup_{x \in X} \int_0^\delta \Phi^{-1} \left( \frac{1}{m(B_{d_{\Phi}}(x, r))} \right) \, dr = 0,$$

then the r.f. $\xi = \xi(x)$ is $d_{\Phi}$ - continuous with probability one.

M.Ledoux and M.Talagrand used in particular the following majorizing measure:

$$m_N := \sum_{l>l_0} 2^{-l+l_0} N \left( X, d_{\Phi}, 2^{-l} \right) \sum_{x \in S(2^{-l})} \delta_x,$$

where $\delta_x$ denotes an usual unit Dirac measure concentrated at the point $x$.

This gives the following entropic estimate:
\[
\sup_{y \in X} \left\{ \mathbb{E} \sup_{x \in X} [\xi(x) - \xi(y)] \right\} \leq K_2(\Phi) \int_0^D \Phi^{-1} \left[ N(X, d_\Phi, r) \right] dr. \tag{5.4}
\]

**Definition 5.2.** The r.f. \( \xi = \xi(x) \) defined on the compact metric set \( X = (X, d) \) is said to be quasi-homogeneous relative some probability Borelian measure \( m \), if

\[
\sup_{r > 0} \left[ h_+(m, r) / h_-(m, r) \right] < \infty. \tag{5.5}
\]

R.M. Dudley in [10], p. 59-62 proved that if the r.f. \( \xi = \xi(x) \) is quasi-homogeneous relative some probability Borelian measure \( m \), then

\[
\int_0^D \Phi^{-1} \left[ N(X, d_\Phi, r) \right] dr < \infty \iff \gamma_m(X, d_\Phi, \Phi) < \infty.
\]

M. Talagrand in [43] proved that if the r.f. \( \xi = \xi(x) \) is Gaussian and centered, then the finiteness of the value \( \gamma_m(X, d_\Phi, \Phi) \) for some Borelian probability measure \( m \) on the set \( X \) is necessary and sufficient condition for it boundedness.

Here evidently

\[
\Phi(z) = \Phi_2(z) \overset{\text{def}}{=} \exp(z^2/2) - 1.
\]

Therefore, if in addition the Gaussian r.f. \( \xi = \xi(x) \) is quasi-homogeneous, the condition

\[
\int_0^D \Phi^{-1} \left[ N(X, d_{\Phi_2}, r) \right] dr < \infty \tag{5.6}
\]

is necessary and sufficient for a.e. boundedness of \( \xi(z) \) and moreover for it continuity relative the natural Dudley’s distance

\[
d_D(x_1, x_2) = [\text{Var}(\xi(x_1) - \xi(x_2))]^{1/2},
\]

which is linear equivalent of course in the considered here Gaussian case to the distance \( d_{\Phi_2} \).

We intent to improve the estimate (5.4) by means of using uniform distribution on the compact space \( (X, d_\Phi) \), as well as in the fourth section.

**Proposition 5.1.** We deduce using again the direct definition of the variable \( h_-(d, m, r) \):

\[
\gamma_m(X, d_\Phi, \Phi) \leq \int_0^D \Phi^{-1} \left( \frac{1}{h_-(d_\Phi, m, r)} \right) dr. \tag{5.7}
\]

**Proposition 5.2.** We conclude using as the capacity of the measure \( m \) described below measure \( \nu_\epsilon \):

\[
\gamma_m(X, d_\Phi, \Phi) \leq \int_\epsilon^D \Phi^{-1} \left( \frac{N(d_\Phi, r, \epsilon)}{N(d_\Phi, \epsilon)} \right) dr, \ \epsilon \in (0, D). \tag{5.8}
\]

As a consequence:
\begin{align*}
\gamma_m(X, d, \Phi) & \leq \inf_{\epsilon \in (0, D)} \int_0^D \Phi^{-1} \left( \frac{N(d, r, \epsilon)}{N(d, \epsilon)} \right) dr, \ \epsilon \in (0, D); \quad (5.8a) \\
\gamma_m(X, d, \Phi) & \leq \lim_{\epsilon \to 0^+} \int_0^D \Phi^{-1} \left( \frac{N(d, r, \epsilon)}{N(d, \epsilon)} \right) dr, \ \epsilon \in (0, D). \quad (5.8b)
\end{align*}

**Proposition 5.3.** Suppose in addition that the (compact) metric space \((X, d)\) is weak homogeneous. If we choose as a capacity of the measure \(m\) arbitrary uniform distribution on the set \(X\), then

\[ \gamma_m(X, d, \Phi) \leq \int_0^D \Phi^{-1} \left( \frac{N(d, r, \epsilon)}{C_-} \right) dr, \ \epsilon \in (0, D). \quad (5.9) \]

Let us show the other approach, which may give us the exponential estimates for distribution of \(\sup_{x \in X} |\xi(x)|\), see e.g. [33], [36].

Let \(\Phi = \Phi(u)\) be again the Young-Orlicz function, generated as before by means of a family of random variables \(\rho(\xi(x_1), \xi(x_2)), \ x_1, x_2 \in X\). We will denote the Orlicz norm by means of the function \(\Phi\) of a r.v. \(\kappa\) defined on our probabilistic space \((\Omega, P)\) as \(\|\|\kappa\|\|L(\Omega, \Phi)\) or for simplicity \(\|\|\kappa\|\|\Phi\).

We can and will suppose without loss of generality

\[ \sup_{x_1, x_2 \in X} \|\|\rho(\xi(x_1), \xi(x_2))\|\|\Phi = 1. \]

We introduce also a new non-random so-called natural bounded: \(v_\Phi(x_1, x_2) \leq 1\) distance on the set \(X\), \(v = v_\Phi(x_1, x_2)\), i.e. generated only by the random field \(\{\xi(x)\}, \ x \in X\) as follows:

\[ v := v_\Phi = v_\Phi(x_1, x_2) := \|\|\rho(\xi(x_1), \xi(x_2))\|\|L(\Omega, \Phi), \ x_1, x_2 \in X. \quad (5.10) \]

**Theorem 5.1.** Let \(m(\cdot)\) be the probabilistic minorizing measure on the set \(X\) relative the distance \(d_\Phi(\cdot, \cdot)\). There exists a non-negative random variable \(Z = Z(v_\Phi, m)\) with unit expectation: \(EZ = 1\) for which

\[ \rho(\xi(x_1), \xi(x_2)) \leq \varpi(x_1, x_2; Z(v_\Phi, m)). \quad (5.11) \]

As a consequence: the r.f. \(\xi = \xi(x)\) is \(d\) - continuous with probability one.

**Proof.** We pick

\[ Z = \int_X \int_X \Phi \left( \frac{\rho(\xi(x_1), \xi(x_2))}{v_\Phi(x_1, x_2)} \right) m(dx_1) m(dx_2). \]

We have by means of theorem Fatou - Tonelli

\[ EZ = E \int_X \int_X \Phi \left( \frac{\rho(\xi(x_1), \xi(x_2))}{v_\Phi(x_1, x_2)} \right) m(dx_1) m(dx_2) = \]

12
\[ \int_X \int_X \mathbf{E} \Phi \left( \frac{\rho(\xi(x_1), \xi(x_2))}{v_\Phi(x_1, x_2)} \right) m(dx_1) m(dx_2) = 1, \tag{5.12} \]

since \( \int_X \int_X m(dx_1) m(dx_2) = 1 \).

It remains to use our proposition 4.1 and apply the L.Arnold and P.Imkeller inequality.

\section{Non-asymptotical exact exponential estimates for distribution of maximum of random fields.}

\subsection*{Grand Lebesgue spaces approach.}

Let \( \xi = \xi(x) \), \( x \in X \) be again separable random field (or process) with values in the real axis \( R \), \( T = \{ x \} \) be arbitrary Borelian subset of \( X \).

Denote

\[ \overline{\xi}_T = \sup_{x \in T} \xi(x), \quad \overline{\xi} = \overline{\xi}_X = \sup_{x \in X} \xi(x), \tag{6.0} \]

\[ Q(u) = Q_\xi(u) = Q(X, u) \overset{def}{=} \mathbf{P}(\sup_{x \in X} \xi(x) > u), \quad u \geq 2; \tag{6.1} \]

\[ Q_+(u) = Q_{\xi,+}(u) = Q_+(X, u) \overset{def}{=} \mathbf{P}(\sup_{x \in X} |\xi(x)| > u), \quad u \geq 2; \tag{6.1a} \]

and for arbitrary Borelian subset \( T \subset X \) we denote

\[ Q(T, u) = \mathbf{P}(\sup_{t \in T} \xi(t) > u), \quad u \geq 2. \tag{6.2} \]

\[ Q_+(T, u) = \mathbf{P}(\sup_{t \in T} |\xi(t)| > u), \quad u \geq 2. \tag{6.2a} \]

Of course, here

\[ Y = R, \quad \rho(y_1, y_2) = |y_1 - y_2|. \]

Our purpose in this section is obtaining an \textit{exponentially exact} as \( u \to \infty \), \( u > u_0 = \text{const} > 0 \) estimation for the probability \( Q(u) \overset{def}{=} Q(X, u) \), in the modern terms of “minorizing measures” and the so-called \( B(\phi) \) spaces, which are a particular case of Grand Lebesgue spaces.

We intent to improve and simplify foregoing results using method of majorizing measures.

In the entropy terms this problem is considered in [10], [12], [13], [30], chapter 3, [34]: in the more modern terms of majorizing measures- in [27], [33], [41] etc. Note that the method of majorizing measures, or equally generic chaining method, gives more strong results as entropy technique in the theory of continuity of random processes and fields, see [11] - [13], [41] - [47], [33] etc.
The estimations of $Q(u) \overset{\text{def}}{=} Q(X,u)$ are used in the Monte-Carlo method, statistics, numerical methods etc., see [14], [17], [30], [36], [37].

A. Preliminaries.

We suppose in this section that the random field $\xi(x)$ to be centered and satisfies the uniform Kramer’s condition, so that the natural function, equally the uniform (exponential) moment generating function

$$\phi(\lambda) = \log \sup_{x \in X} \mathbb{E} \exp(\lambda \xi(x)) \quad (6.3)$$

is finite in some non-trivial interval $\lambda \in (-\lambda_0, \lambda_0)$, $\lambda_0 = \text{const} \in (0, \infty]$.

Then we may introduce the following Young-Orlicz function (up to multiplicative positive constant)

$$\Phi_\phi(u) = \exp(\phi^*(u)) - 1,$$

so that $\sup_{x \in X} ||\xi(x)||B(\phi) = 1$ and following $\sup_{x} ||\xi(x)||(\Phi) < \infty$.

We recall first of all some propositions from the theory of the so-called $B(\phi)$ spaces; more detail descriptions see, e.g. in [24], [30], [33].

Let $\phi = \phi(\lambda), \lambda \in (-\lambda_0, \lambda_0)$, $\lambda_0 = \text{const} \in (0, \infty]$ be some even strong convex which takes positive values for positive arguments twice continuous differentiable function, such that

$$\phi(0) = 0, \; \phi^\prime(0) \in (0, \infty), \; \lim_{\lambda \to \lambda_0} \phi(\lambda)/\lambda = \infty.$$

We say that the centered random variable (r.v) $\xi = \xi(\omega)$ belongs to the space $B(\phi)$, if there exists some non-negative constant $\tau \geq 0$ such that

$$\forall \lambda \in (-\lambda_0, \lambda_0) \Rightarrow \mathbb{E} \exp(\lambda \xi) \leq \exp[\phi(\lambda \tau)]. \quad (6.4)$$

The minimal value $\tau$ satisfying (6.4) is called a $B(\phi)$ norm of the variable $\xi$, write

$$||\xi||B(\phi) = \inf\{\tau, \; \tau > 0 : \forall \lambda \Rightarrow \mathbb{E} \exp(\lambda \xi) \leq \exp(\phi(\lambda \tau))\}.$$ 

This spaces are very convenient for the investigation of the r.v. having a exponential decreasing tail of distribution, for instance, for investigation of the limit theorem, the exponential bounds of distribution for sums of random variables, non-asymptotical properties, problem of continuous of random fields, study of Central Limit Theorem in the Banach space etc.

The space $B(\phi)$ with respect to the norm $|| \cdot ||B(\phi)$ and ordinary operations is a Banach space which is isomorphic to the subspace consisted on all the centered variables of Orlicz’s space $(\Omega, F, \mathbb{P}), N(\cdot)$ with $N$ – function

$$N(u) = \exp(\phi^*(u)) - 1, \; \phi^*(u) = \sup_{\lambda} (\lambda u - \phi(\lambda)).$$

The transform $\phi \to \phi^*$ is called Young-Fenchel transform. The proof of considered assertion used the properties of saddle-point method and theorem of Fenchel-Moraux: $\phi^{**} = \phi$. 

14
The next facts about the $B(\phi)$ spaces are proved in [24], [30], p. 19-40:

1. $\xi \in B(\phi) \iff E\xi = 0$, and $\exists C = \text{const} > 0,$

$$U(\xi, x) \leq \exp(-\phi^*(Cx)), x \geq 0,$$

where $U(\xi, x)$ denotes in this article the tail of distribution of the r.v. $\xi$:

$$U(\xi, x) = \max \left( P(\xi > x), P(\xi < -x) \right), x \geq 0,$$

and this estimation is in general case asymptotically exact.

2. The $B(\phi)$ norm $||\eta||_{B\phi}$ of a random variable $\eta$ is linear equivalent to the following norm

$$||\eta||_{G\psi} \overset{def}{=} \sup_{p \geq 1} \left[ \frac{|\eta|}{\psi(p)} \right]$$

in the Grand Lebesgue space $G\psi$:

$$C_1||\eta||_{G\psi} \leq ||\eta||_{B\phi} \leq C_2||\eta||_{G\psi},$$

where

$$\psi(p) = \frac{p}{\phi^{-1}(p)}.$$

3. For the non-centered random variable $\kappa$ with exponentially decreasing tail of distribution the $B(\phi)$ norm may be defined, for instance, as follows:

$$||\tau||_{B(\phi)} := \left[ ||\tau - E\tau||^2 + (E\tau)^2 \right]^{1/2}.$$

B. Auxiliary constructions.

Let the function $\overline{w} = \overline{w}(x_1, x_2, Z)$ was defined by equality (4.6). Let $m(\cdot)$ be the probabilistic minorizing measure on the set $X$ relative the distance $d_{\phi}(\cdot, \cdot)$, for example, uniform measure. We know, see theorem (5.1) that there exists the non-negative random variable $Z$ with unit expectation: $EZ = 1$ for which

$$\rho(\xi(x_1), \xi(x_2)) \leq \overline{w}(x_1, x_2; Z).$$

Define the random variable (or equally function of $Z$) $\theta = \theta(Z)$ as follows:

$$\theta = \gamma(Z) := \sup_{x_1, x_2 \in X} \overline{w}(x_1, x_2; Z),$$

and introduce the following semi-distance function

$$\zeta(x_1, x_2) := \sup_{Y > 0} \left[ \frac{\overline{w}(x_1, x_2; Y)}{\gamma(Y)} \right],$$

so that
\[ w(x_1, x_2; Y) \leq \zeta(x_1, x_2) \cdot \gamma(Y) \]

or equally

\[ w(x_1, x_2; Z) \leq \zeta(x_1, x_2) \cdot \theta. \]  \hspace{1cm} (6.5)

**C. Lemma 6.1.** Assume in addition that the condition (6.3) is satisfied. Then we have in the representation (6.5)

\[ ||\theta||B(\phi) = K < \infty. \]  \hspace{1cm} (6.6)

**Proof** follows immediately from the main result of paper [38], (see also the article [33],) where using for us result it is formulated in the similar language of \( G\psi \) spaces.

**D. Some estimates.**

Let \( B_\zeta(x_0, \delta) \) be a \( \zeta \) – ball on the set \( X \) with the center at the point \( x_0 \) and radii \( \delta >: 0 \):

\[ B_\zeta(x_0, \delta) = \{ x, \ x \in X, \ \zeta(x, x_0) \leq \delta \}. \]

The diameter of the set \( X \) relative the distance \( \zeta \) will be denoted by \( D_\zeta \); so that we can restrict the values \( \delta \) inside the interval \( \delta \in (0, D_\zeta) \).

We derive using triangle inequality for the values \( x \) inside the ball \( B_\zeta(x_0, \delta) \):

\[ ||\xi(x)||B(\phi) \leq ||\xi(x_0)||B(\phi) + K \cdot \delta \leq 1 + K \cdot \delta, \]

therefore

\[ \sup_{x_0 \in X} Q_{B_\zeta(x_0, \delta)}(u) \leq \exp \left( -\phi^* \left( \frac{u}{1 + K \cdot \delta} \right) \right). \] \hspace{1cm} (6.7)

**E. Concluding exponential estimation.**

Denote by \( S_\zeta(\epsilon) \) the minimal \( \epsilon \) – net for the set \( X \) relative the distance \( d_\zeta \):

\[ S_\zeta(\epsilon) = \{ x_1(\epsilon), x_2(\epsilon), \ldots, x_N(\epsilon) \}; \]

where by our notations (temporarily, in this section)

\[ \text{card} (S_\zeta(\epsilon)) = N(X, \zeta, \epsilon) \overset{\text{def}}{=} N(\epsilon). \]

We have using (6.7):

\[ Q(X, u) = \mathbb{P} \left( \sup_{x \in X} \xi(x) > u \right) = \mathbb{P} \left( \cup_{j=1}^{N(\delta)} \{ \sup_{x \in B(x_j(\delta))} \xi(x) > u \} \right) \leq \sum_{j=1}^{N(\delta)} \mathbb{P} \left( \sup_{x \in B(x_j(\delta))} \xi(x) > u \right) = \sum_{j=1}^{N(\delta)} Q_{B_\zeta(x_j(\delta), \delta)}(u) \leq \]
\[ N(X, \zeta, \delta) \cdot \exp \left( -\phi^* \left( \frac{u}{1 + K \delta} \right) \right). \quad (6.8) \]

**E. Total:**

\[ Q(X, u) \leq \inf_{\delta \in (0, D\zeta)} \left\{ N(X, \zeta, \delta) \cdot \exp \left( -\phi^* \left( \frac{u}{1 + K \delta} \right) \right) \right\}, \quad (6.9) \]

and correspondingly

\[ Q_+(X, u) \leq 2 \inf_{\delta \in (0, D\zeta)} \left\{ N(X, \zeta, \delta) \cdot \exp \left( -\phi^* \left( \frac{u}{1 + K \delta} \right) \right) \right\}. \quad (6.9a) \]

**Examples.**

**Example 6.1.** Suppose in addition to the conditions of proposition 4.2 analogously to the work [33] that the function \( u \to \Phi(u), \ u > 0 \) is logarithmical convex: \[ (\log \Phi)'\prime\prime(u) > 0. \]

Let also \( \gamma = \text{const} \in (0, 1), \ C_1 = \text{const} \in (0, \infty). \) Denote

\[ \delta_0 = \delta_0(u; \gamma, \Phi) = \frac{C_1 \gamma}{u \cdot [\log \Phi]'(u)}. \quad (6.10) \]

We obtain choosing \( \delta = \delta_0 \) substituting into (6.9) that for all sufficiently greatest values \( u : \delta_0(u; \gamma, \Phi) < D\zeta \)

\[ Q(X, u) \leq \frac{(1 - \gamma)^{-1}}{\Phi(u)} \cdot N \left( \frac{C_2 \gamma}{u \cdot [\log \Phi]'(u)} \right). \quad (6.11) \]

**Example 6.2.** Let now \( \Phi(u) = \exp(u^2/2) - 1 \) (subgaussian case). Suppose

\[ N(\epsilon) \leq C_3 \epsilon^{-\kappa}, \; \epsilon \in (0, D), \; \kappa = \text{const} > 0. \quad (6.12) \]

The value \( \kappa \) is said to be majorital dimension of the set \( X \) relative the distance \( w. \)

The optimal value \( \gamma \) in (4.20) if equal to \( \gamma = \gamma_0 := \kappa/(\kappa + 1) \) and we conclude for the values \( u \) such that

\[ \delta_0 = \frac{C_2 \kappa}{(\kappa + 1) u^2} \leq D\zeta : \]

\[ Q(X, u) \leq C_3 \ C_2^{-\kappa} \ k^{-\kappa} \ (\kappa + 1)^{\kappa+1} \ u^{2\kappa} \ e^{-u^2/2}. \quad (6.13) \]

**Example 6.3.** Assume that in the example 6.2 instead the condition (6.12) the following condition holds:

\[ N(\epsilon) \asymp C_4 e^{C_5} \epsilon^{-\beta}, \; \epsilon \in (0, D); \; \beta = \text{const} > 0. \quad (6.14) \]
Then
\[ Q(X, u) \leq e^{-0.5u^2 + C_6 u^{2\beta/(\beta + 1)}}, \quad u \geq C_7. \quad (6.15) \]

Note that in the case \( \beta \geq 2 \) the so-called entropy series
\[ \sum_{n=1}^{\infty} 2^{-n} H^{1/2}(X, w, 2^{-n}) \]
diverges.

7 Concluding remarks.

A. Degrees.
Let \( X = [0, 1]^n, \quad n = 2, 3, \ldots \). In the articles [40], [19] is obtained a multivariate generalization of famous Garsia-Rodemich-Rumsey inequality [16]. Roughly speaking, instead degree ”2” in the inequalities (4.2) and (4.3) stands degree 1 and coefficients dependent on \( d \).

The ultimate value of this degree in general case of arbitrary metric space \((X, d)\) is now unknown; see also [1], [20].

B. Spaces.
Notice that in all considered cases and under our conditions when
\[ \sup_x ||\xi(x)||B(\phi) < \infty, \quad ||\sup_x \xi(x)||B(\phi) < \infty. \]
But in the article [35] was constructed a ”counterexample”: there exists a continuous a.e random process for which
\[ \sup_x ||\xi(x)||B(\phi) < \infty, \quad ||\sup_x \xi(x)||B(\phi) = \infty. \]
This circumstance imply that our conditions are only sufficient but not necessary.

C. Lower bounds.
The lower estimates for probabilities \( Q(u) \) are obtained e.g. in [30], chapter 3, sections 3.5-3.8. They are obtained in entropy terms, all the more so in the terms of minorizing measures.

Note that the lower bounds in [30] may coincide up to multiplicative constants with upper bounds.

D. Other possible applications.
We can obtain having in hand the exact exponential estimations for tail function for maximum distribution of random field, the sufficient conditions for weak compactness of random fields in the space of continuous functions, and in particular derive the sufficient conditions for Central Limit Theorem in Banach space, alike ones in the article [33].
References

[1] Arnold L. and Imkeller P. On the spatial asymptotic Behavior of stochastic Flows in Euclidean Space. Stoch. Processes Appl., 62(1), (1996), 19-54.

[2] Barlow M.T. and Yor M. Semimartingale inequalities via the Garsia-Rodemich-Rumsey lemma, and applications to local times. J. Funct. Anal.; 49(2), (1982), 198-229.

[3] Bednorz W. (2006). A theorem on Majorizing Measures. Ann. Probab., 34, 1771-1781. MR1825156.

[4] Bednorz W. The majorizing measure approach to the sample boundedness. arXiv:1211.3898v1 [math.PR] 16 Nov 2012

[5] Bednorz W. (2010), Majorizing measures on metric spaces. C.R. math. Acad. Sci. Paris, (2010), 348, no. 1-2, 75-78, MR2586748

[6] Bednorz W. Hölder continuity of random processes. arXiv:math/0703545v1 [math.PR] 19 Mar 2007.

[7] Bennett C. and Sharpley R. Interpolation of operators. Orlando, Academic Press Inc.,1988.

[8] Bourbaki N. Commutative Algebra. Elements of Mathematics. Chapters 1-7. Springer Verlag, (1977).

[9] Dmitrovsky V.A. On the maximum distribution and local properties of paths pre - Gaussian random fields. Probab. Theory and Math. Statist., Kiev, KSU, (1981), 25, 154 - 164.

[10] Dudley R.M. Uniform Central Limit Theorem. Cambridge University Press, 1999.

[11] Fernique X. (1975). Regularite des trajectoires des function aleatoires gaussiennes. Ecole de Probablite de Saint-Flour, IV - 1974, Lecture Notes in Mathematic. 480, 1 - 96, Springer Verlag, Berlin.

[12] Fernique X, Caracterisation de processus de trajectoires majores ou continus. Seminaire de Probabilits XII. Lecture Notes in Math. 649, (1978), 691706, Springer, Berlin.

[13] Fernique X. Regularite de fonctions aleatoires non gaussiennes. Ecolee de Ete de Probabilits de Saint-Flour XI-1981. Lecture Notes in Math. 976, (1983), 174, Springer, Berlin.

[14] Frolov A.S., Tchentzov N.N. On the calculation by the Monte-Carlo method definite integrals depending on the parameters. Journal of Computational Mathematics and Mathematical Physics, (1962), V. 2, Issue 4, p. 714-718 (in Russian).
[15] Fiorenza A., and Karadzhov G.E. *Grand and small Lebesgue spaces and their analogs*. Consiglio Nationale Delle Ricerche, Instituto per le Applicazioni del Calcoto Mauro Picone, Sezione di Napoli, Rapporto tecnico n. 272/03, (2005).

[16] Garsia, A. M.; Rodemich, E.; and Rumsey, H., Jr. *A real variable lemma and the continuity of paths of some Gaussian processes*. Indiana Univ. Math. J. 20 (1970/1971), 565-578.

[17] Grigorjeva M.L., Ostrovsky E.I. *Calculation of Integrals on discontinuous Functions by means of depending trials method*. Journal of Computational Mathematics and Mathematical Physics, (1996), V. 36, Issue 12, p. 28-39 (in Russian).

[18] Heinikel B. *Measures majorantes et le theorem de la limite centrale dans C(S)*. Z. Wahrscheinlichkeitstheory. verw. Geb., (1977). 38, 339-351.

[19] Yaozhong Hu and Khoa Le *A multiparameter Garsia-Rodemich-Rumsey inequality and some applications*. arXiv:1211.6809v1 [math.PR] 29 Nov 2012

[20] Imkeller P. and Scheutzov M. *Stratonovich calculus with spatial parameters and anticipative problem in multiplicative ergodic theory*. Ann. Probab. 27(1), (1999), 109-129.

[21] Iwaniec T., P. Koskela P., and Onninen J. *Mapping of finite distortion: Monotonicity and Continuity*. Invent. Math. 144 (2001), 507 - 531.

[22] Kassman M. *A Note on Integral Inequalities and Embedding of Besov Spaces*. Journal of Inequalities in Pure and Applied Mathematics. V.4 Issue 4, article 107, (2003), 47-57.

[23] Kolmogorov, A. N. and Tikhomirov, V. M. (1959). $\epsilon$ – entropy and $\epsilon$ – capacity of sets in a functional space. Uspekhi Mat. Nauk, 14, 3; 86 - 96 (in Russian).

[24] Kozachenko Yu. V., Ostrovsky E.I. (1985). *The Banach Spaces of random Variables of subgaussian type*. Theory of Probab. and Math. Stat. (in Russian). Kiev, KSU, 32, 43 - 57.

[25] Krasnoselsky M.A., Rutizky Ya.B. *Convex function and Orlicz spaces*. GIFML, Moskow, 1958 (in Russian).

[26] Kwapien S. and Rosinsky J. *Sample Hölder continuity of stochastic processes and majorizing measures*. (2004). Seminar on Stochastic Analysis, Random Fields and Applications IV, Progr. in Probab. 58, 155163. Birkhöuser, Basel.

[27] Ledoux M., Talagrand M. (1991) *Probability in Banach Spaces*. Springer, Berlin, MR 1102015.

[28] Liflyand E., Ostrovsky E., Sirota L. *Structural Properties of Bilateral Grand Lebesgue Spaces*. Turk. J. Math.; 34 (2010), 207-219.
[29] Nezza E.D., Palatucci G., Valdinoci E. Hitchhikers guide to the fractional Sobolev spaces. arXiv:1104.4345v3 [math.FA] 19 Nov 2011

[30] Ostrovsky E.I. (1999). Exponential estimations for random Fields and its applications (in Russian). Moscow - Obninsk, OINPE.

[31] Ostrovsky E. AND SIROTA L. Module of continuity for the functions belonging to the Sobolev-Grand Lebesgue Spaces. arXiv:1006.4177v1 [math.FA] 21 Jun 2010

[32] Ostrovsky E., SIROTA L. Continuity of Functions belonging to the fractional Order Sobolev’s-Grand Lebesgue Spaces. arXiv:1301.0132v1 [math.FA] 1 Jan 2013

[33] Ostrovsky E., Rogover E. Exact exponential Bounds for the random Field Maximum Distribution via the Majorizing Measures (Generic Chaining.) arXiv:0802.0349v1 [math.PR] 4 Feb 2008

[34] Ostrovsky E.I. (2002). Exact exponential Estimations for Random Field Maximum Distribution. Theory Probab. Appl. 45 v.3, 281 - 286.

[35] Ostrovsky E., SIROTA L. A counterexample to a hypothesis of light tail of maximum distribution for continuous random processes with light finite-dimensional tails. arXiv:1208.6281v1 [math.PR] 30 Aug 2012

[36] Ostrovsky E., Rogover E. Non-asymptotic exponential bounds for MLE deviation under minimal conditions via classical and generic chaining methods. arXiv:0903.4062v1 [math.PR] 24 Mar 2009

[37] Ostrovsky E., SIROTA L. Monte-Carlo method for multiple parametric integrals calculation and solving of linear integral Fredholm equations of a second kind, with confidence regions in uniform norm. arXiv:1101.5381v1 [math.FA] 27 Jan 2011

[38] Ostrovsky E., SIROTA L. Simplification of the majorizing measure method, with development. arXiv:1302.3202v1 [math.PR] 13 Feb 2013

[39] Prokhorov Yu.V. Convergence of Random Processes and Limit Theorems of Probability Theory. Probab. Theory Appl., (1956), V. 1, 177-238.

[40] Ral’chenko, K. V. The two-parameter Garsia-Rodemich-Rumsey inequality and its application to fractional Brownian fields. Theory Probab. Math. Statist. No. 75 (2007), 167-178.

[41] Talagrand M. (1996). Majorizing measure: The generic chaining. Ann. Probab., 24 1049 - 1103. MR1825156

[42] Talagrand M. (2005). The Generic Chaining. Upper and Lower Bounds of Stochastic Processes. Springer, Berlin. MR2133757.

[43] Talagrand M. (1987). Regularity of Gaussian processes. Acta Math. 159 no. 1-2, 99 149, MR 0906527.
[44] Talagrand M. (1990), Sample boundedness of stochastic processes under increment conditions. Annals of Probability 18, N. 1, 1-49, MR1043935.

[45] Talagrand M. (1992). A simple proof of the majorizing measure theorem. Geom. Funct. Anal. 2, no. 1, 118 - 125. MR 1143666

[46] Talagrand, M. (1994). *Construction of majorizing measures, Bernoulli processes and cotype*. Geom. Funct. Anal. 4, 660 - 717.

[47] Talagrand, M. (1996). *Majorizing measures: the generic chaining*. Ann. Probab., 24, 1049 - 1103.