Modeling the Interaction of an Elastic Collision between two Objects

Alejandro GONZÁLEZ Y HERNÁNDEZ and María del Pilar SEGARRA ALBERÚ

Departamento de Física, Facultad de Ciencias, Universidad Nacional Autónoma de México. Av. Universidad 3000, C.U., Ciudad de México, 04510, México.

agh@ciencias.unam.mx

Abstract. Physics undergraduate students researched the interaction between magnets during an elastic collision of two gliders on an air track. The objective of this research was to achieve students to discover the dynamics of an elastic collision before, during and after the time in which repulsive magnetic forces acted over two gliders on an air track. For several years it has been introduced, in our course of mechanical laboratory, a discovery cycle which is followed by students and has helped them to develop or improve their scientific skills of reasoning. The students began their investigations with an experiment to find the relation between the repulsion magnetic force with the distance of separation between magnets. Then, they videotaped and analyzed the experiment of collisions between magnets attached to gliders on an air track and modeled numerically the forces of interaction during the collision using the relation between the magnetic force with the distance founded previously. At last, they successfully compared the numerical model with the experiment. Here it is reported, how students carried out the experiment of magnetic collisions and its theoretical and numerical modelling and, how they get a better understanding of the concepts, laws, and principles of physics that are immersed in real collisions in the laboratory by the application of the discovery cycle.

1. Introduction

The Laboratory of Mechanics course is taught in the second semester of the Physics career at the Science Faculty of National University Autonomous of Mexico. The purpose of this course is to promote through experimentation, meaningful learning of the principles and laws of Mechanics. An important part of this course is the modelling of the motion phenomena based on Newton's laws and the principles of momentum and mechanics energy, and its comparison with the experimental results. The groups of the Laboratory of Mechanics have a maximum of 24 students divided into six teams of four students, joint in collaborative work. The course has two sessions a week of three hours each one, during the sixteen weeks of the semester. Eleven experimental activities are done during the semester and a final project at the end.

As a strategy of teaching, a discovery cycle [1], is followed by the students in each of the experimental activities done during the course. This discovery cycle consists of four phases: exploration, experimentation, modeling and prediction, and is a research strategy followed by students to acquire knowledge of the dynamic behavior of movement phenomena, developing and applying scientific abilities like experimental skills, theoretical modeling [2], numerically solving of equations of motion and the classification and recognition of patterns.
For many years, work has been done to obtain evidence of whether this cycle promotes the learning sought. As an evidence of the results obtained through it, the work done by students in their final project on the magnetic interaction of an elastic collision \[3, 4\]. on an air track and the results obtained by them in the understanding of the phenomenon as well as working skills, is presented in this article.

2. Theory

The dynamics of phenomena of motion can be modelled by Newton’s Laws and the principles of conservation of momentum and kinetic energy in the elastic collision \[5\], between two gliders on a horizontal air track interacting magnetically.

In an elastic collision and in absence of external net forces, we distinguish three stages in a collision between two bodies: “before collision”, “during the collision”, and “after the collision” as in Figure 1.

![Figure 1. Stages of the elastic collision between two gliders interacting magnetically on a horizontal air track, (a) “after the collision”, (b) “during the collision”, and (c) “after the collision”.
](image)

In an isolated system in one dimension, the principles of conservation of momentum and kinetic energy are valid in a collision before and after the collision, that is:

\[ p_1 + p_2 = p_1' + p_2' = cte, \]  \( (1) \)

and

\[ K_1 + K_2 = K_1' + K_2' = cte, \]  \( (2) \)

where \( p_1, K_1, p_2, K_2, p_1', K_1', p_2', K_2' \) are the momentum and kinetic energy of body 1 and body 2, before and after the collision respectively.

During a magnetic interaction between two bodies 1 and 2, the second and third Newton’s laws\[1\] are applied. First, if we apply the third Newton’s law to bodies 1 and 2, we have:

\[ F_{12} = -F_{21} = -F_m, \]  \( (3) \)

with \( F_{12} \) the force on body 1 due to the body 2, and \( F_{21} = F_m \) (with \( F_m \) the magnetic force on glider 2) the force on body 2 due to the body 1 or the magnetic force on it.

Then, if the second Newton’s law is applied to bodies 1 and 2, considering expression (3), we have:

\[ m_1 a_1 = -F_m, \]  \( (4a) \)

or

\[ m_1 \frac{d^2x_1}{dt^2} = -F_m, \]  \( (4b) \)

and

\[ m_2 a_2 = F_m, \]  \( (5a) \)

or

\[ m_2 \frac{d^2x_2}{dt^2} = F_m, \]  \( (5b) \)

with \( x_1 \) and, \( x_2 \) the displacements of bodies 1 and 2 respect to the origen of positions, and \( m_1, m_2, a_1, \) and \( a_2 \) the masses and accelerations of bodies 1 and 2 respectively.

When the interaction is a magnetic interaction between magnets, it is necessary to measure the magnetic force \( F_m \) between magnets or magnetic dipoles (see Fig. 1b), as a function of separation between magnets (distance between the centres of magnets).
3. Measurement of $F_m$

Students do the following experiment for measuring the repulsive magnetic force between magnets [6,7], with poles opposite.

On an air track inclined an angle $\theta$ are put two gliders with magnets attached to them with opposite poles confronted. One glider is attached to the air track and the other of mass $m$ can move on it freely. Two forces are exerted on this last glider: the magnetic force $F_m$, and the component of the weight of this glider in the direction of the air track $W_{||}$ (Figure 2).

![Figure 2](image.png)

**Figure 2.** The glider of mass $M$ is attached to the inclined air track, and the glider of mass $m$ with additional mass $\Delta m$ moves freely on the air track. The magnetic force on this last glider is balanced by the gravitational force and, the normal force on it.

When the glider of mass $m + \Delta m$ (with $\Delta m$ as an additional mass) to the distance $s$ between magnets is at rest, the magnetic force $F_m$ is balanced by the force $W_{||} = (m + \Delta m)g\text{sen}\theta$, and so, the net force is zero. That is:

$$F_m - (m + \Delta m)g\text{sen}\theta = 0,$$

where $\Delta m$ is an additional mass to the glider of mass $m$.

From expression (6), we get:

$$F_m = (m + \Delta m)g\text{sen}\theta.$$

The expression (7) is a measure of the magnetic force at the distance $s$ between magnets. Now, if we change the mass $\Delta m$ or the inclination angle $\theta$, we can measure the magnetic force $F_m$ for difference distance $s$, so we can find a relation between the magnetic force with the distance $s$ if we do these kinds of changes several times.

4. Magnetic force $F_m$ as a function of $s$

In the experiment on the inclined air track, the students measured the mass $m + \Delta m$ (or the total mass of the glider), the inclination angle $\theta$, and the distance between magnets $s$ for different values of $\Delta m$ or $\theta$. With these measurements, they found a relationship between the magnetic force $F_m$ from expression (7) and the distance $s$.

The relationship found in this work is an empirical function that was got by graphical methods as follows: when the magnetic force $F_m$ is measured, this force increases when $s$ decreases, for this the relation $F_m = F_m(s)$ must be an inverse relation for some power of $s$. If there were magnetic monopoles a law of inverse of the square of $s$ could be a good possibility, but instead of magnetic monopoles, we have magnetic dipoles and exits the possibility that the relationship between the magnetic force with the distance will be an inverse relationship of $s$ of greater power.

After a discussion with the students of how to find the relationship between $F_m$ and $s$, it was concluded that the graph of $F_m$ vs $1/s^2$ (Figure 3) was a good way to find it. In this case, if they got a straight line, there was a proportionality between these two variables, but if they didn’t find this proportionality, students could fit a polynomial of greater grade.
Because the students didn’t find a proportionality between $1/s^2$ and $F_m$, as we can see in Figure 3, they tried a polynomial of higher order as the fitting of this curve. They found that a polynomial of second grade fits well to the curve in Figure 3 with the intersection of the curve with the vertical axis equals to zero.

For the magnetic force relationship between $1/s^2$ and $F_m$, the students report the following linear regression of a quadratic polynomial fit:

$$F_m(s) = a_0 + a_1 \left(\frac{1}{s^2}\right) + a_2 \left(\frac{1}{s^2}\right)^2,$$

or

$$F_m(s) = a_0 + \frac{a_1}{s^2} + \frac{a_2}{s^4},$$

where values of $a_0$, $a_1$, and $a_2$ are in Table 1.

| $a_0$     | $a_1$     | $a_2$     |
|-----------|-----------|-----------|
|   0       | (6.56 ± 0.16) × 10⁻⁹ | (2.76 ± 0.11) × 10⁻⁶ |

Then, the magnetic force of (8a and 8b) is a function inverse of $s$ of different powers.

5. Equations of motion for a magnetic collision of two gliders.

The equations of motion of glider of mass $m_1$ and glider of mass $m_2 = m_1 + \Delta m$ are derived from equations (4b), (5b), and (8b).

However, $F_m(s)$ in (8b) must be expressed in terms of $x_1$ and $x_2$, to have only two differential equations with these two variables.

Figure 4. The glider of mass $m_1$ with velocity $v_1$ collides magnetically with the glider of mass $m_1 = m_1 + \Delta m$ with velocity $v_2$ on a horizontal air track.
The distance $s$ between magnets (see Figure 4) is:

$$ s = (x_2 - x_1) - d - l, \quad (9a) $$

or

$$ s = \Delta x - L, \quad (9b) $$

with $x_1$ and $x_2$ the positions of gliders 1 and 2, $d = (11.9 \pm 0.05)\text{cm}$ and $l = (6 \pm 0.5)\text{mm}$ are the lengths of the gliders and the magnets respectively, $\Delta x = x_2 - x_1$ and $L = d + l$.

So,

$$ F_m(x_1, x_2) = a_0 + \frac{a_1}{(\Delta x - L)^2} + \frac{a_2}{(\Delta x - L)^4} + \frac{a_3}{(\Delta x - L)^6}, \quad (10) $$

the equations of motions of gliders 1 and 2 are:

$$ m_1 \frac{d^2x_1}{dt^2} = -F_m(x_1, x_2), \quad (11a) $$

and

$$ m_2 \frac{d^2x_2}{dt^2} = F_m(x_1, x_2), \quad (11b) $$

Expressions in (11a) and (11b) are differential equations coupled that must be solved simultaneously.

### 6. Numerical method of solution of equations of motion

Analytical solutions of equations (11a) and (11b) are complex to be tried with students of the first year of university, but numerical solutions are a simpler way to be done by the students.

The numerical method done to students was an extended numerical method of Euler. This method needs to know the initial values of gliders 1 and 2 of $x_{10}$, $x_{20}$, $v_{10}$, $v_{20}$, the force $F_{m_{1,0}} = F_m(x_{10}, x_{20})$ in $t_0 = 0$, an interval of time $\Delta t$ that is given by the solver and the iterative algebraic equations in Table 2 with the index $i$ running from 0 to $n$.

**Table 2.** Numerical method to solve the coupled equations of motion of the collision of the two gliders.

| Mass $m_1$ | Mass $m_2$ |
|------------|------------|
| $v'_{1,i+1} = v_{1,i} + \frac{F_{m_{1,i}}}{m_1} \Delta t$ | $v'_{2,i+1} = v_{2,i} + \frac{F_{m_{2,i}}}{m_2} \Delta t$ |
| $v'_{2;i+1}^2 = \frac{v_{1,i} + v_{2;i+1}^2}{2}$ | $v'_{2;i+1}^2 = \frac{v_{2,i} + v_{2;i+1}^2}{2}$ |
| $x_{1,i+1} = x_{1,i} + \frac{v_{1,i}^2 + 1}{2} \Delta t$ | $x_{2,i+1} = x_{2,i} + \frac{v_{2,i}^2 + 1}{2} \Delta t$ |
| $x_{2;i+1}^2 = \frac{x_{1,i} + x_{2,i+1}}{2}$ | $x_{2;i+1}^2 = \frac{x_{2,i} + x_{2,i+1}}{2}$ |
| $F_{m_{1;i+1}^2} = -F_{m_{1}}(x_{1;i+1}, x_{2;i+1}^2)$ | $F_{m_{2;i+1}^2} = -F_{m_{2}}(x_{1;i+1}, x_{2;i+1}^2)$ |
| $v_{1,i+1} = \frac{F_{m_{1,i}}}{m_1} \Delta t$ | $v_{2,i+1} = \frac{F_{m_{2,i}}}{m_2} \Delta t$ |
| $F_{m_{1;1,i+1}} = -F_{m_{1}}(x_{1;i+1}, x_{2;i+1}^2)$ | $F_{m_{2;1,i+1}} = -F_{m_{2}}(x_{1;i+1}, x_{2;i+1}^2)$ |
| $t_{i+1} = t_i + \Delta t$ | $i = i + 1$ |

Students program equations of table 2 in a spreadsheet as Excel to get a data table of times $t_i$, positions $x_{1,i}$, $x_{2,i}$, velocities $v_{1,i}$, $v_{2,i}$, and interaction forces $F_{m_{1;i}}$, $F_{m_{2;i}}$ for $i = 0,1,2,3, \ldots, n$ as numerical solution of equations (11a) and (11b) and all other variables deduced from these as momentum or kinetic energy.

In Table 2, the two columns of iterative algebraic equations are programmed in Excel for the solutions of equations (11a) and (11b). Each step of these two series of equations must be solved simultaneously because they occur at the same time and are coupled by the interactive forces between the two gliders where each one depends on the positions of the two gliders.
The result of the numerical solutions gives a model of the dynamical behavior of the collision of the two gliders on the air track that can be represented graphically. This model represents the three stages of the collision, before, during and after the collision.

7. Model of collision

Although the model of the collision between two gliders with a magnetic interaction is given in a long table of data (Table 3). The variables of the table can be graphed in different kinds of combinations, but the most important relationships between variables are their relationships with time and their graphical interpretations give us the dynamics behavior of the collision.

The columns in table 3 are the calculations of variables of the numerical method given in table 2 and the rows are the steps following in this numerical method, beginning with initial conditions in the first row and in the second row the programming of formulas in Table 2. The third row is the copy of the formulas of the second row that represents the first iteration of the numerical method. The following rows of the numerical method, not shown here for the sake of the economy of space, are the next iterations of the numerical method until to find a condition that stops the process of calculations.

Table 3. The numerical method of Euler extended. The two first rows of the method represent the results of the iterative algebraic equations for \( i = 0 \) and \( i = 1 \), the third row is a copy of the second row and represents the rows \( i = 2, ..., n \).

| Glider 1 | Glider 2 |
|----------|----------|
| \( t \) | \( x_1 \) | \( x_{1,0} \) | \( v_1 \) | \( v_{1,0} \) | \( a_1 \) | \( F_{1,0} \) | \( x_2 \) | \( x_{2,0} \) | \( v_2 \) | \( v_{2,0} \) | \( a_2 \) | \( F_{2,0} \) |
| 0.0008 | 0.0347 | 0.255 | 0.2940 | 0.2940 | -0.4090 | 0.4090 | 0.5993 | 0.5993 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.0008 | 0.0347 | 0.255 | 0.2940 | 0.2940 | -0.4090 | 0.4090 | 0.5993 | 0.5993 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.0005 | 0.0347 | 0.255 | 0.2940 | 0.2940 | -0.4090 | 0.4090 | 0.5993 | 0.5993 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

7.1. Graphical representations.

The relationships between \( x, v, a, \) and \( F_m \) with the time for the two gliders are represented in Figure 5, and these relationships give us the dynamics of the collision between the two gliders, before, during and after the collision.

The graphs \( x vs t, v vs t, a vs t, \) and \( F_m vs t \) represent the positions: \( x_1 \) and \( x_2 \), the velocities: \( v_1 \) and \( v_2 \), the accelerations: \( a_1 \) and \( a_2 \), and the forces: \( F_{m,1} \) and \( F_{m,2} \), of the gliders 1 and 2 respectively as a function of time, that is, \( x_1 = x_1(t), x_2 = x_2(t), v_1 = v_1(t), v_2 = v_2(t), a_1 = a(t), a_2 = a_2(t), F_{m,1} = F_{m,1}(t), \) and \( F_{m,2} = F_{m,2}(t) \).

![Figure 5](image-url)
The collision analysis in Figure 5 was the clash between two gliders of equal mass \((m = 0.224 \text{ kg})\), with a constant velocity of glider 1 \((v_{1b} = 0.36 \text{ m/s})\), and a zero velocity of glider 2 before the collision and after the collision, the velocity of glider 1 results equal to zero and a constant velocity for glider 2 \((v_{2a} = 0.36 \text{ m/s})\) as it was expected. The accelerations of the two gliders and the forces on them are different from zero during the collision, and before and after the collision equal to zero.

During the collision \(x_1(t)\) change from a constant velocity \(v_{1b}\) to a zero velocity, and \(x_2(t)\) change from zero velocity to a constant velocity \(v_{2a}\), with \(v_{1b} = v_{2a}\). The velocity \(v_1(t)\) decreases from \(v_{1b}\) to 0 and the velocity \(v_2(t)\) increases from 0 to \(v_{2a}\), the acceleration \(a_1(t)\) changes from 0 to a minimum value and then to 0, and the acceleration \(a_2(t)\) changes from 0 to a maximum value and then to 0 in a symmetrical form. The force \(F_{m,1}(t)\) and \(F_{m,2}(t)\) change in a similar way as the accelerations of gliders 1 and 2, because during the collision Newton’s second and third laws are valid during the collision in the model.

8. **Comparison between the model and the experiment.**

The functions \(x_1 = x_1(t)\) and \(x_2 = x_2(t)\) model the collisions between gliders 1 and 2, but this model must agree with the experiment of the collision with the glider 1 with velocity \(v_{1b}\) goes against the glider 2 at rest.

![Figure 6](image_url)

**Figure 6.** Comparison between the model and the experiment for the collision of gliders 1 and 2.

The model and the experiment of the collision between gliders 1 and 2 are represented in Figure 6. The functions \(x_1 = x_1(t)\) and \(x_2 = x_2(t)\) are the displacements of these gliders through the time and are represented in Figure 6 by black curves and the corresponding experimental displacements are the red dashed lines.

Because the comparison between these two movements before, during and after the collision for the model and the experiment is of high coincidence, the model is accepted as a good representation of the experimental collision.

9. **Theoretical analysis.**

If it is considered the effect of the force \(F_{jk}\) of the glider \(k\) over the glider \(j\) \((j = 1,2)\) and \((k = 2,1)\) in the interval of time \(\Delta t = t_f - t_i\) during the collision, we have the impulse \(I_{jk}\) of the force exerted on that glider \(j\) by glider \(k\) and this impulse is equal to the charge of the momentum \(\Delta p_j = p_{j,f} - p_{j,i}\) of the glider \(j\), that is:

\[
I_{jk} = \int_{t_i}^{t_f} F_{jk} \, dx = p_{j,f} - p_{j,i}.
\] (12)
Moreover, if it is considered the effect of this force in the displacement \( \Delta x_j = x_{j,f} - x_{j,i} \) of the glider \( j \) during the collision, we have the work \( W_j \) done over the glider \( j \) by the force \( F_{jk} \), and this work is equal to the change of kinetic energy \( \Delta K_j = K_{j,f} - K_{j,i} \) of the body, that is:

\[
W_j = \int_{x_{j,i}}^{x_{j,f}} F_{jk} \, dx = K_{j,f} - K_{j,i}.
\]  

The forces \( F_{jk} \), in expression (12) and (13) are represented in Figure 7.

In expressions (12) and (13), the integrals are the areas under the curves of Figure 7 and these areas were calculated by numerical integration of data in Table 3. These integrations were made as:

\[
I_j^N = \sum F_{jk,1/2} \Delta t,
\]

and,

\[
W_j^N = \sum F_{jk,1/2} \Delta x_j,
\]

Where \( \Delta x_j \), with \( j = 1,2 \), corresponds to the displacements of glider \( j \), and \( N \) indicates the numerical integrations.

**Figure 7.** Graphical representation of forces \( F_{jk} \) as a function of \( t \) or \( x_j(t) \).

In expressions (12) and (13), the integrals are the areas under the curves of Figure 7 and these areas were calculated by numerical integration of data in Table 3. These integrations were made as:

\[
I_j^N = \sum F_{jk,1/2} \Delta t,
\]

and,

\[
W_j^N = \sum F_{jk,1/2} \Delta x_j,
\]

Where \( \Delta x_j \), with \( j = 1,2 \), corresponds to the displacements of glider \( j \), and \( N \) indicates the numerical integrations.

**Table 4.** Numerical integration of impulse and work for gliders 1 and 2 and their comparison with the change of momentum and kinetic energy given by the numerical model.

| \( I_j^N \) | \( \Delta p_1 \) | \( I_j^N \) | \( \Delta p_2 \) | \( W_j^N \) | \( \Delta K_1 \) | \( W_j^N \) | \( \Delta K_2 \) |
|------------|-------------|------------|-------------|-------------|-------------|-------------|-------------|
| -0.893     | -0.892      | 0.893      | 0.893       | -0.0178     | -0.0178     | 0.0178      | 0.0178      |

The results of integrations given in Table 4 are equals for three significant figures for the impulse with the change of the momentum and for the work with the change of kinetic energy for gliders 1 and 2.

10. **Scientific skills developed**

In the study of the magnetic collision of two gliders, the students developed different kinds of scientific skills. We summarize in Table 5 the mean scientific skills developed by students.

The scientific skills of Table 5 were developed by 24 students of the Laboratory of Mechanics course on the second semester of the physics career. They worked in six teams of four students for the investigation of the magnetic forces and magnetic interaction of gliders. The strategy of discovery used in this investigation was applied in previous experimental activities of the course where students had also acquired the techniques used in this project.

However, all the experimental activities and theoretical and numerical deductions of this project had a high degree of complication which served to the assessment of all the items in Table 5. The performance reported here are the average of the scientific skills developed by students grouped in teams of collaborative and independent work under the support of a professor and an assistant.
Table 5. Scientific skills developed by students in the research of the magnetic interaction between two gliders on a horizontal air track.

| Scientific skills                                                                 | Performance % |
|----------------------------------------------------------------------------------|---------------|
| 1. Experimental skills to do and analyze two linked experiments, the first to find a law for the magnetic force between magnets and the second to discover the dynamic behavior of a magnetic collision between two gliders on a horizontal air track. The skills developed in these experiments are: (a) the planning of an experiment and construction of the experimental configuration, (b) the performance of experiments applying control of variables and its measurement with determination of uncertainties, and the data capture using techniques of photography or fast video to get high precision in data capture and (c) the using of video analysis software [8], and spreadsheet to analyze experimental data numerically and graphically, first to find an empirical law for the magnetic force applying change of variables and linear regression analysis to graphical points and second to represent graphically the data of the collision motions of the two gliders graphically and to test experimentally the conservation of momentum and kinetic energy. | 90/95 |
| 2. Theoretical and computational skills to model numerically the magnetic collision between two gliders as (a) applications of second and third Newton’s laws and the law of magnetic forces exerted on gliders 1 and 2 to find the equations of motion for the two gliders in their movements of a collision, (b) design and programming in a spreadsheet of a numerical method of solution of the equations of motion, and (c) graphic representation of the numerical solution data and physical interpretations of the resultant curves. | 85 |
| 3. Classification skills to compare modeling results with experiment data as (a) a positive graphical comparison between the displacements of the gliders in their movements in the collision with the experimental data, (b) testing the accuracy of this comparison, and (c) developing of numerical integration to confirm the validity of principles of conservation of momentum and kinetic energy in the model. | 80 |
| . Pattern recognition skills to predict events didn’t observe experimentally by changing of parameters to study new collision patterns like regular or extreme cases. | 80 |

11. Conclusions
The discovery cycle followed by students in this project helped them to work in the dynamical behavior of a magnetic collision based on the interaction of magnetic dipoles and their theoretical and numerical modeling. During this process students developed several scientific skills given in Table 5. Students’ work indicate that they may investigate and understand step by step the dynamics of an elastic collision as well as the concepts, laws, and principles of physics applied to this collision. The confidence of students in the strategy followed in this research was an additional achievement of this work. As part of the research, a small follow-up of students has been made to find out if in the following years of the degree they continue to work with this learning cycle. Currently we are processing the results.

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