Numerical Radiation Reaction for a Scalar Charge in Kerr Circular Orbit

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We numerically calculate the dissipative part of the self-force on a scalar charge moving on a circular, geodesic, equatorial orbit in Kerr spacetime. The solution to the scalar field equation is computed by separating variables and is expressed as a mode sum over radial and angular modes. The force is then computed in two ways: a direct, instantaneous force calculation which uses the half-retarded-minus-half-advanced field, and an indirect method which uses the energy and angular momentum flux at the horizon and at infinity to infer the force. We are able to show numerically and analytically that the force-per-mode is the same for both methods. To enforce the boundary conditions (ingoing radiation at the horizon and outgoing radiation at infinity for the retarded solution) numerical solutions to the radial equation are matched to asymptotic expansions for the fields at the boundaries. Recursion relations for the coefficients in the asymptotic expansions are given in an appendix.

I. INTRODUCTION: MOTIVATION AND SUMMARY

With sensitive ground-based gravitational wave detectors, such as LIGO, GEO, TAMA and VIRGO in operation and the launch of a space-based gravitational wave detector LISA planned in the next decade, the need to accurately model the waves emitted from inspiraling binary star systems has become acute.

In the case of LIGO and the other ground-based detectors, the sensitivity of the instrument is at a maximum around a few hundred Hertz. This frequency is comparable to the orbital frequency of two neutron stars just prior to their final coalescence, and therefore LIGO is particularly sensitive to these sources. These constituents of these systems have comparable mass and relatively weak gravitational fields; therefore the emitted signal can be computed using a weak-field, slow-motion approximation (i.e. a post-Newtonian approximation). Although this method – and other improvements to the post-Newtonian method – give waveforms that are accurate enough to detect signals buried in the LIGO noise, a more detailed knowledge of late inspiral and merger, accessible only by numerical evolution, is needed to extract from the waves the astrophysics of their sources.

LISA, however, will be sensitive to signals with much lower frequency, in particular, to waves from a stellar-size black hole spiraling into a (super-)massive black hole $10^3 - 10^6 M_\odot$. Detectable sources with this extreme mass-ratio will have periods of (many) minutes and will persist for weeks (or even months). In addition, the smaller mass may spend part of its orbit deep in the strong gravitational field of the larger mass, making a post-Newtonian approximation inappropriate for computing the predicted waveforms. Clearly, this problem is better suited to black hole perturbation theory where the background geometry generated by the larger mass $M$ is treated exactly, and the smaller mass $\mu$ generates a small perturbation of the geometry. To lowest nontrivial order in $\mu/M$, the field generated by the smaller mass causes its trajectory to deviate from a geodesic of the background spacetime. This is the origin of the self-force. Heuristically, one can think of the smaller mass as traveling on a geodesic of the perturbed spacetime. Although this description of the force is intuitive and compelling, computing the self-force entails a number of conceptual and technical difficulties.

At the center of the conceptual difficulties is the renormalization problem: even though the perturbing mass is small compared to the central mass, the perturbed field diverges at the position of the particle. A number of authors have addressed this difficulty and developed formal methods for removing the divergence while preserving the finite parts of the field that give rise to the self-force. These papers solve the problem in principle; there remain, however, a number of difficulties in implementing these prescriptions to find the force and subsequent motion and waveform. In particular, there remains a gauge problem. The self-force equations for a small mass moving in a background spacetime have only been written down in Lorentz gauge, but, unfortunately, most methods for finding the perturbing gravitational fields use a different gauge. Complicating matters further, recent calculations suggest that the gauge transformations that relate gauges of perturbation theory to Lorentz gauge are poorly behaved. Thus it is difficult to use the metric perturbations that are readily available from black hole perturbation theory in the formal equations for the self-force.

One method of side-stepping both the gauge and the renormalization issue is to compute the self-force indirectly by computing the energy and angular momentum flux at infinity. Such techniques usually assume that the particle is in circular geodesic orbit. The particle is then the source of the metric perturbation (or of the electromagnetic or
We consider the Klein-Gordon field $\Phi$ of a point particle of conserved scalar charge $q$, orbiting on a circular, equatorial geodesic of the Kerr geometry. The field $\Phi$ satisfies the massless scalar wave equation

$$\nabla_\alpha \nabla^\alpha \Phi = -4\pi \rho, \tag{2.1}$$

with the scalar charge density $\rho$ a delta function along the trajectory $z^\alpha(\tau)$.

$$\rho = q \int \delta^4(x^\alpha, z^\alpha(\tau)) \, d\tau. \tag{2.2}$$

With this normalization, we have $q = \int_V \rho \, dV$, where $dV$ is the volume element on a spacelike hypersurface $V$ orthogonal to the trajectory. For a circular orbit of radius $r_0$ and frequency $\Omega$, we have

$$\rho = q \int \frac{\delta (t - u^t \tau) \delta (r - r_0) \delta (\theta - \frac{\pi}{2}) \delta (\phi - \Omega \tau)}{\Sigma \sin \theta} \, d\tau, \tag{2.3}$$

where $u^t$ is the (constant) time component of the particle four-velocity, and Eq.(2.1) takes the form,

$$\nabla_\alpha \nabla^\alpha \Phi = -4\pi \frac{q}{r_0^3 u^t} \delta (r - r_0) \delta (\theta - \frac{\pi}{2}) \delta (\phi - \Omega \tau). \tag{2.4}$$

In this paper, we attack a somewhat simpler problem that still puts us on a path to solve the more general problem. We assume that our particle is a scalar charge that acts as the source of a Klein-Gordon field. The self-force arising from the back-reaction of a scalar (rather than a gravitational) field involves no delicate gauge issues. The part of our work reported here avoids the renormalization problem by using the half-retarded-minus-half-advanced field. The divergent structure of the field at the position of the particle is the same for the advanced and retarded solutions, and therefore the difference is smooth at the particle’s position. By evaluating the gradient of the half-retarded-minus-half-advanced field we are able to compute the dissipative (time-antisymmetric) parts of the self-force acting on the particle at any instant; our method does not require time-averaging to compute the force. Although in the present paper we will assume that our source particle is traveling on a circular geodesic orbit, it is not an essential feature of the method. Extending the method to a particle in non-circular orbit would severely complicate the numerical calculation of the field, but it would not appreciably complicate the calculation of the self-force. However, in this special case of circular motion, we are able to show that our direct calculation of the instantaneous force agrees – mode-by-mode – with the indirect calculation based on the average energy loss at infinity and at the horizon.

The next steps in this work are to perform calculations for non-circular orbits and to calculate the conservative as well as the dissipative part of the self-force. As discussed above, the only complication arising in the former task is the numerical calculation of the field from an orbit that has a countably infinite set of frequencies. The latter, however, requires a new procedure that will include a renormalization of the divergent field. It is in anticipation of the delicate subtraction arising in mode-by-mode renormalization that we have taken such care with the accuracy of the radial functions computed in this paper.

Outline and conventions

In section II we use separation of variables to solve the wave equation and write the scalar field $\Phi$ as a sum over radial and angular mode functions. In section III we describe the scheme for the numerical construction of the radial functions which appear in this solution. In section IV we present the argument that the two methods are equivalent, and derive separate expressions for the evolution of the conserved quantities $E$ and $L$ based on each one. Finally in section V we discuss numerical details, show the properties of the radial functions, and present our results for dissipative self-force.

Throughout the paper we use Boyer-Lindquist coordinates $(t, r, \theta, \phi)$ and a metric with signature $(- + + +)$. The mass of the black hole is $M$, its spin is $a$, and the usual abbreviations $\Delta = r^2 + a^2 - 2Mr$ and $\Sigma = r^2 + a^2 \cos \theta$ are used; the radial coordinate of the horizon is written $r_+ = M + \sqrt{M^2 - a^2}$. Our conventions are those of Misner, Thorne and Wheeler [17], except that their symbol for $\Sigma$ is $\rho^2$. 

II. SOLUTION OF THE WAVE EQUATION

The perturbed field is examined at the horizon and infinity and the energy and angular momentum flux is computed. The rate of energy loss is then equated to an orbital energy loss and the rate of inspiral can be inferred. Although such techniques have been broadly applied (e.g. [14, 15]) they have a drawback: They give only the time-averaged dissipative force. [See [16] for a discussion of the shortcomings of the time-averaged force.] When used to infer the rate of orbital decay, these energy balance arguments assume that the field energy on a spatial slice is constant as we go from one orbit to the other. See Section IV A for discussion.
Separating variables yields the solution
\[ \Phi = \frac{1}{2\pi} \frac{q}{u} \sum_{l,m} R_{lm}(r) S_{lm}(\frac{\pi}{2}) S_{lm}(\theta)e^{im(\phi-\Omega t)} , \] (2.5)

where the \( S_{lm} \) and \( R_{lm} \) satisfy angular and radial equations given below. The \( S_{lm} \) are oblate spheroidal harmonics, given by
\[ \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{m_l^2}{\sin^2 \theta} + \lambda_{lm} + m^2 \Omega^2 a^2 \cos^2 \theta \right] S_{lm}(\theta) = 0. \] (2.6)

We take the harmonics to be real, and normalize by
\[ \int_0^\pi S_{lm}(\theta)^2 \sin(\theta) d\theta = 1. \] (2.7)

We fix the sign by demanding that \( S_{lm}e^{im\phi} = \sqrt{2} \pi Y_{lm} \) when \( a = 0 \) (\( \lambda_{lm} \) will reduce to \((l+1)\) in this case). The numerical calculation of the \( S_{lm} \) and their eigenvalues \( \lambda_{lm} \) is a straightforward task, and is not discussed here. (See Numerical Recipes for a treatment similar to ours.) The computation of the \( R_{lm} \) is considerably more complicated and is detailed below.

### III. INTEGRATION OF THE RADIAL EQUATION

The radial functions satisfy
\[ \left[ \frac{\partial}{\partial r} \left( \Delta \frac{\partial}{\partial r} \right) + \frac{m_l^2}{\Delta} \left( \Omega^2 (r^2 + a^2)^2 - 4M ar + a^2 \right) - m^2 \Omega^2 a^2 - \lambda_{lm} \right] R_{lm}(r) = -4\pi \delta(r-r_0). \] (3.1)

First found by Carter, this is also the Teukolsky equation with spin \( s = 0 \), specialized to our source (this equation can be derived from Teukolsky \((\ref{2.20})\) by setting \( \omega = m\Omega \)). To provide accurate initial data for numerical integration of this equation, we solve the equation in asymptotic series valid near the horizon and near infinity. In terms of the angular velocity of the horizon, \( \omega_+ = \frac{a}{2Mr_+} \), and the “tortoise coordinate” \( r^* \), satisfying
\[ \frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta}, \] (3.2)
the series take the form
\[ R_{lm}^+ = \sum_{n=0}^{\infty} C_{n}^{+} (r - r_+)^{n} e^{-im(\Omega - \omega_+)}r^* \] (near \( r_+ \)), \] (3.3)
\[ R_{lm}^\infty = \sum_{n=1}^{\infty} C_{n}^{\infty} \frac{1}{r^n}e^{im\Omega r^*} \] (near \( \infty \)). \] (3.4)

The signs in the exponentials of these series amount to a choice of boundary conditions. Here we have picked ingoing radiation (−) at the horizon and outgoing radiation (+) at infinity, to construct the retarded solution. Note that in the case of circular orbits, where the time dependence is just \( e^{-im\Omega t} \), observers at infinity and the horizon always agree on the direction of the radiation down the hole, and (−) is always the correct sign for this solution.

The coefficients \( C_{n}^{+} \) and \( C_{n}^{\infty} \) satisfy recursion relations given in the appendix, with overall normalization set for each \( l, m \) harmonic by the \( \delta \)-function source. The asymptotic series provide initial values of \( R_{lm} \) and its derivative for numerical integration of the homogeneous radial equation out from the horizon and in from infinity. We thereby obtain homogeneous solutions \( R_{lm}^+ \) (from the horizon to \( r_0 \)) and \( R_{lm}^\infty \) (from \( r_0 \) to infinity). To obtain the solution \( R_{lm} \) to the inhomogeneous wave equation \((\ref{3.11})\), we patch \( R_{lm}^+ \) and \( R_{lm}^\infty \) at \( r = r_0 \). Requiring that \( R_{lm} \) be continuous and that the discontinuity in \( dR_{lm}/dr \) be fixed by the \( \delta \)-function source, we have
\[ R_{lm} = C_{lm} R_{lm}^+(r_<) R_{lm}^\infty(r_+), \] (3.5)
where \( r_\-less \) (\( r_\-greater \)) denotes the lesser (greater) of \( r_0 \) and \( r \), and

\[
C_{lm} = \frac{-4\pi}{\Delta \cdot W(R_{lm}^+, R_{lm}^-)}.
\]  

(3.6)

It is easy to show that the quantity in the denominator, \( \Delta \) times the Wronskian of the integrated solutions, is a constant for any two solutions of the homogeneous radial equation. It will be convenient to define constant factors \( C_{+lm} \) and \( C_{\infty lm} \) for which

\[
R_{lm} = C_{+lm} R_{+lm} \quad \text{when} \quad r < r_0
\]

and

\[
R_{lm} = C_{\infty lm} R_{\infty lm} \quad \text{for} \quad r > r_0,
\]

namely

\[
C_{+lm} = \frac{-4\pi R_{\infty lm}(r_0)}{\Delta W(R_{+lm}, R_{\infty lm})} \quad \text{(3.7)}
\]

and

\[
C_{\infty lm} = \frac{-4\pi R_{lm}^+(r_0)}{\Delta W(R_{lm}^+, R_{\infty lm})} \quad \text{(3.8)}
\]

IV. RADIATION REACTION

A. Equivalence of the Methods

The self-force on a body can be split into two pieces, a conservative force and a dissipative force, by writing the retarded field of the source as a sum of parts that are even and odd under the interchange, \( \text{advanced} \leftrightarrow \text{retarded} \), of ingoing and outgoing radiation.

\[
\Phi_{\text{ret}} = \frac{1}{2}(\Phi_{\text{ret}} + \Phi_{\text{adv}}) + \frac{1}{2}(\Phi_{\text{ret}} - \Phi_{\text{adv}}) \quad \text{(4.1)}
\]

(Because advanced and retarded fields are well-defined on generic, globally hyperbolic spacetimes, this division is generic for the self-force arising from linear fields.) The self-force that we compute is linear in the field \( \Phi \) and can similarly be written as a sum of a part invariant under the interchange advanced \( \leftrightarrow \) retarded – the conservative part of the force, and a part that changes sign under the interchange – the dissipative part of the force.

For a charge sustained in perpetual circular orbit by an external force, the field energy is unchanged from one hypersurface to the next, and the work done by the self-force is therefore equal to the energy radiated to the horizon and to infinity. Because the sign of the radiated energy changes under the advanced \( \leftrightarrow \) retarded interchange, so does the sign of the work done by the self-force. Thus the work done by the self-force is the work done by its dissipative part.

Formally, the stress tensor of a finite test-mass \( \mu \) is the sum of contributions from the mass, the scalar field, and the external force,

\[
T_{\alpha\beta} = T_{\mu\alpha\beta} + T_{\alpha\beta}^S + T_{\alpha\beta}^{\text{ext}},
\]

with

\[
T_{\alpha\beta}^S = \frac{1}{4\pi} \left( \nabla_\alpha \Phi \nabla_\beta \Phi - \frac{1}{2} g_{\alpha\beta} \nabla_\gamma \Phi \nabla_\gamma \Phi \right) \quad \text{(4.2)}
\]

Associated with the timelike Killing vector \( t^\alpha \) is the conserved current \( j^\alpha = -T^{\alpha\beta}_S t^\beta \). The self-force is constructed from \(-\nabla_\beta T_{\alpha\beta}^S = \rho \nabla_\alpha \Phi \), and the work done by the self-force between two \( t \) = constant hypersurfaces \( \Sigma_1 \) and \( \Sigma_2 \) is

\[
W = \int_{\Omega} d^4V \rho t^\alpha \nabla_\alpha \Phi = \int_{\Omega} d^4V \nabla_\alpha j_\alpha^S, \quad \text{(4.3)}
\]

with \( \Omega \) the region of spacetime between \( \Sigma_1 \) and \( \Sigma_2 \). We can write this integral as the sum of a vanishing time derivative and the integral of a 3-dimensional divergence by writing

\[
\nabla_\alpha j_\alpha^S = \partial_\alpha j_\alpha^S + \frac{1}{N} D_a (N j_\alpha^S),
\]

(4.4)

where \( N \) is the lapse, \( D_a \) the covariant derivative operator on the hypersurface, and \( j_\alpha^S \) the projection of \( j_\alpha^S \) into the hypersurface. Using

\[
\int_{\Omega} \partial_\alpha j^\alpha d^4V = \left( \int_{\Sigma_2} - \int_{\Sigma_1} \right) j_\alpha^S N d^3V = 0,
\]

(4.5)
we have
\[ W = \int_{S_\infty} j^a_S dS_a + \int_H j^a_S N dS_a, \]  
(4.5)
where, by \( \int_{S_\infty} \) is meant the limit \( \lim_{r \to \infty} \int_{S_r} \), with \( S_r \) a sphere of constant coordinate \( r \); and \( H \) is the horizon.

Because \( t^a \) is orthogonal to the horizon and to \( S_r \), \( W \) has the form
\[ W = \frac{1}{4\pi} \left[ \int_{S_\infty} \Phi \partial_t \Phi dS_a + \int_H \Phi \partial_t \Phi N dS_a \right] . \]

The change of sign of these flux integrals under the advanced ↔ retarded interchange, clear on general grounds, is apparent from this last form, together with the fact that the interchange is equivalent in the Kerr geometry to the transformation induced by the diffeomorphism \( t, \phi \leftrightarrow -t, -\phi \). The explicit form of these integrals for each mode is given in the next section.

The self-force measured is balanced here by an external force. When no external force is present, and when one can neglect the change in the radiative part of the field energy on successive hypersurfaces, the work \( W \) is equal to the change in the energy of the particle between the hypersurfaces \( \Sigma_1 \) and \( \Sigma_2 \). Modeling the particle by a family of dust balls with stress-energy \( \rho u^\alpha u^\beta \), with \( u^\alpha u^t (t^\alpha + \Omega^\phi) \), we have
\[ W = \int \Omega^\alpha \nabla_\alpha T^\alpha_{\beta \gamma} u^\beta d^4V = \int \Omega^\alpha (\rho u^\alpha u^\beta t^\beta) d^4V \]
(4.6)
If, along a circular geodesic in the mass, we define a momentum for which
\[ p_\alpha t^\alpha = \int_{\Sigma} T^\alpha_{\mu \beta} t^\beta dS_a = \int_{\Sigma} \rho u^\alpha u^\beta t^\beta N dV, \]
(4.7)
then, for a small mass, the integral (4.6) is approximated by \( W = \int \frac{d}{dt} (p_\beta t^\beta) N dt \), and we have
\[ \frac{dW}{dt} = \frac{d}{dt} p_\alpha t^\alpha. \]
(4.8)

Because the dissipative field, \( \Phi_{\text{diss}} := \frac{1}{2} (\Phi_{\text{ret}} - \Phi_{\text{adv}}) \), is regular for a point particle, the dissipative part of the self-force is well-defined for a point particle, without renormalization. The work done by the self-force at the particle, must then, in our case of perpetual circular motion, be identical to the radiative flux of Eq. (4.5).

**B. Energy and Angular Momentum Flux**

We have just argued that we can indirectly find the self-force on the charge by computing the flux integral of Eq. (4.5). The flux of angular momentum to infinity and to the black hole is similarly given by the integral
\[ \int_{S_\infty} \tilde{j}_S^a dS_a + \int_H \tilde{j}_S^a N dS_a, \]
(4.9)
with \( \tilde{j}_S^\beta = T^\beta_\gamma \phi^\gamma \), with \( \phi^\alpha \) the rotational Killing vector \( \partial_\phi \).

Again, when no external torque is present and when the difference between the field angular momentum on successive hypersurfaces can be neglected, the radiated angular momentum is equal to the change in the angular momentum of the particle.

We denote by \( E = -u_t = u_\alpha t^\alpha \) and \( L = u_\phi = u_\alpha \phi^\alpha \) the energy and (z component of) angular momentum per unit rest mass of the charge. In the full dynamical problem, no flux reaches spatial infinity, and one relates a change in energy between successive hypersurfaces to asymptotic flux by choosing a family of asymptotically null hypersurfaces. In our model, however, the charge has been orbiting forever, and the integrals can be evaluated at spatial infinity and at the bifurcation horizon (as in the previous section). Consider first the flux of angular momentum \( L \). We compute
Eq. (4.9) to get \( \frac{dL}{dt} \) (times the rest mass \( \mu \)), for the particle. For a surface of constant \( r \) (and \( t \)), the flux in the direction of increasing \( r \) is

\[
\mathcal{L} = \int T_{\phi}^{\prime} \Sigma d\Omega = \int T_{r} \Delta d\Omega. \tag{4.10}
\]

The real scalar field is a sum \( \Phi = \sum_{lm} \Phi_{lm} \) of complex fields whose angular eigenfunctions \( S_{lm} e^{im\phi} \) are orthogonal. The flux integral is then sum of integrals for each mode, and each mode integral involves a stress-energy tensor of the form

\[
T_{\alpha\beta} = \frac{1}{4\pi} \left( \nabla_{(\alpha} \Phi_{lm}^{*} \nabla_{\beta)} \Phi_{lm} - \frac{1}{2} g_{\alpha\beta} \nabla_{\gamma} \Phi_{lm}^{*} \nabla^{\gamma} \Phi_{lm} \right). \tag{4.11}
\]

The modes have asymptotic behavior

\[
\Phi_{lm} = \frac{q}{2\pi u t} C_{lm}^{\ast} S_{lm} \left( \frac{\pi}{2} \right) S_{lm} (\theta) e^{-im(\Omega - \omega_{+})r} e^{im(\phi - \Omega t)} + O(r - r_{+}) \quad \text{(near } r_{+}) \tag{4.12}
\]

\[
\Phi_{lm} = \frac{q}{2\pi u t} C_{lm}^{\ast} S_{lm} \left( \frac{\pi}{2} \right) S_{lm} (\theta) \frac{1}{r} e^{im\Omega r} e^{im(\phi - \Omega t)} + O(r^{-2}) \quad \text{(near } \infty). \tag{4.13}
\]

The \((r - \phi)\) component of the stress energy in each mode has corresponding behavior

\[
(T_{r\phi})_{lm} = \frac{1}{4\pi} \left( \frac{r^{2} + a^{2}}{\Delta} \right) (\Omega - \omega_{+}) m^{2} \Phi_{lm} [1 + O(r - r_{+})] \quad \text{(near } r_{+}) \tag{4.14}
\]

\[
(T_{r\phi})_{lm} = \frac{-1}{4\pi} \left( \frac{r^{2} + a^{2}}{\Delta} \right) \Omega m^{2} \Phi_{lm} [1 + O(r^{-1})] \quad \text{(near } \infty), \tag{4.15}
\]

and by (4.10) the fluxes are

\[
\mathcal{L}(r_{+}) = \frac{1}{2} \left( \frac{q}{2\pi u t} \right)^{2} (\Omega - \omega_{+}) (2Mr_{+}) m^{2} \left| C_{lm}^{+} \right|^{2} S_{lm} \left( \frac{\pi}{2} \right) \tag{4.16}
\]

\[
\mathcal{L}(\infty) = -\frac{1}{2} \left( \frac{q}{2\pi u t} \right)^{2} \Omega m^{2} \left| C_{lm}^{\ast} \right|^{2} S_{lm} \left( \frac{\pi}{2} \right). \tag{4.17}
\]

These give the rate at which angular momentum crosses the surfaces in the direction of increasing \( r \). The flux integrals (4.14) involve the outward normal of a 3-volume bounded by spatial infinity and the horizon, implying a sign change for the contribution (4.16) at the horizon:

\[
\frac{dL}{dt} = -\frac{q^{2}}{8\pi^{2}\mu (u t)^{2}} \sum_{l,m} m^{2} S_{lm} \left( \frac{\pi}{2} \right)^{2} \left[ 2Mr_{+} (\Omega - \omega_{+}) \left| C_{lm}^{+} \right|^{2} + \Omega \left| C_{lm}^{\ast} \right|^{2} \right]. \tag{4.18}
\]

For \( E \), a similar calculation yields

\[
\frac{dE}{dt} = \Omega \frac{dL}{dt}. \tag{4.19}
\]

This relationship is in fact necessary for the particle to remain in circular orbit as it radiates and spirals inward.

C. Half-Retarded Minus Half-Advanced

Here we calculate the dissipative self-force directly, by taking the gradient of the half-retarded-minus-half-advanced solution. The expression of (4.9) is the retarded solution. As noted above, the advanced solution is obtained from the retarded by the transformation \( \phi \to -\phi \), \( t \to -t \). Denoting the retarded solution by (−) and the advanced solution by (+), we have

\[
\Phi_{\pm} = \frac{q}{2\pi u t} \sum_{l,m} R_{lm}(r) S_{lm}(\pi/2) S_{lm}(\theta) e^{\mp im(\phi - \Omega t)}, \tag{4.20}
\]
and the dissipative solution is then
\[
\Phi_{\text{dis}} = \frac{1}{2} (\Phi_+ - \Phi_-) = \frac{q}{2\pi u^t} \sum_{l,m} i R_{lm}(r) S_{lm}(\pi/2) S_{lm}(\theta) \sin [m(\phi - \Omega t)] . \tag{4.21}
\]

To elucidate this expression, note the \( m \to -m \) symmetry properties of its components. Eqs. (4.12) and (4.13) show that \( R_{l(-m)} = R_{lm}^* \). The \( S_{lm} \) were chosen to have the same sign as the Legendre functions \( P_m^l \), so \( S_{lm}(\pi/2) S_{lm}(\theta) \). The sine function of course changes sign with \( m \). This all implies that the imaginary part of each mode of \( \Phi_{\text{dis}} \) (coming from the real part of \( R_{lm} \)) changes sign with \( m \) while the real part doesn’t. Since \( m = 0 \) does not contribute to the sum by virtue of the sine, the solution is real (like the retarded and advanced solutions), and more importantly comes completely from the imaginary parts of the radial functions, which, as solutions to the homogeneous radial equation, are smooth. We thus obtain the more transparent version of \( \Phi_{\text{dis}} \),
\[
\Phi_{\text{dis}} = -\frac{q}{2\pi u^t} \sum_{l,m} \text{Im} [R_{lm}(r)] S_{lm}(\pi/2) S_{lm}(\theta) \sin [m(\phi - \Omega t)] , \tag{4.22}
\]
each term of which is smooth at the particle. The force is given by minus the orthogonal projection of the gradient of this quantity (evaluated at the location of the particle), times the scalar charge \( q \). To wit,
\[
\frac{d}{d\tau} (p_\alpha) = -q (\delta_\beta^\alpha + u_\alpha u^\beta) \nabla_\beta \Phi_{\text{dis}} . \tag{4.23}
\]

The location of the particle is \( \phi = \Omega t \), so the \( r \) and \( \theta \) components vanish by virtue of the sine in Eq. (4.21). The \( t \) and \( \phi \) components give us the evolution of the conserved quantities \( E \) and \( L \). Because \( \nabla_t = -\Omega \nabla_\phi \), the projection operator is just the identity (ie, \( (\delta_\beta^\alpha + u_\alpha u^\beta) \nabla_\beta = \nabla_\alpha \)), and Eq. (4.23) yields
\[
\frac{dL}{dt} = \frac{-q^2}{2\pi \mu (u^t)^2} \sum_{l,m} m \text{Im} [R_{lm}(r_0)] S_{lm}(\pi/2)^2 \tag{4.24}
\]
\[
(\frac{dE}{dt} = \Omega \frac{dL}{dt} . \tag{4.25}
\]
We have used \( \frac{d}{d\tau} = u^t \frac{d}{dt} \) to rewrite for easy comparison with Eq. (4.18). Eqs. (4.18) and (4.24) must be equal, and furthermore the equality must be mode by mode, since the argument of section IV A is valid mode by mode. We can therefore write
\[
4\pi \text{Im} [R_{lm} (r_0)] = m \left[ 2M r_+(\Omega - \omega_+) |C_{lm}^+|^2 + \Omega |C_{lm}^-|^2 \right] , \tag{4.26}
\]
and in fact this relationship provides a convenient means to estimate the total numerical error involved in the computation of a given \( R_{lm} \).

**V. DISCUSSION**

**A. The Radial Functions**

1. **Computation**

The general scheme for the computation of the radial functions is presented in Section III. To summarize, one integrates in from infinity and out from the horizon, using asymptotic series to provide initial values, and then patches the two solutions together at the particle. Here we discuss our actual implementation of this method. There are two sources of error in the computation: the integration itself, and the initial value that starts it (gotten from the series). We pick a desired maximum error for the whole computation and make sure that the error from neither source exceeds this bound. For the integration, we use the Runga-Kutta Cash-Karp order 4/5 routine, as implemented in the GSL scientific library [21], which reports the error introduced at each step (note that we integrate in \( r \), not \( r^* \)).
The sum of the magnitudes of these errors is a (gross) upper bound on the total numerical error, and we adjust the parameters of the integrator until this falls below our desired maximum. We control the error introduced by the initial value by increasing the value’s accuracy until the result of the integration (at $r_0$) does not change to within the desired error. One can increase the accuracy either by increasing the number of terms in the series used or by moving further away from the particle, where the series are increasingly convergent. We chose to fix the number of terms and increase the distance. For the results accurate to six significant figures presented in this paper, we used fifteen terms from each series, allowing us to begin most infinity integrations somewhere between $r/M = 100$ and $r/M = 1000$, and most horizon integrations at distances of order $10^{-2}$ to $10^{-3}$ away from the horizon. The series appear to converge slowest (i.e., we must begin furthest from the particle) when the difference between $l$ and $|m|$ is largest. The patching of the solutions requires the calculation of the Wronskian; although this can be done at the single point $r_0$, we integrate a little past $r_0$ in either direction to have a range of values over which we can verify that the quantity $\Delta W(R_{lm}^\infty, R_{lm}^\infty)$ of Eq. (3.5) is constant (to the desired accuracy). Finally, at the end of each integration we check that Eq. (4.25) is satisfied (again to the desired accuracy).

2. Properties

Although the field they sum to is real, the radial functions themselves are complex-valued. At the particle, the real parts have cusps while the imaginary parts remain smooth (see Fig. 1 for a picture). This is because the delta function in our source is purely real. The imaginary parts can be thought of as solutions to the source-free wave equation, coupled to our problem via the radiative boundary conditions. Only these smooth functions contribute to the dissipative field (see Eq. (4.22)). Away from the particle, the radial functions can be understood in terms of the asymptotic series Eqs. (3.3) and (3.4). The first terms of these series describe the end behavior,

$$R_{lm}(r \to r_+) \sim e^{-im(\Omega - \omega_+)r^*},$$

$$R_{lm}(r \to \infty) \sim \frac{1}{r} e^{im\Omega r^*}.$$ (5.1)

To understand these we must ‘translate’ from $r^*$ to $r$. As $r \to \infty$, $r^* \to r$, so the behavior at infinity is simply a $1/r$ decaying sinusoid of frequency $m\Omega$. As $r \to r_+$, however, $r^* \to -\infty$. This means that the distance between points in $r$ is increasingly stretched by $r^*$ as one approaches the horizon. The distance between wave crests is constant in $r^*$, so in $r$ it is steadily decreasing. The effect is that wavefronts ‘pile up’ on the horizon, as in Fig. 2(a) below.

![Fig. 1: Two Radial Functions, for $a = 8M$, $r_0 = 7M$ (prograde orbit, $\Omega \approx .05$). The real parts have cusps at the position of the particle, while the imaginary parts are smooth everywhere. The oscillations in (b) are too small compared to the cusp to be seen (we show these below).](image-url)
FIG. 2: Closer looks at $R_{10,10}$ of fig. 1(b) At the horizon wave fronts pile up, and at infinity we have a decaying sinusoid.

B. Dissipative Self-force

The dissipative self-force is characterized by $d\mathcal{L}/dt$, calculated from Eq. (4.18) or equivalently Eq. (4.24). A particle in the adiabatic regime will inspiral along circular orbits according to this quantity until it begins a transition to a 'plunge' orbit near the inner-most stable circular orbit (ISCO), as discussed in [22]. Thus it is only meaningful to compute results for orbits outside the ISCO. The convergence of the $l, m$ sums that give $d\mathcal{L}/dt$ is quite good, until the particle gets too close to the black hole. For most cases, the ISCO is far enough out that all orbits of interest require a small number ($l = 10 - 15$) of modes to converge to six significant figures. However, as a co-rotating hole moves the ISCO inwards (see [23] for a nice discussion of circular orbits in Kerr), more modes can be required. For the extremal co-rotating case, the ISCO approaches the horizon at $r = M$ and the adiabatic regime involves orbits whose convergence is quite bad. For the $r = 2M$ orbit result shown in fig. 3 below, $l = 30$ modes were required for the six significant figures.

FIG. 3: Some values of $d\mathcal{L}/dt \times \mu / q^2$ for various orbits. Negative values of $\alpha$ represent retrograde orbits. Results for orbits within the inner-most stable circular orbit are unphysical and not shown. All results are accurate to the six significant figures displayed.
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VI. APPENDIX–THE RECURSION RELATIONS

The $C_n^+$ and $C_n^\infty$ of Eqs. 3.3 and 3.4 satisfy recursion relations, which were derived with the help of Mathematica. The procedure is to plug in the forms of Eqs. 3.3 and 3.4 to the radial equation 3.1, and rewrite the results with a single summation index, so that one derives an expression that must be zero (the equation is homogeneous away from the particle). These expressions are

$$\sum_{i=0}^{4} f_{i}^+ C_{n-i}^+ = 0 \quad (6.1)$$

$$\sum_{i=0}^{5} f_{i}^\infty C_{n-i}^\infty = 0, \quad (6.2)$$

with $f_{i}^+$ and $f_{i}^\infty$ given below (where we have defined $\Omega_+ \equiv \Omega - \omega_+$).

$$f_0^+ = \frac{-2m^2 M r_+ \Omega + 4M^2 \left[ n^2 - 2imr_+ \Omega_+ + 2m^2 Mr_+ \left( \Omega^2 - \Omega_+^2 \right) \right]}{a^2 \left[ -4n^2 + 8imMn\Omega_+ + m^2 \left( 1 - 4M^2 \left( \Omega^2 - \Omega_+^2 \right) \right) \right]}$$

$$f_1^+ = \frac{2r_+ \left( -1 + 3n - 2n^2 + \lambda + a^2 m^2 \Omega^2 \right) + 4ia^2 m \left( -1 + 2n \right) \Omega_+ - 16m^2 Mr_+ \left( \Omega^2 - \Omega_+^2 \right)}{+ 2M \left[ 1 + 2n^2 - \lambda + 2am^2 \Omega + 3a^2 m^2 \Omega^2 - 4imr_+ \Omega_+ - 4a^2 m^2 \Omega_+^2 + n \left( -3 + 6imr_+ \Omega_+ \right) \right]}$$

$$f_2^+ = \frac{2n^2 - \lambda - 5a^2 m^2 \Omega^2 + 12m^2 Mr_+ \Omega^2 - 4imM \Omega_+ + 10imr_+ \Omega_+ + 4a^2 m^2 \Omega_+^2}{- 12m^2 Mr_+ \Omega_+^2 + in \left[ 3i + 4m \left( M - 2r_+ \right) \Omega_+ \right]}$$

$$f_3^+ = \frac{2m \left[ -i \left( -2 + n \right) \Omega_+ + 2mr_+ \left( \Omega^2 - \Omega_+^2 \right) \right]}{f_4^+ = \frac{m^2 \left( \Omega^2 - \Omega_+^2 \right)}{}}$$

$$f_0^\infty = -2im \left( -1 + n \right) \Omega$$

$$f_1^\infty = 2 + n^2 - \lambda - 8imM \Omega - a^2 m^2 \Omega^2 + n \left( -3 + 4imM \Omega \right)$$

$$f_2^\infty = \frac{2 \left[ ia^2 m \left( 5 - 2n \right) \Omega + M \left( -10 + 9n - 2n^2 + \lambda - 2am^2 \Omega + a^2 m^2 \Omega^2 \right) \right]}{f_3^\infty = 4M^2 \left( -3 + n \right)^2 - a^4 m^2 \Omega^2 + a^2 \left( 18 + m^2 - 12n + 2n^2 - \lambda + 4imM \left( -3 + n \right) \Omega \right)}$$

$$f_4^\infty = 2a^2 \left( -4 + n \right) \left[ M \left( 7 - 2n \right) - ia^2 m \Omega \right]$$

$$f_5^\infty = a^4 \left( -5 + n \right) \left( -4 + n \right)$$
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