Diffeological vector pseudo-bundles

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September 11, 2015

Abstract

We consider a diffeological counterpart of the notion of a vector bundle (we call this counterpart a pseudo-bundle, although in the other works it is called differently; among the existing terms there are a “regular vector bundle” of Vincent and “diffeological vector space over X” of Christensen-Wu). The main difference of the diffeological version is that (for reasons stemming from the independent appearance of this concept elsewhere), diffeological vector pseudo-bundles may easily not be locally trivial (and we provide various examples of such, including those where the underlying topological bundle is even trivial). Since this precludes using local trivializations to carry out many typical constructions done with vector bundles (but not the existence of constructions themselves), we consider the notion of diffeological gluing of pseudo-bundles, which, albeit with various limitations that we indicate, provides when applicable a substitute for said local trivializations. We quickly discuss the interactions between the operation of gluing and typical operations on vector bundles (direct sum, tensor product, taking duals) and then consider the notion of a pseudo-metric on a diffeological vector pseudo-bundle.

MSC (2010): 53C15 (primary), 57R35 (secondary).

Introduction

Diffeology as a subject, introduced by Souriau in the 80’s [10, 11], belongs among various attempts made over the years to extend the usual setting of differential calculus and/or differential geometry. Many of these attempts appeared in the realm of mathematical physics, such as smooth structures `a la Sikorski or `a la Frölicher, and were motivated by the fact that many objects that naturally appear in, for example, noncommutative geometry, such as irrational tori, orbifolds, spaces of connections on principal bundles in Yang-Mills theory, and so on, are not smooth manifolds and cannot be easily treated by similar methods. A rather comprehensive summary of various attempts of extending, in a consistent way, the category of smooth manifolds can be found in [12].

Among such attempts, the diffeology has (at least) the virtue of being an essentially very simple construction, possibly appealing to those not very experienced with heavy analytic matters and more pointed towards geometric setting. As an instance, one finds, at first glance, that many concepts extend to this category almost verbatim (in any case, in an obvious way). But on the other hand, once again with very little effort, one notices, be that with simple examples or a single construction coming from elsewhere, that the trivial extension is unsatisfactory.

The concept of the vector bundle is an excellent instance of this. A trivial extension of this concept just requires substituting each mentioning of a smooth map with “diffeologically smooth”, and maybe choosing a diffeology on the fibre (but then, in the finite-dimensional case, which we are limiting ourselves to, there is a standard choice). Yet, it is easily seen that this is not sufficient; one way to to explain why is to point to the internal tangent bundles of Christensen-Wu [1]. These, frequently enough, turn out not to be vector bundles at all, for the simple reason that they do not always have the same fibre (for reasons related to the underlying topology of the base space), yet, they are more than legitimate candidates to be tangent bundles.

Thus, the aim of this note is to take a closer look at the objects of this type, starting from attempting to define with precision what “of this type” means (apart from the very general definition given in [1]). In particular, we explore the path of constructing these “pseudo-bundles” (as we call them) by a kind of
The specific issues  We start from a specific example, due to Christensen-Wu ([1], Example 4.3); it has a merit that it points out right away what, very informally, can be described as the first main contribution of diffeology to the mathematical landscape: the possibility to treat topological spaces which are in no way smooth manifolds (not even having a manifold’s topology), as if they were such, and to do so in a uniform manner. Namely, the example cited is the space $X$ that consists of the coordinate axes in $\mathbb{R}^2$; it has a kind of smooth structure, an atlas, if you wish, where the charts are restrictions of all usual $\mathbb{R}^2$-valued maps. This sort of structure is an example of a diffeology on a space (in the sense of a diffeological structure). For such, a so-called internal tangent bundle ([1]) is defined, and for this specific $X$ it reveals itself to be something very similar to a typical vector bundle (in fact, it is one everywhere outside the origin, with fibre $\mathbb{R}$), but it is not one because the internal tangent space at the origin is $\mathbb{R}^2$.

This example is significant for more than one reason. Just to mention two specific ones, we observe, first, that the tangent space at the origin being 2-dimensional, while being 1-dimensional elsewhere, seems to be a necessity indeed, since it reflects the usual topological structure of the space; anything else would be counterintuitive. Secondly, the internal tangent bundles possessing a certain multiplicativity property, just starting from this specific space and taking its direct product, not even with itself, but with any $\mathbb{R}^n$, we shall find similar examples, in any dimension, and with a more complicated structure. These observations bring us to the next paragraph.

The aims  What we wish to do in this paper, is to take an abstract look at the vector “bundles” of the above-described type (the precise definition, that of diffeological vector space over $X$, is given later in the paper; it is however almost that of a typical vector bundle, but does not include the requirement of being locally trivial). Informally speaking, we would like to give a more concrete characterization of such pseudo-bundles; to this end, we consider various types of topological operations on them, such as gluing them together. This point of view, which is not entirely general, has to do with possible presentations of the base space as a “simplicial complex”; by this, we do not mean (not necessarily) a simplicial structure in the strict sense, but rather a decomposition of the base space into copies of Euclidean spaces (of different dimensions). The hope is to arrive to some kind of local description, which would be a diffeological counterpart of local trivializations for usual vector bundles.

Acknowledgments  Changing fields and starting anew elsewhere is never easy; and life, in all its complexity, may or may not collaborate with such an endeavour (and frequently does not). At such a moment, especially at the start but even further, it is of the greatest value whoever, at whatever circumstance, shows a support towards this struggle of yours, be that an encouragement a posteriori, or a simple comment that changing things is good. I take advantage of this new piece of work to heartily thank two people who, to me, such support did show, Prof. Riccardo Zucchi and Dr. Elisabetta Chericoni.

1 Diffeology and diffeological vector spaces

To make the paper self-contained, we collect here the definitions of all the main objects that appear in the sequel.

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1 A disclaimer is necessary at this point. First of all, the above is just an opinion by the author. Second, of course manifolds with corners (for instance) already fall under the above description, and are already treated by different methods. The distinction being made is, maybe, that diffeology applies to a much wider range of objects, and has a different point of view on them.

2 This reason is pretty much a statement of the obvious.

3 The usual vector bundle operations, such as taking the direct sum, the tensor product, and the dual bundle were already described in [13]; we briefly recall them.

4 We note right away that this does not yield a satisfactory construction right away; indeed, the diffeology that arises immediately from such gluing, is too much related to the specific decomposition used, and this is one issue that needs to be dealt with.
1.1 Diffeologies on sets

The basic notion is that of a diffeological space and a diffeology on it, together with some particular types of maps between diffeological spaces; we follow [6].

The concept  The definition of a diffeological space and its diffeological structure (or, briefly, its diffeology) is as follows.

Definition 1.1. (11) A diffeological space is a pair \((X, D_X)\) where \(X\) is a set and \(D_X\) is a specified collection of maps \(U \to X\) (called plots) for each open set \(U\) in \(\mathbb{R}^n\) and for each \(n \in \mathbb{N}\), such that for all open subsets \(U \subseteq \mathbb{R}^n\) and \(V \subseteq \mathbb{R}^m\) the following three conditions are satisfied:

1. (The covering condition) Every constant map \(U \to X\) is a plot;
2. (The smooth compatibility condition) If \(U \to X\) is a plot and \(V \to U\) is a smooth map (in the usual sense) then the composition \(V \to U \to X\) is also a plot;
3. (The sheaf condition) If \(U = \bigcup_i U_i\) is an open cover and \(U \to X\) is a set map such that each restriction \(U_i \to X\) is a plot then the entire map \(U \to X\) is a plot as well.

When the context permits, instead of \((X, D_X)\) we simply write simply \(X\).

Definition 1.2. (11) Let \(X\) and \(Y\) be two diffeological spaces, and let \(f : X \to Y\) be a set map. We say that \(f\) is smooth if for every plot \(p : U \to X\) of \(X\) the composition \(f \circ p\) is a plot of \(Y\).

The diffeological counterpart of an isomorphism between diffeological spaces is (expectedly) called a diffeomorphism; there is a typical notation \(C^\infty(X, Y)\) which denotes the set of all smooth maps from \(X\) to \(Y\). An obvious example of a diffeological space is any smooth manifold, whose diffeology consists of all usual smooth maps; then a diffeomorphism in the diffeological sense is the same thing as a diffeomorphism in the usual sense.

The D-topology  There is a canonical topology underlying every diffeological structure on a given set, the so-called D-topology\(^5\). It is defined \(^6\) as the final topology on a diffeological space \(X\) induced by its plots, where each domain is equipped with the standard topology. To be more explicit, if \((X, D_X)\) is a diffeological space then a subset \(A\) of \(X\) is open in the D-topology of \(X\) if and only if \(p^{-1}(A)\) is open for each \(p \in D_X\); such subsets are called D-open. In the case of a smooth manifold with the standard diffeology, the D-topology is the same as the usual topology on the manifold, and this is frequently the case also for non-standard diffeologies. This is due to the fact that, as established in [2], Theorem 3.7, the D-topology is completely determined smooth curves, in the sense that a subset \(A\) of \(X\) is D-open if and only if \(p^{-1}(A)\) is open for every \(p \in C^\infty(\mathbb{R}, X)\).

Comparing diffeologies  In a way somewhat similar as it occurs for the set of all possible topologies on a given set \(X\), the set of all possible diffeologies on \(X\) is partially ordered by inclusion. Specifically, a diffeology \(D\) on \(X\) is said to be finer than another diffeology \(D'\) if \(D \subseteq D'\), while \(D'\) is said to be coarser than \(D\). Among all diffeologies, there is the finest one (the natural discrete diffeology, which consists of all locally constant maps \(U \to X\)) and the coarsest one (which consists of all possible maps \(U \to X\), for all \(U \subseteq \mathbb{R}^n\) and for all \(n \in \mathbb{N}\) and is called the coarsest diffeology). In other words, the set of all diffeologies forms a complete lattice.

Constructing diffeologies by bounds  The above-mentioned structure of a lattice on the set of all diffeologies on a given \(X\) is frequently employed when constructing (or defining) a desired diffeology, for instance, one that contains a given plot, or one that includes only plots that, as maps, enjoy a certain specified property. For such restricted sets of diffeologies, it is frequently possible to claim the existence of the smallest/finest (or the largest/coarsest) diffeology among them; some of the definitions that follow (for example, those of the sum diffeology and of the product diffeology) are instances of this.

\(^5\)A frequent restriction for the choice of a diffeology on a given topological space is that the corresponding D-topology coincide with the given one.
The generated diffeology  This is a particularly important, from the practical point of view, at least, instance of the above-mentioned use of bounds to construct a diffeology. We stress one observation that trivially follows the concept of a generated diffeology: for any set $X$ and any map $p : U \to X$ defined on a domain $U \subset \mathbb{R}^k$ (and for any $k$), there is a diffeology on $X$ for which $p$ is a plot; that is, any map can be seen as a smooth map in the diffeological setting.\[6\]

Let us now state the precise definition. Given a set $X$ and a set of maps $A = \{U \to X\}$, all defined on some domains of some $\mathbb{R}^m$’s, there exists the finest diffeology on $X$ that contains $A$. This diffeology is called the diffeology generated by $A$.

Pushforwards and pullbacks of diffeologies  Let $X$ be a diffeological space, $X'$ an arbitrary set, and let $f : X \to X'$ be any map. Then there exists a finest diffeology on $X'$ that makes the map $f$ smooth; this diffeology is called the pushforward of the diffeology of $X$ by the map $f$ and is denoted by $f_*(D)$, where $D$ stands for the diffeology of $X$. Furthermore, if we have a reverse situation, i.e., if $X$ is just a set and $X'$ is a diffeological space with diffeology $D'$, then there is the pullback of the diffeology $D'$ by a given map $f : X \to X'$: it is the coarsest diffeology on $X$ such that $f$ is smooth. The pullback diffeology is denoted by $f^*(D')$.

The quotient diffeology  A quotient of a diffeological space is always a diffeological space\[7\] for a canonical choice of a diffeology on the quotient. Namely, let $X$ be a diffeological space, let $\cong$ be an equivalence relation on $X$, and let $\pi : X \to Y := X/ \cong$ be the quotient map; the quotient diffeology on $Y$ is the pushforward of the diffeology of $X$ by the natural projection (which is automatically smooth). It can also be described explicitly as follows: $p : U \to Y$ is a plot for the quotient diffeology if and only for each point in $U$ there exist a neighbourhood $V \subseteq U$ and a plot $\tilde{p} : V \to X$ such that $\tilde{p}|_V = \pi \circ p$.

The subset diffeology, inductions and subductions  Let $X$ be a diffeological space, and let $Y \subseteq X$ be its subset. The subset diffeology on $Y$ is the coarsest diffeology on $Y$ making the inclusion map $Y \hookrightarrow X$ smooth. It consists of all maps $U \to Y$ such that $U \to Y \hookrightarrow X$ is a plot of $X$. This notion is frequently used in practice and makes part of further definitions, such as the following ones: for two diffeological spaces $X, X'$ a smooth map $f : X' \to X$ is called an induction if it induces a diffeomorphism $X \to \text{Im}(f)$, where $\text{Im}(f)$ has the subset diffeology of $X$; a map $f : X \to X'$ is said to be a subduction if it is surjective and the diffeology $D'$ of $X'$ is the pushforward of the diffeology $D$ of $X$.

Disjoint sums and products  Let $\{X_i\}_{i \in I}$ be a collection of diffeological spaces, where $I$ is a set of indices. The sum of $\{X_i\}_{i \in I}$ is defined as

$$X = \coprod_{i \in I} X_i = \{(i, x) | i \in I \text{ and } x \in X_i\}.$$ 

The sum diffeology on $X$ is the finest diffeology such that each natural injection $X_i \to \coprod_{i \in I} X_i$ is smooth; it consists of plots that locally are plots of one of the components of the sum. The product diffeology $D$ on the product $\prod_{i \in I} X_i$ is the coarsest diffeology such that for each index $i \in I$ the natural projection $\pi_i : \prod_{i \in I} X_i \to X_i$ is smooth; locally, it consists of tuples of plots of all the components of the product.

Functional diffeology  Let $X, Y$ be two diffeological spaces, and let $C^\infty(X, Y)$ be the set of smooth maps from $X$ to $Y$. Let $\text{ev}$ be the evaluation map, defined by

$$\text{ev} : C^\infty(X, Y) \times X \to Y \text{ and } \text{ev}(f, x) = f(x).$$

The functional diffeology is the coarsest diffeology on $C^\infty(X, Y)$ such that this evaluation map is smooth.

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6Such breadth might have its own disadvantages, of course.

7Unlike smooth manifolds, whose quotients frequently are not manifolds at all.
1.2 Diffeological vector spaces

The concept in itself is quite straightforward: it is a set $X$ that is both a diffeological space and a vector space such that the operations are smooth (with respect to the diffeology).

The concept and some basic constructions Let $V$ be a vector space over $\mathbb{R}$. A vector space diffeology on $V$ is any diffeology of $V$ such that the addition and the scalar multiplication are smooth, that is,

$$[(u, v) \mapsto u + v] \in C^\infty(V \times V, V) \quad \text{and} \quad [(\lambda, v) \mapsto \lambda v] \in C^\infty(\mathbb{R} \times V, V),$$

where $V \times V$ and $\mathbb{R} \times V$ are equipped with the product diffeology. A diffeological vector space over $\mathbb{R}$ is any vector space $V$ over $\mathbb{R}$ equipped with a vector space diffeology.

The following observation could be useful to clarify the concept. Since the constant maps are plots for any diffeology and the scalar multiplication is smooth with respect to the standard diffeology of $\mathbb{R}$, any vector space diffeology on a given $V$ includes maps of form $f(x)v$ for any fixed $v \in V$ and for any smooth map $f: \mathbb{R} \to \mathbb{R}$. Next, since the addition is smooth, any vector space diffeology includes all finite sums of such maps. This immediately implies that any vector space diffeology on $\mathbb{R}^n$ includes all usual smooth maps (since they write as $\sum_{i=1}^n f_i(x)e_i$).

All the usual constructions of linear algebra, such as spaces of (smooth) linear maps, products, subspaces, and quotients, are present in the category of diffeological vector spaces. Obviously, given two diffeological vector spaces $V$ and $W$, one speaks of the space of smooth linear maps between them; this space is denoted by $L^\infty(V, W)$ and is defined simply as:

$$L^\infty(V, W) = L(V, W) \cap C^\infty(V, W);$$

this is an $\mathbb{R}$-linear subspace of $L(V, W)$ and is a priori smaller than the whole space $L(V, W)$. A subspace of a diffeological vector space $V$ is a vector subspace of $V$ endowed with the subset diffeology. It is easy to see (see Section 3.5) that if $V$ is a diffeological vector space and $W \subseteq V$ is a subspace of it then the quotient $V/W$ is a diffeological vector space with respect to the quotient diffeology.

The direct sum/product of diffeological vector spaces Let $\{V_i\}_{i \in I}$ be a family of diffeological vector spaces. Consider the usual direct sum $V = \bigoplus_{i \in I} V_i$ of this family; then $V$, equipped with the product diffeology, is a vector space.

Euclidean structure on diffeological vector spaces The notion of a Euclidean diffeological vector space does not differ much from the usual notion of the Euclidean vector space. A diffeological space $V$ is Euclidean if it is endowed with a scalar product that is smooth with respect to the diffeology of $V$ and the standard diffeology of $\mathbb{R}$; that is, if there is a fixed map $(\cdot, \cdot): V \times V \to \mathbb{R}$ that has the usual properties of bilinearity, symmetricity, and definite-positiveness and that is smooth with respect to the diffeological product structure on $V \times V$ and the standard diffeology on $\mathbb{R}$. We will speak in more detail of this later on, but it is worthwhile pointing out right away that, although many diffeological vector spaces admit plenty of smooth bilinear symmetric forms, a finite-dimensional diffeological vector space admits a smooth scalar product if and only if it is diffeomorphic to some $\mathbb{R}^n$ with the standard diffeology (see Section 3.5). For other finite-dimensional diffeological vector spaces a kind of “minimally degenerate” smooth symmetric bilinear form can be considered (see Section 3.5).

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8Note that $\mathbb{R}$ has standard diffeology here, a fact that has significant implications for what a vector space diffeology could be; the most obvious of those is that the discrete diffeology is never a vector space diffeology (except for the zero space), see the explanation below.

9This is not the case for a non-vector space diffeology of $\mathbb{R}^n$; the simplest example is the already-mentioned discrete diffeology, for which the scalar multiplication is not smooth. A more intricate example is that of the so-called wire diffeology, one generated by the set $C^\infty(\mathbb{R}, \mathbb{R}^n)$. For this diffeology, the scalar multiplication is smooth, but the addition is not.

10It is very easy to give examples where it is strictly smaller; consider, for instance, $\mathbb{R}^n$ with the vector space diffeology generated by a plot of form $\mathbb{R} \ni x \mapsto f(x)e_a$, where $f(x)$ is any non-differentiable function. Then the usual linear dual of $e_a$ is linear but not smooth.

11On the other hand, obtaining a finite-dimensional diffeological vector space where the only smooth bilinear form is the zero form is also easy: take $\mathbb{R}^n$ and $n$ non-differentiable functions $f_1(x), \ldots, f_n(x)$. Then endowing $\mathbb{R}^n$ with the vector space diffeology generated by the $n$ plots $\mathbb{R} \ni x \mapsto f_i(x)e_i$ yields a diffeological vector space where the only smooth (multi)linear map is the zero map.
Fine diffeology on vector spaces  The fine diffeology on a vector space \( \mathbb{R} \) is the finest vector space diffeology on it; endowed with such, \( V \) is called a fine vector space. Note that any linear map between two fine vector spaces is smooth ([6], 3.9). An example of a fine vector space is \( \mathbb{R}^n \) with the standard diffeology, i.e., one that consists of all the usual smooth maps with values in \( \mathbb{R}^n \).\(^{13}\)

The dual of a diffeological vector space  The definition of the diffeological dual was first given in [13] and then in [14]; this concept is a very natural (and obvious) one:

**Definition 1.3.** Let \( V \) be a diffeological vector space. The **diffeological dual** of \( V \), denoted by \( V^* \), is the set \( L^\infty(V, \mathbb{R}) \) of all smooth linear maps \( V \to \mathbb{R} \).

The resulting space is a diffeological vector space for the functional diffeology; in general it is not isomorphic to \( V \). Indeed, as shown in [9], the functional diffeology of the diffeological dual of a finite-dimensional diffeological vector space is always the standard one (in particular, the dual of any space is a fine space). Hence, in the finite-dimensional case the equality \( L^\infty(V, \mathbb{R}) = L(V, \mathbb{R}) \) holds, and so \( V^* \) and \( V \) are isomorphic, if and only if \( V \) is a standard space.\(^{13}\) The matters become less straightforward in the infinite-dimensional case, which in this paper we do not consider.

The tensor product  The definition of the diffeological tensor product was given first in [13] and then in [14] (see Section 3); we recall the latter version. Let \( V_1, \ldots, V_n \) be diffeological vector spaces, let \( T : V_1 \times \ldots \times V_n \to V_1 \otimes \ldots \otimes V_n \) be the universal map onto their tensor product as vector spaces, and let \( Z \leq V_1 \times \ldots \times V_n \) be the kernel of \( T \). The tensor product \( \mathcal{D}_{\otimes} \) on \( V_1 \otimes \ldots \otimes V_n \) is the quotient diffeology on \( V_1 \otimes \ldots \otimes V_n = (V_1 \times \ldots \times V_n)/Z \) coming from the product diffeology on \( V_1 \times \ldots \times V_n \). The diffeological tensor product thus defined possesses the usual universal property (see [13], Theorem 2.3.5):

\[
L^\infty(V_1 \otimes \ldots \otimes V_n, W) \cong \text{Mult}^\infty(V_1 \times \ldots \times V_n, W),
\]

where \( \text{Mult}^\infty(V_1 \times \ldots \times V_n, W) \) is the space of all smooth (with respect to the product diffeology) multilinear maps \( V_1 \times \ldots \times V_n \to W \).

1.3 Diffeological bundles and pseudo-bundles

A smooth surjective map \( \pi : T \to B \) is a **fibration** if there exists a diffeological space \( F \) such that the pullback of \( \pi \) by any plot \( p \) of \( B \) is locally trivial, with fibre \( F \). The latter condition has the obvious meaning, namely that there is a cover of \( B \) by a family of D-open sets \( \{U_i\}_{i \in I} \) such that the restriction of \( \pi \) over each \( U_i \) is trivial with fibre \( F \). For the sake of completeness we mention that there is also another definition of a diffeological fibre bundle ([6], 8.8), which involves the notion of a **diffeological groupoid**; we use the definition given above since it is more practical.

One point that should be stressed right away (even if it is quite obvious) is that the condition of local triviality that \( \pi^{-1}(U_i) \) is diffeomorphic to \( U_i \times F \) as diffeological spaces. The reason why we stress this is that it might easily happen, and it does even for very simple examples (which we present below), that the two spaces are homeomorphic, even diffeomorphic, in the usual sense, but they are not in the diffeological sense. It might even happen that a bundle trivial from the usual point of view is not even locally trivial with respect to diffeologies involved (with, for instance, but not only, an isolated fibre carrying a different diffeology).

**Principal diffeological fibre bundles**  Let \( X \) be a diffeological space, and let \( g : X \to G \) be a smooth action of a diffeological group \( G \) on \( X \), that is, a smooth homomorphism from \( G \) to \( \text{Diff}(X) \). Let \( F \) be the **action map**:

\[
F : X \times G \to X \times X \text{ with } F(x, g) = (x, g_X(x)).
\]

Then the following is true (see the Proposition in Section 8.11 of [6]): if \( F \) is an induction then the projection \( \pi \) from \( X \) to its quotient \( X/G \) is a diffeological fibration, with the group \( G \) as fibre. In this

\(^{12}\)It is easy to see that this set is indeed a (vector space) diffeology. Furthermore, it is the finest one, since, as we have already observed above, it is contained in any other vector space diffeology.

\(^{13}\)Note also that, as shown in [3], Proposition 4.4, if \( V^* \) and \( V \) are isomorphic then they are also diffeomorphic.
case we say that the action of \( G \) on \( X \) is **principal**. Now, if a surjection \( \pi : X \to Q \) is equivalent to \( \text{class} : X \to G/H \), that is, if there exists a diffeomorphism \( \varphi : G/H \to Q \) such that \( \pi = \varphi \circ \text{class} \), we shall say that \( \pi \) is a **principal fibration**, or a **principal fibre bundle**, with structure group \( G \).

**Diffeological vector space over a given \( X \)** What would be a verbatim extension of the concept of a vector bundle into the diffeological setting, which in particular would be a partial case of the definition from the previous paragraph, does not turn out to be sufficient (some reasons for this have been outlined in the introduction). This prospective obvious extension is therefore replaced by the following concept.

**Definition 1.4.** ([P], Definition 4.5) Let \( X \) be a diffeological space. A **diffeological vector space over** \( X \) is a pair \((V, \pi)\) consisting of a diffeological space \( V \) and a smooth map \( \pi : V \to X \) such that each of the fibres \( \pi^{-1}(x) \) is endowed with a vector space structure for which the following properties hold: 1) the addition map \( V \times_X V \to V \) is smooth with respect to the diffeology of \( V \) and the subset diffeology on \( V \times_X V \) coming from the product diffeology on \( V \times V \); 2) the scalar multiplication map \( \mathbb{R} \times V \to V \) is smooth for the product diffeology on \( \mathbb{R} \times V \); 3) the zero section \( X \to V \) is smooth.

Note that if \( X \) is a point, \( V \) is just a diffeological vector space. Furthermore, if \( V \) is a diffeological vector space over \( X \) then each fibre \( p^{-1}(x) \) endowed with the subset diffeology is automatically a diffeological vector space.

**Examples** To illustrate the concept just introduced, we provide two examples. The first one deals with the case of the most standard fibration, that of \( \mathbb{R}^n \) over \( \mathbb{R}^k \) (with \( k < n \)) via the projection onto a subset of the coordinates of the former; the second one is more intricate and is specific to the diffeological version.

**Example 1.5.** Let \( V = \mathbb{R}^n \), and let \( \{e_1, \ldots, e_n\} \) be its canonical basis. Denote by \( X \) the subspace generated by the first \( k \) vectors of this basis, and let \( \pi \) be the projection of \( V \) onto \( X \) (i.e., onto the first \( k \) coordinates). Obviously, the pre-image \( \pi^{-1}(x) \) of any point \( x \in X \) has a natural vector space structure, which is obtained by representing \( \mathbb{R}^n \) as the direct product \( \mathbb{R}^k \times \mathbb{R}^{n-k} \); the fibre \( \pi^{-1}(x) \) has then the form \( \{x\} \times \mathbb{R}^{n-k} \), and the vector space structure is inherited from the second factor.

The space \( X \) being canonically identified with \( \mathbb{R}^k \), we endow it with the standard diffeology. Consider the pullback to \( V \) of this diffeology by the map \( \pi \). Writing a plot \( p : U \to V \) of this diffeology as \( p(u) = (p_1(u), \ldots, p_n(u)) \), and recalling that, one, \( \pi \circ p \) is a plot of \( X \) (so it is a usual smooth map) and, second, the pullback diffeology is the coarsest one with the latter property, we conclude that \( p_k, \ldots, p_n \) must be usual smooth \( \mathbb{R} \)-valued maps, while \( p_{k+1}, \ldots, p_n \) can be any maps. In particular, every fibre has coarse diffeology and has the usual smooth vector space structure.

The second example we provide, stems from the fact that the condition of the local triviality is absent from the definition of a diffeological vector space over a given \( X \); it shows that, in addition to a large diffeology on the fibres, the definition as given allows for topologically complicated (in the sense of the usual topology) total spaces.

**Example 1.6.** Let us describe an example of a diffeological vector space over \( X = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\} \), the union of coordinate axes (the space that appears in the Christensen-Wu example). Let us construct \( V \) as follows. Take three copies of \( \mathbb{R}^2 \) (to distinguish among them, we denote them by \( V_1, V_2, \) and \( V_0 \)). Consider \( f_1 : V_1 \supset \{(0, y)\} \to V_0 \) acting by \( f_1(0, y) = (0, y) \), and, analogously, \( f_2 : V_2 \supset \{(x, 0)\} \to V_0 \) acting by \( f_2(x, 0) = (x, 0) \). Set \( V \) to be the result of gluing of \( V_1 \) and \( V_2 \) to \( V_0 \) via the maps \( f_1 \) and \( f_2 \) respectively.

The resulting space \( V \) has a natural projection \( \pi \) onto \( X := \{(x, 0)\} \cup \{(0, y)\} \), defined by sending every point of \( V_1 \) to its projection on the \( x \)-axis: \( (x, y) \mapsto (x, 0) \), every point of \( V_2 \) to its projection on the \( y \)-axis: \( (x, y) \mapsto (0, y) \), and the whole of \( V_0 \) to the origin: \( (x, y) \mapsto (0, 0) \). Note that \( \pi \) is well-defined.

14 Note that this example is somewhat artificial; the pullback diffeology, as we have just reminded, is the largest possible for which we have a smooth fibration, and presumably the fibrations that arise from, say, applications would carry a smaller (more sensible) diffeology. We put this example to illustrate the extremes of the definition as stated.

15 What we mean here is the usual topological gluing: given two topological spaces \( X \) and \( Y \) and a continuous map \( f : X \to Y \), the result of gluing \( X \) to \( Y \) along \( f \) is the space \( X \cup f Y := \{(x \cup y) \mid x = f(z)\} \).
with respect to the gluing. Observe also that the pre-image of any point of \( X \) has an obvious vector space structure (obtained in the same way as in the previous example).

Now, the space \( X \) is endowed with the subset diffeology \( \mathcal{D}_X \) of \( \mathbb{R}^2 \). Let us now consider the pullback \( \mathcal{D}_V \) of this diffeology by the map \( \pi \). We now show that the subset diffeology on the fibres is the coarse diffeology.

Note first of all that, since \( \pi \circ \rho \) is a plot of \( \mathcal{D}_X \), the image of \( \rho \) is contained in either \( V_1 \cup V_0 \subset V \) or \( V_2 \cup V_0 \subset V \), but not in both. Furthermore, if we assume that its image is wholly contained in \( V_0 \) then it can be any map with values in \( \mathbb{R}^2 \). Thus, the fibre at the origin has the coarse diffeology.

Consider now \( (x, 0) \in X \) with \( x \neq 0 \); let \( \rho : U \to \mathbb{R}^2 = V_1 \) be a plot for the pullback diffeology for which we assume that \( \text{Im}(\rho) \) intersects \( \pi^{-1}(x, 0) \) and is wholly contained in \( V_1 \subset V \). Write \( \rho(u) = (p_1(u), p_2(u)) \). We have \( \pi \circ \rho(u) = p_1(u) \), which by definition of the pullback diffeology and that of the diffeology of \( X \) must be an ordinarily smooth \( \mathbb{R} \)-valued map; no condition is however imposed on \( p_2 \). Actually, since the pullback diffeology is the coarsest one, we should be able to allow for \( p_2 \) to be any \( \mathbb{R} \)-valued map.

An analogous conclusion can be drawn for any plot \( q : U \to \mathbb{R}^2 = V_2 \), i.e., a plot of \( \mathcal{D}_V \) whose image is contained in \( V_2 \subset V \). Namely, if we write \( q(u) = (q_1(u), q_2(u)) \) then \( q_2 \) must be a usual smooth \( \mathbb{R} \)-valued map, while \( q_1 \) could be any map. The observations thus made show that every fibre has the coarse diffeology.

The conclusion drawn in the example is a consequence of endowing the total space \( V \) with the pullback diffeology, which by definition is the coarsest diffeology such that the projection is smooth. Such a diffeology is in general too big; in fact, it is reasonable to at least restrict ourselves to continuous, in the ordinary sense, maps (mostly because we wish to preserve the existing topology of the spaces under consideration). Nonetheless, for the moment it serves us to illustrate the a priori extension of the concept of a diffeological vector space over a given base.

2 Particular cases of vector spaces and vector “bundles”

Before turning to our main subject, we examine here several specific examples, first of vector spaces, then of the diffeological (counterpart of) vector “bundles” which, although simple, are peculiar to diffeology. We have already given two of such examples in the previous section, as a preliminary illustration; the further examples that we are providing now attempt to point towards a coherent picture, starting from ones that are probably always expected to be found among the most basic constructions (they will also serve in the later sections to illustrate the constructions carried out therein).

2.1 The choice of terminology

In the rest of the paper we opt for the term **diffeological vector pseudo-bundle** to denote the same object that is called a regular vector bundle in \[8\] and a diffeological vector space over \( X \) in \[1\] (the definition of which we cited in the previous section). We avoid the former term to distinguish our objects of interest from true diffeological vector bundles (that are locally trivial), while the term of Christensen-Wu can be confused with a diffeological vector space proper (that is, a vector space endowed with a vector space diffeology); besides, it requires to introduce a notation for the base space, something which on occasion might be superfluous or cumbersome. The choice that we favour, that of term pseudo-bundle, also underlines the fact in many natural examples (although that is by no means necessary) are objects are indeed true vector bundles outside of, say, zero measure subset (of the base).

\[16\]It is not immediately clear whether this pullback would necessarily make \( V \) into a diffeological vector space over \( X \); however, whichever the case, there is a standard way, due to Christensen-Wu \[1\], to make it into such, described later in the paper.

\[17\]Note that \( V \) thus being a diffeological vector space over \( X \), namely, carrying a diffeology with respect to which the addition and scalar multiplication on single fibres are smooth, is just a consequence of all fibres having coarse subset diffeology (the coarse diffeology is automatically a vector space diffeology, for any vector space structure).

\[18\]We put the quotations marks, since, as we have seen already and are about to show in more detail, frequently they are not bundles in the usual sense; rightly so, since treating such objects, in a manner consistent with the standard case, is among the aims of diffeology.
2.2 Examples of vector spaces

We start by providing several examples of finite-dimensional diffeological vector spaces that do not have the standard diffeology (but whose diffeology is not particularly large; typically, we take the finest diffeology that contains the standard one, as it must, plus one extra map). These examples will also come into play when we turn to consider vector pseudo-bundles.

Example 2.1. Let \( V = \mathbb{R}^n \), and let \( \{e_1, \ldots, e_n\} \) be its canonical basis. Let \( p : \mathbb{R} \to V \) be the map acting by \( p(x) = |x|e_n \) endow \( V \) with the finest vector space diffeology for which \( p \) is a plot. This example has already been considered in [2] (see also [3]); we briefly recall that for this choice of \( V \), its diffeological dual \( V^* \) has dimension \( n-1 \), as one can see writing an arbitrary element of \( V^* \) as \( \sum_{i=1}^{n} a_i e^i \). Indeed, taking the composition of this sum with \( p \), one obtains the map \( x \mapsto a_n |x| \); this needs to be an ordinary smooth map \( \mathbb{R} \to \mathbb{R} \), which implies that \( a_n = 0 \). Furthermore, the diffeology of \( V \) is obviously not a standard one, which, as has already been mentioned, implies that \( V \) does not admit a smooth scalar product. This can easily be seen directly: if \( A \) is an \( n \times n \) symmetric matrix that defines a smooth bilinear form on \( V \) than composing this form with the plot \( (e_n, p) \) of the product \( V \times V \), where \( v \in V \) is an arbitrary vector and \( e_v : \mathbb{R} \to V \) is the constant map \( e_v(x) = v \), one sees that \( e_n \) is an eigenvector of \( A \) with eigenvalue 0.

The example just described is a kind of basic example for us; we choose it as a simplest possible instance of a diffeological vector space that carries a non-standard diffeologies. It is this example which we will turn to most frequently (usually for such-and-such fixed \( n \)) when we need to illustrate some construction that be specific to diffeology.

There is a variation on this example, which in fact is only different in appearance

\[\text{nevertheless, we describe it for illustrative purposes.}\]

Example 2.2. Let \( V = \mathbb{R}^3 \); endow it with the finest vector space diffeology generated by the map \( p : \mathbb{R} \to V \) acting by \( p(x) = (0, |x|, |x|) \). The space we obtain is quite similar to the previous example; in fact, it becomes precisely the same if, instead of taking the canonical basis, we take any other basis where the third vector is the vector \( e_2 + e_3 \). On the other hand, this example allows to illustrate easily that for diffeological vector spaces there exists a difference between smooth and non-smooth decompositions into direct sums. Namely, if the underlying vector space \( V \) decomposes into a direct sum of two of its vector subspaces, the corresponding direct sum diffeology on \( V \) obtained from the subset diffeologies on \( V_1 \) and \( V_2 \) may be finer than the initial diffeology of \( V \). This is precisely the case for \( V \) in this example (see [2] for details), if we take \( V_1 = \text{Span}(e_1, e_2) \) and \( V_2 = \text{Span}(e_3) \); for both of these the subset diffeology is the standard one, and therefore so is the sum diffeology of their direct sum. On the other hand, the initially chosen diffeology on \( V \) is obviously not the standard one.

Our third example is different from the previous two in that it has a kind of two-dimensional nature; we will use it to illustrate some “non-splitting” (in the purely diffeological sense) properties.

Example 2.3. Let now \( V = \mathbb{R}^2 \); consider a map \( p : \mathbb{R} \to V \) given by the following rule: \( p(x) = (x, 0) \) for \( x \geq 0 \) and \( p(x) = (0, |x|) \) for \( x < 0 \). Let \( \mathcal{D}_V \) be the finest vector space diffeology on \( V \) generated by the map \( p \).\(^{19}\) The differences of this space with respect to the standard \( \mathbb{R}^2 \) are similar to those of the (instance for \( n = 2 \) of) space that appears in the previous example. Specifically, if \( f = (a_1, a_2) \) is a smooth linear map then composing it with \( p \) we get \((f \circ p)(x) = a_1 x \) for \( x \geq 0 \) and \((f \circ p)(x) = a_2 |x| = -a_2 x \) for \( x < 0 \), which implies that for such a map to be smooth in the usual sense we must have \( a_2 = -a_1 \), so once again the diffeological dual has dimension one (it is generated by the map \( e^1 - e^2 \)). Based on a result in \([2]\), we conclude that any smooth bilinear form on such a \( V \) is a multiple of \((e^1 - e^2) \otimes (e^1 - e^2)\) by a smooth real function; it follows that it must be degenerate (the vector \( e_1 + e_2 \) belonging to the kernel of the corresponding quadratic form).

\(^{19}\)In place of \(|x|\), we can take any function that is not differentiable in at least one point; we take \(|x|\), since it is the easiest specific example.

\(^{20}\)The difference is in the choice of a non-canonical basis.

\(^{21}\)We observe, for future use, that the subset \( X = \{(x,y) | xy = 0\} \) (once again, the union of the coordinate axes) considered with the corresponding subset diffeology is different from the same set considered with the subset diffeology relative to the standard one on \( \mathbb{R}^2 \).
The three diffeological vector spaces thus described will provide the main building blocks for our further constructions.

2.3 Diffeological vector pseudo-bundles over finite-dimensional diffeological vector spaces

Let us now turn to the case of a diffeological vector pseudo-bundle $\pi : V \to X$ over a finite-dimensional diffeological vector space $X$, with fibres of finite dimension as well. We assume the underlying vector space of $X$ to be identified with $\mathbb{R}^k$, for appropriate $k$, and we assume that the underlying map between topological spaces is a true bundle (which is the simplest case, obviously). Unless specified otherwise, we denote the diffeologies on $V$ and on $X$ by $\mathcal{D}_V$ and $\mathcal{D}_X$ respectively.

2.3.1 Vector space diffeology and vector pseudo-bundle diffeology on $\mathbb{R}^n \to \mathbb{R}^k$

It stems immediately from the above paragraph that the pseudo-bundle we are to consider are, from the topological point of view, just projections of some $\mathbb{R}^n$ to some $\mathbb{R}^k$; these are quite natural to treat in as much detail as possible, both because they are the simplest ones and because they are precisely the diffeological vector pseudo-bundles that restrict to (said formally, whose images under the forgetful functor into the category of topological vector spaces are) trivial vector bundles (of finite dimension). What we need to specify at this point is how the diffeology on the total space (some $\mathbb{R}^n$) of the pseudo-bundle under consideration is defined.

The vector space structure on fibres This is something we have already mentioned in the examples in Section 1. Typically, the bundle we consider is defined as the projection on the first $k$ coordinates, $k < n$. Then the fibre over a given point $x$ is identified with $\mathbb{R}^{n-k}$ by taking its last $n-k$ coordinates. This allows to pull back the vector space operations, obtaining a natural vector space structure on each fibre. Stated more formally, we represent $\mathbb{R}^n$ as the direct product of $\mathbb{R}^k \times \mathbb{R}^{n-k}$, so that each fibre has form $\{x\} \times \mathbb{R}^{n-k}$ and carries the vector space structure of the second factor.

The projection $\mathbb{R}^n \to \mathbb{R}^k$ and diffeology of $\mathbb{R}^n$ What is said in the previous paragraph is standard, and allows for a rather obvious construction of a vector pseudo-bundle diffeology on a given $\mathbb{R}^n$, writing, again, $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ and taking the product diffeology coming from whatever diffeology the base space ($\mathbb{R}^k$) carries and any vector space diffeology on $\mathbb{R}^{n-k}$ (see more details on this below). What this gives however is a trivial bundle, not only from the topological, but also from the diffeological point of view; in order to have more intricate examples we need to discuss some constructions specific to diffeology.

The one construction that we will frequently need is that of the diffeology generated by a given set of maps; and, for our examples, this, pretty much always, means the diffeology generated by a single map. What we need to discuss here is what this means for diffeology on $\mathbb{R}^n$ that would allow to consider it as a vector pseudo-bundle with respect to the projection $\pi$ on, say, the first $k$ coordinates. Namely, fixed some positive integer $n$ and a map $p : U \to \mathbb{R}^n$, recall that the diffeology generated by $p$ is the diffeology that consists of maps that locally either are constants or filter through $p$; it is quite evident that such a diffeology may not give any vector space structure on the fibres of our prospective pseudo-bundles.

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22One of which is in fact a family of spaces, but they differ by dimension only, so informally we refer to them as a “single space”.

23Formally we should say that their image in the category of topological vector spaces is the projection of form $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k)$.

24Let us see an example of this. Consider $\mathbb{R}^2 \to \mathbb{R}$, the projection of $\mathbb{R}^2$ onto its $x$-axis; let $p : \mathbb{R} \to \mathbb{R}^2$ be given by $p(x) = (x, x)$, and denote by $\mathcal{D}_p$ the diffeology generated by $p$. Let $L_{x_0} = \{(x_0, y) | y \in \mathbb{R}\}$ be any vertical line; then the subset diffeology on $L_{x_0}$ relative to $\mathcal{D}_p$ includes obviously only (locally) constant maps. Indeed, let $\varphi : U \to L_{x_0}$ be a plot of this subset diffeology: if it is not constant then the map $u \mapsto (x_0, \varphi(u))$ filters through $p$ (assuming $U$ is small enough), that is, there is a smooth $f : U \to \mathbb{R}$ such that for all $u \in U$ we have $(x_0, \varphi(u)) = (f(u), f(u))$, that is, $\varphi(u) \equiv x_0$ and so it is a constant map, after all. Finally, recall that the diffeology that consists of locally constant maps only is not a vector space diffeology (this has already been mentioned in the first section), not being closed under the products by smooth maps.

25Note that we make no reference to the diffeology on the base space, namely, we do not consider the question of the projection being smooth, since our aim at the moment is to point out the absence of the structure of diffeological vector space on the fibres (in any case, if the base space is not fixed from the diffeological point of view, it can always be endowed with the pushforward diffeology).
The observation in the previous paragraph is quite evident; let us now consider a slightly trickier question. Denote by \( D^p_p \) the vector space diffeology generated by \( p \); this, by definition, consists of all finite linear combinations, with smooth functional coefficients, of the plots of \( D_p \). Let us consider the question of whether \( D^p_p \) is necessarily a vector pseudo-bundle diffeology for the projection \( \pi \) (the answer depends, obviously, on the choice of \( p \)).

Let us consider the specific map \( p \) described in the footnote to the previous paragraph. A plot \( U \to \mathbb{R}^2 \) of the diffeology \( D^p_p \) writes as \( u \to F(u) + \sum_{i=1}^m (f_i(u), f_j(u)) \), where \( F \) is any smooth (in the usual sense) \( \mathbb{R}^2 \)-valued function. Precisely because it is arbitrary, and all \( f_i \)'s are smooth as well, we conclude that \( D^p_p \) is the standard diffeology of \( \mathbb{R}^2 \) (and in particular is a pseudo-bundle diffeology); and this conclusion does not depend on the specific choice of \( p \), but only on the fact that it is, in turn, a smooth function.

Thus, to obtain an example of a substantially different kind, we actually need to choose for \( p \) some (for instance) non-differentiable function. Let us consider the following example (it is given by one of the simplest non-differentiable functions \( \mathbb{R}^2 \to \mathbb{R}^2 \), the range being the prospective pseudo-bundle).

**Example 2.4.** Let \( V = \mathbb{R}^2 \) endowed with the (finest) vector space diffeology generated by the plot \( p : \mathbb{R}^2 \to V \) acting by \( p(x, y) = (x, |y|); \) if \( \pi \) is the projection of \( V \) onto its first coordinate then \( \pi \circ p \) is obviously smooth for the standard diffeology of \( \mathbb{R} \). Let us first determine the subset diffeology of a generic fibre; let \( x_0 \in \mathbb{R} \). The fibre \( \pi^{-1}(x_0) \) is the set \( Y_0 = \{ (x_0, y) \mid y \in \mathbb{R} \} \) we claim, first of all, that its diffeology includes the plot \( q : \mathbb{R} \to Y_0 \) acting by \( q(y) = (x_0, |y|) \). To show that this is a plot for the subset diffeology of \( Y_0 \) it is sufficient to write it as the composition of \( p \) with a usual smooth function; it suffices to take \( f : \mathbb{R}^2 \to \mathbb{R} \) defined by \( f(x, y) = (x_0, y) \) (obviously smooth) to get \( (p \circ f)(x, y) = p(x_0, y) = (x_0, |y|) = q(y) \).

Thus, the subset diffeology on \( Y_0 \) contains the diffeology generated by \( q \); but a priori it is a diffeology of a differential space, not necessarily a vector space diffeology.

Consider now the question of whether the finest vector space diffeology on \( \mathbb{R}^2 \) generated by \( p \) makes \( \pi : V \to \mathbb{R} \) into a vector pseudo-bundle, or, in other words, if the above subset diffeology on \( Y_0 \) is a vector space diffeology. To answer this question, it is sufficient note that the addition map, which is smooth for the vector space diffeology of \( V \), is not the same as the addition on fibres. Indeed, when we take two points \( (x, y_1) \) and \( (x, y_2) \) in the same fibre, the former yields \((2x, y_1 + y_2)\), the latter, \((x, y_1 + y_2)\), and in particular, the composition with the plot \( p \) in the former case is \((2x, |y_1| + |y_2|)\) and in the latter, \((x, |y_1| + |y_2|)\); so in the neighbourhood of a point \((x, 0)\) with \( x \neq 0 \) the smoothness of the former does not guarantee the smoothness of the latter.

**The pseudo-bundle diffeology on \( \mathbb{R}^n \) generated by a given plot** As follows from the discussion in the previous paragraph, in order to obtain a diffeological vector pseudo-bundle \( \mathbb{R}^n \to \mathbb{R}^k \) which overlies the natural projection, starting from a given plot (or a family of plots; the essence does not change much under such a generalization), we need to introduce a separate notion (even if it is not particularly new; it is based on the existing notions). We give it as follows.

**Definition 2.5.** Let \( X \) be a diffeological space, and let \( \pi : V \to X \) be a surjective map defined on a set \( V \) such that for every \( x \in X \) the pre-image \( \pi^{-1}(x) \) has a vector space structure. Let \( A = \{ p_i : U_i \to V \}_{i \in I} \) be a collection of maps, each defined on a domain \( U_i \), some \( \mathbb{R}^{m_i} \), and such that \( \pi \circ p_i \) is a plot of \( X \) for all \( i \in I \). Let \( D \) be the diffeology on \( V \) generated by \( A \); the pseudo-bundle diffeology on \( V \) generated by \( A \) is the smallest diffeology that contains \( D \) and that makes the fibrewise addition and scalar multiplication \( V \) smooth.

In other words, the pseudo-bundle diffeology generated by \( A \) is the smallest diffeology containing \( A \) and that makes \( \pi : V \to X \) into a diffeological vector pseudo-bundle. Note that, with the way this definition is stated, we need to explain why it makes sense; more precisely, why the pseudo-bundle diffeology exists. It does for essentially the same reason (having to do with the lattice property of diffeologies, see [6], Section 1.25), which is already explained in [H] (see Proposition 4.6, cited also in the present paper, Sect. 3.1). We state this formally.

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26It is obvious that this reasoning is quite general. Namely, if we wish to endow \( \mathbb{R}^n \) with a diffeology such that the subset diffeology on fibres \{ \( x \) \} \( \times \mathbb{R}^{n-k} \) include a given plot \( q : U \to \mathbb{R}^{n-k} \), it is sufficient to endow \( \mathbb{R}^n \) with a diffeology containing a plot \( p : \mathbb{R}^k \times U \to \mathbb{R}^n \) acting by \( p(x, y) = (x, q(y)) \) for \( x \in \mathbb{R}^k \) and \( y \in U \) (a lot of our examples are of this kind). A further extension of this idea could be to define \( p(x, y) = (x, x \cdot q(y)) \); this gives a diffeology which is not a direct product, namely, it is standard over the subspace \{ \( 0 \} \times \mathbb{R}^{n-1} \) and, unless \( q \) smooth, non-standard over the remaining points. See more on this last construction below.
Lemma 2.6. For every \( \pi : V \to X \) and \( A \) as above, the pseudo-bundle diffeology exists and is unique.

Proof. It suffices to consider \( V \) as a diffeological space endowed with the diffeology \( D \); then the hypotheses of Proposition 4.6 of [1] are satisfied, and the proposition affirms precisely the existence and uniqueness of the pseudo-bundle diffeology (although it goes under a different name therein).

Remark 2.7. The construction of the pseudo-bundle diffeology whose existence is guaranteed by the proposition of [1] (see Lemma 2.6) above is rather evident, especially for the type of the examples that we wish to consider; let us outline it for the case of a single generating map (for simplicity, we will restrict ourselves to these types of examples). So, suppose we have the projection \( \pi : \mathbb{R}^n \to \mathbb{R}^k \) of \( \mathbb{R}^n \) onto its first \( k \) coordinates; let us consider the pseudo-bundle diffeology on \( \mathbb{R}^n \) generated by a certain map \( p : U \to \mathbb{R}^n \). Write the map \( p \) in the usual coordinates of \( \mathbb{R}^n \), that is, as \( p(u) = (p_1, \ldots, p_n) \). Then the necessary condition for the existence of the pseudo-bundle diffeology generated by \( p \) is that the map \( \hat{p}_k = (p_1, \ldots, p_k) : U \to \mathbb{R}^k \) be a plot of whatever diffeology we choose to endow the base space \( \mathbb{R}^k \) with. Now, denote by \( \hat{p}_{n-k} : U \to \mathbb{R}^{n-k} \) the map \( (p_{k+1}, \ldots, p_n) \). Then the pseudo-bundle diffeology on \( \mathbb{R}^n \), relative to the projection \( \pi \), is the diffeology generated by the collection of all maps of the following form: \( u \mapsto (\hat{p}_k(u), \hat{p}_{n-k}(u)) \) where \( \hat{p}_{n-k} \) belongs to the subset of maps defined on \( U \), of the vector space diffeology generated by \( \hat{p}_{n-k} \).

2.3.2 The diffeology on the base space

Here we make some remarks concerning the choice of diffeology on the base space of a (prospective) diffeological vector pseudo-bundle; while it is most natural to consider the base space as (an assembling of some copies of) the standard \( \mathbb{R}^k \), this is not a necessity a priori, so we add some precise comments to the matter.

Non-standard diffeology on the base space As we have already done in some examples, and as we will continue to do below, we frequently (but, of course, not always) consider pseudo-bundles whose underlying topological structure is that of a standard projection of \( \mathbb{R}^n \) onto its first \( k \) coordinates; and we typically endow \( \mathbb{R}^k \) with the standard diffeology and \( \mathbb{R}^n \) with the (finest) vector space diffeology generated by a specified plot. These choices do pose a couple of questions, which we do not wish to consider in much detail, but it is worthwhile to say a few words about them. Namely, a minor question is that of diffeology on the base space \( \mathbb{R}^k \); it certainly does not have to be the standard diffeology, and other choices are available, which give the same underlying topological structure, including the almost-classical alternative of wire diffeology (see [6], Section 1.10). The other question regards \( \mathbb{R}^n \), which, for the projection indicated, must carry a diffeology that yields a vector space diffeology on the last \( n-k \) generators; but that does not automatically imply that it must be a vector space diffeology on the whole of \( \mathbb{R}^n \).

The wire diffeology on \( \mathbb{R}^k \) Let us assume that \( \mathbb{R}^k \) is endowed with the wire diffeology, i.e., the diffeology generated by all smooth maps \( \mathbb{R} \to \mathbb{R}^k \) (recall that by Theorem 3.7 of [2] this implies that the underlying D-topology is the usual topology of \( \mathbb{R}^k \)). We wonder what conditions this imposes on the diffeology of \( \mathbb{R}^n \), so that the projection of \( \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k} \) (in the sense of the usual vector spaces, not diffeological ones) onto the first factor be smooth.

Using the usual coordinates, we write an arbitrary plot \( p : U \to \mathbb{R}^n \) as \( p(u) = (p_1(u), \ldots, p_n(u)) \); then \( (\pi \circ p)(u) = (p_1(u), \ldots, p_k(u)) \), and this must be a plot for the wire diffeology on \( \mathbb{R}^k \) (assuming that \( p \) is not constant, and that \( U \) is small enough, this means that \( \pi \circ p \) filters through a smooth function of one variable). This means that the subset diffeology on \( \mathbb{R}^k \subset V \) is contained in the wire diffeology of \( \mathbb{R}^k \) (and it does not have to coincide with it; it might in fact be strictly smaller, for instance, it could be the discrete diffeology, unless we impose topological restrictions which would prevent it from being so). This is the best general conclusion that could be reached under our assumptions.

Vector pseudo-bundle diffeology on \( \mathbb{R}^n \) without vector space structure As follows from the observations in the previous paragraph, a relevant example is quite easy to construct; it suffices to take \( k \geq 2 \), \( n > k \), and endow \( \mathbb{R}^k \) with the aforementioned wire diffeology. After that, present \( \mathbb{R}^n \) as the direct
product $\mathbb{R}^k \times \mathbb{R}^{n-k}$ and endow it with the product diffeology coming from the wire diffeology on $\mathbb{R}^k$ and any vector space diffeology (for instance, the standard one) on $\mathbb{R}^{n-k}$; the projection onto the first factor of the direct product is a diffeological vector pseudo-bundle (a true bundle, actually). As has already been mentioned in Section 1, the wire diffeology is not a vector space diffeology, so we get an example of the kind described in the title of the paragraph.

### 2.3.3 The underlying topological bundle

We collect here some rather simple remarks, which point to a classification of the simplest possible diffeological vector pseudo-bundles, although not in an exhaustive manner. The main point we make here is that, if our base space is $\mathbb{R}^k$ with a diffeology such that the induced topology (the so-called D-topology [2]) is the usual one, then the topological bundle underlying a diffeological vector pseudo-bundle over it is (obviously; this is well-known) trivial, but it might well not be so from the diffeological point of view.

**Topologically and diffeologically trivial bundles** This is the simplest case, corresponding to a usual trivial bundle; we have $\pi : V \to X$ with $X = \mathbb{R}^k$ for an appropriate $k$ and $V$ is diffeomorphic to $X \times V'$ where $V'$ is a finite-dimensional diffeological vector space, with $\pi$ being just the projection of the direct product on the first factor. Identifying the underlying vector space of $V'$ with $\mathbb{R}^m$ for an appropriate $m$ and denoting its diffeology by $D'_m$ (this is a vector space diffeology on $\mathbb{R}^m$, which can be characterized as containing all usually smooth maps with values in $\mathbb{R}^m$ and being closed under linear combinations with functional coefficients), we obtain an identification of $V$ with $\mathbb{R}^{m+k}$ with diffeology that (in the obvious sense) splits as $D_X \times D'_m$.

**Topologically trivial but diffeologically non locally-trivial pseudo-bundles** The question of the existence of objects of which the title of this paragraph speaks can be re-stated as, does the diffeology $D_V$ always “decomposes” as $D_X \times D_m$, for some vector space diffeology $D_m$ on $\mathbb{R}^m$, or does there exist a diffeological vector bundle $\pi : V \to X$ (i.e., diffeologies $D_V$ and $D_X$ on $\mathbb{R}^{m+k}$ and $\mathbb{R}^k$ respectively) such that the underlying topological vector bundle is the projection of $\mathbb{R}^{m+k}$ onto its first $k$ coordinates, but which is not trivial as a diffeological vector bundle, i.e. such that its fibres have different diffeologies? The answer is positive to the latter question (and negative to the former), as the following example shows.

**Example 2.8.** Let $V$ be $\mathbb{R}^2$ endowed with the pseudo-bundle diffeology generated by the map $p : \mathbb{R}^2 \to V$ given by $p(x, y) = (x, |xy|)$, and let $X$ be $\mathbb{R}$, identified with the $x$-axis of $V$ and endowed with the standard diffeology. Let $\pi : V \to X$ be the projection onto the first coordinate; it is obviously smooth, and gives a diffeological vector bundle. However, the diffeology on the fibre at the origin is the standard one, while elsewhere it is not: for $x \neq 0$ the fibre $\pi^{-1}(x)$ carries the vector space diffeology generated by the map $y \mapsto \text{const} \cdot |y|$ (with non-zero constant), which is strictly coarser than the standard one.\(^{27}\)

Informally speaking, the main conclusion that we draw from this example is that the condition that a diffeological vector pseudo-bundle be trivial should be imposed separately, it being independent from other assumptions. Another point that we stress here is that the pseudo-bundle $\pi : V \to X$ of Example 2.8 is not even not locally trivial from the diffeological point of view (there is not a neighbourhood of $0 \in \mathbb{R}$ where the diffeology on the pseudo-bundle be a direct product diffeology). Since this example can be easily extended to any other pair $(\mathbb{R}^n, \mathbb{R}^k)$, we state separately the following:

**Observation 2.9.** For every pair of natural numbers $k < n$, there exists a diffeological structure on $\mathbb{R}^n$ and a smooth projection $\pi : \mathbb{R}^n \to \mathbb{R}^k$ such that $\pi$ is a diffeological vector pseudo-bundle that is locally non-trivial in at least one point.

**Topologically trivial and diffeologically locally trivial non-trivial** This is the final a priori possibility to consider. We’ll leave the question of existence of such in the open (although the answer is probably negative).

\(^{27}\)Once again, we note that the first coordinate of $p$ could be any smooth function.
2.3.4 Under the passage to duals

We discuss the duals (see [13] for the definition) of diffeological vector pseudo-bundles in more detail later on; however, since the passage to duals illustrates particularly well the peculiarities of diffeological pseudo-bundles (as opposed to the usual vector bundles), we put here a preliminary example to the matter. Let us consider again the Example 2.8, i.e., the projection on the $x$-axis $\pi : \mathbb{R}^2 \to \mathbb{R}$, where the diffeology on $\mathbb{R}^2$ is generated by the plot $p(x,y) = (x,|xy|)$. We have observed that the map $\pi$, which obviously defines a topologically trivial vector bundle, gives a non-trivial vector pseudo-bundle from the diffeological point of view, since the fibre at zero is not diffeomorphic as a diffeological vector space to any other fibre. We now observe that this has visible implications if we consider the (diffeological) dual bundle $\pi^* : (\mathbb{R}^2)^* \to (\mathbb{R})^*$.

Note first of all that if $\mathbb{R} \ni x \neq 0$ then $\pi^{-1}(x)$ is an instance (with $n=1$) of a vector space considered in Example 2.1. It was already mentioned there that the diffeological dual of such a space has strictly smaller dimension than the space itself, which in the case we are treating now implies that the dual is just the zero space $\mathbb{R}^0$. On the other hand, the fibre $\pi^{-1}(0)$ has standard diffeology, thus its diffeological dual is $\mathbb{R}$ with the standard diffeology. This easily implies that, from the topological point of view, the total space $(\mathbb{R}^2)^*$ of the dual bundle is the wedge of two copies of $\mathbb{R}$ joined at the origin. Furthermore, its diffeology is equivalent to the subset diffeology of the union of coordinate axes in the standard $\mathbb{R}^2$ (to which $(\mathbb{R}^2)^*$ is naturally identified).

3 Constructing diffeological vector pseudo-bundles

In this section we discuss some issues related to constructing diffeological vector pseudo-bundles, starting with recalling briefly the way described by Christensen-Wu that allows to obtain a vector pseudo-bundle given a smooth surjection $\pi : V \to X$ such that all fibres carry a vector space structure but not necessarily that of a diffeological vector space. We then consider a kind of diffeological gluing of vector (pseudo-)bundles, as a means of obtaining new pseudo-bundles, in particular, non locally trivial ones. This constructive approach of obtaining, by successive gluings of simpler pseudo-bundles (ideally those treated in the preceding section, i.e., with the underlying topological bundles trivial), could be seen as a way to partially compensate for the absence of local trivializations; but we say right away that it is only partial (we show in a section below that there are pseudo-bundles not admitting such a decomposition, for diffeological reasons).

3.1 Obtaining the structure of a diffeological vector space on fibres

One situation that might easily present itself when trying to construct a specific diffeological vector pseudo-bundle is that at some point we get a kind of vector space pre-bundle (in the terminology of [13]), that is, one where the subset diffeology on fibres is actually finer than a vector space diffeology (i.e., the fibres are vector spaces, but the addition and/or scalar multiplication are not smooth in general). That this can actually happen is demonstrated, once again, by the Example 4.3 of [1]: the internal tangent bundle of the coordinate axes in $\mathbb{R}^2$ considered with the Hector’s diffeology (see Definition 4.1 of [1] for details). As is shown in [1], the tangent space at the origin is not a diffeological vector space for the subset diffeology.

In both [13] (Theorem 5.1.6) and [1] (Proposition 4.16), it is shown that the diffeology on the total space can be “expanded” to obtain a diffeological vector space bundle. We now cite the latter result (recall that the term “diffeological vector space over...” means the same thing as our term “diffeological vector pseudo-bundle”).

**Proposition 3.1.** ([1], Proposition 4.6) Let $\pi : V \to X$ be a smooth map between diffeological spaces, and suppose that each fibre of $\pi$ has a vector space structure. Then there is a smallest diffeology $D$ on $V$ which contains the given diffeology and which makes $V$ into a diffeological vector space over $X$.

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28This is also easy to see directly.
29The existence of such examples motivates the introduction of the dvs diffeology on internal tangent bundles by the authors.
The diffeology whose existence is affirmed in this proposition can be described explicitly (see [1], Remark 4.7). It is the diffeology generated by the linear combinations of plots of $V$ and the composite of the zero section with plots of $X$. More precisely, a map $\mathbb{R}^m \ni U \to V$ is a plot of $D$ if and only if it is locally of form $u \mapsto r_1(u)q_1(u) + \ldots + r_k(u)q_k(u)$, where $r_1, \ldots, r_k : U \to \mathbb{R}$ are usual smooth functions (plots for the standard diffeology on $\mathbb{R}$) and $q_1, \ldots, q_k : U \to V$ are plots for the pre-existing diffeology of $V$ such that there is a single plot $p : U \to X$ of $X$ for which $\pi \circ q_i = p$ for all $i$.

### 3.2 Diffeological gluing of vector pseudo-bundles

We now give a precise description of a construction that allows to obtain a wealth of examples of diffeological vector pseudo-bundles by “piecing together” (or “assembling”) some simpler pseudo-bundles. This construction, which we call *diffeological gluing*, is essentially an extension to the present context of the usual topological gluing. It also mimics the “assembly” of the usual vector bundles over smooth manifolds from the trivial bundles over an appropriate $\mathbb{R}^n$ (however, it does not possess the same universality property, meaning that there are plenty of diffeological vector bundles that are not obtained by gluing; see the section that follows for more details on this aspect).

**Gluing of diffeological spaces** Suppose that we have two diffeological vector pseudo-bundles $\pi_1 : V_1 \to X_1$ and $\pi_2 : V_2 \to X_2$. We wish to describe a gluing operation on these, that would give us again a diffeological vector pseudo-bundles. This obviously requires an appropriate gluing operation on diffeological spaces (applied twice, to the pair of the base spaces and to the pair of the total spaces, in the latter case with some further restrictions to preserve the linear structure). This, in turn, requires a smooth map $f : X_1 \supset Y \to X_2$ and its lift to a smooth map $\tilde{f} : \pi_1^{-1}(Y) \to V_2$; this lift should be linear on the fibres (this is the above-mentioned further restriction).

For the sake of clarity, we comment right away how this construction relates to the example of the coordinate axes in $\mathbb{R}^2$. It is not meant to produce it immediately; rather, it describes the first step in the construction, by setting $X_1$ one of the axes with its subset diffeology and the corresponding internal tangent bundle (which is the usual tangent bundle to $\mathbb{R}$), the subset $Y$ is the origin, and finally $X_2$ is a single point and the corresponding bundle is the map $\pi_2$ that sends (another copy of) $\mathbb{R}^2$, with the standard diffeology, to this point. The map $f$ is obvious and sends the origin $(0, 0) \in \mathbb{R}^2$ to the point that composes $X_2$. As far as the lift $\tilde{f}$ is concerned, there are numerous choices. Indeed, this lift is defined on the line $\mathbb{R} = \pi_1^{-1}(0, 0)$, so it is a linear map from $\mathbb{R}$ to $\mathbb{R}^2 = \pi_2^{-1}(X_2)$, thus there are infinite possibilities (we need to specify first the image, which is either the origin or any 1-dimensional linear subspace of $\mathbb{R}^2$, and in this latter case we should specify the action of the map $\tilde{f}$).

What has just been said is sufficient to describe the prospective gluing from the topological point of view; the diffeological aspect can be easily obtained by employing the concept of the quotient diffeology, something that we specify in the paragraph that follows. Assuming this has been done, what we obtain is a map $\pi : V \to X$, where $V$ is the result of gluing $V_1$ and $V_2$ along the map $\tilde{f}$, $X$ is the result of gluing $X_1$ and $X_2$ along the map $f$, and $\pi$ is induced by $\pi_1$ and $\pi_2$ in the obvious way. In what follows we will show that this map is indeed a diffeological vector pseudo-bundle, under the assumption that the map $f$ is injective.

**Diffeology on an assembled space** We use the term “assembled space” to refer to the space resulting from gluing together two other spaces. Suppose we have two diffeological spaces $X_1$ and $X_2$ that are glued together along some smooth map $f : X_1 \supset Y \to X_2$ (the smoothness of $f$ is with respect to the subset diffeology of $Y$). Recall first the standard (topological) definition of gluing, $X_1 \cup_f X_2 = (X_1 \cup X_2)/\sim$, where $\sim$ is the following equivalence relation: $x_1 \sim x_2$ if $x_1 \in X_1 \setminus Y$, $x_2 \sim x_2$ if $x_2 \in X_2 \setminus f(Y)$, and $x_1 \sim x_2$ if $x_1 \in X_1$ and $x_2 = f(x_1)$.

Now, for $X_1$ and $X_2$ diffeological spaces, there is a natural diffeology on $X_1 \cup_f X_2$, namely the sum diffeology; and for whatever equivalence relation exists on a diffeological space (which is $X_1 \cup_f X_2$ in this case) there is the standard quotient diffeology on the quotient space. This is the *gluing* diffeology on $X_1 \cup_f X_2$.

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30Since the diffeology on $\mathbb{R} = \pi_1^{-1}(0, 0)$ is standard, it suffices that it be a linear map; its smoothness then is automatic.
Example 3.2. Let us work out an example that is close to the setting of our interest. Take $X_1 \subset \mathbb{R}^2$, the $x$-coordinate axis and $X_2 \subset \mathbb{R}^2$ the $y$-coordinate axis; both are considered with the subset diffeology of $\mathbb{R}^2$ (so it is the standard diffeology of $\mathbb{R}$). Gluing them at the origin yields, from the topological point of view, the same space that appears in the Christensen-Wu example. The question that we wish to answer now is the following: is its gluing diffeology, as has just been described, the same as the subset diffeology of $\mathbb{R}^2$?

The answer is obviously positive in the neighbourhood of any point that is not the origin, while in a neighbourhood of the latter the plots of the gluing diffeology are precisely those maps that lift to a plot of either $X_1$ or $X_2$ (but not both, which is a crucial point). This means that a plot $p$ of gluing diffeology writes either as $p(u) = (p_1(u), 0)$ or $p(u) = (0, p_2(u))$, where $p_1, p_2$ are $\mathbb{R}$-valued smooth maps (possibly zero maps). This implies that $p$ is indeed a restriction of a smooth $\mathbb{R}^2$-valued map, hence the gluing diffeology does coincide for this space with the subset diffeology (every restriction of a smooth map $U \to \mathbb{R}^2$ is obviously of this form).

Plots of gluing diffeology In the next paragraph we will make use of possibly dubious notation $p_1 \sqcup p_2$ to denote plots of the disjoint union $X_1 \sqcup X_2$, so we need to explain first what we mean. Here $p_1$ and $p_2$ are plots of $X_1$ and $X_2$ respectively; when we say that $p_1 \sqcup p_2$ is a plot of $X_1 \sqcup X_2$, this means that one of these is taken into consideration, and the other is an “empty” map.

Gluing between diffeological vector pseudo-bundles Let us finally consider the gluing of two diffeological vector pseudo-bundles. Let $X_1$ and $X_2$ be two diffeological spaces, and let $\pi_1 : V_1 \to X_1$ and $\pi_2 : V_2 \to X_2$ be two diffeological vector pseudo-bundles over $X_1$ and $X_2$ respectively. Let $f : X_1 \supset Y \to X_2$ be a smooth injective map, and let $\tilde{f} : \pi_1^{-1}(Y) \to V_2$ be a smooth map that is linear on each fibre and such that $\pi_2 \circ \tilde{f} = f \circ (\pi_1)|_{\pi_1^{-1}(Y)}$. The latter property yields an obvious well-defined map

$$\pi : V_1 \sqcup_f V_2 \to X_1 \sqcup_f X_2.$$  

Theorem 3.3. The map $\pi : V_1 \sqcup_f V_2 \to X_1 \sqcup_f X_2$ is a diffeological vector pseudo-bundle.

Proof. The items to check are: 1) that $\pi$ is smooth, 2) that the pre-image of each point is a (diffeological) vector space, 3) that the addition is smooth as a map $(V_1 \sqcup_f V_2) \times_{X_1 \sqcup_f X_2} (V_1 \sqcup_f V_2) \to V_1 \sqcup_f V_2$, 4) that the scalar multiplication is smooth as a map $\mathbb{R} \times (V_1 \sqcup_f V_2) \to V_1 \sqcup_f V_2$; 5) that the zero section $X_1 \sqcup_f X_2 \to V_1 \sqcup_f V_2$ is smooth. We check these conditions one-by-one.

The smoothness of $\pi$ follows directly from the constructions, but for completeness we add details. Take a plot $p : U \to V_1 \sqcup_f V_2$; locally it is a composition of the form $p = \pi_V \circ (p_1 \sqcup p_2)$, where $\pi_V : V_1 \sqcup V_2 \to V_1 \sqcup_f V_2$ is the quotient projection (smooth by definition), $p_1$ for $i = 1, 2$ is a plot of $V_i$, and $p_1 \sqcup p_2$ is the corresponding plot of the disjoint sum. Let also $\pi_X : X_1 \sqcup X_2 \to X_1 \sqcup_f X_2$ be the quotient projection (once again, smooth by definition).

Observe now that there is an obvious equality (by definition of $\pi$) $\pi \circ \pi_V = \pi_X \circ (\pi_1 \sqcup \pi_2)$, and also that $(\pi_1 \sqcup \pi_2) \circ (p_1 \sqcup p_2)$ is some plot $q$ of the disjoint sum $X_1 \sqcup X_2$. Thus,

$$\pi \circ p = \pi \circ \pi_V \circ (p_1 \sqcup p_2) = \pi_X \circ (\pi_1 \sqcup \pi_2) \circ (p_1 \sqcup p_2) = \pi_X \circ q,$$

i.e., a plot of $X_1 \sqcup_f X_2$, since $\pi_X$ is smooth. This establishes the smoothness of $\pi$.

The structure of a vector space on each fibre is inherited from either $V_1$ or $V_2$. More precisely, for $x \in X_1 \sqcup_f X_2$ and the fibre $\pi^{-1}(x)$ it is inherited from $V_1$ if $x \in X_1 \setminus Y$, otherwise it is inherited from $V_2$. Note that by injectivity of $\tilde{f}$ the fibre at a point $y \in Y$ is actually $\pi_2^{-1}(y)$.

The smoothness of the zero section is established by a similar (carried out in reverse) reasoning to the one we used to show the smoothness of $\pi$. Namely, let $q : U \to X_1 \sqcup_f X_2$ be a plot of $X_1 \sqcup_f X_2$, and let $s^0 : X_1 \sqcup_f X_2 \to V_1 \sqcup_f V_2$ be the zero section. Observe that locally $q$ writes as $q = \pi_X \circ (q_1 \sqcup q_2)$, where $q_i$ is a plot of $X_i$ for $i = 1, 2$. Furthermore, let $s^0_i : X_i \to V_i$ be the corresponding zero section; recall that it is smooth by assumption, so the composition $(s^0_1 \sqcup s^0_2) \circ (q_1 \sqcup q_2)$ is some plot $p$ of $V_1 \sqcup V_2$.

Now, it is easy to see that we have the equality $s^0 \circ \pi_X = \pi_V \circ (s^0_1 \sqcup s^0_2)$. Thus, we can put everything together, obtaining

$$s^0 \circ q = s^0 \circ \pi_X \circ (q_1 \sqcup q_2) = \pi_V \circ (s^0_1 \sqcup s^0_2) \circ (q_1 \sqcup q_2) = \pi_V \circ p,$$
which is obviously a plot of $V_1 \cup_f V_2$ by smoothness of $\pi_V$. So $s^0$ is smooth.

Let us now consider the scalar multiplication, i.e., the map $\bullet : \mathbb{R} \times (V_1 \cup_f V_2) \to V_1 \cup_f V_2$. Recall that (by definition of the product diffeology) a plot of the product locally writes as $(f, p)$, where $f : U \to \mathbb{R}$ is a smooth function, and $p : U \to V_1 \cup_f V_2$ is a plot of $V_1 \cup_f V_2$. Furthermore, we can assume that $U$ is small enough, so, as before, $p$ writes as $\pi_V \circ (p_1 \sqcup p_2)$ for $p_1, p_2$ some plots of $V_1, V_2$ respectively. Then we have $(\bullet \circ (f, p))(u) = \pi_V (f(u)p_1(u) + f(u)p_2(u))$, and this is smooth because $\pi_V$ is smooth, and so is the scalar multiplication on each of $V_1, V_2$.

It remains to consider the addition map. Formally, this is a map $+ : (V_1 \cup_f V_2) \times (V_1 \cup_f V_2) \to V_1 \cup_f V_2$. Once again, as above, a plot $p$ of the domain product locally writes as $p = (\pi_V, \pi_V) \circ (p_1 \sqcup p_2, p_1' \sqcup p_2')$, where the plots $p_1 \sqcup p_2$ and $p_1' \sqcup p_2'$ take values in the same fibre of $V_1 \sqcup V_2$. Hence $(+ \circ p)(u) = \pi_V (p_1(u) \sqcup p_2(u)) + \pi_V (p_1'(u) \sqcup p_2'(u))$, so it is smooth because the addition on each individual $V_i$ is smooth.

**Remark 3.4.** The final choice to carry out the gluing along injective maps is inspired by the classical gluing along homeomorphisms and diffeomorphisms.

### 3.3 Smooth maps between pseudo-bundles and gluing

Here we consider the following situation. First of all, let $\pi_1 : V_1 \to X_1$ and $\pi_2 : V_2 \to X_2$ be two diffeological vector pseudo-bundles that we will glue together along a given $f : X_1 \supset Y \to X_2$ with the fixed lift $\tilde{f}$. Let also $\pi'_1 : V'_1 \to X_1$ and $\pi'_2 : V'_2 \to X_2$ be two other diffeological vector pseudo-bundles with the same respective base spaces. Finally, let $F_1 : V'_1 \to V_1$ be a smooth map linear on fibres and such that $\pi'_1 = \pi_1 \circ F_1$. Similarly, let $F_2 : V'_2 \to V_2$ be a smooth map linear on fibres and such that $\pi_2 = \pi'_2 \circ F_2$. We define the following map:

$$\tilde{f}' : (\pi'_1)^{-1}(Y) \to V'_2, \quad \tilde{f}'(u'_1) = F_2(f(F_1(u'_1))).$$

It is obvious that $\tilde{f}'$ is smooth (for the subset diffeology) and linear on fibres, since it is a composition of maps that enjoy these two properties. Furthermore, we easily get the following:

**Lemma 3.5.** The map $\tilde{f}'$ takes values in $(\pi'_2)^{-1}(f(Y))$ and is a lift of $f$ by the maps $\pi'_1$ and $\pi'_2$.

**Proof.** The first part of the statement follows from the definition of $\tilde{f}'$ and from the equalities $\pi'_1 = \pi_1 \circ F_1$ and $\pi_2 = \pi'_2 \circ F_2$. To check the second part of the statement, we need to verify the equality $f \circ \pi'_1 = \pi'_2 \circ \tilde{f}'$ on the appropriate domain of the definition (which is $(\pi'_1)^{-1}(Y)$). Now, it follows from our assumptions on $F_1, F_2$ and $\tilde{f}'$, and from the definition of $\tilde{f}'$ that

$$f \circ \pi'_1 = f \circ \pi_1 \circ F_1 = \pi_2 \circ \tilde{f}' \circ F_1,$$

and

$$\pi'_2 \circ \tilde{f}' = \pi'_2 \circ F_2 \circ f \circ F_1 = \pi_2 \circ f \circ F_1,$$

so the equality does hold.

The main consequence of this lemma is that the map $\tilde{f}'$ can, in its turn, be used to carry out the gluing between the pseudo-bundles $\pi'_1 : V'_1 \to X_1$ and $\pi'_2 : V'_2 \to X_2$. It can also be extended into a more general setting of having $\pi'_1 : V'_1 \to X'_1$ together with a smooth map $f'_1 : X'_1 \to X_1$ that lifts to $F_1 : V'_1 \to V_1$ that is a smooth and linear on fibres, and $\pi'_2 : V'_2 \to X'_2$ together with a smooth map $f'_2 : X'_2 \to X_2$ that lifts to $F_2 : V'_2 \to V_2$, again smooth and linear on fibres. Then defining $f' = f'_2 \circ f \circ f'_1$ and its lift $\tilde{f}' = F_2 \circ f \circ F_1$ allows to carry out the gluing of the pseudo-bundles $\pi'_1 : V'_1 \to X'_1$ and $\pi'_2 : V'_2 \to X'_2$, obtaining a diffeological vector pseudo-bundle $V'_1 \cup_f V'_2 \to X_1 \cup_f X_2$.

### 4 Which diffeological vector pseudo-bundles are the result of gluing?

This brief section collects some preliminary observations regarding what should be a sort of the reverse of gluing, i.e., a kind of diffeological surgery. These are indeed preliminary only; the only precise claim that we make is that the gluing operation, as can be expected from its definition, does not produce all possible diffeologies, not even in very simple cases.
Sating the problem As we can see from Theorem 3.3, diffeological gluing produces a diffeological vector pseudo-bundle, and it is natural then to ask the opposite question: does the gluing procedure allow, starting perhaps from some elementary building blocks, to obtain all diffeological vector pseudo-bundles, at least under some topological restrictions\textsuperscript{43}\textsuperscript{43}.

It is quite easy to show (we do so below) that the answer to this question as stated is negative. In general terms, the reason for this is simply that the gluing diffeology is a rather specific type of a diffeology (and this is an expected conclusion); this starts already at the level of a single assembled space. In this section we give a concrete illustration of this phrase, along with some partial remarks concerning the decomposition of whole pseudo-bundles under some assumptions on the base space.

The gluing of the subset diffeologies on the base space The question that we consider here is the following one. Let \((X, D)\) be a diffeological space, and let \(X_1, X_2 \subset X\) be such that there exists a \((D\)-continuous) map \(f : X_1 \supset Y \rightarrow X_2\) such that \(X\) is \(D\)-homeomorphic to the topological space \(X_1 \cup_f X_2\). Recall that each of \(X_1, X_2\) carries the canonical subset diffeology (which we denote by \(D_1\) and \(D_2\) respectively) relative to \(D\); endow \(X_1 \cup_f X_2\) with the diffeology \(D'\) that is the result of gluing of \(D_1\) and \(D_2\). Does \(D'\) necessarily coincide with \(D\), the initial diffeology of \(X\)?\textsuperscript{32}\textsuperscript{32} The following example shows that the answer is a priori negative.

Example 4.1. Consider again the space \(X\) of the Example 3.3, the union of the two coordinate axes in \(\mathbb{R}^2\), endowed however with another (coarser) diffeology. Namely, we consider \(X\) as a subset of \(\mathbb{R}^2\) that is endowed with the diffeology \(D_X\) of Example 3.3. Let us present \(X\) as the union \(X_1 \cup X_2\), where \(X_1\) is the \(x\)-axis of \(\mathbb{R}^2\) and \(X_2\) is the \(y\)-axis; let us determine first the subset diffeologies, \(D_1\) and \(D_2\) respectively, of \(X_1\) and \(X_2\).

A map \(q : U \rightarrow X_1\) is a plot of \(D_1\) if and only if its composition with the inclusion map is a plot of \(D\), i.e., if the map \(U \rightarrow \mathbb{R}^2\) given by \(u \mapsto (q(u), 0)\) is a plot of \(D\). This means that locally it is a linear combination, with smooth functional coefficients, of maps that either are constant or filter through \(p\) via an ordinary smooth map. It is quite clear then, that in a neighbourhood of any point that is distinct from the origin \(q\) is just an ordinary smooth map. Now, observe that the intersection with \(X_1\) of the image of \(p\) is the positive half-line (with the endpoint at \(0\)), and the intersection of this image with a neighbourhood of \(0\) is a half-interval of form \([0, \varepsilon)\) (in particular, this isn’t an open set). Suppose now that \(q = p \circ f\) for a smooth \(f\) on some small domain \(U\) and that the image \(X' \subset X_1\) contains \(0\); then \(X' \subseteq [0, \varepsilon),\) since it is contained in the image of \(p\). In fact, \(X'\) can be identified with a half-interval \([0, \varepsilon')\) for a suitable \(\varepsilon'\), since \(p\) is bijective on such half-intervals, this means that our smooth \(f\) sends the domain \(U\) to the set \([0, \varepsilon)\), which is obviously impossible. This allows us to conclude that the subset diffeology \(D_1\) of \(X_1\) is the standard one; by analogous reasoning, so is the subset diffeology \(D_2\) of \(X_2\).

On the other hand, we have already established that the gluing of \(X_1\) and \(X_2\) with standard diffeologies at the origin produces the union of the coordinate axes of \(\mathbb{R}^2\) with the subset diffeology of \(\mathbb{R}^2\). This is clearly not the same as our original diffeology \(D\) on \(X\).

What this example illustrates is that already on the base space of a pseudo-bundle, decomposed into a gluing of two its subspaces, the initial diffeology might not be required from gluing\textsuperscript{33}\textsuperscript{33}. For now we limit ourselves to providing this illustration, and avoid to add further generic comments.

\textsuperscript{31} As we have already stated elsewhere, the actual idea behind asking this question is that of looking for an appropriate substitute for local trivializations, such as a statement of the sort: if the base space \(X\) can be covered by copies of standard \(\mathbb{R}^n\)'s (with variable \(n\)) then any finite-dimensional diffeological vector pseudo-bundle can be obtained by gluing of copies of (limited number of models of) diffeological vector pseudo-bundles of the kind described in Section 2. There are ways to phrase this statement such that it becomes quite trivial; otherwise, we are not certain that it is true and provide only preliminary statements pointing in that direction.

\textsuperscript{32} This is analogous in spirit to the case of just a diffeological vector space, which, as shown in [9], might decompose as a direct sum of two subspaces, but the vector space sum diffeology (relative to the subset diffeologies on the subspaces) might be finer than the diffeology of the space itself.

\textsuperscript{33} Not a good news, in the sense of our final purpose.
5 Pseudo-metrics on vector pseudo-bundles

In this section we consider other types of working with diffeological vector pseudo-bundles. In part they are operations typical of usual vector bundles (direct sum, tensor product, dual bundle), in part they deal with the metrics’ issues. Now, the description of the above-listed operations has been available previously and comes from [13]; we provide a complete account, however, also because we will use the details of these constructions to treat the concept of the so-called pseudo-metric on a pseudo-bundle that we introduce.

5.1 Operations on vector pseudo-bundles

Since diffeological vector pseudo-bundles in general are not locally trivial, we cannot use local trivializations to define these operations (in addition to having to specify the diffeology). However, a different description of them exists (see [13], Chapter 5); we recall it in some detail, with a particular attention to showing how one goes round the obstacle of the absence of local trivializations.

Sub-bundles The definition of a sub-bundle is clear, but there is the question why it would be well-posed in reference to the diffeological issues. Specifically, let \( \pi : V \to X \) be a diffeological vector pseudo-bundle, and let \( Z \subset V \) be a subset of \( V \). Endow \( Z \) with the subset diffeology.

**Definition 5.1.** We say that the restriction \( \pi^Z : Z \to X \) of \( \pi : V \to X \) to \( Z \) is a **diffeological vector sub-bundle** (or simply a sub-bundle) of \( \pi : V \to X \) if the following condition holds: for every \( x \in X \) the intersection \( Z \cap \pi^{-1}(x) \) is a vector subspace of \( \pi^{-1}(x) \).

Let us formally prove that this definition is well-posed (although this is rather obvious and is already established in [13]).

**Lemma 5.2.** ([13]) Any diffeological vector sub-bundle \( \pi^Z : Z \to X \) of a diffeological vector pseudo-bundle \( \pi : V \to X \) is itself a diffeological vector pseudo-bundle.

**Proof.** That the restriction \( \pi^Z \) is smooth is a general fact for subsets considered with the subset diffeology. Indeed, the subset diffeology consists of precisely those plots \( p : U \to Z \) whose compositions with the obvious inclusion map is a plot of the ambient space; this essentially means that this is the subset of \( D_V \) consisting of precisely those plots whose range is contained in \( Z \). Since the composition of any plot of \( D_V \) with \( \pi \) is a plot of \( X \) by smoothness of \( \pi \), this holds automatically for the restricted subset of them (considering also that \( \pi^Z \) is just the restriction of \( \pi \) to \( Z \)).

Furthermore, all fibres of \( Z \) carry a vector space structure by the definition given. It remains to check that the two operations are smooth; and, as above, this follows from the definition of the subset diffeology and the fact that the two operations are the restrictions (to \( Z \times_X Z \) and \( Z \) respectively) of the corresponding operations on \( V \). Finally, the smoothness of the zero section is automatic since it takes values in \( Z \) (its composition with any given plot of \( X \) is by assumption a plot of \( V \), which takes values in \( Z \), so it is a plot for the subset diffeology of \( Z \)).

There is also a kind of vice versa version of this lemma, which is actually quite useful, that we now prove.

**Lemma 5.3.** Let \( \pi : V \to X \) be a diffeological vector pseudo-bundle, and let \( W \subset V \) be such that \( \pi|_W : W \to X \) is surjective, and for every \( x \in X \) the pre-image \( W_x = \pi^{-1}(x) \cap W \) is a vector subspace of \( V_x = \pi^{-1}(x) \). Then \( W \) is a diffeological vector pseudo-bundle for the subset diffeology of \( V \).

We note that the surjectivity of \( \pi|_W \) follows from the second condition (a vector subspace is never empty, so neither is \( W_x \)) and therefore is superfluous; we leave it in for reasons of readability.

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34 Along with a precise description of various examples; a number of those given in [13] seem to be just references to pictures, which we do not find satisfactory.

35 In particular, being a vector subspace means that it is non-empty, so the map \( \pi^Z \) is onto \( X \).
Proof. The smoothness of $\pi|_W$ and of the operations follows from the fact that they are restrictions of smooth maps on, respectively, $V$, $V \times_X V$, and $\mathbb{R} \times V$; in the first case it is a direct consequence of the definition of the subset diffeology, and in the other two we use the fact that the subset diffeology is well-behaved with respect to direct products (as follows from the definition of the direct product diffeology). \hfill \Box

The above lemma illustrates once again the extreme flexibility of diffeology\footnote{Coming with its own price, of course.} contrary to what one would normally expect from a differentiable setting, any collection of subspaces of a vector pseudo-bundle, one for each fibre, forms naturally a diffeological sub-bundle.

Quotients Let $\pi : V \to X$ be a diffeological vector pseudo-bundle, and let $\pi^Z : Z \to X$ be a sub-bundle of it. Fibrewise, we can define the quotients $\pi^{-1}(x)/(\pi^Z)^{-1}(x)$ for every $x \in X$ (it makes sense to write here for brevity $V_x/Z_x$), since each space of the sort is the quotient of a diffeological vector space (of $V_x$) over its subspace, $Z_x$. Now, both spaces have subset diffeologies, which we denote by $D^V_x$ and $D^Z_x$ respectively, so their quotient has the corresponding quotient diffeology, denoted by $D^{V/Z}_x$. Let $D^{V/Z}_x$ be the finest diffeology on the set $W = \cup_{x \in X} V_x/Z_x$ such that the subset diffeology on every $V_x/Z_x$ contains the diffeology $D^{V/Z}_x$.

On the other hand, the same structure of a vector space and a subspace of it on each fibre can be seen as an equivalence relation on $V$; the corresponding quotient is again $W$. However, formally at least, the quotient diffeology $D^{V/Z}$ on $W$, relative to the diffeology of $V$ and this equivalence relation, might be different from the diffeology $D^{V/Z}_x$. We now formally establish that they are the same.

Lemma 5.4. Let $\pi : V \to X$ be a diffeological vector pseudo-bundle, and let $\pi^Z : Z \to X$ be a sub-bundle of it. The diffeologies $D^{V/Z}_x$ and $D^{V/Z}$ on the quotient pseudo-bundle $\pi^{V/Z} : W \to X$ coincide.

Proof. Recall that, by definition, $D^{V/Z}$ is the finest diffeology for which $\pi^{V/Z}$ is smooth. In other words, $p' : U \to W$ is a plot for $D^{V/Z}$ if and only if locally it is the composition of some plot $p$ of $D_V$ with the quotient projection $\tilde{\pi} : V \to W$, that is, if $U$ is small enough, we have $p' = \tilde{\pi} \circ p$. Recall also that by definition of $\tilde{\pi}$ we have the equality $\pi = \pi^{V/Z} \circ \tilde{\pi}$ everywhere on $V$.

Now, what we need to show to prove the lemma, is that the restriction of $p'$ on each fibre $(\pi^{V/Z})^{-1} = V_x/Z_x$ of $\pi^{V/Z} : W \to X$ is a plot for the quotient diffeology (of diffeological vector spaces) of $V_x/Z_x$, that is, that locally it is a composition of the projection $V_x \to Z_x$ with a plot of $V_x$. Now, the projection just-mentioned is the restriction to $V_x$ of $\tilde{\pi}$, so to establish the claim it suffices to consider the restriction of $p$ to $V_x$, which is a plot for the latter by definition of the subset diffeology; restricting the equality $p' = \tilde{\pi} \circ p$ to the fibre $V_x$, we get the desired statement. \hfill \Box

Remark 5.5. Putting this lemma together with Lemma 5.3 yields a useful (from practical point of view, at least) conclusion: every quotient of a diffeological vector pseudo-bundle that preserves the operations is a quotient pseudo-bundle over a diffeological vector sub-bundle. Indeed, the condition that operations be preserved simply tells us that on each fibre the kernel of the quotient is a vector subspace. The collection of these subspaces which is considered with the subset diffeology is a sub-bundle by Lemma 5.3 and the choice of diffeologies is consistent everywhere by the lemma established above.

The direct product of pseudo-bundles Let $\pi_1 : V_1 \to X$, $\pi_2 : V_2 \to X$ be two diffeological vector space pseudo-bundles with the same base space. The total space of the product bundle is of course that of the usual product bundle: $V_1 \times_X V_2 = \cup_{x \in X} (V_1)_x \times (V_2)_x$. The product bundle diffeology (see [13], Definition 4.3.1) is the coarsest diffeology such that the fibrewise defined projections are smooth; this diffeology includes, for instance, for each $x \in X$ all maps of form $(p_1, p_2)$, where $p_i : U \to (V_i)_x$ is a plot of $(V_i)_x$ for $i = 1, 2$.

The following is a more concrete description of the product bundle, in the following way. Consider the direct product $V_1 \times V_2$ and endow it with the product diffeology. The direct product bundle $V_1 \times_X V_2$ consists, as a set, of all pairs $(x_1, x_2)$ such that $\pi_1(x_1) = \pi_2(x_2)$. It is endowed with the subset diffeology relative to that of $V_1 \times V_2$: it is precisely the diffeology that is described above. The map $\pi$ is induced
The direct sum of pseudo-bundles
It suffices to add to the above pseudo-bundle $V_1 \times_X V_2 \rightarrow X$ the obvious operations to get the direct sum of pseudo-bundles, which we denote by $\pi_\oplus : V_1 \oplus X V_2 \rightarrow X$ (sometimes writing simply $V_1 \oplus V_2 \rightarrow X$). More precisely, the addition operation is defined as a map $(V_1 \oplus V_2) \times_X (V_1 \oplus V_2) \rightarrow X$ (with the obvious addition on fibres, $(v_1, v_2) + (v'_1, v'_2) = (v_1 + v'_1, v_2 + v'_2)$); the scalar multiplication map is a map $\mathbb{R} \times (V_1 \oplus V_2) \rightarrow X$.

The following statement (also deduced from [13]) is obvious; we cite it for completeness.

**Lemma 5.6.** The pseudo-bundle $\pi_\oplus : V_1 \oplus V_2 \rightarrow X$ is a diffeological vector pseudo-bundle with respect to the operations just described.

**Remark 5.7.** There is the following a priori question. Let $\pi : V \rightarrow X$ be a diffeological vector pseudo-bundle (with finite-dimensional fibres), and let $\pi_\oplus : Z \rightarrow X$ be a sub-bundle of it. Then $Z_x$ is a vector subspace of $V_x$ for every $x$, so we can find a subspace $W_x \subseteq V_x$ such that $V_x = Z_x \oplus W_x$ (as a vector space, without considering the diffeology). As shown above (Lemma 5.5), the collection $W = \cup_{x \in X}$ defines another sub-bundle of $V$, and such that pointwise $V$ splits as a direct sum of these two pseudo-bundles. Does it also split as a diffeological vector pseudo-bundle? Immediately below we give an example that answers this question in the negative.

**Example 5.8.** Here is an example that illustrates that a diffeological vector pseudo-bundle may split as a vector (pseudo-)bundle in the category of vector spaces, but not in the category of diffeological vector spaces. Let $\pi : \mathbb{R}^4 \rightarrow \mathbb{R}$ be the usual projection of $\mathbb{R}^4$ onto its first coordinate (thus, the target space of $\pi$ is identified with the first coordinate axis of $\mathbb{R}^4$). Endow $\mathbb{R}^4$, the total space, with the pseudo-bundle diffeology generated by the plot $p : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ acting by $p(x, y) = (x, 0, x|y|, x|y|)$; let $Z$ be the sub-bundle given, as a subset of $\mathbb{R}^4$, by the equation $x_4 = 0$. Now, observe the following: for each fixed $x \neq 0$ (if $x = 0$ then we just obtain the standard $\mathbb{R}^3$ with the standard $\mathbb{R}^3$ in it, composed of the first three coordinates) the fibre $\pi^{-1}(x)$ is the diffeological vector space with $\mathbb{R}^3$ as the underlying space, endowed with the diffeology generated (in the vector space diffeology sense) by the plot $y \mapsto (0, |y|, |y|)$. This is pretty much the same as an example made in [13], where it is shown that in such a space the subset diffeology of $Z_x$, when summed with the subset diffeology of its standard orthogonal complement $W_x$ (which, for each fixed $x$, is described by setting $x_2 = x_3 = 0$), does not give back the original diffeology of the ambient space $V_x$. Thus, if we define $W$ to be the sub-bundle that as a set is given by the equations $x_2 = x_3 = 0$, then as a pseudo-bundle $\pi : V \rightarrow X$ does split into the direct sum $\pi_\oplus : Z \oplus W \rightarrow X$, but the direct sum diffeology on the latter is the standard one, while that on the former pseudo-bundle is obviously not. We conclude that the splitting of $\pi$ into the direct sum $\pi_\oplus$ does exist but it is not diffeologically smooth.

The tensor product pseudo-bundle
This notion was described in [13] (see Definition 5.2.1); we give a more explicit description than the one that appears therein. Consider the direct product bundle $\pi : V_1 \times_X V_2 \rightarrow X$ defined in the previous paragraph. The pre-image $\pi^{-1}(x) = (V_1 \times_X V_2)_x$ of any point $x \in X$ is obviously the vector space $(V_1)_x \times (V_2)_x$; let $\phi_x : (V_1)_x \times (V_2)_x \rightarrow (V_1)_x \otimes (V_2)_x$ be the universal map onto the corresponding tensor product. The collection of maps $\phi_x$ defines a map $\phi : V_1 \times_X V_2 \rightarrow V_1 \otimes_X V_2 =: \cup_{x \in X}(V_1)_x \otimes_X (V_2)_x$. Let also $Z_x$ be the kernel of $\phi_x$; by Lemma 5.3 the collection $Z$ of subspaces $Z_x$ for all $x \in X$ yields a (diffeological) sub-bundle of $V$, and there is a well-defined quotient pseudo-bundle $(V_1 \times_X V_2)/Z$, where each fibre is the (diffeological) tensor product $(V_1)_x \otimes (V_2)_x$.

**Definition 5.9.** The tensor product bundle diffeology on the tensor product bundle $V_1 \otimes_X V_2$ is the finest diffeology on $V_1 \otimes_X V_2$ that contains the pushforward of the diffeology of $V_1 \times_X V_2$ by the map $\phi$; equivalently, it is the quotient diffeology on the quotient pseudo-bundle $\pi((V_1 \times_X V_2))/Z : (V_1 \times_X V_2)/Z = V_1 \otimes V_2 \rightarrow X$.

Due to the considerations we have made on sub-bundles and quotient bundles, this definition is well-posed, in the sense that the two ways to give it, said to be equivalent, are indeed so.
The dual pseudo-bundle It remains to define the dual bundle (once again, this definition is available in [13], Definition 5.3.1). Let \( \pi : V \to X \) be a diffeological vector space; fibrewise, the dual space of it is constructed by taking the union \( \bigcup_{x \in X} (\pi^{-1}(x))^* \) of the obvious projection, which we denote \( \pi^* \). What is essential here is to define the duality which is endowed (which will also describe the topological structure of the underlying space \( V^* \), via the concept of D-topology underlying the diffeological structure chosen).

**Definition 5.10.** The dual bundle diffeology on \( V^* \) is the finest diffeology on \( V^* \) such that: 1) the composition of any plot with \( \pi^* \) is a plot of \( X \) (equivalently, \( \pi^* \) is smooth); 2) the subset diffeology on each fibre coincides with its diffeology as the diffeological dual \( (\pi^{-1}(x))^* \) of fibre \( \pi^{-1}(x) \).

The following curious example illustrates how much things can change in the diffeological setting.

**Example 5.11.** Let \( \pi : V \to X \) be the diffeological vector pseudo-bundle over the space \( X \) of Example 2.8 that we have constructed in the Example 1.2. Since all fibres have coarse diffeology, their diffeological duals are always zero spaces, which means that the dual bundle in this case is just a trivial covering (in the usual sense) of \( X \) by itself.

One matter that should be discussed on the basis of the above definition is why such diffeology exists. The idea is to start with the pullback of the diffeology of \( X \) by \( \pi \); in this, we shall take the smallest sub-diffeology \( D^* \subset (\pi^*)^* (DX) \) that contains all plots of individual fibres, that is, maps \( U \to (\pi^{-1}(x))^* \) that are plots for the functional diffeology of the diffeological dual \( (\pi^{-1}(x))^* \), for all \( x \), and such that the map \( \pi^* \) be smooth. Applying now the lattice property of the family of all diffeologies on a given set (this is mentioned in Section 1, see [6], Section 1.25 for the detailed treatment), we conclude that the diffeology \( D^* \) is well-defined. In addition, while an a priori question could be, whether the subset diffeology relative to \( D^* \) on each fibre is indeed its functional diffeology (which is the standard diffeology of a diffeological dual, see [13, 14, 17] part (3) of the proof of Proposition 5.3.2 in [13] asserts that the answer is positive (and so the fibres are indeed diffeological duals).

**Explicit description of the dual bundle diffeology** This description is available in [13], see Definition 5.3.1. This definition, together with Proposition 5.3.2 (of the same source), yields the following criterion, a bit cumbersome, but essential from the practical point of view (see examples that follow).

**Lemma 5.12.** Let \( U \) be a domain of some \( \mathbb{R}^l \). A map \( p : U \to V^* \) is a plot for the dual bundle diffeology on \( V^* \) if and only if for every plot \( q : U' \to V \) the map \( Y' \to \mathbb{R} \) acting by \( (u, u') \mapsto p(u)q(u') \), where \( Y' = \{(u, u')|p^*(p(u)) = q((u')) \in X \} \subset U \times U' \), is smooth for the subset diffeology of \( Y' \subset \mathbb{R}^{l+\dim(U')} \) and the standard diffeology of \( \mathbb{R} \).

As an illustration, let us apply this definition to the following example.

**Example 5.13.** Take \( V = \mathbb{R}^2 \) endowed with the pseudo-bundle diffeology generated by the plot \( q \) acting as \( (x, y) \mapsto (x, |x|y) \). Define \( X \) to be the standard \( \mathbb{R} \), and let \( \pi \) be the projection of \( V \) onto its first coordinate.\(^{38}\)

Now let us apply Lemma 5.12 to describe the dual bundle. Let \( p : U \to V^* \) be a (prospective) plot of the dual bundle. It is convenient to write \( p(u) \) as an element of the form \( p(u) = (p_1(u), p_2(u)) \), where \( p_1(u) = \pi^*(p(u)) \) determines the fibre to which \( p(u) \) belongs, while \( p_2(u) \) determines the corresponding element of \( (V_{p_1(u)})^* \). We note right away that \( \pi^* \) being smooth is thus equivalent to \( u \mapsto p_1(u) \) being the usual smooth function.

If for a plot of \( V \) we have a linear combination of constants, this means that we have essentially a usual smooth map \( f : U' \to \mathbb{R}^2 \); we write \( u' \mapsto (f_1(u'), f_2(u')) \). The subset \( Y' \) is composed of all pairs \( (u, u') \) such that \( p_1(u) = f_1(u') \); the corresponding evaluation of \( p \) on \( f \) is \( (u, u') \mapsto p_2(u') \). This should be smooth for the subset diffeology of \( Y' \subset \mathbb{R}^m \) for a suitable \( m \). Since \( f \) can be any smooth map (thus, it could be identically a non-zero constant), this implies that \( p_2 \) must be a smooth function as well.

\(^{37}\)As opposed to being strictly coarser; by construction these are the only options.

\(^{38}\)This is the same pseudo-bundle we have already seen in Example 2.8.
Consider now the evaluation of $p$ on the plot $q$ that defines the diffeology of $V$. We have $p(u)(q(x,y)) = p(u)(x,|xy|) = |xy|p_2(u)$, and this is defined on $Y' = \{(u,(x,y))|p_1(u) = x\}$ and must extend to an ordinary smooth map defined on $U \times \mathbb{R}^2$. This implies that $p_2$ is identically zero outside of the subset $p_1^{-1}(0)$, and it is any smooth function in the interior of this subset. Thus, to define a plot $p$ of the dual bundle we can first choose any smooth function $p_1: U \to \mathbb{R}$, and then choose another smooth function $p_2: U \to \mathbb{R}$ such that $p_2$ has non-zero values only on the interior of $p_1^{-1}(0)$ we obtain in this way $p(u) = (p_1(u), p_2(u))$ which satisfies all the desired conditions.

Finally, do note that if the interior of $p_1^{-1}(0)$ is non-empty then we can find a small enough open set $U_1 \subset p_1^{-1}(0)$ and use an appropriate partition of unity to construct $p_2: U \to \mathbb{R}$ that satisfies all the properties required and whose restriction to $U_1$ is any chosen smooth function. This essentially implies what we wanted, namely, that the diffeology on the fibre at 0 be the fine diffeology of $\mathbb{R}$, while elsewhere it is the obvious coarse diffeology of the one-point space.

5.2 Do the operations commute with gluing?

A natural question that presents itself at this point is whether the operations described in the previous section commute with gluing; or, more precisely, when they do so.

Sub-bundles The easiest case is that of a sub-bundle (it also illustrates why the question is not entirely trivial, since it is rather obvious that the gluing does not necessarily preserve sub-bundles. Here is the precise situation that we wish to consider.

Let $\pi_1: V_1 \to X_1$ and $\pi_2: V_2 \to X_2$ be two dиффологевские vector pseudo-bundles, and let $\pi_2^?: Z_2 \to X_1$ and $\pi_2^?: Z_2 \to X_2$ be their respective vector sub-bundles. Recall that this means that each of $Z_1$, $Z_2$ is a subspace of respectively $V_1$, $V_2$ such that: 1) $Z_i$ has non-empty intersection with each fibre of $\pi_i$ (in other words, the restriction $\pi_i^?$ of $\pi_i$ to $Z_i$ is onto $X_i$), 2) moreover, for $i = 1, 2$ and for every $x \in X_i$, the intersection $Z_i \cap \pi_i^{-1}(x)$ is a vector subspace of $\pi_i^-(x)$, and finally, 3) the diffeology of $Z_i$ is the subset diffeology relative to the diffeology of $V_i$.

In the situation just described it is quite easy to see that in order to even ask the question we must impose the following condition: $\tilde{f}(\pi_1^{-1}(x) \cap Z_1) \subset (\pi_2^{-1}(f(x)) \cap Z_2$ for every $x \in Y$. This condition just states that for every fibre of $Z_1$ in the domain of the definition of $\tilde{f}$ this fibre should be sent to a subspace of the corresponding fibre of $Z_2$; in other words, $\tilde{f}$ must restrict to a gluing between $Z_1$ and $Z_2$. It is quite obvious that this is not automatic, but here is an example where it does not happen.

Example 5.14. Let $\pi_1: \mathbb{R}^3 \to \mathbb{R}$ be the projection to the first coordinate, with both spaces endowed with the standard diffeologies, and let $\pi_2: \mathbb{R}^3 \to \mathbb{R}$ be another copy of the same pseudo-bundle (a true bundle, really). Let $Y = \{0\}$, and let $f$ and $\tilde{f}$ be the obvious identity maps. Let $Z_1$ be the plane given by the equation $z = 0$ (the $(x,y)$-coordinate plane); it suffices to take $Z_2$ to be the $(x,z)$-coordinate plane of the second copy of $\mathbb{R}^3$ to get an example where the image, in the glued pseudo-bundle $\mathbb{R}^3 \cup_f \mathbb{R}^3$, of $Z_1 \cup Z_2$ is not a vector pseudo-bundle at all (indeed, the fibre at 0 is the union of two lines with a one-point intersection, i.e., not a vector space).

Quotients The situation of quotients is similar to that of sub-bundles: the question that we should really ask is, under which conditions a given gluing $\tilde{f}$ of two pseudo-bundles yields a well-defined gluing on their given quotients? This question is a standard, and the answer to it is also standard: this happens if and only if $\tilde{f}$ induces a well-defined gluing over the sub-bundles that are kernels of the quotients. The condition for this has been stated just above, and we do not repeat it.

---

\textsuperscript{39}This interior of course might be empty, in which case we’ll be forced to set $p_2 = 0$ everywhere; on the other hand, there are plenty of smooth functions such that the pre-image of zero has non-empty interior.

\textsuperscript{40}In what concerns the compositions of $q$ with smooth and linear combinations of such, the reasoning just made easily extends to those; we omit the details for reasons of brevity.

\textsuperscript{41}"Sub-pseudo-bundles" would be more precise, but we avoid this term, as we have already done for other terms similar, as it sounds unnecessarily complicated (the object that we are considering is clear from the context).
**Direct product/sum** Let us now consider the behavior of the gluing with respect to the direct product. Suppose we have the following two pairs of vector pseudo-bundles, one composed of \( \pi_1 : V_1 \to X_1 \) and \( \pi_2 : V_2 \to X_2 \) glued together along some appropriate choice of \( f : X_1 \supset Y \to X_2 \) and \( f : \pi_1^{-1}(Y) \to \pi_2^{-1}(f(Y)) \), to form the pseudo-bundle \( \pi : V_1 \cup_f V_2 \to X_1 \cup_f X_2 \). The other pair is \( \pi_1' : V_1' \to X_1 \) and \( \pi_2' : V_2' \to X_2 \), with gluing on the bases along the same map \( f : X_1 \supset Y \to X_2 \) and the lift \( f' : (\pi_1')^{-1}(Y) \to (\pi_2')^{-1}(f(Y)) \); this gives the pseudo-bundle \( \pi' : V_1' \cup_{f'} V_2' \to X_1 \cup_f X_2 \). Let us now take also the direct product bundles \( \pi_{1,x} : V_1 \times_{X_1} V_1' \to X_1 \) and \( \pi_{2,x} : V_2 \times_{X_2} V_2' \to X_2 \), and consider the question whether the two given gluings, \((f,f)\) and \((f',f)\) induce naturally a gluing between these two products.

The gluing of the bases is simply inherited (since the bases are in fact the same); it is given by the same \( f \). Now, if an appropriate lift of it (which we denote by \( f_x \)) exists, it must be, as we know, defined on the whole of \( \pi_{1,x}^{-1}(Y) \), so let us first say what it is. It is quite easy to see that this is \( \pi_{1,x}^{-1}(Y) \times \pi_2^{-1}(Y) \) (since each of the two sets is composed of the whole fibres). Analogously, \( \pi_{2,x}^{-1}(f(Y)) \) decomposes as the (fibrewise) direct product \( (\pi_{1,x}^{-1}(f(Y))) \times f(Y) (\pi_2^{-1}(f(Y))) \). It follows then that \( f_x \) can be defined as the (fibrewise) product of the maps \( f \) and \( f' \), namely, for any \( v_1 \in \pi_1^{-1}(Y) \) and \( v'_1 \in (\pi_1')^{-1}(Y) \) we set \( f_{\times}(v_1,v'_1) = (f(v_1),f'(v'_1)) \). This is well-defined and satisfies all the desired conditions, by definition of the product diffeology (both in the case of pseudo-bundles and in the case of individual vector spaces). Furthermore, if we add the operations so as to obtain the direct sum, these are obviously going to be smooth. What all this means can be summarized as follows:

**Lemma 5.15.** Let \( \pi : V_1 \cup_f V_2 \to X_1 \cup_f X_2 \) be a diffeological vector pseudo-bundle obtained by gluing together pseudo-bundles \( \pi_1 : V_1 \to X_1 \) and \( \pi_2 : V_2 \to X_2 \) along a smooth map \( f : X_1 \supset Y \to X_2 \) and its smooth linear lift \( f : \pi_1^{-1}(Y) \to \pi_2^{-1}(f(Y)) \), and let \( \pi' : V_1' \cup_{f'} V_2' \to X_1 \cup_f X_2 \) be another diffeological vector pseudo-bundle obtained by gluing together pseudo-bundles (over the same respective bases) \( \pi_1' : V_1' \to X_1 \) and \( \pi_2' : V_2' \to X_2 \) along the same smooth map \( f : Y \to X_2 \) and its smooth linear lift \( f' : (\pi_1')^{-1}(Y) \to (\pi_2')^{-1}(f(Y)) \). Then the following two pseudo-bundles are diffeomorphic as pseudo-bundles:

\[
\pi_x : (V_1 \cup_f V_2) \times_{X_1 \cup_f X_2} (V_1' \cup_{f'} V_2') \to X_1 \cup_f X_2,
\]

\[
\pi_{1,x} \cup_{(f_x,f)} \pi_{2,x} : (V_1 \times_{X_1} V_1') \cup_{f_x} (V_2 \times_{X_2} V_2') \to X_1 \cup_f X_2.
\]

The proof is obvious from the construction.

**Tensor product** Let us now turn to the tensor product; by its definition, it suffices to apply Lemma 5.15 and observe that the nuclei are preserved by \( f, f' \), and \( f_x \). So we get an almost complete analogue of Lemma 5.15, namely, the following statement:

**Lemma 5.16.** Let \( \pi : V_1 \cup_f V_2 \to X_1 \cup_f X_2 \) be a diffeological vector pseudo-bundle obtained by gluing together pseudo-bundles \( \pi_1 : V_1 \to X_1 \) and \( \pi_2 : V_2 \to X_2 \) along a smooth map \( f : X_1 \supset Y \to X_2 \) and its smooth linear lift \( f : \pi_1^{-1}(Y) \to \pi_2^{-1}(f(Y)) \), and let \( \pi' : V_1' \cup_{f'} V_2' \to X_1 \cup_f X_2 \) be another diffeological vector pseudo-bundle obtained by gluing together pseudo-bundles (over the same respective bases) \( \pi_1' : V_1' \to X_1 \) and \( \pi_2' : V_2' \to X_2 \) along the same smooth map \( f : Y \to X_2 \) and its smooth linear lift \( f' : (\pi_1')^{-1}(Y) \to (\pi_2')^{-1}(f(Y)) \). Then the following two pseudo-bundles are diffeomorphic as pseudo-bundles:

\[
\pi_\otimes : (V_1 \cup_f V_2) \otimes_{X_1 \cup_f X_2} (V_1' \cup_{f'} V_2') \to X_1 \cup_f X_2,
\]

\[
\pi_{1,\otimes} \cup_{(f_\otimes,f)} \pi_{2,\otimes} : (V_1 \otimes_{X_1} V_1') \cup_{f_\otimes} (V_2 \otimes_{X_2} V_2') \to X_1 \cup_f X_2,
\]

where the map \( f_\otimes \) is induced by \( f \) at the passage to the tensor product.

**Duals** Let us now turn to the question of dual pseudo-bundles. Once again, assume that we have two vector pseudo-bundles, \( \pi_1 : V_1 \to X_1 \) and \( \pi_2 : V_2 \to X_2 \), which are glued together along a smooth map \( f : X_1 \supset Y \to X_2 \) and its smooth (fibrewise)-linear lift \( f : \pi_1^{-1}(Y) \to \pi_2^{-1}(f(Y)) \), yielding the pseudo-bundle \( \pi : V_1 \cup_f V_2 \to X_1 \cup_f X_2 \). We wish to discuss the question of whether (and if so, how) this induces a gluing between the dual bundles.
Let us start by this observation. Consider an arbitrary point \( y \in Y \); the restriction of \( \tilde{f} \) to its pre-image is a smooth linear map \( \pi_1^{-1}(y) \to \pi_2^{-1}(f(y)) \) between two diffeological vector spaces. Then (see [14]) some details can also be found in [8] there is a natural (smooth and linear) dual map \( f^* : (\pi_2^{-1}(f(y)))^* \to (\pi_1^{-1}(y))^* \), which is defined in the usual way (that is, by the rule \( f^*(v_2)(w_2) = v_2^*(f(w_2)) \)). However, since it goes in the opposite (with respect to \( \tilde{f} \)) direction, it obviously cannot be a lift of the existing map \( f : Y \to X_2 \).

The most natural, and the most obvious, way to resolve the situation is to restrict at this point our discussion to gluings along injective \( f \)'s. Indeed, assuming that \( f \) is injective, we easily observe that \( f^* \) is a lift of its inverse \( f^{-1} \).

However, there is still a further condition to impose. Namely, consider some \( y \in Y \); write for brevity \( W_1 \) to denote \( \pi_1^{-1}(y) \) and \( W_2 \) to denote \( \pi_2^{-1}(f(y)) \). Then it is easy to see that in the pseudo-bundle \( \pi^* : (V_1 \cup_f V_2)^* \to X_1 \cup_f X_2 \) the fibre over \( y = f(y) \) is \( (W_2)^* \), while in the pseudo-bundle \( \pi_1^* \cup (\tilde{f}, f^{-1}) \pi_1^* : V_2^* \cup \tilde{f}^* : V_1^* \to X_2 \cup_{f^{-1}} X_1 \) the fibre over the same point is \( (W_1)^* \). It follows that one necessary condition for these two pseudo-bundles to be (fibrewise) diffeomorphic is that for every \( y \in Y \) the fibres \( \pi_1^{-1}(y) \) and \( \pi_2^{-1}(f(y)) \) have diffeomorphic duals.

Thus, the final statement we arrive to is as follows.

**Lemma 5.17.** Let \( \pi_1 : V_1 \to X_1 \) and \( \pi_2 : V_2 \to X_2 \) be two diffeological vector pseudo-bundles, let \( f : Y \to X_2 \) be a smooth injective map defined on a subset \( Y \subset X_1 \), and let \( \tilde{f} : \pi_1^{-1}(Y) \to \pi_2^{-1}(f(Y)) \) be its smooth fibrewise linear lift. Suppose that the restrictions of the corresponding dual bundles \( \pi_1^* \) and \( \pi_2^* \) to \( Y \) and \( f(Y) \) respectively are diffeomorphic. Then the dual map \( f^* : (\pi_2^{-1}(f(Y)))^* \to (\pi_1^{-1}(Y))^* \) is a smooth fibrewise linear lift of the map \( f^{-1} : f(Y) \to Y \) to the dual pseudo-bundles \( \pi_1^* : V_1^* \to X_1 \) and \( \pi_2^* : V_2^* \to X_2 \) and the following two pseudo-bundles are diffeomorphic as pseudo-bundles:

\[
\pi^* : (V_1 \cup_f V_2)^* \to X_1 \cup_f X_2, \quad \text{and} \quad \pi_2^* \cup (\tilde{f}, f^{-1}) \pi_1^* : V_2^* \cup \tilde{f}^* : V_1^* \to X_2 \cup_{f^{-1}} X_1.
\]

**Proof.** We have already recalled (the known fact) that \( \tilde{f}^* \) is smooth and linear on each fibre; let us formally check that it is indeed a lift of \( f^{-1} \). This means that we need to check the equality \( f^{-1} \circ \pi^*_2 = \pi^*_1 \circ \tilde{f}^* \) on the relevant domains of the definition. So let \( y \in Y \), and let \( v_2^* \in (\pi_2^*)^{-1}(f(y)) \); by definition, \( \tilde{f}^*(v_2^*) \) can be viewed as the composition \( v_2^* \circ \tilde{f} \). Thus, evaluated at \( v_2^* \), the left-hand side of the equality that we need to check yields \( y \), while the right-hand side becomes \( \pi_1^*(v_2^* \circ \tilde{f}) \) and thus also gives \( y \) by injectivity of \( f \). The two maps \( \tilde{f}^* \) and \( f^{-1} \) therefore satisfy the conditions necessary so that gluing can be done along them; so we turn to the second part of the statement.

Let us now construct the desired diffeomorphism between the pseudo-bundles. Let us denote for brevity the second map by \( \pi_g^* \). The diffeomorphism \( \varphi \) between the bases is obvious (it is in fact standard); it is given by identity for \( x \in X_1 \setminus Y \) and \( y \in X_2 \setminus f(Y) \), while for \( y = f(y) \) its image is the point that formally writes as \( y' = f^{-1} \in f(Y) / \sim \) for \( y' = f(y) \). Let us construct now its covering \( \tilde{\varphi} : (V_1 \cup_f V_2)^* \to V_2^* \cup \tilde{f}^* : V_1^* \).

The idea behind the construction is immediately clear, of course. The already-existing diffeomorphism between bases gives a one-to-one correspondence between (whole) fibres of the spaces \( (V_1 \cup_f V_2)^* \) and \( V_2^* \cup \tilde{f}^* : V_1^* \). Now, each fibre of the first space is the dual of some fibre \( \pi_1^{-1}(x) \) with \( x \in X_1 \cup_f X_2 \); as we already noted, if, say, \( x \in X_1 \setminus Y \) then \( \pi_1^{-1}(x) \) is a fibre of \( \pi_1 : V_1 \to X_1 \), and its dual is therefore the corresponding fibre of \( \pi_1^* : V_1^* \to X_1 \). Furthermore, the image \( \varphi(x) \) of \( x \) under \( \pi_1^* \) is essentially \( x \) itself, and its pre-image \( (\pi_g^*)^{-1}(\varphi(x)) \) is the corresponding fibre of \( V_1^* \to X_1 \). The same reasoning obviously applies to any point of \( X_2 \setminus f(Y) \); so the construction of \( \tilde{\varphi} \) should only be checked for \( Y / \sim \subset X_1 \cup_f X_2 \).

More precisely, let \( y \in Y \); then by construction \( (\varphi)^{-1}(y) = \left( (\pi_1^{-1}(y)) \cup \pi_2^{-1}(f(y))) \right) / v_1 = f(v_1) \right)^* \), whereas \( (\pi_g^*)^{-1}(y) = ((\pi_2^{-1}(f(y)) \cup (\pi_1^{-1}(y)) / v_2 = f(v_2) \right)^* \). A diffeomorphism between the two is now a consequence of the assumptions of the lemma.

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Recall that this does not imply that the spaces themselves should be diffeomorphic; they may easily not be so, such as in the case of the standard \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) with the diffeology of Example 2.1.
5.3 Pseudo-metrics

As we have already recalled elsewhere, a finite-dimensional diffeological vector space in general does not admit a smooth scalar product, unless it is a standard space. This obviously implies that there is not a straightforward counterpart of the notion of a Riemannian metric on a diffeological space. On the other hand, in the case of a single vector space there is the “best possible” substitute for the notion of a scalar product (which we’ve called a pseudo-metric), a “least degenerate” smooth symmetric bilinear form, and this can be extended to a corresponding notion of a pseudo-metric on a diffeological vector pseudo-bundle.

Definition of a pseudo-metric The formal definition of a pseudo-metric on a single diffeological vector space is as follows.

Definition 5.18. Let $V$ be a diffeological vector space of finite dimension $n$, and let $\varphi : V \times V \to \mathbb{R}$ be a smooth symmetric bilinear form on it. We say that $\varphi$ is a pseudo-metric if the multiplicity of its eigenvalue 0 is equal to $n - \dim(V^*)$.

It is not a priori clear, although it is easy to see (see [9]), why this definition makes sense, that is, why such a pseudo-metric always exists, and why it is the best substitute for the smooth scalar product. It is proven, however, in [9], that for any smooth symmetric bilinear form on $V$ the multiplicity of its eigenvalue 0 is at least $n - \dim(V^*)$. Furthermore, we can always find a smooth symmetric bilinear form such that the multiplicity of 0 be precisely $n - \dim(V^*)$.

Formulating the corresponding notion for diffeological vector pseudo-bundles is then trivial. Stated formally, it is as follows.

Definition 5.19. Let $\pi : V \to X$ be a diffeological vector pseudo-bundle. A pseudo-metric on $V$ is any smooth section $g$ of the diffeological vector pseudo-bundle $\pi^*_\bigstar : V^* \otimes V^* \to X$ such that for every $x \in X$ $g(x)$ is a pseudo-metric on $\pi^{-1}(x)$.

Representing pseudo-metrics In the examples that we provide in the rest of this section, we will need to choose a way to write down pseudo-metrics. What we do is opt for an ad hoc solution (it is not meant to be generally applicable): since our pseudo-bundles are, from the set-map point of view, maps of form $\mathbb{R}^n \to \mathbb{R}^k$ given by the projection onto the first $k$ coordinates, and so the fibres are of form $\{x\} \times \mathbb{R}^{n-k}$ and have a vector space structure with respect to the coordinates $x_{k+1}, \ldots, x_n$, an element of the dual bundle can be naturally written in the form $(x_1, \ldots, x_k, a_{k+1}e^{k+1} + \ldots + a_ne^n)$, where $(x_1, \ldots, x_k)$ is the corresponding point of the base space, $e^{k+1}, \ldots, e^n$ are elements of the basis dual to the canonical one (both in the sense of the standard $\mathbb{R}^n$), and so the element $a_{k+1}e^{k+1} + \ldots + a_ne^n$ has an obvious meaning as an element of the dual space of $(x_1, \ldots, x_k) \times \mathbb{R}^{n-k}$. The possible pseudo-metrics then, being bilinear forms and so elements of the tensor product of the dual with itself, can be written, similarly, as $(x_1, \ldots, x_k), \sum_{i,j=k+1}^n a_{ij}e^i \otimes e^j$ (in both cases, generally speaking, there will be restrictions on coefficients to ensure the smoothness and other conditions).

Examples of pseudo-metrics on topologically trivial pseudo-bundles In this paragraph we provide two examples. The first one is chosen so as to be one of the easiest, but not entirely trivial.

Example 5.20. Let $\pi : \mathbb{R}^3 \to \mathbb{R}$ be given by $\pi(x, y, z) = x$; let us endow $\mathbb{R}$ with the standard diffeology and $\mathbb{R}^3$ with the pseudo-bundle diffeology generated by the map $p : U = \mathbb{R}^2 \to \mathbb{R}^3$ acting by $p(u_1, u_2) = (u_1, 0, |u_2|)$. Defining $g(x) = (x, (x^2 + 1)e^2 \otimes e^2)$ gives a pseudo-metric on this pseudo-bundle (where the meaning of the expression is precisely the one explained in the preceding paragraph). In fact, it is easy to see that any pseudo-metric on $V$ has, in the same notation, the form $g(x) = (x, f(x)e^2 \otimes e^2)$, where $f : \mathbb{R} \to \mathbb{R}$ is a smooth everywhere positive function.

Let us formally show that $g(x) = (x, (x^2 + 1)e^2 \otimes e^2)$ defines a smooth section of $V^* \otimes V^*$. By extension of Lemma 4.12 we need to evaluate it a plot of $V \otimes V$ and show that the resulting ($\mathbb{R}$-valued) function is smooth for the (subset) diffeology of its domain of definition and the standard diffeology of $\mathbb{R}$. What this essentially means (as follows from the definition of the pseudo-bundle diffeology generated by a given plot) and symmetry of each value of $g(x)$, it suffices to consider the pair $p \otimes c_v$, why by
we means a constant map and \( p \) is the generating plot. This evaluation, which formally writes as 
\[ g(p(u_1, u_2)) = (u_1^2 + 1) e^2(0) e^2(0), \]
is obviously the zero map, so it is smooth.

Finally, we comment why \( g \) has the maximal possible rank everywhere. This is simply because for any 
\( x \in \mathbb{R} \) the fibre \( \pi^{-1}(x) \) at \( x \) is, as a diffeological vector space, just \( \mathbb{R}^2 \) endowed with the (non-standard) vector space diffeology generated by the map \( u' \mapsto (0, |u'|) \). This is a specific case of a diffeological vector space seen in Example 2.1 ((corresponding to \( n = 2 \)). As mentioned in the Example, its diffeological dual has (in this specific case) dimension 1; therefore, any smooth bilinear form, being an element of the tensor product of this dual with itself, has rank at most 1. It remains to note that this is precisely the rank of \( g \) at any given point \( x \in X \).

The pseudo-bundle in the previous example is a non-standard one, but it is still a trivial bundle. Let us now consider an instance of a non locally trivial pseudo-bundle, that of Example 2.8.

Example 5.21. Recall that we have \( \pi : \mathbb{R}^2 \rightarrow \mathbb{R} \), where \( \pi \) is the projection on the first coordinate, \( \mathbb{R} = X \) is standard, and the diffeology of \( \mathbb{R}^2 = V \) is the pseudo-bundle diffeology generated by the plot \((x, y) \mapsto (x, |x y|)\). Thus, the fibre at zero has standard diffeology of \( \mathbb{R} \), while elsewhere it has the vector space diffeology generated by plot \( y \mapsto |y| \cdot \text{const} \) (thus, it is again a specific case of a non-standard diffeological vector space described in the Example 2.1.

Now, since all fibres are 1-dimensional, a pseudo-metric is essentially a real-valued function \( f \) on \( X \), measuring the value of the corresponding quadratic form on a chosen basis vector, which in our case we can take, for each fixed \( x \), to be \((x, 1)\). Let us first check that the assignment \( x \mapsto (x, 1) \) defines a smooth section \( s \) of the pseudo-bundle \( V \rightarrow X \). To do so, we need to take an arbitrary plot of \( X \), i.e., a smooth (in the ordinary sense) function \( h : U \rightarrow \mathbb{R} \) and to check that its composition with \( s \) is a plot of \( V \). This is in fact obvious, because \((s \circ h)(u) = (h(u), 1)\), which is a usual smooth function, and we defined the diffeology of \( V \) to be, in particular, a vector space diffeology, which implies that it includes all smooth functions in the usual sense. This section being smooth, we can write any pseudo-metric in the form \( x \mapsto (x, f(x) e^2 \circ e^2) \).

Now, formally a pseudo-metric is a smooth section of the pseudo-bundle \( V^* \otimes V^* \rightarrow X \); in our case, from the description of the diffeologies we see (as also indicated in the Example 2.1) that for \( x \neq 0 \) the fibre at \( x \) has a trivial dual, while for \( x = 0 \) it is the standard \( \mathbb{R} \). This implies that the above-considered function \( f \) is a version of the so-called \( \delta \)-function: \( \delta(0) = 1 \) (or any other positive constant; we set it equal to 1 for technical reasons) and \( \delta(x) = 0 \) for \( x \neq 0 \).

Since \( \delta^2 = \delta \) and by definition of the tensor product diffeology, it is sufficient to check that the assignment \( x \mapsto (x, \delta(x) e^2) \) defines a smooth section of \( V^* \rightarrow X \), in other words, that its composition with any plot of \( X \) yields a plot of \( V^* \). Now, a plot of \( X \) is an ordinary smooth function \( p_1 : U \rightarrow \mathbb{R} \), so the composition we must consider is the map \( p : U \rightarrow V^* \) acting by \( u \mapsto (p_1(u), \delta(p_1(u)) e^2) \).

Let us check that \( p \) is a plot of \( V^* \). We have already characterized these plots in Example 5.13: the condition that we need to check is that the product of the two functions, that is, \( p_1(u) \delta(p_1(u)) \), is identically zero on the whole of \( U \). This follows immediately from the definition of \( \delta \), so we can conclude that setting \( g(x) = (x, \delta(x) e^2 \otimes e^2) \) defines a pseudo-metric on our pseudo-bundle \( \pi : V \rightarrow X \). As an additional observation, we note that, as follows from the above discussion, every pseudo-metric on this pseudo-bundle is of this form, up to a choice of positive constant \( \delta(0) \).

**Pseudo-metrics and gluing** It is quite clear from our above discussion that, since the gluing is well-behaved with respect to bundle maps and the operations on vector pseudo-bundles as soon as appropriate additional conditions are satisfied, it would be so also with respect to pseudo-metrics. Indeed, given a gluing of two pseudo-bundles \( \pi_1 : V_1 \rightarrow X_1 \) and \( \pi_2 : V_2 \rightarrow X_2 \), each endowed with a pseudo-metric \( g_1 \) or, respectively, \( g_2 \), via the maps \( \bar{f} \) and \( f \), it is sufficient (and necessary, of course) to have
\[ g_1(x_1)(v_1, v'_1) = g_2(f(x_1))(\bar{f}(v_1), \bar{f}(v'_1)) \]
for all \( x_1 \in X_1 \) and \( v_1, v'_1 \in \pi_1^{-1}(x_1) \). If this equality is satisfied, there is an obvious corresponding pseudo-metric on the pseudo-bundle \( V_1 \cup_f V_2 \rightarrow X_1 \cup_f X_2 \) obtained by the gluing. Here is a simple example of such.

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43This could be referred to as a \((\bar{f}, f)\)-equivariant choice; we will occasionally say that \( g_1 \) and \( g_2 \) are compatible pseudo-metrics.
Example 5.22. Let us take for \( \pi_1 : V_1 \to X_1 \) and \( \pi_2 : V_2 \to X_2 \) two true bundles, given by taking \( V_1 = V_2 = \mathbb{R}^3 \) with the standard diffeology, \( X_1 = X_2 = \mathbb{R} \) again with the standard diffeology, and finally setting \( \pi_1 \) to be the projection of \( V_1 \) onto its first coordinate, while \( \pi_2 \) is the projection of \( V_2 \) onto its second coordinate. Accordingly, \( X_1 \) is identified with the \( x \)-axis of \( V_1 \), and \( X_2 \) is identified with the \( y \)-axis of \( V_2 \).

The simplest choice of gluing on the bases is to identify the two copies of \( \mathbb{R} \) at the origin, so the map \( f \) is just the point-to-point map \( f : \{0\} \to \{0\} \). The pre-image \( \pi_1^{-1}(0) \), so the domain of definition of \( f \), is the \((y,z)\)-coordinate plane of \( V_1 \), the set \( \{(0,y,z)\} \); the pre-image \( \pi_2^{-1}(0) \), which contains the range of \( f \), is the \((x,z)\)-coordinate plane of \( V_2 \), the set \( \{(x,0,z)\} \). We make the simplest choice possible for \( f \), defining it \( f(0,y,z) = (y,0,z) \); this is obviously linear and smooth.

Finally, we choose two compatible pseudo-metrics \( g_1 \) and \( g_2 \) \[^{44}\] In the form that we have already explained, we can write \( g_1(x) = (x, e^2 \otimes e^2 + e^3 \otimes e^3) \) (note that there are no restrictions on the choice of the first pseudo-metric). Then taking, for instance, \( g_2(y) = (y, e^1 \otimes e^1 + e^3 \otimes e^3) \) satisfies the condition of compatibility.

For completeness, we now add a somewhat less trivial, but still rather simple, example.

Example 5.23. Let \( \pi_1 : V_1 \to X_1 \) be a projection of \( V_1 = \mathbb{R}^3 \) onto its first coordinate, so \( X_1 \), which is again a standard \( \mathbb{R} \), is identified with the \( x \)-axis of \( V_1 \). Let \( V_1 \) be endowed with the pseudo-bundle diffeology generated by the plot \( (x,y,z) \to (x,y,|z|) \). This immediately implies that this bundle is diffeologically trivial with non-standard fibre; this fibre is \( \mathbb{R}^2 \) whose (subset) diffeology is generated by the plot \( z \to |z| \), and from Example \( \ref{ex:1} \) we deduce that its dual is the standard \( \mathbb{R} \).

As for \( \pi_2 : V_2 \to X_2 \), we simply take \( V_2 \) to be the standard \( \mathbb{R}^2 \) and \( \pi_2 \) the projection onto its \( y \)-coordinate.

Note that the assumptions of Lemma \( \ref{lem:5.1} \) are satisfied. Since again we choose to identify \( X_1 \) with \( X_2 \) at their respective origins, \( f \) acts between \( \{(0,y,z)\} \) and \( \{(x,0)\} \). Note that it is essentially a smooth (in the sense of the chosen diffeology on \( \mathbb{R}^2 \)) linear map \( \mathbb{R}^2 \to \mathbb{R} \), where the \( \mathbb{R}^2 \) is endowed with a non-standard diffeology, while \( \mathbb{R} \) is just standard. As has already been mentioned, the only smooth linear maps in this case are smooth multiples of \( e^2 \), so we set \( f(0,y,z) = (y,0) \). Applying the reasoning already made, we choose \( g_1(x) = (x, e^2 \otimes e^2) \) and \( g_2(y) = (y, e^1 \otimes e^1) \).

Existence of pseudo-metrics In this concluding paragraph we turn to the following absolutely natural question: does a pseudo-metric always exist? For the definitions given as of now, we provide an example that shows that the answer is negative.

Example 5.24. Take \( V = \mathbb{R}^4 \) endowed with the pseudo-bundle diffeology generated by the plot \( p : \mathbb{R}^3 \to \mathbb{R}^4 \) that acts by the rule \( (x,y,z) \to (x,y,0,y|z|) \), and take \( X \) to be the standard \( \mathbb{R}^2 \). Let \( \pi \) be the projection of \( V \) onto its first two coordinates; this is obviously smooth, so \( \pi : V \to X \) is a diffeological vector pseudo-bundle.

Now, consider an arbitrary point \( (x_0,y_0) \in X \). It is quite obvious that if \( y_0 = 0 \) then the fibre \( \pi^{-1}(x_0,y_0) \) at this point has standard diffeology of \( \mathbb{R}^2 \), while if \( y_0 \neq 0 \) then it has the vector space diffeology generated by the plot \( z \to (0,y_0|z|) \). The latter is a particular case (for \( n = 2 \)) of the family of spaces described in Example \( \ref{ex:2} \). In particular, we can conclude that for \( y_0 = 0 \) the dual of the corresponding fibre is 2-dimensional, while for \( y_0 \neq 0 \) it is 1-dimensional.

Let us consider a prospective pseudo-metric \( g(x,y) \) on this pseudo-bundle. In our notation \( g(x,y) \) is an element of form \((x,y,a(x,y)e^3 \otimes e^3 + b(x,y)e^3 \otimes e^4 + b(x,y)e^4 \otimes e^3 + c(x,y)e^4 \otimes e^4)\), where the symmetry has already been taken into account. By extension of Lemma \( \ref{lem:5.1} \), we should consider its evaluation on a generic section of \( V \).

Observe first of all that the maps \( p_1, p_2 : \mathbb{R}^2 \to V \) given, respectively, by \( p_1(x,y) = (x,y,1,0) \) and by \( p_2(x,y) = (x,y,0,1) \), are necessarily smooth, to account for constant maps in the subset diffeology of each fibre in the pseudo-bundle diffeology of \( V \). Evaluating our prospective \( g \) on \( p_1 \otimes p_1, p_1 \otimes p_2 \), and \( p_2 \otimes p_2 \) allows us to conclude that the functions \( a, b, c \) must be smooth functions in the ordinary sense \( \[^{45}\] \)

Furthermore, evaluating \( g \) on \( p \otimes p_1 \), we obtain \( b(x,y)|z| \), while evaluating it on \( p \otimes p_2 \), we obtain \( c(x,y)|z| \); both of them must be smooth in the usual sense, that is, \( b \) and \( c \) are identically zero functions.

\[^{44}\] Since the diffeology is standard, they are in fact true metrics.

\[^{45}\] Albeit informally, this was something to expect in view of the fact that the diffeological dual of a finite-dimensional diffeological vector space is always a standard space.
This implies that $g$ never has rank 2 and therefore does not always give a pseudo-metric on fibres; more precisely, it does not on the fibres of form $\pi^{-1}(x,0)$. \[46\]

References

[1] J.D. Christensen – E. Wu, Tangent spaces and tangent bundles for diffeological spaces, arXiv:1411.5425v1.

[2] J.D. Christensen – G. Sinnamon – E. Wu, The D-topology for diffeological spaces, arXiv.math 1302.2935v3.

[3] G. Hector, Géométrie et topologie des espaces difféologiques, in Analysis and Geometry in Foliated Manifolds (Santiago de Compostela, 1994), World Sci. Publishing (1995), pp. 55-80.

[4] G. Hector – E. Macias-Virgos, Diffeological groups, Research and Exposition in Mathematics, 25 (2002), pp. 247-260.

[5] P. Iglesias-Zemmour – Y. Karshon – M. Zadka, Orbifolds as diffeologies, Trans. Amer. Math. Soc. (2010), (6) 362, pp. 2811-2831.

[6] P. Iglesias-Zemmour, Diffeology, Mathematical Surveys and Monographs, 185, AMS, Providence, 2013.

[7] M. Laubinger, Diffeological spaces, Proyecciones (2) 25 (2006), pp. 151-178.

[8] E. Pervova, Multilinear algebra in the context of diffeology, arXiv:1504.08186v2.

[9] E. Pervova, On the notion of scalar product for finite-dimensional diffeological vector spaces, arXiv:1507.03787v1.

[10] J.M. Souriau, Groups différentiels, Differential geometrical methods in mathematical physics (Proc. Conf., Aix-en-Provence/Salamanca, 1979), Lecture Notes in Mathematics, 836, Springer, (1980), pp. 91-128.

[11] J.M. Souriau, Groups différentiels de physique mathématique, South Rhone seminar on geometry, II (Lyon, 1984), Astérisque 1985, Numéro Hors Série, pp. 341-399.

[12] A. Stacey, Comparative smootheology, Theory Appl. Categ., 25(4) (2011), pp. 64-117.

[13] M. Vincent, Diffeological differential geometry, Master Thesis, University of Copenhagen, 2008, available at http://www.math.ku.dk/english/research/top/paststudents/martinvincent.msthesis.pdf

[14] E. Wu, Homological algebra for diffeological vector spaces, arXiv:1406.6717v1.

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\[46\] The closest we arrive to is a smooth rank-1 symmetric form, by choosing $a(x, y)$ to be any smooth everywhere positive function.