UNSTABLE AND STABLE GALAXY MODELS

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ABSTRACT. To determine the stability and instability of a given steady galaxy configuration is one of the fundamental problems in the Vlasov theory for galaxy dynamics. In this article, we study the stability of isotropic spherical symmetric galaxy models $f_0(E)$, for which the distribution function $f_0$ depends on the particle energy $E$ only. In the first part of the article, we derive the first sufficient criterion for linear instability of $f_0(E)$: $f_0(E)$ is linearly unstable if the second-order operator

$$A_0 \equiv -\Delta + 4\pi \int f_0(E)(I - \mathcal{P})dv$$

has a negative direction, where $\mathcal{P}$ is the projection onto the function space $\{g(E, L)\}$, $L$ being the angular momentum [see the explicit formulae (27) and (26)]. In the second part of the article, we prove that for the important King model, the corresponding $A_0$ is positive definite. Such a positivity leads to the nonlinear stability of the King model under all spherically symmetric perturbations.

1. Introduction

A galaxy is an ensemble of billions of stars, which interact by the gravitational field which they create collectively. For galaxies, the collisional relaxation time is much longer than the age of the universe ([8]). The collisions can therefore be ignored and the galactic dynamics is well described by the Vlasov - Poisson system (collisionless Boltzmann equation)

$$\partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0, \quad \Delta U = 4\pi \int_{\mathbb{R}^3} f(t, x, v)dv,$$

where $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$, $f(t, x, v)$ is the distribution function and $U(t, x)$ is its gravitational potential. The Vlasov-Poisson system can also be used to describe the dynamics of globular clusters over their period of orbital revolutions ([11]). One of the central questions in such galactic problems, which has attracted considerable attention in the astrophysics literature, of [7], [8], [11], [31] and the references there, is to determine dynamical stability of steady galaxy models. Stability study can be used to test a proposed configuration as a model for a real stellar system. On the other hand, instabilities of steady galaxy models can be used to explain some of the striking irregularities of galaxies, such as spiral arms as arising from the instability of an initially featureless galaxy disk ([7], [32]).

In this article, we consider stability of spherical galaxies, which are the simplest elliptical galaxy models. Though most elliptical galaxies are known to be non-spherical, the study of instability and dynamical evolution of spherical galaxies could be useful to understand more complicated and practical galaxy models. By Jeans’s Theorem, a steady spherical galaxy is of the form

$$f_0(x, v) \equiv f_0(E, L^2),$$
where the particle energy and total momentum are
\[ E = \frac{1}{2} |v|^2 + U_0(x), \ L^2 = |x \times v|^2, \]
and \( U_0(x) = U_0(|x|) \) satisfies the self-consistent Poisson equation. The isotropic models take the form
\[ f_0(x, v) \equiv f_0(E). \]
The cases when \( f_0'(E) < 0 \) has been widely studied and these models are known to be linearly stable to both radial [9] and non-radial perturbations [2]. The well-known Casimir-Energy functional (as a Liapunov functional)
\[ H(f) \equiv \int \int Q(f) + \frac{1}{2} \int \int |v|^2 f - \frac{1}{8\pi} \int \Delta U f^2 dx, \]
is constant along the time evolution. If \( f_0'(E) < 0 \), we can choose the Casimir function \( Q_0 \) such that
\[ Q_0'(f_0(E)) \equiv -f_0''(E) \]
for all \( E \).

By a Taylor expansion of \( H(f) - H(f_0) \), it follows that formally the first variation at \( f_0 \) is zero, that is,
\[ H^{(1)}(f_0(E)) = 0 \]
and the second order variation of \( H \) at \( f_0 \) is
\[ H^{(2)}[g] \equiv \frac{1}{2} \int \int g^2 \frac{1}{f_0(E)} dx dv - \frac{1}{8\pi} \int |\nabla x U_0| g^2 dx \]
where \( Q''(f_0) = \frac{1}{f_0(E)} \), \( g = f - f_0 \) and \( \Delta U g = \int gdv \). In the 1960s, Antonov [1], [2] proved that
\[ H^{(2)}[Dh] = \int \int |Dh|^2 \frac{1}{f_0(E)} dx dv - \frac{1}{4\pi} \int \Delta \psi_h g^2 dx \]
is positive definite for a large class of monotone models. Here
\[ D = v \cdot \nabla x - \nabla x U_0 \cdot \nabla v, \]
\( h(x, v) \) is odd in \( v \) and \( -\Delta \psi = \int Dh dv \). He showed that such a positivity is equivalent to the linear stability of \( f_0(E) \). In [9], Doremus, Baumann and Feix proved the radial stability of any monotone spherical models. Their proof was further clarified and simplified in [10], [37], [22], and more recently in [33], [21]. In particular, this implies that any monotone isotropic models are at least linearly stable.

Unfortunately, despite its importance and a lot of research (e.g., [20], [9], [9], [13]), to our knowledge, no rigorous and explicit instability criterion of non-monotone models has been derived. When \( f_0'(E) \) changes sign, functional \( H^{(2)}[f_0] \) is indefinite and it gives no stability information, although it seems to suggest that these models are not energy minimizers under symplectic perturbations. In this paper, we first obtain the following instability criterion for general spherical galaxies. For any function \( g \) with compact support within the support of \( f_0(E) \), we define the \( |f_0'(E)| \)-weighted \( L^2(\mathbb{R}^3 \times \mathbb{R}^3) \) space \( L^2_{|f_0'|} \) with the norm \( ||g||_{|f_0'|} \) as
\[ ||h||_{|f_0'|}^2 \equiv \int \int |f_0'(E)| h^2 dx dv. \]
Theorem 1.1. Assume that \( f_0(E) \) has a compact support in \( x \) and \( v \), and \( f'_0 \) is bounded. For \( \phi \in H^1 \), define the quadratic form

\[
(A_0\phi, \phi) = \int |\nabla \phi|^2 dx + 4\pi \int \int f'_0(E) (\phi - P\phi)^2 \, dx \, dv,
\]

where \( P \) is the projector of \( L^2_{|f_0|} \) to

\[\ker D = \{ g (E, L^2) \},\]

and more explicitly \( P\phi \) is given by (18) for radial functions and (26) for general functions. If there exists \( \phi_0 \in H^1 \) such that

\[
(A_0\phi_0, \phi_0) < 0,
\]

then there exists \( \lambda_0 > 0 \) and \( \phi \in H^2 \), \( f(x,v) \) given by (14), such that \( e^{\lambda_0 t}[f,\phi] \) is a growing mode to the Vlasov-Poisson system (1) linearized around \([f_0(E),U_{f_0}]\).

A similar instability criterion can be obtained for symmetry preserving perturbations of anisotropic spherical models \( f_0(E, L^2) \), see Remark 2. We note that the term \( P\phi \) in the instability criterion is highly non-local and this reflects the collective nature of stellar instability. The proof of Theorem 1.1 is by extending an approach developed in [25] for 1D Vlasov-Poisson, which has recently been generalized to Vlasov-Maxwell systems (26), (28). There are two elements in this approach. One is to formulate a family of dispersion operators \( A_\lambda \) for the potential, depending on a positive parameter \( \lambda \). The existence of a purely growing mode is reduced to find a parameter \( \lambda_0 \) such that the \( A_{\lambda_0} \) has a kernel. The key observation is that these dispersion operators are self-adjoint due to the reversibility of the particle trajectories. Then a continuation argument is applied to find the parameter \( \lambda_0 \) corresponding to a growing mode, by comparing the spectra of \( A_\lambda \) for very small and large values of \( \lambda \). There are two new complications in the stellar case. First, the essential spectrum of \( A_\lambda \) is \([0, +\infty)\) and thus we need to make sure that the continuation does not end in the essential spectrum. This is achieved by using some compactness property due to the compact support of the stellar model. Secondly, it is more tricky to find the limit of \( A_\lambda \) when \( \lambda \) tends to zero. For that, we need an ergodic lemma (Lemma 2.4) and use the integrable nature of the particle dynamics in a central field to derive an expression for the projection \( P\phi \) appeared in the limit.

In the second part of the article, we further study the nonlinear (dynamical) stability of the normalized King model:

\[
f_0 = [e^{E_0 - E} - 1]_+
\]

motivated by the study of the operator \( A_0 \). The famous King model describes isothermal galaxies and the core of most globular clusters [24]. Such a model provides a canonical form for many galaxy models widely used in astronomy. Even though \( f'_0 < 0 \) for the King model, it is important to realize that, because of the Hamiltonian nature of the Vlasov-Poisson system (1), linear stability fails to imply nonlinear stability (even in the finite dimensional case). The Liapunov functional is usually required to prove nonlinear stability. In the Casimir-energy functional (2), it is natural to expect that the positivity of such a quadratic form \( \mathcal{H}_{f_0}^{(2)}[g] \) should imply stability for \( f_0(E) \). However, there are at least two serious mathematical difficulties. First of all, it is very challenging to use the positivity of \( \mathcal{H}_{f_0}^{(2)}[g] \) to control higher
order remainder in $\mathcal{H}(f) - \mathcal{H}(f_0)$ to conclude stability \cite{38}. For example, one of the remainder terms is $f^3$ whose $L^2$ norm is difficult to be bounded by a power of the stability norm. The non-smooth nature of $f_0(E)$ also causes trouble here. Second of all, even if one can succeed in controlling the nonlinearity, the positivity of $\mathcal{H}^{(2)}_{f_0}[g]$ is only valid for certain perturbation of the form $g = Dh$ \cite{22}. It is not clear at all if any arbitrary, general perturbation can be reduced to the form $Dh$. To overcome these two difficulties, a direct variational approach was initiated by Wolansky \cite{39}, then further developed systematically by Guo and Rein in \cite{14,15,17,18,19}. Their method avoids entirely the delicate analysis of the second order variation $\mathcal{H}^{(2)}_{f_0}$ in \cite{3}, which has led to first rigorous nonlinear stability proof for a large class of polytropes \cite{18} $f_0(E) = (E_0 - E)^{\gamma}_+$. Their basic idea is to construct galaxy models by solving a variational problem of minimizing the energy under some constraints of Casimir invariants. A concentration-compactness argument is used to show the convergence of the minimizing sequence. All the models constructed in this way are automatically stable.

Unfortunately, despite its success, the King model can not be studied by such a variational approach. The Casimir function for a normalized King model is

\begin{equation}
Q_0(f) = (1 + f) \ln(1 + f) - 1 - f,
\end{equation}

which has very slow growth for $f \to \infty$. As a result, the direct variational method fails. Recently, Guo and Rein \cite{21} proved nonlinear radial stability among a class of measure-preserving perturbations

\begin{equation}
\mathcal{S}_{f_0} \equiv \left\{ f(t,r,v_r,L) \geq 0 : \int Q(f,L) = \int Q(f_0,L), \text{ for } Q \in C^\infty_c \text{ and } Q(0,L) \equiv 0 \right\}.
\end{equation}

The basic idea is to observe that for perturbations in the class $\mathcal{S}_{f_0}$, one can write $g = f - f_0$ as $Dh = \{h, E\}$. Therefore, $\mathcal{H}^{(2)}_{f_0}[g] = \mathcal{H}^{(2)}_{f_0}[Dh]$, for which the positivity was proved in \cite{22} for radial perturbations. To avoid the difficulty of controlling the remainder term by $\mathcal{H}^{(2)}_{f_0}[g]$, an indirect contradiction argument was used in \cite{21}.

As our second main result of this article, we establish nonlinear stability of King’s model for general perturbations with spherical symmetry:

**Theorem 1.2.** The King’s model $f_0 = [e^{E_0 - E} - 1]_+$ is nonlinearly stable under spherically symmetric perturbations in the following sense: given any $\varepsilon > 0$ there exists $\varepsilon_1 > 0$ such that for any compact supported initial data $f(0) \in C^1_c$ with spherical symmetry, if $d(f(0), f_0) < \varepsilon_1$ then

$$
\sup_{0 \leq \varepsilon < \infty} d(f(t), f_0) < \varepsilon,
$$

where the distance functional $d(f, f_0)$ is defined by \cite{32}.

For the proof, we extended the approach in \cite{27} for the $1 + 1D$ Vlasov-Maxwell model. To prove nonlinear stability, we study the Taylor expansion of $\mathcal{H}(f) - \mathcal{H}(f_0)$. Two difficulties as mentioned before are: to prove the positivity of the quadratic form and to control the remainder. We use two ideas introduced in \cite{27}. The first idea is to use any finite number of Casimir functional $Q_i(f, L^2)$ as constraints. The difference from \cite{21} is that we do not impose $Q_i(f, L^2) = Q_i(f_0, L^2)$ in the perturbation class, but expand the invariance equation $Q_i(f(t), L^2) - Q_i(f_0, L^2) =$
\[ Q_i \left( f(0), L^2 \right) - Q_i \left( f_0, L^2 \right) \] to the first order. In this way, we get a constraint for 
\[ g = f - f_0 \] in the form that the coefficient of its projection to \( \partial_1 Q_i \left( f_0, L^2 \right) \) is small. Putting these constraints together, we deduce that a finite dimensional projection of 
\[ g = f - f_0 \] to the space spanned by \( \{ \partial_1 Q_i \left( f_0, L^2 \right) \} \) is small. To control the remainder term, we use a duality argument. Noting that it is much easier to control the potential \( \phi \), we use a Legendre transformation to reduce the nonlinear term in \( g \) to a new one in \( \phi \) only. The key observation is that the constraints on \( g \) in the projection form are nicely suited to the Legendre transformation and yields a non-local nonlinear term in \( \phi \) only with the projections kept. By performing a Taylor expansion of this non-local nonlinear term in \( \phi \), the quadratic form becomes a truncated version of 
\[ (A_0 \phi, \phi) \] defined by (6), whose positivity can be shown to be equivalent to that of Antonov functional. The the remainder term now is only in terms of \( \phi \) and can be easily controlled by the quadratic form. The new complication in the stellar case is that the steady distribution \( f_0(E) \) is non-smooth and compactly supported. Therefore, we split the perturbation \( g \) into inner and outer parts, according to the support of \( f_0 \). For the inner part, we use the above constrained duality argument and the outer part is estimated separately.

2. An Instability Criterion

We consider a steady distribution

\[ f_0(x, v) = f_0(E) \]

has a bounded support in \( x \) and \( v \) and \( f'_0 \) is bounded, where the particle energy 
\[ E = \frac{1}{2} |v|^2 + U_0(x) \]. The steady gravitational potential \( U_0(x) \) satisfies a nonlinear Poisson equation

\[ \Delta U_0 = 4\pi \int f_0 dv. \]

The linearized Vlasov-Poisson system is

\[ \begin{aligned}
\partial_t f + v \cdot \nabla_x f - \nabla_x U_0 \cdot \nabla_v f &= \nabla_x \phi \cdot \nabla_v f_0, \\
\Delta \phi &= 4\pi \int f(t, x, v) dv.
\end{aligned} \]

A growing mode solution \( (e^{\lambda t} f(x, v), e^{\lambda t} \phi(x)) \) to (11) with \( \lambda > 0 \) satisfies

\[ \begin{aligned}
\lambda f + v \cdot \nabla_x f - \nabla_x U_0 \cdot \nabla_v f &= f'_0 v \cdot \nabla \phi.
\end{aligned} \]

We define \([X(s; x, v), V(s; x, v)]\) as the trajectory of

\[ \begin{aligned}
\frac{dX(s; x, v)}{ds} &= V(s; x, v) \\
\frac{dV(s; x, v)}{ds} &= -\nabla_x U_0
\end{aligned} \]

such that \( X(0; x, v) = x \), and \( V(0; x, v) = v \). Notice that the particle energy \( E \) is constant along the trajectory. Integrating along such a trajectory for \(-\infty \leq s \leq 0\), we have

\[ \begin{aligned}
f(x, v) &= \int_{-\infty}^{0} e^{\lambda s} f'_0(E) V(s; x, v) \cdot \nabla_x \phi(X(s; x, v)) ds \\
&= f'_0(E) \phi(x) - f'_0(E) \int_{-\infty}^{0} \lambda e^{\lambda s} \phi(X(s; x, v)) ds.
\end{aligned} \]
Plugging it back into the Poisson equation, we obtain an equation for $\phi$

$$-\Delta \phi + [4\pi \int f'_0(E)dv] \phi - 4\pi \int f'_0(E) \int_{-\infty}^{0} \lambda e^{\lambda s} \phi(X(s; x, v)) ds dv = 0.$$ 

We therefore define the operator $A_\lambda$ as

$$A_\lambda \phi \equiv -\Delta \phi + [4\pi \int f'_0(E)dv] \phi - 4\pi \int f'_0(E) \int_{-\infty}^{0} \lambda e^{\lambda s} \phi(X(s; x, v)) ds dv.$$ 

**Lemma 2.1.** Assume that $f_0(E)$ has a bounded support in $x$ and $v$ and $f'_0$ is bounded. For any $\lambda > 0$, the operator $A_\lambda : H^2 \to L^2$ is self-adjoint with the essential spectrum $[0, +\infty)$.

**Proof.** We denote

$$K_\lambda \phi = -4\pi \int f'_0(E) dv \phi + 4\pi \int f'_0(E) \int_{-\infty}^{0} \lambda e^{\lambda s} \phi(X(s; x, v)) ds dv.$$ 

Recall that $f_0(x, v) = f_0(E)$ has a compact support $\subset S \subset \mathbb{R}^3 \times \mathbb{R}^3$. We may assume $S = S_x \times S_v$, both balls in $\mathbb{R}^3$. Let $\chi = \chi(|x|)$ be a smooth cut-off function for the spatial support of $f_0$ in the physical space $S_x$; that is, $\chi \equiv 1$ on the spatial support of $f_0$ and has compact support inside $S_x$. Let $M_\chi$ be the operator of multiplication by $\chi$. Then $K_\lambda = K_\lambda M_\chi = M_\chi K_\lambda = M_\chi M_\lambda M_\chi$. Indeed,

$$f'_0(x, v) = f'_0(X(s; x, v), V(s; x, v))$$

because of the invariance of $E$ under the flow. So

$$K_\lambda \phi (x) = -4\pi \int f'_0(E) dv \phi + 4\pi \int f'_0(E) \int_{-\infty}^{0} \lambda e^{\lambda s} \phi(X(s; x, v)) ds dv$$

$$= -4\pi \int f'_0(E) dv \phi + 4\pi \int \int_{-\infty}^{0} \lambda e^{\lambda s} (f'_0(E) \phi) (X(s; x, v)) ds dv$$

$$= (M_\chi K_\lambda M_\chi \phi)(x).$$

First we claim that

$$\|K_\lambda\|_{L^2 \to L^2} \leq 8\pi \left| \int f'_0(E) dv \right|_{\infty}.$$

Indeed, the $L^2$ norm for the first term in $K_\lambda$ is easily bounded by $4\pi \left| \int f'_0(E) dv \right|_{\infty}$. For the second term, we have for any $\psi \in L^2$,

$$\left| \int_{-\infty}^{0} \int f'_0(E) \phi(X(s; x, v)) ds dv \psi(x) dx \right|$$

$$\leq 4\pi \int_{-\infty}^{0} \lambda e^{\lambda s} \left( \int |f'_0(E)| \phi^2(X(s; x, v)) dv dx \right)^{\frac{1}{2}} \left( \int |f'_0(E)| \psi^2(x) dx \right)^{\frac{1}{2}} ds$$

$$= 4\pi \int_{-\infty}^{0} \lambda e^{\lambda s} \left( \int |f'_0(E)| \phi^2(x) dv dx \right)^{\frac{1}{2}} \left( \int |f'_0(E)| \psi^2(x) dv dx \right)^{\frac{1}{2}} ds$$

$$= 4\pi \left( \int |f'_0(E)| \phi^2(x) dv dx \right)^{\frac{1}{2}} \left( \int |f'_0(E)| \psi^2(x) dv dx \right)^{\frac{1}{2}}$$

$$\leq 4\pi \left| \int f'_0(E) dv \right|_{\infty} \|\phi\|_2 \|\psi\|_2.$$
Moreover, we have that $K_\lambda$ is symmetric. Indeed, for fixed $s$, by making a change of variable $(y, w) \rightarrow (X(s; x, v), V(s; x, v))$, so that $(z, v) = (X(-s; y, w), V(-s; y, w))$, we deduce that

\[
\int \int 4\pi f_0'(E) \int_{-\infty}^{0} \lambda e^{\lambda s} \phi(X(s; x, v)) dsvd\psi(x) dx
= \int_{-\infty}^{0} \lambda e^{\lambda s} \int \int 4\pi f_0'(E) \phi(y) \psi(X(-s; y, w)) dy dv dw dsdx
= \int \int 4\pi f_0'(E) \int_{-\infty}^{0} \lambda e^{\lambda s} \psi(X(-s; y, -w)) \phi(y) dy dv dw dsdx
= \int \int 4\pi f_0'(E) \int_{-\infty}^{0} \lambda e^{\lambda s} \psi(X(s; x, v)) \phi(x) dv dx dsdx.
\]

Here we have used the fact $[X(s; y, w), V(s; y, w)] = [X(-s; y, -w), -V(s; y, -w)]$ in the last line. Hence

\[(K_\lambda \phi, \psi) = (\phi, K_\lambda \psi).
\]

Since $K_\lambda = K_\lambda M_s$ and $M_s$ is compact from $H^2$ into $L^2$ space with support in $S_x$, so $K_\lambda$ is relatively compact with respect to $-\Delta$. Thus by Kato-Rellich and Weyl’s Theorems, $A_\lambda : H^2 \rightarrow L^2$ is self-adjoint and $\sigma_{ess}(A_\lambda) = \sigma_{ess}(-\Delta)$. □

**Lemma 2.2.** Assume that $f_0'(E)$ has a bounded support in $x$ and $v$ and $f_0'$ is bounded. Let

\[k(\lambda) = \inf_{\phi \in D(A_{\lambda}), ||\phi||_2 = 1} (\phi, A_\lambda \phi),\]

then $k(\lambda)$ is a continuous function of $\lambda$ when $\lambda > 0$. Moreover, there exists $0 < \Lambda < \infty$ such that for $\lambda > \Lambda$

\[(17) \quad k(\lambda) \geq 0.
\]

**Proof.** Fix $\lambda_0 > 0$, $\phi \in D(A_\lambda)$, and $||\phi||_2 = 1$. Then

\[k(\lambda_0) \leq (\phi, A_{\lambda_0} \phi)
\leq (\phi, A_\lambda \phi) + |(\phi, A_{\lambda_0} \phi) - (\phi, A_\lambda \phi)|
\leq (\phi, A_\lambda \phi) + 4\pi \int \int |f_0'(E)| \int_{-\infty}^{0} [\lambda e^{\lambda s} - \lambda_0 e^{\lambda_0 s}] \phi(X(s; x, v)) \phi(x) dsvd\psi(x) dx
\leq (\phi, A_\lambda \phi) + 4\pi \int \int |f_0'(E)| \int_{-\infty}^{0} \int_{\lambda_0}^{\lambda} [\lambda s] e^{\lambda s} + e^{\lambda_0 s} d\lambda d\psi(X(s; x, v)) \phi(x) dsvd\psi(x) dx
\leq (\phi, A_\lambda \phi) + C \lambda_0 \int_{-\infty}^{0} \int_{\lambda_0}^{\lambda} [\lambda s] e^{\lambda s} + e^{\lambda_0 s} d\lambda ds
\leq (\phi, A_\lambda \phi) + C |\ln \lambda - \ln \lambda_0|.
\]

We therefore deduce that by taking the infimum over all $\phi$,

\[k(\lambda_0) \leq k(\lambda) + C |\ln \lambda - \ln \lambda_0|.
\]

Same argument also yields $k(\lambda) \leq k(\lambda_0) + C |\ln \lambda - \ln \lambda_0|$ Thus $|k(\lambda_0) - k(\lambda)| \leq C |\ln \lambda - \ln \lambda_0|$ and $k(\lambda)$ is continuous for $\lambda > 0$.

To prove (17), by (13), we recall from Sobolev’s inequality in $\mathbb{R}^3$. 
\[(K \lambda \phi, \psi) = \left| \int \int 4 \pi f'_0(E) e^{\lambda s} \nabla \phi(X(s;x,v)) V(s) ds dv \psi(x) dx \right| \]
\[\leq \int_{-\infty}^{0} e^{\lambda s} \left( \int \int |\psi|^2 |f'_0(E)| dv dx \right)^{1/2} \cdot \]
\[\times \left[ \int \int |\nabla \phi(X(s))|^2 |f'_0(E)| V(s)^2 dv dx \right]^{1/2} ds \]
\[= \int_{-\infty}^{0} \left( \int \int |\psi|^2 |f'_0(E)| dv dx \right)^{1/2} \int \int v^2 |\nabla \phi(x)|^2 |f'_0(E)| dv dx \right]^{1/2} ds \]
\[\leq \frac{C}{\lambda} |\psi||0||\nabla \phi||2 \leq \frac{C}{\lambda} |\nabla \phi||2, \]

since \(f_0\) has compact support. Therefore, \[(A_\lambda \phi, \phi) = |\nabla \phi|^2 - (K \lambda \phi, \phi) \geq (1 - \frac{C}{\lambda}) |\nabla \phi|^2 \geq 0 \]
for \(\lambda\) large. \(\square\)

We now compute \(\lim_{\lambda \to 0^+} A_\lambda\). We first consider the case when the test function \(\phi\) is spherically symmetric.

**Lemma 2.3.** For spherically symmetric function \(\phi(x) = \phi(|x|)\), we have

\[(18) \quad \lim_{\lambda \to 0^+} (A_\lambda \phi, \phi) = (A_0 \phi, \phi) \equiv \int |\nabla \phi|^2 dx + 4 \pi \int f'_0(E) dv \phi^2 dL \]
\[- 32 \pi^3 \int_{\min U_0}^{E} \int_{0}^{\infty} f'_0(E) \left( \frac{r^2_{(E,L)}}{r_1(E,L)} \frac{\phi dr}{\sqrt{2(E-U_0-L^2/2\pi^2)}} \right)^2 dL dE \]
\[= \int |\nabla \phi|^2 + 32 \pi^3 \int f'_0(E) \int_{r_1(E,L)}^{r_2(E,L)} (\phi - \hat{\phi})^2 \frac{dr dE dL}{\sqrt{2(E-U_0-L^2/2\pi^2)}}. \]

**Proof.** Given the steady state \(f_0(E), U_0(|x|)\) and any radial function \(\phi(|x|)\). To find the limit of

\[(19) \quad (A_\lambda \phi, \phi) = \int |\nabla \phi|^2 dx + 4 \pi \int f'_0(E) dv \phi^2 dx \]
\[- 4 \pi \int f'_0(E) \left( \int_{-\infty}^{0} \lambda e^{\lambda s} \phi(X(s;x,v)) ds \right) \phi(x) dL dE, \]

we study the following

\[(20) \quad \lim_{\lambda \to 0^+} \int_{-\infty}^{0} \lambda e^{\lambda s} \phi(X(s;x,v)) ds. \]

Note that we only need to study \[(20)\] for points \((x, v)\) with \(E = \frac{1}{2} |v|^2 + U_0(|x|) < E_0\) and \(L = |x \times v| > 0\), because in the third integral of \[(19)\] \(f'_0(E)\) has support in
\{E < E_0 \} and the set \{L = 0 \} has a zero measure. We recall the linearized Vlasov-Poisson system in the \( r, v_r, L \) coordinates takes the form

\[
\partial_t f + v_r \partial_r f + \left( \frac{L}{r^3} - \partial_r U_0 \right) \partial_{v_r} f = \partial_r U f \partial_{v_r} f_0,
\]

\[
\partial_{rr} U f + \frac{2}{r} \partial_r U f = 4\pi \int f \, dv.
\]

For the corresponding linearized system, for points \((x, v)\) with \(E < E_0\) and \(L > 0\), the trajectory of \((X(s; x, v), V(s; x, v))\) in the coordinate \((r, E, L)\) is a periodic motion described by the ODE (see [8])

\[
\frac{dr(s)}{ds} = v_r(s),
\]

\[
\frac{dv_r(s)}{ds} = -U_0'(r) + \frac{L^2}{r^3}.
\]

with the period

\[
T(E, L) = 2 \int_{r_1(E, L)}^{r_2(E, L)} \frac{dr}{\sqrt{2(E - U_0 - L^2/2r^2)}},
\]

where \(0 < r_1(E, L) \leq r_2(E, L) < +\infty\) are zeros of \(E - U_0 - L^2/2r^2\). So by Lin’s lemma in [25],

\[
\lim_{\lambda \to 0} \int_{-\infty}^{0} \lambda e^{\lambda s} \phi(X(s; x, v)) ds = \frac{1}{T} \int_{0}^{T} \phi(X(s; x, v)) ds.
\]

Since \(\phi(X(s; x, v) = \phi(r(s))\), a change of variable from \(s \to r(s)\) leads to

\[
\int_{0}^{T} \phi(X(s; x, v)) ds = 2 \int_{r_1}^{r_2} \frac{\phi(r) dr}{\sqrt{2(E - U_0 - L^2/2r^2)}}.
\]

For any function \(g(r, E, L)\), we define its trajectory average as

\[
\bar{g}(E, L) = \frac{\int_{r_1(E, L)}^{r_2(E, L)} g(r, E, L) dr}{\int_{r_1(E, L)}^{r_2(E, L)} \frac{dr}{\sqrt{2(E - U_0 - L^2/2r^2)}}}.
\]

Then

\[
\lim_{\lambda \to 0^+} \int_{-\infty}^{0} \lambda e^{\lambda s} \phi(X(s; x, v)) ds = 2 \int_{r_1}^{r_2} \frac{\phi(r) dr}{\sqrt{2(E - U_0 - L^2/2r^2)}} / T(E, L) = \bar{\phi}(E, L).
\]
and the integrand in third term of (19) converges pointwise to \( f_0'(E)\bar{\phi}\phi \). Thus by the dominated convergence theorem, we have

\[
\lim_{\lambda \to 0^+} (A_{\lambda}\phi, \phi) = \int \nabla \phi^2 dx + 4\pi \int f_0'(E)\phi^2 dx dv - 4\pi \int f_0'(E)\bar{\phi}\phi dx dv
\]

\[
= \int \nabla \phi^2 dx + 4\pi \int f_0'(E)\phi^2 dx dv
\]

\[
- 32\pi^3 \int_{\min U_0}^E \int_0^\infty f_0'(E) \int_{r_1(E,L)}^L \phi (E,L) \phi (r) \frac{dr dEdL}{\sqrt{2(E-U_0-L/2r^2)}}
\]

\[
= \int \nabla \phi^2 dx + 4\pi \int f_0'(E)\phi^2 dx dv
\]

\[
- 32\pi^3 \int_{\min U_0}^E \int_0^\infty f_0'(E) \left( \int_{r_1(E,L)}^L \phi (E,L) \phi (r) \frac{dr dEdL}{\sqrt{2(E-U_0-L/2r^2)}} \right)^2 dEdL
\]

This finishes the proof of the lemma. \( \square \)

To compute \( \lim_{\lambda \to 0^+} (A_{\lambda}\phi, \phi) \) for more general test function \( \phi \), we use the following ergodic lemma which is a direct generalization of the result in [26].

**Lemma 2.4.** Consider the solution \((P(s; p, q), Q(s; p, q))\) to be the solution of a Hamiltonian system

\[
\dot{P} = \partial_q H(P, Q)
\]

\[
\dot{Q} = -\partial_p H(P, Q)
\]

with \((P(0), Q(0)) = (p, q) \in \mathbb{R}^n \times \mathbb{R}^n\). Denote

\[
Q^\lambda m = \int_{-\infty}^0 \lambda e^{\lambda s} m(P(s), Q(s)) ds.
\]

Then for any \((p, q) \in L^2(\mathbb{R}^n \times \mathbb{R}^n)\), we have \( Q^\lambda m \to \bar{\mathcal{P}} m \) strongly in \( L^2(\mathbb{R}^n \times \mathbb{R}^n)\). Here \( \bar{\mathcal{P}} \) is the projection operator of \( L^2(\mathbb{R}^n \times \mathbb{R}^n) \) to the kernel of the transport operator \( D = \partial_q H\partial_p - \partial_p H\partial_q \) and \( \bar{\mathcal{P}} m \) is the phase space average of \( m \) in the set traced by the trajectory.

**Proof.** Denote \( U(s) : L^2(\mathbb{R}^n \times \mathbb{R}^n) \to L^2(\mathbb{R}^n \times \mathbb{R}^n) \) to be the unitary semigroup \((U(s) m = m(P(s), Q(s))\). By Stone Theorem ([40]), \( U(s) \) is generated by \( iR = D \), where \( R = -iD \) is self-adjoint and

\[
U(s) = \int_{-\infty}^{+\infty} e^{ias} dM_\alpha
\]

where \( \{M_\alpha; \alpha \in \mathbb{R}^1\} \) is spectral measure of \( R \). So

\[
\int_{-\infty}^0 \lambda e^{\lambda s} m(P(s), Q(s)) ds = \int_{-\infty}^0 \lambda e^{\lambda s} \int_{\mathbb{R}} e^{ias} dM_\alpha m ds = \int_{\mathbb{R}} \frac{\lambda}{\lambda + i\alpha} dM_\alpha m.
\]
On the other hand, the projection is \( P = M_{\{0\}} = \int_{\mathbb{R}} \xi dM_\alpha \) where \( \xi(\alpha) = 0 \) for \( \alpha \neq 0 \) and \( \xi(0) = 1 \). Therefore

\[
\left\| \int_{-\infty}^{0} \lambda e^{\lambda s} m(P(s), Q(s)) ds - Pm \right\|_{L^2}^2 = \int_{\mathbb{R}} \left| \frac{\lambda}{\lambda + i\alpha} - \xi(\alpha) \right|^2 d\|M_\alpha m\|_{L^2}^2
\]

by orthogonality of the spectral projections. By the dominated convergence theorem this expression tends to 0 as \( \lambda \to 0^+ \), as we wished to prove. The explanation of \( Pm \) as the phase space average of \( m \) is in our remark below. \( \square \)

**Remark 1.** Since \( \int_{-\infty}^{0} \lambda e^{\lambda s} ds = 1 \), the function

\[
(Q^\lambda m)(x, v) = \int_{-\infty}^{0} \lambda e^{\lambda s} m(P(s), Q(s)) ds
\]

is a weighted time average of the observable \( m \) along the particle trajectory. By the same proof of Lemma 2.4, we have

\[
\lim_{T \to -\infty} \frac{1}{T} \int_{0}^{T} m(P(s), Q(s)) ds = Pm.
\]

But from the standard ergodic theory (3) of Hamiltonian systems, the limit of the above time average in (22) equals the phase space average of \( m \) in the set traced by the trajectory. Thus \( Pm \) has the meaning of the phase space average of \( m \) and Lemma 2.4 states that the limit of the weighted time average (21) yields the same phase space average. In particular, if the particle motion is ergodic in the invariant set \( S_I \) determined by the invariants \( E_1, \ldots, I_k \), and if \( d\sigma_I \) denotes the induced measure of \( \mathbb{R}^n \times \mathbb{R}^n \) on \( S_I \), then

\[
\mathcal{P}m = \frac{1}{\sigma_I(S_I)} \int_{S_I} m(p, q) d\sigma_I(p, q).
\]

For integral systems, using action angle variables \((J_1, \ldots, J_n; \varphi_1, \ldots, \varphi_n)\) we have

\[
(\mathcal{P}m)(J_1, \ldots, J_n) = (2\pi)^{-n} \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} m(J_1, \ldots, J_n, \varphi_1, \ldots, \varphi_n) d\varphi_1 \cdots d\varphi_n
\]

for the generic case with independent frequencies (see 1).

Recall the weighted \( L^2 \) space \( L^2_{[f_0]} \) in (5). Then \( U(s) : L^2_{[f_0]} \to L^2_{[f_0]} \) defined by \( U(s)m = m(X(s; x, v), V(s; x, v)) \) is an unitary group, where \( (X(s; x, v), V(s; x, v)) \) is the particle trajectory (8). The generator of \( U(s) \) is \( D = v \cdot \partial_x - \nabla_x U_0 \cdot \nabla_v \) and \( R = -iD \) is self-adjoint by Stone Theorem. By the same proof, Lemma 2.4 is still valid in \( L^2_{[f_0]} \). In particular, for any \( \phi(x) \in L^2(\mathbb{R}^3) \) we have

\[
\int_{-\infty}^{0} \lambda e^{\lambda s} \phi(X(s; x, v)) ds \to \mathcal{P}\phi
\]

in \( L^2_{[f_0]} \), where \( \mathcal{P} \) is the projector of \( L^2_{[f_0]} \) to ker \( D \).

Now we derive an explicit formula for the above limit \( \mathcal{P}\phi \). Note that as in the proof of lemma 2.3 we only need to derive the formula of \( \mathcal{P}\phi \) for points \((x, v)\) with \( E < E_0 \) and \( L > 0 \). Since \( U_0(x) = U_0(r) \), the particle motion (13) in such a center field is integrable and has been well studied (see e.g. 8, 4). For particles with
energy $E < E_0 < 0$, $L > 0$ and momentum $\vec{L} = x \times v$, the particle orbit is a rosette in the annulus

$$A_{E,L} = \{ r_1(E,L) \leq r \leq r_2(E,L) \} = \{ E - U_0 - L^2/2r^2 \geq 0 \},$$

lying on the orbital plane perpendicular to $\vec{L}$. So we can consider the particle motion to be planar. For such case, the action-angle variables are as follows (see e.g. [30]): the actions variables are

$$J_r = \frac{2\pi}{T(E,L)}, \quad J_\theta = L,$$

where

$$T(E,L) = 2 \int_{r_1(E,L)}^{r_2(E,L)} \frac{dr}{\sqrt{2(E - U_0 - L^2/2r^2)}},$$

is the radial period, the angle variable $\varphi_r$ is determined by

$$d\varphi_r = \frac{2\pi}{T(E,L)} \frac{dr}{\sqrt{2(E - U_0 - L^2/2r^2)}},$$

and $\varphi_\theta = \theta - \Delta \theta$ where

$$d(\Delta \theta) = \frac{Lr^{-2} - \Omega_\theta}{\sqrt{2(E - U_0 - L^2/2r^2)}} dr$$

and

$$\Omega_\theta(E,L) = \frac{1}{T(E,L)} \int_{r_1(E,L)}^{r_2(E,L)} \frac{L}{r^2 \sqrt{2(E - U_0 - L^2/2r^2)}} dr$$

is the average angular velocity. For any function $\phi(x) \in H^2(\mathbb{R}^3)$, we denote $\phi_E(r,\theta)$ to be the restriction of $\phi$ in the orbital plane perpendicular to $\vec{L}$. Then by (24), for the generic case when the radial and angular frequencies are independent, we have

$$(P\phi)(E,L) = (2\pi)^{-2} \int_0^{2\pi} \int_0^{2\pi} \phi_E d\varphi_\theta d\varphi_r$$

$$= \frac{1}{\pi T(E,L)} \int_{r_1(E,L)}^{r_2(E,L)} \int_0^{2\pi} \frac{\phi_E(r,\theta) d\theta dr}{\sqrt{2(E - U_0 - L^2/2r^2)}}$$

In particular, for a spherically symmetric function $\phi = \phi(r)$, we recover

$$(P\phi)(E,L) = \frac{2}{T(E,L)} \int_{r_1(E,L)}^{r_2(E,L)} \phi(r) dr \int_0^{2\pi} \frac{\phi(r) d\theta}{\sqrt{2(E - U_0 - L^2/2r^2)}}.$$  

We thus conclude the following

**Lemma 2.5.** Assume that $f_0(E)$ has a bounded support in $x$ and $v$ and $f'_0$ is bounded. For any $\phi \in H^1(\mathbb{R}^3)$, we have

$$(A_\lambda \phi, \phi) = (A_0 \phi, \phi)$$

$$= \int |\nabla \phi|^2 dx + 4\pi \int f'_0(E) dv \phi^2 dx - 4\pi \int f'_0(E) (P\phi)^2 dx dv$$

$$= \int |\nabla \phi|^2 dx + 4\pi \int f'_0(E) (\phi - P\phi)^2 dx dv.$$
where $\mathcal{P}$ is the projector of $L^2_{[0,1]}$ to $\ker D$ and more explicitly $\mathcal{P}\phi$ is given by (26).

The limiting operator $A_0$ is

$$A_0\phi = -\Delta \phi + [4\pi \int f'_0(E) dv] \phi - 4\pi \int f'_0(E) \mathcal{P}\phi dv. \quad (29)$$

Now we give the proof of the instability criterion.

**Proof of Theorem 1.1.** We define

$$\lambda_* = \sup_{k(\lambda)<0} \lambda.$$

By Lemmas 2.1 and 2.5 we deduce that

$$-\infty < \lambda_* \leq \Lambda < \infty.$$

Therefore, by the continuity of $k(\lambda)$, we have

$$k(\lambda_*) = 0.$$

Hence, there exists an increasing sequence of $\lambda_n < \lambda_{n+1} < \lambda_*$ so that $\lambda_n \to \lambda_*$, $k_n \equiv k(\lambda_n) < 0$, and

$$k_n \to k(\lambda_*) = 0.$$

Therefore, $k_n$ are negative eigenvalues. By Lemma 2.2 we get a sequence $\phi_n \in H^2$ such that

$$A_{\lambda_n} \phi_n = k_n \phi_n \quad (30)$$

with $k_n < 0$, $k_n \to 0$ and $\lambda_n \to \lambda_0 > 0$, as $n \to \infty$. Recall $\chi$ the cutoff function of the support of $f_0(E)$ such that $\chi \equiv 1$ for $f_0(E) > 0$. We claim that $\chi \phi_n$ is a nonzero function for any $n$. Suppose otherwise, $\chi \phi_n \equiv 0$, then from the equation (30) we have $(-\Delta - k_n) \phi_n = 0$ which implies that $\phi_n = 0$, a contradiction. Thus we can normalize $\phi_n$ by $\|\chi \phi_n\|_2 = 1$. Taking inner product of (30) with $\phi_n$ and integrating by parts, we have

$$\|\nabla \phi_n\|_2^2 \leq -4\pi \int \int f'_0(E) \phi_n^2 dv dx + \int 4\pi f'_0(E) \int_{-\infty}^{0} \lambda_n e^{\lambda_n s} \phi_n(X(s;x,v)) ds \phi_n(x) dx$$

$$= -4\pi \int \int f'_0(E) (\chi \phi_n)^2 dv dx$$

$$+ \int \int 4\pi f'_0(E) \int_{-\infty}^{0} \lambda_n e^{\lambda_n s} (\chi \phi_n)(X(s;x,v)) ds (\chi \phi_n)(x) dx$$

$$\leq 8\pi \int f'_0(E) dv \int_{-\infty}^{\infty} ||\chi \phi_n||_2^2.$$

Here in the second equality above, we use the fact $\chi = 1$ on the support of $f'_0(E)$ ($f_0(E)$) and that $(\chi \phi_n)(X(s;x,v)) = \phi_n(X(s;x,v))\chi$ due to the invariance of the support under the trajectory flow, as in (13). In the last inequality, we use the same estimate as in (10). Thus,

$$\sup_n ||\phi_n||_{L^6} \leq C \sup_n ||\nabla \phi_n||_2 < C',$$

for some constant $C'$ independent of $n$. Then there exists $\phi \in L^6$ and $\nabla \phi \in L^2$ such that

$$\phi_n \to \phi \text{ weakly in } L^6, \quad \text{and } \nabla \phi_n \to \nabla \phi \text{ weakly in } L^2.$$
This implies that $\chi \phi_n \to \chi \phi$ strongly in $L^2$. Therefore $\|\chi \phi\|_2 = 1$ and thus $\phi \neq 0$. It is easy to show that $\phi$ is a weak solution of $A_{\lambda_0} \phi = 0$ or

\begin{equation}
\Delta \phi = -[4\pi \int f'_0(E) dv] \phi + 4\pi f'_0(E) \int_{-\infty}^{0} \lambda_0 e^{\lambda_0 s} \phi(X(s; x, v)) ds dv = \rho.
\end{equation}

We have that

$$
\int \rho dx = -4\pi \int \int f'_0(E) \phi(x) dx dv + \int_{-\infty}^{0} \lambda_0 e^{\lambda_0 s} \int 4\pi f'_0(E) \phi(X(s; x, v)) dx dv ds
$$

$$
= -4\pi \int \int f'_0(E) \phi(x) dx dv + \int_{-\infty}^{0} \lambda_0 e^{\lambda_0 s} \int 4\pi f'_0(E) \phi(x) dx dv ds = 0
$$

and by (31) $\rho$ has compact support in $S_\ast$, the $x$–support of $f_0(E)$. Therefore from the formula $\phi(x) = \int \frac{\rho(y)}{|x-y|} dy$, we have

$$
\phi(x) = \int \frac{\rho(y)}{|x-y|} dy = \int \frac{\rho(y)}{|x-y|} dy - \int \frac{\rho(y)}{|x|} dy = O \left(|x|^{-2}\right),
$$

for $x$ large, and thus $\phi \in L^2$. By elliptic regularity, $\phi \in H^2$. We define $f(x, v)$ by (14), then $f \in L^\infty$ with the compact support in $S$. Now we show that $e^{\lambda_0 t}[f, \phi]$ is a weak solution to the linearized Vlasov-Poisson system. Since $\phi$ satisfies the Poisson equation (31), we only need to show that $f$ satisfies the linearized Vlasov equation (12) weakly. For that, we take any $g \in C^1_c \left(\mathbb{R}^3 \times \mathbb{R}^3\right)$, and

$$
\int \int_{\mathbb{R}^3 \times \mathbb{R}^3} (Dg) f dx dv
$$

$$
= \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} (Dg) \left(f'_0(E) \phi(x)\right) dx dv - \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} (Dg) f'_0(E) \int_{-\infty}^{0} \lambda_0 e^{\lambda_0 s} \phi(X(s; x, v)) ds dx dv
$$

$$
= I + II.
$$

Since $D$ is skew-adjoint, the first term is

$$
I = - \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} g D \left(f'_0(E) \phi\right) dx dv = - \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f'_0(E) g D \phi dx dv.
$$
For the second term,
\[
II = - \int_{-\infty}^{0} \lambda_0 e^{\lambda_0 s} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f'_0(E) Dg(x, v) \phi (X(s; x, v)) dxdvds
\]
\[
= - \int_{-\infty}^{0} \lambda_0 e^{\lambda_0 s} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f'_0(E) (Dg) (X(-s), V(-s)) \phi (x) dxdvds
\]
\[
= - \int_{\mathbb{R}^3 \times \mathbb{R}^3} f'_0(E) \left\{ \lambda_0 s \phi (x) - \int_{-\infty}^{0} \lambda_0 e^{\lambda_0 s} g (X(-s), V(-s)) ds \right\} \phi (x) dxdv
\]
\[
= \lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left\{ f'_0(E) \phi (x) - f'_0(E) \int_{-\infty}^{0} \lambda_0 e^{\lambda_0 s} \phi (X(s), V(s)) ds \right\} \phi (x) dxdv
\]
\[
= \lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3} g dxdv.
\]
Thus we have
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} (Dg) f dxdv = \int_{\mathbb{R}^3 \times \mathbb{R}^3} (\lambda_0 f - f'_0(E) D\phi) g dxdv
\]
which implies that \( f \) is a weak solution to the linearized Vlasov equation
\[
\lambda_0 f + Df = f'_0(E) v \cdot \nabla x \phi.
\]
\[
\square
\]

**Remark 2.** Consider an anisotropic spherical galaxy with \( f_0 (x, v) = f_0 (E, L^2) \). For a radial symmetric growing mode \( e^{\lambda t} (\phi, f) \) with \( \phi = \phi (|x|) \) and \( f = f (|x|, E, L^2) \). The linearized Vlasov equation (17) becomes
\[
\lambda f + v \cdot \nabla x f - \nabla x U_0 \cdot \nabla_v f = \nabla x \phi \cdot \nabla_v f_0 = \nabla x \phi \cdot \left( \frac{\partial f_0}{\partial E} v + \frac{\partial f_0}{\partial L^2} (|x \times v|) \right)
\]
\[
= \phi (|x|) \frac{x}{|x|} \cdot \left( \frac{\partial f_0}{\partial E} v + 2 \frac{\partial f_0}{\partial L^2} [(x \times v) \times x] \right) = \frac{\partial f_0}{\partial E} v \cdot \nabla x \phi,
\]
which is of the same form as in the isotropic case (20). So by the same proof of Theorem 1.1, we also get an instability criterion for radial perturbations of anisotropic galaxy, in terms of the quadratic form (18) with \( f'_0(E) \) being replaced by \( \frac{\partial f_0}{\partial E} \).

### 3. Nonlinear Stability of the King’s Model

In the second half of the article, we investigate the nonlinear stability of the King model (25). We first establish:

**Lemma 3.1.** Consider spherical models \( f_0 = f_0 (E) \) with \( f'_0 < 0 \). The operator \( A_0 : H^2_r \rightarrow L^2_r \)
\[
A_0 \phi = -\Delta \phi + [4\pi \int f'_0 dv] \phi - 4\pi \int f'_0 P \phi dv
\]
is positive, where $H^2_2$ and $L^2_2$ are spherically symmetric subspaces of $H^2$ and $L^2$, and the projection $\mathcal{P}\phi$ is defined by \cite{27}. Moreover, for $\phi \in H^2_2$ we have

\begin{equation}
(A_0\phi, \phi) \geq \varepsilon \left( |\nabla \phi|_2^2 + |\phi|_2^2 \right)
\end{equation}

for some constant $\varepsilon > 0$.

Proof. Define $k_0 = \inf (A_0\phi, \phi) / (\phi, \phi)$ We want to show that $k_0 > 0$. First, by using the compact embedding of $H^2_2 \hookrightarrow L^2_2$ it is easy to show that the minimum can be obtained and $k_0$ is the lowest eigenvalue. Let $A_0\phi_0 = k_0\phi_0$ with $\phi_0 \in H^2_2$ and $\|\phi_0\|_2 = 1$. The fact that $k_0 \geq 0$ follows immediately from Theorem \cite{12} and the nonexistence of radial modes \cite{13, 22} for monotone spherical models. The proof of $k_0 > 0$ is more delicate. For that, we relate the quadratic form $(A_0\phi, \phi)$ to the Antonov functional \cite{4}. We define $D = v \cdot \partial_x - \nabla_xU_0 \cdot \nabla_v$ to be the generator of the unitary group $U(s):L^2_{\phi_0}[f_0] \rightarrow \mathbb{C}^r_{\phi_0}$ defined by $U(s)m = m(X(s; x, v), V(s; x, v))$.

Here $L^2_{\phi_0}[f_0]$ is the spherically symmetric subspace of $L^2_{\phi_0}[f_0]$, which is preserved under the flow mapping $U(s)$. By the definition of $\mathcal{P}\phi$, we have $\phi_0 - \mathcal{P}\phi_0 \perp \ker D$. By Stone theorem $iD$ is self-adjoint and in particular $D$ is closed. Therefore by the closed range theorem \cite{2}, we have $(\ker D)^\perp = R(D)$, where $R(D)$ is the range of $D$. So there exists $h \in L^2_{\phi_0}[f_0]$ such that $Dh = \phi_0 - \mathcal{P}\phi_0$. Moreover, since $\phi_0 - \mathcal{P}\phi_0$ is even in $v$ and the operator $D$ reverses the parity in $v$, the function $h$ is odd in $v$.

Define $f^- = f^0_h$. We have

\begin{align*}
k_0 &= (A_0\phi_0, \phi_0) = \int |\nabla \phi_0|^2 dx + 4\pi \int \int f^0_h(\phi_0 - \mathcal{P}\phi_0)^2 dx dv \\
&= \int |\nabla \phi_0|^2 dx - 8\pi \int \int |f^0_h| (\phi_0 - \mathcal{P}\phi_0) \phi_0 dx dv \\
&\quad + 4\pi \int \int |f^0_h| (\phi_0 - \mathcal{P}\phi_0)^2 dx dv \\
&= 4\pi \left( \int \int \frac{|Df^+|^2}{|f_0^h|} dx dv + 2 \int \phi_0 \int Df^- dv dx + \frac{1}{4\pi} \int |\nabla \phi_0|^2 dx \right) \\
&= 4\pi \left( \int \int \frac{|Df^+|^2}{|f_0^h|} dx dv + \frac{1}{2\pi} \int \phi_0 \Delta \phi^- dx + \frac{1}{4\pi} \int |\nabla \phi_0|^2 dx \right) \\
&= 4\pi \left( \int \int \frac{|Df^+|^2}{|f_0^h|} dx dv + \frac{1}{4\pi} \int \left( |\nabla \phi_0|^2 - 2\nabla \phi_0 \cdot \nabla \phi^- \right) dx \right) \\
&\geq 4\pi \left( \int \int \frac{|Df^+|^2}{|f_0^h|} dx dv - \frac{1}{4\pi} \int |\nabla \phi^-|^2 dx \right)
\end{align*}

where $\Delta \phi^- = 4\pi \int Df^- dv$. Notice that the last expression above is the Antonov functional $4\pi H(f^-, f^-)$. Since $f^-$ is spherical symmetric and odd in $v$, we have $H(f^-, f^-) > 0$ by the proof in \cite{22} which was further clarified in \cite{33} and \cite{21}. Therefore we get $k_0 > 0$ as desired and $(A_0\phi, \phi) \geq k_0 |\phi|_2^2$. 
To get the estimate (32), we rewrite

\[(A_0 \phi, \phi) = \varepsilon \left( \int |\nabla \phi|^2 \, dx + 4\pi \int f'_0 (\phi - \mathcal{P} \phi)^2 \, dv \right) + (1 - \varepsilon) (A_0 \phi, \phi) \geq \varepsilon \int |\nabla \phi|^2 \, dx - 4\pi \varepsilon \|\phi - \mathcal{P} \phi\|_{L^2_r[|\xi|]}^2 + (1 - \varepsilon) k_0 |\phi|_2^2 \geq \varepsilon \int |\nabla \phi|^2 \, dx + ((1 - \varepsilon) k_0 - C\varepsilon) |\phi|_2^2 \geq \varepsilon \left( \int |\nabla \phi|^2 \, dx + |\phi|_2^2 \right)\]

if \(\varepsilon\) is small enough.

Next, we will approximate the \(\text{ker} D\) by a finite dimensional approximation. Let \(\{\xi_i(E, L) = \alpha_i(E) \beta_i(L)\}_{i=1}^\infty\) be a smooth orthogonal basis for the subspace \(\text{ker} D = \{g(E, L)\} \subset L^2_{[|\xi|]}\). Define the finite-dimensional projection operator \(\mathcal{P}_N : L^2_{[|\xi|]} \to L^2_{[|\xi|]}\) by

\[
\mathcal{P}_N h = \sum_{i=1}^N (h, \xi_i)_{|\xi_i|} \xi_i
\]

and the operator \(A^N : H^2_r \to L^2_r\) by

\[
A^N \phi = -\Delta \phi + [4\pi \int f_0' \, dv] \phi - 4\pi \int f_0' \mathcal{P}_N \phi \, dv.
\]

**Lemma 3.2.** There exists \(K, \delta_0 > 0\) such that when \(N > K\) we have

\[
(A^N \phi, \phi) \geq \delta_0 |\nabla \phi|_2^2
\]

for any \(\phi \in H^2_r\).

**Proof.** First we have \(A^N \to A_0\) strongly in \(L^2\). Indeed, for any \(\phi \in H^2_r\),

\[
\left\| A^N \phi - A_0 \phi \right\|_2 = \left\| 4\pi f_0' (\mathcal{P}_N \phi - \mathcal{P} \phi) \right\|_2 \leq C \|\mathcal{P}_N \phi - \mathcal{P} \phi\|_{L^2_{[|\xi|]}}, \to 0 \quad \text{as } N \to \infty.
\]

We claim that for \(N\) sufficiently large, the lowest eigenvalue of \(A^N\) is at least \(k_0/2\) when \(k_0 > 0\) is the lowest eigenvalue of \(A_0\). Suppose otherwise, then there exists a sequence \(\{\lambda_n\}\) and \(\{\phi_n\} \subset H^2_r\) with \(\lambda_n < k_0/2\), \(\|\phi_n\|_2 = 1\) and \(A^N \phi_n = \lambda_n \phi_n\). This implies that \(\Delta \phi_n\) is uniformly bounded in \(L^2\), by elliptic estimate we have \(\|\phi_n\|_{H^2} \leq C\) for some constant \(C\) independent of \(n\). Therefore there exists \(\phi_0 \in H^2_r\) such that \(\phi_n \to \phi_0\) weakly in \(H^2_r\). By the compact embedding of \(H^2_r \hookrightarrow L^2_r\), we have \(\phi_n \to \phi_0\) strongly in \(L^2_r\) and \(\|\phi_0\|_2 = 1\). The strong convergence of \(A^N \phi_0 \to A_0 \phi_0\) implies that

\[
A^N \phi_n \to A_0 \phi_0
\]

weakly in \(L^2\). Let \(\lambda_n \to \lambda_0 \leq k_0/2\), then we have \(A_0 \phi_0 = \lambda_0 \phi_0\), a contradiction. Therefore we have \((A^N \phi, \phi) \geq k_0/2 |\phi|_2^2\) for \(\phi \in H^2_r\), when \(N\) is large enough. The estimate (34) is by the same proof of (32) in Lemma 3.1. \(\square\)
Recalling (3) with \( f_0 = [e^{E_0 - E} - 1]_+ \) and \( Q_0(f) = (f + 1) \ln(f + 1) - f \), we further define functionals (related to the finite dimensional approximation of \( \ker D \)) as

\[
A_i(f) = \int_0^f \alpha_i(-\ln(s + 1) + E_0) ds,
\]

\[
Q_i(f, L) = A_i(f) \beta_i(L), \quad \text{for } 1 \leq i \leq N.
\]

for \( 1 \leq i \leq N \). Clearly,

\[
\partial_i Q_i(f_0, L) = \alpha_i(-\ln(f_0 + 1) + E_0) \beta_i(L) = \alpha_i(E) \beta_i(L) = \xi_i(E, L),
\]

where \( \{\xi_i(E, L)\}_{i=1}^N \) are used to define \( \mathcal{P}_N \) in Lemma 3.2. Define the Casimir functional \( (E_0 < 0) \)

\[
I(f) = \int [Q_0(f) + \frac{1}{2}|v|^2 f - E_0 f] dxdv - \frac{1}{8\pi} \int |\nabla \phi|^2 dx
\]

which is invariant of the nonlinear Vlasov-Poisson system. We introduce additional \( N \) invariants

\[
J_i(f, L) = \int Q_i(f, L) dxdv.
\]

for \( 1 \leq i \leq N \). We define \( \Omega \) to be the support of \( f_0 (E) \). We first consider

\[
I(f) - I(f_0) = \int [Q_0(f) - Q_0(f_0) + \frac{1}{2}|v|^2 (f - f_0) - E_0 (f - f_0)] dxdv
\]

\[
= \int [Q_0(f) - Q_0(f_0) + (E - E_0) (f - f_0)] dxdv - \frac{1}{8\pi} \int |\nabla (U - U_0)|^2 dx.
\]

We define

\[
g = f - f_0, \quad \phi = U - U_0
\]

and

\[
g_{in} \equiv (f - f_0) 1_{\Omega}, \quad g_{out} \equiv (f - f_0) 1_{\Omega^c}, \quad \Delta \phi_{in} \equiv \int g_{in}, \quad \Delta \phi_{out} \equiv \int g_{out}.
\]

And we define the distance function for nonlinear stability as

\[
d(f, f_0) = \int [Q_0(g_{in} + f_0) - Q_0(f_0) + (E - E_0) g_{in}] dxdv + \frac{1}{8\pi} \int |\nabla \phi_{in}|^2 dx
\]

\[
+ \left\{ \int Q_0(g_{out}) dxdv + \int_{E \geq E_0} (E - E_0) g_{out} dxdv \right\}
\]

\[
= d_{in} + \frac{1}{8\pi} \int |\nabla \phi_{in}|^2 dx + d_{out},
\]

for which each term is non-negative. We therefore split:

\[
I(f) - I(f_0)
\]

\[
= \left\{ \int [Q_0(f_0 + g_{in}) - Q_0(f_0) + (E - E_0) g_{in}] dxdv - \frac{1}{8\pi} \int |\nabla \phi_{in}|^2 dx \right\} +
\]

\[
\left\{ \int Q_0(g_{out}) dxdv + \int_{E \geq E_0} (E - E_0) g_{out} dxdv - \frac{1}{8\pi} \int |\nabla \phi_{out}|^2 dx - \frac{1}{4\pi} \int \nabla \phi_{out} \cdot \nabla \phi_{in} dx \right\}
\]

\[
= I_{in} + I_{out}.
\]
Lemma 3.3. For $\varepsilon > 0$ sufficiently small, we have
\[
\int |\nabla \phi_{\text{out}}|^2 dx \leq C \left( \varepsilon d(f, f_0) + \frac{1}{\varepsilon^{5/4}} [d(f, f_0)]^{5/3} \right).
\]

Proof. In fact, since
\[
\int |\nabla \phi_{\text{out}}|^2 dx \leq C || \int g_{\text{out}} dv ||^2_{L^{6/5}}
\]
\[
\leq C || \int g_{\text{out}} 1_{E_0 \leq E \leq E_0 + \varepsilon} dv ||^2_{L^{6/5}} + C || \int g_{\text{out}} 1_{E > E_0 + \varepsilon} dv ||^2_{L^{6/5}}.
\]
The first term is bounded by
\[
\left[ \int \left[ \int g_{\text{out}}^2 dv \right]^{3/5} \left[ \int 1_{E_0 \leq E \leq E_0 + \varepsilon} dv \right]^{3/5} dx \right]^{5/3}
\]
\[
\leq \left[ \int g_{\text{out}}^2 dv dx \right] \times \left[ \int [\int 1_{E_0 \leq E \leq E_0 + \varepsilon} dv]^{3/2} dx \right]^{2/3}
\]
\[
\leq C \varepsilon [\int g_{\text{out}}^2 dv dx] \leq C \varepsilon [\int g_{\text{out}}^2 dv dx]
\]
\[
\leq C \varepsilon d(f, f_0).
\]
In the above estimates, we use that $\int \int Q_0(g_{\text{out}}) dv dx \geq c \int g_{\text{out}}^2 dv dx$ and
\[
\int 1_{E_0 \leq E \leq E_0 + \varepsilon} dv \leq C \varepsilon,
\]
which can be checked by an explicit computation when $\varepsilon > 0$ is sufficiently small such that $E_0 + \varepsilon \leq 0$.

On the other hand, by the standard estimates (see [12, P. 120-121])
\[
|| \int g_{\text{out}} 1_{E > E_0 + \varepsilon} dv ||^2_{L^{6/5}}
\]
\[
\leq \left[ \int \int g_{\text{out}} 1_{E > E_0 + \varepsilon} dv dx \right]^{\frac{\frac{2}{3}}{\frac{5}{3}}} \times \left[ \int \int |v|^2 g_{\text{out}} 1_{E > E_0 + \varepsilon} dv dx \right]^{\frac{\frac{2}{3}}{\frac{5}{3}}}
\]
\[
\leq \left[ \frac{1}{\varepsilon} \int (E - E_0) g_{\text{out}} 1_{E > E_0 + \varepsilon} dv dx \right]^{\frac{\frac{2}{3}}{\frac{5}{3}}}
\]
\[
\times \left[ \int \int (E - E_0) g_{\text{out}} 1_{E > E_0 + \varepsilon} dv dx + 2 \sup U_0 [\int \int g_{\text{out}} 1_{E > E_0 + \varepsilon} dv dx] \right]^{\frac{\frac{2}{3}}{\frac{5}{3}}}
\]
\[
\leq \left( \frac{1}{\varepsilon} \right)^{\frac{\frac{2}{3}}{\frac{5}{3}}} \left( d + \frac{2 \sup U_0}{\varepsilon} d \right)^{\frac{\frac{2}{3}}{\frac{5}{3}}} \leq \frac{C}{\varepsilon^{5/3}} d^{5/3}.
\]
\[\square\]
By Lemma 3.3 we have
\[ \left| \int \nabla \phi_{\text{out}} \cdot \nabla \phi_{\text{in}} \, dx \right| \leq \| \nabla \phi_{\text{out}} \|_2 \| \nabla \phi_{\text{in}} \|_2 \]
\[ \leq C \left( \varepsilon^{1/3} d(f, f_0) + \frac{1}{\varepsilon^{5/6}} [d(f, f_0)]^{4/3} \right) \]
and therefore for \( \varepsilon \) sufficiently small,
\[ I_{\text{out}} \geq d_{\text{out}} - C \left( \varepsilon^{1/3} d(f, f_0) + \frac{1}{\varepsilon^{5/6}} [d(f, f_0)]^{4/3} + \frac{1}{\varepsilon^{5/3}} [d(f, f_0)]^{5/3} \right). \] (36)

To estimate \( I_{\text{in}} \), we split it into three parts:
\[ \tau \left\{ \int [Q_0(f_0 + g_{\text{in}}) - Q_0(f_0) + (E - E_0)g_{\text{in}} + \phi_{\text{in}}g_{\text{in}}]dxdv + \frac{1}{8\pi} \int |\nabla \phi_{\text{in}}|^2 dx \right\} + \]
\[ (1 - \tau) \left\{ \int [Q_0(f_0 + g_{\text{in}}) - Q_0(f_0) + (E - E_0)g_{\text{in}} + (I - P_N)\phi_{\text{in}}g_{\text{in}}]dxdv + \frac{1}{8\pi} \int |\nabla \phi_{\text{in}}|^2 dx \right\} + \]
\[ (1 - \tau) \int P_N \phi_{\text{in}}g_{\text{in}}dxdv \] (37)
\[ = I_{\text{in}}^1 + I_{\text{in}}^2 + I_{\text{in}}^3, \]
where \( \Delta \phi_{\text{in}} = 4\pi \int g_{\text{in}} \, dv \). We estimate each term in the following lemmas.

Lemma 3.4.
\[ I_{\text{in}}^1 \geq \frac{\tau}{2} d_{\text{in}} - C\tau \int |\nabla \phi_{\text{in}}|^2 dx. \] (38)

Proof. In fact, since the integration region \( \Omega \) is finite, we have
\[ I_{\text{in}}^1 = \tau \left[ \int \int [Q_0(f_0 + g_{\text{in}}) - Q_0(f_0) + (E - E_0)g_{\text{in}} + \phi_{\text{in}}g_{\text{in}}]dxdv + \frac{1}{8\pi} \int |\nabla \phi_{\text{in}}|^2 dx \right] \]
\[ \geq \tau \int \int [Q_0(f_0 + g_{\text{in}}) - Q_0(f_0) + (E - E_0)g_{\text{in}}]dxdv - C\tau \| \phi_{\text{in}} \|_{L^6} \| g_{\text{in}} \|_{L^{6/5}} \]
\[ \geq \tau \int \int [Q_0(f_0 + g_{\text{in}}) - Q_0(f_0) + (E - E_0)g_{\text{in}}]dxdv - C'\tau \| \nabla \phi_{\text{in}} \|_{L^2} \| g_{\text{in}} \|_2 \]
\[ \geq \frac{\tau}{2} d_{\text{in}} - C''\tau \| \nabla \phi_{\text{in}} \|_2^2, \]
since
\[ d_{\text{in}} = \int [Q_0(f_0 + g_{\text{in}}) - Q_0(f_0) + (E - E_0)g_{\text{in}}]dxdv \geq C \| g_{\text{in}} \|_2^2. \]

To estimate \( I_{\text{in}}^2 \), we need the following pointwise duality lemma from elementary calculus.

Lemma 3.5. For any \( c \), and any \( h \), we have
\[ g_{c, f_0}(h) = Q_0(h + f_0) - Q_0(f_0) - Q_0'(f_0)h - ch \geq (f_0 + 1)(1 + c - e^c). \]

Proof. Direct computation yields that the minimizer \( f_c \) of \( g_{c, f_0}(h) \) satisfies the Euler-Lagrange equation
\[ \ln (f_c + f_0 + 1) - \ln (f_0 + 1) - c = 0, \]
Lemma 3.6.

\[ I_{in}^2 \geq \frac{(1 - \tau) \delta_0}{8\pi} \int |\nabla \phi_{in}|^2 dx - C e^{C'\delta_0^2} d^2. \]

Proof. Recall (37). By using Lemma 3.5 for \( c = - (\phi_{in} - P_N \phi_{in}) \) and using the Taylor expansion, we have

\[ I_{in}^2 = (1 - \tau) \int \left[ Q_0(f_0 + g_{in}) - Q_0(f_0) + (E - E_0) g_{in} + (\phi_{in} - P_N \phi_{in}) f_{in} \right] dx dv \]

\[ + \frac{1}{8\pi} (1 - \tau) \int |\nabla \phi_{in}|^2 dx \]

\[ \geq \frac{1}{8\pi} (1 - \tau) \int |\nabla \phi_{in}|^2 dx + (1 - \tau) \int (f_0 + 1) \Omega(1 + \phi_{in} - P_N \phi_{in} - e^{\phi_{in} - P_N \phi_{in}}) dx dv \]

\[ \geq \frac{1}{8\pi} \left\{ \int |\nabla \phi_{in}|^2 dx - 4\pi \int |f'_0(E)| (\phi_{in} - P_N \phi_{in})^2 dx dv \right\} \]

\[ - C e^{(\phi_{in} - P_N \phi_{in})_{\infty}} \int \int |f'_0(E)| |\phi_{in} - P_N \phi_{in}|^3 dx dv \quad (\text{Note } f_0(E) + 1) \Omega = |f'_0(E)|) \]

\[ \geq \frac{(1 - \tau) \delta_0}{8\pi} \int |\nabla \phi_{in}|^2 dx - C e^{(\phi_{in} - P_N \phi_{in})_{\infty}} \int \int |f'_0(E)| |\phi_{in} - P_N \phi_{in}|^3 dx dv. \]

In the last line, we have used Lemma 3.2. To estimate the last term above and conclude our lemma, it suffices to show

\[ |\phi_{in} - P_N \phi_{in}|_{\infty} \leq C_N d^{\frac{3}{2}}. \]

This follows from the facts that for the fixed \( N \) smooth functions \( \xi_i \), we have

\[ |P_N \phi_{in}|_{\infty} \leq \sum_{i=1}^{N} |(\phi_{in}, \xi_i)|_{j_0} |\xi_i|_{\infty} \leq C_N |\phi_{in}|_{\infty}, \]

and since \( \phi \) is spherically symmetric,

\[ |\phi_{in}|(r) = \left| \frac{1}{r} \int_0^r u^2 \rho_{in}(u) du + \int_r^R u \rho_{in}(u) du \right| \leq C' \sqrt{R} |\rho_{in}|_2 \leq C'' \|g_{in}\|_2 \leq C_N d^{\frac{3}{2}} \]

where \( \rho_{in} = \int g_{in} dv \) and \( R \) is the support radius of \( \rho_{in} \). \( \square \)
We now estimate the term \( \int \int P_N \phi_{in} f_{in} dx dv \), for which we use the additional invariants.

**Lemma 3.7.** For any \( \varepsilon > 0 \), we have

\[
|I_{in}^3| \leq C(d^{1/2}(0) + \varepsilon^{1/2}d^{1/2} + \frac{1}{\varepsilon}d)d^{1/2}.
\]

**Proof.** By the definition of \( I_{in}^3 \) in [37], it suffices to estimate \( (g_{in}, \xi_i) \). We expand

\[
J_i(f, L) - J_i(f_0, L) = J_i(f_0 + g_{in}, L) - J_i(f_0, L) + J_i(g_{out}, L)
\]

\[
= (g_{in}, \xi_i) + O(d) + J_i(g_{out}, L).
\]

Notice that

\[
|J_i(g_{out}, L)| \leq C|g_{out}|_{L^1} \leq C\|1_{\{E_0 \leq E \leq E_0 + \varepsilon\}}g_{out}\|_{L^1} + C\|1_{\{E \geq E_0 + \varepsilon\}}g_{out}\|_{L^1}
\]

\[
\leq \varepsilon^{1/2}|g_{out}|_{L^2} + C\varepsilon|1_{\{E \geq E_0 + \varepsilon\}}(E - E_0)g_{out}|_{L^1} \leq C[\varepsilon^{1/2}d^{1/2} + \frac{1}{\varepsilon}d].
\]

It thus follows that

\[
|(g_{in}, \xi_i)| \leq |J_i(f(0), L) - J_i(f_0, L)| + C[\varepsilon^{1/2}d^{1/2} + \frac{1}{\varepsilon}d]
\]

\[
\leq C[d^{1/2}(0) + \varepsilon^{1/2}d^{1/2} + \frac{1}{\varepsilon}d].
\]

Therefore

\[
|I_{in}^3| = (1 - \tau) \left| \int \int P_N \phi_{in} g_{in} dx dv \right| = \left| \int \int \left( \sum_{i=1}^{N} (\phi_{in}, \xi_i)_{f_{0}}^{f_{0}} \right) g_{in} dx dv \right|
\]

\[
\leq \sum_{i=1}^{N} \left| (\phi_{in}, \xi_i)_{f_0} \right| \left| (\xi_i, g_{in}) \right| \leq C\varepsilon \sum_{i=1}^{N} |\phi_{in}|_{\infty} \left| (\xi_i, g_{in}) \right|
\]

\[
\leq Cd^{1/2}[d^{1/2}(0) + \varepsilon^{1/2}d^{1/2} + \frac{1}{\varepsilon}d].
\]

Now we prove the nonlinear stability of King model.

**Proof of Theorem 1.2** The global existence of classical solutions of 3D Vlasov-Poisson system was shown in [34] for compactly supported initial data \( f(0) \in C_1 \). Let the unique global solution be \( (f(t), \phi(t)) \). Let \( d(t) = d(f(t), f_0) \). Combining estimates (36), (38), (39) and (10), we have

\[
I(f(0)) - I(f_0) = I(f(t)) - I(f_0)
\]

\[
\geq d_{out} + \frac{\tau}{2} d_{in} + \left( \frac{1 - \tau}{8\pi} - C\tau \right) \int |\nabla \phi_{in}|^2 dx
\]

\[
- C\varepsilon^{1/3}d(t) + \frac{1}{\varepsilon^{5/6}}d(t)^{4/3} + \frac{1}{\varepsilon^{5/3}}d(t)^{5/3} - C\varepsilon^{C'd(t)^{1/2}}d(t)^{3/2}
\]

\[
- Cd(t)^{1/2} \left[ d^{1/2}(0) + \varepsilon^{1/2}d(t)^{1/2} + \frac{1}{\varepsilon}d(t) \right].
\]
Thus by choosing $\varepsilon$ and $\tau$ sufficiently small, there exists $\delta' > 0$ such that

\begin{equation}
I(0) - I(f_0) \geq \delta' d(t) - C \left( d(t)^{4/3} + d(t)^{5/3} + d(t)^{3/2} \right) - C e^{C t} d(t)^{1/2} d(t)^{3/2} < 0.
\end{equation}

It is easy to show that $I(0) - I(f_0) \leq C'' d(0)$. Define the functions $y_1(x) = \delta' x^2 - C e^{C t} x^3 - C \left( x^{8/3} + x^{10/3} + x^3 \right)$ and $y_2(x) = C d(0)^{1/2} x + C'' d(0)$. Then above estimates implies that $y_1(d(t)^{1/2}) \leq y_2(d(t)^{1/2})$. The function $y_1$ is increasing in $(0, x_0)$ where $x_0$ is the first maximum point. So if $d(0)$ is sufficiently small, the line $y = y_2(x)$ intersects the curve $y = y_1(x)$ at points $x_1, x_2, \ldots$, with $x_1(d(0)) < x_0 < x_2(d(0)) < \cdots$. Thus the inequality $y_1(x) \leq y_2(x)$ is valid in disjoint intervals $[0, x_1(d(0))]$ and $[x_2(d(0)), x_3(d(0))], \cdots$. Because $d(t)$ is continuous, we have that $d(t)^{1/2} < x_1(d(0))$ for all $t < \infty$, provided we choose $d(0)^{1/2} < x_0$. Since $x_1(d(0)) \to 0$ as $d(0) \to 0$, we deduce the nonlinear stability in terms of the distance functional $d(t)^{1/2}$. 

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