MINIMAL GROUPS OF GIVEN REPRESENTATION DIMENSION

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Abstract. For a finite group $G$, let $\text{rdim}(G)$ denote the smallest dimension of a faithful, complex linear representation of $G$. It is clear that $\text{rdim}(H) \leq \text{rdim}(G)$ for any subgroup $H$ of $G$. We consider $G$ with the property that $\text{rdim}(H) < \text{rdim}(G)$ whenever $H$ is a proper subgroup of $G$, in particular proving a classification of such groups when $G$ is abelian or $\text{rdim}(G) \leq 3$.

1. Introduction

We say a finite group $G$ has representation dimension $n$, written $\text{rdim}(G)$, if there is a faithful representation $\rho : G \to \text{GL}_n(\mathbb{C})$ and no faithful representation of dimension $n - 1$. Clearly $\text{rdim}(H) \leq \text{rdim}(G)$ for every subgroup $H$ of $G$. In this article we consider the class of groups $G$ such that $\text{rdim}(H) < \text{rdim}(G)$ for all proper subgroups $H$ of $G$; we call these groups minimally faithful of degree $\text{rdim}(G)$.

If $G$ is abelian or $\text{rdim}(G) = 2$, such groups are straightforward to classify; see Lemma 2.2 and Proposition 3.1. The main result of this paper is a classification of minimally faithful groups of degree 3. We summarize this result here; let $C_n$ be a cyclic group of order $n$ and $Q_8$ the quaternion group of order 8.

**Theorem 1.1.** Let $G$ be a minimally faithful group of degree 3. Then $G$ is one of the following:

a) $C_p \times C_p \times C_p$ for a prime $p$,

b) $(C_p \times C_p) \rtimes C_{3^k}$ for a prime $p \equiv -1 \pmod{3}$, $k \geq 1$, with center $C_{3^{k-1}}$,

c) $C_p \rtimes C_{3^k}$ for a prime $p \equiv 1 \pmod{3}$, $k \geq 1$, with center $C_{3^{k-1}}$,

d) $C_{3^k} \rtimes C_3$ for $k \geq 2$, with center $C_{3^{k-1}}$,

e) The Heisenberg group of order 27,

f) $Q_8 \times C_p \times C_p$ for an odd prime $p$,

g) $(C_{2^k} \rtimes C_2) \times C_p \times C_p$ where $k \geq 2$, $p$ is an odd prime, and $C_2$ acts on $C_{2^k}$ by $g \mapsto g^{1+2^{k-1}}$,,
2. Preliminaries

We write $D_{2n}$ for the dihedral group of order $2n$, $A_n$ for the alternating group of degree $n$, and $S_n$ for symmetric group of degree $n$. Let $G^1 = [G, G]$ be the derived group of $G$, $G^2 = [G^1, G^1]$ for the derived group of $G^1$, and so on. The group $GL_n(\mathbb{C})$ is the group of $n \times n$ invertible matrices over the complex numbers, and its derived group $SL_n(\mathbb{C})$ is the subgroup of matrices with determinant 1. The group $PGL_n(\mathbb{C})$ is the quotient of $GL_n(\mathbb{C})$ by its center. For a subset $S$ of a group $G$ we write $C_G(S)$ for the centralizer of $S$ in $G$ and $N_G(S)$ for its normalizer. We write Aut($G$) for the automorphism group of a group $G$ and Inn($G$) $\cong G/Z(G)$ for the group of inner automorphisms.

**Definition 2.1.** The group $G$ is minimally faithful of degree $n$ if $\text{rdim}(G) = n$ and $\text{rdim}(H) < n$ for all proper subgroups $H$ of $G$.

Our main results classify minimally faithful groups of degree 2 and 3. First we show that the case of abelian minimally faithful groups is straightforward.

**Lemma 2.2.** An abelian group $G$ is minimally faithful of degree $n$ if and only if it is an elementary abelian $p$-group of rank $n$, for some prime $p$.

**Proof.** If $G = C_{n_1} \times \cdots \times C_{n_k}$ then $\text{rdim}(G) \leq k$, with equality if and only if we have written $G$ with a minimal number of factors. Let the $p$-sylow subgroup of $G$ be written as $\prod_{i=1}^{k_p} C_{p^{a_i}}$. The smallest number of factors in an expression for $G$ is $\max_p (k_p)$. If $p$ is such that $k_p = n$, then since $C_{p^n} \subset G$ and $\text{rdim}(C_{p^n}) = n = \text{rdim}(G)$, the conclusion follows from minimal faithfulness. The converse is clear. $\square$
We mention an easy and useful fact: if $G$ is a finite group and $n$ is coprime to $|Z(G)|$ then $\text{rdim}(G \times C_n) = \text{rdim}(G)$. This follows from the fact that any representation of $G$ can be extended to $G \times C_n$ by having $C_n$ act by scalars.

Next, we recall a particular component of the classification of finite nonabelian groups containing only abelian proper subgroups.

**Proposition 2.3** ([MM03]). If $G$ is a finite nonabelian group with only abelian proper subgroups then either $G$ is a $p$-group or else $G = Q \rtimes C_p^m$ where $Q$ is an elementary abelian $q$-group for a prime $q \neq p$, in which case we have $|Q| \equiv 1 \pmod{p}$, and $Z(G) = C_{p^m-1}$.

### 3. Minimally Faithful of Degree 2

**Proposition 3.1.** Suppose $G$ is a nonabelian group and is minimally faithful of degree 2. Then either $G = Q_8$ or $G = \langle a, b \mid a^p = b^{2^m} = bab^{-1}a = 1 \rangle = C_p \rtimes C_{2m}$ for some odd prime $p$ and some $m \geq 1$.

**Proof.** A faithful representation $G \to \text{GL}_2(\mathbb{C})$ must be irreducible, since otherwise it is a sum of two 1-dimensional representations, which would imply that $G$ is abelian. Thus $G$ has even order. Let $P$ be a 2-Sylow subgroup of $G$.

Suppose $G$ is not a 2-group. Then $\text{rdim}(P) = 1$ so $P$ is cyclic. By Cayley’s normal 2-complement theorem, $G = K \rtimes P$ for some subgroup $K$ of odd order. Since $\text{rdim}(K) = 1$, $K$ is cyclic. Let $x \in P$ be a generator. If $x$ centralized $K$ then $G$ would be abelian, and if $x^2$ did not centralize $K$ then the proper subgroup generated by $K$ and $x^2$ would not be cyclic. Therefore $x$ acts on $K$ by an automorphism of order 2. We may decompose $K = K^+ \times K^-$ where $xkx^{-1} = k^{\pm 1}$ for all $k \in K^\pm$. Since $G$ is nonabelian, $K^-$ is nontrivial. If $h \in K^-$ has prime order $p$ then $x$ and $h$ generate a nonabelian group, which must equal $G$. Therefore $G = C_p \rtimes C_{2m}$ as claimed. These groups can be realized inside $\text{GL}_2(\mathbb{C})$: let $H$ be the cyclic group of order $2^{m-1}p$ generated by $x^2$ and $h$, take any faithful character $\chi : H \to \mathbb{C}^*$, and then $\text{Ind}_H^G(\chi)$ is a faithful 2-dimensional representation.

Finally, suppose $G$ is a 2-group. Then $G$ has a cyclic maximal (index-2) subgroup. The $p$-groups with cyclic maximal subgroups have been classified; if $G \neq Q_8$ then one readily sees that $G$ contains $C_2 \times C_2$, and if $G = Q_8$ then it is easy to check that $G$ is minimally faithful of degree 2. $\square$
4. Degree 3

We now begin the classification of minimally faithful groups of degree 3, first proving some easy elementary results.

Lemma 4.1. Suppose $G$ is a nonabelian group.

a) If $G$ has odd order or $Z(G)$ is not cyclic then $\text{rdim}(G) \geq 3$.

b) If $\text{rdim}(G) = 3$ and $G$ has odd order then $3$ divides $|G|$ and $Z(G)$ is cyclic.

Proof. a) Since $G$ is nonabelian, $\text{rdim}(G) \geq 2$. If $G$ has odd order or $Z(G)$ is not cyclic then a faithful representation $\rho : G \rightarrow \text{GL}_2(\mathbb{C})$ would be reducible (the latter case because of Schur’s lemma), hence would factor as a direct sum of two 1-dimensional representations, so $1 \neq [G, G] \subset \ker(\rho)$, a contradiction.

b) If $\text{rdim}(G) = 3$ then a faithful representation $G \rightarrow \text{GL}_3(\mathbb{C})$ must be irreducible: it cannot have a 2-dimensional irreducible summand since $|G|$ is odd and cannot be a sum of 1-dimensional representations since $G$ is nonabelian. Thus $3$ divides $|G|$, and by Schur’s lemma $Z(G)$ is cyclic. \square

Corollary 4.2. Suppose $G$ is a nonabelian group and is minimally faithful of degree 3. Let $H$ be a proper subgroup of $G$.

a) If $H$ is nonabelian then $Z(H)$ is cyclic.

b) If $H$ has odd order then $H$ is abelian. In particular, if $G$ has odd order then $H$ is abelian.

c) If $H$ is abelian then $H$ has at most 2 invariant factors.

d) The order of $G$ is divisible by 2 or 3 (or both).

Proof. The first three parts are immediate since $\text{rdim}(H) \leq 2$. For the last, observe that if $G$ has order coprime to 6 then any representation $G \rightarrow \text{GL}_3(\mathbb{C})$ must be a direct sum of three 1-dimensional representations. \square

Lemma 4.3. Suppose that $G$ is minimally faithful of degree 3. If $Z(G) \cap [G, G]$ has even order then any faithful representation $\rho : G \rightarrow \text{GL}_3(\mathbb{C})$ is reducible and $[G, G]$ is isomorphic to a subgroup of $\text{SL}_2(\mathbb{C})$.

Proof. Suppose $z \in Z(G) \cap [G, G]$ has order 2. If $\rho$ is irreducible then $\rho(z) = -I_3$ by Schur’s lemma. On the other hand, $\rho([G, G]) \subset \text{SL}_3(\mathbb{C})$ and $-I_3 \notin \text{SL}_3(\mathbb{C})$. So $\rho$ is...
reducible, \( G \cong \rho(G) \subset \text{GL}_2(\mathbb{C}) \times \text{GL}_1(\mathbb{C}) \), and \( \rho([G, G]) \) is contained in the derived group of \( \text{GL}_2(\mathbb{C}) \times \text{GL}_1(\mathbb{C}) \), which is \( \text{SL}_2(\mathbb{C}) \times \{1\} \).

\[ \tag*{\Box} \]

**Theorem 4.4.** Let \( G \) be minimally faithful of degree 3. Then \([G, G]\) is abelian.

**Proof.** We may assume \( G \) is nonabelian. Let \( G^1 := [G, G] \).

**STEP 1:** \( G \neq G^1 \).

Suppose \( G = G^1 \). Then any faithful representation \( \rho : G \to \text{GL}_3(\mathbb{C}) \) has image in \( \text{SL}_3(\mathbb{C}) \). We refer to the classification of finite subgroups of \( \text{SL}_3(\mathbb{C}) \), up to \( \text{GL}_3(\mathbb{C}) \)-conjugacy; found in [Serr14]. If \( \rho \) is reducible then (up to conjugacy) \( \rho(G) \) is contained in the image of \( g \mapsto (g, \det(g)^{-1}) : \text{GL}_2(\mathbb{C}) \to (\text{GL}_2(\mathbb{C}) \times \text{GL}_1(\mathbb{C})) \cap \text{SL}_3(\mathbb{C}) \), which would imply \( \text{rdim}(G) = 2 \). Therefore \( \rho \) is irreducible. Among the subgroups of \( \text{SL}_3(\mathbb{C}) \) that act irreducibly and primitively on \( \mathbb{C}^3 \), the groups \( A_5 \), \( PSL_2(7) \), and \( 3A_6 \) (the nontrivial triple cover of \( A_6 \)) all contain \( A_4 \), while the others all have a nonabelian 3-Sylow subgroup (and are not themselves 3-groups). Since \( \text{rdim}(A_4) = 3 \), none of these groups are minimally faithful of degree 3. The same issue occurs if \( G = H \cdot Z(\text{SL}_3(\mathbb{C})) \) where \( H = A_5 \) or \( PSL_2(7) \). Therefore \( G \) acts irreducibly and imprimitively on \( \mathbb{C}^3 \).

Thus \( \rho(G) \) is generated by a finite subgroup \( D \) of the diagonal matrices and a cyclic permutation \( p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \), or else is generated by such a group together with a matrix \( Q = \begin{bmatrix} 0 & a & 0 \\ b & 0 & 0 \\ 0 & 0 & c \end{bmatrix} \) with \( abc = -1 \). In the both cases \( D \) is a normal subgroup of \( \rho(G) \). In the first case \( \rho(G)/D = C_3 \), so \( G \neq G^1 \). In the second case, since \( Q^2 \in D \) and \( QpQ^{-1} \in Dp^{-1} \), we have \( \rho(G)/D = S_3 \), which admits a nontrivial character, so \( G \neq G^1 \).

**STEP 2:** \( G \) is solvable.

If \( G \) is not solvable then \( G^1 \) is not solvable. Since \( G^1 \neq G \) we can regard \( G^1 \) as a subgroup of \( \text{GL}_2(\mathbb{C}) \). Then \( G^2 := [G^1, G^1] \subset \text{SL}_2(\mathbb{C}) \) is not solvable. The unique nonsolvable subgroup of \( \text{SL}_2(\mathbb{C}) \) is the binary icosahedral group, which we will denote by \( BI_{120} \). Thus \( Z(G^2) \cong Z(BI_{120}) \) has order 2 which implies \( Z(G^2) \subset Z(G) \cap [G, G] \). By lemma 4.3, \( \rho(G) \) may be regarded as a subgroup of \( \text{GL}_2(\mathbb{C}) \times \text{GL}_1(\mathbb{C}) \) and \( \rho(G^1) \subset \text{SL}_2(\mathbb{C}) \times \{1\} \) so \( G^1 \cong BI_{120} \). If \( (M, t) \in \rho(G \setminus G^1) \) where \( M \in \text{GL}_2(\mathbb{C}) \), \( t \in \mathbb{C}^\times \), and \( (g, 1) \in \rho(G^1) \), then \( (MgM^{-1}, 1) \in \rho(G^1) \). so \( (M, 1) \) and \( \rho(G^1) \) generate a finite
subgroup of $GL_2(\mathbb{C}) \times \{1\}$ which contains a copy of $BI_{120}$. The only such groups are generated by $BI_{120}$ and a subgroup of scalar matrices, by [Ngu08].

Therefore, since $(M, t) \notin \rho(G^1)$, $(M, t)$ commutes with $\rho(G^1)$, which implies $M$ is a scalar matrix by Schur’s lemma. So $G = Z(G)G^1$ with $Z(G) \cap G^1$ of order 2. If $Z(G)$ is cyclic then $\text{rdim}(G) = 2$ since we can let $Z(G)$ act by scalars on $\mathbb{C}^2$ in a realization of $BI_{120} \subset SL_2(\mathbb{C})$. Suppose $Z(G)$ is not cyclic. Since $Q_8$ is the 2-Sylow subgroup of $BI_{120}$, we have the nonabelian subgroup $Q_8Z(G)$ with noncyclic center. Thus $\text{rdim}(Q_8Z(G)) = 3$. But $Q_8Z(G) \neq G$ since $Q_8$ is not centralized by $BI_{120}$, contradicting minimality of $G$.

**STEP 3:** If $G^1$ is not abelian then $G^1 = Q_8$ or $G^1 = Q_8 \rtimes C_3$.

Suppose $G^1$ is nonabelian. Note $G^2 = [G^1, G^1] \subset SL_2(\mathbb{C})$. The finite subgroups of $SL_2(\mathbb{C})$ are $C_n$, $BD_{4n} := (A, B : A^n = B^2, B^4 = 1 BAB^{-1} = A^{-1})$ (dicyclic or binary dihedral groups of order $4n$, $n \geq 2$), $Q_8 \rtimes C_3$ (the binary tetrahedral group), $BO_{48}$ (the binary octahedral group), and $BI_{120}$ (the binary icosahedral group). Their derived groups are, respectively, $\{1\}$, $C_{2n}$ (respectively, $C_n$) if $n$ is odd (respectively, even), $Q_8$, $Q_8 \rtimes C_3$, and $BI_{120}$. The finite subgroups of $PGL_2(\mathbb{C})$ are $C_n$, $D_{2n}$, $A_4$, $S_4$, and $A_5$. In particular, $G^1/Z(G^1)$ and $G^2/(Z(G^2) \cap \{\pm I_2\}) = [G^1/Z(G^1), G^1/Z(G^1)]$ are such subgroups of $PGL_2(\mathbb{C})$. If $Z(G^2)$ has a unique element $z$ of order 2 then $z \in Z(G) \cap [G, G]$ and lemma 4.3 applies. This holds except when $G^2$ is cyclic of odd order.

Case 1: Since $G$ is solvable, $G^2 \neq BI_{120}$.

Case 2: If $G^2 = BO_{48}$ then $G^2/Z(G^2) \cong S_4$, so $G^1/Z(G^1) \subset PGL_2(\mathbb{C})$ must contain $S_4$, hence equal $S_4$. Thus $G^2/Z(G^2) \subset [S_4, S_4] = A_4$, a contradiction.

Case 3: If $G^2 = Q_8 \rtimes C_3$ then $G^1 \subset SL_2(\mathbb{C})$ equals $BO_{48}$, the only subgroup of $SL_2(\mathbb{C})$ with the correct commutator subgroup. We claim that $BO_{48}$ cannot be realized as a commutator subgroup of any group, hence cannot equal $G^1$. We have $\text{Inn}(BO_{48}) = BO_{48}/(\pm I_2) = S_4$ and $\text{Aut}(BO_{48}) = S_4 \times C_2$. But then we cannot have $\text{Inn}(BO_{48}) \subset \text{Aut}(BO_{48})^1 = A_4$, which needs to occur since the image of the $G^1$ under the natural map $G \to \text{Aut}(G^1)$ equals $\text{Inn}(G^1)$ and lies in $\text{Aut}(G^1)^1$.

Case 4: If $G^2 = BD_{4n}$ with $n > 2$ then $G^1 \subset SL_2(\mathbb{C})$ cannot exist because $BD_{4n}$ is not the commutator subgroup of any finite subgroup of $SL_2(\mathbb{C})$. If $n = 2$ then $G^2 = Q_8$ and $G^1 = Q_8 \rtimes C_3$, since the binary tetrahedral group is the only subgroup of $SL_2(\mathbb{C})$ with the correct commutator subgroup.
Case 5i: If \( G^2 = C_n \) is cyclic of even order \( n > 1 \) then \( G^1 \subset \text{SL}_2(\mathbb{C}) \) and has derived group \( C_n \). Therefore \( G^1 = BD_{4n} \). Realizing \( G \) as a subgroup of \( \text{GL}_2(\mathbb{C}) \times \text{GL}_1(\mathbb{C}) \) (using lemma 4.3), let \( H \) be the subgroup of \( \text{GL}_2(\mathbb{C}) \) generated by the first components of elements of \( G \). Clearly \( H^1 = G^1 \). If \( n \geq 4 \) then \( H/Z(H) \subset \text{PGL}_2(\mathbb{C}) \) cannot have the dihedral group \( D_{2n} \) as its derived group. Since \( G^1/Z(G^1) = D_{2n} \) this is a contradiction. If \( n = 2 \), then \( G^1 = Q_8 \) since this is the unique subgroup of \( \text{SL}_2(\mathbb{C}) \) with commutator subgroup of order 2.

Case 5ii: Now assume \( G^2 = C_n \) is cyclic of odd order \( n > 1 \). Since \( G^1 \) is a nonabelian subgroup of \( G \), \( Z(G^1) \) is cyclic. Suppose \( Z(G^1) \) has even order. Then its unique element of order 2 will lie in \( Z(G) \), so lemma 4.3 applies and \( G^1 \subset \text{SL}_2(\mathbb{C}) \). The only subgroups of \( \text{SL}_2(\mathbb{C}) \) with cyclic derived group are \( BD_{4k} \), but \( BD_{4k} = C_{2k} \) if \( k \) is odd and \( BD_{4k} = C_k \) if \( k \) is even, so the odd-order cyclic group \( G^2 \) is not a derived group of \( G^1 \subset \text{SL}_2(\mathbb{C}) \), which is absurd. Thus \( Z(G^1) \) has odd order. Now \( G^1 \subset \text{GL}_2(\mathbb{C}) \) is nonabelian so \( G^1/Z(G^1) \subset \text{PGL}_2(\mathbb{C}) \), and \((G^1/Z(G^1))^1 = G^2\) so \( G^1/Z(G^1) = D_{2n} \), the dihedral group of order \( 2n \). From the classification of finite subgroups of \( \text{GL}_2(\mathbb{C}) \) in [Ngu08], \( G^1 = C_r \times D_{2n} = C_r \times (G^2 \rtimes C_2) \) where \( r \) is odd (all the other subgroups contain \(-I_2\)). But then \( G^1/G^2 = C_{2r} \) and \( G^2 = C_n \) are both cyclic, which implies that \( G^1 \) is cyclic and \( G^2 \) is trivial [proof: \( G/C_G(G^2) \) is a subgroup of \( \text{Aut}(G^2) \), which is abelian, hence \( G^1 \subset C_G(G^2) \), so \( G^2 \subset Z(G^1) \). But then \( G^1/G^2 \) being cyclic implies \( G^1/Z(G^1) \) is cyclic, so \( G^1 \) is cyclic].

**STEP 4:** \( G^1 \neq Q_8 \) and \( G^1 \neq Q_8 \rtimes C_3 \).

Suppose \( G^1 \) is either of these two groups. We have \( \text{Aut}(Q_8) \cong S_4 \cong \text{Aut}(Q_8 \rtimes C_3) \). Therefore if \( P \) is a \( p \)-Sylow subgroup of \( G \) with \( p \geq 5 \) then \( P \subset C_G(G^1) \). Since \( G^1 \) is nonabelian, and \( P \subset Z(PG^1) \), \( P \) is abelian since it is a proper subgroup of odd order. \( P \) is cyclic. Since \( \rho(G^1) \subset \text{GL}_2(\mathbb{C}) \times \{1\} \) is nonabelian and \( \rho(G) \subset \text{GL}_2(\mathbb{C}) \times \text{GL}_1(\mathbb{C}) \), Schur’s lemma implies that \( \rho(C_G(G^1)) \) is a group of matrices of the form \( \text{diag}(x, x, y) \), so \( P \subset Z(G) \). Because \( P \cap G^1 = 1 \), we have \( G/G^1 \cong F \times P \) for an abelian group \( F \). If \( N \) is the preimage of \( F \) in \( G \), then \( G = N \times P \). Clearly \(|P|\) and \(|Z(N)|\) are coprime, so \( \text{rdim}(N \times P) = \text{rdim}(P) \). Therefore \( P \) is trivial and the only primes that may divide \(|G|\) are 2 and 3.

Since \( G^1 \) contains a unique element \( z \) of order 2, we have \( z \in Z(G) \) and Schur’s lemma forces \( \rho(z) = \text{diag}(-1, -1, 1) \). Let \( K = \ker(\rho(G) \to \text{GL}_2(\mathbb{C})) \), where the map is the projection onto the first factor. The group \( K \) is nontrivial since \( \text{rdim}(G) > 2 \) and \( K \subset \rho(Z(G)) \) since \( K \) consists of elements of the form \( \text{diag}(1, 1, y) \). If \( k \in K \) had order 2 then \( k \) and \( \rho(z) \) would generate a noncyclic subgroup of \( \rho(Z(G)) \). Thus
$G^1 Z(G)$ would be nonabelian with noncyclic center, hence would equal $G$. But then $G^1 = (G^1 Z(G))^1 \subset G^2 Z(G)$, which is false for both $G^1 = Q_8$ and $G^1 = Q_8 \times C_3$. Therefore $K$ has odd order, hence is a 3-group, and if $k \in K$ has order 3 then $k = \text{diag}(1,1,\omega)$ for a primitive 3rd root of unity $\omega$.

Suppose there exists a scalar matrix $xI_3 \in \rho(Z(G))$ for some $1 \neq x \in \mathbb{C}^\times$. Replacing $x$ by a power if necessary, we can assume that $x = -1$ or $x = \omega$. In any case, $xI_3$, $\rho(z)$, and $\text{diag}(1,1,\omega)$ will generate a noncyclic subgroup of $\rho(Z(G))$, yielding the same contradiction as above. Therefore $\rho(G)$ contains no nontrivial scalar matrices. If we write $\rho = \sigma \oplus \chi : G \to \text{GL}_2(\mathbb{C}) \times \text{GL}_1(\mathbb{C})$, then the representation $\sigma \otimes \chi^{-1} : G \to \text{GL}_2(\mathbb{C})$ has trivial kernel, since $\sigma(g)\chi^{-1}(g) = I_2$ if and only if $\sigma(g) = \chi(g)I_2$ if and only if $\rho(g) = \chi(g)I_3$ is a scalar matrix. Thus $\text{rdim}(G) \leq 3$, a contradiction.

Now we will break into cases depending on the rank of the 2-Sylow subgroup of the abelian group $G^1 = [G,G]$. First we prove two auxiliary results.

**Lemma 4.5.** Suppose $G$ is minimally faithful of degree 3, with even order and cyclic 2-Sylow subgroups. Then $G = (C_q \times C_{2^m}) \times C_p \times C_p$ where $q, p$ are distinct odd primes, $m \geq 1$, and a generator of $C_{2^m}$ acts by inversion on $C_q$.

**Proof.** Let $P$ be a 2-Sylow subgroup of $G$ and assume $P$ is cyclic. By Cayley’s normal 2-complement theorem, $G \cong K \times P$, where $K$ is a normal 2-complement. Since $K \neq G$ and has odd order, $K$ is abelian. Let $\phi : P \to \text{Aut}(K)$ be the homomorphism defining $G$, let $x$ be a generator of $P$ and $n$ be the order of $\phi(x)$. Finally, let $H$ be the subgroup generated by $x^2$ and $K$, which is the unique subgroup of index 2 in $G$.

If $n = 2$ then we claim $\text{rdim}(G) \neq 3$ or $Z(G)$ is not cyclic. In this case $H$ is abelian. If $K$ is cyclic then so is $H$, and there is a faithful character $\chi : H \to \mathbb{C}^\times$. But then $\text{Ind}_H^K(\chi)$ is a faithful 2-dimensional representation of $G$. Thus $K$ is not cyclic. Since $K$ has odd order and $x$ acts on $K$ by an automorphism of order 2, we can decompose $K = K^+ \times K^-$ where $K^\pm := \{k \in K : xkx^{-1} = k^{\pm 1}\}$. Since $K^+$ is centralized by $H$ and $x$, it lies in $Z(G)$. Clearly $K^-$ is normalized by $P$, so $G \cong K^+ \times (K^- \rtimes P)$.

Suppose $K^-$ is not cyclic, so it has two invariant factors. For independent generators $a$ and $b$ of $K^-$ we have $xax^{-1} = a^{-1}$ and $xbx^{-1} = b^{-1}$. Suppose $\rho : G \to \text{GL}_3(\mathbb{C})$ was faithful. Since $a$ and $a^{-1}$ are conjugate in $G$, the set of eigenvalues of $\rho(a)$ is equal to the set of its reciprocals. Since $a$ has odd order, a nontrivial eigenvalue of $\rho(a)$ cannot
be equal to its inverse. Thus the eigenvalues of \( \rho(a) \) are \( \zeta_a^\pm 1, 1 \), for some root of unity \( \zeta_a \) of order equal to the order of \( a \) in \( G \). Similar considerations hold for \( \rho(b) \). Since \( K^- \) is not cyclic there is a common prime dividing the orders of \( a \) and of \( b \). This, the commutativity of \( K \), and the assumption that \( \rho \) is faithful means we can assume without loss of generality that \( \rho(a) = \text{diag}(\zeta_a, \zeta_a^{-1}, 1) \) and \( \rho(b) = \text{diag}(\zeta_b, 1, \zeta_b^{-1}) \) (we cannot put the 1 in the same entry for both since then \( \rho \) would not be faithful). But there is no matrix in \( \text{GL}_3(\mathbb{C}) \) that simultaneously conjugates both of the matrices \( \rho(a) \) and \( \rho(b) \) to their inverses: the centralizers of either matrix consist solely of diagonal matrices, so any matrix that conjugates \( \rho(b) \) to \( \rho(b)^{-1} \) is a diagonal matrix times \( \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \), but no such matrix can conjugate \( \rho(a) \) to its inverse. Therefore \( \rho(x) \) cannot be defined, so either \( K^- \) is not cyclic or \( \text{rdim}(G) > 3 \). So \( K^- \) is cyclic. Observe that \( \text{rdim}(K^- \ltimes P) = 2 \): let \( N \) be the cyclic group \((K^-, x^2)\) and \( \chi : N \to \mathbb{C}^\times \) a faithful character. Then \( \text{Ind}^{K^- \ltimes P}_{N}(\chi) \) is a faithful 2-dimensional representation. Therefore \( K^+ \) is nontrivial. If \( K^+ \) is cyclic then \( \text{rdim}(G) = \text{rdim}(K^- \ltimes P) = 2 \). Thus \( K^+ \), and hence \( Z(G) \), is not cyclic. There exists an odd prime \( p \) such that \( C_p \times C_p \subset K^+ \); the group \((C_p \times C_p) \times (K^- \ltimes P)\) is nonabelian with noncyclic center, so has representation dimension at least 3, so equals \( G \) by minimality. Noting that every subgroup of \( K^- \) is normalized by \( P \) by definition, if \( q \) is an odd prime such that \( C_q \subset K^- \), then \((C_p \times C_p) \times (C_q \times P)\) is nonabelian with noncyclic center, hence equals \( G \).

If \( n \geq 4 \) then we claim \( \text{rdim}(G) > 3 \). First assume \( n = 4 \). Suppose there is a cyclic subgroup \( L \subset K \) which is normalized by \( x \) on which conjugation by \( x \) is an automorphism of order 4. We claim that \( \text{rdim}(PL) > 3 \) (in fact it equals 4). If \( y \) generates the cyclic group \( L \) then \( xyx^{-1} = y^t \) where \( t^2 \equiv -1 \pmod{p} \) for some prime \( p \) dividing \( |L| \). Thus \( y \) is conjugate its inverse by \( x^2 \). If \( \rho : PL \to \text{GL}_3(\mathbb{C}) \) is a faithful representation then, as above, the eigenvalues of \( \rho(y) \) are \( \zeta^\pm 1, 1 \) for some root of unity \( \zeta \) of order equal to \( |L| \). But since \( y \) is conjugate to \( y^t \), \( \rho(y^t) \) has the same eigenvalues as \( \rho(y) \), which forces \( \zeta^t = \zeta^{-1} \) or \( \zeta^t = \zeta \) or \( \zeta^t = 1 \). These are all absurd, so \( L \) does not exist. Since \( x^2 \) acts on \( K \) by an automorphism of order 2, we decompose \( K = K^+ \ltimes K^- \) where \( K^\pm := \{ k \in K : x^2kx^{-2} = k^\pm 1 \} \). If \( K^- \) is not cyclic then the proof of the \( n = 2 \) case thus the group generated by \( x^2 \) and \( K^- \) has representation dimension greater than 3. If \( K^- \) is trivial then \( x \) acts on \( K \) by an element of order 2, contradicting the assumption that \( n = 4 \). So \( K^- \) is cyclic and nontrivial. Note that \( K^- \) is normalized by \( x \) since \( x^2(xkx^{-1})x^{-2} = xk^{-1}x^{-1} = (xkx^{-1})^{-1} \) for \( k \in K^- \). But now \( K^- \) satisfies the conditions of the group \( L \) above, so cannot exist. Finally,
assume $n \geq 8$. Then $x^2$ acts on $K$ by an automorphism of order 4, so the previous case shows that $\text{rdim}(H) > 3$, hence $\text{rdim}(G) > 3$ (note that the proof of the $n = 4$ case made no use of the minimality of $G$ to derive $\text{rdim}(G) > 3$).

□

Lemma 4.6. Suppose $G$ is a nonabelian 2-group containing only abelian proper subgroups and $\text{rdim}(G) = 2$. Then either $G = Q_8$ or $G = C_{2k} \rtimes C_2$ where $k \geq 2$ and $C_2$ acts on $C_{2k}$ by $g \mapsto g^{1+2k-1}$.

Proof. If $G$ only has cyclic proper subgroups then $G$ is minimally faithful of degree 2, so by our earlier classification $G = Q_8$. If $G \supset V = C_2 \times C_2$ then we can choose an index-2 subgroup $H \subset G$ with $H \supset V$. The group $H$ is abelian by assumption, and $V$ is the characteristic subgroup of $H$ generated by elements of order 2, so $N_G(V) = G$. The map $G \to \text{Aut}(V) \cong S_3$ factors through $G/H$ and is nontrivial because $Z(G)$ is cyclic. Thus there exists a unique $1 \neq z \in V \cap Z(G)$. Let $v$ and $vz$ be the other nontrivial elements of $V$, and let $g \in G \setminus H$. We have $vvg^{-1} = z$. If $g^2 = 1$ then $g$ and $V$ generate the nonabelian group $C_4 \rtimes C_2$, which must therefore equal $G$. If $g^2 \neq 1$ then some power of $g^{2k-1}$ is nontrivial and lies in $V$, hence is equal to $z$ since $g$ does not commute with $v$ or $vz$. This yields the groups $\langle g, v : g^{2k} = v^2 = 1, vvg = g^{1+2k-1} \rangle$, where $k \geq 2$. Notice these groups have representation dimension 2 since we can induce a faithful character of the index-2 cyclic subgroup generated by $g$ up to $G$. □

Theorem 4.7. Let $G$ be nonabelian and minimally faithful of degree 3, and let $0 \leq r \leq 2$ be the rank of the 2-Sylow subgroup of $G^1$.

A) If $r = 2$ then $G = (C_2 \times C_2) \rtimes C_3$ for some $k \geq 1$.

B) If $r = 1$ then either

i) $G$ is a 2-group,

ii) $G = Q_8 \rtimes C_p \times C_p$ for an odd prime $p$, or

iii) $G = (C_{2k} \rtimes C_2) \rtimes C_p \times C_p$ for an odd prime $p$, where $k \geq 2$ and $C_2$ acts on $C_{2k}$ by $g \mapsto g^{1+2k-1}$.

C) If $r = 0$ then either

i) $G$ has odd order,

ii) $G$ is the group appearing in lemma 4.5, or

iii) $G = (C_p \rtimes C_{2m}) \rtimes C_2$ where $m > 1$, $p$ is an odd prime, and a generator of $C_{2m}$ acts by inversion on $C_p$. 


Proof. Let $\rho : G \to \text{GL}_3(\mathbb{C})$ be a faithful representation of $G$.

A) The elements of order exactly 2 inside $G^1$ generate a subgroup $V \cong C_2 \times C_2$; it is characteristic in $G^1$ so $N_G(V) = G$. Since $\text{Aut}(V) = S_3$, the index $[G : C_G(V)]$ divides 6. The centralizer $C_G(V)$ is abelian since $V \subset Z(C_G(V))$ and $V$ is not cyclic. If $\rho$ was reducible then $G^1$ would be an abelian subgroup of $\text{SL}_2(\mathbb{C})$, hence cyclic. But $G^1 \supset V$, so $\rho$ is irreducible. Thus 3 divides $|G|$ and $G$ contains a nontrivial 3-Sylow subgroup $P$. If $P \subset C_G(V)$ then $[G : C_G(V)] = 2$, since the image of the map $G \to \text{Aut}(V)$ could contain no element of order 3. But if $G$ has an index-2 abelian subgroup then every irreducible representation of $G$ has degree dividing 2, contradicting the irreducibility of $\rho$. Thus $P$ does not centralize $V$. Let $x \in P \setminus C_G(V)$. Then $x$ and $V$ generate a group isomorphic to $R := (C_2 \times C_2) \rtimes C_{3^k}$. Every proper subgroup of $R$ is abelian of rank at most 2, so it suffices to show that $\text{rdim}(R) = 3$.

The group $R$ has a faithful 3-dimensional representation: let $\chi$ be a character of the index-3 abelian subgroup $H := C_2 \times C_2 \times C_{3^{k-1}}$ with $\ker(\chi)$ of order 2, then $\rho = \text{Ind}_H^R(\chi)$ is readily checked to be faithful. We claim that the group $R$ has no faithful 2-dimensional representation. The subgroup $C_2 \times C_2$ is abelian so we can assume it acts diagonally on $\mathbb{C}^2$, but the only diagonal $2 \times 2$ matrices of order 2 are $\text{diag}(1,-1), \text{diag}(-1,1), \text{diag}(-1,-1)$. The third matrix must lie in the center but no order-2 element of $R$ is central. So $\text{rdim}(R) = 3$ and $R = G$.

B) We will first show that either $G$ is a 2-group or $G = P \times C_p \times C_p$ for a 2-Sylow group $P$ and odd prime $p$. Since $r = 1$, $G^1$ has a nontrivial cyclic 2-Sylow subgroup. Hence $G^1$ contains a unique element $z$ of order 2, lemma 4.3 applies: $\rho$ is reducible and $\rho(G^1) \subset \text{SL}_2(\mathbb{C}) \times \{1\}$, so $G^1$ is cyclic. Therefore $\rho(z) = \text{diag}(-1,-1,1)$, where without loss of generality we have assumed $\rho(G) \subset \text{GL}_2(\mathbb{C}) \times \text{GL}_1(\mathbb{C})$.

Suppose that $|G^1| = 2$. Then $G^1 \subset Z(G)$, and $G = P \times K$ where $K$ is abelian of odd order [proof: let $a \in G$ and let $b \in G$ of odd order. If $aba^{-1} = bz$, then the left side has odd order and the right side even order. Thus $aba^{-1}b^{-1} = 1$]. If $K \supset C_p \times C_p$ for an odd prime $p$ then $P \times C_p \times C_p$ is nonabelian with noncyclic center, so has representation dimension 3 and must equal $G$. Otherwise $K$ is cyclic, which implies $\text{rdim}(P \times K) = \text{rdim}(P)$, so by minimality $G = P$.

Now suppose $g$ generates the cyclic group $G^1$ and has (even) order greater than two. Since $G^1$ is abelian we can assume $\rho(G^1)$ consists of diagonal matrices. Thus $\rho(g) = \text{diag}(x, x^{-1}, 1)$ for some $x \neq \pm 1$ and $\rho(C_G(G^1))$ is contained in the abelian group of diagonal matrices. If $n \in G \setminus C_G(G^1)$, then $\rho(n gn^{-1}) = \text{diag}(y, y^{-1}, 1)$, since it must lie in $\rho(G^1)$. The eigenvalues of $\rho(g)$ and $\rho(n gn^{-1})$ are the same but
\(g \neq ngn^{-1}\) so \(y = x^{-1}\). Thus \(n\) acts on \(G^1\) by inversion. Since \(n\) was an arbitrary element of \(G \setminus C_G(G^1)\), we have \([G : C_G(G^1)] = 2\).

Write \(C_G(G^1) = Q \times K = Q \times K^+ \times K^-\) where \(Q\) is the 2-sylow subgroup of \(C_G(G^1)\), \(K\) is its subgroup of odd-order elements, and \(K^\pm := \{k \in K : nkn^{-1} = k^{\pm 1}\}\), noting that the action of \(n\) on \(K\) is by an element of order 2 since \([G : C_G(G^1)] = 2\). Both \(Q\) and \(K^\pm\) are normal in \(G\), with \(Q\) being characteristic in the abelian group \(C_G(G^1)\) and \(K^\pm\) being normalized by definition. Replacing \(n\) by an odd power if necessary, we can assume that the order of \(n\) is a power of 2. Let \(P = \langle Q, n \rangle\), so \(P\) is a 2-Sylow subgroup of \(G\) and

\[
G = K^+ \times (K^- \rtimes P).
\]

If \(k \in K^-\) then \(nkn^{-1}k^{-1} = k^{-2} \in G^1\), but \(k\) has odd order so \(K^- \subset G^1\). Every nontrivial commutator has the form \(ana^{-1}n^{-1}\) for some \(a \in Q \times K^-\). If \(Q \subset Z(P)\) then the nontrivial commutators would all have odd order, contradicting our initial assumption. Thus \(P\) is nonabelian.

If \(K^+\) is not cyclic then it contains \(C_p \times C_p\) for an odd prime \(p\), so \(C_p \times C_p \times P\) is a nonabelian subgroup of \(G\) with noncyclic center, so has representation dimension 3 and must equal \(G\). If \(K^+\) is cyclic and nontrivial then \(\text{rdim}(G) = \text{rdim}(K^- \rtimes P)\) since \(K^- \cap Z(G) = \{1\}\), but \(\text{rdim}(K^- \rtimes P) < \text{rdim}(G)\) by minimality so this is absurd.

Now assume that \(K^+\) is trivial. We will show that \(Z(P)\) is not cyclic, which forces \(G = P\). Let \(L\) be the kernel of the projection \(\rho(G) \to \text{GL}_2(\mathbb{C})\). So \(L \subset \rho(Z(G))\) is those matrices of the form \(\text{diag}(1,1,y)\). If \(L\) was trivial then \(\text{rdim}(G) < 3\). Since \(\rho(K^-) \subset \rho(G^1) \subset \text{SL}_2(\mathbb{C}) \times \{1\}\), we have \(\rho(K^-) \cap \rho(L) = \{1\}\). Hence \(|L|\) is a nontrivial power of 2, so \(L \ni \text{diag}(1,1,-1)\), which together with \(\rho(z) = \text{diag}(-1,-1,1)\) creates a noncyclic subgroup of \(\rho(Z(G) \cap P) \subset \rho(Z(P))\).

We have shown that if \(G\) is not a 2-group then \(G = Q \times C_p \times C_p\) for a 2-group \(Q\) and odd prime \(p\). Suppose \(G\) is not a 2-group. Clearly the 2-Sylow subgroup \(Q\) of \(G\) must be nonabelian. If \(Q\) contained a nonabelian proper subgroup \(Q_1\), then \(Q_1 \times C_p \times C_p\) would be nonabelian with noncyclic center, hence would have representation dimension 3, contradicting minimality of \(G\). Thus every proper subgroup of \(Q\) is abelian. Since \(\text{rdim}(Q) = 2\), lemma 1.6 gives the two possible structures of \(Q\), hence of \(G\). That these groups both have a faithful representation of dimension 3 is simple: let \(\rho = \sigma \oplus \chi : G \to \text{GL}_3(\mathbb{C})\) where \(\sigma : Q \times C_p \to \text{GL}_2(\mathbb{C})\) and \(\chi : C_p \to \text{GL}_1(\mathbb{C})\) are faithful.
C) Let \( P \) be a 2-Sylow subgroup of \( G \), so \( P \) is abelian since \( G^1 \) has odd order. If \( P \) is trivial then \(|G|\) is odd. If \( P \) is cyclic then lemma 4.5 applies. So we may assume \( P \) has two invariant factors. If \( g \in G \) has odd order then \( g \) and \( G^1 \) generate an odd-order subgroup of \( G \). Since \( P \) is nontrivial, any odd-order subgroup of \( G \) is proper, hence abelian. Thus \( g \in C_G(G^1) \) and \( C_G(G^1) \) contains all elements of \( G \) of odd order. Therefore \( G = PC_G(G^1) \). We have \( P \supset V \cong C_2 \times C_2 \) since \( P \) is not cyclic.

Suppose that there exists \( v \in V \setminus Z(G) \). From this we will derive a contradiction. Since \( G^1 \) is abelian of odd order, \( G^1 = K^+ \times K^- \) where \( K^\pm := \{ k \in G^1 : vkv = k^{\pm 1} \} \). Note \( K^- \) is nontrivial since \( v \notin Z(G) \). If \( 1 \neq k \in K^- \) then \( \rho(k) \in \rho(G^1) \subset SL_3(\mathbb{C}) \) is conjugate to its inverse, so its eigenvalues are \( \zeta^{\pm 1}, 1 \) for some odd root of unity \( \zeta \neq 1 \). Choosing a basis so that \( \rho(k) = \text{diag}(\zeta, \zeta^{-1}, 1) \), we see \( \rho(C_G(k)) \) is contained in the abelian subgroup of diagonal matrices. Thus \( C_G(G^1) = C_G(k) \) (both consist under \( \rho \) of the diagonal matrices in \( \rho(G) \)) is an abelian normal subgroup of \( G \). A conjugate of \( \rho(k) \) under \( \rho(G) \) lies in \( \rho(G^1) \) so it is diagonal with the same eigenvalues. The permutations of the diagonal entries of \( \rho(k) \) induces a homomorphism \( G \to S_3 \) with kernel \( C_G(G^1) \). So \([G : C_G(G^1)]\) divides 6 and is divisible by 2. Since \([G : C_G(G^1)] = [P : P \cap C_G(G^1)]\) is a power of 2, we have \([G : C_G(G^1)] = 2\). Now \( v \notin C_G(G^1) \) and \( v \) has order 2 so \( G = C_G(G^1) \rtimes \langle v \rangle \). Let \( x \in C_G(G^1) \cap P \) be the generator of the nontrivial and cyclic 2-Sylow subgroup of \( C_G(G^1) \). If \( T \) is the subgroup of \( C_G(G^1) \) consisting of odd-order elements then \( G = \langle x \rangle \times (T \rtimes \langle v \rangle) \). Since \( Z(T \rtimes \langle v \rangle) \) has odd order, \( \text{rdim}(G) = \text{rdim}(T \rtimes \langle v \rangle) \). But \( T \rtimes \langle v \rangle \) is a proper subgroup of \( G \), contradicting minimality of \( G \).

Thus \( V \subset Z(G) \). In particular, \( Z(G) \) is not cyclic so by Schur’s lemma \( \rho \) is reducible. We can assume without loss of generality that \( \rho(G) \subset GL_2(\mathbb{C}) \times GL_1(\mathbb{C}) \), hence \( \rho(G^1) \subset SL_2(\mathbb{C}) \times \{1\} \). In particular, \( G^1 \) is cyclic. Let \( g \in G^1 \) be nontrivial, so that we may assume \( \rho(g) = \text{diag}(\zeta, \zeta^{-1}, 1) \) for some odd root of unity \( \zeta \neq 1 \). It follows that \( \rho(C_G(G^1)) \) consists of the diagonal matrices in \( \rho(G) \), hence \( C_G(G^1) \) is abelian. Since \( G = PC_G(G^1) \) is nonabelian and \( P \) is abelian, there exists \( y \in P \setminus C_G(G^1) \). Since \( \rho(\langle ygy^{-1} \rangle) \) has the same eigenvalues as \( \rho(g) \), is not equal to \( \rho(g) \), and lies in \( \rho(G^1) \), we must have \( ygy^{-1} = g^{-1} \). The subgroup of \( G \) generated by \( y, g, \) and \( V \) is nonabelian with noncyclic center, so must be equal to \( G \). The same is true if we replace \( g \) by \( g' \neq 1 \), so by minimality \( g \) has prime order \( p \). The group \( H \) generated by \( y \) and \( g \) is isomorphic to \( C_p \rtimes C_{2m} \). Since some power of \( y \) lies in \( V \), the group \( G \), which is generated by \( H \) and \( V \), must be isomorphic to \( (C_p \rtimes C_{2m}) \times C_2 \). If \( m = 1 \) this group has representation dimension 2, so \( m > 1 \). A faithful 3-dimensional representation can be constructed similarly to the previous case. \( \square \)
Remark. The proof of part b) broke into the cases $|G^1| = 2$ and $|G^1| \geq 4$, and showed that $|G^1|$ is always a power of 2. The final classification reveals that in fact we must have $|G^1| = 2$ in the case $r = 1$.

Next we shall show that if $G$ is a minimally faithful 2-group of degree 3 then $Z(G)$ is not cyclic. This will be useful in the subsequent result that classifies the minimally faithful 2-groups of degree 3.

**Lemma 4.8.** Suppose $G$ is a nonabelian 2-group and minimally faithful of degree 3. Then $Z(G)$ is not cyclic.

**Proof.** Suppose $Z(G)$ is cyclic and $\pi : G \to \text{GL}_2(\mathbb{C}) \times \text{GL}_1(\mathbb{C})$ is a faithful 3-dimensional representation, noting that no 3-dimensional representation of $G$ is irreducible since $G$ is a 2-group. Define the two subgroups

\[ N = \ker(G \xrightarrow{\pi} \text{GL}_2(\mathbb{C}) \times \text{GL}_1(\mathbb{C}) \to \text{GL}_1(\mathbb{C})) \]
\[ K = \ker(G \xrightarrow{\pi} \text{GL}_2(\mathbb{C}) \times \text{GL}_1(\mathbb{C}) \to \text{GL}_2(\mathbb{C})). \]

Both $K$ and $N$ are nontrivial since $\text{rdim}(G) = 3$, while $N \cap K = 1$. Clearly $K \subset Z(G)$ since $\pi(K)$ consists of matrices of the form $\text{diag}(1, 1, c) \in Z(\text{GL}_2(\mathbb{C}) \times \text{GL}_1(\mathbb{C}))$. Let $z \in Z(G)$ be the unique element of order 2. For any subgroup $M \subset G$ we have $z \in M$ iff $M \cap Z(G) \neq 1$. Since $K \subset Z(G)$, $z \in K$, so $z \not\in N$ and $N \cap Z(G) = 1$. Since $N$ is a 2-group it has noncyclic center. The subgroup $Z(G)N \cong Z(G) \times N$ therefore has noncyclic center, and must therefore be abelian (it cannot equal $G$ since $Z(G)$ is cyclic). So $N$ is abelian, and in fact $N$ must be cyclic: otherwise $N \supseteq C_2^2$ and $Z(G)N \supseteq C_2^3$. Let $n \in N$ be the unique element in $N$ of order 2. Since $N$ is a normal subgroup of $G$, $n \in Z(G)$, $n \neq z$, a contradiction. \qed

**Proposition 4.9.** Let $G$ be a nonabelian 2-group.

a) If $G$ is minimally faithful of degree 3 and contains a nonabelian proper subgroup then $G = Q_8 \times C_2$.

b) Suppose $G$ contains only abelian proper subgroups. Then $G$ is minimally faithful of degree 3 if and only if every index-2 subgroup of $G$ has two invariant factors.

**Proof.** a) Let $N$ be a proper nonabelian subgroup of $G$, and $Z = Z(G)$, which we have shown is not cyclic. Since $NZ$ is a nonabelian group with noncyclic center, $\text{rdim}(NZ) \geq 3$ and $G = NZ$. This remains true if we replace $Z$ by any noncyclic subgroup of $Z$. So there exists a subgroup $V \cong C_2 \times C_2 \subset Z$ with $G = NV$. If
$N \cap V$ was trivial then any element in $N$ of order 2 would, with $V$, generate $C_2^3$. Thus $N \cap V$ has order 2, since $V \not\subset N$. This implies $G \cong N \times C_2$. If $N$ contained a nonabelian proper subgroup $N_1$, then $N_1 \times C_2$ would be nonabelian with noncyclic center, hence would have representation dimension 3 and contradicting minimality of $G$. Since $\text{rdim}(N) = 2$ and $N \not\cong C_2 \times C_2$, lemma 4.6 forces $N = Q_8$.

b) Let $H \subset G$ be a subgroup of index 2. If $H$ is cyclic then there is a faithful character $\chi : H \to \mathbb{C}^\times$, and then $\text{Ind}_H^G(\chi)$ is a faithful 2-dimensional representation of $G$, which is impossible. The subgroup $H$ cannot have more than 2 invariant factors since $G \not\cong C_2^3$. The first direction is proved.

Now assume that $G$ is a nonabelian 2-group whose index-2 subgroups are all abelian with two invariant factors. Let $H \subset G$ of index 2, so $H = C_{2r} \times C_{2s} = \langle x, y \rangle$ where $r \geq s > 0$. Let $a \in G \setminus H$. Since proper subgroups of $G$ are abelian, $a$ must centralize every proper subgroup of $H$ that it normalizes. So every characteristic subgroup of $H$ is contained in $Z(G)$, and thus $C_{2r-1} \times C_{2s}$ is central if $r > s$ and $C_{2s-1} \times C_{2r-1}$ is central if $r = s$. So $x^2, y^2 \in Z(G)$ if $r = s$ and $x^2, y \in Z(G)$ if $r > s$.

If $a^2 = 1$ then $\langle a, x^{2r-1}, y^{2s-1} \rangle = C_2^3$. This is a proper subgroup of the nonabelian group $G$, contradicting our assumption that all proper subgroups have at most 2 invariant factors. So $a^2 \neq 1$. Now suppose $r = s = 1$. Since $1 \neq a^2 \in H = C_2 \times C_2$, we have $a^4 = 1$ and $|G| = [G : H]|H| = 8$. The subgroup generated by $a$ has index 2 in $G$, contradicting the assumption that all index-2 subgroups have two invariant factors. Thus either $r \geq s > 1$ or $r > s = 1$. In either case $Z(G) \cap H$ contains all elements of $H$ order 2, hence $Z(G)$ is not cyclic and $\text{rdim}(G) \geq 3$. All that remains is to construct a faithful 3-dimensional representation of $G$, since our assumption implies $\text{rdim}(H) = 2$ for all maximal subgroups $H$ of $G$.

We may assume without loss of generality that $x \notin Z(G)$; this is automatic if $r > s$. We have $axa^{-1} = x^iy^j$ for some $i, j \geq 0$. Since $x^2 = ax^2a^{-1} = (x^iy^j)^2 = x^{2i}y^{2j}$, we have $j \in \{0, 2^{s-1}\}$ and $i \in \{1, 1 + 2^{r-1}\}$. In any case, $z := axa^{-1}x^{-1}$ is of order 2, so $z \in Z(G)$. Since $x$ and $a$ do not commute, they generate $G$. Therefore $G/\langle z \rangle$ is abelian, and $z$ generates $[G, G]$.

Let $\chi : H \to \mathbb{C}^\times$ with $\chi(z) = -1$. Then $\rho := \text{Ind}_H^G(\chi)$ is a 2-dimensional representation of $G$. It is clear that $\ker(\rho) \cap aH$ is empty, so $\ker(\rho) \subset H$. If $\ker(\rho)$ was not cyclic then it would contain the three elements in $H$ of order 2, but $z \notin \ker(\rho)$. We have $\ker(\rho) \neq 1$ since $\text{rdim}(G) \geq 3$. Let $k$ be the unique element of order 2 in $\ker(\rho)$ and $\nu : G \to \mathbb{C}^\times$ with $\nu(k) = -1$; such a $\nu$ exists since the image of $k$ in $G/[G, G] = G/\langle z \rangle$ is nontrivial. The representation $\rho \oplus \nu$ is faithful. □
The 2-groups of the form described here have been classified - see \[MM03\] and \[M01\]. For example, the only group of order at most 16 on this list is the nontrivial semidirect product \(C_4 \rtimes C_4\). Since the actual construction of these groups would involve a large detour, we content ourselves to provide the relevant reference.

All that remains is to classify the odd-order groups that are minimally faithful of degree 3. This naturally breaks into the cases of 3-groups and not 3-groups.

**Proposition 4.10.** Suppose \(G\) is a nonabelian 3-group and minimally faithful of degree 3. Then either

a) \(G\) is the Heisenberg group of order 27, or 

b) \(G = \langle a, b | a^{3k} = b^3 = 1, b^{-1}ab = a^{1+3k-1} \rangle = C_3^k \rtimes C_3\) for some \(k \geq 2\).

**Proof.** Let \(H\) be an index-3 subgroup, so \(H\) is abelian and normal. If every index-3 subgroup of \(G\) is cyclic then every proper subgroup is cyclic, which would force \(G\) to be cyclic since it has odd order. So we may assume \(H\) is not cyclic. Let \(B \subset H\) be the subgroup generated by elements of order 3, so \(B \cong C_3 \times C_3\). Since \(Z(G)\) is cyclic, \(B \not\subset Z(G)\). Let \(a \in G \setminus C_G(B)\). Since \(a\) and \(B\) generate a nonabelian group, they generate \(G\). Since \(B\) is characteristic in \(H\), \(a\) normalizes \(B\). We can choose generators \(z\) and \(b\) of \(B\) so that \(az = za\) and \(aba^{-1} = bz\). In particular, \(G\) is generated by \(a\) and \(b\). Suppose \(a\) has order \(3^k\). Since \(a^3, z \in Z(G)\) and \(Z(G)\) is cyclic, either \(k = 1\) or else \(k \geq 2\) and \(z = a^{3k-1}\). The former case yields the Heisenberg group, which is easily shown to be minimally faithful of degree 3. The latter case yields the groups \(C_3^k \rtimes C_3\) with the claimed presentations. Every subgroup is clearly abelian of rank at most 2. Let \(\chi : \langle a \rangle \to \mathbb{C}^\times\) be a faithful character. Then \(\text{Ind}_{\langle a \rangle}^G(\chi)\) is a faithful 3-dimensional representation of \(G\). \(\square\)

**Proposition 4.11.** Suppose \(G\) is a nonabelian group of odd order, is not a 3-group, and is minimally faithful of degree 3. Then there exists a prime \(p \geq 5\) such that either

a) \(G = C_p \rtimes C_{3^n}\) if \(p \equiv 1 \pmod{3}\), where \(C_{3^n-1}\) is the center, or 

b) \(G = (C_p \times C_p) \rtimes C_{3^n}\) if \(p \equiv -1 \pmod{3}\), where \(C_{3^n-1}\) is the center.

**Proof.** We know that 3 divides |\(G|\) and every proper subgroup of \(G\) is abelian. From proposition \([2.3]\) we have |\(G| = 3^ap^b\) for some prime \(p > 3\). Let \(P\) be a \(p\)-Sylow subgroup and \(Q\) a 3-Sylow subgroup. We know one of them is cyclic and the other normal and elementary abelian (of rank at most 2). Since |\(\text{Aut}(C_3^a)| = 2^{a} \cdot 3\) and |\(\text{Aut}(C_3)| = 2\), the subgroup \(Q\) cannot be normalized by \(P\) without being centralized.
by $P$. Thus $Q$ is cyclic and $P$ is elementary abelian. A generator $x$ of $Q$ must act on $P$ by an automorphism of order 3, in order that $G$ contains no proper nonabelian subgroups.

Suppose $P = C_p$. Then 3 divides $|\text{Aut}(P)| = p - 1$ and $G = C_p \rtimes C_3$. One obtains $\text{rdim}(G) = 3$ by inducing a faithful character of the index-3 cyclic subgroup $C_p : Z(G) \cong C_3$. Suppose $P = C_p \times C_p$. Then 3 divides $|\text{Aut}(P)| = (p + 1)p(p - 1)^2$ and $G = (C_p \times C_p) \rtimes C_3$. If $p \equiv 1 \pmod{3}$, let $\alpha, \alpha^2 \in \mathbb{Z}/p\mathbb{Z}$ be the two roots of $t^2 + t + 1 = 0 \pmod{p}$. Then $P = P_0 \times P_1 \times P_2$ where $P_i := \{y \in P : xy^{-1} = y^\alpha\}$. If there exists $1 \neq y \in P_1$ or $1 \neq y \in P_2$ then the subgroup generated by $x$ and $y$ will necessarily be a nonabelian proper subgroup of $G$. But if $P = P_0$ then $G$ is abelian. Thus $p \equiv -1 \pmod{3}$. We need only show that $G$ has a faithful 3-dimensional representation. Let $H = C_p \times C_p \times C_3$ and $\chi : H \to \mathbb{C}^\times$ have kernel of order $p$. Let $\rho := \text{Ind}_H^G(\chi)$. Then $\text{dim}(\rho) = [G : H] = 3$. It is easy to see that $\ker(\rho) \subset H$, and from there that $\ker(\rho) \subset \ker(\chi)$. If $1 \neq y \in \ker(\rho)$ then $xy^{-1} \not\in \ker(\chi)$ since $xy^{-1}$ does not lie in the subgroup generated by $y$. So $xy^{-1} \not\in \ker(\rho)$, which contradicts the fact that $\ker(\rho)$ is a normal subgroup of $G$. Thus $\rho$ is faithful even though $\chi$ is not.

5. Further Work

There are several natural questions that one can pose about the class of minimally faithful finite groups $G$ of degree $n$, for $n > 3$. Are they always solvable? Which $p$-groups are minimally faithful? Which $G$ can be realized inside $\text{SL}_n(\mathbb{C})$? It is perhaps computationally intractable to obtain a full classification for substantially larger $n$.

In another direction, the notion of minimally faithful groups of degree $n$ can be readily defined over an arbitrary field $F$ instead of $\mathbb{C}$. There is still much that can be said for minimally faithful groups of degree 2; we state the results here without proof. Any finite subgroup of $\text{GL}_1(F)$ is cyclic. It follows from this and simple calculations in $\text{GL}_2(F)$ that the only candidates for nonabelian minimally faithful groups of degree 2 are a subset of those for $\mathbb{C}$, namely $Q_8$ and $C_p \rtimes C_2$; these work if and only if $F$ contain a 4th root of unity and a 2nd root of unity, respectively. In particular, if $F$ has characteristic 2 then all minimally faithful groups of degree 2 over $F$ are abelian. An abelian minimally faithful group over $F$ need not be elementary abelian. For example, one can have $G = C_p^a$ with $F$ of characteristic $p$ and containing a $p^{a-1}$-root of unity but not a $p^a$th root of unity, and such that the polynomial $(t^{p^a} - 1)/(t^{p-1} - 1)$ has an irreducible quadratic factor. A root $r$ of such a quadratic factor generates a quadratic extension $F(r)/F$, and multiplication by $r$, as an element of $\text{Aut}_F(F(r)) \cong \text{GL}_2(F)$,
generates a faithful representation of $C_{p^a}$. This arises, for example, if $F = \mathbb{Q}$ and $p^a = 4$. Similarly, if $F$ has characteristic $p$ then $C_p$ is not a subgroup of $\text{GL}_1(F)$ but is a subgroup of $\text{GL}_2(F)$ via the matrices $\begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}$, $0 \leq i \leq p - 1$. Finally, one has the groups $C_p \times C_p$, which will be minimally faithful of degree 2 if and only if $F$ contains a $p$th root of unity.

One could also broaden the class of groups and representations considered beyond the finite class. If $G$ is an arbitrary group, one can define its representation dimension to be the minimum cardinality $\kappa$ such that $G$ admits a faithful representation on a vector space of dimension $\kappa$. Then one could define $G$ to be minimally faithful of degree $\kappa$ if, for all proper subgroups $H$ of $G$, the representation dimension of $H$ is a strictly smaller cardinal than $\kappa$. For example, if $F$ is a finite field and $G = \mathbb{Z}_{p^\infty}$ is the Prüfer $p$-group (the group of $p$-power torsion in $\mathbb{C}^\times$) then $G$ has infinite representation dimension over $F$, while every proper subgroup $H$ of $G$ is finite (and cyclic), so $H$ has finite representation dimension over $F$. Thus $G$ is minimally faithful over any finite field.

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