Rigidity and Parallelism in the spacetime

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The effect of the linear-fractional transformations on the parallel lines in the spacetime has been studied. Fock-Lorentz transformations maps a line to a line, from which one can obtain the combinations rule for the velocities in the Fock-Lorentz transformations. Rigidity is defined as a consequences of holding parallelism under the transformations. The Fock-Lorentz transformations do not preserve rigidity, which leads to some novel results such as growing distances alongside with advancing time. Also, it is shown that the time coordinates of events will come closer to each other in the transformed coordinates by going back in time.

Keywords: Linear-Fractional Transformations, Fock-Lorentz Transformations, Rigidity, Parallelism.

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I. INTRODUCTION

The Fock-Lorentz transformations, which are linear-fractional generalization of the Lorentz transformations, have been obtained by V. A. Fock [1]. Other authors have obtained similar transformations before or after the V. A. Fock. A complete list of references could be found in [2]. Properties of the Fock-Lorentz transformations and their relations with other group theoretical extensions of the special theory of relativity with two fundamental constants have been studied in [3–6].

Studying the geometrical properties of the linear-fractional transformations, and especially the Fock-Lorentz transformations, can help us to have a better visualization of these transformations. These transformations act on the spacetime coordinates, thus we will consider a line or two parallel lines in the spacetime for studying the geometrical properties of the linear-fractional transformations. However, we can assume a straight line or two parallel lines in the true physical space \((x, y, z)\) which in fact can lead to an expanding universe. There are some studies which have related the expansion of the universe with this property of the linear-fractional transformations and the Fock-Lorentz transformations [2, 7, 8].

Mapping a line under the Lorentz or the Fock-Lorentz transformations or any desired transformations in the spacetime can specify the behaviour of a free particle under these transformations. Also, mapping two parallel lines in the spacetime under any transformations can show a better understanding of the concept of simultaneous events and the behavior of these events under these transformations. We will assign a number to the non-rigidity. The sign of this number can show us that these lines will come closer or go away from each other under the transformations. Also, these lines will remain parallel to each other if this number takes a zero value, which is happened obviously for the Lorentz transformations.

II. PARALLEL MOTIONS IN THE SPACETIME

A. The Lorentz Transformation case

Taking \(c = 1\), the standard Lorentz transformation in the spacetime are

\[
\begin{align*}
x' &= \gamma(x - ut) \\
t' &= \gamma(t - ux),
\end{align*}
\]  \(1\)
Consider a free particle $x = vt + x_0$ in the $S$ frame or $(xt)$ plane. The Lorentz transformation maps this line to

$$x' = v't' + \frac{x_0}{\gamma(1 - uv)},$$

where

$$v' = \frac{v - u}{1 - vu},$$

is the velocity of the particle in the $S'$ frame. Note that, this is the Einstein’s combination rule for velocities. Now we consider two particles in a parallel motion in $(xt)$ plane, by which we mean the velocity of the two particles are the same.

$$\begin{align*}
  x &= vt + x_0 \\
  \tilde{x} &= vt + \tilde{x}_0.
\end{align*}$$

The Lorentz transformation maps these parallel lines to

$$\begin{align*}
  x' &= v't' + \frac{x_0}{\gamma(1 - uv)} \\
  \tilde{x}' &= v't' + \frac{\tilde{x}_0}{\gamma(1 - uv)},
\end{align*}$$

which are also parallel lines in the $(x't')$ plane. Note that the Galilean transformations will map Eq.(4) to

$$\begin{align*}
  x' &= v't' + x_0 \\
  \tilde{x}' &= v't' + \tilde{x}_0,
\end{align*}$$

where $v' = v - u$. The main difference, apart from the relativistic addition of velocities, is the factor $\gamma^{-1}(1 - uv)^{-1}$ multiplying the initial positions.

**B. Fock- Lorentz Transformations case**

The Fock-Lorentz transformation in (1+1) dimensions is

$$\begin{align*}
  x' &= \frac{\gamma(x - ut)}{1 + \lambda u \gamma x - \lambda(\gamma - 1)t} \\
  t' &= \frac{\gamma(t - ux)}{1 + \lambda u \gamma x - \lambda(\gamma - 1)t},
\end{align*}$$

where $c = 1$, and $\lambda$ is a very small constant with the dimension of $T^{-2}$. These transformations will map the free particle $x = vt + x_0$ to

$$x' = \left[\frac{v - u - \lambda x_0(\gamma - 1)\gamma}{1 - uv + \lambda ux_0}\right] t' + \frac{x_0}{\gamma(1 - uv + \lambda ux_0)},$$
which is also the equation of a line in the \((x't')\) plane. Thus, the Fock-Lorentz transformations map a free particle motion to a free particle motion. As in the Lorentz case, we can interpret the slope of this line

\[
u' = \frac{v - u - \lambda x_0 (\gamma - 1) / \gamma}{1 - uu + \lambda uu_0},\]

as the composition rule for the velocities in the Fock-Lorentz transformations. This velocity depends on the initial position of the particle \(x_0\). Note also that to obtain (8) we have assumed a free particle with constant velocity in the \((xt)\). We will find a more general expression for the combination rule of the velocities in the next section.

The Fock-Lorentz transformations will map two free particles which are in parallel motions in the \(S\) frame

\[
\begin{align*}
x &= vt + x_0 \\
\tilde{x} &= vt + \tilde{x}_0,
\end{align*}
\]

to

\[
\begin{align*}
x' &= \left[ \frac{v - u - \lambda x_0 (\gamma - 1) / \gamma}{1 - uu + \lambda uu_0} \right] t' + \frac{x_0}{\gamma (1 - uu + \lambda uu_0)} \\
\tilde{x}' &= \left[ \frac{v - u - \lambda \tilde{x}_0 (\gamma - 1) / \gamma}{1 - uu + \lambda uu_0} \right] t' + \frac{\tilde{x}_0}{\gamma (1 - uu + \lambda uu_0)}.
\end{align*}
\]

Although, the velocities of these two particles are the same in the \(S\) frame, their velocities are not the same in the \(S'\) frame. In other words, the Fock-Lorentz transformations do not preserve parallelism. The lines in the \((x't')\) plane, Eq. (11) will cross each other at

\[
(t'_\lambda, x'_\lambda) = \left( \frac{-1}{\lambda}, \frac{\lambda + u - v}{\lambda[1 - uu + \lambda uu_0]} \right).
\]

The time component \(t'_\lambda\) does not depend on \(x_0\) or \(\tilde{x}_0\), which shows that the crossing points are on a line, parallel to the \(x'\) axis. We have obtained Eq. (12) for \(\gamma \gg 1\), but in general the value of \(t'_\lambda\) is independent of the \(x_0\) or \(\tilde{x}_0\). The general expressions for \((t'_\lambda, x'_\lambda)\) are more complex and we only give the value of \(t'_\lambda\), which is

\[
t'_\lambda = \frac{-1}{\lambda} \cdot \frac{1 - uu}{(\gamma - 1) / \gamma - uu}.
\]

III. LINEAR- FRACTIONAL TRANSFORMATIONS IN GENERAL FORM

We consider a general form for the linear-fractional transformations in \((1 + 1)\) dimensions from the \(S\) frame with coordinates \((x, t)\) to the \(S'\) frame with coordinates \((x', t')\).

\[
\begin{align*}
x' &= a_1 + A_{11} x + A_{10} t \\
\frac{1}{1 + a_1 x + a_0 t}, \\
t' &= a_0 + A_{01} x + A_{00} t,
\end{align*}
\]

\[
\frac{1}{1 + a_1 x + a_0 t}.
\]

(14)
Here, $A_{\mu\nu}$ is a general $2 \times 2$ invertible matrix and $\alpha_0$ and $\alpha_1$ are constants.

Taking derivatives of Eq. (14) we obtain the velocity $v' = dx'/dt'$ of the particle in the $S'$ frame as

$$v' = \frac{(A_{11} - \alpha_1 a_1)v + (A_{10} - \alpha_0 a_1) + (A_{10} \alpha_1 - A_{11} \alpha_0)(x - vt)}{(A_{01} - \alpha_1 a_0)v + (A_{00} - \alpha_0 a_0) + (A_{00} \alpha_1 - A_{01} \alpha_0)(x - vt)}.$$  \hspace{1cm} (15)

This formulas can be rewritten in another compact form

$$v' = \frac{A_{11}v + A_{10} - x'(\alpha_1 v + \alpha_0)}{A_{01}v + A_{00} - t'(\alpha_1 v + \alpha_0)}.$$  \hspace{1cm} (16)

We should mention that this formula has been obtained initially by Kerner [9]. Two expressions for the velocity in Eq. (15) and Eq. (16) are equivalent; they differ only in dependency of the velocity of the particle to the $S$ frame coordinates or the $S'$ frame coordinates. It is enough to put the expressions for $x'$ and $t'$ from Eq. (14) in the Eq. (16) to reach to the Eq. (15).

A. Velocity in Fock-Lorentz case

For the Fock-Lorentz transformations we have

$$[A_{\mu\nu}] = \begin{pmatrix} \gamma & -\gamma u \\ -\gamma u & \gamma \end{pmatrix},$$  \hspace{1cm} (17)

where

$$\alpha_1 = \lambda u \gamma, \quad \alpha_0 = -\lambda (\gamma - 1).$$  \hspace{1cm} (18)

Substituting these values for $A_{\mu\nu}$, $\alpha_1$, and $\alpha_0$ in Eq. (16) we obtain the velocity of the particle in the $S'$ frame as

$$v' = \frac{v - u + \lambda \left(\gamma^{-1} - 1\right)(x - vt)}{1 - uv + \lambda u(x - vt)}.$$  \hspace{1cm} (19)

This is a general formula for the velocity. If we take $x = x_0 + vt$ for a free particle with constant speed, we get the same result as in Eq. (9) of section 2.2. Also, from Eq. (16) we get

$$v' = \frac{v - u - \lambda x'(uvx + \gamma^{-1} - 1)}{1 - uv - \lambda t'(uvx + \gamma^{-1} - 1)},$$  \hspace{1cm} (20)

which explicitly shows the dependency of $v'$ on the $(x', t')$ coordinates of the particle in the $S'$ frame.

We have obtained the Einstein’s combination rule for velocities from the slope of the line in the transformed spacetime. We repeated the same procedure for the Fock-Lorentz transformations.
and obtain the combination rule for the velocities. However, the difference between these two combination rules is that in the Fock-Lorentz case the combination rule depends on the initial position of the particle. This is a well known matter, but we have obtained the formulas from mapping a line in the spacetime and this method can lead us to a better visualization. Similar expressions have been obtained initially by Kerner [9], by Manida [2], and Stepanov [7].

IV. RIGIDITY IN THE LINEAR-FRACTIONAL TRANSFORMATIONS OF THE SPACETIME

Using the general forms of transformations in Eq. (14) we can compute the difference

\[ x' - \tilde{x}' = \frac{a_1 + A_{11}x + A_{10}t}{1 + \alpha_1 x + \alpha_0 t} - \frac{a_1 + A_{11}\tilde{x} + A_{10}\tilde{t}}{1 + \alpha_1 \tilde{x} + \alpha_0 \tilde{t}}, \]

(21)

which can be simplified to the first order of \( \alpha \) to

\[ x' - \tilde{x}' \approx \left[ (A_{11} - a_1\alpha_1)(x - \tilde{x}) + (A_{10} - a_1\alpha_0)(t - \tilde{t}) + (A_{11}\alpha_0 - A_{10}\alpha_1)(\tilde{x}t - \tilde{x}t) \right] \left[ 1 - \alpha_1 (x + \tilde{x}) - \alpha_0 (t + \tilde{t}) \right] + O(\alpha^2). \]

For time component the difference will be

\[ t' - \tilde{t}' = \frac{a_0 + A_{01}x + A_{00}t}{1 + \alpha_1 x + \alpha_0 t} - \frac{a_0 + A_{01}\tilde{x} + A_{00}\tilde{t}}{1 + \alpha_1 \tilde{x} + \alpha_0 \tilde{t}}, \]

(22)

or in the simplified form

\[ t' - \tilde{t}' = \left[ (A_{01} - a_0\alpha_1)(x - \tilde{x}) + (A_{00} - a_0\alpha_0)(t - \tilde{t}) + (A_{00}\alpha_1 - A_{01}\alpha_0)(\tilde{x}t - \tilde{x}t) \right] \left[ 1 - \alpha_1 (x + \tilde{x}) - \alpha_0 (t + \tilde{t}) \right] + O(\alpha^2). \]

(23)

Here, for any convenient choices of the \([A_{\mu\nu}]\) and also \(\alpha_0, \alpha_1\), \(a_0, a_1\) we can define some new transformations in the spacetime for which we can check that these transformations will preserve the rigidity or not.

A. Fock-Lorentz transformations case and its visualization

For the specific case of the Fock-Lorentz transformations

\[ x' - \tilde{x}' = \frac{\gamma \left[ 1 + \lambda t(\gamma - 1)/\gamma \right]}{\left[ 1 + \lambda u\gamma x - \lambda(\gamma - 1)t \right] \left[ 1 + \lambda u\gamma\tilde{x} - \lambda(\gamma - 1)\tilde{t} \right]} (x - \tilde{x}), \]

(24)
which in the first order of $\lambda$ can be simplified as

$$x' - \tilde{x}' = \gamma \left[ 1 - \lambda u \gamma (x + \tilde{x}) + \lambda \frac{(\gamma - 1)(2\gamma + 1)}{\gamma} \right] (x - \tilde{x}),$$

(25)

also in the second equation we have put $t = \tilde{t}$.

We have obtained the contraction of the length in the Fock-Lorentz transformations, which has smaller amount with respect to the Lorentz case. If we take $x - \tilde{x} = L$ we will see that $L'$ in the transformed coordinates will depend on the coordinates of the rod in $(xt)$ plane. In other words, the coordinates of the beginning $\tilde{x}$ and the end $x$ of the a rod in $(xt)$ plane and also the time of observation will be entered in this formals. In the Lorentz case, we have assumed the same parallelogram in the $(xt)$ plane and we study the deformations of this parallelogram under the the Fock-Lorentz transformations. We define a new function $f(X, \tilde{X})$ by taking

$$f(X, \tilde{X}) = 1 - \sigma (x + \tilde{x}) + \rho t,$$

(26)

in which

$$\sigma = \lambda \gamma u, \quad \rho = \lambda \frac{(\gamma - 1)(2\gamma + 1)}{\gamma}.$$  

(27)

Thus, Eq. (25) can be rewritten as

$$x' - \tilde{x}' = \gamma f(X, \tilde{X})(x - \tilde{x}).$$

(28)

As indicted two particles are moving with the same velocity in the $(xt)$ plane and by putting the values of $x$ and $\tilde{x}$ in the expression for $f(X, \tilde{X})$ we can relate $f(X, \tilde{X})$ to the $f(X_0, \tilde{X}_0)$ by

$$f(X, \tilde{X}) = f(X_0, \tilde{X}_0) + g(t, t_0),$$

(29)

in which the function $g(t, t_0)$ indicates their difference

$$g(t, t_0) = (\rho - 2\sigma v)(t - t_0).$$

(30)

For high velocities $\gamma \gg 1$ we have approximately

$$g(t, t_0) = 2\lambda \gamma (1 - uv)(t - t_0),$$

(31)

which is obviously greater than zero. Thus,

$$f(X, \tilde{X}) \gg f(X_0, \tilde{X}_0),$$

(32)

which shows that the $x'_0 - \tilde{x}'_0$ interval will grow under the motion of these two particles. Thus, the rigidity is not preserved under the Fock-Lorentz transformations. In fact, going back in time, the
$x'_0 - \tilde{x}'_0$ interval will be shrunk and will go to the zero in $t = -1/\lambda c^2$, which is easily seen from Eq. (25) by putting the numerator of the right hand side of this equation to zero.

If we want to be precise, $x'_0 - \tilde{x}'_0$ can be rewritten in terms of $(x_0 - \tilde{x}_0)$ in the compact form similar to the Eq. (28) as

$$x'_0 - \tilde{x}'_0 = \gamma f(X_0, \tilde{X}_0)(x_0 - \tilde{x}_0).$$  \hspace{1cm} (33)

In the $(x, t)$ plane we have $x - \tilde{x} = x_0 - \tilde{x}_0$, thus by using of the Eq(26) and Eq.(29) we can assign a value $N_{nr}$ for the non-rigidity

$$N_{nr} := (x'_0 - \tilde{x}'_0) - (x_0 - \tilde{x}_0) = \gamma g(t, t_0)(x_0 - \tilde{x}_0).$$  \hspace{1cm} (34)

For $\gamma \gg 1$ by using of the value of $g(t, t_0)$ from Eq. (31) we find

$$N_{nr} = 2\lambda\gamma(1 - uv)(t - t_0)(x_0 - \tilde{x}_0),$$  \hspace{1cm} (35)

which is a positive value as mentioned in the above and will be increased by going forward in time or for the bigger value of $t - t_0$. The negativity or positivity of the value of the rigidity $N_{nr}$ will specify that the parallel line in the $(x't')$ will come close together or not.

On the other hand, for $t' - \tilde{t}'$ from Eq.(22) we have

$$t' - \tilde{t}' = \frac{-u\gamma(1 + \lambda t)}{[1 + \lambda u\gamma x - \lambda(\gamma - 1)t][1 + \lambda u\gamma \tilde{x} - \lambda(\gamma - 1)\tilde{t}]}(x - \tilde{x}),$$  \hspace{1cm} (36)

which can be simplified in first order of $\lambda$ as

$$t' - \tilde{t}' = -\gamma u\left[1 - \lambda u\gamma(x + \tilde{x}) + \lambda(2\gamma - 1)t\right](x - \tilde{x}),$$  \hspace{1cm} (37)

also in the second equation we have put $t = \tilde{t}$. Here, as in the above we introduce

$$\Theta(X, \tilde{X}) = 1 - \sigma(x + \tilde{x}) + \beta t,$$  \hspace{1cm} (38)

in which $\sigma$ is given by Eq. (27) and $\beta = \lambda(2\gamma - 1)$. Thus, we can rewrite Eq. (37) as

$$t' - \tilde{t}' = -u\gamma \Theta(X, \tilde{X})(x - \tilde{x}).$$  \hspace{1cm} (39)

By putting the equations of motions of the assumed two parallel particles we can relate $\Theta(X, \tilde{X})$ to the $\Theta(X_0, \tilde{X}_0)$ as

$$\Theta(X, \tilde{X}) = \Theta(X_0, \tilde{X}_0) + \chi(t, t_0),$$  \hspace{1cm} (40)
in which
\[
\chi(t, t_0) = (\beta - 2\sigma v)(t - t_0).
\] (41)

These equations can help us to compare the \(t' - \tilde{t}'\) with \(t'_0 - \tilde{t}'_0\). For high velocities \(\gamma \gg 1\), we find
\[
\chi(t, t_0) = 2\lambda\gamma(1 - uv)(t - t_0),
\] (42)

which has a positive value. Thus,
\[
|t' - \tilde{t}'| > |t'_0 - \tilde{t}'_0|,
\] (43)

which shows that the value of \(|t'_0 - \tilde{t}'_0|\) will grow by motions of these two particles. In other words, by going back in the time the \(|t' - \tilde{t}'|\) will become smaller and in the \(t = -1/\lambda c^2\) will go to zero as seen from Eq. (36) by putting the numerator of the right side of this equation to zero.

It is also possible to define the rigidity as preserving \(t - \tilde{t} = t_0 - \tilde{t}_0\) under the spacetime transformations which will be brought to \(t' - \tilde{t}' = t'_0 - \tilde{t}'_0\) in the \((x't')\) spacetime. Also, we should take the same initial positions for the particles which have started from different initial times. The Lorentz transformations are symmetric for the time and positions coordinates, and this definition does not lead to significant different aspects. But, the Fock-Lorentz transformations are not symmetric for the time and positions coordinates, and we can encounter with different aspects of the non-rigidity. But, the \(t'_\lambda\) line and crossing of the the particles paths on \(t'_\lambda\) line will be occurred as in the same way. We postpone this study to the future studies.

V. CONCLUSIONS

The Lorentz transformations are affine transformations which map a straight line in the spacetime to a straight line. On the other hand, the affine transformations will preserve the parallelism of the lines and the spacetime is rigid under these transformations. Thus, we can say that the value of non-rigidity is zero for the affine transformations.

In [10], it was shown that the Fock-Lorentz transformations in the spactime and also the MS transformation in the energy- momentum space as projective transformations. The projective transformations also map a straight line to a straight line, but they do not keep the parallelism. The amount of the deviations from the parallelism can be expressed by introducing the rigidity. As mentioned, the positivity and the negativity of the non-rigidity can show that these lines will go away from each other or come close to each other.
We can do the same study for the proposed varying speed of light theories by investigation the effects of these theories on a line or two parallel lines in the spacetime. Also by calculating the non-rigidity we can specify the behaviour of these theories in any region of the spacetime [11, 12]. It may happen that these theories attain different value for the non-rigidity and have different aspects under mapping of the parallel lines. We can do all of these in a very small region of spacetime for comparing the local behaviour of these theories by the Lorentz transformations. Also, one can do the same study for the transformations in the spacetime which has been obtained as a position space images of the doubly special theories by some authors [13–15].

On the other hand, Maguejo-Smolin transformations are linear-fractional transformations in the energy-momentum space [16], and the results of this study is applicable to MS DSR, but with some cares. It should be noted that the concept of the line in the spacetime is not the same in the energy-momentum space.

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