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SEIBERG-WITTEN INVARIANTS AND SURFACE SINGULARITIES II
(SINGULARITIES WITH GOOD $\mathbb{C}^*$-ACTION)

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Abstract. We verify the conjecture formulated in [18] for any normal surface singularity
which admits a good $\mathbb{C}^*$-action. The main result connects the Seiberg-Witten invariant of
the link (associated with a certain “canonical” spin$^c$ structure) with the geometric genus
of the singularity.

As a by-product, we compute the Seiberg-Witten monopoles of the link (associated with
the canonical spin$^c$ structure and the natural Thurston metric) in terms of its Seifert invariants.
Additionally, we also determine in terms of the Seifert invariants all the Reidemeister-
Turaev sign-refined torsion of the link (associated with any spin$^c$ structure).

1. Introduction

The present article is a natural continuation of [18], where the authors formulated a very
general conjecture which relates the topological and the analytical invariants of a complex
normal surface singularity whose link is a rational homology sphere.

Let $(X, 0)$ be a normal two-dimensional analytic singularity. It is well-known that from a
topological point of view, it is completely characterized by its link $M$, which is an oriented
3-manifold. Moreover, by a result of Neumann [20], any decorated resolution graph of $(X, 0)$
carries the same information as $M$. A property of $(X, 0)$ will be called topological if it can
be determined from $M$, or equivalently, from any resolution graph of $(X, 0)$. For example,
for a given resolution, if we take the canonical divisor $K$, and the number $\#V$ of irreducible
components of the exceptional divisor of the resolution, then $K^2 + \#V$ is independent of
the choice of the resolution, it is an invariant of the link $M$ (cf. 2.5).

Our interest is to investigate the possibility to express the geometric genus (which is, by
its very definition, an analytic invariant of the singularity) in terms of topological invariants
of the link.

Let us start with a brief historical survey. M. Artin proved in [1, 2] that the rational
singularities (i.e. $p_g = 0$) can be characterized completely from the graph. In [11], H.
Laufer extended Artin’s results to minimally elliptic singularities, showing that Gorenstein
singularities with $p_g = 1$ can be characterized topologically. Additionally, he noticed that
the program breaks for more complicated singularities (see also the comments in [18] and
[17]). On the other hand, the first author noticed in [17] that Laufer’s counterexamples
do not signal the end of the program. He conjectured that if we restrict ourselves to the
case of those Gorenstein singularities whose links are rational homology spheres then $p_g$
is topological. This was carried out explicitly for elliptic singularities in [17] (partially based
on some results of S. S.-T. Yau, cf. e.g. with [32]).

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For Gorenstein singularities, in the presence of a smoothing with Milnor fiber $F$, the above question can be reformulated in terms of the signature $\sigma(F)$ and/or the topological Euler characteristic $\chi_{\text{top}}(F)$ of $F$. Indeed, via some results of Laufer, Durfee, Wahl and Steenbrink, for Gorenstein singularities, any of $p_g$, $\sigma(F)$ and $\chi_{\text{top}}(F)$ determines the remaining two modulo $K^2 + \#\mathcal{V}$ (see e.g. [14]).

This fact creates the bridge connecting the above problem with the following list of results about the signature $\sigma(F)$. Fintushel and Stern proved in [5] that for a hypersurface Brieskorn singularity whose link is an integral homology sphere, the Casson invariant $\lambda(M)$ of the link $M$ equals $-\sigma(F)/8$. This fact was generalized by Neumann and Wahl in [22].

They proved the same statement for all Brieskorn-Hamm complete intersections and suspensions of plane curve singularities (with the same assumption about the link). Moreover, they conjectured the validity of the formula for any isolated complete intersection singularity (with the same restriction about the link). For some other conjectures about these singularities, the reader can also consult [23].

The goal of [18] was to generalize this conjecture for smoothing of Gorenstein singularities with rational homology sphere link. In fact, more generally (i.e. even if the singularity is not smoothable) [18] conjecturally introduced an “optimal” topological upper bound for $p_g$ in the following sense: it is a topological upper bound for $p_g$ for any normal surface singularity, but additionally, for Gorenstein singularities it yields exactly $p_g$. (Such an “optimal” topological upper bound for elliptic singularities is the length of the elliptic sequence, introduced and studied by S. S.-T. Yau, see e.g. [28], and Laufer.)

The conjecture in [18] replaces the Casson invariant $\lambda(M)$ by a certain Seiberg-Witten invariant of the link, i.e. by the sum of the Casson-Walker invariant and a certain Reidemeister-Turaev sign-refined torsion invariant.

We recall (for details, see the first part [18], and the references listed there) that the set of Seiberg-Witten invariants associates with any spin$^c$ structure $\sigma$ of $M$ a rational number $\text{sw}^0_M(\sigma)$. In [18] we introduced a “canonical” spin$^c$ structure $\sigma_{\text{can}}$ of $M$. This can be done as follows. The (almost) complex structure on $X \setminus \{0\}$ induces a natural spin$^c$ structure on $X \setminus \{0\}$. Then $\sigma_{\text{can}}$, by definition, is its restriction to $M$. (An equivalent definition can be done using a canonical quadratic function constructed by Looijenga and Wahl in [14].) The point is that this structure depends only on the topology of $M$ alone.

We are now ready to recall the conjecture from [18].

1.1. Conjecture. [18] Assume that $(X,0)$ is a normal surface singularity whose link $M$ is a rational homology sphere. Let $\sigma_{\text{can}}$ be the canonical spin$^c$ structure on $M$. Then, conjecturally, the following facts hold.

(1) For any $(X,0)$, there is a topological upper bound for $p_g$ given by:

$$\text{sw}^0_M(\sigma_{\text{can}}) - \frac{K^2 + \#\mathcal{V}}{8} \geq p_g.$$ 

(2) If $(X,0)$ is rational or Gorenstein, then in (1) one has equality.

(3) In particular, if $(X,0)$ is a smoothing of a Gorenstein singularity $(X,0)$ with Milnor fiber $F$, then

$$\text{sw}^0_M(\sigma_{\text{can}}) = -\frac{\sigma(F)}{8}.$$ 

[Notice that if $(X,0)$ is numerically Gorenstein and $M$ is a $\mathbb{Z}_2$–homology sphere then $\sigma_{\text{can}}$ is the unique spin structure of $M$; if $M$ is an integral homology sphere then in the above formulae $\text{sw}^0_M(\sigma_{\text{can}}) = \lambda(M)$, the Casson invariant of $M$.]
In \cite{18}, the conjecture was verified for cyclic quotient and du Val singularities, Brieskorn-Hamm complete intersections, and some rational and minimally elliptic singularities.

In \cite{18} Remark 4.6(2), we noticed that it is very likely that parts (2) and (3) of the conjecture are true even for a larger class of singularities, e.g. for \(\mathbb{Q}\)-Gorenstein singularities, or some families of singularities with some kind of additional analytical rigidity.

In fact, this is exactly the case for normal surface singularities with a (good) \(\mathbb{C}^*\)-action. It is well-known that a complex affine algebraic variety \(X\) admits a \(\mathbb{C}^*\)-action if and only if the affine coordinate ring \(A\) admits a grading \(A = \bigoplus_k A_k\). Following Orlik-Wagreich, we say that the action is good if \(A_k = 0\) for \(k < 0\) and \(A_0 = \mathbb{C}\). This means that the point 0 corresponding to the maximal ideal \(\bigoplus_{k>0} A_k\) is the only fixed point of the action. Additionally, we will assume that \((X, 0)\) is normal.

For these singularities, in this article we prove:

**1.2. Theorem.** Let \((X, 0)\) be a normal surface singularity with a good \(\mathbb{C}^*\)-action. Then

\[
\text{sw}_0^0 M(\sigma_{\text{can}}) - \frac{K^2 + \# V}{8} = p_g.
\]

The proof is based in part on Pinkham’s formula \cite{26} of \(p_g\) in terms of the Seifert invariants of the link (cf. also with Dolgachev’s work about weighted homogeneous singularities; see e.g. \cite{18}). On the other hand, in the proof we use the formulae for \(K^2 + \# V\) and the Reidemeister-Turaev torsion determined in \cite{18}, and a formula for the Casson-Walker invariant proved in \cite{12}. The Reidemeister-Turaev torsion formally shows big similarities with the Poincaré series associated with the graded coordinate ring of the universal abelian cover of \((X, 0)\). In the proof we also borrowed some technique of Neumann applied by him for this Poincaré series \cite{19}.

Notice that in the above theorem we do not require for \((X, 0)\) to be Gorenstein. (This is replaced by the existence of the \(\mathbb{C}^*\)-action.) On the other hand, the theorem has the following corollary (which can also be applied for singularities without \(\mathbb{C}^*\)-action).

**1.3. Corollary.** Assume that the link of a normal surface singularity \((X, 0)\) is a rational homology sphere Seifert 3-manifold. If \((X, 0)\) is rational, or minimally elliptic, or Gorenstein elliptic, then the statements of the above conjecture are true for \((X, 0)\). (I.e., \((X, 0)\) satisfies (1) with equality; and also (3), provided that the additional assumptions of (3) are satisfied.)

Indeed, in the case of these singularities, all the numerical invariants involved in the conjecture are characterized by the link. Moreover, each family contains a special representative which admits a good \(\mathbb{C}^*\)-action. In fact, the above corollary can automatically be extended to any family of singularities with these two properties.

The paper is organized as follows. In section 2 we review the needed definitions and results. For a more complete picture and list of references the reader is invited to consult \cite{18}. Section 3 starts with a theorem (cf. \cite{18}) which connects four topological numerical invariants of the link. These invariants are: the Reidemeister-Turaev torsion, the Casson-Walker invariant, the Dolgachev-Pinkham invariant \(\text{DP}_M\) (which is the topological candidate for \(p_g\)), and finally \(K^2 + \# V\) (which can be identified with the Gompf invariant, cf. \cite{12}). This result implies the above theorem via Pinkham’s result \cite{26} (cf. \cite{26}(10)).
Finally, we have included at the end of section 3 a complete and explicit description of the Reidemeister-Turaev torsion $\mathcal{T}_{M,\sigma}(1)$ for any spin$^c$ structure $\sigma$ in terms of the Seifert invariants of $M$ (the computation follows Neumann’s method mentioned above).

1.4. **Remark.** Theorem 3.1 has a consequence interesting from the point of gauge theory as well. Recall that the modified Seiberg-Witten invariant is the sum $sw_0^M(\sigma) = sw_M(\sigma, u) + KS_M(\sigma, u)/8$ of the Seiberg-Witten monopoles and $(1/8)$ of the Kreck-Stolz invariant (cf. \[24\]). The $(\text{can}_0, u)$-monopoles were described in \[16, 24\], and the Kreck-Stolz invariant was described in \[24\]. Nevertheless, their explicit computation in terms of the Seifert invariants in the general case meets some difficulties. Our theorem 3.1 implies the following.

Assume that $M$ is a rational homology sphere Seifert 3-manifold with $c < 0$. Then, for any good parameter $u$, the signed number of Seiberg-Witten $u$-monopoles can be computed as follows:

$$sw_M(\text{can}_0, u) = -\frac{KS_M(\text{can}_0, u)}{8} + \frac{K^2 + \#V}{8} + DP_M.$$  

Recall that the Seifert manifold $M$ admits a natural metric $g_0$, the so called Thurston metric. If $u = (g_0, 0)$, the invariants appearing on the right hand side have very explicit expression in terms of the Seifert invariants of $M$. Indeed, $KS_M(\text{can}_0, g_0, 0)$ is determined in \[24\] (see also \[18\]), $K^2 + \#V$ is determined in \[18\] (see \[2.7(6)\] here), and $DP_M$ was introduced in \[26\] (it is \[2.7(9)\] here); in fact the last two are $u$-independent. In particular, when the parameter $(g_0, 0)$ is good, the above identity determines $sw_M(\text{can}_0, g_0, 0)$ explicitly in terms of the Seifert invariants.

For results in this direction, see \[16, 24\].

2. **Preliminaries**

2.1. **Definitions.** Let $(X, 0)$ be a normal surface singularity. Consider the holomorphic line bundle $\Omega^2_X \setminus \{0\}$ of holomorphic 2-forms on $X \setminus \{0\}$. If this line bundle is holomorphically trivial then we say that $(X, 0)$ is Gorenstein. Let $\pi : \tilde{X} \to X$ be a resolution over a sufficiently small Stein representative $X$ of the germ $(X, 0)$. Then $p_0(X, 0) := \dim H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is finite and independent of the choice of $\pi$. It is called the geometric genus of $(X, 0)$.

2.2. **The link and its canonical spin$^c$ structure.** Let $(X, 0)$ be a normal surface singularity embedded in $(\mathbb{C}^N, 0)$. Then for $\epsilon$ sufficiently small the intersection $M := X \cap S^{2N-1}_\epsilon$ of a representative $X$ of the germ with the sphere $S^{2N-1}_\epsilon$ (of radius $\epsilon$) is a compact oriented 3-manifold, whose oriented $C^\infty$ type does not depend on the choice of the embedding and $\epsilon$. It is called the link of $(X, 0)$. In this article we will assume that $M$ is a rational homology sphere, and we write $H := H_1(M, \mathbb{Z})$.

The almost complex structure on $X \setminus \{0\}$ determines a spin$^c$ structure on $X \setminus \{0\}$, whose restriction to $M$ will be denoted by $\sigma_{\text{can}} \in \text{Spin}^c(M)$. It turns out that $\sigma_{\text{can}}$ depends only on the oriented $C^\infty$ type of $M$ (for details see \[18\], cf. also with \[14\]).

2.3. **The Seiberg-Witten invariants of $M$.** To describe the Seiberg-Witten invariants one has to consider an additional geometric data belonging to the space of parameters

$$\mathcal{P} := \{ u = (g, \eta); \ g = \text{Riemann metric, } \eta = \text{closed two-form}\}.$$  

Then for each spin$^c$ structure $\sigma$ on $M$ one defines the $(\sigma, g, \eta)$-Seiberg-Witten monopoles. For a generic parameter $u$, the Seiberg-Witten invariant $sw_M(\sigma, u)$ is the signed monopole count. This integer depends on the choice of the parameter $u$ and thus it is not a topological
invariant. To obtain an invariant of $M$, one needs to alter this monopole count. The needed additional contribution is the Kreck-Stolz invariant $KS_M(\sigma, u)$ (associated with the data $(\sigma, u)$), cf. [13] (or see [10] for the original “spin version”). Then, by [3, 13, 15], the rational number

$$\frac{1}{8} KS_M(\sigma, u) + sw_M(\sigma, u)$$

is independent of $u$ and thus it is a topological invariant of the pair $(M, \sigma)$. We denote this modified Seiberg-Witten invariant by $sw^0_M(\sigma)$.

2.4. The Reidemeister-Turaev torsion and the Casson-Walker invariant. For any spin$^c$ structure $\sigma$ on $M$, we denote by

$$\mathcal{T}_{M, \sigma} = \sum_{h \in H} \mathcal{T}_{M, \sigma}(h) h \in \mathbb{Q}[H]$$

the sign refined Reidemeister-Turaev torsion associated with $\sigma$ (for its detailed description, see [29]). It is convenient to think of $\mathcal{T}_{M, \sigma}$ as a function $H \to \mathbb{Q}$ given by $h \mapsto \mathcal{T}_{M, \sigma}(h)$. The augmentation map $aug : \mathbb{Q}[H] \to \mathbb{Q}$ is defined by $\sum a_h h \mapsto \sum a_h$. It is known that $aug(\mathcal{T}_{M, \sigma}) = 0$.

Denote by $\lambda(M)$ the Casson-Walker invariant of $M$ normalized as in [12, §4.7]. Then by a result of the second author [23], one has:

$$sw^0_M(\sigma) = \frac{1}{|H|} \lambda(M) + \mathcal{T}_{M, \sigma}(1).$$

(1)

Below we will present a formula for $\mathcal{T}_{M, \sigma}$ in terms of Fourier transform. For this, consider the Pontryagin dual $\hat{H} := \text{Hom}(H, U(1))$ of $H$. Then a function $f : H \to \mathbb{C}$ and its Fourier transform $\hat{f} : \hat{H} \to \mathbb{C}$ satisfy:

$$\hat{f}(\chi) = \sum_{h \in H} f(h) \bar{\chi}(h); \quad f(h) = \frac{1}{|H|} \sum_{\chi \in H} \hat{f}(\chi) \chi(h).$$

Notice that $\hat{f}(1) = aug(f)$, in particular $\hat{T}_{M, \sigma}(1) = aug(\mathcal{T}_{M, \sigma}) = 0$.

2.5. $M$ as a plumbing manifold. Fix a sufficiently small (Stein) representative $X$ of $(X, 0)$ and let $\pi : \tilde{X} \to X$ be a resolution of the singular point $0 \in X$. In particular, $\tilde{X}$ is smooth, and $\pi$ is a biholomorphic isomorphism above $X \setminus \{0\}$. We will assume that the exceptional divisor $E := \pi^{-1}(0)$ is a normal crossing divisor with irreducible components $\{E_v\}_{v \in V}$. Let $\Gamma(\pi)$ be the dual resolution graph associated with $\pi$ decorated with the self intersection numbers $\{E_v \cdot E_v\}_v$. $\Gamma(\pi)$ can be identified with a plumbing graph, and $M$ with a plumbing 3-manifold constructed from $\Gamma(\pi)$ as its plumbing graph. Since $M$ is a rational homology sphere, all the irreducible components $E_v$ of $E$ are rational, and $\Gamma(\pi)$ is a tree.

Let $D_v$ be a small transversal disc to $E_v$. In fact $\partial D_v$ can be considered as the generic fiber of the $S^1$-bundle over $E_v$ used in the plumbing construction of $M$. Consider the elements $g_v := [\partial D_v] \ (v \in V)$ in $H$. It is not difficult to see that they, in fact, generate $H$.

For the degree of any vertex $v$ (i.e. for $\# \{w : E_w \cdot E_v = 1\}$) we will use the notation $\delta_v$.

Next, we define the canonical cycle $Z_K$ of $(X, 0)$ associated with the resolution $\pi$. This is a rational cycle $Z_K = \sum_{v \in V} r_v E_v$, $r_v \in \mathbb{Q}$, supported by the exceptional divisor $E$, and defined by (the adjunction formula):

$$Z_K \cdot E_v = E_v \cdot E_v + 2 \quad \text{for any } v \in V.$$
Since the intersection matrix \( \{E_v \cdot E_w\}_{v,w} \) is nondegenerate, the above equation has a unique solution. \((X, 0)\) is called numerically Gorenstein if \( r_v \in \mathbb{Z} \) for each \( v \in \mathbb{V} \).

The rational number \( Z_K \cdot Z_K \) will be denoted by \( K^2 \). Let \( \#\mathbb{V} \) denote the number of irreducible components of \( E = \pi^{-1}(0) \). Then \( K^2 + \#\mathbb{V} \) does not depend on the choice of the resolution \( \pi \), it is an invariant of \( M \).

The main object of this paper is a normal surface singularity \((X, 0)\) with a good \( \mathbb{C}^* \)-action. It is well–known that the link of such a singularity is a Seifert 3-manifold, and the minimal resolution graph is a star-shaped graphs. In these case it is convenient to express all the topological invariants of \( M \) in terms of their Seifert invariants. In the next subsections we list briefly the definitions, notations and some of the needed properties.

### 2.6. The Seifert invariants. \([4, 13, 21]\)

Consider a Seifert fibration \( \pi : M \to \Sigma \). In our situation, since \( M \) is a rational homology sphere, the base space \( \Sigma \) has genus zero. Consider a set of points \( \{x_i\}_{i=1}^\nu \) in such a way that the set of fibers \( \{\pi^{-1}(x_i)\}_i \) contains the set of singular fibers. Set \( O_i := \pi^{-1}(x_i) \). Let \( D_i \) be a small disc in \( X \) containing \( x_i \), \( \Sigma' := \Sigma \setminus \cup_i D_i \) and \( M' := \pi^{-1}(\Sigma') \). Now, \( \pi : M' \to \Sigma' \) admits sections, let \( s : \Sigma' \to M' \) be one of them. Let \( Q_i := s(\partial D_i) \) and let \( H_i \) be a circle fiber in \( \pi^{-1}(\partial D_i) \). Then in \( H_1(\pi^{-1}(D_i), \mathbb{Z}) \) one has \( H_i \sim \alpha_i O_i \) and \( Q_i \sim -\beta_i O_i \) for some integers \( \alpha_i > 0 \) and \( \beta_i \) with \((\alpha_i, \beta_i) = 1\). The set \( ((\alpha_i, \beta_i))_{i=1}^\nu \) constitute the set of (unnormalized) Seifert invariants. The number

\[
e = -\sum \frac{\beta_i}{\alpha_i}
\]

is called the (orbifold) Euler number of \( M \). \( M \) is a link of singularity if and only if \( e < 0 \).

Replacing the section by another one, a different choice changes each \( \beta_i \) within its residue class modulo \( \alpha_i \) in such a way that the sum \( e = -\sum_i (\beta_i/\alpha_i) \) is constant.

The set of normalized Seifert invariants \( ((\alpha_i, \omega_i))_{i=1}^\nu \) are defined as follows. Write

\[
e = b + \sum \omega_i/\alpha_i
\]

for some integer \( b \), and \( 0 \leq \omega_i < \alpha_i \) with \( \omega_i \equiv -\beta_i \pmod{\alpha_i} \). Clearly, these properties define \( \{\omega_i\}_i \) uniquely. Notice that \( b \leq e < 0 \). For the uniformity of the notations, in the sequel we assume \( \nu \geq 3 \). (Recall that for cyclic quotient singularities Conjecture was verified in \([18]\).)

For each \( i \), consider the continued fraction \( \alpha_i/\omega_i = b_{i1} - 1/(b_{i2} - 1/(\ldots - 1/b_{i\nu_i}) \ldots) \). Then (a possible) plumbing graph of \( M \) is a star-shaped graph with \( \nu \) arms. The central vertex has decoration \( b \) and the arm corresponding to the index \( i \) has \( \nu_i \) vertices, and they are decorated by \( b_{i1}, \ldots, b_{i\nu_i} \) (the vertex decorated by \( b_{i1} \) is connected by the central vertex).

We will distinguish those vertices \( v \in \mathbb{V} \) of the graph which have \( \delta_v \neq 2 \). We will denote by \( v_0 \) the central vertex (with \( \delta = \nu \)), and by \( \hat{v}_i \) the end-vertex of the \( i \)th arm (with \( \delta = 1 \)) for all \( 1 \leq i \leq \nu \). In this notation, \( g_0^\nu \) is exactly the class of the generic fiber. The group \( H \) has the following representation:

\[
H = ab\langle g_{v_0}^{g_{v_0}}, g_{v_1}, \ldots, g_{v_\nu} \rangle | g_{v_0}^{-b} = \prod_{i=1}^\nu g_{v_i}^{\omega_i}, g_{v_0} = g_{v_i}^{\alpha_i} \text{ for all } i \rangle.
\]

Let \( \alpha := \text{lcm}(\alpha_1, \ldots, \alpha_\nu) \). The order of the group \( H \) and the order \( o \) of the subgroup \( \langle g_{v_0} \rangle \) can be determined by (cf. \([13]\)):

\[
|H| = \alpha \frac{\alpha_1 \cdots \alpha_\nu |e|}{e}, \quad o := |\langle g_{v_0} \rangle| = \frac{\alpha |e|}{e}.
\]
2.7. **Invariants computed from the plumbing graph.** In the sequel we will also use the Dedekind sums. They are defined as follows \([27, 28]\). Let \(\lfloor x \rfloor\) be the integer part, and \(\{x\} := x - \lfloor x \rfloor\) the fractional part of \(x\). Then

\[
s(h, k) = \sum_{\mu=0}^{k-1} \left( \left( \frac{h\mu}{k} \right) \right) \left( \left( \frac{h}{k} \right) \right),
\]

where \(\left( \left( x \right) \right)\) denotes the Dedekind symbol

\[
\left( \left( x \right) \right) = \begin{cases} 
\{x\} - 1/2 & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Z} \\
0 & \text{if} \quad x \in \mathbb{Z}.
\end{cases}
\]

Assume that \(M\) is a Seifert manifold with \(e < 0\). Then one has the following formulae for its invariants.

**• The Casson-Walker invariant.** \([12]\) (6.1.1):

\[
\frac{24}{|H|} \lambda(M) = \frac{1}{e} (2 - \nu + \sum_{i=1}^{\nu} \frac{1}{\alpha_i}) + e + 3 + 12 \sum_{i=1}^{\nu} s(\beta_i, \alpha_i).
\]

**• \(K^2 + \#V\).** \([18]\) (5.4):

\[
K^2 + \#V = \frac{1}{e} (2 - \nu + \sum_{i=1}^{\nu} \frac{1}{\alpha_i})^2 + e + 5 + 12 \sum_{i=1}^{\nu} s(\beta_i, \alpha_i).
\]

**• The Reidemeister-Turaev sign-refined torsion.** For any \(\chi \in \hat{H}\) (and free variable \(t \in \mathbb{C}\)) set

\[
\hat{P}_\chi(t) := \frac{(t^\alpha \chi(g_{\bar{v}_0}) - 1)^{\nu-2}}{\prod_{i=1}^{\nu} (t^\alpha \chi(g_{\bar{v}_i}) - 1)}.
\]

Then, by \([18]\) (5.8), the Fourier transform \(\hat{T}_{M,0}\) of \(T_{M,0}\) is given by

\[
\hat{T}_{M,0}(\chi) = \lim_{t \to 1} \hat{P}_\chi(t) \quad \text{for any} \quad \chi \in \hat{H} \setminus \{1\}.
\]

**• The geometric genus of \((X,0)\).** Let \(M\) be a Seifert manifold with \(e < 0\) and Seifert invariants as above. Define the Dolgachev-Pinkham (topological) invariant of \(M\) by

\[
DP_M := \sum_{l \geq 0} \max \left( 0, -1 + lb - \sum_{i=1}^{\nu} \left\lfloor \frac{l \omega_i}{\alpha_i} \right\rfloor \right).
\]

Assume that \((X,0)\) is a normal surface singularity with a good \(\mathbb{C}^*\)--action (see e.g. \([26]\)) such that its link \(M\) is a rational homology sphere. Then, by \([26]\), (5.7):

\[
p_g(X,0) = DP_M.
\]

3. **The main result**

The key identity of this article is presented in the following theorem.

3.1. **Theorem.** Let \(M\) be a Seifert 3–manifold with \(e < 0\). Then the invariants \(T_{M,0}\), \(\lambda(M)\), \(K^2 + \#V\) and \(DP_M\) are connected by the following identity:

\[
T_{M,0} + \frac{\lambda(M)}{|H|} = \frac{K^2 + \#V}{8} + DP_M.
\]
Remark. Using (2.4(1), for the (modified) Seiberg-Witten invariant one obtains:
\[
\text{sw}^0_M(\sigma_{can}) = \frac{K^2 + \# \mathcal{V}}{8} + D\mathcal{P}. \tag{1}
\]
If \(M\) is a singularity link, one can define on \(M\) a canonical contact structure \(\xi_{can}\) (induced by the natural almost complex structure on \(TM \oplus \mathbb{R}M\); for details, see [8, p. 420] or [18, (4.8)]) with \(c_1(\xi_{can})\) torsion. On the other hand, in [7], Gompf associates with such a contact structure \(\xi\) an invariant \(\theta_M(\xi)\). It turns out that in our case (see [18, 4.8]) \(\theta_M(\xi_{can}) = K^2 + \# \mathcal{V} - 2\). Therefore, for any link of singularity which is a Seifert 3-manifold one has:
\[
\text{sw}^0_M(\sigma_{can}) = \frac{\theta_M(\xi_{can}) + 2}{8} + D\mathcal{P}. \tag{2}
\]
The proof of (3.1) is carried out in several steps.

3.3. Proposition.
\[
\frac{1}{|H|} \sum_{\chi \in H} \hat{P}_\chi(t) = \sum_{l \geq 0} \max \left(0, 1 - lb + \sum_{i=1}^\nu \left\lfloor \frac{-l\omega_i}{\alpha_i} \right\rfloor \right) t^ol.
\]
In particular,
\[
\mathcal{T}_{M,\sigma_{can}}(1) = \lim_{t \to 1} \left( \sum_{l \geq 0} \max \left(0, 1 - lb + \sum_{i=1}^\nu \left\lfloor \frac{-l\omega_i}{\alpha_i} \right\rfloor \right) t^{ol} - \frac{1}{|H|} \cdot \hat{P}_1(t) \right).
\]
Before we start the proof, we draw to the reader’s attention the “mysterious” similarity between our formula (7) for the (Fourier transform) of the Reidemeister-Turaev torsion, and the formula [19, 4.2] of W. Neumann of the Poincaré series of the graded affine ring associated with the universal abelian cover of \((X, 0)\).

In fact, the next proof is based completely on Neumann’s computation about this graded ring on [loc. cit., page 241; (he attributed the idea to D. Zagier).

Proof. Using the identity \(g_\alpha^{\nu_i} = g_\Theta\) in \(H\) (cf. (2.6(3)), first write \(\hat{P}_\chi(t)\) as
\[
(1 - t^\alpha \chi(g_\Theta))^{-2} \prod_i \frac{1 - t^\alpha \chi(g_\Theta)^{\alpha_{s_i}}}{1 - t^\alpha \chi(g_\Theta)} = \left( \sum_{s_0=0}^{\infty} (1 + s_0) \chi(g_\Theta)^{s_0} t^{\alpha s_0} \right) \prod_i \sum_{s_i=0}^{\alpha_i-1} t^{\alpha_i/\alpha_i} (\chi(g_\Theta))^{s_i}
\]
where the (unmarked) sum is over \(s_0 \geq 0\) and \(0 \leq s_i < \alpha_i\) for each \(i\). But \(\sum_{\chi \in H} (h)\) is non-zero only if \(h = 1\), and in that case it is \(|H|\). Using the group structure (2.6(3)) one gets that all the relations in \(H\) have the form
\[
g^{l_1 + \ldots + l_\nu - lb}_{\Theta} \prod_i g^{-\omega_i l - \alpha_i l_i}_{\Theta} = 1,
\]
where \(l_1, \ldots, l_\nu\) and \(l\) are integers. Therefore, \(g^{s_0}_{\Theta} g^{s_i}_{\Theta} \cdots g^{s_\nu}_{\Theta} = 1\) if and only if \(s_0 = l_1 + \ldots + l_\nu - lb\) and \(s_i = -\omega_i l - \alpha_i l_i\) \((1 \leq i \leq \nu)\) for some integers \(l_1, \ldots, l_\nu, l\). Since \(0 \leq s_i < \alpha_i\) one obtains that
\[
l_i = \left\lfloor \frac{-l\omega_i}{\alpha_i} \right\rfloor.
\]
In particular,
\[
1 + s_0 = 1 - lb + \sum_i \left\lfloor \frac{-l\omega_i}{\alpha_i} \right\rfloor,
\]
and only those integers $l$ are allowed for which this number $1 + s_0$ is $\geq 1$. It is easy to see that this cannot happen for $l < 0$. Indeed, for $l < 0$

$$-lb + \sum_i \left\lfloor \frac{-l \omega_i}{\alpha_i} \right\rfloor \leq -lb + \sum_i \left\lfloor \frac{-l \omega_i}{\alpha_i} \right\rfloor = -le < 0.$$ 

Finally notice that the exponent $\alpha (s_0 + \sum_i s_i/\alpha_i)$ of $t$ is $-l\alpha e = lb$ by 2.7(8). This ends the proof of the first formula. For the second part, recall that $\bar{T}_{M, \sigma}(1) = 0$ (cf. 2.4), hence by 2.7(8),

$$\bar{T}_{M, \sigma_{can}}(1) = \lim_{t \to 1} \frac{1}{|H|} \sum_{\chi \in H \setminus \{1\}} \hat{P}_\chi(t).$$

3.4. Remark. Let $(X, 0)$ be a normal singularity with a good $\mathbb{C}^*$-action and affine graded coordinate ring $A = \oplus_k A_k$. Then its Poincaré series is defined by $p_{(X, 0)}(t) = \sum_k \dim(A_k) t^k$. For such an $(X, 0)$, the expression from 3.3 provides exactly $p_{(X, 0)}(t^\nu)$ (cf. 19, page 241). More precisely:

$$p_{(X, 0)}(t) = \sum_{l \geq 0} \max \left( 0, 1 - lb + \sum_{i=1}^\nu \left\lfloor \frac{-l \omega_i}{\alpha_i} \right\rfloor \right) t^l.$$ 

Moreover, if $(X_{ab}, 0)$ denotes the universal abelian cover of $(X, 0)$, then $p_{(X_{ab}, 0)}(t) = \hat{P}_1(t)$ (cf. 19, page 240). Therefore, 3.3 reads as follows:

$$\bar{T}_{M, \sigma_{can}}(1) = \lim_{t \to 1} \left( p_{(X, 0)}(t^\nu) - p_{(X_{ab}, 0)}(t)/|H| \right).$$

Notice that for many special families, the Poincaré series $p_{(X, 0)}(t)$ is computed very explicitly, see e.g. [31].

3.5. Corollary.

$$\bar{T}_{M, \sigma_{can}}(1) - DP_M = \lim_{t \to 1} \left( \sum_{l \geq 0} \left( 1 - lb + \sum_{i=1}^\nu \left\lfloor \frac{-l \omega_i}{\alpha_i} \right\rfloor \right) t^l - \frac{1}{|H|} \cdot \hat{P}_1(t) \right).$$

Proof. Use 2.7(9), 3.3 and the identity $\max(0, x) - \max(0, -x) = x$.

On the right hand side we have a difference of two series, both having poles of order two at $t = 1$. The next results provide their Laurent series at $t = 1$. In fact, we prefer to expand the series in terms of the powers of $t^\nu - 1$ (instead of $t - 1$).

3.6. Proposition. Define $\chi_M := 2 - \sum_{i=1}^\nu (\alpha_i - 1)/\alpha_i$ (cf. e.g. with 19). Then

$$\sum_{l \geq 0} \left( 1 - lb + \sum_{i=1}^\nu \left\lfloor \frac{-l \omega_i}{\alpha_i} \right\rfloor \right) t^l = \frac{-e}{(t^\nu - 1)^2} + \frac{-e - \chi_M/2}{t^\nu - 1} + \frac{2 - \chi_M}{4} + \sum_{i=1}^\nu s(\beta_i, \alpha_i) + R(t),$$

with $\lim_{t \to 1} R(t) = 0$.

Proof. We follow again 19, page 241. The left hand side of 3.6, via 2.6(2) transforms into

$$\sum_{l \geq 0} \left( -le + \frac{\chi_M}{2} \right) t^l + \sum_{i=1}^\nu \left( - \left\lfloor \frac{-l \omega_i}{\alpha_i} \right\rfloor + \frac{\alpha_i - 1}{2 \alpha_i} \right) t^l.$$
Evidently
\[
\sum_{l \geq 0} \left( -le + \frac{\chi M}{2} \right) t^l = \frac{-et^o}{(1-t^o)^2} + \frac{\chi M/2}{1-t^o},
\]
which gives the non-holomorphic part. The second contribution is a sum over \(1 \leq i \leq \nu\).

For each fixed \(i\), write \(l = \alpha_i m + q\) with \(m \geq 0\) and \(0 \leq q < \alpha_i\). Using the notation \(\Sigma_q := \sum_{q=0}^{\alpha_i-1}\) and \(\Sigma'_q := \sum_{q=1}^{\alpha_i-1}\), the \(i^{th}\) summand is
\[
\sum_q \left( -\left\{ \frac{q \omega_i}{\alpha_i} \right\} + \frac{\alpha_i - 1}{2\alpha_i} \right) \sum_{m \geq 0} t^{\alpha_i m + q} = \sum_q \left( -\left\{ \frac{q \omega_i}{\alpha_i} \right\} + \frac{\alpha_i - 1}{2\alpha_i} \right) \frac{t^q}{1-t^{\alpha_i}}.
\]

Separating the two cases \(q = 0\) and \(q > 0\), and using the definition of the Dedekind symbol and the identity \(\{ -x \} = 1 - \{ x \}\) for \(x \notin \mathbb{Z}\), this is transformed into
\[
A(t) := \frac{\alpha_i - 1}{2\alpha_i} + \sum_q \left( \left\{ \frac{q \omega_i}{\alpha_i} \right\} \right) \frac{t^q}{1-t^{\alpha_i}}.
\]

By L’Hospital theorem (and by some simplifications):
\[
\lim_{t \to 1} A(t) = -\sum_q \left( \left\{ \frac{q \omega_i}{\alpha_i} \right\} \right) \frac{q}{\alpha_i} + \frac{\alpha_i - 1}{4\alpha_i}.
\]
Since \(\sum_q \left( \left\{ \frac{q \omega_i}{\alpha_i} \right\} \right) = 0\) and \(\omega_i \equiv -\beta_i \pmod{\alpha_i}\), the result follows from the definition of the Dedekind symbol and the Dedekind sums. \(\square\)

3.7. Remarks. Cf. [19], page 242. In fact,
\[
\sum_{l \geq 0} \left( 1 - lb + \sum_{i=1}^{\nu} \left\lfloor \frac{\omega_i}{\alpha_i} (-l) \right\rfloor \right) t^l = \frac{-e}{(t^o - 1)^2} + \frac{-e - \chi M/2}{t^o - 1} + \sum_{i=1}^{\nu} \frac{1}{\alpha_i} \sum_{z_{\alpha_i}} \frac{1}{1 - \xi (1 - \xi^\alpha_i t^o)},
\]
where the last sum is over \(\xi^\alpha_i = 1, \xi \neq 1\). Since we do not need this statement now, we will skip its proof. The interested reader can prove it easily using the expression \(A(t)\) above and the property (16c) of the Dedekind symbol from [28], page 14.

Paradoxically, the Laurent expansion of the elementary rational fraction \(\hat{P}_1(t)\) is more complicated.

3.8. Proposition.
\[
\frac{\hat{P}_1(t)}{|H|} = \frac{-e}{(t^o - 1)^2} + \frac{-e - \chi M/2}{t^o - 1} + E + Q(t),
\]
where \(\lim_{t \to 1} Q(t) = 0\) and
\[
E := \frac{(e+1)(e+5)}{12e} + \frac{1}{4} \sum_i \left( 1 - \frac{1}{\alpha_i} \right) + \frac{1}{12} \sum_i \left( 1 - \frac{1}{\alpha_i} \right) \left( \frac{1}{4} + \frac{1}{\alpha_i} \right) - \frac{1}{4e} \sum_{i < j} \left( 1 - \frac{1}{\alpha_i} \right) \left( 1 - \frac{1}{\alpha_j} \right).
\]

Proof. First notice that one has the following Taylor expansion;
\[
\frac{t^\gamma - 1}{t^\tau - 1} = \frac{\gamma}{\tau} + \frac{\gamma}{2\tau} (\gamma - \tau) (t^\sigma - 1) + \frac{\gamma}{\sigma \tau} (\gamma - \tau) \left( \frac{2\gamma - \tau}{12\sigma} - \frac{1}{4} \right) (t^\sigma - 1)^2 + \cdots.
\]
Now, use this formula \(\nu + 2\) times in the expression
\[
\hat{P}_1(t) = \frac{1}{(t^o - 1)^2} \cdot \left( \frac{t^o - 1}{t^o - 1} \right)^2 \cdot \prod_i \frac{t^o - 1}{t^o/\alpha_i - 1}.
\]
A long (but elementary) computation, together with \(2.6(4)\), gives the result.

\[\text{□}\]

**Proof of 3.7.** Apply \(2.7(5)\) and \((6)\), respectively \(3.5, 3.6\) and \(3.8\). The verification is elementary.

**3.9.** Remark. The Reidemeister-Turaev torsion revisited. In the above results we needed to determine only \(\mathcal{T}_{M,\sigma_{\text{can}}}(1)\), i.e. only that sign refined torsion which is associated with the canonical \(\text{spin}^c\) structure. But, via similar computations, we can obtain the complete set of invariants \(\{\mathcal{T}_{M,\sigma}(1)\}_{\sigma \in \text{Spin}^c(M)}\).

Indeed, we recall that \(\text{Spin}^c(M)\) is an \(H\)-torsor. Let \(\sigma\) be an arbitrary element of \(\text{Spin}^c(M)\), and take (the unique) \(h_\sigma \in H\) so that \(h_\sigma \cdot \sigma_{\text{can}} = \sigma\). Then, by \([8, 5.7]\), one has:

\[
\hat{\mathcal{T}}_{M,\sigma}(\bar{\chi}) = \bar{\chi}(h_\sigma) \cdot \lim_{t \to 1} \hat{P}_\chi(t) \quad \text{for any } \chi \in \hat{H} \setminus \{1\}.
\]

Using the representation \(2.6(3)\) of \(H\), \(h_\sigma\) evidently can be written as

\[h_\sigma = g_{00}^{a_0} g_{v_1}^{a_1} \cdots g_{v_\nu}^{a_\nu}\]

for some integers \(a_0, a_1, \ldots, a_\nu\).

Then repeating the arguments from the proof of \(3.3\), one gets:

\[
\bar{\chi}(h_\sigma) \hat{P}_\chi(t) = \left( \sum (1 + s_0) t^{\sum \alpha_i s_i + \alpha_i} \chi(h_\sigma^{-1} g_{00}^{a_0} g_{v_1}^{a_1} \cdots g_{v_\nu}^{a_\nu}) \right) = \left( \sum 1 + a_0 - lb + \sum_{i=1}^{\nu} \frac{-l \omega_i + a_i}{\alpha_i} \right) \cdot \hat{P}_1(t),
\]

with \(s_0 \geq 0\) and \(0 \leq s_i < \alpha_i\) for each \(i\). But \(h_\sigma^{-1} g_{00}^{a_0} g_{v_1}^{a_1} \cdots g_{v_\nu}^{a_\nu} = 1\) if and only if \(s_0 = a_0 + l_1 + \cdots + l_\nu - lb\) and \(s_i = a_i - \omega_i l - \alpha_i l_i\) \((1 \leq i \leq \nu)\) for some integers \(l_1, \ldots, l_\nu, l\).

Since \(0 \leq s_i < \alpha_i\) one obtains that

\[
l_i = \left\lfloor \frac{-l \omega_i + a_i}{\alpha_i} \right\rfloor.
\]

In particular,

\[1 + s_0 = 1 + a_0 - lb + \sum_{i=1}^{\nu} \left\lfloor \frac{-l \omega_i + a_i}{\alpha_i} \right\rfloor,
\]

and only those integers \(l\) are allowed for which \(1 + s_0 \geq 1\). Therefore, one gets:

**3.10.** Proposition. Assume that \(M\) is a Seifert 3-manifold with \(e < 0\). Fix an arbitrary \(\text{spin}^c\) structure \(\sigma \in \text{Spin}^c(M)\) characterized by \(h_\sigma \cdot \sigma_{\text{can}} = \sigma\). Write \(h_\sigma\) as \(g_{00}^{a_0} g_{v_1}^{a_1} \cdots g_{v_\nu}^{a_\nu}\) for some integers \(a_0, a_1, \ldots, a_\nu\). Finally, define \(\alpha := \alpha \cdot (a_0 + \sum_{i=1}^{\nu} a_i / \alpha_i)\). Then

\[
\mathcal{T}_{M,\sigma}(1) = \lim_{t \to 1} \left( \sum_{l \in \mathbb{Z}} \max \left(0, 1 + a_0 - lb + \sum_{i=1}^{\nu} \left\lfloor \frac{-l \omega_i + a_i}{\alpha_i} \right\rfloor \right) \cdot t^{\alpha l + \alpha} - \frac{1}{|H|} \cdot \hat{P}_1(t) \right).
\]

(Above, there exists an integer \(m\), such that for \(l < m\) the contribution in the above sum is zero.)

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