LIE ALGEBRAIC CARROLL/GALILEI DUALITY

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Abstract. We characterise Lie groups with bi-invariant bargmannian, galilean or carrollian structures. Localising at the identity, we show that Lie algebras with ad-invariant bargmannian, carrollian or galilean structures are actually determined by the same data: a metric Lie algebra with a skew-symmetric derivation. This is the same data defining a one-dimensional double extension of the metric Lie algebra and, indeed, bargmannian Lie algebras coincide with such double extensions, containing carrollian Lie algebras as an ideal and projecting to galilean Lie algebras. This sets up a canonical correspondence between carrollian and galilean Lie algebras mediated by bargmannian Lie algebras. This reformulation allows us to use the structure theory of metric Lie algebras to give a list of bargmannian, carrollian and galilean Lie algebras in the positive-semidefinite case. We also characterise Lie groups admitting a bi-invariant (ambient) leibnizian structure. Leibnizian Lie algebras extend the class of bargmannian Lie algebras and also set up a non-canonical correspondence between carrollian and galilean Lie algebras.

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1. Introduction

The study of non-lorentzian spacetime geometries is coming of age (see, e.g., the recent reviews [1, 2]), yet there are still some simple questions which have not been asked nor answered. A natural first step when studying an unfamiliar geometric structure is to find examples of such structures with lots of symmetries. A natural class of such examples are homogeneous spaces and, in particular, Lie groups with bi-invariant structures. Bi-invariance is typically quite strong and this often allows one to classify them or at least to characterise them in linear algebraic terms. This is the case, for example, with Lie groups admitting a bi-invariant metric (of any signature), which were studied by Medina [3] and characterised in terms of their Lie algebras by Medina and Revoy [4] (see also [5–7]).
The three protagonists of today’s tale are Bargmann, Carroll and Galilei. They may be used to label Lie groups, Lie algebras, homogeneous spaces and also Cartan geometries (here, equivalently, $G$-structures). It turns out that in all of these settings, objects with these names sit in relation to each other in a way which suggests a correspondence (loosely, a duality) between Carroll and Galilei mediated by Bargmann. This correspondence was first pointed out in a geometric context in [8], but before describing this result let us set the stage by discussing the Lie algebras themselves.

The Carroll and Galilei Lie algebras are examples of kinematical Lie algebras [9, 10]. In spatial dimension $n$, they are spanned by $(L_{ab}, B^a, P^a, H)$, where $L_{ab} = -L_{ba}$ span a Lie subalgebra isomorphic to $so(n)$ under which $B^a, P^a$ are vectors and $H$ a scalar. These conditions translate into the following Lie brackets which are common to all kinematical Lie algebras

\[
\begin{align*}
[L_{ab}, L_{cd}] &= \delta_{bc}L_{ad} - \delta_{ac}L_{bd} - \delta_{bd}L_{ac} + \delta_{bd}L_{ac} \\
[L_{ab}, B^b] &= \delta_{bc}B^a - \delta_{ac}B^b \\
[L_{ab}, P^b] &= \delta_{bc}P^a - \delta_{ac}P^b \\
[L_{ab}, H] &= 0.
\end{align*}
\] (1.1)

The kinematical Lie algebra where all other brackets vanish is called the static kinematical Lie algebra and denoted $s$. All kinematical Lie algebras are deformations of $s$ [11–13]. The subalgebra $s_0$ of $s$ spanned by $(L_{ab}, B^a, P^a)$ will play a rôle in our discussion.

The Carroll Lie algebra $c$ is a central extension of $s_0$, with $H$ the central element and additional nonzero bracket

\[ [B^a, P^b] = \delta_{ab}H. \] (1.2)

In contrast, the Galilei Lie algebra $g$ is an “extension-by-derivation” of $s_0$. The derivation is $ad_H = [H, -]$ and is defined by $ad_H(L_{ab}) = ad_H(P^a) = 0$ and $ad_H(B^a) = -P^a$, resulting in the additional nonzero bracket

\[ [H, B^a] = -P^a. \] (1.3)

We may summarise these observations in the diagrammatical language of (short) exact sequences of Lie algebras as follows:

\[
\begin{array}{ccccccc}
0 & \rightarrow & R & \rightarrow & c & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & s_0 & \rightarrow & g & \rightarrow & R \rightarrow & 0
\end{array}
\] (1.4)

The exact row says that $g$ is an extension-by-derivation of $s_0$, whereas the exact column says that $c$ is a one-dimensional extension of $s_0$. The diagram does not fix the brackets uniquely, since $s_0$ admits many derivations and also other one-dimensional extensions, central or not.

The Bargmann Lie algebra $b$ is a central extension of the Galilei Lie algebra $g$ with additional generator $M$ and additional bracket

\[ [B^a, P^b] = \delta_{ab}M, \] (1.5)

which is reminiscent of the bracket (1.2) with $M$ playing the rôle of $H$. Under this relabelling of bases, we see that the Bargmann Lie algebra $b$ is an extension-by-derivation of the Carroll Lie algebra $c$. The derivation is $ad_H = [H, -]$ where again $ad_H$ annihilates $L_{ab}, P^a, M$ and its action on $B^a$ is given by the bracket (1.3).
This allows us to complete the diagram (1.4) to the following commutative diagram of Lie algebras:

$$
\begin{array}{cccc}
0 & 0 & & \\
\downarrow & & & \\
\mathbb{R} & \mathbb{R} & & \\
\downarrow & & & \\
0 & c & b & \mathbb{R} & 0 \\
\downarrow & & & & \\
0 & s & g & \mathbb{R} & 0 \\
\downarrow & & & & \\
0 & 0 & 0 & \\
\end{array}
$$

(1.6)

which will recur in other contexts in this work with other Lie algebras playing the rôles of $s$, $c$, $g$ and $b$. In fact, this very diagram has already appeared in [14, App. B] in the context of the hamiltonian description of particle dynamics on the (spatially isotropic) homogeneous galilean spacetimes classified in [15]. In that context, the Lie algebras which play the rôles of $s$, $c$, $g$ and $b$ are infinite-dimensional.

The two exact rows in the the diagram (1.6) say that $b$ (resp. $g$) is an extension-by-derivation of $c$ (resp. $s$), whereas the two exact columns say that $b$ (resp. $c$) is a one-dimensional extension of $g$ (resp. $s$).

We could say, borrowing the terminology from the theory of metric Lie algebras, that the Bargmann Lie algebra $b$ is a double extension\(^1\) of $s$. This is more than a mere analogy and we will see that this is exactly right for Lie algebras admitting ad-invariant bargmannian structures (to be defined below).

The Bargmann Lie algebra $b$ can also be defined as the subalgebra of the Poincaré Lie algebra $\mathfrak{iso}(n + 1, 1)$ in one dimension higher which centralises a null translation. Since the Poincaré translations commute, they are contained in $b$ and hence the (simply-connected) Bargmann Lie group acts transitively on Minkowski spacetime $\mathbb{R}^{n+2}$. This fact underlies the geometric Carroll/Galilei duality in [8].

Choosing a Witt frame $(e_0, e_+)$ for Minkowski spacetime $\mathbb{M}^{n+2}$ we may express the generators of the Poincaré algebra as $(L_{ab}, L_{+a}, L_{-a}, L_{++}, P_a, P_{+}, P_{-})$. The centraliser of $P_{+}$ in $\mathfrak{b}$ is spanned by $(L_{ab}, L_{+a}, P_a, P_{+}, P_{-})$ and it is isomorphic to the Bargmann Lie algebra $b$, with $P_{+}$ playing the rôle of $M$. If $X$ is an element of the Poincaré Lie algebra, we let $\xi_X$ denote the corresponding Killing vector field on Minkowski spacetime. The null vector field $\xi_{P_{+}}$ on Minkowski spacetime is not just Killing but actually parallel. It defines a distribution $\xi_{P_{+}} \subset \mathbb{T}M$ which is integrable. The leaves of the corresponding foliation are null hypersurfaces and are copies of the Carroll spacetime $\mathbb{C}^{n+1}$. Indeed, they are homogeneous spaces of the normal subgroup with Lie algebra the ideal of $b$ spanned by $(L_{ab}, L_{+a}, P_a, P_{+})$, which is isomorphic to the Carroll Lie algebra $c$, with $P_{+}$ playing the rôle of $H$. On the other hand, the null reduction [16, 17] of $\mathbb{M}$ by the one-parameter subgroup generated by $\xi_{P_{+}}$ is isomorphic to Galilean spacetime $\mathbb{G}^{n+1}$. Indeed, the quotient is homogeneous under the Lie group whose Lie algebra is the quotient of $b$ by the line spanned by $P_{+}$, which is isomorphic to the Galilei Lie algebra $g$. This results in the following suggestive diagram

$$
\begin{array}{ccc}
\mathbb{C}^{n+1} & \hookrightarrow & \mathbb{M}^{n+2} \\
\downarrow & & \downarrow \\
\mathbb{G}^{n+1} & \hookrightarrow & \mathbb{C}^{n+1} \\
\end{array}
$$

(1.7)

which is the fundamental example of the geometric Carroll/Galilei duality in [8].

This duality seems to be broken when we take all (spatially isotropic) homogeneous kinematical spacetimes into consideration. In the classification of [15] there are three other homogeneous carollian spacetimes besides the Carroll spacetime: the carollian limits $\mathcal{d}S_{C}$ of de Sitter and $\mathcal{d}S_{C}$ of anti de Sitter spacetimes, and the lightcone $\mathcal{L}C$, which can be realised as null hypersurfaces in de Sitter, anti de Sitter and Minkowski spacetimes, respectively [15, 18]. On the other hand, there are two one-parameter families of homogeneous galilean spacetimes. The galilean limit $\mathcal{d}S_{G}$ of de Sitter spacetime is one point ($\gamma = -1$) in a continuum $\mathcal{d}S_{G}$, for $\gamma \in [-1, 1]$, of galilean spacetimes. Similarly, the galilean limit $\mathcal{d}S_{G}$ of anti de Sitter spacetime is one point ($\chi = 0$) in a continuum $\mathcal{d}S_{G_{\chi}}$, for $\chi \geq 0$, of galilean spacetimes. These two continua have a point in common, since $\lim_{\gamma \to 1} \mathcal{d}S_{G} = \lim_{\chi \to \infty} \mathcal{d}S_{G_{\chi}}$. It is often convenient to describe homogeneous spaces infinitesimally via their Klein pairs. The above homogeneous galilean

\(^1\)To be clear, this proposal expands the definition of a double-extension, transcending its origin in the context of metric Lie algebras, to a more general notion in which a double extension is the composition of a one-dimensional extension with an extension-by-derivation; at least in those cases where these two operations commute.
spacetimes have Klein pairs \((g_{\alpha,\beta}, h)\) where \(g_{\alpha,\beta}\) is the kinematical Lie algebra with additional nonzero brackets

\[
[H, B_\alpha] = -P_\alpha \quad \text{and} \quad [H, P_\alpha] = \alpha B_\alpha + \beta P_\alpha,
\]
for some \((\alpha, \beta) \in \mathbb{R}^2\) and \(h\) is the subalgebra spanned by \((L_{ab}, B_a)\). The parameters \(\gamma\) and \(\chi\) labeling the continua of galilean spacetimes serve as convenient parametrisations for the equivalence classes of pairs \((\alpha, \beta)\) under the relation \((\alpha, \beta) \sim (s^2\alpha, s\beta)\) for any nonzero \(s \in \mathbb{R}\). Just like the Galilei Lie algebra \(g = g_{0,0}\), the Lie algebra \(g_{\alpha,\beta}\) also admits a one-dimensional extension \(b_{\alpha,\beta}\) with additional brackets

\[
[B_\alpha, P_\beta] = \delta_{ab} M \quad \text{and} \quad [H, M] = \beta M,
\]
which is central if and only if \(\beta = 0\). In that case, we may distinguish three cases: \(\alpha = 0\), corresponding to the Galilei Lie algebra, \(\alpha > 0\) and \(\alpha < 0\), corresponding to the two Newton–Hooke Lie algebras.

The Lie algebra \(b_{\alpha,\beta}\) can also be described as an extension-by-derivation of the Carroll Lie algebra \(c\). Again we need to relabel the generators \(M \leftrightarrow H\) and the derivation \(ad_H\) is defined by \(ad_H(L_{ab}) = 0\), \(ad_H(B_a) = -P_\alpha\), \(ad_H(P_\alpha) = \alpha B_\alpha + \beta P_\alpha\) and \(ad_H(M) = \beta M\). In other words, we once again arrive at a commutative diagram like (1.6)

\[
\begin{array}{c}
0 \\
\downarrow \\
\mathbb{R} \\
\downarrow \\
\mathbb{R} \\
\downarrow \\
0 \\
\end{array} \quad \begin{array}{c}
0 \\
\downarrow \\
\mathbb{R} \\
\downarrow \\
\mathbb{R} \\
\downarrow \\
0 \\
\end{array} 
\quad \begin{array}{c}
0 \\
\downarrow \\
\mathbb{R} \\
\downarrow \\
\mathbb{R} \\
\downarrow \\
0 \\
\end{array}
\]

\[
0 \to c \to b_{\alpha,\beta} \to \mathbb{R} \to 0
\]

\[
0 \to s_0 \to g_{\alpha,\beta} \to \mathbb{R} \to 0
\]

\[
0 \to 0
\]

In this note we will exhibit yet other avatars of the above diagram. Indeed, our main aim in this note is the classification of (connected, simply-connected) Lie groups admitting bargmannian, carrollian or galilean structures. The Lie correspondence allows us to work at the level of Lie algebras with ad-invariant Bargmannian, Carrollian or Galilean structures. Without aiming to confuse the reader, we shall refer to such Lie algebras as Bargmannian, Carrollian or Galilean, for short. However please note that, paradoxically perhaps, the Bargmann Lie algebra is not Bargmannian, the Carroll Lie algebra is not carrollian and the Galilei Lie algebra is not galilean. Nevertheless Bargmannian, carrollian and galilean Lie algebras sit in relation to each other just like the Bargmann, Carroll and Galilei Lie algebras; namely, in a commutative diagram such as (1.6). Neither should our Bargmannian, carrollian and galilean Lie algebras be confused with the transitive Lie algebras of kinematical spacetimes with such structures. Metric Lie algebras associated with kinematical Lie algebras were studied in [19] in the context of Chern–Simons theories of gravity.

This note is organised as follows. In Section 2 we review the notions of carrollian and galilean structures and their duality via Bargmannian manifolds and specialise to connected Lie groups admitting bi-invariant such structures. This allows us to localise at the identity and discuss ad-invariant Bargmannian, Carrollian and Galilean structures on Lie algebras, which are studied in Section 3. We will see that all three kinds of Lie algebras are characterised in terms of the same data: namely, a metric Lie algebra with a skew-symmetric derivation. This is also the data which defines a one-dimensional double extension of a metric Lie algebra, which is a construction due to Medina and Revoy [4] (see also [5, 6]) which results in a new metric Lie algebra. This will allow us to identify Bargmannian Lie algebras precisely as one-dimensional double extensions of metric Lie algebras. In Section 4 we specialise to strictly carrollian and galilean structures (as opposed to the “pseudo” versions we treated before) and classify the relevant Lie algebras. In Section 5 we discuss Lie groups with bi-invariant leibnizian structures in the sense of [20] and show that they too appear in a commutative diagram such as (1.6). Leibnizian groups (strictly) extend the class of Bargmannian Lie groups and they too mediate a duality between carrollian and galilean Lie group, which is not canonical and hence in principle different from the one discussed in Section 3.5. Finally in Section 6 we offer some concluding remarks. There are two appendices: in Appendix A we give a proof of a result needed in Section 4; and in Appendix B we write some of the Lie algebraic structures in terms of a basis.
2. Bargmannian, carrollian and galilean structures

Let $M$ be a finite-dimensional smooth manifold. Recall that a galilean structure \[21\] on $M$ consists of a nowhere-vanishing one-form $\tau \in \Omega^1(M)$ and a symmetric bivector field $\gamma \in \Gamma(\wedge^2 TM)$, which is everywhere corank-1 as a field of bilinear forms on one-forms and such that its radical is everywhere spanned by $\tau$. Typically one demands that $\gamma$ is positive-semidefinite, but one can also consider pseudo-galilean structures where $\gamma$ has any signature. Galilean structures are particular examples of G-structures \[21, 22\].

Similarly, a carrollian structure \[23, 8\] on $M$ consists of a nowhere-vanishing vector field $\kappa \in X(M)$ and a symmetric $(0, 2)$-tensor field $h \in \Gamma(\wedge^2 T^*M)$ which is everywhere of corank-1 and such that the characteristic distribution is generated by $\kappa$. Again one typically demands that $h$ is positive-semidefinite, but one can also consider pseudo-carrollian structures where $h$ has any signature. For example, the blow-up of spatial infinity of Minkowski spacetime is a pseudo-carrollian manifold \[24\], whereas the blow-up of either past or future timelike infinity is carrollian \[25\]. Carrollian structures are also examples of G-structures \[22\].

One may exhibit a correspondence between carrollian and galilean structures \[8\] by passing to a higher-dimensional manifold $M$ with a bargmannian structure, namely an indefinite metric $g$ together with a nowhere-vanishing null vector field $\xi$. Strictly speaking, bargmannian structures require the metric $g$ to be lorentzian, but one can also have pseudo-bargmannian structures where $g$ is only assumed to be of indefinite signature. In any case, the null vector field $\xi$, foliates $M$ by null (since $\xi \in \xi^+$) hypersurfaces $i: N \to M$ which inherit a carrollian structure $(\xi, \iota^*g)$. If $\xi$ is Killing, then the null reduction \[16, 17\] defines (in the good cases) a fibration $\pi: M \to N$ where $N$ inherits a galilean structure whose clock one-form pulls back to the one-form $\xi^+$ dual to $\xi$ and whose spatial cometric is induced by the inverse of $g$.

In this note we are interested in Lie groups admitting bi-invariant carrollian, galilean or bargmannian structures; that is, triples $(G, \kappa, h)$, $(G, \tau, \gamma)$ or $(G, g, \xi)$ where $G$ is a connected Lie group and $\kappa$ and $h$, $\tau$ and $\gamma$ or $g$ and $\xi$ are invariant under Lie derivatives by both left- and right-invariant vector fields. Such tensor fields are determined uniquely by their value at the identity, where they define tensors of the Lie algebra $\mathfrak{g}$ of $G$ which are invariant under the adjoint representation of $G$ on $\mathfrak{g}$. Since we assume assume that $G$ is connected, it will be enough to demand that the tensors are ad-invariant; that is, invariant under the adjoint action of $g$ on itself.

3. Bargmannian, carrollian and galilean Lie algebras

In this section we characterise Lie algebras admitting ad-invariant bargmannian, carrollian or galilean structures. They turn out to be intimately linked to metric Lie algebras, which we review first.

3.1. Metric Lie algebras and double extensions. Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra. We say that $\mathfrak{g}$ is a metric Lie algebra if it admits an ad-invariant inner product $\langle - , - \rangle$; that is, one for which

$$\langle [W, X], Y \rangle = -\langle X, [W, Y] \rangle$$

(3.1)

for all $W, X, Y \in \mathfrak{g}$. Medina and Reboy \[4\] introduced the notion of a double extension, as a method of constructing new metric Lie algebras from old. In this note we will only need a special case of this construction: namely, a one-dimensional double extension.

The ingredients for a one-dimensional double extension are a metric Lie algebra $\mathfrak{g}_0$ with ad-invariant scalar product $\langle - , - \rangle_0$ and a skew-symmetric derivation $D_0$ of $\mathfrak{g}_0$; that is, for all $X, Y \in \mathfrak{g}_0$,

$$D_0 [X, Y]_0 = [D_0 X, Y]_0 + [X, D_0 Y]_0 \quad \text{and} \quad \langle D_0 X, Y \rangle_0 = -\langle X, D_0 Y \rangle_0,$$

(3.2)

where $\langle - , - \rangle_0$ is the Lie bracket in $\mathfrak{g}_0$. On the vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathbb{R}D \oplus \mathbb{R}Z$, where $D$ and $Z$ are two additional generators, we define the brackets

$$\langle X, Y \rangle = \langle X, Y \rangle_0 + \langle D_0 X, Y \rangle_0 Z$$

(3.3)

$$\langle [D, X], Y \rangle = D_0 X$$

$$\langle Z, - \rangle = 0,$$

for all $X, Y \in \mathfrak{g}_0$. We can define a scalar product on $\mathfrak{g}$ by

$$\langle X, Y \rangle = \langle X, Y \rangle_0 , \quad \langle (X, D), Z \rangle = 0, \quad \langle D, Z \rangle = 1 \quad \text{and} \quad \langle D, D \rangle \in \mathbb{R},$$

(3.4)

for all $X, Y \in \mathfrak{g}_0$, which can be seen to be ad-invariant. We may (and will) perform a Lie algebra automorphism $D \mapsto D - \frac{1}{2} \langle D, D \rangle Z$ which allows us to set $\langle D, D \rangle = 0$. The Lie algebra $\mathfrak{g}$ is said to be the double extension of $\mathfrak{g}_0$ relative to the skew-symmetric derivation $D_0$. This construction is due
to Medina and Revoy [4], refined and applied in the context of two-dimensional conformal field theory in [5, 6], and further refined by Kath and Olbrich [7].

3.2. Bargmannian Lie algebras. A metric Lie algebra \( (\mathfrak{g}, \langle -,- \rangle) \) is said to be bargmannian if there is a nonzero null \( Z \in \mathfrak{g} \), which is central. This of course requires \( \langle -,- \rangle \) to be indefinite, yet still nondegenerate. We will show that bargmannian Lie algebras are one-dimensional double-extensions of metric Lie algebras.

**Proposition 1.** Every bargmannian Lie algebra \( \mathfrak{g} \) is isomorphic to a double extension of a metric Lie algebra \( \mathfrak{g}_0 \) by a skew-symmetric derivation.

**Proof.** Let \( \mathfrak{g} \) be a bargmannian Lie algebra with scalar product \( \langle -,- \rangle \) and central null element \( Z \in \mathfrak{g} \). The one-dimensional isotropic ideal spanned by \( Z \) is clearly minimal and hence by the structure theorem of Medina and Revoy [4] (see also [6, §3]) the Lie algebra \( \mathfrak{g} \) is a double extension of the metric Lie algebra \( \mathfrak{g}_0 := Z^\perp / RZ \), which inherits an inner product from that of \( \mathfrak{g} \), by a skew-symmetric derivation defined as follows. Let \( D \in \mathfrak{g} \) be such that \( \langle D,Z \rangle = 1 \). Then \( \text{ad}_D \) preserves \( Z^\perp \) since \( Z^\perp \) is an ideal of \( \mathfrak{g} \) and \( [D,Z] = 0 \) since \( Z \) is central, hence \( \text{ad}_D \) induces a derivation \( D_0 \) of \( \mathfrak{g}_0 \) as follows: if \( X \in Z^\perp \) and letting \( \overline{X} \in \mathfrak{g}_0 \) denote its projection, we define the derivation \( D_0 \) by \( D_0 \overline{X} = [D,\overline{X}] \), which is well defined. It is clearly a derivation and also skew-symmetric under the inner product \( \langle -,- \rangle_0 \) on \( \mathfrak{g}_0 \) defined by

\[
\langle X,Y \rangle_0 = \langle X,Y \rangle.
\]

The fact that \( D_0 \) is skew-symmetric follows from the fact that so is \( \text{ad}_D \):

\[
\langle D_0X,Y \rangle_0 = \langle [D,X],Y \rangle_0
= \langle D,X,Y \rangle
= -\langle X, [D,Y] \rangle
= -\langle X, D_0Y \rangle_0.
\]

As vector spaces, \( \mathfrak{g} = Z^\perp \oplus R^2D \), where \( Z^\perp = \mathfrak{g}_0 \oplus RZ \) and the Lie brackets are given by

\[
[X,Y] = [X,Y]_0 + \langle D_0X,Y \rangle_0 Z, \quad [D,X] = D_0X \quad \text{and} \quad [Z,-] = 0,
\]

for all \( X,Y \in \mathfrak{g}_0 \), where we have adorned the bracket and the inner product of \( \mathfrak{g}_0 \) with a subscript \( 0 \). The inner product can be brought to the form

\[
\langle X,Y \rangle = \langle X,Y \rangle_0, \quad \langle X,D \rangle = 0, \quad \langle X,Z \rangle = 0, \quad \langle D,Z \rangle = 1, \quad \text{and} \quad \langle D,D \rangle = 0,
\]

for all \( X,Y \in \mathfrak{g}_0 \). In other words, comparing with Section 3.1, we see that \( \mathfrak{g} \) is the double extension of \( \mathfrak{g}_0 \) by the skew-symmetric derivation \( D_0 \). \( \square \)

Notice that if the inner product on \( \mathfrak{g}_0 \) has signature \( (p,q) \), the one on \( \mathfrak{g} \) has signature \( (p+1,q+1) \). Hence if \( \mathfrak{g} \) is lorentzian, then \( \mathfrak{g}_0 \) must have a positive-definite inner product. In that case, it follows from the structure theorem of Medina and Revoy [4] (see also [6]) that \( \mathfrak{g}_0 \) is isomorphic to an orthogonal direct sum of compact simple Lie algebras \( s_i \) (each with a negative multiple of the Killing form) and an abelian Lie algebra \( a \) with a euclidean inner product. We will use this in Section 4 in order to classify (strictly) carrollian and galilean Lie algebras.

3.3. Carollian Lie algebras. Let \( \mathfrak{g} \) be a finite-dimensional real Lie algebra. By an ad-invariant carollian structure on \( \mathfrak{g} \) we mean a pair \( (Z,h) \) consisting of a (nonzero) central element \( Z \in \mathfrak{g} \) and an ad-invariant \( h \in \mathfrak{g}^* \) whose radical is one-dimensional and spanned by \( Z \).

**Proposition 2.** Every carollian Lie algebra \( \mathfrak{g} \) is isomorphic to a one-dimensional central extension of a metric Lie algebra \( \mathfrak{g}_0 \).

**Proof.** Let \( (\mathfrak{g},Z,h) \) be a carollian Lie algebra and let \( \mathfrak{g}_0 = \mathfrak{g}/RZ \). Then \( \mathfrak{g} \) is a central extension of \( \mathfrak{g}_0 \), which we may summarise by the following short exact sequence of Lie algebras:

\[
0 \longrightarrow RZ \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}_0 \longrightarrow 0.
\]

Moreover, \( h \) induces an inner product \( \langle -,- \rangle_0 \) on \( \mathfrak{g}_0 \) by

\[
\langle X,Y \rangle_0 = h(X,Y),
\]

where we mean a pair
where $X \in g_0$ is the image of $X \in g$ under the canonical map $g \to g/\mathbb{R}Z$. The inner product is well-defined since $h(Z, -) = 0$. Since $Z$ spans the radical, $\langle - , - \rangle_0$ is nondegenerate and since $h$ is $g$-invariant, $\langle - , - \rangle_0$ is $g_0$-invariant. In other words, $(g_0, \langle - , - \rangle_0)$ is a metric Lie algebra. □

As vector spaces, $g = g_0 \oplus \mathbb{R}Z$ and the Lie bracket is given by

$$
\langle [X, Y], Z \rangle = [X, Y]_0 + \alpha(X, Y)Z
\langle Z, - \rangle = 0
$$

(3.10)

for all $X, Y \in g_0$ and where $\langle - , - \rangle_0$ is the Lie bracket on $g_0$. It follows that $\alpha \in \wedge^2 g_0^*$ is a 2-cocycle. Corresponding to a 2-cocycle on any metric Lie algebra, there is always a skew-symmetric derivation $D_0$, defined by

$$
\langle D_0X, Y \rangle_0 = \alpha(X, Y)
$$

(3.11)

for all $X, Y \in g_0$. Since $\langle - , - \rangle_0$ is nondegenerate, $D_0$ is uniquely defined. Since $\alpha$ is alternating, $D_0$ is skew-symmetric:

$$
\langle D_0X, Y \rangle_0 = \alpha(X, Y) = -\alpha(Y, X) = -\langle X, D_0Y \rangle_0.
$$

(3.12)

Finally, the cocycle condition for $\alpha$, says that $D_0$ is a derivation: for every $X, Y, U \in g_0$,

$$
\langle D_0[X, Y]_0, U \rangle_0 = \alpha([X, Y]_0, U)
= \alpha(X, [Y, U]_0) + \alpha(Y, [U, X]_0)
= \langle D_0X, [Y, U]_0 \rangle_0 + \langle D_0Y, [U, X]_0 \rangle_0
= \langle D_0X, [Y, U]_0 \rangle_0 - \langle [D_0Y, X]_0, U \rangle_0
= \langle [D_0X, Y]_0, U \rangle_0 + \langle [D_0Y, X]_0, U \rangle_0,
$$

so that $D_0[X, Y]_0 = [D_0X, Y]_0 + [X, D_0Y]_0$.

Let $\hat{g}$ denote the one-dimensional double-extension of $g_0$ relative to the skew-symmetric derivation $D_0$, as discussed in Section 3.1. Then the carrollian Lie algebra $g$ embeds into $\hat{g}$ as the ideal $Z^\perp$, resulting in the following short exact sequence of Lie algebras:

$$
0 \longrightarrow g \longrightarrow \hat{g} \longrightarrow \mathbb{R}D \longrightarrow 0.
$$

(3.13)

### 3.4. Galilean Lie algebras.

Let $g$ be again a finite-dimensional real Lie algebra. By an ad-invariant galilean structure on $g$ we mean a nonzero ad-invariant $\tau \in g^*$ and an ad-invariant $\gamma \in \wedge^2 g$, defining a symmetric bilinear form on $g^*$ which has a one-dimensional radical spanned by $\gamma$.

**Proposition 3.** *Every galilean Lie algebra is an “extension by skew-symmetric derivation” of a metric Lie algebra $g_0$.***

**Proof.** Let $(g, \tau, \gamma)$ be a galilean Lie algebra. Since $\tau$ is ad-invariant, it annihilates brackets $\langle \tau[X, Y] \rangle = 0$ and hence $g_0 := \ker \tau$ is an ideal of $g$. We thus get a short exact sequence of Lie algebras

$$
0 \longrightarrow g_0 \longrightarrow g \longrightarrow \mathbb{R} \longrightarrow 0.
$$

(3.14)

This sequence always splits: let $D \in g$ be any element with $\tau(D) = 1$. Then since $\tau$ annihilates brackets, $\text{ad}_D$ is a derivation of $g_0$. Dualising the above sequence we see that $g_0^* \cong g^*/\mathbb{R}\tau$. The tensor $\gamma$ defines an inner product $\gamma_0$ on $g_0^*$. Since $\gamma$ is $g$-invariant, so is $\gamma_0$. The inverse of $\gamma_0$ is an inner product $\langle - , - \rangle_0$ on $g_0$ relative to which $\text{ad}_D$ is skew-symmetric. In summary, a galilean Lie algebra is an “extension by skew-symmetric derivation” of a metric Lie algebra. □

So just as in the case of carrollian Lie algebras, the underlying data is again a metric Lie algebra and a skew-symmetric derivation. If we again let $\hat{g}$ denote the (one-dimensional) double extension of $g_0$ by the skew-symmetric derivation $\text{ad}_D$, we have that the galilean Lie algebra $g$ is a quotient of $\hat{g}$ by the central line $\mathbb{R}Z$, resulting the short exact sequence

$$
0 \longrightarrow \mathbb{R}Z \longrightarrow \hat{g} \longrightarrow g \longrightarrow 0.
$$

(3.15)
3.5. **Summary.** Let \((g_0, (-,-)_0)\) be a metric Lie algebra and \(D_0\) a skew-symmetric derivation. This data allows us to define three other Lie algebras:

1. a bargmannian Lie algebra \(\mathfrak{g}\), the one-dimensional double extension of \(g_0\) by \(D_0\);
2. a carrollian Lie algebra \(\mathfrak{g}_{\text{car}}\) which is a central extension of \(g_0\) with cocycle given by \(D_0\) via (3.11) or, equivalently, an ideal of \(\mathfrak{g}\); and
3. a galilean Lie algebra \(\mathfrak{g}_{\text{gal}}\) which is an extension by the skew-symmetric derivation \(D_0\) of \(g_0\) or, equivalently, a quotient of \(\mathfrak{g}\).

These Lie algebras fit into the following commutative diagram:

\[
\begin{array}{ccccccccc}
 & & & & 0 & & & & 0 & \\
 & & & & \downarrow & & & & \downarrow & \\
 & & & & \mathbb{R} & & & & \mathbb{R} & \\
 & & & & \downarrow & & & & \downarrow & \\
0 & \rightarrow & \mathfrak{g}_{\text{car}} & \rightarrow & \hat{\mathfrak{g}} & \rightarrow & \mathbb{R} & \rightarrow & 0 \\
& & & & \downarrow & & & & \downarrow & \\
0 & \rightarrow & \mathfrak{g}_0 & \rightarrow & \mathfrak{g}_{\text{gal}} & \rightarrow & \mathbb{R} & \rightarrow & 0 \\
& & & & \downarrow & & & & \downarrow & \\
& & & & 0 & & & & 0 & \\
\end{array}
\]

(3.16)

where the two short exact rows are extensions-by-derivations and the two short exact columns are central extensions. This defines a canonical correspondence between carrollian and galilean Lie algebras. For example, we start with a carrollian Lie algebra \((\mathfrak{g}_{\text{car}}, Z, h)\) given by the data \((g_0, D_0)\). This Lie algebra is an ideal of the double extension \(\hat{\mathfrak{g}}\) of \(g_0\) by \(D_0\) and then we define the galilean dual \(\mathfrak{g}_{\text{gal}}\) of \(\mathfrak{g}_{\text{car}}\) to be the quotient of \(\hat{\mathfrak{g}}\) by the ideal generated by \(Z\). Conversely, let \((\mathfrak{g}_{\text{gal}}, \tau, \gamma)\) be a galilean Lie algebra given by the data \((g_0, D_0)\) and let \(\hat{\mathfrak{g}}\) again denote the double extension of \(g_0\) by \(D_0\). Then \(\mathfrak{g}_{\text{gal}}\) is a quotient of \(\hat{\mathfrak{g}}\) by a one-dimensional ideal spanned by \(Z\) and we define the carrollian dual \(\mathfrak{g}_{\text{car}}\) of \(\mathfrak{g}_{\text{gal}}\) as the ideal \(Z^\perp\) of \(\hat{\mathfrak{g}}\).

In Section 5 we will discuss a non-canonical correspondence between carrollian and galilean algebras which is mediated by a leibnizian Lie algebra (to be defined below), but first we will use the canonical correspondence just established to classify strictly carrollian, galilean and bargmannian Lie algebras.

4. **Classification**

In this section we specialise to strict carrollian and galilean structures, where \(h\) and \(\gamma\) are positive-semidefinite. This means that the metric Lie algebra \(g_0\) has a positive-definite inner product or, equivalently, that the bargmannian Lie algebra \(\mathfrak{g}\) is lorentzian. Lorentzian Lie algebras were classified by Medina [3], but we are interested only in those which are isomorphic to one-dimensional double extensions of a positive signature metric Lie algebra. Since double extensions always make the inner product indefinite, the structure theorem says that metric Lie algebras with positive-definite inner products are necessarily orthogonal direct sums of (compact) simple Lie algebras (with a negative multiple of the Killing form) and an abelian Lie algebra with a (trivially invariant) euclidean inner product. Therefore we will let \(g_0 = s_1 \oplus \cdots \oplus s_k \oplus \mathfrak{a}\), where the \(s_i\) are simple and \(\mathfrak{a}\) is abelian. The direct sums are orthogonal and the inner product on \(s_i\) is given by \(-\lambda_i k_i\), where \(\lambda_i > 0\) and \(k_i\) is the Killing form of \(s_i\), which for compact simple Lie algebras is negative-definite, hence the sign. The inner product on \(\mathfrak{a}\) is any desired euclidean inner product. Up to an isomorphism, we can think of \(\mathfrak{a} = \mathbb{R}^m\) for some \(m\) and the inner product being the standard euclidean inner product on \(\mathbb{R}^m\).

Next we determine the skew-symmetric derivations of \(g_0\). The following proposition, proved in Appendix A, states that any skew-symmetric derivation of \(g_0\) is the sum of an inner derivation of the semisimple part \(s_0 = s_1 \oplus \cdots \oplus s_k\) and a skew-symmetric endomorphism of the abelian part \(\mathfrak{a}\).

**Proposition 4.** Let \(D_0\) be a skew-symmetric derivation of \(g_0\). Then

\[D_0 = T + \sum_x \text{ad}_X,\]

where \(T \in \text{so}(\mathfrak{a})\) and \(X_i \in s_i\).
The double extension $\mathfrak{g}$ of $\mathfrak{g}_0$ by the skew-symmetric derivation $D_0$ has underlying vector space $\mathfrak{g}_0 \oplus \mathbb{R}D \oplus \mathbb{R}Z$ and the brackets are given by (3.3). We may apply the following general linear transformation of the vector space: it is the identity on $\mathfrak{g}_0 \oplus \mathbb{R}Z$ and maps $D \mapsto D - \sum X_i$. In that basis, $[D, X] = T(X_i)$, where $X = X_{ss} + X_a$ with $X_{ss} \in \mathfrak{ss}$ and $X_a \in \mathfrak{a}$. In other words, we may change basis so that effectively $D_0 \in \mathfrak{so}(a)$. Using the euclidean inner product on $a$, we may identify $D_0 \in \mathfrak{so}(a)$ with $\omega \in \wedge^2 \mathfrak{a}^*$ where $\omega(A, B) = (D_0 A, B)_0$ for all $A, B \in \mathfrak{a}$. Let $\omega$ have rank $2\ell$ for some $\ell = 0, 1, \ldots, \lfloor \frac{\dim a}{2} \rfloor$.

If $\ell = 0$, then $D_0 = 0$ and

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathbb{R}D \oplus \mathbb{R}Z$$

where $\mathfrak{a}_2$ is a two-dimensional abelian Lie algebra with a lorentzian inner product.

If $\ell > 0$, we may split $a = a_0 \oplus a_1$, where $a_0 = \ker D_0$ and $a_1 = a_0^\perp$, and $\omega$ defines a symplectic structure on $a_1$. Then the double extension is

$$\mathfrak{g} = \mathfrak{ss} \oplus \mathfrak{a}_0 \oplus \mathfrak{nw}_{2\ell+2},$$

where $\mathfrak{nw}_{2\ell+2}$ is a lorentzian Nappi–Witten algebra [26–28, 5] of dimension $2\ell + 2$ with underlying vector space $\mathfrak{a}_1 \oplus \mathbb{R}D \oplus \mathbb{R}Z$ and brackets

$$[A, B] = \omega(A, B) Z \quad \text{and} \quad [D, A] = D_0 A$$

for all $A, B \in \mathfrak{a}_1$ and with lorentzian inner product

$$\langle A, B \rangle = \langle A, B \rangle'_0, \quad \langle D, Z \rangle = 1 \quad \text{and} \quad \langle D, D \rangle = 0,$$

where the choice $\langle D, D \rangle = 0$ can be arrived at via a Lie algebra automorphism $D \mapsto D - \frac{1}{2} [D, D] Z$.

The corresponding carrollian Lie algebra is the ideal $Z^\perp$ of $\mathfrak{g}$. As a Lie algebra,

$$Z^\perp = \mathfrak{ss} \oplus \mathfrak{a}_0 \oplus \mathfrak{heis}_{2\ell+1}$$

where the Heisenberg Lie algebra $\mathfrak{heis}_{2\ell+1}$ has underlying vector space $\mathfrak{a}_1 \oplus \mathbb{R}Z$ and brackets

$$[A, B] = \omega(A, B) Z$$

for all $A, B \in \mathfrak{a}_1$.

Finally, the corresponding galilean Lie algebra is the quotient $\mathfrak{g}/\mathbb{R}Z$ with underlying vector space $\mathfrak{ss} \oplus \mathfrak{a}_0 \oplus \mathfrak{a}_1 \oplus \mathbb{R}D$, where the bracket on $\mathfrak{a}_1 \oplus \mathbb{R}D$ is given by

$$[A, B] = 0 \quad \text{and} \quad [D, A] = D_0 A,$$

for all $A, B \in \mathfrak{a}_1$.

In summary, the ingredients to construct any bargmannian, carrollian and galilean Lie algebras in diagram (3.16) are as follows:

- a compact semisimple Lie algebra $\mathfrak{ss}$ with a choice of positive-definite ad-invariant inner product: a negative multiple of the Killing form on each of its simple factors;
- a euclidean vector space $\mathfrak{a}_0$, whose only invariant is the dimension; and
- a euclidean vector space $\mathfrak{a}_1$ with a symplectic structure $\omega$.

Any Lie algebra with an ad-invariant bargmannian, carrollian or galilean structure can be constructed out of these ingredients. The metric Lie algebra $\mathfrak{ss} \oplus \mathfrak{a}_0$ is always an orthogonal summand of these Lie algebras, so we may concentrate on the euclidean/symplectic vector space $(\mathfrak{a}_1, \langle - , - \rangle, \omega)$.

The bargmannian Lie algebra is of Nappi–Witten type and has underlying vector space $\mathfrak{a}_1 \oplus \mathbb{R}Z \oplus \mathbb{R}D$ and brackets, for all $A, B \in \mathfrak{a}_1$, given by

$$[A, B] = \omega(A, B) Z \quad \text{and} \quad [D, A] = D_0 A,$$

with $\mathbb{Z}$ central and where $D_0$ is defined by $(D_0 A, B) = \omega(A, B)$. The underlying inner product extends the one on $\mathfrak{a}_1$ by declaring $Z$ to be null and $\langle D, Z \rangle = 1$. We may always redefine $D \mapsto D - \frac{1}{2} [D, D] Z$, which is a Lie algebra automorphism, to ensure that $\langle D, D \rangle = 0$.

The carrollian Lie algebra is a Heisenberg Lie algebra with underlying vector space $\mathfrak{a}_1 \oplus \mathbb{R}Z$ and brackets

$$[A, B] = \omega(A, B) Z,$$

with $\mathbb{Z}$ central.

The galilean Lie algebra is an extension of the abelian Lie algebra $\mathfrak{a}_1$ by the derivation $D_0$. It has underlying vector space $\mathfrak{a}_1 \oplus \mathbb{R}D$ and the only nonzero bracket is the action $[D, A] = D_0 A$, for all $A \in \mathfrak{a}_1$, of the derivation $D_0$.

In Appendix B we describe these Lie algebras in a basis.

Finally, we tackle the isomorphism problem. Which triples $(\mathfrak{a}_1, \langle - , - \rangle, \omega)$ correspond to non-isomorphic Lie algebras. Clearly if two such triples are in the same orbit of $GL(\mathfrak{a}_1)$, then the resulting Lie algebras
(bargmannian, carrollian and galilean) are isomorphic. Hence we need to classify triples \((a_1, \langle -,- \rangle, \omega)\) up to the action of \(\text{GL}(a_1)\). We may use \(\text{GL}(a_1)\) to relate any two scalar products \(\langle -,- \rangle\), leaving the freedom to act with the stabiliser \(O(a_1)\) of the scalar product on the symplectic form. In other words, which are the possible symplectic forms \(\omega\) on \(a_1\) up to orthogonal transformations? This is the same problem as classifying \(D_0 \in \mathfrak{so}(a_1)\) up to the adjoint action of \(O(a_1)\). Using \(O(a_1)\) we may always conjugate to a Cartan subalgebra, or said differently, we can skew-diagonalise \(D_0\) to a matrix of the form

\[
\begin{pmatrix}
0 & \mu_1 & & \\
-\mu_1 & 0 & & \\
& & \ddots & \\
& & & 0 & \mu_n \\
& & & -\mu_n & 0
\end{pmatrix}
\]

where \(\mu_i\) are nonzero and ordered. Conjugating by the full orthogonal group we may assume that \(\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n > 0\), where \(\dim a_1 = 2n\).

We may summarise this discussion as follows.

**Theorem 5.** The isomorphism classes of bargmannian (and hence also carrollian and galilean) Lie algebras are parameterised by the following data:

- a compact Lie algebra with a choice of positive-definite ad-invariant scalar product (i.e., the orthogonal direct sum of a semisimple and an abelian Lie algebras); and
- \((\mu_1, \ldots, \mu_n) \in \mathbb{R}^n\) with \(\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n > 0\), which are the skew-eigenvalues of a symplectic form on the euclidean space \(\mathbb{E}^{2n}\).

The Lie brackets and the ad-invariant bargmannian, carrollian and galilean structures associated to that data can be constructed as explained above.

In other words, the list of bargmannian Lie algebras, from which we can then obtain all carrollian and galilean dual pairs is parametrised by compact semisimple Lie algebras with a choice of positive-definite invariant scalar product, a euclidean space of dimension \(m \geq 0\) and \(n\) positive real numbers \(\mu_1 \geq \cdots \geq \mu_n > 0\).

5. Leibnizian Lie Groups

In this section we characterise Lie groups with bi-invariant leibnizian structures. Such Lie groups extend the class of bargmannian Lie groups and we will see that they also contain carrollian Lie group as a normal subgroup and they quotient to a galilean Lie group. At the Lie algebra level, they too sit in relation to each other in a way described by a diagram isomorphic to (1.6). Hence they define a non-canonical correspondence between carrollian and galilean Lie algebras.

5.1. Leibnizian structures. In [20] Bekãert and Morand introduced the notion of an (ambient) leibnizian structure on a manifold \(M\). This is a triplet \((\xi, \psi, h)\) where \(\xi \in \mathfrak{X}(M)\) is a nowhere-vanishing vector field, \(\psi \in \Omega^2(M)\) is a nowhere-vanishing one-form with \(\psi(\xi) = 0\), and \(h \in \Gamma(\mathcal{O}^2(\ker \psi)*)\) is a corank-1, positive-semidefinite, bilinear form on \(\ker \psi \subset TM\) whose radical is spanned by \(\xi\). Every bargmannian manifold \((M, g, \xi)\) has a natural leibnizian structure given by \(\psi = \xi^0\) and \(h = g|_{\xi^1-\ker \xi^0}\).

A natural question is: which Lie groups admit a bi-invariant leibnizian structure? For the same price, we may (and will) actually consider pseudo-leibnizian structures, where \(h\) is not necessarily positive-semidefinite, but still corank-1 on \(\ker \psi\). We shall refer to such Lie groups and their Lie algebras as leibnizian.

5.2. Leibnizian Lie algebras.

**Definition 6.** A Lie algebra \(g\) is said to be leibnizian if it admits a nonzero central element \(Z \in g\), a nonzero ad-invariant \(\psi \in g^*\) (i.e., \(\psi|_{\ker \psi} = 0\)) with \(\psi(Z) = 0\) and an ad-invariant \(h \in \mathcal{O}^2(\ker \psi)^*\) of corank 1 whose radical is spanned by \(Z\).

It is immediate that if \((g, Z, \psi, h)\) is leibnizian, then \((\ker \psi, Z, h)\) is carrollian. As in Section 3.3, \(g_0 = \ker \psi / \mathbb{R}Z\) is a metric Lie algebra. Let \(\pi : \ker \psi \to g_0\) send \(X \mapsto \overline{X}\). Then the Lie bracket and inner product on \(g_0\) are defined by

\[
[\overline{X}, \overline{Y}]_0 = \overline{[X, Y]} \quad \text{and} \quad \langle \overline{X}, \overline{Y} \rangle_0 = h(X, Y).
\]

As before, \(\ker \psi\) is a central extension of the metric Lie algebra \(g_0\)

\[
0 \to \mathbb{R}Z \to \ker \psi \xrightarrow{\pi} g_0 \to 0,
\]

(5.1)
and the central extension defines a skew-symmetric derivation $D_0$ of $g_0$ as in equation (3.11).

Similarly, if $(g, Z, \psi, h)$ is leibnizian, let $\overline{\psi} := g / RZ$, so that $g$ is a central extension of $\overline{\psi}$:

$$\begin{align*}
0 &\longrightarrow RZ \\ &\longrightarrow g \\ &\longrightarrow \overline{\psi} \\ &\longrightarrow 0,
\end{align*}$$

(5.2)

where $\overline{\pi} : g \to \overline{\psi}$ sends $X \mapsto \overline{X}$. The map $\pi : \ker \psi \to g_0$ above is the restriction of $\overline{\pi}$ to the ideal $\ker \psi$, which explains why we use the same notation. Since $\ker \psi$ is a codimension-1 ideal of $g$, $g$ is an extension-by-derivation of $\ker \psi$:

$$\begin{align*}
0 &\longrightarrow \ker \psi \\ &\longrightarrow g \\ &\longrightarrow \overline{\psi} \\ &\longrightarrow R \\ &\longrightarrow 0
\end{align*}$$

(5.3)

and hence it follows that $g_0$ is a codimension-1 ideal of $\overline{\psi}$, so that $\overline{\psi}$ is an extension by derivation of $g_0$:

$$\begin{align*}
0 &\longrightarrow g_0 \\ &\longrightarrow \overline{\psi} \\ &\longrightarrow R \\ &\longrightarrow 0
\end{align*}$$

(5.4)

where $\overline{\psi}(\overline{X}) = \psi(X)$.

In summary, we have again an instance of our favourite diagram:

$$\begin{array}{ccc}
0 &\longrightarrow & 0 \\
\downarrow & & \downarrow \\
RZ &\longrightarrow & RZ \\
\downarrow & & \downarrow \\
0 &\longrightarrow & \ker \psi \\
\downarrow \pi & & \downarrow \pi \\
g_0 &\longrightarrow & \overline{\psi} \\
\downarrow & & \downarrow \\
0 &\longrightarrow & 0
\end{array}$$

(5.5)

where $\ker \psi$ is carrollian and $g_0$ metric. We will now show that $\overline{\psi}$ is galilean.

Let us split the exact sequences (5.3) and (5.4) by choosing $D \in g$ with $\psi(D) = 1$. Let $\overline{\psi} = \overline{\psi}(D)$ so that $\overline{\psi}(D) = 1$. As vector spaces, $g = \ker \psi \oplus R D$ and hence $\overline{\psi} = g_0 \oplus RD$.

**Lemma 7.** $\text{ad}_D$ induces $\text{ad}_D \overline{\psi}$ which in turn induces a skew-symmetric derivation $\overline{D}_0$ of $(g_0, (-,-)_0)$ defined by

$$\overline{D}_0 \overline{X} := [D, \overline{X}] = [D, X],$$

where the first bracket is in $\overline{\psi}$.

**Proof.** The endomorphism $\overline{D}_0$ is well-defined because $[D, X] \in [g, g] \subset \ker \psi$ and $Z$ is central. We show that it is a derivation. Let $X, Y \in \ker \psi$ and calculate

$$\begin{align*}
\overline{D}_0(X, Y)_0 &= \overline{D}_0[X, Y] \\
&= [D, [X, Y]] \\
&= [D, [X, [D, Y]] + [X, [D, Y]]] \\
&= [D, [X, Y]] + [X, [D, Y]] \\
&= [D, [X, Y]] + [X, \overline{D}_0 Y]_0 \\
&= [D, X, Y]_0 + [X, \overline{D}_0 Y]_0.
\end{align*}$$

(by Jacobi)

To show that it is skew-symmetric we let $X, Y \in \ker \psi$ and calculate

$$\begin{align*}
\langle \overline{D}_0 X, \overline{Y} \rangle_0 &= \langle [D, X], \overline{Y} \rangle_0 \\
&= h([D, X], Y) \\
&= -h(X, [D, Y]) \\
&= -\langle X, [D, Y] \rangle_0 \\
&= -\langle X, \overline{D}_0 Y \rangle_0.
\end{align*}$$

(since $h$ is invariant)
The covector $\mathbf{T} \in \mathfrak{g}^*$, which is induced from $\psi$, is $\mathbf{T}$-invariant since $\psi$ is $\mathfrak{g}$-invariant. The inverse of the $\mathfrak{g}_0$-invariant scalar product on $\mathfrak{g}_0 = \ker \psi$ pushes forward to a symmetric tensor $\gamma \in \mathfrak{so}(\mathbf{T})$. Since the derivation $\mathbf{D}_0$ of $\mathfrak{g}_0$ induced by $\text{ad}_\mathbf{T} \mathbf{T}$ is skew-symmetric, it follows that $\gamma$ is actually $\mathbf{T}$-invariant. In other words, $(\mathbf{T}, \mathbf{T}, \gamma)$ is a galilean Lie algebra.

5.3. A leibnizian Lie algebra which is not bargmannian. Comparing the commutative diagrams (5.5) and (3.16), we see that both $\ker \psi$ and $\mathfrak{g}_{\text{car}}$ are carrollian, $\mathfrak{g}_0$ is metric in both diagrams, $\mathbf{T}$ and $\mathfrak{g}$ are both galilean, but whereas $\mathfrak{g}$ is bargmannian, $\mathfrak{g}$ is leibnizian. It would be tempting to conjecture that leibnizian Lie algebras are actually bargmannian, but this turns out to be false.

In diagram (5.5) there are two skew-symmetric derivations of the metric Lie algebra $\mathfrak{g}_0$ at play: the derivation $\mathbf{D}_0$ associated to the central extension defining $\ker \psi$ and the derivation $\mathbf{D}_0$ by which we extend $\mathfrak{g}_0$ in order to construct $\mathbf{T}$. If $\mathfrak{g}$ were bargmannian, and hence the double extension of $\mathfrak{g}_0$ by a skew-symmetric derivation, both of these derivations would coincide. In the general leibnizian case, however, they need not coincide and this gives us a hint how to prove that the class of leibnizian Lie algebras is a proper subclass of the bargmannian Lie algebras.

We will do this by constructing a leibnizian Lie algebra which is not bargmannian. Let us take $\mathfrak{g}_0$ to be abelian with a euclidean inner product $\langle \cdot , \cdot \rangle_0$. Let $\mathfrak{D}_0 \in \mathfrak{so}(\mathfrak{g}_0)$ and define a Lie algebra $\mathfrak{g}_{\text{car}}$ with underlying vector space $\mathfrak{g}_0 \oplus \mathbb{R} \mathfrak{Z}$ and Lie brackets

$$[X, Y] = (\mathbf{D}_0X, Y)_0 \mathbf{Z} \quad \text{and} \quad \mathbf{Z} \text{ central},$$

(5.6) for all $X, Y \in \mathfrak{g}_0$. Pick $\mathbf{D} \in \mathfrak{so}(\mathfrak{g}_0)$ in a different $SO(\mathfrak{g}_0)$-orbit than $\mathbf{D}_0$. Let $\mathfrak{g}$ be the Lie algebra with underlying vector space $\mathfrak{g}_0 \oplus \mathbb{R} \mathfrak{Z} \oplus \mathbb{R} \mathfrak{W}$ with Lie brackets

$$[X, Y] = (\mathbf{D}_0X, Y)_0 \mathbf{Z}$$

$$[W, X] = DX$$

$$[\mathbf{Z}, -] = 0,$$

(5.7) for all $X, Y \in \mathfrak{g}_0$. We will show one example of this construction for which $\mathfrak{g}$ is not bargmannian. Let $\mathfrak{g}_0$ be four-dimensional, with orthonormal basis $e_i$, $i = 1, \ldots , 4$ and let $\mathfrak{D}_0, \mathbf{D} \in \mathfrak{so}(\mathfrak{g}_0)$ be given relative to this basis by the matrices

$$\mathbf{D}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} 0 & \beta & 0 & 0 \\ -\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & -\gamma & 0 \end{pmatrix}.$$

(5.8)

Let $\mathfrak{g}$ be the six-dimensional Lie algebra spanned by $e_1, \ldots , e_4, e_+, e_-$ with nonzero brackets

$$[e_1, e_j] = (\mathbf{D}_0)_{ij} e_+ \quad \text{and} \quad [e_1, e_-] = D_{ij} e_j,$$

(5.9)
or explicitly,

$$[e_1, e_2] = e_+ \quad [e_3, e_4] = \alpha e_+$$

$$[e_1, e_-] = \beta e_2 \quad [e_3, e_-] = \gamma e_4$$

(5.10)

Let $\theta^1, \ldots , \theta^4, \theta^+, \theta^-$ denote the canonical dual basis for $\mathfrak{g}^*$. Then the Lie algebra $\mathfrak{g}$ is leibnizian with $Z = e_+$, $\psi = \theta^-$ and $h = \sum_{i=1}^4 (\theta^i)^2$. A short calculation shows that the most general ad-invariant symmetric bilinear form on $\mathfrak{g}$ is given up to scale by

$$\begin{pmatrix} \gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & -\beta \gamma \\ 0 & 0 & 0 & 0 & -\beta \gamma & 0 \end{pmatrix}.$$

(5.11)

for any $\lambda \in \mathbb{R}$. This matrix is non-degenerate if and only if $\alpha \beta \gamma \neq 0$. So we can take either of $\alpha$, $\beta$ or $\gamma$ to be zero (with, say, the other two nonzero) and arrive at a leibnizian Lie algebra which is not bargmannian.
6. Conclusion

In this paper we have characterised Lie algebras admitting ad-invariant bargmannian, carrollian, and galilean structures. We have shown that they are all given by the same data: namely a metric Lie algebra and a skew-symmetric derivation. This sets up a canonical correspondence between carrollian and galilean Lie algebras.

We have shown that these Lie algebras, together with the underlying metric Lie algebra, fit into a commutative diagram (3.16) of Lie algebras which is isomorphic to a similar diagram (1.6) involving the Bargmann, Carroll and Galilei Lie algebras (despite them not being bargmannian, carrollian nor galilean) and also a similar diagram [14, App. B] involving some infinite-dimensional Lie algebras playing a rôle in the hamiltonian description of particle dynamics on the homogeneous galilean spacetimes, whose transitive Lie algebras together with their Bargmann extension and the Carroll Lie algebra itself also fit in yet another commutative diagram (1.10) isomorphic to the other commutative diagrams. This diagrammatic coincidence, if indeed it is a coincidence, deserves to be further studied.

For the strict carrollian and galilean cases, the underlying metric Lie algebra is positive-definite and this allows a classification up to isometric isomorphism and the Lie algebras are orthogonal direct sums of a compact Lie algebra with a choice of positive-definite inner product and either a Nappi–Witten Lie algebra (in the bargmannian case), a Heisenberg Lie algebra (in the carrollian case) or an extension by skew-symmetric derivation of an abelian metric Lie algebra (in the galilean case).

We also characterised Lie algebras admitting an ad-invariant leibnizian structure and we saw that they too mediate a Lie algebraic Carroll/Galilei correspondence which is however not canonical. Indeed there is some choice in defining the carrollian dual of a carrollian Lie algebra. This is the phenomenon already observed at the level of the spatially isotropic homogeneous galilean spacetimes. Indeed, as we could see already in diagram (1.10), the Carroll spacetime (as the unique kinematical spacetime associated to the Carroll algebra) is apparently dual to any one of the galilean spacetimes simply because \( c \) is an ideal of any of the \( \mathfrak{b}_{\alpha,\beta} \) which are extensions by (different) derivations of \( c \). Here too, given a carrollian Lie algebra \( \mathfrak{g}_{\text{car}} \) we arrive at a metric Lie algebra \( \mathfrak{g}_0 \) with a skew-symmetric derivation \( D_0 \). We may choose to extend \( \mathfrak{g}_0 \) by this very derivation in order to obtain the galilean dual Lie algebra \( \mathfrak{g}_{\text{gal}} \) in diagram (3.16) (with \( \mathfrak{g}_{\text{bargm}} \) bargmannian) or we could choose any other skew-symmetric derivation and arrive in this way at a leibnizian Lie algebra \( \mathfrak{g} \). In a way, the bargmannian case is canonical, since it is uniquely determined by the carrollian Lie algebra, but the leibnizian case provides added flexibility in the construction.

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Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Appendix A. Skew-symmetric derivations of a reductive metric Lie algebra

In this appendix we give a proof of Proposition 4, which we rephrase slightly.

Let \( \mathfrak{g}_0 = \mathfrak{ss} \oplus \mathfrak{a} \) be an orthogonal direct sum of a compact semisimple Lie algebra \( \mathfrak{ss} \) with a choice of a positive-definite inner product (i.e., a negative multiple of the Killing form on each simple summand) and an abelian Lie algebra \( \mathfrak{a} \) with a choice of euclidean inner product.

**Proposition A.1.** Let \( D \) be a skew-symmetric derivation of \( \mathfrak{g}_0 \). Then \( D = D_1 + D_2 \) where \( D_1 = \text{ad}_\alpha \) for some \( \alpha \in \mathfrak{ss} \) and \( D_2 \in \mathfrak{so}(\mathfrak{a}) \).

**Proof.** Let \( X_1 \in \mathfrak{ss} \) and \( X_2 \in \mathfrak{a} \). Then since they commute and \( D \) is a derivation

\[ 0 = D[X_1, X_2] = [DX_1, X_2] + [X_1, DX_2]. \]
Let $Y_1 \in ss$ and calculate the inner product
\[
0 = \langle DX_1, Y_1 \rangle = \langle DX_1, X_2 \rangle + \langle X_1, DX_2 \rangle = \langle DX_1, Y_1 \rangle - \langle DX_2, X_1 \rangle.
\]
The first term in the RHS is zero because $\langle X_2, Y_1 \rangle = 0$ and hence the second term says that $DX_2$ is perpendicular to $\langle ss, ss \rangle = ss$. In other words, $DX_2 \in a$ and hence $D$ preserves $a$. But then, since $ss$ and $a$ are orthogonal and $D$ is skew-symmetric,
\[
0 = \langle DX_1, X_2 \rangle + \langle X_1, DX_2 \rangle.
\]
where the second term vanishes since $DX_2 \in a = ss$. Therefore $DX_1$ is perpendicular to $a$ and hence $D$ also preserves $ss$. In other words, $D = D_1 + D_2$, where $D_1$ is a skew-symmetric derivation of $ss$ and $D_2$ is a skew-symmetric derivation of $a$. Since $ss$ is semisimple, all derivations are inner and since the inner product is built out of the Killing form, all inner derivations are also skew-symmetric, so that $D_1 = \text{ad}_X$ for some $X \in ss$. Finally, since $a$ is abelian, any endomorphism is a derivation and hence $D_2 \in so(a)$ is any skew-symmetric endomorphism.

Now Proposition 4 follows because if $ss = s_1 \oplus \cdots \oplus s_k$, then $X \in ss$ can be written as $X = \sum X_i$.

**Appendix B. Once more, with indices**

In this appendix we will write down the bargmannian, carrollian and galilean Lie algebras in terms of a basis.

Let $\mathfrak{g}_0$ be a metric Lie algebra with basis $(X_a, \omega^a)$. Lie brackets $[X_a, X_b] = f_{ab}^c X_c$ and scalar product $\langle X_a, X_b \rangle = \eta_{ab}$. Invariance of the inner product simply says that $f_{abc} := f_{ab}^d \eta_{dc}$ is totally skew-symmetric.

Let $D_0 : \mathfrak{g}_0 \to \mathfrak{g}_0$ be a skew-symmetric derivation. Relative to the above basis for $\mathfrak{g}_0$, we write $D_0 X_a = X_b \omega^b \omega^a$, where the skew-symmetry condition says that $\omega_{ab} = \eta_{ac} \omega^c b = -\omega_{ba}$ and the fact that $D_0$ is a derivation says that $\omega^d f_{ab}^c = f_{bc}^d \omega^c a + f_{ac}^d \omega^c b$. (B.1)

The bargainmannian Lie algebra $\mathfrak{g}$ associated to the data $(\mathfrak{g}_0, (-,-), D_0)$ is the double-extension, which has basis $(X_a, X_+, X_-)$ and Lie brackets
\[
[X_a, X_b] = f_{ab}^c X_c + \omega_{ab} X_+, \quad [X_-, X_a] = X_b \omega^b a \quad \text{and} \quad [X_+, -] = 0,
\] (B.2)
and the invariant scalar product can always be brought to a form where the only nonzero entries are:
\[
\langle X_a, X_b \rangle = \eta_{ab} \quad \text{and} \quad \langle X_+, X_- \rangle = 1.
\] (B.3)

The associated carrollian Lie algebra $\mathfrak{g}_{\text{car}}$ is the ideal of $\mathfrak{g}$ spanned by $(X_a, X_+)$ with induced brackets
\[
[X_a, X_b] = f_{ab}^c X_c + \omega_{ab} X_+ \quad \text{and} \quad [X_+, -] = 0.
\] (B.4)

The invariant carrollian structure consists of $X_+$ and the invariant symmetric bilinear form $\langle -,- \rangle$ which is degenerate since $\langle X_+, X_+ \rangle = 0$.

The associated galilean Lie algebra $\mathfrak{g}_{\text{gal}}$ is the quotient of $\mathfrak{g}$ by the ideal spanned by $X_+$. It is spanned by $(X_a, X_-)$ and the brackets are
\[
[X_a, X_b] = f_{ab}^c X_c \quad \text{and} \quad [X_-, X_a] = X_b \omega^b a.
\] (B.5)

Let $(\theta^a, \theta^-)$ be a canonical dual basis for $\mathfrak{g}_{\text{gal}}$. Then the invariant galilean structure is given by $\theta^-$ and $\gamma = \eta_{ab} X_a X_b$, where $\eta_{ab}$ are the entries of the inverse of the restriction of $\eta$ to $\mathfrak{g}_0$.

In the special case where $\eta_{ab}$ is positive-definite we discussed in Section 4, we showed that these Lie algebras are the orthogonal direct sums of certain “primitive” Lie algebras with a compact Lie algebra with a positive-definite invariant inner product (i.e., the direct sum of a compact semisimple Lie algebra with a choice of positive-definite invariant inner product and an abelian Lie algebra with a euclidean inner product). These primitive Lie algebras can be read off from the above expressions by setting $f_{ab}^c = 0$, $\eta_{ab}$ positive-definite and $\omega_{ab}$ symplectic.

Explicitly, the bargainmannian Lie algebra is a Nappi–Witten Lie algebra with brackets
\[
[X_a, X_b] = \omega_{ab} X_+, \quad [X_-, X_a] = X_b \omega^b a \quad \text{and} \quad [X_+, -] = 0,
\] (B.6)
with inner product $\langle X_a, X_b \rangle = \eta_{ab}$ and $\langle X_+, X_- \rangle = 1$. Of course, we can always choose an orthonormal basis where $\eta_{ab} = \delta_{ab}$ and then use orthogonal transformations to bring $\omega_{ab}$ to a normal form
\[
\omega = \lambda_1 \theta^1 \wedge \theta^2 + \lambda_2 \theta^3 \wedge \theta^4 + \cdots + \lambda_{2r+1} \theta^{2r+1} \wedge \theta^{2r}
\] (B.7)
where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t \geq 0$, where the Nappi–Witten Lie algebra has dimension $2t + 2$.

The carrollian Lie algebra is a Heisenberg Lie algebra with brackets

$$[X_a, X_b] = \omega_{ab} X_+ \quad \text{and} \quad [X_+, X_-] = 0,$$

and the only nonzero brackets of the galilean Lie algebra are

$$[X_-, X_a] = X_b \omega^{ba}.$$

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