PERTURBATIONS OF RATIONAL MISIUREWICZ MAPS

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Abstract. In this paper we investigate the perturbation properties of rational Misiurewicz maps, when the Julia set is the whole sphere (the other case is treated in [1]). In particular, we show that if \( f \) is a Misiurewicz map and not a flexible Lattés map, then we can find a hyperbolic map arbitrarily close to \( f \).

1. Introduction

The notion of Misiurewicz maps has its origin from the paper [12] by M. Misiurewicz. In honor of this paper, we proceed with the following definition. First, let \( J(f) \) be the Julia set of \( f \), \( F(f) \) the Fatou set of \( f \) and \( \text{Crit}(f) \) the critical set of \( f \). Let \( \omega(c) \) be the omega limit set of \( c \).

Definition 1.1. A non-hyperbolic rational map \( f \) without parabolic periodic points satisfies the Misiurewicz condition if for every \( c \in \text{Crit}(f) \cap J(f) \), we have \( \omega(c) \cap \text{Crit}(f) = \emptyset \).

In [10], McMullen showed that small Mandelbrot copies are dense in the bifurcation locus in any (non-trivial) analytic family of rational maps. An important element of the proof of this involves the fact that Misiurewicz maps create a family of polynomial-like maps nearby. From this stems the Mandelbrot copies. In this paper we consider rational Misiurewicz maps for which the Julia set is the whole sphere and show that such maps can be perturbed into a rich family of postcritically finite maps. One main ingredient is that we get control of all critical points, which enables perturbations into hyperbolic maps, for instance. We use the the supremum norm on the Riemann sphere to calculate the distance between two rational functions.

The main result of this paper is the following.

Theorem A. Let \( f \) be a rational Misiurewicz map and not a flexible Lattés map. Then we can find a hyperbolic rational map \( g \) arbitrarily close to \( f \).

Moreover, if in addition \( J(f) = \hat{\mathbb{C}} \), then \( g \) can be chosen so that all critical points lie in superattracting cycles.

The case when \( J(f) \neq \hat{\mathbb{C}} \) is treated in [1]. In that paper it is shown that Misiurewicz maps for which \( J(f) \neq \hat{\mathbb{C}} \) are in fact Lebesgue density points of hyperbolic maps. We therefore assume that \( J(f) = \hat{\mathbb{C}} \) to prove the first part of Theorem A. In fact, we will only focus on the second part (which of course implies the first in the case \( J(f) = \hat{\mathbb{C}} \)).

The proof of Theorem A works as long as the given Misiurewicz map admits no quasiconformal deformations. This is why the flexible Lattés maps have to be excluded, because they are the only Misiurewicz maps which admit quasiconformal deformations.

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It will be clear from the construction that in fact \( f \) (having \( J(f) = \hat{C} \)) can be perturbed into any post-critically finite map \( g \) with prescribed repelling orbits on which the critical points shall land on.

The parameter space \( \mathcal{R}^d \) of rational maps of degree \( d \) is a \( 2d + 1 \)-dimensional complex manifold. We will mostly consider normalised families of rational maps, which are \( \mathcal{R}^d \) modulo conjugacy classes of Möbius transformations. The space \( [\mathcal{R}^d] \) of normalised rational maps of degree \( d \geq 2 \) has dimension equal to \( 2d - 2 \).

Misiurewicz maps have good expansion properties, by a Theorem by Mañé [9]. In particular, they satisfy the so called Collet-Eckmann condition, defined as follows.

**Definition 1.2.** A rational map \( f \) satisfies the Collet-Eckmann condition (CE) if there are constants \( C > 0 \) and \( \gamma > 0 \) such that for every critical point \( c \in J(f) \), not containing any other critical point in its forward orbit, we have

\[
| (f^n)'(fc) | \geq C e^{\gamma n},
\]

for all \( n \geq 0 \).

Recently Rivera-Letelier [14] showed that one can perturb a so called backward contractive function to obtain a Misiurewicz map, provided the Julia set is not the whole Riemann sphere. The backward contraction condition is weaker than for example the Collet-Eckmann condition. Hence as a consequence of Rivera-Letelier and Theorem A (or rather [1]), every Collet-Eckmann map for which the Julia set is not the whole Riemann sphere can be approximated by a hyperbolic map. Combining this with [3] (revised version) we obtain the following characterization of rational Misiurewicz maps:

Let \( f \) be a rational Misiurewicz map. Then if \( f \) is not a flexible Lattés map it can be approximated by a hyperbolic map. Moreover,

- if \( J(f) = \hat{C} \), then \( f \) is a Lebesgue density point of CE-maps,
- if \( J(f) \neq \hat{C} \), then \( f \) is a Lebesgue density point of hyperbolic maps.

In the first case above, the CE-maps have their Julia set equal to the whole sphere (which is also a consequence of [3]). In view of the Fatou conjecture, flexible Lattés maps should also be approximable by hyperbolic maps but we have no proof. It seems this question is not much simpler than the Fatou conjecture itself.

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## 2. Preliminaries

Put \( f = f_0 \) and assume that \( f \) is a Misiurewicz map. We assume that \( J(f) = \hat{C} \) unless otherwise stated. The idea is to start with a family \( f_a \) of rational maps parameterized by a small disk \( B(0, r) \subset [\mathcal{R}^d] \), or radius \( r > 0 \), where \( f_0 \) corresponds
to the parameter \(a = 0 \in \mathbb{B}(0, r)\). We then study the iterates of the corresponding critical points for parameters in certain so called Whitney subdisks in \(\mathbb{B}(0, r)\) (see definition below). We will also consider 1-dimensional disks \(B(0, r) \subset \mathbb{B}(0, r)\), where \(\mathbb{B}(0, r)\) is \((2d - 2)\)-dimensional. Let \(c_j(a)\) be the set of critical points for \(f_a\). Define, for any \(a \in \mathbb{B}(0, r)\),
\[
\xi_{n,j}(a) = f^n_j(c_j(a)) = f^n(c_j(a), a).
\]

2.1. Higher order critical points. Of course some critical point might split under perturbation and in this case it is not clear how to define the functions \(c_j(a)\) in (1). In this case one cannot generally hope that the critical points move analytically. If higher order critical points occur we reparameterise the family using a theorem on the resolution of singularities, which goes back to Hironaka's theorem on the resolution of singularities [7]. In fact, what we are looking for is a complex analytic version of resolution of singularities, which goes back to Hironaka's theorem on the resolution of singularities [7]. Based on [4], it is shown that critical points move analytically in a "lifted parameter space" for a real polynomial where the coefficients are real-analytic. Since a direct reference to the complex analytic version of this result was not found, although it seems to be a wellknown folklore result, we outline the argument based on [13] and [4].

Put \(f'(z, a) = G(z, a)\) and consider the analytic set \(\{G(z, a) = 0\}\). This can be viewed as a real-analytic set of double dimension. The singularities of this set appear precisely at the points where \(f\) has a critical point of higher order. To resolve these singularities, we first write \(G = u + iv\), where \(u = u(x, y, a)\) and \(v = v(x, y, a)\) are real-analytic in \(x, y\) and \(a = (a_1, \ldots, a_{d-4})\). By the Resolution Theorem in [4] we get an analytic map \(\phi_u : (y_1, \ldots, y_n) \rightarrow (x, y, a_1, \ldots, a_{n-2})\), such that
\[
u \circ \phi_u = \varepsilon_u \prod_{i=1}^{n} y_i^{k_{i,u}},
\]
where \(\varepsilon_u\) is non zero and \(n = 4d - 2\).

The set \(\{u = 0\}\) has (real) codimension 1 and therefore there must be some \(k_{i,u} > 0\). If we restrict to the set \(B_i = \{y_i = 0\}\) then for \(y \in B_i\) we have \(u \circ \phi_u(y) = 0\). Let us identify \(B_i\) locally with the \(n - 1\) (real) dimensional ball \(\mathbb{B}^{n-1}\).

Now, consider the function \(F(y) = v(\phi_u(y))\), for \(y \in \mathbb{B}^{n-1}\). Using the Resolution Theorem again, we get an analytic function \(\phi_v : (x_1, \ldots, x_{n-1}) \rightarrow \mathbb{B}^{n-1}\) such that
\[
F \circ \phi_v = \varepsilon_v \prod_{i=1}^{n-1} x_i^{k_{i,v}}.
\]

Some \(k_{j,v} > 0\), so (locally) identifying the set \(B_j = \{x_j = 0\}\) with the unit ball \(\mathbb{B}^{n-2}\) (of real dimension \(n - 2\)) we get \(v(\phi_u(\phi_v(x))) = 0\) for \(x \in \mathbb{B}^{n-2}\). Since \(\phi_v(x) \in \mathbb{B}^{n-1}\) we have, for \(x \in \mathbb{B}^{n-2}\),
\[
\begin{align*}
\quad u \circ \phi_u \circ \phi_v(x) = 0 \quad & \\
v \circ \phi_u \circ \phi_v(x) = 0.
\end{align*}
\]

Put \(\phi = \phi_u \circ \phi_v\). Then we see that \(G \circ \phi(x) = 0\), for \(x \in \mathbb{B}^{n-2}\). Let \(\phi(x) = (\psi_0(x), \psi_1(x)) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}\). Then \(\psi_0(x)\) parameterises one critical point. For each choice of \(B_i, B_j\) above we get, in this way, a parameterisation of each critical point. For fixed \(x\), the function \(G(z, \psi_1(x))\) has a zero in \(c_1 = c_1(x)\) (which must be a
critical point for \( f(z, \psi_1(x)) \). If we view \( G \) as a complex function again, we can write
\[
G(z, \psi_1(x)) = (z - c_1(x))G_1(z, x),
\]
where \( z_1 \) and \( G_1 \) are real-analytic in \( x \) and \( G_1 \) is complex analytic in \( z \).

We apply the same argument to \( G_1 \) and obtain a new parameterisation (proper analytic map) \( \psi_2 : x' \to x \) such that
\[
G_1(z, \psi_2(x')) = (z - c_2(x'))G_2(z, x').
\]

Then
\[
G(z, \psi_1 \circ \psi_2(x')) = (z - c_1(\psi_2(x')))(z - c_2(x'))G_2(z, x').
\]

Continuing in the same manner we obtain a proper analytic map \( \psi \), which is a composition of finitely many proper analytic maps \( \psi_1, \psi_2, \ldots \) constructed above, such that
\[
G(z, \phi(a)) = E(z, a)(z - c_1(a)) \cdot \ldots \cdot (z - c_{2d-2}(a)),
\]
where \( E \) is non-vanishing and where \( c_j = c_j(a) \) are real-analytic, \( a \in \mathbb{B}^{n-2} \).

We have to prove that in fact \( c_j(a) \) are complex-analytic. Let \( K \) be the set of parameters in the original parameter space such that there are exactly \( 2d - 2 \) distinct critical points. Then since \( \psi \) is proper, \( K' = \psi^{-1}(K) \) has full measure in \( \mathbb{B}^{n-2} \). Since the critical points are simple in \( K \), we have \( 2d - 2 \) analytic functions \( c_j \) on \( \mathbb{B}^{n-2} \) which are analytic on \( K \) and continuous on \( \mathbb{B}^{n-2} \). Since \( K \) has full measure, the functions \( c_j \) can be continued analytically to the whole set \( \mathbb{B}^{n-2} \) by Rado’s Theorem (p. 301-302 in [5]). Hence the critical points \( c_j \) move analytically as functions of the new variables in \( \mathbb{B}^{n-2} \).

However, the new space \( \mathbb{B}^{n-2} \) is not necessarily normalised in the sense described in the Introduction; the function \( \psi \) need not be injective. This may cause some problems regarding transversality. We will deal with this in the section on Transversality.

2.2. Holomorphic motions and the parameter functions \( x_j \). Let us state the following theorems by Mañé [9]:

**Theorem 2.1** (Mañé’s Theorem I). Let \( f : \hat{C} \to \hat{C} \) be a rational map and \( \Lambda \subset J(f) \) a compact invariant set not containing critical points or parabolic points. Then either \( \Lambda \) is a hyperbolic set or \( \Lambda \cap \omega(c) \neq \emptyset \) for some recurrent critical point \( c \) of \( f \).

**Theorem 2.2** (Mañé’s Theorem II). If \( x \in J(f) \) is not a parabolic periodic point and does not intersect \( \omega(c) \) for some recurrent critical point \( c \), then for every \( \varepsilon > 0 \), there is a neighborhood \( U \) of \( x \) such that

- For all \( n \geq 0 \), every connected component of \( f^{-n}(U) \) has diameter \( \leq \varepsilon \).
- There exists \( N > 0 \) such that for all \( n \geq 0 \) and every connected component \( V \) of \( f^{-n}(U) \), the degree of \( f^n|_V \) is \( \leq N \).
- For all \( \varepsilon_1 > 0 \) there exists \( n_0 > 0 \), such that every connected component of \( f^{-n}(U) \), with \( n \geq n_0 \), has diameter \( \leq \varepsilon_1 \).

An alternative proof of Mañé’s Theorem can also be found by L. Tan and M. Shishikura in [16]. Let us also note that a corollary of Mañé’s Theorem II is that a Misiurewicz map cannot have any Siegel disks, Herman rings or Cremer points (see [9] or [16]).

Since \( f \) is a Misiurewicz map, there is some \( k \geq 0 \) such that the set
\[
P^k(f) = \bigcup_{n > k, c \in \text{Crit}(f) \cap J(f)} f^n(c)
\]
is compact, forward invariant and does not contain any critical or parabolic points. Let us put $\Lambda = P^k(f)$ for the smallest such $k$. By Mañé’s Theorem I it follows that $\Lambda$ is a hyperbolic set. Hence, there exists a holomorphic motion $h : \mathbb{B}(0, r) \times \Lambda \to \hat{\mathbb{C}}$, such that $h_a$ is an injection for each $a \in \mathbb{B}(0, r)$ and $h_a : \Lambda \to \Lambda_a$, where $\Lambda_0 = \Lambda$ and the following holds for $z \in \Lambda$:

$$f_a \circ h_a(z) = h_a \circ f_0(z).$$

Put $v_j(a) = f_a(c_j(a))$ for each marked critical point $c_j(a)$ and $v_j(0) = v_j$.

For $a \in \mathbb{B}(0, r)$, let us introduce the parameter functions $x_j$

$$x_j(a) = v_j(a) - h_a(v(0)).$$

Let us define $\mu_{n,j}(a) = h_a(f_0^n(v_j(0)))$.

**Definition 2.3.** Given a set $E$ and $\delta > 0$, we call the set \{ $x : \text{dist}(x, E) \leq \delta$, $\}$, a $\delta$-neighbourhood of $E$.

### 2.3. Some constants.

We define the constant $\lambda$ to the minimum over $|f'_a(z)|$ over all $(z, a) \in \Lambda_a \times \mathbb{B}(0, r)$. Since $\Lambda = \Lambda_0$ is hyperbolic, in general there is some $N > 0$ such that $|f^n(z, a)| \geq \lambda_0 > 1$ for all $(z, a) \in \Lambda_a \times \mathbb{B}(0, r)$ for some small $r > 0$, but for simplicity assume that $N = 1$ so that $\lambda > 1$.

Let $\mathcal{N}$ be a $\delta'$-neighbourhood around $\Lambda$, such that $|f'_a(z)| \geq \lambda/2$, for all $z \in \mathcal{N}$ and $a \in \mathbb{B}(0, r)$. Moreover, let $U$ be a $\delta$-neighbourhood around the critical points of $f$ so that $U \cap \mathcal{N} = \emptyset$ and $0 < \delta < 1/2$. Choose $r > 0$ such that every $c_j(a)$ belongs to a $\delta^0$-neighbourhood around $Crit(f_0)$ for every $a \in \mathbb{B}(0, r)$. Moreover, let $U_\delta$ be a $\delta_1$-neighbourhood around $Crit(f)$, for some $\delta_1 \leq \delta$. These $U_\delta$ will be defined inductively later.

### 3. Transversality

We will in this section study the functions $x_j(a)$. First, note that $x_j(0) = 0$ for all $j$. We cannot have $x_j(a) \equiv 0$ for all $j$ by Theorem E in [8] (see also Theorem A in [2]), because then all maps in $\mathbb{B}(0, r)$ would be Misiurewicz maps. Hence we may assume that at least one $j$ has that $x_j(a)$ is not identically equal to zero, i.e. $x_j$ has finite order contact (see definition below). In $\mathbb{B}(0, r)$ the function $x_j$ is an analytic function in several variables and has a power series expansion

$$x_j(a) = \sum_{|\alpha| \geq 1} b_\alpha a^\alpha$$

where $\alpha = (\alpha_1, \ldots, \alpha_{2d-2})$ is a multi-index, $\alpha_j \geq 0$ and $|\alpha| = \sum \alpha_j$.

**Definition 3.1.** If $B(0, r)$ is a 1-dimensional disk we say that the critical point $c_j(a)$ has contact of order $k$ if

$$x_j(a) = v_j(a) - h_a(v(0)) = K_1 a^K + \ldots,$$

for some $K_1 \neq 0$. If $x_j(a)$ is identically equal to zero we say that $c_j(a)$ has contact of infinite order in $B(0, r)$.

If we consider $x_j(a)$ as a function of $a \in \mathbb{B} \subset \mathbb{B}(0, r)$ (where $\mathbb{B}$ is an open set of dimension $\leq 2d - 2$), then we say that $c_j$ has finite order contact in $\mathbb{B}$ if $x_j$ is not identically equal to zero in $\mathbb{B}$. 
If it is evident in which set \( x_j \) has finite order contact in, we just say \( x_j \) has finite order contact. Dropping the index and writing \( x(a), v(a) \) we mean \( x_j(a), v_j(a) \) respectively for some index \( j \). This index is chosen so that \( x_j(a) \) has finite order contact unless otherwise stated. Also, we write \( v_a = v(a) \).

3.1. Tangent cones. We want to restrict to parameters where \( x(a) \neq 0 \) and \( x'(a) \neq 0 \). To this end, we construct a cone-like set of the form \( V_0 \times \mathbb{C} \), where \( V_0 \subset \mathbb{P}(\mathbb{C}^{2d-3}) \) is an open ball of directions, such that \( x'(a) \neq 0 \) and \( x(a) \neq 0 \) for all \( a \in V_0 \times B(0,r) \setminus \{0\} \) and for all \( x(a) = x_i(a) \) which have finite order contact in \( B(0,r) \) (here \( B(0,r) \subset \mathbb{C} \)). Let us assume that \( x_i(a) \) has finite order contact for the set of indices \( i \in I \). Each equation from the the set of equations

\[
\begin{align*}
x'_i(a) &= 0, & i &\in I \\
x_i(a) &= 0, & i &\in I
\end{align*}
\]

defines an analytic set. Let us number these analytic sets as \( A_1, A_2, \ldots, A_n \). We now use the standard theory of analytic sets (see e.g. [5]); if \( A \) is an analytic set, then the tangent cone \( C(A,0) \subset \mathbb{C}^{2d-2} \) to \( A \) at 0 is defined as set of vectors \( v \in \mathbb{C}^{2d-2} \) such that there exists \( a_j \to 0, a_j \in A \) and real numbers \( t_j > 0 \) such that \( t_j a_j \to v \) as \( j \to \infty \). By [5] p.83, the set \( C(A,0) \) is an algebraic subset in \( \mathbb{C}^{2d-2} \). Let \( p(v) \) be the projection of \( \mathbb{C}^{2d-2} \) onto \( \mathbb{P}(\mathbb{C}^{2d-3}) \). Now, applying this to the sets \( A_j \), we see that since \( p(C(A_j,0)) \neq \mathbb{P}(\mathbb{C}^{2d-3}) \) and \( C(A_j,0) \) is closed there must be an open ball \( V_0 \subset p(\cap_j C(A_j,0))^c \).

Moreover, any 1-dimensional disk \( B(0,r) \subset \mathbb{B}(0,r) \) is determined by a direction vector \( v \in \mathbb{P}(\mathbb{C}^{2d-3}) \). Let us pick a for \( v \) representative direction vector \( v_0 = (\alpha_1, \alpha_2, \ldots, \alpha_{2d-2}) \) (i.e. so that \( v_0 \) has direction \( v \)). Then the plane in which \( B(0,r) \) lies can be parameterised by

\[
\{a \in \mathbb{C}^{2d-2} : a = (\alpha_1 t, \alpha_2 t, \ldots, \alpha_{2d-2} t), t \in \mathbb{C}\}.
\]

The expansion of \( \theta \) in \( B(0,r) \), where \( \theta \) is either some \( x_j(a) \) or a component of \( x'_j(a) \), becomes

\[
\theta(t) = \sum_{k \geq k_0} p_k t^k + \ldots,
\]

where each \( p_k = p_k(\alpha_1, \ldots, \alpha_{2d-2}) \) is a polynomial in the variables \( (\alpha_1, \ldots, \alpha_{2d-2}) \). By Proposition 1 p. 83 in [5], we have \( \{p_{k_0} = 0\} = C(A_j,0) \). Since \( \cap_j C(A_j,0) \supset C(\cap_j A_j,0) \) we have that for any 1-dimensional disk \( B(0,r) \) with direction \( v \in V_0 \), the expansion of any \( x_j \) or component of \( x'_j \) becomes

\[
\theta(t) = K_1 t^k + \ldots,
\]

where \( K_1 \) is continuous function of the direction and \( k \) is constant for all \( v \in V_0 \). We say that \( k \) is the order of \( f \) in the cone \( W = V_0 \times B(0,r) \). This \( W \) is the starting good cone where the \( x_j \) behaves nicely. The importance of \( W \) is that \( x_j \) has bounded distortion on so called dyadic disks defined as follows.

**Definition 3.2.** A disk \( D \subset \mathbb{R}^n \) of dimension \( \leq n \) is called a \( k \)-Whitney disk or simply a Whitney disk if

\[
diam(D) \geq k \text{dist}(D,0).
\]
We say that \( x' \) has bounded distortion in some set \( D \) if
\[
\|x'(a) - x'(b)\| \leq \varepsilon \|x'(a)\|,
\]
for all \( a, b \in D \), for some small (fixed) \( \varepsilon > 0 \).

By compactness of \( V_1 \) the function \( K_1 \) is uniformly continuous in \( V_1 \). The following lemma now follows easily from (5).

**Lemma 3.3.** There is some \( 0 < k < 1 \) such that every \( x_j(a) \) and every component
of \( x'_j(a) \) has bounded distortion on \( k \)-Whitney disks in \( W \).

Moreover, we will use the following folklore result which follows from [17] by S. van Strien (see also [6] by A. Epstein). In [17] the definition of Misiurewicz maps differs. However, in the case where the Julia set is the whole sphere then our definition coincides with that of van Strien.

**Theorem 3.4.** If \( B(0, r) \) is a \( 2d - 2 \) dimensional ball of normalised rational maps, then the function \( G : a \rightarrow (x_1(a), \ldots, x_n(a)) \) is a local immersion unless \( f \) is a flexible Lattés map.

Hence if \( f \) is not a flexible Lattés map, the tangent vectors \( x'_j(0) \) are all linearly independent. By continuity this also holds in the some small parameter ball \( B(0, r) \).

If critical points of higher order exist for \( f \), then we change to the new variables described in Subsection 2.1. Although it is clear that in this new space \( B^{n-2}, a = 0 \) is still an isolated zero of \( G \) (because \( \psi \) is proper), the new space may not be normalised.

In this case we will perturb our original function \( f \) into a new Misiurewicz map, for which every critical point is simple, and still the Julia set is the whole sphere. To do this, let \( c \) be a higher order critical point and assume that \( c = c_1(0) = \ldots = c_N(0) \), where the functions \( c_j \) are analytic in \( B^{n-2}, 1 < N \leq 2d - 2 \) (i.e. \( c \) has order \( N \)).

For \( 1 \leq i \leq N \), consider the set \( Z_i = \{ a \in B^{n-2} : x_j(a) = 0, \text{for all } j \neq i \} \). By Lemma 2.3 in [17], there is a locally univalent parameterisation of \( Z_i \):

\[
\Psi_i : C \rightarrow Z_i,
\]

where \( \Psi_i(0) = 0 \in B^{n-2} \). Following [17] (Lemma 3.1) we note that there is some periodic repelling point \( q_i(\lambda) \) arbitrarily close to the critical value \( v_i(0) = f^{k+1}(c_i(0), 0) \in \Lambda \). For \( \lambda \in \mathbb{C} \), put

\[
X_i(\lambda) = f^{k+1}_{\Psi(\lambda)}(c_i(\lambda)) - q_i(\lambda).
\]

We now note note that if \( (x_i \circ \Psi_i)'(0) \neq 0 \), for all \( i \), then the tangent vectors \( x'_i(0) \) are linearly independent and \( G \) is an immersion as in Theorem 3.4. In this case we do not need to change our function \( f \).

Hence suppose that \( (x_i \circ \Psi_i)'(0) = 0 \). That means that \( (x_i \circ \Psi_i)(\lambda) = g(\lambda) \lambda^d \), where \( d \geq 2 \) and \( g(\lambda) \neq 0 \) in a neighbourhood \( B(0, \delta) \) of \( 0 \). We claim that \( X_i \circ \Psi_i \) has \( d \) zeros in \( B(0, \delta) \) if \( q_i = q_i(0) \) is chosen sufficiently close to \( v_i(0) \). The proof of this claim is identical to that of [17], pp. 46-48. With \( U_i(\lambda) = q_i(\lambda) - v_i(\lambda) \) we get

\[
t^d g(t) + U_i \circ \Psi_i(t) = X_i \circ \Psi_i(t).
\]

Hence if \( u_i = U_i \circ \Psi_i \) is sufficiently close to zero (i.e. \( q_i \) is sufficiently close to \( v_i \)), the image of \( \partial B(0, \delta) \) under the function \( t^d g(t) / u(t) \) is a curve that encircles \( -1 \) \( d \) times. By the Argument Principle there is at least one solution \( t_0 \in B(0, \delta) \) to

\[
t^d g(t) / u(t) = -1.
\]
For this \( t_0 \), the function \( f_{\Psi(t_0)} \) is a new Misiurewicz map different from \( f_0 \) and where \( c_i(\Psi(t_0)) \) is simple. The other critical points \( c_j(\Psi(t_0)) \), \( j \neq i \) still coincide and form a critical point of order \( N - 1 \).

Repeating this argument \( N - 1 \) times we obtain a Misiurewicz map such that all critical points are simple.

3.2. Outline of proof of Theorem A. To prove the main result we take a full-dimensional Whitney parameter disk \( \mathbb{B}_0 \subset W \). Suppose that \( c_1 \) has finite order contact in \( \mathbb{B}_0 \). We first show that the set \( \xi_{n,1}(\mathbb{B}_0) \) grows up to some definite size before it leaves \( \mathcal{N} \) (Lemma 4.6). By bounded distortion the set \( \xi_{n,1}(\mathbb{B}_0) \) will contain a disk of diameter at least \( S = S_0 \), where \( S \) is some “large scale”.

By normality and compactness we show that for some \( m \), \( \xi_{n+m,1}(B(0,r)) \) covers the whole Riemann sphere apart from at most 2 points. In particular, \( c_1(a) \) and two of its pre-images cannot be omitted by \( \xi_{n+m,1}(B(0,r)) \). Hence \( c_1(B(0,r)) \subset \xi_{n+m,1}(B(0,r)) \) and there is a solution to \( \xi_{n+m,1}(a) - c_1(a) = 0 \) in \( B(0,r) \subset \mathbb{B}(0,r) \).

Next we will pass to the analytic subset \( \mathbb{B}_0 \subset W \) where \( \xi_{n+m,1}(a) - c_1(a) = 0 \), and argue inductively.

In the next step, we consider \( c_2 \), which also has to have finite order contact. In fact, assume that there are no more critical points than \( c_1 \) of finite order contact. Then there would be a small ball \( \mathbb{B} \) around the solution \( a \) to \( \xi_{n+m,1}(a) - c_1(a) = 0 \), where \( \mathbb{B} \) is a family of Misiurewicz maps. This is impossible by [2].

Now, we try to connect this \( c_2 \) with itself also in the same way as we did with \( c_1 \). However, to continue we need to have control of the shape and size of \( \mathbb{B}_1 \). These results are mainly dealt with in the sections 4 and 5. We show that we have good control of the geometry of \( \mathbb{B}_1 \) in Whitney disks, so that the parameter function \( x_2 \) maps \( \mathbb{B}_1 \) onto small circles, so that \( \xi_{n,2}(\mathbb{B}_1) \) in turn grows to another large scale \( S_1 \) (which is typically less than \( S_0 \)). Then, as in the previous case for \( c_1 \), we can use non-normality and compactness again to get another \( N_1 \) for which \( \xi_{n+m,2}(a) - c_2(a) = 0 \), for some \( a \in \mathbb{B}_1 \) and \( m \leq N_1 \). A new manifold \( \mathbb{B}_2 \subset \mathbb{B}_1 \) is thereby formed, where \( \xi_{n+m,2}(a) - c_2(a) = 0 \). We continue in this manner until all critical points are in the Fatou set.

4. Distortion lemmas

In this section we state the necessary distortion lemmas that will be needed to get a Whitney parameter disk to grow to the large scale before it leaves \( \mathcal{N} \). Many lemmas in this section are proven in [2] (and also in [3]) in a one-dimensional version. In this section assume always that \( c = c_j \) is transversal, i.e. \( x(a) = x_j(a) = c_j(a) - v_j(a) \) is not identically equal to zero. Recall the notation \( v_j(a) = v_a \) for the critical values.

Let us start with the following lemma (see [15]).

**Lemma 4.1.** Let \( u_n \in \mathbb{C} \) be complex numbers for \( 1 \leq n \leq N \). Then

\[
(8) \quad \left| \prod_{n=1}^{N} (1 + u_n) - 1 \right| \leq \exp\left( \sum_{n=1}^{N} |u_n| \right) - 1.
\]

The following lemma is a modified version of Lemma 3.2 in [2].

**Lemma 4.2 (Main Distortion Lemma).** For each \( \varepsilon > 0 \) there exists an \( r > 0 \) and \( \delta' > 0 \) such that the following holds. Let \( a, b \in W \). Then as long as \( f_a^j(v_a), f_b^j(v_b) \in \mathcal{N} \)
(recall that $N$ depends on $\delta'$), for all $0 \leq j \leq n$ we have
\[
\left| \frac{(f^n_a)'(v_0)}{(f^n_b)'(v_0)} - 1 \right| \leq \varepsilon.
\]

The same statement holds if one replaces $v(s) = \xi_0(s), s = a, b$, by $\mu_0(t), t = a, b$.

**Proof.** The proof goes in two steps. Let us first show that
\[
\left| \frac{(f^n_a)'(\mu_0(t))}{(f^n_b)'(\xi_0(t))} - 1 \right| \leq \varepsilon_1,
\]
where $\varepsilon_1 = \varepsilon(\delta')$ is close to 0. We have
\[
\sum_{j=0}^{n-1} \left| \frac{f'_j(\mu_j(t)) - f'_j(\xi_j(t))}{f'_j(\xi_j(t))} \right| \leq C \sum_{j=0}^{n-1} |f'_j(\mu_j(t)) - f'_j(\xi_j(t))| \leq C \sum_{j=0}^{n-1} |\mu_j(t) - \xi_j(t)| \leq C \sum_{j=0}^{n-1} \lambda^{j-n} |\mu_n(t) - \xi_n(t)| \leq C(\delta'),
\]
where we used the hyperbolicity of the hyperbolic set $\Lambda_{t}$. By Lemma 4.1, (9) holds of $\delta'$ is small enough. Secondly, we show that
\[
\left| \frac{(f^n_a)'(\mu_0(t))}{(f^n_b)'(\mu_0(s))} - 1 \right| \leq \varepsilon_2,
\]
where $\varepsilon_2 = \varepsilon_2(\delta')$ is close to 0. Put $\lambda_{t,j} = f'_j(\mu_j(t))$. Since $\lambda_{t,j}$ are all analytic in $t$ we have
\[
\lambda_{t,j} = \lambda_{0,j}(1 + \sum_{|\alpha| \geq k_j} c_{\alpha,j} t^\alpha),
\]
where $t = t_1 \cdots t_{2d-2}, \alpha$ is a multi-index and $k_j \geq 1$. Moreover, the condition $R^{j}_a(v_a) \in N$ implies that $n \leq -C \log ||x|| \leq -C' \log ||t||$ for some constant $C'$, where $||t||$ is the norm of $t$ viewed as a vector in $C^{2d-2}$. We have
\[
\frac{(f^n_a)'(\mu_0(t))}{(f^n_b)'(\mu_0(s))} = \prod_{j=0}^{n-1} \frac{\lambda_{t,j}}{\lambda_{s,j}} = \prod_{j=0}^{n-1} \left(1 + \sum_{|\alpha|=k_j} c_{\alpha,j} t^\alpha + \ldots\right) = \prod_{j=0}^{n-1} \left(1 + \sum_{|\alpha|=k_j} c_{\alpha,j} s^\alpha + \ldots\right).
\]
Both the last numerator and denominator in the above equation can be estimated by $1 + C''n ||t||^l$ and $1 + C''n ||s||^l$ respectively, for some constant $C''$ and integer $l \geq 1$. Since $n \leq -C' \log ||t||$ the numerator and denominator are bounded by $1 + O((\log ||t||)||t||^l)$ and $1 + O((\log ||s||)||s||^l)$ respectively, which both can be made arbitrarily close to 1 if $r > 0$ is small enough. From this the lemma follows. \hfill $\square$

We reformulate Lemma 3.3 in [2] in the following vector form.

**Lemma 4.3.** Let $\varepsilon > 0$. If $\delta' > 0$ is sufficiently small, then for every $0 < \delta'' < \delta'$ there exists $r > 0$ such that the following holds. Let $a \in W$ and assume that $\xi_0(a) \in N$, for all $k \leq n$ and $|\xi_n(a) - \mu_n(a)| \geq \delta''$. Then
\[
\|\xi_n'(a) - (f^n_a)'(\mu_0(a)) x'(a)\| \leq \varepsilon \|\xi_n'(a)\|.
\]
Proposition 4.4. Let \( \varepsilon > 0 \). If \( \delta' > 0 \) is sufficiently small, then for every \( 0 < \delta'' < \delta' \), there exists \( r > 0 \) such that the following holds. Let \( a \in W \) and assume that \( \xi_k(a) \in \mathcal{N} \), for all \( k \leq n \) and \( |\xi_k(a) - \mu_n(a)| \geq \delta'' \). Then

\[
\|\xi'_n(a) - (f''_n)^{\prime}(v(a))x'(a)\| \leq \varepsilon\|\xi'_n(a)\|.
\]

More generally, we have a higher-dimensional form of Proposition 4.3 in [3]. Put

\[
Q_n = Q_{n,t}(a) = \xi'_n(a)/(f''_n^{\prime})(v_t(a), a).
\]
Proposition 4.5. For every $\delta > 0$ and sufficiently small $\delta'' > 0$ there is an $r > 0$ such that the following holds. Assume that the parameter $a \neq 0$, $a \in W$, satisfies $\text{dist}(\xi_n(a), \text{Crit}(f_a)) \geq \delta$ for all $n \leq m$. Moreover, assume that $\delta'' \leq |\xi_N(a) - \mu_N(a)| \leq \delta'$ and

\begin{align}
\tag{13} |(f^n)'(\xi_N(a), a)| & \geq C e^n, \quad \text{for } n = 0, \ldots, m, \\
\tag{14} |\partial_a f(\xi_n(a), a)| & \leq B, \quad \forall n > 0, 
\end{align}

where $\gamma \geq (3/4) \log \lambda$ (see Subsection 2.3 for definition of $\lambda$). Then we have, for $n = N, \ldots, m$,

\begin{equation}
\tag{15} ||Q_n(a) - Q_N(a)|| \leq ||Q_N(a)||/1000.
\end{equation}

Proof. First, we prove by induction, that

\begin{equation}
\tag{16} ||\xi_{N+k}(a)|| \geq e^{\gamma'(N+k)},
\end{equation}

where $\gamma' = \min(\log \lambda/(2K), \gamma/2)$ and $K$ is the order of $x(a)$ in $W$. From (3) it is straightforward to show that

$$ ||x'(a)|| \geq C|x(a)|^{\frac{K-1}{K}}, $$

for some constant $C$. Then we get

\begin{equation}
\tag{17} ||\xi_N(a)|| \geq (1/2)|f_N'(v(a))||x'(a)|| \geq (C/2)|f_N'(v(a))||x(a)|^{\frac{K-1}{K}}.
\end{equation}

By Lemma 4.2 and since $\delta'' \leq |\xi_N(a) - \mu_N(a)| \leq \delta'$, we get

$$ ||f_N'(v(a))||x(a)|| \geq \delta''/2. $$

If the perturbation $r > 0$ is chosen sufficiently small, so that $N$ becomes sufficiently large, we deduce that

$$ ||\xi_N'(a)|| \geq (C/2)|f_N'(v(a))||x(a)|^{\frac{K-1}{K}} \geq (C\delta''/4)|f_N'(v(a))|^{1/K} \geq e^{\gamma'}N, $$

where $\gamma' \geq \min(\log \lambda/(2K), \gamma/2)$.

So, assume that

$$ ||\xi_{N+j}'(a)|| \geq e^{\gamma'(N+j)}, \text{ for all } j \leq k. $$

We want to prove that

$$ ||\xi_{N+j}'(a)|| \geq e^{\gamma'(N+j)}, \text{ for all } j \leq k + 1. $$

First note that the assumption $\text{dist}(\xi_n(a), \text{Crit}(f_a)) \geq \delta$, with $\delta = e^{-\Delta}$, implies

\begin{equation}
\tag{18} |f'(\xi_j(a), a)| \geq C_1^{-1} e^{-\Delta K},
\end{equation}

for some $C_1 > 0$. By the Chain Rule we have the recursions

\begin{align}
\tag{19} \frac{\partial f^{n+1}(v(a), a)}{\partial z} &= \frac{\partial f(\xi_n(a), a)}{\partial z} \frac{\partial f^n(v(a), a)}{\partial z}, \\
\tag{20} \frac{\partial f^{n+1}(v(a), a)}{\partial a} &= \frac{\partial f(\xi_n(a), a)}{\partial a} \frac{\partial f^n(v(a), a)}{\partial a} + \frac{\partial f(\xi_n(a), a)}{\partial a}. 
\end{align}
Now, the recursion formulas (19) and (20), together with (18) gives

$$\|\xi'_{N+k+1}(a)\| \geq f'_a(\xi_{N+k}(a))\|\xi'_{N+k}(a)\| \left(1 - \frac{\|\partial_a f_a(\xi_{N+k}(a))\|}{\|f'_a(\xi_{N+k}(a))\|\|\xi'_{N+k}(a)\|}\right)$$

$$\geq \|(f^{k+1}_a)'(\xi_{N}(a))\|\|\xi'_{N}(a)\| \prod_{j=0}^{k} \left(1 - \frac{\|\partial_a f_a(\xi_{N+j}(a))\|}{\|f'_a(\xi_{N+j}(a))\|\|\xi'_{N+j}(a)\|}\right)$$

$$\geq e^{\gamma(k+1)} e^{-\gamma N} \prod_{j=0}^{k} (1 - B'e^{K\Delta} e^{-\gamma(N+j)}) = e^{\gamma(N+k+1)},$$

if $N$ is large enough, since $\gamma' \geq 2\kappa$, (here $B' = BC_1$). Hence (16) follows.

To continue the proof, first note that

$$\|\xi'_{N+k}(a) - \xi'_N(a)\| \leq \sum_{j=0}^{N+k} \|\partial_a f_a(\xi_j(a))\| \prod_{i=j}^{N+k} |f'_a(\xi_{i+1}(a))||.$$  \hfill (21)

Let us put $\lambda_n = f'_a(\xi_n)$, $\mu_n = \partial_a f_a(\xi_n(a))$ and $\xi'_N(a) = \xi'_N$. The first term in (21) is

$$\|\mu_N\| \prod_{i=N+1}^{N+k-1} |\lambda_i| \leq \frac{\|\mu_N\|}{|\lambda_N||\xi'_N|} \prod_{i=N}^{N+k-1} |\lambda_i||\xi'_N| \leq B'e^{\Delta K} e^{-\gamma N} \prod_{i=N}^{N+k-1} |\lambda_i||\xi'_N|.$$  \hfill (22)

Put $C = B'e^{\Delta K}$. The $n$th term in (21) becomes

$$\|\mu_{N+n} \| \prod_{i=N+n}^{N+k-1} |\lambda_i| = \frac{\|\mu_{N+n}\|}{|\lambda_{N+n-1}|\xi'_{N+n-1}|} \prod_{i=N+n-1}^{N+k-1} |\lambda_i||\xi'_{N+n-1}|$$

$$\leq Ce^{-\gamma'N+n-1} \prod_{i=N+n-1}^{N+k-1} |\lambda_i| \left(\prod_{j=N}^{N+n-2} |\lambda_i||\xi'_N| + \sum_{j=N}^{N+n-2} \|\mu_j\| \prod_{i=j+1}^{N+n-2} |\lambda_i|\right)$$

$$\leq Ce^{-\gamma'(N+n-1)} \prod_{i=N}^{N+k-1} |\lambda_i||\xi'_N| + Ce^{-\gamma'(N+n-1)} \sum_{j=N}^{N+n-2} \|\mu_j\| \prod_{i=j+1}^{N+n-2} |\lambda_i|.$$  \hfill (23)

The last sum is the sum of the first $n - 1$ terms in (21). As induction assumption the first $n - 1$ terms in (21) is less than 1. This means that the $n$th term is at most $Cne^{-\gamma'(N+n-1)}$, which is again of course less than 1 if $N$ is chosen sufficiently big.
We get finally
\[
\|\xi_{N+k}(a) - \xi_N(a)\| \leq \frac{Cn}{N+1} \gamma(N+n-1) \|\xi_N(a)\| \prod_{j=n}^{N+k} |f'_{a}(\xi_j(a))|.
\]
So, if \( N \) is big enough,
\[
\|Q_{N+n}(a) - Q_N(a)\| \leq \|Q_N(a)\|/1000.
\]

From Proposition 4.5 and 4.4 we see that the space derivative and parameter derivative are comparable up to a multiplicative quantity, namely \( x'(a) = x'_j(a) \) for some \( j \). However, since \( x'(a) \) generally is not constant, we want to restrict the parameters such that \( x'(a) \) does not vary much. To this end it is naturally to restrict to sets where (7) holds.

If \( x'(t) \) is not constant, then by Lemma 3.3 the set of parameters which satisfies the condition (7) contains a \( k \)-Whitney disk \( D_0 \subset W \) for some \( 0 < k < 1 \) only depending on the function \( x \).

We need to know that a Whitney (parametric) disk grows to the large scale before it leaves \( N \). This is the content of the following lemma. Let the angle between vectors \( x \) and \( y \) be \( \arccos(x \cdot y/\|x\|\|y\|) \in [0, \pi] \). Given two hyper planes \( H_1 \) and \( H_2 \) we say that the angle \( \theta \in [0, \pi] \) between them is defined by
\[
\cos \theta = \inf_{x \in H_1} \sup_{y \in H_2} \frac{x \cdot y}{\|x\|\|y\|}.
\]
Then we can also talk about angles between Whitney disks and hyper planes since every Whitney disk is contained in some unique hyper plane (with dimension equal to the dimension of the Whitney disk). Moreover, if \( D_0 \) is a Whitney disk and \( H \) a hyperplane, then the diameter of the orthogonal projection of \( D_0 \) onto \( H \) is \( \text{diam}(D_0) \cos(\theta) \), where \( \theta \) is the angle between \( D_0 \) and \( H \).

**Lemma 4.6.** If \( r > 0 \) is sufficiently small, there exists a number \( 0 < k < 1 \) only depending on the function \( x \) such that the following holds. Let \( D_0 = B(a_0, r_0) \subset W \) be a \( k \)-Whitney disk (\( D_0 \) has dimension \( \leq 2d - 2 \)) such that the angle \( \theta \) between \( D_0 \) and \( x'(a_0) \) is \( \theta \neq \pi/2 \): There is an \( n > 0 \) and a number \( S = S(\delta', \theta) \), such that the set \( \xi_n(D_0) \subset N \) and has diameter at least \( S \). Moreover, we have low argument distortion, i.e.
\[
\|\xi'_k(a) - \xi'_k(b)\| \leq \|\xi'_k(a)\|/100,
\]
for all \( a, b \in D_0 \) and all \( k \leq n \).

**Proof.** Choose \( n \) maximal such that \( \xi_k(a_0) \in N \) for all \( k \leq n \) and
\[
(\delta' + \delta'')/(2M_0) \leq \|\xi_n(a_0) - \mu_n(a_0)\| \leq (\delta' + \delta'')/2,
\]
where \( M_0 \) is the supremum of \( |f'_a(z)| \) over all \( a \in B(0, r) \) and \( z \in \mathcal{C} \).

Proposition 4.4 holds for all \( a \in W \) satisfying
\[
(23) \quad \delta'' \leq \|\xi_n(a) - \mu_n(a)\| \leq \delta'.
\]
Since \( x'(a) \) has bounded distortion on Whitney disks by Lemma 3.3, for parameters \( a, b \in D_0 \) satisfying (23) we have good control of the geometry:
\[
\|\xi'_n(a) - \xi'_n(b)\| \leq \|\xi'_n(a)\|/100.
\]
Hence, \( \xi_n \) is almost linear in \( D_0 = \mathbb{B}(a_0, r_0) \) if (23) is satisfied.

Assuming that (23) holds for all \( a \in D_0 \), we want to estimate the diameter \( d \) of the set \( \xi_n(D_0) \). It is, up to very low distortion, precisely the orthogonal projection of \( \xi_n(a_0) \) onto the hyperplane where \( D_0 \) lies. The diameter \( d \) can be estimated by

\[
 d \geq \|\xi_n'(a_0)\|\text{diam}(D_0)\cos(\theta) \geq (1/2)\|f^n_a\|(\mu_0(a_0))\|x'(a_0)\|k_0a_0\cos(\theta),
\]

where we also used Proposition 4.4. If the Whitney disk \( D_0 \) is too large such that (23) is not fulfilled, then \( \xi_n(D_0) \) fails to be a subset of

\[
 A(\delta'', \delta', \mu_0(a_0)) = \{z : \delta'' \leq |z - \mu_0(a_0)| \leq \delta'\}
\]

and we may have to diminish \( r_0 \). However, with \( r_0 = k_0\|a_0\| \) we can choose \( \delta'' > 0 \) sufficiently small so that at least if \( k_0 \leq 1/2 \), then \( \xi_n(D_0) \subset A(\delta'', \delta', \mu_0(a_0)) \).

By Lemma 4.2,

\[
 (\delta'' + \delta')/(2M_0) \leq |\xi_n(a_0) - \mu_n(a_0)| \leq 2\|f^n_a\|(\mu_0(a_0))|x(a_0)|.
\]

Thus,

\[
\frac{d}{\delta'} \geq C \frac{\|f^n_a\|(\mu_0(a_0))}{\|f^n_a\|}(\mu_0(a_0)) \frac{|x'(a_0)|}{|x(a_0)|}.
\]

The number \( \|k_0a_0\|/|x(a_0)| \) is bounded from below in \( W \). Hence there is some constant \( C' \) so that \( d/\delta' \geq C' \). So the diameter \( d \) of the set \( \xi_n(D_0) \) is greater than some \( S = S(\delta', \theta) \). Also, by (24), we have bounded argument distortion for all \( a, b \in D_0 \).

Hence, a Whitney disk \( D_0 \subset W \) will grow to size \( S \) under the map \( \xi_n \) before \( \xi_n(D_0) \) leaves \( \mathcal{N} \). At the same time we have strong control over the distortion up to the scale \( S \). Let us formalize and say that we have strong distortion estimates in \( D_0 \) up to time \( n \) if

\[
(25) \quad \|\xi_n'(a) - \xi_k'(b)\| \leq \|\xi_n'(a)\|/100,
\]

holds for all \( a, b \in D_0 \) and for all \( k \leq n \). If it is clear what \( n \) is, we just say strong distortion estimates in \( D_0 \).

Finally we will use the following distortion lemma for the so called free period, i.e. when \( \xi_n(E) \) has left \( \mathcal{N} \), for some set \( E \). The following follows directly.

**Lemma 4.7** (Extended Distortion Lemma). Let \( N \in \mathbb{N} \). For any \( \varepsilon > 0 \) and neighbourhood \( U \) of \( \text{Crit}(f_0) \), there exists an \( r > 0 \) and \( S' > 0 \) such that the following holds. Let \( a, b \in \mathbb{B}(0, r) \) and assume that \( z, w \in \mathcal{N} \) are such that \( f^k(z, a), f^k(w, b) \notin U \) and \( |f^k(z, a) - f^k(w, b)| \leq S' \) for all \( k = 0, \ldots, n \), where \( n \leq N \). Then

\[
\frac{|(f^n)'(z, a)|}{|(f^n)'(w, b)|} - 1 < \varepsilon.
\]

The bound \( N \) will come from the following lemma.

**Lemma 4.8.** There exists an \( r > 0 \) such that the following holds. Fix \( d > 0 \) and let \( \mathcal{S} \) be a family of disks with diameter \( d \) which cover the Julia set \( J(f) \) of the starting function \( f_0 \) and such that each disk \( S \in \mathcal{S} \) is centered at a point in \( J(f) \). Then there exists some constant \( N \) such that

\[
(26) \quad \inf\{m : f^m_0(S) \supset U\} \leq N,
\]

for every disk \( S \in \mathcal{S} \).
such that

\( S \) for all \( F \) for which \( B \). For each \( z \) there is some neighbourhood for which \( N(z) \) is constant. Since \( J(f) \) is compact there is a constant \( N \) such that

\[
\inf\{m : f^m(S) \supseteq \overline{U}\} \leq N,
\]

for any \( S \in \mathcal{S} \). The lemma follows. \( \square \)

Arrange the disks in the family \( \mathcal{S} \) so that any disk \( D \) of diameter \( d \) for which there exists a point \( z \in D \cap J(f) \neq \emptyset \) such that \( \text{dist}(z, \partial D) \geq d/4 \), there exists some \( S \in \mathcal{S} \) such that \( S \subset D \).

5. Closing the critical orbits

Although we have shown that we have strong distortion estimates on small Whitney disks, we start with a full-dimensional disk \( B_0 \subset W \). By Lemma 4.6, \( \xi_n(B_0) \) grows to some large scale size \( S = S_0 \) under strong distortion estimates, for some \( n > 0 \).

Assume that \( x_1(a) \) has finite order contact and assume that we have found a solution \( \xi_{n+1}(a_0) = c_1(a_0) \) for some \( a_0 \in B_0 \). Let \( B_1 \) be the connected component of the set \( \{a \in W : \xi_{n+1}(a) = c_1(a)\} \) containing \( a_0 \). In order to get good geometry control of this manifold, we need to restrict to the set \( B_1 = D \cap B_1 \), where \( D \) is a Whitney disk.

A proof of the following general result can be found in [11] p. 11, for instance.

**Lemma 5.1.** Given an analytic function \( F \) from \( \mathbb{C}^n \) to \( \mathbb{C} \), where \( F(z_0) = w_0 \). Then a relatively open subset \( E \subset F^{-1}(w_0) \) is a submanifold if for all \( z \in E \) we have \( F'(z) \neq 0 \).

Hence, set of parameters \( a \in W \) satisfying \( F_1(a) = \xi_{n+1}(a) - c_1(a) = 0 \) is a submanifold, apart from a set of singularities. In the next lemma we deal the problem of singularities.

**Lemma 5.2.** Assume that \( A \subset W \) is a connected manifold, such that \( \xi_{k,l}(A) \in \mathcal{N} \), for all \( k \leq n \). Assume that \( F_i(a) = \xi_{m+n,l}(a) - c_l(a) = 0 \) for some \( a \in A \) and that \( \xi_{k,l}(a) \cap U_l = \emptyset \), for all \( k \leq n + m - 1 \), where \( U_l \) is a \( \delta_l \)-neighbourhood of \( \text{Crit}(f) \), \( \delta_l \leq \delta \). Assume moreover that \( U_l \) has the property that the first return time into itself is at least \( 2m \) and that every \( c_j(a) \) belongs to a \( \delta^{10}_l \)-neighbourhood of \( \text{Crit}(f) \), for \( a \in B(0, r) \).

Then if \( r > 0 \) is sufficiently small, then \( \|\xi'_{n+m,l}(a)\| > 100\|c_l'(a)\| \) for all zeros of \( F_1 \) inside \( A \). In particular, there are no singularities of \( F_1 \) on its set of zeros inside \( A \).

**Proof.** The condition on \( U_l \) means that any solution to \( F_i(a) = 0 \) for \( a \in A \) must have that \( \xi_{k,l}(a) \cap U_l = \emptyset \) for all \( k \leq n + m - 1 \).

We have \( \|(f^{m}_a)'(v(a))\| \geq e^{\gamma m} \), for some \( \gamma \geq 2\gamma_1 \), by the definition of \( \mathcal{N} \). We can choose \( r \) so that \( m/n \) is arbitrarily small, i.e. during the iterates \( n + 1, \ldots, n + m \) we do not lose much in derivative. In other words,

\[
|(f^{m}_a)'(v(a))| \geq e^{\gamma_1 m} |(f^{m}_a)'(f^{n}_a(v(a)))| \geq e^{\gamma_1 n},
\]

where \( 0 < \gamma_1 < \gamma \). Indeed, we can get \( \gamma_1 \) as close to \( \gamma \) as we want. Choose \( \gamma_1 \) so that \( \gamma_1 \geq \gamma \). By Proposition 4.5,

\[
\|\xi'_{n+m,l}(a)\| \geq e^{\gamma_1 (n + m)} \|x'(a)\| \geq e^{\gamma'(n + m)},
\]
for some $\gamma' \geq (1/k)\gamma_1$ (see proof of Proposition 4.5). Choosing $n$ sufficiently large (i.e. $r > 0$ sufficiently small), we can therefore ensure that

$$\|\xi_{n+m,l}(a)\| > 100\|c_l'(a)\|,$$

for all $a \in V \cap A$, where $V$ is a neighbourhood the solution set $\xi_{n+m,l}(a) - c_l(a) = 0$. Hence $F'(a) \neq 0$ for all $a \in V \cap A$.

Passing on to a certain subset of $B_1' \subseteq W$, we want to show that this set has low curvature viewed as a surface embedded in $W$. To see what conditions are imposed on such a set, we begin with showing that $\xi_{n+m,1}(a)$ has bounded distortion in $B_1'$ if $|\xi_{k,1}(a) - \xi_{k,1}(b)| \leq T$ for all $k \leq n + m$ all $a, b \in B_1'$ and some number $T > 0$, depending on $U = U'$.

**Lemma 5.3.** Let $U' \supset U$ be a $10\delta$-neighbourhood of $\text{Crit}(f) \cap J(f)$ and let $\varepsilon > 0$. Then there exist $r > 0$, $T > 0$ where $T$ only depends on $U$ and $\varepsilon$, such that the following holds. Assume that $a_0 \in W \subseteq B(0, r)$, $\xi_{n,i}(a_0) = c_j(a_0)$ and $\xi_k(i, a_0) \cap U' = \emptyset$ for all $k \leq n - 1$. Then we have

$$|\xi_{n,i}'(a_0) - \xi_{n,i}'(a)| \leq \varepsilon \|\xi_{n,i}'(a)\|,$$

if $|\xi_k(i, a_0) - \xi_k(i, a)| \leq T$ for all $k \leq n$. Moreover, for each such $a$, we have that $\xi_{k, i}(a) \cap U = \emptyset$, for all $k \leq n - 1$.

**Proof.** Write $\xi_k, i = \xi_k$. Let $S > 0$ and $0 < k < 1$ be from Lemma 4.6. Assume that $n_1$ is maximal such that

$$\text{diam}(\xi_{n_1}(D_0)) \leq S, \quad \text{and} \quad \xi_{n_1}(D_0) \subseteq N,$$

where $D_0 \subset W$ is a full-dimensional $k$-Whitney disk with center at $a_0$. Lemma 4.6 implies that (27) holds if $n$ is replaced by $n_1$ for $T = S$ and when $a \in D_0$. We will show that (27) holds after $n$ iterates for some $T \leq S$.

Let $S' > 0$ be the constant in Lemma 4.7, given by $N = n - n_1$, $U$ and some suitable sufficiently small $\varepsilon > 0$. Choose $T \leq S'$ maximal such that the condition $|\xi_k(a_0) - \xi_k(a)| \leq T$ for all $k \leq n$, implies that $\xi_k(a) \cap U = \emptyset$ for all $k \leq n - 1$. Note that $T$ only depends on $U$. Then Lemma 4.7 together with Lemma 4.2 implies that

$$\left|\left(\frac{f_{n,a}}{f_{a,n}}\right)'(v_a) - 1\right| \leq \varepsilon_1,$$

where $\varepsilon_1$ is some suitable sufficiently small positive number ($\varepsilon_1$ depends on $\varepsilon$). Moreover, by the same argument as in the beginning of the proof of Proposition 4.5, we have $|(f_{k,a})'(v_a)| \geq e^{\gamma'k}$, for $x = a, a_0$ and all $k \leq n$ and some $\gamma' \geq \gamma$. Proposition 4.5 and Lemma 3.3 imply that (27) holds since $a \in D_0$. The lemma is proved.

Let us assume that $\xi_{n+m,l}(a) - c_l(a) = 0$ and that $\xi_{n+m,l}(a)$ satisfies the assumptions in the above lemma. Put $F(a) = \xi_{n+m,l}(a) - c_l(a)$. Lemma 5.3 implies immediately that $\|\nabla F(b) - \nabla F(a)\|$ is small if $b$ satisfies $|\xi_k(a) - \xi_k(b)| \leq T$ for all $k \leq m + n - 1$. Hence if that holds for all parameters $a$ in some submanifold in $W$, it means this manifold has low curvature.

**Definition 5.4.** Suppose that $E$ is an open $n$-dimensional connected manifold parameterised by some open set $D \subseteq \mathbb{C}^n$, where $\phi : D \to E$ is a diffeomorphism and $E = \phi(D)$ and $\phi(\partial D) = \overline{E} \setminus E$. We say that $E$ is almost planar if

$$\|\phi'(x) - \phi'(y)\| \leq 1/100,$$
for all $x, y \in D$.

If, in addition, $D$ is a disk and
\[ \text{diam}(E) \geq k \text{dist}(E, 0), \]
then we say that $E$ is an almost planar $k$-Whitney disk. Moreover, if $d = \text{diam}(E)$, then we say that the radius of $E$ is $r = d/2$. For any $0 < r' < r$ by $\text{ddist}(x, \partial E) \geq r'$ we mean the set $\{x \in E : \text{dist}(x, \overline{E}\setminus E) \geq r'\}$. 

Let us now assume that we are in the $l$th step so that we have constructed a nested sequence of almost planar disks $\mathbb{B}_{k+1} \subset \mathbb{B}_k$, $0 \leq k \leq l - 1$, such that each $\mathbb{B}_k$ has that $F = F_k(a) = \xi_{nk+m_k,k}(a) - c_k(a) = 0$ for all $a \in \mathbb{B}_k$. In fact, since the sequence is nested, $F_k(a) = 0$ holds for all $1 \leq k \leq l$ in $\mathbb{B}_l$. Moreover, note that the normal vectors to the solution sets are $F'_k(a) = \xi'_{nk+m_k,k}(a) - c'_k(a)$. These vectors are arbitrarily close to $\xi'_{nk+m_k,k}(a)$ because $\|\xi'_{nk+m_k,k}(a)\|$ is much larger than $\|c'_k(a)\|$, which follows from Lemma 5.2. Also, $\xi'_{nk+m_k,k}(a)$ is well approximated by $(f^n_{a} + m_k)'(v_k(a))x'_k(a)$ by Proposition 4.4. Hence we have the following important fact:

**Fact.** The normal vectors to the solution sets $F_k(a) = 0$ are, up to arbitrary low distortion, parallel to $x'_k(a)$.

A priori, Lemma 3.3 only applies to Whitney disks rather than almost planar Whitney disks. However, since $\mathbb{B}_l$ is almost planar, it is uniformly well approximated by a hyperplane, i.e. the tangent space at points on $\mathbb{B}_l$ does not vary much on $\mathbb{B}_l$. By Lemma 4.6 it then follows that if $c_{l+1}$ has finite order contact, and if $x'_{l+1}(a)$ is not perpendicular to $\mathbb{B}_l$ for $a \in \mathbb{B}_l$, the set $x_{l+1}(\mathbb{B}_l)$ will then grow to the large scale $S = S_l$ before leaving $\mathcal{N}$, i.e. $\xi_{n, l+1}(\mathbb{B}_l) \subset \mathcal{N}$ contains a disk of diameter $S_l$. By Lemma 4.8 there is some $N = N_l$ depending on $S_l$ such that $\xi_{n+m, l+1}(\mathbb{B}_l)$ covers $\mathcal{U}$ for some $m \leq N_l$. Clearly, if $\mathbb{B}_l$ is an almost planar $k_l$-Whitney disk, then $N_l$ depends only on $k_l$.

We now prove that if a given solution is found to $F_l(a) = 0$, then the parameters satisfying the conditions in Lemma 5.3 will contain a $k_{l+1}$-Whitney disk. Recall that the sets $U_l$ are $\delta_l$-neighbourhoods around the critical points on the Julia set for $f$. Let $U'_{l} \supset U_l$ be $10\delta_l$-neighbourhoods around these critical points. Let $M_0 = \max |f'_a(z)|$ where the maximum is taken over all $(z, a) \in \mathcal{C} \times \overline{\mathbb{B}(0, r)}$.

**Lemma 5.5** (Inductive Lemma 1). Assume that $\mathbb{B}_l \subset \mathbb{B}_0 \subset W$ is an almost planar Whitney disk of diameter $2r_l \geq k_l \text{dist}(\mathbb{B}_l, 0)$ and for which every $a \in \mathbb{B}_l$ has that $\xi_{nk+m_k,k}(a) - c_k(a) = 0$ for all $1 \leq k \leq l$ (if $l = 0$ we have no solutions so far). Assume that we have found a solution to
\[ \xi_{n_{l+1}+m_{l+1, l+1}}(a_0) - c_{l+1}(a_0) = 0 \]
for some $a_0 \in \mathbb{B}_l$, such that $\text{ddist}(a_0, \partial \mathbb{B}_l) \geq r_l/2$ and such that $\xi_{n_{l+1}}(\mathbb{B}_l) \subset \mathcal{N}$ and $m_{l+1} \leq N_l$, where $N_l$ only depends on $k_l$.

Then if $r > 0$ is sufficiently small, and if $\text{dim}(\mathbb{B}_l) > 1$, there exists an almost planar $k_{l+1}$-Whitney disk $\mathbb{B}_{l+1} \subset \mathbb{B}_l$ of codimension 1 (in $\mathbb{B}_l$), where $k_{l+1}$ only depends on $k_l$. For every $a \in \mathbb{B}_{l+1}$ we have $\xi_{n_{l+1}+m_{l+1, l+1}}(a) - c_{l+1}(a) = 0$. If $\text{dim}(\mathbb{B}_l) = 1$, the set $\mathbb{B}_{l+1}$ might reduce to a single point.

**Proof.** We can without loss of generality assume that $n_{l+1}$ is the largest integer such that $\xi_{n_{l+1}}(\mathbb{B}_l) \subset \mathcal{N}$ and such that $\xi_{n_{l+1}}(\mathbb{B}_l)$ contains a disk of diameter $S_l$, where
$S_t > 0$ is the large scale from Lemma 4.6. Now choose $U'_t$ such that $\xi_{k,t+1}(a_0) \cap U'_t = \emptyset$ for all $k \leq m_{t+1} + n_{t+1} - 1$ and that the first return time from $U'_t$ to itself is at least $2N_t$. Hence $U_t$ (which is a $\delta_t$ neighbourhood of $\text{Crit}(f_0)$) depends only on $m_{t+1} \leq N_t$, given that $r > 0$ is sufficiently small. The condition on $r > 0$ is that $c(a) \in U(c(0), \delta_t^{(0)})$ for all critical points $c(a)$, $a \in B(0, r)$.

Put $E = \{ a \in W : \xi_{m+n,t+1}(a) - c_{t+1}(a) = 0 \}$. By assumption, the set $E \cap B_t$ is non empty. Since $\dim(B_t) > 1$ and $E$ has codimension 1, using Lemma 5.2 with $A = B_t$, we see that the set $E \cap B_t$ is a smooth manifold. Moreover, we must have $\dim(E \cap B_t) \geq 1$. If $\dim(B_t) = 1$ then $E \cap B_t$ might reduce to a single point.

Let $\xi_{k,t+1} = \xi_k$ and put $m_{t+1} = m$ and $n_{t+1} = n$. According to Lemma 5.3, to have good geometry control of a manifold in $W$, any parameter $b$ in this manifold must satisfy

\[ |\xi_k(a_0) - \xi_k(b)| \leq T_i \]

for some $T_i > 0$ depending on $U_t$ for all $k \leq n + m$. We will show that the set of such parameters $b$ satisfying (28) contains a $k_{t+1}$-Whitney disk $B'$, centered at $a_0$, where $k_{t+1}$ only depends on $k_t$.

Since $m \leq N_t$, the expansion $|(|f^m|)(z)| \leq C_l = C_l(N_t)$ is bounded and depends only on $N_t$. Hence, $|\xi_k(a_0) - \xi_k(b)| \leq T_i$ for all $k \leq n + m$ if $|\xi_k(a_0) - \xi_k(b)| \leq S_{t+1}'$ for all $k \leq n$, where $S_{t+1}' \leq T_t/2C_l$. Now put $S_{t+1}' = S_{t+1}'(2M_0)$ (then $S_{t+1}$ will be the new large scale). We get that (28) holds for a $k_{t+1}$-Whitney $B'$ disk centered at $a_0$, where $k_{t+1}$ is minimal such that $\xi_n(B')$ contains a disk of diameter $S_{t+1}$ (then $\text{diam}(\xi_n(B')) \leq S_{t+1}'$). Since $\xi_n$ is almost linear on $k$-Whitney disks according to Lemma 4.6 (where also $j_{k+1} \leq j_k$, $k = k_0$), we get that $k_{t+1}/k_t - S_{t+1}/S_t$ is arbitrarily close to zero (hence $k_{t+1} \approx k_t S_{t+1}/S_t$).

Moreover, the set $E \cap B'$ must be almost planar in $B'$. It follows that $B_{t+1} = B' \cap (E \cap B_t)$ is an almost planar $k_{t+1}$-Whitney disk in $B_t$. Finally, we see that $k_{t+1}$ only depends on $S_{t+1}$, $k_t$ and $S_t$. Clearly, $S_{t+1}$ depends only on $T_t$ and $C_l$. Now $T_t$ depends on $U_t$ which in turn depends on $N_t$ and moreover $C_l$ depends clearly on $N_t$. Finally, $N_t$ depends on $S_t$ (the previous large scale) which in turn depends on $k_t$. Hence $k_{t+1}$ only depends on $k_t$. The lemma is proved. \[ \square \]

**Lemma 5.6 (Inductive Lemma II).** Assume that we have found an almost planar $k_l$-Whitney disk $B_l$ (of diameter $2r_l$) and a list of critical points $C_l = \{ c_1, \ldots, c_l \}$ depending on the parameter $a$ such that for each $c_k \in C_l$ we have $\xi_{m_k+m_k,k}(a) - c_k(a) = 0$ for all $a \in B_l$ and all $1 \leq k \leq l$. Assume that $n_{l+1}$ is maximal such that $\xi_{n_{l+1},l+1}(B_l) \subset \mathcal{N}$.

Then if $r > 0$ is sufficiently small there exists a solution to

\[ \xi_{n_{l+1}+m_{l+1},l+1}(a) - c_{l+1}(a) = 0 \]

for some $a \in B_l$, such that $\text{dist}(a, \partial B_l) \geq r_l/2$, where $m_{l+1} \leq N_l$, and $N_l$ is an integer which only depends on $k_l$.

**Proof.** Put $\xi_{n_{l+1},l+1} = \xi_n$. It follows from Lemma 4.6 that $\xi_n(B_l)$ contains a disk of diameter at least $S_l$ before leaving $\mathcal{N}$ (where $S_l$ depends only on $k_l$). Indeed, the disk $B_l$ is an intersection of small manifolds determined by $F_j(a) = \xi_{n_j,j}(a) - c_j(a) = 0$. Each of these manifolds has normal vectors equal to $F'(a) = \xi_{0,j}(a) - c_j(a)$. By the Fact on p. 15, we have that $F'(a)$ is almost parallel to $x_j'(0)$. The vectors $x_j'(0)$ are all linearly independent by Theorem 3.4. This implies that if $\theta$ is the angle between a tangent hyper-plane to $B_l$ and a tangent hyper-plane to $x_{l+1}'(a_0)$ (see before Lemma
4.6 for definition) then $\cos(\theta)$ is bounded away from 0 (since both these surfaces are almost planar their tangent hyper-planes do not vary much).

Since $\xi_n(a) \in J(f_0)$ for all $a \in \mathbb{B}_t$ where $\text{dist}(a, \partial \mathbb{B}_t) \geq (3/4)r_I$, there is some (maximal) almost planar disk $\mathbb{B}_t' \subset \mathbb{B}_t$ centered at $a$ such that $\text{dist}(b, \partial \mathbb{B}_t) \geq r_I/2$, for all $b \in \mathbb{B}_t'$. The set $\xi_{n+1}(\mathbb{B}_t')$ will contain a disk of diameter $S_t/8$ centered at the Julia set of $J(f_0)$. By Lemma 4.8 there is an integer $N_t$, where $N_t = N_t(S_t)$ only depends on $S_t$, such that $f_0^m(\xi_n(\mathbb{B}_t')) \supset \bar{U}_t$, $m \leq N_t$. Let $m$ be defined by

$$m = \inf\{k > 0 : f_k^m(\xi_n(\mathbb{B}_t')) \supset \bar{U}_t\}.$$ 

Since the parameter dependence can be made arbitrarily small under $N_t$ iterates, by choosing $r > 0$ sufficiently small, we can also ensure that $f_0^m(\xi_n(\mathbb{B}_t')) = \xi_{n+m}(\mathbb{B}_t') \supset c_{l+1}(\mathbb{B}_t')$. Hence there is a solution to $\xi_{n+1+m+1}(a) - c_{l+1}(a) = 0$ inside $\mathbb{B}_t'$. By the definition of $\mathbb{B}_t'$, we have $\text{dist}(a, \partial \mathbb{B}_t) \geq r_t/2$. \hfill \Box

Remark 5.7. The dependence of the constants $U_t, T_t, S_t, N_t$ and $r > 0$ might seem intricate. Let us clarify the feasibility of choosing these constants in a consistent way. Put $S = S_0$ and $k = k_0$ in Lemma 4.6. The constants $N_j, T_j, S_j, U_j$ depend on each other as follows. The number $T_0$ depends on $U_0$, since the existence of $T_0$ follows from a given $U = U_0$ in Lemma 5.3. From $T_0$ we get some new large scale $S_1$ and its corresponding new Whitney number $k_1$ (see proof of Lemma 5.5). Obviously, $N_t$ depends on $S_t$. The neighbourhood $U_1$ depends on $N_1$ since $U_1$ is defined in terms of the first return time from $U_1$ into itself is at least $2N_t$. Then again $T_1$ depends on $U_1$ and so on. One can write this as a scheme as follows. We write $X \rightarrow Y$ if $Y$ depends on $X$ but not the converse.

$$f \rightarrow S_0 \rightarrow N_0 \rightarrow U_0 \rightarrow T_0 \rightarrow S_1 \rightarrow N_1 \rightarrow U_1 \rightarrow T_1 \rightarrow S_2 \rightarrow \ldots.$$ 

Since there are no loops in this scheme, i.e. there are no two distinct elements $X, Y$ for which both $X \rightarrow Y$ and $Y \rightarrow X$, there is no problem of choosing $S_j, U_j, N_j$.

Moreover, they are independent of $r > 0$, for all $r \leq R$, for some fixed (sufficiently small) $R > 0$.

Lemma 5.8. Assume that $f_0$ is a Misiurewicz map. Then to any compact subset $K$ of the Fatou set $F(f_0)$ there is some $r > 0$ such that $K \subset F(f_a)$ for all $a \in \mathbb{B}(0, r)$.

Proof. Recall that the only Fatou components for Misiurewicz maps are those corresponding to attracting cycles. Assume first that $K$ belongs to a given basin of attraction. That means that in the geometrically attracting case (where the corresponding attracting fixed point is not super-attracting) the conjugating function $\varphi$ can be extended to the whole basin. In the super-attracting case there is a Greens function $\varphi$ arising from the conjugating function. In both cases there are level lines when $|\varphi(z)|$ is constant. Since $K$ is compact there is some $\alpha \in \mathbb{R}$ so that for any $z \in K$, $|\varphi(z)| < \alpha$. Let $N_0 = \{z : |\varphi(z)| < \alpha\}$. Then $N_0$ is open and contains $K$. We have $f(N_0) = N_1 \subset N_0$. Put $N_k = f^k(N_0)$. Therefore,

$$N_0 \supset N_1 \supset \ldots,$$

and $\cap_k N_k$ is the attracting fixed point. Since $\varphi = \varphi_0$ is continuous with respect to the parameter, for some $r > 0$ the set $N_0$ moves continuously in $a$, such that for any $a \in \mathbb{B}(0, r)$, putting $N'_0 = \{z : |\varphi_a(z)| < \alpha\}$, we have $N'_0 \supset K$ and $f_a(N'_0) \subset N'_0$. Again we get a nested sequence of sets $N'_0 \supset N'_1 \ldots$. The intersection $I = \cap_k N'_k$ is an invariant topologically attracting set. It cannot intersect the Julia set since the Julia set is topologically repelling. Hence $I$ is an invariant subset of the Fatou set.
$F(f_a)$ of $f_a$, compactly contained in $F(f_a)$. It follows that $I$ must be a fixed point. From this the lemma follows.

\[\square\]

6. Conclusion and proof of Theorem A

We prove Theorem A by induction finitely many times. Let us start with the given Misiurewicz map $f = f_0$ (not flexible Lattès map) for which $J(f) = \mathbb{C}$. In the end we will find a hyperbolic map arbitrarily close to $f$.

Choose some (sufficiently small) $r > 0$ and some $k_0$-Whitney full dimensional disk $\mathbb{B}_0 \subset W$ (where $k = k_0 \leq 1/2$ from Lemma 4.6). Let us now argue inductively. Assume that we have found solutions to the following equation for $1 \leq k \leq l$ (if $l = 0$ no solution is yet found):

$$\xi_{n+k+m,k}(a) - c_k(a) = 0.$$  \hspace{1cm} (29)

Assume that (29) holds for all $a \in \mathbb{B}_l$ and for all $1 \leq k \leq l$, and that $\mathbb{B}_l$ is an almost planar $k_l$-Whitney disk of radius $r_l$. Let us now process as follows.

Consider the critical point $c_{l+1}$ which has finite order contact (in $\mathbb{B}_l$). Indeed, if no more critical points would have finite order contact, then there would be a small ball $\mathbb{B}$ around any point in $\mathbb{B}_l$ such that all $\mathbb{B}$ are Misiurewicz maps. This is impossible unless $l = 2d - 2$, and then we are done.

Hence assume $l < 2d - 2$. In this case, by Lemma 5.6 there is a solution to $\xi_{n+m,l+1}(a) - c_{l+1}(a) = 0$ for some $a \in \mathbb{B}_l$ such that $ddist(a, \partial \mathbb{B}_l) \geq r_l/2$. By Lemma 5.5 there is a new almost planar $k_{l+1}$-Whitney disk $\mathbb{B}_{l+1} \subset \mathbb{B}_l$, where $\xi_{n+m,l+1}(a) - c_{l+1}(a) = 0$ for all $a \in \mathbb{B}_{l+1}$.

Now, we continue in the same way with $l$ replaced by $l + 1$.

Since the dimension drops 1 in each step, the set of parameters satisfying

$$\xi_{n+k+m,k}(a) - c_k(a) = 0, \quad \text{for all } 1 \leq k \leq l,$$

is a manifold of codimension $l$. Hence $\mathbb{B}_l$ has dimension equal to $2d - 2 - l$. (In the last step, when $l = 2d - 2$ the set $\mathbb{B}_{2d-2}$ might reduce to a single point).

Recall that the parameter space of rational maps of degree $d$ up to conjugacy by a Möbius transformation is equal to $2d - 2$. Hence we can repeat the argument above finitely many times until every critical point lies in the orbit of a super-attracting cycle. Hence, we find a function $f_a$ for some $a \in W \subset \mathbb{B}(0, r)$ which is hyperbolic. In fact every critical point lies in a super-attracting cycle. Since $r > 0$ was arbitrarily small, Theorem A follows.

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