Quantum entanglement can be simulated without communication

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It has recently been shown that all causal correlations between two parties which output each one bit, \(a\) and \(b\), when receiving each one bit, \(x\) and \(y\), can be expressed as convex combinations of local correlations (i.e., correlations that can be simulated with local random variables) and non-local correlations of the form \(a + b = x \cdot y \mod 2\). We show that a single instance of the latter elementary non-local correlation suffices to simulate exactly all possible projective measurements that can be performed on the singlet state of two qubits, with no communication needed at all. This elementary non-local correlation thus defines some unit of non-locality, which we call a \(nl\)-bit.

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The importance of quantum entanglement is by now widely appreciated [1]. Historically, entanglement has first been viewed mainly as a source of paradoxes, most noticeably the Einstein-Podolsky-Rosen (EPR) paradox, which is at the origin of the concept of quantum non-locality [2]. Today, however, entanglement is rather viewed as the resource that makes quantum information science so successful [3, 4, 5]. Indeed, based on entanglement, various informational tasks appear feasible, although they would be impossible using only classical physics.

Following this new trend in quantum information science, a growing community of physicists and computer scientists has started to investigate the resource “entanglement”. Questions like how to manipulate this resource, e.g., how to concentrate or dilute it [6], or how to transform it into secret bits [7], were addressed. Also, a unit of entanglement has been identified and named \(e\)-bit; it consists of a pair of maximally entangled qubits, e.g., a singlet as used in Bohm’s version of the EPR paradox. A few years ago, connections with communication complexity started to be studied [8], with questions like how much classical communication is required to simulate an \(e\)-bit?

Simulating an \(e\)-bit means the following. Two parties, Alice and Bob, receive each a normalized vector \(\vec{\nu}_A\) and \(\vec{\nu}_B\) that characterizes their measurement on the Poincaré sphere, and each has to output a bit, \(A\) and \(B\) [10], see Fig. 1. The statistics of the output bits should exactly reproduce the quantum predictions for all values of \(\vec{\nu}_A\) and \(\vec{\nu}_B\) if Alice and Bob were actually sharing a singlet state \((|01\rangle - |10\rangle)/\sqrt{2}\). For instance, if the vectors are opposite, \(\vec{\nu}_A = -\vec{\nu}_B\), the output bits should always be equal, \(A = B\). From Bell inequality, we know that it is impossible to simulate a singlet without any communication. This is so even if one assumes that both parties share local hidden variables, or in modern terminology, local randomness (that is, they share a non-
finite list of random bits \( \lambda_x \). Of course, if an unlimited amount of communication is allowed, then Alice could simply send her measurement setting \( \nu_A \) to Bob with arbitrary precision, so the simulation of a singlet would become straightforward. But whether such an unlimited amount of communication is necessary was unknown. First answers along this direction were given by A. Tapp, R. Cleve, and G. Brassard [11] in Montreal, and by M. Steiner [12] from the NSA. The Canadian group showed that, quite surprisingly, 8 bits of communication suffice for a perfect (analytic) simulation of the quantum predictions. Steiner, followed by [13], showed that if one allows the number of bits to vary from one instance to another, then 2 bits suffice on average. It was also shown that, with block coding, the number of communicated bits can be reduced to 1.19 bits on average [14]. A few years later, B. Toner and D. Bacon [15] improved on these results and showed that actually a single bit of communication suffices for perfect simulation of a singlet. At this point, the situation was the following: one bit of communication allows one to simulate a singlet, and one singlet provides one secret bit.

Independently of the above story, S. Popescu and D. Rohrlich raised the following question: can there be stronger correlations than the quantum mechanical ones that remain causal (i.e., that do not allow signaling) [16]? Recall that the quantum correlations violate the Bell inequality, but do not allow any faster than light signaling. Popescu and Rohrlich answered by presenting an hypothetical machine that does not allow signaling, yet violates the Clauser-Horne-Shimony-Holt (CHSH) [17] inequality more than quantum mechanics. They concluded by asking why Nature is non-local, but not maximally non-local, where the maximum would only be limited by the no-signaling constraint?

In this Letter, we push this investigation even further by showing that, actually, quantum entanglement can be perfectly simulated by using one instance of this non-local PR machine and no communication at all! Since, as we will show, one instance of the PR machine is a weaker resource than one bit of communication, one is tempted to conclude that Nature may use something like these non-local machines if she is sparing with resources.

**Non-local PR machine.** The non-local PR machine works as follows, see Fig. 2. It admits two input bits \( x \) and \( y \), and yields two output bits \( a \) and \( b \). The bits \( x \) and \( a \) are in Alice’s hands, while \( y \) and \( b \) are on Bob’s side. The machine is such that \( a \) and \( b \) are correlated according to the simple relation (equality modulo 2):

\[
 a + b = x \cdot y 
\]  

(1)

Except for this relation, \( a \) and \( b \) are unbiased random bits. For example, if \( x = y = 0 \), then the machine’s outputs are random but identical: \( a = b = 0 \) or \( a = b = 1 \) with equal probabilities \( \frac{1}{2} \). This implies that the PR machine cannot be used to signal: since the output \( a \) (\( b \)) is locally random, its value cannot convey any information about the input \( y \) (\( x \)) of the other party. This machine is constructed in such a way that the CHSH inequality is violated by the algebraic maximum value of 4, while quantum correlation achieve at most \( 2\sqrt{2} \) [18]. (Remember that with shared randomness only, the maximum allowed value in a local theory is 2.) To see this, let us change the bit values 0 and 1 to the values \( +1 \) and \( -1 \) traditionally used in Bell inequalities. Defines \( a' = 1 - 2a \) and \( b' = 1 - 2b \) and note that

\[
 a' \cdot b' = \begin{cases} 
 1 & \text{if } a + b = 0 \mod 2, \\
 -1 & \text{if } a + b = 1 \mod 2. 
\end{cases} 
\]  

(2)

Denoting by \( E \) the expectation value, one has for the CHSH inequality: \( E(a' \cdot b'|x = 0, y = 0) + E(a' \cdot b'|x = 0, y = 1) + E(a' \cdot b'|x = 1, y = 0) - E(a' \cdot b'|x = 1, y = 1) = 4 > 2 \). The violation of CHSH inequality implies that this PR machine is non-local (even more than quantum physics), so that it cannot be simulated with local variables. Yet, it is causal, like quantum mechanics.

Let us emphasize that the PR machine [11] is not an arbitrary construction. It is, up to elementary symmetries like bit flips, the unique binary causal maximally non-local machine. Indeed, it can be shown that all binary causal correlations can be expressed as convex combinations of local machines (i.e., those which can be simulated with local random variables) and maximally non-local PR machines [14]. The PR machines also have the surprising property that, given an unlimited supply of them, any communication complexity problem can be solved with a single bit of communication [20].

Finally, note that it is straightforward to simulate a PR machine with shared randomness (i.e., local hidden variables) augmented by one bit of communication: the hidden variable \( \lambda \) should then be a random unbiased bit, \( a = \lambda \), and \( x \) should be communicated by Alice to Bob who should output \( b = x \cdot y + \lambda \mod 2 \). But the converse is false: a PR machine cannot be used to communicate since it is causal. Therefore, as already mentioned, the PR machine is a strictly weaker resource than a bit of communication, that is

\[
1 \text{ nl-bit} \prec 1 \text{ bit (supraluminal communication)} 
\]  

(3)

![FIG. 2: Scheme of the PR non-local machine, where \( x \), \( y \) and \( a \), \( b \) denote the input and output bits, respectively.](image-url)
where we have denoted as nl-bit the unit of non-local correlations effected by the PR machine.

**Simulation of a singlet with a non-local PR machine.**

We now show that any projective measurements on a singlet can be perfectly simulated using a single instance of this non-local PR machine, with no communication being necessary. As a consequence of \( \text{E} \), this is a stronger result than the simulation of a singlet with one communicated bit \( \text{E} \). Consider that Alice and Bob share a non-local PR machine as well as shared randomness in the form of pairs of normalized vectors \( l_1 \) and \( l_2 \), randomly and independently distributed over the entire Poincaré sphere. Denote \( \tilde{v}_A \) and \( \tilde{v}_B \) the vectors that determine Alice and Bob measurements, respectively.

The model goes as follows. Alice inputs
\[
x = \text{sg}(\tilde{v}_A \cdot l_1) + \text{sg}(\tilde{v}_A \cdot l_2)
\] into the machine, where
\[
\text{sg}(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}
\]
(Here and now on, all equalities involving bits are taken modulo 2.) She then receives the bit \( a \) out of the machine, and outputs
\[
A = a + \text{sg}(\tilde{v}_A \cdot l_1)
\] as the simulated measurement outcome. Similarly, Bob inputs
\[
y = \text{sg}(\tilde{v}_B \cdot l_+ + \text{sg}(\tilde{v}_B \cdot l_-)
\] into the machine, where \( l_+ = l_1 \pm l_2 \), receives \( b \) out of the machine, and then outputs
\[
B = b + \text{sg}(\tilde{v}_B \cdot l_+) + 1.
\]

Note that since the machine’s outputs \( a \) and \( b \) are random unbiased bits, the simulated measurement outcomes \( A \) and \( B \) are equally random, exactly as for real measurements on a singlet. But the outputs \( a \) and \( b \) are correlated according to relation \( \text{E} \), hence \( A \) and \( B \) are also correlated. The surprising and interesting result is that this correlation is precisely the one predicted by quantum mechanics for the singlet state:

**Theorem:**
\[
E(A + B|\tilde{v}_A, \tilde{v}_B) = \frac{1 + \tilde{v}_A \cdot \tilde{v}_B}{2}
\] \hspace{1cm} (9)

**Proof:** First, compute
\[
A + B = a + b + \text{sg}(\tilde{v}_A \cdot \lambda_1) + \text{sg}(\tilde{v}_B \cdot \lambda_+) + 1
\]
\[
= x \cdot y + \text{sg}(\tilde{v}_A \cdot \lambda_1) + \text{sg}(\tilde{v}_B \cdot \lambda_+) + 1
\]
\[
= z + \text{sg}(\tilde{v}_A \cdot \lambda_1) + \text{sg}(\tilde{v}_B \cdot \lambda_+) + 1
\] \hspace{1cm} (10)

where
\[
z = [\text{sg}(\tilde{v}_A \cdot \lambda_1) + \text{sg}(\tilde{v}_A \cdot \lambda_2) + \text{sg}(\tilde{v}_B \cdot \lambda_+) + \text{sg}(\tilde{v}_B \cdot \lambda_-)]
\]

Next, note that \( \text{E} \) corresponds precisely to the 1-bit communication model \( \text{E} \). Indeed, in this model, Alice outputs \( A = \text{sg}(\tilde{v}_A \cdot \lambda_1) \), communicates the bit \( c = \text{sg}(\tilde{v}_A \cdot \lambda_1) + \text{sg}(\tilde{v}_A \cdot \lambda_2) \) to Bob who outputs \( B = (1 - c) \text{sg}(\tilde{v}_B \cdot \lambda_+) + c \text{sg}(\tilde{v}_B \cdot \lambda_-) + 1 \). The latter can be re-expressed as \( B = z + \text{sg}(\tilde{v}_B \cdot \lambda_+) + 1 \). Thus, \( A + B = z + \text{sg}(\tilde{v}_A \cdot \lambda_1) + \text{sg}(\tilde{v}_B \cdot \lambda_+) + 1 \). Finally, since the expressions for \( A + B \) in our model and the 1-bit communication model are identical and since the latter model satisfies \( \text{E} \), so does our model \( \text{E} \).

**Analogue of entanglement monogamy: the non-local PR machine cannot be shared.** Given the analogy between the entanglement contained in a singlet (1 e-bit) and the non-local but causal correlations produced by the PR machine (1 nl-bit), it is tempting to investigate how deep this analogy can be pushed. One of the key features of entanglement is its monogamy \( \text{E} \). By this one means that if a quantum system \( A \) is strongly entangled with another system \( B \), then \( A \) cannot simultaneously share much entanglement with any third system \( C \). This property is for example at the basis of the quantum no-cloning theorem \( \text{E} \), the monogamy of CHSH inequalities \( \text{E} \), or the security of quantum cryptography \( \text{E} \). We shall see that this same property holds for causal non-local machines.

![FIG. 3: Scheme of a 3-party nonlocal machine.](image-url)
of a singlet state both with Bob and Charles. Then, by measuring her qubit in either the computational basis or the dual basis, Alice would prepare the 2-qubit system shared by Bob and Charles in two different mixtures, which would allow instantaneous signaling between Alice and Bob/Charles. Hence, perfect cloning is impossible, and entanglement must be monogamous. Now, coming back to the monogamy of causal non-local machines, assume Alice holds the two halves of two PR machines, one shared with Bob, the other one shared with Charles (see Fig. 3). Denote by \( z \) and \( c \) Charles’ input and output bits. One has

\[
\begin{align*}
  a + b &= x y \\
  a + c &= x z
\end{align*}
\]  

Therefore, we have \( b + c = x (y + z) \). Assume now that Bob and Charles sit next to each other, at a long distance from Alice. Then if Bob enters \( y = 0 \) and Charles enters \( z = 1 \) in their respective machines, we have \( b + c = x \). This means that, by checking whether their outputs are equal or not, Bob and Charles can know instantaneously whether Alice entered \( x = 0 \) or \( x = 1 \) into the machine. Such a tripartite PR machine would then provide instantaneous signaling between Alice and Bob/Charles. Hence, it cannot exist, and causal non-local machines must be monogamous.

**Conclusion.** Quantum non-locality is one of the most important and amazing discoveries of the 20th century physics. It took a long time to be appreciated, and actually it is still believed to contain deep mysteries. However, today, thanks to quantum information science, entanglement has become better studied and understood. Probably its most remarkable manifestation is quantum teleportation [25], a protocol that allows one to teleport all the characteristics of an object embedded in some energy and matter localized “here” to another piece of energy and matter located at a distance. In this Letter, we contributed to “disentangle” the non-locality inherent to quantum mechanics into its elementary constituent, a unit of non-locality or nl-bit. Surprisingly, the quantum non-locality of a singlet boils down to a rather simple machine, encapsulated by relation (11), which is inspired by the CHSH inequality. We showed that one instance of this non-local machine is sufficient to perfectly simulate a singlet. Since this machine defines a resource that is strictly weaker than any communication while it is sufficient to simulate a singlet, we have in short

\[
1 \text{ e-bit (simulation of)} < 1 \text{ nl-bit} < 1 \text{ bit}
\]

Thus, assuming that Nature is sparing with resources, one is tempted to conclude that the non-local correlations that she exhibits originate from these kinds of machines.

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