Bosonic-type $S$-Matrix, Vacuum Instability and CDD Ambiguities

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Abstract

We consider the simplest bosonic–type $S$-matrix which is usually regarded as unphysical due to the complex values of the finite volume ground state energy. While a standard quantum field theory interpretation of such a scattering theory is precluded, we argue that the physical situation described by this $S$–matrix is of a massive Ising model perturbed by a particular set of irrelevant operators. The presence of these operators drastically affects the stability of the original vacuum of the massive Ising model and its ultraviolet properties.

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1 Introduction

In recent years, remarkable progress has been witnessed in the study of the scaling region nearby the fixed points of the Renormalization Group (RG) of 2-D statistical models. Such progress has been possible thanks to new ideas related to the $S$–matrix theory [1, 2] and there is now a well–defined program to analyse off–critical statistical models. In short, this usually consists in implementing the following steps: (a) a deformation of a Conformal Field Theory (CFT) describing the critical point by means of a relevant integrable operator which drives the model away from criticality; (b) a consequent determination of the exact spectrum of the massive or massless excitations and the elastic $S$–matrix scattering amplitudes thereof; (c) and finally, a reconstruction of the off–critical correlators by means of the Form Factor Approach also accompanied by an analysis of the finite size behaviour of the theory by means of the Thermodynamics Bethe Ansatz (TBA). The scattering theory of the Ising model in a magnetic field [2] together with the calculation of its correlators [3] provide an explicit example of a statistical model solved away from criticality along the line of the above program. Solutions for many other off–critical models have also been found (see for instance [3-19]). Hereinafter the statistical models solved according to an $S$–matrix program will be simply referred as bootstrap models.

All known boostrap models greatly differ from each other by the nature of their spectrum, for the total number of particles and for the detailed structure of their scattering amplitudes. However, looking at them as a whole, they share an important feature, which consists in the fermionic nature of their scattering amplitudes. Namely, the $S$-matrix involving two identical particles takes the value $-1$ at the threshold of their $s$-channel, i.e. $S_{aa}(0) = -1$ (for the notation see below). This condition was proved to be quite important in the further study of their off–shell behaviour. In the Thermodynamics Bethe Ansatz calculations, for instance, fermionic–type $S$-matrices lead to well–defined integral equations which – in all known models – provide real solutions for the ground state energy, in perfect agreement with the CFT prediction [14, 15]. On the other hand, in the Form Factor Approach, fermionic $S$–matrices give rise to spectral series of the correlation functions with fast convergent properties (see for instance, [3, 9, 10, 11, 12]).

Among the set of known solved models the peculiar absence of examples relative to an $S$–matrix of bosonic–type, i.e. an $S$-matrix which satisfies the condition $S_{aa}(0) = 1$, is indeed intriguing. What is the reason? While a widespread suspicion is that such $S$-matrices are unphysical[1], a full understanding of their features and also of their inter-

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1 It is worth mentioning that in an example based on the algebraic Bethe Ansatz [23], which is a different approach to that of this paper, it was shown that the physical vacuum of the theory could only be constructed if the particles were of fermionic type.
pretation are still missing.

Deeply related to the absence of models with a bosonic–type $S$–matrix, there is also the question about the role played in the scattering theory by the so–called CDD factors. It is well known that they give rise to ambiguity in the $S$–matrix of a bootstrap system and it seems therefore quite striking that all consistent bootstrap models have managed to successfully resolve this ambiguity.

The aim of this paper is to address some of the above questions. In particular, we will draw some conclusions about the interpretation of the physical aspects which occur in an integrable theory based on a bosonic–type $S$–matrix. We will consider the simplest non–trivial scattering theory of this kind, consisting of one self–conjugate particle $A$ of mass $m$, with the two–particle $S$–matrix given by

$$S_{AA}(\beta) = -\frac{\sinh \beta - i \sin \pi a}{\sinh \beta + i \sin \pi a} \equiv -f_a(\beta).$$

(1.1)

Unless explicitly stated, $a$ is a real parameter which, for the invariance of the $S$–matrix under $a \to 1 - a$, can be taken in the interval $[0, 1/2]$. In eq. (1.1) $\beta \equiv \beta_1 - \beta_2$, where $\beta_i$ is the rapidity variable of each particle entering its relativistic dispersion relation $E_i = m \cosh \beta_i$, $p_i = m \sinh \beta_i$. Since the Mandelstam variable $s$ is given by $s = (p_1 + p_2)^2 = 2m^2(1 + \cosh \beta)$, the threshold in this channel is reached when $\beta = 0$ and for the $S$–matrix (1.1) we have

$$S_{AA}(0) = \begin{cases} 1 & \text{if } a \neq 0; \\ -1 & \text{if } a = 0, \end{cases}$$

(1.2)

i.e. a bosonic–type $S$–matrix except when $a = 0$. Notice that the asymptotical behaviour of this $S$–matrix is given by $\lim_{\beta \to \infty} S(\beta) = -1$. As it will become clear in the next section, the $S$–matrix (1.1) can also be equivalently regarded as a pure CDD factor added to an initial fermionic–type $S$–matrix.

A word of caution. In the following we will often use the terminology "QFT" to denote briefly and concisely a hypothetical theory underlying the scattering processes. The terminology does not automatically mean that this underlying theory is a consistent Quantum Field Theory, the peculiar features of which are precisely the object of analysis.

The paper is organised as follows. In Section 2 we briefly discuss the CDD factors and their relation with a perturbed CFT. In Section 3 we discuss the difficulties which the bosonic–type $S$–matrix (1.1) pose both in the TBA analysis and in the Form Factor Approach – difficulties which prevents its interpretation in terms of a standard quantum field theory. Section 4 is devoted to an interpretation of the bosonic–type $S$–matrix (1.1) as the one coming from the critical Ising model perturbed by a combination of relevant and irrelevant operators. We will argue that this combination induces an instability in the ground state of the theory, in agreement with the results of the TBA analysis for the bosonic–type $S$–matrix (1.1). Finally our conclusions are summarised in Section 5.
2 CDD Factors and Deformations of CFT

To analyse the scaling region around a critical point described by a CFT, one usually considers the deformation of the conformal action $A_{CFT}$ by means of one of its relevant operators $\Phi(x)$

$$A = A_{CFT} + \lambda \int \Phi(x) d^2x .$$

The resulting action – which describes a RG flow from the original CFT to another fixed point – gives rise in the Minkowski space to the scattering processes among the massless or massive excitations present away from the critical point \[2\]. Important simplifications occur when the action (2.1) defines an integrable quantum field theory: in this case, in fact, all the scattering processes are purely elastic and can be expressed in terms of the two-particle scattering amplitudes $S_{ab}(\beta)$. On a general basis, these amplitudes satisfy the unitarity and crossing symmetry conditions

$$S_{ab}(\beta)S_{ab}(-\beta) = 1 ;$$

$$S_{ab}(\beta) = S_{ab}(i\pi - \beta) .$$

Possible bound states which occur in the scattering processes are promoted to asymptotic states so that the amplitudes satisfy the additional bootstrap equations

$$S_{cd}(\beta) = S_{bd}(\beta - i\bar{u}_{bc})S_{ad}(\beta + i\bar{u}_{ac}) ,$$

where $\bar{u} \equiv \pi - u$ and $iu_{ab}^c$ is the location of the simple pole in $S_{ab}(\beta)$ which identifies the bound state $A_c$ in the scattering channel $A_A A_B \rightarrow A_a A_b$.

The solution of the above equations presents the well–known ambiguity relative to the CDD factors. Let $\hat{S}_{ab}(\beta)$ be a set of functions with a minimum number of poles which solve both eqs. (2.2) and (2.3). Another solution of the first equations (2.2) can easily be obtained by multiplying each amplitude $\hat{S}_{ab}(\beta)$ by the so–called CDD factors, i.e. an arbitrary product of functions $f_a(\beta)$ defined in eq. (1.1), $\hat{S}_{ab}(\beta) \rightarrow C_{ab}(\beta)\hat{S}_{ab}(\beta)$, where

$$C_{ab}(\beta) = \prod_{a_i}^{N_{ab}} f_{a_i}(\beta) .$$

In fact, each $f_a(\beta)$ automatically satisfies eqs. (2.2) and hence their product. Additional constraints on such CDD factors may come from the bootstrap equations (2.3), which they also have to satisfy. Since the CDD factors do not introduce extra poles in the physical sheet $0 \leq \text{Im } \beta \leq \pi$, the conclusion is that the knowledge of the structure of the bound states alone cannot uniquely fix the scattering theory. This ambiguity is however not harmless since scattering theories which differ by CDD factors have a distinct physical content, in particular a different ultraviolet behaviour.
How can we interpret $S$–matrices which differ by CDD factors in terms of an underlying action? First of all, the CDD factors do not spoil – by construction – the integrable nature of the original $S$–matrix, simply because they only add extra phase–shifts to the original elastic scattering amplitudes. Moreover, they also do not alter the structure of the bound states – an infrared property of theory –, because they do not introduce extra poles in the $S$–matrix. Since they mainly influence the ultraviolet behaviour, it seems natural to assume that $S$–matrices related to one another by CDD factors are associated to integrable actions which, in the vicinity of a fixed point, may differ for insertion of irrelevant scalar operators $\eta_i(x)$

$$\mathcal{A} = \mathcal{A}_{CFT} + \lambda \int \Phi(x) \, d^2x + \mu_i \int \eta_i(x) \, d^2x .$$

(2.4)

These irrelevant operators should preserve the set of conserved currents (or part of it) of the original action (2.1) and they should also share the same symmetry properties of the relevant operator $\Phi(x)$. Their presence has however several consequences. In fact, from a RG geometric point of view the irrelevant operators $\eta_i(x)$ are associated to RG trajectories which flow into the fixed point, whereas the relevant operators are associated to those RG trajectories which depart from it. Hence, the simultaneous presence of relevant and irrelevant operators gives rise to a RG trajectory which will pass very closely to the fixed point in question but without stopping at it (Figure 1). In its vicinity, its action can be parameterised as in (2.4). However, in contrast with the QFT defined by the action (2.1), the action given by eq. (2.4) does not automatically ensure the consistency of the relative quantum field theory in the entire RG space of the couplings, i.e. on all possible distance scales. Obviously, there are some preliminary steps in order to give a meaning to the expression (2.4).

First of all, the irrelevant operators initially present in (2.4) will be accompanied by an infinite number of counterterms, so that the final form of the action of the theory will be actually given by the one of (2.4) together with this infinite tower of higher irrelevant operators. This status of art however is not catastrophic as it may appear since the theory can still remain predictive in view of the fact that all counterterms can be constrained by the request of integrability [20, 21]. A real source of problem is another one. In fact, in contrast to those theories which are obtained by perturbing a CFT by means of relevant operators, where their ultraviolet behaviour is completely under control and ruled in fact by the initially CFT, the ultraviolet behaviour of those quantum field theories defined as in (2.4) will be, instead, a–priori unknown. Since at the short distances the effective

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4 It is worth mentioning that the same line of reasoning has been successfully applied in the context of Boundary Quantum Field Theories, with the CDD factors entering the reflection scattering matrix put in correspondance with the irrelevant boundary deformations of the theory, see [19].
coupling constants of the irrelevant operators become extremely large, it should not be a surprise to find out that one may define a consistent quantum field theory in this regime only for special sign of the leading irrelevant coupling. This condition is realised, for instance, for the roaming trajectory of the Toda Field Theories [22] which in each neighborhood of the fixed points it passes, may be regarded as a RG flow induced by a combination of the relevant operator $\Phi = \varphi_{1,3}$ and the irrelevant operator $\eta = \varphi_{3,1}$, for an appropriate set of the couplings [24]. On the contrary, another choice of the sign of the couplings nearby the critical points may induce an ultraviolet instability of the ground state of the theory at certain scale $\Lambda$ which will spoil a quantum field theory interpretation for distance scales $R \leq \Lambda$. As we will see, this seems to be the scenario associated to the $S$–matrix (1.1).

3 Difficulties with the bosonic–type $S$–matrix

A deceptively simple $S$–matrix affected by a CDD ambiguity is obtained by multiplying a scattering amplitude $S_{aa}(\beta)$ by $-1$ (which may be considered as a particular $f_a$ term, i.e. $\lim_{\Delta \to \infty} f_{\frac{1}{2}+\Delta}(\beta)$). If the original scattering theory does not have any bound states, i.e. if there are no constraints coming from the non–linear bootstrap equations (2.3), there is evidently no obstacle to this multiplication. Consequently, by means of this CDD factor we can swap from a fermionic–type $S$–matrix to a bosonic one and vice versa. This is the case, for instance of the $S$–matrix of the Sinh-Gordon model, which is given by $S_{sh} = f_{B}(\beta)$, where $B$ is a function of the coupling constant of the model [5]. The result of this multiplication is in fact the bosonic–type $S$–matrix (1.1). For this reason, the model associated to the $S$–matrix (1.1) may be referred to as “minus Sinh-Gordon model” (mShG).

There is, however, another way of looking at the $S$–matrix (1.1). It may be seen, in fact, as the one obtained by multiplying $S = -1$ by a CDD factor $f_a(\beta)$. $S = -1$ is the $S$–matrix of the thermal Ising model [1] and therefore the analysis of this bosonic–type $S$–matrix may be pursued along the considerations presented in the previous section, as will be done in Section 4. In the meantime, let us investigate what is the nature of the problems which are posed by the bosonic–type $S$–matrix (1.1), regardless of its interpretation. We will initially study its finite size and ultraviolet behaviour by means

\footnote{There is a simple and even trivial analogy of this situation with other quantum field theory problems: let us consider for instance the hamiltonian of massive boson $H = \frac{1}{2}((\nabla \Phi)^2 + m^2\Phi^2) + \lambda \Phi^4$ to which we add the interaction $\mu \Phi^6$. Assume that the solution of the theory at $\mu = 0$ is exactly known. Although a perturbation theory can be formally applied for any sign of the new coupling $\mu$, it is obvious that the new hamiltonian will define a consistent theory only for positive values of this coupling since for $\mu$ negative, no matter how small it may be, there is no longer a stable vacuum.}
of the TBA approach and we will then proceed to the investigation of its Form Factors.

3.1 Thermodynamical Properties

There is a simple way to see that the theory defined by the $S$-matrix \((1.2)\) presents some problematic aspects from the point of view of a quantum field theory. Let us compute in fact the free energy \(F(mR)\) of such a theory on a cylinder of length \(L\) and radius \(R\) (with \(L \gg R\)) by means of the TBA approach \([13, 14]\). As usual, \(R^{-1}\) can be interpreted as the temperature of the one-dimensional system living on an infinite line, \(R^{-1} = T\).

As it is well-known, once the free energy \(F(mR)\) has been computed, the ground state Casimir energy \(E_0(mR)\) – parameterised in terms of the effective central charge \(C(mR)\) as

\[
E_0(mR) \equiv -\frac{\pi}{6R} C(mR) ,
\]

will be given by

\[
E_0(mR) = F(mR) .
\] (3.2)

In the TBA approach, the free energy \(F(mR)\) is initially expressed in terms of the functional

\[
f(\rho, \rho^{(r)}) = \mathcal{E}(\rho^{(r)}) - \frac{1}{R} \mathcal{S}(\rho, \rho^{(r)}) - \mu \mathcal{N}(\rho^{(r)}) .
\] (3.3)

The energy term is given by

\[
\mathcal{E}(\rho^{(r)}) = \int m \cosh \beta \rho^{(r)}(\beta) d\beta ,
\] (3.4)

the entropy term, in the bosonic case, is given by

\[
\mathcal{S}(\rho, \rho^{(r)}) = \int d\beta \left[ (\rho + \rho^{(r)}) \log(\rho + \rho^{(r)}) - \rho \log \rho - \rho^{(r)} \log \rho^{(r)} \right] ,
\] (3.5)

and the particle number is expressed as

\[
\mathcal{N}(\rho^{(r)}) = \int \rho^{(r)}(\beta) d\beta .
\] (3.6)

In the above formulas, \(\rho^{(r)}(\beta)\) is the density of occupied states (roots) of rapidity \(\beta\) whereas \(\rho(\beta)\) is the total density of states (roots and holes), \(\rho = \rho^{(r)} + \rho^{(h)}\). They are related each other by the integral equation

\[
\rho(\beta) = \frac{m}{2\pi} \cosh \beta + \int \varphi(\beta - \beta') \rho^{(r)}(\beta') \frac{d\beta'}{2\pi} ,
\] (3.7)

with the kernel \(\varphi(\beta)\) given in our case by

\[
\varphi(\beta) = -i \frac{d}{d\beta} \log S = \frac{2 \sin \pi a \cosh \beta}{\sinh^2 \beta + \sin^2 \pi a} .
\]
Let us assume that the functional (3.3) admits a minimum with respect to the distributions $\rho(\beta)$ and $\rho^{(r)}(\beta)$, with eq. (3.7) acting as a constraint. If this would be the case, the partition function $Z(L, R)$ in the presence of a chemical potential $\mu$ ($z \equiv e^{\mu R}$) and in the limit $L \to \infty$ can be computed in the saddle point approximation and expressed as

$$Z(L, R) \sim \exp \left[ -mL \hat{F}(R) \right], \quad \hat{F}(R) = \int_{-\infty}^{+\infty} \cosh \beta \log(1 - ze^{-\epsilon(\beta)}) \frac{d\beta}{2\pi}, \quad (3.8)$$

where $\hat{F}(R)$ is the value of the functional $Rf(\rho, \rho^{(r)})$ computed at its minimum and the minimum condition is provided by the integral equation satisfied by the pseudo-energy $\epsilon(\beta)$

$$\epsilon(\beta) = mR \cosh \beta + \int_{-\infty}^{+\infty} \varphi(\beta - \beta') \log(1 - ze^{-\epsilon(\beta')}) \frac{d\beta'}{2\pi}. \quad (3.9)$$

In terms of $\epsilon(\beta)$, the distributions $\rho(\beta)$ and $\rho^{(r)}(\beta)$ which provide the minimum for the functional (3.3) are related each other by

$$\rho^{r} = \rho \frac{ze^{-\epsilon}}{1 - ze^{-\epsilon}}. \quad (3.10)$$

The above discussion briefly summarises the basic ideas of the Thermodynamical Bethe Ansatz approach. Let us now analyse how they are actually implemented in the case of our bosonic model, starting our analysis from the expression of the free–energy obtained in the saddle point approximation of the TBA equations, i.e. from eqs. (3.8) and (3.9). At zero chemical potential $z = 1$, no matter how small $a$ may be (but never zero), the numerical solution of the above equations shows the existence of a critical value of the variable $R$, such that for $R < R_c(a)$, the free energy assumes complex values (Figure 2). It is worth noting the novelty of this situation, which in fact does not occur in all other known (fermionic) models, analysed in the past in terms of the TBA equations (see, for instance, [15]). How can we interpret the occurrence of this critical point in $R$ and the complex values assumed by the free energy for $R < R_c$? What are the consequences of this result? The answer to these questions may be expressed both in physical or in purely mathematical terms.

The simplest physical answer is the following. Since the TBA equations provides the full resummation of the virial expansion of $Z(L, R)$, $R_c^{-1}(a)$ is then identified with the radius of convergence of the corresponding series. The existence of this singularity indicates a non-trivial ultraviolet behaviour of the theory, in particular, it definitly precludes its standard interpretation in terms of a Conformal Field Theory deformed by a relevant operator$^6$.

$^6$ The problem with the ultraviolet behaviour of such theory can also be seen without having recourse
The singularity at \( R = R_c(a) \) may be effectively regarded as a sort of Bose–Einstein condensation phenomenon. To show this, firstly notice that the partition function (3.8) admits an interpretation as that of a free bosonic gas of excitations \( |\chi(\beta)\rangle \) but with dispersion relations which depend on the temperature [13]. In particular, for the energy \( \mathcal{E}(\beta) \) of one of these excitations \( |\chi(\beta)\rangle \) we have \( \mathcal{E}(\beta) \equiv \epsilon(\beta)/R \), where \( \epsilon(\beta) \) is solution of the integral equation (3.9). Hence, in a way completely similar to the fermionic case analysed in [25], one can establish the equivalence between the partition function obtained in eq. (3.8) and the one computed in terms of the following series

\[
Z(L, R) = \sum_{n=0}^{\infty} z^n \langle \chi(\beta_1) \dots \chi(\beta_n) | \chi(\beta_1) \dots \chi(\beta_n) \rangle e^{-\epsilon(\beta_1) - \cdots - \epsilon(\beta_n)},
\]  

(3.11)

where the scalar product of the states is computed according to the commutation rules of free bosonic particles, with a regularization given by

\[
[\delta(\beta - \beta')]^2 \equiv \frac{mL}{2\pi} \cosh(\beta) \delta(\beta - \beta').
\]

(3.12)

In fact, for the partition function in (3.8) we have the following expansion in powers of \((mL)\)

\[
Z(L, R) = 1 - (mL)F(R) + \frac{(mL)^2}{2!}(F(R))^2 + \cdots (-1)^n \frac{(mL)^n}{n!}(F(R))^n + \cdots
\]

(3.13)

where \( F(R) \) admits the representation

\[
F(R) = -\sum_{n=0}^{\infty} \frac{z^n}{n} \mathcal{I}_n(R),
\]

(3.14)

with

\[
\mathcal{I}_n(R) \equiv \int \frac{d\beta}{2\pi} \cosh \beta e^{-n\epsilon(\beta)}.
\]

(3.15)

On the other hand, computing the partition function by using the expression (3.11),

\[
Z(L, R) = 1 + z \mathcal{Z}_1 + z^2 \mathcal{Z}_2 + \cdots z^n \mathcal{Z}_n + \cdots
\]

(3.16)

for the first term we have

\[
\mathcal{Z}_1 = \int \frac{d\beta}{2\pi} \langle \beta | \beta \rangle e^{-\epsilon(\beta)} = \int \frac{d\beta}{2\pi} \delta(\beta - \beta') \langle \beta' | \beta \rangle e^{-\epsilon(\beta')} = \frac{mL}{2\pi} \cosh(\beta) e^{-\epsilon(\beta)} = (mL) I_1;
\]

(3.17)

to numerical integration of the above equations. In fact, in consistent theories, the ultraviolet behaviour is ruled by the so–called kink configuration of eq. (3.9) and the corresponding dilogarithmic functions. In our case the plateau of the kink configuration, solution of algebraic equation \( \epsilon_{pl} = \log(1 - e^{-\epsilon_{pl}}) \) is in fact complex, \( \epsilon_{pl} = \pm i\frac{\pi}{3} \).
and similarly

\[ Z_2 = \frac{1}{2} (mL)^2 I_1^2 + \frac{1}{2} (mL) I_2 ; \]
\[ Z_3 = \frac{(mL)^3}{3!} I_1^4 + \frac{(mL)^2}{2} I_1 I_2 + \frac{(mL)}{3} I_3 , \]
\[ Z_4 = \frac{(mL)^4}{4!} I_1^4 + (mL)^3 I_1^2 I_2 + \frac{(ML)^2}{2} \left[ \frac{2}{3} I_1 I_3 + \left( \frac{I_2}{2} \right)^2 \right] + \frac{(mL)}{4} I_4 , \]

(3.18)

etc. Hence, it is easy to see that the series (3.13) precisely coincides with that of eq. (3.16), the only difference being in the arrangement of the single terms: the terms proportional to \((mL)\) in all \(Z_n\) gives \(F(R)\) as their sum, whereas the sum of the terms proportional to \((mL)^n\) appearing in all \(Z_n\) gives rise to the higher power \((F(R))^n\).

In the light of these considerations, it is then natural to expect that the partition function (3.8) will have a singularity at that value of the temperature \(R_c^{-1}(a)\) where the energy of the excitation (at zero rapidity) vanishes (Figure 3). When this happens, the contribution coming from the average occupation number with \(E = 0\) becomes as important as the entire series, producing a singularity in the free energy which resembles that of the usual Bose–Einstein condensation. One should not be surprised that such effect occurs in this low dimensional system due to the entanglement of the energy \(E(\beta)\) of its excitation with the temperature itself of the system. In conclusion, from the TBA analysis one infers that in our bosonic–type model there is a scale of distance \(\Lambda = R_c\) below which there is a change in the ground state of the theory. Obviously it is not surprising to find out that the evaluation of the series beyond its radius of convergence provides complex values, although no direct physical meaning can be assigned to them.

From a purely mathematical point of view, the occurrence of complex values in \(\hat{F}(R)\) simply indicates the failure of the minimization procedure applied to evaluate the free–energy (3.3). Namely, in our case, it is no longer true that the functional (3.3) admits a minimum with respect to the (positive) real distributions \(\rho(\beta)\) and \(\rho^{(r)}(\beta)\) for any value of \(R\). As a matter of fact, the real minima of the functional (3.3) disappear for \(R \leq R_c\), becoming complex. Hence the value of the functional (3.3) computed at its minima, i.e. \(R\hat{F}(R)\), takes for \(R < R_c\) complex values whereas the functional \(f(\rho, \rho^{(r)})\) itself becomes instead unbounded from below for this range of \(R\). This implies that the saddle point procedure employed by the TBA is no longer justified\[7\]. However, if we still insist on enforcing the validity of the relationship (3.2) between the free–energy and the Casimir energy of the theory, we should conclude that the divergence of the free–energy for \(R \leq R_c\) implies the corresponding divergence of the central charge of the theory. As

\[ ^7 \text{This circumstance is analysed in full detail in Appendix A for simplified version of a TBA system based on a particular bosonic}\ S\text{–matrix.} \]
we will show in the next section, the same conclusion is also reached by using the Form Factor approach.

### 3.2 Form Factors

Let us now analyse the computation of the Form Factors associated to the $S$–matrix (1.1). As we will see, this calculation presents some new and distinct features with respects the analogous calculation for fermionic–type theories.

The definition of the Form Factors of a local hermitian scalar operator $O$ is given as usual by

$$F^O(\beta_1, \ldots, \beta_n) = \langle 0|O|A(\beta_1), \ldots, A(\beta_n) \rangle .$$

(3.19)

These quantities satisfy a set of functional equations [6, 7]

$$F_n(\beta_1, \ldots, \beta_i, \beta_{i+1}, \ldots, \beta_n) = F_n(\beta_1, \ldots, \beta_{i+1}, \beta_i, \ldots, \beta_n) S(\beta_i - \beta_{i+1}) ,$$

$$F_n(\beta_1 + 2\pi i, \ldots, \beta_{n-1}, \beta_n) = F_n(\beta_2, \ldots, \beta_n, \beta_1) = \prod_{i=2}^{n} S(\beta_i - \beta_1) F_n(\beta_1, \ldots, \beta_n)$$

(3.20)

$$\lim_{\beta \to \beta} (\beta - \beta) F^O_n(\beta + i\pi, \beta, \beta_1, \beta_2, \ldots, \beta_n) = i \left(1 - \prod_{i=1}^{n} S(\beta - \beta_i)\right) F^O_n(\beta_1, \ldots, \beta_n) ,$$

which closely resemble those of the Sinh–Gordon theory [26].

For scalar operators, the FF depend solely on the rapidity differences $\beta_{ij} = \beta_i - \beta_j$. As it is evident from the residue equation in (3.20), the chain of FF with even and odd number of particles are decoupled from each other. For simplicity, in the sequel we will consider only FF of even operators under the $Z_2$ symmetry of the model (i.e. those which have only non–vanishing FF on an even number of external states). They can be correspondingly parameterised as

$$F^O_n(\beta_1, \ldots, \beta_n) = H_n \frac{Q_n(x_1, \ldots, x_n)}{(\sigma_n)^{n/2}} \prod_{i<j} \frac{F_{\min}(\beta_{ij})}{x_i + x_j} ,$$

(3.21)

where $x_i = e^{\beta_i}$, $H_n$ is a normalization constant and $F_{\min}(\beta)$ is a function which satisfies the equations

$$F_{\min}(\beta) = S(\beta) F_{\min}(-\beta) ,$$

$$F_{\min}(i\pi - \beta) = F_{\min}(i\pi + \beta) .$$

(3.22)

$Q_n(x_1, \ldots, x_n)$ are symmetric polynomials in the variables $x_1, \ldots, x_n$ of total degree $t_n = \frac{n}{2}(2n - 1)$. They can be expressed in terms of the so–called elementary symmetric polynomials.
polynomials \( \sigma_j^{(n)} \) given by

\[
\sigma_j^{(n)} = \sum_{i_1 < i_2 < \ldots < i_j}^{n} x_{i_1} x_{i_2} \ldots x_{i_j}.
\] (3.23)

Equivalently, they can be obtained in terms of the generating function

\[
\prod_{i=1}^{n} (x + x_i) = \sum_{k=0}^{n} x^{n-j} \sigma_j^{(n)}(x_1, x_2, \ldots, x_n).
\] (3.24)

Although at a formal level, the above formulas of the FF appear identical to those of the Sinh–Gordon model (except for the extra factor \( \sigma_n \frac{n}{2} \) in the denominator), a substantial difference between the two models occurs in the properties of \( F_{\text{min}}(\beta) \). As we will see, this difference has far-reaching consequences.

The two–particle minimal FF of our bosonic–type model may be taken as

\[
F_{\text{min}}(\beta, a) = \exp \left[ -2 \int_0^\infty \frac{dx}{x} \cosh \left( \frac{x}{2}(1 - 2a) \right) \sinh x \cosh \frac{2\hat{\beta}}{2\pi} \right],
\] (3.25)

which can also be expressed as

\[
F_{\text{min}}(\beta, a) = \mathcal{N}(a) \prod_{k=0}^{\infty} \left( \frac{\Gamma \left( k + \frac{1}{2} + \frac{a}{2} + \frac{i\hat{\beta}}{2\pi} \right) \Gamma \left( k + 1 - \frac{a}{2} + \frac{i\hat{\beta}}{2\pi} \right)}{\Gamma \left( k + \frac{3}{2} - \frac{a}{2} + \frac{i\hat{\beta}}{2\pi} \right) \Gamma \left( k + 1 + \frac{a}{2} + \frac{i\hat{\beta}}{2\pi} \right)} \right)^2,
\] (3.26)

where \( \hat{\beta} = i\pi - \beta \). With \( F_{\text{min}}(i\pi, a) = 1 \), we have

\[
\mathcal{N}(a) = \prod_{k=0}^{\infty} \left( \frac{\Gamma \left( k + 1 + \frac{a}{2} \right) \Gamma \left( k + 3/2 - \frac{a}{2} \right)}{\Gamma \left( k + 1 - \frac{a}{2} \right) \Gamma \left( k + 1/2 + \frac{a}{2} \right)} \right)^2.
\] (3.27)

For \( a \neq 0 \), this function has a finite value at the threshold \( \beta = 0 \) given by

\[
F_{\text{min}}(0, a) \equiv \mathcal{F}(a) = \exp \left[ \int_0^\infty \frac{dx}{x} \cosh \left( \frac{x}{2}(1 - 2a) \right) \sinh \frac{x}{2} \cosh \frac{2\hat{\beta}}{2\pi} \right],
\] (3.28)

which is in stark contrast with the value \( F_{\text{min}}(0) = 0 \) of fermionic–type theories, which is forced by the condition \( S(0) = -1 \) of their \( S \)–matrix. The asymptotical behaviour of \( F_{\text{min}}(\beta, a) \) is ruled by

\[
\lim_{\beta \to \infty} F_{\text{min}}(\beta, a) \simeq i \sqrt{2} \sin \pi a \mathcal{F}(a) \exp \left[ -\frac{\hat{\beta}}{2} \right].
\] (3.29)

\(^8\)In the following we will discard the upper index \( n \) in the definition of \( \sigma_k \), since the total number of variables involved will be clear from the context.
Finally, in the limit $a \to 0$, $F_{\text{min}}(\beta, a)$ reduces to

$$F_{\text{min}}(\beta, 0) = \frac{i}{\sinh \frac{\beta}{2}}. \quad (3.30)$$

In order to implement the recursive equations associated to the last equation in (3.20) it is useful to consider the functional equation

$$F_{\text{min}}(\beta + i\pi, a)F_{\text{min}}(\beta, a) = \frac{2i\pi^2 N^2(a)}{\sin \beta + i \sin \pi a}. \quad (3.31)$$

The residue equation in (3.20) implies that the polynomials $Q_n(x_1, \ldots, x_n)$ in eq. (3.21), satisfy the recursive equations

$$Q_{n+2}(-x, x, x_1, \ldots, x_n) = x^3 D(x, x, x_1, \ldots, x_n) Q_n(x_1, \ldots, x_n), \quad (3.32)$$

where

$$D(x, x_1, \ldots, x_n) = 2 (-1)^{\frac{n}{2}} \sum_{\{l,k,p,q\}=0}^n (-1)^{l+q} x^{4n-l-k-p-q} \frac{\sin[\pi a(p-q)]}{\sin \pi a} \sigma_l \sigma_k \sigma_q \sigma_p. \quad (3.33)$$

In writing eq. (3.32) we have posed for convenience

$$H_{n+2} = H_n \frac{\sin \pi a}{(2i \sin \pi a F(a))^n}. \quad (3.34)$$

Since $D(x, x_1, \ldots, x_n)$ contains four elementary symmetric polynomials and each of them is linear in the individual variables $x_i$, the partial degrees $p_n$ of the polynomials $Q_n$ satisfies the condition $p_{n+2} \leq 4 + p_n$.

After these general considerations, let us now attempt to solve the above recursive equations for a particular but significant field. Namely, let us assume that there is an operator in the theory which plays the role of the trace $\Theta(x)$ of the stress–energy tensor $T_{\mu\nu}(x)$. If such an operator exists, its Form Factors have some distinguished properties which may be useful in their explicit determination. First of all, its normalization will be given by

$$F^{\Theta}(i\pi) = \langle A(\beta)|\Theta(0)|A(\beta) \rangle = 2\pi m^2. \quad (3.35)$$

Secondly, from the conservation law $\partial^\mu T_{\mu\nu}(x) = 0$, the polynomials $Q_n(x_1, \ldots, x_n)$ can be factorised as $Q_n(x_1, \ldots, x_n) = \sigma_1 \sigma_{n-1} P_n(x_1, \ldots, x_n)$. Moreover, assuming that this operator is even under the $Z_2$ symmetry of the model, its only non–vanishing FF will be those with an even number of external particles. Finally, the two–point correlation

\footnote{Plugging $\beta = 0$ in (3.31), one obtains the relation $F(a) = \frac{2\pi^2 N^2(a)}{\sin \pi a}$.}
function of $\Theta(x)$ enters a sum–rule which permits evaluation of the difference of the central charges of the theory by going from the large to the short distance scales

$$\Delta C = C_{uv} - C_{ir} = \frac{3}{4\pi} \int d^2 x \, \langle 0|\Theta(x)\Theta(0)|0 \rangle = \int_0^\infty d\mu \, c_1(\mu) \ , \quad (3.36)$$

where $c_1(\mu)$ may be directly expressed in terms of the FF of $\Theta(x)$

$$c_1(\mu) = \frac{12}{\mu^2} \sum_{k=1}^\infty \frac{1}{(2n)!} \int \frac{d\beta_1 \ldots d\beta_{2k}}{(2\pi)^{2n}} \, |F_{2k}^{\Theta}(\beta_1, \ldots, \beta_{2k})|^2 \right) \times \delta\left(\sum_i m \sinh \beta_i\right) \delta\left(\sum_i m \cosh \beta_i - \mu\right) . \quad (3.37)$$

Since at the large distance scale the theory is massive, we will have $C_{ir} = 0$, and the above sum–rule should provide a determination of the central charge $C_{uv}$ of the theory in its ultraviolet regime. The first approximation of the above sum–rule is obtained by the two–particle contribution

$$\Delta C^{(2)}(a) = \frac{3}{2} \int_0^\infty \frac{d\beta}{\cosh^4 \beta} |F_{\text{min}}^{2\beta, a}|^2 . \quad (3.38)$$

With this information on the FF of $\Theta(x)$, let us proceed in the computation of its first representatives. Since in the limit $a \to 0$, our bosonic–type $S$–matrix reduces to the one of thermal Ising model – for which the only non–vanishing FF of $\Theta(x)$ is the one relative to the two–particle state

$$F^{\text{Ising}}_{\text{min}}(\beta) = -2\pi m^2 i \sinh \frac{\beta}{2} , \quad (3.39)$$

we are forced to take as $Q_2(x_1, x_2)$

$$Q_2(x_1, x_2) = \sigma_1 (\sigma_1^2 - 4\sigma_2) , \quad (3.40)$$

and $H_2 = -\frac{3}{2} m^2$. These two quantities play the role of initial values for the recursive equations (3.32).

Notice that by inserting the two–particle FF in (3.38), the corresponding quantity is an increasing function of the parameter $a$, which takes its minimum value $\Delta C^{(2)} = \frac{1}{2}$ at $a = 0$ (as in the thermal Ising model) and reaches its maximum $\Delta C^{(2)} = 0.876...$ at $a = \frac{1}{2}$ (see Figure 4). This increasing monotonic behaviour of $\Delta C^{(2)}(a)$ is in contrast with the decreasing monotonic behaviour presented by the same quantity for fermionic–type theories (see for instance the Sinh-Gordon model [26]).

Using now the factorised form $Q_4(x_1, \ldots, x_4) = \sigma_1 \sigma_3 P_4(x_1, \ldots, x_4)$ and the above expression for $Q_2(x_1, x_2)$, the recursive equation satisfied by $P_4(x_1, \ldots, x_4)$ becomes

$$P_4(-x, x_1, x_2) = -4 \left[ x^8 - x^6 \sigma_1^2 + x^6 \sigma_2 + x^4 \sigma_1^2 \sigma_2 - x^4 \sigma_2^2 - x^2 \sigma_2^2 \right] (\sigma_1^2 - 4\sigma_2) . \quad (3.41)$$

13
In the linear space of the symmetric polynomials with total degree \( t = 10 \) and partial degree \( p \leq 5 \), the above recursive equation does not uniquely fix the polynomial \( P_4(x_1, \ldots, x_4) \), which in fact admits a three–parameter family of solutions

\[
P_4(x_1, \ldots, x_4) = A_1(\sigma_1^3 \sigma_3 \sigma_4 + \sigma_1 \sigma_3^3 - \sigma_1^2 \sigma_2 \sigma_3^2) + A_2(\sigma_1^2 \sigma_2^2 \sigma_4 + \sigma_2^2 \sigma_3^2 - \sigma_1 \sigma_2^3 \sigma_3) + \\
+ A_3(\sigma_1^2 \sigma_4^3 + \sigma_3^2 \sigma_4 - \sigma_1 \sigma_2 \sigma_3 \sigma_4) + 32 \sigma_1 \sigma_2 \sigma_3 \sigma_4 + \\
+ 16 \sigma_3^2 \sigma_4 - 4 \sigma_1^2 \sigma_2 \sigma_3^2 - 4 \sigma_1 \sigma_3^2 \sigma_3 - 64 \sigma_2 \sigma_4^2.
\] (3.42)

This arbitrariness in the four–particle FF of \( \Theta(x) \) may be partially reduced by imposing an additional condition on its FF, namely that they fulfill the “cluster equations” [15, 28, 29]

\[
\lim_{\Lambda \to \infty} F_4^{\Theta}(\beta_1 + \Lambda, \beta_2 + \Lambda, \beta_3, \beta_4) = \frac{1}{\langle \Theta \rangle} F_2^{\Theta}(\beta_1, \beta_2) F_2^{\Theta}(\beta_3, \beta_4).
\] (3.43)

By using the cluster properties of the elementary symmetric polynomials \( \sigma_k \), as determined in [15], together with the asymptotical behaviour eq. (3.29) of \( F_{\min}(\beta, a) \) and the vacuum expectation value \( \langle \Theta \rangle = \frac{\pi m^2}{2 \sin \pi a} \), the cluster equation (3.43) provides the following conditions on the constants \( A_1 \) and \( A_2 \)

\[
A_1 = -1, \quad A_2 = -4.
\] (3.44)

However, the constant \( A_3 \) cannot be fixed in this way, since all terms proportional to it in (3.42) are sub–leading in the above limit (3.43). This arbitrariness of the FF of \( \Theta(x) \), which persists for higher FF, shows that in the mShG model the \( S \)–matrix alone cannot uniquely fix the matrix elements of one of its significant operators. This arbitrariness in the FF of \( \Theta(x) \) should be contrasted with their unique determination in all known examples of fermionic–type \( S \)–matrix [see, for instance \[3, 9, 12, 26\]]

Concerning the \( c \)–theorem sum rule, some conclusions can be also drawn despite the arbitrariness present in the FF of \( \Theta(x) \). Namely, let us assume that the arbitrary constants present at each stage of the recursive equations could be fixed according to some additional principle. What will then be the generic behaviour of the series (3.37) entering the \( c \)–theorem? For the finiteness of the value assumed by the FF at all particle thresholds, an estimate of the integral (3.36) can be provided by only considering the contributions at all thresholds. To do so, let us first observe that the sum (3.37) – apart from the prefactor \( 12/\mu^3 \) –, is an integral of the square of each FF integrated on the phase–space \( \Omega_{2k}(\mu) \) of \( 2k \) particle, defined as

\[
\Omega_{2k}(\mu) = \int \frac{dp_1}{(2\pi)^2 E_1} \cdots \frac{dp_{2k}}{(2\pi)^2 E_{2k}} \delta(\mu - E_1 - \cdots - E_{2k}) \delta(p_1 + \cdots + p_{2k}).
\] (3.45)

\[10\] It is easy to check that also higher Form Factors of \( Z_2 \) odd operators (as the one which creates the particle \( A \) of the mShG model) cannot uniquely fixed.
Near the threshold this quantity may be expressed as

$$\Omega_{2k}(\mu) \simeq \frac{1}{\sqrt{4\pi km}} \left( \frac{1}{8\pi m} \right)^k \frac{1}{\Gamma \left( k - \frac{1}{2} \right)} \left( \mu - 2km \right)^{k - \frac{3}{2}}. \quad (3.46)$$

Concerning the value of the $F_{2k}(\beta_1, \ldots, \beta_{2k})$ at their threshold ($\beta_{ij} = 0$, for all $i$ and $j$), we can split it into two terms: the first one is the product of $H_{2k}$ together with all $F_{\text{min}}(0)$, given by

$$H_{2k} \frac{\sin^2 \pi a}{2 \sin^2 \pi a} (2^k \sin^2 \pi a)^k \left( \frac{\mathcal{F}(a)}{2^k \sin \pi a} \right)^2. \quad (3.47)$$

For the remaining term, we can pose

$$\frac{Q_{2n}(x_1, \ldots, x_n)}{\sigma_{2n}^k \prod_{i<j} (x_i + x_j)} \mid_{\beta_{ij}=0} \simeq \left( \frac{\xi}{2} \right)^{k(2k-1)}. \quad (3.48)$$

An estimate of the ratio $\rho = \frac{\xi}{2}$ may be obtained as follows. From eq. (3.24), the numerical values assumed by the elementary symmetric polynomials $\sigma_j$ at $\beta_i = 0$ (denoted by $\hat{\sigma}_j$) is given by the binomial coefficient $\hat{\sigma}_j = \binom{2k}{j}$. The maximum of these values is for $\sigma_k = \binom{2k}{k} \simeq 4^k$. At the threshold, in the space of symmetric polynomials of total degree $k(4k-1)$ and partial degree $2(2k-1)$, the term which is expected to dominate in the limit $k \to \infty$ is given by $[\hat{\sigma}_k]^{4k-3} \hat{\sigma}_{2k} \simeq 16^{2k-\frac{3}{2}k}$, so that $\xi \sim 16$ and the ratio $\rho$ is therefore expected to be always larger than 1.

Let us now apply the above considerations to the $c$–theorem sum–rule. We will approximate the integral (3.36) by using the “mean theorem”, applied in each interval between all successive thresholds. Apart from some constants, the series entering eq. (3.36) has the following behaviour:

$$\Delta C \simeq \sum_k X^{2k^2}(a) Z^{2k}(a) (2k)! \Gamma \left( k - \frac{1}{2} \right) k, \quad (3.49)$$

where

$$X(a) = \frac{\mathcal{F}(a)}{2 \sin \pi a} \rho^2, \quad (3.50)$$

$$Z(a) = \frac{2 \sin^2 \pi a}{\rho \sqrt{\pi}}.$$

The nature of the series (3.49) is obviously controlled by the parameter $X(a)$: since the ratio $\frac{\mathcal{F}(a)}{2 \sin \pi a}$ entering $X(a)$ is always larger than 1.6. (see Figure 5) and $\rho$ is expected to
be larger than 1, we conclude that, due to the finite value assumed by the FF at the higher thresholds, the series (3.49) is always divergent. This divergence should be regarded as a manifestation of the pathological aspect of the quantum field theory associated to mShG model in its ultraviolet region, as we already learnt by the TBA approach.

4 Irrelevant Deformations of the Ising Model

With the above insights to the nature of the problems presented by the bosonic–type $S$–matrix (1.1), let us come back to the considerations of Section 2 with the aim to identify an underlying action for such an $S$–matrix. We will take the point of view in which $f_a(\beta)$ in eq. (1.1) is seen as a CDD factor for the fermionic $S$–matrix $S = -1$. The latter is identified as the one of the thermal Ising model. Hence, we have to look for irrelevant operators in this model which both preserve its original integrability and its $Z_2$ symmetry such that the extra phase–shift which they induce matches with the CDD factor $f_a(\beta)$.

In the conformal Ising model there are few operators we can play with: in fact, we have the operators of the conformal family of the identity $I$ (which includes the stress–energy tensor $T$), those of the conformal family of the energy field $E$ and finally those of the magnetization field $\sigma$. The operators of the even sector of this model, i.e. those of identity and energy families, can be written in terms of local expressions of the chiral $\Psi(z)$ and anti–chiral $\bar{\Psi}(\bar{z})$ components of a Majorana fermionic field. In particular we have $E(z, \bar{z}) = i\bar{\Psi}(\bar{z})\Psi(z)$ and, for the analytic and anti–analytic part of the stress energy tensor, $T(z) = \frac{1}{\pi} : \Psi \partial \Psi :$ and $\bar{T} = \frac{1}{\pi} : \bar{\Psi} \partial \bar{\Psi} :$.

The original fermionic–type $S$–matrix $S = -1$ originates from the CFT action perturbed by the relevant operator of the energy field

$$A = A_{CFT} + m \int E(x) d^2x .$$

In the euclidean space it may be written as

$$A = \frac{1}{2\pi} \int (\Psi \partial \bar{\Psi} + \bar{\Psi} \partial \Psi + 2im\bar{\Psi}\Psi) d^2x .$$

Let us analyse which operators we can add to this action in order to have an extra phase–shift in the $S$–matrix. None of these operators can be of the magnetic field family as they would spoil both its $Z_2$ symmetry and its integrability [16, 31, 32]. They cannot be either descendent fields of the energy field, since they are all quadratic in the fermionic field: their insertion in the action (4.2) will not induce scattering processes among the fermion particles but will only change their dispersion relations. Hence, the first irrelevant field which can be introduced in (4.1) in order to have extra scattering processes is $TT$. This
operator is in fact quartic in the fermion fields and the new action becomes

$$\mathcal{A} = \frac{1}{2\pi} \int \left( \bar{\Psi} \partial \Psi + \bar{\Psi} \partial \bar{\Psi} + 2im\bar{\Psi}\Psi \right) d^2x + \frac{g}{\pi^2 m^2} \int \bar{\Psi} \partial \bar{\Psi} \partial \bar{\Psi} d^2x + \cdots \quad (4.3)$$

where $g$ is a dimensionless constant. With the insertion of this operator, the theory is still integrable at the lowest order \[33\] although $\mathcal{A}$ in (4.3) should be considered at this stage as an effective action. The coupling constant $g$ can be related to the parameter $a$ entering the $S$–matrix (1.1) as follows. Since the operator $T\bar{T}$ is expected to be effective at very high–energy scales, this suggests matching the lowest order coming from the perturbation theory of (4.3) with the $S$–matrix (1.1), both computed in their high–energy limit. In this kinematical regime the two particles involved in the scattering can be effectively seen as left and right movers. Their momenta $p$ and $q$ may be parameterised in terms of a new rapidity variable $\theta$ as $p = me^{\theta_1}$, $q = -me^{-\theta_2}$, so that the Mandelstam variable $s$ is expressed in this limit as $s = 2m^2e^{\theta_1}$. The lowest order in $g$ of the scattering amplitude in this regime may be computed as in [20] with the result

$$S(s) \simeq -1 + 2ig \frac{s}{m^2} + \cdots \quad (4.4)$$

Let us now compare this expression with the one from the expansion of the $S$–matrix (1.1) around the point $\beta_{12} = \infty$. By setting $1/\sinh \beta_{12} \equiv e^{\theta_{12}}$ as a parameterization around $\beta_{12} = \infty$, the $S$–matrix (1.1) may be written in the vicinity of this point as

$$S(\theta_{12}) = \frac{s + i\frac{m^2}{\sin \pi a}}{s - i\frac{m^2}{\sin \pi a}} \simeq -1 + 2i \sin \frac{\pi a}{m^2} s + \cdots \quad (4.5)$$

so that we have the identification

$$g \simeq \sin \frac{\pi a}{2} \simeq a \quad (4.6)$$

i.e. $g$ is a positive quantity. It is important to note that the sign of the above coupling constant is opposite of the one relative to the roaming model nearby the Ising fixed point \[20\]. This difference in the sign of the coupling relative to the first irrelevant operator seems therefore responsible for the quite distinct ultraviolet behaviour of the two theories: the roaming model has in fact a smooth ultraviolet behaviour controlled by the nearest tricritical Ising point (with central charge $C = \frac{7}{10}$) whereas the QFT associated to our bosonic $S$–matrix develops instead an ultraviolet instability. This situation evidently resembles the two different behaviours which originate from the change of sign in the example of footnote 5.

Let us finally add a few comments on the full hamiltonian of the theory. As discussed in Section 2, the action (4.3) will also contain higher derivative terms $\bar{\Psi} \partial^n \bar{\Psi} \partial^n \Psi$, so
that in general we can write
\[
A = \frac{1}{2\pi} \int (\Psi \partial \Psi + \bar{\Psi} \partial \bar{\Psi} + 2im\bar{\Psi}\Psi) \ d^2x + \frac{g}{\pi^2m^2} \int \Psi \partial \Psi \bar{\Psi} \partial \bar{\Psi} \ d^2x + \quad (4.7)
\]
\[
+ \frac{g}{\pi^2} \sum_{n=2}^{\infty} \frac{a_n}{m^{2n}} \int \Psi \partial^n \bar{\Psi} \partial^n \Psi \ d^2x .
\]

The a–dimensional constants \(a_n\) can be in principle determined by the integrability condition of the theory, i.e. by matching the expansion of the \(S\)–matrix in power of \(s/m^2\) with the Feynman graphs which originate from (4.8). For the presence of the infinite higher derivative terms in the action, the corresponding hamiltonian (with respect to the Ising fixed point) will be then a non–local one. In the Majorana basis\(^\dagger\) denoted as before by \((\Psi(x,t), \bar{\Psi}(x,t))\), it is convenient to write the Hamiltonian in terms of a dimensionless kernel \(V(\hat{z})\), \(\hat{z} = mz\), as
\[
H = \int [\cdots ] : dx + gm^2 \int \int : \bar{\Psi} \Psi : (x)V(\hat{x} - \hat{y}) : \bar{\Psi} \Psi : (y) dx dy
\]
(4.8)
in such way that the coupling constants of the higher derivative terms are nothing but the higher moments of \(V(\hat{z})\). Therefore, once the coefficients \(a_n\) were known, the kernel \(V(\hat{z})\) can be recovered in terms of an inverse Mellin transformation. The advantage of expressing the hamiltonian in this way is twofold. First of all, it shows that the hamiltonian is actually renormalisable albeit obviously non–local. Secondly, it provides an easy argument to show that the non–local term is the one responsible for the instability of the theory. This argument is based on the simple hypothesis that \(\int V(\hat{z})d\hat{z} > 0\). Let us consider in fact the Hamiltonian (4.8) in the Hartree–Fock approximation and let us define an order parameter \(\Delta = -i \langle \Psi \bar{\Psi} \rangle\). For field configurations which do not vary on \(x\), the value of the Hamiltonian (4.8) for unit volume is given by
\[
\frac{H}{L} = m\Delta - gv\Delta^2 ,
\]
(4.9)
where \(v = \int V(\hat{z})d\hat{z} > 0\). When \(g = 0\), we can always subtract from \(H/L\) an infinite constant (due to the Dirac sea of the fermion) such that we can restrict to the positive values of \(\Delta\) and the minimum of \(H/L\) is therefore obtained for \(\Delta = 0\). This subtraction is equivalent to a selection of the vacuum state for the free theory. However, when we switch on a positive value of \(g\), the quantity \(H/L\) becomes once again unbounded from below, so that there would be an instability of the vacuum originally selected by the free theory. Hence the above hypothesis permits to understand in an easy way the nature of the problem posed by our bosonic \(S\)–matrix, i.e. the instability of the vacuum state of the underlying microscopic Hamiltonian.

\(^\dagger\)In this basis \(\gamma^0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\) and \(\gamma^1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\).
5 Conclusions

In this paper we have discussed the physical properties of the simplest bosonic–type $S$–matrix with the aim to clarify the nature of its pathological behaviour, in particular at its short–distance scales. The TBA approach has shown that the difficulties consist in an instability of the vacuum state of the theory which occurs at a certain distance or energy scale, i.e. in the failure of the usual saddle point approach to minimise the free–energy. This quantity in fact for $R < R_c$ becomes in our case unbounded from below. Even in the absence of the above method, the peculiar physical behaviour of the QFT associated to a bosonic $S$–matrix may be inferred by the computation of its Form Factors. We have seen in fact that even with some stringent constraints imposed on the matrix elements they are intrinsically largely undetermined. Moreover, the spectral series which originate from them are generically divergent.

Even after the thorough analysis of this paper, one could be still surprised that an innocent $-1$ in front of an $S$–matrix is the source of all the pathological features of the QFT associated to such an $S$–matrix. The fact is that the $S$–matrix, even for the integrable models where it takes a particularly simple form, is still the final result of an infinite resummation of all microscopic processes dictated by an Hamiltonian. Therefore while it is in general relatively simple to decide whether or not an Hamiltonian may give rise to a consistent Quantum Theory – in terms of stability of the vacuum and the relative excitations thereof – (once again, the simple example of footnote 5 is particularly instructive), the model analysed in this paper shows on the contrary that it may be in general “algorithmically” difficult to trace all this information back and to infer the consistency of a theory starting from the end, i.e. from the knowledge alone of the $S$–matrix.

Several questions arise in relation to the observations of the previous sections. For instance, it would be interesting to analyse the QFT associated to more general CDD factors and characterise those which have a smooth interpolation from the short to the large distance scales, i.e. those ones which identify consistent QFT. Moreover, actions which are initially obtained by a perturbation of irrelevant fields contain an infinite series of corrections due to the higher dimension operators and hence they are generally non–local. In this respect, it would be useful to develop a powerful method for determining in an efficient way all the coupling constants of these operators and to have a criterion to identify those non–local theories which give rise to integrable models.

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Appendix A

Let us consider the calculation of the free-energy of a bosonic type $S$–matrix with a kernel $\varphi(\beta)$ given by

$$\varphi(\beta) = 2\pi \delta(\beta) ,$$  \hspace{1cm} (A.1)

i.e. with a phase–shift defined by

$$-i \ln S(\beta) = \begin{cases} -\pi & \text{if } \beta < 0 \\ 0 & \text{if } \beta = 0 \\ +\pi & \text{if } \beta > 0 \end{cases} $$  \hspace{1cm} (A.2)

In virtue of the locality of the kernel (A.1), the total density of levels $\rho(\beta)$ is simply related to the root density of levels by

$$\rho(\beta) = a(\beta) + \rho^{(r)}(\beta) ,$$  \hspace{1cm} (A.3)

where

$$a(\beta) \equiv \frac{m}{2\pi} \cosh \beta .$$  \hspace{1cm} (A.4)

The functional of the free–energy (at zero chemical potential) is given by

$$f(\rho, \rho^{(r)}) = \mathcal{E}(\rho^{(r)}) - \frac{1}{R} S(\rho, \rho^{(r)})$$  \hspace{1cm} (A.5)

where

$$\mathcal{E}(\rho^{(r)}) = \int m \cosh \beta \rho^{(r)}(\beta) d\beta ,$$  \hspace{1cm} (A.6)

and

$$S(\rho, \rho^{(r)}) = \int d\beta \left[ (\rho + \rho^{(r)}) \log(\rho + \rho^{(r)}) - \rho \log \rho - \rho^{(r)} \log \rho^{(r)} \right] .$$  \hspace{1cm} (A.7)

By plugging the expression (A.3) for $\rho(\beta)$ into eq.(A.5) and minimizing with respect to the distribution $\rho^{(r)}(\beta)$, we have the following condition for the density $\rho^{(r)}$ which minimizes the functional (A.3)

$$\frac{(a + 2\rho^{(r)})^2}{\rho^{(r)} (a + 2\rho^{(r)})^2} = C ,$$  \hspace{1cm} (A.8)

where

$$C = e^{mR \cosh \beta} .$$  \hspace{1cm} (A.9)

For any value of $a$, the left hand side of eq.(A.8) as a function of the positive values of $\rho^{(r)}$ is always larger than 4. Hence a positive real solution does not exist for all values of $R$ but only for those for which it is verified the condition $C \geq 4$. In this case the solution is given by

$$\rho^{(r)}(\beta) = \frac{a}{2} \left[ \sqrt{\frac{C}{C - 4}} - 1 \right] .$$  \hspace{1cm} (A.10)
Correspondingly

\[ \rho(\beta) = \frac{a}{2} \left[ \sqrt{\frac{C}{C-4}} + 1 \right] . \]  

Substituting these expressions into (A.9), the value of the functional at its minimum is given by

\[ \hat{F}(R) = m \int \cosh \beta \ln \left[ \frac{1}{2} \left( 1 + \sqrt{1 - 4e^{-mR\cosh \beta}} \right) \right] . \]  

Hence, within the saddle point solution of the TBA equation we have

\[ E_0(R) = \frac{2\pi}{R} G(mR) , \]  

where

\[ G(x) = \frac{1}{\pi} \int_1^\infty \frac{dt}{\sqrt{t^2 - 1}} \ln \left[ \frac{1}{2} \left( 1 + \sqrt{1 - 4e^{-xt}} \right) \right] . \]  

This expression becomes complex for \( R \leq R_c = \frac{\ln 4}{m} \). \( R_c \) is the value where the first instability of the functional (A.7) shows up, due to the distribution modes \( \rho^{(0)}(\beta) \) and \( \rho(\beta) \) computed at \( \beta = 0 \) (for \( R < R_c \) additional instabilities are induced by other modes).

To see this, first of all notice that, in view of the algebraic equation (A.3), we have a complete decoupling of the contributions due to different \( \beta \) in the free energy. Let us consider then the term \( f^0 \) in (A.8) due to the two distributions computed at \( \beta = 0 \). By using eq.(A.3), with the notation \( \rho = \rho^{(0)}(0) \) and \( a_0 \equiv \frac{m}{2\pi} \) we have

\[ f^0(\rho, R) = \rho - \frac{1}{R} \left[ (a_0 + 2\rho) \ln(a_0 + 2\rho) - (a_0 + \rho) \ln(a_0 + \rho) - \rho \ln \rho \right] . \]  

This expression, as a function of \( \rho \), is plotted in Figure 6 for \( R > R_c, R = R_c \) and \( R < R_c \) respectively. For \( R > R_c \) \( f^0 \) admits a minimum for a positive real value of \( \rho \) and the function is asymptotically positive. Hence for these range of \( R \) the saddle–point approximation results valid. At \( R = R_c \) the minimum has moved at infinity although the value of the function \( f^0 \) has remained finite. For \( R < R_c \) the function \( f^0 \) does not have any longer a minimum and its values are unbounded from below, causing an instability in the corresponding free energy.
References

[1] A.B. Zamolodchikov, Al.B. Zamolodchikov, *Ann. Phys.* **120** (1979), 253.

[2] A.B. Zamolodchikov, in *Advanced Studies in Pure Mathematics, 19* (1989), 641.

[3] G. Delfino and G. Mussardo, *Nucl. Phys.* **B 455** (1995), 724; G. Delfino and P. Simonetti, *Phys. Lett.* **B 383** (1996), 450.

[4] R. Koberle and J.A. Swieca, *Phys. Lett.* **B 86** (1979), 209.

[5] A.E. Arinshtein, V.A. Fateyev and A.B. Zamolodchikov, *Phys. Lett.* **B 87** (1979), 389.

[6] M. Karowski, P. Weisz, *Nucl. Phys.* **B139** (1978), 445.

[7] F.A. Smirnov, *Form Factors in Completely Integrable Models of Quantum Field Theory* (World Scientific) 1992.

[8] G. Mussardo and J.L. Cardy, *Phys. Lett.* **B 225** (1989), 275.

[9] Al.B. Zamolodchikov, *Nucl. Phys.* **B 348** (1991), 619.

[10] V.P. Yurov and Al.B. Zamolodchikov, *Int. J. Mod. Phys.* **A 6** (1991), 3419.

[11] J.L. Cardy and G. Mussardo, *Nucl. Phys.* **B 410** [FS] (1993), 451.

[12] G. Delfino and G. Mussardo, *Phys. Lett.* **B 324** (1994), 40; G. Delfino, G. Mussardo and P. Simonetti, *Phys. Rev. D 51* (1995), R6620.

[13] C.N. Yang and C.P. Yang, *Jour. Math. Phys.* **10** (1969), 1115.

[14] Al.B. Zamolodchikov, *Nucl. Phys.* **B 342** (1990), 695.

[15] T.R. Klassen and E. Melzer, *Nucl. Phys.* **B 338** (1990), 485; *Nucl. Phys.* **B 350** (1991), 635.

[16] G. Mussardo, *Phys. Rep.* **218** (1992), 215 and references therein.

[17] P. Dorey, *Exact S Matrices*, hep-th/9810020; H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, *Nucl. Phys.B 338* (1990), 689.

[18] P. Christe and G. Mussardo, *Nucl. Phys.* **B 330** (1990), 465; P. Christe and G. Mussardo, *Int. J. Mod. Phys.* **A 5** (1990), 4581.

[19] R. Egger, A. Komnik and H. Saleur, *Phys. Rev. B 60* (1999) 5113.
[20] Al.B. Zamolodchikov, *Nucl. Phys.* B 358 (1991), 524.

[21] F. Lesage and H. Saleur, *Perturbation of infra-red fixed points and duality in quantum impurity problems*, cond-mat/9812045.

[22] Al.B. Zamolodchikov, *Resonance Factorized Scattering and Roaming Trajectories*, ENS-LPS-335.

[23] A.G. Izergin and V.E. Korepin, *Lett. Math. Phys.* 6 (1982), 283.

[24] M. Lassig, *Nucl. Phys.* B 380 (1992), 601.

[25] A. LeClair and G. Mussardo, *Finite Temperature Correlation Functions in Integrable QFT*, hep-th/9902075.

[26] A. Fring, G. Mussardo and P. Simonetti, *Nucl. Phys.* B 393 (1993), 413.

[27] A. Koubek and G. Mussardo, *Phys. Lett.* B 311 (1993), 193.

[28] G. Delfino, P. Simonetti and J.L. Cardy, *Phys. Lett.* B 387 (1996), 327.

[29] C. Acerbi, G. Mussardo and A. Valleriani, *J. Phys.* A 30 (1997), 2895.

[30] A.B. Zamolodchikov, *JETP Lett.* 43 (1986), 730; J.L. Cardy, *Phys. Rev. Lett.* 60 (1988), 2709.

[31] G. Delfino, G. Mussardo and P. Simonetti, *Nucl. Phys.* B 473 (1996), 469.

[32] P.D. Fonseca, *Mod. Phys. Lett.* A 13 (1998), 1931.

[33] Al.B. Zamolodchikov, *Nucl. Phys.* B 358 (1991), 497.

[34] V.P. Yurov and Al.B. Zamolodchikov, *Int. J. Mod. Phys.* A 5 (1990), 3221.
Figure Captions

Figure 1. Renormalization Group flow which passes by a fixed point along the directions of the irrelevant and relevant fields $\eta$ and $\Phi$.

Figure 2.a. Plots of the central charge $c(mR) = -\frac{6R}{\pi^2} F(mR)$ for the mShG model at a particular value of $\alpha$, for the thermal Ising model and for a free bosonic model. Below $mR_c$ ($mR_c \sim 1$ in the figure), the central charge of the mShG model assumes complex values.

Figure 3. Profiles of the pseudo–energy $\epsilon$ versus $\beta$ for different values of $R$. $R_c$ corresponds to the value when this function hits the origin.

Figure 4. Two–particle contribution to the $c$–theorem versus the parameter $a$ of the $S$–matrix.

Figure 5. Graph of $\frac{F(a)}{2\sin \pi a}$ versus $a$.

Figure 6.a The free–energy as a function of $\rho^{(r)}(0)$ for $R > R_c$, when there is a minimum.

Figure 6.b The free–energy as a function of $\rho^{(r)}(0)$ for $R = R_c$. The minimum has moved at infinity.

Figure 6.c The free–energy as a function of $\rho^{(r)}(0)$ for $R < R_c$ when there is no longer a minimum.
Figure 1
Figure 2
Figure 3
Figure 6.b
