Global Convergence of Successive Approximations for Non-convex Stochastic Optimal Control Problems

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Abstract. This paper focuses on finding approximate solutions to the stochastic optimal control problems where the state trajectory is subject to controlled stochastic differential equations permitting controls in the diffusion coefficients. An algorithm based on the method of successive approximations is described for finding a set of small measure, in which the control is varied finitely so as to reduce the value of the functional and, as the control domains are not necessarily convex, the second-order adjoint processes are introduced in each minimization step of the Hamiltonian. Under certain convexity conditions, we prove that the values of the cost functional descend to the global minimum as the number of iterations tends to infinity. In particular, a convergence rate for a class of linear-quadratic systems is available.

Key words. Controlled stochastic differential equations; Method of successive approximations; Non-convex control domain; Stochastic maximum principle; Stochastic near-optimal controls

MSC subject classifications. 93E20, 60H10, 60H30, 49M05

1 Introduction

It usually incurs much trouble or is even impossible to precisely calculate analytical solutions to stochastic optimal control problems, which promotes researchers to construct numerical algorithms to obtain approximate solutions in recent years [5, 8, 9]. As one of those algorithms, the one involved the method of successive approximations (MSA for short) is an efficient tool, and the basis of this method is the maximum principle—the necessary condition for the optimality. It includes successive integrations of the state and adjoint equations, and updates the control variables by minimizing the Hamiltonian.

The MSA based on Pontryagin’s maximum principle [3] for seeking numerical solutions to deterministic control systems was first proposed by Krylov et al. [10]. After that, many improved modifications of the MSA have been developed by researchers for a variety of deterministic control systems ([1, 2, 11, 12]). The key idea is to find a set of small measure where the control is varied finitely so as to reduce the value of

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the functional in each iteration. Recently, a modified MSA was first extended to solving the stochastic control problems with convex control domains [9]. Their convergence result shows, roughly speaking, the controls \( \{u^m(\cdot)\}_{m \in \mathbb{N}} \) produced by the MSA algorithm increasingly satisfy the extended stochastic maximum principle as one moves from one iteration to the next. Yet, as Theorem 2.5 in [9] stated, the algorithm converges to a local minimum of the concerned cost functional \( J \). Motivated by this, not only are we aimed at establishing the MSA for the stochastic control problems with general control domains, but also giving a sufficient condition from which, at least, the sequence \( J(u^m(\cdot)) \) is bound to descend to the global minimum \( J(\bar{u}(\cdot)) \) rather than any local minimum, where \( \bar{u}(\cdot) \) is the optimal control. To be more clear, for any \( \delta > 0 \), we prove that there exists a positive integer \( N_\delta \) such that, if \( m \geq N_\delta \), then \( u^m(\cdot) \) minimizes the \( \mathcal{H} \)-function in terms of

\[
\mathbb{E} \left[ \int_0^T \mathcal{H}(t, X^m(t), p^m(t), q^m(t), P^m(t), u^m(t), u^m(t)) dt \right] 
\leq \inf_{u(\cdot)} \mathbb{E} \left[ \int_0^T \mathcal{H}(t, X^m(t), p^m(t), q^m(t), P^m(t), u(t), u^m(t)) dt \right] + \delta, 
\]

where the \( \mathcal{H} \)-function essentially plays a same role as the common Hamiltonian in the stochastic maximum principle. As soon as such deviation is strictly less than \( \delta \), the algorithm can be regarded as the stochastic version of the authors obtained a \( m^{-\frac{1}{2}} \)-order convergence rate for a class of linear-quadratic systems as a special case of the deterministic cases we considered. We’d like to point out that the difference between such two results, we obtain a convergence rate for the linear system with convex and terminal type functionals. Compared with their near-optimal controls are indeed more available than optimal ones as mentioned in [16].

Nevertheless, it is not enough to obtain (1.2). To meet the requirement of reducing the costs of computation in practice, we are more concerned about in which case the convergence rate of \( J(u^m(\cdot)) \) descending to \( J(\bar{u}(\cdot)) \) is available. Recall in the deterministic case (see Theorem 2 in [1]) the authors obtained a \( m^{-1} \)-order convergence rate for the linear system with convex and terminal type functionals. Compared with their result, we obtain a \( m^{-\frac{1}{2}} \)-order convergence rate for a class of linear-quadratic systems as a special case of the stochastic control problems we considered. We’d like to point out that the difference between such two kinds of convergence rate results from the different extent in the sense of decreasing \( J \) in each iteration. For the convenience, let’s explain it by the following error estimate that we have established in section 3,

\[
J(u_{\tau \varepsilon}(\cdot)) - J(u(\cdot)) \leq \mathbb{E} \left[ \int_{E_{\tau \varepsilon}} \Delta_u \mathcal{H}(t) dt \right] + C\varepsilon^{\frac{1}{2}}. 
\]

for given \( \tau \in [0, T] \), \( \varepsilon > 0 \) and admissible control \( u(\cdot) \). (1.3) can be regarded as the stochastic version corresponding to the error estimate (1.8) in [2], where the definition of two-parameter family of admissible controls \( u_{\tau \varepsilon}(\cdot) \) follow the idea to find the set \( E_{\tau \varepsilon} \) with small measure on which \( u_{\tau \varepsilon}(\cdot) \) minimize the \( \mathcal{H} \)-function and keep identical with \( u(\cdot) \) outside \( E_{\tau \varepsilon} \). In (1.3) the term \( \mathbb{E} \left[ \int_{E_{\tau \varepsilon}} \Delta_u \mathcal{H}(t) dt \right] \leq 0 \) that characterizes the extent to which \( u(\cdot) \) deviates from satisfying the necessary condition for optimality, and the equality holds if and only if \( u(\cdot) \) exactly satisfies the stochastic maximum principle. As soon as such deviation is strictly less than 0, we can further understand it to be order of \( \varepsilon \) small enough by Lebesgue’s differentiation theorem. Then, for sufficiently small \( \varepsilon > 0 \), it is not difficult to observe from (1.3) that \( u_{\tau \varepsilon}(\cdot) \) can reduce \( J \) by the
extent of order $O((1 - \varepsilon^2)\varepsilon)$. However, the decreasing of $J$ in the deterministic case is more sharp by the extent of order $O((1 - \varepsilon)\varepsilon)$, since the remainder term on the right-hand side of (1.8) in [2] is of order $O(\varepsilon^2)$ but its counterpart, as we can see in (1.3), is of order $O(\varepsilon^2)$. What should be emphasized is that, since (1.3) is a mechanism to reduce $J$ in each iteration, the multiplier $C$ in the remainder term of (1.3) must be uniform across all admissible controls $u(\cdot)$ and independent of the choices of $\tau$. So in the rigorous proof of (1.3) we essentially show that $C$ only depends on the dimensions of $X$, $W$ and prescribed parameters imposed on the coefficients and control domains. To this end, it needs mild assumptions that the second-order partial derivatives of all the coefficients in the concerned systems are Lipschitz in $x$ uniformly in $(t, u)$.

The rest of the paper is organized as follows. In section 2, preliminaries and the formulation of our problem are given. In section 3, we first establish the MSA algorithm and then study the convergence of it together with (1.3). In particular, we obtain the $m^{-\frac{1}{2}}$-order rate of convergence for a class of linear-quadratic systems. Some technical proofs are listed in section 4.

2 Preliminaries and Problem Formulation

Let $T \in (0, +\infty)$ be a finite time horizon and $n, d, k$ be three positive integers. Denote by $\mathbb{R}^n$ the $n$-dimensional real Euclidean space, $\mathbb{R}^{n \times d}$ the set of all $n \times d$ real matrices and $\mathbb{S}^{n \times n}$ the set of all $n \times n$ real, symmetric matrices. The scalar product (resp. norm) of $A \in \mathbb{R}^{n \times d}$ dimensional real Euclidean space, $(\text{resp. } |A| = \sqrt{\text{tr}(AA^\top)})$, where the superscript $\top$ denotes the transpose of vectors or matrices. Denote by $I_n$ the $n \times n$ identity matrix.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which a standard $d$-dimensional Brownian motion $W = (W^1(t), W^2(t), \ldots, W^d(t))_{t \in [0, T]}$ is defined, and $\mathbb{F} := \{\mathcal{F}_t\}_{t \in [0, T]}$ be the $\mathbb{P}$-augmentation of the natural filtration generated by $W$.

For any given $p, q \geq 1$, we introduce the following spaces and notation.

$L^p_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$: the space of $\mathcal{F}_T$-measurable, $\mathbb{R}^n$-valued random variables $\xi$ such that $\|\xi\|_{L^p} := (\mathbb{E}[|\xi|^p])^{\frac{1}{p}} < \infty$.

$L^\infty_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$: the space of $\mathcal{F}_T$-measurable, $\mathbb{R}^n$-valued random variables $\xi$ such that $\mathbb{P} - \text{ess sup}_{\omega \in \Omega} |\xi(\omega)| < \infty$.

$\mathcal{M}^p_{\mathcal{F}}([0, T]; \mathbb{R}^n)$: the space of $\mathcal{F}$-adapted, $\mathbb{R}^n$-valued processes $\varphi(\cdot)$ on $[0, T]$ such that

$$\|\varphi(\cdot)\|_{p, q} := \left( \mathbb{E}\left[ \left( \int_0^T |\varphi(t)|^p dt \right)^\frac{q}{p} \right] \right)^\frac{1}{q} < \infty.$$  

In particular, we denote by $\mathcal{M}^p_{\mathcal{F}}([0, T]; \mathbb{R}^n)$ the above space when $p = q$.

$L^\infty_{\mathcal{F}}([0, T]; \mathbb{R}^n)$: the space of $\mathcal{F}$-adapted, $\mathbb{R}^n$-valued processes $\varphi(\cdot)$ on $[0, T]$ such that

$$\|\varphi(\cdot)\|_{\infty} := \text{ess sup}_{(t, \omega) \in [0, T] \times \Omega} |\varphi(t, \omega)| < \infty.$$  

$\mathcal{S}^p_{\mathcal{F}}([0, T]; \mathbb{R}^n)$: the space of continuous processes $\varphi(\cdot) \in \mathcal{M}^p_{\mathcal{F}}([0, T]; \mathbb{R}^n)$ such that

$$\|\varphi\|_{\mathcal{S}^p} := \left( \mathbb{E}\left[ \sup_{t \in [0, T]} |\varphi(t)|^p \right] \right)^\frac{1}{p} < \infty.$$
Let $U$ be a nonempty, compact subset of $\mathbb{R}^k$. Consider the following stochastic control problem: over the set of admissible controls
\[
U[0,T] := \{ u(\cdot) | u(\cdot) \text{ is } \mathbb{F}\text{-adapted}, u(t) \in U, \ \mathbb{P} - \text{a.s., } \forall t \in [0,T] \},
\]
minimizing
\[
J(u(\cdot)) := \mathbb{E} \left[ \Phi(X^u(T)) + \int_0^T f(t, X^u(t), u(t)) dt \right]
\]
subject to the controlled stochastic differential equation
\[
\begin{cases}
  dX^u(t) = b(t, X^u(t), u(t)) dt + \sigma(t, X^u(t), u(t)) dW(t), \\
  X^u(0) = x_0,
\end{cases}
\]
where $x_0 \in \mathbb{R}^n$, $b : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^{n \times d}$, $f : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}$ and $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ are deterministic, measurable functions. We assume there exists at least one $u(\cdot) \in U[0,T]$ minimizing (2.2) over $U[0,T]$ and want to find one that reaches the minimum.

We impose the following assumptions on the coefficients of (2.3):

**Assumption 2.1.** Let $L > 0$, $l \in \mathbb{N}_+$ be given.

(i) $b, \sigma, f$ are twice continuously differentiable with respect to $x$; they and their derivatives are continuous in $(t, x, u)$; $b$, $\sigma$, $f_x$ are bounded by $L (1 + |x| + |u|^l)$; $b_x, \sigma_x, b_{xx}, \sigma_{xx}, f_{xx}$ are bounded; $|f(t, 0, u)| \leq L (1 + |u|^l)$, $t \in [0,T]$.

(ii) $\Phi$ is continuous and twice continuously differentiable; $\Phi_x$ is bounded by $L (1 + |x|)$; $\Phi_{xx}$ is bounded.

(iii) $b_{xx}, \sigma_{xx}, f_{xx}, \Phi_{xx}$ are $L$-Lipschitz continuous in $x$, uniformly in $(t, u)$, i.e.
\[
|\psi_{xx}(t, x, u) - \psi_{xx}(t, x', u)| \leq L |x - x'|, \ \forall (t, u) \in [0,T] \times U, \ x, x' \in \mathbb{R}^n,
\]
where $\psi = b, \sigma, f, \Phi$.

**Remark 2.2.** Since $U$ is compact, from the above assumption we actually have: $b, \sigma, f_x$ are bounded by $\tilde{L} (1 + |x|)$ for some $\tilde{L} > 0$ and $f(t, 0, u)$ is bounded for all $t \in [0,T]$.

Put $\alpha = \sup \{|u| : u \in U\}$. The following lemma provides the well-posedness and the $L^p$-estimates of the controlled SDE (2.3).

**Lemma 2.3.** Let Assumption 2.1 hold. Then, for any given $u(\cdot) \in U[0,T]$, (2.3) admits a unique strong solution $X^u(\cdot)$. Moreover, $X^u(\cdot)$ is bounded in $S^\mathbb{F}_T([0,T]; \mathbb{R}^n)$ uniformly across all $u(\cdot) \in U[0,T]$, i.e.
\[
\sup_{a(\cdot) \in U[0,T]} \mathbb{E} \left[ \sup_{t \in [0,T]} |X^u(t)|^8 \right] \leq C,
\]
where $C$ depends on $n, d, T, \alpha, L, \|b_x\|_\infty, \|\sigma_x\|_\infty, x_0$.

**Proof.** For any $u(\cdot) \in U[0,T]$, under Assumption 2.1, there exists a unique strong solution $X^u(\cdot)$ to the SDE in (2.3) ([4], Theorem 1.2; [15], Chapter I, Theorem 6.3) which satisfies the following standard estimate:
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |X^u(t)|^8 \right] \leq C \left( |x_0|^8 + \mathbb{E} \left[ \int_0^T |b(t, 0, u(t))|^4 dt \right]^2 + \left( \int_0^T |\sigma(t, 0, u(t))|^2 dt \right)^4 \right).
\]
Then, under Assumption 2.1, (2.4) follows from (2.5) and the boundedness of $U$. \hfill \square

Set
\[
\begin{align*}
 b(\cdot) &= (b^1(\cdot), b^2(\cdot), \ldots, b^n(\cdot))^T \in \mathbb{R}^n, \\
 \sigma(\cdot) &= (\sigma^1(\cdot), \sigma^2(\cdot), \ldots, \sigma^d(\cdot)) \in \mathbb{R}^{n \times d}, \\
 \sigma^i(\cdot) &= (\sigma^{i1}(\cdot), \sigma^{i2}(\cdot), \ldots, \sigma^{id}(\cdot))^T \in \mathbb{R}^n, i = 1, 2, \ldots, d.
\end{align*}
\]

For $t \in [0, T]$, $\psi = b, \sigma, f$, we simply denote
\[
\psi^n(t) = \psi(t, X^n(t), u(t)), \quad \psi^n_x(t) = \psi_x(t, X^n(t), u(t)), \quad \psi^n_{xx}(t) = \psi_{xx}(t, X^n(t), u(t)). \tag{2.6}
\]

In addition, for $i = 1, \ldots, d$, we denote $\sigma^{ui}(t) = \sigma^i_x(t, X^n(t), u(t))$ and $\sigma^{ui}_x(t) = \sigma^i_{xx}(t, X^n(t), u(t))$.

Given $u(\cdot) \in \mathcal{U}[0, T]$, we call $X^n(\cdot)$ the state trajectory corresponding to $u(\cdot)$. Let $\bar{u}(\cdot)$ be an optimal control, $\bar{X}(\cdot)$ be the optimal state trajectory. Then, in (2.6), we simply write “$\psi^n(t)$”, “$\psi^n_x(t)$”, “$\psi^n_{xx}(t)$” corresponding to the optimal couple $(\bar{X}(\cdot), \bar{u}(\cdot))$.

Now we introduce the adjoint equations of (2.3) by using the notation defined above. The first-order adjoint equation is
\[
\begin{align*}
dp^n(t) &= -\left[ (b^n_x(t))^T p^n(t) + \sum_{i=1}^d (\sigma^{ni}_x(t))^T q^{ni}(t) + f^n_x(t) \right] dt + \sum_{i=1}^d q^{ni}(t) dW_i(t), \quad t \in [0, T], \\
p^n(T) &= \Phi_{xx}(X^n(T)), \tag{2.7}
\end{align*}
\]
and the second-order adjoint equation is
\[
\begin{align*}
dP^n(t) &= -\left[ (b^n_x(t))^T P^n(t) + (P^n(t))^T b^n_x(t) + \sum_{i=1}^d (\sigma^{ni}_x(t))^T P^n(t) \sigma^{ni}_x(t) \right. \\
&\quad + \left. \sum_{i=1}^d [(\sigma^{ni}_x(t))^T Q^{ni}(t) + (Q^{ni}(t))^T \sigma^{ni}_x(t)] \right] dt + \sum_{i=1}^d Q^{ni}(t) dW_i(t), \quad t \in [0, T], \\
P^n(T) &= \Phi_{xx}(X^n(T)), \tag{2.8}
\end{align*}
\]
where the Hamiltonian $H$ is defined by
\[
H(t, x, p, q, u) = p^T b(t, x, u) + \langle q, \sigma(t, x, u) \rangle + f(t, x, u), \tag{2.9}
\]
and $(p^n(\cdot), q^n(\cdot))$ is the solution to (2.7). Here $q^n(\cdot) := (q^{u1}(\cdot), \ldots, q^{ud}(\cdot))$.

Under Assumption 2.1, applying Theorem 5.1 in [6] yields the well-posedness of (2.7).

**Lemma 2.4.** Let Assumption 2.1 hold. Then, for any $u(\cdot) \in \mathcal{U}[0, T]$, (2.7) admits a unique solution $(p^n(\cdot), q^n(\cdot)) \in \mathcal{S}_p^2([0, T]; \mathbb{R}^n) \times \left( \mathcal{M}^{2.8}_F([0, T]; \mathbb{R}^n) \right)^d$.

Similarly, due to the above lemma, applying Theorem 5.1 in [6] again yields the well-posedness of (2.8).
Lemma 2.5. Let Assumption 2.1 hold. Then, for any \( u(\cdot) \in U[0,T] \), (2.8) admits a unique solution \( (P^n(\cdot), Q^n(\cdot)) \in \mathcal{S}_d^J([0,T];\mathbb{R}^n) \times \left( M_{d,1} \times \mathbb{R}^{n \times (n \times n)} \right) \), where \( Q^n_u(\cdot) = (Q^{u,1}(\cdot), \ldots, Q^{u,d}(\cdot)) \).

Define the \( \mathcal{H} \)-function \( \mathcal{H} : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^{n \times n} \times U \times U \rightarrow \mathbb{R} \) by

\[
\mathcal{H}(t,x,p,p,v) = H(t,x,p,p,v) + \frac{1}{2} \sum_{i=1}^{d} \left( \sigma^i(t,x,v) - \sigma^i(t,x,u) \right)^T P \left( \sigma^i(t,x,v) - \sigma^i(t,x,u) \right) - \frac{1}{2} \sum_{i=1}^{d} \left( \sigma^i(t,x,u) \right)^T P \left( \sigma^i(t,x,u) \right).
\]

Then we can rewrite the following stochastic maximum principle ([13], Theorem 3; [15], Theorem 3.2) for (2.2)-(2.3) by \( \mathcal{H} \)-function.

Theorem 2.6. Suppose Assumption 2.1. Let \( (\bar{X}(\cdot), \bar{u}(\cdot)) \) be the optimal couple. Then, \( \forall v \in U \), we have

\[
\mathcal{H}(t,\bar{X}(t),p(t),q(t), P(t), v, \bar{u}(t)) \geq \mathcal{H}(t,\bar{X}(t),p(t),q(t), P(t), \bar{u}(t), \bar{u}(t)), \quad dt \otimes dP \text{-a.e.,}
\]

where \( (p(\cdot),q(\cdot)) \), \( P(\cdot) \) are the solutions to (2.7), (2.8) respectively, corresponding to \( \bar{u}(\cdot) \).

Based on Theorem 2.6, for any \( u(\cdot) \in U[0,T] \), let \( v(\cdot) \) be any process found from the condition

\[
v(t) \in \arg \min_{v \in U} \mathcal{H}(t, X^n(t), p^n(t), q^n(t), P^n(t), v, u(t)), \quad t \in [0,T].
\]

In fact, since the mapping \( v \mapsto \mathcal{H}(t,x,p,p,v,u) \) is continuous and \( U \) is compact, Proposition D.5 in [7] shows that an appropriate measurable selection \( V(t,x,p,p,P,u) \) exists that minimizing \( \mathcal{H}(t,x,p,p,P,\cdot,u) \) over \( U \) for each \( (t,x,p,p,P,u) \). Setting \( v(t) = V(t,X^n(t),p^n(t),q^n(t),P^n(t),u(t)) \), it is easy to verify \( v(\cdot) \in U[0,T] \) since the \( P \)-adaptability of \( v(\cdot) \) can be deduced by Doob’s measurability theorem. Then, for \( t \in [0,T] \), put

\[
\Delta_u \mathcal{H}(t) = \mathcal{H}(t, X^n(t), p^n(t), q^n(t), P^n(t), v(t), u(t)) - \mathcal{H}(t, X^n(t), p^n(t), q^n(t), P^n(t), u(t), u(t)),
\]

\[
\mu(u(\cdot)) = \mathbb{E} \left[ \int_0^T \Delta_u \mathcal{H}(t) dt \right].
\]

In view of (2.12), \( \Delta_u \mathcal{H}(t) \leq 0 \), \( \mu(u(\cdot)) \leq 0 \); if \( \mu(u(\cdot)) = 0 \), this means that \( u(\cdot) \) satisfies the SMP (2.11). Hence we can regard \( \mu(u(\cdot)) \) as characterizing the extent to which the function \( u(\cdot) \) deviates from satisfying the necessary conditions for optimality.

3 The algorithm

In this section, we establish an algorithm and then apply it to finding the near-optimal controls to (2.2)-(2.3). Particularly, we obtain the convergence rate of the algorithm where there are some additional assumptions imposed on the coefficients in (2.3).
3.1 Key estimate to construct the algorithm

Let $\tau \in [0, T]$, $\varepsilon > 0$. Given $u(\cdot) \in \mathcal{U}[0, T]$, put $E_{\tau\varepsilon} = [\tau - \varepsilon, \tau + \varepsilon] \cap [0, T]$ and consider the two-parameter family of admissible controls

$$
u_{\tau\varepsilon}(t) = \begin{cases} v(t), & t \in E_{\tau\varepsilon} \\ u(t), & t \in [0, T] \setminus E_{\tau\varepsilon}, \end{cases} \quad (3.1)$$

where $v(\cdot)$ is introduced by (2.12).

**Proposition 3.1.** Let Assumption 2.1 hold. Then there exists a universal constant $C > 0$ (independent of $u(\cdot), \tau, \varepsilon$) such that

$$J(u_{\tau\varepsilon}(\cdot)) - J(u(\cdot)) \leq \mathbb{E} \left[ \int_{E_{\tau\varepsilon}} \Delta u \mathcal{H}(t) dt \right] + C \varepsilon^2$$

(3.2)

for any $u(\cdot) \in \mathcal{U}[0, T]$ and $u_{\tau\varepsilon}(\cdot)$ defined by (3.1), where $\Delta u \mathcal{H}$ is defined by (2.13).

The construction of $u_{\tau\varepsilon}(\cdot)$ in (3.1) is actually a spike perturbation of the given $u(\cdot)$. So the idea to prove (3.2) is almost same as the one to prove the global stochastic maximum principle for (2.2)-(2.3) except that we further require the constant $C$ is uniform with respect to $u(\cdot), \tau, \varepsilon$. For the convenience of the readers, we put the proof of Proposition 3.1 into section 5.

By definition of $\Delta u \mathcal{H}$, we have $\mathbb{E} \left[ \int_{E_{\tau\varepsilon}} \Delta u \mathcal{H}(t) dt \right] \leq 0$. If $\mathbb{E} \left[ \int_{E_{\tau\varepsilon}} \Delta u \mathcal{H}(t) dt \right] = 0$, then $\Delta u \mathcal{H} = 0$, $dt \otimes d\mathbb{P}$-a.e., which implies $u(\cdot) \equiv v(\cdot)$. Thus it is reasonable to substitute $u(\cdot)$ for $v(\cdot)$ and then (3.2) holds naturally. Otherwise, it follows from (3.2) that, for $\varepsilon$ small enough, it is reasonable to claim that $J(u^{\tau\varepsilon}(\cdot)) - J(u(\cdot)) < 0$ when $\tau$ is chosen to make $\mathbb{E} \left[ \int_{E_{\tau\varepsilon}} \Delta u \mathcal{H}(t) dt \right] \leq \tilde{C} \varepsilon$ hold with a constant $\tilde{C} < 0$ independent of $\tau$ and $\varepsilon$.

Let $N$ be a positive integer. Put $\varepsilon_N = T \cdot 2^{-N}$, $\tau_j^N = (2j - 1)\varepsilon_N$, $j = 1, 2, \ldots, 2^{N-1}$. We simply denote by $E_j^N$ the set $E_{\tau_j^N \varepsilon_N}$ and by $u_{\varepsilon_N}(\cdot)$ the control $u_{\tau_j^N \varepsilon_N}(\cdot)$. The following lemma describes a method to find such a $\tau$ mentioned above.

**Lemma 3.2.** Let Assumption 2.1 hold. Then, given any integer $N \geq 1$, there exists at least one number $j \in \{1, 2, \ldots, 2^{N-1}\}$ such that

$$\mathbb{E} \left[ \int_{E_j^N} \Delta u \mathcal{H}(t) dt \right] \leq 2\varepsilon_N \frac{\mu(u(\cdot))}{T}, \quad (3.3)$$

where $\mu(u(\cdot))$ is defined by (2.14).

**Proof.** The proof is almost same as Lemma 3.2 in [9] so we omit it. \(\square\)

3.2 Construction and convergence of the algorithm

Based on Proposition 3.1 and Lemma 3.2, we establish the algorithm to find the near-optimal control to (2.2)-(2.3).

**Algorithm 1** Algorithm of Modified Successive Approximation for the Optimality of (2.2)-(2.3)

1. Let $u^0(\cdot) \in \mathcal{U}[0, T]$ be an initial approximation.
2. Put $m = 0$. 

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3. For the given \( u^m(\cdot) \), find \( v^m(\cdot), \Delta u^m = \mathcal{H} \) and \( \mu(u^m(\cdot)) \).

4. Put \( N = 1 \).

5. For the given \( N \), find the smallest \( j \in \{1, 2, \ldots, 2^{N-1} \} \) such that (3.3) holds.

6. Evaluate \( J\left(u_{N_j}^m(\cdot)\right) \) for \( \tau_j^N \) found from the step 5.

7. if
   \[
   J\left(u_{N_j}^m(\cdot)\right) - J(u^m(\cdot)) \leq \varepsilon_N \frac{\mu(u^m(\cdot))}{T},
   \]
   then

8. Assign the values \( u_{N_j}^m(t) \) to the control \( u^{m+1}(t) \) for each \( t \in [0, T] \); increase \( m \) by unity: \( m := m + 1 \); proceed to the step 3.

9. else

10. Proceed to the step 12.

11. end if

12. Increase \( N \) by unity: \( N := N + 1 \); proceed to the step 5.

We have the following convergence result.

**Theorem 3.3.** Let Assumption 2.1 hold. Then, for each integer \( m \geq 1 \), we have

\[
J(u^{m+1}(\cdot)) - J(u^m(\cdot)) \leq \frac{\mu^2(u^m(\cdot))}{2C^2T^2},
\]

(3.5)

where \( C \) is the universal constant in (3.2). Moreover,

\[
\lim_{m \to \infty} \mu(u^m(\cdot)) = 0.
\]

(3.6)

**Proof.** Let \( u^m(\cdot) \) be constructed and \( N_m \geq 1 \) be the minimal integer such that

\[
\varepsilon_{N_m} \leq \frac{\mu^2(u^m(\cdot))}{C^2T^2}.
\]

(3.7)

Then \( N_m \) is the minimal number making (3.4) hold. Actually, from (3.2) and (3.3), we have

\[
J\left(u_{\tau_j^N N_m}^m(\cdot)\right) - J(u^m(\cdot)) \leq \left( \frac{2\mu(u^m(\cdot))}{T} + C\sqrt{\varepsilon_{N_m}} \right) \varepsilon_{N_m} \leq \varepsilon_{N_m} \frac{\mu(u^m(\cdot))}{T},
\]

(3.8)

which indicates that \( u^{m+1}(\cdot) \) can be constructed by proceeding Algorithm 1 successfully. Since \( N_m \) is the minimal number in the series \( 1, 2, \ldots \) for which (3.7) holds, we further get

\[
\varepsilon_{N_m} > \frac{\mu^2(u^m(\cdot))}{2C^2T^2}.
\]

(3.9)

Noting that \( \mu(u^m(\cdot)) \leq 0 \), it follows from (3.8) and (3.9) that

\[
J(u^{m+1}(\cdot)) - J(u^m(\cdot)) \leq \frac{\mu^2(u^m(\cdot))}{2C^2T^2},
\]

which is (3.5). Thus we have

\[-\mu^3(u^m(\cdot)) \leq 2C^2T^3 \left[ J(u^m(\cdot)) - J(u^{m+1}(\cdot)) \right].\]
On adding these inequalities, we obtain
\[
\sum_{m=0}^{n-1} (-\mu^3(u^m(\cdot))) \leq 2C^2T^3 \left[ J(u^0(\cdot)) - J(u^m(\cdot)) \right] \\
\leq 2C^2T^3 \left[ J(u^0(\cdot)) - \inf_{u(\cdot) \in \mathcal{U}[0,T]} J(u(\cdot)) \right] \\
< \infty.
\]
Consequently, letting \( n \to \infty \), we have \( \sum_{m=0}^{\infty} (-\mu^3(u^m(\cdot))) < +\infty \), which implies \( \lim_{m \to \infty} \mu(u^m(\cdot)) = 0 \). □

### 3.3 Near-optimality and convergence rate

In this section, we utilize Theorem 3.3 to show that \( J(u^m(\cdot)) \) descend to \( J(\bar{u}(\cdot)) \) with some additional conditions and study the convergence rate in a special case.

For each \( m \in \mathbb{N} \), as mentioned earlier, \( \mu(u^m(\cdot)) \) characterizes the extent to which the resultant control \( u^m(\cdot) \) produced by Algorithm 1 deviating from satisfying the SMP (2.11)—the necessary conditions for optimality. On the other hand, given \( \delta > 0 \) small enough, from Theorem 3.3, we have
\[
-\delta \leq \mu(u^m(\cdot)) \leq 0
\] (3.10)
for sufficiently large \( m \), which means that \( u^m(\cdot) \) satisfies the SMP approximately when \( m \) is large enough. Generally, (3.10) may not imply \( J(u^m(\cdot)) - J(\bar{u}(\cdot)) \leq r(\delta) \) for sufficiently large \( m \), where \( r(\cdot) \) is a is a function of \( \delta \) satisfying \( r(\delta) \to 0 \) as \( \delta \to 0 \). If \( r(\delta) = \tilde{C}\delta^\gamma \) for some \( \gamma > 0 \) independent of the constant \( \tilde{C} \), then \( u^m(\cdot) \) is called near-optimal with order \( \delta^\gamma \). The following result shows that, under certain convex assumptions and for large enough \( m \), (3.10) is sufficient for making \( r(\delta) = \tilde{C}\delta^\gamma \) with a positive constant \( \tilde{C} \) independent of \( \delta \).

**Assumption 3.4.** (i) \( \Phi \) is convex in its argument.

(ii) \( \psi \) is differentiable in \( u \), and there exists a constant \( \tilde{L} > 0 \) such that
\[
|\psi(t,x,u_1) - \psi(t,x,u_2)| + |\psi_u(t,x,u_1) - \psi_u(t,x,u_2)| \leq \tilde{L} |u_1 - u_2|,
\]
where \( \psi = b, \sigma, f \).

**Theorem 3.5.** Let Assumptions 2.1 and 3.4 hold, \( \delta > 0 \) be given. Then there exists a positive integer \( N_\delta \) depending only on \( \delta \) such that, for any \( m \geq N_\delta \), if \( H(t,\cdot, p^m(t), q^m(t), \cdot) \) is convex for a.e. \( t \in [0,T] \), \( \mathbb{P} \)-a.s., we have
\[
J(u^m(\cdot)) - J(\bar{u}(\cdot)) \leq \tilde{C}\delta^\gamma,
\] (3.11)
where \( (p^m(\cdot), q^m(\cdot)) \) is the solution to (2.7) corresponding to \( (X^m(\cdot), u^m(\cdot)) \), \( \tilde{C} > 0 \) is a constant independent of \( \delta \).

**Proof.** By Theorem 3.3, for the given \( \delta > 0 \), there exists a positive integer \( N_\delta \) such that (3.10) holds for each \( m \geq N_\delta \). From (2.14), one can rewrite (3.10) as
\[
\mathbb{E} \left[ \int_0^T \mathcal{H}(t, X^m(t), p^m(t), q^m(t), P^m(t), v^m(t), u^m(t)) dt \right] \\
- \mathbb{E} \left[ \int_0^T \mathcal{H}(t, X^m(t), p^m(t), q^m(t), P^m(t), v^m(t), u^m(t)) dt \right] \geq -\delta.
\] (3.12)
Then, from (2.12), one can verify
\[
\mathbb{E} \left[ \int_0^T \mathcal{H}(t, X^m(t), p^m(t), q^m(t), P^m(t), v^m(t), u^m(t)) dt \right] = \inf_{u(\cdot) \in \mathcal{U}[0,T]} \mathbb{E} \left[ \int_0^T \mathcal{H}(t, X^m(t), p^m(t), q^m(t), P^m(t), u(t), u^m(t)) dt \right].
\]
(3.13)
It follows from (3.12) and (3.13) that
\[
\mathbb{E} \left[ \int_0^T -\mathcal{H}(t, X^m(t), p^m(t), q^m(t), P^m(t), u(t), u^m(t)) dt \right] \geq \sup_{u(\cdot) \in \mathcal{U}[0,T]} \mathbb{E} \left[ \int_0^T -\mathcal{H}(t, X^m(t), p^m(t), q^m(t), P^m(t), u(t), u^m(t)) dt \right] - \delta.
\]
(3.14)
If, for some \( m \geq N_0 \), \( H(t, \cdot, p^m(t), q^m(t), \cdot) \) is convex for a.e. \( t \in [0, T] \), \( \mathbb{P} \)-a.s., then it is equivalent to the concavity of \( -H(t, \cdot, p^m(t), q^m(t), \cdot) \) for a.e. \( t \in [0, T] \), \( \mathbb{P} \)-a.s.. Under Assumption 3.4, this and (3.14) satisfy the conditions in Theorem 5.1 in [16]. Consequently, there exists a constant \( \bar{C} > 0 \) independent of \( \delta \) and \( m \) such that
\[
J(u^m(\cdot)) - \inf_{u(\cdot) \in \mathcal{U}[0,T]} J(u(\cdot)) \leq \bar{C} \delta^\frac{1}{2},
\]
(3.15)
Since we assume the existence of the optimal controls, we finally obtain (3.11).

\[\text{Remark 3.6.} \] From the definition of the \( \mathcal{H} \)-function (2.10), one can rewrite it as
\[
\mathcal{H}(t, x, p, q, P, v, u) = H(t, x, p - P \sigma(t, x, u), v) + \frac{1}{2} \sum_{i=1}^d \left( \sigma_i(t, x, v) \right)^T P \sigma_i(t, x, v),
\]
which is the form adopted by Zhou [16].

\[\text{Remark 3.7.} \] If there is no optimal control to (2.2)-(2.3), then one can only obtain (3.15) instead of (3.11), which implies that \( u^m(\cdot) \) is a \( \delta^\star \)-optimal control as long as \( m \geq N_0 \). Please refer to [16] for the details about the near-optimal controls.

\[\text{Corollary 3.8.} \] Suppose Assumptions 2.1 and 3.4. If \( b, \sigma \) are linear functions with respect to \( (x, u) \) and \( f \) is convex in \( (x, u) \). Then there exist a positive integer \( N_0 \) depending only on \( \delta \) such that, (3.11) holds for all \( m \geq N_0 \).

\[\text{Proof.} \] The proof is similar to that of Theorem 3.5 as, for each \( m \geq N_0 \), the convexity of \( H(t, \cdot, p^m(t), q^m(t), \cdot) \) holds naturally for a.e. \( t \in [0, T] \), \( \mathbb{P} \)-a.s.. \]

Now we provide a case where the convergence rate is available. Let \( b(t, x, u) = b_1(t)x + b_2(t), \sigma(t, x, u) = \sigma(t, u), \Phi(x) = \frac{1}{2} x^T \Gamma x, f(t, x, u) = \frac{1}{2} x^T G(t)x + g(t, u), \) where \( \Gamma \in \mathbb{S}^{n \times n}; b_1, G \) are \( n \times n \) matrix-valued, bounded, deterministic processes; \( b_2 \) is an \( n \)-dimensional, vector-valued, bounded, deterministic process; \( \sigma : [0, T] \times U \rightarrow \mathbb{R}^{n \times d}; g : [0, T] \times U \rightarrow \mathbb{R} \).

\[\text{Theorem 3.9.} \] Let Assumption 2.1 hold and \( b, \sigma, \Phi, f \) be defined as above. Assume \( \tilde{u}(\cdot) \in \mathcal{U}[0,T] \) is an optimal control to (2.2)-(2.3). Then
\[
0 \leq J(u^m(\cdot)) - J(\tilde{u}(\cdot)) \leq \bar{C} m^{-\frac{1}{2}}, \quad m \in \mathbb{N}_+,
\]
(3.16)
where the sequence \( \{u^m(\cdot)\}_m \) is produced by Algorithm 1, \( \bar{C} = \max \{ J(u^1(\cdot)) - J(\tilde{u}(\cdot)), 2C^{-2}T^{-3} \} \). Here \( C > 0 \) is the constant in Theorem 3.3.
To prove Theorem 3.9, we need the following proposition.

**Proposition 3.10.** Let \( \{a_m\}_{m \in \mathbb{N}_+} \) be the sequence of nonnegative numbers such that

\[
a_{m+1} - a_m \leq -Aa_m^3,
\]

where \( A \) is a positive constant. Then \( a_m = O(m^{-\frac{3}{2}}) \).

**Proof.** Let \( a_m = b_m \cdot m^{-\frac{3}{2}} \) for some nonnegative sequence \( \{b_m\}_{m \in \mathbb{N}_+} \). Then it is enough to show that \( b_m \) is bounded for all \( m \in \mathbb{N}_+ \). By (3.17) we have

\[
a_m - a_{m+1} = \frac{b_m}{\sqrt{m}} \left( 1 - \frac{b_{m+1}}{b_m} \sqrt{\frac{m}{m+1}} \right) \geq A \left( \frac{b_m}{\sqrt{m}} \right)^3.
\]

Therefore,

\[
1 - \frac{b_{m+1}}{b_m} \sqrt{\frac{m}{m+1}} \geq \frac{b^2_m}{m}.
\]

After some transformation, we can rewrite the inequality above as

\[
\frac{b_{m+1}}{b_m} \leq \sqrt{1 + \frac{1}{m}} \left( 1 - \frac{b^2_m}{m} \right).
\]

Thus,

\[
\frac{b_{m+1}}{b_m} \leq \left( 1 + \frac{1}{m} \right) \left( 1 - A \frac{b^2_m}{m} \right) = 1 + \frac{1}{m} \left( 1 - Ab^2_m \right) - A \frac{b^2_m}{m^2}.
\]

If \( 1 - Ab^2_m < 0 \), we have

\[
\frac{b_{m+1}}{b_m} \leq 1 + \frac{1}{m} \left( 1 - Ab^2_m \right) - A \frac{b^2_m}{m^2} < 1.
\]

Hence \( b_{m+1} < b_m \). Otherwise, we have \( b_m \leq A^{-\frac{2}{3}} \). Consequently, we conclude that \( b_m \leq \max\{b_1, A^{-\frac{2}{3}}\} \) for all \( m \in \mathbb{N}_+ \). The proof is complete. \( \square \)

**Proof of Theorem 3.9.** Subtracting \( J(\bar{u}(\cdot)) \) from \( J(u^m(\cdot)) \) yields

\[
J(u^m(\cdot)) - J(\bar{u}(\cdot)) = \mathbb{E} \left[ \Phi(X^m(T)) - \Phi(\bar{X}(T)) + \int_0^T \left[ f(t, X^m(t), u^m(t)) - f(t, \bar{X}(t), \bar{u}(t)) \right] dt \right] = \mathbb{E} \left[ (X^m(T))^T \Gamma(X^m(T) - \bar{X}(T)) - \frac{1}{2} (X^m(T) - \bar{X}(T))^T \Gamma(X^m(T) - \bar{X}(T)) \right. \]

\[
+ \int_0^T \left[ (X^m(t))^T G(t)(X^m(t) - \bar{X}(t)) - \frac{1}{2} (X^m(t) - \bar{X}(t))^T G(t)(X^m(t) - \bar{X}(t)) \right] dt
\]

\[
+ \int_0^T \left[ g(t, u^m(t)) - g(t, \bar{u}(t)) \right] dt \right].
\]

Observe that, for any \( u(\cdot) \in \mathcal{U}[0,T] \), (2.7) becomes

\[
p^u(t) = \Gamma X^u(T) + \int_t^T (\bar{b}^u(t) + G(s)X^u(s)) \, ds - \sum_{i=1}^d \int_t^T \bar{q}^{u,i}(s) \, dW^i(s), \quad t \in [0,T],
\]

and (2.8) becomes

\[
P(t) = \Gamma + \int_t^T [\bar{b}^u(t) + \bar{G}(t)b^u(s) + G(s)] \, ds, \quad t \in [0,T].
\]
Then, applying Itô’s lemma yields

\[
0 \leq J(u^m(\cdot)) - J(\bar{u}(\cdot)) = \mathbb{E} \left[ \int_0^T \left[ (q^m(t), \sigma(t, u^m(t)) - \sigma(t, \bar{u}(t)) \right] + g(t, u^m(t)) - g(t, \bar{u}(t)) \right] dt \\
- \frac{1}{2} \sum_{i=1}^d \left[ (\sigma^i(t, u^m(t)) - \sigma^i(t, \bar{u}(t)))^T P^m(t)(\sigma^i(t, u^m(t)) - \sigma^i(t, \bar{u}(t))) dt \right]
\]

(3.19)

\[
\leq \mathbb{E} \left[ \int_0^T \left[ H(t, X^m(t), p^m(t), q^m(t), u^m(t)) - H(t, X^m(t), p^m(t), q^m(t), \bar{u}(t), u^m(t)) \right] dt \right]
\]

\[
= -\mu(u^m(\cdot)).
\]

For each \( m \in \mathbb{N}_+ \), define the nonnegative sequence by

\[
a_m = J(u^m(\cdot)) - J(\bar{u}(\cdot)).
\]

Then, as \(-\mu(u^m(\cdot)) \geq 0\), it follows from (3.5) and (3.19) that

\[
a_{m+1} - a_m = \left[ J(u^{m+1}(\cdot)) - J(\bar{u}(\cdot)) \right] - \left[ J(u^m(\cdot)) - J(\bar{u}(\cdot)) \right] \\
\leq -\frac{a_m^2}{2C^2T^3},
\]

which verifies Proposition 3.10 with \( A = 2C^{-2}T^{-3} \). Hence we have \( a_m \leq \max \{a_1, 2C^{-2}T^{-3}\} m^{-\frac{1}{2}} \) for all \( m \in \mathbb{N}_+ \). The proof is complete. \( \square \)

4 Proof of Proposition 3.1

In this section, we provide a detailed proof of Proposition 3.1. The universal constant \( C \) may depend only on \( n, d, T, \alpha, L, \|b\|_\infty, \|\sigma\|_\infty, \|\Phi_{xx}\|_\infty, \|f_{xx}\|_\infty \), and will change from line to line in our proof.

Recall the notation introduced by (2.6). We first introduce the following two SDEs:

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{dX_1(t)}{dt} = b_x(t)X_1(t)dt + \sum_{i=1}^d \left[ \sigma_x^i(t)X_1(t) + \hat{\sigma}^i(t)1_{E_{x^i}}(t) \right] dW_i(t), \ t \in [0, T], \\
X_1(0) = 0,
\end{array} \right.
\end{aligned}
\]

(4.1)

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{dX_2(t)}{dt} = \left[ b_x(t)X_2(t) + \hat{\delta}(1_{E_{x^i}}(t) + \frac{1}{2} b_{xx}(t)X_1(t)X_1(t)) \right] dt \\
+ \sum_{i=1}^d \left[ \sigma_x^i(t)X_2(t) + \frac{1}{2}\sigma_{xx}^i(t)X_1(t)X_1(t) + \hat{\sigma}^i(t)X_1(t)1_{E_{x^i}}(t) \right] dW_i(t), \ t \in [0, T],
\end{array} \right.
\end{aligned}
\]

(4.2)

where

\[
\hat{\psi}(t) := \psi(t, X_u(t), \nu(t)) - \psi(t, X_u(t), u(t)),
\]

\[
\psi_{xx}(t)X_1(t) := \left( \text{tr} \left\{ \psi_{xx}^1(t)X_1(t)X_1^T(t) \right\}, \ldots, \text{tr} \left\{ \psi_{xx}^d(t)X_1(t)X_1^T(t) \right\} \right)^T,
\]

for \( \psi = b, \{\sigma^i\}_{i=1, \ldots, d} \).
Lemma 4.1. Let Assumption 2.1 hold. Then we have
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |X^u_{\tau\varepsilon}(t) - X^u(t) - X_1(t) - X_2(t)|^2 \right] \leq C \varepsilon^3
\]  
(4.3)
where \(X^u_{\tau\varepsilon}(\cdot)\) is the state trajectory corresponding to \(u_{\tau\varepsilon}(\cdot)\), \(X_1(\cdot)\), \(X_2(\cdot)\) are solutions to (4.1), (4.2) respectively, \(C > 0\) is independent of \(u(\cdot)\), \(v(\cdot)\) and \(\varepsilon\).

Proof. From (2.5) and (3.1), one can verify by a standard estimate for SDEs ([4, 15]) that
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |X_1(t)|^4 + \sup_{t \in [0,T]} |X_2(t)|^4 \right] \leq C \varepsilon^4,
\]  
(4.4)
where \(C > 0\) is independent of \(u(\cdot)\), \(v(\cdot)\), \(\tau\) and \(\varepsilon\).

Set \(\tilde{X}(\cdot) = X_1(\cdot) + X_2(\cdot)\),
\[
\tilde{b}_{xx}(s) = \int_0^s \int_0^1 \theta b_{xx}(s, \tilde{X}(s) + \rho \tilde{X}(s), u_{\tau\varepsilon}(s)) d\rho d\theta,
\]
\[
\tilde{\sigma}_{xx}(s) = \int_0^s \int_0^1 \sigma_{xx}(s, \tilde{X}(s) + \rho \tilde{X}(s), u_{\tau\varepsilon}(s)) d\rho d\theta, \quad i = 1, \ldots, d.
\]

We have
\[
\int_0^t b(s, X^u(s) + \tilde{X}(s), u_{\tau\varepsilon}(s)) ds + \sum_{i=1}^d \int_0^t \sigma_i(s, X^u(s) + \tilde{X}(s), u_{\tau\varepsilon}(s)) dW_i(s)
\]
\[
= \int_0^t \left[ b(s, X^u(s), u_{\tau\varepsilon}(s)) + b_x(s, X^u(s), u_{\tau\varepsilon}(s)) \tilde{X}(s) + \tilde{b}_{xx}(s) \tilde{X}(s) \tilde{X}(s) \right] ds
\]
\[
+ \sum_{i=1}^d \int_0^t \sigma_i(s, X^u(s), u_{\tau\varepsilon}(s)) \tilde{X}(s) + \tilde{\sigma}_{xx}(s) \tilde{X}(s) \tilde{X}(s) dW_i(s)
\]
\[
= \int_0^t \left[ b(s) + b_x(s) \tilde{X}(s) + \frac{1}{2} b_{xx}(s) \tilde{X}(s) \tilde{X}(s) \right] ds
\]
\[
+ \sum_{i=1}^d \int_0^t \sigma_i(s) \tilde{X}(s) + \frac{1}{2} \sigma_{xx}(s) \tilde{X}(s) \tilde{X}(s) dW_i(s)
\]
\[
+ \int_0^t \tilde{b}(s) 1_{E_{\tau\varepsilon}}(s) ds + \int_0^t b_x(s) \tilde{X}(s) 1_{E_{\tau\varepsilon}}(s) ds + \int_0^t \left[ \tilde{b}_{xx}(s) - b_{xx}(s) \right] \tilde{X}(s) \tilde{X}(s) ds
\]
\[
+ \sum_{i=1}^d \int_0^t \tilde{\sigma}_i(s) 1_{E_{\tau\varepsilon}}(s) dW_i(s) + \sum_{i=1}^d \int_0^t \tilde{\sigma}_{xx}(s) \tilde{X}(s) 1_{E_{\tau\varepsilon}}(s) dW_i(s)
\]
\[
+ \sum_{i=1}^d \int_0^t \left[ \tilde{\sigma}_{xx}(s) - \sigma_{xx}(s) \right] \tilde{X}(s) \tilde{X}(s) dW_i(s)
\]
\[
= X^u(t) - x_0 + \tilde{X}(t) + \int_0^t \Pi_{\tau\varepsilon}(s) ds + \sum_{i=1}^d \int_0^t \Lambda_{\tau\varepsilon}^i(s) dW_i(s),
\]
where (using (4.1) and (4.2))
\[
\Pi_{\tau\varepsilon}(s) = \tilde{b}_x(s) X_2(s) 1_{E_{\tau\varepsilon}}(s) + \frac{1}{2} b_{xx}(s) \left[ \tilde{X}(s) \tilde{X}(s) - X_1(t) X_1(t) \right]
\]
\[
+ \left[ \tilde{b}_{xx}(s) - b_{xx}(s) \right] \tilde{X}(s) \tilde{X}(s),
\]
\[
\Lambda_{\tau\varepsilon}^i(s) = \tilde{\sigma}_i(s) X_2(s) 1_{E_{\tau\varepsilon}}(s) + \frac{1}{2} \sigma_{xx}(s) \left[ \tilde{X}(s) \tilde{X}(s) - X_1(t) X_1(t) \right]
\]
\[
+ \left[ \tilde{\sigma}_{xx}(s) - \sigma_{xx}(s) \right] \tilde{X}(s) \tilde{X}(s).\]
Thus we have
\[ X^u(t) + \hat{X}(t) = x_0 + \int_0^t b(s, X^u(s) + \hat{X}(s), u_{\tau\varepsilon}(s))ds + \sum_{i=1}^d \int_0^t \sigma^i(s, X^u(s) + \hat{X}(s), u_{\tau\varepsilon}(s))dW_i(s) - \int_0^t \Pi_{\tau\varepsilon}(s)ds - \sum_{i=1}^d \int_0^t \Lambda^i_{\tau\varepsilon}(s)dW_i(s). \]

Since
\[ X^u_{\tau\varepsilon}(t) = x_0 + \int_0^t b(s, X^u_{\tau\varepsilon}(s), u_{\tau\varepsilon}(s))ds + \sum_{i=1}^d \int_0^t \sigma^i(s, X^u_{\tau\varepsilon}(s), u_{\tau\varepsilon}(s))dW_i(s), \]
we can derive
\[
\left( X^u_{\tau\varepsilon} - X^u - \hat{X} \right)(t) = \int_0^t A_{\tau\varepsilon}(s) \left( X^u_{\tau\varepsilon} - X^u - \hat{X} \right)(s)ds + \sum_{i=1}^d \int_0^t B^i_{\tau\varepsilon}(s) \left( X^u_{\tau\varepsilon} - X^u - \hat{X} \right)(s)dW_i(s) + \int_0^t \Pi_{\tau\varepsilon}(s)ds + \sum_{i=1}^d \int_0^t \Lambda^i_{\tau\varepsilon}(s)dW_i(s),
\]
where
\[ A_{\tau\varepsilon}(s) = \int_0^s b_x \left( s, \tilde{X}(s) + \hat{X}(s) + \theta \left( X^u_{\tau\varepsilon}(s) - X^u(s) - \hat{X}(s) \right), u_{\tau\varepsilon}(s) \right)dt, \]
\[ B^i_{\tau\varepsilon}(s) = \int_0^s \sigma_x^i \left( s, \tilde{X}(s) + \hat{X}(s) + \theta \left( X^u_{\tau\varepsilon}(s) - X^u(s) - \hat{X}(s) \right), u_{\tau\varepsilon}(s) \right)dt, \quad i = 1, \ldots, d. \]

Since \( b_{xx} \) and \( \sigma_{xx} \) are both Lipschitz continuous in \( x \), from (4.4), one can verify that
\[
E \left[ \sup_{t\in[0,T]} \left| \int_0^t \Pi_{\tau\varepsilon}(s)ds \right|^2 + \sum_{i=1}^d \int_0^t \Lambda^i_{\tau\varepsilon}(s)dW_i(s) \right]^2 \leq C\varepsilon^3.
\]

Observe that \( A_{\tau\varepsilon}(\cdot), B^i_{\tau\varepsilon}(\cdot) \) are bounded. From (4.6), applying a standard estimate for (4.5) yields (4.3). □

Since \( \Phi_{xx} \) and \( f_{xx} \) are both Lipschitz continuous, from (4.3) and (4.4), we have
\[
J(u_{\tau\varepsilon}(\cdot)) - J(u(\cdot)) = E \left[ \Phi(X^u_{\tau\varepsilon}(T)) - \Phi(X^u(T)) + \int_0^T [f(t, X^u_{\tau\varepsilon}(t), u_{\tau\varepsilon}(t)) - f(t)]dt \right]
\]
\[
= E \Phi(X^u + X_1 + X_2)(T) - \Phi(X^u(T))
+ E \left[ \int_0^T [f(t, X^u + X_1 + X_2)(t), u_{\tau\varepsilon}(t)) - f(t, X^u + X_1 + X_2)(t), u(t)) - f(t)]dt \right] + R_1(\varepsilon)
\]
\[
= E \left[ \Phi_x(X^u(T))(X_1(T) + X_2(T)) + \frac{1}{2} \Phi_{xx}(X^u(T))X_1(T)X_1(T) \right]
+ E \left[ \int_0^T [f_x(t)(X_1(t) + X_2(t)) + \frac{1}{2} f_{xx}(t)X_1(t)X_1(t)]dt \right] + R_1(\varepsilon) + R_2(\varepsilon)
\]
with
\[
|R_1(\varepsilon)| + |R_2(\varepsilon)| \leq C \left( \varepsilon^\frac{3}{2} + \varepsilon^2 \right),
\]
where $C > 0$ is independent of $u(\cdot), v(\cdot), \tau$ and $\varepsilon$.

From (2.7), (2.8) and (4.7), applying Itô’s lemma to $p^u(t)(X_1(t) + X_2(t)) + \frac{1}{2} \text{tr} \{P^u(t)X_1(t)(X_1(t))^T\}$ on $[0, T]$ yields

\[
J(u_{\tau \varepsilon}(\cdot)) - J(u(\cdot)) = E \left[ \int_0^T \left[ H(t, X^u(t), p^u(t), q^u(t), u_{\tau \varepsilon}(t)) - H(t, X^u(t), p^u(t), q^u(t), u(t)) \right] dt \right] \\
+ \frac{1}{2} E \left[ \sum_{i=1}^d \int_0^T \left( \sigma^i(t, X^u(t), u_{\tau \varepsilon}(t)) - \sigma^i(t) \right)^T P^u(t) \left( \sigma^i(t, X^u(t), u_{\tau \varepsilon}(t)) - \sigma^i(t) \right) dt \right] \\
+ R_1(\varepsilon) + R_2(\varepsilon)
\]

\[
= E \left[ \int_0^T \left[ H(t, X^u(t), p^u(t), q^u(t), v(t), u(t)) - H(t, X^u(t), p^u(t), q^u(t), u(t), u(t)) \right] 1_{E_{\tau \varepsilon}}(t) dt \right] \\
+ R_1(\varepsilon) + R_2(\varepsilon)
\]

\[
= E \left[ \int_{E_{\tau \varepsilon}} \Delta u H(t) dt \right] + R_1(\varepsilon) + R_2(\varepsilon),
\]

which together with (4.8) implies (3.2). The proof is complete.

References

[1] A. A. Lyubushin, *Modifications and convergence of method of successive approximations for optimal control problems*. Computational Mathematics and Mathematical Physics, 19(6) (1979), pp. 53-61.

[2] F. L. Chernousko and A. A. Lyubushin, *Method of successive approximations for solution of optimal control problems*. Optimal Control Applications and Methods, 3 (1982), pp. 101-114.

[3] V. G. Boltyanski, R. V. Gamkrelidze and L. S Pontryagin, *On the theory of optimal processes*. Dokl. Akad. Nauk SSSR, 10 (1956), pp. 7-10.

[4] R. Carmona, *Lectures on BSDEs, Stochastic Control, and Stochastic Differential Games with Financial Applications*. Ringgold, Inc, 2016.

[5] R. Carmona and M. Laurière, *Convergence analysis of machine learning algorithm for the numerical solution of mean field control and games: II—The finite horizon case*. arXiv:1908.01613.

[6] N. El Karoui, S. Peng and M. C. Quenez, *Backward stochastic differential equations in finance*. Mathematical Finance, 7(1) (1997), pp. 1-71.

[7] O. Hernandez-Lerma and J. Lasserre, *Discrete-Time Markov Control Processes*, Springer, New York, 1996.

[8] B. Kerimkulov, D. Šiška and Ł. Szpruch, *Exponential convergence and stability of Howard’s policy improvement algorithm for controlled diffusions*. SIAM Journal on Control and Optimization, 58(3) (2020), pp. 1314-1340.
[9] B. Kerimkulov, D. Šiška and Ł. Szpruch, *A modified MSA for stochastic control problems*. Applied Mathematics and Optimization, 84(3) (2021), 3417–3436.

[10] I. A. Krylov and F. L. Chernousko, *On the method of successive approximations for solution of optimal control problems*. Computational Mathematics and Mathematical Physics, 2(6) (1962), pp. 1371-1382.

[11] I. A. Krylov and F. L. Chernousko, *Algorithm of the method of successive approximations for optimal control problems*. Computational Mathematics and Mathematical Physics, 12(1) (1972), pp. 15-38.

[12] Q. Li, L. Chen, C. Tai, and W. E, *Maximum principle based algorithms for deep learning*. Journal of Machine Learning Research, 18(165) (2018), pp. 1-29.

[13] S. Peng, *A general stochastic maximum principle for optimal control problems*. SIAM Journal on Control and Optimization, 28(4) (1990), pp. 966-979.

[14] S. Peng, *Backward stochastic differential equations and applications to optimal control*. Applied mathematics & optimization, 27(2) (1993), pp. 125-144.

[15] J. Yong and X. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*. Springer, 1999.

[16] X. Zhou, *Stochastic near-optimal controls: necessary and sufficient conditions for near-optimality*. SIAM Journal on Control and Optimization, 36(3) (1998), pp. 929-947.