The Cheeger Inequality and Coboundary Expansion: Beyond Constant Coefficients

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Abstract

The Cheeger constant of a graph, or equivalently its coboundary expansion, quantifies the expansion of the graph. This notion assumes an implicit choice of a coefficient group, namely, $\mathbb{F}_2$. In this paper, we study Cheeger-type inequalities for graphs endowed with a generalized coefficient group, called a sheaf; this is motivated by applications to locally testable codes. We prove that a graph is a good spectral expander if and only if it has good coboundary expansion relative to any (resp. some) constant sheaf, equivalently, relative to any ‘ordinary’ coefficient group. We moreover show that sheaves that are close to being constant in a well-defined sense are also good coboundary expanders, provided that their underlying graph is an expander, thus giving the first example of good coboundary expansion in non-constant sheaves. By contrast, for general sheaves on graphs, it is impossible to relate the expansion of the graph and the coboundary expansion of the sheaf.

We shall say that $(X, w)$ is a $\lambda$-spectral expander $(\lambda \in [-1, 1])$ if all the eigenvalues of its adjacency matrix except for 1 (counted with multiplicity 1) lie in the interval $[-\lambda, \lambda]$, and write $\lambda(X, w)$ for the largest $\lambda$ for which this holds. Thus, $h(X, w) \geq 1 - \lambda(X, w)$.

1 Introduction

Expander Graphs

Informally, a (finite) graph is called an expander if relatively many edges cross between every set of vertices and its complement. More precisely, if $X$ is a graph and $w : X \to \mathbb{R}_+$ is a function assigning non-negative weights to the vertices and edges of $X$, then the expansion of the weighted graph $(X, w)$ is quantified by its Cheeger constant,

$$h(X, w) = \min_{\emptyset \neq S \subseteq X(0)} \frac{w(E(S, X(0) - S))}{\min\{w(S), w(X(0) - S)\}}.$$

(1.1)

Here, $X(0)$ is the set of vertices of $X$ and $E(A, B)$ denotes the set of edges with one vertex in $A$ and the other in $B$. One says that $(X, w)$ is an $\varepsilon$-combinatorial expander if $h(X) \geq \varepsilon$.

In what follows, we shall assume that the weight function $w$ satisfies some normalization conditions that are listed in §2B. In particular, we require that $w(X(0)) = w(X(1)) = 1$, where $X(1)$ is the set of edges of $X$. For example, when $X$ is a regular graph, one can set $w(v) = \frac{1}{|X(0)|}$ for every vertex $v \in X(0)$ and $w(e) = \frac{1}{|X(1)|}$ for every edge $e \in X(1)$.

It is a celebrated fact that $h(X, w)$ can be bounded from below using the eigenvalues of the normalized adjacency matrix of $(X, w)$; we recall its definition in §2C. In more detail, if $\lambda_2(X, w)$ is the second-largest eigenvalue of this matrix (the largest is 1), then $h(X, w) \geq 1 - \lambda_2(X, w)$ (T. H. Theorem 4.4(1)), for instance. We shall say that $(X, w)$ is a $\lambda$-spectral expander $(\lambda \in [-1, 1])$ if all the eigenvalues of its adjacency matrix except for 1 (counted with multiplicity 1) lie in the interval $[-\lambda, \lambda]$, and write $\lambda(X, w)$ for the largest $\lambda$ for which this holds. Thus, $h(X, w) \geq 1 - \lambda(X, w)$. 

It approaches a constant as $q$ grows.
Coboundary Expansion

Meshulam–Wallach [23] and Gromov [11], following the earlier work of Linial–Meshulam [19], observed that the $\varepsilon$-expansion condition for graphs can be restated in terms of cohomology with $\mathbb{F}_2$-coefficients, and thus be generalized to higher dimensions if $X$ is a (weighted) simplicial complex, rather than a graph. This type of expansion is quantified by the coboundary expansion of $X$ in the relevant dimension, and coincides with the Cheeger constant in dimension 0. Recent works studying the coboundary expansion of simplicial complexes in dimensions $> 0$ include [3, 21, 14, 22, 20, 4, 17].

Recall that the 0-dimensional coboundary expansion of a weighted graph $(X, w)$ can be defined as follows: First, view $X$ as a 1-dimensional simplicial complex, which means that we add an empty face of dimension $-1$ to $X$. We write $X(i) \ (i \in \{-1, 0, 1\})$ for the set of $i$-dimensional faces of $X$. For every edge $e \in X(1)$, choose one its vertices, denote it as $e^-$ and denote the other vertex as $e^+$. Recall that an $i$-cochain on $X$ with coefficients in $\mathbb{F}_2$ is an assignment of an element of $\mathbb{F}_2$ to every $i$-face of $X$, i.e., a vector $f \in \mathbb{F}_2^{X(i)}$. We write $C^i = C^i(X, \mathbb{F}_2) = \mathbb{F}_2^{X(i)}$ and denote the $x$-coordinate of $f \in C^i$ as $f(x)$. The coboundary maps $d_{-1} : C^{-1} \to C^0$ and $d_0 : C^0 \to C^1$ are now defined by

$$(d_{-1}f)(v) = f(\emptyset),$$
$$(d_0f)(e) = f(e^+) - f(e^-).$$

Clearly, $d_0 \circ d_{-1} = 0$. Thus, $B^0 = B^0(X, \mathbb{F}_2) := \text{im}(d_{-1})$ — called the space of 0-boundaries on $X$ — is contained in $Z^0 = Z^0(X, \mathbb{F}_2) := \ker(d_0)$ — the space of 0-cochains on $X$. The coboundary expansion of $X$ in dimension 0 measures the expansion of 0-cochains under $d_0$, taking into account that $B^0$ must be mapped to 0. Formally, this is the largest non-negative real number $\text{cb}_0(X, w)$ such that

$$\|d_0f\| \geq \text{cb}_0(X, w) \cdot \text{dist}(f, B^0) \quad \forall f \in C^0,$$

where $\| \cdot \|$ and $\text{dist}(\cdot)$ are the weighted Hamming norm and distance (in $\mathbb{F}_2^{X(0)}$ or $\mathbb{F}_2^{X(1)}$) given by $\|f\| = w(\text{supp}f)$ and $\text{dist}(f, g) = w(\text{supp}(f - g))$. We say that $(X, w)$ is an $\varepsilon$-coboundary expander in dimension 0 if $\varepsilon \leq \text{cb}_0(X)$.

It is straightforward to see that $\text{cb}_0(X, w)$ coincides with the Cheeger constant $h(X, w)$. However, the description of $h(X, w)$ via cochains reveals that we have made an implicit choice of a coefficient group, namely, $\mathbb{F}_2$. It is now natural to ask what happens if we replace $\mathbb{F}_2$ with another abelian group.

Our first result (Theorem 5.2) answers this question. Let $R$ be a nontrivial abelian group, and let $\text{cb}_0(X, w, R)$ be the 0-dimensional coboundary expansion of $(X, w)$ when the coefficient group is taken to be $R$. Then, in the same way that $h(X, w) = \text{cb}_0(X, w) \geq 1 - \lambda_2(X, w)$, we have

$$\text{cb}_0(X, w, R) \geq 1 - \lambda_2(X, w).$$

Moreover, while $\text{cb}_0(X, w, R)$ may vary with $R$, we always have

$$\frac{1}{2} h(X, w) \leq \text{cb}_0(X, w, R) \leq h(X, w),$$

so $(X, w, R)$ is a good coboundary expander whenever $(X, w)$ is a good expander (Corollary 5.4).

That said, the goal of this paper is to establish lower bounds on the 0-dimensional coboundary expansion w.r.t. even more general coefficient systems, called sheaves; this problem was not previously studied. Exploring such generalized coefficient systems has implications to locally testable codes that we explain later on. Our results give rise to new examples of locally testable codes, although with poor rate, and are also used in [9] for constructing good locally testable codes in an indirect manner.

Sheaves

The common meaning of a sheaf in the literature is a sheaf on a topological space; such sheaves are ubiquitous to topology and algebraic geometry. The sheaves that we consider here are discrete, more elementary analogues that are defined over cell complexes and are better known in the literature as cellular sheaves. They

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1Our paper [9] will be subsumed by other papers featuring stronger results.
were first introduced by Shepard [25] and studied further by Curry [2]; a concise treatment can be found in [12] or [9]. The authors observed in [9] that coboundary expansion is actually a property of sheaves on simplicial complexes, rather than a property of the complex, so one may also think of sheaves as generalizing the coefficient group that is used in the definition of coboundary expansion.

For simplicity, we restrict our discussion here to sheaves on graphs; the general definition over simplicial complexes is given in [4A]. Unlike [25, 2] and similar sources, we shall need to take the empty face into account when defining a sheaf; we call such a structure an augmented sheaf to avoid confusion.

Let $X$ be a graph. An augmented sheaf $F$ on $X$ consists of

1. an abelian group $F(x)$ for every $x \in X = X(−1) ∪ X(0) ∪ X(1)$, and
2. a group homomorphism $\text{res}_{y \to x}^F : F(x) \to F(y)$ for all $x \subset y \in X$

such that

$$\text{res}_{x \to y}^F \circ \text{res}_{y \to z}^F = \text{res}_{x \to z}^F$$

for every edge $e$ and vertex $v \subset e$. The maps $\text{res}_{e \to v}^F$ are the restriction maps of $F$. We will usually drop the superscript $F$ from $\text{res}_{e \to v}^F$ when there is no risk of confusion.

A sheaf on $X$ is an augmented sheaf $F$ such that $F(\emptyset) = 0$. For a sheaf, the condition (1.2) holds automatically.

The simplest example of an augmented sheaf on $X$ is obtained by choosing an abelian group $R$ and setting $F(x) = R$ for every $x \in X$, and $\text{res}_{y \to x}^F = \text{id}_R$ for every $x \subset y \in X$. We denote this augmented sheaf by $R_X^\infty$. Augmented sheaves of this form (up to isomorphism) are called constant augmented sheaves.

### Coboundary Expansion of Sheaves on Graphs

Let $(X, w)$ be a weighted graph. We can replace the role of the coefficient group $R$ in the definition of $\text{cb}_0(X, w, R)$ with a general augmented sheaf on $X$.

In more detail, let $F$ be an augmented sheaf on $X$ such that $F(x) \neq 0$ for every nonempty face $x \in X$. The $i$-cochains $(i \in \{-1, 0, 1\})$ of $X$ with coefficients in $F$ are members of $C_i = C^i(X, F) := \prod_{x \in X} F(x)$. We define the coboundary maps $d_{−1} : C^1 → C^0$ and $d_0 : C^0 → C^1$ as in the case of $\mathbb{F}_2$-coefficients, but with the difference that the restriction maps of $F$ are invoked:

$$(d_{−1}f)(v) = \text{res}_{v \to 0} f(0) \quad \forall v \in X(0),$$

$$(d_0f)(e) = \text{res}_{e \to +} f(e^+) - \text{res}_{e \to −} f(e^-) \quad \forall e \in X(1).$$

Again, we have $d_0 \circ d_{−1} = 0$, and so $B^0(X, F) := \text{im} d_{−1} \subseteq \text{ker} d_0 =: Z^0(X, F)$. The quotient $H^0(X, F) := Z^0(X, F) / B^0(X, F)$ is the 0-th cohomology group of $F$. The 0-dimensional coboundary expansion of $(X, w, F)$, or just $F$, is the smallest non-negative real number $\text{cb}_0(X, w, F)$ such that

$$\|d_0f\| \geq \text{cb}_0(X, w, F) \text{dist}(f, B^0) \quad \forall f \in C^0(X, F),$$

where again, $\text{dist}(,)$ is the weighted Hamming distance on $C^0$ and $C^1$ given by $\text{dist}(f, g) = \text{supp}(f - g)$.

Our earlier discussion of coboundary expansion with coefficients in an abelian group $R$ can now be seen as addressing the special case of a constant augmented sheaf $R_X^\infty$.

### Cosystolic Expansion of Sheaves on Graphs and Locally Testable Codes

Cosystolic expansion is a more lax version of coboundary expansion. It was introduced in [6], [14], [8] in order to extend the reach of Gromov’s work on the minimal overlap forced by mapping a simplicial complex into $\mathbb{R}^d$ [11]. These work just mentioned use $\mathbb{F}_2$ as the implicit coefficient group $\mathbb{F}_2$, but, as observed by the authors in [6], cosystolic expansion can be defined for any sheaf on a simplicial complex.

Restricting to the case of weighted graphs, an augmented sheaf $F$ on a weighted graph $(X, w)$ is said to be an $(\varepsilon, \delta)$-cosystolic expander in dimension 0 if

1. $\|d_0f\| \geq \varepsilon \text{dist}(f, Z^0(X, F))$ for all $f \in C^0(X, F)$, and
2. $\|g\| \geq \delta$ for all $g \in Z^0(X, F) - B^0(X, F)$. 

Here, as before, \( \text{dist}(f,g) = w(\text{supp}(f-g)) \).

As noted in [9] and earlier in [13] in the special case \( \mathcal{F} = (F_2)_X^+ \), cosystolic expansion can be restated in the language of \textit{locally testable codes}. Specifically, suppose that there is a finite abelian group \( \Sigma \) such that \( \mathcal{F}(v) \) is a subgroup of \( \Sigma \) for every \( v \in X(0) \). Then we may think of \( Z^0(X,\mathcal{F}) \) as a code inside \( C^0(X,\mathcal{F}) \subseteq \Sigma^X(0) \). Moreover, this code has a natural 2-query tester: given access to some \( f \in C^0 \), the tester chooses an edge \( e \in X(1) \) uniformly at random, reads \( f(e^+) \) and \( f(e^-) \), and accepts if \( f(e^+) \in \mathcal{F}(e^+) \), \( f(e^-) \in \mathcal{F}(e^-) \) and \( \text{res}_{e^+ \rightarrow e^-} f(e^+) - \text{res}_{e^- \rightarrow e^+} f(e^-) = (d_0 f)(e) = 0 \). Now, if \( \mathcal{F} \) is a sheaf, i.e. \( \mathcal{F}(\emptyset) = 0 \), then up to scaling of the constants, \( (X,w,\mathcal{F}) \) is an \((\varepsilon,\delta)\)-coboundary expander in dimension 0 if and only if the code \( Z^0(X,\mathcal{F}) \subseteq \Sigma^X(0) \) has relative distance \( \geq \delta \) and its natural tester has \textit{soundness} \( \geq \varepsilon \). Likewise, an augmented sheaf \( \mathcal{F} \) is an \( \varepsilon \)-coboundary expander in dimension 0 if and only if \( B^0(X,\mathcal{F}) = Z^0(X,\mathcal{F}) \), and the natural tester of \( Z^0(X,\mathcal{F}) \) has soundness \( \geq \varepsilon \).

From this point of view, studying the coboundary and cosystolic expansion of augmented sheaves in dimension 0 is a natural step toward solving the open problem of constructing good\( ^2 \) 2-query locally testable codes.

The main result of this work bounds the 0-dimensional coboundary expansion of some special \textit{non-constant} augmented sheaves using the spectral expansion \( \lambda(X,w) \) of their underlying weighted graph \( (X,w) \). This gives rise to the first examples of \textit{non-constant} sheaves with good coboundary expansion. Before describing this result, we first explain why the non-constant case is difficult to handle.

\textbf{Coboundary Expansion of Sheaves: What Cannot Be Said In General} 

Let \( \mathcal{F} \) be an augmented sheaf on a weighted graph \( (X,w) \). In contrast with constant case \( \mathcal{F} = R^X_+ \), when \( \mathcal{F} \) is general, using (some function of) \( \lambda(X,w) \) in order to bound \( \text{cb}_0(X,w,\mathcal{F}) \) from below is impossible, even if we impose the requirement \( Z^0(X,\mathcal{F}) = B^0(X,\mathcal{F}) \). Indeed, since the restriction maps are not required to be injective, there is no reason to expect that the boundary of a 0-cochain \( f \in C^0(X,\mathcal{F}) \) will have support that is proportional in weight to that of \( f \). The following example makes this intuition precise.

\textbf{Example 1.1.} Let \( (X, w) \) be any weighted graph, and let \( R \) be a nonzero abelian group. Define a \textit{sheaf} \( \mathcal{F} \) on \( X \) by setting:

- \( \mathcal{F}(x) = R \) for every \( \emptyset \neq x \in X \),
- \( \mathcal{F}(\emptyset) = R^X(0) \),
- \( \text{res}_{e^+ \rightarrow e^-} = 0 \) for every \( e \in X(1) \) and \( v \in X(0) \) with \( v \subseteq e \).
- \( \text{res}_{e \rightarrow \emptyset} : R^X(0) \rightarrow R \) is the projection onto the \( v \)-th component for every \( v \in X(0) \).

One readily checks that \( B^0(X,\mathcal{F}) = Z^0(X,\mathcal{F}) = R^X(0) \) and that \( d_0(f) = 0 \) for every \( f \in C^0(X,\mathcal{F}) \). Thus, \( \text{cb}_0(X,w,\mathcal{F}) = 0 \), regardless of what \( \lambda(X,w) \) or \( h(X,w) \) are.

It is tempting to hope that the problem highlighted in the example would be solved if we would require all the restriction maps \( \text{res}_{e^+ \rightarrow e^-} (v \in X(0), e \in X(1)) \) to be injective. However, we show that this is still not the case in Example 5.6.

\textbf{The Sheaves which We Study} 

Since addressing the general case is futile, we focus in this work on a special kind of augmented sheaves, namely, \textit{quotients} of constant augmented sheaves. This case is further motivated by [9], because understanding the coboundary expansion of such augmented sheaves is necessary in order to apply Theorem 9.5 in \textit{op. cit.}

In more detail, let \( (X, w) \) be a weighted graph and let \( \mathcal{F} \) be an augmented sheaf on \( X \). As expected, a \textit{subsheaf} of \( \mathcal{F} \) is an augmented sheaf \( \mathcal{G} \) on \( X \) such that \( \mathcal{G}(x) \subseteq \mathcal{F}(x) \) for every \( x \in X \), and the restriction maps of \( \mathcal{G} \) agree with that of \( \mathcal{F} \). In this case, one can form the quotient sheaf \( \mathcal{F}/\mathcal{G} \), which assigns every \( x \in X \) the abelian group \( \mathcal{F}(x)/\mathcal{G}(x) \), and has the evident restriction maps; see [13] Example 4.1 for further details. In the special case where \( \mathcal{F} = R^X_+ \) for an abelian group \( R \), specifying a subsheaf of \( \mathcal{F} \) amounts merely

\footnote{As usual, a code \( C \subseteq \Sigma^n \) (varying within a family) is called \textit{good} if it has linear distance and its rate is bounded away from 0.}
to specifying a subgroup $G(x)$ of $R$ for every $x \in X$, subject to the requirement that $G(x) \subseteq G(y)$ whenever $x \subseteq y$. This can be simplified even further: choose a subgroup $R_x \subseteq R$ for every $x \in X$ (including $x = \emptyset$), and then put $G(x) = \sum_{y \subseteq x} R_y$. That is, put:

1. $G(\emptyset) = R_\emptyset$,

2. $G(v) = R_\emptyset + R_v$ for all $v \in X(0)$, and

3. $G(e) = R_\emptyset + R_u + R_v + R_e$ for all $e \in X(1)$, where $u, v$ are the vertices of $e$.

One can quickly reduce to the case where $R_\emptyset = 0$, so we will assume this henceforth.

Now consider the quotient sheaf $R_X^+/G$. As the following example shows, even in this restricted setting, $\text{cb}_0(X, w, R_X^+/G)$ may be 0 if no assumption is made on the subgroups $\{R_x\}_{x \in X}$.

**Example 1.2.** Suppose that $(X, w)$ is a weighted graph and $R$ is a vector space $V$ of some large dimension over a field $\mathbb{F}$. Choose some nonconstant $f \in C^0(X, V_X^+) = V^X(0)$ and put $g = df \in C^1(X, R_X^+)$. Set $R_v = 0$ for every vertex $v \in X(0)$, and for every $e \in X(1)$, let $R_e = \mathbb{F} \cdot g(e)$ — a subspace of $V$ of dimension 1 or 0. While $V_X^+/G$ may seem very “close” to $V_X^+$, we actually have $\text{cb}_0(X, wV_X^+/G) = 0$ even when $\text{cb}_0(X, wV_X^+/G) \geq \frac{1}{2} h(X, w)$ is large. Indeed, we may view $f$ as an element of $C^0(X, V_X^+) - B^0(X, V_X^+/G)$, and by construction $df = 0$ in $C^1(X, V_X^+/G)$ (we annihilated the coordinates of $df$ by passing from $V_X^+$ to $V_X^+/G$).

The problem demonstrated in the example can be overcome by imposing some **linear disjointness** assumptions on the $\{R_x\}_{x \in X}$. Here, a finite collection of subgroups $\{R_i\}_{i \in I}$ of $R$ is said to be **linearly disjoint** if the summation map $(r_i)_{i \in I} \mapsto \sum_i r_i : \prod_{i \in I} R_i \to R$ is injective. For instance, in Example 1.2, if the edges $e_1, \ldots, e_\ell$ form a cycle in $X$, then $g(e_1) + \cdots + g(e_\ell) = 0$, which means that $R_{e_1}, \ldots, R_{e_\ell}$ are not linearly disjoint. Our main result says that if we do impose some linear disjointness assumptions, then $\text{cb}_0(X, w, V_X^+/G)$ will be large provided that $(X, w)$ is a good spectral expander.

**The Main Result**

Let $(X, w)$ be a weighted graph, let $R$ be an abelian group and let $\{R_x\}_{x \in X}$ be subgroups of $R$ with $R_\emptyset = 0$. Define the subsheaf $G$ of $R_X^+$ as before, and suppose that the following linear disjointness assumptions hold:

1. For every subgraph $Y$ of $X$ which is either a cycle of length $\leq \lceil \frac{2}{3} |X(0)| \rceil$ or a path of a length $\leq 2$ (see §2A), the summation map $\prod_{y \in Y(0), y \not\in Y(1)} R_y \to R$ is injective.

2. For every distinct $u, v \in X(0)$, we have $R_u \cap R_v = 0$.

We show in Theorem 6.1 that under these hypotheses, we have

$$\text{cb}_0(X, w, R_X^+/G) \geq \frac{2}{5} - O(\lambda(X, w)) - O(t),$$

(1.3)

where $t$ is a constant depending on how uniform the weight function $w$ is. For example, $t = \frac{2}{5}$ if $X$ is a $k$-regular graph and $w$ is given by $w(v) = \frac{1}{|X(0)|}$ for $v \in X(0)$ and $w(e) = \frac{1}{|X(1)|}$ for $e \in X(1)$.

We also prove a variant of this result for $(r + 1)$-partite weighted graphs, i.e., the underlying graphs of $(r + 1)$-partite weighted simplicial complexes. Such weighted graphs always have $-\frac{1}{r}$ as an eigenvalue, which makes the right hand side of (1.3) negative if $r$ is too small, e.g., if $X$ is a bipartite graph. In Theorem 6.2 we show that the eigenvalue $-\frac{1}{r}$ can be ignored at the expense of getting a slightly lower bound on $\text{cb}_0(X, w, R_X^+/G)$.

**Remark 1.3.** Write $n = |X(0)|$. We expect that the length of the cycles considered in condition (1) can be made lower than $\lceil \frac{2}{5} n \rceil$. However, as demonstrated in Example 1.2, it cannot be lowered below the girth of $X$. Since there are good expander graphs with $n$ vertices and girth $\Theta(\log n)$, we cannot expect a general lower bound such as (1.3) to hold if we do not assume that (1) holds for cycles of length $\leq \Theta(\log n)$.
Spherical buildings we can always embed

The proof of Theorems 6.1 and 6.2 is quite involved. Broadly speaking, given a 0-cochain $f \in C^0(X, R_X^+/G)$ such that its coboundary has small support, we restrict $f$ to special subgraphs $Y$ of $X$, e.g. cycles, showing that $f$ agrees with some $g \in B^0(Y, R_X^+/G)$ on that $Y$ (but $g$ depends on $Y$). We then observe that the special subgraphs can be clustered into larger subgraphs, and eventually show that at least one such cluster has a large weight. This allows us to bound $\text{dist}(f, B^0)$ from above, and that is enough to get (1.3) with a little more work.

In order to make our argument work, we first had to establish strong variants of the Expander Mixing Lemma for weighted graphs (Theorem 3.2) and weighted $(r + 1)$-partite graphs (Theorem 3.2), which were not available in the literature. These variants of the Expander Mixing Lemma will likely be useful elsewhere.

New Examples of Locally Testable Codes

Our main Theorems 6.1 and 6.2 give rise to new examples of locally testable codes, although with poor rate. For example, take $R$ to be an $\mathbb{F}_2$-vector space of dimension $m$, and let $e_1, \ldots, e_m$ be a basis for $R$. Let $X$ be a $k$-regular expander graph $X$ with a uniform weight function $w$, and let $v_1, \ldots, v_n$ ($n \geq m$) be the vertices of $X$. Now, with notation as before, put

- $R_{e_i} = \mathbb{F}_2 e_i$ for $i \in \{1, \ldots, m\}$,
- $R_{e_i} = 0$ for $i \in \{m + 1, \ldots, n\}$, and
- $R_e = 0$ for every $e \in X(1)$.

Putting $F = R_X^+/G$, we see that $F(v)$ is an $\mathbb{F}_2$-vector space of dimension $m - 1$ or $m$ for every $v \in X(0)$, so we can always embed $F(v)$ in $\Sigma := \mathbb{F}_{2^n}$ and view $Z^0(X, F)$ as a code inside $\Sigma^{X(0)} \cong \Sigma^n$. By our Theorem 6.2, the natural 2-query tester of $Z^0(X, F)$ has soundness $2/3 - O(\lambda(X, w)) - O(\frac{1}{n})$. One can further show that $Z^0(X, F)$ has relative distance $1 - \frac{1}{n}$ (Corollary 6.12), so by letting $n$ grow, we get a family of locally testable codes with linear distance. However, the rate of these codes is $\frac{1}{n-1}$; this is the same as the rate of the constant-word code $\{(a, \ldots, a) \mid a \in \Sigma\} \subseteq \Sigma^n$, but $Z^0(X, F)$ is in general not the constant-word code. In the special case $m = n$, we actually have $\dim F(v) = m - 1$ for all $v \in X(0)$, so we can take $\Sigma := \mathbb{F}_2^{m-1}$ and get the slightly larger rate $\frac{1}{n-1}$. 

About The Proof, Useful Side-Results

Spherical buildings are an important class of simplicial complexes admitting special structural properties. Gromov conjectured that the coboundary expansion of all spherical buildings of dimension $\leq d$ is bounded away from zero in all dimensions if the coefficient group is $\mathbb{F}_2$, and this was affirmed in [22] for the coefficient group $\mathbb{F}_2$ (with a natural choice of weights), and for a general constant coefficient group in [24]. We improve these results for coboundary expansion in dimension 0.

In more detail, we show (Theorem 7.2) that if $(X, w)$ is the weighted graph underlying a finite $r$-dimensional $q$-thick spherical building, then $\lambda_2(X, w) = O(\frac{1}{\sqrt{q} - 3r})$ and $\lambda(X, w) = O(\frac{r}{\sqrt{q} - 3r})$, provided $q > (3r)^2$. By plugging this into our main results, we conclude that $\text{cb}_0(X, w, R_X^+/G) \geq 1 - O(\frac{2^r}{\sqrt{q} - 3r})$ for every abelian group $R \neq 0$ (Corollary 7.4), and for a subsheaf $G$ of $R_X^+/G$ as before, we have

$$\text{cb}_0(X, w, R_X^+/G) \geq \frac{2^r}{5r + 2} - O(\frac{r^2}{\sqrt{q} - 3r}),$$

provided (1) and (2) hold (Corollary 7.6). These results apply to finite simplicial complexes covered by a $q$-thick affine building of dimension $r \geq 2$.

We also note that our bound $\lambda_2(X, w) = O(\frac{1}{\sqrt{q} - 3r})$ (for $q > 9r^2$) implies that the $q$-thick spherical building $X$ is an $O(\frac{1}{\sqrt{q} - 3})$-skeleton expander (see [21] §3.4, for instance). This improves a result of Evra and the second author [3, Theorem 5.19] who showed that $X$ is an $O(\frac{r^{(r+1)!}}{\sqrt{q}^{r+1}})$-skeleton expander.
Further Implications to Locally Testable Codes

Our Theorem 6.2 and its specialization to spherical buildings (Corollary 7.6) have indirect applications towards construction of good 2-query locally testable codes from sheaves on 2-dimensional simplicial complexes in [9]; see Theorems 8.1 and 9.5 in op. cit.

Outline

The paper is organized as follows: Section 2 is preliminary and recalls necessary facts about simplicial complexes, weight functions, spectral expansion and partite graphs. In Section 3 we prove weighted versions of the Cheeger Inequality and the Expander Mixing Lemma. Section 4 recalls sheaves on graphs, their cohomology and their expansion. In Section 5 we show that an expander graph with a constant augmented sheaf is a good coboundary expander in dimension 0, but that this may fail for locally constant sheaves. The subject matter of Section 6 is proving that taking a quotient of a constant augmented sheaf by a small subsheaf still results in a good coboundary expander in dimension 0, provided the underlying graph is a sufficiently good expander. This result is applied to finite spherical buildings in Section 7. Finally, in Section 8 we raise some questions about possible extensions of our results.

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2 Preliminaries

2A Simplicial Complexes

If not indicated otherwise, simplicial complexes and graphs are assumed to be finite. A graph is a 1-dimensional simplicial complex. In particular, we do not allow graphs to have loops or double edges (but double edges can be accounted for using weight functions discussed in §2B). Let $X$ be a simplicial complex. We write $V(X)$ for the vertex set of $X$ and $X(i) (-1 \leq i \in \mathbb{Z})$ for the set of $i$-faces of $X$. Note that a vertex and a 0-face are not the same thing, namely, $v \in V(X)$ if and only if $\{v\} \in X(0)$. When there is no risk of confusion, we will abuse the notation and treat vertices as 0-faces and vice versa. The $k$-dimensional skeleton of $X$ is the simplicial complex $X(\leq k) := \bigcup_{i=0}^{k} X(i)$. Given $z \in X$ and $A \subseteq X$, we write

$$A_{\geq z} = \{ y \in A : y \supseteq z \},$$
$$A_{\leq z} = \{ y \in A : y \subseteq z \},$$
$$A_z = \{ y - z | y \in A_{\geq z} \}.$$  

For example, if $x \in X(0)$ is a 0-face, then $X(1)_{\geq x}$ is the set of edges (i.e. 1-faces) containing $x$, and $X(1)_{\leq x}$ is the set 0-faces adjacent to $x$. Also, for any $z \in X$, the set $X_z$ is the link of $X$ at $z$; it is a simplicial complex.

An ordered edge in $X$ is a pair $(u, v)$ such that $\{u, v\}$ is an edge in $X$. The set of ordered edges in $X$ is denoted $X_{\text{ord}}(1)$. If $A, B \subseteq X(0)$, then we write $E(A, B)$ for the set of edges in $X$ with one vertex in $A$ and the other in $B$. Likewise, $E_{\text{ord}}(A, B)$ is the set of ordered edges $(u, v) \in X_{\text{ord}}(1)$ with $\{u\} \in A$ and $\{v\} \in B$. We also let $E(A) = E(A, A)$ and $E_{\text{ord}}(A) = E_{\text{ord}}(A, A)$.

The simplicial complex $X$ is said to be pure of dimension $d$ ($0 \leq d \in \mathbb{Z}$) if every face of $X$ is contained in a $d$-face. We then say that $X$ is a $d$-complex for short. The $k$-dimensional skeleton of a $d$-complex is a $k$-complex for all $k \in \{0, \ldots, d\}$.

Suppose now that $X$ is a graph and let $Y \subseteq X$ be a subset. We say that $Y$ is a cycle of length $\ell$ ($\ell \geq 3$) if $Y$ is a subgraph of $X$ that is isomorphic to the cycle graph on $\ell$ vertices. Given $x, y \in X(0)$ ($x = y$ is allowed), we say that $Y$ is a closed path of length $\ell$ ($\ell \geq 0$) from $x$ to $y$ if $Y$ is a subgraph of $X$ such that $x, y \in Y(0)$, $Y(0) - \{x, y\}$ contains exactly $\ell - 1$ distinct 0-faces $x_1, \ldots, x_{\ell-1}$, and $Y(1)$ consists of exactly $\ell$-faces adjacent to $x_1, \ldots, x_{\ell-1}$.

3Here, “closed” should be understood as topologically closed, rather than having the same start and end point.
edges which are \( x \cup x_1, x_1 \cup x_2, \ldots, x_{d-2} \cup x_{d-1}, x_{d-1} \cup y \). An open path from \( x \) to \( y \) is a subset \( Z \subseteq X \) of the form \( Y - \{x, y\} \), where \( Y \) is a closed path from \( x \) to \( y \) in \( X \); the length of \( Z \) is defined to be the length of \( Y \). An open path \( Z \) is not a subgraph of \( X \), but we shall still write \( Z(i) = Z \cap X(i) \) for \( i \in \{0, 1\} \).

The following easy lemma, whose proof is left to the reader, will be needed in the sequel.

**Lemma 2.1.** Let \( X \) be a graph, let \( X' \) be a subgraph of \( X \) and let \( C \) be a cycle in \( X \) meeting \( X' \). Then \( C - X' \) is a disjoint union of open paths.

### 2B Weights

A weight function on \( d \)-complex \( X \) is a function \( w : X \to \mathbb{R}_+ \) such that

1. \( \sum_{y \in X(d)} w(y) = 1 \),
2. \( w(x) = \left( \frac{d+1}{\dim x+1} \right)^{-1} \sum_{y \in X(d)_{\geq x}} w(y) \) for all \( x \in X \) with \( \dim x < d \).

We then say that \( (X, w) \) is a weighted \( d \)-complex; if \( d = 1 \) we say that \( (X, w) \) is a weighted graph. If we strengthen (W1) to \( w(y) = |X(d)|^{-1} \) for all \( y \in X(d) \), then this recovers the weight functions considered in [22] and [10].

Given a weight function \( w : X \to \mathbb{R}_+ \) and \( A \subseteq X \), we write \( w(A) = \sum_{a \in A} w(a) \). A similar convention applies to subsets of \( X_{\text{ord}}(1) \), where the weight of an ordered edge is defined to be the weight of its underlying unordered edge.

The conditions (W1) and (W2) imply that \( w \) defines a probability measure on \( X(d) \), and that for \( x \in X(i) \) with \( i \in \{-1, \ldots, d-1\} \), the value \( w(x) \) is the probability of obtaining \( x \) by choosing a \( d \)-face \( y \subseteq X(d) \) according to \( w \) and then choosing an \( i \)-face of \( y \) randomly uniformly. Consequently, \( w(X(i)) = 1 \) for all \( i \in \{-1, \ldots, d-1\} \). It is also straightforward to check that (W2) implies

\[
 w(X(\ell)_{\geq x}) = \binom{\ell+1}{k+1} w(x)
\]

for all \(-1 \leq k \leq \ell \leq d\) and \( x \in X(k) \). As a result, if \( (X, w) \) is a weighted \( d \)-complex, then for every \( k \in \{0, \ldots, d\} \), its \( k \)-dimensional skeleton \( (X(\leq k), w|_{X(\leq k)}) \) is a weighted \( k \)-complex.

**Example 2.2.** (i) Let \( X \) be a \( d \)-complex. The canonical weight function \( w_X : X \to \mathbb{R}_+ \) is defined by putting \( w_X(y) = \frac{1}{|X(d)|} \) for all \( y \in X(d) \) and defining \( w_X \) on the other faces using the formula in (W2).

(ii) If \( X \) is a \( k \)-regular graph on \( n \) vertices, then \( X \) is a pure 1-dimensional simplicial complex, and the canonical weight function assigns every edge of \( X \) the weight \( \frac{2}{kn} \), every vertex of \( X \) the weight \( \frac{1}{n} \) and the value 1 to the empty face.

(iii) If \( X \) is a \( d \)-complex, and \( 0 \leq k < d \), then \( (X(\leq k), w_X|_{X(\leq k)}) \) is a weighted \( k \)-complex, but \( w_X|_{X(\leq k)} \) is in general different from \( w_{X(\leq k)} \).

**Remark 2.3.** In [24] Definition 2.1 and [13] Definition 3.2, a balanced weight function on a \( d \)-complex is defined to be a function \( m : X \to \mathbb{R}_+ \) such that \( m(x) = (d - \dim x)! \cdot m(X(d)_{\geq x}) \) for all \( x \in X \). If \( (X, w) \) in a weighted \( d \)-complex in our sense, then \( m : X \to \mathbb{R}_+ \) defined by \( m(x) = \frac{(d+1)!}{(\dim x+1)!} \cdot w(x) \) is a balanced weight function in the sense of [24] and [13].

### 2C Expansion of Weighted Graphs

Let \( (X, w) \) be a weighted graph. Given \( i \in \{0, 1\} \), write \( C^i(X, \mathbb{R}) \) for the set of functions \( f : X(i) \to \mathbb{R} \). We endow \( C^i(X, \mathbb{R}) \) with the inner product defined by

\[
\langle f, g \rangle = \frac{1}{(i+1)!} \sum_{x \in X(i)} f(x)g(x)w(x)
\]

for all \( f, g \in C^i(X, \mathbb{R}) \). Given \( A \subseteq X(0) \), we write \( 1_A \) for the function in \( C^0(X, w) \) taking the value 1 on \( A \) and 0 on \( X(0) - A \).
The weighted adjacency operator of $(X, w)$, denoted $A = A_{X,w}$, and the weighted Laplacian of $(X, w)$, denoted $\Delta = \Delta_{X,w}$, are the linear operators from $C^0(X, \mathbb{R})$ to itself defined by

\[
(Af)(x) = \sum_{e \in X(1) \geq x} \frac{w(e)}{2w(x)} f(e - x),
\]

\[
(\Delta f)(x) = f(x) - \sum_{e \in X(1) \geq x} \frac{w(e)}{2w(x)} f(e - x),
\]

for all $f \in C^0(X, \mathbb{R})$ and $x \in X(0)$. We have $A = \text{id} - \Delta$. For example, if $X$ is a $k$-regular graph and $w$ is the canonical weight function of $X$, then $A_{X,w}$ is the ordinary adjacency operator of $X$ scaled by $\frac{1}{k}$, cf. Example 2.2(ii).

Follows from [24, Proposition 2.11]. (Consult Remark 2.3. For a general weighted $\lambda$-expander for short, if $\text{Spec}(A|_{C^0(X, \mathbb{R})}) \subseteq [\lambda, \mu]$. We also write $\lambda(X, w)$ for the smallest $\lambda \geq 0$ such that $\text{Spec}(A|_{C^0(X, \mathbb{R})}) \subseteq \lambda, -\lambda$.)

Fix a linear ordering $L$ on $V(X)$. Given an edge $e = \{u, v\} \in X(1)$ with $u < v$ relative to $L$, we write $e^+ = \{v\}$ and $e^- = \{u\}$. The 0-coboundary map is the linear operator $d_0 = d_0^L : C^0(X, \mathbb{R}) \to C^1(X, \mathbb{R})$ defined by

\[
(d_0 f)(e) = f(e^+) - f(e^-),
\]

for all $f \in C^0(X, \mathbb{R})$, $e \in X(1)$. The weighted 1-boundary map is the dual operator $d_0^* : C^1(X, \mathbb{R}) \to C^0(X, \mathbb{R})$ relative to the inner products of $C^0(X, \mathbb{R})$ and $C^1(X, \mathbb{R})$.

**Lemma 2.4.** Under the previous assumptions:

(i) $\Delta = d_0^* d_0$. In particular, $\Delta$ is positive semidefinite and $A$ is self-adjoint.

(ii) $\text{Spec} \Delta \subseteq [0, 2]$ and $\text{Spec} A \subseteq [-1, 1]$.

(iii) $\Delta_{1X(0)} = 0$ and $A_{1X(0)} = 1_{X(0)}$.

**Proof.** (i) This follows from [24, Proposition 2.11]. (Consult Remark 2.3. For a general weighted $d$-complex $(X, w)$, the inner product of $C^0(X, \mathbb{R})$ used in op. cit. is given by $\langle f, g \rangle = \frac{(d+1)!}{(1+1)!} \sum_{x \in X(i)} f(x)g(x)w(x)$. The Laplacian $\Delta$ is denoted $\Delta^D$ in op. cit.)

(ii) By (i), it is enough to prove that $\|A\| \leq 1$. Let $f \in C^0(X, \mathbb{R})$ and $x \in X(0)$. Then $\sum_{e \in X(1) \geq x} \frac{w(e)}{2w(x)} = 2w(x) = 1$, so by Jensen’s inequality,

\[
\sum_{e \in X(1) \geq x} \frac{w(e)}{2w(x)} f(e - x)^2 \leq \sum_{e \in X(1) \geq x} \frac{w(e)}{2w(x)} f(e - x)^2.
\]

Using this, we get

\[
\|Af\|^2 = \sum_{x \in X(0)} w(x) \sum_{e \in X(1) \geq x} \frac{w(e)}{2w(x)} f(e - x)^2
\]

\[
\leq \sum_{x \in X(0)} w(x) \sum_{e \in X(1) \geq x} \frac{w(e)}{2w(x)} f(e - x)^2 = \sum_{x \in X(0)} \sum_{e \in X(1) \geq x} \frac{w(e)}{2} f(e - x)^2
\]

\[
= \sum_{y \in X(0)} \sum_{e \in X(1) \geq y} \frac{w(e)}{2} f(y)^2 = \sum_{y \in X(0)} w(y) f(y)^2 = \|f\|^2,
\]

which is what we want.

(iii) Let $x \in X(0)$. Then $(A_{1X(0)})'(x) = \sum_{e \in X(0) \geq x} \frac{w(e)}{2w(x)} = 1 = 1_{X(0)}(x)$.

Let $(X, w)$ be a weighted graph. We define

\[
C^0(X, \mathbb{R}) = 1_{X(0)} = \{ f \in C^0(X, \mathbb{R}) : \sum_{x \in X(0)} w(x) f(x) = 0 \}.
\]

By Lemma 2.4, $\Delta$ and $A$ take $C^0(X, \mathbb{R})$ to itself. Given $\mu \leq \lambda$ in $\mathbb{R}$, we say that $(X, w)$ is a $[\mu, \lambda]$-spectral expander, or just $[\mu, \lambda]$-expander for short, if $\text{Spec}(A|_{C^0(X, \mathbb{R})}) \subseteq [\mu, \lambda]$. We also write $\lambda(X, w)$ for the smallest $\lambda \geq 0$ such that $\text{Spec}(A|_{C^0(X, \mathbb{R})}) \subseteq \lambda, -\lambda$. 9
Remark 2.5. (i) At this level of generality, a weighted graph can be a $[\mu, \lambda]$-expander even when both $\mu$ and $\lambda$ are negative. For example, consider a complete graph $X$ on $n+1$ vertices with its canonical weight function $w = w_X$. It is well-known that eigenvalues of $A_{X,w}$ are $1, -\frac{1}{n}, \ldots, -\frac{1}{n}$ (including multiplicities), so $(X, w)$ is a $[-\frac{n}{r}, -\frac{1}{r}]$-expander.

(ii) We shall see below that if $(X, w)$ is a bipartite weighted graph, then $-1 \in \text{Spec} \ A_{X,w}$, so such a weighted graph is a $[\mu, \lambda]$-expander only when $\mu \leq -1$.

We now recall Kaufman and Oppenheim's version of the Cheeger Inequality for graphs [18, Theorem 4.4(1)], which relates $[-1, \lambda]$-expansion and the Cheeger constant $h(X, w)$ (see the Introduction). To that end, it is convenient to introduce the following variation on the Cheeger constant $h(X, w)$, namely,

$$h'(X, w) = \min_{\emptyset \neq A \subseteq X(0)} \frac{w(E(A, X(0) - A))}{2w(A)w(X(0) - A)}$$

(this is denoted $h_G$ in op. cit.). Informally, $h'(X, w)$ is minimum possible ratio between the weight of the edges leaving $A$, and the expected weight if $(X, w)$ were to behave like a random graph. Since $\min\{\alpha, 1 - \alpha\} \leq 2\alpha(1 - \alpha) \leq 2\min\{\alpha, 1 - \alpha\}$ for all $\alpha \in [0, 1]$, we have

$$\frac{1}{2} h(X, w) \leq h'(X, w) \leq h(X, w). \quad (2.2)$$

Theorem 2.6 ([18 Theorem 4.4(1)]). Let $(X, w)$ be a weighted graph which is also a $[-1, \lambda]$-expander. Then

$$h'(X, w) \geq 1 - \lambda.$$ 

That is, for every $A \subseteq X$, we have $w(E(A, X(0) - A)) \geq (1 - \lambda) \cdot 2w(A)(1 - w(A))$.

A theorem of Friedland and Nabben [10, Theorem 2.1] implies a converse to the theorem, namely, if $h(X, w) \geq \varepsilon$, then $(X, w)$ is a $[-1, \sqrt{1 - \frac{\varepsilon^2}{2}}]$-expander. (Specifically, assuming $V(X) = \{1, \ldots, n\}$, apply [10, Theorem 2.1] with $w_{i,j} = w(\{i, j\})$ and $d_i = 2w(\{i\})$. The numbers $d_i$ defined op. cit. are $2w(\{i\})$ in our notation, and the constant $i(X, w)$ considered there equals $\frac{1}{2} h(X, w)$ in our notation.)

2D Partite Simplicial Complexes

Recall that an $(r+1)$-partite simplicial complex is a tuple $(X, V_0, \ldots, V_r)$ such that $X$ is a simplicial complex, $V_0, \ldots, V_r$ is a partition of the set of vertices $V(X)$ (in particular, $V_i \neq \emptyset$ for all $i$), and every face $x \in X$ contains at most one vertex from each $V_i$, i.e., $|x \cap V_i| \leq 1$ for all $i \in \{0, \ldots, r\}$. We then write $X_{\langle i \rangle}$ for the set of 0-faces having their vertex in $V_i$. We say that $(X, V_0, \ldots, V_r)$ is pure if every face of $X$ is contained in an $(r+1)$-face; in this case $\dim X = r$. A bipartite graph is just 2-partite simplicial complex.

A weighted $(r+1)$-partite simplicial complex is a tuple $(X, w, V_0, \ldots, V_r)$ such that $(X, V_0, \ldots, V_r)$ is a pure $(r+1)$-partite simplicial complex and $w$ is a weight function on the $r$-complex $X$.

When there is no risk of confusion, we will suppress the partition $(V_0, \ldots, V_r)$ from the notation, writing simply that $X$ is an $(r+1)$-partite complex, or that $(X, w)$ is a weighted $(r+1)$-partite simplicial complex.

Lemma 2.7. Let $(X, w, V_0, \ldots, V_r)$ be a weighted $(r+1)$-partite simplicial complex. Then:

(i) $w(X_{\langle i \rangle}) = \frac{1}{r+1}$ for all $i \in \{0, \ldots, r\}$.

(ii) $w(E(X_{\langle i \rangle}, X_{\langle j \rangle})) = \frac{2}{r(r+1)}$ for all distinct $i, j \in \{0, \ldots, r\}$.

(iii) The subspace $L$ of $C^0(X, \mathbb{R})$ spanned by $1_{X_{\langle 0 \rangle}}, \ldots, 1_{X_{\langle r \rangle}}$ is invariant under $A = A_{X,w}$. The $r + 1$ eigenvalues of $A$ on this subspace are $1, -\frac{1}{r}, \ldots, -\frac{1}{r}$ (including multiplicities). If the link $X_z$ is connected for all $z \in X \setminus (\leq r - 2)$, then $L$ is the sum of the 1-eigenspace and $-\frac{1}{r}$-eigenspace of $A$. 

10
Proof. We prove (i) and (ii) together. Given $I \subseteq \{0,\ldots,r\}$, let $X_I$ denote the set of $(|I|-1)$-faces of $X$ having a vertex in $V_i$ for all $i \in I$. We claim that $w(X_I) = \left(\frac{r+1}{|I|}\right)^{-1}$. Indeed,

$$w(X_I) = \sum_{x \in X_I} w(x) = \sum_{x \in X_I} \left(\frac{r+1}{|I|}\right)^{-1} \sum_{y \in X(r+1)_{x}} w(y)$$

$$= \left(\frac{r+1}{|I|}\right)^{-1} \sum_{y \in X(r+1)_{x}} \sum_{x \in X_I} w(y) = \left(\frac{r+1}{|I|}\right)^{-1} \sum_{y \in X(r+1)} 1 \cdot w(y)$$

$$= \left(\frac{r+1}{|I|}\right)^{-1} w(X(r+1)) = \left(\frac{r+1}{|I|}\right)^{-1},$$

where the fourth equality holds because every $(r+1)$-face contains exactly one face in $X_I$. (i) and (ii) now follow by taking $I = \{i\}$ and $I = \{i,j\}$, respectively.

Part (iii) follows from \cite{24} Proposition 5.2 and its proof.

Given a weighted pure $(r+1)$-partite simplicial complex $(X, w, V_0,\ldots,V_r)$, we write $C^0_\mu(X, \mathbb{R})$ for $L^\perp$, where $L = \text{span}_\mathbb{R}\{1_{X(0)},\ldots,1_{X(r)}\}$. In view of Lemma \cite{27} iii), when regarding the spectrum of $A = A_{X,w}$ on $C^0(X, \mathbb{R})$, it is reasonable to set aside the eigenvalues occurring on the subspace $L$. Thus, we say that $(X, w, V_0,\ldots,V_r)$ is an $(r+1)$-partite $[\mu, \lambda]$-expander if the spectrum of $A$ on $C^0_\mu(X, \mathbb{R})$ is contained in the interval $[\mu, \lambda]$.

Oppenheim \cite{24} Lemma 5.5\footnote{There is a typo in this source: the last inequality should be “$1 + \frac{1}{n}(1-\lambda(X)) \leq \kappa(X) \leq 1 + \kappa(1-\lambda(X))$.”} observed that if $(X, w)$ is a weighted $(r+1)$-partite $[-1, \lambda]$-expander with $0 \leq \lambda \leq \frac{1}{r}$, then $(X, w)$ is in fact an $r$-partite $[-r\lambda, \lambda]$-expander. In fact, when $(X, w)$ is a weighted bipartite graph, Spec $A_{X,w}$ is symmetric around 0; this follows from the following lemma, whose straightforward proof is left to the reader.

**Lemma 2.8.** Let $(X, w, V_0, V_1)$ be a weighted bipartite graph, let $\lambda \in \mathbb{R}$ and let $f \in C^0_\mu(X, \mathbb{R})$ be a $\lambda$-eigenfunction of $A = A_{X,w}$. Define $f' \in C^0_\mu(X, \mathbb{R})$ by $f'(x) = f(x)$ if $x \in X_0$ and $f'(x) = -f(x)$ otherwise. Then $Af' = -\lambda f'$.

## 3 Mixing Lemmas for Weighted Graphs

In this section, we prove two fine versions of the Expander Mixing Lemma applying to weighted graphs, and $(r+1)$-partite weighted simplicial complexes, respectively. The former result is well-known for non-weighted graphs, but we were unable to find the weighted version considered here in the literature. Both results will be needed in the sequel to establish our main results about coboundary expansion.

Recall that given a weighted graph $(X, w)$ and $A \subseteq X(0)$, we write $1_A \in C^0(X, \mathbb{R})$ for the function taking the value 1 on $A$ and 0 elsewhere.

**Lemma 3.1.** Let $(X, w)$ be a weighted graph and let $A, B \subseteq X(0)$. Then

$$\langle A_{X,w} 1_A, 1_B \rangle = \frac{1}{2} w(E_{\text{ord}}(A, B)).$$

**Proof.** By unfolding the definitions, $(A1_A, 1_B)$ evaluates to

$$\sum_{x \in X(0)} (A1_A)(x) 1_B(x) w(x) = \sum_{x \in X(0)} \sum_{e \in X(1)_{\geq x}} \frac{w(e)}{2w(x)} 1_A(e-x) 1_B(x) w(x)$$

$$= \frac{1}{2} \sum_{x \in X(0)} \sum_{e \in X(1)_{\geq x}} 1_A(e-x) 1_B(x) w(e) = \frac{1}{2} w(E_{\text{ord}}(A, B)).$$

**Theorem 3.2** (Weighted Expander Mixing Lemma). Let $(X, w)$ be a weighted graph, which is also a $[\mu, \lambda]$-expander. Let $A, B \subseteq X(0)$ and put $\alpha = w(A)$, $\beta = w(B)$. Then:

(i) $|\frac{1}{2} w(E_{\text{ord}}(A, B)) - \alpha \beta| \leq \max\{|\lambda|, |\mu|\} \sqrt{\alpha \beta (1-\alpha)(1-\beta)}$. 

Proof. By unfolding the definitions, $(A1_A, 1_B)$ evaluates to

$$\sum_{x \in X(0)} (A1_A)(x) 1_B(x) w(x) = \sum_{x \in X(0)} \sum_{e \in X(1)_{\geq x}} \frac{w(e)}{2w(x)} 1_A(e-x) 1_B(x) w(x)$$

$$= \frac{1}{2} \sum_{x \in X(0)} \sum_{e \in X(1)_{\geq x}} 1_A(e-x) 1_B(x) w(e) = \frac{1}{2} w(E_{\text{ord}}(A, B)).$$

**Theorem 3.2** (Weighted Expander Mixing Lemma). Let $(X, w)$ be a weighted graph, which is also a $[\mu, \lambda]$-expander. Let $A, B \subseteq X(0)$ and put $\alpha = w(A)$, $\beta = w(B)$. Then:

(i) $|\frac{1}{2} w(E_{\text{ord}}(A, B)) - \alpha \beta| \leq max\{|\lambda|, |\mu|\} \sqrt{\alpha \beta (1-\alpha)(1-\beta)}$. 

Proof. By unfolding the definitions, $(A1_A, 1_B)$ evaluates to

$$\sum_{x \in X(0)} (A1_A)(x) 1_B(x) w(x) = \sum_{x \in X(0)} \sum_{e \in X(1)_{\geq x}} \frac{w(e)}{2w(x)} 1_A(e-x) 1_B(x) w(x)$$

$$= \frac{1}{2} \sum_{x \in X(0)} \sum_{e \in X(1)_{\geq x}} 1_A(e-x) 1_B(x) w(e) = \frac{1}{2} w(E_{\text{ord}}(A, B)).$$

$$\sum_{x \in X(0)} (A1_A)(x) 1_B(x) w(x) = \sum_{x \in X(0)} \sum_{e \in X(1)_{\geq x}} \frac{w(e)}{2w(x)} 1_A(e-x) 1_B(x) w(x)$$

$$= \frac{1}{2} \sum_{x \in X(0)} \sum_{e \in X(1)_{\geq x}} 1_A(e-x) 1_B(x) w(e) = \frac{1}{2} w(E_{\text{ord}}(A, B)).$$
Theorem 3.3

where in the third equality we used Lemma 2.4(iii). By Lemma 3.1, we have

\[ \mu \|f\|^2 \leq \langle Af, f \rangle \leq \lambda \|f\|^2, \]

(3.1)

\[ |\langle Af, g \rangle| \leq \max\{\|\lambda\|, \|\mu\|\}\|f\|\|g\|. \]

(3.2)

Put \( f := 1_A - \alpha 1_{X(0)} \) and \( g := 1_B - \beta 1_{X(0)} \). We first check that \( f, g \in C_0^0(X, \mathbb{R}) \) and \( \|f\|^2 = \alpha(1 - \alpha) \), \( \|g\|^2 = \beta(1 - \beta) \). Indeed,

\[ \langle f, 1_{X(0)} \rangle = \langle A(1_{X(0)}), f + g \rangle = \alpha \langle 1_{X(0)}, 1_{X(0)} \rangle + \langle Af, g \rangle = \alpha \beta + \langle Af, g \rangle, \]

where in the third equality we used Lemma 2.4(iii). By Lemma 3.1, \( \langle A1_A, 1_B \rangle = \frac{1}{2} w(E_{\text{ord}}(A, B)) \), so we get

\[ \frac{1}{2} w(E_{\text{ord}}(A, B)) - \alpha \beta = \langle Af, g \rangle. \]

(3.3)

Now, by (3.1),

\[ |\frac{1}{2} w(E_{\text{ord}}(A, B)) - \alpha \beta| \leq \max\{\|\lambda\|, \|\mu\|\}\|f\|\|g\| = \max\{\|\lambda\|, \|\mu\|\} \sqrt{\lambda(1 - \alpha)(1 - \beta)}, \]

which proves (i). Also, taking \( A = B \) in (3.3) and using (3.2) gives

\[ \mu \alpha(1 - \alpha) = \mu \|f\|^2 \leq \frac{1}{2} w(E_{\text{ord}}(A)) - \alpha \beta \leq \lambda \|f\|^2 = \lambda \alpha(1 - \alpha), \]

and (ii) follows because \( \frac{1}{2} w(E_{\text{ord}}(A)) = w(A) \). \( \square \)

Recall from Lemma 2.7 that if \( (X, w, V_0, \ldots, V_r) \) is a weighted \((r + 1)\)-partite simplicial complex, then \(- \frac{1}{r}\) is an eigenvalue of \( A \). As a result, the constant \( \max\{\|\lambda\|, \|\mu\|\} \) appearing in Theorem 3.2(i) is at least \( \frac{1}{r} \). In particular, when \( X \) is a bipartite graph, this constant is 1, and Theorem 3.2(i) gives almost no information about \( w(E_{\text{ord}}(A, B)) \). We remedy this in the following theorem.

Theorem 3.3 (Expander Mixing Lemma for Weighted \((r + 1)\)-Partite Graphs). Let \( (X, w) \) be a weighted pure \((r + 1)\)-partite simplicial complex that is an \((r + 1)\)-partite \([-\lambda, \lambda]\)-expander. Let \( A, B \subseteq X(0) \) and put \( \alpha = w(A), \beta = w(B), \alpha_i = w(A \cap X(i)) \) and \( \beta_i = w(B \cap X(i)) \) \((i \in \{0, \ldots, r\}) \). Then:

(i) If there are \( T, S \subseteq \{0, \ldots, r\} \) such that \( A \subseteq \bigcup_{i \in T} X(i), B \subseteq \bigcup_{j \in S} X(j) \) and \( S \cap T = \emptyset \), then

\[ \frac{1}{2} w(E(A, B)) - \frac{r + 1}{r} \alpha \beta \leq \lambda (r + 1) \sqrt{\alpha \beta \left( \frac{|T|}{r + 1} - \alpha \right) \left( \frac{|S|}{r + 1} - \beta \right)}, \]

(ii) In general, \( \frac{1}{2} w(E_{\text{ord}}(A, B)) - \frac{r + 1}{r} |\alpha \beta - \sum_{i=0}^{r} \alpha_i \beta_i| \leq \lambda r \sqrt{\alpha \beta (1 - \alpha)(1 - \beta)}. \) In particular,

\[ \frac{1}{2} w(E_{\text{ord}}(A, B)) \leq \frac{r + 1}{r} \left|\alpha \beta - \sum_{i=0}^{r} \alpha_i \beta_i\right| + \lambda r \sqrt{\alpha \beta (1 - \alpha)(1 - \beta)} \]

\[ \leq \frac{r + 1}{r} \alpha \beta + \lambda r \sqrt{\alpha \beta (1 - \alpha)(1 - \beta)}. \]

12
Proof. For $i \in \{0, \ldots, r\}$, let $A_i = A \cap X_{(i)}$ and $B_i = B \cap X_{(i)}$. Define $f_i = 1_{A_i} - (r+1)\alpha_i1_{X_{(i)}}$ and $g_i = 1_{B_i} - (r+1)\beta_i1_{X_{(i)}}$. Since $\text{supp}(f_i) \subseteq X_{(i)}$, we have $\langle 1_{X_{(i)}}, f_i \rangle = 0$ for all $j \in \{0, \ldots, r\} - \{i\}$, whereas by Lemma 2.7(i), we also have $\langle 1_{X_{(i)}}, f_i \rangle = \langle 1_{X_{(i)}}, 1_{A_i} \rangle - (r+1)\alpha_i\langle 1_{X_{(i)}}, 1_{X_{(i)}} \rangle = w(A_i) - (r+1)\alpha_i \frac{1}{r+1} = 0$, so $f_i \in C^0_c(X, \mathbb{R})$. We further note that

$$
\|f_i\|^2 = w(A_i)(1-(r+1)\alpha_i)^2 + w(X_{(i)} - A_i)((r+1)\alpha_i)^2
= \alpha_i(1-(r+1)\alpha_i)^2 + (\frac{1}{r+1} - \alpha_i)(r+1)^2 \alpha_i^2 = \alpha_i(1-(r+1)\alpha_i).
$$

Fix some distinct $i, j \in \{0, \ldots, r\}$. By Lemmas 3.1 and Lemma 2.7(ii), we have $\langle A_1 \{X_i\}, 1_{\{X_j\}} \rangle = \frac{1}{2}w(E(X_i, X_j)) = \frac{1}{2}\frac{2}{r(r+1)} = \frac{1}{r(r+1)}$. Now, using Lemma 3.1 again, we find that

$$
\frac{1}{2}w(E(A_i, B_j)) = \langle A_1 \{A_i\}, 1_{B_j} \rangle = \langle A((r+1)\alpha_i1_{X_{(i)}}) + Af_i, (r+1)\beta_j1_{X_{(i)}} + g_j \rangle = \langle (r+1)\alpha_iA1_{X_{(i)}}, (r+1)\beta_j1_{X_{(i)}} \rangle + \langle f_i, g_j \rangle = (r+1)^2\alpha_i\beta_j \frac{1}{r(r+1)} + \langle Af_i, g_i \rangle = \frac{r+1}{r} - \alpha_i\beta_j + \langle Af_i, g_i \rangle.
$$

Since $(X, w)$ is an $(r+1)$-partite $[-\lambda, \lambda]$-expander, we have

$$
\langle Af_i, g_j \rangle \leq \lambda \|f_i\|\|g_j\| = \lambda \sqrt{\alpha_i\beta_i(1-(r+1)\alpha_i)(1-(r+1)\beta_j)}.
$$

Together with (3.4), this implies that

$$
\frac{1}{2}w(E(A_i, B_j)) - \frac{r+1}{r} - \alpha_i\beta_j \leq \lambda \sqrt{\alpha_i\beta_i(1-(r+1)\alpha_i)(1-(r+1)\beta_j)}.
$$

(3.5)

It is routine, yet tedious, to check that the real two-variable function $h(x, y) = \sqrt{xy(1-(r+1)x)(1-(r+1)y)}$ is concave on $[0, \frac{1}{r+1}] \times [0, \frac{1}{r+1}]$. Thus, by Jensen’s inequality, for any sequence of points $\{(x_k, y_k)\}_{k=1}^{\infty}$ in the square $[0, \frac{1}{r+1}]^2$, we have $\sum_k h(x_k, y_k) \leq th(\sum k \frac{x_k}{T}, \sum k \frac{y_k}{T})$. We apply this with together with (3.5) to prove (i) and (ii).

To prove (i), we consider the points $\{(\alpha_i, \beta_j)\}_{i \in S, j \in T}$. By (3.5), Jensen’s inequality and our assumptions on $A$ and $B$, we have

$$
\frac{1}{2}w(E(A, B)) - \frac{r+1}{r} - \alpha \beta = |\sum_{i \in T} \sum_{j \in S} \frac{1}{2}w(E(A_i, B_j)) - \frac{r+1}{r} \sum_{i \in T} \sum_{j \in S} \alpha_i\beta_j|
\leq \sum_{i \in T} \sum_{j \in S} \left|\frac{1}{2}w(E(A_i, B_j)) - \frac{r+1}{r} - \alpha_i\beta_j\right|
\leq \lambda |T||S| \sqrt{\frac{\alpha}{|T|} \frac{\beta}{|S|} (1 - \frac{(r+1)\alpha}{|T|})(1 - \frac{(r+1)\beta}{|S|})}
\leq \lambda (r+1) \sqrt{\alpha \beta (r+1 - \alpha)(\frac{|S|}{r+1} - \beta)},
$$

which is what we want.

5Indeed, writing $k = r+1$, the determinant of the Hessian matrix of $h$ evaluates to $\frac{k^3}{(r+1)^2} \frac{1}{r} + \frac{k^2}{r} + \frac{k}{r} - 4k$, which is positive on the interior of $[0, \frac{1}{r+1}]^2$. In addition, $\frac{\partial^2 h}{(\partial x)^2} = -\frac{\sqrt{(1-kx)}}{x(1-kx)^{3/2}}$ is negative on that domain, so the Hessian matrix is negative semidefinite.
To prove (ii), we consider all the points \((\alpha_i, \beta_j)\) with \(i, j \in \{0, \ldots, r\}\) and \(i \neq j\). By \([3.5]\) and Jensen’s inequality, we have

\[
\left| \frac{1}{2}w(E(A, B)) - \frac{r+1}{r} \left[ \alpha \beta - \sum_{i=0}^{r} \alpha_i \beta_i \right] \right| = \left| \sum_{i \neq j} \left( \frac{1}{2}w(E(A, B)) - \frac{r+1}{r} \alpha_i \beta_i \right) \right| \\
\leq \sum_{i \neq j} \lambda \sqrt{\alpha_i \beta_i (1 - (r+1)\alpha_i)(1 - (r+1)\beta_j)} \\
\leq \lambda r (r+1) \sqrt{\frac{\alpha \beta}{r+1}} \frac{1}{r+1} (1 - \frac{(r+1)\alpha}{r+1})(1 - \frac{(r+1)\beta}{r+1}) \\
= \lambda r \sqrt{\alpha \beta (1-\alpha)(1-\beta)},
\]

so we are done. \(\Box\)

4 Sheaves on Simplicial Complexes

We recall from \([9]\) the definition of augmented sheaves on simplicial complexes as well as their cohomology and coboundary expansion, focusing particularly in the case of augmented sheaves on graphs.

4A Sheaves on Simplicial Complexes

Let \(X\) be a simplicial complex. Following \([9]\), an augmented sheaf \(\mathcal{F}\) on \(X\) consists of

1. an abelian group \(\mathcal{F}(x)\) for every \(x \in X\) (including the empty face), and
2. a group homomorphism \(\text{res}_{y \leftarrow x}^\mathcal{F} : \mathcal{F}(x) \to \mathcal{F}(y)\) for all \(x \subseteq y \in X\) such that

\[
\text{res}_{z \leftarrow y}^\mathcal{F} \circ \text{res}_{y \leftarrow x}^\mathcal{F} = \text{res}_{z \leftarrow x}^\mathcal{F} \quad (4.1)
\]

whenever \(x \subseteq y \subseteq z \in X\). This generalizes the case of graphs considered in the introduction. We will usually drop the superscript \(\mathcal{F}\) from \(\text{res}_{y \leftarrow x}^\mathcal{F}\) when there is no risk of confusion.

A sheaf on \(X\) is an augmented sheaf \(\mathcal{F}\) with \(\mathcal{F}(\emptyset) = 0\). In this case, if \(X\) is a graph, then condition (4.1) always holds.

Example 4.1. Let \(X\) be a simplicial complex and let \(R\) be an (additive) abelian group.

The constant augmented sheaf associated to \(R\) is the augmented sheaf \(\mathcal{F}_c\) on \(X\) determined by \(\mathcal{F}(x) = R\) for all \(x \in X\) and \(\text{res}_{y \leftarrow x}^{\mathcal{F}_c} = \text{id}_R\) for all \(x \subseteq y \in X\). We denote this \(\mathcal{F}\) by \(R_X\), or \(R^+\) when \(X\) is clear from the context.

The constant sheaf associated to \(R\) is the sheaf \(\mathcal{F}'\) determined by setting \(\mathcal{F}'(x) = R\) for \(x \in X - \{\emptyset\}\), \(\text{res}_{y \leftarrow x}^{\mathcal{F}'} = \text{id}_R\) for all \(\emptyset \neq x \subseteq y \in X\), \(\mathcal{F}'(\emptyset) = 0\) and \(\text{res}_{y \leftarrow \emptyset}^{\mathcal{F}'} = 0\). We denote \(\mathcal{F}'\) by \(R_X\).

There are obvious notions of subsheaves, quotient sheaves, and homomorphisms between sheaves, see \([9]\).

An augmented sheaf (resp. sheaf) \(\mathcal{F}\) on a graph \(X\) is said to be constant if it is isomorphic to the constant augmented sheaf (resp. sheaf) associated to some abelian group \(R\). A sheaf \(\mathcal{F}\) is called locally constant if the restriction map \(\text{res}_{y \leftarrow x}^\mathcal{F} : \mathcal{F}(x) \to \mathcal{F}(y)\) is an isomorphism for all \(\emptyset \neq x \subseteq y \in X\).

4B Sheaf Cohomology

Let \(\mathcal{F}\) be an augmented sheaf on a simplicial complex \(X\). Fix a linear ordering \(L\) on the vertices of \(X\) and, for every \(i \in \mathbb{N} \cup \{-1, 0\}\), let \(C^i(X, \mathcal{F})\) denote \(\prod_{x \in X(i)} \mathcal{F}(x)\). Elements of \(C^i(X, \mathcal{F})\) are called \(i\)-chains with coefficients in \(\mathcal{F}\). Given \(f \in C^i(X, \mathcal{F})\), we write the \(x\)-component of \(f\) as \(f(x)\). Writing \(x = \{v_0, \ldots, v_i\}\) with \(v_0 < v_1 < \cdots < v_i\), we let \(x_j\) denote \(x - \{v_j\}\). As in \([9]\), the \(i\)-th coboundary map \(d_i : C^i(X, \mathcal{F}) \to C^{i+1}(X, \mathcal{F})\) is defined by

\[
(d_i f)(y) = \sum_{j=0}^{i+1} (-1)^j \text{res}_{y \leftarrow x_j} f(x_j)
\]
for all \( f \in C^i(X, \mathcal{F}) \), \( y \in X(i+1) \). For example, \( d_{-1} \) and \( d_0 \) are given by the formulas from the introduction:

\[
(d_{-1}f)(e) = \text{res}_{e\to \emptyset} f(\emptyset) \quad \forall y \in X(0),
\]
\[
(d_0f)(e) = \text{res}_{e\to e^+} f(e^+) - \text{res}_{e\to e^-} f(e^-) \quad \forall e \in X(1),
\]

where here, \( e^+ \) is the larger 0-face of \( e \) and \( e^- \) is a smaller 0-face of \( e \), relative to \( L \). We have \( d_i \circ d_{i-1} = 0 \).

As usual, the \( i \)-coboundaries, \( i \)-cycles, and \( i \)-th cohomology of \((X, \mathcal{F})\) are defined to be

\[
B^i(X, \mathcal{F}) := \text{im} \ d_{i-1}, \quad Z^i(X, \mathcal{F}) := \text{im} \ d_i, \quad H^i(X, \mathcal{F}) = \frac{Z^i(X, \mathcal{F})}{B^i(X, \mathcal{F})},
\]

respectively.

For example, if \( \mathcal{F} = R^1_X \) for an abelian group \( R \) (see Example 4.1), then \( B^0(X, R^+) \) is the set of constant functions from \( X(0) \) to \( R \), \( Z^0(X, R^+) \) is the set of functions \( f : X(0) \to R \) which care constant on each connected component of \( X \) and \( H^0(X, R^+) \) is isomorphic to \( R^{\pi(X)|-1} \) and coincides with the reduced singular cohomology group \( \tilde{H}^0(X, R) \).

### 4C Coboundary and Cosystolic Expansion

Let \((X, w)\) be a weighted \( d \)-complex (see §2B) and let \( \mathcal{F} \) be an augmented sheaf on \( X \). For \( i \in \{-1, 0, \ldots, d\} \), the \textit{\( w \)-support norm} on \( C^i(X, \mathcal{F}) \) is the function \( \| \cdot \|_w : C^i(X, \mathcal{F}) \to \mathbb{R} \) defined by

\[
\|f\|_w = w(\text{supp} f),
\]

where \( \text{supp} f = \{ x \in X(i) : f(x) \neq 0 \} \).

The corresponding metric on \( C^i(X, \mathcal{F}) \) is

\[
\text{dist}_w(f, g) := \|f - g\|_w.
\]

The subscript \( w \) will sometimes be dropped from \( \| \cdot \|_w \) and \( \text{dist}_w \) when there is no risk of confusion.

Let \( \varepsilon, \delta \in [0, \infty) \) and \( i \in \{-1, 0, \ldots, d-1\} \). Following §2B, we define the \( i \)-coboundary expansion of \((X, w, \mathcal{F})\), denoted

\[
\text{cb}_i(X, w, \mathcal{F})
\]

to be the supremum of the set of \( \varepsilon \in [0, \infty) \) for which

\[
\|d_i f\|_w \geq \varepsilon \text{dist}_w(f, B^i(X, \mathcal{F})) \quad \forall f \in C^i(X, \mathcal{F}).
\]

We further say that \((X, w, \mathcal{F})\) is an \( \varepsilon \)-\textit{coboundary expander in dimension} \( i \) if \( \text{cb}_i(X, w, \mathcal{F}) \geq i \). In addition, we call \((X, w, \mathcal{F})\) an \( (\varepsilon, \delta) \)-\textit{cosystolic expander in dimension} \( i \) if

(C1) \( \|d_i f\|_w \geq \varepsilon \text{dist}_w(f, Z^i(X, \mathcal{F})) \) for all for all \( f \in C^i(X, \mathcal{F}) \), and

(C2) \( \|f\|_w \geq \delta \) for all \( f \in Z^i(X, \mathcal{F}) - B^i(X, \mathcal{F}) \).

Note that both definitions do not depend on the linear ordering on \( V(X) \) chosen to define \( d_i \). Also, if \( \text{cb}_i(X, w, \mathcal{F}) > 0 \), then we must have \( H^i(X, \mathcal{F}) = 0 \).

**Example 4.2.** Let \((X, w)\) be a weighted graph. We noted in the introduction that \( \text{cb}_0(X, w, (F_2)_X^+) = h(X, w) \) (see Example 4.1 for the definition of \((F_2)_X^+\)). If \( X \) is moreover \( k \)-regular, \( w \) is its canonical weight function (Example 2.2), and we consider the non-weighted Cheeger constant

\[
\tilde{h}(X) = \min_{0 \neq S \subseteq X(0)} \frac{|E(S, X(0) - S)|}{\min\{|S|, |X(0) - S|\}},
\]

then \( \tilde{h}(X) = \frac{k}{2} \text{cb}_0(X, w, (F_2)_X^+) \).

The triple \((X, w, (F_2)_X^+)\) is an \( (\varepsilon, \delta) \)-cosystolic expander in dimension 0 if and only if each connected component of \( X \) is an \( \varepsilon \)-combinatorial expander in the sense of the introduction, and the weight of the vertices of every connected component is at least \( \delta \).

\(^6\)Warning: When \( \mathcal{F} \) is the constant sheaf \( \mathbb{R} \), the norm \( \| \cdot \|_w \) is not the norm induced from the inner products we defined on \( C^0(X, \mathbb{R}) \) and \( C^1(X, \mathbb{R}) \) in §2C. Moreover, \( (C^i(X, \mathbb{R}), \| \cdot \|_w) \) is not a normed \( \mathbb{R} \)-vector space.

\(^7\)This definition slightly differs from the one in §2F if \( \mathcal{F}(x) = 0 \) for some nonempty faces \( x \in X \).
The main results of the following sections — Theorems 5.2, 6.1, 6.2 and Corollary 7.6 — will only concern with coboundary expansion in dimension 0. However, they can be converted to statements about cosystolic expansion by means of the following remark.

**Remark 4.3.** Let \((X, w)\) be a weighted graph, let \(F\) be an augmented sheaf on \(X\) and let \(\varepsilon > 0\). Denote by \(F_0\) the subsheaf of \(F\) determined by \(F_0(0) = 0\) and \(F_0(x) = F(x)\) for \(x \neq 0\), and put \(\delta = \max \{ ||d_{-1}f||_w \mid f \in F(0)\}\). If \((X, w, F)\) is an \(\varepsilon\)-coboundary expander in dimension 0, then \((X, w, F_0)\) is a \((\varepsilon, \delta)\)-coboundary expander in dimension 0. The converse holds if \(H^0(X, F) = 0\).

5 Coboundary Expansion of Constant Augmented Sheaves on Graphs

We now show that if a weighted graph \((X, w)\) is a \([-1, \lambda]\)-spectral expander (see §2C), then \((X, w, F)\) is a good coboundary expander in dimension 0 for every constant augmented sheaf \(F\). This is well-known when \(X\) is a \(k\)-regular graph with its canonical weight function and \(F = (F_2)^+\) (notation as in Example 4.1); use Example 4.2 and the Cheeger Inequality for Graphs (§3 Theorem 1.2.3) or Theorem 2.6 above.

In contrast, we show that under mild assumptions, \(X\) admits locally constant sheaves \(F\) such that \((X, w, F)\) has poor coboundary expansion in dimension 0, regardless of how small \(\lambda(X, w)\) is.

**Lemma 5.1.** Suppose that \(\alpha_0, \ldots, \alpha_t \in [0, 1]\) satisfy \(\alpha_0 \geq \max \{\alpha_1, \ldots, \alpha_t\}\) and \(\sum_{i=0}^{t} \alpha_i \leq 1\). Then \(\sum_{i=0}^{t} \alpha_i(1 - \alpha_i) \geq \sum_{i=1}^{t} \alpha_i\).

**Proof.** After rearranging, the inequality becomes \(\sum_{i=1}^{t} \alpha_i^2 \leq \alpha_0(1 - \alpha_0)\). Since \(\alpha_0 \geq \max \{\alpha_1, \ldots, \alpha_t\}\), we have \(\sum_{i=1}^{t} \alpha_i^2 \leq \sum_{i=1}^{t} \alpha_i \alpha_i \leq \alpha_0(1 - \alpha_0)\).

Recall from §2C that \(h'(X, w) = \min_{\phi \neq A \subseteq X(\emptyset)} \frac{w(E(A, X(\emptyset) - A))}{2w(X(\emptyset) - A)}\).

**Theorem 5.2.** Let \((X, w)\) be a weighted graph and let \(R\) be a nontrivial (additive) abelian group. Then:

(i) \((X, w, R_\lambda^X)\) is an \(h'(X, w)\)-coboundary expander in dimension 0.

(ii) If \((X, w, R_\lambda^X)\) is an \(\varepsilon\)-coboundary expander in dimension 0, then \(h(X, w) \geq \varepsilon\) and \(h'(X, w) \geq \frac{\varepsilon}{2}\).

If \(X\) is connected, then Remark 4.3 and (i) imply that \((X, w, R)\) is an \((h'(X, w), 1)\)-cosystolic expander in dimension 0.

**Proof.** (i) Given \(a \in R\), write \(f_a\) for the element of \(C^0(X, R_\lambda^X)\) defined by \(f_a(x) = a\) for all \(x \in X(0)\). We abbreviate \(\| \cdot \|_w\) to \(\| \cdot \|\) and \(h'(X, w)\) to \(h'\).

Let \(f \in C^0(X, R_\lambda^X)\). We need to show that \(\|d_0f\| \geq h'\|f - f_a\|\) for some \(a \in R\). Let \(a_0 = 0, a_1, \ldots, a_t \in R\) be the values attained by \(f\) together with 0, and write \(A_i = \{x \in X(0) : f(x) = a_i\}\) and \(\alpha_i = w(A_i)\). Then \(\sum_{i=0}^{t} \alpha_i = 1\) and \(\sum_{i=1}^{t} \alpha_i = \|f\|\). Choose \(j \in \{0, \ldots, t\}\) such that \(\alpha_j = \max\{\alpha_0, \ldots, \alpha_t\}\). By replacing \(f\) with \(f - f_{a_j}\), we may assume that \(j = 0\).

Let \(e \in X(1)\) and let \(u, v\) be the 0-faces of \(e\). If \(u \in A_i\) and \(v \in A_j\) for distinct \(i\) and \(j\), then \(f(x) \in \{a_i - a_j, a_j - a_i\}\), and otherwise \(f(x) = 0\). This means that \(\|d_0f\| = \sum_{0 \leq i < j \leq t} w(E(A_i, A_j))\).

Writing \(A_i^c := X(0) - A_i\), we have \(\sum_{0 \leq i < j \leq t} w(E(A_i, A_j)) = \sum_{i=0}^{t} \frac{1}{2} w(E(A_i, A_i^c))\). By the definition of \(h' = h'(X, w)\),

\[
\sum_{i=0}^{t} \alpha_i(1 - \alpha_i) \geq h' \cdot 2 \alpha_i(1 - \alpha_i)
\]

Thus, \(\|d_0f\| \geq h' \sum_{i=0}^{t} \alpha_i(1 - \alpha_i) \geq h' \sum_{i=1}^{t} \alpha_i = h'||f||\), where the second inequality is Lemma 5.1.

(ii) Fix a nontrivial element \(a \in R\). For every \(A \subseteq X(0)\), let \(f_A \in C^0(X, R^+\lambda)\) denote the function from \(X\) to \(R\) taking the value \(a\) on \(A\) and 0 elsewher. Writing \(A^c := X(0) - A\), we have \(\sup \|d_0f_A = E(A, A^c)\) and \(\text{dist}(f_A, B^0(X, R_\lambda^X)) = \min \{w(A), w(A^c)\}\). Since \((X, w, R_\lambda^X)\) is an \(\varepsilon\)-coboundary expander in dimension 0, we have \(w(E(A, A^c)) \geq \varepsilon \min \{w(A), w(A^c)\}\). As \(A \subseteq X(0)\) was arbitrary, this means that \(h(X, w) \geq \varepsilon\), and by (2.2), \(h'(X, w) \geq \frac{\varepsilon}{2}\).
Corollary 5.3. Let \((X, w)\) be a weighted graph which is a \([-1, \lambda]\)-expander, and let \(R\) be a nontrivial abelian group. Then \((X, w, R^+)\) is a \((1 - \lambda)\)-coboundary expander in dimension 0.

Proof. This follows from Theorems 2.6 and 5.2(i). \(\square\)

Corollary 5.4. Let \((X, w)\) be a weighted graph, let \(R, S\) be nontrivial abelian groups and let \(\varepsilon \geq 0\). If \((X, w, R^+)\) is an \(\varepsilon\)-coboundary expander in dimension 0, then \((X, w, S^+)\) is an \(\frac{\varepsilon}{2}\)-coboundary expander in dimension 0.

Proof. Combine parts (i) and (ii) of Theorem 5.2. \(\square\)

Example 5.5. Let \(X\) be a complete graph on 3 vertices and let \(w\) be its canonical weight function. It is routine to check that the coboundary expansion of \((X, w, (\mathbb{F}_2)^X)\) in dimension 0 is 2 while the coboundary expansion of \((X, w, R^+)\) is \(\frac{3}{2}\) for every abelian group \(R\) admitting at least 3 elements. (The latter can be checked either directly, or using Corollary 5.3 and the fact that \((X, w)\) is a \([-1, -\frac{1}{2}]\)-expander.)

We now demonstrate that locally constant sheaves on good expander graphs may have poor coboundary expansion.

Example 5.6. Let \((X, w)\) be a connected weighted graph such that every vertex in \(X\) is contained in at least 2 edges. Let \(e_0\) be an edge of minimal weight in \(X(1)\), and let \(v_0\) be one of the 0-faces of \(e_0\). Let \(\mathbb{F}\) be a field with more than \(2\) elements, and let \(\alpha \in \mathbb{F} - \{0, 1\}\). We define a locally constant sheaf \(\mathcal{F}\) on \(X\) as follows: put \(\mathcal{F}(x) = \mathbb{F}\) for all \(x \in X - \{\emptyset\}\), and for all \(\emptyset \neq v \subseteq x \in X(1)\), let
\[
\text{res}_x^{\mathcal{F}} = \begin{cases} 
\text{id}_{\mathbb{F}} & (e, v) \neq (e_0, v_0) \\
\alpha \text{id}_{\mathbb{F}} & (e, v) = (e_0, v_0).
\end{cases}
\]
We claim that the coboundary expansion of \((X, w, \mathcal{F})\) in dimension 0 is at most \(w(e_0) \leq \frac{1}{|X(0)|}\), regardless of how large \(\lambda(X, w)\) or \(h(X, w)\) are. In particular, Theorem 5.2(i) and Corollary 5.3 fail for locally constant sheaves \(\mathcal{F}\). To see this, note first that \(Z^0(X, \mathcal{F}) = 0\); this can be seen by inspecting the values of \(f \in Z^0(X, \mathcal{F})\) on a cycle subgraph containing \(e_0\) (such a cycle exists by our assumption on \(X\)) and using the connectivity of \(X\). Thus, \(\text{dist}_w(f, B^0(X, \mathcal{F})) = \|f\|_w\) for all \(f \in C^0(X, \mathcal{F})\). Taking \(f = (1_{\mathbb{F}})_{x \in X(0)}\), we get \(\text{supp}(d_0f) = \{e_0\}\), so \(\|d_0f\|_w = w(e_0)\) while \(\text{dist}_w(f, B^0(X, \mathcal{F})) = 1\).

6 Coboundary Expansion of Quotients of Constant Sheaves on Graphs

In this section, we show that taking the quotient of a constant augmented sheaf on a weighted graph by a “small” subsheaf still results in a good coboundary expander, provided that the weighted graph is a good enough spectral expander.

Recall that given an abelian group \(R\), a collection of subgroups \(\{R_i\}_{i \in I}\), is said to be linearly disjoint (in \(R\)) if the summation map \((r_i)_{i \in I} \mapsto \sum_i r_i : \bigoplus_{i \in I} R_i \to R\) is injective. For example, if \(R\) is a vector space over a field \(\mathbb{F}\) and \(R_i = \mathbb{F}v_i\) for some \(v_i \in R\), then \(\{R_i\}_{i \in I}\) are linearly disjoint if and only if the vectors \(\{v_i\}_{i \in I}\) (including repetitions) are linearly independent in \(R\).

Theorem 6.1. Let \((X, w)\) be a weighted graph with \(n\) vertices and let \(R\) be an (additive) abelian group. Suppose that we are given subgroups \(R_x\) of \(R\) for every nonempty \(x \in X\), and set \(R_0 = \{0_R\}\). Define a subsheaf \(\mathcal{G}\) of \(R^+\) by setting \(\mathcal{G}(x) = \sum_{y \subseteq x} R_y\) \((x \in X)\), and put
\[
t = \max \left\{ \frac{w(e)}{w(x)} \mid x \in X(0), e \in X(1)_{\geq x} \right\} \quad \text{and} \quad s = \max \{w(e) \mid e \in X(1)\}.
\]
Suppose that

(1) for every subgraph \(Y\) of \(X\) which is either a cycle of length \(\leq \lfloor \frac{2}{3}|X(0)|\rfloor\) or a path of a length \(\leq 2\) (in the sense of §2A), the subgroups \(\{R_y\}_{y \in Y}\) are linearly disjoint in \(R\), and

(2) for every distinct \(u, v \in X(0)\), the subgroups \(R_u\) and \(R_v\) are linearly disjoint.
If $X$ is a $[\mu, \lambda]$-spectral expander ($\mu, \lambda \in \mathbb{R}$), then

$$c_b(X, w, R_X^+/G) \geq \frac{2 - 4\lambda - 4\max(|\lambda|, |\mu|) - 5t - 2s}{5 - 2\lambda}.$$  

**Theorem 6.2.** Let $(X, w)$ be an $(r + 1)$-partite weighted simplicial complex. Let $R$, $\{R_x\}_{x \in X}$, $G$, $s$, $t$ be as in Theorem 6.1 and assume that conditions (1) and (2) of that theorem are fulfilled. If $(X, w)$ is an $(r + 1)$-partite $[\mu, \lambda]$-expander (see §2D) with $\lambda \geq -\frac{1}{2}$, then

$$c_b(X, w, R_X^+/G) \geq \frac{2r - 4r\lambda - 4r\max(|\lambda|, |\mu|) - (5r + 2)t - 2rs}{5r + 2 - 2r\lambda}.$$  

In both theorems, if all edges in $X$ have the same weight, or if all the subgroups $\{R_x\}_{x \in V(e)}$ are linearly disjoint, then we can eliminate the term $-2s$, resp. $-2rs$, in the numerator of $\varepsilon$; see Remark 6.11 below. We also note that elementary analysis shows that artificially increasing $\lambda$ will only decrease the resulting coboundary expansion. For a statement regarding the cosystolic expansion of the sheaf $R/G$, see Corollary 6.12 below.

**Example 6.3.** Suppose that $X$ is a connected $k$-regular graph on $n$ vertices and $w$ is its canonical weight function (Example 2.2). Let $R$, $\{R_x\}_{x \in X}$, $G$, $t$ and $s$ be as in Theorem 6.1. Then $t = \frac{2}{kn} = \frac{2}{k}$ and $s = \frac{2}{c_b}$; in fact, we can ignore $s$ because all edges have the same weight. Let $\rho \in [0, 1]$ be a number with $\lambda \leq \rho$ for any eigenvalue $\lambda \neq \pm 1$ of $A_{X,w}$. Recall that $X$ is called a Ramanujan graph if we can take $\rho = \frac{2\sqrt{k-1}}{k}$; see [31] (for instance) for details and motivation for this definition.

If $X$ is not bipartite, then $(X, w)$ is a $[-\rho, \rho]$-expander (see §2C), and Theorem 6.1 implies that

$$c_b(X, w, R^+/G) \geq \frac{2 - 8\rho - 10/k}{2 - 8\rho - 2/k}.$$  

If $X$ is a Ramanujan graph, for instance, then

$$c_b(X, w, R^+/G) \geq \frac{2 - 8\rho - 14/k}{2 - 8\rho - 2/k}.$$  

If $X$ is bipartite, then $(X, w)$ is a 2-partite $[-\rho, \rho]$-expander and Theorem 6.2 says that

$$c_b(X, w, R^+/G) \geq \frac{2 - 8\rho - 14/k}{2 - 8\rho - 2/k}.$$  

Again, taking $X$ to be a bipartite Ramanujan graph gives

$$c_b(X, w, R^+/G) \geq \frac{2 - 8\rho - 14/k}{2 - 8\rho - 2/k}.$$  

The rest of this section is dedicated to proving Theorems 6.1 and 6.2. We prove both theorems together in a series of lemmas.

**Lemma 6.4.** Let $(X, w)$ be a weighted graph and let $F$ be an augmented sheaf on $X$. Let $\varepsilon \in \mathbb{R}_+$, $\lambda \in \mathbb{R}$ and suppose that $(X, w)$ is a $[-1, \lambda]$-expander, and for every $v \in X(0)$ and $h \in F(v) - \{0\}$, we have

$$w(\{ e \in X(1)_{\geq v} : \text{res}_{e \leftarrow v} h(v) \neq 0 \}) \geq \varepsilon w(v).$$

Let $f \in C^0(X, F)$ and $\alpha \in [0, \infty]$. If $\|f\|_w \leq \alpha$, then

$$(\varepsilon - 2\lambda - (2 - 2\lambda)\alpha)\|f\|_w \leq \|d_0f\|_w.$$  

**Proof.** Write $A = \text{supp} f$ and $A^c = X(0) - A$. Since decreasing $\alpha$ increases the left hand side of the desired inequality, it is enough to prove the lemma for $\alpha = \|f\|_w$.

Fix some $v \in X(0)$ with $f(v) \neq 0$. We claim that for every $e \in X(1)_{\geq v}$ with $\text{res}_{e \leftarrow v} f(v) \neq 0$, at least one of the following hold:

(i) $e \in E(A),$

(ii) $e \in \text{supp}(d_0f) \cap E(A, A^c).$

Indeed, we have $(d_0f)(e) = \pm \text{res}_{e \leftarrow v} f(v) \mp \text{res}_{e \leftarrow v} f(e - v)$. If $f(e - v) \neq 0$, then $v$ and $e - v$ are in $\text{supp} f$, and $e \in E(A)$. Otherwise $f(e - v) = 0$, so $(d_0f)(e) = \pm \text{res}_{e \leftarrow v} f(v) \neq 0$ and $e \in \text{supp}(d_0f) \cap E(A, A^c)$. This means that

$$w(\{ e \in X(1)_{\geq v} : \text{res}_{e \leftarrow v} f(v) \neq 0 \}) \leq w(E(A)_{\geq v}) + w(\text{supp}(d_0f) \cap E(A, A^c))_{\geq v}.$$
By assumption, the left hand side of the last inequality is at least \( \varepsilon w(v) \). Summing over all \( v \in \text{supp} f \), we get
\[
\varepsilon \|f\|_w \leq \sum_{v \in A} w(E(A) \geq v) + \sum_{v \in A} w([\text{supp}(d_0 f) \cap E(A, A^c)] \geq v) \\
\leq 2w(E(A)) + w(\text{supp } d_0 f) \leq 2(\alpha^2 + \lambda(1 - \alpha)) + \|d_0 f\|_w.
\]
Here, the second inequality holds because every edge in \( E(A) \) is counted exactly twice and every edge in \( \text{supp}(d_0 f) \) is counted at most once, whereas the third inequality follows from Theorem 6.2(ii). By rearranging, we find that
\[
\|d_0 f\|_w \geq \varepsilon \|f\|_w - 2\alpha^2 - 2\alpha(1 - \alpha) = (\varepsilon - 2\lambda - 2\alpha + 2\lambda)\|f\|_w.
\]

Lemma 6.5. Let \((X, w), R, \{R_x\}_{x \in X}, G, t \) be as in Theorem 6.4 or Theorem 6.2, and let \( \mathcal{F} = R^+ / \mathcal{G} \). If \( \{R_y\}_{y \in Y} \) are linearly disjoint in \( R \) for every path \( Y \subseteq X \) of length 2, then for every \( x \in X(0) \) and \( h \in \mathcal{F}(x) - \{0\} \), we have
\[
w(\{e \in X(1) \geq x : \text{res}_{e \leftarrow x} h \neq 0\}) \geq (2 - t)w(x).
\]

Proof. Let \( x \in X(0) \) and \( h \in \mathcal{F}(x) - \{0\} \). If \( \text{res}_{e \leftarrow x} h \neq 0 \) for all \( e \in X(1) \geq x \), then \( w(\{e \in X(1) \geq x : \text{res}_{e \leftarrow x} h \neq 0\}) = w(X(1) \geq x) = 2w(x) \) because \( w \) is a weight function (see 2.3), and the lemma holds.

Suppose now that there exists \( y \in X(1) \geq x \) such that \( \text{res}_{e \leftarrow x} h = 0 \). We claim that \( \text{res}_{e \leftarrow x} h \neq 0 \) for all \( z \in X(1) \geq x \) different from \( y \). Fix such \( z \) and let \( x' = y - x \) and \( x'' = z - x \). Then \( Y = \{0, x, x', x'', y, z\} \) is a path of length 2 in \( X \), meaning that \( R_0 = 0, R_x, R_x', R_x'' \), and \( R_y, R_z \) are linearly disjoint in \( R \). Choose \( g \in R \) with \( h(x) = g + R_x \) (note that \( G(x) = R_x + R_0 = R_x \)). Since \( \text{res}_{y \leftarrow x} h = 0 \), we have \( g \in R_y + R_x + R_x' \). Likewise, if \( \text{res}_{x' \leftarrow x} h = 0 \), then \( g \in R_z + R_x + R_x'' \). Consequently, \( g \in (R_y + R_z + R_x') \cap (R_z + R_x + R_x'') = R_x = \mathcal{G}(x) \), which contradicts our assumption that \( h = g + \mathcal{G}(x) \neq 0 \) in \( \mathcal{F}(x) \). We conclude that \( \{e \in X(1) \geq x : \text{res}_{e \leftarrow x} h \neq 0\} \neq X(1) \geq x - \{y\} \). Since \( w(X(1) \geq x) - w(y) = 2w(x) - w(y) \geq 2w(x) - tw(x) = (2 - t)w(x) \), we are done.

Lemma 6.6. Let \((X, w)\) be a weighted graph with \( n \) vertices. Define \( t \) and \( s \) as in Theorem 6.4 and let \( T \) be a subgraph of \( X \).

(i) If \( w(T(1)) \geq t \), then \( T \) contains a cycle.

(ii) If \( w(T(1)) \geq t + s \), then \( T \) contains a cycle of length \( \leq \lceil \frac{2}{3} n \rceil \).

Proof. (i) It is enough to show that if \( T \) contains no cycles, then \( w(T(1)) < t \). In this case, \( T \) is a forest, i.e., every connected component of \( T \) is a tree. Choose roots for the trees in \( T \) and denote the set of roots by \( R \subseteq V(X) \). For \( v \in V(X) - R \), let \( p(v) \in V(X) \) denote the parent of \( v \) in \( T \). Then
\[
w(T(1)) = \sum_{v \in V(X) - R} w\{p(v), p(v')\} \leq \sum_{v \in V(X) - R} tw\{v\} < \sum_{x \in X(0)} tw(x) = t.
\]

(ii) By (i), \( T \) contains a cycle \( C_1 \). Choose an edge \( e_1 \) in \( C_1 \). Then \( w(e_1) \leq s \). Let \( T' = T - \{e_1\} \), i.e., the graph obtained from \( T \) by removing the edge \( e_1 \). Then \( w(T'(1)) = w(T(1)) - w(e) \geq t + s - s = t \), so by (i), \( T' \) contains another cycle, \( C_2 \), and \( e_1 \notin C_2 \). If \( C_1(0) \cap C_2(0) = \emptyset \), then one of \( C_1 \), \( C_2 \) has less than \( \frac{1}{2} n \) vertices, and is therefore the required cycle.

Suppose now that \( C_1(0) \cap C_2(0) \neq \emptyset \). By Lemma 2.1, \( C_1 - C_2 \) is a nonempty disjoint union of open paths. Let \( Z \) be one of these paths, and let \( x, y \) denote its end points. If \( x = y \), then \( C_1 \) and \( C_2 \) share exactly one vertex. Thus, \( |C_1(0)| + |C_2(0)| \leq n + 1 \), and again, one of \( C_1 \), \( C_2 \) is the required cycle. Assume \( x \neq y \). Then \( x, y \in C_2(0) \). Let \( P \) and \( Q \) denote the two paths from \( x \) to \( y \) contained in \( C_2 \) and put \( p = |P(0)|, q = |Q(0)|, z = |Z(0)| \). Since \( P(0) - \{x, y\}, Q(0) \), and \( Z(0) \) are pairwise disjoint, we have \( p + q + z \leq n + 2 \). Let \( R_1 := P \cup Q, R_2 := R_1 \cup Z, R_3 := R_1 \cup Z \). Then \( R_1, R_2, R_3 \) are cycles, and \( |R_1(0)| + |R_2(0)| + |R_3(0)| = (p + q - 2) + (p + z) + (q + z) = 2p + q + z - 2 \leq 2n + 1 \). This means that there is \( i \in \{1, 2, 3\} \) such that \( |R_i(0)| \leq \lceil \frac{2}{3} (n + 1) \rceil = \lceil \frac{2}{3} n \rceil \), so we are done.

Remark 6.7. The proof of Lemma 6.6(ii) also shows that the girth of a graph with \( n \) vertices and \( n + 1 \) edges is at most \( \lceil \frac{2}{3} n \rceil \). This bound is tight, e.g., consider a graph \( X \) obtained by gluing 3 closed paths of length \( k \) or \( k - 1 \) at their endpoints.

Lemma 6.8. Let \( X \) be a connected graph which is a union of its cycle subgraphs. Then every two vertices in \( X \) can be connected by a path of length \( \leq \frac{2}{3}(|X(0)| - 1) \).
Proof. Write \( n = |X(0)| \), and let \( x, y \in X(0) \) be distinct 0-faces. Since \( X \) is connected, there is a path from \( x \) to \( y \), and our assumption on \( X \) implies that this path is contained in a union of cycles. This means that there are cycles \( R_1, \ldots, R_t \) such that \( x \in R_i(0), y \in R_i(0) \) and there exists \( x_i \in R_i(0) \cap R_{i+1}(0) \) for all \( i \in \{1, \ldots, t-1\} \). Set \( x_0 = x \) and \( x_t = y \). We prove the lemma by induction on \( t \).

If \( t = 1 \), then \( x, y \in R_1 \), so there is a path from \( x \) to \( y \) of length at most \( |R_1(0)| \leq |X(0)| \leq \frac{2}{3}(n-1) \) (because \( R_1 \), and hence \( X \), has at least 3 vertices). Suppose henceforth that \( t > 1 \).

If \( R_t \cap R_j \neq \emptyset \) for some \( i, j \in \{1, \ldots, t\} \) with \( i+2 < j \), then we can remove \( R_{i+1}, \ldots, R_{j-1} \) from \( R_1, \ldots, R_t \) and finish by the induction hypothesis. We may therefore assume that \( R_i \cap R_j = \emptyset \) whenever \( i + 2 \leq j \).

Next, if \( |R_t(0) \cap R_{i+1}(0)| = 1 \) for all \( i \in \{1, \ldots, t-1\} \), then \( R_t(0) \cap R_{i+1}(0) = \{x_i\} \) for all \( i \in \{1, \ldots, t-1\} \). Since \( R_t \cap R_j = \emptyset \) when \( i + 2 \leq j \), this means that the sets \( R_t(0) - \{x_i\}, \ldots, R_{i+1}(0) - \{x_i\} \) are pairwise disjoint. Thus, letting \( r_i = |R_t(0)| - 1 \), we have \( \sum_{i=1}^{t-1} r_i = |X(0) - \{x_i\}| = n - 1 \). Let \( i \in \{1, \ldots, t\} \). Since \( R_t \) is a cycle with \( r_i + 1 \) vertices, there is a closed path \( P_i \) from \( x_i \) to \( x_i \) of length at most \( \frac{1}{3}(r_i + 1) \leq \frac{2}{3}r_i \) (because \( r_i \geq 2 \)). The union \( P_1 \cup \cdots \cup P_t \) is a path from \( x_0 \in x_t \) of length at most \( \sum_{i=1}^{t} \frac{2}{3}r_i \leq \frac{2}{3}(n-1) \).

Finally, if there is \( i \in \{1, \ldots, t-1\} \) such that \( R_i \) and \( R_{i+1} \) share at least 2 vertices, then \( R_t - R_{i+1} \) is a disjoint union of open paths and each of these paths has distinct end points (cf. Lemma 2.1). Of these open paths, let \( P \) denote the one containing \( x_{i-1} \) and let \( z, w \) be its endpoints. Then \( z, w \in R_{i+1}(0) \). Let \( Q \) denote a closed path from \( z \) to \( w \) in \( R_{i+1} \) which also includes \( x_i \). Then \( R' := P \cup Q \) is a cycle containing both \( x_{i-1} \) and \( x_{i+1} \). We replace \( R_i, R_{i+1} \) with \( R' \) and proceed by induction on \( t \).

Lemma 6.9. Let \( X \) be a cycle graph, let \( R, \{R_x\}_{x \in X} \) and \( \mathcal{G} \) be as in Theorem 6.1, and suppose that all the \( \{R_x\}_{x \in X} \) are linearly disjoint in \( R \). Then for every \( f \in Z^0(X, R^+ / \mathcal{G}) \), there exists \( h \in R \) such that \( f(v) = h + R_v \) for all \( v \in X \).

Proof. Let \( v_0, \ldots, v_{t-1} \) be the 0-faces of \( X \) and let \( e_0, \ldots, e_{t-1} \) be the edges of \( X \). We may choose the numbering so that \( v_i, v_{i+1} \mod \ell \subset c_i \) for all \( i \). We write \( v_\ell = v_0 \) for convenience.

Let \( f \in Z^0(X, R^+ / \mathcal{G}) \), and choose \( g \in C^0(X, R^+) \) projecting onto \( f \), i.e., \( f(v) = g(v) + R_v \) for all \( v \in X(0) \).

For every \( i \in \{0, \ldots, \ell - 1\} \), write \( g_i = g(v_i) \), and let \( g'_i = g_i - g_0 \). Unfolding the definitions, the assumption \( d_0 f = 0 \) is equivalent to having \( g'_{i+1} - g'_i = g_{i+1} - g_i \in R_{v_i} + R_{v_{i+1}} + R_{e_i} \) for all \( i \in \{0, \ldots, \ell - 1\} \).

We claim that there exist \( c_0 \in R_{v_0}, \cdots, c_{\ell-2} \in R_{v_{\ell-2}}, c_{\ell-1} \in R_{v_{\ell-1}} \) and \( u_i \in R_{v_i} \) \( (i \in \{0, \ldots, \ell - 2\}) \) such that \( g'_i = c_0 + u_0 + c_1 + \cdots + c_{i-1} + u_{i-1} + c_i \) for all \( i \in \{1, \ldots, \ell - 1\} \). The proof is by induction on \( i \). For the case \( i = 1 \), note that \( g'_1 = g_1 - g_0 \in R_{v_0} + R_{v_1} + R_{u_0} \), so we can choose \( c_0 \in R_{v_0}, c_1 \in R_{v_0} \), and \( u_0 \in R_{u_0} \) such that \( g'_1 = c_0 + u_0 + c_1 \). Suppose now that \( i \in \{1, \ldots, \ell - 2\} \), and \( c_0, c_1, \ldots, c_{i-1}, c_i, u_0, \ldots, u_{i-1} \) were chosen so that \( g'_j = c_0 + u_0 + \cdots + c_{j-1} + u_{j-1} + c_j \) for all \( j \in \{1, \ldots, i\} \). Then

\[
g'_{i+1} - c_0 - u_0 - \cdots - c_{i-1} - u_{i-1} - c_i = g'_{i+1} - g'_i \in R_{v_i} + R_{v_{i+1}} + R_{e_i}.
\]

Since \( R_{v_0}, \ldots, R_{v_{i-1}}, R_{v_i}, \ldots, R_{v_{\ell-1}} \) are linearly disjoint in \( R \), we can choose \( c_i \in R_{v_i}, u_i \in R_{e_i} \), and \( c_{i+1} \in R_{v_{i+1}} \) such that \( g'_{i+1} = c_0 + u_0 + \cdots + c_{i-1} + u_{i-1} + c_i \in R_{v_i} \), hence our claim.

Now, we have \( c_0 + u_0 + \cdots + c_{\ell-2} + u_{\ell-2} + c_{\ell-1} = g'_{\ell-1} = g_{\ell-1} - g_\ell \in R_{v_0} + R_{v_\ell-1} \), and so, since \( \{R_x\}_{x \in X(0) \cup X(1)} \) are linearly disjoint, we must have \( c_0 = \cdots = c_{\ell-2} = 0 \) and \( u_0 = \cdots = u_{\ell-2} = 0 \). This means that \( g'_i = c_0 + c_i \) for all \( i \in \{1, \ldots, \ell - 1\} \). Define \( h = g_\ell + c_0 \). Then for all \( i \in \{1, \ldots, \ell - 1\} \), we have \( g_i = g'_i + g_\ell - c_0 = h + R_{v_i} \), and \( g_0 = h - c_0 \in R_{v_0} \). This means that \( f(v_i) = h + R_{v_i} \) for all \( i \), so we proved the existence of \( h \).

We are now ready to prove the following key lemma.

Lemma 6.10. Under the assumptions of Theorem 6.1, put \( F = R^+ / \mathcal{G} \) and let \( f \in C^0(X, F) \) and \( \beta \in [0, 1] \). If \( \|d_0 f\|_w \leq \beta \), then

\[
dist_w(f, B^0(X, F)) < \frac{2}{3} + \frac{1}{3} \left[ \beta + t + s + \lambda + 2 \max\{|\lambda|, |\mu|\} \right].
\]

If, instead, we use the assumptions of Theorem 6.2, then

\[
dist_w(f, B^0(X, F)) < \frac{2r + 2}{3r + 2} + \frac{r}{3r + 2} \left[ \beta + t + s + \lambda + 2r \max\{|\lambda|, |\mu|\} \right].
\]
Proof. Step 1. Let $n = |X(0)|$. We call a subgraph $Y$ of $X$ an $f$-blob, or just a blob for short, if:

(b1) $Y$ is connected and equals to the union of its cycle subgraphs, and

(b2) there exists $g \in R$ such that $f(v) = g + R_v$ for all $v \in Y(0)$.

Note that condition (b1) implies that $|Y(0)| > 2$, because a cycle has at least 3 vertices. We denote an element $g \in G$ as in (b2) by $g_Y$; we will see below that $g_Y$ is uniquely determined by $Y$.

Denote the set of $f$-blobs by $B$. By Lemma 6.9 and assumption (1) of Theorem 6.1, every cycle $Y \subseteq X$ of length $\lceil \frac{3}{2}n \rceil$ or less is a blob. Note that condition (b1) implies that $g_Y$ vanishes on $Y(0)$.

Step 2. Observe that if $Y$ and $Z$ are blobs such that $|Y(0) \cap Z(0)| > 1$, then $Y \cup Z$ is also a blob. Indeed, (b1) holds for $Y \cup Z$ because it holds for $Y$ and $Z$, and $Y \cap Z \neq \emptyset$. To see that (b2) holds, fix a choice of $g_Y$ and $g_Z$. Then for every $v \in Y(0) \cap Z(0)$, we have $g_Y + R_v = f(v) = g_Z + R_v$, so $g_Y - g_Z \in R_v$. Choosing distinct $u, v \in Y(0) \cap Z(0)$, we get $g_Y - g_Z \in R_u \cap R_v = 0$, because $R_u$ and $R_v$ are linearly disjoint (assumption (2) of Theorem 6.1). This means that $g_Y = g_Z$ and (b2) holds for $Y \cup Z$ by taking $g := g_Y = g_Z$.

Applying the previous paragraph with $Y = Z$ shows that $g_Y$ is uniquely determined by $Y$.

Step 3. Write $M = \bigcup_{Y \in B} Y$. Then $M$ is a subgraph of $X$.

We claim that $M(1) \subseteq X(1) - \text{supp}(d_0 f)$. To show this, it is enough to prove that $Y(1) \subseteq X(1) - \text{supp}(d_0 f)$ for any blob $Y$. Let $e$ be an edge in $Y$ and let $u$ and $v$ be its 0-faces. Then $f(u) = g_Y + R_u$ and $f(v) = g_Y + R_v$. As a result, $(d_0 f)(e) = g_Y - g_Y + (R_u + R_v + R_e) = 0$ in $F(e)$, meaning that $e \notin \text{supp}(d_0 f)$, hence our claim.

We observed in Step 1 that every cycle of length $\leq \lceil \frac{3}{2}n \rceil$ in $X$ is a blob, and thus contained in $M$. It follows that the graph underlying $X(1) - \text{supp}(d_0 f) - M(1)$ contains no cycles of length $\leq \lceil \frac{3}{2}n \rceil$. Thus, by Lemma 6.6, $w(X(1) - \text{supp}(d_0 f) - M(1)) < t + s$. Since $\|d_0 f\|_w \leq \beta$, it follows that

$$w(M(1)) > 1 - \beta - t - s.$$ 

Step 4. A blob is called maximal if it is not properly contained in any other blob. Write $M$ for the set of maximal blobs. Since every blob is contained in a maximal blob, $M = \bigcup_{Y \in M} Y$. Step 2 tells us that for every $Y \in M$ and $Z \in B$, either $Y = Z$, or $|Y(0) \cap Z(0)| \leq 1$.

Let $N$ denote the set of 0-faces of $X$ belonging to more than one blob in $M$. We define a graph $\Gamma$ as follows: The vertices of $\Gamma$ are $M \cup N$ and the edges of $\Gamma$ are pairs $\{x, Y\}$ such that $x \in N$, $Y \in M$ and $x \in Y(0)$. See Figure 1 for an illustration.

Figure 1: An illustration of the collection of blobs $M$ (left), the associated graph $\Gamma$ (middle), and the partition $\{Y^* \mid Y \in M\}$ (right). The blobs are labelled $A$–$E$. The black vertices are those in living in $N$. The set of roots taken on the right is $R = \{A\}$.

We claim that the graph $\Gamma$ has no cycles. For the sake of contradiction, suppose otherwise. Then there exist $\ell \geq 2$ and distinct $x_0, \ldots, x_{\ell - 1} \in N$, $Y_0, \ldots, Y_{\ell - 1} \in M$ such that $\{x_i\} = Y_i \cap Y_{i+1}$ for all $i \in \{0, \ldots, \ell - 1\}$ (with the convention that $Y_\ell = Y_0$). By applying Lemma 6.8 to $Y_i$, we see that there is a closed path $P_i \subseteq Y_i$
of length $\leq \frac{3}{2}\sum_{i=0}^{\ell}(|Y_i(0)| - 1)$ from $x_{i-1}$ to $x_i$. Since the sets $Y_0(0) - \{x_0\}, \ldots, Y_{\ell-1}(0) - \{x_{\ell-1}\}$ are pairwise disjoint, the union $P = \bigcup_{i=0}^{\ell-1} Y_i$ is a cycle of length at most $\frac{3}{2}\sum_{i=0}^{\ell}(|Y_i(0)| - 1) \leq \frac{3}{2}n$. This means that $P$ is a blob — see Step 1. By construction, $P$ and each of the $Y_i$ share at least two 0-faces, namely, $x_i$ and $x_{i+1}$, so we must have $Y_0 = \cdots = Y_{\ell-1} = P$. This contradicts our assumption that $Y_0, \ldots, Y_{\ell-1}$ are distinct.

**Step 5.** By the previous step, $\Gamma$ is a forest. We choose a set of roots $\mathcal{R}$ for the trees in $\Gamma$. By the definition of $N$, every connected component of $\Gamma$ contains a vertex in $M$, so we may choose $\mathcal{R}$ to be contained in $M$. We denote the parent of every $Y \in M - \mathcal{R}$ by $x(Y)$. Since $x(Y)$ is not a root, it has a parent, which we denote by $g(Y)$ (the grandparent of $Y$). For every blob $Y \in M$, define:

$$Y^* = \begin{cases} Y(0) & Y \in \mathcal{R} \\ Y(0) - \{x(Y)\} & Y \in M - \mathcal{R}. \end{cases}$$

We claim that $\{Y^* \mid Y \in M\}$ is a partition of $M(0)$ (see Figure 1 for an illustration). To see this, note first that $\bigcup_{Y \in M} Y^* \subseteq \bigcup_{Y \in M} Y(0) = M(0)$. To see that $M(0) \subseteq \bigcup_{Y \in M} Y^*$, observe that every $x \in M(0)$ belongs to $Y(0)$ for some $Y \in M$. If $Y \in \mathcal{R}$, then $Y^* = Y(0)$ and $x \in Y^*$. Otherwise, $Y^* = Y(0) - \{x(Y)\}$, so $x \in Y^*$ provided $x \neq x(Y)$. If $x = x(Y)$, then $x \in g(Y(0))$ by the definition of $\Gamma$. Since $g(Y)$ is the parent of $x$ in $\Gamma$, the 0-face $x$ cannot be the parent of $g(Y)$, so $x \neq x(g(Y))$ if $g(Y) \notin \mathcal{R}$, which means that $x \in g(Y)^*$. This proves that $x \in \bigcup_{Y \in M} Y^*$. To finish, we need to show that $Y^* \cap Z^* = 0$ for any distinct $Y, Z \in M$. For the sake of contradiction, suppose there is $x \in Y^* \cap Z^*$. Then $x \in N$ and $x$ is adjacent to $Y$ and $Z$ in $\Gamma$. This means that at least one of $Y, Z$ is not a parent of $x$, say it is $Y$. Then $x$ must be the parent of $Y$, which means that $x \notin Y - \{x(Y)\} = Y^*$, a contradiction.

Next, let $e \in M(1)$. We claim that one of the following holds:

(i) $e \in E(Y^*)$ for some $Y \in M$,

(ii) $e \in E(Y^*, g(Y)^*)$ for some $Y \in M - \mathcal{R}$.

Indeed, let $y, z$ denote the 0-faces of $e$. Since $M = \bigcup_{Y \in M} Y$, there is $Y \in M$ such that $e \in Y(1)$, and thus $y, z \in Y(0)$. If both $y$ and $z$ are different from $x(Y)$, then $y, z \in Y^*$ and (i) holds. Otherwise, exactly one of $y, z$ equals $x(Y)$, say $y = x(Y)$ and $z \in Y^*$. Then $y$ is the parent of $Y$ in $\Gamma$. As explained in the previous paragraph, $y \in g(Y)$ and $y$ is not a parent of $g(Y)$, so $y \in g(Y)^*$ and (ii) holds.

**Step 6.** At this point, we assume that we are in the setting of Theorem 6.1. For every $Y \in M$, put $\alpha_Y = w(Y^*)$ and set

$$\alpha = \max\{\alpha_Y \mid Y \in M\}.$$

Since $\{Y^* \mid Y \in M\}$ is a partition of $M(0)$ (Step 5), we have $\sum_{Y \in M} \alpha_Y = w(M(0)) \leq 1$.

Let $g^{-1}(Y)$ denote the set of blobs $Z \in M$ with $g(Z) = Y$. By Step 3 and the last paragraph of Step 5, we have

$$1 - \beta - t - s < w(M(1)) \leq \sum_{Y \in M} w(E(Y^*)) + \sum_{Y \in M} w(E(Y^*, \bigcup_{Z \in g^{-1}(Y)} Z^*)).$$

Put $\theta = \max\{|\mu|, |\lambda|\}$. By Theorem 3.2, the right hand side is at most

$$\sum_{Y \in M} (\alpha_Y^2 + \lambda \alpha_Y) + \sum_{Y \in M} (2\alpha_Y \sum_{Z \in g^{-1}(Y)} \alpha_Z + 2\theta \sqrt{\alpha_Y \sum_{Z \in g^{-1}(Y)} \alpha_Z}) \leq \sum_{Y \in M} \alpha_Y (\alpha + \lambda) + \sum_{Y \in M} (2\alpha_Y \sum_{Z \in g^{-1}(Y)} \alpha_Z + \theta (\alpha_Y + \sum_{Z \in g^{-1}(Y)} \alpha_Z)) = \sum_{Y \in M} \alpha_Y (\alpha + \lambda) + \theta \sum_{Y \in M} \alpha_Y + \sum_{Y \in M} \sum_{Z \in g^{-1}(Y)} (2\alpha_Y + \theta) \alpha_Z \leq \alpha + \lambda + \theta + \sum_{Z \in M - \mathcal{R}} \alpha_Z (2\alpha g(Z) + \theta) \leq \alpha + \lambda + \theta + (2\alpha + \theta) = 3\alpha + 2\theta + \lambda.$$
Thus, \(1 - \beta - t - s < 3\alpha + 2\theta + \lambda\), and by rearranging, we get
\[
\alpha > \frac{1}{3} - \frac{\beta}{3} - \frac{t}{3} - \frac{s}{3} - \frac{\lambda}{3} + 2\max\{|\lambda|, |\mu|\}.
\]

By the definition of \(\alpha\), there exists a blob \(Y\) with \(w(Y(0)) \geq w(Y^*) = \alpha\). As explained in Step 1, it follows that \(\text{dist}_w(f, B^0(X, F)) \leq 1 - \alpha\), so this completes the proof of the lemma under the assumptions of Theorem 6.4

**Step 7.** Finally, suppose that we are in the setting of Theorem 6.2. As in Step 6, we find that
\[
1 - \beta - t - s < \sum_{Y \in \mathcal{M}} w(E(Y^*)) + \sum_{Y \in \mathcal{M}} w(E(Y^* \cup Z \in g^{-1}(Y) Z^*)).
\]

Using Theorem 3.3(ii) and Theorem 3.2(ii) (the assumption \(\lambda \geq -\frac{1}{2}\) guarantees that \((X, w)\) is a \([-1, 1]\)-expander if we forget the \((r + 1)\)-partite structure of \(X\)) we see that the right hand side is at most
\[
\sum_{Y \in \mathcal{M}} (\alpha^2 + \lambda \alpha_Y) + \sum_{Y \in \mathcal{M}} (\frac{2r + 2}{r} \alpha_Y \sum_{Z \in g^{-1}(Y)} \alpha_Z + 2r \theta \sqrt{\alpha_Y \sum_{Z \in g^{-1}(Y)} \alpha_Z})
\]
and a computation similar to the one in Step 6 shows that this expression is bounded by
\[
\alpha + \lambda + r \theta + (\frac{2r + 2}{r} \alpha + r \theta) = \frac{3r + 2}{r} \alpha + 2r \theta + \lambda.
\]
As a result,
\[
\alpha > \frac{1 - \beta - t - s - \lambda - 2r \max\{|\lambda|, |\mu|\}}{3r + 2}
\]
and we conclude the proof as in Step 6.

We are now ready to prove Theorems 6.1 and 6.2.

**Proof of Theorem 6.1.** Fix some \(\beta \in [0, 1]\), and let \(f \in C^0(X, R^+ / \mathcal{G})\) be an 0-cochain such that \(\|f\|_w = \text{dist}_w(f, B^0(X, R^+ / \mathcal{G}))\). If \(\|d_0 f\|_w \geq \beta\), then we have \(\|d_0 f\|_w \geq \beta \|f\|_w\). On the other hand, if \(\|d_0 f\|_w < \beta\), then by Lemma 6.10, we have \(\|f\|_w = \text{dist}_w(f, B^0(X, R^+ / \mathcal{G})) \leq \frac{2}{3} + \frac{1}{3} [\beta + t + s + \lambda + 2 \max\{|\lambda|, |\mu|\}]\). By Lemmas 6.4 and 6.5 this means that
\[
\|d_0 f\|_w \geq \left[2 - t - 2\lambda - (2 - 2\lambda)\left(\frac{2}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \lambda + 2 \max\{|\lambda|, |\mu|\}\right)\right] \|f\|_w
\geq \left[\frac{1}{3} - \frac{1}{3} \lambda - \frac{1}{3} \max\{|\lambda|, |\mu|\} - \frac{2}{3} t - \frac{2}{3} s\right] \|f\|_w
\]
As a result, \((X, w, R^+ / \mathcal{G})\) is an \(\varepsilon\)-coboundary expander in dimension 0 for
\[
\varepsilon = \min\{\beta, \frac{2}{3} - \frac{1}{3} \lambda - \frac{1}{3} \max\{|\lambda|, |\mu|\} - \frac{2}{3} t - \frac{2}{3} s\} - (\frac{2 - 2\lambda}{3}) \beta
\]
The right hand side is minimized when \(\varepsilon = \beta = \frac{2 - 4\lambda - 4 \max\{|\lambda|, |\mu|\} - 5t - 2s}{5 - 2\lambda}\), and theorem follows.

**Proof of Theorem 6.2.** Similarly to the proof of Theorem 6.1, we find that for every \(\beta \in [0, 1]\), the triple \((X, w, R^+ / \mathcal{G})\) is an \(\varepsilon\)-coboundary expander in dimension 0 for
\[
\varepsilon = \min\{\beta, \frac{2r}{3} - \frac{1}{3} \lambda - \frac{1}{3} \max\{|\lambda|, |\mu|\} - \frac{2r}{3} t - \frac{2r}{3} s\} - (\frac{2 - 2\lambda}{3}) \beta,
\]
where \(\theta = \max\{|\lambda|, |\mu|\}\). The minimum of this expression is attained for \(\varepsilon = \beta = \frac{2r - 4r\lambda - 4\theta - (5r + 2)t - 2rs}{5r + 2 - 2\lambda}\), hence the theorem.

**Remark 6.11.** In Theorems 6.1 and 6.2, if all edges in \(X\) have the same weight, or if all the subgroups \(\{R_x\}_{x \in X}\) are linearly disjoint, then we can eliminate the terms \(-2s\), resp. \(-2rs\), in the expression for \(\varepsilon\), or equivalently take \(s = 0\).

Indeed, if all edges in \(X\) have the same weight, then the assertion \(w(M(1)) > 1 - \beta - t - s\) in Step 3 of the proof of Lemma 6.10 implies that \(w(M(1)) \geq 1 - \beta - t\). Likewise, if all the subgroups \(\{R_x\}_{x \in X}\) of \(R\) are linearly disjoint, then in the same place in the proof, we can apply Lemma 6.6(i) instead of Lemma 6.6(ii) and get that \(w(M(1)) > 1 - \beta - t\). Carrying the entire proof of Lemma 6.10 using the inequality \(w(M(1)) \geq 1 - \beta - t\) in place of \(w(M(1)) > 1 - \beta - t - s\) allows us to eliminate \(s\) at the cost of replacing the strict inequalities in the proof with non-strict inequalities. This, in turn, eliminates \(s\) from Theorems 6.1 and 6.2.
Corollary 6.12. Let $X, w, R, \{R_v\}_{v \in X}, \mathcal{G}, \varepsilon$ be as in Theorem 6.1 or Theorem 6.2. Put $\eta = \max \{w(x) \mid x \in X(0)\}$ (we always have $\eta \leq \frac{\varepsilon}{2}$). Then, regarding $\mathcal{G}$ as a subsheaf of the constant sheaf $R$ on $X$, the triple and $(X, w, R/\mathcal{G})$ is an $(\varepsilon, 1 - \eta)$-cosystolic expander in dimension 0.

Proof. By Theorems 6.1 and 6.2 and Remark 4.3 it is enough to show that for all $f \in R = C_{-1}(X, R^+ / \mathcal{G})$, we have $\|d_{-1}f\| \geq 1 - \eta$. Observe that $(d_{-1}f)(v) = f + R_v \in R/R_v$ for all $v \in X(0)$. Thus, $\|d_{-1}f\| = 1 - w(A)$, where $A$ is the set of 0-faces $v$ such that $f \in R_v$. Since every two of the groups $\{R_v\}_{v \in X(0)}$ are linearly disjoint, $A$ is either empty or a singleton, so $w(A) \leq \eta$ and the lemma follows. (It is worth noting that, if $R_v \neq 0$ for all $v$, then by choosing $f \in R_v$ with $w(v)$ maximal, we get $\|d_{-1}f\| = 1 - \eta$, so the constant $1 - \eta$ cannot be lowered in general.) \( \square \)

7 The Case of Finite Buildings

We now apply Corollary 5.3 and Theorem 6.2 to the 0-skeleton of finite buildings admitting a strongly transitive group action, giving upper bounds on the coboundary expansion in terms of the thickness and the type of the building. This is the main reason why we have taken care to treat the case of $(r + 1)$-partite weighted simplicial complexes. Applications of the results of this section to locally testable codes appear in [9].

We refer the reader to [1] for an extensive discussion of buildings. Here we satisfy with saying that a building is a possibly-infinite simplicial complex $X$ equipped with a collection $\mathcal{E}$ of subcomplexes called apartments satisfying a list of axioms. All the apartments of $X$ are isomorphic to each other and to a Coxeter complex $\Sigma$. The data of $\Sigma$ (up to isomorphism) is encoded by a Coxeter diagram $T$, which is a finite graph with edges labelled by elements from the set $\{3, 4, 5, \ldots \} \cup \{\infty\}$ (unlabelled edges are given the label 3 by default). We write $T = T(X)$ and say that $T$ is the type of $X$. There is a labelling of the vertices of $X$ by the vertices of $T$ making $X$ into a pure $(r + 1)$-partite simplicial complex, where $r = \dim X = |V(T)| - 1$. The type of a face $z \in X$ is the set $t(z) \subseteq V(T)$ consisting of the types of the vertices of $z$. If $\dim z \leq \dim X - 1$, then the link $X_z$ is also a building and its type $T(X_z)$ is the Coxeter diagram obtained from $T = T(X)$ by removing the vertices in $t(z)$.

Let $B$ denote the $V(T) \times V(T)$ real matrix determined by

$$B_{u,v} = \begin{cases} 1 & u = v \\ \frac{\pi}{m} & \{u,v\} \in T(1) \text{ and } m \text{ is the label of } \{u,v\} \\ 0 & \{u,v\} \notin T(1). \end{cases}$$

The building $X$ is called spherical if $B$ is positive definite, or equivalently, if its apartments are finite. Following [1, Chapter 10], we call $X$ affine if $B$ is positive semidefinite and rank $B = |V(T)| - 1$ (consult [1, Proposition 10.44]). See [1, p. 50, Remark 10.33(b)] for a complete list of the possible spherical and affine Coxeter diagrams. If $X$ is an affine building and $z$ is a face with $0 \leq \dim z < \dim X$, then the link $X_z$ is a spherical building.

A building $X$ of dimension $r$ is called $q$-thick if every $(r - 1)$-face of $X$ is contained in at least $q$ $r$-faces. We say that $X$ is thick if it is 3-thick.

Recall from [1, §6.1.1] that an $r$-dimensional building $X$ is said to posses a strongly transitive action if there is a group $G$ acting on $X$ via type-preserving simplicial automorphisms such that $G$ takes apartments to apartments and acts transitively on the set of pairs $(A, x)$ consisting of an apartment $A \in \mathcal{E}$ and an $r$-face $x$ in $A$. Since $G$ is type preserving, this means that $G$ acts transitively on the set of faces of a fixed type $t \subseteq T(X)(0)$. Moreover, for every $z \in X$ of dimension $\dim X - 2$ or less, the building $X_z$ also possesses a strongly transitive action. It follows from Tits’ classification of thick spherical and affine buildings, see [1, Chapter 9, §11.9] for a survey, that all thick finite buildings of dimension $\geq 2$ and all locally-finite thick affine buildings of dimension $\geq 3$ admit a strongly transitive action.

Example 7.1. Let $\mathbb{F}$ be a field and let $n \in \mathbb{N}$. The incidence complex of nontrivial subspaces of $\mathbb{F}^{n+1}$, denoted $A_n(\mathbb{F})$, is an $(n - 1)$-dimensional building of type $A_n$, where $A_n$ is the Coxeter diagram consisting of a single path of length $n - 1$ with all edges labeled 3. In more detail, the vertices of $A_n(\mathbb{F})$ are the nontrivial subspaces of $\mathbb{F}^{n+1}$ and its faces are the sets of vertices which are totally ordered by inclusion. The apartments
of $A_n(F)$ are induced from bases of $F^{n+1}$ as follows: If $E$ is a basis of $F^{n+1}$, then the collection of faces $x = \{V_0, \ldots, V_i\} \in A_n(F)$ for which each $V_j$ is spanned by a subset of $E$ is an apartment.

The group $GL_{n+1}(F)$ acts on $A_n(F)$ via its standard action on $F^{n+1}$. This action is strongly transitive because $GL_{n+1}(F)$ acts transitively on the set of bases of $F^{n+1}$.

When $n = 2$, the graph $A_2(F)$ is nothing but the incidence graph of points and lines in the 2-dimensional projective plane over $F$.

Let $X$ be a building of type $T = T(X)$. We define

$$m(X) = m(T) = \max\left(\{2\} \cup \{n \mid T \text{ has an edge labelled } n\}\right).$$

Observe that $m(X) \geq m(X_i)$ for every face $z \in X$ of dimension $< \dim X - 1$. This definition is motivated by the following theorem, which we derive from results of Evra–Kaufman [8] (see also [7]) and Oppenheim [24].

**Theorem 7.2.** Let $X$ be a (finite) $r$-dimensional simplicial complex such that one of the following hold:

1. $X$ is a $q$-thick spherical building admitting a strongly transitive action;
2. the universal covering of $X$ is a $q$-thick affine building of dimension $\geq 2$ admitting a strongly transitive action.

In both cases, $X$ has the structure of a pure $(r + 1)$-partite simplicial complex. Let $w$ denote the canonical weight function of $X$ (Example [7.2] let $T$ denote the Coxeter diagram of the building mentioned in (1) or (2) and put $m = m(T)$. Then $(X, w)$ is an $(r + 1)$-partite $[-r\lambda, \lambda]$-expander for

$$\lambda = \frac{\sqrt{m - 2}}{\sqrt{q} - (r - 1)\sqrt{m - 2}},$$

provided $q \geq r^2(m - 2)$. Furthermore, $m \leq 8$ when (1) holds, and $m \leq 6$ when (2) holds.

**Proof.** Suppose first that $r = 1$. Then only case (1) is possible. Thus, $X$ is a 1-dimensional building admitting a strongly transitive action by a group $G$. The Coxeter graph of $X$ consists of two vertices and one edge labelled $m$, so each apartment in $X$ is a cycle graph of length $2m$. We label the vertices of $T = T(X)$ by 0 and 1 and write $X_{\{i\}}$ ($i \in \{0, 1\}$) for the 0-faces of type $\{i\}$ in $X$. Since $G$ acts transitively on $X_{\{0\}}$ and $X_{\{1\}}$, $X$ is a biregular graph. Write $n_i = |X_{\{i\}}|$ and let $k_i$ denote the number of edges containing a 0-face in $X_{\{i\}}$. Then $|X(1)| = n_0k_0 = n_1k_1$, which means that $w(e) = \frac{1}{n_0k_0} = \frac{1}{n_1k_1}$ for all $e \in X(1)$ and $w(x) = \frac{1}{2m}$ for all $x \in X_{\{i\}}$. We may assume without loss of generality that $k_0 \leq k_1$. Note also that $q \leq k_0$. We now adapt the proofs of [3] Propositions 5.21, 5.22 to our weighted graph situation, and also improve the expansion constants, in order to show that $(X, w)$ is a bipartite $[-\frac{m - 2}{q}, \frac{m - 2}{q}]$-expander.

Let $f' \in C^0(X, \mathbb{R})$ (notation as in [21]) denote a nonzero eigenfunction of $\mathcal{A} := \mathcal{A}_{X, w}$ with eigenvalue $\lambda$. We need to prove that $\lambda^2 \leq \frac{m - 2}{q}$. It is enough to consider the case $\lambda \neq 0$. In this case, $f'$ cannot vanish on $X_{\{0\}}$. Thus, we may choose $s \in X_{\{0\}}$ with $f'(s) \neq 0$ and put $K = \{g \in G : gs = s\}$. By applying Lemma 2.8 to $f'$, we see that there exists $f'' \in C^0(X, \mathbb{R})$ with $Af'' = -\lambda f''$ and $f''(s) \neq 0$.

Given 0-faces $x, y \in X(0)$, we denote the $K$-orbit of $x \in X(0)$ by $[x]$, and write $d(x, y)$ for the length of the shortest path from one face to $y$ in $X$. We claim that following hold (cf. [3] Definition 5.20):

1. The number of $K$-orbits in $X(0)$ is exactly $m + 1$.
2. For all $i \in \{0, 1\}$, the 0-faces in $X_{\{i\}}$ of maximal distance from $s$ form a $G$-orbit. This maximal distance is $m - (m + i \mod 2)$.
3. If $x \in X(0) - \{s\}$ is not of maximal distance from $s$, then there is exactly one 0-face $y$ adjacent to $x$ with $d(s, y) < d(s, x)$.

This is similar to the proof of [3] Propositions 5.22: To see (i), fix an apartment $E$ containing $s$ and apply [3] Lemma 5.16 to conclude that every $K$-orbit in $X(0)$ meets $E$. Let $x, y \in E(0)$ be the 0-faces adjacent to $s$. By the strong transitivity of the $G$-action, there is $g \in G$ such that $g(E) = E$ and $g\{s, x\} = g\{s, y\}$. Since $G$ preserves types, $g(s) = s$, so $g$ is a reflection of the 2m-cycle $E$ fixing $s$. This means that all except possibly
2 $K$-orbits meet $E$ in at least two vertices, so there can be at most $m + 1$ $K$-orbits. On the other hand, since the action of $G$ preserves distances and there are 0-faces of distance $m$ in $X$, the number of $K$-orbits on $X(0)$ must be at least $m + 1$. This proves (i). The first assertion of (ii) is shown exactly as in op. cit., and the second assertion follows from the fact that $s \in X(0)$ and each apartment is a cycle of length $2m$. As for (iii), let $D$ be the set of 0-faces $y \in X(0)$ adjacent to $x$ with $d(s, y) < d(s, x)$. It is shown in the proof [op. cit.] that there is an apartment $E$ of $X$ with $s, x \in E$ and $D \subseteq E$. Since $E$ is a cycle graph, $D$ must be a singleton.

Next, let us regard $A = A_{X, w}$ as an operator from the complex vector space $C^0(X, \mathbb{C})$ to itself; this does not affect the spectrum of $A$. Put $Y = K \setminus X$ and let $C^0(Y, \mathbb{C})$ denote the set of functions from $Y$ to $\mathbb{C}$ (note that $Y$ is not a graph in general). We define a linear operator $B : C^0(Y, \mathbb{C}) \to C^0(Y, \mathbb{C})$ by

$$(B g)[x] = \sum_{y \in X(1)_x} \frac{w(x \cup y)}{2w(x)} g[y].$$

Note that this does not depend on the representative $x$ in the orbit $[x]$.

We claim that $\text{Spec } B \subseteq \text{Spec } A$; in particular, $\text{Spec } B \subseteq \mathbb{R}$. Indeed, given $\mu \in \mathbb{C}$ and $g \in C^0(Y, \mathbb{C})$ with $B g = \mu g$, the function $f \in C^0(X, \mathbb{C})$ defined by $f(x) = g[x]$ satisfies $Af = \mu f$ because, for all $x \in X(0)$,

$$(Af)(x) = \sum_{y \in X(0)_x} \frac{w(x \cup y)}{w(x)} f(y) = \sum_{y \in X(0)_x} \frac{w(x \cup y)}{w(x)} g[y] = (Bg)[x] = \mu g[x] = \mu f(x).$$

We further claim that the $A$-eigenvalue $\lambda$ corresponding to $f' \in C^0(X, \mathbb{R})$ is in $\text{Spec } B$. Indeed, define $g' \in C^0(Y, \mathbb{C})$ by $g'[y] = \sum_{k \in K} f'(ky)$ and note that $g'[s] = |K| f'(s) \neq 0$. Then, for every $x \in X(0)$,

$$(Bg')[x] = \sum_{y \in X(1)_x} \frac{w(x \cup y)}{2w(x)} g'[y] = \sum_{y \in X(1)_x} \sum_{k \in K} \frac{w(x \cup y)}{2w(x)} f'(ky) = \sum_{k \in K} \sum_{y \in X(1)_x} \frac{w(kx \cup ky)}{2w(kx)} f'(ky) = \sum_{k \in K} (Af)(kx) = \sum_{k \in K} \lambda f(kx) = \lambda g'[x].$$

Thus, $\lambda g' = g'$ and $\lambda \in \text{Spec } B$. Applying this argument for the $A$-eigenfunctions $f''$, $1_{X(0)}$ and $1_{X(1)}$ shows that we also have $1, -1, -\lambda \in \text{Spec } B$. Since $\lambda \neq -\lambda$ (because $\lambda \neq 0$) and $\text{Spec } B \subseteq \mathbb{R}$, this means that

$$2 + 2\lambda^2 = (-1)^2 + (-\lambda)^2 + \lambda^2 + 1^2 \leq \text{Tr}(B^2).$$

We now bound $\text{Tr}(B^2)$ from above. Fix a set of representatives $U$ for $K \setminus X(0)$ and write $1_{[x]} \in C^0(Y, \mathbb{C})$ for the characteristic function of $\{x\}$. Then

$$\text{Tr}(B^2) = \sum_{x \in U} (B^2 1_{[x]})[x] = \sum_{x \in U} \sum_{y \in X(1)_x} \frac{w(x \cup y)}{2w(x)} (B 1_{[x]})[y]$$

$$= \sum_{x \in U} \sum_{y \in X(1)_x} \sum_{z \in X(1)_y} \frac{w(x \cup y)w(y \cup z)}{4w(x)w(y)} 1_{[x]}[z] = \frac{1}{k_0 k_1} \sum_{x \in U} |L(x)|,$$

where $L(x)$ is the set of triples $(x, y, z) \in X(0)^3$ with $(x, y), (y, z) \in X(1)$ and $z \in [x]$. Let $s'$ be the unique $0$-face in $U$ with $d(s, s') = m$ and let $s''$ be the unique $0$-face in $U$ with $d(s, s'') = m - 1$; they exist by (ii) above. Put $\epsilon = m \mod 2$ and note that $s'' \in X(\epsilon)$. We analyze the size of $L(x)$ by splitting into four cases:

I) $x = s'$: Let $(s', y, z) \in L(s')$. There are $k_\epsilon$ possibilities for $y$. Since $d(s, y) = m - 1 > 0$, there is some $z' \in X(1)_y$ with $d(s, z') < d(s, y) < d(s, s') = d(s, z)$, so $z' \neq z$. This means that, for each $y$, there are at most $k_1 - \epsilon - 1$ possibilities for $z$. Thus, $|L(s')| \leq k_\epsilon (k_1 - \epsilon - 1) = k_0 k_1 - k_1 - \epsilon$.

II) $x = s''$: $|L(s'')| \leq k_0 k_1$; in fact, this holds for any $x \in U$.
III) \( x = s \): Since \( s = \{s\} \), we have \( L(s) = \{(s, y, s) \mid y \in X(1)\} \), so \( |L(s)| = k_0 \).

IV) \( x \neq s, s', s'' \): Write \( t(x) = \{i\} \). If \((x, y, z) \in L(x)\), then both \( x \) and \( y \) are not of maximum distance from \( s \) in \( X(0) \). Noting that \( d(s, x) = d(s, z) \) (because \( z \in [x] \)), (iii) implies that \( y \) is uniquely determined by \( x \) if \( d(s, y) < d(s, x) \) and \( z \) is uniquely determined by \( y \) in if \( d(s, y) > d(s, z) \). Thus, \( |L(x)| \leq 1 \cdot k_{l-1} + k_i \cdot 1 = k_0 + k_1 \).

By (i), \( |U| \leq m + 1 \), so we conclude that

\[
\text{Tr}(B^2) \leq \frac{(k_0k_1 - k_{l-1}) + k_0k_1 + k_0 + (m - 2)(k_0 + k_1)}{k_0k_1} \leq 2 + \frac{2(m - 2)}{q},
\]

where the inequality holds because \( q \leq k_0 \leq k_{l-1} \). Combining this with (7.1) gives the desired conclusion

\[
\lambda^2 \leq \frac{m - 2}{q}.
\]

This proves the theorem when \( r = 1 \).

Suppose now that \( r > 1 \). Assumptions (1) and (2) imply that all the positive-dimensional links of \( X \) are connected, and for every \( z \in X(r - 2) \), the complex \( X_z \) is a 1-dimensional spherical building admitting a strongly transitive action and having \( m(X_z) \leq m \). Since we proved the theorem when \( r = 1 \), the weighted graph \((X_z, w_{X_z})\) is a 2-partite \([-\sqrt{\frac{m-2}{q}}, \sqrt{\frac{m-2}{q}}]\)-expander. Moreover, our assumption \( q \geq r^2(m - 2) \) implies that \( \sqrt{\frac{m-2}{q}} \leq \frac{1}{r} \). Thus, by a theorem of Oppenheim [24 Corollary 5.6] (see also the formula at the end of [24 Theorem 1.4]), \((X, w)\) is an \((r + 1)\)-partite \([-r\lambda, \lambda]\)-expander for

\[
\lambda = 1 - \frac{r(1 - \sqrt{\frac{m-2}{q}}) - (r - 1)}{(r - 1)(1 - \sqrt{\frac{m-2}{q}}) - (r - 2)} = \frac{\sqrt{m-2}}{\sqrt{q} - (r - 1)\sqrt{m-2}}.
\]

\[\square\]

**Example 7.3.** Let \( \mathbb{F}_q \) be a finite field with \( q \) elements and let \( X = A_2(\mathbb{F}_q) \) (notation as in Example 7.1). Then \( X \) is a \((q + 1)\)-thick building admitting a strongly transitive action, and \( m(X) = m(A_2) = 3 \). Thus, by Theorem 7.2, \((X, w_{X})\) is a bipartite \([-\frac{1}{\sqrt{q+r}}, \frac{1}{\sqrt{q+r}}]\)-expander. This agrees with the well-known fact that \( \text{Spec}(A_{X,w}) = \{\pm 1, \pm \frac{\sqrt{q}}{\sqrt{q+r}}\} \) with 1 and \(-1\) occurring with multiplicity 1. (In fact, the counting argument in cases I)-IV) in the proof can be slightly improved to make the bounds match.)

**Corollary 7.4.** Let \( X, w, r, q, m \) be as in Theorem 7.3 and assume \( q \geq r^2(m - 2) \). Then \((X, w, R^+)\) is a \((1 - \frac{\sqrt{m-2}}{\sqrt{q} - (r - 1)\sqrt{m-2}})\)-coboundary expander in dimension 0 for every nontrivial abelian group \( R \).

**Proof.** This follows from Theorem 7.2 and Corollary 6.3 \(\square\)

**Lemma 7.5.** Let \( X \) be a pure \( r \)-dimensional (finite) simplicial complex and let \( w \) be its canonical weight function. Suppose that \( X \) is \( q \)-thick, i.e., every \((r - 1)\)-face of \( X \) is contained in at least \( q \) \( r \)-faces. Then, for every \( e \in X(1) \) and \( x \in X(0)_{\leq e} \), we have \( \frac{w(e)}{w(x)} \leq \frac{2}{q+r-1} \).

**Proof.** Fix \( e \in X(1) \) and let \( x, y \) be the \( 0 \)-faces of \( e \). It follows readily from the defining properties of \( w \) in §2.2 that \( \frac{w(e)}{w(x)} = 2|X(r)_{\geq l}| \). Let \( c \in X(r)_{\geq l} \). By assumption, the set \( N(c) := X(r)_{\geq l} - \{c\} \) has at least \( q - 1 \) members. Note that the members of \( N(c) \) are \( r \)-faces containing \( x \) but not \( e \). Let \( c' \in X(r)_{\geq l} - X(r)_{\geq e} \). If \( c \in X(r)_{\geq e} \) is an \( r \)-face such that \( c' \in N(c) \), then \( c \cup c' = c' \cup y \), which means that \( e \subseteq c \subseteq c' \cup y \). Since \( |c' \cup y| = r + 2 \) and \( |c| = 2 \), there are at most \( r \) faces \( c \in X(r)_{\geq e} \) with \( c' \in N(c) \). As a result,

\[
r |X(r)_{\geq e} - X(r)_{\geq 2}| \geq \sum_{c \in X(r)_{\geq e}} |N(c)| \geq (q - 1)|X(r)_{\geq e}|.
\]

By rearranging, we find that

\[
|X(r)_{\geq 2}| \leq \frac{(1 + \frac{q-1}{r})^{-1}}{\frac{2}{q+r-1}} \leq \frac{r}{q+r-1},
\]

so \( \frac{w(e)}{w(x)} \leq \frac{2}{q+r-1} \). \(\square\)

\(8\)There is a typo in [24 Corollary 5.6]: the expression \( 1 - (n - k) f^{n-k-2}(\lambda) \) should be \( 1 + (n - k - 1) f^{n-k-2}(\lambda) \).
Corollary 7.6. Let $X, w, r, q, m$ be as in Theorem 7.2 and assume $q \geq r^2(m - 2)$. Let $R$ be an abelian group and let $\{R_x\}_{x \in X - \{0\}}$ be subgroups of $R$ satisfying conditions (1) and (2) of Theorem 6.1 (after setting $R_0 = \{0\}$, e.g., such that the summation map $\bigoplus_{x \in X} R_x \to R$ is injective. For every $x \in X$, put $G(x) = \sum_{y \in x} R_y$. Then $G$ is a subsheaf of $R^+$ and

$$
cb_0(X, w, R^+_X/G) \geq \frac{2r}{5r^2 + 2} - \frac{(4r^3 + 4r)\sqrt{m - 2}}{(5r^2 + 2)(\sqrt{q} - (r - 1)\sqrt{m - 2})} - \frac{14r + 4}{(5r^2 + 2)(q + r - 1)}. \tag{1}
$$

Proof. By Theorem 7.2, $(X, w)$ is an $(r + 1)$-partite $[-r\lambda, \lambda]$-expander for $\lambda = \frac{\sqrt{m - 2}}{q - (r - 1)\sqrt{m - 2}}$, and by Lemma 7.5 we have $t := \max\{\frac{|w(e)|}{w(x)} \mid e \in X(1), x \in X(0)\} \leq \frac{2}{q + r - t}$. In particular, $s := \max\{|w(e)| \mid e \in X(1)\} \leq t = \frac{2}{q + r - 1}$. Now apply Theorem 6.2 (The expansion constant guaranteed by the theorem is slightly better than the simpler expression used in the corollary.)

\[ \square \]

Remark 7.7. When $r = \dim X > 1$, the constants $t$ and $s$ of Theorem 6.2 are often much smaller than the bounds used in the proof of Corollary 7.6. Using a better upper bound on $t$ and $s$ will decrease the absolute value of the right fraction in the lower bound for $\cb_0$ in Corollary 7.6. Also, when $r = \dim X = 1$, we may eliminate $s$ by Remark 6.11, thus changing the right fraction in the lower bound for $\cb_0$ to $-t = -\frac{2}{q + r - 1}$.

Example 7.8. The lower bound on $\cb_0(X, w, R^+_X/G)$ provided by Corollary 7.6 and Remark 7.7 is detailed in the following table for some spherical $q$-thick buildings of dimensions 1 and 2 admitting a strongly transitive action.

| dim $X$ | $T(X)$ | $\cb_0(X, w, R^+_X/G)$ | > 0 if $q \geq$ |
|---------|--------|------------------------|-----------------|
| 1       | $A_2$  | $\frac{7}{9} - \frac{\sqrt{7}}{9} - \frac{2}{9}$ | 29               |
|         | $C_2$  | $\frac{7}{10} - \frac{\sqrt{10}}{10} - \frac{2}{10}$ | 45               |
|         | $G_2$  | $\frac{7}{10} - \frac{\sqrt{10}}{10} - \frac{2}{10}$ | 78               |
| 2       | $A_3$  | $\frac{1}{3} - \frac{2}{3(\sqrt{5} - 1)} - \frac{8}{3(5 + 1)}$ | 136              |
|         | $C_3$  | $\frac{1}{3} - \frac{2}{3(\sqrt{5} - 2)} - \frac{8}{3(5 + 1)}$ | 257              |

8 Further Questions

We finish with several questions about possible extensions of Theorems 6.1 and 6.2.

If not indicated otherwise, $(X, w)$ is assumed to be a weighted graph (resp. $(r + 1)$-partite weighted simplicial complex) which is a $[-\lambda, \lambda]$-expander (resp. $(r + 1)$-partite $[-\lambda, \lambda]$-expander) for some $\lambda > 0$. We let $R$ be a nontrivial abelian group, $\{R_x\}_{x \in X}$ be subgroups of $R$ with $R_0 = \{0\}$, and define the subsheaf $G$ of $R^+_x$ as in Theorem 6.1.

As $\lambda$ approaches 0, the lower bound on the 0-dimensional coboundary expansion of $(X, w, R^+_x/G)$ provided by Theorem 6.1 (resp. Theorem 6.2) approaches $\frac{2}{5}$ (resp. $\frac{2r}{5r^2 + 2}$). We expect that this could be improved.

Question 8.1. Provided that all the $\{R_x\}_{x \in X}$ are linearly disjoint in $R$, is it the case that the coboundary expansion of $(X, w, R^+_x/G)$ in dimension 0 approaches 1 as $\lambda \to 0$?

Next, we ask whether condition (1) of Theorem 6.1 can be relaxed.

Question 8.2. Let $m : \mathbb{N} \to \mathbb{N} \cup \{0\}$ be a function satisfying $0 \leq m(n) \leq n$ for all $n \in \mathbb{N}$. Assuming $\lambda$ is fixed and sufficiently small, does the coboundary expansion of $(X, w, R^+_x/G)$ in dimension 0 remains bounded away from 0 if (instead of condition (1) of Theorem 6.1) we require that every $m(|X(0)|)$ of the $\{R_x\}_{x \in X}$ are linearly disjoint in $R$? More specifically:

(a) Can we take $m = o(n)$? Can we take $m = O(1)$?

(b) What if we also require that $R_x = 0$ when $x$ is not a 0-face?
The motivation behind (b) is that in [9], we only need Theorems 6.1 and 6.2 in the special case where \( R_x = 0 \) for all \( x \in X - X(0) \). Also, if one allows \( R_x \) to be nonzero when \( x \) is an edge, then it is impossible to take \( m = o(\log n) \) as noted in Remark 1.3. However, the answer might still be “yes” under the assumptions of (b).

Finally, we ask whether Theorems 6.1 and 6.2 extend to higher dimensions.

**Question 8.3.** Suppose that \((X, w)\) is a weighted simplicial complex (resp. \((r + 1)\)-partite weighted simplicial complex), and we are given subgroups \( \{R_x\}_{x \in X} \) of \( R \) which are all linearly disjoint. Form the subsheaf \( \mathcal{G} \) of \( R^+ \) as in Theorem 6.1, and suppose that the underlying graph of \( X_z \) is a \([-\lambda, \lambda]\)-expander (resp. \((\dim X - \dim z)\)-partite \([-\lambda, \lambda]\)-expander) for every \( z \in X \) with \( \dim z \leq \dim X - 2 \). Provided \( \lambda \) is sufficiently small, is there an \( \varepsilon > 0 \), depending only on \( \lambda \), \( \dim X \) and \( r \), such that \((X, w, R^+/\mathcal{G})\) is an \( \varepsilon \)-coboundary expander in dimensions \( 0, \ldots, \dim X - 1 \)?

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