EQUIVARIANT HEAT ASYMPTOTICS
ON SPACES OF AUTOMORPHIC FORMS

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Abstract. Let $G$ be a connected, real, semisimple Lie group with finite center and $K$ a compact subgroup of $G$. In this paper, we derive $K$-equivariant asymptotics for heat traces with remainder estimates on compact Riemannian manifolds carrying a transitive $G$-action. In particular, if $K$ is a maximal compact subgroup, we recover the leading coefficient in the Minakshisundaram-Pleijel expansion of the $K$-equivariant heat trace of the Laplace-Beltrami operator on spaces of automorphic forms for arbitrary rank.

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1. Introduction

Let $G$ be a connected, real, semisimple Lie group with finite center, acting smoothly and transitively on a compact, $n$-dimensional Riemannian manifold $M$. Further let $K$ be a compact subgroup of $G$ acting isometrically on $M$. In this paper, we derive $K$-equivariant asymptotics for traces of heat semigroups associated to strongly elliptic operators on $M$ with remainder estimates. In particular, if $M = \Gamma \backslash G$, where $\Gamma$ is a discrete, uniform subgroup of $G$, and $K$ is a maximal compact subgroup, we recover the leading coefficient in the Minakshisundaram-Pleijel expansion of the $K$-equivariant heat trace of the Laplace-Beltrami operator on $L^2(\Gamma \backslash G)$, together with an estimate for the remainder.

The study of the asymptotic behavior of heat semigroups and their kernels has a long history. One of the pioneering works in this direction was the derivation of an asymptotic expansion for the fundamental solution of the heat equation on a compact manifold by Minakshisundaram and Pleijel [19]. The first three coefficients in this expansion were computed by McKean and Singer [17] in terms of geometric

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quantities, yielding corresponding expansions of heat traces. This culminated in a heat theoretic proof of the index theorem by Atiyah, Bott and Patodi \cite{1}. In the case of Riemannian symmetric spaces, an explicit expression for the fundamental solution of the heat equation was given by Gangolli \cite{12} using Harish-Chandra’s Plancherel theorem. Later, Donnelly \cite{11} generalized the constructions in \cite{19} and \cite{3} to Riemannian manifolds admitting a properly discontinuous group of isometries with compact quotient. Equivariant heat traces for the Beltrami-Laplace operator on a compact Riemannian manifold with a compact isometry group and no singular orbits were established first by Donnelly and Br"{u}ning-Heintze \cite{6}. Following these developments, Miatello \cite{18}, and DeGeorge and Wallach \cite{10} established asymptotic expansions for heat traces of Bochner-Laplace operators on homogeneous vector bundles over compact, locally symmetric spaces of rank one. Holomorphic semigroups generated by strongly elliptic operators on Lie groups have been studied systematically by Langlands \cite{16} and Robinson and ter Elst \cite{27}, \cite{29}, giving lower and upper bounds for their kernels. For further references, see also \cite{9} and \cite{4}.

To illustrate our results, let $(\pi, L^2(M))$ be the right regular representation of $G$ on the Hilbert space of square integrable functions on $M$ with respect to the canonical density. Since $K$ acts on $M$ by isometries, $\pi$ is a unitary representation of $K$. Consider further the group kernel $f_t$ of a strongly elliptic operator $\Omega$ of order $q$ associated to the representation $\pi$, where $t > 0$. The corresponding heat operator is then given by $e^{-t\Omega} = \pi(f_t)$, and characterized in Theorem 1 as a pseudodifferential operator of order $-\infty$. Due to the compactness of $M$, this implies that $\pi(f_t)$ is of trace class. Using this characterization, we consider the decomposition

$$L^2(M) = \bigoplus_{\sigma \in \hat{K}} L^2(M)_\sigma$$

of $L^2(M)$ into $K$-isotypic components, and derive asymptotics with remainder estimates for the trace

$$\text{tr}(P_\sigma \circ \pi(f_t) \circ P_\sigma)$$

of the restriction of $\pi(f_t)$ to the isotypic component $L^2(M)_\sigma = P_\sigma(L^2(M))$ as $t$ goes to zero, $P_\sigma$ being the corresponding projector; see Theorem 1. In order to do so, one has to describe the asymptotic behavior of certain oscillatory integrals, which has been determined before in \cite{26} while studying the spectrum of an invariant elliptic operator. The difficulty here resides in the fact that, since the critical sets of the corresponding phase functions are not smooth, a desingularization procedure is required in order to apply the method of the stationary phase in a suitable resolution space. In the case that $f_t$ has an asymptotic expansion of the form

$$f_t(g) \sim \frac{1}{td^{d/4}} e^{-b \left( \frac{d(g,e)^2}{1} \right)^{1/(q-1)}} \sum_{j=0}^{\infty} c_j(g) t^j, \quad b > 0,$$

near the identity $e \in G$ with analytic coefficients $c_j(g)$, where $d = \dim G$, and $d(g,e)$ denotes the distance of $g \in G$ from the identity with respect to the canonical left-invariant metric on $G$, we show in Corollary 4 as our main result that

$$\text{tr}(P_\sigma \circ \pi(f_t) \circ P_\sigma) = \frac{d_\sigma \left[ \pi_{\sigma}\|_{H} : 1 \right]}{(2\pi)^{n-k}\kappa(n-k)/q} c_0(e) \sim \text{vol}(\Xi/K) + O(t^{-(n-k-1)/q}(\log t)^{\Lambda-1}),$$
where $(\pi_\sigma, V_\sigma) \in \sigma$, $\kappa$ denotes the dimension of a $K$-orbit of principal type, and $H \subset K$ is a principal isotropy group, while $\Lambda$ is the maximal number of elements of a totally ordered subset of the set of $K$-isotropy types. Further, $\text{vol}(\Xi/K)$ represents a Gaussian volume of the symplectic quotient $\Xi/K$ of the zero level set $\Xi = J_{K}^{-1}(0)$ of the momentum map $J_{K} : T^*M \to \mathfrak{g}^*$ of the Hamiltonian action of $K$ on $M$ given by

$$\tilde{\text{vol}}(\Xi/K) = \int_{\text{Reg} \Xi} \left( \int_{\mathfrak{g}} e^{-i\mathcal{J}(p, \xi)(X)} e^{-b\|X\|^2/(q-1)} dX \right) \frac{d(\text{Reg} \Xi)(p, \xi)}{\text{vol} \mathcal{O}(p, \xi)},$$

where $\| \cdot \|_\theta$ is the norm given by the canonical inner product $\langle \cdot, \cdot \rangle_\theta$ on $\mathfrak{g}$ and $dX$ the corresponding Euclidean measure on $\mathfrak{g}$, while $\mathcal{O}(p, \xi)$ denotes a $K$-orbit in $T^*M$ and $J_G : T^*M \to \mathfrak{g}^*$ the momentum map of the Hamiltonian action of $G$ on $T^*M$. The expression $d(\text{Reg} \Xi)$ denotes the volume measure induced on the regular part of $\Xi$ by the Sasaki metric on $T^*M$. Note that in the considered situations $G$ itself could be compact. Actually, an examination of the proofs shows that our results also hold for connected, real, unimodular groups $G$ carrying a $K$-bi-invariant metric, and therefore also for connected, compact, real Lie groups.

As an application, we consider in Theorem 5 the case $M = \Gamma \backslash G$, where $\Gamma \subset G$ is a discrete, co-compact subgroup. In the particular case where $p_t$ is the classical heat kernel of the Laplace-Beltrami operator on $G$, and $K$ is a maximal compact subgroup, we recover the known asymptotic

$$\text{tr}(P_\sigma \circ \pi(p_t) \circ P_\sigma) = \frac{d_\sigma[\pi_\sigma | H : 1]}{(4\pi t)^{\frac{\dim G}{2}}} \text{vol}(\Gamma \backslash G)' / K + O(t^{-\frac{\dim G}{2}(\log t)^{-1}}),$$

where $(\Gamma \backslash G)'$ denotes the union of $K$-orbits of principal type; see Corollary 4.

In addition, the previous results, combined with Selberg’s trace formula, yield an asymptotic description of $L_\sigma f_t$ at the identity, where $L_\sigma$ denotes the projector onto the isotypical component $L^2(G)_\sigma$ of the left-regular representation $(L, L^2(G))$ of $G$; see Proposition 2. Finally, for torsion-free $\Gamma$ and maximal $K$ we recover with our techniques the first coefficient in the Minakshisundaram-Pleijel expansion, together with an estimate for the remainder, of vector valued heat kernels on the compact locally symmetric space $\Gamma \backslash G / K$, generalizing the work in [13] and [10] to arbitrary rank.

This paper is organized as follows. The microlocal structure of general convolution operators with rapidly decaying group kernels on paracom pact, smooth manifolds is described in Section 2. In Section 3, the Langlands kernel of a semigroup generated by a strongly elliptic operator on a closed Riemannian manifold $M$ is considered, and its equivariant heat trace is expressed in terms of oscillatory integrals. Since the occurring phase functions do have singular critical sets, the stationary phase principle cannot immediately be applied to describe the asymptotic behavior of those integrals. Instead, we rely on the results in [20], where resolution of singularities was used to partially resolve the singularities of the relevant critical sets. This yields short-time asymptotics with remainder estimates for equivariant heat traces in Section 4. Finally, in Section 5, we consider the particular case $M = \Gamma \backslash G$, where $\Gamma$ denotes a uniform lattice in $G$. If $\Gamma$ is torsion-free, we recover asymptotics for heat traces of Bochner-Laplace operators on compact locally symmetric spaces of arbitrary rank.
2. Convolution operators

Let $G$ be a connected, real, semisimple Lie group with finite center and Lie algebra $\mathfrak{g}$. Denote by $\langle X, Y \rangle = \text{tr}(\text{ad} X \circ \text{ad} Y)$ the Cartan-Killing form on $\mathfrak{g}$ and by $\theta$ a Cartan involution of $\mathfrak{g}$. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be the Cartan decomposition of $\mathfrak{g}$ into the eigenspaces of $\theta$, corresponding to the eigenvalues $+1$ and $-1$, respectively. Put $\langle X, Y \rangle_\theta := -\langle X, \theta Y \rangle$. Then $\langle \cdot, \cdot \rangle_\theta$ defines a left-invariant Riemannian metric on $G$. With respect to this metric, we define $d(g, h)$ as the geodesic distance between two points $g, h \in G$, and set $|g| = d(g, e)$, where $e$ is the identity element of $G$. Note that $d(g_1 g, g_1 h) = d(g, h)$ for all $g, g_1, h \in G$. In contrast to the Killing form, $\langle \cdot, \cdot \rangle_\theta$ is no longer $\text{Ad}(G)$-invariant, but still $\text{Ad}(K)$-invariant, where $K$ denotes the maximal compact subgroup of $G$ corresponding to $\mathfrak{k}$, so that $d(gk, hk) = d(g, h)$ for all $g, h \in G$ and $k \in K$. Indeed, one has the following:

**Lemma 1.** The modified Killing form $\langle \cdot, \cdot \rangle_\theta$ is $\text{Ad}(K)$-invariant, which implies that the corresponding Riemannian distance $d$ on $G$ is right $K$-invariant. In particular, $|g| = |kgk^{-1}|$ for all $g \in G$ and $k \in K$.

**Proof.** This is a well-known fact, but for completeness, we include a proof here. First, note that for $k \in K$ the morphisms $\text{Ad}(k)$ and $\theta$ commute. Indeed, the inclusions $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}$, and $[\mathfrak{t}, \mathfrak{p}] \subset \mathfrak{p}$ together with the relation $\text{Ad}(e^X) = e^{\text{ad} X}$, $X \in \mathfrak{g}$, imply that $\text{Ad}(K) \mathfrak{k} \subset \mathfrak{k}$, $\text{Ad}(K) \mathfrak{p} \subset \mathfrak{p}$. Hence, $\text{Ad}(k) \theta X = \theta \text{Ad}(k) X$ for all $X \in \mathfrak{g}$. But then

$$\langle \text{Ad}(k) X, \text{Ad}(k) Y \rangle_\theta = -\langle \text{Ad}(k) X, \theta \text{Ad}(k) Y \rangle = -\langle \text{Ad}(k) X, \text{Ad}(k) \theta Y \rangle = -\langle X, \theta Y \rangle = \langle X, Y \rangle_\theta$$

for all $X, Y \in \mathfrak{g}$, $k \in K$, showing the $\text{Ad}(K)$-invariance of $\langle \cdot, \cdot \rangle_\theta$. This implies that the left-invariant Riemannian metric $\langle \cdot, \cdot \rangle$ on $G$ defined by $\langle \cdot, \cdot \rangle_\theta$ is right $K$-invariant. Indeed, if $i_k : G \to G$, $h \mapsto khk^{-1}$, its differential at the identity $e$ is by definition $\text{Ad}(k) = (di_k)_e : \mathfrak{g} \to \mathfrak{g}$. Furthermore, $\text{Ad}(k) = (dL_k)_{k^{-1}} \circ (dR_{k^{-1}})_e$, $R_g$ and $L_g$ being right- and left-translation by $g \in \mathfrak{g}$, respectively, so that

$$\langle (dR_{k^{-1}})_e(X), (dR_{k^{-1}})_e(Y) \rangle_{k^{-1}} = \langle \text{Ad}(k) X, \text{Ad}(k) Y \rangle_\theta = \langle \langle X, Y \rangle \rangle_e$$

for arbitrary $X, Y \in \mathfrak{g}$ and $k \in K$. The right $K$-invariance of $d$ now follows. □

Let us consider next a paracompact $C^\infty$-manifold $M$ of dimension $n$, and assume that $G$ acts on $M$ in a smooth and transitive way. Let $C(M)$ be the Banach space of continuous, bounded, complex valued functions on $M$, equipped with the supremum norm, and let $(\pi, C(M))$ be the corresponding continuous right regular representation of $G$ given by

$$\pi(g) \varphi(p) = \varphi(p \cdot g), \quad \varphi \in C(M), \quad g \in G, \quad p \in M.$$

The representation of the universal enveloping algebra $\mathfrak{U}$ of the complexification $\mathfrak{g}_C$ of $\mathfrak{g}$ on the space of differentiable vectors $C(M)_{\infty}$ will be denoted by $d\pi$. We shall also consider the right regular representation of $G$ on $C^\infty(M)$ which, equipped with the topology of uniform convergence on compacta, becomes a Fréchet space. This representation will be denoted by $\pi$ as well. Let $(L, C^\infty(G))$ and $(R, C^\infty(G))$ be the left, respectively right, regular representation of $G$. A function $f$ on $G$ is said
to be of at most exponential growth if there exists $\kappa > 0$ such that $|f(g)| \leq Ce^{\kappa |g|}$ for some constant $C > 0$ and all $g \in G$. Let $dg$ be the Haar measure on $G$ corresponding to the Riemannian structure induced by $\langle \cdot, \cdot \rangle_g$. We introduce now the Casselman-Wallach space of rapidly decaying functions on $G$.

**Definition 1.** The Casselman-Wallach space of rapidly decreasing functions on $G$, denoted by $\mathcal{S}(G)$, is given by all functions $f \in C^\infty(G)$ satisfying the following conditions:

i) for every $\kappa \geq 0$ and $X \in \mathcal{U}$, there exists a constant $C > 0$ such that

$$|dL(X)f(g)| \leq Ce^{-\kappa |g|};$$

ii) for every $\kappa \geq 0$ and $X \in \mathcal{U}$, one has $dL(X)f \in L^1(G, e^{\kappa |g|} \, dg)$.

**Remark 1.** 1) The space $\mathcal{S}(G)$ was introduced first in [31] in a slightly different way. As noted already in [8], it is quite different from the Harish-Chandra space of rapidly decaying functions on $G$ introduced in [13]. In our context, the consideration of the space $\mathcal{S}(G)$ was motivated by the study of strongly elliptic operators and the decay properties of the semigroups generated by them; see [25] and Section 3.

2) Note that condition ii) in the previous definition is already implied by condition i). Indeed, let $c > 0$ be such that $e^{-c|g|} \in L^1(G, dg)$. Let $\kappa \geq 0$ and $X \in \mathcal{U}$ be given, and assume that $f \in C^\infty(G)$ fulfills i), so that in particular $|e^{(\kappa+c) |g|} dL(X)f(g)| \leq C$ for all $g \in G$ and a suitable constant $C > 0$. Then

$$\|dL(X)f e^{\kappa |\cdot|}\|_{L^1(G, dg)} \leq C \|e^{-c |\cdot|}\|_{L^1(G, dg)} < \infty,$$

showing that $dL(X)f \in L^1(G, e^{\kappa |g|} \, dg)$.

3) If $f \in \mathcal{S}(G)$, $dR(X)f$ satisfies conditions i) and ii) of Definition 1 as well. Furthermore, if one compares the space $\mathcal{S}(G)$ with the Fréchet spaces $\mathcal{S}_{a,b}(G)$ defined in [30] Section 7.1.1, where $a$ and $b$ are smooth, positive, $K$-bi-invariant functions on $G$ satisfying certain properties, one easily sees that $a(g) = e^{|g|}$ and $b(g) = 1$ satisfy the selfsame properties, except for the smoothness at $g = e$ and the $K$-bi-invariance of $a$.

Let us now associate to $f \in \mathcal{S}(G)$ and $\varphi \in C(M)$ the vector valued integral

$$\int_G f(g)\pi(g)\varphi \, dg \in C(M),$$

yielding a continuous linear operator

$$\pi(f) = \int_G f(g)\pi(g) \, dg$$

on $C(M)$. Its restriction to $C^\infty_c(M)$ induces a continuous linear operator

$$\pi(f) : C^\infty_c(M) \rightarrow C(M) \subset \mathcal{D}'(M),$$

with Schwartz kernel given by the distribution section $\mathcal{K}_f \in \mathcal{D}'(M \times M, 1 \otimes \Omega_M)$, where $\Omega_M$ denotes the density bundle of $M$. In what follows, we shall show that $\pi(f)$ is an operator with smooth kernel. As we shall see, the smoothness of the operators $\pi(f)$ is a direct consequence of the fact that $G$ acts transitively on $M$.

Thus, let $\left\{ (\bar{W}_\iota, \varphi_\iota) \right\}_{\iota \in I}$ be a locally finite atlas of $M$. By [15] page 273, there exists a locally finite refinement $\left\{ \bar{W}_\iota \right\}_{\iota \in I}$ with the same index set such that $\bar{W}_\iota \subset \bar{W}_\iota'$ for every $\iota \in I$. Assume that the $\bar{W}_\iota$ are compact, and let $\{\alpha_\iota\}_{\iota \in I}$ be a partition
of unity subordinate to the atlas \(\{ (\tilde{W}_i, \varphi_i) \}_{i \in I} \), meaning that

(a) the \(\alpha_i\) are smooth functions, and \(0 \leq \alpha_i \leq 1\);
(b) \(\text{supp} \alpha_i \subset \tilde{W}_i\);
(c) \(\sum_{i \in I} \alpha_i = 1\).

Further let \(\{ \alpha'_i \}_{i \in I} \) be another set of functions satisfying condition (a) and, in addition,

(b') \(\text{supp} \alpha'_i \subset \tilde{W}'_i\);
(c') \(\alpha'_i \vert_{\tilde{W}_i} \equiv 1\).

Consider now the localization of \(\pi(f)\) with respect to the latter atlas

\[
A_f u = [\pi(f)|_{\tilde{W}_i}(u \circ \varphi_i)] \circ \varphi_i^{-1}, \quad u \in C^\infty_c(W_i), \quad W_i = \varphi_i(\tilde{W}_i) \subset \mathbb{R}^n,
\]

corresponding to the diagram

\[
\begin{array}{ccc}
C^\infty_c(\tilde{W}_i) & \xrightarrow{\pi(f)|_{\tilde{W}_i}} & C^\infty(\tilde{W}_i) \\
\varphi_i^* & \uparrow & \varphi_i^* \\
C^\infty_c(W_i) & \xrightarrow{A_f^i} & C^\infty(W_i).
\end{array}
\]

Let \(p \in \tilde{W}_i\). Writing \(\varphi_i^0 = \varphi_i \circ g \circ \varphi_i^{-1}\) and \(x = \varphi_i(p) = (x_1, \ldots, x_n) \in W_i\), we obtain

\[
A_f^i u(x) = \int_G f(g)[(u \circ \varphi_i) \alpha'_i](\varphi_i^{-1}(x) \cdot g) \, dg = \int_G f(g) c_i(x, g)(u \circ \varphi_i^0)(x) \, dg,
\]

where we put \(c_i(x, g) = \alpha'_i(\varphi_i^{-1}(x) \cdot g)\). Next, define the functions

\[
a'^i_f(x, \xi) = e^{-ix \cdot \xi} \int_G e^{i\varphi_i^0(x) \cdot \xi} c_i(x, g) f(g) \, dg.
\]

Since \(f\) is rapidly falling, differentiation under the integral yields \(a'^i_f(x, \xi) \in C^\infty(W_i \times \mathbb{R}^n)\). We can now state

**Theorem 1** (Structure theorem). Let \(M\) be a paracompact \(C^\infty\)-manifold of dimension \(n\) and \(G\) a connected, real, semisimple Lie group with finite center acting on \(M\) in a smooth and transitive way. Further let \(f \in S(G)\) be a rapidly decaying function on \(G\). Then the operator \(\pi(f)\) is a pseudodifferential operator of class \(L^{-\infty}(M)\); that is, it is locally of the form\(^1\)

\[
A_f^i u(x) = \int e^{ix \cdot \xi} a'^i_f(x, \xi) \hat{u}(\xi) \, d\xi, \quad u \in C^\infty_c(W_i),
\]

where the symbol \(a'^i_f(x, \xi) \in S^{-\infty}(W_i, \mathbb{R}^n)\) is given by (3), and \(d\xi = (2\pi)^{-n} \, d\xi\). In particular, the kernel of the operator \(A_f^i\) is given by the oscillatory integral

\[
K_{A_f^i}(x, y) = \int e^{i(x-y) \cdot \xi} a'^i_f(x, \xi) \, d\xi \in C^\infty(W_i \times W_i).
\]

\(^1\)Here and in what follows we use the convention that, if not specified otherwise, integration is to be performed over whole Euclidean space.
Proof. Our considerations will essentially follow the proof of Theorem 4 in [25]. For a thorough treatment of pseudodifferential operators, the reader is referred to [28]. Fix a chart \((\tilde{W}_i, \varphi_i)\), and let \(p \in \tilde{W}_i\), \(x = (x_1, \ldots, x_n) = \varphi_i(p) \in \mathbb{R}^n\). In what follows we shall show that \(a_j'(x, \xi)\) belongs to the symbol class \(\mathcal{S}^{-\infty}(\tilde{W}_i \times \mathbb{R}^n)\). For later purposes, we shall actually consider the slightly more general amplitudes

\[
(6) \quad a_j'(x, \xi; k_1, k_2) = e^{-i\varphi_i^{-1}(x) \cdot k_1 k_2} \int_G e^{i\varphi_i^{-1}(x) \cdot k_1 k_2} \xi c_i(x, \xi; k_1, k_2) f(g) \, dg
\]

where \(k_1, k_2 \in G\). Here we shall take into account the unimodularity of \(G\). In particular, \(a_j'(x, \xi) = a_{ij}'(x, \xi; e, e)\). Denote by \(V_{i,p}\) the set of all \(g \in G\) such that \(p \cdot g \in \tilde{W}_i\). Assume that \(g \in V_{i,p}\), and write

\[
\psi_{\xi,x}^i(g) = e^{i\varphi_i'(x) \cdot \xi}.
\]

For \(X \in \mathfrak{g}\) one computes that

\[
dR(X)\psi_{\xi,x}^i(g) = \frac{d}{ds} e^{i\varphi_i'(x) \cdot X} \psi_{\xi,x}^i(g) \big|_{s=0} = i\psi_{\xi,x}^i(g) \sum_{i=1}^n \xi_i dR(X) x_{i,p}(g),
\]

where we put \(x_{i,p}(g) = x_i(p \cdot g)\). Let \(\{X_1, \ldots, X_d\}\) be a basis of \(\mathfrak{g}\). Since \(G\) acts locally transitively on \(\tilde{W}_i\), the \(n \times d\) matrix

\[
(dR(X_j) x_{i,p}(g))_{i,j}
\]

has maximal rank. As a consequence, there exists a neighborhood \(\tilde{U}_p\) of \(p\) and indices \(j_1, \ldots, j_n\) such that

\[
det \ (dR(X_{j_k}) x_{i,p'}(g))_{i,k} \neq 0 \quad \forall p' \in \tilde{U}_p.
\]

Hence,

\[
(7) \quad \begin{pmatrix} dR(X_{j_1}) \psi_{\xi,x}^i(g) \\ \vdots \\ dR(X_{j_n}) \psi_{\xi,x}^i(g) \end{pmatrix} = i\psi_{\xi,x}^i(g) \mathcal{M}(x', g) \xi,
\]

where \(\mathcal{M}(x', g) = \left( dR(X_{j_k}) x_{i,i'}^{-1}(x'(g)) \right)_{i,k} \in \text{GL}(n, \mathbb{R})\) is an invertible matrix for all \(x' \in \varphi_i(\tilde{U}_p)\). Consider now the extension of \(\mathcal{M}(x', g)\) as an endomorphism in \(\mathcal{C}^1[\mathbb{R}^n_{\xi}]\) to the symmetric algebra \(S(\mathcal{C}^1[\mathbb{R}^n_{\xi}] \simeq \mathbb{C}[\mathbb{R}^n_{\xi}]\). Since \(\mathcal{M}(x', g)\) is invertible, its extension to \(S^N(\mathcal{C}^1[\mathbb{R}^n_{\xi}])\) is also an automorphism for any \(N \in \mathbb{N}\). Regarding the polynomials \(\xi_1, \ldots, \xi_n\) as a basis in \(\mathcal{C}^1[\mathbb{R}^n_{\xi}]\), let us denote the image of the basis vector \(\xi_j\) under the endomorphism \(\mathcal{M}(x', g)\) by \(\mathcal{M}_{\xi_j}\), so that by (7)

\[
\mathcal{M}_{\xi_k} = -i\psi_{\xi,x}^i(g) dR(X_{j_k}) x_{i,i'}^{-1}(x'(g)), \quad 1 \leq k \leq n.
\]

In this way, each polynomial \(\xi_{j_1} \otimes \cdots \otimes \xi_{j_n} \equiv \xi_{j_1} \cdots \xi_{j_n}\) can be written as a linear combination

\[
(8) \quad \xi^\alpha = \sum_{\beta} \Lambda^\alpha_{\beta}(x', g) \mathcal{M}_{\xi_{j_1}} \cdots \mathcal{M}_{\xi_{j_n}},
\]
where the $\Lambda^\alpha_{\beta}(x',g)$ are smooth functions given in terms of the matrix coefficients of $\mathcal{M}(x',g)$. We now have for arbitrary indices $\beta_1, \ldots, \beta_r$ and all $x' \in \varphi_1(\tilde{U}_p)$,

$$i^r \psi^i_{\xi',x'}(g) \mathcal{M}_{\beta_1} \cdots \mathcal{M}_{\beta_r} = dR(X_{\beta_1} \cdots X_{\beta_r}) \psi^i_{\xi',x'}(g)$$

(9)

$$+ \sum_{s=1}^{r-1} \sum_{\alpha_1, \ldots, \alpha_s} d^\alpha_{\beta_1, \ldots, \beta_s}(x',g) dR(X_{\alpha_1} \cdots X_{\alpha_s}) \psi^i_{\xi',x'}(g),$$

where the coefficients $d^\alpha_{\beta_1, \ldots, \beta_s}(x',g)$ are smooth functions given by the matrix coefficients of $\mathcal{M}(x',g)$ which are at most of exponential growth in $g$ and independent of $\xi$; see Lemma 4 in [23]. The key step in proving the theorem is that, as an immediate consequence of equations (8) and (9), we can express $(1 + |\xi|^2)^N$ as a linear combination of derivatives $dR(X^\alpha) \psi^i_{\xi',x'}(g)$, obtaining for arbitrary $N \in \mathbb{N}$ and $x' \in \varphi_1(\tilde{U}_p)$ the equality

$$\psi^i_{\xi',x'}(g)(1 + |\xi|^2)^N = \sum_{r=0}^{2N} \sum_{|\alpha|=r} b^\alpha_{\xi}(x',g) dR(X^\alpha) \psi^i_{\xi',x'}(g),$$

(10)

where the coefficients $b^\alpha_{\xi}(x',g)$ are at most of exponential growth in $g$. Let us now show that $a^\alpha_{ij}(x,\xi; k_1, k_2) \in S^{-\infty}(W_x \times \mathbb{R}^n_\xi)$ for each fixed $k_1, k_2 \in K$. Note that $a^\alpha_{ij}(x,\xi; k_1, k_2) \in C^{\infty}(W_x \times \mathbb{R}^n_\xi \times K \times K)$. While differentiation with respect to $\xi$ does not alter the growth properties of the functions $a^\alpha_{ij}(x,\xi; k_1, k_2)$, differentiation with respect to $x$ yields additional powers in $\xi$. As one computes, $(\partial^\alpha_x \partial^\beta_x a^\gamma_{ij})(x,\xi; k_1, k_2)$ is a finite sum of terms of the form

$$\xi^\delta e^{-i\xi^2 k_1 k_2}(x) \int_G \psi^i_{\xi,\varphi}(g) (L(k_1)R(k_2)^{-1}) f(g) d^\delta_{\gamma,\beta''}(x,k_1, k_2,g)(\partial^\alpha_x \partial^\beta_x c_i)(x, g)$$

$$\cdot \partial^\beta_x [a^\delta_{x'}(\varphi^{-1}(x) \cdot k_1 k_2)]dg,$$

the functions $d^\delta_{\gamma,\beta''}(x,k_1, k_2,g)$ being at most of exponential growth in $g$. Next let $f_1 \in S(G)$, and assume that $f_2 \in C^\infty(G)$, together with all its derivatives, is at most of exponential growth. Then, by [23 Proposition 1], we have

$$\int_G f_1(g)dL(X^i)f_2(g)dg = (-1)^{|\gamma|} \int_G dL(X^i) f_1(g)f_2(g)dg,$$

(11)

where for $X^i = X^i_{x_1} \ldots X^i_{x_\gamma}$ we wrote $X_\gamma = X^i_{x_1} \ldots X^i_{x_\gamma}$, $i$ being an arbitrary multi-index, and similar expressions for the derivatives $dR$. Now let $\mathcal{O}$ denote an arbitrary compact set in $W_x$. By Heine–Borel, $\varphi^{-1}(\mathcal{O})$ can be covered by a finite number of neighborhoods $\tilde{U}_p$. Making use of equation (10) and integrating according to (11), we obtain for arbitrary multi-indices $\alpha, \beta$ the estimate

$$|(\partial^\alpha_x \partial^\beta_x a^\gamma_{ij})(x,\xi; k_1, k_2)| \leq \frac{1}{(1 + |\xi|^2)^N} C_{\alpha,\beta,\xi} \varphi^{-1}(\mathcal{O}), \quad x \in \mathcal{O},$$

where $N \in \mathbb{N}$, since $L(k_1)R(k_2)^{-1} f \in S(G)$. This proves that $a^\alpha_{ij}(x,\xi; k_1, k_2) \in S^{-\infty}(W_x \times \mathbb{R}^n_\xi)$ for each fixed $k_1, k_2 \in K$. Since equation (11) is an immediate consequence of the Fourier inversion formula, the proof of the theorem is now complete. \qed

Let $dM$ be a fixed density on $M$, and denote by $L^2(M)$ the space of square integrable functions on $M$. In the case that $M$ is compact, the fact that the integral
operators $\pi(f)$ have smooth kernels implies that they are trace-class operators in $L^2(M)$. Indeed, one has the following:

**Lemma 2.** Let $X$ be a compact manifold of dimension $n$ with volume form $dX$. Let $k : X \times X \to \mathbb{C}$ be a kernel function of class $C^{(n+1)}(X \times X)$. Then the operator $$(Kf)(p) = \int_X k(p, q)f(q)dX(q), \quad f \in L^2(X, dX),$$ is trace class, and $tr K = \int_X k(p, p)dX(p)$.

**Proof.** See [18, Lemma 2.2]. \qed

In our situation, we obtain

**Corollary 1.** Let $M$ be a compact $C^\infty$-manifold of dimension $n$, and $G$ a connected, real, semisimple Lie group with finite center acting on $M$ in a transitive way. If $f \in S(G)$, then $\pi(f)$ is a trace class operator in $L^2(M)$ and

$$tr \pi(f) = \sum \int_{W_i} (\alpha_i \circ \varphi_i^{-1})(x)K_{\alpha_i}(x, x)dx$$

$$= \sum \int_{M} \alpha_i(p)K_{\alpha_i}(\varphi_i(p), \varphi_i(p))j_i(p)dM(p),$$

where $dx$ denotes Lebesgue measure in $\mathbb{R}^n$, and $(\varphi_i^{-1})^*(dM) = (j_i^{-1} \circ \varphi_i^{-1})dx$.

**Proof.** By Theorem 1, $K_f \in C^\infty(M \times M, \mathbb{C} \otimes \Omega_M)$. Locally, the kernel $K_f$ is determined by the smooth functions $\phi$. Restricting the latter to the respective diagonals in $W_i$, one obtains a family of functions on $M$,

$$k_f(p) = K_{\alpha_i}(\varphi_i(p), \varphi_i(p)), \quad p \in \tilde{W}_i,$$

which define a density $k_f dM \in C^\infty(M, \Omega_M)$ on $M$. Since $M$ is compact, it can be integrated, and by Lemma 2 we get

$$tr \pi(f) = \int_{M} k_f(p)dM(p) = \sum \int_{W_i} (\alpha_i \circ \varphi_i^{-1})(x)K_{\alpha_i}(x, x)dx$$

$$= \sum \int_{\tilde{W}_i} \alpha_i(p)k_f(p)j_i(p)dM(p),$$

where we wrote $(\varphi_i^{-1})^*(dM) = (j_i^{-1} \circ \varphi_i^{-1})dx$. \qed

3. **Equivariant heat asymptotics**

From now on, let $M$ be a closed Riemannian manifold of dimension $n$ and $G$ a connected, real, semisimple Lie group with finite center acting smoothly and transitively on $M$. Denote by $dM$ the canonical density on $M$ induced by the Riemannian metric. Consider further an arbitrary compact subgroup $K$ of $G$ acting isometrically on $M$, and let $\tilde{K}$ denote the set of all equivalence classes of unitary irreducible representations of $K$. Let $(\pi, V_\sigma)$ be a unitary, irreducible representation of $K$ of dimension $d_\sigma$ belonging to $\sigma \in \tilde{K}$, and $\chi_\sigma(k) = tr \pi(k)$ the corresponding character. As a unitary representation of $K$, $(\pi, L^2(M))$ decomposes into isotypic components according to

$$L^2(M) \simeq \bigoplus_{\sigma \in \tilde{K}} L^2(M)_\sigma,$$
where \( L^2(M)_\sigma = P_\sigma(L^2(M)) \), and \( P_\sigma = d_\sigma \int_K \overline{\chi_\sigma(k)} \pi(k) \, dk \) is the corresponding projector in \( L^2(M) \), \( dk \) being the Haar measure on \( K \) induced by the restriction of \( \langle \cdot, \cdot \rangle_\theta \) to \( \mathfrak{k} \times \mathfrak{k} \). Take \( f \in S(G) \), and consider the restriction \( P_\sigma \circ \pi(f) \circ P_\sigma \) of the integral operator \( \pi(f) \) to the isotypic component \( L^2(M)_\sigma \). As one computes, for \( \varphi \in L^2(M) \),

\[
[P_\sigma \circ \pi(f) \circ P_\sigma] \varphi(p) = d_\sigma \int_K \overline{\chi_\sigma(k)} \pi(f) \circ P_\sigma \varphi(p \cdot k) \, dk
\]

\[
= d_\sigma \int_K \int_G \overline{\chi_\sigma(k)} f(g) P_\sigma \varphi(p \cdot kg) \, dg \, dk
\]

\[
= d^2_\sigma \int_K \int_K \int_K \overline{\chi_\sigma(k)} f(g) \overline{\chi_\sigma(k_1)} \varphi(p \cdot kgk_1) \, dg \, dk \, dk_1.
\]

Since \( G \) is unimodular, one obtains

\[
P_\sigma \circ \pi(f) \circ P_\sigma = \pi(H_f^g),
\]

where \( H_f^g \in S(G) \) is given by

\[
H_f^g(g) = d^2_\sigma \int_K \int_K f(k^{-1}gk^{-1}) \overline{\chi_\sigma(k)} \chi_\sigma(k) \, dk \, dk_1;
\]

compare [2, Section 2]. Note that if \( f \) is \( K \)-bi-invariant, \( \pi(f) \) commutes with \( P_\sigma \), so that \( P_\sigma \circ \pi(f) \circ P_\sigma = P_\sigma \circ \pi(f) = \pi(f) \circ P_\sigma \). In Section 5, we shall also consider kernels of the form

\[
\int_K \int_K f(k^{-1}gk^{-1}) \sigma_{ij}(k) \sigma_{lm}(k_1) \, dk \, dk_1
\]

where \( \sigma_{ij}(k) = \langle \pi_\sigma(k) e_i, e_j \rangle \) are matrix elements of \( \pi \) with respect to a basis \( \{ e_i \} \) of \( V_\sigma \). Consider now the cotangent bundle \( \pi : T^*M \to M \), as well as the tangent bundle \( \tau : T(T^*M) \to T^*M \), and define on \( T^*M \) the Liouville form

\[
\Theta(\xi) = \tau(\xi) [\pi_\sigma(\xi)], \quad \xi \in T(T^*M).
\]

We then regard \( T^*M \) as a symplectic manifold with the standard symplectic form \( \omega = d\Theta \) and canonical volume form \( d(T^*M) = \sigma^n/n! \). Further, we will endow \( T^*M \) with the Sasaki metric, so that the corresponding Riemannian volume form coincides with \( d(T^*M) \). Let \( \{ (\tilde{W}_i, \varphi_i) \} \) be the atlas of \( M \) of the previous section.

Introducing on \( T^*\tilde{W}_i \) the coordinates

\[
\eta = \sum_i \xi_i (dx_i)_p \in T^*_p \tilde{W}_i, \quad \varphi_i(p) = (x_1, \ldots, x_n), \quad p \in \tilde{W}_i,
\]

one computes

\[
\omega \equiv \sum_i d\xi_i \wedge dx_i, \quad d(T^*\tilde{W}_i) \equiv d\xi_1 \wedge dx_1 \wedge \cdots \wedge d\xi_n \wedge dx_n.
\]

In particular, on \( \tilde{W}_i \) one has \( d(T^*\tilde{W}_i)(p, \xi) \equiv j_i(p) dM(p) d\xi \). With the notation as in the previous section we then have the following:

**Proposition 1.** Let \( f \in S(G) \) and \( \sigma \in \hat{K} \). Then \( \pi(H_f^g) \) is of trace class, and

\[
\text{tr} \, \pi(H_f^g) = \frac{d_\sigma}{(2\pi)^n} \sum_{i,\ell} \int_K \int_{T^*M} e^{i\Phi_i(p,\xi,k)} \alpha_i(p) \alpha_\ell(p \cdot k) \overline{\chi_\sigma(k)}
\]

\[
\cdot a^i_\ell(\varphi_i(p), \xi; k) d(T^*M)(p, \xi) \, dk,
\]
where we set \( \Phi_{\epsilon, i}(p, \xi, k) = (\varphi_i(p \cdot k) - \varphi_i(p)) \cdot \xi \), while \( a^i_{\alpha}(x, \xi; k_1, k_2) \in S^{-\infty}(W_1 \times \mathbb{R}^n) \) was defined in (6).

**Proof.** By Corollary 1, \( \pi(H^\sigma_f) \) is of trace class, and at the microlocal level one has

\[
\left[ \pi(H^\sigma_f)(u \circ \varphi_i) \right](\varphi_i^{-1}(x)) = A^i_{H^\sigma_f} u(x), \quad u \in C^\infty_c(W_i),
\]

where \( A^i_{H^\sigma_f} \) is given by (1). By the unimodularity of \( G \), together with (3) and (5), we get

\[
K^i_{A_{H^\sigma_f}}(x, y) = \int e^{i(x-y) \cdot \xi} a^i_{H^\sigma_f}(x, \xi) \, d\xi = \int \left[ \int_G e^{i(\varphi_i(x) - y) \cdot \xi} c_1(x, g) H^\sigma_f(g) \, dg \right] \, d\xi
= d^2 \int \left[ \int_G \int_K \int_K f(g) e^{i(\varphi_i(x) - y) \cdot \xi} c_1(x, k_1 g k) \chi_\sigma(k_1) \chi_\sigma(k) \, dk \, dk \, dg \right] \, d\xi.
\]

Let \( \psi \in C^\infty_c(\mathbb{R}^n, \mathbb{R}^+ \) be equal to 1 near the origin and \( \epsilon > 0 \). By Lebesgue’s theorem on bounded convergence,

\[
K^i_{A_{H^\sigma_f}}(x, y) = \lim_{\epsilon \to 0} \int e^{i(x-y) \cdot \xi} a^i_{H^\sigma_f}(x, \xi) \psi(\xi) \, d\xi,
\]

since \( a^i_{H^\sigma_f}(x, \xi) \) is rapidly falling in \( \xi \). Similar considerations hold for the local kernels of \( \pi(f) \circ P_\sigma \). Noting that the trace is invariant under cyclic permutations, and arguing as in the proof of Corollary 1, one obtains for

\[
\text{tr} \pi(H^\sigma_f) = \text{tr}(P_\sigma \circ \pi(f) \circ P_\sigma) = \text{tr}(\pi(f) \circ P_\sigma)
\]

the expression

\[
\lim_{\epsilon \to 0} d^2 \sum_{i, k} \int_{\tilde{W}} \int_G \int_K e^{i(\varphi_i(p \cdot g k) - \varphi_i(p)) \cdot \xi} f(g) c_1(x, g) \chi_\sigma(k) \psi(\xi) \, dk \, dg \, d\xi \, dM(p)
= \lim_{\epsilon \to 0} d^2 \sum_{i, k} \int_{\tilde{W}} \int_G \int_K e^{i(\varphi_i(p \cdot g k) - \varphi_i(p)) \cdot \xi} f(g) c_1(x, g) \chi_\sigma(k) \psi(\xi) \, dk \, dg \, d\xi \, dM(p)
= \lim_{\epsilon \to 0} d^2 \sum_{i, k} \int_{\tilde{W}} \int_G \int_K e^{i(\varphi_i(p \cdot g k) - \varphi_i(p)) \cdot \xi} f(g) c_1(x, g) \chi_\sigma(k) \psi(\xi) \, dk \, dg \, d\xi \, dM(p)
\]

where the change of order of integration is permissible, since everything is absolutely convergent. Note that we used the equality

\[
1 = \sum_{i} \alpha_i(p \cdot k) \alpha'_i(p \cdot k).
\]

Finally, it was shown in the proof of Theorem 1 that \( a^i_{\sigma_f}(\varphi_i(p), \xi; k_1, k) \) is rapidly falling in \( \xi \), so that we can pass to the limit under the integral sign, and the assertion follows.

In what follows, we shall address the case where \( f = f_t \in \mathcal{S}(G) \), \( t > 0 \), is the Langlands kernel of a semigroup generated by a strongly elliptic operator associated to the representation \( \pi \). Our main goal will be the derivation of asymptotics for

\[
\text{tr} \pi(H^\sigma_{f_t}) = \text{tr}(P_\sigma \circ \pi(f_t) \circ P_\sigma)
\]
as \( t \to 0^+ \). Thus, let \( \mathcal{G} \) be a Lie group and \((\pi, \mathcal{B})\) a continuous representation of \( \mathcal{G} \) in some Banach space \( \mathcal{B} \). Denote by \( \mathfrak{g} \) the Lie algebra of \( \mathcal{G} \) and by \( X_1, \ldots, X_d \) a basis of it. Consider further a strongly elliptic differential operator of order \( q \) associated to \( \pi \),

\[
\Omega = \sum_{|\alpha| \leq q} c_{\alpha} d\pi(X^\alpha),
\]

meaning that \( \text{Re}(-1)^{q/2} \sum_{\alpha=q} c_{\alpha} \xi^\alpha \geq \kappa |\xi|^q \) for all \( \xi \in \mathbb{R}^d \) and some \( \kappa > 0 \). The general theory of strongly continuous semigroups establishes that its closure generates a strongly continuous holomorphic semigroup of bounded operators which is given by

\[
S_t = \int_{\mathcal{G}} \pi(g) d\mu_t(g),
\]

the representation \( \pi \) being measurable with respect to the measures \( \mu_t \). The \( \mu_t \) are absolutely continuous with respect to Haar measure \( d\mathcal{G} \) on \( \mathcal{G} \) so that if we denote by \( f_t(g) \in L^1(\mathcal{G}, d\mathcal{G}) \) the corresponding Radon-Nikodym derivatives, one has an expression

\[
S_t = \pi(f_t) = \int_\mathcal{G} f_t(g) \pi(g) d\mathcal{G}(g), \quad t > 0.
\]

The function \( f_t(g) \in L^1(\mathcal{G}, d\mathcal{G}) \) is analytic in \( t \in \mathbb{R}^+ \) and \( g \in \mathcal{G} \), universal for all Banach representations, and one can show that \( f_t \in S(\mathcal{G}) \). Moreover, it satisfies the following \( L^\infty \) upper bounds. There exist constants \( a, b, c_1, c_2 > 0 \) and \( \omega \geq 0 \) such that

\[
|\langle dR(X^\alpha) \partial_t^l H_{f_t} \rangle|_{t=\tau} \leq ac_1^{|\alpha|} c_2^l |\alpha|! l! \tau^{-|\alpha|+d} \int_0^\tau e^{\omega \tau} e^{-b(|\theta|^q/\tau)^{1/(q-1)}} d\tau,
\]

for all \( \tau > 0, g \in \mathcal{G}, l \in \mathbb{N}, \) and multi-indices \( \alpha \). For a complete exposition of these facts the reader is referred to [27] pages 30, 152, and 209] or [29]. In what follows, we shall call \( f_t \) the \textit{Langlands, or group kernel} of the holomorphic semigroup \( S_t \). Returning to our situation, let \( \mathcal{G} = G \), and \( \pi \) be the regular representation of \( G \) on \( L^2(M) \). Let us mention that as a consequence of the bounds (19), we have the following:

**Corollary 2.** Let \( K \) be a maximal compact subgroup. Then there exist constants \( a, b, c_1, c_2 > 0 \) and \( \omega \geq 0 \) such that

\[
|\langle dR(X^\alpha) \partial_t^l H_{f_t} \rangle|_{t=\tau} \leq ac_1^{|\alpha|} c_2^l |\alpha|! l! \tau^{-|\alpha|+d} \int_0^\tau e^{\omega \tau} e^{-b(|\theta|^q/\tau)^{1/(q-1)}} d\tau,
\]

for all \( \tau > 0, g \in G, l \in \mathbb{N}, \) and multi-indices \( \alpha \). Here \( d(gK, hK) \) denotes the geodesic distance in \( G/K \) induced by the Killing form on \( \mathfrak{g} \).
Proof. Clearly,

$$|H_{f_t}(g)|_{t=\tau} \leq d_\sigma^2 \int_K \int_K |f_t(k^{-1} g k^{-1})| \, dk \, dk_1.$$  

According to (19) we therefore have

$$|H_{f_t}^\sigma(g)| \leq d_\sigma^2 a t^{-\frac{d}{q}} e^{\omega t} \int_K \int_K e^{-b\left(\frac{|k^{-1} g k^{-1}|}{d}\right)^{\frac{1}{q}}} \, dk \, dk_1,$$

where $|k^{-1} g k^{-1}| = d(k^{-1} g k^{-1}, e) = d(gk^{-1}, k_1)$. Put $\mathbb{X} = G/K$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. By restriction of the Killing form to $T_e \mathbb{X} \simeq \mathfrak{p}$ one obtains an invariant Riemannian metric on $\mathbb{X}$ such that the canonical projection map $G \rightarrow \mathbb{X}$ becomes a Riemannian submersion. Now, if $d(gK, hK)$ denotes the geodesic distance on $\mathbb{X},$

$$|g| = d(g, e) \geq d(gK, K), \quad g \in G;$$

compare [21] Theorem 3.1]. By applying similar arguments to the derivatives, the corollary follows.

We arrive now at the following:

**Theorem 2.** Consider a strongly elliptic differential operator $\Omega$ of order $q \geq 2$ associated to $(\pi, L^2(M))$ and the corresponding semigroup $S_t = \pi(f_t)$ with Langlands kernel $f_t$, $t > 0$. Further let $\beta \in C^\infty_c(G)$, $0 \leq \beta \leq 1$, have support in a sufficiently small neighborhood $U$ of $e \in G$ satisfying $U = U^{-1}$, and assume that $\beta = 1$ on a ball of radius $R > 0$ around $e$. Then, for $\sigma \in \hat{K}$,

$$\text{tr} \pi(H_{f_t}^\sigma) = \text{tr} \pi(H_{f_t}^\sigma_{\beta}) + O(t^\infty),$$

where

$$\text{tr} \pi(H_{f_t}^\sigma_{\beta}) = \frac{d_\sigma}{(2\pi)^{n+q/2}} \sum_i \int_K \int_{T^*M} e^{\Phi_i(p, \xi, k)/t^{1/2}} \alpha_i(p) \chi_{\sigma}(k)$$

$$\cdot b_{f_t}(\varphi_i(p), \xi/t^{1/2}, e, k) \, d(T^* M)(p, \xi) \, dk, \quad t > 0,$$

and

$$b_{f_t}(\varphi_i(p), \xi; k_1, k) = e^{-i\Phi_i(p, k_1 k)} \int_U e^{i\Phi_i(p, k_1 k)} \cdot c_{\beta}(\varphi_i(p), k_1 g k) f_i(g) \beta(g) \, dg$$

is rapidly decaying in $\xi$ and vanishes if $p \cdot k_1 k \notin \bar{W}_i$. Furthermore, for any multi-indices $\alpha, \beta, \delta_1, \delta,

$$|\partial_x^\alpha \partial_\xi^\beta \partial_{k_1}^\delta \partial_k^\delta[b_{f_t}(x, \xi/t^{1/2}; k_1, k)]| \leq C$$

for some constant $C > 0$ independent of $0 < t < 1$.

**Proof.** To determine the asymptotic behavior of $\text{tr} \pi(H_{f_t}^\sigma)$ as $t \to 0$ by means of Proposition [11] we first have to examine the $t$-dependence of the amplitude $a_{f_t}^\xi(\varphi_i(p), \xi; k_1, k)$ as $t \to 0$ for fixed $k, k_1 \in K$. Consider therefore for $f \in S(G)
the amplitudes
\[
1 \alpha_{f}^{\xi}(x, \xi; k_{1}, k_{2}) = e^{-i\varphi_{1}^{k}k_{2}(x)\xi} \alpha_{t}(\varphi_{t}^{-1} \cdot k_{1}k_{2}) \int_{G} e^{i\varphi_{1}^{k}k_{2}(x)\xi} c_{i}(x, k_{1}gk_{2}) f(g) \\
\cdot (1 - \beta)(g) dg,
\]
\[
2 \alpha_{f}^{\xi}(x, \xi; k_{1}, k_{2}) = e^{-i\varphi_{1}^{k}k_{2}(x)\xi} \alpha_{t}(\varphi_{t}^{-1} \cdot k_{1}k_{2}) \int_{G} e^{i\varphi_{1}^{k}k_{2}(x)\xi} c_{i}(x, k_{1}gk_{2}) f(g) \\
\cdot \beta(g) dg.
\]
Similarly to (10), one has for arbitrary $N \in \mathbb{N}$ the equality
\[
(20) \quad \psi_{\xi, x}^{i}(k_{1}gk_{2})(1 + |\xi|^{2})^{N} = \sum_{r=0}^{2N} \sum_{|\alpha|=r} b_{\alpha}^{N}(x, g, k_{1}, k_{2}) dR(X^{a})[\psi_{\xi, x}^{i}(k_{1}gk_{2})],
\]
where the coefficients $b_{\alpha}^{N}(x, g, k_{1}, k_{2})$ are at most of exponential growth in $g$. With (11) and (19) we obtain
\[
|1 \alpha_{f}^{\xi}(\varphi_{1}(p), \xi; k_{1}, k)| \leq c(1 + |\xi|^{2})^{-N} e^{\omega t_{t}^{2} - \frac{d}{2} |\xi|^2} e^{-bR^{\sigma/(q-1)}} [1 - q - 1] \\
\cdot \int_{G} e^{-|g|^{2}/(q-1)} e^{\kappa|g|} dg
\]
for small $t > 0$ and constants $b, c > 0, \kappa, \omega > 0$. Consequently, $1 \alpha_{f}^{\xi}(x, \xi; k_{1}, k)$ vanishes to all orders as $t \to 0$ or $|\xi| \to \infty$, provided that $\theta > 2$, and with Proposition 11 we obtain the equality
\[
\text{tr} \pi(H_{f_{,\beta}}^{\sigma}) = \text{tr} \pi(H_{f_{,\beta}}) + O(t^{N})
\]
\[
= \frac{d_{\sigma}}{(2\pi)^{n}} \sum_{i, \xi, \epsilon} \int_{K} \int_{T^{*}M} e^{i\Phi_{i}(p, \xi, k)} \alpha_{i}(p) \alpha_{i}(p \cdot k) \chi_{\sigma}(k) \\
\cdot 2 \alpha_{f}^{\xi}(\varphi_{1}(p), \xi; e, k) d(T^{*}M)(p, \xi) dk + O(t^{N})
\]
for any $N \in \mathbb{N}$. Let $\psi \in C_{c}^{\infty}(\mathbb{R}^{n}, \mathbb{R}^{+})$, be equal to 1 near the origin and $\varepsilon > 0$. Repeating the arguments in the proof of Proposition 11 with $f$ replaced by $f_{i, \beta}$, one obtains for $\text{tr} \pi(H_{f_{,\beta}}^{\sigma})$ the expression
\[
\lim_{\varepsilon \to 0} \frac{d_{\sigma}}{(2\pi)^{n}} \sum_{\xi} \int_{U} \int_{K} \int_{W_{i}} e^{i(\varphi_{1}(p \cdot gk) - \varphi_{i}(p \cdot k))\xi} e^{i(\varphi_{1}(p \cdot k) - \varphi_{i}(p))\xi} \\
\cdot f_{i}(g) \beta(g) \alpha_{i}(p \cdot gk) \overline{\chi_{\sigma}(k)} \psi(\varepsilon \xi) j_{i}(p) \ d\xi dM(p) dk dg.
\]
Here we took into account that $U \subset G$ can be chosen so small that for all $k_{1}, k_{2} \in K$, $g \in U$, and $\xi \in I$, $p \cdot k_{1}gk_{2} \in \text{supp} \alpha_{i}' \implies p \cdot k_{1}k_{2} \in \overline{W_{i}}'$, since $I$ can be assumed to be finite due to the compactness of $M$. Consequently $b'_{f}(x, \xi; k_{1}, k_{2}) = e^{-i\varphi_{1}^{k}k_{2}(x)\xi} \int_{U} e^{i\varphi_{1}^{k}k_{2}(x)\xi} c_{i}(x, k_{1}gk_{2}) f(g) \beta(g) dg$, $f \in S(G)$, is well defined, and $b'_{f}(\varphi_{1}(p), \xi; k_{1}, k) = 0$ for $p \cdot k_{1}k \notin \overline{W_{i}}'$. From the considerations in the proof of Theorem 11 and (20) it follows that $b'_{f}(x, \xi; k_{1}, k_{2}) \in S_{-\infty}(W_{i} \times \mathbb{R}^{n})$.
for arbitrary \( k_1, k_2 \in K \). Thus, we arrive at
\[
\text{tr} \pi (H_{t,\beta}) = \lim_{\varepsilon \to 0} d_{\sigma} \int_K \int_{W_t} \int e^{i(p, \xi - \Phi, p, \Phi)} \alpha_{\xi} (p) \chi (k) \psi (\varepsilon \xi) 
\cdot b_{f_t} (\xi, \xi ; \varepsilon) f_0 (p) d\xi dM (p) dk.
\]
By passing to the limit under the integral sign and performing the substitution \( \xi \to \xi / t^{1/q} \), one finally arrives at the desired result. To examine the \( t \)-dependence of the amplitude \( b_{f_t} (\xi, \xi / t^{1/q}; k_1, k) \) as \( t \to 0 \), introduce canonical coordinates on \( U \) according to
\[
\Psi: \mathbb{R}^d \ni \xi = (\zeta_1, \ldots, \zeta_d) \mapsto g = e^{\sum \zeta_i x_i} \in U.
\]
Consider further the Taylor expansion at \( \zeta = 0 \) of order \( N \),
\[
[\varphi_k (x) - \varphi_k (x)] = \sum_{|\alpha| > 0} c_\alpha (x, k_1, k) \zeta^\alpha,
\]
where the coefficients \( c_\alpha (x, k_1, k) \) depend smoothly on \( x, k_1 \), and \( k \). Furthermore, the coefficients of order \( N \) are measurable functions in \( \zeta \). Write \( \Psi^* (dg) = J (\zeta) d\zeta \). Performing the substitution \( \zeta \to t^{1/q} \zeta \) and taking into account the bounds \( [19] \), one computes with \( \tau = t^{1/q} \):
\[
|b_{f_t} (x, \xi / \tau; k_1, k)| = e^{\int_{\tau^{-1} \psi^{-1} (U)} e^{-i \sum_{|\alpha| > 0, j} c_\alpha (x, k_1, k) \zeta_j} f_0 (x, k_1 e^\tau \sum \zeta_i x_i, k)}
\cdot (f_1) (e^{\tau \sum \zeta_i x_i}, J (\zeta) d\zeta) \leq e^{\omega t} \int_{\mathbb{R}^d} e^{-b_1 |\zeta|^{q/(q-1)}} d\zeta
\]
for some constants \( b', c' > 0 \) and \( \omega > 0 \), where we took into account that there exists some constant \( C > 0 \) such that \( C^{-1} |\zeta| \leq |g| \leq C |\zeta| \) for all \( g \in U \). A similar examination of the derivatives finally yields the estimate
\[
|\partial_{x}^\alpha \partial_{\zeta}^\beta \partial_{k_1}^\gamma \partial_{k_2}^\delta [b_{f_t} (x, \xi / t^{1/q}; k_1, k)]| \leq C
\]
for some constant \( C > 0 \) independent of \( 0 < t < 1 \) and arbitrary indices \( \alpha, \beta, \delta_1, \delta \).

**Remark 2.** Note that since \( b_{f_t} \) is rapidly decaying in \( \xi \), for any \( N \in \mathbb{N} \) there exists a constant \( c_N > 0 \) such that
\[
|b_{f_t} (\xi, \xi / t^{1/q}; k_1, k)| \leq \frac{c_N}{(1 + |\xi|^{1/q})^N} = \frac{c_N t^{2N/q}}{(t^{2/q} + |\xi|)^N} \leq \frac{c_N t^{2N/q}}{|\xi|^{2N}}.
\]
Therefore, if \( \theta \in C_c^\infty (\mathbb{R}^n, [0, 1]) \) is a cut-off function such that \( \theta (\xi) = 1 \) for \( |\xi| \leq 1 \) and \( \theta (\xi) = 0 \) for \( |\xi| \geq 2 \), then
\[
\int_K \int_K \int_{T^{*} M} |b_{f_t} (\varphi, \xi / t^{1/q}; k_1, k)| (1 - \theta (\xi)) d(T^{*} M) (p, \xi) dk dk_1 \leq C_N t^{2N/q}
\]
for any \( N \in \mathbb{N} \) and suitable constants \( C_N \).
Summing up, we arrive at

**Corollary 3.** Let \( \sigma \in \hat{K} \), and \( \psi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^+ ) \) be equal to 1 near the origin. Further let \( t, \varepsilon > 0 \). Then

\[
\text{tr } \pi(H_{f_t}^\sigma) = \lim_{\varepsilon \to 0} \frac{d\sigma}{(2\pi)^{n/2}} \sum_i \int_K \int_{T^* M} e^{i\Phi_i(p,\xi,k)/t^{1/\eta}} \alpha_i(p) \frac{\chi_\sigma(k)}{t^{n/q}}
\]

\[
\cdot b^i_j(\varphi_i(p),\psi(p); e, k) \psi(\varepsilon \xi) d(T^* M)(p,\xi) dk + O(t^{\infty}).
\]

**Proof.** This is an immediate consequence of Theorem 2 and Lebesgue's theorem on bounded convergence. \( \square \)

4. **Singular equivariant asymptotics and resolution of singularities**

Corollary 3 showed that in order to describe the traces

\[
\text{tr } \pi(H_{f_t}^\sigma) = \text{tr}(P_\sigma \circ \pi(f_t) \circ P_\sigma) \quad \text{as } t \to 0^+,
\]

one has to study the asymptotic behavior of oscillatory integrals of the form

\[
I(\mu) = \int_K \int_{T^* \tilde{W}} e^{i\Phi(p,\xi,k)/\mu} a(p \cdot k, p,\xi,k) d(T^* M)(p,\xi) dk
\]

as \( \mu \to 0^+ \) by means of the stationary phase principle, where \( (\varphi, \tilde{W}) \) are local coordinates on \( M \), while \( a \in C_c^\infty(\mathring{W} \times T^* \mathring{W} \times K) \) is an amplitude which might depend on \( \mu \) and

\[
\Phi(p,\xi,k) = (\varphi(p \cdot k) - \varphi(p)) \cdot \xi.
\]

For this, let us note that \( K \) acts on \( T^* M \) in a Hamiltonian way, meaning that the infinitesimal generators of the action are Hamiltonian vector fields. More precisely, define for any element \( X \) in the Lie algebra \( \mathfrak{k} \) of \( K \) the function

\[
J_X : T^* M \to \mathbb{R}, \quad \eta \mapsto \Theta(\tilde{X})(\eta),
\]

where \( \tilde{X} \) denotes the fundamental vector field on \( T^* M \), respectively \( M \), generated by \( X \). Then \( \mathcal{L}_X \Theta = dJ_X + \iota_{\tilde{X}} \omega = 0 \). The corresponding symplectic momentum map is then given by

\[
\mathbb{J}_K : T^* M \to \mathfrak{k}^*, \quad \mathbb{J}_K(\eta)(X) = J_X(\eta);
\]

see [23, Section 4.5]. Let us next compute the critical set of the phase function \( \Phi \). Clearly, \( \partial_\xi \Phi(p,\xi,k) = 0 \) if, and only if, \( p \cdot k = p \). Write \( \varphi(p) = (x_1, \ldots, x_n) \), \( \eta = \sum \xi_i(dx_i)_p \in T^*_p \mathring{W} \). Assuming that \( p \cdot k = p \), one computes for any \( X \in \mathfrak{k} \),

\[
\frac{d}{ds} \left( \varphi(p \cdot ke^{-tX}) \cdot \xi \right)_{t=0} = \sum \xi_i \tilde{X}_p(x_i) = \sum \xi_i(dx_i)_p(\tilde{X}_p) = \eta(\tilde{X}_p) = \Theta(\tilde{X})(\eta) = \mathbb{J}_K(\eta)(X),
\]
so that \( \partial_{(k_1,k)} \Phi(p, \xi, k) = 0 \) if, and only if, \( J_K(\eta) = 0 \). A further computation shows that
\[
\partial_x \Phi(\varphi^{-1}(x), \xi, k) = [\partial_1(\varphi \circ k \circ \varphi^{-1})]_{x, \xi} - 1 |\xi = [(k)_{x}^* - 1] |\xi,
\]
so that \( \partial_p \Phi(p, \xi, k) = 0 \) amounts precisely to the condition \( k^* \xi = \xi \). Collecting everything together one obtains
\[
\text{Crit}(\Phi) = \{(p, \xi, k) \in T^* \tilde{W} \times K : (\Phi_p)(p, \xi, k) = 0\}\}
\]
(23)
where \( \Xi = J_K^{-1}(0) \) denotes the zero level of the momentum map of \( K \). Now, the major difficulty resides in the fact that, unless the \( K \)-action on \( T^*M \) is free, the considered momentum map is not a submersion, so that \( \Xi \) and \( \text{Crit}(\Phi) \) are not smooth manifolds. The stationary phase theorem can therefore not immediately be applied to the integrals \( I(\mu) \). Nevertheless, it was shown in [26] that by constructing a strong resolution of the set
\[
\mathcal{N} = \{(p, k) \in M \times K : p \cdot k = p\}
\]
a partial desingularization \( Z : \tilde{X} \to X = T^* M \times K \) of the set
\[
\mathcal{C} = \{(p, \xi, k) \in \Xi \times K : k \cdot (p, \xi) = (p, \xi)\}
\]
can be achieved, and applying the stationary phase theorem in the resolution space, an asymptotic description of \( I(\mu) \) can be obtained. More precisely, the map \( Z \) yields a partial monomialization of the local ideal \( I_{\Phi} = (\Phi) \) generated by the phase function (22) according to
\[
Z^*(I_{\Phi}) \cdot \mathcal{E}_{\tilde{X}, \tilde{X}} = \prod_j \sigma_j^{-l_j} \cdot Z^*_p(I_{\Phi}) : \mathcal{E}_{\tilde{X}, \tilde{X}},
\]
where \( \mathcal{E}_{\tilde{X}} \) denotes the structure sheaf of rings of \( \tilde{X} \), \( \sigma_j \) are local coordinate functions near each \( \tilde{x} \in \tilde{X} \) and \( l_j \) natural numbers. As a consequence, the phase function factorizes locally according to \( \Phi \circ Z \equiv \prod_j \sigma_j^{-l_j} \cdot \tilde{\Phi}^{wk} \), and one shows that the weak transforms \( \tilde{\Phi}^{wk} \) have clean critical sets. Asymptotics for the integrals \( I(\mu) \) are then obtained by pulling them back to the resolution space \( \tilde{X} \) and applying the stationary phase theorem to the \( \tilde{\Phi}^{wk} \) with the variables \( \sigma_j \) as parameters. As a consequence, one obtains

**Theorem 3.** Let \( M \) be a connected, closed Riemannian manifold and \( K \) a compact, connected Lie group acting isometrically on \( M \). Consider the oscillatory integral
\[
I(\mu) = \int_K \int_{T^* \tilde{W}} e^{i \Phi(p, \xi, k)} / a(p \cdot k, p, \xi, k) d(T^* M)(p, \xi) dk, \quad \mu > 0,
\]
where \( (\varphi, \tilde{W}) \) are local coordinates on \( M \), while \( a \in \mathcal{C}_c^\infty(\tilde{W} \times T^* \tilde{W} \times K) \) is an amplitude which might depend on the parameter \( \mu \) and \( \Phi(p, \xi, k) = (\varphi(p \cdot k) - \varphi(p)) \cdot \xi \). Furthermore, assume that for all multi-indices one has \( |\partial^\alpha a| \leq C \) with a constant \( C > 0 \) independent of \( \mu \). Then \( I(\mu) \) has the asymptotic expansion
\[
I(\mu) = (2\pi \mu)^{\kappa} \mathcal{L}_0 + O(\mu^{\kappa+1}(\log \mu^{-1})^{\Lambda-1}), \quad \mu \to 0^+,
\]
where $\kappa$ is the dimension of a $K$-orbit of principal type in $M$, $\Lambda$ the maximal number of elements of a totally ordered subset of the set of $K$-isotropy types, and the leading coefficient is given by

$$L_0 = \int_{\text{Reg} C} \frac{a(p \cdot k, p, \xi, k)}{|\det \Phi''(p, \xi, k) N(p, \xi, k) \text{Reg} C|^{1/2}} d(\text{Reg} C)(p, \xi, k),$$

where $\text{Reg} C$ denotes the regular part of $C$ and $d(\text{Reg} C)$ the induced volume density.

In particular, the integral over $\text{Reg} C$ exists.

Proof. See [26, Theorem 9.1].

As a consequence, one obtains the following asymptotic description as $t \to 0$ for $\text{tr} \pi(H_{f_t}^\sigma) = \text{tr}(P_\sigma \circ \pi(f_t) \circ P_\sigma)$.

**Theorem 4.** Let $M$ be a closed Riemannian manifold of dimension $n$, $G$ a connected, real, semisimple Lie group with finite center acting smoothly and transitively on $M$, and $K$ an arbitrary compact subgroup $K \subset G$ acting isometrically on $M$. Consider further a strongly elliptic differential operator $\Omega$ of order $q \geq 2$ associated to the regular representation $(\pi, L^2(M))$ of $G$ with heat semigroup $S_t = \pi(f_t)$, $t > 0$. Then, for $\sigma \in \hat{K}$,

$$\text{tr} \pi(H_{f_t}^\sigma) = \frac{d_\sigma}{(2\pi)^{n-\kappa t(n-\kappa)/q}} \sum_i \int_{\text{Reg} C} \alpha_i(p) b_i^t(\varphi_i(p), \xi / t^{1/q}; e, k) \chi_\sigma(k) \cdot \frac{d(\text{Reg} C)(p, \xi, k)}{|\det \Phi''(p, \xi, k) N(p, \xi, k) \text{Reg} C|^{1/2}} + O(t^{-(n-\kappa-1)/q} (\log t)^{\Lambda-1}),$$

where $\kappa$ is the dimension of a $K$-orbit of principal type in $M$ and $\Lambda$ the maximal number of elements of a totally ordered subset of the set of $K$-isotropy types.

Proof. This is an immediate consequence of Corollary 3 and Theorem 3, together with Lebesgue’s theorem on bounded convergence.

In general, it is not possible to obtain more explicit expressions for the leading term unless one has more knowledge about the Langlands kernel $f_t$ as $t \to 0$. In particular, the bounds (19) are not sufficient for this purpose. We shall therefore make the following assumption, which is likely to hold in many cases.

**Assumption 1.** In a small neighborhood of $e \in G$, $f_t$ has an asymptotic expansion of the form

$$f_t(g) \sim \frac{1}{td^{1/q}} e^{-b(\frac{|g|}{t})^{1/(q-1)}} \sum_{j=0}^{\infty} c_j(g) t^j, \quad |g| \ll 1,$$

where $b > 0$, and the coefficients $c_j$ are analytic in $g$.

Let $H \subset K$ be a principal isotropy subgroup, and denote by

$$[\pi_\sigma|_H : 1]$$

the multiplicity of the trivial representation in the restriction of the unitary irreducible representation $\pi_\sigma$ to $H$. We then have the following corollary.
Corollary 4. Let Assumption \(\mathbb{1}\) be fulfilled, and let \(J_G : T^*M \to \mathfrak{g}^*\) be the momentum map of the \(G\)-action on \(T^*M\). Then
\[
\text{tr } \pi(H^n_J) = \frac{d_\sigma c_0(\sigma)}{(2\pi)^{n-\kappa}t^{n-\kappa}/q} \text{vol}(\Xi/K) + O(t^{-(n-\kappa-1)/q})(\log t)^{\Lambda-1},
\]
where
\[
\tilde{\text{vol}}(\Xi/K) = \int_{\text{Reg} \Xi} \left( \int_{\mathfrak{g}} e^{-i\xi G(p,\xi)(X)} e^{-b\|X\|^q/(q-1)} dX \right) \frac{d(\text{Reg} \Xi)(p,\xi)}{\text{vol}(O(p,\xi))}
\]
represents a Gaussian volume of the symplectic quotient \(\Xi/K\). Here \(O(p,\xi)\) denotes the \(K\)-orbit in \(T^*M\) through \((p,\xi)\) and \(dX\) the Euclidean measure in \(\mathfrak{g}\) induced by \(\langle \cdot, \cdot \rangle_{\theta}\).

Remark 3. Note that the integral over \(\mathfrak{g}\) in \(\tilde{\text{vol}}(\Xi/K)\) corresponds to the composition of the \(\mathfrak{g}\)-Fourier transform of the Gaussian \(X \mapsto e^{-b\|X\|^q/(q-1)}\) with the momentum map \(J_G\). It is \(K\)-invariant due to the equivariance of \(J_G\) and the \(\text{Ad}(K)\)-invariance of \(\langle \cdot, \cdot \rangle_{\theta}\). Therefore, \(\text{vol}(\Xi/K)\) can be reduced to an integral over \(\text{Reg} \Xi/K\) and, by cotangent bundle reduction, to an integral over \(T^*(M'/K)\), where \(M'\) denotes the union of principal orbits in \(M\).

Proof. On \(\text{Reg} C\) we have \(p \cdot k = p\), so that
\[
b_f^i(\varphi_i(p), \xi; e, k) = e^{-i\varphi_i(p)\xi} \int_U e^{i\varphi_i(p-k^{-1}gk)\xi} \alpha_i'(p \cdot k^{-1}gk) f_i(g) \beta(g) \, dg = \int_U e^{i(\varphi_i(p-g) - \varphi_i(p))\xi} \alpha_i'(p \cdot g)(f_i \beta)(kgk^{-1}) \, dg,
\]

since we can assume that \(U\) is invariant under conjugation by \(K\), the notation being the same as in the proof of Theorem \(\mathbb{2}\). Consider now geodesic normal coordinates on \(U\),
\[
\Psi : \mathbb{R}^d \ni \zeta = (\zeta_1, \ldots, \zeta_d) \mapsto g = \exp_e \left( \sum \zeta_i X_i \right) \in U,
\]

with respect to the left-invariant Riemannian metric on \(G\) given by \(\langle \cdot, \cdot \rangle_{\theta}\). We then have \(|g| = \|\sum \zeta_i X_i\|_{\theta}\) for \(g \in U\). With respect to these coordinates, Taylor expansion at \(\zeta = 0\) of order \(N\) yields
\[
[\varphi_i(p \cdot g) - \varphi_i(p)]_j = \sum_{|\alpha| > 0} c^j_\alpha(p) \zeta^\alpha, \quad g \in U, \quad p \in \widetilde{W},
\]

where the coefficients \(c^j_\alpha(p)\) depend smoothly on \(p\), and the ones of order \(N\) depend measurably on \(\zeta\). Write \(\Psi^*(dg) = J(\zeta) d\zeta \equiv J(X) dX\). Expanding again in
\( \tau = t^{1/q} \) at \( \tau = 0 \) gives, with Assumption 1

\[
\begin{align*}
 b_{f_t}^\prime (\varphi_t(p), \xi/\tau; e, k) &= \int_{\tau^{-1} \Psi^{-1}(U)} e^{i \sum_{|i| > 0, j} c_{i,j}(p)(\tau \xi_j/\tau) \alpha_i'(p \cdot \exp_e (\tau \sum \zeta_i X_i))} \\
&\cdot \tau^d (f_t \beta) \left( k \exp_e (\tau \sum \zeta_i X_i) k^{-1} \right) J(\tau \xi) d\zeta \\
&= \int_{\mathbb{R}^d} e^{i \sum_{|i| > 0, j} c_{i,j}(p)(\tau \xi_j/\tau) \alpha_i'(p \cdot \exp_e (\tau \sum \zeta_i X_i))} \\
&\cdot (c_0 \beta) \left( k \exp_e (\tau \sum \zeta_i X_i) k^{-1} \right) e^{-b \| \sum \zeta_i X_i \|_{\theta}^{q/(q-1)}} J(\tau \xi) d\zeta + O(t) \\
&= \alpha_i'(p) c_0(e) \int_{\mathbb{R}^d} e^{i \sum_{i,j} c_{i,j}(p) \xi_j} e^{-b \| \sum \zeta_i X_i \|_{\theta}^{q/(q-1)}} d\zeta + O(\tau),
\end{align*}
\]

all integrands involved being measurable and, due to the presence of the Gaussian factor, integrable. In addition, we took into account that by Lemma 1 we have \( |g| = |kgk^{-1}| \) for all \( g \in G \) and \( k \in K \) and that the differential of \( \exp_e \) at 0 is given by the identity; compare also \([14\text{, Chapter 1, Theorem 6.5}]\). Next, consider the symplectic momentum map of the group \( G \) given by

\[
\mathbb{J}_G : T^* M \to \mathfrak{g}^*, \quad \mathbb{J}_G(\eta)(X) = \Theta(\tilde{X})(\eta)
\]

(see \([23\text{, Section 4.5}]\)), where \( \Theta \) denotes the Liouville form on \( T^* M \) introduced in \([15]\). With respect to the coordinates \( \varphi_t(p) = (x_1, \ldots, x_n), \eta = \sum \xi_i(d x_i) p \in T^*_p \tilde{W}_t \), one computes for any \( X \in \mathfrak{g} \):

\[
\mathbb{J}_G(\eta)(X) = \Theta(\tilde{X})(\eta) = \eta(\tilde{X}_p) = \sum \xi_i(d x_i) p(\tilde{X}_p) = \sum \xi_i \tilde{X}_p(x_i).
\]

Since \( c_{i,j}^\prime(p) = (\tilde{X}_i)_p(x_j) \), one obtains

\[
\begin{align*}
 b_{f_t}^\prime (\varphi_t(p), \xi/t^{1/q}; e, k) &= \alpha_i'(p) c_0(e) \int_{\mathbb{R}^d} e^{i \sum \zeta_i \tilde{X}_i(p, \xi_i) e^{-b \| \sum \zeta_i X_i \|_{\theta}^{q/(q-1)}}} d\zeta
\end{align*}
\]

up to terms of order \( O(t^{1/q}) \). Note that due to the transitivity of the \( G \)-action, the matrix \( (\tilde{X}_i)_p(x_j) \) has maximal rank, so the leading coefficient is, indeed, rapidly falling in \( \xi \). Let us now remark that for any smooth, compactly supported function \( u \) on \( \Xi \cap T^* \tilde{W}_t \) and any \( v \in C^\infty(K) \) one has the formula

\[
\int_{\text{Reg} \mathcal{C}} \frac{v(k) u(p, \xi) d(\text{Reg} \mathcal{C})(p, \xi, k)}{\det \Phi_{\mathfrak{g}}(p, \xi, k) |\mathcal{N}_{(p, \xi, k)} \text{Reg} \mathcal{C}|^{1/2}}
\]

\[
= \int_{\mathcal{H}} v(k) dk \cdot \int_{\text{Reg} \Xi} u(p, \xi) d(\text{Reg} \Xi)(p, \xi) \frac{d(\text{Reg} \Xi)(p, \xi)}{\text{vol} \mathcal{O}(p, \xi)}
\]

(compare \([7\text{, Lemma 7}]\)), where \( H \subset K \) is a principal isotropy group of the \( K \)-action on \( M \) and \( \mathcal{O}(p, \xi) \) the \( K \)-orbit in \( T^* M \) through \( (p, \xi) \). In particular,

\[
\int_{\mathcal{H}} \chi_{\sigma}(k) dk = [\pi_{\sigma}|_H : 1],
\]

where \([\pi_{\sigma}|_H : 1]\) denotes the multiplicity of the trivial representation in the restriction to \( H \) of the unitary irreducible representation \( \pi_{\sigma} \). The assertion now follows from Theorem 4. \( \square \)
To motivate Assumption 1 and to illustrate our results, let us consider the classical heat kernel on $G$. Thus, consider a Cartan decomposition of $g$ as in (1), and let $X_1, \ldots, X_m$ be an orthonormal basis of $\mathfrak{p}$ and $Y_1, \ldots, Y_l$ an orthonormal basis for $\mathfrak{k}$ with respect to $\langle \cdot, \cdot \rangle_\theta$. If $\Omega$ and $\Omega_K$ denote the Casimir elements of $G$ and $K$, $K$ being here the maximal compact subgroup corresponding to $\mathfrak{k}$, one has

$$
\Omega = \sum_{i=1}^m X_i^2 - \sum_{i=1}^l Y_i^2, \quad \Omega_K = -\sum_{i=1}^l Y_i^2.
$$

Let

$$
P = -\Omega + 2\Omega_K = -\sum_{i=1}^m X_i^2 - \sum_{i=1}^l Y_i^2.
$$

As it turns out, $dR(P)$ is the Beltrami-Laplace operator $\Delta_G$ on $G$ with respect to the left invariant metric defined by $\langle \cdot, \cdot \rangle_\theta$. $dR(P)$ is a strongly elliptic operator associated to $R$, and it generates a strongly continuous semigroup which coincides with the classical heat semigroup $e^{-t\Delta_G}$, whose kernel $p_t$ is given by the corresponding universal Langlands kernel. In particular,

$$(27) \quad e^{-t\Delta_G} = R(p_t);$$

see [21, Section 3]. Let us now recall that on Riemannian manifolds admitting a properly discontinuous group of isometries with compact quotient, a fundamental solution of the heat equation with Gaussian bounds can be constructed explicitly [11]. Furthermore, every real, semisimple Lie group possesses a discrete, torsion-free subgroup with compact quotient [5]. If therefore $H(t, g, h)$ is the fundamental solution of the heat equation $\partial / \partial t + \Delta_G$ on $G$ constructed in this way, the Gaussian bounds imply that it is essentially given by the Langlands kernel $p_t(g^{-1}h)$. Furthermore, one has an asymptotic expansion of the form

$$
H(t, g, h) \sim (4\pi t)^{-d/2} e^{-\frac{d^2(g, h)}{4}} \sum_{j=0}^\infty t^j u_j(g, h),
$$

valid in a sufficiently small neighborhood of the diagonal in $G \times G$; see [11] Theorem 3.3. As before, $d(g, h)$ denotes the geodesic distance between two points with respect to the left-invariant metric on $G$, and $u_0(g, g) = 1$. Corollary 4 then implies

**Corollary 5.** Let $\Delta_G$ be the Laplace-Beltrami operator on $G$, and let $p_t \in \mathcal{S}(G)$ be its heat kernel. Consider further an arbitrary compact subgroup $K$ of $G$ and $\sigma \in \hat{K}$. Then

$$
\text{tr} \pi(H^\sigma_{p_t}) = \frac{d_\sigma(4\pi)^{-d/2}}{(2\pi)^{n-k} t^{(n-k)/2}} |\pi_\sigma[H : 1]| \tilde{\text{vol}}(\Xi/K) + O(t^{-(n-k-1)/2}(\log t)^{\lambda-1}),
$$

where the notation is as in Corollary 4. Note that in this case

$$
\tilde{\text{vol}}(\Xi/K) = (4\pi)^{d/2} \int_{\text{Reg} \Xi} e^{-\|\mathcal{B}_G(p, \xi)\|^2} \frac{d(\text{Reg} \Xi)(p, \xi)}{\text{vol} \mathcal{O}(p, \xi)}.
$$

**5. Spaces of automorphic forms**

In this section, we apply the previous analysis to spaces of automorphic forms. In particular, we recover asymptotics for heat traces of Bochner-Laplace operators on compact, locally symmetric spaces. In the rank one case, this problem was already considered by Miatello [18] and DeGeorge and Wallach [10]. As before, let $G$ denote
a connected, real, semisimple Lie group with finite center and $\Gamma$ a discrete, uniform subgroup of $G$. Consider $M = G \backslash G$, endowed with the invariant measure induced by the measure $dg$ on $G$, and denote by $\pi_{\Gamma}(g)\varphi(h) = \varphi(hg)$, $g, h \in G$, the right regular representation of $G$ in the space $L^2(\Gamma\backslash G)$ of square integrable functions on $\Gamma\backslash G$. Since $\Gamma \backslash G$ is compact, the right regular representation decomposes discretely according to

$$\pi_{\Gamma} \simeq \bigoplus_{\varphi \in \hat{G}} m_\varphi \varphi,$$

where $\hat{G}$ stands for the set of equivalence classes of irreducible unitary representations of $G$, $(\pi_\varphi, H_\varphi) \in \varphi$, and $m_\varphi < \infty$ denotes the multiplicity of $\varphi$ in $(\pi_{\Gamma}, L^2(\Gamma\backslash G))$. For $f \in C_c^\infty(G)$, the Bochner integral $\pi_{\Gamma}(f) = \int_G f(g)\pi_{\Gamma}(g) \, dg$ defines a bounded operator on $L^2(\Gamma\backslash G)$ whose kernel is given by the $C^\infty$ function

$$k_f(g, h) = \sum_{\gamma \in \Gamma} f(g^{-1}\gamma h), \quad g, h \in G,$$

the series converging uniformly on compacta. The regularity of the kernel implies that $\pi_{\Gamma}(f)$ is of trace class, and

$$\text{tr} \, \pi_{\Gamma}(f) = \int_{\Gamma \backslash G} k_f(g, g) \, dg = \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} f(g^{-1}\gamma g) \, dg.$$

Note that by slightly abusing notation, we are denoting the invariant measure on $\Gamma \backslash G$ also by $dg$. If $f \in L^1(G)$, the operator $\pi_{\Gamma}(f)$ is still defined, but might not be of trace class. If $f \in S(G)$ is rapidly falling, it was shown in Theorem 1 that $\pi_{\Gamma}(f)$ is a smooth operator, which by Corollary 1 implies that it has a well-defined trace. As in the case of a compactly supported $f$, one can show that for $f \in S(G)$ the kernel of $\pi_{\Gamma}(f)$ is given globally by the expression (28) and that it satisfies Selberg’s trace formula. Indeed, one has the following:

**Lemma 3.** Let $f \in S(G)$ be a rapidly decaying function on $G$. Then the series $k_f(g, h) = \sum_{\gamma \in \Gamma} f(h^{-1}\gamma g)$ converges uniformly on compacta to a $C^\infty$ function and represents the integral kernel of the bounded operator $\pi_{\Gamma}(f) : L^2(\Gamma\backslash G) \to L^2(\Gamma\backslash G)$.

**Proof.** By Definition 1 for all $\kappa > 0$ we have the inequality

$$|f(h^{-1}\gamma g)| \leq C_\kappa e^{-\kappa |h^{-1}\gamma g|}, \quad g, h \in G,$$

as well as for all derivatives of all orders of $f$. Consequently,

$$\sum_{\gamma \in \Gamma} |f(h^{-1}\gamma g)| \leq \sum_{\gamma \in \Gamma} e^{-\kappa d(h^{-1}\gamma g, e)} = \sum_{\gamma \in \Gamma} e^{-\kappa d(\gamma g, h)},$$

since left-translation by $h$ is an isometry. Now, recall that for a metric space $(X, d)$ and a discrete, infinite subgroup $\Gamma' \subset \text{Iso}(X)$ of the isometry group of $X$ the corresponding Poincaré series is defined by

$$P(s, p, q) = \sum_{\gamma \in \Gamma'} e^{-s d(p, \gamma q)}, \quad p, q \in X, \quad s > 0.$$

By general theory [22], for each discrete subgroup $\Gamma'$, there exists a $\delta_{\Gamma'} > 0$, called the critical exponent of $\Gamma'$, such that $P(s, p, q)$ converges for $s > \delta_{\Gamma'}$ and diverges for $s < \delta_{\Gamma'}$. Furthermore, the exponent $\delta_{\Gamma'}$ does not depend on $p$ or $q$. The estimate

More precisely, $\pi_{\Gamma}(g)\varphi(\Gamma h) = \varphi(\Gamma hg)$, where $\Gamma h \in \Gamma \backslash G$, $g \in G$. 

$$\sum_{\gamma \in \Gamma} |f(h^{-1}\gamma g)| \leq \sum_{\gamma \in \Gamma} e^{-\kappa d(h^{-1}\gamma g, e)} = \sum_{\gamma \in \Gamma} e^{-\kappa d(\gamma g, h)},$$

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\[\sum_{\gamma \in \Gamma} |f(h^{-1}\gamma g)| \leq \sum_{\gamma \in \Gamma} e^{-\kappa d(h^{-1}\gamma g, e)} = \sum_{\gamma \in \Gamma} e^{-\kappa d(\gamma g, h)},\]

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$$P(s, p, q) = \sum_{\gamma \in \Gamma'} e^{-s d(p, \gamma q)}, \quad p, q \in X, \quad s > 0.$$
Corollary 6. Let \( f \in \mathcal{S}(G) \). Then \( \text{tr} \pi_f \) satisfies the Selberg trace formula

\[
(33) \quad \bigoplus_{\gamma \in \hat{G}} m_\gamma \text{tr} \pi_\gamma(f) = \sum_{[\gamma]} \text{vol}(\Gamma \setminus G, \gamma) \int_{G,\gamma} f(g^{-1} \gamma g) dg,
\]

where \([\gamma]\) denotes the conjugacy class of \( \gamma \) in \( \Gamma \), and \( \Gamma, \gamma \) and \( G, \gamma \) are the centralizers of \( \gamma \) in \( \Gamma \) and \( G \), respectively.

Proof. Lemmas 2 and 3 yield

\[
\text{tr} \pi_f = \int_{\Gamma \setminus G} k_f(g, g) dg = \int_{\Gamma \setminus G} \sum_{\gamma \in \Gamma} f(g^{-1} \gamma g) dg.
\]
Denoting by $[\gamma]$ the conjugacy class of $\gamma$ and by $\Gamma_\gamma$ the centralizer of $\gamma$ in $\Gamma$, one deduces
\[
\text{tr} \, \pi_\Gamma(f) = \int_{\Gamma \setminus G} \sum_{[\gamma]} \sum_{\delta \in \Gamma_\gamma \setminus \Gamma} f(g^{-1} \delta^{-1} \gamma \delta g) \, dg = \sum_{[\gamma]} \int_{\Gamma \setminus G} \sum_{\delta \in \Gamma_\gamma \setminus \Gamma} f(g^{-1} \delta^{-1} \gamma \delta g) \, dg,
\]
everything being uniformly convergent. Replacing the inner sum by an integral with a counting measure $d\delta$ yields
\[
\text{tr} \, \pi_\Gamma(f) = \sum_{[\gamma]} \int_{\Gamma \setminus G} \int_{\Gamma_\gamma \setminus \Gamma} f(g^{-1} \delta^{-1} \gamma \delta g)d\delta \, dg = \sum_{[\gamma]} \int_{\Gamma_\gamma \setminus G} f(y^{-1} \gamma y)dy,
\]
where we took into account that for any sequence $G_1 \subset G_2 \subset G$ of unimodular groups, an invariant measure on $G_1 \setminus G$ can be written as the product of invariant measures on $G_2 \setminus G$ and $G_1 \setminus G_2$, respectively. With the same argument, the above equality can be rewritten as
\[
\text{tr} \, \pi_\Gamma(f) = \sum_{[\gamma]} \int_{G_\gamma \setminus G} \int_{\Gamma_\gamma \setminus G} f(v^{-1} u^{-1} \gamma uv)dudv,
\]
where $G_\gamma$ denotes the centralizer of $\gamma$ in $G$. Since $u^{-1} \gamma u = \gamma$ and $\Gamma_\gamma \setminus G_\gamma$ is compact, one finally obtains the geometric side of the trace formula
\[
\text{tr} \, \pi_\Gamma(f) = \sum_{[\gamma]} \text{vol}(\Gamma \setminus G_\gamma) \int_{G_\gamma \setminus G} f(g^{-1} \gamma g) \, dg.
\]
To obtain the spectral side, note that according to the decomposition $\mathbb{K}$, we have
\[
\text{tr} \, \pi_\Gamma(f) = \bigoplus_{\varrho \in \hat{G}} m_\varrho \text{tr} \, \pi_\varrho(f),
\]
where $\pi_\varrho(f) = \int_G f(g) \pi_\varrho(g) \, dg$ is of trace class and defines a distribution
\[
\theta_\varrho : \text{C}_c^\infty(G) \ni f \mapsto \text{tr} \, \pi_\varrho(f) \in \mathbb{C}
\]
on $G$ which represents the global character of $\varrho$. Selberg’s trace formula for $f \in \mathcal{S}(G)$ now follows. \qed

Consider next an arbitrary compact subgroup $K$ of $G$ and $\sigma \in \hat{K}$. As was shown in Lemma 1 with respect to the Riemannian structure induced by $\langle \cdot, \cdot \rangle_\theta$, one can assume that $K$ acts on $G$, and consequently on $\Gamma \setminus G$, from the right in an isometric way. Theorem and Corollary of the previous section then immediately imply

**Theorem 5.** Let $\pi_\Gamma$ be the right regular representation of $G$ on $L^2(\Gamma \setminus G)$ and $f_t \in \mathcal{S}(G)$, $t > 0$, the Langlands kernel of a semigroup generated by a strongly elliptic operator associated to the representation $\pi_\Gamma$. Then
\[
\text{tr} \, \pi_\Gamma(H^\sigma_{f_t}) = \frac{d_\sigma}{(2\pi)^{\dim_G/K} \ell^{\dim_G/K}} \sum_{\ell} \int_{\text{Reg} \mathbb{C}} \alpha_\ell(\Gamma g) b_{f_t}(\varphi_\ell(\Gamma g), \xi/t^{1/q}; e, k)
\]
\[
\cdot \frac{d(\text{Reg} \mathbb{C})(\Gamma g, \xi, k)}{\det \Phi''_{\ell \ell}(\Gamma g, \xi, k) N_{(\Gamma g, \xi, k)} \text{Reg} \mathbb{C}|^{1/2}}
\]
\[
+ O(t^{-(\dim_G/K - 1)/q} (\log t)^{A-1}).
\]
If $f_t$ satisfies Assumption 1, the leading term of $\text{tr} \, \pi_{1}(H_{f_t}^{\gamma})$ is given by

$$
d_{\sigma} c_0(e)[\pi_{\sigma}|_{H} : \mathbf{1}] \sim \frac{\text{vol}(\Xi/K)}{(2\pi)^{\dim G/K} t^{\dim G/K}}.
$$

Next, Selberg’s formula allows us to describe $L_{\sigma} f_t$ at the identity asymptotically as $t \to 0$, where $L_{\sigma}$ denotes the projector onto the isotypic component $L_{\sigma}^{\gamma}(G)$. 

**Proposition 2.** Let $f_t \in S(G)$, $t > 0$, be the Langlands kernel of a semigroup generated by a strongly elliptic operator associated to the representation $\pi_{1}$. Then

$$
(L_{\sigma} f_t)(e) = \frac{d_{\sigma}}{(2\pi)^{\dim G/K} \text{vol}(\Gamma \backslash G)} \prod_{\ell} \int_{\text{Reg} C} \alpha_{\ell}(\varphi_{t}(\Gamma g, \xi, t^{1/q}; e, k)) \cdot \chi_{\sigma}(k) \det \Phi_{\ell t}^{\gamma}(\Gamma g, \xi, k) \, \text{vol}(\Xi/K),
$$

up to terms of order $O(t^{-\dim G/K - 1}/(\log t)^{\Lambda - 1})$. If, in addition, Assumption 1 is satisfied, the leading term of $(L_{\sigma} f_t)(e)$ is given by

$$
\frac{d_{\sigma} c_0(e)[\pi_{\sigma}|_{H} : \mathbf{1}]}{(2\pi)^{\dim G/K} \text{vol}(\Gamma \backslash G)} \sim \frac{\text{vol}(\Xi/K)}{t^{\dim G/K}},
$$

the notation being as in Corollary 4.

**Proof.** By Theorem 2, $\text{tr} \, \pi_{1}(H_{f_t, \gamma}) = \text{tr} \, \pi_{1}(H_{f_t}^{\gamma}) + O(t^{\infty})$, where $0 \leq \gamma \leq 1$ is a test function on $G$ with support in a sufficiently small neighborhood $U$ of $e \in G$ and which is equal to 1 close to $e$. Next, recall that for any $r \in G$, the $G$-conjugacy class $[\gamma]_{G}$ is closed. Furthermore, every compactum in $G$ meets only finitely many $[\gamma]_{G}$; see Lemma 8.1. Consequently, by choosing the support of $\gamma$ sufficiently small, we obtain with Corollary 7,

$$
\text{tr} \, \pi_{1}(H_{f_t, \gamma}) = \text{vol}(\Gamma \backslash G) \cdot H_{f_t}^{\gamma}(e),
$$

and the assertion follows from the previous theorem. □

From now on, let $K$ be a maximal compact subgroup of $G$ and $\sigma \in \hat{K}$. In agreement with known results for the equivariant heat trace of the Beltrami-Laplace operator of a compact Riemannian manifold with a compact isometry group whose orbits do have all the same dimension 6, we now obtain

**Corollary 7.** Let $p_t$ denote the heat kernel of the Laplace-Beltrami operator on $G$ introduced in Theorem 2. We then have the expansion

$$
\text{tr} \, \pi_{1}(H_{p_t}^{\gamma}) = \frac{d_{\sigma} c_0(e)[\pi_{\sigma}|_{H} : \mathbf{1}]}{(4\pi t)^{\dim G/K}} \frac{\text{vol}(\Gamma \backslash G)}{K} + O(t^{-\dim G/K - 1}/(\log t)^{\Lambda - 1}),
$$

where $(\Gamma \backslash G)'$ denotes the union of $K$-orbits of principal type in $\Gamma \backslash G$.

**Proof.** We begin by noting that with respect to the identification $T^{*}G \simeq G \times g^{*}$, the momentum map on $T^{*}G$ is given by $(g, \mu) \mapsto \text{Ad}^{*}_{\mu \cdot g} \mu$. Now $(\cdot, \cdot)_{g}$ coincides with the Cartan-Killing form on $p \times p$. Taking into account the $\text{Ad}(G)$-invariance of the
latter and performing the inverse $p$-Fourier transform one computes $e^{-\|\sigma(Gg,\mu)\|_h^2}$ on $\Xi$. Using cotangent bundle reduction to $T^*((\Gamma\backslash G)^{\prime}/K)$ we obtain with Corollary 5

$$\widetilde{\text{vol}}(\Xi/K) = (4\pi)^{d/2} \int_{\text{Reg} \Xi} e^{-\|\sigma\|_h^2} \frac{d(\text{Reg} \Xi)(Gg,\mu)}{\text{vol} \mathcal{O}(Gg,\mu)} = (4\pi)^{d/2} \frac{\pi^{\dim G/K}}{2^d \pi^{\dim G/K}} \text{vol}(\Gamma\backslash G)^{\prime}/K,$$

and the assertion follows. \hfill \Box

We now apply our results to heat kernels of Bochner-Laplace operators on compact, locally symmetric spaces. Let $(\pi_{\sigma}, V_{\sigma})$ be an irreducible unitary representation of $K$ of class $\sigma \in \tilde{K}$. Consider the associated homogeneous vector bundle $\tilde{E}_{\sigma} = (G \times V_{\sigma})/K$ over $G/K$, and endow it with the $G$-invariant Hermitian metric induced by the inner product in $V_{\sigma}$. Let $g = \mathfrak{g} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ as in \cite{[1]}, and consider the unique $G$-invariant connection $\nabla$ on $\tilde{E}_{\sigma}$ given by the condition that if $s$ is a smooth cross-section, $Y \in \mathfrak{p}$, and $\Pi : G \to G/K$ is the canonical projection, then

$$\nabla_{\Pi^*} (Y) (s) = \frac{d}{dt} s(e^{tY} K)|_{s=0},$$

$\Pi^*$ being the differential of $\Pi$ at $e \in G$. Further let $\tilde{\Delta}_{\sigma} = \nabla^* \nabla$ be the Bochner-Laplace operator of $\nabla$, and denote by $C^\infty(\tilde{E}_{\sigma})$, $C^\infty_c(\tilde{E}_{\sigma})$ and $L^2(\tilde{E}_{\sigma})$ the usual spaces of sections of $\tilde{E}_{\sigma}$. With respect to the identification

$$C^\infty(\tilde{E}_{\sigma}) = (C^\infty(G) \otimes V_{\sigma})^K,$$

where $(C^\infty(G) \otimes V_{\sigma})^K = \{ \varphi : G \to V_{\sigma} \text{ is smooth and } \varphi(gk) = \pi_{\sigma}(k)^{-1} \varphi(g), k \in K, g \in G \},$ and the corresponding identifications for $C^\infty_c(\tilde{E}_{\sigma})$ and $L^2(\tilde{E}_{\sigma})$, one has

$$\tilde{\Delta}_{\sigma} = -dR(\Omega) \otimes \text{id} + \text{id} \otimes d\pi_{\sigma}(\Omega_K) = -dR(\Omega) \otimes \text{id} + \lambda_{\sigma} \text{id}$$

for some $\lambda_{\sigma} \geq 0$, $\Omega$ and $\Omega_K$ being the Casimir elements of $G$ and $K$, respectively; see \cite{[13] Proposition 1.1}. As it turns out, the operator $\tilde{\Delta}_{\sigma} : C^\infty_c(\tilde{E}_{\sigma}) \to L^2(\tilde{E}_{\sigma})$ is essentially self-adjoint and has a unique self-adjoint extension which we shall also denote by $\tilde{\Delta}_{\sigma}$. It is a positive operator, and we denote the corresponding heat semigroup by $e^{-t\tilde{\Delta}_{\sigma}}$. It is given by

$$(e^{-t\tilde{\Delta}_{\sigma}} \varphi)(g) = \int_G h_t^\sigma(g_1) \varphi(gg_1) \, dg_1, \quad \varphi \in (L^2(G) \otimes V_{\sigma})^K,$$

where $h_t^\sigma : G \to \text{End}(V_{\sigma})$ is square integrable and has the covariance property

$$h_t^\sigma(g) = \pi_{\sigma}(k) h_t^\sigma(k^{-1} g k_1) \pi_{\sigma}(k_1)^{-1}, \quad g \in G, k, k_1 \in K.$$ 

As one can show, $h_t^\sigma$ is actually given in terms of the classical heat kernel $p_t$ introduced in \cite{[27]} according to

$$(e^{t\lambda_{\sigma}}) \int_K \int_K p_t(k^{-1} g k_1) \pi_{\sigma}(k k_1^{-1}) \, dk_1 \, dk;$$

see \cite{[2]} and \cite{[21] Section 3}. Now let $\Gamma$ be a discrete, uniform, torsion-free subgroup of $G$. Then $\Gamma$ acts freely on $G/K$, and $\Gamma\backslash G/K$ constitutes a compact, locally symmetric space. Let $E_{\sigma} = \Gamma\backslash \tilde{E}_{\sigma} \to \Gamma\backslash G/K$ be the pushdown of the homogenous vector bundle $\tilde{E}_{\sigma} \to G/K$. Again, we have the identification

$$C^\infty(E_{\sigma}) = (C^\infty(\Gamma\backslash G) \otimes V_{\sigma})^K,$$
and similarly for $C_c^\infty(E_\sigma)$ and $L^2(E_\sigma)$. Since $\Delta_\sigma$ is $G$-invariant, it induces an elliptic, essentially self-adjoint operator $\Delta_\sigma = \nabla^* \nabla : C_c^\infty(E_\sigma) \to L^2(E_\sigma)$, where $\nabla$ is the pushdown of the canonical connection $\nabla$. Let $e^{-t\Delta_\sigma}$ be the corresponding heat semigroup. With respect to a basis $\{e_i\}$ of $V_\sigma$, we obtain with (35) and (36)

$$[(e^{-t\Delta_\sigma} \varphi)(g)]_{ij} = \sum_{k=1}^{\dim \sigma} \pi_T(j)^k H^\sigma_{ij}([\varphi(g)]_k, \varphi \in (L^2(\Gamma \setminus G) \otimes V_\sigma)^K),$$

where

$$j^k H^\sigma_{ij}(g) = e^{t\lambda_\sigma} \int_K \int_K p_t(k^{-1}gk_1)(\pi_\sigma(kk_1^{-1}))_{jk} dk_1 dk.$$ 

Thus, $e^{-t\Delta_\sigma}$ is given by the matrix of convolution operators $\pi_T(j^k H^\sigma_{ij})$. The group kernels $j^k H^\sigma_{ij}$ are essentially of the same form as the kernels $H^\sigma_{ij}$ defined in (14), and we arrive at

**Theorem 6.** Let $\sigma \in \hat{K}$ and $\Delta_\sigma$ be the Bochner-Laplace operator on the homogeneous vector bundle $E_\sigma = \Gamma \setminus (G \times V_\sigma)/K \to \Gamma \setminus G/K$. Then

$$\text{tr } e^{-t\Delta_\sigma} = \frac{e^{t\lambda_\sigma} \dim G/K}{(4\pi t)^{\dim G/K}} \text{vol } \Gamma \setminus G/K + O(e^{t\lambda_\sigma} t^{-(\dim G/K - 1)/2}).$$

**Proof.** With $\sigma_{ij}(k) = (\pi_\sigma(k)e_i, e_j)$ one computes

$$j^j H^\sigma_{ij}(g) = \sum_i e^{t\lambda_\sigma} \int_K \int_K p_t(k^{-1}gk_1^{-1})\sigma_{ij}(k_1) \sigma_{ji}(k_1) dk_1 dk.$$ 

Hence,

$$\text{tr } e^{-t\Delta_\sigma} = \text{tr } \pi_T \left( \sum_j j^j H^\sigma_{ij} \right) = e^{t\lambda_\sigma} \sum_{i,j} \text{tr } (Q_{ij} \circ \pi_T(p_t) \circ Q_{ji})$$

$$= e^{t\lambda_\sigma} \sum_{i,j} \text{tr } (\pi_{\gamma}(p_t) \circ Q_{ij} \circ Q_{ji}),$$

where we put $Q_{ij} = \int_K \alpha_{ij}(k) dk$; compare (13). As in Corollary 7 one then deduces

$$\text{tr } e^{-t\Delta_\sigma} = \frac{e^{t\lambda_\sigma} \dim G/K}{(4\pi t)^{\dim G/K}} \text{vol } (\Gamma \setminus G)^{\prime}/K \sum_{i,j} \int_H (\sigma_{ji} * \sigma_{ij})(k) dk$$

up to terms of order $O(e^{t\lambda_\sigma} t^{-(\dim G/K - 1)/2}(\log t)^{\Lambda - 1})$. Let us now note that the isotropy group of a point $\Gamma g \in \Gamma \setminus G$ of the $K$-action is conjugate to the finite group $gKg^{-1} \cap \Gamma$. If $G$ is torsion-free, meaning that no non-trivial element $\gamma \in \Gamma$ is conjugate in $G$ to an element of $K$, all isotropy groups of the $K$-action are trivial, so that $K$ acts freely on $\Gamma \setminus G$. This implies that the isotropy group of a point $\Gamma g \in \Gamma \setminus G$ of the $K = K \times K$ action is given by $H = \{e\}$. Consequently, $K$ acts on $\Gamma \setminus G$ with only one orbit type, so that $\Lambda = 1$. The assertion now follows, since $(\Gamma \setminus G)^{\prime} = \Gamma \setminus G$ and

$$\sum_{i,j} (\sigma_{ji} * \sigma_{ij})(e) = \sum_{i,j} \int_K \alpha_{ij}(k) \alpha_{ij}(k) dk = d_\sigma$$

by the orthogonality relations for matrix coefficients. □
Remark 4. Note that for the proof of Theorem 6, no resolution of singularities is needed, since $K$ acts on $\Gamma \backslash G$ with just one orbit type in case that $\Gamma$ has no torsion. Therefore, instead of Theorem 4, one could have used directly the generalized stationary phase theorem. Indeed, for generalized Laplace operators on Hermitian vector bundles over compact manifolds a full asymptotic expansion of the heat trace is known [4, Theorem 2.41], and Theorem 6 is consistent with this expansion. Nevertheless, if $\Gamma$ is allowed to have torsion as in Theorem 5 and Corollary 7, $K$-orbits in $\Gamma \backslash G$ are either principal or exceptional, and a desingularization process is necessary. Moreover, it is likely that Theorem 6 can be generalized to this situation, in which case $\Gamma \backslash G/K$ is an orbifold.

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