Is Cabibbo-Kobayasi-Maskawa Matrix Unitary?

C. S. Kim\textsuperscript{a}

\textit{Dept. of Physics and IPAP, Yonsei Univ, Seoul 120-749, Korea}

H. Yamamoto\textsuperscript{b}

\textit{Dept. of Physics & Astrophysics, Univ of Hawaii, HI 96822, USA}

First, we give summary of the present values of CKM matrix elements. Then, we discuss whether CKM matrix is unitary or not, and how we can find out if it is not unitary.

1 Introduction

The Cabibbo-Kobayashi-Maskawa (CKM) matrix\footnote{kim@kimcs.yonseiac.kr, http://phya.yonsei.ac.kr/~cskim/} in three generation is

\[ V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{ub} \\ V_{cd} & V_{cb} \\ V_{td} & V_{tb} \end{pmatrix} \]

\[ = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}. \]

Directly measured values\footnote{hitoshi@phys.hawaii.edu, http://www.phys.hawaii.edu/~hitoshi/} of the elements of $2 \times 2$ Cabibbo matrix, $V_C$, are

\[ V_C = \begin{pmatrix} V_{ud} & V_{us} \\ V_{cd} & V_{cs} \end{pmatrix} = \begin{pmatrix} 0.9740 \pm 0.0010 & 0.2196 \pm 0.0023 \\ 0.224 \pm 0.016 & 1.04 \pm 0.16 \end{pmatrix}. \]

Only after assuming 3 generation unitarity, $V_C$ becomes well known, and can be parametrized with one parameter, $\lambda = \sin \theta_c \approx 0.22$, within 90\% CL

\[ V_C = \begin{pmatrix} 0.9745 \sim 0.0010 & 0.217 \sim 0.224 \\ 0.217 \sim 0.224 & 0.9737 \sim 0.9753 \end{pmatrix} \approx \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda \\ -\lambda & 1 - \frac{\lambda^2}{2} \end{pmatrix}. \]

We note that the directly measured values (2) still have relatively large uncertainties of $\sim 5\%$. Extension to (unitarized) three generation by Kobayashi and Maskawa leads one non-trivial phase angle $\delta_{\text{KM}}$. In Wolfenstein parametrization\footnote{kim@kimcs.yonseiac.kr, http://phya.yonsei.ac.kr/~cskim/}, it can be approximated as

\[ V_{\text{KM}} = \begin{pmatrix} V_{ub} \\ V_{cb} \\ V_{tb} \end{pmatrix} \approx \begin{pmatrix} A\lambda^3(\rho - i\eta) \\ A\lambda^2 \\ 1 \end{pmatrix}, \]

\[ \text{where } A = \frac{1}{\sqrt{2}}, \rho = \frac{1}{2}, \text{ and } \eta = 0, \text{ for } \lambda = 0.22. \]
where $\rho + i\eta \approx \sqrt{\rho^2 + \eta^2}\exp(i\delta_{13})$, and we assumed $V_{tb} = 1$. In coming discussions, we first assume $V_{CKM}$ being unitary, but later we will investigate possible non-unitarity of $V_{CKM}$.

2 Theoretical Determination of Elements of $|V_{CKM}|$

2.1 $|V_{cb}| = A\lambda^2$

From the exclusive $B \to D^*\ell\nu$ decay,

$$
\frac{d\mathcal{B}}{dw}(B \to D^*\ell\nu) = \tau_B \frac{d\Gamma}{dw} = \tau_B \times [...] \times \sqrt{w^2 - 1}|V_{cb}|^2 \mathcal{F}(w),
$$

(5)

where

$$
\mathcal{F}(w) = \mathcal{F}(1)(1 - \hat{\rho}^2(w - 1) + ...), \quad \mathcal{F}(1) = 0.924 \pm 0.027,
$$

(6)

with $w \equiv v_B \cdot v_D^*$. LEP and CLEO measured $d\mathcal{B}/dw |_{w \to 1}$, $\hat{\rho}^2$, $\tau_B$ to obtain the value

$$
|V_{cb}|(B \to D^*\ell\nu) = 0.0387 \pm 0.0031, \quad (i.e. A \sim 0.8).
$$

Inclusive measurement of semileptonic total decay rate, $\Gamma_{s.l.}(B \to X_c l\nu)$, can also give the value $|V_{qb}|$ from

$$
\Gamma_{s.l.}(B \to X_q \ell \nu) = \gamma_{q}^{th} \times |V_{qb}|^2
$$

(7)

$$
= \left( \frac{G_F^2 m_B^2}{192\pi^3} \right) \left[ z_0 \left( 1 - \frac{2\alpha_s}{3\pi} g(\epsilon_q) \right) + \frac{1}{2} \right] \left( G_b - K_b \right) - 2z_1 G_b + ..., \quad |V_{qb}|^2,
$$

where

$$
g(\epsilon) = (\pi^2 - \frac{31}{4})(1 - \epsilon^2)^2 + \frac{3}{2},
$$

$$
z_0(\epsilon) = 1 - 8\epsilon^2 + 8\epsilon^6 - \epsilon^8 - 24\epsilon^4 \ln \epsilon, \quad z_1(\epsilon) = (1 - \epsilon^2)^4,
$$

$$
K_b = -\lambda_1/m_b^2, \quad G_b = 3\lambda_2/m_b^2, \quad \alpha_s = \alpha_s(m_b^2),
$$

with $\epsilon_q = m_q/m_B$, $-\lambda_1 = \mu_2^2 = (0.1 \sim 0.6) \text{ GeV}^2$, $\lambda_2 = \frac{1}{4}(m_B^2 - m_B^2) = 0.12 \text{ GeV}^2$. Now from the total semileptonic decay width, we can measure

$$
|V_{cb}| = \left[ \frac{\Gamma_{s.l.}(B \to X_q \ell \nu)}{\Gamma_{tot}(B)\exp\tau_B^{\exp}^{\ell \nu}} \right]^{1/2} = 0.0419 \sqrt{\frac{\mathcal{B}(B \to X_q \ell \nu)}{0.105}} \sqrt{\frac{1.55}{\tau_B}(1 \pm 0.04)}.
$$

We remark that the uncertainty in determination of $\lambda_1 = -\mu_2^2$, the average kinetic energy of $b$-quark in $B$ meson, is still large.
2.2 \(|V_{ub}| = A\lambda^3 \sqrt{\rho^2 + \eta^2}\)

This is probably the most important element, and at the same time one of the most difficult to be measured. Exclusively, we use the semileptonic decays, \(B \to \rho l\nu, \pi l\nu, \ldots\), which invoke large theoretical uncertainties from hadronic form factors, and their model dependences. Recently by using data of large \(q^2\) (14 < \(q^2\) < 21 GeV^2) and large \(E_l > 2.3\) GeV in \(B \to \rho l\nu\), CLEO derived

\[|V_{ub}| = (3.23 \pm 0.24 \pm 0.25 \pm 0.58) \times 10^{-3},\]

where the last error is from model dependence.

As we can easily see from Eq. (7), if we also measure the total decay width of \(\Gamma_{s,l}(B \to X_u l\nu)\), then we can extract \(|V_{ub}|\) from

\[\left| \frac{V_{ub}}{V_{cb}} \right| \simeq (0.81 \pm 0.06) \times \left[ \frac{BR(B \to X_u l\nu)}{BR(B \to X_c l\nu)} \right]^{1/2},\]

where the error is from \(m_b, \alpha_s, \mu_e^2\). However, the separation of \(B \to X_u\) from the dominant \(B \to X_c\) is experimentally difficult. The promising method is to use \(M_X\), hadronic invariant recoiled mass. This is because

\[m_u \ll m_c \implies m_{\pi,\rho} \ll m_{D,D^*} \implies m_{X_u} \ll m_{X_c}.\]

In Fig. 1 we show the double differential distribution \(\frac{d\Gamma}{dm_X dE_l}\).
2.3 \(|V_{ts}| \text{ and } |V_{td}|\)

\(B_d - \bar{B}_d\) and \(B_s - \bar{B}_s\) mixings can give values of \(|V_{td}|\) and \(|V_{ts}|\) from

\[
\Delta m_q = m(B_q^H) - m(B_q^L) = \left( \frac{G_F f_{B_q}^2}{(2\pi)^2 m_W^2 m_{B_q}} \right) |f_{B_q}| |\eta_{QCD} F_{sd}^q | |V_{td}^* V_{tb}|^2, \tag{8}
\]

where \(f_{B_d} \approx (170 \sim 180) \pm 30 \text{ MeV}, f_{B_s} \approx (195 \sim 205) \pm 30 \text{ MeV}, \frac{f_{B_d}}{f_{B_s}} = 1.15 \pm 0.05, B_{B_d} = 0.75 \pm 0.15 \approx B_{B_s}, \) and \(f_{B_s} \sqrt{B_{B_s}/f_{B_d}} \sqrt{B_{B_d}} = 1.14 \pm 0.06.\)

The recent experimental values\(^1\) are
\[\Delta m(B_d) = 0.471 \pm 0.016 \text{ ps}^{-1}, \quad \Delta m(B_s) > 14.3 \text{ ps}^{-1} \text{ (95% CL)}. \tag{9}\]

If we assume the Standard Model short distance \((sd)\) interaction, \(F_{sd}^d = F_{sd}^s\), then:

- \(\Delta m(B_d)_{\exp} \) gives \(|V_{td}| \approx (8.1 \pm 1.8) \times 10^{-3}.\)
- Assuming \(|V_{ts}| = |V_{cb}|\) and \(|V_{td}| = (0.004 \sim 0.012),\) we get

\[
\left( \frac{\Delta m(B_d)}{\Delta m(B_s)} \right)_{\text{SM}} = 0.056 \pm 0.08,
\]

which is quite larger than the present experimental ratio, \(< 0.0341 \text{ (95% CL)}.\)

- Assuming the Standard Model is correct, then the present data \((9)\) gives \(|V_{td}/V_{ts}| < 0.217 \text{ (95% CL)}.\)

As is well known, the mass difference, \(\Delta m_q\), can be easily polluted by new physics, and so we can approach differently to find new physics, instead of determining the Standard Model parameters, like \(V_{ts}, V_{td}.\) Any new physics, that has different short distance \((sd)\) interaction structure, or has spectator-quark depending interactions, is likely to show up in \(\Delta m_q.\)

3 Generalization of \(V_{\text{CKM}}\) and Unitary Conditions

3.1 Generalization of \(V_{\text{CKM}}\)

If we are considering the possibility of non-unitary CKM matrix, Eq. \((1)\) can be generalized now with 4 phase angles (after absorbing 5 phases to quark fields),
and nine real mixing angles parametrized by $|V_{ij}|$, resulting to 13 independent parameters in total:

$$V_{\text{general}} = \left( \begin{array}{ccc} |V_{ud}| & |V_{us}| & |V_{ub}|e^{i\delta_3} \\ |V_{cd}| & |V_{cs}|e^{i\delta_2} & |V_{cb}| \\ |V_{td}|e^{i\delta_31} & |V_{ts}| & |V_{tb}|e^{i\delta_33} \end{array} \right).$$

(10)

Based on present experimental constraints on the values of CKM matrix elements, and starting from the approximate Wolfenstein parameterization (4), which is already non-unitary, the generalized CKM matrix can be parametrized as

$$V_{\text{general}} \approx \left( \begin{array}{ccc} 1 - \frac{A_1^2}{2} & \lambda A_1 \lambda_3 (\rho_1 - i\eta_1) \\ -\lambda & 1 - \frac{A_2^2}{2} & A_1 \lambda_2^2 \\ A_1 \lambda_3^2 (1 - \rho_2 - i\eta_2) & A_2 \lambda_2 & 1 \end{array} \right).$$

(11)

We note that

- By assuming that $V_{tb} = 1$ and $V_{cs}$ is real as in the case of Wolfenstein parametrization, in Eq. (11) we now have only 5 real parameters, $\lambda, A_1, A_2, \rho_1, \rho_2$ and 2 phase parameters, $\eta_1, \eta_2$. Except for $V_{tb}$, the values of elements in the right-most column or the bottom row are not constrained by the parametrization.

- $|V_{cb}|$ and $|V_{ts}|$ can be different because of $A_1 \neq A_2$. This will produce consequently many new results in analysis of CKM constraints, such as $B_s - \bar{B}_s$ mixing, $B \to X_s + \gamma$ decay, $b \to s$ penguin, etc.

- In order to check the unitarity of CKM matrix, we have to use the general parametrization (10) or (11) instead of the already unitarized parametrization (1) or (4) in extracting CKM parameters from the experimental observables.

- If there exists a massive singlet down quark $b'$, then the columns of $V_{\text{general}}$ remain to be unitary, but the rows do not.

Usual unitary relation for the matrix (11) becomes

$$V_{k1} V_{k3}^* = V_{11} V_{13}^* + V_{21} V_{23}^* + V_{31} V_{33}^* = 0 ?$$

$$0 \simeq V_{13}^* - A_1 \lambda^3 + V_{33},$$

$$0 \approx (A_1 \lambda^3) \times \{(\rho_1 + i\eta_1) - 1 + (1 - \rho_2 - i\eta_2)\},$$

(13)

and now we have only 5 unknowns, $A_1 \lambda^3, \rho_1, \eta_1, \rho_2$ and $\eta_2$. 

5
or equivalently

\[ \lambda |V_{cb}|, \ V_{ub}, \ \sin \gamma, \ |V_{td}| \ \text{and} \ \sin \beta. \]

Those 5 unknowns are exactly the same as the 5 sufficient conditions for the drawing a triangle uniquely – three sides and two angles. As is well known, the usual 3 conditions, two sides and one angle, which are \( A_1 \lambda^3, \ \rho, \ \eta \) (or \( \lambda |V_{cb}|, \ |V_{ub}|, \ \sin \gamma \) or equivalently \( A_1 \lambda^3, \ |V_{td}|, \ \sin \beta \) in unitarized CKM matrix, are only the necessary conditions for drawing a triangle. In order to see that Eq. (13) equals zero, we have to measure those 5 independent observables.

3.2 (Minimum and Complete) Unitary Conditions

As is explained, we have to measure precisely 5 observables to check if just (most popular) one of unitary triangles is really triangle. Eq. (13) can be written in two equations by using the usual sine and cosine rules,

\[ (A) \ \frac{|V_{ub}|}{\sin \beta} = \frac{|V_{td}|}{\sin \gamma} \quad \text{or} \quad \frac{|V_{td}|}{|V_{ub}|} = \frac{\sin \gamma}{\sin \beta}, \]

\[ (B) \ \frac{\lambda |V_{cb}|}{|V_{ub}|} = \cos \gamma + \frac{|V_{td}|}{|V_{ub}|} \cos \beta. \]

Here we have, instead of one sine (or cosine) rule, two equations because the third angle \( \alpha \) is not independent due to the relation \( \alpha + \beta + \gamma = \pi \). We remark here a few comments on Eq. (14):

- The angles \( \beta, \ \gamma \) need to be measured independently without assuming unitarity of CKM matrix. Recent direct measurement of \( \beta \) at CDF still relies on the presumed unitary assumption of \( |V_{cb}| = |V_{ts}| \) in the analysis of \( B \to J/\psi K_s \) decay. If we ignore the small unitary violation in \( |V_{cb}| \neq |V_{ts}| \), CDF’s direct result \[\beta_{\text{CDF}} = 0.79 \pm 0.43.\]

- Future measurements on \( \frac{|V_{ub}|}{|V_{cb}|} \) at Babar and Belle, as explained in section 2, would be one of the most important ingredient in the test of Eq. (14).

- For the ratio \( \frac{|V_{td}|}{|V_{ub}|} \), we may use the relation

\[ \frac{BR(B^+ \to \rho^0 \bar{\nu})}{BR(B^+ \to \rho^2 e^+ \bar{\nu})} = 6 \left( \frac{\alpha^2}{4 \pi^2} \right) |C_{10}^\prime|^2 \times \frac{|V_{td}|^2}{|V_{ub}|^2}, \]
with

$$C_{10} = \frac{X(m_t^2/m_W^2)}{\pi \sin^2 \theta_W},$$

where $X(x_t)$ is Inami Lim function.

In Eq. (15), the constant 6 comes from 3 neutrino species and from isospin relation in form factors of $B \to \rho^\pm$, $B \to \rho^0$. Because of complete cancelation of the hadronic form factors in the ratio of branching fractions of those two decays, there is not any theoretical uncertainties in Eq. (15), though it would be an experimental challenge to measure the small branching fraction, $\text{BR}(B \to \rho\nu\nu) \sim 4 \times 10^{-7}$.

Measuring angle $\gamma$ would be very difficult, if we do not assume any unitarity in the analyses. However, if we assume the unitarity priorly, then we can calculate it from the relation (14A), and then compare the value with the independently measured values from, as an example, Neubert-Rosner bound to check if the unitarity holds.

3.3 Comments on the Discrepancy in recently extracted Values of $\gamma$

As is well known, the discrepancy in extracted values of $\gamma$ from CKM-fitting at $\rho - \eta$ plane and from the $\chi^2$ analysis of non-leptonic $B$ decays has aroused hot debates over what is going on underneath of the unitary triangle. The value of $\gamma$ has been obtained as

$$\gamma = 60^0 \sim 80^0 \quad \text{(from unitary triangle fitting)},$$

$$= 90^0 \sim 140^0 \quad \text{(from nonleptonic analysis)}.$$ 

In both analyses, the unitary conditions have been extensively assumed. If we believe both analyses are correctly performed and the theoretical assumptions used (including the factorization assumption) are correct, one of the most plausible answers would be “non-unitarity of CKM matrix”. An easy answer from our previous argument, explained in the second item after Eq. (11), is that $|V_{cb}| \neq |V_{ts}|$. If those two elements are not equal, then:

- We cannot simply add the constraint from $B_s - \bar{B}_s$ mixing result on $\rho - \eta$ plane to get the allowed smaller circular region of $|V_{td}|$.
- Adding of two circular regions (from measurements $|V_{ub}|$, $|V_{td}|$) does not reduce to the overlapped small region. Instead, we will have the summed large region in the unitary plane. Therefore, without the direct measurement on $\gamma$ we cannot decide which value is correct.
Very soon, we will have a flood of experimental results on CKM elements from many experiments, asymmetric and symmetric $e^+e^-$ colliders, and hadronic machines. Without having correct (theoretical) strategy of coping the data, we will be easily fooled. We argue the present discrepancy on $\gamma$ is just an early example.

Acknowledgments

CSK was supported in part by Grant No. 1999-2-111-002-5 from the Interdisciplinary research program of the KOSEF, in part by the BSRI Program of MOE, Project No. 99-015-D10032, and in part by the KRF Sughak-research program, Project No. 1997-011-D00015. HY was supported by the Department of Energy Grant DE-FG02-91ER40654.

References

1. N. Cabibbo, *Phys. Rev. Lett.* 10, 531 (1963); M. Kobayashi, T. Maskawa, *Prog. Theo. Phys.* 49, 652 (1973).
2. Particle Date Group, *Eur. Phys. J.* C 3, 1 (1998).
3. L. Wofenstein, *Phys. Rev. Lett.* 51, 1945 (1983).
4. M. Neubert, [hep-ph/9801260](http://arxiv.org/abs/hep-ph/9801260).
5. P. Drell, [hep-ex/9711026](http://arxiv.org/abs/hep-ex/9711026).
6. M. Luke et al., *Phys. Lett.* B 321, 88 (1994); M. Shifman et al., *Phys. Rev. D* 51, 2217 (1995); P. Ball et al., *Phys. Rev. D* 52, 3929 (1995).
7. CLEO Collab., *Phys. Rev. D* 61, 052001 (2000).
8. N. Uraltzev, *Int. J. Mod. Phys.* A 11, 515 (1996); C.S. Kim, *Phys. Lett.* B 414, 347 (1997).
9. V. Barger, C.S. Kim, R.J.N. Phillips, *Phys. Lett.* B 251, 629 (1990).
10. K.K. Jeong, C.S. Kim, Y.G. Kim, work in progress.
11. $|V_{ts}|, |V_{td}|$ from Lepton-Photon Symposium (1999).
12. CDF Collab., [hep-ex/9900003](http://arxiv.org/abs/hep-ex/9900003).
13. T. Aliev, C.S. Kim, *Phys. Rev. D* 58, 013003 (1998).
14. M. Neubert, J.R. Rosner, *Phys. Rev. Lett.* 81, 5076 (1998).
15. A. Stocchi, talk given in this Conference.
16. N.G. Deshpande, X. He, W. Hou, S. Pakvasa, *Phys. Rev. Lett.* 82, 2240 (1999).