ABSTRACT In this work, we examine a mixed oligopoly model within the framework of consistent conjectural variations, in which one of the producers is a labor-managed company competing with other private firms, and consumer demand is given by a discontinuous function. Private firms attempt to maximize their net profit while the labor-managed company attempts to maximize the convex combination of its net profit with an income-per-worker function, which describes the mixed nature of the model. Each producer conjectures the dependence of the market-clearing price on its own production volume and then selects its most suitable conjecture based on a verification procedure. We introduce the notions of exterior and interior equilibrium and prove the existence and uniqueness of the conjectural variation equilibrium, as well as the existence of the particular equilibrium states known as consistent. Finally, we analyze the behavior of the market’s consistent equilibrium state in response to changes in consumer demand.

INDEX TERMS Consistent conjectural variations, discontinuous demand, game theory, labor-managed company, mixed oligopoly.

I. INTRODUCTION

Recently, monographs and papers (see, e.g., [1], [2]) that address agents’ behavioral patterns in mixed markets have become very popular. Whenever a governmental agent that maximizes welfare competes against private firms that maximize their profits, a homogeneous commodity market (oligopoly) is referred to as a mixed market. The main difference between the models studied is the objective (utility) function that the public company maximizes. Kalashnikov et. al. (see, e.g., [1], [2]) examined a mixed oligopoly in which the public agent intends to maximize social surplus.

There is another concept concerning a special agent whose objective represents the company’s income per worker. Such an agent is called a labor-managed firm. Ward, in [3], demonstrated this concept in 1958. It was considered “the first swallow”. Thereafter, many researchers (see, e.g., [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18]) have investigated the conjectural variation equilibrium in such fields of mixed oligopoly.

Some scholars (see, e.g., [19], [20], [21]) have examined a conjectural variation oligopoly model that models the influence that the agents have on the market through some special parameters. Moreover, a new class of conjectural variation equilibria (CVE), concerning the “influence” of agents, redefined the structure of the Cournot-Nash equilibrium. This
means that various results concerning equilibrium existence and uniqueness were obtained. That is to say, they consider not only classical Cournot competition (in which all the influence coefficient values are equal to 1) but also a Cournot-type model that has influence coefficients different from 1.

The consistency (or, sometimes, “rationality”) of equilibrium is defined as the coincidence between the conjectured reaction function of each agent and the conjectural best response of the same agent. A conceptual problem arises when considering consistency for many agents, as seen in [1]. Bulavsky proposed a completely new approach in [22] to address this conceptual difficulty that appears in many-player models. It is assumed that each decision-maker decides the conjectures, not to describe the (optimal) response of the other decision-makers but to describe the variations in the market price relying on his or her output variations. Assuming that all players know their rivals’ conjectures (hereafter, called influence coefficients), each agent can check whether his or her influence coefficient is consistent with those of others by applying a certain verification procedure. The same formulas appearing in the latter verification procedure were independently obtained in [23], where they established the uniqueness and existence of the equilibrium with consistent conjectures in an electricity market.

The authors (see, e.g., [1] and [24]) extended the results from [22] to a mixed oligopoly model, considering a continuously differentiable objective function. In the paper [2], the restriction was eased, and the demand function was permitted to be discontinuous in a finite number of points where not only its derivative but also the function itself may experience a jump. Regardless, the same results from the papers [1], [24] were obtained under these weaker conditions.

In this paper, we extend the results from [2] to an oligopoly where a labor-managed company competes against private agents maximizing their profits. The rest of this paper is organized as follows. Section II illustrates the mixed oligopoly model where the special agent is a labor-managed agent with a not necessarily smooth demand function, similar to the mathematical model in [2]. Section III explains the idea of exterior equilibrium, i.e., the CVE, where the influence coefficients are determined exogenously, providing the uniqueness and existence theorem for the CVE. In Section IV a further advanced concept of equilibrium, referred to as interior, is described as an exterior equilibrium when the conjectures are consistent. Results regarding the existence of the interior equilibrium are also formulated in Section IV, together with the definition of consistency (verification procedure). In Section V, we present numerical experiments to demonstrate the agents’ behavior when active demand increases (or decreases). At the end of this paper, the acknowledgments and references are listed.

II. MODEL DESCRIPTION

In this paper, we extend the results from [25] and [26] to the case of a partially mixed oligopoly. Thus, let us consider an oligopoly market of a homogeneous good, consisting of a single public company and \( n \) private firms, \( n \in \{1, 2, \ldots \} \). We denote the public company with the index \( i = 0 \), while private firms are denoted with the indices \( i \in \{1, \ldots, n\} \). Each private firm \( i \in \{1, \ldots, n\} \) estimates its production expenses with a cost function, \( f_i(q_i) \), where \( q_i \) is its production volume sent to the market. Public company \( i = 0 \) also estimates its expenses with a function but takes into account the number of workers required for the production process; thus, its cost function is denoted as the product of two functions \( f(q_0) = \ell(q_0)f_0(q_0) \), where \( q_0 \) is the production volume.

Following the ideas from previous works (see, e.g., [27]), we divide market demand into two types: passive demand, which represents consumers that take into account the price of the product before buying, and active demand, which represents consumers that buy the product regardless of price. Passive demand is given by a function, \( G(p) \), where \( p \) is the price of the product, while active demand is given by a constant, \( D \).

The properties of the functions introduced above are stated in the following assumptions.

Assumption A1: The passive demand function \( G(p) \) is defined for every \( p > 0 \), being nonnegative, nonincreasing, and piecewise continuously differentiable. The set of points (if any) where \( G(p) \) is not continuously differentiable is finite, and at those points, both \( G(p) \) and its derivative \( G'(p) \) may only have jump discontinuities. In addition, if \( n = 1 \), then there exists \( \epsilon > 0 \) such that \( |G'(p)| > \epsilon \) for all \( p > 0 \) (where \( G(p) \) is differentiable). Moreover, active demand \( D \) is a nonnegative constant.

Assumption A2: For each \( i \in \{0, 1, \ldots, n\} \), the cost function \( f_i(q_i) \) is defined for every \( q_i \geq 0 \), being nonnegative, twice continuously differentiable, and satisfying that \( f_i'(0) > 0 \) and \( f''_i(q_i) > 0 \) for all \( q_i \geq 0 \). Moreover, function \( \ell(q_0) \) is given by

\[
\ell(q_0) = aq_0 + b, \quad a, b > 0,
\]

thus, the total cost function for the public company takes the following form:

\[
f(q_0) = \ell(q_0)f_0(q_0) = (aq_0 + b)f_0(q_0),
\]

which satisfies the same properties as the function \( f_0(q_0) \).

Under Assumption A1, we have that for every \( p > 0 \), both \( G(p) \) and its derivative \( G'(p) \) have (finite) right-hand limits, which we denote by \( G(p+) \) and \( G'(p+) \), respectively, as well as left-hand limits, denoted by \( G(p-) \) and \( G'(p-) \), respectively. Hence, the equilibrium between demand and supply is achieved when the following inequality holds:

\[
G(p+) + D \leq \sum_{i=0}^{n} q_i \leq G(p-) + D,
\]

which we call a balance inequality. Moreover, since \( G(p) \) is nonincreasing, we have that \( G'(p) \leq 0 \) if it exists, and \( G'(p+), G'(p-) \leq 0 \) for every \( p > 0 \).

The behavior of producers in the market is described below.
Producers $i \in \{1, \ldots, n\}$ are private firms that select their production volume $q_i \geq 0$ to maximize their own net profit given by the following function:

$$\pi_i(p, q_i) = pq_i - f_i(q_i), \quad i \in \{1, \ldots, n\}. \quad (4)$$

Moreover, producer $i = 0$ is a public company that selects its production volume $q_0 \geq 0$ to maximize the convex combination of the average income per worker and its own net profit, given by the following function:

$$S(\beta, p, q_0) = \beta \left( \frac{pq_0 - \ell(q_0)f_0(q_0)}{\ell(q_0)} \right) + (1 - \beta)(pq_0 - \ell(q_0)f_0(q_0)) \quad (5)$$

where (following the ideas from [28]) $\beta \in [0, 1]$, the parameter of the convex combination, is chosen beforehand by the public company.

From function (5), we can see that if the public company selects the value $\beta = 0$, then it solely maximizes its net profit; conversely, if $\beta = 1$ is selected, then only the average income per worker is maximized.

Now, we say that each producer $i$, $i \in \{0, 1, \ldots, n\}$, accepts that the variations in its own production volume $q_i$ affect the price $p$ of the product in the market; thus, making use of the first-order necessary conditions, we obtain the following relationships.

For private firms $i \in \{1, \ldots, n\}$,

$$\begin{align*}
\frac{\partial \pi_i}{\partial q_i} &= p + q_i \frac{\partial p}{\partial q_i} - f_i'(q_i) \\
&= 0, \quad \text{if } q_i > 0, \\
&\leq 0, \quad \text{if } q_i = 0,
\end{align*} \quad (6)$$

and for the public company,

$$\begin{align*}
\frac{\partial S}{\partial q_0} &= \beta \left( \frac{p + q_0 \frac{\partial p}{\partial q_0} - \ell(q_0)f_0(q_0)}{\ell(q_0)} \right) + (1 - \beta)\left(p + q_0 \frac{\partial p}{\partial q_0} - \ell(q_0)f_0(q_0)\right) \\
&- \ell(q_0)f_0'(q_0) \\
&= 0, \quad \text{if } q_0 > 0, \\
&\leq 0, \quad \text{if } q_0 = 0.
\end{align*} \quad (7)$$

From conditions (6) and (7), we can see that each individual producer $i \in \{0, 1, \ldots, n\}$ needs to know the value of the (partial) derivative $\frac{\partial p}{\partial q_i}$ to evaluate the optimality of its production volume, which is not possible since the price $p$ depends on the market equilibrium between every producer; however, using the idea of conjectural variations (see [29], [30]), we assume that every producer conjectures the value of its corresponding derivative to be a function of its own production volume; i.e.,

$$\frac{\partial p}{\partial q_i} = -v_i(q_i), \quad i \in \{0, 1, \ldots, n\}, \quad (8)$$

where $v_i = v_i(q_i)$ is the function conjectured by producer $i$.

Since an increase in the total production volume in the market does not increase the value of the price, the derivatives $\frac{\partial p}{\partial q_i}$ are nonpositive; thus, in relationship (8), we introduce the negative sign to work with nonnegative values for the conjectures $v_i$.

In addition, to evaluate optimality conditions (6) and (7), each producer only needs to know the value of its corresponding derivative at the equilibrium state; thus, we say that each producer $i$, $i \in \{0, 1, \ldots, n\}$, assumes that its conjectured function $v_i$ is a nonnegative constant (similar to the Cournot and perfect competition conjectures), which we refer to as the $i$-th producer’s influence coefficient.

Under this assumption that the conjectures (influence coefficients) $v_i$ are nonnegative constants, we can calculate the second derivatives for the private firms’ objective functions

$$\frac{\partial^2 \pi_i}{\partial q_i^2} = -2v_i - f_i''(q_i) < 0, \quad i \in \{1, \ldots, n\}, \quad (9)$$

and the public company’s objective function

$$\frac{\partial^2 S}{\partial q_0^2} = -\beta \left( \frac{2bv_0}{\ell(q_0)^2} + \frac{2abp}{\ell(q_0)^2} + f_0''(q_0) \right) - (1 - \beta)\left(2v_0 + 2af_0''(q_0) \right) + \ell(q_0)f_0'(q_0)) < 0, \quad (10)$$

which are strictly negative, thus providing the concavity of every producer’s objective function. Hence, first-order optimality conditions (6) and (7) are also sufficient.

By substituting all conjectures from (8) into optimality conditions (6) and (7), we rewrite the first-order necessary and sufficient optimality conditions as follows:

For the public company,

$$\begin{align*}
p &= \Theta_0(\beta, q_0, v_0), & \text{if } q_0 > 0, \\
p &\leq \Theta_0(\beta, 0, v_0), & \text{if } q_0 = 0,
\end{align*} \quad (11)$$

where

$$\Theta_0(\beta, q_0, v_0) = \frac{\beta \ell(q_0)(q_0v_0 + \ell(q_0)f_0'(q_0))}{\beta b + (1 - \beta)\ell(q_0)^2} + \frac{(1 - \beta)\ell(q_0)^2(a_0q_0 + q_0v_0 + \ell(q_0)f_0'(q_0))}{\beta b + (1 - \beta)\ell(q_0)^2}, \quad (12)$$

and for the private firms,

$$\begin{align*}
p &= \Theta_i(q_i, v_i), & \text{if } q_i > 0, \\
p &\leq \Theta_i(0, v_i), & \text{if } q_i = 0,
\end{align*} \quad i \in \{1, \ldots, n\}, \quad (13)$$

where

$$\Theta_i(q_i, v_i) = q_i v_i + f_i'(q_i), \quad i \in \{1, \ldots, n\}. \quad (14)$$

Since the influence coefficients $v_i$ can take any nonnegative value, we have to determine, for every $i \in \{0, 1, \ldots, n\}$, which value of $v_i$ is the most suitable for producer $i$. To do this, we use the approach from [22], where these influence coefficients are determined together with the values of $p$ and $q_i$, $i \in \{0, 1, \ldots, n\}$, by finding the market’s equilibrium.
state. The latter equilibrium is based on a verification procedure described in Section IV and is referred to as **interior equilibrium**. However, before introducing such concepts, we first need to introduce another concept of equilibrium known as **exterior equilibrium**.

### III. EXTERIOR EQUILIBRIUM

As mentioned in the previous section, for every producer $i$, the dependence of the market price $p$ on its own production volume $q_i$ is unknown; thus, according to the framework of conjectural variations, each producer estimates the value of the derivative $\frac{\partial p}{\partial q_i} = -v_i$ (its influence coefficient) to decide its optimal production volume.

Hence, to evaluate market equilibrium, we present the following results concerning producers’ optimal response functions.

**Lemma 1:** The function $\Theta_0(\beta, q_0, v_0)$ is strictly increasing with respect to $q_0 \geq 0$, nondecreasing with respect to $v_0 \geq 0$ and monotone with respect to $\beta \in [0, 1]$; in particular, $\Theta_0(\beta, 0, v_0)$ is nonincreasing with respect to $\beta \in [0, 1]$.

Moreover,

$$\lim_{q_0 \to +\infty} \Theta_0(\beta, q_0, v_0) = +\infty, \quad \beta \in [0, 1], \quad v_0 \geq 0. \quad (15)$$

**Lemma 2:** For each $i \in \{1, \ldots, n\}$, the function $\Theta_i(q_i, v_i)$ is strictly increasing with respect to $q_i \geq 0$ and nondecreasing with respect to $v_i \geq 0$; moreover, the following limit exists (being either finite or infinite)

$$p_i^1(v_i) = \lim_{v_i \to +\infty} \Theta_i(q_i(v_i), v_i), \quad v_i \geq 0. \quad (16)$$

**Proposition 1:** There exist the optimal response functions $q_0(\beta, p, v_0)$ for any $\beta \in [0, 1]$, $v_0 \geq 0$ and $p > 0$, which satisfies optimality condition (11), and $q_i(p, v_i), \ i \in \{1, \ldots, n\}$, for any $v_i \geq 0$ and $p \in (0, p_i^1(v_i))$, which satisfies optimality condition (13).

Proposition 1 tells us that each producer $i$ is able to calculate its optimal production volume $q_i$ for any given price $p$ based on its influence coefficient $v_i$. Now, we define the equilibrium state achieved by the producers due to their conjectures.

**Definition 1:** Let the values $\beta \in [0, 1]$ and $v_i \geq 0, \ i \in \{0, 1, \ldots, n\}$, be fixed. A vector, $(p, q_0, q_1, \ldots, q_n)$, which consists of the market price $p > 0$ and the production volumes $q_i \geq 0, i \in \{0, 1, \ldots, n\}$, is called **exterior equilibrium** if the market is balanced, i.e., balance inequality (3) is satisfied, and every producer’s output volume is optimal, i.e., first-order optimal conditions (11) and (13) hold.

From the optimality conditions (11) and (13), and the behavior of the functions $\Theta_0$ and $\Theta_i, \ i \in \{1, \ldots, n\}$, described by Lemmas 1 and 2, we can see that for every producer to make a profit, the price that appears in the market must be at least higher than the following value:

$$p_0 = \max \left\{ \Theta_0(0, 0, 0), \max_{i \in \{1, \ldots, \nu\}} \Theta_i(0, 0) \right\}$$

$$= \max \left\{ af_0(0) + bf_0(0), \max_{i \in \{1, \ldots, \nu\}} f_i'(0) \right\}. \quad (17)$$

The latter is stated in the following lemma.

**Lemma 3:** Let Assumptions A1 and A2 hold, and let $(p, q_0, q_1, \ldots, q_n)$ be the exterior equilibrium for the fixed values $\beta \in [0, 1]$ and $v_i \geq 0, \ i \in \{0, 1, \ldots, \nu\}$. If $p > p_0$, then, $q_i > 0$ for all $i \in \{0, 1, \ldots, \nu\}$.

Thus, if the market price is less or equal to the value $p_0$, then the optimal productions for some producers may be zero; in such a case, we can assume that such producers are not part of the market in the first place. In theory, it is possible for producers to re-enter the market if the price increases enough; however, studying the impact of a producer leaving or re-entering the market is beyond the scope of this work.

Now, by Lemma 1, there exist the values $q_i^0$ and $q_i^1$ such that

$$\Theta_0(0, q_i^0, 0) = p_0 \quad \text{and} \quad \Theta_0(1, q_i^0, 0) = p_0. \quad (18)$$

thus, to guarantee that the parameters of the model allow for every producer to always make a profit, regardless of the influence coefficients selected, we introduce the following assumption:

**Assumption A3:** There exist the values $q_i^0 \geq 0, \ i \in \{1, \ldots, \nu\}$, such that

$$f_i'(q_i^0) = p_0, \quad i \in \{1, \ldots, \nu\},$$

which together with $q_i^0$ and $q_i^1$, satisfy the following inequality:

$$\max\{q_i^0, q_i^1\} + \sum_{i=1}^{\nu} q_i^0 < G(p_0^+). \quad (20)$$

Now, we can prove the main result of this section.

**Theorem 1:** Under Assumptions A1, A2, and A3, for any $\beta \in [0, 1], \ v_i \geq 0, \ i \in \{0, 1, \ldots, \nu\}$, and $D \geq 0$, there exists a unique exterior equilibrium $(p_1^0, q_0^0, q_1^0, \ldots, q_n^0)$, where the functions $p^{e_1} = p^{e_1}(\beta, v_0, v_1, \ldots, v_{\nu}, D)$ and $q_i^{e_1} = q_i^{e_1}(\beta, v_0, v_1, \ldots, v_{\nu}, D), \ i \in \{0, 1, \ldots, \nu\}$, are continuous in their domain. Moreover, $p^{e_1} > p_0$ and $q_i^{e_1} > 0$ for all $i \in \{0, 1, \ldots, \nu\}$, and equilibrium price $p^{e_1}$ has right- and left-hand partial derivatives with respect to $D$, given by the following formulas:

$$\frac{\partial p^{e_1}}{\partial D^+} = \begin{cases} \frac{1}{\Omega - G(p^{e_1})}, & \text{if } \sum_{i=0}^{\nu} q_i^{e_1} = G(p^{e_1}) + D, \\ 0, & \text{if } \sum_{i=0}^{\nu} q_i^{e_1} > G(p^{e_1}) + D, \end{cases} \quad (21)$$

and

$$\frac{\partial p^{e_1}}{\partial D^-} = \begin{cases} \frac{1}{\Omega - G(p^{e_1})}, & \text{if } \sum_{i=0}^{\nu} q_i^{e_1} = G(p^{e_1}) - D, \\ 0, & \text{if } \sum_{i=0}^{\nu} q_i^{e_1} < G(p^{e_1}) - D, \end{cases} \quad (22)$$
where
\[
\Omega = \frac{1}{\partial \Theta_0 / \partial q_0} (\beta, q_0^x, v_0) + \sum_{i=1}^{n} \frac{1}{\partial \Theta_i / \partial q_i} (q_i^x, v_i).
\] (23)

Now, we are going to reduce formulas (21) and (22) into a single equation. To do this, let us consider the graph of \(G(p)\) in \(\mathbb{R}^2_+\), and for every discontinuity at a point \(\hat{p}\), let us connect the points \((\hat{p}, G(\hat{p}+))\) and \((\hat{p}, G(\hat{p}−))\) with a vertical line, thus, obtaining a new curve \(L\) in \(\mathbb{R}^2_+\), which we call (passive) demand curve. We can see that each point \((\hat{p}, G)\in L\) satisfies that \(G(\hat{p}+) \leq G \leq G(\hat{p}−);\) then, at the discontinuity points \(\hat{p}\) of \(G(p)\), we redefine the right-hand derivative of \(G(p)\) as \(G'(\hat{p}+) = −\infty\) if \(\hat{G} > G(\hat{p}+)\) and the left-hand derivative of \(G(p)\) as \(G'(\hat{p}−) = −\infty\) if \(\hat{G} < G(\hat{p}−)\).

Hence, the exterior equilibrium \((p^x, q^x_0, q^x_1, \ldots, q^x_n)\) defines the point \((\hat{p}, G^x)\in L\), where \(G^x = \sum q^x_i - D\); and formulas (21)-(22) are reduced into the single formula:
\[
\frac{\partial p^x}{\partial D^x} = \frac{1}{\frac{1}{\partial q_0} \sum_{i=1}^{n} \frac{1}{\partial q_i} G'(p^x) \pm}.
\] (24)

Using formula (24), we can finally deduce the real value of each producer’s influence coefficient and introduce the concept of interior (consistent) equilibrium.

### IV. INTERIOR EQUILIBRIUM

First, we describe the procedure of verification for the influence coefficients from [22]. Let \((p^x, q^x_0, q^x_1, \ldots, q^x_n)\) be the exterior equilibrium for the influence coefficients \(v_i \geq 0\) and certain values of \(\beta\) and \(D\). Now, suppose that producer \(k, k \in \{0, 1, \ldots, n\}\), wants to estimate the real value of its influence coefficient; thus, it refrains from maximizing its benefit and instead starts to make small variations to its production volume \(q_k\) around its equilibrium value \(q^x_k\). In this case, producer \(k\) is temporarily outside of the market’s equilibrium and its production has to be subtracted from consumer demand, particularly subtracted from active demand. Since active demand \(D\) is a constant, if we consider the difference \(D_k = D - q_k\) as the new active demand in a market where producer \(k\) is absent, then the variations in \(q_k\) will be equivalent to the variations in \(D_k\) but in the opposite direction. Therefore, producer \(k\) can estimate the right- and left-hand (partial) derivatives of the equilibrium price with respect to its own production volume using the relationship:
\[
\frac{\partial p^x}{\partial D^x_k} = \frac{\partial p^x}{\partial (D - q^x_k)^\pm} = -\frac{\partial p^x}{\partial q^x_k},
\] (25)
i.e., the right- and left-hand limits of its influence coefficient.

By Theorem 1, we can apply formula (24) to calculate the one-sided derivatives \(\frac{\partial p^x}{\partial D^x_k}\); however, since producer \(k\) is outside the market equilibrium, we have to exclude from formula (24) the terms corresponding to index \(i = k\).

Hence, we arrive at the following consistency criterion.

**Definition 2 (Consistency Criterion):** Let \((p^x, q^x_0, q^x_1, \ldots, q^x_n)\) be the exterior equilibrium for the values \(v_i \geq 0, i \in \{0, 1, \ldots, n\}\), \(D \geq 0\), and \(\beta \in [0, 1]\). The influence coefficient \(v_i, i \in \{0, 1, \ldots, n\}\), is referred to as consistent if there exists \(r_i\) such that
\[
\min\{G'(p^x+), G'(p^x−)\} \leq r_i \leq \max\{G'(p^x+), G'(p^x−)\}
\] (26)
and the following relationship holds:
\[
v_i = \frac{1}{\sum_{j=1}^{n} \frac{1}{q_j} - r_i}, \quad \text{if } i = 0,
\]
\[
v_i = \frac{1}{\sum_{j=1}^{n} \frac{1}{q_j} - r_i - 1}, \quad \text{if } i \in \{1, \ldots, n\}.
\] (27)

In the above definition, the values of \(G'(p^x+)\) and \(G'(p^x−)\) are calculated at the point \((p^x, \sum q^x_i - D) \in L\), as defined at the end of Section III; thus, it may happen that \(r_i = −\infty\), in which case identity (27) is reduced to \(v_i = 0\).

In other words, we define the influence coefficient \(v_i\) as consistent if producer \(i\) conjectures the (negative) value of \(v_i\) between the right- and left-hand limits of the rate of change of the equilibrium price with respect to its own production.

Of course, one can think that each producer should select the influence coefficient that provides the highest benefit instead; however, as shown in [31], for a classical oligopoly market, the influence coefficients that provide the highest profit for each individual producer always satisfy the consistency criterion, which serves as justification for choosing the consistent influence coefficients as optimal conjectures.

**Definition 3:** Let the value \(\beta \in [0, 1]\) be fixed. A vector \((p, q_0, q_1, \ldots, q_n, v_0, v_1, \ldots, v_n)\), where \((p, q_0, q_1, \ldots, q_n)\) is the exterior equilibrium for the influence coefficients \(v_i \geq 0, i \in \{0, 1, \ldots, n\}\), is called an interior equilibrium if every influence coefficient is consistent, i.e., for every \(i \in \{0, 1, \ldots, n\}\) there exists the value \(r_i\) that satisfies relationships (26) and (27) from the consistency criterion. In addition, if the values of \(r_i\) are all the same, i.e., \(r_i = r_j\) for all \(i, j \in \{0, 1, \ldots, n\}\), then the interior equilibrium is referred to as a strong interior equilibrium.

From inequality (26), we can see that the consistency criterion allows for different values of \(r_i\) whenever \(G'(p^x+) \neq G'(p^x−)\); however, if the interior equilibrium corresponds to a smooth point of demand curve \(L\), then \(G'(p^x+) = G'(p^x−)\), implying that the values \(r_i, i \in \{0, 1, \ldots, n\}\), are all the same. Therefore, from now on, we consider only the strong interior equilibrium states.

**Theorem 2:** Under Assumptions A1, A2, and A3, for any \(\beta \in [0, 1]\) and \(D \geq 0\), there exists a strong interior equilibrium \((p^*, q^*_0, q^*_1, \ldots, q^*_n, v^*_0, v^*_1, \ldots, v^*_n)\).
In the next section, we analyze the behavior of the strong interior equilibrium in terms of the discontinuities of consumer demand.

V. STRUCTURE OF DEMAND AND EQUILIBRIUM

The aim of this section is to study the behavior of the strong interior equilibrium’s price and supply under variations in consumer demand. We are interested not in the quantitative changes but rather in the qualitative picture described by the model. The accurate reflection of a real-life situation in this picture can serve as an argument in favor of the model being adequate.

For the experiment, we consider an oligopoly market consisting of a public company and 2 private firms. The cost function for public company $i = 0$ is given by

$$f(q_0) = \ell(q_0)f_0(q_0),$$

(28)

where

$$\ell(q_0) = 0.003q_0 + 2,$$

(29)

$$f_0(q_0) = 0.012q_0^2 + 1.5q_0,$$

(30)

and the cost functions for private firms $i \in \{1, 2\}$ are given by

$$f_1(q_1) = 0.043q_1^2 + 2.5q_1,$$

(31)

and

$$f_2(q_2) = 0.038q_2^2 + 3.5q_2.$$  

(32)

The parameter for the convex combination of the public company’s objective function is $\beta = 1$, and consumers’ passive demand is given by the step function

$$G(p) = \begin{cases} 
400, & \text{if } p \leq 30, \\
0, & \text{if } p > 30, 
\end{cases}$$

(33)

while active demand has the value $D_0 = 600$.

We are interested in the strong interior equilibrium; thus, for this particular case, the consistency criterion’s system of equations given by (27) is reduced as follows:

$$v_0 = \frac{1}{1} + \frac{1}{1} - r,$$

(34)

$$v_i = \frac{\partial \Theta_0}{\partial q_0}(\beta, q_0^*, v_0) + \frac{\partial \Theta_2}{\partial q_2}(q_2^*, v_2),$$

$$v_j = \frac{\partial \Theta_0}{\partial q_0}(\beta, q_0^*, v_0) + \frac{\partial \Theta_2}{\partial q_2}(q_2^*, v_2),$$

(35)

$$i \in \{1, 2\}, j = 3 - i.$$

Since the exterior equilibrium’s functions are continuous with respect to the influence coefficients (by Theorem 1), we can easily find the solutions for the system of equations given by (34)-(35) using the methodology presented in [27].

Therefore, for each pair of values $r \in [-\infty, 0]$ and $D \geq 0$, we can find the solution $v_i(r, D), i \in [0, 1, 2]$, for the system (34)-(35) and construct the total supply function

$$Q(p; r, D) = q_0(1, p, v_0(r, D)) + q_1(p, v_1(r, D)) + q_2(p, v_2(r, D)).$$

(36)

If the value of $D$ is fixed, then for each $r \in [-\infty, 0]$ the curve $Q(p; r, D)$ depicts the total supply in the market as a function of price $p$. In particular, if $r = 0$, then the curve $Q(p; 0, D)$ corresponds to the case when producers conjecture their maximum (possible) influence on market price. Moreover, if $r = -\infty$, then, regardless of the value of $D$, the solution for system (34)-(35) is $v_i = 0, i \in [0, 1, 2]$, and the curve $Q(p; -\infty, D)$ represents the case of perfect competition, i.e., when producers assume that none of them have a significant influence on market price.

The graphs of these curves for our example are shown in Figure 1.

From Figure 1, we can see that the total supply function $Q(p; -\infty, D_0)$ intersects the demand function $G(p) + D_0$ at the point $(p, G+D) \approx (30, 929.03)$, which satisfies $G'(p+) = G'(p-) = -\infty$; thus, this point represents a strong interior equilibrium. Moreover, the point $(p, G+D) = (30, 600)$ satisfies that $G'(p+) = 0$ and $G'(p-) = -\infty$; thus, this point represents a second strong interior equilibrium whenever any of the total supply functions $Q(p; r, D_0)$ pass through it, which, in this case, is the total supply function corresponding to the value $r \approx -7.3492$. A third (and last) strong interior equilibrium is at the point $(p, G+D) \approx (43.372, 600)$, where the total supply function $Q(p; 0, D_0)$ intersects the demand function $G(p) + D_0$ (indeed, at this point, $G'(p) = 0$).

We continue to refer to these 3 interior equilibriums in the order that they are mentioned above.

Now, let the economy stay in the third (strong) interior equilibrium at $(p, G+D) \approx (43.372, 600)$, and suppose that active demand starts to grow from its original value $D_0 = 600$, elevating the demand curve as shown in Figure 2.

From Figure 2 we can see that the market price starts to increase as demand grows since the market, in its current
From Figure 4, we can see that when active demand drops to $D_1$, the second and third interior equilibriums meet at the point $(p, G + D) \approx (30, 432.62)$. If active demand keeps decreasing beyond $D_1$, then the interior equilibrium at $(p, G + D) \approx (30, 432.62)$ disappears, forcing the economy to jump to the remaining interior equilibrium (the first interior equilibrium), resulting in the market’s total supply jumping from the value $D_1 \approx 432.62$ to a new value, $G(p) + D > 832.62$, satisfying both passive and active demand, while the price remains at $p_1 = 30$, as shown in Figure 5.

Now, assume that active demand now has the value $D_2 = 400 < D_1$; thus, the economy is still in the first interior equilibrium (which is the only interior equilibrium at the moment) at point $(p, G + D) = (30, 800)$; and suppose that active demand starts to recover by increasing to the value $D_3 \approx 529.03$, as shown in Figure 6.

From Figure 6, we can see that as active demand increases from $D_2$ to $D_3$, the first interior equilibrium changes continuously, holding the price at the value $p_1 = 30$ and completely satisfying consumer demand $G(p) + D$. In this situation, supply increases under the same price at the cost of diminishing...
FIGURE 7. Behavior of the interior equilibrium when active demand increases beyond $D_3$.

FIGURE 8. Behavior of the interior equilibrium when active demand changes from $D_3$ to $D_4$.

FIGURE 9. Behavior of the interior equilibrium when active demand increases beyond $D_4$.

From Figures 7 and 8, we can see that if active demand belongs to the interval $[D_3, D_4]$, then the first interior equilibrium is fixed at point $(p, G + D) \approx (30, 929.03)$, under the supply regime of perfect competition. In this interval, neither the market price nor the total supply depend on active demand; thus, a deficit appears since consumer demand $G(p) + D$ is no longer satisfied. As we can see in Figure 8, if active demand increases to the value $D_4$, then the deficit increases to the value of passive demand $G(p_1)$; therefore, only active demand is satisfied. At this stage, the first and second interior equilibriums are the same, and any further increase will cause this interior equilibrium $(p, G + D) \approx (30, 929.03)$ to disappear, as shown in Figure 9.

From Figure 9, we see that the economy finally jumps back to its original state, the third interior equilibrium (which is now the only interior equilibrium), but only after active demand grows beyond the value $D_4 \approx 929.03$, which is larger than its original value $D_0 = 600$. At this stage, there is no more deficit in consumer demand because passive demand has fallen to zero again, which results in the market price jumping from the value $p_1 = 30$ to a new value, $p > p_2 \approx 73.799$.

Finally, we note that if the demand curve has two or more steps, then the picture will be similar. The main difference would be that the process described above would be repeated as many times as the number of steps between the curves $Q(p; -\infty, D)$ and $Q(p; 0, D)$. Moreover, if the demand curve is not discontinuous but has somewhat smoothed steps with a high slope, then the process will be almost the same; the perfect competition regime will not appear, but its role will be played, instead, by a close production regime with small influence coefficients defined by the steepness of the demand curve in its almost vertical parts.

VI. CONCLUSION

We present a mixed oligopoly model with conjectural variations equilibrium (CVE) in this paper. The CVE model does not necessarily have a continuous demand function and the special agent is a labor-managed company. The agents’ conjectures refer to price variations that depend on whether their production output increases or decreases. We establish results concerning uniqueness and existence for the CVE (exterior equilibrium) through any selection of possible conjectures. We build a criterion to define the consistency of conjectures, which we call influence coefficients. We also provide an existence theorem concerning the interior equilibrium, which is defined as the CVE with consistent conjectures.

Furthermore, we illustrate an experiment on the behavior of the (strong) interior equilibrium under changes in consumer demand, depicting similarities with the actual behavior of the market in real-life situations.

In our future works, we will prepare the basic study of dynamics in demand and supply by exploring the behavior
of the consistent conjectures as functions of a parameter based on the demand function’s (sub)derivative with respect to price.

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N. Kalashnykova et al.: Consistent Conjectural Variations Equilibrium in a Mixed Oligopoly Model
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