Diverse applications of the gauging of equations method

B. Blankleider¹, A. N. Kvinikhidze¹,², and T. Skawronski¹

¹School of Chemical and Physical Sciences, Flinders University, Bedford Park, SA 5042, Australia,
²The Mathematical Institute of Georgian Academy of Sciences, Tbilisi, Georgia

Abstract.

The gauging of equations method was developed some years ago in order to extract gauge invariant electromagnetic currents from strongly interacting few-body systems described by integral equations. Although the method is by now well-established for this purpose, it is not so well known that the same approach can be used to solve a wide variety of problems. The purpose of this paper is to demonstrate the true power of this method by using it to solve key problems in three very different areas of study; in particular, we show (i) how generalized parton distributions (GPDs) can be determined in the case where hadrons are described in terms of their partonic degrees of freedom through solutions of dynamical equations, (ii) how to construct gauge invariant two-nucleon currents in cutoff effective field theory (EFT), even though the cutoff usually violates the gauge invariance of the underlying Lagrangian, and (iii) how to construct a potential model of \(\pi N\) scattering that is crossing symmetric and approximately unitary.

It is noteworthy that in each case, the gauging of equations method produces results that are consistent with the relevant requirements of quantum field theory. For example, in extracting GPDs from the model of strong interactions defined by dynamical equations, all possible mechanisms contributing to the GPDs are taken into account, so that all GPD sum rules are satisfied automatically. Similarly, in the construction of electromagnetic currents in cutoff EFT, gauge invariance is a consequence of the fact that an external photon is effectively coupled to every part of every strong interaction diagram in the model.

1. Introduction

Experimentally, photons have long been used as probes to study the structure of hadronic systems. Theoretically, the corresponding reactions are described by matrix elements of the electromagnetic (EM) current operator, \(J^\mu\), and it must ultimately be the goal of the theorist to construct models where such matrix elements describe the experimental data as accurately as possible. In this respect, an essential constraint on \(J^\mu\) is that it should be introduced in a gauge invariant way. Note that gauge invariance is a stronger constraint than just current conservation, \(\partial_\mu J^\mu(x) = 0\): beyond current conservation, it requires commutation relations of the form \([J^0, \Psi] = \hat{e}\Psi\), where the matrix \(\hat{e}\) extracts the charge of the field \(\Psi\). This difference is particularly important for an effective field theory (EFT), where an additional constraint increases its predictive power.

Gauge invariance is a fundamental property of any theory at the Lagrangian level, and it is essential to preserve this gauge invariance in the formalism leading to matrix elements that
describe physical strong interaction processes and their coupling to an external electromagnetic field. Here we are concerned with descriptions where the strong interaction processes are described by dynamical integral equations. These may be, for example, four-dimensional Bethe-Salpeter type equations that sum up, non-perturbatively, an infinite number of Feynman diagrams, or they may be their properly reduced three-dimensional versions. Gauge invariance in such equations is manifested in the form of Ward-Takahashi identities between certain Green functions. Because one uses models to approximate the full theory, gauge invariance is not guaranteed \textit{a priori}; for this reason, much effort has been devoted to the question of how to impose this invariance within such strong interaction models [1, 2, 3, 4, 5, 6, 7, 8, 9].

Here we present a method for constructing gauge invariant currents that is applicable to any type of strongly interacting systems which are described nonperturbatively by integral equations. This method involves a direct gauging of the equations themselves in such a way that one gets a linear equation for a Green function of the strongly interacting system with the external photon effectively attached everywhere, so that gauge invariance and thereby current conservation is guaranteed.

A notorious problem plaguing many field-theoretical approaches based on the nonperturbative summing of Feynman diagrams is that of overcounting [10, 11, 6]. An important feature of the gauging of equations method is that it not only guarantees gauge invariance, but it also attaches photons everywhere without introducing overcounting. In this sense the actual effect of the method exceeds expectations, and perhaps it is for this reason that this additional feature of the method has been somewhat under-appreciated. For example, a group intensively engaged in calculations of nucleon form-factors in the quark-diquark model fell victim to the overcounting problem [12, 13], despite the fact that such overcounting was discussed in the context of the gauging of equations method a few years earlier [11, 5]. Even a specific Comment [14] on the overcounting in Refs. [12, 13] turned out not to be convincing, and it took another 5 years until the problem was finally recognised and the corresponding expressions corrected in further publications since 2005 [15, 16, 17].

Given the power of the gauging of equations method just outlined, it is gratifying that it can be used on a wide variety of problems. Its use in generating electromagnetic currents of standard three-dimensional (3D) and four-dimensional (4D) few-body problems is by now well documented [2, 3, 4, 5, 18, 6, 7, 8, 19, 20, 9]. The purpose of the present paper is to demonstrate the versatility of this method by applying it to three very different areas of intense current interest - generalized parton distributions (GPDs), nuclear effective field theory (EFT), and potential models of $\pi N$ scattering. The paper therefore has three main parts. The first part, Section II, is devoted to the construction of the GPDs along with the gauge invariant currents within the model of strong interactions defined by dynamical equations. Here the gauging of equations method provides a powerful approach where gauge invariance and GPD sum rules are implemented in a theoretically correct fashion, and with all possible overcounting avoided. In the second part, Section III, for the first time a gauge invariant approach to cutoff-regularized two-nucleon ($NN$) EFT is constructed. It is noteworthy that our approach is able to construct gauge invariant currents and preserve power counting, in spite of the fact that such regularization destroys the gauge invariance built into the Lagrangian. Finally in Section IV, the latest application of the gauging of equations method is presented: we provide an approximate solution to the long-standing problem of constructing a non-perturbative potential model of $\pi N$ scattering that, simultaneously, respects both crossing symmetry and unitarity. In particular, the derived $\pi N$ amplitude has exact crossing symmetry while the accuracy of its unitarity depends on the numerical ability of the model to fit data. The key to this achievement is the idea of ”double-gauging” a dressed nucleon propagator with pions.
2. Generalized parton distributions.

The structure of hadrons is comprehensively described through their GPDs [21, 22, 23, 24, 25, 26, 27, 28] (for reviews see Refs. [29, 30]). GPDs are usually defined in terms of light front correlation functions

\[ \langle P' | \bar{q}_\beta(0)q_\alpha(y) | P \rangle \]

where the bilocal operator \( \bar{q}_\beta(0)q_\alpha(y) \) is a product of quark, gluon, or other parton fields, and \( | P \rangle \) is a hadronic state with momentum \( P \). However, perhaps the most generally applicable way to specify GPDs is through the density matrix defined as [22, 23]

\[ \rho_{\alpha\beta}(P', P, k) = \int d^4 y e^{ik \cdot y} \langle P' | T \bar{q}_\beta(0)q_\alpha(y) | P \rangle_C \delta(y^+) \]  

(1)

where \( 'T' \) stands for usual time \( (y^0) \) ordering, and subscript \( 'C' \) indicates that only connected contributions are retained. Eq. (1) describes the virtual (off-shell) scattering of partons off the hadronic state, as illustrated in Fig. 1. Then, for the study of deeply virtual Compton scattering (DVCS) where one is restricted to the light front \( (y^+ = 0) \), Eq. (1) can be used to define the light front distribution function

\[ \rho_{\alpha\beta}(P', P, k) \equiv \int \frac{dk^-}{2\pi} \rho_{\alpha\beta}(P', P, k) = \int d^4 y e^{ik \cdot y} \langle P' | T \bar{q}_\beta(0)q_\alpha(y) | P \rangle_C \delta(y^0). \]  

(2)

where \( k = (k^+, k^\perp) \). A further integration over \( d^2 k^\perp \) then leads to the usual definition of GPDs.\(^1\)

On the other hand, for studies where rotational invariance is important, like investigations of the shape of a nucleon [33, 34, 35], Eq. (1) can be used to define a rotationally invariant distribution function on the \( y^0 = 0 \) surface

\[ \rho_{\alpha\beta}(P', P, k) \equiv \int \frac{dk^0}{2\pi} \rho_{\alpha\beta}(P', P, k) = \int d^4 y e^{ik \cdot y} \langle P' | T \bar{q}_\beta(0)q_\alpha(y) | P \rangle_C \delta(y^0). \]  

(3)

Also, the same expression for \( \rho_{\alpha\beta} \) as Eq. (1), but where \( \bar{q}_\beta \) and \( q_\alpha \) are nucleon fields and \( | P \rangle \) is the state vector corresponding to an atomic nucleus, can be similarly used to describe the nucleonic structure of nuclei.

Using Eq. (1) as the basic starting point for the description of GPDs and other quantities, this paper addresses the question of how to extract the density matrix \( \rho_{\alpha\beta} \) in the case where the hadron’s structure is modeled by a dynamical equation describing the mutual scattering of its constituents. As a concrete example, we consider the case of a meson or diquark modeled by the Bethe-Salpeter equation, as well as the case of a baryon, modeled as the bound state of three quarks, whose bound state wave function is found by solving a Faddeev-like three-body scattering equation. For this purpose we consider both a four-dimensional (4D) formulation of the three-body scattering equations, and a covariant three-dimensional (3D) one using the "spectator approach" of Gross [36, 37, 38].

\(^1\) Here one also needs to replace the time-ordered product of fields by an ordinary product, a step justified for the diagonal case \( P = P' \) in Ref. [31], and for the general case in Ref. [32].
An important requirement in any extraction of $\rho_{\alpha\beta}$ is the preservation of sum rules which relate GPDs to electromagnetic form factors. In this regard, the density matrix defined by Eq. (1) must satisfy the sum rule of Eq. (25) which relates $\rho_{\alpha\beta}$ to the electromagnetic vertex function $\Gamma^\mu$. Charge and current conservation then lead to two further sum rules: Eq. (28), and Eq. (29) - we shall refer to these three sum rules collectively as 'GPD sum rules'. In any given model, of course, these sum rules are not guaranteed. Indeed, the nonperturbative nature of the model considered here (nonperturbative solutions of scattering equations), makes the task of extracting $\rho_{\alpha\beta}$, while preserving the GPD sum rules, particularly challenging.

Some years ago a similar problem presented itself: how to determine the electromagnetic currents of hadronic systems described by nonperturbative solutions of scattering equations. Here, of course, the currents had to obey charge and current conservation. The solution to this problem came with the development of the 'gauging of equations method' [2, 3, 4, 5, 6, 7]. This method not only solves the problem of extracting charge and current conserving electromagnetic currents, but it does so in accordance with strict adherence to theory: the external electromagnetic field is attached to all possible places in the nonperturbative strong interaction processes defined by the dynamic equation model. A further feature of the method is that it automatically takes care of all the overcounting problems that plague 4D approaches [10, 8, 4]. Indeed, the gauging of equations method has since been instrumental in the construction of current conserving electromagnetic interactions [39, 40, 41, 9], and in enabling careful analyses of the overcounting problem [14, 42, 9].

In the present paper we exploit the fact that the gauging of equations method can be applied not only to matrix elements of the electromagnetic current operator, but to all other field operators as well. In particular, by applying this method to the case of matrix elements of bilocal operators, we are able to construct the density matrix $\rho_{\alpha\beta}$, and thus GPDs, of hadronic systems described by dynamical equations. Moreover, just like 'gauging equations with a local electromagnetic operator' results in currents where the electromagnetic field is attached to all possible places in the strong interaction model, 'gauging equations with a bilocal operator' results in GPDs where the two external legs $\bar{q}_\beta$ and $q_\alpha$ (bilocal field) originate from cutting all possible bare quark propagators in the strong interaction model. It is this completeness of 'cutting bare propagators' that makes the resulting $\rho_{\alpha\beta}$ satisfy the GPD sum rules.

More specifically, by applying the gauging of equations method for bilocal fields to a dynamical equation describing a hadronic state, we obtain an expression for the density matrix $\rho_{\alpha\beta}$ that obeys the GPD sum rules whenever the input quantities - the distribution functions that result from cutting all bare propagators in (i) the dressed quark propagator, and (ii) the quark-quark potential - themselves satisfy corresponding sum rules. Because the gauging of equations method for bilocal fields cuts all bare propagators in the model, the resulting density matrix not only satisfies the GPD sum rules, but it does so in the theoretically correct way.

For the concrete model of a baryon described by Faddeev-like equations, the gauging of equations method leads to Eq. (42) for the density matrix $\rho_{\alpha\beta}$. This expression satisfies the GPD sum rules whenever the inputs satisfy corresponding sum rules, and can be used directly for practical calculations.

2.1. Gauging

By 'gauging' we shall mean the transformation $G \rightarrow G^U$ where $G$ is the $n$-point Green function

$$G(x_1,\ldots,x_n) = \langle 0|Tq(x_1)\ldots\bar{q}(x_n)|0\rangle,$$

and $G^U$ is the 'gauged' Green function defined as the corresponding $(n+2)$-point function

$$G^U(x_1,\ldots,x_n;x,y) = \langle 0|Tq(x_1)\ldots\bar{q}(x)q(y)\ldots\bar{q}(x_n)|0\rangle_c.$$
where subscript \( c \) indicates that all contributions having a disconnected piece \((0|T\bar{q}(x)q(y)|0)\) are excluded. To be definite, we shall refer to quark parton distributions in hadrons and thus take all the \( q(x_1), \ldots, q(x_n) \), \( q(x) \) and \( q(y) \) to represent quark fields; however, it should be understood that our discussion applies equally well, for example, to gluon distributions in which case \( q(x) \) and \( q(y) \) would represent gluon fields, and to nucleon distributions within a nucleus, in which case \( q(x_1), \ldots, q(x_n) \), \( q(x) \) and \( q(y) \) would all represent spinor nucleon fields. To make the connection with GPDs more clear, it is also useful to write Eq. (5) with two of the spinor components made explicit:

\[
G^U_{\alpha\beta}(x_1, \ldots, x_n; x, y) = (0|T\bar{q}(x_1)\cdots\bar{q}(x)q_\alpha(y)\cdots q(y)|0)_{\epsilon}. \tag{6}
\]

The definition of the gauged Green function given in Eq. (5) can be considered as an extension to bilocal fields of the definition used in the case of coupling to a local field, for example the electromagnetic field, which involves the \((n+1)\)-point function

\[
G^\mu(x_1, \ldots, x_n; z) = (0|Tq(x_1)\cdots\bar{q}(z)\tilde{\Gamma}^\mu q(z)\cdots q(x_n)|0) \tag{7}
\]

where \( J^\mu(z) = \bar{q}(z)\tilde{\Gamma}^\mu q(z) \) is the electromagnetic current operator. Indeed the gauging method used in this paper is closely based on the one developed for the \((n+1)\)-point function of Eq. (7) \[4, 5, 6, 7\].

We are interested in the case where the Green function \( G \) is modelled nonperturbatively as the solution to an integral equation of the form

\[
G = G_0^P + G_0 VG \tag{8}
\]

where \( G_0 \) is a product of dressed single particle propagators, \( G_0^P \) is the antisymmetrized version of \( G_0 \), and \( V \) is the interaction kernel (in the 3 \( \rightarrow \) 3 processes of main interest here, \( V \) consists of all possible 3-particle irreducible diagrams). In the ‘gauging of equations method’, the \((n+1)\)-point Green function \( G^\mu \) is obtained by ‘gauging’ Eq. (8) with a local (vector) field as \[4, 5, 6, 7\]

\[
G^\mu = G_0^{\mu P} + G_0^\mu VG + G_0 V^\mu G + G_0 VG^\mu. \tag{9}
\]

Similarly, we obtain the \((n+2)\)-point Green function \( G^U \) by ‘gauging’ Eq. (8) with a bilocal (spinor) field as

\[
G^U = G_0^{U P} + G_0^U VG + G_0 V^U G + G_0 VG^U. \tag{10}
\]

The one-to-one correspondence between these two types of gauging is self-evident, with each of the above two equations leading to solutions (for \( G^\mu \) and \( G^U \)) that are of identical form. When necessary to distinguish between these two types of gauging, we shall refer to the transformation \( G \rightarrow G^U \) according to Eq. (10) as ‘U-gauging’, and the transformation \( G \rightarrow G^\mu \) according to Eq. (9) as ‘\( \mu \)-gauging’.

Although this method of gauging is designed specifically for nonperturbative approaches, its idea is rooted in perturbation theory: for example, to any diagram \( D \) contributing to

\[
\begin{align*}
&\text{(a)} & &\text{(b)} & &\text{(c)}
\end{align*}
\]

\[\text{Figure 2.} \quad \text{Pictorial representation of 'U-gauging': (a) Bare particle propagator. (b) Bare particle propagator cut through the middle. (c) Inner cut legs form a bilocal external field.}\]
Figure 3. Example of diagrams which cannot be obtained by gauging. (a) One gluon exchange contribution. Joining the end points α and β gives a tadpole diagram which is zero. (b) A three-pion (dashed lines) contribution which, upon joining together the end-points α and β, gives a 2 → 2 diagram which is forbidden by G-parity.

\[ G(x_1, \ldots, x_n), \] there corresponds a sum of diagrams \( D_{\alpha\beta}^{U} \) belonging to \( G_{\alpha\beta}^{U}(x_1, \ldots, x_n; x, y) \), each of which can be obtained by replacing a bare propagator \( d_0(u - v) = \langle 0_f | T q_f(u) \bar{q}_f(v) | 0_f \rangle \) in \( D \) (subscript \( f \) indicates a free field or state) by \( d_0^{U}(u, v; x, y) = \langle 0_f | T q_f(u) \bar{q}_\alpha(x) q_\beta(y) | 0_f \rangle \) where subscript \( c \) ensures that only the contribution corresponding to Fig. 2(c) is retained. Thus Eq. (10) is just the statement that \( G^{U} \) is obtained from \( G \) by inserting operator \( \bar{q}_\alpha(x) q_\beta(y) \) within every bare propagator of every Feynman diagram of the theory. This insertion corresponds pictorially to cutting the bare propagator into two, as illustrated in Fig. 2. Unfortunately not every bare propagator of every Feynman diagram of the perturbation theory generated by \( S \) can be obtained by gauging \( G \) in this way. However, in many problems, including those discussed in the present paper, either only diagrams obtained by gauging are of interest, or the missing ones are easily taken into account. Two examples of diagrams that contribute to \( G_{\alpha\beta}^{U} \) but that cannot be obtained by \( U \)-gauging \( G \) are given in Fig. 3

For those used to the functional integral approach it might be more transparent to define \( U \)-gauging by a functional derivative with respect to an external bilocal field \( B(x, y) \) introduced into the action \( S[q, A] \) (where \( A \) represents gluons or other fields) as \( S[q, A] \rightarrow S[q, A] + \int d^4x d^4y \bar{q}(x) B(x, y) q(y) \) [43]. This modification amounts to the replacement of the inverse bare quark propagator as

\[ d_0^{-1}(x - y) \rightarrow d_{0B}^{-1}(x, y) = d_0^{-1}(x - y) - iB(x, y) \] (11)

in the perturbation theory generated by \( S[q, A] \), in order to obtain the perturbation theory generated by \( S[q, A, B] \). The usual Green function, the vacuum expectation of the time ordered vacuum loops, can be written in terms of a functional integral as

\[ G(\{x_i\}, \{y_j\}, \{z_k\}) = \langle 0| T q(x_i) \ldots \bar{q}(y_j) \ldots A(z_k) \ldots |0 \rangle = \frac{\int Dq \ D\bar{q} \ DA \ e^{iS[q, A]} \cdot q(x_i) \ldots \bar{q}(y_j) \ldots A(z_k) \ldots}{\int Dq \ D\bar{q} \ DA \ e^{iS[q, A]}}. \] (12)

When modified by the presence of the external bilocal field \( B \), this Green function becomes

\[ G_B(\{x_i\}, \{y_j\}, \{z_k\}) = \frac{\int Dq \ D\bar{q} \ DA \ e^{iS[q, A, B]} \cdot q(x_i) \ldots \bar{q}(y_j) \ldots A(z_k) \ldots}{\int Dq \ D\bar{q} \ DA \ e^{iS[q, A, B]}}. \] (13)

Introducing the generating functionals

\[ Z[B] = \int Dq \ D\bar{q} \ DA \ e^{iS[q, A, B]}, \quad Z = Z[0] = \int Dq \ D\bar{q} \ DA \ e^{iS[q, A]}, \] (14)
the gauged Green function can then be defined as
\[ \mathcal{G}^U(\{x_i\}, \{y_j\}, \{z_k\}; x, y) = -i \left. Z^{-1} \frac{\delta Z[B] \mathcal{G}_B(\{x_i\}, \{y_j\}, \{z_k\})}{\delta \mathcal{B}(x, y)} \right|_{c, \mathcal{B}=0} \]
\[ = -i \left. Z^{-1} Z[B] \frac{\delta \mathcal{G}_B(\{x_i\}, \{y_j\}, \{z_k\})}{\delta \mathcal{B}(x, y)} \right|_{c, \mathcal{B}=0} \]
\[ = -i \left. Z^{-1} \frac{\delta Z[B]}{\delta \mathcal{B}(x, y)} \mathcal{G}_B(\{x_i\}, \{y_j\}, \{z_k\}) \right|_{c, \mathcal{B}=0} \]
\[ = -i \left. \frac{\delta \mathcal{G}_B(\{x_i\}, \{y_j\}, \{z_k\})}{\delta \mathcal{B}(x, y)} \right|_{\mathcal{B}=0} + d(x, y) \mathcal{G}(\{x_i\}, \{y_j\}, \{z_k\})|_c. \tag{15} \]

The last term in the Eq. (15) should be discarded as it contributes only to those disconnected terms which are forbidden by the meaning of subscript \(c\). Note that there are still other disconnected contributions to Eq. (15) which should be kept. So the proper functional definition of gauging is just the functional derivative:
\[ \mathcal{G}^U(\{x_i\}, \{y_j\}, \{z_k\}; x, y) = -i \left. \frac{\delta \mathcal{G}_B(\{x_i\}, \{y_j\}, \{z_k\})}{\delta \mathcal{B}(x, y)} \right|_{\mathcal{B}=0}. \tag{16} \]

In this sense Eq. (10) is just a statement of the product rule for derivatives. Note that Eq. (11) for the modified bare propagator is obvious, but can be derived from Eq. (13) as well. The gauged bare propagator can be derived using
\[ \frac{\delta d_{0B}^{-1}(x_1, x_2)}{\delta \mathcal{B}(x, y)} = -i \left. \frac{\delta \mathcal{B}(x_1, x_2)}{\delta \mathcal{B}(x, y)} \right|_{\mathcal{B}=0} = -i \delta(x_1 - x) \delta(y - x_2) \tag{17} \]
and
\[ \frac{\delta (d_{0B} d_{0B}^{-1})}{\delta \mathcal{B}} = \frac{\delta d_{0B}}{\delta \mathcal{B}} d_{0B}^{-1} + d_{0B} \frac{\delta d_{0B}^{-1}}{\delta \mathcal{B}} = 0. \tag{18} \]

One finds that
\[ d_{0U}^U(x_1, x_2; x, y) = -i \left. \frac{\delta d_{0B}(x_1, x_2)}{\delta \mathcal{B}(x, y)} \right|_{\mathcal{B}=0} = d_{0B}(x_1 - x) d_{0B}(y - x_2) \]
\[ d_{0U}^U(x_1, x_2; x, y) = -i \left. \frac{\delta d_{0B}(x_1, x_2)}{\delta \mathcal{B}(x, y)} \right|_{\mathcal{B}=0} = d_0(x_1 - x) d_0(y - x_2). \tag{19} \]

As expected from the discussion above, the antisymmetrizing contribution, \(d_0(x_1 - x_2) d_0(y - x)\), does not appear in the expression for the gauged free propagator \(d_{0U}^U(x_1, x_2; x, y)\). It is also clear from Eq. (19) how the gauging of the bare quark propagator corresponds, diagrammatically, to cutting the bare propagator into two pieces, as illustrated in Fig. 2. Finally, Eq. (16), together with the product rule for derivatives, enables one to see that gauging any complicated diagram corresponds to cutting bare propagators entering this diagram in all possible ways.

2.2. Extracting GPDs from three-body scattering equations

2.2.1. GPD sum rules

As discussed above, all GPDs can be obtained from the density matrix \(\rho_{\alpha \beta}\) defined in Eq. (1). In turn, \(\rho_{\alpha \beta}\) can be found from the \((n + 2)\)-point Green function \(\mathcal{G}_{\alpha \beta}^U\), as given by Eq. (6), by inserting a complete set of states on either side of the operator \(\tilde{q}_\beta(x) q_\alpha(y)\) and then taking residues at the bound state poles corresponding to the physical states \(|P\rangle\) and
By writing \( G'^{U}_{\alpha\beta} = G U^{U}_{\alpha\beta} G \) and recognizing that \( G \sim i \Psi_P \bar{\Psi}_P / (P^2 - M^2) \) in the vicinity of the \( P^2 = M^2 \) pole, one obtains

\[
\rho_{\alpha\beta}(P', P, k) = \int d^4y e^{ik\cdot y} \langle P' | T \bar{q}_\beta(0) q_\alpha(y) | P \rangle c = \bar{\Psi}_P \Gamma_{\alpha\beta}^U \Psi_P
\]

where \( \Psi_P \) is the bound state wave function corresponding to state \( |P\rangle \):

\[
\Psi_P(x_1, x_2, x_3) = \langle 0 | T q(x_1) q(x_2) q(x_3) | P \rangle,
\]

and

\[
\Gamma_{\alpha\beta}^U = G^{-1} G_{\alpha\beta}^U G^{-1}
\]

is the corresponding bound state vertex function. To find \( G_{\alpha\beta}^U \) we shall use the \( U \)-gauging method described in the previous section. This is in direct analogy to what was done in Refs. [4, 5, 6, 7] to find the electromagnetic bound state current which is given as

\[
j^\mu(P', P) = \langle P' | \bar{q}(0) \hat{\Gamma}^\mu q(0) | P \rangle = \bar{\Psi}_P \Gamma^\mu \Psi_P
\]

where

\[
\Gamma^\mu = G^{-1} G^\mu G^{-1}.
\]

The close similarity between the definitions of \( G^U \) in Eq. (5) and \( G^\mu \) in Eq. (7) is embodied in the sum rule

\[
\sum_{\alpha,\beta} \int \frac{d^4k}{(2\pi)^4} \rho_{\alpha\beta}(P', P, k) \hat{\Gamma}^\mu_{\beta\alpha} = j^\mu(P', P),
\]

which follows from Eq. (20) and Eq. (23). Furthermore, since the bound state current satisfies current conservation,

\[
(P' - P)^\mu j^\mu(P', P) = 0,
\]

and charge conservation,

\[
j^\mu(P, P) = 2QP^\mu,
\]

where \( Q \) is the total charge of the three-body bound state, the density matrix \( \rho_{\alpha\beta} \) satisfies two further sum rules,

\[
(P' - P)^\mu \sum_{\alpha,\beta} \int \frac{d^4k}{(2\pi)^4} \rho_{\alpha\beta}(P', P, k) \hat{\Gamma}^\mu_{\beta\alpha} = 0,
\]

and

\[
\sum_{\alpha,\beta} \int \frac{d^4k}{(2\pi)^4} \rho_{\alpha\beta}(P, P, k) \hat{\Gamma}^\mu_{\beta\alpha} = 2QP^\mu.
\]

Note that \( |P\rangle \) is an eigenstate of the conserved charge operator \( \hat{Q} = \int d^3x J^0(x) \) with corresponding eigenvalue \( Q \); that is, \( Q \) is a physical quantity corresponding to the conserved Noether current \( J^\mu(x) = \bar{q}(x) \hat{\Gamma}^\mu q(x) \). Although we take \( J^\mu \) to be the conserved electromagnetic current operator, it is clear that essentially the same expressions will hold for conserved isotopic vector currents (CVC), conserved axial currents (CAC), and partially conserved axial currents (PCAC) for which the right-hand side (RHS) of Eq. (26) and Eq. (28) would be non-zero [44].
2.2.2. GPDs of two-body bound states  Before discussing the GPDs of three-body bound states described by Faddeev-like equations, it is useful to first demonstrate the main ideas of this approach on the simpler case of two-body bound states described by the Bethe-Salpeter (BS) equation. In this respect we note that GPDs in Bethe-Salpeter approaches have already received some attention in the literature [45, 46, 47].

To describe the scattering of two distinguishable particles (e.g., a quark and an anti-quark) one can use the integral equation for the two-body Green function 

\[ G = \Gamma G, \]

(30)

where \( \Gamma = \Gamma_1 d_1^{-1} + \Gamma_2^{-1} + V \),

(31)

and \( d_i \) is the propagator of particle \( i \), \( \Gamma_i \) is its electromagnetic vertex function, and \( V \) is the gauged two-body potential. The two-body bound state current can then be found by taking left and right residues of \( G \) at the bound state poles:

\[ j^\mu = \bar{\phi} (d_1^\mu d_2 + d_1^\mu d_2 V d_1^\mu) \phi \]

(32)

where \( \psi = d_1 d_2 \phi \) defines the two-body bound state vertex function \( \phi \) in terms of the two-body bound state wave function \( \psi \), and \( d_i^\mu = d_i \Gamma_i^\mu d_i \) is the \( \mu \)-gauged propagator of particle \( i \). This agrees with the result first derived by Gross and Riska [1]. An alternative but equivalent approach is to start with the Bethe-Salpeter equation for \( \phi \):

\[ \phi = V d_1 d_2 \phi. \]

(33)

This equation can then be gauged to obtain [6]

\[ \phi^\mu = T(d_1 d_2)^\mu \phi + (1 + T d_1 d_2) V^\mu d_1 d_2 \phi \]

(34)

where \( T \) is two-body t matrix. Taking the residue at the bound state pole of \( T \) one again obtains Eq. (32). As shown in Ref. [1], the \( j^\mu \) of Eq. (32) obeys charge and current conservation if the inputs \( d_i^\mu \) and \( V^\mu \) obey both Ward and Ward-Takahashi identities.

Replacing \( \mu \)-gauging by \( U \)-gauging in the above derivations, one obtains the equation corresponding Eq. (32):

\[ \rho = \bar{\phi} (d_1^\rho d_2 + d_1^\rho d_2 V U d_1 d_2) \phi \]

(35)

where \( \rho \) is the matrix whose components are \( \rho_{\alpha\beta} \). It is clear that the \( \rho \) of Eq. (35) and the \( j^\mu \) of Eq. (32) will obey the GPD sum rule of Eq. (25) if the input pairs \( (d_i^\mu, d_i^\rho) \), and \( (V^\mu, V U) \) each obey corresponding sum rules. This aspect is an essential ingredient of our approach and will be discussed in more detail below. The other two GPD sum rules, Eq. (28) and Eq. (29), are then automatically satisfied because of the charge and current conservation properties of \( j^\mu \). Eq. (35) is illustrated in Fig. 4.

2.2.3. GPDs of three-body bound states  The sum rules for GPDs, derived in subsection A above, constitute important constraints satisfied by the exact theory of strong interactions. As such, it is desirable that the same sum rules be also satisfied by models that seek to approximate the exact theory. In Ref. [6, 7] we showed that for nonperturbative strong interaction models described by integral equations, the gauging of equations method, when applied to obtain \( G^\mu \) and therefore the bound state electromagnetic current \( j^\mu \), results in both charge and current conservation being satisfied. We shall now show that the same gauging method, when applied
to obtain $G^U$ and therefore $\rho_{\alpha\beta}$, results additionally in the GPD sum rules of Eq. (25), Eq. (28), and Eq. (29) being satisfied.

Here we apply the U-gauging procedure to the case of three identical particles, so that our expression for $\rho_{\alpha\beta}$ can be used directly for calculations of GPDs in the nucleon or nucleon distributions in $^3$He. We assume that the interaction kernel $V$ defining $G$ [Eq. (8)] is given as a sum of only two-particle interactions, so that

$$V = \sum_{i=1}^{3} \frac{1}{2} v_i d_i^{-1}$$  \hspace{1cm} (36)

where $v_i$ is the (fully antisymmetric) interaction potential between particles $j$ and $k$ (where $ijk$ is a cyclic permutation of 123), $d_i$ is the dressed propagator of particle $i$, and the $1/2$ is a factor arising from antisymmetry [7]. As shown in Ref. [7], the $\mu$-gauging of Eq. (8) then gives the following expression for $G^\mu$:

$$G^\mu = G \Gamma^\mu G,$$  \hspace{1cm} (37)

$$\Gamma^\mu = \frac{1}{6} \sum_{i=1}^{3} \left( \Gamma_i^\mu d_i^{-1} d_k^{-1} + \frac{1}{2} v_i^\mu d_i^{-1} - \frac{1}{2} v_i d_i^\mu \right).$$  \hspace{1cm} (38)

Taking left and right residues of $G^\mu$ then gives the bound state electromagnetic current of three identical particles:

$$j^\mu(P', P) = \frac{1}{6} \sum_{i=1}^{3} \Phi_{P'} d_j d_k \left( d_i^\mu d_j^{-1} d_k^{-1} + \frac{1}{2} v_i^\mu d_i^{-1} - \frac{1}{2} v_i d_i^\mu \right) d_j d_k \Phi_{P}$$  \hspace{1cm} (39)

where $\Psi_{P} = d_1 d_2 d_3 \Phi_{P}$ defines the three-body bound state vertex function $\Phi_{P}$ in terms of the three-body bound state wave function $\Psi_{P}$. Replacing $\mu$-gauging by $U$-gauging in the above gives the three-body density matrix

$$\rho(P', P, k) = \frac{1}{6} \sum_{i=1}^{3} \Phi_{P'} d_j d_k \left( d_i^U d_j^{-1} d_k^{-1} + \frac{1}{2} v_i^U d_i^{-1} - \frac{1}{2} v_i d_i^U \right) d_j d_k \Phi_{P}.$$  \hspace{1cm} (40)

Again, it is clear that the $\rho$ of Eq. (40) and the $j^\mu$ of Eq. (39) will obey the sum rule of Eq. (25) if the input pairs ($d_i^\mu$, $d_i^U$), and ($v_i^\mu$, $v_i^U$) each obey corresponding sum rules, and that the sum rules of Eq. (28) and Eq. (29) are also satisfied because the $j^\mu$ of Eq. (39) satisfies both charge and current conservation. The contributions to $\rho$, as given by Eq. (40), are illustrated in Fig. 5. It is noteworthy that the last term of Eq. (40) (last term of Fig. 5), comes with a negative sign; as discussed in Ref. [7], this subtraction term is necessary to remove overcounted contributions that are present in the first term.
Figure 5. Contributions to the three-body density matrix of Eq. (40) specifying the GPD’s of three-body bound states.

As in the two-body case discussed above, one can give an alternative derivation of $j^\mu$ and $\rho$ by gauging the bound state equation for the vertex function. This has the advantage that one gauges directly the equation that one actually solves numerically; this way, for example, any approximations used in the solution of the equation will appropriately be taken into account in the gauged result. Indeed, because of the numerical difficulty of solving 4D integral equations, one is often interested to perform a 3D reduction of the original 4D approach. If one invokes such a reduction, it turns out that it is better to reduce the dimension of 4D integral equations first, and then gauge the resulting 3D bound state equations in order to deduce the expressions for $j^\mu$ and $\rho$. This point was examined in detail in Ref. [48] in the context of Gross’s 3D spectator approach [36, 37, 38]. Here we shall present the results of gauging only the 4D bound state equation, and refer the reader to Ref. [48] for the analogous 3D case.

As is well known from three-body theory, the bound state vertex function $\Phi$ can be specified by writing it as a sum of Faddeev components, $\Phi = \Phi_1 + \Phi_2 + \Phi_3$, where

$$
\Phi_1 = -t_1d_2d_3P_{12}\Phi_1. \tag{41}
$$

In Eq. (41) $t_1$ is the two-body t matrix between particles 2 and 3 and $P_{12}$ is the operator that interchanges particles 1 and 2. The two-body potential $v_1$ and corresponding t matrix $t_1$ are defined to be antisymmetric ($P_{23}v_1 = -v_1$ and $P_{23}t_1 = -t_1$) so that full antisymmetry of the three-body wave function $\Psi = d_1d_2d_3\Phi$ is ensured.

In order to find the expression for $\rho_{\alpha \beta}$, as defined above, we can follow Ref. [5] and directly U-gauge Eq. (41). After some algebra we obtain

$$
\rho(P', P, k) = \Phi_1' d_1 (d_2'^\mu d_3 + d_2 d_3'^\mu) \left( \frac{1}{2} - P_{12} \right) \Phi_1'
+ \Phi_1' \left( P_{12} - \frac{1}{2} \right) d_2d_3d_1'v_1'^\mu d_2d_3(P_{12} - \frac{1}{2})\Phi_1', \tag{42}
$$

with a corresponding bound state electromagnetic current

$$
j^\mu(P', P) = \Phi_1' d_1 (d_2'^\mu d_3 + d_2 d_3'^\mu) \left( \frac{1}{2} - P_{12} \right) \Phi_1'
+ \Phi_1' \left( P_{12} - \frac{1}{2} \right) d_2d_3d_1'v_1'^\mu d_2d_3(P_{12} - \frac{1}{2})\Phi_1'. \tag{43}
$$

The exact expression for $\rho$, Eq. (42), corresponds to cutting (as in Fig. 2) every one of the infinite number of bare particle propagators that exist in the nonperturbative expression for the strong interaction Green function $G$, Eq. (8). In the same way the corresponding expressions for the electromagnetic current $j^\mu$, Eq. (43), corresponds to attaching a photon to all the bare propagators in the nonperturbative theory. It is this completeness of the photon attachment
that guarantees the charge and current conservation properties of $j^\mu$. It is evident that the $\rho$ of Eq. (42) and the $j^\mu$ of Eq. (43) will fulfill the GPD sum rules whenever the gauged input quantities, $\Gamma^\mu$ and $\bar{\Gamma}_J^\mu$, satisfy corresponding sum rules.

2.2.4. Gauged propagator The gauged one-particle propagators $d^\mu$ and $\bar{\Gamma}_J^\mu$ form one of the basic inputs to the equations describing the three-body bound state current $j^\mu$, Eq. (39), and the density matrix $\rho$, Eq. (40), respectively. In momentum space these gauged propagators are defined as

$$d^\mu(p', p) = \int d^4x \; d^4y \; e^{i(p' - p \cdot y)} \langle 0 | T q(x) \tilde{q}(0) \hat{\Gamma}^\mu(0) \tilde{q}(y) | 0 \rangle$$

and

$$d^\mu_{\alpha\beta}(p', p, k) = \int d^4x \; d^4y \; d^4z \; e^{i((p' + k) - p \cdot z)} \langle 0 | T q(x) \tilde{q}_\beta(0) q_\alpha(z) \tilde{q}(y) | 0 \rangle.$$  \hspace{1cm} (44)

Writing the spinor indices of $q(x)$ and $\tilde{q}(y)$ explicitly, the subscript $c$ in Eq. (45) means that the disconnected part $\langle 0 | T q_\alpha(x) \tilde{q}_\beta(y) | 0 \rangle$ of Eq. (40) is excluded. On the other hand, the remaining disconnected part, $\langle 0 | T q_\alpha(x) \tilde{q}_\beta(0) | 0 \rangle$, does contribute to the U-gauged propagator as

$$d^\mu_{\alpha\beta}(p', p, k)_{\text{disc}} = (2\pi)^4 \delta^4(k - p) d^\mu_{\alpha\beta}(p') d^\mu_{\alpha\beta}(p).$$  \hspace{1cm} (46)

Similarly to Eq. (25), the two types of gauged propagator are related by a sum rule:

$$\sum_{\alpha, \beta} \int \frac{d^4k}{(2\pi)^4} \; d^\mu_{\alpha\beta}(p', p, k) \hat{\Gamma}^\mu_{\beta\alpha} = d^\mu(p', p).$$  \hspace{1cm} (47)

In addition, $d^\mu$ satisfies the Ward-Takahashi (WT) identity

$$\langle p' - p)^\mu d^\mu(p', p) = i e [d(p) - d(p')] \hspace{1cm} (48)$$

and the Ward identity

$$d^\mu(p', p) = -ie \frac{\partial d(p)}{\partial p_\mu},$$  \hspace{1cm} (49)

where $e$ is the charge of the particle (for isodoublet particle fields $e$ is a $2 \times 2$ matrix). We thus can specify two further sum rules for $d^\mu_{\alpha\beta}$ as

$$\langle p' - p)^\mu \sum_{\alpha, \beta} \int \frac{d^4k}{(2\pi)^4} \; d^\mu_{\alpha\beta}(p', p, k) \hat{\Gamma}^\mu_{\beta\alpha} = i e [d(p) - d(p')]$$  \hspace{1cm} (50)

and

$$\sum_{\alpha, \beta} \int \frac{d^4k}{(2\pi)^4} \; d^\mu_{\alpha\beta}(p', p, k) \hat{\Gamma}^\mu_{\beta\alpha} = -ie \frac{\partial d(p)}{\partial p_\mu}.\hspace{1cm} (51)$$

Also, writing

$$d^\mu(p', p) = d(p') \Gamma^\mu d(p),$$  \hspace{1cm} (52a)

$$d^\mu_{\alpha\beta}(p', p, k) = d(p') \Gamma^\mu_{\alpha\beta} d(p),$$  \hspace{1cm} (52b)

where $\Gamma^\mu$ and $\Gamma^\mu_{\alpha\beta}$ are one-body vertex functions, the WT and Ward identities give

$$\langle p' - p)^\mu \Gamma^\mu(p', p) = i e [d^{-1}(p') - d^{-1}(p)]$$  \hspace{1cm} (53)
and
\[ \Gamma^\mu(p,p) = ie \frac{\partial d^{-1}(p)}{\partial p_\mu} \] (54)
respectively, while Eq. (47) implies that
\[ \sum_{\alpha,\beta} \int \frac{d^4k}{(2\pi)^4} \Gamma^U_{\alpha\beta}(p',p,k) \hat{\Gamma}_\beta = \Gamma^\mu(p',p). \] (55)

Note that, to save on notation, we use the same symbols to denote the one-body vertex functions of Eq. (52) and the three-body bound state vertex functions of Eq. (22) and Eq. (24) - the type of vertex function meant should be clear from the context; moreover, it is easy to see that Eq. (55) in fact holds true for both cases. Combining Eq. (53) and Eq. (54) with Eq. (55) gives the sum rules
\[ (p' - p)^\mu \sum_{\alpha,\beta} \int \frac{d^4k}{(2\pi)^4} \Gamma^U_{\alpha\beta}(p',p,k) \hat{\Gamma}_\beta = ie \left[ d^{-1}(p') - d^{-1}(p) \right], \] (56)
\[ \sum_{\alpha,\beta} \int \frac{d^4k}{(2\pi)^4} \Gamma^U_{\alpha\beta}(p,p,k) \hat{\Gamma}_\beta = ie \frac{\partial d^{-1}(p)}{\partial p_\mu}, \] (57)
which relate the vertex function \( \Gamma^U \) to the propagator \( d \).

In the absence of dressing, in which case we write Eqs. (52) as
\[ d^U_0(p',p) = d_0(p') \Gamma^0 d_0(p), \] (58a)
\[ d^U_{0,\alpha\beta}(p',p,k) = d_0(p') \Gamma^U_{0,\alpha\beta} d_0(p), \] (58b)
where \( d_0 = i/(\not{p} - m) \) is the bare particle propagator, and Eq. (58b) with all spinor indices revealed reads
\[ d^U_{0,\alpha\beta}(p',p,k) = \sum_{k,l} d_0(ik) \Gamma^U_{0,k\alpha,l\beta} d_0(\alpha j), \] (59)

one finds that
\[ \Gamma^\mu_0 = \hat{\Gamma}^\mu, \] (60a)
\[ \Gamma^U_{0,k\alpha,l\beta} = (2\pi)^4 \delta(k - p) \delta_{k\beta} \delta_{l\alpha}. \] (60b)

As mentioned previously, the density matrix \( \rho \), as specified by Eq. (42), will satisfy the GPD sum rules only if the gauged inputs \( d^U \) and \( v^U \), will themselves satisfy corresponding sum rules. For the case of \( d^U \), the above discussion shows that, in the exact theory, \( d^U \) does indeed satisfy such sum rules, namely, Eq. (47), Eq. (50), and Eq. (51). However, what is of practical interest in our approach is to construct strong interaction models of \( d^U \) such that Eq. (47), Eq. (50), and Eq. (51) are still satisfied. In this respect, one might think of modeling \( d^U \) by just its disconnected part, as given by Eq. (46):
\[ d^U_{\alpha\beta}(p',p,k) \equiv (2\pi)^4 \delta^4(k - p) d_{\alpha\beta}(p') d_{\alpha j}(p). \] (61)

In this case the left hand side of the sum rule of Eq. (47) would give
\[ \sum_{\alpha,\beta} \int \frac{d^4k}{(2\pi)^4} d^U_{\alpha\beta}(p',p,k) \hat{\Gamma}_\beta = d(p') \hat{\Gamma} d(p). \] (62)
Since $\hat{\Gamma}^{\mu} \neq \Gamma^{\mu}$, the RHS of Eq. (62) is not equal to $d^{\mu}(p', p)$, so the sum rules of Eq. (47), Eq. (50) and Eq. (51) will not be satisfied; indeed, only in the case of no dressing, for which Eq. (60a) holds, does such a model for $d^{U}$ satisfy the required sum rules.

This shows that the connected part of $d^{U}$ needs to be taken into account if we want the GPD sum rules for $\rho$ to be satisfied in a model with dressed propagators. One way that this can be achieved is to construct a purely phenomenological $d^{U}$ which satisfies the sum rules of Eq. (47), Eq. (50), and Eq. (51). However, a theoretically more rigorous way would be to first construct a model for the dressed propagator $d$ using the Dyson-Schwinger (DS) equation, and then $U$-gauging the DS equation. The details of constructing $d^{U}$ in this way will be discussed elsewhere (although Fig. 6, discussed later, provides a graphical example of such an approach).

2.2.5. Gauged potential The last inputs needed to be considered are the gauged two-body potentials $v^{\mu}$ and $v^{U}$. As for the gauged propagators, their construction will depend on the nature of the model chosen for potential $v$. However, independently of the model chosen for $v$, it is a requirement of our approach that $v^{\mu}$ is constructed so that it satisfy the WT identity

$$q_{\mu}v^{\mu}(p_{1}'p_{2};p_{1}p_{2}) = i(\epsilon_{1}v(p_{1}' - q_{1}', p_{2};p_{1}p_{2}) - v(p_{1}'p_{2}; p_{1} + q, p_{2})\epsilon_{1} + \epsilon_{2}v(p_{1}'p_{2}; p_{1}p_{2} - q)\epsilon_{2} - v(p_{1}'p_{2}; p_{1}p_{2} + q)\epsilon_{2}) \tag{63}$$

and the Ward identity

$$v^{\mu}(p', P - p'; p, P - p) = -i \left[ \epsilon_{1} \frac{\partial v(p', P - p'; p, P - p)}{\partial p_{\mu}} + \epsilon_{2} \frac{\partial v(p', P - p'; p, P - p)}{\partial P_{\mu}} \right] \tag{64}$$

where $\epsilon_{i}$ is the charge of particle $i$. In the case where $v$ is defined through a finite sum of Feynman diagrams or through a dynamical equation, applying the $\mu$-gauging procedure to the equation defining $v$ will give a $v^{\mu}$ that automatically satisfies Eq. (63) and Eq. (64).

With an appropriately constructed $v^{\mu}$, all that is left is to construct the $U$-gauged potential $v^{U}$ so that it satisfy the sum rule

$$\sum_{\alpha, \beta} \int \frac{d^{4}k}{(2\pi)^{4}} v^{U}_{\alpha \beta}(p_{1}'p_{2}', p_{1}p_{2}, k) \hat{\Gamma}^{\mu}_{\alpha} = v^{\mu}(p_{1}'p_{2}, p_{1}p_{2}). \tag{65}$$

Again, if $v$ is defined through a finite sum of Feynman diagrams or through a dynamical equation, applying the $U$-gauging procedure to the equation defining $v$ will give a $v^{U}$ that automatically satisfies Eq. (65).

In the case that $v$ is not given in terms of Feynman diagrams or a dynamical equation, one can easily construct $v^{\mu}$ and $v^{U}$ ad hoc, in order to satisfy Eq. (63), Eq. (64), and Eq. (65).

2.3. Discussion In this section we have demonstrated our $U$-gauging method by obtaining the density matrix $\rho_{\alpha \beta}$ of Eq. (1), and therefore GPDs, for the specific cases of a two-quark bound state described by the Bethe-Salpeter equation, and a three-quark bound state described by 4D Faddeev-like equations. The method corresponds to cutting all possible bare quark propagators in the strong interaction model, and ensures that GPD sum rules are satisfied automatically. Also, because our formulation is Lorentz covariant, the derived GPDs obey polynomiality [30]. However, it is important to emphasize that our procedure is general, applying to any type of dynamical equation, as well as to various types of distributions. For example, the $U$-gauging
Figure 6. (a) The dressed propagator $d$ in rainbow approximation. (b) The distribution function $dU = dΓ^U d$ obtained by cutting all possible bare propagators in the rainbow perturbation series for $d$ shown in (a). The propagator $d$ defined by (a) and the vertex function $Γ^U$ defined by (b) will satisfy the GPD sum rules of Eq. (55), Eq. (56) and Eq. (57).

procedure can be applied to effectively cut all the bare propagators existing within a given dynamical equation model, not just those of quarks. In this way one could derive the model’s generalized gluon distribution [30], pion distribution, etc. With sum rules corresponding to current conservation being ensured, one can, for example, selectively turn on and off $U$-gauging of different constituents in order to study the fraction of a hadron’s quantum number (e.g. spin) carried by a quark, gluon, meson, etc.

The whole approach holds also for the 3D spectator formalism [48], and furthermore, can be applied to nucleon distributions in nuclei, i.e., with nucleons replacing quarks in the above formulation. Similarly, one could consider matrix elements as in Eq. (1), but taken between three-particle states other than $⟨P'|$ and $|P⟩$. Apart from these three-body bound states, one can have states of three free particles and those where one particle is free while the other two form a bound sub-system. All these transitions are considered in Ref. [4, 5, 6, 7] for the case of electromagnetic currents, and can be used directly to find GPDs simply through the replacements

$$dμ → d^U, \quad vμ → v^U.$$  \hspace{1cm} (66)

Off-diagonal transitions $N → πN$ in context of GPDs have also been suggested and their importance emphasized in Ref. [29]. In this regard we should mention that applying
techniques of the present paper to the $\pi NN$ equations derived in Ref. [49, 50, 10], and their electromagnetic currents [8], would lead to a comprehensive description of the GPDs for the transitions $NN \to \pi NN, d \to \pi d$, etc. With the same techniques, mesonic correction to nucleon GPDs can be calculated. For example, within the NJL model of the nucleon, one can use the static approximation where the mass of the exchanged quark is taken to infinity. Then the quark-diquark Green function, which enters the expression for the mesonic corrections [51], is calculated algebraically, making the calculation of corrections very practical. In Ref. [51], we have already shown how to calculate mesonic corrections to electromagnetic currents of the NJL model, and using the $U$-gauging method proposed here one could calculate the mesonic corrections to GPDs so that all sum rules are satisfied. The important point is that our approach enables these mesonic corrections to enter calculations of GPDs and currents consistently.

3. Gauge invariant currents in $NN$ EFT with cutoff regularization.

In this section we shall apply the gauging of equations method to two-nucleon ($NN$) EFT, a subject at the heart of modern approaches to nuclear physics [52], and show that it is instrumental in constructing gauge invariant currents when a momentum cutoff is introduced into the theory [53]. To this end we first briefly summarise the main ideas behind $NN$ EFT as first proposed by Weinberg [54, 55, 56].

Nuclear forces cannot be as yet derived directly from the fundamental theory of strong interactions, quantum chromodynamics (QCD), due to that part of low energy physics which should be treated non-perturbatively. Presently, nuclear forces are most rigorously studied within the framework of chiral EFT whose fundamental degrees of freedom are pionic and nucleonic fields. EFT is based on the most general Lagrangian which approximately satisfies chiral and other symmetries of QCD, and provides a systematic and model independent way of analyzing the properties of hadronic systems at low energies. The pion and single-baryon sector are treated purely perturbatively in EFT, whereas for the two (or more) nucleon sector a non-perturbative problem is involved. The reason for this was pointed out by Weinberg [54, 55, 56]: the scattering amplitude experiences an enhancement stemming from the propagators in the purely nucleonic intermediate states. As a result, perturbation theory is applicable only to the kernel of the corresponding equation for the scattering amplitude, while the equation itself needs to be solved non-perturbatively.

The non-perturbative nature of the scattering problem makes it impossible to implement a symmetry preserving regularization such as dimensional regularization. The latter is the method of dealing with infinities which appear in perturbative quantum field theory - it eliminates power-law divergences and isolates all logarithmic divergences so that gauge invariance and chiral symmetry are preserved.

Thus, for calculations of $NN$ scattering (in which case the perturbative series for the T-matrix is useless due to its divergence at the energies of interest), the use of cut-off regularization cannot be avoided, despite the fact that it destroys the gauge invariance of the theory at the level of the Lagrangian. Here we will show how cutoff regularization can nevertheless be carried out in the framework of the chiral effective theory without destroying gauge invariance of the matrix elements corresponding to the processes involving electromagnetic interaction.

An important aspect of our approach is that the used cut-off regularized EFT is explicitly constructed to be equivalent to EFT in the renormalization scheme: both the $NN$ scattering amplitude and the $NN$ EM currents in the cutoff scheme are identical to the ones in the renormalization scheme. This equivalence ensures that the potentials and current vertices have the same decomposition into long and short range terms (corresponding to the Goldstone meson exchange, and contact interaction parts, respectively) as in the renormalization scheme, and that power counting applies to the new $NN$ potentials and current vertices. The currents
in the new approach are manifestly gauge invariant, i.e., the current vertices obey usual WT identities at each stage of the calculation. The gauging of equations method is instrumental in this achievement.

3.1. Derivation of currents using the gauging of equations method

Here we show how to derive the exact currents of cutoff-regularized EFT; that is, the currents which we derive using cutoff EFT are identical to those which would be produced in EFT within the standard subtractive renormalization scheme. We achieve our formulation utilising the gauging of equations method [6, 7].

The exact current is defined via the quantity $T^\mu$ which results from attaching an external photon to all possible places within the scattering amplitude $T$. To be concrete, we take $T$ to be the $NN$ scattering amplitude based on an underlying chiral Lagrangian with explicit nucleon and pion degrees of freedom; moreover, we assume the scattering amplitude satisfies the Lippmann-Schwinger (LS) equation in the subtractive renormalization scheme:

$$T = K + KG_0T$$  \hspace{1cm} (67)

where $G_0$ is the non-relativistic propagator of two non-interacting nucleons, and $K$ is the $NN$ potential expressed in terms of bare coupling constants, i.e., those which determine the Lagrangian. To be more clear, the scattering amplitude cannot be written as a solution of the LS equation whose driving term is the renormalized potential because the renormalization procedure distorts the form of the LS equation. For this reason, we start with the LS equation where the potential $K$ is extracted directly from the Lagrangian, and includes an infinite number of counter terms which cancel infinities of the loop integrals. In our approach, the kernel $K$ used to generate the finite scattering amplitude $T$ is meant to be the total sum of the $NN$ irreducible diagrams (not a part of them). For the purposes of introducing a cutoff, we then construct an exact effective potential $V$ for which it is apparent that all the infinities are cancelled.

Although, as mentioned above, the kernel $K$ extracted from the chiral Lagrangian contains pion exchanges and contact interaction terms (the latter encoding the heavier meson exchanges and other integrated out higher energy physics), our derivation of the cutoff scheme is applicable also for the phenomenological approach where the contact term in $K$ (short range part of the interaction) is mimicked by usual finite $\rho$ meson exchange, i.e. $K = K_\pi + K_\rho$. We emphasize that in this section of the paper a symbolic equation like that of Eq. (67) represents a non-relativistic three-dimensional integral equation; in particular, we take all quantities $T, K, G_0$, etc., as operators in relative three-momentum space with an overall total four-momentum delta function removed. Thus Eq. (67) is shorthand for

$$T(p', p, P) = K(p', p, P) + \int d\mathbf{k}' d\mathbf{k} K(p', \mathbf{k}', P) G_0(\mathbf{k}', \mathbf{k}; P) T(\mathbf{k}, \mathbf{p}, P)$$  \hspace{1cm} (68)

where $P = (E, \mathbf{P})$ is the total four-momentum (made up of the off-shell total energy $E$ and total momentum $\mathbf{P} = p_1 + p_2 = p_1' + p_2'$, $p = \frac{1}{2}(p_1 - p_2)$ is the relative momentum of the two nucleons in initial state (similarly $p', \mathbf{k}$, and $\mathbf{k}'$), and

$$T(p', p, P) \equiv \langle p'|T(P)|p \rangle, \quad K(p', p, P) \equiv \langle p'|K(P)|p \rangle, \quad (69a)$$  

$$G_0(\mathbf{k}', \mathbf{k}; P) \equiv \langle \mathbf{k}'|G_0(P)|\mathbf{k} \rangle = \delta(\mathbf{k}'_1 - \mathbf{k}_1) \frac{1}{E - k'^2/2M - k^2/2M + i\eta}. \quad (69b)$$

The chiral effective Lagrangian also involves the electromagnetic vector-potential $A_\mu$ in a gauge invariant way (in a way more general than that due to minimal coupling). In general, this means that the strong interaction vertices in the Lagrangian, and the corresponding ones with an
attached external photon, are related to each other via WT identities. Given such a Lagrangian, attaching an external photon everywhere to the solution of Eq. (67) is carried out by gauging the LS equation, Eq. (67):

$$T^\mu = K^\mu + K^\mu G_0 T + KG_0^\mu T + KG_0 T^\mu.$$  \hspace{1cm} (70)

This equation is likewise shorthand for a three-dimensional equation, similar to Eq. (68), but involves the gauged quantities

$$T^\mu(p', P'; p, P) \equiv \langle p' | T^\mu(P', P) | p \rangle, \quad K^\mu(p', P'; p, P) \equiv \langle p' | K^\mu(P', P) | p \rangle,$$

$$G_{0}^\mu(k', P'; k, P) \equiv \langle k' | G_{0}^\mu(P', P) | k \rangle = \langle k' | G_{0}(P') \Gamma_0^\mu(P', P) G_{0}(P) | k \rangle \hspace{1cm} (71a)$$

$$G_{0}^\mu(k', P'; k, P) \equiv \langle k' | G_{0}^\mu(P', P) | k \rangle = \langle k' | G_{0}(P') \Gamma_0^\mu(P', P) G_{0}(P) | k \rangle \hspace{1cm} (71b)$$

where \( P' = P + q \) for an incoming photon of momentum \( q \), and \( \Gamma_0^\mu = G^{-1}_0 G_{0}^\mu G^{-1}_0 \) is the single nucleon current as extracted from the initial Lagrangian:

$$\langle p' | \Gamma_0^\mu(P', P) | p \rangle = \delta(p'_2 - p_2) \Gamma_1^\mu(p'_1, p_1) + \delta(p'_1 - p_1) \Gamma_2^\mu(p'_2, p_2) \hspace{1cm} (72)$$

where \( \Gamma_i^\mu \) is the electromagnetic vertex function of nucleon \( i \):

$$\Gamma_i^\mu(p', p) = ie_i \left( 1, \frac{p' + p + i\sigma_i \times q}{2M} \right) \hspace{1cm} (73)$$

where \( q = p' - p \). Solving Eq. (70), one derives the gauged scattering amplitude \( T^\mu \):

$$T^\mu = (1 + TG_0) K^\mu (1 + G_0 T) + TG_0^\mu T. \hspace{1cm} (74)$$

As in the previous sections, the quantity obtained through the use of the gauging of equations method, \( T^\mu \) in this case, is guaranteed to satisfy the WT identity. This identity has the form of Eq. (63) and can be conveniently written in symbolic form as

$$q_\mu T^\mu = [\Gamma_0, T] \hspace{1cm} (75)$$

where

$$\langle p' | \Gamma_0(P', P) | p \rangle \equiv ie_1 \delta(p'_2 - p_2) + ie_2 \delta(p'_1 - p_1) = ie_1 \delta(p' - p - q/2) + ie_2 \delta(p' - p + q/2). \hspace{1cm} (76)$$

It is obvious that a naive introduction of a cutoff into Eq. (67) and Eq. (74) will destroy the gauge invariance [Eq. (75) will no longer hold]. It is the goal of this section to introduce a cutoff in a way that preserves the gauge invariance.

We begin by introducing an effective potential \( V \) in the cutoff scheme, by requiring that it generate, via the LS equation, the same off-shell \( NN \) scattering amplitude, \( T \), as the one determined in the underlying theory by the solution of Eq. (67). This requirement is important to enable the cutoff theory to describe not only elastic \( NN \) scattering, but \( NN \) currents as well, because the latter are determined in the underlying theory (at least partly) by the half-off-shell solution of Eq. (67). We achieve this requirement by expressing the LS equation, Eq. (67), as a set of two equations

$$T = V + VG_0 \Theta T \hspace{1cm} (77a)$$

$$V = K + KG_0 \Theta V \hspace{1cm} (77b)$$

where \( \langle p' | \Theta(P) | p \rangle = \delta(p' - p) \theta(\Lambda - |p|) \) is the sharp cutoff function with cutoff momentum \( \Lambda \). Note that \( \Theta + \Theta = 1 \) and \( [G_0, \Theta] = 0 \).
Although the couple of equations, Eqs. (77), result only from simple algebra, a non-trivial aspect of this decomposition is that it preserves the power counting properties of the potentials. That is, according to Eqs. (77b), for sufficiently small relative momenta, $|p'|, |p| < \Lambda$, potentials $V(E, p', p)$ and $\tilde{K}(E, p', p)$ have the same long range parts (only their short range parts are different - see Ref. [53]). This fact is at the heart of our derived EFT theory with cutoff, namely, in that this theory has the same predictive power as the underlying theory given by the initial Lagrangian.

Although Eq. (77b) may be of no use to derive the exact effective potential $V$ for the EFT with cutoff [one cannot use perturbation theory, because high momenta are involved in Eq. (77b)], if the potential of the underlying theory, $K$, is given, then for sufficiently small momenta $|p|$ and $|p'|$, the unknown part of $V$ is restricted to only zero-range potentials (consisting of the delta function and its derivatives in coordinate space) with the same number of coupling constants as in the underlying theory (these are not known a priori and are supposed to be extracted from experimental data.)

As a next step, we need to gauge both Eq. (77a) and Eq. (77b) to derive the five-point function $T^\mu$ of Eq. (74). Gauging Eqs. (77) we obtain

$$\tilde{T}^\mu = V^\mu + V^\mu G_0 \Theta T + V(G_0 \Theta)^\mu T + VG_0 \Theta \tilde{T}^\mu$$  \hspace{1cm} (78a)

$$V^\mu = K^\mu + K^\mu G_0 \Theta V + K(G_0 \Theta)^\mu V + KG_0 \Theta \tilde{V}^\mu$$  \hspace{1cm} (78b)

where, for the moment, we use a tilde notation to distinguish $\tilde{T}^\mu$ from the $T^\mu$ of Eq. (74), until it is explicitly shown, below, that the two are in fact equal to each other. The solution of Eqs. (78) is

$$\tilde{T}^\mu = (1 + TG_0 \Theta)V^\mu (1 + G_0 \Theta T) + T(G_0 \Theta)^\mu T$$  \hspace{1cm} (79)

where

$$V^\mu = (1 + VG_0 \Theta)K^\mu (1 + G_0 \Theta V) + V(G_0 \Theta)^\mu V.$$  \hspace{1cm} (80)

In general, even if the input to a dynamical equation is specified, once the equation is gauged, one will still need to specify how exactly to attach a "photon" to the input. Thus, although the input to Eqs. (77) is known, consisting of the kernel $K$ and the cutoff modified free Green functions $G_0\Theta$ and $G_0\bar{\Theta}$, the corresponding gauged inputs $K^\mu$, $(G_0\Theta)^\mu$ and $(G_0\bar{\Theta})^\mu$, appearing in the gauged result of Eq. (79) and Eq. (80), still need to be defined. In the case of $K^\mu$ this is easy, as it is specified by the underlying Lagrangian and will automatically satisfy local gauge invariance, i.e., it will fulfill the WT identity $q_\mu K^\mu = [\Gamma_0, K]$. It is less obvious what to take for the gauged input $(G_0\Theta)^\mu$; however, we shall require it to likewise obey the WT identity, as only in this way can we ensure the necessary WT identity for $V^\mu$ [via Eq. (80)]. We shall also require that

$$(G_0\Theta)^\mu + (G_0\bar{\Theta})^\mu = G_0^\mu$$  \hspace{1cm} (81)

if only to guarantee that $\tilde{T}^\mu = T^\mu$. Indeed, let us now derive $\tilde{T}^\mu = T^\mu$ using Eq. (81) (even though from the above described gauging procedure, this may be obvious). Using Eq. (79) and Eq. (80) we have that

$$\tilde{T}^\mu = (1 + TG_0 \Theta)V^\mu (1 + G_0 \Theta T) + T(G_0 \Theta)^\mu T$$  

$$= (1 + TG_0 \Theta)[(1 + VG_0 \Theta)K^\mu (1 + G_0 \Theta V) + V(G_0 \Theta)^\mu V]$$  

$$\times (1 + G_0 \Theta T) + T(G_0 \Theta)^\mu T$$  

$$= (1 + TG_0 \Theta + TG_0 \bar{\Theta})K^\mu (1 + G_0 \Theta T + G_0 \bar{\Theta} T) + T(G_0 \Theta)^\mu T + T(G_0 \bar{\Theta})^\mu T$$  

$$= (1 + TG_0)K^\mu (1 + G_0 T) + TG_0^\mu T = T^\mu.$$  \hspace{1cm} (82)

We can derive the WT identity for $\tilde{T}^\mu$ if we assume that $(G_0\Theta)^\mu$ satisfies the WT identity

$$q_\mu (G_0\Theta)^\mu = [\Gamma_0, G_0\Theta].$$  \hspace{1cm} (83)
and if we can prove the WT identity for $V$: $q_{\mu}V^\mu = [\Gamma_0, V]$. Of course, we already know that $\tilde{T}^\mu$ must satisfy the WT identity because, as just shown, $T^\mu = T^\mu$ and $T^\mu$ satisfies the WT identity of Eq. (75). Nevertheless, we shall still derive the WT identity for $\tilde{T}^\mu$ just to see it as a result of our gauging, i.e., without using the above derivation of $\tilde{T}^\mu = T^\mu$. Indeed, combining the fact that $q_{\mu}G_0^\mu = [\Gamma_0, G_0]$ with Eq. (81) and Eq. (83), we obtain the WT for $(G_0\Theta)^\mu$,

\[ q_{\mu}(G_0\Theta)^\mu = [\Gamma_0, G_0\Theta], \tag{84} \]

and hence the WT identity for $V^\mu$:

\[
q_{\mu}V^\mu = (1 + VG_0\Theta)q_{\mu}K^\mu(1 + G_0\Theta V) + Vq_{\mu}(G_0\Theta)^\mu V
= (1 + VG_0\Theta)[\Gamma_0, K](1 + G_0\Theta V) + V[\Gamma_0, G_0\Theta]V
= (1 + VG_0\Theta)\Gamma_0 V - \Gamma_0(1 + G_0\Theta V) + V[\Gamma_0, G_0\Theta]V
= [\Gamma_0, V]. \tag{85}
\]

The WT identity for $\tilde{T}^\mu$ then follows from Eq. (79).

All that is now left, is the explicit construction of the gauged input $(G_0\Theta)^\mu$. In this respect, we note that Eq. (79) and the WT relation between $V^\mu$ and $V$, Eq. (85), are important to the construction of effective currents with maximal predictive power. It is then crucial that the single nucleon current of the theory with cutoff, $(G_0\Theta)^\mu$, not only satisfy the WT identity of Eq. (83), but that it also preserve the predictive power of the underlying theory. It is found that the following form for $(G_0\Theta)^\mu$ satisfies both these requirements and provides for the effective interaction current, $V^\mu$ to have the same long-short range decomposition as $K^\mu$ [53]:

\[
\langle p'\mid G_0^{-1}(G_0\Theta)^\mu G_0^{-1}\mid p \rangle = \frac{1}{2} \delta(p'_2 - p_2) A^\mu_i + (1 \leftrightarrow 2) \tag{86}
\]

where

\[
A^\mu_i = \Gamma^\mu_i(p'_1, p_1) \left[ \theta(\Lambda - p) + \theta(\Lambda - p') \right] + [0, \Gamma_1(p', p)] \frac{\theta(\Lambda - p') - \theta(\Lambda - p)}{p'^2 - p^2} (E'_{cm} M - p'^2 + E_{cm} M - p^2) \tag{87}
\]

In the above expression, $p = |p|, p' = |p'|$, and $E'_{cm}$ ($E_{cm}$) is the energy of the two initial (final) nucleons in their own centre of mass system. The detailed derivation of Eq. (87) can be found in Ref. [53], and is based on a number of constraints, one of which is the WT identity, Eq. (83).

### 3.2. How to calculate different, photon involving, reactions

The full dynamics of $NN$ reactions involving an external photon is contained in the gauged $NN$ scattering amplitude, $T^\mu$, discussed above. The electromagnetic bound state form factors, the $NN$ bound state photodisintegration amplitude, and the $NN$ bremsstrahlung amplitude, can all be derived from Eq. (79) by using the bound-state pole structure of the scattering amplitude,

\[
T(p', p, P) \sim \frac{\phi_P(p')\phi_P(p)}{E - E_b} \quad \text{as} \quad E \rightarrow E_b, \tag{88}
\]

where $E_b$ is the bound-state energy, and the equation for the bound state vertex function $\phi$,

\[
\phi = V G_0 \Theta \phi. \tag{89}
\]
3.2.1. The $NN$ bound state current: $B + \gamma \to B'$ Although the physical $NN$ system has only one bound state, the deuteron, we shall keep the presentation more general by formally allowing different two-body bound states, $B$ and $B'$, in the initial and final channels, respectively. Taking residues of Eq. (79) at the initial and final bound state poles, we obtain the bound state current

$$ \langle P' | J^\mu(0) | P \rangle = \bar{\phi}_P [G_0 \Theta V^\mu \Theta + (G_0 \Theta)^\mu] \phi_P. \tag{90} $$

Using the explicit form for the single nucleon EM vertex, Eq. (87), one can easily write the expanded form of Eq. (90); for example, for the zeroth component of the current

$$ \langle P' | J^0(0) | P \rangle = \int d\mathbf{p}' d\mathbf{p} \bar{\phi}_P(\mathbf{p}') \frac{\theta(\Lambda - p')}{(E_b' - p'^2/M)(E_b - p^2/M)} \phi_P(\mathbf{p}) \tag{91} $$

where $E_b'$ and $E_b$ are binding energies in the final and initial state, respectively. We note that the bound state vertex functions, $\phi_P(\mathbf{p})$, in fact do not depend on the total momentum $P$ due to Galillei invariance (i.e., they depend only on the relative momentum of the nucleons, $\mathbf{p}$); also, the first (6-dimensional) integral of Eq. (91) corresponds to exchange currents with $\mathbf{p}'$, $\mathbf{p}$ being independent integration variables, while the second (3-dimensional) integral corresponds to the impulse approximation with $\mathbf{p}' = \mathbf{p} + \mathbf{q}/2$.

3.2.2. The $NN$ bound state photodisintegration current: $B + \gamma \to N + N$ To find the bound state photodisintegration current, $j_0^\mu$, in addition to the photon attachments contained in $T^\mu$, one also needs photon attachments to the two free final state nucleons. To include the latter, it is sufficient to gauge the quantity $G_0 T$:

$$ \langle G_0 T \rangle^\mu = G_0^\mu T + G_0 T^\mu, \tag{92} $$

take the residue at the initial bound state pole, and then to "chop off" the final legs:

$$ j_0^\mu = G_0^{-1}[G_0^\mu + G_0(1 + TG_0 \Theta)V^\mu \Theta + G_0 T(G_0 \Theta)^\mu] \phi = [G_0^{-1} G_0^\mu + (1 + TG_0 \Theta)V^\mu \Theta + T(G_0 \Theta)^\mu] \phi. \tag{93} $$

3.2.3. The $NN$ Bremsstrahlung current: $N + N \to N + N + \gamma$ To find the Bremsstrahlung current, $j_{B\gamma}^\mu$, in addition to the photon attachments contained in $T^\mu$, one also needs photon attachments to the free initial- and final-state nucleons. For this, it is sufficient to gauge the quantity $G_0 T G_0$ and the to "chop off" both the initial and final legs:

$$ j_{B\gamma}^\mu = G_0^{-1}(G_0 T G_0)^\mu G_0^{-1} = T^\mu + G_0^{-1} G_0^\mu T + T G_0^\mu G_0^{-1} \tag{94} $$

$$ = (1 + TG_0 \Theta)V^\mu(1 + G_0 \Theta T) + T(G_0 \Theta)^\mu T + \Gamma_0^\mu G_0 T + TG_0 \Gamma_0^\mu. $$

3.3. Discussion

In the context of two-nucleon EFT, we have developed a cutoff scheme for strong interaction amplitudes and their EM currents that is equivalent to, and has the same predictive power as, two-nucleon EFT in the renormalization scheme.

In the renormalization scheme, we have an $NN$ potential $K$ with long and short range parts, whose short range couplings are determined from a comparison of the generated amplitude with the effective range expansion. We also have the interaction current $K^\mu$ which is given by the
initial Lagrangian, and is restricted only by local gauge invariance, i.e. by the WT identity \( q_{\mu}K^\mu = [\Gamma_0, K] \). The physical \( NN \) scattering amplitude and \( NN \) currents then are derived via Eq. (67) and Eq. (74) respectively.

In the cutoff scheme we also have an \( NN \) potential \( V \) with long and short range parts, whose short range couplings are determined from a comparison with the effective range expansion. We also have an interaction current \( V^\mu \) which is restricted only by local gauge invariance, i.e., by the WT identity \( q_{\mu}V^\mu = [\Gamma_0, V] \). The physical \( NN \) scattering amplitude and \( NN \) currents are derived via the cutoff versions of Eq. (67) and Eq. (74), namely, Eq. (77a) and Eq. (79), respectively. The currents in our cutoff scheme are conserved.

The essential difference between the approaches with interactions \( K \) and \( V \), is in that \( K \) is given directly by the initial Lagrangian, whereas \( V \) is related to \( K \) in a complicated way. This difference is of no consequence in the philosophy of EFT because \( K \) and \( V \) have the same long range parts. The short range parts of \( K \) and \( V \) are determined by comparing the \( NN \) scattering amplitude with predictions of experiment, while the only actual way of determining both \( K^\mu \) and \( V^\mu \) is by relating them to \( K \) and \( V \) via their respective WT identities.

The new technical element of our EFT with cutoff is the single nucleon current, \((G_0\Theta)^\mu\), which satisfies the WT identity, Eq. (83), and whose specific form is given by Eq. (87). It prescribes a specific gauge invariant way for implementing the cutoff in the impulse approximation.

Eq. (87) and its derivation in Ref. [53] can be used as a guide for the gauge invariant treatment of other theories where a cutoff is a necessary attribute for actual calculations (see for example the NJL model of Ref. [57]). Gauge invariance can be maintained by special regularization [analogous to the one of Eq. (87)] of the integrals corresponding to loops with attached photons, as is the case of the present paper. This regularization will depend on the way the loops without photons are regularized.

4. Crossing symmetric model of pion-nucleon scattering

Our final example demonstrates how the gauging of equations method can be used to attach external particles other than photons to non-perturbative processes. In particular, we use it to obtain a complete attachment of two external pions to a dressed nucleon propagator of an underlying nonperturbative \( \pi N \) potential model, and in this way, generate a crossing symmetric \( \pi N \) amplitude. The formulation automatically provides expressions also for the crossing symmetric and gauge invariant \( \pi N \) photoproduction and Compton scattering amplitudes. We show that our amplitudes are also unitary if they coincide on-shell with the amplitudes obtained by attaching one pion to the dressed \( \pi NN \) vertex of the same potential model. It is left to future numerical calculations to determine the accuracy with which this coincidence can be achieved by adjusting parameters of the underlying \( \pi N \) model. To this extent, our gauging method provides an approximate solution to the long-standing problem of constructing a potential model that is at the same time unitary and crossing symmetric [58].

4.1. Underlying strong interaction \( \pi N \) model

The \( \pi N \) scattering amplitude \( t \) is often described using the set of equations [59, 60, 61]

\[
\begin{align*}
t &= fg\tilde{f} + t_b, \\
\tilde{f} &= f_0 + f_0G_0t_b, \\
g &= g_0 + g_0\Sigma g, \\
t_b &= v + t_bG_0v, \\
f &= f_0 + t_bG_0f_0, \\
\Sigma &= \tilde{f}_0G_0f,
\end{align*}
\]

\( (95a) \quad (95b) \quad (95c) \)

where \( f \) \((f_0)\) is the dressed (bare) \( N \to \pi N \) vertex, \( \tilde{f} \) \((\tilde{f}_0)\) is the dressed (bare) \( \pi N \to N \) vertex, \( g \) \((g_0)\) is the dressed (bare) nucleon propagator, \( G_0 \) is the disconnected \( \pi N \) propagator, and \( t_b \) \((v)\) is the "non-pole" \( \pi N \) t-matrix (potential) with the pole term \( fg\tilde{f} \) \((f_0g_0\tilde{f}_0)\) removed. Although these equations provide an exact description in full field theory, their main feature is...
that they allow one to preserve unitarity when making models for the potential $v$ and bare vertex $f_0$. However, like all potential models, these equations suffer from a lack of crossing symmetry, a property whose importance has been emphasized for more than 50 years [62, 63, 64].

Similarly, the pion photoproduction amplitude $t^\gamma$ is often described by a set of equations that essentially result from Eq. (95a) and Eq. (95b) by replacing the initial pion with a photon [65, 66, 67]:

$$t^\gamma = fg\bar{f}_0^\gamma + t^\gamma_b,$$

where $\bar{f}^\gamma$ ($f_0^\gamma$) is the dressed (bare) $\gamma N \rightarrow N$ vertex, and $t^\gamma_b$ ($v^\gamma$) is the pion photoproduction amplitude (Born term) with the pole term $fg\bar{f}_0^\gamma (f_0g_0\bar{f}_0^\gamma)$ removed. Once again the feature of these equations is that they respect unitarity. This time, however, these equations suffer not only from a lack of crossing symmetry, but also from the breaking of manifest gauge invariance (because the photon is not coupled to all places in the underlying field theory). We shall refer to Eqs. (95) and Eqs. (96) as the standard description.

Below we shall present new equations for $\pi N$ scattering, pion photoproduction, and Compton scattering, that are based on the potential model of Eqs. (95), but that preserve crossing symmetry and manifest gauge invariance. Our approach is based on the idea of coupling external pions and photons to all possible places in the dressed propagator $g$ of Eq. (95c), and is achieved using the gauging of equations method [6, 68]. Like the standard description, our approach is exact in full field theory; however, just opposite to the standard description, when models are made for the potential $v$ and bare vertex $f_0$, our approach preserves crossing symmetry and gauge invariance at the expense of unitarity. The lack of built-in unitarity is not surprising since our approach effectively sums the full perturbation series in a way that is different from the usual method of iterating a kernel. Nevertheless, we show that our amplitudes will satisfy unitarity whenever the crossing symmetric $\pi N$ amplitude coincides, on-shell, with the one obtained by attaching one pion to the dressed $\pi NN$ vertex $f$ of Eq. (95b).

4.2. Single-gauged amplitude

In Refs. [6, 68] we introduced a technique for attaching an external photon to all possible places (vertices, propagators, potentials, etc.) within a strongly interacting system described by dynamical equations. The completeness of the attachment led to the gauge invariance of the resulting electromagnetic currents. Here we use the same technique to also attach first one external pion, and then in the next section, a second external pion, in order to achieve our goal of deriving a crossing symmetric $\pi N$ scattering amplitude.

We begin by applying our gauging technique to the $\pi N$ Green function $G$ generated by the non-pole potential $v$:

$$G = G_0 + G_0vG.$$  

(97)

Denoting by $G^\mu$ the 5-point function resulting from a complete attachment of an external pion to $G$, Eq. (97) is ”gauged” to obtain

$$G^\mu = G^\mu_0 + G^\mu_0vG + G_0v^\mu G + G_0vG^\mu$$  

(98)

which is easily solved to get

$$G^\mu = G\Lambda^\mu G, \qquad \Lambda^\mu = \Lambda^\mu_0 + v^\mu$$  

(99)

2 For simplicity of presentation, we ignore any terms that cannot be obtained by the attachment of external pions or photons. Such contributions, if present, are gauge invariant and crossing symmetric on their own, and can therefore be separately added to our derived amplitudes.

3 We shall use ”gauging” to mean the process of attaching any external particle, not just a gauge boson.
where \( \Lambda_0^\mu \equiv G_0^{-1}G_0^\mu G_0^{-1} \) is a vertex function derived by attaching an external pion to the disconnected \( \pi N \) propagator \( G_0 \). Note that \( G_0 = g_\pi g_N \) where \( g_\pi \) is the pion propagator and \( g_N \) is the effective nucleon propagator (in exact field theory \( g_N = g \), but in practical calculations where there are no explicit two-pion states, one usually takes \( g_N \) to be just the pole part of \( g \)). As G-parity conservation forbids a three-pion vertex, \( \Lambda_0^\mu = \Gamma_N^\mu g_\pi^{-1} \) where \( \Gamma_N^\mu \equiv g_N^{-1}g_N^\mu g_N^{-1} \).

Thus

\[
\Lambda^\mu = \Gamma_N^\mu g_\pi^{-1} + v^\mu.
\]

The gauged potential \( v^\mu \) can be constructed phenomenologically, or derived by gauging a specific model for \( v \). Similarly, the gauging of the dressed nucleon propagator \( g \) of Eq. (95c), gives the 3-point function \( g^\mu \):

\[
g^\mu = g\Gamma^\mu g, \quad \Gamma^\mu = \Gamma_0^\mu + \Sigma^\mu
\]

where \( \Gamma^\mu \) is the dressed \( \pi NN \) vertex function, \( \Gamma_0^\mu \equiv g_0^{-1}g_0^\mu g_0^{-1} \) is the bare vertex function, and \( \Sigma^\mu \) is the gauged dressing. Gauging \( \Sigma = f_0Gf_0 \) then leads to a simple intuitive expression for the \( \pi NN \) dressed vertex function:

\[
\Gamma^\mu = \Gamma_0^\mu + \bar{f}_0^\mu G_0f_0 + \bar{f}G_0f_0^\mu + \bar{f}G_0v^\mu G_0f,
\]

which is illustrated in Fig. (1).

The main results of this section come from the gauging of the dressed \( \pi NN \) vertex \( f \):

\[
(G_0 f g)^\nu = (Gf_0g)^\nu = G^\nu f_0g + Gf_0^\nu g + Gf_0^\nu = G\Lambda^\nu g_0 + Gf_0^\nu g + Gf_0^\nu \Gamma g^\nu g.
\]

Cutting off the external legs immediately gives the amplitude \( T^\nu \equiv G_0^{-1}(Gf g)^\nu g^{-1} \):

\[
T^\nu = \bar{f}G_0(\Lambda_0^\nu g_0f + v^\nu G_0f + f_0^\nu) + fg_0^\nu \Gamma.
\]

If superscript \( \nu \) corresponds to an external photon, then \( T^\nu Z \) is the properly normalized manifestly gauge invariant pion photoproduction amplitude, originally derived in Ref. [69] using a more involved approach. If superscript \( \nu \) corresponds to an external pion, then \( T^\nu Z \) is a "hybrid" \( \pi N \) scattering amplitude where the initial state pion is due to gauging and the final state pion is due to the original standard description. In a similar way, vertex \( \bar{f} \) can be gauged to obtain the hybrid amplitude \( \bar{T}^\nu \) where the final state pion is due to gauging. One can express these amplitudes in a "pole plus non-pole" form analogous to Eq. (95a):

\[
T^\nu = fg_0^\nu + T^\nu_b, \quad T_b^\nu = (1 + t_b G_0)\bar{V}^\nu, \quad V^\nu = f_0^\nu + \Lambda^\nu G_0f, \quad \bar{V}^\nu = \bar{V}_0^\nu + \bar{f}G_0\Lambda^\nu.
\]

\[
\bar{T}^\nu = \bar{f}g_0^\nu + \bar{T}_b^\nu, \quad \bar{T}_b^\nu = \bar{V}_0^\nu(1 + G_0t_b), \quad V_0^\nu = f_0^\nu + \Lambda^\nu G_0f + \bar{f}G_0\Lambda^\nu.
\]
In exact field theory, relations connecting the bare $\pi NN$ vertex $f_0$ and "non-pole" $\pi N$ potential $v$ of the standard description, Eqs. (95), to the corresponding gauged quantities $f_0^\mu$ and $v^\mu$ of the single-gauged description - see the first two of Eqs. (108).

It follows that amplitudes $T^\nu$ and $\bar{T}^\mu$ satisfy non-standard Bethe-Salpeter equations

$$T^\nu = V^\nu + vG_0T^\nu, \quad \bar{T}^\mu = \bar{V}^\mu + \bar{f}G_0f,$$

where the kernel and inhomogeneous terms involve $\pi N$ potentials of different origin. The amplitudes are clearly not crossing symmetric.

We can now express Eq. (102) in the following two ways analogous to Eq. (95b):

$$\Gamma^\nu = \bar{f}G_0V^\nu, \quad \bar{f}^\nu = \Gamma^\nu_0 + \bar{f}G_0f, \quad \bar{f}^\nu = \Gamma^\nu_0 + \bar{f}G_0f^\mu,$$

where Eq. (107a) and Eq. (107b) are to be used to describe pion (or photon) absorption and creation vertices, respectively. In exact field theory, $\Gamma^\mu$ can be identified with function $f$ of the standard description. Eqs. (107) then imply the following identities (in exact field theory) relating the standard description quantities $f_0$ and $v$ (typically the model inputs) to their gauged counterparts:

$$f_0 = F_0^\mu, \quad v = \bar{V}^\mu; \quad \bar{f}_0 = \bar{F}_\nu^\nu, \quad v = V^\nu.$$

The first two of Eqs. (108) are illustrated in Fig. 8.

4.2.1. Unitarity For ease of presentation, we discuss unitarity within the framework of time ordered perturbation theory for which

$$G_0(E^+) - G_0(E^-) = -2\pi i\delta(E - H_0)$$

where $H_0$ is the free Hamiltonian and $E^\pm = E \pm i\epsilon$. We consider only 2-body unitarity and thus restrict the discussion to energies $E$ below the two-pion threshold. In this energy region the potential $v$ and bare vertex $f_0$ are real, and the standard description of Eqs. (95) will therefore satisfy the following unitarity relations:

\begin{align*}
t - t^\dagger &= t_0\delta t, \quad t_0 - t_0^\dagger = t_b^\dagger \delta t_b, \quad (110a) \\
f - f^\dagger &= t_0^\dagger \delta f, \quad \bar{f} - \bar{f}^\dagger = \bar{f}^\dagger \delta t_b, \quad (110b) \\
g - g^\dagger &= g^\dagger (\Sigma - \Sigma^\dagger) g, \quad \Sigma - \Sigma^\dagger = \bar{f} \delta f, \quad (110c) \\
G - G^\dagger &= (1 + G_0^\dagger t_0^\dagger) \delta (1 + t_bG_0), \quad (110d)
\end{align*}
where $\delta$ is shorthand for $-2\pi i\delta(E-H_0)$ and where identical quantities with and without a dagger represent the same functions of $E^-$ and $E^+$, respectively (i.e., a dagger does not mean Hermitian conjugate, but rather, $T \equiv T(E^+)$ and $T^\dagger \equiv T(E^-)$). Applying these relations to Eqs. (105) and Eqs. (107) one obtains the analogous unitarity relations for the hybrid amplitudes:

\begin{align}
T^\nu - T^\nu_\dagger &= t^\dagger \delta T^\nu,
T^\mu_b - T^\mu_b_\dagger &= t^\dagger_0 \delta T^\mu_b,
T^\mu - T^\mu_\dagger &= \bar{T}^\mu_\dagger \delta t,
\Gamma^\mu - \Gamma^\mu_\dagger &= \bar{T}^\mu_\dagger_\dagger \delta f,
\end{align}

(111a)

(111b)

(111c)

In the case of gauging with photons, Eq. (111a) provides just the usual statement of Watson’s theorem for the gauge invariant pion photoproduction amplitude $T^\nu$. However, in the case of gauging with pions, Eq. (111a) differs from the usual statement of unitarity for the hybrid amplitude in that a $T^\nu_\dagger$ has been replaced with a $t^\dagger_0$ from the standard description. Thus the only way for a hybrid $\pi N$ amplitude $T^\nu$ to be unitary is for it to coincide, on-shell, with the standard amplitude $t$.

4.3. Double-gauged amplitude

To obtain a crossing symmetric $\pi N$ scattering amplitude, we attach two external pions to the dressed nucleon propagator $g$; that is, we first gauge $g$ to obtain $g^\mu$ as in Eq. (101), and then gauge Eq. (101) to obtain $g^{\mu\nu}$ with indices $\nu$ and $\mu$ denoting the initial- and final-state pion, respectively. Thus

\[ g^{\mu\nu} = g T^{\mu\nu} g, \]

\[ T^{\mu\nu} = \Gamma^\mu g \Gamma^\nu + \Gamma^\nu g \Gamma^\mu + \Sigma^{\mu\nu} \]

(112)

where $ZT^{\mu\nu}$ is the properly normalized crossing symmetric $\pi N$ amplitude. In Eq. (112)

\[ \Gamma^\mu_0 = (\bar{g}^{-1} g_0^{-1})^\mu = g_0^{-1} g_0^{-1} - \Gamma^\mu g_0 \Gamma^\mu_0 - \Gamma^\mu_0 g_0 \Gamma^\mu, \]

\[ \Sigma^{\mu\nu} = \bar{f}^{\mu\nu} G_0 f + \bar{f} G_0 f^{\mu\nu} + \bar{f} G_0 \Lambda^{\mu\nu} G_0 f + A^{\mu\nu} + A_{\mu\nu} \]

(113a)

(113b)

where

\[ \Lambda^{\mu\nu} = \Gamma^{\mu\nu} N g_0^{-1} + v^{\mu\nu}, \]

\[ A^{\mu\nu} = \bar{V}^{\mu} G_0 V^{\nu} = \bar{V}^{\mu} G_0 T^{\nu}_b = \bar{T}^\mu_0 G_0 V^{\nu}. \]

(114a)

(114b)

One can thus write the crossing symmetric $T^{\mu\nu}$ in terms of pole and non-pole parts as

\[ T^{\mu\nu} = \Gamma^{\mu} g \Gamma^{\nu} + T^{\mu\nu}_b, \]

\[ T^{\mu\nu}_b = V^{\mu\nu} + \bar{T}^\mu_0 G_0 V^{\nu}, \]

\[ V^{\mu\nu} = \Gamma^{\nu} g \Gamma^{\mu} + \Gamma^{\mu}_0 + \bar{f}^{\mu\nu} G_0 f + \bar{f} G_0 f^{\mu\nu} + \bar{f} G_0 \Lambda^{\mu\nu} G_0 f + A^{\nu\mu} \]

(115a)

(115b)

We note that the second of Eqs. (115a) still has the basic structure of a Bethe-Salpeter equation although neither the two non-pole $\pi N$ potentials $V^{\mu\nu}$ and $V^{\nu}$, nor the two non-pole $t$ matrices $T^{\mu\nu}_b$ and $T^{\mu\nu}_b$ are of the same origin. It is also important to note that Eqs. (115) apply also to the cases where either one or both of the superscripts $\mu$ and $\nu$ refer to the gauging by photons. That is, these equations provide a unified, crossing symmetric description of pion-nucleon elastic scattering, pion photoproduction, and Compton scattering. Moreover, the electromagnetic amplitudes are due to a complete attachment of photons and are therefore manifestly gauge invariant.
4.3.1. Unitarity  In the crossing symmetric formulation, the $\pi N$ potential $V^{\mu\nu}$ of Eq. (115b) is real below two-pion threshold, as is the hybrid potential $V^{\nu}$. It is thus straightforward to obtain the 2-body unitarity relations from Eq. (115a) and the unitarity relations for the hybrid amplitudes, Eqs. (111). One obtains

$$T^{\mu\nu} - T^{\mu\nu\dagger} = \bar{T}^{\mu\nu\dagger} T^{\nu},$$

$$T_b^{\mu\nu} - T_b^{\mu\nu\dagger} = \bar{T}_b^{\mu\nu\dagger} T_b^{\nu}.$$  

(116)

As for the hybrid case, these have the same form as usual unitarity relations, and differ from them only in that they contain $\pi N$ $t$ matrices of different origin. The task of achieving exact 2-body unitarity for the crossing symmetric amplitudes is therefore a numerical one - the standard potential model of Eqs. (95), and its parameters, need to be adjusted so as to ensure that the single- and double-gauged amplitudes coincide on-shell. Finally, it is worth pointing out that 3-body unitarity could be obtained in a similar fashion by gauging a standard Faddeev-like description of the $\pi N N$ system [70].

References

[1] F. Gross and D. O. Riska, Phys. Rev. C 36, 1928 (1987).
[2] A. N. Kvinikhidze and B. Blankleider, in invited talk given at the Joint Japan Australia Workshop: Quarks, Hadrons and Nuclei (unpublished, Adelaide, November 15-24, 1995).
[3] H. Haberzettl, Phys. Rev. C 56, 2041 (1997), Preprint nucl-th/9704057.
[4] A. N. Kvinikhidze and B. Blankleider, Phys. Rev. C 56, 2963 (1997), Preprint nucl-th/9706051.
[5] A. N. Kvinikhidze and B. Blankleider, Phys. Rev. C 56, 2973 (1997), Preprint nucl-th/9706052.
[6] A. N. Kvinikhidze and B. Blankleider, Phys. Rev. C 60, 044003 (1999), Preprint nucl-th/9901001.
[7] A. N. Kvinikhidze and B. Blankleider, Phys. Rev. C 60, 044004 (1999), Preprint nucl-th/9901002.
[8] A. N. Kvinikhidze and B. Blankleider, Phys. Rev. C 59, 1263 (1999), Preprint nucl-th/9810025.
[9] F. Gross, A. Stadler, and M. T. Pena, Phys. Rev. C 69, 034007 (2004), Preprint nucl-th/0311095.
[10] A. N. Kvinikhidze and B. Blankleider, Nucl. Phys. A574, 788 (1994), Preprint nucl-th/9402010.
[11] A. N. Kvinikhidze and B. Blankleider, Nucl. Phys. A631, 559c (1998), Preprint nucl-th/9710639.
[12] J. C. R. Bloch, C. D. Roberts, S. M. Schmidt, A. Bender, and M. R. Frank, Phys. Rev. C 60, 062201 (1999), Preprint nucl-th/9907120.
[13] J. C. R. Bloch, C. D. Roberts, and S. M. Schmidt, Phys. Rev. C 61, 065207 (2000), Preprint nucl-th/9911068.
[14] B. Blankleider and A. N. Kvinikhidze, Phys. Rev. C 62, 039801 (2000), Preprint nucl-th/9912003.
[15] R. Alkofer, A. Holl, M. Kloker, A. Krassnigg, and C. D. Roberts, Few Body Syst. 37, 1 (2005), Preprint nucl-th/0412046.
[16] C. D. Roberts, M. S. Bhagwat, A. Holl, and S. V. Wright, Eur. Phys. J. ST 140, 53 (2007), Preprint 0802.0217.
[17] I. Cloet, G. Eichmann, B. El-Bennich, T. Klahn, and C. Roberts, Few Body Syst. 46, 1 (2009), Preprint 0812.0416.
[18] D. R. Phillips and S. J. Wallace, Few Body Syst. 24, 175 (1998), Preprint nucl-th/9708027.
[19] R. Alkofer and L. von Smekal, Phys. Rept. 353, 281 (2001), Preprint hep-ph/0007355.
[20] A. N. Kvinikhidze and B. Blankleider, Phys. Rev. D 68, 025021 (2003), Preprint hep-th/0303038.
[21] D. Mueller, D. Robaschik, B. Geyer, F. M. Dittes, and J. Horejsi, Fortschr. Phys. 42, 101 (1994), Preprint hep-ph/9812448.
[22] X.-D. Ji, Phys. Rev. Lett. 78, 610 (1997), Preprint hep-ph/9603249.
[23] X.-D. Ji, Phys. Rev. D 55, 7114 (1997), Preprint hep-ph/9609381.
[24] A. V. Radyushkin, Phys. Lett. B380, 417 (1996), Preprint hep-ph/9604317.
[25] A. V. Radyushkin, Phys. Lett. B385, 333 (1996), Preprint hep-ph/9605431.
[26] A. V. Radyushkin, Phys. Rev. D 56, 5524 (1997), Preprint hep-ph/9704207.
[27] M. Burkardt, Phys. Rev. D 62, 071503 (2000), Preprint hep-ph/0005108.
[28] M. Burkardt, Phys. Rev. D 66, 119903 (2002).
[29] K. Goeke, M. V. Polyakov, and M. Vanderhaeghen, Prog. Part. Nucl. Phys. 47, 401 (2001), Preprint hep-ph/0106012.
[30] M. Diehl, Phys. Rept. 388, 41 (2003), Preprint hep-ph/0307382.
[31] R. L. Jaffe, Nucl. Phys. B229, 205 (1983).
[32] M. Diehl and T. Gousset, Phys. Lett. B428, 359 (1998), Preprint hep-ph/9801233.
[33] X.-d. Ji, Phys. Rev. Lett. 91, 062001 (2003), Preprint hep-ph/0304037.
[34] G. A. Miller, Phys. Rev. C 68, 022201 (2003), Preprint nucl-th/0304076.
[35] F. Gross and P. Agbakpe, Phys. Rev. C 73, 015203 (2006), Preprint nucl-th/0411090.
[36] F. Gross, Phys. Rev. 186, 1448 (1969).
[37] F. Gross, Phys. Rev. C 26, 2203 (1982).
[38] F. Gross, Phys. Rev. C 26, 2226 (1982).
[39] M. Oettel, Ph.D. thesis, T"ubingen University (2000), Preprint nucl-th/0012067.
[40] N. Ishii, Nucl. Phys. A689, 793 (2001), Preprint nucl-th/0004063.
[41] D. R. Phillips, S. J. Wallace, and N. K. Devine, Phys. Rev. C 58, 2261 (1998), Preprint nucl-th/9802067.
[42] M. Oettel, M. Pichowsky, and L. von Smekal, Eur. Phys. J. A 8, 251 (2000), Preprint nucl-th/9909082.
[43] R. T. Cahill and S. M. Gunner, Phys. Rev. C 26, 171 (1998), Preprint hep-ph/9812491.
[44] B. Blankleider and A. N. Kvinikhidze, Few Body Syst. Suppl. 12, 223 (2000), Preprint nucl-th/9912075.
[45] B. C. Tiburzi and G. A. Miller, Phys. Rev. D 67, 054014 (2003), Preprint hep-ph/0210304.
[46] B. C. Tiburzi and G. A. Miller, Phys. Rev. D 67, 054015 (2003), Preprint hep-ph/0210305.
[47] S. Noguera, L. Theussl, and V. Vento, Eur. Phys. J. A 20, 259 (2007), Preprint hep-ph/0412409.
[48] A. N. Kvinikhidze and B. Blankleider, Phys. Lett. B307, 7 (1993).
[49] B. Blankleider and A. N. Kvinikhidze, in Proceedings of the Sixth International Symposium on Meson-Nucleon Physics and the Structure of the Nucleon, edited by D. Drechsel, G. Höler, W. Kluge, and B. M. K. Neffens (Blaubeuren/Tübingen, Germany, 1995), vol. 11 of PiN Newslett., p. 96, Preprint nucl-th/9508027.
[50] A. N. Kvinikhidze, M. C. Birse, and B. Blankleider, Phys. Rev. C 66, 045203 (2002), Preprint hep-ph/0110060.
[51] S. Weinberg, Phys. Rev. D 251, 288 (1990).
[52] S. Weinberg, Nucl. Phys. B363, 3 (1991).
[53] S. Weinberg, Phys. Lett. B295, 114 (1992), Preprint hep-ph/9209257.
[54] H. Asami, N. Ishii, W. Bentz, and K. Yazaki, Phys. Rev. C 51, 3388 (1995).
[55] A. Martin, Phys. Rev. 161, 1528 (1967).
[56] S. Moro, S. A. Afnan, and B. Blankleider, Phys. Rev. C 66, 044004 (2002), Preprint nucl-th/0004063.
[57] A. N. Kvinikhidze, B. Blankleider, E. Epelbaum, C. Hanhart, and M. P. Valderrama, Phys. Rev. C 80, 044004 (2009), Preprint 0904.4258.
[58] A. M. Birse, B. Blankleider, E. Epelbaum, C. Hanhart, and M. P. Valderrama, Phys. Rev. C 80, 044004 (2009), Preprint nucl-th/0509032.
[59] A. N. Kvinikhidze, B. Blankleider, E. Epelbaum, C. Hanhart, and M. P. Valderrama, Phys. Rev. C 80, 044004 (2009), Preprint nucl-th/0509032.
[60] G. F. Chew and F. E. Low, Phys. Rev. 101, 1570 (1956).
[61] R. J. McLeod, Phys. Rev. C 29, 1098 (1984).
[62] C. Fernandez-Ramirez, E. Moya de Guerra, and J. M. Udias, Phys. Lett. B660, 188 (2008).
[63] S. Nozawa, B. Blankleider, and T. S. H. Lee, Nucl. Phys. A513, 459 (1990).
[64] Y. Surya and F. Gross, Phys. Rev. C 53, 2422 (1996).
[65] V. Pascualtsa and J. A. Tjon, Phys. Rev. C 61, 054003 (2000), Preprint nucl-th/0003050.
[66] G. F. Chew and F. E. Low, Phys. Rev. 101, 1570 (1956).
[67] R. J. McLeod, Phys. Rev. C 29, 1098 (1984).
[68] C. Fernandez-Ramirez, E. Moya de Guerra, and J. M. Udias, Phys. Lett. B660, 188 (2008).
[69] S. Nozawa, B. Blankleider, and T. S. H. Lee, Nucl. Phys. A513, 459 (1990).
[70] Y. Surya and F. Gross, Phys. Rev. C 53, 2422 (1996).
[71] V. Pascualtsa and J. A. Tjon, Phys. Rev. C 60, 045209 (2004), Preprint nucl-th/0407068.
[72] A. N. Kvinikhidze and B. Blankleider, Phys. Rev. C 60, 044004 (1999), Preprint nucl-th/9901002.
[73] C. H. M. van Antwerpen and I. R. Afnan, Phys. Rev. C 52, 554 (1995), Preprint nucl-th/9407038.
[74] I. R. Afnan and B. C. Pearce, Phys. Rev. C 35, 737 (1987).