Induced matchings in graphs of degree at most 4

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Abstract

We show that if $G$ is a connected graph of maximum degree at most 4, which is not $C_{2,5}$, then the strong matching number of $G$ is at least $\frac{1}{4}n(G)$. This bound is tight and the proof implies a polynomial time algorithm to find an induced matching of this size.

1 Introduction

An induced matching of a graph $G = (V, E)$ is an edge set $M \subseteq E$ such that each vertex of $G$ is incident to at most one edge in $M$ (i.e., $M$ is a matching) and if $ab, cd \in M$ then none of the edges $ac, ad, bc, bd$ is in $E$. The maximum size of an induced matching in $G$ is called the strong matching number of $G$ and is denoted by $\nu_s(G)$.

While a maximum matching can be efficiently found in every graph [10], the problem of computing the strong matching number is NP-hard even in quite restricted classes. It is proved to be NP-hard for subcubic bipartite graphs [21, 4, 19], $C_4$-free bipartite graphs [19], line graphs [17] or cubic planar graphs [9]. In fact, even for 3-regular bipartite graphs, there is some constant $c > 1$ such that the problem cannot be approximated within a factor of $c$ unless $P = NP$ [22].

On the positive side, a maximum induced matching can be found efficiently in several classes of graphs such as weakly chordal graphs [6], AT-free graphs [7], graphs of bounded clique-width [17], and several other classes [2, 3, 5, 12, 13, 19].

One direction in recent research on induced matching is to lower bound the strong matching number of a graph $G$ in terms of its maximum degree $\Delta(G)$ and its order $n(G)$ or its number of edges $m(G)$. Let $G$ be a connected graph, an easy observation [22] yields

$$\nu_s(G) \geq \frac{n(G)}{2(2\Delta(G)^2 - 2\Delta(G) + 1)}.$$  

Joos [15] proved a sharp bound for $\Delta$ sufficiently large

$$\nu_s(G) \geq \frac{n(G)}{([\frac{\Delta}{2}] + 1)([\frac{\Delta}{2}] + 1)}.$$  

He conjectured that this bound holds for all $\Delta \geq 3$ except for $G \in \{C_{2,5}, K_{3,3}^+\}$. Here $K_{3,3}^+$ is the graph obtained from $K_{3,3}$ by subdividing an edge by a new vertex and $C_{2,5}$ is the graph obtained from $C_5$ by separating each vertex into two non adjacent vertices (see Figure 1). It is easy to see that $K_{3,3}^+$ is a subcubic graph, $C_{2,5}$ is a 4-regular graph and $\nu_s(K_{3,3}^+) = \nu_s(C_{2,5}) = 1$.
For connected subcubic graphs, Joos, Rautenbach and Sasse [15] showed that $\nu_s(G) \geq \frac{n(G)}{6}$ if $G \neq K_{3,3}$. This result, proved by simple local reduction, strengthens an earlier lower bound $\nu_s(G) \geq \frac{1}{9}m(G)$ in [18] for subcubic planar graphs.

This research direction seems to be inspired by a conjecture of Erdős and Nešetřil on the strong chromatic number $\chi'_s(G)$, i.e., the minimum number of induced matchings of $G$ into which $G$ can be partitioned.

**Conjecture 1.** If $G$ is a connected graph with maximum degree $\Delta$ then

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & \text{if } \Delta \text{ is even;} \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1), & \text{if } \Delta \text{ is odd.} \end{cases}$$

The currently best known upper bound for the strong chromatic number is $\chi'_s(G) \leq 1.998\Delta^2$ when $\Delta$ is sufficiently large, due to Molloy and Reed [20]. Conjecture 1 is proved for subcubic graphs in [1] [14].

The conjecture of Erdős and Nešetřil, if true, implies that for a regular graph $G$ of even degree $\Delta$ we have $\nu_s(G) \geq \frac{2\Delta}{5\Delta}$, as observed in [16]. Note that Joos’ conjectured bound strengthens this bound for $\Delta = 4$ and the result of Joos, Rautenbach and Sasse [16] confirms Joos’ conjectured bound for $\Delta = 3$.

In this paper we prove the conjecture of Joos for $\Delta = 4$, namely, we prove the following.

**Theorem 1.** Let $G \neq C_{2,5}$ be a connected graph with maximum degree at most $4$. Then the strong matching number of $G$ is at least $\frac{1}{2}n(G)$.

### 2 Proof of the main theorem

We first need some notations. For a subset $X$ of $V$ we denote by $G[X]$ the subgraph of $G$ induced by the vertices of $X$ and we use $G - X$ to denote $G[V - X]$. The number of isolated vertices of a subgraph $H$ of $G$ is denoted by $i(H)$. For $X \subseteq V$ we denote by $d^{\text{out}}(X)$ the number of edges between $X$ and $V - X$. When $X = \{v\}$, $d^{\text{out}}(X)$ is simply the degree of $v$ in $G$ and is written as $d(v)$. The set of vertices adjacent to a vertex $v$ in $G$ is denoted by $N(v)$, noting that $|N(v)| = d(v)$. The minimum degree of a vertex in $G$ is denoted by $\delta(G)$.

In the remainder of this section we prove the following equivalent form of Theorem 1.
**Theorem 1’**. Let $G$ be a graph with maximum degree at most 4 and $G$ has no connected component which is $C_{2,5}$. Then

$$\nu_s(G) \geq \frac{1}{9}(n(G) - i(G)).$$

We proceed by contradiction. Suppose that $G$ is a counterexample to Theorem 1’ of minimum order, namely, $\Delta(G) \geq 4$.

$$\nu_s(G) < \frac{1}{9}n(G), \quad (1)$$

and

$$\text{for every proper subgraph } G' \text{ of } G, \nu_s(G') \geq \frac{1}{9}((n(G') - i(G'))) \quad (2)$$

It is worth remarking that the minimality of $G$ implies that $G$ is connected and every subgraph of $G$ is not $C_{2,5}$.

The main point of our proof is to show that $G$ satisfies

$$G \text{ has girth at least } 5 \text{ and } \delta(G) \geq 3. \quad (3)$$

First, let us see how (3) implies contradiction. Suppose that $G$ satisfies (3). Let $uv$ be any edge of $G$. Then $G$ is connected by the assumption on minimality. Let $X = N(u) \cup N(v)$. Then $|X| \leq 8$ as the maximum degree of $G$ is at most 4. Since the girth of $G$ is at least 5, each vertex in $V(G) - X$ is adjacent to at most 2 vertices in $X$. Combining this with the assumption $\delta(G) \geq 3$ we obtain that there is no isolated vertex in $G' = G - X$. Moreover, $G' \neq C_{2,5}$ as remarked above. Therefore, $\nu_s(G') \geq \frac{1}{9}n(G')$ holds. It is easy to see that if $M$ is an induced matching of $G'$ then $M \cup \{uv\}$ is an induced matching of $G$. Therefore,

$$\nu_s(G) \geq 1 = \nu_s(G') + 1 \geq \frac{1}{9}n(G') + 1 \geq \frac{1}{9}(n(G) - 8) > \frac{1}{9}n(G)$$

holds, contradicting (1).

Next, we will prove (3) through a sequence of claims using local reduction, similar to (16). We call a vertex of degree 1 in $G$ an end-vertex. For an induced subgraph $G'$ of $G$ we denote by $I(G')$ the set of isolated vertices in $G'$ and $I_j(G') \subseteq I(G')$ the set of isolated vertices in $G'$ which has degree $j$ in $G$. The cardinalities of $I(G'), I_j(G')$ are denoted by $i(G'), i_j(G')$, for $j \in \{1, 2, 3, 4\}$, respectively. Then we have

$$i(G') = i_1(G') + i_2(G') + i_3(G') + i_4(G'). \quad (4)$$

For $X \subseteq V$, if $G' = G - X$ then, since each vertex in $I(G')$ is adjacent only to vertices in $X$, one can see that

$$d^{out}(X) \geq i_1(X) + 2i_2(X) + 3i_3(X) + 4i_4(X). \quad (5)$$

We also have

$$d^{out}(X \cup I(G')) = d^{out}(X) - (i_1(X) + 2i_2(X) + 3i_3(X) + 4i_4(X)). \quad (6)$$

**Claim 1.** The neighbor of an end-vertex has degree 4.
Proof. Let $u$ be an end-vertex and $v$ its unique neighbor. Suppose to the contrary that $d(v) \leq 3$. Let $X = \{v\} \cup N(v)$ and $G' = G - X$. Then a simple counting shows that $|X| \leq 4$ and $d^{out}(X) \leq 6$, thus $i(G') \leq 6$, by (4) and (5).

If both $i(G') = 6$ and $|X| = 4$ hold, $G$ is the graph in Figure 2 and it is easy to see that

$$\nu_s(G) = 2 > \frac{1}{9}n(G),$$

a contradiction to (1).

If $i(G') < 6$ or $|X| < 4$, then, noting that $n(G') = n(G) - |X|$, we have $\nu_s(G) \geq 1 + \frac{1}{9}(n(G') - i(G')) \geq 1 + \frac{1}{9}(n(G) - 9) = \frac{1}{9}n(G)$, again a contradiction to (1). $\square$

![Figure 2: The graph in the proof of Claim 1; thick edges indicate the induced matching.](image)

**Claim 2.** No two end-vertices have a common neighbor.

**Proof.** Suppose to the contrary that two end-vertices $u_1, u_2$ have a common neighbor $v$. Then $d(v) = 4$ by Claim 1. Let $X = \{v\} \cup N(v) = \{v, u_1, u_2, w_1, w_2\}$ and $G' = G - X$. Then $|X| = 5$ and $d^{out}(X) \leq 6$.

If $i(G') \leq 4$ then $\nu_s(G) \geq 1 + \frac{1}{9}(n(G') - i(G')) \geq 1 + \frac{1}{9}n(G)$, a contradiction. Therefore, let us suppose that $i(G') \geq 5$. Then both $w_1, w_2$ must be adjacent to some end-vertices, say $t_1, t_2$, and moreover $w_1$ and $w_2$ are not adjacent, otherwise $d^{out}(X) \leq 4$, which implies $i(G') \leq 4$, a contradiction.

Let $X' = X \cup N(w_1) \cup N(w_2)$ and $G'' = G - X'$. Then $|X'| \leq 11$ and $d^{out}(X') \leq 3$, which implies $i(G'') \leq 3$ (see Figure 3). Since for every induced matching $M$ of $G''$, $M \cup \{w_1t_1, w_2t_2\}$ is an induced matching of $G$, we have

$$\nu_s(G) \geq 2 + \frac{1}{9}(n(G'') - i(G'')) \geq 2 + \frac{1}{9}(n(G) - 11 - 3) > \frac{1}{9}n(G),$$

a contradiction. $\square$

![Figure 3: An illustration for the proof of Claim 2](image)

**Claim 3.** No two end-vertices have distance 4 in $G$.

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Proof. Suppose to the contrary that two end-vertices $u_1$ and $u_2$ have distance 4 in $G$ and $u_1v_1wv_2u_2$ is a path of length 4 linking $u_1$ and $u_2$. Consider $X = \{v_1, v_2\} \cup N(v_1) \cup N(v_2)$. Then $|X| \leq 9$, and $d^{\text{out}}(X) \leq 14$. From (4) and (5), we have $d^{\text{out}}(X) \geq 2i(G') - i_1(G')$. Thus, if $i(G') \geq 10$ then $i_1(G') \geq 2i(G') - d^{\text{out}}(X) \geq 6$. However, vertices in $I_1(G')$ are only adjacent to vertices in set $X - \{v_1, v_2, u_1, v_1\}$, which consists of at most 5 vertices. Hence, there are two end-vertices in $I_1(G')$ that have a common neighbor, contradicting Claim 2. Therefore, $i(G') \leq 9$, and, noting that each induced matching $M$ of $G'$ can be extended to an induced matching $M \cup \{u_1v_1, u_2v_2\}$ of $G$, we derive that

$$\nu_s(G) \geq 2 + \nu_s(G') \geq 2 + \frac{1}{9}(n(G') - i(G')) \geq 2 + \frac{1}{9}(n(G) - 9) = \frac{1}{9}n(G),$$

a contradiction to (1). (See Figure 4)

![Figure 4: An illustration for the proof of Claim 3](image)

Claim 4. $\delta(G) \geq 2$.

Proof. Suppose to the contrary that $u$ is a vertex of degree 1 and $v$ is its unique neighbor in $G$. Then, by Claim 1, $v$ has 3 other neighbors $w_1, w_2, w_3$. Let $X = \{u, v, w_1, w_2, w_3\}$ and $G' = G - X$. Since each induced matching $M$ of $G'$ can be extended to an induced matching $M \cup \{uv\}$ of $G$, if $i(G') \leq 4$ then

$$\nu_s(G) \geq 1 + \nu_s(G') \geq 1 + \frac{1}{9}(n(G') - i(G')) \geq 1 + \frac{1}{9}(n(G) - 5 - 4) = \frac{1}{9}n(G)$$

holds, contradicting (1). Therefore, let us suppose that $i(G') \geq 5$. Since

$$2i(G') - i_1(G') \leq d^{\text{out}}(X) \leq 9,$$

we must have $i_1(G') \geq 1$. By Claim 2 and 3, if $s$ and $t$ are two vertices in $I_1(G')$ then they are adjacent to two distinct vertices $w_i$ and $w_j$ and furthermore, $w_i$ and $w_j$ are adjacent. Hence,

$$d^{\text{out}}(X) \leq 9 - 2(i_1(G') - 1) = 11 - 2i_1(G').$$

Therefore, by (5), $11 \geq 3i_1(G') + 2i_2(G') + 3i_3(G') + 4i_4(G')$ holds, which, together with $i_1(G') \geq 1$ and $i_2(G') \geq 5$, implies that $i_1(G') = 1$, $i_2(G') = 4$, $i_3(G') = i_4(G') = 0$ and furthermore $w_1, w_2, w_3$ all have degree 4 and are not adjacent. But then $d^{\text{out}}(X \cup I(G')) \geq 5$, and hence $n(G) = 10$. Let $s$ be the unique vertex in $I_1(G')$ and suppose without loss of generality that its unique neighbor in $G$ is $w_1$. Since $d(w_1) = 4$ and $i_2(G') = 4$, there is a vertex $t$ in $I_2(G')$ that is not adjacent to $w_1$. Then, $\{w_1s, w_2t\}$ is an induced matching of $G$, and thus $\nu_s(G) \geq 2 > \frac{1}{9}n(G)$, a contradiction.
Claim 5. No two vertices of degree 2 are adjacent.

Proof. Suppose to the contrary that $u, v$ are two adjacent vertices of degree 2. Let $X = N(u) \cup N(v)$ and $G' = G - X$. Then $|X| = 6$ and $d_{\text{out}}(X) \leq 10$. By Claim 4, we see that $d_{\text{out}}(X) \leq 6$. Moreover, since for each induced matching $M$ of $G'$, $M \cup \{uv\}$ is an induced matching of $G$, it follows that

$$\nu_s(G) \geq 1 + \nu_s(G') \geq 1 + \frac{1}{9}(n(G) - 4 - 3) \geq \frac{1}{9}n(G)$$

holds, a contradiction.

Claim 6. No vertex of degree 2 is contained in a triangle.

Proof. Suppose to the contrary that $u$ is a vertex of degree 2 which is contained in a triangle and $v$ is one of its neighbors. Let $X = N(u) \cup N(v)$ and $G' = G - X$. Then $|X| = 5$ and $d_{\text{out}}(X) \leq 8$. On the other hand, by Claim 4 and (5), $d_{\text{out}}(X) \geq 2i_2(G') + 3i_3(G') + 4i_4(G')$ and $i(G') = i_2(G') + i_3(G') + i_4(G')$ hold. Hence $i(G') \leq d_{\text{out}}(X)/2 \leq 4$. Therefore,

$$\nu_s(G) \geq 1 + \nu_s(G') \geq 1 + \frac{1}{9}(n(G) - 5) = \frac{1}{9}n(G),$$

a contradiction.

Claim 7. No vertex of degree 2 is contained in a cycle of length 4.

Proof. Suppose that $u$ is a vertex of degree 2 which is contained in a cycle $uvwv$ of length 4. Let $X = N(u) \cup N(v)$ and $G' = G - X$. Then $|X| = 6$ and $d_{\text{out}}(X) \leq 10$, thus $i(G') \leq d_{\text{out}}(X)/2 \leq 5$.

If $i(G') \leq 3$ then $\nu_s(G) \geq 1 + \nu_s(G') \geq \frac{1}{9}n(G)$, contradicting (4). Hence we may suppose that $i(G') \geq 4$. Then there exists a vertex $s$ in $I(G')$ that is not adjacent to $t$. Since $\delta(s) \geq 2$ by Claim 4, $s$ is adjacent to a vertex $r$ in $N(v) - \{u, w\}$. Also we have $|N(X \cup I(G'))| \leq d_{\text{out}}(X \cup I(G')) \leq d_{\text{out}}(X) - 2i(G') \leq 2$, where the second inequality follows from (4), (5), (6) and Claim 4.

Now let $X' = X \cup I(G') \cup N(X \cup I(G'))$ and $G'' = G - X'$. Then $|X'| \leq |X| + i(G') + |N(X \cup I(G'))| \leq 6 + 5 + 2 = 13$. Moreover, $d_{\text{out}}(X') \leq 3|N(X \cup I(G'))| \leq 6$. Hence $i(G'') \leq d_{\text{out}}(X)/2 \leq 3$. Since for each induced matching $M$ of $G''$, $M \cup \{ut, rs\}$ is an induced matching of $G''$, it follows that

$$\nu_s(G) \geq 2 + \nu_s(G'') \geq \frac{1}{9}n(G),$$

contradicting (4). (see Figure 5)

Claim 8. $\delta(G) \geq 3$.

Proof. Suppose to the contrary that $u$ is a vertex of degree 2 in $G$ and $v, w$ are its neighbors. Let $X = N(u) \cup N(v)$ and $G' = G - X$. Then $|X| \leq 6$ and $d_{\text{out}}(X) \leq 12$. By Claim 4, no vertex of degree 2 is contained in a cycle of length 4, thus $w$ is not adjacent to any neighbor of $v$ other than $u$. Now if $i(G') \leq 3$ then $\nu_s(G) \geq 1 + \nu_s(G') \geq 1 + \frac{1}{9}(n(G) - 6 - 3) = \frac{1}{9}n(G)$, a contradiction.

So let us suppose that $i(G') \geq 4$. Then there is a vertex $s \in I(G')$ that is not adjacent to $w$, note that then $s$ is adjacent only to vertices in $N(v) - \{u\}$, so $s$ is contained in a cycle

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(Continued on the next page)
of length 4. Therefore \( d(s) \geq 3 \) by Claim 7. Let \( t \in N(v) \) be one of its neighbors. Then \( \{st, uw\} \) is an induced matching of \( G \) (see Figure 6).

Since vertices in \( I(G') \) are adjacent only to vertices in \( X \) we have \( d^{\text{out}}(X \cup I(G')) \leq d^{\text{out}}(X) - \sum_{x \in I(G')} d(x) \). Since \( d(x) \geq 2 \) for all \( x \in I(G') \), by Claim 4 \( d(s) \geq 3 \) and \( i(G') \geq 4 \), we have \( \sum_{x \in I(G')} d(x) \geq 3 \times 2 + 3 = 9 \). Thus \( d^{\text{out}}(X \cup I(G')) \leq 3 \).

Let \( X' = X \cup I(G') \cup N(X \cup I(G')) \) and \( G'' = G - X' \). Then one can easily see that \( d^{\text{out}}(X') \leq 3|N(X \cup I(G'))| \leq 3d^{\text{out}}(X \cup I(G')) \leq 9 \), so \( i(G'') \leq \left\lfloor \frac{d^{\text{out}}(X')}{2} \right\rfloor = 4 \), by Claim 4 and 5. Now let us upper bound \( |X'| \). We have

\[
|X'| \leq |X| + i(G') + d^{\text{out}}(X \cup I(G')) \\
\leq |X| + i(G') + d^{\text{out}}(X) - 2i(G') \\
= |X| + d^{\text{out}}(X) - i(G') \\
\leq 6 + 12 - 4 \\
= 14.
\]

Noting that every matching \( M \) of \( G'' \) can be extended to a matching \( M \cup \{st, uw\} \) of \( G \), we derive

\[
\nu_s(G) \geq 2 + \nu_s(G'') \geq 2 + \frac{1}{9}(n(G) - 14 - 4) = \frac{1}{9}n(G),
\]

a contradiction.

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**Claim 9.** \( G \) contains no triangle.

**Proof.** Suppose to the contrary that\( uvw \) is a triangle in \( G \). Let \( X = N(u) \cup N(v) \) and \( G' = G - X \). Then \( |X| \leq 7 \). So if \( i(G') \leq 2 \) we have \( \nu_s(G) \geq 1 + \nu_s(G') \geq 1 + \frac{1}{9}(n(G) - 7 - 2) = \frac{1}{9}n(G) \), a contradiction.
Let us suppose that \( i(G') \geq 3 \). It is easy to see that \( d^{\text{out}}(X) \leq 14 \). Hence, by Claim \( \text{[8]} \), \( i(G') = i_3(G') + i_4(G') \leq \lfloor d^{\text{out}}(X)/3 \rfloor \leq 4 \). Since \( |N(w) - X| \leq 2 \), and \( \delta(G) \geq 3 \) by Claim \( \text{[8]} \) there exists a vertex \( s \in I(G') \) such that \( s \) is not adjacent to \( w \). Then \( s \) is adjacent to a vertex \( r \in N(v) - \{u, w\} \) and \{\( uw, sr \)\} is an induced matching of \( G \).

Let \( X' = X \cup I(G') \cup N(w) \cup N(r) \) and \( G'' = G - X' \). Let \( Y = (N(w) \cup N(r)) - (X \cup I(G')) \). Then by simple counting one can see that \( |Y| \leq 4 \) and \( |X'| = |X| + i(G') + |Y| \leq 11 + i(G') \).

Since each vertex in \( Y \) is adjacent to at least one vertex in \( X \) and to at most 3 vertices outside \( X \), we have

\[
\begin{align*}
d^{\text{out}}(X') & \leq d^{\text{out}}(X \cup I(G')) - |Y| + 3|Y| \\
& \leq d^{\text{out}}(X) - 3i(G') + 2|Y|,
\end{align*}
\]

where the last equality follows from \( \text{[9]} \) and Claim \( \text{[8]} \). Therefore,

\[
|X'| + i(G'') \leq 11 + i(G') + \left\lfloor \frac{d^{\text{out}}(X')}{3} \right\rfloor \leq 11 + \left\lfloor \frac{d^{\text{out}}(X) + 2|Y|}{3} \right\rfloor \leq 11 + \left\lfloor \frac{14 + 2 \times 4}{3} \right\rfloor = 18.
\]

Since each induced matching \( M \) of \( G'' \) can be extended to an induced matching \( M \cup \{uw, sr\} \) of \( G \), it follows that

\[
\nu_s(G) \geq 2 + \nu_s(G'') \geq 2 + \frac{1}{9}(n(G) - 18) = \frac{1}{9}n(G),
\]

holds, contradicting \( \text{[1]} \).

\( \square \)

\textbf{Claim 10.} If a vertex \( u \) is contained in a cycle of length 4 then \( d(u) \geq 4 \).

\textbf{Proof.} Suppose to the contrary that \( u \) is contained in a cycle \( C = uwxy \) of length 4 and \( d(u) = 3 \). Let \( v \) be the neighbor of \( u \) that is not contained in \( C \). Let \( X = N(u) \cup N(v) \) and \( G' = G - X \). Then \( |X| \leq 7 \) and \( d^{\text{out}}(X) \leq 13 \). Hence, if \( i(G') \leq 2 \) then \( \nu_s(G) \geq 1 + \nu_s(G') \geq \frac{1}{5}n(G) \) holds, a contradiction. So let us suppose that \( i(G') \geq 3 \).

If \( d(v) = 3 \) then \( |X| = 6 \) and \( d^{\text{out}}(X) \leq 10 \), thus \( i(G') \leq 3 \). Hence \( \nu_s(G) \geq 1 + \nu_s(G') \geq \frac{1}{5}n(G) \) holds, again a contradiction. So we may suppose that \( d(v) = 4 \) and \( N(v) = \{u, x, t, r\} \).

We will use the following assertion.

\textbf{Assertion 1.} If there is an induced matching in \( G[X \cup I(G')] \) then \( \nu_s(G) \geq \frac{1}{5}n(G) \).

\textbf{Proof.} Let \( \{ab, cd\} \) be an induced matching in \( G[X \cup I(G')] \). Let \( X_1 = X \cup I(G') \cup N(\{a, b, c, d\}) \) and \( Y_1 = N(\{a, b, c, d\}) - (X \cup I(G')) \). Then since each vertex in \( Y_1 \) is adjacent to a vertex in \( \{a, b, c, d\} \subseteq X \cup I(G') \) and \( \delta(G) \geq 3 \) by Claim \( \text{[8]} \), \( \nu_s(G) \) we have \( |Y_1| \leq d^\text{out}(X \cup I(G')) - 3i(G') \leq 13 - 3 \times 3 = 4 \). Using a similar argument as in the proof of Claim \( \text{[9]} \), we obtain

\[
d^{\text{out}}(X_1) \leq d^{\text{out}}(X) - 3i(G') - |Y_1| + 3|Y_1| = d^{\text{out}}(X) - 3i(G') + 2|Y_1|,
\]

\( \text{[10]} \).
and $$|X_1| = |X| + |I(G')| + |Y_1|.$$ Therefore,

$$|X_1| + i(G_1) \leq |X| + i(G') + |Y_1| + \frac{d_{out}(X_1)}{3}$$

$$= |X| + \frac{5}{3}|Y_1| + \frac{d_{out}(X)}{3}$$

$$\leq 7 + \frac{5}{3} \times 4 + \frac{13}{3}$$

$$= 18.$$ 

Hence, since for each induced matching $$M$$ of $$G''$$, $$M \cup \{ab, cd\}$$ is an induced matching of $$G$$, we have

$$\nu_s(G) \geq 2 + \nu_s(G_1) \geq 2 + \frac{1}{9}(n(G) - 18) = \frac{1}{9}n(G),$$

a contradiction. 

Now we are ready to complete the proof of Claim 10. Since $$i(G') \geq 3$$ and $$|N(r) - X|, |N(t) - X| \leq 2$$, by observing that each vertex in $$I(G')$$ must be adjacent to either $$r$$ or $$t$$, we deduce that there are two vertices $$s_1, s_2 \in I(G')$$ such that $$s_1$$ is adjacent to $$r$$ but not to $$t$$ and $$s_2$$ is adjacent to $$t$$ but not to $$r$$. Then since $$r$$ and $$t$$ are not adjacent by Claim 9, $$\{s_1r, s_2t\}$$ is an induced matching of $$G[X \cup I(G')]$$. So Assertion 1 implies contradiction. 

Claim 11. There is no cycle of length 4.

Proof. Suppose that $$C = uvxy$$ is a cycle of length 4 in $$G$$. Let $$X = N(u) \cup N(v)$$ and $$G' = G - X$$. Then $$|X| \leq 8$$ and $$d_{out}(X) \leq 16$$. If $$i(G') \leq 1$$ then a similar argument as in previous claims yields $$\nu_s(G) \geq \frac{1}{9}n(G)$$, a contradiction. Therefore, we may suppose that $$i(G') \geq 2$$.

Since there are no three vertices in $$X$$ with pairwise distance 3 and each vertex in $$I(G')$$ has degree at least 3 we obtain that each vertex of $$I(G')$$ lies on a cycle of length 4. Therefore, by Claim 10 we have

all vertices in $$I(G')$$ have degree 4. (7)

We will use the following assertion, which is similar to Assertion 1.

Assertion 2. If there is an induced matching in $$G[X \cup I(G')]$$ then $$\nu_s(G) \geq \frac{1}{9}n(G)$$.

Proof. Let $$\{ab, cd\}$$ be an induced matching in $$G[X \cup I(G')]$$. Let $$X_1 = X \cup I(G') \cup N(\{a, b, c, d\})$$ and $$Y_1 = N(\{a, b, c, d\}) - (X \cup I(G'))$$.

We first prove that if $$|Y_1| \leq 4$$ then $$\nu_s(G) \geq \frac{1}{9}n(G)$$, a contradiction. In fact, a similar argument as in the proof of Assertion 1 yields

$$d_{out}(X_1) \leq d_{out}(X) - 4i(G') - |Y_1| + 3|Y_1| = d_{out}(X) - 4i(G') + 2|Y_1|,$$

where the multiplicity 4 for $$i(G')$$ is due to (7). We also have

$$|X_1| = |X| + |I(G')| + |Y_1|.$$
Therefore,
\[
|X_1| + i(G_1) \leq |X| + i(G') + |Y| + \frac{d^{\text{out}}(X_1)}{4} \\
= |X| + \frac{3}{2}|Y_1| + \frac{d^{\text{out}}(X)}{4} \\
\leq 8 + \frac{3}{2} \times 4 + \frac{16}{4} \\
= 18.
\]
It follows that
\[
\nu_s(G) \geq 2 + \nu_s(G_1) \geq 2 + \frac{1}{9}(n(G) - 18) = \frac{1}{9}n(G)
\]
holds.

It remains to prove that \(|Y_1| \leq 4\). Indeed, if \(|\{a, b, c, d\} \cap I(G')| \geq 2\), then it is easy to see that \(|Y_1| = |N(\{a, b, c, d\} - (X \cup I(G'))| \leq 4\). On the other hand, if \(\{a, b, c, d\} \subset X\) then a simple counting shows that \(d^{\text{out}}(X) \leq 12\) and hence \(|Y_1| \leq d^{\text{out}}(X) - 4i(G') \leq 4\), where the coefficient 4 of \(i(G')\) is due to \([7]\). Thus we may suppose that \(a \in I(G')\) and \(\{b, c, d\} \subset X\), note that then \(b \notin \{u, v\}\). Let \(t \in I(G') - s\). First consider the case \(\{c, d\} \cap \{u, v\} = \emptyset\). Since \(d(s) = d(t) = 4\) and \(|X - \{u, v, b, c, d\}| = 3\) we have that both \(s\) and \(t\) have a neighbor in \(\{b, c, d\}\). This implies that \(|Y_1| = |N(\{a, b, c, d\}) - (X \cup I(G'))| \leq 4\), as desired. Next, consider the case \(\{c, d\} = \{u, v\}\), it is easy to see that then \(|Y_1| \leq 2\). Finally, consider the case \(|\{c, d\} \cap \{u, v\}| = 1\), say \(d \in \{u, v\}\) and \(c \notin \{u, v\}\). Then since \(|N(t) \cap \{x, y\}| \leq 1\), as \(G\) contains no triangle, \(d(t) = 4\) and \(|X - \{u, v, x, y\}| = 4\), we have that \(t\) must be adjacent to \(b\) or \(c\). Hence again we obtain \(|Y - 1| \leq 4\), as desired. \(\square\)

Now suppose that there is no induced matching of size 2 in \(G[X \cup I(G')]\). We prove that \(G = C_{2,5}\). Let \(N(v) = \{u, x, a, b\}\) and \(N(u) = \{v, y, c, d\}\). Since each vertex \(s\) in \(I(G')\) is not adjacent to both \(x\) and \(y\) at the same time as \(G\) has no triangle, we may assume without loss of generality that there is an \(s \in I(G')\) that is adjacent to \(a, c\) but not to \(y\). Then since \(\{sa, yu\}\) is not an induced matching of \(G[X \cup I(G')]\) and since \(ua \notin E\) by Claim \([8]\), we must have \(ya \in E\). (See Figure \(7\) for an illustration.) Now by considering pair \(\{du, sa\}\), since \(ua, us \notin E\), we see that either \(da \in E\) or \(sd \in E\), but not both of them belong to \(E\).

![Figure 7: An illustration of Case 1; dashed edges indicate newly “found” edges and thick edges indicate considered pair in each step.](image)

**Case 1**: \(da \in E\) and \(sd \notin E\). 
(See an illustration for this case in Figure \(8\))

Since \(d(s) = 4\) we derive that \(sx, sb \in E\). Consider pair \(\{sx, ud\}\), we see that \(dx \in E\) since \(ux, us, ud \notin E\). Now let \(t\) be a vertex in \(I(G') - \{s\} \neq \emptyset\), due to the assumption that
I(G') ≥ 2. Then since \( d(a) = d(x) = 4 \), we know that \( N(a) = \{ s, t, x, y \} \), \( N(x) = \{ b, d, y, v \} \) and \( ta \notin E \). Therefore, \( N(t) \subseteq \{ b, c, d, x, y \} \) holds. However, by considering pair \( \{ sb, ud \} \) we derive that \( bd \in E \) which implies that \( G \) contains a triangle \( tbd \), contradicting Claim 9.

**Case 2:** \( sd \in E \) and \( da \notin E \).

Consider pair \( \{ sc, bv \} \), using the fact that \( G \) contains no triangle from Claim 9, we see that either \( bs \) or \( bc \) is in \( E \) but not both of them. Hence we consider these cases separately.

**Case 2.1:** \( bc \in E \) and \( bs \notin E \).

(See Figure 9 for an illustration.)

Then considering pair \( \{ sb, uy \} \) we derive that \( yb \in E \). Now let \( t \in I(G') - \{ s \} \neq \emptyset \). Since \( N(y) = \{ x, u, a, b \} \) and \( tb, tc \) do not both belong to \( E \), we must have \( tx, td, ta \in E \). Considering pair \( \{ uc, tx \} \), noting that \( xc \notin E \) since otherwise \( asc \) is a triangle in \( G \), we derive that \( tc \in E \). Finally, by considering pair \( \{ by, sd \} \), we obtain that \( bd \in E \). However then \( V(G) = X \) and \( G = C_{1,5} \), where the pairs corresponding to the vertices of \( C_5 \) are \( \{ a, x \}, \{ v, y \}, \{ b, u \}, \{ c, d \}, \{ s, t \} \), a contradiction to the assumption that \( G \neq C_{2,5} \).

**Case 2.2:** \( bs \in E \) and \( bc \notin E \).

(See Figure 10 for an illustration.)

Then considering pair \( \{ sb, uy \} \) we derive that \( yb \in E \). Now let \( t \in I(G') - \{ s \} \), then we may assume without loss of generality that \( ta, tc \in E \). Considering pair \( \{ tc, by \} \), we have that \( tb \in E \). Considering pair \( \{ ta, ud \} \), we obtain that \( td \in E \). By considering pair \( \{ xy, tc \} \) we deduce that \( xc \in E \). Finally, by considering pair \( \{ td, xv \} \) we derive that \( xd \in E \). Therefore, \( V(G) = X \cup I(G') \) and \( G = C_{2,5} \) where the pairs corresponding to the vertices of \( C_5 \) are \( \{ x, u \}, \{ v, y \}, \{ a, b \}, \{ c, d \}, \{ s, t \} \), a contradiction to the assumption that \( G \neq C_{2,5} \).

The example in Figure 11 due to Joos [15], shows that the lower bound in Theorem 1 is tight. It is easy to see that the proof of Theorem 1 implies a polynomial time algorithm to find an induced matching of size \( \frac{1}{2} n(G) \).

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Figure 9: An illustration for Case 2.1; dashed edges indicate newly “found” edge and thick edges indicate considered pair in each step.

Figure 10: An illustration for Case 2.2; dashed edges indicate newly “found” edges and thick edges indicate considered pair in each step.

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