ASYMPTOTIC FLUCTUATIONS OF REPRESENTATIONS OF THE UNITARY GROUPS

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ABSTRACT. We study asymptotics of representations of the unitary groups $U(n)$ in the limit $n \to \infty$ and we show that in many aspects they behave like large random matrices. In particular, we show that the highest weight of a random irreducible component in the Kronecker tensor product of two irreducible representations behaves asymptotically in the same way as the spectrum of the sum of two large random matrices with prescribed eigenvalues. This agreement happens not only on the level of the mean values (and thus can be described within Voiculescu’s free probability theory) but also on the level of fluctuations (and thus can be described within the framework of higher order free probability).

1. INTRODUCTION

1.1. Asymptotics of representations of the unitary groups. In general, questions concerning representations of the unitary groups $U(n)$ and manipulations with them (e.g. the problem of decomposing the Kronecker tensor product of two irreducible representations into a sum of irreducible components) have a well-known answer given by some algorithm involving some combinatorial objects, such as Young tableaux, weights or Littelmann paths [Lit95]. However, in the limit $n \to \infty$ the computational complexity of such combinatorial algorithms becomes very quickly intractable. It is therefore natural to ask for some partial or approximate answers which would be useful and meaningful asymptotically. For similar problems in relation to the symmetric groups, we refer to [Bia98].

The first result in this direction was due to Biane [Bia95] who proved that a typical irreducible component of a representation of the unitary group $U(n)$ resulting from some natural representation-theoretic operations can be asymptotically described in the language of Voiculescu’s free probability theory [VDN92]. This highly non-commutative probability theory was known to describe the asymptotic behavior of large random matrices [Voi91].

In this paper, we revisit the work of Biane [Bia95] and generalize his results. We show that the fact that both representations and random matrices are asymptotically described by Voiculescu’s free probability can be
explained by the fact that representations behave asymptotically in the same way as large random matrices. This behavior concerns not only the mean value (as in the original work of Biane [Bia95]) but also more refined equality with respect to fluctuations around the mean values. This kind of results can be naturally expressed within the higher order free probability theory [MS06, MSS07, CMSS07] which was developed as a framework capable of describing fluctuations of random matrices in an abstract manner, just like the original Voiculescu’s free probability provides an abstract description of the average behavior of random matrices. Analogous results hold true for representations of the other classical series of compact Lie groups.

We also show that the technical assumption from the original paper of Biane [Bia95] concerning the growth speed of a typical highest weight can be significantly weakened.

The above mentioned results reduce the original problem of the asymptotics of representations of the unitary groups to the better and more widely understood problem of large random matrices spectra.

The main method of proof is to treat a representation of $U(n)$ as an $n \times n$ random matrix, the entries of which do not commute and to show that under some mild assumptions this non-commutativity asymptotically tends to zero hence a representation of $U(n)$ can be treated for $n \to \infty$ as a classical random matrix. Essentially the same line of proof was used in our previous paper [CS09] in order to study asymptotics of representations of a fixed compact Lie group.

In the remaining part of this section we introduce the basic notations and present in more detail the main results of the paper.

1.2. The canonical random matrix associated to a representation. If $\rho$ is an irreducible representation of $U(n)$ corresponding to the highest weight $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbb{Z}^n$ we define its shifted highest weight $l = (l_1 > \cdots > l_n) \in \mathbb{Z}^n$ by

$$l_i = \lambda_i + (n - i).$$

For the purposes of this article, it is more convenient to index irreducible representations by their shifted highest weights instead of, as usually, the highest weights; for this reason we use the symbol $\rho_l$ to denote the corresponding irreducible representation.

To an irreducible representation $\rho_l$ we associate a random matrix

$$X = X(\rho_l) = U \begin{bmatrix} l_1 & \cdots & \cdot & \\
\cdot & \cdot & \cdots & \cdot \\
\cdots & \cdots & \cdots & \cdots \\
l_n & \cdots & \cdots & l_n \end{bmatrix} U^{-1},$$
where $U$ is a random unitary matrix, distributed according to the Haar measure on $U(n)$. Another way of defining this random matrix is to say that its distribution is the uniform measure on the manifold of all hermitian matrices with the prescribed eigenvalues.

If $\rho$ is reducible, we consider its decomposition into irreducible components

$$\rho = \bigoplus_l n_l \cdot \rho_l,$$

where $n_l \in \{0, 1, \ldots \}$ denote the multiplicities and we consider a probability measure on the set of all shifted weights given as follows:

$$P(l) = \frac{n_l \cdot (\text{dimension of } \rho_l)}{\text{(dimension of } \rho)}.$$  

More generally, to a reducible representation $\rho$, we associate a random matrix $X(\rho)$ whose law is determined by the fact that it is unitarily invariant and that its eigenvalues are distributed according to the probability measure $P(l)$.  

The random matrix $X = X(\rho)$ contains essentially all information about the representation $\rho$. In our previous paper [CS09] we studied applications of this matrix in the study of the asymptotics of representations of a fixed unitary group $U(n)$ (and in fact, representations of any fixed compact Lie group). In this article we study asymptotics of representations $\rho_n$ of the unitary groups $U(n)$ in the limit $n \to \infty$, therefore we will have to replace the random matrix $X$ by a sequence of random matrices $(X(\rho_n))$ with their sizes tending to infinity.

1.3. The main result. The main result of this paper can be stated as follows:

**Theorem 1.** For each $n$ let $\rho_n$ be a representation of the unitary group $U(n)$ and assume that $\varepsilon_n = o\left(\frac{1}{n}\right)$. Then the sequence of rescaled representations $(\varepsilon_n \rho_n)$ viewed as a sequence of random matrices with non-commutative entries converges in distribution (in the sense of higher order free probability) if and only if the sequence of random matrices $(\varepsilon_n X(\rho_n))$ converges in distribution (in the sense of higher order free probability). If the limits exist, they are equal.

The formulation of the above theorem contains several notions which will be defined throughout the paper. For the purposes of this introduction it is enough to think that the above theorem says that the rescaled representations $(\varepsilon_n \rho_n)$ behave asymptotically in the same way as the sequence $(\varepsilon_n X(\rho_n))$ of random matrices. In Section 1.5 we show concrete applications to representation theory of this abstract result.
1.4. Spectral measure for representations and random matrices. Let $X$ be an $n \times n$ hermitian random matrix; we denote by $l = (l_1 \geq \cdots \geq l_n) \in \mathbb{R}^n$ the set of its eigenvalues (counted with multiplicities). We define the spectral measure of $X$ as the random probability measure on the real line

$$\mu_X = \sum \frac{1}{n} \delta_{l_i}.$$  

For an irreducible representation $\rho = \rho_l$ of $U(n)$ corresponding to the shifted highest weight $l = (l_1 \geq \cdots \geq l_n) \in \mathbb{Z}^n$ we define its spectral measure

$$\hat{\mu}_\rho = \hat{\mu}_l = \sum \frac{1}{n} \delta_{l_i}$$  

which is a probability measure on $\mathbb{R}$. In order to distinguish it from another definition of the spectral measure for a representation which will be introduced later, we shall call it also naive spectral measure.

If $\rho$ is a reducible representation, we define its spectral measure by the same formula (5) but now $l$ is a random shifted highest weight as defined by (3). In this case $\hat{\mu}_\rho$ is a random probability measure on $\mathbb{R}$.

Notice that the random probability measure $\hat{\mu}_\rho$ is nothing else but the spectral measure of the random matrix $X(\rho)$ associated to $\rho$ therefore the definitions of the spectral measure for random matrices and for representations have something in parallel.

If $\mu$ is a probability measure on $\mathbb{R}$ and $\varepsilon$ is a real number we shall denote by $D_\varepsilon \mu$ the dilation of the measure $\mu$; in other words it is the distribution of the random variable $\varepsilon X$, where $X$ is a random variable with the distribution $\mu$. We use the notational shorthands

$$\varepsilon l = (\varepsilon l_1, \ldots, \varepsilon l_n) \quad \text{for } l = (l_1, \ldots, l_n),$$

$$\hat{\mu}_{\varepsilon \rho} = D_\varepsilon \hat{\mu}_\rho,$$

$$X(\varepsilon \rho) = \varepsilon X(\rho).$$

and we think about the formal expression $\varepsilon \rho$ as about some kind of a rescaled representation.

1.5. Applications of the main result. Let us present here a few concrete consequences of Theorem 1. A more complete collection of its applications is be given in Section 5 together with the proofs.

We start with a solution to the problem mentioned in the beginning of Section 1.1, the decomposition of Kronecker tensor products into irreducible components.

**Corollary 2.** Let $$(\varepsilon_n)$$ be a sequence of real numbers such that $\varepsilon_n = o \left( \frac{1}{n} \right)$. For each $i \in \{1, 2\}$ and $n \geq 1$ let $\rho_n^{(i)}$ be an irreducible representation of
Assume that for each $i \in \{1, 2\}$ the sequence $\hat{\mu}_{\varepsilon_n \rho_n^{(i)}}$ of the (rescaled) spectral measures converges in moments to some probability measure $\mu^{(i)}$. We consider the Kronecker tensor product $\rho_n^{(3)} = \rho_n^{(1)} \otimes \rho_n^{(2)}$. Then the (rescaled) spectral measure $\hat{\mu}_{\varepsilon_n \rho_n^{(3)}}$ converges in moments almost surely to the Voiculescu’s free convolution $\mu^{(1)} \boxplus \mu^{(2)}$ [VDN92].

A similar result was proved by Biane [Bia95] under much stronger assumptions on decay of $\varepsilon$, namely that $\varepsilon = o\left(\frac{1}{n^\alpha}\right)$ for all values of the exponent $\alpha$.

The Corollary 2 is formulated in terms of free additive convolution which belongs to the language of Voiculescu’s free probability theory [VDN92]. It can be strengthened by establishing a direct bridge with the theory of unitarily invariant random matrices as in the following corollary:

**Corollary 3.** Let the assumptions of Corollary 2 be fulfilled. For $i \in \{1, 2\}$ we denote by $X_n^{(i)} = X(\varepsilon_n \rho_n^{(i)})$ the random matrix corresponding to the (rescaled) representation $\rho_n^{(i)}$ and define $X_n^{(3)} = X_n^{(1)} + X_n^{(2)}$, where random matrices $X_n^{(1)}$ and $X_n^{(2)}$ are chosen to be independent.

Then, the spectral measures of rescaled representations $\varepsilon_n \rho_n^{(3)}$ and the spectral measures of random matrices $X_n^{(3)}$ asymptotically have the same mean and their fluctuations around this mean are asymptotically identical and Gaussian.

In the light of Corollary 3 the contents of Corollary 2 should not come as a surprise since it is well known [Voi91] that Voiculescu’s free convolution describes asymptotics of the spectrum of sum of two independent random matrices.

Similarly, we can handle the problem of restriction to a unitary subgroup:

**Corollary 4.** Let $(\varepsilon_n)$ be a sequence of real numbers such that $\varepsilon_n = o\left(\frac{1}{n}\right)$, for each $n \geq 1$ let $\rho_n$ be an irreducible representation of $U(n)$ such that the sequence of (rescaled) spectral measures $\hat{\mu}_{\varepsilon_n \rho_n}$ converges in moments to some limit $\mu$. Let $(m_n)$ be a sequence of integers such that $1 \leq m_n \leq n$ and such that the limit $\alpha = \lim_{n \to \infty} \frac{m_n}{n} > 0$ exists and is positive. For each $n$ we define $\rho_n' = \rho_n|_{U(m_n)}$ to be a representation of $U(m_n)$ given by the restriction of $\rho_n$ to the subgroup.

Then the sequence of (rescaled) spectral measures $\hat{\mu}_{\varepsilon_n \rho_n'}$ converges almost surely in moments to the free compression of $\mu$ by a free projector of trace $\alpha$ [VDN92].

In addition the spectral measures of the representation $\varepsilon_n \rho_n'$ and the spectral measure of the random matrix $X_n'$ asymptotically have the same mean and their fluctuations around this mean are asymptotically identical and
Gaussian, where $X'_n$ denotes the $m_n \times m_n$ upper-left corner of the random matrix $X_n$.

1.6. **Element of proof: Representations of Lie groups.** In our previous paper [CS09] we studied the asymptotics of a sequence $(\rho_n)$ of representations of a fixed compact Lie group $G$. The main idea was that each representation $\rho : g \rightarrow \text{End}(V)$ of the corresponding Lie algebra $g$ can be equivalently viewed as $\rho \in g^* \otimes \text{End}(V)$. Since $\text{End}(V)$ equipped with the normalized trace $\text{tr}$ can be viewed as a non-commutative probability space, $\rho$ becomes a non-commutative random vector in $g^*$. Our problem is therefore reduced to studying the sequence $(\varepsilon_n \rho_n)$ of non-commutative random vectors in $g^*$, where $(\varepsilon_n)$ is some suitably chosen sequence of numbers which takes care of the right normalization. We proved that in many situations the distribution of $\varepsilon_n \rho_n$ converges to a classical (commutative) probability distribution on $g^*$ which, when the group $G$ has some matrix structure, can be interpreted as some random matrix. In this way several problems of the asymptotic representation theory of Lie group $G$ have answers in terms of certain random matrices and their eigenvalues.

In this paper the fixed group $G$ is replaced by a sequence of compact Lie groups $G_1, G_2, \ldots$ and we study the asymptotic properties of the sequence $(\rho_n)$, where $\rho_n$ is a representation of $G_n$. Our previous paper [CS09] is not directly applicable because each $\rho_n$ is a non-commutative random vector in a different space, namely $g^*_n$, and it is not possible to consider the limit of the distributions. In particular, we are not able to compare the distributions of representations of different groups, unless they have some common structure. In the following we show how to overcome this difficulty and how to find a substitute of the notion of convergence in distribution which will allow us to speak about asymptotic distribution of a sequence of representations.

In Lemma 13 we prove that the $k$-th moment of the representation $\rho$ of $G$

$$M_k(\rho) = \mathbb{E} \left( \rho^{\otimes k} \right) \in (g^*)^{\otimes k}$$

(for the exact definition, see Section 2.5) is invariant under the coadjoint action of $G$. The set of such $G$-invariant elements of $(g^*)^{\otimes k}$ is denoted by $[(g^*)^{\otimes k}]_G$. For many groups $G$ the corresponding invariant spaces $[(g^*)^{\otimes k}]_G$ are surprisingly nice.

The common structure of the groups $(G_n)$ which turns out to be sufficient for our purposes is the following one: we assume that for each $k$ the spaces $[(g^*_n)^{\otimes k}]_{G_n}$ (possibly, except for finitely many values of $n$) are all isomorphic in some nice canonical way to (a subspace of) some vector
space denoted by $[(g^*)^\otimes k]_G$; in this way we can regard

$$\mathcal{M}_k(\rho_n) \in [(g^*_n)^\otimes k]_{G_n} \subseteq [(g^*)^\otimes k]_G.$$  

For each value of $k$ in the invariant space $[(g^*)^\otimes k]_G$, we choose some basis. Now it makes sense to speak about the asymptotic behavior of the coordinates of $\mathcal{M}_k(\varepsilon_n \rho_n)$ in this basis, for some suitably chosen sequence $(\varepsilon_n)$ and we are able to compare the distributions of representations of different groups.

In concrete examples of the series of the unitary groups $(U(n))$, the corresponding invariants are given by the vector spaces given by the symmetric groups algebras $\mathbb{C}[\mathfrak{S}(k)]$. For the orthogonal groups $(O(n))$ and the symplectic groups $(Sp(n))$, the invariants are given by Brauer algebras. In the sequel, we focus on the first of the above mentioned cases, namely on the sequence $(U(n))$. Nevertheless our methods of proof can be readily extended to give similar results on the two other series as well.

1.7. **Higher order free probability.** For a Lie algebra representation $\rho_n : \mathfrak{u}(n) \to \text{End}(V_n)$ the corresponding moment

$$\mathcal{M}_k(\varepsilon_n \rho_n) \in \left[ (\mathfrak{u}(n))^\otimes k \right]_{U(n)} = \left[ (\mathbb{M}_n(\mathbb{C}))^\otimes k \right]_{U(n)} \subseteq \mathbb{C}[\mathfrak{S}(k)]$$

can be identified with a function on the symmetric group which is given explicitly as

$$\left( \mathcal{M}_k(\varepsilon_n \rho_n) \right)(\pi) = \varepsilon_n^k \text{tr} [\rho_n(e_{1 \pi_1}) \cdots \rho_n(e_{k \pi_k})].$$

The above quantities (6) contain complete information about representation $\rho_n$; the study of asymptotics of representations is therefore reduced to studying asymptotics of $\mathcal{M}_k(\varepsilon_n \rho_n)$ in the limit $n \to \infty$. It remains to determine which asymptotics will be most convenient.

The same problem appears in the random matrix theory, where analogous quantities $\mathcal{M}_k(X_n)$ can be considered for a unitarily invariant random matrix $X_n$. This problem has been studied in the context of the theory of higher order free probability which was introduced by Mingo and Speicher and later on was further developed also by the authors of this article [MS06, MSS07, CMSS07]. The main goal of this theory is to give an abstract framework which would be able to describe asymptotics of fluctuations of random matrices in a similar way as Voiculescu’s original free probability [VDN92] describes the mean behavior of random matrices. This goal was achieved by the notions of higher order moments and higher order free cumulants which on one side have very nice probabilistic interpretations for a given sequence of random matrices and on the other side are abstract quantities which concern abstract objects modeling limits of random matrices.
Our paper here gives applications of [CMSS07] to representation theory, and therefore stands as a first example of uses of the theory of higher order freeness beyond random matrix theory.

1.8. **Organization of the paper.** In Section 2 we recall the notations related to non-commutative random variables and non-commutative random vectors. In Section 3 we study unitarily invariant random matrices with non-commutative entries. In Section 4 we study representations as random matrices with non-commutative entries and prove our main result. In Section 5 we present applications of the main result and provide proofs of the results presented in Section 1.5.

2. NON-COMMUTATIVE PROBABILITY

2.1. **Non-commutative probability spaces.** Let \((\Omega, \mathcal{M}, P)\) be a Kolmogorov probability space. We consider the algebra 
\[
\mathcal{L}^\infty-(\Omega) = \bigcap_{n \geq 1} \mathcal{L}^n(\Omega)
\]
of random variables with all moments finite. This algebra is equipped with a functional \(E: \mathcal{L}^\infty-(\Omega) \rightarrow \mathbb{C}\) which to a random variable associates its mean value.

We consider a generalization of the above setup in which the commutative algebra \(\mathcal{L}^\infty-(\Omega)\) is replaced by any (possibly non-commutative) \(*\)-algebra \(\mathfrak{A}\) with a unit and \(E: \mathfrak{A} \rightarrow \mathbb{C}\) is any linear functional which is normalized (i.e. \(E(1) = 1\)) and positive (i.e. \(E(x^*x) > 0\) for all \(x \in \mathfrak{A}\) such that \(x \neq 0\)). The elements of \(\mathfrak{A}\) are called *non-commutative random variables* and the functional \(E\) is called the mean value or expectation. We also say that \((\mathfrak{A}, E)\) is a non-commutative probability space [VDN92, Mey93].

2.2. **Partitions and partitioned permutations.** The set of partitions of the set \(\{1, \ldots, l\}\) is endowed with the order defined as follows: \(\mathcal{V} \leq \mathcal{W}\) if every block of partition \(\mathcal{V}\) is contained in some block of partition \(\mathcal{W}\).

For a permutation \(\pi\) we denote by \(C(\pi)\) the partition corresponding to the cycles of \(\pi\). We write \(\pi \leq \mathcal{W}\) if every cycle of permutation \(\pi\) is contained in some block of partition \(\mathcal{W}\) or, in other words, if \(C(\pi) \leq \mathcal{W}\).

We denote by \(#\mathcal{V}\) the number of blocks of a partition \(\mathcal{V}\). We also denote by \(#\pi = #C(\pi)\) the number of cycles of \(\pi\).

The set of partitions carries a lattice structure \(\vee, \wedge\) where the smallest element is the discrete partition \(0_l = \{\{1\}, \ldots, \{l\}\}\) and the largest element is the rough partition \(1_l = \{\{1, \ldots, l\}\}\).

A *partitioned permutation* of \(Z = \{1, \ldots, l\}\) is a pair \((\mathcal{V}, \pi)\), where \(\mathcal{V}\) is a partition of the set \(Z\) and \(\pi\) is a permutation of the same set such that
\( \pi \leq \mathcal{V} \). For a given permutation \( \pi \) we denote by \((0, \pi) := (C(\pi), \pi)\) the partitioned permutation with the smallest possible partition for \( \pi \).

We define the length of the permutation \( \pi \in \mathcal{S}(l) \) as \(|\pi| = l - \# \pi \). We also define the length of the partitioned permutation \((\mathcal{V}, \pi)\) of the set \( \{1, \ldots, l\} \) as

\[ |(\mathcal{V}, \pi)| = |\pi| + 2(\# \pi - \# \mathcal{V}) \]

and the length of a partition \(|\mathcal{V}|\) of the same set as \(|\mathcal{V}| = l - \# \mathcal{V} \).

We say that \((\mathcal{V}_1, \pi_1) \cdot (\mathcal{V}_2, \pi_2) = (\mathcal{V}_3, \pi_3)\) if \(\mathcal{V}_1 \vee \mathcal{V}_2 = \mathcal{V}_3\) and \(\pi_1 \pi_2 = \pi_3\) and \(|(\mathcal{V}_1, \pi_1)| + |(\mathcal{V}_2, \pi_2)| = |(\mathcal{V}_3, \pi_3)|\). Notice that with this definition the product of two partitioned permutations is not always defined. The definition of the product of a (partitioned) permutation is rather natural since one can show that it is equal to the minimal number of factors necessary to write a given (partitioned) permutation as a product of (partitioned) transpositions.

We say that \((\mathcal{V}_1, \pi_1) \leq (\mathcal{V}_2, \pi_2)\) if \((\mathcal{V}_1, \pi_1) \cdot (0, \pi_1^{-1} \pi_2) = (\mathcal{V}_2, \pi_2)\). This relation is in general not transitive.

We say that partitioned permutations \((\mathcal{V}_1, \pi_1)\) and \((\mathcal{V}_2, \pi_2)\) are conjugate by a permutation \(\sigma\) if they are equal after relabeling the elements of \(\{1, \ldots, l\}\) given by \(\sigma\). Formally speaking, this means that \(\pi_2 = \sigma \pi_1 \sigma^{-1}\) and for each pair \(a, b \in \{1, \ldots, l\}\), elements \(a, b\) belong to the same block of \(\mathcal{V}_2\) if and only if \(\sigma^{-1}(a), \sigma^{-1}(b)\) belong to the same block of \(\mathcal{V}_1\).

### 2.3. Tensor independence and non-commutative cumulants

Let \((\mathfrak{A}_i)\) be a (finite or infinite) sequence of subalgebras of the non-commutative probability space \(\mathfrak{A}\). They are said to be tensor independent iff they commute and \(\mathbb{E}(a_1 a_2 \ldots) = \mathbb{E}(a_1) \mathbb{E}(a_2) \ldots\) holds for all sequences \((a_i)\) which contain only finitely many elements different from 1 and such that \(a_i \in \mathfrak{A}_i\). Tensor independence can be regarded as a substitute of the usual independence of classical random variables in the non-commutative setup.

Let \(\widetilde{\mathfrak{A}} = \bigotimes_{n \in \mathbb{N}} \mathfrak{A}\) be the inductive limit of algebraic tensor products. This is a non-commutative probability space together with the infinite tensor product state \(\mathbb{E}^{\otimes \infty}\). Clearly, the subalgebras

\[ \mathfrak{A}^{(i)} = \bigotimes_{i - 1 \text{ times}} 1 \otimes \ldots \otimes 1 \otimes \mathfrak{A} \otimes 1 \otimes \ldots \subset \widetilde{\mathfrak{A}} \]

are tensor independent. We will regard \((\mathfrak{A}^{(i)})_i\) as a family of tensor independent copies of the algebra \(\mathfrak{A}\). Given \(a \in \mathfrak{A}\), we define its \(i\)-th tensor independent copy \(a^{(i)} \in \mathfrak{A}^{(i)}\) by

\[ a^{(i)} = 1 \otimes i^{-1} \otimes a \otimes 1 \otimes \ldots \]

With this material we can introduce the notion of non-commutative cumulant. For each \(i \in \{1, \ldots, l\}\) let \(a_i \in \mathfrak{A}\) be a non-commutative random
variable. For any partition $\mathcal{V}$ we can define a multilinear moment map

$$E_{\mathcal{V}} : \mathbb{A} \times \cdots \times \mathbb{A} \rightarrow \mathbb{C}$$

by

$$E_{\mathcal{V}}(a_1, \ldots, a_l) = E^{\otimes \infty} \left( a_1^{(b(1))} \cdots a_l^{(b(l))} \right),$$

where $b : \{1, \ldots, l\} \rightarrow \mathbb{N}$ is any function defining the partition $\mathcal{V}$, i.e. $b(i) = b(j)$ if and only if $i$ and $j$ belong to the same block of $\mathcal{V}$. Following the classical scheme, we define tensor cumulants to be multilinear maps

$$k_{\mathcal{V}} : \mathbb{A} \times \cdots \times \mathbb{A} \rightarrow \mathbb{C}$$

such that

$$\sum_{\mathcal{W} \subseteq \mathcal{V}} k_{\mathcal{W}} = E_{\mathcal{V}}$$

for every partition $\mathcal{V}$.

A special role is played by the cumulant corresponding to the maximal partition; we will use a special notation for it:

$$k(a_1, \ldots, a_l) := k_{1^l}(a_1, \ldots, a_l).$$

Observe that this definition is actually Lehner’s cumulant in case of the tensor independence case, cf \cite{Leh04}. When $\mathbb{A} = L^\infty(-\Omega)$, this corresponds to the classical probability space, and tensor cumulants coincide with the classical cumulants of random variables.

Notice that the family $(E_{\mathcal{V}})$ is multiplicative in the sense that $E_{\mathcal{V}}(a_1, \ldots, a_l)$ is a product of the expressions $E(a_{i_1} \cdots a_{i_m})$ over the blocks $\{i_1 < \cdots < i_m\}$ of the partition $\mathcal{V}$. It follows immediately that the family $(k_{\mathcal{V}})$ is multiplicative as well, hence it is uniquely determined by the cumulants of the form (8). Therefore $(k_{\mathcal{V}})$ can be alternatively defined as the unique family which is multiplicative and for which the weaker version of condition (7) holds true, namely for any value of $l$

$$\sum_{\mathcal{W}} k_{\mathcal{W}} = E,$$

where the sum runs over all partitions of $\{1, \ldots, l\}$. For more on this topic of multiplicative functions on partitions and their applications to free probability theory we refer to \cite{NS06}.
2.4. Cumulants and commutators. In the following we will use the following shorthands:

\[ k(\ldots, a_i, a_{i+1}, \ldots) := k(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, a_{i+2}, \ldots, a_n), \]
\[ k(\ldots, a_{i+1}, a_i, \ldots) := k(a_1, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_n), \]
\[ k(\ldots, [a_i, a_{i+1}], \ldots) := k(a_1, \ldots, a_{i-1}, [a_i, a_{i+1}], a_{i+2}, \ldots, a_n), \]
and similar ones.

**Lemma 5.** For any elements \( a_1, \ldots, a_n \in A \), any \( 1 \leq i \leq n-1 \) and any partition \( \mathcal{W} \) of \( \{1, \ldots, n\} \) such that \( i \) and \( i+1 \) are connected by \( \mathcal{W} \)

\[ k_{\mathcal{W}'}(\ldots, a_i, a_{i+1}, \ldots) - k_{\mathcal{W}}(\ldots, a_{i+1}, a_i, \ldots) = k_{\mathcal{W}'}(\ldots, [a_i, a_{i+1}], \ldots), \]

where \( \mathcal{W}' \) denotes the partition of \( \{1, \ldots, n-1\} \) resulting from \( \mathcal{W} \) by merging \( i \) and \( i+1 \) into one element and by relabeling elements \( i+2, \ldots, n \) into elements \( i+1, \ldots, n-1 \).

**Proof.** This is an application of the non-commutative version of the formula of Leonov and Shiraev for cumulants of products to the right hand side of the above equality. We provide an alternative proof below.

Let \( \mathcal{V} \) be any partition of \( \{1, \ldots, n\} \) such that \( i \) and \( i+1 \) are connected by \( \mathcal{V} \). From the defining relations for cumulants it follows that

\[ E_{\mathcal{V}'}(\ldots, [a_i, a_{i+1}], \ldots) = E_{\mathcal{V}}(\ldots, a_i, a_{i+1}, \ldots) - E_{\mathcal{V}}(\ldots, a_{i+1}, a_i, \ldots) = \sum_{\mathcal{W} \leq \mathcal{V}} k_{\mathcal{W}}(\ldots, a_i, a_{i+1}, \ldots) - k_{\mathcal{W}}(\ldots, a_{i+1}, a_i, \ldots). \]

From the multiplicativity of cumulants it follows that if \( \mathcal{W} \) does not connect \( i \) and \( i+1 \) then the corresponding summand on the right hand side is equal to zero. It follows that the sum on the right hand side can be written as

\[ \sum_{\mathcal{W}' \leq \mathcal{V}} k_{\mathcal{W}'}(\ldots, a_i, a_{i+1}, \ldots) - k_{\mathcal{W}'}(\ldots, a_{i+1}, a_i, \ldots). \]

It follows that the multiplicative function

\[ k_{\mathcal{W}'}(\ldots, [a_i, a_{i+1}], \ldots) := k_{\mathcal{W}'}(\ldots, a_i, a_{i+1}, \ldots) - k_{\mathcal{W}'}(\ldots, a_{i+1}, a_i, \ldots) \]

fulfills the defining property of cumulants. \( \square \)

2.5. Non-commutative random vectors. Let \((A, \mathbb{E})\) be a non-commutative probability space and \( V \) be a vector space; the elements of \( V \otimes A \) will be called non-commutative random vectors in \( V \) (over a non-commutative probability space \((A, \mathbb{E})\)).

Given \( v_1 = x_1 \otimes a_1 \in V_1 \otimes A \) and \( v_2 = x_2 \otimes a_2 \in V_2 \otimes A \) we define

\[ v_1 \otimes v_2 = (x_1 \otimes a_1) \otimes (x_2 \otimes a_2) = (x_1 \otimes x_2 \otimes a_1 a_2) \in V_1 \otimes V_2 \otimes A \]
and its linear extension to non-elementary tensors. Whenever \( v_1 = v_2 \) with \( V_1 = V_2 \) one shortens the notation as \( v^\otimes 2 \in V^\otimes 2 \otimes \mathfrak{A} \) and one extends it by recursion to the definition of

\[
v^\otimes l \in V^\otimes l \otimes \mathfrak{A}.
\]

Observe that this definition is reminiscent of the definition of tensor product of representations of compact quantum groups of Woronowicz [Wor87], provided that \( \mathfrak{A} \) is a quantum group and \( V \) a representation of \( \mathfrak{A} \).

For a non-commutative random vector \( v \) we define its \( l \)-th order vector moment \( \mathcal{M}_l(v) \) to be

\[
\mathcal{M}_l(v) = \langle \mathbf{1} \otimes E \rangle v^\otimes l \in V^\otimes l.
\]

We define the distribution of a non-commutative random vector as its sequence \( \langle \mathcal{M}_l(v) \rangle_{l=1,2,...} \) of moments.

The above definitions can be made more explicit as follows: let \( e_1, \ldots, e_d \) be a base of the finite-dimensional vector space \( V \). Then a (classical) random vector \( v \) in \( V \) can be viewed as

\[
v = \sum_i a_i e_i,
\]

where \( a_i \) are the (random) coordinates. Then a non-commutative random vector can be viewed as the sum (9) in which \( a_i \) are replaced by non-commutative random variables. One can easily see that the sequence of moments

\[
\mathcal{M}_l(v) = \sum_{i_1,...,i_l} \mathbb{E}(a_{i_1} \cdots a_{i_l}) e_{i_1} \otimes \cdots \otimes e_{i_l}
\]

contains nothing else but the information about the mixed moments of the non-commutative coordinates \( a_1, \ldots, a_d \) and the convergence of moments is equivalent to the convergence of the mixed moments of \( a_1, \ldots, a_d \).

In the sequel of the paper, we pay special attention to the case when the vector space \( V = \mathbb{M}_n(\mathbb{C}) \) is the matrix algebra. In this case the non-commutative random vectors, elements of \( \mathbb{M}_n(\mathbb{C}) \otimes \mathfrak{A} = \mathbb{M}_n(\mathfrak{A}) \) can be also called random matrices with non-commutative entries.

3. Unitarily invariant matrices with quantum entries and higher-order probability spaces

3.1. Schur-Weyl duality. We recall the following result, known as Schur-Weyl duality theorem:

**Theorem 6.** Let \( \rho \) be the diagonal action of the unitary group \( U(n) \) on \( (\mathbb{C}^n)^\otimes k \). Let \( \hat{\rho} \) be the action of the symmetric group \( \mathfrak{S}(k) \) on \( (\mathbb{C}^n)^\otimes k \) by permutation of elementary tensors.
The actions of $\mathfrak{g}(k)$ and of $U(n)$ commute, therefore $\rho \times \tilde{\rho}$ is a representation of $\mathfrak{g}(k) \times U(n)$ on $(\mathbb{C}^n)^{\otimes k}$. This representation is multiplicity free. Equivalently, the commutant of $\rho$ in $(\mathbb{C}^n)^{\otimes k}$ is $\tilde{\rho}$, and the converse is true as well.

3.2. Random vectors in $u(n)^*$ and spectral measure. For $G = U(n)$, $\mathfrak{g} = u(n)$, we elaborate on the discussion of Section 1.6 and describe the the invariant space $\left[(\mathfrak{g}^*)^{\otimes k}\right]_G$ to which the moments $\mathbb{M}_k(\rho)$ belong.

We use the fact that the Lie algebra complexification $u(n) \otimes_R \mathbb{C} = \mathfrak{gl}(n) = \mathbb{M}_n(\mathbb{C})$ has a matrix structure. We equip $\mathbb{M}_n(\mathbb{C})$ with a bilinear form $\langle A, B \rangle = \text{Tr} \, A^T B$ which gives an isomorphisms allowing to identify $\mathbb{M}_n(\mathbb{C})^* \cong \mathbb{M}_n(\mathbb{C})$. The coadjoint action of $U(n)$ on $\mathbb{M}_n(\mathbb{C})^*$ is given explicitly as follows: for any $f \in \mathbb{M}_n(\mathbb{C})^*$ and $x \in \mathbb{M}_n(\mathbb{C})$ we have

$$(U \cdot f)(x) = \left(\text{Ad}^*_{U^{-1}}(f)\right)(x) = f\left(\text{Ad}_{U^{-1}}(x)\right),$$

$$\text{Tr} \, f^T U^{-1} x U = \text{Tr} \, U f^T U^{-1} x = \text{Tr} \, [\left(U^{-1}\right)^T f U^T] \cdot x = \left[ U f U^{-1} \right]^T x,$$

where on the right-hand side we view $f$ as an element of $\mathbb{M}_n(\mathbb{C})$ thanks to the isomorphism $\mathbb{M}_n(\mathbb{C}) \cong \mathbb{M}_n(\mathbb{C})$. In other words, the coadjoint action of $U(n)$ on $u(n)^*$ corresponds to the adjoint action on $\mathbb{M}_n(\mathbb{C})$ by the complex conjugate matrix.

From the above discussion it follows that we can view any $Z \in \left[(u(n)^*)^{\otimes k}\right]_{U(n)}$ as endomorphism of $(\mathbb{C}^n)^{\otimes k}$ which commutes with the diagonal action of $U(n)$. From Schur-Weyl duality (Theorem 6) it follows that $Z$ can be identified with an element of $\mathbb{C}[\mathfrak{S}(n)]$. A way to make this identification straightforward is to consider the function $\text{Tr}_{\sigma} Z \in \mathbb{C}[\mathfrak{S}(k)]$ defined by

$$\text{Tr}_{\sigma} Z = \text{Tr} (\sigma Z) \quad \text{for any } \sigma \in \mathfrak{S}(k),$$

where on the right-hand side we view $\sigma$ as endomorphism of $(\mathbb{C}^n)^{\otimes k}$ given by permutation of the factors. It is not very difficult to show that $\text{Tr}_{\bullet} Z$ gives a complete information about any $Z \in \left[ \mathbb{End} \left((\mathbb{C}^n)^{\otimes k}\right) \right]_{U(n)}$.

If a $U(n)$-invariant (classical) random element $X$ in $u(n)^*$ is viewed as a random matrix in $\mathbb{M}_n(\mathbb{C})$ then $\text{Tr}_{\bullet} X^{\otimes k} \in \mathfrak{S}(k)$ is a function on the symmetric group (with values being random variables). It is central and multiplicative with respect to the cycle decomposition of permutations; it follows that the family $\left(\text{Tr}_{\bullet} \mathbb{M}_k(X)\right)$ can be interpreted as the collection of mixed moments of the random variables corresponding to the cycles:

$$\text{(10)} \quad \text{Tr}_{(1,2,\ldots,l)} \mathbb{M}_l(X) = \text{Tr} \, X^l = \text{tr} \, X^l = n \int_{\mathbb{R}} z^l \, d\mu_X.$$
In other words, all the information about the distribution of \( X \) (from the viewpoint of non-commutative probability theory) is contained in the family \((10)\) (notice that the definition of the spectral measure \( \mu_X \) has to be modified for \( X \in u(n)^* \) since the latter corresponds to an antihermitian matrix hence its spectral measure is supported not on the real line \( \mathbb{R} \) but on the imaginary line \( i\mathbb{R} \)). The above quantities \((10)\) are random variables which have a very simple interpretation as random moments of the spectral measure of \( X \) viewed as a random matrix. Thus the study of a unitarily invariant (classical) random element in \( u(n)^* \) is reduced to studying the joint distribution of the family \((10)\) or, equivalently, to studying the behavior of its random spectral measure \( \mu_X \).

In this article we are concerned about a non-commutative vector in \( u(n)^* \) which corresponds to some representation of \( u(n) \); due to this noncommutativity the discussion from the previous paragraph does not apply directly. However, the scaling of the representations considered in this article is such that asymptotically this noncommutativity becomes in some sense neglectable, therefore the spectral measure \( \mu_X \) and its moments still remain very useful notions. Nevertheless we need to explain how to define the spectral measure for a random matrix with non-commutative entries and we shall do it in the following.

3.3. Random matrices with non-commutative entries and their spectral measures. Let \( X \in M_n(\mathfrak{a}) \) be a non-commutative random matrix. If the joint distribution of the collection of random variables

\[
(\text{tr} \: X^k)
\]

coincides with the joint distribution of classical random variables of the form

\[
\int_{\mathbb{R}} x^k \: d\mu_X,
\]

where \( \mu_X \) is a random probability measure on \( \mathbb{R} \), we say that \( \mu_X \) is the spectral measure of \( X \). Clearly, for classical random matrices the above definition coincides with the usual definition of the spectral measure \((4)\). In general the existence and the uniqueness of the spectral measure are not obvious.

3.4. Other classical series of Lie groups. The method outlined in Section\( 3.2 \) can be adapted to the other classical series of compact Lie groups, namely the orthogonal and the symplectic groups, as follows. One has to adjust Schur-Weyl duality; in particular the commutants are no longer given by the symmetric group algebras but by Brauer algebras. For this reason, at first sight it might seem that the right analogue of the quantities \((10)\) is the
collection of traces

\[ \text{Tr} \left[ X \cdots X^T \cdots X \cdots X^T \cdots \right] \]

over all words in \( X \) and \( X^T \). However, once we take into account that in the case of the orthogonal groups the element \( X \) we are studying is not an arbitrary random element in \( \mathbb{M}_n(\mathbb{C}) \) but as an element of the dual of the Lie algebra fulfills an additional relation \( X^T = -X \), it becomes clear that quantities (10) are applicable without any changes also in this new setup and the notion of the spectral measure is applicable without any modifications as well. The case of the symplectic groups is analogous.

3.5. **Unitarily invariant random matrices.** Let \((\mathfrak{A}, \mathbb{E})\) be a non-commutative probability space. We say that a random matrix with non-commutative entries \( X \in \mathbb{M}_n(\mathfrak{A}) \) is **unitarily invariant** if for every \( U \in U(n) \) the joint distribution of the entries the matrix \( X = (x_{ij})_{1 \leq i,j \leq n} \) coincides with the joint distribution of the entries of the matrix \( X' = (x'_{ij})_{1 \leq i,j \leq n} = UXU^{-1} \).

**Proposition 7.** If \( X \) is a unitarily invariant \( n \times n \) random matrix with non-commutative entries then for each integer \( 1 \leq k \leq n \) and each partition \( \mathcal{V} \) of the set \( \{1, \ldots, k\} \), there exists a unique function \( \kappa_{\mathcal{V}} \in \mathbb{C}[\mathfrak{S}(k)] \) with the property that for all choices of the indices \( i_1, \ldots, i_k, j_1, \ldots, j_k \in \{1, \ldots, n\} \), we have

\[ k_{\mathcal{V}}(X_{i_1j_1}, \ldots, X_{i_kj_k}) = \sum_{\pi \in \mathfrak{S}(k)} [j_1 = i_{\pi(1)}] \cdots [j_n = i_{\pi(n)}] \kappa_{(\mathcal{V}, \pi)} \]

and such that \( \kappa_{(\mathcal{V}, \pi)} \) is non-zero only for \( \pi \leq \mathcal{V} \).

This function is explicitly given by

\[ \kappa_{(\mathcal{V}, \pi)} = k_{\mathcal{V}}(X_{1\pi(1)}, \ldots, X_{k\pi(k)}). \]

**Proof.** By sorting the factors we may view \( X^{\otimes k} \in (\mathbb{M}_n(\mathfrak{A}))^{\otimes k} \otimes \mathfrak{A}^{\otimes k} \). The multilinear map \( k_{\mathcal{V}} \) gives rise to a functional \( k_{\mathcal{V}} : \mathfrak{A}^{\otimes k} \to \mathbb{C} \). The element \((\text{Id} \otimes k_{\mathcal{V}})(X^{\otimes k}) \in (\mathbb{M}_n(\mathfrak{A}))^{\otimes k}\) is invariant under the adjoint action of the unitary group hence from Schur-Weyl duality (Theorem 6) it follows that it can be identified with an element of the symmetric group algebra \( \kappa_{\mathcal{V}} \in \mathbb{C}[\mathfrak{S}(k)] \); the equality (11) follows immediately. Equation (12) follows by appropriate choice of the indices in (11).

An analogue of (11) holds true for \( \mathbb{E}(X_{i_1j_1} \cdots X_{i_kj_k}) \) as well; it follows that if \( \pi \not\subseteq \mathcal{V} \) then for every \( \mathcal{W} \leq \mathcal{V} \) we also have \( \pi \not\subseteq \mathcal{W} \) hence

\[ \mathbb{E}_{\mathcal{W}}(X_{1\pi(1)}, \ldots, X_{k\pi(k)}) = 0. \]

Relation (7) can be regarded as an upper triangular system of linear equations; it follows that \( k_{\mathcal{V}} \) is a linear combination of \( (\mathbb{E}_{\mathcal{W}})_{\mathcal{W} \leq \mathcal{V}} \). It shows that

\[ \kappa_{(\mathcal{V}, \pi)} = k_{\mathcal{V}}(X_{1\pi(1)}, \ldots, X_{k\pi(k)}) = 0 \quad \text{if} \; \pi \not\subseteq \mathcal{V} \]
which finishes the proof. □

3.6. Higher order free probability. The concept of higher order free probability was introduced in a series of papers [MS06, MSS07, CMSS07]. In this article we deal with a simplified problem of fluctuations of a single random matrix (as opposed to fluctuations of several random matrices). In this section we present the necessary notions and notations of higher order free probability in this simplified setup.

Assume that for each \( n \geq 1 \), an \( n \times n \) random matrix \( X^{(n)} \) is given. When there is no possible confusion, we omit the explicit dependence on \( n \) and we will simply write \( X = X^{(n)} = (x_{ij})_{1 \leq i,j \leq n} \). We systematically assume that \( X \) is unitarily invariant.

Two kinds of quantities can be used to describe properties of the random matrix \( X \). The macroscopic quantities describe the probabilistic behavior of the family of the traces \((\text{Tr} X^k)_{k \geq 1}\). We are mainly interested, up to some normalization, in the tensor cumulants of the form:

\[
(13) \quad k(\text{Tr} X^{p_1}, \ldots, \text{Tr} X^{p_l}).
\]

As we will see, in the setup of the representations of the unitary groups one can treat \((\text{Tr} X^p)\) as a family of classical random variables therefore the tensor cumulant in (13) is in fact a classical cumulant.

The microscopic quantities describe the probabilistic behavior of the entries of the random matrix \( X \); in particular we study the tensor cumulants

\[
(14) \quad \kappa_{p_1, \ldots, p_l} := k(X_{1\gamma(1)}, \ldots, X_{k\gamma(k)}),
\]

where \( k = p_1 + \cdots + p_l \) and \( \gamma \) is the following permutation:

\[
(15) \quad \gamma = (1, 2, \ldots, p_1)(p_1 + 1, p_1 + 2, \ldots, p_1 + p_2) \cdots \left(p_1 + \cdots + p_{l-1} + 1, p_1 + \cdots + p_{l-1} + 2, \ldots, p_1 + \cdots + p_l\right).
\]

In the usual context of random matrix theory where the entries of the matrix \( X \) commute, the quantities \( \kappa_{p_1, \ldots, p_l} \) and their products are sufficient to describe the joint distribution of the entries of \( X \). In order to deal with the case of random matrices with non-commutative entries we need more information. It turns out that it is enough to consider the family of quantities \( \kappa_{(\mathcal{V}, \pi)} \) given by \((11)\). In particular, for an appropriate choice of \((\mathcal{V}, \pi)\) they coincide with the quantities \((14)\):

\[
\kappa_{(1, \gamma)} = \kappa_{p_1, \ldots, p_l}.
\]

Higher order free probability theory studies the limits of the quantities \((13)\) and \((14)\) after appropriate normalization, as the size \( n \) of the matrix \( X \) tends to infinity. We need to revisit the proofs from the paper [CMSS07] in order to ensure they also apply in our non-commutative situation.
3.7. **Relation between macroscopic and microscopic quantities.** The following theorem gives the key relation between the macroscopic and microscopic quantities describing a random matrix with non-commuting entries.

**Theorem 8.** If $X$ is an $n \times n$ unitarily invariant random matrix with non-commuting entries then

\[
 k_1(\text{Tr} \ X^{p_1}, \ldots, \text{Tr} \ X^{p_l}) = \sum_{(V, \pi)} \kappa(V, \pi) \ n^{\#(\gamma_\pi^{-1})},
\]

where $\gamma$ is given by (15) and the sum runs over partitioned permutations $(V, \pi)$ of the set $\{1, \ldots, k\}$ such that $V \vee \gamma = 1_k$, where $k = p_1 + \cdots + p_l$.

**Proof.** This result follows from Equation (22) in [CMSS07]; however for the sake of completeness and since in the aforementioned paper the cumulants were defined in a seemingly different way via Möbius inversion formula, we present an alternative proof here.

There is a bijective correspondence between partitions of the set $\{1, \ldots, l\}$ and partitions $W$ of the set $\{1, \ldots, k\}$ such that $W \geq \gamma$ given by replacing each element of the set $\{1, \ldots, l\}$ by the block corresponding to the appropriate cycle of $\gamma$. It will be clear from the context which of the above two interpretations we use. We have

\[
 \mathbb{E}_W[\text{Tr} \ X^{p_1}, \ldots, \text{Tr} \ X^{p_l}] = \sum_{1 \leq i_1, \ldots, i_k \leq n} \mathbb{E}_W[X_{i_1 i_{\gamma(1)}}, \ldots, X_{i_n i_{\gamma(n)}}] \\
 = \sum_{V \leq W} \sum_{1 \leq i_1, \ldots, i_k \leq n} k_V[X_{i_1 i_{\gamma(1)}}, \ldots, X_{i_k i_{\gamma(k)}}] \\
 = \sum_{\pi \leq V \leq W} \sum_{1 \leq i_1, \ldots, i_k \leq n} [i_{\gamma(1)} = i_{\pi(1)}] \cdots [i_{\gamma(k)} = i_{\pi(k)}] \kappa(V, \pi) \\
 = \sum_{\pi \leq V \leq W} n^{\#(\gamma_{\pi}^{-1})} \kappa(V, \pi),
\]

where the third equality follows from Proposition 7. For a partition $U \geq \gamma$ we define (on the left hand side we view $U$ as a partition of the set $\{1, \ldots, l\}$):

\[
 k_U(\text{Tr} \ X^{p_1}, \ldots, \text{Tr} \ X^{p_l}) = \sum_{\pi \leq V \leq U \atop \gamma \vee V = U} n^{\#(\gamma_{\pi}^{-1})} \kappa(V, \pi).
\]

It is clear that so defined $k_U$ is a multiplicative function on partitions and fulfills the moment-cumulant formula (7) which finishes the proof. \qed
3.8. Decay of the cumulants of entries. All considerations in this paper so far are non-asymptotic. In this section we study asymptotics of random matrices with non-commutative entries as the size of the matrix tends to infinity.

For each \( n \geq 1 \) let \( X^{(n)} \) be an \( n \times n \) unitarily invariant random matrix with non-commuting entries. As before, we make the dependence in \( n \) implicit and instead of \( X^{(n)} \) we simply write \( X \). This notation applies to other quantities as well (for example \( \kappa_{(\mathcal{V},\pi)} \) depends implicitly on \( n \)).

The following result is a theorem-and-definition. It provides a definition of the quantities \( K_{(\mathcal{V},\pi)} \) and \( M_{p_1,\ldots,p_l} \):

**Theorem 9.** Assume that for every partitioned permutation \( (\mathcal{V}, \pi) \) the limit

\[
K_{(\mathcal{V},\pi)} := \lim_{n \to \infty} n^{|(\mathcal{V},\pi)|} K_{(\mathcal{V},\pi)}
\]

exists and is finite. Then

\[
M_{p_1,\ldots,p_l} := \lim_{n \to \infty} n^{2(l-1)} k_l(\text{tr} X^{p_1}, \ldots, \text{tr} X^{p_l}) = \sum_{(\mathcal{V},\pi) \leq (1_k,\gamma)} K_{(\mathcal{V},\pi)},
\]

where \( \gamma \) is given by (15).

**Proof.** This is a special case of Equation (35) from the paper [CM´SS07]. The only difficulty is that the paper [CM´SS07] deals with random matrices with commuting entries. So one has to revisit the original proof in order to ensure that it applies to the non-commutative situation. This is indeed the case thanks to Theorem 8.

The proof from [CM´SS07] relies in the fact that one can write Equation (16) in the form

\[
n^{2(l-1)} k_l(\text{tr} X^{p_1}, \ldots, \text{tr} X^{p_l}) = \sum_{(\mathcal{V},\pi) \leq (1_k,\gamma)} \frac{1}{n^{|(\mathcal{V},\pi)|} K_{(\mathcal{V},\pi)}}
\]

The result follows from the fact that the following triangle inequality holds true

\[
|(0,\gamma^{-1})| + |(\mathcal{V},\pi)| - |(1_k,\gamma)| \geq 0
\]

with the equality holding if and only if \( (\mathcal{V},\pi) \leq (1_k,\gamma) \). \( \square \)

If the above limits exist, it is convenient to think that the sequence \( X^{(n)} \) of random matrices converges to some (abstract) limit object \( X^{(\infty)} \). In the context of higher order free probability the quantities \( K_{(\mathcal{V},\pi)} \) are called higher order free cumulants of \( X^{(\infty)} \) and the quantities \( M_{(\mathcal{V},\pi)} \) are called higher order moments of \( X^{(\infty)} \) [CM´SS07].
The above theorem shows that the microscopic quantities describing random matrix uniquely determine the macroscopic quantities. For our purposes it is necessary to have also the opposite and to express the microscopic quantities in terms of their macroscopic counterparts. However, in the non-commutative case this is not possible in general since the microscopic quantities \( \kappa(V, \pi) \) contain much more information than the macroscopic quantities (13), as can be seen by a cardinality argument. In order to have the description in the opposite direction, one needs to assume that the entries of the matrices under consideration asymptotically commute.

3.9. **Converse of the condition from Section 3.8.** We say that a sequence \( (X) = (X^{(n)}) \) of unitarily invariant random matrices (with non-commutative entries) has asymptotically vanishing commutators up to degree \( m_0 \) if

\[
(20) \quad k_{V'}(X_{1\pi(1)}, \ldots, X_{i-1,\pi(i-1)}, [X_{i,\pi(i)}, X_{i+1,\pi(i+1)}], \ldots, X_{k\pi(k)}) = o\left(\frac{1}{n|V, \pi|}\right),
\]

holds true for any partitioned permutation \((V, \pi)\) of the set \(\{1, \ldots, m\}\), for \(m \leq m_0\) and any value of \(i\) such that \(i\) and \(i+1\) are connected by \(V\) and where \(V'\) should be understood as in Lemma 5.

The following lemma and theorem provide the key induction step for the proof of the main result of this paper, Theorem 17.

**Lemma 10.** Let \( (X) \) be a sequence of random matrices which has asymptotically vanishing commutators up to degree \( m_0 \) and assume that the limits (17) exist and are finite for all partitioned permutations of the sets \(\{1, \ldots, m\}\) for every \(m < m_0\).

Then

\[
\lim_{n \to \infty} n|V, \pi|\left(\kappa(V, \pi) - \kappa(W, \sigma)\right) = 0
\]

whenever \((V, \pi)\) and \((W, \sigma)\) are conjugate partitioned permutations of the set \(\{1, \ldots, m\}\) for \(m \leq m_0\).

**Proof.** From the multiplicativity of cumulants it follows that it is enough to prove the lemma in the case when \(V = W = 1\) is the partition consisting of only one block. Also, it is enough to show the lemma under additional assumption that \(\pi\) and \(\sigma\) are conjugate by a transposition \((i, i + 1)\) interchanging two neighboring elements. But under the above assumptions this is a direct application of Lemma 5 and Equation (12).

**Theorem 11.** Let \( (X) \) be a sequence of random matrices which has asymptotically vanishing commutators up to degree \( m_0 \). Assume that the limit (17)
exists for all partitioned permutations \((\mathcal{V}, \pi)\) of the set \(\{1, \ldots, m\}\) for all \(m < m_0\). Assume also that the limit
\[
M_{p_1, \ldots, p_l} := \lim_{n \to \infty} n^{2l-2k} \left( \text{tr} X^{p_1}, \ldots, \text{tr} X^{p_l} \right)
\]
exists and is finite for all integer \(p_1, \ldots, p_l \geq 1\) such that \(p_1 + \cdots + p_l \leq m_0\).

Then the limit (17) exists for any partitioned permutation \((\mathcal{V}, \pi)\) of the set \(\{1, \ldots, m\}\) for \(m \leq m_0\). Furthermore, \(K(\mathcal{V}, \pi)\) depends only on the conjugacy class of the partitioned permutation \((\mathcal{V}, \pi)\).

**Proof.** We prove the claim by induction with respect to \(m_0\). Looking at Equation (19), one notices that from the inductive hypothesis and multiplicativity of cumulants, every summand on the right hand side which corresponds to \(\mathcal{V}\) consisting of more than one block converges to a finite limit. Thanks to Lemma 10 each summand for which \(\mathcal{V} = 1\) consists of one block can be rewritten in the form
\[
\left[ n_{(1, \gamma)} K_{(1, \gamma)} + o(1) \right] \frac{1}{n^{1(0, \gamma \pi^{-1})+1(|(\mathcal{V}, \pi)|-|1, \gamma|)}},
\]
where \(\gamma = \gamma_{p_1, \ldots, p_k}\) with \(p_1 \geq \cdots \geq p_k\) given by (15) is a permutation conjugate to \(\pi\) with cycles arranged in a special way.

Thus we can view the collection of equations (19) over \(p_1 \geq \cdots \geq p_l\) such that \(p_1 + \cdots + p_l = m_0\) as a system of equations with the variables \(Z_{p_1, \ldots, p_k} = n^{1(1, \gamma p_1, \ldots, p_k)} K_{(1, \gamma p_1, \ldots, p_k)}\) over \(p_1 \geq \cdots \geq p_k\) with \(p_1 + \cdots + p_k = m_0\). In the limit \(n \to \infty\) this system of equations has a particularly simple form given by (18) hence it is upper-triangular. Therefore it is non-singular and by continuity it remains non-singular for \(n\) in some neighborhood of infinity. Solving this system of equations thanks to Cramer formulas shows that the limits
\[
\lim_{n \to \infty} n^{1(1, \gamma p_1, \ldots, p_k)} K_{(1, \gamma p_1, \ldots, p_k)}
\]
exist.

For arbitrary partitioned permutations \((\mathcal{V}, \pi)\) the existence of the limits follows from Lemma 10 and multiplicativity of cumulants. Lemma 10 also implies that the limits depend only on the conjugacy class. \(\square\)

### 3.10. Stability of decay.

The decay of cumulants of random matrix moments seen in (21) is rather typical. The following lemma shows that this kind of decay is stable under taking polynomial functions.

**Lemma 12.** For each \(\alpha\) in some index set and \(n \geq 1\) let \(I^{(n)}_{\alpha}\) be a sequence of random variables. Assume that for any \(l \geq 1\) and any choice of \(\alpha_1, \ldots, \alpha_l\) the limit
\[
\lim_{n \to \infty} n^{2l-2k} (I_{\alpha_1}, \ldots, I_{\alpha_l})
\]
exists and is finite.
Then the limit
\[
\lim_{n \to \infty} n^{2l-2} k(P_1, \ldots, P_l)
\]
exists and is finite for any polynomials \( P_1, \ldots, P_l \) in variables \( (I_\alpha) \).

**Proof.** The assumption reads:
\[
\lim_{n \to \infty} n^{2|V|} k_V(I_{\alpha_1}, \ldots, I_{\alpha_l})
\]
exists and is finite for any choice of partition \( V \).

It is enough to show that the lemma holds true if each polynomial \( P_i \) is a monomial. Therefore it is enough to study the asymptotics of the expression
\[
(22) \quad k(I_{\alpha_1} \cdot \cdots \cdot I_{\alpha_{p_1}}, I_{\alpha_{p_1+1}} \cdot \cdots \cdot I_{\alpha_{p_1+p_2}}, \ldots, I_{\alpha_{p_1+\cdots+p_{l-1}+1}} \cdot \cdots \cdot I_{\alpha_{p_1+\cdots+p_l}}).
\]
We denote by \( V = \{ \{1, \cdots, p_1\}, \{p_1 + 1, \ldots, p_1 + p_2\}, \ldots, \{p_1 + \cdots + p_{l-1} + 1, p_1 + \cdots + p_l\} \} \) the corresponding partition. From the formula of Leonov and Siraev [LS59] it follows that (22) is equal to
\[
\sum_{W : V \cup W = 1} k_W(I_{\alpha_1}, I_{\alpha_2}, \ldots).
\]
Due to a simple combinatorial inequality \(|V \cup W| \leq |V| + |W|\) it follows that \(|W| \geq l\) which finishes the proof. \(\square\)

3.11. **Convergence in distribution in the sense of higher order free probability.** We say that a sequence \((X)\) of random matrices converges in distribution in the macroscopic sense of higher order free probability if the limit \(M_{p_1, \ldots, p_l}\) exists and is finite for any choice of integers \(p_1, \ldots, p_l \geq 1\).

We say that a sequence \((X)\) of random matrices converges in distribution in the microscopic sense of higher order free probability if the limit \(K_{(V, \pi)}\) exists and is finite for any choice of a partitioned permutation \((V, \pi)\).

With these notions we can reformulate the results of this section. Theorem 9 shows in particular that the convergence in the microscopic sense implies the convergence in the macroscopic sense while Theorem 11 shows that (under some additional assumptions) the converse implication holds true as well. In particular, in the case of classical random matrices both notions are equivalent and there is no need to make a distinction between them.

4. REPRESENTATIONS AND RANDOM MATRICES WITH NON-COMMUTING ENTRIES

4.1. **Representation as a random matrix with non-commutative entries.** Let \( g \) be the Lie algebra of \( G \). Its representation \( \rho : g \to \text{End}(V) \) can be alternatively viewed as \( \rho \in g^* \otimes \text{End}(V) \), i.e. as a non-commutative random vector in \( g^* \) over the non-commutative probability space \( (\text{End}(V), \text{tr}) \).
We consider the coadjoint action $(\text{Ad}_{g^{-1}})^*$ of $G$ on $\mathfrak{g}^*$ given explicitly by $g \cdot x = (\text{Ad}_{g^{-1}})^*(x)$ for $g \in G$ and $x \in \mathfrak{g}^*$. This action extends to an action of $G$ on $(\mathfrak{g}^*)^\otimes k$.

**Lemma 13.** If $\rho : \mathfrak{g} \to \text{End}(V)$ is a representation viewed as a non-commutative random vector and $k \geq 1$ is an integer then

$$M_k(\rho) = \mathbb{E} \left( \rho^\otimes k \right) \in (\mathfrak{g}^*)^\otimes k$$

is invariant under the coadjoint action of $G$.

**Proof.** For any $x_1, \ldots, x_k \in \mathfrak{g}$

$$(g \cdot M_k(\rho))(x_1 \otimes \cdots \otimes x_k)$$

$$= M_k(\rho)(\text{Ad}_{g^{-1}}(x_1) \otimes \cdots \otimes \text{Ad}_{g^{-1}}(x_k))$$

$$= \text{tr} \left[ \rho(\text{Ad}_{g^{-1}}(x_1)) \cdots \rho(\text{Ad}_{g^{-1}}(x_k)) \right]$$

$$= \text{tr} \left[ \rho(g^{-1})\rho(x_1) \cdots \rho(x_k)\rho(g) \right]$$

$$= \text{tr} \left[ \rho(x_1) \cdots \rho(x_k) \right] = M_k(\rho)(x_1 \otimes \cdots \otimes x_k).$$

□

Under the notations from Section 3.2 we may view $\rho \in \mathfrak{u}(n)^* \otimes \text{End}(V)$ as an $n \times n$ matrix with entries in the non-commutative probability space $(\text{End}(V), \text{tr})$ given explicitly as

$$\rho = \begin{bmatrix} \rho(e_{11}) & \cdots & \rho(e_{1n}) \\ \vdots & \ddots & \vdots \\ \rho(e_{n1}) & \cdots & \rho(e_{nn}) \end{bmatrix} \in \mathbb{M}_n(\mathbb{C}) \otimes \text{End}(V).$$

(23)

The remainder of this section can be skipped during a first reading. Unlike in the case of the Lie algebra $\mathfrak{u}(n)$, there is no obvious canonical choice of the matrix structure on its dual $\mathfrak{u}(n)^*$. In Section 3.2 this structure was chosen based on a bilinear form $\langle A, B \rangle = \text{Tr} A^T B$. One can argue however, that a bilinear form $\langle A, B \rangle = \text{Tr} AB$ would be equally natural. This new way of choosing the matrix structure on $\mathfrak{u}(n)^*$ would have some advantages: for example the coadjoint action of $U(n)$ on it corresponds to the usual adjoint action on $\mathbb{M}_n(\mathbb{C})$ (without the somewhat artificial complex conjugation). With respect to this new convention, representation $\rho$ viewed as a matrix becomes

$$\rho = \begin{bmatrix} \rho(e_{11}) & \cdots & \rho(e_{1n}) \\ \vdots & \ddots & \vdots \\ \rho(e_{1n}) & \cdots & \rho(e_{nn}) \end{bmatrix} \in \mathbb{M}_n(\mathbb{C}) \otimes \text{End}(V).$$

(24)
Matrices (23) and (24) differ just by transposition. The advantage of the notation (23) is that it coincides with the notation of Želobenko [Žel73] which will be useful later on in the calculation of the spectral measure.

The calculation of the spectral measure of (24) can be done by the analogous methods to those of Želobenko [Žel73]; the only difference is that instead of considering the tensor product with the canonical representation one should consider the tensor product with the contragradient one. An analogue of Lemma 13 holds true also in this case which shows that, in fact, for the purposes of this article it does not really matter which of the two definitions is being used.

4.2. Representations of the unitary groups. For a complete introduction to the representation theory of Lie groups we refer to [BtD95, FH91]. For our purposes it is enough to know that the irreducible representations of $U(n)$ are in a bijective correspondence with the shifted highest weights $l = (l_1 > \cdots > l_n) \in \mathbb{Z}^n$ (the relation with the more common usual highest weights $\lambda$ is given by (1)).

For any $k \geq 1$, we denote

$$Z_k = \sum_{1 \leq i_1, \ldots, i_k \leq n} e_{i_1 i_2} e_{i_2 i_3} \cdots e_{i_k i_1} \in \mathfrak{u}(u(n))$$

in the universal enveloping algebra. We will need the following result, due to Želobenko [Žel73, Theorem 2, p. 163].

**Proposition 14.** Let $\rho$ be an irreducible representation of $u(n)$ corresponding to the shifted highest weight $l = (l_1 > \cdots > l_n)$. Then

$$\rho(Z_k) = \sum_{1 \leq i_1, \ldots, i_k \leq n} \rho(e_{i_1 i_2}) \rho(e_{i_2 i_3}) \cdots \rho(e_{i_k i_1}) = \sum_{i=1}^n \gamma_i l_i^k,$$

where the number on the right-hand side should be understood as a multiple of the identity matrix and

$$\gamma_i = \frac{1}{n} \prod_{j \neq i} \left(1 - \frac{1}{l_i - l_j}\right).$$

4.3. Spectral measure of a representation. Two definitions of the spectral measure for a representation were given: the naive definition of $\hat{\mu}$ from Section 1.4 and the definition of the spectral measure $\mu$ for non-commutative random matrices in Section 3.3. In this section we will compare these two non-equivalent definitions.

Firstly, observe that for any representation $\rho : u(n) \to \operatorname{End}(V)$

$$\int_{\mathbb{R}} z^k \, d\mu_\rho(z) = \operatorname{tr}[X(\rho)]^k = \frac{1}{n} \rho(Z_k) \in \operatorname{End}(V),$$

(25)
which we view as a non-commutative random variable. Assume now that \( \rho = \rho_l \) is irreducible; since \( Z_k \in \mathbb{C}[\mathcal{S}(k)] \) is a central element it follows that \( (25) \) is a multiple of identity. In other words, the random variables from this collection are constant hence the spectral measure \( \mu_{\rho_l} \) is deterministic. Furthermore, this spectral measure \( \mu_{\rho_l} \) is nothing else but the spectral measure of the matrix \( (23) \) of the size \( (n \dim V) \times (n \dim V) \). It remains to find out this measure explicitly.

**Proposition 15.** If \( \rho_l \) is the irreducible representation corresponding to the shifted highest weight \( l \) then its spectral measure (viewed as in Section 3.3) is given by the probability measure

\[
\mu_{\rho_l} = \mu_l = \sum_i \gamma_i \delta_{l_i}.
\]

**Proof.** From Proposition 14 and (25) it follows that the (possibly signed) probability measure

\[
(26) \quad \mu'_{\rho_l} = \left( 1 - \sum_i \gamma_i \right) \delta_0 + \sum_i \gamma_i \delta_{l_i}
\]

fulfills

\[
(27) \quad \int P(x) \, d\mu'_{\rho_l} = \text{tr} P(X(\rho))
\]

for every polynomial \( P \). Since both \( \mu'_{\rho_l} \) given by (26) and the spectral measure of \( \rho_l \) are finitely supported, it follows immediately from (27) that they are equal.

It remains to show that \( \sum_i \gamma_i = 1 \) hence the first summand in (26) vanishes. This can be done by a careful analysis of the proof of Želobenko [Žel73]; we provide an alternative proof below.

For \( l = (l_1, \ldots, l_n) \) and any integer \( s \) we denote \( l+s = (l_1+s, \ldots, l_n+s) \). Notice that the irreducible representation of the unitary group \( \rho_{l+s} \) can be explicitly written as \( \rho_{l+s}(U) = (\det U)^s \rho_l(U) \) for any \( U \in U(n) \) hence the corresponding representation of the Lie algebra fulfills \( \rho_{l+s}(x) = s \text{Tr} x \cdot 1 + \rho_l(x) \) for any \( x \in u(n) \). Therefore, if we view \( \rho_l \) and \( \rho_{l+s} \) as random matrices with non-commutative entries then

\[
\rho_{l+s} = s 1 + \rho_l.
\]

It follows that the spectral measure of \( \rho_{l+s} \) is just the spectral measure of \( \rho_l \) shifted by \( s \); on the other hand the measure \( \rho_{l+s} \) given by (26) is equal to the shifted measure \( \rho_l \) only if \( \sum_i \gamma_i = 1 \). \( \square \)
In the case when the representation $\rho$ is not irreducible its spectral measure is a random probability measure on the real line which can be interpreted as the spectral measure of a random irreducible representation $\rho_{l}$ distributed according to (3).

It becomes clear that the naive definition of the spectral measure $\hat{\mu}$ and the natural definition of the spectral measure $\mu$ do not coincide. Nevertheless the following lemma shows that they coincide asymptotically (under some mild technical assumptions).

**Lemma 16.** For each $k \geq 1$ there exist polynomials $P_k$ and $Q_k$ in $k$ variables such that for any shifted highest weight $l$

\begin{equation}
\begin{align*}
m_k(\hat{\mu}_l) &= P_k(n, m_1(\mu_l), \ldots, m_{k-1}(\mu_l)), \\
m_k(\mu_l) &= Q_k(n, m_1(\hat{\mu}_l), \ldots, m_{k-1}(\hat{\mu}_l)),
\end{align*}
\end{equation}

where

$$m_k(\mu) = \int_{\mathbb{R}} z^k \, d\mu(z)$$

denotes the $k$-th moment of a given measure $\mu$.

We define a degree on polynomials by assigning the degree 1 to the variable $n$ and the degree $i$ to the variables $m_i(\mu_l), m_i(\hat{\mu}_l)$. Then the polynomials $P_k$ and $Q_k$ have degree $k$ and their leading terms are given by

$$m_k(\mu_l) + (\text{terms of degree } k \text{ which contain at least one factor } n)$$

and

$$m_k(\hat{\mu}_l) + (\text{terms of degree } k \text{ which contain at least one factor } n)$$

respectively.

**Proof.** We denote

$$L = \begin{bmatrix} l_1 & & \\ & l_2 & \\ & & \ddots \\ & & & l_n \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 & -1 & \cdots & -1 \\ 0 & -1 & \cdots & -1 \\ 0 & \ddots & \vdots \\ 0 & \cdots & -1 \\ 0 & & & 0 \end{bmatrix}.$$
The (rescaled) moments of the spectral measure $\mu_l$ are given in the book of Želobenko [Žel73]:

$$n \ m_k(\mu_l) = \sum_{1 \leq i, j \leq n} \left[ (L + J)^k \right]_{ij} = \sum_{\alpha_1, \ldots, \alpha_q \geq 0, \ \alpha_1 + \cdots + \alpha_q + l - 1 = k} \ \sum_{1 \leq i, j \leq n} \left[ L^{\alpha_1} J L^{\alpha_2} \cdots J L^{\alpha_q} \right]_{ij}$$

$$= \sum_{\alpha_1, \ldots, \alpha_q \geq 0, \ \alpha_1 + \cdots + \alpha_q + l - 1 = k} \ \sum_{1 \leq i_1 < \cdots < i_q \leq n} (-1)^{q-1} l_{i_1}^{\alpha_1} \cdots l_{i_q}^{\alpha_q}$$

$$= \sum_{\alpha_1 \geq \cdots \geq \alpha_q \geq 0, \ \alpha_1 + \cdots + \alpha_q + q - 1 = k} (-1)^{q-1} m_{(\alpha)}(l_1, \ldots, l_n),$$

where $m_{(\alpha)}$ denotes the monomial symmetric polynomial. We allow here a small abuse of notation, namely we allow some of the elements of $(\alpha)$ to be equal to zero; this does not lead to problems since we treat $m_{(\alpha)}$ not as a symmetric function but as a polynomial in a finite, fixed number of variables. The right hand side can be therefore expressed as a polynomial in power-sum symmetric polynomials $p_i(l_1, \ldots, l_n)$ over $i \geq 0$. The existence of the polynomials $P_k$ follows now from the observation that

$$p_0(l_1, \ldots, l_n) = n \ m_0(\mu) = n,$$

$$p_i(l_1, \ldots, l_n) = n \ m_i(\mu) \quad \text{for} \ i \geq 1.$$

It is easy to check that by assigning to the expression $m_{(\alpha)}$ the degree $(\alpha_1 + 1) + \cdots + (\alpha_q + 1)$ one gets a filtration. With respect to this filtration $p_i$ has degree $i + 1$, therefore the passage from quantities $(p_i)$ to $n$ and $(m_i(\mu))$ corresponds to assigning to variable $n$ degree 1 and to $m_i(\mu)$ degree $i$ which coincides with the choice of degrees in the formulation of the lemma. The proof of the required properties of the polynomials $(P_k)$ is finished by the observation that the right-hand side of (29) has degree $k + 1$.

The family of equations (28) can be solved with $(m_k(\mu))$ as unknowns which shows existence of polynomials $(Q_k)$ and their required properties.

4.4. Proof of the main result. We come to the main result of this paper, Theorem 17, which we state in the following, more precise form:

**Theorem 17.** For each $n$ let $\rho_n$ be a representation of the unitary group $U(n)$ and assume that $\varepsilon_n = \mathcal{O} \left( \frac{1}{n} \right)$. Then the following conditions are equivalent:
(a) the sequence of rescaled representations \((\varepsilon_n \rho_n)\) viewed as a sequence of random matrices with non-commutative entries converges in distribution in the macroscopic sense of higher order free probability,

(b) the sequence of rescaled representations \((\varepsilon_n \rho_n)\) viewed as a sequence of random matrices with non-commutative entries converges in distribution in the microscopic sense of higher order free probability,

(c) the sequence of random matrices \((\varepsilon_n X(\rho_n))\) converges in the sense of higher order free probability.

If the limits exist, they are equal (in the sense that they describe the same limiting object in the sense of higher order free probability) and, furthermore, the limit \(K(\mathcal{V}, \pi)\) in (b) depends only on the conjugacy class of \((\mathcal{V}, \pi)\).

The fact that the limit \(K(\mathcal{V}, \pi)\) in (b) depends only on the conjugacy class of \((\mathcal{V}, \pi)\) can be informally interpreted as asymptotic commutativity of the entries of the matrices.

Proof. Assume that Condition (a) holds true. We will use induction over \(m_0\) in order to prove (b): assume that the limit (17) exists for all partitioned permutations \((\mathcal{V}, \pi)\) of the set \(\{1, \ldots, m\}\) for all \(m < m_0\). For \(X = \varepsilon \rho\) we can write down explicitly the form of the commutator on the left hand side of (20); it follows that the sequence \((X)\) has asymptotically vanishing commutators up to the order \(m_0\). Therefore Theorem 11 can be applied and the limit (17) exists for all partitioned permutations \((\mathcal{V}, \pi)\) of the set \(\{1, \ldots, m\}\) for all \(m \leq m_0\), thus we finished the proof of the induction step.

The opposite implication (b) \(\implies\) (a) is much simpler and is given directly by Theorem 9.

In order to show the implication (c) \(\implies\) (a) we need to show that the cumulant

\[
k(\varepsilon^{k_1} m_{k_1}(\mu_\rho), \ldots, \varepsilon^{k_l} m_{k_l}(\mu_\rho))
\]

sufficiently quickly converges to zero. In order to do this we use Lemma 16 and express (30) in terms of the cumulants of polynomials in \((m, \hat{\mu})\). Lemma 12 finishes the proof.

The opposite implication (a) \(\implies\) (c) can be proved in an analogous way. 

\]

5. Applications to asymptotic representation theory

5.1. Gaussian fluctuations of measures. Let \((\mu_n)\) be a sequence of random probability measures on \(\mathbb{R}\). We will say that the fluctuations of \((\mu_n)\)
are asymptotically Gaussian (with covariance decay $\frac{1}{n^2}$) if the limit
\begin{equation}
\lim_{n \to \infty} \mathbb{E} \int_{\mathbb{R}} z^k \, d\mu_n
\end{equation}
exists for any $k \geq 1$ and the joint distribution of the family of random variables
\begin{equation}
n \left( \int_{\mathbb{R}} z^k \, d\mu_n - \mathbb{E} \int_{\mathbb{R}} z^k \, d\mu_n \right)
\end{equation}
converges in moments to some Gaussian distribution (in the sense that the distribution of any finite family converges).

We say that two such sequences of random probability measures have asymptotically the same Gaussian fluctuations (with covariance decay $\frac{1}{n^2}$) if they are asymptotically Gaussian, their corresponding limits (31) are equal and the fluctuations (32) converge to the same Gaussian limit.

5.2. Gaussianity of fluctuations for representations. We prove Corollary 3 in the following, more precise form.

**Corollary 18.** Let $(\varepsilon_n)$ be a sequence of real numbers such that $\varepsilon_n = o \left( \frac{1}{n} \right)$. For each $i \in \{1, 2\}$ and $n \geq 1$ let $\rho_n^{(i)}$ be an irreducible representation of $U(n)$. Assume that for each $i \in \{1, 2\}$ the sequence $\hat{\mu}_{\varepsilon_n \rho_n^{(i)}}$ of the (rescaled) spectral measures converges in moments to some probability measure $\mu^{(i)}$.

We consider the Kronecker tensor product $\rho_n^{(3)} = \rho_n^{(1)} \otimes \rho_n^{(2)}$. For $i \in \{1, 2, 3\}$ we denote by $X_n^{(i)} = X(\varepsilon_n \rho_n^{(i)})$ the random matrix corresponding to the representation $\rho_n^{(i)}$ and define $\tilde{X}_n^{(3)} = X_n^{(1)} + X_n^{(2)}$, where random matrices $X_n^{(1)}$ and $X_n^{(2)}$ are chosen to be independent.

Then the spectral measures of rescaled representations $\varepsilon_n \rho_n^{(3)}$ and the spectral measures of random matrices $\tilde{X}_n^{(3)}$ have asymptotically the same Gaussian fluctuations with covariance decay $\frac{1}{n^2}$.

Note that the conclusion of the theorem is not affected depending on whether the spectral measure that we choose is the natural one or the naive one.

**Proof.** From the assumptions it follows immediately that for every $i \in \{1, 2\}$ the sequence $\left( \varepsilon_n \rho_n^{(i)} \right)$ converges in the macroscopic sense of higher order probability theory. Therefore, Theorem 17 shows that the sequence $\left( \varepsilon_n \rho_n^{(i)} \right)$ converges also in the microscopic sense of higher order probability theory and that this microscopic limit is the same as for the sequence of random matrices $\left( X_n^{(i)} \right)$. 

For Lie groups representations \( \rho^{(i)} : \mathfrak{u}(n) \to \text{End}(V^{(i)}) \), \( i \in \{1, 2\} \) it follows that
\[
\rho^{(3)}_n(x) = \rho^{(1)}_n(x) \otimes 1 + 1 \otimes \rho^{(2)}_n(x) \in \text{End}(V^{(1)} \otimes V^{(2)})
\]
for any \( x \in \mathfrak{u}(n) \), hence
\[
(33) \quad \rho^{(3)}_n = \rho^{(1)}_n \otimes 1 + 1 \otimes \rho^{(2)}_n \in \text{End}(V^{(1)} \otimes V^{(2)}) \otimes \mathfrak{u}(n)^*.
\]

On the other hand, if \( X^{(i)} \in \mathcal{L}^\infty-(\Omega^{(i)}) \otimes \mathbb{M}_n(\mathbb{C}) \), \( i \in \{1, 2\} \) are random matrices, the sum of their independent copies can be realized on the product probability space \( \Omega^{(1)} \times \Omega^{(2)} \) as
\[
(34) \quad \tilde{X}^{(3)} = \rho^{(1)}_n \otimes 1 + 1 \otimes \rho^{(2)}_n \in \mathcal{L}^\infty-(\Omega^{(1)} \times \Omega^{(2)}) \otimes \mathfrak{u}(n)^*.
\]

Each of the expressions (33) and (34) is a sum of two (non-commutative) random vectors in \( \mathfrak{u}(n)^* \) which have tensor independent coordinates. If follows immediately that also \( \varepsilon_n \rho_n \) and \( \tilde{X}^{(3)} \) converge in the microscopic sense of higher order free probability theory and that the limits are equal.

We apply Theorem 17 again and show that \( \varepsilon_n \rho_n, X^{(3)} \) and \( \tilde{X}^{(3)} \) converge in the macroscopic sense of higher order free probability theory and that their limits are equal.

For a sequence \((X_n)\) of random matrices the convergence in the macroscopic sense of higher order free probability theory is equivalent to existence of the limits (21) and implies that the limits
\[
(35) \quad M_l = \lim_{n \to \infty} \mathbb{E} \text{tr} X^l,
\]
\[
(36) \quad M_{l_1, l_2} = \lim_{n \to \infty} \text{Cov} \left( n \text{tr} X^{l_1}, n \text{tr} X^{l_2} \right),
\]
\[
M_{l_1, \ldots, l_i} = \lim_{n \to \infty} k \left( n \text{tr} X^{l_1}, \ldots, n \text{tr} X^{l_i} \right) = 0 \quad \text{for } i \geq 3
\]
exist; in particular it shows that the spectral measure of \( X_n \) has asymptotically Gaussian fluctuations which finishes the proof that the spectral measures (both the naive and the natural ones) have the same Gaussian fluctuations as random matrices \( \tilde{X}^{(3)} \). \( \square \)

5.3. Almost surely convergence.

Proof of Corollary 2. For a sequence \((X)\) of random matrices which converges in the macroscopic sense of higher order free probability, Equation (36) shows that for every value of \( l \geq 1 \)
\[
\text{Var } \text{tr } X^l = O \left( \frac{1}{n^2} \right)
\]
so Chebyshev’s inequality together with Borel-Cantelli lemma show that \( \text{tr } X^l \) converges to (35) almost surely.
Since the spectral measure of the sum of independent random matrices concentrates around Voiculescu’s free convolution of their spectral measures [Voi91], the results presented in the above proof of Corollary [18] finish the proof.

We skip the proof of Corollary [4] since it follows very closely the above proofs for the Kronecker tensor product.

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