On the second Robin eigenvalue of the Laplacian

Xiaolong Li 1 · Kui Wang 2 · Haotian Wu 3

Received: 6 August 2020 / Accepted: 4 October 2023 / Published online: 26 October 2023
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2023

Abstract
We study the Robin eigenvalue problem for the Laplace–Beltrami operator on Riemannian manifolds. Our first result is a comparison theorem for the second Robin eigenvalue on geodesic balls in manifolds whose sectional curvatures are bounded from above. Our second result asserts that geodesic balls in nonpositively curved space forms maximize the second Robin eigenvalue among bounded domains of the same volume.

Mathematics Subject Classification 35P15 · 49R05 · 58C40 · 58J50

1 Introduction
Let \((M^n, g)\) be a complete Riemannian manifold of dimension \(n\) and \(\Omega \subset M^n\) be a bounded domain with Lipschitz boundary. The Robin eigenvalue problem for the Laplace operator on \(\Omega\) is
\[
\begin{cases}
-\Delta u = \lambda u & \text{in } \Omega, \\
\frac{\partial u}{\partial v} + \alpha u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where \(\Delta\) denotes the Laplace–Beltrami operator, \(v\) denotes the outward unit normal to \(\partial \Omega\), and \(\alpha \in \mathbb{R}\) is the Robin parameter. The eigenvalues, denoted by \(\lambda_{k,\alpha}(\Omega)\) for \(k = 1, 2, \ldots\), are increasing and continuous in \(\alpha\), and for each \(\alpha\) satisfy
\[
\lambda_{1,\alpha}(\Omega) < \lambda_{2,\alpha}(\Omega) \leq \lambda_{3,\alpha}(\Omega) \leq \cdots \to \infty.
\]
where each eigenvalue is repeated according to its multiplicity. The first eigenvalue is simple if \( \Omega \) is connected; the first eigenfunction is positive.

The theory of self-adjoint operators yields variational characterizations of the Robin eigenvalues. In particular, the first two eigenvalues are characterized by

\[
\lambda_{1,\alpha}(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^2 \, d\mu_g + \alpha \int_{\partial \Omega} u^2 \, dA_g}{\int_{\Omega} u^2 \, d\mu_g} : u \in W^{1,2}(\Omega) \setminus \{0\} \right\}
\]

(1.2)

and

\[
\lambda_{2,\alpha}(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^2 \, d\mu_g + \alpha \int_{\partial \Omega} u^2 \, dA_g}{\int_{\Omega} u^2 \, d\mu_g} : u \in W^{1,2}(\Omega) \setminus \{0\}, \int_{\Omega} uu_1 \, d\mu_g = 0 \right\}
\]

(1.3)

respectively, where \( d\mu_g \) is the Riemannian measure induced by the metric \( g \), \( dA_g \) is the induced measure on \( \partial \Omega \), and \( u_1 \) is the first eigenfunction associated with \( \lambda_{1,\alpha}(\Omega) \).

The Robin eigenvalue problem (1.1) generates a global picture of the spectrum of the Laplace operator. Indeed, the Neumann (\( \alpha = 0 \)), the Steklov (\( \lambda = 0 \)) and the Dirichlet (\( \alpha \to \infty \)) eigenvalue problems are all special cases of the Robin eigenvalue problem. Hence, existing results on the Dirichlet, Neumann or Steklov eigenvalues naturally motivate the investigation on the Robin eigenvalues.

The classical eigenvalue comparisons of Cheng [7] state that the first Dirichlet eigenvalue of a geodesic ball in \( (M^n, g) \) with Ricci curvature \( \text{Ric}_g \geq (n-1)\kappa g \), where \( \kappa \in \mathbb{R} \), is less than or equal to that of a geodesic ball in a space form of constant sectional curvature \( \kappa \), and that the reverse inequality holds if the Ricci lower bound is replaced by the sectional curvature upper bound \( \text{Sect}_g \leq \kappa \) and the radius of the geodesic ball is no larger than the injectivity radius at its center. Recently, the first two authors [16, Theorem 1.1] have obtained Cheng type comparison theorems for the first Robin eigenvalue when \( \alpha > 0 \) and have proved comparison theorems with the reverse inequalities when \( \alpha < 0 \); moreover, the comparison theorems hold for the first Robin eigenvalue of the \( p \)-Laplacian for \( p \in (1, \infty) \).

Motivated by Cheng’s eigenvalue comparison theorems, Escobar [11] proved that in two and three dimensions, the first nonzero Steklov eigenvalue of a geodesic ball, whose radius is assumed to be less than or equal to the injectivity radius at the its center, in a complete manifold with \( \text{Sect}_g \leq \kappa \), where \( \kappa \in \mathbb{R} \), does not exceed the first nonzero Steklov eigenvalue of a geodesic ball of the same radius in a complete simply connected space form of constant sectional curvature \( \kappa \); equality holds if and only if the geodesic balls are isometric.

Inspired by Escobar’s work on the first nonzero Steklov eigenvalue, our first result is a comparison theorem for the second Robin eigenvalue.

**Theorem 1.1** Let \( (M^n, g) \) be a complete Riemannian manifold with sectional curvature \( \text{Sect}_g \leq \kappa \) for \( \kappa \in \mathbb{R} \), \( B(R) \) be an injective geodesic ball of radius \( R \) (i.e., \( R \) does not exceed the injectivity radius at its center) in \( M^n \), and \( B_\kappa(R) \) be a geodesic ball in an \( n \)-dimensional complete simply connected space form of constant sectional curvature \( \kappa \). If \( n = 2 \) or \( n = 3 \), and \( \alpha \leq 0 \), then

\[
\lambda_{2,\alpha}(B(R)) \leq \lambda_{2,\alpha}(B_\kappa(R)).
\]

Equality holds if and only if \( B(R) \) is isometric to \( B_\kappa(R) \).

An analogue of Theorem 1.1 in higher dimensions holds under additional symmetry assumption, cf. Theorem 3.1.

By taking \( \alpha = -\sigma_1(B_\kappa(R)) \), Theorem 1.1 recovers Escobar’s result in [11].
Corollary 1.2  Under the hypotheses of Theorem 1.1, we have

$$\sigma_1(B(R)) \leq \sigma_1(B_\kappa(R)).$$

Equality holds if and only if $B(R)$ is isometric to $B_\kappa(R)$.

In the second part of this paper, we investigate the shape optimization problem for the Robin eigenvalues.

In Euclidean space, the classical Faber-Krahn inequality asserts that the ball uniquely minimizes the first Dirichlet eigenvalue among bounded domains with the same volume. When $\alpha > 0$, the ball is also the unique minimizer of the first Robin eigenvalue $\lambda_{1,\alpha}(\Omega)$ among domains of the same volume in $\mathbb{R}^n$, as was shown in dimension two by Bossel [5] in 1986 and extended to all dimensions $n \geq 2$ by Daners [8] in 2006. An alternative approach via the calculus of variations was found by Bucur and Giacomini [3, 4] later. For negative values of $\alpha$, it was conjectured by Bareket [2] in 1977 that the ball would be the maximizer among domains in $\mathbb{R}^n$ with the same volume. However, in 2015, Freitas and Krejčířík [12] disproved Bareket’s conjecture by showing that the ball is not a maximizer for sufficiently negative values of $\alpha$. In the same paper, the authors showed that in dimension two, the disk uniquely maximizes $\lambda_{1,\alpha}(\Omega)$ for $\alpha < 0$ with $|\alpha|$ sufficiently small, and conjectured that the maximizer still has radial symmetry whenever $\alpha < 0$ and should switch from a ball to a shell at some critical value of $\alpha$.

Let us turn to the shape optimization problem for the second Robin eigenvalue $\lambda_{2,\alpha}(\Omega)$. Suppose for the moment that $\Omega \subset \mathbb{R}^n$. When $\alpha > 0$, both the second Dirichlet eigenvalue and the second Robin eigenvalue are uniquely minimized by the disjoint union of two equal balls among bounded Lipschitz domains of the same volume. This was proved by Kennedy [15]. When $\alpha = 0$, we have $\lambda_{2,0}(\Omega) = \mu_1(\Omega)$, the first nonzero Neumann eigenvalue, for which the classical Szegö-Weinberger inequality states that among domains with the same volume, the ball uniquely maximizes $\mu_1(\Omega)$. When $\alpha < 0$, it is expected, cf. [14, Problem 4.41], that $\lambda_{2,\alpha}(\Omega)$ should be maximal on the ball for a range of Robin parameters. This expectation has recently been confirmed by Freitas and Laugesen, who proved in [13] that the ball uniquely maximizes $\lambda_{2,\alpha}(\Omega)$ among domains of the same volume provided that $\alpha$ lies in a regime connecting the first nonzero Neumann eigenvalue $\mu_1$ and the first nonzero Steklov eigenvalue $\sigma_1$, namely $\alpha \in [-\frac{n+1}{n}R^{-1}, 0]$, where $R$ is the radius of the ball of the same volume as $\Omega$. Taking $\alpha = 0$ and $\alpha = -1/R$ recovers the Szegö-Weinberger inequality for $\mu_1(\Omega)$ and the Brock-Weinstock inequality for $\sigma_1(\Omega)$ respectively, and both classical inequalities assert that the ball is the unique maximizer among domains with the same volume in Euclidean space.

It is well-known that the Faber–Krahn inequality holds in any Riemannian manifold in which the isoperimetric inequality holds, see [6]. Also, the Szegö-Weinberger inequality holds for domains in the hemisphere and in the hyperbolic space [1]. Therefore, it is a natural question to extend the result of Freitas and Laugesen [13] to space forms. In this direction, our second result states that in complete simply connected nonpositively curved space forms, geodesic balls uniquely maximize the second Robin eigenvalue among domains with the same volume.

Theorem 1.3  Let $(M_n^\kappa, g_\kappa)$ be a complete simply connected $n$-dimensional space form of constant sectional curvature $\kappa$, where $\kappa \leq 0$, and $\Omega \subset M_n^\kappa$ be a bounded domain with Lipschitz boundary. Let $\Omega^* \subset M_n^\kappa$ be a geodesic ball of the same volume as $\Omega$, and $\sigma_1(\Omega^*)$ be the first nonzero Steklov eigenvalue on $\Omega^*$. If $\alpha \in [-\sigma_1(\Omega^*), 0]$, then

$$\lambda_{2,\alpha}(\Omega) \leq \lambda_{2,\alpha}(\Omega^*).$$
Equality holds if and only if $\Omega$ is a geodesic ball.

When $\kappa = 0$, $\sigma_1(\Omega^*) = R^{-1}$, where $R$ is the radius of the ball $\Omega^*$ in $\mathbb{R}^n$. Then Theorem 1.3 says that the ball maximizes the second Robin eigenvalue $\lambda_{2,\alpha}$, where the Robin parameter $\alpha \in [-R^{-1}, 0]$, among domains of the same volume in Euclidean space. In comparison, the same result has been proved in [13] for the larger interval, i.e., $[-\frac{n+1}{n}, R^{-1}]$, of Robin parameters. We note that these two intervals agree asymptotically as $n \to \infty$. The proof in [13] used scaling arguments, which are special to Euclidean space. In contrast, we give a uniform proof for all space forms with nonpositive curvature.

By taking $\alpha = -\sigma_1(\Omega^*)$, Theorem 1.3 implies that geodesic balls uniquely maximize the first nonzero Steklov eigenvalue among domains of the same volume in complete simply connected nonpositively curved space forms, which recovers a result of Escobar in [10]. The result in [10] has recently been generalized to Riemannian manifolds by the authors of this paper in [17].

This paper is organized as follows. In Sect. 2, we set up the notation and recall some facts on the eigenfunctions for the second Robin eigenvalue. In Sect. 3, we prove Theorem 1.1 and its higher dimensional analogue assuming additional symmetries. Section 4 is devoted to the proof of Theorem 1.3.

2 Preliminaries

Throughout the paper, the function $sn_k$ is defined by

$$sn_k(t) := \begin{cases} \frac{1}{\sqrt{k}} \sin(\sqrt{k}t), & \text{if } k > 0, \\ t, & \text{if } k = 0, \\ \frac{1}{\sqrt{-k}} \sinh(\sqrt{-k}t), & \text{if } k < 0. \end{cases}$$

We fix some notation. For any bounded Lipschitz domain $\Omega \subset M := M^n$, we denote by $\text{diam}(\Omega)$ the diameter of $\Omega$, by $|\Omega|$ and $|\partial \Omega|$ the $n$-dimensional volume of $\Omega$ and the $(n-1)$-dimensional Hausdorff measure of $\partial \Omega$ respectively, each taken with respect to the Riemannian metric $g$. Let $(M_k, g_k)$ denote the $n$-dimensional complete simply connected space form of constant sectional curvature $\kappa$ and $\Omega^*_q$ be a geodesic ball in $M_k$ centered at $q$ with $|\Omega^*_q|_\kappa = |\Omega|$, where $|\Omega^*_q|_\kappa$ is the $n$-dimensional volume of $\Omega^*$ with respect to $g_k$.

We collect some facts about the Robin eigenfunctions. Assume $\alpha \leq 0$, and denote by $\lambda_{2,\alpha}(B_k(R))$ the second Robin eigenvalue for a geodesic ball $B_k(R)$ of radius $R$ in the space form $M_k$, and the corresponding eigenfunctions are given by

$$u_i(x) = F(r) \psi_i(\theta), \quad 1 \leq i \leq n,$$

where $\psi_i(\theta)$’s are the linear coordinate functions restricted to $\mathbb{S}^{n-1}$, and $F(r) : [0, R] \to [0, \infty)$ solves the ODE initial value problem

$$F'' + (n - 1) \frac{sn_k'}{sn_k} F' + \left( \lambda_{2,\alpha}(B_k(R)) - \frac{n-1}{sn_k^2} \right) F = 0, \quad F(0) = 0, \quad F'(0) = 1,$$

see [18, Proposition 3.1]. We have $F'(R) = -\alpha F(R)$, and $\lambda_{2,\alpha}(B_k(R))$ is characterized by

$$\lambda_{2,\alpha}(B_k(R)) = \inf \left\{ \int_0^R \left( (v')^2 + \frac{n-1}{sn_k^2} v^2 \right) sn_k^{n-1} \, dt + \alpha v^2(R) sn_k^{n-1}(R) / \int_0^R v^2 sn_k^{n-1} \, dt \right\}$$

for any solution $v$ of the above ODE.
for \( v \in W^{1,2}([0, R]) \setminus \{0\} \) with \( v(0) = 0 \).

**Proposition 2.1** Let \( F(r) \) be the solution to (2.2). If \( \alpha < 0, \) then

1. \( F'(r) > 0 \) for \( r \in [0, R] \).
2. Assume further that \( \alpha \geq -\frac{s_n'(r)}{sn_k(r)} \). Then \( \frac{F'(r)}{F(r)} \geq -\alpha \) for \( r \in (0, R] \).

**Proof** Let \( N(r) = sn_k^{n-1}(r)F'(r) \), then direct calculation gives

\[
N'(r) = \left(\frac{n - 1}{sn_k^2(r)} - \lambda_2,\alpha(B_k(R))\right)sn_k^{n-1}(r)F'(r),
\]

from which it follows that \( N'(r) \) has at most one zero in \([0, R]\) and is positive near 0. Since that \( N(0) = 0 \) and \( N(R) = -\alpha sn_k^{n-1}(R)F(R) > 0 \), we then have \( N(r) > 0 \) for \( r \in (0, R] \), proving (1).

To prove (2), let \( v(r) = \frac{F(r)}{F(\alpha)} \). Then \( v(R) = -\alpha, v(r) > 0 \) for \( r \in (0, R] \) and \( \lim_{r \to 0^+} v(r) = +\infty \). Rewriting equation (2.2) as an ODE for \( v \) yields

\[
v' + v^2 + (n - 1)\frac{s_n'(r)}{sn_k(r)}v + \left(\lambda_2,\alpha(B_k(R)) - \frac{n - 1}{sn_k^2(r)}\right) = 0. \tag{2.4}
\]

We claim that \( v(r) \geq -\alpha \) on \((0, R]\). Suppose not, then there exists \( r_0 \in (0, R) \) such that

\[
v'(r_0) = 0, \quad v''(r_0) \geq 0, \quad \text{and} \quad v(r_0) < -\alpha.
\]

Then differentiating (2.4) in \( r \) and using \( sn_k sn_k'' - (sn_k')^2 = -1 \), we have at \( r = r_0 \) that

\[
0 = v''(r_0) - (n - 1)\frac{v(r_0)}{sn_k^2(r_0)} + 2(n - 1)\frac{s_n'(r_0)}{sn^2_k(r_0)}
\]

\[
> (n - 1)\frac{\alpha}{sn_k^2(r_0)} + 2(n - 1)\frac{s_n'(r_0)}{sn^2_k(r_0)}
\]

\[
= \frac{n - 1}{sn_k^2(r_0)} \left(\alpha + 2\frac{s_n'(r_0)}{sn_k(r_0)}\right). \tag{2.5}
\]

Again using \( sn_k sn_k'' - (sn_k')^2 = -1 \), we see that \( sn'(r)/sn(r) \) is monotonically decreasing in \( r \), so (2.5) implies that

\[
\alpha < -2\frac{s_n'(r)}{sn_k(r)},
\]

which contradicts the assumption in (2). Therefore, (2) is proved. \( \square \)

Recall that the first nonzero Steklov eigenvalue \( \sigma_1(\Omega) \) is characterized variationally by

\[
\sigma_1(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^2 d\mu_g \setminus \left\{ \int_{\partial \Omega} u dA_g = 0 \right. \int_{\partial \Omega} u dA_g = 0 \} \right\}. \tag{2.6}
\]

**Proposition 2.2** If \( \alpha \geq -\sigma_1(B_k(R)) \), then \( \lambda_2,\alpha(B_k(R)) \geq 0. \)

**Proof** Since

\[
\int_{\partial B_k(R)} F(r) \psi_1(\theta) dA = 0, \quad 1 \leq i \leq n,
\]

\[\text{ Springer}\]
the functions $u_i = F(r) \psi_i(\theta)$ $(1 \leq i \leq n)$ are test functions for $\sigma_1(B_{\kappa}(R))$. Therefore, from (2.6) we get

$$\sum_{i=1}^{n} \int_{B_{\kappa}(R)} |\nabla u_i|^2 d\mu \geq \sigma_1(B_{\kappa}(R)) \sum_{i=1}^{n} \int_{\partial B_{\kappa}(R)} |u_i|^2 dA$$

$$\geq -\alpha \sum_{i=1}^{n} \int_{\partial B_{\kappa}(R)} |u_i|^2 dA.$$ 

Recall that

$$\tilde{\lambda}_{2,\alpha}(B_{\kappa}(R)) = \frac{\sum_{i=1}^{n} \int_{B_{\kappa}(R)} |\nabla u_i|^2 d\mu + \alpha \sum_{i=1}^{n} \int_{\partial B_{\kappa}(R)} |u_i|^2 dA}{\sum_{i=1}^{n} \int_{B_{\kappa}(R)} u_i^2 d\mu},$$

so then $\tilde{\lambda}_{2,\alpha}(B_{\kappa}(R)) \geq 0$. 

\[3 \text{ Comparison theorem for } \lambda_{2,\alpha} \]

In this section we prove Cheng type comparison theorem for the second Robin eigenvalue.

**Proof of Theorem 1.1** Suppose $\alpha \leq 0$. Let $u_1$ be a positive first eigenfunction for $\lambda_{1,\alpha}(B(R))$ and $\psi_i$ $(1 \leq i \leq n)$ be the restriction of the linear coordinate functions on $\mathbb{S}^{n-1}$ $(n \geq 2)$. Since $\psi_1, \psi_2, \ldots, \psi_n$ are linearly independent, there exists $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ such that

$$\sum_{i=1}^{n} a_i \int_{B(R)} F(r) \psi_i(\theta) u_1(x) d\mu_g = 0,$$

where $F(r)$ is defined by (2.2). Let

$$\Psi(\theta) := \sum_{i=1}^{n} a_i \psi_i(\theta),$$

then $u(x) := F(r) \Psi(\theta)$ is a smooth function on $B(R)$ satisfying

$$\int_{B(R)} F(r) u(x) u_1(x) d\mu_g = 0.$$

So $u(x)$ is a test function for $\lambda_{2,\alpha}(B(R))$, and hence

$$\lambda_{2,\alpha}(B(R)) \leq \frac{\int_{B(R)} |\nabla u|^2 d\mu_g + \alpha \int_{B(R)} u^2 dA_g}{\int_{B(R)} u^2 d\mu_g}.$$ 

For $n = 2$ or $n = 3$, the Riemann metric in the (geodesic) polar coordinates has the form $dr^2 + f^2(r, \theta) d\theta^2$, where $d\theta^2$ is the standard metric on $\mathbb{S}^{n-1}$. So then

$$\int_{B(R)} |\nabla u|^2 d\mu_g = \int_{B(R)} |F'(r)|^2 \Psi^2(\theta) d\mu_g + \int_{B(R)} \frac{F^2(r)}{f^2(r, \theta)} |\nabla_{\mathbb{S}^{n-1}} \Psi(\theta)|^2 d\mu_g. \quad (3.1)$$

**Case 1.** Two dimensional case.
When \( n = 2 \), we have \( f(r, \theta) = J(r, \theta) \), where \( J(r, \theta) \) is the volume element at \( (r, \theta) \). Computing in the polar coordinates and using integration by parts, we get

\[
I = \int_{B(R)} |F'(r)|^2 \Psi^2(\theta) \, d\mu_g
\]

\[
= \int_{\mathbb{S}^1} \int_0^R |F'(r)|^2 \Psi^2(\theta) J(r, \theta) \, dr \, d\theta
\]

\[
= \int_{\mathbb{S}^1} \Psi^2(\theta) \left( \int_0^R F'(r) J(r, \theta) \, dF(r) \right) \, d\theta
\]

\[
= \int_{\mathbb{S}^1} \Psi^2(\theta) \left( F'(R) F(R) J(R, \theta) - \int_0^R (F'(r) J(r, \theta))' F(r) \, dr \right) \, d\theta
\]

\[
= -\alpha \int_{\mathbb{S}^1} F^2(R) \Psi^2(\theta) J(R, \theta) \, d\theta - \int_{\mathbb{S}^1} \Psi^2(\theta) \left( \int_0^R (F'(r) J(r, \theta))' F(r) \, dr \right) \, d\theta,
\]

where we used the boundary condition \( F'(R) = -\alpha F(R) \) in integration by parts.

Since \( \text{Sect}_g \leq \kappa \), by the comparison theorem there holds

\[
\frac{J'(r, \theta)}{J(r, \theta)} \geq \frac{J'_k(r)}{J_k(r)},
\]

where \( J_k(r) = s n^{n-1}_k(r) \) is the volume element in \( M_k \). This inequality, together with the ODE (2.2) for \( F \), and \( F' > 0 \) on \([0, R]\) proved in Proposition 2.1, implies that

\[
III = \int_0^R (F'(r) J(r, \theta))' F(r) \, dr
\]

\[
= \int_0^R \left( F''(r) + \frac{J'(r, \theta)}{J(r, \theta)} F'(r) \right) F(r) J(r, \theta) \, dr
\]

\[
\geq \int_0^R \left( F''(r) + \frac{J'_k(r)}{J_k(r)} F'(r) \right) F(r) J(r, \theta) \, dr
\]

\[
= \int_0^R \left( \frac{F^2(r)}{J_k^2(r)} - \lambda_{2,\omega}(B_k(R)) F^2(r) \right) J(r, \theta) \, dr
\]

\[
\geq \int_0^R \frac{F^2(r)}{J_k(r)} \, dr - \lambda_{2,\omega}(B_k(R)) \int_0^R F^2(r) J(r, \theta) \, dr.
\]

Therefore, we obtain

\[
I = \int_{B(R)} |F'(r)|^2 \Psi^2(\theta) \, d\mu_g
\]

\[
\leq -\alpha \int_{\partial B(R)} u^2 \, dA_g + \lambda_{2,\omega}(B_k(R)) \int_{B(R)} u^2 \, d\mu_g - \int_{\mathbb{S}^1} \int_0^R \frac{F^2(r)}{J_k(r)} \Psi^2(\theta) \, dr \, d\theta. \quad (3.2)
\]

Using the comparison result \( J(r, \theta) \geq J_k(r) \) and again computing in the polar coordinates, we have
\[ II = \int_{B(R)} \frac{F^2(r)}{J^2(r, \theta)} |\nabla S^1 \Psi(\theta)|^2 d\mu_g \]

\[ = \int_{S^1} \int_0^R \frac{F^2(r)}{J(r, \theta)} |\nabla S^1 \Psi(\theta)|^2 dr d\theta \]

\[ \leq \int_{S^1} \int_0^R \frac{F^2(r)}{J_k(r)} |\nabla S^1 \Psi(\theta)|^2 dr d\theta \]

\[ = \int_{S^1} \int_0^R \frac{F^2(r)}{J_k(r)} \Psi^2(\theta) dr d\theta, \tag{3.3} \]

where in the last equality we used

\[ \int_{S^{n-1}} |\nabla S^{n-1} \Psi(\theta)|^2 d\theta = (n - 1) \int_{S^{n-1}} \Psi(\theta)^2 d\theta \quad \text{for} \quad n \geq 2, \tag{3.4} \]

which follows from the facts that

\[ \sum_{i=1}^n \psi_i^2 = 1 \quad \text{and} \quad \sum_{i=1}^n |\nabla S^{n-1} \psi_i|^2 = n - 1. \]

Putting together equality (3.1), inequalities (3.2) and (3.3), we obtain

\[ \int_{B(R)} |\nabla u|^2 d\mu_g \leq -\alpha \int_{\partial B(R)} u^2 dA_g + \lambda_{2,\alpha}(B_k(R)) \int_{B(R)} u^2(x) d\mu_g, \]

thus proving \( \lambda_{2,\alpha}(B(R)) \leq \lambda_{2,\alpha}(B_k(R)). \)

**Case 2.** Three dimensional case.

When \( n = 3 \), we have \( f(r, \theta) = \sqrt{J(r, \theta)} \). By similar calculations as in Case 1, we get

\[ I = \int_{B(R)} |F'(r)|^2 \Psi^2(\theta) d\mu_g \]

\[ = -\alpha \int_{S^2} F^2(R) \Psi^2(\theta) J(R, \theta) d\theta - \int_{S^2} \Psi^2(\theta) \left( \int_0^R (F'(r)J(r, \theta))' F(r) dr \right) d\theta, \]

where we used the boundary condition \( F'(R) = -\alpha F(R) \) in integration by parts.

Using the comparison results \( \frac{J'(r, \theta)}{J(r, \theta)} \geq \frac{J_k'(r)}{J_k(r)} \) and \( J(r, \theta) \geq J_k(r) \), equation (2.2) for \( F \), and \( F' > 0 \) on \([0, R] \), we obtain

\[ IV = \int_0^R (F'(r)J(r, \theta))^2 F(r) dr \]

\[ = \int_0^R \left( F''(r) + \frac{J'(r, \theta)}{J(r, \theta)} F'(r) \right) F(r) J(r, \theta) dr \]

\[ \geq \int_0^R \left( F''(r) + \frac{J_k'(r)}{J_k(r)} F'(r) \right) F(r) J(r, \theta) dr \]

\[ = \int_0^R \left( 2 \frac{F^2(r)}{J_k(r)} - \lambda_{2,\alpha}(B_k(R)) F^2(r) \right) J(r, \theta) dr \]

\[ \geq \int_0^R 2F^2(r) dr - \lambda_{2,\alpha}(B_k(R)) \int_0^R F^2(r) J(r, \theta) dr. \]
Therefore, we have
\[ I = \int_{B(R)} |F'(r)|^2 \Psi^2(\theta) \, d\mu_g \]
\[ \leq -\alpha \int_{\partial B(R)} u^2 \, dA_g + \lambda_{2,\alpha}(B_k(R)) \int_{B(R)} u^2 \, d\mu_g - 2 \int_{S^2} \int_0^R F^2(r) \Psi^2(\theta) \, dr \, d\theta. \]
(3.5)

Direct calculation gives
\[ II = \int_{B(R)} \frac{F^2(r)}{f^2(r, \theta)} |\nabla S^2 \Psi(\theta)|^2 \, d\mu_g \]
\[ = \int_{S^2} \int_0^R \frac{F^2(r)}{J(r, \theta)} |\nabla S^2 \Psi(\theta)|^2 J(r, \theta) \, dr \, d\theta \]
\[ = \int_{S^2} \int_0^R F^2(r)|\nabla \Psi(\theta)|^2 \, dr \, d\theta \]
\[ = 2 \int_{S^2} \int_0^R F^2(r)\Psi^2(\theta) \, dr \, d\theta, \]
(3.6)

where in the last equality we used (3.4).

Putting together equalities (3.1) and (3.6), and inequality (3.5), we obtain
\[ \int_{B(R)} |\nabla u|^2 \, d\mu_g \leq -\alpha \int_{\partial B(R)} u^2 \, dA_g + \lambda_{2,\alpha}(B_k(R)) \int_{B(R)} u^2(x) \, d\mu_g, \]
thus implying \( \lambda_{2,\alpha}(B(R)) \leq \lambda_{2,\alpha}(B_k(R)) \).

The inequalities in the arguments above become equalities if and only if \( B(R) \) is isometric to \( B_k(R) \). Therefore, the proof of Theorem 1.1 is now complete.

Setting \( \alpha = -\sigma_1(B_k(R)) \), comparison for the second Robin eigenvalue implies comparison for the first nonzero Steklov eigenvalue.

**Proof of Corollary 1.2** The first nontrivial Neumann eigenvalue \( \mu_1(B(R)) \) is positive. Since \( \lambda_{2,0}(B(R)) = \mu_1(B(R)) > 0 \), \( \lambda_{2,-\sigma_1(B_k(R))}(B(R)) \leq \lambda_{2,-\sigma_1(B_k(R))}(B_k(R)) = 0 \) by Theorem 1.1, and \( \lambda_{2,\alpha}(B(R)) \) is continuous in \( \alpha \), there exists \( \alpha_0 \in [-\sigma_1(B_k(R)), 0) \) such that \( \lambda_{2,\alpha_0}(B(R)) = 0 \). Let \( u \) be an eigenfunction for \( \lambda_{2,\alpha_0}(B(R)) \), then the variational characterization (1.3) implies
\[ \frac{\int_{B(R)} |\nabla u|^2 \, d\mu_g}{\int_{\partial B(R)} u^2 \, dA_g} = -\alpha_0. \]

On the other hand, for \( \lambda_{2,\alpha_0}(B(R)) = 0 \) and the associated eigenfunction \( u \), (1.1) becomes
\[ \begin{cases} \Delta u = 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial v} = -\alpha_0 u \quad \text{on } \partial \Omega. \end{cases} \]

So using the divergence theorem, we get
\[ 0 = \int_{B(R)} \Delta u \, d\mu_g = \int_{\partial B(R)} \frac{\partial u}{\partial v} \, dA_g = -\alpha_0 \int_{\partial B(R)} u \, dA_g. \]
which implies that \( u \) is a test function for \( \sigma_1(B(R)) \). By the variational characterization (2.6), we then have

\[
\sigma_1(B(R)) \leq \frac{\int_{\Omega} |\nabla u|^2 \, d\mu_g}{\int_{\partial \Omega} u^2 \, dA_g} = -\alpha_0 \leq \sigma_1(B_\kappa(R)),
\]

which proves Corollary 1.2.

\[\square\]

In dimension four or higher, an analogue of Theorem 1.1 holds under additional symmetry assumption.

**Theorem 3.1** Let \((M^n, g)\), where \(n \geq 4\), be a complete Riemannian manifold with sectional curvature \(\text{Sect}_g \leq \kappa\) for \(\kappa \in \mathbb{R}\), and \(B(R)\) be an injective geodesic ball\(^1\) of radius \(R\). Assume that the metric \(g\) is centrally symmetric, namely, there is an isometry fixing the center \(o\) of \(B(R)\) and mapping \(\gamma(t)\) to \(\gamma(-t)\) for any minimizing geodesic \(\gamma\) with \(\gamma(0) = o\). If \(\alpha \leq 0\), then

\[
\lambda_{2,\alpha}(B(R)) \leq \lambda_{2,\alpha}(B_\kappa(R)).
\]

Equality holds if and only if \(B(R)\) is isometric to \(B_\kappa(R)\).

**Proof** By the symmetry assumption on the metric \(g\), we have

\[
\int_{\mathbb{S}^{n-1}} \psi_i(\theta) J(r, \theta) \, d\theta = 0, \quad 1 \leq i \leq n,
\]

and the first Robin eigenfunction corresponding to \(\lambda_{1,\alpha}(B(R))\) is also centrally symmetric. So \(u_i(x) = F(r)\psi_i(\theta)\) can be used as test functions for \(\lambda_{2,\alpha}(B(R))\). As a result,

\[
\lambda_{2,\alpha}(B(R)) \leq \frac{\sum_{i=1}^{n} \left( \int_{B(R)} |\nabla u_i|^2 \, d\mu_g + \alpha \int_{\partial B(R)} u_i^2 \, dA_g \right)}{\sum_{i=1}^{n} \int_{B(R)} u_i^2 \, d\mu_g}
\]

\[
= \frac{\sum_{i=1}^{n} \int_{B(R)} |\nabla u_i|^2 \, d\mu_g + \alpha \int_{\partial B(R)} F^2 \, dA_g}{\int_{B(R)} F^2 \, d\mu_g}.
\]

(3.7)

Using the Rauch comparison theorem for manifolds with \(\text{Sect}_g \leq \kappa\), we estimate

\[
\sum_{i=1}^{n} \int_{B(R)} |\nabla u_i|^2 \, d\mu_g \leq \int_{B(R)} |F'(r)|^2 \, d\mu_g + \frac{1}{sn^2_\kappa(r)} \sum_{i=1}^{n} |\nabla^{\mathbb{S}^{n-1}} \psi_i(\theta)|^2 F^2(r) \, d\mu_g
\]

\[
= \int_{\mathbb{S}^{n-1}} \int_{0}^{R} |F'(r)|^2 J(r, \theta) + \frac{(n-1)F^2(r)}{sn^2_\kappa(r)} J(r, \theta) \, dr \, d\theta.
\]

Recall that

\[
\int_{0}^{R} |F'(r)|^2 J(r, \theta) \, dr \leq -\alpha F^2(R) J(R, \theta) - \int_{0}^{R} \frac{(n-1)F^2(r)}{sn^2_\kappa(r)} J(r, \theta) \, dr
\]

\[- \lambda_{2,\alpha}(B_\kappa(R)) \int_{0}^{R} F^2(r) J(r, \theta) \, dr.
\]

\(\square\)

---

\(^1\) As defined in Theorem 1.1.
so then
\[
\sum_{i=1}^{n} \int_{B(R)} |\nabla u_i|^2 \, d\mu_g \leq -\alpha F^2(R) \int_{\mathbb{S}^{n-1}} J(R, \theta) \, d\theta + \lambda_{2,\alpha}(B_R) \int_{B(R)} F^2(r) \, d\mu_g
\]
\[
= -\alpha \int_{\partial B(R)} F^2 \, dA_g + \lambda_{2,\alpha}(B_R) \int_{B(R)} F^2(r) \, d\mu_g,
\]
Therefore, we conclude from (3.7) that
\[
\lambda_{2,\alpha}(B(R)) \leq \lambda_{2,\alpha}(B_R).
\]
The inequalities in the arguments above are equalities if and only if \(B(R)\) is isometric to \(B_R\). Therefore, Theorem 3.1 is proved. \(\square\)

In the same way Corollary 1.2 follows from Theorem 1.1, Theorem 3.1 has the following implication.

**Corollary 3.2** Under the hypotheses of Theorem 3.1, there holds
\[
\sigma_1(B(R)) \leq \sigma_1(B_R).
\]
Equality holds if and only if \(B(R)\) is isometric to \(B_R\).

**4 Shape optimization of \(\lambda_{2,\alpha}\)**

In this section, we prove that geodesic balls maximize the second Robin eigenvalue among domains with the same volume in nonpositively curved space forms.

From here on, we assume \(\Omega\) to be a bounded domain with Lipschitz boundary in the complete simply connected space form \((M_\kappa, g_\kappa)\) with \(\text{Sect}_{g_\kappa} = \kappa\). Let \(\Omega^* \subset M_\kappa\) be a geodesic ball such that \(|\Omega|^\kappa = |\Omega^*|^\kappa\). We write \(d\mu_{g_\kappa}\) and \(dA_{g_\kappa}\) as \(d\mu\) and \(dA\) respectively for short.

Let \(R\) be the radius of \(\Omega^*\), and we extend the function \(F\) defined in (2.2) by
\[
F_1(r) := \begin{cases} F(r), & r \leq R, \\ F(R) e^{-\alpha(r-R)}, & r > R. \end{cases} \tag{4.1}
\]
Then for simplicity, we denote \(F_1(r)\) by \(F(r)\) in the following computations. By definition, \(F\) is continuously differentiable on \((0, \infty)\). If \(\alpha \leq 0\), then \(F\) is non-decreasing on \([0, \infty)\). In below, \(\sigma_1(\Omega^*)\) denotes the first nonzero Steklov eigenvalue of \(\Omega^*\).

**Proposition 4.1** Assume that \(\alpha \in [-\sigma_1(\Omega^*), 0]\). Define \(H : [0, \infty) \to \mathbb{R}\) by
\[
H(r) := (F'(r))^2 + \frac{n-1}{sn^2_\kappa} F^2(r) + 2\alpha F(r) F'(r) + \alpha \frac{(n-1)sn_\kappa'(r)}{sn_\kappa(r)} F^2(r), \tag{4.2}
\]
where \(F(r)\) is defined in (4.1). Then \(H\) is monotonically decreasing on \((0, \infty)\).

**Proof** By assumption, \(\alpha \geq -\sigma_1(\Omega^*) = -\sigma_1(B_\kappa(R))\), so Proposition 2.2 applies and hence \(\lambda_{2,\alpha}(\kappa, R) \geq 0\).

We claim that \(\sigma_1(\Omega^*) \leq 2\frac{sn_\kappa'}{sn_\kappa}\). Indeed, by Corollary 3.2, \(\sigma_1(\Omega^*) \leq \sigma_1(B_\kappa(R))\) for \(\kappa \leq 0\); in particular, \(\sigma_1(\Omega^*) \leq \sigma_1(B_0(0)) = R^{-1}\). The claim follows from the elementary inequality \(R^{-1} \leq 2\frac{sn_\kappa'}{sn_\kappa}\) for \(\kappa \leq 0\). So part (2) of Proposition 2.1 applies and hence \(F' \geq -\alpha F\) on \([0, R]\).
Case 1. $\alpha \in [-\sigma_1(\Omega_\kappa), 0)$.

If $0 < r \leq R$, then we compute that

$$H'(r) = 2F'F'' - \frac{2(n-1)sn'_k}{sn_k^3} F^2 + \frac{2(n-1)}{sn_k^2} FF'$$

$$+ 2\alpha(F')^2 + 2\alpha FF'' - \alpha \frac{n-1}{sn_k^2} F^2 + 2\alpha(n-1) \frac{sn'_k}{sn_k} FF'.$$

Using the ODE (2.2) of $F(r)$, we have

$$I = 2F'F'' - \frac{2(n-1)sn'_k}{sn_k^3} F^2 + \frac{2(n-1)}{sn_k^2} FF'$$

$$= 2F' \left(-\left(\frac{sn'_k}{sn_k} F' - \left(\lambda_{2,\alpha}(B_k(R)) - \frac{n-1}{sn_k^2}\right) F\right) - \frac{2(n-1)sn'_k}{sn_k^3} F^2\right)$$

$$+ \frac{2(n-1)}{sn_k^2} FF'$$

$$= -\frac{2(n-1)}{sn_k^3} \left( sn_k^2 sn'_k (F')^2 - 2sn_k^2 FF' + sn_k^2 F^2 \right) - 2\lambda_{2,\alpha}(B_k(R)) FF'$$

$$\leq -\frac{2(n-1)}{sn_k^3} \left( sn_k F' - F \right)^2 - 2\lambda_{2,\alpha}(B_k(R)) FF'$$

$$\leq -2\lambda_{2,\alpha}(B_k(R)) FF' ,$$

where in the first inequality we used $sn'_k(r) \geq 1$ for $\kappa \leq 0$. Using (2.2) again,

$$II = 2\alpha(F')^2 + 2\alpha FF'' - \alpha \frac{n-1}{sn_k^2} F^2 + 2\alpha(n-1) \frac{sn'_k}{sn_k} FF'$$

$$= 2\alpha(F')^2 + 2\alpha F \left(-\left(\frac{sn'_k}{sn_k} F' - \left(\lambda_{2,\alpha}(B_k(R)) - \frac{n-1}{sn_k^2}\right) F\right) - \frac{2(n-1)sn'_k}{sn_k^3} F^2\right)$$

$$- \alpha \frac{n-1}{sn_k^2} F^2 + 2\alpha(n-1) \frac{sn'_k}{sn_k} FF'$$

$$= 2\alpha(F')^2 + \alpha \frac{n-1}{sn_k^2} F^2 - 2\alpha \lambda_{2,\alpha}(B_k(R)) F^2$$

$$< -2\alpha \lambda_{2,\alpha}(B_k(R)) F^2 .$$

So we conclude

$$H'(r) < -2\lambda_{2,\alpha}(B_k(R)) FF' - 2\alpha \lambda_{2,\alpha}(B_k(R)) F^2$$

$$\leq 2\alpha \lambda_{2,\alpha}(B_k(R)) F^2 - 2\alpha \lambda_{2,\alpha}(B_k(R)) F^2$$

$$= 0 ,$$

where in the second inequality we used $F' \geq -\alpha F$ on $(0, R]$ and $\lambda_{2,\alpha}(\kappa, R) \geq 0$. Therefore, $H(r)$ is monotonically decreasing on $(0, R]$. If $r \geq R$, then by definition (4.1), $F(r) = F(R)e^{-\alpha(r-R)}$. So then

$$H(r) = \left(-\alpha^2 + \frac{n-1}{sn_k^2(r)} + (n-1)\alpha \frac{sn'_k(r)}{sn_k(r)}\right) F^2(r) .$$
Differentiating $H$ in $r$ and using $sn_k' \geq 1$ for $\kappa \leq 0$ and $sn_k''sn_k - (sn_k')^2 = -1$, we have
\[
H'(r) = \left(2\alpha^3 - \frac{2(n-1)sn_k'(r)}{sn_k^2(r)} - 3\frac{(n-1)\alpha}{sn_k^2(r)} - \frac{2(n-1)\alpha^2}{sn_k(r)}\right)F^2(r)
\leq \left(2\alpha^3 - \frac{2(n-1)}{sn_k^3(r)} - 3\frac{(n-1)\alpha}{sn_k^2(r)} - \frac{2(n-1)\alpha^2}{sn_k(r)}\right)F^2(r)
= -\frac{2(n-1)}{sn_k^3(r)} \left(1 + \frac{3}{2}\alpha sn_k(r) + \alpha^2 sn_k^2(r)\right)F^2(r) + 2\alpha^3 F^2(r)
= -\frac{2(n-1)}{sn_k^3(r)} \left(1 + \frac{3}{4}\alpha sn_k(r)\right)^2 + \frac{7}{16}\alpha^2 sn_k^2(r)F^2(r) + 2\alpha^3 F^2(r)
< 0,
\]
thus proving that $H(r)$ is monotonically decreasing on $[R, \infty)$.

**Case 2.** $\alpha = 0$.

By the same argument as in Case 1, and using that $\lambda_{2,0}(\kappa, R)$ is the first nonzero Neumann eigenvalue of $B_\kappa(R)$, which is positive, we reach the same conclusion that $H' < 0$ on $(0, \infty)$.

Therefore, we have proved the proposition. \qed

We have the following center of mass lemma.

**Lemma 4.1** There exists a point $p \in \text{hull}(\Omega)$, the closed geodesic convex hull of $\Omega$, such that
\[
\int_{\Omega} F(r_p(x)) \frac{\exp_p^{-1}(x)}{r_p(x)} u_1(x) \, d\mu = 0, \tag{4.3}
\]
where $F$ is defined in (4.1), $r_p(x) = \text{dist}_K(p, x)$, $\exp_p^{-1}$ is the inverse of the exponential map $\exp_p : T_pM_\kappa \to M_\kappa$, and $u_1$ is a first eigenfunction for $\lambda_{1,\kappa}(\Omega)$.

**Proof** The proof is similar to [9, Lemma 4.1]. Define the vector field
\[
X(p) = \int_{\Omega} F(r_p(x)) \frac{\exp_p^{-1}(x)}{r_p(x)} u_1(x) \, d\mu.
\]
Then the integral curves of $X$ define a mapping from hull($\Omega$) to itself. Since hull($\Omega$) is convex and contained in the injectivity radius, hull($\Omega$), it is a topological ball. Therefore, $X$ must have a zero by the Brouwer fixed point theorem. \qed

The proof of Theorem 1.3 now proceeds in four propositions.

From here on, we fix the point $p$ according to Lemma 4.1 so that (4.3) holds. Let $(r, \theta)$ denote the polar coordinates centered at $p$ and $J(r, \theta)$ denote the volume element at $(r, \theta)$. Then we have
\[
\frac{\exp_p^{-1}(x)}{r_p(x)} = (\psi_1(\theta), \psi_2(\theta), \ldots, \psi_n(\theta)),
\]
where $\psi_i$’s are the restrictions of the linear coordinate functions on $\mathbb{S}^{n-1}$. We define
\[
v_i(x) := F(r_p(x)) \psi_i(\theta), \quad 1 \leq i \leq n,
\]
and rewrite (4.3) as
\[
\int_{\Omega} v_i(x) u_1(x) \, d\mu = 0, \quad 1 \leq i \leq n.
\]
So \( v_i \)'s are test functions for \( \lambda_{2,\alpha}(\Omega) \).

**Proposition 4.2** Under the hypotheses of Theorem 1.3, there holds

\[
\lambda_{2,\alpha}(\Omega) \leq \frac{\int_\Omega |F'(r_p)|^2 + \frac{n-1}{sn_k^2(r_p)} F^2(r_p) \, d\mu + \alpha \int_{\partial\Omega} F^2(r_p) \, dA}{\int_\Omega F^2(r_p) \, d\mu}. \tag{4.4}
\]

**Proof** We denote by \( \nabla^{S^{n-1}} \) the covariant derivative with respect to the standard metric on \( S^{n-1} \), and by \( \nabla \) the covariant derivative with respect to the metric \( g_\kappa = dr^2 + sn_k^2(r)d\theta^2 \) on \( M_\kappa \). Using

\[
\sum_{i=1}^n \psi_i^2 = 1 \quad \text{and} \quad \sum_{i=1}^n |\nabla^{S^{n-1}} \psi_i|^2 = n - 1,
\]

we compute that

\[
\sum_{i=1}^n \int_\Omega |\nabla v_i|^2 \, d\mu = \sum_{i=1}^n \int_\Omega |\nabla (F(r_p)\psi_i)|^2 \, d\mu = \sum_{i=1}^n \int_\Omega \left( |F'(r_p)|^2 \psi_i^2 + \frac{F^2(r_p)}{sn_k^2(r_p)} |\nabla^{S^{n-1}} \psi_i|^2 \right) \, d\mu = \int_\Omega \left( |F'(r_p)|^2 + \frac{n-1}{sn_k^2(r_p)} F^2(r_p) \right) \, d\mu. \tag{4.5}
\]

On the other hand,

\[
\sum_{i=1}^n \int_{\partial\Omega} v_i^2 \, dA = \sum_{i=1}^n \int_{\partial\Omega} |F(r_p)|^2 \psi_i^2 \, dA = \int_{\partial\Omega} |F(r_p)|^2 \, dA. \tag{4.6}
\]

So using the averaging of Rayleigh quotients for \( v_i \)'s, and (4.5) and (4.6), we obtain

\[
\lambda_{2,\alpha}(\Omega) \leq \frac{\sum_{i=1}^n \int_\Omega |\nabla v_i|^2 \, d\mu + \alpha \sum_{i=1}^n \int_{\partial\Omega} v_i^2 \, dA}{\sum_{i=1}^n \int_\Omega v_i^2 \, d\mu} = \frac{\int_\Omega |F'(r_p)|^2 + \frac{n-1}{sn_k^2(r_p)} F^2(r_p) \, d\mu + \alpha \int_{\partial\Omega} F^2(r_p) \, dA}{\int_\Omega F^2(r_p) \, d\mu}.
\]

This proves the proposition. \( \square \)

**Proposition 4.3** Under the hypotheses of Theorem 1.3, there holds

\[
\lambda_{2,\alpha}(\Omega) \leq \frac{\int_\Omega H(r_p) \, d\mu}{\int_\Omega F^2(r_p) \, d\mu}. \tag{4.7}
\]

**Proof** Using \( |\nabla r_p| = 1 \), we have

\[
\int_{\partial\Omega} F^2(r_p) \, dA \geq \int_{\partial\Omega} F^2(r_p) \langle \nabla r_p, v \rangle \, dA = \int_\Omega \text{div} (F^2(r_p)\nabla r_p) \, d\mu
\]

Springer
\[ \frac{\int_{\Omega} H(r_p) \, d\mu}{\int_{\Omega} F^2(r_p) \, d\mu} \leq \frac{\int_{\Omega_p^*} H(r_p) \, d\mu}{\int_{\Omega_p^*} F^2(r_p) \, d\mu}. \] (4.9)

Equality holds if and only if \( \Omega = \Omega_p^* \).

**Proof** Recall that \( F \) defined in (4.1) is non-decreasing, we have

\[
\int_{\Omega} F^2(r_p) \, d\mu = \int_{\Omega \cap \Omega_p^*} F^2(r_p) \, d\mu + \int_{\Omega \setminus \Omega_p^*} F^2(r_p) \, d\mu
\geq \int_{\Omega \cap \Omega_p^*} F^2(r_p) \, d\mu + \int_{\Omega \setminus \Omega_p^*} F^2(R) \, d\mu
\geq \int_{\Omega_p^*} F^2(r_p) \, d\mu. \] (4.10)

By Proposition 4.1, \( H \) is monotonically decreasing, so then

\[
\int_{\Omega} H(r_p) \, d\mu = \int_{\Omega \cap \Omega_p^*} H(r_p) \, d\mu + \int_{\Omega \setminus \Omega_p^*} H(r_p) \, d\mu
\leq \int_{\Omega \cap \Omega_p^*} H(r_p) \, d\mu + \int_{\Omega \setminus \Omega_p^*} H(R) \, d\mu
\leq \int_{\Omega_p^*} H(r_p) \, d\mu. \] (4.11)

Inequality (4.9) follows from (4.10) and (4.11). In particular, equality in (4.9) holds if and only if both (4.10) and (4.11) are equalities, which occurs if and only if \( \Omega = \Omega_p^* \).

Therefore, the proposition is proved. \( \square \)

**Proposition 4.5** Under the hypotheses of Theorem 1.3, there holds

\[
\lambda_{2,\alpha}(\Omega_p^*) = \frac{\int_{\Omega_p^*} H(r_p) \, d\mu}{\int_{\Omega_p^*} F^2(r_p) \, d\mu}.
\]

**Proof** Recall that \( F(r)\psi_1(\theta) \) are the eigenfunctions corresponding to \( \lambda_{2,\alpha}(\Omega_p^*) \), so then

\[
\lambda_{2,\alpha}(\Omega_p^*) = \frac{\int_{\Omega_p^*} |F'|^2(r_p) + \frac{n-1}{sn_k(r_p)} F^2(r_p) \, d\mu + \alpha \int_{\partial \Omega_p^*} F^2(r_p) \, dA}{\int_{\Omega_p^*} F^2(r_p) \, d\mu}.
\]
and
\[
\int_{\partial \Omega_p^*} F^2(r_p) \, dA = \int_{\partial \Omega_p^*} \langle (F^2)(r_p) \nabla r_p, v \rangle \, dA = \int_{\Omega_p^*} \text{div} \left( F^2(r_p) \nabla r_p \right) \, d\mu = \int_{\Omega_p^*} \left( (F^2)' + F^2 \Delta r_p \right) \, d\mu = \int_{\Omega_p^*} \left( (F^2)' + \frac{(n-1)sn'_{\kappa}}{sn_{\kappa}} F^2 \right) \, d\mu.
\]

Therefore, we have proved the proposition. \(\square\)

**Proof of Theorem 1.3** The theorem follows from combining Propositions 4.3–4.5. \(\square\)

**Acknowledgements** We thank Professors Richard Schoen, Lei Ni and Zhou Zhang for their encouragement and support. We also thank the anonymous referee for valuable comments on the manuscript. X. Li is partially supported by Simons Collaboration Grant #962228 and a start-up grant at Wichita State University; K. Wang is partially supported by NSFC No.11601359; H. Wu is supported by ARC Grant DE180101348. Both K. Wang and H. Wu acknowledge the excellent work environment provided by the Sydney Mathematical Research Institute.

**References**

1. Ashbaugh, M.S., Benguria, R.D.: Sharp upper bound to the first nonzero Neumann eigenvalue for bounded domains in spaces of constant curvature. J. London Math. Soc. 52(2), 402–416 (1995)
2. Bareket, M.: On an isoperimetric inequality for the first eigenvalue of a boundary value problem. SIAM J. Math. Anal. 8(2), 280–287 (1977)
3. Bucur, D., Giacomini, A.: A variational approach to the isoperimetric inequality for the Robin eigenvalue problem. Arch. Ration. Mech. Anal. 198(3), 927–961 (2010)
4. Bucur, D., Giacomini, A.: Faber-Krahn inequalities for the Robin-Laplacian: a free discontinuity approach. Arch. Ration. Mech. Anal. 218(2), 757–824 (2015)
5. Bossel, M.-H.: Membranes élastiquement liées: extension du théorème de Rayleigh-Faber-Krahn et de l’inégalité de Cheeger. C. R. Acad. Sci. Paris Sér. I. Math. 302(1), 47–50 (1986)
6. Chavel, I.: Eigenvalues in Riemannian geometry, volume 115 of Pure and Applied Mathematics. Academic Press, Inc., Orlando, FL, (1984). Including a chapter by Burton Randol, With an appendix by Jozef Dodziuk
7. Cheng, S.Y.: Eigenvalue comparison theorems and its geometric applications. Math. Z. 143(3), 289–297 (1975)
8. Daners, D.: A Faber-Krahn inequality for Robin problems in any space dimension. Math. Ann. 335(4), 767–785 (2006)
9. Edelen, N.: The PPW conjecture in curved spaces. J. Funct. Anal. 272(3), 849–865 (2017)
10. Escobar, J.F.: An isoperimetric inequality and the first Steklov eigenvalue. J. Funct. Anal. 165(1), 101–116 (1999)
11. Escobar, J.F.: A comparison theorem for the first non-zero Steklov eigenvalue. J. Funct. Anal. 178(1), 143–155 (2000)
12. Freitas, P., Krejčířík, D.: The first Robin eigenvalue with negative boundary parameter. Adv. Math. 280, 322–339 (2015)
13. Freitas, P., Laugesen, R.S.: From Neumann to Steklov and beyond, via Robin: the Weinberger way. Am. J. Math. 143(3), 969–994 (2021)
14. Henrot, A. (ed.): Shape Optimization and Spectral Theory. De Gruyter Open, Warsaw (2017)
15. Kennedy, J.: An isoperimetric inequality for the second eigenvalue of the Laplacian with Robin boundary conditions. Proc. Am. Math. Soc. 137(2), 627–633 (2009)
16. Li, X., Wang, K.: First Robin eigenvalue of the \(p\)-Laplacian on Riemannian manifolds. Math. Z. 298(3–4), 1033–1047 (2021)
17. Li, X., Wang, K., Wu, H.: An upper bound for the first nonzero Steklov eigenvalue. Preprint, (2020). arXiv:2003.03093 [math.DG]

18. Li, X., Wang, K., Wu, H.: The second Robin eigenvalue in non-compact rank-1 symmetric spaces. Preprint, (2022). arXiv:2208.07546 [math.DG]

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.