A STUDY OF FINITELY GENERATED FREE GROUPS VIA THE FUNDAMENTAL GROUPS

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June 2017
Free groups have many applications in Algebraic Topology. In this paper I specifically study the finitely generated free groups by using the covering spaces and fundamental groups. By the Van Kampen’s theorem, we have a famous fact that $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} = F_2$. Therefore, to study $F_n$, we could try to figure out the covering spaces of $S^1 \vee S^1$ or even $\sqcup_i S^1$. And in the appendix B, we prove the Nielsen-Schreier theorem which we will use this to study finitely index subgroups of $F_n$. 
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Free groups have many applications in Algebraic Topology. In this paper I specifically study the finitely generated free groups by using the covering spaces and fundamental groups. To review the definition of Free groups, you can read [1] or the Appendix A.

By the Van Kampen’s theorem, we have a famous fact that \( \pi_1(S^1 \vee S^1) = \mathbb{Z} \ast \mathbb{Z} = F_2 \). Therefore, to study \( F_n \), we could try to figure out the covering spaces of \( S^1 \vee S^1 \) or even \( \bigvee_i S^1 \). And in the appendix B, we use prove the Nielsen-Schreier theorem which we will use this to study finitely index subgroups of \( F_n \).

**Theorem 1.** For \( m \) and \( n \) non-negative integers, the rank of a free group is well-defined. That is, if \( F_n \), the free group of rank \( n \), is isomorphic to \( F_m \), the free group of rank \( m \), then \( m = n \).

**Proof.** Suppose we have an isomorphism \( \phi : F_m \to F_n \). Let’s denote \( \{a_1, \ldots, a_m\} \) be the free generating set of \( F_m \) and \( \{b_1, \ldots, b_n\} \) be the free generating set of \( F_n \). As shown in the figure below, consider the canonical surjection \( F_n \xrightarrow{Ab} \bigoplus_i \mathbb{Z} < b_i > \), we get a homomorphism \( \phi \circ Ab : F_m \to \bigoplus_j \mathbb{Z} < a_j > \). Then consider the canonical surjection \( F_m \xrightarrow{Ab} \bigoplus_j \mathbb{Z} < a_j > \), because \( \bigoplus_i \mathbb{Z} < b_i > \) is abelian, there exists a unique homomorphism \( F : \bigoplus_j \mathbb{Z} < a_j > \to \bigoplus_i \mathbb{Z} < b_i > \) s.t. the following diagram commutes.

\[
\begin{array}{ccc}
F_m & \xrightarrow{\phi} & F_n \\
& & \xrightarrow{Ab} \\
\bigoplus_j \mathbb{Z} < a_j > & \xrightarrow{F} & \bigoplus_i \mathbb{Z} < b_i >
\end{array}
\]

Notice that because both \( \phi \) and \( Ab \) are surjective, so is the map \( F \). So we get a surjective homomorphism \( \mathbb{Z}^m \to \mathbb{Z}^n \). Now we use the map \( F_n \xrightarrow{\phi^{-1}} F_m \), then by the same reason, we get a map \( G : \bigoplus_i \mathbb{Z} < b_i > \to \bigoplus_j \mathbb{Z} < a_j > \) s.t. which is also
a surjective homomorphism. So we have an isomorphism $\mathbb{Z}^m \to \mathbb{Z}^n$. So we have $m = n$. That is to say, the rank of a free group is well-defined. □

**Theorem 2.** For $m$ and $n$ non-negative integers, for which $(m, n)$ does $F_m$ have a subgroup isomorphic to $F_n$?

**Proof.** At first suppose $n = 1$, then $F_1 \cong \mathbb{Z}$ is abelian. So we can let $m = 1$ and then we have $F_1 \cong F_1$. Notice that if $m > 1$, $F_m$ is not an abelian group, so it cannot have a subgroup isomorphic to $F_1$.

Now let $n \geq 2$. Notice that for the map $F_1 \to F_n$ sends the generating element of $F_1$ to the generating element of $F_n$. This is an injective homomorphism, so $F_1$ has a subgroup isomorphic to $F_n$. Similarly for the map $F_2 \to F_n$ sends the two elements of the generating set of $F_2$ to the generating set of $F_n$. Notice that this is also an injective homomorphism, so $F_2$ has a subgroup isomorphic to $F_n$. Now we claim that $F_m$ has subgroup isomorphic to $F_2$ for all $m > 1$. Let consider the fundamental group of $S^1 \vee S^1$. Notice that it is just $F_2$. By **Theorem 10**, the map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ induced by the covering map is injective. So we suffice to show that the space $S^1 \vee \cdots \vee S^1$ (with $m$ copies of $S^1$) is a covering space of $S^1 \vee S^1$ (because the fundamental group of $S^1 \vee \cdots \vee S^1$ is just $F_m$, this show $F_m$ has a subgroup isomorphic to $F_2$).

For $m = 2$, $S^1 \vee S^1$ is obviously a covering space of $S^1 \vee S^1$ (by the identity map as the covering map).

For $m = 3$, consider the figure below,
This is a covering space of $S^{1} \vee S^{1}$ (by the covering map defined as the arrows). Notice that this is deformation retract to $S^{1} \vee S^{1} \vee S^{1}$ (when we quotient out one of edges as the figure above). So the fundamental group of $S^{1} \vee S^{1} \vee S^{1} (F_{3})$ has a subgroup isomorphic $p_{*}$ to $F_{2}$.

For $m > 3$, similar as the above pattern, consider the figure below,

This is a covering space of $S^{1} \vee S^{1}$ (by the covering map defined as the arrows). Notice that this is deformation retract to $S^{1} \vee S^{1} \vee S^{1}$ (as the figure above). So the fundamental group of $S^{1} \vee S^{1} \vee S^{1} (F_{m})$ has a subgroup isomorphic $p_{*}$ to $F_{2}$.

Then because $F_{2}$ has a subgroup isomorphic to $F_{n}$, so $F_{m}$ has a subgroup isomorphic to $F_{n}$ for $m > 1$. Then overall for $n \geq 2$, $F_{m}$ will directly have subgroup isomorphic to $F_{n}$ (for all $m > 0$, when $m = 0$, trivial). \hfill \Box

**Theorem 3.** For $m$ and $n$ non-negative integers, for which $(m, n)$ does $F_{m}$ have a normal subgroup isomorphic to $F_{n}$?

**Proof.** At first suppose $n = m = 1$, that is trivial because they are isomorphism. If $n = 1, m > 1$, $F_{m}$ is not even subgroup isomorphic to a subgroup of $F_{1}$. If $m = 1$ and $n > 1$, then obviously $F_{1}$ is abelian and so the homomorphism sends the generating element of $F_{1}$ to a generating of $F_{n}$ sends $F_{1}$ an abelian subgroup isomorphic to $F_{n}$, so it is normal.
Next, suppose \( m, n > 1 \), we claim the **Theorem 14**. If \( F \) is finitely generated free group and \( N \) is a nontrivial normal subgroup of infinite index, then \( N \) is not finitely generated. Now suppose \( F_m \) has a subgroup isomorphic to the \( F_n \), then the claim implies that \( F_m \) can be isomorphism to a subgroup with finite index. Because a subgroup of a free group is free, the normal subgroup is also free and by part (1), the rank of that normal group is also \( m \). And then by the Nielsen-Schreier formula **Theorem 15**, we have \( m = 1 + e(n - 1) \) where \( e \) is the index of that normal subgroup. That is to say, a necessary condition that \( F_m \) has a subgroup isomorphic to \( F_n \) is \( n - 1 \mid m - 1 \).

Next we show that if \( n - 1 \mid m - 1 \), then \( F_m \) has a normal subgroup isomorphic to \( F_n \). If \( n = m \), that is straightforward. Let \( k(n - 1) = m - 1 \) for \( k \geq 2 \). Let \( S^1 \vee \cdots \vee S^1 \) with \( m \) copies is the space \( \tilde{X} \) and \( S^1 \vee \cdots \vee S^1 \) with \( m \) copies be the base space. As the figure below, we define a map \( p : \tilde{X} \to X \) as the figure to make \( \tilde{X} \) be a covering space of \( X \).

![Diagram](image)

Notice that for each \( x \in X \) and each pair \( \tilde{x}, \tilde{x}' \) in the fiber of \( x \), there is a deck transformation taking \( \tilde{x} \) to \( \tilde{x}' \) by the rotation map with feasible degree. Therefore, \( \tilde{X} \) is a normal covering space. Then by **Theorem 13**, \( p_\ast(\pi_1(\tilde{X}, \tilde{x}_0)) \) is a normal subgroup of \( \pi_1(X, x_0) \simeq F_n \). That is to say \( F_m \) has a normal subgroup isomorphic to \( F_n \). \( \square \)

**Theorem 4.** For \( m \) and \( n \) non-negative integers, for which \( (m, n) \) does \( F_m \) have a quo-
tient group isomorphic to \( F_n \)?

**Proof.** Claim: \( F_m \) has a quotient group isomorphic to \( F_n \) iff \( m \geq n \).

To the forward direction, suppose we have a normal subgroup \( N \) s.t. \( F_m/N \cong F_n \). Let’s denote the isomorphism by \( \phi : F_m/N \to F_n \) and denote the quotient map \( \pi : F_m \to F_m/N \). So we have a (surjective) homomorphism \( F_m \xrightarrow{\phi \circ \pi} F_n \).

Similar as part (1), Let’s denote \( \{a_1, \cdots, a_m\} \) be the free generating set of \( F_m \) and \( \{b_1, \cdots, b_n\} \) be the free generating set of \( F_n \). As shown in the figure below, denote \( \psi := \phi \circ \pi \) and consider the canonical surjection \( F_n \xrightarrow{Ab} \bigoplus_i \mathbb{Z} < b_i > \), we get a homomorphism \( \psi \circ Ab : F_m \to \bigoplus_i \mathbb{Z} < b_i > \). Then consider the canonical surjection \( F_m \xrightarrow{Ab} \bigoplus_j \mathbb{Z} < a_j > \), because \( \bigoplus_i \mathbb{Z} < b_i > \) is abelian, there exists a unique homomorphism \( F : \bigoplus_j \mathbb{Z} < a_j > \to \bigoplus_i \mathbb{Z} < b_i > \) s.t. the following diagram commutes.

\[
\begin{array}{ccc}
F_m & \xrightarrow{\psi} & F_n \\
\downarrow^{\phi \circ \pi} & & \downarrow^{\phi \circ \pi} \\
\bigoplus_j \mathbb{Z} < a_j > & \xrightarrow{F} & \bigoplus_i \mathbb{Z} < b_i >
\end{array}
\]

Notice that because both \( \psi \) and \( Ab \) are surjective, so is the map \( F \). So we get a surjective homomorphism \( \mathbb{Z}^m \to \mathbb{Z}^n \). That is to say, \( m \geq n \).

Conversely, suppose \( m \geq n \). Then there is a natural surjection from \( F_m \to F_n \) by sending the first \( n \) generating element from \( F_m \) to the generating set of \( F_n \). Then by the 1st Isomorphism Theorem of groups, this surjection means \( F_m \) have a quotient group isomorphic to \( F_n \). \( \square \)

**Theorem 5.** For \( m \) and \( n \) non-negative integers, list all the index 3 subgroups of \( F_2 \) and indicate which ones are conjugates. For each one, give a list of its free generators.

**Proof.** Consider \( X = S^1 \vee S^1 \). By the **Theorem 11** the number of sheets of a path-connected covering space equals the index of \( p_* (\pi_1 (\tilde{X}, x_0)) \) in \( \pi_1 (X, x_0) \). The
following is all the connected 3-sheeted covering spaces of $S^1 \vee S^1$, up to iso-
morphism of covering spaces without basepoints. As shown below.

By the Theorem 1.38 on page 67, there exists an one to one corresponding be-
tween the isomorphism classes of path-connected covering spaces and conju-
gacy classes. So none of $p_*\langle \pi_1(\tilde{X}, \tilde{x}_0) \rangle$ for covering space $\tilde{X}$ above conjugates. □

**Theorem 6.** For $m$ and $n$ non-negative integers, does $F_2$ have any other rank four
subgroups other than the ones listed above? If so, are any of them normal?

**Proof.** Let's consider in two cases. At first suppose the subgroup has finite in-
dex, then by the Nielsen-Schreier formula (Theorem 15), the index of those sub-
groups has index 3. Then they are exactly $p_*\langle \pi_1(\tilde{X}, \tilde{x}_0) \rangle$ where $\tilde{X}$ are the covering
spaces above. Notice that as the figure above, there are four of them are normal
covering space (as the argument in part (3) for each $x \in X$ and each pair $\tilde{x}, \tilde{x}'$ in
the fiber of $x$, for that four covering space, there is a deck transformation taking
$\tilde{x}$ to $\tilde{x}'$ by the rotation map with feasible degree. Therefore, $\tilde{X}$ is a normal cov-
ering space. so by the Theorem 13 $p_*\langle \pi_1(\tilde{X}, \tilde{x}') \rangle$ is a normal subgroup). Next
consider the infinite index case. Consider the figure below.
It has rank four but it is infinitely sheeted, so it is infinite index. Notice that by Theorem 14, if $p_\ast(p_1(\bar{X}, \bar{x}))$ for that covering space above is a normal subgroup, then $p_\ast(p_1(\bar{X}, \bar{x}))$ cannot be finitely generated. That is to say, for all infinite index subgroup, it cannot be normal. □
APPENDIX A

AN INTRODUCTION TO FREE GROUPS AND COVERING SPACES

Let $S$ be a nonempty set, we denote $S^{-1}$ be any set disjoint from $S$ s.t. the cardinality are equal. Let’s denote $s \in S$ its corresponding element $s^{-1}$ in $S^{-1}$. Take a singleton set which is not contained in $S \cup S^{-1}$ (let’s denote it $\{e\}$). Then we define a word on $S$.

**Definition 1.** A word on $S$ is a sequence $(s_1, \cdots, s_n)$ where $s_i \in S \cup S^{-1} \cup \{e\}$ for $1 \leq i \leq n$. A reduced word is a word s.t.

1. $s_{i+1} \neq s_i^{-1}$ for all $i$ with $s_i \neq e$

2. If $s_k = e$ for some $k$, then the word is $e$.

So then we define the set of reduced words as $F(S)$.

**Theorem 7.** $F(S)$ is a group under concatenation. We define $F(S)$ a free group.

**Theorem 8.** (The Universal Property of free groups) Let $G$ be a group, with a map $\psi : S \rightarrow G$, then there is a unique group homomorphism $\Psi : F(S) \rightarrow G$ s.t. the following diagram commutes:

$$
\begin{array}{ccc}
S & \overset{\psi}{\longrightarrow} & F(S) \\
\downarrow & & \downarrow \Psi \\
G & & 
\end{array}
$$

**Corollary 1.** $F(S)$ is unique up to isomorphism.

**Theorem 9.** Every subgroup of a free group is free.

**Corollary 2.** If $F$ is a finitely generated free group and $N$ is a nontrivial normal subgroup of infinite index, then $N$ is not finitely generated.

Next we review some basic facts we need about covering spaces.
Definition 2. A covering space of a space $X$ is a space $\tilde{X}$ together with a map $p : \tilde{X} \to X$ s.t. for each $x \in X$, there exists an open neighborhood $U$ in $X$ s.t. $p^{-1}(U)$ is a union of disjoint open sets in $\tilde{X}$, each of which is mapped homeomorphically onto $U$ by $p$.

Theorem 10. The map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ induced by a covering space $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is injective. The image subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \in \pi_1(X, x_0)$ consists of the homotopy classes of loops in $X$ based at $x_0$ whose lifts to $X$ starting at $\tilde{x}_0$ are loops.

Theorem 11. The number of sheets of a covering space $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ with $X$ and $\tilde{X}$ path-connected equals the index of $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \in \pi_1(X, x_0)$.

Theorem 12. If $X$ is path-connected and locally path-connected, and semilocally simply-connected. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$, obtained by associating the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ to the covering space $(\tilde{X}, \tilde{x}_0)$. If basepoints are ignored, this correspondence gives a bijection between isomorphism classes of path-connected covering spaces $p : \tilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$.

Theorem 13. Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a path-connected covering space of the path-connected, locally path-connected space $X$, and let $H$ be the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subset \pi_1(X, x_0)$. Then this covering space is normal iff $H$ is a normal subgroup of $\pi_1(X, x_0)$.

The proofs of the above theorems can be found by [3], [1]. Overall, we will use that facts about free groups.
APPENDIX B

NIELSEN-SCHREIER THEOREM

Theorem 14. If $F$ is finitely generated free group and $N$ is a nontrivial normal subgroup of infinite index, then $N$ is not finitely generated.

Theorem 15. (Nielsen-Schreier theorem) If $G$ is a free group on $n$ generators, and $H$ is a subgroup of finite index $e$, then $H$ is free of rank $1 + e(n - 1)$.

Proof. Let’s prove it using algebraic topology again. By the theorem above, we have an $e$-sheeted (path-connected) covering space (denote it $\tilde{X}$) of $X = \bigvee_i S^1$.

Claim. For $X$ a finite CW complex and $p : \tilde{X} \to X$ an $e$-sheeted covering space, show that $\chi(\tilde{X}) = e\chi(X)$.

Proof. Consider the characteristic maps $\varphi_i : D^k \to X$, because $D^k$ is contractible, we can lift it by the lifting criterion. Because $\tilde{X}$ is $n$-sheeted. There are exactly $e$ liftings of $\varphi_i$ to $Y$. So for each $k$-cell $e^k$ in $X$, there exists $e$ $k$-cells in the lifted CW-structure of $\tilde{X}$. Now let $a_i$ be the number of $i$-cells of $\tilde{H}$, and $b_i$ be the number of $i$-cells in $X$. We have $a_i = e \cdot b_i$. And $\chi(\tilde{H}) = \sum_{i=0}^n (-1)^ia_i = \sum_{i=0}^n (-1)^i eb_i = e\chi(X)$. □

Let’s figure out $\chi(X)$. Notice that the CW complex construction of $X$ is 1 0-cell and $n$ 1-cells. So $\chi(X) = 1 - n$. On the other hand, let’s denote $\tilde{X}$ has 1 0-cell and $x$ 1-cells. Because $X$ is path-connected and orientable, so is it path-connected covering space $\tilde{X}$. So $H_1(\tilde{X})$ is free. Then by the Euler-Characteristic formula from [3], we have $\chi(\tilde{X}) = \sum_{i=0}^n (-1)^i \text{rank } H_i(\tilde{X}) = 1 - x$. That is to say $\chi(\tilde{X}) = e\chi(X) = e(1 - n) = 1 - x$. So $\text{rank } H_1(\tilde{X}) = x = 1 + e(n - 1)$. I.e., $H_1(\tilde{X}) = \mathbb{Z}^{1+e(n-1)}$. Notice that $H_1$ is an abelianization of the fundamental group. We have $\pi_1(\tilde{X}) = *_{1+e(n-1)} \mathbb{Z} = F_{1+e(n-1)}$. □
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