Which Regular Languages can be Efficiently Indexed?

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Abstract

Consider the problem of matching a pattern $P$ of length $\pi$ against the elements of a given regular language $L$ in the setting where $L$ can be pre-processed off-line in a fast data structure (an index). Regular expression matching is an ubiquitous problem in computer science, finding fundamental applications in areas including, but not limited to, natural language processing, search engines, compilers, and databases. Recent results have settled the exact complexity of this problem: $\Theta(\pi m)$ time is necessary and sufficient for indexed pattern matching queries, where $m$ is the size of an NFA recognizing $L$. This, however, does not mean that all regular languages are hard to index: for instance, for the sub-class of Wheeler languages [SODA’20] we can reduce query time to the optimal $O(\pi)$. A Wheeler language admits a total order of a finite refinement of its Myhill-Nerode equivalence classes reflecting the co-lexicographic order of their elements. This boosts indexing performance because classes whose elements are suffixed by $P$ form a range in this order. In [SODA’21], this technique was extended to arbitrary NFAs by allowing the order to be partial. This line of attack suggested that the width $p$ of such an order is the parameter ultimately capturing the fine-grained complexity of the problem: (i) indexed pattern matching can always be solved in $O(\pi p^2)$ time, (ii) the standard powerset construction algorithm always produces an output whose size is exponential in $p$ rather than in the input’s size, and (iii) $p$ even determines how succinctly NFAs can be encoded.

In the present work, we take a step further and study the hierarchy of $p$-sortable languages: regular languages accepted by automata of width $p$. Our main contributions are the following: (i) we show that the hierarchy is strict and does not collapse, (ii) we provide (exponential) upper and lower bounds relating the minimum widths of equivalent NFAs and DFAs, and (iii) we characterize DFAs of minimum $p$ for a given $L$ via a co-lexicographic variant of the Myhill-Nerode theorem. Our findings imply that in polynomial time we can build an index breaking the worst-case conditional lower bound of $\Omega(\pi m)$, whenever the input NFA’s width is at most $\epsilon \log_2 m$, for any constant $0 \leq \epsilon < 1/2$.

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1 Introduction

String indexing is the algorithmic problem of building a small data structure (an index) over a given string supporting fast substring search queries [15]. Building efficient string indexes is a challenging problem which finds important applications in several areas, notably bioinformatics [14] [13]. Lifting this problem to a regular collection $L$ of strings is an even more challenging problem and naturally calls into play finite state automata. As a matter of fact, Regular expression matching is an ubiquitous problem in computer science, finding fundamental applications in areas including, but not limited to, natural language processing,
search engines, compilers, and databases. When $\mathcal{L}$ is represented as an NFA (equivalently, a regular expression) of size $m$, existing on-line algorithms [3] solve the problem in $O(\pi m)$ time, $\pi$ being the length of the query pattern. Recent lower bounds by Backurs and Indyk [4], Equi et al. [3, 7], Potechin and Shallit [10], and Gibney [11] show that, unless important conjectures such as the Strong Exponential Time Hypothesis (SETH) [12] fail, this complexity cannot be significantly improved. This holds even in the off-line setting (the subject matter of our work) where $\mathcal{L}$ can be pre-processed in an index in polynomial time and the complexity is measured in terms of query times [9]. As pointed out by Backurs and Indyk [4], Gagie et al. [10], Alanko et al. [1, 2], and Cotumaccio and Prezza [5], however, this does not necessarily mean that all regular languages are hard to index.

Indeed, in [1, 2] we tackled the task of characterising regular languages admitting a direct generalization of known string indexing techniques—those accepted by so-called Wheeler automata introduced in [10]. More specifically, recalling that a state $q$ of an NFA can be seen as the collection $S_q$ of strings labeling the paths that connect the start state with $q$, we proved that Wheeler automata are those whose states $q$ (i) have $S_q$ that is a convex set $I_q$ in the co-lexicographically ordered set of strings read on the automaton’s paths, and (ii) are such that the family of these $I_q$ enjoys the so-called prefix/suffix property: the only way $I_q$ can be intersect another $I_{q'}$ is that a suffix of the former coincides with a prefix of the latter (or vice versa).

The co-lexicographic order over strings can be naturally lifted to the elements of a family of convex sets enjoying such property. In turn, this defines an order over the automaton’s states which enables pattern matching queries in optimal $O(\pi)$ time: states reached by a path labeled with a given string $P$ form an interval in this order [10].

Let $\text{Pref}(\mathcal{L}(A))$ denote the prefix closure of the language $\mathcal{L}(A)$ recognized by $A$. Given a finite-state automaton $A$ and representing all $\alpha \in \text{Pref}(\mathcal{L}(A))$ on a line where $\alpha \preceq \beta$ means that $\alpha$ is co-lexicographically smaller than or equal to $\beta$, we can depict a Wheeler automaton/language as follows:

![Figure 1](image-url)  

**Deterministic** Wheeler automata are simply those whose $I_q$’s have pairwise empty intersection. From the computational point of view, the above characterisation has the interesting consequence that turning a nondeterministic Wheeler automaton into a deterministic one takes polynomial time with the classic powerset construction algorithm, in contrast with the general (exponential) case of regular languages. In fact, we proved an even stronger property: any Wheeler NFA with $n$ states admits an equivalent Wheeler DFA with at most $2^n$ states.

Since, clearly, not all (interesting, regular) languages admit a Wheeler accepting automaton, the next natural question is: what if we want to index a general regular language? can we say something on the language’s propensity to be indexed? can we give directions/bounds on the complexity of such indexing task?

In this paper, elaborating on the idea put forward in [6], we prove that the above picture is a sort of one-dimensional version of a more general one. In this more general view, the set of the $I_q$’s is (always, for any automaton) partially ordered and all its elements end up in a
collection of \( p \) linearly ordered components, where \( p \) is the order’s width. In other words, we study the scenario in which we do not put any constraint on the language and prove that the picture becomes:

\[
\text{linear component 1} \\
\vdots \\
\text{linear component } p
\]

![Figure 2](image)

Figure 2 Family of convex sets \( I_q \) (on the linear order \((\text{Pref}(\mathcal{L}(A)), \prec))\) corresponding to the states of an arbitrary NFA. The family can be partitioned into \( p \) linear components, each enjoying the prefix/suffix property. However, the union of any two (distinct) components may not satisfy this property.

Looking at the projection of the language on a single linear component, we almost see a Wheeler language: its elements \( I_q \) form a prefix/suffix family. In fact, the picture is a bit more complex as transitions can let us move among different components. As it turns out, the order’s width \( p \) is a fundamental measure of NFA complexity [5]: (i) indexed pattern matching can always be solved in \( O(\pi p^2) \) time (the Wheeler case corresponding to \( p = 1 \)), (ii) the standard powerset construction algorithm always produces an output whose size is exponential in \( p \), rather than in the input’s size, and (iii) \( p \) even determines how succinctly NFAs can be encoded.

Within this framework, our main contribution is to begin the study of the hierarchy of \( p \)-sortable languages: regular languages accepted by automata of width \( p \) (for the minimum such \( p \)). In this hierarchy, regular languages are sorted according to the new fundamental measure of NFA complexity \( p \). More in detail:

1. We show that the hierarchy is strict and does not collapse: a \( p \)-sortable language exists for all \( p \geq 1 \).
2. We explore the effect that determinism has on the automaton’s width. We prove upper- and lower-bounds showing that determinism forces an exponentially-large \( p \) in the worst case.
3. We characterize DFAs of minimum \( p \) for a given regular language via a co-lexicographic variant of the Myhill-Nerode theorem. Interestingly, we show that the smallest minimum-width automaton is not unique.
4. We furthermore give a more general self-contained proof of the exponential dependency in \( p \) between the input and output sizes of the standard powerset construction algorithm.

Our findings have also important algorithmic consequences. For instance, let \( 0 < \epsilon < 1 \) be a constant and \( A \) be an NFA of size \( m \) and width at most \( p \leq (\epsilon/2) \log_2 m \). Contributions (2) and (4) — when combined with the polynomial-time DFA indexing algorithms of [5] — imply that in polynomial time we can index \( A \) so that pattern matching queries on \( \mathcal{L}(A) \) are solved in \( O(\pi m^\epsilon) \) time. This breaks asymptotically the conditional lower bound \( \Omega(\pi m^\epsilon) \) of Equi et al. [9] holding in the worst-case even when polynomial preprocessing time is allowed.

The paper is organized as follows: after giving some definitions and notations in Sections 2 and 3 in Section 4 we discuss the notion of \textit{width} of a regular languages and the two hierarchies—deterministic/nondeterministic—based on this notion. In Section 5 we prove a Myhill-Nerode theorem for each level of the deterministic hierarchy.

Due to limited space, most of the proofs can be found in the appendix.
2 Notation

2.1 Orders

We say that \((V, \leq)\) is a partial order if \(V\) is a set and \(\leq\) is a binary relation on \(V\) being reflexive, antisymmetric, and transitive. Any \(u, v \in V\) are said to be \(\leq\)-comparable if either \(u \leq v\) or \(v \leq u\). We write \(u < v\) when \(u \leq v\) and \(u \neq v\). We write \(u \parallel v\) if \(u\) and \(v\) are not \(\leq\)-comparable. Note that for every \(u, v \in V\) exactly one of the following hold true: (1) \(u = v\), (2) \(u < v\), (3) \(v < u\), (4) \(u \parallel v\). We say that \((V, \leq)\) is a total order if \((V, \leq)\) is a partial order and every pair on elements in \((V, \leq)\) are \(\leq\)-comparable. A subset \(Z \subseteq V\) is a \(\leq\)-chain if \((Z, \leq)\) is a total order, and a family \(\{V_i\}_{i=1}^{p}\) is a \(\leq\)-chain partition if \(\{V_i\}_{i=1}^{p}\) is a partition of \(V\) and each \(V_i\) is a \(\leq\)-chain. The width of \((V, \leq)\) is the smallest integer \(p\) for which there exists a chain partition \(\{Q_i\}_{i=1}^{p}\). We say that \(U \subseteq V\) is an \(\leq\)-antichain if every pair of elements in \(U\) are not \(\leq\)-comparable. Dilworth’s theorem \([6]\) states that the width of \((V, \leq)\) is the cardinality of a largest \(\leq\)-antichain. A subset \(C\) of a partial order \((V, \leq)\) is \(\leq\)-convex if for every \(u, v, z \in V\), if \(u, z \in C\) and \(u \leq v \leq z\), then \(v \in C\). If \(C\) is a finite \(\leq\)-convex set we call it an \(\leq\)-interval. If the order is deducible from the context, we drop the prefix \(\leq\). A monotone sequence in a partial order \((V, \leq)\) is a sequence \((v_n)_{n \in \mathbb{N}}\) with \(v_n \in V\) and either \(v_i \leq v_{i+1}\), for all \(i\), or \(v_i \geq v_{i+1}\), for all \(i\).

2.2 Automata

A nondeterministic finite automaton (NFA) is a 5-tuple \(A = (Q, s, \Sigma, \delta, F)\), where \(Q\) is the set of states, \(s\) is the initial state, \(\Sigma\) is the alphabet, \(\delta : Q \times \Sigma \to \text{Pow}(Q)\) is the transition function, and \(F \subseteq Q\) is the set of final states. An automaton \(A\) is deterministic (a DFA), if \(|\delta(q, a)| \leq 1\), for any \(q \in Q\) and \(a \in \Sigma\). As customary, we extend \(\delta\) to operate on strings as follows: for all \(q \in Q, a \in \Sigma\), and \(\alpha \in \Sigma^*:\)

\[
\delta(q, \epsilon) = \{q\}, \quad \delta(q, \alpha a) = \bigcup_{v \in \delta(q, \alpha)} \delta(v, a).
\]

We say that a state \(q'\) is reachable from a state \(q\) if there exists \(\alpha \in \Sigma^*\) with \(q' \in \delta(q, \alpha)\). If the automaton is deterministic we write \(\delta(q, \alpha) = q'\) for the unique \(q'\) such that \(\delta(q, \alpha) = \{q'\}\) (if defined). We denote by \(L(A) = \{\alpha \in \Sigma^* | \delta(s, \alpha) \cap F \neq \emptyset\}\).

Throughout this paper we assume that every NFA \(A\) satisfies the following properties: (1) \(L(A) \neq \emptyset\), (2) every state is reachable from the initial state, (3) every state is either final or it allows to reach a final state, (4) the initial state is not reachable from any other state, (5) if \(q \in \delta(q', \alpha) \cap \delta(q'', b)\) then \(a = b\) (input-consistency).

As a consequence of the above assumptions, for every state \(u \neq s\) there exists a unique letter \(\lambda(u)\) such that \(u \in \delta(q, \lambda(u))\), for some \(q\). For the initial state \(s\), we define \(\lambda(s) = \#\), where \(# \not\in \Sigma\). Moving labels from an edge to its target state, input-consistent automata can be described as state-labeled automata and this is what we do in our examples. It is also convenient to define an edge of an automaton as a triple \((u, v, a)\) with \(v \in \delta(u, a)\) (or simply \((u, v)\), since \(a = \lambda(v)\)) and denote the set of edges of the automaton as \(E_A\) — simply \(E\) when \(A\) is clear from the context.

The class of NFAs satisfying (1-5) is fully general when languages are concerned. In particular, it can be easily seen that any automaton can be converted into an input-consistent one recognizing the same language, possibly at the price of increasing \(|Q|\) by a multiplicative factor \(|\Sigma|\).

We assume that on the alphabet \(\Sigma\) there is a fixed, predetermined total order \(\preceq\). We also assume that \(# \prec a\) for every \(a \in \Sigma\).
If $A = (Q, s, \Sigma, \delta, F)$ is an NFA, we denote with $\text{Pref}(\mathcal{L}(A))$ the set of all prefixes of some word in $\mathcal{L}(A)$. For every $\alpha \in \text{Pref}(\mathcal{L}(A))$, let $I_\alpha = \{ q \in Q \mid \delta(s, \alpha) = q \}$. For every $q \in Q$, let $I_q = \{ \alpha \in \text{Pref}(\mathcal{L}(A)) \mid \delta(s, \alpha) = q \}$.

The Myhill-Nerode equivalence $\equiv_\Sigma$ for $\mathcal{L} \subseteq \Sigma^*$ is the equivalence relation on $\text{Pref}(\mathcal{L})$ such that for every $\alpha, \beta \in \text{Pref}(\mathcal{L})$:  

$$\alpha \equiv_\Sigma \beta \text{ if and only if } (\forall \gamma \in \Sigma^*)(\alpha\gamma \in \mathcal{L} \iff \beta\gamma \in \mathcal{L}).$$

## 3 Preliminary Definitions

In this section we recall basic definitions and results from [5]. We consider orders on the set of the automaton’s states reflecting the co-lexicographic order of the words spelling paths from the source. As stated in the previous section, we always refer to a fixed linear order $\prec$ on the alphabet $\Sigma$, co-lexicographically extended to words.

**Definition 1.** ([5, Def. 3.1]) Let $A = (Q, s, \Sigma, \delta, F)$ be an NFA. A co-lexicographic order on $A$ is a partial order $(Q, \leq)$ such that:

1. (Axiom 1) For every $u, v \in Q$, if $\lambda(u) \prec \lambda(v)$, then $u < v$ (hence $s \leq u$ for every $u \in U$).
2. (Axiom 2) For every $(u', u), (v', v) \in E_A$, if $\lambda(u) = \lambda(v)$ and $u < v$, then $u' \leq v'$.

Wheeler automata [10] are precisely those for which the order $\leq$ of Definition 1 is total. Since not all automata admit a Wheeler order, the totality requirement restricts the class of automata we are allowed to use when considering a language. On the other hand, if we drop the linearity requirement, we may consider the whole class of finite automata and use the width of the partial order (intuitively, the “distance” from being a linear order) to classify automata and the languages they accept (see Definitions 5 and 6).

The following lemma from [5] describes how a co-lexicographic order on the states of an automaton is related to the co-lexicographic order on the words that can be read on its paths.

**Lemma 2.** ([5, Lem. 3.1]) Let $A = (Q, s, \Sigma, \delta, F)$ be an NFA, and let $\leq$ be a co-lexicographic order on $A$. Let $u, v \in Q$ and $\alpha, \beta \in \text{Pref}(\mathcal{L}(A))$ be such that $u \in I_\alpha$, $v \in I_\beta$ and $\{u, v\} \not\subseteq I_\alpha \cap I_\beta$ (or equivalently, $\alpha \in I_u$, $\beta \in I_v$ and $\{\alpha, \beta\} \not\subseteq I_u \cap I_v$). Then:

1. If $\alpha \prec \beta$, then $u < v$ or $u \parallel v$.
2. If $u < v$, then $\alpha \prec \beta$.

Automata can be classified according to the width of their co-lexicographic orders:

**Definition 3.** ([5, Def. 3.3]) Let $A = (Q, s, \Sigma, \delta, F)$ be an NFA.

1. We say that $A$ is $p$-sortable if there exists a co-lexicographic order $\leq$ on $A$ such that $Q$ admits a $\leq$-chain partition $\{Q_i\}_{i=1}^p$.
2. The width of an automaton $A$, denoted by $\text{width}(A)$, is the smallest integer $p$ for which $A$ is $p$-sortable.

In Fig. 3 we present an NFA $A$ with $\text{width}(A) = 2$ (with the natural alphabetic order over the letters, except that $\#$ is the smallest). The set of states is partitioned in two classes $Q_1, Q_2$, where states in $Q_1$ are colored dark grey, states in $Q_2$ are colored light grey, and inside a class states are ordered from left to right. Notice that this NFA is not 1-sortable: let $q, q'$ be the states labeled by $x$ with $q$ above $q'$; then $ax, axx \in I_q$, $bx \in I_q \setminus I_q$ and, since $ax \prec bx \prec axx$, using Lemma 2 we see that the $q$ and $q'$ cannot be compared in any co-lexicographic order.
An NFA can have many co-lexicographic orders and the larger the order (that is, the larger the number of comparable state pairs is), the more insight we gain on the language it recognizes. In [5, Thm. 6.2] it was shown that DFAs admit a unique maximal co-lexicographic order, while for NFAs the maximal order is, in general, not unique. Here we give an explicit definition of the maximal co-lexicographic order of a DFA: two distinct states are comparable if and only if the strings reaching the former are co-lexicographically smaller than those reaching the latter. This order will be particularly important to obtain our results.

\[ \text{Lemma 4. Let } A = (Q, s, \Sigma, \delta, F) \text{ be a DFA. For every } u, v \in Q, \text{ let:} \]
\[ u \leq v \iff u = v \lor (\forall \alpha \in I_u)(\forall \beta \in I_v)(\alpha \prec \beta). \]

\[ \text{Then, } \leq \text{ is a co-lexicographic order. Moreover, for every co-lexicographic order } \leq' \text{ on } A \text{ and for every } u, v \in Q, \text{ if } u \leq' v, \text{ then } u \leq v. \text{ We say that } \leq \text{ is the maximal co-lexicographic order on } A. \]

It is well known that nondeterministic automata can be exponentially more succinct (as far as the number of states is concerned) than equivalent deterministic ones. However, for specific classes of automata, this gap can be considerably reduced. For example, in [1] it is proved that for nondeterministic Wheeler automata the gap is linear. This result is generalized in [5] as follows, substantiating the importance of the notion of automata’s width. In the appendix we give an independent, self-contained, and more general proof of the following bound:

\[ \text{Lemma 5. (}[5, \text{ Thm. 5.1]}\] Let \( A = (Q, s, \Sigma, \delta, F) \) be an NFA, with \( |Q| = n \) and width\( (A) = p \). Let \( A^* = (Q^*, E^*, \Sigma, s^*, F^*) \) be the powerset automaton of \( A \). Then, \( |Q^*| \leq 2^p(n - p + 1) - 1 \).

The above lemma states that the well-known exponential explosion of NFA determinization occurs in the width \( p \), rather than in the number \( n \) of states. This fact has important consequences to the study of efficient algorithms on automata: for example, it implies that the PSPACE-complete NFA equivalence problem is fixed-parameter tractable with respect to \( p \) (via powerset construction followed by DFA equivalence checking).

4 \hspace{1cm} \textbf{Width of Regular Languages}

On the grounds of the basic definition of automata’s co-lexicographic width, we start studying its implications for the theory of regular languages. In this section, we define the “width of a regular language”, based on co-lexicographic orders for the automata recognizing it.
We prove that the states which is very peculiar of Wheeler languages, as the gap from nondeterministic to deterministic width for regular languages is, in general, exponential. However, as we shall see, this is a characteristic which is very peculiar of Wheeler languages, as the gap from nondeterministic to deterministic width for regular languages is, in general, exponential.

In Lemma 3 we check that every level of both the above hierarchies is non-empty. To see this we shall use the following lemma:

**Lemma 7.** Let \( A = (Q, s, \Sigma, \delta, F) \) be an NFA. Assume that \( A \) contains a simple cycle of length \( m \) such that all edges of the cycle are equally labeled. Then, \( \text{width}(A) \geq m \).

**Proof.** Let \( e \in \Sigma \) and \( u_0, \ldots, u_{m-1} \) \( m \) pairwise distinct states such that:

\[
\delta(u_0, e) = u_1, \delta(u_1, e) = u_2, \ldots, \delta(u_{m-1}, e) = u_0.
\]

We prove that the states \( u_0, \ldots, u_{m-1} \) are pairwise incomparable in any co-lexicographic order \( \leq \) on \( A \). Suppose there are \( 0 \leq i < i + h \leq m - 1 \) such that \( u_i, u_{i+h} \) are comparable. Assume w.l.o.g. that \( u_i < u_{i+h} \); since the \( u_j \)'s are pairwise distinct, denoting by 

\[
[i - nh] < \cdots < u_{i-h} < u_i < u_{i+h}.
\]

Since there are exactly \( m \) possible elements modulo \( m \), there exist \( j < k \) with \( [i - jh] = [i - kh] \), a contradiction. Hence, the states \( u_0, \ldots, u_{m-1} \) are pairwise incomparable in any co-lexicographic order on \( A \). This proves that \( \text{width}(A) \geq m \).  

**Lemma 8.** For every integer \( m \geq 1 \), there exists \( L \) such that \( \text{width}^N(L) = \text{width}^D(L) = m \).

**Proof.** Define:

\[
L_m = \{ a^{km} | k \geq 0 \}.
\]

Notice that it will suffice to prove that \( L_m \) is recognized by a \( m \)-sortable DFA but it cannot be recognized by any \((m-1)\)-sortable NFA. Consider the DFA \( A_m \), having \( m+1 \) nodes \( s, q_1, \ldots, q_m \) (with \( q_m \) final), \( \delta(s, a) = q_1, \delta(q_i, a) = q_{i+1} \), for \( i = 1, \ldots, m-1 \), and \( \delta(q_m, a) = q_1 \), endowed with the maximal co-lexicographic order defined as in Lemma 4. Under this order the states \( q_1, \ldots, q_m \) are pairwise incomparable, so that \( A_m \) is \( m \)-sortable.

Next, consider any NFA \( A \) that recognizes \( L_m \), whose alphabet must be \( \Sigma = \{ a \} \). Since \( L_m \) is an infinite language any NFA \( A \) recognizing \( L_m \) must contain a simple cycle \( C \). Let \( u \) be any node in \( C \) and let \( a^c \) with \( c \geq 1 \) be the label of the cycle. If \( a^k, a^b \) are the labels of some paths from \( s \) to \( u \) and from \( u \) to a final state, respectively, we must have \( m|b \) and \( m|(b + k + c) \); hence \( c \) is a non-zero multiple of \( m \) and by Lemma 7 we conclude that \( A \) cannot be \((m-1)\)-sortable.

Clearly, for every regular language \( L \) we have \( \text{width}^N(L) \leq \text{width}^D(L) \). Moreover, for languages with \( \text{width}^N(L) = 1 \), the so called Wheeler languages, it is known that the nondeterministic and deterministic width coincide (see [1]). However, as we shall see, this is a characteristic which is very peculiar of Wheeler languages, as the gap from nondeterministic to deterministic width for regular languages is, in general, exponential.

In order to prove the exponential gap between nondeterministic and deterministic width, we first prove the following.
Which Regular Languages can be Efficiently Indexed?

Lemma 9. Let \( p_1, \ldots, p_k \) be distinct primes. Then, there exists a language \( \mathcal{L} \) such that \( \text{width}^D(\mathcal{L}) \geq \prod_{i=1}^{k} p_i \) and \( \text{width}^N(\mathcal{L}) \leq \sum_{i=1}^{k} p_i \).

Proof. Consider the language:
\[
\mathcal{L} = \{a^r \mid (\exists i \in \{1, \ldots, k\}) (p_i | r)\}.
\]

It is easy to explicitly define an NFA recognizing \( \mathcal{L} \) with \( n = 1 + \sum_{i=1}^{k} p_i \) states: the source state and \( k \) disjoint cycles, of length \( p_1, \ldots, p_k \), connected to the source state via an edge. Since the source state can always be compared to any other state, the width of this NFA is at most \( n - 1 \).

In order to prove that \( \text{width}^D(\mathcal{L}) \geq \prod_{i=1}^{k} p_i \), consider a DFA \( \mathcal{A} \) recognizing \( \mathcal{L} \). Any such DFA is composed by a line of states, \( \delta(s_0, a) = s_1, \delta(s_1, a) = s_2, \ldots, \delta(s_{k-1}, a) = s_k \), terminating in a cycle of length, say, \( \ell \). We prove that \( \prod_{i=1}^{k} p_i \) divides \( \ell \). Consider \( m \) such that the word \( a^{u_0} \) reaches a final state inside the cycle, where \( u_0 = (\prod_{i=1}^{k} p_i)^m \). If \( u_1 = u_0 + \ell \), the word \( a^{u_1} \) arrives to the same final state, hence it belongs to \( \mathcal{L} \); let \( p_i \) such that \( p_i \) divides \( u_1 \) (say, \( p_1 = p_1 \)). It follows that \( p_1 \) divides \( \ell \). Consider now \( m' \) such that the word \( a^{u_2} \) arrives to a final state of the cycle, where \( u_2 = (\prod_{i=2}^{k} p_i)^m' + \ell \). Hence, there exists \( p_i \) such that \( p_i \) divides \( u_2 \). Since \( p_1 \) divides \( \ell \) but not \( (\prod_{i=2}^{k} p_i)^m \), we have \( p_i \neq p_1 \), say, \( p_i = p_2 \). It follows that \( p_2 \) divides \( \ell \).

Iterating the above argument we obtain that \( \ell \) is divided by any \( p_i \), hence by their product. From Lemma 7 it then follows that \( \text{width}(\mathcal{A}) \geq \prod_{i=1}^{k} p_i \).

Since this is true for any DFA recognizing \( \mathcal{L} \), we conclude that \( \text{width}^D(\mathcal{L}) \geq \prod_{i=1}^{k} p_i \).

Our lower bound immediately follows:

Lemma 10. There exists a language \( \mathcal{L} \) such that \( \text{width}^D(\mathcal{L}) \geq e^{\sqrt{\text{width}^N(\mathcal{L})}} \).

Proof. Let \( p_1, \ldots, p_k \) be all primes no larger than a fixed \( n \). The primorial function grows asymptotically as \( \prod_{i=1}^{k} p_i = e^{(1+o(1))n} \geq e^n \) and the sum of the primes no larger than \( n \) grows asymptotically as \( \sum_{i=1}^{k} p_i \in O(n^2 / \log n) \) and it is, in fact, never larger than \( n^2 \). Now, consider the language \( \mathcal{L} \) of Lemma 9. Combining \( \text{width}^N(\mathcal{L}) \leq n^2 \) with \( \text{width}^D(\mathcal{L}) \geq e^n \), we obtain the claimed lower bound.

We now complement the above lower bound with an upper bound which can be proved by means of the usual powerset construction:

Lemma 11. Let \( \mathcal{A} \) be an NFA and let \( \mathcal{A}^* \) be the powerset automaton obtained from \( \mathcal{A} \). Then, \( \text{width}(\mathcal{A}^*) \leq 2^{\text{width}(\mathcal{A})} - 1 \).

From the above lemma we immediately get the following result.

Corollary 12. Let \( \mathcal{L} \) be a language. Then, \( \text{width}^D(\mathcal{L}) \leq 2^{\text{width}^N(\mathcal{L})} - 1 \).

Notice that, when \( \text{width}^N(\mathcal{L}) = 1 \), that is, when \( \mathcal{L} \) is Wheeler, we obtain that \( \text{width}^D(\mathcal{L}) = \text{width}^N(\mathcal{L}) \) (as already proved in 11).

An important consequence of the above bounds is that, in polynomial time, we can index an interesting subset of the regular languages (represented as NFAs) such that pattern matching query times break the indexability lower bound of Equi et al. 9 (holding in the worst-case even when polynomial preprocessing time is allowed):

Corollary 13. Let \( \mathcal{A} \) be an NFA with \( m \) edges and \( \text{width}(\mathcal{A}) \leq (e/2) \log_2 m \) for any constant \( 0 < \epsilon < 1 \). We can index \( \mathcal{A} \) in polynomial time so that pattern matching queries on \( \mathcal{L}(\mathcal{A}) \) are solved in \( O(pn^2) \) time, \( p \) being the pattern's length.
Proof. Let $\sigma$ be the alphabet’s size, $n$ be the number of states of $A$, and $p = \text{width}(A)$. First, note that $\sigma \leq m$ holds after a (polynomial-time) re-mapping of the alphabet to the interval $[1, m]$ so that all symbols label at least one edge. The first step is to run the powerset construction algorithm and build a DFA $A^*$ equivalent to $A$. An analysis of the powerset construction algorithm’s complexity [for example, see [5] Lem. 5.1], combined with Lemma 5, shows that in $O(2^p(n-p+1)n^2\sigma) = O(m^{1+\epsilon/2}n^3)$ time the powerset construction algorithm generates $A^*$, which has at most $\bar{n} \leq 2^p(n-p+1) \leq m^{\epsilon/2}n$ states and at most $\bar{m} \leq \bar{n} \cdot \sigma \leq m^{1+\epsilon/2}n$ edges. By Lemma 11, the width of $A^*$ is at most $\bar{p} < 2^p \leq m^{\epsilon/2}$. At this point, [5] Cor. 6.1 states that we can build the generalized FM-index [3] Thm. 4.2 of $A^*$ in $O(\bar{m}^2 + \bar{n}^{5/2}) = O((m^{1+\epsilon/2}n)^2 + (m^{\epsilon/2}n)^{5/2})$ time. This index supports pattern matching queries in $O(\pi \cdot \bar{p}^2 \cdot \log(\bar{p} \cdot \sigma)) = O(\pi m^\epsilon \cdot \log m)$ time, which can be made $O(\pi m^\epsilon)$ by infinitesimally adjusting $\epsilon$. All running times for building the index are polynomial in $m$ and $n$, that is, in the size of the input. □

Before Corollary 13, only the case $p = 1$ (Wheeler languages [10,2]) admitted a polynomial-time indexing strategy (by the indexing algorithms of Alanko et al. [1]) beating the indexability lower bound [9] of $\Omega(\pi m)$. Corollary 13 extends this result to a larger class of NFAs.

5 The co-lexicographic Myhill-Nerode theorem

The Myhill-Nerode Theorem for regular languages states that there is an exact correspondence between DFAs recognizing a regular language $L$ and right invariant equivalences on words with finite index, realizing $L$ as a union of classes. Moreover, among such equivalences, there is one realizing the DFA with minimum number of states recognizing $L$ and this automaton is unique up to isomorphism. As proved in [1], these results carry over to Wheeler languages and automata, so that we can speak about the minimum Wheeler DFA recognizing a Wheeler language. In this section we prove that the picture is more complex in the sortability context. First, we prove that, given a regular language $L$, it is not always the case that among the DFAs recognizing $L$ and having minimum width, there is a unique DFA with minimum number of states, up to isomorphism.

Lemma 14. There exists a regular language $L$ such that:
1. $\text{width}^N(L) = \text{width}^D(L) = 2$.
2. There exist two non-isomorphic DFAs $A$ and $B$ with the same number of states such that $L(A) = L(B) = L$, $\text{width}(A) = \text{width}(B) = 2$, and no 2-sortable DFA with fewer states recognizes $L$.

However, as we shall see in Corollary 25, if we fix the partition of $\text{Pref}(L)$ induced by a $p$-sortable DFA recognizing $L$, among all $p$-sortable DFA recognizing $L$ and inducing the same partition there is a unique DFA with minimum number of states. More generally, we will relate DFA sortability and convexity in equivalence relations, thereby deriving a co-lexicographic Myhill-Nerode theorem for regular languages.

First, let us recall the definition of right-invariant equivalence relation, which is at the root of the classical Myhill-Nerode theorem.

Definition 15. Let $L \subseteq \Sigma^*$ be a regular language and let $\sim$ be an equivalence relation on $\text{Pref}(L)$.
1. We say that $\sim$ respects $\text{Pref}(L)$ if:

\[(\forall \alpha, \beta \in \text{Pref}(L))(\forall \phi \in \Sigma^*)(\alpha \sim \beta \land \alpha \phi \in \text{Pref}(L) \rightarrow \beta \phi \in \text{Pref}(L)).\]
23:10 Which Regular Languages can be Efficiently Indexed?

2. If \( \sim \) respects \( \text{Pref}(\mathcal{L}) \), we say that \( \sim \) is right-invariant if for every \( \alpha, \beta \in \text{Pref}(\mathcal{L}) \) and for every \( \phi \in \Sigma^* \), if \( \alpha \sim \beta \) and \( \alpha \phi \in \text{Pref}(\mathcal{L}) \) (and so \( \beta \phi \in \text{Pref}(\mathcal{L}) \)), then \( \alpha \phi \sim \beta \phi \).

The next step is to focus on co-lexicographic equivalence relations that are consistent with respect to a given partition of \( \text{Pref}(\mathcal{L}) \). More specifically, with the following definitions we introduce equivalence relations that (1) refine the partition in a forward-stable manner (that is, by extending two equivalent words with any fixed word \( \phi \) we end up in the same partition’s class), and (2) whose classes form co-lexicographic convex sets.

**Definition 16.** Let \( \mathcal{L} \subseteq \Sigma^* \) be a regular language, and let \( \sim \) be an equivalence relation on \( \text{Pref}(\mathcal{L}) \). Let \( \mathcal{P} = \{U_1, \ldots, U_p\} \) be a partition of \( \text{Pref}(\mathcal{L}) \). For every \( \alpha \in \text{Pref}(\mathcal{L}) \), let \( U_\alpha \) be the unique element \( U_i \) of \( \mathcal{P} \) such that \( \alpha \in U_i \).

**Definition 17.** Let \( \mathcal{L} \subseteq \Sigma^* \) be a regular language. Let \( \mathcal{P} = \{U_1, \ldots, U_p\} \) be a partition of \( \text{Pref}(\mathcal{L}) \).

1. Let \( \sim \) be an equivalence relation on \( \text{Pref}(\mathcal{L}) \) that respects \( \text{Pref}(\mathcal{L}) \). We say that \( \sim \) is \( \mathcal{P} \)-consistent if for every \( \alpha, \beta \in \text{Pref}(\mathcal{L}) \) and for every \( \phi \in \Sigma^* \), if \( \alpha \sim \beta \) and \( \alpha \phi \in \text{Pref}(\mathcal{L}) \) (and so \( \beta \phi \in \text{Pref}(\mathcal{L}) \)), it holds \( U_{\alpha \phi} = U_{\beta \phi} \). Note that, in particular, \( U_\alpha = U_\beta \).

2. Let \( \sim \) be a \( \mathcal{P} \)-consistent equivalence relation on \( \text{Pref}(\mathcal{L}) \). We say that \( \sim \) is \( \mathcal{P} \)-convex if for every \( \alpha \in \text{Pref}(\mathcal{L}) \) we have that \([\alpha]_\sim \) is a convex set in \((U_\alpha, \Delta)\).

In the following, we will be interested in the coarsest equivalence relation refining a given \( \mathcal{P} \)-consistent equivalence relation.

**Lemma 18.** Let \( \mathcal{L} \subseteq \Sigma^* \) be a regular language, and let \( \mathcal{P} = \{U_1, \ldots, U_p\} \) be a partition of \( \text{Pref}(\mathcal{L}) \). Let \( \sim \) be an equivalence relation on \( \text{Pref}(\mathcal{L}) \) that respects \( \text{Pref}(\mathcal{L}) \). For every \( \alpha, \beta \in \text{Pref}(\mathcal{L}) \), define:

\[
\alpha \sim_\mathcal{P} \beta \iff (\alpha \sim \beta) \land (\forall \phi \in \Sigma)(\alpha \phi \in \text{Pref}(\mathcal{L}) \Rightarrow U_{\alpha \phi} = U_{\beta \phi}).
\]

Then, \( \sim_\mathcal{P} \) is the coarsest \( \mathcal{P} \)-consistent equivalence relation on \( \text{Pref}(\mathcal{L}) \) that refines \( \sim \). We say that \( \sim_\mathcal{P} \) is the \( \mathcal{P} \)-consistent refinement of \( \sim \).

At this point, we add to the previous definitions one additional ingredient: we embed our equivalence relation in the co-lexicographic order by requiring that the relation is also \( \mathcal{P} \)-convex.

**Lemma 19.** Let \( \mathcal{L} \subseteq \Sigma^* \) be a regular language, and let \( \mathcal{P} = \{U_1, \ldots, U_p\} \) be a partition of \( \text{Pref}(\mathcal{L}) \). Let \( \sim \) be a \( \mathcal{P} \)-consistent equivalence relation on \( \text{Pref}(\mathcal{L}) \). For every \( \alpha, \gamma \in \text{Pref}(\mathcal{L}) \), define:

\[
\alpha \sim^c \gamma \iff (\alpha \sim \gamma) \land \\
(\forall \beta, \phi \in \Sigma^*)(\alpha \phi, \beta \phi \in \text{Pref}(\mathcal{L})) \land (U_{\alpha \phi} = U_{\beta \phi}) \land (\min\{\alpha, \gamma\} \prec \beta \prec \max\{\alpha, \gamma\}) \Rightarrow \alpha \phi \sim \beta \phi).
\]

Then, \( \sim^c \) is a \( \mathcal{P} \)-consistent and \( \mathcal{P} \)-convex equivalence relation on \( \text{Pref}(\mathcal{L}) \) being the coarsest \( \mathcal{P} \)-convex equivalence relation on \( \text{Pref}(\mathcal{L}(A)) \) that refines \( \sim \). We say that \( \sim^c \) is the \( \mathcal{P} \)-convex refinement of \( \sim \).

**Definition 20.** Let \( \mathcal{L} \) be a regular language. Let \( \mathcal{P} = \{U_1, \ldots, U_p\} \) be a partition of \( V \).

1. We define \( \equiv_{\mathcal{L}, \mathcal{P}} \) to be the \( \mathcal{P} \)-consistent refinement of \( \equiv_{\mathcal{L}} \).

2. We define \( \equiv^c_{\mathcal{L}, \mathcal{P}} \) to be the \( \mathcal{P} \)-convex refinement of \( \equiv_{\mathcal{L}, \mathcal{P}} \).
The last steps consist in establishing a map from the linear components of an NFA recognizing \( L \) (i.e. the elements of a chain partition for one of its valid co-lexicographic orders) to subsets of \( \text{Pref}(L) \). First, we note that a chain partition of the NFA naturally induces a cover of \( \text{Pref}(L) \).

\[ \text{Definition 21.} \quad \text{Let } A = (Q, s, \Sigma, \delta, F) \text{ be an NFA, let } \preceq \text{ be a co-lexicographic order on } A \text{ and let } \{Q_i\}_{i=1}^{p} \text{ be a } \leq \text{-chain partition of } Q. \text{ For every } i \in \{1, \ldots, p\}, \text{ define:} \]

\[ \text{Pref}(L(A))^i = \{\alpha \in \text{Pref}(L(A)) \mid I_{\alpha} \cap Q_i \neq \emptyset\}. \]

\[ \text{Remark 22.} \quad \text{In general, } \{\text{Pref}(L(A))^i\}_{i=1}^{p} \text{ is not a partition of } \text{Pref}(L(A)), \text{ but just a cover of } \text{Pref}(L(A)), \text{ that is, } \cup_{i \in \{1, \ldots, p\}} \text{Pref}(L(A))^i = \text{Pref}(L(A)). \text{ Nonetheless, note that } \{\text{Pref}(L(A))^i\}_{i=1}^{p} \text{ is a partition of } \text{Pref}(L(A)) \text{ if } A \text{ is a DFA.} \]

Then, a \( \mathcal{P} \)-sortable NFA (for a given partition \( \mathcal{P} \) of \( \text{Pref}(L(A)) \)) is defined as one admitting a chain partition of its states that is mapped (in the sense of the above definition) to \( \mathcal{P} \).

\[ \text{Definition 23.} \quad \text{Let } A = (Q, s, \Sigma, \delta, F) \text{ be an NFA, and let } \mathcal{P} = \{U_1, \ldots, U_p\} \text{ be a partition of } \text{Pref}(L(A)). \text{ We say that } A \text{ is } \mathcal{P} \text{-sortable if there exists a co-lexicographic order } \leq \text{ on } A \text{ and a } \leq \text{-chain partition } \{Q_i\}_{i=1}^{p} \text{ such that for every } i \in \{1, \ldots, p\}: \]

\[ \text{Pref}(L(A))^i = U_i. \]

We can finally state our co-lexicographic extension of the Myhill-Nerode theorem.

\[ \text{Theorem 24 (Co-lexicographic Myhill-Nerode theorem).} \quad \text{Let } L \text{ be a language. Let } \mathcal{P} \text{ be a partition of } \text{Pref}(L). \text{ The following are equivalent:} \]

1. \( L \) is recognized by a \( \mathcal{P} \)-sortable NFA.
2. \( \equiv_{\preceq, \mathcal{P}} \) has finite index.
3. \( L \) is the union of some classes of a \( \mathcal{P} \)-convex, right invariant equivalence relation on \( \text{Pref}(L) \) of finite index.
4. \( L \) is recognized by a \( \mathcal{P} \)-sortable DFA.

By taking \( \mathcal{P} = \{\text{Pref}(L)\} \), the above theorem generalizes the one devised for the Wheeler case in [14, Thm 2.1]. In particular, the theorem implies once again that if \( \text{width}^N(L) = 1 \), then \( \text{width}^D(L) = 1 \). However, the above theorem tells us more: the reason why in general \( \text{width}^N(L) \) is strictly smaller than \( \text{width}^D(L) \) is that in general, given an NFA \( A \), a co-lexicographic order \( \leq \) on \( A \) and a \( \leq \)-chain partition \( \{Q_i\}_{i=1}^{p} \), we have that \( \{\text{Pref}(L(A))^i\}_{i=1}^{p} \) need not be a partition of \( \text{Pref}(L(A)) \). On the other hand, if in our NFA all equally-labeled edges leaving the same \( \leq \)-chain end in the same \( \leq \)-chain, then \( \{\text{Pref}(L(A))^i\}_{i=1}^{p} \) is a partition of \( \text{Pref}(L(A)) \). This means that the inequality \( \text{width}^N(L) < \text{width}^D(L) \) does not depend on the expressive power of nondeterminism itself, but it depends on the nondeterministic behaviour of the elements in the \( \leq \)-chain partition.

We notice that our characterization implies the existence of a unique canonical minimum \( \mathcal{P} \)-sortable DFA.

\[ \text{Corollary 25.} \quad \text{Let } L \text{ be a language. Let } \mathcal{P} \text{ be a partition of } \text{Pref}(L). \text{ If } L \text{ is recognized by some } \mathcal{P} \text{-sortable DFA, then there exists a } \mathcal{P} \text{-sortable DFA } A \text{ such that all } \mathcal{P} \text{-sortable DFAs recognizing } L \text{ and non-isomorphic to } A \text{ have a larger number of states. In other words, } A \text{ is the minimum } \mathcal{P} \text{-sortable DFA recognizing } L. \]
We can finally merge the results of the previous sections by observing that \( width^D(\mathcal{L}) \) is nothing but the cardinality of a (smallest) partition of \( Pref(\mathcal{L}) \) mapping to the linear components of a co-lexicographic order for some automaton recognizing \( \mathcal{L} \).

**Corollary 26.** Let \( \mathcal{L} \) be a language. The following are equivalent:
1. \( width^D(\mathcal{L}) = p \).
2. There exists a partition \( \mathcal{P} \) of \( Pref(\mathcal{L}) \) having cardinality \( p \) such that \( \equiv^L_{\mathcal{P}} \) has finite index.
3. There exists a partition \( \mathcal{P} \) of \( Pref(\mathcal{L}) \) having cardinality \( p \) such that \( \mathcal{L} \) is the union of some classes of a \( \mathcal{P} \)-convex, right invariant equivalence relation on \( Pref(\mathcal{L}) \) of finite index.

### 6 Conclusions and further developments

We proved that the concept of automaton’s width allows us to define two non collapsing hierarchies of regular language, and that the levels of such hierarchies are meaningful complexity measures. Although levels 1 in each of the two hierarchies denote the same class of languages, the Wheeler ones, where we can also find unique minimal automata modulo isomorphism \([1]\), we proved that this is no longer true for higher levels, where we have an exponential gap between the nondeterministic and the deterministic hierarchy. Moreover, we also proved that even the deterministic levels above level 1 lack the uniqueness of minimal automata. However, by fixing certain parameters (a partition of the prefixes of the language) we can retrieve an uniqueness result.

Our language-theoretic results find important applications to the study of regular expression matching algorithms: we showed that regular languages represented as NFAs in the low (logarithmic) levels of the nondeterministic hierarchy admit indexes supporting fast pattern matching queries.

In a paper in preparation we shall consider the following questions on the width notion.

1. Given a regular language \( \mathcal{L} \) (say, by giving its minimum DFA) can we calculate its width in an effective way? Notice that the width of the minimum DFA does not, in general, reflect the width of the language already at level one: there are Wheeler languages for which the minimum DFA is not Wheeler (see \([1]\)), so the question is not trivial.
2. As for other interesting subclasses of regular languages, Wheeler languages admit an automata free characterization: a language \( \mathcal{L} \) is Wheeler if and only if every monotone sequence in \((Pref(\mathcal{L}), \preceq)\) is “thin”, i.e. it ends definitely in at most one Myhill-Nerode class \([1]\). Can we find a similar characterization for languages of width \( p \), for \( p > 1 \)?
3. Is it possible to derive the width of a language directly from some combinatorial/graph-theoretical property of the minimum DFA accepting \( \mathcal{L} \)?
4. Our lower bound of Lemma \([10]\) and upper bound of Lemma \([11]\) do not match. Can we improve this result by providing tight bounds for the separation between the deterministic and nondeterministic hierarchies?

In addition, our work opens further intriguing questions of more algorithmic flavor. For instance, can we devise a fast algorithm that, given a DFA, outputs the equivalent DFA of minimum width? Can we extend our efficient indexing strategies to higher levels of the nondeterministic hierarchy? Can we prove conditional lower bounds for the regular expression matching problem as a function of the language’s width?
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A Proofs of Section 3

Lemma 4. Let $A = (Q, s, \Sigma, \delta, F)$ be a DFA. For every $u, v \in Q$, let:

$$u \leq v \iff u = v \lor (\forall \alpha \in I_u)(\forall \beta \in I_v)(\alpha < \beta).$$

Then, $\leq$ is a co-lexicographic order. Moreover, for every co-lexicographic order $\leq'$ on $A$ and for every $u, v \in Q$, if $u \leq' v$, then $u \leq v$. We say that $\leq$ is the maximal co-lexicographic order on $A$.

Proof. It is straightforward to check that $\leq$ is a partial order. Let us prove Axiom 1. If for some $u, v \in Q$ it holds $\lambda(u) < \lambda(v)$, then every string in $I_u$ ends with $\lambda(v)$ and every strings in $I_v$ ends with $\lambda(v)$, hence we conclude $u < v$ (in particular, this works for $u = s$ also). Let us prove Axiom 2. Consider two edges $(u', u), (v', v) \in E$ such that $\lambda(u) = \lambda(v)$ and $u < v$. We want to prove that $u' < v'$. Fix $\alpha' \in I_u$ and $\beta' \in I_v$; we must prove that $\alpha' < \beta'$. Let $c = \lambda(u) = \lambda(v)$. We have $\alpha'c \in I_u$ and $\beta'c \in I_v$, so from $u < v$ it follows $\alpha'c < \beta'c$ and so $\alpha' < \beta'$.

Finally, let us prove that $\leq$ is the maximal co-lexicographic order. Let $\leq'$ be a co-lexicographic order on $A$, and assume that $u \leq' v$; we must prove that $u < v$. Fix $\alpha \in I_u$ and $\beta \in I_v$; we must prove that $\alpha < \beta$. Since $I_u \cap I_v = \emptyset$ (being $A$ a DFA), the conclusion follows from lemma 2. □

In the remaining part of this section we give a proof of Lemma 5. We start with some general consideration on partitions on the domain of a binary relation.

Definition 27. Let $R$ be a binary relation on the set $V$, and let $\{V_i\}_{i=1}^p$ be a partition of the set $V$. We say that $\{V_i\}_{i=1}^p$ is a $R$-comparable partition of $V$ if for every $i \in \{1, \ldots, p\}$ and for every $x_1, x_2 \in V_i$, with $x_1 \neq x_2$ it holds $(x_1, x_2) \in R \lor (x_2, x_1) \in R$.

Remark 28. If $(V, \leq)$ is a partial order, then a partition of $V$ is a $\leq$-comparable partition of $V$ if and only it is a $\leq$-chain partition of $V$.

Lemma 29. Let $R$ be a binary relation on $V$, and let $U$ be a family of nonempty subsets of $V$. Let $S$ be a relation on $U$ such that:

$$(\exists x \in U_1)(\exists y \in U_2)((x, y) \not\subseteq U_1 \cup U_2 \land (x, y) \in R) \implies (U_1, U_2) \in S.$$ 

If $V$ admits a $R$-comparable partition of cardinality $p$, then $U$ admits a $S$-comparable partition of cardinality at most $2^p - 1$.

Proof. Let $\{V_i\}_{i=1}^p$ be a $R$-comparable partition of $V$ of cardinality $p$. For every nonempty set $K \subseteq \{1, \ldots, p\}$, define:

$$U_K = \{U \in U \mid (\forall i \in \{1, \ldots, k\})(U \cap V_i \neq \emptyset \iff i \in K)\}.$$ 

Notice that every $U \in U$ belongs to exactly one $U_K$, so $\{U_K \mid \emptyset \not\subseteq K \subseteq \{1, \ldots, p\}, U_K \neq \emptyset\}$ is a partition of $U$ having cardinality at most $2^p - 1$. As a consequence, we are just left with proving that each nonempty $U_K$ consists of $S$-comparable elements. Fix $U_1, U_2 \in U_K$, with $U_1 \neq U_2$. We must prove that $U_1$ and $U_2$ are $S$-comparable. Since $U_1 \neq U_2$, there exists either $x \in U_1 \setminus U_2$ or $y \in U_2 \setminus U_1$. Assume that there exists $x \in U_1 \setminus U_2$ (the other case is analogous). In particular, let $i \in \{1, \ldots, p\}$ be the unique integer such that $x \in V_i$. Since $U_1, U_2 \in U_K$, from the definition of $U_K$ it follows that there exists $y \in U_2 \cap V_i$. Notice that $\{x, y\} \not\subseteq U_1 \cap U_2$ (so in particular $x \neq y$), and since $x, y \in V_i$ we conclude that $x$ and $y$ are $R$-comparable. If $(x, y) \in R$, we conclude $(U_1, U_2) \in S$, and if $(y, x) \in R$, we conclude $(U_2, U_1) \in S$, so $U_1$ and $U_2$ are $S$-comparable. □
Lemma 30. Let \((V, \leq)\) be a finite partial order, with \(n = |V|\), and let \(U\) be a family of nonempty subsets of \(V\). Let \(S\) be an antisymmetric relation on \(U\) such that:

\[(\exists x \in U_1)(\exists y \in U_2)((x, y) \not\in U_1 \cap U_2 \land x < y) \implies (U_1, U_2) \in S.\]

If \((V, \leq)\) admits an \(\leq\)-chain partition of cardinality \(p\), then \(|U| \leq 2^p(n - p + 1) - 1\).

Proof. From lemma 29 we know that:

\[|U| = \sum_{\#K \leq 1} |U_K|.\]

Moreover, from lemma 29 we also know for every \(U_1, U_2 \in U_K\), with \(U_1 \neq U_2\), we have that at least one between \((U_1, U_2) \in S\) and \((U_2, U_1) \in S\) holds true. Since \(S\) is antisymmetric, we conclude that for every \(U_1, U_2 \in U_K\), with \(U_1 \neq U_2\), exactly one between \((U_1, U_2) \in S\) and \((U_2, U_1) \in S\) holds true.

Fix \(K\). For every \(U \in U_K\) and for every \(i \in K\), let \(m_i\) be the smallest element of \(U \cap V_i\) (this makes sense because \((V_i, \leq)\) is totally ordered) and let \(|m_i|\) be the position of \(m_i\) in the total order \((V_i, \leq)\) (in particular, this means that \(|m_i| \in \{1, \ldots, |V_i|\}\)). Similarly, let \(M_i\) be the largest element of \(U \cap V_i\) and let \(|M_i|\) be the position of \(M_i\) in the total order \((V_i, \leq)\).

Fix \(U_1, U_2 \in U_K\), with \(U_1 \neq U_2\). Note the following:

1. Assume that for some \(i \in K\) it holds \(m_i \leq M_i\). Then, it must be \((U_1, U_2) \in S\). Indeed, assume that \(m_i \leq M_i\) (the other case is analogous). Then, we have \(m_i \in U_1\), \(m_i \in U_2\), \(\{m_i, M_i\} \subseteq U_1 \cap U_2\) and \(m_i < M_i\), so we conclude \((U_1, U_2) \in S\). Notice that we can equivalently state that if \((U_1, U_2) \in S\), then \((\forall i \in K)(|m_i| \leq |M_i| \land |M_i| < |M_i|)\).

2. Assume that for some \(i \in K\) it holds \(|m_i| \leq |M_i| \land |M_i| = |M_i|\). Then, it must be \(U_1 \cap V_i = U_2 \cap V_i\). Indeed, suppose by contradiction that \(U_1 \cap V_i \neq U_2 \cap V_i\). This means that there exists \(x \in (U_1 \cap V_i) \setminus U_2\) or \(y \in (U_2 \cap V_i) \setminus U_1\). Assume that there exists \(x \in (U_1 \cap V_i) \setminus U_2\) (the other case is analogous). Notice that \(m_i = m_i \land M_i = M_i\), so it must be \(m_i < x < M_i = M_i\). Now, we have \(x \in U_1\), \(m_i \in U_2\), \(\{x, m_i\} \not\subseteq U_1 \cup U_2\) and \(m_i < x\), so \((U_2, U_1) \in S\). On the other hand, we have \(x \in U_1\), \(M_i \in U_2\), \(\{x, M_i\} \not\subseteq U_1 \cup U_2\) and \(x < M_i\), so \((U_1, U_2) \in S\), a contradiction.

3. Assume that \((\forall i \in K)(|m_i| \not\leq |M_i| \lor |M_i| = |M_i|)\). Then, it must be \(U_1 = U_2\).

Indeed, from point 2 we obtain \((\forall i \in K)(U_1 \cap V_i = U_2 \cap V_i)\), so \(U_1 = \cup_{i \in K}(U_1 \cap V_i) = U_2\). Notice that we can equivalently state that if \(U_1 \neq U_2\), then \((\exists i \in K)(|m_i| \not\leq |M_i| \lor |M_i| = |M_i|)\).

Now it is easy to show that:

\[(U_1, U_2) \in S \iff (\forall i \in K)(|m_i| \leq |M_i| \land |M_i| < |M_i|)\land
\&(\exists i \in K)(|m_i| < |M_i| \lor |M_i| < |M_i|).\]

Indeed, \((\iff)\) follows from point 1. As for \((\implies)\), notice that \((\forall i \in K)(|m_i| \leq |M_i| \land |M_i| < |M_i|)\) again follows from point 1, whereas \((\exists i \in K)(|m_i| < |M_i| \lor |M_i| < |M_i|)\) follows from point 3.

For every \(U \in U_K\), define:

\[T(U) = \sum_{i \in K}(|m_i| + |M_i|).\]
Pick any $U_1, U_2 \in \mathcal{U}_K$, with $U_1 \neq U_2$. By equation 2 if $(U_1, U_2) \in S$, then $T(U_1) < T(U_2)$, and if $(U_2, U_1) \in S$, then $T(U_2) < T(U_1)$. In particular, we always have $T(U_1) \neq T(U_2)$. Since for every $U \in \mathcal{U}_K$, we have $2|K| < T(U) < 2 \sum_{v \in K} |V|$, then:

$$|U_K| \leq 2 \sum_{v \in K} |V| - 2|K| + 1.$$ \hspace{1cm} (3)

From equations 1 and 3, we obtain:

$$|U| \leq \sum_{\emptyset \neq K \subseteq \{1, \ldots, p\}} (2 \sum_{v \in K} |V| - 2|K| + 1) = 2 \sum_{\emptyset \neq K \subseteq \{1, \ldots, p\}} \sum_{v \in K} |V| - 2 \sum_{\emptyset \neq K \subseteq \{1, \ldots, p\}} |K| + \sum_{\emptyset \neq K \subseteq \{1, \ldots, p\}} 1.$$

Notice that $\sum_{\emptyset \neq K \subseteq \{1, \ldots, p\}} \sum_{v \in K} |V| = 2^{p-1} \sum_{i \in \{1, \ldots, p\}} |V_i| = 2^{p-1} n$ because every $i \in \{1, \ldots, p\}$ occurs in exactly $2^{p-1}$ subsets of $\{1, \ldots, p\}$. Similarly, we obtain $\sum_{\emptyset \neq K \subseteq \{1, \ldots, p\}} |K| = 2^{p-1} p$ and $\sum_{\emptyset \neq K \subseteq \{1, \ldots, p\}} 1 = 2^p - 1$. We conclude:

$$|U| \leq 2^n p - 2^{p+1} + 2^p - 1 = 2^p n - p + 1 - 1.$$ 

Let $A = (Q, s, \Sigma, \delta, F)$ be an NFA. Recall that the powerset automaton $A^* = (Q^*, E^*, \Sigma, s^*, F^*)$ of $A$ is the DFA defined as follows: (i) $Q^* = \{ I_\alpha \mid \alpha \in \text{Pref} (\mathcal{L}(A)) \}$, (ii) $E^* = \{ (I_\alpha, I_\alpha, e) \mid \alpha \in \Sigma^*, e \in \Sigma, \alpha e \in \text{Pref} (\mathcal{L}(A)) \}$, (iii) $s^* = \{ s \}$, and (iv) $F^* = \{ I_\alpha \mid \alpha \in \mathcal{L}(A) \}$. In particular, $\mathcal{L}(A^*) = \mathcal{L}(A)$.

**Lemma 5** Let $A = (Q, s, \Sigma, \delta, F)$ be an NFA, with $|Q| = n$ and $\text{width}(A) = p$. Let $A^* = (Q^*, E^*, \Sigma, s^*, F^*)$ be the powerset automaton of $A$. Then, $|Q^*| \leq 2^p (n - p + 1) - 1$.

**Proof.** Let $\leq$ be a co-lexicographic order on $A$ such that there exists a $\leq$-chain partition of cardinality $p$. Let $\leq^*$ be the maximal co-lexicographic order on $A^*$, that is, for every $I_\alpha, I_\beta \in Q^*$:

$$I_\alpha <^* I_\beta \iff (\forall \alpha', \beta' \in \text{Pref} (\mathcal{L}(A))) ((I_\alpha, = I_\alpha) \land (I_\beta, = I_\beta) \rightarrow \alpha' \prec \beta').$$

We check that the condition of Lemma 30 are satisfied with $V = Q$, $U = \{ I_\alpha \mid \alpha \in \text{Pref} (\mathcal{L}(A)) \}$, and $\leq$, $S = \leq^*$ as above. Fix $I_\alpha, I_\beta \in Q^*$, and assume that there exist $u \in I_\alpha$ and $v \in I_\beta$ such that $\{ u, v \} \not\subseteq I_\alpha \cap I_\beta$ and $u < v$. We want to prove that $I_\alpha <^* I_\beta$. Fix any $\alpha', \beta' \in \text{Pref} (\mathcal{L}(A))$ such that $I_\alpha = I_\alpha$ and $I_\beta = I_\beta$. From Lemma 2 we conclude $\alpha' \prec \beta'$, as desired. This means that we can apply Lemma 30 so the conclusion follows.

**B Proofs of Section 4**

**Lemma 11** Let $A$ be an NFA and let $A^*$ be the powerset automaton of $A$. Then, $\text{width}(A^*) \leq 2^{\text{width}(A)} - 1$.

**Proof.** The proof is analogous to that of lemma 5, the only difference is that now we must use lemma 29 instead of lemma 30.

$\blacktriangle$
 Lemma 14. There exists a regular language $L$ such that
1. $\text{width}^N(L) = \text{width}^D(L) = 2$.
2. There exist two non-isomorphic DFAs $A_1$ and $A_2$ with the same number of states such that $L(A_1) = L(A_2) = L$, $\text{width}(A_1) = \text{width}(A_2) = 2$ and no 2-sortable DFA with fewer states recognizes $L$.

Proof. In the proof we shall use the following remark:

**Remark 31.** A partition $Q_1, \ldots, Q_p$ of the states of a DFA $A = (Q, s, \Sigma, \delta, F)$ is a chain partition w.r.t. the maximal co-lexicographic order if and only if the following holds: for any pair $q, q'$ of distinct states belonging to the same component of the partition, if $\lambda(q) = \lambda(q')$ then either $I_q \prec I_{q'}$ or $I_{q'} \prec I_q$.

Consider the regular language $L$ accepted by the (input-consistent) automaton $A$ in Figure 4 where the initial state is labelled by $\#$, all states are final, and the letters in $\Sigma$ are ordered alphabetically. Notice that the automaton $A$ is the minimum automaton for the language accepted (one can easily check that for each pair of states there is a word which is readable from one state and not from the other). Using [2, Thm. 3.1] we easily see that $L$ is not Wheeler, since the infinite monotone sequence

$$ac < bc < acc < bcc < acc << \ldots < ac^n < bc^n \prec \ldots$$

flips indefinitely through two different states of the minimum automaton. It follows that the width (deterministic and nondeterministic) of $L$ is greater than one.

![Figure 4](image-url) The minimum automaton for $L$.

Let us check that there is no 2-sortable automaton with just one state more than the minimum automaton recognizing the same language. By the Myhill-Nerode Theorem for regular languages we know that any automaton recognizing a language can be obtained from the minimum automaton by “dividing” some states. Let $q_1, q_2, q_3, q_4$ be the nodes of the minimum automaton labelled by $d$, from left to right. Then

$$I_{q_1} = \{d, fd\}, \ I_{q_2} = \{ed, ffd\}, \ I_{q_3} = \{eed, fffd\}, \ I_{q_4} = \{ceed\}.$$
Notice that the states $q_1, q_2, q_3$ are pairwise incomparable with respect to the maximal co-lexicographic order. Hence, one among $q_1, q_2, q_3$ should be “divided” if we want to obtain a 2-sortable automaton. However, if we divide just one of these states, the remaining two and the state $q_4$ would still be reached by the same words as in the minimum automaton and would be still pairwise incomparable w.r.t. the maximal co-lexicographic order of the DFA.

It follows that a two sortable automaton recognizing the language should have at least 2 states more than the minimum automaton.

We next show that we can indeed realize 2-sortability by dividing two states of the minimum automaton (in two different ways).

Consider the automata in figures 5 and 6; these automata recognize the language $L(A)$, and they are 2-sortable, because the maximal co-lexicographic order among their states is so. Indeed, using Remark 31, we can easily check the 2-sortability of the automaton; in the pictures, some states are colored either with light gray or dark gray depending on the component to which they belong (the non colored states can be easily assigned to one of the two components in such a way that Axiom 2 holds true). The reader is invited to check that, in both automata, all relevant states are colored and that states with the same color can be linearly ordered by the maximal co-lexicographic ordering.

![Figure 5](image-url) A 2-sortable automaton recognizing $L$.

To finish the proof of the lemma, notice that both automata have two states more than the minimum automaton, but they are not isomorphic.

We now state and prove some lemmas preliminary to our co-lexicographic extension of Myhill-Nerode theorem (Thm. 24).

**Lemma 18.** Let $\mathcal{L} \subseteq \Sigma^*$ be a regular language, and let $\mathcal{P} := \{U_1, \ldots, U_p\}$ be a partition of $\text{Pref}(\mathcal{L})$. Let $\sim$ be an equivalence relation on $\text{Pref}(\mathcal{L})$ that respects $\text{Pref}(\mathcal{L})$. For every $\alpha, \beta \in \text{Pref}(\mathcal{L})$, define:

$$\alpha \sim_\mathcal{P} \beta \iff (\alpha \sim \beta) \land (∀\phi ∈ \Sigma)(\alpha\phi ∈ \text{Pref}(\mathcal{L}) \rightarrow U_{\alpha\phi} = U_{\beta\phi}).$$

Then, $\sim_\mathcal{P}$ is the coarsest $\mathcal{P}$-consistent equivalence relation on $\text{Pref}(\mathcal{L})$ that refines $\sim$. We say that $\sim_\mathcal{P}$ is the $\mathcal{P}$-consistent refinement of $\sim$. 

Then, \( \sim \) must also refine \( \sim_R \), because for every refinement \( \mathcal{P} \) of \( \sim \) we have that \( \mathcal{P} \) is an equivalence relation.

It is immediate to check that every \( \mathcal{P} \)-consistent equivalence relation on \( \text{Pref}(\mathcal{L}) \) that refines \( \sim \) must also refine \( \sim_R \), so we only have to show check that \( \sim_R \) is \( \mathcal{P} \)-consistent.

Assume that \( \alpha, \beta \in \text{Pref}(\mathcal{L}) \) satisfy \( \alpha \sim_R \beta \), and assume that \( \alpha \phi \in \text{Pref}(\mathcal{L}) \). Then, \( \beta \phi \in \text{Pref}(\mathcal{L}) \) follows from the assumptions on \( \sim \), and \( U_{\alpha \phi} = U_{\beta \phi} \) follows from \( \alpha \sim \beta \). □

**Lemma 19.** Let \( \mathcal{L} \subseteq \Sigma^* \) be a regular language, and let \( \mathcal{P} = \{ U_1, \ldots, U_p \} \) be a partition of \( V \). Let \( \sim \) be a \( \mathcal{P} \)-consistent equivalence relation on \( \text{Pref}(\mathcal{L}) \). For every \( \alpha, \gamma \in \text{Pref}(\mathcal{L}) \), define:

\[
\alpha \sim^c \gamma \iff (\alpha \sim \gamma) \land \\
\land (\forall \beta, \phi \in \Sigma^*)((\alpha \phi, \beta \phi \in \text{Pref}(\mathcal{L})) \land (U_{\alpha \phi} = U_{\beta \phi}) \land (\min\{\alpha, \gamma\} < \beta < \max\{\alpha, \gamma\}) \rightarrow \alpha \phi \sim \beta \phi).
\]

Then, \( \sim^c \) is a \( \mathcal{P} \)-consistent and \( \mathcal{P} \)-convex equivalence relation on \( \text{Pref}(\mathcal{L}) \) being the coarsest \( \mathcal{P} \)-consistent equivalence relation on \( \text{Pref}(\mathcal{L}(A)) \) that refines \( \sim \). We say that \( \sim^c \) is the \( \mathcal{P} \)-convex refinement of \( \sim \).

**Proof.** Let us prove that \( \sim^c \) is an equivalence relation. First, notice that \( \sim^c \) is reflexive, because for every \( \alpha \in \text{Pref}(\mathcal{L}) \) we have that \( \min\{\alpha, \alpha\} = \max\{\alpha, \alpha\} = \alpha \), so one immediately obtains \( \alpha \sim^c \alpha \). Now, let us prove that \( \sim^c \) is symmetric. Assume that \( \alpha \sim^c \gamma \); we want to prove that \( \gamma \sim^c \alpha \). To this end, it suffices to observe that if \( \gamma, \beta \in \Sigma^* \) satisfy \( \gamma \phi, \beta \phi \in \text{Pref}(\mathcal{L}) \) and \( U_{\gamma \phi} = U_{\beta \phi} \), then we obtain that \( \alpha \phi \sim \beta \phi \) because \( \sim \) is \( \mathcal{P} \)-consistent. Let us prove that \( \sim^c \) is transitive. Assume that \( \alpha \sim^c \gamma \) and \( \gamma \sim^c \delta \). We want to prove that \( \alpha \sim^c \delta \). First, \( \alpha \sim \delta \) follows from the transitivity of \( \sim \). Now, assume that \( \beta, \phi \in \Sigma^* \) satisfy \( \alpha \phi, \beta \phi \in \text{Pref}(\mathcal{L}) \) and \( U_{\alpha \phi} = U_{\beta \phi} \) and \( \min\{\alpha, \delta\} < \beta < \max\{\alpha, \delta\} \). We must prove that \( \alpha \phi \sim \beta \phi \). Since \( \sim \) is \( \mathcal{P} \)-consistent, we obtain that \( \alpha \phi, \beta \phi, \gamma \phi, \delta \phi \in \text{Pref}(\mathcal{L}) \) and \( U_{\alpha \phi} = U_{\beta \phi} = U_{\gamma \phi} = U_{\delta \phi} \). Notice that either \( \min\{\alpha, \gamma\} < \beta < \max\{\alpha, \gamma\} \) or \( \min\{\alpha, \delta\} < \beta < \max\{\gamma, \delta\} \), so the conclusion follows from either \( \alpha \sim^c \gamma \) or \( \gamma \sim^c \delta \).

First, observe that \( \sim^c \) is \( \mathcal{P} \)-consistent because it is a refinement of \( \sim \), which is \( \mathcal{P} \)-consistent.
Notice that every \(P\)-convex equivalence relation on \(\text{Pref}(L)\) that refines \(\sim^c\) must also refine \(\sim^e\). As a consequence, we only have to prove that \(\sim^e\) is \(P\)-convex.

Fix \(\alpha \in \text{Pref}(L)\). We must prove that \([\alpha]_{\sim^e}\) is a convex set in \((U_\alpha, \leq)\). First, notice that \([\alpha]_{\sim^e} \subseteq U_\alpha\) because \(\sim^e\) is \(P\)-consistent. Now, fix \(\beta, \gamma, \delta \in U_\alpha\) such that \(\beta \prec \gamma \prec \delta\) and \(\beta, \delta \in [\alpha]_{\sim^e}\). We must prove that \(\gamma \in [\alpha]_{\sim^e}\). In other words, starting from \(U_\beta = U_\gamma = U_\delta\), \(\beta \prec \gamma \prec \delta\) and \(\beta \sim^e \delta\) we must prove that \(\beta \sim^e \gamma\). First, notice that \(\beta \sim^e \gamma\) follows from \(\beta \sim^e \delta\). Now, fix \(\beta', \phi \in \Sigma^*\) such that \(\beta \phi, \beta' \phi \in \Sigma^*, U_{\beta \phi} = U_{\beta' \phi}\) and \(\beta \prec \beta' \prec \gamma\) (and so \(\beta \prec \beta' \prec \delta\)). Then, \(\beta \phi \sim^e \beta' \phi\) follows once again from \(\beta \sim^e \delta\).

From lemmas \[18\] and \[19\] we immediately obtain the following corollary.

\textbf{Corollary 32.} Let \(L \subseteq \Sigma^*\) be a regular language, and let \(P = \{U_1, \ldots, U_p\}\) be a partition of \(\text{Pref}(L)\). Let \(\sim\) be an equivalence relation on \(\text{Pref}(L)\) that respects \(\text{Pref}(L)\). Let \(\sim_P\) be the \(P\)-consistent refinement of \(\sim\) and let \(\sim^c_P\) be the \(P\)-convex refinement of \(\sim\). Then, \(\sim^c_P\) is the coarsest refinement of \(\sim\) being both \(P\)-consistent and \(P\)-convex.

Notice that both in the definition of \(P\)-consistent refinement (lemma \[18\]) and in the definition of \(P\)-convex refinement (lemma \[19\]) we have introduced a string \(\phi \in \Sigma^*\) that "propagates" the considered properties. The intuition is that we want to ensure that right-invariance is preserved (basically, we are defining the right-invariant refinement). More precisely, we have the following corollary, which is easily proved.

\textbf{Corollary 33.} The \(P\)-consistent refinement of a right-invariant equivalence relation is right-invariant. The \(P\)-convex refinement of a right-invariant equivalence relation is right-invariant.

The following definition interprets the requirement that automata must be input-consistent from an equivalence relation perspective.

\textbf{Definition 34.} Let \(L \subseteq \Sigma^*\) be a regular language, and let \(\sim\) be an equivalence relation on \(\text{Pref}(L)\). We say that \(\sim\) is input-consistent if for every \(\alpha, \beta \in \text{Pref}(L)\), if \(\alpha \sim \beta\), then \(\text{end}(\alpha) = \text{end}(\beta)\). Note that \([e]_{\sim} = \{e\}\).

It is immediate to derive the following lemma.

\textbf{Lemma 35.} Let \(L \subseteq \Sigma^*\) be a regular language, and let \(\sim\) be an equivalence relation on \(\text{Pref}(L)\). For every \(\alpha, \beta \in \text{Pref}(L)\), define:

\[
\alpha \sim^* \beta \iff (\alpha \sim \beta) \land (\text{end}(\alpha) = \text{end}(\beta)).
\]

Then, \(\sim^*\) is the coarsest input-consistent equivalence relation on \(\text{Pref}(L)\) refining \(\sim\). We say that \(\sim^*\) is the input-consistent refinement of \(\sim\).

\textbf{Remark 36.} Assume that \(\sim\) has some of the following properties: finite index, right-invariance, \(P\)-convexity (for some partition \(P\) of \(\text{Pref}(L)\)). Then, it is easy to check that these properties are inherited by \(\sim^*\).

Let \(\mathcal{A} = (Q, s, \Sigma, \delta, F)\) be an NFA, let \(\leq\) be a co-lexicographic order on \(\mathcal{A}\) and let \(\{Q_i\}_{i=1}^p\) be a \(\leq\)-chain partition of \(Q\). For every \(\alpha \in \text{Pref}(\mathcal{L}(\mathcal{A}))\) and for every \(i \in \{1, \ldots, p\}\), we define:

\[
I^i_\alpha = I_\alpha \cap Q_i.
\]

In particular, we can restate definition \[21\] as follows:

\[
\text{Pref}(\mathcal{L}(\mathcal{A}))^i = \{\alpha \in \text{Pref}(\mathcal{L}(\mathcal{A})) \mid I^i_\alpha \neq \emptyset\}.
\]
Note that for every $\alpha \in \text{Pref}(\mathcal{L}(A))$ we have that $\{I^i_\alpha \mid i \in \{1,\ldots,p\}, I^i_\alpha \neq \emptyset\}$ is a partition of $I_\alpha$.

**Remark 37.** Let $A = (Q, s, \Sigma, \delta, F)$ be a $P$-sortable NFA, where $P = \{U_1, \ldots, U_p\}$ is a partition of $\text{Pref}(\mathcal{L}(A))$. Let $\preceq$ be a co-lexicographic order on $A$ and let $\{Q_i\}_{i=1}^p$ be a $\preceq$-chain partition witnessing that $A$ is $P$-sortable. Then, for every $\alpha \in \text{Pref}(\mathcal{L}(A))$ there exists exactly one $i \in \{1,\ldots, p\}$ such that $I^i_\alpha \neq \emptyset$ (otherwise $P$ would not be a a partition).

**Lemma 38.** Let $A = (Q, s, \Sigma, \delta, F)$ be an NFA, let $\preceq$ be a co-lexicographic order on $A$, and let $\{Q_i\}_{i=1}^p$ be a $\preceq$-chain partition of $Q$. Fix $i \in \{1,\ldots, p\}$ and fix $u \in Q_i$. Then, $I_u$ is a convex set in $(\text{Pref}(\mathcal{L}(A)))^i$.

**Proof.** Fix $\alpha, \beta, \gamma \in \text{Pref}(\mathcal{L}(A))^i$ such that $\alpha \preceq \beta \preceq \gamma$ and $\alpha, \gamma \in I_u$. We must prove that $\beta \in I_u$. Assume by contradiction that $\beta \notin I_u$. Since $\beta \in \text{Pref}(\mathcal{L}(A))^i$, then there exists $v \in Q_i$ such that $\beta \in I_v$. Notice that we have $\alpha \in I_u$, $\beta \in I_v$ and $\{\alpha, \beta\} \not\subseteq I_u \cap I_v$ and $\alpha \preceq \beta$, so by Lemma 2 we conclude $u \preceq v$ or $u \parallel v$. Since $u, v \in Q_i$ and $Q_i$ is a $\preceq$-chain, we conclude $u \prec v$. By using $\beta \prec \gamma$, one analogously obtains $v \prec u$, a contradiction.

**Lemma 39.** Let $A = (Q, s, \Sigma, \delta, F)$ be an NFA, let $\preceq$ be a co-lexicographic order on $A$, and let $\{Q_i\}_{i=1}^p$ be a $\preceq$-chain partition of $Q$. Fix $i \in \{1,\ldots, p\}$ and fix $\alpha \in \text{Pref}(\mathcal{L}(A))$. Then, the set $\{\gamma \in \text{Pref}(\mathcal{L}(A))^i \mid I^i_\alpha = I^i_\gamma\}$ is a convex set in $(\text{Pref}(\mathcal{L}(A)))^i$.

**Proof.** We have to prove that if $\alpha, \beta, \gamma \in \text{Pref}(\mathcal{L}(A))^i$ satisfy $\alpha \preceq \beta \preceq \gamma$ and $I^i_\alpha = I^i_\beta$, then $I^i_\alpha = I^i_\gamma$. First, let us show that $I^i_\alpha \subseteq I^i_\beta$. Pick $u \in I^i_\alpha$. From $I^i_\alpha = I^i_\gamma$ it follows $\alpha, \gamma \in I_u$, so since $\alpha \preceq \beta \preceq \gamma$ and $\beta \in \text{Pref}(\mathcal{L}(A))^i$ we conclude $\beta \in I_u$ (or equivalently, $u \in I^i_\beta$) by Lemma 38.

Now suppose by contradiction that $I^i_\beta \nsubseteq I^i_\alpha$. Let $v \in I^i_\beta \setminus I^i_\alpha$. Since $\alpha \in \text{Pref}(\mathcal{L}(A))^i$, we can pick $u \in I^i_\alpha$. Notice that $u \in I^i_\alpha$, $v \in I^i_\beta$, $\{u, v\} \not\subseteq I^i_\alpha \cap I^i_\beta$ and $\alpha \preceq \beta$, so by 2 we conclude $(u \preceq v) \lor (u \parallel v)$. Since $u, v \in Q_i$ and $Q_i$ is a $\preceq$-chain, we conclude $u \prec v$. By using $\beta \prec \gamma$, one analogously obtains $v \prec u$, a contradiction.

**Definition 40.** Let $A = (Q, s, \Sigma, \delta, F)$ be an NFA. Define the equivalence relation $\sim_A$ on $\text{Pref}(\mathcal{L}(A))$ as follows. For every $\alpha, \beta \in \text{Pref}(\mathcal{L}(A))$, define:

$$\alpha \sim_A \beta \iff I^i_\alpha = I^i_\beta.$$

**Lemma 41.** Let $A = (Q, s, \Sigma, \delta, F)$ be a $P$-sortable NFA, where $P = \{U_1, \ldots, U_p\}$ is a partition of $\text{Pref}(\mathcal{L}(A))$. Then, $\sim_A$ respects $\text{Pref}(\mathcal{L})$, and it is $P$-consistent and $P$-convex. Moreover, $\sim_A$ is right-invariant, input consistent and its index is finite.

**Proof.** Assume that $\alpha \sim_A \beta$. This means that if we read $\alpha$ and $\beta$ on $A$, then we reach the same set of states. As a consequence, if $\phi \in \Sigma^*$ satisfies $a \phi \in \text{Pref}(\mathcal{L}(A))$, then it must be $b \phi \in \text{Pref}(\mathcal{L}(A))$ and $I^i_{\alpha \phi} = I^i_{\beta \phi}$, so proving that $\sim_A$ respects $\text{Pref}(\mathcal{L}(A))$ and it is $P$-consistent. Finally, $\sim_A$ is $P$-convex because for every $\alpha \in \text{Pref}(\mathcal{L}(A))$ we have that $\{\gamma \in \text{Pref}(\mathcal{L}(A)) \mid I^i_\alpha = I^i_\gamma\}$ is a convex set in $(U_{\alpha}, \preceq)$ by Lemma 39 (see Remark 37). The remaining properties are straightforward to prove.

Considering now a $P$-sortable NFA $A$, for a given partition $P$ of $\text{Pref}(\mathcal{L}(A))$, we compare the equivalence $\sim_A$ with the $P$-consistent, $P$-convex refinement $\equiv^c_{\mathcal{L}(A), P}$ of the Myhill-Nerode equivalence $\equiv_{\mathcal{L}(A)}$.

**Lemma 42.** Let $A = (Q, s, \Sigma, \delta, F)$ be a $P$-sortable NFA, where $P = \{U_1, \ldots, U_p\}$ is a partition of $\text{Pref}(\mathcal{L}(A))$. Then, $\sim_A$ is a refinement of $\equiv^c_{\mathcal{L}(A), P}$. 

CVIT 2016
23:22 Which Regular Languages can be Efficiently Indexed?

**Proof.** It is immediate to realize that \( \sim_A \) is a refinement of \( \equiv \). Moreover \( \sim_A \) is \( \mathcal{P} \)-consistent and \( \mathcal{P} \)-convex by lemma 41. On the other hand, by corollary 32 we have that \( \equiv_{\mathcal{L}(A)} \) is the coarsest refinement of \( \equiv_{\mathcal{L}(A)} \) being both \( \mathcal{P} \)-consistent and \( \mathcal{P} \)-convex, so the conclusion follows.

▶ **Lemma 43.** Let \( L \subseteq \Sigma^* \) be a language, and let \( \mathcal{P} = \{U_1, \ldots, U_p\} \) be a partition of \( \text{Pref}(L) \). Assume that \( \mathcal{L} \) is the union of some classes of a \( \mathcal{P} \)-convex, input-consistent, right invariant equivalence relation \( \sim \) on \( \text{Pref}(L) \). Then, \( \mathcal{L} \) is recognized by a \( \mathcal{P} \)-sortable DFA \( A_{\sim} = (Q_{\sim}, s_{\sim}, \Sigma, E_{\sim}, F_{\sim}) \) such that:

1. \( |Q_{\sim}| \) is equal to the index of \( \sim \);
2. \( \sim_{A_{\sim}} \) and \( \sim \) are the same equivalence relation (in particular, \( |Q_{\sim}| \) is equal to the index of \( \sim_{A_{\sim}} \)).

Moreover, if \( B \) is a \( \mathcal{P} \)-sortable DFA that recognizes \( \mathcal{L} \), then \( A_{\sim_B} \) is isomorphic to \( B \).

**Proof.** Define the DFA \( A_{\sim} = (Q_{\sim}, E_{\sim}, \Sigma, s_{\sim}, F_{\sim}) \) as follows.

- \( Q_{\sim} = \{[\alpha]_{\sim} | \alpha \in \text{Pref}(\mathcal{L}) \} \);
- \( s_{\sim} = [\epsilon]_{\sim} \), where \( \epsilon \) is the empty string;
- \( E_{\sim} = \{(a, [\alpha]_{\sim}) | \alpha \in \Sigma^*, a \in \Sigma, \alpha a \in \text{Pref}(\mathcal{L}) \} \);
- \( F_{\sim} = \{[\alpha]_{\sim} | \alpha \in \mathcal{L} \} \).

It is easy to show (see e.g. [2] Thm 2.16) that \( A_{\sim} \) is well-defined \( A \) that satisfies the properties assumed throughout this paper (input-consistency...) and such that \( \mathcal{L}(A_{\sim}) = \mathcal{L} \).

In particular, for all \( \alpha, \beta \in \text{Pref}(\mathcal{L}) \), it holds:

\[
\alpha \in [\beta]_{\sim} \iff \delta_{\sim}(s_{\sim}, \alpha) = [\beta]_{\sim}.
\]

(4)

Note that so far we do not need \( \mathcal{P} \)-convexity yet.

Let \( \preceq \) be the maximal co-lexicographic order on \( A \) (see Lemma 4). Notice that from equation 4 it follows that for every \( \alpha, \beta \in \text{Pref}(\mathcal{L}) \):

\[
\alpha \in [\beta]_{\sim} \iff (\forall \alpha' \in [\alpha]_{\sim}) (\forall \beta' \in [\beta]_{\sim}) (\alpha' \prec \beta')
\]

For every \( i \in \{1, \ldots, p\} \), define:

\[
Q_i = \{[\alpha]_{\sim} | U_\alpha = U_i \}.
\]

Since \( \sim \) in \( \mathcal{P} \)-convex, then \( Q_i \) is well-defined and it is a \( \preceq \)-chain, so \( \{Q_i\}_{i=1}^p \) is a \( \preceq \)-chain partition of \( Q_{\sim} \).

Finally from equation 4 we obtain:

\[
\text{Pref}(\mathcal{L}(A_{\sim}))^i = \{\alpha \in \text{Pref}(\mathcal{L}(A_{\sim})) | \delta_{\sim}(s_{\sim}, \alpha) \in Q_i \} = \{\alpha \in \text{Pref}(\mathcal{L}(A_{\sim})) | (\exists [\beta]_{\sim} \in Q_i \alpha \in [\beta]_{\sim}) \} = \{\alpha \in \text{Pref}(\mathcal{L}(A_{\sim})) | U_\alpha = U_i \} = U_i.
\]

In other words, \( A_{\sim} \) witnesses that \( \mathcal{L} \) is recognized by a \( \mathcal{P} \)-sortable DFA. Moreover:

1. One immediately notices that the number of states of \( A_{\sim} \) is equal to the index of \( \sim \).
2. By equation 4

\[
\alpha \sim_{A_{\sim}} \beta \iff \delta_{\sim}(s_{\sim}, \alpha) = \delta_{\sim}(s_{\sim}, \beta) \iff [\alpha]_{\sim} = [\beta]_{\sim} \iff \alpha \sim \beta
\]

so \( \sim_{A_{\sim}} \) and \( \sim \) are the same equivalence relation.
Finally, suppose $B$ is a $\mathcal{P}$-sortable DFA that recognizes $L$. Notice that by lemma $[11]$ we have that $\sim_B$ is a $\mathcal{P}$-convex, input consistent, right invariant equivalence relation on $\text{Pref}(L)$ of finite index such that $L$ is the union of some $\sim_B$-classes, so $A_{\sim_B}$ is well defined. Call $Q_B$ the set of states of $B$, and let $\phi : Q_{\sim_B} \to Q_B$ be the function sending $[\alpha]_{\sim_B}$ into the state in $Q_B$ reached by reading $\alpha$. Notice that $\phi$ is well-defined because by the definition of $\sim_B$ we obtain that all strings in $[\alpha]_{\sim_B}$ reach the same state of $B$. It is easy to check that $\phi$ determines an isomorphism between $A_{\sim_B}$ and $B$. ◀

Theorem 24 (Co-lexicographic Myhill-Nerode theorem). Let $L$ be a language. Let $\mathcal{P}$ be a partition of $\text{Pref}(L)$. The following are equivalent:

1. $L$ is recognized by a $\mathcal{P}$-sortable NFA.
2. $\equiv_{L,\mathcal{P}}$ has finite index.
3. $L$ is the union of some classes of a $\mathcal{P}$-convex, right invariant equivalence relation on $\text{Pref}(L)$ of finite index.
4. $L$ is recognized by a $\mathcal{P}$-sortable DFA.

Proof. (1) $\to$ (2) Let $A$ be a $\mathcal{P}$-sortable NFA. Since $\sim_A$ has finite index (Lemma $[11]$), by Lemma $[12]$ we conclude that $\equiv_{L,\mathcal{P}}$ has finite index.

(2) $\to$ (3) The desired equivalence relation is $\equiv_{L,\mathcal{P}}$. Indeed, by definition $\equiv_{L,\mathcal{P}}$ is $\mathcal{P}$-convex. Since $\equiv_L$ is right-invariant, then $\equiv_{L,\mathcal{P}}$ is right-invariant by Corollary $[33]$. Similarly one obtains that $L$ is the union of some $\equiv_{L,\mathcal{P}}$-classes.

(3) $\to$ (4) Let $\sim$ be a $\mathcal{P}$-convex, right invariant equivalence relation on $\text{Pref}(L)$ of finite index such that $L$ is the union of some $\sim$-classes. By Remark $[30]$ we can also assume that $\sim$ is input-consistent. The conclusion follows from Lemma $[33]$.

(4) $\to$ (1) Trivial. ◀

Corollary 25. Let $L$ be a language. Let $\mathcal{P}$ be a partition of $\text{Pref}(L)$. If $L$ is recognized by some $\mathcal{P}$-sortable DFA, then there exists a $\mathcal{P}$-sortable DFA $A$ such that all $\mathcal{P}$-sortable DFAs recognizing $L$ and non-isomorphic to $A$ have a larger number of states. In other words, $A$ is the minimum $\mathcal{P}$-sortable DFA recognizing $L$.

Proof. The minimum automaton is $A_{\sim^*}$, where $\sim^*$ in the input-consistent refinement of $\equiv_{L,\mathcal{P}}$ and where $A_{\sim^*}$ is built as in Lemma $[33]$. Indeed, the number of states of $A_{\sim^*}$ is equal to the index of $\sim^*$, or equivalently, of $\sim_{A_{\sim^*}}$. On the other hand, if $B$ is any distinct $\mathcal{P}$-sortable DFA recognizing $L$, then $\sim_B$ is $\mathcal{P}$-convex, right invariant and input-consistent by Lemma $[11]$, so it must be a (strict) refinement of $\sim^*$ and we conclude that $B$ has more states than $A_{\sim^*}$. ◀