GLOBAL SOLUTION AND TIME DECAY OF THE COMPRESSIBLE EULER-MAXWELL SYSTEM IN $\mathbb{R}^3$

ZHONG TAN, YANJIN WANG, AND YONG WANG

Abstract. We study the global existence and large time behavior of solutions near a constant equilibrium to the compressible Euler-Maxwell system in $\mathbb{R}^3$. The previous works mainly assumed that the $H^N$ ($N \geq 3$) and $L^1$ norms of the initial data are sufficiently small. In this paper, we first construct the global unique solution by assuming that the $H^3$ norm of the initial data is small, but the higher order derivatives can be arbitrarily large. If further the initial data belongs to $H^{-s}$ ($0 \leq s < 3/2$) or $B^{-1}_{2,\infty}$ ($0 < s \leq 3/2$), by a regularity interpolation trick, we obtain the various decay rates of the solution and its higher order derivatives. As an immediate byproduct, the $L^p$-$L^2$ ($1 \leq p \leq 2$) type of the decay rates follow without requiring that the $L^p$ norm of initial data is small.

1. Introduction

The dynamics of the electrons interacting with their self-consistent electromagnetic field can be described by the compressible Euler-Maxwell system \cite{12}:

$$\begin{cases}
\partial_t \tilde{n} + \text{div}(\tilde{n}\tilde{u}) = 0, \\
\partial_t (\tilde{n}\tilde{u}) + \text{div}(\tilde{n}\tilde{u} \otimes \tilde{u}) + \nabla p(\tilde{n}) = -\tilde{n}(\tilde{E} + \varepsilon \tilde{u} \times \tilde{B}) - \frac{1}{\tau} \tilde{n}, \\
\varepsilon \lambda^2 \partial_t \tilde{E} - \nabla \times \tilde{B} = \varepsilon \tilde{n}, \\
\varepsilon \partial_t \tilde{B} + \nabla \times \tilde{E} = 0, \\
\lambda^2 \text{div} \tilde{E} = n_{\infty} - \tilde{n}, \quad \text{div} \tilde{B} = 0, \\
(\tilde{n}, \tilde{u}, \tilde{E}, \tilde{B})|_{t=0} = (\tilde{n}_0, \tilde{u}_0, \tilde{E}_0, \tilde{B}_0).
\end{cases} \tag{1.1}$$

The unknown functions $\tilde{n}, \tilde{u}, \tilde{E}, \tilde{B}$ represent the electron density, electron velocity, electric field and magnetic field, respectively. We assume the pressure $p(\tilde{n}) = A\tilde{n}^\gamma$ with constants $A > 0$ and $\gamma \geq 1$ the adiabatic exponent. $1/\gamma > 0$ is the relaxation time. $\lambda > 0$ is the Debye length, and $\varepsilon = 1/c$ with $c$ the speed of light. In the motion of the fluid, due to the greater inertia the ions merely provide a constant background $n_{\infty} > 0$.

Despite its physical importance, due to the complexity there are only few mathematical studies on the Euler-Maxwell system. In one space dimension, Chen, Jerome and Wang \cite{1} proved the global existence of entropy weak solutions to the initial-boundary value problem for arbitrarily large initial data in $L^\infty$. Since the Euler-Maxwell system is a symmetric hyperbolic system, the Cauchy problem in $\mathbb{R}^3$ has a local unique smooth solution when the initial data is smooth, see Kato \cite{10} and Jerome \cite{9} for instance. Recently, there are some results on the global existence and the large time behavior of smooth solutions with small perturbations, see Duan \cite{2}, Ueda and Kawashima \cite{21}, Ueda, Wang and Kawashima \cite{22}. For the asymptotic limits that derive simplified models starting from the Euler-Maxwell system, we refer to \cite{8} \cite{15, 25} for the relaxation limit; \cite{24} for the non-relativistic limit; \cite{13, 14} for the quasi-neutral limit; \cite{19, 20} charged for WKB asymptotics; and references therein.

The main purposes of this paper are to refine a global existence of smooth solutions near the constant equilibrium $(n_{\infty}, 0, 0, B_{\infty})$ to the compressible Euler-Maxwell system and to derive some various time decay rates of the solution as well as its spatial derivatives of any order.

2010 Mathematics Subject Classification. 83C22; 82D37; 76N10; 35Q35; 35B40.

Key words and phrases. Compressible Euler-Maxwell system; Global solution; Time decay rate; Energy method; Interpolation.

Corresponding author: Yong Wang, wangyongxmu@163.com.
Supported by the National Natural Science Foundation of China-NSAF (No. 10976026) and the Natural Science Foundation of Fujian Province of China (No. 2012J05011).
We should emphasize that our results highly rely on that we consider the relaxation case. The non-relaxation case is much more difficult, we refer to [3, 5] for such direction. It turns out that it is more convenient to reformulate the compressible Euler-Maxwell system (1.1) as follows.

Without loss of generality, we take the constants \( \tau, \varepsilon, \lambda, A, n_c \) in (1.1) to be one. We define

\[
\begin{align*}
\left\{ \begin{array}{l}
n(x, t) = \frac{2}{\gamma - 1} \left\{ \left[ \frac{n(x, \xi)}{\sqrt{\gamma}} \right]^{\gamma - 1} \right\}^{\frac{1}{\gamma - 1}} - 1, \quad u(x, t) = \frac{1}{\sqrt{\gamma}} \hat{n}(x, \frac{\xi}{\sqrt{\gamma}}), \\
E(x, t) = \frac{1}{\sqrt{\gamma}} \hat{E}(x, \frac{\xi}{\sqrt{\gamma}}), \\
B(x, t) = \frac{1}{\sqrt{\gamma}} \hat{B}(x, \frac{\xi}{\sqrt{\gamma}}) - B_{\infty}.
\end{array} \right.
\end{align*}
\]

Then the Euler-Maxwell system (1.1) is reformulated equivalently as

\[
\begin{align*}
&\partial_t n + \text{div} u = - u \cdot \nabla n - \nu \text{div} v, \\
&\partial_t u + \nu u + u \times B_{\infty} + \nabla n + \nu E = - u \cdot \nabla u - \mu n \nabla n - u \times B, \\
&\partial_t E - \nu \nabla \times B - \nu u = \nu f(n) u, \\
&\partial_t B + \nu \nabla \times E = 0, \\
&\text{div} E = - \nu f(n), \quad \text{div} B = 0,
\end{align*}
\]

Here \( \mu := \frac{\gamma - 1}{2} \), \( \nu := \frac{1}{\sqrt{\gamma}} \), and the nonlinear function \( f(n) \) is defined by

\[
f(n) := \left( 1 + \frac{\gamma - 1}{2} n \right)^{\frac{2}{\gamma - 1}} - 1. \tag{1.4}
\]

Notice that we have assumed \( \gamma > 1 \). If \( \gamma = 1 \), we instead define

\[
n := \sqrt{A} (\ln \tilde{n} - \ln n_{\infty}) = \sqrt{A} \ln \tilde{n}. \tag{1.5}
\]

In this paper, we only consider the case \( \gamma > 1 \), and the case \( \gamma = 1 \) can be treated in the same way by using the reformulation in terms of the new variables correspondingly.

**Notation.** In this paper, we use \( H^s(\mathbb{R}^d) \), \( s \in \mathbb{R} \) to denote the usual Sobolev spaces with norm \( \| \cdot \|_{H^s} \) and \( L^p(\mathbb{R}^d) \), \( 1 \leq p \leq \infty \) to denote the usual \( L^p \) spaces with norm \( \| \cdot \|_{L^p} \). \( \nabla^\ell \) with an integer \( \ell \geq 0 \) stands for the usual any spatial derivatives of order \( \ell \). When \( \ell < 0 \) or \( \ell \) is not a positive integer, \( \nabla^\ell \) stands for \( \Lambda^\ell \) defined by \( \Lambda^\ell f := \mathcal{F}^{-1}(\xi^\ell \hat{f}) \), where \( \mathcal{F} \) is the usual Fourier transform operator and \( \mathcal{F}^{-1} \) is its inverse. We use \( H^s(\mathbb{R}^d) \), \( s \in \mathbb{R} \) to denote the homogeneous Sobolev spaces on \( \mathbb{R}^d \) with norm \( \| \cdot \|_{H^s} \) defined by \( \| f \|_{H^s} := \| \Lambda^s f \|_{L^2} \). We then recall the homogeneous Besov spaces. Let \( \phi \in C_c^\infty(\mathbb{R}^d) \) be such that \( \phi(\xi) = 1 \) when \( |\xi| \leq 1 \) and \( \phi(\xi) = 0 \) when \( |\xi| \geq 2 \). Let \( \varphi_j(\xi) = \varphi(2^j \xi) \) and \( \varphi_j(\xi) = \varphi(2^{-j} \xi) \) for \( j \in \mathbb{Z} \). Then by the construction, \( \sum_{j \in \mathbb{Z}} \varphi_j(\xi) = 1 \) if \( \xi \neq 0 \). We define \( \Delta_j f := \varphi_j^* f \), then for \( s \in \mathbb{R} \) we define the homogeneous Besov spaces \( \dot{B}^s_{2,\infty}(\mathbb{R}^d) \) with norm \( \| \cdot \|_{\dot{B}^s_{2,\infty}} \) defined by

\[
\| f \|_{\dot{B}^s_{2,\infty}} := \sup_{j \in \mathbb{Z}} 2^{sj} \| \Delta_j f \|_{L^2}. \tag{1.6}
\]

Throughout this paper we let \( C \) denote some positive (generally large) universal constants and \( \lambda \) denote some positive (generally small) universal constants. They do not depend on either \( k \) or \( N \); otherwise, we will denote them by \( C_k, C_N \), etc. We will use \( a \lesssim b \) if \( a \leq Cb \), and \( a \sim b \) means that \( a \lesssim b \) and \( b \lesssim a \). We use \( C_0 \) to denote the constants depending on the initial data and \( k, N, s \). For simplicity, we write \( \| (A, B) \|_X := \| A \|_X + \| B \|_X \) and \( f := \int_{\mathbb{R}^3} f \, dx \).

For \( N \geq 3 \), we define the energy functional by

\[
\mathcal{E}_N(t) := \sum_{l=0}^{N} \left\| \nabla^l (n, u, E, B) \right\|_{L^2}^2 \tag{1.7}
\]

and the corresponding dissipation rate by

\[
\mathcal{D}_N(t) := \sum_{l=0}^{N} \left\| \nabla^l (n, u) \right\|_{L^2}^2 + \sum_{l=0}^{N-1} \left\| \nabla^l E \right\|_{L^2}^2 + \sum_{l=1}^{N-1} \left\| \nabla^l B \right\|_{L^2}^2. \tag{1.8}
\]
Our first main result about the global unique solution to the system (1.3) is stated as follows.

**Theorem 1.1.** Assume the initial data satisfy the compatible conditions
\[
\text{div}E_0 = -\nu f(n_0), \quad \text{div}B_0 = 0.
\]
There exists a sufficiently small \( \delta_0 > 0 \) such that if \( E_0(0) \leq \delta_0 \), then there exists a unique global solution \( (n, u, E, B)(t) \) to the Euler-Maxwell system (1.3) satisfying
\[
\sup_{0 \leq t \leq \infty} E_3(t) + \int_0^\infty D_3(\tau) \, d\tau \leq C E_3(0).
\]
Furthermore, if \( E_N(0) < +\infty \) for any \( N \geq 3 \), there exists an increasing continuous function \( P_N(\cdot) \) with \( P_N(0) = 0 \) such that the unique solution satisfies
\[
\sup_{0 \leq t \leq \infty} E_N(t) + \int_0^\infty D_N(\tau) \, d\tau \leq P_N(E_N(0)).
\]

The proof of Theorem 1.1 is inspired by the recent works of Guo [6] and Wang [23]. The major difficulty here is the regularity-loss of the electromagnetic field. We will do the refined energy estimates stated in Lemma 2.8–2.9, which allow us to deduce
\[
d\frac{d}{dt} E_3 + D_3 \lesssim \sqrt{E_3} D_3
\]
and for \( N \geq 4 \),
\[
d\frac{d}{dt} E_N + D_N \leq C_N D_{N-1} E_N.
\]

Then Theorem 1.1 follows in the fashion of [6, 23].

Our second main result is on some various decay rates of the solution to the system (1.3) by making the much stronger assumption on the initial data.

**Theorem 1.2.** Assume that \( (n, u, E, B)(t) \) is the solution to the Euler-Maxwell system (1.3), constructed in Theorem [1.4] with \( N \geq 5 \). There exists a sufficiently small \( \delta_0 = \delta_0(N) \) such that if \( E_N(0) \leq \delta_0 \), and assuming that \( (u_0, E_0, B_0) \in \dot{H}^{-s} \) for some \( s \in [0, 3/2) \) or \( (u_0, E_0, B_0) \in \dot{B}^{-s}_{2, \infty} \) for some \( s \in (0, 3/2] \), then we have
\[
\|(u, E, B)(t)\|_{\dot{H}^{-s}} \leq C_0
\]
or
\[
\|(u, E, B)(t)\|_{\dot{B}^{-s}_{2, \infty}} \leq C_0.
\]

Moreover, for any fixed integer \( k \geq 0 \), if \( N \geq 2k + 2 + s \), then
\[
\left\| \nabla^k (n, u, E, B)(t) \right\|_{L^2} \leq C_0 (1 + t)^{-\frac{k-s}{2}}.
\]

Furthermore, for any fixed integer \( k \geq 0 \), if \( N \geq 2k + 4 + s \), then
\[
\left\| \nabla^k (n, u, E)(t) \right\|_{L^2} \leq C_0 (1 + t)^{-\frac{k+1+s}{2}};
\]
if \( N \geq 2k + 6 + s \), then
\[
\left\| \nabla^k n(t) \right\|_{L^2} \leq C_0 (1 + t)^{-\frac{k+s+4}{2}};
\]
if \( N \geq 2k + 10 + s \) and \( B_{\infty} = 0 \), then
\[
\left\| \nabla^k (n, \text{div} u)(t) \right\|_{L^2} \leq C_0 (1 + t)^{-\frac{k+s+4}{2}}.
\]

The proof of Theorem 1.2 is based on the regularity interpolation method developed in Strain and Guo [17], Guo and Wang [7] and Sohinger and Strain [16]. To prove the optimal decay rate of the dissipative equations in the whole space, Guo and Wang [7] developed a general energy method of using a family of scaled energy estimates with minimum derivative counts and interpolations among them. Note that the homogeneous Sobolev space \( \dot{H}^{-s} \) was introduced there to enhance the decay rates. By the usual embedding theorem, we know that for \( p \in (1, 2] \), \( L^p \subset \dot{H}^{-s} \) with \( s = 3 \left( \frac{1}{p} - \frac{1}{2} \right) \in [0, 3/2) \). Hence the \( L^p-L^2 \) type of the optimal decay results follows as a corollary. However, this does not cover the case \( p = 1 \). To amend this, Sohinger...
and Strain [10] instead introduced the homogeneous Besov space $B^{-s}_{2,\infty}$ due to the fact that the endpoint embedding $L^1 \subset B^{-\frac{3}{2}}_{2,\infty}$ holds. The method of [7, 10] can be applied to many dissipative equations in the whole space, however, it cannot be applied directly to the compressible Euler-Maxwell system which is of regularity-loss. Based on the refined energy estimates stated in Lemma 2.8–2.9, we deduce

$$\frac{d}{dt} L^{k+2} + D^{k+2} \leq C_k \| (n, u) \|_{L^\infty} \| \nabla^{k+2} (n, u) \|_{L^2} \| \nabla^{k+2} (E, B) \|_{L^2},$$

(1.20)

where $L^{k+2}$ and $D^{k+2}$ with minimum derivative counts are defined by (3.9) and (3.10) respectively. Then combining the methods of [7, 10] and a trick of Strain and Guo [17] to treat the electromagnetic field, we are able to conclude the decay rate (1.16). If in view of the whole solution, the decay rate (1.16) can be regarded as be optimal. The faster decay rates (1.17)–(1.19) follow by revisiting the equations carefully. In particular, we will use a bootstrap argument to derive (1.19).

As quoted above, by Theorem 1.2 we have the following corollary of the usual $L^p - L^2$ type of the decay results:

**Corollary 1.3.** Under the assumptions of Theorem 1.2 except that we replace the $H^{-s}$ or $B^{-s}_{2,\infty}$ assumption by that $(u_0, E_0, B_0) \in L^p$ for some $p \in [1, 2]$, then for any fixed integer $k \geq 0$, if $N \geq 2k + 2 + s_p$, then

$$\| \nabla^k (n, u, E, B) (t) \|_{L^2} \leq C_0 (1 + t)^{-\frac{k + s_p}{2}}.$$  

(1.21)

Here the number $s_p := 3 \left( \frac{1}{p} - \frac{1}{2} \right)$.

Furthermore, for any fixed integer $k \geq 0$, if $N \geq 2k + 4 + s_p$, then

$$\| \nabla^k (n, u, E) (t) \|_{L^2} \leq C_0 (1 + t)^{-\frac{k + s_p}{2}};$$

(1.22)

if $N \geq 2k + 6 + s_p$, then

$$\| \nabla^k n (t) \|_{L^2} \leq C_0 (1 + t)^{-\frac{k + 6 + s_p}{2}};$$

(1.23)

if $N \geq 2k + 10 + s_p$ and $B_\infty = 0$, then

$$\| \nabla^k (n, \text{div} u) (t) \|_{L^2} \leq C_0 (1 + t)^{-\frac{k + 6 + s_p}{2}}.$$  

(1.24)

The followings are several remarks for Theorem 1.1, Theorem 1.2 and Corollary 1.3

**Remark 1.4.** In Theorem 1.1, we only assume the $H^3$ norm of the initial data is small, but the higher order derivatives can be arbitrarily large. Notice that in Theorem 1.2, the $H^{-s}$ and $B^{-s}_{2,\infty}$ norms of the solution are preserved along the time evolution, however, in Corollary 1.3 it is difficult to show that the $L^p$ norm of the solution can be preserved. Note that the $L^2$ decay rate of the higher order spatial derivatives of the solution are obtained. Then the general optimal $L^q (2 \leq q \leq \infty)$ decay rates of the solution follow by the Sobolev interpolation.

**Remark 1.5.** We remark that Corollary 1.3 not only provides an alternative approach to derive the $L^p - L^2$ type of the optimal decay results but also improves the previous results of the $L^p - L^2$ approach in Ueda and Kawashima [21] and Duan [2]. In Ueda and Kawashima [21], the decay rates (1.21)–(1.23) with $p = 2$ were proved by using the time weighted energy method, and when $p = 1$ they were proved by combining the time weighted energy method and the linear decay analysis but under the stronger assumption that $\| (n_0, u_0, E_0, B_0) \|_{L^1}$ is sufficiently small. In Duan [2], assuming that $B_\infty = 0$ and $\| (n_0, u_0, B_0) \|_{L^1}$ is sufficiently small, by combining the energy method and the linear decay analysis, Duan proved that

$$\| n(t) \|_{L^2} \leq C_0 (1 + t)^{-\frac{1}{4}}, \quad \| (u, E)(t) \|_{L^2} \leq C_0 (1 + t)^{-\frac{3}{4}} \quad \text{and} \quad \| B(t) \|_{L^2} \leq C_0 (1 + t)^{-\frac{7}{4}}.$$  

(1.25)

Notice that for $p = 1$, our decay rate of $n(t)$ in (1.21) is $(1 + t)^{-13/4}$.

The rest of our paper is organized as follows. In section 2 we establish the refined energy estimates for the solution and derive the negative Sobolev and Besov estimates. Theorem 1.1 and Theorem 1.2 are proved in section 3.
2. Nonlinear energy estimates

In this section, we will do the a priori estimate by assuming that $\|n(t)\|_{H^3} \leq \delta \ll 1$. Recall the expression (1.4) of $f(n)$. Then by Taylor’s formula and Sobolev’s inequality, we have

$$f(n) \sim n \text{ and } |f^{(k)}(n)| \leq C_k \text{ for any } k \geq 1.$$  \hspace{1cm} (2.1)

2.1. Preliminary. In this subsection, we collect the analytic tools which will be used in the paper and prove a basic estimate for the nonlinear function $f(n)$.

**Lemma 2.1.** Let $2 \leq p \leq +\infty$ and $\alpha, \mu, \ell \geq 0$. Then we have

$$\|\nabla^\alpha f\|_{L^p} \leq C_p \|\nabla^\mu f\|_{L^2}^{1-\theta} \|\nabla^\ell f\|_{L^2}^\theta.$$  \hspace{1cm} (2.2)

Here $0 \leq \theta \leq 1$ (if $p = +\infty$, then we require that $0 < \theta < 1$) and $\alpha$ satisfy

$$\alpha + 3 \left( \frac{1}{2} - \frac{1}{p} \right) = m(1 - \theta) + \ell \theta.$$  \hspace{1cm} (2.3)

**Proof.** For the case $2 \leq p < +\infty$, we refer to Lemma A.1 in [7]; for the case $p = +\infty$, we refer to Exercise 6.1.2 in [4] (pp. 421).

**Lemma 2.2.** For any integer $k \geq 0$, we have

$$\|\nabla^k f(n)\|_{L^\infty} \leq C_k \|\nabla^k \nabla n\|_{L^2}^{1/4} \|\nabla^{k+2} n\|_{L^2}^{3/4},$$  \hspace{1cm} (2.4)

and

$$\|\nabla^k f(n)\|_{L^2} \leq C_k \|\nabla^k \nabla n\|_{L^2}.$$  \hspace{1cm} (2.5)

**Proof.** The proof is based on Lemma 2.1. For (2.4), we refer to Lemma 3.1 in [7]. For (2.5), in light of (2.1), it suffices to prove that when $k \geq 1$, (2.3) holds for all $f(n)$ with bounded derivatives. We will use an induction on $k \geq 1$. If $k = 1$, we have

$$\|\nabla f(n)\|_{L^2} = \|f'(n)\|_{L^2} \lesssim \|\nabla n\|_{L^2}.$$  \hspace{1cm} (2.6)

Assume (2.5) holds for from 1 to $k - 1$. We use the Leibniz formula to have

$$\|\nabla^k f(n)\|_{L^2} = \|\nabla^{k-1}(f'(n)\nabla n)\|_{L^2} \leq C_k \left( \|f'(n)\nabla^k n\|_{L^2} + \|\nabla f'(n)\nabla^{k-1} n\|_{L^2} + \sum_{\ell=2}^{k-1} \|\nabla^\ell f'(n)\nabla^{k-\ell} n\|_{L^2} \right).$$  \hspace{1cm} (2.7)

Here if $k = 2$, then the summing term in (2.7) is nothing, etc. By Hölder’s inequality and Sobolev’s inequality, we have

$$\|f'(n)\nabla^k n\|_{L^2} + \|\nabla f'(n)\nabla^{k-1} n\|_{L^2} \lesssim \|\nabla^k n\|_{L^2} + \|\nabla n\|_{L^3} \|\nabla^{k-1} n\|_{L^6} \lesssim \|\nabla^k n\|_{L^2}. \hspace{1cm} (2.8)$$

For the summing term we use the induction hypothesis to obtain that for $2 \leq \ell \leq k - 1$,

$$\|\nabla^\ell f'(n)\nabla^{k-\ell} n\|_{L^2} \lesssim \|\nabla^\ell f'(n)\|_{L^2} \|\nabla^{k-\ell} n\|_{L^\infty} \lesssim \|\nabla^\ell n\|_{L^2} \|\nabla^{k-\ell} n\|_{L^\infty}. \hspace{1cm} (2.9)$$

By Lemma 2.1 if $\ell \leq \lfloor \frac{k-1}{2} \rfloor$, then we have

$$\|\nabla^\ell n\|_{L^2} \|\nabla^{k-\ell} n\|_{L^\infty} \lesssim \|\nabla^\alpha n\|_{L^2}^{\frac{k-\ell-\frac{3}{2}}{k}} \|\nabla^k n\|_{L^2}^{\frac{\ell + \frac{3}{2}}{k}} \|\nabla^k n\|_{L^2}^{\frac{k-\ell + \frac{3}{2}}{k}} \lesssim \|\nabla^k n\|_{L^2}, \hspace{1cm} (2.10)$$

where $\alpha$ is defined by

$$\ell = \alpha \times \frac{k - \ell + \frac{3}{2}}{k} + k \times \frac{\ell - \frac{3}{2}}{k} \Rightarrow \alpha = \frac{3k}{2(k-\ell) + 3} < 3; \hspace{1cm} (2.11)$$

if $\ell \geq \lfloor \frac{k-1}{2} \rfloor + 1$, then we have

$$\|\nabla^\ell n\|_{L^2} \|\nabla^{k-\ell} n\|_{L^\infty} \lesssim \|n\|_{L^2}^{1-\frac{\ell}{k}} \|\nabla^k n\|_{L^2} \|\nabla^\alpha n\|_{L^2} \|\nabla^k n\|_{L^2} \lesssim \|\nabla^k n\|_{L^2}, \hspace{1cm} (2.12)$$
Lemma 2.5. Let \( k \leq 0 \) be an integer and define the commutator
\[
\begin{align*}
[\nabla^k, g] h = \nabla^k (gh) - g \nabla^k h.
\end{align*}
\] (2.14)
Then we have
\[
\| [\nabla^k, g] h \|_{L^2} \leq C_k \left( \| \nabla g \|_{L^\infty} \left\| \nabla^{k-1} h \right\|_{L^2} + \left\| \nabla^k g \right\|_{L^2} \| h \|_{L^\infty} \right).
\] (2.15)
Proof. It can be proved by using Lemma 2.1, see Lemma 3.4 in [11] (pp. 98) for instance. □

We have the \( L^p \) embeddings:

Lemma 2.4. Let \( 0 \leq s < 3/2 \), \( 1 < p < 2 \) with \( 1/2 + s/3 = 1/p \), then
\[
\| f \|_{H^{-s}} \lesssim \| f \|_{L^p}.
\] (2.16)
Proof. It follows from the Hardy-Littlewood-Sobolev theorem, see [4]. □

Lemma 2.5. Let \( 0 < s \leq 3/2 \), \( 1 < p < 2 \) with \( 1/2 + s/3 = 1/p \), then
\[
\| f \|_{B_{2,\infty}^s} \lesssim \| f \|_{L^p}.
\] (2.17)
Proof. See Lemma 4.6 in [16]. □

It is important to use the following special interpolation estimates:

Lemma 2.6. Let \( s \geq 0 \) and \( \ell \geq 0 \), then we have
\[
\left\| \nabla^\ell f \right\|_{L^2} \leq \left\| \nabla^{\ell+1} f \right\|_{L^2}^{1-\theta} \| f \|_{H^{-s}}^\theta, \text{ where } \theta = \frac{1}{\ell + 1 + s}.
\] (2.18)
Proof. It follows directly by the Parseval theorem and Hölder’s inequality. □

Lemma 2.7. Let \( s > 0 \) and \( \ell \geq 0 \), then we have
\[
\left\| \nabla^\ell f \right\|_{L^2} \leq \left\| \nabla^{\ell+1} f \right\|_{L^2}^{1-\theta} \| f \|_{B_{2,\infty}^s}^\theta, \text{ where } \theta = \frac{1}{\ell + 1 + s}.
\] (2.19)
Proof. See Lemma 4.5 in [16]. □

2.2. Energy estimates. In this subsection, we will derive the basic energy estimates for the solution to the Euler-Maxwell system (1.3). We begin with the standard energy estimates.

Lemma 2.8. For any integer \( k \geq 0 \), we have
\[
\frac{d}{dt} \sum_{l=k}^{k+2} \left\| \nabla^l (n, u, E, B) \right\|_{L^2}^2 + \lambda \sum_{l=k}^{k+2} \left\| \nabla^l u \right\|_{L^2}^2 \\
\lesssim C_k \left( \| (n, u) \|_{H^{k+1} \cap H^2} + \| \nabla B \|_{L^2} \right) \left( \sum_{l=k}^{k+2} \left\| \nabla^l (n, u) \right\|_{L^2}^2 + \sum_{l=k}^{k+1} \left\| \nabla^l E \right\|_{L^2}^2 + \left\| \nabla^{k+1} B \right\|_{L^2}^2 \right) \\
+ \| (n, u) \|_{L^\infty} \left\| \nabla^{k+2} (n, u) \right\|_{L^2} \left\| \nabla^{k+2} (E, B) \right\|_{L^2}.
\] (2.20)
Proof. The standard $\nabla^l$ ($l = k, k + 1, k + 2$) energy estimates on the system (1.3) yield
\[
\frac{1}{2} \frac{d}{dt} \int |\nabla^l(n, u, E, B)|^2 + \nu \|\nabla^l u\|^2_{L^2} = -\mu \int \nabla^l (n \text{div } u) \nabla^l n + \nabla^l (n \nabla n) \cdot \nabla^l u - \int \nabla^l (u \cdot \nabla n) \nabla^l n + \nabla^l (u \cdot \nabla u) \cdot \nabla^l u
\]
\[
- \int \nabla^l (u \times B) \cdot \nabla^l u + \nu \int \nabla^l (f(n) u) \cdot \nabla^l E
\]
:= I_1 + I_2 + I_3 + I_4.
\]
We now estimate $I_1 \sim I_4$. First, we use the commutator notation (2.14) to rewrite $I_1$ as
\[
I_1 = -\mu \int (n \text{div } \nabla^l u + [\nabla^l, n] \text{div } u) \nabla^l n + (n \nabla \nabla^l n + [\nabla^l, n] \nabla n) \cdot \nabla^l u
\]
\[
= -\mu \int n \text{div}(\nabla^l u \nabla^l n) + [\nabla^l, n] \text{div } \nabla^l n + [\nabla^l, n] \nabla n \cdot \nabla^l u.
\]
By integrating by parts, we have
\[
- \int n \text{div}(\nabla^l u \nabla^l n) = \int \nabla n \cdot \nabla^l u \nabla^l n \leq \|\nabla n\|_{L^\infty} \|\nabla^l u\|_{L^2} \|\nabla^l n\|_{L^2}.
\]
We use the commutator estimate of Lemma 2.3 to bound
\[
- \int [\nabla^l, n] \text{div } \nabla^l n \leq C_1 (\|\nabla n\|_{L^\infty} \|\nabla^{l-1} \text{div } u\|_{L^2} + \|\nabla^l n\|_{L^2} \|\text{div } u\|_{L^\infty}) \|\nabla^l n\|_{L^2}
\]
\[
\leq C_1 \|\nabla (n, u)\|_{L^\infty} \|\nabla^{l-1} (n, u)\|_{L^2} \|\nabla^l n\|_{L^2}.
\]
Bounding the last term of $I_1$ similarly, and then applying the same arguments to the term $I_2$, by Sobolev’s and Cauchy’s inequalities, we deduce
\[
I_1 + I_2 \leq C_1 \|(n, u)\|_{H^3} \|\nabla^l (n, u)\|_{L^2}^2.
\]
Next, we estimate the term $I_3$, and we must be much more careful with this term since the magnetic field $B$ has the weakest dissipative estimates. First of all, we have
\[
I_3 = -\sum_{\ell=1}^l C_\ell \int \nabla^{l-\ell} u \times \nabla^l B \cdot \nabla^l u \leq C_1 \sum_{\ell=1}^l \|\nabla^{l-\ell} u \nabla^l B\|_{L^2} \|\nabla^l u\|_{L^2}.
\]
Here, if $l < 1$, then it’s nothing, and etc. We have to distinct the arguments by the value of $l$. First, let $l = k$. We take $L^3 - L^6$ and then apply Lemma 2.1 to have
\[
\|\nabla^{k-\ell} u \nabla^l B\|_{L^2} \lesssim \|\nabla^{k-\ell} u\|_{L^3} \|\nabla^l B\|_{L^6}
\]
\[
\lesssim \|\nabla^\alpha u\|_{L^2}^{\frac{\ell}{2}} \|\nabla^k u\|_L^{1-\frac{\ell}{2}} \|\nabla B\|_{L^2}^{\frac{1}{2}} \|\nabla B\|_L^{1-\frac{\ell}{2}} \|\nabla B\|_{L^2}^{\frac{\ell}{2}}.
\]
where $\alpha$ is defined by
\[
k - \ell + \frac{1}{2} = \alpha \times \frac{\ell}{k} + k \times \left(1 - \frac{\ell}{k}\right) \Rightarrow \alpha = \frac{k - 2 \ell}{2}.
\]
Hence by Young’s inequality, we have that for $l = k$,
\[
I_3 \leq C_k \left(\|u\|_{H^\frac{\ell}{2}} + \|B\|_{L^2}\right) \left(\|\nabla^k u\|_{L^2}^{\frac{\ell}{2}} + \|\nabla^{k+1} B\|_{L^2}^{\frac{\ell}{2}}\right).
\]
We then let $l = k + 1$. If $1 \leq \ell \leq k$, we take $L^3 - L^6$ and by Lemma 2.1 again to obtain
\[
\|\nabla^{k+1-\ell} u \nabla^l B\|_{L^2} \lesssim \|\nabla^{k+1-\ell} u\|_{L^3} \|\nabla^l B\|_{L^6}
\]
\[
\lesssim \|\nabla^\alpha u\|_{L^2}^{\frac{\ell}{2}} \|\nabla^{k+1} u\|_{L^2}^{1-\frac{\ell}{2}} \|\nabla B\|_{L^2}^{\frac{\ell}{2}} \|\nabla B\|_{L^2}^{1-\frac{\ell}{2}} \|\nabla B\|_{L^2}^{\frac{\ell}{2}}.
\]
where $\alpha$ is defined by
\[ k + 1 - \ell + \frac{1}{2} = \alpha \times \frac{\ell}{k} + (k + 1) \times \left( 1 - \frac{\ell}{k} \right) \implies \alpha = 1 + \frac{k}{2\ell} \leq \frac{k}{2} + 1; \] (2.31)
if $\ell = k + 1$, we take $L^\infty - L^2$ to get
\[ \left\| u \nabla^{k+1} B \right\|_{L^2} \lesssim \left\| u \right\|_{L^\infty} \left\| \nabla^{k+1} B \right\|_{L^2}. \] (2.32)
We thus have that for $l = k + 1$, by Sobolev’s inequality,
\[ I_3 \leq C_k \left( \left\| u \right\|_{H^{\frac{k}{2} + 1} H^2} + \left\| \nabla B \right\|_{L^2} \right) \left( \left\| \nabla^{k+1} u \right\|_{L^2}^2 + \left\| \nabla^{k+1} B \right\|_{L^2}^2 \right). \] (2.33)
We now let $l = k + 2$. If $1 \leq \ell \leq k$, we take $L^3 - L^6$ and by Lemma 2.1 again to have
\[ \left\| \nabla^{k+2-\ell} u \nabla^\ell B \right\|_{L^2} \lesssim \left\| \nabla^{k+2-\ell} u \right\|_{L^3} \left\| \nabla^\ell B \right\|_{L^6} \lesssim \left\| \nabla^\alpha u \right\|_{L^2} \left\| \nabla^{k+2} u \right\|_{L^2} \left\| \nabla B \right\|_{L^2} \left\| \nabla^{k+1} B \right\|_{L^2}. \] (2.34)
where $\alpha$ is defined by
\[ k + 2 - \ell + \frac{1}{2} = \alpha \times \frac{\ell}{k} + (k + 2) \times \left( 1 - \frac{\ell}{k} \right) \implies \alpha = 2 + \frac{k}{2\ell} \leq \frac{k}{2} + 2; \] (2.35)
if $\ell = k + 1$ or $k + 2$, we take $L^\infty - L^2$ to get
\[ \left\| \nabla u \nabla^{k+1} B \right\|_{L^2} \lesssim \left\| u \right\|_{L^\infty} \left\| \nabla^{k+1} B \right\|_{L^2}, \] (2.36)
and
\[ \left\| u \nabla^{k+2} B \right\|_{L^2} \lesssim \left\| u \right\|_{L^\infty} \left\| \nabla^{k+2} B \right\|_{L^2}. \] (2.37)
We thus have that for $l = k + 2$,
\[ I_3 \leq C_k \left( \left\| u \right\|_{H^{\frac{k}{2} + 2} H^3} + \left\| \nabla B \right\|_{L^2} \right) \left( \left\| \nabla^{k+2} u \right\|_{L^2}^2 + \left\| \nabla^{k+1} B \right\|_{L^2}^2 \right) \] (2.38)
We now estimate the last term $I_4$. We again have to distinguish the arguments by the value of $l$. First, for $l = k + 1$, we have
\[ I_4 = \sum_{\ell=0}^{l} C_\ell \int \nabla^\ell f(n) \nabla^{l-\ell} u \cdot \nabla^\ell E \leq C_l \sum_{\ell=0}^{l} \left\| \nabla^\ell f(n) \nabla^{l-\ell} u \right\|_{L^2} \left\| \nabla^\ell E \right\|_{L^2}. \] (2.39)
If $0 \leq \ell \leq l - 2$, we take $L^\infty - L^2$ and by Lemma 2.1 and the estimate (2.4) of Lemma 2.2 to obtain
\[ \left\| \nabla^\ell f(n) \nabla^{l-\ell} u \right\|_{L^2} \leq \left\| \nabla^\ell f(n) \right\|_{L^\infty} \left\| \nabla^{l-\ell} u \right\|_{L^2} \leq C_l \left\| \nabla^\ell f \right\|_{L^2} \left\| \nabla^{l+2} n \right\|_{L^2} \left\| \nabla^{l-\ell} u \right\|_{L^2} \leq C_l \left( \left\| n \right\|_{L^2} \right)^{\frac{4}{2} - \frac{l+2}{2}} \left\| \nabla^\ell n \right\|_{L^2} \left\| \nabla^{l-\ell} u \right\|_{L^2} \] (2.40)
where $\alpha$ is defined by
\[ l - \ell = \alpha \times \frac{2\ell + 3}{2l} + l \times \left( 1 - \frac{2\ell + 3}{2l} \right) \implies \alpha = \frac{3l}{2\ell + 3} \leq l; \] (2.41)
if $\ell = l - 1$, we take $L^6 - L^3$ and by the estimate (2.5) of Lemma 2.2 to have
\[ \left\| \nabla^{l-1} f(n) \nabla u \right\|_{L^2} \leq \left\| \nabla^{l-1} f \right\|_{L^6} \left\| \nabla u \right\|_{L^3} \leq C_l \left\| \nabla^l n \right\|_{L^2} \left\| u \right\|_{H^2}; \] (2.42)
if $\ell = l$, we take $L^2 - L^\infty$ and by the estimate (2.21) of Lemma 2.2 to have
\[ \left\| \nabla^l f(n) \right\|_2 \leq \left\| \nabla^l u \right\|_{L^\infty} \leq C_l \left\| \nabla^l n \right\|_2 \left\| u \right\|_{H^2}. \] (2.43)
We thus have that for $l = k$ or $k + 1$,
\[ I_4 \leq C_l \left\| (n, u) \right\|_{H^l \cap H^2} \left( \left\| \nabla^l (n, u) \right\|^2_2 + \left\| \nabla^l E \right\|^2_2 \right). \] (2.44)

Now for $l = k + 2$, we rewrite $I_4$ as
\[ I_4 = -\sum_{\ell=0}^{k+2} C_{k+2}^\ell \int \nabla^\ell f(n) \nabla^{k+2-\ell} u \cdot \nabla^{k+2} E \\
= -\int \left( f(n) \nabla^{k+2} u + \nabla^{k+2} f(n) u \right) \cdot \nabla^{k+2} E \\
+ \sum_{\ell=1}^{k+1} C_{k+2}^\ell \int \nabla \left( \nabla^{k+2-\ell} f(n) \nabla^\ell u \right) \cdot \nabla^{k+1} E \\
= -\int \left( f(n) \nabla^{k+2} u + \nabla^{k+2} f(n) u \right) \cdot \nabla^{k+2} E \\
+ (k + 2) \int \left( \nabla^{k+2} f(n) \nabla u + \nabla f(n) \nabla^{k+2} u \right) \cdot \nabla^{k+1} E \\
+ \sum_{\ell=2}^{k+1} C_{k+2}^\ell \int \nabla^{k+3-\ell} f(n) \nabla^\ell u \cdot \nabla^{k+1} E + \sum_{\ell=1}^{k} C_{k+2}^\ell \int \nabla^{k+2-\ell} f(n) \nabla^{\ell+1} u \cdot \nabla^{k+1} E \\
:= I_{41} + I_{42} + I_{43}. \]

By Lemma 2.2, we have
\[ I_{41} \leq C_k \left( \left\| f(n) \right\|_{L^\infty} \left\| \nabla^{k+2} u \right\|_2 + \left\| \nabla^{k+2} f(n) \right\|_2 \left\| u \right\|_{L^\infty} \right\| \nabla^{k+2} E \|_2 \] (2.46)
and
\[ I_{42} \leq C_k \left( \left\| \nabla^{k+2} f(n) \right\|_2 \left\| \nabla u \right\|_{L^\infty} + \left\| \nabla f(n) \right\|_{L^\infty} \left\| \nabla^{k+2} u \right\|_2 \right\| \nabla^{k+1} E \|_2 \] (2.47)
As for the cases $l = k, k + 1$ for $I_4$, we can bound $I_{43}$ by
\[ I_{43} \leq C_k \left\| u \right\|_{H^{k+2} \cap H^2} \left( \left\| \nabla^{k+1} (n, u) \right\|^2_2 + \left\| \nabla^{k+1} E \right\|^2_2 \right). \] (2.48)
Hence, we have that for $l = k + 2$,
\[ I_4 \leq C_k \left\| u \right\|_{H^{k+1} \cap H^3} \left( \left\| \nabla^{k+1} (n, u) \right\|^2_2 + \left\| \nabla^{k+2} (n, u) \right\|^2_2 + \left\| \nabla^{k+1} E \right\|^2_2 \right) \] (2.49)
Consequently, plugging the estimates for $I_1 \sim I_4$ into (2.21) with $l = k, k + 1, k + 2$, and then summing up, we deduce (2.20).

Note that in Lemma 2.8 we only derive the dissipative estimate of $u$. We now recover the dissipative estimates of $n, E$ and $B$ by constructing some interactive energy functionals in the following lemma.
Lemma 2.9. For any integer \( k \geq 0 \), we have that for any small fixed \( \eta > 0 \),

\[
\frac{d}{dt} \left( \sum_{l=k}^{k+1} \int \nabla^l u \cdot \nabla \nabla^l u + \sum_{l=k}^{k+1} \int \nabla^l u \cdot \nabla^l u - \eta \int \nabla^k E \cdot \nabla \nabla^k E \times B \right) \\
+ \lambda \left( \sum_{l=k}^{k+2} \left\| \nabla^l u \right\|_{L^2}^2 + \sum_{l=k}^{k+1} \left\| \nabla^l E \right\|_{L^2}^2 + \left\| \nabla^{k+1} E \right\|_{L^2}^2 \right) \\
\leq C \sum_{l=k}^{k+2} \left\| \nabla^l u \right\|_{L^2}^2 + C \left( \left\| (n, u) \right\|_{H^{k+1} \cap H^3} + \left\| \nabla B \right\|_{L^2} \right) \left( \sum_{l=k}^{k+2} \left\| \nabla^l (n, u) \right\|_{L^2}^2 + \left\| \nabla^{k+1} B \right\|_{L^2}^2 \right).
\]  

(2.50)

Proof. We divide the proof into several steps.

Step 1: Dissipative estimate of \( n \).

Applying \( \nabla^l \) \((l = k, k + 1)\) to (1.3) and then taking the \( L^2 \) inner product with \( \nabla \nabla^l n \), we obtain

\[
\int \partial_t \nabla^l u \cdot \nabla \nabla^l u + \left\| \nabla \nabla^l n \right\|_{L^2}^2 \leq -\nu \int \nabla^l E \cdot \nabla \nabla^l n + C \left\| \nabla^l u \right\|_{L^2} \left\| \nabla^{l+1} n \right\|_{L^2} \\
+ \left\| \nabla^l (u \cdot \nabla n + \mu n \nabla n + u \times B) \right\|_{L^2} \left\| \nabla^{l+1} n \right\|_{L^2}.
\]  

(2.51)

The delicate first term on the left-hand side of (2.51) involves \( \partial_t \nabla^l u \), and the key idea is to integrate by parts in the \( t \)-variable and use the continuity equation (1.3). Thus integrating by parts for both the \( t \)- and \( x \)-variables, we obtain

\[
\int \nabla^l \partial_t u \cdot \nabla \nabla^l u = \frac{d}{dt} \int \nabla^l u \cdot \nabla \nabla^l u - \int \nabla^l u \cdot \nabla \nabla^l \partial_t n \\
= \frac{d}{dt} \int \nabla^l u \cdot \nabla \nabla^l u + \int \nabla^l \text{div} u \nabla^l \partial_t n \\
= \frac{d}{dt} \int \nabla^l u \cdot \nabla \nabla^l u - \left\| \nabla^l \text{div} u \right\|_{L^2}^2 - \int \nabla^l \text{div} u \nabla^l (u \cdot \nabla + \mu n \nabla u) \\
\geq \frac{d}{dt} \int \nabla^l u \cdot \nabla \nabla^l u - C \left\| \nabla^{l+1} u \right\|_{L^2}^2 - C \left\| \nabla^l (u \cdot \nabla n) \right\|_{L^2}^2 - C \left\| \nabla^l (\text{div} u) \right\|_{L^2}^2.
\]  

(2.52)

Using the commutator estimate of Lemma 2.3, we have

\[
\left\| \nabla^l (u \cdot \nabla n) \right\|_{L^2} \leq \left\| u \right\|_{L^\infty} \left\| \nabla^l n \right\|_{L^2} + \left\| \left[ \nabla^l , u \right] \cdot \nabla n \right\|_{L^2} \\
\leq \left\| u \right\|_{L^\infty} \left\| \nabla^{l+1} n \right\|_{L^2} + C_l \left\| \nabla^l u \right\|_{L^\infty} \left\| \nabla^l u \right\|_{L^2} + C_l \left\| \nabla^l u \right\|_{L^2} \left\| \nabla n \right\|_{L^\infty} \\
\leq C_l \left\| (n, u) \right\|_{H^3} \left( \left\| \nabla^l (n, u) \right\|_{L^2} + \left\| \nabla^{l+1} n \right\|_{L^2} \right).
\]  

(2.53)

Similarly,

\[
\left\| \nabla^l (\text{div} u) \right\|_{L^2} \leq C_l \left\| (n, u) \right\|_{H^3} \left( \left\| \nabla^l (n, u) \right\|_{L^2} + \left\| \nabla^{l+1} u \right\|_{L^2} \right).
\]  

(2.54)

Hence, we obtain

\[
\int \nabla^l \partial_t u \cdot \nabla \nabla^l u \geq \frac{d}{dt} \int \nabla^l u \cdot \nabla^l u - C \left\| \nabla^{l+1} u \right\|_{L^2}^2 \\
- C_l \left\| (n, u) \right\|_{H^3}^2 \left( \left\| \nabla^l (n, u) \right\|_{L^2}^2 + \left\| \nabla^{l+1} (n, u) \right\|_{L^2}^2 \right).
\]  

(2.55)
Next, integrating by parts and using the equation \((1.3)_3\), we have
\[
-\nu \int \nabla^l E \cdot \nabla \nabla^l n = \nu \int \nabla^l \text{div} E \nabla^l n = -\nu^2 \int \nabla^l f(n) \nabla^l n
\]
\[
= -\nu^2 \int \nabla^l (n + f(n) - n) \nabla^l n
\]
\[
\leq -\nu^2 \left\| \nabla^l n \right\|_{L^2}^2 + C_l \left\| n \right\|_{H^3} \left\| \nabla^l n \right\|_{L^2}^2.
\] (2.56)

Here we have used the estimate \(\left\| \nabla^l (f(n) - n) \right\|_{L^2} \leq C_l \left\| n \right\|_{H^3} \left\| \nabla^l n \right\|_{L^2}\), which follows by noticing that \(f(n) - n \sim n^2\) and the similar arguments presented in Lemma 2.2.

Lastly, as in \((2.53)-(2.54)\), we have
\[
\left\| \nabla^l (u \cdot \nabla u + \mu n \nabla n) \right\|_{L^2} \leq C_l \left\| (n, u) \right\|_{H^3} \left( \left\| \nabla^l (n, u) \right\|_{L^2} + \left\| \nabla^{l+1} (n, u) \right\|_{L^2} \right).
\] (2.57)

From the estimate of \(I_3\) in Lemma 2.8, we have that for \(l = k\) or \(k + 1\),
\[
\left\| \nabla^l (u \times B) \right\|_{L^2} \leq C_k \left( \left\| u \right\|_{H^{k+1} \cap H^2} + \left\| B \right\|_{L^2} \right) \left( \left\| \nabla^l u \right\|_{L^2} + \left\| \nabla^{k+1} B \right\|_{L^2} \right).
\] (2.58)

Plugging the estimates \((2.53)-(2.58)\) into \((2.51)\), by Cauchy’s inequality, we obtain
\[
\frac{d}{dt} \sum_{l=k}^{k+2} \int \nabla^l u \cdot \nabla \nabla^l n + \lambda \sum_{l=k}^{k+2} \left\| \nabla^l n \right\|_{L^2}^2 \leq C \sum_{l=k}^{k+2} \left\| \nabla^l u \right\|_{L^2}^2 + C_k \left( \left\| (n, u) \right\|_{H^{k+1} \cap H^2} + \left\| B \right\|_{L^2} \right) \left( \sum_{l=k}^{k+2} \left\| \nabla^l (n, u) \right\|_{L^2}^2 + \left\| \nabla^{k+1} B \right\|_{L^2}^2 \right).
\] (2.59)

This completes the dissipative estimate for \(n\).

**Step 2: Dissipative estimate of \(E\).**

Applying \(\nabla^l (l = k, k + 1)\) to \((1.3)_2\) and then taking the \(L^2\) inner product with \(\nabla^l E\), we obtain
\[
\int \nabla^l \partial_t u \cdot \nabla^l E + \nu \left\| \nabla^l E \right\|_{L^2}^2 \leq -\nu \int \nabla^l \nabla^l n \cdot \nabla^l E + C \left\| \nabla^l u \right\|_{L^2} \left\| \nabla^l E \right\|_{L^2}
\]
\[
+ \left\| \nabla^l (u \cdot \nabla u + \mu n \nabla n + u \times B) \right\|_{L^2} \left\| \nabla^l E \right\|_{L^2}.
\] (2.60)

Again, the delicate first term on the left-hand side of \((2.60)\) involves \(\partial_t \nabla^l u\), and the key idea is to integrate by parts in the \(t\)-variable and use the equation \((1.3)_3\) in the Maxwell system. Thus we obtain
\[
\int \nabla^l \partial_t u \cdot \nabla^l E = \frac{d}{dt} \int \nabla^l u \cdot \nabla^l E - \int \nabla^l u \cdot \nabla^l \partial_t E
\]
\[
= \frac{d}{dt} \int \nabla^l u \cdot \nabla^l E - \nu \left\| \nabla^l u \right\|_{L^2}^2 - \nu \int \nabla^l u \cdot \nabla^l (f(n)u + \nabla \times B).
\] (2.61)

From the estimates of \(I_4\) in Lemma 2.3 we have that
\[
\left\| \nabla^l (f(n)u) \right\|_{L^2} \leq C_l \left\| u \right\|_{H^{l+1} \cap H^2} \left\| \nabla^l (n, u) \right\|_{L^2}.
\] (2.62)

We must be much more careful with the remaining term in \((2.61)\) since there is no small factor in front of it. The key is to use Cauchy’s inequality and distinct the cases of \(l = k\) and \(l = k + 1\) due to the weakest dissipative estimate of \(B\). For \(l = k\), we have
\[
\nu \int \nabla^k u \cdot \nabla \nabla^k B \leq \varepsilon \left\| \nabla^{k+1} B \right\|_{L^2}^2 + C_\varepsilon \left\| \nabla^k u \right\|_{L^2}^2;
\] (2.63)

for \(l = k + 1\), integrating by parts, we obtain
\[
\nu \int \nabla^{k+1} u \cdot \nabla \nabla^{k+1} B = \nu \int \nabla \nabla^{k+1} u \cdot \nabla^{k+1} B \leq \varepsilon \left\| \nabla^{k+1} B \right\|_{L^2}^2 + C_\varepsilon \left\| \nabla^{k+2} u \right\|_{L^2}^2.
\] (2.64)
Plugging the estimates (2.61)–(2.64) and (2.57)–(2.58) from Step 1 into (2.60), by Cauchy’s inequality, we then obtain
\[
\frac{d}{dt} \sum_{l=k}^{k+1} \int \nabla^l u \cdot \nabla^l E + \lambda \sum_{l=k}^{k+1} \left\| \nabla^l E \right\|_{L^2}^2 \\
\leq \varepsilon \left\| \nabla^{k+1} B \right\|_{L^2}^2 + C \varepsilon \sum_{l=k}^{k+2} \left\| \nabla^l u \right\|_{L^2}^2 \\
+ C_k \left( \left\| (n, u) \right\|_{H^{k+1} \cap H^3}^2 + \left\| \nabla B \right\|_{L^2}^2 \right) \left( \sum_{l=k}^{k+2} \left\| \nabla^l (n, u) \right\|_{L^2}^2 + \left\| \nabla^{k+1} B \right\|_{L^2}^2 \right). 
\]

This completes the dissipative estimate for \( E \).

**Step 3: Dissipative estimate of \( B \).**

Applying \( \nabla^k \) to (1.3) and then taking the \( L^2 \) inner product with \( -\nabla \times \nabla^k B \), we obtain
\[
- \int \nabla^k \partial_t E \cdot \nabla \times \nabla^k B + \nu \left\| \nabla \times \nabla^k B \right\|_{L^2}^2 \\
\leq \nu \left\| \nabla^k u \right\|_{L^2} \left\| \nabla \times \nabla^k B \right\|_{L^2} + \nu \left\| \nabla^k (f(n)u) \right\|_{L^2} \left\| \nabla \times \nabla^k B \right\|_{L^2}. 
\]

Integrating by parts for both the \( t \)- and \( x \)-variables and using the equation (1.3) \(_4\), we have
\[
- \int \nabla^k \partial_t E \cdot \nabla \times \nabla^k B = - \frac{d}{dt} \int \nabla^k E \cdot \nabla \times \nabla^k B + \int \nabla \times \nabla^k E \cdot \nabla^k \partial_t B \\
= - \frac{d}{dt} \int \nabla^k E \cdot \nabla \times \nabla^k B - \nu \int \left\| \nabla \times \nabla^k E \right\|_{L^2}^2. 
\]

From the estimates of \( I_4 \) in Lemma 2.8, we have that
\[
\left\| \nabla^k (f(n)u) \right\|_{L^2} = C_k \left\| u \right\|_{H^k \cap H^2} \left\| \nabla^k (n, u) \right\|_{L^2}. 
\]

Plugging the estimates (2.61)–(2.68) into (2.66) and by Cauchy’s inequality, since \( \text{div} B = 0 \), we then obtain
\[
- \frac{d}{dt} \int \nabla^k E \cdot \nabla \times \nabla^k B + \lambda \left\| \nabla^{k+1} B \right\|_{L^2}^2 \\
\leq C \left\| \nabla^k u \right\|_{L^2}^2 + C \left\| \nabla^{k+1} E \right\|_{L^2}^2 + C_k \left\| u \right\|_{H^k \cap H^2} \left\| \nabla^k (n, u) \right\|_{L^2}^2. 
\]

This completes the dissipative estimate for \( B \).

**Step 4: Conclusion.**

Multiplying (2.69) by a small enough but fixed constant \( \eta \) and then adding it with (2.65) so that the second term on the right-hand side of (2.69) can be absorbed, then choosing \( \varepsilon \) small enough so that the first term in (2.69) can be absorbed; we obtain
\[
\frac{d}{dt} \left( \sum_{l=k}^{k+1} \int \nabla^l u \cdot \nabla^l E - \eta \int \nabla^k E \cdot \nabla \nabla^k B \right) + \lambda \sum_{l=k}^{k+1} \left\| \nabla^l E \right\|_{L^2}^2 + \left\| \nabla^{k+1} B \right\|_{L^2}^2 \\
\leq C \sum_{l=k}^{k+2} \left\| \nabla^l u \right\|_{L^2}^2 + C_k \left( \left\| (n, u) \right\|_{H^{k+1} \cap H^3}^2 + \left\| \nabla B \right\|_{L^2}^2 \right) \left( \sum_{l=k}^{k+2} \left\| \nabla^l (n, u) \right\|_{L^2}^2 + \left\| \nabla^{k+1} B \right\|_{L^2}^2 \right). 
\]

Adding the inequality above with (2.59), we get (2.50).

**2.3. Negative Sobolev estimates.** In this subsection, we will derive the evolution of the negative Sobolev norms of \( (u, E, B) \). In order to estimate the nonlinear terms, we need to restrict ourselves to that \( s \in (0, 3/2) \). We will establish the following lemma.

\[ \square \]
Lemma 2.10. For $s \in (0, 1/2]$, we have
\[
\frac{d}{dt} \left\{(n, u) \right\}_{\mathcal{H}^{-s}}^2 + \Lambda \|u\|_{\mathcal{H}^{-s}}^2 \lesssim \left(\|n, u\|_{\mathcal{H}^1}^2 + \|\nabla B\|_{\mathcal{H}^1}^2\right) \|(n, u, B)\|_{\mathcal{H}^{-s}} + \|E\|_{\mathcal{H}^2}^2.
\] (2.71)
and for $s \in (1/2, 3/2)$, we have
\[
\frac{d}{dt} \|(n, u, B)\|_{\mathcal{H}^{-s}}^2 + \Lambda \|u\|_{\mathcal{H}^{-s}}^2 \lesssim \left(\|n, u\|_{\mathcal{H}^1}^2 + \|B\|_{L^2}^{2-1/2} \|\nabla B\|_{L^2}^{3/2-s} \|u\|_{L^2}^2\right) \|(n, u, B)\|_{\mathcal{H}^{-s}} + \|E\|_{\mathcal{H}^2}^2.
\] (2.72)

Proof. The $\Lambda^{-s}$ ($s > 0$) energy estimate of (1.3.2)–(1.3.4) yield
\[
\frac{1}{2} \frac{d}{dt} \|u, E, B\|_{\mathcal{H}^{-s}}^2 + \nu \|u\|_{\mathcal{H}^{-s}}^2
\]
\[
= - \int \Lambda^{-s} (u \cdot \nabla u + \mu \nabla \nabla u + u \times B) \cdot \Lambda^{-s} u + \nu \int \Lambda^{-s} f(n) u \cdot \Lambda^{-s} E - \int \Lambda^{-s} \nabla n \cdot \Lambda^{-s} u
\]
\[
\lesssim \|u \cdot \nabla u + \mu \nabla \nabla u + u \times B\|_{\mathcal{H}^{-s}} \|u\|_{\mathcal{H}^{-s}} + ||f(n) u||_{\mathcal{H}^{-s}} \|E\|_{\mathcal{H}^{-s}} + ||\nabla n\|_{\mathcal{H}^{-s}} \|u\|_{\mathcal{H}^{-s}}.
\] (2.73)

We now restrict the value of $s$ in order to estimate the other terms on the right-hand side of (2.73). If $s \in (0, 1/2]$, then $1/2 + s/3 < 1$ and $3/s \geq 6$. Then applying Lemma 2.4 together with Hölder’s, Sobolev’s and Young’s inequalities, we obtain
\[
\|u \cdot \nabla u\|_{\mathcal{H}^{-s}} \lesssim \|u \cdot \nabla \|_{L^{\frac{1}{1+s}}(\mathbb{R}^3)} \lesssim \|u\|_{L^{3/s}} \|\nabla u\|_{L^2}
\]
\[
\lesssim \|\nabla u\|_{L^2}^{1/2+s} \|\nabla u\|_{L^2}^{1/2-s} \|\nabla u\|_{L^2}
\]
\[
\lesssim \|\nabla u\|_{H^1}^2 + \|\nabla u\|_{L^2}^2.
\] (2.74)

Similarly, we can bound
\[
\|n \nabla n\|_{\mathcal{H}^{-s}} \lesssim \|\nabla n\|_{H^1}^2 + \|\nabla n\|_{L^2}^2;
\]
\[
\|u \times B\|_{\mathcal{H}^{-s}} \lesssim \|B\|_{L^2}^{s-1/2} \|\nabla B\|_{L^2}^{3/2-s} \|u\|_{L^2}
\]
\[
\lesssim \|u\|_{H^1}^2 + \|u\|_{L^2}^2.
\] (2.75)

Now if $s \in (1/2, 3/2)$, we shall estimate the right-hand side of (2.73) in a different way. Since $s \in (1/2, 3/2)$, we have that $1/2 + s/3 < 1$ and $2 < 3/s < 6$. Then applying Lemma 2.4 and using (different) Sobolev’s inequality, we have
\[
\|u \cdot \nabla u\|_{\mathcal{H}^{-s}} \lesssim \|u\|_{L^{3/s}} \|\nabla u\|_{L^2} \lesssim \|u\|_{L^2}^{s-1/2} \|\nabla u\|_{L^2}^{3/2-s} \|\nabla u\|_{L^2}
\]
\[
\lesssim \|u\|_{H^1}^2 + \|\nabla u\|_{L^2}^2;
\]
\[
\|n \nabla n\|_{\mathcal{H}^{-s}} \lesssim \|n\|_{H^1}^2 + \|\nabla n\|_{L^2}^2;
\]
\[
\|u \times B\|_{\mathcal{H}^{-s}} \lesssim \|B\|_{L^2}^{s-1/2} \|\nabla B\|_{L^2}^{3/2-s} \|u\|_{L^2};
\]
\[
\|f(n) u\|_{\mathcal{H}^{-s}} \lesssim \|u\|_{H^1}^2 + \|n\|_{L^2}^2.
\] (2.78)

Note that we fail to estimate the remaining last term on the right-hand side of (2.73) as above. To overcome this obstacle, the key point is to make full use of the equation (1.3.5) to rewrite $n = n - f(n) + \nu^{-1} \text{div} E$. This idea was also used in [18–24]. Indeed, using (1.3.5), we have
\[
\|\nabla n\|_{\mathcal{H}^{-s}} \lesssim \|\Lambda^{-s} \nabla \text{div} E\|_{L^2} + ||\nabla (f(n) - n)\|_{\mathcal{H}^{-s}}
\]
\[
\lesssim ||E\|_{H^2}^2 + ||\nabla (f(n) - n)\|_{\mathcal{H}^{-s}}.
\] (2.82)

Here we have used the facts that $s < 3/2$ and $f(n) - n = O(n^2)$. Estimating the last term in (2.82) as before, and then collecting all the estimates we have derived, by Cauchy’s inequality, we deduce (2.71) for $s \in (0, 1/2]$ and (2.72) for $s \in (1/2, 3/2)$. □
2.4. Negative Besov estimates. In this section, we will derive the evolution of the negative Besov norms of \((u, E, B)\). The argument is similar to the previous subsection.

**Lemma 2.11.** For \(s \in (0, 1/2]\), we have
\[
\frac{d}{dt} \| (u, E, B) \|_{B^{-s}_{2, \infty}}^2 + \lambda \| u \|_{B^{-s}_{2, \infty}}^2 \lesssim \left( \| (n, u) \|_{H^1}^2 + \| \nabla B \|_{H^1}^2 \right) \| (u, E, B) \|_{B^{-s}_{2, \infty}}^2 + \| E \|_{H^2}^2; \tag{2.83}
\]
and for \(s \in (1/2, 3/2]\), we have
\[
\frac{d}{dt} \| (u, E, B) \|_{B^{-s}_{2, \infty}}^2 + \lambda \| u \|_{B^{-s}_{2, \infty}}^2 \lesssim \left( \| (n, u) \|_{H^1}^2 + \| B \|_{L^2}^{s-1/2} \| \nabla B \|_{L^2}^{3/2-s} \| u \|_{L^2} \right) \| (u, E, B) \|_{B^{-s}_{2, \infty}}^2 + \| E \|_{H^2}^2. \tag{2.84}
\]

**Proof.** The \(\hat{\Delta}_j\) energy estimates of (1.3) and (1.4) yield, with multiplication of \(2^{-2s_j}\) and then taking the supremum over \(j \in \mathbb{Z}\),
\[
\frac{1}{2} \frac{d}{dt} \| (u, E, B) \|_{B^{-s_{j}}_{2, \infty}}^2 + \nu \| u \|_{B^{-s_{j}}_{2, \infty}}^2 \lesssim \sup_{j \in \mathbb{Z}} 2^{-2s_j} \left( - \int \hat{\Delta}_j (u \cdot \nabla u + \mu n \nabla n + u \times B) \cdot \Delta_j u \right) + \sup_{j \in \mathbb{Z}} 2^{-2s_j} \left( \nu \int \hat{\Delta}_j (\delta (n) u) \cdot \Delta_j E - \int \hat{\Delta}_j \nabla n \cdot \hat{\Delta}_j u \right) \lesssim \| u \cdot \nabla u + \mu n \nabla n + u \times B \|_{B^{-s_{j}}_{2, \infty}} \| u \|_{B^{-s_{j}}_{2, \infty}} + \| \nabla \delta (n) u \|_{B^{-s_{j}}_{2, \infty}} \| E \|_{B^{-s_{j}}_{2, \infty}} + \| \nabla n \|_{B^{-s_{j}}_{2, \infty}} \| u \|_{B^{-s_{j}}_{2, \infty}}. \tag{2.85}
\]

Then the proof is exactly same as the proof of Lemma 2.10 except that we should apply Lemma 2.5 instead to estimate the \(B^{-s_{j}}_{2, \infty}\) norm. Note that we allow \(s = 3/2\). \(\square\)

### 3. Proof of Theorems

3.1. **Proof of Theorem 1.1.** In this subsection, we will prove the unique global solution to the system (1.3), and the key point is that we only assume the \(H^3\) norm of initial data is small.

**Step 1. Global small \(E_3\) solution.**

We first close the energy estimates at the \(H^3\) level by assuming a priori that \(\sqrt{\mathcal{E}_3(t)} \leq \delta\) is sufficiently small. Taking \(k = 0, 1\) in (2.20) of Lemma 2.8 and then summing up, we obtain
\[
\frac{d}{dt} \sum_{l=0}^{3} \| \nabla^l (n, u, E, B) \|_{L^2}^2 + \lambda \sum_{l=0}^{3} \| \nabla^l u \|_{L^2}^2 \lesssim \sqrt{\mathcal{E}_3} \mathcal{D}_3 + \sqrt{\mathcal{D}_3} \sqrt{\mathcal{E}_3} \lesssim \delta \mathcal{D}_3. \tag{3.1}
\]

Taking \(k = 0, 1\) in (2.50) of Lemma 2.9 and then summing up, we obtain
\[
\frac{d}{dt} \left( \sum_{l=0}^{2} \int \nabla^l u \cdot \nabla \nabla^l n + \sum_{l=0}^{2} \int \nabla^l u \cdot \nabla^l E - \eta \sum_{l=0}^{1} \int \nabla^l E \cdot \nabla \nabla^l B \right) + \lambda \left( \sum_{l=0}^{3} \| \nabla^l n \|_{L^2}^2 + \sum_{l=0}^{2} \| \nabla^l E \|_{L^2}^2 + \sum_{l=1}^{2} \| \nabla^l B \|_{L^2}^2 \right) \lesssim \sum_{l=0}^{3} \| \nabla^l u \|_{L^2}^2 + \delta^2 \mathcal{D}_3. \tag{3.2}
\]

Multiplying (3.2) by a sufficiently small but fixed factor \(\epsilon\) and then adding it with (3.1), since \(\delta\) is small, we deduce that there exists an instant energy functional \(\tilde{\mathcal{E}}_3\) equivalent to \(\mathcal{E}_3\) such that
\[
\frac{d}{dt} \tilde{\mathcal{E}}_3 + \mathcal{D}_3 \leq 0. \tag{3.3}
\]

Integrating the inequality above directly in time, we obtain (1.10). By a standard continuity argument, we then close the a priori estimates if we assume at initial time that \(\mathcal{E}_3(0) \leq \delta_0\) is sufficiently small. This concludes the unique global small \(E_3\) solution.
Step 2. Global $E_N$ solution.

We shall prove this by an induction on $N \geq 3$. By (1.10), then (1.11) is valid for $N = 3$. Assume (1.11) holds for $N - 1$ (then now $N \geq 4$). Taking $k = 0, \ldots, N-2$ in (2.20) of Lemma 2.8 and then summing up, we obtain

$$
\frac{d}{dt} \sum_{l=0}^{N} \left\| \nabla^l (n, u, E, B) \right\|^2_{L^2} + \lambda \sum_{l=0}^{N} \left\| \nabla^l u \right\|^2_{L^2} \leq C_N \sqrt{D_{N-1}} \sqrt{\mathcal{E}_N} \sqrt{D_N} + C_N \sqrt{D_{N-1}} \sqrt{\mathcal{E}_N} \sqrt{D_N} \leq C_N \sqrt{D_{N-1}} \sqrt{\mathcal{E}_N} \sqrt{D_N}.
$$

(3.4)

Here we have used the fact that $3 \leq \frac{N-2}{2} + 2 \leq N - 2 + 1$ since $N \geq 4$. Note that it is important that we have put the two first factors in (2.20) into the dissipation.

Taking $k = 0, \ldots, N-2$ in (2.50) of Lemma 2.9 and then summing up, we obtain

$$
\frac{d}{dt} \sum_{l=0}^{N-1} \int \nabla^l u \cdot \nabla \nabla^l n + \sum_{l=0}^{N-1} \int \nabla^l u \cdot \nabla^l E - \eta \sum_{l=0}^{N-2} \int \nabla^l E \cdot \nabla \times \nabla^l B \right)
+ \lambda \left( \sum_{l=0}^{N} \left\| \nabla^l n \right\|^2_{L^2} + \sum_{l=0}^{N-1} \left\| \nabla^l E \right\|^2_{L^2} + \sum_{l=0}^{N-1} \left\| \nabla^l B \right\|^2_{L^2} \right) \leq C \sum_{l=0}^{N} \left\| \nabla^l u \right\|^2_{L^2} + C_N \sqrt{D_{N-1}} \sqrt{\mathcal{E}_N} \sqrt{D_N}.
$$

(3.5)

Multiplying (3.5) by a sufficiently small factor $\varepsilon$ and then adding it with (3.4), we deduce that there exists an instant energy functional $\tilde{\mathcal{E}}_N$ equivalent to $\mathcal{E}_N$ such that, by Cauchy’s inequality,

$$
\frac{d}{dt} \tilde{\mathcal{E}}_N + D_N \leq C_N \sqrt{D_{N-1}} \sqrt{\tilde{\mathcal{E}}_N} \sqrt{D_N} \leq \varepsilon D_N + C_N \varepsilon D_{N-1} \mathcal{E}_N.
$$

(3.6)

This implies

$$
\frac{d}{dt} \tilde{\mathcal{E}}_N + \frac{1}{2} D_N \leq C_N \mathcal{D}_{N-1} \mathcal{E}_N.
$$

(3.7)

We then use the standard Gronwall lemma and the induction hypothesis to deduce that

$$
\mathcal{E}_N(t) + \int_0^t D_N(\tau) d\tau \leq C \mathcal{E}_N(0) e^{C_N \int_0^t \mathcal{D}_{N-1}(\tau) d\tau} \leq C \mathcal{E}_N(0) e^{C_N \mathcal{P}_{N-1}(\mathcal{E}_{N-1}(0))} \leq C \mathcal{E}_N(0) e^{C_N \mathcal{P}_{N-1}(\mathcal{E}_{N}(0))} = P_N(\mathcal{E}_N(0)).
$$

(3.8)

This concludes the global $\mathcal{E}_N$ solution. The proof of Theorem 1.1 is completed. $\square$

3.2. Proof of Theorem 1.2. In this subsection, we will prove the various time decay rates of the unique global solution to the system (1.3) obtained in Theorem 1.1. Fix $N \geq 5$. We need to assume that $\mathcal{E}_N(0) \leq \delta_0 = \delta_0(N)$ is small. Then Theorem 1.1 implies that there exists a unique global $\mathcal{E}_N$ solution, and $\mathcal{E}_N(t) \leq P_N(\mathcal{E}_N(0)) \leq \delta_0$ is small for all time $t$. Since now our $\delta_0$ is relative small with respect to $N$, we just ignore the $N$ dependence of the constants in the energy estimates in the previous section.

Step 1. Basic decay.

For the convenience of presentations, we define a family of energy functionals and the corresponding dissipation rates with minimum derivative counts as

$$
\mathcal{E}^{k+2} = \sum_{l=k}^{k+2} \left\| \nabla^l (n, u, E, B) \right\|^2_{L^2}
$$

(3.9)

and

$$
\mathcal{D}^{k+2} = \sum_{l=k}^{k+2} \left\| \nabla^l (n, u) \right\|^2_{L^2} + \sum_{l=k}^{k+1} \left\| \nabla^l E \right\|^2_{L^2} + \left\| \nabla^{k+1} B \right\|^2_{L^2}.
$$

(3.10)
By Lemma 2.8, we have that for $k = 0, \ldots, N - 2$,
\[
\frac{d}{dt} \left( \sum_{l=k}^{k+2} \left\| \nabla^l (n, u, E, B) \right\|_{L^2}^2 + \sum_{l=k}^{k+2} \left\| \nabla^l u \right\|_{L^2}^2 \right) \leq \sqrt{\delta_0 D_k^{k+2} + \|(n, u)\|_{L^\infty} \left\| \nabla^{k+2} (n, u) \right\|_{L^2} \left\| \nabla^{k+2} (E, B) \right\|_{L^2}}.
\] (3.11)

By Lemma 2.9, we have that for $k = 0, \ldots, N - 2$,
\[
\frac{d}{dt} \left( \sum_{l=k}^{k+2} \left\| \nabla^l u \right\|_{L^2}^2 \right) 
+ \lambda \left( \sum_{l=k}^{k+2} \left\| \nabla^l n \right\|_{L^2}^2 + \sum_{l=k}^{k+2} \left\| \nabla^l E \right\|_{L^2}^2 + \left\| \nabla^{k+1} B \right\|_{L^2}^2 \right) 
\leq \sum_{l=k}^{k+2} \left\| \nabla^l u \right\|_{L^2}^2 + \delta_0 \left( \sum_{l=k}^{k+2} \left\| \nabla^l (n, u) \right\|_{L^2}^2 \right). \quad (3.12)
\]

Multiplying (3.12) by a sufficiently small but fixed factor $\varepsilon$ and then adding it with (3.11), since $\delta_0$ is small, we deduce that there exists an instant energy functional $\tilde{E}_k^{k+2}$ equivalent to $E_k^{k+2}$ such that
\[
\frac{d}{dt} \tilde{E}_k^{k+2} + D_k^{k+2} \lesssim \|(n, u)\|_{L^\infty} \left\| \nabla^{k+2} (n, u) \right\|_{L^2} \left\| \nabla^{k+2} (E, B) \right\|_{L^2}. \quad (3.13)
\]

Note that we cannot absorb the right-hand side of (3.13) by the dissipation $D_k^{k+2}$ since it does not contain $\left\| \nabla^{k+2} (E, B) \right\|_{L^2}^2$. We will distinguish the arguments by the value of $k$. If $k = 0$ or $k = 1$, we bound $\left\| \nabla^{k+2} (E, B) \right\|_{L^2}$ by the energy. Then we have that for $k = 0, 1$,
\[
\frac{d}{dt} \tilde{E}_k^{k+2} + D_k^{k+2} \lesssim \sqrt{D_k^{k+2}} \sqrt{D_k^{k+2}} \sqrt{E_3} \lesssim \sqrt{\delta_0 D_k^{k+2}}, \quad (3.14)
\]
which implies
\[
\frac{d}{dt} \tilde{E}_k^{k+2} + D_k^{k+2} \leq 0. \quad (3.15)
\]

If $k \geq 2$, we have to bound $\left\| \nabla^{k+2} (E, B) \right\|_{L^2}$ in term of $\left\| \nabla^{k+1} (E, B) \right\|_{L^2}$ since $\sqrt{D_k^{k+2}}$ cannot control $\|(n, u)\|_{L^\infty}$. The key point is to use the regularity interpolation method developed in [17]. By Lemma 2.11, we have
\[
\|(n, u)\|_{L^\infty} \left\| \nabla^{k+2} (n, u) \right\|_{L^2} \left\| \nabla^{k+2} (E, B) \right\|_{L^2} \lesssim \|(n, u)\|_{L^2} \left( \frac{1}{L^2} \right) \left\| \nabla^k (n, u) \right\|_{L^2} \left\| \nabla^{k+2} (n, u) \right\|_{L^2} \left\| \nabla^{k+1} (E, B) \right\|_{L^2} \left( \frac{1}{L^2} \right) \left\| \nabla^\alpha (E, B) \right\|_{L^2} \left( \frac{1}{L^2} \right),
\] (3.16)

where $\alpha$ is defined by
\[
k + 2 = (k + 1) \times \left( 1 - \frac{3}{2k} \right) + \alpha \times \frac{3}{2k} \implies \alpha = \frac{5}{3} k + 1. \quad (3.17)
\]

Hence, for $k \geq 2$, if $N \geq \frac{5}{3} k + 1 \iff 2 \leq k \leq \frac{3}{5} (N - 1)$, then by (3.13), we deduce from (3.13) that
\[
\frac{d}{dt} \tilde{E}_k^{k+2} + D_k^{k+2} \lesssim \sqrt{E_N D_k^{k+2}} \lesssim \sqrt{\delta_0 D_k^{k+2}}, \quad (3.18)
\]
which allow us to arrive at that for any integer $k$ with $0 \leq k \leq \frac{3}{5} (N - 1)$ (note that $N - 2 \geq \frac{3}{5} (N - 1) \geq 2$ since $N \geq 5$), we have
\[
\frac{d}{dt} \tilde{E}_k^{k+2} + D_k^{k+2} \leq 0. \quad (3.19)
\]

The fact that $D_k^{k+2}$ is weaker than $E_k^{k+2}$ prevents the exponential decay of the solution. In order to effectively derive the decay rate from (3.19), we still manage to bound the missing terms in the energy, that is, $\left\| \nabla^k B \right\|_{L^2}^2$ and $\left\| \nabla^{k+2} (E, B) \right\|_{L^2}^2$ in terms of $E_k^{k+2}$ in (3.19). We again
use the regularity interpolation method, but now we need to also do the Sobolev interpolation between the negative and positive Sobolev norms. Assuming for the moment that we have proved (1.14) or (1.15). Using Lemma 3.6 we have that for \( s \geq 0 \) and \( k + s > 0 \),

\[
\left\| \nabla^k B \right\|_{L^2} \leq \left\| \nabla^k B \right\|_{H^{-s}} \leq C_0 \left\| \nabla^{k+1} B \right\|_{L^2}. \tag{3.20}
\]

Similarly, using Lemma 2.7, we have that for \( s > 0 \) and \( k + s > 0 \),

\[
\left\| \nabla^k B \right\|_{L^2} \leq \left\| \nabla^{k+1} B \right\|_{L^2} \leq C_0 \left\| \nabla^k B \right\|_{L^2}. \tag{3.21}
\]

On the other hand, for \( k + 2 < N \), we have

\[
\left\| \nabla^{k+2}(E, B) \right\|_{L^2} \leq \left\| \nabla^{k+1}(E, B) \right\|_{L^2} \left\| \nabla^N(E, B) \right\|_{L^2} \leq C_0 \left\| \nabla^{k+1}(E, B) \right\|_{N^{-2}}. \tag{3.22}
\]

Then we deduce from (3.19) that

\[
\frac{d}{dt} \mathcal{E}^{k+2} + \left\{ \mathcal{E}^{k+2}_k \right\}^{1+\vartheta} \leq 0, \tag{3.23}
\]

where \( \vartheta = \max \left\{ \frac{1}{k+s}, \frac{1}{N-k-2} \right\} \). Solving this inequality directly, we obtain in particular that

\[
\mathcal{E}^{k+2}_k(t) \leq \left\{ \left( \mathcal{E}^{k+2}_k(0) \right)^{-\vartheta} + \vartheta t \right\}^{-1/\vartheta} \leq C_0(1 + t)^{-1/\vartheta} = C_0(1 + t)^{-\min\{k+s,N-k-2\}}. \tag{3.24}
\]

Notice that (3.24) holds also for \( k + s = 0 \) or \( k + 2 = N \). So, if we want to obtain the optimal decay rate of the whole solution for the spatial derivatives of order \( k \), we only need to assume \( N \) large enough (for fixed \( k \) and \( s \)) so that \( k + s \leq N - k - 2 \). Thus we should require that

\[
N \geq \max \left\{ k + 2, \frac{5}{3}k + 1, 2k + 2 + s \right\} = 2k + 2 + s. \tag{3.25}
\]

This proves the optimal decay (1.16).

Finally, we turn back to prove (1.14) and (1.15). First, we prove (1.14) by using Lemma 3.10. However, we are not able to prove them for all \( s \in [0,3/2] \) at this moment. We must distinguish the arguments by the value of \( s \). First, for \( s \in (0,1/2] \), integrating (2.71) in time, by (1.10) we obtain that for \( s \in (0,1/2] \),

\[
\left\| (u, E, B)(t) \right\|_{H^{-s}}^2 \leq \left\| (u_0, E_0, B_0) \right\|_{H^{-s}}^2 + \int_0^t D_3(\tau) \left( 1 + \left\| (u, E, B)(\tau) \right\|_{H^{-s}} \right) d\tau \\lesssim C_0 \left( 1 + \sup_{0 \leq \tau \leq t} \left\| (u, E, B)(\tau) \right\|_{H^{-s}} \right). \tag{3.26}
\]

By Cauchy’s inequality, this together with (1.10) gives (1.14) for \( s \in [0,1/2] \) and thus verifies (1.10) for \( s \in [0,1/2] \). Next, we let \( s \in (1/2, 1) \). Observing that we have \( (u_0, E_0, B_0) \in \dot{H}^{-1/2} \) since \( \dot{H}^{-s} \cap L^2 \subset \dot{H}^{-s'} \) for any \( s' \in [0, s] \), we then deduce from what we have proved for (1.10) with \( s = 1/2 \) that the following decay result holds:

\[
\left\| \nabla^k (n, u, E, B)(t) \right\|_{H^2} \leq C_0(1 + t)^{-\frac{k+1/2}{2}} \quad \text{for} \quad k = 0, 1. \tag{3.27}
\]
Here, since we have required $N \geq 5$ and now $s = 1/2$, so we can have taken $k = 1$ in (1.16). Thus by (3.27), (1.10) and Hölder’s inequality, we deduce from (2.72) that for $s \in (1/2, 1)$,

$$\|(u, E, B)(t)\|_{\dot{H}^{-s}}^2 \lesssim \|(u_0, E_0, B_0)\|_{\dot{H}^{-s}}^2 + \int_0^t \mathcal{D}_3(\tau) \left(1 + \|(u, E, B)(\tau)\|_{\dot{H}^{-s}}\right) d\tau$$

$$+ \int_0^t \|B(\tau)\|_{L^2}^{2-s} \sqrt{\mathcal{D}_3(\tau)} \|(u, E, B)(\tau)\|_{\dot{H}^{-s}} d\tau$$

$$\leq C_0 \left(1 + \left(1 + \int_0^t (1 + \tau)^{-2(1-s/2)} d\tau\right) \sup_{0 \leq \tau \leq t} \|(u, E, B)(\tau)\|_{\dot{H}^{-s}}\right)$$

$$\leq C_0 \left(1 + \sup_{0 \leq \tau \leq t} \|(u, E, B)(\tau)\|_{\dot{H}^{-s}}\right).$$

(3.28)

Here we have used the fact $s \in (1/2, 1)$ so that the time integral in (3.28) is finite. This gives (1.14) for $s \in (1/2, 1)$ and thus verifies (1.16) for $s \in (1/2, 1)$. Now let $s \in [1,3/2)$. We choose $s_0$ so that $s - 1/2 < s_0 < 1$. Hence, $(u_0, E_0, B_0) \in \dot{H}^{-s_0}$. We then deduce from what we have proved for (1.16) with $s = s_0$ that the following decay result holds:

$$\left\|\nabla^k (u, E, B)(t)\right\|_{\dot{H}^s} \leq C_0 (1 + t)^{-\frac{k+s_0}{2}}$$

for $k = 0, 1$.

(3.29)

Here, since we have required $N \geq 5$ and now $s = s_0 < 1$, so we can have taken $k = 1$ in (1.16). Thus by (3.28) and Hölder’s inequality, we deduce from (2.72) that for $s \in [1,3/2)$, similarly as in (3.28),

$$\|(u, E, B)(t)\|_{\dot{H}^{-s}}^2 \leq C_0 \left(1 + \left(1 + \int_0^t (1 + \tau)^{-(s_0+3/2)} d\tau\right) \sup_{0 \leq \tau \leq t} \|(u, E, B)(\tau)\|_{\dot{H}^{-s}}\right)$$

$$\leq C_0 \left(1 + \sup_{0 \leq \tau \leq t} \|(u, E, B)(\tau)\|_{\dot{H}^{-s}}\right).$$

(3.30)

Here we have used the fact $s - s_0 < 1/2$ so that the time integral in (3.30) is finite. This gives (1.14) for $s \in [1,3/2)$ and thus verifies (1.16) for $s \in [1,3/2)$. Note that (1.15) can be proved similarly except that we use instead Lemma 2.11.

Step 2. Further decay.

We first prove (1.17) and (1.18). First, noticing that $\nu f(n) = \text{div} E$, by (1.16) and Lemma 2.2 if $N \geq 2k + 4 + s$, then

$$\left\|\nabla^k n(t)\right\|_{L^2} \lesssim \left\|\nabla^k f(n)(t)\right\|_{L^2} \lesssim \left\|\nabla^{k+1} E(t)\right\|_{L^2} \lesssim C_0 (1 + t)^{-\frac{k+s_0}{2}},$$

(3.31)

where we have used $n = f^{-1}(f(n))$.

Next, applying $\nabla^k$ to (1.30), (1.33) and then multiplying the resulting identities by $\nabla^k u$, $\nabla^k E$ respectively, summing up and integrating over $\mathbb{R}^3$, we obtain

$$\frac{1}{2} \frac{d}{dt} \int \left\|\nabla^k (u, E)\right\|^2 + \nu \left\|\nabla^k u\right\|_{L^2}^2$$

$$= - \int \nabla^k \left(\nabla n + u \cdot \nabla u + \mu n \nabla n + u \times B\right) \cdot \nabla^k u + \nu \int \nabla^k \left(\nabla \times B + f(n) u\right) \cdot \nabla^k E$$

$$\lesssim \left\|\nabla^{k+1} n\right\|_{L^2} \left\|\nabla^k u\right\|_{L^2} + \left\|\nabla^k (u \cdot \nabla u + \mu n \nabla n + u \times B)\right\|_{L^2} \left\|\nabla^k u\right\|_{L^2}$$

$$+ \left\|\nabla^k \left(\nabla \times B + f(n) u\right)\right\|_{L^2} \left\|\nabla^k E\right\|_{L^2}.$$ 

(3.32)

On the other hand, taking $l = k$ in (2.60), we may have

$$\int \nabla^k \partial_t u \cdot \nabla^k E + \nu \left\|\nabla^k E\right\|_{L^2}^2$$

$$\lesssim \left(\left\|\nabla^{k+1} n\right\|_{L^2} + \left\|\nabla^k u\right\|_{L^2}\right) \left\|\nabla^k E\right\|_{L^2} + \left\|\nabla^k (u \cdot \nabla u + \mu n \nabla n + u \times B)\right\|_{L^2} \left\|\nabla^k E\right\|_{L^2}.$$ 

(3.33)
Substituting (2.61) with \( l = k \) into (3.31), we may then have
\[
\frac{d}{dt} \int \nabla^k u \cdot \nabla^k E + \nu \| \nabla^k E \|_{L^2}^2 \\
\leq C \| \nabla^k u \|_{L^2}^2 + C \left( \left\| \nabla^{k+1} n \right\|_{L^2} + \left\| \nabla^k u \right\|_{L^2} \right) \| \nabla^k E \|_{L^2}^2 \\
+ \left\| \nabla^k \left( u \cdot \nabla u + \mu n \nabla n + u \times B \right) \right\|_{L^2} \| \nabla^k E \|_{L^2} + \left\| \nabla^k \left( \nabla \times B + f(n) u \right) \right\|_{L^2} \| \nabla^k u \|_{L^2}^2.
\]
(3.34)

Multiplying (3.34) by a sufficiently small but fixed factor \( \varepsilon \) and then adding it with (3.32), since \( \varepsilon \) is small, we deduce that there exists \( \mathcal{F}_k(t) \) equivalent to \( \| \nabla^k (u, E)(t) \|_{L^2} \) such that, by Cauchy’s inequality, (2.25), (2.29), (2.41), (1.16) and (3.31),
\[
\frac{d}{dt} \mathcal{F}_k(t) + \mathcal{F}_k(t) \\
\lesssim \| \nabla^{k+1} n \|_{L^2}^2 + \left\| \nabla^{k+1} B \right\|_{L^2}^2 + \left\| \nabla^k \left( u \cdot \nabla u + \mu n \nabla n + u \times B \right) \right\|_{L^2}^2 + \left\| \nabla^k f(n) u \right\|_{L^2}^2 \\
\lesssim \| \nabla^{k+1} (n, B) \|_{L^2}^2 + \left( \| u \|_{H^{1/2}} + \| \nabla B \|_{L^2} \right)^2 \| \nabla^{k+1} B \|_{L^2}^2 + \| (n, u) \|_{L^\infty}^2 \| \nabla^{k+1} (n, u) \|_{L^2}^2 \\
\leq C_0 (1 + t)^{-(k+1+s)}.
\]
(3.35)

where we required \( N \geq 2k + 4 + s \). Applying the standard Gronwall lemma to (3.35), we obtain
\[
\mathcal{F}_k(t) \leq \mathcal{F}_k(0) e^{-t} + C_0 \int_0^t e^{-(t-t')} (1 + \tau)^{-(k+1+s)} \, d\tau \lesssim C_0 (1 + t)^{-(k+1+s)}.
\]
(3.36)

This implies
\[
\| \nabla^k (u, E)(t) \|_{L^2} \lesssim \sqrt{\mathcal{F}_k(t)} \lesssim C_0 (1 + t)^{-\frac{k+1+s}{2}}.
\]
(3.37)

We thus complete the proof of (1.17). Notice that (1.18) now follows by (3.31) with the improved decay rate of \( E \) in (1.17), just requiring \( N \geq 2k + 6 + s \).

Now we prove (1.19). Assuming \( B_\infty = 0 \), then we can extract the following system from (1.3)\_2 - (1.3)\_3, denoting \( \psi = \text{div} u \),
\[
\begin{align*}
\partial_t n + \psi &= -u \cdot \nabla n - \mu n \text{div} u, \\
\partial_t \psi + \nu \psi - \nu^2 n &= -\Delta n - \text{div}(u \cdot \nabla u + \mu n \nabla n + u \times B) + \nu^2 (f(n) - n).
\end{align*}
\]
(3.38)

Applying \( \nabla^k \) to (3.38) and then multiplying the resulting identities by \( \nu^2 \nabla^k n \), \( \nabla^k \psi \), respectively, summing up and integrating over \( \mathbb{R}^3 \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \int \nu^2 \| \nabla^k n \|^2 + \| \nabla^k \psi \|^2 + \nu \| \nabla^k \psi \|_{L^2}^2 \\
= -\nu^2 \int \nabla^k (u \cdot \nabla n + \mu n \text{div} u) \nabla^k n - \int \nabla^k \Delta n \nabla^k \psi \\
- \int \nabla^k \left[ \text{div}(u \cdot \nabla u + \mu n \nabla n + u \times B) - \nu^2 (f(n) - n) \right] \nabla^k \psi.
\]
(3.39)

Applying \( \nabla^k \) to (3.38)\_1 and then multiplying by \( -\nabla^k n \), as before integrating by parts over \( t \) and \( x \) variables and using the equation (3.38)\_1, we may obtain
\[
-\frac{d}{dt} \int \nabla^k \psi \nabla^k n + \nu^2 \| \nabla^k n \|_{L^2}^2 \\
= \| \nabla^k \psi \|_{L^2}^2 + \nu \int \nabla^k n \nabla^k \psi + \int \nabla^k (u \cdot \nabla n + \mu n \text{div} u) \nabla^k \psi \\
+ \int \nabla^k \left[ \Delta n + \text{div}(u \cdot \nabla u + \mu n \nabla n + u \times B) - \nu^2 (f(n) - n) \right] \nabla^k n.
\]
(3.40)
Applying the Gronwall lemma again, we obtain
\[ \frac{d}{dt} G_k(t) + G_k(t) \lesssim \left\| \nabla^{k+2} n \right\|_{L^2}^2 + \left\| \nabla^{k+1} (u \cdot \nabla u) \right\|_{L^2}^2 + \left\| \nabla^{k+1} (n \nabla n) \right\|_{L^2}^2 + \left\| \nabla^{k+1} (u \times B) \right\|_{L^2}^2 + \left\| \nabla^{k} (u \cdot \nabla n) \right\|_{L^2}^2 + \left\| \nabla^{k} (n \div u) \right\|_{L^2}^2. \] (3.41)

Notice that we have used the following estimates, by using the arguments in Lemma 2.2.2
\[ \left\| \nabla^{k} (f (n) - n) \right\|_{L^2}^2 \lesssim C_k \left\| n \right\|_{H^5}^2 \left\| \nabla^{k} n \right\|_{L^2}^2 \lesssim \delta_0 \left\| \nabla^{k} n \right\|_{L^2}^2. \] (3.42)
By Lemma 2.3 and Cauchy’s inequality, we obtain
\[ \left\| \nabla^{k+1} (u \times B) \right\|_{L^2}^2 = \left\| u \times \nabla^{k+1} B + \nabla^{k+1} (u) \times B \right\|_{L^2}^2 \lesssim \left\| u \times \nabla^{k+1} B \right\|_{L^2}^2 + \left\| \nabla^{k+1} (u) \times B \right\|_{L^2}^2 \lesssim \| u \|_{L^\infty}^2 \left\| \nabla^{k+1} B \right\|_{L^2}^2 + \left\| \nabla^{k} B \right\|_{L^2}^2 + \left\| \nabla^{k+1} u \right\|_{L^2}^2 \left\| B \right\|_{L^\infty}^2. \] (3.43)

The other nonlinear terms on the right-hand side of (3.41) can be estimated similarly. Hence, we deduce from (3.41) that, by (1.16)–(1.18),
\[ \frac{d}{dt} G_k(t) + G_k(t) \lesssim \left\| \nabla^{k+2} n \right\|_{L^2}^2 + \left\| u \right\|_{L^\infty}^2 \left\| \nabla^{k+1} B \right\|_{L^2}^2 + \left\| \nabla u \right\|_{L^\infty}^2 \left\| \nabla^{k} B \right\|_{L^2}^2 + \left\| B \right\|_{L^\infty}^2 \left\| \nabla^{k+1} u \right\|_{L^2}^2 \]
\[ + \left\| (n, u) \right\|_{L^\infty}^2 \left\| \nabla^{k+2} (n, u) \right\|_{L^2}^2 + \left\| \nabla (n, u) \right\|_{L^\infty}^2 \left\| \nabla^{k+1} (n, u) \right\|_{L^2}^2 \]
\[ \leq C_0 \left( (1 + t)^{-(k+3+s)} + (1 + t)^{-(k+7/2+2s)} + (1 + t)^{-(k+11/2+2s)} \right)
\[ \leq C_0 (1 + t)^{-(k+3+s)}, \]
where we required \( N \geq 2k + 6 + s \). Applying the Gronwall lemma to (3.44) again, we obtain
\[ G_k(t) \leq G_k(0) e^{-t} + C_0 \int_0^t e^{-(t-\tau)} (1 + \tau)^{-(k+3+s)} d\tau \leq C_0 (1 + t)^{-(k+3+s)}. \] (3.45)
This implies
\[ \left\| \nabla^{k} (n, \psi) (t) \right\|_{L^2} \lesssim \sqrt{G_k(t)} \leq C_0 (1 + t)^{-\frac{k+3+s}{2}}. \] (3.46)
If required \( N \geq 2k + 10 + s \), then by (3.46), we have
\[ \left\| \nabla^{k+2} n (t) \right\|_{L^2} \lesssim C_0 (1 + t)^{-\frac{k+3+s}{2}}. \] (3.47)
Having obtained such faster decay, we can then improve (3.44) to be
\[ \frac{d}{dt} G_k(t) + G_k(t) \leq C_0 \left( (1 + t)^{-(k+5+s)} + (1 + t)^{-(k+7/2+2s)} \right) \leq C_0 (1 + t)^{-(k+7/2+2s)}. \] (3.48)
Applying the Gronwall lemma again, we obtain
\[ \left\| \nabla^{k} (n, \psi) (t) \right\|_{L^2} \lesssim \sqrt{G_k(t)} \leq C_0 (1 + t)^{-\frac{k+7/4+s}{2}}. \] (3.49)
We thus complete the proof of (1.19). The proof of Theorem 1.2 is completed.
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