A Superconducting “Dripping Faucet”

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When a current is applied to a type-I superconducting strip containing a narrow channel across its width, magnetic flux spots nucleate at the edge and are then driven along the channel by the current. These flux “drops” are reminiscent of water drops dripping from a faucet, a model system for studying low-dimensional chaos. We use a novel high-bandwidth Hall probe to detect in real time the motion of individual flux spots moving along the channel. Analyzing the time series consisting of the intervals between successive flux drops, we find distinct regions of chaotic behavior characterized by positive Lyapunov exponents, indicating that there is a close analogy between the dynamics of the superconducting and water drop systems.

Water dripping from a faucet is an everyday phenomenon that illustrates many of the basic ideas of deterministic chaos [1,2,3]. As the drip rate is increased, the intervals between successive drops pass through regimes of periodic, multiply-periodic, and chaotic behavior. In the chaotic regime, the drop dynamics is deterministic, in that next several drop intervals can be predicted from previous intervals. There is, however, no long-term predictability of the system; a small change in an earlier interval will lead to an exponentially different set of future intervals. In a simple model [4], a drop grows until it reaches a threshold size; it then detaches and falls under the influence of gravity. The remaining water oscillates as a drop begins to grow again. A key element of this model is that the time that any drop detaches is causally related to the time the previous drop detached, via the “memory” of that previous time stored in the oscillations. It is in this way that the time interval between two drops is deterministically related to earlier drop intervals. Yet, because of the delicacy of the threshold condition, small changes in the state of an earlier drop can have a profound effect on later ones.

Interestingly, very similar physics may govern the current-driven nucleation, growth, and breakoff of magnetic flux spots or drops in type-I superconductors. When current is passed along a thin-film strip of type-I material containing a narrow channel across its width, flux begins to enter the channel at its ends [5,6,7,8,9]. This region of flux grows to some critical size and then breaks off as a flux spot, containing \( \sim 100 \) flux quanta \( \Phi_0 \), which is then driven down the channel by the current. The analogy with water drops is readily apparent, and suggests that in this superconducting dripping faucet chaotic dynamics might be observable for flux drops as well.

In the experiment we report here, a high-bandwidth Hall sensor is used to directly measure the passage of individual flux spots along a channel formed in a lead strip. Just as for water drops falling from a faucet, we find that the flux spot dynamics can be periodic, with single or multiple periods, or nonperiodic, with a broad distribution of drop intervals. Applying the tools of nonlinear time-series analysis to the data, we find that these nonperiodic regimes are in fact chaotic, characterized by positive Lyapunov exponents, allowing the prediction of the drop sequence roughly five drops into the future.

The sample geometry used is similar to that of Chimienti et al. [3]. In a two-step process, a 1-µm-thick lead film is first evaporated onto a sapphire substrate, leading to a 1-mm-wide strip bridged by a 3-µm-wide gap defined by liftoff. Then a second lead strip, 4 µm thick and 160 µm wide, is evaporated on top of the first. As shown schematically in Fig. 1(a), this results in a channel that is 3 µm wide, 1 µm deep, and 160 µm long. The films are of high quality, with \( T_c \approx 7.2 \) K and \( R_{300 \text{ K}}/R_{10 \text{ K}} > 500 \).

Hall sensors, with active areas of \( \approx 1.5 \times 1.5 \mu m \), were fabricated from high-quality GaAs/AlGaAs heterostructures. The Hall voltage was detected using a cooled JFET preamplifier mounted near the Hall probe. At room temperature, the Hall signal was further amplified and then passed through an antialiasing filter with a bandwidth of 5 MHz. The Hall probes were mounted on a scanning head that allowed the probe to be moved over any point of the channel, or to be scanned over the surface for magnetic imaging.

In dripping faucet experiments, the drop dynamics are typically investigated as a function of the average drip rate, a rate usually controlled by varying the pressure.
behind the nozzle. In our experiment, we can vary the
flux-spot nucleation rate by changing the external current
flowing through the strip. Figure 1(b) shows a magnetic
image, taken with the Hall probe in scanning mode, of
the strip with a current flowing along it. The bright band
along the upper edge of the sample, and the dark band
along its lower edge, reflect the magnetic field due to
this applied current. When the current exceeds a critical
value $I_c$, this magnetic field becomes large enough to al-
low the nucleation and subsequent breakoff of flux drops
from the channel ends. Because the field is oppositely
directed at the two edges, the flux drops from each edge
have opposite signs as well. As Fig. 1(b) shows, these
drops of opposite sign are then driven by the current to-
ward the middle of the channel, where they annihilate.

In order to detect the motion of individual flux spots in
real time, the Hall probe was held fixed over one point of
the channel, about 40 $\mu$m (or 1/4 of its length) from the
channel’s end. Here, we are well away from the very edge
where the flux spots nucleate and break off, so that we
observe the dynamics of only well-formed flux spots. At
the same time, this position is far from the annihilation
point, so that we avoid the complicating effects of the
annihilation process.

The results reported here were obtained in zero applied
field at a temperature of 4.5 K. At higher temperatures
close to $T_c$, the magnetic field of the flux spots is weak
and difficult to observe; at low temperatures the criti-
cal current becomes impractically large. The results at
other temperatures, or for other samples, look qualita-
tively similar, but differ in their details. At 4.5 K, the
current was swept from the critical current $I_c = 497$ mA
to a threshold current $I_t = 590$ mA at which continuous
flux flow occurred in the channel. The Hall probe signal
was digitized at a rate of $10^7$ samples/s, with a total of
16 $\times$ $10^6$ points taken in one run.

Figure 2 shows short segments of an entire data run.
The signal consists of well-defined Hall voltage pulses
as each flux spot passes beneath the probe. In Fig. 2(a),
taken at a current somewhat above the critical current,
we see a train of pulses with two distinct periods. As the
current is increased, the behavior changes to that shown
in Fig. 2(b). Here we observe a complex train of larger
and smaller pulses, with no evident periodicity. As the
current is swept through this low-current Region I, the
behavior changes alternately between the periodic type
behavior seen in Fig. 2(a) and the more complex behavior
in Fig. 2(b). A fundamental question to be addressed
concerns the nature of the complex dynamical behavior
in Fig. 2(b). Are the pulse times and sizes only the result
of some stochastic process, or is the underlying dynamics
in fact deterministic?

As the current is further increased, the behavior changes
Suddenly to the purely period-one behavior shown in Fig. 2(c). We call the fairly wide range of cur-
rents over which this periodic behavior is observed Region
II. Finally, at the highest currents, the flux-flow dynam-
ics enters Region III. As shown in Fig. 2(d), the Hall
voltage consists of fairly flat-topped pulses interspersed
with occasional short pulses. We interpret this pattern as
representing elongated flux “sausages” interspersed with
more circular flux spots. As the current is further in-
creased, Region III ends when the flux spots merge and
continuous flux flow occurs in the channel.

One run of 16 million data points contains some
600,000 pulses representing a variety of dynamical
regimes. In order to analyze this large data set we need a
way to characterize the data in a concise way. For water
drop experiments a commonly used measure is the time
interval $\Delta t$ between successive drops. Each drop interval
can then be plotted versus the driving parameter, such
as the water pressure at the nozzle. We have found it
useful to plot our data in a similar way. The time at
which a flux “drop” occurs is determined by when the
voltage crosses a certain threshold. The resulting bifur-
cation diagram is shown in Fig. 3(a), in which we plot the
time intervals $\Delta t$ between the 638,848 individual drops
observed during the run. The gray-scale intensity of the
image is proportional to the probability of finding, at a
given driving current, a particular value of $\Delta t$. The three
regions I–III previously described are readily apparent.

Region I is characterized by regions of singly- and
multiply-periodic behavior interspersed with more com-
plex regions distinguished by broad distributions of
“dripping” time intervals. We’ll discuss this interesting
region in more detail below. In Region II, purely periodic
behavior is observed, as indicated by the single $\Delta t$ ob-
served at any given current. As the current is increased,
this period decreases, indicating higher-frequency flux
nucleation. Finally, at high currents, Region III emerges.
This region is similar in appearance to Region I, with
some periodic sections mixed with more complex regimes.
(At the tail end of Region II there is a short section with
continuous flux flow.)
In Fig. 3(b) is shown an enlarged view of the most interesting section of Region I. At this level of detail it is clear, for example, that the pulse train in Fig. 2(a) is not truly periodic. First, the shorter drop interval observed in Fig. 2(a) can be seen to actually consist of two possible intervals of 2.11 and 2.25 µs. Second, the longer time interval of about 4.2 µs is broadened by about 0.16 µs. At the current corresponding to the pulse train shown in Fig. 2(b), a very broad range of drop intervals is apparent, reflecting the multitude of pulse intervals observed in Fig. 2(b). Other regions in Fig. 3(b) also exhibit multiple periodicity, quasiperiodicity, and broad distributions of ∆t with no evident periodicity.

We have discussed how the dynamics of a real dripping faucet is governed by highly nonlinear processes. Our qualitative analysis of our “superconducting dripping faucet” suggests that nonlinear processes are at play here as well. And, since flux drops represent a driven, dissipative system it seems possible that the system’s behavior is determined by a chaotic attractor rather than just being a stochastic manifestation of flux spot nucleation. To investigate this we have analyzed the sequence of 638,847 drop intervals using the TISEAN [10] package for nonlinear time series analysis, calculating specific quantities such as the largest Lyapunov exponent and the correlation fractal dimension. In order to make such an analysis, stationary sequences—those in which the underlying governing dynamics is unchanging—should be used. We have chosen the finite sequences 1–3 indicated in Fig. 3(b) as approximations of true stationary sequences. These sequences contain 4000, 2000, and 3000 drop intervals, respectively. Our quantitative analysis is performed in the associated phase space constructed using the method of time delay reconstruction [11], and a nonlinear noise reduction algorithm was applied to the data before further analysis.

We begin our time-series analysis by computing Lyapunov exponents, which characterize the evolution of the separation between two nearby trajectories in phase space. If the dynamics is governed by deterministic chaos then nearby trajectories diverge exponentially and the largest Lyapunov exponent is positive. We have used the algorithm of Kantz [10, 12] to study this divergence. For sequence 1 at the left of Fig. 4, the average separation between points in phase space, starting with an average separation of about 0.008 (ln(0.008) = −4.8), increases linearly on this log-lin graph, so that there is indeed an exponential divergence of nearby trajectories. This linear increase extends over about 4–5 consecutive steps, indicating that weak correlations still exist between a given drop interval and one four or five drops later. Similar results hold for sequences 2 and 3. At large enough time steps the originally nearby points become completely uncorrelated, and the curve begins to approach the size of the attractor. The curves with unfilled markers in Fig. 4 are calculated for surrogate data obtained by phase-randomizing the data [13] from sequences 1 and 3. In this case, the slope of the average expansion rate is almost vertical: Any two points are completely uncorrelated, and their average distance immediately jumps to the average size of the (randomized) attractor. This clear distinction between the original and the surrogate data proves that our dynamics is incompatible with a linear stochastic process, but instead is well-described by a nonlinear deterministic process.

The correlation dimension [14] quantifies the self-similarity exhibited by the attractor’s structure in phase space. By counting the points inside of a ball which is moved along the phase-space trajectory we obtain the correlation sum $C(\varepsilon)$ as a function of the ball radius $\varepsilon$. 

![FIG. 3: Bifurcation diagram for “dripping” time intervals. (a) Drop intervals over the entire range of currents for which pulses were observed. Arrows a–d denote the currents at which the segments of pulses shown in Fig. 2 were taken. (b) An expanded section of Regions I and II. Short sequences 1–3 are analyzed in more detail using the tools of nonlinear analysis.](image-url)
In Fig. 4 we show the divergence of trajectories from sequences 1–3 of Fig. 2. The graphs for sequences 2 and 3 are offset horizontally by 7 and 14 units respectively. The embedding dimension for this analysis was $m = 5$.

In Fig. 5 we show the phase-space distance-dependence of the correlation sum for sequence 1, calculated for different embedding dimensions $m$. The linear scaling region between $0.034 \mu s$ and $0.09 \mu s$ is due to the points in the attractor formed in a “sea” of continuous positive flux, in which case the breakoff time of the next drop is sensitively dependent on the position and size of previous drops. This leads to the chaotic behavior observed in Region I. As the current is further increased, the drop intervals shrink and the interactions become stronger. It appears that in this regime the interactions act to “lock” together successive drops in a highly periodic way, leading to our observed Region II. At the tail end of Region II the drops are so closely spaced that they merge into continuous flux flow. At the highest currents, this continuous flow begins to break up again, but as Fig. 2(d) showed, the signal consists regions of flux separated by short segments of zero flux. We can think of these short segments as negative-going pulses moving in a “sea” of continuous positive flux, in which case the dynamics should be similar to that in Region I, as is in fact observed.

Although no general theory exists for calculating this nucleation, growth, and breakoff mechanism, the remarks just given are enough to envision the kinds of correlations that such a theory would yield. At low driving currents, drops are well-separated and, experimentally, their nonlinear interactions are evidently such that the breakoff time of the next drop is sensitively dependent on the position and size of previous drops. This leads to the chaotic behavior observed in Region I. As the current is further increased, the drop intervals shrink and the interactions become stronger. It appears that in this regime the interactions act to “lock” together successive drops in a highly periodic way, leading to our observed Region II. At the tail end of Region II the drops are so closely spaced that they merge into continuous flux flow. At the highest currents, this continuous flow begins to break up again, but as Fig. 2(d) showed, the signal consists regions of flux separated by short segments of zero flux. We can think of these short segments as negative-going pulses moving in a “sea” of continuous positive flux, in which case the dynamics should be similar to that in Region I, as is in fact observed.

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[1] P. Martien, S. C. Pope, P. L. Scott, and R. S. Shaw, Phys. Lett. A 110, 399 (1985).
[2] X. Wu and Z. Schelly, Physica D 40, 433 (1989).
[3] K. Dreyer and F. Hickey, Am. J. Phys. 59, 619 (1991).
[4] R. Shaw, The Dripping Faucet as a Model Chaotic System (Aerial Press, Santa Cruz, 1984).
[5] D. Chimenti, H. Watson, and R. Huebener, J. Low Temp. Phys. 23, 303 (1976).
[6] D. Chimenti and R. Huebener, Solid State Commun. 21, 467 (1977).
[7] D. Chimenti and J. R. Clem, Phil. Mag. 38, 635 (1978).
[8] V. Hurm, K.-P. Selig, and R. P. Huebener, Z. Physik B 32, 175 (1979).
[9] B. Mühlemeier, J. Parisi, R. Huebener, and W. Buck, J. Low Temp. Phys. 64, 131 (1986).
[10] R. Hegger, H. Kantz, and T. Schreiber, Chaos 9, 413 (1999).
[11] F. Takens, Lect. Not. Math. 898, 366 (1981).
[12] H. Kantz, Phys. Lett. A 185, 77 (1994).
[13] T. Schreiber and A. Schmitz, Physica D 142, 346 (2000).
[14] P. Grassberger and I. Procaccia, Physica D 9, 189 (1983).
[15] W. Buck and J. Parisi, Z. Naturforsch. 44a, 247 (1989).