On Banach spaces with the Tsirelson property

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Dedicated to the memory of S. Banach.

Abstract. A Banach space \( X \) is said to have the Tsirelson property if it does not contain subspaces that are isomorphic to \( l_p \) \((1 \leq p < \infty) \) or \( c_0 \). The article contains a quite simple method to producing Banach spaces with the Tsirelson property.

A Banach space \( X \) is said to have the Tsirelson property if it does not contain subspaces that are isomorphic to \( l_p \) \((1 \leq p < \infty) \) or \( c_0 \).

The first example of a Banach space with such property was constructed by B.S. Tsirelson [1].

The article contains a quite simple method to producing Banach spaces with the Tsirelson property. Results were communicated at the Ukrainian Mathematical Congress (August 2001, Kiev) and was announced in the Book of Abstracts of the International Conference on Functional Analysis that take place as a part of the Congress [2].

1. Definitions and notations

Let \( \mathcal{B} \) be a (proper) class of all Banach spaces.

**Definition 1.** Let \( X, Y \in \mathcal{B} \). \( X \) is finitely representable in \( Y \) (in symbols: \( X <^f Y \)) if for each \( \varepsilon > 0 \) and for every finite dimensional subspace \( A \) of \( X \) there exists a subspace \( B \) of \( Y \) and an isomorphism \( u : A \to B \) such that

\[
\|u\| \|u^{-1}\| \leq 1 + \varepsilon.
\]

Spaces \( X \) and \( Y \) are said to be finitely equivalent if \( X <^f Y \) and \( Y <^f X \).

Any Banach space \( X \) generates classes

\[
X^f = \{ Y \in \mathcal{B} : X \sim_Y Y \} \quad \text{and} \quad X <^f = \{ Y \in \mathcal{B} : Y <^f X \}
\]

For any two Banach spaces \( X, Y \) their Banach-Mazur distance is given by

\[
d(X, Y) = \inf\{ \|u\| \|u^{-1}\| : u : X \to Y \},
\]

where \( u \) runs all isomorphisms between \( X \) and \( Y \) and is assumed, as usual, that \( \inf \emptyset = \infty \).

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It is well known that \( \log d(X, Y) \) defines a metric on each class of isomorphic Banach spaces. A set \( M_n \) of all \( n \)-dimensional Banach spaces, equipped with this metric, is a compact metric space that is called the Minkowski compact \( M_n \). A disjoint union \( \cup \{ M_n : n < \infty \} = M \) is a separable metric space, which is called the Minkowski space.

Consider a Banach space \( X \). Let \( H(X) \) be a set of all its different finite dimensional subspaces (isometric finite dimensional subspaces of \( X \) in \( H(X) \) are identified). Thus, \( H(X) \) may be regarded as a subset of \( M \), equipped with the restriction of the metric topology of \( M \).

Of course, \( H(X) \) need not to be a closed subset of \( M \). Its closure in \( M \) will be denoted \( \overline{H(X)} \). From definitions follows that \( X \sim_f Y \) if and only if \( \overline{H(X)} \subseteq \overline{H(Y)} \).

Spaces \( X \) and \( Y \) are finitely equivalent (in symbols: \( X \sim_f Y \)) if simultaneously \( X \sim_f Y \) and \( Y \sim_f X \). Therefore, \( X \sim_f Y \) if and only if \( \overline{H(X)} = \overline{H(Y)} \).

There is a one to one correspondence between classes of finite equivalence \( X^f = \{ Y \in B : X \sim_f Y \} \) and closed subsets of \( M \) of kind \( \overline{H(X)} \).

Indeed, all spaces \( Y \) from \( X^f \) have the same set \( \overline{H(X)} \). This set, uniquely determined by \( X \) (or, equivalently, by \( X^f \)), will be denoted by \( M(X^f) \) and will be referred as to the Minkowski’s base of the class \( X^f \).

**Definition 2.** For a Banach space \( X \) its \( l_p \)-spectrum \( S(X) \) is given by

\[
S(X) = \{ p \in [0, \infty] : l_p \sim_f X \}.
\]

Certainly, if \( X \sim_f Y \) then \( S(X) = S(Y) \). Thus, the \( l_p \)-spectrum \( S(X) \) may be regarded as a property of the whole class \( X^f \). So, notations like \( S(X^f) \) are of obvious meaning.

**Definition 3.** Let \( X \) be a Banach space. It is called:

* c-convex, if \( \infty \notin S(X) \);
* B-convex, if \( 1 \notin S(X) \);
* Finite universal, if \( \infty \in S(X) \).

Let \( \{ X_i : i \in I \} \) be a collection of Banach spaces. A space

\[
l_p(X_i, I) = \left( \sum \oplus \{ X_i : i \in I \} \right)_p
\]

is a Banach space of all families \( \{ x_i \in X_i : i \in I \} = x \), with a finite norm

\[
\| x \|_p = \left( \sum \{ \| x_i \|_{X_i} : i \in I \} \right)^{1/p}.
\]

If \( I = \mathbb{N} \), then instead \( l_p(X_i, \mathbb{N}) \) it will be written \( l_p(X_i) \). If all \( X_i \)'s are equal to a given Banach space \( X \), then the notation \( l_p(X) \) is used.

If \( I \) consists of two elements only (say, \( I = \{ 1, 2 \} \)), then \( l_p(X_i, I) \) is denoted by \( X_1 \oplus_p X_2 \).

**Definition 4.** Let \( I \) be a set; \( D \) be an ultrafilter over \( I \); \( \{ X_i : i \in I \} \) be a family of Banach spaces. An ultraproduct \( (X_i)_D \) is a quotient space

\[
(X)_D = l_\infty(X_i, I)/N(X_i, D),
\]

where \( l_\infty(X_i, I) \) is a Banach space of all families \( x = \{ x_i \in X_i : i \in I \} \), for which

\[
\| x \| = \sup \{ \| x_i \|_{X_i} : i \in I \} < \infty;
\]
N (X_i, D) is a subspace of \( l_\infty (X_i, I) \), which consists of such \( x \)'s that
\[
\lim_D \| x_i \|_{X_i} = 0.
\]

If all \( X_i \)'s are all equal to a space \( X \in B \) then the ultraproduct is said to be the \textit{ultrapower} and is denoted by \((X)_D\).

An operator \( d_X : X \to (X)_D \) that asserts to any \( x \in X \) an element \((x)_D \in (X)_D\), which is generated by a stationary family \( \{ x_i = x : i \in I \} \), is called the \textit{canonical embedding} of \( X \) into its ultrapower \((X)_D\).

It is well-known that a Banach space \( X \) is finitely representable in a Banach space \( Y \) if and only if there exists such ultrafilter \( D \) (over \( I = \cup D \)) that \( X \) is isometric to a subspace of the ultrapower \((Y)_D\).

In what follows a notion of inductive (or direct) limit will be used. Recall a definition.

Let \( \langle I, \ll \rangle \) be a \textit{partially ordered set}. It said to be \textit{directed} (to the right hand) if for any \( i, j \in I \) there exists \( k \in I \) such that \( i \ll k \) and \( j \ll k \).

Let \( \{ X_i : i \in I \} \) be a set of Banach spaces that are indexed by elements of a directed set \( \langle I, \ll \rangle \). Let \( m_{i,j} : X_i \to X_j \) be isomorphic embeddings.

**Definition 5.** A system \( \{ X_i, m_{i,j} : i, j \in I; i \ll j \} \) is said to be an inductive (or direct) system if
\[
m_{i,k} = m_{i,j} \circ m_{j,k}
\]
for all \( i \ll j \ll k \) \((\text{Id}_Y \text{ denotes the identical operator on } Y)\).

Let
\[
X = \cup \{ X_i \times \{ i \} : i \in I \}
\]

Elements of \( X \) are pairs \((x, i)\), where \( x \in X_i \). Let \( =_{eq} \) be a relation of equivalence of elements of \( X \), which is given by the following rule:
\[
(x, i) =_{eq} (y, j) \text{ if } m_{i,k}x = m_{j,k}y \text{ for some } k \in I.
\]

A class of all elements of \( X \) that are equivalent to a given \((x, i)\) is denoted as
\[
[x, i] = \{ (y, j) : (y, j) =_{eq} (x, i) \}.
\]

A set of all equivalence classes \([x, i]\) is denoted \( X_\infty \). Clearly, \( X_\infty \) is a linear space. Let \( \| [x, i] \| = \lim I \| m_{i,j}x \|_{X_j} \) be a semi-norm on \( X_\infty \). Let
\[
\text{Null}(X) = \{ [x, i] : \| [x, i] \| = 0 \}.
\]

**Definition 6.** A \textit{direct limit of the inductive system} \( \{ X_i, m_{i,j} : i, j \in I; i \ll j \} \) is a quotient space
\[
\lim_D X_i = X_\infty / \text{Null}(X).
\]

Let \( X \in B; \omega \) be an infinite cardinal; \( \dim (X) = \omega \); \( \alpha \) be an infinite limit ordinal (so, \( \alpha \) may be considered as a cardinal); \( \alpha \leq \omega \).

**Definition 7.** An \( \alpha \)-sequence \( \{ x_\beta : \beta < \alpha \} \) of elements of \( X \) is said to be

- **Spreading, if for every \( n < \omega \) (\( \omega \) denotes the first infinite cardinal or, equivalently, ordinal), every \( \varepsilon > 0 \), every set of scalars \( \{ a_k : k < n \} \) and any choosing of \( i_0 < i_1 < ... < i_{n-1} < \alpha; j_0 < j_1 < ... < j_{n-1} < \alpha \)
\[
\left\| \sum_{k<\alpha} a_k x_{i_k} \right\| \leq \left\| \sum_{k<\alpha} a_k x_{j_k} \right\|.
\]
• C-unconditional, where $C < \infty$ is a constant, if

$$C^{-1} \left\| \sum_{k<n} a_k \epsilon_k x_{ik} \right\| \leq \left\| \sum_{k<n} a_k x_{ik} \right\| \leq C \left\| \sum_{k<n} a_k \epsilon_k x_{ik} \right\|$$

for any choosing of $n < \omega$; $\{a_k : k < n\}$; $i_0 < i_1 < \ldots < i_{n-1} < \alpha$ and of signs $\{\epsilon_k \in \{+, -\} : k < n\}$.

• Unconditional, if it is C-unconditional for some $C < \infty$.

• Symmetric, if for any $n < \omega$, any finite subset $I \subset \alpha$ of cardinality $n$, any rearrangement $\varsigma$ of elements of $I$ and any scalars $\{\alpha_i : i \in I\}$,

$$\left\| \sum_{i \in I} \alpha_i z_i \right\| = \left\| \sum_{i \in I} \alpha_{\varsigma(i)} z_i \right\|$$.

• Subsymmetric, if it is both spreading and 1-unconditional.

Let $C < \infty$ be a constant. Two $\alpha$-sequences $\{x_\beta : \beta < \alpha\}$ and $\{y_\beta : \beta < \alpha\}$ are said to be C-equivalent if for any finite subset $I = \{i_0 < i_1 < \ldots < i_{n-1}\}$ of $\alpha$ and for any choosing of scalars $\{a_k : k < n\}$

$$C^{-1} \left\| \sum_{k<n} a_k x_{ik} \right\| \leq \left\| \sum_{k<n} a_k y_{ik} \right\| \leq C \left\| \sum_{k<n} a_k x_{ik} \right\| .$$

Two $\alpha$-sequences $\{x_\beta : \beta < \alpha\}$ and $\{y_\beta : \beta < \alpha\}$ are said to be equivalent if they are C-equivalent for some $C < \infty$.

**Remark 1.** The same definitions may be used in a case when instead of $\alpha$-sequences will be regarded families $\{x_i : i \in I\} \subset X$ indexed by elements of a linearly ordered set $(I, \prec)$. In such a case it will be said about spreading families, unconditionals families and so on. $\omega$-sequences will be called sequences and may be denoted like $(x_n)$.

### 2. Indices of divisibility

**Definition 8.** Let $1 \leq p \leq \infty$, $X^f$ be a class of finite equivalence. $X^f$ is said to be p-divisible if for some $Y \in X^f$ the space $l_p(Y)$ belongs to $X^f$.

**Remark 2.** It is easy to see that if $X^f$ is p-divisible then for every $Z \in X^f$ a space $l_p(Z)$ belongs to $X^f$ too. Indeed, since $l_p(Y)$ belongs to $X^f$, then for any ultrafilter $D$ ($l_p(Y)_D \in X^f$). Certainly, $l_p(Y)_D$ is isometric to a subspace of $(l_p(Y))_D$. If $D$ is such that $Z$ is isometric to a subspace of $(Y)_D$ then

$$l_p(Z) \hookrightarrow l_p((Y)_D) \hookrightarrow (l_p(Y))_D$$

and, hence, $l_p(Z) \in X^f$.

**Remark 3.** A simple criterion on $X^f$ to be p-divisible is: for any pair $A$, $B \in \mathfrak{M}(X^f)$ their $l_p$-sum $A \oplus_p B$ belongs to $\mathfrak{M}(X^f)$. Clearly, this condition satisfies when $X^f$ is p-divisible. Conversely, if $A$, $B \in \mathfrak{M}(X^f)$ implies $A \oplus_p B \in \mathfrak{M}(X^f)$ for any $A$, $B$, then the space $W = l_p(A_i)_i$, where $\{A_i : i \in I\}$ is a numeration of all spaces from $\mathfrak{M}(X^f)$, belonging to $X^f$. Obviously, $l_p(W) \in X^f$ and, thus, $X^f$ is p-divisible.

Let $X, Y$ be Banach spaces. It will be said that $X$ is $Y$-saturated, if any infinite-dimensional subspace of $X$ contains a subspace, which is isomorphic to $Y$.

**Theorem 1.** Every $p$-divisible class $X^f$ contains a separable $l_p$-saturated space.
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Proof. Let \((A_n)_{n<\infty}\) be a dense subset of \(\mathfrak{M}(X^f)\). Obviously, \(l_p(A_n) \in X^f\) and is \(l_p\)-saturated.

This simple result will play an important role in the following result.

Definition 9. Let \(X^f\) be a class of finite equivalence. Its index of divisibility, \(\text{Index}(X^f)\) is a set of all such \(p \in [1, \infty]\) that \(X^f\) is \(p\)-divisible:

\[
\text{Index}(X^f) = \{p \in [1, \infty] : W \in X^f \implies l_p(W) \in X^f\}.
\]

For some classes \(X^f\) their index of divisibility may be empty, \(\text{Index}(X^f) = \emptyset\).

E.g., \(\text{Index}((l_p \oplus l_q)^f) = 0\) if \(p \neq q\).

Sometimes \(\text{Index}(X^f)\) consists of a single point:

\[
\text{Index}((l_p)^f) = \{p\}.
\]

The maximal set \(\text{Index}(X^f)\) has the class \((l_\infty)^f\):

\[
\text{Index}((l_\infty)^f) = [1, \infty].
\]

Indeed, the space \(l_p(l_\infty) \in (l_\infty)^f\) for any \(p\).

Theorem 2. If \(\text{Card}(\text{Index}(X^f)) \geq 2\) then the class \(X^f\) contains a space that has the Tsirelson property.

Proof. Let \(p, q \in \text{Index}(X^f); p \neq q\). Then \(X^f\) contains two separable spaces: \(X_p\), which is \(p\)-saturated and \(X_q\), which is \(q\)-saturated. Clearly, both and \(X_q\) may be represented as closure of unions of chains of their finite dimensional subspaces:

\[
X_p = \bigcup A_k; A_1 \hookrightarrow A_2 \hookrightarrow ... \hookrightarrow A_n \hookrightarrow ...;
\]

\[
X_q = \bigcup B_k; B_1 \hookrightarrow B_2 \hookrightarrow ... \hookrightarrow B_n \hookrightarrow ....
\]

Chose a sequence \(\varepsilon_n \downarrow 0\) and define inductively a sequence of isomorphic embeddings.

At the first step find the least number \(n(1)\) such that there exists an isomorphic embedding \(u_1: A_1 \rightarrow B_{n(1)}\) with \(\|u_1\|\|u_1^{-1}\| \leq 1 + \varepsilon_1\); at the same step choose the minimal \(m(2)\) such that there exists an isomorphic embedding \(u_2: B_{n(1)} \rightarrow A_{m(2)}\) with \(\|u_2\|\|u_2^{-1}\| \leq 1 + \varepsilon_2\).

If operators \(u_1, u_2, \ldots, u_{2k}\) and numbers \(n(1), m(2), n(3), \ldots, m(2k)\) are already chosen, then \(n(2k+1)\) is the least number such that there exists an isomorphic embedding \(u_{2k+1}: A_{m(2k)} \rightarrow B_{n(2k+1)}\) with \(\|u_{2k+1}\|\|u_{2k+1}^{-1}\| \leq 1 + \varepsilon_{2k+1}\).

Also, chose \(m(2k+2)\) as the least number such that there exists an isomorphic embedding \(u_{2k+2}: B_{n(2k+1)} \rightarrow A_{m(2k+2)}\) with \(\|u_{2k+2}\|\|u_{2k+2}^{-1}\| \leq 1 + \varepsilon_{2k+2}\).

As a result it will be obtained a chain of finite dimensional spaces from \(\mathfrak{M}(X^f)\):

\[
cA_1 \rightarrow B_{n(1)} \rightarrow A_{m(2)} \rightarrow ... \rightarrow A_{m(2k)} \rightarrow B_{n(2k+1)} \rightarrow ...
\]

which may be regarded as a direct system.

Let \(Y\) be a direct limit of a given direct system. Obviously, \(Y\) may be represented as the closure of a chain

\[
A'_1 \hookrightarrow B'_1 \hookrightarrow A'_m(2) \hookrightarrow ... \hookrightarrow A'_m(2k) \hookrightarrow B'_m(2k+1) \hookrightarrow ...Y
\]

of its finite dimensional subspaces, such that

\[
d(A'_m(2k), A_m(2k)) \leq 1 + \varepsilon_m(2k); \quad d(B'_m(2k+1), B_m(2k+1)) \leq 1 + \varepsilon_m(2k+1).
\]

Assume that \(l_r\) is isomorphic to a subspace of \(Y\). Let \(j: l_r \rightarrow Y\).
Then a chain

\[ jl \cap A_1' \rightarrow jl \cap B_{n(1)}' \leftarrow ... \rightarrow jl \cap A_{m(2k)}' \leftarrow jl \cap B_{n(2k+1)}' \rightarrow \ldots \]

contains a sub-chain

\[ C_1 \leftarrow C_2 \leftarrow ... \leftarrow C_k \leftarrow \ldots \]

where \( C_k \) is \( \lambda \)-isomorphic to a \( l_{r}^{(s_k)} \) for some \( s_k \) and some \( \lambda < \infty \), which does not depend on \( k \).

However, this is impossible: if \( \{ jl \cap B_{n(2k+1)}' : k < \infty \} \) contains such a chain, then \( l_r \) must be isomorphic to a subspace of \( X_q \). If \( \{ jl \cap A_{m(2k)}' : k < \infty \} \) contains such a chain, then \( l_r \) must be isomorphic to a subspace of \( X_p \). Since \( X_p \) is \( p \)-saturated and \( X_q \) is \( q \)-saturated, this is impossible.

\[ \square \]

**Corollary 1.** There exists a finite universal Banach space \( Z \) that has the Tsirelson property.

**Proof.** As was noted, \( \text{Index}(l_{\infty}) = [1, \infty] \) and, hence,

\[ \text{Card} \left( \text{Index}(l_{\infty}) \right) \geq 2. \]

\[ \square \]

**Remark 4.** Recall that the original B. Tsirelson’s space [1] also was finitely universal.

**Remark 5.** It may be given a more short proof of the theorem 2.

Indeed, let \( p, q \in \text{Index}(X^f) \). Let \( \{ A_i : i < \infty \} \) be a dense in \( \mathfrak{M}(X^f) \) sequence. Consider a sequence of spaces \( (B_k)_{k<\infty} \) that are defined inductively:

\[
\begin{align*}
B_1 &= A_1; & B_2 &= B_1 \oplus_p A_2; & B_3 &= B_2 \oplus_q A_3; & \ldots \\
B_{2k} &= B_{2k-1} \oplus_p A_{2k}; & B_{2k+1} &= B_{2k} \oplus_q A_{2k+1}; & \ldots 
\end{align*}
\]

It easy to see that \( \lim B_n \) has the Tsirelson property.

Nevertheless, the proof of the theorem 2 (in difference from the given above one) may be applied in more general cases, as it will be shown below.

It may be presented other examples of classes \( X^f \), such that \( \text{Index}(X^f) \) is more then one single point.

**Theorem 3.** For any closed subset \( e \subset [1, \infty) \) there exists a class \( (X_e)^f \) such that

\[ \text{Index}((X_e)^f) = e. \]

**Proof.** Let \( \{ p_i : i < \infty \} \) be a countable dense subset of \( e \). Consider an infinite matrix

\[
\begin{pmatrix}
p_1 & p_2 & p_3 & \ldots \\
p_1 & p_2 & p_3 & \ldots \\
p_1 & p_2 & p_3 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

Let \( (q_i) \) be an enumeration of it in a sequence, such that each of \( p_k \)'s occurs among \( q_i \)'s infinitely many times. It may be assumed that \( p_1 \neq 2. \)
A class \((X_\varepsilon)^f\) will be constructed by induction. Let \(X_0\) be a Banach space that generates a class \((X_0)^f\) with \(\text{Index}((X_0)^f) = \emptyset\). Let \(X_0 = l_p, \oplus l_2\)

\[ \begin{align*}
X_1 &= l_q(X_0); \quad X_2 = l_{q_2}(X_1); \quad \ldots; \quad X_{n+1} = l_{q_{n+1}}(X_n); \quad \ldots
\end{align*} \]

Certainly, \(X_i\) may be regarded as a subspace of \(X_{i+1}\) and, thus, the direct chain

\[ X_1 \hookrightarrow X_2 \hookrightarrow \ldots \hookrightarrow X_{n+1} \hookrightarrow \ldots \]

has a direct limit \(X_\varepsilon = \bigcup X_i\). It is clear that the class \((X_\varepsilon)^f\) has desired properties. Indeed, for any \(A, B \in M((X_\varepsilon)^f)\) and any \(p_i\), their \(l_{p_i}\)-sum \(A \oplus_{p_i} B\) belongs to \(M((X_\varepsilon)^f)\), as it follows from the construction. Hence,

\[ \{p_i : i < \infty\} \subseteq \text{Index}((X_\varepsilon)^f). \]

If \(A \subseteq \text{Index}((X_\varepsilon)^f)\) then its closure \(\overline{A}\) also belongs to \(\text{Index}((X_\varepsilon)^f)\). This fact easily follows from the closedness of \(M((X_\varepsilon)^f)\). If \(p \not\in \varepsilon\) then \(p \not\in \text{Index}((X_\varepsilon)^f)\) since in a contrary case any sum \(A \oplus_{p} B\), where \(A, B \in M((X_\varepsilon)^f)\) belongs to \(M((X_n)^f)\) for some \(n\). Certainly, this is impossible if \(p \not\in \varepsilon\).

\(\square\)

**Remark 6.** Of course, \(\text{Index}(X^f) \subseteq S(X^f)\) for any Banach space \(X\). The method, presented above, allow to construct classes \(X^f\) with \(\text{Index}(X^f) = S(X^f)\).

E.g. for a two-point set \(\{2, q\}\), where \(q > 2\) it may be considered:

- as \(X_0\) the space \(l_q\) (for which \(S(l_q) = \{2, q\}\));
- as \(X_1\) the space \(l_2(l_q)\);
- \(X_3 = l_q(l_2(l_q))\);
- \(X_4 = l_2(l_q(l_2(l_q)))\) etc.

Clearly, the direct limit of \(X_i\)'s has the two-pointed \(l_p\)-spectrum and the same index of divisibility.

## 3. Superstable classes of finite equivalence

It was shown that some classes of finite equivalence contain spaces with the Tsirelson properties.

From the other hand, there are classes \(X^f\) that has "anti-Tsirelson property": every representative of a such class contains some \(l_p\). Some of these classes may be pick out by using of stable Banach spaces, which was introduced by J.-L. Krivine and B. Maurey [3] and their generalization - superstable Banach spaces that were defined by J. Reyaund [4].

**Definition 10.** A Banach space \(X\) is said to be stable provided for any two sequences \((x_n)\) and \((y_m)\) of its elements and every pair of ultrafilter \(D, E\) over \(\mathbb{N}\)

\[ \lim_{D(n)} \lim_{E(m)} \|x_n + y_m\| = \lim_{E(m)} \lim_{D(n)} \|x_n + y_m\|. \]

The notations \(D(n)\) and \(E(m)\) are used here (instead of \(D\) and \(E\)) to underline the variable \((n\) or \(m\) respectively) in expressions like \(\lim_{D(n)} f(n, m)\).

**Definition 11.** (Cf. [4]). A Banach space \(X\) is said to be superstable if every its ultrapower \((X)_{D}\) is stable.

**Theorem 4.** Let \(X^f\) be a class of finite equivalence. \(X^f\) contains a superstable Banach space if and only if every space \(Y \in X^f\) (and, as a consequence, every space \(W\), which is finitely representable in \(X\)) is stable.
PROOF. If \( Y \in X^f \) is superstable then each its subspace is stable because of the property of a Banach space to be stable is inherited by its subspaces. Hence, each subspace of every ultrapower \((Y)_D\) is stable too, because of \( \{Z : Z <_f Y\} \) is coincide with the set
\[ \{Z : Z \text{ is isometric to a subspace of some ultrapower } (Y)_D\} \]

Conversely, if every \( Y \in X^f \) is stable, then all ultrapowers of \( Y \) are stable too. \( \square \)

**Definition 12.** A class \( X^f \) of finite equivalence that contains a superstable space will be called a superstable class.

In [3] it was shown:
1. Any stable Banach space \( X \) is weakly sequentially complete.
2. Every subspace of \( X \) contains a subspace isomorphic to some \( l_p \) (1 \( \leq p < \infty \)).

**Theorem 5.** Let \( X^f \) be a superstable class. Then there exists such \( p \in [1, \infty) \) that every \( Y \in X^f \) contains a subspace, which is isomorphic to \( l_p \).

**Proof.** Let \( X \in \mathcal{B} \). Let
\[ T(X) = \{p : l_p \text{ is isomorphic to a subspace of } X\}. \]

Let \( Y, Z \in X^f \). Let both \( Y \) and \( Z \) are separable and stable. Thus, \( T(Y) \neq \emptyset \) and \( T(Z) \neq \emptyset \). Assume that \( T(Y) \cap T(Z) = \emptyset \). Similarly to the proof of the theorem 2 may be constructed a space \( W \in X^f \) that has the Tsirelson property. However this conflicts with the superstability of \( X^f \). Hence \( \cap \{T(Y) : Y \in X^f\} \neq \emptyset \). \( \square \)

**Remark 7.** There are classes \( X^f \) such that the intersection \( \cap \{T(Y) : Y \in X^f\} \) consists exactly of \( n \) points.

Indeed, let \( X = l_{q_1} \oplus l_{q_2} \oplus ... \oplus l_{q_n} \), where \( 2 < q_1 < q_2 < ... < q_n \). Clearly,
\[ \cap \{T(Y) : Y \in X^f\} = \{q_1, q_2, ..., q_n\}. \]

**Problem 1.** Whether there exists such a class \( X^f \) that \( \cap \{T(Y) : Y \in X^f\} \) is infinite?

In what follows will be needed a definition.

**Definition 13.** (Cf. [5]) Let \( X \) be a Banach space, \( (x_n) \subset X \) be a nontrivial normed sequence of elements of \( X \) (i.e. \( (x_n) \) contains no Cauchy subsequences); \( D \) be an ultrafilter over \( \mathbb{N} \). For a finite sequence \((a_i)_{i=1}^n \subset \mathbb{R}^n \) let
\[ l((a_i)_{i=1}^n) = \lim_{D(m_1)} \lim_{D(m_2)} ... \lim_{D(m_n)} \left\{ \left\| \sum_{k=1}^n a_k x_{m_k} \right\| : m_1 < m_2 < ... < m_n \right\}. \]

Let \( sm(X, (x_n), D) \) be a completion of a linear space \( c_00 \) of all sequences \((a_i)_{i=1}^\infty \) of real numbers such that all but finitely many \( a_i \)'s are equal to zero.
The space \( sm(X, (x_n), D) \) is said to be a spreading model of the space \( X \), which is based on the sequence \((x_n)\) and on the ultrafilter \( D \).

Clearly, any spreading model of a given Banach space \( X \) has a spreading basis and is finitely representable in \( X \).

In [3] it was shown that every spreading model of a stable Banach space \( X \) has a symmetric basis. The following result shows that in a superstable class \( X^f \) the converse is also true.
DEFINITION 14. Let $X$ be a Banach space. Its IS-spectrum $IS(X)$ is a set of all (separable) spaces $\langle Y, (y_i) \rangle$ with a spreading basis $(y_i)$ which are finitely representable in $X$.

Notice that if $(y'_i)$ and $(y''_i)$ are different spreading bases of $Y$ then $\langle Y, (y'_i) \rangle$ and $\langle Y, (y''_i) \rangle$ are regarded as different members of $IS(X)$.

The last reservation may be omitted if one assumes that members of $IS(X)$ are nonseparable spaces $\langle Y, (y_\alpha)_{\alpha < \omega_1} \rangle$ with uncountable spreading bases $(y_\alpha)_{\alpha < \omega_1}$ (here and below $\omega_1$ denotes the first uncountable cardinal). The proof of this assertion will be given below.

THEOREM 6. A class $X^f$ is superstable if and only if every member $\langle Y, (y_i) \rangle$ of its IS-spectrum has a symmetric basis.

PROOF. Let $X^f$ be superstable; $Y \in X^f$. According to [3] every spreading model of $Y$ has a symmetric basis. Let $(Y)_D$ be an ultrapower by a countably incomplete ultrafilter. Then (see [6]) $(Y)_D$ contains any separable Banach space which is finitely representable in $Y$. In particular, any space $\langle Z, (z_i) \rangle$ of $IS(X^f)$ is isometric to a subspace of $(Y)_D$. Since $(z_i)$ is a spreading sequence, $sm((Y)_D, (z_i), E)$ is isometric to $\langle Z, (z_i) \rangle$ for any ultrafilter $E$. Hence, $(z_i)$ is a symmetric sequence.

Conversely, assume that every $\langle Z, (z_i) \rangle$ has a symmetric basis. Assume that $X$ is not superstable. Then there exists a space from $X^f$ which is not stable (it may be assumed that $X$ is not stable itself). According to [3] there are such sequences $(x_n)$ and $(y_m)$ of elements of $X$ that

$$\sup_{m<n} \|x_n + y_m\| > \inf_{m>n} \|x_n + y_m\|.$$

Let $D$ be a countably incomplete ultrafilter over $\mathbb{N}$. Let $X_0 \overset{df}{=} X$;

$$X_n \overset{df}{=} (X_{n-1})_D; \quad n = 1, 2, \ldots; \quad X_\infty = \bigcup_{n \geq 1} X_n.$$

Here is assumed that $X_n$ is a subspace of $X_{n+1} = (X_n)_D$ under the canonical embedding $d_{X_n} : X_n \rightarrow (X_n)_D$.

Let $D, E$ be ultrafilters over $\mathbb{N}$. Their product $D \times E$ is a set of all subsets $A$ of $\mathbb{N} \times \mathbb{N}$ that are given by

$$\{ j \in \mathbb{N} : \{ i \in \mathbb{N} : (i, j) \in A \} \in D \} \in E.$$

Certainly, $D \times E$ is an ultrafilter and for every Banach space $Z$ the ultrapower $(Z)_{D \times E}$ may be in a natural way identified with $((Z)_D)_E$.

So, the sequence $(x_n) \subset X$ defines elements

$$\mathfrak{f}_1 = (x_n)_D \in (X)_D;$$

$$\mathfrak{f}_2 = (x_n)_{D \times D} \in (((X)_D)_D;$$

$$\mathfrak{f}_k = \underbrace{d_X \times D \times \ldots \times D}_{k \text{ times}} \in \underbrace{((X)_D)_D \ldots}_{k \text{ times}};$$

Notice that $\mathfrak{f}_k \in X_k \setminus X_{k-1}$. It is easy to verify that $(\mathfrak{f}_k)_{k<\infty} \subset X_\infty$ is a spreading sequence. Since $X_\infty \in X^f$, it is symmetric. Moreover, for any $z \in X$, where $X$ is regarded as a subspace of $X_\infty$ under the direct limit of compositions

$$d_{X_n} \circ d_{X_{n-1}} \circ \ldots \circ d_{X_0} : X \rightarrow X_n,$$
the following equality is satisfied: for any pair \( m, n \in \mathbb{N} \)
\[
\|x_n + z\| = \|x_m + z\|.
\]
Since \((x_n)\) and \(z\) are arbitrary elements of \(X\), this contradicts to the inequality
\[
\sup_{m<n} \|x_n + y_m\| > \inf_{m>n} \|x_n + y_m\|.
\]
\(\square\)

This result may be generalized to classes of crudely finite equivalence.

**Definition 15.** Let \(X, Y\) be Banach spaces. \(X\) is said to be crudely finitely representable in \(Y\) (in symbols: \(X <_F Y\)) if \(X\) is isomorphic to a some space which is finite representable in \(Y\) (it is easy to see that this definition is equivalent to the standard one).

Let \(X \approx Y\) denotes that Banach spaces \(X\) and \(Y\) are isomorphic.

A class \(X^F\) of crudely finite equivalence, which is generated by a Banach space \(X\) is given by
\[
X^F = \{Y \in \mathcal{B} : Y <_F X \text{ and } X <_F Y\} = \bigcup \{Y^f : Y \approx X\}.
\]

**Definition 16.** A class \(X^F\) of crudely finite equivalence is said to be crudely superstable if it contains a superstable space.

In other words, \(X^F\) is crudely superstable if one of classes \(Y^f\) that form \(X^F\) (i.e., one of classes of the union \(\bigcup \{Y^f : Y \approx X\} = X^F\)) is superstable.

Certainly, any crudely superstable class \(X^F\) has the property:
Every Banach space \(Y\), which is finitely representable in a some space \(Z \in X^F\) contains a subspace that is isomorphic to some \(l_p\) \((1 < p < \infty)\).

**Theorem 7.** A class \(X^F\) is crudely superstable if and only if for every \(Z \in X^F\) its IS-spectrum \(IS(Z)\) consists of spaces \(\langle W, (w_n) \rangle\), whose natural bases \((w_n)\) are \(c_Z\)-equivalent to symmetric bases where the constant \(c_Z\) depends only on \(Z\).

**Proof.** Let \(X^F\) be crudely superstable. Then some \(Y \in X^F\) is superstable and for any \(Z\), which is crudely finitely representable in \(Y\), and for any space \(\langle W, (w_n) \rangle\) from the IS-spectrum \(IS(Z)\), its natural basis \((w_n)\) is \(d(Z, Y_1)\)-equivalent to a symmetric one, where \(Y_1 \in Y^f\).

Conversely, let \(X_0\) be such that every space \(\langle W, (w_n) \rangle \in IS(X_0)\) has a basis \((w_n)\), which is equivalent to a symmetric one (certainly, this is equivalent to the assertion that for every \(Z \in X^F\) its IS-spectrum \(IS(Z)\) consists of spaces \(\langle W', (w'_n) \rangle\), whose natural bases \((w'_n)\) are equivalent to symmetric bases. It is easy to show that there exists a constant \(c_Z\) such that every space \(\langle W, (w_n) \rangle \in IS(X_0)\) has a basis \((w_n)\) which is \(c_X\)-equivalent to a symmetric one.

Indeed, let \((c_k)\) be a sequence of real numbers with a property: for every \(k < \infty\) there exists \(\langle W_k, (w^k_n) \rangle \in IS(X_0)\) such that \((w^k_n)\) is \(c_k\)-equivalent to a symmetric basis and is not \(c_{k-1}\)-equivalent to any symmetric basis. Without loss of generality it may be assumed that all spaces \((W_k)\) are subspaces of a space from \((X_0)^f\), e.g. \(W_k \hookrightarrow X_0\). Consider an ultrapower \(\langle X_0 \rangle_D\) and its elements \(w_k = (w^k_n)_{D(k)}\). Clearly \((w_k)_{k<\infty} \subseteq (X_0)_D\) is a spreading sequence that is not equivalent to any symmetric sequence.

Consider some \(Z \in X^F\), such that \(Z\) contains any space from its IS-spectrum. Let \(c_Z\) be the corresponding constant which was defined above. Let \(\{Z_\alpha : \alpha < \kappa\}\) \((\kappa\) is a cardinal number) be a numeration of all subspaces of \(Z\) that may be represented
as \( \text{span}(z_i^{(\alpha)}) \) (i.e. as a closure of linear span of \( \{z_i^{(\alpha)} : i < \infty\} \)) for such sequences \( \{z_i^{(\alpha)} : i < \infty\} \) (not necessary spreading ones) that are \( c_2 \)-equivalent to symmetric sequences.

Using the standard procedure of renorming, due to A. Pełczyński [12], it may be constructed a space \( Z_\infty \approx Z \) such that \( Z_\infty \) contains as a subspace every space \( W_\infty \) from \( IS(Z_\infty) \), (by the renorming procedure) has a symmetric basis.

From the theorem 6 follows that \( Z_\infty \) (and, hence, the whole class \( (Z_\infty) ^{\ell} \)) is superstable. Since \( Z_\infty \in X^F \) the class \( X^F \) is crudely superstable.

Indeed, it is sufficient to choose as a unit ball \( B(Z_\infty) = \{w \in Z_\infty : \|w\| \leq 1\} \) a convex hull of the union of a set \( B(Z) \) with sets \( \{c_Z^{−1}j_\alpha B(i_\alpha Z_\alpha) : \alpha < \kappa\} \), where \( i_\alpha : Z_\alpha \to W_\alpha \) is an isomorphism between \( Z_\alpha \) and a space \( W_\alpha \) with a symmetric basis \( (w_n^{(\alpha)}) \), which is given by \( i_\alpha (w_n^{(\alpha)}) = w_n^{(\alpha)} \) for all \( n < \infty; \|i_\alpha \| \|i_\alpha^{−1}\| \leq c_Z; j_\alpha \) is an embedding of the unit ball \( B(W_\alpha) \) in a set \( c_Z B(Z_\alpha) \), which is given by \( c_Z B(Z_\alpha) = \{z \in Z_\alpha : c_Z^{−1}z \in B(Z_\alpha)\} \). Namely,

\[
B(Z_\infty) = \text{conv}\{B(Z) \cup (\cup\{c_Z^{−1}j_\alpha B(i_\alpha Z_\alpha) : \alpha < \kappa\})\}.
\]

\[\square\]

4. Spaces spanned by spreading uncountable sequences

From results of the previous section follows that spreading sequences play an important role in the study of the Tsirelson property.

Ordinals will be understands in the von Neuman sense: an ordinal \( \alpha \) will be regarded as a set \( \{\beta : \beta < \alpha\} \) of all ordinals that are less then \( \alpha \).

Small Greece letters \( \alpha, \beta, \gamma, \delta, \zeta \) denote ordinals; \( \kappa, \tau \) be cardinals. In what follows cardinals will be identified with initial ordinals. Finite ordinals and cardinals may be denoted also by small Latin letters \( i, j, k, m, n \).

The least infinite ordinal (cardinal) is denoted by \( \omega \); the first uncountable ordinal (cardinal) - by \( \omega_1 \).

For \( \alpha, \beta \) are ordinals, the symbol \( \alpha^\beta \) denotes the ordinal degree.

A symbol \( 2^\tau \) denotes the cardinal degree - a cardinality of a set \( \exp \tau \) of all subsets of a cardinal \( \tau \).

Let \( X \) be a Banach space, \( A \subset X \).

The linear span \( \text{lin}(A) \), i.e. a set of all linear combinations of elements of \( A \) need not to be a closed subspace of \( X \). A closure \( \overline{\text{lin}(A)} \) will be denoted by \( \text{span}(A) \).

A dimension of a Banach space \( X \), \( \dim(X) \) is the least cardinality of a subset \( A \subset X \) such that \( \text{span}(A) = X \).

Let \( \{x_n : n < \omega\} \) be a spreading sequence.

It generates a sequence \( \mathcal{N} = (\mathcal{N}_m)_{m<\omega} \) of norms: \( \mathcal{N}_m \) is a norm on \( \mathbb{R}^m \), which is given by

\[
\mathcal{N}_m ((a_i)_{i=1}^m) = \left\| \sum_{i=1}^m a_i x_i \right\|.
\]

Let \( \langle I, \ll \rangle \) be a linearly ordered set.

Consider a vector space \( c_{00} (I, \ll) \) of all families \( \{a_i : i \in I\} \) of scalars all but finitely many elements of which are vanished.

\( c_{00} (I, \ll) \) may be equipped with a norm \( \mathcal{N} \) which is given by

\[
\mathcal{N} (\{a_i : i \in I\}) = \mathcal{N}_m (a_{i_1}, a_{i_2}, ..., a_{i_m}),
\]
where \(m = \text{card}\{i \in I : a_i \neq 0\}; i_1 \ll i_2 \ll \ldots \ll i_m\) and all \(\{a_{ik} : k = 1, 2, \ldots, m\}\) are differ from 0.

The completion of the normed space \(\langle c_{00}(I, \ll), \mathcal{N}\rangle\) is a Banach space, which will be denoted by \(X_{\mathcal{N}}(I, \ll)\).

So, the spreading sequence \(\{x_n : n < \omega\} = x\) generates a class of Banach spaces of kind \(X_{\mathcal{N}}(I, \ll)\), which will be called the tower \([x]\), generated by \(x\).

The following result is obvious ant its proof is omitted.

**Theorem 8.** Let \(\{x_n : n < \omega\} = x\) be a spreading sequence; \([x]\) be the corresponding tower.

If \(x\) is symmetric, then all spaces \(Y \in [x]\) of the dimension \(\tau\) are pairwise isometric, where \(\tau\) is an arbitrary infinite cardinal.

If \(x\) is equivalent to a symmetric sequence then all \(Y \in [x]\) of the dimension \(\tau\) are pairwise isomorphic.

Let
\[
\mathcal{B}_x = \{X \in \mathcal{B} : \text{dim}(X) = \tau\};
\]
\[
X^\approx = \{Y \in \mathcal{B} : Y \approx X\}.
\]

Let \(\mathcal{K}\) be a class of Banach spaces, which is closed under isomorphisms (i.e. \(X \in \mathcal{K}\) implies that \(X^\approx \subset \mathcal{K}\). Let
\[
\mathcal{K}^\approx = \{X^\approx : X \in \mathcal{K}\}.
\]

It may be shown that if \(x\) is not isometric (resp., is not equivalent to a symmetric sequence) then the cardinality \(\text{card}(\{x\} \cap \mathcal{B}_x) = 2^\tau\) (resp., the cardinality \(\text{card}(\{x\} \cap \mathcal{B}_x)^\approx) = 2^\tau\).

Let \(\approx\) be a cardinal; \(\sigma : \approx \to \approx\) be a transposition (i.e. one-to-one mapping of \(\approx\) onto \(\approx\)).

It will be said that \(\sigma\) is almost identical if the correlation
\[
\gamma_1 < \gamma_2 \Rightarrow \sigma\gamma_1 < \sigma\gamma_2,
\]
where \(\gamma_1, \gamma_2 \in \approx\) may get broken at most finitely many times.

**Lemma 1.** Let \(\{x_n : n < \omega\}\) be a spreading sequence that is not symmetric.

Let \(\sigma : \omega \to \omega\) be a transposition, which is not almost identical.

If sequences \(\{x_n : n < \omega\}\) and \(\{x_{\sigma n} : n < \omega\}\) are equivalent then both of them are equivalent to a symmetric sequences.

**Proof.** Consider a sequence \(\{y_{\alpha} : \alpha < \omega^2\}\), which is given by
\[
\{y_{\omega \cdot 2k+n} = x_n; \ y_{\omega \cdot (2k+1)+n} = x_{\sigma n} : k < \omega; \ n < \omega\}.
\]
This sequence is equivalent to a sequence \(\{y'_{\alpha} : \alpha < \omega^2\}\) that belongs to the tower \([x]\), generated by \(\{x_n : n < \omega\} = x\).

Certainly, this is equivalent to
\[
C^{-1} \left\| \sum_{k<n} a_k x_{i_k} \right\| \leq \left\| \sum_{k<n} a_k y_{\alpha_k} \right\| \leq C \left\| \sum_{k<n} a_k x_{i_k} \right\|
\]
for every \(n < \omega\); every scalars \((a_k)_{k<n}\) every choice \(\alpha_0 < \alpha_1 < \ldots < \alpha_{n-1} < \omega^2\) and \(i_0 < i_1 < \ldots < i_{n-1} < \omega\) and some \(C < \infty\) that depends only on equivalence constant between \(\{x_n : n < \omega\}\) and \(\{x_{\sigma n} : n < \omega\}\). Since \(\sigma\) has only a finite number of inversions, our definition of \(\{y_{\alpha} : \alpha < \omega^2\}\) implies that \(x\) is equivalent to a symmetric sequence.

\(\square\)
THEOREM 9. Let \( \alpha, \beta \) be ordinals; \( \omega \leq \beta \leq \alpha \). Let \( \{x_\gamma : \gamma < \alpha\} \) be a subsymmetric \( \alpha \)-sequence which is not equivalent to any \( \alpha \)-symmetric sequence. Let 

\[
X_\alpha = \text{span}\{x_\gamma : \gamma < \alpha\}; \quad X_\beta = \text{span}\{x_\gamma : \gamma < \beta\}.
\]

If \( \beta^\omega < \alpha \) then spaces \( X_\alpha \) and \( X_\beta \) are not isomorphic.

**Proof.** Assume that \( X_\alpha \) is isomorphic to a subspace \( Z \) of \( X_\beta \).

Let \( I : X_\alpha \to X_\beta \) be the corresponding operator of isomorphic embedding. Without loss of generality it may be assumed that an image of element \( x_\gamma (\gamma < \alpha) \) in \( X_\beta \) is a finite linear combination with rational coefficients of some \( x_\zeta \)'s (\( \zeta < \beta \)):

\[
Ix_\gamma = \sum_{k=0}^{n(\gamma)} a_k^\gamma x_{\zeta_k(\gamma)}; \quad \zeta_0(\gamma) < \zeta_1(\gamma) < \ldots < \zeta_{n(\gamma)}(\gamma) < \beta.
\]

Thus, to any \( x_\gamma \) corresponds a finite sequence of rational numbers \( (a_k^\gamma)_{k=0}^{n(\gamma)} \).

Let \( (p_n)_{n<\omega} \) be a numeration of all finite sequences of rationals. The set \( (p_n)_{n<\omega} \) generates a partition of \( \alpha \) on parts \( (P_n)_{n<\omega} \) in a following way: an ordinal \( \gamma < \alpha \) belongs to \( P_n \) if \( (a_k^\gamma)_{k=0}^{n(\gamma)} = p_n \).

Since \( \alpha > \beta^\omega \geq \omega^\omega \), one of \( P_n \)'s should contain a sequence \( \{\gamma_i : i < \delta\} \) of ordinals, which order type \( \delta \) (in a natural order : \( i < j \) implies that \( \gamma_i < \gamma_j \)) is greater then \( \beta^\omega \). Clearly, for all such \( \gamma_i \),

\[
Ix_{\gamma_i} = \sum_{k=0}^{n} a_k x_{\zeta_k(\gamma_i)},
\]

where \( n \) and \( (a_k^\gamma)_{k=1}^{n} \) do not depend on \( i \).

A set of all sequences \( \zeta_0(\zeta_i) < \zeta_1(\zeta_i) < \ldots < \zeta_n(\zeta_i) < \beta \) can be ordered to have only the order type \( \leq \beta^n \). Hence, conditions \( \gamma_{i_1} < \gamma_{i_2} \Rightarrow \zeta_0(\gamma_{i_1}) < \zeta_0(\gamma_{i_2}) \) must get broken for infinite many pairs \( \gamma_{i_1}, \gamma_{i_2} \).

The inequality

\[
C^{-1} \norm{\sum_{k<m} b_k x_{\gamma_k}} \leq \norm{\sum_{k<m} b_k \left( \sum_{k=0}^{n} a_k x_{\zeta_k(\gamma_k)}\right)} \leq C \norm{\sum_{k<m} b_k x_{\gamma_k}},
\]

where \( C = d(X_\alpha, Z) \), shows that \( \{x_\gamma : \gamma < \alpha\} \) is equivalent to a symmetric sequence.

**Theorem 10.** Let \( \mathfrak{r} = \{x_n : n < \omega\} \) be a subsymmetric sequence, which is not isomorphic to a symmetric one. Then for any cardinal \( \kappa \geq \omega \) there exists \( 2^\kappa \) pairwise non isomorphic Banach spaces of dimension \( \kappa \), which belongs to the same tower \( \mathfrak{r} \).

**Proof.** Let \( \{x_\gamma : \gamma < \kappa\} \) be a subsymmetric \( \kappa \)-sequence. Let \( I = (I, \ll) \) be a linearly ordered set of cardinality \( \kappa \); \( J = (J, \ll') \) - another linear ordering of \( I \).

Consider families \( \{x_i : i \in I\} \) and \( \{x_j : j \in J\} \) that are indexed (and ordered) by elements of \( I \) and \( J \) respectively.

Let \( X_I = \text{span}\{x_i : i \in I\}; \quad X_J = \text{span}\{x_j : j \in J\} \). Certainly, \( X_I \) and \( X_J \) are isomorphic only if there exists one-to-one mappings of embedding \( u : I \to J \) and \( w : J \to I \), which are almost monotone in a following sense:

\[
i_1 \ll i_2 \Rightarrow u(i_1) \ll' u(i_2) \quad \text{for all but finitely many pairs } i_1, i_2 \in I;
\]

\[
j_1 < j_2 \Rightarrow w(j_1) \ll w(j_2) \quad \text{for all but finitely many pairs } j_1, j_2 \in J.
\]

Since there exists \( 2^\kappa \) orderings of \( I \) for any pair of which such almost monotone mappings do not exist, this prove the theorem. \( \square \)
Corollary 2. The cardinality of the set of all Banach spaces of dimension \( \kappa \) is
\[
\text{card} (\mathcal{B}_\kappa) = 2^{\kappa}.
\]

Proof. The inequality \( \text{card} (\mathcal{B}_\kappa) \geq 2^{\kappa} \) follows from the previous theorem. The inverse inequality is obvious. \( \square \)

Remark 8. It is of interest that different sets \( \mathcal{B}_\kappa \) and \( \mathcal{B}_\tau \) may be of the same cardinality. The appearance of such a case depends on the model of the set theory that lies in the base of all functional analysis.

E.g., if one assumes the Martin axiom MA with the negation of continuum hypothesis \( \neg \text{CH} \) then for all cardinals \( \kappa \) such that \( \omega < \kappa < 2^{\omega} \)
\[
\text{card} \mathcal{B}_\kappa = \text{card} \mathcal{B}_\omega = 2^{\omega}.
\]

It may be given an interesting cardinal criterion of superstability.

Theorem 11. Let \( X \) be a Banach space; \( X^f \) and \( X^F \) be corresponding classes of finite and crudely finite equivalence respectively, which are generated by \( X \).

If \( \text{card} (X^f \cap \mathcal{B}_\kappa) < 2^{\kappa} \) then \( X^f \) is a superstable class.

If \( \text{card} \left( (X^F \cap \mathcal{B}_\kappa) \right) < 2^{\kappa} \) then \( X^F \) is crudely superstable.

Proof. According to [8] if \( IS(X) \) contains a space \( \langle W, (w_n) \rangle \) with a spreading basis then for every cardinal \( \kappa \) and every transposition \( \sigma \) of \( \tau \) there are spaces \( X \) and \( X_\sigma \), which are finitely equivalent to \( X \) and such that \( \dim (X) = \dim (X_\sigma) = \kappa \); \( X \) contains a subspace isometric to \( W_\kappa = \text{span} \{ w_\alpha : \alpha < \kappa \} \); \( X_\sigma \) contains a subspace isometric to \( W_\sigma = \text{span} \{ w_\beta : \alpha < \kappa \} \). Improving arguments of [8] it may be shown that \( X \) and \( X_\sigma \) are isometric (resp., isomorphic) if and only if \( W_\kappa \) and \( W_\sigma \) are isometric (resp., isomorphic). Hence, if \( X \) is not superstable then \( \text{card} (X^f \cap \mathcal{B}_\kappa) = 2^{\kappa} \). Similarly for the second part of the theorem. \( \square \)

It is known that a separable Banach space may have a lot of pairwise non equivalent symmetric bases (e.g. the classical A. Pelczyński’s space \( P_\sigma \), complementarily universal in the class of all Banach spaces with unconditional bases [9] has a continuum number of pairwise non equivalent symmetric bases; cf. [10]).

However, if a nonseparable Banach space has a symmetric or subsymmetric (uncountable) basis, this basis is unique up to equivalence.

Theorem 12. Let \( \kappa > \omega \) be a cardinal; \( \{ x_\alpha : \alpha < \kappa \} \) and \( \{ y_\alpha : \alpha < \kappa \} \) be subsymmetric sequences; \( X = \text{span} \{ x_\alpha : \alpha < \kappa \} \); \( Y = \text{span} \{ y_\alpha : \alpha < \kappa \} \). If spaces \( X \) and \( Y \) are isomorphic then \( \kappa \)-sequences \( \{ x_\alpha : \alpha < \kappa \} \) and \( \{ y_\alpha : \alpha < \kappa \} \) are equivalent.

Proof. Let \( u : X \to Y \) be an isomorphism; \( \| u \| \| u^{-1} \| = c \). It may be assumed that \( ux_\alpha \in Y \) is represented as a block
\[
ux_\alpha = \sum_{k=1}^{n(\alpha)} a_{k}^x y_{\beta_k(\alpha)}
\]
for some sequence of rational scalars \( (a_{k}^x)_{k \leq n(\alpha)} \) and finite \( n(\alpha) \). Moreover, it may be assumed that this blocks are not intersected, i.e. that any member of a given sequence \( (\beta_k(\alpha))_{k \leq n(\alpha)} \) belongs only to this block.
Since $\mathcal{X}$ is uncountable, among such blocks there is an infinite number of identical ones, that differs only in sequences $(\beta_k \langle \alpha \rangle)_{k \leq n(\alpha)}$. Let $A \subset \mathcal{X}$ be a such that all elements $\{ux_\alpha : \alpha \in A\}$ are represented by those identical blocks:

$$ux_\alpha = \sum_{k=1}^{n} a_k y_{\beta_k}(\alpha) \quad \text{for} \quad \alpha \in A.$$  

Let $(b_i)$ be a sequence of scalars; $a = \max_{1 \leq k \leq n} \{a_k\}$.  

Then for any finite subset $A' \subset A$, $A' = \{\alpha_i\}_{i=1}^{m}$, because of unconditionality of sequences $(x_\alpha)$ and $(y_\alpha)$,

$$\left\| \sum_{i=1}^{m} b_i x_\alpha \right\| \geq c \left\| \sum_{i=1}^{m} b_i \left( \sum_{k=1}^{n} a_k y_{\beta_k}(i) \right) \right\| \geq c \left\| \sum_{i=1}^{m} b_i y_{\beta_i}(i) \right\|.$$  

Analogously, the converse inequality that proves the theorem may be obtained. \hfill \square

5. Further constructions of spaces with the Tsirelson properties  

It was noted that superstable classes are only a constituent part of all classes of finite equivalence. There exist classes of finite equivalence, in which every space fails to have the Tsirelson property, and which, at the same time, are not superstable.

Example 1. Let $2 < p < q < \infty$. Consider a sequence of Banach spaces

$$Y_1 = l_2; \ Y_2 = l_p (Y_1); \ Y_3 = l_2 (Y_2); \ldots; \ Y_{2n} = l_p (Y_{2n-1}); \ Y_{2n+1} = l_2 (Y_{2n}); \ldots$$

Similarly to the theorem 3 it may be shown that for a limit space $Y_\infty = \lim_{n \to \infty} Y_n$ its index of divisibility $\text{Index}(Y_\infty) = \{2, p\}$. According to the theorem 2 there exists a space $Z \sim_f Y_\infty$ that has the Tsirelson property. Hence, $Y_\infty$ is not isomorphic to a stable space.

Consider a space $X_{pq} = Y_\infty \oplus_q l_q$. Certainly, this space is not superstable. However, every space $Z$ from the class $(X_{pq})'\ell$ contains a subspace, isomorphic to $l_r$. Indeed, if $Z \in (X_{pq})'\ell$ then $Z$ is of kind $Z = Z_1 \oplus Z_2$, where $Z_1 \in (Y_\infty)'\ell$; $Z_2 \in (l_q)'\ell$ (since $S(Z) = \{2, p, q\}$ and $q \notin S(Y_\infty)$). Obviously, $Z_2$ contains a subspace, isomorphic to $l_q$.

Theorem 13. Let $X$ be a Banach space. If the corresponding class $X^\ell$ of crudely finite equivalence is not crudely superstable then there exists a Banach space $Y$, which is finite representable in $X$ and has the Tsirelson property.

Proof. According to the proof of theorem 2, it is enough to check a such space $Z \sim_f X$ that the corresponding class $Z^\ell$ contains two spaces: $Z_1$ and $Z_2$ with $T(Z_1) \cap T(Z_2) = \emptyset$. Recall that $T(Z)$ denotes a set of all $p \in [1, \infty]$ such that $Z$ contains a subspace isomorphic to $l_p$.

Since $X^\ell$ is not crudely superstable, there exists $(W, (w_n)) \in IS(X)$ such that $w = (w_n)$ is a spreading sequence, non-equivalent to any symmetric one.

Consider a tower $[\mathcal{W}]$ and all separable spaces from it. By the results of previous section, there are an uncountable set of such spaces that are pairwise non-isomorphic.

Let $\{w_\sigma : \sigma \in \Sigma_0\}$ be their numeration, where $\Sigma_0 \subset \Sigma$ is a set of all rearrangements of $\omega$, that are not almost identical.
It will be assumed that \( \mathfrak{w}_\sigma = \{ w_{\sigma n} : n < \omega \} \) forms a spreading basis of a space \( \mathcal{W}_\sigma \).

It will be shown that \( \cap \{ T(\mathfrak{w}_\sigma) : \sigma \in \Sigma_0 \} = \emptyset \). Since every \( \mathcal{W}_\sigma < f X \), this will prove the theorem.

Suppose that this intersection is not empty, i.e. that there exists such \( p \in [1, \infty) \) that \( l_p \) is isomorphic to a subspace of every \( \mathcal{W}_\sigma (\sigma \in \Sigma_0) \).

The natural basis \( (e_n) \) of \( l_p \) is reproducible in the terminology of [11], i.e. if \( l_p \) is isomorphic to a subspace of a Banach space \( Z \) with a basis \( (z_n) \) then there exists a such isomorphical embedding \( u : l_p \to Z \) that

\[
u (e_k) = \sum_{i=m_k+1}^{m_{k+1}} a_i z_i
\]

for a sequence of scalars \( (a_i) \) and a sequence of naturals \( m_1 < \ldots < m_k < \ldots \).

Hence there exists a such isomorphic embedding \( u_\sigma : l_p \to \mathcal{W}_\sigma \) that

\[
u_\sigma (e_k) = \sum_{i=m_k+1}^{m_{k+1}(\sigma)} a_i w_{\sigma_1}.\]

Because of \( \text{card}(\Sigma_0) > \omega \), and \( \{ \mathfrak{w}_\sigma : \sigma \in \Sigma_0 \} \) are spreading sequences, it may be chosen an uncountable subset \( \Sigma_0(1) \subseteq \Sigma_0 \) such that for all \( \sigma \in \Sigma_0(1) \)

\[
u_\sigma (e_1) = \sum_{i=1}^{m_1} a_i w_{\sigma_1},\]

where \( m_1 \) and \( (a_i)_{i=1}^{m_1} \) does not depend on \( \sigma \in \Sigma_0(1) \).

Proceeding by induction, it may be chosen an uncountable subset \( \Sigma_0(2) \subseteq \Sigma_0(1) \) such that for all \( \sigma \in \Sigma_0(2) \)

\[
u_\sigma (e_k) = \sum_{i=m_k+1}^{m_{k+1}} a_i w_{\sigma_1},\]

etc. As a result it will be obtained a sequence \( \Sigma_0^{(1)} \supset \ldots \supset \Sigma_0^{(n)} \supset \ldots \) of uncountable subsets of \( \Sigma \), whose intersection is not empty (since \( \omega \) is not confinal with \( \omega_1 \)) such that for all \( \sigma \in \Sigma_0^\infty = \cap_n \Sigma_0^{(n)} \)

\[
u_\sigma (e_k) = \sum_{i=m_k+1}^{m_{k+1}} a_i w_{\sigma_1},\]

where \( (n_k)_{k<\omega} \) and \( (a_k)_{k<\omega} \) does not depend on \( \sigma \in \Sigma_0^\infty \).

It was assumed that all sequences \( \{ u_\sigma (e_k) : k < \omega \} : \sigma \in \Sigma_0^\infty \) are equivalent to the natural basis \( (e_n) \) of \( l_p \). Hence there exists a such constant \( c \in (0, \infty) \) that for every finite sequence \( (\xi_j)_{j=1}^n \) of scalars and every \( \sigma \in \Sigma_0^\infty \)

\[
\begin{align*}
c^{-1} \left\| \sum_{k=1}^n \xi_k \left( \sum_{i=m_k+1}^{m_{k+1}} a_i w_{\sigma_1} \right) \right\| &\leq \left\| \sum_{k=1}^n \xi_k \left( \sum_{i=m_k+1}^{m_{k+1}} a_i w_i \right) \right\| \\
&\leq c \left\| \sum_{k=1}^n \xi_k \left( \sum_{i=m_k+1}^{m_{k+1}} a_i w_{\sigma_1} \right) \right\|
\end{align*}
\]

Certainly, this implies that the sequence \( \{ w_i : i < \omega \} \) is equivalent to a symmetric sequence.

Indeed, assume that it is not equivalent to any symmetric sequence. Then there exists a double sequence \( \{ \xi_k^{(n)} : k \leq m_n \} : n < \infty \) such that for all \( n < \infty \)

\[
\begin{align*}
\left\| \sum_{k=1}^n \xi_k^{(n)} w_k \right\| &= 1 \quad \text{and} \quad \lim_{n \to \infty} \left\| \sum_{k=1}^n \xi_k^{(n)} w_{\sigma k} \right\| = 0 \text{ for some rearrangement } \sigma.
\end{align*}
\]

The expression

\[
\left\| \sum_{k=1}^n \xi_k \left( \sum_{i=m_k+1}^{m_{k+1}} a_i w_i \right) \right\| = \left\| \sum_{k=1}^n \xi_k a_{m_k} w_k + z \right\|
\]

may be presented in a form
Similarly,
\[ \left\| \sum_{k=1}^{n} \xi_k \left( \sum_{i=m_k+1}^{m_{k+1}} a_i w_{ai} \right) \right\| = \left\| \sum_{k=1}^{n} \xi_k a_{m_k} w_{\sigma_k} + z' \right\| . \]

Let \( \xi_k a_{m_k} = \xi_k^{(n)} \). This is possible since \( (\xi_k) \) is an arbitrary finite sequence of scalars. Then
\[ c^{-1} \left\| \sum_{k=1}^{n} \xi_k a_{m_k} w_{\sigma_k} + z \right\| \leq \left\| \sum_{k=1}^{n} \xi_k a_{m_k} w_{\sigma_k} \right\| \leq c \left\| \sum_{k=1}^{n} \xi_k a_{m_k} w_{\sigma_k} + z \right\| , \]
where \( z \) in both cases may be assume to be the same element. So,
\[ c^{-1} \left\| \sum_{k=1}^{n} \xi_k^{(n)} w_{\sigma_k} + z \right\| \leq \left\| \sum_{k=1}^{n} \xi_k^{(n)} w_{\sigma_k} \right\| \leq c \left\| \sum_{k=1}^{n} \xi_k^{(n)} w_{\sigma_k} + z \right\| , \]
i.e. \( c^{-1} \| e_n + z \| \leq \| e'_n + z \| \), where \( \| e'_n \| = 1 \) and \( \| e_n \| \) infinitely increased. Certainly, this is impossible. \( \square \)

6. Ultracommutative Banach spaces

Let \( X \) be a Banach space; \( D, E \) be ultrafilters. It will be said that ultrapowers \( (X)_D \) and \( (X)_E \) are strongly identical if there exists an isometry \( i : (X)_D \to (X)_E \) such that \( i \circ d_D = d_E \), where \( d_D \) and \( d_E \) are canonical embeddings of \( X \) into \( (X)_D \) and \( (X)_E \) respectively. In a such case it will be written \( \langle (X)_D, d_D \rangle \equiv \langle (X)_E, d_E \rangle \).

**Definition 17.** A Banach space \( X \) is said to be ultracommutative if for any pair of ultrafilters \( D, E \) over \( \mathbb{N} \) ultrapowers \( (X)_{D \times E} \) and \( (X)_{E \times D} \) are strongly identical.

**Theorem 14.** Any ultracommutative Banach space \( X \) is stable.

**Proof.** Let \( (x_n) \) and \( (y_m) \) be two sequences of elements of \( X \). The double sequence \( z_{nm} = x_n + y_m \) generates a pair of elements: \( (z_{nm})_{D(n) \times E(m)} \in (X)_{D \times E} \) and \( (z_{nm})_{E(m) \times D(n)} \in (X)_{E \times D} \). Since \( X \) is ultracommutative, these elements are of equal norms. Indeed, let \( i : (X)_{D \times E} \to (X)_{E \times D} \) be an isometry. Then, by the ultracommutativity, \( i z_{nm} = z_{nm} \) and \( i(z_{nm})_{D(n) \times E(m)} = (z_{nm})_{E(m) \times D(n)} \). Certainly, this implies that \( \lim_{D(n)} \lim_{E(m)} \| x_n + y_m \| = \lim_{E(m)} \lim_{D(n)} \| x_n + y_m \| \). \( \square \)

**Corollary 3.** In a general case iterated ultrapowers \( (X)_{D \times E} \) and \( (X)_{E \times D} \) are not strongly identical.

The converse is not true.

**Theorem 15.** Let \( X \) be an ultracommutative Banach space. Then each its spreading model is isometric to a some \( l_p \) \( (1 \leq p < \infty) \).

**Proof.** As was noted in the proof of theorem 6 (cf. also [12]) every spreading model \( sm(X, (x_n), D) \) of \( X \) may be obtained by using iterated ultrapowers; its natural basis is constructed by induction:
\[ e_1 = (x_n)_D \in (X)_D; \ e_2 = (x_n)_{D \times D} \in ((X)_D)_D, \text{ etc.} \]
Assume that $X$ is ultracommutative. Then, by the preceding theorem it is stable, in particular every its spreading model has a symmetric basis.

For $e_1, e_2$ as above and scalars $a$ and $b$ put:

$$f(a, b) = \|ae_1 + be_2\|.$$  

Certainly, the function $f$ is homogeneous: $f(\lambda a, \lambda b) = \lambda f(a, b)$; symmetric: $f(a, b) = f(b, a)$; monotone: $f(a, b) \leq f(c, c)$ provided $a < c$ and $b < d$; satisfies the norming condition $f(0, 1) = 1$ and, at least, from the ultracommutativity follows:

$$f(a, f(b, c)) = f(f(a, b), c).$$

Hence, according to the Kolmogoroff-Nagumo theorem (cf. [13] and [14]) or from the same result, obtained independently by Bohnenblust [15], either there exists such $p < \infty$ that $f(a, b) = (a^p + b^p)^{1/p}$ or $f(a, b) = \max(a, b)$. Since $X$ is stable, the last case is impossible. So, $sm(X, (x_n), D) = l_p$.

**Corollary 4.** Spaces $L_p$ ($1 \leq p \neq 2 \leq \infty$) are not ultracommutative.

**Proof.** According to [16] for $1 < p < 2$ every $L_p$ contains a subspace $Y_p$ with a symmetric basis that is complementably universal in the class of all subspaces of $L_p$ with an unconditional basis. Certainly, $Y_p$ is not isomorphic to any $l_r$. For $p > 2$ the result follows by duality.

Indeed, for superreflexive Banach spaces $(X^*)_D = ((X)_D)^*$, that implies that $X$ and $X^*$ in a superreflexive case either both are non-ultracommutative, or both enjoy this property.

To close the proof notice that $L_1$ for every $p \in (1, 2)$ contains a subspace, isometric to $L_p$. □

**Remark 9.** In a general case even if $X$ is ultracommutative, its ultrapower $(X)_D$ (by a countably incomplete ultrafilter) does not have this property: if its $l_p$-spectrum contains a point $p \neq 2$ then $(X)_D$ contains a subspace, isometric to $L_p$. However, there are ultracommutative spaces $X$ with ultracommutative ultrapowers that are not isomorphic to $L_2$.

**Example 2.** Consider a space $Z_\infty = \left(\sum \oplus_{p_i}^{(n_i)}\right)_2$, where $p_i \to 2$; $n_i \to \infty$; $|2 - p_i| \log n_i \to \infty$. Certainly, $Z_\infty$ is stable; every space $X$, which is finitely equivalent to $Z_\infty$ is of kind $X = Z_\infty \oplus_2 l_2(\dim(X))$. In particular, all separable spaces from the class $(Z_\infty)^f$ are pairwise isometric. It is of interest to point out that the space $Z_\infty$ is stable (and also is superstable) by the cardinal criterion (theorem 11); card $\left((Z_\infty)^f \cap B_\infty\right) = 1 < 2^\kappa$ for all cardinals $\kappa$.

Notice, that there are continuum of paired sequences $r = \{p_i, n_i\}_{i<\infty}$ that generate nonisomorphic spaces of kind $Z_\infty$.

**Example 3.** It is easy to check that any space of kind $\left(\sum \oplus A_k\right)_p$, where $(A_k)$ is a sequence of finite dimensional Banach spaces and $1 \leq p < \infty$, is ultracommutative.
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