ÉTALE COVERINGS IN CODIMENSION 1 WITH APPLICATIONS TO MORI DREAM SPACES

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Abstract. The present paper is devoted to developing relations between Galois étale coverings in codimension 1 and étale fundamental groups in codimension 1 of algebraic varieties, aimed to studying the topology of Mori dream spaces. In particular, the universal étale covering in codimension 1 of a non-degenerate toric variety and a canonical Galois étale covering in codimension 1 of a Mori dream space (MDS) are exhibited. Sufficient conditions for the latter being either still a MDS or the universal étale covering in codimension 1 are given. As an application, a canonical toric embedding of K3 universal coverings, of Enriques surfaces which are Mori dream, is described.

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Introduction

The main topics of the present paper are étale coverings in codimension 1 between algebraic varieties, in the following simply called 1-coverings, aimed to studying the topology of Mori dream spaces (MDS). A 1-covering is a finite morphism, étale over an open Zariski subset of the domain, whose complementary closed subset has codimension strictly greater than 1 (see Definition 1.1).

As far as I know, there is no too much literature on this topic. They were studied in some detail by F. Catanese in [5], although in the slightly broader sense of quasi-étale morphisms, i.e. quasi-finite morphisms, étale in codimension 1. More recently they have been employed by D. Grab, S. Kebekus and T. Peternell in [14], as a useful tool to describing how extending étale coverings of the regular locus of a complex, quasiprojective algebraic variety, admitting at most Kawamata log-terminal singularities, to coverings of the whole variety. Moreover in 2008, W. Buczynska posted the still unpublished paper [4], where she studied relations between 1-coverings and the topological fundamental group in codimension 1 of a complex algebraic variety, with applications to toric varieties. In all the three papers above, the ground field is that of complex numbers, $\mathbb{C}$, in a rather essential sense, due to special techniques the authors employed.

Here, I resume the Buczynska’s approach, by revising her topological results in [4] from the algebraic-étale point of view. Then what has been here obtained about relations between 1-coverings and the algebraic fundamental group in codimension 1 is holding on a general algebraically closed field $K = \mathbb{C}$, with char $K = 0$. This is the content of §1.3 and §1.4. As observed in Remarks 1.11 and 1.26 results here obtained, like e.g. Theorem 1.10, Corollary 1.24 and Theorem 1.25 do note imply their topological analogous statements proved by Buczynska in [4], unless the involved fundamental groups are finite, as in the importante case of toric varieties.

Consequently §2 is devoted to apply results of previous sections to toric varieties, so obtaining a natural field extension of results proved in [4, §4]. In particular, Theorem 2.15 shows that a non-degenerate toric variety always admits a universal 1-covering, which is still a non-degenerate toric variety: this is an extension of [32, Thm. 2.2] in which the same statement was proved for a complex, complete and $\mathbb{Q}$-factorial toric variety. Let me here recall that, as for the universal covering, in general, an algebraic variety does not admit a universal 1-covering. Then the main interest of Theorem 2.15 resides in defining a class of algebraic varieties, namely non-degenerate toric varieties, giving an exception toward such a general fact.

Recalling that a MDS has a canonical toric embedding, what proved in §2 applies to give interesting consequences on the topology of a MDS. This is the content of §3 where we considered a slightly broader (with respect to MDS) category of spaces called, coherently with [29], weak Mori dream spaces (wMDS). A wMDS
admitting a projective closed embedding is a MDS in the usual Hu-Keel sense [21]. Probably the main result here obtained is the construction of a canonical 1-covering $\tilde{X}$ of a wMDS $X$, given by Theorem 3.17. In particular, such a canonical 1-covering comes with a canonical closed embedding into the universal 1-covering $\tilde{W}$ of the the canonical ambient toric variety $W$ of $X$, whose existence is guaranteed by the previous Theorem 2.15. Unfortunately, this canonical embedding between 1-coverings does not turn out to be a neat embedding (see Def. 3.12), in general: but the latter is shown to be equivalent with the condition of being a wMDS for the canonical 1-covering $\tilde{X}$.

The following §3.4 and §3.5 are dedicated to studying properties of the canonical embedding $\tilde{X} \hookrightarrow \tilde{W}$ and the topology of $\tilde{X}$ itself, respectively. In particular, as a consequence of results of M. Artebani and A. Laface [1], S.-Y. Jow [22] and G. Ravindra and V. Srinivas [27], Proposition 3.23 gives some sufficient conditions for $\tilde{X} \hookrightarrow \tilde{W}$ being a neat embedding, hence the canonical 1-covering $\tilde{X}$ still being a wMDS. On the other hand, by applying deep results of M. Goresky and R. MacPherson [12], Theorem 3.27 gives a sufficient condition for the canonical 1-covering $\tilde{X} \rightarrow X$ being the universal one, in the complex case $\mathbb{K} = \mathbb{C}$.

The present paper is organized as follows. §1.1 1.2 1.3 are dedicated to recall standard facts on étale coverings and étale fundamental groups. In particular, we give sketches of proof of properties like e.g. Prop. [1.6] and Prop. [1.8] to introduce techniques useful for the following analogous treatment of étale 1-coverings and étale fundamental groups in codimension 1. Theorem 1.10 gives an algebraic-étale counterpart of [4, Thm. 3.4] (see Remark 1.11). The following §1.4 introduces the (local) theory of Galois 1-coverings and étale fundamental groups in codimension 1. Main result of this section is Theorem 1.25 relating the étale fundamental group in codimension 1 of a normal variety with the étale fundamental group of its regular locus, so giving an algebraic-étale counterpart of [4, Cor. 3.10]. Then §1.5 and §1.6 ends up §1 by fixing notation on divisors' pull back and 1-coverings between complete orbifolds. As already described above, §2 and §3 are devoted to applying results and techniques, developed in §1 to toric varieties and wMDS, respectively. In the last §4 I give evidences of both positive and negative occurrences in Theorem 3.17 by means of two interesting example. The former is given by Example 4.1, describing a case in which the canonical 1-covering is still a wMDS (actually a MDS): this example was borrowed from id no. 97 in [19]. The latter is given by very special families of Enriques surfaces which are Mori dream spaces. Their canonical 1-covering is also their universal étale covering, hence a K3 surface which can never be a MDS, as admitting an infinite automorphism group. In this case Theorem 3.17 gives interesting information about this kind of special Enriques surfaces, their K3 universal coverings and the associated canonical toric embeddings (see Cor. [4.3] and Rem. 4.4).

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1. Étale covering in codimension 1 (1-covering)

The present section is devoted to recall and extend to any algebraically closed field $\mathbb{K}$, with $\text{char } \mathbb{K} = 0$, concepts and results introduced in [4, § 3], under the assumption $\mathbb{K} = \mathbb{C}$. Notice that results here given cannot in general replace Buczyńska’s results in [4] about the fundamental group in codimension 1 of a complex algebraic variety, since known conditions on the pro-finite completion $\widehat{G}$ of a group $G$ do not transfer to the group $G$ itself, except for the particular case $G$ finite.

Notation. Throughout the presente paper a small closed subset $C$ of an algebraic variety $X$ is a Zariski closed $C \subset X$ such that $\text{codim}_X C > 1$. The complementary set $X \setminus C$ is called a big open subset of $X$.

Moreover, a morphism of algebraic varieties $\phi : Y \to X$ with $X$ irreducible, is called an étale covering if it is a finite étale morphism; since $X$ is irreducible than $\phi$ is surjective with finite fibres of constant cardinality called the degree of $\phi$ (deg $\phi$).

The following is the key definition of the present paper: what is meant by étale covering in codimension 1 of an algebraic variety $X$. Here, $X$ is assumed irreducible although connected should be enough: in fact one can apply the following definition to every irreducible component.

Definition 1.1 (1-covering). Let $\phi : Y \to X$ be a morphism of irreducible algebraic varieties over $\mathbb{K}$. Then $\phi$ is called an étale covering in codimension 1 (or simply a 1-covering) if it is finite and étale in codimension 1, that is, there exists a small Zariski closed subset $C \subset X$ such that

$$\phi|_{Y_C} : Y_C := \phi^{-1}(X \setminus C) \longrightarrow X \setminus C$$

is a finite and étale morphism onto the the complementary big open subset $X \setminus C$. The small closed $C$ is called the branching locus of $\phi$ and denoted by $C = \text{Br } \phi$.

The degree of the étale covering $\phi|_{Y_C}$ is called the degree of the 1-covering $\phi$, that is $\deg \phi := \deg(\phi|_{Y_C})$.

Recall that the automorphism group $\text{Aut}(\phi)$ of an étale covering $\phi : Y \to X$ is the group of isomorphisms $\varphi : Y \to Y$ such that $\phi = \phi \circ \varphi$. An étale covering is called Galois if $|\text{Aut}(\phi)| = \deg \phi$. By the following Proposition 1.5 this is the same of asking that $\text{Aut}(\phi)$ acts transitively over the fibres.

A Galois 1-covering is a 1-covering $\phi : Y \to X$ such that $|\text{Aut}(\phi|_{Y_C})| = \deg \phi$, where $C = \text{Br } \phi$. This means that $\text{Aut}(\phi|_{Y_C})$ acts transitively over the fibres of points in $X \setminus C$. In the following we will denote

$$\text{Aut}^{(1)}(\phi) := \text{Aut}(\phi|_{Y_C})$$

A universal covering in codimension 1 (or simply a universal 1-covering) is a Galois 1-covering $\varphi : \overline{X} \to X$ such that, for every Galois 1-covering $\phi : Y \to X$, there exists a 1-covering $f : \overline{X} \to Y$ with $\varphi = \phi \circ f$.

Lemma 1.2. Let $X$ be an irreducible and reduced algebraic variety and $C \subset X$ be a small closed subset. Then an étale covering $\phi : U \to X \setminus C$ can be always extended to a 1-covering $\overline{\phi} : \overline{U} \to X$. In particular if $X$ is smooth then $\overline{\phi}$ is an étale covering, that is $\text{Br } \overline{\phi} = \emptyset$.

Proof. This is an improvement of [4, Lemma 3.15]: see the following Remark 1.3. Up to an affine open cover of $X$, one can assume $X$ affine, that is $X = \text{Spec } A$. Let
\[ I = (f_1, \ldots, f_c) \subset A \text{ be the ideal defining the small closed subset } C = \text{Spec}(A/I). \]

Then
\[ X \setminus C = \bigcup_{i=1}^{c} X_{f_i}, \quad X_{f_i} := \text{Spec} A_{f_i} \]

where \( A_{f_i} \) is the localization \( S_i^{-1}A \) with respect to \( S_i := \{ f_i^n \mid n \in \mathbb{N} \} \). Consider \( U_i := \phi^{-1}(X_{f_i}) \) and set \( \phi_i := \phi|_{U_i} : U_i \to X_{f_i} \). Since \( U = \bigcup_i U_i \), the extension of \( \phi \) can then be preformed by extending every \( \phi_i \). The latter is a finite morphism over the affine open \( X_{f_i} \). Then
\[ U_i = \text{Spec} B_i, \quad \phi_{i,*} : A_{f_i} \hookrightarrow B_i \]

where \( B_i \) is a finitely generated \( \phi_{i,*}(A_{f_i}) \)-module. Set
\[ \overline{U}_i = \text{Spec} \overline{B}_i, \quad \overline{B}_i := B_i \otimes_{\phi_{i,*}(A_{f_i})} A \]

Then \( \phi_{i,*} \) admits a natural lifting \( \overline{\phi}_{i,*} : A \hookrightarrow \overline{B}_i \) making commutative the following diagram
\[
\begin{array}{ccc}
A & \xrightarrow{\overline{\phi}_{i,*}} & \overline{B}_i \\
\downarrow & & \downarrow \\
A_{f_i} & \xrightarrow{\phi_{i,*}} & B_i
\end{array}
\]

By construction the induced morphism \( \overline{\phi}_i : \overline{U}_i \rightarrow X_{f_i} \) is finite and étale, possibly ramified along the small closed \( V(f_i) \). Patching all together, we get a finite étale morphism \( \overline{\phi} : \overline{U} := \bigcup_i \overline{U}_i \rightarrow X \setminus C \), possibly ramified over the small closed subset \( C = \bigcap_i V(f_i) \), so giving a 1-covering of \( X \).

Assume \( X \) be smooth. Then the singular locus \( \text{Sing} \overline{U} \) is a small closed subset of \( \overline{U} \). Let \( \psi : \overline{U} \rightarrow \overline{U} \) be the normalization of \( \overline{U} \); it is a finite map which is an isomorphism outside of the small closed subset \( \overline{C} := \psi^{-1}(\text{Sing} \overline{U}) \). Then \( \overline{\phi} \circ \psi : \overline{U} \rightarrow X \) is a morphism from a normal variety to a smooth one, which is étale outside of \( C \); this means that its branch locus is a small closed subset of \( X \) included in \( C \). By the Zariski-Nagata purity theorem (see e.g. [35, Thm. 5.2.13]) this means that \( \text{Br}(\overline{\phi} \circ \psi) = \emptyset \) and \( \overline{\phi} \circ \psi \) is an étale covering of \( X \). Then also \( \text{Br}(\overline{\phi}) = \emptyset \) and \( \overline{\phi} \) is an étale covering of \( X \).

\[ \square \]

Remark 1.3. Consider the case \( \mathbb{K} = \mathbb{C} \) and let \( X \) be a smooth complex irreducible algebraic variety. Let \( X^{an} \) be the corresponding complex manifold endowed with the analytic topology, with respect to which \( X^{an} \) turns out to be path-connected and semi-locally simply connected. Then the Riemann Existence Theorem [16 Thm. XII.5.1] establishes a categorical equivalence between the category of étale coverings of \( X \) and the category of finite topological coverings of \( X^{an} \) [13, 24 thm. 3.4]. In particular, this implies that the analytic counterpart of the previous Definition [1.1] is [1 Def. 3.13]. Then the previous Lemma [1.2] implies and extends [4 Lemma 3.15].

1.1. The étale fundamental group of an algebraic variety. Recall that the étale (or algebraic) fundamental group of a connected algebraic variety \( X \), with a chosen base point \( x \in X \), is defined as the automorphism group of the fiber functor \( F^x \) assigning to each étale covering \( \phi : Y \rightarrow X \) the finite set given by its fibre \( F^x(\phi) := \phi^{-1}(x) \) over the base point \( x \) (see e.g. [35 Def. 5.4.1]). Then the étale
fundamental group is a functor from the category of étale coverings to the category of groups. Grothendieck proved that it is pro-representable [16], [35, Prop. 5.4.6], that is it can be represented as the inverse limit
\[
\pi^\text{et}_1(X, x) := \lim_{\text{←−}} \text{Aut}(\phi_i)
\]
running through all the Galois étale coverings \(\{X_i \xrightarrow{\phi_i} X\}_{i \in \mathcal{I}}\).

**Remark 1.4.** To emphasize the importance of the base point \(x \in X\) in the definition (1), notice that the direct system of Galois étale coverings can be represented as a direct system of finite sets by setting
\[
\{X_i \xrightarrow{\phi_i} X\} \quad \text{is represented by a surjection } \phi_i^{-1}(x) \twoheadrightarrow \phi_j^{-1}(x),
\]
where the fibred product on the right is obtained from the inclusion morphism \(\{x\} \hookrightarrow X\). Then the partial order relation on Galois étale coverings induced by covering morphisms as follows
\[
\{X_j \xrightarrow{\phi_j} X\} \leq \{X_i \xrightarrow{\phi_i} X\} \quad \text{iff } \exists \phi_{ij} : X_i \longrightarrow X_j \quad : \quad \phi_i = \phi_j \circ \phi_{ij}
\]
is represented by a surjection \(\phi_i^{-1}(x) \twoheadrightarrow \phi_j^{-1}(x)\), among the corresponding finite sets. This is actually a direct system because the fibred product of Galois étale coverings is a Galois étale covering [25, IV (4.1)] giving the following commutative diagram of Galois étale coverings of \(X\)
\[
\forall i, j \in \mathcal{I}
\]
\[
\begin{array}{ccc}
X_i & \xrightarrow{\phi_i} & X_j \\
\downarrow{\phi_i} & & \downarrow{\phi_j} \\
X & \xrightarrow{\phi_{ij}} & Y
\end{array}
\]
Notice that, given a morphism of Galois étale coverings \(\phi_{ij} : X_i \longrightarrow X_j\), the choice of base points \(x_i \in X_i\) and \(x_j \in X_j\), such that \(\phi_i(x_i) = x = \phi_j(x_j)\) and \(\phi_{ij}(x_i) = x_j\), induces a morphism \(\phi_{ij,*} : \text{Aut}(\phi_i) \longrightarrow \text{Aut}(\phi_j)\), over which the inverse limit (1) is performed.

Recall the following key fact about étale morphisms:

**Proposition 1.5** ([24], Cor. 2.16; [35], Cor. 5.3.3). Let \(\phi : Y \longrightarrow X\) and \(f : Z \longrightarrow X\) be morphisms of algebraic varieties over an algebraically closed field. Assume \(\phi\) is étale and \(Z\) is connected. Let \(\varphi, \varphi' : Z \longrightarrow Y\) be morphisms lifting \(f\), that is such that
\[
\phi \circ \varphi = f = \phi \circ \varphi' : Y \longrightarrow Z \xrightarrow{\varphi} X \xrightarrow{\phi} Y
\]
If there exists \(z \in Z\) such that \(\varphi(z) = \varphi'(z)\) then \(\varphi = \varphi'\).
A first consequence of Proposition 1.5 is that the transitive action of the Galois group \( \text{Aut}(\phi_i) \) can be represented by acting on \( \phi_i^{-1}(x) \) with a subgroup of the group \( \mathfrak{S}_i^\ast \) of cyclic permutations. In fact, every non-trivial automorphism of the representing fibre \( \phi_i^{-1}(x) \) cannot fix any point.

**Proposition 1.6** ([35], Cor. 5.5.2). For any \( x, x' \in X \) there exists an isomorphism

\[
\pi^\text{\`et}_1(X, x) \cong \pi^\text{\`et}_1(X, x')
\]

well defined up to conjugation.

**Proof.** Let us give here an argument proving the statement, as it will be useful in the sequel (e.g. when proving Cor. 1.28). By choosing base points

\[
\forall i, j \in \mathcal{I} \quad x_i, x'_i \in X_i : \quad \phi_i(x_i) = x, \ \phi_i(x'_i) = x', \ \phi_{ij}(x_i) = x_j, \ \phi_{ij}(x'_i) = x'_j
\]

one immediately gets bijections \( f_i : \phi_i^{-1}(x) \xrightarrow{1:1} \phi_i^{-1}(x') \), for every \( i \in \mathcal{I} \), well defined as \( X_i \xrightarrow{\phi_i} X \) are Galois. In particular

\[
\forall i, j \in \mathcal{I} \quad \{X_j \xrightarrow{\phi_j} X\} \subseteq \{X_i \xrightarrow{\phi_i} X\} \implies \phi_i^{-1}(x) \xrightarrow{f_i} \phi_j^{-1}(x)
\]

The choice of two collections of bijections \( \{f_i\}_{i \in \mathcal{I}} \), \( \{g_i\}_{i \in \mathcal{I}} \) gives rise to two isomorphisms \( \mathfrak{S}_i^\ast \cong \mathfrak{S}_i^\ast' \) as follows:

\[
\forall i \in \mathcal{I} \quad \phi_i : \sigma_x \in \mathfrak{S}_i^\ast \xrightarrow{f_i \circ \sigma_x \circ f_i^{-1}} \mathfrak{S}_i^\ast'
\]

\[
\forall i \in \mathcal{I} \quad \phi_i : \sigma_x \in \mathfrak{S}_i^\ast \xrightarrow{g_i \circ \sigma_x \circ g_i^{-1}} \mathfrak{S}_i^\ast'
\]

Then \( \psi_i(\sigma_x) = \tau_i \circ \phi_i(\sigma_x) \circ \tau_i^{-1} \), with \( \tau_i = g_i \circ f_i^{-1} \in \mathfrak{S}_i^\ast' \), for every \( i \in \mathcal{I} \). The statement is then obtained by passing to the inverse limit on \( \mathcal{I} \).

**Remark 1.7.** A collection of bijections \( \{f_i : \phi_i^{-1}(x) \xrightarrow{1:1} \phi_i^{-1}(x')\}_{i \in \mathcal{I}} \), as constructed in the previous proof, is usually called a path joining \( x \) to \( x' \), due to the analogous construction on topological fundamental groups, when \( K = \mathbb{C} \), as described at the end of the following Remark 1.9.

**Proposition 1.8.** Let \( f : (Y, y) \rightarrow (X, x) \) be a morphism of pointed irreducible algebraic varieties, that is \( x = f(y) \). Then there exists an induced homomorphism of étale fundamental groups:

\[
f_* : \pi^\text{\`et}_1(Y, y) \rightarrow \pi^\text{\`et}_1(X, x)
\]

**Proof.** This is a standard construction (see e.g. [25] Chap. V and [35] § 5.5, pg. 178) often useful in the sequel: for this reason it is here reported.

Let \( \phi' : X' \rightarrow X \) be a Galois étale covering of \( X \). Consider the fibred product commutative diagram

\[
\begin{array}{c}
Y \times_X X' \xrightarrow{pr_2} X' \\
pr_1 \downarrow \quad \downarrow \phi' \\
Y \xrightarrow{f} X
\end{array}
\]
Then \( f^\#: \mathcal{C} \times X \longrightarrow \mathcal{C} \) turns out to be a Galois étale covering of \( X \). This gives a functor \( f^\# \) from the category of étale coverings of \( X \) to that of étale coverings of \( Y \) such that, when applying the corresponding fibre functors, one gets an isomorphism \( f^* \) of fibers over the base points, namely:

\[
f^*: \mathcal{P}^{-1}(y) = F_y^\mathcal{P}(\mathcal{P}) = F_y^\mathcal{P} \circ f^\#(\phi') \cong F_f^\mathcal{P}(\phi') = \phi'^{-1}(x)
\]

Then \( f^* \) induces an homomorphism between fundamental groups as automorphisms groups of fibres functors, as follows

\[
f_* : \sigma \in \text{Aut}(F_y^\mathcal{P}) = \pi^\text{ét}(Y, y) \longrightarrow f^* \circ \sigma \circ (f^*)^{-1} \in \text{Aut}(F_f^\mathcal{P}) = \pi^\text{ét}(X, x)
\]

\[
\square
\]

**Remark 1.9.** For \( K = \mathbb{C} \) the Riemann Existence Theorem [10 Thm. XII.5.1] gives a canonical isomorphism between the étale fundamental group \( \pi^\text{ét}(X, x) \) and the pro-finite completion of the fundamental group \( \pi_1(X^\text{an}, x) \), that is

\[
\pi^\text{ét}(X, x) \cong \hat{\pi}_1(X^\text{an}, x) := \lim_{\longrightarrow} \left( \pi_1(X^\text{an}, x) / N \right)
\]

where \( N \) ranges through all the normal subgroups with finite index of \( \pi_1(X^\text{an}, x) \) [10 Cor. 5.2]. Notice that \( \pi_1(X^\text{an}, x) \) naturally maps onto each of its quotients, giving rise to a canonical map

\[
\pi : \pi_1(X^\text{an}, x) \longrightarrow \hat{\pi}_1(X^\text{an}, x)
\]

If \( \pi_1(X^\text{an}, x) \) is a finite group then \( \pi \) is an isomorphism.

The previous Propositions [1.3.1.6 and 1.3.8] are generalizations, to every algebraic closed field \( K \) with \( \text{char} \ K = 0 \), of well known topological analogous results. In particular, for \( K = \mathbb{C} \), Prop. 1.6.0 can be obtained as an immediate consequence, passing to pro-finite completions, of the isomorphism \( \pi_1(X^\text{an}, x) \cong \pi_1(X^\text{an}, x') \) obtained by choosing a path connecting \( x \) and \( x' \).

The previous Lemma [1.2] is the key ingredient to show the following excision property for the étale fundamental group of a smooth variety.

**Theorem 1.10.** Let \( C \) be a small closed subset of a smooth and irreducible algebraic variety \( X \), that is \( \text{codim}_X C \geq 2 \). Let \( x \in X \setminus C \) be a fixed base point. Then \( \pi^\text{ét}(X \setminus C, x) \cong \pi^\text{ét}(X, x) \).

**Remark 1.11.** In [4 Thm. 3.4] Buczynska proved a statement which is the analogue of Theorem 1.10 in the particular case \( K = \mathbb{C} \) and for the fundamental group \( \pi_1(X^\text{an}, x) \), under the further hypothesis that \( C \) is also smooth: in fact her proof is essentially based on differential-topological technics. In the Appendix of [4] she sketched a road map to dropping such a smoothness condition on \( C \).

Notice that, if \( K = \mathbb{C} \) then Theorem 1.10 does not imply in general [4 Thm. 3.4], unless \( X^\text{an} \) admits a finite fundamental group \( \pi_1(X^\text{an}, x) \); in this case the Buczynska’s result is obtained without any smoothness assumption on \( C \). In fact, in this case \( \pi_1(X^\text{an}, x) \cong \hat{\pi}_1(X^\text{an}, x) \cong \pi^\text{ét}(X, x) \). On the other hand, since \( X \) is smooth (hence normal) the inclusion \( X \setminus C \hookrightarrow X \) induces a surjection

\[
\pi_1((X \setminus C)^\text{an}, x) \twoheadrightarrow \pi_1(X^\text{an}, x)
\]
(see e.g. [7 Thm. 12.1.5]). Finally there is a canonical surjection onto a group $G$ from its pro-finite completion $\hat{G}$, so giving

$$\pi^\text{et}_1(X, x) \cong \pi^\text{et}_1(X \setminus C, x)$$

$$\Rightarrow \Rightarrow$$

$$\pi_1((X \setminus C)^\text{an}, x) \Rightarrow \pi_1(X^\text{an}, x) \cong \pi_1((X \setminus C)^\text{an}, x)$$

**Proof of Thm. 1.10.** Clearly a Galois étale covering $\phi_i : X_i \rightarrow X$ restricts to give a Galois étale covering of $U = X \setminus C$, namely $\phi'_i : U_i \rightarrow U$, where $U_i = \phi_i^{-1}(X_i \setminus C)$ and $\phi'_i = \phi_i|_{U_i}$. Conversely, Lemma 1.2 shows that every Galois étale covering $\phi'_i : U_i \rightarrow U$ can be extended to a Galois étale covering $\overline{\phi}_i : \overline{U}_i \rightarrow X$. Notice that, up to isomorphism, these procedures are inverse to each other. In fact $\phi_i$ and $\overline{\phi}_i$ matches on the Zariski open $U_i$. Moreover they are étale morphisms, meaning that for every $y \in C$ there exists a Zariski open $V \subseteq X$ such that

$$\phi_i^{-1}(V) \cong V \cong \overline{\phi}_i^{-1}(V)$$

$C$ can be covered by a finite number of such open subsets $V$, gluing together to give a global matching $\phi_i = \overline{\phi}_i$. Moreover, for every $i$, there is an isomorphism $\text{Aut}(\phi'_i) \cong \text{Aut}(\phi_i)$. The statement is then proved by passing to inverse limits. ☐

**Remark 1.12.** The previous Theorem 1.10 is a consequence of Lemma 1.2 Conversely, the isomorphism $\pi^\text{et}_1(X \setminus C, x) \cong \pi^\text{et}_1(X, x)$, induced by the inclusion $X \setminus C \hookrightarrow X$, implies that every finite étale covering of $X \setminus C$ extends to giving a finite étale covering of $X$, as a consequence of Grothendieck’s equivalence (see e.g. [35 Thm. 5.4.2], [24 Thm. 3.1]). Then:

**Theorem 1.10 is equivalent to the smooth part of Lemma 1.2.**

An interesting generalization of this fact, in the complex projective setup, has been recently given by Greb, Kebekus and Peternell who proved, as a particular case of a deeper result, that a normal, complex, projective algebraic variety $X$ with at most Kawamata log-terminal (Klt) singularities, admits a 1-covering $\gamma : X' \rightarrow X$ such that every finite étale covering of the regular locus $X'_\text{reg}$ extends to a finite étale covering of $X'$, or, equivalently, that the inclusion $X'_\text{reg} \hookrightarrow X'$ induces an isomorphism $\pi^\text{et}_1(X'_\text{reg}, x) \cong \pi^\text{et}_1(X', x)$ [14 Thm. 1.5].

1.2. **The universal étale covering.** Recall that the **universal étale covering** of an irreducible algebraic variety $X$ is a Galois étale covering dominating every element in the direct system of Galois étale covering of $X$. In general it does not exists as it is a pro-finite covering, pro-representable as the inverse limit of Galois étale covering. By construction, the Galois group of the universal étale covering of $X$ (if existing!) is the étale fundamental group of $X$.

**Definition 1.13.** By analogy with the complex case, as recalled in the previous Remark 1.5, an irreducible algebraic variety is called **simply connected** if $\pi^\text{et}_1(X, x)$ is trivial, for some (hence for every) point $x \in X$.

Notice that, for $\mathbb{K} = \mathbb{C}$, $\pi^\text{et}_1(X, x) \cong \{1\}$ if and only if $\pi_1(X^\text{an}, x) \cong \{1\}$. 
Proposition 1.14. A Galois étale covering \( \phi' : X' \to X \) is the universal étale covering of \( X \) if and only if \( X' \) is simply connected.

Proof. Let \( \phi' : X' \to X \) be the universal étale covering of \( X \). Let \( \psi : X'' \to X' \) be a Galois étale covering of \( X' \). Then \( \phi' \circ \psi : X'' \to X \) is a Galois étale covering of \( X \) such that

\[
\{ X' \xrightarrow{\phi'} X \} \leq \{ X'' \xrightarrow{\phi' \circ \psi} X \}
\]

\( \phi' \) is universal \( \implies \) \( \{ X' \xrightarrow{\phi'} X \} \cong \{ X'' \xrightarrow{\phi' \circ \psi} X \} \)

\( \implies \) \( \operatorname{Aut}(\phi') \cong \{1\} \)

Therefore \( \pi_1^{\text{et}}(X', x') \cong \{1\} \), for \( x' \in X' \), since \( X'' \) is arbitrary.

For the converse, assume \( \pi_1^{\text{et}}(X', x') \cong \{1\} \) and consider any further Galois étale covering \( \phi'' : X'' \to X \). Then the commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\phi'} & X'' \\
\downarrow{\phi'} & & \downarrow{\phi''} \\
X & \xrightarrow{\phi''} & X
\end{array}
\]

exhibits \( \text{pr}_1 : X' \times_X X'' \to X' \) as a Galois étale covering of \( X' \). By the inverse limit pro-representation of \( \pi_1^{\text{et}}(X', x') \) one gets a natural surjection

\[
\{1\} \cong \pi_1^{\text{et}}(X', x') \twoheadrightarrow \operatorname{Aut}(\phi' \times_X \phi'') \twoheadrightarrow \operatorname{Aut}(\phi' \times_X \phi'') \cong \{1\}
\]

Then \( \text{pr}_1 \) is an isomorphism and \( \text{pr}_2 \circ \text{pr}_1^{-1} : X' \to X'' \) is a morphism of Galois étale covering of \( X \), so giving that \( \{ X'' \xrightarrow{\phi''} X \} \leq \{ X' \xrightarrow{\phi'} X \} \) and showing that the latter is the universal étale covering of \( X \). \( \square \)

Proposition 1.15. If a Galois 1-covering \( \phi' : X' \to X \) is universal then \( X' \) is simply connected.

Proof. Let \( \phi'' : X'' \to X' \) be any Galois étale covering of \( X' \). Then

\[
\phi := \phi' \circ \phi'' : X'' \to X
\]

is a Galois 1-covering of \( X \) such that \( \text{Br} \phi = \text{Br} \phi' =: C \). \( \phi' \) is universal, meaning that there exists a Galois 1-covering \( \psi : X' \to X'' \) such that \( \phi \circ \psi = \phi' \), that is the following diagram commutes

\[
\begin{array}{ccc}
X'' & \xrightarrow{\psi} & X' \\
\downarrow{\phi''} & & \downarrow{\phi'} \\
X' & \xrightarrow{\phi'} & X
\end{array}
\]

Then \( \phi'' \circ \psi \in \operatorname{Aut}(\phi'|_{X''}) \), where \( X''_{C} := \phi'^{-1}(X \setminus C) \). Then \( \phi'' \) restricts to give an isomorphism on the big open subset \( X''_{C} := \phi'^{-1}(X \setminus C) \subseteq X'' \), meaning that \( \phi'' \) gives actually an isomorphism \( X'' \cong X' \), as \( \phi'' \) is étale. Then \( \operatorname{Aut}(\phi'') \cong \{1\} \).

Passing to the inverse limit on the direct system of Galois étale coverings of \( X' \), one gets \( \pi_1^{\text{et}}(X', x') \cong \{1\} \), for every \( x' \in X' \). \( \square \)
1.3. The étale fundamental group in codimension 1. Let $X$ be an irreducible and reduced algebraic variety and $x \in X$ a fixed point. Consider the collection of big Zariski open neighborhoods of $x$ in $X$

$$\Omega^{(1)}_x := \{U \subseteq X \mid U \text{ is open, } x \in U \text{ and } \text{codim}_X(X \setminus U) > 1\}$$

Consider the partial order relation $\preceq$ on $\Omega^{(1)}_x$ given by setting: $U \preceq V :\Leftrightarrow U \supseteq V$. Then $(\Omega^{(1)}_x, \preceq)$ is a direct system because any two elements are dominated by their intersection.

**Proposition 1.16.** Consider $U, V \in \Omega^{(1)}_x$ such that $U \preceq V$. Then there exists a well defined homomorphism $\pi^\text{ét}(V, x) \rightarrow \pi^\text{ét}(U, x)$.

**Proof.** Apply Proposition 1.8 to the open embedding $V \hookrightarrow U$. □

**Definition 1.17** (The étale fundamental group in codimension 1). Let $X$ be an irreducible and reduced algebraic variety and $x \in X$ a base point. The following inverse limit

$$\pi^\text{ét}_1(X, x)^{(1)} := \lim_{U \in \Omega^{(1)}_x} \pi^\text{ét}_1(U, x)$$

is called the étale fundamental group in codimension 1 of $X$ centered at $x$.

**Remark 1.18.** For $K = \mathbb{C}$, by the Riemann Existence Theorem of Grothendieck, the étale fundamental group defined in Definition 1.17 is the pro-finite completion of the fundamental group in codimension 1 $\pi_1^1(X^\text{an}, x)$ defined in [4, Def. 3.1], that is

$$\pi^\text{ét}_1(X, x)^{(1)} := \lim_{U \in \Omega^{(1)}_x} \pi_1(U^\text{an}, x) = \hat{\pi}_1^1(X^\text{an}, x)$$

Therefore if $\pi_1^1(X^\text{an}, x)$ is finite then $\pi^\text{ét}_1(X, x)^{(1)} \cong \pi_1^1(X^\text{an}, x)$.

It makes then sense to set the following definition even when $K$ is an arbitrary algebraically closed field with $\text{char} K = 0$:

**Definition 1.19** ($x$-1-connectedness). Let $X$ be an irreducible and reduced algebraic variety and $x \in X$ be a fixed base point. Then $X$ is called locally connected in codimension 1 near to $x$ (or $x$-1-connected for ease) if $\pi^\text{ét}_1(X, x)^{(1)}$ is trivial.

1.4. The direct system of local Galois 1-coverings. Consider the collection

$$\{\phi_i : X_i \rightarrow X\}_{i \in I_x^{(1)}}$$

of all Galois 1-coverings of $X$ such that $x \in X \setminus \text{Br} \phi_i$, for every $i \in I_x^{(1)}$. Call such a 1-covering a local Galois 1-covering of $X$ centered at $x$.

**Proposition 1.20.** Let $X$ be an irreducible algebraic variety and $x \in X$ a base point. Then the set of all local Galois 1-coverings of $X$ centered at $x$ is a direct system and

$$\pi^\text{ét}_1(X, x)^{(1)} = \lim_{i \in I_x^{(1)}} \text{Aut}^{(1)}(\phi_i)$$

where $\text{Aut}^{(1)}(\phi_i)$ is defined in Definition 1.1.
Proof. As for the direct system of étale coverings, set
\[ \forall i, j \in \mathbb{N} \setminus 0 \quad \{ X_j \xrightarrow{\phi_j} X \} \subseteq \{ X_i \xrightarrow{\phi_i} X \} \iff \exists \phi_{ij} : X_i \to X_j : \phi_i = \phi_j \circ \phi_{ij} \]
defining an order relation on the considered set of local Galois 1-coverings. Moreover, it turns out to be a direct system since the fibred product
\[ \phi_j \times X \phi_i : X_j \times X_i \to X \]
is still a local Galois 1-covering of \( X \) centered at \( x \), as
\[ \text{Br}(\phi_j \times X \phi_i) = \text{Br} \phi_j \cup \text{Br} \phi_i \]
(recall the commutative diagram 2).

Moreover, the 1-covering morphism \( \phi_{ij} : X_i \to X_j \) clearly induces a surjection on fibres \( \phi_i^{-1}(x) = F^x(\phi_i) \to F^x(\phi_j) = \phi_j^{-1}(x) \) and then a morphism on the associated automorphism groups
\[ \text{Aut}^{(1)}(\phi_i) \to \text{Aut}^{(1)}(\phi_j) \]
where \( U_i := X \setminus \text{Br} \phi_i \) and \( U_j := X \setminus \text{Br} \phi_j \). Then their inverse limits are well defined and the statement follows immediately by Definition 1.17.

Definition 1.21 (Universal local 1-covering). Let \( X \) be an irreducible and reduced algebraic variety and \( x \in X \) be a fixed base point. A local Galois 1-covering centered at \( x \) is called universal if it dominates every element in the direct system of local Galois 1-coverings of \( X \) centered at \( x \).

Remark 1.22. Let \( \tilde{\phi} : \tilde{X} \to X \) be the universal 1-covering of \( X \), as defined in Definition 1.14. Then, for every \( x \in X \setminus \text{Br}(\tilde{\phi}) \) it is also the universal local 1-covering of \( X \) centered at \( x \).

Proposition 1.23. A local Galois 1-covering \( \phi' : X' \to X \) centered at \( x \in X \) is universal if and only if \( X' \) is \( x' \)-1-connected for some (hence every) \( x' \in \phi'^{-1}(x) \), that is \( \pi_1^{\text{ét}}(X', x')^{(1)} \cong \{1\} \).

Proof. The proof is the analogue of that proving Proposition 1.14 by replacing “étale covering of \( X \)” (resp. “of \( X' \)”’) with “local Galois 1-covering of \( X \) centered at \( x \)” (resp. “of \( X' \) centered at \( x' \)”’). Notice that the choice of \( x' \in \phi'^{-1}(x) \) is actually irrelevant for the proof working.

We are now in a position to giving some further analogous results to those given in [4, § 3].

Corollary 1.24 (Compare with Cor. 3.9 in [4]). If \( X \) is a smooth irreducible algebraic variety then \( \pi_1^{\text{ét}}(X, x)^{(1)} \cong \pi_1^{\text{ét}}(X, x), \) for every \( x \in X \).

Proof. By definition
\[ \pi_1^{\text{ét}}(X, x)^{(1)} = \lim_{\text{small} \ C \subseteq X} \pi_1^{\text{ét}}(X \setminus C, x) \leq \lim_{\text{small} \ C \subseteq X} \pi_1^{\text{ét}}(X, x) \cong \pi_1^{\text{ét}}(X, x) \]

Theorem 1.25 (Compare with Cor. 3.10 in [4]). Let \( X \) be a normal irreducible algebraic variety and \( X_{\text{reg}} \subseteq X \) the Zariski open subset of regular points of \( X \). Then \( \pi_1^{\text{ét}}(X, x)^{(1)} \cong \pi_1^{\text{ét}}(X_{\text{reg}}, x), \) for every regular point \( x \in X_{\text{reg}} \).
Proof. This proof is similar to the one of Theorem 1.10.

On the one hand, consider the direct system \( \{ X_i \xrightarrow{\phi_i} X \}_{i \in \mathcal{I}} \) of local Galois 1-coverings of \( X \) centered at \( x \). On the other hand, let \( \{ U_j \xrightarrow{\psi_j} X_{\text{reg}} \}_{j \in \mathcal{J}} \) be the direct system of Galois étale covering of \( X_{\text{reg}} \).

A local Galois 1-covering \( \phi_i : X_i \longrightarrow X \) with branching locus \( C_i := \text{Br} \phi_i \) restricts to give a Galois local 1-covering \( \phi'_i : X'_i \longrightarrow X_{\text{reg}} \) whose branching locus is given by \( C'_i := \text{Br} \phi'_i = C_i \cap X_{\text{reg}} \) and \( X'_i = \phi'^{-1}_i(X_{\text{reg}}) \), \( \phi'_i = \phi_i|_{X'_i} \). Since \( X_{\text{reg}} \) is smooth and \( C'_i \) is a small closed subset of \( X_{\text{reg}} \), Lemma 1.2 ensures that \( \phi'_i \) is actually a Galois étale covering of \( X_{\text{reg}} \), so giving \( \phi'_i = \psi_j \) and \( X'_i = U_j \) for some \( j \in \mathcal{J} \).

Conversely a Galois étale covering \( \psi_j : U_j \longrightarrow X_{\text{reg}} \) extends to giving a local Galois 1-covering \( \psi_j : U_j \longrightarrow X \) branched along \( \text{Br} \psi_j = \text{Sing}(X) \), then centered at \( x \in X_{\text{reg}} \), meaning that \( \overline{\psi}_j = \phi_i \) and \( \overline{U}_j = X_i \) for some \( i \in \mathcal{I} \).

Reasoning as in the proof of Theorem 1.10 these two processes turns out to be inverse to each other. Finally, since \( \phi_i \) and \( \psi_j \) agree on \( U_j = X'_i \) one gets an isomorphism of automorphisms group \( \text{Aut}^{(1)}(\phi_i) \cong \text{Aut}(\psi_j) \). Then passing to inverse limits one gets

\[
\pi_1^{\text{ét}}(X, x)^{(1)} = \lim_{\rightarrow} \text{Aut}^{(1)}(\phi_i) \cong \lim_{\rightarrow} \text{Aut}(\psi_j) = \pi_1^{\text{ét}}(X_{\text{reg}}, x)
\]

\[\blacksquare\]

Remark 1.26. For \( K = \mathbb{C} \), what observed in Remark 1.11 with respect to the excision property given by Theorem 1.10 applies also to previous Corollary 1.24 and Theorem 1.25. In general they do not imply the analogous Buczyńska’s results, unless when \( \pi_1^{\text{ét}}(X, x)^{(1)} \), \( \pi_1^{\text{ét}}(X, x) \) and \( \pi_1^{\text{ét}}(X_{\text{reg}}, x) \) are assumed to be finite groups.

Remark 1.27. As already observed in Remark 1.12 by Grothendieck’s equivalence, the previous Theorem 1.25 is equivalent to saying that every finite étale covering of the regular locus \( X_{\text{reg}} \) of a normal, irreducible algebraic variety \( X \) extends to giving a finite étale 1-covering of \( X \), which is Lemma 1.2 applied to a normal variety \( X \), with \( C = \text{Sing}(X) \).

The previous Theorem 1.25 allows us to dropping local conditions for 1-coverings of a normal variety \( X \), when base points are chosen in the big open \( X_{\text{reg}} \) of regular points. Namely we get the following consequences.

Corollary 1.28. Let \( X \) be a normal and irreducible algebraic variety. Then

\[
\pi_1^{\text{ét}}(X, x)^{(1)} \cong \pi_1^{\text{ét}}(X, x')^{(1)}
\]

for every \( x, x' \in X_{\text{reg}} \).

Proof. By Theorem 1.25 and Proposition 1.6 one has

\[
\pi_1^{\text{ét}}(X, x)^{(1)} \cong \pi_1^{\text{ét}}(X_{\text{reg}}, x) \cong \pi_1^{\text{ét}}(X_{\text{reg}}, x') \cong \pi_1^{\text{ét}}(X, x')^{(1)}
\]

\[\blacksquare\]

Proposition 1.29. Let \( \phi' : X' \longrightarrow X \) be a Galois 1-covering of a normal irreducible algebraic variety \( X \). Then \( \phi' \) is the universal 1-covering of \( X \) if and only the open subset \( X'_{\text{reg}} \subseteq X' \) of regular points is simply connected, that is \( \pi_1^{\text{ét}}(X'_{\text{reg}}, x') \cong \{1\} \) for some (hence every) \( x' \in X'_{\text{reg}} \). In particular \( X' \) is normal, too.
In other words, \( \varphi' : X' \rightarrow X \) is the universal 1-covering if and only it is the universal local Galois 1-covering of \( X \) centered at any regular point of \( X \).

Proof. Assume \( \varphi' : X' \rightarrow X \) be the universal 1-covering of \( X \). Then \( X' \) is normal, otherwise the normalization of \( X' \) gives a further finite map \( \varphi : \tilde{X}' \rightarrow X' \) which is an isomorphism, hence étale, outside of the closed subset \( C' \subseteq \text{Sing}(X') \subset X' \), where \( C' := \varphi'^{-1}(\text{Br}(\varphi')) \). Notice that \( \text{Sing}(X') \) is small since \( X \) is normal and \( \varphi'|_{X'_{C'}} : X'_{C'} \rightarrow X' \setminus C' \) is étale, where \( X'_{C'} := \varphi'^{-1}(X' \setminus C') \) as in Definition 1.1. Then \( \varphi' \circ \varphi : \tilde{X}' \rightarrow X \) is a 1-covering of \( X \) dominating the universal one. Then it has to be trivial and actually \( X' \cong \tilde{X}' \) is normal.

Consider a Galois étale covering \( \varphi : U \rightarrow X'_{\text{reg}} \). Lemma 1.2 allows us to extend \( \varphi \) to giving a Galois 1-covering \( \overline{\varphi} : \overline{U} \rightarrow X' \) whose branching locus is given by \( \overline{\text{Br}}(\overline{\varphi}) \subseteq \text{Sing}(X') \), which is a small closed as \( X' \) is normal. Since \( \text{Br}(\varphi' \circ \overline{\varphi}) \subseteq \varphi'(\text{Sing}(X')) \cup \text{Br}(\varphi') \) is a small closed in \( X \), the morphism \( \varphi' \circ \overline{\varphi} : U \rightarrow X \) turns out to be a Galois 1-covering of \( X \) dominating the universal one. Then \( \overline{\varphi} \) is an isomorphism and the same holds for \( \varphi \). This gives \( \text{Aut}(\varphi) \cong \{1\} \). Since \( \varphi \) is arbitrary, one has \( \pi_1^\text{ét}(X'_{\text{reg}}, x') \cong \{1\} \) for every \( x' \in X'_{\text{reg}} \).

For the converse, consider a further Galois 1-covering \( \varphi'' : X'' \rightarrow X \). The fibred product \( X' \times_X X'' \) gives a Galois 1-covering \( \text{pr}_1 : X' \times_X X'' \rightarrow X' \) (recall the commutative diagram (3)) branched along the small closed subset \( \varphi'^{-1}(\text{Br}(\varphi'')) \) of \( X' \). Restrict \( \text{pr}_1 \) to admit target in \( X'_{\text{reg}} \); this induces a Galois 1-covering \( \varphi : U \rightarrow X'_{\text{reg}} \) with \( U := \text{pr}_1^{-1}(X'_{\text{reg}}), \varphi = \text{pr}_1|_U \) and \( \text{Br} \varphi \subseteq \varphi'^{-1}(\text{Br}(\varphi'')) \cap X'_{\text{reg}} \), which is a small closed subset of \( X'_{\text{reg}} \). Since \( X'_{\text{reg}} \) is smooth, Lemma 1.2 implies that \( \text{Br} \varphi = \emptyset \), so giving a Galois étale covering of \( X'_{\text{reg}} \). Choose \( x' \in X'_{\text{reg}} \). Then

\[
\{1\} \cong \pi_1^\text{ét}(X'_{\text{reg}}, x') \xrightarrow{\text{Aut}(\varphi)} \text{Aut}(\varphi) = \text{Aut}(\text{pr}_1)
\]

so giving that \( X' \cong X' \times_X X'' \). Therefore \( \text{pr}_2 \circ \text{pr}_1^{-1} : X' \rightarrow X'' \) gives a morphism of Galois 1-coverings, meaning that

\[
\{X'' \xrightarrow{\varphi''} X\} \leq \{X' \xrightarrow{\varphi'} X\}
\]

Since \( \varphi'' \) is arbitrary, this shows that \( \varphi' \) is universal.

Remark 1.30. For \( K = \mathbb{C} \), the analogous property of Corollary 1.28 on the fundamental groups of \( X^{\text{an}} \) with different base points, is not directly implied by the algebraic statement on their pro-finite completions. Anyhow, it is a straightforward consequence of path connectedness of \( X^{\text{an}} \).

On the contrary, Proposition 1.29 implies the analogous statement on topological 1-coverings of \( X^{\text{an}} \) since \( \pi_1^\text{top}(X'_{\text{reg}}, x') \cong \{1\} \) if and only if \( \pi_1(X'_{\text{reg}}, x') \cong \{1\} \). Then Proposition 1.29 proves a proof of what stated in [4] Rem. 3.14, under the further hypothesis that \( X \) is normal.

1.5. Pull back of divisors. Let \( X \) be an irreducible and normal, algebraic variety of dimension \( n \) over the complex field \( \mathbb{C} \). The group of Weil divisors on \( X \) is denoted by \( \text{Div}(X) \); it is the free group generated by prime divisors of \( X \). For \( D_1, D_2 \in \text{Div}(X), D_1 \sim D_2 \) means that they are linearly equivalent. The subgroup of Weil divisors linearly equivalent to 0 is denoted by \( \text{Div}_0(X) \leq \text{Div}(X) \). The quotient group \( \text{Cl}(X) := \text{Div}(X)/\text{Div}_0(X) \) is called the class group, giving the
following short exact sequence of \( \mathbb{Z} \)-modules

\[ 0 \longrightarrow \text{Div}_0(X) \longrightarrow \text{Div}(X) \xrightarrow{d_X} \text{Cl}(X) \longrightarrow 0 \]

Given a divisor \( D \in \text{Div}(X) \), its class \( d_X(D) \) is often denoted by \([D]\), when no confusion may arise.

Consider a dominant morphism \( \phi : Y \rightarrow X \) of normal irreducible algebraic varieties. Then a pull back \( \phi^{\#} \) is well defined on Cartier divisors by pulling back local equations. This procedure clearly sends principal divisors to principal divisors, so defining a pull back homomorphism \( \phi^\ast : \text{Pic}(X) \rightarrow \text{Pic}(Y) \), where Pic denotes the group of linear equivalence classes of Cartier divisors. The given hypotheses on \( \phi, Y \) and \( X \) allow us to extending the definition of \( \phi^{\#} \) to every Weil divisor as follows:

\[ \forall D \in \text{Div}(X) \quad \phi^{\#}(D) := \phi^{\#}(D \cap X_{\text{reg}}) \in \text{Div}(Y). \]

Notice that \( D \cap X_{\text{reg}} \) is a Cartier divisor on \( X_{\text{reg}} \); then \( \phi^{\#}(D \cap X_{\text{reg}}) \) is a Cartier divisor in \( Y_{\text{reg}} \cap \phi^{-1}(X_{\text{reg}}) \) which is a Zariski open subset of \( Y \). Clearly \( \phi^{\#} : \text{Div}(X) \rightarrow \text{Div}(Y) \), as defined in (5), sends Cartier divisors to Cartier divisors and principal divisors to principal divisors, so giving a well defined pull back homomorphism \( \phi^\ast : \text{Cl}(X) \rightarrow \text{Cl}(Y) \) such that \( \phi^\ast|_{\text{Pic}(X)} \) is the pull back of Cartier divisors defined above.

In the case \( \phi : Y \rightarrow X \) is a 1-covering of normal and irreducible algebraic varieties, the pre-image \( \phi^{-1}(D) \subseteq Y \) of a Weil divisor \( D \in \text{Div}(X) \) is still a Weil divisor of \( Y \), meaning that the pull back defined by (5) can be easily rewritten by setting

\[ \phi^{\#}(D) = \phi^{-1}(D). \]

1.6. 1-coverings of complete orbifolds. Let \( X \) be a complete orbifold. Then one can easily deduce the following property of 1-coverings of \( X \).

**Proposition 1.31.** Let \( \phi : Y \rightarrow X \) be a 1-covering of a complete orbifold \( X \). Then \( Y \) is a complete orbifold, too.

This is a specialization of the following property, holding for finite morphisms.

**Proposition 1.32.** Let \( \phi : Y \rightarrow X \) be a finite morphism of irreducible and reduced algebraic varieties. Then \( Y \) is complete if and only if \( X \) is complete. Moreover if \( X \) is an orbifold then also \( Y \) is an orbifold.

**Proof.** Given an algebraic variety \( Z \), consider the following commutative diagram

\[
\begin{array}{ccc}
Y \times Z & \xrightarrow{\phi \times id} & Y \\
\downarrow{\phi \times id} & & \downarrow{\phi} \\
X \times Z & \xrightarrow{\pi_X} & Z \\
\end{array}
\]

where \( \pi_X \) and \( \pi_Y \) are natural projections on the second factor. The map \( \phi \times id \) is closed since it is a finite morphism. On the one hand, if \( X \) is complete then \( \pi_X \) is a closed map and \( \pi_Y = \pi_X \circ (\phi \times id) \) is closed, so giving that \( Y \) is complete. On the other hand, if \( Y \) is complete then \( \pi_Y \) is a closed map and, given a closed subset \( C \subseteq X \times Z \), its image \( \pi_X(C) = \pi_Y \circ (\phi \times id)^{-1}(C) \) is closed, as \( \phi \times id \) is continuous.

Then \( \pi_X \) is a closed map and \( X \) is complete.
Being an orbifold is a local property, then we can reduce to consider a Zariski open subset $U \subseteq X$ which is the quotient of an affine space $\widehat{U} = \text{Spec}(A)$ by a finite group $G$ i.e. $U \cong \widehat{U}/G$ and $U = \text{Spec}(A^G)$. Set $V := \phi^{-1}(U)$. Since $\phi$ is a finite morphism $V = \text{Spec}(\mathcal{B})$ where $\mathcal{B}$ is a a finitely generated $A^G$-module. Consider the fibred product

\[
\begin{array}{ccc}
\hat{V} := V \times_U \widehat{U} & \xrightarrow{\pi} & \widehat{U} \\
V \xrightarrow{\phi} & & \xrightarrow{\pi_G} \U
\end{array}
\]

Notice that the morphism $\pi$ is the quotient projection by the extended action of $G$ over $\hat{V}$, defined by setting $g \cdot (v, \hat{u}) := (v, g \cdot \hat{u})$. Then $\hat{V} = \text{Spec}(\mathcal{B} \otimes_{A^G} A)$ and $\mathcal{B} = (\mathcal{B} \otimes_{A^G} A)^G$, so giving $V \cong \hat{V}/G$.

For toric varieties, being a complete orbifold is equivalent to being a complete and $\mathbb{Q}$-factorial variety: then every toric 1-covering (see the following Definition 2.6) of a $\mathbb{Q}$-factorial and complete toric variety is still a $\mathbb{Q}$-factorial and complete toric variety. In this case some stronger fact holds.

## 2. Application to toric varieties

The present section is meant to applying results of the previous section to the case of toric varieties, so generalizing to every algebraically closed field $K$, with $\text{char } K = 0$, results given in [4, § 4] and in [32] under the assumption $K = \mathbb{C}$.

### 2.1. Preliminaries and notation on toric varieties

Throughout the present paper we will adopt the following definition of a toric variety:

**Definition 2.1** (Toric variety). A toric variety is a term $(X, \mathbb{T}, x_0)$ such that:

(i) $X$ is an irreducible, normal, $n$-dimensional algebraic variety over an algebraically closed field $K$ with $\text{char } K = 0$,

(ii) $\mathbb{T} \cong (\mathbb{K}^*)^n$ is a $n$-torus freely acting on $X$,

(iii) $x_0 \in X$ is a special point called the base point, such that the orbit map

$t \in \mathbb{T} \mapsto t \cdot x_0 \in \mathbb{T} \cdot x_0 \subseteq X$ is an open embedding.

For standard notation on toric varieties and their defining fans we refer to the extensive treatment [7].

**Definition 2.2** (Morphism of toric varieties). Let $Y$ and $X$ be toric varieties with acting tori $\mathbb{T}_Y$ and $\mathbb{T}_X$ and base points $y_0$ and $x_0$, respectively. A morphism of algebraic varieties $\phi : Y \longrightarrow X$ is called a morphism of toric varieties if

(i) $\phi(y_0) = x_0$,

(ii) $\phi$ restricts to give a homomorphism of tori $\phi_\mathbb{T} : \mathbb{T}_Y \longrightarrow \mathbb{T}_X$ by setting

$\phi_\mathbb{T}(t) \cdot x_0 = \phi(t \cdot y_0)$

The previous conditions (i) and (ii) are equivalent to require that $\phi$ induces a morphism between underling fans, as defined e.g in [7] § 3.3.
2.1.1. List of notation.

\( M, N, M_R, N_R \) denote the group of characters of \( \mathbb{T} \), its dual group and their tensor products with \( \mathbb{R} \), respectively;

\( \Sigma \subseteq \mathfrak{P}(N_R) \) is the fan defining \( X \);

\( \mathfrak{P}(N_R) \) denotes the power set of \( N_R \);

\( \Sigma(i) \) is the \( i \)-skeleton of \( \Sigma \);

\( \langle v_1, \ldots, v_s \rangle \subseteq N \) cone generated by \( v_1, \ldots, v_s \in N_R \);

if \( s = 1 \) this cone is called the ray generated by \( v_1 \);

\( L(v_1, \ldots, v_s) \subseteq N \) sublattice spanned by \( v_1, \ldots, v_s \in N \);

\( L_r(A) \subseteq \mathbb{Z}^m \) is the sublattice spanned by the rows of \( A \);

\( L_c(A) \subseteq \mathbb{Z}^d \) is the sublattice spanned by the columns of \( A \);

\( A_I, A^I \) \( \forall I \subseteq \{1, \ldots, m\} \) the former is the submatrix of \( A \) given by the columns indexed by \( I \) and the latter is the submatrix of \( A \) whose columns are indexed by the complementary subset \( \{1, \ldots, m\} \setminus I \);

positive a matrix (vector) whose entries are non-negative.

Let \( A \in M(d, m; \mathbb{Z}) \) be a \( d \times m \) integer matrix, then

\( L_r(A) \subseteq \mathbb{Z}^m \) is the sublattice spanned by the rows of \( A \);

\( L_c(A) \subseteq \mathbb{Z}^d \) is the sublattice spanned by the columns of \( A \);

\( A_I, A^I \) \( \forall I \subseteq \{1, \ldots, m\} \) the former is the submatrix of \( A \) given by the columns indexed by \( I \) and the latter is the submatrix of \( A \) whose columns are indexed by the complementary subset \( \{1, \ldots, m\} \setminus I \);

positive a matrix (vector) whose entries are non-negative.

Given a matrix \( V = (v_1 \cdots v_m) \in M(n, m; \mathbb{Z}) \), then

\( \langle V \rangle = \langle v_1, \ldots, v_m \rangle \subseteq N_R \) is the cone generated by the columns of \( V \);

\( SF(V) = SF(v_1, \ldots, v_m) \) is the set of all rational simplicial fans \( \Sigma \) such that \( \Sigma(1) = \{\langle v_1 \rangle, \ldots, \langle v_m \rangle\} \subseteq N_R \) and \( |\Sigma| = \langle V \rangle \) [31, Def. 1.3].

\( I_\Sigma := \{I \subseteq \{1, \ldots, m\} \mid \langle V_I \rangle \in \Sigma\} \)

\( G(V) \) is a Gale dual matrix of \( V \) [31 § 3.1].

Given a fan \( \Sigma \) in \( N_R \cong \mathbb{R}^n \), the integer matrix \( V = (v_1 \cdots v_m) \in M(n, m; \mathbb{Z}) \), whose columns are primitive generators of the 1-skeleton \( \Sigma(1) = \{\langle v_1 \rangle, \ldots, \langle v_m \rangle\} \), is called a fan matrix of the toric variety \( X(\Sigma) \). If every column of \( V \) is composed by coprime entries then \( V \) is called a reduced fan matrix. The Gale dual \( Q = G(V) \) of a fan matrix is called a weight matrix of \( X(\Sigma) \). \( Q \) is called a reduced weight matrix when \( V \) is a reduced fan matrix.

2.2. \( F, CF, W \)-matrices and poly weighted spaces (PWS).

**Definition 2.3** (\( F, CF \)-matrices, Def. 3.10 in [31]). An \( F \)-matrix is a \( n \times m \) matrix \( V \) with integer entries, satisfying the conditions:

(a) \( \text{rk}(V) = n \);

(b) \( V \) is \( F \)-complete i.e. \( \langle V \rangle = N_R \cong \mathbb{R}^n \) [31 Def. 3.4];

(c) all the columns of \( V \) are non zero;

(d) if \( v \) is a column of \( V \), then \( V \) does not contain another column of the form \( \lambda v \) where \( \lambda > 0 \) is real number.
A CF–matrix is a F–matrix satisfying the further requirement

(e) the sublattice \( L_c(V) \subseteq \mathbb{Z}^n \) is cotorsion free, that is, \( L_c(V) = \mathbb{Z}^n \) or, equivalently, \( L_r(V) \subseteq \mathbb{Z}^m \) is cotorsion free.

A F–matrix \( V \) is called reduced if every column of \( V \) is composed by coprime entries [31, Def. 3.13].

The most significant example of an F-matrix is given by the fan matrix \( V \) of a rational and complete fan \( \Sigma \).

**Definition 2.4** (W-matrix, Def. 3.9 in [31]). A W–matrix is an \( r \times m \) matrix \( Q \) with integer entries, satisfying the following conditions:

(a) \( \text{rk}(Q) = r \);
(b) \( L_r(Q) \) does not have cotorsion in \( \mathbb{Z}^m \);
(c) \( Q \) is W–positive, that is, \( L_r(Q) \) admits a basis consisting of positive vectors [31, Def. 3.4];
(d) Every column of \( Q \) is non-zero.
(e) \( L_r(Q) \) does not contain vectors of the form \( (0, \ldots, 0, 1, 0, \ldots, 0) \).
(f) \( L_r(Q) \) does not contain vectors of the form \( (0, a, 0, \ldots, 0, b, 0, \ldots, 0) \), with \( ab < 0 \).

A W–matrix is called reduced if \( V = G(Q) \) is a reduced F–matrix [31, Def. 3.14, Thm. 3.15]

The most significant example of a W-matrix \( Q \) is given by the weight matrix of a rational and complete fan \( \Sigma \).

**Definition 2.5** (Poly weighted space, Def. 2.7 in [31]). A poly weighted space (PWS) is a \( n \)–dimensional \( \mathbb{Q} \)–factorial complete toric variety \( X(\Sigma) \), whose reduced fan matrix \( V \) is a CF–matrix i.e. if

- \( V = (v_1, \ldots, v_m) \) is an \( n \times m \) CF–matrix,
- \( \Sigma \in SF(V) \).

2.3. 1-coverings of toric varieties. A priori, a 1-covering \( \phi : Y \rightarrow X \) of a toric variety \( X \) need not be an equivariant morphism of toric varieties and \( Y \) may not even be a toric variety. A posteriori, we will see that, actually, this is not the case when \( X \) is a non-degenerate toric variety (see the following Remark 2.12 for a discussion of such an hypothesis). Let us then start by setting the following

**Definition 2.6** (toric 1-covering). A 1-covering \( \phi : Y \rightarrow X \) between toric varieties \( Y \) and \( X \) is called a toric 1-covering if \( \phi \) is a morphism of toric varieties in the sense of Definition 2.2

**Proposition 2.7** (see e.g. Thm. 3.2.6 in [7]). Let \( X(\Sigma) \) be a toric variety and consider the torus embedding \( T \hookrightarrow T \cdot x_0 \subseteq X \). Let \( x_{\rho} \) be the distinguished point of a ray \( \rho \in \Sigma(1) \) (see e.g. [7 § 3.2]). Let \( D_{\rho} \) be the associated torus invariant divisor i.e. \( D_{\rho} = T \cdot x_{\rho} \subseteq X \). Then \( \bigcup_{\rho \in \Sigma(1)} D_{\rho} = X \setminus T \cdot x_0 \).

**Theorem 2.8.** Let \( X(\Sigma) \) be a non-degenerate toric variety, \( Y \) be a normal irreducible algebraic variety and \( \phi : Y \rightarrow X \) be a Galois 1-covering. Then \( Y \) is a non-degenerate toric variety and \( \phi \) is a toric 1-covering with branching locus

\[
C = \text{Br}(\phi) \subseteq \bigcup_{\rho \in \Sigma(1)} D_{\rho}
\]
A proof of this result is deferred to § 2.6.1, after the proof of the following theorem.

2.4. The étale fundamental group of a toric variety. Let us start by recalling the following Grothendieck’s remark.

Theorem 2.9 (Cor. 1.2 in Exp. XI, [16]). A normal, rational and complete algebraic variety is simply connected.

Corollary 2.10. A complete toric variety is simply connected.

More general results on the computation of the étale fundamental group of a toric variety were obtained by Danilov.

Theorem 2.11 (Prop. 9.3 in [8]). Let \( X(\Sigma) \) be a toric variety such that the support \(|\Sigma|\) spans \( N_\mathbb{R} \). Then, for every \( x \in X \),

\[
\pi_1^\text{ét}(X, x) \cong N / N_\Sigma
\]

where \( N_\Sigma \subseteq N \) is the sublattice spanned by elements in \(|\Sigma| \cap N\).

Remark 2.12. Recall that a toric variety \( X(\Sigma) \) is complete if and only if \(|\Sigma| = N_\mathbb{R}\). Then Danilov’s Theorem 2.11 implies the previous Corollary 2.10, as a particular case.

Moreover, notice that asking for the fan’s support to span \( N_\mathbb{R} \) is actually not too restrictive. In fact, the following facts are equivalent (see e.g. [7, Prop. 3.3.9]):

1. the support \(|\Sigma|\) spans \( N_\mathbb{R}\),
2. the 1-skeleton \( \Sigma(1) \) spans \( N_\mathbb{R}\),
3. \( H^0(X, \mathcal{O}_X^*) \cong \mathbb{K}^*\),
4. \( X(\Sigma) \) has no torus factors.

A toric variety of this kind is usually called non-degenerate. Then, up to torus factors, Danilov’s Theorem 2.11 applies to every toric variety.

Finally, notice that, up to torus factors, a toric variety turns out to admit finite étale fundamental group, since \( N_\Sigma \) is a full sublattice of \( N \). Then, for \( \mathbb{K} = \mathbb{C} \), results of the previous section § 1 apply as well to the topological fundamental group of the associated analytic variety \( X^\text{an} \).

2.5. The étale fundamental group in codimension 1 of a toric variety. We are now in a position to applying results of § 1 and computing the étale fundamental group of a toric variety without torus factors.

Theorem 2.13. Let \( X(\Sigma) \) be a non-degenerate toric variety and let \( X_1 = X(\Sigma(1)) \) the toric variety whose fan is given by the 1-skeleton \( \Sigma(1) \) of \( \Sigma \). Then \( X_1 \) is a big open subset of the regular locus \( X_{\text{reg}} \) of \( X \) and, for every point \( x \in X_1 \),

\[
\pi_1^\text{ét}(X, x)^{(1)} \cong \pi_1^\text{ét}(X_{\text{reg}}, x) \cong \pi_1^\text{ét}(X_1, x) \cong N / N_1
\]

where \( N_1 \subseteq N \) is the sublattice spanned by \( \Sigma(1) \cap N \).

Proof. Since \( X \) is a normal irreducible algebraic variety, Theorem 1.25 gives the following isomorphism

\[
(7) \quad \pi_1^\text{ét}(X, x)^{(1)} \cong \pi_1^\text{ét}(X_{\text{reg}}, x)
\]

for every regular point \( x \in X_{\text{reg}} \). Notice that \( X_1 \) is smooth: its fan \( \Sigma(1) \) is regular as consisting of 1-dimensional cones, only. Moreover, \( X_1 \) turns out to be a big open
subset of $X$. Then $X_1 \subseteq X_{\text{reg}}$ is a big open subset of $X_{\text{reg}}$, too. By the excision property given by Theorem 1.10, one has

$$\pi_1^{\text{et}}(X_{\text{reg}}, x) \cong \pi_1^{\text{et}}(X_1, x)$$

for every $x \in X_1$. Finally, since $\Sigma(1)$ spans $N_\mathbb{R}$, one applies Danilov’s Theorem 2.11 to get

$$\pi_1^{\text{et}}(X_1, x) \cong N / N_1$$

The proof ends up by putting together (7), (8) and (9).

Remark 2.14. Notice that, by Theorem 2.13, the étale fundamental group in codimension 1 of a non-degenerate toric variety turns out to be a finite group. Then for $K = \mathbb{C}$, Theorems 2.11 and 2.13 suffice to proving analogous statements for the fundamental group in codimension 1 of the associated analytic variety $X_{\text{an}}$, so directly implying Buczyńska’s results given in [4, § 4] for any complex toric variety, by obviously adding the contribution of any torus factor.

2.6. The universal 1-covering of a non-degenerate toric variety. It is a well known fact, already observed in the beginning of § 1.2, that in general the universal étale covering of an algebraic variety does not exist. The same clearly holds for the universal (local) 1-covering. Therefore exhibiting a class of algebraic varieties admitting either a universal étale covering or a universal (local, in case) 1-covering, is always of some interest. Recently, jointly with Lea Terracini, we proved that $\mathbb{Q}$-factorial and complete toric varieties, over the complex field $\mathbb{C}$, always admit a universal 1-covering [32, Thm. 2.2], which turns out to be still a $\mathbb{Q}$-factorial and complete toric variety, coherently with the previous Proposition 1.31 and Theorem 2.8. In particular a universal 1-covering of this kind is always a PWS (in the sense of Definition 2.5) canonically determined by the initially given $\mathbb{Q}$-factorial complete toric variety.

The present section is meant to generalizing this result over the ground field and to extending it to the bigger range of non-degenerate toric varieties, so dropping both hypothesis of completeness and $\mathbb{Q}$-factoriality.

Theorem 2.15 (Compare with Thm. 2.2 and Rem. 2.3 in [32]). A non-degenerate toric variety $X$ over an algebraically closed field $K$ with char $K = 0$, admits a universal 1-covering $\varphi : \tilde{X} \to X$ which is a toric 1-covering of non-degenerate toric varieties. The induced pull-back on divisors gives a group epimorphism $\varphi^* : \text{Cl}(X) \to \text{Cl}(\tilde{X})$ whose kernel is

$$\ker(\varphi^*) \cong \text{Tors}(	ext{Cl}(X)) \cong \pi_1^{\text{et}}(X_{\text{reg}}, x)^{(1)} \cong \pi_1^{\text{et}}(X_{\text{reg}}, x)$$

for every regular point $x \in X_{\text{reg}}$.

In particular every non-degenerate toric variety $X$ can be canonically described as a finite geometric quotient $X \cong \tilde{X} / \pi_1^{\text{et}}(X_{\text{reg}}, x)^{(1)}$ of the universal 1-covering $\tilde{X}$ by the torus-equivariant action of $\pi_1^{\text{et}}(X_{\text{reg}}, x)^{(1)} \cong \text{Tors}(	ext{Cl}(X))$ on the fibers of $\varphi$.

Moreover, if $V$ is a fan matrix of $X$ then $\tilde{V} = G(G(V))$ is a fan matrix of $\tilde{X}$.

By construction $\tilde{X}$ is $\mathbb{Q}$-factorial (complete) if and only if $X$ is $\mathbb{Q}$-factorial (complete). In particular, if $X$ is both complete and $\mathbb{Q}$-factorial then its universal 1-covering $\tilde{X}$ is a PWS.

Corollary 2.16 (Rem. 2.4 in [32], Prop. 3.1.3 in [31]). Consider a toric 1-covering $\phi : Y \to X$ of a non-degenerate toric variety $X$ over an algebraically closed field
K with char K = 0. If V and W are fan matrices of X and Y, respectively, then there exists a unique matrix β ∈ GLn(Q) ∩ M(n, n; Z) such that V = β · W.

Moreover if X is Q-factorial then also Y is, and φ∗ : Cl(X) → Cl(Y) is a group epimorphism inducing a Q-module isomorphism

\[ \text{Pic}(X) ⊗_Z Q \cong \text{Cl}(X) ⊗_Z Q, \quad \phi^*_Q \cong \text{Cl}(Y) ⊗_Z Q \cong \text{Pic}(Y) ⊗_Z Q. \]

**Proof of Thm. 2.15.** Calling \( n = \dim X \) and \( r = \text{rk} \text{Cl}(X) \), recall the definition of \( S_\Sigma \subseteq \mathcal{P}\{1, \ldots, n + r\} \) given in \[2.1.1\]. Let V be a fan matrix of X. Then \( \Sigma(1) = \{(v_i) | v_i \text{ is the } i\text{-th column of } V\} \). Consider the sublattice \( N_1 \subseteq N = \mathbb{Z}^n \) spanned by the \( v_i \)'s. Since X is non-degenerate, the lattice \( N_1 \) is a full sublattice of N and \( N/N_1 \) is a finite abelian group. Let \( \tilde{V} = \mathcal{G}(\mathcal{G}(V)) \) be a double Gale dual matrix of V and consider the fan

\[ \tilde{\Sigma} := \{(\tilde{V}_I) | I ∈ S_\Sigma \} ⊆ \mathcal{P}(N_1) \]

defining a toric variety \( \tilde{X} = \tilde{X}(\tilde{\Sigma}) \). The natural inclusion \( N_1 ↪ N = \mathbb{Z}^n \) induces a surjection \( \tilde{X} \twoheadrightarrow X \) which turns out to be the canonical projection of the quotient of \( \tilde{X} \) by the action of the finite abelian group \( N/N_1 \). Theorem \[2.13\] gives that

\[ \pi_1^\text{ét}(X, x) \cong \pi_1^\text{ét}(X_{\text{reg}}, x) \cong N/N_1 \]

for every \( x ∈ X_{\text{reg}} \). The following Lemma \[2.17\] shows that \( N/N_1 \cong \text{Tors} \text{Cl}(X) \). The same argument applied to \( \tilde{X} \) shows that it is 1-connected and \( \tilde{X} \twoheadrightarrow X \) turns out to be the universal 1-covering of X. Moreover \( \text{Tors} \text{Cl}(X) = 0 \) and \( \text{rk} \text{Cl}(\tilde{X}) = \text{rk} \text{Cl}(X) = r \). By the construction \[10\] of the fan \( \tilde{\Sigma} \), one clearly see that X is Q-factorial (complete) if and only if \( \tilde{X} \) is.

**Lemma 2.17** (Compare with Thm. 2.4 in \[31\]). Let \( X(\Sigma) \) be a non-degenerate toric variety and \( N_1 \subseteq N \) be the sublattice spanned by primitive generators of rays in \( \Sigma(1) \). Then

\[ \text{Tors} \text{Cl}(X) \cong N/N_1 \]

**Proof.** The proof is the same as in \[31\] Thm. 2.4. Anyway it is here reported to adapting the key argument to the current weaker hypotheses.

Let \( \text{Div}_T(X) \) denotes the group of torus invariant Weil divisors. Then there is the following well known short exact sequence (see e.g. \[7\] Thm. 4.1.3)

\[ 0 \longrightarrow M \overset{\text{div}}{\longrightarrow} \text{Div}_T(X) \overset{d}{\longrightarrow} \text{Cl}(X) \longrightarrow 0 \]

Adopting the same notation as in the proof of Thm. \[2.15\] this gives

\[ \text{Cl}(X) \cong \text{Div}_T(X) / \text{Im}(\text{div}) \cong \mathbb{Z}^{n+r} / L_r(V) \]

where V is a fan matrix of X (recall notation introduced in \[2.1.1\]). Then

\[ \text{Tors} \text{Cl}(X) \cong \text{Tors} \mathbb{Z}^{n+r}/L_r(V) \cong \text{Tors} \mathbb{Z}^n/L_c(V) \cong \mathbb{Z}^n/L_r(T_n) \]

where \( \begin{pmatrix} T_n \\ 0 \end{pmatrix} \) is the Hermite normal from of \( N_1 \), meaning that \( N/N_1 \cong \mathbb{Z}^n/L_r(T_n) \). □

**Proof of Cor. 2.16.** The first part of the statement follows immediately by \[31\] Prop. 3.1.3 (see also \[32\] Rem. 2.4) whose argument is completely \( \mathbb{Z} \)-linear. The second part is then an immediate consequence of the previous Theorem \[2.15\]. □
2.6.1. A proof of Theorem 2.8. By Theorem 2.15, $X$ admits a universal 1-covering $\varphi : \tilde{X} \to X$ which is a toric 1-covering of non-degenerate toric varieties. Then there exists a Galois 1-covering $f : \tilde{X} \to Y$ such that $\varphi = \phi \circ f$. In particular this means that there exists a (normal) subgroup $H \leq \text{Aut}(\varphi)$ such that $Y \cong \tilde{X}/H$ and $\phi$ is the associated quotient projection \cite[Prop. 5.3.8]{35}. Again Theorem 2.15 gives that \[ \text{Aut}(\varphi) \cong \text{Tors}(\text{Cl}(X)) \cong \pi_1^\text{ét}(X, x)^{(1)} \cong N/N_1 \] meaning that $H$ corresponds to a sublattice $N_H \leq N$ such that $N_1 \leq N_H$, $H \cong N_H/N_1 \cong \text{Tors}(\text{Cl}(Y)) \cong \pi_1^\text{ét}(Y, y)^{(1)}$ for some base point $y \in \varphi^{-1}(x)$. Then \cite[Rem. 2.4]{32} shows that there exists an integer matrix $\eta \in \text{GL}_n(\mathbb{Q}) \cap \text{M}_n(\mathbb{Z})$ such that $Y$ is the non-degenerate toric variety whose fan matrix is given by $V^\eta := \eta \cdot V$ and determined by the following fan \[ \Sigma_\eta := \{ \langle V^\eta_I \rangle | I \in \Sigma \} \subseteq \mathbb{P}(N_H) \] By construction, $\phi$ is clearly equivariant giving rise to a toric 1-covering.

3. Application to Mori dream spaces

The present section is meant to applying results of the previous sections \cite{1} and \cite{2} to the case of Mori dream spaces. Actually varieties here considered are more general algebraic varieties than Mori dream spaces as introduced by Hu and Keel in \cite{21}, as we will not require neither any projective embedding nor completeness when showing main applications. These varieties will be called \emph{weak} Mori dream spaces (wMDS) to distinguishing them from the usual Hu-Keel Mori dream spaces (MDS) (see Definition 3.4).

Next subsections § 3.1 and § 3.2 will be devoted, the former, to recalling main notation on Cox rings, essentially following \cite{2}, and the latter, to quickly explaining main results about the toric embedding properties of a wMDS, as studied in \cite{29}.

3.1. Cox sheaf and algebra of an algebraic variety. For what concerning the present topic we will essentially adopt the approach described in the extensive book \cite{2} and notation introduced in \cite[§ 1.3]{29}. The interested reader is referred to those sources for any further detail.

3.1.1. Assumption. In the following, Cl$(X)$ is assumed to be a \emph{finitely generated} (f.g.) abelian group of rank $r := \text{rk}(\text{Cl}(X))$. Then $r$ is called either the Picard number or the rank of $X$. Moreover we will assume that every invertible global function is constant i.e. \[ H^0(X, \mathcal{O}_X^*) \cong \mathbb{K}^*. \] Recall Remark 2.12 for equivalent interpretations of assumption (11).

3.1.2. Choice. Choose a f.g. subgroup $K \leq \text{Div}(X)$ such that \[ d_K := d_X|_K : K \to \text{Cl}(X) \] is an \emph{epimorphism}. Then $K$ is a free group of rank $m \geq r$ and \cite{41} induces the following exact sequence of $\mathbb{Z}$-modules \[ 0 \to K_0 \to K \to \text{Cl}(X) \to 0 \] where $K_0 := \text{Div}_0(X) \cap K = \ker(d_K)$. 

Definition 3.1 (Sheaf of divisorial algebras, Def. 1.3.1.1 in [2]). The sheaf of divisorial algebras associated with the subgroup $K \leq \text{Div}(X)$ is the sheaf of $K$-graded $\mathcal{O}_X$-algebras

$$S := \bigoplus_{D \in K} S_D, \quad S_D := \mathcal{O}_X(D),$$

where the multiplication in $S$ is defined by multiplying homogeneous sections in the field of functions $\mathbb{K}(X)$.

3.1.3. Choice. Choose a character $\chi : K_0 \to \mathbb{K}(X)^*$ such that

$$\forall D \in K_0 \quad D = (\chi(D))$$

where $(f)$ denotes the principal divisor defined by the rational function $f \in \mathbb{K}(X)^*$. Consider the ideal sheaf $\mathcal{I}_\chi$ locally defined by sections $1 - \chi(D)$ i.e.

$$\Gamma(U, \mathcal{I}_\chi) = ((1 - \chi(D))|_U \mid D \in K_0) \subseteq \Gamma(U, S).$$

This induces the following short exact sequence of $\mathcal{O}_X$-modules (13)

$$0 \longrightarrow \mathcal{I}_\chi \longrightarrow S \overset{\pi_\chi}{\longrightarrow} S/\mathcal{I}_\chi \longrightarrow 0$$

Definition 3.2 (Cox sheaf and Cox algebra, Construction 1.4.2.1 in [2]). Keeping in mind the exact sequence (13), the Cox sheaf of $X$, associated with $K$ and $\chi$, is the quotient sheaf $\text{Cox} := S/\mathcal{I}_\chi$ with the $\text{Cl}(X)$-grading

$$\text{Cox} := \bigoplus_{\delta \in \text{Cl}(X)} \text{Cox}_\delta, \quad \text{Cox}_\delta := \pi_\chi \left( \bigoplus_{D \in d_{K\chi}(\delta)} S_D \right).$$

Passing to global sections, one gets the following Cox algebra (usually called Cox ring) of $X$, associated with $K$ and $\chi$,

$$\text{Cox}(X) := \text{Cox}(X) = \bigoplus_{\delta \in \text{Cl}(X)} \Gamma(X, \text{Cox}_\delta).$$

Remarks 3.3.

1. [2 Prop. 1.4.2.2] Depending on choices 3.1.2 and 3.1.3, both Cox sheaf and algebra are not canonically defined. Anyway, given two choices $K, \chi$ and $K', \chi'$ there is a graded isomorphism of $\mathcal{O}_X$-modules

$$\text{Cox}(K, \chi) \cong \text{Cox}(K', \chi').$$

2. For any open subset $U \subseteq X$, there is a canonical isomorphism

$$\Gamma(U, S)/\Gamma(U, \mathcal{I}_\chi) \cong \Gamma(U, \text{Cox}).$$

In particular $\text{Cox}(X) \cong H^0(X, S)/H^0(X, \mathcal{I}_\chi)$. This fact gives a precise meaning to the usual ambiguous writing

$$\text{Cox}(X) \cong \bigoplus_{[D] \in \text{Cl}(X)} H^0(X, \mathcal{O}_X(D)).$$
3.2. Weak Mori dream spaces (wMDS) and their embedding. In the literature Mori dream spaces (MDS) come with a required projective embedding essentially for their optimal behavior with respect to the termination of Mori program. As explained in [29], this assumption is not necessary to obtain main properties of MDS, like e.g. their toric embedding, chamber decomposition of their moving and pseudo-effective cones and even termination of Mori program, for what this fact could mean for a complete and non-projective algebraic variety.

According to notation introduced in [29], we set the following

**Definition 3.4** (wMDS). An irreducible, \(\mathbb{Q}\)-factorial algebraic variety \(X\) satisfying assumption \(3.1.1\) is called a weak Mori dream space (wMDS) if \(\text{Cox}(X)\) is a finitely generated \(\mathbb{K}\)-algebra. A projective wMDS is called a Mori dream space (MDS).

3.2.1. Total coordinate and characteristic spaces. Consider a wMDS \(X\) and its Cox sheaf \(\text{Cox}\). The latter is a locally of finite type sheaf, that is there exists a finite affine covering \(\bigcup U_i = X\) such that \(\text{Cox}(U_i)\) are finitely generated \(\mathbb{K}\)-algebras [2, Propositions 1.6.1.2, 1.6.1.4]. The relative spectrum of \(\text{Cox}\) [18, Ex. II.5.17],

\[
\hat{X} = \text{Spec}(\text{Cox}(X)) \xrightarrow{p_X} X
\]

is an irreducible normal and quasi-affine variety \(\hat{X}\), coming with an actions of the quasi-torus \(G := \text{Hom}(\text{Cl}(X), \mathbb{K}^*)\), whose quotient map is realized by the canonical morphism \(p_X\) in [13] \(2\ §1.3.2, \) Construction 1.6.1.5]. \(\hat{X}\) is called the characteristic space of \(X\) and \(G\) is called the characteristic quasi-torus of \(X\).

Moreover consider

\[
\bar{X} := \text{Spec}(\text{Cox}(X))
\]

which is an irreducible and normal, affine variety, called the total coordinate space of \(X\). Then there exists an open embedding \(j_X : \hat{X} \hookrightarrow \bar{X}\). The action of the quasi-torus \(G\) extends to \(\bar{X}\) in such a way that \(j_X\) turns out to be equivariant.

**Theorem 3.5** (Cox Theorem for a wMDS). Let \(X\) be a wMDS and consider the natural action of the quasi-torus \(G := \text{Hom}(\text{Cl}(X), \mathbb{K}^*)\) on the total coordinate space \(\bar{X}\). Then the loci of stable and semi-stable points coincide with the open subset \(j_X(\hat{X}) \subseteq \bar{X}\), which is the characteristic space of \(X\). Then the canonical morphism \(p_X : \hat{X} \rightarrow X\) is the associated 1-free and geometric quotient. In particular

\[
(p_X)_* : (\mathcal{O}_{\hat{X}}) \cong \text{Cox} \quad , \quad (p_X)^* : \mathcal{O}_X \xrightarrow{\sim} \text{Cox}^G := (p_X)_* \mathcal{O}_\hat{X}.
\]

For a definition of used notation and a sketch of proof we refer the interested reader to Definitions 2.3,4,5 and Theorem 2.6 in [29].

3.2.2. Irrelevant loci and ideals. \(\text{Cox}(X)\) is a finitely generated \(\mathbb{K}\)-algebra. Then, up to the choice of a set of generators \(\mathfrak{x} = (x_1, \ldots, x_m)\), we get

\[
\text{Cox}(X) \cong \mathbb{K}[\mathfrak{x}] / I
\]

being \(I \subseteq \mathbb{K}[\mathfrak{x}] := \mathbb{K}[x_1, \ldots, x_m]\) a suitable ideal of relations. Calling \(\overline{W} := \text{Spec} \mathbb{K}[\mathfrak{x}] \cong \mathbb{K}^m\), the canonical surjection

\[
\pi_X : \mathbb{K}[\mathfrak{x}] \longrightarrow \text{Cox}(X)
\]

gives rise to a closed embedding \(\tilde{\pi} : \bar{X} \hookrightarrow \overline{W} \cong \mathbb{K}^m\), depending on the choice of \((\mathbb{K}, \chi, \mathfrak{x})\).
Definition 3.6 (Irrelevant loci and ideals). Let $X$ be a wMDS. The irrelevant locus of a total coordinate space $\overline{X}$ of $X$ is the Zariski closed subset given by the complement $B_X := \overline{X} \setminus j_X(\hat{X})$. Since $\overline{X}$ is affine, the irrelevant locus $B_X$ defines an irrelevant ideal of the Cox algebra $\text{Cox}(X)$, as
\[
\mathcal{I}_{rr}(X) := \{ f \in \text{Cox}(X) \mid \delta \in \text{Cl}(X) \text{ and } f|_{B_X} = 0 \} \subseteq \text{Cox}(X).
\]
Analogously, after the choice of a set $\mathfrak{X}$ of generators of $\text{Cox}(X)$, consider the lifted irrelevant ideal of $X$
\[
\mathcal{I}_{rr} := \pi_X^{-1}(\mathcal{I}_{rr}(X)) \subseteq \mathbb{K}[\mathfrak{X}].
\]
The associated zero-locus of the Cox algebra $\text{Cox}(X)$ of $\text{Cox}(X)$, an element $\mathfrak{y} \in \mathfrak{X}$ such that $\mathfrak{y} \in \mathfrak{X}$ is affine, the irrelevant locus $B_X$ is affine, and if its class $\delta$ is $\text{Cl}(X)$-prime if there exists $\delta \in \text{Cl}(X)$ such that $y \in \text{Cox}(X)_{\delta}$ (i.e. $y$ is homogeneous) and, for $i = 1, 2,$ $\forall \delta_i \in \text{Cl}(X), \forall f_i \in \text{Cox}(X)_{\delta_i} \ y | f_1 f_2 \implies y | f_1$ or $y | f_2$.

Definition 3.8 (Cox generators and bases). Given a wMDS $X$ and a set $\mathfrak{X}$ of generators of $\text{Cox}(X)$, an element $x \in \mathfrak{X}$ is called a Cox generator if its class $\overline{x}$ is $\text{Cl}(X)$-prime. If $\mathfrak{X}$ is entirely composed by Cox generators then it is called a Cox basis of $\text{Cox}(X)$ if it has minimum cardinality.

Theorem 3.9 (Canonical toric embedding). Let $X$ be a wMDS and $\mathfrak{X}$ be a Cox basis of $\text{Cox}(X)$. Then there exists a closed embedding $i : X \hookrightarrow W$ into a $Q$-factorial and non-degenerate toric variety $W$, fitting into the following commutative diagram
\[
\begin{array}{ccc}
X & \xrightarrow{i} & \mathcal{W} \\
\downarrow{p_X} & & \downarrow{p_W} \\
\overline{X} & \xrightarrow{\hat{i}} & \overline{W}
\end{array}
\]
where
1. $\overline{W} = \text{Spec} \mathbb{K}[\mathfrak{X}]$,
(2) $\hat{W} := W \setminus \hat{B}$ is a Zariski open subset and $j_W : \hat{W} \hookrightarrow W$ is the associated open embedding,

(3) $\hat{i} := i|_{\hat{X}}$,

(4) $p_W : \hat{W} \rightarrow W$ is a 1-free geometric quotient by an action of the characteristic quasi-torus $G = \text{Hom}(\text{Cl}(X), \mathbb{K}^*)$ on the affine variety $W$, with respect to $i$ turns out to be equivariant and $j_W(\hat{W})$ is the locus of stable and semi-stable points. Moreover $(p_W)^* : \mathcal{O}_W \xrightarrow{\cong} (p_W)_* \mathcal{O}_W^G$.

For a proof of this theorem we refer the interested reader to [29, Thm. 2.10, Cor. 2.15]. Here we just recall that, given the Cox basis $X = \{x_1, \ldots, x_n\}$, the embedding, canonically determined by the surjection (16) between the associated algebras, can be concretely described by evaluating the Cox generators as follows

$$x \in X \mapsto \tilde{t}(x) := (x_1(x), \ldots, x_m(x)) \in \mathbb{K}^m.$$  

Moreover the $G$-action on $\hat{W}$ is defined by observing that the class $\tilde{t}_i$ is homogeneous, that is there exists a class $\delta_i \in \text{Cl}(X)$ such that $\tilde{t}_i \in \text{Cox}(X)_{\delta_i}$. Then one has

$$(g, z) \in G \times \hat{W} \mapsto g : z := (\chi_1(g) z_1, \ldots, \chi_m(g) z_m) \in \hat{W}$$

where $\chi_i : G \rightarrow \mathbb{K}^*$ is the character defined by $\chi_i(g) = g(\delta_i)$.

Remarks 3.10. (1) The ambient toric variety $W$, defined in Theorem 3.9, only depends on the choices of the Cox basis $X$ and no more on $K$ and $\chi$, as given in (3.1.2) and (3.1.3). In fact, for different choices $K', \chi'$ we get an isomorphic Cox ring, as observed in Remark 3.3 (1). Then it still admits the same presentation $\mathbb{K}[X]/I$, meaning that the toric embedding $i : X \hookrightarrow W$ remains unchanged, up to isomorphism.

Actually the toric embedding exhibited in Theorem 3.9 only depends on the cardinality $|X|$. One can then fix a canonical toric embedding $i : X \hookrightarrow W$ as that associated, up to isomorphisms, to a Cox basis of minimum cardinality.

(2) Varieties $\hat{W}$ and $\hat{W}$, exhibited in Theorem 3.9, are called the characteristic space and the total coordinate space, respectively, of the canonical toric ambient variety $W$. In particular, the geometric quotient $p_W : \hat{W} \rightarrow W$ is precisely the classical Cox’s quotient presentation of a non-degenerate (i.e. not admitting torus factors) toric variety [6].

3.2.4. The canonical toric embedding is a neat embedding. Let $X$ be a wMDS and $i : X \hookrightarrow W$ be its canonical toric embedding constructed in Theorem 3.9. Let $V = (v_1, \ldots, v_m)$ be a fan matrix of $W$, which is a representative matrix of the dual morphism

$$\text{Hom}(\text{Div}_T(W), \mathbb{Z}) \xrightarrow{\text{div}_W^\vee} N := \text{Hom}(M, \mathbb{Z}).$$

In the following we will then denote $D_i := D_{v_i}$ the prime torus invariant associated with the ray $(v_i) \in \Sigma(1)$, for every $1 \leq i \leq m$.

Proposition 3.11 (Pulling back divisor classes). Let $i : X \hookrightarrow W$ be a closed embedding of a normal irreducible algebraic variety $X$ into a toric variety $W(\Sigma)$ with acting torus $T$. Let $D_{\rho} = T \cdot \overline{x_\rho}$, for $\rho \in \Sigma(1)$, be the invariant prime divisors of $W$ and assume that $\{i^{-1}(D_{\rho})\}_{\rho \in \Sigma(1)}$ is a set of pairwise distinct irreducible
hypersurfaces in $X$. Then it is well defined a pull back homomorphism $i^*: \text{Cl}(W) \to \text{Cl}(X)$.

For a proof, the interested reader is referred to [29, Prop. 2.12].

**Definition 3.12 (Neat embedding).** Let $X$ be an irreducible and normal algebraic variety and $W(\Sigma)$ be a toric variety. Let $\{D_\rho\}_{\rho \in \Sigma(1)}$ be the torus invariant prime divisors of $W$. A closed embedding $i : X \hookrightarrow W$ is called a neat (toric) embedding if

1. $\{i^{-1}(D_\rho)\}_{\rho \in \Sigma(1)}$ is a set of pairwise distinct irreducible hypersurfaces in $X$,
2. the pullback homomorphism defined in Proposition 3.11, $i^*: \text{Cl}(W) \cong \rightarrow \text{Cl}(X)$, is an isomorphism.

**Proposition 3.13.** The canonical toric embedding $i : X \hookrightarrow W$, of a wMDS $X$, is a neat embedding. Moreover the isomorphism $i^*: \text{Cl}(W) \cong \rightarrow \text{Cl}(X)$ restricts to give an isomorphism $\text{Pic}(W) \cong \text{Pic}(X)$.

For a proof, the interested reader is referred to [29, Prop. 2.14].

3.2.5. **Sharp completions of the canonical ambient toric variety.** Every algebraic variety can be embedded in a complete one, by Nagata’s theorem [26, Thm.]. For those endowed with an algebraic group action Sumihiro provided an equivariant version of this theorem [33, 34]. In particular, for toric varieties, it corresponds with the Ewald-Ishida combinatorial completion procedure for fans [11, Thm. III.2.8], recently simplified by Rohrer [28]. Anyway, all these procedures in general require the adjunction of some new ray into the fan under completion, that is an increasing of the Picard number. This is necessary in dimension $\geq 4$: there are examples of 4-dimensional fans which cannot be completed without the introduction of new rays. Consider the Remark ending up § III.3 in [10] and references therein, for a discussion of this topic; for explicit examples consider [30, Ex. 2.16] and the canonical ambient toric variety presented in [29, Ex. 2.40].

In the following, a completion not increasing the Picard number will be called sharp. Although a sharp completion of a toric variety does not exist in general, Hu and Keel showed that the canonical ambient toric variety $W$, of a MDS $X$, always admits sharp completions, which are even projective, one for each Mori chamber contained in $\text{Nef}(W) \cong \text{Nef}(X)$ [21, Prop. 2.11]. Unfortunately this is no more the case for a general wMDS: a counterexample exhibiting a wMDS whose canonical ambient toric variety does not admit any sharp completion is given in [29, Ex. 2.40].

Theorem 2.33 in [29] characterizes those weak Mori dream spaces $X$ whose canonical ambient toric variety $W$ admits a sharp completion $Z$, as those admitting a filling cell inside the nef cone $\text{Nef}(X)$: a filling cell is a cone of the secondary fan of $X$ arising as the common intersection of all the cones of a saturated bunch of cones containing the bunch of cones associated with $W$ and giving rise to the nef cone of a complete toric variety [29, Def. 2.28].

**Definition 3.14 (Fillable wMDS).** A wMDS $X$ is called fillable if $\text{Nef}(X)$ contains a filling cell $\gamma$.

**Theorem 3.15 (see Thm. 2.33 in [29]).** A wMDS $X$ with canonical ambient toric variety $W$ is fillable if and only if there exists a sharp completion $W \hookrightarrow Z$. In particular, if $X$ is complete then the induced closed embedding $X \hookrightarrow Z$ is neat.
3.3. The canonical 1-covering of a wMDS. Let $X$ be a wMDS and consider:

- its canonical toric embedding $i : X \hookrightarrow W(\Sigma)$, as given in Theorem 3.9,
- a toric completion $\iota : W \hookrightarrow Z(\gamma, \Sigma')$ of $W$, if existing, as given in Theorem 3.15, and corresponding to the choice of a filling cell $\gamma \subseteq \text{Nef}(X) \cong \text{Nef}(W)$ arising from a filling fan $\Sigma'$ of $\Sigma$, that is $\Sigma' \in SF(V)$ and $\Sigma \subseteq \Sigma'$, being $V$ a fan matrix of $W$ (and $Z$).

Notice that both $W$ and its completion $Z$ are non-degenerate toric varieties. Then Theorem 2.15 guarantees the existence of universal 1-coverings $\varphi : \tilde{W} \twoheadrightarrow W$ and $\psi : \tilde{Z} \twoheadrightarrow Z$.

Remark 3.16. Since the fan $\Sigma'$ of $Z$ is a filling fan of the fan $\Sigma$ of $W$, recalling the construction (10) of the covering fans $\tilde{\Sigma}'$ of $\tilde{Z}$ and $\tilde{\Sigma}$ of $\tilde{W}$, one immediately concludes that $\tilde{\Sigma}'$ is a filling fan of $\tilde{\Sigma}$, that is $\tilde{Z}$ is a completion of $\tilde{W}$, giving rise to the following commutative diagram

\[
\begin{array}{ccc}
\tilde{W} & \xrightarrow{i} & \tilde{Z} \\
\downarrow \varphi & & \downarrow \psi \\
X & \xrightarrow{i} & W
\end{array}
\]

Moreover:

1. $\text{Cl}(\tilde{W}) \cong \text{Cl}(\tilde{Z})$ is free and $\text{rk}(\text{Cl}(\tilde{W})) = \text{rk}(\text{Cl}(W)) = \text{rk}(\text{Cl}(Z)) = \text{rk}(\text{Cl}(\tilde{Z}))$;
2. $\text{Cox}(\tilde{W}) \cong \text{Cox}(W) \cong K[X] \cong \text{Cox}(Z) \cong \text{Cox}(\tilde{Z})$, where the left and right isomorphisms are $K$-algebras isomorphisms and not isomorphisms of graded algebras; in fact $\text{Cox}(\tilde{W})$ and $\text{Cox}(\tilde{Z})$ are graded on $\text{Cl}(\tilde{W}) \cong \text{Cl}(\tilde{Z})$, while $\text{Cox}(W)$ and $\text{Cox}(Z)$ are graded on $\text{Cl}(W) \cong \text{Cl}(Z)$;
3. $\tilde{W}$ and $\tilde{Z}$ are 1-connected, hence they are simply connected by Proposition 1.15.

We are now in a position to present and prove the main result of the present paper.

**Theorem 3.17.** A wMDS $X$ admit a canonical 1-covering $\phi : \tilde{X} \rightarrow X$ and a canonical closed embedding $\tilde{i} : \tilde{X} \hookrightarrow \tilde{W}$ into the universal 1-covering $\tilde{W}$ of $W$. They fit into the following commutative diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{i}} & \tilde{W} \\
\downarrow \phi & & \downarrow \varphi \\
X & \xrightarrow{i} & W
\end{array}
\]

Moreover, the following facts are equivalent:

1. $\tilde{i}$ is neat,
2. $\text{Cl}(\tilde{X})$ is free and $\text{rk}(\text{Cl}(\tilde{X})) = \text{rk}(\text{Cl}(X))$,
3. $\tilde{X}$ is a wMDS and $\text{Cox}(X) \cong \text{Cox}(\tilde{X})$ are isomorphic as $K$-algebras, differing from each other only by their graduation over $\text{Cl}(X)$ and $\text{Cl}(\tilde{X})$, respectively.
Finally, if \( X \) is fillable, there is a canonical open embedding \( \tilde{i} : \tilde{W} \hookrightarrow \tilde{Z} \) into the canonical universal 1-covering \( \psi : \tilde{Z} \to Z \), completing diagram (19) as follows

\[
\begin{array}{ccc}
\tilde{X} & \overset{i}{\hookrightarrow} & \tilde{W} \\
\downarrow \phi & & \downarrow \phi \\
X & \overset{i}{\hookrightarrow} & W \\
\end{array}
\]

Definition 3.18. In the same notation of Theorem 3.17, \( \phi : \tilde{X} \to X \) is called the canonical 1-covering of \( X \) and we say that \( \tilde{X} \) is a torsion-free, rank-preserving, 1-covering wMDS of \( X \) when the equivalent conditions (1), (2), (3) hold.

Proof of Theorem 3.17. Given the universal 1-covering \( \varphi : \tilde{W} \to W \), we get the following short exact sequence of abelian groups, associated with the canonical torsion subgroup \( \text{Tors}(\text{Cl}(W)) \leq \text{Cl}(W) \)

\[
0 \to \text{Tors}(\text{Cl}(W)) \to \text{Cl}(W) \overset{\varphi^*}{\to} \text{Cl}(\tilde{W}) \to 0
\]

Since \( \mathbb{K}^* \) is reductive, dualizing over \( \mathbb{K}^* \) gives the short exact sequence

\[
1 \to \text{Hom}(\text{Cl}(\tilde{W}), \mathbb{K}^*) \to \text{Hom}(\text{Cl}(W), \mathbb{K}^*) \overset{\varphi^*}{\to} \text{Hom}(\text{Tors}(\text{Cl}(W)), \mathbb{K}^*) \to 1
\]

Since \( \text{Cl}(\tilde{W}) \) is free, \( H := \text{Hom}(\text{Cl}(\tilde{W}), \mathbb{K}^*) \) turns out to be a full subtorus of the quasi-torus \( G := \text{Hom}(\text{Cl}(W), \mathbb{K}^*) \cong \text{Hom}(\text{Cl}(X), \mathbb{K}^*) \), giving rise to the finite quotient

\[
\mu := \text{Hom}(\text{Tors}(\text{Cl}(W)), \mathbb{K}^*) \cong G/H.
\]

By item (2) in the previous Remark 3.16, one has

\[
\overline{W} = \text{Spec}(\text{Cox}(W)) \cong \text{Spec} \mathbb{K}[\mathcal{X}] \cong \mathbb{K}^m \cong \text{Spec}(\text{Cox}(\tilde{W})) = \tilde{W}
\]

where \( m = |\mathcal{X}| \). Under this identification of Cox rings and total coordinate spaces, also irrelevant ideals and loci of \( W \) and \( \tilde{W} \) coincide, by definition (10) of the fan \( \tilde{\Sigma} \). Recalling diagram (17), one then has the following quotient description of the 1-covering \( \varphi : \tilde{W} \to W \)

\[
\tilde{W} \cong j_W(\tilde{W})/H \overset{\varphi}{\to} j_W(\tilde{W})/G \cong W
\]

and of the canonical toric embedding

\[
X \cong j_W \circ \tilde{i}(\tilde{X})/G \overset{\mu}{\to} j_W(\tilde{W})/G \cong W
\]

Define

\[
(21) \quad \tilde{X} := j_W \circ \tilde{i}(\tilde{X})/H.
\]
This comes with an associated closed embedding $\tilde{X} \overset{\tilde{i}}{\rightarrow} \tilde{W}$, equivariant with respect to the $H$-action, and the following commutative diagram

\[
\begin{array}{ccc}
\tilde{X} = j_W \circ \hat{i} \left( \hat{X} \right)/H & \cong & j_W(\hat{W})/H \cong \hat{W} \\
\downarrow \phi / \mu & & \downarrow \phi / \mu \\
X = j_W \circ \hat{i} \left( \hat{X} \right)/G & \cong & j_W(\hat{W})/G \cong W
\end{array}
\]

which is precisely the commutative diagram (19). Let us show that $\phi : \tilde{X} \to X$ is a 1-covering. In fact $\phi : \tilde{W} \to W$ is a toric 1-covering and $W$ is non-degenerate. Since $\phi$ is unramified in codimension 1, Theorem 2.8 implies that $Br(\phi) \subseteq R := \bigcup_{1 \leq i < j \leq m} D_i \cap D_j$

Proposition 3.13 shows that $i$ is a neat closed embedding. Then $Br(\phi) \subseteq X \cap R$ still has codimension greater than 1 in $X$.

Notice now that

$$\forall \ j = 1, \ldots, m \quad \phi^{-1}(i^{-1}(D_j)) = (i \circ \phi)^{-1}(D_j) = (\varphi \circ \tilde{i})^{-1}(D_j) = \tilde{i}^{-1}(\varphi^{-1}(D_j)).$$

Since $i$ is a neat embedding and $\phi$ is a 1-covering, then $\{\phi^{-1}(i^{-1}(D_j))\}_{j=1}^m$ is a set of pairwise distinct hypersurfaces of $\tilde{X}$. On the other hand, $\{\varphi^{-1}(D_j)\}_{j=1}^m$ is the set of torus invariant prime divisors of $\tilde{W}$. Then the closed toric embedding $\tilde{i}$ satisfies hypotheses of Proposition 3.11 so giving a well defined pull back homomorphism $\tilde{i}^* : Cl(\tilde{W}) \to Cl(\tilde{X})$. Consider the following commutative diagram of group homomorphisms

\[
\begin{array}{ccc}
Cl(W) & \overset{i^*}{\longrightarrow} & Cl(X) \\
\downarrow \phi^* & & \downarrow \varphi^* \\
Cl(\tilde{W}) & \overset{\tilde{i}^*}{\longrightarrow} & Cl(\tilde{X})
\end{array}
\]

being the pull back $\phi^* : Cl(X) \to Cl(\tilde{X})$ well defined by (6) in § 1.5. Assume the following fact, whose proof is postponed.

**Lemma 3.19.** $\ker \phi^* = Tors(Cl(X))$

Therefore $\text{rk}(\text{Im} \phi^*) = \text{rk}(Cl(X)) = \text{rk}(Cl(W)) = \text{rk}(Cl(\tilde{W}))$, meaning that $\tilde{i}$ is neat if and only if $\phi^*$ is surjective, that is if and only if $Cl(\tilde{X})$ is free and $\text{rk}(Cl(\tilde{X})) = \text{rk}(Cl(X))$, proving that (1) $\Leftrightarrow$ (2).

To show that (2) $\Leftrightarrow$ (3), notice that by construction we have the following commutative diagram

\[
\begin{array}{ccc}
\tilde{X} & \overset{\tilde{i}}{\rightarrow} & \tilde{W} \\
p_X & & p_W \\
\tilde{X} & \overset{\tilde{i}}{\rightarrow} & \tilde{W} \\
p_X & & p_W \\
X & \overset{i}{\rightarrow} & W
\end{array}
\]
Define $\widetilde{\text{Cox}} := (p_\tilde{X}), \mathcal{O}_{\tilde{X}}$. Recall that the canonical morphism $p_X$ of the relative spectrum construction give the following isomorphism

$$\text{Cox} \cong (p_X)_* \mathcal{O}_X = \phi_*( (p_\tilde{X}), \mathcal{O}_{\tilde{X}} ) = \phi_* \widetilde{\text{Cox}}.$$ 

Passing to global sections and observing that $\phi^{-1}(X) = \tilde{X}$, we get that

$$\text{Cox}(X) = \Gamma(X, \text{Cox}) \cong \Gamma(\tilde{X}, \widetilde{\text{Cox}}).$$

This is not an isomorphism of graded algebras, but it suffices to prove that $\Gamma(\tilde{X}, \widetilde{\text{Cox}})$ is a finitely generated algebra.

For what concerning their graduations, notice that

$$\text{Cox} = \bigoplus_{\delta \in \text{Cl}(X)} \text{Cox}_\delta \cong \phi_* \widetilde{\text{Cox}} = \bigoplus_{\eta \in \text{Im} \phi^*} \phi_{\eta} \widetilde{\text{Cox}}_{\eta} \cong \bigoplus_{\delta \in (\text{Im} \phi^*)^{-1}(\eta)} \text{Cox}_\delta.$$

Call $\tilde{X}'$ the wMDS admitting Cox sheaf and class group given by $\widetilde{\text{Cox}}$ and $\text{Cl}(\tilde{X}') = \text{Im} \phi^*$, respectively. Applying Theorem 3.9 and Proposition 3.13 to $\tilde{X}$ and $\tilde{X}'$, by replacing the quasi-torus action of $G$ with the torus action of $H$ and $H' := \text{Hom}(\text{Im} \phi^*, \mathbb{K})$, respectively, one gets

$$\left( \tilde{X}' = \text{Spec}_X(\text{Cox}) / H' \right) \cong \left( \tilde{X} / H = \tilde{X} \right) \iff \text{Im} \phi^* = \text{Cl}(\tilde{X}).$$

For what concerning the last part of the statement, notice that $X$ is fillable if and only if $\tilde{X}$ is fillable. In particular, recalling diagram 18, the previous commutative diagram 22 extends to give the following one

$$\text{X} = j_W \circ \iota \left( \tilde{X} \right) / H \cong j_W (\tilde{W}) / H \cong \tilde{W} / \mathcal{O}_X \cong Z,$$

which is precisely the commutative diagram 20.

**Proof of Lemma 3.19**

$\text{Tors}(\text{Cl}(X)) \subseteq \ker \phi^*$. In fact, if $\delta \in \text{Tors}(\text{Cl}(X))$ then $(i^*)^{-1}(\delta) \in \text{Tors}(\text{Cl}(W))$. Therefore $\phi^*((i^*)^{-1}(\delta)) = 0$, so giving $\phi^* (\delta) = \tilde{i}^* \circ \phi^* \circ (i^*)^{-1}(\delta) = 0$.

$\ker \phi^* \subseteq \text{Tors}(\text{Cl}(X))$. Consider $\delta \in \text{Cl}(X)$ such that $\phi^*(\delta) = 0$. Then, for every $D \in d^{-1}_K(\delta)$ the divisor $\phi_#(D) = \phi^{-1}(D)$ is principal. In particular it is an invariant divisor with respect to the action of $\mu$, meaning that $\phi_#(D) = (f)$ for some $\mu$-homogeneous function $f \in \mathbb{K}(X)^*$. Consider the $|\mu|$-power $q : \mathbb{K} \to \mathbb{K}$, such that $q(z) = z^{|\mu|}$, and define $\overline{f} \in \mathbb{K}(X)^*$ by setting

$$\forall x \in X \quad \overline{f}(x) := q(f(y)) \text{ for some } y \in \phi^{-1}(x).$$

$\overline{f}$ is well defined because $f$ $\mu$-homogeneous gives

$$\forall \zeta \in \mu \quad q(f(\zeta \cdot y)) = q(f(y)).$$

Notice that $|\mu|D = \phi(\phi_#(D)) = (\overline{f}) \sim 0$, so giving that $D \in \text{Tors}(\text{Cl}(X))$. 

\qed
Remark 3.20. Notice that the 1-covering $\phi : \widetilde{X} \to X$ is canonical, in the sense that it does not depend on the choice of the set of generators $X$. In fact, for a different choice $X'$, let $i' : X \hookrightarrow W'$ be the $X'$-canonical toric embedding. By Proposition 3.13

$$G := \text{Hom}(\text{Cl}(X), \mathbb{K}^*) \cong \text{Hom}(\text{Cl}(W), \mathbb{K}^*) \cong \text{Hom}(\text{Cl}(W'), \mathbb{K}^*).$$

Then every free part of $G$ is isomorphic to $H$, that is

$$\text{Hom}(\text{Cl}(... \cong \text{Hom}(\text{Cl}(\widetilde{W}', \mathbb{K}^*)).$$

and the same holds for the torsion subgroup

$$\text{Hom}(\text{Tors}(\text{Cl}(W)), \mathbb{K}^*) \cong \mu \cong \text{Hom}(\text{Tors}(\text{Cl}(W')), \mathbb{K}^*).$$

On the other hand $\widehat{X} = \text{Spec}_X(C_\text{ox}) \cong \widehat{X}'$. Therefore the 1-covering $\widetilde{X} = jW \circ i(\widehat{X}) / H \cong jW' \circ i'(\widehat{X}') / H \xrightarrow{\phi/\mu} X$

is canonically fixed, up to isomorphisms.

### 3.4. When the canonical embedding of the canonical 1-covering is neat?

Given a wMDS $X$ with canonical toric embedding $i : X \hookrightarrow W$, let $\phi : \widetilde{X} \to X$ be the canonical 1-covering, constructed in Theorem 3.17, and $\tilde{i} : \tilde{X} \hookrightarrow \tilde{W}$ be its canonical closed toric embedding giving rise to the commutative diagram (19). Keeping in mind the equivalent conditions (1), (2), (3) in the statement of Theorem 3.17, being neat for $\tilde{i}$ is a sort of extension to $\mathbb{Q}$-factorial varieties of the Grothendieck-Lefschetz theorem [15, Exp. XI], for the class group morphism $\tilde{i}^* : \text{Cl}(\widehat{W}) \to \text{Cl}(\widehat{X})$. Following [22, 1], and [27], we can obtain sufficient conditions to get neatness of $\tilde{i}$. At this purpose we need to introduce the following.

**Definition 3.21.** A $\mathbb{Q}$-factorial toric variety $W = W(\Sigma)$ (or equivalently its simplicial fan $\Sigma$) is called $k$-neighborly if for any $k$ rays in $\Sigma(1)$ the convex cone they span is in $\Sigma(k)$. Equivalently, by Gale duality, this means that

$$\text{Nef}(X) \subseteq \bigcap_{1 \leq i_1 < \cdots < i_k \leq |\Sigma(1)|} \langle Q^{(i_1, \ldots, i_k)} \rangle$$

The following characterization of a $k$-neighborly toric variety follows by the inclusion [23], recalling the natural correspondence between the bunch of cones of $W$ and the generators of its irrelevant ideal $\mathcal{I}\text{rr}(W)$. See also [22, Prop. 10, Rmk. 11] for further details.

**Proposition 3.22.** A $\mathbb{Q}$-factorial toric variety $W$ is $k$-neighborly if and only if the irrelevant locus $B \subseteq \overline{W}$ has codimension $\dim B > k$.

We are now in a position of giving the following sufficient conditions for the neatness of $\tilde{i}$.

**Proposition 3.23.** Let $X, \tilde{X}, W, \widetilde{W}$ as above, then the canonical closed embedding $\tilde{i} : \tilde{X} \to \widetilde{W}$ is neat if one of the following happens:

1. if $X$ is a smooth complete intersection of codimension $\ell$ in $\tilde{W}$ and the latter is a smooth, projective, $(1+\ell)$-neighborly toric variety, with $\dim(\tilde{W}) \geq 3+\ell$;
2. if $X$ is a complete intersection of codimension $\ell$ in $W$ and the irrelevant locus $B_X \subset \overline{X}$ has codimension $\geq 1+\ell$;
(3) if $\tilde{X} \in |D|$ is a general element, with $D$ an ample divisor of $\tilde{W}$ and the latter is projective with $\dim(\tilde{W}) \geq 4$.

Proof. (1) is an iterated application of [22, Thm. 6], keeping in mind the equivalence established by Proposition 3.22 and recalling the equivalence $(1) \iff (3)$ in Theorem 3.17. For (2) notice that by the commutative diagram (19), $X$ is a complete intersection of codimension $l$ in $W$ if and only if $\tilde{X}$ is a complete intersection of codimension $\tilde{l}$ in $\tilde{W}$ and $\text{codim}(\tilde{X}) = \text{codim}(X)$. Then apply [1, Thm. 2.1] and equivalence $(1) \iff (2)$ in Theorem 3.17 to get the neatness of $\tilde{i}$. Finally (3) is a direct application of [27, Thm. 1], recalling equivalence $(1) \iff (2)$ in Theorem 3.17. □

3.5. When the canonical 1-covering is the universal 1-covering? Let $X$ be a fillable wMDS and $\tilde{X}$ be its canonical 1-covering. Let $X \hookrightarrow Z$ and $\tilde{X} \hookrightarrow \tilde{Z}$ be complete toric embeddings assigned by the choice of a filling chamber $\gamma \subseteq \text{Nef}(W)$, as in Theorem 3.17 diagram (20). Proposition 1.29 allows us to conclude that

- $\phi : \tilde{X} \rightarrow X$ is the universal 1-covering of $X$ if and only if the open subset $X_{\text{reg}}$, of regular points of $\tilde{X}$, is simply connected i.e. $\pi^*\tilde{X}(X_{\text{reg}}, x) = 0$ for every regular point $x \in X_{\text{reg}}$.

Notice that $\varphi : \tilde{Z} \rightarrow Z$ is the universal 1-covering of $Z$, that is $\pi^*\tilde{Z}(Z_{\text{reg}}, z) = 0$, for every regular point $z \in Z_{\text{reg}}$. Therefore asking for simply connectedness of $\tilde{X}_{\text{reg}}$ translates in a sort of Weak Lefschetz Theorem on the étale fundamental groups of smooth loci in $\tilde{X} \rightarrow \tilde{Z}$. Clearly we cannot hope this result holding in general. In the following we consider the particular case $\mathbb{K} = \mathbb{C}$ with, in addition, some strong hypotheses on singularities of $\tilde{X}$ and the embedding $\tilde{X} \hookrightarrow \tilde{Z}$.

Definition 3.24. Let $X$ be a wMDS and $i : X \hookrightarrow W$ be its canonical toric embedding. $X$ is called quasi-smooth if the singular locus of $X$ is included in the singular locus of the ambient toric variety $W$, that is

$$i(\text{Sing}(X)) \subseteq \text{Sing}(W) \cap i(X) \iff W_{\text{reg}} \cap i(X) \subseteq i(X_{\text{reg}})$$

Moreover, $X$ is called a complete intersection if the relations’ ideal $I \subset \mathbb{K}[X]$, such that $\text{Cox}(X) \cong \mathbb{K}[X]/I$, is generated by exactly $l := \text{codim}(X)$ polynomials.

Definition 3.25 (Small Q-factorial modification). A birational map $f : X \dashrightarrow Y$, between irreducible, complete and Q-factorial algebraic varieties, is called a small Q-factorial modification (sQM) if it is biregular in codimension 1 i.e. there exist Zariski open subsets $U \subseteq X$ and $V \subseteq Y$ such that $f|_U : U \cong V$ is biregular and $\text{codim}(X \setminus U) \geq 2, \text{codim}(Y \setminus V) \geq 2$.

Remark 3.26. Notice that a Q-factorial and complete algebraic variety $X$ is a wMDS if and only if there exists a sQM $X \dashrightarrow X'$ such that $X'$ is a MDS [29, Lemma 3.2].

Theorem 3.27. Assume $\mathbb{K} = \mathbb{C}$ and $X$ be a complete and fillable wMDS, which is a complete intersection and admitting a sQM $X \dashrightarrow X'$ to a quasi-smooth MDS. Then the canonical torsion free 1-covering $\tilde{X} \rightarrow X$ is the universal 1-covering of $X$. In particular, a MDS which is a quasi-smooth complete intersection is simply connected and always admits a universal 1-covering.

After a sQM and an iterated application of a Veronese embedding, the previous statement is obtained by the following result of Goresky and MacPherson
**Theorem 3.28** (see §II.1.2 in [12]). Let $Y$ be the complement of a closed subvariety of an $n$-dimensional complex analytic variety $Y$ and $j : Y \to \mathbb{P}^N$ be a proper embedding. Let $H \subseteq \mathbb{P}^N$ be a hyperplane. Then the homomorphism induced by inclusion on the fundamental groups $\pi_1((j|_Y)^{-1}(H)) \to \pi_1(Y)$ is an isomorphism for $n \geq 2$.

This statement is deduced from the theorem opening §II.1.2 in [12], by assuming the therein immediately following assumption (1), since $j$ is proper, and assumption (2).

**Proof of Thm. 3.27**. The $\mathbb{Q}$m $X \to X'$ fits into the following 3-dimensional commutative diagram

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}|_{\tilde{X}}} & \tilde{X}' \\
\phi & \downarrow & \phi' \\
\tilde{Z} & \xrightarrow{\tilde{f}} & \tilde{Z}' \\
\end{array}
\quad
\begin{array}{ccc}
\phi & \downarrow & \phi' \\
Z & \xrightarrow{f} & Z' \\
\end{array}
$$

where:

- vertical maps $\phi$ and $\phi'$ are canonical torsion-free 1-coverings,
- vertical maps $\varphi$ and $\varphi'$ are canonical universal 1-coverings,
- diagonal maps are complete toric embeddings associated with the choice of a filling cell $\gamma \subseteq \text{Nef}(X) \cong \text{Nef}(W)$,
- horizontal maps $f, \tilde{f}, f|_{X}, \tilde{f}|_{\tilde{X}}$ are small $\mathbb{Q}$-factorial modifications: in particular $X', \tilde{X}'$ are MDS and $Z', \tilde{Z}'$ are projective $\mathbb{Q}$-factorial toric varieties

$$(24) \quad \pi_1(\tilde{Z}'_{reg}) \cong \{1\}. $$

Let us first assume that $X$ is an hypersurface of its canonical ambient toric variety $W$, hence of its completion $Z$. Then, by definition (21), $\tilde{X}$ is an hypersurface of $\tilde{Z}$. After the $\mathbb{Q}$m $\tilde{f}, \tilde{X}'$ turns out to be a quasi-smooth hypersurface of $\tilde{Z}'$. Recall that $\tilde{Z}'$ is projective, so giving the following projective embedding of $\tilde{X}'$

$$(25) \quad \tilde{X}' \hookrightarrow \mathbb{P}^N_{\mathbb{Q},d} \to \mathbb{P}(N^d+1)'$$

so that $\tilde{X}' = h^{-1}(F_d) = j^{-1}(H)$, where:

- $F_d \subseteq \mathbb{P}^N$ is a suitable hypersurface of degree $d$,
- $v_{N,d}$ is the Veronese embedding,
- $H \subseteq \mathbb{P}(N^d+1)$ is the hyperplane such that $v_{N,d}^{-1}(H) = F_d$. 
Apply now Theorem 3.28 by setting \( Y = \bar{Y}', Y = \bar{Z}_{reg}' \). Quasi-smoothness of \( \bar{X}' \) implies that
\[
(j|_{\bar{Z}_{reg}'})^{-1}(H) = \bar{Z}_{reg}' \cap \bar{X}' \subseteq \bar{X}_{reg}'
\]
The latter inclusion induces a covariant surjection on associated fundamental groups (see e.g. [7, Thm. 12.1.5] and references therein), so giving
\[
\{1\} \cong \pi_1(\bar{Z}_{reg}') \cong \pi_1(\bar{Z}_{reg}' \cap \bar{X}') \twoheadrightarrow \pi_1(\bar{X}_{reg}') \quad \Rightarrow \quad \pi_1(\bar{X}_{reg}') \cong \{1\}
\]
by relation (24) and Theorem 3.28. The last step is proving that \( \pi_1(\bar{X}_{reg}) \cong \pi_1(\bar{X}_{reg}') \cong \{1\} \).

Let us now assume \( X \) be a complete intersection of \( c \geq 2 \) hypersurfaces of \( W \), hence of its completion \( Z \). This means that \( I = (f_1, \ldots, f_c) \) in \( \text{Cox}(W) \cong \mathbb{C}[x_1, \ldots, x_m] \), where \( \mathbb{C}[x_1, \ldots, x_m]/I \cong \text{Cox}(X) \). Then \( X \) is an hypersurface of the complete intersection \( Y = Y' \) of \( W \) associated with the ideal \( (f_1, \ldots, f_{c-1}) \) in the construction given by Theorem 3.9. Then by definition (21), \( X \) is an hypersurface of the complete intersection \( Y \subseteq Z \). After the s \( \text{Qm} f |_Y \), \( \bar{X}' \) turns out to be an hypersurface of the complete intersection \( \bar{Y}' \subseteq \bar{Z}' \). In particular, \( \bar{Y}' \) is projective and, by induction on \( c \), we can assume
\[
\pi_1(\bar{Y}_{reg}') \cong \{1\}.
\]
Diagram (25) can be replaced by the following projective embedding of \( \bar{X}' \)

\[
\begin{array}{c}
\bar{X}' \xrightarrow{h} \mathbb{P}^{N_d} \\
\bigtriangleup \downarrow j \\
\bar{Y} \xrightarrow{\pi_{\bar{Y}_{reg}'} \cap \bar{X}' \subseteq \bar{X}_{reg}'} \end{array}
\]
so that \( \bar{X}' = h^{-1}(F_d) = j^{-1}(H) \), for a suitable hypersurface \( F_d \subseteq \mathbb{P}^{N_d} \). Apply now Theorem 3.28 by setting \( Y = \bar{Y}', Y = \bar{Y}_{reg}' \). Quasi-smoothness of \( \bar{X}' \) implies that
\[
(j|_{\bar{Y}_{reg}'})^{-1}(H) = \bar{Y}_{reg}' \cap \bar{X}' \subseteq \bar{X}_{reg}'
\]
Therefore
\[
\{1\} \cong \pi_1(\bar{Y}_{reg}') \cong \pi_1(\bar{Y}_{reg}' \cap \bar{X}') \twoheadrightarrow \pi_1(\bar{X}_{reg}') \quad \Rightarrow \quad \pi_1(\bar{X}_{reg}') \cong \{1\}.
\]
The last step, proving that \( \pi_1(\bar{X}_{reg}) \cong \pi_1(\bar{X}_{reg}') \cong \{1\} \), proceeds exactly as in case \( c = 1 \).

If \( X \) is a MDS which is a quasi-smooth complete intersection, one can run the previous argument by taking \( f \) as the identity. Finally, the simply connectedness of the MDS \( X \) is proved by setting \( Y = \bar{Y} \), that is, by assuming the closed subvariety in the statement of Theorem 3.28 as empty. Then apply the same inductive argument by starting with \( Y = Z \) and recalling that a complete toric variety is always simply connected, by Corollary 2.10. \( \square \)

**Remark 3.29.** The previous Theorem 3.27 can be certainly generalized to admitting some further singularity for either \( X \) or \( X' \): in fact Goresky-MacPherson results are more general than Theorem 3.28, which presents a statement adapted to the case here considered. However, any such generalization strongly depends on the
kind of admitted singularities for $X$ and needs a careful application of deep and more general results due to Goresky-MacPherson and Hamm- Lê (see [12], [17]).

4. Examples and Further Applications

This section is devoted to present examples of Mori dream spaces whose canonical 1-covering, in a case, admits a neat embedding in its canonical ambient toric variety, as it is still a MDS, and, in the other case, does not admit a neat embedding in a toric variety, as it is no more a MDS. We will start with an evidence of the first kind, by revising an example of a MDS already studied by Hausen and Keicher in [20, Ex. 2.1]. Then we will exhibit an interesting evidence of the second kind, given by Enriques surfaces which are Mori dream spaces.

4.1. An example by Hausen and Keicher. Example here presented is obtained by considering, up to isomorphism, the Cox ring studied in [20, Ex. 2.1] and also listed in the Cox ring database [19], where it is reported as the id no. 97.

Consider the grading map $d_K : K = \mathbb{Z}^8 \rightarrow \mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z}$, whose free part is represented by the weight matrix

$$Q = \begin{pmatrix}
2 & 1 & 0 & 2 & 0 & 2 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 2 & 2 & 2 & 2
\end{pmatrix} = (q_1 \cdots q_8)$$

and whose torsion part is represented by the torsion matrix

$$T = (\bar{0} \bar{0} \bar{0} \bar{0} 1 1 1 1)$$

Then, consider the quotient algebra

$$R = \mathbb{K}[x_1, \ldots, x_8]/(x_1x_8 + x_2x_7 + x_3x_6 + x_4x_5)$$

graded by $d_K$. This is consistent since the relation defining $R$ is homogeneous with respect to such a grading. Moreover $R$ turns out to be a Cox ring with $\mathfrak{X} := \{\mathbf{x}_1, \ldots, \mathbf{x}_8\}$ giving a Cox basis of $R$. Then $\mathfrak{X} := \text{Spec}(R) \subseteq \text{Spec}\mathbb{K}[x] =: \mathbb{W}$ defines the total coordinate space of a wMDS $X := \tilde{X}/G$ and its canonical ambient toric variety $W := \tilde{W}/G$, where

$$\begin{align*}
\tilde{X} &= \mathfrak{X}/B_{\mathfrak{X}} \text{ being } B_{\mathfrak{X}} = \mathcal{V}(\mathcal{Irr}(X)) \\
\tilde{W} &= \mathbb{W}/\tilde{B} \text{ being } \tilde{B} = \mathcal{V}(\mathcal{Irr})
\end{align*}$$

$$\begin{align*}
\mathcal{Irr}(X) &= \begin{pmatrix}
\mathbf{x}_1\mathbf{x}_3\mathbf{x}_7, \mathbf{x}_1\mathbf{x}_5\mathbf{x}_6, \mathbf{x}_1\mathbf{x}_5\mathbf{x}_7, \mathbf{x}_2\mathbf{x}_4\mathbf{x}_8, \mathbf{x}_2\mathbf{x}_5\mathbf{x}_6, \mathbf{x}_2\mathbf{x}_5\mathbf{x}_7, \mathbf{x}_3\mathbf{x}_4\mathbf{x}_8 \\
\mathbf{x}_1\mathbf{x}_2\mathbf{x}_7\mathbf{x}_8, \mathbf{x}_1\mathbf{x}_3\mathbf{x}_6\mathbf{x}_8, \mathbf{x}_1\mathbf{x}_4\mathbf{x}_5\mathbf{x}_8, \mathbf{x}_2\mathbf{x}_3\mathbf{x}_6\mathbf{x}_7, \mathbf{x}_2\mathbf{x}_4\mathbf{x}_5\mathbf{x}_7, \mathbf{x}_3\mathbf{x}_4\mathbf{x}_5\mathbf{x}_6
\end{pmatrix} \\
\mathcal{Irr} &= \begin{pmatrix}
\mathbf{x}_1\mathbf{x}_3\mathbf{x}_7, \mathbf{x}_1\mathbf{x}_5\mathbf{x}_6, \mathbf{x}_1\mathbf{x}_5\mathbf{x}_7, \mathbf{x}_2\mathbf{x}_4\mathbf{x}_8, \mathbf{x}_2\mathbf{x}_5\mathbf{x}_6, \mathbf{x}_2\mathbf{x}_6\mathbf{x}_8, \mathbf{x}_3\mathbf{x}_4\mathbf{x}_7, \mathbf{x}_3\mathbf{x}_4\mathbf{x}_8 \\
\mathbf{x}_1\mathbf{x}_2\mathbf{x}_7\mathbf{x}_8, \mathbf{x}_1\mathbf{x}_3\mathbf{x}_6\mathbf{x}_8, \mathbf{x}_1\mathbf{x}_4\mathbf{x}_5\mathbf{x}_8, \mathbf{x}_2\mathbf{x}_3\mathbf{x}_6\mathbf{x}_7, \mathbf{x}_2\mathbf{x}_4\mathbf{x}_5\mathbf{x}_7, \mathbf{x}_3\mathbf{x}_4\mathbf{x}_5\mathbf{x}_6
\end{pmatrix} \\
G &= \text{Hom}(\text{Cl}(W), \mathbb{K}^*) \cong \text{Hom}(\mathbb{Z}^3 \oplus \mathbb{Z}/2, \mathbb{K}^*)
\end{align*}$$

A Gale dual matrix of $Q$ is given by the following $CF$-matrix

$$\bar{V} = \begin{pmatrix}
1 & 0 & 0 & 0 & -2 & 0 & -2 & 3 \\
0 & 1 & 0 & 0 & -2 & 0 & -1 & 2 \\
0 & 0 & 1 & 0 & -2 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & -1 & 0 & -2 & 2 \\
0 & 0 & 0 & 0 & 1 & -2 & 1
\end{pmatrix}$$
Notice that $Q$ is a Gale dual matrix of both $\tilde{V}$ and the following $F$-matrix

$$V = \begin{pmatrix}
1 & 0 & 0 & 1 & -3 & 0 & -4 & 5 \\
0 & 1 & 0 & 1 & -3 & 0 & -3 & 4 \\
0 & 0 & 1 & 1 & -3 & 0 & -2 & 3 \\
0 & 0 & 0 & 2 & -2 & 0 & -4 & 4 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1
\end{pmatrix} = (v_1 \cdots v_8)$$

Moreover, it turns out that $T \cdot V^T = 0$. Then $V$ is a fan matrix of $W$, while $\tilde{V}$ is a fan matrix of the universal 1-covering $\tilde{W}$ of $W$. In particular,

$$\tilde{W} = \hat{W}/H \quad \text{where} \quad H := \text{Hom}(\text{Cl}(\tilde{W}), K^*) \cong \text{Hom}(\mathbb{Z}^3, K^*)$$

The canonical torsion free 1-covering $\tilde{X}$ of $X$ is the given by $\tilde{X} = \hat{X}/H$. It is a MDS whose canonical ambient toric variety is given by $\hat{W}$. In particular, $\tilde{i} : \tilde{X} \hookrightarrow \tilde{W}$ is a neat embedding. Notice that $X$ is quasi-smooth and satisfies hypotheses of Theorem 3.27.

4.2. Mori Dream Enriques surfaces. An Enriques surface is a complex projective smooth surface $X$ with $q(X) = p_g(X) = 0, 2K_X \sim 0$ but $K_X \not\sim 0$. There are several well known facts about Enriques surfaces, few of them are here recalled:

**Proposition 4.1** (§ VIII.15 in [3]). Let $X$ be an Enriques surface. Then

1. $\Cl(X) \cong \mathbb{Z}^1 \oplus \mathbb{Z}/2\mathbb{Z}$, the torsion part being generated by the canonical class $[K_X]$; then $X$ has Picard number $r = 10$;
2. the fundamental group of $X$ is $\pi_1(X) \cong \mathbb{Z}/2\mathbb{Z}$;
3. if $\tilde{X} \rightarrow X$ is the universal covering space of $X$, then $\tilde{X}$ is a K3 surface, that is a complex smooth projective surface with $K_X \sim 0$ and $q(X) = 0$.

Enriques surfaces which are MDS are very special inside the 10-dimensional moduli space of Enriques surfaces. In fact they correspond to those admitting a finite automorphism group [2, Thm. 5.1.3.12] and explicitly classified by Kondo [23]: namely they consist of two 1-dimensional families and five 0-dimensional families (see also [2, Thm. 5.1.6.1]). The following result was firstly conjectured by Dolgachev [9, Conj. 4.7] and then proved by Kondo [23, Cor. 6.3].

**Theorem 4.2** (Dolgachev-Kondo). Let $X$ be an Enriques surface and $\tilde{X}$ its K3 universal covering. Then $\text{Aut}(\tilde{X})$ is infinite.

Since an Enriques surface $X$ is smooth, the canonical 1-covering of $X$, whose existence is guaranteed by Theorem 3.17 when $X$ is a MDS, is actually unramified by Lemma 1.2 so giving precisely the universal topological covering $\phi : \tilde{X} \rightarrow X$. Then the previous Dolgachev-Kondo Theorem implies that $\tilde{X}$ cannot be a MDS, by [2] Thm. 5.1.3.12], that is the canonical closed embedding $\tilde{X} \hookrightarrow \tilde{W}$ cannot be a neat embedding.

Anyway, Theorem 3.17 allows us to conclude some interesting properties of the canonical toric embedding $X \hookrightarrow W$, of a Mori Dream Enriques surface $X$, and its lifting to canonical 1-coverings $\tilde{X} \hookrightarrow \tilde{W}$, summarized as follows:

**Corollary 4.3.** Let $X$ be a Mori Dream Enriques surface, $i : X \hookrightarrow W$ its canonical toric embedding and consider the natural commutative diagram of embeddings and
I-coverings:

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{i}} & \tilde{W} \\
\downarrow \phi & & \downarrow \phi \\
X & \xrightarrow{i} & W
\end{array}
\]

Then:

1. the canonical 1-covering \( \phi : \tilde{X} \to X \) is the universal (1-)covering of \( X \),
2. \( \text{Cl}(\tilde{X}) \) and \( \text{Cl}(\tilde{W}) \) are free groups,
3. \( \tilde{r} := \text{rk}(\text{Cl}(\tilde{X})) > r := \text{rk}(\text{Cl}(X)) = \text{rk}(\text{Cl}(W)) = 10 \),
4. both \( X \) and \( W \) have torsion Picard group,
5. both the toric ambient varieties \( W \) and \( \tilde{W} \) do not admit any fixed point by the torus action.

Proof. (1) follows by the smoothness of \( X \) and Lemma 3.2.

(2) follows by Theorem 2.15 for what’s concerning the universal 1-covering \( \tilde{W} \), while it is a classically well known fact for what’s concerning the universal topological \( K3 \) covering \( \tilde{X} \).

(3) follows by the Dolgachev-Kondo Theorem 4.2, keeping in mind the equivalent conditions (2) and (3) in the statement of Theorem 3.17.

(4) is the previous Proposition 4.1 (1), for what’s concerning \( X \), and follows by Proposition 3.13 when recalling that the canonical toric embedding \( X \hookrightarrow W \) is neat.

(5) for \( W \) is a consequence of the previous item (2). In fact, since \( \text{Pic}(W) \) admits a non-trivial torsion subgroup, the fan \( \Sigma \) of \( W \) cannot admit maximal cones of full dimension \( \text{dim}(W) \), that is \( W \) cannot admit any fixed point under the torus action. This fact lifts to the universal 1-covering \( \tilde{W} \) by the construction of its fan \( \tilde{\Sigma} \) as explained by (10) in the proof of Theorem 2.15.

\( \square \)

Remark 4.4. Let us emphasize that the previous Corollary 4.3 implies that

- the universal \( K3 \) covering of a Mori Dream Enriques surface (that is an Enriques surface with finite automorphism group) admits a canonical embedding as a smooth subvariety of a \( \mathbb{Q} \)-factorial toric variety, whose class group is a free abelian group of rank 10 and whose torus action does not admit any fixed point.

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