Online Resource Allocation with Time-Flexible Customers

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In classic online resource allocation problems, a decision-maker tries to maximize her reward through making immediate and irrevocable choices regarding arriving demand points (agents). However, in many settings such as hospitals, services, ride-sharing, and cloud computing platforms, some arriving agents may be patient and willing to wait a short amount of time for the resource. Motivated by this, we study the online resource allocation problem in the presence of time-flexible agents under an adversarial online arrival model. We present a setting with flexible and inflexible agents who seek a resource or service that replenishes periodically. Inflexible agents demand the resource immediately upon arrival while flexible agents are willing to wait a short period of time. Our work presents a class of POLYtope-based Resource Allocation (POLYRA) algorithms that achieve optimal or near-optimal competitive ratios under an adversarial arrival process. Such POLYRA algorithms work by consulting a particular polytope and only making decisions that guarantee the algorithm’s state remains feasible in this polytope. When the number of agent types is either two or three, POLYRA algorithms can obtain the optimal competitive ratio. We design these polytopes by constructing an upper bound on the competitive ratio of any algorithm, which is characterized via a linear program (LP) that considers a collection of overlapping worst-case input sequences. Our designed POLYRA algorithms then mimic the optimal solution of this upper bound LP via its polytope’s definition, obtaining the optimal competitive ratio. When there are more than three types, we show that our overlapping worst-case input sequences do not result in an attainable competitive ratio, adding an additional challenge to the problem. Considering this, we present a near-optimal nested POLYRA algorithm which achieves at least 80% of the optimal competitive ratio while having a simple and interpretable structure.

Key words: Resource Allocation, Online Algorithm Design, Competitive Ratio, Time-flexible Customers, Polytope-based Resource Allocation Algorithms

History:

1. Introduction

Online resource allocation is one of the fundamental problems faced by almost any industry, including governments (e.g., during a pandemic crisis), hospitals, e-commerce, and airlines to name a few. In this problem, agents arrive over time, and upon arrival, we need to make an irrevocable decision about whether or not to allocate a resource to them. Typically, we assume that arriving
agents require the resource immediately and so accepted agents immediately consume a unit of our resource, and rejected agents are lost forever. However, in many realistic applications, certain agents are flexible with the timing of the resource and may be willing to wait a short period of time for it. For example, in healthcare, while some patients need to receive medical attention immediately, others are more flexible. For cloud computing services, again not all jobs have the same urgency level. In ride-sharing apps and grocery delivery services, the fact that some customers are flexible with the timing of their service has led to new features by Lyft (ride-sharing) and Instacart (grocery delivery). Lyft’s “Wait and Save” option asks the user to wait up to 25 minutes for a ride rather than finding them the nearest driver, while Instacart’s “Fast and Flexible” option gives the user a 3-day delivery window rather than a few hours. These applications motivate the key research question for our work: How can we design algorithms that optimally take advantage of such time-flexible customers?

To answer this question, we present a novel online resource allocation model which captures the idea that some agents are flexible in the timing of their service. Our model consists of \( T \) time periods and \( C \) capacity of a single service or resource that is replenished up to capacity at the beginning of each time period. In each time period, agents arrive one by one in an online fashion and we (the service provider) must decide whether or not to accept the agent. Agents can fall into one of two categories: inflexible and flexible. Inflexible agents represent typical agents in online resource allocation settings in that they demand the resource immediately. On the other hand, flexible agents are willing to wait for the resource capacity to replenish once as long as we make an upfront commitment to serving them by then. In other words, if we accept a flexible agent in period \( t \in [T] \), that agent just requires the resource by the end of time period \( t + 1 \). In addition to heterogeneity in timing flexibility, agents can also differ in the reward (or revenue) they give to the service provider. We assume that we have \( K \geq 2 \) types with rewards \( r_1 < r_2 < \ldots < r_K \), among which the first \( M \in [K - 1] \) types are flexible. This combination of heterogeneity in two dimensions (timing and reward) makes our model rich but at the same time challenging to solve.

Our goal in this work is to present algorithms that earn a high total reward in every time period under any arbitrary arrival process. In other words, we consider an adversarial arrival model and design algorithms with optimal or near-optimal competitive ratios in each time period. Adversarial arrival models have been studied widely in revenue management, online matching, and resource allocation (see e.g. [Ball and Queyranne (2009)], [Golrezaei et al. (2014)], [Elmachtoub and Levi (2015)],
Ashlagi et al. (2019), Ma et al. (2021)). Considering adversarial models lead to robust algorithms that perform well under any arbitrary arrival process.

1.1. Our Contributions

In addition to our modeling contribution, our work makes the following contributions.

**Polytope-based resource allocation (polyra) algorithm.** We present the class of POLYtope-based Resource Allocation (POLYRA) algorithms, parametrized by a feasibility polytope $B$. These algorithms make decisions by keeping track of a state consisting of $K + M$ buckets representing the decisions our algorithms have made thus far. Upon the arrival of an agent, the algorithm accepts the agent if they fit into one of its buckets without violating the feasibility conditions enforced by the polytope $B$. We show that this class of algorithms—via the design of these buckets and the choice of feasibility polytope $B$—can gracefully take advantage of agents’ temporal flexibility.

**Optimality of POLYRA algorithms with two types.** When there are two types of agents ($K = 2$) with one flexible and one inflexible type, we show that there exists a simple POLYRA algorithm that achieves the optimal competitive ratio (Theorem 1 and 2). This POLYRA algorithm makes use of a nested feasibility polytope which places booking limits on the total number of each type of agent that can be accepted. We highlight that the structure of our nested POLYRA algorithm bears some resemblance to the simple nested booking limit algorithm by Ball and Queyranne (2009), which is designed for a setting without flexibility. Our optimal 2-type algorithm, however, achieves a competitive ratio of $\frac{2}{3 - r_1/r_2}$, which is strictly greater than the $\frac{1}{2 - r_1/r_2}$ from Ball and Queyrenne’s algorithm and hence shows the value of flexibility.

**Optimality of POLYRA algorithms with three types.** When there are three types of agents ($K = 3$) with one flexible or two flexible types, we present two optimal POLYRA algorithms with feasible polytopes $B^{(1)}$ (Theorem 4) and $B^{(2)}$ (Theorem 5). To design both these feasible polytopes, we first present an upper bound on the competitive ratio of any non-anticipating resource allocation algorithm (Theorem 3). This upper bound, which is characterized by a linear program (LP), is constructed by considering four overlapping arrival sequences (across two time periods) that are truncated versions of a particular worst-case arrival sequence. Our POLYRA algorithms then mimic the optimal solution to this LP in constructing their feasible polytopes, yielding the optimal competitive ratio. Interestingly, unlike the case when $K = 2$, the feasible polytopes $B^{(1)}$ and $B^{(2)}$ are not necessarily of the simple nested structure (as per Definition 2), highlighting the challenges of exploiting agents’ temporal flexibility optimally.
Near optimality of simple nested POLYRA algorithms for more than three types.
When there are at least four types of agents, we show that a generalization of our upper bound LP from three types is no longer tight as it is difficult to characterize the worst-case input sequence. Considering this challenge, we turn back to our simple POLYRA algorithms with nested polytope and show that they are near-optimal. To do so, we first characterize a loose, but simple upper bound $\Gamma_{up}$ on the competitive ratio of any non-anticipating algorithm for any number of types (Theorem 6). Then, we show how to compute the competitive ratio of any POLYRA algorithm with nested polytope using a simple LP (Theorem 7). We then present an explicit instance of a nested polytope that is inspired by this LP, whose competitive ratio is $\Gamma_{nest}$. Finally, show that $\Gamma_{nest}/\Gamma_{up} \geq 0.8$, which proves that the class of nested polytopes is near-optimal. (Theorem 8).

We highlight that the proof of the near-optimality of our nested POLYRA algorithm, which is quite involved, uses novel ideas. It turns out that directly comparing $\Gamma_{nest}$ and $\Gamma_{up}$ is challenging because it is hard to relate the two quantities: $\Gamma_{nest}$ is given by an LP geared towards nested polytopes while $\Gamma_{up}$ is an upper bound on any algorithm’s performance and unrelated to nested polytopes. To bridge $\Gamma_{nest}$ and $\Gamma_{up}$, we define an intermediate value, denoted by $\Gamma$, which is an upper bound on the competitive ratio of any nested polytope. To bound the ratio of $\Gamma_{nest}/\Gamma_{up}$, we then bound the two following ratios: $\Gamma_{nest}/\Gamma$ and $\Gamma/\Gamma_{up}$. The first ratio ($\Gamma_{nest}/\Gamma$) measures how good our nested algorithm is, compared to the best nested algorithm that one can hope to design, and the second ratio ($\Gamma/\Gamma_{up}$) measures how well the class of nested algorithms can approximate the optimal algorithm.

Managerial insights. Our work sheds light on the value of flexibility by characterizing an upper bound on the competitive ratio of any algorithm, which rises as the number of flexible types increases; see Figure 1 (The figure shows the optimal competitive ratio with and without flexible agents when the number of types $K$ is 2 or 3). As more types become flexible, we can design algorithms that better approximate a clairvoyant algorithm. In addition, our work shows that agents’ time-flexibility can be exploited via easy-to-implement POLYRA algorithms that consult their acceptance and rejection decisions with feasible polytopes. These polytopes, which are established with the help of our constructed upper bounds, enable our POLYRA algorithms to gracefully hedge against the uncertainty in the arrival process while taking advantage of agents’ flexibility.

2. Related Works
Our work fits in with existing literature in the following research areas:
Figure 1 For $K = 2$ (left) and $K = 3$ (right), we plot the optimal competitive ratio in the case where $\frac{r_1}{r_2} = \gamma$ (for $K = 2$) and $\frac{r_1}{r_2} = \frac{r_2}{r_3} = \gamma$ (for $K = 3$). For each $\gamma$, we plot the optimal competitive ratio when the first $M$ out of $K$ types are flexible. The gap between the above curves highlights the value of flexibility, i.e., the increase in competitive ratio we can achieve by having more types be flexible.

**Single-leg revenue management.** Our work is an extension of the single-leg Revenue Management (RM) problem to a setting where agents not only have different rewards, but also different temporal flexibility levels. The single-leg RM problem, which is a fundamental RM problem, involves allocating multiple units of a single resource to arriving agents of various types (i.e., demand), each with some fixed reward. The main challenge is deciding in an online fashion how many low-reward agents to accept while making sure there is enough capacity for potential higher-reward agents. A classic result from Littlewood (1972) shows that the optimal number of lower type agents to accept can be given by a simple threshold-based policy. Extensive further work has been done on this model, with numerous variations such as adding a network resource structure and the addition of choice modeling (see e.g., Talluri and van Ryzin (1998, 2004), Perakis and Roels (2010), Golrezai et al. (2014), Gallego et al. (2015), Ke et al. (2019), Ma et al. (2020), Sun et al. (2020), and Strauss et al. (2018) for a survey).

**Adversarial demand modeling.** Most previous work in the single-leg RM problem and its variations model demand using a stochastic process (e.g., Asadpour et al. (2020), Talluri and van Ryzin (1998, 2004)) or robust uncertainty sets (e.g., Perakis and Roels (2010), Birbil et al. (2006)). In both cases, we must make certain assumptions about how demand behaves. However, in an adversarial framework with competitive ratio analysis, no assumptions about demand are made as the performance of any algorithm is given by its worst-case performance over any possible demand.
arrival. One of the first lines of work in adversarial demand modeling was \cite{Ball2009}, whose work is a special case of our model. The authors show that the optimal policy initially accepts all arriving agents, but gradually only accepts higher reward agents once certain capacity thresholds are reached. Their solution structure is similar to that of Littlewood in that both reserve a certain fraction of the capacity exclusively for higher type reward agents, but differ in that the thresholds are not set based on knowing the demand distribution, but rather on the ratios between agent rewards. Other works in this area have explored variations such as setting posted prices instead of making binary accept/reject decisions in \cite{Eren2010}, relaxing the adversarial arrival order by assuming that part of the arrivals is stochastic in \cite{Hwang2018} or by removing the assumption of independent demand classes (the idea that higher type agents never choose the lower-priced option) in \cite{Ma2021}. Our work contributes to the literature on adversarial single-leg RM problems by studying how to optimally exploit demand time-flexibility in adversarial single-leg RM problems.

Our work is also related to the literature on flexibility. Papers in this area can be divided into three streams: supply side, demand side, and time-based flexibility.

**Supply-side Flexibility.** Supply side flexibility refers to the ability of a firm to satisfy demand for a product with more than a single combination of raw goods or underlying resources. This kind of production-side flexibility has been well studied in the literature. In particular, it has been shown that a limited amount of flexibility (e.g. the long chain design) is almost as good as a fully flexible system (see e.g. \cite{Jordan1995, Simchi-Levi2012, Simchi-Levi2015, Désir2016, Chen2019, DeValve2021}). (See Petrick\ et\ al.\ (2012)\ for\ a\ survey). While most work understanding production-side flexibility has been in an offline setting, the adage “a little flexibility goes a long way” also seems to be true in the online setting from \cite{Asadpour2020}.

**Demand-side Flexibility.** On the other hand, demand-side flexibility refers to customers’ willingness to accept one of many products that are usually considered substitutes (e.g. two flights that depart within an hour). The main insights from consumer flexibility is that when customers reveal their indifference to a firm (sometimes for a discount or incentive), it greatly improves capacity utilization by helping the firm overcome uncertainty as seen in \cite{Gallego2004}.

\footnote{See also \cite{Mak2009, Lu2015, Shen2019} for works that study supply-side flexibility in the presence of disruptions to demand and supply and \cite{Armony2004, Castro2021} for works that study flexibility in queuing systems.}
In addition, the potential discount offered also induces more demand as a larger segment of the population may now be able to afford the good or service. One concrete example of demand-side flexibility is the idea of opaque products in which customers specify a bundle of specific goods that they are indifferent between (such as the color of a notebook) and the firm chooses the exact good for them Elmachtoub and Wei (2015), Elmachtoub et al. (2019).

**Time-flexible Customers.** Time-flexible (patient) customers have been studied in many applications healthcare (e.g. scheduling non-urgent elective surgeries, see Gerchak et al. (1996), Huh et al. (2013) or organ transplantation (e.g.,Bertsimas et al. (2013)) and ride-sharing (e.g. pooled rides, see Dogan and Jacquillat (2019), Jacob and Roet-Green (2017)). Time-flexible (patient) customers have also studied in the context of pricing problems (see, e.g., Katta and Sethuraman (2005), Chen and Farias (2018), Golrezaei et al. (2020), Lobel (2020), Abhishek et al. (2021)). In dynamic pricing literature with time-flexible customers, it is assumed that customer’s valuation for an item decays with their waiting time. These customers, however, time their purchasing decision strategically and the goal is to identify an optimal sequence of prices over time. This line of work investigates the impact of having time-flexible customers in pricing decisions, while in our work we investigate the benefit of time-flexible customers in resource allocation problems.

Another related line of work involves online job scheduling with deadlines (e.g. Ashlagi et al. (2019), Azar et al. (2015)), which is related to our work in that less urgent jobs will have longer deadlines. (See also Chawla et al. (2017) for work that studies pricing problems in the presence of customers with deadlines.) However, one major difference between our work and that of online job scheduling is that we assume that a commitment must be made immediately, whereas in online job scheduling, a job is allowed to be partially processed or remain idle and then rejected any time before the deadline. Finally, time-flexibility has been studied in matching problems under stochastic setting, rather than adversarial setting that we consider (see, e.g., Ashlagi et al. (2017), Aouad and Saritaç (2020), Akbarpour et al. (2020), Hu and Zhou (2021)). In these papers, customers arrive and depart according to some stochastic arrival and abandonment patterns. To design effective algorithms, one needs to optimize the time of matching while striking a balance between increasing market thickness and mitigating the risk of customers abandoning the system. One of the main differences between these papers and our work is the timing of acceptance decisions. In our setting, we need to make immediate acceptance/rejection decisions and once we accept a
customer, we are fully committed to serving the customer. In the aforementioned papers, however, both acceptance and serving decisions can be delayed as no commitment is made upon customer arrival.

The remainder of our paper is organized as follows. In Section 3 we present the model and define our notion of competitive ratio. In Section 4 we define the idea of a POLYRA algorithm and feasible polytope, which is the basis of all algorithms in this paper. In Sections 5, 6 and 7 we discuss our contributions for two types, three types, and beyond three types respectively.

3. Problem Setup
3.1. Model Definitions
Consider a service-provider who allocates her resources (i.e., drivers, server memory, or doctors’ time) to agents who arrive over the course of $T$ time periods, where $T$ is unknown to the service-provider. In each of $T$ time periods, the service-provider begins with $C$ units of a single perishable resource that do not carry over between periods. There are $K$ types of agents with rewards (or revenues) $0 < r_1 < \ldots < r_K$ where the first $M \in [K - 1]$ types are flexible, while the remaining $K - M$ types are inflexible. All agents demand a single unit of our resource upon arrival, but flexible and inflexible customers differ in when they require the resource by. The precise definition of flexible and inflexible will be discussed later. We make the assumption that the reward for any flexible agent is less than that of any inflexible agent, which is a reasonable assumption as agents who more urgently need a resource also value it more. We also assume that $M < K$ meaning that there is always at least 1 inflexible type as in most applications there is always a type of agent who urgently needs the resource.

In each time period $t \in [T]$, agents arrive one at a time in an online fashion. Let $I_t = \{z_{t,1}, z_{t,2}, \ldots\}$ be this arrival sequence in the $t$-th time period, where $z_{t,n} \in [K]$ is the type of the $n$-th arriving agent within time period $t$. We make no assumptions about $I_t$, which can be adversarially chosen. Upon the arrival of the $n$-th agent, we get to observe their type $z_{t,n} \in [K]$ and we must make an irrevocable decision of whether to accept or reject this agent. If $z_{t,n} \in \{M + 1, \ldots, K\}$ (meaning they are inflexible), then accepting the agent means that we allocate one unit of our resource to this agent and collect the associated reward $r_{z_{t,n}}$. If $z_{t,n} \in [M]$ (meaning they are flexible), then accepting the agent means that we make a commitment to allocate a resource to this agent, but it can be either one from the current time period or one of the replenished resources from the next time period $t + 1$. The key property that makes flexible agents useful is that we do not have to make
this decision immediately upon acceptance. The order of events in each time period is described more precisely as follows:

**Step 1.** Time period $t$ begins and resources replenish to $C$ capacity.

**Step 2.** Flexible agents accepted in time period $t-1$ who have not yet received a resource are each allocated a resource.

**Step 3.** Agents arrive and we make accept or reject decisions after observing their type.
- **Inflexible agents:** Accepted agents are allocated a resource from the current time period $t$.
- **Flexible agents:** Accepted agents are guaranteed a resource, but are indifferent between one from time period $t$ or $t+1$. Such agents are willing to wait until Step 2. of time period $t+1$ to receive a resource.

**Step 4.** Time period $t$ ends and leftover resources are allocated to some flexible agents accepted in this time period who have not yet been allocated a resource.

We discuss two technical assumptions about our model. First, in order to allow algorithms to take advantage of flexibility in the last period $T$, we add a dummy period $T+1$ where $I_{T+1} = \emptyset$, but we still have $C$ capacity available to serve flexible agents from period $T$. This period is necessary for otherwise flexible agents that arrive in the last period would not be any different from inflexible agents and so an adversary could choose $T = 1$, reducing us to a problem where all agents are inflexible. Second, this work will focus on the continuous version of the resource allocation problem, where we are allowed to partially accept and serve agents. Although the continuous problem is less realistic than the discrete one, its analysis is much simpler and we can still capture all the core ideas regarding time-flexibility. For the discrete model, all of the algorithms from this paper can still be applied and are asymptotically correct as $C \to \infty$. That is, if we directly apply our algorithms to the discrete setting, our guarantees hold as $C$ gets asymptotically large. In addition, to deal with small values of $C$, one can apply a technique used in Ball and Queyranne (2009) to randomly accept or reject agents, obtaining the same performance guarantees in expectation.

### 3.2. Objective and Benchmark

Our goal is to develop algorithms which achieve a high performance (rewards) under any arbitrary arrival sequence. For an algorithm $A$, arrival sequence $\{I_t\}_{t \in \{T\}}$, and $\tau \in \{T\}$, let $\text{Rew}_{A,\tau} (\{I_t\}_{t \in \{T\}})$ be the reward of algorithm $A$ in time period $\tau$. Recall from our model definition that an agent’s reward is obtained when a resource is allocated, which for flexible agents may be different from when they are accepted. As for the benchmark, we consider an *inflexible optimal clairvoyant benchmark*. 


For any \( t \in [T] \) and arrival sequence \( \{I_t\}_{t \in [T]} \), let \( \text{OPT}(I_t) \) be the optimal reward that can be obtained from \( I_t \) using \( C \) capacity. In other words, \( \text{OPT}(I_t) \) is the maximum reward that can be obtained in time period \( t \) under the arrival sequence \( I_t \) if all agents are treated as inflexible. The benchmark has full knowledge of the arrival process and selects a subset of agents in \( I_t \) that leads to the highest rewards in time period \( t \). We measure the performance of an algorithm \( \mathcal{A} \) using the following competitive ratio (CR) definition, which compares our algorithm to the inflexible optimal clairvoyant benchmark in each time period:

\[
\text{CR}_\mathcal{A} = \inf_{T, \{I_t\}_{t \in [T]}, \tau \in [T]} \left\{ \frac{\text{Rew}_{\mathcal{A}, \tau}(\{I_t\}_{t \in [T]})}{\text{OPT}(I_\tau)} \right\}.
\]

In order for an algorithm \( \mathcal{A} \) to obtain a high CR, it must have a good performance in every time period \( t \in [T] \), as \( \text{CR}_\mathcal{A} \) is governed by the worst time period.

**Benchmark Justification.** At first glance, our benchmark may seem unconventional. Typically, in online analysis, the benchmark is the optimal clairvoyant solution that knows the entire arrival sequence in advance (i.e., \( \{I_t\}_{t \in [T]} \)) and can also take advantage of agents’ flexibility. Our benchmark, as defined by equation (1), compares our algorithm’s performance to an inflexible optimal clairvoyant solution that is not able to take advantage of agents’ flexibility. We choose this benchmark because the optimal or near-optimal algorithms induced by this benchmark are able to demonstrate the value of flexibility whereas those under the conventional benchmark cannot.

Consider the case of \( M = 0 \), meaning that there are no flexible type agents. This case corresponds exactly to the model from \textbf{Ball and Queyranne (2009)}, which states that no algorithm can have a CR higher than \( L = \left( K - \sum_{i=1}^{K-1} \frac{r_i}{r_{i+1}} \right)^{-1} \). Suppose that instead of our inflexible optimal clairvoyant benchmark, we consider the flexible optimal clairvoyant benchmark. We then would expect that as \( M \) goes from 0 to \( K - 1 \) (i.e., as more agents become flexible), there would exist algorithms that break this \( L \) upper bound. However, it turns out that even when \( M = K - 1 \), when almost all the types are flexible, we cannot achieve higher CR than \( L \), which is the upper bound on CR when all types are inflexible. See Section \textbf{A} for a formal statement and details. This highlights that if we allow the benchmark to use flexibility, we make the benchmark so powerful that we cannot see the value of flexibility. Considering these limitations of using the conventional CR definition, we use our own CR-like definition from equation (1) which compares our algorithm’s performance to an inflexible benchmark. This way, no matter the number of flexible types \( M \), the benchmark’s performance for any arrival sequence stays the same, so we are able to measure to what extent the addition of flexible types improves an algorithm’s performance.
Type | 1  2  ...  M  M+1  ...  K  \\
---|---|---|---|---|---|---|
Row 1 | B_{1,1} B_{2,1} ... B_{M,1} B_{M+1} ... B_K  \\
Row 2 | B_{1,2} B_{2,2} ... B_{M,2}  \\

Figure 2  The $K+M$ buckets in our state $B$. Buckets in the first row contain agents that have been allocated a resource from the current time period, while buckets in the second row contain only flexible agents who have been accepted but not received the resource.

In addition to comparing our algorithm’s performance to an inflexible benchmark, our definition of CR from equation (1) looks at the per-period CR rather than comparing the total reward that $\mathcal{A}$ attains compared to the optimal. This choice is mostly for analytical convenience as the resulting algorithms are much easier to describe and interpret. Additionally, considering this benchmark results in an algorithm that has an equally good performance across all the time periods. We note that if $\mathcal{A}$ achieves a per-period CR of $\Gamma$ by equation (1), then it is also the case that:

$$\inf_{T, \{I_t\}_{t \in [T]}} \frac{\sum_{\tau \in [T]} \text{REW}_{\mathcal{A}, \tau} (\{I_t\}_{t \in [T]})}{\sum_{\tau \in [T]} \text{OPT}(I_{\tau})} \geq \Gamma,$$

implying that the cumulative reward of $\mathcal{A}$ is also at least $\Gamma$ times that of the optimal.

We now proceed to describe the general structure of all algorithms we present in this work.

4. Polytope-based Resource Allocation (POLYRA) Algorithms

In this work, we present a class of algorithms called Polytope-based resource allocation (POLYRA) algorithms. The key to such algorithms is a well defined state $B \in \mathbb{R}_{+}^{K+M}$ that encapsulates all the decisions our algorithms have made so far in past time periods. This state must be able to keep track of a number of variables such as the number of resources remaining, how resources have already been allocated, and the number of flexible type agents that have been accepted and have not yet received their resource. Each POLYRA algorithm instance is parameterized by a feasible polytope $\mathcal{B} \subset \mathbb{R}_{+}^{K+M}$ which intuitively describe the set of all states that our algorithm is allowed to visit. A POLYRA algorithm will make all of its decisions by consulting the current state $B$ and the feasible polytope $\mathcal{B}$ while maintaining the invariant that $B \in \mathcal{B}$. Note that for any given number of types $K$ and flexible types $M$, the state definition is the same regardless of the rewards $r_1, \ldots, r_K$, but the specific feasible polytope $\mathcal{B}$ will certainly depend on the rewards. We now describe in detail the state, as well as, how we use the state to make decisions.

**State.** At any point in time, our algorithm keeps track of a state $B$ which consists of the following $K+M$ dimensional vector of non-negative numbers:

$$B = (B_{1,1}, B_{2,1}, \ldots, B_{M,1}, B_{M+1}, \ldots, B_K; B_{1,2}, B_{2,2}, \ldots, B_{M,2}).$$
We will refer to each index of the state as a bucket. These buckets are illustrated more visually in Figure 2. As agents are accepted from the arrival sequence, they are assigned to these buckets. Note that $B_{k,j}$, $k \in [M]$ and $j \in [2]$, or $B_k$, $k \in \{M + 1, \ldots, K\}$, simply refers to the number of agents that have been assigned to that particular bucket.

- **First row.** In any given time period $t$, the first row of buckets (i.e., $(B_{1,1}, B_{2,1}, \ldots, B_{M,1}, B_{M+1,1}, \ldots, B_K)$) keeps track of the number of agents of the $K$ types that have been allocated one of the $C$ resources from this time period. For $k \in [M]$, the bucket $B_{k,1}$ contains type $k$ agents who have been allocated a resource from the current time period. Some of these agents may have been accepted in time period $t - 1$ and allocated the resource from step 2 of time period $t$. Others may be flexible agents that arrived in time period $t$ whom we decided to allocate a resource to, even though they would be willing to wait until time period $t + 1$. For $k \in \{M + 1, \ldots, K\}$, the bucket $B_k$ simply contains type $k$ agents whom we have allocated one of this period’s $C$ resources to. Since such agents are inflexible, they must have been accepted in the current time period $t$.

- **Second row.** Each bucket in the second row $B_{k,2}$, $k \in [M]$, is the number of agents of flexible type $k$ that we have accepted but have not yet allocated a resource to. These buckets only exist for the flexible types, and it is clear that any agent assigned to $B_{k,2}$ must have arrived in the current time period $t$.

Recall from our 4 step process in Section 3.1 that any algorithm must describe how it performs steps 3 and 4. Step 3 represents the acceptance decision, which deals with this idea of a feasible polytope. Step 4 involves deciding how to allocate leftover resources at the end of each time period.

**Acceptance Rule and Feasible Polytope.** Upon the arrival of an agent of type $k$, we will accept them if and only doing so can lead to a state that remains feasible in $B$. We define that more precisely as follows:

- **Inflexible Types.** If type $k \in \{M + 1, \ldots, K\}$, we accept them if bucket $B_k$ can be incremented such that the new state is feasible. If this is possible, then we accept this type $k$ agent, allocate them a resource from the current time period $t$ and then update the state accordingly by increasing $B_k$ by one.\footnote{More precisely, under our continuous model, we find the largest value of $\epsilon \leq 1$ such that: $B + \epsilon \cdot e_k \in B$, where $e_k$ is a length $K + M$ vector with all 0’s except a 1 at the $k$-th index.}

- **Flexible Types.** If type $k \in [M]$, then we consider two possibilities, which correspond to assigning this agent to bucket $B_{k,2}$ or $B_{k,1}$. First, we consider assigning the agent to bucket
which corresponds to seeing whether the state $B' = B + e_{K+k}$ is feasible (i.e., $B' \in \mathcal{B}$). If so, we accept this type $k$ agent, assign them to bucket $B_{k,2}$, but we do not allocate a resource to them at the moment (which is in line with the definition of all agents in the second row, i.e., $B_{k,2}$). The state updates to $B'$. If $B' \notin \mathcal{B}$, then we proceed to the second option which is to try assigning this incoming type $k$ agent to bucket $B_{k,1}$. If $B'' = B + e_k$ is feasible, then we assign this agent to $B_{k,1}$, allocate a resource from the current time period to them, and then update the state to $B''$. If both options fail, then we reject the agent.

**Leftover Resource Allocation and Updating Bucket State $B$.** At the end of time period $t$, our algorithm will use any remaining resources it has, which is equal to $C - (B_{i,1} + \ldots + B_{M,1} + B_{M+1} + \ldots + B_K)$, to serve flexible agents assigned to buckets $B_{1,2}, \ldots, B_{M,2}$ (i.e. the second row) by prioritizing agents with higher rewards. (Recall that we already allocated resources to any agents assigned to the buckets in the first row.) In other words, we go in order from $B_{M,2}, \ldots, B_{1,2}$ and allocate resources to as many agents as possible until we run out of either resources or agents. Let $\hat{B}_{i,2}, i \in [M]$, be the number of agents in $B_{i,2}$ who have not received a resource in this process. As we transition from time period $t$ to time period $t+1$, our resources replenish to $C$ capacity. We then allocate a resource to each of the $\hat{B}_{i,2}$ agents of type $i$ (Step 3 from Section 3.1). We then update our state $B$ in the following way:

$$\hat{B} = (\hat{B}_{1,2}, \hat{B}_{2,2}, \ldots, \hat{B}_{M,2}, 0, \ldots, 0, \ldots, 0).$$

We then allocate a replenished resource to all the agents assigned to $\hat{B}_{1,2}, \hat{B}_{2,2}, \ldots, \hat{B}_{M,2}$ and collect the corresponding rewards. Note that this reward is collected in time period $t+1$. Our new state $\hat{B}$ is consistent with our state definition, that all agents assigned to the first row have already been allocated a resource from the current time period $t+1$. We highlight that the updated buckets state $\hat{B}$ should be also feasible, i.e., $\hat{B} \in \mathcal{B}$. Such a crucial property may not be satisfied for any feasible polytope $\mathcal{B}$, but it is satisfied by all of the feasibility polytopes we consider in this paper. Another properties of the polytopes we consider in this paper is that if we have two states $B$ and $B'$ such that $B \preceq B'$ (coordinate-wise), then $B' \in \mathcal{B}$ implies $B \in \mathcal{B}$. That is, we would like $\mathcal{B}$ to be a downward closed polytope. This property says that our polytopes are designed so that decreasing

---

3 More precisely under our continuous model, we will try to first maximize $\epsilon_1 \leq 1$ such that $B + e_{k+K} \cdot \epsilon_1$ is feasible. We accept $\epsilon_1$ fraction of this incoming agent to bucket $B_{k,2}$. If $\epsilon_1 < 1$, then we proceed to do the same for bucket $B_{k,1}$ by finding the largest $\epsilon_2 \leq 1 - \epsilon_1$ such that $B + e_k \cdot \epsilon_2$ is feasible. We accept $\epsilon_2$ fraction of our incoming agent into bucket $B_{k,1}$. Any remaining fractional part of this agent is rejected.
any coordinate of a feasible state maintains feasibility. One consequence of this property is that if \( B \) cannot accommodate a positive amount of an incoming type \( j \) agent, then we can be certain that \( B' \) also cannot. In other words, if a particular type \( j \) cannot be accommodated in \( B \) early on in the arrival sequence \( I_t \), then certainly it will also not be accommodated later in \( I_t \) when more agents of other types may have been accepted. The two properties discussed above are summarized in the definition below.

**Definition 1 (Consistent Polytopes).** A polytope \( B \) is consistent if:

(i) For any time period \( t \in [T] \), and (feasible) buckets state \( B \), the updated buckets state after applying our serving rule (i.e., \( \hat{B} \) presented in equation (2)) remains feasible, i.e., \( \hat{B} \in B \).

(ii) For any state \( B \) and \( B' \) such that \( B \preceq B' \) (coordinate-wise), if \( B' \in B \), then \( B \in B \). In other words, lowering the values in a feasible \( B' \) continues to make the state feasible.

With the definition of POLYRA algorithms in hand, we now proceed to discuss how POLYRA algorithms are optimal for our model with two types.

5. Optimal Algorithm with Two Types

Consider a special case of our model with \( K = 2 \) types, where type 1 is flexible and type 2 is inflexible. We will show that the optimal algorithm, in this case, is indeed a POLYRA algorithm with a very simple feasible polytope \( B \). The proof of all the statements in this section is presented in Section B.

5.1. Upper Bound

We start with the following theorem that provides an upper bound on the CR of any non-anticipating algorithm.

**Theorem 1 (Upper Bound on the Competitive Ratio with Two Types).** Suppose that there are two types with reward of \( r_i \), \( i \in [2] \), where \( r_1 < r_2 \). If type 1 is flexible and type 2 is inflexible, then for any non-anticipating algorithm \( A \), we have \( CR_A \leq \frac{2}{3-r_1/r_2} \).

This upper bound is constructed using a technique that will be elaborated in the proof sketch of Theorem [2] in which we present a similar upper bound for 3 types of agents. We now show that this upper bound is tight by presenting a POLYRA algorithm which attains the CR of \( \frac{2}{3-r_1/r_2} \).
5.2. Optimal Algorithm

Consider a POLYRA algorithm with a feasible polytope $B^f$, defined below:

$$B^f = \left\{ (b_{1,1}, b_{2}, b_{1,2}) \in \mathbb{R}_+^3 : b_{1,i} \leq \frac{C}{3 - r_1/r_2}, \ i \in [2], \ b_{1,1} + b_2 \leq C \right\}.$$  \hspace{1cm} (3)

Here, superscript $f$ in $B^f$ stands for “flexible.” Note that we use the lowercase letters $b_{1,1}, b_{2}, b_{1,2}$ as dummy variables in the definition of our polytopes, while the capital letters $B_{1,1}, B_2, B_{1,2}$ are reserved when we talk about states or the buckets themselves.

**Theorem 2 (Optimal Algorithm with Two Types).** Suppose that there are two types with reward of $r_i$, $i \in [2]$, where $r_1 < r_2$ with type 1 being flexible and type 2 being inflexible. Then, the POLYRA algorithm with feasible polytope $B^f$ obtains a CR of $\frac{2}{3 - r_1/r_2}$, and hence is optimal.

We can interpret the polytope $B^f$ in the following way. Recall that $B_{1,1}$ and $B_2$ keep track of the number of resources (out of $C$) from the current time period that we have already allocated to type 1 and 2 agents respectively, while $B_{1,2}$ keeps track of the number of type 2 agents that we have accepted but not yet allocated a resource to. The constraint $b_{1,1} \leq \frac{C}{3 - r_1/r_2}$ makes sure that in any given time period, we do not allocate more than a $\frac{1}{3 - r_1/r_2}$ fraction of our resources to type 1 agents. We also have the same condition for $b_{1,2}$ in order to make sure we do not commit to too many flexible agents that may be served in time period $t+1$.

To build some insights, we now compare this POLYRA algorithm with polytope $B^f$ (when $K = 2$ and $M = 1$) with the optimal algorithm from Ball and Queyranne (2009) (when $K = 2$ and $M = 0$). Based on Theorem 1 from Ball and Queyranne (2009), the optimal algorithm for $K = 2$ and $M = 0$ is also a POLYRA algorithm with the following polytope:

$$B^i = \left\{ (b_1, b_2) \in \mathbb{R}_+^2 : b_1 \leq \frac{C}{2 - r_1/r_2}, \ b_1 + b_2 \leq C \right\}.$$  \hspace{1cm} (4)

and achieves a CR of $\frac{1}{2 - r_1/r_2}$. Here, the superscript $i$ in $B^i$ stands for “inflexible.”

We see that when type 1 agents are flexible (i.e. $M = 1$), we are able to achieve a $\frac{2}{3 - r_1/r_2}$ CR through polytope $B^f$, while when type 1 agents are not flexible (i.e. $M = 0$), we can only obtain $\frac{1}{2 - r_1/r_2}$ through polytope $B^i$. The secret to $B^f$ lies in the bucket $B_{1,2}$. When type 1 agents are assigned to bucket $B_{1,2}$, we are tentatively scheduling those agents to be served with the next time period capacity, meaning they do not preclude type 2 agents from being served in the current time period. If many type 2 agents arrive, then bucket $B_2$ will be full of type 2 agents and we will end up having very few leftover resources at the end of the time period and truly delaying the
type 1 agents in $B_{1,2}$ to the next time period. On the other hand, if few type 2 agents arrive and $B_{1,1} + B_2 < C$, we can allocate the remaining resources to the type 1 agents in bucket $B_{1,2}$. This highlights the value of flexibility: the type 1 agents assigned to bucket $B_{1,2}$ allow our algorithm to hedge against the uncertainty in the number of arriving high-reward agents (i.e., type 2 agents).

We further highlight that the feasible polytopes $B^f$ and $B^i$, both have a nice simple structure by enforcing a cap on the number of type 1 agents as well as the total number of type 1 and type 2 agents in each row. Both polytopes restrict the number of type 1 agents that can be accepted in order to leave enough room for type 2 agents. This kind of structure can be extended to an arbitrary number of types $K$ and flexible types $M$ as we do in the next section.

5.3. Nested Polytopes

**Definition 2 (Nested Polytopes).** When there are $K$ types with the first $M$ flexible, for any $(n_1, \ldots, n_K)$ with $0 \leq n_1 \leq \ldots \leq n_K = C$, we define $B_{\text{nest}}(n_1, n_2, \ldots, n_K)$ to be the set of

$$(b_1, b_{2,1}, \ldots, b_{M,1}, b_{M+1,1}, \ldots, b_{K,1}; b_{1,2}, b_{2,2}, \ldots, b_{M,2}) \subset \mathbb{R}^{K+M}$$

that satisfy the following $K+M$ constraints:

$$\sum_{i=1}^{k} b_{i,j} \leq n_k \quad k \in [M], j \in [2]$$

$$\sum_{i=1}^{M} b_{i,1} + \sum_{i=M+1}^{k} b_{i} \leq n_k \quad k \in \{M+1, \ldots, K\} \quad (B_{\text{nest}}(n_1, n_2, \ldots, n_K))$$

A nested polytope $B_{\text{nest}}(n_1, n_2, \ldots, n_K)$ essentially imposes booking limits on the $C$ resources we have in each time period, restricting type $k$ agents to only $n_k$ out of the $C$ resources. To enforce this, the polytope $B_{\text{nest}}(n_1, n_2, \ldots, n_K)$ must impose constraints on both the number of assigned agents to the buckets in the first row, i.e., $b_{i,1}, i \in [M]$ and $b_{i}, i \in \{M+1, \ldots, K\}$, as well as, the number of assigned agents to the buckets in the second row, i.e., $b_{i,2}, i \in [M]$. Since the first row of buckets captures exactly agents that have been allocated a resource from the current time period, we certainly need the constraints $\sum_{i=1}^{k} b_{i,1} \leq n_k$ and $\sum_{i=1}^{M} b_{i,1} + \sum_{i=M+1}^{k} b_{i} \leq n_k$.

Not only are nested polytopes easy to interpret, but also they are parameterized by just $K$ variables $(n_1, n_2, \ldots, n_K)$. As we have seen above, when $K = 2$, regardless of $M = 0$ or $M = 1$, the optimal algorithm is a POLYRA algorithm with nested polytope. Furthermore, Theorem 4 from Ball and Queyranne (2009) implies that nested polytopes are indeed optimal when $M = 0$ (i.e. there are no flexible types) for any number of types. A natural question to ask is whether nested
polytopes are optimal in general when flexible agents are present. Unfortunately, as we show in the next section, even with three types \((K = 3)\) and one flexible type \((M = 1)\), the nested POLYRA algorithm is no longer optimal. Nonetheless, as we show in Section 7, such a simple POLYRA algorithm enjoys a strong performance guarantee for an arbitrary number of types.

**Lemma 1 (Consistency of Nested Polytopes).** Nested polytopes are consistent as per Definition 7.

*Proof of Lemma 7.* Nested polytopes satisfy property (i) from Definition 1 because nested polytopes have the same constraints duplicated across the two rows of buckets (see the first constraint from Definition 2). Recall that property (i) requires that for any feasible state \(B = (B_{1,1}, \ldots, B_{M,1}, B_{M+1,1}, \ldots, B_{K,B_1}, B_{1,2}, \ldots, B_{M,2}) \in B\), that the state \(\hat{B} = (B_{1,2}, \ldots, B_{M,2}, 0, 0, \ldots, 0)\) to be feasible as well. The first constraint from Definition 2 (i.e., \(\sum_{i=1}^{k} b_{i,j} \leq n_{k}, k \in [M], j \in [2])\) is clearly satisfied as the same constraint are duplicated cross the first two rows of \(B\). The second constraint (i.e., \(\sum_{i=1}^{M} b_{i,1} + \sum_{i=M+1}^{k} b_{i} \leq n_{k}, k \in \{M + 1, \ldots, K\}\)) is satisfied because \(B_{1,2} + \ldots + B_{M,2} \leq n_{M}\) and that \(n_{M} \leq n_{k}\) for \(k \geq M\).

Furthermore, it is clear that nested polytopes satisfy property (ii) from Definition 1 because all the constraints of a nested polytope restrict the sum of the \(b_{i,j}\) or \(b_{i}\)'s to be at most a certain value, so clearly reducing \(b_{i,j}\) or \(b_{i}\) maintains feasibility. \(\square\)

6. Optimal Algorithm with Three Types

In this section, we consider a special case with \(K = 3\) and \(M \in \{1, 2\}\). Our main results in this section is an explicit characterization of two polytopes \(B^{(1)}\) and \(B^{(2)}\) which give optimal POLYRA algorithms for the case when \(M = 1\) and \(M = 2\), respectively. We present an innovative technique where we first construct an upper bound on the CR of any online algorithm through a linear program (LP), and then use its optimal solution values to build our polytopes \(B^{(1)}\) and \(B^{(2)}\).

6.1. An Upper Bound for Three Types

In order to prove the optimality of our algorithms for \(K = 3\), we first need to establish an upper bound. To do so, for each of \(M = 1\) and \(M = 2\), we construct worst-case arrival sequences, which best exemplify the main challenge that an online algorithm must face: the trade-off between a moderate reward now versus waiting for a potentially higher future reward. We then argue that to obtain a CR of \(\Gamma\) on these worst-case arrival sequences, the behavior of non-anticipating algorithm needs to satisfy certain conditions. These conditions lead to an LP construction, where the objective of the LP maximizes the CR \(\Gamma\) and the constraints enforce the aforementioned conditions.
**Theorem 3 (Upper Bound on the CR with Three Types).** Suppose that there are three types with \( M \in \{1, 2\} \) and rewards \( r_1 < r_2 < r_3 \). Then, the CR of any non-anticipating algorithm \( A \), i.e., \( CR_A \), is at most the optimal value of the following LP, denoted by \( \Gamma^* \).

\[
\Gamma^* := \max_{\Gamma, \{s_{i,t}\} \in [3], t \in [2]} \Gamma
\]

s.t. \( s_{1,t} + s_{2,t} + s_{3,t} \leq C \) \( t \in [2] \) \hfill (5)

\[
\Gamma \cdot r_3 \cdot C \leq s_{1,t} \cdot r_1 + s_{2,t} \cdot r_2 + s_{3,t} \cdot r_3 \quad t \in [2] \hfill (6)
\]

(upper3)

\[
\Gamma \cdot r_1 \cdot C \leq (s_{1,1} + s_{1,2}) \cdot r_1 \hfill (7)
\]

\[
\Gamma \cdot r_2 \cdot C \leq \left(C - \sum_{t=1}^{M} s_{2,t}\right) \cdot r_1 + \left(\sum_{t=1}^{M} s_{2,t}\right) \cdot r_2 \hfill (8)
\]

\[
\Gamma \cdot r_2 \cdot C \leq (s_{1,1} + s_{1,2}) \cdot r_1 + \left(\sum_{t=1}^{M} s_{2,t}\right) \cdot r_2 \hfill (9)
\]

\[
\Gamma \cdot r_2 \cdot C \leq s_{1,1} \cdot r_1 + s_{1,2} \cdot r_2 \quad \text{if } M = 1 \hfill (10)
\]

Note that in LP \((\text{upper3})\) and all other LPs presented in this paper, we assume that all decision variables are non-negative. Next, we present a proof sketch of Theorem 3. The complete proof of this theorem and the proof of all other statements in this section are presented in Section C.1.

**Proof Sketch of Theorem 3.** In this proof sketch, we outline the general structure of all the upper bounds presented in this paper. Our upper bound construction takes advantage of the fact that any non-anticipating algorithm \( A \) does not know the future, and therefore is unable to differentiate between two arrival sequences \( I \) and \( I' \) until the first index at which they differ. Using this idea, we build our upper bounds around a collection of overlapping multi-period arrival sequences \( I \), and argue that any non-anticipating algorithm \( A \) that claims to be \( \Gamma \)-competitive must satisfy certain conditions when dealing with each arrival sequence \( \{I_t\}_{t \in [T]} \in I \). We write down these conditions in terms of a linear program, maximize the value of \( \Gamma \), and get our upper bound.

To demonstrate our idea, let us consider an example where \( K = 3, M = 1, C = 3 \). Our proof can be extended to any value of \( C \) and also for \( M = 2 \). Consider the following collection of arrival sequences: \( \{I_1, I_2\} \), as well as its truncated forms \( \{I_1^1, I_2^1\}, \{I_1^2, I_2^2\}, \{I_1^3, I_2^3\}, \{I_1^4, I_2^4\} \) (each row is single arrival sequence over two time periods)

\[
I_1 = \{1,1,1,2,2,2,3,3,3\} \quad I_2 = \{2,2,2,3,3,3\}
\]

\[
I_1^1 = \{1,1,1\} \quad I_2^1 = \{}
\]

\[
I_1^2 = \{1,1,1\} \quad I_2^2 = \{}
\]
\[
I_1^2 = \{1,1,1,2,2,2\} \quad I_2^2 = \{\}
\]
\[
I_1^3 = \{1,1,1,2,2,2,3,3,3\} \quad I_2^3 = \{\}
\]
\[
I_1^4 = \{1,1,1,2,2,2,3,3,3\} \quad I_2^4 = \{2,2,2\}
\]

We chose the worst-case arrival sequence \(\{I_1, I_2\}\) such that all arriving agents must be served within two time periods (hence not having flexible type 1 in \(I_2\)). In each of these two time periods, we send agents in increasing order of reward and \(C\) copies at a time. This allows our worst-case arrival sequence to capture the core challenge of online resource allocation: the tension between committing to lower reward agents early on and waiting for higher reward agents that might not arrive. The purpose of the truncated arrival sequences \(\{I_1^j, I_2^j\}\) for \(j \in [4]\) is to capture this core challenge by considering overlapping arrival sequences. For example, suppose \(\mathcal{A}\)'s behavior on \(\{I_1, I_2\}\) is to simply accept no type 1 or type 2 agents and accept all 3 type 3 agents in both time periods achieving exactly the optimal reward. However, such a behavior would achieve a reward of 0 if the arrival sequence were instead \(\{I_1^1, I_2^3\}\). This is because \(\mathcal{A}\) has no way of differentiating between \(\{I_1^1, I_2^3\}\) and \(\{I_1, I_2\}\) until after all type 1 agents arrive, so we can assume that the number of type 1 agents that \(\mathcal{A}\) accepts from \(I_1\) or \(I_1^1\) is the same.

Using the ideas from the example above, we now give a sketch of how the constraints in \(\text{(upper3)}\) come about. Suppose we are given an algorithm \(\mathcal{A}\) which is \(\Gamma\)-competitive. Let \(s_{i,t}, i \in [3], t \in [2]\), be the number of type \(i\) agents that have received one of the \(C\) resources in time period \(t\) under \(\mathcal{A}\). Since types 2 and 3 are inflexible, \(s_{i,t}\) also represents the number of type 2 and 3 agents that \(\mathcal{A}\) accepts from \(I_t\), while for type 1, \(s_{1,1} + s_{1,2}\) is the number of type 1 agents that \(\mathcal{A}\) accepts from \(I_1\). Since \(\text{OPT}_{i}(\{I_1, I_2\}) = r_3 \cdot C\), to be \(\Gamma\)-competitive, we must have \(\text{REW}_{\mathcal{A},t}(\{I_1, I_2\}) = r_1 \cdot s_{1,t} + r_2 \cdot s_{2,t} + r_3 \cdot s_{3,t} \geq \Gamma \cdot r_3 \cdot C\), which gives rise to constraint \(\text{(6)}\). In addition, we certainly must have \(s_{1,t} + s_{2,t} + s_{3,t} \leq C\), as given in \(\text{(5)}\). We apply a similar reasoning to arrival sequences \(\{I_1^1, I_2^3\}, \{I_1^2, I_2^2\}\) and \(\{I_1^4, I_2^4\}\), to get the \(\Gamma \cdot r_i \cdot C\) (for \(i = 1\) or \(2\)) reward lower bounds in constraints \(\text{(7)}, \text{(9)}, \text{(8)}, \text{and} \text{(10)}\). Very crucially, we use the fact that since these arrival sequences are truncated versions of \(\{I_1, I_2\}\), and so \(s_{i,t}\) also define \(\mathcal{A}\)'s behavior on \(\{I_1^1, I_2^3\}, \{I_1^2, I_2^2\}\) and \(\{I_1^4, I_2^4\}\).

\(\Box\)

We will now state two POLYRA algorithms which achieve the \(\Gamma^*\) CR from the upper bound, one for the case where \(M = 1\) and another for \(M = 2\). Both of these optimal algorithms require us to

\footnote{Note that \(\{I_1^1, I_2^3\}\) is actually not useful as it would imply the constraint \(r_1 \cdot s_{1,1} + r_2 \cdot s_{2,1} + r_3 \cdot s_{3,1} \geq \Gamma \cdot r_3 \cdot C\), which was already captured by \(\{I_1, I_2\}\).}
first compute the upper bound presented above, get an optimal solution \((\Gamma^*, \{s_{i,t}^*\}_{i \in [3], t \in [2]}\) and use some of these values as constants in the feasible polytopes \(B^{(1)}\) for \(M = 1\) and \(B^{(2)}\) for \(M = 2\). We will require some properties of this optimal upper bound solution, which are described in Lemmas \([6, 7]\) in Sections \([C.2.1, C.2.2]\). We highlight that the upper bound only characterizes an optimal algorithm’s behavior against the worst-case arrival sequences, not any arbitrary arrival sequences over \(T\) periods. Nonetheless, we show that how to use the upper bound solution, which represents a particular way of dealing with the worst-case arrival sequence, to construct an optimal algorithm for any arrival sequence.

6.2. Optimal Three Type Algorithm for \(M = 1\)

With \(M = 1\), type 1 agents are flexible while types 2 and 3 are inflexible. Hence, our state \(B\) contains 4 buckets as shown below:

| Type | 1      | 2      | 3      |
|------|--------|--------|--------|
| Row 1| \(B_{1,1}\) | \(B_{2}\) | \(B_{3}\) |
| Row 2| \(B_{1,2}\) |

Let \(\Delta_{i,j} := r_i - r_j\) and \((\Gamma^*, \{s_{i,t}^*\}_{i \in [3], t \in [2]}\) be the optimal solution to the upper bound LP \([\text{UPPER3}]\). We define our feasible polytope \(B^{(1)}\) as follows

\[
B^{(1)} = \left\{(b_{1,1}, b_{1,2}, b_2, b_3) \in \mathbb{R}_+^4 : \begin{array}{l}
\frac{b_{1,1} + b_2 + b_3}{b_{1,1}} \leq s_{1,1}^* + s_{2,1}^* + s_{3,1}^* \\
\Delta_{3,1} \cdot b_{1,1} + \Delta_{3,2} \cdot b_2 \leq \Delta_{3,1} \cdot s_{1,1}^* + \Delta_{3,2} \cdot s_{2,1}^* \\
\frac{b_{1,2}}{i \in [2]} \leq s_{1,i}^*
\end{array} \right\}
\]

Theorem 4 (Optimality of POLYRA Algorithm with Feasible Polytope \(B^{(1)}\)). When \(K = 3\) and \(M = 1\), the POLYRA algorithm with feasible polytope \(B^{(1)}\) achieves a CR of \(\Gamma^*\), and hence, is optimal. Here, \(\Gamma^*\) is the optimal objective value of the LP \([\text{UPPER3}]\).

Discussion on Feasible Polytope \(B^{(1)}\). Here, we give some intuition about the constraints of \(B^{(1)}\) and why it achieves a CR of \(\Gamma^*\). The first constraint, i.e., \(b_{1,1} + b_2 + b_3 \leq s_{1,1}^* + s_{2,1}^* + s_{3,1}^*\) simply enforces that we do not allocate more resources in a time period than \(s_{1,1}^* + s_{2,1}^* + s_{3,1}^*\) which is bounded above by \(C\) as per constraint \([5]\) of the upper bound LP \([\text{UPPER3}]\). In fact, as we show in property \([a]\) of Lemma \([6]\) we actually have \(s_{1,1}^* + s_{2,1}^* + s_{3,1}^* = C\). The purpose of the second constraint, i.e., \(\Delta_{3,1} \cdot b_{1,1} + \Delta_{3,2} \cdot b_2 \leq \Delta_{3,1} \cdot s_{1,1}^* + \Delta_{3,2} \cdot s_{2,1}^*\), is to leave enough space in bucket \(B_3\) for type 3 agents, which we do so by enforcing that a linear combination of the number of agents assigned to buckets \(B_{1,1}\) and \(B_{1,2}\) is not too large. Finally, the last constraint, i.e., \(b_{1,2} \leq s_{1,2}^*, i \in [2]\), simply makes sure that our buckets do not contain too many type 1 agents in order to leave enough room for the type 2 and 3 agents. Observe that the feasible polytope \(B^{(1)}\) is not a nested polytope.
as per Definition 2. In particular, in order to obtain the optimal CR $\Gamma^*$, the algorithm has to keep track of a linear combination of agents assigned to buckets $B_{1,1}$ and $B_2$ (i.e., $\Delta_{3,1} \cdot B_{1,1} + \Delta_{3,2} \cdot B_2$), rather than simply the sum of the agents assigned to these two buckets (i.e., $B_{1,1} + B_2$). Recall that in a nested polytope, one only needs to keep track of the total number of agents assigned to the first $i \in [3]$ buckets.

It can be shown that if we apply our POLYRA algorithm with feasible polytope $\mathcal{B}^{(1)}$ to the worst-case arrival sequence from $\{I_1, I_2\}$ as defined in the proof sketch of Theorem 3, then the behavior of this algorithm will exactly correspond to the optimal solution $\{s^*_i,t\}$ in that our algorithm will serve exactly $s^*_i,t$ agents of type $i$ in period $t \in [2]$. In this way, our POLYRA algorithm mimics the upper bound when it encounters the worst-case arrival sequence. It turns out from Theorem 4 our algorithm that mimics the upper bound optimal solution via the feasible polytope $\mathcal{B}^{(1)}$ also achieves a $\Gamma^*$ CR on any arrival sequence.

We now comment on the proof idea of Theorem 4. First, we note that the polytope $\mathcal{B}^{(1)}$ is consistent as per Lemma 8 which is proven in Section C.3.1. This is crucial because recall that POLYRA algorithms are only well defined when the feasible polytope is consistent. Since our polytope $\mathcal{B}^{(1)}$ contains many terms from the optimal upper bound LP solution (i.e. $\{s^*_i,t\}_{i \in [3], t \in [2]}$), we also need to show some properties of this optimal upper bound solution, which are stated in Lemmas 6, 7 and proven in Sections C.2.1, C.2.2. We then show that our proposed POLYRA algorithm obtains the optimal CR against any arrival sequence through the following casework. Suppose that at the end of time period $t$, before allocating the leftover resources, that our state is $\mathcal{B}$ and let $\ell \in \{0, 1, 2, 3\}$ be the highest type of agent for which all the available space in $\mathcal{B}$ for type $\ell$ agents is occupied. In other words, if we were to encounter one more type $\ell$ agent, we would completely reject that agent. We define $\ell = 0$ if $\mathcal{B}$ can accommodate another agent of any type and still remain feasible. By our definition of $\ell$, it must be the case that $\mathcal{B}$ is able to accommodate an agent of type $k$ for all $k > \ell$. By property (ii) of consistency (Definition 1), this implies that our algorithm has not rejected a single agent of type $k$, which means that our algorithm is doing just as well as the optimal solution when it comes to type $k$ agents. For types $k' < \ell$, we may have rejected some agents that the optimal clairvoyant solution accepted, but we show that we have accepted enough such agents of type $k' < \ell$ to be $\Gamma^*$-competitive.

We now describe our polytope construction for three types when $M = 2$. 
6.3. Optimal Three Type Algorithm for $M = 2$

With $M = 2$, types 1, 2 agents are flexible while type 3 agents are inflexible. Hence, our state $B$ contains 5 buckets as shown below:

| Type  | 1 | 2 | 3 |
|-------|---|---|---|
| Type 1| $B_1,1$ | $B_2,1$ | $B_3$ |
| Type 2| $B_1,2$ | $B_2,2$ |

To construct the optimal algorithm for three types with two flexible types, we again first compute the optimal solution $(\Gamma^*, \{s_{i,t}^*\}_{i \in [3], t \in [2]})$ from the upper bound LP in Theorem 3 and use it to construct our polytope $B^{(2)}$, which is defined in the following theorem.

**Theorem 5.** For the case where $K = 3$ and $M = 2$, the POLYRA algorithm with the nested polytope $B^{(2)}$, defined below, achieves a CR of $\Gamma^*$, and hence, is optimal:

$$B^{(2)} := B_{nest} \left( \frac{1}{2} \cdot (s_{1,1}^* + s_{1,2}^*), \frac{1}{2} \cdot (s_{1,1}^* + s_{1,2}^* + s_{2,1}^* + s_{2,2}^*), C \right).$$

Here, $(\Gamma^*, \{s_{i,t}^*\}_{i \in [3], t \in [2]})$ is the optimal solution to the upper bound LP and nested polytopes $B_{next}$ are defined in Definition 2.

**Discussion on the nested polytope $B^{(2)}$ and the optimal algorithm.** With two flexible types (i.e., $M = 2$), the optimal algorithm takes a simple form in the sense that it can be represented by a nested POLYRA algorithm as per Definition 2. This polytope guarantees that all feasible states have $B_{1,t} \leq \frac{1}{2} \cdot (s_{1,1}^* + s_{1,2}^*), t \in [2]$ and $B_{1,t} + B_{2,t} \leq \frac{1}{2} \cdot (s_{1,1}^* + s_{1,2}^* + s_{2,1}^* + s_{2,2}^*), t \in [2]$, meaning that it imposes a limit on the total number of type 1 agents in each row, as well as the total number of type 1 and 2 agents. Just like with 1 flexible type, one can show that if we apply our POLYRA algorithm with feasible polytope $B^{(2)}$ to the corresponding worst-case arrival sequence for $M = 2$, our POLYRA algorithm will exactly serve $s_{i,t}^*$ agents of type $i \in [3]$ in time period $t \in [2]$. In this sense, we again mimic our upper bound solution as a template for making sure our algorithm is $\Gamma^*$-competitive against the worst-case arrival. It turns out that this worst-case arrival sequence is indeed the worst-case as achieving a $\Gamma^*$ CR on it implies a $\Gamma^*$ CR on all arrival sequences. To prove that such a nested polytope achieves the desired CR of $\Gamma^*$, we will apply Theorem 7 which uses an LP to characterize the CR of any nested polytope, including ones for $K > 3$ types.

We now see that while $B^{(1)}$ is not a nested polytope, $B^{(2)}$ is. Recall from the earlier discussion that $B^{(1)}$ is not nested because it has a constraint of the form $c_1 \cdot b_{1,1} + c_2 \cdot b_2 \leq c_3$ where $c_1 \neq c_2$. On the other hand, all of $B^{(2)}$’s constraints involve a linear combination of variables with coefficients all equal to 1. This can be explained by the following intuition: Both polytopes $B^{(1)}$ and $B^{(2)}$ have
constraints which make sure that we do not accept too many type 1 or 2 agents, thus making sure enough space is reserved for type 3 agents. When type 1 is flexible and type 2 is not (i.e. \( M = 1 \)), this constraint is of the form \( \Delta_{3,1} \cdot b_{1,1} + \Delta_{3,2} \cdot b_2 \) where the coefficients \( \Delta_{3,1} \) and \( \Delta_{3,2} \) are different. This is because type 1 and 2 differ in their flexibility so our linear combination must weigh the two differently. On the other hand, when both type 1 and type 2 are flexible (i.e \( M = 2 \)), restricting the simple sum of \( b_{1,1} \) and \( b_{2,1} \) is sufficient.

7. A Near-optimal Nested POLYRA Algorithm for \( K > 3 \)

We proceed with the case where \( K > 3 \) and \( M \in [K - 1] \). We will show by example that for \( K > 3 \), our intuitive overlapping worst-case arrival sequences from Theorem 3 do not lead to an attainable CR. Considering this, we present a simple POLYRA algorithm with nested polytope which is near-optimal, achieving a CR of at least a 0.8 fraction of the optimal CR. The proof of all the statements of this section is presented in Section D.

7.1. Our Upper Bound is No Longer Tight

The crux of our upper bound constructions have been the choice of the worst-case arrival sequence and the resulting collection of truncations that it generates. For \( K = 2 \) and 3, by presenting algorithms whose CR matches exactly those upper bounds, we showed that the upper bound is tight/attainable.

For \( K > 3 \) types, our upper bound construction can easily be generalized, but we will see from the following example that doing so may not lead to a tight upper bound. To show this, we present alternative worst-case arrival sequences under which we may get a tighter upper bound than the bound obtained from extending our worst-case arrival sequences in Theorem 3.

**Example 1 (Overlapping arrival sequences do not lead to a tight upper bound).**

Suppose that we have \( K = 4 \) types with rewards \( r_1 < r_2 < r_3 < r_4 \) and \( M = 2 \) (i.e., types 1 and 2 are flexible, while types 3 and 4 are inflexible). For notational convenience, we choose \( C = 3 \). If we use the same reasoning as before, our worst-case arrival sequence \( \{I_1, I_2\} \) is

\[
I_1 = \{1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4\} \quad I_2 = \{3, 3, 3, 4, 4, 4\}
\]

and its collection of truncations is

\[
\begin{align*}
I_1^1 &= \{1, 1, 1\} & I_1^2 &= \{\} \\
I_1^2 &= \{1, 1, 1, 2, 2, 2\} & I_1^3 &= \{\} \\
I_1^3 &= \{1, 1, 1, 2, 2, 2, 3, 3, 3\} & I_1^4 &= \{\} \\
I_1^4 &= \{1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4\} & I_1^5 &= \{\} \\
I_1^5 &= \{1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4\} & I_1^6 &= \{3, 3, 3\}
\end{align*}
\]

We present two alternative choices of worst-case arrival sequences as follows:
• **Removing type(s).** Consider using the following worst-case arrival sequence

\[
\mathcal{T}_1 = \{1, 1, 1, 3, 3, 3, 4, 4, 4\}, \mathcal{T}_2 = \{3, 3, 3, 4, 4, 4\}
\]

The difference between \(\{\mathcal{T}_1, \mathcal{T}_1\}\) and \(\{I_1, I_2\}\) is that this new worst-case arrival sequence (and its truncations) do not send any type 2 agents. This worst-case arrival sequence leads to a tighter upper bound when \(r_2\) is close to \(r_3\). Consider an extreme case where \(r_2 = r_3\) (or are arbitrarily close). If that is the case, then an adversary should not send any type 2 agents because type 2 agents are flexible, while type 3 agents are not, yet they achieve (nearly) the same reward. In general for \(K > 4\), for some reward values \(r_1, \ldots, r_K\) we can construct a tighter worst-case arrival sequence by choosing an index \(j \in [M]\) and removing types \(j, j+1, \ldots, M\) from the original worst-case arrival sequence.

• **Considering three time periods, rather than two.** Consider the following worst-case arrival sequence.

\[
\tilde{I}_1 = \{1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4\}, \tilde{I}_2 = \{1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4\}, \tilde{I}_3 = \{3, 3, 3, 4, 4, 4\}
\]

The difference between \(\{\tilde{I}_1, \tilde{I}_2, \tilde{I}_3\}\) and \(\{I_1, I_2\}\) is that this new worst-case arrival sequence sends agents in ascending order of reward over three time periods rather than two. Let \(\Gamma^*\) be the upper bound given by considering the original worst case arrival sequence \(\{I_1, I_2\}\) while \(\tilde{\Gamma}\) is the upper bound when we consider \(\{\tilde{I}_1, \tilde{I}_2, \tilde{I}_3\}\) as the worst case. Notice that \(\tilde{I}_1 = I_1\) and so any non-anticipating algorithm must behave the same in the first time period against both worst-case arrival sequences.

Suppose \(\mathcal{A}\) is an algorithm that is \(\Gamma^*\) competitive. After accepting and serving some subset of the 12 agents in \(I_1\), \(\mathcal{A}\) begins \(t = 2\) with say \(c_1\) unserved type 1 agents and \(c_2\) unserved type 2 agents (i.e. the accepted flexible agents from \(t = 1\) that we did not serve). Under our original second time period arrival sequence \(I_2\), we enforce that this commitment of \(c_1\) type 1 and \(c_2\) type 2 agents does not preclude \(\mathcal{A}\) from achieving a \(\Gamma^*\) CR when the worst case arrival sequences continues as \(I_2 = \{3, 3, 3, 4, 4, 4\}\). However, it may be the case that if the arrival sequence were to continue as \(\tilde{I}_2\) and \(\tilde{I}_3\), then this commitment of \(c_1\) type 1 agents and \(c_2\) type 2 agents makes it impossible for any algorithm to be \(\Gamma^*\) competitive anymore against all the truncations of \(\{\tilde{I}_2, \tilde{I}_3\}\): perhaps we can only hope to be \(\tilde{\Gamma} < \Gamma^*\) competitive. In other words, depending on the commitment \(c_1, c_2\), it is possible that starting in that state, \(\{\tilde{I}_2, \tilde{I}_3\}\) more...
accurately depicts the worst-case scenario than \{I_2\}. In general, it is unclear if there is an upper bound to the number of time periods we need to consider in order to get the tightest upper bound.

As a concrete numerical example, if we take the case where \( r_1 = 1, r_2 = 3, r_3 = 4, r_4 = L \), then as \( L \to \infty \) our old upper bound gives \( \frac{20}{90} = 0.45 \) whereas taking the tighter of the two new worst-case arrival sequences shown above yields \( \frac{15}{34} \approx 0.4411 < 0.45 \).

Remark. The upper bounds created by these new worst-case arrival sequences may be only tighter than our original upper bound only when \( K > 3 \) because for \( K = 2 \) and \( K = 3 \), there are too few types for some of these nuances to show. These new worst-case arrival sequences may be tighter when there are tensions between multiple flexible and inflexible types. With \( K = 2 \) or \( K = 3 \), since there is always either just 1 flexible type or 1 inflexible type, we do not see such nuances.

Seeing how a similar upper bound to the ones presented for \( K = 2 \) and \( K = 3 \) is not tight for \( K > 3 \), we turn out attention to presenting a POLYRA algorithm with nested polytope which has a CR at least \( 0.8 \cdot \Gamma^* \). Here, \( \Gamma^* \) is the best CR that any algorithm can obtain. To do this, we will do the following:

- In Section 7.2, we present a different (looser) upper bound \( \Gamma_{up} \geq \Gamma^* \), which admits a nice closed-form solution that will be useful for analysis (Theorem 6).
- In Section 7.3, we present a formula (in terms of an LP) which computes the CR for any instance of the nested POLYRA algorithm (Theorem 7).
- In Section 7.4, inspired by the LP from Theorem 7, we present a specific instance of the nested POLYRA algorithm that has a simple closed-form solution for its nest sizes \( (\pi_1, \ldots, \pi_K) \). We show that this particular instance of the nested polytope attains at least 80% of the optimal CR given by the upper bound from Theorem 6.

7.2. Simplified Upper Bound with a Closed-form Solution

Since our goal for \( K > 3 \) is to present an instance of the nested polytope that achieves at least 80% of the optimal CR, we present a general upper bound for the CR of any online algorithm with \( K > 3 \) types. As we saw from Example 11 it is difficult to come up with a tight upper bound when \( K > 3 \). As a result, we propose an upper bound in Theorem 6 which is not tight, but is simple in that it exhibits a closed-form solution that is convenient for analysis.
**Theorem 6 (Simple Upper Bound on the CR with K Types).** Suppose that there are \( K \) types with \( M \in [K-1] \) flexible agent types and reward of \( r_i, i \in [K] \), where \( 0 < r_1 < r_2 < \ldots < r_K \).

Let \( G \in (1, K - M) \) be defined as follows:

\[
G := K - M - \sum_{i=M+2}^{K} \frac{r_{i-1}}{r_i}.
\]  

(12)

The CR of any non-anticipating algorithm \( A \), i.e., \( \text{CR}_A \), is less than or equal to the following quantity

\[
\Gamma_{\text{up}} := \min\left( \Gamma_{LP}, \frac{1}{G} \right),
\]

(13)

where \( \Gamma_{LP} \) is defined as the optimal value of the following LP, denoted by \((\text{SIMPLE-UPPER})\).

\[
\Gamma_{LP} := \max_{\Gamma} \Gamma \cdot r_k \cdot C \leq \sum_{i=1}^{k} r_i \cdot s_i
\]

\[
s.t. \quad \Gamma \cdot r_k \cdot C \leq \sum_{i=1}^{M} r_i \cdot s_i + \sum_{i=M+1}^{k} r_i \cdot s_i \quad k \in [M]
\]

\[(\text{SIMPLE-UPPER})\]

\[
2 \cdot \Gamma \cdot r_k \cdot C \leq 2 \cdot \sum_{i=1}^{M} r_i \cdot s_i + \sum_{i=M+1}^{k} r_i \cdot s_i \quad k = M + 1, \ldots, K - 1
\]

\[
2 \cdot \Gamma \cdot r_K \cdot C \leq \sum_{i=1}^{M} r_i \cdot s_i + \sum_{i=M+1}^{K} r_i \cdot s_i
\]

\[
\sum_{i=1}^{K} s_i \leq 2 \cdot C
\]

(16)

(17)

Furthermore, \( \Gamma_{LP} \) admits the following closed-form solution

\[
\Gamma_{LP} = \frac{2}{2 \cdot G + M - \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i} - \frac{r_M}{r_{M+1}} + \frac{r_M}{r_K}}.
\]

(18)

**Intuition for Theorem 6.** Theorem 6 shows that the optimal CR of any algorithm cannot exceed \( \Gamma_{up} = \min\left( \Gamma_{LP}, \frac{1}{G} \right) \), where \( G = K - M - \sum_{i=M+2}^{K} \frac{r_{i-1}}{r_i} \) and \( \Gamma_{LP} \) is defined in equation \((\text{SIMPLE-UPPER})\).

The term \( \frac{1}{G} \) in the definition of \( \Gamma_{up} \) comes from Theorem 4 of Ball and Queyranne (2009), which shows that \( \frac{1}{G} \) is an upper bound on the CR of any online algorithm in the situation when you have \( K - M \) types with rewards \( r_{M+1}, \ldots, r_K \) all of which are inflexible. For our model with \( M \) flexible types and \( K - M \) inflexible types, an upper bound on the performance of any online algorithm can be obtained by an adversary who ignores the \( M \) flexible types, reducing our problem to the aforementioned case with \( K - M \) inflexible types. Therefore, \( \frac{1}{G} \) is an upper bound on the performance of any online algorithm.
We now discuss the intuition behind the term $\Gamma_{LP}$ and why the corresponding LP $\textit{(SIMPLE-UPPER)}$ is truly an upper bound on the CR of any non-anticipating algorithm. We further provide insights into how we construct LP $\textit{(SIMPLE-UPPER)}$ so that it admits a closed-form solution.

Recall the intuition from the proof sketch of the 3-type upper bound (Theorem 3). We considered worst-case arrival sequences which send agents in ascending order of reward over $T = 2$ time periods. We then used truncated versions of these arrival sequences to argue that any non-anticipating algorithm cannot fully reject the lower reward agents that arrive initially. The LP $\textit{(SIMPLE-UPPER)}$ is constructed using similar worst-case arrival sequences. In particular, we consider the following worst-case arrival sequence $\{I_1, I_2\}$ and their truncated versions:

- $I_1 = \{1, \ldots, 1, \ldots, K, \ldots, K\}$
- $I_2 = \{M + 1, \ldots, M + 1, \ldots, K, \ldots, K\}$.

Although the worst-case arrival sequences in constructing LP $\textit{(SIMPLE-UPPER)}$ are similar to those used in construction of LP $\textit{(UPPER3)}$, there is an important difference between these two LPs. To construct LP $\textit{(SIMPLE-UPPER)}$ for a non-anticipating algorithm $\mathcal{A}$, we define $s_i$ for $i \in [K]$ to be the total number of type $i$ agents that algorithm $\mathcal{A}$ allocates a resource to over the $T = 2$ time periods. This differs from the our variables from $\textit{(UPPER3)}$ in that the latter defined variables $s_{i,t}$, $i \in [3]$, $t \in [2]$ for the number of resources from time period $t$ that are allocated to type $i$ agents.

This important change of variables in LP $\textit{(SIMPLE-UPPER)}$ allows us to relax some of the constraints imposed in LP $\textit{(UPPER3)}$ leading to a closed-form solution in equation (18). In fact, this closed-form solution is obtained by noticing that the constraint matrix for $\textit{(SIMPLE-UPPER)}$ is upper triangular. This implies that we can solve for its optimal solution in closed form by iteratively solving for $s_1, s_2, \ldots, s_K$.

### 7.3. An LP Characterization of the CR of any Nested POLYRA Algorithms

From Theorem 1 ($K = 2$) and Theorem 5 ($K = 3, M = 2$), we have seen two examples of nested polytopes in action. In each theorem, we gave a particular instance of the nested polytope and showed that it achieves the optimal CR. Now, with $K > 3$ types, we present a general way of computing the CR of any given instance of the nested polytope. That is, for any given values of $n_1, \ldots, n_K$ such that $0 \leq n_1 \leq n_2 \leq \ldots \leq n_K$, we show how to compute the CR achieved by a POLYRA algorithm with feasible polytope $\mathcal{B}_\text{nest}(n_1, \ldots, n_K)$. 
Theorem 7 (CR of Nested POLYRA Algorithms). For any given values of \( n_1, \ldots, n_K \) such that \( 0 \leq n_1 \leq n_2 \leq \cdots \leq n_K = C \), the CR of the nested POLYRA algorithm with polytope \( B_{nest}(n_1, \ldots, n_K) \) is given by \( \Gamma_{nest}(\{ n_i \}_{i \in [K]}) \), which is defined by the following LP

\[
\Gamma_{nest}(\{ n_i \}_{i \in [K]}) := \max_{\Gamma, \{ s_i^j \}_{j \in [K], i \in [j]}} \Gamma \\
\text{s.t.} \quad \Gamma \cdot C \cdot r_j \leq \sum_{i=1}^j s_i^j \cdot r_i \quad j \in [K] \\
(NEST) \quad \sum_{i=1}^j s_i^j \leq C \quad j \in [K] \\
\Delta n_j \leq s_i^j \leq 2 \cdot \Delta n_j \quad j \leq M, i \in [j] \\
\Delta n_j \leq s_i^j \leq 2 \cdot \Delta n_j \quad j > M, i \in [j].
\]

Here, \( \Delta n_i = n_i - n_{i-1} \) and \( n_0 \) is defined to be zero. When \( \{ n_i \}_{i \in [K]} \) is understood from the context, we omit it and just write \( \Gamma_{nest} \).

Intuition for Theorem 7 and its proof sketch. We argue that we can characterize the CR of any POLYRA algorithm with nested polytope \( B_{nest}(n_1, \ldots, n_K) \) by considering \( K \) worst-case scenarios. We argue in the full proof that if our POLYRA algorithm with nested polytope performs well with respect to each of these \( K \) worst-case scenarios (i.e., obtains a CR of \( \Gamma \) in these worst-case scenarios), each associated with a state \( B^j \), \( j \in [K] \), then it is guaranteed to perform well with respect to any arbitrary arrival sequence (i.e., obtains a CR of \( \Gamma \) for any arbitrary arrival sequence).

We define these states \( B^j \) and then provide some intuition about their construction. The state \( B^j \) associated with scenario \( j \in [K] \) is defined as follows:

- If \( j \in [M] \), then the state \( B^j_{i,1} = B^j_{i,2} = \Delta n_i \) for \( i \in [j] \) and all other buckets are empty.
- If \( j \in \{ M + 1, \ldots, K \} \), then we have \( B^j_{i,1} = \Delta n_i \) for all \( i \in [M] \), \( B^j_i = \Delta n_i \) for all \( i \in \{ M + 1, \ldots, j \} \) and all other buckets are empty.
For each $j \in [K]$, scenario $B^j$ is a state which cannot accommodate any more type $j$ agents, and out of all states for which this is true, $B^j$ contains a set of agents that the minimizes its reward by maxing out on the number of type 1, 2, ..., $j - 1$ agents in that order. Given such a state $B^j$, imagine that time period $t$ has just ended and we are in state $B^j$. We now can allocate the $C - B^j_{1,1} - \cdots - B^j_{M,1} - B^j_{M+1} - B^j_K$ remaining resources to agents assigned to $B^j_{1,2}, \ldots, B^j_{M,2}$ and collect their reward. Now, let $s^j_i$ be the total number of type $i$ agents that have been allocated one of the current time period’s resources in scenario $j$. Then, the reward that is obtained in this time period can be written as $\sum_{i=1}^j r_i \cdot s^j_i$. In order for our choice of nest sizes $n_1, \ldots, n_K$ to produce a $\Gamma$-competitive algorithm, we claim in the theorem that $\sum_{i=1}^j r_i \cdot s^j_i \geq \Gamma \cdot C \cdot r_j$ must be true, leading to constraint (19). In addition, we must have that $\sum_{i=1}^j s^j_i \leq C$ as we can allocate no more than $C$ resources in total in scenario $j$, which gives rise to constraint (20). Finally, the remaining two constraints (21) and (22) are just bounds on the range of $s^j_i$.

We demonstrate these worst-case scenarios using an example in Figure 3. Here, $K = 4, M = 2$ and $C = 5$ with $(n_1, n_2, n_3, n_4) = (2, 3, 4, 5)$ and thus $(\Delta n_1, \Delta n_2, \Delta n_3, \Delta n_4, \Delta n_5) = (2, 1, 1, 1)$. The four worst-case scenario states $B^1, B^2, B^3, B^4$ are shown, where the numbers in the 6 boxes for each state are exactly the values of $B^j_{1,1}, B^j_{2,1}, B^j_{3,1}, B^j_{1,2}, B^j_{2,2}, j \in [3]$ and the blue shaded boxes are simply buckets with non-zero agents assigned to them. For each of these 4 states, we can compute the revenue that an algorithm would achieve in a time period if $B^j, j \in [4]$ were the final state (before leftover capacity is allocated). For example, with state $B^2 = (2, 1, 0, 0, 2, 1)$, the first row, i.e., $(2, 1, 0, 0)$, tells us we allocated 2 resources to type 1 agents and 1 resource to type 2 agents, and thus have $5 - 2 - 1 = 2$ remaining resources, which we greedily allocate to the one type 2 agent from the second row and one of the two type 1 agents from the second row. This gives a total revenue of $3 \cdot r_1 + 2 \cdot r_2$, which must be at least $\Gamma \cdot r_2 \cdot 5$ for our nested polytope to be $\Gamma$ competitive.

### 7.4. A Near-Optimal Nested POLYRA Algorithm for More than Three Types.

We present the following instance of a nested POLYRA algorithm, which we show is near optimal by comparing its performance to that of the upper bound from Theorem 6. Our near-optimal nested POLYRA algorithm has a closed form solution for the size of its nests, denoted by $\pi_1, \ldots, \pi_K$. In particular, in our nested POLYRA algorithm, $\Delta n_i = \pi_i - \pi_{i-1}$, $i \in [K]$, is given as follows:

$$
\Delta n_i := \begin{cases} 
\frac{1}{2} \cdot \Gamma \cdot \left(1 - \frac{r_{i+1}}{r_i}\right) \cdot C & i \in [M] \\
\Gamma \cdot \left(1 - \frac{r_{i+1}}{2r_i}\right) \cdot C & i = M + 1 \\
\Gamma \cdot \left(1 - \frac{r_{i+1}}{r_i}\right) \cdot C & i = M + 2, \ldots, K
\end{cases}
$$

(23)
where we define
\[ \Gamma := \frac{2}{2 \cdot G + M - \sum_{i=1}^{M+1} \frac{r_{i+1}}{r_i}}. \tag{24} \]

Here, we define \( r_0 = 0 \) and recall that \( G = K - M - \sum_{i=M+2}^{K} r_i \).

**Intuition for the near-optimal nested algorithm.** We now provide some intuition on where the specific values of \( n_i, i \in [K] \), come from. Recall that our LP \( \text{LP} \) can be used to calculate the CR of any instance of the nested polytope. That is, by plugging in \( \{n_i\}_{i \in [K]} \) as constants into \( \text{LP} \), the optimal objective value gives us exactly \( \Gamma_{\text{nest}}(\{n_i\}_{i=1}^K) \), which is the CR of a POLYRA algorithm with nested polytope \( \mathcal{B}_{\text{nest}}(n_1, \ldots, n_K) \). On the other hand, if we treat \( \{n_i\}_{i \in [K]} \) as variables instead of constants in \( \text{LP} \), then we can solve the following optimization problem to compute the choice of \( 0 \leq n_1 \leq \ldots \leq n_K = C \) that achieves the highest CR \( \Gamma^*_{\text{nest}} \).

\[
\Gamma^*_{\text{nest}} := \max_{n_1 \leq \ldots \leq n_K} \Gamma_{\text{nest}}(n_1, \ldots, n_K) \tag{25}
\]

The second line above holds because our LP \( \text{LP} \) is linear in \( \Delta n_1, \ldots, \Delta n_K \) and \( n_1, \ldots, n_K \).

Unfortunately, \( \Gamma^*_{\text{nest}} \) does not admit any kind of closed-form solution and is thus inconvenient for analysis. However, if we remove constraint (20) from the above LP, then the resulting LP solution does have a nice closed-form solution, which is precisely \( \Gamma \). That is, \( \Gamma \) and \( \{\pi_i\}_{i=1}^K \) are optimal solutions to the following LP:

\[
\Gamma = \max_{\Gamma, \{n_i\}_{i=1}^K, \{s'_{ij}\}_{j \in [K], i \in [j]}} \Gamma \text{ s.t. } (19), (21), (22), \ n_1 \leq \ldots \leq n_K = C. \tag{27}
\]

Although by relaxing constraint (20), we get an LP that admits a closed-form solution, its optimal objective value \( \Gamma \) only provides an upper bound for the CR of any nested polytope instance. This is stated formally in Lemma 2 below, whose proof can be found in Section D.3.

**Lemma 2 (Upper Bound on the CR of any Nested Alg.).** Let \( \Gamma \) and \( \{\pi_i\}_{i=1}^K \) be defined as per equation (24) and \( \Delta \pi_i = \pi_i - \pi_{i-1}, i \in [K] \), is defined in equation (23). Then, \( \Gamma \) is the optimal solution to the LP in equation (27) and hence \( \Gamma \) is an upper bound on the CR of any nested algorithm.

We highlight with the help of \( \Gamma \), we show the main result of this section, that the CR of our nested POLYRA algorithm is at least \( 0.8 \cdot \Gamma_{up} \).
Theorem 8 (A Near-optimal Nested POLYRA Algorithm). Consider a nested POLYRA
algorithm with the feasible polytope $B_{\text{nest}}(\pi_1, \ldots, \pi_K)$, where $\Delta_{\pi_i} = \pi_i - \pi_{i-1}, i \in [K]$, is defined in equation (23). Then, the CR of this algorithm is at least 80% of the optimal CR. More precisely, $\Gamma_{\text{nest}}(\pi_1, \ldots, \pi_K)$ can be lower bounded as follows
\[
\Gamma_{\text{nest}}(\pi_1, \ldots, \pi_K) \geq 0.8 \Gamma_{\text{up}},
\]
where $\Gamma_{\text{up}}$, defined in Theorem 6, is an upper bound on the CR of any non-anticipating algorithm.

Proof of Theorem 8. For notational convenience, we abbreviate $\Gamma_{\text{nest}}(\pi_1, \ldots, \pi_K)$ as just $\Gamma_{\text{nest}}$. To show that $\Gamma_{\text{nest}} \geq 0.8 \Gamma_{\text{up}}$, as stated earlier, we crucially take advantage of the intermediate quantity $\overline{\Gamma}$, defined in (24). We show that the following two inequalities hold
\[
\frac{\Gamma_{\text{nest}}}{\overline{\Gamma}} \geq f_1(G) \quad \frac{\overline{\Gamma}}{\Gamma_{\text{up}}} \geq f_2(G)
\]
for some functions $f_1, f_2$ which will be defined in Lemmas 3 and 4 below. The ratio $\Gamma_{\text{nest}}/\overline{\Gamma}$ measures how close the CR of our nested algorithm ($\Gamma_{\text{nest}}$) is to $\overline{\Gamma}$, which is the upper bound on the CR that any nested algorithm can obtain. Furthermore, the ratio $\overline{\Gamma}/\Gamma_{\text{up}}$ measures how well the class of nested algorithms approximates the optimal algorithm. We highlight that as one of our novel technical contributions, the lower bounds $f_1, f_2$ we provide for the aforementioned ratios are both function of the variable $G = K - M - \sum_{i=M+2}^{K} \frac{r_{i-1}}{r_i}$, rather than being constants. (Recall that $1/G$ is an upper bound on the CR of any online algorithm in the situation when you have $K - M$ types with rewards $r_{M+1}, \ldots, r_K$ all of which are inflexible.) Bounding these ratios as a function of $G$ is necessary as $\min_G f_1(G) \approx 0.816$ and $\min_G f_2(G) = 0.8$, which implies that simply bounding the two ratios with constants would not be sufficient. (See Figure 4 for illustrations of these functions.) We show in Lemma 5 that $f_1(G) \cdot f_2(G) \geq 0.8$ for all $G$ which completes the proof. The statements of Lemmas 3, 4 and 5 are below and their proofs can be found in Section D.3.

Lemma 3 ($\Gamma_{\text{nest}}$ vs. the Upper Bound on the CR of any Nested Alg. ($\overline{\Gamma}$)). Let $\overline{\Gamma}$ be the quantity defined in equation (24). Then,
\[
\frac{\Gamma_{\text{nest}}}{\overline{\Gamma}} \geq f_1(G) := \begin{cases} 
1 & 2 \cdot \overline{\pi}_M \leq C \\
1 - \frac{1}{2} e^{-2 \cdot G + 1} & 2 \cdot \overline{\pi}_M > C
\end{cases}
\]
where $G$ is defined in equation (12).
Figure 4  Plots of $f_1(G)$ and $f_2(G)$ in the two cases: $2 \cdot \pi_M \leq C$ and $2 \cdot \pi_M > C$. We can see that $f_1(G) \cdot f_2(G) \geq 0.8$ for all values of $G$, whose domain is $(1, K - M)$. In this example, we have $K - M = 5$.

**Lemma 4 (Upper Bound on the CR of any Nested Alg. ($\Gamma$) vs. that of any Alg. ($\Gamma_{up}$)).** Let $\hat{\Gamma}$ be the quantity defined in equation (24) and $\Gamma_{up} = \min (\Gamma_{LP}, \frac{1}{C})$, defined in equation (12), be an upper bound on the CR of any non-anticipating algorithm. Then,

$$\frac{\Gamma}{\Gamma_{up}} \geq f_2(G) := \begin{cases} 1 - \frac{1 - \max(2 - G, 0)}{2G + 1 - \max(2 - G, 0)} & 2 \cdot \pi_M \leq C \\ 1 - \frac{1 - \max(2 - G, 0)}{4G - 2} & 2 \cdot \pi_M > C \end{cases}$$

(29)

where $G$ is defined in equation (12).

**Lemma 5 (Near-optimality of our Nested Alg.).** For any value of $G \in (1, K - M)$, we have that $f_1(G) \cdot f_2(G) \geq 0.8$, where $G$ is defined in equation (12).

**Discussion of Lemmas 3, 4, and 5** These lemmas crucially use two functions $f_1(\cdot), f_2(\cdot)$ that are depicted in Figure 4. Recall that $f_1(G)$ is a lower bound on the ratio $\Gamma_{nest}/\Gamma$ and $f_2(G)$ is a lower bound on the ratio $\Gamma/\Gamma_{up}$. The definitions of these functions are given in a piecewise fashion based on the relationship between $2 \cdot \pi_M$ and $C$. Recall from our near-optimal nested polytope definition in equation (23) that $\pi_M$ is the amount of each time period’s resources that we allow flexible agents of type $k \in [M]$ to receive. Thus, $2 \cdot \pi_M$ is the maximum number of type $k \in [M]$ agents that we could have in our state $B$ at any time. Our definition of $f_1(\cdot)$ says that if $2 \cdot \pi_M \leq C$, then $f_1(G) = 1$ for all $G$ meaning that $\Gamma_{nest} = \Gamma$. This is true because when $2 \cdot \pi_M \leq C$—as we show in the proof of Lemma 3—constraint (20), which differentiates $\Gamma_{nest}$ and $\Gamma$, is not binding.

In this case, by achieving this upper bound of $\Gamma$, we know that our nested POLYRA algorithm with

---

5 In this case, in LP [NEST] we can set $s'_j = 2 \cdot \Delta n_i$ for all $i \leq j \leq M$ and $s'_i = \Delta n_i$ for all $i \leq j$ and $j > M$ while maintaining feasibility of constraint (20) (i.e. $\sum_{i=1}^n s'_i \leq C$).
nested polytope $B_{\text{nest}}(\bar{\pi}_1, \ldots, \bar{\pi}_K)$ achieves the highest competitive ratio over all nested polytopes. Otherwise, if $2 \cdot \bar{\pi}_M > C$, then $\Gamma_{\text{nest}} < \Gamma$ and $f_1(G)$ characterizes that gap induced by adding constraint (20), which shrinks exponentially as $G$ increases. To see this, we observe that $2 \cdot \bar{\pi}_M$ is inversely proportional to $G^4$. Therefore, as $G$ increases, $2 \cdot \bar{\pi}_M$ decreases, which brings us closer to the case where $2 \cdot \bar{\pi}_M \leq C$ in which $f_1(\cdot) = 1$.

As for $f_2(G)$, which is a lower bound on the ratio $\Gamma / \Gamma_{\text{up}}$, note that $f_2(\cdot)$ measures how well the class of POLYRA algorithms with nested polytopes can approximate the optimal algorithm. Figure 3 shows that when $G \in (1, K - M)$ is either close to 1 or $K - M$, $f_2(G)$ gets large values, implying that for very small or large values of $G$, nested algorithms can better approximate the optimal algorithm. In the following, we discuss each case separately.

Recall that $G = K - M - \sum_{i=M+2}^{K} \frac{r_i}{r_i}$. When $G$ is small (i.e., close to 1), then $r_i, i \in \{M + 1, \ldots, K\}$ (i.e. the rewards of the inflexible types) are all very close to each other, which is similar to having just a single inflexible type. It turns out that the class of nested algorithms performs well when the number of inflexible types is small. We saw an example of this for 3 types. When only one type inflexible (i.e. $M = 2$), a nested POLYRA algorithm is optimal, while when two types is inflexible (i.e $M = 1$), then the optimal polytope was not nested. On the flip side, when $G$ is large, then $\Gamma_{\text{up}} = \min(\Gamma_{\text{LP}}, 1/G)$ is small. This means it is difficult for any algorithm to perform well and makes it easier for nested POLYRA algorithms to perform relatively well.

8. Conclusion
Taking advantage of various types of flexibility has been shown to be effective in many settings such as job scheduling, online retail, and ride-sharing. Our work presents agents who are patient in terms of timing of their service and give the service provider more flexibility in how to allocate their scarce resources. While it is certain that flexibility is beneficial to the service provider, our work seeks to answer the question of how to optimally take advantage of this flexibility. To do so, we proposed an online resource allocation model with a single replenishing resource and heterogeneous agent types, some of which require a resource immediately upon arrival while others are willing to wait for a short time. Our problem setting is unique in that for flexible agents, we must make an upfront commitment as to whether they will receive the resource, but we can delay the decision of

$^6$ We can write out $2 \cdot \bar{\pi}_M$ explicitly as: $2 \cdot \sum_{i=M+1}^{M} \frac{r_i}{r_i} = \Gamma \cdot (M - \sum_{i=1}^{M} \frac{r_i-1}{r_i}) \cdot C$. We notice that $\Gamma$ is inversely proportional to $G$ by equation (24) and $(M - \sum_{i=1}^{M} \frac{r_i-1}{r_i})$ does not overlap in any of the $\{r_i\}_{i \in [K]}$ with $G$. (Recall that $G$ only depends on $r_i, i \geq M + 1$). Thus, $2 \cdot \bar{\pi}_M$ is inversely proportional to $G$. 

which exact resource to allocate to them. We explored an adversarial arrival model and proposed a class of polyra algorithms that perform optimally or near-optimally under a CR analysis. When the number of agent type is small (i.e. 2 or 3), we are able to achieve the optimal CR through a tight upper bound, which guides the construction of optimal polytopes. When the number of types exceeds 3, we show that a simple class of polytopes, the nested polytopes, achieve a near-optimal CR and provide an explicit characterization of the near-optimal instance.

Our work also opens up a number of avenues for future investigation. While flexible customers in our model are willing to wait only a single time period, one can imagine extending our model so that certain flexible customers are willing to wait two or more time periods. As another natural future direction, one can consider a setting in which agents are not inherently flexible or inflexible, but make a choice between the two depending on the price of each service. While the exact details of the algorithm for these new directions may be context-specific, the general insights provided in this work could be valuable in these settings as well.

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Appendix A: Benchmark Justification

For any input sequence \( \{I_t\}_{t \in [T]} \), we define \( \text{opt-total}(\{I_t\}) \) to be the maximum total reward that a clairvoyant algorithm could achieve on the input \( \{I_t\}_{t \in [T]} \) where this clairvoyant algorithm is also able to utilize agents’ flexibility. Then, the more conventional CR definition for an algorithm \( A \) is given by:

\[
\text{CR}_A = \inf_{\{I_t\}_{t \in [T]}} \left\{ \frac{\sum_{\tau=1}^T \text{REW}_{A,\tau}(\{I_t\}_{t \in [T]})}{\text{opt-total}(\{I_t\}_{t \in [T]})} \right\}.
\]  

(30)

Here, the numerator is calculating the total reward over \( T \) periods of our algorithm \( A \), while the denominator is the total reward of the optimal clairvoyant solution that knows \( \{I_t\}_{t \in [T]} \) in advance. More importantly, this clairvoyant solution utilizes agents flexibility.

Fix revenue values \( r_1 < \ldots < r_K \) and define

\[
L = \frac{1}{K - \sum_{i=1}^{K-1} \frac{r_i}{r_{i+1}}}
\]

to be the upper bound from Theorem 4 of Ball and Queyranne (2009), which states that \( L \) is an upper bound on the performance of any non-anticipating algorithm when there are no flexible types (i.e. \( M = 0 \)).

We show the following theorem, states that under the benchmark given by equation (30).

**Theorem 9.** For any \( K \geq 2 \) and \( M = K - 1 \), no non-anticipating algorithm \( A \) can have CR (as given by (30)) greater than \( L \), i.e. \( \text{CR}_A \leq L \), where \( L = \frac{1}{K - \sum_{i=1}^{K-1} \frac{r_i}{r_{i+1}}} \).

This theorem implies that under this benchmark defined above, we cannot see any value in flexibility in the case where almost every type is flexible. By making our benchmark able to use flexibility, it becomes such a strong benchmark that it neutralizes any additional competitive that a flexible algorithm might hope to obtain.

**Proof of Theorem 9.** For \( j \in [K] \), let \( I_j \) consists of simply \( 2 \cdot C \) copies of all the types from 1 through \( j \) inclusive in ascending order. For example, \( I_2 \) would consist of \( 2 \cdot C \) type 1 agents followed by \( 2 \cdot C \) type 2 agents.

We will use the following \( K \) worst-case input sequences \( I^j = \{I^j_1, I^j_2\} \):

\[
I^1_1 = I_1 \quad I^1_2 = \emptyset \\
I^2_1 = I_2 \quad I^2_2 = \emptyset \\
\vdots \\
I^{K-1}_1 = I_{K-1} \quad I^{K-1}_2 = \emptyset \\
I^K_1 = I_K \quad I^K_2 = I_K
\]

We can see that for each \( j \in [K] \), we have that:

\[
\text{opt-total}(I^j) = 2 \cdot r_j \cdot C.
\]

This is because for types \( j \in [K-1] \), the optimal solution is simply to accept the last \( 2 \cdot C \) type \( j \) agents, serve them on both days and achieve a reward of \( 2 \cdot C \cdot r_j \). For type \( K \) agents, the optimal reward is obtained by accepting and serving \( C \) of them in each of period 1 and 2 for a total reward of \( 2 \cdot C \cdot r_K \).
Given a non-anticipating algorithm \( A \) which achieves a CR of \( \Gamma \), let’s define \( \{a_i\}_{i \in [K]} \) to be the total number of type \( i \) agents that \( A \) accepts when faced with the input sequence \( I^K \). For \( j < K \), since \( I^j \) and \( I^K \) are the same in the same \( 2 \cdot j \cdot C \) agents in the first period, this implies that \( a_1, \ldots, a_j \) is also the number of type 1 through \( j \) agents accepted in the first period when \( A \) faces input \( I^j \). Given that \( A \) is \( \Gamma \) competitive against all of \( I^1, \ldots, I^K \), we must have:

\[
\sum_{i=1}^k a_i \cdot r_i \geq \Gamma \cdot \text{OPT-TOTAL}(I^k) = 2 \cdot \Gamma \cdot r_k \cdot C
\]

for all \( k \in [K] \). In addition, the total number of agents we accept out of \( I^K \) cannot be larger than the total capacity of \( 2 \cdot C \) and so we have that \( \sum_{i=1}^k a_i \leq 2 \cdot C \).

Therefore, an upper bound on the performance of any online algorithm \( A \) is given by the following LP:

\[
\max_{\{a_k\}_{k \in [K]}} \Gamma \\
\text{s.t.} \quad \sum_{i=1}^k a_i \cdot r_i \geq 2 \cdot \Gamma \cdot r_k \cdot C \\
\sum_{i=1}^K a_i \leq 2 \cdot C \\
\Gamma, a_k \geq 0 \quad k \in [K]
\] (31)

We will show that choosing \( \Gamma = L \) and \( a_k = 2 \cdot \Gamma \cdot \left( 1 - \frac{r_{k-1}}{r_k} \right) \cdot C \) for all \( k \in [K] \) (define \( r_0 = 0 \)) is a feasible solution, and that all inequalities above hold with equality, meaning that the solution is optimal (as we have \( K + 1 \) variable and \( K + 1 \) linearly independent constraints). For constraint (31), we have:

\[
\sum_{i=1}^k a_i \cdot r_i = \sum_{i=1}^k 2 \cdot \Gamma \cdot \left( 1 - \frac{r_{i-1}}{r_i} \right) \cdot r_i \cdot C = 2 \cdot \Gamma \cdot r_k \cdot C.
\]

For constraint (32), we have:

\[
\sum_{i=1}^K a_i = \sum_{i=1}^K 2 \cdot \Gamma \cdot \left( 1 - \frac{r_{i-1}}{r_i} \right) \cdot C = 2 \cdot C.
\]

Hence, we have the desired result, that \( \Gamma = B \) is the optimal solution of the LP above. This shows that no non-anticipating algorithm can achieve a CR of more than \( B \), hence proving our theorem.

Appendix B: Proof of Statements from Section 5

B.1. Proof of Theorem 1

**Proof of Theorem 1** Consider the following two input sequences:

1. \( I^a_1 = [1,1,\ldots,1], I^a_2 = \emptyset \)
2. \( I^b_1 = [1,1,\ldots,1,2,2,\ldots,2], I^b_2 = [2,2,\ldots,2] \)

Consider any non-anticipating algorithm \( A \). Such an algorithm should accept the same number of agents of type 1 in both input sequences \( a \) and \( b \). Let \( x \) be the faction of agents of type 1 that algorithm \( A \) accepts in the first time period. Then, in the input sequence \( a \), the CR of this algorithm in the first time period, as well as across two time periods, is \( x \). Now consider how such an algorithm would behave on input \( b \). Given
that $x$ fractions of agents of type 1 is accepted, the maximum possible total reward of this algorithm over
the two time periods of input $b$ is given by $C \cdot (x \cdot r_1 + (2 - x) \cdot r_2)$ while the maximum revenue is $2C \cdot r_2$, leading to the overall competitive ratio of $(x \cdot r_1 + (2 - x) \cdot r_2)/(2r_2)$. To obtain the best CR on both input
sequences $a$ and $b$, we set this ratio equal to $x$ (the CR of the algorithm in the first time period under input $a$). This leads to

$$\frac{x \cdot r_1 + (2-x) \cdot r_2}{2 \cdot r_2} = x \Rightarrow x = \frac{2}{3 - r_1/r_2}.$$ 

We have just showed that the cumulative CR is at most $\frac{2}{3 - r_1/r_2}$. Then, considering the fact that the optimal
inflexible reward in both time periods of input $b$ is equal to $C \cdot r_2$, we can conclude that per-period CR of
algorithm $A$ is at most $\frac{2}{3 - r_1/r_2}$. This completes the proof.

B.2. Proof of Theorem 2

Proof of Theorem 2. Let $B$ be the bucket state at the end of a time period $t$. To show the result, we will perform case-work based on whether an incoming agent of type $k \in \{1, 2\}$ would be accepted or rejected at state $B$. We say that our state $B$ rejects an incoming agent of type $k$ if an additional unit of type $k$ arriving to this state $B$ would be rejected (i.e. no positive fractional amount of that agent can fit in our state $B$).

Case 1: An incoming agent of type 2 would be rejected. If our algorithm rejects an agent of type 2, it must be the case that the first row is full (i.e., $B_{1,1} + B_2 = C$). This implies that the total reward in this time period is at least:

$$r_1 \cdot B_{1,1} + r_2 \cdot B_{1,2} = B_{1,1} \cdot r_1 + (C - B_{1,1}) \cdot r_2 \geq \left(\frac{1}{3 - \frac{r_1}{r_2}} \cdot r_1 + \left(1 - \frac{1}{3 - \frac{r_1}{r_2}}\right) \cdot r_2\right) \cdot C$$

$$= \frac{2}{3 - \frac{r_1}{r_2}} \cdot r_2 \cdot C,$$

where the first inequality follows from the fact that $B_{1,1} \leq C \cdot \frac{1}{3 - \frac{r_1}{r_2}}$ and coefficient of $B_{1,1}$ in the above expression is $C \cdot (r_1 - r_2) < 0$. Now, considering that the optimal clairvoyant reward in this time period is at most $C \cdot r_2$, we can conclude that the CR of our algorithm in this time period is at most $\frac{2}{3 - r_1/r_2}$, which is the desired result.

Case 2: An incoming agent of type 2 would be accepted, but an incoming agent of type 1 be rejected. Given that an incoming type 2 agent would be accepted, we have that $B_{1,1} + B_2 < C$ as that is the only constraint involving $B_2$. The fact that an incoming type 1 agent is rejected means that for each of $B_{1,1}$ and $B_{1,2}$, there is some constraint in $B$ involving them that is tight. Since $B_{1,1} + B_2 < C$, this implies that we must have $B_{1,1} = C \cdot \frac{1}{3 - \frac{r_1}{r_2}}$ and $B_{1,2} = C \cdot \frac{1}{3 - \frac{r_1}{r_2}}$.

More importantly, the fact that an incoming type 2 agent is accepted implies that no type 2 agents from $I_t$ were rejected. This is the fact that nested polytopes are consistent (Lemma 1) and property (ii) from the definition of consistency (Definition 1). Hence, we conclude that $B_2$ is precisely the number of type
2 agents present in $I_t$, which in turn is at least the number optimal solution because no type 2 agents have been rejected. This means that the optimal clairvoyant reward for this time period is no greater than $B_2 \cdot r_2 + (C - B_2) \cdot r_1$. 


Recall that our algorithm’s serving rule will choose to serve everyone in $B_{1,1}$ and $B_2$ (i.e. the first row) and use the remaining space to serve as many agents from $B_{1,2}$ as possible. Based on this rule, the following expression exactly computes the reward our algorithm achieves:

$$B_2 \cdot r_2 + \min\{(C - B_2), B_{1,1} + B_{1,2}\} \cdot r_1.$$

We notice that if $C - B_2 \leq B_{1,1} + B_{1,2}$, then our algorithm’s reward is exactly the same as the upper bound on the optimal reward, giving us a CR of 1. Now consider the case of $C - B_2 > B_{1,1} + B_{1,2}$. Since we have an expression for our reward as well as an upper bound on the optimal reward, we look at their ratio with a little algebra:

$$\frac{B_2 \cdot r_2 + (B_{1,1} + B_{1,2}) \cdot r_1}{B_2 \cdot r_2 + (C - B_2) \cdot r_1} \geq \frac{(B_{1,1} + B_{1,2}) \cdot r_1}{(C - B_2) \cdot r_1} = \frac{2 \cdot \frac{c}{3 - r_1/r_2}}{C - B_2} \geq \frac{2}{3 - r_1/r_2}.$$  

Here, the first inequality holds because for any $a, b > 0$ with $a < b$ and $c > 0$, we have $\frac{a + c}{b + c} \geq \frac{a}{b}$, while the first equality holds because $B_{1,1} = B_{1,2} = \frac{c}{3 - r_1/r_2}$.

**Case 3: Any incoming agent can be accepted.** In this case, no agent have been rejected by our algorithm. Our algorithm’s serving rule will exactly serve all the type 2 agents, and as many type 1 agents as possible, which is the optimal solution. Hence in this case, we have a CR of 1.

**Appendix C: Proof of Statements from Section [6]**

**C.1. Full Proof of Theorem 3**

**Proof of Theorem 3** To show the result, we consider the following four input sequences.

- **Input a.** $I^a_1 = [1,1,\ldots,1,2,2,\ldots,2,3,3,\ldots,3], I^2_1 = [M+1, M+1, \ldots, M+1, \ldots, 3,3,\ldots,3]$. Note that if $M = 2$, then $I^2_2$ contains only $C$ type 3 agents.

- **Input b.** $I^b_1 = [1,1,\ldots,1,2,2,\ldots,2,3,3,\ldots,3]$ and $I^b_2 = [M+1, M+1, \ldots, M+1]$.

- **Input c.** $I^c_1 = [1,1,\ldots,1,2,2,\ldots,2]$ and $I^c_2 = \emptyset$.

- **Input d.** $I^d_1 = [1,1,\ldots,1]$ and $I^d_2 = \emptyset$.

Observe that all of the input sequences $\{I^a_1, I^a_2\}, \{I^b_1, I^b_2\}$ and $\{I^c_1, I^c_2\}$ are simply truncated versions of $\{I^a_1, I^a_2\}$ where the input sequence suddenly ends after a certain $C$ copies of a type arrive. In order for any non-anticipating algorithm $A$ to be $\Gamma$ competitive, it must be robust towards these sudden truncation of the input sequence.

**Definitions of Variables.** We now start with a few definitions that describes the behavior of any non-anticipating algorithm $A$ against the input sequence $a$, i.e., $\{I^a_1, I^a_2\}$. The $s_{i,t}$’s correspond to the variables in the LP, while the $a_{i,t}$’s are defined for clarity in the arguments to follow.

- **$\{s_{i,t}\}_{i \in [2], t \in [2]}$: The number of type $i$ agents that $A$ serves at the end of time period $t$ when all the agents have arrived.**

- **$\{a_{i,t}\}_{i \in [2], t \in [2]}$: The number of type $i \in [2]$ agents that $A$ accepts in $I^a_t$ for $t \in [2]$. Note that if $I^a_t$ does not contain agents of type $i$, then $a_{i,t} := 0$. Further, we only define $a_{i,t}$ for $i \in [2]$ because type 3 is always inflexible, so $s_{3,t}$ would always be equal to $s_{3,t}$.**
We show some simple relations between the $a_{i,t}$’s and $s_{i,t}$ that will be used in the arguments to follow. It is easy to verify that

\[ a_{1,1} = s_{1,1} + s_{1,2}; \tag{33} \]

that is, the number of served agents of type 1 across two time periods is equal to the number of accepted agents of type 1 in the first time period. This is because agents of type 1 only appears in the first time period. Similarly, we have

\[ a_{2,1} = \sum_{j=1}^{M} s_{2,j} \quad \text{and} \quad a_{2,2} = \begin{cases} s_{2,2} & M = 1 \\ 0 & M = 2. \end{cases} \tag{34} \]

To see why, first consider the case of $M = 1$. Then, the number of accepted agents of type 2 in the first time period, i.e., $a_{2,1}$, is equal to $\sum_{t=1}^{M} s_{2,t} = s_{2,1}$. (Recall that with $M = 1$, type 2 is inflexible.) Similarly, the number of accepted agents of type 2 in the second time period, i.e., $a_{2,2}$, is equal to $s_{2,2}$. Now, consider the case of $M = 2$. The number of accepted agents of type 2 in the first time period, i.e., $a_{2,1}$, is equal to $\sum_{j=1}^{M} s_{2,j} = s_{2,1} + s_{2,2}$. In addition, the number of accepted agents of type 2 in the second time period, i.e., $a_{2,2}$, is equal to 0. This is because with $M = 2$, under input $a$, no agents of type 2 arrive in the second time period.

Since we assumed that $A$ achieves at least a $\Gamma$ fraction of the optimal revenue in each time period for any input sequence $I_t, t \in [T]$, $A$ must satisfy certain requirements against the 4 input sequences $a, b, c, d$ defined above.

**Obtaining the CR of $\Gamma$ against input $a$.** Under input $a$, we have that $\text{OPT}_{\tau} \{ I^a_t \}_{t \in [2]} = r_3 \cdot C$ for both $\tau \in [2]$ whereas $A$ will serve $s_{i,\tau}$ copies of agent type $i$ for $i \in [3]$. Thus, in order to achieve the CR of $\Gamma$ in both time periods $t \in [2]$, we must have $\Gamma \cdot r_3 \cdot C \leq r_1 \cdot s_{1,t} + r_2 \cdot s_{2,t} + r_3 \cdot s_{3,t}$, for $t = 1, 2$. In addition, $s_{1,t} + s_{2,t} + s_{3,t} \leq C$ must be true because of our capacity constraint. This leads to constraints [0] and [1] in the LP.

**Obtaining the CR of $\Gamma$ against input $b$.** This input is only useful when $M = 1$. This is because with $M = 2$, both inputs $a$ and $b$ lead to the same constraints. Assuming that $M = 1$, we now use the fact that algorithm $A$ is non-anticipating. Since input $a$ and input $b$ are identical for $t = 1$, $A$ makes the exact same decisions on both for $t = 1$, while for $t = 2$ we know that $A$ accepts $a_{2,2}$ type 2 agents on both inputs. In particular, this means that $s_{1,2}$ (the number of type 1 agents that were accepted but not served in the first time period) must be the same under inputs $a$ and $b$. Similarly, $a_{2,2} = s_{2,2}$ (the number of type 2 agents that were served in the second time period) must be the same under inputs $a$ and $b$. Hence, the total reward in the second time period under input $b$ is $r_1 \cdot s_{1,2} + r_2 \cdot s_{2,2}$.

This implies that algorithm $A$ obtains the CR of $\Gamma$ under input $b$ if $\Gamma \cdot \text{OPT}_2 \{ I^a_t \}_{t \in [2]} = \Gamma \cdot C \cdot r_2 \leq r_1 \cdot s_{1,2} + r_2 \cdot s_{2,2}$, giving rise to constraint [10] in the LP.

**Obtaining the CR of $\Gamma$ against input $c$.** Again using the fact that $A$ is non-anticipating, $a_{1,1}$ and $a_{2,1}$ (the number of accepted agents of type 1 and 2, respectively, in the first time period) must be the same under input $a$ and $c$. Now consider input $c$ and observe that under this input, we have $\text{OPT}_1 \{ I^a_t \}_{t \in [2]} = C \cdot r_2$. This implies that in order to obtain the CR of $\Gamma$ under input $c$, there must be some way to choose $s_{1} \leq a_{1,1}$ type 1 agents (out of $a_{1,1}$ type 1 agents) and $s_{2} \leq a_{2,1}$ type 2 agents (out of $a_{2,1}$ type 2 agents), such that
For each of properties (e), (b), (c) and (d), we will argue by contradiction that for any optimal solution \( \Gamma^* \) to the LP \((\text{upper3})\), the existence of \( s_1 \) and \( s_2 \) implies the above two conditions because

\[
a_{1,1} \cdot r_1 + a_{2,1} \cdot r_2 \geq \Gamma \cdot r_2 \cdot C
\]

\[
(C - a_{2,1}) \cdot r_1 + a_{2,1} \cdot r_2 \geq \Gamma \cdot r_2 \cdot C.
\]

Substituting \( a_{1,1} = s_{1,1} + s_{1,2} \) and \( a_{2,1} = \sum_{i=1}^{M} s_{2,i} \) in equations (35) and (36), we get exactly constraints (9) and (10) in the LP.

**Obtaining the CR of \( \Gamma \) against input \( d \).** Since \( \mathcal{A} \) is non-anticipating, its decisions, and in particular, \( a_{1,1} \) (the number of accepted agents of type 1 in the first time period), must be the same under input \( a \) and \( d \). This implies that \( \mathcal{A} \) obtains the CR of \( \Gamma \) under input \( d \) if \( r_1 \cdot a_{1,1} \geq \Gamma \cdot \text{OPT}_1(\{I^d_t\}_{t \in \{2\}}) = \Gamma \cdot C \cdot r_1 \), which is constraint (7) in the LP.

### C.2. Properties of the 3-Class Upper Bound

We present many properties of the 3-class upper bound that will be crucial in proving the optimality of our 3-class polytopes \( \mathcal{B}^{(1)} \) and \( \mathcal{B}^{(2)} \).

#### C.2.1. Statement and Proof of Lemma 6

**Lemma 6 (Tight Constraints in the 3-Type Upper Bound \((\text{upper3})\).** Let \( (\Gamma^*, \{s^*_t\}_{t \in \{3\}, i \in \{2\}}) \) be any optimal solution to the LP \((\text{upper3})\) presented in Theorem 5. Then the following five properties must be true:

(a) \( s_{1,t} + s_{2,t} + s_{3,t} = C \) for \( t \in \{2\} \) (i.e. constraint (51) is tight).

(b) \( r_1 \cdot (s_{1,t} + s_{2,t}) = \Gamma^* \cdot r_1 \cdot C \) (i.e. constraint (7) is always tight).

(c) \( r_1 \cdot \min(s_{1,t} + s_{2,t}, C - \sum_{i=1}^{M} s_{3,i}^*) + r_2 \cdot \sum_{i=1}^{M} s_{3,i}^* = \Gamma^* \cdot r_2 \cdot C \) (i.e. at least one of constraints (8) and (9) is tight).

(d) If \( M = 1 \), then \( r_1 \cdot s_{1,2}^* + r_2 \cdot s_{2,2}^* = \Gamma^* \cdot r_2 \cdot C \) (i.e. constraint (10) is tight).

(e) \( \Gamma^* \cdot r_3 \cdot C = r_1 \cdot s_{1,t}^* + r_2 \cdot s_{2,t}^* + r_3 \cdot s_{3,t}^* \) for \( t \in \{2\} \) (i.e. constraint (9) is tight).

**Proof of Lemma 6** We will first show property (a) holds for every optimal solution to the LP \((\text{upper3})\). Then, for each of properties (e), (b), (c) and (d) we will argue by contradiction that for any optimal solution \( (\Gamma^*, \{s^*_i\}_{t \in \{3\}, i \in \{2\}}) \) for which the corresponding constraint is not tight, we can construct a new feasible solution \( (\Gamma^*, \{s^*_i\}_{t \in \{3\}, i \in \{2\}}) \) such that \( \Gamma^* = \Gamma^* \), but \( s_{1,t} + s_{2,t} + s_{3,t} < C \) for some \( t \in \{2\} \), which would imply that \( \Gamma^* \) is not the optimal objective value by property (a).

(a) The main idea behind this proof is that if \( s_{1,t}^* + s_{2,t}^* + s_{3,t}^* < C \) for some \( t \in \{2\} \), then we can change the variables \( \{s^*_i\}_{t \in \{3\}, i \in \{2\}} \) in a way such that all constraints of the LP \((\text{upper3})\) involving \( \Gamma^* \) are not tight, implying that \( \Gamma^* \) can be increased and contradicting the fact that \( \Gamma^* \) is optimal.
Fix the value of $t$ such that $s_{1,t}^* + s_{2,t}^* + s_{3,t}^* + 2 \cdot \delta = C$ for some $\delta > 0$. We construct a new solution $\{s_{i,t}' \in [3], i \in [2], \Gamma^*\}$ to (\text{UPPER3}) as follows:

$$
\begin{align*}
    s_{1,t}' &= s_{1,t}^* + \delta \\
    s_{1,2-t}' &= s_{1,2-t}^* - \delta/2 \\
    s_{2,t}' &= s_{2,t}^* + \delta \\
    s_{2,2-t}' &= s_{2,2-t}^* \\
    s_{3,t}' &= s_{3,t}^* \\
    s_{3,2-t}' &= s_{3,2-t}^* 
\end{align*}
$$

We now need to check that this new solution $\{s_{i,t}' \in [3], i \in [2], \Gamma^*\}$ is feasible and that all constraints involving $\Gamma^*$ are not tight.

- **Constraint (5):** By our construction, we have that $s_{1,t}^* + s_{2,t}^* + s_{3,t}^* = C$ and $s_{1,2-t}^* + s_{2,2-t}^* + s_{3,2-t}^* = s_{1,2-t}^* + s_{2,2-t}^* + s_{3,2-t}^* \leq C$.

- **Constraint (6):** This constraint is clearly satisfied for $t$ as $s_{i,t}' \geq s_{i,t}^*$ for all $i \in [3]$. For $2-t$, we have that $r_1 \cdot s_{1,2-t}^* + r_2 \cdot s_{2,2-t}^* + r_3 \cdot s_{3,2-t}^* = r_1 \cdot s_{1,2-t}^* + r_2 \cdot s_{2,2-t}^* + r_3 \cdot s_{3,2-t}^* + \delta \cdot (r_2 - r_1)$ which is strictly greater than $\Gamma \cdot r_3 \cdot C$ by the fact that $\{s_{i,t}^* \in [3], i \in [2], \Gamma^*\}$ is feasible. Hence, this constraint is satisfied and not tight.

- **Constraint (7):** This is satisfied and not tight because $s_{1,t}' + s_{1,2-t}' > s_{1,t}^* + s_{1,2-t}^*$.

- **Constraint (8):** Notice that we can re-write this constraint in the LP as:

$$
\Gamma \cdot r_2 \cdot C \leq C \cdot r_1 + (r_2 - r_1) \cdot \left( \sum_{t=1}^{M} s_{2,t} \right). \tag{37}
$$

Our new solution $\{s_{i,t}' \in [3], i \in [2], \Gamma^*\}$ is feasible and makes this constraint not tight because $\sum_{t=1}^{M} s_{2,t}' > \sum_{t=1}^{M} s_{2,t}^*$ for both $M = 1$ and $M = 2$.

- **Constraint (9):** By the same reasoning as in the argument of constraint (8), we have that $r_1 \cdot s_{1,t}^* + r_2 \cdot s_{2,t}^* > r_1 \cdot s_{1,t}' + r_2 \cdot s_{2,t}'$ for both $\tau = t$ and $\tau = 2-t$.

- **Constraint (10):** This is for the same reason as that of constraint (9).

Therefore, we have proven that $\{s_{i,t}' \in [3], i \in [2], \Gamma^*\}$ is feasible and that all constraints involving $\Gamma^*$ are not tight, which contradicts the optimality of $\Gamma^*$. Hence, all optimal solutions must have the property that constraint (5) is tight, i.e. $s_{1,t}^* + s_{2,t}^* + s_{3,t}^* = C$.

(b) Assume for contradiction that constraint (7) is not tight; that is, our optimal solution has $(s_{1,1,t}^* + s_{1,2,t}^*) > \Gamma^* \cdot r_1 \cdot C$. We then know that one of $s_{1,1,t}^* > 0$ for some $t \in [2]$. Fix that particular value of $t$ and construct our new solution $(\Gamma', \{s_{i,t}' \in [3], i \in [2]\})$ with the following modifications:

$$
\begin{align*}
    s_{1,t}' &= s_{1,t}^* - \epsilon \\
    s_{2,t}' &= s_{2,t}^* + \frac{r_1}{r_2} \cdot \epsilon
\end{align*}
$$

where we choose $\epsilon > 0$ sufficiently small such that $r_1 \cdot (s_{1,1,t}^* + s_{1,2,t}^*) \geq \Gamma^* \cdot r_1 \cdot C$ (constraint (7) is satisfied) and $s_{1,1,t}^* \geq 0$. It is easy to see that $s_{1,t}^* + s_{2,t}^* + s_{3,t}^* < C$ now, which is what we want to do for the contradiction. All we need to do now is verify that this new solution $(\Gamma', \{s_{i,t}' \in [3], i \in [2]\})$ is feasible by checking the remaining constraints. To do so, we can group constraints (4), (8), (9) and (10) into categories based on the coefficients of $s_{1,t}$ and $s_{2,t}$. Note that all other constraints involving $s_{1,t}$ and $s_{2,t}$ are greater-than-or-equal-to (i.e., $\geq$) inequalities.
• Constraints (8), (9) and (10): these constraints all involve the term \( r_1 \cdot s_{1,t} + r_2 \cdot s_{2,t} \) on the right hand side of the \( \leq \) inequality. Our new solution has the property that

\[
r_1 \cdot s'_{1,t} + r_2 \cdot s'_{2,t} = r_1 \cdot s'_{1,t} + r_2 \cdot s^*_{2,t},
\]

and so all the constraints of this form remain satisfied.

• Constraint (5): Recall from equation (37) that this constraint can be re-written such that the right hand side of the \( \leq \) includes only a term \( p \cdot \sum_{\tau = 1}^{M} s_{2,\tau} \) and our new solution has the property that \( \sum_{\tau = 1}^{M} s'_{2,\tau} \geq \sum_{\tau = 1}^{M} s^*_{2,\tau} \), meaning that this constraint continues to be satisfied.

Thus, we have shown a contradiction by property [a] of this Lemma and so it must be the case that every optimal solution has the property that constraint (7) is tight.

(c) Suppose for contradiction that neither constraint (8) nor (9) is tight for our solution \( (\Gamma^*, \{s^\prime_{1,t}\}_{\tau \in [\mathcal{\theta}], t \in [2]}), \) meaning that

\[
r_1 \cdot (s^*_{1,1} + s^*_{1,2}) + r_2 \cdot \sum_{t = 1}^{M} s^*_{2,t} > \Gamma^* \cdot r_2 \cdot C \tag{38}
\]

\[
r_1 \cdot \left( C - \sum_{t = 1}^{M} s^*_{2,t} \right) + r_2 \cdot \sum_{t = 1}^{M} s^*_{2,t} > \Gamma^* \cdot r_2 \cdot C \tag{39}
\]

By property (b), we know that \( r_1 \cdot (s^*_{1,1} + s^*_{1,2}) = \Gamma^* \cdot r_1 \cdot C < \Gamma^* \cdot r_2 \cdot C \), which means from inequality (38) that \( \sum_{t = 1}^{M} s^*_{2,t} > 0 \) must be true. This implies that there exists \( t \in [2] \) for which \( s^*_{2,t} > 0 \). Fix this particular value of \( t \) and consider a new solution \( (\Gamma^\prime, \{s^\prime_{1,t}\}_{\tau \in [\mathcal{\theta}], t \in [2]}), \) where the modifications we make are as follows

\[
s'_{2,t} = s^*_{2,t} - \epsilon \quad s^\prime_{3,t} = s^*_{3,t} + \epsilon \cdot \frac{r_2}{r_3}
\]

Given that inequalities (38) and (39) are both not binding and \( s^*_{2,t} \) appears in both with a positive coefficient, we can find some \( \epsilon > 0 \) such that our new solution variables \( s^\prime_{2,t} \) and \( s^\prime_{3,t} \) also satisfy (38) and (39). Furthermore, we can choose \( \epsilon \) small enough such that \( s^\prime_{2,t} \geq 0 \). Now, we just need to verify that \( (\Gamma^\prime, \{s^\prime_{1,t}\}_{\tau \in [\mathcal{\theta}], t \in [2]}), \) is feasible.

• Constraint (3) (i.e., \( s_{1,t} + s_{2,t} + s_{3,t} \leq C \)). By our construction, we have

\[
s'_{1,t} + s^\prime_{2,t} + s^\prime_{3,t} = s^*_{1,t} + s^*_{2,t} + s^*_{3,t} - \epsilon \cdot \left( \frac{r_2}{r_3} \right) < C
\]

• Constraint (9) (i.e., \( \Gamma \cdot r_3 \cdot C \leq s_{1,t} \cdot r_1 + s_{2,t} \cdot r_2 + s_{3,t} \cdot r_3 \)). By our construction, we have

\[
r_1 \cdot s'_{1,t} + r_2 \cdot s^\prime_{2,t} + r_3 \cdot s^\prime_{3,t} = r_1 \cdot s^*_{1,t} + r_2 \cdot s^*_{2,t} + r_3 \cdot s^*_{3,t}
\]

Thus, this constraint continues to be satisfied.

• Constraint (7) (i.e., \( \Gamma \cdot r_1 \cdot C \leq (s_{1,1} + s_{1,2}) \cdot r_1 \)) and constraint (10) (i.e., \( \Gamma \cdot r_2 \cdot C \leq s_{1,2} \cdot r_1 + s_{2,2} \cdot r_2 \)). Constraint (7) continues to be satisfied by our new solution because \( s^*_{1,1} = s'_{1,1} \) and \( s^*_{1,2} = s'_{1,2} \). A similar argument can be applied for constraint (10).

• Constraints (8) and (9) are satisfied because we chose \( \epsilon \) small enough so that these constraints continue to be satisfied.
By property (a) of this lemma, we get the desired contradiction, proving that at least one of constraints (8) and (9) must be tight.

(d) For \( M = 1 \), suppose for contradiction that constraint (10) (i.e., \( \Gamma \cdot r_2 \cdot C \leq s_{1,2} \cdot r_1 + s_{2,2} \cdot r_2 \)) is not tight for our solution \((\Gamma^*, \{s^*_{1,t}\})_{t \in [3], t \in [2]}\). Similar to our proof for property (c) we have that \( s^*_{2,2} > 0 \). This is because we know that \( s^*_{1,2} \leq \Gamma^* \cdot C \) by property (b) and as a result, it would be impossible to have \( r_1 \cdot s^*_{1,2} + r_2 \cdot s^*_{2,2} \geq \Gamma^* \cdot r_2 \cdot C \) without \( s^*_{2,2} > 0 \). We create our new solution \((\Gamma^*, \{s'_{1,t}\})_{t \in [3], t \in [2]}\) with the following modifications

\[
\begin{align*}
    s'_{2,2} &= s^*_{2,2} - \epsilon \\
    s'_{3,2} &= s^*_{3,2} + \epsilon \cdot \frac{r_2}{r_3},
\end{align*}
\]

where we choose \( \epsilon \) small enough so that our new solution continues to satisfy constraint (10) and such that \( s^*_{2,2} \geq 0 \). We check that this new solution is feasible and therefore optimal. For \( M = 1 \), the only constraints that involve \( s^*_{2,2} \) and \( s^*_{3,2} \) are (3), (6), and (10). We have already assumed constraint (10) is satisfied, so we just check the remaining two. With respect to constraint (3) (i.e., \( \Gamma \cdot r_3 \cdot C \leq s_{1,1} \cdot r_1 + s_{2,1} \cdot r_2 + s_{3,1} \cdot r_3 \)), using similar reasoning as in part (c) we see that our construction preserves the value of \( r_1 \cdot s^*_{1,2} + r_2 \cdot s^*_{2,2} + r_3 \cdot s^*_{3,2} \) (that is, \( r_1 \cdot s^*_{1,2} + r_2 \cdot s^*_{2,2} + r_3 \cdot s^*_{3,2} = r_1 \cdot s'_{1,2} + r_2 \cdot s'_{2,2} + r_3 \cdot s'_{3,2} \)), and hence constraint (3) still holds. With respect to constraint (6) (i.e., \( s_{1,1} + s_{2,1} + s_{3,1} \leq C \)), the value of \( s^*_{1,2} + s^*_{2,2} + s^*_{3,2} \) is slightly lower than \( s^*_{1,2} + s^*_{2,2} + s^*_{3,2} \), implying constraint (5) is not tight. This gives us the desired contradiction per property (a) of this lemma.

(e) Contrary to our claim, suppose that constraint (6) is not tight for our solution \((\Gamma^*, \{s^*_{1,t}\})_{t \in [3], t \in [2]}\) for either \( t = 1 \) or \( t = 2 \). Fix that particular value of \( t \). We claim that \( s^*_{3,t} > 0 \) because of properties (c) and (d). From those properties, we get that \( r_1 \cdot s^*_{1,t} + r_2 \cdot s^*_{2,t} \leq \Gamma^* \cdot r_2 \cdot C \) for both \( t = 1 \) and \( t = 2 \), which means that if we want \( r_1 \cdot s^*_{1,t} + r_2 \cdot s^*_{2,t} + r_3 \cdot s^*_{3,t} \geq \Gamma^* \cdot r_3 \cdot C \), then \( s^*_{3,t} \) must not be 0. Consider a new solution \((\Gamma', \{s'_{1,t}\})_{t \in [3], t \in [2]}\) where we make the following change

\[
    s'_{3,t} = s^*_{3,t} - \epsilon,
\]

where we choose \( \epsilon \) small enough such that our new solution continues to satisfy constraint (4) and \( s^*_{3,t} \geq 0 \). (The rest of the variables remain the same.) Our new solution clearly also satisfies constraint (5) (i.e., \( s^*_{1,t} + s^*_{2,t} + s^*_{3,t} \leq C \)) since we have decreased \( s^*_{3,t} \) while leaving \( s^*_{1,t} \) and \( s^*_{2,t} \) unchanged. All other constraints from LP (UPPER3) continue to be satisfied because they do not involve \( s^*_{3,t} \). We get that our new solution is optimal, which gives us a contradiction again because \((\Gamma', \{s'_{1,t}\})_{t \in [3], t \in [2]}\) is optimal yet does not not have constraint (5) tight. Thus, we get that constraint (6) must be tight for all optimal solutions.

\[\square\]

C.2.2. Statement and Proof of Lemma 7 The following lemma is used to show that the polytopes \( B^{(1)} \) and \( B^{(2)} \) satisfy property (i) of consistency, defined in Definition 1.

**Lemma 7 (Properties of 3-Type Upper Bound Solution).** Let \((\Gamma^*, \{s^*_{1,t}\})_{t \in [3], t \in [2]}\) be an optimal solution to the \( B_{(1)} \) \( B^{(2)} \). Then the following three properties hold.
(a) For both $M = 1$ and $M = 2$, every optimal solution must have the property that
\[ \Delta_{3,1} \cdot s_{1,t}^* + \Delta_{3,2} \cdot s_{2,t}^* = (1 - \Gamma^*) \cdot r_3 \cdot C. \]

(b) For $M = 1$, there exists an optimal solution such that $s_{1,1}^* \geq s_{1,2}^*$, and

(c) For $M = 2$, it is always the case that $C - (s_{2,1}^* + s_{2,2}^*) \geq \frac{1}{2} \left( s_{1,1}^* + s_{1,2}^* \right)$. 

Here, $\Delta_{i,j} := r_i - r_j$ for any $i, j \in [3], i \neq j$.

**Proof of Lemma 7** We show each of the three properties separately.

(a) By properties (a) and (e) of Lemma 6, we have $s_{1,1}^* + s_{2,1}^* + s_{3,t}^* = C$ for $t \in [2]$ and $\Gamma^* \cdot r_3 \cdot C = r_1 \cdot s_{1,1}^* + r_2 \cdot s_{2,1}^* + r_3 \cdot s_{3,t}^*$ for $t \in [2]$. We then immediately get the desired result by multiplying the former by $r_3$ and then subtracting the two equations.

(b) Contrary to our claim, suppose that $s_{1,1}^* < s_{1,2}^*$. Then, we present another optimal solution $(\Gamma^*, \{s_{i,t}^*\}_{i \in [3], t \in [2]})$ for which $s_{1,1}^* \geq s_{1,2}^*$. We construct our new solution $(\Gamma', \{s_{i,t}^*\}_{i \in [3], t \in [2]})$ identical to our old solution $(\Gamma^*, \{s_{i,t}^*\}_{i \in [3], t \in [2]})$ except with the following changes:

- $s_{1,t}^* = s_{1,3-t}^*$
- $s_{2,t}^* = s_{2,3-t}^*$
- $s_{3,1}^* = s_{3,3-t}^*$

Our new solution is constructed by taking the old solution and swapping the first and second rows. It is clear that $s_{1,1}^* \geq s_{1,2}^*$ now, so we just need to verify feasibility of the new solution.

Clearly, constraints (5) (i.e., $s_{1,1}^* + s_{2,t}^* + s_{3,t}^* \leq C$) and (6) (i.e., $\Gamma^* \cdot r_3 \cdot C \leq s_{1,t}^* \cdot r_1 + s_{2,t}^* \cdot r_2 + s_{3,t}^* \cdot r_3$) are still satisfied, as these constraints were valid for both $t = 1$ and $t = 2$. In addition, constraint (7) (i.e., $\Gamma^* \cdot r_1 \cdot C \leq (s_{1,1}^* + s_{1,2}^*) \cdot r_1$) is still satisfied since it is symmetric between $t = 1$ and $t = 2$. We just need to verify constraints (8), (9), and (10). For constraint (5) (i.e., $\Gamma^* \cdot r_2 \cdot C \leq (C - s_{2,1}^*) \cdot r_1 + s_{2,1}^* \cdot r_2$) and constraint (9) (i.e., $\Gamma^* \cdot r_2 \cdot C \leq (s_{2,1}^* + s_{2,2}^*) \cdot r_1 + s_{2,1}^* \cdot r_2$), we verify them together as follows

\[ r_1 \cdot \min(s_{1,1}^* + s_{1,2}^* - s_{2,1}^*, r_1 \cdot s_{1,1}^* + r_2 \cdot s_{1,2}^* - C) \geq \Gamma^* \cdot r_2 \cdot C \]

Finally, for constraint (10) (i.e., $\Gamma^* \cdot r_2 \cdot C \leq s_{2,1}^* \cdot r_1 + s_{2,2}^* \cdot r_2$), we use property (a) of this lemma which says that $\Delta_{3,1} \cdot s_{1,1}^* + \Delta_{3,2} \cdot s_{2,1}^* = \Delta_{3,1} \cdot s_{1,1}^* + \Delta_{3,2} \cdot s_{2,1}^*$. By our definitions, we then have $\Delta_{3,1} \cdot s_{1,1}^* + \Delta_{3,2} \cdot s_{2,1}^* = \Delta_{3,1} \cdot s_{1,1}^* + \Delta_{3,2} \cdot s_{2,1}^*$, this equality implies that $s_{2,1}^* \leq s_{2,2}^*$ Taking this equality (i.e., $\Delta_{3,1} \cdot s_{1,1}^* + \Delta_{3,2} \cdot s_{2,1}^* = \Delta_{3,1} \cdot s_{1,1}^* + \Delta_{3,2} \cdot s_{2,1}^*$), multiply through by $\frac{\Delta_{3,1}}{s_{3,1}^*}$ and then adding/subtracting a term on both sides, we get

\[ r_1 \cdot s_{1,1}^* + r_2 \cdot s_{2,1}^* + \left( r_1 \cdot \frac{\Delta_{3,2}}{\Delta_{3,1}} - r_2 \right) \cdot s_{2,1}^* = r_1 \cdot s_{1,1}^* + r_2 \cdot s_{2,1}^* + \left( r_1 \cdot \frac{\Delta_{3,2}}{\Delta_{3,1}} - r_2 \right) \cdot s_{2,2}^* \]

We notice that $\left( r_1 \cdot \frac{\Delta_{3,2}}{\Delta_{3,1}} - r_2 \right)$ is negative. This along with the fact $s_{2,1}^* \leq s_{2,2}^*$, we get that $r_1 \cdot s_{1,1}^* + r_2 \cdot s_{2,1}^* \leq r_1 \cdot s_{1,2}^* + r_2 \cdot s_{2,2}^*.$

We have already shown in the verification of constraints (5) and (7) that $r_1 \cdot s_{1,1}^* + r_2 \cdot s_{2,1}^* \geq \Gamma^* \cdot r_2 \cdot C$ and so we get the desired result for constraint (10) that $r_1 \cdot s_{1,2}^* + r_2 \cdot s_{2,2}^* \geq \Gamma^* \cdot r_2 \cdot C$. Thus, we have shown that this new solution $(\Gamma', \{s_{i,t}^*\}_{i \in [3], t \in [2]})$ is feasible, optimal, and has the desired property.

---

7 This simplifies to $r_1 \cdot r_3 - r_1 \cdot r_2 < r_2 \cdot r_3 - r_1 \cdot r_2$ implying that $r_1 < r_2$, which is true.
Our goal is to show that $C - (s_{2,1}^* + s_{2,2}^*) \geq \frac{1}{2} (s_{1,1}^* + s_{1,2}^*)$ is always true. To do so, we will prove a lower bound on $C - (s_{2,1}^* + s_{2,2}^*)$ and an upper bound on $\frac{1}{2} (s_{1,1}^* + s_{1,2}^*)$. We show that the lower bound on $C - (s_{2,1}^* + s_{2,2}^*)$ is greater and equal to the upper bound on $\frac{1}{2} (s_{1,1}^* + s_{1,2}^*)$. This gives us the desired result.

Lower bound on $C - (s_{2,1}^* + s_{2,2}^*)$. Here, we show that $C - (s_{2,1}^* + s_{2,2}^*) \geq (1 - \Gamma^*) \cdot \Gamma^* \cdot r_2 \cdot C$. To present this upper bound, we give a lower bound on $(s_{2,1}^* + s_{2,2}^*)$ using properties (b) and (c) of Lemma 6 which state that

$$r_1 \cdot (s_{1,1}^* + s_{1,2}^*) = \Gamma^* \cdot r_1 \cdot C \quad \text{Property (b) of Lemma 6}$$

$$r_1 \cdot \min(s_{1,1}^* + s_{1,2}^*, C - s_{2,1}^* - s_{2,2}^*) + r_2 \cdot (s_{2,1}^* + s_{2,2}^*) = \Gamma^* \cdot r_2 \cdot C \quad \text{Property (c) of Lemma 6.}$$

The first equation clearly gives us $s_{1,1}^* + s_{1,2}^* = \Gamma^* \cdot C$. We plug this into the second equation. For the second equation, we perform case-work based on which term of the minimum is taken and solve for $s_{2,1}^* + s_{2,2}^*$ to get:

$$s_{2,1}^* + s_{2,2}^* = \begin{cases} 
\Gamma^* \cdot \left(1 - \frac{r_1}{r_2}\right) \cdot C & \Gamma^* \cdot \left(1 - \frac{r_1}{r_2}\right) + \Gamma^* \leq 1 \\
\Gamma^* \cdot \frac{r_1 - r_2}{r_2} \cdot C & \text{otherwise}
\end{cases}$$

We claim that $s_{2,1}^* + s_{2,2}^* \leq \Gamma^* \cdot \frac{r_1 - r_2}{r_2} \cdot C$ is always true. This is equivalent to showing $\Gamma^* \cdot \left(1 - \frac{r_1}{r_2}\right) + \Gamma^* \leq 1$. This is easy to show with some algebra. Therefore, we have the following lower bound on $C - (s_{2,1}^* + s_{2,2}^*)$, which is the desired result:

$$C - (s_{2,1}^* + s_{2,2}^*) \geq C - \frac{\Gamma^* \cdot r_2 - r_1}{r_2 - r_1} \cdot C = \frac{(1 - \Gamma^*) \cdot r_2}{r_2 - r_1} \cdot C. \quad (40)$$

Upper bound on $\frac{1}{2} \cdot (s_{1,1}^* + s_{1,2}^*)$. For the upper bound, we have:

$$\frac{1}{2} \cdot (s_{1,1}^* + s_{1,2}^*) \leq \frac{1}{2} \left( \frac{(1 - \Gamma^*) \cdot r_3}{r_3 - r_1} \cdot C + \frac{(1 - \Gamma^*) \cdot r_3}{r_3 - r_1} \cdot C \right) = \frac{(1 - \Gamma^*) \cdot r_3}{r_3 - r_1} \cdot C, \quad (41)$$

where the first inequality is true by property (a) of this lemma, where we show $\Delta_{3,1} \cdot s_{1,1}^* + \Delta_{3,2} \cdot s_{2,2}^* = \Delta_{3,1} \cdot s_{1,1}^* + \Delta_{3,2} \cdot s_{2,1}^* = (1 - \Gamma^*) \cdot C \cdot r_3$ and hence $s_{1,1}^* \leq \frac{(1 - \Gamma^*) \cdot C \cdot r_3}{\Delta_{3,1}}$ for $t \in [2]$.

Comparing the bounds. Observe the lower bound on $C - (s_{2,1}^* + s_{2,2}^*)$, presented in (40), is smaller than the upper bound on $\frac{1}{2} \cdot (s_{1,1}^* + s_{1,2}^*)$, presented in (41). That is, we have

$$\frac{r_2}{r_2 - r_1} \geq \frac{r_3}{r_3 - r_1} \iff -r_1 \cdot r_2 \geq -r_1 \cdot r_3 \iff r_2 \leq r_3.$$

This proves the desired result that $C - (s_{2,1}^* + s_{2,2}^*) \geq \frac{1}{2} \cdot (s_{1,1}^* + s_{1,2}^*)$.

C.3. Proof of Statements about the 3-Class Polytopes

C.3.1. Statement and Proof of Lemma 8

**Lemma 8 (Consistency of $\mathcal{B}^{(1)}$).** The polytope $\mathcal{B}^{(1)}$ is consistent as per Definition [7].

---

Note that $\Gamma^* \cdot \left(\frac{r_2 - r_1}{r_2}\right) \cdot C \leq \frac{r_2 - r_1}{r_2 - r_1} \cdot C$ holds if and only if the following inequality holds $\Gamma^* \left(\frac{r_2 - 2r_2r_1 + r_1^2}{r_2 - r_1} \right) \leq \Gamma^* \frac{r_2^2 - r_1r_2}{r_2 - r_1}$ which is equivalent to the condition $\Gamma^* \cdot \left(1 - \frac{r_1}{r_2}\right) + \Gamma^* \leq 1$. 

---
Proof of Lemma 8. Recall that
\[ B^{(1)} = \left\{ (b_{1,1}, b_{1,2}, b_{2}, b_{3}) \in \mathbb{R}_+^4 : \begin{align*}
\Delta_{3,1} \cdot b_{1,1} + \Delta_{3,2} \cdot b_2 &\leq \Delta_{3,1} \cdot s_{1,1} + \Delta_{3,2} \cdot s_{2,1} \\
b_{1,i} &\leq s_{1,i} \quad i \in [2]
\end{align*} \right\} \]
In order to show that \( B^{(1)} \) is consistent, we need to prove the two properties from Definition 4:

For property (i), we must show that for any value of \( b_{1,2} \) that if such a value were moved to \( b_{1,1} \), then no constraints involving \( b_{1,1} \) would be broken (assuming \( b_2 = b_3 = 0 \)). The upper bound on \( b_{2,1} \) is \( s_{1,2}^{(1)} \) and so proving this lemma involves showing that (1) \( s_{1,2}^{(1)} \leq s_{1,1}^{(1)} \) and (2) \( \Delta_{3,1} \cdot s_{1,1}^{(1)} + \Delta_{3,2} \cdot s_{2,1}^{(1)} \). The first condition is a result of property \( (b) \) of Lemma 7, the second condition is true because of property \( (a) \) of Lemma 7 while the third condition is by constraint \( (5) \) from \([\text{UPPER3}]\).

For property (ii), we see that all the constraints limit a positive linear combination of \( b_{1,1}, b_{1,2}, b_2 \) and \( b_3 \) to be no more than some constant. Thus, \( B^{(1)} \) is clearly a downward closed polytope.

C.3.2. Proof of Theorem 4

Proof of Theorem 4. Throughout the proof, we take advantages of properties of the optimal solution to LP \([\text{UPPER3}]\) which are presented in a series of lemmas (Lemmas 6 and 7). Per property \( (a) \) of Lemma 6, any optimal solution \( \{r^*_t\}_{t \in [3]} \) to LP \([\text{UPPER3}]\) must have the property that constraint \( (5) \) is tight for both \( t = 1 \) and \( t = 2 \). That is, \( s_{1,t}^* + s_{2,t}^* + s_{3,t}^* = \Gamma^* \), \( t \in [2] \). Hence, the constraints of \( B^{(1)} \) can be written as follows:

\begin{align*}
&b_{1,1} + b_2 + b_3 \leq C \quad (42) \\
&\Delta_{3,1} \cdot b_{1,1} + \Delta_{3,2} \cdot b_2 \leq (1 - \Gamma^*) \cdot \Gamma^* \cdot r_3 \cdot C \quad (43) \\
&b_{1,i} \leq s_{1,i}^* \quad i \in [2] \quad (44)
\end{align*}

Consider an arbitrary input sequence \( \{I_t\}_{t \in [3]} \) and a particular time period \( \tau \in [T] \). Our goal is to show:

\[ \frac{\text{Rew}_{B^{(1)}, \tau}(\{I_t\}_{t \in [T]})}{\text{OPT}_{\tau}(\{I_t\}_{t \in [T]})} \geq \Gamma^* , \]

Here, with a slight abuse of notation, \( \text{Rew}_{B^{(1)}, \tau}(\{I_t\}_{t \in [T]}) \) is the reward of the POLYRA algorithm with feasible polytope \( B^{(1)} \) in time period \( \tau \). Recall that in our model, reward is earned whenever an agent is served as opposed to accepted, implying that all of \( \text{Rew}_{B^{(1)}, \tau}(\{I_t\}_{t \in [T]}) \) is earned by agents served in time period \( \tau \). We focus on \( I_\tau \) and define the following notation for our analysis:

- Let \( x_1, x_2, x_3 \) be the number of type 1, 2 and 3 agents appearing in \( I_\tau \) respectively.
- Let \( y_1, y_2, y_3 \) be the number of type 1, 2 and 3 agents respectively that the optimal inflexible benchmark serves out of \( I_\tau \). In other words, we have
  \[ \text{OPT}_{\tau}(\{I_t\}_{t \in [T]}) = r_1 \cdot y_1 + r_2 \cdot y_2 + r_3 \cdot y_3 , \]
  where \( y_i \leq x_i \) for \( i \in [3] \).
- Let \( B = (B_{1,1}, B_{2}, B_{3}, B_{1,2}) \) be the final number of agents assigned to each bucket at the end of the input sequence \( I_\tau \). By definition of our POLYRA algorithm and by Lemma 8, we know that \( B \in B^{(1)} \). (Lemma 8 shows that \( B^{(1)} \) is a consistent polytope.)
We show the result based on the following case-work, where in case 1, we assume that an incoming type 3 agent would be rejected. That is, an incoming type 3 agent could not be fitted to the final state $B$; see Section $H$ for details. In case 2, we assume that an incoming type 2 agent would be rejected while an incoming type 3 agent would be accepted, and so on.

**Case 1:** *Incoming type 3 would be rejected.* If an incoming type 3 agent would be rejected, then it must be the case $B_1 + B_2 + B_3 = C$ because if we had $B_1 + B_2 + B_3 < C$, then it would be possible to add a non-zero amount of an incoming type 3 agent to the bucket $B_3$ while still satisfying all the constraints of $B^{(1)}$. Given that $B_1 + B_2 + B_3 = C$, we know that our serving rule will simply serve all agents in the first row. Therefore, we can write the reward of our algorithm as:

$$\text{Rew}_{B^{(1)}, r}(\{I_t\}_{t \in [T]}) = r_1 \cdot B_1 + r_2 \cdot B_2 + r_3 \cdot B_3$$

$$= r_1 \cdot B_1 + r_2 \cdot B_2 + r_3 \cdot (C - B_1 - B_2)$$

$$= r_3 \cdot C - \Delta_{3, 1} \cdot B_1 - \Delta_{3, 2} \cdot B_2$$

$$\geq r_3 \cdot C - (1 - \Gamma^*) \cdot r_3 \cdot C$$

$$= \Gamma^* \cdot r_3 \cdot C$$

$$\geq \Gamma \cdot \text{opt}_{r}(\{I_t\}_{t \in [T]}).$$

The final inequality is true because the most reward the optimal algorithm can achieve in a single time period is $C \cdot r_3$. The last inequality is the desired result. Note that the only assumption we really needed to make this case work is $B_1 + B_2 + B_3 = C$, for which an incoming type 3 agent being rejected is a sufficient condition. Thus, for the remainder of this proof, we assume that $B_1 + B_2 + B_3 < C$.

**Case 2:** *Incoming type 3 would be accepted, but an incoming type 2 would be rejected.* The fact that an incoming type 3 agent at the end of $I_r$ could be accepted means that no type 3 agents in $I_r$ were rejected at all. By the design of our polytope $B^{(1)}$, we see that if a constraint involving a particular bucket is tight, then that constraint will remain tight as the variables $B_1, B_{1,2}, B_2$ and $B_3$ only increase as we accept more agents. This observation gives us the important relationship, that $B_3 = x_3$. (Recall $x_i$ is the number of type $i$ agents appearing in $I_r$.) As all type 3 agents from $I_r$ have been accepted into $B_3$. This along with the fact that $x_3 \geq y_3$ imply the following upper bound on the optimal revenue:

$$\text{opt}_{r}(\{I_t\}_{t \in [T]}) = r_1 \cdot y_1 + r_2 \cdot y_2 + r_3 \cdot y_3$$

$$\leq r_2 \cdot (C - y_3) + r_3 \cdot y_3$$

$$\leq r_2 \cdot (C - B_3) + r_3 \cdot B_3$$

(Recall that $y_i$, $i \in [3]$, is the number of type $i$ that the optimal inflexible benchmark serves out of $I_r$.) The fact that an incoming agent of type 2 would be rejected implies that some constraint in $B^{(1)}$ involving $B_2$ holds with equality. Since we already assumed that $B_1 + B_2 + B_3 < C$, the only other possible constraint is (13), implying that

$$\Delta_{3, 1} \cdot B_1 + \Delta_{3, 2} \cdot B_2 = (1 - \Gamma^*) \cdot r_3 \cdot C.$$
With this equation in hand, our goal will be to show the following inequality

\[
\text{Rew}_{B^{(1)}, r}(\{I_t\}_{t \in [T]}) \geq \Gamma^* \cdot r_2 \cdot C + \Delta_{3,2} \cdot B_3.
\]  

(50)

Showing this inequality completes the proof for this case because the CR can be bounded as follows:

\[
\frac{\text{Rew}_{B^{(1)}, r}(\{I_t\}_{t \in [T]})}{\text{OPT}_r(\{I_t\}_{t \in [T]})} \geq \frac{\Gamma^* \cdot r_2 \cdot C + \Delta_{3,2} \cdot B_3}{r_1 \cdot y_1 + r_2 \cdot y_2 + r_3 \cdot B_3} \geq \frac{\Gamma^* \cdot r_2 \cdot C}{r_2 \cdot (C - B_3) + r_3 \cdot B_3} = \frac{\Gamma^* \cdot r_2 \cdot C + \Delta_{3,2} \cdot B_3}{r_2 \cdot C + \Delta_{3,2} \cdot B_3} \geq \frac{\Gamma^* \cdot r_2 \cdot C}{r_2 \cdot C} = \Gamma^*.
\]

where the last inequality is true because of the fact that if \(a < b\) and \(c > d\), then \(\frac{a + c}{b + d} > \frac{a}{b}\). The last equality is the desired result.

To show inequality (50), we consider the following two cases: (a) \(B_{1,1} = 0\), and (b) \(B_{1,1} > 0\).

- **Case 2a**: \(B_{1,1} = 0\). Using equation (49), we get that \(B_2 = \frac{\Gamma^*}{\Delta_{3,2}} \cdot (1 - \Gamma^*) \cdot r_3 \cdot C\). By property (a) of Lemma 7, we have \(\Delta_{3,1} \cdot s_{1,1}^* + \Delta_{3,2} \cdot s_{2,1}^* = (1 - \Gamma^*) \cdot r_3 \cdot C\), which gives us

\[
B_2 = \frac{1}{\Delta_{3,2}} \cdot (\Delta_{3,1} \cdot s_{1,1}^* + \Delta_{3,2} \cdot s_{2,1}^*) \geq s_{1,1}^* + s_{2,1}^*.
\]

(51)

A lower bound on \(\text{Rew}_{B^{(1)}, r}(\{I_t\}_{t \in [T]})\) can be achieved by just considering the revenue from \(B_{1,1}, B_2\), and \(B_3\), which we are guaranteed to serve.

\[
\text{Rew}_{B^{(1)}, r}(\{I_t\}_{t \in [T]}) \geq r_1 \cdot B_{1,1} + r_2 \cdot B_2 + r_3 \cdot B_3
\]

\[
\geq r_1 \cdot 0 + r_2 \cdot (s_{1,1}^* + s_{2,1}^*) + r_3 \cdot B_3 \geq r_1 \cdot s_{1,1}^* + r_2 \cdot s_{2,1}^* + r_3 \cdot B_3 \geq \Gamma^* \cdot r_2 \cdot C + r_3 \cdot B_3.
\]

The last inequality is the desired result (see inequality (50)).

- **Case 2b**: \(B_{1,1} > 0\). For this case, we will take advantage of the fact that in our POLYRA algorithms, we prioritize assigning incoming flexible agents to buckets in the second row over the first row. In our three type problem with \(M = 1\), we know that if \(B_{1,1} > 0\), then it must mean that \(B_{1,2}\) is full. Applying this reasoning to this particular case gives us the assumption that \(B_{1,2} = s_{1,2}^*\).

Recall that all agents in the first row (i.e., \(B_{1,1}, B_2, B_3\)) have already been allocated the resource and we use the leftover \(C - B_{1,1} - B_2 - B_3\) resources to serve as many agents in \(B_{1,2}\) as possible. Therefore, we can write our reward as follows

\[
\text{Rew}_{B^{(1)}, r}(\{I_t\}_{t \in [T]}) = r_1 \cdot B_{1,1} + r_1 \cdot \min(B_{1,2}, C - B_{1,1} - B_2 - B_3) + r_2 \cdot B_2 + r_3 \cdot B_3.
\]

(52)
With respect to “min” in the above equation note that \( B_{1,2} \) is the number of type 1 agents remaining after serving all the agents in the first row, while \( C - B_{1,1} - B_2 - B_3 \) is the capacity remaining. We will now show that choosing \( B_{1,1} = s_{1,1}^* \) and \( B_2 = s_{2,1}^* \) gives an upper bound on equation (52). To see this, recall the following formula which relates \( B_{1,1} \) and \( B_2 \), which is obtained by solving for \( B_2 \) in terms of \( B_{1,1} \) in equation (49):

\[
B_2 = \frac{1}{\Delta_{3,2}} ((1 - \Gamma^*) \cdot r_3 \cdot C_3 - \Delta_{3,1} \cdot B_{1,1}) .
\]

We plug this value of \( B_2 \) into equation (52) and we consider how the resulting equation changes with \( B_{1,1} \). The coefficient of \( B_{1,1} \) depends whether \( \min(B_{1,2}, C - B_{1,1} - B_2 - B_3) \) is \( B_{1,2} \) or \( C - B_{1,1} - B_2 - B_3 \):

- If \( B_{1,2} \leq C - B_{1,1} - B_2 - B_3 \), by replacing \( B_2 \) by the expression in equation (53), the coefficient of \( B_{1,1} \) in equation (52) is

\[
r_1 + r_2 \cdot \frac{\Delta_{3,1}}{\Delta_{3,2}} + r_2 \cdot \frac{\Delta_{3,1}}{\Delta_{3,2}} = -(r_1 - r_1) \cdot \frac{\Delta_{3,1}}{\Delta_{3,2}} = 0 .
\]

Here, we used some simple algebra to get the first equality.

- If \( B_{1,2} > C - B_{1,1} - B_2 - B_3 \), by replacing \( B_2 \) by the expression in equation (53), the coefficient of \( B_{1,1} \) is

\[
r_1 - r_1 + r_1 \cdot \frac{\Delta_{3,1}}{\Delta_{3,2}} - r_2 \cdot \frac{\Delta_{3,1}}{\Delta_{3,2}} = -(r_2 - r_1) \cdot \frac{\Delta_{3,1}}{\Delta_{3,2}} < 0 .
\]

This shows that the coefficient of \( B_{1,1} \) in equation (52) is always negative, meaning that a lower bound is given by setting \( B_{1,1} \) as large as possible. By the last constraint in \( B^{(1)} \), i.e., equation (44) with \( i = 1 \), we see that \( B_{1,1} \leq s_{1,1}^* \), so a lower bound is obtained by choosing \( B_{1,1} = s_{1,1}^* \). This also implies that \( B_2 = s_{2,1}^* \). This is because we know (1) from equation (49) that \( \Delta_{3,1} \cdot B_{1,1} + \Delta_{3,2} \cdot B_2 = (1 - \Gamma^*) \cdot r_3 \cdot C \) and (2) from property [a] of Lemma 7 that \( \Delta_{3,1} \cdot s_{1,1}^* + \Delta_{3,2} \cdot s_{2,1}^* = (1 - \Gamma^*) \cdot r_3 \cdot C \).

At this point, we have concluded that choosing \( B_{1,1} = s_{1,1}^* \) and \( B_3 = s_{2,1}^* \) gives an upper bound on the reward. We now piece everything together by using equation (52), substituting \( B_{1,1} = s_{1,1}^* \), \( B_{1,2} = s_{1,2}^* \) and \( B_{2,1} = s_{2,1}^* \) to get:

\[
\text{Rew}_{B^{(1)}}(\{I_t\}_{t \in [r]}) = r_1 \cdot \min (B_{1,1} + B_{1,2}, C - B_{2,1} - B_3) + r_2 \cdot B_{2,1} + r_3 \cdot B_3 \\
= r_1 \cdot \min (s_{1,1}^* + s_{1,2}^*, C - s_{2,1}^* - B_3) + r_2 \cdot s_{2,1}^* + r_3 \cdot B_3 \\
\geq r_1 \cdot \min (s_{1,1}^* + s_{1,2}^* - B_3, C - s_{2,1}^* - B_3) + r_2 \cdot s_{2,1}^* + r_3 \cdot B_3 \\
\geq r_1 \cdot \min (s_{1,1}^* + s_{1,2}^* - B_3, C - s_{2,1}^*) + r_2 \cdot s_{2,1}^* + \Delta_{3,1} \cdot B_3 \\
\geq \Gamma \cdot C \cdot r_2 + \Delta_{3,2} \cdot B_3 . \quad \text{by constraints (9), (8) and } \Delta_{3,1} \geq \Delta_{3,2}
\]

This completes the proof of Case 2 as we have shown the desired result from equation (50).

**Case 3: Incoming type 2 and 3 are accepted, but incoming type 1 is rejected.** Using a similar argument as above, we get that \( B_2 = x_2 \geq y_2 \) and \( B_3 = x_3 \geq y_3 \) as no type 2 or type 3 agents in \( I_s \) have been rejected. Furthermore, this means that constraints (12) and (13) of \( B^{(1)} \) are not tight. Therefore, in order
for an incoming type 1 to be rejected, it must be the case that constraints \((44)\) is tight for any \(i = \{1, 2\}\), implying that \(B_{1,1} = s_{1,1}^*\) and \(B_{1,2} = s_{1,2}^*\). Hence, we have

\[
\text{Rew}_{\mathcal{B}_{(1)}}(\{I_i\}_{i \in \{2\}}) = r_1 \cdot \min (B_{1,1} + B_{1,2}, C - B_2 - B_3) + r_2 \cdot B_2 + r_3 \cdot B_3
\]

\[
= r_1 \cdot \min (s_{1,1}^*, s_{1,2}^*, C - B_2 - B_3) + r_2 \cdot B_2 + r_3 \cdot B_3
\]

\[
\geq r_1 \cdot \min (\Gamma^*, C - B_2 - B_3) + r_2 \cdot B_2 + r_3 \cdot B_3 \quad \text{by constraint } \(7\)
\]

\[
\geq r_1 \cdot \min (\Gamma^*, C - B_2 - B_3) + (r_2 - r_1) \cdot B_2 + (r_3 - r_1) \cdot B_3 + r_1 (B_2 + B_3)
\]

\[
\geq r_1 \cdot (\Gamma^* + \Delta_{2,1} \cdot B_2 + \Delta_{3,1} \cdot B_3)
\]

Therefore, we have the desired inequality.

**All incoming agent types are accepted.** Since no incoming agents are rejected, our POLYRA algorithm with feasible set \(\mathcal{B}^{(1)}\) has accepted all agents in \(I_s\) and will serve all the type 2 and 3 agents along with as many type 1 agents as possible. This is clearly at least as much reward as what the clairvoyant optimal algorithm would serve, so we achieve a CR of 1.

### C.3.3. Proof of Theorem 5

**Proof of Theorem 5** Recall that \(\mathcal{B}^{(2)} := \mathcal{B}_{nest} (n_1, n_2, n_3)\), where

\[
n_1 = \frac{1}{2} (s_{1,1}^* + s_{1,2}^*), \quad n_2 = \frac{1}{2} (s_{1,1}^* + s_{1,2}^* + s_{2,1}^* + s_{2,2}^*), \quad n_3 = C.
\]

For any \(i \in [3]\), define \(\Delta n_i = n_i - n_{i-1}\), where we set \(n_0 = 0\). By property \((a)\) of Lemma \(6\) which states that \(s_{1,t}^* + s_{2,t}^* + s_{3,t}^* = C\), \(t \in [2]\), we have \(\Delta n_3 = n_3 - n_2 = \frac{1}{2} (s_{1,1}^* + s_{1,2}^*)\), resulting in the following values of \(\Delta n_1, \Delta n_2, \Delta n_3\)

\[
\Delta n_1 = \frac{1}{2} (s_{1,1}^* + s_{1,2}^*), \quad \Delta n_2 = \frac{1}{2} (s_{2,1}^* + s_{2,2}^*), \quad \Delta n_3 = \frac{1}{2} (s_{3,1}^* + s_{3,2}^*).
\]

We now apply Theorem \(7\) which gives an LP that calculates the CR of any particular instance of the nested polytope. We use that LP to calculate \(\Gamma_{nest}\), the CR of \(\mathcal{B}_{nest} (n_1, n_2, n_3)\) as follows

\[
\Gamma_{nest} := \max_{\Gamma : (\mathcal{S}_i', \mathcal{S}_j') \in \{3\} \times \{2\}} \Gamma \quad \text{s.t.} \quad \Gamma \cdot C \cdot r_1 \leq s_{1,1}^* \cdot r_1 \quad (54)
\]

\[
\Gamma \cdot C \cdot r_2 \leq s_{1,2}^* \cdot r_1 + s_{2,2}^* \cdot r_2 \quad (55)
\]

\[
\Gamma \cdot C \cdot r_3 \leq s_{1,1}^* \cdot r_1 + s_{2,3}^* \cdot r_2 + s_{3,3}^* \cdot r_3 \quad (56)
\]

\[
\Delta n_1 \leq s_{1,1}^* \cdot s_{1,2}^* \leq 2 \cdot \Delta n_1 \quad (57)
\]

\[
\Delta n_2 \leq s_{2,1}^* \leq 2 \cdot \Delta n_2 \quad (58)
\]

\[
s_{i,j}^3 \leq \Delta n_i \quad i \in [3] \quad (59)
\]

\[
\sum_{i=1}^j s_i^j \leq C \quad j \in [3] \quad (60)
\]

In order to show that a POLYRA algorithm with feasible region \(\mathcal{B}^{(2)}\) achieves a \(\Gamma^*\) CR, it would be sufficient to show that \(\Gamma_{nest} \geq \Gamma^*\). To do so, we start with the optimal solution to the upper bound \((\Gamma^*, \{s_{i,j}^*\}_{i \in [3], j \in [2]})\)
and use it to create a feasible instance of the aforementioned LP for which $\Gamma = \Gamma^*$. We set the variables $s_i^j$ as follows:

\[
s_1^1 = 2 \cdot \Delta n_1, \quad s_1^2 = \min(C - 2 \cdot \Delta n_2, 2 \cdot \Delta n_1), \quad s_2^2 = 2 \cdot \Delta n_2, \quad s_3^3 = \Delta n_i, i \in [3]
\]  

(61)

We verify that such an assignment is feasible. We have that constraints (54), (55) and (56) are satisfied by looking at the corresponding constraints (7), (9), and (6) respectively from the upper bound. To see why, consider constraint (54). This constraint holds if $\Gamma^* \cdot C \cdot r_1 \leq s_1^1 \cdot r_1$, which by our construction, it is equivalent to $\Gamma^* \cdot C \cdot r_1 \leq (s_{1,1}^* + s_{1,2}^* \cdot r_1$. Then, clearly $\Gamma^* \cdot C \cdot r_1 \leq (s_{1,1}^* + s_{1,2}^*) \cdot r_1$ holds by constraint (7) in (UPPER3).

Next consider constraint (55). This constraint holds if $\Gamma^* \cdot C \cdot r_2 \leq s_1^2 \cdot r_1 + s_2^2 \cdot r_2$. By our construction, this is the same as:

\[
\Gamma^* \cdot C \cdot r_2 \leq \min(C - 2 \cdot \Delta n_2, 2 \cdot \Delta n_1) \cdot r_1 + 3 \cdot \Delta n_2 \cdot r_2 = \min(C - (s_{2,1}^* + s_{2,2}^*), (s_{1,1}^* + s_{1,2}^*)) \cdot r_1 + (s_{2,1}^* + s_{2,2}^*) \cdot r_2
\]

The above inequality holds per constraints (8) and (9) in (UPPER3). To show constraint (56) holds, note that this constraint is equivalent to

\[
\Gamma \cdot C \cdot r_3 \leq s_1^3 \cdot r_1 + s_2^2 \cdot r_2 + s_3^3 \cdot r_3 = \frac{1}{2} (s_{1,1}^* + s_{1,2}^*) \cdot r_1 + \frac{1}{2} (s_{2,1}^* + s_{2,2}^*) \cdot r_2 + \frac{1}{2} (s_{3,1}^* + s_{3,2}^*) \cdot r_3.
\]

The above inequality holds because of constraint (9) in (UPPER3).

Now, consider constraint (57). For both $s_1^1$ and $s_1^2$, the upper bound of $2 \cdot \Delta n_1$ is immediately satisfied, while the lower bound of $\Delta n_1$ is trivially satisfied by $s_1^1$. For the lower bound on $s_1^2 = \min(C - 2 \cdot \Delta n_2, 2 \cdot \Delta n_1)$ if $2 \cdot \Delta n_1$ is taken in the minimum, then we are done. Otherwise, if $C - 2 \cdot \Delta n_2$ is the minimum, then the lower bound is also satisfied because

\[
C - 2 \cdot \Delta n_2 = C - (s_{2,1}^* + s_{2,2}^*) \geq \frac{1}{2} (s_{1,1}^* + s_{1,2}^*) = \Delta n_1,
\]

where the inequality is true by property (b) from Lemma 7. For constraint (58) and (59), they are all trivially satisfied by our assignment in equation (61). Finally, we argue that constraint (60) is satisfied. For $j = 1$, constraint (60) required us to show $s_1^1 = 2 \cdot \Delta n_1 = s_{1,1}^* + s_{1,2}^* \leq C$, which is true because of property (b) from Lemma 7 which states that $r_1 \cdot (s_{1,1}^* + s_{1,2}^*) = r_1 \cdot \Gamma^* \cdot C$. For $j = 2$, constraint (60) we need to show that $s_1^2 + s_2^1 \leq C$, which is true by the definition of $s_1^1$ and $s_2^2$ from equation (61). For $j = 3$, constraint (60) required us to show that

\[
s_1^3 + s_2^3 + s_3^3 = \Delta n_1 + \Delta n_2 + \Delta n_3 = \frac{1}{2} (s_{1,1}^* + s_{1,2}^* + s_{2,1}^* + s_{2,2}^* + s_{3,1}^* + s_{3,2}^*) \leq \frac{1}{2} \cdot 2 \cdot C,
\]

where the inequality holds because of constraint (5) from LP (UPPER3).

We have thus shown that the assignment of $s_i^j$ from (61), along with $\Gamma = \Gamma^*$ leads to a feasible solution, meaning that $\Gamma_{\text{feas}} \geq \Gamma^*$. This allows us to conclude by Theorem 7 that this particular instance of the nested algorithm as given by (61) has a CR at least $\Gamma^*$, and hence is optimal.
Appendix D: Proof of Statements from Section 7

D.1. Proof of Theorem 6

Proof of Theorem 6 As mentioned in the proof sketch, the proof of this theorem consists of 3 parts: (1) Arguing why we have $\frac{1}{C}$ in $\min (\Gamma_{LP}, \frac{1}{C})$, (2) Arguing that $\Gamma_{LP}$ is a valid upper bound, and (3) Showing that $\Gamma_{LP}$ has a nice closed form expression as stated in the theorem.

Taking the minimum of $\Gamma_{LP}$ and $\frac{1}{C}$. From Theorem 4 of Ball and Queyranne (2009), we have that $\frac{1}{C}$ is an upper bound on the CR of any non-anticipating algorithm when faced with $K - M$ types or revenue $r_M < r_{M+1} \ldots < r_K$ where all the types are inflexible. For our model (which includes $K - M$ inflexible types), it is clear that this quantity is also an upper bound as the adversary can choose not to include any flexible agents in the input sequence.

$\Gamma_{LP}$ is a valid upper bound. To show that $\Gamma_{LP}$ is an upper bound on the CR of any non-anticipating algorithm, we will use a similar argument as in the proof of Theorem 3 where we present a number of overlapping input sequences and argue that the behavior of any non-anticipating algorithm must be similar across these input sequences.

We start out with a primary input sequence and we consider truncated versions of it. Define our primary input sequence $\{I_1, I_2\}$ as follows:

\[ I_1 = \underbrace{\{1, \ldots, 1, \ldots, 1\}}_{C} \underbrace{\{2, \ldots, 2\}}_{C} \ldots \underbrace{\{K, \ldots, K\}}_{C} \]
\[ I_2 = \underbrace{\{M+1, \ldots, M+1\}}_{C} \underbrace{\{M+2, \ldots, M+2\}}_{C} \ldots \underbrace{\{K, \ldots, K\}}_{C} \]

Our input sequence $\{I_1, I_2\}$ consists of $C \cdot (K + M)$ agents in total over the course of the 2 time periods. We define $2 \cdot K - M$ truncated versions of this input sequence, denoted by $\{I_i, I_2\}$ for $i \in [2K - M]$ which consist of sending only the first $C \cdot i$ of these $C \cdot (2 \cdot K - M)$ agents. For example, the input sequence $\{I_1, I_2\}$ would contain all $C$ copies of all $K$ types in the first time period, and then just $C$ copies of type $M + 1$ in the second time period.

Consider any algorithm $\mathcal{A}$ that claims to be $\Gamma_{LP}$ competitive and how it behaves when faced with our primary input sequence $\{I_1, I_2\}$. Let $s_{i,t}$ be the number of type $i$ agents that $\mathcal{A}$ accepts from $I_t$ and define $s_i = s_{i,1} + s_{i,2}$ to be the total number of type $i$ agents accepted in $\{I_1, I_2\}$. Note that for $t = 2$, since type $i \in [M]$ agents do not appear in $I_2$, we have that $s_{i,2} = 0$. In order for $\mathcal{A}$ to be truly $\Gamma_{LP}$ competitive, it must achieve this CR against not only $\{I_1, I_2\}$, but also $\{I_i, I_2\}$ for all $i \in [2K + M]$. For each of these input sequences, we consider what must be true about the $s_{i,1}$’s in order for $\mathcal{A}$ to be $\Gamma_{LP}$ competitive against them.

- $\Gamma_{LP}$ competitive against $\{I_1, I_2\}$. The optimal clairvoyant revenue for each of $I_1$ and $I_2$ is $r_K \cdot C$. In order for our algorithm to achieve an $\Gamma_{LP}$ CR against that, a necessary condition is that

\[ \sum_{t=1}^{2} \sum_{i=1}^{K} r_{i,t} \cdot s_{i,t} \geq 2 \cdot \Gamma_{LP} \cdot r_K. \]

The left hand side can also be written as $\sum_{i=1}^{K} \sum_{t=1}^{2} r_{i,t} \cdot s_i$. This results in constraint (16) from the LP.
• \( \Gamma_{LP} \) competitive against \( \{I_1^i, I_2^i\} \) for \( i \in [M] \). For such values of \( i \), we only look at the optimal reward for \( I_1^i \) since \( I_2^i = \emptyset \). The optimal algorithm achieves a reward of \( C \cdot r_i \) on \( I_1^i \). In order for \( A \) to be \( \Gamma_{LP} \) competitive, certainly a necessary condition is that the total revenue from the agents we have accepted from \( I_1^i \) must be at least \( \Gamma_{LP} \cdot r_i \cdot C \). Note that this is a necessary, and not a sufficient condition—it is possible to have accepted enough agents, but we may not be able to serve all of them given the \( C \) capacity and recall that reward is earned when the agent is served. Since \( I_1^i \) and \( I_1 \) are identical for the first \( C \cdot i \) agents, when \( A \) is run on \( I_1^i \), it must have the same behavior as on the first \( C \cdot i \) agents of \( I_1 \), meaning \( A \) accepts \( s_{j,1} \) copies of type \( j \) for \( j \in [i] \). Therefore, the following must necessarily be true for \( A \) to truly be \( \Gamma_{LP} \) competitive:

\[
\sum_{j=1}^{i} r_j \cdot s_{j,1} \geq \Gamma_{LP} \cdot r_i \cdot C
\]

Since \( s_{i,2} = 0 \) for \( i \in [M] \), we can replace \( s_{j,1} \) with \( s_j \) and this gives constraint (14) in the LP.

• \( \Gamma_{LP} \) competitive against \( \{I_1^i, I_2^i\} \) and \( \{I_1^{K+i}, I_2^{K+i}\} \) for \( i \in \{M+1, \ldots, K\} \). Here, we consider these two input sequences together because they are both truncated after \( C \) copies of type \( i \) agents have arrived, meaning that the optimal revenue for the time period that the truncation happens is \( C \cdot r_i \).

For \( \{I_1^i, I_2^i\} \), the same reasoning as the \( i \in [M] \) case applies, so we must have that

\[
\sum_{j=1}^{i} r_j \cdot s_{j,1} \geq \Gamma_{LP} \cdot r_i \cdot C
\]

(62)

For \( \{I_1^{K+i}, I_2^{K+i}\} \), we claim that the following is a lower bound on the total reward of the agents that \( A \) has accepted at the end of \( I_2^{K+i} \)

\[
\sum_{j=1}^{M} r_j \cdot s_{j,1} + \sum_{j=M+1}^{i} r_j \cdot s_{j,2}
\]

(63)

This is true because at the end of \( I_2^{K+i} \), the agents that \( A \) has accepted can only contain flexible agents from \( t = 1 \) and inflexible agents that arrived in \( I_2^{K+i} \). Therefore, the quantity above must also be at least \( \Gamma_{LP} \cdot r_i \cdot C \). Adding up the inequality (62) with the fact that expression (63) must be at least \( \Gamma_{LP} \cdot r_i \cdot C \) yields

\[
\sum_{j=1}^{i} r_j \cdot s_{j,1} + \sum_{j=1}^{M} r_j \cdot s_{j,1} + \sum_{j=M+1}^{i} r_j \cdot s_{j,2}
\]

\[
= \left( \sum_{j=1}^{M} r_j \cdot s_{j,1} + \sum_{j=M+1}^{i} r_j \cdot s_{j,1} \right) + \sum_{j=1}^{M} r_j \cdot s_{j,1} + \sum_{j=M+1}^{i} r_j \cdot s_{j,2}
\]

\[
= 2 \sum_{j=1}^{M} r_j \cdot s_{j,1} + \sum_{j=M+1}^{i} r_j \cdot (s_{j,1} + s_{j,2})
\]

\[
= 2 \cdot \sum_{j=1}^{M} r_j \cdot s_j + \sum_{j=M+1}^{i} r_j \cdot s_j
\]

\[
\geq 2 \cdot \Gamma_{LP} \cdot C
\]

This gives us constraint (15) in the LP.
Finally, we know that since all the agents in \( \{I_1, I_2\} \) must be served within these 2 time periods, the total number of agents accepted cannot exceed \( 2 \cdot C \). This gives us constraint (17). Thus, we have verified that any \( \Gamma_{LP} \) is an upper bound on the CR of any non-anticipating algorithm \( \mathcal{A} \).

**Verifying Optimality.** Now that we have shown \( \Gamma_{LP} \) is indeed an upper bound, we proceed to verify that the given formula of \( \Gamma_{LP} \) in equation (18) is correct. We notice that the linear program (SIMPLE-UPPER) is one with \( K + 1 \) variables and \( K + 1 \) constraints, meaning that if we can find a particular assignment of the \( s_i \)'s and \( \Gamma \) such that all inequalities are satisfied with equality, then we can be sure that it is the optimal solution. Let \( \Gamma_{LP} \) be defined as in equation (18), and consider the following choice of \( s_i \) for \( i \in [K] \):

\[
s_i^* = \begin{cases} 
\Gamma_{LP} \cdot \left(1 - \frac{r_{i-1}}{r_i}\right) & i \in \{1, 2, \ldots, M\} \\
2 \cdot \Gamma_{LP} \cdot \left(1 - \frac{r_{i-1}}{r_i}\right) & i \in \{M + 1, \ldots, K - 1\} \\
2 \cdot \Gamma_{LP} \cdot \left(1 - \frac{r_{i-1}}{r_i} + \frac{1}{2} \cdot \frac{r_{M}}{r_K}\right) & i = K.
\end{cases}
\]

We verify that this choice of \( s_i^* \) along with \( \Gamma_{LP} \) satisfy all constraints from (SIMPLE-UPPER) with equality.

- **Constraint (14):** For any \( k \in [M] \), we have

\[
\sum_{i=1}^{k} r_i \cdot s_i^* = \sum_{i=1}^{k} r_i \cdot \Gamma_{LP} \cdot \left(1 - \frac{r_{i-1}}{r_i}\right) = \Gamma_{LP} \cdot \sum_{i=1}^{k} (r_i - r_{i-1}) = \Gamma_{LP} \cdot r_k.
\]

- **Constraint (15):** From constraint (14) for \( k = M \), we know that the term \( 2 \cdot \sum_{i=1}^{M} r_i \cdot s_i^* = 2 \cdot \Gamma_{LP} \cdot r_M \).

This implies that

\[
2 \cdot \sum_{i=1}^{M} r_i \cdot s_i^* + \sum_{i=M+1}^{K} r_i \cdot s_i^* = 2 \cdot \Gamma_{LP} \cdot r_M + 2 \cdot \Gamma_{LP} \cdot \sum_{i=M+1}^{K} r_i \cdot \left(1 - \frac{r_{i-1}}{r_i}\right) = 2 \cdot \Gamma_{LP} \cdot \left( r_M + \sum_{i=M+1}^{K-1} (r_i - r_{i-1}) \right) = 2 \cdot \Gamma_{LP} \cdot r_K.
\]

- **Constraint (16):** By definition, and our previous results that

\[
\sum_{i=1}^{M} r_i \cdot s_i^* = \Gamma_{LP} \cdot r_M \quad \text{and} \quad 2 \cdot \sum_{i=1}^{M} r_i \cdot s_i^* + \sum_{i=M+1}^{K-1} r_i \cdot s_i^* = 2 \cdot \Gamma_{LP} \cdot r_K - 1,
\]

we have

\[
\sum_{i=1}^{M} r_i \cdot s_i^* + \sum_{i=M+1}^{K-1} r_i \cdot s_i^* + r_K \cdot s_K^* = 2 \cdot \Gamma_{LP} \cdot \left( \frac{1}{2} \cdot r_M + (r_K - r_K - 1 - \frac{1}{2}r_M) \right) = 2 \cdot \Gamma_{LP} \cdot r_K.
\]

- **Constraint (17):** Again by definition, we have

\[
\sum_{i=1}^{K} s_i^* = \sum_{i=1}^{M} s_i^* + \sum_{i=M+1}^{K-1} s_i^* + s_K^* = \Gamma_{LP} \cdot \left[ \sum_{i=1}^{M} \left(1 - \frac{r_{i-1}}{r_i}\right) + \sum_{i=M+1}^{K-1} 2 \cdot \left(1 - \frac{r_{i-1}}{r_i}\right) + 2 \cdot \left(1 - \frac{r_{K-1}}{r_K} + \frac{1}{2} \cdot \frac{r_M}{r_K}\right) \right]
\]

\[
= \Gamma_{LP} \cdot \left[ M + 2 \cdot (K - M - 1) + 2 - \sum_{i=1}^{M} \frac{r_{i-1}}{r_i} - 2 \sum_{i=M+1}^{K} \frac{r_{i-1}}{r_i} + \frac{r_M}{r_K} \right] = 2,
\]

where the last equation holds because \( \Gamma_{LP} = \frac{2}{2K - M - \sum_{i=1}^{M} r_{i-1} - 2 \sum_{i=M+1}^{K} \frac{r_{i-1}}{r_i} + \frac{r_M}{r_K}} \). This completes the proof that our choice of \( s_i^* \) is optimal.

Thus, we have shown that \( \Gamma_{up} \) is indeed an upper bound on the performance of any non-anticipating algorithm.
D.2. Proof of Theorem 7

Proof of Theorem 7: Fix an optimal solution to the LP in Theorem 7 denoted by \((\Gamma_{\text{nest}}, \{s_i^j\}_{j \in [K], i \in [I]})\). Throughout this proof, when we refer to \(s_i^j\), it will be this particular optimal solution (we avoid giving it a \(*\) due to the existing superscript). In the proof, we will greatly leverage the fact that this solution \((\Gamma_{\text{nest}}, \{s_i^j\}_{j \in [K], i \in [I]})\) satisfies all the constraints in the LP in Theorem 7.

Fix a particular time step \(\tau \in [T]\) and consider the final state \(B\) that our POLYRA algorithm serves from \(B_{\text{nest}}(n_1, \ldots, n_K)\) has at the end of the input sequence. For type \(i \in [K]\), define \(x_i\) to be the number of copies of type \(i\) that appeared in \(I_\tau\), while let \(y_i\) be the number of copies of type \(i\) that the optimal clairvoyant algorithm serves from \(I_\tau\). Given that our optimal clairvoyant algorithm is not flexible, we have that \(y_i \leq x_i\) for all \(i \in [K]\). For notational convenience, for each \(i \in [K]\), let \(z_i\) be the number of type \(i\) agents that are from the first row of buckets in this final state \(B\) and for \(i \in [M]\), let \(w_i\) be the number of type \(i\) agents that are served in \(B_{i,2}\) where we define \(w_i = 0\) if \(i > M\). From our serving rule, it is clear that \(z_i = B_{i,1}\) for \(i \leq M\) and \(z_i = B_i\) otherwise, and that \(w_i \leq B_{i,2}\) for all \(i \in [M]\).

Let \(k_0\) be the maximum index such that our algorithm serves as many type \(k\) agents as have arrived in the input sequence. Mathematically, we have

\[
k_0 := \max\{i \in [K] \mid z_i + w_i < x_i\},
\]

where we define \(k_0 = 0\) if the set we take the max over is empty. (Note that \(k_0 = 0\) is not very interesting: it means that for every type \(i\), we are serving as many copies of that type as have arrived in the input sequence, meaning that algorithm will certainly be 1 competitive. Thus, we can simply assume that \(k_0 \neq 0\).) By this definition of \(k_0\), we know that \((z_i + w_i) \geq x_i \geq y_i\) for all \(i > k_0\). Our goal will be to show a lower bound on \(\text{Rew}_{B_{\text{nest}},\tau}(\{I_t\}_{t \in [T]}\) and an upper bound on \(\text{OPT}_{\tau}(\{I_t\}_{t \in [T]}\), which will allow us to lower bound their ratio. We start with presenting an upper bound on \(\text{OPT}_{\tau}(\{I_t\}_{t \in [T]}\):

\[
\text{OPT}_{\tau}(\{I_t\}_{t \in [T]}\) = \sum_{i=1}^{k_0} r_i \cdot y_i + \sum_{i=k_0+1}^{K} r_i \cdot y_i = \sum_{i=1}^{k_0} r_i \cdot y_i + r_{k_0} \cdot \left(\sum_{i=k_0+1}^{K} y_i\right) + \sum_{i=k_0+1}^{K} \Delta_i \cdot x_i \leq C \cdot r_{k_0} + \sum_{i=k_0+1}^{K} \Delta_i \cdot x_i.
\]

Next, we will show the following lower bound on \(\text{Rew}_{B_{\text{nest}},\tau}(\{I_t\}_{t \in [T]}\):

\[
\text{Rew}_{B_{\text{nest}},\tau}(\{I_t\}_{t \in [T]}\) \geq \Gamma_{\text{nest}} \cdot r_{k_0} \cdot C + \sum_{i=k_0+1}^{K} \Delta_i \cdot x_i.
\]

Inequalities (66) and (67) together would give us the desired result as the first terms of are off by a factor of \(\Gamma_{\text{nest}}\) while the second term (i.e., \(\sum_{i=k_0+1}^{K} \Delta_i \cdot x_i\)) is the same.

In the rest of this proof, we show the lower bound on \(\text{Rew}_{B_{\text{nest}},\tau}(\{I_t\}_{t \in [T]}\) in equation (67). To do so, we focus on type \(k_0\). There are two reasons why our algorithm does not serve all the type \(k_0\) agents that have arrived in \(I_\tau:\)

9 To make the definition of \(k_0\) more clear, consider an example with \(k_0 = 3\) and the number of type \(K = 5\). Then, we know that no agents of type 4 and 5 are rejected while some of the agents of type 3 are rejected. Further, we cannot make such statement for agents of type \(k < k_0 = 3\), i.e., types 1 and 2.
Case 1. In this case, at some point in \( I_r \), our algorithm rejected an agent of type \( k_0 \).

1. Case 2. Our algorithm accepted all type \( k_0 \) agents from \( I_r \), but it is not able to serve all of them because some of them are in bucket \( B_{k_0,2} \) and we do not have enough capacity remaining after serving all the agents in buckets \( B_{1,1}, \ldots, B_{M,1}, B_{M+1}, \ldots, B_K \).

We consider these two reasons separately as case 1 and case 2.

Case 1. Since all nested polytopes are consistent as per Definition 1, we know that if our algorithm rejected a type \( k_0 \) agent at some point in \( I_r \), then the final state \( B \) also has the property that it would reject an incoming type \( k_0 \) agent. This implies that some constraint involving each of the 1 or 2 bucket(s) where type \( k_0 \) agents are allowed to go must be tight. We consider two sub-cases based on whether \( k_0 \leq M \) or \( k_0 > M \).

- Case 1a: \( k_0 \leq M \). In this case, for each of \( B_{k_0,1} \) and \( B_{k_0,2} \), there must be at least one constraint involving them that is tight in \( B \). We see that all constraints involving \( B_{k_0,1} \) are of the form

\[
\sum_{i=1}^\ell_1 B_{i,1} \leq n_{\ell_1}, \quad \ell_1 \geq k_0, \quad \text{or} \quad \sum_{i=1}^M B_{i,1} + \sum_{i=M}^\ell_1 B_i \leq n_{\ell_1}, \quad \ell_1 \geq M + 1
\]

while for \( B_{k_0,2} \), the only constraints involving it are of the form

\[
\sum_{i=1}^{\ell_2} B_{i,2} \leq n_{\ell_2}, \quad \ell_2 \geq k_0.
\]

Let \( \ell_1, \ell_2 \in \{k_0, k_0 + 1, \ldots, K\} \) be indices for which the above constraints involving \( B_{k_0,1} \) and \( B_{k_0,2} \) respectively are tight, meaning we have

\[
\sum_{i=1}^{\ell_1} z_i = n_{\ell_1}, \quad \text{and} \quad \sum_{i=1}^{\ell_2} B_{i,2} = n_{\ell_2}, \quad \text{(68)}
\]

where the first equality is true because by our serving rule \( z_i \) is equal to \( B_i \) or \( B_{i,1} \).

For notational convenience, define \( Z, Z', W, \) and \( W' \) to be the following:

\[
Z = \sum_{i=1}^{k_0} z_i, \quad Z' = \sum_{i=k_0+1}^{K} z_i, \quad W = \sum_{i=1}^{k_0} w_i, \quad W' = \sum_{i=k_0+1}^{M} w_i. \quad \text{(69)}
\]

(Recall that we are in the case of \( k_0 \leq M \) and hence \( W' \) is well-defined. Furthermore, since \( w_i = 0 \) for \( i > M \), \( W' = \sum_{i=k_0+1}^{K} w_i \).) Furthermore define the following quantities for the rewards from serving such agents.

\[
R_Z = \sum_{i=1}^{k_0} r_i \cdot z_i, \quad R_{Z'} = \sum_{i=k_0+1}^{K} r_i \cdot z_i, \quad R_W = \sum_{i=1}^{k_0} r_i \cdot w_i, \quad R_{W'} = \sum_{i=k_0+1}^{M} r_i \cdot w_i. \quad \text{(70)}
\]

It is clear that \( \text{Rew}_{B_{ Brent,k_0}}(\{I_i\}_{i \in [r]}) = R_Z + R_{Z'} + R_W + R_{W'} \). Our goal is to show inequality \( (67) \). To do so, we will decompose \( R_{Z'} \) and \( R_{W'} \) into terms which have \( \Delta_{i,k_0} \) as in \( (67) \). We see that:

\[
R_{Z'} + R_{W'} = \sum_{k_0+1}^{K} r_i \cdot z_i + \sum_{k_0+1}^{M} r_i \cdot w_i = r_{k_0} \cdot (Z' + W') + \sum_{k_0+1}^{K} \Delta_{i,k_0} \cdot z_i + \sum_{k_0+1}^{M} \Delta_{i,k_0} \cdot w_i \\
\geq r_{k_0} \cdot (Z' + W') + \sum_{k_0+1}^{K} \Delta_{i,k_0} \cdot x_i.
\]
where the last inequality is true because by definition of \( k_0 \), we have that for types \( i \geq k_0, w_i + z_i \geq x_i \). This gives us

\[
\text{Rew}_{\text{nest}} \left( \{ I_t \} \in [T] \right) \geq R_Z + R_W + r_{k_0} \cdot (Z' + W') + \sum_{k_0+1}^K \Delta_{i,k_0} \cdot x_i. \tag{71}
\]

We provide a lower bound on \( R_Z + r_{k_0} \cdot Z' \) as follows:

\[
R_Z + r_{k_0} \cdot Z' = \sum_{i=1}^{k_0} r_i \cdot z_i + r_{k_0} \cdot \left( \sum_{i=k_0+1}^{K} z_i + \sum_{i=1}^{K} z_i \right) \\
\geq \sum_{i=1}^{k_0} r_i \cdot \Delta n_i + r_{k_0} \cdot \left( n_{k_0} + \sum_{i=k_0+1}^{K} z_i \right) \\
= \sum_{i=1}^{k_0} r_i \cdot \Delta n_i + r_{k_0} \cdot (Z + Z' - n_{k_0}), \tag{72}
\]

where the inequality holds because of equation (68) that \( \sum_{i=1}^{k_0} z_i = n_{k_0} \). To see why, observe that by the structure of the feasibility polytope (\( B_{\text{nest}}(n_1, \ldots, n_K) \)), we have \( \sum_{i=1}^{k_0} z_i \leq n_i, i \in [K] \). This and the fact that \( \sum_{i=1}^{k_0} z_i = n_{k_0} \) and \( r_1 < r_2 < \ldots < r_K \) imply that \( \sum_{i=1}^{k_0} r_i \cdot z_i + r_{k_0} \cdot \sum_{i=k_0+1}^{K} z_i \geq \sum_{i=1}^{k_0} r_i \cdot \Delta n_i + r_{k_0} \cdot (n_{k_0} - n_{k_0}) \). Put differently, since the sum (i.e., \( \sum_{i=1}^{k_0} z_i = n_{k_0} \)) is fixed and \( r_1 < r_2 < \ldots < r_{k_0} \), we get a lower bound by iteratively choosing \( z_1 \) to be as large as possible, followed by \( z_2 \), etc. Doing so sets all the \( z_i \)'s to be equal to \( \Delta n_i \).

We take the result of inequality (72) and plug it back into equation (71) to get the following lower bound

\[
\text{Rew}_{\text{nest}} \left( \{ I_t \} \in [T] \right) \geq \sum_{i=1}^{k_0} r_i \cdot \Delta n_i + R_W + r_{k_0} \cdot (W' + Z + Z' - n_{k_0}) + \sum_{k_0+1}^{K} \Delta_{i,k_0} \cdot x_i \tag{73}
\]

\[
\geq \sum_{i=1}^{k_0} r_i \cdot \Delta n_i + \sum_{i=1}^{k_0} r_i \cdot (s_{i}^{k_0} - \Delta n_i) + \sum_{k_0+1}^{K} \Delta_{i,k_0} \cdot x_i \tag{74}
\]

\[
\geq \Gamma_{\text{nest}} \cdot r_{k} \cdot C + \sum_{k_0+1}^{K} \Delta_{i,k_0} \cdot x_i \tag{75}
\]

where inequality (74) holds because of Lemma 9 stated below, which we can apply because the total number of served agents in our algorithm \( Z + Z' \geq n_{k_1} \geq n_{k_0} \) by equation (68). By this lemma, we know \( R_W + r_{k_0} \cdot (W' + Z + Z' - n_{k_0}) \geq \sum_{i=1}^{k_0} r_i \cdot (s_{i}^{k_0} - \Delta n_i) \). Inequality (73) holds because of constraint (19) in the LP [NEST]. By this constraint, we have \( \sum_{i=1}^{k_0} r_i \cdot \Delta n_i + \sum_{i=1}^{k_0} r_i \cdot (s_{i}^{k_0} - \Delta n_i) \geq \Gamma_{\text{nest}} \cdot r_{k} \cdot C \) for any \( k \leq M \). Observe that the last equation is the desired result, and completes the proof for this case.

**Lemma 9.** Recall that \( k_0 := \max \{ i \in [K] \mid z_i + w_i < x_i \} \), \( R_Z = \sum_{i=1}^{k_0} r_i \cdot z_i \), \( R_{Z'} = \sum_{i=k_0+1}^{K} r_i \cdot z_i \), \( R_W = \sum_{i=1}^{M} r_i \cdot w_i \), \( R_{W'} = \sum_{i=k_0+1}^{M} r_i \cdot w_i \), \( Z = \sum_{i=1}^{k_0} z_i \) and \( Z' = \sum_{i=k_0+1}^{K} z_i \). Suppose that \( k_0 \leq M \) and \( Z + Z' - n_{k_0} \geq 0 \). Then, we have

\[
R_W + r_{k_0} \cdot (W' + Z + Z' - n_{k_0}) \geq \sum_{i=1}^{k_0} r_i \cdot (s_{i}^{k_0} - \Delta n_i),
\]

where \( (\Gamma_{\text{nest}}, \{ s_{j}^{k_0} \})_{j \in [K], i \in [J]} \) is the optimal solution to the LP [NEST].
The proof of this lemma is presented at the end of this section.

- **Case 1b:** $k_0 > M$. For this case, some constraint involving $B_{k_0}$ must be tight and such constraints are of the form

\[
\sum_{i=1}^{M} B_{i, 1} + \sum_{i=M}^{\ell} B_i \leq n_\ell \quad \ell \in \{k_0, k_0 + 1, \ldots, M\}.
\]  

(76)

Let $\ell \in \{k_0, k_0 + 1, \ldots, M\}$ be an index for which the above is tight, meaning that we have: $\sum_{i=1}^{\ell} z_i = n_\ell$.

Given this equality, we can compute a lower bound on our algorithm’s reward as follows by considering only the reward from the first row of buckets

\[
\text{Rew}_{B_{\text{next}, \tau}}(\{I_t\}_{t \in \{T\}}) \geq \sum_{i=1}^{k_0} r_i \cdot z_i + \sum_{i=k_0+1}^{K} r_i \cdot z_i
\]

\[= \sum_{i=1}^{k_0} r_i \cdot z_i + r_{k_0} \cdot \left( \sum_{i=k_0+1}^{K} z_i \right) + \sum_{i=k_0+1}^{K} \Delta_{i, k_0} \cdot z_i
\]

\[\geq \sum_{i=1}^{k_0} r_i \cdot \Delta n_i + \sum_{i=k_0+1}^{K} \Delta_{i, k_0} \cdot z_i
\]

\[\geq \Gamma_{\text{next}} \cdot r_{k_0} \cdot C + \sum_{i=k_0+1}^{K} \Delta_{i, k_0} \cdot z_i \quad \text{by constraint (19)}
\]

The second-to-last inequality is true because $\sum_{i=1}^{\ell} z_i = n_\ell$ and the fact that $\ell \in \{k_0, k_0 + 1, \ldots, K\}$ give us a lower bound on $\sum_{i=1}^{K} r_i \cdot z_i + r_{k_0} \cdot \left( \sum_{i=k_0+1}^{K} z_i \right)$ by setting $z_i$ to be as large as possible, followed by $z_2$, etc. The last inequality follows from constraint (19) in [NEST] (i.e., the fact that $\Gamma_{\text{next}} \cdot C \cdot r_j \leq \sum_{i=1}^{j} s_i \cdot r_i$) and constraint (22) that for any $j \geq M$, $s'_j \leq \Delta n_j$. These two constraints lead to $\sum_{i=1}^{k_0} r_i \cdot \Delta n_i \geq \Gamma_{\text{next}} \cdot r_{k_0} \cdot C$ for any $k \geq M$. The last inequality is the desired result.

**Case 2:** Recall that in this case, we do not have $z_{k_0} + w_{k_0} \geq x_{k_0}$ because we do not have the capacity to serve all the agents in $B_{k_0, 2}$ (i.e. $w_{k_0} < B_{k_0, 2}$). Clearly this can only happen for $k_0 \leq M$ as accepted inflexible agents are always served. The fact that we ran out of capacity to serve all of $B_{k_0, 2}$ implies that

\[
\sum_{i=1}^{K} z_i + \sum_{i=k_0}^{M} w_i = C
\]  

(77)

In other words, the total amount of agents we are serving must be exactly at capacity. Using this, we get the following lower bound on the reward:

\[
\text{Rew}_{B_{\text{next}, \tau}}(\{I_t\}_{t \in \{T\}}) \geq \sum_{i=1}^{k_0} r_i \cdot z_i + \sum_{i=k_0+1}^{K} r_i \cdot z_i + \sum_{i=k_0}^{M} r_i \cdot w_i
\]

\[= \sum_{i=1}^{k_0} r_i \cdot z_i + r_{k_0} \cdot \left( \sum_{i=k_0+1}^{K} z_i + \sum_{i=k_0}^{M} w_i \right) + \sum_{i=k_0+1}^{K} \Delta_{i, k_0} \cdot z_i + \sum_{i=k_0}^{M} \Delta_{i, k_0} \cdot w_i
\]

\[\geq \sum_{i=1}^{k_0} r_i \cdot \Delta n_i + r_{k_0} \cdot \left( C - n_{k_0} \right) + \sum_{i=k_0+1}^{K} \Delta_{i, k_0} \cdot z_i + \sum_{i=k_0}^{M} \Delta_{i, k_0} \cdot w_i
\]

\[\geq \sum_{i=1}^{k_0} r_i \cdot \Delta n_i + r_{k_0} \cdot \left( C - n_{k_0} \right) + \sum_{i=k_0+1}^{K} \Delta_{i, k_0} \cdot x_i.
\]
Here, the second-to-last inequality is true because when equation (74) holds, a lower bound on $\sum_{i=1}^{k_0} r_i \cdot z_i + r_{k_0} \cdot (\sum_{i=k_0+1}^{K} z_i + \sum_{i=k_0}^{M} w_i)$ is attained by greedily choosing to put as much weight as possible in $z_1$ followed by $z_2$, etc. and this is done by choosing $z_i = \Delta n_i$ for $i \in [k_0]$. The last inequality is true because by definition of $k_0$, all types $i$ for which $i > k_0$ have the property that our algorithm serves at least $x_i$ copies of them.

Given that our goal is inequality (67) (i.e., $\text{Rew} \cdot \Gamma_{\text{nest}} \cdot \tau \left( \{I_i\}_{i \in [\tau]} \right) \geq \Gamma_{\text{nest}} \cdot r_{k_0} \cdot C + \sum_{i=k_0+1}^{K} \Delta \cdot k_0 \cdot x_i$), all that is left to show is that

$$\sum_{i=1}^{k_0} r_i \cdot \Delta n_i + r_{k_0} \cdot (C - n_{k_0}) \geq \Gamma_{\text{nest}} \cdot r_{k_0} \cdot C.$$  

To show this, we use the following chain of inequalities:

$$\sum_{i=1}^{k_0} r_i \cdot \Delta n_i + r_{k_0} \cdot (C - n_{k_0}) \geq \Gamma_{\text{nest}} \cdot r_{k_0} \cdot C - r_{k_0} \cdot \sum_{i=1}^{k_0} (s_i^{k_0} - \Delta n_i) + r_{k_0} \cdot (C - n_{k_0})$$

$$\geq \Gamma_{\text{nest}} \cdot r_{k_0} \cdot C - r_{k_0} \cdot \sum_{i=1}^{k_0} s_i^{k_0} + r_{k_0} \cdot (C - n_{k_0})$$

$$= \Gamma_{\text{nest}} \cdot r_{k_0} \cdot C - r_{k_0} \cdot \sum_{i=1}^{k_0} s_i^{k_0} + r_{k_0} \cdot (C - n_{k_0})$$

$$\geq \Gamma_{\text{nest}} \cdot r_{k_0} \cdot C,$$

where the first inequality follows from constraint (19) in the LP (NEST) that we have $\sum_{i=1}^{k_0} r_i \cdot \Delta n_i + \sum_{i=1}^{k_0} r_i \cdot (s_i^{k_0} - \Delta n_i) \geq \Gamma_{\text{nest}} \cdot r_{k_0} \cdot C$ for any $k \leq M$. The second inequality follows from constraint (21) that $\Delta n_j \leq s_i^j$ for any $i \leq j \leq M$. The equality holds because $\sum_{i=1}^{k_0} \Delta n_i = n_{k_0}$, and finally, the last inequality holds because by constraint (20) in the LP (NEST) $\sum_{i=1}^{k_0} s_i^j \leq C$. Thus, we have shown that inequality (67) holds and we have argued earlier why this implies a $\Gamma_{\text{nest}}$ CR for this case.

D.2.1. Proof of Lemma 9

Here, we will show that $R_W + r_{k_0} \cdot (W' + Z + Z' - n_{k_0}) \geq \sum_{i=1}^{k_0} r_i \cdot (s_i^{k_0} - \Delta n_i)$.

To do, we start with making the following two claims about the right and left hands of the aforementioned inequality.

**Claim 1:** The expression $R_W + r_{k_0} \cdot (W' + Z + Z' - n_{k_0})$ is greater than or equal to the optimal objective value of the following optimization problem evaluated at $n_{k_0}$, i.e., $R_W + r_{k_0} \cdot (W' + Z + Z' - n_{k_0}) \geq \text{LOWER}(n_{k_0})$, where

$$\text{LOWER}(x) = \max_{\omega_1, \ldots, \omega_{k_0}} \sum_{i=1}^{k_0} r_i \cdot \omega_i + r_{k_0} \cdot (W' + x - n_{k_0})$$

(s.t $\omega_i \leq B_{i,2}$, $i \in [k_0]$)

$$W' + x + \sum_{i=1}^{k_0} \omega_i \leq C$$

(78)

(79)

**Proof of Claim 1:** We show the claim in two steps. In the first step, we show that $R_W + r_{k_0} \cdot (W' + Z + Z' - n_{k_0}) \geq \text{LOWER}(Z + Z')$ and in the second step, we show that $\text{LOWER}(Z + Z') \geq \text{LOWER}(n_{k_0})$.

This completes the proof. Step 1: To show this step, we verify that $\omega_i = w_i$, $i \in [k_0]$, is the optimal solution to problem $\text{LOWER}(Z + Z')$. Recall that $w_i$ is the number of type $i$ agents that are served in $B_{i,2}$ where we define $w_i = 0$ if $i > M$. From our serving rule, it is clear that our algorithm chooses $w_i$ using the $C - W' - Z - Z'$
capacity remaining to maximize \( \sum_{i=1}^{k_0} r_i \cdot \omega_i \) subject to \( \omega_i \leq B_{i,2} \). This is precisely the LP presented, i.e, \( \text{LOWER}(Z + Z') \). (Note that the expression “\( r_{k_0} \cdot (W' + Z + Z' - n_{k_0}) \)” in the objective value of the LP does not impact its optimal solution as it is not a function of \( \omega_i \)’s. It is become more clear later why we include this expression in the objective function.)

Step 2: Here, we show that \( \text{LOWER}(Z + Z') \geq \text{LOWER}(n_{k_0}) \). To do so, we consider three cases, based on whether or not the second constraints in these two problems (i.e., constraints \( \Omega \)) are binding. Observe that when the second constraint in \( \text{LOWER}(n_{k_0}) \) is binding (i.e., \( W' + n_{k_0} + \sum_{i=1}^{k_0} \omega_i = C \)), the second constraint in \( \text{LOWER}(Z + Z') \) is also binding (i.e., \( W' + Z + Z' + \sum_{i=1}^{k_0} \omega_i = C \)). This is because we have \( Z + Z' \geq n_{k_0} \) by our assumption. In the first case, in the optimal solution of both \( \text{LOWER}(Z + Z') \) and \( \text{LOWER}(n_{k_0}) \), the second constraint is not binding. This leads to the optimal solution of \( \omega_i^* = B_{i,2} \) for any \( i \in [k_0] \) for both problems. It is then easy to see that the objective value of \( \text{LOWER}(n_{k_0}) \) (i.e., \( \sum_{i=1}^{k_0} r_i \cdot B_{i,2} + r_{k_0} \cdot W' \)) is at most that of \( \text{LOWER}(Z + Z') \) (i.e., \( \sum_{i=1}^{k_0} r_i \cdot B_{i,2} + r_{k_0} \cdot (W' + Z + Z' - n_{k_0}) \)). Again recall that \( Z + Z' \geq n_{k_0} \).

Now, consider the second case in which the second constraint in \( \text{LOWER}(n_{k_0}) \) is not binding while the second constraint in \( \text{LOWER}(Z + Z') \) is binding. The optimal objective value of \( \text{LOWER}(n_{k_0}) \) is then given by

\[
\sum_{i=1}^{k_0} r_i \cdot B_{i,2} + r_{k_0} \cdot W' < \sum_{i=1}^{k_0} r_i \cdot B_{i,2} + r_{k_0} \cdot (C - n_{k_0} - \sum_{i=1}^{k_0} B_{i,2}) = \sum_{i=1}^{k_0} (r_i - r_{k_0}) \cdot B_{i,2} + r_{k_0} \cdot (C - n_{k_0}),
\]

where the inequality holds because the second constraint in \( \text{LOWER}(n_{k_0}) \). Next, let \( \omega_{i,1}^*, i \in [k_0] \) be the optimal solution to \( \text{LOWER}(Z + Z') \). The optimal objective value of \( \text{LOWER}(Z + Z') \) is then given by

\[
\sum_{i=1}^{k_0} r_i \cdot \omega_{i,1}^* + r_{k_0} \cdot (W' + Z + Z' - n_{k_0}) = \sum_{i=1}^{k_0} r_i \cdot \omega_{i,1}^* + r_{k_0} \cdot (C - Z - Z' - \sum_{i=1}^{k_0} \omega_{i,1}^* + Z + Z' - n_{k_0})
\]

\[
= \sum_{i=1}^{k_0} r_i \cdot \omega_{i,1}^* + r_{k_0} \cdot (C - \sum_{i=1}^{k_0} \omega_{i,1}^* - n_{k_0})
\]

\[
= \sum_{i=1}^{k_0} (r_i - r_{k_0}) \cdot \omega_{i,1}^* + r_{k_0} \cdot (C - n_{k_0}),
\]

where the first equation holds because we assumed that the second constraint in \( \text{LOWER}(Z + Z') \) is binding. Then, since \( r_1 < r_2 < \ldots < r_{k_0} \) and \( \omega_{i,1}^* \leq B_{i,2} \), we have \( \sum_{i=1}^{k_0} (r_i - r_{k_0}) \cdot \omega_{i,1}^* + r_{k_0} \cdot (C - n_{k_0}) \geq \sum_{i=1}^{k_0} (r_i - r_{k_0}) \cdot B_{i,2} + r_{k_0} \cdot (C - n_{k_0}). \) Recall that the right hand side is an upper bound on the optimal objective value of \( \text{LOWER}(n_{k_0}) \), and hence, we obtain the desired result. Now, consider the final case where the second constraint in both \( \text{LOWER}(Z + Z') \) and \( \text{LOWER}(n_{k_0}) \) are binding and let \( \omega_{i,2}, i \in [k_0] \) be the optimal solution to A2. Then, using the previous arguments, we can show that the optimal objective value of \( \text{LOWER}(Z + Z') \) and \( \text{LOWER}(n_{k_0}) \) are respectively \( \sum_{i=1}^{k_0} (r_i - r_{k_0}) \cdot \omega_{i,2}^* + r_{k_0} \cdot (C - n_{k_0}) \) and \( \sum_{i=1}^{k_0} (r_i - r_{k_0}) \cdot \omega_{i,1}^* + r_{k_0} \cdot (C - n_{k_0}) \). It is then easy to see that the optimal value of \( \text{LOWER}(Z + Z') \) is greater than that of \( \text{LOWER}(n_{k_0}) \). This is because we have

\[
\sum_{i=1}^{k_0} \omega_{i,1}^* = C - W' - Z - Z', \quad \text{and} \quad \sum_{i=1}^{k_0} \omega_{i,2}^* = C - W' - n_{k_0}.
\]

This and the fact that \( Z + Z' \geq n_{k_0} \) give us \( \sum_{i=1}^{k_0} \omega_{i,2}^* > \sum_{i=1}^{k_0} \omega_{i,1}^* \). Then, observe that to maximize the objective values of both \( \text{LOWER}(Z + Z') \) and \( \text{LOWER}(n_{k_0}) \), we start to ‘fill’ \( \omega_i \)'s in a decreasing order of
Our goal is to prove that \( \sum \) constraint (79). where (a) holds because by definition of \( n \) result optimal solutions will have a certain structure. For example, with \( i > 64 \) following optimization problem.

Starting with the definition of \( W \) to each LP, we first prove a crucial property for our analysis.

Comparing \( \text{LOWER}(n_{k_0}) \) and \( \text{UPPER} \). Our goal is to show that the optimal objective value for the former is at least that of the latter. Showing this completes the proof. Before we reason about the optimal solutions to each LP, we first prove a crucial property for our analysis.

Claim: \( W' \geq \sum_{i=1}^{k_0} (\Delta n_i - B_{i,2}) \)

Proof: Starting with the definition of \( W' \), we have:

\[
W' = \sum_{i=k_0+1}^{M} w_i \geq \sum_{i=k_0+1}^{\ell_2} w_i \overset{(a)}{=} \sum_{i=k_0+1}^{\ell_2} B_{i,2} = \sum_{i=1}^{\ell_2} B_{i,2} - \sum_{i=1}^{k_0} B_{i,2} \geq n_{\ell_2} - \sum_{i=1}^{k_0} B_{i,2} \geq n_{k_0} - \sum_{i=1}^{k_0} B_{i,2} = \sum_{i=1}^{k_0} (\Delta n_i - B_{i,2}) \tag{82}
\]

where (a) holds because by definition of \( k_0 \), \( w_i = B_{i,2} \) for all \( i \geq k_0 \). Inequality (b) holds by definition \( \ell_2 \in \{k_0, k_0+1, \ldots, M\} \); we have: \( \sum_{i=1}^{\ell_2} B_{i,2} = n_{\ell_2} \). Inequality (c) holds because \( w_i \geq 0 \) and \( \ell_2 \geq k_0 \), and as a result \( n_{\ell_2} \geq n_{k_0} \). This concludes the proof of the claim.

Consider how the two LP’s \( \text{LOWER}(n_{k_0}) \) and \( \text{UPPER} \) are different. The former is transformed into the latter by changing \( B_{i,2} \) to \( \Delta n_i \) in constraint (78) and removing \( W' \) from both the objective value and constraint (79).

Let \( \{\omega^*_i\}_{i=1}^{k_0} \) be an optimal solution to \( \text{LOWER}(n_{k_0}) \) while let \( \{\hat{\omega}^*_i\}_{i=1}^{k_0} \) be an optimal solution to \( \text{UPPER} \). Our goal is to prove that \( \sum_{i=1}^{k_0} r_i \cdot \omega^*_i + r_{k_0} \cdot W' \geq \sum_{i=1}^{k_0} r_i \cdot \hat{\omega}^*_i \) which can be done by showing that

\[
\sum_{i=1}^{k_0} (\hat{\omega}^*_i - \omega^*_i) \leq W'. \tag{83}
\]

Since both \( \text{LOWER}(n_{k_0}) \) and \( \text{UPPER} \) are instances of a continuous knapsack problem, we know that both optimal solutions will have a certain structure. For example, with \( \text{LOWER}(n_{k_0}) \), we have a knapsack with \( C - W' - n_{k_0} \) capacity and we have access to \( B_{i,2} \) copies of type \( i \) for \( i \in [k_0] \), which achieves a reward of \( r_i > 0 \). We know that the optimal solution will have one of two characteristics: either (1) we use up all the \( C - W' - n_{k_0} \) capacity, meaning that constraint (79) is binding, or (2) Constraint (79) is not binding, but we have used up all \( B_{i,2} \) copies of all types \( i \in [k_0] \) and so constraint (78) is binding for all \( i \). We now perform case-work on which of the constraints in \( \text{LOWER}(n_{k_0}) \) is binding, as we just described.
• **Case 1.** Given that (23) is tight for lower($n_{k_0}$) and (81) is certainly satisfied in (upper) we have:

$$W' + n_{k_0} + \sum_{i=1}^{k_0} \omega_i^* = C \geq \sum_{i=1}^{k_0} \omega_i + n_{k_0}$$

Re-arranging gives us the desired result in (83).

• **Case 2.** In this case, we assume that $\omega_i^* = B_i, 2$ for all $i \in [k_0]$. This implies that:

$$\sum_{i=1}^{k_0} (\omega_i - \omega_i^*) = \sum_{i=1}^{k_0} (\omega_i - B_i, 2) \leq \sum_{i=1}^{k_0} (\Delta n_i - B_i, 2) \leq W'$$

which is the desired result in (83). Here, first inequality is true from constraint (81), while the second inequality is from our useful property (82) (i.e., $W' \geq \sum_{i=1}^{k_0} (\Delta n_i - B_i, 2)$) proved earlier.

Therefore, we have shown that the optimal objective value of lower($n_{k_0}$) is always at least that of (upper). This completes our proof.

D.3. Proof of the Lemmas Used in the Proof of Theorem 8

D.3.1. Proof of Lemma 2

*Proof of Lemma 2* The optimization problem in equation (27) can be written as

$$\Gamma := \max_{\Gamma, \{n_i\}_{i=1}^{K}, \{s_i\}_{j \in [K], i \in [j]}} \Gamma$$

s.t

$$\Gamma \cdot C \cdot r_j \leq \sum_{j=1}^{j} s_i \cdot r_i$$

$$\Delta n_j \leq s_i \leq 2 \cdot \Delta n_j$$

$$i \leq j \leq M$$

$$s_i \leq \Delta n_j$$

$$i \leq j \text{ and } j > M$$

$$n_1 \leq \ldots \leq n_K = C.$$.

It is clear that the optimal solution will simply take each $s_i$ to be as large as possible, meaning that $s_i = 2 \cdot \Delta n_j$ if $i \leq j \leq M$ and $s_i = \Delta n_j$ if $i \leq j$ and $j > M$. This effectively removes the variables $\{s_i\}_{j \in [K], i \in [j]}$ from the above LP. We show that if we define $\Gamma$ and $\sum n_i$, as in equations (29) and (24), then the constraint $\Gamma \cdot C \cdot r_j \leq \sum_{i=1}^{j} s_i \cdot r_i$ will be tight for all $j \in [K]$ thereby proving optimality. We split this proof into two parts: (To simplify notation, here we denote $\sum n_i$, $i \in [K]$, by $\Delta n_i$.)

- **j ≤ M:**

  $$\sum_{i=1}^{j} s_i \cdot r_i = \sum_{i=1}^{j} 2 \cdot \Delta n_i \cdot r_i$$

  $$= \Gamma \cdot C \cdot \sum_{i=1}^{j} \left(1 - \frac{r_{i-1}}{r_i}\right) \cdot r_i$$

  $$= \Gamma \cdot C \cdot r_j.$$

- **j > M:**

  $$\sum_{i=1}^{j} s_i \cdot r_i = \sum_{i=1}^{M} \Delta n_i \cdot r_i + r_{M+1} \cdot \Delta n_{M+1} + \sum_{i=M+2}^{j} \Delta n_i \cdot r_i$$

  $$= \frac{1}{2} \Gamma \cdot C \cdot r_M + \Gamma \cdot \left(r_{M+1} - \frac{1}{2} r_M\right) \cdot C + \Gamma \cdot C \cdot \sum_{i=M+2}^{j} (r_i - r_{i-1})$$

  $$= \Gamma \cdot C \cdot r_j.$$
Finally, we need to make sure that \( n_K = C \) is satisfied through the following calculation:
\[
\sum_{i=1}^{K} \Delta n_i = \Gamma \cdot C \cdot \left( \frac{1}{2} \sum_{i=1}^{M} \left( 1 - \frac{r_{i-1}}{r_i} \right) + \frac{1}{2} \frac{r_M}{r_{M+1}} + \sum_{i=M+2}^{K} \left( 1 - \frac{r_{i-1}}{r_i} \right) \right)
\]
\[
= \Gamma \cdot C \cdot \left( \frac{1}{2} \cdot M + K - M - \frac{1}{2} \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i} - \sum_{i=M+2}^{K} \frac{r_{i-1}}{r_i} \right)
\]
\[
= \Gamma \cdot C \cdot \left( \frac{1}{2} \cdot M - \frac{1}{2} \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i} + G \right)
\]
\[
= C.
\]

Hence, we have proven that our definitions of \( \Gamma \) and \( \overline{\Delta n_i} \) are indeed closed-form solutions of the optimal solution to the given LP. \( \square \)

D.3.2. Proof of Lemma 3

Proof of Lemma 3. The goal of this lemma is to show that our instance of the nested polytope, which is constructed using \( \Gamma \) has a CR, denoted by \( \Gamma_{nest} \), which is not too far from \( \Gamma \) itself. Recall that
\[
\max_{\Gamma, \{n_i\}_{i=1}^K, \{s_i\}_{j \in [K], i \in [j]}} \Gamma \text{ s.t. } (19), (21), (22), \; n_1 \leq \ldots \leq n_K = C
\]

where (19), (21), and (22) are constraints in the LP \( (NEST) \). That is, we would like to show \( \Gamma_{nest} \geq f_1(G) \cdot \Gamma \), where \( \Gamma_{nest} \) is the CR of our nested polytope with nest size defined as
\[
\Delta n_i = \begin{cases} 
\frac{1}{2} \cdot \Gamma \cdot \left( 1 - \frac{r_{i-1}}{r_i} \right) \cdot C & i \in [M] \\
\Gamma \cdot \left( 1 - \frac{r_{i-1}}{r_i} \right) \cdot C & i = M + 1 \\
\Gamma \cdot \left( 1 - \frac{r_{i-1}}{r_i} \right) \cdot C & i = M + 2, \ldots, K
\end{cases}
\]

To show this, we will use the LP \( (NEST) \) from Theorem 7 which tells us how to compute the CR of any arbitrary instance of the nested polytope with given values of \( (\Delta n_1, \ldots, \Delta n_K) \). Instead of solving LP \( (NEST) \) at given values of \( (\Delta n_1, \ldots, \Delta n_K) \), we find a feasible solution \( (\Gamma, \{s_i\}_{j \in [K], i \in [j]})) \) to the LP with \( \Gamma = f_1(G) \cdot \Gamma \). This gives us the desired result. The feasible solution needs to satisfy constraints of LP \( (NEST) \) i.e., constraints (19), (20), (21), and (22).

Since our function \( f_1(G) \) is piecewise, we consider two cases based on the relationship between \( 2 \cdot n_M \) and \( C \). Recall that
\[
f_1(G) := \begin{cases} 1 & 2 \cdot n_M \leq C \\
1 - \frac{1}{2} e^{-2G+1} & 2 \cdot n_M > C
\end{cases}
\]

Case 1: \( 2 \cdot n_M \leq C \): For this case, we see that \( f_1(G) = 1 \). \( \{s_i\}_{j \in [K], i \in [j]} \) which means that we need to exhibit a collection of feasible \( \{s_i^j\}_{j \in [K], i \in [j]} \) such that \( \Gamma \cdot r_j \cdot C \leq \sum_{i=1}^{M} r_i \cdot s_i^j \) for all \( j \in [K] \). We consider two sub-cases based on whether \( j \leq M \) or \( j > M \).

- Case 1a: \( j \leq M \). In this case, we have that \( s_i^j \in [\Delta n_j, 2 \cdot \Delta n_j] \). We take \( s_i^j = 2 \cdot \Delta n_i \). Since \( \sum_{i=1}^{j} s_i^j = 2 \cdot \Delta n_i \), we know that the capacity constraint (20) is also satisfied. Thus, we have:
\[
\sum_{i=1}^{j} r_i \cdot s_i^j = 2 \sum_{i=1}^{j} r_i \cdot \frac{1}{2} \Gamma \cdot \left( 1 - \frac{r_{i-1}}{r_i} \right) \cdot C
\]
\[
= \Gamma \cdot C \cdot \sum_{i=1}^{j} r_i - r_{i-1}
\]
\[
= \Gamma \cdot r_j \cdot C. \quad (84)
\]
The first equation holds because by our definition and nest sizes, we have \( s'_i = 2 \cdot \Delta n_i = \frac{1}{2} \cdot \Gamma \cdot (1 - r_{i-1}/r_i) \).

- **Case 1b:** \( j > M \). In this case, to have a feasible solution, for any \( j > M \), we must have that \( s'_i \in [0, \Delta n_j] \). So, we take \( s'_i = \Delta n_j, j > M \). (Recall that, for \( j \leq M \), we set \( s'_i = 2 \cdot \Delta n_i \).) We note that \( \sum_{i=1}^j s'_i = \sum_{i=1}^j \Delta n_i = n_j \leq n_K \leq C \). Thus, by our nest sizes, we get:

\[
\sum_{i=1}^j r_i \cdot s'_i = \Gamma \cdot C \cdot \left[ \frac{1}{2} \sum_{i=1}^M r_i \cdot \left( 1 - \frac{r_{i-1}}{r_i} \right) + r_{M+1} \cdot \left( 1 - \frac{1}{2} \frac{r_{M}}{r_{M+1}} \right) + \sum_{i=M+2}^K \cdot r_i \cdot \left( 1 - \frac{r_{i-1}}{r_i} \right) \right] = \Gamma \cdot C \cdot \left[ \frac{1}{2} \cdot r_M + \left( r_{M+1} - \frac{1}{2} \frac{r_M}{r_{M+1}} \right) + \sum_{i=M+2}^K \right] = \Gamma \cdot r_j \cdot C .
\]

Note that only in the \( j \leq M \) case did we use the assumption that \( 2 \cdot n_M \leq C \). This means that for Case 2, the same argument holds for \( j > M \) and we only need to work with \( j \leq M \).

**Case 2:** \( 2 \cdot n_M > C \). Our goal is this case is to prove that for all \( j \in [K] \), we have can find a choice of \( s'_i \) for \( i \in [j] \) such that \( f_1(G) \cdot \Gamma \leq \sum_{i=1}^j r_i \cdot s'_i \). As we just mentioned, we only look at the case where \( j \in [M] \) because otherwise the same argument from Case 1b can be applied. Fix a particular value of \( j \in [M] \). If \( 2 \cdot n_j \leq C \), then we are done because setting \( s'_i = 2 \cdot \Delta n_i \) for \( i \in [j] \) is feasible, and we can apply the same set of equalities as in (83).

Otherwise, assume that \( 2 \cdot n_j > C \) and so setting \( s'_i = 2 \cdot \Delta n_i \) for \( i \in [j] \) is not feasible as it breaks the capacity constraint on \( s'_i \) (i.e. constraint (20) in (NEST)). Since we can no longer choose \( s'_i = 2 \cdot \Delta n_i \) for all \( i \in [j] \), we choose to *greedily* set them so that \( \sum_{i=1}^j r_i \cdot s'_i \) is maximized. This involves first setting all the \( s'_i \)'s , \( i \in [j] \), to be \( \Delta n_i \) (the lower bound of \( s'_i \)) and then using the remaining \( C - \sum_{i=1}^j \Delta n_i = C - n_j \) capacity to greedily make increase each of \( s'_i, s'_{j-1}, s'_{j-2}, \ldots, s'_1 \) by as much as possible up to \( 2 \cdot \Delta n_i \) in that order. More formally, let \( \ell \) be defined as follows:

\[
\ell := \max \left\{ k \in [j] : n_j + \sum_{i=k}^j \Delta n_i \cdot C \right\} .
\]

Note that \( \ell \) is well defined because we assumed that \( 2 \cdot n_j > C \) so certainly the index 1 is in the set and hence we are not taking the maximum of an empty set. We choose our \( s'_i \) as follows:

\[
s'_i = \begin{cases} 
\Delta n_i & i < \ell \\
\Delta n_i + C - n_j - \sum_{i'=\ell+1}^j \Delta n_i & i = \ell \\
2 \cdot \Delta n_i & j \geq i > \ell 
\end{cases}
\]

(85)

For \( i > \ell \), our construction chooses \( s'_i = 2 \cdot \Delta n_i \) (the upper bound of each \( s'_i \)), while for \( i < \ell \), we choose \( s'_i \) to be \( \Delta n_i \) (the lower bound for \( s'_i \)). For \( i = \ell \), we choose \( s'_i \) to use up the remaining capacity so that \( \sum_{i=1}^j s'_i = C \) is satisfied. To see why this is true, note that

\[
\sum_{i=1}^j s'_i = \sum_{i=1}^{\ell-1} \Delta n_i + \Delta n_\ell + C - n_j - \sum_{i=\ell+1}^j \Delta n_i + 2 \cdot \sum_{i=\ell+1}^j \Delta n_i \\
= \sum_{i=1}^j \Delta n_i + C - n_j = C .
\]
This shows that constraint (20) is satisfied by our choice of \(s_i^e\). Now, we verify that \(s_i^e \in [\Delta n_i, 2 \cdot \Delta n_i]\) (constraint (21)). This is clear for \(i \neq \ell\). For \(i = \ell\), we need to show that \(C - n_j - \sum_{\ell = \ell+1}^{j} \Delta n_{\ell} \in [0, \Delta n_{\ell}]\). The upper bound of \(\Delta n_{\ell}\) is true by definition of \(\ell\). Recall that by definition of \(\ell\), we have that \(n_j + \sum_{\ell = \ell}^{j} \Delta n_{\ell} > C\), which implies that \(C - n_j - \sum_{\ell = \ell+1}^{j} \Delta n_{\ell} \leq \Delta n_{\ell}\). For the lower bound of 0, we consider two small cases. If \(\ell = j\), then we see that \(C - n_j - \sum_{\ell = j+1}^{j} \Delta n_{\ell} = C - n_{j} \geq 0\). Otherwise, if \(\ell < j\), then what we want to show (i.e. \(C - n_j - \sum_{\ell = \ell+1}^{j} \Delta n_{\ell} \geq 0\)) is true because \(\ell\) was chosen to be the maximum of the set, meaning that \(\ell + 1 \in [j]\) has the property that \(n_j + \sum_{\ell = \ell+1}^{j} \Delta n_{\ell} \leq C\). This is exactly what we want to show.

Having verified constraints (20) and (21), we now proceed to show that for this fixed value of \(j\), we have \(\sum_{i=1}^{j} r_i \cdot s_i^e \geq f_1(G) \cdot \Gamma \cdot C\).

\[
\sum_{i=1}^{j} r_i \cdot s_i^e = \sum_{i=1}^{\ell} r_i \cdot \Delta n_i + r_{\ell} \cdot \left( C - n_j - \sum_{i=\ell+1}^{j} \Delta n_i \right) + \frac{1}{2} \sum_{i=\ell+1}^{j} r_i \cdot \Delta n_i
\]

\[
= \frac{1}{2} \cdot \Gamma \cdot r_{\ell} \cdot C + r_{\ell} \cdot \left( C - n_j - (n_j - n_{\ell}) \right) + \frac{1}{2} \cdot \Gamma \cdot (r_{\ell} - r_{\ell}) \cdot C
\]

\[
= \Gamma \cdot r_{\ell} \cdot C + r_{\ell} \cdot \left( C - 2n_j + n_{\ell} - \frac{1}{2} \cdot \Gamma \cdot C \right).
\]

By definition of \(f_1(G)\), our goal is to show that \(\frac{\sum_{i=1}^{j} r_i \cdot s_i^e}{\Gamma \cdot r_{\ell} \cdot C} \geq \frac{1}{2} \cdot e^{-2 \cdot G + 1}\). Dividing the above equation through by \(\Gamma \cdot r_{\ell} \cdot C\), it would be sufficient to have

\[
\frac{r_{\ell}}{\Gamma \cdot r_{\ell} \cdot C} \cdot \left( \frac{1}{2} \cdot \Gamma \cdot C - C + 2n_j - n_{\ell} \right) \leq \frac{1}{2} \cdot e^{-2 \cdot G + 1}.
\]

Using the fact that \(n_k = \frac{1}{2} \cdot \Gamma \cdot C \cdot \sum_{i=1}^{k} \left( 1 - \frac{r_{i-1}}{r_i} \right)\) for any \(k\), we re-write the left hand side of the above as:

\[
\frac{1}{2} \frac{r_{\ell}}{r_j} \cdot \left( 1 - 2 + 2 \cdot \sum_{i=1}^{\ell} \left( 1 - \frac{r_{i-1}}{r_i} \right) - \sum_{i=1}^{\ell} \left( 1 - \frac{r_{i-1}}{r_i} \right) \right) = \frac{1}{2} \frac{r_{\ell}}{r_j} \cdot \left( 1 - 2 + 2j - 2 \cdot \sum_{i=1}^{\ell} \frac{r_{i-1}}{r_i} + \sum_{i=1}^{\ell} \frac{r_{i-1}}{r_i} \right)
\]

Using the definition of \(\Gamma = \frac{2 \cdot G + M - 2 \cdot \sum_{i=1}^{M-1} \frac{r_{i-1}}{r_i}}{2 \cdot M - 2 \cdot \sum_{i=1}^{M-1} \frac{r_{i-1}}{r_i}}\) from equation (21), we have that that the above expression is upper bounded by

\[
\frac{1}{2} \frac{r_{\ell}}{r_j} \cdot \left( 1 - 2 \cdot G - M + \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i} + 2j - \ell - \sum_{i=\ell+1}^{j} \frac{r_{i-1}}{r_i} + \sum_{i=\ell+1}^{j} \frac{r_{i-1}}{r_i} \right)
\]

\[
= \frac{1}{2} \frac{r_{\ell}}{r_j} \cdot \left( 1 - 2 \cdot G - M + 2j - \ell - \sum_{i=\ell+1}^{j} \frac{r_{i-1}}{r_i} + \sum_{i=\ell+1}^{j} \frac{r_{i-1}}{r_i} \right)
\]

\[
= \frac{1}{2} \frac{r_{\ell}}{r_j} \cdot \left( 1 - 2 \cdot G - M + 2j - \ell - \sum_{i=\ell+1}^{j} \frac{r_{i-1}}{r_i} + \sum_{i=\ell+1}^{j} \frac{r_{i-1}}{r_i} - (M - j + 1) + (M - j + 1) \right)
\]

\[
\leq \frac{1}{2} \frac{r_{\ell}}{r_j} \cdot \left( 2 - 2 \cdot G + j - \ell - \sum_{i=\ell+1}^{j} \frac{r_{i-1}}{r_i} \right)
\]

\[
\leq \frac{1}{2} \cdot e^{-2 \cdot G + 1} \quad \text{By Lemma 10}
\]

In the last step, we applied Lemma 10 stated below, with \(x_i = \frac{r_{i+1}}{r_{i+1}}\) for \(x \in [j - \ell]\), \(c = 2 \cdot G + j - \ell\) and \(n = j - \ell\). That gives us the desired result.
**Lemma 10.** Consider the following multivariate function:

\[ f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i \cdot \left( c - \sum_{i=1}^{n} x_i \right), \]

where \( x_i \in (0, 1) \) for \( i \in [n] \). Then, we have

\[ f(x_1, \ldots, x_n) \leq e^{-(n+1-c)} . \]

**Proof of Lemma 10.** Let \( s = \sum_{i=1}^{n} x_i \). We assume that \( c - s > 0 \). Otherwise the statement is trivially true. For any fixed value of \( s \), by the inequality of arithmetic and geometric mean (AM–GM) \(^{10}\) we have that

\[ \sqrt[n]{\prod_{i=1}^{n} x_i} \leq \frac{s}{n}, \]

which implies that

\[ f(x_1, \ldots, x_n) \leq \left( \frac{s}{n} \right)^n \cdot \left( c - n \cdot \frac{s}{n} \right) \]

If we define \( x := \frac{s}{n} \), then we have that \( f(x_1, \ldots, x_n) \leq g(x) \) where \( g(x) = x^n \cdot (c - n \cdot x) \) for \( x \in (0, 1) \). Our assumption that \( c - s > 0 \) now carries over and becomes \( c - n \cdot x > 0 \) implying that \( x \in (0, \frac{c}{n}) \). Our goal now is to show that \( g(x) \leq e^{-(n+1-c)} \) for any \( x \) in that interval. To do so, we observe that the zeros of \( g \) are precisely \( x = 0 \) and \( x = \frac{c}{n} \) and it is clear that \( g(x) \) is positive for all \( x \in (0, \frac{c}{n}) \). This means that \( g(x) \), a polynomial in \( x \), must have a local maximum in this interval. By a simple derivative calculation, we see that the only critical point of \( g \) in that interval is at \( x^* = \frac{c}{n+1} \) and so it must be a local maximum. Therefore, we have

\[ g(x) \leq g(x^*) = \left( \frac{c}{n+1} \right)^{n+1} \cdot \left( 1 - \frac{n+1-c}{n+1} \right)^{n+1} \leq e^{-(n+1-c)}. \]

This completes the proof.

**D.4. Proof of Lemma 4**

**Proof of Lemma 4.** We begin by one useful property that we will use multiple times in this proof.

**Claim:** Recall that \( G := K - M - \frac{\frac{r_{M+1}}{r_{M+2}}}{r_{M+3}} - \frac{\frac{r_{M+2}}{r_{M+3}}}{r_{M+4}} - \ldots - \frac{\frac{r_{K-1}}{r_K}}{r_K} \) as per equation \([12]\). For any value of \( G \), whose domain is \((1, K - M)\), we have:

\[ \frac{\frac{r_{M+1}}{r_K}}{r_K} \geq \max(2 - G, 0). \] (86)

**Proof:** If \( 2 - G \leq 0 \) (i.e. \( G \geq 2 \)), then the statement is trivially true as \( \frac{\frac{r_{M+1}}{r_K}}{r_K} \geq 0 \). Thus, we only focus on the case where \( G \in (1, 2) \). Recall that the definition of \( G \) is \( K - M - \frac{\frac{r_{M+1}}{r_{M+2}}}{r_{M+3}} - \frac{\frac{r_{M+2}}{r_{M+3}}}{r_{M+4}} - \ldots - \frac{\frac{r_{K-1}}{r_K}}{r_K} \) and note that \( G \) is \( K - M \) minus the sum of \( K - M - 1 \) terms, each of which is between 0 and 1 exclusive. In addition, we notice that the product of these \( K - M - 1 \) terms is exactly \( \frac{\frac{r_{M+1}}{r_K}}{r_K} \).

\[ \frac{\frac{r_{M+1}}{r_K}}{r_K} \geq 2 - G. \]

\(^{10}\)The inequality of arithmetic and geometric means states that for any non-negative real numbers \( x_1, \ldots, x_n \), the following is true: \( \frac{x_1 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n} \), where the left side is the arithmetic mean while the right side is the geometric mean.
This inequality is true because when we fix the sum of the $K - M - 1$ terms to be $K - M - G$ and each term is in $(0, 1)$ and $G \in (1, 2)$, then the minimum value of their product occurs when all but one of them is $1$ for a sum of $K - M - 2$ and the last one is $2 - G$. This is formally proven in Lemma 11 stated below and proven in Section 3.6.

**Lemma 11.** Given $x_1, \ldots, x_n \in (0, 1)$ such that $\sum_{i=1}^{n} x_i = n - 1 + \epsilon$ for some $\epsilon \in (0, 1)$, we have that $\prod_{i=1}^{n} x_i \leq \epsilon$.

Having presented the useful property, we now show the result: Here we would like to compare two quantities $\Gamma$ and $\Gamma_{up}$ that both have closed form solutions from equations (29) and (33). Our goal here is to show that $\frac{\Gamma}{\Gamma_{up}} \geq f_2(G) := \begin{cases} 1 - \frac{1 - \max(2-G, 0)}{2} & 2 \cdot n_M \leq C \\ 1 - \frac{1 - \max(2-G, 0)}{4} & 2 \cdot n_M > C \end{cases}$

**Case 1:** $2 \cdot n_M < C$. We split our analysis into sub-cases based on the minimum in the definition of $\Gamma_{up}$.

(Recall that $\Gamma_{up} = \min(\Gamma_L, \frac{1}{G})$.)

- **Case 1a:** $\Gamma_L \leq \frac{1}{G}$. Recall that $\Gamma_L = \frac{2}{2 \cdot G + M - \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i}}$ and $G = K - M - \sum_{i=M+2}^{K} \frac{r_{i-1}}{r_i}$.

  Hence, the condition $\Gamma_L \leq \frac{1}{G}$ can be written as

  $$2 \cdot G \leq 2 \cdot G + M - \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i} - \frac{r_M}{r_{M+1}} + \frac{r_M}{r_K} \quad \Rightarrow \quad M - \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i} \geq \frac{r_M}{r_{M+1}} - \frac{r_M}{r_K}.$$

  Also, recall that $\Gamma = \frac{2}{2 \cdot G + M - \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i}}$. We then bound our desired expression as follows:

  $$\frac{\Gamma}{\Gamma_L} = \frac{2 \cdot G + M - \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i} - \frac{r_M}{r_{M+1}} + \frac{r_M}{r_K}}{2 \cdot G + M - \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i}}$$

  $$= 1 - \frac{\frac{r_M}{r_{M+1}} - \frac{r_M}{r_K}}{2 \cdot G + M - \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i}}$$

  $$\geq 1 - \frac{\frac{r_M}{r_{M+1}} - \frac{r_M}{r_K}}{2 \cdot G + M - \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i}}.$$

  Here, in the first inequality, we used inequality (87) in order to give an upper bound to $\sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i}$ in the denominator. The second inequality is true because we see that $\frac{r_M}{r_{M+1}} - \frac{r_M}{r_K}$ appears in both the numerator and denominator of our fraction. We multiply both of those terms by $\frac{r_{M+1}}{r_M} > 1$, increasing the numerator and denominator of the fraction by the same amount, making it larger and closer to 1.

  We apply the useful property in equation (86), and immediately get our desired result:

  $$\frac{\Gamma}{\Gamma_L} \geq 1 - \frac{1 - \frac{r_M}{r_{M+1}}}{2 \cdot G + M - \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i}} \geq 1 - \frac{1 - \max(2-G, 0)}{2 \cdot G + 1 - \max(2-G, 0)}$$

- **Case 1b:** $\Gamma_L > \frac{1}{G}$. In this case, the negation of inequality (87) is true, which allows us to bound the desired expression in the following way

  $$\frac{\Gamma}{\Gamma_L} = \frac{\Gamma}{1/G} = \frac{2 \cdot G}{2 \cdot G + M - \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i}}$$
\[
\begin{align*}
&= 1 - \frac{M - \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i}}{2 \cdot G + M - \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i}} \\
&\geq 1 - \frac{r_{M+1}}{2 \cdot G + \frac{r_M}{r_{M+1}} - \frac{r_M}{r_K}}.
\end{align*}
\]

The last inequality is obtained by applying the inverse of inequality (87) and noticing that by replacing \(M - \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i}\) in both the numerator and denominator with a larger term, we make the fraction term larger. This resulting expression is exactly the same as the second-to-last expression in inequality (88).

Thus, we conclude that regardless of whether we are in Case 1a or Case 1b, the same bound form Case 1a holds.

This concludes the proof for Case 1. Notice that we did not use the fact that \(2 \cdot n_M \leq C\). The statements we make are actually true for any relationship between \(2 \cdot n_M\) and \(C\). In case 2, we show that if \(2 \cdot n_M > C\), then we can strengthen our bounds.

**Case 2:** \(2 \cdot n_M > C\). We use the fact that \(n_M = \frac{1}{2} \cdot \Gamma \cdot \sum_{i=1}^{M} \left(1 - \frac{r_{i-1}}{r_i}\right) \cdot C\) and \(\Gamma = \frac{2}{2 \cdot G + M - \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i}}\) to simplify this condition into the following

\[
2 \cdot n_M > C \quad \Rightarrow \quad \Gamma \cdot \sum_{i=1}^{M} \left(1 - \frac{r_{i-1}}{r_i}\right) > 1 \quad \Rightarrow \quad 2 \cdot \left(M - \sum_{i=1}^{M} \frac{r_{i-1}}{r_i}\right) > 2 \cdot G + M - \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i}.
\]

This is further simplified as

\[
M - \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i} > 2 \cdot G - 2 \cdot \frac{r_M}{r_{M+1}}.
\]

Using inequality (89), we construct a lower bound as follows (note that we do not perform case-work on \(\Gamma_{LP}\)). Rather we just use the fact that \(\Gamma_{LP} \leq \Gamma\):

\[
\frac{\Gamma}{\Gamma_{LP}} \geq \frac{2 \cdot G + M - \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i} \cdot \frac{r_M}{r_{M+1}}}{2 \cdot G + M - \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i}} = 1 - \frac{r_{M+1}}{2 \cdot G + M - \sum_{i=1}^{M+1} \frac{r_{i-1}}{r_i}} \geq 1 - \frac{r_{M+1}}{2 \cdot G + 2 \cdot G - 2 \cdot \frac{r_M}{r_{M+1}}} = 1 - \frac{r_{M+1}}{4 \cdot G - 2 \cdot \frac{r_M}{r_{M+1}}} \geq 1 - \frac{1 - \frac{r_M}{r_{M+1}}}{4 \cdot G - 2}.
\]

The first inequality is true by inequality (89), while for the last inequality we take \(\frac{r_M}{r_{M+1}} = 1\) to give a lower bound. This is a valid lower bound because of the fact that the function \(h(x) = \frac{x - \frac{r_M}{r_{M+1}}}{4 \cdot G - 2 \cdot x}\) for \(x \in (0, 1)\) is increasing as \(x\) approaches the discontinuity point \(x = 2 \cdot G > 2\) (as \(G > 1\)). Therefore, we give a lower bound on \(1 - h(x)\) by choosing \(x = 1\). Now, we use our useful property that \(\frac{r_{M+1}}{r_K} \geq \max(2 - G, 0)\) to get our desired result

\[
1 - \frac{1 - \frac{r_M}{r_{M+1}}}{4 \cdot G - 2} \geq 1 - \frac{1 - \max(2 - G, 0)}{4 \cdot G - 2}.
\]
D.5. Proof of Lemma 5

Proof of Lemma 5 Recall that
\begin{align*}
f_1(G) = \begin{cases} 
1 & 2 \cdot n_M \leq C \\
1 - \frac{1}{2} e^{-2} G + 1 & 2 \cdot n_M > C
\end{cases}
\end{align*}

and
\begin{align*}
f_2(G) = \begin{cases} 
1 - \frac{1 - \max(2 - G, 0)}{2 + 1 - \max(2 - G, 0)} & 2 \cdot n_M \leq C \\
1 - \frac{1 - \max(2 - G, 0)}{4 G - 2} & 2 \cdot n_M > C
\end{cases}
\end{align*}

For $2 \cdot n_M \leq C$, we have that $f_1(G) \cdot f_2(G) = 1 - \frac{1 - \max(2 - G, 0)}{2 + 1 - \max(2 - G, 0)}$. This function is increasing for $G > 2$ and decreasing for $G \in (1, 2)$, meaning that its minimum value is at $G = 2$ for a value of $\frac{4}{5}$.

For $2 \cdot n_M < C$, we consider the function
\[
f_1(G) \cdot f_2(G) = \left(1 - \frac{1}{2} e^{-2} G + 1\right) \left(1 - \frac{1 - \max(2 - G, 0)}{4 G - 2}\right).
\]

We claim that the minimum of this function occurs at $G = 2$, for which the function value is $(1 - \frac{1}{2} e^{-2}) \cdot \frac{5}{6} \approx 0.8125 > \frac{4}{5}$. For $G > 2$, it is easy to see that $f_1(G) \cdot f_2(G)$ is an increasing function. For $G < 2$, we see that $f_1(G) \cdot f_2(G)$ has critical points at $G \approx 1.1394$ and $G \approx 1.5496$ for which the function values are 0.8139 and 0.8158, both of which are significantly larger than $\frac{4}{5}$. Therefore, we have proven that $f_1(G) \cdot f_2(G) \geq \frac{4}{5}$ is true for any value of $G$.

D.6. Proof of Lemma 11

Proof of Lemma 11 Here, we show that given $x_1, \ldots, x_n \in (0, 1)$ such that $\sum_{i=1}^{n} x_i = n - 1 + \epsilon$ for some $\epsilon \in (0, 1)$, we have that $\prod_{i=1}^{n} x_i \leq \epsilon$. To do so, we consider the following minimization problem, where we relax $x_i \in (0, 1)$ to $x_i \in [0, 1]$. Clearly this new minimization problem’s objective is a lower bound on that of our original problem.

\[
\min_{x_1, \ldots, x_n} \prod_{i=1}^{n} x_i \\
\text{s.t} \sum_{i=1}^{k} x_i = n - 1 + \epsilon \quad x_i \in [0, 1]
\]

We claim that the minimum above occurs when all but one of the $x_i$’s are equal to 1 and the last is equal to $\epsilon$. To see this, we first notice that given that $\epsilon > 0$, any feasible solution cannot choose $x_i = 0$ for any $i \in [n]$ for otherwise there is no way for the remaining $n - 1$ of the $x_i$’s to add up to something greater than $n - 1$. Now, consider any optimal solution $(x_i^*)_{i=1}^{n}$ to the above minimization problem. For any two indices $i \neq j$, we claim that one of $x_i^*$ or $x_j^*$ must be equal to 1. Suppose for contradiction that there exists indices $i \neq j$ such that $0 < x_i^* < 1$ and $x_j^* < 1$. Then, we can increase $x_j^*$ by $\delta > 0$ and decrease $x_i^*$ by $\delta$ and this results in a smaller objective value because:

\[
x_i^* \cdot x_j^* \geq x_i^* \cdot x_j^* + (x_i^* - x_j^*) \cdot \delta - \delta^2 = (x_i^* + \delta)(x_j^* - \delta)
\]
and all other terms $x_k^*$ for $k \notin \{i, j\}$ remain the same. Given that $0 < x_i^* < x_j^* < 1$, it is always possible to find such a $\delta > 0$ so that the $(x_i^* + \delta)$ and $(x_j^* - \delta)$ are both in $[0, 1]$. Notice that their sum is the same as $x_i^* + x_j^*$ meaning that this new solution is feasible. Therefore, we get a contradiction and so it must be the case that one of $x_i^*$ or $x_j^*$ is equal to 1. Applying this argument to all pairs of indices in $[n]$ gives us the conclusion that all but one of the $x_1^*, \ldots, x_n^*$'s is equal to 1, which shows our desired result.