Abstract

In this paper we discuss the strong coupling limit of chiral \( N = 1 \) supersymmetric gauge theory via their embedding into M-theory. In particular we focus on the brane box models of Hanany and Zaffaroni and show that after a T-duality transformation their M-theory embedding is described by supersymmetric 3-cycles; its geometry will encode the holomorphic non-perturbative information about the gauge theory.
Contents

1 Introduction 3

2 N=2 Gauge Theories and 2-cycles 4

3 N=1 Gauge Theories and Supersymmetric 3-cycles 9
   3.1 N = 1 brane boxes .................................................. 9
   3.2 T-duality, M-theory embedding and the emergence of the 3-cycles ... 13
   3.3 Brane Cubes and M-theory ........................................... 15
   3.4 The supersymmetric d-cycles ........................................ 16
      3.4.1 The d-cycle equations ............................................. 16
      3.4.2 Supersymmetric 2-cycles ....................................... 19
      3.4.3 Supersymmetric 3-cycles ....................................... 21
   3.5 Supersymmetric 3-cycles for intersecting branes and N = 1 brane boxes . 22
      3.5.1 Branes as quaternionic coordinates ............................ 22
      3.5.2 Uniform Bending – Sewing of N = 2 models .................. 24
      3.5.3 General N = 1 brane boxes .................................... 27

4 Conclusions 29

Appendix A: Derivation of the d-cycle equations 29

Appendix B: Some facts from Hamiltonian dynamics 33
1 Introduction

Recently it was demonstrated that many interesting non-perturbative results in superstring theory and supersymmetric field theories can be derived from 11-dimensional M-theory. In particular Witten \cite{1} has shown that $N = 2$ supersymmetric gauge theories can be solved via M-theory by lifting the corresponding 10-dimensional type II brane configurations \cite{2} to 11 dimensions. As a result the intersection of $n$ parallel NS 5-branes and $k$ suspended D 4-branes is described in 11 dimensions by an one-dimensional complex curve, which can be seen as a two-dimensional supersymmetric cycle embedded in the flat four-dimensional space $\mathbb{R}^3 \times S^1$. This curve precisely agrees with the famous Seiberg-Witten curves \cite{3} of $N = 2$ supersymmetric field theories.

Non-chiral $N = 1$ gauge theories can be obtained by rotating one or several of the NS 5-branes such that they intersect by a certain angle. The corresponding continuous parameter can be regarded in field theory as a mass parameter which explicitly breaks $N = 2$ supersymmetry down to $N = 1$. The M-theory embedding of the non-chiral $N = 1$ models, constructed in this way, leads again to supersymmetric 2-cycles, now embedded in the flat six-dimensional space $\mathbb{R}^5 \times S^1$ \cite{4, 5, 6}. Analyzing these curves, the form of the corresponding $N = 1$ superpotentials can be derived.

A generic way to construct chiral $N = 1$ gauge theories in four dimensions is provided by the brane box models of Hanany and Zaffaroni \cite{7}. Here one deals with a type IIB configuration of intersecting NS and D 5-branes. We will show that upon a T-duality transformation to the IIA superstring, the M-theory lifting of the chiral $N = 1$ brane box configurations leads to supersymmetric 3-cycles suitably embedded into $\mathbb{C}^3$. These supersymmetric 3-cycles precisely correspond to the $SU(3)$ special Lagrangian calibrations (SLAG), discussed e.g. in \cite{10, 11}, which reduce the original supersymmetry by a factor 1/8. This is just the right amount of supersymmetry breaking for a generic chiral $N = 1$ gauge theory which cannot be smoothly deformed into a $N = 2$ gauge theory.

Following \cite{1} one can study quantum effects in the corresponding gauge theory by analyzing how the branes bend each other. Especially it is easy to study the $\beta$ function: the gauge coupling is encoded in the distance between two NS branes or in the area of the box in the case of boxes respectively. So the shape of the bent branes directly gives some information about the running gauge coupling. In order to obtain the exact quantum information (or at least the information protected by holomorphy) one can lift the brane configuration to M-theory and use 11d SUGRA to solve it. For IIA brane configurations like those studied in \cite{1} this is straightforward, since 11d SUGRA on a circle is dual to IIA.

The boxes of \cite{7} live in IIB theory, so in order to perform the M-theory lift on has to use the relation that IIB on a circle is M-theory on a torus, so we have to compactify one of

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*An alternative, but more restricted construction of chiral $N = 1$ models via orientifolds was introduced in \cite{8, 9}.
the common wordvolume directions. Like in the 5d case studied in [12] this means that we are really solving the 4d theory on $R^3 \times S^1$. In the limit where the area of the M-theory torus shrinks to zero or grows to infinity we recover the $N = 1$ $d = 4$ and $N = 2$ $d = 3$ gauge theories respectively. We will show that this way all the holomorphic information about these gauge theories is encoded in the geometry of a SUSY 3-cycle. Especially we expect the superpotential to correspond to the volume and the gauge couplings on the Coulomb branches to be related to periods of the cycle. The possible 3-cycles for given boundary values will encode the vacuum structure of the theory. Gauge theories which dynamical SUSY breaking will correspond to situation, where the minimal area cycle for the given boundary problem is not a SUSY 3-cycle.

In Section 2 we will review Witten’s rediscovery of the Seiberg-Witten curve in terms of the lift to M-theory. In Section 3 we will introduce the classical brane box setups and show how they relate to 3-cycles. We will develop some tools which we believe are very helpful in constructing the 3-cycles. For the special case of finite theories and theories that satisfy the uniform bending requirement of [13] we are able to perform the lift explicitly. However the corresponding cycles turn out to be superpositions of special 3-cycles that are of the form 2-cycle times line. We end with some preliminary results about more general cases.

2 N=2 Gauge Theories and 2-cycles

In order to understand better the supersymmetric 3-cycles in the $N = 1$ brane box models, let us first recall the M-theory embedding of the $N = 2$ models which leads to supersymmetric 2-cycles. The four-dimensional $N = 2$ gauge theories are based on the following brane set up in type IIA superstring theory on $R^{10}$:

- $n$ NS 5-branes with world volumes parametrized by $x^0, x^1, x^2, x^3, x^4$ and $x^5$; they are located at $x^7 = x^8 = x^9 = 0$ and at some fixed value of $x^6_\alpha$ ($\alpha = 1, \ldots, n$), at least in classical approximation.

- $k_\alpha$ ($\alpha = 0, \ldots, n$) D 4-branes with world volumes in the $x^0, x^1, x^2, x^3$ and $x^6$ directions, being suspended in the $x^6$ direction between the $(\alpha)^{th}$ and $\alpha^{th} + 1$ NS 5-brane. For $\alpha = 0$ or $\alpha = n$ the D 4-branes are semi-infinite to the left of the first NS 5-brane or, respectively, to the right of the $n^{th}$ NS 5-brane. The 4-branes are located in $x^4-x^5$ plane at the complex coordinate $v = x^4 + ix^5$.

This $N = 2$ brane configuration, which preserves 1/4 of the original supersymmetries, is summarized in figure [1].

**Alternatively, within the so called elliptic models, the coordinate $x^6$ is compact, such that $k_0 = k_n$ and the correspond D 4-branes are also finite and suspended between the first and the $n^{th}$ NS 5-brane.
Figure 1: The N=2 brane configuration

The four-dimensional gauge group is given by

$$G = \prod_{\alpha=1}^{n-1} SU(k_\alpha).$$  \hspace{1cm} (1)

In addition one finds hypermultiplets in the representations:

$$(k_0, \bar{k}_1) \oplus (k_1, \bar{k}_2) \oplus \cdots \oplus (k_{n-1}, \bar{k}_n).$$  \hspace{1cm} (2)

(Here, $k_0$ and $\bar{k}_n$ act as global flavor representations.)

The gauge coupling constants $g_\alpha$ of every $SU(k_\alpha)$ group factor are determined by the differences between the positions of the NS 5-branes:

$$\frac{1}{g_\alpha^2} = \frac{x_{\alpha+1}^6 - x_\alpha^6}{g_s},$$  \hspace{1cm} (3)

where $g_s$ is the string coupling constant.

After having discussed the classical picture, let us now come to solution of the model after taking into account the quantum corrections. The D 4-branes exert a force on the NS 5-branes causing them to bend. Since the tensions of the NS 5-brane and the tensions of the D branes have a different behavior in terms of the string coupling constant, $T_{NS} \sim g_s^{-2}$ compared to $T_D \sim g_s^{-1}$, this bending is a quantum effect. As a result, a NS 5-brane on which 4-branes end does not have a definite value of $x^6$ in contrast to the classical picture. More specifically, when there is a bending which moves the two NS branes towards each other, the coupling becomes strong at high energies, i.e. we deal with an IR free theory. Conversely, if the bending is outwards, there is an asymptotically free theory. In four dimensions the bending is logarithmic with the distance $r$, whereas in $d$ dimensions the local bending of the NS branes goes like $r^{d-4}$, $r$ being the ‘position’ of the D brane on the
world volume of the NS brane. This shows that in dimensions $d < 4$ all gauge theories are asymptotically free, and all gauge theories with finite gauge coupling are IR free for $d > 4$. The bending is absent if the number of left D branes ending on a given NS 5-brane is equal to the number of D branes ending from the right.

In one-loop perturbation theory in four dimensions, the bending of the NS 5-branes leads to a logarithmic variation of $x_6^\alpha$ in terms of $v$:

$$x_6^\alpha = (k_\alpha - k_{\alpha-1}) \log |v|. \quad (4)$$

Inserting this into eq.(3), the logarithmic running of the gauge coupling $g_\alpha$ is determined as

$$\frac{1}{g_\alpha^2} = \frac{-2k_\alpha + k_{\alpha+1} + k_{\alpha-1}}{g_s} \log |v|. \quad (5)$$

Note that the prefactor $b_{N=2}^{N=2} = -2k_\alpha + k_{\alpha+1} + k_{\alpha-1}$ precisely agrees with the $N = 2$ $\beta$-function coefficient of the gauge group $SU(k_\alpha)$ with $N_f = k_{\alpha+1} + k_{\alpha-1}$ fundamental matter fields. In this way the shape of the branes incorporates the 1-loop effects in field theory.

In $N = 2$ field theory there are no higher loop effects. However there are still non-perturbative effects due to instantons. These instantons can be seen directly in the brane picture, namely the D 0-branes are instantons within D 4-branes. The problem is now to solve the theory by including all these effects. This can be done by “lifting” the IIA configuration to M-theory \[1\]. The advantage of considering the above configuration of branes in M-theory is that the D 4-branes and NS 5-branes are in fact the same object, the M 5-brane. The intersection of the D 4- and NS 5-branes in IIA was singular but this is smoothed out in M-theory and in fact it is possible to consider all the D 4-branes and NS 5-branes as a single M 5-brane with complicated worldvolume. However the conditions for preserving $N=2$ supersymmetry restrict the embedding of the M 5-brane worldvolume and it is possible to find the function describing this embedding explicitly. Clearly this must incorporate the classical IIA brane set up as well as the field theory 1-loop corrections through the shape of the M 5-brane. However the non-perturbative instantons are also automatically included since D 0-branes are simply Kaluza-Klein momentum modes of compactified M-theory.

Let us discuss in slightly more detail how the M-theory embedding of the $N = 2$ brane configurations is constructed. The lifting to M-theory is performed by adding the 11th coordinate $x^{10}$ which is periodic with period $2\pi R_{11}$. Now, the complex coordinate $s_\alpha = (x_6^\alpha + ix^{10}_\alpha)/R_{11}$ describes the asymptotic positions of the M 5-branes in the $x^6$-$x^{10}$ plane, whereas as before $v$ denotes the asymptotic positions of the M 5-branes in the $x^4$-$x^5$ plane (all M 5-branes have common world volume in the 0123-directions, and are fixed at $x^7 = x^8 = x^9 = 0$). Regarding $x^{10}$ as $\theta$-parameter one can introduce a complex coupling constant

$$\tau_\alpha = \frac{\theta_\alpha}{2\pi} + i \frac{4\pi}{g_\alpha^2} = i(s_{\alpha+1} - s_\alpha). \quad (6)$$
Concentrating on the directions $x^4, x^5, x^6$ and $x^{10}$ we see that the M 5-brane world volume spans a two-dimensional surface $\Sigma^{(2)}_{n,k\alpha}$ in the four-manifold $\mathbb{R}^3 \times S^1$. A priori, $\Sigma^{(2)}_{n,k\alpha}$ is determined by two real functions of the coordinates $x^4, x^5, x^6$ and $x^{10}$:

\[
\begin{align*}
 f_{n,k\alpha}(x^4, x^5, x^6, x^{10}) &= 0, \\
 g_{n,k\alpha}(x^4, x^5, x^6, x^{10}) &= 0.
\end{align*}
\] (7)

However $N = 2$ space-time supersymmetry requires that $s$ varies holomorphically with $v$, such that $\Sigma^{(2)}_{n,k\alpha}$ is a Riemann surface in $\mathbb{R}^3 \times S^1$:

\[
\Sigma^{(2)}_{n,k\alpha} : F_{n,k\alpha}(s, v) = f_{n,k\alpha} + ig_{n,k\alpha} = 0.
\] (8)

This means that $\Sigma^{(2)}_{n,k\alpha}$ is a supersymmetric 2-cycle, or in terms of $[10]$, $\Sigma^{(2)}_{n,k\alpha}$ describes a $SU(2)$ Kähler calibration. The holomorphy of the equation $F_{n,k\alpha}(s, v)$ implies that the real functions have to obey the Cauchy-Riemann differential equations, where $f_i$ denotes $\partial f/\partial x_i$:

\[
\begin{align*}
 f_4 &= g_5, & f_6 &= g_{10}, \\
 f_5 &= -g_4, & f_{10} &= -g_6.
\end{align*}
\] (9)

As explained in $[1]$, it is very useful to perform an holomorphic change of variables, $t = \exp(-s)$, in order to describe the asymptotic positions of the M 5-branes in the correct way. Specifically consider the complex equation

\[
F(t, v) = 0.
\] (10)

At a given value of $v$, the roots of $F$ in $t$ are the positions of the NS 5-branes, i.e. $F$ is a polynomial of degree $n$ in $t$. On the other hand, for fixed $t$, the roots of $F$ in $v$ are the positions of the IIA D 4-branes. Therefore $n$ parallel NS 5-branes with positions at $t_i$ ($i = 1, \ldots, n$) are simply described by the function $F(t, v) = \prod_{i=1}^{n}(t - t_i)$, and $k$ parallel D 4-branes, positioned at values $v_j$ ($j = 1, \ldots k$), correspond to the choice $F(t, v) = \prod_{j=1}^{k}(v - v_j)$. Then it immediately follows that $n$ NS 5-branes intersected by $k$ D 4-branes correspond to

\[
F(t, v) = \prod_{i=1}^{n}(t - t_i) \prod_{j=1}^{k}(v - v_j).
\] (11)

In this case, since the number of D 4-branes to the left and to the right of each NS 5-brane is the same, there is no bending of the NS 5-branes by the D 4-branes.

Next, let us briefly discuss, how the perturbative, one-loop bending can be described $[1]$. Consider the situation of one NS 5-brane with $k_0$ ($k_1$) D 4-branes ending on it from the left (right). Then the induced bending in the region, where $t$ is very large, corresponds to the following choice for $F$:

\[
F(t, v) = v^{k_0}(t - e^{k_1-k_0}),
\] (12)
where $\epsilon$ is a constant. Introducing back the variable $s$, this equation immediately leads to eq.(4), namely the logarithmic bending of $s$ in terms of the Higgs field $v$:

$$s = (k_1 - k_0) \log v. \quad (13)$$

Putting all these informations from eqs.(11,12) together, the supersymmetric 2-cycle equation for a general $N = 2$ brane configuration takes the form

$$\sum_{n,k_{\alpha}}^{(2)}: F_{n,k_{\alpha}}(t, v) = p_{k_0}(v)t^n + p_{k_1}(v)t^{n-1} + \cdots + p_{k_{n-1}}(v)t + p_{k_n}(v), \quad (14)$$

where the $p_{k_{\alpha}}(v)$ are polynomials in $v$ of degree $k_{\alpha}$. The up to now unspecified parameters of the polynomials $p_{k_{\alpha}}(v)$ appear as the moduli of the gauge theory. This describes the non-perturbative solution of the model with gauge group $\prod_{n=1}^{n-1} SU(k_{\alpha})$ with hyper-multiplets in bi-fundamental representations. For example, the pure $SU(k)$ gauge theory without matter fields, i.e $n = 2$, $k_0 = k_2 = 0$, $k_1 = k$, is described by the curve

$$F(t, v) = t^2 + p_k(v)t + 1. \quad (15)$$

This is nothing else than the famous Seiberg-Witten curve of genus $k - 1 \quad [3, 14, 15, 16]$. Why does this procedure work? The solution of the gauge theory is found by using the duality between 11d SUGRA and IIA string theory. When $R_{11}$ is large, 11d SUGRA is valid. Solving for the exact shape of the M5 brane means solving the classical minimal area problem for a tensile brane (the shape of a soap bubble) with given boundary conditions. The requirement of supersymmetry tells us that we should only consider special minimal area configurations, the SUSY 2-cycles.

On the other hand we identified the gauge group and matter content at small $R_{11}$, where we have weakly coupled IIA string theory and the analysis of [2] applies. In order to decouple the bulk modes from the gauge theory we have to take the string scale $M_s \to \infty$ and $M_{pl} \to \infty$ and in order to decouple the KK modes from the finite interval, we have to also send $L \to 0$, where $L$ is the length of the $x^6$ intervals. This all has to be done holding $g_{YM}^2 = \frac{g_s^2}{M_s L}$ fixed. In 11d units $g_{YM}^2 = \frac{R_{11}}{L}$. In order to keep the interacting gauge theory (fixed $g_{YM}^2$) while decoupling the KK modes ($L \to 0$) $R_{11}$ has to go to zero! This is the limit in which the brane setup reduces to 4d SYM. But this is the opposite limit of the one we were able to solve, where $R_{11}$ and hence $L$ have to be very large and we can use 11d SUGRA. So we should only expect quantities protected by holomorphy, like the $N = 2$ prepotential, which can’t depend on the real parameter $R_{11}$, to come out correctly. Indeed it was shown in [17] that unprotected quantities like the 4-derivative terms in the effective action disagree with field theory results.
# N=1 Gauge Theories and Supersymmetric 3-cycles

## 3.1 $N=1$ brane boxes

Now let us discuss the brane boxes of \[7\] which can be used to construct $N=1$ supersymmetric gauge theories with chiral matter content. The starting point is now a type IIB superstring with the following branes included (see figure 2):

- $n$ parallel NS 5-branes with world volumes along the (012345)-directions. These 5-branes are fixed at $x^7 = x^8 = x^9 = 0$ and are placed at arbitrary positions in $x^6$.
- $n'$ parallel NS' 5-branes with world volumes along the (012367)-directions. The NS' branes are fixed at $x^5 = x^8 = x^9 = 0$ and are placed at arbitrary positions in $x^4$.
- D 5-branes with world volumes along the (012346)-directions. The D 5-branes can take different position on the NS 5-branes in the $x^5$-direction and also different positions on the NS' 5-branes in the $x^7$-direction. Depending on the specific model one likes to discuss, the directions $x^4$ and $x^6$ can be either uncompactified or periodic (elliptic models). We will concentrate on the non-elliptic models. It follows that the D 5-branes are finite in the directions $x^6$, $x^4$ in case they are placed inside the ‘inner’ boxes. However they are semi-infinite in case they end only on one NS (NS') brane (‘outer’ boxes).

\[\ast\] Like for the $N=2$ brane models, the $N=1$ brane box models can be related to fractional branes via T-duality [18].

![Figure 2: The N=1 brane configuration](image)

| NS | NS | NS | NS | NS | NS | NS |
|----|----|----|----|----|----|----|
| $k_{0,0}$ | $k_{1,0}$ | | $k_{n-1,0}$ | $k_{n,0}$ | | |
| $k_{0,1}$ | | | | | | |
| $k_{0,n'-1}$ | $k_{n',0}$ | | $k_{n,n'-1}$ | | | |

\[\ast\] Like for the $N=2$ brane models, the $N=1$ brane box models can be related to fractional branes via T-duality [18].
This brane configuration preserves 1/8 of the original supersymmetry. We see that a generic configurations consists of a grid of \((n+1)(n'+1)\) boxes built by \(n\) NS 5-branes and \(n'\) NS' 5-branes in the \(x^4\)-\(x^6\) plane. We are labelling the boxes by the two indices \(\alpha, \alpha'\) where \(\alpha = 0, \ldots, n\) and \(\alpha' = 0, \ldots, n'\). The \((n-1)(n'-1)\) ‘inner’ boxes with \(\alpha = 1, \ldots, n-1, \alpha' = 1, \ldots, n' -1\) have always finite area whereas the remaining ‘outer’ boxes have infinite size in case of uncompactified directions \(x^4\) and \(x^6\).

Now, let \(k_{\alpha,\alpha'}\) denote the number of D 5-branes which are placed in the box \([\alpha, \alpha']\). The suspended D 5-branes inside the inner boxes give rise to the following gauge group in four dimensions:

\[
G = \prod_{\alpha=1}^{n-1} \prod_{\alpha'=1}^{n'-1} SU(k_{\alpha,\alpha'}). \tag{16}
\]

The associated classical gauge coupling constants are given by the area of the corresponding box \([\alpha, \alpha']\):

\[
\frac{1}{g_{a,a'}^2} = \frac{(x_{\alpha+1}^4 - x_\alpha^4)(x_{\alpha'+1}^6 - x_{\alpha'}^6)}{g_s}. \tag{17}
\]

The matter content of the model consists of three types of chiral \(N = 1\) representations. First they are ‘horizontal’ chiral bi-fundamentals \(H_{\alpha,\alpha'}\) in the representations \((k_{\alpha,\alpha'}, k_{\alpha+1,\alpha'})\) of \(SU(k_{\alpha,\alpha'}) \times SU(k_{\alpha+1,\alpha'})\). Second there exist ‘vertical’ chiral bi-fundamentals \(V_{\alpha,\alpha'}\) in the representations \((k_{\alpha,\alpha'}, k_{\alpha,\alpha'+1})\) of \(SU(k_{\alpha,\alpha'}) \times SU(k_{\alpha,\alpha'+1})\); finally we have ‘diagonal’ chiral fields \(D_{\alpha,\alpha'}\) in the representations \((k_{\alpha,\alpha'}, k_{\alpha-1,\alpha'-1})\) of \(SU(k_{\alpha,\alpha'}) \times SU(k_{\alpha-1,\alpha'-1})\) \((\alpha, \alpha' > 1)\). In this context the groups \(SU(k_{\alpha,\alpha'})\) with \(\alpha = 0, n\) or \(\alpha' = 0, n'\) act as global flavor symmetries if the directions \(x^4\) and \(x^6\) are uncompactified. Note that the choices for the \(k_{\alpha,\alpha'}\) are severely constrained by the requirement of absence of anomalies. If all three types of matter multiplets are present then there exists a classical superpotential of the following form:

\[
W = \sum_{\alpha,\alpha'} H_{\alpha,\alpha'} V_{\alpha+1,\alpha'} D_{\alpha+1,\alpha'+1} - \sum_{\alpha,\alpha'} H_{\alpha,\alpha'+1} V_{\alpha,\alpha'} D_{\alpha+1,\alpha'+1}. \tag{18}
\]

One of the simplest (non-elliptic) models is given by the choice \(n = n' = 2\), \(k_{1,1} = N_c\) and \(k_{0,1} = k_{2,1} = N_f\), whereas \(k_{\alpha,\alpha'} = 0\) for \(\alpha' = 0, 2\). This choice of brane boxes corresponds to the supersymmetric QCD with \(G = SU(N_c)\) and with \(N_f\) fundamental plus antifundamental chiral fields. A second way to obtain SUSY QCD with \(N_f\) fundamental plus antifundamental matter fields is given by the choice \(k_{1,1} = N_c\) and \(k_{0,1} = k_{2,1} = k_{1,0} = k_{1,2} = N_f/2\) and zero otherwise. Finally the same spectrum can be realized by the choice \(k_{1,1} = N_c\) and \(k_{\alpha,\alpha'} = N_f/3\) \((\alpha, \alpha' \neq 1)\). However in this case a superpotential of the type eq. (18) is present.

So far we have only discussed the classical field theory. Of course it is essential to understand the quantum features as well. Chiral \(N = 1\) exhibit a huge variety of interesting
quantum phenomena. Especially the generic theory will have an anomaly which should show up as an inconsistency of the brane box as a string background. In these general cases the bending of the brane boxes isn’t well understood yet, but some special cases can be analyzed.

It is clear that the bending of the NS and NS’ branes depends on the number $k_{\alpha,\alpha'}$ of D 5-branes in each box. A very special class of $N = 1$ gauge theories is given by the finite models for which all $\beta$-functions and all anomalous dimensions vanish to all orders in perturbation theory [19, 20]. This condition includes the vanishing of the one-loop $\beta$-functions. In the brane picture complete finiteness means that all NS and NS’ branes do not bend at all, i.e. if the number of D 5-branes in every box is the same [20]. Then obviously, $N_f = 3N_c$ for every gauge group factor, and the one-loop $\beta$-functions are zero.

The corresponding brane setup consists of several branes put on top of each other. Each of the branes preserves 1/2 of the supersymmetries, together they still preserve 1/8, so the intersection is BPS. This ensures that the static branes don’t exert any force on each other. We can freely move the constituent branes since they don’t feel the presence of the other branes at all. Motions of the branes in the 46 plane corresponds to changing the areas of the various boxes and hence to changing the gauge couplings. Taking NS (NS’) branes away along the $x^7$ ($x^5$) direction destroys the box structure and corresponds to turning on FI terms.

Another special situation is that of uniform bending. The condition of uniform bending was first introduced by [13]. There it was argued to be necessary for consistency. As we will see this is too stringent. However uniform bent setups are very special and allow for a precise treatment of the quantum properties. To motivate the uniform bending requirement consider the basic cross configuration of figure 3.

For $x^6 \rightarrow -\infty$ (to the far left) the effects of the NS brane on the bending of the NS’ brane should be negligible. The D5 ending on the NS’ brane just looks like a 5d gauge
theory with 8 supercharges and leads to the standard linear bending \cite{21}. The slope of
the bending is given by the difference of branes ending from either side, hence
\[
slope_{x^6\to-\infty} = k_{\alpha,\alpha'} - k_{\alpha,\alpha'}+1. \tag{19}
\]
For the same reason we will have linear bending to the far right, that is for \(x^6\to\infty\), the
slope this time given by
\[
slope_{x^6\to\infty} = k_{\alpha+1,\alpha'} - k_{\alpha+1,\alpha'}+1. \tag{20}
\]
The observation of \cite{13} was that if
\[
k_{\alpha,\alpha'} - k_{\alpha,\alpha'}+1 = k_{\alpha+1,\alpha'} - k_{\alpha+1,\alpha'}+1 \tag{21}
\]
the bending on the far left is the same as on the far right and one may expect that the
shape of the NS’ in fact does not change at all as a function of \(x^6\). In \cite{18} it was observed
that the most general setup compatible with condition (21) can be achieved by “sewing”
together \(N = 2\) models, that is we take branes corresponding to 5d gauge theory built
out of NS and D5 branes and move them on top of a similar setup build out of NS’ and
D5 branes\footnote{In \cite{18} they also allowed sewing in a third kind of \(N = 2\) system connected to diagonal lines in the
box setup. This doesn’t lead to uniform bent models anymore and should hence be treated separately.} as illustrated in figure 4.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{sewing.png}
\caption{Sewing of N=2 models.}
\end{figure}

This sewing can be taken quite literally: as in the finite case there is a no force condition
between the constituent pieces, since their intersection still preserves some supersymmetry.
So we are free to move them independently. These deformations should correspond
to marginal couplings in the field theory. Since we are not just tuning the distance be-
tween two NS branes but are actually moving around compound systems, these marginal
operators won’t just be the gauge coupling as in the finite case but will also involve the superpotential couplings. Using the methods of [19], [20] were indeed able to show that the field theory has these marginal operators if the conditions (21) are satisfied for all boxes. Since the subsystems don’t influence each other, the exact bending is given in terms of the linear bending of the subsystems. All uniform bent systems are anomaly free.

The question remains, how to decide whether the more general setups are anomaly free. So far we were only able to discuss some very special cases, e.g. not including pure SYM. Some progress in understanding the bending in these cases has been made in [22], however without reaching a final answer. Several aspects of this problem can be easier understood in a T-dual picture. Consider an elliptic model, that is we take the $x^4 - x^6$ plane to be compactified on a torus. For a finite model, T-dualizing these two compact directions turns the D5 branes into D3 branes on an $\mathbb{C}^3/\Gamma$ orbifold background. Following [23] it was argued in [18] that the generic box T-dualizes into an orbifold with fractional branes, that is where the orbifold group is embedded into the gauge group via some other representation than the regular one. On the orbifold side one immediately faces the issue of tadpole cancellations. Non-vanishing tadpoles signal the presence of a source, that is a net charge sitting in the internal space. So in orbifold compactifications the tadpoles have to vanish for consistency. Since we are dealing with non-compact backgrounds some non-vanishing tadpoles may be tolerated. The relevant space is the FP set of the orbifold action transverse to the D3 brane. For the $N = 2$ case (a D3 brane in 0123 with an orbifold acting on 6789 space) the FP set is 2 dimensional. A net charge in 2d will give rise to a logarithmic singularity. In [23] it was shown that this divergence is nothing else but the running of the gauge coupling. Tadpole cancellation is hence not required for consistency. Vanishing of the tadpoles is equivalent to finiteness of the gauge theory.

In the same spirit Leigh and Rozali analyzed the orbifold dual of the brane boxes [24]. A generic orbifold element will leave a 0d FP set. These tadpoles have to vanish. Otherwise the gauge theory is anomalous. However some orbifold elements will leave a 2d FP set untouched. The corresponding tadpoles will only lead to logarithmic divergences which once more can be identified as the running gauge coupling. So vanishing of all tadpoles again implies finiteness of the gauge theory. Vanishing of the tadpoles for the 0d FP is required for anomaly freedom. Leigh and Rozali indeed showed that for all brane boxes leading to anomaly free gauge theories these 0d tadpoles vanish.

3.2 T-duality, M-theory embedding and the emergence of the 3-cycles

Now let us describe the the strong coupling limit of the $N = 1$ via embedding the brane boxes into M-theory. Since our original brane configurations is in the type IIB string, we have first to perform a T-duality transformation to the type IIA superstring before we can perform the M-theory embedding. We do not want to touch the NS and NS’ 5-branes, and we also do not want to create any D6-branes; therefore we will T-dualize over one
of the spatial directions common to all branes. To be specific we now assume that $x^3$ is periodic with radius $R_3^{IIB}$ and we perform the T-duality with respect to the $x^3$-direction. This leads to the following brane configuration:

- $n$ parallel NS 5-branes with world volumes along the (012345)-directions. These 5-branes are fixed at $x^7 = x^8 = x^9 = 0$ and are placed at arbitrary positions in $x^6$.
- $n'$ parallel NS' 5-branes with world volumes along the (012367)-directions. The NS' branes are fixed at $x^5 = x^8 = x^9 = 0$ and are placed at arbitrary positions in $x^4$.
- D 4-branes with world volumes along the (01246)-directions. These D 4-branes take different $x^5$ positions on the NS 5-branes and also different $x^7$ positions on the NS' 5-branes. In addition the D 4-branes can have arbitrary positions in the compactified spatial $x^3$-direction.

This configuration preserves like before 1/8 of the original supersymmetries and corresponds to a three-dimensional gauge theory with $N = 2$ space-time supersymmetry. The three-dimensional gauge theory can be simply obtained from the four-dimensional $N = 1$ models by circle compactification on $S^1$ in the $x^3$-direction. In the decompactification limit, $R_3^{IIB} \to \infty$, the four-dimensional $N = 1$ gauge theories are rediscovered. On the hand, for $R_3^{IIB} \to 0$, the theory is truly three-dimensional. Note that in three dimensions, a new Coulomb branch can be opened, since the three-dimensional vector multiplets contain one real scalar degree of freedom. The corresponding modulus $v$ is associated in the brane picture with the positions of the D 4-branes in the $x^3$-direction.

The three-dimensional gauge coupling is classically related to the 4d gauge coupling as $1/g_3^2 = R_3^{IIB}/g_4^2$. So in the limit $R_3^{IIB} \to \infty$ one must send $g_3 \to 0$ in order to have a finite coupling $g_4$. The scalar field in the vector multiplet live on a dual circle with radius $R_3^{IIA} = 1/R_3^{IIB}$. So in the 4d limit, $R_3^{IIA} \to 0$, one has to integrate out the fields with masses of order $v$ corresponding to the Coulomb branch. In this way we can regard $v$ as the parameter which sets the scale $\Lambda$ of the four-dimensional gauge theory. So in order to determine the logarithmic ‘running’ of the four-dimensional gauge coupling $g_4$ in terms of $\Lambda$, $1/g_3^2 = b^{N=1} \log \Lambda$, we will be in particular interested in the bending of the coordinates $x^4$ and $x^6$ in terms of $x^3$. (This precisely corresponds to the logarithmic running of $x^6$ in terms of the Higgs field vev $v = x^4 + ix^5$ in case of the $N = 2$ brane models.) This will be further discussed in section 3.5.2. Note that in the 3-dimensional limit, $R_3^{IIB} \to 0$, the pure Yang-Mills gauge theory has no stable supersymmetric groundstate unlike the 4d theory. Many more details of the dynamics and superpotentials of three-dimensional, $N = 2$ supersymmetric gauge theories can be found in [25].

After the duality in $x^3$ we are now ready for lifting the above configuration to M-theory by adding the period direction $x^{10}$ with radius $R_{11}$.

Then the intersection of all branes is described by the smooth configurations of M 5-branes. Like in the $N = 2$ case the singular intersections of the NS, NS’ and D 4-branes
are described in M-theory by a single smooth M 5-brane. Asymptotically, this M 5-brane takes the shape of the classical IIA branes:

- The NS 5-branes asymptotically correspond to M 5-branes which extend in the \((012345)\)-space and take different positions in the \(x^6, x^7, x^8, x^9\) and \(x^{10}\) directions.
- The NS’ 5-branes asymptotically look like M 5-branes with world volumes along the \((012367)\)-space and positions in \(x^4, x^5, x^8, x^9\) and \(x^{10}\).
- Finally, the asymptotics of the D 4-branes is given by M 5-branes with world volumes in \(x^0, x^1, x^2, x^4, x^6\) and \(x^{10}\) and positions in \(x^3, x^5, x^7, x^8\) and \(x^9\).

So all branes have common world volumes in the \((012)\)-space and are all located at \(x^8 = x^9 = 0\). Therefore, to characterize the M-theory configurations we have to focus on the six-dimensional space spanned by the coordinates \(x^3, x^4, x^5, x^7\) and \(x^{10}\). Each asymptotic brane fills three particular directions in this space. This means that the general embedding of the M-theory 5-branes is described by a three-dimensional submanifold in \(\mathbb{R}^4 \times T^2\). Supersymmetry requires this to be a supersymmetric 3-cycle. In the language of \([10]\) this is called a \(SU(3)\) special Lagrangian calibration (SLAG) which breaks 7/8 of the supersymmetries.

Let us again briefly consider the validity of our approach. In 11d units the 3d gauge coupling is given by \(g_{YM}^2 = \frac{R_{11}}{A}\) where \(A\) is the area of the box. As in the \(N = 2\) case there are two distinct limits if we want to keep \(g_{YM}^2\) fixed: for small \(R_{11}\) and \(A\) the KK modes decouple and the brane setup reproduces the gauge theory. For large \(R_{11}\) we can solve using 11d SUGRA, that is by solving for the SUSY 3-cycle. So we are again only solving MQCD and not really the gauge theory. However, as for \(N = 2\), all holomorphic quantities should be encoded in the geometry of the 3-cycle. That is of the terms in the 2-derivative approximation of the effective action, the holomorphic gauge coupling and the superpotential should be encoded in the 3-cycle, whereas the Kähler potential probably escapes our control.

### 3.3 Brane Cubes and M-theory

As already mentioned in the original work of \([7]\) the idea of brane boxes can naturally be extended to brane cubes and brane hypercubes. Each time we add one more NS brane with yet another orientation breaking another half of the supersymmetry. The D brane ends on these new NS branes as well, so that the brane spans a 3d cube or 4d hypercube instead of the 2d box we considered so far. Let us briefly discuss, how these configurations are lifted to M-theory. We will find that the situation is especially interesting in the case of brane cubes, where one can find two distinct models, one with chiral and one with non-chiral SUSY.
For simplicity let us consider the brane cubes directly on the IIA side with the D4 brane suspended between the NS branes. This is the setup that lifts to M-theory in a straightforward fashion. The third NS brane that we add should have a fixed $x^2$ position, so that the D4 brane is finite in this direction as well as in $x^4$ and $x^6$. There are two distinct possibilities to do so. The first is to add an NS brane along $014567$. This is the setup considered in [26]. It leads to a chiral $N = (2, 0)$ supersymmetric gauge theory in $d = 2$. Rotations in $89$ space give rise to the $U(1)_R$ symmetry. The dual orbifold consists of D1 branes living on top of an $\mathbb{C}^4/\Gamma$ orbifold, where $\Gamma$ is a subgroup of $SU(4)$. By the same reasoning as for the brane box we find that this chiral brane cube should lift to M-theory via a SUSY 4-cycle in the 7d space $234567$, that is via a SUSY 4-cycle associated with $G_2$ holonomy.

The second possibility is to have the NS brane in $012468$. This leaves us with $N = (1, 1)$ in 2 dimensions. Since this time all three types of NS branes have a common direction ($x^3$) we can perform a T-duality to type IIB as for the box, leading to a 3d $N = 1$ theory. This time the dual orbifold is a $G_2$ orbifold while the lift to M-theory has to be performed via an $SU(4)$ SLAG 4-cycle in $\mathbb{C}^4$. Therefor this non-chiral cube requires the same techniques as the SLAG 3-cycle. By simply adding another NS along 23469 we find the brane hypercube and its lift via an $SU(5)$ SLAG.

### 3.4 The supersymmetric $d$-cycles

#### 3.4.1 The $d$-cycle equations

A d-dimensional ‘curve’ $\Sigma^{(d)}$, embedded into 2d-dimensional flat space $\mathbb{R}^{2d}$ with coordinates $x^i$ ($i = 1, \ldots, 2d$) can be described at least locally by the zero locus of $d$ real functions $f^m(x^1, \ldots, x^{2d})$:

$$\Sigma^{(d)} = \mathbb{V}(f^1, \ldots, f^d) = \{(x_1, \ldots, x_{2d}) \mid f^m(x^1, \ldots, x^{2d}) = 0, \ m = 1, \ldots, d\}.$$  

If one wants to deal with a so called supersymmetric $d$-cycle, the choice of the functions $f^m$ is highly constrained. To study these restrictions we first introduce $d$ real coordinates $\xi_i$ ($i = 1, \ldots, d$) which parametrize the curve $\Sigma^{(d)}$. Furthermore we consider complex coordinates $u^i, u^i = x^{2i-1} + ix^{2i},$ of $\mathbb{C}^d$. Then the $d$-cycle can be characterized by making the complex $u^i$ to be functions of the real coordinates $\xi_i$, i.e. by the following embedding map $i$ from $\Sigma^{(d)}$ into $\mathbb{C}^d$:

$$i : \Sigma^{(d)} \longrightarrow \mathbb{C}^d : \xi_i \longrightarrow u^i(\xi_i), \ i = 1, \ldots, d. \hspace{1cm} (22)$$

The intersection configuration (for the case $d = 3$) is depicted in figure [2]. Now by applying the partial derivative $\partial_{\xi_k}$ to the defining equations $f^m$ of the $d$-cycle, we get the following relations:

$$\sum_{n=1}^d (f^m_{u^n} u^n_{\xi_k} + f^m_{u^n} u^n_{\xi_k}) = 0 \hspace{1cm} (23)$$
We will denote the left matrix by $M$ and the right matrix as $\bar{M}$, henceforth. Note, the sign in front of $\bar{M}$ depends on the dimension $d$ of the cycle. With the help of these matrices we can express the bared derivatives by the unbared ones in the following way:

$$\partial_k \bar{U} = (-1)^d \bar{M}^{-1} M \cdot \partial_k U = N \cdot \partial_k U. \quad (24)$$

By definition $N$ shares the properties:

1. $N^{-1} = (-1)^d M^{-1} \bar{M} = \bar{N}$
2. $|\det N| = 1$
3. If $N = N^T \Rightarrow N \in U(d)$
4. If further $\det N = 1 \Rightarrow N \in SU(d)$

Remembering the $d$-cycle should be supersymmetric we can ask for restrictions of the matrix $N$ following from this condition. It is well known that the notion of supersymmetric
cycles [27] coincides with the notion of special Lagrangian submanifolds [28] which can be rephrased in terms of the embedding map \( i : d\text{-cycle} \rightarrow \mathbb{C}^d \) and the two conditions:

\[
i^* \Im \Omega = 0 \quad \text{volume minimizing}
i^* \omega = 0 \quad \text{Lagrangian submanifold}
\]

Here \( \Omega \) is the complex structure of \( \mathbb{C}^d \), which we can choose to be \( \Omega = du^1 \wedge du^2 \wedge \cdots \wedge du^d \); \( \omega \) is the canonical Kähler form, \( \omega = \frac{1}{2i} \sum_i du^i \wedge d\bar{u}^i \). As it is shown in appendix A for the case of \( d \)-cycles, from the first equation we derive straightforward that \( N \) restricted to the \( d \)-cycle must be of unit determinant.

\[
\det N |_{\mathbb{V}(f^1, \ldots, f^d)} = 1
\]

But then to utilize this for computation in the embedding space we reformulate that condition a little bit. If

\[
I(\mathbb{V}) = \{ f \in C^\infty(\mathbb{R}^{2d}) \mid f|_{\mathbb{V}} = 0 \}
\]

denotes the ideal of functions vanishing on \( \mathbb{V}(f^1, \ldots, f^d) \), the above equation can be rewritten as

\[
\det N = 1 + [\text{some } g \in I(\mathbb{V})]
\]

This kind of non uniqueness is apparent throughout the equations. To get a handle for that is the main obstruction for concrete computations.

To keep that difference in mind but deal with the equations as there is no difference at the same time, we replace the equality sign by the congruence symbol \( (\equiv) \), i.e.

\[
\det N \equiv 1. \tag{27}
\]

By close inspection of the second equation in (25) (see again appendix A) one is led to a further restriction on \( N \), namely

\[
N \equiv N^T. \tag{28}
\]

So, in this way, we have translated the conditions of having a supersymmetric cycle to restrictions on our defining equations \( f^m \).

In summary, all what we have done so far can be formulated in a short but important proposition which is the starting point for all further computations:

**Proposition:** A \( d \)-cycle, represented as an intersection of \( d \) real valued functions is supersymmetric, iff \( N \equiv N^T \) and \( \det N \equiv 1 \).
It will turn out to be very useful to reformulate the last proposition \( N \equiv N^T \) in a different, but equivalent way. Namely, it is not difficult to show that the requirement \( N \equiv N^T \) is equivalent to the condition that the matrix \( MM^+ \) should be real modulo \( I(V) \). To prepare this reformulation we remark that by the split of the coordinates of \( \mathbb{R}^{2d} \) into the coordinates of \( \mathbb{C}^d \) they inherit an intrinsic meaning as the spatial and momentum variables of symplectic geometry. This is given by

\[
    u^i = q^i + i p^i, \tag{29}
\]

i.e. the real part of \( u^i \) gets the meaning of a spatial coordinate whereas the \( p^i \) is a momentum variable. Then we are free to define the convenient Poisson brackets of phase-space functions \( \{f, g\} \). This is done in the standard way as

\[
    \{f, g\} = \sum_{i=1}^{d} \left( \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i} \frac{\partial g}{\partial q^i} \right) = \sum_{i=1}^{d} (f_{2i-1} g_{2i} - f_{2i} g_{2i-1}), \tag{30}
\]

where \( f_{2i-1} = \frac{\partial f}{\partial q^{2i-1}} \) and \( f_{2i} = \frac{\partial f}{\partial p^{2i}} \). Then the matrix \( MM^+ \) reads

\[
    (MM^+)^{mn} = \langle \nabla f^m, \nabla f^n \rangle \pm i \cdot \{f^m, f^n\}. \tag{31}
\]

So \( MM^+ \) is a real matrix modulo \( I(V) \), i.e. \( N \equiv N^T \), if all Poisson brackets among the defining functions \( f^m \) and \( f^n \) vanish:

\[
    \{f^m, f^n\} \equiv 0. \tag{32}
\]

So we get a more suitable set of equations for concrete calculations. On the other side the last equations can be understood very natural (see appendix [3]).

**Corollary:** A \( d \)-cycle, represented as an intersection of \( d \) real valued functions is supersymmetric, iff \( \{f^i, f^j\} \equiv 0 \) and \( \det N \equiv 1^\ast \).

### 3.4.2 Supersymmetric 2-cycles

Now we want to rederive the known result for the case of supersymmetric two-cycles to give a simple check of our formalism and to establish our point of view on the meaning of the defining equations of the \( d \)-cycle. If we look at the brane configuration, we parametrise 4 space by \( u^1 = x_4 + i x_{10} \) and \( u^2 = x_5 + i x_6 \). Note that D 4-brane positions \( x^4 \) and \( x^5 \) correspond to the \( q^i \)-variables, whereas the NS 5-brane positions \( x^{10} \) and \( x^6 \) are the conjugated \( p^i \) variables. Now we work out the two-cycle conditions on the two real function \( f^1 = f(x^4, x^5, x^6, x^{10}) \) and \( f^2 = g(x^4, x^5, x^6, x^{10}) \). With

\[
    M = \begin{pmatrix}
        f_{u^1} & f_{u^2} \\
        g_{u^1} & g_{u^2}
    \end{pmatrix}
\]

\*After finishing and submitting our paper we became aware of [28] where it was already stated, however without detailed proofs, that a special Lagrangian submanifold is determined by the eqs. [27, 32]
one can calculate the two-cycle equations which result in:

\[
\begin{align*}
\{f,g\} &\equiv 0 \quad \Rightarrow \quad 0 \equiv g_4 f_{10} - g_{10} f_4 + g_5 f_6 - g_6 f_5, \\
\det N &\equiv 1 \quad \Rightarrow \quad 0 \equiv g_6 f_4 - g_{10} f_5 - g_4 f_6 + g_5 f_{10}.
\end{align*}
\]

(33) (34)

In analysing these equations it is a simple task to verify that all functions \(f\) and \(g\) satisfying

\[
\begin{align*}
f_6 &= g_{10} \\
f_4 &= g_5 \\
f_{10} &= -g_6 \\
f_5 &= -g_4
\end{align*}
\]

do solve our equations. These are the “Cauchy-Riemann” differential equations which state that \(f\) and \(g\) are the real and imaginary part of a holomorphic function in the variables \(v = x_4 + ix_5\) and \(s = x_6 + ix_{10}\), respectively. Then we choose as our coordinates

\[
\begin{align*}
v &= x_4 + i \cdot x_5 \\
t &= e^{-s} = e^{-(x_6 + ix_{10})}
\end{align*}
\]

to respect the compactness of the \(x_{10}\) direction. Hence \(f\) and \(g\) fit into a holomorphic function in two variables \(v\) and \(t\).

Up to now we have shown, that there is a subclass of solutions to our equations, which coincides with the well known holomorphicity argument. But as they stand our equations are more general and we have to think about that difference. Nevertheless in a certain sense every geometrical two-cycle should be described by a holomorphic function, yet. That is to say in the whole variety of functions specifying the same geometrical two-cycle, there is a distinguished holomorphic function, i.e. there is a lot of redundancy in the description, which could be exploited for constructing solutions, maybe. Therefore we are looking for a way of to mod out that redundancy.

This will be done by imposing some additional differential constraints in a generic way. To do that, recall the following properties of the matrix \(N\):

1. By definition: \(N = (-1)^d \tilde{M}^{-1} M\).

2. As shown in appendix A \(N\) can be factorized through \(a \in U(d)\)

\[
N = \lambda^{-1} = \bar{a} a^{-1}.
\]
Note that this does not imply that \( M \) must be unitary. But we want to choose \( M \) as close as possible to being unitary. We hope that this resolves the redundancy problem.

A unitary matrix \( M \) does have orthonormalized rows and columns. Thus to begin with the construction of an unitary \( M \) we want to orthogonalize the rows and columns of \( M \). Since we know the expression for \( MM^+ \) in geometrical this is straightforward. Simply we have to require orthogonality of the gradients of \( F \) and \( G \)

\[
< \nabla F, \nabla G > = 0.
\]

Of course one has to be careful in doing that. One has to ensure that by requiring these additional properties, the common zero set is unchanged. In fact, this can be done without getting into trouble. If the length of these both gradients coincides, our equations reduce to the Cauchy-Riemann equations, indeed.

There are problems in generalizing this nice looking observation to higher dimensional cycles but we hope that this construction works, too.

### 3.4.3 Supersymmetric 3-cycles

Recall the characteristic \( N = 1 \) brane configuration:

| NS   | 0 1 2 3 4 5 |
|------|-------------|
| NS’  | 0 1 2 3 6 7 |
| D4   | 0 1 2 4 6 10 |

Table 2: brane configuration.

Since the NS branes together with the D 4-branes build a \( N = 2 \) subsystem, two conjugated \((q,p)\) pairs are given by \((q_1, p_1) = (x^{10}, x^3)\) and \((q_2, p_2) = (x^6, x^5)\). Then a single NS brane as well as a single D 4-brane is automatically a supersymmetric 3-cycle, namely a supersymmetric 2-cycle in the \( x^3 \cdot x^5 \cdot x^6 \cdot x^{10} \) space times the line \( x^7 = \text{const} \) in \( x^4 \cdot x^7 \) space. The last pair of coordinates is fixed by the requirement that also the NS’ brane is a supersymmetric 3-cycle: \((q_3, p_3) = (x^4, x^7)\). Note that with this choice the three coordinates \( x^3, x^5, x^7 \) of the D 4-branes are all momentum variables. In summary, the complex structure of \( \mathbb{C}^3 \) takes the following form:

\[
\begin{align*}
u_1 &= x^{10} + i \cdot x^3 \\
u_2 &= x^4 + i \cdot x^7 \\
u_3 &= x^6 + i \cdot x^5
\end{align*}
\]

Now we can work out the supersymmetric 3-cycle conditions on the three functions \( f^1 = f(x^3, x^4, x^5, x^6, x^7, x^{10}), f^2 = g(x^3, x^4, x^5, x^6, x^7, x^{10}) \) and \( f^3 = h(x^3, x^4, x^5, x^6, x^7, x^{10}) \).
First, the three Poisson brackets are given by the following set of equations:

\[
\begin{align*}
0 &\equiv \{f, g\} = f_{10}g_3 - f_3g_{10} + f_4g_7 - f_7g_4 + f_6g_5 - f_5g_6, \\
0 &\equiv \{f, h\} = f_{10}h_3 - f_3h_{10} + f_4h_7 - f_7h_4 + f_6h_5 - f_5h_6, \\
0 &\equiv \{g, h\} = g_{10}h_3 - g_3h_{10} + g_4h_7 - g_7h_4 + g_6h_5 - g_5h_6.
\end{align*}
\]

The \( \det N \equiv 1 \) equation takes the following form:

\[
0 \equiv (f_4g_6 - f_7g_5 - f_6g_4 + f_5g_7)h_{10} + (g_{10}f_6 - g_3f_5 - f_{10}g_6 + f_3g_5)h_4 \\
+ (f_{10}g_4 - f_3g_7 - g_{10}f_4 + g_3f_7)h_6 + (g_3f_4 + g_{10}f_7 - f_10g_7 - f_3g_4)h_5 \\
+ (f_{10}g_5 + f_3g_6 - g_3f_6 - g_{10}f_5)h_7 + (f_6g_7 - f_7g_6 - f_{10}g_5 + f_5g_4)h_3.
\]

For a supersymmetric 3-cycles these four equations must be zero, but not necessarily identically, but in general only on the 3-cycle, i.e. modulo the ideal of vanishing functions determined by \( f, g \) and \( h \).

One particular class of solutions for these equations is of course given by all 3-cycles which are a supersymmetric 2-cycle in the \( x^3-x^5-x^6-x^{10} \) space times the line \( x^7 = \text{const} \) in \( x^4-x^7 \) space: \( \Sigma^{(3)} = \Sigma^{(2)} \times \mathbb{R} \). The corresponding choice of functions is \( f = f(x^3, x^5, x^6, x^{10}), \) \( g = g(x^3, x^5, x^6, x^{10}) \), \( f \) and \( g \) being real and imaginary parts of a holomorphic function \( F(x^3 + ix^5, x^6 + ix^{10}) \), and \( h = x^7 - \text{const} \).

As a first and very simple check we can verify that flat, parallel M5-branes in their three possible asymptotic limits, namely being NS, NS’ or D4-branes, are indeed supersymmetric 3-cycles. For example consider the \( n \) parallel NS 5-brane, positioned at \( x_i^6, x_i^7 \) and \( x_i^{10} \) \((i = 1, \ldots, n) \). Hence the three functions \( f, g \) and \( h \) are given as

\[
\begin{align*}
f &= \prod_{i=1}^{n}(x^6 - x_i^6), \\
g &= \prod_{i=1}^{n}(x^7 - x_i^7), \\
h &= \prod_{i=1}^{n}(x^{10} - x_i^{10}).
\end{align*}
\]

It is easy to show that all eqs.\((35)\) and \((36)\) are identically zero. The same is of course true for \( n' \) parallel NS’ 5-branes and \( k \) parallel D 4-branes. In the following sections we will discuss more complicated brane intersections and bent brane configurations.

### 3.5 Supersymmetric 3-cycles for intersecting branes and \( N = 1 \) brane boxes

#### 3.5.1 Branes as quaternionic coordinates

In the following sections we like to construct the defining equations \( f, g \) and \( h \) for those supersymmetric 3-cycles, which correspond to intersecting NS, NS’ and D 4-branes, and
in particular for those, which correspond to \( N = 1 \) brane box configurations. For this purpose we would like to introduce three types of ‘coordinates’, called \( s, s' \) and \( v \), which denote the asymptotic positions in \( \mathbb{C}^3 \) of the NS, NS’ and D 4-branes respectively. These ‘coordinates’ should be on the same footing as the complex variables \( s = x^6 + ix^{10} \) and \( v = x^4 + ix^5 \) of the \( N = 2 \) (NS-D4) brane configurations.

To achieve this aim we will now extend the dimension of the space by including also the directions \( x^2 \) and \( x^8 \). This means that we are now dealing with supersymmetric 4-cycles which are embedded into the space \( \mathbb{C}^4 \), which is spanned by the directions \((2,3,4,5,6,7,8,10)\). All our branes now fill 4 dimensions of this eight dimensional space: their world volumes completely fill \( x^2 \), and they are all positioned at \( x^8 = 0 \). That means that the 4-cycles, which correspond to the brane boxes of the NS, NS’ and D 4-branes are in fact nothing else than supersymmetric 3-cycles times the line \( x^8 = 0 \). As discussed in detail above, we could add yet another type of NS-branes, called NS” branes, with world volumes along the \((3,4,6,8)\)-directions and positions in the \((2,5,7,10)\) space. Considering intersections of all four types of branes (NS-NS’-NS”-D4) one can construct brane cube configurations along the \((3,4,6,8)\)-directions and positions in the \((2,5,7,10)\) space. Considering intersections of all four types of branes (NS-NS’-NS”-D4) one can construct brane cube configurations, where the D 4-branes are now finite in the directions \((2,3,4,5,6,7,8,10)\). All our branes now fill 4 dimensions of this eight dimensional space: their positions in \( C\) are on the same footing as the complex variables \( s, v \). Note that the 4-cycles, which correspond to the brane boxes of the NS, NS’ and D 4-branes are completely filled by the directions \((3,4,6,8)\), and \( x^8 \). These brane cube configurations provide two-dimensional gauge theories with \((1,1)\) supersymmetry. A generic brane cube configuration corresponds to a supersymmetric 4-cycle, which is not a direct product of supersymmetric 3-cycle times \( \mathbb{R} \).

The positions of the branes in \( \mathbb{C}^4 \) can now nicely described by introducing quaternionic numbers. A general quaternion \( q \in \mathbb{H} \) has the structure

\[
q = q^0\sigma_0 + q^1\sigma_1 + q^2\sigma_2 + q^3\sigma_3,
\]

where \( \sigma_0 = 1 \) and the \( \sigma_i \) \( (i = 1, 2, 3) \) are the Pauli matrices, satisfying \( \sigma_i\sigma_j = \epsilon_{ijk}\sigma_k \). Clearly, a quaternion is zero, \( q = 0 \), if all its components \( q_i \) \( (i = 0, \ldots, 3) \) are vanishing. Alternatively, we can also define the quaternions via two complex numbers \( z_1 = q^0 + iq^1 \) and \( z_2 = q^2 - iq^3 \) as \( q = z_1 + jz_2 \), where \( i = \sigma_1, j = \sigma_2 \) and \( k = i \cdot j = \sigma_3 \).

Now we can associate to every brane a particular quaternion \( q \), which describes its asymptotic position in \( \mathbb{C}^4 \), and hence is a function of the position variables of every brane:

\[
\begin{align*}
NS : & \quad q_{NS} = q(x^6, x^7, x^8, x^{10}), \\
NS' : & \quad q_{NS'} = q(x^4, x^5, x^8, x^{10}), \\
D4 : & \quad q_{D4} = q(x^3, x^5, x^7, x^8).
\end{align*}
\]

The four defining equations \( f^m(x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^{10}) \) \( (m = 1, \ldots, 4) \) for the 4-cycle can be now simply written in terms of a single quaternionic function function \( F(q_{NS}, q_{NS'}, q_{D4}) \):

\[
F(q_{NS}, q_{NS'}, q_{D4}) = f^1(x^i) + i f^2(x^i) + j f^3(x^i) + k f^4(x^i).
\]

Of course, for a general function \( F(q_{NS}, q_{NS'}, q_{D4}) \) one still has to verify whether the 4-cycle is supersymmetric. This is not automatic unlike the case of the supersymmetric 2-cycles,
where every holomorphic function corresponds to a supersymmetric 2-cycle. Specifically, as discussed in section [3.4.1], the supersymmetry conditions are given by the requirement that six Poisson brackets \( \{ f^m, f^n \} \) plus \((\det N - 1)\) have to vanish (modulo the ideal of vanishing functions determined by the zero locus of the \( f^m \)). In addition, since we want the supersymmetric 4-cycle \( \Sigma^{(4)} \) to be of the form \( \Sigma^{(4)} = \Sigma^{(3)} \times \mathbb{R}_{x^8=0} \), the function \( F(s, s', v) \) has to be chosen in such a way that the common zero locus of the \( f^m \) always contains the line \( x^8 = 0 \). In principle it is also possible to obtain the three 3-cycle equations \( f, g \) and \( h \) by solving one of the four equations \( f^m \) with respect to \( x^8 \) and substituting the result into the remaining equations.

To understand this procedure of constructing supersymmetric 3-cycles let us first consider case of classical brane configurations which are not bent by quantum effects. To describe flat branes we introduce the following quaternionic coordinates in analogy to the complex variables \( s \) and \( v \):

\[
\begin{align*}
NS &: \quad s &= x^6 + ix^{10} + jx^7 - kx^8, \\
NS' &: \quad s' &= x^4 + ix^5 + jx^{10} - kx^8, \\
D4 &: \quad v &= x^3 + ix^5 + jx^7 - kx^8.
\end{align*}
\]  

(41)

A single NS brane is a supersymmetric 4-cycle simply defined by the equation \( F = s = 0 \) and likewise for the other branes. Next consider the triple intersection of \( n \) parallel NS branes with \( n' \) parallel NS' branes and \( k \) parallel D 4-branes. This configuration corresponds to un-bent NS and NS' branes. In the language of field theory it leads to finite \( N = 1 \) gauge theories. The associated quaternionic function \( F \) is given by the following polynomial:

\[
F(s, s', v) = \prod_{i=1}^{n}(s - s_i) \prod_{j=1}^{n'}(s' - s'_j) \prod_{l=1}^{k}(v - v_l).
\]  

(42)

Here \( s_i, s'_j \) and \( v_l \) are constant quaternionic numbers with zero \( \sigma_3 \)-component, which denote the positions of the three types of branes. It is not a difficult but a tedious calculation to show that this function \( F \) corresponds to a supersymmetric 4-cycle. However note that the supersymmetric 4-cycle equations are not identically satisfied but only on the branes themselves, i.e. only modulo the ideal of vanishing functions of the \( f^m \).

### 3.5.2 Uniform Bending – Sewing of \( N = 2 \) models

Now we will construct the supersymmetric 3-cycles which correspond to those \( N = 1 \) brane boxes which can be obtained via the sewing or superposition of two \( N = 2 \) subsystems. As explained in section (3.1), this means that all the NS branes as well as all the NS’ branes are bent in an uniform way.\(^{10}\) The NS''-brane corresponds to the quaternion \( s'' = x^{10} + ix^5 + jx^7 - kx^2 \).

\(^{10}\)The NS''-brane corresponds to the quaternion \( s'' = x^{10} + ix^5 + jx^7 - kx^2 \).
In general, the bending of the NS and NS’ branes should be parametrized by the $x^3$ position of the D 4-branes, where $x^3$ is nothing else that the parameter which is associated to the Coulomb branch in three dimensions. In addition, we roughly expect that the bending of the NS brane is encoded in the functions $x^6(x^3)$ and $x^{10}(x^3)$, and analogously, the bending of the NS’ branes is determined by $x^4(x^3)$ and $x^{10}(x^3)$. Since $x^3$ takes in four dimensions the role of $\Lambda_{QCD}$, $x^4$, $x^6$ and $\cos x^{10}$ ($x^{10}$ is periodic!) should be logarithmic functions of $x^3$.

For the case of uniform bending we can be much more explicit. Consider first the uniform bending of the NS brane caused by $k$ D 4-branes. From the $N = 2$ models we know that the perturbative bending is described by a two dimensional Laplace equation with the holomorphic, logarithmic solution $x^6 + ix^{10} = k \log(x^3 + ix^5)$. In the same way, for the other $N = 2$ subsystem, NS’– k’D4, the following perturbative solution for the bending holds: $x^4 + i x^{10} = k' \log(x^7 + ix^3)$. This behaviour now suggest that we define the following quaternionic coordinates which describe the asymptotic positions of the bent branes in a correct way:

$$
\begin{align*}
NS : & \quad t = e^{x^6} \cos x^{10} + ie^{x^6} \sin x^{10}, \\
NS' : & \quad t' = e^{x^4} \cos x^{10} + je^{x^4} \sin x^{10}, \\
D4 : & \quad v = x^3 + ix^5 + jx^7 - kx^8.
\end{align*}
$$

Sewing together the perturbative bending of the two $N = 2$ subsystems provides us with the following quaternionic function for the supersymmetric 3-cycle, which corresponds to the simple brane box shown in figure 6: For $k = k' = 1$ the quaternionic function simply takes the form

$$
F(t, t', v) = [t - v][t' - v] = 0.
$$

Figure 6: A simple brane box with uniform bending
which is defined by $F(t, t', v)$ is a true 3-cycle; it consists out of two branches, namely the superposition of the curve $t - v^k = 0$, which is a 2-cycle in the $3 - 5 - 6 - 10$-directions times the $x^4$-axis, together with the curve $t' - v^{k'} = 0$, which represents a 2-cycle, now in the directions $3 - 4 - 7 - 10$ times the $x^6$-axis.

After having understood the most simple $N = 1$ brane box with uniform bending (see figure 6) we can now construct the non-perturbative, supersymmetric 3-cycle equations which describe the generic $N = 1$ brane box with uniform bending situation. It is given by the superposition of two $N = 2$ subsystems: the first one consists out of $n$ NS 5-branes with $k_\alpha$ D 4-branes suspended between the NS branes (see figure 1). The second $N = 2$ subsystem has the same structure, but now $n'$ NS' branes with $k'_{\alpha'}$ suspended D 4-branes. After sewing together these two subsystems, the $N = 1$ brane box has the form shown in figure 4. Now recall that, non-perturbatively, every $N = 2$ system of this kind is characterized by the complex 2-cycle polynomial eq. (14). Then the sewing procedure simply corresponds to the multiplication of the two $N = 2$ polynomials, where we replace the complex variables by the corresponding quaternionic variables. In this way we get a supersymmetric 3-cycle which consists out of two branches, namely the direct sum

$$\Sigma^{(3)} = \Sigma^{(2)} \times \mathbb{R} \oplus \Sigma^{(2)} \times \mathbb{R}. \quad (45)$$

Note that the two superposed 3-cycles have a common volume in the 3-10 space. In general the quaternionic 3-cycle equations will have the following structure:

$$\Sigma^{(3)} : F(t, t', v) = \left[ p_{k_\alpha}(v) t^n + \cdots + p_{k_n}(v) \right] \left[ p_{k'_{\alpha'}}(v) t'^{n'} + \cdots + p_{k'_{n'}}(v) \right]. \quad (46)$$

This expression can be expanded and one obtains a polynomial of the following structure:

$$F(t, t', v) = \sum_{\alpha = 0}^n \sum_{\alpha' = 0}^{n'} p_{k_\alpha}(v) p_{k'_{\alpha'}}(v) t^{n-\alpha} t'^{n'-\alpha'}. \quad (47)$$

Note that the degree of the polynomial in $v$ in front of each term $t^{n-\alpha} t'^{n'-\alpha'}$ precisely agrees with the number of D 4-branes in each box $[\alpha, \alpha']$.

For example the sewing of two pure $N = 2$ gauge theories with $G = SU(k)$ and $G' = SU(k')$ leads to a $N = 1$ gauge theory with $N_c = N_f = k + k'$ (see figure 4). The corresponding 3-cycle equations are then simply given in terms of the product of two Seiberg-Witten elliptic curves of genus $(k - 1)$ resp. $(k' - 1)$. This strongly suggests that the instanton numbers of the pure $N = 2$ Yang-Mills theory with gauge group $SU(k)$ are intimately related to those of SUSY QCD with $G = SU(2k)$ and $N_f = 2k$.

At the end of this section let us compute the perturbative running of the $N = 1$ gauge coupling constant. A priori we deal with two different Coulomb branches parametrized by $x^3 + ix^5$ resp. $x^3 + ix^7$. In the following we will consider the common direction, $x^3$, and freeze the other directions, i.e. $x^5 = x^7 = 0$. Now consider the box $[\alpha, \alpha']$ with the
Figure 7: Sewing of two pure $N = 2$ Yang-Mills theories

The corresponding gauge group $SU(k_{\alpha} + k'_{\alpha'})$ follows from eq. (44) we derive that

$$
\begin{align*}
    x_{\alpha'+1}^4 - x_{\alpha'}^4 &= L + (k'_{\alpha'+1} + k'_{\alpha'-1} - 2k'_{\alpha'}) \log x^3, \\
    x_{\alpha+1}^6 - x_{\alpha}^6 &= L + (k_{\alpha+1} + k_{\alpha-1} - 2k_{\alpha}) \log x^3,
\end{align*}
$$

where $L$ is the classical distance between the NS and NS' branes. Then using eq. (17), the gauge coupling constants exhibits the following running behaviour:

$$
\frac{1}{g_{\alpha,\alpha'}^2} = (g_s)^{-1} \left( L^2 + L(k'_{\alpha'+1} + k'_{\alpha'-1} + k_{\alpha+1} + k_{\alpha-1} - 2k'_{\alpha'} - 2k_{\alpha}) \log x^3 \\
\quad + (k'_{\alpha'+1} + k'_{\alpha'-1} - 2k'_{\alpha'})(k_{\alpha+1} + k_{\alpha-1} - 2k_{\alpha})(\log x^3)^2 \right). \tag{49}
$$

Since $N_c = k_{\alpha} + k'_{\alpha'}$ and $N_f = k'_{\alpha'+1} + k'_{\alpha'-1} + k'_{\alpha'} + k_{\alpha+1} + k_{\alpha-1} + k_{\alpha}$, the coefficient in front of log $x^3$ precisely agrees with the one-loop $N = 1$ $\beta$-function coefficient $b_{N=1} = -3N_c + N_f^{11}$.

### 3.5.3 General $N = 1$ brane boxes

In the last section we have discussed already a quite large class of $N = 1$ gauge theories, namely those $N = 1$ models with $N_f \geq N_c$ can be obtained by sewing $N = 2$ brane configurations. On the other hand, $N = 1$ gauge theories with $N_f < N_c$ like pure $N = 1$ Yang-Mills and also models with chiral fermions are more general and cannot be obtained by the $N = 2$ sewing procedure. Of course these models are very interesting to study dynamical supersymmetry breaking and the effect of anomalies. In general, we expect that a brane box which corresponds in field theory to a model without vacuum at finite

\footnote{It was already observed in refs. [18, 29] that the brane box models with uniform bending lead to the correct $N = 1$ $\beta$-function coefficients.}
value of the moduli, like supersymmetric QCD with $0 < N_f < N_c$, leads to a 3-cycle which does not satisfy the minimal area requirement. In this case the 3 Poisson brackets may still be zero, i.e. $N \equiv N^T$, but $Re(\det N)$ is non-vanishing. Similarly, in chiral $N = 1$ gauge theories with dynamical supersymmetry breaking we expect a stable non-supersymmetric ground state. That is the \textit{det} requirement will be satisfied, while the cycle won’t be a Lagrangian submanifold anymore. If furthermore the model is anomalous, the 3-cycle should not exist at all.

In the following we like to propose a specific ansatz for the 3-cycle equations for a general $N = 1$ brane box model. We will again use the quaternionic formalism with quaternions $t$, $t'$ and $v$ (see eq.(43)). Motivated by the previous discussions, our ansatz will consist out of a polynomial in these variables, where the degree of the polynomial in $t$ ($t'$) corresponds to the number of NS (NS') branes in the corresponding brane box. Hence, for a general brane box as shown in figure (2) the quaternionic 3-cycle equations are assumed to take the following form

$$
\Sigma^{(3)} : \quad F(t, t', v) = \sum_{\alpha=0}^{n} \sum_{\alpha'=0}^{n'} p_{k_{\alpha,\alpha'}}(v) t^{n-\alpha} t'^{n'-\alpha'}.
$$

(50)

$p_{k_{\alpha,\alpha'}}(v)$ is a polynomial in $v$ whose degree is given by the number $k_{\alpha,\alpha'}$ of D 4-branes in each box $[\alpha, \alpha']$. As already said, for a brane box which corresponds in field theory to an anomaly free gauge theory with supersymmetric vacuum, one should be able to proof [30] that this polynomial provides a supersymmetric 3-cycle. At the moment it is however not possible for us to show this in general; the main technical difficulty is the observation that the supersymmetric 3-cycle equations must be satisfied only modulo the ideal $I(V)$ of functions vanishing on the 3-cycle $V(f, g, h)$. Note however that in case of uniform bending the polynomial eq.(50) takes the form of eq.(47), namely it factorizes as in eq.(46), and the supersymmetric 3-cycle equations are satisfied.

A particularly interesting case is pure supersymmetric QCD with gauge group $SU(k)$. Here the 3-cycle polynomial should have the following structure.

$$
F(t, t', v) = \sum_{\alpha=0}^{2} \sum_{\alpha'=0}^{2} t^{2-\alpha} t'^{2-\alpha'} + p_k(v) t t',
$$

(51)

where $p_k(v)$ is a polynomial in $v$ of degree $k$. For finite $R^{IIB}_3$ which includes the decompactification limit to four dimensions, $R^{IIB}_3 \rightarrow \infty$, there exist a supersymmetric vacuum in field theory such the supersymmetric 3-cycle equations should be satisfied for this ansatz. On the other hand, in the 3-dimensional limit, $R^{IIB}_3 \rightarrow 0$, the supersymmetric 3-cycle equations should be violated, since there is no supersymmetric vacuum in 3-dimensional pure Yang-Mills gauge theory [30].
4 Conclusions

We have shown that SUSY 3-cycles play a similar role in \( N = 1 \) SUSY gauge theories as the Seiberg-Witten curve in \( N = 2 \) in the sense that their geometry encodes the holomorphic information about the gauge theory. Especially we expect the superpotential to correspond to the volume and the couplings on the Coulomb branch (if present) to the periods of the cycle \([30]\). We were able to construct these cycles for gauge theories that satisfy the uniform bending requirement of \([13]\). The tools we used in establishing these cycles should be useful for the more general cases as well.

Note added: See the footnote on page \([30]\).

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Appendix A: Derivation of the d-cycle equations

Consider the embedding map \( i : d \text{ - cycle} \rightarrow M_{2d}^2 \) and the two conditions:

\[
i^*3m\Omega = 0 \quad \text{volume minimizing} \\
i^*\omega = 0 \quad \text{Lagrangian submanifold}
\]

With \( \Omega = du^1 \wedge \ldots \wedge du^d \) the requirement of minimal volume reads

\[
0 = i^*3m\Omega = 3m(du^1(\xi_1, \ldots, \xi_d) \wedge \ldots \wedge du^d(\xi_1, \ldots, \xi_d))
\]

\[
= 3m(\epsilon_{i_1 \ldots i_d} u^1_{i_1} u^2_{i_2} \ldots u^d_{i_d}) \, d\xi^1 \wedge \ldots \wedge d\xi^d
\]

\[
\Rightarrow 0 = \frac{1}{2i} (\epsilon_{i_1 \ldots i_d} u^1_{i_1} \ldots u^d_{i_d} - \epsilon_{i_1 \ldots i_d} \bar{u}^1_{i_1} \ldots \bar{u}^d_{i_d})
\]

\[
= \frac{1}{2i} (\epsilon_{i_1 \ldots i_d} u^1_{i_1} \ldots u^d_{i_d} - \epsilon_{i_1 \ldots i_d} N^1_{j_1} \ldots N^d_{j_d} u^1_{i_1} \ldots u^d_{i_d})
\]

\[
= \frac{1}{2i} (\epsilon_{i_1 \ldots i_d} - \epsilon_{j_1 \ldots j_d} N^1_{i_1} \ldots N^d_{i_d}) u^1_{i_1} \ldots u^d_{i_d}
\]

\[
= \frac{1}{2i} \left(1 - \det N\right) \frac{\partial(u^1, \ldots, u^d)}{\partial(\xi_1, \ldots, \xi_d)}
\]

which yields

\[
\det N|_{V(f_1, \ldots, f_n)} = 1 \quad \text{or for short} \quad \det N \equiv 1.
\]
For the calculation of the det-equation the following relation is useful.

\[ \det N \equiv 1 \iff \det M - (-1)^d \det \bar{M} \equiv 0 \]

Now we turn to the second equation. Choosing the canonical Kähler (symplectic) form \( \omega = \frac{1}{2i} \sum_i du^i \wedge d\bar{u}^i \), the pull back operation results in

\[ 0 = i^* \omega = \frac{1}{2i} \sum_i du^i (\xi_1 \ldots \xi_d) \wedge d\bar{u}^i (\xi_1 \ldots \xi_d) \]

\[ = \frac{1}{2i} \sum_i \left( \sum_k u^i_{\xi_k} d\xi_k \right) \wedge \left( \sum_l \bar{u}^i_{\xi_l} d\xi_l \right) \]

\[ = \frac{1}{2i} \sum_{k<l} \sum_i \left[ u^i_{\xi_k} \bar{u}^i_{\xi_l} - u^i_{\xi_l} \bar{u}^i_{\xi_k} \right] d\xi_k \wedge d\xi_l \]

\[ \Rightarrow \quad 0 = \sum_i \left[ u^i_{\xi_k} \bar{u}^i_{\xi_l} - u^i_{\xi_l} \bar{u}^i_{\xi_k} \right] \]

\[ = \sum_i \left[ u^i_{\xi_k} \left( \sum_m N^i_{m \xi_l} u^m_{\xi_l} \right) - u^i_{\xi_l} \left( \sum_m N^i_{m \xi_k} u^m_{\xi_k} \right) \right] \]

\[ = \sum_i u^i_{\xi_k} \left( \sum_m N^i_{m \xi_l} u^m_{\xi_l} \right) - \sum_i u^i_{\xi_l} \left( \sum_m N^i_{m \xi_k} u^m_{\xi_k} \right) \]

\[ = \sum_i u^i_{\xi_k} \left( \sum_m N^i_{m \xi_l} u^m_{\xi_l} \right) - \sum_m \left( \sum_i u^i_{\xi_l} N^i_{m \xi_k} \right) u^m_{\xi_k} \]

\[ = \sum_i u^i_{\xi_k} \left( \sum_m N^i_{m \xi_l} u^m_{\xi_l} \right) - \sum_i \left( \sum_m N^{T^i}_{m \xi_l} u^m_{\xi_l} \right) u^i_{\xi_k} \]

\[ = \sum_{i,m} \left( N^i_{m} - N^{T^i}_{m} \right) u^i_{\xi_k} u^m_{\xi_l} \]

which is satisfied if we set \( N \equiv N^T \). However, as it stands, this requirement is sufficient, only. Now we intend to give a proof that the condition is necessary, too.

To proof \( N \equiv N^T \) we remember some facts from symplectic geometry especially various ways of characterising Lagrangian planes in symplectic vector spaces. The utility of this investigation rests on the simple observation that our conditions on the d-cycle to be a special Lagrangian submanifolds are in fact conditions on its tangent bundle, i.e. Lagrangian planes locally.

To begin with, we consider a complex vector space \( \mathbb{C}^d \) furnished with a Hermitian structure

\[ \langle x, y \rangle = \sum_i x_i \bar{y}_i = g(x, y) + i \sigma(x, y) \]
which splits into an Euclidean metric $g$ and a symplectic form $\sigma$. One can check that $\sigma$ coincides with

$$\omega = \frac{1}{2i} \sum_i du^i \wedge d\bar{u}^i.$$ 

given before. Therefore we identify both objects. The two-form $\omega$ is non degenerated, antisymmetric and bilinear. With help of $\omega$ we can define the notion of symplectic orthogonality.

**Definition 1.** The orthogonal complement of a vector subspace $E \in \mathbb{C}^d$ is defined by

$$E^\perp = \{ x \in \mathbb{C}^d \mid \omega(x, E) = 0 \}$$

In the special case that $E = E^\perp$ we call $E$ a Lagrangian plane. Obviously on a Lagrangian plane the symplectic form restricts to zero. So we recognize the content of the constraint $i^*\omega = 0$. It simply states that all tangent spaces to the supersymmetric cycle are Lagrangian planes embedded in the tangent space of the embedding space. Here we collect some facts:

1. $Sp(E)$ operates transitively on Lagrangian planes
2. Since $U(d)$ preserves the Hermitian form, it is contained in $Sp(E)$.
3. By $\Lambda(d)$ we denote the Graßmannian of Lagrangian planes
4. $\lambda \in \Lambda(d)$ is characterized by choosing an orthonormal basis $(a_1, \ldots, a_n)$ with respect to the Euclidean metric $g$. But then it is orthonormal with respect to the Hermitian form, too:

$$<a_i, a_j> = g(a_i, a_j) + i \omega(a_i, a_j) = \delta_{ij},$$

i.e. the matrix $a = (a_1, \ldots, a_n)$ is unitary. The other direction works, too. Hence

$$\lambda \in \Lambda(d) \iff \exists a \in U(d), \lambda = a(\mathbb{R}^d)$$

5. Obviously each Lagrangian plane will be stabilized by any element in $O(n)$, i.e. we can regard the Graßmannian of Lagrangian planes as the quotient space

$$\Lambda(d) = \frac{U(d)}{O(d)}$$
How can we define a projection from $U(d)$ onto $\Lambda(d)$? We observe that two elements $a$ and $a'$ determine the same Lagrangian plane, iff

$$\lambda = a(\mathbb{R}^d) = a'(\mathbb{R}^d) \iff a\bar{a}^{-1} = a'\bar{a}'^{-1},$$

which is constant on the $O(d)$-orbits of the fibration. Now we can identify $\Lambda(d)$ with the image of the projection map

$$\pi : U(d) \rightarrow \Lambda(d)$$

$$a \mapsto \lambda = a\bar{a}^{-1}$$

By abuse of language we denote the matrix representative $a\bar{a}^{-1}$ of the Lagrangian plane $\lambda = a(\mathbb{R}^n)$ by $\lambda$ again. But how can we associate the geometrical object with this artificial matrix representative? The connection between the matrix $\lambda$ on the one side and the concrete Lagrangian plane $\lambda$ on the other side is given through the central equation

$$x \in \lambda \iff x = \lambda \bar{x}$$

In the last formula we recognize the familiar equation (24). But now we know, that we can represent $\lambda$ as $\lambda = a\bar{a}^{-1}$ and this yields straight forward

$$\lambda^+ = \bar{a}^{-1}a^+ = a^{-1T}a^{-1} = a\bar{a}^{-1} = \bar{\lambda}$$

$$\Rightarrow \lambda^T = \lambda$$

But then we can finally conclude by identifying $\lambda = N^{-1}$ and performing some mild manipulations that

$$N \equiv N^T$$
Appendix B: Some facts from Hamiltonian dynamics

Theorem 1. (Liouville) Suppose \((f^1, \ldots, f^d)\) is a set of smooth functions on a symplectic manifold \(M^{2d}\) that are pairwise in involution, i.e. \(\{f^i, f^j\} = 0\). Let \(M_\xi\) be the joint level surface determined by a system of equations \(f^1(x) = \xi_1, \ldots, f^d(x) = \xi_d\). Suppose the functions are functionally independent on \(M_\xi\) (that is, the gradients of the functions are linearly independent at each point of \(M_\xi\)). Then the following assertions are true:

1. The level surface \(M_\xi\) is a smooth \(n\)-dimensional submanifold that is invariant with respect to the flows determined by the vector fields \(X_{f^i}\).
2. The connected components of \(M_\xi\) are diffeomorphic to \(T^k \times \mathbb{R}^{d-k}\).
3. If \(M_\xi\) is compact and connected, then it is diffeomorphic to the \(d\)-dimensional torus \(T^d\).

Now we want to show how we utilize this theorem for our purposes. At first we identify the functions \((f^1, \ldots, f^d)\) as the defining equations of our searched for intersection. Then we start with such functions \(f_i\), such that the gradients are linear independent everywhere. Then the gradients span the normal directions to our \(d\)-cycle. Can we construct in a canonical way a set of vector fields, which form a basis for the orthogonal complement of these normal directions? To answer this questions we have a look on some simple properties of these vector fields and very natural associated objects. We start with a simple but important definition:

Definition 2. (Hamiltonian vector field) A Hamiltonian vector field \(X\) is defined by the property

\[
d(X \iota \omega) = 0,
\]

i.e. \(\mathcal{L}_X \omega = 0\) which reflects the property of the Hamiltonian flow to preserve the symplectic form. In the case of mild topology closeness implies exactness and we can write

\[
df + X_f \iota \omega = 0
\]

assigning to the Hamiltonian vector field its generating function \(f\).

We will show that the Hamiltonian vector fields corresponding to the \(f_i\) do span the orthogonal complement mentioned before. At first we observe that \(X_f \perp \text{grad } f\) by construction. By using the “symplectic involution” \(\sigma\) given by

\[
\sigma = \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix}, \quad \sigma^2 = -I_{2d}
\]
we can write $X_f$ as $X_f = \sigma \cdot \text{grad} f$. Sometimes $X_f$ is called the symplectic gradient, therefore. Now we calculate

$$<\text{grad} f, \sigma \cdot \text{grad} f> = -<\sigma^2 \cdot \text{grad} f, \sigma \cdot \text{grad} f>$$
$$= -<\sigma \cdot \text{grad} f, \sigma^+ \sigma \cdot \text{grad} f>$$
$$= -<\text{grad} f, \sigma \cdot \text{grad} f>$$

Evidently $\text{grad} f$ and $X_f$ are orthogonal vectors. But is $X_f$, orthogonal to all gradients $\text{grad} f^i$? Now we exploit the integrability condition. Since all functions do commute with respect to the Poisson bracket we conclude:

$$0 = \{ f^i, f^j \} \overset{\text{Def}}{=} \omega(X_{f^i}, X_{f^j}) = X_{f^j} \lrcorner X_{f^i} \lrcorner \omega = -X_{f^j} \lrcorner df^i = -df^i(X_{f^j})$$
$$= -<\text{grad} f^i, \sigma \cdot \text{grad} f^j>$$

So we recognize that our integrability condition guarantees the orthogonality of the span $X_{f^i}$ to the normal directions. Since the normal span is linear independent and

$$<X_{f^i}, X_{f^j}> = <\nabla f^i, \nabla f^j>$$

the Hamiltonian vector fields are independent, too. Now we have to care for the Lagrangian property. Does the symplectic form $\omega$ vanish on the subspace spanned by the $X_{f^i}$? Reading the last formula in the other direction

$$\omega(X_{f^i}, X_{f^j}) = \{ f^i, f^j \}$$

this wish becomes true, too. The next question touches the sore spot of the whole business. Is the space spanned by the $X_{f^i}$ tangent to the level surface $M_\xi$? We want to investigate the Hamiltonian flow generated by $f^i$. Obviously the Hamiltonian $f^i$ is a constant of motion. Further since the other $f^j$ are in involution with $f^i$ they are constants of motion, too. Hence the level surface $M_\xi$ is preserved by all Hamiltonian flows corresponding to the associated Hamiltonian vector fields $X_{f^i}$. But for $f^i$ to be a constant of motion

$$X_{f^i}(f^i) \overset{1}{=} 0,$$

i.e. $X_{f^i}$ is tangent to $M_\xi$ everywhere.

**Corollary 1.** The level surface $M_\xi$ is a Lagrangian submanifold.
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36