Bergman Projection on the Symmetrized Bidisk

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Abstract
We apply the Bekollé–Bonami estimate for the (positive) Bergman projection on the weighted \(L^p\) spaces on the unit disk. As the consequences, we obtain the boundedness of the Bergman projection on the weighted Sobolev space on the symmetrized bidisk. We also improve the boundedness result of the Bergman projection on the unweighted \(L^p\) space on the symmetrized bidisk in Chen et al. (J Funct Anal 279(2):108522, 2020).

Keywords Bergman kernel · Bergman projection · Symmetrized bidisk

Mathematics Subject Classification 32A25 · 32A36

1 Introduction

Let \(\Omega\) be a bounded domain in \(C^n\). The Bergman projection \(B_{\Omega}\) on \(\Omega\) is the orthogonal projection from \(L^2(\Omega)\) to its subspace \(A^2(\Omega)\)—the set of square integrable holomorphic functions, defined by

\[
B_{\Omega}(f)(w) = \int_{\Omega} B(w, \eta) f(\eta) d\nu(\eta)
\]

Dedicated to the memory of Nessim Sibony.

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for any \( f \in L^2(\Omega) \), where \( B(w, \eta) \) is the Bergman kernel on \( \Omega \times \Omega \).

The Bergman projection \( B_\Omega \) in \( L^p(\Omega) \) space, Sobolev spaces \( W^{k,p}(\Omega) \) and corresponding weighted spaces is closely related to the \( \bar{\partial} \)-Neumann problem on \( \Omega \) (cf. [10, 15, 31]), and the regularity problem of \( B_\Omega \) is one of the classical problems in several complex variables. It is well known that the \( L^2 \) Sobolev regularity of the Bergman projection implies the \( L^2 \) Sobolev regularity of the \( \bar{\partial} \)-Neumann operator on a bounded smooth pseudoconvex domain \( \Omega \) [9]. On the other hand, on the worm domain \( \Omega \), Barrett showed that \( B_\Omega \) does not preserve \( W^{k,2} \) for large \( k \) [4]. This was used by Christ to prove the failure of the global regularity of the \( \bar{\partial} \)-Neumann operator [16].

When \( \Omega \) is a bounded smooth domain with various convexity conditions on the boundary, the \( L^p \) Sobolev regularity of \( B_\Omega \) has been intensively studied for general \( p \in (1, \infty) \) (cf. [25, 27, 28]). When \( \Omega \) is not smooth, the study of the \( L^p \) regularity of \( B_\Omega \) has also attracted substantial attention in recent years (see for example [7, 11, 13, 17, 23, 24, 32]).

The symmetrized bidisk is an interesting model of non-smooth domains and various analytic and geometric properties have been studied intensively (see for example [1–3]). Let \( \Phi \) be the rational proper holomorphic map from the bidisk \( \mathbb{D} \times \mathbb{D} \) to \( \mathbb{C}^2 \) given by \( \Phi(w_1, w_2) = (w_1 + w_2, w_1 w_2) \) with the determinant of Jacobian \( J_\mathbb{C} \Phi(w) = w_1 - w_2 \).

The symmetrized bidisk is then the image of \( \mathbb{D} \times \mathbb{D} \) under \( \Phi \) given by

\[
\mathbb{G} = \{(w_1 + w_2, w_1 w_2) \in \mathbb{C}^2 \mid (w_1, w_2) \in \mathbb{D} \times \mathbb{D}\}.
\]

Let \((z_1, z_2)\) be the coordinate on \( \mathbb{G} \) and let \( \delta(z_1, z_2) = -\log |z_1^2 - 4z_2| \) be the weight function on \( \mathbb{G} \) with \( \Phi^* \delta = -2 \log |w_1 - w_2| \). The norm of weighted Sobolev space \( W^{k,p}(\mathbb{G}, l \delta) \) on \( \mathbb{G} \) is given by

\[
\|f\|_{W^{k,p}(\mathbb{G}, l \delta)}^p = \sum_{|\alpha| \leq k} \int_{G} |D_{z, \bar{z}}^\alpha(f(z))|^p e^{-l \delta} \, d v(z) = \sum_{|\alpha| \leq k} \int_{\mathbb{G}} |D_{z, \bar{z}}^\alpha(f(z))|^p |z_1^2 - 4z_2|^l \, d v(z), \tag{1.1}
\]

where \( D_{z, \bar{z}}^\alpha = \frac{\partial^{\alpha_1} + \partial^{\alpha_2} + \partial^{\alpha_3} + \partial^{\alpha_4}}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \partial \bar{z}_1^{\alpha_3} \partial \bar{z}_2^{\alpha_4}} \) with multi-index \( \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \). When \( k = 0 \), the norm of weighted \( L^p \) space is defined similarly.

In [14], the first author, Krantz, and the third author derived the \( L^p \) regularity of the Bergman projection \( B_\mathbb{G} \) on the symmetrized bidisk \( \mathbb{G} \) by considering the boundedness of Bergman projection on \( L^p \) functions over upper-half complex plane. In this article, we study the \( L^p \) regularity of \( B_\mathbb{G} \) by considering Bekollé–Bonami constant of certain weight on the unit disk \( \mathbb{D} \) in complex plane. For the Sobolev regularity, we follow the strategy of holomorphic integration by parts (cf. [8, 12, 18, 30]). In the earlier works (cf. [12]), the holomorphic integration by parts will reduce the Sobolev estimates of the Bergman projection to the \( L^p \) estimate of the Bergman projection as the weight function is annihilated by the tangential vector field. However, in the case of the symmetrized bidisk, the weight function cannot be annihilated by the tangential vector field. Instead we reduce the Sobolev regularity of \( B_\mathbb{G} \) to the boundedness of \( B_\mathbb{D}^+ \) on...
the weighted $L^p$ space on the unit disk $\mathbb{D}$ (see the definition of $\mathcal{B}_D^+$ in Sect. 2) and study those weighted $L^p$ regularities by checking the Bekollé–Bonami constant of the corresponding weights.

We would like to point out that the method in this paper would have been used to derive the weighted Sobolev estimates of the Bergman projection on other domains, for instance, the symmetrized polydisk or even a domain covered by the product of planar domains through a holomorphic rational proper map. However, we choose to merely treat the symmetrized bidisk here in order to demonstrate the idea.

The main result is the following theorem on the weighted Sobolev regularity.

**Theorem 1.1** Given $|w_1 - w_2|^2$ as a weight function on $w_1 \in \mathbb{D}$, if the norm of $\mathcal{B}_D^+ : L^p(\mathbb{D}, |w_1 - w_2|^2) \to L^p(\mathbb{D}, |w_1 - w_2|^2)$ is uniformly bounded independent of $w_2 \in \mathbb{D}$ for some $p \in (1, \infty)$, then the Bergman projection $\mathcal{B}_G$ on the symmetrized bidisk $G$ is bounded from $W^{k,p}(G)$ to $W^{k,p}\left(\mathbb{G}, \frac{3kp\delta}{2}\right)$ for any positive integer $k$.

In order to obtain an explicit range in $p$ such that $\mathcal{B}_G$ is bounded between weighted Sobolev spaces and $L^p$ spaces, we obtain the following results.

**Proposition 1.2** Let $2 < p < +\infty$. For any fixed $w_2 \in \mathbb{D}$, $\mathcal{B}_D^+ : L^p(\mathbb{D}, |w_1 - w_2|^2) \to L^p(\mathbb{D}, |w_1 - w_2|^2)$ is bounded. Moreover, the norm is uniformly bounded independent of $w_2$.

**Proposition 1.3** Let $\frac{4}{3} < p < 4$. For any fixed $w_2 \in \mathbb{D}$, $\mathcal{B}_D : L^p(\mathbb{D}, |w_1 - w_2|^{2-p}) \to L^p(\mathbb{D}, |w_1 - w_2|^{2-p})$ is bounded. Moreover, the norm is uniformly bounded independent of $w_2$.

As the straightforward corollary, we have:

**Corollary 1.4**

- For $2 < p < +\infty$, the Bergman projection $\mathcal{B}_G$ is bounded from $W^{k,p}(G)$ to $W^{k,p}\left(\mathbb{G}, \frac{3kp\delta}{2}\right)$ for any positive integer $k$.

- For $\frac{4}{3} < p < 4$, the Bergman projection $\mathcal{B}_G$ is bounded from $L^p(G)$ to $L^p(G)$.

- For $\frac{4}{3} < p < \infty$, the Bergman projection $\mathcal{B}_G$ is bounded from $L^p(G)$ to $L^p\left(\mathbb{G}, \frac{\delta}{2}\right)$.

**Remark 1.5** By the similar method, one can prove that the Bergman projection on the $n$-dimensional symmetrized polydisk $G^n$ is $L^p$ bounded for $p \in \left(1 + \frac{n-1}{n+1}, 1 + \frac{n+1}{n-1}\right)$, which is a slight improvement of Theorem 4.9 in [14]. Since the case $n = 2$ is proved in details, we only demonstrate the difference in the general dimension as follows. Firstly, notice that Proposition 4.1 for $G^n$ still holds with the weight $|w_1 - w_2|^{2-p}$ replaced by $\left|\prod_{i<j}(w_i - w_j)\right|^{2-p}$. Secondly, combining Lemma 2.4 in [14] and Theorem 2.1, it suffices to show $\mathcal{B}_D : L^p(\mathbb{D}, |w - \xi|^{\frac{2-p}{n+1}}) \to L^p(\mathbb{D}, |w - \xi|^{\frac{2-p}{n+1}})$ is bounded for any $\xi \in \mathbb{D}$ and the norm is uniformly bounded independent of $\xi$. Lastly, for $p \in \left(1 + \frac{n-1}{n+1}, 1 + \frac{n+1}{n-1}\right)$, the boundedness of $\mathcal{B}_D$ in $L^p(\mathbb{D}, |w - \xi|^{\frac{2-p}{n+1}})$ can be proved using the similar argument as in the proof of Proposition 1.3.
2 The Bergman Projection on the Weighted $L^p$ Space

Let $B(w, \eta)$ be the Bergman kernel on $\Omega \times \Omega$, then define the operator $B^+_\Omega$ by

$$B^+_\Omega(f)(w) = \int_\Omega |B(w, \eta)| f(\eta)dv(\eta),$$

for any $f \in L^2(\Omega)$. For the purpose of the present article, we only consider Bekollé–Bonami’s result of the Bergman projection on the weighted $L^p$ space over the unit disk $\mathbb{D}$. Let $T_z$ denote the Carleson tent defined as:

$$T_z := \{ w \in \mathbb{D} : \left| 1 - \frac{w z}{|z|} \right| < 1 - |z| \} \text{ for } z \neq 0,$$

and $T_z := \mathbb{D}$ when $z = 0$. Note that for $z \neq 0$, $T_z$ is the intersection of the unit disk and a disk centered at a point $\frac{z}{|z|}$ on the unit circle with radius $R = 1 - |z| < 1$. By elementary geometry, it can be shown that $\int_{T_z} dA(w) \approx (1 - |z|)^2 = R^2$ (cf. Lemma 2.1 in [20]).

In [5], Bekollé and Bonami proved a well-known regularity result of the Bergman projection on weighted $L^p$ space over the unit disk. Here we will apply the following formulation due to [29]. Note that there are extensive recent studies on the Bekollé–Bonami estimates (cf. [20–22] and the references therein).

**Theorem 2.1** [29] Let the weight $\sigma$ be a positive, local integrable function on $\mathbb{D}$ and let $1 < p < \infty$. Then

$$\left( B_p(\sigma) \right)^\frac{1}{p} \lesssim \| B_D : L^p(\mathbb{D}, \sigma) \rightarrow L^p(\mathbb{D}, \sigma) \| \leq \| B^+_D : L^p(\mathbb{D}, \sigma) \rightarrow L^p(\mathbb{D}, \sigma) \| \lesssim \left( B_p(\sigma) \right)^{\max\{1, \frac{1}{p-1}\}},$$

where

$$B_p(\sigma) := \sup_{z \in \mathbb{B}_n} \frac{\int_{T_z} \sigma(w) dA(w)}{\int_{T_z} dA(w)} \left( \frac{\int_{T_z} \sigma^{-\frac{1}{p-1}}(w) dA(w)}{\int_{T_z} dA(w)} \right)^{p-1}.$$

Now we are going to prove Propositions 1.2 and 1.3 by checking the corresponding Bekollé–Bonami constants. Both of the proofs are similar to the corresponding arguments in §4 in [14], but we still include them here for the completeness.

**Proof of Proposition 1.2** We only show the case $z \neq 0$ as the case $z = 0$ is similar. Let $L = \text{dist} \left( w_2, \frac{z}{|z|} \right)$. We will prove the uniform boundedness of $B_p \left( |w_1 - w_2|^2 \right)$ for different types of disks.
Assume \( L \geq 10R \), then \( 9R \leq |w_1 - w_2| \leq 11R \) for any \( w_1 \in T_z \). It follows that
\[
\frac{\int_{T_z} |w_1 - w_2|^2 dA(w_1)}{\int_{T_z} dA(w_1)} \cdot \left( \frac{\int_{T_z} |w_1 - w_2|^{-\frac{2}{p-1}} dA(w_1)}{\int_{T_z} dA(w_1)} \right)^{p-1} \leq \left( \frac{11}{9} \right)^2.
\]

Assume \( L < 10R \). We split our argument into two different cases. For \( 0 < R < \delta \), where \( \delta > 0 \) is sufficiently small, then \( T_z \subset D' := D(w_2; 20R) \), the disc centered at \( w_2 \) with radius \( 20R \). It follows that
\[
\int_{T_z} |w_1 - w_2|^2 dA(w_1) \leq \int_{D'} |w_1 - w_2|^2 dA(w_1) \approx R^4;
\]
and
\[
\int_{T_z} |w_1 - w_2|^{-\frac{2}{p-1}} dA(w_1) \leq \int_{D'} |w_1 - w_2|^{-\frac{2}{p-1}} dA(w_1) \approx R^\frac{2p-4}{p-1}
\]
if \( p > 2 \). Therefore
\[
\frac{\int_{T_z} |w_1 - w_2|^2 dA(w_1)}{\int_{T_z} dA(w_1)} \cdot \left( \frac{\int_{T_z} |w_1 - w_2|^{-\frac{2}{p-1}} dA(w_1)}{\int_{T_z} dA(w_1)} \right)^{p-1} \approx \frac{R^4}{R^2} \cdot \left( \frac{R^\frac{2p-4}{p-1}}{R^2} \right)^{p-1} = 1,
\]

independent of \( w_2 \in \mathbb{D} \) provided \( p > 2 \). On the other hand, for \( \delta < R < 1 \), then \( T_z \subset \mathbb{D} \subset D := D(w_2; 2) \) and \( \int_{T_z} dV(w_1) \geq C_\delta > 0 \). So
\[
\int_{T_z} |w_1 - w_2|^2 dA(w_1) \leq \int_D |w_1 - w_2|^2 dA(w_1) \lesssim 1;
\]
and
\[
\int_{T_z} |w_1 - w_2|^{-\frac{2}{p-1}} dA(w_1) \leq \int_D |w_1 - w_2|^{-\frac{2}{p-1}} dA(w_1) \lesssim 1
\]
if \( p > 2 \). Therefore,
\[
\frac{\int_{T_z} |w_1 - w_2|^2 dA(w_1)}{\int_{T_z} dA(w_1)} \cdot \left( \frac{\int_{T_z} |w_1 - w_2|^{-\frac{2}{p-1}} dA(w_1)}{\int_{T_z} dA(w_1)} \right)^{p-1} \leq \frac{\text{Constant}}{C_\delta^p}
\]
independent of \( w_2 \in \mathbb{D} \) provided \( p > 2 \).

Hence, for \( 2 < p < \infty \), the proposition is proved as the Bekollé–Bonami constant is uniformly bounded independent of \( w_2 \in \mathbb{D} \) by Theorem 2.1. \( \square \)
**Proof of Proposition 1.3** We only show the case \( z \neq 0 \) as the case \( z = 0 \) is similar. Let \( L = \text{dist} \left( w_2, \frac{z}{|z|} \right) \). We will prove the uniform boundedness of \( B_p \left( |w_1 - w_2|^{2-p} \right) \) for different types of disks.

Assume \( L \geq 10R \), then \( 9R \leq |w_1 - w_2| \leq 11R \) for any \( w_1 \in T_z \). When \( \frac{4}{3} < p \leq 2 \), \(|w_1 - w_2|^{2-p} \leq (11R)^{2-p} \) and \(|w_1 - w_2|^{-\frac{2-p}{p-1}} \leq (9R)^{-\frac{2-p}{p-1}} \). So

\[
\frac{\int_{T_z} |w_1 - w_2|^{2-p} \, dA(w_1)}{\int_{T_z} dA(w_1)} \cdot \left( \frac{\int_{T_z} |w_1 - w_2|^{-\frac{2-p}{p-1}} \, dA(w_1)}{\int_{T_z} dA(w_1)} \right)^{p-1} \leq \left( \frac{11}{9} \right)^{2-p}.
\]

When \( 2 \leq p < 4 \), \(|w_1 - w_2|^{2-p} \leq (9R)^{2-p} \) and \(|w_1 - w_2|^{-\frac{2-p}{p-1}} \leq (11R)^{-\frac{2-p}{p-1}} \). So

\[
\frac{\int_{T_z} |w_1 - w_2|^{2-p} \, dA(w_1)}{\int_{T_z} dA(w_1)} \cdot \left( \frac{\int_{T_z} |w_1 - w_2|^{-\frac{2-p}{p-1}} \, dA(w_1)}{\int_{T_z} dA(w_1)} \right)^{p-1} \leq \left( \frac{9}{11} \right)^{2-p}.
\]

Assume \( L < 10R \). Similarly, we split our argument into two cases. For \( 0 < R < \delta \), where \( \delta > 0 \) is a sufficiently small constant, then \( T_z \subset D' := D(w_2; 20R) \). It follows that

\[
\int_{T_z} |w_1 - w_2|^{2-p} \, dA(w_1) \leq \int_{D'} |w_1 - w_2|^{2-p} \, dA(w_1) \approx R^{4-p}
\]

if \( p < 4 \); and

\[
\int_{T_z} |w_1 - w_2|^{-\frac{2-p}{p-1}} \, dA(w_1) \leq \int_{D'} |w_1 - w_2|^{-\frac{2-p}{p-1}} \, dA(w_1) \approx R^{\frac{3p-4}{p-1}}
\]

if \( p > \frac{4}{3} \); and thus

\[
\frac{\int_{T_z} |w_1 - w_2|^{2-p} \, dA(w_1)}{\int_{T_z} dA(w_1)} \cdot \left( \frac{\int_{T_z} |w_1 - w_2|^{-\frac{2-p}{p-1}} \, dA(w_1)}{\int_{T_z} dA(w_1)} \right)^{p-1} \approx \frac{R^{4-p}}{R^2} \cdot \left( \frac{R^{\frac{3p-4}{p-1}}}{R^2} \right)^{p-1} = 1,
\]

independent of \( w_2 \in D \) for \( \frac{4}{3} < p < 4 \). On the other hand, for \( \delta < R < 1 \), then \( T_z \subset D' \subset D := D(w_2; 2) \) and \( \int_{T_z} dV(w_1) \geq C_\delta > 0 \). It follows that

\[
\int_{T_z} |w_1 - w_2|^{2-p} \, dA(w_1) \leq \int_D |w_1 - w_2|^{2-p} \, dA(w_1) \leq 1
\]
if \( p < 4 \); and

\[
\int_{T_z} |w_1 - w_2|^{-\frac{2-p}{p-1}} dA(w_1) \leq \int_D |w_1 - w_2|^{-\frac{2-p}{p-1}} dA(w_1) \lesssim 1
\]

if \( p > \frac{4}{3} \); and hence,

\[
\int_{T_z} |w_1 - w_2|^{-2p} dA(w_1) \leq \frac{\left( \int_{T_z} |w_1 - w_2|^{-\frac{2-p}{p-1}} dA(w_1) \right)^{p-1}}{\text{Constant} \ C^p_{\delta}},
\]

independent of \( w_2 \in \mathbb{D} \) for \( \frac{4}{3} < p < 4 \). Therefore, the proposition is proved.

\( \square \)

### 3 Weighted Sobolev Regularity of \( B_G \)

#### 3.1 Transferring the Data to the Product Space

Functions on \( G \) can be transferred to functions on \( \mathbb{D} \times \mathbb{D} \) by the pull-back by \( \Phi \). Also the Bergman kernel \( B_G \) on \( G \times G \) has the following representation under the change of coordinates (cf. [14], or [19, 26]).

**Proposition 3.1** The Bergman kernel \( B_G \) on \( G \times G \) has the following representation with coordinate \((w_1, w_2, \eta_1, \eta_2) \in \mathbb{D} \times \mathbb{D} \times \mathbb{D} \times \mathbb{D} \).

\[
B_G(\Phi(w), \Phi(\eta)) = \frac{1}{2\pi^2} \cdot \frac{1}{w_1 - w_2} \cdot \frac{1}{\eta_1 - \eta_2} \left[ \frac{1}{(1 - w_1 \eta_1)^2} \cdot \frac{1}{(1 - w_2 \eta_2)^2} \right] = \frac{1}{(1 - w_1 \eta_2)^2} \cdot \frac{1}{(1 - w_2 \eta_1)^2}.
\]

(3.1)

On the other hand, since we are going to consider the Sobolev norm of the Bergman projection of functions, any differential operator with the anti-holomorphic direction acting on the holomorphic functions vanishes. Thus we only need to consider the holomorphic differential operators \( D^\alpha_z := \frac{\partial^{\alpha_1 + \alpha_2}}{\partial z_1^\alpha_1 \partial \bar{z}_1^\alpha_2} \) on \( \mathbb{G} \) with multi-index \( \alpha = (\alpha_1, \alpha_2) \).

**Lemma 3.2** For any multi-index \( \alpha = (\alpha_1, \alpha_2) \) with \( \alpha_1 \geq 0, \alpha_2 \geq 0, |\alpha|:=\alpha_1 + \alpha_2 \geq 1, \)

\[
D^\alpha_z = \frac{1}{(w_1 - w_2)^{2|\alpha|-1}} \sum_{1 \leq |\beta| \leq |\alpha|} P_{\alpha, \beta}(w_1, w_2) \frac{\partial^{\beta_1 + \beta_2}}{\partial w_1^{\beta_1} \partial w_2^{\beta_2}},
\]

(3.2)

where \( P_{\alpha, \beta}(w_1, w_2) \) are holomorphic monomials in \( w_1, w_2 \) with degree at most \( 2|\alpha|-1 \). One the other hand, for any multi-index \( \beta = (\beta_1, \beta_2, \beta_3, \beta_4) \),

\[
D^\beta_{\bar{w}}, \bar{w} = \sum_{0 \leq |\alpha| \leq |\beta|} \bar{P}_{\alpha, \beta}(w_1, \bar{w}_1, w_2, \bar{w}_2) \frac{\partial^{\alpha}}{\partial z_1^{\alpha_1} \partial \bar{z}_1^{\alpha_2} \partial z_2^{\alpha_3} \partial \bar{z}_2^{\alpha_4}},
\]

(3.3)
where \( P_{\alpha, \beta}(w_1, \bar{w}_1, w_2, \bar{w}_2) \) are polynomials in \( w_1, \bar{w}_1, w_2, \bar{w}_2 \) with degree at most \(|\beta|\).

**Proof** By the change of coordinates under the holomorphic mapping \( \Phi \), we have

\[
\begin{aligned}
\frac{\partial}{\partial w_1} &= \frac{\partial}{\partial z_1} + w_2 \frac{\partial}{\partial z_2}, \\
\frac{\partial}{\partial w_2} &= \frac{\partial}{\partial z_2} + w_1 \frac{\partial}{\partial z_1}.
\end{aligned}
\]

Then (3.4) implies

\[
\begin{aligned}
\frac{\partial}{\partial z_1} &= \frac{w_1}{w_1-w_2} \frac{\partial}{\partial w_1} - \frac{w_2}{w_1-w_2} \frac{\partial}{\partial w_2}, \\
\frac{\partial}{\partial z_2} &= -\frac{1}{w_1-w_2} \frac{\partial}{\partial w_1} + \frac{1}{w_1-w_2} \frac{\partial}{\partial w_2}.
\end{aligned}
\]

This shows the case when \(|\alpha| = 1\). Suppose (3.2) holds when \(|\alpha| = j\), we prove the case of \(|\alpha| = j + 1\) by induction.

\[
\begin{aligned}
\frac{\partial}{\partial z_1} \frac{\partial^{\alpha_1+\alpha_2}}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2}} &= \left( \frac{w_1}{w_1-w_2} \frac{\partial}{\partial w_1} - \frac{w_2}{w_1-w_2} \frac{\partial}{\partial w_2} \right) \\
&\quad \times \left( \frac{1}{(w_1-w_2)^{2(\alpha_1+\alpha_2)}} \sum_{0<|\beta|\leq|\alpha|} P_{\alpha, \beta}(w_1, w_2) \frac{\partial^{\beta_1+\beta_2}}{\partial w_1^{\beta_1} \partial w_2^{\beta_2}} \right) \\
&\quad + \frac{1}{(w_1-w_2)^{2(\alpha_1+\alpha_2)}} \sum_{0<|\beta|\leq|\alpha|} w_1 \frac{\partial}{\partial w_1} P_{\alpha, \beta}(w_1, w_2) \frac{\partial^{\beta_1+\beta_2}}{\partial w_1^{\beta_1} \partial w_2^{\beta_2}} \\
&\quad + \frac{1}{(w_1-w_2)^{2(\alpha_1+\alpha_2)}} \sum_{0<|\beta|\leq|\alpha|} w_1 P_{\alpha, \beta}(w_1, w_2) \frac{\partial^{\beta_1+\beta_2+1}}{\partial w_1^{\beta_1+1} \partial w_2^{\beta_2}} \\
&\quad + \frac{1}{(w_1-w_2)^{2(\alpha_1+\alpha_2)+1}} \sum_{0<|\beta|\leq|\alpha|} (-2(\alpha_1 + \alpha_2) + 1) w_2 \\
&\quad \times \frac{\partial^{\beta_1+\beta_2}}{\partial w_1^{\beta_1} \partial w_2^{\beta_2}} \\
&\quad + \frac{1}{(w_1-w_2)^{2(\alpha_1+\alpha_2)}} \sum_{0<|\beta|\leq|\alpha|} (-w_2) \frac{\partial}{\partial w_2} P_{\alpha, \beta}(w_1, w_2) \frac{\partial^{\beta_1+\beta_2}}{\partial w_1^{\beta_1} \partial w_2^{\beta_2}} \\
&\quad + \frac{1}{(w_1-w_2)^{2(\alpha_1+\alpha_2)}} \sum_{0<|\beta|\leq|\alpha|} (-w_2) P_{\alpha, \beta}(w_1, w_2) \frac{\partial^{\beta_1+\beta_2+1}}{\partial w_1^{\beta_1} \partial w_2^{\beta_2+1}}.
\end{aligned}
\]
This finishes the induction after factoring out \( \frac{1}{(w_1 - w_2)^{2\alpha_1 + \alpha_2 + \tau}} \). Equation (3.3) can be proved in a similar manner by induction.

\[]

\section*{3.2 Holomorphic Integration by Parts}

The holomorphic derivatives of the Bergman kernel can be transformed to anti-holomorphic derivative as follows.

\textbf{Lemma 3.3} When \( w_i \neq 0 \),
\[ \frac{\partial^\beta}{\partial w_i^\beta} \left( \frac{1}{(1-w_i \bar{\eta}_j)^2} \right) = \frac{\bar{\eta}_j^\beta}{w_i^\beta} \cdot \frac{\partial^\beta}{\partial \bar{\eta}_j^\beta} \left( \frac{1}{(1-w_i \bar{\eta}_j)^2} \right) \] for any \( i, j = 1, 2. \)

\textbf{Proof} Let \( r = w_i \bar{\eta}_j \). By applying the chain rule \( \beta \) times, we have
\[ \frac{\partial^\beta}{\partial w_i^\beta} \left( \frac{1}{(1-w_i \bar{\eta}_j)^2} \right) = \frac{\partial^\beta}{\partial r^\beta} \left( \frac{1}{(1-r)^2} \right) \cdot \bar{\eta}_j^\beta. \]

Similarly,
\[ \frac{\partial^\beta}{\partial \bar{\eta}_j^\beta} \left( \frac{1}{(1-w_i \bar{\eta}_j)^2} \right) = \frac{\partial^\beta}{\partial r^\beta} \left( \frac{1}{(1-r)^2} \right) \cdot w_i^\beta. \]

The lemma is thus proved.

\[]

The next lemma implies that \( \frac{\partial}{\partial \bar{\eta}} \) can be replaced by the tangential operator and it follows from the straightforward calculation.

\textbf{Lemma 3.4} Let \( T_\eta = \eta \frac{\partial}{\partial \eta} - \bar{\eta} \frac{\partial}{\partial \bar{\eta}} \) be the tangential operator on the disc \( \mathbb{D} \) and \( f \) be anti-holomorphic. Then for \( \beta \in \mathbb{Z}^+ \cup \{0\} \), we have
\[ \eta^\beta \frac{\partial}{\partial \eta^\beta} (f) = T_\eta^\beta (f). \] (3.6)

For \( \beta \geq 1 \), following the idea of the partial Bergman kernel in [18], one can define \( K_\beta(w, \eta) = \frac{1}{(1-w \bar{\eta})^\beta} - \sum_{j=0}^{\beta-1} (j + 1)(w \bar{\eta})^j \) as Bergman kernel subtracting the first \( \beta \) terms in its Taylor series in \( w \bar{\eta} \). Then one obtains
\[ \eta^\beta \frac{\partial}{\partial \eta^\beta} \left( \frac{1}{(1-w \bar{\eta})^2} \right) = \frac{\partial}{\partial \eta^\beta} K_\beta(w, \eta). \] (3.7)
Moreover,

\[
K_\beta(w, \eta) = \frac{\partial}{\partial (w\bar{\eta})} \left( \sum_{j=\beta}^{\infty} (w\bar{\eta})^{j+1} \right)
= \frac{\partial}{\partial (w\bar{\eta})} \left( \frac{(w\bar{\eta})^{\beta+1}}{1-w\bar{\eta}} \right)
= \frac{(\beta + 1)(w\bar{\eta})^\beta - \beta(w\bar{\eta})^{\beta+1}}{(1-w\bar{\eta})^2}.
\]

It follows that

\[
|K_\beta(w, \eta)| = \left| \frac{(\beta + 1)(w\bar{\eta})^\beta - \beta(w\bar{\eta})^{\beta+1}}{(1-w\bar{\eta})^2} \right| \lesssim \left| \frac{w^\beta}{(1-w\bar{\eta})^2} \right|.
\] (3.8)

**Corollary 3.5** Assume \( f(\eta_1, \eta_2) \in W^{k,2}(\mathbb{D} \times \mathbb{D}) \) and \( w_1, w_2 \in \mathbb{D}^* = \mathbb{D} \setminus \{0\} \), then

\[
\int_{\mathbb{D} \times \mathbb{D}} \left[ \frac{\partial^{\beta_1}}{\partial w_1^{\beta_1}} \left( \frac{1}{(1-w_1\bar{\eta}_1)^2} \right) \cdot \frac{\partial^{\beta_2}}{\partial w_2^{\beta_2}} \left( \frac{1}{(1-w_2\bar{\eta}_2)^2} \right) \right] f(\eta_1, \eta_2) \, d\nu(\eta)
= \frac{1}{w_1^{\beta_1} w_2^{\beta_2}} \int_{\mathbb{D} \times \mathbb{D}} K_{\beta_1}(w_1, \eta_1) K_{\beta_2}(w_2, \eta_2) T_{\eta_1}^{\beta_1} T_{\eta_2}^{\beta_2} f(\eta_1, \eta_2) \, d\nu(\eta).
\]

**Proof** By Lemmas 3.3, 3.4, (3.7) and Fubini theorem, we have

\[
\int_{\mathbb{D} \times \mathbb{D}} \left[ \frac{\partial^{\beta_1}}{\partial w_1^{\beta_1}} \left( \frac{1}{(1-w_1\bar{\eta}_1)^2} \right) \cdot \frac{\partial^{\beta_2}}{\partial w_2^{\beta_2}} \left( \frac{1}{(1-w_2\bar{\eta}_2)^2} \right) \right] f(\eta_1, \eta_2) \, d\nu(\eta)
= \int_{\mathbb{D}} \frac{1}{w_1^{\beta_1}} T_{\eta_1}^{\beta_1} \left( K_{\beta_1}(w_1, \eta_1) \right) \left( \frac{1}{w_2^{\beta_2}} \int_{\mathbb{D}} T_{\eta_2}^{\beta_2} \left( K_{\beta_2}(w_2, \eta_2) \right) f(\eta_1, \eta_2) \, d\nu(\eta_2) \right) \, d\nu(\eta_1)
= \int_{\mathbb{D}} \frac{1}{w_1^{\beta_1} w_2^{\beta_2}} \int_{\mathbb{D}} T_{\eta_1}^{\beta_1} \left( K_{\beta_1}(w_1, \eta_1) \right) \left( \int_{\mathbb{D}} T_{\eta_2}^{\beta_2} \left( K_{\beta_2}(w_2, \eta_2) \right) f(\eta_1, \eta_2) \, d\nu(\eta_2) \right) \, d\nu(\eta_1)
= \int_{\mathbb{D}} \frac{1}{w_1^{\beta_1} w_2^{\beta_2}} \int_{\mathbb{D}} K_{\beta_1}(w_1, \eta_1) T_{\eta_1}^{\beta_1} \left( \int_{\mathbb{D}} K_{\beta_2}(w_2, \eta_2) T_{\eta_2}^{\beta_2} \left( f(\eta_1, \eta_2) \right) \, d\nu(\eta_2) \right) \, d\nu(\eta_1)
= \int_{\mathbb{D} \times \mathbb{D}} K_{\beta_1}(w_1, \eta_1) K_{\beta_2}(w_2, \eta_2) T_{\eta_1}^{\beta_1} T_{\eta_2}^{\beta_2} f(\eta_1, \eta_2) \, d\nu(\eta),
\]

where the second and the third equalities follow from the integration by parts. \( \square \)
4 Proof of the Main Results

Proof of Theorem 1.1 For \( f \in W^{k, p}(\mathbb{D}) \), by Lemma 3.2, one sees

\[
\|B_G(f)\|_{W^{k, p}(\mathbb{D}, (\frac{3kp}{2})\delta)}^p = \sum_{0 \leq |\alpha| \leq k} \int_{\mathbb{D}} |D^\alpha_z (B_G(f))|^p |z_1^2 - 4z_2^2|^{\frac{3kp}{2}} d\nu(z)
\]

\[
\leq \int_{\mathbb{D}} |B_G(f)|^p |z_1^2 - 4z_2^2|^{\frac{3kp}{2}} d\nu(z)
\]

\[
+ \sum_{1 \leq |\alpha| \leq k} \int_{\mathbb{D} \times \mathbb{D}} \frac{1}{|(w_1 - w_2)^2|^{p|\alpha|-1}} \sum_{1 \leq |\beta| \leq |\alpha|} |P_{\beta, \alpha}(w_1, w_2)|^p (4.1)
\]

\[
\leq \sum_{|\beta| \leq k} \int_{\mathbb{D} \times \mathbb{D}} |D^\beta_w (B_G(f) \circ \Phi)|^p |w_1 - w_2|^{kp+p+2} d\nu(w)
\]

\[
= \frac{1}{2^p \pi^{2p}} \sum_{|\beta| \leq k} \int_{\mathbb{D} \times \mathbb{D}} \int_{\mathbb{D} \times \mathbb{D}} \frac{\partial^\beta_1 + \partial^\beta_2}{\partial w_1^{\beta_1} \partial w_2^{\beta_2}} (B_G(f) \circ \Phi)^p \cdot \frac{1}{w_1 - w_2} \cdot \frac{1}{\eta_1 - \eta_2}
\]

\[
\left[ \frac{1}{(1 - w_1 \eta_1)^2} \cdot \frac{1}{(1 - w_2 \eta_2)^2} - \frac{1}{(1 - w_1 \eta_2)^2} \cdot \frac{1}{(1 - w_2 \eta_1)^2} \right] f(\Phi(\eta))|\eta_1 - \eta_2|^{kp+p+2} d\nu(w)
\]

When the derivative applies to \( \frac{1}{w_1 - w_2} \), the degree will be no less than \(-k - 1\), which will be absorbed by the weight \( |w_1 - w_2|^{kp+p+2} \). Here we only consider the one term
in (4.2), and the other term can be handled by the same argument.

\[
\begin{align*}
& \int_{D \times D} \left| \frac{\partial \beta_1}{\partial w_1^\beta_1} \left( \frac{1}{(1 - w_1 \bar{\eta}_1)^2} \right) \cdot \frac{\partial \beta_2}{\partial w_2^\beta_2} \left( \frac{1}{(1 - w_2 \bar{\eta}_2)^2} \right) \right| f(\Phi(\eta))(\eta_1 - \eta_2) \, dv(\eta) |^p \\
& \quad \cdot |w_1 - w_2|^2 \, dv(w) \\
& = \int_{D \times D} \left| \frac{1}{w_1^\beta_1 w_2^\beta_2} \int_{D \times D} K_{\beta_1}(w_1, \eta_1) K_{\beta_2}(w_2, \eta_2) T_{\eta_1 \beta_1}^\eta \cdot T_{\eta_2 \beta_2}^\eta \\
& \quad (f(\Phi(\eta))(\eta_1 - \eta_2)) \, dv(\eta) |^p \\
& \quad \cdot |w_1 - w_2|^2 \, dv(w) \\
& \lesssim \int_{D \times D} \left| \int_{D} \left( \frac{1}{1 - w_2 \bar{\eta}_2} \right)^2 \int_{D} \left( \frac{1}{1 - w_1 \bar{\eta}_1} \right)^2 \left| T_{\eta_1 \beta_1}^\eta \cdot T_{\eta_2 \beta_2}^\eta (f(\Phi(\eta))(\eta_1 - \eta_2)) \right| \\
& \quad dA(\eta_1) \, dA(\eta_2) |^p \\
& \quad \cdot |w_1 - w_2|^2 \, dv(w) \\
& = \int_{D \times D} \left| \mathcal{B}_{D, \eta_1}^+ \mathcal{B}_{D, \eta_2}^+ \left| T_{\eta_1 \beta_1}^\eta \cdot T_{\eta_2 \beta_2}^\eta (f(\Phi(\eta))(\eta_1 - \eta_2)) \right| |^p \cdot |w_1 - w_2|^2 \, dv(w) \\
& \lesssim \int_{D} \int_{D} \left| T_{w_1 \beta_1}^\eta \cdot T_{w_2 \beta_2}^\eta (f(\Phi(\eta))(\eta_1 - \eta_2)) \right| |^p \cdot |w_1 - w_2|^2 \, dv(w_1) \, dv(w_2) \\
& \lesssim \int \int \left| T_{w_1 \beta_1}^\eta \cdot T_{w_2 \beta_2}^\eta (f(\Phi(w))(w_1 - w_2)) \right| |^p \cdot |w_1 - w_2|^2 \, dv(w_1) \, dv(w_2) \\
& \lesssim \sum_{|\alpha| \leq k} \sum_{|q| \leq k_2} \int_{D} |D_{\alpha \beta_1}^\eta \cdot D_{\alpha \beta_2}^\eta (f \circ \Phi(\eta))|^p \cdot |w_1 - w_2|^2 \, dv(w) \\
& \lesssim \sum_{|\alpha| \leq k} \int_{G} |D_{\alpha \beta_1}^\eta (f)|^p \, dv(z) = \| f \|_{W^k,p(G)}^p,
\end{align*}
\]

where the first equality follows from Corollary 3.5, the first inequality follows from (3.8), the second and the third inequalities follow from the assumption on $\mathcal{B}_{D}^+$, and the last inequality follows from Lemma 3.2.

For the first term on the right hand side of (4.1), by Bell’s transformation formula,

\[
B_{D \times D}(J_{C} \Phi \cdot (h \circ \Phi)) = J_{C} \Phi \cdot (B_{D}(h) \circ \Phi),
\]

(4.4)
where \( h \in L^2(\mathbb{G}) \) ([6]), one sees

\[
\int_\mathbb{G} |B_G(f)|^p |z_1^2 - 4z_2|^3 d\nu(z) = \int_{\mathbb{D} \times \mathbb{D}} |B_D B_D ((w_1 - w_2)(f \circ \Phi)(w))|^p |w_1 - w_2|^{(3k-1)p+2} d\nu(w) \\
\lesssim \int_{\mathbb{D} \times \mathbb{D}} (B_D^+ B_D^+ (|f \circ \Phi|))^p |w_1 - w_2|^2 d\nu(w) \\
\lesssim \int_{\mathbb{D} \times \mathbb{D}} |f \circ \Phi|^p |w_1 - w_2|^2 d\nu(w) \\
= \int_\mathbb{G} |f|^p d\nu(z) = \|f\|_{L^p(\mathbb{G})}.
\]

The first inequality is due to the boundedness of \(|w_1 - w_2|\) and \(3k - 1 \geq 0\), and the second inequality follows from the assumption on \(B_D^+\) as in the proof of (4.3). Combining two parts, Theorem 1.1 is proved. \(\square\)

**Proof of Corollary 1.4** The first part follows by combining Theorem 1.1 and Proposition 1.2. The second part follows by combining Propositions 1.3 and 4.1. For the third part, when \(\frac{4}{3} < p < 4\), it follows from Part 2 since the weight function \(e^{-\frac{p}{2}}\) is bounded from above. When \(p > 2\), it follows from Part 1. The only difference is that it suffices to have \(|z_1^2 - 4z_2|^2 = |w_1 - w_2|^p\) as the weight function in (4.5) to cancel out the term of \(|J_C\Phi|^{-p} = |w_1 - w_2|^{-p}\) arisen in (4.4). \(\square\)

**Proposition 4.1** If the norm of \(B_D : L^p(\mathbb{D}, |w_1 - w_2|^2-p) \to L^p(\mathbb{D}, |w_1 - w_2|^{2-p})\) is uniformly bounded independent of \(w_2 \in \mathbb{D}\) for some \(p \in (1, \infty)\), then the Bergman projection \(B_G\) on the symmetrized bidisk \(\mathbb{G}\) is bounded from \(L^p(\mathbb{G})\) to itself.

The proposition is implicit proved in [14] (cf. Theorem 3.1) and we also include the proof here for the completeness.

**Proof** By Bell’s transformation formula (4.4), to prove the \(L^p\) boundedness of \(B_G\):

\[
\|B_G(h)\|_{L^p(\mathbb{G})} \lesssim \|h\|_{L^p(\mathbb{G})}
\]

for \(h \in L^p(\mathbb{G})\), it is equivalent to prove

\[
\int_{\mathbb{D} \times \mathbb{D}} |B_D \cdot (J_C \Phi \cdot (h \circ \Phi)) \cdot (J_C \Phi)^{-1}|^p \cdot |J_C \Phi|^2 dV \lesssim \int_{\mathbb{D} \times \mathbb{D}} |h \circ \Phi|^p \cdot |J_C \Phi|^2 dV.
\]

Let \(g = J_C \Phi \cdot (h \circ \Phi)\), it suffices to prove that, for any \(g \in L^p(\mathbb{D} \times \mathbb{D}, |J_C\Phi|^2-p)\),

\[
\int_{\mathbb{D} \times \mathbb{D}} |B_D \cdot (g)\|^p \cdot |J_C \Phi|^{2-p} dV \lesssim \int_{\mathbb{D} \times \mathbb{D}} |g|^p \cdot |J_C \Phi|^{2-p} dV.
\]
By plugging in $J_C\Phi$ and applying the Fubini’s Theorem, one obtains

$$
\int_{D \times D} |B_D \times D(g)|^p \cdot |w_1 - w_2|^{2-p} dV = \int_{D} \int_{D} |B_D(g(w_1, w_2))|^p \cdot |w_1 - w_2|^{2-p} dA(w_1) dA(w_2)
$$

$$
\lesssim \int_{D} \int_{D} |g(w_1, w_2)|^p \cdot |w_1 - w_2|^{2-p} dA(w_1) dA(w_2).
$$

The last inequality is due to the boundedness of $B_D$ from $L^p(D, |w_1 - w_2|^{2-p})$ to itself, the independence and the symmetry in $w_1, w_2$. 

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Declarations

Final Remark
Right before this manuscript is accepted for publication, Zhenghui Huo and Brett D. Wick showed in a preprint (cf. arXiv:2303.10002) that the sufficient condition in Remark 1.5 is in fact the necessary condition as well for the Bergman projection to be $L^p$ bounded on the symmetrized polydisc.

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