Radial deformation of the Kerr spacetime

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Abstract

Since the Kerr metric is an idealized vacuum solution, the spacetime around accreting rotating black holes is certainly a non-vacuum solution and thus deviates from the Kerr metric. In the absence of an exact interior Kerr solution, we propose a radial deformation of the Kerr metric which leads to a non-vacuum geometry and study the spacetime as a function of the deformation parameter. The status of energy conditions is discussed in detail. It is shown that the resulting spacetime leads to the appearance of anisotropic wormholes in certain cases.

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I. INTRODUCTION

Black holes are perhaps the most strange and fascinating objects known to exist in the universe that are predicted by Einstein’s theory of gravity. Though these objects made their first appearance in the famous exact spherically symmetric solution found by Karl Schwarzschild [1], the concept of black hole has been crystallized by physicists for many years [2]. For a long time, a part of the physics community was rather sceptical about the actual existence of black holes, but the situation has changed in recent years, notably because of the wide variety of phenomena of astronomical observations: from X-ray binary systems to active galactic nuclei (including our home, the Milky Way) [3].

The problem of finding a vacuum gravitational field which describes a stationary rotating black hole was solved by Kerr [4] and his solution remains of considerable interest in general relativity. In astrophysics, it is very important to have a metric for the interior of a rotating star, which should match to the solution of Kerr, since we usually take the solution of Kerr as an exterior solution. Many people tried to find interior solutions but what they found had in general some problem. For example, the matching was approximate [5, 7]. It was found that regular disks as sources of the Kerr metric give energy-momentum tensors which do not satisfy the dominant energy conditions [8]. The question if a perfect fluid can be the source of the metric of Kerr is an open one [9, 10].

There are strong evidences that black holes and wormholes are closely related. A new frame for the unity of the black hole with the wormhole’s dynamics is presented by Hayward [11]. Black holes are described with an outer null trapped surface and wormholes with an outer timelike trapped surface that the incoming light beam from the surface will start to diverge [12]. Wormholes are tunnels in the geometry of space and time that connect two separate and distinct regions of spacetimes. Although such objects were long known to be solutions of Einstein equation, a renaissance in the study of wormholes took place during 80’s motivated by the possibility of quick interstellar travel [13].

Currently, there exist some activities in the field of wormhole physics following, particularly, the seminal works of Morris, Thorne and Yurtsever [14]. Morris and Thorne assumed that their traversable wormholes were time independent, non-rotating, and spherically symmetric bridges between two universes. The manifold of interest is thus a static spherically symmetric spacetime possessing two asymptotically flat regions. These kinds of wormholes
could be threaded both by quantum and classical matter fields that violate certain energy
in the vicinity of the throat known as exotic matter. On general grounds, it has recently
been shown that the amount of exotic matter needed at the wormhole throat can be made
arbitrarily small, thereby facilitating an easier construction of wormholes [15].

Previously, Damour and Solodukhin investigated a radial deformation of Schwarzschild
spacetime, showing that the resulting wormholes can mimic many observational features of
black holes [16]. A radial deformation of the Reissner-Nordstrom metric which leads to the
appearance of charged, traversable wormholes was investigated by Mehdizadeh, Ebrahimi
and Riazi [17]. Lemos et al. [18] studied extremal ε-wormholes on the threshold of the
formation of an event horizon, quasi-black holes, and wormholes on the basis of quasi-black
holes. They investigated whether or not the resulting objects remain regular in the near-
horizon limit. They showed that the requirement of full regularity, i.e., finite curvature and
absence of naked behavior, up to an arbitrary neighborhood of the gravitational radius of
the object implies ruling out potential mimickers in most of the cases.

In this article, we apply a radial deformation to the Kerr metric and the corresponding
energy-momentum tensor treading this metric is calculated which gives the required type of
matter distribution and energy flux for this deformation. Energy conditions and the question
whether this new structure is a deformed black hole or it is a wormhole, is investigated. In
other words, we present a family of rotating anisotropic asymptotically flat fluid solutions,
which depend on three parameters including the parameters of mass and angular momentum.
For certain values of these parameters, although \( \rho > 0 \), the weak (and therefore null and
strong) energy conditions are violated in the whole physical spacetime. All solutions are
singular on a ring lying in the equatorial plane (like the ring singularity of Kerr solution),
while the deformed metric has one more singularity.

II. DEFORMING THE KERR SPACETIME

Our motivation for such a deformation is the following: the presence of a rotating fluid
leads to a deviation from the vacuum Kerr metric. Since it has been proved difficult (or
perhaps impossible) to construct an exact interior Kerr solution, we apply a radial deforma-
tion to the Kerr metric which is supposed to be caused by the presence of the rotating fluid.
In other words, since in the physical world rotating black holes are not isolated and accrete
matter, the metric will be different from the vacuum of Kerr metric. Since the investigation of the general form of the deformed metric is difficult, we study a special kind of radial deformation.

We consider a metric which in Boyer-Lindquist coordinates \[19\] has the form

\[
ds^2 = -(1 - \frac{2mr}{r^2 + a^2 \cos^2 \theta})dt^2 + \left(\frac{r^2 + a^2 \cos^2 \theta}{r^2 + 2r^2 \varepsilon + r^2 \varepsilon^2 - 2mr - 2mr \varepsilon + a^2}\right)dr^2
+ (r^2 + a^2 \cos^2 \theta)d\theta^2 + \sin^2 \theta(r^2 + a^2 + \frac{2mra^2 \sin^2 \theta}{r^2 + a^2 \cos^2 \theta})d\varphi^2
- \frac{4mra \sin^2 \theta}{r^2 + a^2 \cos^2 \theta}dt d\varphi
\]  

(1)

where \(\rho^2 = r^2 + a^2 \cos^2 (\theta)\). Here, \(m\) and \(a\) denote the mass and the rotation parameters, respectively. This metric differs from the standard Kerr metric \[2, 20, 21\] only through the presence of the dimensionless parameter \(\varepsilon\), where \(\varepsilon = 0\) corresponds to the Kerr spacetime (endowed with a horizon). By contrast, for \(\varepsilon \neq 0\) the structure of the spacetime is dramatically different: there is no event horizon, instead there is a throat that joins two anisotropic and asymptotically flat regions. This spacetime is an example of a Lorentzian wormhole \[22\].

The asymptotic expansion of the deformed metric (1) against \(r\) is

\[
ds^2 = -dt^2 + \frac{1}{1 + 2\varepsilon + \varepsilon^2}dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2.
\]

(2)

this relation imply that the deformed metric (1) is asymptotically flat for all values of \(\varepsilon\). In principle, we must used tetrad component or curvature invariants to discuss the behavior of deformed metric at infinity. The relevant components (see \[2\]) in this case is (according to notations of MTW)

\[
F = R_{\theta\varphi}^{\theta\varphi}
\]

(3)

and we have

\[
R_{\theta\varphi}^{\theta\varphi} \propto \frac{1}{r^2} \rightarrow 0 \quad \text{if} \quad r \rightarrow \infty
\]

(4)

therefor we conclude that the spacetime is asymptotically flat for all values of \(\varepsilon\).

The Kretschmann scalar \((R_{\mu\nu\lambda\kappa} R^{\mu\nu\lambda\kappa})\) of the deformed metric (1) is

\[
K = \frac{F(r, \varepsilon, m, a)}{(r^2 + a^2 \cos^2 \theta)^6(r^2 + a^2 - 2mr)^4},
\]

(5)

where \(F\) is a lengthy expression not useful to be reproduced here (we note that the zeros of \(F\) can’t compensate the zeros of the denominator. ). The deformed metric (1) has two
intrinsic singularities which correspond to
\[ \rho^2 = r^2 + a^2 \cos^2 \theta = 0 \] (6)

and
\[ \Delta = r^2 - 2mr + a^2 = 0 \] (7)

these occur when
\[ x^2 + y^2 = 0, \quad z = 0 \quad (i.e. \quad x = y = z = 0) \] (8)

and
\[ r^\pm = m \pm \sqrt{m^2 - a^2} \] (9)

Therefore from (8) we find that all solutions are singular on a ring lying in the equatorial plane (the ring singularity of Kerr solution), while from (9) we find that those have two more singularities which corresponds to equation (9).

From the deformed metric (1), the equation
\[ g^{rr} = 0 \] (10)

has two solutions
\[ r^\pm = \frac{1}{1 + \varepsilon} (m \pm \sqrt{m^2 - a^2}), \] (11)

while the two solutions of \( g_{tt} = 0 \) are
\[ r^\pm = m \pm \sqrt{m^2 - a^2 \cos^2 \theta} \] (12)

In order to have real roots, we should have
\[ m > |a| \] (13)

a condition already known for the Kerr black holes (\( m < |a| \) corresponds to naked singularities in the Kerr metric which are ruled out by cosmic censorship hypothesis [23]). We will see later that the spacetime can be extended through the throat hypersurface in such a way that these singularities are excluded.

The eigenvalues of the Kerr metric tensor (in an optional point with coordinates \((t, r, \theta, \varphi)\)) are
\[ \lambda_1 = \frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2mr + a^2} \]
\[ \lambda_2 = r^2 + a^2 \cos^2 \theta \]
\[ \lambda_3 = -\frac{Y(r, \theta)}{r^2 + a^2 \cos^2 \theta} \] (14)
\[ \lambda_4 = -\frac{Z(r, \theta)}{r^2 + a^2 \cos^2 \theta} \]
where

\[ Y(r, \theta) = \frac{1}{2} \left( (a^4(r^2 - 2mr + a^2)^2 \sin^8 \theta - 2a^2(a^2 + a + r^2)(a^2 - a + r^2)(r^2 - 2mr + a^2) \right. \\
+ a^2) \sin^6 \theta(a^8 + (4r^2 - 4)a^6 + (8mr + 1 - 8r^2 + 6r^4)a^4 + (8mr^3 - 4r^4) \\
+ 4r^6 + 8m^2r^2)a^2 + r^8) \sin^4 \theta + 2(a^2 + a + r^2)(a^2 - a + r^2)(r^2 - 2mr + a^2) \\
+ a^2) \sin^2 \theta + (r^2 - 2mr + a^2)^{1/2} + (a^4 + (r^2 - 2mr)a^2) \sin^4 \theta + (-a^4 \\
+ (-2r^2 - 1)a^2 - r^4) \sin^2 \theta + a^2 - 2mr + r^2), \tag{15} \]

and

\[ Z(r, \theta) = \frac{1}{2} \left( -(a^4(r^2 - 2mr + a^2)^2 \sin^8 \theta - 2a^2(a^2 + a + r^2)(a^2 - a + r^2)(r^2 - 2mr + a^2) \right. \\
+ a^2) \sin^6 \theta + (a^8 + (4r^2 - 4)a^6 + (8mr + 1 - 8r^2 + 6r^4)a^4 + (8mr^3 - 4r^4) \\
+ 4r^6 + 8m^2r^2)a^2 + r^8) \sin^4 \theta + 2(a^2 + a + r^2)(a^2 - a + r^2)(r^2 - 2mr + a^2) \\
+ a^2) \sin^2 \theta + (r^2 - 2mr + a^2)^{1/2} + (a^4 + (r^2 - 2mr)a^2) \sin^4 \theta + (-a^4 \\
+ (-2r^2 - 1)a^2 - r^4) \sin^2 \theta + a^2 - 2mr + r^2), \tag{16} \]

while the eigenvalues of the deformed metric \( \text{[1]} \) are

\[ \lambda_1' = \frac{r^2 + a^2 \cos^2 \theta}{r^2 + \varepsilon^2 r^2 + 2r\varepsilon - 2mr - 2mr\varepsilon + a^2} \]
\[ \lambda_2' = r^2 + a^2 \cos^2 \theta \]
\[ \lambda_3' = -\frac{Y(r, \theta)}{r^2 + a^2 \cos^2 \theta} \]
\[ \lambda_4' = -\frac{Z(r, \theta)}{r^2 + a^2 \cos^2 \theta} \tag{17} \]

Comparing relations (14) and (17) we fined that

\[ \lambda_1 \neq \lambda_1', \]

while

\[ \lambda_2 = \lambda_2', \]
\[ \lambda_3 = \lambda_3', \]

and

\[ \lambda_4 = \lambda_4'. \]
FIG. 1: $\lambda_1$ and $\lambda'_1$ are plotted against $r$ for $\varepsilon = -0.3$, $m = 2$ and $a = 1$. The solid line denotes $\lambda_1$ and the dashed line corresponds to $\lambda'_1$.

The signature of the deformed metric (1) depends on the sign of the $\lambda'_1$, $\lambda_1$, which together with $\lambda'_1$ is shown in Fig. 1 for $m = 2$, $a = 1$, $\varepsilon = -0.3$ and $\theta = \frac{\pi}{2}$. It is clear that the region $r > r'_\varepsilon$ has Lorentzian signature and we have finite redshift at $r = r'_\varepsilon$. The signature of the metric becomes improper for $r^+ < r < r'_\varepsilon$, which is excluded from the spacetime in the extension proposed here. Using the famous technique of cut and paste [24–26], the spacetime can be extended from $r = r'_\varepsilon$ in such a way the resulting wormhole structure excludes the intrinsic singularities $r^\pm$ and also the intrinsic ring singularity. Therefore the physical spacetime includes the region $r > r'_\varepsilon$. Also one should note that the Kretschman scalar $(R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa})$ has a finite value at the throat, implying a finite gravity at that hypersurface. Since the point $(t, r, \theta, \varphi)$ is an optional point, therefore these conclusion are extendable for whole of spacetime.

III. ENERGY-MOMENTUM CONDITIONS

Using the Einstein’s equations, we readily obtain the energy-momentum tensor with components:

$$T_{tt} = \frac{1}{8\pi G}G_{tt} = -\frac{1}{8\pi G}\varepsilon \frac{X(r, \theta)}{(a^2 - 2mr + r^2)^2(a^2 \cos^2 \theta + r^2)^4} \tag{18}$$

$$T_{rr} = \frac{1}{8\pi G}G_{rr} = \frac{1}{8\pi G}\varepsilon \frac{Y(r, \theta)}{(r^2 + a^2 \cos^2 \theta)^2((1 + \varepsilon)^2 \varepsilon^2 - 2mr(1 + \varepsilon) + a^2)(a^2 - 2mr + r^2)} \tag{19}$$
where $X, Y, Z, W, V$ are functions of coordinate $r$ and $\theta$. The eigenvalues of the matrix $T_{\mu\nu}$ are

$$\lambda_1 = \frac{1}{16\pi G \varepsilon} \frac{X'(r, \theta)}{(a^2 - 2mr + r^2)^2(a^2 \cos^2 \theta + r^2)^4}$$

$$\lambda_2 = \lambda_+ = \frac{1}{8\pi G \varepsilon} \frac{Y'(r, \theta)}{(r^2 + a^2 \cos^2 \theta)^2((1 + \varepsilon)^2r^2 - 2mr(1 + \varepsilon) + a^2)(a^2 - 2mr + r^2)}$$

$$\lambda_3 = \frac{1}{8\pi G \varepsilon} \frac{Z'(r, \theta)}{(a^2 - 2mr + r^2)^2(r^2 + a^2 \cos^2 \theta)^2}$$

$$\lambda_4 = \lambda_- = \frac{1}{16\pi G \varepsilon} \frac{W'(r, \theta)}{(a^2 - 2mr + r^2)^2(a^2 \cos^2 \theta + r^2)^4}$$

where $X', Y', Z'$ and $W'$ are functions of coordinate $r$ and $\theta$. What can be inferred from this eigenvalues and is important to us is that first; all the eigenvalues are proportional to the parameter $\varepsilon$, therefore the eigenvalues tend to zero if $\varepsilon \to 0$ (as the fluid becomes diluted). Second; due to the non-zero spatial eigenvalues, the distribution can not be a dust. Third; because of the unequal pressure components, this tensor does not describe a perfect fluid. Therefore, the proposed spacetime, contains an imperfect and anisotropic fluid.

Energy-momentum conditions and in particular their status in wormhole spacetimes are extensively discussed in Morris and Thorne [13] and Visser [22]. Here, we apply the general procedure to the deformed metric (1). Calculating the eigenvectors of the matrix $T_{\mu\nu}$ we find that the eigenvector which corresponds to the eigenvalue $\lambda_-$ is timelike. Then the energy density $\rho$ is given by the relation

$$\rho = \lambda_-$$

and the components of pressure are

$$p_1 = \lambda_2$$

$$p_2 = \lambda_3$$

$$p_3 = \lambda_+$$

For any non-spacelike vector $v^\mu$, the weak energy condition reads [22, 27]

$$T_{\mu\nu} v^\mu v^\nu \geq 0 \implies \rho \geq 0 \quad and \quad \rho + p_i \geq 0, \quad i = 1, 2, 3$$
Using (27) and (28), it can be seen that although $\rho \geq 0$, $\rho + p_2 \geq 0$ and $\rho + p_3 \geq 0$, the weak (and therefore null and strong) energy conditions are violated in the whole physical spacetime due to the fact that $\rho + p_1 < 0$ everywhere in the spacetime.

IV. SUMMARY AND CONCLUSION

In this paragraph we summarize the results of our calculation for the radial deformation of the Kerr metric ($g_{rr}(r) = \frac{\rho^2(r,\theta)}{\Delta(r)} \rightarrow \frac{\rho^2(r,\theta)}{\Delta((1+\epsilon)r)}$). It is shown that the resulting spacetime, represents a family of exact anisotropic rotating fluid solutions. All solutions are singular on a ring lying in the equatorial plane (like the ring singularity of Kerr solution), while the deformed metric has one more singularity. The curvature invariants at large $r$ tend to zero for the deformed metric (1), therefore the spacetime is asymptotically flat.

We introduced wormhole geometries that arise for certain values of the parameters (for example $m = 2$, $a = 1$ and $\epsilon = -0.1$). This happened because in this range of parameters, coordinate and intrinsic singularities are pushed below the physical domain of interest (i.e. the domain with Lorentzian signature), and the spacetime is extended beyond $r > r^+_\epsilon$ (In fact, the new metric (1) doesn’t cover whole spacetime covered by the Kerr metric). We showed that the resulting wormhole can be thought of as being formed by two distinct but identical asymptotically flat spacetimes joined at the throat ($r^+_\epsilon$) and we note that the redshift function, $g_{tt}$, is finite at $r = r^+_\epsilon$. Extending the spacetime in this way will exclude the inner horizon and ring singularity present in the Kerr spacetime. Also one should note that the Kretschman scalar ($R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa}$) has a finite value at the throat, implying a finite gravity at that hypersurface.

We discussed the energy conditions for the supporting energy-momentum tensor and showed that like most other wormholes, null, weak and strong energy conditions are violated. The energy density, as well as $\rho + p_i$ ($i = 2, 3$), however, are positive everywhere. As a prospect for future work, it would be useful to investigate in more detail the causal structure of the deformed Kerr spacetime and try to find a deeper description of the matter.
source in terms of known physical fields or fluids.

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