Linear Conjugacy of Chemical Reaction Networks

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1 Background

- Chemical Reactions
- Mass-Action Kinetics
- Weakly Reversible Networks
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2 Linearly Conjugacy
- Main Theorem
- Examples
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An elementary reaction consists of a set of reactants which turn into a set of products, e.g.

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\[ 2\text{H}_2 + \text{O}_2 \xrightarrow{k} 2\text{H}_2\text{O} \]

Species/Reactants
An elementary reaction consists of a set of reactants which turn into a set of products, e.g.

\[ 2\text{H}_2 + \text{O}_2 \overset{k}{\rightarrow} 2\text{H}_2\text{O} \]

Reactant Complex
An elementary reaction consists of a set of reactants which turn into a set of products, e.g.

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**Product Complex**
An elementary reaction consists of a set of reactants which turn into a set of products, e.g.

$$2H_2 + O_2 \xrightarrow{k} 2H_2O$$

Reaction Constant
An elementary reaction consists of a set of reactants which turn into a set of products, e.g.

\[ 2\text{H}_2 + \text{O}_2 \overset{k}{\rightarrow} 2\text{H}_2\text{O} \]

Chemical kinetics is the study of the *rates/dynamics* resulting from systems of such reactions.
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Chemical kinetics is the study of the *rates/dynamics* resulting from systems of such reactions.

In order to build a model, we assume the mixture is spatially homogeneous, temperature and volume are held constant, and the law of mass action applies.
Consider the general network $\mathcal{N}$ given by

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Under mass-action kinetics, this network is governed by the system of autonomous, polynomial, ordinary differential equations

$$\dot{x} = \sum_{i=1}^{r} k_i (z_i' - z_i) x^{z_i}$$

(1)

where $x_i, \ i = 1, \ldots, m$, are the reactant concentrations.
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where $x_i, \ i = 1, \ldots, m$, are the reactant concentrations.

Model is applied to systems biology, enzyme kinetics, industrial reactors, neural networks, atmospherics, etc., and is related to predator-prey and epidemic growth models in biology.
The particular class of networks which I have been interested in are *weakly reversible networks*. 
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C_1 & \xrightarrow{k_1} C_2 \\
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\[ C_1 \xrightarrow{k_1} C_2 \]
\[ k_3 \leftarrow k_2 \]
\[ C_3. \]

Under the assumption of mass-action kinetics, strong properties are known about the dynamics of weakly reversible networks.
Mass-action systems are often very difficult to analyze and many types of behaviour are possible (stable, multi-stable, oscillatory, chaotic behaviours, etc.)
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However, many classes of systems with strongly predictable behaviour are known (e.g. weakly reversible systems).

**CHALLENGE:** Determine conditions under which a system with unknown dynamics can be related to a system with known behaviour.
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   ▪ Mass-Action Kinetics
   ▪ Weakly Reversible Networks

2 Linearly Conjugacy
   ▪ Main Theorem
   ▪ Examples
In [1], G. Craciun and C. Pantea give necessary and sufficient conditions under which two different reaction networks \( \mathcal{N} \) and \( \mathcal{N}' \) generate the same set of differential equations under the assumption of mass-action kinetics.
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The qualitative dynamics of $\mathcal{N}'$ are transferred to $\mathcal{N}$, even if the graph structure is wildly different!
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The qualitative dynamics of $\mathcal{N}'$ are transferred to $\mathcal{N}$, even if the graph structure is wildly different!

Further work has been done by G. Szederkényi et al. in developing computer algorithms which determine such equivalent networks [2, 3].
We extend this work to networks which do not necessarily generate the same mass-action kinetics.
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Let $\Phi(x_0, t)$ denote the flow associated with $\mathcal{N}$ and $\Psi(y_0, t)$ denote the flow associated with $\mathcal{N}'$. 
We extend this work to networks which do not necessarily generate the same mass-action kinetics.

We rely on the well-known theory of conjugacy of dynamical systems.

Let $\Phi(x_0, t)$ denote the flow associated with $N$ and $\Psi(y_0, t)$ denote the flow associated with $N'$.

We will say $N$ and $N'$ are linearly conjugate if there exists a linear mapping $h : \mathbb{R}^m_{>0} \mapsto \mathbb{R}^m_{>0}$ such that

$$h(\Phi(x_0, t)) = \Psi(h(x_0), t)$$

for all $x_0 \in \mathbb{R}^m_{>0}$. 

Let $C_{react}$ denote the set of reactant complexes in either the complex set $C$ or the complex set $C'$.

**Theorem**

*Suppose that for the rate constants $k_i > 0$, $i = 1, \ldots, r$, there exist constants $b_i > 0$, $i = 1, \ldots, \tilde{r}$, and $c_j > 0$, $j = 1, \ldots, m$, such that, for every $C^0 \in C_{react}$,*

\[
\sum_{i=1}^{r} k_i (z_i' - z_i) = T \sum_{i=1}^{\tilde{r}} b_i (\tilde{z}_i' - \tilde{z}_i)
\]

*where $T = \text{diag} \{ c_j \}_{j=1}^{m}$. Then $N$ is linearly conjugate to $N'$ with rate constants*

\[
\tilde{k}_i = b_i \prod_{j=1}^{m} c_j^{\tilde{z}_{ij}}, \quad i = 1, \ldots, \tilde{r}.
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$$

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$$
\tilde{k}_i = b_i \prod_{j=1}^{m} c_j^{\tilde{z}_{ij}}, \quad i = 1, \ldots, \tilde{r}.
$$
Example 1:

Consider the chemical reaction network

\[ N : \quad \begin{align*}
A_1 + 2A_2 & \xrightarrow{k_1} A_1 + 3A_2 \\
2A_1 & \xrightarrow{k_4} A_2.
\end{align*} \]
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This is linearly conjugate to

$$\mathcal{N}' : \quad A_1 + 2A_2 \xleftrightarrow{\tilde{k}_1} A_1 + 3A_2 \xleftrightarrow{\tilde{k}_2} A_1 + A_2 \xleftrightarrow{\tilde{k}_3} 2A_1.$$

Source complexes are conserved!
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This is linearly conjugate to

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\[ \quad A_1 + A_2 \xleftrightarrow{\tilde{k}_2, \tilde{k}_3} 2A_1. \]

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Source complexes are conserved!
Example 2:

Consider the chemical reaction network

\[ A_1 + 2A_2 \xrightarrow{\epsilon} A_1 \]

\[ \mathcal{N} : \quad 2A_1 + A_2 \xrightarrow{1} 3A_2 \]

\[ A_1 + 3A_2 \xrightarrow{1} A_1 + A_2 \xrightarrow{1} 3A_1 + A_2 \]

for \( \epsilon > 0 \).
Example 2:

Consider the chemical reaction network

\[ \mathcal{A}_1 + 2\mathcal{A}_2 \xrightarrow{\epsilon} \mathcal{A}_1 \]

\[ \mathcal{N} : \quad 2\mathcal{A}_1 + \mathcal{A}_2 \xrightarrow{1} 3\mathcal{A}_2 \]

\[ \mathcal{A}_1 + 3\mathcal{A}_2 \xrightarrow{1} \mathcal{A}_1 + \mathcal{A}_2 \xrightarrow{1} 3\mathcal{A}_1 + \mathcal{A}_2 \]

for \( \epsilon > 0 \). This is linearly conjugate to

\[ \mathcal{N}^\prime : \]

\[ \mathcal{A}_1 + 2\mathcal{A}_2 \xrightarrow{\tilde{k}_1} \mathcal{A}_1 + \mathcal{A}_2 \]

\[ \mathcal{A}_1 + 3\mathcal{A}_2 \xleftarrow{\tilde{k}_3} 2\mathcal{A}_1 + \mathcal{A}_2 \]

\[ \tilde{k}_4 \uparrow \tilde{k}_5 \uparrow \downarrow \tilde{k}_2 \]
Example 2:

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N : \begin{align*}
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\end{align*}
\]

for \( \epsilon > 0 \). This is linearly conjugate to

\[
N' : \begin{align*}
A_1 + 2A_2 & \xrightarrow{\tilde{k}_1} A_1 + A_2 \\
2A_1 + A_2 & \xrightarrow{\tilde{k}_4} \xrightarrow{\tilde{k}_5} \xrightarrow{\tilde{k}_2} \\
A_1 + 3A_2 & \xleftarrow{\tilde{k}_3} 2A_1 + A_2
\end{align*}
\]

The third reaction is split into two!
Example 3:

Consider the chemical reaction network

\[ A_1 \xrightarrow{k_1} 2A_1 + 2A_2 \xrightarrow{k_2} A_2 \xrightarrow{k_3} A_1 + A_2. \]
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\[ A_1 + A_2 \overset{\tilde{k}_3}{\leftrightarrow} A_2 \]

\[ N' : \]

\[ \tilde{k}_4 \downarrow \tilde{k}_5 \downarrow \uparrow \tilde{k}_2 \]

\[ A_1 \xrightarrow{\tilde{k}_1} 2A_1 + 2A_2. \]
Example 3:

Consider the chemical reaction network

\[ \mathcal{A}_1 \xrightarrow{k_1} 2\mathcal{A}_1 + 2\mathcal{A}_2 \xrightarrow{k_2} \mathcal{A}_2 \xrightarrow{k_3} \mathcal{A}_1 + \mathcal{A}_2. \]

This is linearly conjugate to

\[ \mathcal{N}' : \]

\[ \mathcal{A}_1 + \mathcal{A}_2 \xleftrightarrow{\tilde{k}_3} \mathcal{A}_2 \]

\[ \tilde{k}_4 \downarrow \hspace{1cm} \tilde{k}_5 \downarrow \hspace{1cm} \uparrow \tilde{k}_2 \]

\[ \mathcal{A}_1 \xrightarrow{\tilde{k}_1} 2\mathcal{A}_1 + 2\mathcal{A}_2. \]

Under certain conditions, strictly product complexes can become source complexes!
Outstanding! But what are the next steps?
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1. Generally need to find suitable target networks $\mathcal{N}'$ - computer programs are necessary for all but the simplest cases.
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2. What about non-linear transformations?
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1. Generally need to find suitable target networks $\mathcal{N}'$ - computer programs are necessary for all but the simplest cases.

2. What about non-linear transformations?

3. What about other (i.e. non-mass-action) dynamics?
Thanks for coming out!
G. Craciun and C. Pantea, *Identifiability of chemical reaction networks*, J. Math. Chem. 44 (2008), no. 1, pp. 244–259.

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