Probability representation entropy for spin-state tomogram.
O. V. Man'ko, and V. I. Man'ko
Lebedev Physical Institute, Moscow, Leninskie pr., 53

Abstract

Probability representation entropy (tomographic entropy) of arbitrary quantum state is introduced. Using the properties of spin tomogram to be standard probability distribution function the tomographic entropy notion is discussed. Relation of the tomographic entropy to Shannon entropy and von Neumann entropy is elucidated.

1 Introduction

There exists formulation of quantum mechanics (see, e.g. [1, 2]) where quantum system states are associated with probabilities (instead of wave functions or density matrix). One can use this formulation to develop naturally notion of entropies related to the probabilities. The probabilities determining quantum states are called state tomograms. Since arbitrary quantum state of any quantum system is determined by the probability the entropy (which we call probability representation entropy) is always associated with the quantum state. In [3, 4] tomograms of spin states were introduced. The star-product formalism for tomographic symbols of spin operators was introduced in [5, 6]. Spin tomography was developed in [2, 7, 8, 9, 10]. The tomography for two spins was introduced in [11, 12]. The tomograms for spin were also considered in [13, 14, 15]. Since tomograms of quantum states are standard probability distribution functions, all the characteristics of the probability distributions known in probability theory can be applied also in the framework of tomographic probability representation of quantum states. The most important characteristics related to probability distributions are entropy and information (see, e.g. [16]). For symplectic tomograms [17] the notion of entropy has been shortly discussed in [18].

The aim of this work is to introduce the notion of tomographic entropy and information and to consider these notion for spin states. We consider both one particle and multiparticle cases. In classical probability theory the notion of Shannon entropy [19] is the basic one. In quantum mechanics von Neumann [20] introduced entropy related to density operator (see, e.g. [21]). We will obtain the relation of the introduced tomographic entropy and information to the Shannon and von Neumann entropies. The article is organised as follows. In Section II the standard notions of probability theory are reviewed. In Section III the spin tomography of one particle is reviewed and entropy of spin states is discussed. In Section IV spin tomograms for two particles are discussed and corresponding entropies are introduced. In Section V the relations of tomographic entropies to von Neumann entropy are discussed. Conclusions and perspectives are presented in Section VI.

2 Properties of entropy

In this section we review standard properties of entropy following presentation and notation of probability theory given in [16]. In quantum mechanics a pure state is associated with a vector
|ψ⟩ in the Hilbert space. A mixed state is associated with a density matrix ρ. In probability representation of quantum mechanics one describes the state by a probability distribution depending on extra parameters. This probability distribution is either function of discrete random variable (spin) or a function of continuous random variable (position). Extra parameters determine reference frame where these variables are measured. Thus one has one-to-one correspondence (invertable map)

\[ ρ \leftrightarrow ω \]

where ω is the discussed probability distribution. With each probability distribution one can associate entropy following the Shannon prescription. For any quantum state (given by ω) we associate entropy S. This entropy S we call probability representation entropy of quantum state. This entropy is the function of discussed extra parameters (reference frame parameters). Below we review generic properties of an entropy H used in probability theory.

Given discrete random variable of the system X, which has states \( x_i \). The Shannon entropy \( H(X) \) is defined as

\[ H(X) = -\sum_{i=1}^{n} P_i \ln P_i, \]

where \( P_i \) is probability distribution function for the random variable (probability of the state \( x_i \)). Entropy of a bipartite system with discrete random variables describing subsystems \( X \) and \( Y \) is defined as

\[ H(X,Y) = -\sum_{i=1}^{n} \sum_{j=1}^{m} P_{ij} \ln P_{ij}, \]

where \( P_{ij} \) is joint probability distribution of two random variables (joint probabilities to have states \( x_i \) and \( y_j \)). Conditional entropy is defined as

\[ H(Y|x_i) = -\sum_{j=1}^{m} P(y_j|x_i) \ln P(y_j|x_i). \]

Here \( P(y_j|x_i) \) is probability for system Y to be in state \( y_j \), if the system X is in the state \( x_i \). Let us define the complete conditional entropy by relation

\[ H(Y|X) = -\sum_{i=1}^{n} \sum_{j=1}^{m} P_{ij} \ln P(y_j|x_i). \]

It is known that

\[ H(X,Y) = H(X) + H(Y|X), \quad H(X,Y) \leq H(X). \]

Also

\[ H(X,Y) \leq H(X) + H(Y) \]

Information on the system X obtained due to observation of the system Y is defined by the relation

\[ I_{Y \to X} = H(X) - H(X|Y). \]

One has definition of complete mutual information obtained on systems X and Y

\[ I_{X \leftrightarrow Y} = I_{X \to Y} = I_{Y \to X}. \]
Also the information is given by the formula

\[ I_{Y+X} = H(X) + H(Y) - H(X,Y) \geq 0 \]  \hspace{1cm} (10)

Let us apply these general relations to the spin systems using tomographic probabilities. For bipartite spin system the subsystem \( X \) is identified with first particle with spin \( j_1 \) and the subsystem \( Y \) with second particle with spin \( j_2 \) for given group element \( U(n) \) which determines the basis in the space of spin quantum states. The measured variables \( x_i, y_j \) are spin projections on \( z \)-axis \( m_1 \) and \( m_2 \), respectively. For spin system we denote entropy by capital letter \( S \).

### 3 Spin states in the tomographic representation

According to formalism of probability representation of quantum spin states the state of system with spin \( j \) with density matrix \( \rho \) is associated with probability distribution function \( \omega^{(j)}(m, \vec{n}) \), where \( m = -j, -j + 1, \ldots, j \), the unit vector \( \vec{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \) determines the point on the sphere. This probability distribution (called spin state tomogram) was introduced in [3, 4] and it is normalized for each vector \( \vec{n} \), i.e.

\[ \sum_{m=-j}^{j} \omega(m, \vec{n}) = 1. \]  \hspace{1cm} (11)

It is related to density operator \( \rho \) by the formula

\[ \omega(m, \vec{n}) = \langle jm | D^{(j)^+}(u) \rho D^{(j)}(u) | jm \rangle. \]  \hspace{1cm} (12)

Here states \( |jm \rangle \) are standard states with spin projection \( m \) on the \( z \)-axis, i.e.,

\[ \hat{j}_z |jm \rangle = m |jm \rangle. \]  \hspace{1cm} (13)

The matrix \( D^{(j)}(u) \) is the matrix of irreducible representation of the SU(2) group. The 2x2 -matrix \( u \) is the element of this group. We use Euler angles \( \phi, \theta, \psi \) as parameters of this element \( u \).

\[ u(\phi, \theta, \psi) = \begin{pmatrix} \cos(\frac{\theta}{2})e^{\frac{i}{2}(\phi+\psi)} & \sin(\frac{\theta}{2})e^{-\frac{i}{2}(\phi-\psi)} \\ -\sin(\frac{\theta}{2})e^{\frac{i}{2}(\phi-\psi)} & \cos(\frac{\theta}{2})e^{\frac{i}{2}(\phi+\psi)} \end{pmatrix}. \]  \hspace{1cm} (14)

Due to the structure of the formula (12) the tomogram does not depend on Euler angle \( \psi \). The density matrix \( \rho^{(j)} \) with matrix elements \( \rho^{(j)}_{mm'} \) can be constructed if one knows the tomogram according to relation [4]

\[ (-1)^{m'} \rho^{(j)}_{mm'} = \sum_{k=0}^{2j} \sum_{i=-k}^{k} (2k + 1)^2 \sum_{i=-j}^{j} (-1)^i \omega(i, u) D_{01}^{k}(u) \times \left( \begin{array}{ccc} j & j & k \\ i & -i & 0 \end{array} \right) \left( \begin{array}{ccc} j & j & k \\ m & -m' & l \end{array} \right) \frac{d\Omega}{8\pi^2}, \]  \hspace{1cm} (15)

Here \( m, m' = -j, -j + 1, \ldots, j \); the integration is performed with respect to the angular variables \( \phi, \theta, \psi \), i.e.,

\[ \int d\Omega = \int_{0}^{2\pi} d\phi \int_{0}^{2\pi} d\psi \int_{0}^{\pi} \sin \theta d\theta. \]
The Shannon entropy [19] associated to any probability distribution function provides the entropy associated to spin tomogram. Thus we define the tomographic entropy as the function on the sphere

\[ S(\vec{n}) = - \sum_{m=-j}^{j} \omega(m, \vec{n}) \ln \omega(m, \vec{n}). \]  

(16)

The tomography of quark states suggested in [8] was generalized and the unitary spin-tomography was discussed in [22]. The unitary spin tomogram is defined as follows

\[ \omega(m, U(n)) = \langle jm|U^+(n)\rho U(n)|jm\rangle. \]  

(17)

Here \( n = 2j + 1 \), \( n \)-dimensional matrix \( U(n) \) is element of unitary group. Since \( D^{(j)}(u) \) is subgroup of the unitary group, the tomogram (17) determines the density matrix of spin state. If one takes as matrix \( U(n) \) the matrix \( D^{(j)}(u) \) the unitary spin tomogram becomes the spin tomogram. The function (17) is the probability distribution function which is normalised for each group element \( U(n) \), i.e.,

\[ \sum_{m=-j}^{j} \omega(m, U(n)) = 1. \]  

(18)

In fact the tomogram is probability to get the spin projection \( m \) in the “rotated” basis

\[ |jm\bar{m}⟩ = U(n)|jm⟩. \]  

(19)

The application of Shannon entropy formalism to the unitary spin tomogram defines the unitary tomographic entropy which is the function on unitary group, i.e.,

\[ S(U(n)) = - \sum_{m=-j}^{j} \omega(m, U(n)) \ln \omega(m, U(n)). \]  

(20)

For \( U(n) \) taken as \( D^{(j)}(u) \) the entropy (20) coincides with entropy (16).

## 4 Tomography for two-spin particle

Below we apply the results discussed in Section 2 for two subsystems \( X \) and \( Y \) to the case of two spins. Let us consider now two particles with spin \( j_1 \) and \( j_2 \), respectively. The basis in the space of states can be given by product vector

\[ |m_1m_2⟩ = |j_1m_1⟩|j_2m_2⟩. \]  

(21)

The density matrix \( \rho \) of a system state can be mapped onto spin-tomogram [10, 22]

\[ \omega(m_1, m_2, \vec{n}_1, \vec{n}_2) = \langle m_1m_2|D^{(j_1)+}(u_1) \otimes D^{(j_2)+}(u_2)\rho D^{(j_2)}(u_2) \otimes D^{(j_1)}(u_1)|m_1m_2⟩. \]  

(22)

This is joint probability distribution function for two discrete random variables \( m_1 \) and \( m_2 \) which are spin projections on the directions \( \vec{n}_1 \) and \( \vec{n}_2 \), respectively. The function is normalised

\[ \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \omega(m_1, m_2, \vec{n}_1, \vec{n}_2) = 1. \]  

(23)
Tomographic entropy \( S(\vec{n}_1, \vec{n}_2) \) can be associated with this probability distribution function

\[
S(\vec{n}_1, \vec{n}_2) = - \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \omega(m_1, m_2, \vec{n}_1, \vec{n}_2) \ln \omega(m_1, m_2, \vec{n}_1, \vec{n}_2).
\]  

(24)

This tomographic entropy depends on the points on two spheres determined by unit vectors \( \vec{n}_1 \) and \( \vec{n}_2 \). The tomographic probability for two particles determines the tomographic probability for one particle, e.g.,

\[
\omega(m_1, \vec{n}_1) = \sum_{m_2=-j_2}^{j_2} \omega(m_1, m_2, \vec{n}_1, \vec{n}_2).
\]  

(25)

In \cite{22} unitary spin tomogram was introduced by relation

\[
\omega(m_1, m_2, U(n)) = \langle m_1 m_2 | U^+(n) \rho U(n) | m_1 m_2 \rangle, \quad n = (2j_1 + 1)(2j_2 + 1).
\]  

(26)

This tomogram is joint probability distribution function depending on unitary group element \( U(n) \). It is normalised for each group element, i.e.,

\[
\sum_{m_1=j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \omega(m_1, m_2, U(n)) = 1.
\]  

(27)

The joint tomographic probability (26) determines the tomographic probability for one particle depending on unitary group element

\[
\omega_1(m_1, U(n)) = \sum_{m_2=-j_2}^{j_2} \omega_1(m_1, m_2, U(n)).
\]  

(28)

Analogous probability can be obtained for the second spin. We can associate with the tomographic probability the entropy which is the function on the unitary group

\[
S_1(U(n)) = - \sum_{m_1=-j_1}^{j_1} \omega_1(m_1, U(n)) \ln \omega_1(m_1, U(n)).
\]  

(29)

Also the tomographic entropy related to the tomogram (26) determines the tomographic probability for one particle depending on unitary group element

\[
S(U(n)) = - \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \omega(m_1, m_2, U(n)) \ln \omega(m_1, m_2, U(n)).
\]  

(30)

This entropy depends on unitarity group parameters. For the matrix \( U(n) = D^{(j_1)}(u_1) \otimes D^{(j_2)}(u_2) \) the entropy (30) coincides with the entropy (24).

We can construct also conditional probability distribution for first spin

\[
\omega_{C1}(m_1, U(n)) = \frac{\omega(m_1, m_2, U(n))}{\sum_{m_1=-j_1}^{j_1} \omega(m_1, m_2, U(n))}.
\]  

(31)

Analogous formula can be written for second spin. Tomographic information \( I_{Y \leftrightarrow X} \), where \( X \) is \( j_1 \)-spin system and \( Y \) is \( j_2 \)-spin system is given by the function on the unitary group

\[
I_{j_1 \leftrightarrow j_2}(U(n)) = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \omega(m_1, m_2, U(n)) \ln \frac{\omega(m_1, m_2, U(n))}{\omega_1(m_1, U(n)) \omega_2(m_2, U(n))}.
\]  

(32)

Here \( \omega_2(m_2, U(n)) \) is given by (28) with replacement \( 1 \leftrightarrow 2 \).
5 Information and Relation to von Neumann entropy

The unitary spin entropy (20) for the case of spin state of single particle defines the von Neumann entropy of this state

\[ S_N = -Tr [\rho \ln \rho]. \]  

(33)

In fact, there exist elements of unitary group \( U(0)(n) \), \( n = 2j + 1 \), which diagonalize the density matrix \( \rho \). For these elements \( U(0)(n) \) the tomogram is equal to probability distribution function which coincides exactly with eigenvalues of the density matrix. It means that the tomographic entropy (20) for these values of unitary group elements is equal to von Neumann entropy of the spin state, i.e.,

\[ S(U(0)(n)) = S_N. \]  

(34)

On the other hand it is obvious that for the elements \( U(0)(n) \) the tomographic entropy takes minimal possible value. It follows from the property that the probability distributions determined by density matrix diagonal elements for the matrix obtained by means of unitary rotation of the basis are smoother than distributions provided by eigenvalues of the density matrix [22]. Thus the von Neumann entropy is the minimum of tomographic entropy.

One can introduce complete mutual information for two spins using the relation

\[ I_{j_2+j_1} (U(n)) = S_1 (U(n)) + S_2 (U(n)) - S (U(n)). \]  

(35)

Here the tomographic entropies are functions on the unitary group and the introduced information is also the function on the unitary group. The discussed entropies and information can be used to study the properties of multipartite quantum states from new viewpoint.

6 Conclusion

We summarize the main results of the work. We introduced concept of probability representation entropy of quantum states. This entropy (we also call this entropy tomographic entropy) is defined via tomographic probability distribution function determining the quantum state in the approach which is called probability representation of quantum mechanics. The tomographic probability was defined using standard Shannon relation of entropy and probability distribution. We applied this approach to spin systems. Since for spin system the tomographic probability of quantum state is the function on unitary group the tomographic entropy is also the function on unitary group. The von Neumann entropy of quantum state was shown to be the minimal value of the tomographic entropy. This minimum of tomographic entropy is realised for a set of unitary group elements which are diagonal unitary matrices commuting with density matrix considered in the basis where it is also diagonal. Using standard relation of entropy and information the notion of probability representation information (tomographic information) was introduced. Some standard relations known in probability theory were applied to the tomographic entropy and tomographic information of quantum states. The construction of tomographic entropies and informations for multipartite system is straightforward. The introduced notions of tomographic entropy and tomographic information elucidate new aspects of quantum states which are naturally appear in probability representation of quantum mechanics*. 
*When the paper was finished authors became aware of [23, 24], where some aspects of other entropies related to discussed ones in this work were considered (without the tomographic framework).

7 Acknowledgement

This study was supported by the Russian Foundation for Basic Research under Project No 03-02-16408.

References

[1] S. Mancini, V. I. Man’ko, P. Tombesi, Phys. Lett., 213 A, 1 (1996)
[2] Olga Man’ko, and V. I. Man’ko, J. Russ. Laser Res., 18, 407 (1997)
[3] V. V. Dodonov, and V. I. Man’ko, Phys. Lett., 239A, 335 (1997)
[4] V. I. Man’ko, and O. V. Man’ko, JETP, 85, 430 (1997)
[5] O. V. Man’ko, V. I. Man’ko, and G. Marmo, Phys. Scripta, 62, 446 (2000)
[6] O. V. Man’ko, V. I. Man’ko, and G. Marmo, J. Phys., 35 A, 699 (2002)
[7] S. Mancini, O. V. Man’ko, V. I. Man’ko, P. Tombesi, J. of Physics A, 34, 3461 (2001)
[8] A. B. Klimov, O. V. Man’ko, V. I. Man’ko, Yu. F. Smirnov, and V. N. Tolstoy, J. Phys., 35A, 6101 (2002)
[9] V. A. Andreev, O. V. Man’ko, V. I. Man’ko, and S. S. Safonov, J. Russ. Laser Res., 19, 340 (1998)
[10] O. V. Man’ko, V. I. Man’ko, and S. S. Safonov, Theor. Math. Phys., 115, 185 (1998)
[11] V. I. Man’ko, S. S. Safonov, Yad. Fiz., 4, 658 (1998)
[12] V. A. Andreev, V. I. Man’ko, JETP, 87, 239 (1998)
[13] S. Weigert, Phys. Rev. Lett., 84, 802 (2000)
[14] O. Castanos, R. Lopes-Pena, M. A. Man’ko, and V. I. Man’ko, J. Phys., 36 A, 4677 (2003)
[15] U. Leonhardt, Phys. Rev., 53 A, 2998 (1996)
[16] E. C. Wentzel, Probability Theory (Nauka, Moscow) (1969), 4th edition
[17] S. Mancini, V. I. Man’ko, P. Tombesi, Quantum Semiclass. Opt., 7, 615 (1995)
[18] M. A. Man’ko, J. Russ. Laser Res., **22**, 168 (2001); S. De Nicola, R. Fedele, M. A. Man’ko, V. I. Man’ko, *Eur. Phys. J. B*, **36**, 385 (2003)

[19] C. E. Shannon, *Bell Systems Technical Journal*, **27**, 379 (1948)

[20] J. von Neumann, *Matematische Grundlagen der Quantenmechanik*, (Springer) (1932)

[21] A. S. Holevo, *Statistical structure of quantum theory* (Springer LNP 67); *Russ. Math. Surveys*, **53**:6, 1295 (1998)

[22] V. I. Man’ko, G. Marmo, E. C. G. Sudarshan, F. Zaccaria, J. Russ. Laser Res., **24**, 507 (2003)

[23] A. Stotland, A. A. Pomeransky, E. Bachmat, D. Cohen, ”The information entropy of quantum mechanical states,” quant-ph/040121, (2004).

[24] D. Collins, S. Popescu ”Frames of Reference and the Intrinsic Directional Information of a Particle with Spin,” quant-ph/0401096,(2004).