Domain Wall Fermions and MC Simulations of Vector Theories.

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It is known that domain wall fermions may be used in MC simulations of vector theories. The practicality and usefulness of such an implementation is investigated in the context of the vector Schwinger model, on a 2+1 dimensional lattice. Preliminary results of a Hybrid Monte Carlo simulation are presented.

1. Introduction

Domain Wall Fermions (DWF) for lattice gauge theories were developed in [1]. Since then a wealth of activity has followed with the main focus being the application of DWF to chiral lattice gauge theories, but also to vector lattice gauge theories (see [2] and references therein). As a result of these works little doubt remains that DWF can be used to regularize vector gauge theories on the lattice. Because DWF have many attractive features it is surprising that little work has been done in the past few years investigating their practicality and usefulness in numerical simulations of lattice vector gauge theories. A preliminary investigation of this, in the context of the vector Schwinger model, is presented here.

2. The model

The massless case of the one flavor model is described by the lattice Euclidean action:

\[ S[\bar{\psi}, \psi, A_\mu] = \beta \sum_p \text{Re}(U_p) - \bar{\psi} \mathcal{M}(U) \Psi \]

\[ \mathcal{M}(U)_{\mu \nu} = \delta_{\mu \nu} \sum_{p} \left[ \frac{1 + \gamma_{\mu}}{2} U_{n \mu, \mu \nu} \delta_{n + \mu, n'} - \frac{1 - \gamma_{\mu}}{2} U_{n \mu, -\mu \nu} \delta_{n - \mu, n'} \right] \]

\[ + \delta_{\mu \nu} \left[ \frac{1 + \gamma_{5}}{2} \delta_{s+1, s'} + \frac{1 - \gamma_{5}}{2} \delta_{s-1, s'} \right] \]

\[ - \delta_{\mu \nu} \delta_{s, s'} [3 - m(s)] \]

In the above \( U \) is the U(1) gauge field, \( U_p \) the standard plaquette, \( \beta = g_0^2 \) with \( g_0 \) the gauge coupling, and \( n \) is a collective space time coordinate. The photon mass is \( m_\gamma = g_0/\sqrt{2} \) and the space time volume is \( V = L^2 \). The key ingredient of DWF is the introduction of the extra direction \( s \), with periodic boundary conditions, size \( L_s \), and a mass defect at \( s = 0 \) and \( s = L_s/2 \) \( (m(s) = +m0, m(s) = -m0) \) for \( L_s/2 \leq s < L_s, 0 < m_0 < 1 \). In the free case a Dirac fermion appears with its right component exponentially peaked at \( s = 0 \) and its left one at \( s = L_s/2 \). The overlap of the left and right components is small and decreases exponentially with increasing \( L_s \). In the limit \( L_s \to \infty \) one obtains a single massless Dirac fermion. The introduction of the extra direction also introduces heavy modes. Their contribution needs to be subtracted \[ \mathcal{M} \]. This is done by dividing the fermionic determinant, \( \sqrt{\det \mathcal{M}} \), by \( \sqrt{\det \mathcal{M}_+} \det \mathcal{M}_- \), where \( \mathcal{M}_+ \), \( \mathcal{M}_- \) are the same as \( \mathcal{M} \) but with \( m(s) = +m0, m(s) = -m0 \) respectively. These determinants can be produced by introducing appropriate auxiliary bosonic fields.

There are two implementations of DWF that can be used in numerical simulations:

I The overlap formalism \[ \mathcal{M} \]. A transfer matrix is constructed along the extra direction and from it the corresponding Hamiltonian \( \mathcal{H} \) is extracted. This formulation allows for the strict \( L_s \to \infty \) limit to be taken. Observables can be calculated by obtaining all the eigenvalues and eigenvectors of \( \mathcal{H} \) (\( \mathcal{H} \) is a matrix of size \( V \times V \)). The lattice vector Schwinger model was simulated successfully using this method \[ \mathcal{M} \]. However, the method requires large amounts of computer time and high statistics simulations in four dimensions and large volumes may not be possible at present.
A direct simulation of (1) with \( L_s \) finite. The chiral limit is obtained by extrapolating to the \( L_s \to \infty \) limit. This method is very attractive for several reasons:

a) Unlike Wilson fermions there is no fine tuning involved; the chiral limit is the \( L_s \to \infty \) limit. At the \( L_s \to \infty \) limit the quark mass is multiplicatively renormalized \[5\] (for a recent investigation of this using Hybrid Monte Carlo (HMC) see \[6\]). Although for finite \( L_s \) there will always be an additive component its size is expected to be decreasing exponentially with \( L_s \).

b) Unlike staggered fermions there is no breaking of flavor symmetry on the lattice.

c) Because the overlap of the left and right components is exponentially small in the free case, one would expect that very small explicit breaking of chiral symmetry can be obtained for a relatively small \( L_s \). If this is the case, then the method will be very valuable in studies involving spontaneous chiral symmetry breaking.

d) Anomalous symmetry breaking has a natural interpretation \[1\] and one may expect that this will facilitate related studies.

Of course, there is a price to be paid for these nice features. The theory has one more dimension and therefore demands more computer resources.

Given the above considerations several questions should be answered before DWF can be used to simulate lattice vector gauge theories and in particular lattice QCD. Are there any hidden difficulties in an HMC simulation of the model? Does the method work and give the correct answers? How does the computational difficulty depend on \( L_s \) and how large should \( L_s \) be? How good are DWF in addressing questions related to anomalous breaking of axial symmetries and spontaneous breaking of chiral symmetries? Preliminary answers to a few of these questions are presented below.

3. The size of the extra dimension

The \( L_s \) dependence can be investigated by direct comparison with the \( L_s \to \infty \) result obtained using the overlap formalism. In the massless case the fermionic effective action \( S_{eff} \) is finite in the zero topological sector and \( S_{eff} \to -\infty \) otherwise. On an \( 8 \times 8 \) lattice, \( S_{eff} \) is calculated for several \( L_s \) for a background gauge field with topological charge 0 in figure 1, and with charge 1 in figure 2. The very different \( L_s \) dependence in the two topological sectors is evident. This indicates that DWF correctly reproduce topological effects already at \( L_s = 10 - 12 \). Furthermore, the overlap result in the zero topological sector (dashed line in figure 1) is also reached at \( L_s = 10 - 12 \). Similar results for the size of \( L_s \) have been obtained from studies of the pion mass in the 0 sector \[6\].

![Figure 1. \( S_{eff} \) in topological sector 0](image1)

![Figure 2. \( S_{eff} \) in topological sector 1](image2)

4. Hybrid Monte Carlo and anomalous symmetry breaking

A variant of (1) developed in \[5\] is most suited for use with the HMC algorithm. The basic difference is that \( m(s) = m_0 \) but free boundary conditions are used along the extra direction.
The two chiral modes are now bound to the two free boundaries. A mass term connects the free boundaries and introduces a bare quark mass $m_f$. 

The advantage of this is that it only needs half the size of $L_s$ and also only a single bosonic field to perform the subtraction mentioned earlier.

The massive $N_f = 2$ flavor vector Schwinger model is simulated using this formulation and the HMC algorithm. The expectation value of the operator $W = \prod_{i=1}^{N_f} \bar{\Psi}_R^i \Psi_L^i + \prod_{i=1}^{N_f} \bar{\Psi}_L^i \Psi_R^i$ is calculated. If the volume is kept fixed and $m_f$ is made very small the effect of the zero modes coming from different topological sectors becomes important [7]. In particular, as can be seen from the overlap implementation [3] [4], in the massless and $L_s \to \infty$ limit the operator $W$ receives contributions only from sectors $\pm 1$, while the fermion Boltzman weight (fermion determinant) is not zero only in sector 0. As $m_f$ is turned on (and/or $L_s$ is decreased from infinity) the fermionic determinant becomes non zero in sectors other than 0 and the operator $W$ receives contributions from sectors other than $\pm 1$. Therefore for small $m_f$ the HMC algorithm will mostly sample the sector 0 where the observable $W$ receives small contributions. The algorithm will infrequently visit the sectors $\pm 1$ but when it does the observable $W$ will receive large contributions to make up for the small sampling rate. As a result, when $m_f$ is decreased a larger number of HMC iterations will be needed to sample the $\pm 1$ sectors correctly.

5. Preliminary results

Preliminary results are presented for $m_0 = 0.9$, $L = 6$, $m_f L = 3.0$. The HMC sweeps vary from 6,000 for $m_f = 0.01$ to 2,000 for $m_f = 0.5$. The trajectory length is 1 and the step size is 0.02 for $m_f = 0.01$ and 0.04 for the rest. The average conjugate gradient iterations vary from 56 to 86. The time history of $W$ for $m_f = 0.01$ is shown in fig. 3. The large “spikes” are related to configurations with charge $\pm 1$ (measured with the geometric method). $\langle W \rangle$ vs. $m_f$ is in fig. 4 and agrees roughly with the $m_f = 0$ overlap result [4]. For comparison with the exact answer see [4].

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