Inverting Sets And The Packing Problem

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Abstract

Given a set $V$, a subset $S$, and a permutation $\pi$ of $V$, we say that $\pi$ permutes $S$ if $\pi(S) \cap S = \emptyset$. Given a collection $S = \{V; S_1, \ldots, S_m\}$, where $S_i \subseteq V$ ($i = 1, \ldots, m$), we say that $S$ is invertible if there is a permutation $\pi$ of $V$ such that $\pi(S_i) \subseteq V - S_i$. In this paper, we present necessary and sufficient conditions for the invertibility of a collection and construct a polynomial algorithm which determines whether a given collection is invertible. For an arbitrary collection, we give a lower

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bound for the maximum number of sets that can be inverted. Finally, we consider the problem of constructing a collection of sets such that no sub-collection of size three is invertible. Our constructions of such collections come from solutions to the packing problem with unbounded block sizes. We prove several new lower and upper bounds for the packing problem and present a new explicit construction of packing.

1 Introduction

The notion of invertibility arises as a tool for approaching other combinatorial problems, for example, the problem of constructing a minimal size edge-set in a Cayley graph which intersects every cycle of a given length. We explain this connection using the hypercube as an example.

1.1 Square-Blocking Edge-Sets in a Hypercube

Let \( f(n) \) denote the minimal number of edges in a hypercube \( Q_n \) such that their removal from the hypercube yields a square-free graph. Evaluating \( f(n) \) or computing the asymptotics for it is a long-standing open problem. Erdős ([10]) conjectured that \( f(n) \approx n2^{n-2} \) (see also [1], [2], [3], [4], [5], [11], [12], [13]). The best known lower bound on \( f(n) \) has been obtained by F.R.K. Chung [2] and is given by 

\[
f(n) \geq (\alpha - o(1))n2^{n-1},
\]

where \( \alpha \) is about 0.377.

For every \( n \geq 0 \), we view \( Q_{n+1} \) as the union of two copies of \( Q_n \), denoted respectively \( Q' \) and \( Q'' \), and let \( W \) denote the set of edges between vertices in \( Q' \) and vertices in \( Q'' \). Obviously, \( W \) is a matching with \( 2^n \) edges; it can be viewed as a one-to-one mapping from \( V(Q'') \) onto \( V(Q') \) as well as from \( V(Q'') \) onto \( V(Q') \). These mappings are naturally expanded to one-to-one mappings of the corresponding edge-sets. If \( K \subseteq E(Q'') \), then \( W(K) \) denotes the image of \( K \) under the mapping \( W \). Given \( W' \subseteq W \), let \( C(W') \) denote the set of vertices in \( Q' \) that are incident to the edges in \( W' \). Then the following statement can be easily proved.

**Proposition.** Let \( N' \subseteq E(Q') \), \( N'' \subseteq E(Q'') \), and \( W' \subseteq W \). Then \( N' \cup N'' \cup W' \) is a square-blocking set in \( Q_{n+1} \) if and only if \( N' \) and \( N'' \) are square-blocking in \( Q' \) and \( Q'' \), respectively, and \( C(W') \) is a vertex cover of the subgraph \( E(Q') - N' - W(N'') \).
Since $Q_n$ is a connected bipartite graph, either of the partitions can serve as a vertex cover $C(W')$ for the corresponding $W' \subset W$. With that, the choice of $N'$ and $N''$ is arbitrary, as long as both are square-blocking sets for $Q'$ and $Q''$, respectively. Depending on the choice of the construction for small values of $n$, this construction leads to a square blocking set of size $(n-2)2^{n-2}$.

A smaller square-blocking set can be obtained if we try to construct $N'$ and $N''$ to minimize the intersection of $N' \cap W(N'')$. Thus, we may try to construct $N''$ to be an image of $N'$ under some permutation of the hypercube. One way to construct such a permutation is to focus on the vertices of $Q'$ that are incident to at least $n/2$ edges in $N'$. Let $v' \in Q', N'(v') \geq n/2$ and $v'' = W(v')$. Let $E'$ (respectively $E''$) be the edges in $N'$ (respectively in $N''$) that are adjacent to $v'$ (respectively $v''$). We will be able to save at least one edge in $W'$ if

$$F' \cap F'' = \emptyset,$$

where $F' = E - E'$ and $F'' = E - E''$. In order to save more edges, we may try to find a permutation of all directions in $Q$ such that the condition above holds for as many vertices as possible. If there are $m$ vertices $v_1, \ldots, v_m$ whose degrees $\geq n/2$, and $S_i$ ($i = 1, \ldots, m$) is the set of directions of $Q_n$ for which the corresponding edges are not in $N'$, then our goal is to permute the set of all directions so that as many of $S'_i$'s are inverted as possible.

In Section 2, we establish a necessary and sufficient condition for a given collection of sets to be inverted by a single permutation, and give a bound for the number of sets that can be inverted in every collection $S$ with a given distribution of set sizes. An interesting reverse question is as follows: how many sets can a collection have if no sub-collection containing a given number is invertible? It turns out that even when the bound is three, the number is exponential, e.g., there are exponentially large collections of sets such that the maximal invertible sub-collection contains only two sets. Construction of such a collection comes from solution of a packing problem for which the block sizes are unbounded.

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1It is easy to see that a collection of any two subsets of size not more than $n/2$ is invertible.
1.2 Packing Problem

Let \( n \) be a positive integer, and let \( c \) and \( \alpha \) be two reals in the interval \([0, 1]\). The version of the packing problem considered in this paper is defined as the construction of the maximum number of \( cn \)-subsets of a set \([1, \ldots, n]\), such that any two of the sets intersect in fewer than \( \alpha cn \) elements.

Proving that a certain “big” packings exist is often done using the probabilistic method. The only algorithm that the method suggests is then the brute-force search, which is doubly-exponential for this problem. For example, here is a result related to Shannon’s theorem (Z. Füredi, private communication):

**Theorem 1.1** For every \( 0 < d, c < 1 \), if \( d > c^2 \), then there exists \( \epsilon > 0 \) such that there is a collection of \((1 + \epsilon)^n\) \( cn \)-sets such that the intersection of any two contains \( < dn \) elements.

**Proof.** (sketch) Let \((1 + \epsilon)^n = N\). Choose randomly \( N \) sets of size \( cn \) each. Then the expected size of the intersection of any two is \( c^2 n \). By Chernoff’s inequality

\[
\text{Prob} [ |A \cap A'| = c^2 n + x \sqrt{n}] \lesssim \exp(-x^2).
\]

Since

\[
\text{Prob} [ |A \cap A'| > d n] \leq \exp(-(d - c)^2 n),
\]

one can delete a small number of bad elements. \( \square \)

Since the probabilistic proof above does not provide an efficient way to construct a packing, it is reasonable to seek algorithms that would construct a packing with a sufficiently large number of blocks. Since the output would have an exponentially large collection of sets, the running time of the algorithm is inevitably an exponential function of \( n \). Finally, if a solution to the packing problem is given by an explicit construction we may expect that the size of the collection is even smaller than that guaranteed by an algorithm.

**Definition 1.1** A graph \( \mathcal{G}(n, c, \alpha) \) is defined as follows: the vertices of \( \mathcal{G} \) are the \( cn \)-subsets of \([1, \ldots, n]\); two vertices are adjacent if and only if the corresponding sets intersect in \( \geq \alpha cn \) elements. The number of vertices, the degree of a vertex, and the maximal size of an independent set of \( \mathcal{G} \) is denoted \( N = N(n, c, \alpha) \), \( D = D(n, c, \alpha) \), and \( P = P(n, c, \alpha) \), respectively.
The Packing Problem is then to evaluate the size of the maximum independent set in $G(n, c, \alpha)$.

**Definition 1.2** Given a set $V$, a collection $S = S_1, \ldots, S_m$ of subsets of $V$, and a set $\Gamma$ of permutations of $V$, $\kappa(S, \Gamma)$ is defined to be the maximum $k$ such that there exists a permutation $\pi \in \Gamma$ which inverts $k$ members of $S$. If $\Gamma$ is the set of all permutations of $V$, then we write $\kappa(S)$ instead of $\kappa(S, \Gamma)$. Given $S \subset V$ and a set $\Gamma$ of permutations of $V$, $\lambda(S, \Gamma)$ denotes the number of permutations in $\Gamma$ that invert $S$.

## 2 Inverting Subsets of a Given Set

It turns out that there is a simple, necessary, and sufficient condition for a collection $S = \{V; S_1, S_2, \ldots, S_m\}$ to be invertible. We define a bipartite graph $G = G(S)$ with a bipartition $(V_1, V_2)$ as follows. Each of the sets $V_i$ ($i = 1, 2$) is in one-to-one correspondence with $V$; two vertices $i \in V_1$ and $j \in V_2$ are adjacent if and only if no set $S_k$, ($k = 1, \ldots, m$) contains both $i$ and $j$.

**Theorem 2.1** A collection $S = \{V; S_1, \ldots, S_m\}$ is invertible if and only if $G(S)$ has a perfect matching.

**Proof.** If $S$ is invertible and $\pi$ is a permutation which inverts each $S_i$'s, then $\pi$ can also be viewed as the perfect matching of $G$. The reverse is also straightforward. $\blacksquare$

Two immediate corollaries from the theorem above are:

**Corollary 2.1** There is a polynomial algorithm which checks if a collection is invertible, and if it is, outputs an inverting permutation. $\blacksquare$

**Corollary 2.2** If sets $\{S_1, \ldots, S_m\}$ are disjoint subsets of $V$, then $S$ is invertible if and only if for every $i = 1, \ldots, m$, $|S_i| \leq |V|/2$.

**Proof.** Use the previous theorem together with the König condition on bipartite graphs with a perfect matching. $\blacksquare$
Corollary 2.3 If $|V| = 2k$, then $S = \{V, S_1, \ldots, S_m\}$ with $|S_i| = k$ \((i = 1, \ldots, m)\) is invertible if and only if
\[
|\cap_{i \in I} S_i \cap (\cap_{i \in J} \bar{S}_i)| = |\cap_{i \in I} \bar{S}_i \cap (\cap_{i \in J} S_i)|.
\]
for every index set $I \subseteq [1, m]$.

Proof. If $\pi$ inverts $S$, then $\pi^{-1}$ inverts $\{V, \bar{S}_1, \ldots, \bar{S}_m\}$. Thus $|\cap_{i \in I} S_i| = |\cap_{i \in I} \bar{S}_i|$ for any index set $I$. The corollary follows by inclusion-exclusion.

Remark. There are only at most $|V|$ non-empty conditions in Corollary 2.3.

Theorem 2.2 Let $S = \{V; S_1, S_2, S_3\}$ be an invertible collection satisfying $|S_i| = k$ \((i = 1, 2, 3)\). Then $S$ is invertible if and only if
\[
|S_1 \cap S_2 \cap S_3| \leq |\bar{S}_1 \cap \bar{S}_2 \cap \bar{S}_3| \leq |S_1 \cap S_2 \cap S_3| + \frac{3}{2}(|V| - 2k). \quad (*)
\]

Proof. The proof uses inclusion-exclusion to check the conditions of Theorem 2.1. We omit the details.

Corollary 2.4 Let $S$ be a collection of sets such that every element of $V$ belongs to at most two sets of the collection. Then $S$ is invertible if and only if every three sets in $S$ satisfy the condition of $(*)$ of Theorem 2.2.

2.1 Inverting Large Sub-Collections

Our goal is to establish a bound on the number of sets in a given collection $S$ that can be inverted by a single permutation. We prove the existence of a permutation using a standard counting technique. The crucial detail in our case is that we consider a special class of permutations, so called simple permutations. This restriction substantially increases the lower bound. Given $S = \{V; S_1, \ldots, S_m\}$, and a permutation $\pi$ of $V$, we denote $\kappa(S, \pi)$ the number of sets in the collection that are inverted by $\pi$. Then $\kappa(S) = \max_{\pi} \kappa(S, \pi)$. If $\Pi$ is a given class of permutations of $V$, $\lambda(S, \Pi)$ denotes the number of permutations in $\Pi$ that invert the set $S$. A permutation $\pi$ of a set with $n$ elements is called simple, if it has $\lfloor n/2 \rfloor$ disjoint cycles of length two; $\sigma(n)$ denotes the number of simple permutations of a set with $n$ elements.

By extending the proof in [9], we get the following
Lemma 2.1

\[ \sigma(n) = \frac{n!}{2^{\lceil n/2 \rceil} \lceil n/2 \rceil!}. \]

Using Lemma 2.1 and a simple counting argument, we have the following lemma.

Lemma 2.2 Let \( S \) be a subset of \( V \). Then there are

\[ \frac{(n - i)!}{2^{\lceil n/2 - i \rceil} \lceil n/2 - i \rceil!} \]

simple permutations that invert \( S \), where \( i = |S| \leq n/2 \).

Theorem 2.3 Let a collection \( S \) contain \( m_i \) sets of cardinality \( i \) (\( i = 1, \ldots, \lfloor n/2 \rfloor \)). Then

\[ \kappa(S) \geq \frac{\lfloor n/2 \rfloor!}{n!} \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{(n - i)2^i}{\lfloor n/2 - i \rfloor!} m_i. \]

Proof. Let \( \Pi \) be the class of simple permutations of \( V \). Obviously,

\[ \kappa(S) \geq \kappa(S, \Pi) \geq \frac{1}{\sigma(n)} \sum_{\pi \in \Pi} \kappa(S, \pi). \]

On the other hand,

\[ \sum_{\pi \in \Pi} \kappa(S, \pi) = \sum_{S \in \mathcal{S}} \lambda(S, \Pi). \]

Using the two previous lemmas we get the following:

\[ \kappa(S) \geq \frac{2^{\lfloor n/2 \rfloor} \lfloor n/2 \rfloor!}{n!} \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{(n - i)!}{2^{\lfloor n/2 \rfloor - i} \lfloor n/2 - i \rfloor!} m_i = \frac{\lfloor n/2 \rfloor!}{n!} \sum_{i=1}^{\lfloor n/2 \rfloor} 2^i \frac{(n - i)!}{\lfloor n/2 - i \rfloor!} m_i. \]

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Corollary 2.5 If there exists $\epsilon > 0$ such that for all $i = 1, \ldots, \lfloor n/2 \rfloor$, $m_i \geq (1 - \epsilon)^i (n)_i$, then

$$\kappa(S) \geq (1 - \epsilon)^{3n/2}.$$ 

Proof. Using the previous theorem,

$$\kappa(S) \geq \frac{(\lfloor n/2 \rfloor)!}{n!}(1 - \epsilon) \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{2^i(n - i)!}{[n/2 - i]!} \times \frac{n!}{i!(n - i)!}$$

$$= (1 - \epsilon) \sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n/2}{i} 2^i = (1 - \epsilon)^{3n/2}. \quad \Box$$

Corollary 2.6 There exists a sequence $\{M_n\}$ of square-blocking sets in the hypercube $Q_n$, such that $|M_{n+1}| \leq 2|M_n| + 2^n - f2^{n/3}$ where $f$ is a constant.

Proof. Using the previous results, one can construct square-blocking sets in a $Q_n$ that save $O(2^{n/3})$ edges of the hypercube.

Remark. Erdős asks:
Given $n > 0$, what is the largest $m$ such that there exists an invertible collection $S = \{V; S_1, \ldots, S_m\}$ with $|V| = n$?

2.2 Set Collections with No Three Invertible

Given that $n$ is even, what is the largest number of subsets of $[n]$ of size $n/2$ such that no three are invertible? It turns out that such collections can be exponentially large.

Lemma 2.3 Let $k < n/2$, $K = [1, n/2 - k]$, and let $\mathcal{P} = \{R_i\}$ be a collection of $k$-subsets of $[1, n] - K$ such that the intersection of any two of them contain $< k/3$ elements. Then for the collection $S = \{K \cup R_i\}$, no sub-collection of size three is invertible.

Proof. If $S_i = K \cup R_i$ ($i = 1, 2, 3$) is a collection of three sets from $S$, then

$$\left| \bigcap_{i=1}^{3} S_i \right| = \frac{n}{2} - k + \left| \bigcap_{i=1}^{3} R_i \right|, \text{ and } \left| \bigcap_{i=1}^{3} \bar{S}_i \right| = \frac{n}{2} + k - \left| \bigcup_{i=1}^{3} R_i \right|. \quad 8$$
On the other hand, since \(|\bigcup_{i=1}^3 R_i| = 3k - \sum_{i,j} |R_i \cap R_j| + |\bigcap_{i=1}^3 R_i| > 2k + |\bigcap_{i=1}^3 R_i|\), we see that the necessary condition of invertibility from Theorem 2.2 does not hold for the collection \(S_1, S_2, S_3\).

Thus, any packing \(\Pi(n/2, c, 1/s)\) with exponentially many blocks implies the existence of an exponentially large collection of sets such that no three of them are invertible.

3 Packing with Unbounded Blocks

3.1 Lower Bounds for the Packing Problem

While there is a vast literature devoted to the packing problem with bounded block sizes (see [6] for references), there has been relatively modest progress in the area of packing with unbounded block sizes. As noted in the introduction, the packing problem \(\Pi(n, c, \alpha)\) is equivalent to evaluating the maximal size of an independent set in the graph \(\mathcal{G}(n, c, \alpha)\). Our first bound follows from Turán’s theorem.

**Theorem 3.1** (Turán [14]). Every graph with \(N\) vertices and average degree \(D\) contains an independent set of size \(\geq N/(D + 1)\).

Below, we use \(N\) and \(D\) to denote the vertex number and vertex degree of the graph \(\mathcal{G}(n, c, \alpha)\); \(S(c)\) denotes \(c^{-c}(1 - c)^{-1+c}\) for a given \(c(0 < c < 1)\).

**Lemma 3.1** There exists \(A > 0\) such that

\[
\binom{n}{cn} \asymp \frac{A}{\sqrt{n}} S^n(c).
\]

**Proof.** Use the Stirling formula.

**Lemma 3.2** If \(\alpha > c\), then there exists \(q > 1\) such that for every \(i > \alpha cn\),

\[
\binom{cn}{i} \binom{n-cn}{n-cn-i} \binom{cn}{i-1} \binom{n-cn}{n-cn+i+1} > q.
\]
Proof. The following transformations are readily checked.

\[
\frac{(\binom{cn}{i})(\binom{n-cn}{i})}{(\binom{n}{i+1})(\binom{n-cn}{i+1})} = \frac{(i+1)!(cn-i-1)!^2(n-2cn+i+1)!}{i!(cn-i)!^2(n-2cn+i)!}
\]

\[
= \frac{(i+1)(n-2cn+i+1)}{(cn-i)^2} > \frac{i(n-2cn+i)}{(cn-i)^2} > \frac{\alpha(1-2c+\alpha c)}{c(1-\alpha)^2} > 1.
\]

The last inequality is equivalent to \(\alpha > c\). \(\blacksquare\)

Lemma 3.3 Let \(\alpha > c\). Then there exists positive constants \(A'\) and \(A''\) such that

\[
N \asymp \frac{A'}{\sqrt{n}} \frac{1}{c^{cn}(1-c)(1-c)n};
\]

\[
D \asymp \frac{A''}{n} \frac{(1-c)^{(1-c)n}}{\alpha^{cn}(1-\alpha)^{2(1-\alpha)cn}c^{(1-\alpha)n}(1-2c+\alpha c)(1-2c+\alpha c)n}.
\]

Proof. The asymptotic for \(N\) follows directly from Lemma 3.1. From the definition of the graph \(G(n, c, \alpha)\), \(D = \sum_{i \geq \alpha cn} \binom{cn}{i} \binom{n-cn}{cn-\alpha cn}\). Then by Lemma 3.2, the first term of the summation is the largest, and every other term is at least a constant smaller than the previous. Thus, up to a constant,

\[
D = \left(\binom{cn}{\alpha cn}\right) \binom{n-cn}{cn-\alpha cn}.
\]

Using Lemma 3.1 again, we have

\[
D = \frac{1}{n} \left(\frac{1}{\alpha^{n}(1-\alpha)^{1-n}}\right)^{cn} \left(\binom{n}{1-\alpha}\frac{1}{1-\alpha} \frac{1}{1-c} \frac{1}{1-\alpha} \frac{1}{1-c} \frac{1}{1-\alpha} \frac{1}{1-c} \frac{1}{1-\alpha} \frac{1}{1-c} \frac{1}{1-\alpha} \frac{1}{1-c}\right)^{1-c}
\]

\[
= \frac{1}{n} \alpha^{cn} (1-\alpha)^{1-\alpha(cn)(1-\alpha)n-(1-\alpha)cn}
\]

\[
= \frac{1}{n} \alpha^{cn} (1-\alpha)^{2(1-\alpha)cn} c^{(1-\alpha)cn}(1-2c+\alpha c)(1-2c+\alpha c)n
\]

\(\blacksquare\)

Theorem 3.2 Let

\[
T(n, c, \alpha) = \frac{\alpha^{cn} (1-\alpha)^{2(1-\alpha)cn} (1-2c+\alpha c)^{(1-2c+\alpha c)n}}{c^{cn}(1-c)^{2(1-c)n}}.
\]

Then, if \(\alpha > c\), then there is a packing \(\Pi(n, c, \alpha)\) with at least \(T(n, c, \alpha)\) blocks. \(\blacksquare\)
The next theorem shows how to compute the value of $c$ which maximizes $T(n,c,\alpha)$ for a given $\alpha$.

**Theorem 3.3** Given $\alpha > 0$, the value of $c$ which maximizes $T(n,c,\alpha)$ is the positive root of the following equation

$$\alpha^a(1-\alpha)^{2(1-\alpha)}(1-c)^2 = c^\alpha(1-2c+\alpha c)^{2-\alpha}.$$  

**Proof.** Let $f(c) = \log(T(n,c,\alpha))/n$. We have

$$f(c) = \alpha c \log(\alpha/c) + 2(1-\alpha)\log(1-\alpha) + (\alpha-2)\log(1-2c+\alpha c)$$

$$+ 2\log(1-c) + \log(1-2c+\alpha c) - 2\log(1-c) - \alpha c \log(c).$$

Isolating the terms that are multiples of $c$, we get

$$f(c) = c(\alpha \log(\alpha/c) + 2(1-\alpha)\log(1-\alpha) + (\alpha-2)\log(1-2c+\alpha c)$$

$$+ 2\log(1-c) + \log(1-2c+\alpha c) - 2\log(1-c) - \alpha c \log(c).$$

Note that if $c = \alpha$ or as $c \to 0$ the logarithm is 0. Thus the maximum is in the range $0 < c < \alpha$. After differentiation and simplification,

$$f'(c) = (\alpha \log(\alpha/c) + 2(1-\alpha)\log(1-\alpha)$$

$$+ (\alpha-2)\log(1-2c+\alpha c) + 2\log(1-c)).$$

We can rewrite this as

$$f'(c) = \log \left( \frac{\alpha^a(1-\alpha)^{2(1-\alpha)}(1-c)^2}{c^\alpha(1-2c+\alpha c)^{2-\alpha}} \right).$$

Thus $f'(c) = 0$ when

$$\alpha^a(1-\alpha)^{2(1-\alpha)}(1-c)^2 = c^\alpha(1-2c+\alpha c)^{2-\alpha}.$$  

**Corollary 3.1** For any particular $\alpha$ we can find the optimum $c$ by solving the equation above numerically. For example, if $\alpha = 1/3$, then the optimal value of $c$ is close to 0.082508, which yields the base of the exponent in $T(n,c,\alpha)$ close to 1.0245.
3.2 Explicit Constructions

We would like to construct a family of sets without big intersections, instead of just proving that such a thing exists.

Recall that we are interested in the packing problem \( \Pi(n, c, \alpha) \). Let us assume that \( k = 1/\alpha \) is an integer. One way to recursively construct a packing is to divide the \( n \) elements into \( 2^k \) equal sized disjoint subsets \( A_i \). Now we recursively construct packings \( \Pi_i(n/2^k, c, \alpha/2) \) on each subset \( A_i \). Each set \( S_j \) of the new packing is the union on one set from each of the \( \Pi_i \). We choose the \( S_j \) so that no two of them have more than one set in common. The intersection of two of these sets has size at most

\[
\frac{cn}{2k} + 2k(n/2k)c(\alpha/2) = cn/2 + nca/2 = cn\alpha.
\]

How many sets did we construct? First we need to count the number of \( S_j \) as a function of \( |\Pi_j| \). Clearly the upper bound is \( |\Pi_j|^2 \). In most cases this bound can be achieved. Let \( q = |\Pi_j| \) and assume that there are more than \( 2k \) integers \( a_i \) in the range \( 1 < a_i < q \) such that whenever \( j \neq j' \) the difference \( j - j' \) is relatively prime to \( q \). Number the elements of \( \Pi_j \) from 0 to \( q \). There will be one set \( S_{lm} \) in the constructed family \( \Pi \) for each pair \((l, m)\) satisfying \( 0 \leq l, m < q \). The set \( S_{lm} \) will contain the set numbered \( l \) from \( \Pi_1 \) and the set numbered \( m \) from \( \Pi_2 \). The contribution from \( \Pi_j \) is the set numbered \( l + a_jm \) (provided that \( j > 2 \)).

Now we want to see that two of the \( S_{lm} \) share at most one set. If they don’t, then \( l + a_jm = l' + a_jm' \) and \( l + a_jm = l' + a_jm' \). This is equivalent to \( l - l' = a_j(m' - m) \) and \( l - l' = a_j'(m' - m) \). This means \( a_j(m' - m) = a_j'(m' - m) \), so \( m = m' \) since \( a_j - a_j' \) is not a zero divisor. Therefore, both sets are the same and no two distinct \( S_{lm} \) have two or more sets from the \( \Pi_j \) in common.

Let \( F(n, c, \alpha) \) represent the size of the constructed family as well as the family itself. As long as we avoid the base case, we have

\[
F(n, c, \alpha) = (F(n\alpha/2, c, \alpha/2))^2.
\]

We will say that the base case occurs when \( n\alpha/4 \leq 1 \). We will choose \( c \) so that the base case construction consists of \( n \) sets, each containing one element. For some reason we are lead to conjecture that the solution to this recurrence is

\[
\log(F(n, c, \alpha)) = 2A\sqrt{B \log n + C \log^2 \alpha + D \log \alpha}.
\]
Substituting, we get
\[ 2^{\sqrt{B \log n + C \log^2 \alpha + D \log \alpha}} = 2^{\sqrt{B \log(n \alpha/2) + C \log^2(\alpha/2) + D \log(\alpha/2) + 1}}. \]

Taking logarithms and expanding, we get
\[
A \sqrt{B \log n + C \log^2 \alpha + D \log \alpha} \\
= A \sqrt{B \log n + B \log \alpha - B + C \log^2 \alpha - 2C \log \alpha + 1} \\
+ D \log \alpha - D + 1
\]

If we choose \( B = 1 \) and \( C = 1/2 \), the right side simplifies to
\[
A \sqrt{\log n + (1/2) \log^2 \alpha + D \log \alpha} \\
= A \sqrt{\log n + (1/2) C \log^2 \alpha} \\
+ D \log \alpha - D + 1
\]

Finally, we see that \( D = 1 \) and that we can choose \( A \) to make the base case work. Therefore
\[
F = 2^{O(\sqrt{\log n})}. \quad \Box
\]

### 3.3 Upper Bounds

**Theorem 3.4** For \( c > \alpha \), \( P(n, c, \alpha) \leq \frac{1 - \alpha}{c - \alpha} \).

**Proof.** Let \( S = \{X_1, \ldots, X_m\} \) be an independent subset of \( \mathcal{G}(n, c, \alpha) \). The size of \( \bigcup S \) can be bounded from below by applying the Schwarz inequality to indicator functions of sets as done by Chung and Erdős in [3]:
\[
\left( \sum_{1 \leq i \leq m} |X_i| \right)^2 \leq |\bigcup S| \sum_{1 \leq i, j \leq m} |X_i \cap X_j|.
\]
This yields
\[
(mc)^2 \leq n(m(m - 1) \alpha c n + mc n),
\]
and solving for \( m \) gives \( m \leq \frac{1 - \alpha}{c - \alpha} \). \quad \Box

The next result enables us to use Theorem 3.4 to obtain bounds on \( P(n, c, \alpha) \) for \( c \leq \alpha \).
Theorem 3.5 Let $1 \geq e \geq c$ and $\alpha c \geq d \geq 0$. Then
\[
\binom{(e - d)n}{(e - c)n} P(n, c, \alpha) \leq \binom{n}{cn} P\left(\frac{(e - d)n}{e - d}, \frac{\alpha c - d}{c - d}\right).
\]

Proof. Let $S$ be an independent subset of $G(n, c, \alpha)$. For $U \subset V \subseteq [1, \ldots, n]$ with $|U| = dn$ and $|V| = en$, let $S(U, V) = \{X \in S \mid U \subseteq X \subseteq V\}$. For $X, Y \in S(U, V)$, we have $|(X \setminus U) \cap (Y \setminus U)| < (\alpha c - d)n$. Hence $|S(U, V)| \leq P((e - d)n, \frac{\alpha c - d}{e - d}).$ There are $\binom{n}{cn}$ many choices for $U \subset V \subseteq [1, \ldots, n]$ with $|U| = dn$ and $|V| = en$. Each $X \in S$ is a member of $\binom{(1-c)n}{(e-c)n}$ many $S(U, V)$. This gives
\[
|S| \binom{(1-c)n}{(e-c)n} \binom{cn}{dn} \leq \binom{n}{cn} \binom{en}{dn} P\left(\frac{(e - d)n}{e - d}, \frac{\alpha c - d}{c - d}\right).
\]
The result follows. □

Let $N(c, \alpha) = \frac{1}{c - \alpha}$. Combining Theorems 3.4 and 3.5 yields the following corollary:

Corollary 3.2 If $\frac{\alpha c - d}{e - d} < \frac{d}{e - d}$, then
\[
P(n, c, \alpha) \leq \binom{n}{cn} N\left(\frac{c - d}{e - d}, \frac{\alpha c - d}{c - d}\right) / \binom{(e - d)n}{(e - c)n}.
\]

We can now obtain good asymptotic bounds on $P(n, c, \alpha)$. Let $I(x) = -x \log(x) - (1 - x) \log(1 - x)$. Corollary 3.2 implies
\[
\frac{\log P(n, c, \alpha)}{n} \leq I(e) - (e - d)I\left(\frac{e - c}{e - d}\right) + o(1), \quad (*)
\]
provided that $\frac{\alpha c - d}{e - d} < \frac{d}{e - d}$. Let $B(e, d) = (e - d)I\left(\frac{e - c}{e - d}\right)$. To minimize the bound on $\log P(n, c, \alpha)$, we find the maximum of $B(e, d)$ with the given constraints. Note that the constraints are linear in $d$ and $e$, and $B(e, d)$ is increasing in $e$ and decreasing in $d$. By continuity, the bound of $(*)$ holds for $\frac{\alpha c - d}{e - d} = \frac{d}{e - d}$ and is minimized when this identity holds.
Let $e' = \frac{(1-\alpha)c}{e-c}$ and $d' = \frac{(1-\alpha)c}{e-d}$. In terms of $e'$ and $d'$, the constraint is $1 - d' = \frac{e'}{d'+e'}$, which implies that $e' + d' = 1$. Additional constraints on $e'$ and $d''$ are obtained from the inequalities $1 \geq e' \geq c \geq \alpha c \geq d \geq 0$.

If $d = 0$, then $d' = (1 - \alpha)$. If $e = 1$, then $d' = \frac{1-2c+\alpha}{1-c}$. We now have

$$B(e, d) = \frac{c(1 - \alpha)}{d'(1 - d')} I(d')$$

to be maximized for $1 - \alpha \leq d' \leq \frac{1-2c+\alpha}{1-c}$. The function $\frac{I(d')}{d'(1-d')}$ is given by $-\log(d') - \log(1-d') - \log(1-d') - \log(1-d')$, which is the sum of two convex functions on $(0, 1)$. (To see that $f(x) = -\log(1-x)$ is convex, write $f(x) = \sum_{i \geq 1} x^{n-1}$.)

It follows that $B(e, d)$ is maximized on the boundary. Thus our best asymptotic bounds on $\log P(n, c, \alpha)$ are obtained from

$$\frac{\log P(n, c, \alpha)}{n} \leq I(c) - \frac{c(1 - \alpha)}{d'(1 - d')} I(d') + o(1)$$

with $d' = 1 - \alpha$ or $d' = \frac{1-2c+\alpha}{1-c}$. The value of $d'$ which yields the smaller bound depends on $c$ and $\alpha$. To compare this to the lower bounds obtained earlier, consider $c = 0.0825$ and $\alpha = \frac{1}{3}$. Then $\exp\left(\frac{\log P(n, c, \alpha)}{n}\right) \leq 1.0655 + o(1)$. For $\alpha = \frac{1}{3}$, the largest bound obtained for all $c$ occurs for $c = 0.1476$ and gives $\exp\left(\frac{\log P(n, c, \alpha)}{n}\right) \leq 1.0766 + o(1)$.

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