TEICHMÜLLER STRUCTURES AND DUAL GEOMETRIC GIBBS TYPE MEASURE THEORY FOR CONTINUOUS POTENTIALS

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Abstract. The Gibbs measure theory for smooth potentials is an old and beautiful subject and has many important applications in modern dynamical systems. For continuous potentials, it is impossible to have such a theory in general. However, we develop a dual geometric Gibbs type measure theory for certain continuous potentials in this paper following some ideas and techniques from Teichmüller theory for Riemann surfaces. Furthermore, we prove that the space of those continuous potentials has a Teichmüller structure. Moreover, this Teichmüller structure is a complete structure and is the completion of the space of smooth potentials under this Teichmüller structure. Thus our dual geometric Gibbs type theory is the completion of the Gibbs measure theory for smooth potentials from the dual geometric point of view.

1. Introduction

Starting from the celebrated work of Sinai [34, 35] and Ruelle [30, 31], a mathematical theory of Gibbs states, an important idea originally from physics, became an important research topic in modern dynamical systems. Later, Bowen [5] brought Sinai and Ruelle’s work into the study of Axiom A dynamical systems. Their work finally led to a definition of an SRB measure for a dynamical system. A very important feature of a Gibbs measure (or an SRB measure) is that it is an equilibrium state.

In the original study of Gibbs measures, a potential must be smooth, which means it must be at least $C^\alpha$ for some $0 < \alpha \leq 1$. Later the smoothness condition was relaxed to the summability condition in Walters’ paper [39] (see also [10]) but it is essentially the same as the smooth case. For a long time, I have been interested in a study of a Gibbs type theory for continuous potentials. But this is impossible in general. However, we will show that it is possible for a certain class of continuous potentials if we bring in some ideas and techniques from Teichmüller theory and quasiconformal mapping theory.

Key words and phrases. circle endomorphism, symbolic space, dual symbolic space, dual derivative, dual Gibbs measure, quasisymmetric homeomorphism, symmetric homeomorphism.

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A basic idea in the Teichmüller theory is to use measurable coordinates to view Riemann surfaces. That is, by fixing a Riemann surface, all other Riemann surfaces homeomorphic to this Riemann surface can be viewed from measurable coordinates on this Riemann surface up to isotopy. A fundamental result in the study of Teichmüller theory for Riemann surfaces is the measurable Riemann mapping theorem as we describe below.

A measurable function \( \mu \) on the Riemann sphere \( \hat{\mathbb{C}} \) is called a Beltrami coefficient if its \( L^\infty \)-norm \( k = \| \mu \|_\infty < 1 \). The corresponding equation \( H_\mu H_z = \mu H_z \) is called the Beltrami equation. The measurable Riemann mapping theorem says that the Beltrami equation has a solution \( H \) which is a quasiconformal homeomorphism of \( \hat{\mathbb{C}} \) whose quasiconformal dilatation is less than or equal to \( K = (1 + k)/(1 - k) \). It is called a \( K \)-quasiconformal homeomorphism.

The study of the measurable Riemann mapping theorem has a long history since Gauss considered in the 1820’s the connection with the problem of finding isothermal coordinates for a given surface. As early as 1938, Morrey [28] systematically studied homeomorphic \( L^2 \)-solutions of the Beltrami equation. But it took almost twenty years until in 1957 Bers [3] observed that these solutions are quasiconformal (refer to [27, p. 24]). Finally the existence of a solution to the Beltrami equation under the most general possible circumstance, namely, for measurable \( \mu \) with \( \| \mu \|_\infty < 1 \), was shown by Bojarski [4] and by Ahlfors and Bers [2]. In this generality the existence theorem is sometimes called the measurable Riemann mapping theorem.

In this paper, we will borrow many ideas and techniques in the Teichmüller theory and the quasiconformal mapping theory to develop a Gibbs type measure theory for certain continuous potentials. We will prove that the space of these continuous potentials have Teichmüller structures. We will prove that for such a continuous potential, there is a Gibbs type measure which is an equilibrium state. Some properties about these Gibbs type measure are also studied.

We organize the paper as follows. In §2, we define a uniformly symmetric circle endomorphism and prove three examples. In particular, for the third example, we mention and prove a more general version of the result which we proved in [23]. This more general result (Theorem 1) says that a \( C^1 \) circle endomorphism Hölder conjugate to a topologically expanding circle endomorphism itself is expanding.

In §3, we review some classic results in dynamical systems which eventually imply that there is only one topological model for the dynamics of all circle endomorphisms of the same degree. In the same section, we study the bounded nearby geometric property. The conclusion of this property is that a conjugacy is quasisymmetric. This enables us to define a Teichmüller structure on a space of circle endomorphisms.
In §4, we define the dual symbolic space and geometrical models defined on it which we call dual derivatives.

In §5, we define the Teichmüller space of smooth expanding circle endomorphisms and the Teichmüller space of uniformly symmetric circle endomorphisms. Furthermore, we prove that the first Teichmüller space equals the space of all Hölder continuous dual derivatives and the second Teichmüller space equals the space of all continuous dual derivatives. Moreover, the second one is the completion of the first one under the Teichmüller metric. To prove this result and make this paper self-contained, we state a special case (Theorem 5) of the result about differentiable rigidity, which has been developed in [15, 17, 18, 19] for a more general situation. For the sake of the completeness of this paper, we give a detailed proof. We also state the main result (Theorem 10) in [11] in §6. That is, in §6, we first define an asymptotically conformal circle endomorphism and prove that a circle endomorphism is uniformly symmetric if and only if it is asymptotically conformal. After this, we prove the completion result in the end of §6.

In §7, we prove that the Teichmüller space of uniformly symmetric circle endomorphisms is contractible. In a remark in this section, we also state and give a outline of the proof about the contractility of the space of all $C^{1+\alpha}$ circle expanding circle endomorphisms and the Teichmüller space of smooth expanding circle endomorphisms.

In §8 and §9, we define the linear model for a uniformly symmetric circle endomorphism. We study the relation between the linear model and the dual derivative. We use this relation to set up a one-to-one correspondence between the Teichmüller space of uniformly symmetric circle endomorphisms and the space of all continuous dual derivatives. Furthermore, we give a characterization of a dual derivative.

In §10, we define the maximum distance on the Teichmüller space of uniformly symmetric circle endomorphisms and compare this maximum distance with the Teichmüller distance.

In §11, we give a brief review of the Gibbs measure theory for the smoothness case and define a dual invariant measure. In the same section, we post several questions which we study in this paper. In §12, we give a review of the $g$-measure theory. In §13, we returned to the Gibbs measure theory for the smoothness case but from the dual geometric point of view.

Finally, in §14, we prove the existence of a dual geometric Gibbs type measure for every continuous potential in the Teichmüller space of uniformly symmetric circle endomorphisms. This measure can be viewed as a coordinate structure such that the dynamical system is smooth under this structure. Note that we start from a uniformly symmetric circle endomorphism which may be very singular. Most important, this measure is an equilibrium state. This
result could be served as the role of the Riemann mapping theorem on the dual symbolic space.

In §15, we study values of metric entropy for the Teichmüller space of uniformly symmetric circle endomorphisms. The maximum value of the metric entropy is $\log d$, which is the topological entropy. We prove that the infimum of the metric entropy for the Teichmüller space of uniformly symmetric circle endomorphisms is zero.

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2. Circle endomorphisms

Let $T = \{ z \in \mathbb{C} \mid |z| = 1 \}$ be the unit circle in the complex plane $\mathbb{C}$. Suppose

$$f : T \to T$$

is an orientation-preserving covering map of degree $d \geq 2$. We call it in this paper a circle endomorphism. Suppose

$$h : T \to T$$

is an orientation-preserving homeomorphism. We call it in this paper a circle homeomorphism.

For a circle endomorphism $f$, it has a fixed point. We will assume throughout this paper that $f(1) = 1$.

The universal cover of $T$ is the real line $\mathbb{R}$ with a covering map

$$\pi(x) = e^{2\pi ix} : \mathbb{R} \to T.$$ 

Then every circle endomorphism $f$ can be lifted to an orientation-preserving homeomorphism

$$F : \mathbb{R} \to \mathbb{R}, \quad F(x + 1) = F(x) + d, \quad \forall x \in \mathbb{R}.$$ 

We will assume throughout this paper that $F(0) = 0$. Then there is a one-to-one correspondence between $f$ and $F$. Therefore, we also call such an $F$ a circle endomorphism.

Every orientation-preserving circle homeomorphism $h$ can be lifted to an orientation-preserving homeomorphism

$$H : \mathbb{R} \to \mathbb{R}, \quad H(x + 1) = H(x) + 1.$$
We will assume throughout this paper that $0 \leq H(0) < 1$. Then there is a one-to-one correspondence between $h$ and $H$. Therefore, we also call such an $H$ a circle homeomorphism.

A circle endomorphism $f$ is $C^k$ for $k \geq 1$ if the $k^{th}$-derivative $F^{(k)}$ exists and is continuous. And, furthermore, it is called $C^{k+\alpha}$ for some $0 < \alpha \leq 1$ if $F^{(k)}$ is $\alpha$-Hölder continuous, that is,

$$\sup_{x \neq y \in \mathbb{R}} \frac{|F^{(k)}(x) - F^{(k)}(y)|}{|x - y|^{\alpha}} = \sup_{x \neq y \in [0,1]} \frac{|F^{(k)}(x) - F^{(k)}(y)|}{|x - y|^{\alpha}} < \infty.$$  

A $C^1$ circle endomorphism $f$ is called expanding if there are constants $C > 0$ and $\lambda > 1$ such that

$$(F^n)'(x) \geq C\lambda^n, \quad n = 1, 2, \ldots.$$  

A circle homeomorphism $h$ is called quasisymmetric if there is a constant $M \geq 1$ such that

$$M^{-1} \leq \frac{|H(x + t) - H(x)|}{|H(x) - H(x - t)|} \leq M, \quad \forall x \in \mathbb{R}, \forall t > 0.$$  

Furthermore, it is called symmetric if there is a bounded function $\varepsilon(t) > 0$ for $t > 0$ such that $\varepsilon(t) \to 0^+$ as $t \to 0^+$ and such that

$$\frac{1}{1 + \varepsilon(t)} \leq \frac{|H(x + t) - H(x)|}{|H(x) - H(x - t)|} \leq 1 + \varepsilon(t), \quad \forall x \in \mathbb{R}, \forall t > 0.$$  

**Example 1.** A $C^1$-diffeomorphism of $T$ is symmetric.

However, the class of symmetric homeomorphisms is larger than the class of $C^1$-diffeomorphisms. For example, a symmetric homeomorphism may not necessarily be absolutely continuous.

**Definition 1.** A circle endomorphism $f$ is called uniformly symmetric if there is a bounded function $\varepsilon(t) > 0$ for $t > 0$ such that $\varepsilon(t) \to 0^+$ as $t \to 0^+$ and such that

$$\frac{1}{1 + \varepsilon(t)} \leq \frac{|F^{-n}(x + t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x - t)|} \leq 1 + \varepsilon(t)$$  

for all $x \in \mathbb{R}$, all $t > 0$, and all $n > 0$.

**Example 2.** A $C^{1+\alpha}$, for some $0 < \alpha \leq 1$, circle expanding endomorphism $f$ is uniformly symmetric. Furthermore, $\varepsilon(t) \leqDt^\alpha$ for some constant $D > 0$ and $0 \leq t \leq 1$.

**Proof.** Since $F(x+1) = F(x) + d$, then $F'(x+1) = F'(x)$ is a periodic function. Since $F$ is $C^{1+\alpha}$, we have a constant $C_1 > 0$ such that

$$|F'(x) - F'(y)| \leq C_1|x - y|^{\alpha}, \quad \forall x, y \in \mathbb{R}.$$
Since $F$ is expanding, we have a constant $C_2 > 0$ and $\lambda > 1$ such that
\[
(F^n)'(x) \geq C_2 \lambda^n, \quad \forall x \in \mathbb{R}, \; n > 0.
\]

For any $x, y \in \mathbb{R}$ and $n > 0$, let $x_k = F^{-k}(x)$ and $y_k = F^{-k}(y)$, $0 \leq k \leq n$. Then
\[
\left| \log \frac{(F^{-n})'(x)}{(F^{-n})'(y)} \right| = \left| \log \frac{(F^n)'(y_n)}{(F^n)'(x_n)} \right| \leq \sum_{k=1}^{n} \left| \log F'(x_k) - \log F'(y_k) \right|
\]
\[
\leq \frac{1}{C_2 \lambda} \sum_{k=1}^{n} |F'(x_k) - F'(y_k)| \leq \frac{C_1}{C_2 \lambda} \sum_{k=1}^{n} |x_k - y_k|^\alpha \leq \frac{C_1}{C_2^{1+\alpha} \lambda} \sum_{k=1}^{n} \lambda^{-\alpha k} |x - y|^\alpha.
\]

Let
\[
C = \frac{C_1 \lambda^\alpha}{C_2^{1+\alpha} (\lambda^\alpha - 1) \lambda}.
\]

Then we have the following Hölder distortion property:
\[
e^{-C|x-y|^\alpha} \leq \frac{(F^{-n})'(x)}{(F^{-n})'(y)} \leq e^{C|x-y|^\alpha}, \quad \forall x, y \in \mathbb{R}, \; \forall n > 0.
\]

Furthermore, let
\[
\varepsilon(t) = \begin{cases} 
Ct^\alpha - 1, & 0 < t \leq 1, \\
C - 1, & t > 1.
\end{cases}
\]

Then $\varepsilon(t) > 0$ is a bounded function such that $\varepsilon(t) \to 0$ as $t \to 0^+$ and such that
\[
\frac{1}{1 + \varepsilon(t)} \leq \frac{(F^{-n})'(\xi)}{(F^{-n})'(\eta)} = \frac{|F^{-n}(x + t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x - t)|} \leq 1 + \varepsilon(t)
\]
for all $x \in \mathbb{R}$, all $t > 0$, and all $n > 0$, where $\xi$ and $\eta$ are two numbers in $[0, 1]$. Thus $F$ is uniformly symmetric. Furthermore, one can see that $\varepsilon(t) \leq Dt^\alpha$ for some constant $D > 0$ and $0 \leq t \leq 1$. We proved the example. \(\square\)

**Remark 1.** The uniformly symmetric condition is a weaker condition than the $C^{1+\alpha}$ expanding condition for some $0 < \alpha \leq 1$. For example, a uniformly symmetric circle endomorphism could be totally singular, that is, it could map a set with positive Lebesgue measure to a set with zero Lebesgue measure. But we will see in the rest of the paper, many dynamical aspects, from the dual geometric point of view, of a $C^{1+\alpha}$ expanding circle endomorphism for some $0 < \alpha \leq 1$ will be preserved by a uniformly symmetric circle endomorphism.

Another example of a uniformly symmetric circle endomorphism is a $C^1$ Dini expanding circle endomorphism as follows. Suppose $f$ is a $C^1$ circle endomorphism. The function
\[
\omega(t) = \sup_{|x - y| \leq t} |F'(x) - F'(y)|, \quad t > 0,
\]
is called the modulus of continuity of $F'$. Then $f$ is called $C^1$ Dini if
\[ \int_0^1 \frac{\omega(t)}{t} dt < \infty. \]

Suppose $f$ is a $C^1$ Dini expanding circle endomorphism. Let $C > 0$ and $\lambda > 1$ be two constants such that
\[ (F^n)'(x) \geq C\lambda^n, \quad x \in \mathbb{R}, \quad n \geq 1. \]
Define
\[ \tilde{\omega}(t) = \sum_{n=1}^{\infty} \omega(C^{-1}\lambda^{-n}t). \]
Then
\[ \tilde{\omega}(t) \leq \int_0^\infty \omega(C^{-1}\lambda^{-x}t)dx = \frac{1}{\log \lambda} \int_0^{C^{-1}\lambda^{-1}t} \frac{\omega(y)}{y} dy < \infty \]
for all $0 \leq t \leq 1$ and $\tilde{\omega}(t) \to 0$ as $t \to 0$.

**Example 3.** A $C^1$ Dini circle expanding endomorphism $f$ is uniformly symmetric. Furthermore, $\varepsilon(t) \leq D\tilde{\omega}(t)$ for some constant $D > 0$ and $0 \leq t \leq 1$.

**Proof.** Since $F(x+1) = F(x)+d$, then $F'(x+1) = F'(x)$ is a periodic function. Since $f$ is $C^1$ expanding, there are two constants $C_1 > 0$ and $\lambda > 1$ such that
\[ (F^n)'(x) \geq C_1\lambda^n, \quad \forall x \in \mathbb{R}, \quad n > 0. \]

For any $x, y \in \mathbb{R}$ and $n > 0$, let $x_k = F^{-k}(x)$ and $y_k = F^{-k}(y), 0 \leq k \leq n$. Then
\[ |\log \frac{(F^n)'(x)}{(F^n)'(y)}| = |\log \frac{(F^{-n})'(x)}{(F^{-n})'(y)}| \leq \sum_{k=1}^{n} |\log F'(x_k) - \log F'(y_k)| \leq \frac{1}{C_1\lambda} \sum_{k=1}^{n} \omega(C^{-1}\lambda^{-k}|x-y|). \]
Let $C = 1/(C_1\lambda)$. Then we have the following Dini distortion property:
\[ e^{-C\tilde{\omega}(|x-y|)} \leq \frac{(F^{-n})'(x)}{(F^{-n})'(y)} \leq e^{C\tilde{\omega}(|x-y|)}, \quad \forall x, y \in \mathbb{R}, \quad \forall n > 0. \]
Furthermore, let
\[ \varepsilon(t) = \begin{cases} \frac{e^{C\tilde{\omega}(t)} - 1}{e^{C\tilde{\omega}(1)} - 1}, & 0 < t \leq 1 \\ e^{C\tilde{\omega}(1)} - 1, & t > 1. \end{cases} \]
Then $\varepsilon(t) > 0$ is a bounded function such that $\varepsilon(t) \to 0$ as $t \to 0^+$ and such that
\[ \frac{1}{1 + \varepsilon(t)} \leq \frac{(F^{-n})' (\xi)}{(F^{-n})' (\eta)} = \frac{|F^{-n}(x + t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x - t)|} \leq 1 + \varepsilon(t) \]
for all $x \in \mathbb{R}, t > 0$, and $n > 0$, where $\xi$ and $\eta$ are two numbers in $[0, 1]$. Thus $F$ is uniformly symmetric. Furthermore, we have a constant $D > 0$ such that $\varepsilon(t) \leq D\tilde{\omega}(t)$ for all $0 \leq t \leq 1$. We proved the example. \qed
A circle homeomorphism $h$ is called Hölder if there is a constant $0 < \alpha \leq 1$ such that
\[
\sup_{x \neq y} \frac{|H(x) - H(y)|}{|x - y|^{\alpha}} < \infty.
\]
We say that a circle endomorphism $f$ is Hölder conjugate to another circle endomorphism $g$ if there is a Hölder circle homeomorphism $h$ such that
\[
f \circ h = h \circ g.
\]
A circle endomorphism $g$ is called topologically expanding if there are constants $C > 0$ and $\lambda > 1$ such that
\[
|G^n(x) - G^n(y)| \geq C\lambda^n|x - y|, \quad \forall x, y \in [0, 1], \quad \forall n \geq 0.
\]
Following the proof of our result in [23] about Katok’s conjecture, we have the following more general result.

**Theorem 1.** Suppose that $f$ is a $C^1$ circle endomorphism and suppose that $f$ is Hölder conjugate to a topologically expanding circle endomorphism $g$. Then $f$ itself is expanding.

Combining this theorem and Example 3, we have that

**Example 4.** A $C^1$ Dini circle endomorphism $f$ which is Hölder conjugate a topologically expanding circle endomorphism $g$ is uniformly symmetric.

**Proof of Theorem 1.** The proof of the theorem is almost similar to the proof given in [23]. However, for the sake of the completeness of this paper, we give a detailed proof.

Suppose the degree of $g$ is $d$. Since $f$ is topologically conjugate to $g$, its degree is also $d$. The preimage $g^{-1}(1)$ contains $d$ points and cuts $T$ into $d$ closed intervals $\varpi_{0,g} = \{J_{0,g}, \ldots, J_{d-1,g}\}$. Actually $\varpi_{0,g}$ is a Markov partition in the meaning that
\begin{enumerate}
  \item $T = \bigcup_{k=0}^{d-1} J_{k,g}$,
  \item $J_{i,g}$ and $J_{j,g}$ have disjoint interiors for $0 \leq i \neq j \leq d - 1$,
  \item the restriction of $g$ on the interior of $J_{i,g}$ is one to one for every $0 \leq i \leq d - 1$,
  \item $g(J_{i,g}) = T$ for every $0 \leq i \leq d - 1$.
\end{enumerate}
Thus we can generate a sequence of Markov partitions

\[
\varpi_{n,g} = g^{-n}\varpi_{0,g}
\]
for $n = 1, 2, \ldots$. The set $\varpi_{n,g}$ contains all intervals $J$ such that $g^n : J \to J_{k,g}$ for some $1 \leq k \leq d$ is a homeomorphism.

From [23], we have constants $C_0 > 0$ and $0 \leq \tau_0 < 1$ such that
\[
\max_{J \in \varpi_{n,g}} |J| \leq C_0\tau_0^n, \quad \forall n \geq 0.
\]
Since \( f \) is Hölder conjugate to \( g \), we have a homeomorphism \( h \) satisfying such that \( f \circ h = h \circ g \). Let

\[
\varpi_{n,f} = \{ h(J) \mid J \in \varpi_{n,g} \}.
\]

Then we have a constant \( C_1 > 0 \) and \( \tau_1 = \tau_0^a \) such that

\[
|J| \leq C_1 \tau_1^n, \quad \forall J \in \varpi_{n,f}, \quad \forall n \geq 0.
\]

We use \( I \) to denote the lift interval of \( J \) in the unit interval \([0, 1]\). Given any interval \( J \in \varpi_{n,f}, F^n(I) = [m, m + 1] \) for some integer \( m \geq 0 \). For any \( x, y \in I \),

\[
A_n(x, y) = \log \left( \frac{F^n)'(x)}{F^n)'(y)} \right) = \sum_{i=0}^{n-1} \left( \log F'(F^i(x)) - \log F'(F^i(y)) \right).
\]

Let

\[
a_n = \max_{J \in \varpi_{n,f}} \left\{ \max_{x \in I} \log F'(x) - \min_{x \in I} \log F'(x) \right\}
\]

and

\[
D_n = \sum_{i=1}^{n} a_k \quad \text{and} \quad E_n = e^{-D_n}.
\]

Then

\[
|A_n(x, y)| \leq D_n
\]

since \( F' \) is a periodic function of period 1.

Since \( \log F' \) is uniformly continuous on \([0, 1]\), we have that \( a_n \to 0 \) as \( n \to \infty \). This implies that

\[
\frac{D_n}{n} \to 0 \quad \text{as} \quad n \to \infty
\]

and

\[
\sqrt[n]{E_n} = e^{-\frac{D_n}{n}} \to 1 \quad \text{as} \quad n \to \infty.
\]

Since \( F^n(I) = [m, m + 1] \), by the mean value theorem, we have a point \( y_n \in I \) such that

\[
(F^n)'(y_n) = 1/|J| \geq C_1^{-1} \tau_1^{-n}, \quad \forall n \geq 0.
\]

This implies that

\[
(F^n)'(x) \geq E_n(F^n)'(y_n) \geq E_n C_1^{-1} \tau_1^{-n} = C_1^{-1} \left( \sqrt[n]{E_n} \tau_1^{-1} \right)^n, \quad \forall n \geq 0.
\]

Thus we have constants \( C > 0 \) and \( \lambda > 1 \) such that

\[
(F^n)'(x) \geq C \lambda^n, \quad \forall n \geq 0.
\]

That is, \( f \) is expanding. We proved the theorem. \( \square \)
3. Symbolic space and topological representation

Suppose $f$ is a circle endomorphism of degree $d \geq 2$ with $f(1) = 1$. Consider the preimage $f^{-1}(1)$. As we have seen in the proof of Theorem 1, $f^{-1}(1)$ cuts $T$ into $d$ closed intervals $J_0, J_1, \ldots, J_{d-1}$, ordered by the counter-clockwise order of $T$. Suppose $J_0$ has an endpoint 1. Then $J_{d-1}$ also has an endpoint 1. Let

$$\varpi_0 = \{J_0, J_1, \ldots, J_{d-1}\}.$$ 

Then it is a Markov partition, that is,

i. $T = \bigcup_{k=0}^{d-1} J_k$,

ii. the restriction of $f$ to the interior of $J_i$ is injective for every $0 \leq i \leq d - 1$,

iii. $f(J_0) = T$ for every $0 \leq i \leq d - 1$.

Let $I_0, I_1, \ldots, I_{d-1}$ be the lifts of $J_0, J_1, \ldots, J_{d-1}$ in $[0, 1]$. Then we have that

i) $[0, 1] = \bigcup_{k=0}^{d-1} I_k$,

ii) $F(I_i) = [i, i + 1]$ for every $0 \leq i \leq d - 1$.

Let

$$\eta_0 = \{I_0, I_1, \ldots, I_{d-1}\}.$$ 

Then it is a partition of $[0, 1]$.

Consider the pull-back partition $\varpi_n = f^{-n} \varpi_0$ of $\varpi_0$ by $f^n$. It contains $(d - 1)^n$ intervals and is also a Markov partition of $T$. Intervals $J$ in $\varpi_n$ can be labeled as follows. Let $w_n = i_0 i_1 \cdots i_{n-1}$ be a word of length $n$ of 0’s, 1’s, $\cdots$, and $(d - 1)$’s. Then $J_{w_n} \in \varpi_n$ if $f^k(J_{w_n}) \subset J_{i_k}$ for $0 \leq k \leq n - 1$. Then

$$\varpi_n = \{J_{w_n} \mid w_n = i_0 i_1 \cdots i_{n-1}, i_k \in \{0, 1, \ldots, d - 1\}, k = 0, 1, \ldots, d - 1\}.$$ 

Let $\eta_n$ be the corresponding lift partition of $\varpi_n$ in $[0, 1]$ with the same labelings. Then

$$\eta_n = \{I_{w_n} \mid w_n = i_0 i_1 \cdots i_{n-1}, i_k \in \{0, 1, \ldots, d - 1\}, k = 0, 1, \ldots, d - 1\}.$$ 

Consider the space

$$\Sigma = \prod_{n=0}^{\infty} \{0, 1, \ldots, d - 1\} = \{w = i_0 i_1 \cdots i_k \cdots i_{n-1} \cdots \mid i_k \in \{0, 1, \ldots, d - 1\}, k = 0, 1, \ldots\}$$ 

with the product topology. It is a compact topological space. A left cylinder for a fixed word $w_n = i_0 i_1 \cdots i_{n-1}$ of length $n$ is

$$[w_n] = \{w' = i_0 i_1 \cdots i_{n-1} i'_{n+1} \cdots \mid i'_{n+k} \in \{0, 1, \cdots, d - 1\}, k = 0, 1, \cdots\}$$

All left cylinders form a topological basis of $\Sigma$. We call it the left topology. The space $\Sigma$ with this left topology is called the symbolic space.
For any \( w = i_0i_1 \cdots i_{n-1}i_n \cdots \), let
\[
\sigma(w) = i_1 \cdots i_{n-1}i_n \cdots
\]
be the left shift map. Then \((\Sigma, \sigma)\) is called a symbolic dynamical system.

For a point \( w = i_0 \cdots i_{n-1}i_n \cdots \in \Sigma\), let \( w_n = i_0 \cdots i_{n-1} \). Then
\[
\cdots \subset J_{w_n} \subset J_{w_{n-1}} \subset \cdots \subset J_1 \subset T.
\]

Since each \( J_{w_n} \) is compact,
\[
J_w = \cap_{n=1}^{\infty} J_{w_n} \neq \emptyset.
\]

If every \( J_w = \{x_w\} \) contains only one point, then we define the projection \( \pi_f \) from \( \Sigma \) onto \( T \) as
\[
\pi_f(w) = x_w.
\]

The projection \( \pi_f \) is 1-1 except for a countable set
\[
B = \{ w = i_0i_1 \cdots i_{n-1}1000 \cdots , i_0i_1 \cdots i_{n-1}(d-1)(d-1) \cdots \}.
\]

From our construction, one can check that
\[
\pi_f \circ \sigma(w) = f \circ \pi_f(w), \quad w \in \Sigma.
\]

For any interval \( I = [a,b] \) in \([0,1]\), we use \( |I| = b - a \) to mean its Lebesgue length. Let
\[
\iota_{n,f} = \max_{w_n} |I_{w_n}|,
\]
where \( w_n \) runs over all words of \( \{0,1,\cdots,d-1\} \) of length \( n \).

Two circle endomorphisms \( f \) and \( g \) are topologically conjugate if there is an orientation-preserving circle homeomorphism \( h \) of \( T \) such that
\[
f \circ h = h \circ g.
\]

The following result is first proved by Shub in [32] for \( C^2 \) expanding circle endomorphisms by using the contracting mapping theorem.

**Theorem 2.** Let \( f \) and \( g \) be two circle endomorphisms such that both \( \iota_{n,f} \) and \( \iota_{n,g} \) tend to zero as \( n \to \infty \). Then \( f \) and \( g \) are topologically conjugate if and only if their topological degrees are the same.

**Proof.** The topological conjugacy preserves the topological degree. Thus if \( f \) and \( g \) are topologically conjugate, then their topological degrees are the same.

Now suppose \( f \) and \( g \) have the same topological degree. Then they have the same symbolic space. Since both sets \( J_{w,f} = \{x_w\} \) and \( J_{w,g} = \{y_w\} \) contain only a single point for each \( w \), we can define
\[
h(x_w) = y_w.
\]

One can check that \( h \) is an orientation-preserving homeomorphism with the inverse
\[
h^{-1}(y_w) = x_w.
\]
Therefore, for a fixed degree \(d \geq 2\), there is only one topological model \((\Sigma, \sigma)\) for dynamics of all circle endomorphisms of degree \(d\) with \(\iota_n \to 0\) as \(n \to \infty\).

**Definition 2.** The sequence \(\{\varpi_n\}_{n=0}^\infty\) of nested partitions of \(T\) is said to have bounded nearby geometry if there is a constant \(C > 0\) such that for any \(n \geq 0\) and any two intervals \(I, I' \in \eta_n\) with a same endpoint or one has an endpoint 0 and the other has an endpoint 1 (in which case we say they have a common endpoint by modulo 1),

\[
C^{-1} \leq \frac{|I'|}{|I|} \leq C.
\]

The sequence \(\{\varpi_n\}_{n=0}^\infty\) of nested partitions of \(T\) is said to have bounded geometry if there is a constant \(C > 0\) such that

\[
\frac{|L|}{|I|} \geq C, \quad \forall L \subset I, \; L \in \eta_{n+1}, \; I \in \eta_n, \quad \forall n \geq 0.
\]

The bounded nearby geometry implies the bounded geometry since each interval \(I \in \eta_n\) is divided into \(d\) subintervals in \(\eta_{n+1}\). But it is not true for the other direction.

**Theorem 3.** Suppose \(f\) is a uniformly symmetric circle endomorphism. Then the sequence \(\{\varpi_n\}_{n=0}^\infty\) of nested partitions of \(T\) has bounded nearby geometry and thus bounded geometry.

**Proof.** Let \(F\) with \(F(0) = 0\) be the lift of \(f\). Define

\[G_k(x) = F^{-1}(x + k) : [0, 1] \to [0, 1], \quad \text{for} \quad k = 0, 1, \ldots, n - 1.\]

For any word \(w_n = i_0 i_1 \cdots i_{n-1}\), define

\[G_{w_n} = G_{i_0} \circ G_{i_1} \circ \cdots \circ G_{i_{n-1}}.\]

Then

\[I_{w_n} = G_{w_n}([0, 1]) = F^{-n}([m, m + 1]),\]

where \(m = i_{n-1} + i_{n-2}d + \cdots + i_0 d^{n-1}\). Suppose \(I'_{w_n}\) is an interval in \(\eta_n\) having a common endpoint with \(I_{w_n}\) modulo 1. Then

\[I'_{w_n} = F^{-n}([m + 1, m + 2]) \text{ or } F^{-n}([m - 1, m]).\]

Thus

\[
\frac{1}{1 + \varepsilon(1)} \leq \frac{|I_{w_n}|}{|I'_{w_n}|} \leq 1 + \varepsilon(1).
\]

Let \(C = 1 + \varepsilon(1)\). Then we have that

\[C^{-1} \leq \frac{|I|}{|I'|} \leq C.\]
for any intervals $I, I' \in \eta_n$ with a common endpoint modulo 1, $n = 0, 1, \cdots$. This means that $\{\pi_n\}_{n=0}^\infty$ has the bounded nearby geometry. We proved the theorem. \hfill \square

**Corollary 1.** Any two uniformly symmetric circle endomorphisms $f$ and $g$ of the same degree $d \geq 2$ are topologically conjugate and the conjugacy is a quasisymmetric homeomorphism.

**Proof.** From $f \circ h = h \circ g$ and $g(1) = 1$, $h(1)$ is a fixed point of $f$, that is, $f(h(1)) = h(1)$. Let $k(z) = z/h(1)$ and $\tilde{f} = k \circ f \circ k^{-1}$. Then $\tilde{f}(1) = 1$. Take $\tilde{h} = k \circ h$. We have that $\tilde{h}(1) = 1$ and $\tilde{f} \circ \tilde{h} = \tilde{h} \circ g$. So $\tilde{h}$ is quasisymmetric if and only if $h$ is quasisymmetric. So, without loss of generality, we assume that $h(1) = 1$.

Suppose

$$\eta_{n, f} = \{I_{w_n, f}\} \text{ and } \eta_{n, g} = \{I_{w_n, g}\}, \quad n = 1, 2, \cdots$$

are two sequences of Markov partitions for $f$ and $g$, respectively.

From the bounded geometry property (Theorem 3), we have a constant $0 < \tau < 1$ such that

$$\varsigma_{n, f} = \max_{w_n} |I_{w_n, f}|, \quad \varsigma_{n, g} = \max_{w_n} |I_{w_n, g}| \leq \tau^n, \quad \forall \ n = 1, 2, \cdots$$

Then Theorem 2 implies that $f$ and $g$ are topologically conjugate.

Suppose $h$ is the topological conjugacy between $f$ and $g$ and $H$ is its lift to $\mathbb{R}$. By adding all integers, the sequence of partitions $\eta_{n, f}$ and $\eta_{n, g}$ induce two sequences of partitions of $\mathbb{R}$, which we still denoted as $\eta_{n, f}$ and $\eta_{n, g}$. Both of these sequences of partitions have bounded nearby geometry.

Let $\Omega$ be the set of all endpoints of intervals $I \in \eta_n$, $n = 0, 1, \cdots, \infty$. Then it is dense in $\mathbb{R}$.

For $x \in \Omega$. Consider the interval $[x - t, x]$. There is a largest integer $n \geq 0$ such that there is an interval $I = [a, x] \in \eta_{n, f}$ satisfying $[x - t, x] \subseteq I$. Suppose $J = [b, x] \in \eta_{n+1, f}$. Then $J \subseteq [x - t, x]$. Let $J' = [x, c] \in \eta_{n+1, f}$. From Theorem 3 there is a constant $C > 0$ such that

$$C^{-1} \leq \frac{|J'|}{|J|} \leq C.$$

If $|J'| > t$, we have $|J'| \leq Ct$. Let $J'_k = [x, c_k] \in \eta_{n+k+1, f}$ for $k > 0$. From the bounded geometry, there is a $0 < \tau < 1$ such that

$$|J'_k| \leq \tau^k Ct.$$

Let $n$ be the smallest integer greater than $-\log C / \log \tau$. Then $|J'_k| \leq t$. This implies that $J'_k \subseteq [x, x + t]$. So we have

$$\frac{|H(J'_k)|}{|H(I)|} \leq \frac{|H(x + t) - H(x)|}{|H(x) - H(x - t)|} \leq \frac{|H(J')|}{|H(J)|},$$
where \( H(I) \in \eta_{n,g} \), \( H(J), H(J') \in \eta_{n+1,g} \), and \( H(J'_k) \in \eta_{n+k+1,g} \). Now from the bounded geometry for \( g \), we have a constant, still denote as \( C > 0 \), such that
\[
C^{-1} \leq \frac{|H(J'_k)|}{|H(J)|} \leq \frac{|H(x + t) - H(x)|}{|H(x) - H(x - t)|} \leq \frac{|H(J)|}{|H(J)|} \leq C.
\]

If \( |J'| \leq t \), we have \( |J'| \geq C^{-1} t \). Let \( J'_k = [x, c_{-k}] \in \eta_{n-k+1,f} \) for \( k \geq 0 \).
Then from the bounded geometry, there is a constant, which we still denote as \( 0 < \tau < 1 \), such that \( |J'_k| \geq \tau^{-k} C^{-1} t \). Let \( k \) be the smallest integer greater than \(-\log C/\log \tau \). Then \( |J'_k| \geq t \). This implies that \( J'_k \supseteq [x, x + t] \). So we have
\[
\frac{|H(J')|}{|H(J)|} \leq \frac{|H(x + t) - H(x)|}{|H(x) - H(x - t)|} \leq \frac{|H(J'_k)|}{|H(J)|},
\]
where \( H(I) \in \eta_{n,g}, H(J), H(J') \in \eta_{n+1,g}, \) and \( H(J'_k) \in \eta_{n-k+1,g} \). Now from the bounded geometry for \( g \), we have a constant, which we still denote as \( C > 0 \), such that
\[
C^{-1} \leq \frac{|H(J)|}{|H(I)|} \leq \frac{|H(x + t) - H(x)|}{|H(x) - H(x - t)|} \leq \frac{|H(J'_k)|}{|H(J)|} \leq C.
\]

For any \( x \in \mathbb{R} \), since \( \Omega \) is dense in \([0,1]\), we have a sequence \( x_n \in \Omega \) such that \( x_n \to x \) as \( n \to \infty \). For any \( t > 0 \), we have that
\[
C^{-1} \leq \frac{|H(x_n + t) - H(x_n)|}{|H(x_n) - H(x_n - t)|} \leq C.
\]

Since \( H \) is uniformly continuous on \( \mathbb{R} \), we get that
\[
C^{-1} \leq \frac{|H(x + t) - H(x)|}{|H(x) - H(x - t)|} \leq C.
\]

We proved the theorem. \( \Box \)

Remark 2. The bounded nearby geometry and the quasisymmetric property for a conjugacy have been also studied for one-dimensional maps with critical points in [15, 16, 20].

4. Dual symbolic space and geometric representation

Suppose \( f \) is a circle endomorphism. Suppose \( \{\eta_n\}_{n=0}^{\infty} \) is the sequence of partitions of \([0,1]\). As we have seen in the previous section, for each interval in \( \eta_n \), there is a labeling \( w_n = i_0 i_1 \cdots i_{n-1} \). One can think of this kind of labelings as the left topology: read ordered digits from the left to the right. Now we read from the same ordered digits from the right to the left, that is,
\[
w_n^* = j_{n-1} \cdots j_1 j_0
\]
where \( j_{n-1} = i_0, \cdots, j_1 = i_{n-2}, \) and \( j_0 = i_{n-1} \). Thus we consider the dual symbolic space
\[
\Sigma^* = \{w^* = \cdots j_{n-1} \cdots j_k \cdots j_1 j_0 \mid j_k \in \{0,1,\cdots,d-1\}, k = 0,1,\cdots\}
\]
equipped with the right topology which is generated by all right cylinders
\[ [w_n^*] = \{ w^* = j_n j_{n-1} \cdots j_1 j_0 \mid j'_k \in \{0, 1, \cdots, d-1\}, k = 0, 1, \cdots \}, \]
where \( w_n^* = j_{n-1} \cdots j_1 j_0 \) is a fixed word of \( \{0, 1, \cdots, d-1\} \) of length \( n \).

Consider the right shift map
\[ \sigma^*: w^* = \cdots j_{n-1} \cdots j_1 j_0 \rightarrow \sigma^*(w^*) = \cdots j_{n-1} \cdots j_1. \]
Then we call \( (\Sigma^*, \sigma^*) \) the dual symbolic dynamical system for \( f \).

The dual derivative of \( f \) is defined on the dual symbolic space \( \Sigma^* \) as follows. For any \( w^* = \cdots j_{n-1} \cdots j_1 j_0 \in \Sigma^* \), let
\[ w_n^* = j_{n-1} \cdots j_1 j_0 \quad \text{and} \quad v_{n-1}^* = \sigma^*(w_n^*) = j_{n-1} \cdots j_1. \]
Then
\[ I_{w_n} \subset I_{v_{n-1}}. \]
Define
\[ (5) \quad D^*(f)(w_n^*) = \frac{|I_{v_{n-1}}|}{|I_{w_n}|}. \]

**Definition 3.** If for every \( w^* \in \Sigma^* \),
\[ D^*(f)(w^*) = \lim_{n \to \infty} D^*(f)(w_n^*) \]
eexists, then we define a function
\[ (6) \quad D^*(f)(w^*) : \Sigma^* \to \mathbb{R}^+. \]
We call this function the dual derivative of \( f \).

**Remark 3.** We used to call one divided by a dual derivative a scaling function. The notion of the scaling function is first introduced into the study of geometric Cantor sets on the line by Sullivan in [36] where a scaling function is used to define differentiable structures of geometric Cantor sets on the line. A general version of scaling functions for any Markov maps is defined in [15] (see also [16]). This general notion of scaling function has been used extensively in [17, 18, 19] as a complete smooth invariant in the smooth classification of one-dimensional maps with critical points. Since a circle endomorphism of degree \( d \geq 2 \) can be thought as a Markov map, we use the definition in [15] (see also [16]). However, the notion of the dual derivative is a more appropriate term in this paper for the study of dual geometric Gibbs measure theory.

A function \( \phi^*(w^*) \) on \( \Sigma^* \) is called Hölder continuous if there are constants \( C > 0 \) and \( 0 < \tau < 1 \) such that
\[ |\phi^*(w^*) - \phi^*(\tilde{w}^*)| \leq C \tau^n \]
as long as the first $n$ digits of $w^*$ and $\tilde{w}^*$ from the right are the same. If we consider a metric
\[ d(w^*, \tilde{w}^*) = \sum_{k=0}^{\infty} \frac{|j_k - j'_k|}{d^k} \]
on $\Sigma^*$, then $\phi^*(w^*)$ being Hölder continuous is equivalent to the condition that there are two constants $C > 0$ and $0 < \beta \leq 1$ such that
\[ |\phi^*(w^*) - \phi^*(\tilde{w}^*)| \leq C (d(w^*, \tilde{w}^*))^\beta, \quad w, \tilde{w}^* \in \Sigma^*. \]

**Theorem 4.** Suppose $f$ is a uniformly symmetric circle endomorphism. Then its dual derivative
\[ D^*(f)(w^*) : \Sigma^* \to \mathbb{R}^+ \]
exists and is a continuous function. Furthermore, if $f$ is $C^{1+\alpha}$, then $D^*(f)(w^*)$ is Hölder continuous. Actually when $f$ is $C^1$ Dini expanding, the modulus of continuity of $D^*(f)(w^*)$ is controlled by $\tilde{\omega}(t)$.

We first prove the following lemma. Suppose $Q : [0, 1] \to [0, 1]$ is a homeomorphism such that $Q(0) = 0$ and $Q(1) = 1$. Let $M \geq 1$. We say that $Q$ is $M$-quasisymmetric on $[0, 1]$ if
\[ M^{-1} \leq \frac{|Q(x+t) - Q(x)|}{|Q(x) - Q(x-t)|} \leq M, \quad \forall x - t, x, x + t \in [0, 1], t > 0. \]

**Lemma 1.** There is a function $\zeta(M) > 0$ satisfying $\zeta(M) \to 0$ as $M \to 1$ such that for any $M$-quasisymmetric homeomorphism $Q$ on $[0, 1]$ such that $Q(0) = 0$ and $Q(1) = 1$,
\[ |Q(x) - x| \leq \zeta(M), \quad \forall x \in [0, 1]. \]

**Proof.** Consider points $x_n = 1/2^n$, $n = 0, 1, \cdots$. $M$-quasisymmetry and the normalization $Q(0) = 0, Q(1) = 1$ imply that
\[ \frac{1}{1 + M} \frac{1}{2^{n-1}} \leq Q(\frac{1}{2^n}) \leq \frac{1}{1 + M^{-1}} Q(\frac{1}{2^n}). \]
Similarly,
\[ \left( \frac{1}{1 + M} \right)^n \leq Q(\frac{1}{2^n}) \leq \left( \frac{1}{1 + M^{-1}} \right)^n, \quad \forall n \geq 1. \]
Furthermore, by $M$-quasisymmetry and induction on $n = 1, 2, \cdots$, yield
\[ \left( \frac{1}{1 + M} \right)^n \leq Q(\frac{i}{2^n}) - Q(\frac{i-1}{2^n}) \leq \left( \frac{1}{1 + M^{-1}} \right)^n, \quad \forall n \geq 1, \quad 1 \leq i \leq 2^n. \]
Let
\[ \tau_n = \max \left\{ \left( \frac{M}{M + 1} \right)^n - \frac{1}{2^n}, \frac{1}{2^n} - \left( \frac{1}{M + 1} \right)^n \right\}, \quad n = 1, 2, \cdots. \]
Then for $n = 1$,
\[ |Q(\frac{1}{2}) - \frac{1}{2}| \leq \tau_1 = \frac{1}{2M + 1}, \]
and for any \( n > 1 \), we have
\[
\max_{0 \leq i \leq 2^n} |Q\left(\frac{i}{2^n}\right) - \frac{i}{2^n}| \leq \max_{0 \leq i \leq 2^{n-1}} |Q\left(\frac{i}{2^{n-1}}\right) - \frac{i}{2^{n-1}}| + \tau_n
\]
By summing over \( k \) for \( 1 \leq k \leq n \), we obtain
\[
\max_{0 \leq i \leq 2^n} |Q\left(\frac{i}{2^n}\right) - \frac{i}{2^n}| \leq \delta_n = \sum_{k=1}^{n} \tau_k.
\]
If we put \( \zeta(M) = \sup_{1 \leq n < \infty} \{\delta_n\} \), by summing geometric series, we obtain
\[
\zeta(M) = \max_{1 \leq n < \infty} \left\{ M - 1 + \frac{1}{2^n} - M\left(\frac{1}{1 + M}\right)^n, 1 - \frac{1}{M} + \frac{1}{M}\left(\frac{1}{M}\right)^n - \frac{1}{2^n} \right\}.
\]
Clearly, \( \zeta(M) \to 0 \) as \( M \to 1 \), and since the dyadic points
\[\{i/2^n \mid n = 1, 2, \cdots ; 0 \leq i \leq 2^n\}\]
are dense in \([0, 1]\), we conclude
\[|Q(x) - x| \leq \zeta(M) \quad \forall \ x \in [0, 1],\]
which proves the lemma.

**Proof of Theorem [4]** Suppose \( w^* = \cdots j_{n-1} \cdots j_1 j_0 \in \Sigma^* \). Let
\[w_n^* = j_{n-1} \cdots j_1 j_0 \quad \text{and} \quad v_{n-1}^* = j_{n-1} \cdots j_1.\]
By definition,
\[D^*(f)(w_n^*) = \frac{|I_{v_{n-1}}|}{|I_{w_n}|},\]
where \( I_{w_n} \subset I_{v_{n-1}} \). Consider the sequence \( \{D^*(f)(w_n^*)\}_{n=1}^{\infty} \).
Let \( 0 < \tau < 1 \) be a constant such that
\[\tilde{\epsilon}_n = \max_{w_n} |I_{w_n}| \leq \tau^n, \quad \forall n \geq 1.\]
For any \( \epsilon > 0 \), let \( n_0 > 0 \) be an integer such that \( \zeta(1 + \epsilon(\tau^{n-1})) \leq \epsilon \) for all \( n > n_0 \). Then for any \( m > n > n_0 \), we have that
\[F^{-m-n}(I_{v_{n-1}}) = I_{v_{n-1}} \quad \text{and} \quad F^{-m-n}(I_{w_m}) = I_{w_n}\]
Since \( F^{-m-n}I_{v_{n-1}} \) is a \((1 + \epsilon(\tau^{n-1}))\)-quasisymmetric homeomorphism, from Lemma [4] (by normalizing \( I_{v_{n-1}} \) to \([0, 1]\) and \( I_{w_m} \) to \([0, x]\) by a linear transformation),
\[|D^*(f)(w_m^*) - D^*(f)(w_n^*)| = \frac{|F^{-m-n}(I_{v_{n-1}})|}{|F^{-m-n}(I_{w_n})|} \cdot \frac{|I_{v_{n-1}}|}{|I_{w_n}|} \leq \zeta(1 + \epsilon(\tau^{n-1})) \leq \epsilon.\]
This implies that \( \{D^*(f)(w_n^*)\}_{n=1}^{\infty} \) is a Cauchy sequence. Thus the limit
\[D^*(f)(w^*) = \lim_{n \to \infty} D^*(f)(w_n^*)\]
exists.
Now consider two points 

\[ w^* \cdots j_{m-1} \cdots j_n j_{n-1} \cdots j_0 \quad \text{and} \quad \tilde{w}^* = \cdots j_{m-1} \cdots j'_n j_{n-1} \cdots j_0. \]

Let \( w^*_m = j_{m-1} \cdots j_n j_{n-1} \cdots j_0 \) and \( \tilde{w}^*_m = j_{m-1} \cdots j'_n j_{n-1} \cdots j_0. \) Then \( w^*_n = \tilde{w}^*_n. \) For any \( m > n, \)

\[
|D^*(f)(w^*_m) - D^*(f)(\tilde{w}^*_m)| \\
\leq |D^*(f)(w^*_m) - D^*(f)(w^*_n)| + |D^*(f)(\tilde{w}^*_m) - D^*(f)(w^*_n)| \leq 2\zeta(1 + \varepsilon(\tau^{n-1})).
\]

So by taking a limit,

\[
|D^*(f)(w^*) - D^*(f)(\tilde{w}^*)| \leq 2\zeta(1 + \varepsilon(\tau^{n-1})).
\]

Thus we have that

\[
D^*(f)(w^*) : \Sigma^* \to \mathbb{R}^+
\]

is a continuous function whose modulus of continuity is bounded by \( 2\zeta(1 + \varepsilon(\tau^{n-1})). \)

Moreover, if \( f \) is a \( C^{1+\alpha} \) expanding circle endomorphism for some \( 0 < \alpha \leq 1, \)
from the Hölder distortion property \( \text{[I]} \), there is a constant \( C > 0 \) such that

\[
|D^*(f)(w^*) - D^*(f)(\tilde{w}^*)| \leq C\tau^\alpha(\tau^{n-1}).
\]

This implies that the dual derivative \( D^*(f)(w^*) \) is Hölder continuous.

When \( f \) is \( C^1 \) Dini, then there is a constant \( C > 0 \) such that

\[
|D^*(f)(w^*) - D^*(f)(\tilde{w}^*)| \leq C\varepsilon(\tau^{n-1}).
\]

Thus the dual derivative \( D^*(f)(w^*) \) is continuous and its modulus of continuity is controlled by \( \varepsilon(\tau^{n-1}) \). We proved the theorem. \( \square \)

5. Teichmüller spaces and dual derivatives

For a fixed integer \( d \geq 2, \) let \( C^{1+} \) be the space of all \( C^{1+\alpha}, \) \( 0 < \alpha \leq 1, \)
expanding circle endomorphisms of degree \( d. \) Take \( q_d(z) = z^d \) as a basepoint in \( C^{1+}. \) A marked \( C^{1+} \) circle endomorphism by \( q_d \) is a pair \((f, h_f)\) where \( f \in C^{1+} \) and \( h_f \) is the orientation-preserving homeomorphism of \( T \) such that \( h_f(1) = 1 \) and

\[
f \circ h_f = h_f \circ q_d.
\]

From Corollary \( \text{[I]} \) for any marked \( C^{1+} \) circle endomorphism \((f, h_f)\) by \( q_d, \)
\( h_f \) is quasisymmetric. Thus we can define Teichmüller equivalence relation \( \sim_T, \) Teichmüller space, and Teichmüller type metric as follows.

**Definition 4.** Two marked \( C^{1+} \) circle endomorphisms are equivalent, denoted as \((f, h_f) \sim_T (g, h_g)\), if \( h_f \circ h_g^{-1} \) is a \( C^1 \)-diffeomorphism.

**Definition 5.** The Teichmüller space

\[
\mathcal{T}C^{1+} = \{ [([f, h_f]) : f \in C^{1+}, \text{ with the basepoint } ([q_d, id]) \}
\]

is defined as the space of all \( \sim_T \)-equivalence classes \([([f, h_f])\) in the space of all marked \( C^{1+} \) circle endomorphisms by \( q_d. \)
Now let us define the Teichmüller type metric $d_T(\cdot, \cdot)$ on $\mathcal{T}^1$. We first consider the universal Teichmüller space. We refer to \[1, 12, 27\] as standard references for this subject. Let $\mathcal{QS}$ be the set of all quasisymmetric orientation-preserving homeomorphisms of the unit circle $T$ factored by the space of all Möbius transformations of the circle. (Then $\mathcal{QS}$ may be identified with the set of all quasisymmetric orientation-preserving homeomorphisms of the unit circle fixing three points). Let $\mathcal{S}$ be the subset of $\mathcal{QS}$ consisting of all symmetric orientation-preserving homeomorphisms of the unit circle $T$. The space $\mathcal{S}$ is a subgroup of $\mathcal{QS}$ closed in the Teichmüller topology. For any $h \in \mathcal{QS}$, let $\mathcal{E}_h$ be the set of all quasiconformal extensions of $h$ into the unit disk. For each $\tilde{h} \in \mathcal{E}_h$, let

$$\mu_{\tilde{h}} = \frac{\tilde{h}_z}{\tilde{h}_z}$$

be its complex dilatation. Let

$$k_{\tilde{h}} = \|\mu(z)\|_\infty \quad \text{and} \quad K_{\tilde{h}} = \frac{1 + k_{\tilde{h}}}{1 - k_{\tilde{h}}}.$$ 

Here $K_{\tilde{h}}$ is called the quasiconformal dilatation of $\tilde{h}$. Using quasiconformal dilatation, we can define a distance in $\mathcal{QS}$ by

$$d_T(h_1, h_2) = \frac{1}{2} \inf \{ \log K_{\tilde{h}_1, \tilde{h}_2^{-1}} \mid \tilde{h}_1 \in \mathcal{E}_{h_1}, \tilde{h}_2 \in \mathcal{E}_{h_2} \}.$$ 

Here $(\mathcal{QS}, d)$ is called the universal Teichmüller space. It is a complete metric space and a complex manifold with complex structure compatible with the Hilbert transform.

The topology coming from the metric $d_T$ on $\mathcal{QS}$ induces a topology on the factor space $\mathcal{QS} \mod \mathcal{S}$. Given two cosets $\mathcal{S}f$ and $\mathcal{S}g$ in this factor space, define a metric by

$$d_T(f, g) = \inf_{A, B \in \mathcal{S}} d(Af, Bg).$$

The quotient space $\mathcal{QS} \mod \mathcal{S}$ with this metric is a complete metric space and a complex manifold. The topology on $(\mathcal{QS} \mod \mathcal{S}, d_T)$ is the finest topology which makes the projection $\pi : \mathcal{QS} \rightarrow \mathcal{QS} \mod \mathcal{S}$ continuous, and $\pi$ is also holomorphic.

An equivalent topology on the quotient space $\mathcal{QS} \mod \mathcal{S}$ can be defined as follows. For any $h \in \mathcal{QS}$, let $\tilde{h}$ be a quasiconformal extension of $h$ to a small neighborhood $U$ of $T$ in the complex plane. Let

$$\mu_{\tilde{h}} = \frac{\tilde{h}_z}{\tilde{h}_z}, \quad z \in U$$

and

$$k_{\tilde{h}} = \|\mu(z)\|_{\infty, U} \quad \text{and} \quad B_{\tilde{h}} = \frac{1 + k_{\tilde{h}}}{1 - k_{\tilde{h}}}.$$
Then the boundary dilatation $h$ is defined as

$$B_h = \inf_{U,h} B_{\tilde{h}},$$

where the infimum is taken over all quasiconformal extensions $\tilde{h}$ of $h$ in a neighborhood $U$ of $T$. It is known that $h$ is symmetric if and only if $B_h = 1$.

Define

$$\tilde{d}(h_1, h_2) = \frac{1}{2} \log B_{h_1^{-1} h_2}^{-1}.$$

The two metrics $\overline{d}$ and $\tilde{d}$ on $\mathcal{QS}$ mod $\mathcal{S}$ are equal.

The Teichmüller type metric $d_T(\cdot, \cdot)$ on $\mathcal{T}C^{1+}$ is defined similarly as follows. Let $\Pi$ and $\Pi'$ be two points in $\mathcal{T}C^{1+}$. Then

$$d_T(\Pi, \Pi') = \frac{1}{2} \log B_{h^{-1} \circ \tau g},$$

where $\Pi, \Pi' \in \mathcal{T}C^{1+}$ and $(f, h_f) \in \Pi$ and $(g, \tau_g) \in \Pi'$. Since $d_T(\cdot, \cdot)$ is defined by $\tilde{d}(\cdot, \cdot)$, it easy to check it satisfies the symmetric condition and the triangle inequality. If we have that $d_T(\Pi, \Pi') = 0$ if and only if $\Pi = \Pi'$, then $d_T(\cdot, \cdot)$ is indeed a metric. To prove this property, we need the following rigidity result.

Suppose $f, g \in C^{1+}$ are conjugate by an orientation-preserving homeomorphism $h$, that is,

$$f \circ h = h \circ g.$$

If $h$ is differentiable at $p \in T$, then, from the last equation, $h$ is differentiable at all points in

$$\text{BI}(p) = \bigcup_{n=0}^{\infty} f^{-n}(p),$$

the set of all backward images of $p$.

**Definition 6.** We call $h$ differentiable at $p \in T$ with uniform bound if there are a small neighborhood $Z$ of $p$ and a constant $C > 0$ such that

$$C^{-1} \leq |h'(q)| \leq C, \quad q \in \text{BI}(p) \cap Z.$$

**Theorem 5.** Suppose $f, g \in C^{1+}$ are conjugate by an orientation-preserving homeomorphism $h$, that is, $f \circ h = h \circ g$. Then $h$ is a $C^1$-diffeomorphism if and only if $h$ is differentiable at one point with uniform bound.

**Proof.** Note that $h$ is differentiable if and only if its lift $H$ is differentiable. If $H$ is a $C^1$-diffeomorphism, then

$$1 = H(1) - H(0) = \int_0^1 H'(x)dx.$$

So there is at least one point in $[0, 1]$ such that $H'(x) \neq 0$. This is the “only if” part.
To prove the “if” part, suppose $H$ is differentiable at $x_0$ with uniform bound. Let $p_0 = \pi(x_0)$. Then the set of all backward images $BI(p_0)$ of $p_0$ is dense in $T$. The lift set $\tilde{B}I(x_0)$ of $BI(p_0)$ to $[0,1]$ are all points

$$x_{nm} = F^{-n}(x_0 + m), \ n = 0, 1, \cdots, \ m = 0, 1, \cdots, d^n - 1.$$ 

Since

$$H(F^n(x_{nm})) = G^n(H(x_{nm})) \pmod 1,$$

$$H'(x_{nm}) = \frac{H'(x_0)(F^n)'(x_{nm})}{(G^n)'(H(x_{nm}))}.$$ 

So $H$ is differentiable at every point in $\tilde{S}$ with non-zero derivatives. Thus we can take $x_0 \in (0,1)$.

Let

$$x_0 \in \cdots \subset I_{w_k} \subset I_{w_{k-1}} \subset \cdots I_{w_1} \subset [0,1]$$

be a sequence of nested intervals in the sequence of Markov partitions $\{\eta_k\}_{k=0}^\infty$. Assume $I = \overline{Z} = I_{w_0}$ is the closure of a neighborhood of $x_0$ in Definition 6, that is, there is a constant $C_0 > 0$ such that

$$C_0^{-1} \leq |H'(x)| \leq C_0, \ x \in \tilde{B}I(x_0) \cap I.$$ 

Consider the set $S(I)$ of all intervals $J \in \eta_{w_0+k}$ such that $J \subset I$ and $F^k(J) = I \pmod 1$ for $k = 1, 2, \cdots$. Let $\Omega(I)$ be the union of all these intervals. Then, just by the expanding property of $f$, the set $\Omega(I)$ has a full Lebesgue measure in $I$.

For any $J \in S(I)$, $F^k(J) = I \pmod 1$ and $G^k(H(J)) = H(I) \pmod 1$ for some $k \geq 1$. We have

$$\frac{|H(J)|}{|J|} = \frac{(F^k)'(\xi)|H(I)|}{(G^k)'(\eta)|I|}.$$ 

Take $x \in \tilde{B}I(x_0) \cap J$. Then $y = F^k(x) \in \tilde{B}I(x_0) \cap I$ and

$$\frac{(F^k)'(x)}{(G^k)'(H(x))} = \frac{H'(x)}{H'(y)}.$$ 

Thus

$$C_0^{-2} \leq \frac{(F^k)'(x)}{(G^k)'(H(x))} \leq C_0^2.$$ 

This implies

$$C_0^{-2}(F^k)'(x) \frac{(G^k)'(\eta)|H(I)|}{(F^k)'(\xi)(G^k)'(x)|I|} \leq \frac{|H(J)|}{|J|} \leq C_0^2 \frac{(F^k)'(\xi)(G^k)'(x)|H(I)|}{(F^k)'(x)(G^k)'(\eta)|I|}.$$ 

From the Hölder distortion property (1), there is a constant $C_1 > 1$ such that

$$C_1^{-1} \leq \frac{|H(J)|}{|J|} \leq C_1.$$ 

Since both $\Omega(I)$ and $H(\Omega(I))$ have full measures in $I$ and $H(I)$, respectively, from the additive formula, this implies that $H|I$ is bi-Lipschitz.
Since $H|I$ is bi-Lipschitz, $H'$ exists a.e. in $I$ and is integrable. Since $(H|I)'(x)$ is measurable and $H|I$ is a homeomorphism, we can find a point $y_0$ in $I$ and a subset $E_0$ containing $y_0$ such that

1. $H|I$ is differentiable at every point in $E_0$;
2. $y_0$ is a density point of $E_0$;
3. $H'(y_0) \neq 0$; and
4. the derivative $H'|E_0$ is continuous at $y_0$.

Since $[0,1]$ is compact, there is a subsequence $\{F^{n_k}(y_0) \mod 1\}_{k=1}^{\infty}$ converging to a point $z_0$ in $[0,1]$. Without loss of generality, assume $z_0 \in (0,1)$. Let $I_0 = (a,b)$ be an open interval about $z_0$. There is a sequence of intervals $\{I_k\}_{k=1}^{\infty}$ such that $y_0 \in I_k \subseteq I$ and $F^{n_k} : I_k \rightarrow I_0 \ (\mod 1)$ is a $C^{1+\alpha}$ diffeomorphism. Then $|I_k|$ goes to zero as $k$ tends to infinity.

From the Hölder distortion property (I), there is a constant $C_2 > 0$ such that

$$\left| \log \left( \frac{(F^{n_k})'(w)}{(F^{n_k})'(z)} \right) \right| \leq C_2, \quad \forall w, z \in I_k, \ \forall k \geq 1.$$ 

Since $y_0$ is a density point of $E_0$, for any integer $s > 0$, there is an integer $k_s > 0$ such that

$$\frac{|E_0 \cap I_k|}{|I_k|} \geq 1 - \frac{1}{s}, \quad \forall k \geq k_s.$$ 

Let $E_k = F^{n_k}(E_0 \cap I_k) \ (\mod 1)$. Then $H$ is differentiable at every point in $E_k$ and, from the Hölder distortion property (I), there is a constant $C_3 > 0$ such that

$$\frac{|E_k \cap I_0|}{|I_0|} \geq 1 - \frac{C_3}{s}, \quad \forall k \geq k_s.$$ 

Let

$$E = \cap_{s=1}^{\infty} \cup_{k \geq k_s} E_k.$$ 

Then $E$ has full measure in $I_0$ and $H$ is differentiable at every point in $E$ with non-zero derivative.

Next, we are going to prove that $H'|E$ is uniformly continuous. For any $x$ and $y$ in $E$, let $z_k$ and $w_k$ be the preimages of $x$ and $y$ under the diffeomorphism $F^{n_k} : I_k \rightarrow I_0 \ (\mod 1)$. Then $z_k$ and $w_k$ are in $E_0$. From $H \circ F = G \circ H \ (\mod 1)$, we have that

$$H'(x) = \frac{(G^{n_k})'(H(z_k))}{(F^{n_k})'(z_k)} H'(z_k)$$

and

$$H'(y) = \frac{(G^{n_k})'(H(w_k))}{(F^{n_k})'(w_k)} H'(w_k).$$

So

$$\left| \log \left( \frac{H'(x)}{H'(y)} \right) \right| \leq \left| \log \left( \frac{(G^{n_k})'(H(z_k))}{(G^{n_k})'(H(w_k))} \right) \right| + \left| \log \left( \frac{(F^{n_k})'(z_k)}{(F^{n_k})'(w_k)} \right) \right| + \left| \log \left( \frac{H'(z_k)}{H'(w_k)} \right) \right|.$$
Suppose both $f$ and $g$ are $C^{1+\alpha}$ for some $0 < \alpha \leq 1$. From the Hölder distortion property (1), there is a constant $C_4 > 0$ such that
\[
|\log \left| \frac{(F^{n_k})'(w_k)}{(F^{n_k})'(z_k)} \right| \leq C_4 |x - y|^{\alpha}
\]
and
\[
|\log \left| \frac{(G^{n_k})'(H(z_k))}{(G^{n_k})'(H(w_k))} \right| \leq C_4 |H(x) - H(y)|^{\alpha}
\]
for all $k \geq 1$. Therefore,
\[
|\log \left( \frac{H'(x)}{H'(y)} \right) | \leq C_4 \left( |x - y|^\alpha + |H(x) - H(y)|^\alpha \right) + \left| \log \left( \frac{H'(z_k)}{H'(w_k)} \right) \right|
\]
for all $k \geq 1$. Since $H'|E_0$ is continuous at $y_0$, the last term in the last inequality tends to zero as $k$ goes to infinity. Hence
\[
|\log \left( \frac{H'(x)}{H'(y)} \right) | \leq C_4 \left( |x - y|^\alpha + |H(x) - H(y)|^\alpha \right).
\]
This means that $H'|E$ is uniformly continuous. So it can be extended to a continuous function $\phi$ on $I_0$. Because $H|I_0$ is absolutely continuous and $E$ has full measure,
\[
H(x) = H(a) + \int_a^x H'(x)dx = H(a) + \int_a^x \phi(x)dx
\]
on $I_0$. This implies that $H|I_0$ is actually $C^1$. (This, furthermore, implies that $H|I_0$ is $C^{1+\alpha}$).

Now for any $x \in [0, 1]$, let $J$ be an open interval about $x$. By the expanding condition on $f$, there is an integer $n > 0$ and an open interval $J_0 \subset I_0$ such that $F^n : J_0 \to J$ (mod 1) is a $C^{1+\alpha}$ diffeomorphism. By the equation $H \circ F = G \circ H$, we have that $H|J$ is $C^{1+\alpha}$. Therefore, $H$ is $C^{1+\alpha}$. We proved the theorem. \(\square\)

**Remark 4.** This kind of the rigidity phenomenon has been also studied for one-dimensional dynamical systems with critical points in [15, 16, 17, 18, 19].

**Remark 5.** If $f$ and $g$ in Theorem 2 are both $C^{1+1}$, then one can prove that $h$ is bi-Lipschitz by using a different argument which was given by Sullivan in his lectures at the CUNY Graduate Center during 1986-1989 [38] as follows.

Suppose $h$ is differentiable at a point $x_0$ on the circle. Then
\[
h(x) = h(x_0) + h'(x_0)(x - x_0) + o(|x - x_0|)
\]
for $x$ close to $x_0$. Suppose
\[
f \circ h = h \circ g.
\]
Consider $\{x_n = f^n(x_0)\}_{n=0}^\infty$. Let $0 < a < 1$ be a fixed number. Consider the interval $I_n = (x_n, x_n + a)$. Let $J_n = (x_0, z_n)$ be an interval such that
\[
f^n : J_n \to I_n
\]
is a $C^{1+1}$ diffeomorphism. Let $f^{-n} : I_n \to J_n$ denote its inverse. Since $f$ is expanding, the length $|J_n| \to 0$ as $n \to \infty$. Similarly, we have that

$$g^n : h(J_n) \to h(I_n)$$

is a $C^{1+1}$ diffeomorphism. Let $g^{-n} : h(I_n) \to h(J_n)$ be its inverse. Then we have that

$$h(x) = g^n \circ h \circ f^{-n}(x), \quad x \in I_n.$$  

Let

$$\alpha_n(x) = \frac{x - x_0}{x_n - x_0} : J_n \to (0, 1)$$

and

$$\beta_n(x) = \frac{x - h(x_0)}{h(x_n) - h(x_0)} : h(J_n) \to (0, 1).$$

Then

$$h(x) = (g^n \circ \beta_n^{-1}) \circ (\beta_n \circ h \circ \alpha_n^{-1}) \circ (\alpha_n \circ f^{-n})(x), \quad x \in I_n.$$  

From the Hölder distortion property \([7]\) for $\alpha = 1$, we get

$$|\log \frac{(f^{-n})'(x)}{(f^{-n})'(y)}| \leq C|x - y|, \quad \forall \ x, y \in I_n$$

and

$$|\log \frac{(g^{-n})'(x)}{(f^{-n})'(y)}| \leq C|x - y|, \quad \forall \ x, y \in h(I_n).$$

This implies that $g^n \circ \beta_n^{-1}$ and $\alpha_n \circ f^{-n}$ are a sequence of $b$-Lipschitz homeomorphisms with a uniform Lipschitz constant. Therefore, they have convergent subsequences. The map $\beta_n \circ h \circ \alpha_n^{-1}$ converges to a linear map. Since the unit circle is compact and all $I_n$ with a fixed length $a$, $\bigcap_{n=1}^{\infty} I_n$ contains an interval $I$. Thus $h$ is a bi-Lipschitz homeomorphism on $I$. Since $f$ and $g$ are expanding, this implies that $h$ is bi-Lipschitz on the whole unit circle $T$.

However, this argument does not work for the case when $0 < \alpha < 1$. The reason is that in this case, we have only

$$|\log \frac{(f^{-n})'(x)}{(f^{-n})'(y)}| \leq C|x - y|^{\alpha}, \quad \forall \ x, y \in I_n$$

and

$$|\log \frac{(g^{-n})'(x)}{(f^{-n})'(y)}| \leq C|x - y|^{\alpha}, \quad \forall \ x, y \in h(I_n)$$

from the Hölder distortion property \([7]\). Therefore, $g^n \circ \beta_n^{-1}$ and $\alpha_n \circ f^{-n}$ are only a sequence of $\alpha$-Hölder homeomorphisms with a uniform Hölder constant. We cannot conclude that $h$ is bi-Lipschitz. The method developed in \([15, 16, 17, 18, 19]\) (which is presented in the proof of Theorem \([3]\)) is, in particular, useful for maps having only $C^{1+\alpha}$ smoothness for $0 < \alpha < 1$.

**Theorem 6.** Suppose $f, g \in C^{1+}$. Then $(f, h_f) \sim_T (g, h_g)$ if and only if $D^*(f) = D^*(g)$. Furthermore, $d_T(\Pi, \Pi') = 0$ if and only if $\Pi = \Pi'$. 
Proof. Let \( \{\eta_{n,f}\}_{n=1}^{\infty} \) and \( \{\eta_{n,g}\}_{n=1}^{\infty} \) be the corresponding sequences of nested partitions on \([0, 1]\) for \( f \) and \( g \). Let

\[ h = h_f \circ h_g^{-1}. \]

Then

\[ f \circ h = h \circ g. \]

Suppose \( H \) is the lift of \( h \) such that \( H(0) = 0 \). Then for any interval \( I_{w_n} \in \eta_{n,f} \),

\[ H(I_{w_n}) \in \eta_{n,g}. \]

Suppose \( (f, h_f) \sim_T (g, h_g) \). Then \( h \) is a \( C^1 \)-diffeomorphism of \( T \). For any \( w^* = \cdots j_{n-1} \cdots j_0 \in \Sigma^* \), let \( w^*_n = j_{n-1} \cdots j_0 \) and \( v^*_{n-1} = j_{n-1} \cdots j_1 \). Then

\[ D^*(g)(w^*_n) = \frac{|H(I_{w_n})|}{|H(I_{w_n})|} = \frac{H'(\xi)}{H'(\xi)} |I_{v_{n-1}}| = H'(\xi) \frac{D^*(f)(w^*_n)}{\tau}. \]

This implies that

\[ D^*(g)(w^*_n) = D^*(f)(w^*_n). \]

Now suppose \( D^*(g)(w^*_n) = D^*(f)(w^*_n) \). Since \( f \) and \( g \) are both \( C^{1+\alpha} \)-expanding for some \( 0 < \alpha \leq 1 \), there are constants \( C > 0 \) and \( 0 < \tau < 1 \) such that

\[ |D^*(f)(w^*_n) - D^*(f)(w^*_n)| \leq C \tau^n \quad \text{and} \quad |D^*(g)(w^*_n) - D^*(g)(w^*_n)| \leq C \tau^n. \]

This implies that there is a constant \( C' > 0 \) such that

\[ \frac{D^*(g)(w^*_n)}{D^*(f)(w^*_n)} \leq 1 + C' \tau^n, \quad \forall n > 0. \]

Let \( C'' = \prod_{n=0}^{\infty} (1 + C' \tau^n) \). Then

\[ \frac{|H(I_{w_n})|}{|I_{w_n}|} = \prod_{k=0}^{n} \frac{D^*(g)(w^*_{n-k})}{D^*(f)(w^*_{n-k})} \leq C'', \quad \forall w_n, \forall n > 0. \]

From the additive formula, we conclude that \( H \) is Lipschitz continuous. But a Lipschitz continuous function is absolutely continuous (at this point, we can also use a theorem of Shub and Sullivan [33] to show that \( h \) is a \( C^1 \)-diffeomorphism), so it is differentiable almost everywhere. Since \( H \) is a homeomorphism, it must have a differentiable point with non-zero derivative. Now Theorem [5] implies that that \( h \) is \( C^1 \)-diffeomorphism.

From the definition, \( d_T(\Pi, \Pi') = 0 \) if and only if \( h = h_f^{-1} \circ h_g \) is symmetric. If \( \Pi = \Pi' \), then \( h = h_f^{-1} \circ h_g \) is \( C^1 \)-diffeomorphism. So it is symmetric.

On the other hand, if \( h = h_f^{-1} \circ h_g \) is symmetric, then from Lemma[1] there is a bounded function \( \varepsilon(t) > 0 \) such that \( \varepsilon(t) \to 0 \) as \( t \to 0 \) and a constant \( 0 < \tau < 1 \) such that

\[ |D^*(g)(w^*_n) - D^*(f)(w^*_n)| = \frac{|H(I_{v_{n-1}})|}{|H(I_{w_n})|} \leq \varepsilon(\tau^n). \]

We get

\[ D^*(g)(w^*_n) = D^*(f)(w^*_n). \]
This further implies that $h$ is a $C^1$-diffeomorphism. So $\Pi = \Pi'$.

**Definition 7.** We call $d_T(\cdot, \cdot)$ the Teichmüller metric on $T \mathcal{C}^{1+}$.

Following Theorem 6, we can set up a one-to-one correspondence between the Teichmüller space $T \mathcal{C}^{1+}$ and the space of all Hölder continuous dual derivatives:

$$
\Pi = [(f, h_f)] \to D^*(f)(w^*).
$$

Therefore,

$$
T \mathcal{C}^{1+} = \{ D^*(f)(w^*) \mid f \in C^{1+} \}
$$
equipped with the Teichmüller metric $d_T(\cdot, \cdot)$. However, this is not a complete space. Next we will study the completion of this space.

Let $d \geq 2$ be the same fixed integer. Suppose $\mathcal{U}S$ is the space of all uniformly symmetric circle endomorphisms of degree $d$. We define the Teichmüller space for $\mathcal{U}S$ as we did for $C^{1+}$.

Let $q_d(z) = z^d$ be the basepoint in $\mathcal{U}S$. A marked circle endomorphism by $q_d$ is a pair $(f, h_f)$, where $f \in \mathcal{U}S$ and $h_f$ is the orientation-preserving homeomorphism of $T$ such that $h_f(1) = 1$ and

$$
f \circ h_f = h_f \circ q_d.
$$

From Corollary 1, for any marked circle endomorphism $(f, h_f)$ by $q_d$, $h_f$ is quasisymmetric. Thus we can define Teichmüller equivalence relation $\sim_T$, Teichmüller space, and Teichmüller metric as follows.

**Definition 8.** Two marked circle endomorphisms are equivalent, denoted as $(f, h_f) \sim_T (g, h_g)$, if $h_f \circ h_g^{-1}$ is a symmetric homeomorphism.

**Definition 9.** The Teichmüller space

$$
T \mathcal{U}S = \{ [(f, h_f)] \mid f \in \mathcal{U}S, \text{ with the basepoint } [(q_d, id)] \}
$$
is the space of all $\sim_T$-equivalence classes $[(f, h_f)]$ in the space of all marked circle endomorphisms by $q_d$. Teichmüller metric $d_T(\cdot, \cdot)$ is defined as

$$
d_T(\Psi, \Psi') = \frac{1}{2} \log B_{h_f^{-1} \circ h_g}
$$
where $(f, h_f) \in \Psi$ and $(g, h_g) \in \Psi'$.

If $f, g \in C^{1+}$ and if the conjugacy $h$ between $f$ and $g$ is symmetric, then from Theorem 6, $h$ must be a $C^1$-diffeomorphism. This implies that the Teichmüller space $T \mathcal{C}^{1+}$ is indeed a subspace of the Teichmüller space $T \mathcal{U}S$. Furthermore, we have that

**Theorem 7.** The space $(T \mathcal{U}S, d_T(\cdot, \cdot))$ is a complete space and is the completion of the space $(T \mathcal{C}^{1+}, d_T(\cdot, \cdot))$. 
Our proof of this theorem needs some result for asymptotically conformal circle endomorphisms in [11]. For the purpose of self-contained of this paper and for the convenience of the reader, we includes some materials from [11] in the next section. Therefore, we delay the proof of Theorem 7 into the next section.

6. ASYMPTOTICALLY CONFORMAL CIRCLE ENDOMORPHISMS

Suppose $g$ is a quasiconformal homeomorphism defined on a plane domain $\Omega$. Let

$$\mu(z) = \frac{g_z}{g_z}$$

for $z \in \Omega$ and let

$$K_z(g) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}.$$

Here $K_z(g)$ is called the dilatation of $g$ at $z$.

Suppose $H$ is a quasisymmetric homeomorphism of the real line. Define the skew quasisymmetric distortion function as

$$\rho(x, y, k) = \frac{|H(x + ky) - H(x)|}{|H(y) - H(x)|}.$$ 

In particular, let $\rho(x, y) = \rho(x, y, 1)$. The Beurling-Ahlfors extension procedure provides a canonical extension $\tilde{H}$ of any quasisymmetric homeomorphism $H$ to the whole complex plane such that the Beltrami coefficient $\mu$ of $\tilde{H}$ satisfies $\|\mu\|_\infty < 1$. Furthermore, it satisfies the following well-known theorem (see [12]).

**Theorem 8.** The Beurling-Ahlfors extension of a quasisymmetric self-mapping $H$ of the real axis has a Beltrami coefficient $\mu$ with $|\mu(x + iy)| \leq \eta(y)$ for some vanishing function $\eta(y)$ if, and only if, there is a vanishing function $\epsilon(y)$ such that

$$\frac{1}{1 + \epsilon(y)} \leq \rho_H(x, y) \leq 1 + \epsilon(y).$$

A generalization of Theorem 8 can be founded in [6] and in [11] with a complete proof.

**Theorem 9.** Suppose the skew quasisymmetric distortion functions $\rho_0(x, y, k)$ and $\rho_1(x, y, k)$ of $H_0$ and $H_1$ satisfy the inequality

$$|\rho_0(x, y, k) - \rho_1(x, y, k)|, \quad |\rho_0(x, -y, k) - \rho_1(x, -y, k)| \leq \epsilon(y)$$

for $x, y > 0 \in \mathbb{R}$ and $0 < k \leq 1$, where $\epsilon(y)$ is a vanishing function, that is, $\epsilon(y) \to 0$ as $y \to \mathbb{R}$. Suppose furthermore that $\mu_0$ and $\mu_1$ are the Beltrami coefficients of the Beurling-Ahlfors extensions $\tilde{H}_0$ and $\tilde{H}_1$, that is,

$$\mu_0(z) = \frac{\tilde{H}_0}{H_0z} \quad \text{and} \quad \mu_1(z) = \frac{\tilde{H}_1}{H_1z}.$$
Then there is a vanishing function \( \eta(y) \) depending only on \( \epsilon(y) \) such that
\[
|\mu_0(x + iy) - \mu_1(x + iy)| \leq \eta(y).
\]

Conversely, given two quasiconformal maps \( \tilde{H}_0 \) and \( \tilde{H}_1 \) preserving the real axis and a vanishing function \( \eta(y) \) such that
\[
|\mu_0(z) - \mu_1(z)| \leq \eta(y),
\]
then there is a vanishing function \( \epsilon(y) \) such that
\[
|\rho_0(x, y, k) - \rho_1(x, y, k)|, \quad |\rho_0(x, -y, k) - \rho_1(x, -y, k)| \leq \epsilon(y)
\]
for \( x, y > 0 \in \mathbb{R} \) and \( 0 < k \leq 1 \), where \( H_0 \) and \( H_1 \) are the restrictions of \( \tilde{H}_0 \) and \( \tilde{H}_1 \) to the real axis.

Proof. We adapt the proof from [11]. We take the following formulas as the definition of the Beurling-Ahlfors extension:
\[
\tilde{H} = U + iV,
\]
where
\[
U(x, y) = \frac{1}{2y} \int_{x-y}^{x+y} H(s) ds = \frac{1}{2} \int_{-1}^{1} H(x + ky) dk
\]
and
\[
V(x, y) = \frac{1}{y} \int_{x}^{x+y} H(s) ds - \frac{1}{y} \int_{x-y}^{x} H(s) ds.
\]
In (8) and (9) we have chosen a normalization slightly different from the one given in [11]. It has the property that the extension of the identity is the identity and the extension is affinely natural, by which we mean that for affine maps \( A \) and \( B \),
\[
\tilde{id}_{\mathbb{R}} = id_{\mathbb{C}}
\]
and
\[
A \circ \tilde{H} \circ B = A \circ \tilde{H} \circ B.
\]

Note that
\[
\int_{0}^{1} \rho(x, y, k) dk = \frac{1}{H(x) - H(x-y)} \left( \frac{1}{y} \int_{x}^{x+y} H(s) ds - H(x) \right)
\]
and
\[
\int_{0}^{1} \rho(x, -y, k) dk = \frac{1}{H(x+y) - H(x)} \left( H(x) - \frac{1}{y} \int_{x-y}^{x} H(s) ds \right).
\]

Let
\[
\begin{align*}
L &= H(x) - H(x-y) \\
R &= H(x+y) - H(x) \\
L' &= H(x) - \frac{1}{y} \int_{x-y}^{x} H(s) ds, \\
R' &= \frac{1}{y} \int_{x}^{x+y} H(s) ds - H(x).
\end{align*}
\]
and let $\rho_+(x, y) = \int_0^1 \rho(x, y, k) dk$ and $\rho_-(x, y) = \int_0^1 \rho(x, -y, k) dk$. Let $\rho(x, y) = \rho_H(x, y)$. Then

$$\begin{align*}
\rho(x, y) &= R/L \\
\rho_+(x, y) &= R'/L \\
\rho_-(x, y) &= L'/R.
\end{align*}$$

Notice that for symmetric homeomorphisms the quantity $\rho$ approaches 1 and the two quantities $\rho_+$ and $\rho_-$ approach $1/2$ as $y$ approaches zero. The complex dilatation of $\tilde{H}$ is given by

$$\mu(z) = \frac{K(z) - 1}{K(z) + 1}$$

where

$$K(z) = \frac{\tilde{H}_z + \tilde{H}_{\bar{z}}}{{\tilde{H}_z - \bar{H}_{\bar{z}}}} = \frac{(U + iV)_x + (U + iV)_{\bar{x}}}{(U + iV)_x - (U + iV)_{\bar{x}}}$$

$$= \frac{(U + iV)_x - i(U + iV)_y + (U + iV)_x + i(U + iV)_y}{(U + iV)_x - i(U + iV)_y - (U + iV)_x + i(U + iV)_y} \quad \frac{U_x + iV_x}{V_y - iU_y}.$$ 

Thus

$$K(z) = \frac{1 + ia}{b - ic},$$

where $a = V_x/U_x$, $b = V_y/U_x$ and $c = U_y/U_x$.

To find estimates for these three ratios we must find expressions for the four partial derivatives of $U$ and $V$ in (8) and (9). In the notation of (12)

$$\begin{align*}
U_x &= \frac{1}{2y} (R + L), \\
V_x &= \frac{1}{y} (R - L), \\
V_y &= \frac{1}{y} (R + L) - \frac{1}{y} (R' + L'), \\
U_y &= \frac{1}{2y} (R - L) - \frac{1}{2y} (R' - L').
\end{align*}$$

Thus

$$\begin{align*}
a(1 + \rho) &= \frac{2R - L \cdot R + L}{R + L}, \\
b(1 + \rho) &= \frac{2R' - L' \cdot R + L}{R + L}, \\
c(1 + \rho) &= \frac{R - L \cdot R' + L'}{R + L}.
\end{align*}$$

Finally, we obtain

$$\begin{align*}
a &= \frac{2(\rho - 1)}{\rho^1 + \rho_{-1} \cdot \rho_+}, \\
b &= \frac{2(\rho^1 - \rho_{-1} \cdot \rho_+)}{\rho^1 + \rho_{-1} \cdot \rho_+}, \\
c &= \frac{\rho - 1 + \rho_+ \cdot \rho_0}{\rho^1 + \rho_{-1} \cdot \rho_+}.
\end{align*}$$
Since $K(z) = (1 + ia)/(b - ic)$, $K(z) + 1 = (1 + ia + b - ic)/(b - ic)$, we have

$$\mu_1(z) - \mu_0(z) = \frac{K_1(z) - 1}{K_1(z) + 1} - \frac{K_0(z) - 1}{K_0(z) + 1} = 2\frac{K_1(z) - K_0(z)}{(K_1(z) + 1)(K_0(z) + 1)} =$$

$$\frac{2(1 + ia_1)(b_0 - ic_0) - (1 + ia_1)(b_1 - ic_1)}{(1 + ia_1 + b_1 - ic_1)(1 + ia_0 + b_0 - ic_0)}.$$

(15) $$\frac{2(a_1 - a_0)(ib_1 + c_1) + (b_0 - b_1)(1 + ia_1) + (c_1 - c_0)(i - a_1)}{(1 + ia_1 + b_1 - ic_1)(1 + ia_0 + b_0 - ic_0)}.$$

From the equation for $b$ in (14) and the inequalities $\rho_+ < \rho$ and $\rho \rho_- < 1$, we see that $b > 0$. Since this inequality is true for $b_1$ and for $b_0$, it follows that the denominator of (15) is greater than 1. These equations show that if $a_0, b_0, c_0$ converge to $a_1, b_1, c_1$, as $y$ approaches zero, then $\mu_0$ approaches $\mu_1$. Clearly $\rho_0$ approaches $\rho_1$, implies $a_0$ approaches $a_1$.

From the hypothesis

$$|\rho_0(x, y, k) - \rho_1(x, y, k)|, \ |\rho_0(x, -y, k) - \rho_1(x, -y, k)| \leq \epsilon(y)$$

for $x, y > 0 \in \mathbb{R}$ and $0 < k \leq 1$, we have that

$$|\rho_{1+}(x, y) - \rho_{0+}(x, y)| \leq \int_0^1 |\rho_1(x, y, k) - \rho_0(x, y, k)|dk \leq \epsilon(y)$$

and

$$|\rho_{1-}(x, y) - \rho_{0-}(x, y)| \leq \int_0^1 |\rho_1(x, -y, k) - \rho_0(x, -y, k)|dk \leq \epsilon(y).$$

This implies that $b_0, c_0$ converge to $b_1, c_1$, as $y$ approaches zero. This completes the proof of the first half of the theorem.

Since the subsequent arguments do not require the second half, we only sketch the proof. Notice that if $\tilde{H}_0$ and $\tilde{H}_1$ are quasiconformal self-maps of the complex plane preserving the real axis with Beltrami coefficients $\mu_0$ and $\mu_1$ satisfying

$$|\mu_0(z) - \mu_1(z)| \leq \epsilon(y)$$

for a vanishing function $\epsilon(y)$, then the quasiconformal map $\tilde{H}_1 \circ (\tilde{H}_0)^{-1}$ has Beltrami coefficient $\sigma$ with

$$|\sigma(z)| \leq \epsilon'(y)$$

for another vanishing function $\epsilon'(y)$. Then $\tilde{H}_1 \circ (\tilde{H}_0)^{-1}$ carries the extremal length problem for the family of curves joining $[-\infty, \tilde{H}_0(x-y)]$ to $[\tilde{H}_0(x), \tilde{H}_0(x+ky)]$ to the extremal length problem for the family of curves joining $[-\infty, \tilde{H}_1(x-y)]$ to $[\tilde{H}_1(x), \tilde{H}_1(x+ky)]$. If $\Lambda_0(x, y, k)$ and $\Lambda_1(x, y, k)$ are these two extremal lengths, then by the Grötzsch argument there is another vanishing function $\epsilon''(y)$ such that

$$\left| \log \frac{\Lambda_0(x, y, k)}{\Lambda_1(x, y, k)} \right| \leq \epsilon''(y).$$
In [1] Ahlfors shows that if $\Lambda$ is the extremal length of the curve family that joins the interval $[-\infty, -1]$ to $[0, m]$, $\Lambda$ is an increasing real analytic function of $m$. In particular,

\[(16) \quad \left| \log \frac{m_0}{m_1} \right| < \epsilon \text{ if and only if } \left| \log \frac{\Lambda_0}{\Lambda_1} \right| < \epsilon' \]

and $\epsilon$ and $\epsilon'$ approach zero simultaneously.

Hence by (16) there is another vanishing function $\eta(y)$ such that

\[\frac{H_0(x + ky) - H_0(x)}{H_0(x) - H_0(x - y)} - \frac{H_1(x + ky) - H_1(x)}{H_1(x) - H_1(x - y)} \leq \eta(y).\]

Similarly, we have that

\[\frac{H_0(x) - H_0(x - ky)}{H_0(x + y) - H_0(x)} - \frac{H_1(x) - H_1(x - ky)}{H_1(x + y) - H_1(x)} \leq \eta(y).\]

This completes the proof of the second half of the theorem.

**Definition 10.** We call a circle endomorphism $f$ of degree $d \geq 2$ a uniformly asymptotically conformal if it has a reflection invariant extension $\tilde{f}$ defined in a small annulus $r < |z| < 1/r'$ such that

\[\tilde{f}(1/z) = 1/\tilde{f}(z),\]

and such that for every $\epsilon > 0$ there exists a possibly smaller annulus $U = \{z : r' < |z| < 1/r'\}$ such that

\[(17) \quad K_z(\tilde{f}^{-n}) < 1 + \epsilon \]

for almost all $z$ in $U$ and all $n > 0$.

From the quasiconformal mapping theory (see [1]), if $\tilde{f}$ is a uniformly asymptotically conformal, then the restriction $\tilde{f}$ of $\tilde{f}$ to the unit circle $T$ is uniformly symmetric. It is also easy to see that if $\tilde{f}$ acting on a neighborhood of the unit circle with $\tilde{f}(1) = 1$ is uniformly asymptotically conformal if and only if there is a unique lift $\tilde{F}$ to an infinite strip containing $\mathbb{R}$ and bounded by lines parallel to $\mathbb{R}$ such that

1) $\pi \circ \tilde{F} = \tilde{f} \circ \pi,$
2) $\tilde{F}(0) = 0,$
3) $\tilde{F}(z + 1) = \tilde{F}(z) + d,$ and
4) $\tilde{F}$ preserves the real axis and $\tilde{F}(\overline{z}) = \overline{\tilde{F}(z)}$.

In light of the above, we have an equivalent definition.

**Definition 11.** We call a circle endomorphism $f$ a uniformly asymptotically conformal if for every $\epsilon > 0$, there is a $\delta > 0$ such that if the absolute value of $y = \text{Im} \ z$ is less than $\delta$, then

\[(18) \quad K_z(\tilde{F}^{-n}) < 1 + \epsilon \]
for all \( n > 0 \).

The following theorem is proved in [11]. For the purpose of self-contained of this paper, we include the proof.

**Theorem 10.** A circle endomorphism \( f \) of degree \( d \geq 2 \) is uniformly symmetric if and only if it is uniformly asymptotically conformal.

Before to prove this theorem, we prove the following lemma. Let \( \zeta(M) \) be the function in Lemma 1.

**Lemma 2.** Let \( \vartheta(M) = M - 1 + M \zeta(M) \). Then for any homeomorphism \( H \) of \( \mathbb{R} \) and any \( x, y > 0 \in \mathbb{R} \), if \( H \) restricted to the interval \([x - y, x + y]\) is \( M \)-quasisymmetric, then

\[
\max \{|\rho_H(x, y, k) - k|, |\rho_H(x, -y, k) - k|\} \leq \vartheta(M), \quad \forall 0 < k \leq 1.
\]

**Proof.** Consider \( \hat{H}(k) = (H(x + ky) - H(x))/ (H(x + y) - H(x)) \). Then \( \hat{H}(1) = 1 \) and \( \hat{H}(0) = 0 \). Also, \( \hat{H} \) is quasisymmetric because

\[
\frac{\hat{H}(k + j) - \hat{H}(k)}{\hat{H}(k) - \hat{H}(k - j)} = \frac{H(x + ky + jh) - H(x + ky)}{H(x + ky) - H(x + ky - jy)}
\]

for any \( 0 \leq k \leq 1 \) and \( j > 0 \) such that \([k - j, k + j] \subset [0, 1]\) and this is bounded above by \( M \) and below by \( 1/M \) because \( H \) is \( M \)-quasisymmetric. So, from Lemma 1

\[
k - \zeta(M) \leq \frac{H(x + ky) - H(x)}{H(x + y) - H(x)} \leq k + \zeta(M).
\]

Thus

\[
(k - \zeta(M))\rho_H(x, y) \leq \frac{H(x + ky) - H(x)}{H(x) - H(x - y)} \leq (k + \zeta(M))\rho_H(x, y).
\]

Since \( 1/M \leq \rho_H(x, y) \leq M \) and we are assuming that \( 0 < k \leq 1 \), this implies that

\[
|\rho_H(x, y, k) - k| \leq \vartheta(M) = M - 1 + M \zeta(M).
\]

Similarly, we have that

\[
|\rho_H(x, -y, k) - k| \leq \vartheta(M) = M - 1 + M \zeta(M).
\]

\( \square \)

**Proof of Theorem 10.** We only need to prove the ”only if” part. Let \( F \) be the lift to the real axis of \( f \) such that \( F(0) = 0, F(x + 1) = F(x) + d \) and such that \( \pi \circ F = f \circ \pi \). By Theorem 1 there is a quasisymmetric homeomorphism \( H \) of \( \mathbb{R} \) fixing 0 and 1 such that

i) \( H \circ P \circ H^{-1} = F \) where \( P(x) = dx \), and

ii) \( H \circ T \circ H^{-1} = T \) where \( T(x) = x + 1 \).

It will suffice to find an extension \( \tilde{F} \) of \( F \) such that
(1) \( \tilde{F} \circ T(z) = T^d \circ \tilde{F}(z) \) and
(2) the Beltrami coefficients \( \mu_{\tilde{F}^{-n}} \) of \( \tilde{F}^{-n} \) satisfy
\[
|\mu_{\tilde{F}^{-n}}(x + iy)| \leq \epsilon(y)
\]
where \( \epsilon(y) \) is independent of \( n \) and \( x \).

Let \( \tilde{H} \) be a reflection invariant quasiconformal extension of \( H \). We define
\[
\tilde{F} = \tilde{H} \circ P \circ \tilde{H}^{-1}
\]
since \( \tilde{H} \) extends \( H \), clearly \( \tilde{F} \) extends \( F \) and is a reflection invariant extension.

Suppose \( \rho_0(x, y, k) \) and \( \rho_1(x, y, k) \) are the skew quasisymmetric distortions of \( F^{-n} \circ H \) and \( H \). By Lemma 2 there is a vanishing function \( \epsilon(y) \) such that
\[
|\rho_0(x, y, k) - \rho_1(x, y, k)|, \quad |\rho_0(x, -y, k) - \rho_1(x, -y, k)| \leq \epsilon(y)
\]
for all real numbers \( x \), all \( y > 0 \), all \( k \) with \( 0 < k \leq 1 \) and all \( n \geq 1 \). Applying Theorem 3 there is another vanishing function \( \eta(y) \) such that the Beltrami coefficients \( \mu_{\tilde{F}^{-n} \circ H} \) and \( \mu_{\tilde{H}} \) satisfy
\[
|\mu_{\tilde{F}^{-n} \circ H}(z) - \mu_{\tilde{H}}(z)| = \eta(y), \quad \forall n > 0.
\]

Since
\[
\tilde{F}^{-n} \circ H = H \circ P^{-n} = \tilde{H} \circ P^{-n},
\]
we conclude that
\[
|\mu_{\tilde{H}}(d^{-n}z) - \mu_{\tilde{H}}(z)| \leq \eta(y).
\]
Also, since the Beurling-Ahlfors extension is affinely natural, \( \mu_{\tilde{H}}(T(z)) = \mu_{\tilde{H}}(z) \) and \( \tilde{H} \circ T \circ \tilde{H}^{-1}(z) = T(z) \). We conclude that \( \tilde{F} = \tilde{H} \circ P \circ \tilde{H}^{-1} \) is uniformly asymptotically conformal.

**Proof of Theorem 7** Suppose \( \{\kappa_n\}_{n=1}^\infty = \{(f_n, h_n)\}_{n=1}^\infty \) is a Cauchy sequence in \( TUS \). Then
\[
d_T(S_{h_n}, S_{h_m}) \to 0 \quad \text{as} \quad m, n \to \infty.
\]
We may assume by working modulo \( \mathcal{S} \) that \( h_n^{-1}h_m \) tends to the identity map as \( m \) and \( n \) go to infinity. Therefore, \( \{h_n\}_{n=1}^\infty \) is a Cauchy sequence in the universal Teichmüller space and \( h_n \) tends to a quasisymmetric map \( h \) as \( n \) goes to infinity.

Since \( f_n = h_n^{-1}qdh_n \) for all \( n \geq 1 \), \( f_n = h_n^{-1}h_m f_m h_n^{-1}h_n \) for all \( n, m \geq 1 \). Let \( g_{n,w_k} \) be inverse branches of \( f_n^k \) defined on \( T \setminus \{1\} \). By considering their lifts to \( \mathbb{R} \), we can think of them as defined on the whole circle \( T \). Then
\[
g_{n,w_k} = h_n^{-1}h_m g_{m,w_k} h_m^{-1}h_n.
\]
Let \( k_{nm} = h_n^{-1}h_m \) and let
\[
\rho_{nm} = \sup_{x \in T, t > 0} \frac{|k_{nm}(x + t) - k_{nm}(x)|}{|k_{nm}(x) - k_{nm}(x - t)|}
\]
be its quasisymmetric distortion. Then \( \rho_{nm} \to 1 \) as \( n, m \to \infty \). Let
\[
\rho(g_{n,w_k}, t) = \sup_{x \in T} \frac{|g_{n,w_k}(x + t) - g_{n,w_k}(x)|}{|g_{n,w_k}(x) - g_{n,w_k}(x - t)|}, \quad t > 0,
\]
be the quasisymmetric distortion of \( g_{n,w_k} \). Then we have that
\[
\rho(g_{n,w_k}, t) \leq \rho_{nm}^2 \rho(g_{m,w_k}, t), \quad \forall n, m \geq 1.
\]
So there is a positive bounded function \( \epsilon(t) \to 1 \) as \( t \to 0 \) such that
\[
\rho(g_{n,w_k}, t) \leq \epsilon(t), \quad \forall n \geq 1, \quad \forall w_k, \quad \forall t > 0.
\]
Define \( f = h^{-1} q_h \). Let \( g_{w_k} \) be inverse branches of \( f^k \) defined on \( T \setminus \{1\} \).
By considering their lifts to \( \mathbb{R} \), we think of them as defined on the whole circle \( T \). Let
\[
\rho(g_{w_k}, t) = \sup_{x \in T} \frac{|g_{w_k}(x + t) - g_{w_k}(x)|}{|g_{w_k}(x) - g_{w_k}(x - t)|}, \quad t > 0,
\]
be the quasisymmetric distortion of \( g_{w_k} \).
Let \( l_n = h^{-1} h_n \). Then \( f = l_n f_n l_n^{-1} \) for all \( n > 0 \). Let
\[
\rho(l_n) = \sup_{x \in T, t > 0} \frac{|l_n(x + t) - l_n(x)|}{|l_n(x) - l_n(x - t)|}
\]
be the quasisymmetric constant of \( l_n \). Then \( \{\rho(l_n)\} \to 1 \) is a bounded sequence.
(Actually, \( \rho_n \to 1 \) as \( n \to \infty \).)
Since \( g_{w_k} = l_n g_{n,w_k} l_n^{-1} \),
\[
\rho(g_{w_k}, t) \leq (\rho(l_n))^2 \epsilon(t) \leq \sup_{n \geq 1} \{(\rho(l_n))^2\} \epsilon(t), \quad \forall w_k, \quad \forall n > 0, \quad \forall t > 0.
\]
This means that \( f \) is uniformly symmetric, so \([f, h]) \in \mathcal{TUS} \). But \( f_n \to f \) as \( n \to \infty \) in \( \mathcal{TUS} \). Thus \( \mathcal{TUS} \) is complete.
For any \([f, h]) \in \mathcal{TUS} \) and any \( \epsilon > 0 \), we will prove that there is an analytic circle map \( f_\epsilon \in \mathcal{C}^{1+\alpha} \) such that \([f_\epsilon, h_\epsilon]) \) is in the \( \epsilon \)-neighborhood of \([f, h]) \) in \( \mathcal{TUS} \). We use a technique in complex dynamics (refer to \cite{9}) to construct \( f_\epsilon \) as follows.
Consider a quasiconformal extension \( \tilde{h} \) of \( h \) to the complex plane. Then \( \tilde{f} = \tilde{h} q_\tilde{h} \tilde{h}^{-1} \) is a quasiregular map of the complex plane. Let
\[
\mu_{\tilde{f}_n}(z) = \frac{(\tilde{f}_n)^{\tilde{\pi}}(z)}{(\tilde{f}_n)^{\tilde{\pi}}(\bar{z})}
\]
be the Beltrami coefficient of \( \tilde{f}_n \). Assume \( \mu_{\tilde{f}_n}(z) \) is symmetric about the unit circle, that is, \( \mu_{\tilde{f}_n}(z) = \mu_{\tilde{f}_n}(1/\bar{z}) \).

Since \( f \) is uniformly symmetric, from Theorem \cite{10}, we can pick an extension \( \tilde{f} \) (equivalently, pick an extension \( \tilde{h} \) of the conjugacy \( h \)) such that there is a
function $\gamma(t) \to 0$ as $t \to 0$ and such that $|\mu \tilde{f}_n(z)| \leq \gamma(|z|^{2n} - 1)$ for all $n > 0$ and a.e. $z$. From calculus,

$$\mu \tilde{f}_n(z) = \frac{\mu_h^n(q_d^n(z)) - \mu_h(z)}{1 + \mu_h^n(q_d^n(z)) \mu_h(z)} \Theta(z), \quad \text{where } |\Theta(z)| = 1.$$ 

This implies that

$$|\mu_h^n(q_d^n(z)) - \mu_h(z)| \leq C \gamma(|z|^{2n} - 1)$$

for all $n > 0$ and a.e. $z$ where $C > 0$ is a constant. For any $\epsilon > 0$, we have a $\delta > 0$ such that $\gamma(t) < \epsilon/C$ for all $0 \leq t < \delta$. Let

$$A_0 = \{ z \in \mathbb{C} \mid 1 - \delta < |z| < (1 - \delta)^{1/2} \} \cup \{ z \in \mathbb{C} \mid (1 + \delta)^{1/2} < |z| < 1 + \delta \}$$

and set $A_n = q_d^{-n}(A_0)$. Define $\mu(z) = \mu_h(z)$ for $z \in \mathbb{C} \setminus (\cup_{n=1}^{\infty} A_n)$ and $\mu = \mu_h^n(q_d^n(z))$ for $z \in A_n$ and $n > 0$. Then $\mu$ is a Beltrami coefficient defined on the complex plane and symmetric about the unit circle. Let $\varphi$ be a quasiconformal homeomorphism solving the Beltrami equation $\varphi_\tau = \mu(z) \varphi_z$. Then $\varphi|T$ is a homeomorphism of $T$. Define

$$\tilde{f}_\epsilon = \varphi q_d \varphi^{-1}.$$ 

From calculus,

$$\mu \tilde{f}_\epsilon(z) = \frac{\mu(q_d(z)) - \mu(z)}{1 + \mu(q_d(z)) \mu(z)} \Theta(z), \quad \text{where } |\Theta(z)| = 1.$$ 

So $(\tilde{f}_\epsilon)_\tau = 0$ for $(1 - \delta)^{1/2} < |z| < (1 + \delta)^{1/2}$, that is, $f_\epsilon = \tilde{f}_\epsilon|T$ is analytic. Because $|\mu(z) - \mu_f(z)| < \epsilon$ for all $z \in \mathbb{C}$, $f_\epsilon$ is $\epsilon$-approximate to $f$ in the metric $d_T(\cdot, \cdot)$.

The sequence of Markov partitions $\{ \omega_{n,f} \}_{n=0}^{\infty}$ is just an image of the sequence of Markov partitions $\{ \omega_{n,q_d} \}_{n=0}^{\infty}$ under $\varphi|T$. Since $\varphi|T$ is quasisymmetric and $\{ \eta_{n,q_d} \}_{n=0}^{\infty}$ has bounded geometry, then $\{ \omega_{n,f} \}_{n=0}^{\infty}$ has bounded geometry. A real analytic circle endomorphism having bounded geometry is expanding (refer to [16 Chapter 3] or [23]). Thus $f_\epsilon \in C^1$. This completes the proof.

7. Contractibility

By definition, a topological space $X$ is contractible if there is a continuous map

$$\Psi(x, t) : X \times [0, 1] \to X$$

such that $\Psi(x, 0) = x$ and $\Psi(x, 1) = x_0$ for all $x \in X$ where $x_0$ is a fixed point in $X$. In this section, we prove the following theorem.

Theorem 11. The space $TUS$ is contractible.
Proof. Let \( x_0 = [(q_d, id)] \) be the basepoint of \( \mathcal{TUS} \). For any \( x \in \mathcal{TUS} \), let \((f, h_f)\) be a representation in \( x \). From Theorem \( \text{[III]} \) \((f, h_f)\) has an extension \((\tilde{f}, \tilde{h}_f)\) over an annulus neighborhood \( \{z \mid 1/r < |z| < r\} \) for some \( r > 1 \) such that \( \tilde{f} \) is symmetric about \( T \) and such that

\[
|\mu_{\tilde{f}^{-n}}(z)| \leq \eta(y), \quad z = x + yi
\]

for a vanishing function \( \eta(y) \), where \( \tilde{f}^{-n} \) means any inverse branch of \( \tilde{f}^n \). Since

\[
\tilde{f}^n = \tilde{h}_f \circ q_d^n \circ \tilde{h}_f^{-1},
\]

we have that

\[
\mu_{\tilde{f}^{-n}}(z) = \theta(z) \frac{\mu_{\tilde{h}_f}(q_d^{-n}(z)) - \mu_{\tilde{h}_f}(z)}{1 - \mu_{\tilde{h}_f}(q_d^{-n}(z))\mu_{\tilde{h}_f}(z)}
\]

where \( |\theta(z)| = 1 \) and \( \|\tilde{h}_f\|_\infty \leq k < 1 \). (Again \( \tilde{f}^{-n} \) means any inverse branch of \( \tilde{f}^n \) and \( q^{-n} \) means the corresponding inverse branch of \( q^n \).) This implies that

\[
|\mu_{\tilde{h}_f}(q_d^{-n}(z)) - \mu_{\tilde{h}_f}(z)| \leq \hat{\eta}(y), \quad z = x + iy
\]

for a vanishing function \( \hat{\eta}(y) \).

Let \( \mu = \mu_{\tilde{h}_f} \) and \( h^{t\mu} \) be the unique solution of the Beltrami equation with Beltrami coefficient \( t\mu \) for \( 0 \leq t \leq 1 \). From the measurable Riemann mapping theorem, we know that \( h^{t\mu} \) depends on \( t \) and \( \mu \) continuously. (Actually, if we consider \( t \) as a complex parameter, then \( h^{t\mu} \) depends on \( t \) and \( \mu \) holomorphically.) Then \( \tilde{f}_t = h^{t\mu} \circ q_d \circ (h^{t\mu})^{-1} \) is a continuous family of circle endomorphisms such that

\[
\mu_{\tilde{f}_t^{-n}}(z) = \theta_t(z) \frac{t(\mu_{\tilde{h}_f}(q_d^{-n}(z)) - \mu_{\tilde{h}_f}(z))}{1 - t^2\mu_{\tilde{h}_f}(q_d^{-n}(z))\mu_{\tilde{h}_f}(z)} \leq \hat{\eta}(y), \quad z = x + iy
\]

for a vanishing function \( \hat{\eta}(y) \). Thus \((\tilde{f}_t, h^{t\mu})\) is a continuous family of uniformly asymptotically conformal maps.

Let \( f_t = \tilde{f}_t|T \) and \( h_t = h^{t\mu}|T \). Then \((f_t, h_t)\) is a continuous family of marked uniformly symmetric circle endomorphism. Thus \( \tau_t = [(f_t, h_t)] \) is a continuous path in \( \mathcal{TUS} \) connecting \( \tau \) and the basepoint \([q_d]\).

Define

\[
\Psi(\tau, t) = \tau_t : \mathcal{TUS} \times [0, 1] \to \mathcal{TUS}
\]

is a continuous homotopy map moving every point to the basepoint. So \( \mathcal{TUS} \) is contractible. \( \square \)

Remark 6. For any fixed \( 0 < \alpha \leq 1 \), let \( C^{1+\alpha} \) be the space of all \( C^{1+\alpha} \) circle expanding endomorphisms. Then \( C^{1+\alpha} \) is also a contractible space. It is a fact communicated to me by Anatole Katok. The proof can be as follows. Let \( f \) be a \( C^{1+\alpha} \) circle expanding endomorphism. There is a \( C^{1+\alpha} \) circle diffeomorphism \( h \) such that \( g = h^{-1} \circ f \circ h \) preserves the Lebesgue measure (see,
for example, [22]). The derivative $h'$ is the unique fixed point of the positive transfer operator (or called Ruelle’s Perron-Frobius operator)

$$\mathcal{L}\phi(z) = \sum_{w \in f^{-1}(z)} \frac{\phi(w)}{f'(w)}$$

from the space of all $\alpha$-Hölder continuous functions into itself. Since the fixed point can be obtained from the contracting fixed point theorem (see, for example, [21]), the fixed point depends on $f$ continuously. Let $H$ be the corresponding map for $h$ on the real line. Define

$$H_t'(x) = tH'(x) + (1 - t), \quad 0 \leq t \leq 1.$$

Then we have that $H_t'(x)$ is an $\alpha$-Hölder continuous positive periodic function of period 1 and that $H_t(x) = \int_0^x H_t'(\xi)d\xi$ is a $C^{1+\alpha}$ diffeomorphism of the real line satisfying $H_t(x + 1) = H_t(x) + 1$ and $H_t(0) = 0$. Thus it a $C^{1+\alpha}$ circle diffeomorphism. Let $h_t$ fixing 1 be the corresponding $C^{1+\alpha}$ diffeomorphism of $T$. Define $f_t = h_t^{-1} \circ f \circ h_t$. Since $h_t$ is $C^{1+\alpha}$ circle diffeomorphism, $f_t$ is a $C^{1+\alpha}$ expanding circle endomorphism for any $0 \leq t \leq 1$. Then $\gamma_1(t) = \{f_t\}_{0 \leq t \leq 1}$ is a continuous curve in $C^{1+\alpha}$ connecting $f$ and $g$.

Let $G$ be the corresponding map for $g$ on the real line. We have that $G(x + 1) = G(x) + d$ and $G'(x + 1) = G'(x)$. The fact that $g$ preserves the Lebesgue measure is equivalent to that

$$\sum_{0 \leq i \leq d-1} \frac{1}{G'(G^{-1}(x+i))} = 1, \quad \forall x \in [0,1].$$

This implies that $G'(x) > 1$ for all $x \in [0,1]$. Define

$$G_t'(x) = tG'(x) + (1 - t)d, \quad x \in \mathbb{R}, \quad 0 \leq t \leq 1.$$

For any $0 \leq t \leq 1$, we have that $G_t'(x + 1) = G_t'(x)$ for all $x \in \mathbb{R}$ and that $G_t'(x) > 1$ for all $x \in [0,1]$. Define $G_t(x) = \int_0^x G_t'(\xi)d\xi$. Then $G_t(x + 1) = G_t(x) + d$ for all $x \in \mathbb{R}$. Thus $G_t$ is a $C^{1+\alpha}$ expanding circle endomorphism. Let $g_t$ be the corresponding one on $T$. Then $\gamma_2(t) = \{g_t\}_{0 \leq t \leq 1}$ is a continuous curve in $C^{1+\alpha}$ connecting $g$ and $q_d(z) = z^d$.

Define

$$\Psi(f, t) = \begin{cases} \gamma_1(2t), & 0 \leq t \leq \frac{1}{2}; \\ \gamma_2(2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then $\Psi(f, t) : C^{1+\alpha} \to C^{1+\alpha}$ is a continuous homotopy map moving every point to the point $q_d(z) = z^d$. So $C^{1+\alpha}$ is contractible.

Following the above argument, we have also that $C^{1+}$ and $\mathcal{T}C^{1+}$ are both contractible spaces.
8. Linear Model and Dual Derivative

Suppose $f$ is a uniformly symmetric circle endomorphism of degree $d \geq 2$. Let $D^*(f)(w^*)$ be its dual derivative.

$$
\delta = D^*(f)(\cdots 000) = \lim_{n \to \infty} \frac{F^{-n}(1)}{F^{-(n+1)}(1)}.
$$

Let $\Upsilon(x) = x + 1$ be the translation by 1 on the real line $\mathbb{R}$. For any $x \in \mathbb{R} = (-\infty, \infty)$, let

$$
[x] = \{y \in \mathbb{R} \mid \Upsilon^n(x) = y, \text{ for some integer } n\}
$$

Then the unit circle can be thought as a topological space

$$
\mathbb{R}/\Upsilon = \{[x]\}
$$

with linear Lebesgue metric introduced from $\mathbb{R}$. The copies of the unit circle are $[k, k+1)$ for all integers $k$. The circle endomorphism $f$ can be thought of as a map

$$
[x] \to [F(x) \pmod{1}].
$$

For each $n > 0$, consider the homeomorphism

$$
\vartheta_n(x) = \frac{F^{-n}(x)}{F^{-n}(1)} : \mathbb{R} \to \mathbb{R}.
$$

The $\{\vartheta_n(x)\}$ is a sequence of uniformly symmetric homeomorphisms of $\mathbb{R}$. We would like to show that $\{\vartheta_n(x)\}$ is a convergent sequence and uniformly on any compact set of $\mathbb{R}$ as follows.

For any $\epsilon > 0$, there is an $n_0 > 0$ such that $F^{-m}$ on $[0, F^{-n}(1)]$ is $(1 + \epsilon)$-quasisymmetric for any $m > n \geq n_0$. Then

$$
H(y) = \frac{F^{-m+n}(F^{-n}(1)y)}{F^{-m}(1)} : [0, 1] \to [0, 1]
$$

is a $(1 + \epsilon)$-quasisymmetric homeomorphism with $H(0) = 0$ and $H(1) = 1$. From Lemma 1

$$
|H(y) - y| \leq \frac{\epsilon}{2}, \forall y \in [0, 1].
$$

Thus for $y = F^{-n}(x)/F^{-n}(1)$ for $x \in [0, 1]$, we have

$$
|\vartheta_{n+m}(x) - \vartheta_n(x)| < \epsilon, \forall m > n \geq n_0.
$$

This implies that $\{\vartheta_n(x)\}_{n=0}^{\infty}$ is a uniformly convergent Cauchy sequence on $[0, 1]$. Thus it converges uniformly to a function $\vartheta(x)$ on $[0, 1]$. Similarly the sequence of inverses $\{\vartheta^{-1}_n(y) = F^n(F^{-n}(1)y)\}_{n=0}^{\infty}$ is also a uniformly convergent Cauchy sequence, it converges uniformly to a function which is the inverse of $\vartheta(x)$. So $\vartheta(x)$ is a homeomorphism. A direct calculation implies that $\vartheta(x)$ is symmetric on $[0, 1]$. For any fixed $k > 0$, $F^{-k}$ maps $[0, d^k]$ onto $[0, 1]$. Using the relation

$$
\vartheta_{n+k}(x) = \frac{F^{-n}(1)}{F^{-n-k}(1)} \vartheta_n(F^{-k}(x)), \ x \in [0, d^k],
$$

we have

$$
\vartheta_n(F^{-k}(x)) = \frac{F^{-n}(1)}{F^{-n-k}(1)} \vartheta_{n+k}(x), \ x \in [0, d^k].
$$
we get that \( \{ \vartheta_n(x) \}_{n=0}^{\infty} \) is a uniformly convergent Cauchy sequence on \([0, d^k]\) and converges to a uniformly symmetric homeomorphism \( \vartheta(x) \), since \( \{ F^{-k} \}_{k=0}^{\infty} \) is uniformly symmetric. We conclude that \( \{ \vartheta_n(x) \} \) is a convergent sequence and converges uniformly on any compact set of \( \mathbb{R}^+ = [0, \infty) \); and the limit function is a uniformly symmetric homeomorphism \( \vartheta \) of \( \mathbb{R}^+ \). Moreover, \( \vartheta(x) \) conjugates \( F \) to a linear map \( x \to \delta x \) on \( \mathbb{R}^+ \), that is,

\[
\vartheta \circ F \circ \vartheta^{-1}(x) = \delta x, \quad \forall x \geq 0.
\]

Similar, we can prove the above on \( \mathbb{R}^- = (-\infty, 0] \).

The linear model of \( f \) is the conjugate function of the linear equivalence \( \Upsilon \) by \( \vartheta \), that is,

\[
L(x) = \vartheta \circ \Upsilon \circ \vartheta^{-1}(x) = \vartheta(\vartheta^{-1}(x) + 1).
\]

Since

\[
F \circ \Upsilon(x) = \Upsilon^d \circ F,
\]

We have

\[
L(0) = 1, \quad \text{and} \quad L(x) = \delta^{-1} L^d(\delta x), \quad \forall x \in \mathbb{R}.
\]

Now we have a new point of view for the unit circle and the circle endomorphism \( f \): For any \( x \in [0, 1) \), let

\[
[x]_L = \{ y \in \mathbb{R} \mid L^n(x) = y, \text{ for some integer } n \}.
\]

The unit circle can be thought as a topological space

\[
\mathbb{R}/L = \{ [x]_L \}
\]

with the metric introduced from \( L \). Copies of the unit circle now are all intervals \( [\vartheta(k), \vartheta(k + 1)] = [L^k(0), L^{k+1}(0)] \) for all integers \( k \). The circle endomorphism \( f \) can be thought of as a map

\[
[x]_L \to [\delta x \mod L].
\]

**Theorem 12.** Suppose \( f \) and \( g \) are two uniformly symmetric circle endomorphisms of the same degree \( d \geq 2 \). Let \( h \) be the conjugacy between \( f \) and \( g \), that is,

\[
h \circ f = g \circ h.
\]

Then \( h \) is a symmetric homeomorphism if and only if the linear models of \( f \) and \( g \) are the same and \( D^*(f)(\cdots 000) = D^*(g)(\cdots 000) \).

**Proof.** Suppose \( h \) is symmetric. Applying Lemma 11 we have that

\[
D^*(f)(\cdots 000) = D^*(g)(\cdots 000).
\]

Let \( F \), \( G \), and \( H \) be the lifts of \( f \), \( g \), and \( h \). Then

\[
F^{-n}(x) = H(G^{-n}(H^{-1}(x))).
\]
Since $H(1) = 1$, we get
\[
\frac{F^{-n}(x)}{F^{-n}(1)} = \frac{H \circ G^{-n} \circ H^{-1}(x)}{H \circ G^{-n}(1)}.
\]
Since $H$ is symmetric, we get, by using Lemma 1,
\[
\vartheta_f(x) = \vartheta_g \circ H^{-1}(x).
\]
So
\[
L_f(x) = \vartheta_f(\vartheta_f^{-1}(x) + 1) = \vartheta_g(H^{-1}(H(\vartheta^{-1}_g(x) + 1)))
\]
\[
= \vartheta_g(H^{-1}(H(\vartheta^{-1}_g(x))) + 1) = \vartheta_g(\vartheta^{-1}_g(x) + 1) = L_g(x).
\]
Conversely, suppose $L_f = L_g$ and
\[
\delta = D^s(f)(\cdots 000) = D^s(g)(\cdots 000).
\]
Then
\[
L_f(x) = \vartheta_f(\vartheta_f(x) + 1) = \vartheta_g(\vartheta_g(x) + 1) = L_g(x)
\]
Let
\[
H(x) = \vartheta^{-1}_g \circ \vartheta_f(x).
\]
We have $H(x + 1) = H(x) + 1$. So $H$ is a symmetric circle homeomorphism.
Since
\[
F(x) = \vartheta^{-1}_f(\delta \vartheta_f(x)) \quad \text{and} \quad G(x) = \vartheta^{-1}_g(\delta \vartheta_g(x)),
\]
we get that
\[
F(x) = H^{-1} \circ G \circ H(x).
\]
So $f$ and $g$ are symmetrically conjugate. We proved the theorem. \qed

Suppose $\{\eta_n\}_{n=0}^\infty$ is the sequence of nested partitions on $[0,1]$ for $f$. For any $w^* \in \Sigma^*$, let $w^*_n = j_{n-1} \cdots j_1 j_0$ and $v^*_n = j_{n-1} \cdots j_1$. Since $\vartheta(x)$ is symmetric on $[0,1]$ with $\vartheta(0) = 0$ and $\vartheta(1) = 1$, from Lemma 1
\[
D^s(f)(w^*) = \lim_{n \to \infty} \frac{|I_{v_{n-1}}|}{|I_{w_n}|} = \lim_{n \to \infty} \frac{|\vartheta(I_{v_{n-1}})|}{|\vartheta(I_{w_n})|}.
\]
Consider non-negative integers
\[
k = j_0 + j_1 d + \cdots + j_{n-1} d^{n-1} \quad \text{and} \quad l = j_1 + j_2 d + \cdots + j_{n-1} d^{n-2}.
\]
Then $k = dl + j_0$ and
\[
I_{w_n} = F^{-n}([k, k + 1]) \quad \text{and} \quad I_{v_{n-1}} = F^{-(n-1)}([l, l + 1]).
\]
Since $\vartheta(F^{-n}(x)) = \delta^{-n} \vartheta(x)$ and since $\delta \vartheta(l) = \vartheta(F(l)) = \vartheta(dl)$,
\[
\frac{|\vartheta(I_{v_{n-1}})|}{|\vartheta(I_{w_n})|} = \frac{\delta|\vartheta([l, l + 1])|}{|\vartheta([k, k + 1])|} = \frac{|\vartheta([k - j_0, k + d - j_0])|}{|\vartheta([k, k + 1])|}.
\]
This implies that
\[
\frac{|\vartheta([k - j_0, k + d - j_0])|}{|\vartheta([k, k + 1])|} = D^s(f)(\cdots 000 j_{n-1} \cdots j_1 j_0).
\]
Since $L^k(0) = \vartheta(k)$, the above equality says that all values of $L^k(0)$ are uniquely determined by

$$\{D^*(f)(\cdots 000j_{n-1}\cdots j_1j_0) \mid j_k = 0, 1, \cdots (d-1), k = 0, 1, \cdots, n - 1\}.$$  

Using Equation (20), we get that the linear model $L$ is uniquely determined by the dual derivative $D^*(f)(w^*)$. Thus we have a corollary of Theorem [19].

**Corollary 2.** Suppose $f$ and $g$ are two uniformly symmetric circle endomorphisms of the same degree $d \geq 2$. Let $h$ be the conjugacy between $f$ and $g$ such that $h(1) = 1$, that is,

$$h \circ f = g \circ h.$$  

Then $h$ is a symmetric homeomorphism if and only if the dual derivatives of $f$ and $g$ are the same, that is,

$$D^*(f)(w^*) = D^*(g)(w^*), \quad \forall w^* \in \Sigma^*.$$  

Following the above theorem we set a one-to-one correspondence between the Teichmüller space $\mathcal{TUS}$ and the space of all continuous dual derivatives:

$$\Pi = [(f, h_f)] \rightarrow D^*(f)(w^*).$$  

Therefore,

(21)  

$$\mathcal{TUS} = \{D^*(f)(w^*) \mid f \in \mathcal{US}\}$$

equipped with the Teichmüller metric $d_T(\cdot, \cdot)$. This is a complete space.

9. Characterization of dual derivatives

Assume $d = 2$ in this section. Suppose $f \in \mathcal{US}$. Let $D^*(f)(w^*)$ be its dual derivative. Then it is easy to see the following summation condition:

(22)  

$$\frac{1}{D^*(f)(w^*0)} + \frac{1}{D^*(f)(w^*1)} = 1, \quad \forall w^* \in \Sigma^*.$$  

Another non-trivial condition is the following compatibility condition:

(23)  

$$\prod_{n=0}^{\infty} \frac{D^*(f)(w^*0^1\cdots1)}{D^*(f)(w^*10^\cdots0)} = const, \quad \forall w^* \in \Sigma^*.$$  

The convergence is uniform. And moreover, if $f \in \mathcal{C}^{1+}$, then the convergence is exponential. We give a proof of this non-trivial condition as follows.

First let us set up a relation between the dual derivative $D^*(f)(w^*)$ and the linear model $L$. Suppose $\vartheta$ is the symmetric homeomorphism such that

$$L(x) = \vartheta Y \vartheta^{-1}(x) \quad \text{and} \quad \delta x = \vartheta F \vartheta^{-1}(x).$$

Then $L^k([0, 1]) = [\vartheta(k), \vartheta(k+1)]$ for every integer $k$.  

For any \( w^* = \cdots j_{n-1} \cdots j_0 \in \Sigma^* \), let \( w^*_n = j_{n-1} \cdots j_0 \) and define integers
\[
k = k(w^*_n) = \sum_{q=0}^{n-1} j_q 2^q \quad \text{and} \quad l = k(\sigma^*(w^*_n)) = \sum_{q=0}^{n-2} j_{q+1} 2^q.
\]
Then \( k = 2l + j_0 \). By the definitions,
\[
D^*(f)(w^*) = \lim_{n \to \infty} \frac{|I_{w^*_n}|}{|I_{\sigma^*(w^*_n)}|} = \lim_{n \to \infty} \frac{\delta(I_{w^*_n})}{\delta(I_{\sigma^*(w^*_n)})}.
\]
Note that
\[
I_{w^*_n} = G_{j_{n-1}} \circ \cdots \circ G_{j_0}(I) = F^{-1} \circ \Upsilon^{j_{n-1}} \circ \cdots \circ F^{-1} \circ \Upsilon^{j_0}([0, 1])
\]
\[
= F^{-n}(\Upsilon^{j_0 + 2j_1 + \cdots + 2^{n-1}j_{n-1}}([0, 1])) = F^{-n}([k, k+1]),
\]
and, similarly,
\[
I_{\sigma^*(w^*_n)} = F^{-n+1}([l, l+1]).
\]
Therefore, since \( \delta(F^{-n}(x)) = \delta^{-n}(x) \) and since
\[
\delta(I) = \delta(F(I)) = \delta(F(\Upsilon^l(0))) = \delta(\Upsilon^{2l}(F(0)) = \delta(2l),
\]
we have
\[
\frac{|\delta(I_{w^*_n})|}{|\delta(I_{\sigma^*(w^*_n)})|} = \frac{|\delta(F^{-n}([k, k+1]))|}{|\delta(F^{-n+1}([l, l+1]))|} = \frac{|\delta([k, k+1])|}{|\delta([l, l+1])|} = \frac{|\delta([k, k+1])|}{|\delta([k-j_0, k-j_0+2])|}.
\]
Let \( I = [0, 1] \). Since \( \delta I = I \cup L(I) \) and \( \vartheta(k) = L_k(0) \), we can rewrite
\[
\frac{|\delta([k, k+1])|}{|\delta([k-j_0, k-j_0+2])|} = \frac{|L_k(I)|}{|L_{k-j_0}^{}(I + L(I))|} = \left(1 + \frac{|L_k(I)|}{|L^{(-1)^{j_0}}_k(I)|}\right)^{-1}.
\]
Thus we get
\[
D^*(f)(w^*) = \lim_{n \to \infty} \frac{|L_k(I)|}{|L_{k-j_0}^{}(I + L(I))|} = \lim_{n \to \infty} \left(1 + \frac{|L^{(-1)^{j_0}}_k(I)|}{|L_k(I)|}\right)^{-1}.
\]
For any \( w^* = \cdots w^*_n = \cdots j_{n-1} \cdots j_1 j_0 \in \Sigma^* \), define
\[
sol(w^*) = \lim_{n \to \infty} \frac{|L_k(I)|}{|L^{(k-1)}_k(I)|}.
\]
(This is similar to a solenoid function defined in \([37][29]\)). Then, by considering \( w^* = v^* j_0 \), we have
\[
(24) \quad D^*(f)(v^*0) = \lim_{n \to \infty} \left(1 + \frac{|L^{k+1}_k(I)|}{|L_k(I)|}\right)^{-1} = (1 + \sol(v^*)^{-1})^{-1}
\]
and
\[
(25) \quad D^*(f)(v^*1) = \lim_{n \to \infty} \left(1 + \frac{|L^{k-1}_k(I)|}{|L_k(I)|}\right)^{-1} = \left(1 + \frac{1}{\sol(v^*)^{-1}}\right)^{-1}.
\]
These two equations combining with the summation condition \([22]\) imply that
\[
(26) \quad \sol(v^*1) = \frac{D^*(f)(v^*0)}{D^*(f)(v^*1)}.
\]
Since
\[
\frac{\delta L_k(I)}{\delta L_{k-1}(I)} = \frac{L^{2k}(\delta I)}{L^{2k-2}(\delta I)} = \frac{L^{2k}(I) + L^{2k+1}(I)}{L^{2k-2}(I) + L^{2k-1}(I)},
\]
we have the following formula:
\[
sol(w^0) = \lim_{n \to \infty} \frac{L^{2k-2}(I) + L^{2k-1}(I)}{L^{2k}(I)} = \frac{1 + L^{2k-2}(I)}{1 + sol(w^1)}.
\]
Equations (24), (25), and (26) imply that
\[
D^*(f)(w^{01}) = \frac{1 + [sol(w^{01})]^{-1}}{sol(w^{10})} = \frac{1 + [sol(w^{01})]^{-1}}{sol(w^{10})}.
\]
(\text{Note that for } w^1, 2k - 1 \text{ corresponds to } w^{01}.)
Similarly,
\[
D^*(f)(w^{011}) = \frac{1 + [sol(w^{011})]^{-1}}{sol(w^{100})} = \frac{1 + [sol(w^{011})]^{-1}}{sol(w^{10})}.
\]
Proceeding by induction, we conclude
\[
(27) \quad \text{sol}(w^*10\cdots0) = \prod_{i=0}^{n-1} \frac{D^*(f)(w^*01\cdots1)}{D^*(f)(w^*10\cdots0)}.
\]
From the quasymmetric distortion property (Lemma 1), (27) converges uniformly to \(sol(\cdot \cdot \cdot 0 \cdot \cdot \cdot 0)\) for all \(w^* \in \Sigma^*\). If \(f \in C^1\) from the Hölder distortion property (1), (27) converges exponentially to \(sol(\cdot \cdot \cdot 0 \cdot \cdot \cdot 0)\) for all \(w^* \in \Sigma^*\). We proved the compatibility condition (23).

In the paper [7, 8, 21], we further proved that the conditions (22) and (23) are also sufficient as follows.

**Theorem 13.** Let \(\Psi(w^*)\) be a positive continuous function on \(\Sigma^*\). Then it is a dual derivative of an \(f \in US\) if and only if it satisfies the conditions (22) and (23). Furthermore, if \(\Psi(w^*)\) is a Hölder continuous function on \(\Sigma^*\), then it is a dual derivative of an \(f \in C^1\) if and only if it satisfies the conditions (22) and (23).

The proof of this theorem is technical so we will not include in this paper. The reader who is interested in this topic can refer to [7, 8, 21].

**Remark 7.** A similar result to the second half of Theorem 13 was studied by Pinto and Sullivan in [20], for a solenoid function. They introduced a matching condition for a function on \(\Sigma^*\) and proved that a Hölder continuous function on \(\Sigma^*\) is a solenoid function of an \(f \in C^1\) if and only if it satisfies the matching condition. Furthermore, using some relation between the solenoid function and the linear model for an \(f \in US\), Cui proved in [6] that two uniformly symmetric circle endomorphisms are symmetric conjugate if and only if they...
have the same eigenvalues at the corresponding periodic points. We would like to note that [21] contains a much easy and straightforward understanding to these results.

From Theorem 13, we have the following representations for the Teichmüller spaces when $d = 2$:

$$\mathcal{T}^1 = \{ \Psi^*(w^*) | \Psi^*(w^*) \text{ is Hölder continuous and satisfies (22) and (23)} \}$$

and

$$\mathcal{TUS} = \{ \Psi^*(w^*) | \Psi^*(w^*) \text{ is continuous and satisfies (22) and (23)} \}$$

10. THE MAXIMUM DISTANCE AND THE TEICHMÜLLER DISTANCE.

We have introduce a Teichmüller metric $d_T(\cdot, \cdot)$ on $\mathcal{TUS}$ which is a complete metric. We also showed that $\mathcal{TUS}$ can be represented by continuous functions $\Psi^*(w^*)$ on $\Sigma^*$. For a function $\Psi^*(w^*)$, we can define the maximum norm

$$||\Psi^*|| = \max_{w^* \in \Sigma^*} |\Psi^*(w^*)|.$$

This gives a distance

$$d_{\max}(\Psi^*, \tilde{\Psi}^*) = ||\Psi^* - \tilde{\Psi}^*||$$

on $\mathcal{TUS}$. We call it the maximum metric. Since $\mathcal{TUS}$ contains all positive continuous functions satisfying (22) and (23) taking values in $(1, \infty)$. We have following theorems.

**Theorem 14.** The identity map

$$id_{TM} : (\mathcal{TUS}, d_T(\cdot, \cdot)) \rightarrow (\mathcal{TUS}, d_{\max}(\cdot, \cdot))$$

is uniformly continuous.

**Theorem 15.** The identity map

$$id_{MT} : (\mathcal{TUS}, d_{\max}(\cdot, \cdot)) \rightarrow (\mathcal{TUS}, d_T(\cdot, \cdot))$$

is continuous.

**Corollary 3.** The topologies induced by the Teichmüller metric $d_T(\cdot, \cdot)$ and by the maximum metric $d_{\max}(\cdot, \cdot)$ are the same.

**Proof of Theorem 14.** Suppose $\Pi, \Pi' \in \mathcal{TUS}$. Suppose $K = \exp(2d_T(\Pi, \Pi')) \geq 1$. For any $\epsilon > 0$, we have two marked circle endomorphisms $(f, h_f) \in \Pi$ and $(g, h_g) \in \Pi'$ such that

$$h = h_g \circ h_f^{-1} : T \rightarrow T$$

can be extended to a $K(1 + \epsilon)$-quasiconformal map $\tilde{h}$ defined on an annulus $\{ z \in \mathbb{C} | \frac{1}{r} < |z| < r \}$ for some $r > 1$. This implies that there is a $\delta > 0$ and
$M = M(K, \epsilon) > 0$ such that $M \rightarrow 1$ as $K \rightarrow 1$ and $\epsilon \rightarrow 0$ and such that $H$, which is a lift of $h$, is a $(\delta, M)$-quasisymmetric, that is,

$$M^{-1} \leq \frac{|H(y) - H(\frac{x+y}{2})|}{|H(\frac{x+y}{2}) - H(x)|} \leq M$$

for any $x, y$ with $|x - y| \leq \delta$. Note that $h \circ f = g \circ h$ and $H \circ F = G \circ H \pmod{1}$.

For any point $w^* = \cdots w_n^* \in \Sigma_A$, we have that

$$I_{w^*_n, f} \in \eta_{n, f} \quad \text{and} \quad I_{\sigma^*(w^*_n), f} \in \eta_{n-1, f}$$

and

$$I_{w^*_n, g} = H(I_{w^*_n, g}) \in \eta_{n, g} \quad \text{and} \quad I_{\sigma^*(w^*_n), g} = H(I_{\sigma^*(w^*_n), g}) \in \eta_{n-1, g}.$$

Note that

$$I_{w^*_n, f} \subset I_{\sigma^*(w^*_n), f} \quad \text{and} \quad I_{w^*_n, g} \subset I_{\sigma^*(w^*_n), g}.$$

Let $n_0 > 0$ be an integer such that

$$|I_{\sigma^*(w^*_n), f}| \leq \delta$$

for all $n \geq n_0$. Then $H|I_{\sigma^*(w^*_n), f}$ is a $M$-quasisymmetric homeomorphism.

By rescaling $I_{\sigma^*(w^*_n), f}$ and $I_{\sigma^*(w^*_n), g}$ into the unit interval $[0, 1]$ by linear maps, we can think $H|I_{\sigma^*(w^*_n), f}$ as a $M$-quasisymmetric homeomorphism of $[0, 1]$ and fixes $0$ and $1$. Then Lemma 1 implies that

$$\frac{1}{D^*(g)(w^*_n)} - \frac{1}{D^*(f)(w^*_n)} = \left| \frac{|H(I_{w^*_n, f})|}{|H(I_{\sigma^*(w^*_n), f})|} - \frac{|I_{w^*_n, f}|}{|I_{\sigma^*(w^*_n), f}|} \right| \leq \zeta(M).$$

This implies that

$$|D^*(g)(w^*) - D^*(f)(w^*)| \leq \zeta(M).$$

Therefore,

$$d_{\text{max}}(D^*(g), D^*(f)) \leq \zeta(M(d_T(\Pi, \Pi')(1 + \epsilon)).$$

This proved the theorem. \qed

Proof of Theorem 1. Suppose $id_{MT}$ is not continuous. That is, we have a real number $\epsilon > 0$ and a point $\Psi^*$ and a sequence of points $\{\Psi^*_m\}_{m=1}^{\infty}$ in the Teichmüller space $TU(S)$ such that

$$d_{\text{max}}(\Psi^*_m, \Psi^*) = \|\Psi^*_m - \Psi^*\| \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty$$

but

$$d_T(\Pi^m, \Pi^*) \geq \epsilon, \quad \forall \, m,$$

where $\Pi$ and $\Pi^m$ are the corresponding points for $\Psi^*$ and $\Psi^*_m$. Let $(f, h_f) \in \Pi$ be a fixed representation and $(f_m, h_{f_m}) \in \Pi_m$ for each $m$ be a representation.

Let $F$ and $F_m$ be the corresponding circle endomorphisms of the real line. Let $\{\eta_n\}_{n=0}^{\infty}$ and $\{\eta_{m,n}\}_{n=0}^{\infty}$ be the corresponding sequences of nested Markov partitions. Since $\|D^*(f_m) - D^*(f)\| \rightarrow 0$ as $m \rightarrow \infty$, we have a constant
\[ a = a(D^*(f)) > 0 \text{ such that } D^*(f_m)(w^*) \geq a \text{ for sufficient large } m \text{ and all } w^* \in \Sigma^*. \] Let us assume this true for all \( m \). Since \( \Sigma^* \) is a compact set, we have that there is another constant \( b = b(a) > 0 \) such that \( D^*(f_m)(w^*_n) \geq b \) (see (5)) for all \( m \) and all \( n \). This says that the collection of the sequences \( \{\eta_{m,n}\}_{n=0}^{\infty} \) of nested Markov partitions has uniformly bounded geometry. From a method in [20], which is also shown in the proof of Corollary 1 and which gives a calculation of quasisymmetric dilatation from bounded geometry, we have a constant \( M > 0 \) such that the quasisymmetric dilatations of all \( H_m \) are less than or equal to \( M \), that is, Since

\[ QS_m = \sup_{x \in \mathbb{R}, t > 0} \frac{|H_m(x + t) - H_m(x)|}{|H_m(x) - H_m(x - t)|} \leq M. \]

for every \( m > 0 \). This says that by modulo all M"obius transformations, \( \{H_m\}_m \) is a compact subset in the universal Teich"uller space. Thus \( \{(f_m, h_m)\}_{m=1}^{\infty} \) is a compact set in \( TUS \). So it has a convergent subsequence. Let us assume that \( \{(f_m, h_m)\}_{m=1}^{\infty} \) itself is convergent and converges to \( [(f_0, h_0)] \) as \( m \to \infty \). Let \( H_0 \) be a lift of \( h_0 \). Then \( H_m \) tends to \( H_0 \) modulo all M"obius transformations under the maximal norm on the real line. Assume \( H_m \to H_0 \) uniformly on the real line as \( m \to \infty \).

For any \( w^*_n \), let \( I_{w^*_n, f_m} \in \eta_{m,n} \) and \( I_{w^*_n, f_0} \in \eta_{n,f_0} \). We have that \( |I_{w^*_n, f_m}| \to |I_{w^*_n, f_0}| \) as \( m \to \infty \) for each fixed \( n \) and \( w^*_n \).

Since the sequences \( \{\eta_{n,m}\}_{n=0}^{\infty} \) of nested Markov partitions have uniformly bounded geometry, this again says that there are constants \( C = C(S) > 0 \) and \( 0 < \mu = \mu(D^*(f)) < 1 \) such that \( \nu_{n,m} \leq C \mu^n \) for all \( n \) and \( m \), where

\[ \nu_{n,m} = \max_{I \in \eta_{m,n}} |I|. \]

This implies that \( D^*(f_m)(w^*_n) \to D^*(f_m)(w^*) \) and \( D^*(f_0)(w^*_n) \to D^*(f_0)(w^*) \) as \( n \to \infty \) uniformly on \( m \geq 1 \) and \( w^* \in \Sigma^* \). Thus we can change double limits for each \( w^* \in \Sigma^* \),

\[ D^*(f)(w^*) = \lim_{m \to \infty} D^*(f_m)(w^*) = \lim_{m \to \infty} \lim_{n \to \infty} D^*(f_m)(w^*_n) \]

\[ = \lim_{n \to \infty} \lim_{m \to \infty} D^*(f_m)(w^*_n) = \lim_{n \to \infty} D^*(f_0)(w^*_n) = D^*(f_0)(w^*). \]

From Corollary 2 this implies that \( [(f,h)] = [(f_0,h_0)] = \Pi \). This is a contradiction to our original assumption. The contradiction says that \( id_{MT} \) is continuous at each point \( \Psi^* \).

11. \( \sigma \)-INVARINTE MEASURES AND DUAL \( \sigma^* \)-INVARINTE MEASURES

Consider the symbolic dynamical system \((\Sigma,\sigma)\) and a positive Hölder continuous function \( \psi(w) \). The standard Gibbs theory (refer to [5, 30, 31, 34, 35])
implies that there is a number $P = P(\log \psi)$ called the pressure and a $\sigma$-invariant probability measure $\mu = \mu_\psi$ such that

$$C^{-1} \leq \frac{\mu([w_n])}{\exp(-P_n + \sum_{i=0}^{n-1} \log \psi(\sigma^i(w)))} \leq C$$

for any left cylinder $[w_n]$ and any point $w = w_n \ldots$, where $C$ is a fixed constant. Here, $\mu$ is a $\sigma$-invariant probability measure means that $\mu(\sigma^{-1}(A)) = \mu(A)$ for all Borel sets of $\Sigma$. A $\sigma$-invariant probability measure satisfying the above inequalities is called the Gibbs measure with respect to the given potential $\log \psi$.

Two positive H"older continuous functions $\psi_1$ and $\psi_2$ are said to be cohomologous equivalent if there is a continuous function $u = u(w)$ on $\Sigma$ such that

$$\log \psi_1(w) - \log \psi_2(w) = u(\sigma(w)) - u(w).$$

If two functions are cohomologous to each other, they have the same Gibbs measure. Therefore, the Gibbs measure can be thought of as a representation of a cohomologous class.

The Gibbs measure $\mu$ for a given potential $\log \phi$ is also an equilibrium state for this potential as follows. Consider the measure-theoretical entropy $h_{\mu}(\sigma)$. Since the Borel $\sigma$-algebra of $\Sigma$ is generated by all left cylinders, then $h_{\mu}(\sigma)$ can be calculated as

$$h_{\mu}(\sigma) = \lim_{n \to \infty} \frac{1}{n} \sum_{w_n} \left( \frac{\mu([w_n])}{\mu(\sigma([w_n]))} \log \mu([w_n]) \right)$$

$$= \lim_{n \to \infty} \sum_{w_n} \left( -\mu([w_n]) \log \frac{\mu([w_n])}{\mu(\sigma([w_n]))} \right),$$

where $w_n$ runs over all words $w_n = i_0 \cdots i_{n-1}$ of $\{0, 1, \ldots, d-1\}$ of length $n$. Then $\mu$ is an equilibrium state in the sense that

$$P(\log \psi) = h_{\mu}(\sigma) + \int_{\Sigma} \log \psi(w) d\mu(w) = \sup \{ h_{\nu}(\sigma) + \int_{\Sigma} \log \psi(w) d\nu(w) \},$$

where $\nu$ runs over all $\sigma$-invariant probability measures. The measure $\mu$ is unique in this case.

There is a natural way to transfer a $\sigma$-invariant probability measure $\mu$ (not necessarily a Gibbs measure) to a $\sigma^*$-invariant probability measure $\mu^*$ as follows:

Given any right cylinder $[w_n^*]$ in $\Sigma^*$ where $w_n^* = j_{n-1} \cdots j_0$, then

$$w_n = i_0 \cdots i_{n-1} = j_{n-1} \cdots j_0 = w_n^*,$$

which defines a left cylinder

$$[w_n] = \{ w' = i'_0 \cdots i'_{n-1} i'_n \cdots | i'_0 = i_0, \ldots, i'_{n-1} = i_{n-1} \}.$$
Define 

\[ \mu^*([w_n^*]) = \mu([w_n]). \]

Then 

\[ \mu^*([w_n^*]) = \mu([w_n]) = \mu(\sigma^{-1}([w_n])) \]

\[ = \mu(\cup_{i=0}^{d-1}[iw_n]) = \sum_{i=0}^{d-1} \mu([iw_n]) = \sum_{j=0}^{d-1} \mu^*([jw_n^*]) \]

This implies that \( \mu^* \) satisfies the finite additive law for all cylinders, i.e., if \( A_1, \cdots, A_k \) are finitely many pairwise disjoint right cylinders in \( \Sigma^* \), then

\[ \mu^*(\cup_{i=1}^{k} A_k) = \sum_{i=1}^{k} \mu^*(A_i). \]

Also \( \mu^* \) satisfies the continuity law in the sense that if \( \{A_n\}_{n=1}^{\infty} \) is a sequence of decreasing cylinders and tends to the empty set, then \( \mu^*(A_n) \) tends to zero as \( n \) goes to \( \infty \). The reason is that since a cylinder of \( \Sigma^* \) is a compact set, a sequence of decreasing cylinders tending to the empty set must be eventually all empty. The Borel \( \sigma \)-algebra in \( \Sigma^* \) is generated by all right cylinders. So \( \mu^* \) extends to a measure on \( \Sigma^* \). We have the following proposition.

**Proposition 1.** The probability measure \( \mu^* \) is a \( \sigma^* \)-invariant probability measure.

**Proof.** We have seen that \( \mu^* \) is a measure on \( \Sigma^* \). Since \( \mu^*(\Sigma^*) = 1 \), it is a probability measure. For any right cylinder \([w_n^*]\),

\[ \mu^*((\sigma^*)^{-1}([w_n])) = \mu^*(\cup_{j=0}^{d-1}[w_n^*j]) = \sum_{j=0}^{d-1} \mu^*([w_n^*j]) = \sum_{i=0}^{d-1} \mu([w_n^*i]) = \mu(\cup_{i=0}^{d-1}[w_n^*i]) = \mu^*([w_n^*]). \]

So \( \mu^* \) is \( \sigma^* \)-invariant. We proved the proposition. \( \square \)

We call \( \mu^* \) a dual \( \sigma^* \)-invariant probability measure. A natural question now is as follows.

**Question 1.** Is a dual invariant probability measure a Gibbs measure with respect to some continuous or Hölder continuous potential on \( \Sigma^* \)?

Some more interesting geometric questions from the Teichmüller point of view are the followings. Consider a metric induced from the dual probability invariant measure \( \mu^* \) (in the case that \( \mu^* \) is supported on the whole \( \Sigma^* \) and has no atomic point), that is,

\[ d(w^*, \tilde{w}^*) = \mu^*([w_n^*]) \]

where \([w_n^*]\) is the smallest right cylinder containing both \( w^* \) and \( \tilde{w}^* \).
Question 2. Is $\sigma^*$ differentiable under the metric $d(\cdot, \cdot)$? More precisely, does
the limit
$$\frac{d\sigma^*}{dw^*}(w^*) = \lim_{n \to \infty} \frac{\mu^*(\sigma^*([w_n^*]))}{\mu^*([w_n^*])}$$
exists for every $w^* = \cdots w_n^* \in \Sigma^*$? If it exists, is the limiting function continuous or Hölder continuous on $\Sigma^*$?

Question 3. Given a positive continuous or Hölder continuous function $\psi^*(w^*)$ on $\Sigma^*$. Can we find a $\sigma^*$-invariant measure $\mu^*$ on $\Sigma^*$, such that the right shift map $\sigma^*$ under the metric $d(\cdot, \cdot)$ induced from this measure is $C^1$ with the derivative $\psi^*(w^*)$?

Actually, there is a measure-theoretical version related to these questions. I will first give a review of this theory.

12. $g$-measures

Let $X = \Sigma^*$ (or $\Sigma$) and let $f$ be $\sigma^*$ (or $\sigma$). Let $\mathcal{B}$ be the Borel $\sigma$-algebra of $X$. Let $\mathcal{M}(X)$ be the space of all finite Borel measures on $X$. Let $\mathcal{M}(X, f)$ be the space of all $f$-invariant probability measures in $\mathcal{M}(X)$. Let $\mathcal{C}(X)$ be the space of all continuous real functions on $X$. Then $\mathcal{M}(X)$ is the dual space of $\mathcal{C}(X)$. Denote

$$<\phi, \mu> = \int_X \phi(x) d\mu, \quad \phi \in \mathcal{C}(X) \text{ and } \mu \in \mathcal{M}(X).$$

A real non-negative continuous function $\psi$ on $X$ is called a $g$-function [25] if

$$\sum_{f(y) = x} \psi(y) = 1.$$ 

For a $g$-function $\psi(x)$, define the transfer operator $\mathcal{L}_\psi$ from $\mathcal{C}(X)$ into itself as

$$\mathcal{L}_\psi \phi(x) = \sum_{f(y) = x} \phi(y) \psi(y), \quad \phi \in \mathcal{C}(X).$$

One can check that $\mathcal{L}_\psi \phi = \mathcal{L}_1(\psi \phi)$ and if $\psi$ is a $g$-function, then $\mathcal{L}_\psi 1 = 1$. Let $\mathcal{L}^*_\psi$ be the dual operator of $\mathcal{L}_\psi$, that is, $\mathcal{L}_\psi^*$ is the operator from $\mathcal{M}(X)$ into itself satisfying

$$<\phi, \mathcal{L}_\psi^* \mu> = <\mathcal{L}_\psi^* \phi, \mu>, \quad \forall \phi \in \mathcal{C}(X) \text{ and } \forall \mu \in \mathcal{M}(X).$$

Definition 12. Suppose $\psi$ is a $g$-function. Then a probability measure $\mu \in \mathcal{M}(X)$ is called a $g$-measure for $\psi$ if it is a fixed point of $\mathcal{L}_\psi^*$, that is,

$$\mathcal{L}_\psi^* \mu = \mu.$$ 

Lemma 3. Suppose $\psi$ is a $g$-function. Then any $g$-measure $\mu$ for $\phi$ is an $f$-invariant measure.
Proof. For any Borel set $B \in \mathcal{B}$,
\[
\mu(f^{-1}(B)) = \langle 1_{f^{-1}(B)}, \mu \rangle = \langle \mathcal{L}_1^B \circ f, \mu \rangle = \mu(B).
\]
So $\mu$ is $f$-invariant. \qed

For any $\mu \in \mathcal{M}(X)$, let $\tilde{\mu} = \mathcal{L}_1^* \mu$.

**Lemma 4.**
\[
\tilde{\mu}(B) = \sum_{j=0}^{d-1} \mu(f(B \cap [j])),
\]
where $B$ is any Borel subset in $\mathcal{B}$ and $[j]$ is the right (or left) cylinder of $j$. Moreover, if $\mu \in \mathcal{M}(X,f)$, $\mu$ is absolutely continuous with respect to $\tilde{\mu}$.

Proof. For any Borel subset $B \in \mathcal{B}$,
\[
\tilde{\mu}(B) = \langle 1_B, \mathcal{L}_1^* \mu \rangle = \langle \mathcal{L}_1^B, \mu \rangle.
\]
But
\[
\mathcal{L}_1^B(x) = \sum_{j=0}^{d-1} 1_B(xj) = \sum_{j=0}^{d-1} 1_{f(B \cap [j])}(x).
\]
So we have that
\[
\tilde{\mu}(B) = \sum_{j=0}^{d-1} \mu(f(B \cap [j])).
\]

If $\mu$ is $f$-invariant, then we have that
\[
\tilde{\mu}(B) = \sum_{j=0}^{d-1} \mu(f(B \cap [j])) = \sum_{j=0}^{d-1} \mu(f^{-1}(f(B \cap [j]))) \geq \sum_{j=0}^{d-1} \mu(B \cap [j]) = \mu(B).
\]
Therefore, $\mu(B) = 0$ whenever $\tilde{\mu}(B) = 0$. So $\mu$ is absolutely continuous with respect to $\tilde{\mu}$. \qed

Suppose $\mu \in \mathcal{M}(X,f)$. Then $\mu$ is absolutely continuous with respect to $\tilde{\mu}$. So the Radon-Nikodým derivative
\[
D_\mu(x) = \frac{d\mu}{d\tilde{\mu}}(x), \quad \tilde{\mu} - a.e. \ x
\]
of $\mu$ with respect to $\tilde{\mu}$ exists $\tilde{\mu}$-a.e. and is a $\tilde{\mu}$-measurable function.

The following theorem is in Ledrappier’s paper [26] and is used in Walters’ paper [39] for the study of a generalized version of Ruelle’s theorem. We give a complete proof here.

**Theorem 16.** Suppose $\psi \geq 0$ is a $g$-function and $\mu \in \mathcal{M}(X)$ is a probability measure. The followings are equivalent:

i) $\mu$ is a $g$-measure for $\psi$, i.e., $\mathcal{L}_\psi^* \mu = \mu$.

ii) $\mu \in \mathcal{M}(X,f)$ and $D_\mu(x) = \psi(x)$ for $\tilde{\mu}$-a.e. $x$. 

iii) $\mu \in \mathcal{M}(X,f)$ and
\[ E[\phi|f^{-1}(B)](x) = \mathcal{L}_\psi \phi(fx) = \sum_{fy=fx} \psi(y)\phi(y), \text{ for } \mu\text{-a.e. } x, \]
where $E[\phi|f^{-1}(B)]$ is the conditional expectation of $\phi$ with respect to $f^{-1}(B)$.

iv) $\mu \in \mathcal{M}(X,f)$ and is an equilibrium state for the potential $\log \psi$ with the meaning that
\[ 0 = h_\mu(f) + \int_X \log \psi \, d\mu = \sup \{ h_\nu(f) + \int_X \log \psi \, d\nu \mid \nu \in \mathcal{M}(X,f) \}. \]
(Note that the pressure $P(\log \psi) = 0$ for a $g$-function $\psi$.)

Proof. We first note that since $C(X)$ is dense in the space $L^1(\tilde{\mu})$ of all $\tilde{\mu}$-measurable and integrable functions (as well as in the space $L^1(\mu)$), then $<\cdot,\cdot>$ can be extended to $L^1(\tilde{\mu})$ (as well as to $L^1(\mu)$). We already know that a $g$-measure for $\psi$ is $f$-invariant.

First, we prove i) implies ii). For any $\phi(x) \in C(X)$,
\[ <\phi\psi,\tilde{\mu}] = <\phi\psi, L_1^\ast \mu > = < \mathcal{L}_1(\phi\psi), \mu > = < \mathcal{L}_1^\ast \phi, \mu > = < \phi, \mu >, \]
Thus $D_\mu = \psi$ for $\tilde{\mu}$-a.e. $x$.

Second, we prove that ii) implies i). Since, for any $\phi(x) \in C(X)$,
\[ <\phi, \mu > = < \phi D_\mu, \tilde{\mu}] = <\phi, \mu > < \phi, \mathcal{L}_1^\ast \mu > = < \mathcal{L}_1(\phi\psi), \mu > < \mathcal{L}_1^\ast \phi, \mu > = < \phi, \mathcal{L}_1^\ast \phi >. \]
This implies $\mathcal{L}_1^\ast \mu = \mu$. Thus $\mu$ is a $g$-measure for $\psi$.

We prove i) implies iii). For any Borel set $B \in \mathcal{B}$,
\[ < (L_\psi \phi) \circ f \cdot 1_{f^{-1}(B)}, \mu > = < (L_1(\psi \phi)) \circ f \cdot 1_B \circ f, \mu > = < \mathcal{L}_1(\psi \phi) \cdot 1_B, \mu > = < \mathcal{L}_1(\psi \phi_1 B \circ f), \mu > = < \mathcal{L}_\psi(\phi_1 B \circ f), \mu > = < \phi_1 B \circ f, \mu > = < \phi 1_{f^{-1}(B)}, \mu > \]
That is,
\[ E[\phi|f^{-1}(B)] = (L_\psi \phi) \circ f, \mu\text{-a.e. } x. \]

Note that
\[ ((L_\psi \phi) \circ f)(x) = \sum_{y \in f^{-1}(fx)} \psi(y)\phi(y). \]

We now prove that iii) implies i). Since, for any $\phi \in C(X)$,
\[ E[\phi|f^{-1}(B)] = \mathcal{L}_\psi \phi(fx), \mu\text{-a.e. } x, \]
then,
\[ < \phi, \mu > = < (L_\psi \phi) \circ f, \mu > = < \mathcal{L}_\psi \phi, \mu > = < \phi, \mathcal{L}_1^\ast \mu >. \]
Thus $\mathcal{L}_1^\ast \mu = \mu$. 
We prove that \( ii \) implies \( iv \). For any \( \nu \in \mathcal{M}(X, f) \), let
\[
D_\nu = \frac{d\nu}{d\tilde{\nu}}, \quad \tilde{\nu} - a.e.,
\]
be the Radon-Nikodým derivative. We claim that
\[
h_\nu(f) = -\int_X \log D_\nu d\nu.
\]
We prove this claim. Since \( -\log D_\nu \) is a non-negative \( \tilde{\nu} \)-measurable function and since \( \nu \) is absolutely continuous with respect to \( \tilde{\nu} \), it is also a \( \nu \)-measurable function. Thus
\[
\int_X -\log D_\nu d\nu = \int_X -D_\nu \log D_\nu d\tilde{\nu}.
\]
By the definition,
\[
D_\nu(x) = \lim_{n \to \infty} \frac{\nu([i_0i_1 \cdots i_{n-1}])}{\nu([i_1 \cdots i_{n-1}])}, \quad \tilde{\nu} - a.e. \quad x = i_0i_1 \cdots i_{n-1} \cdots.
\]
Let
\[
D_{n,\nu}(x) = \frac{\nu([i_0i_1 \cdots i_{n-1}])}{\nu([i_1 \cdots i_{n-1}])}.
\]
Then
\[
D_\nu(x) = \lim_{n \to \infty} D_{n,\nu}(x), \quad \tilde{\nu} - a.e. \quad x.
\]
and
\[
-D_\nu(x) \log D_\nu(x) = \lim_{n \to \infty} (-D_{n,\nu}(x) \log D_{n,\nu}(x)), \quad \tilde{\nu} - a.e. \quad x.
\]
Since \(-t \log t\) is a positive bounded function on \([0, 1]\), by the Lebesgue control convergence theorem,
\[
\int_X \lim_{n \to \infty} (-D_{n,\nu}(x) \log D_{n,\nu}(x)) \, d\tilde{\nu} = \lim_{n \to \infty} \int_X -D_{n,\nu}(x) \log D_{n,\nu}(x) d\tilde{\nu}.
\]
However,
\[
\int_X -D_{n,\nu}(x) \log D_{n,\nu}(x) d\tilde{\nu}
\]
\[
= \sum_{[i_0 \cdots i_{n-1}]} -\nu([i_0i_1 \cdots i_{n-1}]) \log \left( \frac{\nu([i_0i_1 \cdots i_{n-1}])}{\nu([i_1 \cdots i_{n-1}])} \right) \tilde{\nu}([i_0i_1 \cdots i_{n-1}])
\]
\[
= \sum_{[i_0 \cdots i_{n-1}]} -\nu([i_0i_1 \cdots i_{n-1}]) \log \left( \frac{\nu([i_0i_1 \cdots i_{n-1}])}{\nu([i_1 \cdots i_{n-1}])} \right).
\]
Note that \( \tilde{\nu}([i_0i_1 \cdots i_{n-1}]) = \nu([i_1 \cdots i_{n-1}]). \) But we know that
\[
h_\nu(f) = \lim_{n \to \infty} \frac{1}{n} \sum_{[i_0 \cdots i_{n-1}]} -\nu([i_0i_1 \cdots i_{n-1}]) \log \nu([i_0i_1 \cdots i_{n-1}])
\]
\[
= \lim_{n \to \infty} \sum_{[i_0 \cdots i_{n-1}]} -\nu([i_0i_1 \cdots i_{n-1}]) \log \left( \frac{\nu([i_0i_1 \cdots i_{n-1}])}{\nu([i_1 \cdots i_{n-1}])} \right).
\]
We proved the claim.
The claim says that $h_\nu(f) = - <\log D_\nu, \nu >$ for any $\nu \in \mathcal{M}(X, f)$. Then

$$h_\nu(f) + <\log \psi, \nu > = <\log \frac{\psi}{D_\nu}, \nu > \leq <\frac{\psi}{D_\nu} - 1, \nu > = <\frac{\psi}{D_\nu}, \nu > - 1$$

$$= <\psi, \hat{{\nu}} > - 1 = <\mathcal{L}_1\psi, \nu > - 1 = <1, \nu > - 1 = 1 - 1 = 0.$$  

Note that here we use the inequality

$$(28) \quad \log t \leq t - 1 \quad \text{and} \quad \log t = t - 1 \quad \text{if and only if} \quad t = 1.$$  

The assumption in (ii) is that $D_\mu = \psi, \hat{\mu} - a.e. x$. But $\mu \ll \hat{\mu}, D_\mu = \psi, \mu$-a.e. $x$ too. So we have that

$$h_\mu + \int_X \log \psi \, d\mu = 0.$$  

So $\mu$ is an equilibrium state for the potential $\log \psi$ in the meaning that

$$0 = h_\mu(f) + \int_X \log \psi \, d\mu = \sup \{ h_\nu(f) + \int_X \log \psi \, d\nu \mid \nu \in \mathcal{M}(X, f) \}.$$  

Last we prove that (iv) implies (i). Suppose $\mu \in \mathcal{M}(X, f)$ is an equilibrium state for the potential $\log \psi$. We have that

$$h_\mu(f) + <\log \psi, \mu > = 0.$$  

We already know that

$$h_\mu(f) + <\log D_\mu, \mu > = 0.$$  

So we have that

$$h_\mu(f) + <\log \psi, \mu > = h_\mu(f) + <\log D_\mu, \mu >.$$  

Therefore,

$$0 = <\log \psi - \log D_\mu, \mu > = <\frac{\psi}{D_\mu} - 1, \mu >$$

$$\leq <\frac{\psi}{D_\mu} - 1, \mu > = <\frac{\psi}{D_\mu}, \mu > - 1 = <\psi, \hat{\mu} > - 1$$

$$= <\psi, \mathcal{L}_1\mu > = <\mathcal{L}_1\psi, \mu > - 1 = <1, \mu > - 1 = 1 - 1 = 0.$$  

Formula (28) implies that

$$(29) \quad \frac{\psi(x)}{D_\mu(x)} = 1, \quad \mu - a.e. x.$$  

**Remark 8.** This cannot implies that

$$\frac{\psi(x)}{D_\mu(x)} = 1, \quad \hat{\mu} - a.e. x,$$

since $\hat{\mu}$ may not be absolutely continuous with respect to $\mu$. So this will not implies (ii). However, if $\psi(x) > 0$ for all $x \in X$, then

$$D_{\hat{\mu}}(x) = \frac{d\hat{\mu}}{d\mu}(x) = \frac{1}{D_\mu(x)} = \frac{1}{\psi(x)}, \quad \mu - a.e. x.$$
This implies that \( \tilde{\mu} \) is absolutely continuous with respect to \( \mu \). Then Equation (29) implies \( ii \).

For any \( \phi(x) \in C(X) \),

\[
\langle \phi, L^*_{\psi} \mu \rangle = \langle L_{\psi} \phi, \mu \rangle = \langle L_1 (\psi \phi) \mu, \mu \rangle = \langle \psi \phi, L^*_{\tilde{\mu}} \mu \rangle = \langle \phi \psi, \tilde{\mu} \rangle = \langle \phi \psi, D^* \mu \rangle = \langle \phi, \mu \rangle.
\]

This says that \( L^*_{\psi} \mu = \mu \). We proved \( i \).

\( \square \)

For any \( \sigma \)-invariant probability measure \( \mu \), let \( \mu^* \) be the dual \( \sigma^* \)-invariant probability measure which we have constructed in the previous section. Then we have a \( \tilde{\mu} \)-measurable function

\[
D_{\mu}(w) = \lim_{n \to \infty} \frac{\mu([w_n])}{\mu([\sigma(w_n)])}, \quad \text{for } \tilde{\mu}\text{-a.e. } w = w_n \cdots \in \Sigma
\]

and a \( \tilde{\mu}^* \)-measurable function

\[
D_{\mu^*}(w^*) = \lim_{n \to \infty} \frac{\mu^*([w^*_n])}{\mu^*([\sigma^*(w^*_n)])}, \quad \text{for } \tilde{\mu}^*\text{-a.e. } w^* = \cdots w^*_n \in \Sigma^*.
\]

Now a question related to those questions in the end of the previous section is as follows.

**Question 4.** Can \( D_{\mu^*}(w^*) \) (or \( D_{\mu}(w) \)) be extended to a continuous or Hölder continuous \( g \)-function?

In the next two sections, we give an affirmative answer to this question.

13. **Gibbs measures and dual geometric Gibbs measures**

Consider \( f \in C^{1+} \). One over the derivative \( 1/f'(x) \) can be lifted to a positive Hölder continuous function

\[
\psi(w) = \psi_f(w) = \frac{1}{f'(\pi_f(w))}
\]

on the symbolic space \( \Sigma \). By thinking of \( \log \psi \) as a potential on \( (\Sigma, \sigma) \), there is a unique \( \sigma \)-invariant measure \( \mu = \mu_{\psi} \) (Gibbs measure for the potential \( \log \psi \)) as we have mentioned in the previous section such that

\[
C^{-1} \leq \frac{\mu([w_n])}{\exp(\sum_{i=0}^{n-1} \log \psi(\sigma^i(w)))} \leq C
\]

for any left cylinder \([w_n]\) and any \( w = w_n \cdots \in [w_n] \), where \( C \) is a fixed constant. (Note that \( P = P(\log \psi) = 0 \) in this case.)
Every element \( \Phi = [(f,h)] \) in the Teichmüller space \( T^{C_{1+}} \) can also be represented by the Gibbs measure \( \mu \) for the potential \( \log \psi(w) \). The reason is that for every \((g,h) \in \Psi\), \( h = h_f \circ h_g^{-1} \) is a \( C^1 \) diffeomorphism of \( T \) such that
\[
 f(h(x)) = h(g(x)).
\]
Then
\[
 f'(h(x))h'(x) = h'(g(x))g'(x).
\]
Therefore,
\[
 \log \psi_f(w) - \log \psi_g(w) = \log h'(w) - \log h'(\sigma(w)).
\]
So \( \psi_g \) and \( \psi_f \) are cohomologous to each other.

The Gibbs measure \( \mu \) in this context enjoys the following geometric property too: The push-forward measure \( \mu_{geo} = (\pi f)_* \mu \) is a \( C^{1+\alpha} \) smooth \( f \)-invariant measure for some \( 0 < \alpha \leq 1 \). This means that there is a \( C^\alpha \) function \( \rho \) on \( T \) such that
\[
 \mu_{geo}(A) = \int_A \rho(x) dx, \quad \text{for all Borel subsets } A \text{ on } T.
\]

There is another way to find the density \( \rho \). First it is a standard method to find an invariant measure for a dynamical system \( f \). Let \( \mu_0 \) be the Lebesgue measure on \( T \). Consider the push-forward measure \( \mu_n = (f^n)_* \mu_0 \) by the \( n^{th} \) iterates of \( f \). Sum up these measures to get
\[
 \nu_n = \frac{1}{n} \sum_{k=0}^{n-1} \mu_n.
\]
Any weak limit of a subsequence of \( \{\nu_n\} \) will be an \( f \)-invariant measure. Since we start with an \( f \in C^{1+} \), we can prove that the sequence \( \{\mu_n\} \) is actually convergent in the \( C^1 \) topology to a \( C^{1+\alpha} \) smooth measure \( \mu_{geo} \) for some \( 0 < \alpha \leq 1 \) as follows: Each \( \mu_n = (f^n)_* \mu_0 \) has an \( \alpha \)-Hölder continuous density
\[
 \rho_n(x) = \sum_{f^n(y)=x} \frac{1}{(f^n)'(y)}.
\]
Following the theory of transfer operators (refer to [22]), \( \rho_n(x) \) converges uniformly to an \( \alpha \)-Hölder continuous function \( \rho(x) \). Thus
\[
 \mu_{geo}(A) = \int_A \rho(x) dx
\]
is the limit of \( \mu_n \) and is a \( C^{1+\alpha} \) smooth \( f \)-invariant probability measure.

Let \( y = h(z) = \mu_{geo}([1,z]) \) be the distribution function of \( \mu_{geo} \), where \([1,z] \) is the oriented arc on \( T \) from 1 to \( z \). Then \( \varsigma = h(z) \) is a \( C^{1+\alpha} \)-diffeomorphism of \( T \). Let
\[
 g(\varsigma) = h \circ f \circ h^{-1}(\varsigma), \quad z = h^{-1}(\varsigma).
\]
(Note that $g$ here means a circle endomorphism, not a $g$-function!) Then $g$ preserves the Lebesgue measure $d\varsigma$ (which means that $g_\ast(d\varsigma) = d\varsigma$, or equivalently, the Lebesgue measure is $g$-invariant). Since the Lebesgue measure is an ergodic $g$-invariant measure, $(g, h_g)$ is unique in the Teichmüller point $\Pi = [(f, h_f)]$.

By considering $\psi_g(w) = 1/g'(\pi_g(w))$, then $\psi_g(w)$ is a $g$-function on $\Sigma$ and $\mu$ is a $g$-measure. Thus $\mu$ is an equilibrium state for the potential $-\log f'(x)$, that is,

$$0 = P(-\log f'(x)) = h_{\mu_{\text{geo}}}(f) - \int_T \log f'(x) d\mu_{\text{geo}}$$

$$= h_{\mu_{\text{geo}}}(f) - \int_T \log f'(x) \rho(x) dx$$

$$= \sup \{ h_\nu(f) - \int_T \log f'(x) dv \mid \nu \text{ is an } f\text{-invariant probability measure} \}$$

$$= h_{\text{Leb}}(g) - \int_T \log g'(y) dy,$$

where $h_{\mu_{\text{geo}}}$ and $h_{\text{Leb}}(g)$ denote the measure-theoretical entropies with respect to $\mu_{\text{geo}}$ and the Lebesgue measure. The equilibrium state $\mu_{\text{geo}}$ is unique in this case.

Now by considering the dual invariant probability measure $\mu^*$ for this Gibbs measure $\mu$, we have that

**Theorem 17.** Suppose $f \in C^{1+}$. Consider $\Sigma^*$ with the metric $d(\cdot, \cdot)$ induced from $\mu^*$ on $\Sigma^*$. Then the right shift $\sigma^*$ is $C^{1+}$ differentiable with respect to $d(\cdot, \cdot)$ and its derivative is the dual derivative $D^*(f)(w^*)$ of $f$, i.e.,

$$\frac{d\sigma^*}{dw^*}(w^*) = D^*(f)(w^*), \quad \forall w^* \in \Sigma^*.$$

(Note that $\sigma^*$ is $C^{1+}$ differentiable means that it is differentiable and the derivative is a Hölder continuous function.)

**Proof.** Suppose $w^* = \cdots j_{n-1} \cdots j_1 j_0$ is a point in $\Sigma^*$. Let $w^*_n = j_{n-1} \cdots j_1 j_0$ and $v^*_n = j_{n-1} \cdots j_1$. Let $I_{w_n}$ and $I_{v_{n-1}}$ be the corresponding intervals in the $n^{th}$-partition $\eta_n$ and the $(n-1)^{th}$-partition $\eta_{n-1}$.

From the definition,

$$\mu^*([w^*_n]) = \mu([w_n]) = \mu_{\text{geo}}(I_{w_n})$$

and

$$\mu^*([v^*_n]) = \mu([v_{n-1}]) = \mu_{\text{geo}}(I_{v_{n-1}}).$$

Consider the ratio

$$\frac{\mu^*([v^*_{n-1}])}{\mu^*([w^*_n])} = \frac{\mu_{\text{geo}}(I_{v_{n-1}})}{\mu_{\text{geo}}(I_{w_n})} = \frac{h'(\xi)}{h(\xi')} D^*(f)(w^*_n).$$
Since the distribution function of $\mu_{geo}$ is a $C^{1+\alpha}$-diffeomorphism, the ratio $h'(\xi)/h'(\xi')$ converges to 1 exponentially as $n \to \infty$. We also know that $D^*(f)(w_n^*)$ converges $D^*(f)(w^*)$ exponentially as $n \to \infty$. So there are two constants $C > 0$ and $0 < \tau < 1$ such that

$$\left| \frac{\mu^*(\{v_{n-1}^*\})}{\mu^*(\{w_n^*\})} - D^*(f)(w^*) \right| \leq C\tau^n, \quad \forall n > 0.$$  

This implies that

$$\frac{d\sigma^*}{dw^*}(w^*) = \lim_{n \to \infty} \frac{\mu^*(\{v_{n-1}^*\})}{\mu^*(\{w_n^*\})} = D^*(f)(w^*).$$

So $\sigma^*$ is $C^{1+\alpha}$ smooth whose derivative is $D^*(f)(w^*)$. We proved the theorem. 

Since the convergence in the proof is exponential and $D^*(f)(w^*)$ is a strictly positive function and $\Sigma^*$ is a compact space, we have the Gibbs inequalities:

$$C^{-1} \leq \frac{\mu^*(\{w_n^*\})}{\exp(\sum_{l=0}^{n-1} - \log D^*(f)((\sigma^*)^l(w^*)))} \leq C$$

for any right cylinder $[w_n^*]$ and any $w^*$ in this cylinder, where $C > 0$ is a fixed constant.

**Corollary 4.** The measure $\mu^*$ is the Gibbs measure for the potential $-\log D^*(f)(w^*)$.

Thus we call $\mu^*$ a dual geometric Gibbs measure for the potential $-\log D^*(f)(w^*)$ in this paper. Let $h_{\mu^*}(\sigma^*)$ be the measure-theoretic entropy of $\sigma^*$ with respect to $\mu^*$. Since the Borel $\sigma$-algebra of $\Sigma^*$ is generated by all right cylinders, then $h_{\mu^*}(\sigma^*)$ can be calculated as

$$h_{\mu^*}(\sigma^*) = \lim_{n \to \infty} \frac{1}{n} \sum_{w_n^*} \left( -\mu([w_n^*]) \log \mu([w_n^*]) \right)$$

$$= \lim_{n \to \infty} \sum_{w_n^*} \left( -\mu([w_n^*]) \log \frac{\mu([w_n^*])}{\mu(\sigma([w_n^*]))} \right),$$

where $w_n^*$ runs over all words $w_n^* = j_{n-1} \cdots j_0$ of $\{0,1,\cdots,d-1\}$ of length $n$.

**Corollary 5.** The dual geometric Gibbs measure $\mu^*$ for the potential $-\log D^*(f)(w^*)$ is a $g$-measure with respect to the $g$-function $1/D^*(f)(w^*)$ whose pressure

$$P(-\log D^*(f)) = 0.$$  

Moreover, the Radon-Nikodým derivative

$$D_{\mu^*}(w^*) = \frac{1}{D^*(f)(w^*)}, \quad \text{for } \tilde{\mu^*}-\text{a.e. } w^*,$$
\( \mu^* \) is a unique equilibrium state for the potential \(- \log D^*(f)(w^*)\) in the sense that

\[
0 = P(- \log D^*(f)) = h_{\mu^*}(\sigma^*) - \int_{\Sigma^*} \log D^*(f)(w^*)d\mu^*(w^*)
= \sup \{ h_\nu(\sigma^*) - \int_{\Sigma^*} \log D^*(f)(w^*)d\nu(w^*) \mid \nu \text{ is a } \sigma^*\text{-invariant measure} \}.
\]

Now following Theorem 17 and Corollary 5, we conclude one of the main results in this paper, which is in some sense similar to the measurable Riemann mapping theorem for smooth Beltrami coefficients in the real one-dimensional case.

**Theorem 18.** Suppose \( \Psi^*(w^*) \in TL^1 \). Then there is a unique non-atomic measure \( \mu^* \) whose support is the whole \( \Sigma^* \) such that consider the metric \( d(\cdot, \cdot) \) induced from \( \mu^* \) on \( \Sigma^* \), the right shift \( \sigma^* \) is \( C^1 \) differentiable and \( \Psi^*(w^*) \) is the derivative, that is,

\[
\frac{d\sigma^*}{dw^*}(w^*) = \Psi^*(w^*), \quad \forall w^* \in \Sigma^*.
\]

Moreover, by considering the dynamical system

\[
\sigma^*: \Sigma^* \rightarrow \Sigma^*
\]

and the given potential \(- \log \Psi(w^*)\), the dual invariant measure \( \mu^* \) (or the induced metric \( d(\cdot, \cdot) \)) is an equilibrium state for the potential \(- \log \Psi(w^*)\).

Suggested by Theorem 18, we have the following definition.

**Definition 13.** Suppose \( \Psi^*(w^*) \) is a positive continuous function defined on \( \Sigma^* \). A non-atomic probability measure \( \mu^* \) with support on the whole \( \Sigma^* \) is a dual geometric Gibbs type measure for the potential \(- \log \Psi^*(w^*)\) if

\[
\frac{d\sigma^*}{dw^*}(w^*) = \Psi^*(w^*), \quad \forall w^* \in \Sigma^*.
\]

In the last section, we will discuss the existence of a dual geometric Gibbs type measure for a continuous potential \(- \log \Psi^*(w^*)\).

14. Dual geometric Gibbs type measures for continuous potentials.

A map \( f \in US \) may not be differentiable everywhere (it may not even be absolutely continuous). There is no suitable Gibbs theory to be used in the study of geometric properties of a \( \sigma\)-invariant measure. We thus turn to the dual symbolic dynamical system \((\Sigma^*, \sigma^*)\) and produce a similar dual geometric Gibbs type measure theory as we did in the previous section.

Suppose \( \mu \) is a probability measure on \( T \). We call it a symmetric measure if its distribution function \( h(z) = \mu([1, z]) \) is a symmetric circle homeomorphism, where \([1, z]\) means the oriented arc on \( T \) from 1 to \( z \).
An $f$-invariant measure $\mu$ can be found as we did in the previous section. Let $\mu_0$ be the Lebesgue measure. Consider the push-forward measures $\mu_n = (f^n)_* \mu_0$ and sum them up to get

$$\nu_n = \frac{1}{n} \sum_{i=0}^{n-1} \mu_n.$$ 

Take a weak limit $\mu_{\text{geo}}$ of a subsequence of $\{\nu_n\}$. Then $\mu_{\text{geo}}$ is an $f$-invariant probability measure. In the following we will prove that $\mu_{\text{geo}}$ is a symmetric $f$-invariant probability measure.

Actually we will prove that the sequence of the distribution functions $\{h_n(z)\}_{n=0}^{\infty}$ of $\{\nu_n\}_{n=0}^{\infty}$ has a convergent subsequence. And every convergent subsequence converges to the distribution function $h(z)$ of $\mu_{\text{geo}}$ and $h(z)$ is symmetric.

Let $H_n(x)$ be the lift of $h_n(z)$ to the real line $\mathbb{R}$. Then

$$H_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} \sum_{l=0}^{d_k-1} |F^{-k}([l, l + \frac{1}{2}])|.$$ 

**Theorem 19.** Suppose $f$ is a uniformly symmetric circle endomorphism. Then the sequence $\{H_n(x)\}_{n=0}^{\infty}$ has a convergent subsequence in the maximal norm on $\mathbb{R}$. Every convergent subsequence converges in the maximal norm on $\mathbb{R}$ to a symmetric circle homeomorphism. Thus the sequence $\{h_n(z)\}_{n=0}^{\infty}$ has a convergent subsequence in the maximal norm on $T$. Every convergent subsequence converges to a symmetric circle homeomorphism $h(z)$ and the corresponding subsequence of probability measures $\{\mu_n\}_{n=0}^{\infty}$ converges in the weak topology to an $f$-invariant symmetric probability measure $\mu_{\text{geo}}$ whose distribution function is $h(z)$.

**Proof.** Since $f$ is uniformly symmetric, there is a bounded positive function $\epsilon(t) > 0$ with $\epsilon(t) \to 0$ as $t \to 0^+$ such that

$$\frac{1}{1 + \epsilon(t)} \leq \frac{|F^{-n}(x + t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x - t)|} \leq 1 + \epsilon(t), \quad \forall x \in \mathbb{R}, t > 0.$$ 

Let $C > 0$ be an upper bound of $\epsilon(t)$.

From the definition of $H_n$,

$$H_n\left(\frac{1}{2}\right) = \frac{1}{n} \sum_{k=0}^{n-1} \sum_{l=0}^{d_k-1} |F^{-k}([l, l + \frac{1}{2}])|.$$ 

Since $H_n(0) = 0$ and $H_n(1) = 1$,

$$\frac{1}{1 + C^{-1}} \leq \frac{|F^{-k}([l, l + \frac{1}{2}])|}{|F^{-k}([l, l + 1])|} \leq \frac{1}{1 + C}.$$ 

This implies that

$$\frac{1}{1 + C^{-1}} \leq H_n\left(\frac{1}{2}\right) \leq \frac{1}{1 + C}.$$
Similarly,
\[
\frac{1}{1+C^{-1}} \leq \frac{H_n(\frac{1}{4})}{H_n(\frac{1}{2})} \leq \frac{1}{1+C}.
\]

Since \( \{H_n(x)\}_{n=0}^{\infty} \) is a sequence of quasisymmetric circle homeomorphisms whose quasisymmetric constants are bounded uniformly by \( C \), and since the distances of the images of any two points in \( \{0, 1/4, 1/2, 1\} \) under \( H_n \) are greater than a constant uniformly on \( n \), \( \{H_n(x)\}_{n=0}^{\infty} \) is in a compact set in the space of all quasisymmetric circle homeomorphisms. Thus \( \{H_n(x)\}_{n=0}^{\infty} \) has a convergent subsequence \( \{H_{n_i}(x)\}_{i=0}^{\infty} \) in the maximal norm on \( \mathbb{R} \) whose limiting function \( H(x) \) is a circle homeomorphism. Furthermore, since the sequence \( \{H_n\}_{n=0}^{\infty} \) is uniformly symmetric, that is,
\[
\frac{1}{1 + \varepsilon(t)} \leq \frac{|H_n(x + t) - H_n(x)|}{|H_n(x) - H_n(x - t)|} \leq 1 + \varepsilon(t), \quad \forall x \in \mathbb{R}, \ t > 0,
\]
the limiting circle homeomorphism \( H(x) \) is also symmetric, and
\[
\frac{1}{1 + \varepsilon(t)} \leq \frac{|H(x + t) - H(x)|}{|H(x) - H(x - t)|} \leq 1 + \varepsilon(t), \quad \forall x \in \mathbb{R}, \ t > 0.
\]

Since \( H_n(x) \) is the lift of \( h_n \), \( \{h_{n_i}(x)\}_{i=0}^{\infty} \) is a convergent subsequence in the maximal norm on \( T \) whose limiting function \( h(x) \) is a circle homeomorphism whose lift is \( H(x) \). Since \( h_n(z) \) is the distribution function of \( \nu_n \), so \( \{\nu_{n_i}\}_{i=0}^{\infty} \) is a convergent subsequence in the weak topology and converges to \( \mu_{\text{geo}} \) whose distribution function is \( h(z) \). So \( \mu_{\text{geo}} \) is a symmetric measure.

We now lift \( \mu_{\text{geo}} \) to \( \Sigma \) to get a \( \sigma \)-invariant measure \( \mu \) as follows. For any finite word \( w_n = i_0 \cdots i_{n-1} \), consider the left cylinder \([w_n]\). Define
\[
\mu([w_n]) = \mu_{\text{geo}}(I_{w_n}),
\]
where \( I_{w_n} \) is the interval in \( \eta_n \) labeled by \( w_n \). One can check that it satisfies the finite additive law and the continuity law. So it can be extended to a \( \sigma \)-invariant probability measure \( \mu \) on \( \Sigma \) such that
\[
(\pi_f)_* \mu = \mu_{\text{geo}}.
\]
For \( \mu \), we can construct its dual invariant measure \( \mu^* \) on \( \Sigma^* \) as we did in the previous two sections. Then we have the following dual geometric Gibbs type property as we had before in the smooth case:

**Theorem 20.** Suppose \( f \in \mathcal{US} \). Consider \( \Sigma^* \) with the metric \( d(\cdot, \cdot) \) induced from \( \mu^* \) on \( \Sigma^* \). Then the right shift \( \sigma^* \) is \( C^1 \) differentiable with respect to \( d(\cdot, \cdot) \) and its derivative is the dual derivative \( D^*(f)(w^*) \) of \( f \), i.e.,
\[
\frac{d\sigma^*}{dw^*}(w^*) = D^*(f)(w^*), \quad \forall w^* \in \Sigma^*.
\]
Proof. Suppose \( w^* = \cdots j_{n-1} \cdots j_1 j_0 \) is a point in \( \Sigma^* \). Let \( w^*_n = j_{n-1} \cdots j_1 j_0 \) and \( v^*_{n-1} = j_{n-1} \cdots j_1 \). Let \( I_{w_n} \) and \( I_{v_{n-1}} \) be the corresponding intervals in the \( n^{th} \)-partition \( \eta_n \) and the \((n - 1)^{th}\)-partition \( \eta_{n-1} \).

From the definition,
\[
\mu^*([w^*_n]) = \mu([w_n]) = \mu_{\text{geo}}(I_{w_n})
\]
and
\[
\mu^*([v^*_{n-1}]) = \mu([v_{n-1}]) = \mu_{\text{geo}}(I_{v_{n-1}}).
\]

Consider the ratio
\[
\frac{\mu^*([v^*_{n-1}])}{\mu^*([w^*_n])} = \frac{\mu_{\text{geo}}(I_{v_{n-1}})}{\mu_{\text{geo}}(I_{w_n})}.
\]

Since the distribution function \( h(z) \) of \( \mu_{\text{geo}} \) is symmetric, from the quasisymmetric distortion property (Lemma 1), the sequence
\[
\frac{\mu_{\text{geo}}(I_{v_{n-1}})}{|I_{v_{n-1}}|} = \frac{\mu_{\text{geo}}(I_{v_{n-1}})}{|I_{w_n}|} - D^*(f)(w^*_n)
\]
converges to 0 as \( n \to \infty \) uniformly on \( w^* \). This implies that
\[
\frac{d\sigma^*}{dw^*}(w^*) = \lim_{n \to \infty} \frac{\mu^*([v^*_{n-1}])}{\mu^*([w^*_n])} = D^*(f)(w^*).
\]

So \( \sigma^* \) is \( C^1 \) under the metric \( d(\cdot, \cdot) \) induced from \( \mu^* \) whose derivative is \( D^*(f)(w^*) \). We proved the theorem. \( \square \)

Finally, we conclude one of the main results in this paper, which is in some sense similar to the measurable Riemann mapping theorem for general Beltrami coefficients in the one-dimensional case.

**Theorem 21.** Suppose \( \Psi^*(w^*) \in TUS \). Then there is a dual geometric Gibbs type measure \( \mu^* \) for the continuous potential \( -\log \Psi^*(w^*) \). It is a \( g \)-measure for the \( g \)-function \( 1/\Psi^*(f)(w^*) \) whose pressure
\[
P(-\log D^*(f)) = 0.
\]

Moreover, the Radon-Nikodým derivative
\[
D_{\mu^*}(w^*) = \frac{1}{\Psi^*(f)(w^*)}, \quad \text{for } \tilde{\mu}^*\text{-a.e. } w^*.
\]

And, furthermore, the dual invariant measure \( \mu^* \) is an equilibrium state for the continuous potential \( -\log \Psi^*(f)(w^*) \) in the sense that
\[
0 = P(-\log D^*(f)) = h_{\mu^*}(\sigma^*) - \int_{\Sigma^*} \log \Psi^*(f)(w^*)d\mu^*(w^*)
\]
\[
= \sup \left\{ h_{\nu}(\sigma^*) - \int_{\Sigma^*} \log \Psi^*(f)(w^*)d\nu(w^*) \mid \nu \text{ is a } \sigma^*-\text{invariant measure} \right\}.
\]
15. Symmetric invariant measure and metric entropy

Suppose \( \tau \in \mathcal{TUS} \) and suppose \( f \in \tau \). From Theorem 19, there is an \( f \)-invariant symmetric measure \( \mu_{\text{geo}} \). Let \( h(z) = \mu_{\text{geo}}([1, z]) \) be the distribution function of \( \mu_{\text{geo}} \). Then

\[
\tilde{f} = h \circ f \circ h^{-1} \in \tau
\]

preserves the Lebesgue measure \( \text{Leb} \) on \( T \). This means that the Lebesgue measure \( \text{Leb} \) is \( \tilde{f} \)-invariant. Let \( h_{\mu_{\text{geo}}}(f) \) be the metric entropy of \( f \) with respect to \( \mu_{\text{geo}} \). Then we have that

\[
h_{\mu_{\text{geo}}}(f) = h_{\text{Leb}}(\tilde{f}) = h_{\mu^*}(\sigma^*).
\]

From Theorem 21, \( h_{\mu_{\text{geo}}}(f) \) is a positive number.

If the topological degree of \( f \) is \( d \geq 2 \). Then the topological entropy of \( f \) is \( \log d \), which is the maximum value of the metric entropy \( h_{\mu_{\text{geo}}}(f) \) over all \( \tau \in \mathcal{TUS} \) and all \( f \in \tau \) with their symmetric \( f \)-invariant measures \( \mu_{\text{geo}} \).

**Theorem 22.** The infimum of the metric entropy \( h_{\mu_{\text{geo}}}(f) \) over all \( \tau \in \mathcal{TUS} \) and all \( f \in \tau \) with their symmetric \( f \)-invariant measures \( \mu_{\text{geo}} \) is zero.

**Proof.** To prove this theorem, we construct a family \( \{f_s\}_{0 < s < 1} \) of orientation-preserving circle endomorphisms such that each of them is \( C^{1+\alpha} \) expanding for some \( 0 < \alpha \leq 1 \) and preserves the Lebesgue measure. Then the equivalent class \([f_s]\) is a point in \( \tau \in \mathcal{TUS} \). Moreover, we prove that the metric entropy \( h_{\text{Leb}}(f_s) \) tends to 0 as \( s \to 1^- \). Without loss of generality, we prove this theorem for \( d = 2 \) as follows.

First let us consider the unit circle \( T \) as \( \mathbb{R}/\mathbb{Z} \). Let \([0, 1]\) be a copy of \( T \) such that \( 0 = 1 \). Consider a piecewise smooth expanding map for any \( 0 < s < 1 \),

\[
L(x) = \begin{cases} 
\frac{x}{s}, & x \in [0, s); \\
\frac{x-s}{1-s}, & x \in [s, 1]
\end{cases}
\]

The Lebesgue measure \( \text{Leb} \) on \([0, 1]\) is the unique smooth \( L \)-invariant measure and the metric entropy

\[
h_{\text{Leb}}(L) = s \log s + (1-s) \log(1-s).
\]

Thus \( h_{\text{Leb}}(L) \to 0 \) as \( s \to 1^- \). Next we will smooth \( L \) such that the resulting map is a \( C^{1+\alpha} \) circle expanding endomorphism \( f \) and still preserves the Lebesgue measure \( \text{Leb} \) on \( T \).

Let \( r = 1 - s \). Then

\[
s + r = 1.
\]

Let \( 0 < \alpha \leq 1 \). Consider the interval \([-r, s]\) and construct a \( \alpha \)-Hölder continuous function \( \phi(x) \) on it such that

\[
i) \quad \phi(x) = \begin{cases} 
-r^{-1}, & x \in [-r, -r/2] \\
s^{-1}, & x \in [0, s]
\end{cases}
\]
ii) \( \int_0^x \phi(\xi) d\xi = 1, \)

iii) \( r^{-1} \leq \phi(x) \leq Mr^{-1}, \quad \forall x \in [-r, -r^2], \)

iv) \( s^{-1} \leq \phi(x) \leq r^{-1}, \quad x \in [-r^2, 0]. \)

Then

\[
\tilde{f}_0(x) = \int_0^x \phi(\xi) d\xi = \frac{x}{s} : [0, s] \to [0, 1]
\]

is a \( C^{1+\alpha} \)-diffeomorphism and

\[
\tilde{f}_1(x) = \int_s^x \phi(\xi - 1) d\xi : [s, 1] \to [0, 1]
\]

is a \( C^{1+\alpha} \)-diffeomorphism. Furthermore,

\[
\tilde{f}_1'(1-) = \phi(0-) = \tilde{f}_0'(0+) = \phi(0+) = s^{-1}.
\]

So we define a circle endomorphism \( \tilde{f} \) which is \( C^{1+\alpha} \) on \([0, 1] \setminus \{s\}. \) Moreover, the derivative \( \tilde{f}'(x) \geq \min\{s^{-1}, r^{-1}\} > 1 \) for any \( x \in [0, 1] \setminus \{s\}. \)

Let \( \tilde{g}_0 \) and \( \tilde{g}_1 \) be the inverses of \( \tilde{f}_0 \) and \( \tilde{f}_1. \) Consider the interval \( I_0 = [1/2, 1]. \)

Since \( f_1'(x) = r^{-1} \) on \([s, (1 + s)/2], \) we have \( f_1((1 + s)/2) = 1/2. \) Therefore, the preimage of \( I_0 \) under \( \tilde{f} \) is the union of two intervals \([s/2, s] = \tilde{g}_0(I_0) \) and \([(1 + s)/2, 1] = \tilde{g}_1(I_0) \)

Define \( g_1(x) = \tilde{g}_1(x) \) on \([0, 1]. \) Then its inverse \( f_1 = \tilde{f}_1 : [s, 1] \to [0, 1] \) is a \( C^{1+\alpha} \) diffeomorphism. Denote

\[
\psi_1(x) = g_1'(x) = \frac{1}{\phi(g_1(x) - 1)}, \quad x \in [0, 1].
\]

Then

\[
g_1(x) = s + \int_0^x \psi_1(\xi) d\xi, \quad x \in [0, 1].
\]

Define

\[
\psi_0(x) = 1 - \psi_1(x) = 1 - \frac{1}{\phi(g_1(x) - 1)}, \quad x \in [0, 1].
\]

Then \( \psi_0(x) = s \) for \( x \in [0, 1/2] \) since in this case \( g_1(x) \in [s, (1 + s)/2] \) and \( g_1(x) - 1 \in [-r, -r^2]. \) Define

\[
g_0(x) = \int_0^x \psi_0(\xi) d\xi, \quad x \in [0, 1].
\]

It is clearly that \( g_0(0) = 0 \) and \( g_0(1) = s. \) So

\[
g_0 : [0, 1] \to [0, s]
\]

is a \( C^{1+\alpha} \)-diffeomorphism. Let

\[
f_0(x) : [0, s] \to [0, 1]
\]

be the inverse of \( g_0(x). \) Then it is a \( C^{1+\alpha} \)-diffeomorphism. Furthermore,

\[
f_0'(s-) = \frac{1}{g_0'(1-)} = \frac{1}{1 - \phi(0-)} = r^{-1} = f_1'(s+).
\]
Thus
\[ f(x) = \begin{cases} f_0(x), & x \in [0, s]; \\ f_1(x), & x \in [s, 1] \end{cases} \]
is a $C^{1+\alpha}$ expanding circle endomorphism.

For any $x \in T$, let \{x_0, x_1\} = $f^{-1}(x)$ such that $x_0 \in [0, s]$ and $x_1 \in [s, 1]$. Then we have that
\[ \frac{1}{f'(x_0)} + \frac{1}{f'(x_1)} = g'_0(x) + g'_1(x) = \psi_0(x) + \psi_1(x) = 1. \]
This implies that for any interval $J$ of $T$,
\[ \text{Leb}(f^{-1}(J)) = \text{Leb}(g_0(J)) + \text{Leb}(g_1(J)) = \text{Leb}(J). \]
So $f$ preserves the Lebesgue measure.

Now we prove that the metric entropy $h_{\text{Leb}}(f_s)$ tends to 0 as $s \to 1^-$.

Let $a = 1 - f(1 - r^2)$. That is $1 - a = f(1 - r^2)$. Since $\phi(x) \leq r^{-1}$ for $x \in [-r^2, 0]$,
\[ a = f(1) - f(1 - r^2) = \int_{1-r^2}^1 \phi(\xi - 1) d\xi \leq r^2 r^{-1} = r. \]
If $1 - a \leq x \leq 1$, then $1 - r^2 \leq g_1(x) \leq 1$. This implies that $\phi(g_1(x) - 1) \leq r^{-1}$, that is,
\[ g'_1(x) = \psi_1(x) \geq r, \quad \forall x \in [1 - a, 1]. \]
Hence
\[ g'_0(x) = 1 - g_1(x) \leq 1 - r = s, \quad \forall x \in [1 - a, 1]. \]
This implies that
\[ s - g_0(1 - a) = g_0(1) - g_0(1 - a) = \int_{1-a}^1 g'_0(\xi) d\xi \leq sa. \]
Furthermore,
\[ g_0(1 - a) \geq s - sa \geq s - sr. \]
On the other hand, if $0 \leq x \leq 1 - a$, then $s \leq g_1(x) \leq 1 - r^2$ and $-r \leq g_1(x) \leq r^{-2}$. This implies that
\[ \psi_1(x) = \frac{1}{\phi(g_1(x) - 1)} \geq r, \quad \forall x \in [0, 1 - a]. \]
Hence
\[ g'_0(x) = \psi_0(x) = 1 - \psi_1(x) \geq 1 - r, \quad \forall x \in [0, 1 - a]. \]
It follows that for $0 \leq x \leq g_0(1 - a)$,
\[ f'(x) = \frac{1}{g'_0(f_0(x))} \leq (1 - r)^{-1}. \]
Note that for all $x \in T$, $\phi(x) \leq Mr^{-1}$. It implies that $f'(x) \leq Mr^{-1}$ for all $x \in T$. 
Since the Lebesgue measure $Leb$ is $f$-invariant, by Rohlin’s formula
\[
\begin{align*}
    h_{Leb}(f) &= \int_{T} \log f'(\xi) d\xi = \int_{T}^{g_0(1-a)} \log f'(\xi) d\xi + \int_{T}^{1} \log f'(\xi) d\xi \\
    &\leq g_0(1-a) \log (1-r)^{-1} + (1 - g_0(1-a)) \log (Mr^{-1}) \\
    &\leq - \log s + (1 - (s-sr)) \log (Mr^{-1}) \leq - \log s - r(1+s) \log (M^{-1}r). 
\end{align*}
\]
When $s \to 1^-$, $r \to 0^+$. This implies
\[
    h_{Leb}(f_s) \to 0 \quad \text{as} \quad s \to 1^-.
\]
We have completed the proof. \qed

Let $\Psi^*(w^*)$ be the function model of $\tau \in TUS$. Let $\mu^*$ be a dual geometric Gibbs type measure for the continuous potential $- \log \Psi^*(w^*)$. Let $\mu_{geo}$ be the corresponding symmetric measure for an $f \in \tau$. Then $h_{\mu^*}(\sigma^*) = h_{\mu_{geo}}(f)$. Finally, as a consequence of Theorem 22, we have that

**Theorem 23.** The maximum value of the metric entropy $h_{\mu^*}(\sigma^*)$ over all $\Psi^*(w^*) \in TUS$ with their dual geometric Gibbs type measures $\mu^*$ is logd. And the infimum of the metric entropy $h_{\mu^*}(\sigma^*)$ over all $\Psi^*(w^*) \in TUS$ with their dual geometric Gibbs type measures $\mu^*$ is 0.

**Remark 9.** The infimum of the metric entropy for all Anosov diffeomorphisms of a smooth manifold with their SRB measures has been studied in a recent paper [13]. The infimum of the metric entropy for all area-preserving Anosov diffeomorphisms of a smooth manifold with their SRB measures is still an open problem. Theorem 22 answers this problem in the one-dimensional case too.

**Remark 10.** The family $\{f_s\}_{0<s<1}$ constructed in the proof of Theorem 22 is totally degenerate. This means that $\tau_s = [f_s]$ tends to the boundary of the Teichmüller space but its limit can not be seen on the boundary. This is different from the family constructed in [14] where the family tends to the boundary of the space of all Anosov diffeomorphisms with a limiting point on the boundary. The limiting point is an almost hyperbolic diffeomorphism. This is also an interesting problem in the Teichmüller theory of Riemann surfaces, that is, which curve is totally degenerate and which curve tends to a surface with a parabolic node. The construction of the families in the proof of Theorem 22 and in [14] provides some idea to study this problem.

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