Partial characterization of graphs having a single large Laplacian eigenvalue

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Abstract

The parameter $\sigma(G)$ of a graph $G$ stands for the number of Laplacian eigenvalues greater than or equal to the average degree of $G$. In this work, we address the problem of characterizing those graphs $G$ having $\sigma(G) = 1$. Our conjecture is that these graphs are stars plus a (possible empty) set of isolated vertices. We establish a link between $\sigma(G)$ and the number of anticomponents of $G$. As a by-product, we present some results which support the conjecture, by restricting our analysis to some classes of graphs.

1 Introduction

Let $G$ be a graph on $n$ vertices and $m$ edges and let $d_1 \geq \cdots \geq d_n$ be its degree sequence. We denote by $A(G)$ its adjacency matrix and by $D(G)$ the diagonal matrix having $d_i$ in the diagonal entry $(i, i)$, for every $1 \leq i \leq n$, and 0 otherwise. The Laplacian matrix of $G$ is the positive semidefinite matrix $L(G) = D(G) - A(G)$. The eigenvalues of $L(G)$ are called Laplacian eigenvalues of $G$; the spectrum of $L(G)$ is the Laplacian spectrum of $G$ and will be denoted by $L\text{spec}(G)$. Since it is easily seen that 0 is a Laplacian eigenvalue and it is well-known that Laplacian eigenvalues are less than or equal to $n$ it turns out that $L\text{spec}(G) \subset [0, n]$. From now on, if $L\text{spec}(G) = \{\mu_1, \mu_2, \ldots, \mu_n\}$, we will assume that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, where $\mu_n = 0$.

Understanding the distribution of Laplacian eigenvalues of graphs is a problem that is both relevant and difficult. It is relevant due to the many applications related to Laplacian matrices (see, for example [13, 14]). It seems to be difficult because little is known about how the Laplacian eigenvalues are distributed in the interval $[0, n]$.

Our main motivation is understanding the structure of graphs that have few large Laplacian eigenvalues. In particular, we would like to characterize graphs that have a single large Laplacian eigenvalue. What do we mean by a large Laplacian eigenvalue? A reasonable measure is to compare this eigenvalue with the average of all eigenvalues. Since the average of Laplacian eigenvalues equals the average degree $\overline{d}(G) = \frac{2m}{n}$ of $G$, we say that a Laplacian eigenvalue is large if it is greater than or equal to the average degree.

Inspired by this idea, the paper [3] introduces the spectral parameter $\sigma(G)$ which counts the number of Laplacian eigenvalues greater than or equal to $\overline{d}(G)$. Equivalently, $\sigma(G)$ is the largest index $i$ for which $\mu_i \geq \frac{2m}{n}$. Since the greatest Laplacian eigenvalue $\mu_1$ is at least $\frac{2m}{n}$ then it follows that $\sigma(G) \geq 1$.

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There is evidence that $\sigma(G)$ plays an important role in defining structural properties of a graph $G$. For example, it is related to the clique number $\omega$ of $G$ (the number of vertices of the largest induced complete subgraph of $G$) and it also gives insight about the Laplacian energy of a graph [15, 3]. Moreover, several structural properties of a graph are related to $\sigma$ (see, for example [2, 3]).

In this paper we are concerned with furthering the study of $\sigma(G)$. In particular, we deal with a problem posed in [3] which asks for characterizing all graphs $G$ having $\sigma(G) = 1$; i.e., having only one large Laplacian eigenvalue. Our conjecture is that the only connected graph on $n$ vertices having $\sigma = 1$ is the star $K_{1,n-1}$ and that the only nonconnected graph on $n$ vertices having $\sigma = 1$ is a star together with some isolated vertices. More precisely, we conjecture that graphs having $\sigma = 1$ are some stars plus a (possibly empty) set of isolated vertices. From now on, $K_{1,r} + sK_1$ denotes the star on $r+1$ vertices plus $s$ isolated vertices.

**Conjecture 1.** Let $G$ be a graph. Then $\sigma(G) = 1$ if and only if $G$ is isomorphic to $K_1$, $K_2 + sK_1$ for some $s \geq 0$, or $K_{1,r} + sK_1$ for some $r \geq 2$ and $0 \leq s < r - 1$.

In this work, we show that this conjecture is true if it holds for graphs which are simultaneously connected and co-connected (Conjecture 12) and prove that Conjecture 1 is true for forests and extended $P_4$-laden graphs [5] (a common superclass of split graphs and cographs). The main tool for proving our results is an interesting link we have found between $\sigma$ and the number of anticomponents of $G$ (see Section 2). The interesting feature of this result is that it relates a spectral parameter with a classical structural parameter. Studying structural properties of the anticomponents of $G$ may shed light on the distribution of Laplacian eigenvalues and, reciprocally, the distribution of Laplacian eigenvalues should give insight about the structure of the graph.

This article is organized as follows. In Section 2 we state definitions and previous results concerning Laplacian eigenvalues. In Section 3, we present some new results which establish the connection between $\sigma$ and the number of nonempty anticomponents of $G$. In Section 4, we present some evidence on the validity of Conjecture 1 by proving that the conjecture is true when $G$ is either a forest, or a $P_4$-laden graph.

## 2 Definitions

In this article, all graphs are finite, undirected, and without multiple edges or loops. All definitions and concepts not introduced here can be found in [17]. We say that a graph is empty if it has no edges. A trivial graph is a graph with precisely one vertex; every trivial graph is isomorphic to the graph which we will denote by $K_1$. A graph is nontrivial if it has more than one vertex.

We use the standard notation $\Delta(G)$ to denote the maximum degree of a graph $G$.

Let $G_1$ and $G_2$ be two graphs such that $V(G_1) \cap V(G_2) = \emptyset$. The disjoint union of $G_1$ and $G_2$, denoted $G_1 \cup G_2$, is the graph whose vertex set is $V(G_1) \cup V(G_2)$, and its edge set is $E(G_1) \cup E(G_2)$. We write $kG$ to represent the disjoint union $G + \cdots + G$ of $k$ copies of a graph $G$. The join of $G_1$ and $G_2$, denoted $G_1 \vee G_2$, is the graph obtained from $G_1 + G_2$ by adding new edges from each vertex of $G_1$ to every vertex of $G_2$.

A vertex $v$ of a graph $G$ is a twin of another vertex $w$ of $G$ if they both have the same neighbors in $V(G) \setminus \{v, w\}$. We say that a graph $G'$ is obtained from $G$ by adding a twin $v'$ to a vertex $v$ of $G$ if $V(G') = V(G) \cup \{v'\}$, $v'$ is a twin of $v$ in $G'$, and $G' - v'$ is isomorphic to $G$.

By $G[S]$ we denote the subgraph of $G$ induced by a subset $S \subseteq V(G)$.

We use $\overline{G}$ to denote the complement graph of a graph $G$. An anticomponent of a graph $G$ is the subgraph of $G$ induced by the vertex set of a connected component of $\overline{G}$. More precisely, an induced subgraph $H$ of $G$ is an anticomponent if $\overline{H}$ is a connected component of $\overline{G}$. Notice that if $G_1, G_2, \ldots, G_k$ are the anticomponents of $G$, then $G = G_1 \vee \cdots \vee G_k$. A graph $G$ is co-connected if $\overline{G}$ is connected.

A forest is a graph with no cycles and a tree is a connected forest. The complete graph on $n$ vertices is denoted by $K_n$. A universal vertex of a graph $G$ is a vertex $v$ adjacent to every vertex $w$ different from $v$. A star is a graph isomorphic to $K_1$ or to a tree with a universal vertex. We use $K_{1,n-1}$ to denote the star on $n$ vertices, where $K_{1,0}$ is isomorphic to $K_1$ and $K_{1,1}$ is isomorphic to $K_2$. The chordless path (respectively, cycle) on $k$ vertices is denoted by $P_k$ (respectively, $C_k$).
A stable set of a graph is a set of pairwise nonadjacent vertices. A clique of a graph is a set of pairwise adjacent vertices.

Throughout this article, given two graphs $G$ and $H$, we write $G = H$ to point out that $G$ and $H$ belong to the same isomorphism class.

The following well-known result provides a lower bound for the largest Laplacian eigenvalue of a graph with at least one edge in terms of the maximum degree of the graph.

**Lemma 2 ([7]).** Let $G$ be a graph on $n$ vertices with at least one edge. Then $\mu_1(G) \geq 1 + \Delta(G)$.

The second largest Laplacian eigenvalue of a graph is lower bounded by the second term of the degree sequence of the graph.

**Lemma 3 ([10]).** Let $G$ be a graph with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$ and spectrum $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. Then $\mu_2 \geq d_2$.

It is worth mentioning that Brouwer and Haemers [1] generalized the above result by presenting a lower bound for the $k$th greatest Laplacian eigenvalue in terms of $d_k$, answering a conjecture raised by Guo [8].

It is easy to prove that the Laplacian spectrum of the disjoint union $G_1 + G_2$ is the union of the Laplacian spectrums of $G_1$ and $G_2$. The next result allows to determine the Laplacian spectrum of the join $G_1 \vee G_2$, from those of $G_1$ and $G_2$.

**Theorem 4 ([12, Theorem 2.20]).** Let $G_1$ and $G_2$ be two graphs with Laplacian spectrums $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n_1} = 0$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_2} = 0$, respectively. Then the Laplacian eigenvalues of $G_1 \vee G_2$ are $n_1 + n_2$; $n_2 + \mu_i$, for $1 \leq i \leq n_1 - 1$; $n_1 + \lambda_i$, for $1 \leq i \leq n_2 - 1$ and $0$.

### 3 Relating $\sigma$ and the number of anticomponents

This section is devoted to establish a link between $\sigma(G)$ and the number of anticomponents of $G$.

In virtue of Theorem 4, the following result immediately holds.

**Lemma 5.** If $G = G_1 \vee \cdots \vee G_k$, with $k \geq 1$, is a graph on $n$ vertices, then $n$ is a Laplacian eigenvalue of $G$ with multiplicity at least $k - 1$.

**Lemma 6.** If $G$ has $k$ anticomponents, then $k \leq \sigma(G) + 1$.

**Proof.** Let $G = G_1 \vee \cdots \vee G_k$ where $G_1, \ldots, G_k$ are the anticomponents of $G$. For any graph $G$ with at least one vertex we have that $\sigma(G) \geq 1$ and thus the assertion follows when $k = 1$. We may assume that $k \geq 2$. Lemma 5 implies that $n$ is a Laplacian eigenvalue of $G$ with multiplicity at least $k - 1$ in $G$. Thus $\mu_{k-1}(G) = n$ which implies that $\sigma(G) \geq k - 1$.

**Remark 7.** The upper bound given by Lemma 6 is sharp when $\sigma(G) > 1$. Indeed, for $s \geq 2$ consider the graph $G = 4K_2 \vee K_1 \vee \cdots \vee K_1$, where $s$ is the number of $K_1$’s. The average degree of $G$ is $s + 7 - \frac{4s}{s+8}$ and it has $s + 1$ anticomponents. Since its Laplacian eigenvalues are $s + 8$, $s + 2$, $s$, and $0$ with multiplicities $s$, $4$, $3$, and $1$, respectively, it follows that $\sigma(G) = s$.

We use $\ell(G)$ to denote the number of nonempty anticomponents of a graph $G$. Recall that a nontrivial graph has at least two vertices. The following result looks further into the case where equality holds in Lemma 6 showing that $\sigma(G)$ is an upper bound for $\ell(G)$.

**Theorem 8.** Let $G$ be a graph having $k = \sigma(G) + 1$ anticomponents. Then $\ell(G) \leq \sigma(G)$. Moreover, if $\sigma(G) = \ell(G)$, then the remaining anticomponent of $G$ is empty but nontrivial.

**Proof.** Write $G = G_1 \vee \cdots \vee G_k$ where $G_1, \ldots, G_k$ are the anticomponents of $G$. Since $\sigma(G) \geq 1$ then $k \geq 2$.

We set the following notations for each $i \in \{1, \ldots, k\}$:

\[ n_i = |V(G_i)|, \quad m_i = |E(G_i)|, \quad \mu_1^{(i)} = \mu_1(G_i). \]
Assume that \( G_1, \ldots, G_k \) are the nonempty anticomponents. Since \( k \geq 2 \) and we are assuming that \( \sigma(G) = k - 1 \) it turns out that \( \mu_k(G) < \overline{d}(G) \). Therefore, for each \( i \in \{1, \ldots, k\} \) such that \( n_i > 1 \) we have that

\[
n - n_i + \mu_1^{(i)} \leq \mu_k(G) < \frac{2m}{n} = \frac{2 \sum_{j=1}^{k} m_j + 2 \sum_{1 \leq i < j \leq k} n_i n_j}{n},
\]

the first inequality holds by Theorem 4. Equivalently,

\[
\mu_1^{(i)} < \frac{2 \sum_{j=1}^{k} m_j - (n^2 - 2 \sum_{1 \leq i < j \leq k} n_i n_j)}{n} + n_i = \frac{2 \sum_{j=1}^{k} m_j - \sum_{j=1}^{k} n_j^2 + n n_i}{n}.
\]

As a consequence of Lemma 2, we obtain the following lower bound for each \( i \in \{1, \ldots, \ell\} \):

\[
\mu_1^{(i)} \geq \Delta(G_i) + 1 \geq \overline{d}(G_i) + 1 = \frac{2m_i}{n_i} + 1.
\]  

Combining (1) and (2), we deduce that, for each \( i \in \{1, \ldots, \ell\} \),

\[
2n_i \sum_{j=1}^{k} m_j - n_i \sum_{j=1}^{k} n_j^2 + n n_i^2 - 2n m_i - n n_i > 0.
\]

Arguing towards a contradiction, suppose that \( \ell(G) = k \). If we sum up the left-hand side of (3) for each \( i \in \{1, \ldots, k\} \), we obtain

\[
2n \sum_{j=1}^{k} m_j - n \sum_{j=1}^{k} n_j^2 + n \sum_{i=1}^{k} n_i^2 - 2n \sum_{i=1}^{k} m_i - n^2 = -n^2
\]

which is not a positive quantity. This contradiction proves that \( G \) has at most \( k - 1 = \sigma(G) \) nonempty anticomponents and our first assertion follows.

Assume now that \( \ell(G) = k - 1 \). Suppose that \( G_k \) is trivial. Hence \( n_k = 1 \) and \( m_k = 0 \). Summing up to the left-hand side of (3) for each \( i \in \{1, \ldots, k - 1\} \), we obtain that

\[
-2 \sum_{j=1}^{k-1} m_j + \sum_{j=1}^{k} n_j^2 - n^2 = -2 \sum_{j=1}^{k-1} m_j - 2 \sum_{1 \leq i < j \leq k} n_i n_j
\]

should be a positive number. This contradiction shows that \( G_k \) must be nontrivial.

Recall that a bipartite graph is a graph whose set of vertices can be partitioned into two (possibly empty) stable sets called partite sets of the bipartite graph. A complete bipartite graph is a bipartite graph isomorphic to \( rK_1 \lor sK_1 \) for two positive integers \( r \) and \( s \). We denote by \( K_{r,s} \) the complete bipartite graph isomorphic to \( rK_1 \lor sK_1 \). The upper bound \( \sigma(G) \) on \( \ell(G) \) for those graphs having exactly \( \sigma(G) + 1 \) anticomponents is not tight when \( \sigma(G) = 1 \). Indeed, the following result shows that if a graph \( G \) has \( \sigma(G) = 1 \), then \( G \) has no nonempty anticomponents.

**Corollary 9.** If \( G \) is a graph with \( \sigma(G) = 1 \) and \( \overline{G} \) is disconnected, then \( G \) is a complete bipartite graph.

**Proof.** In virtue of Lemma 6, the number of anticomponents of \( G \) is at most 2. Since \( \overline{G} \) is disconnected, we conclude that \( G \) has precisely two anticomponents \( G_1 \) and \( G_2 \) and thus \( G = G_1 \lor G_2 \).

Suppose, for a contradiction, that \( G_1 \) is a nonempty anticomponent of \( G \). Because of Theorem 8, we conclude that \( G_2 \) is empty but nontrivial. Following the notation used in the proof of Theorem 8, we have that \( m_2 = 0 \). For \( i = 1 \), inequality (3) becomes

\[
-2n_2 m_1 - n_1 n_2^2 + n_2 n_1^2 - n_1^2 - n_1 n_2 > 0.
\]
Since $G_2$ is a nontrivial empty graph it follows that $\mu_1^{(2)} = 0$ and hence, for $i = 2$, inequality (1) becomes
\[ 2m_1 - n_2^2 + n_1 n_2 > 0. \] (5)
Summing up (4) and $n_2$ times (5) gives
\[ -n_2^2 - n_1 n_2 > 0. \]
This contradiction arose from supposing that $G$ has some nonempty anticomponent. Hence, both anticomponents of $G$ are empty; i.e., $G$ is a complete bipartite graph.

4 Graphs with $\sigma = 1$

In this section we provide some evidence in order to make plausible Conjecture 1. We first verify Conjecture 1 for graphs having disconnected complement; namely, we prove that the only graphs having $\sigma = 1$ and disconnected complement are the stars (including the trivial star $K_1$). Then, we prove that Conjecture 1 can be reduced to proving that the only connected and co-connected graph with $\sigma = 1$ is $K_1$. We then verify Conjecture 1 for extended $P_4$-laden graphs, a common superclass of the classes of cographs and split graphs.

4.1 Reduction to co-connected graphs

We first obtain a result which proves the validity of Conjecture 1 for graphs having disconnected complement.

**Theorem 10.** Let $G$ be a graph on $n$ vertices such that $\overline{G}$ is disconnected. Then $\sigma(G) = 1$ if and only if $G = K_{1,n-1}$.

**Proof.** Assume first that $G = K_{1,n-1}$. Then $\overline{d}(G) = 2 - \frac{2}{n}$. If $n = 2$, the Laplacian eigenvalues of $G$ are 2 and 0. If $n \geq 3$, the Laplacian eigenvalues of $G$ are $n$, 1 and 0, each with multiplicity 1, $n-2$ and 1, respectively. In any case we have that $\sigma(G) = 1$.

Conversely, assume that $\sigma(G) = 1$. Corollary 9 implies that $G = K_{n_1,n_2}$, where $n_2 \geq n_1 \geq 1$ and $n = n_1 + n_2$. The average degree of $G$ is equal to $\frac{2n_1 n_2}{n}$. In virtue of Theorem 4, the Laplacian eigenvalues of $K_{n_1,n_2}$ are $n$, $n_2$, $n_1$ and 0, each with multiplicity $n_1$, $n_2-1$, $n_1-1$ and 1, respectively.

Arguing towards a contradiction, suppose that $n_1 \geq 2$. Hence $\mu_2(G) = n_2$. Since $2n_1 \leq n$ we deduce that $\overline{d}(G) = \frac{2n_1 n_2}{n} \leq \mu_2(G)$, which contradicts the fact that $\sigma(G) = 1$. This contradiction proves that $n_1 = 1$ and therefore we conclude that $G = K_{1,n-1}$.

As a consequence of Theorem 10, Conjecture 1 is equivalent to the validity of the following weaker conjecture.

**Conjecture 11.** Let $G$ be a graph with connected complement. Then, $\sigma(G) = 1$ if and only if $G$ is isomorphic to $K_1$, $K_2 + sK_1$ for some $s > 0$, or $K_{1,r} + sK_1$ for some $r \geq 2$ and $0 < s < r-1$.

4.2 Reduction to connected and co-connected graphs

We next show that the validity of Conjectures 1 and 11 can be reduced to the validity of the following even weaker conjecture.

**Conjecture 12.** Let $G$ be a connected graph with connected complement. Then, $\sigma(G) = 1$ if and only if $G$ is isomorphic to $K_1$.

A graph class $G$ is **closed by taking components** if every connected component of every graph in $G$ also belongs to $G$. In particular, the class of all graphs is closed by taking components. Below we prove that the reduction from Conjecture 1 to Conjecture 12 holds even when restricted to any graph class closed by taking components.
**Theorem 13.** Let $G$ be a graph class closed by taking components. If Conjecture 12 holds for $G$, then Conjecture 1 also holds for $G$.

**Proof.** Let $G$ be a graph in $G$ with $\sigma(G) = 1$. Assume first that $G$ is connected. If $G$ is co-connected, by hypothesis, $G$ is isomorphic to $K_1$. If $G$ is not co-connected, then $G$ is isomorphic to $K_{1,r}$ for some $r \geq 1$, by virtue of Theorem 10.

Assume now that $G$ is disconnected and let $G = G_1 + G_2$, where each of $G_1$ and $G_2$ has at least one vertex. We can assume, without loss of generality, that $G_1$ is connected and $\mu_1(G_1) \geq \mu_1(G_2)$. If $G_1$ were empty, then $G_2$ would also be empty, contradicting $\sigma(G) = 1$. Hence we can assume, without loss of generality, that $G_1$ is nonempty. Let $n_i$ and $m_i$ denote the number of vertices and edges of $G_i$, respectively, for each $i \in \{1, 2\}$. Since $\sigma(G) = 1$,

$$\frac{2m_2}{n_2} \leq \mu_1(G_2) < \overline{d}(G) = \frac{2(m_1 + m_2)}{n_1 + n_2},$$

This implies that

$$\frac{2m_2}{n_2} < \frac{2m_1 + 2m_2}{n_1 + n_2} < \frac{2m_1}{n_1}.$$  \hfill (6)

As a consequence of (6) we have that

$$\mu_2(G_1) < \frac{2m_1 + 2m_2}{n_1 + n_2} < \frac{2m_1}{n_1} = \overline{d}(G_1).$$

We conclude that $\sigma(G_1) = 1$. Since $G$ is closed by taking components, $G_1 \in G$. Thus, if $G_1$ were co-connected, then $G_1 = K_1$, contradicting the assumption that $G_1$ is nonempty. Hence $G_1$ is not co-connected and, by Theorem 10, we have that $G_1 = K_{1,r}$ for some $r \geq 1$.

From (6) we deduce that

$$\mu_1(G_2) < \frac{2m_1 + 2m_2}{n_1 + n_2} < \frac{2m_1}{n_1} = \frac{2r}{r + 1} < 2,$$

and hence, by virtue of Lemma 2, we conclude that $G_2$ must be empty. Then there exists an integer $s \geq 1$ such that $G_2 = sK_1$ and therefore it turns out that $G = K_{1,r} + sK_1$. The average degree of $G$ is $\overline{d}(G) = \frac{2r}{r + 1 + s}$. If $r = 1$, then $\sigma(G) = 1$ because the second largest Laplacian eigenvalue of $G$ is 0. If $r \geq 2$, then, as the second largest eigenvalue of $G$ is 1 it follows that $\sigma(G) = 1$ if and only if $s < r - 1$. \hfill $\square$

A **cograph** is a graph with no induced $P_4$. It is well-known that the only connected and co-connected cograph is $K_1$ [16]. Hence, Conjecture 12 holds trivially for cographs and, by Theorem 13, Conjecture 1 holds for cographs.

### 4.3 Characterizing forests and extended $P_4$-laden graphs with $\sigma = 1$

In this section, we verify Conjecture 1 for forests and extended $P_4$-laden graphs (a common superclass of cographs and split graphs).

A graph class $G$ is **monotone** if $G \in G$ implies that every subgraph of $G$ also belongs to $G$. Notice that every monotone graph class is closed by taking components. It can be easily seen that the class of all forests is monotone and thus it is closed by taking components.

**Theorem 14.** Conjecture 1 holds for forests.

**Proof.** Notice that if $T$ is a connected and co-connected forest, then $T$ is either $K_1$ or a tree with diameter greater than two. By virtue of Theorem 13, it suffices to show that if $T$ is a tree with diameter greater than two, then $\sigma(T) \geq 2$. Assume that $T$ is a tree with diameter greater than two. Hence there exists two vertices $v_1$ and $v_2$ such that $d(v_1) \geq d(v_2) \geq 2 > \frac{2}{n} = \overline{d}(T)$. By Lemma 3, $\mu_2(T) \geq d_2(T) \geq 2 > \overline{d}(T)$. Therefore, $\sigma(T) \geq 2$. \hfill $\square$
Let \( \mathcal{H} \) be a set of graphs. We use the term \( \mathcal{H}\text{-free} \) for referring to the family of those graphs having no graph in \( \mathcal{H} \) as induced subgraph. If \( \mathcal{H} \) has just one element \( H \), we write \( H\text{-free} \) for simplicity. A split graph \cite{hole-free} is a graph whose vertex set can be partitioned into a clique \( C \) and a stable set \( S \), such a partition \((C,S)\) of its vertices is called a split partition. It is well known that the class of split graphs coincides with the class of \( \{2K_2,C_4,C_5\}\)-free graphs. A pseudo-split graph \cite{pseudo-split} is a \( \{2K_2,C_4\}\)-free graph. Hence the class of pseudo-split graphs is a superclass of split graphs \cite{hole-free}. An extended \( P_4\)-laden graph \cite{P4-laden} is a graph such that every induced subgraph on at most six vertices that contains more than two induced \( P_4 \)'s is a pseudo-split graph. By definition, the class of extended of \( P_4\)-laden graphs is a superclass of the class of pseudo-split graphs and hence also of split graphs. Moreover, the class of extended \( P_4\)-laden graphs is a superclass of different superclasses of cographs defined by restricting the number of induced \( P_4 \)'s, including \( P_4 \)-lite graphs \cite{P4-lite} and \( P_4 \)-tidy graphs \cite{P4-tidy}. A spider \cite{spider} is a graph whose vertex set can be partitioned into three sets \( S, C, \) and \( R \), where \( S = \{s_1, \ldots, s_k\} \) \((k \geq 2)\) is a stable set; \( C = \{c_1, \ldots, c_k\} \) is a clique; \( s_i \) is adjacent to \( c_j \) if and only if \( i = j \) \((\text{a thin spider})\), or \( s_i \) is adjacent to \( c_j \) if and only if \( i \neq j \) \((\text{a thick spider})\); \( R \) is allowed to be empty and all the vertices in \( R \) are adjacent to all the vertices in \( C \) and nonadjacent to all the vertices in \( S \). The sets \( C, S \) and \( R \) are called body, legs and head of the spider, respectively. In order to characterize those extended \( P_4\)-laden graphs with \( \sigma(G) = 1 \), we rely on the following structural result.

**Theorem 15** \cite{P4-laden}. Each connected and co-connected extended \( P_4\)-laden graph \( G \) satisfies one of the following assertions:

1. \( G \) is isomorphic to \( K_1, P_5, \overline{P}_5, \) or \( C_5 \);
2. \( G \) is a spider or arises from a spider by adding a twin to a vertex of the body or the legs; or
3. \( G \) is a split graph.

We first obtain the following result which is concerned with spiders or graphs arising from a spider by adding a twin to a vertex of the body or the legs.

**Lemma 16.** If \( G \) is a spider or a graph that arises from a spider by adding a twin of a vertex of the body or the legs, then \( \sigma(G) \geq 2 \).

*Proof.* We will prove that \( d_2(G) \geq 7(G) \). We consider four cases. In each case we denote by \( k \) and \( n_H \) the number of vertices in the body and in the head of the corresponding spider, respectively. Recall that \( k \geq 2 \).

1. Assume that \( G \) is a thin spider. By construction, \( d_2(G) = k + n_H \) and \( |E(G)| \geq 2k + n_H \). Hence

\[
\overline{d}(G) \leq \frac{k^2 + k + 2kn_H + n_H^2 - n_H}{2k + n_H} = \frac{2k^2 + kn_H + 2kn_H + n_H^2}{2k + n_H} - \frac{k^2 - k + n_H k + n_H}{2k + n_H} \\
\leq \frac{(k + n_H)(2k + n_H)}{2k + n_H} = d_2(G).
\]

2. Assume that \( G \) arises from a thin spider by adding a twin to a vertex of the body or the leg. By construction, \( d_2(G) \geq k + n_H \) and \( |E(G)| \geq 2k + n_H + 1 \). Hence

\[
\overline{d}(G) \leq \frac{k^2 + k + 2kn_H + n_H^2 - n_H + 2k + 2n_H + 2}{2k + n_H + 1} = \frac{2k^2 + kn_H + 2kn_H + n_H^2 + n_H}{2k + n_H + 1} - \frac{k^2 - 2k - 2k n_H}{2k + n_H + 1} \\
\leq \frac{(k + n_H)(2k + n_H + 1)}{2k + n_H + 1} = d_2(G).
\]

Notice that the second inequality holds, whenever \( k > 2 \) or \( n_H \neq 0 \). However, when \( k = 2 \) and \( n_H = 0 \), it can be verified by inspection that \( \overline{d}(G) \leq d_2(G) \) holds.
3. Assume that $G$ is a thick spider. By construction, $d_2 \geq 2\sigma(G) + n_H$ and $|E(G)| \geq 2\sigma(G) + n_H$. Hence
\[
\overline{d}(G) \leq \frac{k^2 + k + 2kn_H + n_H^2 - n_H + 2k(k - 2)}{2k + n_H} = \frac{k^2 - k + 2k^2 - 2k + 2kn_H + n_H^2 - n_H}{2k + n_H} = \frac{4(k^2 - k) + (4k - 2)n_H + n_H^2}{2k + n_H} - \frac{k^2 - k + (2k - 1)n_H}{2k + n_H} \leq \frac{2(k - 1) + n_H}{2k + n_H} = d_2(G).
\]

4. Assume that $G$ arises from a thick spider by adding a twin to a vertex of the body or the leg. By construction, $d_2 \geq 2\sigma(G) + n_H$ and $d_2 \geq 2\sigma(G) + n_H + 1$. Hence
\[
\overline{d}(G) \leq \frac{k^2 - k + 2k^2 - 2k + 2kn_H + n_H^2 - n_H + 2k(k - 2) + 2n_H}{2k + n_H + 1} = \frac{3k^2 + k - 2 + (2k + 1)n_H + n_H^2}{2k + n_H + 1} = \frac{4k^2 - 2k - 2 + (4k - 2)n_H + n_H^2}{2k + n_H + 1} - \frac{k^2 - 3k + (2k - 2)n_H}{2k + n_H + 1} \leq \frac{2(k - 1) + n_H}{2k + n_H + 1} = d_2(G).
\]

Notice that the second inequality holds, whenever $k > 2$ or $n_H \neq 0$. However, if $k = 2$ and $n_H = 0$, it can be verified that $\overline{d}(G) \leq d_2(G)$ holds by inspection.

We have shown that in all possible cases, $d_2(G) \geq \overline{d}(G)$. Hence, by virtue of Lemma 3, we conclude that $\mu_2(G) \geq \overline{d}(G)$ which means that $\sigma(G) \geq 2$. □

**Theorem 17.** Conjecture 1 holds for split graphs.

**Proof.** Let $(C, S)$ be a split partition of the graph on $n$ vertices $G$ such that $|C| = c$ and $|S| = n - c$. We label the vertices of $G$ so that $C = \{v_1, \ldots, v_c\}$ and $S = \{v_{c+1}, \ldots, v_n\}$. We can assume, without loss of generality, that $C$ is a maximal clique of $G$ under inclusion and $d_i \geq d_{i+1}$, for each $i \in \{1, \ldots, n-1\}$.

We claim that if $G$ is a split graph with $\sigma(G) = 1$, then $G$ is isomorphic to $K_{1,r-1} + (n-r)K_1$ for some $r$ such that $2 \leq r \leq n$.

In order to prove our claim we assume that $G$ is nonisomorphic to $K_{1,r-1} + (n-r)K_1$, for each $r \in \{2, \ldots, n\}$ and we will prove that $\sigma(G) \geq 2$. By virtue of Lemma 3, it suffices to prove that $d_2 \geq \overline{d}(G)$ or equivalently that
\[
\sum_{i=3}^{n} (d_2 - d_i) \geq d_1 - d_2.
\]

We will consider two cases.

1. Assume that $d_2 \geq c$. Since $C$ is a maximal clique, $d_2 - d_i \geq 1$ for each $i \in \{c+1, \ldots, n\}$. Hence
\[
\sum_{i=3}^{n} (d_2 - d_i) \geq \sum_{i=c+1}^{n} (d_2 - d_i) \geq n - c \geq d_1 - d_2.
\]

2. Assume that $d_2 = c - 1$. Our assumption on $G$ implies that $c > 2$. Moreover, we have that $d_i \leq 1$ for each $i \in \{c+1, \ldots, n\}$. Consequently, $d_2 - d_i \geq 1$ for each such $i$ and the reasoning follows as above.

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Thus we have proved our claim. In particular, the only connected and co-connected split graph with \( \sigma = 1 \) is \( K_1 \); i.e., Conjecture 12 holds for split graphs. Therefore, by virtue of Theorem 13, Conjecture 1 holds for split graphs.

By combining Theorem 15, Lemma 16, and Theorem 17, we obtain the following result.

**Theorem 18.** Conjecture 1 holds for extended \( P_4 \)-laden graphs.

**Proof.** Let \( G \) be a connected and co-connected graph extended \( P_4 \)-laden graph with \( \sigma(G) = 1 \). The proof of the theorem will follow by considering different cases depending on which of the assertions of Theorem 15 hold. Because of Lemma 16, \( G \) does not satisfy assertion 2. If \( G \) satisfies assertion 1, then \( G \) is isomorphic to \( K_1 \) because \( \sigma(C_5) = \sigma(P_5) = \sigma(P_5) = 2 \). If \( G \) satisfies assertion 3, then \( G \) is isomorphic to \( K_1 \) because of Theorem 17. We conclude that Conjecture 12 holds. Therefore, Theorem 13 implies that Conjecture 1 holds for all extended \( P_4 \)-laden graphs \( G \).

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