Propagation of wave packets along intensive simple waves
A. M. Kamchatnov$^{1,2}$ and D. V. Shaykin$^{1,2}$

$^{1)}$Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow, 108840, Russia
$^{2)}$Moscow Institute of Physics and Technology, Institutsky lane 9, Dolgoprudny, Moscow region, 141700, Russia

We consider propagation of high-frequency wave packets along a smooth evolving background flow whose evolution is described by a simple-wave type of solutions of hydrodynamic equations. In geometrical optics approximation, the motion of the wave packet obeys the Hamilton equations with the dispersion law playing the role of the Hamiltonian. This Hamiltonian depends also on the amplitude of the background flow obeying the Hopf-like equation for the simple wave. The combined system of Hamilton and Hopf equations can be reduced to a single ordinary differential equation whose solution determines the value of the background amplitude at the location of the wave packet. This approach extends the results obtained in Ref. $^7$ for the rarefaction background flow to arbitrary simple-wave type background flows. The theory is illustrated by its application to waves obeying the KdV equation.

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I. INTRODUCTION

As is known, there is deep analogy between propagation of high-frequency wave packets and motion of particles in classical mechanics. This analogy was discovered by Hamilton and now it is widely used as a geometrical optics approximation to the description of wave propagation in various non-uniform and non-stationary media (see, e.g., Refs. $^1$–$^3$ and references therein). In this approach, the wavelength of a linear wave is supposed to be much smaller than any other characteristic size of the problem, hence such linear waves can form a wave packet whose size is also very small and its propagation can be represented as a motion of a point particle with the space and time dependent Hamiltonian. In particular, this method allows one to solve various problem on propagation of high-frequency water waves in different situations (see, e.g., $^4$–$^6$). As a result, the paths of the wave packets along the background mean flow can be found as well as the variations of the wave number and the amplitude.

In typical problems mentioned above, the mean flow or the medium non-homogeneity were prescribed externally, that is, if it is characterized by a single parameter $u$, then in the simplest one-dimensional situation the function $u = u(x, t)$ is considered as known. Linear waves propagating along such a background flow have a dispersion law $\omega = \omega(u, k, x, t)$ which relates the frequency $\omega$ of the linear wave with its wave number $k$ in vicinity of the point $x$ at the moment of time $t$. However, instead of external prescription of the background flow $u(x, t)$, one may suppose that the initial distribution $u(x, t = 0)$ is given, so that the background evolves dynamically and we want to consider the propagation of high-frequency wave packets along such a non-stationary background flow. Generally speaking, this is a quite difficult problem, but it has recently been noticed in Ref. $^7$ that if one confines oneself to the simple-wave type of the background evolution, then the problem greatly simplifies and becomes tractable analytically. In particular, the propagation of wave packets along rarefraction wave was studied in Ref. $^7$ and two different possible regimes were found—transmission of the packet through the finite rarefaction wave or trapping of the packet inside it. In this paper, we extend the approach of Ref. $^7$ to arbitrary smooth profile of the simple-wave type. To this end, we use the consistency condition that the wave packet propagates with the group velocity along the simple-wave solution of the dynamical dispersionless equation $^8$–$^9$. Recently, such a generalization permitted us to develop a new approach to derivation of the number of solitons produced by an intensive initial pulse at asymptotically large time $^{10}$–$^{12}$. In these papers, the wave packet was identified with the small amplitude pulse at asymptotically large time $^{10}$–$^{12}$. In these papers, the wave packet was identified with the small amplitude pulse at asymptotically large time $^{10}$–$^{12}$. In these papers, the wave packet was identified with the small amplitude pulse at asymptotically large time $^{10}$–$^{12}$.
Thus, we arrive at the Hopf equation
\[ u_t + V_0(u)u_x = 0, \]  
which governs evolution of the smooth background flow.

A typical wavelength \( \sim 2\pi/k \) of the wave packet propagating along the background flow is much smaller than the characteristic size \( l \) of variation of \( u(x,t) \). We assume that the size of the wave packet made from these linear waves with wave numbers close to \( k \) is also much smaller than \( l \). Therefore, in the geometric optics approximation, we can define the “mean” coordinate \( x(t) \) of the packet and the “mean” or “carrier” wave number \( k(t) \) around which the spectrum of the packet is concentrated. Then, according to Hamilton’s optical-mechanical analogy (see, e.g., Ref. \( \text{[13]} \) ) \( x(t) \) and \( k(t) \) obey the Hamilton equations
\[
\frac{dx}{dt} = \frac{\partial \omega(u,k)}{\partial k}, \quad \frac{dk}{dt} = -\frac{\partial \omega(u,k)}{\partial x}. \tag{3}
\]

In our case the dependence of the Hamiltonian \( \omega \) on \( x \) and \( t \) is determined by the solution \( u = u(x,t) \) of the equation (2) for the background flow. This leads immediately to an important consequence: from (2) and (3) we get at once
\[
\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt}, \quad \frac{du}{dt} = -\frac{\partial u}{\partial x} \frac{\partial \omega}{\partial k} \frac{dk}{dt} = -\left( V_0 - \frac{\partial \omega}{\partial k} \right) \frac{\partial u}{\partial x},
\]
and the ratio of these expressions yields
\[
\frac{dk}{du} = \frac{\partial \omega/\partial u}{V_0 - \partial \omega/\partial k}. \tag{4}
\]

This equation was derived in Ref. \( \text{[14]} \) as a consequence of the Whitham modulation equations at the small-amplitude edge of dispersive shock waves. Here the right-hand side only depends on \( u \) and \( k \), so the solution of this equation can be written as
\[
k = k(u,q), \tag{5}
\]
where \( q \) is the integration constant which can be found, for example, from the condition that the wave packet enters into the region of a non-uniform flow with the wave number \( k_0 \).

Now we take into account that the solution of the Hopf equation (2) is given by the expression (see, e.g., Ref. \( \text{[2]} \))
\[
x - V_0(u)t = \mathcal{F}(u), \tag{6}
\]
where \( \mathcal{F}(u) \) denotes the function inverse to the initial distribution \( u = u_0(x) \). If the initial pulse has a form of a hump on the constant background (see, e.g., Fig. 1(a)), then the inverse function has two branches shown in Fig. 1(b), and Eq. (6) defines the solution \( u = u(x,t) \) for each branch in an implicit form. In a similar way we can consider a “negative” background pulse with \( u_0(x) \leq 0 \).

As soon as we find the function \( u = u(x,t) \) as a solution of Eq. (6) and the function \( k = k(u) \) as a solution (5) of Eq. (4), we can find the law of motion of the wave packet in the following way\( ^{\text{[3]}} \). The first Hamilton equation (3) says that the wave packet moves with the group velocity \( v_g(u,k) = \partial \omega(u,k)/\partial k \) and in time interval \( dt \) it passes the distance \( dx = v_g dt \) along the “surface” \( u = u(x,t) \) of the smooth solution of Eq. (6). We assume that the packet’s path is determined by parametric formulas \( t = t(u) \), \( x = x(u) \) and find the derivate of Eq. (6) with respect to \( u \). Then elimination of \( dx/du = (dx/dt)(dt/du) = v_g(dt/du) \) from the expression for this derivative yields a linear differential equation
\[
[v_g(u,k(u)) - V_0(u)]\frac{dt}{du} - V'_0(u)t = \mathcal{F}'(u). \tag{7}
\]

for the function \( t(u) \) which can be easily solved with the initial condition \( t = 0 \) at some initial value of \( u \) corresponding to the initial location of the wave packet. Substitution of \( t(u) \) into Eq. (6) gives \( x = x(u) \) and as a result we obtain the law of motion \( x = x(t) \) in a parametric form. If the function \( \mathcal{F}(u) \) is two-valued, then the solution \( t = t_1(m) \) found in this way is correct up to the moment \( t_1(u_m) \) when the packet reaches the maximal (or minimal) value of the background distribution \( u = u_m \) (according to the solution (6) \( u_m \) does not depend on time). After that the packet moves along the second branch of the smooth solution and Eq. (7) with \( \mathcal{F} = \mathcal{F}_2(u) \) must be solved with the initial condition \( t = t_1(u_m) \) at \( u = u_m \) which gives \( t = t_2(u) \). Thus, we obtain the function \( t = t(u) \) for the entire motion of the packet propagation along the evolving pulse \( u = u(x,t) \). In the next Section we shall illustrate this method by its application to two typical problems.

### III. example: Korteveg-de Vries equation

To illustrate the formulated above approach, we shall apply it to waves whose propagation obeys the Korteweg-de Vries (KdV) equation
\[
u_t + 6uu_x + u_{xxx} = 0. \tag{8}
\]
It appears in a number of physical applications for description of weakly nonlinear and weakly dispersive unidirectional wave propagation. It was first derived\(^\text{15}\) for explanation of soliton propagation on shallow water and in this paper we shall assume this physical application of the theory.

If we neglect the last dispersive term, it reduces to the Hopf equation (see (2))

\[
u_t + 6uu_x = 0 \tag{9}
\]

with \(V_0(u) = 6u\). It describes large scale evolution of smooth waves with a characteristic dimension \(l\). We are interested in propagation along such an evolving wave of a wave packet with the carrier wave number \(k \ll 1/l\). Within such a packet the background amplitude can be assumed constant and the linearized KdV equation

\[
u_t' + 6uu'_x + u'''' = 0
\]

yields for harmonic linear waves \(u' \propto \exp[i(kx - \omega t)]\) the dispersion law

\[
\omega(u, k) = 6uk - k^3. \tag{10}
\]

Hence, the wave packet moves with the velocity

\[
v_g(u, k) = \frac{\partial \omega}{\partial k} = 6u - 3k^2. \tag{11}
\]

At first, we shall briefly consider the propagation of such a wave packet along a rarefaction wave. This problem was discussed in much detail in Ref.\(^7\). We shall show that the Hamiltonian approach permits one to simplify the solution and to obtain some additional results which will help us to compare them with more general situations considered in the subsequent Subsections.

A. Propagation of a wave packet along a rarefaction wave

Let the initial distribution have the form

\[
u_0(x) = \begin{cases}
0, & x > 0, \\
x/(6t_0), & 6u_-t_0 \leq x \leq 0, \\
u_-, & x < 6u_-t_0,
\end{cases} \tag{12}
\]

where \(u_- < 0\) (see Fig. 2). Then the solution of Eq. (9) is a rarefaction wave

\[
u(x, t) = \frac{x}{6(t + t_0)} \quad \text{for} \quad 6u_- (t + t_0) \leq x \leq 0, \tag{13}
\]

which connects the plateau regions \(u = 0\) for \(x > 0\) and \(u = u_-\) for \(x < 6u_- (t + t_0)\). The left edge of the rarefaction wave propagates along the constant background \(u = u_-\) to the left with the “sound velocity” \(6u_-.\)

Now let a high-frequency wave packet with the wave number \(k_0\) enter into the rarefaction wave region from the right at the moment \(t = 0\), so that its initial coordinate is \(x(0) = 0\) and the initial velocity equals to \(v_g(0) = -3k_0^2/2\).

The wave number \(k\) varies during the propagation and we find the law of this variation by solving Eq. (4), which in our case takes the form

\[
\frac{dk}{du} = \frac{2}{k}; \tag{14}
\]

with the initial condition \(k(0) = k_0\) to obtain

\[
k(u) = \sqrt{4u + k_0^2}. \tag{15}
\]

The function inverse to the initial distribution (12) is given by \(\tau(u) = 6t_0u\), so Eq. (7) transforms to

\[
2 \left( u + \frac{k_0^2}{4} \right) \frac{dt}{du} = -(t + t_0) \tag{16}
\]

and we easily find its solution with the initial condition \(t(0) = 0\):

\[
t(u) = \frac{k_0t_0}{\sqrt{4u + k_0^2}} - t_0. \tag{17}
\]

Consequently, along the packet’s path the background amplitude changes with time as

\[
u(t) = \frac{k_0^2}{4} \cdot \frac{t(t + 2t_0)}{(t + t_0)^2} \tag{18}
\]

and its substitution into Eq. (6) yields the explicit formula for the path:

\[
x(t) = -\frac{3k_0^2}{2} \cdot \frac{t(t + 2t_0)}{t + t_0}. \tag{19}
\]

If \(|u_-|\) is large enough (\(|u_-| > k_0^2/4\)), then in the limit \(t \to \infty\) the packet moves with constant velocity

\[
v_g(\infty) = -\frac{3k_0^2}{2}. \tag{20}
\]
smaller in the absolute value than the velocity $6|u_-|$ of the left edge of the rarefaction wave, hence the packet remains forever inside the rarefaction wave region. The wave number (15) depends on time as

$$k(t) = \frac{k_0 t_0}{t + t_0}, \quad (21)$$

and $k \to 0$ as $t \to \infty$. This means that the packet disperses with time and we go beyond applicability of the geometric optics Hamiltonian approximation.

If $|u_-| < k_0^2/4$, then the packet passes through the rarefaction wave region and goes out from it with the wave number

$$k_- = \sqrt{k_0^2 + 4u_-} < k_0 \quad (22)$$

at the moment of time

$$t_- = \left(\frac{k_0}{k_-} - 1\right) t_0. \quad (23)$$

The exit coordinate of the packet on the plateau with $u = u_-$ is given by

$$x(t_-) = \frac{6k_0 u_-}{k_-} t_0. \quad (24)$$

If there were no the rarefaction wave, the packet would move to the point with the coordinate $-3k_0^2 t_-$ shifted with respect to (24) by the distance

$$\Delta x = -\frac{3k_0(k_0 - k_-)^2}{2k_-} t_0, \quad (25)$$

which can be called the “phase shift” caused by interaction of the packet with the rarefaction wave.

These analytical results agree very well with the results of numerical solution of the KdV equation and some details can be found in Ref.7.

**B. Propagation of a wave packet along a negative pulse**

Now we shall turn to the motion of a high-frequency wave packet along a more general simple-wave profile which occupies a finite region in the space. We shall start with a “negative” pulse for which $u_0(x) < 0$ and for definiteness we shall take the initial profile in the form (see Fig. 3)

$$u_0(x) = -4u_m \left(\frac{x}{l} + \sqrt{-\frac{x}{l}}\right), \quad -l \leq x \leq 0, \quad (26)$$

which admits a simple enough explicit analytical solution of the problem. In this case we have two branches of the inverse function

$$\pi_1(u) = \frac{l}{4} \left( -\frac{u}{u_m} - 2 + 2\sqrt{1 + \frac{u}{u_m}}\right), \quad \pi_2(u) = \frac{l}{4} \left( -\frac{u}{u_m} - 2 - 2\sqrt{1 + \frac{u}{u_m}}\right), \quad (27)$$

and for both branches the solution of the equation $x - 6ut = \pi(u)$ with respect to $u(x,t)$ yields the same expression

$$u(x,t) = \frac{4u_m}{(l - 24u_m t)^2} \left[ 12u_m t(l + 2x) - lx \right. - l\sqrt{24u_m t(l + 2x) - lx}]. \quad (28)$$

This dispersionless solution is single-valued up to the wave breaking moment $t_b$ determined by the condition $\partial u/\partial x \to \infty$ as $u \to 0$ which gives

$$t_b = \frac{l}{12u_m}. \quad (29)$$

As before, we assume that the wave packet with the wave number $k_0$ enters into the pulse’s region at the point $x = 0$ at the moment of time $t = 0$, and we are interested in its motion along the evolving profile (28).
FIG. 5. (a) The initial profile of the wave packet at $0 < x < 100$ (red) and its profile for a large value of time at $-600 < x < -400$ (blue), when it is practically blocked inside the smooth pulse. The wavelength considerably increases with time. (b) The initial spectrum of the wave packet centered around $k_0 = 1.7$ (red) and its spectrum for a large value of time $t = 100$ (blue) when the carrier wave number is close to zero according the formula (15) with $u \rightarrow -k_0^2/4$. Since the packet’s amplitude is not infinitely small, the propagation of the packet is accompanied by generation of higher harmonics with corresponding peaks in the spectrum. The peak at the local carrier wave number $k = 0.28$ and two additional peaks at $k = 0.56$ and $k = 0.84$ are clearly seen in our numerical solution of the KdV equation.

Equation (14) does not depend on the form of the profile, so the dependence of the wave number $k(u)$ on the background amplitude is still given by Eq. (15). However, the right-hand side of Eq. (7) equals now to one of the following expressions

$$\begin{align*}
\bar{\mathcal{F}}_1(u) &= -\frac{t}{4u_m} \left( 1 - \frac{1}{\sqrt{1 + u/u_m}} \right), \\
\bar{\mathcal{F}}_2(u) &= -\frac{t}{4u_m} \left( 1 + \frac{1}{\sqrt{1 + u/u_m}} \right),
\end{align*}$$

depending on along which branch the packet moves. Solution of the linear equation

$$\frac{12}{t} \left( u + \frac{k_0^2}{4} \right) \frac{dt}{du} + 6t = -\bar{\mathcal{F}}(u)$$

with $\bar{\mathcal{F}}(u) = \bar{\mathcal{F}}_1(u)$ and the initial condition $t(0) = 0$ gives for the motion along the first branch the expression

$$t_1(u) = \frac{l}{48u_m \sqrt{u + k_0^2/4}} \times$$

$$\int_0^u \left( 1 - \frac{1}{\sqrt{1 + u/u_m}} \right) \frac{du}{\sqrt{u + k_0^2/4}}$$

$$= \frac{l}{24u_m \sqrt{k_0^2 + 4u}} \left[ \frac{k_0^2 + 4u - k_0^2}{\sqrt{k_0^2 + 4u}} \right] + 2\sqrt{u_m} \ln \frac{k_0 + 2\sqrt{u_m}}{\sqrt{k_0^2 + 4u} + 2\sqrt{u_m} + u}. \quad (32)$$

If $k_0 < 2\sqrt{u_m}$, then $t_1 \rightarrow \infty$ as $u \rightarrow -k_0^2/4$, consequently $u$ does not reach the minimal value $u_m$ and the packet moves all the time along the first branch corresponding to the right side of the smooth distribution point according to the law

$$x(u) = 6ut_1(u) + \bar{x}_1(u). \quad (33)$$

Formulas (32), (33) determine the packet’s path $x = x(t)$ in a parametric form. In Fig. 4(a) we compare the analytical theory with the numerical solution of the KdV equation. We take the initial pulse in the form (26) with $u = \sqrt{u_0}$ on the left-hand side of the smooth distribution and the initial condition $x_0 = 55$ and the carrier wave number equal to $k_0 = 1.7$. The group velocity tends in this limit to the value

$$v_g \approx -\frac{3}{2} \frac{k^2}{k_0^2} \quad \text{as} \quad t \rightarrow \infty \quad (34)$$

in agreement with the numerical solution (see Fig. 4(b)).

In Fig. 5(a) we show the packet’s profile at two moments of time — at the initial one and at large time close to the asymptotic regime (34) when the packet is practically blocked within the smooth profile and moves with the velocity (34). To find these profiles, we solve numerically the KdV equation for two types of the initial conditions—with and without the initial wave packet contribution. Then subtraction of the second profile from the first one yields the packet’s amplitude $u_{\text{linear}}(x,t)$ without contribution of the smooth pulse. We see that the wavelength of the carrier wave increases considerably in the asymptotic regime. This qualitative observation is confirmed quantitatively by the plots of the spectrum

$$F(k) = \int_{-\infty}^{\infty} u_{\text{linear}}(x,t) e^{-ikx} dx \quad (35)$$

shown in Fig. 5(b). It is worth noticing that since in our numerical simulations the packet’s amplitude is not
infinitely small, the propagation of the packet is accompanied by generation of higher harmonics. As a result, the spectrum acquires additional peaks in the wavenumber distribution and two such peaks are clearly seen in Fig. 5(b).

Formula (33) together with Eq. (15) yields the dependence $k = k(x)$ of the carrier wave number along the packet’s path. This wave number vanishes in the asymptotic limit $u \to -k_0^2/4$ and its dependence on time is given in a parametric form by the formulas (15) and (32). In Fig. 6 we compare the analytical theory with the results extracted from the numerical solution of the KdV equation and find very good agreement.

The asymptotic behavior at $t \to \infty$ can be interpreted in the following way. The first branch of the dispersionless solution (28) not too close to the minimum point approaches in the limit $t \to \infty$ to the rarefaction wave solution $u(x, t) \approx x/(6t)$, so that for small $k_0 < \sqrt{u_m}$ the packet’s motion practically coincides with the motion considered in the preceding Subsection. The asymptotic law of motion is obtained from Eq. (19) with $t_0 = 0$:

$$ x(t) \approx -\frac{3u_m^2}{2} t, \quad (36) $$

that is the wave packet moves asymptotically with the velocity (34). However, one has to keep in mind that after the wave breaking moment (29) the dispersive shock wave appears at the left edge of the pulse (28) and the right edge of the shock propagates according to the law (see Ref. 16)

$$ x_{DSW} = \frac{3A^{2/3}}{2^{1/3}} t^{1/3}, \quad A = \int \sqrt{u_0(x)} \, dx. \quad (37) $$

Consequently, after the moment $t \approx 2A/k_0^3$ the packet enters into the dispersive shock region and propagates along its averaged profile rather than along the dispersionless solution (28). Thus, the above theory is applicable for the time $t < 2A/k_0^3$ which may be very large for long enough initial pulse length $l$.

If $k_0 > 2\sqrt{u_m}$, then the packet reaches the minimal point $u_m$ of the background profile at the moment

$$ t_m = t_1(-u_m) = \frac{l}{24u_m\sqrt{k_0^2 - 4u_m}} \times$$

$$ \times \left[\sqrt{k_0^2 - 4u_m} - k_0 + \sqrt{u_m \ln \frac{k_0 + 2\sqrt{u_m}}{k_0 - 2\sqrt{u_m}}} \right]. \quad (38) $$

After that it starts its motion along the second branch for which we easily obtain

$$ t_2(u) = \frac{l}{24u_m\sqrt{k_0^2 + 4u}} \left[\sqrt{k_0^2 + 4u} - k_0 +$$

$$ + 2\sqrt{u_m \ln \frac{\sqrt{k_0^2 + 4u} + 2\sqrt{u + u_m}}{k_0 - 2\sqrt{u_m}}} \right]. \quad (39) $$

If it arrives to the left edge $u = 0$ of the dispersionless solution at the moment

$$ t_2(0) = \frac{l}{12u_m k_0} \ln \frac{k_0 + 2\sqrt{u_m}}{k_0 - 2\sqrt{u_m}} \quad (40) $$

before the wave breaking time (29), $t_2(0) \leq t_b$, that is if $k_0 \geq k_0'$, where $k_0'$ is the root of the equation

$$ \sqrt{u_m} \ln \frac{k_0' + 2\sqrt{u_m}}{k_0' - 2\sqrt{u_m}} = 1, \quad \text{or} \quad k_0' \approx 2.399 \sqrt{u_m}. \quad (41) $$

then the packet moves all the time along the smooth profile without getting into the dispersive shock region. For the wave number in the interval $2\sqrt{u_m} < k_0 < k_0'$ the packet’s path goes partly through the dispersive shock region and this part of the path is outside of the applicability domain of our theory. In Fig. 7 we show the

![Fig. 7](image-url)

**FIG. 7.** (a) The total numerical profiles $u(x, t)$ obtained by numerical solution of the KdV equation for $u_m = 1$, $k_0 = 2.4$ at the moments of time $t = 0$ (red), $t = 35$ (blue), $t = 69$ (black). (b) The profiles of the linear wave packets obtained by subtraction of the smooth evolution of the background pulse at the same moments of time.

![Fig. 8](image-url)

**FIG. 8.** The spectrum of the wave packet at the same moments of time which are indicated in Fig. 7. At the moment of time $t = 69$ the packet is located partly outside the smooth pulse and partly inside it what causes two humps in the distribution. The narrow peak in the spectral distribution corresponds to restoration of the initial carrier wave number during the exit of the packet from the smooth pulse region.
The theory of propagation of a high-frequency wave packet along a positive pulse is somewhat simpler. If we take the initial background profile in the form shown in Fig. 11, then it breaks at once at $t = 0$, but the packet with the wave number $k_0$ propagates to the left faster than the left edge of the appearing here dispersive shock wave, so the packet considered as a point-like structure moves along a smooth profile. Now for illustration of the theory we take the initial distribution in the form

$$u_0(x) = 4u_m \left( \frac{x}{l} + \sqrt{-\frac{x}{l}} \right), \quad -l \leq x \leq 0,$$

C. Propagation of a wave packet along a positive pulse

The theory of propagation of a high-frequency wave packet along a positive pulse is somewhat simpler. If we take the initial background profile in the form shown in Fig. 11, then it breaks at once at $t = 0$, but the packet with the wave number $k_0$ propagates to the left faster than the left edge of the appearing here dispersive shock wave, so the packet considered as a point-like structure moves along a smooth profile. Now for illustration of the theory we take the initial distribution in the form

$$u_0(x) = 4u_m \left( \frac{x}{l} + \sqrt{-\frac{x}{l}} \right), \quad -l \leq x \leq 0,$$
so that

\[
\varphi_1(u) = \frac{l}{4} \left( \frac{u}{u_m} - 2 + 2 \sqrt{1 - \frac{u}{u_m}} \right),
\]
\[
\varphi_2(u) = \frac{l}{4} \left( \frac{u}{u_m} - 2 - 2 \sqrt{1 - \frac{u}{u_m}} \right),
\]

and

\[
\varphi_1'(u) = \frac{l}{4u_m} \left( 1 - \frac{1}{\sqrt{1 - u/u_m}} \right),
\]
\[
\varphi_2'(u) = \frac{l}{4u_m} \left( 1 + \frac{1}{\sqrt{1 - u/u_m}} \right).
\]

The dispersionless solution \( x - 6ut = \varphi(u) \) can be transformed to the explicit formula for \( u(x, t) \):

\[
u(x, t) = \frac{4u_m}{(l + 24u_m t)^2} \left[ 12 u_m t (l + 2x) + lx + l \sqrt{24u_m t (6u_m t - x) - lx} \right],
\]

and the packet’s motion along its two branches \((45)\) is described by the solutions of Eq. \((31)\) with the right-hand sides given by Eqs. \((46)\). As a result we obtain the expressions for the moments of time at which the packet reaches the points with the amplitude \( u \) of the evolving smooth background along each branch of the solution:

\[
t_1(u) = \frac{l}{24u_m \sqrt{k_0^2 + 4u}} \left\{ k_0 - \sqrt{k_0^2 + 4u} + 2 \arcsin \sqrt{\frac{k_0^2 + 4u}{k_0^2 + 4u_m}} - 2 \arcsin \frac{k_0}{\sqrt{k_0^2 + 4u_m}} \right\},
\]
\[
t_2(u) = \frac{l}{24u_m \sqrt{k_0^2 + 4u}} \left\{ k_0 - \sqrt{k_0^2 + 4u + 2\pi} - 2 \arcsin \sqrt{\frac{k_0^2 + 4u}{k_0^2 + 4u_m}} + 2 \arcsin \frac{k_0}{\sqrt{k_0^2 + 4u_m}} \right\}.
\]

The formula \( x(u) = 6ut(u) + \varphi(u) \) for the location of the packet together with Eqs. \((48)\) determines the law of motion of the point-like packet. The positive background accelerates the packet and it reaches the opposite edge of the background distribution at the moment \( t_2(0) \). Hence, in this case the phase shift is positive and equal to

\[
\Delta x_L = l \left\{ 1 - \frac{k_0}{4\sqrt{u_m}} \left( \pi - 2 \arctan \frac{k_0}{2\sqrt{u_m}} \right) \right\}
\]
\[
= l \left\{ 1 - \frac{ik_0}{4\sqrt{u_m}} \ln \frac{ik_0 + 2\sqrt{u_m}}{ik_0 - 2\sqrt{u_m}} \right\}.
\]

The last expression can be obtained from Eq. \((43)\) by means of the replacement \( k_0 \rightarrow ik_0 \).

Interaction of the wave packet with the positive smooth pulse is qualitatively the same for any large enough value of \( k_0 \). However, in comparison with numerical results we often have to take into account a finite size of the packet and therefore it is necessary to make the following reservation related with a specific form of the initial distribution \((44)\) which breaks at the same moment of time \( t = 0 \) as the packet starts its motion according to Fig. 11. Consequently, at the initial stage of propagation the packet with \( k_0 \sim 1 \) overlaps due to its finite size with the dispersive shock wave. This leads to noticeable deformation of the spectrum of the packet which introduces some uncertainty in numerical evaluation of the carrier wave number. To avoid such inessential complications, we have chosen in our numerics a large enough initial wave number \( k_0 = 5 \), so that the duration of interaction of the packet with the dispersive shock wave becomes

![Image](attachment:figure12.png)

FIG. 12. (a) The total numerical profiles \( u(x, t) \) obtained by numerical solution of the KdV equation for \( u_m = 1, l = 800, k_0 = 5 \) at the moments of time \( t = 0 \) (red), \( t = 4 \) (blue), \( t = 12 \) (black). Oscillatory region at the right edge of the wave structure for \( t = 12 \) corresponds to the dispersive shock wave generated after the wave breaking moment. (b) The profiles of the linear wave packets obtained by subtraction of the smooth evolution of the background pulse at the same moments of time.

![Image](attachment:figure13.png)

FIG. 13. The numerically obtained spectrum \((35)\) of the linear packet for the three moment of time \( t = 0 \) (red), \( t = 4 \) (blue), \( t = 12 \) (black). As we see, at the moment \( t = 12 \) when the packet leaves the non-uniform smooth pulse region, the carrier wave number returns to its initial value \( k_0 = 5 \) in agreement with the theory.
very small. We show in Fig. 12 the full amplitude $u(x,t)$ for three moments of time $t = 0, 4, 12$ (a) and the linear packet’s amplitude for the same moments of time (b) obtained by means of the subtraction procedure described above for the propagation of the packet along the negative background pulse. Although the packet widens during its propagation, the spectrum changes very little and it remains being localized around the initial value $k_0 = 5$ after the packet’s exit from the smooth pulse region: see Fig. 14. The packet’s coordinate and its velocity agree very well with theoretical predictions; see Fig. 15.

**IV. CONCLUSION**

In this paper, we have developed the theory of propagation of short wavelength packets along the evolving smooth background under supposition that this evolution belongs to the simple-wave type of dispersionless flows. Large difference of wavelength scales in the packet and in the smooth pulse allows one to combine the Hamilton equations describing propagation of packets in the geometrical optics approximation with the Hopf equation describing the dispersionless evolution of the simple wave. As an immediate consequence of this combined system, we obtain Eq. (4) derived earlier by G. A. El in Ref. 14 from the small-amplitude limit of Whitham equations which describe evolution of dispersive shock waves in the Gurevich-Pitaevskii approach to the dispersive shocks theory developed in Ref. 17 (see also Ref. 18 and references therein). Solution of this equation with the initial condition $k = k_0$ at $u = u_0$ yields the dependence $k = k(u)$ of the carrier wave number $k$ on the value $u$ of the background flow amplitude at the instant location of the packet. Then, as was shown in Ref. 17, the path of the packet can be found from the condition that the packet’s motion with the group velocity must be consistent with the known solution of the Hopf equation for the background flow. This method was applied earlier to description of motion of small-amplitude edges of dispersive shock waves in wave systems obeying non-integrable equations (the shallow water Serre equation in Ref. 9 and equations for ion-sound nonlinear plasma waves in Ref. 11), and here we extend it arbitrary localized wave packets.

The developed in this paper approach is illustrated by its application to wave systems whose evolution is described by the KdV equation. This type of problems has been recently discussed in Ref. 7 for a particular case of a rarefaction wave background flow. We have extended the theory to the general form of simple wave flows. The analytical theory was illustrated by examples of the initial distributions which admit simple and explicit analytical solutions. Comparison of our analytical theory with the results of numerical solutions of the KdV equation with appropriate initial conditions demonstrates quite good accuracy of our approximation. One may suppose that this approach will find application to various problems of wave packets propagation along evolving smooth pulses of a simple wave type.

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