On errors in Euler’s complex exponent and formula for solving ODEs

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Abstract. There are some mistakes in Euler’s works. Some of them form the basis of most of the sciences, including differential equations and complex analysis. We discuss them here.

1. Introduction
Leonhard Euler was an eighteenth Swiss mathematician, astronomer, physicist, engineer and logician. He made extensive contributions to the sciences. We address only a tip of this iceberg here. That being some of the mistakes that can be found on his works on differential equations and complex exponent expansions.

In the next section, Section 2, we elaborate on the errors. First on differential equations, then complex exponent expansions.

Section 3 is on our approach to Euler’s objectives. We show how the differential equation he made the mistake on can be solved through quadrature. That is, solved the only natural way. Next, we correct Euler’s error on the expansion.

The third error is on the famous formula $\exp(i\pi) = -1$, which is a part of complex exponential expansion error.

2. The errors

2.1. The error in the second-order ordinary differential equation solution
Euler proposed the formula

$$y = e^x,$$  \hspace{1cm} (1)

to solve the differential equation
Here and are real constants. The function \( y = y(x) \) is also real. The usage of this formula can be found in the most elementary text that treat differential equations [1], to the most advanced [2].

### 2.1.1. The procedure

First the derivatives are obtained. That is,

\[
\frac{dy}{dx} = re^x, 
\]

and

\[
\frac{d^2y}{dx^2} = r^2e^x. 
\]

Substituting these derivatives into (2) yields

\[
ar^2 + br + c = 0, 
\]

a quadratic equation, with the solution

\[
r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. 
\]

The case \( b^2 - 4ac > 0 \), according to Euler, leads to the solution

\[
y = A \exp \left( \frac{-b - \sqrt{b^2 - 4ac}}{2a} x \right) + B \exp \left( \frac{-b + \sqrt{b^2 - 4ac}}{2a} x \right), 
\]

where both \( A, B \) are constants. For the case \( b^2 - 4ac < 0 \), Euler suggested

\[
y = \exp \left( \frac{-b}{2a} x \right) \left[ C \cos \left( \frac{-\sqrt{b^2 - 4ac}}{2a} x \right) + D \sin \left( \frac{\sqrt{b^2 - 4ac}}{2a} x \right) \right], 
\]

where both \( C \) and \( D \) re constants. For the case \( b^2 - 4ac = 0 \), the solution follows naturally without Euler’s assistance. And is,

\[
y = E x + F. 
\]

Here the constants are \( E \) and \( F \).

### 2.1.2. The error

It is natural to expect that the two outer solutions to not only be equal to one another, but equal to (2) too. Note that (7) equated to (9) and (10) when \( b^2 - 4ac = 0 \) leads to

\[
A \exp \left( \frac{-b}{2a} x \right) = E x + F = C \exp \left( \frac{-b}{2a} x \right). 
\]
This is a contradiction, hence the error we allege.

2.2. The error in Euler’s expansion of the complex exponential function.
Euler used Taylor’s theorem for expanding real functions to arrive at the formula

\[ e^{ix} = \cos(x) + i \sin(x). \]  

(11)

This formula can be found in every textbook that deals with complex functions, and in many research articles. One demonstration of its usage is the famous formula

\[ e^{i\pi} = -1. \]

(12)

It results from letting \( x \) in (11) to assume the value \( \pi \). It is an error, for \( x \) can never assume real values, as we now demonstrate.

2.2.1. Euler’s expansion. Note that Taylor’s theorem for expanding real function states

**Theorem 1**: Suppose \( f: (a, b) \to \mathbb{R} \) is a function on \((a, b)\), where \( a, b \in \mathbb{R} \) with \( a < b \). Assume that for some positive integer \( n \), \( f \) is \( n \)–times differentiable on the open interval \((a, b)\), and that \( f, f', f'', \cdots, f^{n-1} \) all extend continuously to the closed interval \([a, b]\). Then there exists \( c \in (a, b) \) such that

\[ f(\xi) = \sum_{k=0}^{n-1} \frac{(f^{(k)}(a))}{k!} (\xi - a)^k + \frac{(f^{(n)}(a))}{n!} (\xi - a)^n. \]

Note that the theorem does not deviate from asserting that

\[ \xi = R \]

(13)

and \( a, b, c \) and \( f \) are all real. These can be found in [3], [4], [5], [6] and [7], and countlessly many others. From the theorem, Euler then deduced the expansion

\[ e^{\xi} = 1 + \frac{\xi}{1!} + \frac{\xi^2}{2!} + \frac{\xi^3}{3!} + \frac{\xi^4}{4!} + \cdots. \]

(14)

He then let \( \xi = ix \), from which he deduced that
Next he noted that the cosine expansion is
\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots, \] (16)
and sine expansion is
\[ \sin x = i \left( \frac{x}{1!} - \frac{x^3}{3!} + \cdots \right), \] (17)
from these he concluded that
\[ e^{ix} = \cos x + i \sin x. \] (18)

This conclusion appears to be perfectly fine. But it is not. There is an error.

2.2.2 The error. After (14), when he let \( \xi = lx \), he remained silent on the fact \( \xi \in \mathbb{R} \). Had he been vocal, he could have realized that his assumption suggested that \( lx \in \mathbb{R} \), a simple replacement, leading to \( x \in i\mathbb{R} \), which is not a real number as everybody believes. The expression in (16) then would never have resulted. Which is the third error.

3. Our approach
Here we address the errors we pointed out.

3.1. The solution for equation (2)
First note that (2) can be reduced to the simplified case
\[ \frac{d^2y}{dx^2} = -\lambda y. \] (19)

Separating the variables and introducing integral signs gives
\[ \int \frac{(y')^2}{2} \, dx = -\lambda \int \frac{y^2}{2} \, dx. \] (20)
Integration yields
where $E$ is the constant of integration. Separating the variables again in (21) and integrating, yields

$$y = \sqrt{2E}\sin\left(\sqrt{\lambda x + \phi}\right),$$

where $\phi$ is the constant of integration.

3.2. *An expansion of exp*(*) in terms of cosines.* After having deduced

$$e^{ix} = \cos(x) + i \sin(x),$$

Euler should have simply stated that

$$x \in iR.$$  

That is, $x$ is not a real variable, and need not assume real values. A proper formula arising from his analyses should have been

$$e^\theta = \cos(i\theta) - i \sin(i\theta),$$

with $\theta$ taking on real values.

4. **Conclusion**

If one compares Euler’s work on the differential and the exponent expansion, one cannot help but conclude that Euler was eager to resolve the complex results from the differential equation, that he decided to keep quite on the contradiction. Unfortunately, the error of this convenience is beginning to be felt in this century that does not tolerate shortcuts.

The complex exponent $\text{exp}(ix)$ can however be expanded properly. Unfortunately the analyses thereof leads to a very challenging differential equation, but should be resolved because it finds applications in very important fields like the case where Kdv equations arise in plasma flows.

Lastly, Euler’s tendency to a throw in formulas that appear to fit, but do not, has affected many other scholars who came after him. We can add the French mathematician Joseph Liouville (1809–1882) on his treatment of the stability of differential equations, and Gustav de Vries (1866–1934) and Diederik Korteweg (1848–1941) for their solution for the equations bear their names.
References

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