Towards Gaussian states for loop quantum gravity

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An important challenge in loop quantum gravity is to find semiclassical states – states that are as close to classical as quantum theory allows. This is difficult because the states in the Hilbert space used in LQG are excitations over a vacuum in which geometry is highly degenerate. Additionally, fluctuations are distributed very unevenly between configuration and momentum variables. Coherent states that have been proposed to balance the uncertainties more evenly can, up to now, only do this for finitely many degrees of freedom. Our work is motivated by the desire to obtain Gaussian states that encompass all degrees of freedom. To obtain a toy-model we reformulate the U(1) holonomy-flux algebra in any dimension as a Weyl algebra, and discuss generalisations to SU(2). We then define and investigate a new class of states on these algebras which behave like quasifree states on the momentum variables.

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I. INTRODUCTION

In quantum field theory (QFT), the states can often be regarded as excitations over a special state, such as a ground state or thermal state. This state encodes the physical circumstances such as the strength of the fluctuations or the temperature, and the excitations over it inherit many of its basic properties. In the following, we will loosely refer to such a special state as a vacuum.
Loop quantum gravity (LQG) is a QFT without classical background geometry. The basic fields are a SU(2) connection one-form $A$ and a tensor density $E$ [1, 2]:

$$[A_i^a(x), E^b_j(y)] = i\hbar\beta_a^i\delta^b_j\delta(x, y)$$  \hspace{1cm} (1)

The field algebra consists of parallel transporters and flux-like variables

$$h_e = \mathcal{P} \exp(-\int_e A), \quad E_S(f) = \frac{1}{2}\int_S f^j(x)E^a_j(x)\epsilon_{abc}dx^b \wedge dx^c$$  \hspace{1cm} (2)

where $e$ is a path and $S$ a surface in space. The commutation relations between these operators are purely topological in nature. The resulting algebra is called holonomy-flux (HF) algebra. Consequently the natural vacuum state has very different properties than those in QFT on Minkowski space. Physically, this Ashtekar-Lewandowski (AL) state [3] corresponds to a degenerate spatial metric $g_{ab} = 0$ and a canonically conjugate extrinsic geometry with infinite fluctuations. This state is a natural ground state when general covariance is at the forefront. In fact, it is the unique diffeomorphism invariant state [4, 5] (see however [6, 7]). The AL state is a special case in a whole family of states, all peaked on spatial geometry [8, 9]. Apart from the AL state, these states are not invariant under spatial diffeomorphisms.

At the other end of the spectrum, there is a construction of a vacuum state due to Dittrich and Geiller (albeit on a modified algebra) which is dual to the AL state in the sense that it is peaked on flat extrinsic geometry, while fluctuations in the spatial metric are maximal. [10, 11]. See also [12] for an elegant formulation of this idea in the Abelian case.

When it comes to the description of classical spacetime, neither excitations above the AL vacuum nor above the Dittrich-Geiller vacuum are particularly suitable, due to the uneven distribution of fluctuations. AL excitations have been used to construct coherent states [13–15], but these states have semiclassical properties only for finitely many degrees of freedom. Going over to infinitely many degrees of freedom leads to new measures on the space of connection fields. [16, 17]. It is also possible to transfer Gaussian measures from background dependent QFT to the space of connections used in LQG. [18, 19], but the resulting Hilbert spaces so far do not support the holonomy flux algebra of LQG [20].

The present work is also concerned with finding new states for LQG. For this it is necessary to first define the algebras carefully. We do this for the HF algebra in section VI A We should point out that we use a definition that is less strict than that of [3, 21], in that does not contain all relations among iterated commutators that are present in the AL representation. We also consider the case of the structure group U(1) [22] as a toy-model for the more complicated SU(2) HF algebra. The U(1) case is interesting, because the relations of the HF algebra can be brought into the form of a Weyl algebra, by going over to the algebra elements [23]

$$W(e, S) = e^{\frac{1}{2}(e, S)}h_e e^{iE_S}.$$  \hspace{1cm} (3)

This formulation makes contact with free quantum fields on a fixed geometric background.

As others before us, we are unable to find states for the HF algebra that are Gaussian with uncertainties split between the canonical variables. But we will describe a new type of state in which the flux operators have Gaussian fluctuations, whereas the properties of the holonomies are those of the AL representation. To be precise, the product of a holonomy and a flux has a vacuum expectation value

$$\langle 0 | h_e e^{iE_S} | 0 \rangle = \delta_{e, 0} e^{-\frac{1}{2}\alpha_S f} e^{i\beta_S f},$$  \hspace{1cm} (4)

where $\alpha_S$ is an $S$-dependent bilinear form, and $\beta_S$ a linear one. This kind of state gives a new representation of the HF algebra, in which the spatial geometry fluctuates around an average value given by $\beta_S$. If one chooses the covariance $\alpha$ to be vanishing, one obtains states of the type considered in [3, 6, 24, 26].

One motivation for the consideration of these states is the quantum origin of the primordial perturbations. The current observations of the CMB can be described by saying that primordial perturbations of density and spatial metric seem to be described by a Gaussian random field with a certain covariance. A state of the form (4) describes the quantum geometry of the early universe.

We will start the discussion by reviewing some aspects of LQG and the definition and properties of Weyl algebras, in sections I and II respectively. Then we will discuss the U(1) model in detail, defining the Weyl algebra in section IV and the new states in section V. The situation for SU(2) is discussed in section VI with a precise definition of the HF algebra in section VI A, the new states in VI B and a discussion of the changes of the area spectrum due to the fluctuations in VIC. We end with a short summary and outlook, section VII.
II. LOOP QUANTUM GRAVITY

We want to present the basics of the canonical quantization of Yang-Mills type GR, known as loop quantum gravity (LQG), following [27, 28]. To this end, we presuppose that the spacetime is globally hyperbolic, and a Cauchy surface has been chosen.

The canonically conjugate variables of LQG are holonomies along one dimensional paths and fluxes through two dimensional hypersurfaces, both contained in the Cauchy surface. The holonomies are path ordered exponentials of an $\mathfrak{su}(2)$ connection $A$ related to the spatial spin connection of gravity:

$$h_e = \mathcal{P} \exp (- \int_e A). \quad (5)$$

They take values in $\text{SU}(2)$. The fluxes of the Yang-Mills electric field $E$ are smeared against $\mathfrak{su}(2)$-valued functions $f$ with support on a two-dimensional, orientable hypersurface $S$, given by

$$E_S(f) = \int_S \frac{1}{2} f^j(x) E^j_b(x) \epsilon_{abc} dx^b \wedge dx^c. \quad (6)$$

The canonical commutation relation of holonomies and fluxes is only sensitive to intersection points of the corresponding edge and surface. For a pair of edge and surface intersecting in only one of the endpoints of $e$ it is given by

$$[h_e, E_S(f)] = \frac{\hbar k \beta}{4 i} \kappa(e, S) \begin{cases} h_e f(p) & p = e \cap S \text{ source of } e, \\ -f(p) h_e & p = e \cap S \text{ target of } e. \end{cases} \quad (7)$$

with the Barbero-Immirzi parameter $\beta$. Here, $\kappa(e, S)$ encodes partly the relative orientation of edge and surface. It is $\pm 1$ for an edge above or below the surface.

For the algebra underlying loop quantum gravity, one needs the notion of cylindrical functions. Therefore, one has to establish abstract graphs which are a finite collection of edges that are allowed to build out vertices by intersections of their beginning and final points, i.e. $\gamma = \{e_1, e_2, \ldots, e_n\}$. One further speaks of the set of edges $E(\gamma)$ and the set of vertices $V(\gamma)$ of a given graph $\gamma$. Now we call $\mathcal{A}$ the space of smooth connections and look at functions on this space,

$$F: \mathcal{A} \rightarrow \mathbb{C}. \quad (8)$$

Such a function is said to be cylindrical w. r. t. a given graph $\gamma$ if there is a function

$$F_\gamma: \text{SU}(2)^{|E(\gamma)|} \rightarrow \mathbb{C}, \quad (9)$$

such that the function on $\mathcal{A}$ can be expressed by this function of powers of $\text{SU}(2)$, i.e. by setting for $A \in \mathcal{A}$

$$F(A) = F_\gamma \left( \{h_e(A)\}_{e \in E(\gamma)} \right). \quad (10)$$

The cylindrical functions form an algebra called Cyl. The fluxes $E_S(f)$ can be used to define the Hamiltonian vector fields

$$X_{S,f} = [E_S(f), \cdot], \quad (11)$$

which act on Cyl.

The $*$-Lie-algebra which is quantized for LQG is the algebra generated by smooth cylindrical functions and the flux vector fields by commutators as multiplication, subject to complex conjugation as involution. The restriction to $\text{Cyl}^\infty$ ensures that $X_{S,f}[F]$ is still a smooth cylindrical function. The resulting quantum $*$-algebra is known as the holonomy-flux algebra. For the precise definition that we will use, see section VI A.

There is a unique diffeomorphism invariant representation of the HF algebra [4, 5].

**Theorem II.1.** The Ashtekar-Isham-Lewandowski state

$$\varphi_{\text{AL}}(FX_{S_1,f_1}X_{S_2,f_2} \cdots X_{S_n,f_n}) = \begin{cases} 0 & n > 0 \\ \mu_0(F) & n = 0 \end{cases} \quad (12)$$

is the only diffeomorphism invariant state on the HF algebra. Here, the state acts on elements of $\text{Cyl}^\infty$ as

$$\mu_0(F) = \int_{\text{SU}(2)^{|E(\gamma)|}} \prod_{e \in E(\gamma)} d\mu_H(g_e) F_\gamma \left( \{g_e\}_{e \in E(\gamma)} \right). \quad (13)$$
A sequence of holonomies and flux vector fields can be always brought into a normal ordered form – flux vector fields to the right and holonomies to the left – by using the commutation relations. The resulting expression is by linearity of the states a sum of terms of the form used in the theorem. Furthermore, the measure that is used in the integration over the powers of SU(2) is just the product of the Haar measures for the individual copies of SU(2).

The representation that arises from this state via the GNS construction is called Ashtekar-Lewandowski (AL) representation \[3, 29\]. The Hilbert space is the space of square integrable functions over the space of generalized – specifically distributional – connections:

\[ H = L^2(\mathcal{A}, d\mu_{\text{AL}}). \] (14)

The representation of holonomies and cylindrical functions and fluxes is similar to the ordinary Schrödinger representation, i.e. as a multiplication operator and a derivative. For the cylindrical functions one sets

\[ (\pi_{\text{AL}}(F)\Psi)(A) = F(A)\Psi(A). \] (15)

The AL representation of the fluxes that act on cylindrical functions is basically the following:

\[ (\pi_{\text{AL}}(E_S(f))\Psi)(A) = X_{S,f}[\Psi] = \frac{\hbar\kappa_\beta}{2} \sum_{v \in V(\gamma)} f^j(v) \sum_{e \in E(v)} \kappa(e, S) J^{(v,e)}_j\Psi(A). \] (16)

The first sum runs over all vertices of the graph \( \gamma \) underlying the cylindrical function \( \Psi \). Here it is assumed that every intersection point between edges and the surface is a – possibly trivial – vertex considered in \( V(\gamma) \). At these vertices one evaluates the smearing function of the flux vector fields. The second sum now runs over all edges \( e \) that begin or end at the vertex \( v \). The object \( J^{(v,e)}_j \) encodes the left- and right-invariant vector fields acting on copies of SU(2) assigned to the specific edges of a vertex, for out- and in-going edges, respectively.

### III. REVIEW OF WEYL ALGEBRAS

In the following we want to give a short introduction to the topic of Weyl algebras of canonical commutation relations and quasi free states. We base our discussion on \[30\] and \[31\].

A Weyl algebra is a \( C^* \)-Algebra, solely constructed from a canonical commutation relation (CCR). It is sensible to present the construction in several steps.

**Definition III.1.** The CCR \( * \)-algebra over the pre-symplectic space \((H, \sigma)\), denoted by \( \text{CCR}(H, \sigma) \), is the algebra generated by the Weyl elements \( W(X) \), \( X \in H \) that satisfy the Weyl relations

\[ W(X)W(Y) = e^{-\frac{i}{\hbar} \sigma(X,Y)} W(X + Y) \] (17)

and are subject to the involution \( * \), such that

\[ W(X)^* = W(-X) = W(X)^{-1}. \] (18)

The Weyl elements are unitary with respect to the involution and hence a representation of this algebra has to be a unitary representation.

**Definition III.2.** Let \( \mathcal{H} \) be a Hilbert space and let \( \pi : H \to U(\mathcal{H}) \subset \mathcal{B}(\mathcal{H}) \), \( X \mapsto \pi(W(X)) \) be a map from the real vector space \( H \) into the unitary operators on \( \mathcal{H} \). Then \( (\pi, \mathcal{H}) \) is a representation of a canonical commutation relation in terms of a pre-symplectic form or equivalently a CCR representation over the pre-symplectic space \((H, \sigma)\), if the so called Weyl operators satisfy the Weyl relations

\[ \pi(W(X))\pi(W(Y)) = e^{-\frac{i}{\hbar} \sigma(X,Y)} \pi(W(X + Y)) \] (19)

and \( \pi \) is a \( * \)-homomorphism, i.e. \( \pi(W(X)^*) = \pi(W(X))^\dagger \).

In principle, \( \text{CCR}(\hbar, \sigma) \) is homomorphic to a \( C^* \)-algebra of bounded operators on a given representation Hilbert space, because of the unitarity of the Weyl elements. In order to promote the \( * \)-algebra to a \( C^* \)-algebra one exploits the fact that with \( (\pi, \mathcal{B}^2(H)) \) there is always a \( * \)-representation, which allows for the definition of a norm that is highly dependent on CCR \( * \)-representations. It is known as the minimal regular norm \[30\].
Definition III.4. Let field operators separate into partitions of two-point correlation functions. Gaussian states, i.e. expectation values of Weyl elements are Gaussian functions, or non-degenerate, not only exists but is also unique up to isomorphism. One uses the minimal regular norm, since there is always a CCR representation as one can consider the so called Hilbert space of square summable sequences indexed by $H$.

The important feature of a Weyl algebra, as just defined, over a symplectic space, i.e. for a non-degenerate $\sigma$, is that all of the norms for different representations in the minimal regular norm are just the same (see (30)). Furthermore, there is a uniqueness theorem for this case. The Weyl algebra $\text{CCR}^\text{Weyl}(H, \sigma)$ over a symplectic space, i.e. $\sigma$ is non-degenerate, not only exists but is also unique up to isomorphism.

The notion of quasifree states has been introduced by Araki in the 1960’s [32–34]. This is the most general class of Gaussian states, i.e. expectation values of Weyl elements are Gaussian functions, or $n$-point correlation functions of field operators separate into partitions of two-point correlation functions.

Definition III.3. We equip the $*$-algebra $\text{CCR}(H, \sigma)$ with the minimal regular norm $\|A\| := \sup \{ \|\pi(A)\|_\pi \mid (\pi, H) \text{ is a } \text{CCR }*\text{-representation} \}$.

The $C^*$-algebra, obtained by the completion of $\text{CCR}(H, \sigma)$ w. r. t. the minimal regular norm, is referred to as Weyl algebra or Weyl CCR algebra over the pre-symplectic space $(H, \sigma)$:

$$\text{CCR}^\text{Weyl}(H, \sigma) := \overline{\text{CCR}(H, \sigma)}.$$ (21)

The existence of such quasifree states is tied to a condition on the covariance and the symplectic form, namely

$$\sqrt{\alpha(X, X)} \sqrt{\alpha(Y, Y)} \geq \frac{1}{2} |\sigma(X, Y)| \quad \forall X, Y \in H. \quad \text{(23)}$$

With this, the existence is reduced to finding a covariance that satisfies the determining inequality. Looking at the inequality, one can realize that the positive symmetric form should rather be seen as an inner product on $\Omega$.

The separation of correlation functions into two-point functions follows from Stone’s theorem since

$$\langle B_\pi(X_1)B_\pi(X_2)\ldots B_\pi(X_n) \rangle_\Omega = \frac{1}{i^n} \frac{\partial^n}{\partial t_1 \partial t_2 \ldots \partial t_n} \varphi(W(t_1X_1)W(t_2X_2)\ldots W(t_nX_n))\bigg|_{t=0}$$

$$= \frac{1}{i^n} \frac{\partial^n}{\partial t_1 \partial t_2 \ldots \partial t_n} \exp \left\{ -\frac{1}{2} \sum_{j=1}^{n} (t_j)^2 \alpha(X_j, X_j) - \sum_{j<k} t_j t_k \left( \alpha(X_j, X_k) + \frac{i}{2} \sigma(X_j, X_k) \right) \right\} \bigg|_{t=0}. \quad \text{(27)}$$

Clearly this vanishes for odd $n$. For $n$ even in contrast, there is still only one non-vanishing term, which carries a linear contribution for every single $t_k$. This is exactly the term that is separated into two point function. Summarizing, we have

$$\langle B_\pi(X_1)B_\pi(X_2)\ldots B_\pi(X_n) \rangle_\Omega = \begin{cases} 0 \prod_{\text{partitions}} \langle B_\pi(X_i)B_\pi(X_j) \rangle_\Omega & n \text{ odd} \\ \sum_{\text{partitions}} \langle B_\pi(X_i)B_\pi(X_j) \rangle_\Omega & n \text{ even} \end{cases} \quad \text{(28)}$$
IV. WEYL ALGEBRA FOR GAUGE GROUP U(1)

The quantum electrodynamics analog of loop quantum gravity is the situation where one considers the kinematics of a U(1) Yang-Mills theory and quantizes it in the spirit of diffeomorphism invariant quantum gravity [22, 35]. We want to discuss the toy-model in general and look at a formulation in \( D + 1 \) dimensions and hence a \( D \)-dimensional spatial manifold.

We consider a pair of a U(1) connection or vector potential \( A_a(x) \) and an electric field \( E^a(x) \), set coupling constants, including \( \hbar \), to 1 and stipulate the following CCR:

\[
\begin{align*}
[A_a(x), E^b(y)] &= i \delta^b_a \delta^{(D)}(x, y), \\
[A_a(x), A_b(y)] &= 0 = [E^a(x), E^b(y)]. 
\end{align*}
\] (29)

The smeared connection is defined completely analogous to the SU(2) case. For the smeared flux, however, we go without an additional smearing function and consider only the integration over the surface. Smeared connection and flux therefore are given by

\[
\begin{align*}
A(e) &= \int_e A = \int_0^1 dt \, A_a(e(t)) \dot{e}^a(t), \\
E(S) &= \int_S (\ast E) = \int_S \frac{1}{(D-1)!} E^a_{\alpha_1\alpha_2...\alpha_{D-1}} dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_{D-1}}. 
\end{align*}
\] (30)

The commutation relation of these variables is characterized by the oriented intersection number \( I(e, S) \) of a surface \( S \) and a path \( e \). This object has already been looked at in the context of LQG in [36, 37] under the name Gauss linking number. For the CCR, we find

\[
[A(e), E(S)] = i I(e, S). \] (31)

The actual commutation relation that is analogous to SU(2) considers holonomies. For U(1) there is no need for path ordering in the holonomies and we can simply look at the object

\[
h_e = e^{i A(e)}. \] (32)

As in the non-abelian theory, the holonomies are elements of the gauge group and for that reason elements of U(1). The commutation relation for holonomies and fluxes becomes

\[
[h_e, E(S)] = -I(e, S) \, h_e. \] (33)

In this toy-model, the fluxes commute and the same applies to holonomies.

A. Distributional form factors for edges and surfaces

For the Weyl algebra, we need at least a pre-symplectic space and hence an explicit vector space. This will not be an ordinary vector space but a certain space of edges and surfaces. The formulation that turns out to be the most suitable for such a vector space is in terms of distributional objects, which we want to call form factors, describing edges and surfaces. A similar notion for edges has already been used in [36, 37], although the treatment of surfaces is somewhat different, since they are described by their closed boundary curves.

**Definition IV.1.**

1. Let \( e : [0, 1] \rightarrow \sigma \) be an embedding of an analytic, oriented path into the spatial \( D \)-manifold \( \sigma \). Then

\[
F_e^a(x) = \int_0^1 dt \, \dot{e}^a(t) \delta^{(D)}(x, e(t))
\] (34)

is a called distributional form factor for the edge \( e \). The dot indicates the derivative w. r. t. the curve parameter \( t \). This can be used to smear one-forms along this specific edge.
2. Let \( S : [0,1]^{D-1} \to \sigma \) be an embedding of an analytic, oriented surface into \( \sigma \). Then

\[
F_{S_a}(x) = \frac{1}{2\pi i} \int_0^1 \int_0^1 \cdots \int_0^1 dt_D \cdots dt_1 \epsilon_{a_1 a_2 \cdots a_D} S_{i_1}^{a_1}(t_1) S_{i_2}^{a_2}(t_2) \cdots S_{D-1}^{a_{D-1}}(t_D) \delta^{(D)}(x,S(t))
\]  

is called distributional form factor for the surface \( S \). Here, \( t = (t_1, t_2, \ldots, t_{D-1}) \) is a right-handed parametrization of \( S \) and the comma indicates the derivative w.r.t. the parametrization, i.e.

\[
S_{i_1}^{a_1}(t_1) = \frac{\partial S_{a_1}}{\partial t_{i_1}}.
\]

This can be used to smear vector densities over this specific surface.

In term of form factors for edges, the smeared connection is

\[
A(e) = \int_\sigma d^Dx A_a(x) F_e^a(x).
\]

The same can be done with flux variable:

\[
E(S) = \int_\sigma d^Dx E^a(x) F_{S_a}(x).
\]

The oriented intersection number can be deduced from the commutator of holonomy and flux, expressed via form factors. This yields

\[
I(e, S) = \frac{1}{i} [A(e), E_S] = \int_\sigma d^Dx F_e^a(x) F_{S_a}(x)
\]

Having established the above notion of distributional form factors, we want to construct a vector space. In principle, each surface or edge is described by a \( D \)-tuple of individual form factors, to which we will refer to by omitting the index, i.e. \( F_e \) or \( F_S \). Since we can add distributions and multiply them by scalars it is trivial to see that they possess a vector space structure. In this sense we can look at finite linear combinations of form factors like e.g.

\[
F := n_1 F_{e_1} + n_2 F_{e_2} + \cdots + n_k F_{e_k}, \quad n_k \in \mathbb{Z}
\]

for some edges. The index structure of the form factors now tells us that it is not very reasonable to add form factors for edges and surfaces. We can only add form factors of the same kind if we want to integrate them against objects that have an opposite index structure.

In principle, we would like to consider only linear combinations w.r.t. integer scalars as indicated above, since they could be interpreted as representation labels of U(1). However, when doing so, we run into a problem. The integer numbers \( \mathbb{Z} \) are not a field since they lack a multiplicative inverse. As a consequence it is not possible to have a vector space over the integers. When we want to work at the level of integers, the structure we have at hand is a \( \mathbb{Z} \)-module. In order to be able to work with such an object when considering Weyl algebras and quasifree states we would have to generalize Weyl algebras to symplectic \( \mathbb{Z} \)-modules, which we do not want to do here. Therefore, we generalize the linear combinations to coefficients in \( \mathbb{R} \). The price to pay is that we actually cannot work with U(1) anymore but rather have to deal with the Bohr compactification of the real line, \( \mathbb{R}_{\text{Bohr}} \). However, the integer linear combinations of form factors are part of the symplectic vector space corresponding to \( \mathbb{R}_{\text{Bohr}} \) and hence it is at this point convenient to work in this setup. Furthermore, a possible Weyl algebra constructed subsequently will be a \( \mathbb{R}_{\text{Bohr}} \) Weyl algebra and hence a U(1) HF Weyl algebra will be embedded into this larger algebra. Every representation of the \( \mathbb{R}_{\text{Bohr}} \) algebra gives rise to a representation of the U(1) algebra. We will consider a U(1) HF algebra that is embedded into a \( \mathbb{R}_{\text{Bohr}} \) HF algebra and consider only representations of the U(1) algebra that come from the \( \mathbb{R}_{\text{Bohr}} \) representation.

We define different vector spaces of distributional form factors as follows:

**Definition IV.2.** Let \( F_e \) and \( F_S \) be distributional form factors for edges and surfaces.

1. The vector space of distributional form factors of paths is denoted by

\[
H_e = \left\{ \sum_{k=1}^m \lambda_k F_{e_k} \, \middle| \, m < \infty, \lambda_k \in \mathbb{R} \right\}.
\]

2. The vector space of distributional form factors of surfaces is denoted by

\[
H_S = \left\{ \sum_{k=1}^m \lambda_k F_{S_k} \, \middle| \, m < \infty, \lambda_k \in \mathbb{R} \right\}.
\]
3. The vector space of distributional form factors of paths and surfaces is then

\[ H = H_e \oplus H_S. \] (42)

The elements in \( H \) are denoted by \((e, S)\) and stand for the pair of linear combinations of form factors \( F_e \in H_e \) and \( F_S \in H_S \), which correspond to collections of edges and surfaces, respectively.

By the addition of form factor tuples like

\[ (e_1, S_1) + (e_2, S_2) = (e_1 + e_2, S_1 + S_2) \] (43)

we actually mean addition of the form factors for the individual distributions for the edges and surfaces. The neutral element of the addition, needed for the vector space structure, is inherently given by the form factor that is constantly zero everywhere, because there is no edge or surface, denoted by \((0, 0)\). Scalar multiplication will be interpreted as multiplying the individual form factors by a scalar and is denoted as

\[ n(e, S) = (ne, nS). \] (44)

What we also need for a vector space is the inverse of addition. This can be simply set to

\[ (-e, -S) = -(e, S). \] (45)

The appearing minus sign has a slightly more convenient interpretation, which can be seen when looking the form factor for the inverse of the generic edge \( e \):

\[ F_{e^{-1}}(x) = -F_e(x) \] (46)

and hence the additive inverse for edges only is given by the form factor for the inverse edge.

The same holds true for surfaces. Multiplying the form factor by a minus sign can be interpreted as a change of orientation.

Given this vector space structure, we can look at the oriented intersection number that becomes a map

\[ I : H_e \oplus H_S \rightarrow \mathbb{Z}_2, \quad (e, S) \mapsto I(e, S), \] (47)

that is bilinear, as can be easily seen from the integral form of the intersection number:

\[ I(e + e', S + S') = I(e, S) + I(e, S') + I(e', S) + I(e', S'). \] (48)

For concatenated edges of the form \( e = e_2 \circ e_1 \), the form factors separate according to

\[ F_e(x) = F_{e_1}(x) + F_{e_2}(x). \] (49)

Consequently, the smeared connections break up in parts belonging to the segments, i.e.

\[ A(e) = A(e_1) + A(e_2). \] (50)

This reproduces the decomposition property for holonomies as

\[ h_e = e^{iA(e)} = e^{iA(e_2) + iA(e_1)} = e^{iA(e_2)}e^{iA(e_1)} = h_{e_2}h_{e_1}. \] (51)

For surfaces a similar behaviour arises. A sum of form factors, such that the surfaces consists of a set of faces \( \{S_k\} \), decomposes and for the electric fluxes we find

\[ E(S) = \sum_k E(S_k). \] (52)
B. Construction of the Weyl algebra

We want to construct a Weyl algebra for the U(1) toy-model. This is done by deriving the form of the Weyl elements and then analyzing the symplectic structure that emerges from this. Some of the results we present here were first obtained for the 3-dimensional case in [23].

As the holonomies are already the exponentiated versions of the connection, we only need to consider exponentiated fluxes, by setting

\[ V_S := e^{i \mathcal{E}(S)}. \]  

(53)

In principle, the exponentiated fluxes can also be seen as U(1) elements. The fluxes are real and complex conjugation yields the inverse exponentiated flux. Hence, they are unitary.

The exponentiated version of the commutation relation can be readily obtained by the Baker-Campbell-Hausdorff decomposition, since at the level of smeared connections and fluxes all higher order iterated commutators vanish:

\[ h_e V_S = e^{-i \mathcal{H}(e, S)} V_S h_e. \]

(54)

We propose that the identification

\[ W(e, S) = e^{\frac{i}{2} \mathcal{H}(e, S)} h_e V_S \]

(55)

is a suitable candidate for a U(1) Weyl element, i.e. obeys a set of Weyl relations according to the just mentioned definition. We find that

\[ W(e, S) = W(-e, -S). \]

(56)

Here, we used that both holonomies and exponentiated fluxes are unitary and the relation (54).

For the Weyl relations, we have to consider multiplication of two of the proposed Weyl elements. Using relation (54), we have for pairs of form factors \((e_1, S_1)\) and \((e_2, S_2)\) that indeed

\[ W(e_1, S_1) W(e_2, S_2) = e^{-i \frac{1}{2} (\mathcal{H}(e_1, S_2) - \mathcal{H}(e_2, S_1))} W(e_1 + e_2, S_1 + S_2). \]

(57)

Again, addition of edges and surface is supposed to be interpreted as addition of the corresponding form factors. First of all, this equation implies

\[ W(e_1, S_1) W(e_2, S_2) = e^{-i \frac{1}{2} \mathcal{H}(e_1, S_2) - \mathcal{H}(e_2, S_1)} W(e_2, S_2) W(e_1, S_1). \]

(58)

Furthermore, the neutral element of the vector space \(H\) is mapped to the Weyl element \(W(0, 0) = 1\).

Having established this exponentiated commutation relation, we need to extract a symplectic form, which is needed for the Weyl algebra. This is given by the exponent in the last line of (57), which leads to the form

\[ \sigma((e_1, S_1), (e_2, S_2)) = I(e_1, S_2) - I(e_2, S_1). \]

(59)

The linearity of the intersection number in edges and surfaces extends here to bilinearity of \(\sigma\). Since \(\sigma\) is furthermore anti-symmetric in its arguments and hence is a pre-symplectic form. Therefore, we can state a definition of the Weyl algebra corresponding to the canonical commutation relation (29).

Definition IV.3. The CCR algebra over the pre-symplectic space \((H, \sigma)\), with \(H\) the vector space of pairs of distributional form factors for edges and surfaces and \(\sigma((e_1, S_1), (e_2, S_2)) = I(e_1, S_2) - I(e_2, S_1)\), generated by the Weyl elements

\[ W(e, S) = e^{\frac{i}{2} \mathcal{H}(e, S)} h_e V_S \]

(60)

that obey the Weyl relations

\[ W(e_1, S_1) W(e_2, S_2) = e^{-i \frac{1}{2} \sigma((e_1, S_1), (e_2, S_2))} W(e_1 + e_2, S_1 + S_2), \]

\[ W(e, S) = W(-e, -S), \]

(61)
is called the U(1) HF Weyl algebra.

It should be pointed out that according to the discussion preceding definition \[\text{IV.2}\] we consider only Weyl elements that belong to the \(\mathbb{Z}\)-module contained in \(H\) as physically relevant.

In the following, we want to show that the pre-symplectic form \(\sigma\) is in fact a symplectic form, i.e. \(\sigma\) is non-degenerate.

**Proposition IV.4.** The pre-symplectic form

\[
\sigma((e, S), (e', S')) = I(e, S') - I(e', S)
\]

is non-degenerate, i.e.

\[
\sigma((e, S), (e', S')) = 0 \quad \forall (e', S') \in H
\]

implies that \((e, S) = (0, 0)\).

**Proof.** We want to argue by contradiction and thus suppose that eq. \[\text{63}\] holds true and \((e, S) \neq (0, 0)\). Consequently there are three cases, which we have to look at.

For the beginning we look at the situation

\[(e, S) \equiv (e, 0)\].

If the pre-symplectic form is non-degenerate, it must hold that

\[I(e, S') = I(e', S)\]

for all pairs \((e', S')\). In the just stated situation with only the edge part being non-trivial, the right hand side of the above equation vanishes identically and the requirement on the intersection numbers becomes

\[I(e, S') = 0 \quad \forall S'.\]

But for a non-trivial path or a linear combination of paths there is surely an at least infinitesimal surface that intersects \(e\). Hence, this results in a contradiction.

Secondly, we look at the opposite situation where there is only a non-trivial surface. i.e.

\[(e, S) \equiv (0, S)\].

Analogously to the previous case now the left hand side of \[\text{65}\] vanishes, which yields

\[0 = I(e', S) \quad \forall e'.\]

Again, for a non-trivial surface or a linear combination of such, there is surely an at least infinitesimal edge \(e'\) that causes an intersection. Also here, we run into a contradiction.

The third situation is that both \(e\) and \(S\) are non-trivial. However, we have to distinguish again two cases. Let us suppose that \(I(e, S) \neq 0\). The choice of the pair of edge and surface that leads into a contradiction is

\[(e', S') = (-e, S),\]

which is just the original element, but with e.g. inverse orientation for the edge. Equation \[\text{65}\] hence turns into

\[I(e, S) = I(-e, S) = -I(e, S)\]

and, because we required the intersection number to be non-vanishing, this is a contradiction.

Finally, the situation in which the intersection number of \(e\) and \(S\) is vanishing has to be considered: \(I(e, S) \equiv 0\). Regarding a contradiction, we take a look at the situation

\[(e', S') = (e', S),\]

where we keep \(e'\) generic but fix the surface part to \(S' = S\). This results in

\[0 = I(e, S) = I(e', S) \quad \forall e'.\]

As we already argued, there is certainly an edge that intersect the non-trivial surface \(S\) and we are again confronted with a contradiction.

After all, every assumption that \((e, S)\) is non-trivial resulted in a contradiction, which implies that

\[(e, S) = 0.\]

This completes the proof and \(\sigma((e, S), (e', S')) = I(e, S') - I(e', S)\) is in fact a symplectic form.

This result implies that the Weyl algebra constructed above indeed exists and is unique.
V. ALMOST QUASIFREE STATES

What distinguishes the CCR of U(1) LGQ from other field theories is its purely topological nature. Holonomies and fluxes do not commute if there is an intersection point between the corresponding edge and surface. A covariance satisfying the inequality \( \sum_j \frac{1}{2} \sigma(S_j, S_k) e^{i \beta(S)} \) would have in any case to surpass the number of intersection points.

For this reason, it seems very unlikely that it is possible to find a quasifree state for the U(1) toy-model and LQG in general. In the framework of projective loop quantum gravity \(^{28, 39}\), there is a result about exactly the type of inequalities needed for quasifree states. It states that there cannot be a non-singular covariance that satisfies an inequality similar to \( (23) \) and hence there cannot exist such states. This seems to be a feature that comes from the structure of the underlying algebra itself.

Nevertheless, it is possible to construct a new type of state for the U(1) HF Weyl algebra, which can be interpreted as a hybrid of the Ashtekar-Lewandowki state for the HF algebra and a quasifree state for Weyl algebras.

A. Almost quasifree states for the U(1) holonomy-flux Weyl algebra

In addition to the Gaussian fluctuations of quasifree states, we want to include a controllable peak position into the new state. This leads, alongside the fluctuations given by two-point correlation functions, to non-vanishing one-point correlation functions, i.e. a condensate contribution.

**Proposition V.1.** For a symmetric, positive semi-definite bilinear form (covariance) \( \alpha : H_S \times H_S \rightarrow \mathbb{R} \) and a linear function (condensate contribution) \( \beta : H_S \rightarrow \mathbb{R} \) with \( \beta(S)(0) = 0 \), the linear functional

\[
\varphi(W(e, S)) = \delta_{e,0} e^{-\frac{1}{2} \alpha(S, S)} e^{i \beta(S)}
\]  

(74)

is a state on CCR\(^{\text{Weyl}}(H, \sigma)\), which is called almost quasifree.

**Proof.** In order to show that \( \varphi \) is a state, we show that it is normalized and positive. We have
\[
\varphi(W(0, 0)) = \delta_{0,0} e^{-\frac{1}{2} \alpha(0, 0)} e^{i \beta(0)(0)} = 1.
\]  

(75)

For algebra elements \( x = \sum_{k=1}^{n} b_k W(e_k, S_k) \), with \( n \in \mathbb{N} \) and \( b_k \in \mathbb{C} \), we have that
\[
\varphi(xx^*) = \sum_{j,k=1}^{n} b_j \overline{b}_k \varphi(W(e_j, S_j)W(e_k, S_k))
\]
\[
= \sum_{j,k=1}^{n} b_j \overline{b}_k \varphi(W(e_j - e_k, S_j - S_k)) e^{-\frac{1}{2} \sigma((e_j, S_j) - (e_k, S_k))}
\]
\[
= \sum_{j,k=1}^{n} b_j \overline{b}_k e^{-\frac{1}{2} \alpha(S_j - S_k, S_j - S_k)} e^{i \beta(S_j - S_k)} e^{-\frac{1}{2} \sigma((e_j, S_j) - (e_k, S_k))}
\]
\[
= \sum_{j,k=1}^{n} c_j c_k e^{-\frac{1}{2} \alpha(S_j - S_k, S_j - S_k)} e^{-\frac{1}{2} \sigma((e_j, S_j) - (e_k, S_k))}
\]  

(76)

and introduced in the last line the coefficients \( c_j = b_j e^{i \beta(S)} \). The resulting sum can be split into three different contributions, depending on the configuration of edges.

The sum can run over pairs of non-trivial edges, so all \( e_j \neq 0 \). With this
\[
\sum_{j,k=1}^{n} d_j d_k c_j c_k e^{-\frac{1}{2} \alpha(S_j - S_k, S_j - S_k)} e^{-\frac{1}{2} \sigma((e_j, S_j) - (e_k, S_k))} =
\]
\[
= \sum_{j,k=1}^{n} c_j c_k e^{-\frac{1}{2} \alpha(S_j - S_k, S_j - S_k)} e^{-\frac{1}{2} \sigma((e_j, S_j) - (e_k, S_k))} = \sum_j |c_j|^2 \geq 0,
\]  

(77)

where we used the Kronecker delta to collapse the sum over \( k \) and from the third to the fourth line that the symplectic form is anti-symmetric and hence its diagonal elements vanish.
The second contribution are terms where non-trivial and trivial edged meet in the Kronecker delta. These terms vanish immediately.

For the third contribution, we have to consider terms with only trivial edges. Hence, the Kronecker delta gives one and the sums do not collapse. Realizing that the symplectic form does also vanish, this yields

\[ \sum_{j,k=1}^{n} c_j \bar{c}_k e^{-\frac{1}{2} \left( \alpha(S_j,S_j) + \alpha(S_k,S_k) - 2 \alpha(S_j,S_k) \right)} = \sum_{j,k=1}^{n} d_j \bar{d}_k e^{\alpha(S_j,S_k)}. \]  

Here, we introduced \( d_j = c_j e^{-\frac{1}{2} \alpha(S_j,S_j)} \). Since the Hadamard product of two semi-definite \( n \times n \) matrices is also positive semi-definite, as well as the sum of positive semi-definite matrices, the exponential of the covariance is positive semi-definite. Hence,

\[ \sum_{j,k=1}^{n} d_j \bar{d}_k e^{\alpha(S_j,S_k)} \geq 0 \]  

and the positivity of the state, i.e. \( \varphi(xx^*) \geq 0 \), follows.

The almost quasifree state \( \varphi \) behaves like the AL state for holonomies, i.e. is only non-vanishing if there are none, but shows a different behaviour concerning the fluxes. The flux part is a Gaussian function. Being differentiable only in the surface variable, there are only field operators corresponding to fluxes. Similar to the AL representation, there cannot be field operators for the connection.

Given a representation of this almost quasifree state, e.g. the GNS representation, the representation of fluxes is determined by Stone’s theorem:

\[ \pi(E(S)) = \left. \frac{1}{i} \frac{d}{dt} \pi(W(0,tS)) \right|_{t=0}. \]  

The holonomies can directly be represented by the Weyl elements:

\[ \pi(h_e) = \pi(W(e,0)). \]  

Similar to the situation for quasifree states, the commutator of holonomies and fluxes is given by the symplectic form. Hence, we find that

\[ [\pi(h_e), \pi(E(S))] = -i (e, S) \pi(h_e), \]  

such that this a is in fact a representation of the CCR of the toy-model.

While the vacuum expectation values containing holonomies still vanish, the one- and two-point function of fluxes do not anymore. Using the representation \([0]\) and the state \([2]\), the peak for fluxes lies at

\[ \langle \pi(E(S)) \rangle_\Omega = \left. \frac{1}{i} \frac{d}{dt} \varphi_S(W(0,tS)) \right|_{t=0} = \beta(S), \]  

i.e. the condensate contribution \( \beta \). The two-point functions, describing the fluctuations include the surface covariance:

\[ \langle \pi(E(S_1))\pi(E(S_2)) \rangle = \left. \frac{1}{i^2} \frac{d^2}{dt_1 dt_2} \varphi_S(W(0,t_1 S_1 + t_2 S_2)) \right|_{t_1,t_2=0} = \alpha(S_1, S_2) + \beta(S_1) \beta(S_2). \]  

Similarly, all higher \( n \)-point functions decompose into products of covariances and condensate contributions. For vanishing \( \beta \), the only the two-point functions contribute and reproduce the behaviour of quasifree states.

Because of the GNS construction, each choice of \( \alpha \) and \( \beta \) yields, up to isomorphisms, a different almost quasifree representation of the holonomy-flux Weyl algebra. Since there is no default object like the oriented intersection number \( H_S \) that allows for the definition of an inner product one has to use additional structures.

### B. A first representation of almost quasifree states

In this and the following section we want to introduce representations of CCR\(^{\text{Weyl}}(H,\sigma)\). Similar to the nature of the almost quasifree state, which resembles in some aspects the AL state, the representations will be constructed as an augmented version of the AL representation.
The Hilbert space for this first representation is

$$\mathcal{H}_\mathcal{E} = \mathcal{H}_{\text{AL}} \otimes L^2(\mathbb{C}),$$

(85)
i.e. the AL Hilbert space times the Hilbert space of square summable, complex sequences, which is the Hilbert space of the harmonic oscillator. The orthonormal basis of $L^2(\mathbb{C})$ is denoted by $\{\Omega_n\}$, such that $\langle \Omega_i, \Omega_j \rangle = \delta_{ij}$. The representation is characterized by the flux through a classical background electric field $E^{(0)}$

$$\mathcal{E}_S = \int d^Dx E^{(0)\alpha}(x) F_{S\alpha}(x),$$

(86)

which gives the peak position of the Gaussian part of the almost quasifree state. The fluctuations and hence the covariance of the representation are determined by functions $f^\alpha(x)$ integrated against a surface form factor, giving rise to the flux like object

$$f_S = \int d^Dx f^\alpha(x) F_{S\alpha}(x).$$

(87)

We will refer to this representation as the classical flux representation.

Instead of giving a representation of the Weyl elements right from the beginning, we present the representations of holonomies and fluxes, and show that this gives rise to the desired representation. As the almost quasifree state behaves like the AL state for holonomies, we define the representation of holonomies accordingly:

$$\pi_\mathcal{E}(h_c) = h_c \otimes 1_{L^2(\mathbb{C})}. $$

(88)

Holonomies act only on the AL part of the Hilbert space. The flux representation on this Hilbert space is given by

$$\pi_\mathcal{E}(E(S)) = X_S \otimes 1_{L^2(\mathbb{C})} + 1_{\text{AL}} \otimes (f_S(a + a^\dagger) + \mathcal{E}_S),$$

(89)

with the usual annihilation and creation operators of the harmonic oscillator, i.e. $[a, a^\dagger] = 1$.

At first, we want to check the commutation relations for holonomy and flux representation. For the fluxes we have

$$[\pi_\mathcal{E}(E(S)), \pi_\mathcal{E}(E(S'))] = [X_S, X_{S'}] \otimes 1_{L^2(\mathbb{C})} + 1_{\text{AL}} \otimes f_S f_{S'} [(a + a^\dagger), (a + a^\dagger)] = 0,$$

(90)

since the flux operators for U(1) commute. Also for the holonomies, the commutator vanishes:

$$[\pi_\mathcal{E}(h_c), \pi_\mathcal{E}(h_{c'})] = [h_c, h_{c'}] \otimes 1_{L^2(\mathbb{C})} = 0.$$

(91)

The remaining commutator is the one between holonomies and fluxes:

$$[\pi_\mathcal{E}(h_c), \pi_\mathcal{E}(E(S))] = -X_S[h_c] \otimes 1_{L^2(\mathbb{C})} = \pi_\mathcal{E}(-X_S[h_c]) = \pi_\mathcal{E}([h_c, E(S)])$$

(92)

Here, we used that $X_S[h_c]$ is a cylindrical function again and hence can be represented in this diagonal form.

The next step is to exponentiate the flux operators. By means of the binomial theorem, the Cauchy product and the fact that the both contributions to $\pi_\mathcal{E}(E(S))$ commute, it is easy to see that

$$e^{i\pi_\mathcal{E}(E(S))} = e^{iX_S} \otimes e^{if_S(a + a^\dagger) + i\mathcal{E}_S},$$

(93)

We want to fix some notation for holonomies and exponentiated fluxes in this classical flux representation:

$$h_\mathcal{E} = h_c \otimes 1,$$

$$V_\mathcal{S} = e^{iX_S} \otimes e^{if_S(a + a^\dagger) + i\mathcal{E}_S}.$$  

(94)

We can now define Weyl operators corresponding to the Weyl elements (55), namely

$$W^\mathcal{E}(e, S) = e^{i\pi_\mathcal{E}(e(S))} h_\mathcal{E} V_\mathcal{S} = e^{ix} e^{i\mathcal{E}_S} h_e e^{iX_S} \otimes e^{if_S(a + a^\dagger) + i\mathcal{E}_S}.$$  

(95)
The Weyl operators are subject to the Weyl relations
\[ W^E(e_1, S_1)W^E(e_2, S_2) = e^{-\frac{i}{\hbar}\left(\mathcal{I}(e_1, S_2) - \mathcal{I}(e_2, S_1)\right)} W^E(e_1 + e_2, S_1 + S_2), \]
\[ W^E(e, S)\dagger = W^E(-e, -S). \]

(96)

Since the Weyl operators are unitary, they are necessarily bounded and live in the \( C^* \)-algebra of bounded operators on \( \mathcal{H}_E \).

Having established these Weyl operators and their commutation relations, we see that we have in fact a CCR representations, as in definition III.2 of the canonical commutation relation of holonomies and fluxes, when considering the Hilbert space \( \mathcal{H}_E \) and the same symplectic form \( \mathcal{I}(e_1, S_1, e_2, S_2) \). We have a representation of the \( U(1) \) HF Weyl algebra from definition IV.3 on the classical flux Hilbert space \( \mathcal{H}_E \). By setting the edge to zero in the Weyl operator, we recover of course the exponentiated flux operator and by taking the derivative according to Stones theorem we have again access to the flux operator.

The vacuum of \( \mathcal{H}_E \) is the product of the vacua of the individual Hilbert spaces, which is
\[ \Omega_E = \Omega_{AL} \otimes \Omega_0, \]
(97)

where \( \Omega_0 \) is the cyclic vector of the harmonic oscillator Hilbert space.

In order to determine the vacuum expectation value of the Weyl operators we have to determine their action on \( \Omega_E \). This is furthermore very instructive to see, since it tells us how the states generated by \( W^E(e, S) \) actually look like. Because of the tensor product structure we can look at the individual factors separately. We begin with the AL-part. Every cylindrical function can be expressed in terms of holonomies, so it is sufficient to look only at a single holonomy that acts on the AL-vacuum in order to understand the structure of Weyl operator generated states. Therefore, we consider
\[ e^{\frac{i}{\hbar}\mathcal{I}(e,S)}h_e e^{iX_S} \Omega_{AL} = e^{\frac{i}{\hbar}\mathcal{I}(e,S)} h_e. \]

This is the case because the AL-vacuum state is basically the constant function \( \Omega_{AL} = 1 \) and hence is killed by acting on it with the derivative operator \( X_S \). There is an additional phase factor, which is a remnant of the flux and knows about the intersection structure of \( e \) and \( S \). Therefore the AL-part of the Weyl operator creates cylindrical functions from the vacuum that have additional information about intersections with surfaces.

For the harmonic oscillator part we make use of the notion of coherent states. We realize that
\[ e^{if_S(a^\dagger + a)} = e^{if_S a^\dagger - \bar{f}_S a}, \]

(99)

which is the coherent state operator for the harmonic oscillator for a purely imaginary coherent state parameter \( if_S \). Hence the action of this on the harmonic oscillator vacuum generates a coherent state w.r.t. the function \( f_S \), describing the fluctuations, i.e.
\[ e^{if_S(a^\dagger + a)}\Omega_0 = e^{-\frac{i}{\hbar}f_S^2} \sum_{n=0}^{\infty} \frac{(if_S)^n}{\sqrt{n!}} (a^\dagger)^n \Omega_n =: \Omega_{if_S}. \]

(100)

Putting both tensor factors together and including the condensate contribution yields the action of the Weyl operators on the vacuum:
\[ W^E(e, S)\Omega_E = e^{\frac{i}{\hbar}\mathcal{I}(e,S)} e^{ixS} (h_e \otimes \Omega_{if_S}). \]

(101)

This is in fact a product of a holonomy encoding the edge \( e \) and coherent state of the harmonic oscillator that knows about the classical flux through the surface \( S \) and the corresponding fluctuations. Additionally the phase factor at the beginning is aware of the intersection structure of \( e \) and \( S \).

Finally we can consider the vacuum expectation value. The inner product on the Hilbert space \( \mathcal{H}_E = \mathcal{H}_{AL} \otimes L^2(\mathbb{C}) \) is given by the product of the ones of the individual Hilbert spaces, since there is no entanglement:
\[ \langle \cdot, \cdot \rangle_{\mathcal{H}_E} = \langle \cdot, \cdot \rangle_{\mathcal{H}_{AL}} \langle \cdot, \cdot \rangle_{L^2(\mathbb{C})}. \]

(102)

Again, we look at the contributions separately. For holonomies and exponentiated fluxes we have
\[ \langle \Omega_{AL}, e^{\frac{i}{\hbar}\mathcal{I}(e,S)} h_e e^{ixS} \Omega_{AL} \rangle_{\mathcal{H}_{AL}} = \delta_{e,0}. \]

(103)
Although there is a remnant of the surface in the vacuum expectation value, it does not contribute to it, since the expectation value is only non-trivial if the edge is trivial. The vacuum expectation value of harmonic oscillator coherent state operator is

\[ \langle \Omega_0, e^{i f_S(a + a^\dagger)} \Omega_0 \rangle_{\mathcal{H}} = e^{-\frac{1}{2} f_S^2}. \]  

(104)

Putting things together yields the desired result. We find

\[ \langle \Omega_\xi, W^\xi (e, S) \Omega_\xi \rangle_{\mathcal{H}_\xi} = \delta_{e,0} e^{\frac{1}{2} f_S^2} e^{i \xi S}. \]

(105)

The vacuum expectation value of the Weyl operators hence is highly peaked at trivial holonomies and is a Gaussian which is bilinear and symmetric. Further, we can identify the classical flux

\[ F \]

specific functions of representations of holonomies but without losing contact between the individual representations, by choosing

\[ \alpha(S, S') = f_S f_{S'}, \]

(106)

which is bilinear and symmetric. Further, we can identify the classical flux \( \xi_S \) with a condensate contribution \( \beta(S) \).

The almost quasifree state, corresponding to this covariance and condensate, is given by

\[ \varphi(W(e, S)) = \delta_{e,0} e^{\frac{1}{2} \alpha(S, S')} e^{i \beta(S)} = \delta_{e,0} e^{\frac{1}{2} f_S^2} e^{i \xi S}, \]

(107)

which matches the vacuum expectation value of the Weyl operators in the classical flux representation. Therefore, by means of the GNS-construction, the classical flux representation \( (\pi_\xi, \mathcal{H}_\xi, \Omega_\xi) \) is unitarily equivalent to the GNS representation of the almost quasifree state \( |107\rangle \). The classical flux representation can be interpreted as a representation of the almost quasifree state \( |107\rangle \).

There is in fact a certain similarity to the Koslowski-Sahlmann (KS) representation and actually the considerations in [8, 9]. There, the extension of the AL representation by a classical flux is considered. However, everything takes place on \( \mathcal{H}_{AL} \) only. This allows for a shift of the flux peak, not for fluctuations in the fluxes.

The representation we present here is closely related to the results about the representation theory of the HF algebra developed in [40]. There, the main result is that it is possible to extend the representation theory of the HF algebra. The representation \( (\mathcal{H}, \pi) \) is split up into a direct sum of representations, which means that

\[ \mathcal{H} \cong \bigoplus_\nu \mathcal{H}_\nu, \]

\[ \pi \cong \bigotimes_\nu \pi_\nu, \]

(108)

where the individual Hilbert spaces are \( \mathcal{H}_\nu \cong L_2(\mathbb{R}, d\mu_\nu) \). There is also an inclusion map \( I_\nu : \mathcal{H}_\nu \hookrightarrow \mathcal{H} \) and a projection map \( P_\nu : \mathcal{H} \rightarrow \mathcal{H}_\nu \). Under some rather mild assumptions, for whose details we refer to [40], it is possible to extend the representation such that – adopted for our U(1) considerations – one finds

\[ \pi(E(S)) I_\nu(F) = I_\nu(\pi_{AL}(E(S)) F) + \sum_i I_i(F_{i\nu}(S)). \]

(109)

Here \( F \) is a cylindrical function whose representation is \( \pi_{AL}(F) = F \) and \( F_{i\nu}(S) \) are functions that have to have similar properties as the fluxes. Also for U(1) the AL-representation of fluxes \( \pi_{AL}(E(S)) \) are implemented as derivatives analogously to the SU(2) formulation when replacing the left and right invariant derivatives by their U(1) counterparts. The outstanding feature of this result is that it is in fact possible to split the representation into a direct sum of representations of holonomies but without loosing contact between the individual representations, by choosing specific functions \( F_{i\nu}(S) \).

C. Almost quasifree representation with respect to a Fock space

In this section, we will present that it is possible to generalize the preceding discussion to a tensor product of \( \mathcal{H}_{AL} \) and a bosonic Fock space

\[ \mathcal{F} = \bigotimes_{k=0}^\infty \mathcal{H}^\otimes k. \]

(110)
Here, $\hbar$ denotes the one-particle Hilbert space. For $f \in \mathcal{h}$, the creation and annihilation operators of this field theory are subject to the commutation relations

$$
[a(f), a^\dagger(g)] = \langle f, g \rangle_{\mathcal{h}} \mathbb{1}_\mathcal{F},
$$
$$
[a(f), a(g)] = 0 = [a^\dagger(f), a^\dagger(g)].
$$

With this, the selfadjoint field operators of the theory are denoted by

$$
\phi(f) = a^\dagger(f) + a(f).
$$

The commutation relations for creation and annihilation operators extend to $\phi(f)$, i.e.

$$
[\phi(f), \phi(g)] = 2i \text{Im} \langle f, g \rangle_{\mathcal{h}}.
$$

For general one-particle Hilbert space elements $f$ and $g$, this is not forced to vanish. However, for well defined field operators that are smeared with real test functions the commutator vanishes identically. The two-point functions w.r.t. to the Fock vacuum $\Omega_0$ are

$$
\langle \phi(f) \phi(g) \rangle_{\Omega_0} = \langle f, g \rangle_{\mathcal{h}}.
$$

The total Hilbert space we want to consider for almost quasifree states is now the tensor product space

$$
\mathcal{H}_F = \mathcal{H}_{\text{AL}} \otimes \mathcal{F}
$$

with the cyclic vacuum state

$$
\Omega_F = \Omega_{\text{AL}} \otimes \Omega_0.
$$

The main difference to the harmonic oscillator approach of last section is how the information about surfaces and the Fock space structure are interwoven. In the last section we just multiplied the harmonic oscillator annihilation and creation operators by the constant classical flux, which was basically just a number. Here, we want to go a slightly different way. Let us suppose we have real and linear map

$$
\Gamma : H_S \rightarrow \mathcal{h},
$$

which allows to relate a surface and hence its form factor to an element of the one-particle Hilbert space. The linearity is necessary in order to have $\Gamma(S + S') = \Gamma(S) + \Gamma(S')$, such that the field operators are linear in the surfaces and behave similar to the AL flux operators. We want to consider the following representation of holonomies and fluxes on the tensor product Hilbert space:

$$
\pi_F(h_e) = h_e \otimes \mathbb{1}_\mathcal{F},
$$
$$
\pi_F(E(S)) = X_S \otimes \mathbb{1}_\mathcal{F} + \mathbb{1}_{\text{AL}} \otimes (\phi(\Gamma(S)) + \mathcal{E}_S).
$$

This is supposed to be a Fock space generalization of eq. (112). In order to check if this is a representation of the HF algebra we consider the commutation relations of the object defined above. For the fluxes we find

$$
[\pi_F(E(S)), \pi_F(E(S'))] = [X_S, X_{S'}] \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\text{AL}} \otimes [\phi(\Gamma(S)), \phi(\Gamma(S'))] = 0,
$$

since the flux operators still commute and for real $\Gamma$ the commutator of field operators vanishes, too. Also the commutator of holonomies still vanishes:

$$
[\pi_F(h_e), \pi_F(h_{e'})] = [h_e, h_{e'}] \otimes \mathbb{1}_{\mathcal{F}} = 0.
$$

The commutator of holonomy and flux representations is also recovered as

$$
[\pi_F(h_e), \pi_F(E(S))] = [h_e, X_S] \otimes \mathbb{1}_{\mathcal{F}} = \pi_F([h_e, E(S)]).
$$

We indeed found a representation of the HF algebra on $\mathcal{H}_F = \mathcal{H}_{\text{AL}} \otimes \mathcal{F}$. Analogously to the previous discussion, we find that

$$
V_S^F = e^{i \pi_F(E(S))} = e^{i X_S} \otimes e^{i \phi(\Gamma(S)) + i \mathcal{E}_S}
$$

(122)
and set
\[ h_c^F = h_c \otimes 1_F. \] (123)

With this, we can determine commutation relations for \( h_c^F \) and \( V_S^F \), analogously to the first representation, and hence define Weyl elements
\[ W^F(e, S) = e^{\frac{i}{2}I(e, S)} h_c e^{iX_S} \otimes e^{i\phi(\Gamma(S)) + i\mathcal{E}_S}, \] which satisfy the Weyl relations
\[ W^F(e_1, S_1)W^F(e_2, S_2) = e^{-\frac{i}{2}(I(e_1, S_2) - I(e_2, S_1))} W^F(e_1 + e_2, S_1 + S_2), \]
\[ W^F(e, S)^\dagger = W^F(-e, -S). \] (125)

When considering the corresponding vacuum expectation value, the inner product splits into two contributions and hence the vacuum expectation value splits into
\[ \langle W^F(e, S) \rangle_{\Omega_F} = \langle e^{i\mathcal{H}(e, S)} h_c e^{iX_S} \rangle_{\Omega_A^L} \langle e^{i\phi(\Gamma(S)) + i\mathcal{E}_S} \rangle_{\Omega_0}. \] (126)

The vacuum expectation value on \( \mathcal{H}_A^L \) is given by
\[ \langle e^{i\mathcal{H}(e, S)} h_c e^{iX_S} \rangle_{\Omega_A^L} = e^{i\mathcal{H}(e, S)} \delta_{e,0} = \delta_{e,0}. \] (127)

Using the Baker-Campbell-Hausdorff decomposition of the annihilation and creation part of \( \phi(\Gamma(S)) \), we find
\[ \langle e^{i\phi(\Gamma(S)) + i\mathcal{E}_S} \rangle_{\Omega_0} = e^{-\frac{i}{2}(\Gamma(S), \Gamma(S))_h} e^{i\mathcal{E}_S}. \] (128)

Consequently, when combining the contributions, we again end up with an expression that is singular in the fluxes but Gaussian in the surfaces:
\[ \langle W^F(e, S) \rangle_{\Omega_F} = \delta_{e,0} e^{-\frac{i}{2}(\Gamma(S), \Gamma(S))_h} e^{i\mathcal{E}_S}. \] (129)

This motivates the interpretation of the just presented representation of the HF algebra as a almost quasifree representation of the HF Weyl algebra determined by the covariance
\[ \alpha(S, S') = \langle \Gamma(S), \Gamma(S') \rangle_h \] (130)
an the condensate contribution \( \mathcal{E}_S \).

VI. THE SITUATION FOR SU(2)

For full loop quantum gravity, we have to consider the structure group SU(2). The main difference to the toy-model is the non-commutativity of fluxes and that holonomies are elements of SU(2). For holonomies it is only possible to combine two of them in a single exponential if they represent parts of concatenated edges. Exponentiated fluxes can in general not be combined in a single exponential describing a sum of fluxes, since the commutator of fluxes does not yield a flux again.

Another problem is the localization of the commutation relation. In the U(1) toy-model all information about the intersection behaviour of edge and surface, coming from the commutation relation, is encoded in the exponential of the oriented intersection number and can be separated from the actual holonomy and flux. This is not possible for SU(2). Here, the commutator inserts into the holonomy \( \mathfrak{su}(2) \)-valued functions at the intersection point with the surface. The exponentiated intersection number of the toy-model, becomes a complicated combination of segments of the holonomy and SU(2) elements at the intersection points.

As a result, it is impossible to extract the symplectic form from this kind of expressions. It is equally not managable to directly construct a general symplectic form from the commutation relation. Therefore, we cannot construct a Weyl algebra formulation of LQG.

These problems notwithstanding, we can find a representation of the HF algebra of LQG, which resembles a representation of an almost quasifree state. The basic idea is to modify the flux representation in a fashion similar to the toy-model.
A. The holonomy-flux algebra

To properly talk about different representations, we have to precisely define the algebra that we will be working with in the SU(2) case. Here the correct algebra cannot be a Weyl algebra due to the non-Abelian nature of SU(2). A precise definition of the HF algebra as a quotient of a free algebra can be found in [21], and a similar in [20]. Both definitions take into account most of the relations that are present in the AL representation of holonomies and fluxes. In particular, the fluxes generate an infinite dimensional Lie algebra with additional relations among its elements. For example, it was pointed out in [21] that the remarkable identity

\[ [E_S(f), [E_S(f'), E_S(f'')]] = \frac{1}{4} E_S([f, [f', f'']]) \]  

(131)

holds on certain states in the AL representation, and by extension in the Lie-algebra generated by the fluxes.

On the other hand, it was noted in [20] that only few of the relations are really necessary to show uniqueness of diffeomorphism-invariant representations. More generally, only a small subset of relations of a given classical (Poisson) algebra can be respected by the commutation relations in a quantum theory of the given system, as was first pointed out by Groenewold and Van Hove. Thus one can argue (see for example [20]) that only the relations that are absolutely characteristic for the LQG approach should be taken over to the quantum algebra. This would mean requiring the basic commutator

\[ [E_S(f), F] = X_{S,f}[F] \]  

(132)

as a relation in the quantum algebra, but not a more complicated relation like (131). Here, we will follow [20], in taking less relations into account. Besides linear and adjointness relations, we require only the multiplicative structure among cylindrical functions and (132). These relations imply further relations via the Jacobi identities. We will comment on this below.

We will mathematically define the algebra by starting with a free algebra over a certain set of symbols and then dividing by the two-sided ideal generated by the relations listed above. This has the advantage that we automatically impose all the relations that are implied by those that we list explicitly.

Let us first define the label set for the quantum algebra \( \mathfrak{A}_{HF} \). On the one hand we use the cylindrical functions \( \text{Cyl} \) as labels. On the other hand, let \( \mathfrak{F} \) be the set of linear combinations of elementary flux variables, i.e. functionals of the form \( E_S(f) \) for a surface \( S \) and a smearing field \( f \). In particular, these variables satisfy

\[ E_S(f + f') = E_S(f) + E_S(f'), \quad E_{-S}(f) = E_S(-f), \]  

(133)

where \( -S \) is the surface \( S \) as a manifold, but with the opposite orientation. Now we are in a position to define the free algebra we will be starting from. Take algebra \( \mathfrak{A}_{free} \) of formal linear combinations of sequences of elements of \( \text{Cyl} \cup \mathfrak{F} \), with multiplication and \( * \) defined by

\[
\begin{align*}
(a_1, a_2, \ldots, a_m) \cdot (b_1, b_2, \ldots, b_n) &:= (a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n), \\
(a_1, a_2, \ldots, a_m)^* &:= (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_m),
\end{align*}
\]

(134, 135)

and linear extension. Here all entries \( a_1, b_1, \ldots \) are from \( \text{Cyl} \cup \mathfrak{F} \). The algebra \( \mathfrak{A}_{HF} \) that we will be working with for the SU(2) case is the algebra of equivalence classes under the relations defined by a certain ideal in \( \mathfrak{A}_{free} \). Let

\[ \mathfrak{A}_{HF} = \mathfrak{A}_{free}/I \]  

(136)

where \( I \) is the two-sided ideal generated by the following classes of elements:

\[
\begin{align*}
\alpha(a) - (\alpha a), & \\
(a) + (b) - (a + b), & \\
(F_1, F_2) - (F_1 F_2), & \\
(E_S(f), F) - (E_S(f), F) - (X_{S,f}[F]).
\end{align*}
\]

(137, 138, 139, 140)

Here \( \alpha \in \mathbb{C}, a, b \in \text{Cyl} \cup \mathfrak{F}, \) and \( F_1, F_2 \in \text{Cyl} \). This means that elements of \( \mathfrak{A}_{HF} \) are equivalence classes of elements in \( \mathfrak{A}_{free} \), multiplied by

\[ [x][y] = [xy], \quad x, y \in \mathfrak{A}_{free}. \]  

(141)

The ideal \( I \) is closed under the \( * \)-operation in \( \mathfrak{A}_{free} \), so the \( * \)-operation defined by

\[ [x]^* := [x^*], \quad x \in \mathfrak{A}_{free} \]  

(142)
is well defined on $\mathfrak{A}_{\text{HF}}$.

How can we define representations of $\mathfrak{A}_{\text{HF}}$? A practical way to do this is to first specify a representation $\tilde{\pi}$ of $\mathfrak{A}_{\text{free}}$, by specifying $\tilde{\pi}$ on a basis of $\text{Cyl}$ and one of $\mathfrak{F}$. Then we have the following result.

**Lemma VI.1.** A representation $\tilde{\pi}$ of $\mathfrak{A}_{\text{free}}$ defines a representation of $\mathfrak{A}_{\text{HF}}$ by

$$
\pi([x]) := \tilde{\pi}(x)
$$

if the generators $\{137\} - \{140\}$ of the ideal $I$ are in the kernel of $\pi$.

**Proof.** A basis of the ideal $I$ is given by elements of the form

$$x = (a_1, \ldots, a_m) \cdot \iota \cdot (b_1, \ldots, b_n), \quad a_1, \ldots, a_m, b_1, \ldots, b_n \in \text{Cyl} \cup \mathfrak{F},$$

where $\iota$ is one of the generators $\{137\} - \{140\}$ of $I$. Per definition

$$\tilde{\pi}(x) = \tilde{\pi}(a_1) \ldots \tilde{\pi}(a_m) \tilde{\pi}(\iota) \tilde{\pi}(b_1) \ldots \tilde{\pi}(b_n) = \tilde{\pi}(a_1) \ldots \tilde{\pi}(a_m) \cdot 0 \cdot \tilde{\pi}(b_1) \ldots \tilde{\pi}(b_n) = 0$$

since, by assumption $\iota$ is in the kernel of $\tilde{\pi}$. Thus $\tilde{\pi}(I) = 0$. The first question is now, whether $\{143\}$ well defines $\pi$ on $\mathfrak{A}_{\text{HF}}$. We immediately confirm

$$\pi(x + \iota) = \pi(x) + \pi(\iota) = \pi(x), \quad x \in \mathfrak{A}_{\text{free}}, \iota \in I,$$

since $I$ is in the kernel of $\pi$, whence $\pi$ is indeed well defined on $\mathfrak{A}_{\text{HF}}$. Now, checking the representation property

$$\pi([x][y]) = \pi([xy]) = \tilde{\pi}(xy) = \tilde{\pi}(x)\tilde{\pi}(y) = \pi([x])\pi([y]).$$

$\square$

In [21] some doubts regarding the consistency of the results of [9] were expressed. These doubts also have relevance for the contents of the current work, so we would like to explicitly address them. The doubts have to do with the relations implied by those generating the ideal $I$. Indeed, it is not easy to enumerate the relations contained in $I$, and there are many. In particular, there are all the relations following from Jacobi identities for commutators, since the Jacobi identities are implied by the definition

$$[[a], [b]] := [a][b] - [b][a], \quad a, b \in \mathfrak{A}_{\text{free}}.$$

One can for example check that, as was pointed out in [21],

$$[[E_S(f), [E_S(f'), E_S(f'')]], F] = \frac{1}{4}[[E_S([f, [f', f'']]), F]]$$

in $\mathfrak{A}_{\text{HF}}$, where here and in the following we drop the brackets $[\cdot]$ denoting equivalence classes for better readability. Note that in contrast the relation $\{131\}$ does not hold in $\mathfrak{A}_{\text{HF}}$ as defined above.

We also note that there cannot be any inconsistency in the relations. The worst that can happen is that (i) $I = \mathfrak{A}_{\text{free}},$ and hence $\mathfrak{A}_{\text{HF}}$ becomes trivial, or (ii) that there does not exist representations $\tilde{\pi}$ that have the generators $\{137\} - \{140\}$ of $I$ in the kernel. The existence of the AL-representation $\{15\}, \{16\}$ shows however, that (i) does not happen, as the operators in that representation fulfill the relations $\{137\} - \{140\}$ and are not trivial. Indeed, because of this the AL-representation is a representation of $\mathfrak{A}_{\text{HF}}$ in the literal sense, and hence also (ii) is not a problem.

The representations of $\{9\}$ fulfill the assumptions of lemma VI.1. Therefore they are representations of $\mathfrak{A}_{\text{HF}}$ as defined above. The same holds for the class of representations that we will propose below. One problem with $\{9\}$ is that the construction of the quotient algebra $\{136\}$ is invoked without giving much detail and hence inviting misunderstandings.

**B. An almost quasifree representation**

For a representation of the HF algebra, which mimicks the properties of an almost quasifree state, we consider the Hilbert space.

$$\mathcal{H}_F = \mathcal{H}_{\text{AL}} \otimes \mathcal{F},$$

(145)
where $F \equiv F(\hbar)$ and with the cyclic vector
\[ \Omega_F = \Omega_\text{AL} \otimes \Omega_0. \] (146)

The AL vacuum is still the constant identity function and $\Omega_0$ is the Fock vacuum.

The very idea is again to take the AL representation of the flux variable and augment it by a surface dependent operator on the Fock space, while keeping the representation of holonomies or here rather cylindrical functions as it is. Likewise to the toy-model, we can define a classical background flux
\[ \mathcal{E}_S(f) = \int_\mathcal{S} \frac{1}{2} f^j(x) E^{(a)}_j(x) \epsilon_{abc} dx^b \wedge dx^c \] (147)
which serves as the condensate contribution. We consider therefore the following representation, at first just of the free algebra $\mathfrak{A}_\text{free}$.
\[ \tilde{\pi}_F((E_S(f))) = X_{S,f} \otimes \mathbb{1}_F + \mathbb{1}_\text{AL} \otimes (\phi(\Gamma(S,f)) + \mathcal{E}_S(f)), \] (148)
\[ \tilde{\pi}_F((F)) = F \otimes \mathbb{1}_F. \]

This can be extended linearly and over products to a representation of $\mathfrak{A}_\text{free}$. Here, we require the map $(S,f) \mapsto \Gamma(S,f)$ again to be real and now linear in both, $S$ and $f$.

Now we have to check whether $\tilde{\pi}_F(\cdot)$ defines a representation of $\mathfrak{A}_\text{HF}$ as well. To do that, we use lemma [41]. We just have to check that the generators of the ideal $I$ are in the kernel of $\tilde{\pi}_F(\cdot)$. The only non-obvious one is the one involving the commutator of $E_S(f)$ and a cylindrical function. We calculate
\[ \tilde{\pi}_F((E_S(f), F) - (F, E_S(f)) - (X_{S,f}[F])) = \tilde{\pi}_F((E_S(f))) \tilde{\pi}_F((F)) - \tilde{\pi}_F((F)) \tilde{\pi}_F((E_S(f))) - \tilde{\pi}_F((X_{S,f}[F])) \]
\[ = [\tilde{\pi}_F((E_S(f))), \tilde{\pi}_F((F))] - \tilde{\pi}_F((X_{S,f}[F])) \]
\[ = X_{S,f}[F] \otimes \mathbb{1}_F - X_{S,f}[F] \otimes \mathbb{1}_F \]
\[ = 0. \]

The other generators are also in the kernel by similar calculations. Therefore $\tilde{\pi}_F(\cdot)$ yields a representation of $\mathfrak{A}_\text{HF}$.

One of the characteristic features of the almost quasifree representations was factorization of flux operator $n$-point functions. This can be also directly recovered here. The one-point correlation function of flux operators is given by
\[ \langle \pi_F(E_S(f)) \rangle_{\Omega_F} = \mathcal{E}_S(f), \] (149)
since the field one-point functions vanish. Considering a product of two flux representations, i.e.
\[ \pi_F(E_S(f)) \pi_F(E_{S'}(f')) = \]
\[ = (X_{S,f} \otimes \mathbb{1}_F + \mathbb{1}_\text{AL} \otimes (\phi(\Gamma(S,f)) + \mathcal{E}_S(f)))(X_{S',f'} \otimes \mathbb{1}_F + \mathbb{1}_\text{AL} \otimes (\phi(\Gamma(S',f')) + \mathcal{E}_{S'}(f'))) \]
\[ = X_{S,f}X_{S',f'} \otimes \mathbb{1}_F + X_{S,f} \otimes (\phi(\Gamma(S',f')) + \mathcal{E}_{S'}(f')) + X_{S',f'} \otimes (\phi(\Gamma(S,f)) + \mathcal{E}_S(f)) \]
\[ + \mathbb{1}_\text{AL} \otimes (\phi(\Gamma(S,f)) + \mathcal{E}_S(f))(\phi(\Gamma(S',f')) + \mathcal{E}_{S'}(f')), \] (150)
we realize that, upon taking the vacuum expectation value, only the last term survives, since there is no operator that kills the AL vacuum. This yields the two-point correlation function
\[ \langle \pi_F(E_S(f)) \pi_F(E_{S'}(f')) \rangle_{\Omega_F} = \langle \Gamma(S,f), \Gamma(S',f') \rangle_\hbar + \mathcal{E}_S(f)\mathcal{E}_{S'}(f'). \] (151)

For cylindrical functions only, there is no significant change to the pure AL representation, exactly as in the toy-model.

Finally, to demonstrate that there are possible maps $\Gamma$, we will be specific about the scalar field that we added to the formalism. We consider the one-particle Hilbert space
\[ \mathcal{H} = L^2(\mathbb{R}^3, d^3x). \] (152)
The augmentations we added to the flux representation, hence, can then interpreted as a smeared scalar field on $\mathbb{R}^3$.

The smeared scalar field operator is of the form
\[ \phi(g) = \int_{\mathbb{R}^3} d^3x g(x) \phi(x), \] (153)
with $\phi(x)$ being an operator valued distribution and $g(x)$ a test function that is used to cast $\phi(x)$ into a well defined operator on the Fock space. Considering the pair $(S, f)$, we have to keep in mind that the Lie algebra valued smearing function only has support on the surface. Therefore we can work with $f$ actually only when we integrate it over the intrinsic parametrization of $S$. Furthermore we need an embedding of the surface into $\mathbb{R}^3$ in order to be able to integrate it against the scalar field. A possible definition is thus

$$\phi(\Gamma(S, f)) = \int_{\mathbb{R}^3} d^3x \Gamma(S, f)\phi(x) = \int_{\mathbb{R}^3} d^3x \int_S d^2u K_{S\tau}(x, u)f^i(u)\phi(x).$$

In this, $(u_1, u_2)$ is an intrinsic parametrization of $S$ and $K_{S\tau}(x, u)$ is an integral kernel that relates points on the surface to points in $\mathbb{R}^3$. As an explicit example for $\Gamma(S, f)$ one might consider

$$\Gamma(S, f) = (e^\Delta \sum_i f^i F_S)(x),$$

where $\Delta = \partial_i \partial^i$ is the negative-definite Laplacian on $\mathbb{R}^3$.

At the end of the day we have a representation of the holonomy flux algebra – underlying loop quantum gravity – at disposal which is probably as close as possible to an actual almost quasifree representation of a theoretical Weyl algebra in the sense of definition III.3. The presumably most important characteristic of this representation is the fact that it is Gaussian for the fluxes in a non-extremal way.

C. Revisiting the area operator

In this section we consider the derivation of the area operator of loop quantum gravity for the representation introduced in the previous section.

In the context of the KS representation in [9], there is a similar augmentation to the flux representation of by a classical flux. Also in [9], there is an analysis of the area operator in terms of the extended representation. So it is possible to compare the area operator for the almost quasifree representation to both, the AL and the KS representation. It turns out that, in fact, the similarities to the representation in [9] are sufficient to adopt their procedure and especially some crucial details to our situation. For reasons of simplicity, we do not consider the full augmented flux representation in (148), but drop the classical flux part. Since this contribution to the AL representation is thoroughly dealt with in [9], we expect a similar behaviour here.

For the derivation we follow closely the steps in both, [41] and [9]. Regarding the details of the original derivation of the area operator we refer to [41]. The area operator is a quantization of the classical area functional

$$A_S = \int_S d^2x \sqrt{E_i(x)E^i(x)}.$$

In accordance with [41] we want to use a parametrization such that the surface $S$ is determined by $x^3 = 0$ and is parametrized by the other two components $x^1$ and $x^2$. The single component of the electric field we have to integrate over is $E_i \equiv E^i_3$. The main issue of the derivation of the area operator is now to carefully regularize this integral and replace the classical expressions by their quantum counterparts. For a regularization one considers a family of non-negative densities $f_\epsilon(x, y)$, which, upon taking away the regularization parameter $\epsilon$, become Dirac deltas, i.e.

$$\lim_{\epsilon \to 0} f_\epsilon(x, y) = \delta^{(2)}(x, y).$$

In the original derivation, this regularization allows for a point-splitting of the area operator into a sum of operators that only acts at intersection point the graph w.r.t. which we want determine the area of the surface. The same is considered here.

In every representation at hand we do not have direct access to operator valued distributions for the electric field, but only to operators for fluxes. These can nevertheless be evaluated at certain points of the surface using a regularization as introduced above. One considers an SU(2) smearing function of the form $f \equiv f_\epsilon(x, y)\tau_i$ and determines the flux operators with respect to this. They can be denoted by

$$X_{x\tau}(x) := X_{S, f_\epsilon(x, \cdot)\tau_i},$$

where the second argument of $f_\epsilon(x, y)$ is omitted because it is subject to some internal integration[4]. The surface is also omitted since the considerations refer to only a single surface. The AL area operator then turns out to be the

\[^1\text{For the detailed form of the operators we refer to [41] and the notation therein.}\]
\[
A_{\text{AL}}(S) = \lim_{\epsilon \to 0} \int_S d^2x \sqrt{X_{\epsilon i}(x)X_{\epsilon i}(x)} = \sum_{v \in V(\gamma)} A_{\text{AL}, \gamma}(S),
\]

splitting into an expression of local area operators at the individual vertices of \( \gamma \). The graph \( \gamma \) is assumed to be adapted to the surface in the sense that there is a vertex at every intersection point.

As shown in \([41]\), the regularized fluxes act on a cylindrical function \( \Psi_\gamma \) as

\[
X_{\epsilon i}(x)\Psi_\gamma = \frac{\ell_P^2}{2} \sum_{v \in V(\gamma)} f_\epsilon(x, v) \sum_{J_v} \kappa_{J_v} X_{J_v} x \Psi_\gamma.
\]

Here \( J_v \) is a label for the edges beginning or ending at vertex \( v \) of \( \gamma \). The operators \( X_{J_v} \) act for ingoing / outgoing edges as right- / left-invariant derivative w. r. t. \( e_{J_v} \) and \( \kappa_{J_v} \equiv \kappa(e_{J_v}, \gamma) \). A straightforward calculation shows that at each vertex

\[
[X^i_{J_v}, X^j_{J_v}] = 2i\alpha \delta_{J_v} e_{J_v} \epsilon^{ij} X^k_{J_v},
\]

where \( \alpha \) depends on the choice of basis of \( \mathfrak{su}(2) \). This implies that the left- / right-invariant vector fields \( X^i_{J_v} \) individually satisfy – at a fixed vertex and for each edge – the algebra relation of \( \mathfrak{su}(2) \) and hence can be treated like spin operators. Operators for different edges commute. The sum over the edges at a vertex splits, via the sign of \( \kappa_{J_v} \), into a total contribution of type up and total contribution of type down.

We now have to evaluate the scalar field operator at exactly the same smearing function. Equation (154) therefore turns into

\[
\phi_{\epsilon i}(x) := \phi(\Gamma(S, f_\epsilon(x, \cdot ) \tau_i)) = \int_{\mathbb{R}^3} d^3X \int_S d^2y K_{\epsilon i}(X, y) f_\epsilon(x, y) \phi(X).
\]

The full augmented representation of fluxes hence is

\[
E_{\epsilon i}(x) := X_{\epsilon i}(x) \otimes 1_F + 1_{\text{AL}} \otimes \phi_{\epsilon i}(x).
\]

These are the object we want to re-derive the area operator for.

Analogously to \([41]\) and \([2]\) we consider the area operator as the limit of the regularized object

\[
A_{S, \epsilon} = \int_S d^2x \sqrt{E_{\epsilon i}(x)E_{\epsilon i}(x)} =: \int_S d^2x \sqrt{g_{S, \epsilon}(x)}.
\]

The task is now to analyze the object under the square root, i.e. \( g_{S, \epsilon}(x) \). To this end we expand the product in order consider the individual contributions:

\[
g_{S, \epsilon}(x) = E_{\epsilon i}(x)E_{\epsilon i}(x) = X_{\epsilon i}(x)X_{\epsilon i}(x) \otimes 1_F + 1_{\text{AL}} \otimes \phi_{\epsilon i}(x) + 2 X_{\epsilon i}(x) \otimes \phi_{\epsilon i}(x).
\]

The first term \([165]\) is exactly the term leading to the AL area operator, while the second term \([166]\) is of the same form but only depends on the scalar field. Only in the third contribution \([167]\), there is an interaction between the AL and the scalar field parts of the flux representation.

Following the ideas of \([3]\), we realize that the individual contributions \([165]\), \([166]\) and \([167]\) mutually commute. This is obvious for considering the first and the second contribution. For the second and the third contribution it follows from the fact that \( \{ \phi(X), \phi(Y) \} = 0 \). Finally, for the first and the third contribution the vanishing commutator follows from \([161]\). As a consequence, there is a complete set of states that are eigenstates of all three terms. Having established this, we can now analyze the contributions individually.

**AL contribution:** The result of the analysis in \([41]\) for the piece \( X_{\epsilon i}(x)X_{\epsilon i}(x) \) when acting on an eigenstate \( \Psi_\gamma \) with underlying spin network \( \gamma \) is

\[
X_{\epsilon i}(x)X_{\epsilon i}(x)\Psi_\gamma = \sum_{v \in V(\gamma)} (f_\epsilon(x, v))^2 (a_v)^2 \Psi_\gamma,
\]

\[
a_v = \frac{\ell_P^2}{2} \sqrt{2j_v^u (j_v^u + 1) + 2j_v^d (j_v^d + 1) - j_v^{a+d} (j_v^{a+d} + 1)}.
\]
At a vertex \( v \) the actual form of the contribution \( a_v \) shows up from a recoupling of all edges of type up with total spin \( j^\text{up}_v \) and all edges of type down with total spin \( j^\text{down}_v \). The third contribution arises from coupling the up and down contributions.

**Scalar field contribution:** The treatment of this is a little tricky. In the first place we are dealing with operator valued distributions integrated against test functions and need to consider eigenstates of such objects. As the operator valued distribution \( \phi(X) \) is the quantum field theoretical analog of the position operator in quantum mechanics it is clear that there are no proper eigenstates of \( \phi(X) \). One consequently has to go over to a formulation in terms of generalized eigenstates. For quantum mechanics this can even be formulated mathematically precise when considering the framework of rigged Hilbert spaces. Here one works with a triple \( \mathcal{S} \subset \mathcal{H} \subset \mathcal{S}' \), which consists of the actual Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^3, d^3x) \), the space \( \mathcal{S} \) of test functions on \( \mathbb{R}^3 \) and its dual \( \mathcal{S}' \), the space of tempered distributions. The idea is now to transfer this procedure to the scalar field on a Fock space in order to be able to work with states that satisfy an eigenvalue equation of the form

\[
\phi(f)|F\rangle = F(f)|F\rangle, \tag{170}
\]

with the generalized eigenstate with respect to \( F \in \mathcal{S}' \) denoted by \(|F\rangle\) and \( F(f) \) interpreted in a distributional sense, i.e.

\[
F(f) = \int_{\mathbb{R}^3} d^3X F(X)f(X). \tag{171}
\]

A technical argument which we present in appendix A shows that it is reasonable to assume that there is a sufficiently large class of real valued functions \( F \) that give rise to a tempered distribution and allow to span the Hilbert space \( \mathcal{H} \). In a distributional sense eq. \((170)\) can be seen as

\[
\phi(X)|F\rangle = F(X)|F\rangle. \tag{172}
\]

With this at hand we can start to analyze the piece \( \phi_ei(x)\phi_e^i(x) \) by acting with a single field on a generalized eigenstate:

\[
\phi_ei(x)|F\rangle = \int_{\mathbb{R}^3} d^3X \int_S d^2y K_{Si}(X,y)f_e(x,y)\phi(X)|F\rangle
= \int_{\mathbb{R}^3} d^3X \int_S d^2y K_{Si}(X,y)f_e(x,y)F(X)|F\rangle
=: F_{ei}(x)|F\rangle. \tag{173}
\]

The whole contribution then is of course

\[
\phi_ei(x)\phi_e^i(x)|F\rangle = F_{ei}(x)F_{e^i}(x)|F\rangle. \tag{174}
\]

**Mixed contribution:** For the final piece, we have to consider an eigenstate \( \Psi_\gamma \otimes |F\rangle \). Acting on this yields

\[
(2X_e^i(x) \otimes \phi_ei(x)) \Psi_\gamma \otimes |F\rangle = \left( \ell_P^2 \sum_{v \in V(\gamma)} f_e(x,v) \left( X_v^u - X_v^d \right) \otimes F_{ei}(x) \right) \Psi_\gamma \otimes |F\rangle, \tag{175}
\]

where we already split up the contributions for edges of type up and down. For each vertex we now chose a coordinate system such that the \( z \)-component of the spin operators is aligned with \( F_{ei}(x) \). This allows to identify

\[
\left( \left( X_v^u - X_v^d \right) \otimes F_{ei}(x) \right) \Psi_\gamma \otimes |F\rangle = \left( m_v \sqrt{F_{ei}(x)F_{e^i}(x)} \right) \Psi_\gamma \otimes |F\rangle \tag{176}
\]

and \( m_v = m_v^u - m_v^d \) denotes the difference of total magnetic quantum numbers of up and down contributions. In total we have

\[
(2X_e^i(x) \otimes \phi_ei(x)) \Psi_\gamma \otimes |F\rangle = \left( \ell_P^2 \sum_{v \in V(\gamma)} f_e(x,v)m_v \sqrt{F_{ei}(x)F_{e^i}(x)} \right) \Psi_\gamma \otimes |F\rangle. \tag{177}
\]

Putting everything together, we find the following as the action of \( g_{S,e} \) on the common eigenvector

\[
g_{S,e}(x)\Psi_\gamma \otimes |F\rangle = \\
= \left( \sum_{v \in V(\gamma)} \left( f_e(x,v) \right)^2 (a_v)^2 + F_{ei}(x)F_{e^i}(x) + \ell_P^2 \sqrt{F_{ei}(x)F_{e^i}(x)} \sum_{v \in V(\gamma)} f_e(x,v)m_v \right) \Psi_\gamma \otimes |F\rangle. \tag{178}
\]
Following [9], we want to perform a completion of the square w. r. t. the first two terms in the bracket. To this is end we have to take a look at the first term. At first we want to calculate the following:

\[
\left( \sum_{v \in V(\gamma)} f_\epsilon(x, v)a_v \right)^2 = \sum_{v, v' \in V(\gamma)} f_\epsilon(x, v)f_\epsilon(x, v')a_va_{v'} = \sum_{v \in V(\gamma)} (f_\epsilon(x, v))^2 (a_v)^2. \tag{179}
\]

This is in fact possible since we can choose \( \epsilon \) to be so small that \( f_\epsilon(x, v)f_\epsilon(x, v') = 0 \) if the vertices do not coincide. As it will turn out, there is an ambiguity in completing the square. We have to choose if we want to have \((a + b)^2\) or \((a - b)^2\). Since we do not want to consider the ambiguity at this point, we introduce \( s \in \{-1, 1\} \) and add this to the square. Hence we rewrite

\[
g_{S, \epsilon}(x) \Psi_\gamma \otimes |F\rangle = \left( \sum_{v \in V(\gamma)} f_\epsilon(x, v)a_v + s \sqrt{F_{\epsilon i}(x)F_{\epsilon i}(x)} \right)^2
+ \sum_{v \in V(\gamma)} f_\epsilon(x, v)\sqrt{F_{\epsilon i}(x)F_{\epsilon i}(x)} \left( \epsilon_1^2 m_v - s2a_v \right) \Psi_\gamma \otimes |F\rangle \tag{180}
\]

and so have found the eigenvalue of \( g_{S, \epsilon}(x) \).

However, the above eigenvalue is not the desired result. For this we have to take the square root and integrate over the surface. The suggestion is again to follow the descriptions in [9]. We consider two real, positive variables \( a \geq b \). As a matter of fact the variables satisfy the inequalities

\[
\sqrt{a} \leq \sqrt{a + b} \leq \sqrt{a} + \sqrt{b}, \quad \sqrt{a} - \sqrt{b} \leq \sqrt{a - b} \leq \sqrt{a}. \tag{181}
\]

The quadratic part of (180) is manifestly positive and the absolute value of the remaining part has to smaller compared to quadratic part in order to ensure positivity of the operator \( g_{S, \epsilon} \). Hence, the inequalities are employable. The advantage of this is that \( a \) and \( b \) depend on the regulator \( \epsilon \). If we can show that

\[
\lim_{\epsilon \to 0} \int \sqrt{b} = 0, \tag{182}
\]

it holds furthermore, by means of the inequalities, that

\[
\lim_{\epsilon \to 0} \int \sqrt{a \pm b} = \lim_{\epsilon \to 0} \sqrt{a}. \tag{183}
\]

Before we consider the square root of the second term in (180) we take a closer look at

\[
F_{\epsilon i}(x)F_{\epsilon i}(x) = \int_{\mathbb{R}^3} d^3X \int_{\mathbb{R}^3} d^3Y \int_S d^2y \int_S d^2z K_{S_i}(X, y)K_{S_i}(Y, z)f_\epsilon(x, y)F(X)f_\epsilon(x, z)F(Y)
\]

\[
= \lim_{\epsilon \to 0} \int_{\mathbb{R}^3} d^3X \int_{\mathbb{R}^3} d^3Y \int_S d^2y \int_S d^2z K_{S_i}(X, y)K_{S_i}(Y, z)\delta^{(2)}(x, y)F(X)\delta^{(2)}(x, z)F(Y) = \tag{184}
\]

\[
\int_{\mathbb{R}^3} d^3X \int_{\mathbb{R}^3} d^3Y K_{S_i}(X, x)K_{S_i}(Y, x)F(X)F(Y) =: F_\phi(x)
\]

and realize that in the limit where we remove the regulator this and hence also its square root are bounded functions on the surface, by means of the integral kernel and the generalized eigenvalue being test functions. Due to this we want to remove the regulator for this object already now.

The final thing to realize is that if \( f_\epsilon(x) \) is a density that converges to the delta function, its square root converges to zero. Now we are able to encounter the considered square root. We already take along the limit and the integration:

\[
\lim_{\epsilon \to 0} \int_S d^2x \left| \sum_{v \in V(\gamma)} f_\epsilon(x, v)\sqrt{F_\phi(x)}(\epsilon_1^2 m_v - s2a_v) \right| = \tag{185}
\]

\[
= \lim_{\epsilon \to 0} \int_S d^2x \sum_{v \in V(\gamma)} \sqrt{f_\epsilon(x, v)}\sqrt{F_\phi(x)}\sqrt{|\epsilon_1^2 m_v - s2a_v|} = 0
\]
since we can choose $\epsilon$ to be small enough to perform the summation outside of square root and the absolute value.

According to this result, we finally can consider taking the square root of $g_{S,\epsilon}$, perform the integration and remove the regulator:

$$\lim_{\epsilon \to 0} \int_S d^2x \sqrt{g_{S,\epsilon}} \Psi_\gamma \otimes |F\rangle = \lim_{\epsilon \to 0} \int_S d^2x \sum_{v \in V(\gamma)} f_\epsilon(x,v) a_v + s \sqrt{F_{\epsilon i}(x) F_{\epsilon i}^*(x)} |\Psi_\gamma \otimes |F\rangle. \quad (186)$$

This is now the point where the ambiguity in completing the square comes into play. Both terms are manifestly positive. So if $s = +1$ we can simply remove the absolute value, for $s = -1$ we cannot. It is not clear if we can drag limit and integration inside the absolute value. Since we are at this point satisfied with the fact that we are able derive a notion of the area operator for this almost quasifree representation, we choose $s = +1$. Hence,

$$\lim_{\epsilon \to 0} \int_S d^2x \sqrt{g_{S,\epsilon}} \Psi_\gamma \otimes |F\rangle = \left( \lim_{\epsilon \to 0} \int_S d^2x \sum_{v \in V(\gamma)} f_\epsilon(x,v) a_v + \lim_{\epsilon \to 0} \int_S d^2x \sqrt{F_{\epsilon i}(x) F_{\epsilon i}^*(x)} \right) |\Psi_\gamma \otimes |F\rangle \quad (187)$$

In the limit the integral takes away the Dirac delta and the first term of the above equation is just the original area operator. Replacing the eigenvalues again by the corresponding operators, the area operator for this almost quasifree representation reads

$$A(S) = A_{AL}(S) \otimes 1_F + 1_{AL} \otimes \int_S d^2x \sqrt{\phi_i(K_S(x)) \phi_i^*(K_S(x))}. \quad (188)$$

Here we furthermore use some notation for the scalar fields once the regulator is removed:

$$\phi_i(K_S(x)) = \int_{\mathbb{R}^3} d^3Y K_{Si}(Y, x) \phi(Y). \quad (189)$$

This result is in fact somewhat similar to \[7\]. There, the original area operator is extended by the classical area of the surface. Another similarity is that there are only additive changes that increase the quantum area. This however might be crucially dependent on our resolution of the ambiguity. In any case, the addition here is an operator which carries its own quantum mechanical fluctuations.

### VII. CONCLUSIONS AND OUTLOOK

In this work, we presented a new type of state and the corresponding representations of the HF algebra $\mathcal{A}_{HF}$ underlying loop quantum gravity, and also for a toy model with structure group $U(1)$. The key feature is a Gaussian vacuum expectation value for fluxes, encoding spatial geometry, which is characterized by a condensate contribution, e.g. a background flux, and fluctuations determined by a covariance of surfaces, here determined by e.g. a scalar field.

For the $U(1)$ toy-model of LQG, we reformulated the HF algebra in terms of a Weyl algebra. On this, we introduced a new class of almost quasifree states, which behaves like the AL state for holonomies and cylindrical functions, while it is Gaussian for fluxes. We worked out the representation in two examples, utilizing coherent states of the harmonic oscillator and the Fock representation of a scalar field.

For the HF algebra $\mathcal{A}_{HF}$ for $SU(2)$ defined in section VIIA we introduced a new class of representation, with the behaviour described above. In particular there are non-vanishing contributions for the $n$-point correlation functions. For a concrete example, we demonstrated that this change of representation leads to a significant change in the area spectrum of surfaces.

It might be tempting to interpret the presence of a scalar field in the fluxes of the new representation as a toy-model for matter coupling. Especially, because the geometric correlation functions are determined by the scalar field part. However, including the scalar field into the flux operators would lead to non-vanishing commutators between fluxes and the momentum conjugate to the scalar field. As geometric and matter variables have to have trivial commutation relations, the scalar field we introduced cannot be interpreted as a physical matter field. Rather, it only serves to introduce the Gaussianity in the representation.

How can we extend and apply and extend the results contained in this work?
One area of application for the new states is the quantum origin of the primordial perturbations. The current observations of the CMB suggest that primordial perturbations of the spatial metric (and matter density) are described well by a Gaussian random field with a certain covariance. Thus the states that we describe might be well suited to describe the quantum geometry of the early universe. In the standard picture, the covariance of the fluctuations is determined by following an initial quantum state through inflation. The new states allow to think about a quantum gravitational origin of the fluctuations.

The results we presented strongly suggest that

$$\pi_F \left( (E_S(f)) \right) = X_{S,f} \otimes \mathbb{1}_H + \mathbb{1}_{AL} \otimes \mathbb{1}_{S,f}$$

where gives $\Xi_{S,f}$ is a classical random field with suitable dependence on $S, f$ gives a representation of $\mathbb{A}_{HF}$. Thus the fluctuations do not seem to have to be Gaussian.

Are there states that are Gaussian in both variables? This question still stands and should eventually be resolved, either positively or negatively. There are some indications that it is not possible to find such states. In a slightly different framework, there is indeed a no-go result [38, 39]. Also, for the U(1) theory, there are some indications that no representations of that type can be found [23]. On the other hand, if such states do exist, then the almost quasifree states of the present work might be a stepping stone to reach them.

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Appendix A: Eigenstates of a quantum field

In this appendix we will show that under reasonable assumptions there is a sufficiently large class of real valued functions $F$ such that there are generalized eigenstates $|F\rangle$ for a scalar field that fulfill

$$\phi(X)|F\rangle = F(X)|F\rangle.$$  \hfill (A1)

A rigged Hilbert space is a Hilbert space $\mathcal{H}$, together with a dense, continuously embedded topological vector space $\Phi \subset \mathcal{H}$. As a consequence, $\mathcal{H}$ is contained in the topological dual of $\Phi$, $\mathcal{H} \subset \Phi'$. The improper eigenstates of selfadjoint operators can find their home in such duals:

**Theorem A.1** ([42, 43]). Any self-adjoint operator $A$ mapping $\Phi$ continuously (in the topology of $\Phi$) onto itself possesses a complete system of generalized eigenfunctions $(F_\alpha)$, i.e. elements $F_\alpha \in \Phi'$ such that for any $\phi \in \Phi$,

$$F_\alpha(A\phi) = \lambda_\alpha F_\alpha(\phi), \quad \alpha \in \mathbb{A},$$  \hfill (A2)

where the set of values of the function $\alpha \mapsto \lambda_\alpha$, $\alpha \in \mathbb{A}$, is contained in the spectrum of $A$ and has full measure with respect to the spectral measure $\sigma_f(\lambda)$, of any element $f \in \mathcal{H}$. The completeness of the system means that $F_\alpha(\phi) \neq 0$ for any $\phi \in \Phi$, $\phi \neq 0$, for at least one $\alpha \in \mathbb{A}$.

We will now show how this could be applied to a quantum field. For definiteness, we work with a scalar field on Minkowski space. We write

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \sqrt{2\omega_p} \left( a_p e^{ip \cdot x - \omega_p t} + a_p^\dagger e^{-ip \cdot x + \omega_p t} \right)$$

$$= \frac{1}{\sqrt{2}} \left( \phi_-(x,t) + \phi_+(x,t) \right).$$  \hfill (A3)

$\omega_p$ are the eigenvalues

$$\omega_p = \sqrt{p^2 + m^2}$$  \hfill (A4)

of the operator

$$D = \sqrt{-\Delta + m^2},$$  \hfill (A5)

where

$$\pi_F \left( (E_S(f)) \right) = X_{S,f} \otimes \mathbb{1}_H + \mathbb{1}_{AL} \otimes \Xi_{S,f}$$

and

$$\pi_F \left( (F \otimes \mathbb{1}_H) \right) = F \otimes \mathbb{1}_H.$$  \hfill (190)
and \( a_p, a_p^\dagger \) are standard momentum space annihilation/creation operators with
\[
[a_p, a_q^\dagger] = (2\pi)^3 \delta(p, q).
\] (A7)

In the following, we will set \( t = 0 \) and drop the time argument from all the functions. Consequently,
\[
a(x) := (D\phi_-)(x), \quad a^\dagger(x) := (D\phi_+)(x)
\] (A8)
are standard momentum space annihilation/creation operators with
\[
[a(f_1), a^\dagger(f_2)] = \langle f_1 | f_2 \rangle \hbar I
\] (A9)
over the Fock space
\[
\mathcal{H} = F(\hbar), \quad \hbar = L^2(\mathbb{R}^3, d^3x).
\] (A10)

The operators \( \phi(f) \) for smooth, real valued functions \( f \) of compact support are mutually commuting and selfadjoint on \( \mathcal{H} \). Therefore they must have a common set of generalized eigenstates. In the following, we want to investigate such states \( |F \rangle \) with the property
\[
\phi(f)|F \rangle = F(f)|F \rangle.
\] (A11)
for a suitable class of real valued functions \( F \). It will be useful to work with the dense domain \( \mathcal{D} \subset \mathcal{H} \)
\[
\mathcal{D} = \text{span}\{ \prod_{\text{finite}} a^\dagger(f_i) |0\rangle \mid f_1, f_2, \ldots \in \mathcal{S}(\mathbb{R}^3) \}
\] (A12)
where \( \mathcal{S}(\mathbb{R}^3) \) denotes the Schwartz functions. \( \mathcal{D} \) is contained in the domain of the \( \phi(f) \). Consider the following operator
\[
O(F) = \pi^{-\frac{1}{2}} e^{\frac{1}{2}(DF | DF)\hbar} e^{-\frac{i}{2}(D\phi_+ - \sqrt{2} D\phi_-) | D\phi_+ - \sqrt{2} D\phi_-)^2 d^3x} e^{\sqrt{2} a^\dagger(DF)}.
\] (A13)
\[
= \pi^{-\frac{1}{2}} e^{\frac{i}{2}(DF | DF)\hbar} e^{f(a^\dagger(x))^2 d^3x} e^{\sqrt{2} a(DF)}.
\] (A14)

This definition requires, at minimum that \( F \) is such that \( DF \in \mathcal{H} \). We have that

**Lemma A.2.** Formally, i.e. without consideration of domains,
\[
\phi(f)O(F) = F(f)O(F) + \frac{1}{\sqrt{2}} O(F)a(D^{-1} f).
\] (A15)

**Proof.** One can do a direct calculation, but it is easier to realize that
\[
[a(f), \Gamma[a^\dagger(\cdot)]] = \int f(x) \frac{\delta \Gamma[a^\dagger(\cdot)]}{\delta a^\dagger(x)} d^3x
\] (A16)
where \( \Gamma[\cdot] \) is a functional which we assume to be differentiable. Then, noting
\[
\frac{\delta}{\delta a^\dagger(x)} O(F) = -(a^\dagger(x) - \sqrt{2} DF(x))O(F),
\] (A17)
the assertion is a straightforward calculation. \( \square \)

This lemma shows that \( O(F) |0\rangle \) are formally the sought-for eigenstates (A11). But these are obviously not normalizable, so in what sense do they even exist?

**Lemma A.3.** For \( F \in \mathcal{S}'(\mathbb{R}^3), DF \in \mathcal{H}, \) the objects
\[
|F \rangle = O(F) |0\rangle
\] (A18)
define linear forms over the domain \( \mathcal{D} \) (A12).
Proof. We attempt to define the linear form
\[ \langle X | F \rangle = \langle X | O(F) | 0 \rangle, \quad X \in \mathcal{D} \quad (A19) \]
by expanding the exponentials in a Taylor series and taking the limit. Since we assume \( DF \in \mathfrak{h} \), the first exponential in \((A14)\) is no problem, and since \( \mathcal{D} \) only contains elements of finite particle number, also the third exponential represents no problem. But we have to consider the definition of the operator
\[ a_2 = \int (a(x))^2 \, d^3x \quad (A20) \]
and its adjoint which is used in the definition of \( O(F) \). A short calculation shows that
\[ \langle \prod_k a_\dagger(f_k) \Omega | a_2 \Psi \rangle = \sum_{(l,m), l \neq m} \langle f_l | f_m \rangle_h \langle \prod_{k \neq l,m} a_\dagger(f_k) \omega | \Psi \rangle \quad (A21) \]
where the products, and consequently the sums are finite, and \( \Omega = |0\rangle \) the vacuum. This shows that \( a_2 \) and its powers are well defined on \( \mathcal{D} \) and, since \( \mathcal{D} \) only contains elements of finite particle number, also its exponential represents no problem. \( \square \)

Now note that the operators \( \phi(f), f \in \mathcal{S}(\mathbb{R}^3) \) map \( \mathcal{D} \) into itself, because \( D^{-1} \) maps \( \mathcal{S}(\mathbb{R}^3) \) into itself. We strongly suspect that \( \mathcal{D} \) can be used to create a rigged Hilbert space, suitable for application of theorem [A1].

**Conjecture A.4.** There is a topology on \( \mathcal{D} \) that

1. is stronger than that induced from \( \mathcal{H} \),
2. is strong enough such that \( |F\rangle \in \mathcal{D}' \quad \forall F : DF \in \mathfrak{h} \).
3. is weak enough such that \( |F\rangle \in \mathcal{D}' \quad F : DF \in \mathfrak{h} \) comprise all generalized eigenstates.

If this conjecture is true, then

**Corollary A.5.**

1. The joint eigenstates of \( \phi(f), f \in \mathcal{S}(\mathbb{R}^3) \) are
\[ |F\rangle \quad \text{with } F \in \mathcal{S}'(\mathbb{R}^3), \quad DF \in L^2(\mathbb{R}^3) \quad (A22) \]
2. Functions of \( \phi(f) \) have
\[ A(\phi(f))|F\rangle = A(F(f))|F\rangle \quad (A23) \]
3. If for two operators \( A, B \) on \( \Phi \)
\[ A|F\rangle = B|F\rangle \quad \forall F : F \in \mathcal{S}'(\mathbb{R}^3), \quad DF \in L^2(\mathbb{R}^3) \quad (A24) \]
then \( A = B \).

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[1] A. Ashtekar, New Variables for Classical and Quantum Gravity, *Phys. Rev. Lett.* **57**, 2244 (1986).
[2] J. F. Barbero G., Real Ashtekar variables for Lorentzian signature space times, *Phys. Rev.* **D51**, 5507 (1995), arXiv:gr-qc/9410014 [gr-qc].
[3] A. Ashtekar and J. Lewandowski, Projective techniques and functional integration for gauge theories, *J. Math. Phys.* **36**, 2170 (1995), arXiv:gr-qc/9411046 [gr-qc].
[4] C. Fleischhack, Representations of the Weyl algebra in quantum geometry, *Commun. Math. Phys.* **285**, 67 (2009), arXiv:math-ph/0407006 [math-ph].
[5] J. Lewandowski, A. Okolow, H. Sahlmann, and T. Thiemann, Uniqueness of diffeomorphism invariant states on holonomy-flux algebras, *Commun. Math. Phys.* **267**, 703 (2006), arXiv:gr-qc/0504147 [gr-qc].
[6] M. Dziendzikowski and A. Okolow, New diffeomorphism invariant states on a holonomy-flux algebra, *Class. Quant. Grav.* **27**, 225005 (2010), arXiv:0912.1278 [math-ph].
[7] M. Varadarajan, Towards new background independent representations for loop quantum gravity, *Class. Quant. Grav.* **25**, 105011 (2008), arXiv:0709.1680 [gr-qc].
[8] T. A. Koslowski, Dynamical Quantum Geometry (DQG Programme), (2007). arXiv:0709.3465 [gr-qc]
[9] H. Sahlmann, On loop quantum gravity kinematics with non-degenerate spatial background, Class. Quant. Grav. 27, 125007 (2010) arXiv:1006.0388 [gr-qc]
[10] B. Dittrich and M. Geiller, A new vacuum for Loop Quantum Gravity, Class. Quant. Grav. 32, 112001 (2015), arXiv:1401.6441 [gr-qc]
[11] B. Bahr, B. Dittrich, and M. Geiller, A new realization of quantum geometry, (2015). arXiv:1506.08571 [gr-qc]
[12] P. Drobiski and J. Lewandowski, Continuum approach to the BF vacuum: The U(1) case, Phys. Rev. D96, 126011 (2017) arXiv:1705.09836 [gr-qc]
[13] T. Thiemann, Gauge field theory coherent states (GCS): 1. General properties, Class. Quant. Grav. 18, 2025 (2001) arXiv:hep-th/0005233 [hep-th]
[14] T. Thiemann and O. Winkler, Gauge field theory coherent states (GCS). 2. Peakedness properties, Class. Quant. Grav. 18, 2561 (2001) arXiv:hep-th/0005247 [hep-th]
[15] T. Thiemann and O. Winkler, Gauge field theory coherent states (GCS): 3. Ehrenfest theorems, Class. Quant. Grav. 18, 4629 (2001) arXiv:hep-th/0005234 [hep-th]
[16] T. Thiemann and O. Winkler, Gauge field theory coherent states (GCS): 4. Infinite tensor product and thermodynamical limit, Class. Quant. Grav. 18, 4997 (2001) arXiv:hep-th/0005235 [hep-th]
[17] T. Thiemann, Complexifier coherent states for quantum general relativity, Class. Quant. Grav. 23, 2063 (2006) arXiv:gr-qc/0206037 [gr-qc]
[18] M. Varadarajan, Fock representations from U(1) holonomy algebras, Phys. Rev. D61, 104001 (2000) arXiv:gr-qc/0001050 [gr-qc]
[19] A. Ashtekar and J. Lewandowski, Relation between polymer and Fock excitations, Class. Quant. Grav. 18, L117 (2001) arXiv:gr-qc/0107043 [gr-qc]
[20] H. Sahlmann, When do measures on the space of connections support the triad operators of loop quantum gravity?, J. Math. Phys. 52, 012503 (2011) arXiv:gr-qc/0207112 [gr-qc]
[21] A. Stottmeister and T. Thiemann, Structural aspects of loop quantum gravity and loop quantum cosmology from an algebraic perspective, (2013). arXiv:1312.3657 [gr-qc]
[22] A. Corichi and K. V. Krasnov, Ambiguities in loop quantization: Area vs. electric charge, Modern Physics Letters A 13, 1339 (1998), http://www.worldscientific.com/doi/pdf/10.1142/S0217732398001406
[23] S. Nekovar, Gaussian Measures on Spaces of Connections, and Representation Theory of Holonomy-Flux Algebras, Master thesis, Friedrich-Alexander-Universität Erlangen-Nürnberg (2014).
[24] M. Varadarajan, The generator of spatial diffeomorphisms in the Koslowski-Sahlmann representation, Class. Quant. Grav. 30, 175017 (2013) arXiv:1306.6126 [gr-qc]
[25] M. Campiglia and M. Varadarajan, The Koslowski-Sahlmann representation: gauge and diffeomorphism invariance, Class. Quant. Grav. 31, 075002 (2014) arXiv:1311.6117 [gr-qc]
[26] M. Campiglia and M. Varadarajan, The Koslowski-Sahlmann representation: Quantum Configuration Space, Class. Quant. Grav. 31, 175009 (2014) arXiv:1406.0579 [gr-qc]
[27] A. Ashtekar and J. Lewandowski, Background independent quantum gravity: A Status report, Class. Quant. Grav. 21, R53 (2004) arXiv:gr-qc/0404018 [gr-qc]
[28] T. Thiemann, Modern Canonical Quantum General Relativity (Cambridge University Press, Cambridge, 2007).
[29] A. Ashtekar and J. Lewandowski, Differential geometry on the space of connections via graphs and projective limits, J. Geom. Phys. 17, 191 (1995) arXiv:hep-th/9412073 [hep-th]
[30] P. Drobiski and J. Lewandowski, Continuum approach to the BF vacuum: The U(1) case, Phys. Rev. D96, 126011 (2017) arXiv:1705.09836 [gr-qc]
[31] A. Ashtekar and A. Corichi, Photon inner product and the Gauss linking number, Class. Quant. Grav. 14, A43 (1997) arXiv:gr-qc/9608017 [gr-qc]
[32] A. Ashtekar and A. Corichi, Gauss linking number and electromagnetic uncertainty principle, Phys. Rev. D56, 2073 (1997) arXiv:hep-th/9701136 [hep-th]
[33] S. Lanéry and T. Thiemann, Projective Loop Quantum Gravity I. State Space, J. Math. Phys. 57, 122304 (2016) arXiv:1411.3592 [gr-qc]
[34] S. Lanéry and T. Thiemann, Projective Loop Quantum Gravity II. Searching for Semi-Classical States, J. Math. Phys. 58, 052302 (2017) arXiv:1510.01925 [gr-qc]
[35] H. Sahlmann, Some results concerning the representation theory of the algebra underlying loop quantum gravity, J. Math. Phys. 52, 012502 (2011) arXiv:gr-qc/0207111 [gr-qc]
[36] A. Ashtekar and J. Lewandowski, Quantum theory of geometry. 1: Area operators, Class. Quant. Grav. 14, A55 (1997) arXiv:gr-qc/9602046 [gr-qc]
[37] Encyclopedia of Mathematics: Rigged Hilbert Space, http://www.encyclopediaofmath.org
[30] index.php?title=Rigged Hilbert space&oldid=36747, accessed: 2018-01-07.

[43] P. Blanchard and E. Brüning, *Spectral Analysis in Rigged Hilbert Spaces*, Vol. vol 69. Birkhäuser, Cham (2015).