Fully-Dynamic Submodular Cover with Bounded Recourse

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Abstract

In submodular covering problems, we are given a monotone, nonnegative submodular function $f : 2^N \to \mathbb{R}_+$ and wish to find the min-cost set $S \subseteq N$ such that $f(S) = f(N)$. When $f$ is a coverage function, this captures SetCover as a special case. We introduce a general framework for solving such problems in a fully-dynamic setting where the function $f$ changes over time, and only a bounded number of updates to the solution (a.k.a. recourse) is allowed. For concreteness, suppose a nonnegative monotone submodular integer-valued function $g_t$ is added or removed from an active set $G^{(t)}$ at each time $t$. If $f^{(t)} = \sum_{g \in G^{(t)}} g$ is the sum of all active functions, we wish to maintain a competitive solution to SubmodularCover for $f^{(t)}$ as this active set changes, and with low recourse. For example, if each $g_t$ is the (weighted) rank function of a matroid, we would be dynamically maintaining a low-cost common spanning set for a changing collection of matroids.

We give an algorithm that maintains an $O(\log(f_{\text{max}}/f_{\text{min}}))$-competitive solution, where $f_{\text{max}}, f_{\text{min}}$ are the largest/smallest marginals of $f^{(t)}$. The algorithm guarantees a total recourse of $O(\log(c_{\text{max}}/c_{\text{min}}) \cdot \sum_{t \leq T} g_t(N))$, where $c_{\text{max}}, c_{\text{min}}$ are the largest/smallest costs of elements in $N$. This competitive ratio is best possible even in the offline setting, and the recourse bound is optimal up to the logarithmic factor. For monotone submodular functions that also have positive mixed third derivatives, we show an optimal recourse bound of $O(\sum_{t \leq T} g_t(N))$. This structured class includes set-coverage functions, so our algorithm matches the known $O(\log n)$-competitiveness and $O(1)$ recourse guarantees for fully-dynamic SetCover. Our work simultaneously simplifies and unifies previous results, as well as generalizes to a significantly larger class of covering problems. Our key technique is a new potential function inspired by Tsallis entropy. We also extensively use the idea of Mutual Coverage, which generalizes the classic notion of mutual information.

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1 Introduction

In the SubmodularCover problem, we are given a monotone, nonnegative submodular function \( f : 2^\mathcal{N} \to \mathbb{Z}_+ \), as well as a linear cost function \( c \), and we wish to find the min-cost set \( S \subseteq \mathcal{N} \) such that \( f(S) = f(\mathcal{N}) \). This is a classical NP-hard problem: e.g., when \( f \) is a coverage function we capture the SetCover problem. Moreover, the greedy algorithm is known to be an \((1 + \ln f_{\text{max}})\)-approximation, where \( f_{\text{max}} \) is the maximum value of any single element [Wol82]. This bound is tight assuming \( P \neq \text{NP} \), even for the special case of SetCover [Fei98, DS14].

We consider this SubmodularCover problem in a fully-dynamic setting, where the notion of coverage changes over time. At each time, the underlying submodular function changes from \( f(t) \) to \( f(t+1) \), and the algorithm may have to change its solution from \( S_t \) to \( S_{t+1} \) to cover this new function. We do not want our solutions to change wildly if the function changes by small amounts. The goal of this work is to develop algorithms where this “churn” \(|S_t \triangle S_{t+1}|\) is small (perhaps in an amortized sense), while maintaining the requirement that each solution \( S_t \) is a good approximate solution to the function \( f(t) \). The change \(|S_t \triangle S_{t+1}|\) is often called recourse in the literature.

This problem has been posed and answered in the special case of SetCover—a.k.a. the HittingSet problem. In this problem, hyperedges arrive to and depart from an active set over time, and we must maintain a small set of vertices that hit all active hyperedges. We know an algorithm that maintains an \( O(\log n_t) \)-approximation which has constant amortized recourse [GKKP17]; here \( n_t \) refers to the number of active hyperedges at time \( t \). In other words, the total recourse—the total number of changes over \( T \) edge arrivals and departures—is only \( O(T) \). The algorithm and analysis are based on a delicate token-based argument, which gives each hyperedge a constant number of tokens upon its arrival, and moves these tokens in a careful way between edges to account for the changes in the solution. What do we do in the more general SubmodularCover case, where there is no notion of sets any more?

In this work we study the model where a submodular function is added or removed from the active set \( G(t) \) at each timestep: this defines the current submodular function \( f(t) := \sum_{g \in G_t} g \) as the sum of functions in this active set. The algorithm must maintain a subset \( S_t \subseteq \mathcal{N} \) such that \( f(t)(S_t) = f(t)(\mathcal{N}) \) with cost \( c(S_t) \) being within a small approximation factor of the optimal SubmodularCover for \( f(t) \), such that the total recourse \( \sum_t |S_t \triangle S_{t+1}| \) remains small. To verify that this problem models dynamic HypergraphVertexCover, each arriving/departing edge \( A_t \) should correspond to a submodular function \( g_t \) taking on value 1 for any set \( S \) that hits \( A_t \), and value zero otherwise (i.e., \( g_t(S) = 1[S \cap A_t \neq \emptyset] \)).

1.1 Our Results

Our main result is the following:

**Theorem 1.1** (Informal). There is a deterministic algorithm that maintains an \( e^2 \cdot (1 + \log f_{\text{max}}) \)-competitive solution to Submodular Cover in the fully-dynamic setting where functions arrive/depart over time. This algorithm has total recourse:

\[
O\left( \sum_t g_t(\mathcal{N}) \ln \left( \frac{c_{\text{max}}}{c_{\text{min}}} \right) \right),
\]

\(^1\)In the introduction we restrict to integer-valued functions for simplicity; all results extend to general submodular functions with suitable changes. See the technical sections for the full results and nuanced details.
where \( g_t(N) \) is the value of the function considered at time \( t \), and \( c_{\text{max}}, c_{\text{min}} \) are the maximum and minimum element costs.

Let us parse this result. Firstly, the approximation factor almost matches Wolsey’s result up to the multiplicative factor of \( e^2 \); this is best possible in polynomial time unless \( P = \text{NP} \) even in the offline setting. Secondly, the amortized recourse bound should be thought of as being logarithmic—indeed, specializing it to HITTINGSET where each \( g_t(N) = 1 \), we get a total recourse of \( O(T \log(c_{\text{max}}/c_{\text{min}})) \) over \( T \) timesteps, or an amortized recourse of \( O(\log(c_{\text{max}}/c_{\text{min}})) \) per timestep. Hence this recourse bound is weaker by a log-of-cost-spread factor, while generalizing to all monotone submodular functions. Finally, since we are allowed to give richer functions \( g_t \) at each step, it is natural that the recourse scales with \( \sum_t g_t(N) \), which is the total “volume” of these functions. In particular, this problem captures fully-dynamic HYPERGRAPHVERTEXCOVER where at each round a batch of \( k \) edges appears all at once (this is in contrast to the standard fully-dynamic model where hyperedges appear one at a time). In this case the algorithm may have to buy up to \( k = g_t(N)/f_{\text{min}} \) new vertices in general to maintain coverage.

We next show that for coverage functions (and hence for the HYPERGRAPHVERTEXCOVER problem), a variation on the algorithm from Theorem 1.1 can remove the log-of-cost-spread factor in terms of recourse, at the cost of a slightly coarser competitive ratio. E.g., for HYPERGRAPHVERTEXCOVER the new competitive ratio corresponds to an \( O(\log n) \) guarantee versus an \( O(\log D_{\text{max}}) \) guarantee, where \( D_{\text{max}} \) being the largest degree of any vertex.

**Theorem 1.2 (Informal).** There is a deterministic algorithm that maintains an \( O(\log f(N)) \)-competitive solution to SUBMODULARCOVER in the fully-dynamic setting where functions arrive/depart over time, and each function is \( 3 \)-increasing in addition to monotone and submodular. Furthermore, this algorithm has total recourse:

\[
O\left( \sum_t g_t(N) \right).
\]

Indeed, this result holds not just for coverage functions, but for the broader class of \( 3 \)-increasing, monotone, submodular functions [FH05], which are the functions we have been considering, with the additional property that have positive discrete mixed third-derivatives. At a high level, these are functions where the mutual coverage does not increase upon conditioning.

### 1.2 Techniques and Overview

The most widely known algorithm for SUBMODULARCOVER is the greedy algorithm; this repeatedly adds to the solution an element maximizing the ratio of its marginal coverage to its cost. It is natural to try to use the greedy algorithm in our dynamic setting; the main issue is that out-of-the-box, greedy may completely change its solution between time steps. In their result on recourse-bounded HYPERGRAPHVERTEXCOVER, [GKKP17] showed how to effectively imitate the greedy algorithm without sacrificing more than a small amount of recourse. A barrier to making greedy algorithms dynamic is their sequential nature, and hyperedge inserts/deletes can play havoc with this. So they gave a local-search algorithm that skirts this sequential nature, while simultaneously retaining the key properties of the greedy algorithm. Unfortunately, their analysis hinges on delicately assigning edges to vertices. In the more general SUBMODULARCOVER setting, there are no edges to track, and this approach breaks down.

Our first insight is to return to the sequential nature of the greedy algorithm. Our algorithm maintains an ordering \( \pi \) on the elements of \( N \), which induces a solution to the SUBMODULARCOVER
problem: we define the output of the algorithm as the prefix of elements that have non-zero marginal value. We maintain this permutation dynamically via local updates, and argue that only a small amount of recourse is necessary to ensure the solution is competitive. To bound the competitive ratio, we imagine that the permutation corresponds to the order in which an auxiliary offline algorithm selects elements, i.e. a stack trace. We show that our local updates maintain the invariant that this is the stack trace of an approximate greedy algorithm for the currently active set of functions. We hope that this general framework of doing local search on the stack trace of an auxiliary algorithm finds uses in other online/dynamic algorithms.

Our main technical contribution is to give a potential function to argue that our algorithm needs bounded recourse. The potential measures the (generalized) entropy of the coverage vector of the permutation $\pi$. This coverage vector is indexed by elements of the universe $\mathcal{N}$, and the value of each coordinate is the marginal coverage (according to permutation $\pi$) of the corresponding element. Entropy is often used as a potential function (notably, in recent developments in online algorithms) but in a qualitatively different way. In many if not all cases, these algorithms follow the maximum-entropy principle and seek high entropy solutions subject to feasibility constraints; the potential function then tracks the convergence to this goal. On the other hand, in our setting the cost function is the support size of the coverage vector, and minimizing this roughly corresponds to low entropy. We show entropy decreases sufficiently quickly with every change performed during the local search, and increases by a controlled amount with insertion/deletion of each function $g$, thus proving our recourse bounds.

We find our choice of entropy to be interesting for several reasons. For one, we use a suitably chosen Tsallis entropy [Tsa88], which is a general parametrized family of entropies, instead of the classical Shannon entropy. The latter also yields recourse bounds for our problem, but they are substantially weaker (see Appendix D). Tsallis entropy has appeared in several recent algorithmic areas, for example as a regularizer in bandit settings [SS14], and as an approximation to Shannon entropy in streaming contexts [HNO08]. Secondly, it is well known that for certain problems, minimizing an $\ell_1$-objective is an effective proxy for minimizing sparsity [CRT06]. In our dynamics, the $\ell_1$ mass stays constant since total coverage does not change when elements are reordered. However, entropy is a good proxy for sparsity, in the sense that it decreases monotonically (and quickly!) as our algorithm moves within the $\ell_1$ level set towards sparse vectors.

In Section 2, we study the unit cost case to lay out our framework and highlight the main ideas. We show a $\log f_{\max}/f_{\min}$ competitive algorithm for fully-dynamic $\text{SUBMODULARCOVER}$ with $O(\sum_t g_t(\mathcal{N})/f_{\min})$ total recourse. Then in Section 3, we turn to general cost functions. We again show a $\log(f_{\max}/f_{\min}) \cdot \sum_t g_t(\mathcal{N})/f_{\min}$ recourse. The algorithm template is the same as before, but with a suitably generalized potential function and analysis. Here we also require a careful choice of the Tsallis entropy parameter, near (but not quite at) the limit point at which Tsallis entropy becomes Shannon entropy.

In Section 4, we show how to remove the $\log(c_{\max}/c_{\min})$ dependence and achieve optimal recourse for a structured family of monotone, submodular functions: the class of 3-increasing (monotone, submodular) set functions [FH05]. These are set functions, all of whose discrete mixed third derivatives are nonnegative everywhere. Submodular functions in this class include measure coverage functions, which generalize set coverage functions, as well as entropies for distributions with positive interaction information (see, e.g., [Jak05] for a discussion). Since this class includes $\text{SETCOVER}$, this recovers the optimal $O(1)$ recourse bound of [GKKP17]. For this result we use a more interesting generalization of the potential function from Section 2 that reweights the coverage of elements in the permutation non-uniformly as a function of their mutual coverage with other elements of the
permutation.

In Appendix A, we show how to get improved randomized algorithms when the functions $g_t$ are assumed to be $r$-juntas. This is in analogy to approximation algorithms for SetCover under the frequency parametrization. In Appendix B, we also show how to run an online “combiner” algorithm that gets the best of all worlds, with a competitive ratio of $O(\min(r, \log f_{\max}/f_{\min}))$. Finally, in Appendix C, we demonstrate the generality of our framework by using it to recover known recourse bounded algorithms for fully-dynamic MinimumSpanningTree and MinimumSteinerTree. These achieve $O(1)$ competitive ratios, and recourse bounds of $O(\log D)$ where $D$ is the aspect ratio of the metric. Our proofs here are particularly simple and concise.

1.3 Related Work

Submodular Cover. While we introduced the problem for integer-valued functions, all results can be extended to real-valued settings by adding a dependence on $f_{\min}$, the smallest marginal value. Wolsey [Wol82] showed that the greedy algorithm, repeatedly selecting the element maximizing marginal coverage to cost ratio, gives a $1 + \ln(f_{\max}/f_{\min})$ approximation for SubmodularCover; this guarantee is tight unless $P = \text{NP}$ even for the special case of SetCover [Fei98, DS14]. Fujito [Fuj00] gave a dual greedy algorithm that generalizes the $F$-approximation for SetCover [Hoc82] where $F$ is the maximum-frequency of an element.

SubmodularCover has been used in many applications to resource allocation and placement problems, by showing that the coverage function is monotone and submodular, and then applying Wolsey’s greedy algorithm as a black box. We automatically port these applications to the fully-dynamic setting where coverage requirements change with time. E.g., in selecting influential nodes to disseminate information in social networks [GBLV13, LG16, TWPD17, IIOW10], exploration for robotics planning problems [KMGG08, JCMP17, BMKB13], placing sensors [WCZW15, RP15, ZPW+17, MW16], and other physical resource allocation objectives [YCDW15, LCL+16, TRPJ16]. The networking community has been particularly interested in SubmodularCover recently, since SubmodularCover models network function placement tasks [AGG+09, LPS13, KN15, LRS18, CWJ18]. E.g., [LRS18] want to place middleboxes in a network incrementally, and point out that avoiding closing extant boxes is a huge boon in practice.

The definition of $m$-increasing functions is due to Foldes and Hammer [FH05]. Bach [Bac13] gave a characterization of the class of measure coverage functions (which we define later) in terms of its iterated discrete derivatives. This class generalizes coverage functions, and is contained in the class of 3-increasing functions. [IKBA20] give several additional examples of 3-increasing functions. Several papers [IKBA20, CM18, WMWP15, WWLP13] have given algorithms specifically for 3-increasing submodular function optimization.

Online and Dynamic Algorithms. There is a still budding series of work on recourse-bounded algorithms. Besides [GKKP17] which is most directly related to our work, researchers have studied the Steiner Tree problem [IW91, GK14, GGG16, LOP+15], clustering [GLKX20, CAHP+19], matching [GKKV95, CDKL09, BLSZ14], and scheduling [PW93, Wes00, AGZ99, SS09, SV10, EL14, GKS14].

A rich parallel line of work has studied how to minimize update time for problems in the dynamic or fully-dynamic setting. In [GKKP17], the authors give an $O(\log n)$ competitive and $O(F \log n)$ update time algorithm for fully-dynamic HypergraphVertexCover. An ongoing program of research for the frequency parametrization of HypergraphVertexCover [BHI18, BCH17, BK19, AAG+19, BHN19, BHNW20] has so far culminated in an $F(1 + \epsilon)$ competitive algorithm with $\text{poly}(F, \log c_{\max}/c_{\min}, 1/\epsilon)$ update time (where $F$ is the frequency).
In recent work, [GL20] studied the problem of maintaining a feasible solution to Submodular-Cover in a related online model. That setting is an insertion-only irrevocable analog of this work, where functions $g_t$ may never leave the active set. Our results can be seen as an extension/improvement when recourse is allowed: not only can our algorithm handle the fully-dynamic case with insertions and deletions, but we improve the competitive ratio from $O(\log n \cdot \log f_{\text{max}} / f_{\text{min}})$ to $O(\log f_{\text{max}} / f_{\text{min}})$, which is best possible even in the offline case.

Our work is related to work on convex body chasing (e.g., [AGGT20, Sel20]) in spirit but not in techniques. For an online/dynamic covering problems, the set of feasible fractional solutions within distance $\alpha$ of the optimal solution at a given time step form a convex set: our goal is similarly to “chase” these convex bodies, while limiting the total movement traversed. The main difference is that we seek absolute bounds on the recourse, instead of recourse that is competitive with the optimal chasing solution. (We can give such bounds because our feasible regions are structured and not arbitrary convex bodies).

### 1.4 Notation and Preliminaries

A set function $f : 2^N \rightarrow \mathbb{R}^+$ is submodular if $f(A \cap B) + f(A \cup B) \leq f(A) + f(B)$ for any $A, B \subseteq N$. It is monotone if $f(A) \leq f(B)$ for all $A \subseteq B \subseteq N$. We assume access to a value oracle for $f$ that computes $f(T)$ given $T \subseteq N$. The contraction of $f : 2^N \rightarrow \mathbb{R}^+$ onto $N \setminus T$ is defined as $f_T(S) = f(S | T) := f(S \cup T) - f(T)$. If $f$ is submodular then $f_T$ is also submodular for any $T \subseteq N$. We use the following notation.

\[
\begin{align*}
    f^{(t)}_{\text{max}} &:= \max \{ f^{(t)}(j) \mid j \in N \}, \\
    f^{(t)}_{\text{min}} &:= \min \{ f^{(t)}(j \mid S) \mid j \in N, S \subseteq N, f^{(t)}(j \mid S) \neq 0 \}, \\
    f^{(t)}_{\text{max}} &:= \max_t f^{(t)}_{\text{max}}, \\
    f^{(t)}_{\text{min}} &:= \min_t f^{(t)}_{\text{min}}.
\end{align*}
\]

Also we let $c_{\text{max}}$ and $c_{\text{min}}$ denote the largest and smallest costs of elements respectively.

We will sometime use the simple and well known inequalities:

**Fact 1.3.** Given positive numbers $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$:

\[
\min_i \frac{a_i}{b_i} \leq \frac{\sum_i a_i}{\sum_i b_i} \leq \max_i \frac{a_i}{b_i}.
\]

Throughout this paper, we will use the convention that $1 : k$ denotes that range of indices from 1 to $k$.

**Mutual Coverage.** We will use the notion of mutual coverage defined in [GL20]. Independently, [IKBA20] defined and studied the same quantity under the slightly different name submodular mutual information.

**Definition 1.4 (Mutual Coverage).** The mutual coverage and conditional mutual coverage with respect to a set function $f : 2^N \rightarrow \mathbb{R}^+$ are defined as:

\[
\begin{align*}
    I_f(A; B) &:= f(A) + f(B) - f(A \cup B), \\
    I_f(A; B \mid C) &:= f_C(A) + f_C(B) - f_C(A \cup B).
\end{align*}
\]
We may think of $\mathcal{I}_f(A; B \mid C)$ intuitively as being the amount of coverage $B$ “takes away” from the coverage of $A$ (or vice-versa, since the definition is symmetric in $A$ and $B$), given that $C$ was already chosen. This generalizes the notion of mutual information from information theory: if $\mathcal{N}$ is a set of random variables, and $S \subseteq \mathcal{N}$, and if $f(S)$ denotes the joint entropy of the random variables in the set $S$, then $\mathcal{I}$ is the mutual information.

**Fact 1.5** (Chain Rule). Mutual coverage respects the identity:

$$\mathcal{I}_f(A; B_1 \cup B_2 \mid C) = \mathcal{I}_f(A; B_1 \mid C) + \mathcal{I}_f(A; B_2 \mid C \cup B_1).$$

This neatly generalizes the chain rule for mutual information.

### 2 Unit Cost Submodular Cover

#### 2.1 The Algorithm

We now present our first algorithm for unit-cost fully-dynamic \textsc{SubmodularCover}. We will show the following:

**Theorem 2.1.** For any $\gamma > e$, there is a deterministic algorithm that maintains a $\gamma (\log f_{\max} / f_{\min} + 1)$-competitive solution to unweighted \textsc{SubmodularCover} in the setting where functions arrive/depart over time. Furthermore, this algorithm has total recourse:

$$2 \cdot \frac{e \cdot \ln \gamma}{\gamma - e \cdot \ln \gamma} \cdot \frac{\sum g_t(\mathcal{N})}{f_{\min}} = O\left(\frac{\sum g_t(\mathcal{N})}{f_{\min}}\right).$$

The algorithm and its analysis are particularly clean; we will build on these in the following sections. We begin by describing the algorithm. We maintain a permutation $\pi$ of the elements in $\mathcal{N}$, and assign to each element its marginal coverage given what precedes it in the permutation. We write this marginal value assigned to element $\pi_i$ as

$$\mathcal{F}_\pi(\pi_i) := f(\pi_i \mid \pi_{1:i-1}).$$

We consider two kinds of local search moves:

1. **Swaps:** transform $\pi$ to $\pi'$ by moving an element at position $i$ to position $i - 1$ on the condition that $\mathcal{F}(\pi_i) \geq \mathcal{F}(\pi_{i-1})$. In words, this is a bubble operation (as in bubble-sort).

2. **$\gamma$-moves:** transform the permutation $\pi$ to $\pi'$ by moving an element $u$ from a position $q$ to some other position $p < q$ on the condition that for all $i \in \{p, \ldots, q - 1\}$,

$$\mathcal{F}_{\pi'}(\pi'_p) \geq \gamma \cdot \mathcal{F}_\pi(\pi_i).$$

In words, when $u$ moves ahead in line, it “steals” coverage from other elements along the way; we require that the amount covered by $u$ after the move (which is given by $\mathcal{F}_{\pi'}(\pi'_p)$ since $u$ now sits at position $p$) to be at least a $\gamma$ factor larger than the coverage before the move of any element that $u$ jumps over. (See Figure 1.)

The dynamic algorithm is the following. When a new function $g^{(t)}$ arrives or departs, update the coverages $\mathcal{F}_\pi$ of all the elements in the permutation. Subsequently, while there is a local search move available, perform the move. Output the prefix of $\pi$ of all elements with non-zero coverage. This is summarized in Algorithm 1, with a setting of $\gamma > e$. 

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Algorithm 1 FullyDynamicSubmodularCover

1: \( \pi \leftarrow \) arbitrary initial permutation of elements \( N \).
2: for \( t = 1, 2, \ldots, T \) do
3: \( t \)th function \( g_t \) arrives/departs.
4: while there exists a legal \( \gamma \)-move or a swap for \( \pi \) do
5: \( \) Perform the move, and update \( \pi \).
6: Output the collection of \( \pi_i \) such that \( F(\pi_i) > 0 \).

2.2 Bounding the Cost

Let us start by arguing that if the algorithm terminates, it must produce a competitive solution.

Lemma 2.2. Suppose no \( \gamma \)-moves are possible, then for every index \( i \) such that \( F(\pi_i) > 0 \), and for every index \( i' > i \), the following holds. Let \( \pi' \) be the permutation where \( \pi_{i'} \) is moved to position \( i \). Then

\[
F(\pi_i) > \frac{F(\pi_{i'})}{\gamma}.
\] (2.2)

Proof. Suppose there are elements \( \pi_i \) and \( \pi_{i'} \) such that condition (2.2) does not hold, i.e. \( F(\pi_{i'}) \geq \gamma \cdot F(\pi_i) \). Since by assumption there are no swaps available, the permutation \( \pi \) is in non-increasing order of \( F(\pi_i) \) values, so for all indices \( j > i \) it also holds that \( F(\pi_{i'}) \geq \gamma \cdot F(\pi_j) \). Hence moving \( i' \) from its current position to position \( i \) is a legal \( \gamma \)-move, which is a contradiction. \( \square \)

Corollary 2.3. The output at every time step is \( \gamma \cdot (\log \frac{f_{\max}}{f_{\min}} + 1) \)-competitive.

Proof. Lemma 2.2 implies that the solution is equivalent to greedily selecting an element whose marginal coverage is within a factor of \( 1/\gamma \) of the largest marginal coverage currently available. Given this, the standard analyses of the greedy algorithm for SubmodularCover [Wol82] imply that the solution is \( \gamma \cdot (\log \frac{f_{\max}}{f_{\min}} + 1) \) competitive. \( \square \)

2.3 Bounding the Recourse

We move on to showing that the algorithm must terminate with \( O(g(N)/f_{\min}) \) amortized recourse. For this analysis, we define a potential function parametrized by a number \( \alpha \in (0, 1) \) to be fixed.
As noted in the introduction, up to scaling and shifting, this is the Tsallis entropy with parameter \( \alpha \). We show several properties of this potential:

### Properties of \( \Phi_\alpha \):

1. **(I)** \( \Phi_\alpha \) increases by at most \( g_\ell(N) \cdot (f_{\min})^{\alpha-1} \) with the addition of function \( g_\ell \) to the active set.

2. **(II)** \( \Phi_\alpha \) does not increase with deletion of functions from the system.

3. **(III)** \( \Phi_\alpha \) does not increase during swaps.

4. **(IV)** For an appropriate range of \( \gamma \), the potential \( \Phi_\alpha \) decreases by at least \( \gamma/(e \ln \gamma) - 1 \) \cdot (\( f_{\min} \))\(^\alpha = \Omega((f_{\min})^\alpha) \) with every \( \gamma \)-move.

#### Lemma 2.4

If \( \alpha = (\ln \gamma)^{-1} \), then \( \Phi_\alpha \) respects properties (I)–(IV).

**Proof.** For brevity, define \( h(z) := z^\alpha \). Since \( \alpha \in (0, 1) \), this function is concave and non-decreasing.

We start with property (I). When a function \( g \) is added to the system, for some set of \( i \in [n] \), it increases \( k_i := \pi_\alpha(\pi(i)) \) by some amount \( \Delta_i \). Observe that \( \sum_i \Delta_i = g(N) \). By the concavity of \( h \):

\[
\sum_i h(k_i + \Delta_i) - \sum_i h(k_i) \leq \sum_i h(\Delta_i) \leq \frac{\sum \Delta_i}{f_{\min}} \cdot h(f_{\min}) = g(N) \cdot (f_{\min})^{\alpha-1}.
\]

Property (II) follows since \( h \) is non-decreasing.

For property (III), we wish to show that if \( u \) immediately precedes \( v \) in \( \pi \) and \( \pi_\alpha(u) \leq \pi_\alpha(v) \), then swapping \( u \) and \( v \) does not increase the potential. Let \( \hat{\pi} \) denote the permutation after the swap. Note that \( \pi_\alpha(u) \leq \pi_\alpha(v) \) and \( \pi_\alpha(v) \geq \pi_\alpha(v) \), since \( u \) may only have lost some amount of coverage to \( v \). Suppose this amount is \( k \), i.e. \( k = \pi_\alpha(u) - \pi_\alpha(v) = \pi_\alpha(v) - \pi_\alpha(v) \). Then:

\[
\Phi_\alpha(f, \pi) - \Phi_\alpha(f, \pi) = h(\pi_\alpha(u)) + h(\pi_\alpha(v)) - h(\pi_\alpha(u)) - h(\pi_\alpha(v)) = h(\pi_\alpha(u) - k) + h(\pi_\alpha(v) + k) - h(\pi_\alpha(u)) - h(\pi_\alpha(v))
\]

which is non-positive due to the concavity of \( h \).

It remains to prove property (IV). Suppose we perform a \( \gamma \)-move on a permutation \( \pi \). Let \( u \) be the element moving to some position \( p \) from some position \( q > p \), and let \( \pi' \) denote the permutation after the move. For convenience, also define:

\[
v_i := \pi_\alpha(\pi(i)), \quad \text{(the original coverage of the } i^{th} \text{ set)}
\]

\[
a_i := I_f(\pi_i \mid \pi_{1:i-1}) = \pi_\alpha(\pi_i) - \pi_\alpha'(\pi_i). \quad \text{(the loss in coverage of the } i^{th} \text{ set)}
\]

Note that for all \( i \not\in \{p, \ldots, q\} \), we necessarily have \( a_i = 0 \). Also note that \( \pi_\alpha'(S) = \sum_i a_i \), and by the definition of a \( \gamma \)-move, for any \( j \) we have \( \sum_i a_i \geq \gamma \cdot v_j \). Then the change in potential is:

\[
\Phi_\alpha(f, \pi') - \Phi_\alpha(f, \pi) = \left( \sum_i a_i \right)^\alpha + \sum_i (v_i - a_i) \alpha - \sum_i v_i^\alpha
\]

\[
\leq \left( \sum_i a_i \right)^\alpha - \sum_i a_i \cdot \alpha \cdot v_i^{\alpha-1} \quad \text{(2.3)}
\]
Above, (2.3) holds since \( h \) is concave and thus \( h(v_i - a_i) - h(v_i) \leq \nabla h(v_i) \cdot (-a_i) \). Line (2.4) holds by the definition of a \( \gamma \)-move, since \( \sum_i a_i \geq \gamma v_j \) for every \( j \in \{p, \ldots, q\} \). Finally, (2.5) comes from our choice of \( \alpha = (\ln \gamma)^{-1} \) and the fact that \( \sum_i a_i \geq f_{\min} \).

Finally, we return to proving the main theorem.

**Proof of Theorem 2.1.** By Lemma 2.2, if Algorithm 1 (using Definition 2.1 for \( \mathcal{F}_\pi \)) terminates then it is \( \gamma \cdot (\log f_{\max}/f_{\min} + 1) \)-competitive.

By (I)–(IV), the potential \( \Phi_\alpha \) increases by at most \( g_t(N)(f_{\min})^{\alpha - 1} \) for every function \( g_t \) inserted to the active set, decreases by \( (f_{\min})^\alpha \cdot (\gamma/(e \ln \gamma) - 1) \) per \( \gamma \)-move, and otherwise does not increase. By inspection, \( \Phi_\alpha \geq 0 \). The number of elements \( e \) with \( \mathcal{F}_\pi(e) > 0 \) grows by 1 only during \( \gamma \)-moves in which \( \mathcal{F}_\pi(e) \) was initially 0. Otherwise, this number never grows. We account for elements leaving the solution by paying recourse 2 upfront when they join the solution.

Hence, the number of changes to the solution is at most:

\[
2 \cdot \sum_t g_t(N) \cdot \frac{e \ln \gamma}{(f_{\min})^\alpha (\gamma - e \ln \gamma)} = O \left( \frac{\sum_t g_t(N)}{f_{\min}} \right).
\]

Our algorithm gives a tunable tradeoff between approximation ratio and recourse depending on the choice of \( \gamma \). Note that if we wish to optimize the competitive ratio, setting \( \gamma = e(1 + \delta) \) gives a recourse bound of

\[
\left[ 1 + \frac{\ln(1 + \delta)}{\delta - \ln(1 + \delta)} \right] \sum_t g_t(N)/f_{\min} = O \left( \frac{1}{\delta^2} \right) \sum_t g_t(N)/f_{\min}
\]

as \( \delta \) approaches 0. For simplicity one can set \( \gamma = e^2 \) to get the bound in Theorem 2.1.

## 3 General Cost Submodular Cover

### 3.1 The Algorithm

We now turn to the general costs case and show the main result of our paper:

**Theorem 3.1.** There is a deterministic algorithm that maintains an \( e^2 \cdot (\log f_{\max}/f_{\min} + 1) \)-competitive solution to SubmodularCover in the setting where functions arrive/depart over time. Furthermore, this algorithm has amortized recourse:

\[
O \left( \frac{g(N)}{f_{\min}} \ln \left( \frac{c_{\max}}{c_{\min}} \right) \right)
\]

per function arrival/departure.

Given the last section, our description of the new algorithm is very simple. We reuse Algorithm 1, except we redefine:

\[
\mathcal{F}_\pi(\pi_i) := \frac{f(\pi_i | \pi_{1:i-1})}{e(\pi_i)}.
\]

We will specify the last free parameter \( \gamma \) shortly.
3.2 Bounding the Cost

To bound the competitive ratio, note that Eq. (2.2) did not use any particular properties of $\mathcal{F}_\pi$, except that in the solution output by the algorithm, there are no $\gamma$-moves or swaps with respect to $\mathcal{F}_\pi$ in permutation $\pi$. The analog of Corollary 2.3 is nearly identical:

**Corollary 3.2.** The output at every time step is $\gamma \cdot (\log f^{(t)}_{\text{max}}/f^{(t)}_{\text{min}} + 1)$-competitive.

*Proof.* Lemma 2.2 implies that the solution is equivalent to greedily selecting an element whose marginal coverage/cost ratio is within a factor of $1/\gamma$ of the largest marginal coverage/cost ratio currently available. The standard analyses of the greedy algorithm for SUBMODULARCOVER [Wol82] imply that the solution is $\gamma \cdot (\log f^{(t)}_{\text{max}}/f^{(t)}_{\text{min}} + 1)$ competitive. 

3.3 Bounding the Recourse

To make our analysis as modular as possible, we will write a general potential function, parametrized by a function $h : \mathbb{R} \to \mathbb{R}$:

$$\Phi_h(f, \pi) := \sum_{i \in N} c(\pi_i) \cdot h \left( \mathcal{F}_\pi(\pi_i) \right).$$

With foresight, we require several properties of $h$:

**Properties of $h$:**

(i) $h$ is monotone and concave.

(ii) $h(0) = 0$.

(iii) $h$ satisfies $x \cdot h'(x/\gamma) \geq (1 + \epsilon_\gamma)h(x)$.

(iv) $h$ satisfies $y \cdot h(x/y)$ is monotone in $y$.

To bound the recourse, our goal will again be to show the following properties of our potential function $\Phi_h$:

**Properties of $\Phi_h$:**

(I) $\Phi_h$ increases by at most

$$\frac{g_t(N)}{f^{(t)}_{\text{min}}} \cdot c_{\max} \cdot h \left( \frac{f^{(t)}_{\text{min}}}{c_{\max}} \right)$$

with the addition of function $g_t$ to the active set.

(II) $\Phi_h$ does not increase with deletion of functions from the system.

(III) $\Phi_h$ does not increase during sorting.

(IV) For an appropriate range of $\gamma$, the potential $\Phi_h$ decreases by at least

$$\epsilon_\gamma \cdot c_{\min} \cdot h \left( \frac{f^{(t)}_{\text{min}}}{c_{\min}} \right)$$

with every $\gamma$-move.
Together, the statements imply a total recourse bound of:

\[
\sum_{i} g_t(N) \cdot \frac{c_{\text{max}}}{c_{\text{min}}} \cdot \frac{h(f_{\text{min}}/c_{\text{max}})}{h(f_{\text{min}}/c_{\text{min}})}
\]

**Lemma 3.3.** If \( h \) respects properties (i)-(iv) then \( \Phi_h \) respects properties (I)-(IV).

**Proof.** We start with property (I). When a function \( g_t \) is added to the system, for some set of \( i \in [n] \), it increases \( k_i := \mathcal{F}_\pi(\pi(i)) \) by some amount \( \Delta_i \). Then the potential increase is:

\[
\Phi_h(f^{(t)}, \pi) - \Phi_h(f^{(t-1)}, \pi) = \sum_{i \in [n]} c(\pi_i) \cdot h \left( \frac{k_i + \Delta_i}{c(\pi_i)} \right) - \sum_{i \in [n]} c(\pi_i) \cdot h \left( \frac{k_i}{c(\pi_i)} \right)
\]

\[
\leq \sum_{i \in [n]} c(\pi_i) \cdot h \left( \frac{\Delta_i}{c(\pi_i)} \right) \quad (3.1)
\]

\[
\leq \sum_{i \in [n]} c_{\text{max}} \cdot h \left( \frac{\Delta_i}{c_{\text{max}}} \right) \quad (3.2)
\]

\[
\leq \sum_{i \in [n]} \Delta_i \cdot c_{\text{max}} \cdot h \left( \frac{f_{\text{min}}}{c_{\text{max}}} \right) \quad (3.3)
\]

\[
= \frac{g_t(N)}{f_{\text{min}}} \cdot c_{\text{max}} \cdot h \left( \frac{f_{\text{min}}}{c_{\text{max}}} \right).
\]

Above step (3.1) is by properties (i) and (ii), step (3.2) is by property (iv) and step (3.3) is by property (i).

Property (II) follows since \( h \) is non-decreasing.

The proof of property (III) is similar to the one in Lemma 2.4. Suppose \( u \) immediately precedes \( v \) in \( \pi \) but \( \mathcal{F}_\pi(u) \leq \mathcal{F}_\pi(v) \), and let \( \hat{\pi} \) denote the permutation after the swap. We have that \( \mathcal{F}_\pi(u) \leq \mathcal{F}_\pi(v) \) and \( \mathcal{F}_\pi(v) \geq \mathcal{F}_\pi(v) \), since \( u \) may only have lost some amount of coverage to \( v \). Suppose this amount is \( k \), i.e. \( k = \mathcal{F}_\pi(u) - \mathcal{F}_\pi(u) = \mathcal{F}_\pi(v) - \mathcal{F}_\pi(v) \). Then:

\[
\Phi_h(f, \hat{\pi}) - \Phi_h(f, \pi) = c(u) \cdot h \left( \frac{\mathcal{F}_\pi(u) - k}{c(u)} \right) - h(\mathcal{F}_\pi(u)) - c(v) \cdot h(\mathcal{F}_\pi(v))
\]

\[
= c(u) \left( h \left( \frac{\mathcal{F}_\pi(v)}{c(u)} \right) - h(\mathcal{F}_\pi(u)) \right)
\]

\[
+ c(v) \left( h \left( \frac{\mathcal{F}_\pi(u) + k}{c(v)} \right) - h(\mathcal{F}_\pi(v)) \right)
\]

\[
\leq k \cdot h'(\mathcal{F}_\pi(v)) - h'(\mathcal{F}_\pi(u))
\]

which is non-positive due to the concavity of \( h \) and the fact that \( \mathcal{F}_\pi(v) \geq \mathcal{F}_\pi(u) \).

Finally the proof of property (IV) is also similar to the version in the last section. Suppose we perform a \( \gamma \)-move on a permutation \( \pi \). Let \( u \) be the element moving to some position \( p \) from some position \( q > p \), and let \( \pi' \) denote the permutation after the move. Then:

\[
\Phi_h(f, \pi') - \Phi_h(f, \pi) = c(u) \cdot h \left( \frac{\sum_{i \in [n]} a_i}{c(u)} \right) + \sum_{i \in [n]} c(\pi_i) \cdot h \left( \frac{v_i - a_i}{c(\pi_i)} \right) - \sum_{i \in [n]} c(\pi_i) \cdot h \left( \frac{v_i}{c(\pi_i)} \right)
\]

\[
\leq c(u) \cdot h \left( \frac{\sum_{i \in [n]} a_i}{c(u)} \right) - \sum_{i \in [n]} a_i \cdot h' \left( \frac{v_i}{c(\pi_i)} \right) \quad (3.4)
\]
\[ \leq c(u) \cdot h \left( \sum_{i \in [n]} \frac{a_i}{c(u)} \right) - c(u) \sum_{i \in [n]} \frac{a_i}{c(u)} \cdot h' \left( \frac{\sum_{i \in [n]} a_i}{\gamma \cdot c(u)} \right) \]  
(3.5)

\[ \leq c(u) \cdot h \left( \sum_{i \in [n]} \frac{a_i}{c(u)} \right) - (1 + \epsilon_{\gamma}) \cdot c(u) \cdot h \left( \frac{\sum_{i \in [n]} a_i}{c(u)} \right) \]  
(3.6)

\[ \leq -\epsilon_{\gamma} \cdot c(u) \cdot h \left( \frac{\sum_{i \in [n]} a_i}{c(u)} \right) \]  
(3.7)

Above step (3.4) uses the concavity of \( h \). Step (3.5) uses the fact that moving \( u \) was a legal \( \gamma \)-move and thus \( \sum_{i \in [n]} a_i / c(u) \geq \gamma v_i / c(\pi_i) \). Step (3.6) follows from property (iii). Finally, (3.7) uses property (iv) again.

With this setup, the proof of the main theorem boils down to identifying an appropriate function \( h \), and a suitable choice of \( \gamma \).

**Proof of Theorem 3.1.** Recall the general recourse bound is

\[ \sum_{t=1}^{T} g_t(N) \epsilon_{\gamma} \cdot f_{\min} / c_{\min} \cdot h(f_{\min} / c_{\max}) / h(f_{\min} / c_{\min}) \]

A good choice for \( h \) is \( h(x) = x^{1-\delta} / (1-\delta) \) for \( \delta = (\ln(c_{\max} / c_{\min}) + 1)^{-1} \), and with \( \gamma = e^2 \). Properties (i)–(iv) are easy to verify. To see that this implies the bound

\[ O \left( \frac{\sum_{t} g_t(N)}{f_{\min}} \ln \left( \frac{c_{\max}}{c_{\min}} \right) \right) , \]

note that \( \gamma = e^2 \geq (1 + \delta) / (1 - \delta) \)\(^{1/\delta} \), in which case \( \epsilon_{\gamma} = \gamma^{\delta - 1} - 1 \geq \delta \). Furthermore, \( (c_{\max} / c_{\min})^\delta = O(1) \).

### 4 Improved bounds for 3-increasing functions

In this section we show to remove the \( \log(c_{\max} / c_{\min}) \) dependence from the recourse bound when \( f \) is assumed to be from a structured class of set functions: the class of 3-increasing functions. We start with some definitions.

#### 4.1 Higher Order Monotonicity of Boolean Functions

Following the notation of [FH05], we define the *derivative of a set function* \( f \) as:

\[ \frac{df}{dx}(S) = f(S \cup \{x\}) - f(S \setminus \{x\}) \]

We notate the \( m \)-th order derivative of \( f \) with respect to the subset \( A = \{i_1, \ldots, i_m\} \) as \( df/dA \), and this quantity has the following concise expression:

\[ \frac{df}{dA}(S) = \sum_{B \subseteq A} (-1)^{|B|} f((S \cup A) \setminus B) \].

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Definition 4.1 ($m$-increasing [FH05]). We say that a set function is $m$-increasing if all its $m$-th order derivatives are nonnegative, and $m$-decreasing if they are nonpositive. We denote by $D_m^+$ and $D_m^-$ the classes of $m$-increasing and $m$-decreasing functions respectively.

Note that $D_1^+$ is the class of monotone set functions, and $D_2^-$ is the class of submodular set functions. When $f$ is the joint entropy set function, the $m$-th derivative above is also known as the interaction information, which generalizes the usual mutual information for two sets of variables, to $m$ sets of variables.

We will soon show improved algorithms for the class $D_3^+$, that is the class of 3-increasing set functions. The following derivation gives some intuition for these functions:

\[
\frac{df}{d\{x,y,z\}}(S) = \sum_{B \subseteq \{x,y,z\}} (-1)^{|B|} f((S \cup \{x,y,z\}) \setminus B)
\]

\[
= f(x \mid S) - f(x \mid S \cup \{y\}) - (f(x \mid S \cup \{y\}) - f(x \mid S \cup \{y,z\}))
\]

\[
= I_f(x, y \mid S) - I_f(x, y \mid S \cup \{z\}). \quad (4.1)
\]

Thus a function $f$ is contained in $D_3^+$ if and only if mutual coverage decreases after conditioning.

Bach [Bac13, Section 6.3] shows that a nonnegative function is in $D_{2m-1}^+ \cap D_{2m}^-$ simultaneously for all $m \geq 1$ if and only if it is a measure coverage function: each element $i \in \mathcal{N}$ is associated with some measurable set $S_i$ under a measure $\mu$, and $f(S_1, \ldots, S_t) = \mu(\bigcup_{i=1}^t S_i)$.

4.2 The Algorithm

We now show our second main result:

**Theorem 4.2.** There is a deterministic algorithm that maintains an $O(\log f(N)/f_{\min})$-competitive solution to SubmodularCover in the fully-dynamic setting where functions arrive/depart over time, and each function is $3$-increasing in addition to monotone and submodular. Furthermore, this algorithm has total recourse:

\[
O \left( \sum_t g_t(N) \frac{1}{f_{\min}} \right).
\]

We reuse Algorithm 1 from previous sections, only this time we redefine $\mathcal{F}_\pi$ more substantially. Given two permutations $\alpha$ and $\beta$ on $\mathcal{N}$, we define the $(i, j)$ mutual affinity of $(\alpha, \beta)$ (and its conditional variant) as

\[
\mathcal{I}_{\alpha,\beta}(\alpha_i, \beta_j) := I_f(\alpha_i, \beta_j \mid \alpha_{1:i-1} \cup \beta_{1:j-1}),
\]

\[
\mathcal{I}_{\alpha,\beta}(\alpha_i, \beta_j \mid S) := I_f(\alpha_i, \beta_j \mid \alpha_{1:i-1} \cup \beta_{1:j-1} \cup S).
\]

Recall that $I_f$ denotes the Mutual Coverage. To give some insight into these definitions, observe that mutual affinity telescopes cleanly:

**Observation 4.3.** The chain rule implies that

\[
\sum_{j \in [n]} \mathcal{I}_{\alpha,\beta}(\alpha_i, \beta_j) = f(\alpha_i \mid \alpha_{1:i-1}),
\]

\footnote{We avoid the term set coverage function used by [Bac13], since we already use this terminology to mean the special case of a counting measure defined by a finite set system, as in SetCover.}
we observe that:

\[
\sum_{i \in [n]} J_{\alpha,\beta}(\alpha_i, \beta_j) = f(\beta_j | \beta_{1:j-1}), \\
\sum_{i,j \in [n]} J_{\alpha,\beta}(\alpha_i, \beta_j) = f(\mathcal{N}).
\]

Let \( \psi \) denote the ordering of \( \mathcal{N} \) in increasing order of cost. Then:

\[
\tilde{F}_\pi(\pi_i) := \sum_{j \in [n]} \frac{J_{\pi,\psi}(\pi_i, \psi_j)}{c(\pi_i) \cdot c(\psi_j)}.
\]

(4.2)

The following observation gives some intuition for this definition.

**Observation 4.4.** The expression \( J_{\alpha,\beta}(\alpha_i, \beta_j) \) is nonzero only if (a) element \( \alpha_i \) precedes element \( \beta_j \) in permutation \( \alpha \), and also (b) element \( \beta_j \) precedes element \( \alpha_i \) in permutation \( \beta \).

In light of these remarks, \( \tilde{F}_\pi(\pi_i) \) decomposes the marginal coverage of \( \pi_i \) into its mutual affinity with all the elements \( \psi_j \in \mathcal{N} \) that are simultaneously cheaper than \( \pi_i \) and that follow \( \pi_i \) in the permutation \( \pi \), and weights each of these affinities by \( (c(\pi_i)c(\psi_j))^{-1} \).

With this re-definition of \( \tilde{F}_\pi \), Algorithm 1 is fully specified (though it is still parametrized by \( \gamma \)). We move to proving formal guarantees.

### 4.3 Bounding the Cost

Since our definition of \( \tilde{F}_\pi \) (and thus the behavior of the algorithm) has significantly changed, we need to reprove the competitive ratio guarantee. Our goal will be the following Lemma:

**Lemma 4.5.** If no swaps or \( \gamma \)-moves are possible, the solution is \( \gamma^2 \cdot \log(f(t)^{(N)}/f_{\min}^{(t)}) \)-competitive.

For the remainder of Section 4.3, we drop the superscript and refer to \( f(t) \) as simply \( f \).

Define a level \( \mathcal{L}_\ell \) be the collection of elements \( u \) such that \( \tilde{F}_\pi(u) \in [\gamma^\ell, \gamma^{\ell+1}) \). Note that since no swaps were possible, permutation \( \pi \) is sorted in decreasing order of \( \tilde{F}_\pi \), and thus \( \mathcal{L}_\ell \) forms a contiguous interval of indices in \( \pi \). Our proof strategy will be to show that for any level \( \ell \), the total cost of all elements in \( \mathcal{L}_\ell \) is at most \( \gamma^2 \cdot c(\text{OPT}) \). Subsequently, we will argue that there are at most \( O(\log f(N)/f_{\min}) \) non-trivial levels.

**Lemma 4.6 (Each Level is Inexpensive).** If there are no swaps or \( \gamma \)-moves for \( \pi \), then for any \( \ell > 0 \), the total cost of all elements in \( \mathcal{L}_\ell \) is at most \( \gamma^2 \cdot c(\text{OPT}) \).

**Proof.** Suppose some level \( \mathcal{L}_\ell \) has cost \( c(\mathcal{L}_\ell) \geq \gamma^2 \cdot c(\text{OPT}) \). We will argue that in this case there must be a legal \( \gamma \)-move available. To start, using Fact 1.3 we observe that:

\[
\gamma^\ell \leq \min_{u \in \mathcal{L}_\ell} \tilde{F}_\pi(u) \stackrel{(\text{def})}{=} \min_{u \in \mathcal{L}_\ell} \frac{\sum_{j \in [n]} J_{\pi,\psi}(u, \psi_j)/c(\psi_j)}{c(u)} \leq \frac{\sum_{u \in \mathcal{L}_\ell} \sum_{j \in [n]} J_{\pi,\psi}(u, \psi_j)/c(\psi_j)}{\sum_{u \in \mathcal{L}_\ell} c(u)},
\]

which by rearranging means:

\[
\sum_{u \in \mathcal{L}_\ell} \sum_{j \in [n]} \frac{J_{\pi,\psi}(u, \psi_j)}{c(\psi_j)} \geq \gamma^\ell \cdot c(\mathcal{L}_\ell) > \gamma^{\ell+2} \cdot c(\text{OPT}).
\]

(4.3)

Let \( \pi_< \) be the set of items preceding \( \mathcal{L}_\ell \) in \( \pi \), and \( \pi_> \) be the set of items in or succeeding \( \mathcal{L}_\ell \). We also define \( \text{OPT}_\geq := \text{OPT} \cap \pi_> \).
First, we imagine moving all the elements of \( \text{OPT}_{\geq} \), simultaneously and in order according to \( \pi \), to positions just before \( \mathcal{L}_\ell \). Let \( \pi' \) be this new permutation. We first show that some \( o \in \text{OPT}_{\geq} \) has a high value \( \mathcal{F}_{\pi'}(o) \) after this move. Then we show that this element \( o \) also constitutes a potential \( \gamma \)-move in \( \pi \), a contradiction.

For the first claim, we start by using Fact 1.3 again:

\[
\max_{o \in \text{OPT}_{\geq}} \mathcal{F}_{\pi'}(o) \geq \sum_{o \in \text{OPT}_{\geq}} \sum_{j \in [n]} \mathcal{F}_{\pi',\psi}(o, \psi_j | \pi_{<}) / c(\psi_j) \\
\geq \sum_{o \in \text{OPT}_{\geq}} \sum_{j \in [n]} \mathcal{F}_{\pi,\psi}(u, \psi_j | \pi_{<}) / c(\psi_j) \\
= \sum_{j \in [n]} \mathcal{F}_{\pi,\psi}(u, \psi_j | \pi_{<}) / c(\psi_j) \\
\geq \frac{1}{c(\text{OPT}_{\geq})} \cdot \sum_{j \in [n]} \mathcal{F}_{\pi,\psi}(u, \psi_j | \pi_{<}) / c(\psi_j) \\
> \gamma^{\ell+2},
\]

The lines (4.4) and (4.5) above follow by Observation 4.3 (note that (4.5)) is an inequality because \( \mathcal{L}_\ell \) may be a strict subset of \( \pi_{\geq} \). Since \( u \in \mathcal{L}_\ell \subseteq \pi_{\geq} \), this is:

\[
\mathcal{F}_{\pi'}(o) = \sum_{j \in [n]} \mathcal{F}_{\pi',\psi}(o, \psi_j | \pi_{<}) / c(\psi_j) \\
\geq \sum_{u \in \mathcal{L}_\ell} \sum_{j \in [n]} \mathcal{F}_{\pi,\psi}(u, \psi_j | \pi_{<}) / c(\psi_j) \\
= \frac{1}{c(\text{OPT}_{\geq})} \cdot \sum_{j \in [n]} \mathcal{F}_{\pi,\psi}(u, \psi_j | \pi_{<}) / c(\psi_j) \\
> \gamma^{\ell+2},
\]
\[
\sum_{j \in [n]} \left( J_{\pi',\psi}(o, \psi_j) + (J_{\pi'',\psi}(o, \psi_j) - J_{\pi'',\psi}(o, \psi_j | \{d\})) \right) c(o) c(\psi_j) \\
\geq \sum_{j \in [n]} \left( J_{\pi',\psi}(o, \psi_j) \right) c(o) c(\psi_j),
\]

where the second equality used \( J_{\pi',\psi}(o, \psi_j) = J_{\pi'',\psi}(o, \psi_j | \{d\}) \), and where the inequality used equation (4.1), that \( I_f(a, b) - I_f(a, b | c) \geq 0 \) for 3-increasing functions. Repeating this process inductively until all elements but \( o \) have been returned to their original positions in \( \pi \) yields the permutation \( \pi^o \), which proves the claim.

Now we can complete the proof of Lemma 4.6. This means that if \( \mathfrak{S}_{\pi'}(o) \geq \gamma^L + 2 \), then there is a legal \( \gamma \)-move (namely the one which moves \( o \) ahead of \( L_i \)), because by assumption every \( u \in \pi_o \) has \( \mathfrak{S}_{\pi}(u) \leq \gamma^L + 1 \). This contradicts the assumption that none existed.

Next, we argue that the cost of all elements with very high or very low values of \( \mathfrak{S}_{\pi} \) is small.

**Lemma 4.8** (Extreme Values Lemma). If there are no swaps or \( \gamma \)-moves for \( \pi \), the following hold:

(i) There are no elements \( u \) such that \( 0 < \mathfrak{S}_{\pi}(u) \leq f_{\min} / (\gamma \cdot (c(\text{OPT}))^2) \).

(ii) The total cost of all elements \( u \) such that \( \mathfrak{S}_{\pi}(u) \geq (f(N))^2 / (f_{\min} \cdot (c(\text{OPT}))^2) \) is at most \( \sqrt{\gamma} \cdot \text{OPT} \).

**Proof of Lemma 4.8.** To prove item (i), we observe that if permutation \( \pi \) has no local moves, then every element \( u \) must have high enough \( \mathfrak{S}_{\pi}(u) \) to prevent any elements that follow it from cutting it in line.

**Claim 4.9.** If there are no swaps or \( \gamma \)-moves for \( \pi \), then for every \( \pi_i \in \mathcal{N} \), and every \( j \in [n] \) such that \( J_{\pi,\psi}(\pi_i, \psi_j) > 0 \) we have:

\[
\mathfrak{S}_{\pi}(\pi_i) \geq f(\psi_j | \pi_{1:i-1}) / \gamma \cdot (c(\psi_j))^2.
\]

**Proof.** If there are no swaps or \( \gamma \) moves, then \( \mathfrak{S}_{\pi'}(\psi_j) < \gamma \cdot \mathfrak{S}_{\pi_i} \) (where \( \pi' \) is the permutation obtained from \( \pi \) by moving the element \( \psi_j \) to the position ahead of \( u \)). Expanding definitions:

\[
\mathfrak{S}_{\pi'}(\psi_j) \overset{\text{def}}{=} \sum_{j' \in [n]} J_{\pi',\psi}(\psi_j, \psi_{j'}) c(\psi_j) c(\psi_{j'}) \geq \sum_{j' \in [n]} J_{\pi',\psi}(\psi_j, \psi_{j'}) c(\psi_j) c(\psi_{j'}) \overset{\text{factors}}{=} f(\psi_j | \pi_{1:i-1}) / (c(\psi_j))^2.
\]

In the first inequality we used that \( c(\psi_{j'}) \leq c(\psi_j) \) by Observation 4.4, in the second equality we used Observation 4.3. Rearranging terms yields the claim.

Now item (i) follows by setting \( \psi_j \) to be the cheapest element that succeeds \( u \) in the permutation. Note that \( c(\psi_j) \leq c(\text{OPT}) \).

For item (ii), let \( S \) be the set of indices \( i \) with \( \mathfrak{S}_{\pi}(\pi_i) \geq (f(N))^2 / (f_{\min} \cdot (c(\text{OPT}))^2) \). Then by Fact 1.3:

\[
\frac{f(N)}{c(\text{OPT}) \sqrt{f_{\min}}} \leq \min_{i \in S} \sqrt{\mathfrak{S}_{\pi}(\pi_i)} = \min_{i \in S} \sqrt{\mathfrak{S}_{\pi}(\pi_i)} \overset{\text{def}}{=} \left( \frac{1}{c(\pi_i)} \sum_{j \in [n]} J_{\pi,\psi}(\pi_i, \psi_j) \right) c(\psi_j) \sqrt{\mathfrak{S}_{\pi}(\pi_i)} \overset{\text{factors}}{=} \left( \frac{1}{c(S)} \sum_{i \in S} \sum_{j \in [n]} J_{\pi,\psi}(\pi_i, \psi_j) \right) c(\psi_j) \sqrt{\mathfrak{S}_{\pi}(\pi_i)} \overset{\text{bound}}{=} \left( \frac{1}{c(S)} \sum_{i \in S} \sum_{j \in [n]} J_{\pi,\psi}(\pi_i, \psi_j) \right). \quad (4.8)
\]

\[
\frac{f(N)}{c(\text{OPT}) \sqrt{f_{\min}}} \leq \min_{i \in S} \sqrt{\mathfrak{S}_{\pi}(\pi_i)} = \min_{i \in S} \sqrt{\mathfrak{S}_{\pi}(\pi_i)} \overset{\text{def}}{=} \left( \frac{1}{c(\pi_i)} \sum_{j \in [n]} J_{\pi,\psi}(\pi_i, \psi_j) \right) c(\psi_j) \sqrt{\mathfrak{S}_{\pi}(\pi_i)} \overset{\text{factors}}{=} \left( \frac{1}{c(S)} \sum_{i \in S} \sum_{j \in [n]} J_{\pi,\psi}(\pi_i, \psi_j) \right) c(\psi_j) \sqrt{\mathfrak{S}_{\pi}(\pi_i)} \overset{\text{bound}}{=} \left( \frac{1}{c(S)} \sum_{i \in S} \sum_{j \in [n]} J_{\pi,\psi}(\pi_i, \psi_j) \right). \quad (4.9)
\]
Rearranging to bound the cost:

\[
c(S) \leq \frac{c(\text{OPT})\sqrt{f_{\text{min}}}}{f(N)} \cdot \sum_{i \in S} \sum_{j \in [n]} \frac{\mathcal{I}_{\pi,\psi}(\pi_i, \psi_j)}{c(\psi_j)\sqrt{\mathcal{F}_\pi(\pi_i)}}
\]

\[
\leq \sqrt{\gamma} \cdot c(\text{OPT})\sqrt{f_{\text{min}}} \cdot \sum_{i \in S} \sum_{j \in [n]} \frac{\mathcal{I}_{\pi,\psi}(\pi_i, \psi_j)}{\sqrt{f_{\min}}}
\]

\[
\leq \sqrt{\gamma} \cdot c(\text{OPT})\sqrt{f_{\text{min}}} \cdot \sum_{i \in S} \sum_{j \in [n]} \mathcal{I}_{\pi,\psi}(\pi_i, \psi_j)
\]

\[
= \sqrt{\gamma} \cdot c(\text{OPT}).
\]

Above, step (4.10) is due to Claim 4.9 again and noting that if the denominator is 0 then the numerator is also 0, and step (4.11) is due to Observation 4.3.

Using these ingredients, we can now wrap up the proof of the competitive ratio.

**Proof of Lemma 4.5.** The number of levels \( \ell \) such that \( \gamma^\ell \) lies between \( f_{\text{min}}/(\gamma(c(\text{OPT}))^2) \) and \( (f(N))^2/(f_{\text{min}} \cdot (c(\text{OPT}))^2) \) is \( O(\log_\gamma(f(N)/f_{\text{min}})^2) = O(\log(f(N)/f_{\text{min}})). \) By Lemma 4.6, each level in this range has cost at most \( O(c(\text{OPT})). \) By Lemma 4.8, there are no elements with non-zero coverage in levels below this range, and the total cost of all elements above this range is \( O(c(\text{OPT})). \) Thus the total cost of the solution is \( O(\log(f(N)/f_{\text{min}})) \cdot c(\text{OPT}). \)

### 4.4 Bounding the Recourse

We follow our recipe of using the modified Tsallis entropy as a potential (with a reminder that the underlying definitions of \( \mathcal{F}_\pi \) have changed) with \( \alpha \) fixed to 1/2:

\[
\Phi_{1/2}(\pi) := \sum_{i=1}^{n} c(\pi_i)\sqrt{\mathcal{F}_\pi(\pi_i)}.
\]

It is worthwhile to interpret these quantities in the context of HypergraphVertexCover. In this case, \( \psi_i \) and \( \psi_j \) correspond to vertices. Let \( \Gamma(v) \) (and \( \Gamma(V) \)) denote the edge-neighborhood of \( v \) (and the union of the edge neighborhoods of vertices in the set \( V \)), then

\[
\mathcal{I}_{\pi,\psi}(\pi_i, \psi_j) = |[\Gamma(\pi_i) \cap \Gamma(\psi_j)] \setminus [\Gamma(\pi_{1:i-1}) \cup \psi_{1:j-1})]|,
\]

and if we use \( c(e) \) to denote the cost of the cheapest vertex hitting edge \( e \), we can simplify

\[
c(\pi_i) \cdot \sqrt{\mathcal{F}_\pi(\pi_i)} := \sqrt{\sum_{e \in \Gamma(\pi_i) \setminus \Gamma(\pi_{1:i-1})} \frac{c(\pi_i)}{c(e)}}.
\]

In words, we are reweighting each hyperedge by the ratio of costs between the current vertex covering it, and its cheapest possible vertex that could cover it. Intuitively, this means the potential will be high when many elements are in sets that are significantly more expensive than the cheapest sets they could lie in.

This time the properties of \( \Phi_{1/2} \) we show are:
$\Phi_{1/2} = \sum \sqrt{\sum \frac{c(\pi_i)}{c(\psi_1)} \cdot J_\pi(\pi_i; \psi_1) + \frac{c(\pi_i)}{c(\psi_2)} \cdot J_\pi(\pi_i; \psi_2) + \frac{c(\pi_i)}{c(\psi_3)} \cdot J_\pi(\pi_i; \psi_3) + \frac{c(\pi_i)}{c(\psi_4)} \cdot J_\pi(\pi_i; \psi_4) + \ldots}$

**Figure 3:** Illustration of $\Phi_{1/2}$. Elements are arranged in order of $\pi$.

---

**Properties of $\Phi_{1/2}$:**

(I) $\Phi_{1/2}$ increases by at most $g(N)/\sqrt{f_{\text{min}}}$ with every addition of a function to the system.

(II) $\Phi_{1/2}$ does not increase with deletion of functions from the system.

(III) $\Phi_{1/2}$ does not increase during swaps.

(IV) If $\gamma > 4$, then $\Phi_{1/2}$ decreases by at least $\Omega(\sqrt{f_{\text{min}}})$ with every $\gamma$-move.

Together, these will yield a recourse bound of $\sum t g_t(N)/f_{\text{min}}$.

**Lemma 4.10.** $\Phi_{1/2}$ satisfies property (I).

**Proof.** Consider a step in which $f$ changes to $f'$ because the function $g$ was added to the system. For convenience, let $\widehat{J}_\pi$ and $\widehat{J}_{\pi,\psi}$ denote the quantities $\widetilde{J}_\pi$ and $J_{\pi,\psi}$ after $g$ has been added. Then the potential increase is:

$$
\Phi_{1/2}(f(t), \pi) - \Phi_{1/2}(f(t-1), \pi) = \sum_{i \in [n]} c(\pi_i) \frac{\widehat{J}_\pi(\pi_i)}{\sqrt{\widehat{J}_\pi(\pi_i)}} - \sum_{i \in [n]} c(\pi_i) \frac{\widetilde{J}_\pi(\pi_i)}{\sqrt{\widetilde{J}_\pi(\pi_i)}}
$$

$$
= \sum_{i \in [n]} \sum_{j \in [n]} c(\psi_j) \frac{\widehat{J}_{\pi,\psi}(\pi_i, \psi_j)}{\sqrt{\widehat{J}_\pi(\pi_i)}} - \sum_{i \in [n]} \sum_{j \in [n]} c(\psi_j) \frac{J_{\pi,\psi}(\pi_i, \psi_j)}{\sqrt{\widetilde{J}_\pi(\pi_i)}}
$$

Since $\widehat{J}_\pi(\pi_i)$ can only increase when a function $g$ is added to the active set, this is:

$$
\leq \sum_{i \in [n]} \sum_{j \in [n]} \frac{\widehat{J}_{\pi,\psi}(\pi_i, \psi_j) - J_{\pi,\psi}(\pi_i, \psi_j)}{c(\psi_j) \sqrt{\widehat{J}_\pi(\pi_i)}}
$$

$$
= \sum_{j \in [n]} g(\psi_j) \psi_{j-1} \left( \sum_{j' \in [n]} \frac{c(\psi_j) J_{\pi,\psi}(\psi_j, \psi_{j'})}{c(\psi_{j'})} \right)^{-1}
$$

(4.12)
4.12, for all indices (III) (IV) 4.14

4.13

4.4 these terms telescopes such that:

\[ \frac{g(N)}{\sqrt{\min}}. \]

Step (4.12) uses the relationship \( \mathcal{H}_\pi(p_i) \geq \mathcal{H}_\pi(p_j) \), which holds because summands are nonzero only if \( p_j \) succeeds \( p_i \) in permutation \( \pi \) (by Observation 4.4), or the numerator is 0, and the fact that elements are sorted in decreasing order of \( \mathcal{H}_\pi \). Step (4.13) uses \( c(p_j') \leq c(p_j) \), also by Observation 4.4. Finally step (4.14) uses Observation 4.3.

**Lemma 4.11.** \( \Phi_{1/2} \) satisfies property (III).

**Proof.** Consider an index \( i \) such that swapping \( p_i \) and \( p_{i+1} \) increases the potential. Then for some quantity \( \delta := \sum_{j \in [n]} (\mathcal{I}_{\pi,\psi}(p_i, p_j) - \mathcal{I}_{\pi',\psi}(p_i, p_j)) / c(j) > 0 \) (since \( f \in D_3^d \)) we have:

\[
0 < \Delta \Phi_{1/2} \\
0 < c(p_{i+1}) \left( \sqrt{\mathcal{H}_\pi(p_{i+1}) + \frac{\delta}{c(p_{i+1})}} - \sqrt{\mathcal{H}_\pi(p_{i+1})} \right) + c(p_i) \left( \sqrt{\mathcal{H}_\pi(p_i)} - \frac{\delta}{c(p_i)} - \sqrt{\mathcal{H}_\pi(p_i)} \right) \\
0 < \frac{\delta}{2\sqrt{\mathcal{H}_\pi(p_{i+1})}} - \frac{\delta}{2\sqrt{\mathcal{H}_\pi(p_i)}} \Rightarrow \mathcal{H}_\pi(p_{i+1}) \leq \mathcal{H}_\pi(p_i). \]

Above, (4.15) holds since square root is a concave function and thus \( \sqrt{a+b} - \sqrt{a} \leq b/(2\sqrt{a}) \). This implies that the local move was not a legal swap.

**Lemma 4.12.** If \( \gamma > 4 \), then \( \Phi_{1/2} \) satisfies property (IV).

**Proof.** Suppose the local move changes the permutation \( \pi \) to \( \pi' \) by moving element \( u \) from position \( q \) to \( p \). For notational convenience, define the following quantities:

\[
v_i := \mathcal{H}_\pi(p_i), \\
a_{ij} := \mathcal{I}_{\pi}(p_i, p_j) - \mathcal{I}_{\pi'}(p_i, p_j) \overset{\text{def}}{=} 1[p \leq i \leq q] \cdot (\mathcal{I}_{\pi}(p_i, p_j) - \mathcal{I}_{\pi}(p_i, p_j | \{u\})�.

Recall that by (4.1), the quantity \( a_{ij} \) is (crucially) nonnegative when \( f \) is 3-increasing. Also, by expanding the definition of Mutual Coverage, for all indices \( i \in \{p, \ldots, q\} \) we can rewrite \( a_{ij} \) as:

\[
a_{ij} = f(p_j | p_{1:i-1} \cup p_{1:j-1}) - f(p_j | p_{1:i} \cup p_{1:j-1}) \\
- f(p_j | p_{1:i-1} \cup p_{1:j-1} \cup \{u\}) + f(p_j | p_{1:i} \cup p_{1:j-1} \cup \{u\}) \\
= \mathcal{I}_f(u, p_j | p_{1:i-1} \cup p_{1:j-1}) - \mathcal{I}_f(u, p_j | p_{1:i} \cup p_{1:j-1}).
\]

Thus, by the Chain Rule these terms telescopes such that:

\[
\sum_{i \in \{p, \ldots, q\}} a_{ij} = \mathcal{I}_f(u, p_j | p_{1:p-1}, p_{1:j-1}) \overset{\text{def}}{=} \mathcal{I}_{\pi',\psi}(u, p_j).
\]

With this, we can bound the potential change after a local move as

\[
\Phi_{1/2}(f, \pi') - \Phi_{1/2}(f, \pi)
\]
\[\begin{align*}
&= c(u) \sqrt{\tilde{f}_{\pi'}(u)} + \sum_{i \in [n]} c(\pi_i) \sqrt{\nu_i - \sum_{j \in [n]} \frac{a_{ij}}{c(\pi_i)c(\psi_j)}} - \sum_{i \in [n]} c(\pi_i) \sqrt{\nu_i} \\
&\leq c(u) \sqrt{\tilde{f}_{\pi'}(u)} - c(\pi_i) \sum_{i \in [n]} \frac{1}{2 \sqrt{\nu_i}} \cdot \sum_{j \in [n]} \frac{a_{ij}}{c(\pi_i)c(\psi_j)} \\
&\leq c(u) \sqrt{\tilde{f}_{\pi'}(u)} - \frac{\sqrt{\gamma}}{2} \cdot \frac{c(\pi_i)}{\sqrt{\tilde{f}_{\pi'}(u)}} \cdot \sum_{i \in [n]} \sum_{j \in [n]} \frac{a_{ij}}{c(\pi_i)c(\psi_j)} \\
&= c(u) \sqrt{\tilde{f}_{\pi'}(u)} - \frac{\sqrt{\gamma}}{2} \cdot \frac{c(u)}{\sqrt{\tilde{f}_{\pi'}(u)}} \cdot \sum_{j \in [n]} \frac{\mathcal{J}_{\pi',\psi}(u, \psi_j)}{c(\pi_i)c(\psi_j)} \\
&\quad \text{(def)} \\
&\quad \equiv - \left( \frac{\sqrt{\gamma}}{2} - 1 \right) \sqrt{\sum_{j \in [n]} \frac{c(u) \cdot \mathcal{J}_{\pi',\psi}(u, \psi_j)}{c(\psi_j)}} \\
&\leq - \left( \frac{\sqrt{\gamma}}{2} - 1 \right) \sqrt{\sum_{j \in [n]} \mathcal{J}_{\pi',\psi}(u, \psi_j)} \\
&\quad \text{(4.18)} \\
&\leq - \left( \frac{\sqrt{\gamma}}{2} - 1 \right) \sqrt{f_{\min}}. \quad \text{(4.19)}
\end{align*}\]

Above, step (4.16) holds since square root is a concave function and thus \(\sqrt{a + b} - \sqrt{a} \leq b/(2\sqrt{a})\). The next step (4.17) is due to the definition of \(\gamma\)-moves which ensure that \(\nu_i \leq \tilde{f}_{\pi'}(u)/\gamma\) (note that this step also makes use of the nonnegativity of \(a_{ij}\)). Using the telescoping property of the \(a_{ij}\), we can continue:

\[\begin{align*}
&= c(u) \sqrt{\tilde{f}_{\pi'}(u)} - \frac{\sqrt{\gamma}}{2} \cdot \frac{c(u)}{\sqrt{\tilde{f}_{\pi'}(u)}} \cdot \sum_{j \in [n]} \mathcal{J}_{\pi',\psi}(u, \psi_j) \\
&\quad \text{(def)} \\
&\equiv - \left( \frac{\sqrt{\gamma}}{2} - 1 \right) \sqrt{\sum_{j \in [n]} \frac{c(u) \cdot \mathcal{J}_{\pi',\psi}(u, \psi_j)}{c(\psi_j)}} \\
&\leq - \left( \frac{\sqrt{\gamma}}{2} - 1 \right) \sqrt{\sum_{j \in [n]} \mathcal{J}_{\pi',\psi}(u, \psi_j)} \\
&\quad \text{(4.18)} \\
&\leq - \left( \frac{\sqrt{\gamma}}{2} - 1 \right) \sqrt{f_{\min}}. \quad \text{(4.19)}
\end{align*}\]

Step (4.18) comes from Observation 4.4, and finally step (4.19) follows by Observation 4.3 and the fact that \(f_{\min}\) is a lower bound on marginal coverage for any element with nonzero marginal coverage. \(\square\)

We wrap up with the proof of the main theorem.

\textit{Proof of Theorem 4.2.} Set \(\gamma = 5 > 4\). By Lemma 4.5, if Algorithm 1 (using Definition 4.2 for \(\tilde{f}_{\pi}\)) terminates then it is \(O(\log f(N)/f_{\min})\)-competitive.

By (I)–(IV), the potential \(\Phi_{1/2}\) increases by at most \(g_t(N)/\sqrt{f_{\min}}\) for every function \(g_t\) inserted to the active set, decreases by \(\sqrt{f_{\min}} \cdot (\sqrt{\gamma}/2 - 1)\) per \(\gamma\)-move, and otherwise does not increase. By inspection, \(\Phi_{\alpha} \geq 0\). The number of elements \(e\) with \(\tilde{f}_{\pi}(e) > 0\) grows by 1 only during \(\gamma\)-moves in which \(\tilde{f}_{\pi}(e)\) was initially 0. Otherwise, this number never grows. We account for elements leaving the solution by paying recourse 2 upfront when they join the solution.

Hence, the number of changes to the solution is at most:

\[2 \cdot \frac{\sum_t g_t(N)}{\sqrt{f_{\min}}} \cdot \frac{2}{\sqrt{f_{\min}(\sqrt{\gamma} - 2)}} = O\left(\frac{\sum_t g_t(N)}{f_{\min}}\right).\] \(\square\)

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Algorithm for $r$-bounded instances

We can achieve a better approximation ratio if each function $g$ is an $r$-junta for small $r$. Recall that an $r$-junta is a function that depends on at most $r$ variables. In this section we prove the theorem:

**Theorem A.1.** There is a randomized algorithm that maintains an $r$-competitive solution in expectation to Fully Dynamic SUBMODULARCOVER in the setting where functions arrive/depart over time, and these functions are each $r$-juntas. Furthermore this algorithm has total recourse:

$$\sum_{t} g_t(N) / f_{\min}.$$
Our proof follows the framework of [GKKP17], which is itself a dynamic implementation of the algorithm of [Pit85].

We start with some notation. Let \( V_g = \{ u \in \mathcal{N} \mid g(u) \neq 0 \} \) be the elements influencing \( g \). Our assumption says that \( |V_g| \leq r \). Let \( S \) be the solution maintained by the algorithm. Each function \( g \in G^{(t)} \) maintains a set \( U_g \subseteq V_g \) of elements assigned to it. We say that \( g \) is responsible for \( U_g \). We also define the following operation:

**Probing a function.** Sample one element \( u \in V_g \setminus S \) with probability:

\[
\frac{1}{c(u)} \frac{1}{\sum_{v \in V_g \setminus S} c(v)}.
\]

Add the sampled element \( u \) to the current solution \( S \), and to \( U_g \).

Given these definitions, we are ready to explain the dynamic algorithm.

**Function arrival.** When a function \( g \) arrives, initialize its element set \( U_g \) to \( \emptyset \). Then, while \( g(S) \neq g(\mathcal{N}) \), probe \( g_t \).

**Function departure.** When a function \( g \) departs, remove all its assigned elements \( U_g \) from \( S \). This may leave some set of functions \( g_1, \ldots, g_s \) uncovered. For each of these functions \( g_t \) in order of arrival, while \( g_t(S) \neq g(\mathcal{N}) \), probe \( g_t \).

The \( \sum g_t(\mathcal{N})/f_{\min} \) recourse bound is immediate, since the total number of probes can be at most \( \sum g_t(\mathcal{N})/f_{\min} \) in total. It remains to bound the competitive ratio.

We prove the following:

**Lemma A.2.** For any element \( u \in \mathcal{N} \):

\[
E \left[ \sum_{g \in G^{(t)} : v \in U_g} \sum_{u \in V_g} c(v) \right] \leq r \cdot c(u).
\]

This will imply as a consequence:

\[
E[c(S)] \leq \sum_{o \in \text{OPT}_t} E \left[ \sum_{g \in G^{(t)} : o \in V_g} \sum_{v \in U_g} c(v) \right] \leq r \cdot \sum_{o \in \text{OPT}_t} c(o) = r \cdot c(\text{OPT}_t).
\]

*Proof of Lemma A.2.* The proof is by induction. Fix \( u \in \mathcal{N} \), and consider the functions \( g \in G^{(t)} \) for which \( u \in V_g \). Let \( X_i \) be the random variable that is the \( i^{\text{th}} \) function probed. Let \( Y_i \) be the random variable that is the element of \( N \) sampled during the \( i^{\text{th}} \) probe. With this notation, the inductive hypothesis is:

\[
E \left[ \sum_{i \geq j} c(Y_i) \mid X_1, Y_1, \ldots, X_{j-1}, Y_{j-1} \right] \leq r \cdot c(u).
\]
For the base case \( i = m \), note that given \( X_1, Y_1, \ldots, X_{m-1}, Y_{m-1} \), the variable \( X_m \) is determined. Suppose \( X_m = g \). Then:

\[
\mathbb{E}[c(Y_m) \mid X_1, Y_1, \ldots X_{m-1}, Y_{m-1}] = \mathbb{E}[c(Y_m) \mid X_1, Y_1, \ldots X_{m-1}, Y_{m-1}, X_m = g]
\]

\[
= \sum_{v \in V_g} \frac{1}{c(v)} \left( \sum_{v' \in V_g \setminus S_t} \frac{1}{c(v')} \right) c(v)
\]

\[
\leq \frac{|V_g|}{\sum_{v' \in V_g \setminus S_t} \frac{1}{c(v')}}
\]

\[
\leq r \cdot c(u).
\]

For the inductive step, suppose the claim holds for \( j + 1 \), and consider the case for \( j \):

\[
\mathbb{E}\left[ \sum_{i \geq j} c(Y_i) \mid X_1, Y_1, \ldots X_{j-1}, Y_{j-1} \right]
\]

\[
= \mathbb{E}[c(Y_j) \mid X_1, Y_1, \ldots X_{j-1}, Y_{j-1}] + \mathbb{E}\left[ \sum_{i \geq j+1} c(Y_i) \mid X_1, Y_1, \ldots X_{j-1}, Y_{j-1} \right]
\]

\[
\leq \frac{r}{\sum_{v' \in V_g \setminus S_t} \frac{1}{c(v')}} + \sum_{u' \neq u} \mathbb{E}\left[ \sum_{i \geq j+1} c(Y_i) \mid X_1, \ldots Y_{j-1}, Y_j = u' \right] \cdot \mathbb{P}(Y_j = u' \mid X_1, \ldots, Y_{j-1})
\]

\[
\leq \frac{r}{\sum_{v' \in V_g \setminus S_t} \frac{1}{c(v')}} + r \cdot c(u) \cdot \left( 1 - \frac{1}{c(u) \sum_{v' \in V_g \setminus S_t} \frac{1}{c(v')}} \right)
\]

\[
= r \cdot c(u).
\]

\[\square\]

**B Combiner Algorithm**

We show that we can adapt the combiner algorithm of [GKKP17] to our general problem.

**Theorem B.1.** Let \( A_G \) be an \( O(\log f(N)/f_{\min}) \)-competitive algorithm for fully-dynamic SUBMODULARCOVER with amortized recourse \( R_G \). Let \( A_{PD} \) be an \( O(r) \)-competitive algorithm for fully-dynamic SUBMODULARCOVER when all functions are \( r \)-juntas with amortized recourse \( R_{PD} \). Then there is an algorithm achieving an approximation ratio of \( O(\min(\log(f(N)/f_{\min}), r)) \) for fully-dynamic SUBMODULARCOVER when all functions are \( r \)-juntas, and it has total recourse \( O(R_G + R_{PD}) \).

**Proof.** The idea is to partition the functions into different buckets based on their junta-arity, in powers of 2 up to \( \log f(N)/f_{\min} \). We run a copy of \( A_{PD} \) which we call \( A_{PD}^{(l)} \) on each bucket \( B_\ell \) separately, and run \( A_G \) one single time on the set of remaining functions.

Formally, for every index \( 0 < \ell < \lfloor \log \log(f(N)/f_{\min}) \rfloor \), maintain a bucket \( B_\ell \) representing the set of functions \( g \) such that \( g \) is a \( k \)-junta, for \( k \in [2^\ell, 2^{\ell+1}) \). Also maintain the bucket \( B_G \) for any remaining functions. When a functions arrives, we insert it into exactly one appropriate bucket and update the appropriate algorithm.

**Lemma B.2.** The total cost of the solution maintained by the algorithm is \( O(\min(\log f(N), r)) \).

**Proof.** If \( r \leq \log(f(N)/f_{\min}) \), the algorithm never runs \( A_G \). Each algorithm \( A_{PD}^{(l)} \) is \( O(2^{\ell+1}) \)-competitive, and thus maintains a solution of cost no more than \( O(2^{\ell+1})c(\text{OPT}) \). The largest
bucket index is \( \ell_{\text{max}} = \lceil \log r \rceil \). Hence the total cost of the solution is:

\[
\sum_{\ell=1}^{\ell_{\text{max}}} O(2^{\ell+1}) \cdot c(\text{OPT}) = O(r) \cdot c(\text{OPT}).
\]

If on the other hand, \( r > \log(f(N)/f_{\text{min}}) \), then the largest bucket index is \( \ell_{\text{max}} = \lceil \log \log(f(N)/f_{\text{min}}) \rceil \).

The total cost of the \( A_{PD} \) algorithms is then

\[
\sum_{\ell=1}^{\ell_{\text{max}}} O(2^{\ell+1}) \cdot c(\text{OPT}) = O(\log f(N)/f_{\text{min}}) \cdot c(\text{OPT}).
\]

Meanwhile, the total cost of the solution maintained by \( A_G \) on the remaining functions has cost \( O(\log f(N)/f_{\text{min}}) \cdot c(\text{OPT}) \). Thus the global solution maintained by the combiner algorithm is also \( O(\log f(N)/f_{\text{min}}) \)-competitive.

The recourse bound is immediate since each function \( g \) arrives to/departs from exactly one bucket, so at most one algorithm among \( \{A^{(\ell)}_{PD}\}_{\ell} \cup \{A_G\} \) has to update its solution at every time step.

\[ \square \]

## C Further Applications

In this section, we show how to recover several known results on recourse bounded algorithms using our framework. We hope this is a step towards unifying the theory of low recourse dynamic algorithms.

### C.1 Online Metric Minimum Spanning Tree

In this problem, vertices in a metric space are added online to an active set. Let \( A_t \) denote the active set at time \( t \). After every arrival, the algorithm must add/remove edges to maintain a spanning tree \( S_t \) for \( A_t \) that is competitive with the `MinimumSpanningTree`. We show:

**Theorem C.1.** There is a deterministic algorithm for `Online Metric MinimumSpanningTree` that achieves a competitive ratio of \( O(1) \) and an amortized recourse bound of \( O(\log D) \), where \( D \) is the ratio of the maximum to minimum distance in the metric.

[GGK16] show how to get an \( O(1) \) worst case recourse bound.

`MinimumSpanningTree` is a special case of `SubmodularCover` in which \( \mathcal{N} \) is the set of edges of the graph, and \( f \) is the rank function of a graphic matroid. The main difference between the dynamic version of this problem and our setting is that here vertices arrive online *along with all their incident edges*. Hence not only is the submodular function changing, but \( \mathcal{N} \) is also growing. We show that this detail can be handled easily.

We define the submodular function \( f^{(t)} \) to be the rank function for the current graphic matroid, i.e. \( f(S) = |A_t| - c_t \), where \( c_t \) is the number of connected components induced by \( S \) on the set of vertices seen thus far. Note that \( f_{\text{max}} = f_{\text{min}} = 1 \), so an edge having nonzero coverage is equivalent to the edge being in our current solution, \( S_t \).

Now the algorithm is:
Algorithm 2 FULLY_DYNAMIC_MST

1: $\pi \leftarrow$ arbitrary initial permutation of edges.
2: for $t = 1, 2, \ldots, T$ do
3:     When vertex $v_t$ arrives, add edges incident to $v_t$ to tail of permutation in arbitrary order, and update $f(t)$.
4:     while there exists a legal $\gamma$-move or a swap for $\pi$ do
5:         Perform the move, and update $\pi$.
6:     Output the collection of $\pi_i$ such that $F_{\pi}(\pi_i) > 0$.

As in Corollary 2.3, if the algorithm terminates then it represents the stack trace of an approximate greedy algorithm for MINIMUM_SPANNING_TREE. Hence the solution is $O(1)$ competitive. To bound the recourse, we use the general potential $\Phi_h$ from Section 3. As before, local moves decrease the potential $\Phi_h$ by $\epsilon_\gamma \cdot c_{\min} \cdot h\left(\frac{1}{c_{\min}}\right)$, so it suffices to show that the potential does not increase by too much when $f(t)$ is updated. Exactly one new edge will have increased marginal coverage, and its coverage will increase from 0 to 1. Thus the increase in potential is at most $c_{\max} \cdot h(1/c_{\max})$. Together, these imply an amortized recourse bound of:

$$\frac{1}{\epsilon_\gamma} \cdot \frac{c_{\max}}{c_{\min}} \cdot \frac{h(1/c_{\max})}{h(1/c_{\min})}.$$

Setting $h(x) = x^{1-\delta}/(1-\delta)$ along with $\delta = (\ln(c_{\max}/c_{\min}+1))^{-1}$ and $\gamma = \epsilon^2$ as in Theorem 3.1, we have $\epsilon_\gamma \geq \delta$, and hence we get a recourse bound of $O(\ln(c_{\max}/c_{\min})) = O(\ln D)$.

C.2 Fully-Dynamic Metric Minimum Steiner Tree

We show that we can also fit into our framework the harder problem of maintaining a tree that spans a set of vertices in the fully-dynamic setting where vertices can both arrive and depart. We must produce a tree $S_t$ that spans the current set of active vertices $A_t$, but we allow ourselves to use Steiner vertices that are not in the active set. We show:

**Theorem C.2.** There is a deterministic algorithm for Fully-Dynamic Metric MINIMUM STEINER TREE that achieves a competitive ratio of $O(1)$ and an amortized recourse bound of $O(\log D)$, where $D$ is the ratio of the maximum to minimum distance in the metric.

This guarantee matches that of [LOP+15]. Separately [GK14] showed how to improve the bound to $O(1)$ amortized recourse.

Our algorithm is the same local search procedure as before, with one twist. We maintain a set of vertices $L$ we call the live set. This set is the union of the active terminals we need to span, and any Steiner vertices currently being used. We define $f(t)$ similarly to before as $f(t)(S) = |L| - c_t$, where $c_t$ is the number of connected components induced by the edge set $S$ on the set of vertices in $L$. Note that this function is submodular, because it is the rank function of the graphic matroid on the live vertex set $L$.

Now when a vertex $v$ departs, we mark it as a Steiner vertex but leave it in the live set. If at any point during the local search $\deg(v) = 2$, we replace $v$ with the edge that shortcuts between $v$’s two neighbors. If at any point point $\deg(v) = 1$, we delete $v$ and its neighboring edge.

To show the competitive ratio we can rely on known results [IW91, GK14]. If Algorithm 3 terminates, the output tree is known as a $\gamma$-stable extension tree for the terminal set $S$.  

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Algorithm 3 FullyDynamicSteinerTree

1: $\pi \leftarrow$ arbitrary initial permutation of edges.
2: for $t = 1, 2, \ldots, T$ do
3:     if vertex $v_t$ arrives then
4:         Add edges incident to $v_t$ to tail of permutation in arbitrary order, and update $f^{(t)}$.
5:     else if vertex $v_t$ departs then
6:         Mark $v_t$ as a Steiner vertex. Run CleanSteinerVertices.
7: while there exists a legal $\gamma$-move or a swap for $\pi$ do
8:     Perform the move, and update $\pi$.
9:     Run CleanSteinerVertices.
10: Output the collection of $\pi_i$ such that $\mathcal{F}_{\pi}(\pi_i) > 0$.

procedure CleanSteinerVertices

1: while there is a Steiner vertex $v$ with $\deg(v) = 2$ do
2:     Let $u_1$ and $u_2$ be the neighbors of $v$.
3:     Add the edge $(u_1, u_2)$ to the position of $(v, u_1)$ in $\pi$. // this shortcuts $v$
4:     Delete all edges incident to $v$ from $\mathcal{N}$, remove $v$ from the live set, and update $f^{(t)}$.
5: while there is a Steiner vertex $v$ with $\deg(v) = 1$ do
6:     Delete all edges incident to $v$ from $\mathcal{N}$, remove $v$ from the live set, and update $f^{(t)}$.

Lemma C.3 (Lemma 5 of [IW91]). If $T$ is a $\gamma$-stable extension tree for $A_t$, then:

$$c(T) \leq 4\gamma \cdot c(\text{Opt}(A_t))$$

where $\text{Opt}(A_t)$ is the optimal Steiner tree for terminal set $A_t$.

Since we set $\gamma = e^2$, this gives us a competitive ratio of $4e^2 = O(1)$.

It remains to show the recourse bound. Deleting degree 1 and 2 vertices requires a constant number of edge changes, so this can be charged to each vertex’s departure. We show that the potential argument from before is not hampered by the changes to the algorithm.

Claim C.4. The procedure CleanSteinerVertices does not increase the potential.

Proof. When a degree 1 Steiner vertex is deleted, the incident edge is removed from the permutation and no other edge’s marginal coverage changes.

When a degree 2 Steiner vertex is deleted, the edges $(v, u_1)$ and $(v, u_2)$ are replaced by the edge $(u_1, u_2)$. Recall that our choice of potential is:

$$\Phi_h(\pi) = \sum_{e \in S_t} c(e)^\delta$$

for $0 < \delta < 1$. By triangle inequality, and concavity of $h$:

$$d(u_1, u_2)^\delta \leq (d(v, u_1) + d(v, u_2))^\delta \leq d(v, u_1)^\delta + d(v, u_2)^\delta.$$ 

Thus this replacement only decreases the potential. \hfill \square

Otherwise, the potential increases during vertex arrivals and decreases during $\gamma$-moves exactly as in Appendix C.1. We are left with the same recourse bound of $O(\ln D)$. 

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D Bounds using the Shannon entropy potential

We show that Shannon entropy also works as a potential, albeit with the weaker recourse bound of:

\[
O\left( \frac{\sum t g_t(N)}{f_{\text{min}}} \ln \left( \frac{c_{\text{max}}}{c_{\text{min}} f_{\text{max}}} \right) \right).
\]

Define the Shannon Entropy potential to be the expression:

\[
\Phi_1(f, \pi) := \sum_{i \in N} \mathfrak{f}_\pi(\pi_i) \log \frac{c(\pi_i)}{\mathfrak{f}_\pi(\pi_i)}.
\]

In order to ensure that \( \Phi_1 \) remains nonnegative and monotone in each \( \mathfrak{f}_\pi(\pi_i) \), scale \( c \) by \( 1/c_{\text{min}} \) and \( f \) by \( 1/(e \cdot f_{\text{max}}) \) such that all costs are greater than 1 and all coverages are less than \( 1/e \). We will account for this scaling at the end.

Note that \( \Phi_1 \) is \( \Phi_h \) from Section 3 with \( h(x) = x \log(1/x) \). This \( h \) satisfies properties (i), (ii) and (iv) but not (iii).

Properties of \( \Phi_1 \):

(I) \( \Phi_1 \) increases by at most \( g_t(N) \cdot \ln(c_{\text{max}}/f_{\text{min}}) \) with the addition of function \( g_t \) to the active set.

(II) \( \Phi_1 \) does not increase with deletion of functions from the system.

(III) \( \Phi_1 \) does not increase during swaps.

(IV) If \( \gamma > e \), then \( \Phi_1 \) decreases by at least \( f_{\text{min}} \ln(\gamma/e) \) with every \( \gamma \)-move.

The proofs that \( \Phi_1 \) satisfies properties (I)–(III) follows directly from Lemma 3.3, since these do not use property (iii). It remains to show the last property.

**Lemma D.1.** If \( \gamma > e \), every \( \gamma \)-move decreases \( \Phi_1 \) by at least \( f_{\text{min}} \ln(\gamma/e) \).

**Proof.** Suppose we perform a \( \gamma \)-move on a permutation \( \pi \). Let \( u \) be the element moving to some position \( p \) from some position \( q > p \), and let \( \pi' \) denote the permutation after the move. For convenience, also define:

\[
v_i := \mathfrak{f}_\pi(\pi_i), \quad \text{(the original coverage of the } i^{\text{th}} \text{ set})
\]

\[
a_i := I_f(\pi_i; u \mid \pi_{1:i-1}) = \mathfrak{f}_\pi(\pi_i) - \mathfrak{f}_{\pi'}(\pi_i). \quad \text{(the loss in coverage of the } i^{\text{th}} \text{ set})
\]

Then:

\[
\Phi_1(f, \pi') - \Phi_1(f, \pi)
\]

\[
= \sum_{i=1}^n (v_i - a_i) \ln \frac{c(\pi_i)}{v_i - a_i} + \sum_{i=1}^n a_i \ln \frac{c(u)}{\sum_{i=1}^n a_i} - \sum_{i=1}^n v_i \ln \frac{c(\pi_i)}{v_i}
\]

\[
\leq -\sum_{i=1}^n a_i \ln \left( \frac{c(\pi_i)}{e \cdot v_i} \right) + \sum_{i=1}^n a_i \ln \frac{c(u)}{\sum_{i=1}^n a_i} \quad \text{(D.1)}
\]
\[
\begin{align*}
\leq & - \sum_{i=1}^{n} a_i \ln \left( \frac{\gamma}{e \cdot \sum_i a_i} \right) + \sum_{i=1}^{n} a_i \ln \frac{c(u)}{\sum_{i=1}^{n} a_i} \\
= & - \sum_{i=1}^{n} a_i \ln \left( \frac{2}{e} \right) \\
= & - f_{\min} \cdot \ln \left( \frac{\gamma}{e} \right).
\end{align*}
\] (D.2)

Step (D.1) follows because, by concavity of the function \( h(x) = x \log x \), we have \( h(a + b) - h(a) \leq b \cdot h'(a) \). Step (D.2) follows because \( u \) moving to position \( p \) is a \( \gamma \)-move, hence \( \sum_j a_j / c(u) \geq \gamma \cdot v_i / c(\pi_i) \).

We now show the weaker recourse bound. By (I), the potential \( \Phi_h \) increases by at most

\[ g_t(N) \cdot \ln \left( \frac{c_{\max}}{f_{\min}} \right) \]

with the addition of function \( g_t \) to the active set. By (IV), it decreases by \( \Omega(f_{\min}) \) with every move that costs recourse 1, and otherwise does not increase. Since we scaled costs by \( 1/c_{\min} \) and coverages by \( 1/(e \cdot f_{\max}) \), this implies a recourse bound of:

\[ O\left( \sum_t g_t(N) \cdot \ln \left( \frac{c_{\max}}{c_{\min} \cdot f_{\min}} \right) \right). \]