A polynomial identity via differential operators

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Dedicated to Professor Winfried Bruns, on the occasion of his 70th birthday

Abstract We give a new proof of a polynomial identity involving the minors of a matrix, that originated in the study of integer torsion in a local cohomology module.

1 Introduction

Our study of integer torsion in local cohomology modules began in the paper [Si], where we constructed a local cohomology module that has $p$-torsion for each prime integer $p$, and also studied the determinantal example $H^3_{I_2}(\mathbb{Z}[X])$ where $X$ is a $2 \times 3$ matrix of indeterminates, and $I_2$ the ideal generated by its size 2 minors. In that paper, we constructed a polynomial identity that shows that the local cohomology module $H^3_{I_2}(\mathbb{Z}[X])$ has no integer torsion; it then follows that this module is a rational vector space. Subsequently, in joint work with Lyubeznik and Walther, we showed that the same holds for all local cohomology modules of the form $H^k_{I_t}(\mathbb{Z}[X])$, where $X$ is a matrix of indeterminates, $I$ the ideal generated by its size $t$ minors, and $k$ an integer with $k > \text{height } I$.[LSW, Theorem 1.2]. In a related direction, in joint work with Bhatt, Blickle, Lyubeznik, and Zhang, we proved that the local cohomology of a polynomial ring over $\mathbb{Z}$ can have $p$-torsion for at most finitely many $p$; we record a special case of [BBLSZ, Theorem 3.1]:

Theorem 1. Let $R$ be a polynomial ring over the ring of integers, and let $f_1, \ldots, f_m$ be elements of $R$. Let $n$ be a nonnegative integer. Then each prime integer that is a nonzerodivisor on the Koszul cohomology module $H^n(f_1, \ldots, f_m; R)$ is also a nonzerodivisor on the local cohomology module $H^n_{(f_1, \ldots, f_m)}(R)$.

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These more general results notwithstanding, a satisfactory proof or conceptual understanding of the polynomial identity from [Si] had previously eluded us; extensive calculations with Macaulay2 had led us to a conjectured identity, which we were then able to prove using the hypergeometric series algorithms of Petkovšek, Wilf, and Zeilberger [PWZ], as implemented in Maple. The purpose of this note is to demonstrate how techniques using differential operators underlying the papers [BBLSZ] and [LSW] provide the “right” proof of the identity, and, indeed, provide a rich source of similar identities.

We remark that there is considerable motivation for studying local cohomology of rings of polynomials with integer coefficients such as $H_k^I(t)(\mathbb{Z}[X])$: a matrix of indeterminates X specializes to a given matrix of that size over an arbitrary commutative noetherian ring (this is where $\mathbb{Z}$ is crucial), which turns out to be useful in proving vanishing theorems for local cohomology supported at ideals of minors of arbitrary matrices. See [LSW] Theorem 1.1] for these vanishing results, that build upon the work of Bruns and Schwänzl [BS].

\section{Preliminary remarks}

We summarize some notation and facts. As a reference for Koszul cohomology and local cohomology, we mention [BH]; for more on local cohomology as a $\mathfrak{D}$-module, we point the reader towards [Ly1] and [BBLSZ].

\textbf{Koszul and Čech cohomology}

For an element $f$ in a commutative ring $R$, the Koszul complex $K^\bullet(f; R)$ has a natural map to the Čech complex $C^\bullet(f; R)$ as follows:

\[
\begin{array}{c}
K^\bullet(f; R) := 0 \longrightarrow R \overset{f}{\longrightarrow} R \longrightarrow 0 \\
C^\bullet(f; R) := 0 \longrightarrow R \longrightarrow R_f \longrightarrow 0.
\end{array}
\]

For a sequence of elements $f = f_1, \ldots, f_m$ in $R$, one similarly obtains

\[
K^\bullet(f; R) := \bigotimes_j K^\bullet(f_j; R) \longrightarrow \bigotimes_j C^\bullet(f_j; R) =: C^\bullet(f; R),
\]

and hence, for each $n \geq 0$, an induced map on cohomology modules

\[
H^n(f; R) \longrightarrow H^n_{(f)}(R).
\]
Now suppose $R$ is a polynomial ring over a field $F$ of characteristic $p > 0$. The Frobenius endomorphism $\varphi$ of $R$ induces an additive map

$$H^n_{f,\varphi}(R) \longrightarrow H^n_{f \varphi}(R) = H^n_{f}(R),$$

where $f^p = f_1^p, \ldots, f_n^p$. Set $R\{\varphi\}$ to be the extension ring of $R$ obtained by adjoining the Frobenius operator, i.e., adjoining a generator $\varphi$ subject to the relations $\varphi r = r^p \varphi$ for each $r \in R$; see [Ly2] Section 4. By an $R\{\varphi\}$-module we will mean a left $R\{\varphi\}$-module. The map displayed above gives $H^n_{f,\varphi}(R)$ an $R\{\varphi\}$-module structure. It is not hard to see that the image of $H^n_{f,\varphi}(R)$ in $H^n_{f}(R)$ generates the latter as an $R\{\varphi\}$-module; what is much more surprising is a result of Álvarez, Blickle, and Lyubeznik, [ABL, Corollary 4.4], by which the image of $H^n_{f,\varphi}(R)$ in $H^n_{f}(R)$ generates the latter as a $D(R, F)$-module; see below for the definition. The result is already notable in the case $m = 1 = n$, where the map (1) takes the form

$$H^1_{f}(f; R) = R/fR \longrightarrow R/fR = H^1_{f,\varphi}(R) \quad [1] \longrightarrow [1/f].$$

By [ABL], the element $1/f$ generates $R_f$ as a $D(R, F)$-module. It is of course evident that $1/f$ generates $R_f$ as an $R\{\varphi\}$-module since the elements $\varphi^e(1/f) = 1/f^{pe}$ with $e \geq 0$ serve as $R$-module generators for $R_f$. See [BDV] for an algorithm to explicitly construct a differential operator $\delta$ with $\delta(1/f) = 1/f^{pe}$, along with a Macaulay2 implementation.

**Differential operators**

Let $A$ be a commutative ring, and $x$ an indeterminate; set $R = A[x]$. The divided power partial differential operator

$$\frac{1}{k!} \frac{\partial^k}{\partial x^k}$$

is the $A$-linear endomorphism of $R$ with

$$\frac{1}{k!} \frac{\partial^k}{\partial x^k}(x^m) = \binom{m}{k} x^{m-k} \quad \text{for } m \geq 0,$$

where we use the convention that the binomial coefficient $\binom{m}{k}$ vanishes if $m < k$. Note that

$$\frac{1}{r!} \frac{\partial^r}{\partial x^r} \cdot \frac{1}{s!} \frac{\partial^s}{\partial x^s} = \binom{r+s}{r} \frac{1}{(r+s)!} \frac{\partial^{r+s}}{\partial x^{r+s}}.$$

For the purposes of this paper, if $R$ is a polynomial ring over $A$ in the indeterminates $x_1, \ldots, x_d$, we define the ring of $A$-linear differential operators on $R$, de-
noted $\mathcal{D}(R,A)$, to be the free $R$-module with basis
\[ \frac{1}{k_1!} \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{1}{k_d!} \frac{\partial^{k_d}}{\partial x_d^{k_d}} \quad \text{for } k_i \geq 0, \]
with the ring structure coming from composition. This is consistent with more general definitions; see [Gr, 16.11]. By a $\mathcal{D}(R,A)$-module, we will mean a left $\mathcal{D}(R,A)$-module; the ring $R$ has a natural $\mathcal{D}(R,A)$-module structure, as do localizations of $R$. For a sequence of elements $f$ in $R$, the Čech complex $C^\bullet(f; R)$ is a complex of $\mathcal{D}(R,A)$-modules, and hence so are its cohomology modules $H^n_{f \cdot}(R)$. Note that for $m \geq 1$, one has
\[ \frac{1}{k_n!} \frac{\partial^k}{\partial x^k} \left( \frac{1}{x^m} \right) = (-1)^k \binom{m+k-1}{k} \frac{1}{x^{m+k}}. \]

We also recall the Leibniz rule, which states that
\[ \frac{1}{k!} \frac{\partial^k}{\partial x^k} (fg) = \sum_{i+j=k} \frac{1}{i!} \frac{\partial^i}{\partial x^i} (f) \frac{1}{j!} \frac{\partial^j}{\partial x^j} (g). \]

### 3 The identity

Let $R$ be the ring of polynomials with integer coefficients in the indeterminates $(u\ v\ w\ x\ y\ z)$. The ideal $I$ generated by the size 2 minors of the above matrix has height 2; our interest is in proving that the local cohomology module $H^3_I(R)$ is a rational vector space. We label the minors as $\Delta_1 = vz - wy$, $\Delta_2 = wx - uz$, and $\Delta_3 = uy - vx$. Fix a prime integer $p$, and consider the exact sequence
\[ 0 \to R \xrightarrow{p} R \to R \to \overline{R} \to 0, \]
where $\overline{R} = R/pR$. This induces an exact sequence of local cohomology modules
\[ \to H^2_I(R) \xrightarrow{\pi} H^2_I(\overline{R}) \to H^3_I(R) \xrightarrow{p} H^3_I(R) \to H^3_I(\overline{R}) \to 0. \]

The ring $\overline{R}/p\overline{R}$ is Cohen-Macaulay of dimension 4, so [PS, Proposition III.4.1] implies that $H^3_I(\overline{R}) = 0$. As $p$ is arbitrary, it follows that $H^3_I(R)$ is a divisible abelian group. To prove that it is a rational vector space, one needs to show that multiplication by $p$ on $H^3_I(R)$ is injective, equivalently that $\pi$ is surjective. We first prove this using the identity (2) below, and then proceed with the proof of the identity.

For each $k \geq 0$, one has
and the cohomology class \( \eta \)

The Koszul cohomology module \( H \)

Since \( \bigwedge^2 \) is a regular ring of positive characteristic, \( H \) is generated, as an \( R \)-module, by \( \Delta_1 \), \( \Delta_2 \), and \( \Delta_3 \).

The Koszul cohomology module \( H \) is readily seen to be generated, as an \( R \)-module, by elements corresponding to the relations

\[
u \Delta_1 + v \Delta_2 + w \Delta_3 = 0 \quad \text{and} \quad x \Delta_1 + y \Delta_2 + z \Delta_3 = 0.
\]

These two generators of \( H \) map, respectively, to

\[
\alpha := \begin{pmatrix} w \\ \Delta_1 \Delta_2, -v \\ \Delta_1 \Delta_3, u \\ \Delta_2 \Delta_3 \end{pmatrix} \quad \text{and} \quad \beta := \begin{pmatrix} z \\ \Delta_1 \Delta_2, -y \\ \Delta_1 \Delta_3, x \\ \Delta_2 \Delta_3 \end{pmatrix}
\]
in $H^2_I(R)$. Thus, $H^2_I(R)$ is generated over $R$ by $\varphi^e(\alpha)$ and $\varphi^e(\beta)$ for $e \geq 0$. But

$$\varphi^e(\alpha) = \pi(\eta^e_p - 1)$$

is in the image of $\pi$, and hence so is $\varphi^e(\beta)$ by symmetry. Thus, $\pi$ is surjective.

**The proof of the identity**

We start by observing that $C^2(\Delta_1, \Delta_2, \Delta_3; R)$ is a $\mathcal{D}(R, \mathbb{Z})$-module. The element

$$\left( \begin{array}{ccc} w & -v & u \\ \Delta_1 \Delta_2 & -\Delta_1 \Delta_3 & \Delta_2 \Delta_3 \end{array} \right)$$

is a 2-cocycle in $C^2(\Delta_1, \Delta_2, \Delta_3; R)$ since

$$\frac{w}{\Delta_1 \Delta_2} + \frac{-v}{\Delta_1 \Delta_3} + \frac{u}{\Delta_2 \Delta_3} = 0. \quad (3)$$

We claim that the identity (2) is simply the differential operator

$$D = \frac{1}{k!} \frac{\partial^k}{\partial u^k} - \frac{1}{k!} \frac{\partial^k}{\partial y^k} + \frac{1}{k!} \frac{\partial^k}{\partial z^k}$$

applied termwise to (3); we first explain the choice of this operator: set $k = p^e - 1$, and consider $\overline{D} = D \mod p$ as an element of

$$\mathcal{D}(R, \mathbb{Z})/p\mathcal{D}(R, \mathbb{Z}) = \mathcal{D}(R/pR, \mathbb{Z}/p\mathbb{Z}).$$

It is an elementary verification that

$$\overline{D}(u \Delta_2^{p^e - 1} \Delta_3^{p^e - 1}) \equiv u^{p^e} \mod p,$$

$$\overline{D}(v \Delta_3^{p^e - 1} \Delta_1^{p^e - 1}) \equiv v^{p^e} \mod p,$$

$$\overline{D}(w \Delta_1^{p^e - 1} \Delta_2^{p^e - 1}) \equiv w^{p^e} \mod p.$$ 

Since $k < p^e$, the differential operator $\overline{D}$ is $\overline{R}^{p^e}$-linear; dividing the above equations by $\Delta_2^{p^e-1} \Delta_3^{p^e-1}$, $\Delta_3^{p^e-1} \Delta_1^{p^e-1}$, and $\Delta_1^{p^e-1} \Delta_2^{p^e-1}$ respectively, we obtain

$$\overline{D} \left( \frac{w}{\Delta_1 \Delta_2}, \frac{-v}{\Delta_1 \Delta_3}, \frac{u}{\Delta_2 \Delta_3} \right) \equiv \left( \frac{w^{p^e}}{\Delta_1 \Delta_2}, \frac{-v^{p^e}}{\Delta_1 \Delta_3}, \frac{u^{p^e}}{\Delta_2 \Delta_3} \right) \mod p,$$

which maps to the desired cohomology class $\varphi^e(\alpha)$ in $H^2_I(R)$. Of course, the operator $D$ is not unique in this regard.

Using elementary properties of differential operators recorded in §2 we have
It remains to evaluate $D \left( \frac{v}{A_3^2 A_4^2} \right) = 1 \frac{\partial^k}{k! \partial u^k} 1 \frac{\partial^k}{k! \partial y^k} 1 \frac{\partial^k}{k! \partial z^k} \left[ \frac{v}{(uy - vx)(vz - wy)} \right] v(-v)^k = 1 \frac{\partial^k}{k! \partial u^k} \left[ \frac{v(-v)^k}{(uy - vx)(vz - wy)^k+1} \right]
= 1 \frac{\partial^k}{k! \partial y^k} \left[ \frac{v(-v)^k}{(uy - vx)(vz - wy)^k+1} \right]
= 1 \frac{\partial^k}{k! \partial z^k} \left[ \frac{v(-v)^k}{(uy - vx)(vz - wy)^k+1} \right]
= w^{k+1} \sum_{i,j} \left( \sum_{i,j} k+i \right) \left( \sum_{i,j} k+j \right) \left( \sum_{i,j} \frac{(-u)^i}{\Delta_1^{k+1+i} \Delta_2^{k+1+j}} \right)
= u^{k+1} \sum_{i,j} \left( \sum_{i,j} k+i \right) \left( \sum_{i,j} k+j \right) \left( \sum_{i,j} \frac{(-w)^i}{\Delta_2^{k+1+i} \Delta_3^{k+1+j}} \right)

A similar calculation shows that $D \left( \frac{w}{A_3^2 A_4^2} \right) = w^{k+1} \sum_{i,j} \left( \sum_{i,j} k+i \right) \left( \sum_{i,j} k+j \right) \left( \sum_{i,j} \frac{(-u)^i}{\Delta_1^{k+1+i} \Delta_2^{k+1+j}} \right)$.

It remains to evaluate $D \left( \frac{u}{A_2 A_3} \right)$; we reduce this to the previous calculation as follows. First note that the differential operators $\frac{\partial}{\partial u} \frac{\partial}{\partial y} \frac{\partial}{\partial v} \frac{\partial}{\partial x}$ commute; it is readily checked that they agree on $\frac{u}{A_2 A_3}$. Consequently the operators $\frac{\partial}{\partial u} \frac{\partial}{\partial y} \frac{\partial}{\partial v} \frac{\partial}{\partial x}$ agree on $\frac{u}{A_2 A_3}$ as well. But then

$$D \left( \frac{u}{A_2 A_3} \right) = 1 \frac{\partial^k}{k! \partial u^k} 1 \frac{\partial^k}{k! \partial v^k} 1 \frac{\partial^k}{k! \partial x^k} \left[ \frac{u}{(wx - uz)(uy - vx)} \right]$$

which, using the previous calculation and symmetry, equals

$$u^{k+1} \sum_{i,j} \left( \sum_{i,j} k+i \right) \left( \sum_{i,j} k+j \right) \left( \sum_{i,j} \frac{(-w)^i}{\Delta_2^{k+1+i} \Delta_3^{k+1+j}} \right).$$
Identities in general

Suppose \( f = f_1, \ldots, f_m \) are elements of a polynomial ring \( R \) over \( \mathbb{Z} \), and \( g_1, \ldots, g_m \) are elements of \( R \) such that

\[
g_1 f_1 + \cdots + g_m f_m = 0.
\]

Then, for each prime integer \( p \) and \( e \geq 0 \), the Frobenius map on \( \overline{R} = R/pR \) gives

\[
g_1^p f_1^p + \cdots + g_m^p f_m^p \equiv 0 \mod p.
\] (4)

Now suppose \( p \) is a nonzerodivisor on the Koszul cohomology module \( H^m(f; R) \). Then Theorem [1] implies that (4) lifts to an equation

\[
G_1 f_1^N + \cdots + G_m f_m^N = 0
\] (5)

in \( R \) in the sense that the cohomology class corresponding to (5) in \( H^{m-1}_f(R) \) maps to the cohomology class corresponding to (4) in \( H^{m-1}_f(\overline{R}) \).

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