Combined Mean Field Limit and Non-relativistic Limit of Vlasov-Maxwell Particle System to Vlasov-Poisson System

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Abstract

In this paper we consider the mean field limit and non-relativistic limit of relativistic Vlasov-Maxwell particle system to Vlasov-Poisson equation. With the relativistic Vlasov-Maxwell particle system being a starting point, we carry out the estimates (with respect to $N$ and $c$) between the characteristic equation of both Vlasov-Maxwell particle model and Vlasov-Poisson equation, where the probabilistic method is exploited. In the last step, we take both large $N$ limit and non-relativistic limit (meaning $c$ tending to infinity) to close the argument.

Keywords: probabilistic method, mean field limit, non-relativistic limit, Vlasov-Maxwell equation, Vlasov-Poisson equation.

AMS Classification:

1 Introduction

The time evolution of plasmas is a very important topic of physics. In many cases, for example when considering the plasma in a nuclear fusion reactor, the temperature of the particles forming the plasma is sufficiently high to neglect quantum effects. Given that the number of particles forming the plasma is very high, also a mean-field approximation for the internal electromagnetic forces of the system can be argued [11], thus the system is in

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good approximation given by the relativistic Vlasov-Maxwell equations:

\[
\begin{aligned}
\frac{\partial f_m}{\partial t} + \hat{v} \cdot \nabla_x f_m + (E_m + c^{-1} \hat{v} \times B_m) \cdot \nabla_v f_m &= 0, \\
\frac{\partial E_m}{\partial t} &= c \nabla \times B_m - j_m, \quad \nabla \cdot E_m = \rho_m, \\
\frac{\partial B_m}{\partial t} &= -c \nabla \times E_m, \quad \nabla \cdot B_m = 0,
\end{aligned}
\]

(1.1)

where \( \hat{v} = \frac{v}{\sqrt{1 + c^{-2} v^2}} \), \( \rho_m(t, x) = \int_{\mathbb{R}^3} f_m(t, x, v) \, dv \) and \( j_m(t, x, v) = \int_{\mathbb{R}^3} \hat{v} f_m(t, x, v) \, dv \). The parameter \( c \) is the speed of light, \( (E_m, B_m) \) is the electro-magnetic field, and the distribution function \( f_m(t, x, v) \geq 0 \) describes the density of particles with position \( x \in \mathbb{R}^3 \) and velocity \( v \in \mathbb{R}^3 \). Assuming that there are no external electromagnetic fields, the initial data

\[
\begin{aligned}
f_m(0, x, v) &= f_0(x, v), \\
E_m(0, x) &= E_0(x), \\
B_m(0, x) &= B_0(x),
\end{aligned}
\]

(1.2)

satisfy the compatibility conditions \( \nabla \cdot E_0(x) = \rho_0(x) = \int_{\mathbb{R}^3} f_0(x, v) \, dv, \quad \nabla \cdot B_0(x) = 0 \).

Local existence and uniqueness of classical solutions to this initial value problem for smooth and compactly supported data was established in [9]. These solutions can be extended globally in time provided the momentum support can be controlled assuming certain conditions on the initial data, e.g. smallness [10] closeness to neutrality [8] or closeness to spherical symmetry [23]. It is worth to mention that different approaches to the results in [9] were given in [2, 16]. In order to obtain global existence of solutions, DiPerna and Lions restricted the solution concept to weak solutions. We refer to [5].

As was shown in [25] using an integral representation for the electric and magnetic field due to Glassey and Strauss [9], the solutions of relativistic Vlasov-Maxwell system converge in the point-wise sense to solutions of the non-relativistic Vlasov-Poisson system (below) at the rate of \( 1/c \) as \( c \) tends to infinity. The Vlasov-Poisson system reads

\[
\begin{aligned}
\frac{\partial f_p}{\partial t} + v \cdot \nabla_x f_p + E_p \cdot \nabla_v f_p &= 0, \\
E_p(t, x) &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \rho_p(t, y) \frac{x - y}{|x - y|^3} \, dy, \\
\rho_p(t, y) &= \int_{\mathbb{R}^3} f_p(t, y, v) \, dv,
\end{aligned}
\]

(1.3)

with the initial data \( f_p(0, x, v) = f_0(x, v) \). Here the indexes \( m \) and \( p \) in (1.1) and (1.3) stand for Maxwell and Poisson respectively. We note that there are global existence results for classical solutions of the Vlasov-Poisson system [20, 21, 26].

However, a more interesting and challenging question is to consider what the corresponding particle model of relativistic Vlasov-Maxwell equation might be and whether we
can prove the validity of the mean field description rigorously in the limit \( N \to \infty \). Up to our knowledge and at the time of this writing, taking both the mean field limit and the non-relativistic limit (or classical limit) of the Vlasov-Maxwell system into account is rare in the literatures. Concerning the mean field limit, Braun and Hepp [4] and Dobrushin [6] have proposed rigorous derivations of a system analogous to the Vlasov-Poisson system with a twice differentiable mollification of the Coulomb potential. Hauray and Jabin [12] have succeeded in treating the case of singular potentials, but not including the Coulomb singularity yet. Recently, Lazarovici and Pickl [19] gave a probabilistic proof of the validity of the mean field limit and propagation of chaos for the \( N \)-particle systems in three dimensions with Coulomb potential with \( N \)-dependent cutoff, which provides us with a very constructive idea of method. Lazarovici generalized this result including electromagnetic fields, proving the validity of the relativistic Vlasov-Maxwell equation [18] considering charges of radius \( N^{-\delta} \) with \( \delta < 1/12 \).

Writing down the corresponding \( N \)-particle model of the non-relativistic Vlasov-Maxwell system is a perplexing task because one needs to find a suitable description for the electromagnetic self-interaction within the theory of classical electrodynamics [7, 14, 27]. The problem of deriving a regularized version of the Vlasov-Maxwell system from a particle model was explicitly mentioned by Kiessling in [15]. Only after several years did Golse [11] establish the mean field limit of a \( N \)-particle system towards a regularized version of the relativistic Vlasov-Maxwell system with the help of [7] by Elsken, Kiessling and Ricci.

In the present work, we want to combine the mean field limit and non-relativistic limit of the regularized relativistic Vlasov-Maxwell particle model to Vlasov-Poisson equation. The method we apply here is more or less along the line of [9, 11, 19] using a mollifier for regularization removes the difficulties caused by the electromagnetic self-interaction forces. Unlike regularizing the Coulomb potential in the mean field limit established in [4, 6], the regularization of the self-interaction force in the Vlasov-Maxwell system is more difficult since the electromagnetic field involves both a scalar and vector potentials [11]. The solutions of the relativistic Vlasov-Maxwell system, as was discussed by Glassey and Strauss in [9], are closely related to the wave equation. This connection uses Kirchhoff’s formula, which we also used in this paper. We would like to mention that there are other representations of the solutions of the relativistic Vlasov-Maxwell system, for example [2, 3], but they are all in fact equivalent.

This paper is organized as follows: in Section 2, we prepare the regularization-procedure of both, the relativistic Vlasov-Maxwell and the Vlasov-Poisson system, and provide estimates between the solutions of these two systems. In Section 3, we introduce the particle model of the relativistic Vlasov-Maxwell system and apply the probabilistic method to
carry out the estimates between the characteristic equation of the particle model and that of the relativistic Vlasov-Maxwell system. We summarize our results in Section 4.

2 Regularization of the Vlasov-Maxwell and the Vlasov-Poisson Systems

Let \( \chi \in C_0^\infty \) satisfy

\[
\chi(x) = \chi(-x) \geq 0, \quad \text{supp}(\chi) \subset B_1(0), \quad \int_{\mathbb{R}^3} \chi(x) \, dx = 1,
\]

and define the regularizing sequence

\[
\chi^N(x) = N^{3\theta} \chi \left( N^\theta x \right). \tag{2.1}
\]

The regularized version of the Vlasov-Maxwell System (VMN) with unknown \((f_m^N, B_m^N, E_m^N)\) is given by:

\[
\begin{align*}
\partial_t f_m^N + \hat{v} \cdot \nabla_x f_m^N + \left( E_m^N + c^{-1} \hat{v} \times B_m^N \right) \cdot \nabla_v f_m^N &= 0, \\
\partial_t E_m^N &= c \nabla \times B_m^N - \chi^N *_x \chi^N *_x J_m^N, \\
\partial_t B_m^N &= -c \nabla \times E_m^N, \\
\nabla \cdot E_m^N &= \chi^N *_x \chi^N *_x \rho_m^N, \\
\nabla \cdot B_m^N &= 0,
\end{align*}
\]

and initial data (IVMN)

\[
\begin{align*}
f_m^N(0, x, v) &= f_0(x, v), \\
E_m^N(0, x) &= \chi^N *_x \chi^N *_x E_0(x), \\
B_m^N(0, x) &= \chi^N *_x \chi^N *_x B_0(x),
\end{align*}
\]

The regularized version of the Vlasov-Poisson System (VPN) with unknown \((f_p^N, E_p^N)\) is given by:

\[
\begin{align*}
\partial_t f_p^N + v \cdot \nabla_x f_p^N + E_p^N \cdot \nabla_v f_p^N &= 0, \\
E_p^N(t, x) &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\chi^N(p) \chi^N(z) \rho_p^N(t, y - p - z)}{|x - y|^3} \, dy \, dz, \\
\rho_p^N(t, y) &= \int_{\mathbb{R}^3} f_p^N(t, y, v) \, dv,
\end{align*}
\]

with the initial data \( f_p^N(0, x, v) = f_0(x, v) \).

**Theorem 2.1.** Let \( f_0 \) be a nonnegative \( C^1 \)-function with compact support in \( \mathbb{R}^6 \) and \( B_0 \) be in \( C_0^2(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3) \cap W^{2,1}(\mathbb{R}^3) \). Assume further that

\[
\| \nabla_x B_0 \|_{L^\infty(\mathbb{R}^3)} + \| B_0 \|_{L^\infty(\mathbb{R}^3)} \leq \frac{1}{c^2},
\]
and

\[
E_0(x) = \frac{1}{4\pi} \iint_{\mathbb{R}^6} \frac{x - y}{|x - y|^3} f_0(y, v) \, dv dy.
\]

Then

1. There exists a \( T > 0 \) such that (VPN) with initial data \( f_p^N(0, x, v) = f_0(x, v) \) admits a unique \( C^1 \)-solution \((f_p^N, E_p^N)\) on the time interval \([0, T])\).

2. There exists a \( T^* > 0 \) (independent of \( c \)) such that for \( c \geq 1 \), (VMN) with the initial condition (IVMN) has a unique \( C^1 \)-solution \((f_m^N, B_m^N, E_m^N)\) on the time interval \([0, T^*])\). Furthermore there exist nondecreasing functions (independent of \( c \) and \( N \)) \( q(t) : [0, T^*) \to \mathbb{R} \) and \( H(t) : [0, T^*) \to \mathbb{R} \) such that

\[
f_m^N = 0, \quad \text{if} \quad |v| \geq q(t),
\]

\[
\|E_m^N(t, x)\|_{L^\infty([0, T^*) \times \mathbb{R}^3)} + \|B_m^N(t, x)\|_{L^\infty([0, T^*) \times \mathbb{R}^3)} \leq H(t).
\]

3. Let \( \tilde{T} = \min(T, T^*) \), then for every \( T \in [0, \tilde{T}) \) there exists a constant \( M \) (depending on \( T \) and the initial data, but not on \( c \)) such that for \( c \geq 1 \)

\[
\|f_m^N - f_p^N\|_{L^\infty([0, T]) \times \mathbb{R}^3 \times \mathbb{R}^3)} + \|E_m^N - E_p^N\|_{L^\infty([0, T]) \times \mathbb{R}^3)} + \|B_m^N\|_{L^\infty([0, T]) \times \mathbb{R}^3)} \leq \frac{M}{c}.
\]

The proof of the Theorem involves many long and tedious calculations since it includes many cut-offs and mollifications, however, no technical difficulties other than presented in the paper [25] appear. Therefore we deliver the proof in the appendix at the end of the paper for those readers who want to take a closer look at the details.

**Remark 2.1.** In the current setting, i.e., repulsive particle interactions, both \( \tilde{T} \) and \( T^* \) can be global. But in the attractive case, there might be lack of global existence of solutions. Therefore both existence results we give are locally in time. The limits \( N \to \infty \) and \( c \to \infty \) in our paper are taken in the time interval where both solutions exist.

We assume in Theorem 2.1 that \( f_0 \) has compact support, so let

\[
q_0 = \sup\{|v| : \text{there exists } x \in \mathbb{R}^3 \text{ such that } f_0(x, v) \neq 0\}.
\]

Further, we define the characteristic curves \((x(t, x_0, v_0, t_0), v(t, x_0, v_0, t_0))\) (or in short \((x(t), v(t))\)) by

\[
\begin{align*}
\frac{dx}{dt} &= \hat{v}, \\
\frac{dv}{dt} &= E_m^N + c^{-1} \hat{v} \times B_m^N.
\end{align*}
\]
Therefore $f^N_m$ remains non-negative if $f_0$ is non-negative and
\[
\sup \{ f^N_m(t, x, v) : x \in \mathbb{R}^3, v \in \mathbb{R}^3, t \in [0, T^*) \} = \| f_0 \|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)}.
\]

We also define
\[
p_0 = \sup \{|x| : \text{there exists } v \in \mathbb{R}^3 \text{ such that } f_0(x, v) \neq 0\}.
\]

Hence $f^N_m(t, x, v) = 0$, if $|x| \geq p_0 + tq(t)$.

Before we prove the Theorem, we write the second order form of Maxwell’s equation:

\[
\begin{aligned}
\begin{cases}
\partial_t E^N_m - c^2 \Delta E^N_m = -\chi^N \ast_x \chi^N \ast_x (c^2 \nabla_x \rho^N_m + \partial_t j^N_m), \\
\partial_t B^N_m - c^2 \Delta B^N_m = c \chi^N \ast_x \chi^N \ast_x \nabla \times j^N_m, \\
E^N_m(0, x) = \chi^N \ast_x \chi^N \ast_x E_0, \\
B^N_m(0, x) = \chi^N \ast_x \chi^N \ast_x B_0, \\
\partial_t E^N_m(0, x) = c \nabla \ast_B B^N_m(0, x) - \chi^N \ast_x \chi^N \ast_x j^N_m(0, x), \\
\partial_t B^N_m(0, x) = -c \nabla \ast_B E^N_m(0, x).
\end{cases}
\end{aligned}
\] (2.6)

**Proposition 2.1.** Let $Y(t, x) \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^3)$ satisfy

\[
\begin{aligned}
\begin{cases}
\partial_t Y - c^2 \Delta Y = \delta_{(t,x)=(0,0)}, \\
\text{supp } Y \subset \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \ |x| \leq ct\},
\end{cases}
\end{aligned}
\] (2.7)

then $Y(t, x) = \frac{1_{t>0}}{4\pi ct|x|} \delta(|x| - ct)$.

The proof of this Proposition is standard. $Y(t, x)$ is called the fundamental solution of the wave equation. Set $Y^N = \chi^N \ast_x \chi^N \ast_x Y$, then the solutions of (2.6) are given in terms of

\[
\begin{aligned}
\begin{cases}
E^N_m = \partial_t Y^N \ast_x E_0 + Y^N \ast_x (c \nabla \ast_B B_0 - j^N_m(0, \cdot)) - Y^N \ast_{t,x} (c^2 \nabla \rho^N_m + \partial_t j^N_m), \\
B^N_m = \partial_t Y^N \ast_x B_0 - c Y^N \ast_x \nabla \ast E_0 + c Y^N \ast_{t,x} \nabla \ast j^N_m.
\end{cases}
\end{aligned}
\] (2.8)

Using that
\[
\int_{|y-x| \leq ct} h(ct - |y - x|, y) dy = c^2 \int_0^t \int_{|\omega|=1} (t - \tau)^2 h(c\tau, x + c(t - \tau)\omega) d\omega d(\tau),
\]
we can also write the solutions of (2.6) in the form

\[
\begin{aligned}
\begin{cases}
E^N_m = \mathbb{E}_0 - \frac{1}{4\pi c^2} \iint_{\mathbb{R}^3} dpdz d\chi^N(p) \chi^N(z) \frac{(c^2 \nabla_y \rho^N_m \ast \partial_t j^N_m)(t - c^{-1}|x - y|, y - p - z)}{|x - y|}, \\
B^N_m = \mathbb{E}_0 + \frac{1}{4\pi c} \iint_{\mathbb{R}^3} dpdz d\chi^N(p) \chi^N(z) \frac{\nabla_y \ast_B j^N_m(y - p - z, t - c^{-1}|x - y|)}{|x - y|},
\end{cases}
\end{aligned}
\] (2.9)
where

\[
\begin{align*}
\mathbb{E}_0 &= \mathbb{E}_0(t) = \frac{t}{4\pi} \int_{|\omega|=1} E_m^N(0, x + ct\omega) \, d\omega + \frac{t}{4\pi} \int_{|\nu|=1} E_m^N(0, x + ct\omega) \, d\omega, \\
\mathbb{B}_0 &= \mathbb{B}_0(t) = \frac{t}{4\pi} \int_{|\nu|=1} B_m^N(0, x + ct\omega) \, d\nu + \frac{t}{4\pi} \int_{|\nu|=1} B_m^N(0, x + ct\omega) \, d\omega.
\end{align*}
\] (2.10)

3 Combined Mean Field Limit and Non-relativistic Limit

3.1 Regularized Vlasov-Maxwell Particle System

The regularized Vlasov-Maxwell system is given by

\[
\begin{align*}
\partial_t f_m^N(x,v) + \hat{v} \cdot \nabla_x f_m^N(x,v) + \left( E_m^N + c^{-1} \hat{v} \times B_m^N \right) \cdot \nabla_v f_m^N &= 0, \\
\partial_t E_m^N(x,v) &= c\nabla \times B_m^N - \chi^N * \chi^N * f_m^N, \\
\partial_t B_m^N(x,v) &= -c\nabla \times E_m^N, \\
\nabla \cdot E_m^N &= \rho_m^N, \\
\nabla \cdot B_m^N &= 0,
\end{align*}
\] (3.1)

where \( \hat{v} = \frac{v}{\sqrt{1 + c^{-2}v^2}} \), \( \rho_m^N(t,x) = \int_{\mathbb{R}^3} f_m^N(t,x,v) \, dv \), \( j_m^N(t,x) = \int_{\mathbb{R}^3} \hat{v} f_m^N(t,x,v) \, dv \) and the initial data

\[
\begin{align*}
f_m^N(0,x,v) &= f_0(x,v), \\
E_m^N(0,x) &= \chi^N * f_m^N(0,x), \\
B_m^N(0,x) &= \chi^N * f_m^N(0,x),
\end{align*}
\] (3.2)

satisfy the compatibility conditions \( \nabla \cdot E_0(x) = \rho_m^N(0,x) = \rho_0(x) \), \( \nabla \cdot B_0(x) = 0 \).

We consider the corresponding interacting particle system with position \( x_i \in \mathbb{R}^3 \), \( i = 1, \ldots, N \). The equations of the characteristics read

\[
\begin{align*}
\frac{dx_i}{dt} &= \hat{v}(v_i) = \frac{v_i}{\sqrt{1 + c^{-2}v_i^2}}, \\
\frac{dv_i}{dt} &= E_m^N(t,x_i) + c^{-1} \hat{v}(v_i) \times B_m^N(t,x_i),
\end{align*}
\] (3.3)

where

\[
\begin{align*}
E_m^N &= \partial_t Y^N *_x E_0 + Y^N *_x \left( c\nabla \times B_0 - j_m^N(0,.) \right) - Y^N *_{t,x} \left( c^2 \nabla \rho_m^N + \partial_t j_m^N \right), \\
B_m^N &= \partial_t Y^N *_x B_0 - c Y^N *_x \nabla \cdot E_0 + c Y^N *_{t,x} \nabla \times j_m^N.
\end{align*}
\] (3.4)

Before we present the analytical results in this section, we introduce the following notations:
Definition 3.1. 1. For any $1 \leq i \leq N$ (labeling the particle with position $x_{m}^{i,N} \in \mathbb{R}^{3}$ and velocity $v_{m}^{i,N} \in \mathbb{R}^{3}$) we denote the pair-interaction force by

$$F_{m}^{1,N}(t,x_{m}^{i,N}) = -\frac{1}{N-1} \sum_{j=1,j\neq i}^{N} \int_{0}^{t} (\hat{v}(v_{j,m}^{i,N}(s)) \partial_{t} + c^{2} \nabla_{x}Y^{N}(t-s,x_{m}^{i,N}(t) - x_{m}^{j,N}(s)) \, ds,$$

$$F_{m}^{2,N}(t,x_{m}^{i,N},v_{m}^{i,N}) = -\frac{1}{N-1} \sum_{j=1,j\neq i}^{N} \int_{0}^{t} \hat{v}(v_{m}^{i,N}(t)) \times \left( \hat{v}(v_{j,m}^{i,N}(s)) \times \nabla_{x}Y^{N}(t-s,x_{m}^{i,N}(t) - x_{m}^{j,N}(s)) \right) \, ds,$$

and the mean-field force of the Vlasov system by

$$F_{m}^{3,N}(t,x_{m}^{i,N},v_{m}^{i,N}) = E_{0} \ast_{x} \partial_{t}Y^{N}(t,x_{m}^{i,N}) + (c \nabla \times B_{0} - j_{m}^{N}(0,\cdot)) \ast_{x}Y^{N}(t,x_{m}^{i,N})$$

$$+ c^{-1} \hat{v}(v_{m}^{i,N}) \times B_{0} \ast_{x} \partial_{t}Y^{N}(t,x_{m}^{i,N})$$

$$- \hat{v}(v_{m}^{i,N}) \times (\nabla \times E_{0}) \ast_{x}Y^{N}(t,x_{m}^{i,N}).$$

2. Let $(X_{m}^{N}(t),V_{m}^{N}(t))$ be the trajectory on $\mathbb{R}^{6N}$ which evolves according to the Newtonian equation of motion for the regularized Vlasov-Maxwell system, i.e.,

$$\begin{cases}
\frac{d}{dt}X_{m}^{N}(t) = \hat{V}(V_{m}^{N}(t)), \\
\frac{d}{dt}V_{m}^{N}(t) = \Psi_{m}^{1,N}(t,X_{m}^{N}(t),V_{m}^{N}(t)) + \Psi_{m}^{2,N}(t,X_{m}^{N}(t),V_{m}^{N}(t)) + \Gamma_{m}^{N}(t,X_{m}^{N}(t),V_{m}^{N}(t)),
\end{cases}$$

(3.5)

where $\Psi_{m}^{1,N}(t,X_{m}^{N}(t),V_{m}^{N}(t))$ and $\Psi_{m}^{2,N}(t,X_{m}^{N}(t),V_{m}^{N}(t))$ denote the total interaction force with

$$\left( \Psi_{m}^{1,N}(t,X_{m}^{N}(t),V_{m}^{N}(t)) \right)_{i} = F_{m}^{1,N}(t,x_{m}^{i,N})$$

$$= -\frac{1}{N-1} \sum_{j=1,j\neq i}^{N} \int_{0}^{t} (\hat{v}(v_{j,m}^{i,N}(s)) \partial_{t} + c^{2} \nabla_{x}Y^{N}(t-s,x_{m}^{i,N}(t) - x_{m}^{j,N}(s)) \, ds,$$

$$\left( \Psi_{m}^{2,N}(t,X_{m}^{N}(t),V_{m}^{N}(t)) \right)_{i} = F_{m}^{2,N}(t,x_{m}^{i,N},v_{m}^{i,N})$$

$$= -\frac{1}{N-1} \sum_{j=1,j\neq i}^{N} \int_{0}^{t} \hat{v}(v_{m}^{i,N}(t)) \times \left( \hat{v}(v_{j,m}^{i,N}(s)) \times \nabla_{x}Y^{N}(t-s,x_{m}^{i,N}(t) - x_{m}^{j,N}(s)) \right) \, ds$$

while $\Gamma_{m}^{N}(t,X_{m}^{N}(t),V_{m}^{N}(t))$ stands for the self-driven force with

$$\left( \Gamma_{m}^{N}(t,X_{m}^{N}(t),V_{m}^{N}(t)) \right)_{i} = F_{m}^{3,N}(t,x_{m}^{i,N},v_{m}^{i,N})$$

$$= E_{0} \ast_{x} \partial_{t}Y^{N}(t,x_{m}^{i,N}) + (c \nabla \times B_{0}^{N} - j_{m}^{N}(0,\cdot)) \ast_{x}Y^{N}(t,x_{m}^{i,N})$$

$$+ c^{-1} \hat{v}(v_{m}^{i,N}) \times B_{0} \ast_{x} \partial_{t}Y^{N}(t,x_{m}^{i,N}) - \hat{v}(v_{m}^{i,N}) \times (\nabla \times E_{0}) \ast_{x}Y^{N}(t,x_{m}^{i,N}).$$

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3. Let \((\overline{X}_m(t), \overline{V}_m(t))\) be the trajectory on \(\mathbb{R}^{6N}\) which evolves according to the regularized Vlasov-Maxwell equation

\[
\partial_t f_m^N + \hat{v} \cdot \nabla_x f_m^N + (E_m^N + c^{-1} \hat{v} \times B_m^N) \cdot \nabla_v f_m^N = 0,
\]

i.e.,

\[
\begin{cases}
\frac{d}{dt} \overline{X}_m(t) = \hat{V}(\overline{V}_m(t)), \\
\frac{d}{dt} \overline{V}_m(t) = \overline{\Psi}_m^1(t, \overline{X}_m(t)) + \overline{\Psi}_m^2(t, \overline{X}_m(t), \overline{V}_m(t)) + \Gamma_m(t, \overline{X}_m(t), \overline{V}_m(t)),
\end{cases}
\]

where

\[
(\overline{\Psi}_m^1(t, \overline{X}_m(t)))_i = F_m^{1,N}(t, \overline{x}_m^i,N) = -(c^2 \nabla \rho_m^N + \partial_x j_m^N) *_{t,x} Y^N(t, \overline{x}_m^i,N)
\]

\[
= - \int \int_{\mathbb{R}^6} \int_0^t \text{d}s \text{d}y \text{d}v
\]

\[
(c^2 \nabla + \hat{v}(v) \partial_s)f_m^N(s, \overline{x}_m^i,N - y, v)Y^N(t - s, y),
\]

\[
(\overline{\Psi}_m^2(t, \overline{X}_m(t), \overline{V}_m(t)))_i = F_m^{2,N}(t, \overline{x}_m^i,N, \overline{v}_m^i,N)
\]

\[
= - \int \int_{\mathbb{R}^6} \int_0^t \text{d}s \text{d}y \text{d}v
\]

\[
\hat{v}(\overline{v}_m^i,N) \times \hat{v}(v) \times \nabla_x f_m^N(s, \overline{x}_m^i,N - y, v)Y^N(t - s, y),
\]

\[
(\Gamma_m^N(t, \overline{X}_m(t), \overline{V}_m(t)))_i = F_m^{3,N}(t, \overline{x}_m^i,N, \overline{v}_m^i,N)
\]

\[
= E_0^N *_{x} \partial_t Y^N(t, \overline{x}_m^i,N) + (c \nabla \times B_0^N - j_m^N(0, \cdot)) *_{x} Y^N(t, \overline{x}_m^i,N)
\]

\[
+ c^{-1} \hat{v}(\overline{v}_m^i,N) \times B_0^N *_{x} \partial_t Y^N(t, \overline{x}_m^i,N)
\]

\[
- \hat{v}(\overline{v}_m^i,N) \times (\nabla \times E_0^N) *_{x} Y^N(t, \overline{x}_m^i,N).
\]

represent the total interaction forces and the self-driven force, respectively.

\((X(t), V(t))\) and \((\overline{X}(t), \overline{V}(t))\) without superscript \(N\) denote the particle configurations driven by the force without cut-off. \((X, V)\) and \((\overline{X}, \overline{V})\), without the argument \(t\), stand for the stochastic initial data, which are independent and identically distributed. Note that we always consider the same initial data for both systems, that means \((X, V) = (\overline{X}, \overline{V})\).

The following lemma gives us and estimates on the interaction forces, which will be used in the limiting procedure.

**Lemma 3.1.** Let \(F_m^{1,N}(t, x)\) and \(F_m^{2,N}(t, x, v)\) be defined as in (3.8) and (3.9). Then there exists a constant \(M\) such that

\[
\|F_m^{1,N}(t, x)\|_{L^\infty([0,T] \times \mathbb{R}^3)} + \|F_m^{2,N}(t, x, v)\|_{L^\infty([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)} \leq cM.
\]
Proof. By definition, we know that
\[
\| F_{m}^{1,N}(t, x) \|_{L^\infty([0, T] \times \mathbb{R}^3)}
= \| - (c^2 \nabla_x f_m^N + \partial_t j_m^N) *_{t,x} Y^{N}(t, x) \|_{L^\infty([0, T] \times \mathbb{R}^3)}
= \left\| \int_0^t \int_{\mathbb{R}^6} (c^2 \nabla_x + \dagger(v) \partial_s) f_m^N(s, x - y, v) Y^{N}(t - s, y) \, ds \, dy \, dv \right\|_{L^\infty([0, T] \times \mathbb{R}^3)},
\]
where for \( s < t \)
\[
Y^{N}(t - s, y) = \int_{\mathbb{R}^3} \int_{|z| \leq c(t-s)} \frac{1}{4\pi c |z|} \delta(|z| - c(t-s)) \chi^N(y - p - z) \chi^N(p) \, dz \, dp
= \int_{\mathbb{R}^3} \int_{|\omega| = 1} \int_0^{c(t-s)} \frac{\tau}{4\pi c} \delta(\tau - c(t-s)) \chi^N(y - p - \tau \omega) \chi^N(p) \, d\tau \, d\omega \, dp
= \int_{\mathbb{R}^3} \int_{|\omega| = 1} \frac{t - s}{4\pi} \chi^N(y - p - c(t-s) \omega) \chi^N(p) \, d\omega \, dp.
\]
and
\[
\| Y^{N}(t - s, y) \|_{L^\infty([0, T] \times \mathbb{R}^3)} \leq \frac{M}{4\pi c} \int_{\mathbb{R}^3} \chi^N(p) \, dp = \frac{M}{4\pi c}.
\]
So
\[
\left(3.10\right)
= \left\| \frac{1}{4\pi} \int_0^t \int_{\mathbb{R}^3} \int_{|\omega| = 1} \int_0^{c(t-s)} \frac{\tau}{4\pi c} \delta(\tau - c(t-s)) \chi^N(y - p - \tau \omega) \chi^N(p) \, d\tau \, d\omega \, dp \int_{\mathbb{R}^3} d\nu \right\|_{L^\infty([0, T] \times \mathbb{R}^3)}
\leq cM \left( \sup_{0 \leq t \leq T} \| \partial_t f_m^N(t, \cdot, \cdot) \|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} + \sup_{0 \leq t \leq T} \| \nabla_x f_m^N(t, \cdot, \cdot) \|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \right).
\]
And similarly we have
\[
\| F_{m}^{2,N}(t, x, v) \|_{L^\infty([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3)}
= \left\| - \int_0^t \int_{\mathbb{R}^6} \dagger(v) \times \hat{v}(z) \times \nabla_x f_m^N(s, x - y, z) Y^{N}(t - s, y) \, ds \, dy \, dz \right\|_{L^\infty([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3)}
\leq cM \left( \sup_{0 \leq t \leq T} \| \nabla_x f_m^N(t, \cdot, \cdot) \|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \right).
\]

3.2 Regularized Vlasov-Poisson Particle Model

In this section, we consider the Vlasov-Poisson particle model and deduce estimates of the distance between the solutions of Vlasov-Maxwell and Vlasov-Poisson.
The regularization of the Vlasov-Poisson System (VPN) with unknown \((f^N_p, E^N_p)\) is given by

\[
\begin{cases}
\partial_t f^N_p + v \cdot \nabla_x f^N_p + E^N_p \cdot \nabla_v f^N_p = 0, \\
E^N_p(t, x) = \frac{1}{4\pi} \int \int \int_{\mathbb{R}^3} dydpdz \frac{x-y}{|x-y|^3} \chi^N(p)\chi^N(z)\rho^N_p(t, y-p-z), \\
\rho^N_p(t, y) = \int_{\mathbb{R}^3} f^N_p(t, y, v) dv,
\end{cases}
\tag{3.11}
\]

with the initial data \(f^N_p(x, v, 0) = f_0(x, v)\). Thus, the corresponding Vlasov-Poisson equations of characteristics read

\[
\begin{cases}
\frac{d}{dt} X^N_p = V^N_p, \\
\frac{d}{dt} V^N_p = E^N_p(t, X^N_p) = \frac{1}{4\pi} \int \int \int_{\mathbb{R}^3} dydpdz \frac{X^N_p-y}{|X^N_p-y|^3} \chi^N(p)\chi^N(z)\rho^N_p(t, y-p-z).
\end{cases}
\tag{3.12}
\]

Now we can compare the solutions of the Vlasov-Maxwell and Vlasov-Poisson equations. This requires a more detailed estimate on the respective solutions, which we denote by \(f^N_m\) and \(f^N_p\). Using the results in section 3, we know that

\[
\|E^N_m + c^{-1} \hat{v} \times B^N_m - E^N_p\|_{L^\infty(\mathbb{R}^3)} \leq \|E^N_m - E^N_p\|_{L^\infty(\mathbb{R}^3)} + \|c^{-1} \hat{v} \times B^N_m\|_{L^\infty(\mathbb{R}^3)} \leq c^{-1} M.
\tag{3.13}
\]

We will next compare the \(N\)-particle Vlasov-Maxwell equation with the Vlasov-Poisson equation. Since the \(N\)-body system is subject to a regularized force it is most natural to introduce that regularization also for the Vlasov-Poisson system. The translation of the one-body Vlasov-Poisson system to an \(N\)-body dynamics is straight forward: each particle moves with the same flow given by the Vlasov-Poisson equation. This allows now comparison with the \(N\)-body characteristics coming from the \(N\)-particle Vlasov-Maxwell equation.

**Definition 3.2.** Let \((X^N_p(t), V^N_p(t))\) be the trajectory on \(\mathbb{R}^{6N}\) which evolves according to the regularized Vlasov-Poisson equation

\[
\partial_t f^N_p + v \cdot \nabla_x f^N_p + E^N_p \cdot \nabla_v f^N_p = 0,
\tag{3.14}
\]

i.e.,

\[
\begin{cases}
\frac{d}{dt} X^N_p(t) = V^N_p(t), \\
\frac{d}{dt} V^N_p(t) = \nabla^N_p(t, X^N_p(t)),
\end{cases}
\tag{3.15}
\]
where

\[
\left( \Psi_p^N(t, X_p^N(t)) \right)_i = F_p^N(t, \pi_p^i) = E_p^N(t, \pi_p^i) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} dy dp dz \frac{\pi_p^N - y}{|\pi_p^N - y|^3} \chi^N(p) \chi^N(z) \rho_p^N(t, y - p - z).
\]

### 3.3 Estimates for the Mean Field Limit

In this section, we present our key results in full detail. To show the desired convergence, our method can be summarized as follows. First, we start from the Newtonian system with carefully chosen cut-off and meanwhile introduce an intermediate system which involves a convolution-type interaction with cut-off, respectively mollifier given in (2.1). Then, we show convergence of the intermediate system to the final mean field limit, where the law of large number comes into play. The crucial point of this method is that we apply stochastic initial data or in other words we consider a stochastic process. This enables us to use tools from probability theory, which helps to better understand the mean field process. The overall procedure can be summarized as follows:

The following assumptions are used throughout this section.

**Assumption 3.1.** We assume that

(a) \( E_0 \) and \( B_0 \) are all Lipschitz continuous functions.

(b) \( \alpha \in \left( 0, \frac{1}{8} \right), \beta \in \left( \alpha, \frac{1}{4} - \frac{\alpha}{4} \right) \) and \( \theta \in \left( 0, \frac{1}{16} - \alpha - 4\beta \right) \).

**Definition 3.3.** Let \( S_t : \mathbb{R}^{6N} \times \mathbb{R} \rightarrow \mathbb{R} \) be the stochastic process given by

\[
S_t = \min \left\{ 1, N^\alpha, \sup_{0 \leq s \leq t} \left| (X_m^N(s), V_m^N(s)) - (\overline{X}_m^N(s), \overline{V}_m^N(s)) \right| \right\}.
\]

The set, where \( |S_t| = 1 \), is defined as \( \mathcal{N}_\alpha \), i.e.,

\[
\mathcal{N}_\alpha := \left\{ (X, V) : \sup_{0 \leq s \leq t} \left| (X_m^N(s), V_m^N(s)) - (\overline{X}_m^N(s), \overline{V}_m^N(s)) \right|_\infty > N^{-\alpha} \right\}.
\]

Here and in the following we use \( \cdot \) as the supremum norm on \( \mathbb{R}^{6N} \). Note that

\[
\mathbb{E}_0(S_{t+dt} - S_t | \mathcal{N}_\alpha) \leq 0,
\]

since \( S_t \) takes the value of one for \( (X, V) \in \mathcal{N}_\alpha \).
Theorem 3.1. Let $f^N_m(t,x,v)$ be a solution of the regularized Vlasov-Maxwell equation (3.6). Suppose that Assumptions 3.1 are satisfied. Then there exists a constant $M$ such that

$$
P_0 \left( \sup_{0 \leq s \leq t} \left| (X^N_m(s), V^N_m(s)) - (\bar{X}^N_m(s), \bar{V}^N_m(s)) \right| \right)_\infty > N^{-\alpha} \leq e^{Mt} \cdot c^4 N^{-(1-4\beta-16\theta)}.
$$

The proof of the theorem will be presented later in this section.

Definition 3.4. The sets $\mathcal{N}_\beta$ and $\mathcal{N}_\gamma$ are characterized by

$$\mathcal{N}_\beta := \left\{ (X_m, V_m) : \left| N^\beta \Psi^1_m \left( \bar{X}^N_m(t), \bar{V}^N_m(t) \right) - N^\beta \overline{\Psi}^1_m \left( \bar{X}^N_m(t), \bar{V}^N_m(t) \right) \right|_\infty > N^{-\beta} \right\}, (3.17)$$

$$\mathcal{N}_\gamma := \left\{ (X_m, V_m) : \left| N^\beta \Psi^2_m \left( \bar{X}^N_m(t), \bar{V}^N_m(t) \right) - N^\beta \overline{\Psi}^2_m \left( \bar{X}^N_m(t), \bar{V}^N_m(t) \right) \right|_\infty > N^{-\gamma} \right\}. (3.18)$$

Next, we will see that the probability of both sets $\mathcal{N}_\beta$ and $\mathcal{N}_\gamma$ tends to 0 as $N$ goes to infinity. We prove the following two lemmas:

Lemma 3.2. There exists a constant $M < \infty$ such that

$$P_0(\mathcal{N}_\beta) \leq Mc^4 \cdot N^{-(1-4\beta-16\theta)}.$$

Proof. First, we let the set $\mathcal{N}_\beta$ evolve along the characteristics of the regularized Vlasov-Maxwell equation

$$\mathcal{N}_{\beta,t} := \left\{ (\bar{X}^N_m(t), \bar{V}^N_m(t)) : \left| N^\beta \Psi^1_m \left( \bar{X}^N_m(t), \bar{V}^N_m(t) \right) - N^\beta \overline{\Psi}^1_m \left( \bar{X}^N_m(t), \bar{V}^N_m(t) \right) \right|_\infty > 1 \right\}$$

and consider the following fact

$$\mathcal{N}_{\beta,t} \subseteq \bigoplus_{i=1}^N \mathcal{N}^i_{\beta,t},$$

where

$$\mathcal{N}^i_{\beta,t} := \left\{ (\bar{x}^i_m, \bar{v}^i_m) : \left| N^\beta \frac{1}{N-1} \sum_{j=1,j\neq i}^N \int_t^0 (\hat{v}(\nu^j_m(s))\partial_t + c^2 \nabla_x)Y^N(t-s, \bar{x}^i_m(t) - \bar{x}^j_m(s)) ds \right. \right.$$
We therefore get
\[ P_t(\mathcal{N}_{\beta,t}) \leq \sum_{i=1}^{N} P_t(\mathcal{N}_{\beta,i}) = N P_t(\mathcal{N}_{\beta,t}), \]
where in the last step we used symmetry in exchanging any two coordinates.

Using Markov inequality gives
\[ P_t(\mathcal{N}_{\beta,t}^j) \leq \mathbb{E}_t \left[ \left( \frac{N^\beta}{N-1} \right)^4 \sum_{j=2}^{N} F^N_1(t, \overline{x}_m^1, \overline{x}_j^N) - N^\beta \mathcal{F}^N_1(t, \overline{x}_m^1) \right]^4 \]
\[ = \left( \frac{N^\beta}{N-1} \right)^4 \mathbb{E}_t \left[ \left( \sum_{j=2}^{N} F^N_1(t, \overline{x}_m^1, \overline{x}_j^N) - (N-1) \mathcal{F}^N_1(t, \overline{x}_m^1) \right)^4 \right]. \]

Let \( h_j := F^N_1(t, \overline{x}_m^1, \overline{x}_j^N) - \mathcal{F}^N_1(t, \overline{x}_m^1) \). Then, each term in the expectation (3.19) takes the form of \( \prod_{j=2}^{N} h_j^k \) with \( \sum_{j=1}^{N} k_j = 4 \), and more importantly, the expectation assumes the value of zero whenever there exists a \( j \) such that \( k_j = 1 \). This can be easily verified by integrating over the \( j \)-th variable first or, in other words, by acknowledging the fact that \( \forall j = 2, \ldots, N \), there holds
\[ \mathbb{E}_t \left[ F^N_1(t, \overline{x}_m^1, \overline{x}_j^N) - \mathcal{F}^N_1(t, \overline{x}_m^1) \right] = 0. \]

Then, we can simplify the estimate (3.19) to
\[ P_t(\mathcal{N}_{\beta,t}^j) \leq \left( \frac{N^\beta}{N-1} \right)^4 \mathbb{E}_t \left[ \sum_{j=2}^{N} h_j^4 + \sum_{2 \leq m < n} \left( \frac{4}{2} \right) h_m^2 h_n^2 \right]. \]

Since
\[ \| F^N_1 \|_{L^\infty([0,T] \times \mathbb{R}^3)} \leq M c^2 N^{4\theta}, \]
and
\[ \| F^N_1 \|_{L^\infty([0,T] \times \mathbb{R}^3)} \leq c M \left( \sup_{0 \leq t \leq T} \| f^N_m(t, \cdot, \cdot) \|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} + \sup_{0 \leq t \leq T} \| \nabla_x f^N_m(t, \cdot, \cdot) \|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \right) \]
we thus have for any fixed \( j \)
\[ |h_j| \leq |F^N_1(t, \overline{x}_m^1, \overline{x}_j^N)| + |\mathcal{F}^N_1(t, \overline{x}_m^1)| \]
\[ \leq c M \left( N^{4\theta} + \sup_{0 \leq t \leq T} \| f^N_m(t, \cdot, \cdot) \|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} + \sup_{0 \leq t \leq T} \| \nabla_x f^N_m(t, \cdot, \cdot) \|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \right). \]

Therefore \( |h_j| \) is bounded to any power and we obtain
\[ \mathbb{E}_t \left[ h_m^2 h_n^2 \right] \leq M c^4 N^{16\theta} \quad \text{and} \quad \mathbb{E}_t \left[ h_j^4 \right] \leq M c^4 N^{16\theta} \]
and consequently
\[
\mathbb{P}_t(N_{\beta,t}^1) \leq \left( \frac{N^\beta}{N-1} \right)^4 \cdot \left( Mc^4 \cdot (N-1) + Mc^4 N^{16\theta} \cdot \frac{(N-1)(N-2)}{2} \right) \\
\leq Mc^4 N^{16\theta} \cdot N^{-(2-4\beta-16\theta)}.
\]

By noticing the fact that
\[
\mathbb{P}_0(N_{\beta}) = \mathbb{P}_t(N_{\beta,t}) \leq N \mathbb{P}_t(N_{\beta,t}^1) \\
\leq N \cdot Mc^4 \cdot N^{-(2-4\beta-16\theta)} = Mc^4 \cdot N^{-(1-4\beta-16\theta)},
\]
we obtain the desired result.

\[\blacksquare\]

In fact, this result holds for any \( \beta \) if we change the power in the proof to be another even number (depending on \( \beta \)) greater than four. So with similar estimates we get:

**Lemma 3.3.** There exists a constant \( M < \infty \) such that
\[
\mathbb{P}_0(N_\gamma) \leq Mc^4 \cdot N^{-(1-4\gamma-10\theta)}.
\]

**Lemma 3.4.** Let \( N_\alpha, N_\beta, N_\gamma \) be defined as in (3.16)-(3.18). Suppose that \( f_m^N(t,x,v) \) is a solution of the regularized Vlasov-Maxwell equation and Assumption 3.1 is satisfied. Then there exists a constant \( M < \infty \) such that
\[
\left| \left( \hat{V}(V_m^N(t)), \Psi_m^{1,N}(t), X_m^N(t), V_m^N(t) \right) + \Psi_m^{2,N}(t,X_m^N(t),V_m^N(t)) + \Gamma_m^N(t,X_m^N(t),V_m^N(t)) \right|_\infty \\
\leq MS_t(X,V)N^{-\alpha} + N^{-\beta}
\]
for all initial data \( (X,V) \in (N_\alpha \cup N_\beta \cup N_\gamma)^c \).
Proof. Applying triangle inequality gives
\[
\left| \left( \hat{V}(V_m(t), \Psi_m(t, X_m(t), V_m(t)) + \Psi^2_m(t, X_m(t), V_m(t)) + \Gamma_m(t, X_m(t), V_m(t)) \right) \right| \\
- \left( \hat{V}(\nabla_m(t), \Psi_m(t, X_m(t), \nabla_m(t)) + \Psi^2_m(t, X_m(t), \nabla_m(t)) + \Gamma_m(t, X_m(t), \nabla_m(t)) \right) \right| \infty \\
\leq \left| \hat{V}(V_m(t)) - \hat{V}(\nabla_m(t)) \right| \infty + \left| \Psi_m(t, X_m(t), V_m(t)) - \Psi_m(t, X_m(t), \nabla_m(t)) \right| \infty \\
+ \left| \Psi^2_m(t, X_m(t), V_m(t)) - \Psi^2_m(t, X_m(t), \nabla_m(t)) \right| \infty \\
+ \left| \Gamma_m(t, X_m(t), V_m(t)) - \Gamma_m(t, X_m(t), \nabla_m(t)) \right| \infty \\
=: \left| I_1 \right| + \left| I_2 \right| + \left| I_3 \right| + \left| I_4 \right| + \left| I_5 \right| + \left| I_6 \right|. 
\]

Next, we estimate term by term.

- Since \((X, V) \notin N_\alpha\),
\[
\left| I_1 \right| := \left| \hat{V}(V_m(t)) - \hat{V}(\nabla_m(t)) \right| \infty \leq MS_t(X, V).N^{-\alpha}. 
\]

- Note, that \(F^1_m\) is Lipschitz continuous in \(x\). We denote \(L\) as the global Lipschitz constant for all the Lipschitz continuous functions in this paper. Thus we obtain
\[
\left| \frac{1}{N-1} \sum_{i \neq j} F^1_m(t, x_m^i, \bar{x}_m^i) - \frac{1}{N-1} \sum_{i \neq j} F^1_m(t, \bar{x}_m^i, \bar{x}_m^i) \right| \infty \leq \frac{1}{N-1} \sum_{i \neq j} L \cdot 2|x_m^i - \bar{x}_m^i|. \tag{3.20} 
\]

Since \((X, V) \notin N_\alpha\), it follows in particular for any \(1 \leq i \leq N\) that
\[
|x_m^i - \bar{x}_m^i| \leq N^{-\alpha}. 
\]

So together with (3.20), we have
\[
\left| \Psi_m(t, X_m(t), V_m(t)) - \Psi_m(t, \bar{X}_m(t), \bar{V}_m(t)) \right| \infty \leq 2LN^{-\alpha}, 
\]
and thus
\[ |I_2| := \left| \Psi_m^{1,N}(t, X_m^N(t), V_m^N(t)) - \Psi_m^{1,N}(t, \bar{X}_m^N(t), \bar{V}_m^N(t)) \right|_\infty \leq MS_t(X, V)N^{-\alpha}. \]

Similarly
\[ |I_4| := \left| \Psi_m^{2,N}(t, X_m^N(t), V_m^N(t)) - \Psi_m^{2,N}(t, \bar{X}_m^N(t), \bar{V}_m^N(t)) \right|_\infty \leq MS_t(X, V)N^{-\alpha}. \]

- Since \((X, V) \notin \mathcal{N}_\beta\), it follows directly
\[ |I_3| := \left| \Psi_m^{1,N}(t, \bar{X}_m^N(t), \bar{V}_m^N(t)) - \Psi_m^{1,N}(t, \bar{X}_m^N(t), \bar{V}_m^N(t)) \right|_\infty \leq N^{-\beta}. \]

- Since \((X, V) \notin \mathcal{N}_\gamma\), it follows directly
\[ |I_5| := \left| \Psi_m^{2,N}(t, \bar{X}_m^N(t), \bar{V}_m^N(t)) - \Psi_m^{2,N}(t, \bar{X}_m^N(t), \bar{V}_m^N(t)) \right|_\infty \leq N^{-\beta}. \]

- Since \(E_0\) and \(B_0\) under Assumption 4.1(a) are Lipschitz continuous, we have for each \(1 \leq i \leq N, (x_m^{i,N}, v_m^{i,N}) = ((X_m^N(t), V_m^N(t)))_1, (\bar{x}_m^{i,N}, \bar{v}_m^{i,N}) = ((\bar{X}_m^N(t), \bar{V}_m^N(t)))_1\) and together with the fact that \((X, V) \notin \mathcal{N}_\alpha\), there holds
\[ |I_6| := \left| \Gamma_m^N(t, X_m^N(t), V_m^N(t)) - \Gamma_m^N(t, \bar{X}_m^N(t), \bar{V}_m^N(t)) \right|_\infty \leq LS_t(X, V)N^{-\alpha}. \]

Combining all the six terms, we end up with
\[
\left| \left( \dot{V}(V_m^N(t)), \Psi_m^{1,N}(t, X_m^N(t), V_m^N(t)) + \Psi_m^{2,N}(t, X_m^N(t), V_m^N(t)) + \Gamma_m^N(t, X_m^N(t), V_m^N(t)) \right) \\
- \left( \dot{V}(\bar{V}_m^N(t)), \Psi_m^{1,N}(t, \bar{X}_m^N(t), \bar{V}_m^N(t)) + \Psi_m^{2,N}(t, \bar{X}_m^N(t), \bar{V}_m^N(t)) + \Gamma_m^N(t, \bar{X}_m^N(t), \bar{V}_m^N(t)) \right) \right|_\infty \\
\leq MS_t(X, V)N^{-\alpha} + N^{-\beta}
\]
for all \((X, V) \in (\mathcal{N}_\alpha \cup \mathcal{N}_\beta \cup \mathcal{N}_\gamma)^c\).

Using all the Lemmas above we can now prove Theorem 3.1:

**Proof of Theorem 3.1**

From the definition of the Newtonian flow (3.5) and the characteristics of the Vlasov equation (3.7), we know that
\[
(X_m^N(t + dt), V_m^N(t + dt)) = (X_m^N(t), V_m^N(t)) \\
+ \left( \dot{V}(V_m^N(t)), \Psi_m^{1,N}(t, X_m^N(t), V_m^N(t)) + \Psi_m^{2,N}(t, X_m^N(t), V_m^N(t)) + \Gamma_m^N(t, X_m^N(t), V_m^N(t)) \right) dt + o(dt),
\]

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and
\[
(X_m(t + dt), V_m(t + dt)) = (X_m(t), V_m(t)) + \left(\hat{V}(\overline{V}_m(t)), \overline{V}_m(t, X_m(t)) + \overline{V}_m(t, X_m(t), \overline{V}_m(t)) + \Gamma_m(t, X_m(t), V_m(t))\right)dt + o(dt).
\]

Thus
\[
\left| (X_m(t + dt), V_m(t + dt)) - (X_m(t + dt), V_m(t + dt)) \right| \leq \left| (X_m(t), V_m(t)) - (X_m(t), V_m(t)) \right| + \left| \left(\hat{V}(\overline{V}_m(t)), \overline{V}_m(t, X_m(t)) + \overline{V}_m(t, X_m(t), \overline{V}_m(t)) + \Gamma_m(t, X_m(t), V_m(t))\right) - \left(\hat{V}(\overline{V}_m(t)), \overline{V}_m(t, X_m(t)) + \overline{V}_m(t, X_m(t), \overline{V}_m(t)) + \Gamma_m(t, X_m(t), V_m(t))\right) \right| dt + o(dt),
\]
i.e.,
\[
S_{t+dt} - S_t \leq \left| \left(\hat{V}(\overline{V}_m(t)), \overline{V}_m(t, X_m(t)) + \overline{V}_m(t, X_m(t), \overline{V}_m(t)) + \Gamma_m(t, X_m(t), V_m(t))\right) - \left(\hat{V}(\overline{V}_m(t)), \overline{V}_m(t, X_m(t)) + \overline{V}_m(t, X_m(t), \overline{V}_m(t)) + \Gamma_m(t, X_m(t), V_m(t))\right) \right| dt + o(dt),
\]
Taking the expectation over both sides yields
\[
\mathbb{E}_0 \left[ S_{t+dt} - S_t \right] = \mathbb{E}_0 \left[ S_{t+dt} - S_t \mid \mathcal{N}_\alpha \right] + \mathbb{E}_0 \left[ S_{t+dt} - S_t \mid \mathcal{N}_\alpha^c \right]
\]
\[
\leq \mathbb{E}_0 \left[ S_{t+dt} - S_t \mid (\mathcal{N}_\beta \cup \mathcal{N}_\gamma) \setminus \mathcal{N}_\alpha \right] + \mathbb{E}_0 \left[ S_{t+dt} - S_t \mid (\mathcal{N}_\alpha \cup \mathcal{N}_\beta \cup \mathcal{N}_\gamma)^c \right]
\]
\[
\leq 0 \quad \mathbb{E}_0 \left[ (\mathcal{N}_\beta \cup \mathcal{N}_\gamma) \setminus \mathcal{N}_\alpha \right] N^\alpha dt
\]
\[
+ \mathbb{E}_0 \left[ \overline{V}_m(t, X_m(t), V_m(t)) - \overline{V}_m(t, X_m(t), \overline{V}_m(t)) \right] \mathbb{E}_0 \left[ (\mathcal{N}_\beta \cup \mathcal{N}_\gamma) \setminus \mathcal{N}_\alpha \right] N^\alpha dt
\]
\[
+ \mathbb{E}_0 \left[ \overline{V}_m(t, X_m(t), V_m(t)) - \overline{V}_m(t, X_m(t), \overline{V}_m(t)) \right] \mathbb{E}_0 \left[ (\mathcal{N}_\beta \cup \mathcal{N}_\gamma) \setminus \mathcal{N}_\alpha \right] N^\alpha dt
\]
\[
+ \mathbb{E}_0 \left[ (\mathcal{N}_\beta \cup \mathcal{N}_\gamma) \setminus \mathcal{N}_\alpha \right] N^\alpha dt
\]
\[
=: J_1 + J_2 + J_3 + J_4 + J_5 + o(dt),
\]
where in the second step we use \(\mathbb{E}_0(S_{t+dt} - S_t \mid \mathcal{N}_\alpha) \leq 0\) and decompose the set \(\mathcal{N}_\alpha^c\) into \((\mathcal{N}_\beta \cup \mathcal{N}_\gamma) \setminus \mathcal{N}_\alpha\) and \((\mathcal{N}_\alpha \cup \mathcal{N}_\beta \cup \mathcal{N}_\gamma)^c\).

Since \((X, V) \notin \mathcal{N}_\alpha\), it follows
\[
J_1 = \mathbb{E}_0 \left[ \hat{V}(\overline{V}_m(t)) - \hat{V}(\overline{V}_m(t)) \right] \mathbb{E}_0 \left[ (\mathcal{N}_\beta \cup \mathcal{N}_\gamma) \setminus \mathcal{N}_\alpha \right] N^\alpha dt
\]
\[
\leq L(\mathbb{P}_0(\mathcal{N}_\beta) + \mathbb{P}_0(\mathcal{N}_\gamma)) dt.
\]
Due to the definition of $\Psi_{1,N}^{m}$, $\Psi_{2,N}^{m}$, and $\Gamma_{m}^{N}$, we obtain
\[ J_2 + J_3 + J_4 \leq M \left( \mathbb{P}_0(\mathcal{N}_\beta) + \mathbb{P}_0(\mathcal{N}_\gamma) \right) N^\alpha dt. \]

Thanks to Lemma 4.1 and Lemma 4.2, we get
\[ J_1 + J_2 + J_3 + J_4 \leq M \left( \mathbb{P}_0(\mathcal{N}_\beta) + \mathbb{P}_0(\mathcal{N}_\gamma) \right) N^\alpha dt \leq Me^4 \cdot N^{-(1-4\beta-16\theta)} N^\alpha dt. \]

On the other hand, Lemma 3.4 states that
\[ J_5 = \mathbb{E}_0 \left[ S_{t+dt} - S_t \mid (\mathcal{N}_\alpha \cup \mathcal{N}_\beta \cup \mathcal{N}_\gamma)^c \right] \leq (M \cdot \mathbb{E}_0[S_t] N^{-\alpha} + N^{-\beta}) \cdot N^\alpha dt + o(dt) \]
\[ = M \cdot \mathbb{E}_0[S_t] dt + N^{\alpha-\beta} dt + o(dt). \]

Therefore, we can determine the estimate
\[ \mathbb{E}_0[S_{t+dt}] - \mathbb{E}_0[S_t] \leq \mathbb{E}_0[S_{t+dt} - S_t] \leq M \cdot \mathbb{E}_0[S_t] dt + M \cdot c^4 \cdot N^{-(1-4\beta-16\theta)} dt + o(dt). \]

Equivalently, we have
\[ \frac{d}{dt} \mathbb{E}_0[S_t] \leq M \cdot \mathbb{E}_0[S_t] + M \cdot c^4 \cdot N^{-(1-4\beta-16\theta)}. \]

Gronwall’s inequality yields
\[ \mathbb{E}_0[S_t] \leq e^{Mt} \cdot c^4 \cdot N^{-(1-4\beta-16\theta)}. \]

The proof is completed by the following Markov inequality
\[ \mathbb{P}_0 \left( \sup_{0 \leq s \leq t} \left| \left( X_{m}^N(s), V_{m}^N(s) \right) - \left( X_{p}^N(s), V_{p}^N(s) \right) \right|_\infty > N^{-\alpha} \right) = \mathbb{P}_0(S_t = 1) \leq \mathbb{E}_0[S_t]. \]

3.4 Estimates for the Non-relativistic Limit

Due to the key estimate (3.13), it is easy to repeat the whole procedure in the previous subsection to obtain

**Theorem 3.2.** Let $f_{m}^N(t,x,v)$ and $f_{p}^N(t,x,v)$ be the solutions to the regularized Vlasov-Maxwell equation (3.6) and (3.14) respectively with the same initial data $f_0$. Suppose that Assumptions 3.1 are satisfied. Then
\[ \mathbb{P}_0 \left( \sup_{0 \leq s \leq t} \left| \left( X_{m}^N(s), V_{m}^N(s) \right) - \left( X_{p}^N(s), V_{p}^N(s) \right) \right|_\infty > N^{-\alpha} \right) \leq e^{Mt} \frac{M}{c}. \]

**Remark 3.1.** We point out that the proof is straightforward when we use the flows of (3.7) and (3.15).
3.5 Combined Limit

Now with all the estimates we achieved above, we take $c = N^{\eta}$, $\eta \in (0, \frac{1-\alpha-4\beta-16\theta}{4})$.

**Theorem 3.3.** Let $f_{m}^{N}(t,x,v)$ and $f_{p}^{N}(t,x,v)$ be the solutions to the regularized Vlasov-Maxwell equation (3.6) and (3.14) respectively with the same initial data $f_{0}$. Suppose that Assumption 3.1 is satisfied. Then

$$\lim_{N \to \infty, c \to \infty} P_{0} \left( \sup_{0 \leq s \leq t} \left| (X_{m}^{N}(s), V_{m}^{N}(s)) - (X_{p}^{N}(s), V_{p}^{N}(s)) \right|_{\infty} > N^{-\alpha} \right) = 0.$$

4 Summary

In this paper we compared the time evolution of the one particle density of the $N$-particle Vlasov-Maxwell system with the Vlasov-Poisson equation. We showed closeness of both time evolutions for $N$ and $c$ being large enough.

5 Appendix: Proof of Theorem 2.1

*Proof.*

1. Using the same method as Kurth, R. in [17], it is easy to prove that (VPN) has a unique $C^{1}$-solution $(f_{p}^{N}, E_{p}^{N})$ on the time interval $[0, T > 0)$.

2. The proof of existence of solutions of (VMN) is similar to Glassey, R., Strauss, W [9], while the proof of existence of functions $q(t)$ and $F(t)$ with the respective properties follows the ideas of Jack Schaeffer as given in [25]. Therefore we omit the proof in this manuscript.

3. Next we prove the third part of the theorem. Similar to (A13) and (A14) in the
Appendix of [25], we use the convenient notation \( \nu = \frac{y - x}{|y - x|}, x, y \in \mathbb{R}^3 \). We obtain

\[
E^N_m(t, x) = \mathbb{E}_0 - \frac{1}{4\pi ct} \int \int \int_{\mathbb{R}^3} dvpdz \int_{|x - y| = ct} dS_y \chi^N(p) \chi^N(z) f_0(y - p - z, v) \frac{\nu - c^{-2} \hat{v} \cdot \nu \hat{v}}{(1 + c^{-1} \hat{v} \cdot \nu)}
\]

\[
- \frac{1}{4\pi} \int \int \int_{\mathbb{R}^3} dvpdz \int_{|x - y| < ct} dy \chi^N(p) \chi^N(z) \frac{f^N_m(t - c^{-1} |x - y|, y - p - z, v)}{|x - y|^2} \frac{(1 - c^{-2} |\hat{v}|^2)(\nu + c^{-1} \hat{v})}{(1 + c^{-1} \hat{v} \cdot \nu)^2}
\]

\[
- \frac{1}{4\pi c^2} \int \int \int_{\mathbb{R}^3} dvpdz \int_{|x - y| < ct} dy \chi^N(p) \chi^N(z) \frac{f^N_m(t - c^{-1} |x - y|, y - p - z, v)}{|x - y|} \frac{1}{(1 + c^{-1} \hat{v} \cdot \nu)^2(1 + c^{-2} |\hat{v}|^2)^{\frac{3}{2}}}
\]

\[
\times \left[ (1 + c^{-1} \hat{v} \cdot \nu)(E^N_m + c^{-1} \hat{v} \times B^N_m) + c^{-2}(\hat{v} \cdot \nu \nu - \hat{v} \cdot \nu) \hat{v} \cdot E^N_m \right. \\
\left. - (\nu + c^{-1} \hat{v}) \nu \cdot (E^N_m + c^{-1} \hat{v} \times B^N_m) \right] \bigg|_{t - c^{-1} |x - y|, y - p - z}
\]

\[
B^N_m = \mathbb{B}_0 + \frac{1}{4\pi ct} \int \int \int_{\mathbb{R}^3} dvpdz \int_{|x - y| = ct} dS_y \chi^N(p) \chi^N(z) f_0(y - p - z, v) \frac{(\nu \times c^{-1} \hat{v})}{(1 + c^{-1} \hat{v} \cdot \nu)}
\]

\[
+ \frac{1}{4\pi c} \int \int \int_{\mathbb{R}^3} dvpdz \int_{|x - y| < ct} dy \chi^N(p) \chi^N(z) \frac{f^N_m(t - c^{-1} |x - y|, y - p - z, v)}{|x - y|^2} \frac{(\nu \times c^{-1} \hat{v})}{(1 + c^{-1} \hat{v} \cdot \nu)^2}
\]

\[
+ \frac{1}{4\pi c^2} \int \int \int_{\mathbb{R}^3} dvpdz \int_{|x - y| < ct} dy \chi^N(p) \chi^N(z) \frac{f^N_m(t - c^{-1} |x - y|, y - p - z, v)}{|x - y|} \frac{1}{(1 + c^{-1} \hat{v} \cdot \nu)^2(1 + c^{-2} |\hat{v}|^2)^{\frac{3}{2}}}
\]

\[
\times \left[ (1 + c^{-1} \hat{v} \cdot \nu) \times (E^N_m + c^{-1} \hat{v} \times B^N_m) \\
\right. \\
\left. - c^{-2}(\nu \times \hat{v})(\hat{v} + c\nu) \cdot (E^N_m + c^{-1} \hat{v} \times B^N_m) \right] \bigg|_{t - c^{-1} |x - y|, y - p - z}
\]

where \( \bigg|_{t - c^{-1} |x - y|, y - p - z} \) means \( E^N_m(t - c^{-1} |x - y|, y - p - z) \) and \( B^N_m(t - c^{-1} |x - y|, y - p - z) \). In order to prove Theorem 2.1, we note that the core of the proof consists in comparing the integral representation of \( (E^N_m, B^N_m) \) given above with the one of \( E^N_p \) given in (VPN) that is

\[
E^N_p(t, x) = \frac{1}{4\pi} \int \int \int_{\mathbb{R}^3} dvpdz \chi^N(p) \chi^N(z) f^N_p(t, y - p - z) \frac{x - y}{|x - y|^3}.
\]
To obtain uniform convergence, we will thoroughly calculate $E^N_m$ and $B^N_m$. First, we consider $E^N_m$.

**Lemma 5.1.** ([23], Lemma 1) Let $g$ be a continuous function of compact support on $\mathbb{R}^3$, then there exists a constant $M > 0$ such that

$$r \int_{|\omega|=1} |g(x + r\omega)| d\omega \leq M.$$ 

for all $r > 0$.

Note that for $|v| \leq q(t)$, with $q(t) \geq 1$,

$$|\hat{v}| \leq \frac{q(t)}{(1 + c^{-2}|v|^2)^{\frac{1}{2}}} \leq q(t),$$

and

$$\frac{1}{1 + c^{-1}\hat{v} \cdot \nu} \leq 2c^{-2}(c^2 + q^2(t)) \leq 4q^2(t).$$

From the proposition and the above two inequalities, we get $\forall x \in \mathbb{R}^3$, $t \in [0, T]$

$$\left| \frac{1}{4\pi ct} \iint_{\mathbb{R}^9} dv dp dz \int_{|x-y|=ct} dS_y \chi^N(p) \chi^N(z) f_0(y - p - z, v) \frac{e^{-\hat{v} \cdot \nu}}{(1 + c^{-1}\hat{v} \cdot \nu)} \right| \leq M \frac{tq^4(t)}{c^2} \int_{\mathbb{R}^9} dv dp dz \chi^N(p) \chi^N(z) \int_{|\omega|=1} d\omega f_0(x - p - z + ct\omega, v)$$

$$\leq Mq^4(t)c^{-2} = O(c^{-2})$$

and

$$\left| \frac{1}{4\pi ct} \iint_{\mathbb{R}^9} dv dp dz \int_{|x-y|=ct} dS_y \chi^N(p) \chi^N(z) f_0(y - p - z, v) \frac{e^{-\hat{v} \cdot \nu}}{(1 + c^{-1}\hat{v} \cdot \nu)} \right| \leq M \frac{tq^3(t)}{c} \int_{\mathbb{R}^9} dv dp dz \chi^N(p) \chi^N(z) \int_{|\omega|=1} d\omega f_0(x - p - z + ct\omega, v)$$

$$\leq Mq^3(t)c^{-2} = O(c^{-1}).$$

Hence $\forall x \in \mathbb{R}^3$, $t \in [0, T]$

$$E_1(t, x) = \frac{1}{4\pi ct} \iint_{\mathbb{R}^9} dv dp dz \int_{|x-y|=ct} dS_y \chi^N(p) \chi^N(z) f_0(y - p - z, v)\nu + O(c^{-1}).$$

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As
\[
\left| \frac{1}{4\pi} \iiint_{\mathbb{R}^9} dvdpdz \int_{|x-y|<ct} dy \chi^N(p)\chi^N(z) \frac{f^N_m(t-c^{-1}|x-y|,y-p-z,v)}{|x-y|^2} \frac{\hat{v}^2(\nu + c^{-1}\hat{v})}{(1+c^{-1}\hat{v} \cdot \nu)^2c^2} \right| 
\leq \frac{1}{4\pi c^2} \int_{|y|<P_0+q(t)} \int_{|v|<q(t)} (4q^2(t)^2q^2(t)(1+c^{-1}q(t)) \|f_0\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \frac{1}{|x-y|^2} dvdy 
\leq \frac{M}{c^2} q^6(t)(1+c^{-1}q(t))q^3(t) \int_{|y|<P_0+q(t)} \frac{1}{|x-y|^2} dy 
\leq \frac{M}{c^2},
\]
where we have in the last step used the fact that
\[
\sup_x \int_{|y|<P_0+q(t)} \frac{1}{|x-y|^2} dy < M(P_0, q(t)). \quad (5.1)
\]

In the same way, we obtain
\[
\left| \frac{1}{4\pi} \iiint_{\mathbb{R}^9} dvdpdz \int_{|x-y|<ct} dy \chi^N(p)\chi^N(z) \frac{f^N_m(t-c^{-1}|x-y|,y-p-z,v)}{|x-y|^2} \frac{\hat{v}}{(1+c^{-1}\hat{v} \cdot \nu)^2c^2} \right| 
\leq \frac{M}{c},
\]
so we have
\[
\mathbb{E}_2(t, x) = \frac{1}{4\pi} \iiint_{\mathbb{R}^9} dvdpdz \int_{|x-y|<ct} dy \chi^N(p)\chi^N(z) \frac{f^N_m(t-c^{-1}|x-y|,y-p-z,v)}{|x-y|^2} \frac{\nu}{(1+c^{-1}\hat{v} \cdot \nu)^2} + O(c^{-1}) 
\leq \frac{1}{4\pi} \iiint_{\mathbb{R}^9} dvdpdz \int_{|x-y|<ct} dy \chi^N(p)\chi^N(z) \frac{f^N_m(t-c^{-1}|x-y|,y-p-z,v)}{|x-y|^2} \nu + O(c^{-1}),
\]
where the following estimate has been used
\[
\left| \frac{1}{(1+c^{-1}\hat{v} \cdot \nu)^2} - 1 \right| = \left| \frac{2c^{-1}\hat{v} \cdot \nu + c^{-2}(\hat{v} \cdot \nu)^2}{(1+c^{-1}\hat{v} \cdot \nu)^2} \right| \leq \frac{M}{c} q^4(t) (q(t) + c^{-1}q^2(t)) \leq \frac{M}{c}.
\]
Recalling Theorem 2.1 and $|\hat{v}| < c$, we get

$$\left| E_0 \right| \leq \frac{1}{4\pi c^2} \int \int \int_{R^9} dv dp dz \int_{|x-y|<ct} dy \left(4q^2(t)\right)^2 \chi^N(p)\chi^N(z)6H(t-c^{-1}|x-y|, y-p-z, v)\frac{f^N_m(t-c^{-1}|x-y|, y-p-z, v)}{|x-y|}$$

$$\leq M \frac{c^2}{c^2} \int_{|y|<P_0+\eta_0(t)} \int_{|v|<\eta_0(t)} \frac{1}{|x-y|} dy \int_{|v|<\eta_0(t)} \|f_0\|_{L^\infty(R^3 \times R^3)} dv \leq \frac{M}{c}.$$

**Lemma 5.2.** ([23], Lemma 2) Let $g \in C^2(R^3)$. Assume that $\Delta g$ has compact support for $c > 0$ and $t \geq 0$,

$$\frac{\partial t}{\partial t} \left( t \int_{|\omega|=1} g(x+ct\omega) d\omega \right) = -\int_{|x-y|>ct} \frac{\Delta g(y)}{|x-y|} dy.$$

Now using this lemma, we estimate $E_0$. We know

$$E_0 = \partial_t \int_{|\omega|=1} t \int_{|\omega|=1} E^N_m(0, x+ct\omega) d\omega + \frac{t}{4\pi} \int \int_{R^6} dp dz \int_{|\omega|=1} d\omega \chi^N(p)\chi^N(z) \left(c \nabla \times B_0(x-p-z+ct\omega) - \int_{R^3} \hat{v} f_0(x-p-z+ct\omega, v) dv \right).$$

From Lemma 3.1, we get

$$\frac{t}{4\pi} \left| \int \int_{R^6} dp dz \int_{|\omega|=1} d\omega \chi^N(p)\chi^N(z) (c \nabla \times B_0(x-p-z+ct\omega) \right| \leq \frac{M}{c}$$

and by Lemma 3.2, we obtain

$$\frac{t}{4\pi} \left| \int \int_{R^6} dp dz \int_{|\omega|=1} d\omega \chi^N(p)\chi^N(z) \int_{R^3} \hat{v} f_0(x-p-z+ct\omega, v) dv \right|$$

$$= \frac{1}{4\pi c} \left| \int \int_{R^6} dp dz \int_{|\omega|=1} d\omega \chi^N(p)\chi^N(z) \int_{R^3} \hat{v} c f_0(x-p-z+ct\omega, v) dv \right| \leq \frac{M}{c},$$

thus

$$E_0 = \partial_t \int_{|\omega|=1} t \int_{|\omega|=1} E^N_m(0, x+ct\omega) d\omega + O(c^{-1}).$$

Now, in order to further calculate $E_0$, we set

$$g(x) := \frac{1}{4\pi} \int \int \int_{R^3} dv dy dp dz \chi^N(p)\chi^N(z) f_0(y-p-z, v) \frac{y-p-z, v}{|x-y|}.$$
Note that $\nabla g(x) = -E_m^N(0, x)$ and $\Delta g(x) = \iiint_{\mathbb{R}^9} dv dp dz \chi^N(p) \chi^N(z) f_0(x-p-z, v)$.

Using Lemma 3.2, we get

$$
\partial_t \int_{|x|=1} \frac{t}{4\pi} E_m^N(0, x + ct \omega) \ d\omega
= -\partial_t \int_{|x|=1} \frac{t}{4\pi} \nabla g(x + ct \omega) \ d\omega
= -\frac{1}{4\pi} \nabla \int_{|x-y|>ct} \frac{\Delta g(y)}{|x-y|} \ dy
= -\frac{1}{4\pi} \nabla \left( \int_{|x-y|>ct} dv dp dz \int_{|x-y|>ct} dy \chi^N(p) \chi^N(z) \frac{f_0(y-p-z, v)}{|x-y|} \right)
= -\frac{1}{4\pi} \int_{|x-y|>ct} dv dp dz \int_{|x-y|>ct} dy \chi^N(p) \chi^N(z) \frac{\nabla f_0(y-p-z, v)}{|x-y|}.
$$

Recall that $f_0$ has compact support, so by the divergence theorem, we have

$$
-\int_{|x-y|>ct} dv dp dz \int_{|x-y|=ct} dS_y \chi^N(p) \chi^N(z) \frac{f_0(y-p-z, v)}{|x-y|}
= \int_{|x-y|>ct} dv dp dz \int_{|x-y|>ct} dy \nabla_y \left( \chi^N(p) \chi^N(z) \frac{f_0(y-p-z, v)}{|x-y|} \right)
= \int_{|x-y|>ct} dv dp dz \int_{|x-y|>ct} dy \chi^N(p) \chi^N(z) \frac{\nabla f_0(y-p-z, v)}{|x-y|}

\frac{1}{4\pi} \left( \int_{|x-y|>ct} dv dp dz \int_{|x-y|=ct} dS_y \chi^N(p) \chi^N(z) \frac{f_0(y-p-z, v)}{|x-y|} \right)
\frac{1}{4\pi} \int_{|x-y|>ct} dv dp dz \int_{|x-y|=ct} dS_y \chi^N(p) \chi^N(z) \frac{f_0(y-p-z, v)}{|x-y|}

$$

Hence

$$
\mathbb{E}_0 = \frac{1}{4\pi} \int_{|x-y|>ct} dv dp dz \int_{|x-y|=ct} dS_y \chi^N(p) \chi^N(z) \frac{f_0(y-p-z, v)}{|x-y|}
\frac{1}{4\pi} \int_{|x-y|>ct} dv dp dz \int_{|x-y|=ct} dS_y \chi^N(p) \chi^N(z) \frac{f_0(y-p-z, v)}{|x-y|}
+ O(c^{-1}).
$$

Therefore

$$
E_m^N(t, x)
= -\frac{1}{4\pi} \int_{|x-y|<ct} dv dp dz \int_{|x-y|<ct} dy \chi^N(p) \chi^N(z) \frac{f_m^N(t-c^{-1}|x-y|, y-p-z, v)}{|x-y|^2}
-\frac{1}{4\pi} \int_{|x-y|>ct} dv dp dz \int_{|x-y|>ct} dy \chi^N(p) \chi^N(z) \frac{f_0(y-p-z, v)}{|x-y|^2}
+ O(c^{-1})
= -\frac{1}{4\pi} \int_{|x-y|>ct} dv dp dz \int_{|x-y|>ct} \chi^N(p) \chi^N(z) \frac{f_m^N(t-c^{-1}|x-y|, y-p-z, v)}{|x-y|^2} + O(c^{-1}).
$$
From the representation of $E_{p}^{N}(t, x)$ from (VPN), we have

$$|E_{m}^{N}(t, x) - E_{p}^{N}(t, x)|$$

$$= \frac{1}{4\pi} \int_{\mathbb{R}^{12}} d v d p d z d y \chi^{N}(p) \chi^{N}(z) \frac{\nu}{|x-y|^{2}}$$

$$\times (f_{m}^{N}(\max\{0, t - c^{-1}|x-y|\}, y - p - z, v) - f_{p}^{N}(t, y - p - z, v)) \bigg| + O(c^{-1})$$

$$\leq \frac{M}{c} + \frac{1}{4\pi} \int_{\mathbb{R}^{12}} d v d p d z d y$$

$$\chi^{N}(p) \chi^{N}(z) \frac{|f_{m}^{N} - f_{p}^{N}|(\max\{0, t - c^{-1}|x-y|\}, y - p - z, v)}{|x-y|^{2}}$$

$$+ \frac{1}{4\pi} \int_{\mathbb{R}^{12}} d v d p d z d y$$

$$\chi^{N}(p) \chi^{N}(z) \frac{|f_{p}^{N}(\max\{0, t - c^{-1}|x-y|\}, y - p - z, v) - f_{p}^{N}(t, y - p - z, v)|}{|x-y|^{2}}.$$ 

Recall that $(f_{p}^{N}, E_{p}^{N})$ is a $C^{1}$-solution of (VPN). Now since $E_{p}^{N}$ is $C^{1}$ and $f_{0}$ has compact support, it follows that

$$q_{p}^{N} = \sup\{|v|: \exists x \in \mathbb{R}^{3}, \tau \in [0, t] \text{ s.t. } f_{p}^{N}(\tau, x, v) \neq 0\}$$

is finite on $[0, T]$. Also $\partial_{t} f_{p}^{N}$ is bounded on $[0, T] \times \mathbb{R}^{6}$. Let

$$Q := \max\{q(T), q_{p}^{N}(T)\}$$

and

$$G(t) := \sup \left\{|f_{m}^{N}(\tau, x, v) - f_{p}^{N}(\tau, x, v)|: x \in \mathbb{R}^{3}, v \in \mathbb{R}^{3} \text{ and } \tau \in [0, t]\right\}.$$ 

Then

$$|E_{m}^{N}(t, x) - E_{p}^{N}(t, x)|$$

$$\leq \int_{|y|<P_{0}+TQ} \int_{|v|<Q} G(\max\{0, t - c^{-1}|x-y|\}) \frac{d v d y}{|x-y|^{2}}$$

$$+ \int_{\mathbb{R}^{12}} d v d p d z d y$$

$$\chi^{N}(p) \chi^{N}(z) \frac{1}{|x-y|^{2}} \int_{\max\{0, t-c^{-1}|x-y|\}}^{t} |\partial_{t} f_{p}^{N}(\tau, y - p - z, v)| d \tau + \frac{M}{c}$$

$$\leq G(t)MQ^{3} \int_{|y|<P_{0}+TQ} \frac{1}{|x-y|^{2}} d y + MQ^{3} \int_{|y|<P_{0}+TQ} \frac{c^{-1}|x-y|}{|x-y|^{2}} d y + \frac{M}{c}$$

$$\leq MG(t) + \frac{M}{c},$$

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where we have used (5.1). Now we begin to estimate $B^N_m$. Using Lemma 3.2, we get for the first term $\mathcal{B}_1$

$$
|\mathcal{B}_1| = \frac{1}{4\pi ct} \left| \int \int \int_{\mathbb{R}^6} dvdpdz \int_{|x-y|=ct} dS_y \chi^N(p) \chi^N(z) f_0(y - p - z, v) \frac{(\nu \times c^{-1} \hat{v})}{(1 + c^{-1} \hat{v} \cdot \nu)} \right|

\leq \frac{1}{4\pi} \int \int \int_{\mathbb{R}^6} dvdpdz \chi^N(p) \chi^N(z) 4q^2(t) c^{-1} q(t) \int_{|\omega|=1} ctf_0(x - p - z + ct\omega, v) d\omega

\leq \frac{M}{c} \int_{\mathbb{R}^6} \chi^N(p) \chi^N(z) \, dpdz = \frac{M}{c}.

$$

Secondly we look into $\mathcal{B}_2$.

$$
|\mathcal{B}_2| = \frac{1}{4\pi c} \left| \int \int \int_{\mathbb{R}^6} dvdpdz \int_{|x-y|<ct} dy \chi^N(p) \chi^N(z) f_1^N(t - c^{-1}|x-y|, y - p - z, v) \frac{(1 - c^{-2}|\hat{v}|^2)(\nu \times \hat{v})}{|x-y|^2} \right|

\leq \frac{1}{4\pi c} \int \int \int_{\mathbb{R}^6} dvdpdz \int_{|x-y|<ct} dy \chi^N(p) \chi^N(z) \frac{\|f_0\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} (4q^2(t))^2 q(t)}{|x-y|^2}

\leq \frac{M}{c} \int_{|y|<P_0+TQ} \int_{|v|<Q} \frac{1}{|x-y|^2} dvdy \leq \frac{M}{c},

$$

where (5.1) has been used again. The last term $\mathcal{B}_3$ can be shown to be $O(c^{-2})$ in the same way as $\mathcal{E}_3$. Now, what is left is $\mathcal{B}_0$. It is easy to calculate that $\partial_t B_0 = -c \nabla \times E_0 = 0$. Using Lemma 3.2 and Theorem 2.1, we get

$$
|\partial_t \int_{|\omega|=1} \frac{t}{4\pi} B^N_m(0, x + ct\omega) \, d\omega| = \left| \partial_t \int \int_{\mathbb{R}^6} dpdz \int_{|\omega|=1} d\omega \frac{t}{4\pi} \chi^N(p) \chi^N(z) B_0(x - p - z + ct\omega) \right|

\leq \int \int_{\mathbb{R}^6} dpdz \frac{1}{4\pi ct} \chi^N(p) \chi^N(z) \int_{|\omega|=1} ct|B_0(x - p - z + ct\omega)| \, d\omega

+ \int \int_{\mathbb{R}^6} dpdz \frac{1}{4\pi} \chi^N(p) \chi^N(z) \int_{|\omega|=1} ct|\nabla B_0(x - p - z + ct\omega)| \, d\omega

\leq M \int \int_{\mathbb{R}^6} \frac{1}{4\pi ct} \chi^N(p) \chi^N(z) \, dpdz + \int \int_{\mathbb{R}^6} \frac{1}{4\pi} \chi^N(p) \chi^N(z) ct \frac{1}{c^2} \, d\omega

\leq \frac{M}{c}.

$$

Hence

$$
B^N_m = \mathcal{B}_0 + \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 = O(c^{-1}).

(5.2)

$$
Combing (5.2) and [5.2], we know that
\[ |E_p^N - E_m^N - c^{-1} \hat{v} \times B_m^N| \leq MG(t) + \frac{M}{c}, \quad t < T, \]  
for \( |\hat{v}| < c \).

It remains to estimate \( f_m^N - f_p^N \). For ease of notation, we define \( g = f_m^N - f_p^N \). It is not difficult to calculate that
\[
\partial_t g + \hat{v} \cdot \nabla_x g + (E_m^N + c^{-1} \hat{v} \times B_m^N) \cdot \nabla_v g
= (v - \hat{v}) \cdot \nabla_x f_p^N + (E_p^N - E_m^N - c^{-1} \hat{v} \times B_m^N) \cdot \nabla_v f_p^N
= \frac{|v|^2 \hat{v}}{c^2 (1 + \sqrt{1 + c^{-2} |v|^2})} \cdot \nabla_x f_p^N + (E_p^N - E_m^N - c^{-1} \hat{v} \times B_m^N) \cdot \nabla_v f_p^N
\]  
(5.4)

Note that both \( |\nabla_x f_p^N| \) and \( |\nabla_v f_p^N| \) are bounded on \([0, T] \times \mathbb{R}^6\) and \( \nabla_x f_p^N(t, x, v) = 0 \) if \( |v| > q_p^N(t) \). Hence
\[
|\partial_t g + \hat{v} \cdot \nabla_x g + (E_m^N + c^{-1} \hat{v} \times B_m^N) \cdot \nabla_v g|
\leq \frac{M}{c^2} + M|E_p^N - E_m^N - c^{-1} \hat{v} \times B_m^N|, \quad 0 \leq t \leq T.
\]  
(5.5)

For any \( x \in \mathbb{R}^3, \ v \in \mathbb{R}^3, t \in [0, T] \), we define \((x(t), v(t))\) as in (2.5) and calculate
\[
\left| \frac{d}{dt} g(t, x(t), v(t)) \right| = |\partial_t g + \hat{v} \cdot \nabla_x g + (E_m^N + c^{-1} \hat{v} \times B_m^N) \cdot \nabla_v g|
\leq \frac{M}{c} + MG(t), \quad 0 \leq t \leq T.
\]  
(5.7)

Note that \( g(t, x(t), v(t))|_{t=0} = 0 \), so \( \forall x, v, t \), let \((x(0), v(0))\) be the corresponding initial data of (2.5). Then
\[
|g(t, x, v)| = |g(t, x(t), v(t)) - g(0, x(0), v(0))|
\leq \int_0^t \frac{d}{ds} g(s, x(s), v(s)) \, ds
\leq \int_0^t \left( \frac{M}{c} + MG(s) \right) \, ds
\leq \frac{Mt}{c} + \int_0^t MG(s) \, ds, \quad 0 \leq t \leq T.
\]

By the definition of \( g \) and \( G(t) \) we get
\[
G(t) \leq \frac{M}{c} + M \int_0^t G(s) \, ds, \quad 0 \leq t \leq T.
\]

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Using the Gronwall’s inequality, we get
\[ G(t) \leq \frac{M}{c} \exp(Mt) \leq \frac{M}{c}, \quad 0 \leq t \leq T. \]
Therefore
\[ \| f_m^N - f_p^N \|_{L^\infty([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)} + \| E_m^N - E_p^N \|_{L^\infty([0,T] \times \mathbb{R}^3)} + \| B_m^N \|_{L^\infty([0,T] \times \mathbb{R}^3)} \leq \frac{M}{c}.
\]
for all \( c \geq 1 \). This completes the proof of Theorem.

\[ \blacksquare \]

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