NON-VANISHING COHOMOLOGY CLASSES IN UNIFORM LATTICES OF $\text{SO}(n,\mathbb{H})$ AND AUTOMORPHIC REPRESENTATIONS

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ABSTRACT. Let $X$ denote the non-compact globally Hermitian symmetric space of type $DIII$, namely, $\text{SO}(n,\mathbb{H})/U(n)$. Let $\Lambda$ be a uniform torsionless lattice in $\text{SO}(n,\mathbb{H})$. In this note we construct certain complex analytic submanifolds in the locally symmetric space $X := \Gamma\backslash \text{SO}(n,\mathbb{H})/U(n)$ for certain finite index sub lattices $\Gamma \subset \Lambda$ and show that their dual cohomology classes in $H^*(X_\Gamma;\mathbb{C})$ are not in the image of the Matsushima homomorphism $H^*(X_u;\mathbb{C}) \to H^*(X_\Gamma;\mathbb{C})$, where $X_u = \text{SO}(2n)/U(n)$ is the compact dual of $X$. These submanifold arise as sub-locally symmetric spaces which are totally geodesic, and, when $\Lambda$ satisfies certain additional conditions, they are non-vanishing ‘special cycles’. Using the fact that $X_\Lambda$ is a Kähler manifold, we deduce the occurrence in $L^2(\Lambda\backslash \text{SO}(n,\mathbb{H}))$ of a certain irreducible representation $(\mathcal{A}_q,\mathcal{A}_q)$ with non-zero multiplicity when $n \geq 9$. The representation $\mathcal{A}_q$ is associated to a certain $\theta$-stable parabolic subalgebra $q$ of $\mathfrak{g}_0 := \text{so}(n,\mathbb{H})$. Denoting the smooth $U(n)$-finite vectors of $\mathcal{A}_q$ by $\mathcal{A}_q,\mathcal{U}(n)$, the representation $\mathcal{A}_q$ is characterised by the property that $H^{p,p}(\mathfrak{g}_0 \otimes \mathcal{U},U(n);\mathcal{A}_q,\mathcal{U}(n)) \cong H^{p-n,2+p-n}(\text{SO}(2n-2)/U(n-1);\mathbb{C})$, $p \geq 0$, for $n \geq 9$.

1. Introduction

Let $G$ be a connected non-compact real semi simple Lie group with finite centre and let $K \subset G$ be a maximal compact subgroup. Denote by $\mathfrak{g}_0$ the Lie algebra of $G$ and by its complexification respectively. Let $\theta : \mathfrak{g}_0 \to \mathfrak{g}_0$ denote the Cartan involution that fixes $\mathfrak{t}_0$, so that $\mathfrak{g}_0 = \mathfrak{t}_0 \oplus \mathfrak{p}_0$ where $\mathfrak{p}_0$ is the $-1$-eigenspace of $\theta$. We denote by $\mathfrak{p}$ the complexification of $\mathfrak{p}_0$ so that $\mathfrak{g}_0 = \mathfrak{t}_0 \oplus \mathfrak{p}$. The Killing form restricted to $\mathfrak{p}_0$ is a positive definite inner product and defines a Riemannian metric on $X = G/K$ making it a globally symmetric space. If $\Gamma$ is a torsionless discrete subgroup of $G$, then the locally symmetric space $X_\Gamma := \Gamma\backslash X = \Gamma\backslash G/K$ is a smooth aspherical manifold.

Throughout this paper we assume that $\Gamma$ is a uniform lattice in $G = \text{SO}(n,\mathbb{H}) \cong \text{SO}^*(2n), n \geq 4$. (Note that $\Gamma$ is irreducible since $G$ is simple.)

Our aim is to construct closed oriented totally geodesic submanifolds of $X_\Gamma$, which are known as special cycles, such that their Poincaré duals are non-zero cohomology classes in $H^*(X_\Gamma;\mathbb{R})$ in the case when $G = \text{SO}(n,\mathbb{H})$.

They arise as sub locally symmetric spaces $C(\sigma) = X(\sigma)_{\Gamma(\sigma)}$ where $X(\sigma) = G(\sigma)/K(\sigma)$, $G(\sigma) \subset G$ is the fixed point of an algebraically defined involutive automorphism $\sigma$ of $G$ so that $\Gamma(\sigma) = \Gamma \cap G(\sigma)$ is again a lattice in $G(\sigma)$.

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When $C(\sigma)$ is a complex analytic submanifold of $X_\Gamma$, then, in view of the fact that $X_\Gamma$ is a Kähler manifold—it is even a complex projective variety—it follows immediately that $[C(\sigma)]$ is non-zero. In the more general case when $C(\sigma)$ is not complex analytic, to show that the Poincaré dual to the fundamental class of $X(\sigma)_{\Gamma(\sigma)} \hookrightarrow X_\Gamma$ is non-zero, one constructs another special cycle $C(\tau)$ of complementary dimension and shows that the cup product $[C(\sigma)], [C(\tau)]$ in $H^*(X_\Gamma; \mathbb{R})$ is non-zero. The existence of such pairs of special cycles of complementary dimensions imply, not only the non-vanishing of their Poincaré duals, but also the stronger condition that their Poincaré duals are cohomology classes that cannot be representable by $G$-invariant forms on $G/K$. Equivalently, these cohomology classes are not in the image of the Matsushima homomorphism $[13]$ $H^*(X_u; \mathbb{C}) \to H^*(X_\Gamma; \mathbb{C})$ where $X_u$ denotes the compact dual of $X$. The method of constructing such pairs of special cycles are well understood; see [15] and [22].

Special cycles were first constructed by Millson and Raghunathan [15] in the case of $G = SU(p,q), SO_0(p,q), Sp(p,q)$. Schwermer and Waldner [25] considered the case $G = SU^*(2n)$, and Waldner [30] the case when $G$ is the non-compact real form of the exceptional group $G_2$. Recently the groups $SL(n, \mathbb{R}), SL(n, \mathbb{C}), n \geq 3$, have been considered by Schimpf [24]. In all these cases, the choice of the uniform lattice under consideration had to be suitably restricted.

We too need to restrict the class of lattices in $G = SO(n, \mathbb{H})$. Let $F$ be a totally real number field and let $O_F$ denote its ring of integers. Let $G$ be an $F$-algebraic group such that the $\mathbb{R}$-points $G(\mathbb{R})$ is isomorphic as a Lie group to $G = SO(n, \mathbb{H})$ and such that the $O_F$-points $G(O_F)$ is a uniform lattice in $G(\mathbb{R})$. (Up to commensurability any uniform lattice of $G$ arises in this manner, thanks to the arithmeticity theorem of Margulis.) We shall refer to such an $F$-structure on $G$ as type $DIII_u$.

Our basic hypothesis is a certain condition (see Theorem 2.4) on the $F$-structure which guarantees the existence of a Cartan involution $\theta = \theta_\mathbb{R}$ induced by an $F$-involution $\theta_F$ on $G$ given by conjugation by a diagonal matrix in $G(F)$. Such an $F$-rational Cartan involution will be said to be of diagonal type.

Suppose that $\Gamma$ is commensurable with $G(O_F)$. Whenever required, $\Gamma$ may be replaced by a finite index subgroup which is stable under finitely many pairwise commuting collection of involutions $\sigma = \sigma_\mathbb{R}$ induced by $F$- involutions $\sigma_F$, including a Cartan involution $\theta$ defined over $F$ of diagonal type. Once the former condition is met, the latter is readily achieved—one need only consider the intersection of $\Gamma$ with the members of the collection $\sigma(\Gamma)$ as $\sigma$ varies in the finite collection of commuting involutions.

Our first main result is the following:

**Theorem 1.1.** Let $n \geq 8$. Suppose that $\theta$ is an $F$-rational Cartan involution on $G = SO(n, \mathbb{H})$ of diagonal type where the $F$-structure on $G$ is of type $DIII_u$. Suppose that $\Lambda$ is any torsionless lattice in $SO(n, \mathbb{H}), n \geq 4$, that is commensurable with $G(O_F)$. Then there exist cohomology classes in $H^{2p}(X_\Lambda)$ which are not in the image of the Matsushima homomorphism $H^*(X_u; \mathbb{C}) \to H^*(X_\Lambda)$, in all even dimensions $2p$ between $2n - 4$ and
\(n(n-1) - (2n - 4). \) Moreover, when \( n \equiv 1 \mod 2, \) there exists a cofinal family \( \{ \Gamma \} \) of finite index subgroups \( \Gamma \hookrightarrow \Lambda \) such that \( H^{n(n-1)/2}(X_{\Gamma}; \mathbb{C}) \) contains classes not belonging to the image of the Matsushima homomorphism.

In order to state our next main result, which may be viewed as an application of the above theorem, we need to recall certain facts about \( L^2(\Gamma \backslash G) \) as well as relative Lie algebra cohomology of unitary representations. (Here \( \Gamma \) is a uniform lattice.) To a Haar measure above theorem, we need to recall certain facts about the work of Gelfand and Pyatetskii-Shapiro ([7, Ch. 1, §2.3], [8]). \( L^2(\Gamma \backslash G) \) is a completed direct sum of irreducible unitary \( G \)-representations \( (\pi, H_\pi) \) each of which occurs with finite multiplicity \( m(\pi, \Gamma) \). The representations of \( \pi \) with \( m(\pi, \Gamma) > 0 \) are called automorphic representations. Denoting the space of smooth \( K \)-finite vectors of a \( G \)-representation \( V \) by \( V_K \), one has the Matsushima isomorphism \( H^*(X_{\Gamma}; \mathbb{C}) = H^*(g, K; L^2(\Gamma \backslash G)_K) \cong \bigoplus_{\pi} m(\pi, \Gamma)H^*(g, K; H_{\pi,K}). \) See [14]. By a result of D. Wigner \( H^*(g, K; H_{\pi,K}) \) is non-zero only if, the infinitesimal character of \( \pi \) equals that of the trivial representation of \( G \). The representations \( (\pi, H_\pi) \) with non-vanishing \( (g, K) \)-cohomology have been classified (up to unitary equivalence) in terms of the \( \theta \)-stable parabolic subalgebras \( q \) of \( g \). Denote the representation corresponding to \( q \) by \( (A_q, A_q) \). If \( H^*(g, K; H_{\pi,K}) \neq 0 \) then \( (\pi, H_\pi) \) is unitarily equivalent to \( (A_q, A_q) \) for some \( \theta \)-stable parabolic subgroup of \( g_0 \).

When \( X \) is a Hermitian symmetric space, \( X_\Gamma \) is a Kähler manifold, which is compact since \( \Gamma \) is uniform. One has the Hodge decomposition \( H^*(X_{\Gamma}; \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(X_{\Gamma}; \mathbb{C}). \) Also one has the Hodge decomposition \( H^*(g, K; V_K) = \bigoplus_{p+q=r} H^{p,q}(g, K; V_K) \) where \( V \) is any unitary \( G \)-representation; see [4, Ch. II §4]. Corresponding to each \( q \), there exists a pair of numbers, \( (R_+(q), R_-(q)) \), called the type of \( q \), such that \( H^{p,q}(g, K; A_{q,K}) = 0 \) unless \( p \geq R_+(q), q \geq R_-(q), p - q = R_+(q) - R_-(q). \) Moreover \( H^{R_+(q), R_-(q)}(g, K; A_{q,K}) \cong \mathbb{C}. \)

We shall show, when \( G, \Gamma \) are as in Theorem 1.1, that, for \( n \geq 8 \) there exists a unitary representation \( A_q \), where the \( \theta \)-stable parabolic subalgebra \( q \) is of type \( (n-2, n-2) \), and none of type \( (n-l, n-l) \) for \( 2 < l < n \) or \( l = 1 \). This representation is unique up to unitary equivalence when \( n \geq 9 \).

It is an important open problem in the study of cohomology of locally symmetric spaces, to determine, for a given \( \Gamma \subset G \) and \( q \), whether \( m(A_q, \Gamma) \) is positive, let alone its value. We have the following result.

**Theorem 1.2.** Let \( n > 8. \) Let \( G \) be an \( F \)-structure on \( SO(n, \mathbb{H}) \) of type \( DIII_u \) admitting an \( F \)-rational Cartan involution as in Theorem 1.1. Then, for any torsionless lattice \( \Gamma \) commensurable with \( G(O_F) \) we have \( m(A_q, \Gamma) > 0 \) where \( q \) is of type \( (n-2, n-2) \).

The above result seems to be a new addition to the vast body of work on the occurrence of \( H_\pi \) in \( L^2(\Gamma \backslash G) \) as well as the asymptotics of \( m(\pi, \Gamma) \) in various settings (not restricted to the case of \( SO(n, \mathbb{H}) \)) such as when \( \pi = A_q \) is holomorphic (i.e. of type \((p,0)\)) due to Anderson [1], or a discrete series representation due to Clozel [5], DeGeorge and Wallach [6], or when \( \pi = A_q \) where the real reductive subgroup \( L \subset G \) with \( l = q \cap \overline{q} \) is isomorphic
to a group of the form \((S^1)^r \times SO(n - r, \mathbb{H})\) due to Li [12]. See also [4, Chapter VIII], [20, §6] and [10].

The paper is organised as follows. In §2 we consider \(F\)-structures on \(SO(n, \mathbb{H})\), construct \(F\)-involutions, including \(F\)-rational Cartan involution, under certain hypotheses on the \(F\)-structure, and construct special cycles, in §2.7, associated to these involutions. In §3 we consider \(\theta\)-stable parabolic subalgebras of \(SO(n, \mathbb{H})\) and compute their relative Lie algebra cohomology in terms of the certain combinatorial data associated to them. We also classify, in Proposition 3.7, \(\theta\)-stable parabolic subalgebras of \(\mathfrak{so}(n, \mathbb{H})\) of type \((p, q)\) when \(p, q \leq n - 1\). Theorems 1.1 and 1.2 will be proved in §4.

2. Arithmetic lattices in \(SO(n, \mathbb{H})\)

2.1. \(SO(n, \mathbb{H})\) and \(SO(2n, \mathbb{C})\). Let \(\mathbb{H}\) denote the division algebra of real quaternions with standard \(\mathbb{R}\)-basis \(1, i, j, k = ij\). We denote by \(\tau_c : \mathbb{H} \rightarrow \mathbb{H}\) the canonical conjugation defined as \(\tau_c(q) = \bar{q} := q_0 - q_1 i - q_2 j - q_3 k\) where \(q = q_0 + q_1 i + q_2 j + q_3 k\). We denote by \(N(q)\) the norm of \(q\), namely, \(N(q) = q \bar{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2\). The conjugation \(\tau_c\) is an anti-automorphism of the division algebra \(\mathbb{H}\). One has also the anti-automorphism \(\tau_r : \mathbb{H} \rightarrow \mathbb{H}\) defined as \(\tau_r(q) = q_0 + q_1 i - q_2 j + q_3 k = j\tau_c(q)j^{-1}\), known as the reversion. Applying \(\tau_r\) to each entry we obtain an \(\mathbb{R}\)-vector space automorphism \(M_n(\mathbb{H}) \rightarrow M_n(\mathbb{H})\) again denoted \(\tau_r\). Explicitly, \(\tau_r(Q) = Q_0 + Q_1 i - Q_2 j + Q_3 k\) where \(Q = Q_0 + Q_1 i + Q_2 j + Q_3 k\). Since transposition \(P \mapsto \tau P\) is an \(\mathbb{R}\)-algebra anti-automorphism of \(M_n(\mathbb{R})\), and since \(\tau_r\) is an anti-automorphism of \(\mathbb{H}\), it is follows that \(Q \mapsto \tau r(Q)\) is an \(\mathbb{R}\)-algebra anti-automorphism of \(M_n(\mathbb{H})\).

Note that \(\mathbb{H}\) splits over \(\mathbb{C}\), that is, we have an isomorphism of \(\mathbb{C}\)-algebras \(\mathbb{H} \otimes \mathbb{R} \mathbb{C} \cong M_2(\mathbb{C})\), where \(q \otimes 1 = q = z + w j \in \mathbb{H}, z, w \in \mathbb{C}\) is sent to \((\bar{z} w, z w)\). Under this isomorphism, \(\tau_r(q)\) maps to \((\bar{z} w, -\bar{z} w)\).

The inclusion \(\mathbb{H} \hookrightarrow \mathbb{H} \otimes \mathbb{C} \cong M_2(\mathbb{C})\) yields an obvious embedding \(M_n(\mathbb{H}) \hookrightarrow M_{2n}(\mathbb{C})\) where each quaternion in the domain is viewed as a \(2 \times 2\) matrix with entries in \(\mathbb{C}\). However, it is more convenient for our purposes to use a different embedding, namely, \(\psi : M_n(\mathbb{H}) \rightarrow M_{2n}(\mathbb{C})\) defined as \(Q \mapsto (Z, W)\) where \(Z, W \in M_n(\mathbb{C})\) are defined by \(Q = Z + W j\). It is easy to see that \(\psi\) is an \(\mathbb{R}\)-algebra homomorphism. Note that \(\psi(t Q) \neq t \psi(Q)\), indeed it is readily verified that \(\psi(t \tau r(Q)) = t \psi(Q)\). This shows that the anti-automorphism \(\tau r\) on \(M_n(\mathbb{H})\) corresponds, under \(\psi\), to transposition in \(M_{2n}(\mathbb{C})\). In particular \(\tau r(Q) = Q\) if and only if \(\psi(Q)\) is symmetric if and only if \(Z\) is symmetric and \(W\) is skew-hermitian. Similarly \(\tau r(Q) = -Q\) if and only if \(Z\) is skew-symmetric and \(W\), hermitian.

The group \(SO(n, \mathbb{H})\) is defined as \(\{Q \in SL(n, \mathbb{H}) \mid \tau r(Q)Q = I_n\}\) where \(I_n\) denotes the \(n \times n\) identity matrix. By the above discussion, it is clear that \(\psi\) restricts to a monomorphism of Lie groups \(SO(n, \mathbb{H}) \rightarrow SO(2n, \mathbb{C})\), again denoted \(\psi\). We shall identify \(SO(n, \mathbb{H})\) with its image under \(\psi\). The Lie algebra of \(SO(n, \mathbb{H})\) is seen to be \(\mathfrak{so}(n, \mathbb{H}) = \{Q \in M_n(\mathbb{H}) \mid \tau r(Q) + Q = 0\} = \{(Z, W) \in M_{2n}(\mathbb{C}) \mid Z + t Z = 0, W - t W = 0\}\). It is
clear from this description that $\mathfrak{so}(n, \mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{so}(2n, \mathbb{C})$, the Lie algebra of $\text{SO}(n, \mathbb{C})$. Thus $\text{SO}(n, \mathbb{H})$ is a real form of $\text{SO}(2n, \mathbb{C})$. The subgroup $K = \{ Z + Wj \in \text{SO}(n, \mathbb{H}) \mid Z, W \in M_n(\mathbb{R}) \}$ is isomorphic to $U(n)$ since $^t\tau_r(Z + Wj) = ^tZ - ^tWj = ^r(Z + Wj)$ if $Z, W \in M_n(\mathbb{R})$ where bar denotes conjugation in $M_n(\mathbb{R} \oplus \mathbb{R}j) \cong M_n(\mathbb{C})$. We also have $\mathfrak{k}_0 = \{ A + Bj \in \mathfrak{so}(n, \mathbb{H}) \mid A, B \in M_n(\mathbb{R}) \}$, which corresponds under $\psi$ to $\{ (A, B) \mid A, B \in M_n(\mathbb{R}), A + ^tA = 0, B = ^tB \}$. The subgroup $K$ is a maximal compact subgroup of $\text{SO}(n, \mathbb{H})$ and will be referred to as the standard maximal compact subgroup of $\text{SO}(n, \mathbb{H})$. The corresponding Cartan involution $\theta : \text{SO}(n, \mathbb{H}) \to \text{SO}(n, \mathbb{H})$ is $Q \mapsto JQJ^{-1}$, where $J = jI_n \in M_n(\mathbb{H})$ and its differential at the identity element, again denoted by the same symbol, is $\theta : \mathfrak{so}(n, \mathbb{H}) \to \mathfrak{so}(n, \mathbb{H})$ is again conjugation by $J$. Thus $\mathfrak{p}_0$, the $-1$ eigenspace of $\theta$, equals $\{ Ai + Bjk \in \mathfrak{so}(n, \mathbb{H}) \mid A, B \in M_n(\mathbb{R}) \}$ and corresponds under $\psi$ to the subspace $\{ i(A, B) \in \psi(\mathfrak{so}(2n, \mathbb{H})) \mid A, B \in M_n(\mathbb{R}) \}$. Therefore $\mathfrak{p} = \mathfrak{p}_0 \otimes_{\mathbb{R}} \mathbb{C} \cong \{ \begin{pmatrix} Z & W \\ \bar{W} & \bar{Z} \end{pmatrix} \in \mathfrak{so}(2n, \mathbb{C}) \}$. Note that multiplication on the right by $J$ yields a complex structure on $\mathfrak{p}_0$.

2.2. $F$-structures on $\text{SO}(n, \mathbb{H})$. Our aim in this section is to construct linear algebraic groups defined over number fields $F \subset \mathbb{R}$ such that the $\mathbb{R}$-points isomorphic, as a Lie group, to $\text{SO}(n, \mathbb{H})$. This is most conveniently achieved by starting with quaternion algebras over number fields which do not split over $\mathbb{R}$.

Let $F$ be a subfield of $\mathbb{R}$ and let $\alpha, \beta \in F^\times$. We denote by $\mathbb{D} := \mathbb{H}^{\alpha,\beta}_F$ the quaternion algebra over $F$ generated by $i, j$ where $i^2 = \alpha, j^2 = \beta, ij = -ji =: k$, so that $k^2 = -\alpha\beta$. Sending $i$ to $i$ and $j$ to $k/\alpha$ yields an $F$-algebra isomorphism $\mathbb{H}^{\alpha,\beta}_F \cong \mathbb{H}^\alpha_F$. Also $\mathbb{H}^{\alpha,\beta}_F \cong \mathbb{H}^\beta_F$. In view of this, we assume, without loss of generality, that $\beta < 0$.

One has an involutive anti-automorphism $\tau_\psi$ on $\mathbb{D}$ sending $q = q_0 + iq_1 + jq_2 + kj_3$ to $\bar{q} = q_0 - iq_1 - jq_2 - kq_3$. We have also the norm $N(q) = q\bar{q} = q_0^2 - \alpha q_1^2 - \beta q_2^2 + \alpha\beta q_3^2$. The $F$-linear map $\tau_\psi : \mathbb{D} \to \mathbb{D}$ is the reversion $\tau_\psi(q) = q_0 + iq_1 - jq_2 + kj_3$.

If $E \subset \mathbb{C}$ is an extension field of $F$ we denote $\mathbb{D} \otimes_F E = \mathbb{H}^{\alpha,\beta}_E$ by $\mathbb{D}_E$. Note that $\mathbb{D}$ splits over any field $E$ that contains $F(a) \subset \mathbb{C}$ where $a^2 = \alpha$. In particular, $\mathbb{D}_E \cong M_2(\mathbb{R})$ if $\alpha > 0$. If both $\alpha$ and $\beta$ are negative, then $\mathbb{D}_E$ is a division algebra over any extension field $E \subset \mathbb{R}$ of $F$ and we have an isomorphism $\mathbb{D}_E = \mathbb{H}^{\alpha,\beta}_E \cong \mathbb{H}^{\alpha\beta,\alpha\beta}_E$ where $a, b \in (E^*)^2$.

Suppose that $E \subset \mathbb{C}$ contains $F(a, b)$ where $a^2 = \alpha, b^2 = -\beta$, then $i \mapsto \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}, j \mapsto \begin{pmatrix} b & 0 \\ 0 & -a \end{pmatrix}$, defines an $E$-algebra isomorphism $\mathbb{D}_E \to M_2(\mathbb{C})$. Thus we have an $F$-algebra embedding $\psi_E : \mathbb{D} \hookrightarrow M_2(\mathbb{E})$. The embedding $\psi_E$ defines an $F$-algebra embedding (denoted by the same symbol) $\psi_E : M_n(\mathbb{D}) \to M_{2n}(\mathbb{E})$ where $Q = Q_0 + Q_1i + Q_2j + Q_3k \mapsto \begin{pmatrix} Q_0 + aQ_1 & bQ_2 + abQ_3 \\ -bQ_2 + abQ_3 & Q_0 - aQ_1 \end{pmatrix}$. (As $\psi_E$ depends not only on $F$ but also on the choice of square roots of $\alpha, -\beta$, the notation is somewhat imprecise.) Again reversion on $\mathbb{D}$ induces an involutive anti-automorphism $Q \mapsto ^t\tau_\psi(Q)$ of the $F$-algebra $M_n(\mathbb{D})$. One has the relation $\psi_E(^t\tau_\psi(Q)) = ^t\psi_E(Q)$ for all $Q$ in $M_n(\mathbb{D})$. A matrix $A \in M_n(\mathbb{D})$ is said to be $\tau_\psi$-hermitian if $^t\tau_\psi(A) = A$. Writing $A = A_0 + iA_1 + jA_2 + kA_3$, we see that $A$ is $\tau_\psi$-hermitian if and only if $A_2 = -^tA, ^tA_1 = A_3, r = 0, 1, 3$. If $A$ is $\tau_\psi$-hermitian and
α > 0, then $\psi_F(A)$ is real symmetric. (Recall that $\beta < 0$ by hypothesis and so $b$ is real whereas $a$ is real if $\alpha > 0$ and is purely imaginary if $\alpha < 0$.)

Let $A \in M_n(\mathbb{D})$ be $\tau_r$-hermitian and invertible. The group $SO(A, \mathbb{D}) := \{X \in M_n(\mathbb{D}) \mid \tau_r(X)AX = A, \ \det(X) = 1\}$ is the $F$-points of an $F$-algebraic group $G$ such that $G(\mathbb{C}) \cong SO(2n, \mathbb{C})$. It is known that all $F$-structures on $SO(n, \mathbb{C})$ arise this way when $n > 4$. (See [18, Ch. 18].) When $\alpha > 0$ we have $G(\mathbb{R}) = SO(A, \mathbb{D}_\mathbb{R}) \cong SO(p, 2n-p)$ where the symmetric matrix $\psi_F(A) \in M_{2n}(\mathbb{R})$ has exactly $p$ positive eigenvalues. In particular if $\psi_F(A)$ is positive or negative definite, $G(\mathbb{R})$ is compact. When $\alpha < 0$ (and since $\beta < 0$ by assumption) we have $\mathbb{D}_\mathbb{R} \cong \mathbb{H}$ and the group $G(\mathbb{R})$ is isomorphic to the type AIII real form $SO(n, \mathbb{H}) \cong SO^+(2n)$ of $SO(2n, \mathbb{C})$.

Suppose that $P \in M_n(\mathbb{D})$ is a non-singular matrix such that $\tau_r(P)AP = \lambda A$ for some $\lambda \in F$. Then it is easily verified that $Q \mapsto PQP^{-1}$ defines an $F$-automorphism of $SO(A, \mathbb{D})$.

2.3. $F$-rational Cartan involution. Recall that $J = jI_n \in M_n(\mathbb{H})$ and that $Q \mapsto JQJ^{-1}$ is the Cartan involution $\theta$ of $SO(n, \mathbb{H})$ that fixes the standard maximal compact subgroup $K = \{Z \in SO(n, \mathbb{H}) \mid Z, W \in M_n(\mathbb{R})\}$. Assume that $\alpha, \beta \in F$ are negative and let $\mathbb{D} = \mathbb{H}^{\alpha, \beta}$. It is readily seen that $\theta$ is also the Cartan involution of $SO(I_n, \mathbb{D}_\mathbb{R})$ that fixes the standard maximal compact subgroup $K = \{Z + W j \in SO(n, \mathbb{D}_\mathbb{R}) \mid Z, W \in M_n(\mathbb{R})\}$. In fact $\theta$ is induced from an $F$-rational involution $\theta : SO(I_n, \mathbb{D}) \to SO(I_n, \mathbb{D})$ which fixes the subgroup $K_F := \{Z + W j \in SO(I_n, \mathbb{D}) \mid Z, W \in M_n(F)\}$. The group $K_F$ is an $F$-algebraic group whose $\mathbb{R}$-points are $K$.

Let $A = \tau_r(P)P$ where $P \in M_n(\mathbb{D})$ is non-singular. Then $A$ is $\tau_r$-hermitian and non-singular. One has an $F$-isomorphism $SO(A, \mathbb{D}) \to SO(I_n, \mathbb{D})$ defined as $Q \mapsto PQP^{-1}$. It follows that $K_A := P^{-1}K_FP$ is an $F$-algebraic group whose $\mathbb{R}$-points is the maximal compact subgroup $P^{-1}KP \subset SO(A, \mathbb{D}_\mathbb{R})$. We shall prove the existence of an $F$-rational Cartan involution for a more general class of $\tau_r$-hermitian matrices. This will be preceded by some preliminary observations.

**Lemma 2.1.** Let $\alpha, \beta \in F^\times$ with $\beta < 0$, $\mathbb{D} = \mathbb{H}_F^{\alpha, \beta}$ and let $A$ be any $\tau_r$-hermitian matrix. There exists a non-singular matrix $P \in M_n(\mathbb{D})$ and a diagonal matrix $D = \text{diag}(d_1, \ldots, d_n) \in M_n(\mathbb{D})$ such that $\tau_r(P)AP = D$.

**Proof.** Regard $V = \mathbb{D}^n$ as a right $\mathbb{D}$-module. One has a $\tau_r$-hermitian pairing $\langle u, v \rangle : V \times V \to \mathbb{D}$ defined as $\langle u, v \rangle := \tau_r(u)Av$. It is $\mathbb{D}$-linear in the second argument and $\tau_r$-twisted $\mathbb{D}$-linear in the first, that is, $\langle uq, vq' \rangle = \tau_r(q)\langle u, v \rangle q'$ for $q, q' \in \mathbb{D}$. If $e_1, \ldots, e_n \in F^n \subset \mathbb{D}^n$ denotes the standard basis of $V$ over $\mathbb{D}$, then $A$ equals the matrix $(\langle e_i, e_j \rangle)$. Since $A$ is non-singular, the pairing is non-degenerate. It is easy to show, using a Gram-Schmid orthogonalisation argument starting with the standard basis, the existence of an orthogonal basis $\mathcal{B} := \{v_1, \ldots, v_n\}$ for $V$ (as a right $\mathbb{D}$-module), although it is not in general possible to obtain an orthonormal basis. Let $d_i := \langle v_i, v_i \rangle = \tau_r(v_i)Av_i$. We set $P \in M_n(\mathbb{D})$ to have columns the vectors $v_1, \ldots, v_n$. Then $\tau_r(P)AP = \text{diag}(d_1, \ldots, d_n) =: D$. \qed
The $v_l$—and hence the matrix $P$—is uniquely determined by the orthogonalisation process. Inductively, the $v_l$ are determined by the following requirements: $v_1 = e_1$ and $v_l \perp v_k, k < l$, $v_l \perp e_l$ belongs to the $\mathbb{D}$-span of $e_1, \ldots, e_{l-1}$.

The group $\text{SO}(A, \mathbb{D})$ is $F$-isomorphic to $\text{SO}(D, \mathbb{D})$ where $D$ is as in the above lemma. An $F$-isomorphism $\text{SO}(A, \mathbb{D}) \to \text{SO}(D, \mathbb{D})$ is given by $Q \mapsto P^{-1}QP$.

**Lemma 2.2.** Suppose that $\alpha, \beta < 0$. Let $D = \text{diag}(d_1, \ldots, d_n) \in M_n(\mathbb{D})$ be non-singular and $\tau_r$-hermitian. Then there exists a $\tau_r$-hermitian diagonal matrix $P \in M_n(\mathbb{D}_R)$ such that $D = c\tau_r(P)P = P^2$.

**Proof.** It suffice to show that any $d \in \mathbb{D}$ such that $\tau_r(d) = d = a + bi + ck$ is expressible as $d = \tau_r(u).u = u^2$ with $u = \tau_r(u) \in \mathbb{D}_R$. We need to find $u = x + yi + wk$ such that $d = (x + yi + wk)(x + yi + wk)$. We may (and do) assume that $bc \neq 0$. This is equivalent to solving the following system of equations over $\mathbb{R}$:

$$x^2 + \alpha y^2 - \alpha \beta w^2 = a,$$

$$2xy = b,$$

$$2xw = c.$$  

We obtain $2w = c/x, 2y = b/x$ and so substituting in the first equation implies that $f(x) := 4x^2 + \alpha b^2/x^2 - \alpha \beta c^2/x^2 = a$. Since $\alpha, -\alpha \beta < 0$, the function $f : \mathbb{R}_{>0} \to \mathbb{R}$ is continuous and onto. Let $f(x_0) = a$. Then $u = x_0 + bi/(2x_0) + ck/(2x_0)$ is a solution. □

**Remark 2.3.** The value of $x_0 \in \mathbb{R}_{>0}$ in the above proof is unique. This means that $u$ is unique up to sign. The latter statement is evidently valid if $bc = 0$. It follows that each diagonal entry of $P$ uniquely determined up to a sign $\pm$.

**Theorem 2.4.** Let $A$ be any $\tau_r$-hermitian non-singular matrix in $M_n(\mathbb{D})$ where $\mathbb{D} = \mathbb{H}_F^{\alpha, \beta}$ with $\alpha, \beta \in F$ both negative. Suppose that $\tau_r(P_0)AP_0 = D = \text{diag}(d_1, \ldots, d_n)$ where $N(d) \equiv N(d_1) \mod (F^+)2 \forall l \leq n$. Then there exists an $F$-involution $\theta_A$ of $\text{SO}(A, \mathbb{D})$ which defines a Cartan involution of $\text{SO}(A, \mathbb{D}_R)$. In fact $\theta_A$ is the restriction to $\text{SO}(A, \mathbb{D})$ of $\iota_Y$ where $Y = cPJP^{-1} \in M_n(\mathbb{D})$ for a suitable $P \in M_n(\mathbb{D}_R)$ and $c \in \mathbb{R}$.

**Proof.** Since $\text{SO}(A, \mathbb{D})$ is $F$-isomorphic to $\text{SO}(D, \mathbb{D})$ it suffices to show that $\text{SO}(D, \mathbb{D})$ admits an $F$-rational Cartan involution $\theta_D$ with $\text{Fix}(\theta_D) = K_D$. By Lemma 2.2, there exists diagonal matrix $P = \text{diag}(p_1, \ldots, p_r) \in M_n(\mathbb{D}_R)$ such that $\tau_r(P)P = P, D = P^2 = \tau_r(P)P$. In particular, since $\iota_J$ is the Cartan involution that fixes $K \subset \text{SO}(I_n, \mathbb{H}_R^{\alpha, \beta})$, it follows that $\iota_{J^P}J_P$ is the Cartan involution that fixes $P^{-1}KP$. It remains to find a $c \in \mathbb{R}$ so that $\iota_Y = \iota_{J^P}J_P$ is an $F$-involution where $Y := cP^{-1}JP \in M_n(\mathbb{D})$. By our hypothesis on $D$, there exist positive numbers $t_l \in F^+, 1 \leq l \leq n$, with $t_l = 1, t_l > 0 \forall l$ such that $t_l^2 N(d_l) = N(d_1) \forall l$. Let $c_l = t_l N(p_l) \in F^+, 1 \leq l \leq n$. Then $c_l^2 = t_l^2 N(p_l)^2 = t_l^2 N(d_l) = N(d_1) = c_1^2$. As $c_1 > 0$, we have $c_l = c_1 =: c$ for all $l \leq n$. It remains to show that the diagonal matrix $Y = cP^{-1}JP$ is in $M_n(\mathbb{D})$. We compute $l$-th diagonal entry—call it $y_l$—of $Y$ using $p_l = \tau_r(p_l) = j\tau_c(p_l)j/\beta$. We have $y_l = c p_l^{-1} j p_l = c j_{N(p_l)}^\beta j p_l = c j_{N(p_l)}^\beta j \tau_c(p_l) j p_l = j t_l p_l^2 = j t_l d_l \in \mathbb{D}$. Thus we may take $\theta_A$ to be the restriction of $\iota_Y$. □
2.4. Commuting pairs of $F$-Involutions. We keep the notations of §2.2. We shall now construct $F$-involutions of $G = \text{SO}(A, \mathbb{D})$ where $\mathbb{D} = \mathbb{H}^n_{F}$, with $\alpha, \beta < 0$, and $A \in M_n(\mathbb{D})$ is $\tau_r$-hermitian. The $F$-involutions which commute with the $F$-rational Cartan involution $\theta_A$ would play a crucial role in the sequel.

Let $A \in M_n(\mathbb{D})$ be $\tau_r$-hermitian and non-singular. In view of Lemma 2.1, we may assume, without loss of generality, that $A = \text{diag}(a_1, \ldots, a_n) \in M_n(\mathbb{D})$ is diagonal. Then $a_i = x_i + y_i + z_i k \forall i$. It is readily checked that for $S := \text{diag}(\epsilon_1, \ldots, \epsilon_n), \epsilon_i \in \{-1,1\}$, satisfies the conditions $S^2 = I, \tau_r(S) = S$, thus $\tau_r(S)AS = SAS$ and so $Q \mapsto S^{-1}QS = SQS$ defines an involutive $F$-automorphism of $\text{SO}(A, \mathbb{D})$. These involutions will be referred to as the sign involutions.

More generally, $Q \mapsto S^{-1}QS$ is an involutive $F$-automorphism if $S \in M_n(\mathbb{D})$ satisfies the following conditions:

\begin{align*}
S^2 &= -\lambda I_n, \text{ for some } \lambda \in F^\times \quad (1) \\
\tau_r(S)AS &= \mu A, \text{ for some } \mu \in F^\times. \quad (2)
\end{align*}

First we shall classify all such $S = \text{diag}(s_1, \ldots, s_n)$ that are diagonal. By (1), $s_k^2 = -\lambda, \forall k$. If $s_k \in F^\times$ for some $k$, then $\lambda < 0$. So for any $l$, $s_l^2 > 0$ and so $s_l \in F^\times$. Thus $s_i = \pm s_j \forall i, j$ and so that $S = \text{diag}(\epsilon_1 s, \ldots, \epsilon_n s)$ where $s = |s_1|$ and each $\epsilon_i$ equals 1 or $-1$. The involution defined by $S$ is therefore a sign involution.

Suppose that $s_k \notin F^\times$ for some $k$. Then $s_k^2 = -\lambda \in F^\times$ implies that $s_k \in Fi + Fj + Fk$ and $\lambda = -s_k^2 = N(s_k) > 0$. Therefore for any $l \leq n$, $s_l^2 < 0$ and so $s_l \in Fi + Fj + Fk, N(s_l) = \lambda$. Now equation (2) says that $\tau_r(s_k)ak s_k = \mu ak$ and so $\mu^2 N(ak) = N(\tau_r(s_k)ak s_k) = N(ak)N(s_k)^2 = \lambda^2 N(ak)$ and so $\mu = \pm \lambda$. Using $\tau_r(s) = j^{-1} \tau_r(s)j = -j^{-1}s j \forall s \in Fi + Fj + Fk$, we obtain that $\mu ak = \tau_r(s_k)ak s_k = -j^{-1} s_j ak s_k$ which implies $\mu s_k j ak = s_k (-s_k) j ak s_k = \lambda j ak s_k = \pm \mu j ak s_k$, that is, either $s_k j ak = j ak s_k \forall k$ in which case $\lambda = \mu$, or, $s_k j ak = -j ak s_k \forall k$ in which case $\lambda = -\mu$. Equivalently $S j A = j AS$ or $S j A = -j AS$.

Since $s_k \in Fi + Fj + Fk$, if $s_k$ commutes with $j ak$ then $s_k \in F(j ak)$. In this case $S = D j A$ where $D$ is $M_n(F)$ is a diagonal matrix.

It is clear that $s_k$ anti-commutes with $j ak$ if and only if $s_k \perp (j ak)$ in $Fi + Fj + Fk \subset \mathbb{D}$ with respect to the inner product on the $F$-vector space $\mathbb{D}$, which is associated to the quadratic form $q \mapsto N(q)$, which is positive definite since $\alpha, \beta < 0$. This is immediate from comparing the coefficient of 1 on both sides of $s_k(j ak) = -(j ak)s_k$; observe that $\tau_r(A) = A$ implies that $j a_l \in Fi + Fj + Fk$ for all $l$. The involution on $\text{SO}(A, \mathbb{D})$ induced by $S$ will be called an involution of even type if $S j A = j AS$, equivalently $\lambda = \mu$. If $S j A = -j AS$, then the involution induced by $S$ will be referred to as an involution of odd type, equivalently $\lambda = -\mu$.

Lemma 2.5. Let $A = \text{diag}(a_1, \ldots, a_n) \in M_n(\mathbb{H}^n_{F}), \alpha, \beta < 0$ be a non-singular diagonal $\tau_r$-hermitian matrix. (i) An $F$-involution of $\text{SO}(A, \mathbb{D})$ of even type exists if and only if $N(a_j) \equiv N(a_k) \mod (F^\times)^2$ for all $1 \leq j, k \leq n$. 


(ii) An involution of odd type exists if $N(a_j) \equiv N(a_k) \mod (F^\times)^2$ for all $j, k$.

(iii) Any two $F$-involutions of even type (possibly for different values of $\lambda$) commute. Also any $F$-involution of even type commutes with any $F$-involution of odd type.

Proof. (i) Let $N(a_k) = t_k^2N(a_1), k \leq n$ for suitable $t_k \in F^\times$. Set $S = \text{diag}(s_1, \ldots, s_n)$ with $s_k := t_k^{-1}ja_k \forall k \leq n$. Then $S$ satisfies equations (1) and (2) with $\lambda = N(ja_1) = \mu$ and so $S$ induces an involution of even type. The discussion preceding the statement of the lemma establishes the converse part.

(ii) Suppose that $N(a_k) = t_k^2N(a_1), 1 \leq k \leq n$. Then $N(t_k^{-1}ja_k) = N(ja_1)$ for all $k \leq n$. Thus $ja_k \mapsto t_kja_1$ defines an isomorphism of inner product spaces $Fja_k \to Fja_1$. It follows their orthogonal complements $(Fja_1)^\perp, (Fja_k)^\perp$ in $Fi + Fj + Fk$ are isometric, that is, they are isomorphic as inner product spaces. (See [11, Ch. XV, Theorem 10.2].) Fix linear isometries $f_k : (Fja_1)^\perp \to (Fja_k)^\perp$ and choose $s_1 \in (Fja_1)^\perp$ to be any non-zero element. If $s_k := f_k(s_1), 2 \leq k \leq n$, then $S := \text{diag}(s_1, \ldots, s_n)$ satisfies (1) and (2) with $-\mu = \lambda = N(s_1)$, and so $S$ induces an involution of odd type.

(iii) If $S$ determines an $F$-involution $\sigma$ of even type and $S'$ an $F$-involution $\sigma'$ of odd type, then, by the discussion preceding the statement of the lemma, we have a diagonal matrix $D \in M_n(F)$ such that $S = D\sigma A$. Since $D$ commutes with any diagonal matrix in $M_n(\mathbb{D})$ we have $SS' = D\sigma' A = -D\sigma'jA = -S'D\sigma A = -SS'$. It follows that $tstS' = tS's = t_{-S'S} = tsS's = tstS'$ and so $\sigma$ and $\sigma'$ commute. We leave to the reader the verification that any two involutions of even type commute. \hfill \Box

**Remark 2.6.** (i) We observe that, assuming $A = \text{diag}(a_1, \ldots, a_n)$, the Cartan involution $\theta_A$ constructed in Theorem 2.4 is of even type. This is because, in the notation of that theorem, $\theta_A$ is the restriction of $\iota_Y$ to $SO(A, \mathbb{D})$ where $Y = \text{diag}(y_1, \ldots, y_l)$ with $y_l = t_ljd_l \in Fjd_l \forall l \leq n$. The value of $\lambda$ for which equation (1) satisfied by $\theta_A$ is obtained as $\lambda = N(y_1) = -\beta N(d_1)$. In particular any involution of even or odd type commutes with $\theta_A$.

(ii) Suppose that $\sigma$ is an involution of even type induced by a diagonal matrix $S$ that satisfies equations (1) and (2). Then $S = CY$ where $C \in M_n(F)$ is a diagonal matrix of the form $\text{diag}(\epsilon_1c, \ldots, \epsilon_n c), \epsilon_k \in \{1, -1\}$. It follows that $\sigma = \varepsilon \circ \theta_A$ where $\varepsilon$ is a sign involution.

**2.5. Fixed points of involutions.** Let $F$ be a real number field. Let $A \in M_n(\mathbb{D})$ be $\tau_r$-hermitian and non-singular, where $\mathbb{D} = \mathbb{H}_F^{\alpha, \beta}, \alpha, \beta \in F$ both negative. Without loss of generality we may (and do) assume that $A$ is diagonal, say, $A = \text{diag}(a_1, \ldots, a_n)$; see §2.3. Assume that $N(a_1) \equiv N(a_1) \mod (F^\times)^2$. Let $S \in M_n(\mathbb{D})$ be non-singular diagonal matrix $\text{diag}(s_1, \ldots, s_n)$ that satisfies (1), (2) of §2.4. Denote by $\sigma$ the involution on $G := SO(A, \mathbb{D})$ defined by conjugation by $S$.

The group $C := \langle \sigma \rangle$ is the $F$-group $\text{Spec}(F[t]/(t^2 - 1))$. We denote by $G(\sigma) := \text{Fix}(\sigma)$, the fixed subgroup of $G$ fixed by the action of $C$ on $G$. It is an $F$-algebraic subgroup of
whose $F$-algebra of regular functions is the algebra of coinvariants $F[G]/\langle f - \sigma(f) \mid f \in F[G] \rangle$ for the action of $C$ on $F[G]$.

The $\mathbb{R}$-points $G(\sigma)(\mathbb{R})$ of the $F$-group $G(\sigma)$ is the fixed subgroup of $G(\mathbb{R})$ for the action of $C_\mathbb{R} = \text{Spec}(\mathbb{R}[t]/(t^2 - 1))$ on $G(\mathbb{R})$. That is, $G(\sigma)(\mathbb{R}) = \{Q \in G(\mathbb{R}) \mid \sigma(Q) = Q\}$.

For our applications it is important to know if the group $G(\sigma)(\mathbb{R})$ acts by preserving the orientation on the symmetric space $X(\sigma) := G(\sigma)(\mathbb{R})/K(\sigma)$, where $K(\sigma) = K \cap G(\sigma)(\mathbb{R})$. Of course it is enough to check this for a set of elements belonging to each connected component of $G(\sigma)(\mathbb{R})$. The requirement that $G(\sigma)(\mathbb{R})$ act by preserving the orientation on $X(\sigma)$ is called condition $Or$ in [22, Theorem 4.11].

We first consider $S$ so that $\sigma$ is a sign involution. If suffices to consider the case when $S = S_l := \begin{pmatrix} I_l & -I_{n-l} \\ I_{n-l} & I_{n-l} \end{pmatrix}$. In this case the $F$-points of $G(\sigma)$ is easily determined: An element $Q \in SO(A, \mathbb{D})$ is fixed by $\sigma$ if and only if $Q$ is block diagonal $(X Y)$ where $X \in SO(A_l, \mathbb{D}), Y \in SO(A_{n-l}', \mathbb{D})$. Here $A_l = \text{diag}(a_1, \ldots, a_l), A_{n-l}' = \text{diag}(a_{l+1}, \ldots, a_n)$. In this case the group $G(\sigma)(\mathbb{R})$ is the connected group $SO(A_l, \mathbb{D}) \times SO(A_{n-l}', \mathbb{D}) \cong SO(l, \mathbb{H}) \times SO(n - l, \mathbb{H})$, and $X(\sigma) \cong SO(l, \mathbb{H})/U(l) \times SO(n - l, \mathbb{H})/U(n - l)$. Since $G(\sigma)(\mathbb{R})$ is connected, the condition $Or$ is satisfied. We remark that $X(\sigma)$ is hermitian symmetric and in fact the inclusion $X(\sigma) \hookrightarrow X = G(\mathbb{R})/K = SO(A, \mathbb{D})/U(n)$ is complex analytic. This is because $\sigma(z) = z$ for all $z \in Z_K \cong \mathbb{S}^4$ the centre of $K = U(n)$. (Cf. [22, Remark 4.8(ii)].)

Next we turn to $F$-involutions of the even type. In this case, in view of Remark 2.6, we see that any such involution $\sigma$ equals $\varepsilon \theta_A$ where $\varepsilon$ is a sign involution. For notational convenience we assume that $\varepsilon$ is induced by $S_l = \begin{pmatrix} I_l & -I_{n-l} \\ I_{n-l} & I_{n-l} \end{pmatrix}$.

The following observation reduces computation of $G(\sigma)(\mathbb{R})$ to the special case when $A = I$ and $\alpha = \beta = -1$.

In view of Lemma 2.2, we have an isomorphism of Lie groups: $SO(A, \mathbb{D}) \rightarrow SO(I_n, \mathbb{D})$ given by $Q \mapsto PQP^{-1}$ where $P$ is an invertible $\tau_r$-hermitian in $SO(A, \mathbb{D})$ such that (the diagonal matrix) $A = \tau_r(\varepsilon)P = P^2$. Also this isomorphism is $\mathbb{Z}/2\mathbb{Z}$-equivariant where the action on $SO(A, \mathbb{D})$ is given by $\sigma$ and on $SO(I_n, \mathbb{D})$ by $\varepsilon \theta_{I_n}$. This is because $\theta_A$ is the conjugation by $cP^{-1}JP$ with $c \in \mathbb{F}^\times$ (see Theorem 2.4), and $S_l$ commutes with $P$ (since $P$ is diagonal). Therefore $G(\sigma)(\mathbb{R}) \rightarrow \text{Fix}(\varepsilon \theta_{I_n}), Q \mapsto PQP^{-1}$ is an isomorphism. Finally $SO(I_n, \mathbb{D}) \cong SO(I_n, \mathbb{H}/\mathbb{R}^{-1}) = SO(n, \mathbb{H})$ where $i \mapsto ai, j \mapsto bj, a, b \in \mathbb{R}_{>0}$ with $a^2 = -\alpha, b^2 = -\beta$. Clearly this isomorphism is again equivariant with respect to the action of $\langle \varepsilon \theta_{I_n} \rangle$ on the domain and that of $\langle \varepsilon \theta \rangle$ the target. (Recall from §2.1 that $\theta$ is the Cartan involution that fixes the standard maximal compact subgroup $K = \{X + Zj \in SO(n, \mathbb{H}) \mid X, Z \in M_n(\mathbb{R})\}$. Hence $G(\sigma)(\mathbb{R}) \cong \text{Fix}(\varepsilon \theta)$.

We now compute $\text{Fix}(\varepsilon \theta)$. Write $X \in M_n(\mathbb{R})$ as a block matrix $(X_{11} \ X_{12})$ where $X_{11} \in M_l(\mathbb{R}), X_{22} \in M_{n-l}(\mathbb{R})$. Then $\varepsilon(X) = S_lXS_l^{-1} = \begin{pmatrix} X_{11} & -X_{12}^T \\ -X_{21} & X_{22} \end{pmatrix}$. Thus $X = \varepsilon(X)$ if and only if $X_{12} = 0, X_{21} = 0$ and $X = -\varepsilon(X)$ if and only if $X_{11} = 0, X_{22} = 0$. 
Now let \( Q = X + Yi + Zj + Wk \in \text{SO}(n, \mathbb{H}) \) with \( X, Y, Z, W \in M_n(\mathbb{R}) \). Then \( \theta(Q) = X - Yi + Zj - Wk \). So \( \varepsilon(\theta(Q)) = Q \) if and only if, writing \( X, Y, Z, W \in M_n(\mathbb{R}) \) as block matrices as in the previous paragraph, we have \( X_{12} = Z_{12} = 0, X_{21} = Z_{21} = 0, Y_{11} = W_{11} = 0, Y_{22} = W_{22} = 0 \). Hence \( K(\varepsilon) := \text{Fix}(\varepsilon) \cap K = \{ X + Zj \in \text{SO}(n, \mathbb{H}) \mid X_{12} = Z_{12} = 0, X_{21} = Z_{21} = 0 \} \cong U(l) \times U(n - l) \) since \( X \) is skew symmetric and \( Z \) is symmetric as \( X + Zj \in \text{SO}(n, \mathbb{H}) \). Since \( K(\varepsilon) \) is connected and is a maximal compact subgroup of \( \text{Fix}(\varepsilon) \), the latter group is connected.

We claim that \( \text{Fix}(\varepsilon) = U(l, n - l) \). We need only show that the Lie algebra of \( \text{Fix}(\varepsilon) \) is isomorphic to \( u(l, n - l) \). To establish the claim, consider the Cartan decomposition \( \mathfrak{so}(n, \mathbb{H}) = \mathfrak{t}_0 \oplus \mathfrak{p}_0 \), defined by \( \theta \). As \( \varepsilon \) commutes with \( \theta \), the spaces \( \mathfrak{t}_0, \mathfrak{p}_0 \subset \mathfrak{so}(n, \mathbb{H}) \) are stable by \( \varepsilon \), and, \( \text{Lie}(\text{Fix}(\varepsilon)) \subset \mathfrak{so}(n, \mathbb{H}) \) decomposes as a direct sum of \((+1)\)-eigenspaces of \( \varepsilon|_{\mathfrak{t}_0} \), and the \((-1)\)-eigenspace of \( \varepsilon|_{\mathfrak{p}_0} \). This is in fact the Cartan decomposition of \( \text{Lie}(\text{Fix}(\varepsilon)) \). Denote these subspaces by \( \mathfrak{t}_0(\varepsilon), \mathfrak{p}_0(\varepsilon) \) respectively. A direct calculation shows that \( \mathfrak{t}_0(\varepsilon) \) consists of \( X + Zj \in \mathfrak{so}(n, \mathbb{H}) \) with \( X = (X_{11}, X_{22}) \) skew symmetric and \( Z = (Z_{11}, Z_{22}) \) symmetric, (where diagonal blocks have sizes \( l \) and \( n - l \)). Similarly \( \mathfrak{p}_0(\varepsilon) \) consists of \( Yi + Wk \in \mathfrak{so}(I_n, \mathbb{D}_\mathbb{R}) \) with \( Y = (Y_{12}), W = (W_{12}) \); thus \( Y, W \) are real skew symmetric matrices. Now it is readily verified that the Lie algebra of \( \text{Fix}(\varepsilon) \) is isomorphic to \( u(l, n - l) \). Thus \( \mathfrak{g}_0(\sigma) \cong u(l, n - l) \).

It remains to consider \( \text{Fix}(\sigma) \) when \( \sigma \) is an involution of odd type. As in the case of involutions of even type, it suffices to consider the case when \( A = I_n, \alpha = \beta = -1 \), \( \sigma \) is induced by conjugation by a diagonal matrix \( S = \text{diag}(s_1, \ldots, s_n) \in M_n(\mathbb{H}) \) such that \( S^2 = -I_n, SJ = -JS \) where \( J := jI_n \) and \( N(s_1) = 1 \forall l \). Write \( s_l := ie(\lambda_l) \in \mathbb{R}i + \mathbb{R}k \), where \( e(\lambda) := \cos(\lambda) + j \sin(\lambda) \in \mathbb{R}[j] \).

Suppose that \( Q = (q_{lm}) \in M_n(\mathbb{H}) \) is such that \( \sigma(Q) = SQS^{-1} = Q \). Then, writing \( q_{lm} = z_{lm} + iw_{lm} \) with \( z_{lm}, w_{lm} \in \mathbb{R}[j] \), a straightforward computation yields, for \( 1 \leq l, m \leq n \), \( z_{lm} = x_{lm}e(\frac{\lambda_m - \lambda_l}{2}), w_{lm} = y_{lm}e(\frac{\lambda_m + \lambda_l}{2}) \) where \( X = (x_{lm}), Y = (y_{lm}) \in M_n(\mathbb{R}) \). Set \( T := \text{diag}(e(\lambda_1/2), \ldots, e(\lambda_n/2)) \). Then we have \( Q = T^{-1}XT + iTYT = T^{-1}XT + T^{-1}YT = T^{-1}(X + iY)T \). Conversely, if \( Q = T^{-1}(X + iY)T \) with \( X, Y \in M_n(\mathbb{R}) \), then \( \sigma(Q) = Q \) using \( S = iT^2 = -T^{-2}i \). Thus \( V := \{ Q \in M_n(\mathbb{H}) \mid \sigma(Q) = Q \} = T^{-1}M_n(\mathbb{R}[i])T \). Since \( T \tau_r(T) = T^{-1}, \tau_r(Q) = T^{-1}T(X + iY)T^{-1} \) for all \( Q \in V \).

We are ready to compute the group \( G(\sigma)(\mathbb{R}) \subset \text{SO}(n, \mathbb{H}) \). Clearly \( G(\sigma)(\mathbb{R}) = \text{SO}(n, \mathbb{H}) \cap V = \{ T^{-1}CT \in V \mid \tau_r(T^{-1}CT), T^{-1}CT = I_n, C \in M_n(\mathbb{R}[i]) \} = \{ T^{-1}CT \in V \mid \tau_r(T^{-1}CT) = I_n \} = T^{-1}O(n, \mathbb{R}[i])T \cong O(n, \mathbb{C}) \). We conclude that the group \( G(\sigma)(\mathbb{R}) \) has exactly two components. The Cartan involution \( \theta \) restricts to a Cartan involution on \( G(\sigma)(\mathbb{R}) \) and we see that \( \mathfrak{p}_0(\sigma) = \{ T^{-1}iT^2 \mid T^{-1} Y = -Y \in M_n(\mathbb{R}) \} \).

**Lemma 2.7.** The action of the group \( G(\sigma)(\mathbb{R}) \cong O(n, \mathbb{C}) \) on \( X(\sigma) \cong O(n, \mathbb{C})/O(n) \) preserves the orientation if and only if \( n \) is odd. Thus involutions of odd type satisfies condition Or of [22, Theorem 4.11] precisely when \( n \) is odd.

**Proof.** It suffices to show that the action of the element \( g = T^{-1}\text{diag}(-1, 1, \ldots, 1)T \in G(\sigma)(\mathbb{R}) \), given by the isotropy representation on \( \mathfrak{p}_0(\sigma) = \{ T^{-1}iT^2 \mid T^{-1} Y = -Y \in M_n(\mathbb{R}) \} \),
is orientation reversing if and only if \( n \) is even. This is equivalent to showing that the element \( \text{diag}(-1,1,\ldots,1)=g \) reverses the orientation on the space of all \( n \times n \) real skew symmetric matrices if and only if \( n \) is even. The lemma now follows by a straightforward computation. \( \square \)

**Remark 2.8.** When \( \sigma \) is a sign involution or an involution of even type, \( X(\sigma) \) is hermitian symmetric. On the other hand, if \( \sigma \) is of odd type, then \( X(\sigma) \) is a Lagrangian submanifold of \( X=\text{SO}(I_n,\mathbb{D}_\mathbb{R})/K \). This is because the \( G(\mathbb{R}) \)-invariant integrable almost complex structure \( \mathcal{J} \) on \( X \) is obtained by conjugation by \( Z:=(I_n-J)/\sqrt{2}=e(-\pi/4)I_n \in K \) on the tangent space at the identity coset, namely \( p_0 \). If \( Q=iQ_1+kQ_3 \in p_0 \), then \( \mathcal{J}(iQ_1+kQ_3)=e(-\pi/4)Qe(\pi/4)=Qe(\pi/2)=Q.J=-iQ_3+kQ_1 \). So if \( Q=T^{-1}iYT \in p_0(\sigma) \), as \( Z \) commutes with \( T \), we have \( \mathcal{J}(Q)=T^{-1}iYT.J=T^{-1}kYT \). Since \( iX \) and \( kY \) are orthogonal for the Riemannian metric on \( p_0 \), and since conjugation by \( T \in K \) is an isometry of \( p_0 \), we conclude that \( X(\sigma) \) is a Lagrangian submanifold of \( X \). In view of the \( G \)-invariance of the Riemannian metric, it follows that \( C(\sigma) \subset X_A \) is a Lagrangian submanifold.

### 2.6. Restriction of scalars and arithmetic lattices

Let \( F \subset \mathbb{R} \) be a number field, i.e. a finite extension of \( \mathbb{Q} \). As usual \( \bar{\mathbb{Q}} \) denotes the field of algebraic numbers.

We shall denote by \( V_\infty \) the set of all real or complex embeddings of \( F \) and by \( S_\infty \) the set of all real embeddings. The inclusion \( F \hookrightarrow \mathbb{R} \) will be denoted by \( \iota \in V_\infty \). We denote by \( \mathcal{O}_F \) the ring of integers in \( F \). Let \( G \) be a semisimple algebraic group defined over \( F \), we denote by \( G(\mathcal{O}_F) \) the arithmetic subgroup of \( G(\mathbb{R}) \) namely the \( \mathcal{O}_F \)-points of \( G \). For \( \sigma \in V_\infty \), we denote by \( G^\sigma \) the corresponding algebraic group defined over \( \sigma(F) \). Let \( \mathcal{G} \) be the \( \mathbb{Q} \)-algebraic group got by restriction of scalars from \( F \) to \( \mathbb{Q} \), that is, \( \mathcal{G}:=\text{Res}_{F/\mathbb{Q}}(G) \). Then by a theorem of Borel and Harish-Chandra, \( \mathcal{G}(\mathbb{Z}) \), the group of \( \mathbb{Z} \)-points in \( \mathcal{G} \), is a lattice in \( G(\mathbb{R}) \). The \( \mathbb{R} \)-points of \( \mathcal{G} \), which is a real Lie group with finitely many connected components, is obtained as:

\[
\mathcal{G}(\mathbb{R}) = \prod_{\sigma \in S_\infty} G^\sigma(\mathbb{R}) \times \prod_{\{ \sigma, \delta \}, \sigma \neq \delta} G^\sigma(\mathbb{C}) \cong (\prod_{\sigma \in S_\infty} G^\sigma(\mathbb{R})) \times G(\mathbb{C})^{d_2}
\]

where \( d_2 \) is the number of pairs of conjugate complex embeddings of \( F \). The second factor is understood to be trivial if \( F \) is totally real, that is, if \( V_\infty = S_\infty \).

Suppose that \( A \in M_n(\mathbb{D}) \) is \( \tau_r \)-hermitian and non-singular. Assume that \( \alpha > 0, \beta < 0 \). We have an embedding \( \psi_\mathbb{R} : M_n(\mathbb{D}) \to M_{2n}(\mathbb{R}) \), after fixing positive square roots of \( \alpha, -\beta \). See \( \S 2.2 \). Then \( \psi_\mathbb{R}(A) \) is symmetric. If \( \psi_\mathbb{R}(A) \) has exactly \( p \) positive eigenvalues, then \( \text{SO}(n, \mathbb{H}^\alpha_{\mathbb{R}}) \cong \text{SO}(p, 2n-p) \). In particular, when \( p = 0,2n \), then \( A \) is definite and the group \( \text{SO}(I_n, \mathbb{H}^\alpha_{\mathbb{R}}) \cong \text{SO}(2n) \) is the compact form of \( \text{SO}(2n, \mathbb{C}) \). If \( \alpha, \beta < 0 \), then \( \mathbb{H}^\alpha_{\mathbb{R}} \cong \mathbb{H} \) and \( \text{SO}(A, \mathbb{H}^\alpha_{\mathbb{R}}) \cong \text{SO}(n, \mathbb{H}) \).

Now suppose that \( F \) is a totally real number field, \( F \neq \mathbb{Q} \); thus \( \sigma(F) \subset \mathbb{R} \) for all \( \sigma \in V_\infty \). Denote by \( \mathbb{D}_\sigma \) the quaternion algebra \( \mathbb{H}^\sigma_{\mathbb{R}}(\sigma(F)) \). Suppose that \( \alpha, \beta \in F \) are such that \( \alpha, \beta < 0 \) and \( \sigma(\alpha) > 0, \sigma(\beta) < 0 \) for all \( \sigma \in V_\infty, \sigma \neq \iota \). In view of the fact that \( \sigma \) fixes all the rationals, it is easily seen that such elements \( \alpha, \beta \in F \) do exist.
Fix square roots \( a, b \in \overline{\mathbb{Q}} \) for \( \alpha, -\beta \) respectively so that we have an embedding \( \psi_{\overline{\mathbb{Q}}} : M_n(\mathbb{D}) \to M_{2n}(\overline{\mathbb{Q}}) \). For any \( \sigma \in V_\infty, \sigma \neq \iota \), choose an automorphism \( \tilde{\sigma} \) of \( \overline{\mathbb{Q}} \) that extends the isomorphism \( \sigma : F \to \sigma(F) \). We shall denote by \( \psi_{\overline{\mathbb{Q}}}^\sigma : M_n(\mathbb{D}_\sigma) \to M_{2n}(\overline{\mathbb{Q}}) \) the embedding corresponding to the choice \( \tilde{\sigma}(a), \tilde{\sigma}(b) \) of square roots of \( \sigma(\alpha), \sigma(-\beta) \in \sigma(F) \) respectively.

Let \( A \in M_n(\mathbb{D}) \) be a non-singular \( \tau_r \)-hermitian matrix, that is, \( \psi_{\iota}(A) = A \), \( \det(A) \neq 0 \). Since \( \tilde{\sigma}(a), \tilde{\sigma}(b) \) are real and since \( \sigma(F) \subset \mathbb{R} \), the matrix \( \psi_{\overline{\mathbb{Q}}}^\sigma(\sigma(A)) \) is real symmetric for all \( \sigma \neq \iota \). Let \( \lambda_1, \ldots, \lambda_{2n} \in \overline{\mathbb{Q}} \) be the eigenvalues of \( \psi_{\overline{\mathbb{Q}}}(A) \in M_{2n}(\overline{\mathbb{Q}}) \). Then \( \tilde{\sigma}(\lambda_1), \ldots, \tilde{\sigma}(\lambda_{2n}) \) are the eigenvalues of \( \psi_{\overline{\mathbb{Q}}}^\sigma(\sigma(A)) \) for all \( \sigma \neq \iota \) in \( V_\infty \). We choose \( A \) such that for all \( \sigma \neq \iota \) the eigenvalues \( \tilde{\sigma}(\lambda_j) \) are positive for \( 1 \leq j \leq 2n \), so that \( \psi_{\overline{\mathbb{Q}}}^\sigma(\sigma(A)) \) is positive definite. In view of the fact that \( \tilde{\sigma}(\lambda_j) \) is a conjugate of \( \lambda_j, 1 \leq j \leq 2n \), it is clear that such matrices \( A \) exist. In fact, starting with any \( \tau_r \)-hermitian matrix \( A \), the matrix \( rI + A \in M_n(\mathbb{D}) \) has the required property for any rational number \( r > 0 \) which is larger than the absolute value of any conjugate of \( \lambda_j, 1 \leq j \leq 2n \).

**Theorem 2.9.** Let \( \iota : F \hookrightarrow \mathbb{R} \) be a totally real number field, \( F \neq \overline{\mathbb{Q}} \). Let \( \alpha, \beta \in F^\times \). Suppose that \( \alpha, \beta \) are negative and that \( \sigma(\alpha) > 0, \sigma(\beta) < 0 \) for all \( \sigma \neq \iota \) in \( V_\infty \). Let \( A \in M_n(\mathbb{H}_F^{\alpha,\beta}) \) be \( \tau_r \)-hermitian and non-singular. Suppose that \( \psi_{\overline{\mathbb{Q}}}^\sigma(\sigma(A)) \in M_{2n}(\mathbb{R}) \) is (positive or negative) definite for all \( \sigma \neq \iota \). Then the \( \mathcal{O}_F \)-points of \( SO(A, \mathbb{H}_F^{\alpha,\beta}) \cong SO(n, \mathbb{H}) \) is a uniform arithmetic lattice.

**Proof.** Since \( F \neq \overline{\mathbb{Q}}, V_\infty \) contains at least two elements. Let \( G \) be the \( F \)-algebraic group \( SO(A, \mathbb{D}) = SO(A, \mathbb{H}_F^{\alpha,\beta}) \). Our hypotheses that, for \( \sigma \in V_\infty, \sigma \neq \iota, \sigma(\alpha) > 0, \sigma(\beta) < 0 \) and \( \psi_{\overline{\mathbb{Q}}}^\sigma(\sigma(A)) \in M_{2n}(\mathbb{R}) \) is definite, imply that \( G^\sigma(\mathbb{R}) = SO(\sigma(A), \mathbb{H}_R^{\alpha(\sigma)\sigma(\beta)}) \) is compact. Also since \( F \) is totally real, it is clear from (3) that the only non-compact factor in \( G(\mathbb{R}) = G^\iota(\mathbb{R}) = SO(A, \mathbb{H}_R^{\alpha,\beta}) \cong SO(n, \mathbb{H}) \). By the above discussion, we see that \( \Gamma := G(\mathcal{O}_F), \) which is the image of \( G(\mathbb{Z}) \) under the projection \( G(\mathbb{R}) \to G(\mathbb{R}) \) is a lattice in \( G(\mathbb{R}) \). Since \( G(\mathbb{R}) \) has a non-trivial compact factor, \( \Gamma \) has no unipotent elements and we conclude that \( \Gamma \) is uniform. (See [21].)

**Remark 2.10.** When \( F = \overline{\mathbb{Q}} \) the above proof fails. In fact, let \( A = \text{diag}(a_1, \ldots, a_n) \) where the \( a_k \) are all non-zero integers. The \( \tau_r \)-hermitian inner product on \( \mathbb{D}^n \) defined by \( A \) when restricted to the subspace \( \mathbb{Q}^n \) corresponds to the integral quadratic form \( B(u) := \langle u, u \rangle = \sum_{1 \leq k \leq n} a_k u_k^2 \). If not all the \( a_k \) are of the same sign, then \( B \) is an indefinite form and hence it represents a real zero. It is a classical result that, if \( n \geq 5 \), then \( B \) has an integral zero, say, \( u \). (See [26, Ch. IV, §3].) From this it is not difficult to see that \( G(\mathbb{Z}) \) has unipotent element and hence cannot be a uniform lattice in \( G(\mathbb{R}) \). If \( a_k > 0 \) for all \( k \), then the quadratic form on the \( \mathbb{Q} \)-vector subspace of \( \mathbb{D}^n \) spanned by \( ie_1, e_2, \ldots, e_n \) is integral and indefinite. Again, as before, it has an integral zero, leading to the conclusion that \( G(\mathbb{Z}) \) is not uniform.

We do not know any example of a \( \mathbb{Q} \)-algebraic group \( G \) with \( G(\mathbb{R}) = SO(n, \mathbb{H}) \) for which \( G(\mathbb{Z}) \) is a uniform lattice.
2.7. Special cycles in \( X_\Gamma \). We now put together the results of §2.6 and §2.5 to construct special cycles in \( \Gamma \backslash SO(n, \mathbb{H})/U(n) \) for lattices \( \Gamma \).

Let \( A \in M_n(\mathbb{H}_F^{a,b}) \) be a non-singular \( \tau \)-hermitian matrix where \( F \neq \mathbb{Q} \) is any totally real number field and \( \alpha, \beta \in F \) satisfy the hypotheses of Theorem 2.9. Thus \( \sigma(\alpha) > 0, \sigma(\beta) < 0 \) for all \( \sigma \in \mathcal{V}_\infty = S_\infty \) except the inclusion \( \iota : F \hookrightarrow \mathbb{R} \). Let \( \mathbf{G} = SO(A, \mathbb{H}_F^{a,b}) \) so that \( G(\mathbb{R}) \cong SO(n, \mathbb{H}) \). With notation as in §2.6, by Theorem 2.9, \( G(O_F) \subset G(\mathbb{R}) \) is an arithmetic lattice. Our hypotheses on \( \alpha, \beta \) and on \( A \) imply that \( G(O_F) \) is uniform.

We assume that \( A \) is a diagonal matrix \( \text{diag}(a_1, \ldots, a_n) \) such that \( N(a_1) \equiv N(a_l) \mod (F^\times)^2 \) for all \( l \leq n \). Let \( \theta \) denote the Cartan involution of \( G(\mathbb{R}) \) arising from the \( F \)-Cartan involution constructed in Theorem 2.4 and let \( \sigma \) be a sign involution or an involution of even or odd type that commutes with \( \theta \). (See §2.5.) We let \( \Gamma \) be any torsionless finite index subgroup of \( G(O_F) \) that is stable by \( \sigma \) and \( \theta \). In fact any torsionless finite index subgroup of \( G(O_F) \) contains such a lattice \( \Gamma \). Let \( \Gamma(\sigma) = \Gamma \cap G(\sigma)(\mathbb{R}) \). Clearly it is a discrete subgroup of \( G(\sigma)(\mathbb{R}) \) and we have a closed embedding \( \Gamma(\sigma) \backslash G(\sigma)(\mathbb{R}) \hookrightarrow \Gamma \backslash G(\mathbb{R}) \). It follows that \( \Gamma(\sigma) \backslash G(\sigma)(\mathbb{R}) \) is compact and so \( \Gamma(\sigma) \) is a uniform lattice in \( G(\sigma)(\mathbb{R}) \).

Let \( C(\sigma) = \Gamma(\sigma) \backslash X(\sigma) \) where \( X(\sigma) = G(\sigma)(\mathbb{R})/K(\sigma) \subset X := G(\mathbb{R})/K \cong SO(n, \mathbb{H})/U(n) \). Since \( X(\sigma) \) is connected, so is \( C(\sigma) \). Thus \( C(\sigma) \) is a special cycle in \( X_\Gamma \). In the case when \( \sigma \) is a sign involution or an involution of even type, both \( C(\sigma), C(\sigma \theta) \) are locally hermitian symmetric and the inclusion \( C(\sigma) \hookrightarrow X_\Gamma \) is holomorphic. When \( \sigma \) is of odd type, the special cycles \( C(\sigma), C(\sigma \theta) \) are Lagrangian submanifolds of \( X_\Gamma \). As remarked in the introduction, \([C(\sigma)]\), the dual cohomology class of \( C(\sigma) \) is non-zero, and, is in fact not in the image of the Matsushima homomorphism \( H^*(X_u; \mathbb{C}) \to H^*(X_\Gamma; \mathbb{C}) \) by [15, Theorem 2.1].

Suppose that \( \sigma \) is an involution of even type or of sign type. Then \( C(\sigma) \) is a complex submanifold of the compact Kähler manifold \( X_\Gamma \). If \( C(\sigma) \) is of complex codimension \( p \) in \( X_\Gamma \), then \( [C(\sigma)] \) is of Hodge type \( (p, p) \), that is, \([C(\sigma)] \in H^{p,p}(X_\Gamma; \mathbb{C})\). Moreover, if \( \Lambda \) is a torsionless lattice in \( G(\mathbb{R}) \) that contains \( \Gamma \), as the covering projection \( \pi : X_\Gamma \to X_\Lambda \) is holomorphic, the image of \( C(\sigma) \)—denoted \( C_\Lambda(\sigma) \)—is an analytic cycle in \( X_\Lambda \). Again, as \( X_\Lambda \) is a Kähler manifold, it represents a non-zero dual cohomology class \([C_\Lambda(\sigma)] \in H^{p,p}(X_\Lambda; \mathbb{C})\) which is again not in the image of the Matsushima homomorphism \( H^*(X_u; \mathbb{C}) \to H^*(X_\Lambda; \mathbb{C}) \). For, otherwise \([C(\sigma)] = \deg(\pi).\pi^*[C_\Lambda(\sigma)] \in H^*(X_\Gamma; \mathbb{C})\) would be in the image of the Matsushima homomorphism, in view of the following commutative diagram

\[
\begin{array}{ccc}
H^*(X_u; \mathbb{C}) & \to & H^*(X_\Lambda; \mathbb{C}) \\
\downarrow \text{id} & & \downarrow \pi^* \\
H^*(X_u; \mathbb{C}) & \to & H^*(X_\Gamma; \mathbb{C})
\end{array}
\]

where the horizontal arrows are the Matsushima homomorphisms.

Next assume that \( \sigma \) is of odd type and \( n \) is odd. Then, by Lemma 2.7, \( \sigma \) satisfies the condition \( Or \) of [22, Theorem 4.11]. Therefore there exists a finite index subgroup \( \Lambda \subset \Gamma \) which is stable by \( \sigma \) and \( \theta \) such that the submanifold \( C_\Lambda(\sigma) \), more briefly denoted
$C(\sigma)$, defined as $C(\sigma) := \Lambda \setminus G(\sigma)(\mathbb{R})/K(\sigma)$ is a special cycle and $[C(\sigma)] \cdot [C(\sigma')] \neq 0$ in $H^*(X_\Lambda; \mathbb{C})$. Note that $\dim C(\sigma) = \binom{n}{2} = (1/2) \dim X_\Gamma$ so that $[C(\sigma)] \in H^G(X_\Lambda; \mathbb{C})$.

**Example 2.11.** Taking $\sigma$ to be the involution corresponding to $S_{1,n-1} = \text{diag}(1, -1, \ldots, -1)$, we obtain that $G(\sigma)(\mathbb{R}) \cong SO(1, \mathbb{H}) \times SO(n-1, \mathbb{H}) \cong S^1 \times SO(n-1, \mathbb{H})$ and the special cycle $C_\Lambda(\sigma)$ has (complex) codimension $n-1$ in $X_\Lambda$.

3. $L^2(\Gamma \setminus G)$ and $\theta$-stable parabolic subalgebras

3.1. Relative Lie algebra cohomology and Matsushima isomorphism. Let $\Gamma$ be a uniform lattice in a linear connected semisimple group $G$. We keep the notations of §1. Thus $K$ denotes a maximal compact subgroup of $G$, $\theta$ the corresponding Cartan involution, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition.

Recall that a $\theta$-stable parabolic subalgebra of $\mathfrak{g}_0$ is a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ such that $\theta(\mathfrak{q}) = \mathfrak{q}$ and $\mathfrak{q} \cap \mathfrak{q}' =: \mathfrak{l}$ is the Levi subalgebra of $\mathfrak{q}$. Here, the bar refers to conjugation of $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ with respect to the real form $\mathfrak{g}_0$. As recalled in §1, a result of Gelfand and Piatetskii-Shapiro [8], [7] states that $L^2(\Gamma \setminus G)$ decomposes into a Hilbert sum $\hat{\bigoplus}_{\pi \in \mathcal{G}} m(\pi, \Gamma) H_{\pi}$ of irreducible unitary representations $H_{\pi}$ with finite multiplicities $m(\pi, \Gamma)$. One has the Matsushima isomorphism $H^*(\Gamma; \mathbb{C}) \cong H^*(\mathfrak{g}, K; L^2(\Gamma \setminus G)_K) = \bigoplus_{\pi \in \mathcal{G}} m(\pi, \Gamma) H^*(\mathfrak{g}, K; H_{\pi,K})$. Since $\Gamma \setminus G$ is compact, there are only finitely many $\pi \in \hat{\mathcal{G}}$ with $m(\pi, \Gamma) \neq 0$ having non-vanishing $(\mathfrak{g}, K)$-cohomology. In fact, to each $\theta$-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}_0$ there is an irreducible unitary $G$-representation $(\mathcal{A}_q, \mathcal{A}_q)$ associated to $\mathfrak{q}$ with $H^*(\mathfrak{g}, K; \mathcal{A}_q, K) \neq 0$. If $V$ is a $(\mathfrak{g}, K)$-module with $H^*(\mathfrak{g}, K; V) \neq 0$, then $V$ is isomorphic, as a $(\mathfrak{g}, K)$-module to the space of (smooth) $K$-finite vectors of $\mathcal{A}_q$ for some $\theta$-stable parabolic subalgebra of $\mathfrak{g}_0$. Up to unitary equivalence, there are only finitely many representations of the form $(\mathcal{A}_q, \mathcal{A}_q)$. Hence we have

$$H^*(\Gamma; \mathbb{C}) \cong \bigoplus_{[q]} m(\mathcal{A}_q, \Gamma) H^*(\mathfrak{g}, K; \mathcal{A}_q, K)$$

where the sum is over all equivalence classes of $\theta$-stable parabolic subalgebras $\mathfrak{q}$ of $\mathfrak{g}_0$; here $\mathfrak{q}$ and $\mathfrak{q}'$ belong to the same equivalence class $[\mathfrak{q}]$ if $\mathcal{A}_q$ and $\mathcal{A}_{q'}$ are unitarily equivalent.

It is known that two irreducible unitary representations $(\pi, H_\pi)$, $(\sigma, H_\sigma)$ of $G$ are unitarily equivalent if their Harish-Chandra modules $H_{\pi,K}$ and $H_{\sigma,K}$ are isomorphic as $(\mathfrak{g}, K)$-representations. Moreover, any irreducible unitary $(\mathfrak{g}, K)$-representation arises as the Harish-Chandra module of a *unique* irreducible unitary $G$-representation. (See [9, Ch. IX].) Thus, in order to describe $\mathcal{A}_q$, it suffices to describe its Harish-Chandra module. We shall do this in the special case when the (complex) rank of $G$ equals the rank of $K$ as this condition holds in the case when $G = SO(n, \mathbb{H})$. Fix a maximal torus $T \subset K$. In view of our assumption on $G$, $\mathfrak{t} = \text{Lie}(T) \otimes \mathbb{C} = \mathfrak{t}_0 \otimes \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$. We assume that $\mathfrak{t} \subset \mathfrak{q}$. Write $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ where $\mathfrak{u}$ is the nilradical of $\mathfrak{q}$. Choose a positive system $(\mathfrak{l} \cap \mathfrak{t}, \mathfrak{t})$ and extend it to a positive system for $(\mathfrak{t}, \mathfrak{t})$ so that the weights of $\mathfrak{u} \cap \mathfrak{t}$ are all positive roots of $\mathfrak{t}$. Then $\mathcal{A}_q$ is determined, up to unitary equivalence, by the set of weights in $\mathfrak{u} \cap \mathfrak{p}$. (See Remarkequivalentq(i).)
Now let $L \subset G$ be the Lie subgroup corresponding to the Lie subalgebra $l_0 := l \cap g_0$. Then $K \cap L$ is a maximal compact subgroup of $L$. Let $Y_q'$ denote the compact dual of $L/K \cap L$. It turns out that $H^r(g, K; A_{q,K}) = \text{Hom}_K(\Lambda^r(p), A_{q,K}) \cong H^{r-R(q)}(Y_q', \mathbb{C})$, where $R(q) := \text{dim}_\mathbb{C}(p \cap u)$.

When $\text{rank}(K) = \text{rank}(G)$ and $q$ is a Borel subalgebra, $A_q$ is a discrete series representation. In this case, $L = T$ and $R(q) = (1/2) \dim G/K$ and $Y_q$ is a point.

Suppose that $G/K$ is a Hermitian symmetric space, equivalently the centre of $T$ is non-discrete. Then the space $X_T = \Gamma \backslash G/K$ admits the structure of a smooth projective variety arising from a $G$-invariant complex structure on $G/K$. The tangent space $p_0$ being a complex vector space, $p = p_+ \oplus p_-$ where $p_+$ and $p_-$ are conjugate complex vector spaces. The real tangent space $p_0$ may be identified with the holomorphic tangent space $p_+$ as $K$-representations. Note that $p_-$ is the dual of $p_+$ as a $K$-module. Also $p_+, p_-$ are abelian subalgebras of $g$. The Hodge structure on $H^r(\Gamma; \mathbb{C})$ arises from the decomposition $\Lambda^p(p) = \Lambda^p(p_+ \oplus p_-) = \oplus_{a+b=r} \Lambda^a(p_+) \otimes \Lambda^b(p_-)$. More precisely, $H^{a,b}(g, K; A_{q,K}) \cong \text{Hom}_K(\Lambda^a(p_+) \otimes \Lambda^b(p_-), A_{q,K})$ and we have

$$H^{a,b}(\Gamma; \mathbb{C}) = \oplus_{a+b=r} m(q, \Gamma) H^{a,b}(g, K; A_{q,K}),$$

(4) where $m(q, \Lambda)$ stands for $m(A_q, \Lambda)$. See [4, Ch. VII, §§2.3]. Let $R_\pm(q) = \text{dim}_\mathbb{C}(p_+ \cap u)$ so that $R(q) = R_+(q) + R_-(q)$. Then

$$H^{a,b}(g, K; A_{q,K}) = 0 \text{ if } a - b \neq R_+(q) - R_-(q).$$

(5)

See [29, Proposition 6.19].

The construction of the Harish-Chandra module of $A_q$ was originally due to Parthasarathy [19]. Vogan-Zuckerman [29], Vogan [27] gave a construction in terms of cohomological induction and showed that they are unitarizable. A very readable account explaining the basic properties of $A_q$ is given in [28].

### 3.2. The $\theta$-stable parabolic subalgebras of $SO(n, \mathbb{H})$

Recall from §2.1 that $SO(n, \mathbb{H})$ is a real form of $SO(2n, \mathbb{C})$. We shall identify $SO(n, \mathbb{H})$ with its image under $\psi : SO(n, \mathbb{H}) \to SO(2n, \mathbb{C})$. Write $G := SO(n, \mathbb{H}), K = \left\{ \begin{pmatrix} Z & W \\ -W & Z \end{pmatrix} \in G \mid Z, W \in M_n(\mathbb{R}) \right\} \cong U(n)$, a maximal compact subgroup of $G$. (See §2.1.) Let $T \subset K$ be the maximal torus $\left\{ \begin{pmatrix} Z & W \\ -W & Z \end{pmatrix} \in K \mid Z, W \text{ diagonal} \right\}$. Note that $\text{rank}(K) = \text{rank}(G(\mathbb{C}))$. So $t$ is a Cartan subalgebra of $g = so(2n, \mathbb{C})$.

Denote by $B$ the matrix $\begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}$ where $B \in M_n(\mathbb{C})$. The Lie algebra $t_0$ of $T$ is $\{ B \in M_{2n}(\mathbb{R}) \mid B \text{ is diagonal} \}$. We set $\epsilon_j : t \to \mathbb{C}$ be the $\mathbb{C}$-linear form defined as $\epsilon_j(B) = -ib_j \in \mathbb{C}$ where $B = \text{diag}(b_1, \ldots, b_n)$. Note that $\epsilon_j$ takes real values on $it_0$. Then $\Phi = \{ \pm(\epsilon_i \pm \epsilon_j) \mid 1 \leq i < j \leq n \}, \Phi_{t} = \{ \epsilon_i - \epsilon_j \mid 1 \leq i, j \leq n, i \neq j \}$ are the set of roots of $g$, respectively, of $t$. We take $\Phi^+$ to be $\{ \epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n \}$ and $\Phi^+_t = \Phi_t \cap \Phi^+$. We denote by $\Phi_n$ the set of non-compact roots $\{ \pm(\epsilon_p + \epsilon_q) \mid 1 \leq p < q \leq n \}$ and by $\Phi^+_n$ the set $\Phi^+ \cap \Phi_n$ of positive non-compact roots.
Let \( x \in \mathfrak{t}_0 \), where \( x := iX, X = \text{diag}(x_1, \ldots, x_n) \in M_n(\mathbb{R}) \). Applying \( Ad(g) \) for a suitable \( g \in K \) if necessary, we may (and do) assume that \( x_1 \geq \ldots \geq x_n \), so that \( \alpha(x) \geq 0 \ \forall \alpha \in \Phi^+ \). If \( (\epsilon_\rho + \epsilon_q)(x) \geq 0, p > q \), then \( (\epsilon_\rho + \epsilon_q)(x) \geq 0 \ \forall r \leq p, s \leq q \). The centralizer \( \mathfrak{l}_x \) of \( x \in \mathfrak{g} \) is a reductive subalgebra of \( \mathfrak{g} \) that contains \( \mathfrak{t} \). It contains a root space \( \mathfrak{g}_\alpha \) whenever (a) \( \alpha = \epsilon_i - \epsilon_j \) and \( x_i = x_j \), or (b) \( \alpha = \pm(\epsilon_i + \epsilon_j) \) and \( x_i = -x_j \).

We have \( \mathfrak{g} = \mathfrak{l}_x \oplus \mathfrak{u} \oplus \mathfrak{u}_- \), where \( \mathfrak{u} = \mathfrak{u}_x \) is the nilpotent subalgebra \( \oplus_{\alpha \in \Phi, \alpha(x) > 0} \mathfrak{g}_\alpha \) and \( \mathfrak{u}_- = \oplus_{\alpha \in \Phi, \alpha(x) < 0} \mathfrak{g}_\alpha \). Then \( \mathfrak{q} = \mathfrak{q}_x := \mathfrak{l}_x \oplus \mathfrak{u}_x \) is a \( \theta \)-stable parabolic subalgebra of \( \mathfrak{g}_0 \). Up to conjugation by \( K \), all \( \theta \)-stable parabolic subalgebras in \( \mathfrak{g} \) arise this way.

Consider the equivalence relation \( p \sim q \) if \( |x_p| = |x_q| \) on the set \( \{1 \leq j \leq n\} \). If \( x_p = 0 \) we denote the corresponding equivalence class \([p]\) by \( N_p \) and its cardinality \( \#N_p \) by \( N_p \).

If \( x_p \neq 0 \), we define two subsets \( I_p := \{ j \in [p] \mid x_j > 0 \}, J_p := \{ j \in [p] \mid x_j < 0 \} \) of \([p]\). Note that \( I_p, J_p \) are disjoint sets of consecutive integers, at least one of which is non-empty and \( I_p \cup J_p = [p] \). The sets \( N_p, I_p, J_p, 1 \leq p \leq n, x_p \neq 0 \), form a partition of the integers 1 up to \( n \). Denote by \( L(x) \) the set \( \{ \max I_p \mid \exists q \ x_p = -x_q \} \), and by \( R(x) \) the set \( \{ \min J_p \mid \exists q \ x_p = -x_q \} \). Clearly there is a bijection \( L(x) \rightarrow R(x) \) sending \( p \in L(x) \) to the unique \( p' \in R(x) \) such that \( x_p = -x_{p'} \).

Also, let \( s := \max N_p \) if \( N_p \) is non-empty. We shall now describe the root system of \( \mathfrak{t} \) with respect to \( \mathfrak{t} \). Let \( \Phi(x) := \{ \epsilon_i - \epsilon_j \mid x_i = x_j \} \cup \{ \pm(\epsilon_i + \epsilon_j) \mid x_i = -x_j \} \) and let \( \Delta_\pm = \Delta_+ \cup \Delta_- \cup \Delta^0 \) where \( \Delta_+ := \{ \epsilon_i - \epsilon_i+1 \mid x_i = x_{i+1} \geq 0 \} \cup \{ \epsilon_i - \epsilon_{i+1} \mid i, i+1 \in J_i, I_i \neq \emptyset \} \) and \( \Delta^- := \{ \epsilon_i + \epsilon_j \mid p \in L(x) \} \cup \{ \epsilon_{m-1} + \epsilon_m \mid x_{m-1} = x_m = 0 \} \); thus \( \epsilon_{m-1} + \epsilon_m \in \Delta^0 \) if and only if \( N_p \geq 2 \). Then \( \mathfrak{l}_x = \mathfrak{t} \oplus_{\alpha \in \Phi(x)} \mathfrak{g}_\alpha \). Also, \( \Delta_\pm \) is a set of simple roots for a positive root system \( \Phi(x) \subset \Phi(x) \). Write \( \Phi(x) = \{ \alpha \in \Phi(x) \mid \alpha = \pm(\epsilon_i \pm \epsilon_j), i, j \in [p] \} \) and \( \Delta_\pm = \Delta_+ \cup \Phi(x) \). Then \( (\Phi_\pm(x), \Delta_\pm) \) is an irreducible root system except when \( x_p = 0 \), and \( \#N_p = N_p = 2 \). Also \( \Phi(x) = \bigcup [p]\left[ \Phi_\pm(x) \right] \) with \( \Phi_\pm(x), \Phi_\mp(x) \) being orthogonal if \([p] \neq [q] \). Moreover \( [\mathfrak{l}_x, \mathfrak{l}_x] \) is a direct sum of simple ideals \( \mathfrak{s}_x \) whose root system (relative to \( \mathfrak{t}_x := \mathfrak{t} \cap \mathfrak{s}_x \)) is given by restriction of elements of \( \Phi_\pm(x) \) to \( \mathfrak{t}_x \). The Killing-Cartan type of the Lie algebra \( \mathfrak{s}_x \) is (a) \( A_{\#[p]-1} \) if \( x_p \neq 0 \), (b) type \( D_N \) when \( N_p \geq 2 \). (Of course, a genuine type \( D \) factor occurs only when \( N_p \geq 4 \).) The radical of \( \mathfrak{l}_x \) is isomorphic to \( \mathbb{C}^s \) where \( s \) equals the number of singletons among the sets \( N_p, 1 \leq p \leq n \).

Let \( R(q) = \dim_{\mathbb{C}}(p \cap \mathfrak{u}) \). As remarked in \S 3.1, the first non-vanishing \((g, K)\)-cohomology group \( H^j(g, K; A_{q, K}) \) occurs in dimension \( j = R(q) \). The following proposition gives a combinatorial formula for \( R(q) \).

**Proposition 3.1.** Let \( x = (x_1, \ldots, x_n) \) where \( x_i \geq x_j, 1 \leq i < j \leq n \). With the above notations, we have

\[
[I_x, L_\pm] \cong \oplus_{[p], x_p \neq 0}(\#[p], \mathbb{C}) \oplus \mathfrak{so}(2N_x, \mathbb{C})
\]

where the last summand occurs only when \( N_x \geq 2 \). Also,

\[
R(q_x) = \dim_{\mathbb{C}}(u \cap p) = \left( \frac{n}{2} \right) - \left( \frac{N_x}{2} \right) - \sum_{[p], x_p \neq 0} \#I_p \cdot \#J_p.
\]

**Proof.** In view of the above discussion, we need only establish the asserted formula for \( R(q_x) \). Note that \( R(q_x) = \#\Phi^+_x - \#\{ \alpha \in \Phi^+_x \mid \alpha, -\alpha \in \Phi_x \} \). Note that \( \alpha, -\alpha \in \Phi_x \) if and
only if $\alpha \in \Phi(1, t)$. So we need to count the number of $\alpha \in \Phi_n^+$ which are roots of $l$. Let $\alpha := \epsilon_i + \epsilon_j$ and let $\alpha, -\alpha \in \Phi_x, i < j$; thus $x_i = -x_j$.

Case 1. If $x_i = 0 = x_j$, the root space $g_\alpha$ is contained in the ideal corresponding to the summand $\mathfrak{so}(2N_x, \mathbb{C})$. There are exactly $\binom{n^+}{2}$ many such positive roots.

Case 2. Let $x_i \neq 0$. Then $0 = \alpha(x) = x_i + x_j = 0$ implies that $x_i > 0, x_j < 0$ and so there exists a unique $p \in L_x$ such that $p \sim i \sim j$. So $k \in I_{\{q\}}$ for $i \leq k \leq p$ and $l \in J_{\{q\}}$ for $p' \leq l \leq j$. We see that $\alpha = \sum_{i \leq k < j}(\epsilon_k - \epsilon_{k+1}) + \epsilon_p + \epsilon_l + \sum_{p' \leq l < j}(\epsilon_l - \epsilon_l)$ is an expression of $\alpha$ as a sum of simple roots of $(\Phi_x, \Delta_x)$. Hence $\alpha \in \Phi_x^+$ and it occurs in the simple ideal $\mathfrak{s}_p$ isomorphic to $\mathfrak{so}(\#I_{\{q\}}, \#J_{\{p\}}) = \mathfrak{so}(\#\{q\})$. It is clear that there are exactly $\#I_{\{q\}} \cdot \#J_{\{p\}}$ such positive roots in $\mathfrak{so}(\{q\})$. It follows that the set $\{\alpha \in \Phi_n^+ | \alpha \in \Phi(1, t)\}$ has cardinality equal to $\binom{n^+}{2} + \sum_{x_i \neq 0} \#I_p \cdot \#J_p$. This completes the proof. □

Recall that $R_\pm(q) = \text{dim}_\mathbb{C}(u \cap p_\pm)$. The following lemma provides a formula for $R_+(q_x)$. An analogous formula for $R_-(q)$ can be readily obtained. We omit the proof of this lemma, which is immediate from the definitions of $I_{\{q\}}, J_{\{p\}}$.

**Lemma 3.2.** Let $x = (x_1, \ldots, x_n)$ be a monotone decreasing sequence of real numbers. For $1 \leq p < n$, let $q(p)$ denote the largest integer in $[p, n]$ such that $x_{q(p)} > -x_p$. We have $R_+(q_x) = \sum_{x_p > 0, 1 \leq p < n} q(p) - p$.

Suppose that $x_p > 0$. If $J_{\{p\}} \neq \emptyset$, then $q(p) = \min(J_{\{p\}}) - 1$; if $J_{\{r\}} = \emptyset$ for all $r \geq p$, then $q(p) = n$.

**Lemma 3.3.** Let $x = (x_1, \ldots, x_n)$ be a monotone decreasing sequence of real numbers. Let $\iota(x) := (-x_n, \ldots, -x_1)$. Then $R_\pm(q_x) = R_\pm(q_{\iota(x)})$.

**Proof.** It is readily checked that the root space $g_{\iota(x)} \subset q_{\iota(x)}$ if and only if $g_{-\alpha} \subset q_x$ where $\iota : \Phi \rightarrow \Phi$ is the bijection induced by $\iota(\epsilon_p) = \epsilon_{n+1-p}$. In particular, when $i \neq j$, $g_{\epsilon_{n+1-j} + \epsilon_{n+1-i}} \subset q_x$ if and only if $-x_j - x_i > 0$ if and only if $x_i + x_j < 0$ if and only if $g_{-\epsilon_i - \epsilon_j} \subset q_x$. Since $g_{\iota(x)} \subset p_+$ if and only if $g_{-\alpha} \subset p_-$, it follows that $R_+(q_{\iota(x)}) = R_-(q_x)$. This completes the proof. □

In the examples below, we often write $R_+$ for $R_+(q)$, etc., when there is no possibility of confusion.

**Example 3.4.** (i) Let $s = s(k, l) = (1, \ldots, 1, 0, 0, \ldots, 0, -1, -1, \ldots, -1)$, where there are $k$ many $1$s and $l$ many $-1$s. Then $R_+(q_s) = (n-l-1) + (n-l-2) + \ldots + (n-l-k) = k(n-l-k) + \binom{k}{2}$; similarly $R_- = l(n-k-l) + \binom{l}{2}$. (ii) More generally, if $x_i = -x_n > 0$, then $R_+ \geq k(n-l-k) + \binom{k}{2}$ where $k = \#J_{[1]}, l = \#J_{[1]} \geq 1$.

Recall that $Y_q$ is the compact dual of the symmetric space $L/L \cap K$ where $L \subset G$ is the Lie subgroup whose Lie algebra is $\mathfrak{l}_0 = \mathfrak{l} \cap \mathfrak{g}_0 \subset \mathfrak{g}_0$. The roots of $\mathfrak{l}_0 \cap \mathfrak{t}_0$ with respect to $\mathfrak{t}_0$ are precisely the compact roots in $\Phi_x$ (restricted to $\mathfrak{t}_0$). The root space of $\mathfrak{t}_0$ corresponding to a compact root $\alpha$ will be denoted $\mathfrak{t}_{0, \pm, \alpha} := \mathfrak{t}_0 \cap (\mathfrak{t}_+ + \mathfrak{t}_- \alpha)$. Thus $\mathfrak{t}_0 \cap \mathfrak{l}_0 = \mathfrak{t}_0 \oplus (\oplus_{\pm, \alpha} \mathfrak{t}_{0, \pm, \alpha})$. 
where the sum is over pairs of roots $\pm \alpha$ in $\Phi_x \cap \Phi_\mathfrak{t}$. Therefore the set of compact roots of $[t_0, t_0]$ are the restrictions of elements of $\Phi_x \cap \Phi_\mathfrak{t}$ to $t_0 \cap [t_0, t_0]$. We shall make no distinction in the notation between a root of $(t_0, t_0)$ and its restriction to $t_0 \cap [t_0, t_0]$.

**Lemma 3.5.** The derived algebra $[l, l]$ is a direct sum of the following simple Lie algebras $\mathfrak{s}[p], 0$, the sum being over the set of all equivalence classes $[p] \subset \{1, 2, \ldots, n\}$:

1. $\mathfrak{su}(\# [p])$ if $I_{[p]}$ or $J_{[p]}$ is empty,
2. $\mathfrak{su}(\# I_{[p]}, \# J_{[p]})$ if both $I_{[p]}, J_{[p]}$ are non-empty,
3. $\mathfrak{so}(2N_x, \mathbb{H})$, when $N_x \geq 2$.

We omit the proof. In fact, it is possible to describe the simple factors of the reductive Lie group $L$ which implies the above lemma. Alternatively an easy computation yields the compact and non-compact roots of the real form $\mathfrak{s}[p], 0$ of $\mathfrak{s}[p]$, which allows one to determine the Lie algebra $\mathfrak{s}[p], 0$.

We are now ready to describe the symmetric space $Y_q$, the compact dual of the symmetric space $L/L \cap K \cong [L, L]/([L, L] \cap K)$. We have $Y_q = M/[L, L] \cap K$ where $M$ is the maximal compact subgroup of the complex semisimple Lie group with Lie algebra $[l, l]$. From Proposition 3.1 and the above lemma we see that $M = (\prod_{[p], x_p \neq 0} SU(\# [p])) \times SO(2N_x), [L, L] \cap K = (\prod_{[p], x_p \neq 0} S(U(\# I_{[p]}) \times U(\# J_{[p]}))) \times U(N_x)$. It is understood that $U(0)$ is trivial and that the last factor in $M$ and $[L, L] \cap K$ is present only if $N_x \geq 2$. Hence $Y_q = (\prod_{[p], x_p \neq 0} \mathbb{C}G_{\# [p], \# I_{[p]}}) \times SO(2N_x)/U(N_x)$, where $\mathbb{C}G_{k+l,t}$ denotes the complex Grassmann manifold of $l$-planes in $\mathbb{C}^{k+l}$.

In view of the fact that $H^j(\mathfrak{g}, K; A_q, K) \cong H^{j-R_q}(Y_q; \mathbb{C})$ we obtain an explicit formula for the Poincaré polynomial of $H^*(\mathfrak{g}, K; A_q, K)$.

Recall, from [2], that the Poincaré polynomial $P_t(X)$ (where cohomology with coefficients in a field of characteristic zero is understood) of a homogeneous manifold of the form $X = M/H$ where $M$ is a compact connected Lie group and $H$ is a connected subgroup having the same rank as $M$ is given by the formula of Hirsch. This is applicable to the complex Grassmann manifold $\mathbb{C}G_{k+l,t}$ and to $SO(2k)/U(k)$. We have

$$P_t(\mathbb{C}G_{k+l,t}) = (1 - v^{l+1}) \cdots (1 - v^{k+l})/((1 - v) \cdots (1 - v^k))$$

where $v := t^2$. (See [16].) Also, the Poincaré polynomial of $SO(2k)/U(k)$ is

$$P_t(SO(2k)/U(k)) = (1 + v)(1 + v^2) \cdots (1 + v^{k-1}).$$

Recall that $R = R_q = \dim_{\mathbb{C}}(u_x \cap p)$.

**Theorem 3.6.** We keep the notations as above and set $q := q_x$. The Poincaré polynomial $P_t(A_q)$ of $H^*(\mathfrak{g}, K; A_q, K)$ is given by

$$P_t(A_q) = t^{R_q} \left( \prod_{[p], x_p \neq 0} P_t(\mathbb{C}G_{\# [p], \# I_{[p]}}) \times P_t(SO(2N_x)/U(|N_x|)) \right).$$
3.3. **Representations $A_q$ with $R_{\pm}(q) \leq n-1$**. We consider the equivalence relation on the set of all $\theta$-stable parabolic subalgebras of $g_0 = \mathfrak{so}(n, \mathbb{H})$ defined as $q \sim q'$ if $A_q$ and $A_{q'}$ are unitarily equivalent, or equivalently, if the corresponding Harish-Chandra modules are isomorphic as $(g, K)$-modules. Clearly $q \sim q'$ if they belong to the same $K$-conjugacy class. Thus each conjugacy class is represented by a $\theta$-stable parabolic subalgebras of the form $q_x$ with $x = (x_1, \ldots, x_n) \in i\mathfrak{q}$ monotone decreasing. Two such elements $x, x'$ define equivalent representations $A_{q_x}, A_{q_x'}$ if and only if $u_x \cap p = u_{x'} \cap p$. (See Remark 3.9(i).)

We shall classify all $R_q$ with $R_{\pm}(q) \leq n-1$ assuming $n > 8$. In view of Lemma 3.3 we assume without loss of generality that $R_{-}(q_x) \leq R_{-}(q_x) \leq n-1$. We shall write $R_{\pm}$ to denote $R_{\pm}(q_x)$. We ignore the case $R_{+} = R_{-} = 0$ which corresponds to $q = g$. So $x_1 > 0$.

**Proposition 3.7.** Let $n > 8$. Let $x = (x_1, \ldots, x_n)$ be a monotone decreasing with $x_1 > 0$. Suppose that $0 \leq R_{-} \leq R_{+} \leq n-1$. Then the following statements hold:

1. If $R_{+} > 0$, then $R_{+} \geq n-2$.
2. If $R_{+} = n-1$, then $R_{-} \in \{0, n-2\}$. Moreover $q_x = q_s$ where $s = (1,0,\ldots,0,0)$ or $(2,0,\ldots,0,-1)$.
3. If $R_{+} = n-2$, then $R_{-} = n-2$ and $x_1 = -x_n, x_j = 0, 2 \leq j < n$.

**Proof.** The proof is broken up into three cases depending on whether $x_1 + x_n$ is positive, negative, or zero.

**Case 1:** Suppose that $x_1 + x_n > 0$. Then $R_{+} \geq n-1$ with equality if and only if $x_2 + x_3 \leq 0$. Our assumption that $R_{-} \leq n-1$ implies that we must have $R_{+} = n-1$ and $x_2 + x_{n-1} \leq 0$. If $x_2 + x_{n-1} < 0$, then $x_i + x_j < 0$, $2 \leq i < n-1 \leq j \leq n$. In particular $R_{-} \geq 2n-5$. Since $R_{-} \leq n-1$, this is impossible if $n > 4$. So $x_2 + x_{n-1} = 0$. This implies that $x_2 = -x_j, 3 \leq j \leq n-1$.

If $x_2 \neq 0$, then $x_j < 0 \forall j \geq 3$. Hence $x_i + x_j < 0$ for $3 \leq i < j \leq n$. Thus $R_{-} \geq (\binom{n-2}{2}) \geq 2n-7$. As $R_{-} \leq n-1$, we must have $n-1 \geq 2n-7$ contradicting $n > 6$. So $x_2 = 0$. This means that $x_j = 0 \forall j \leq n-1$. Now $x_1 + x_n > 0$ implies that $R_{-} = n-2$ if $x_n < 0$ and $R_{-} = 0$ if $x_n = 0$. Thus, we conclude that if $n > 6, x_1 + x_n > 0$, then $R_{+} = n-1, R_{-} \in \{0, n-2\}$. Moreover, $q_x = q_s$ where $s$ equals $(1,0,\ldots,0)$ or $(2,0,\ldots,0,-1)$.

**Case 2:** Suppose that $x_1 + x_n < 0$. Then $R_{-} \geq n-1$. Since $n-1 \geq R_{+} \geq R_{-}$ by hypothesis, it follows that $R_{+} = n-1 = R_{-}$. Now we proceed, using Lemma 3.3, as in the previous case and conclude that if $n > 6$, then there is no such $x$.

**Case 3:** Suppose that $x_1 + x_n = 0$. Set $a := \#I_{[1]}, b := \#J_{[1]}$. We have $a, b \geq 1$. There are three subcases to consider:

- **Subcase (i):** Let $a = 1$. We claim that $b = 1$ and that $R_{\pm} = n-2$ if $n \geq 7$.

  Clearly $R_{-} \geq n-2$ as $x_i + x_n < 0 \forall i > 1$. This implies that $R_{-} \geq n-2$. If $b > 1$, then $x_i + x_n < 0, x_j + x_{n-1} < 0 \forall 2 \leq i \leq n-1, 2 \leq j \leq n-2$ and so $R_{-} \geq 2n-5$ implies that $n-1 \geq 2n-5$ and so $n \leq 4$, contrary to our hypothesis. Hence we must
have $b = 1$. Let $R_+ = n - 1$ and $n \geq 7$. We will arrive at a contradiction. Since $x_1 + x_n = 0, a = b = 1$, $R_+ \leq n - 1$, we must have $x_2 + x_3 \geq 0, x_2 + x_j \leq 0$ for $4 \leq j \leq n$ and $x_1 + x_j \leq 0$ for $3 \leq i \leq n$. Since $R_- \leq n - 1$, we must have $x_{n-3} + x_{n-1} = 0$ which implies that $x_2 + x_4 = 0, x_2 + x_5 = 0, x_3 + x_4 = 0, x_3 + x_5 = 0, x_4 + x_5 = 0$. (For the last equality we have used $n - 1 > 5$.) Therefore $x_2 = x_3 = x_4 = x_5 = 0$. In particular $x_2 + x_3 = 0$ and so $R_+ = n - 2$, establishing our claim.

Subcase (ii): Let $b = 1$. If $a > 1$, again arguing as above we have $n - 1 \geq R_+ \geq 2n - 5$, which implies $n \leq 6$. So if $n \geq 7$ we are reduced to the previous subcase, namely $a = 1$, and so conclude that $R_+ = n - 2 = R_-$. 

Subcase (iii): Suppose that $a, b > 1$. Firstly, we have $a + b \leq n$. Since $a, b \geq 2$, we have $x_i + x_j > 0, 1 \leq i \leq 2, i < j \leq n - b$. Therefore $n - 1 \geq R_+ \geq n - 1 - b + n - 2 - b$. Hence $b \geq (n - 2)/2$. Write $n = 2m + 1$ or $n = 2m$, according as $n$ is odd or even, so that $b \geq m$ or $m - 1$, depending on the parity of $n$.

Suppose that $n = 2m + 1$. Then Example 3.4(ii) yields $n - 1 \geq R_+ \geq a(n - a - b) + \binom{m}{2}$. So $2m \geq (m^2 - m)/2$, which implies that $m \leq 5$ and so $n \leq 11$. Suppose that $n = 11$. If $a + b < 11$, then $a = b = m = 5$ and so we have the estimate $R_+ \geq a(n - a - b) + \binom{m}{2} = 15$. If $a + b = 11$, then one of them, say $a$ equals $6$. In this case $R_+ \geq \binom{6}{2} = 15$. Similarly, the possibility that $n = 9$ is also eliminated.

Suppose that $n = 2m$. Proceeding as in the case $n$ odd, we obtain that $a, b \geq m - 1$, $2m - 1 \geq (m - 1)(m - 2)/2$ which implies $m \leq 6$. When $n = 10$, it is readily verified that $9 \geq a(n - a - b) + \binom{a}{2}$ has no solution when $a, b \geq 4$. Similarly, when $n = 12$, the inequality $11 \geq a(n - a - b) + \binom{a}{2}$ has no solution when $a, b \geq 5$. This establishes our claim. 

**Remark 3.8.** When $4 \leq n \leq 8$, there are more possibilities for the $\theta$-stable parabolics with $R_+, R_- \leq n - 1$. The following list of sequences gives the complete list of the ‘exceptional’ $\theta$-stable parabolic subalgebras $q = q_x$ where $1 \leq R_- \leq R_+ \leq n - 1$, and the values of $R_+(q_x)$. In all these cases $R_+ = R_-$. 

$n = 8$: The only exceptional $q$ corresponds to $x = (1, 1, 1, 1, -1, -1, -1, -1)$ where $R_+ = 6$.

$n = 7$: The only exceptional $q$ corresponds to $x = (1, 1, 1, 0, -1, -1, -1, -1)$, where $R_+ = 6$.

$n = 6$: There two exceptions corresponding to $x = (1, 1, 1, -1, -1, -1)$ in which case $R_+ = 3$, and to $x = (2, 2, 1, -1, -2, -2)$ where $R_+ = 5$.

$n = 5$: There are two exceptional cases corresponding to $x = (1, 1, 0, -1, -1)$ with $R_+ = 3; x = (2, 1, 0, -1, -2)$ with $R_+ = 4$.

$n = 4$: The only exceptional case corresponds to $x = (1, 1, -1, -1)$ with $R_+ = 1$.

**Remark 3.9.** (i) Suppose that $\Phi(q)$ is the set of weights of $q = q_x$. Choose a positive system for $(\mathfrak{t}, \mathfrak{t})$ with respect to which the roots in $\Phi^+_x \cap \Phi_\mathfrak{t}$ and the weights of $\mathfrak{u} \cap \mathfrak{t}$ are positive. Such a positive system of $(\mathfrak{t}, \mathfrak{t})$ is said to be compatible with $\Phi(q)$. In what
follows, we talk of highest weight of a $K$-representation with respect to this positive system. Let $\mu = \mu(q)$ be the sum of all the $t$-weights of $u \cap p$. Then it is known that the irreducible $K$-representation $V_\mu$ with highest weight $\mu$ occurs in the $(g, K)$-module $A_{q,K}$ with multiplicity one. Any other $K$-type occurring in $A_{q,K}$ has highest weight equal to a sum $\mu + \sum a_\beta \beta$ where $a_\beta \geq 0$ and $\beta$ varies over the weights of $u \cap p$. Thus $V_\mu$ is the lowest $K$-type occurring in $A_q$. Moreover, if $q' := q_{x'}$ and the positive system of $(t, t)$ is compatible with $\Phi(q')$, then $A_q$ and $A_{q'}$ are unitarily equivalent if and only if $u \cap p = u' \cap p$. This is a particular case of a very general statement proved in [23, Proposition 4.5].

(ii) In view of the fact that the $(g, K)$-cohomology of $A_{q,K}$ depends only on $R(q)$ and the compact symmetric space $Y_q$ associated to $q$, it is possible that the Harish-Chandra modules associated to two non-isomorphic representations $A_q, A_{q'}$ can have isomorphic $(g, K)$-cohomology. For example, if $A_q$ is a discrete series representation we have $R(q) = (1/2) \dim(G/K)$ and $H^*(g, K; A_{q,K}) \cong \mathbb{C}$. In this case, however, distinct such discrete series have distinct pairs $(R^+(q), R^−(q))$. It is easy to construct examples of inequivalent representations whose $(g, K)$-cohomologies have isomorphic Hodge types for all pairs $(p, q)$. For example, this happens when $x = (2, 1, 1, −1, −1, −2)$, $x' = (2, 2, 1, −1, −2, −2)$.

(iii) Proposition 3.7, among many others, has been obtained by an entirely different approach, by Arghya Mondal [17] using a combinatorial model referred to as the decorated staircase diagram.

4. Proofs of Theorems 1.1 and 1.2

We are now ready to prove the main results of the paper. If $q$ is a $\theta$-stable parabolic subalgebra, we shall denote by $[q]$ the equivalence class of $q$ where $q \sim q'$ if the irreducible $G$-representations $(A_q, A_q)$ and $(A_{q'}, A_{q'})$ are unitarily equivalent. When $q \sim g$, we have $q = g$ and the representation $A_q$ is the trivial (one-dimensional) representation.

Proofs of Theorem 1.1 and 1.2: Let $u \in H^{p,p}(X_\Gamma; \mathbb{C})$. Recall the Matsushima isomorphism $H^{p,p}(X_\Gamma) = \bigoplus_{[q]} m(q, \Gamma) H^{p,p}(g, K; A_{q,K})$. The summand corresponding to $q = g$ is the image of the Matsushima homomorphism $H^*(X_u; \mathbb{C}) \to H^*(X_\Gamma; \mathbb{C})$. Write $u = \sum_{[q]} u_{[q]}$ where $u_{[q]} \in H^*(g, K; A_{q,K})$. Then $u_{[q]} = 0$ for all $q \neq g$ if and only if $u$ is in the image of the Matsushima homomorphism.

Any sign involution or an involution of even type $\sigma$ (constructed in §2.4) defines an analytic cycle $C_\Lambda(\sigma)$ in the Kähler manifold $X_\Lambda$ and hence a non-zero cohomology class $[C_\Lambda(\sigma)] \in H^{p,p}(X_\Lambda; \mathbb{C})$ where $p$ is the (complex) codimension of $C_\Lambda(\sigma) \subset X_\Lambda$. See §2.7. Since the cohomology class represented by a special cycle is not in the image of the Matsushima homomorphism, it does not belong to the component $H^*(g, K; \mathbb{C}) \subset H^*(X_\Lambda; \mathbb{C})$. It follows that $[C_\Lambda(\sigma)]_{[q]} \neq 0$ for some $q \neq g$. For any such $q$, we must have $R_+(q) = R_−(q) \leq p$.

Taking $C := C_\Lambda(\sigma)$ to be as in Example 2.11 we have $p = n−1$. It follows that $[C]_{[q_0]} \neq 0$ for some $q_0$ of type $(R_+, R_+)$ with $R_+ \leq n−1$. Hence $m(q_0, \Lambda) \neq 0$. Since $Y_{q_0}$ is Hermitian symmetric by Lemma 3.5, we see that $H^{j,j}(Y_{q_0}; \mathbb{C}) \neq 0$ for $0 \leq j \leq \dim \mathbb{C} Y_{q_0}$. 


Since $H^{r-r}(g, K; A_{q_0, K}) \cong H^{r-R_+ + r-R_+}(Y_{q_0}; \mathbb{C}) \neq 0$ for $R_+ \leq r \leq \dim_{\mathbb{C}} X_{\Lambda} - R_+ = \binom{n}{2} - R_+$, we conclude that there are cohomology classes of Hodge type $(r, r)$ in $X_{\Lambda} \setminus Y_{q_0}$ which are not in the image of the Matsushima map for all $r$ such that $R_+ \leq r \leq \binom{n}{2} - R_+$.

When $n$ is odd we observed in §2.7 that the dual cohomology class of a special cycle $[C(\sigma)]$ where $\sigma$ is of odd type is a non-zero cohomology class in the middle dimension $(n^2)$.

By Proposition 3.7, when $n > 8$ and $1 \leq r \leq n - 1$, there is exactly one class of $\theta$-stable parabolic subalgebra $[q_0]$ of type $(r, r)$ corresponding to $r = n - 2$. So $[C_{\Lambda}(\sigma)]_{q_0} \neq 0$. In follows that $m(q_0, \Lambda) \neq 0$ for any lattice $\Lambda$ as in the statement of Theorem 1.1. From what has been established already, we see that there are cohomology classes of Hodge type $(r, r)$ for $n - 2 \leq r \leq \binom{n}{2} - (n - 2)$ which are not in the image of the Matsushima homomorphism $H^*(X_{\Lambda}; \mathbb{C}) \to H^*(X_{\gamma}; \mathbb{C})$. This proves Theorem 1.1.

\[\square\]

Remark 4.1. Our method of proofs of the main results of this paper is applicable in a more general setting. Analogous results for the case $G = Sp(n, \mathbb{R})$ are under preparation.

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