High-frequency homogenization of nonstationary periodic equations

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ABSTRACT
In $L^2(\mathbb{R})$, we consider an elliptic differential operator $A_\varepsilon, \varepsilon > 0$, of the form $A_\varepsilon = -\frac{d}{dx} g(x/\varepsilon) \frac{d}{dx} + \varepsilon^{-2} V(x/\varepsilon)$ with periodic coefficients. For the nonstationary Schrödinger equation with the Hamiltonian $A_\varepsilon$ and for the hyperbolic equation with the operator $A_\varepsilon$, analogs of homogenization problems, related to the edges of the spectral bands of the operator $A_\varepsilon$, are studied (the so-called high-frequency homogenization). For the solutions of the Cauchy problems for these equations with special initial data, approximations in $L^2(\mathbb{R})$-norm for small $\varepsilon$ are obtained.

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1. Introduction
1.1. Periodic homogenization
The study of the wave propagation in periodic structures is of significant interest both for applications and from the theoretical point of view. Direct numerical simulations of such processes may be difficult. One of the approaches to study these problems is application of homogenization theory. The aim of homogenization is to describe the macroscopic properties of inhomogeneous media by taking into account the properties of the microscopic structure. An extensive literature is devoted to homogenization problems. First of all, we mention the books [1–3].

Let us discuss a typical problem of homogenization theory. Let $\Gamma$ be a lattice in $\mathbb{R}^d$, and let $\Omega$ be the cell of $\Gamma$. For any $\Gamma$-periodic function $F(x)$, we denote $F^\varepsilon(x) := F(\varepsilon^{-1} x)$, where $\varepsilon > 0$ is a (small) parameter. In $L^2(\mathbb{R}^d)$, consider a differential operator (DO) formally given by

$$\widehat{A}_\varepsilon = -\text{div} g^\varepsilon(x) \nabla,$$

where $g(x)$ is a Hermitian $\Gamma$-periodic $(d \times d)$-matrix-valued function, bounded and positive definite. Operator (1) models the simplest cases of microinhomogeneous media with $\varepsilon \Gamma$-periodic structure. Let $u_\varepsilon(x)$ be a (weak) solution of the elliptic equation

$$-\text{div} g^\varepsilon(x) \nabla u_\varepsilon(x) + u_\varepsilon(x) = f(x),$$

where $f \in L^2(\mathbb{R}^d)$. For $\varepsilon \to 0$, the solution $u_\varepsilon$ converges to the solution $u_0$ of the ‘homogenized’ equation:

$$-\text{div} g^0 \nabla u_0(x) + u_0(x) = f(x).$$
The operator $\tilde{A}_{\text{hom}} = -\text{div} g^0 \nabla$ is called the effective operator for $\tilde{A}_{\varepsilon}$. The matrix $g^0$ is determined by a well-known procedure (see, e.g. [1, Chapter 2, §3], [4, Chapter 3, §1]) that requires solving an auxiliary boundary value problem on the cell $\Omega$. Besides finding the effective coefficients, the following questions are of great interest. What is the type of convergence $u_{\varepsilon} \to u_0$? What is an estimate for $u_{\varepsilon} - u_0$?

There are various ways to prove the convergence. The classical method is to use asymptotic expansions in two scales (see, e.g. the books [1,2] and [3, Chapter 1, Section 4]). Another way to pass to the limit is using the two-scale convergence technique (see, e.g. [5]). In the present paper, we deal with error estimates in the high-frequency homogenization problem for the nonstationary Schrödinger equation and the hyperbolic equation. We use a spectral approach to homogenization. The spectral approach goes back to papers [6,7] (see also [3, Chapter 2, Section 6]), where the asymptotic behavior of the fundamental solution of elliptic and parabolic equations was studied. Subsequently, there was the paper [8], where the classical results for elliptic equations [2, Theorem 9.1] were proved using spectral method. Later, the results with correctors were obtained in [9] in the case of the whole space $\mathbb{R}^d$ and in [10] in the case of a bounded domain.

1.2. Operator error estimates in homogenization

M. Birman and T. Suslina [4] suggested the operator-theoretic approach (the variant of the spectral approach) to homogenization problems in $\mathbb{R}^d$, based on the scaling transformation, the Floquet–Bloch theory, and the analytic perturbation theory.

Let $u_{\varepsilon}$ be the solution of Equation (2), and let $u_0$ be the solution of Equation (3). In [4], it was proved that

$$\|u_{\varepsilon} - u_0\|_{L_2(\mathbb{R}^d)} \leq C\varepsilon \|f\|_{L_2(\mathbb{R}^d)},$$

(4)

Since $u_{\varepsilon} = (\tilde{A}_{\varepsilon} + I)^{-1}f$ and $u_0 = (\tilde{A}_{\text{hom}} + I)^{-1}f$, estimate (4) can be rewritten in operator terms:

$$\| (\tilde{A}_{\varepsilon} + I)^{-1} - (\tilde{A}_{\text{hom}} + I)^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C\varepsilon.$$

(5)

Parabolic equations were studied in [11,12]. In operator terms, the following approximation for the parabolic semigroup $e^{-t\tilde{A}_{\varepsilon}}$, $t > 0$, was obtained:

$$\|e^{-t\tilde{A}_{\varepsilon}} - e^{-t\tilde{A}_{\text{hom}}}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C\varepsilon (t + \varepsilon^2)^{-1/2}, \quad t > 0.$$

(6)

Estimates (5) and (6) are order-sharp; the constants $C$ are controlled explicitly in terms of the problem data. These estimates are called operator error estimates in homogenization. More accurate approximations for the resolvent and the exponential with correctors taken into account were found in [13–16].

A different approach to operator error estimates (the shift method) for the elliptic and parabolic problems was suggested by V. Zhikov and S. Pastukhova in the papers [17–19]. See also the survey [20].

The situation with homogenization of nonstationary Schrödinger-type equations and hyperbolic equations is quite different. The papers [21–28] were devoted to such problems. In operator terms, the behavior of the operator functions $e^{-it\tilde{A}_{\varepsilon}}$ and $\cos(t\tilde{A}_{\varepsilon}^{1/2})$, $\tilde{A}_{\varepsilon}^{1/2} \sin(t\tilde{A}_{\varepsilon}^{1/2})$ (where $t \in \mathbb{R}$) for small $\varepsilon$ was studied. For these operator functions, it is impossible to obtain approximations in the operator norm on $L_2(\mathbb{R}^d)$, and we are forced to consider the norm of operators acting from the Sobolev space $H^q(\mathbb{R}^d)$ (with a suitable $q$) to $L_2(\mathbb{R}^d)$. In [21], the following sharp-order estimates were proved:

$$\|e^{-it\tilde{A}_{\varepsilon}} - e^{-it\tilde{A}_{\text{hom}}}\|_{H^3(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(1 + |t|)\varepsilon,$$

(7)

$$\|\cos(t\tilde{A}_{\varepsilon}^{1/2}) - \cos(t(\tilde{A}_{\text{hom}})^{1/2})\|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(1 + |t|)\varepsilon.$$

(8)
In [22], the result for the operator $\hat{A}_e^{-1/2} \sin(t \hat{A}_e^{1/2})$ was obtained:

$$\| \hat{A}_e^{-1/2} \sin(t \hat{A}_e^{1/2}) - (\hat{A}^\text{hom})^{-1/2} \sin(t (\hat{A}^\text{hom})^{1/2})\|_{H^1(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(1 + |t|) \varepsilon.$$  \(9\)

Moreover, in [22], an approximation of the operator $\hat{A}_e^{-1/2} \sin(t \hat{A}_e^{1/2})$ for a fixed $t$ in the $(H^2 \to H^1)$-norm with error of order $O(\varepsilon)$ (with a corrector taken into account) was obtained. Next, in [23–26], it was shown that these results are sharp with respect to the norm type as well as with respect to the dependence on $t$ (for large $t$). On the other hand, it was shown that under some additional assumptions (e.g. if the matrix $g(x)$ has real entries) estimates (7)–(9) can be improved:

$$\|e^{-it\hat{A}_e} - e^{-it\hat{A}_e^\text{hom}}\|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(1 + |t|^{1/2}) \varepsilon,$$

$$\|\cos(t \hat{A}_e^{1/2}) - \cos(t (\hat{A}^\text{hom})^{1/2})\|_{H^{1/2}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(1 + |t|^{1/2}) \varepsilon,$$

$$\|\hat{A}_e^{-1/2} \sin(t \hat{A}_e^{1/2}) - (\hat{A}^\text{hom})^{-1/2} \sin(t (\hat{A}^\text{hom})^{1/2})\|_{H^{1/2}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(1 + |t|^{1/2}) \varepsilon.$$ \(10\)

More accurate approximations of the operator $e^{-it\hat{A}_e}$ with correctors taken into account were found in [27] (see also [28]).

Note that in [4,11–16,21–28], a much broader class of operators than (1) (including matrix DOs) was studied. In particular, operators of the form

$$\mathcal{A}_e = - \text{div}\hat{g}(x) \nabla + \varepsilon^{-2} V^e(x)$$ \(12\)

were considered. Here $\hat{g}(x)$ is a $\Gamma$-periodic positive definite and bounded $(d \times d)$-matrix-valued function with real entries, the potential $V(x)$ is a $\Gamma$-periodic real-valued function, $V \in L^p(\Omega)$ with a suitable $p$ (and it is assumed that $\inf \text{spec } A_1 = 0$). For operator (12), it is impossible to find an operator $\hat{A}^\text{hom}$ with constant coefficients such that the corresponding operator functions converge to the operator functions of $\hat{A}^\text{hom}$. However, some approximations can be found if we ‘border’ operator functions of $\hat{A}^\text{hom}$ by appropriate rapidly oscillating factors. In particular, an analog of (5) is as follows:

$$\|(A_e + I)^{-1} - [\omega^e] (\hat{A}^\text{hom} + I)^{-1} [\omega^e]\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C \varepsilon,$$

where $\omega(x)$ is a positive $\Gamma$-periodic solution of the equation

$$- \text{div}\hat{g}(x) \nabla \omega(x) + V(x) \omega(x) = 0$$

satisfying the normalization condition $\|\omega\|^2_{L_2(\Omega)} = |\Omega|$, and $\hat{A}^\text{hom}$ is the effective operator for operator (1) with the matrix $g(x) = \hat{g}(x)\omega^2(x)$.

Let us explain the method using the example of operator (1). The scaling transformation reduces investigation of the behavior of the operator $(\hat{A}_e + I)^{-1}$, $\varepsilon \to 0$, to studying the operator $(\hat{A} + \varepsilon^2 I)^{-1}$, where $\hat{A} = \hat{A}_1 = - \text{div} g(x) \nabla$. Next, by the Floquet–Bloch theory, the operator $\hat{A}$ expands in the direct integral of the operators $\hat{A}(k)$ acting in the space $L_2(\Omega)$. The operator $\hat{A}(k)$ is defined by the differential expression $- \text{div}_k g(x) \nabla_k$, where $\nabla_k = \nabla + ik$, $\text{div}_k = \text{div} + i(k, \cdot)$, with periodic boundary conditions. The spectrum of the operator $\hat{A}(k)$ is discrete. It turns out that the behavior of the resolvent $(\hat{A} + \varepsilon^2 I)^{-1}$ can be described in terms of the threshold characteristics of $\hat{A}$ at the edge of the spectrum, i.e. it is sufficient to know the spectral decomposition of $\hat{A}$ only near the lower edge of the spectrum. In particular, the effective matrix $g^0$ is a Hessian of the first band function $E_1(k)$ at the point $k = 0$.

Finally, we mention the recent paper [29], where the authors investigated the problem of convergence rates for a solution of the initial Dirichlet boundary value problem for a wave equation; analogs of estimates (10), (11) as well as results with the Dirichlet corrector were obtained.
1.3. High-frequency homogenization

As stated above, only a small neighborhood of the bottom of the spectrum (i.e. waves with low frequencies) contributes to homogenization. However, we can consider problems of wave propagation when the frequency is proportional to $\varepsilon^{-1}$ or $\varepsilon^{-2}$ (the high-frequency mode). In this case, even the leading order of the asymptotics oscillates rapidly. These problems were studied in [2, Chapter 4] using WKB-ansatz.

Traditional methods of homogenization theory, related to asymptotic expansions in two scales, were applied to these problems in [30,31]. We also cite the paper [32], where the application of the results of [30] to photonic crystals was considered. In [30], an asymptotic expansion for solutions of the equation

$$\text{div} g^\varepsilon(x) \nabla u^\varepsilon(x) + v^2 \rho^\varepsilon(x) u^\varepsilon(x) = 0,$$

which are perturbations of the standing waves, was obtained (the functions $g(x)$, $\rho(x)$ were supposed to be sufficiently smooth and $\Gamma$-periodic). In [31], a similar problem for travelling waves was considered. In these papers, a convergence of solutions has not been studied.

For a nonstationary Schrödinger equation results of this kind are called effective mass theorems (see, e.g. the course [33] and references therein). In the paper [34], homogenization of the Cauchy problem for a nonstationary Schrödinger equation with well-prepared initial data concentrating on a Bloch eigenfunction was studied using techniques of two-scale convergence and suitable oscillating test functions; a rigorous derivation of effective mass theorems was obtained (in terms of the strong two-scale convergence). In [35], the effective mass approximation and the $k \cdot p$ multi-band models, well known in solid-state physics, were discussed. Such homogenization asymptotics were investigated by using the envelope-function decomposition. These models were proved to be close (in the strong sense) to the exact dynamics. Moreover, the position density was proved to converge weakly to its effective mass approximation.

Finally, we also mention the papers [36,37], where asymptotics of Green’s function for different values of the spectral parameter has been studied.

Now, let us discuss error estimates for high-frequency homogenization. This topic has been studied in [38–41] in the one-dimensional case ($d = 1$) and in [42–44] in the case of arbitrary dimension $d$. It is well known that the spectrum of $A$ has a band structure and may have gaps. For the sake of simplicity, we consider the case where $d = 1$ and $\Gamma = \mathbb{Z}$; in this case we shall use the notation $A_\varepsilon$ for operator (12). Let $\sigma > 0$ be a (non-degenerate) left edge of a band with an odd number ($\geq 3$) in the spectrum of the operator $A = A_1$. Then for $A_\varepsilon$, this edge ‘moves’ to the point $\varepsilon^{-2}\sigma$ (to the high-frequency (high-energy) region). Instead of (2), we consider the equation

$$- \frac{d}{dx} g^\varepsilon(x) \frac{d}{dx} u^\varepsilon(x) - (\varepsilon^{-2}\sigma - x^2) u^\varepsilon(x) = f(x),$$

where $f \in L_2(\mathbb{R})$. It is supposed that $\varepsilon > 0$ is such that the point $\varepsilon^{-2}\sigma - x^2$ belongs to the gap in the spectrum of the operator $A_\varepsilon$. Similarly to (5), the question is reduced to studying the operator $(A_\varepsilon - (\varepsilon^{-2}\sigma - x^2) I)^{-1}$. In [38], the following result was proved:

$$\| (A_\varepsilon - (\varepsilon^{-2}\sigma - x^2) I)^{-1} - [\psi_{\sigma}^{\text{hom}}(A_{\sigma}^{\text{hom}} + x^2 I)^{-1} \psi_{\sigma}^{\text{hom}}] \|_{L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R})} \leq C \varepsilon.$$

(14)

Here $A_{\sigma}^{\text{hom}} = -b_\sigma \frac{d^2}{dx^2}$ is the corresponding effective operator, $b_\sigma > 0$ is the coefficient in the asymptotics of the band function $E(k)$ corresponding to the band for which $\sigma$ is the left edge; $E(k) \sim \sigma + b_\sigma k^2$, $k \sim 0$; and $\psi_{\sigma}$ is a real-valued periodic solution of the equation $A\psi_{\sigma} = \sigma \psi_{\sigma}$, normalized in $L_2(0, 1)$. Consequently, the possibility of homogenization for Equation (13) is a threshold effect near the edge of an internal gap.

Estimate (14) was obtained in [38] in the case where $V(x) = 0$. In [42], an analog of estimate (14) was proved for operators (12) in arbitrary dimension $d \geq 1$. More accurate approximations with correctors were obtained in [39,40,43].
Parabolic equations in the one-dimensional case were studied in [41]. It was proved that
\[ \|e^{-tA_\varepsilon}E_{A_\varepsilon}[e^{-2\sigma}, \infty) - e^{-t\sigma/\varepsilon^2}[\varphi_{\lambda}^\varepsilon]e^{-tA_{\lambda}^{\text{hom}}}[\varphi_{\lambda}^\varepsilon]\|_{L^2(\mathbb{R})} \rightarrow L^2(\mathbb{R}) \]
\[ \leq C e^{-t\sigma/\varepsilon^2}(t + \varepsilon^2)^{-1/2}, \quad t > 0, \]
and a more accurate approximation with a corrector was found. Here \( E_{A_\varepsilon}[e^{-2\sigma}, \infty) \) is the spectral projection of the operator \( A_\varepsilon \) corresponding to the interval \( [e^{-2\sigma}, \infty) \). The generalization of this result for the case of arbitrary dimension was obtained in [44].

### 1.4. Main results

In this paper, we study error estimates for high-frequency homogenization of nonstationary Schrödinger equations and hyperbolic equations in the one-dimensional case \((d = 1)\). Main results of this paper are formulated in Section 6. (Note that in Section 6, it is convenient for us to use slightly different notations.) In the introduction, we consider again only the case where \( \sigma \) is a (non-degenerate) left band edge with an odd number \( s \) in the spectrum of the operator \( A \).

Let \( f, g \in L^2(\mathbb{R}) \). Consider the Cauchy problems
\[
\begin{aligned}
\begin{cases}
\frac{\partial}{\partial t} u_\varepsilon(x, t) = (A_\varepsilon u_\varepsilon)(x, t), \\
u_\varepsilon(x, 0) = (\Upsilon_\varepsilon f)(x),
\end{cases}
\end{aligned}
\begin{aligned}
\begin{cases}
\frac{\partial^2}{\partial t^2} v_\varepsilon(x, t) = -(A_\varepsilon v_\varepsilon)(x, t) + \varepsilon^{-2}\sigma v_\varepsilon(x, t), \\
v_\varepsilon(x, 0) = (\Upsilon_\varepsilon f)(x), \\
(\partial_t v_\varepsilon)(x, 0) = (\Upsilon_\varepsilon g)(x),
\end{cases}
\end{aligned}
\tag{15}
\]
where
\[
(\Upsilon_\varepsilon f)(x) := (2\pi)^{-1/2} \int_{\mathbb{R}} (\Phi f)(k) \sum_{j=s}^{\infty} e^{ikx} \varphi_j(x/\varepsilon, \varepsilon k) \chi_{\Omega_{j-\sigma+1}}(\varepsilon k) \, dk.
\]
Here \( \{e^{ikx} \varphi_j(x, k)\}_{j=s}^{\infty} \) are the Bloch waves corresponding to the bands with the numbers \( j \geq s \);
\[
\Omega_j = (-j\pi, -(j-1)\pi) \cup ((j-1)\pi, j\pi), \quad j \in \mathbb{N},
\]
are the Brillouin zones. (To capture the high-frequency regime according to the edge \( \sigma \) the Bloch expansion starts not at the lowest but at the \( s \)th band.) The initial data of problems (15) are superpositions of the Bloch waves with the amplitudes, which are equal to the Fourier images \( (\Phi f)(k), (\Phi g)(k) \) of the functions \( f(x), g(x), \) and belong to the subspace \( E_{A_\varepsilon}[e^{-2\sigma}, \infty) L^2(\mathbb{R}) \). Moreover, in the case of the hyperbolic equation an additional spectral shift by \( \varepsilon^{-2}\sigma \) is included. Main results of the paper are the following estimates:
\[
\begin{aligned}
\|u_\varepsilon(\cdot, t) - e^{-it\sigma/\varepsilon^2}[\varphi_{\lambda}^\varepsilon]u_\varepsilon(\cdot, t)\|_{L^2(\mathbb{R})} \leq C(1 + |t|^{1/2})\varepsilon \|f\|_{H^2(\mathbb{R})}, \quad f \in H^2(\mathbb{R}),
\end{aligned}
\tag{17}
\]
\[
\begin{aligned}
\|v_\varepsilon(\cdot, t) - \varphi_{\lambda}^\varepsilon v_\varepsilon(\cdot, t)\|_{L^2(\mathbb{R})} \leq C(1 + |t|^{1/2})\varepsilon (\|f\|_{H^{3/2}(\mathbb{R})} + \|g\|_{H^{1/2}(\mathbb{R})}), \quad f \in H^{3/2}(\mathbb{R}), \quad g \in H^{1/2}(\mathbb{R}).
\tag{18}
\end{aligned}
\]
Here \( u_0 \) and \( v_0 \) are the solutions of the effective problems
\[
\begin{aligned}
\begin{cases}
\frac{\partial}{\partial t} u_0(x, t) = (A_{\sigma}^{\text{hom}} u_0)(x, t), \\
u_0(x, 0) = f(x),
\end{cases}
\end{aligned}
\begin{aligned}
\begin{cases}
\frac{\partial^2}{\partial t^2} v_0(x, t) = -(A_{\sigma}^{\text{hom}} v_0)(x, t), \\
v_0(x, 0) = f(x), \quad (\partial_t v_0)(x, 0) = g(x),
\end{cases}
\end{aligned}
\tag{15}
\]
and \( A_{\sigma}^{\text{hom}}, \varphi_{\lambda} \) are the same as in (14). Estimates (17) and (18) can be formulated in operator terms, but we postpone the discussion until Section 6.
The cases of other spectral edges are discussed as well. We also obtain some results in the case where \( \sigma \) is the point where two bands touch (see Remark 7.5). The results presented here are restricted to one dimension, which allows us to use the analyticity of the Bloch eigenvalues and eigenfunctions. In the multi-dimensional setting nonstationary Schrödinger equations were studied in [45] using a different approach.

Note that some estimate similar to (17) was proved in [35, Theorem 4.2], but (17) is better for large values of time.

1.5. Notation

Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be complex separable Hilbert spaces. If \( A : \mathcal{H}_1 \to \mathcal{H}_2 \) is a closed linear operator, then \( \text{Dom} A \) stands for its domain, the adjoint operator is denoted by \( A^* \). The symbol \( \| \cdot \|_{\mathcal{H}_1 \to \mathcal{H}_2} \) denotes the norm of a linear bounded operator from \( \mathcal{H}_1 \) to \( \mathcal{H}_2 \). Next, if \( A \) is a self-adjoint operator in some Hilbert space, then we use the notation \( \text{spec} A \) for the spectrum of \( A \), \( E_A(\delta) \) stands for the spectral projection of the operator \( A \) corresponding to the Borel set \( \delta \subset \mathbb{R} \).

The standard \( L^p \) classes of functions on an interval \((a, b) \subset \mathbb{R}\) are denoted by \( L^p(a, b) \), \( 1 \leq p \leq \infty \).

If \( f \) is a measurable function, then \([f] \) or \( f(x) \) denote the operator of multiplication by the function \( f \) in the space \( L^2 \). Next, \( H^s(\mathbb{R}) \) is the Sobolev class of order \( s \in \mathbb{R} \) and integrability index 2; and \( \tilde{H}^1(0, 1) \) is the subspace formed by the functions from \( H^1(0, 1) \) whose 1-periodic extensions belong to \( H^1_{\text{loc}}(\mathbb{R}) \).

If \( F(x) \) is an 1-periodic function, then we put \( F_\varepsilon(x) := F(\varepsilon^{-1}x) \). By \( \Phi_k \) we denote the Fourier transform on \( \mathbb{R} \) defined on the Schwartz class by the formula

\[
(\Phi v)(k) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ikx} v(x) \, dx, \quad v \in \mathcal{S}(\mathbb{R}),
\]

and extended by continuity up to the unitary mapping \( \Phi : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \). For the characteristic function of a set \( \delta \subset \mathbb{R} \), we use the notation \( \chi_\delta \). The range of values of an arbitrary function \( f \) is denoted by \( R(f) \).

2. The operator \( A \)

Let \( A \) be a self-adjoint operator in \( L^2(\mathbb{R}) \) generated by the differential expression

\[
A = -\frac{d}{dx} \tilde{g}(x) \frac{d}{dx} + V(x),
\]

(19)

where

\[
\begin{aligned}
\tilde{g} & \text{ is a real-valued measurable function,} \\
0 < \alpha_0 \leq \tilde{g}(x) \leq \alpha_1 < \infty, & \tilde{g}(x + 1) = \tilde{g}(x), \ x \in \mathbb{R},
\end{aligned}
\]

(20)

and \( V(x) \) is a real-valued potential such that \( V \in L^1(0, 1) \), \( V(x + 1) = V(x), \ x \in \mathbb{R} \). The precise definition of the operator \( A \) is given in terms of the semi-bounded closed quadratic form

\[
a[u, u] = \int_{\mathbb{R}} \left( \tilde{g}(x)|u'(x)|^2 + V(x)|u(x)|^2 \right) \, dx, \quad u \in H^1(\mathbb{R}).
\]

(21)

Adding an appropriate constant to \( V \), we assume that \( \inf \text{spec} A = 0 \). Under this assumption, the operator \( A \) admits a convenient factorization (see, e.g. [46], [4, Chapter 6, Section 1.1]). To describe
this factorization, we consider the equation

$$-(\tilde{g}(x)\omega'(x))' + V(x)\omega(x) = 0$$

(which is understood in the weak sense). There exists an 1-periodic solution \(\omega \in \tilde{H}^1(0, 1)\) of this equation defined up to a constant factor. This factor can be fixed so that \(\omega(x) > 0\) and

$$\|\omega\|_{L^2(0, 1)} = 1.$$  \hspace{1cm} (22)

It turns out that the solution \(\omega(x)\) is positive definite and bounded:

$$0 < \beta_0 \leq \omega(x) \leq \beta_1 < \infty.$$  \hspace{1cm} (23)

The function \(\omega\) is Lipschitz and is a multiplier in \(H^1(\mathbb{R})\) and in \(\tilde{H}^1(0, 1)\). The substitution \(u = \omega \phi\) transforms form (21) to the form

$$a[u, u] = \int_{\mathbb{R}^d} \omega^2(x)\tilde{g}(x)|\phi'(x)|^2 \, dx, \quad u = \omega \phi, \quad \phi \in H^1(\mathbb{R}).$$  \hspace{1cm} (24)

This means that operator (19) admits the following factorization:

$$A = -\omega(x)^{-1} \frac{d}{dx} g(x) \frac{d}{dx} \omega(x)^{-1}, \quad g = \omega^2 \tilde{g}.$$  \hspace{1cm} (25)

We take representation (25) of the operator \(A\) as the initial definition, i.e. we assume that \(A\) is the operator generated by form (24), where \(\tilde{g}\) and \(\omega\) are 1-periodic functions satisfying (20), (22), (23). We can return to representation (19) putting \(V(x) = (\tilde{g}(x)\omega'(x))/\omega(x)\). However, the potential \(V(x)\) may be highly singular.

3. Spectral decomposition of operator (25)

We need to describe the spectral properties of operator (25). For this, let us introduce the objects associated with the spectral resolution of operator (25). In \(L_2(0, 1)\), consider the family of quadratic forms

$$a(k)[u, u] = \int_0^1 g(x)|\phi' + ik\phi|^2 \, dx, \quad \phi = \omega^{-1} u \in \tilde{H}^1(0, 1), \quad k \in \mathbb{R}.$$  \hspace{1cm} (26)

The operator generated by form (26) is denoted by \(A(k)\). The parameter \(k \in \mathbb{R}\) is called the quasi-momentum. Let \(E_l(k)\), \(l \in \mathbb{N}\), be consecutive (counted with multiplicities) eigenvalues of the operator \(A(k)\), and let \(\varphi_l(\cdot, k)\), \(l \in \mathbb{N}\), be the corresponding normalized eigenfunctions. The functions \(E_l(k)\) are called band functions; they are \((2\pi)\)-periodic. Next, \(E_l(x + 1, k) = E_l(x, k)\), and the functions \(e^{ikx}\varphi_l(x, k)\) can be chosen to be \((2\pi)\)-periodic in \(k\).

Denote by \(\Omega = \Omega_1 = (-\pi, \pi]\) the central (first) Brillouin zone. Since the functions \(E_l(k)\) and \(e^{ikx}\varphi_l(x, k)\) are periodic, it suffices to consider \(k \in \Omega\) only. However, it will be sometimes convenient to assume that \(k \in \mathbb{R}\).

Consider the function \(E_s(k)\) for some \(s \in \mathbb{N}\). The following facts are well known (see, e.g. [47,XIII.16]).

1°. The function \(E_s\) is Lipschitz and even.

2°. The mapping \(k \mapsto E_s(k), k \in \Omega\), covers the band \(R(E_s)\) twice.

3°. The function \(E_s\) is piecewise real-analytic, its smoothness may be lost at the points where \(E_{s+1}(k) = E_s(k)\) or \(E_{s-1}(k) = E_s(k)\).

4°. The equality \(E_{s+1}(k) = E_s(k), k \in \Omega\), is possible only if \(k = \pi\) (\(s\) is an odd number).
The equality $E_{s-1}(k) = E_s(k)$, $k \in \tilde{\Omega}$, is possible only if $k = 0$ ($s$ is an odd number).

For $0 \leq k \leq \pi$ the function $E_s(k)$ is strictly monotone.

If the number $s$ is odd, then the function $E_s(k)$, $k \in \tilde{\Omega}$, has its minimum value at the point $k = 0$ and its maximum value at the point $k = \pi$.

If the number $s$ is even, then the function $E_s(k)$, $k \in \tilde{\Omega}$, has its maximum value at the point $k = 0$ and its minimum value at the point $k = \pi$.

We need estimates for the band functions $E_l(k)$, $l \in \mathbb{N}$. Consider the form

$$a^\circ(k)[\phi, \phi] = \int_0^1 |\phi' + ik\phi|^2 \, dx, \quad \phi \in H^1(0, 1), \quad k \in \mathbb{R}.$$ 

Let $E_l^*(k)$, $l \in \mathbb{N}$, be consecutive (counted with multiplicities) eigenvalues of the corresponding operators. They are $(2\pi)$-periodic and are equal to

$$E_0^*(k) = k^2, \quad k \in \tilde{\Omega},$$

$$E_{2j}^*(k) = (2\pi - |k|)^2, \quad E_{2j+1}^*(k) = (2\pi + |k|)^2, \quad k \in \tilde{\Omega}, \quad j \in \mathbb{N}.$$ 

From periodicity of the functions $\{E_l^*(k)\}_{l \in \mathbb{N}}$ it follows that

$$E_l^*(k)\chi_{\tilde{\Omega}_l}(k) = k^2\chi_{\tilde{\Omega}_l}(k), \quad k \in \mathbb{R}, \quad l \in \mathbb{N}. \tag{27}$$

Here $\tilde{\Omega}_l$ are the Brillouin zones (16). Next, using the Fourier series, one can show that

$$\alpha_0\beta_0^2a^\circ(k)[\phi, \phi] \leq a(k)[u, u] \leq \alpha_1\beta_1^2a^\circ(k)[\phi, \phi], \quad \phi = \omega^{-1}u \in \tilde{H}^1(0, 1), \quad k \in \mathbb{R},$$

whence

$$\alpha_0\beta_0^2\beta_1^{-2}a^\circ(k)[\phi, \phi] \leq a(k)[u, u] \leq \alpha_1\beta_0^2\beta_1^{-2}a^\circ(k)[\phi, \phi],$$

$$\phi = \omega^{-1}u \in \tilde{H}^1(0, 1), \quad k \in \mathbb{R}. \tag{28}$$

Using the variational principle, periodicity of the functions $\{E_l(k)\}_{l \in \mathbb{N}}$, and (27), we get

$$\alpha_0\beta_0^2\beta_1^{-2}k^2\chi_{\tilde{\Omega}_l}(k) \leq E_l(k)\chi_{\tilde{\Omega}_l}(k) \leq \alpha_1\beta_0^2\beta_1^{-2}k^2\chi_{\tilde{\Omega}_l}(k), \quad k \in \mathbb{R}, \quad l \in \mathbb{N}. \tag{29}$$

Now we introduce integral operators

$$\Psi_l: L_2(\mathbb{R}) \to L_2(S^1), \quad l \in \mathbb{N}, \tag{30}$$

defined on the Schwartz class by the following formula:

$$(\Psi_l v)(k) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ikx} \frac{1}{\varphi_l(x, k)} v(x) \, dx, \quad v \in \mathcal{S}(\mathbb{R}), \quad l \in \mathbb{N}. \tag{31}$$

Points of the circle $k \in S^1$ can be realized, for example, as points from $\tilde{\Omega}$. However, other realizations will be also convenient for us. Operators (30) extend by continuity up to partial isometries 'onto'. The operators $\Psi_l^*\Psi_l$ are orthogonal projections of $L_2(\mathbb{R})$ onto $E_\Delta(R(E_l))L_2(\mathbb{R})$, and $\sum_{l \in \mathbb{N}} \Psi_l^*\Psi_l = I$. The
following representation (the Floquet–Bloch decomposition) is true:

\[ A = \sum_{l=1}^{\infty} \Psi_l^\ast [E_l] \Psi_l. \] (31)

From (31) it follows that the spectrum of the operator \( A \) is the union of intervals (bands), which are the ranges of the functions \( E_l(k) \):

\[ \sigma(A) = \bigcup_{l=1}^{\infty} R(E_l) = [E_1(0), E_1(\pi)] \cup [E_2(\pi), E_2(0)] \cup [E_3(0), E_3(\pi)] \cup \ldots \]

In the one-dimensional case, the spectral bands cannot overlap. The intervals

\[ (\infty, E_1(0)), (E_1(\pi), E_2(\pi)), (E_2(0), E_3(0)) \ldots \]

are called spectral gaps. Note that some bands may touch, i.e. they may intersect at the boundary points. This means that some gaps may be empty.

### 4. A gap

Let us fix some \( s \in \mathbb{N} \). Here and throughout the paper, we agree to denote \( E(k) := E_s(k) \), \( \varphi(x, k) := \varphi_s(x, k) \), \( \Psi := \Psi_s \). Put

\[ \tilde{H}^1(0,1) := \{ f : \omega^{-1} f \in H^1(0,1) \}, \quad \| f \|_{\tilde{H}^1(0,1)} := \| \omega^{-1} f \|_{H^1(0,1)}. \]

We suppose that at least one of the following four conditions is fulfilled.

**Condition 4.1:** Let \( s \) be an odd number. There is the gap \( (E_{s-1}(0), E(0)) \neq \emptyset \) in the spectrum of the operator \( A \).

**Condition 4.2:** Let \( s \) be an even number. There is the gap \( (E(0), E_{s+1}(0)) \neq \emptyset \) in the spectrum of the operator \( A \).

Under the assumptions of Conditions 4.1 or 4.2, we have

\[ E(k) = \sigma_0 \pm b_0 k^2 + k^4 \gamma_0(k), \quad |k| \leq \pi, \quad b_0 > 0, \] (32)

\[ \varphi(x, k) = \varphi_0(x) + k \theta_0(x, k), \quad |k| < \pi. \] (33)

In equality (32), the sign ‘+’ corresponds to the case where Condition 4.1 is fulfilled, ‘−’ to the case where Condition 4.2 is fulfilled. The function \( \gamma_0(k) \) is continuous, and for \( |k| < \pi \) this function is real analytic; \( \sigma_0 := E(0), \varphi_0(x) := \varphi(x, 0); \varphi(\cdot, k) \) and \( \theta_0(\cdot, k) \) are real-analytic functions of \( k \in (-\pi, \pi) \) with values in \( \tilde{H}^1(0,1) \).

**Condition 4.3:** Let \( s \) be an odd number. There is the gap \( (E(\pi), E_{s+1}(\pi)) \neq \emptyset \) in the spectrum of the operator \( A \).

**Condition 4.4:** Let \( s \) be an even number. There is the gap \( (E_{s-1}(\pi), E(\pi)) \neq \emptyset \) in the spectrum of the operator \( A \).
Under the assumptions of Conditions 4.3 or 4.4, we have
\[ E(k) = \sigma_k \pm b_k (k - \pi)^2 + (k - \pi)^4 \gamma_k(k), \quad \sigma_k \pm b_k > 0, \quad 0 \leq k \leq 2\pi, \quad b_k > 0, \tag{34} \]
\[ \varphi(x, k) = \psi_k(x) + (k - \pi) \theta_k(x, k), \quad 0 < k < 2\pi. \tag{35} \]
In equality (34), the sign ‘+’ corresponds to the case where Condition 4.4 is fulfilled, ‘−’ to the case where Condition 4.3 is fulfilled. The function \( \gamma_k(k) \) is continuous, and for \( 0 < k < 2\pi \) this function is real-analytic; \( \varphi_k(x) := \varphi(x, \pi); \varphi(\cdot, k) \) and \( \theta_k(\cdot, k) \) are real-analytic functions of \( k \in (0, 2\pi) \) with values in \( \mathcal{H}_1(0, 1) \).

**Remark 4.1:** To include the case of the semi-infinite gap \((-\infty, 0)\) under the assumptions of Condition 4.1, we formally put \( E_0(0) = -\infty \).

**Remark 4.2:** The functions \( \varphi_0 \) and \( \varphi_k \) in (33), (35) belong to \( \mathcal{H}_1(0, 1) \) and therefore are bounded. We suppose that they are 1-periodically extended to \( \mathbb{R} \).

**Remark 4.3:** The coefficients \( b_0, b_k \) in (32) and (34) can be expressed in terms of solutions of some auxiliary boundary value problems on the interval of periodicity \((0, 1)\). See, e.g., [41, Remarks 2.2, 2.4].

### 5. Auxiliary statements

Put \( k_0 := 0 \) if Condition 4.1 or 4.2 is fulfilled, and put \( k_0 := \pi \) if Condition 4.3 or 4.4 is fulfilled. Here and throughout the paper we assume that
\[ \gamma_k(k_0) \neq 0. \tag{36} \]
Suppose that Condition 4.1 or 4.4 is satisfied. Consider the expression \((E(k) - \sigma_k(k))^1/2\). Using the Taylor formula for the function \( \sqrt{1 + x} \), we have
\[ (E(k) - \sigma_k(k))^1/2 = b_k^{1/2}|k - k_0| + |k - k_0|^3 \gamma_k(k), \tag{37} \]
where \( \gamma_k(k) = \frac{1}{2}b_k^{-1/2}\gamma_k(k)(1 + O((k - k_0)^2)), k \sim k_0 \). Similarly, if Condition 4.2 or 4.3 is satisfied, we have
\[ (\sigma_k - E(k))^1/2 = b_k^{1/2}|k - k_0| - |k - k_0|^3 \gamma_k(k). \]
Let us fix \( 0 < \kappa < \pi \) so that
\[
\frac{1}{2}|\gamma_k(k_0)| \leq |\gamma_k(k)| \leq \frac{3}{2}|\gamma_k(k_0)|, \tag{38}
\]
\[
\frac{1}{2}|\gamma_k(k_0)| \leq |\gamma_k(k)| \leq \frac{3}{2}|\gamma_k(k_0)|, \tag{39}
\]
\[
(k - k_0)^2|\gamma_k(k)| \leq \frac{1}{2}b_k^{1/2}, \tag{40}
\]
where \( |k - k_0| \leq \kappa \), and denote \( \mathcal{R} := \{k: |k| \leq \kappa\} \).

**Lemma 5.1:** Suppose that one of Conditions 4.1–4.4 is fulfilled. Let \( q \geq 0, r \geq -1 \). Then for \( 0 < \varepsilon \leq 1 \) we have
\[
\left\| (\Psi^* - [\varphi_k] \Phi^*) \left[ \frac{\varepsilon^q}{(|k - k_0|^2 + \varepsilon^2)^{q/2}} \chi_{\mathcal{R}}(k - k_0) \right] \right\|_{L^2(\mathbb{R})} \leq C_{\min(1,q)} \varepsilon, \tag{41}
\]
\[
\left\| (\Psi^* - [\varphi_k] \Phi^*) \left[ \frac{\varepsilon^r}{(|k - k_0|^2 + \varepsilon^2)^{r/2}} |k - k_0|^{-1} \chi_{\mathcal{R}}(k - k_0) \right] \right\|_{L^2(\mathbb{R})} \leq C_{\min(0,r)} \varepsilon. \tag{42}
\]
The constant \( C \) from (41) depends on \( q, \kappa, \|\theta_k\|_{M_k(k_0)} \); the constant \( C \) from (42) depends on \( r, \kappa, \|\theta_k\|_{M_k(k_0)} \); the quantity \( \|\theta_k\|_{M_k(k_0)} \) is defined below in (44).
Proof: Let us prove (41). Consider the adjoint operator
\[
[e^q (|k - k_0|^2 + \varepsilon^2)^{-q/2} \chi_R(k - k_0)] (\Psi - \Phi[\varphi_{k_0}]).
\]
By (33), (35), we conclude that the operator \([\chi_R(k - k_0)] (\Psi - \Phi[\varphi_{k_0}])\) is the integral operator with the kernel
\[
(2\pi)^{-1/2} \chi_R(k - k_0) e^{-i x k} (\varphi(x, k) - \varphi_{k_0}(x)) = (2\pi)^{-1/2} \chi_R(k - k_0) e^{-i x k} (k - k_0) \theta_{k_0}(x, k).
\]
Kernel (43) differs from the kernel of the Fourier operator by the factor \((k - k_0) \chi_R(k - k_0) \theta_{k_0}(x, k)\).
The function \(\theta_{k_0}(x, k)\) is a multiplier on the set of kernels of bounded integral operators that take \(L_2(\mathbb{R})\) to \(L_2(0 - \kappa, 0 + \kappa)\) (see [38, §2, Sec. 2]), and its norm in the class of multipliers is estimated by
\[
\|\theta_{k_0}\|_{M_{\kappa}(0)} := \text{ess-sup}_{x \in \mathbb{R}} \|\theta_{k_0}(x, \cdot)\|_{C^1[0 - \kappa, \kappa]} < \infty.
\]
Together with the inequalities
\[
e^q |k - k_0| \left((k - k_0)^2 + \varepsilon^2\right)^{-q/2} \chi_R(k - k_0) \leq \kappa^{1-q} e^q, \quad \text{for } 0 \leq q \leq 1,
\]
\[
e^q |k - k_0| \left((k - k_0)^2 + \varepsilon^2\right)^{-q/2} \chi_R(k - k_0) \leq \varepsilon, \quad \text{for } q > 1,
\]
this yields (41).
Estimate (42) is proved in a similar way taking into account the inequalities
\[
|k - k_0| \cdot |k - k_0|^{-1} e^r \left((k - k_0)^2 + \varepsilon^2\right)^{-r/2} \chi_R(k - k_0) \leq \frac{e^r}{(\kappa^2 + 1)^{r/2}}, \quad \text{for } -1 \leq r < 0,
\]
\[
|k - k_0| \cdot |k - k_0|^{-1} e^r \left((k - k_0)^2 + \varepsilon^2\right)^{-r/2} \chi_R(k - k_0) \leq 1, \quad \text{for } r \geq 0.
\]

Lemma 5.2: Suppose that one of Conditions 4.1–4.4 is fulfilled, and condition (36) is also fulfilled. For \(t \neq 0\) we have
\[
\left\| \left[ \frac{e^q ((k - k_0)^2 + \varepsilon^2)^{q/2} \sin \left( \frac{1}{2} t e^{-2} (k - k_0)^4 \gamma_{k_0}(k) \right) \chi_R(k - k_0) \right] \right\|_{L_2(\mathbb{R}) \to L_2(\mathbb{R})}.
\]
\[
\leq \left\{ \begin{array}{ll}
\frac{e^{q/2} |t|^{q/4}}{\left|\gamma_{k_0}(k)\right|^{-1/2} + \varepsilon |t|^{1/2} y^{q/2}}, & \text{for } 0 \leq q \leq 4 \text{ (and } 0 < \varepsilon |t|^{-1/2} \leq \varepsilon), \\
\left|\gamma_{k_0}(k)\right| \varepsilon^2 |t|, & \text{for } q > 4 \quad \left(\text{and } 0 < \varepsilon \leq \frac{\kappa \sqrt{q - 4}}{2}\right),
\end{array} \right.
\]
(45)
\[
\left\| \left[ \frac{e^q ((k - k_0)^2 + \varepsilon^2)^{q/2} \sin \left( \frac{1}{2} t e^{-1} (k - k_0)^3 \gamma_{k_0}(k) \right) \chi_R(k - k_0) \right] \right\|_{L_2(\mathbb{R}) \to L_2(\mathbb{R})}.
\]
\[
\leq \left\{ \begin{array}{ll}
\frac{e^{q/3} |t|^{q/3}}{\left|\gamma_{k_0}(k)\right|^{-2/3} + \varepsilon^{4/3} |t|^{2/3} y^{q/2}}, & \text{for } 0 \leq q \leq 3 \text{ (and } 0 < \varepsilon |t|^{-1} \leq \varepsilon), \\
\left|\gamma_{k_0}(k)\right| \varepsilon^2 |t|, & \text{for } q > 3 \quad \left(\text{and } 0 < \varepsilon \leq \frac{\kappa \sqrt{q - 3}}{\sqrt{3}}\right),
\end{array} \right.
\]
Consider the sets $k$ and $r$.

Let us estimate the norm in the right-hand side of (48). Introduce the functions $K_0$, and $\tilde{\gamma}_0(k)$ periodically extend them to $\mathbb{R}$.

Here the notation $X \asymp Y$ means that $c_1 Y \leq X \leq c_2 Y$ with some constants $c_1$ and $c_2$, which may depend only on $q$ and $r$; $\varepsilon := (2\pi)^{-1/2}|\gamma_0(k_0)|^{1/2}k^2$, $\tilde{\varepsilon} := (2\pi)^{-1}|\gamma_0(k_0)|k^3$.

**Proof**: Let us prove estimate (45). Estimates (46), (47) can be proved in a similar way. We have

$$
\left\| \frac{\varepsilon q^{q/2}}{(k - k_0)^2 + \varepsilon^2} \sin \left( \frac{1}{2} t \varepsilon^{-2} (k - k_0)^4 \gamma_0(k) \right) \chi(R(k - k_0)) \right\|_{L_2(\mathbb{R}) \to L_2(\mathbb{R})} = \left\| \frac{\varepsilon q^{q/2}}{(k - k_0)^2 + \varepsilon^2} \sin \left( \frac{1}{2} t \varepsilon^{-2} (k - k_0)^4 \gamma_0(k) \right) \chi(R(k - k_0)) \right\|_{L_\infty}.
$$

(48)

Let us estimate the norm in the right-hand side of (48). Introduce the functions

$$
h_1(y) = \begin{cases} \frac{2}{\pi} y, & \text{for } y \in [0, \pi/2], \\ \frac{2}{\pi} (\pi - y), & \text{for } y \in (\pi/2, \pi], \\ \end{cases} \quad h_2(y) = \begin{cases} y, & \text{for } y \in [0, \pi/2], \\ \pi - y, & \text{for } y \in (\pi/2, \pi], \\ \end{cases}
$$

and $\pi$-periodically extend them to $\mathbb{R}$. We have

$$
h_1(y) \leq |\sin y| \leq h_2(y).
$$

We shall estimate maxima of the functions

$$
\varepsilon q(k - k_0)^2 + \varepsilon^2 q/2 h_j \left( \frac{1}{2} t \varepsilon^{-2} (k - k_0)^4 \gamma_0(k) \right) \chi(R(k - k_0)), \quad j = 1, 2,
$$

(49)

from below (for $j = 1$) and from above (for $j = 2$). Since $\varepsilon q(k^2 + \varepsilon^2) - q/2$ is monotone decreasing in $k > 0$, and $h_1, h_2$ are periodic, maxima of functions (49) are reached on the set

$$
K = \left\{ k: \frac{1}{2} |t| \varepsilon^{-2} (k - k_0)^4 |\gamma_0(k)| \leq \frac{\pi}{2} \right\}.
$$

Consider the sets

$$
K'_1 = \left\{ k: \frac{3}{4} |t| \varepsilon^{-2} \cdot |\gamma_0(k) (k - k_0)^4 \leq \frac{\pi}{2} \right\},
$$

(50)

$$
K'_2 = \left\{ k: \frac{1}{4} |t| \varepsilon^{-2} \cdot |\gamma_0(k) (k - k_0)^4 \leq \frac{\pi}{2} \right\}.
$$

(51)

From (38) it follows that $K'_1 \subset K \subset K'_2$. Let $\tilde{k} := \max K$ and $\tilde{k}'_{1,2} := \max K'_1,2$. We have

$$
\tilde{k}'_1 = k_0 + (2\pi / 3)^{1/4} |\gamma_0(k_0)|^{-1/4} \varepsilon^{1/2} |t|^{-1/4},
$$

$$
\tilde{k}'_2 = k_0 + (2\pi)^{1/4} |\gamma_0(k_0)|^{-1/4} \varepsilon^{1/2} |t|^{-1/4}
$$

(maxima are reached when we have equalities in (50), (51)), and

$$
|\tilde{k}'_1 - k_0| \leq |\tilde{k} - k_0| \leq |\tilde{k}'_2 - k_0|.
$$

(52)
Consider function (49) with \( j = 2 \). The right inequality in (38) implies

\[
\varepsilon^q ((k - k_0)^2 + \varepsilon^2)^{-q/2} h_2 \left( \frac{1}{2} t \varepsilon^{-2} (k - k_0)^4 \gamma(k) \right) \chi_R(k - k_0)
\]

\[
= \varepsilon^q ((k - k_0)^2 + \varepsilon^2)^{-q/2} \cdot \frac{1}{2} |t| \varepsilon^{-2} (k - k_0)^4 |\gamma(k)| \chi_R(k - k_0)
\]

\[
\leq \varepsilon^q ((k - k_0)^2 + \varepsilon^2)^{-q/2} \cdot \frac{3}{4} |t| \varepsilon^{-2} |\gamma(k)| |(k - k_0)^4 \chi_R(k - k_0)|.
\]

The function \( k^4 (k^2 + \varepsilon^2)^{-q/2} \) is even and monotonically increases in \( k \geq 0 \) for \( 0 \leq q \leq 4 \). We substitute \( k = \tilde{k} \) in the right-hand side of the inequality obtained above and estimate it using the right inequality in (52):

\[
\varepsilon^q ((\tilde{k} - k_0)^2 + \varepsilon^2)^{-q/2} \leq \frac{3\pi}{2} \cdot \frac{\varepsilon^{q/2} |t|^{q/4}}{(\varepsilon^{-2} |\gamma(k_0)|)^{-1/2} + |t|^{1/2})^{q/2}}, \quad 0 < \varepsilon |t|^{-1/2} \leq \epsilon, \quad 0 \leq q \leq 4.
\]

Here the inclusion \( K' \subset [k_0 - \kappa, k_0 + \kappa] \) (which follows from \( 0 < \varepsilon |t|^{-1/2} \leq \epsilon \)) is taken into account. For \( q > 4 \) the function \( k^4 (k^2 + \varepsilon^2)^{-q/2}, k \geq 0 \), has the maximum \( 16q^{-q/2}(q - 4)^{q/2-2}e^{-q} \), which is reached at the point \( k_0 = 2(q - 4)^{-1/2}\epsilon \). Then

\[
\varepsilon^q ((k - k_0)^2 + \varepsilon^2)^{-q/2} \leq \frac{3\pi}{2} \cdot \frac{\varepsilon^{q/2} |t|^{q/4}}{(\varepsilon^{-2} |\gamma(k_0)|)^{-1/2} + |t|^{1/2})^{q/2}}, \quad 0 < \varepsilon |t|^{-1/2} \leq \epsilon, \quad 0 \leq q \leq 4.
\]

As a result, we have obtained the upper estimate in (45).

Now we consider function (49) with \( j = 1 \). From the left inequality in (38) it follows that

\[
\varepsilon^q ((k - k_0)^2 + \varepsilon^2)^{-q/2} h_1 \left( \frac{1}{2} t \varepsilon^{-2} (k - k_0)^4 \gamma(k) \right) \chi_R(k - k_0)
\]

\[
\geq \varepsilon^q ((k - k_0)^2 + \varepsilon^2)^{-q/2} \cdot \frac{1}{2\pi} |t| \varepsilon^{-2} |\gamma(k)| |(k - k_0)^4 \chi_R(k - k_0)|.
\]

For \( 0 \leq q \leq 4 \) we substitute \( k = \tilde{k} \) in the right-hand side of this inequality and estimate it using the left inequality in (52):

\[
\varepsilon^q ((\tilde{k} - k_0)^2 + \varepsilon^2)^{-q/2} \cdot \frac{1}{2\pi} |t| \varepsilon^{-2} |\gamma(k)| |(\tilde{k} - k_0)^4 | \chi_R(\tilde{k} - k_0)|
\]

\[
\geq \frac{1}{3} \cdot \frac{\varepsilon^{q/2} |t|^{q/4}}{(\varepsilon^{-2} |\gamma(k_0)|)^{-1/2} + |t|^{1/2})^{q/2}}, \quad 0 < \varepsilon |t|^{-1/2} \leq \epsilon, \quad 0 \leq q \leq 4,
\]

and for \( q > 4 \) we have

\[
\max_k \varepsilon^q ((k - k_0)^2 + \varepsilon^2)^{-q/2} h_1 \left( \frac{1}{2} t \varepsilon^{-2} (k - k_0)^4 \gamma(k) \right) \chi_R(k - k_0)
\]

\[
\geq \frac{8}{\pi} q^{-q/2}(q - 4)^{q/2-2} |\gamma(k)| |e^2 |t|, \quad q > 4.
\]

Thereby, the lower estimate in (45) is proved.
6. Main results of this paper

In this section, we formulate the main results of this paper. We give detailed proofs in the cases of Conditions 4.1 and 4.3, and we only formulate the results in the cases of Conditions 4.2 and 4.4.

6.1. The case where Condition 4.1 is fulfilled

Let $\varepsilon > 0$ be a small parameter. In $L_2(\mathbb{R})$, we consider the operator formally defined by the differential expression

$$A_\varepsilon = -(\omega^\varepsilon)^{-1} \frac{d}{dx} g^\varepsilon \frac{d}{dx} (\omega^\varepsilon)^{-1}, \quad g = \tilde{g} \omega^2.$$  \hspace{1cm} (53)

Here $\tilde{g}$ and $\omega$ are 1-periodic functions satisfying conditions (20), (22), and (23). The precise definition of the operator $A_\varepsilon$ is given in terms of the corresponding quadratic form (cf. (24)). Operators (25) and (53) satisfy the following relation:

$$A_\varepsilon = \varepsilon^{-2} T_\varepsilon^* A T_\varepsilon,$$

where $T_\varepsilon$ is the operator of scaling transformation: $(T_\varepsilon u)(x) = \varepsilon^{1/2} u(\varepsilon x)$.

Now it is convenient to realize points of the circle in (29) in the following way:

$$\Psi_l: L_2(\mathbb{R}) \to L_2(\tilde{\Omega}_{l-s+1}), \quad l \geq s.$$  \hspace{1cm} (54)

Recall that $\tilde{\Omega}_j$, $j \in \mathbb{N}$, were defined in (16). Let $f \in L_2(\mathbb{R})$. We study the behavior of the solution $u_\varepsilon(x,t)$, $\varepsilon \to 0$, of the following Cauchy problem for the nonstationary Schrödinger equation

$$\left\{ \begin{array}{l}
  i \frac{\partial}{\partial t} u_\varepsilon(x,t) = (A_\varepsilon u_\varepsilon)(x,t), \\
  u_\varepsilon(x,0) = (\Upsilon_\varepsilon^{(+)} f)(x),
\end{array} \right.$$  \hspace{1cm} (55)

where

$$\Upsilon_\varepsilon^{(+)} f \in \mathcal{E}_{A_\varepsilon} \{ e^{-2} \sigma_0, \infty \} L_2(\mathbb{R}).$$

In $L_2(\mathbb{R})$, we consider the operator $A_0^{\text{hom}} = -b_0 \frac{d^2}{dx^2}$, $\text{Dom} A_0^{\text{hom}} = H^2(\mathbb{R})$, which is called the effective operator at the left edge $\sigma_0 = E(0)$ of the band $R(\tilde{E})$. Let $u_0(x,t)$ be the solution of the corresponding 'homogenized' problem

$$\left\{ \begin{array}{l}
  i \frac{\partial}{\partial t} u_0(x,t) = (A_0^{\text{hom}} u_0)(x,t), \\
  u_0(x,0) = f(x).
\end{array} \right.$$  \hspace{1cm} (56)

The solutions of problems (54) and (55) can be represented as follows:

$$u_\varepsilon = e^{-itA_\varepsilon} \Upsilon_\varepsilon^{(+)} f, \quad u_0 = e^{-itA_0^{\text{hom}}} f,$$

where $\Upsilon_\varepsilon^{(+)} := \sum_{j=s}^{\infty} T_\varepsilon^{s} \Psi_l^{*} R_{l+1-j} \Phi T_\varepsilon$, and $R_l$ is the operator of restriction to $\tilde{\Omega}_l$.

**Theorem 6.1:** Suppose that Conditions 4.1 and (36) are fulfilled. Let $u_\varepsilon$ be the solution of problem (54), and let $u_0$ be the solution of problem (55). Let $t \neq 0$, $f \in H^q(\mathbb{R})$, $0 \leq q \leq 2$. We have

$$\|u_\varepsilon(t) - e^{-it} \varphi_0 \|_{L_2(\mathbb{R})} \leq C(1 + |t|^{q/4}) e^{q/2} \|\varphi\|_{H^q(\mathbb{R})},$$

$$0 < \varepsilon \leq 1, \quad 0 < \varepsilon |t|^{-1/2} \leq \varepsilon,$$

with the constant $C = C(q, \kappa, \|\varphi_0\|_{L_\infty}, |\gamma_0(0)|, \|\theta_0\|_{M_\kappa(0)})$.  \hspace{1cm} (57)
**Proof:** By (56), estimate (57) can be reformulated in the operator terms:

\[
\left\| e^{-itA_e} \gamma_e^{(+)\epsilon} - e^{-ite^{-2}\sigma_0} [\varphi_0^e] e^{-itA_0^{\text{hom}}} \right\|_{H^q(\mathbb{R}) \to L_2(\mathbb{R})} \leq C (1 + |t|^{q/4}) \epsilon^{q/2},
\]

(58)

where \( t \neq 0, 0 < \epsilon \leq 1, 0 < \epsilon |t|^{-1/2} \leq \epsilon, 0 \leq q \leq 2 \). Thus our aim is to prove (58). Since the operator \((-d_x^2 + I)^{q/2}\) is an isometric isomorphism of the Sobolev space \(H^q(\mathbb{R})\) onto \(L_2(\mathbb{R})\),

(59)

we have

\[
\left\| e^{-itA_e} \gamma_e^{(+)\epsilon} - e^{-ite^{-2}\sigma_0} [\varphi_0^e] e^{-itA_0^{\text{hom}}} \right\|_{H^q(\mathbb{R}) \to L_2(\mathbb{R})} = \left\| \left( e^{-itA_e} \gamma_e^{(+)\epsilon} - e^{-ite^{-2}\sigma_0} [\varphi_0^e] e^{-itA_0^{\text{hom}}} \right) \left( -d_x^2 + I \right)^{q/2} \right\|_{L_2(\mathbb{R}) \to L_2(\mathbb{R})},
\]

From the unitarity of the scaling transformation it directly follows that

\[
\left\| \left( e^{-itA_e} \gamma_e^{(+)\epsilon} - e^{-ite^{-2}\sigma_0} [\varphi_0^e] e^{-itA_0^{\text{hom}}} \right) \left( -d_x^2 + I \right)^{q/2} \right\|_{L_2(\mathbb{R}) \to L_2(\mathbb{R})} = \left\| \left( e^{-ite^{-2}A} \gamma^{(+)\epsilon} - e^{-ite^{-2}\sigma_0} [\varphi_0] e^{-ite^{-2}A_0^{\text{hom}}} \right) \epsilon q \left( -d_x^2 + \epsilon^2 I \right)^{q/2} \right\|_{L_2(\mathbb{R}) \to L_2(\mathbb{R})},
\]

(60)

where \( \gamma^{(+)\epsilon} := \sum_{j=1}^{\infty} \Psi_j^* R_{j-1+i} \Phi \). Introduce the projection \( F_{\mathcal{R}} = \Phi^* [\chi_{\mathcal{R}}] \Phi \). Obviously,

\[
\| \epsilon q \left( -d_x^2 + \epsilon^2 I \right)^{-q/2} (I - F_{\mathcal{R}}) \|_{L_2(\mathbb{R}) \to L_2(\mathbb{R})} \leq \kappa^{-q} \epsilon q,
\]

(61)

whence

\[
\left\| \left( e^{-ite^{-2}A} \gamma^{(+)\epsilon} - e^{-ite^{-2}\sigma_0} [\varphi_0] e^{-ite^{-2}A_0^{\text{hom}}} \right) \epsilon q \left( -d_x^2 + \epsilon^2 I \right)^{-q/2} (I - F_{\mathcal{R}}) \right\|_{L_2(\mathbb{R}) \to L_2(\mathbb{R})} \leq (1 + \| \varphi_0 \|_{\infty}) \kappa^{-q} \epsilon q.
\]

(62)

Consider the operator

\[
\left( e^{-ite^{-2}A} \gamma^{(+)\epsilon} - e^{-ite^{-2}\sigma_0} [\varphi_0] e^{-ite^{-2}A_0^{\text{hom}}} \right) \epsilon q \left( -d_x^2 + \epsilon^2 I \right)^{-q/2} F_{\mathcal{R}}.
\]

The introduction of the projection \( F_{\mathcal{R}} \) will allow us to control the approximation error by restricting the quasimomentum to a small neighborhood \( \mathcal{R} \) of \( k_0 = 0 \) and applying Lemmata 5.1, 5.2 based on the Taylor expansions (32), (33) of the \( \mathcal{R} \)th eigenvalue and eigenfunction around \( k_0 = 0 \).

The following identities are true:

\[
e^{-ite^{-2}A} \gamma^{(+)\epsilon} \epsilon q \left( -d_x^2 + \epsilon^2 I \right)^{-q/2} F_{\mathcal{R}} = \Psi^* [e^{-ite^{-2}E(k)}] \epsilon q (k^2 + \epsilon^2)^{-q/2} \chi_{\mathcal{R}}(k) \Phi,
\]

(63)

\[
[\varphi_0] e^{-ite^{-2}A_0^{\text{hom}}} \epsilon q (d_x^2 + \epsilon^2 I)^{-q/2} F_{\mathcal{R}} = [\varphi_0] \Phi^* [e^{-ite^{-2}b_0 k^2}] \epsilon q (k^2 + \epsilon^2)^{-q/2} \chi_{\mathcal{R}}(k) \Phi.
\]

(64)

By (41), we have

\[
\| (\Psi^* - [\varphi_0] \Phi^*) [e^{-ite^{-2}b_0 k^2}] \epsilon q (k^2 + \epsilon^2)^{-q/2} \chi_{\mathcal{R}}(k) \Phi \| \leq C \epsilon^{\min(1,q)}, \quad 0 < \epsilon \leq 1.
\]

(65)
It remains to estimate
\[
\|\Psi^*[(e^{-it\varepsilon^2E(k)} - e^{-it\varepsilon^2\sigma_0}e^{-it\varepsilon^2b_0k^2})e^q(k^2 + \varepsilon^2)^{-q/2}\chi_R(k)]\Phi\|_{L_2(\mathbb{R}) \to L_2(\mathbb{R})}.
\]
From (32) it is seen that
\[
e^{-it\varepsilon^2E(k)} - e^{-it\varepsilon^2\sigma_0}e^{-it\varepsilon^2b_0k^2} = e^{-it\varepsilon^2(\sigma_0 + b_0k^2)}(e^{-it\varepsilon^2k^4\gamma_0(k)} - 1)
= -e^{-it\varepsilon^2(\sigma_0 + b_0k^2)}e^{-\frac{1}{2}it\varepsilon^2k^4\gamma_0(k)} \cdot 2i \sin \left(\frac{1}{2}t\varepsilon^2k^4\gamma_0(k)\right).
\] (66)
Applying upper estimate (45) and taking into account that
\[
\left|e^{-\frac{1}{2}it\varepsilon^2k^4\gamma_0(k)}\right| = 1, \quad \left|e^{-it\varepsilon^2(\sigma_0 + b_0k^2)}\right| = 1,
\]
we get
\[
\|\Psi^*[(e^{-it\varepsilon^2E(k)} - e^{-it\varepsilon^2\sigma_0}e^{-it\varepsilon^2b_0k^2})e^q(k^2 + \varepsilon^2)^{-q/2}\chi_R(k)]\Phi\|_{L_2(\mathbb{R}) \to L_2(\mathbb{R})} \leq \frac{C\varepsilon^{q/2}|t|^q/4}{(|\gamma_0(0)|^{-1/2} + \varepsilon|t|)^{1/2}q^{7/2}}, \quad 0 < \varepsilon|t|^{-1/2} \leq \varepsilon.
\] (67)
Combining (60), (62)–(65), and (67), we arrive at the required estimate (58).

Let \(f, g \in L_2(\mathbb{R})\). Now we consider the behavior of the solution \(v_\varepsilon(x, t), \varepsilon \to 0\), of the Cauchy problem for the hyperbolic equation
\[
\begin{cases}
\frac{\partial^2}{\partial t^2}v_\varepsilon(x, t) = -(A_\varepsilon v_\varepsilon)(x, t) + \varepsilon^{-2}\sigma_0 v_\varepsilon(x, t), \\
v_\varepsilon(x, 0) = (\gamma_\varepsilon(+)f)(x), \quad (\partial_tv_\varepsilon)(x, 0) = (\gamma_\varepsilon(+)g)(x).
\end{cases}
\] (68)
Let \(v_0(x, t)\) be the solution of the corresponding ‘homogenized’ problem
\[
\begin{cases}
\frac{\partial^2}{\partial t^2}v_0(x, t) = -(A_{\sigma_0,0}^\text{hom} v_0)(x, t), \\
v_0(x, 0) = f(x), \quad (\partial_tv_0)(x, 0) = g(x).
\end{cases}
\] (69)
The solutions of problems (68) and (69) satisfy the following representations:
\[
\begin{align*}
v_\varepsilon &= \cos \left(tA_{\sigma_0,0}^{1/2}\right)_{\sigma_0,\varepsilon}^{(+)f} + A_{\sigma_0,0}^{1/2} \sin \left(tA_{\sigma_0,0}^{1/2}\right)_{\sigma_0,\varepsilon}^{(+)g}, \\
v_0 &= \cos(t(A_{\sigma_0,0}^{\text{hom}})^{1/2})f + (A_{\sigma_0,0}^{\text{hom}})^{-1/2} \sin(t(A_{\sigma_0,0}^{\text{hom}})^{1/2})g.
\end{align*}
\] (70)
where \(A_{\sigma_0,0} := (A_\varepsilon - \varepsilon^{-2}\sigma_0 I)E_{A_\varepsilon}[\varepsilon^{-2}\sigma_0, \infty]\).

**Theorem 6.2**: Suppose that Conditions 4.1 and (36) are fulfilled. Let \(v_\varepsilon\) be the solution of problem (68), and let \(v_0\) be the solution of problem (69). Let \(t \neq 0, f \in H^q(\mathbb{R}), g \in H^r(\mathbb{R})\). We have
\[
\|v_\varepsilon(\cdot, t) - \varphi_0^{\varepsilon}v_0(\cdot, t)\|_{L_2(\mathbb{R})} \leq C \left(1 + |t|^{q/3}\varepsilon^{2q/3}\|f\|_{H^q(\mathbb{R})} + (1 + |t|^{q/3})\varepsilon^{2q/3}\|g\|_{H^r(\mathbb{R})}\right),
\] (0 < \varepsilon \leq 1, \quad 0 < |t|^{-1} \leq \tilde{\varepsilon}, \quad 0 \leq q \leq 3/2, \quad 0 \leq r \leq 1/2, (71)
\[
\|v_\varepsilon(\cdot, t) - \varphi_0^{\varepsilon}v_0(\cdot, t)\|_{L_2(\mathbb{R})} \leq C \left(1 + |t|^{q/3}\varepsilon^{2q/3}\|f\|_{H^q(\mathbb{R})} + (1 + |t|^{q/3})\varepsilon^{2q/3}\|g\|_{H^r(\mathbb{R})}\right),
\] (0 < \varepsilon \leq 1, \quad 0 < |t|^{-1} \leq \tilde{\varepsilon}, \quad 0 \leq q \leq 3/2, \quad -1 \leq r < 0, (72)
with the constants \(C = C(q, r, \kappa, \alpha_0, \beta_0, \beta_1, \sigma_0, b_0, \|\varphi_0\|_{L_\infty}, |\gamma_0(0)|, \|\theta_0\|_{M_\varepsilon(0)})\).
Proof: By (70), we have to prove the estimates

$$\left\| \cos \left( t A_{\sigma_0, \varepsilon}^{1/2} \right) \mathcal{Y}_e^{(+)} - \left[ \varphi_0^e \right] \cos(t A_0^{\text{hom}})^{1/2} \right\|_{H^r(\mathbb{R}) \to L_2(\mathbb{R})} \leq C(1 + |t|^{q/3}) e^{2q/3}, \quad 0 \leq q \leq 3/2,$$

$$\left\| A_{\sigma_0, \varepsilon}^{-1/2} \sin \left( t A_{\sigma_0, \varepsilon}^{1/2} \right) \mathcal{Y}_e^{(+)} - \left[ \varphi_0^e \right](A_0^{\text{hom}})^{-1/2} \sin(t A_0^{\text{hom}})^{1/2} \right\|_{H^r(\mathbb{R}) \to L_2(\mathbb{R})} \leq C(1 + |t|^{(r+1)/3}) e^{(2r+2)/3}, \quad 0 \leq r \leq 1/2,$$

$$\left\| A_{\sigma_0, \varepsilon}^{-1/2} \sin \left( t A_{\sigma_0, \varepsilon}^{1/2} \right) \mathcal{Y}_e^{(+)} - \left[ \varphi_0^e \right](A_0^{\text{hom}})^{-1/2} \sin(t A_0^{\text{hom}})^{1/2} \right\|_{H^r(\mathbb{R}) \to L_2(\mathbb{R})} \leq C((1 + |t|^{(r+1)/3}) e^{(2r+2)/3} + |t|^{1/3} e^{2/3}), \quad -1 \leq r < 0,$$

where $t \neq 0$, $0 < \varepsilon \leq 1$, $0 < \varepsilon |t|^{-1} \leq \bar{\varepsilon}$. Using (59) and the unitarity of the scaling transformation, we have

$$\left\| \cos \left( t A_{\sigma_0, \varepsilon}^{1/2} \right) \mathcal{Y}_e^{(+)} - \left[ \varphi_0^e \right] \cos(t A_0^{\text{hom}})^{1/2} \right\|_{H^r(\mathbb{R}) \to L_2(\mathbb{R})} \leq \left\| \left( \cos \left( t \varepsilon^{-1} A_{\sigma_0}^{1/2} \right) \mathcal{Y}_e^{(+)} - \left[ \varphi_0 \right] \cos(t \varepsilon^{-1} (A_0^{\text{hom}})^{1/2}) \right) \varepsilon^q (-d_x^2 + \varepsilon^2 I) - q/2 \varepsilon^2 \right\|_{L_2(\mathbb{R}) \to L_2(\mathbb{R})} \quad \text{for} \quad 0 < \varepsilon \leq 1. \quad (76)$$

and

$$\left\| A_{\sigma_0, \varepsilon}^{-1/2} \sin \left( t A_{\sigma_0, \varepsilon}^{1/2} \right) \mathcal{Y}_e^{(+)} - \left[ \varphi_0^e \right](A_0^{\text{hom}})^{-1/2} \sin(t A_0^{\text{hom}})^{1/2} \right\|_{H^r(\mathbb{R}) \to L_2(\mathbb{R})} \leq \varepsilon \left( \left\| A_{\sigma_0}^{-1/2} \sin \left( t \varepsilon^{-1} A_{\sigma_0}^{1/2} \right) \mathcal{Y}_e^{(+)} \right\|_{H^r(\mathbb{R}) \to L_2(\mathbb{R})} - \left\| \left[ \varphi_0 \right] \cos \left( t \varepsilon^{-1} (A_0^{\text{hom}})^{1/2} \right) \varepsilon^q (-d_x^2 + \varepsilon^2 I) - q/2 \right\|_{L_2(\mathbb{R}) \to L_2(\mathbb{R})} \right), \quad \text{for} \quad 0 < \varepsilon \leq 1. \quad (77)$$

where $A_{\sigma_0} := (A - \sigma_0 I) \mathcal{E}_A(\sigma_0, \infty)$.

Let us estimate expression (76). Similarly to the proof of Theorem 6.1, one obtains

$$\left\| \left( \cos \left( t \varepsilon^{-1} A_{\sigma_0}^{1/2} \right) \mathcal{Y}_e^{(+)} - \left[ \varphi_0 \right] \cos(t \varepsilon^{-1} (A_0^{\text{hom}})^{1/2}) \right) \varepsilon^q (-d_x^2 + \varepsilon^2 I) - q/2 \varepsilon^2 \right\|_{L_2(\mathbb{R}) \to L_2(\mathbb{R})} \leq (1 + \left\| \varphi_0 \right\|_{L_\infty}) \kappa^{-q} \varepsilon^q,$$

$$\cos \left( t \varepsilon^{-1} A_{\sigma_0}^{1/2} \right) \mathcal{Y}_e^{(+)} \varepsilon^q (-d_x^2 + \varepsilon^2 I) - q/2 \varepsilon^2 \mathcal{E}_R \varepsilon^q \left( \cos \left( t \varepsilon^{-1} (E(k) - \sigma_0)^{1/2} \right) \varepsilon^q (k^2 + \varepsilon^2) - q/2 \chi_R(k) \right) \Phi,$$

$$\left[ \varphi_0 \right] \cos \left( t \varepsilon^{-1} (A_0^{\text{hom}})^{1/2} \right) \varepsilon^q (-d_x^2 + \varepsilon^2 I) - q/2 \mathcal{E}_R \varepsilon^q \left( \cos \left( t \varepsilon^{-1} (b_0 k^2)^{1/2} \right) \varepsilon^q (k^2 + \varepsilon^2) - q/2 \chi_R(k) \right) \Phi,$$

$$\| (\Psi^* - \left[ \varphi_0 \right] \Phi^*) \left[ \cos \left( t \varepsilon^{-1} (E(k) - \sigma_0)^{1/2} \right) \varepsilon^q (k^2 + \varepsilon^2) - q/2 \chi_R(k) \right] \Phi \| \leq C \varepsilon^{\min(1,q)}, \quad 0 < \varepsilon \leq 1.$$

It remains to estimate the norm

$$\left\| \Psi^* \left( \cos \left( t \varepsilon^{-1} (E(k) - \sigma_0)^{1/2} \right) - \cos \left( t \varepsilon^{-1} (b_0 k^2)^{1/2} \right) \right) \right\|_{L_2(\mathbb{R}) \to L_2(\mathbb{R})}.$$
Transform the difference of the cosines using the identity
\[
\cos \left( t e^{-1} (E(k) - \sigma_0)^{1/2} \right) - \cos \left( t e^{-1} (b_0 k^2)^{1/2} \right) = \\
= -2 \sin \left( \frac{1}{2} t e^{-1} \left( (E(k) - \sigma_0)^{1/2} + (b_0 k^2)^{1/2} \right) \right) \\
\times \sin \left( \frac{1}{2} t e^{-1} \left( (E(k) - \sigma_0)^{1/2} - (b_0 k^2)^{1/2} \right) \right).
\]

Obviously, \(|\sin(\frac{1}{2} t e^{-1} ((E(k) - \sigma_0)^{1/2} + (b_0 k^2)^{1/2}))| \leq 1\). Applying upper estimate (46) and taking into account (37), we obtain that
\[
\left\| \Psi^* [\cos(t e^{-1} (E(k) - \sigma_0)^{1/2}) - \cos(t e^{-1} (b_0 k^2)^{1/2}))] \varepsilon_0^q(k^2 + \varepsilon^2)^{-q/2} \chi_\mathcal{R}(k) \varepsilon \right\|_{L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R})} \\
\leq \frac{C \varepsilon^{q/3} |t|^{q/3}}{(|\tilde{\gamma}_0(0)|^{-2/3} + \varepsilon^{4/3} |t|^{2/3})^{q/2}}, \quad 0 < \varepsilon |t|^{-1} \leq \tilde{\varepsilon},
\]
which completes the proof of (73).

Now we turn to the proof of estimates (74), (75). First, we estimate the operator under the norm sign in (77) multiplied by the projection \((I - \mathcal{F}_\mathcal{R})\) on the right:
\[
\varepsilon \left( A_{\sigma_0}^{-1/2} \sin \left( t e^{-1} A_{\sigma_0}^{1/2} \right) \mathcal{Y}^{(+)} \right.
- [\varphi_0] (A_0^{hom})^{-1/2} \sin \left( t e^{-1} (A_0^{hom})^{1/2} \right) \varepsilon \mathcal{I} \left( -d_x^2 + \varepsilon^2 I \right)^{-r/2} (I - \mathcal{F}_\mathcal{R})
\]
\[
= \varepsilon \sum_{j=s}^{\infty} \Psi_j^*[R_{j-1} \left( (E_j(k) - \sigma_0)^{-1/2} \sin \left( t e^{-1} (E_j(k) - \sigma_0)^{1/2} \right) \chi_{\Omega_{j-1}}(k) \right]
- [\varphi_0] \Phi^* \left( (b_0 k^2)^{-1/2} \sin \left( t e^{-1} (b_0 k^2)^{1/2} \right) \right) \left[ \varepsilon \mathcal{I} (k^2 + \varepsilon^2)^{-r/2} (1 - \chi_\mathcal{R}(k)) \right] \Phi.
\]
(78)

We use the elementary inequality \(|\sin x| \leq 1, x \in \mathbb{R}\), and the estimates
\[
(E_j(k) - \sigma_0)^{-1/2} \chi_{\Omega_{j-1}}(k)(1 - \chi_\mathcal{R}(k)) \leq (E(k) - \sigma_0)^{-1/2}, \quad j \geq s,
\]
\[
(b_0 k^2)^{-1/2} (1 - \chi_\mathcal{R}(k)) \leq b_0^{-1/2} k^{-1}.
\]

Next, if \(r \geq 0\), then \(\varepsilon \mathcal{I} (k^2 + \varepsilon^2)^{-r/2} \leq 1\), and the \((L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R}))\)-norm of operator (78) is estimated by \(C\varepsilon\). Consider the case where \(-1 \leq r < 0\). Let \(N \geq 1\). Then
\[
\varepsilon \mathcal{I} (k^2 + \varepsilon^2)^{-r/2} \chi_{[-N,N]}(k) \leq \varepsilon \mathcal{I} (N^2 + 1)^{-r/2}, \quad -1 \leq r < 0,
\]
\[
(b_0 k^2)^{-1/2} \varepsilon \mathcal{I} (k^2 + \varepsilon^2)^{-r/2} (1 - \chi_{[-N,N]}(k)) \leq b_0^{-1/2} N^{-1} (N^2 + 1)^{-r/2} \varepsilon^r, \quad -1 \leq r < 0.
\]

Here we have taken into account that the function \((k^2 + \varepsilon^2)^{-r/2}\) is even and monotonically increasing in \(k \geq 0\), and the function \(|k|^{-1} (k^2 + \varepsilon^2)^{-r/2}\) is even and monotonically decreasing in \(k \geq 1\). Next, from lower estimate (28) and periodicity of the functions \(E_j(k)\) we deduce
\[
E_j(k) \chi_{\Omega_{j-1}}(k) = \alpha_0 \beta_0^2 \beta_1^{-2} (|k| + \pi (s - 1))^2 \chi_{\Omega_{j-1}}(k)
\]
\[
\geq \alpha_0 \beta_0^2 \beta_1^{-2} k^2 \chi_{\Omega_{j-1}}(k), \quad j \geq s,
\]
and therefore

$$
(E(k) - \sigma_0)^{-1/2} (1 - \chi_{[-N,N]}(k)) \chi_{\Omega_j - r^+}(k) \epsilon^r (k^2 + \epsilon^2)^{-r/2}
\leq (\alpha_0 \beta_0^2 \beta_1^{-2} k^2 - \sigma_0)^{-1/2} (1 - \chi_{[-N,N]}(k)) \epsilon^r (k^2 + \epsilon^2)^{-r/2}
\leq C \epsilon^r, \quad C = (\alpha_0 \beta_0^2 \beta_1^{-2} N^2 - \sigma_0)^{-1/2} (N^2 + 1)^{-r/2},
$$

$$
j \geq s, \quad -1 \leq r < 0.

(79)

Here we have taken into account that the function in the middle part of (79) monotonically decreases in $k \geq N$. The number $N$ is chosen such that

$$
\alpha_0 \beta_0^2 \beta_1^{-2} N^2 - \sigma_0 > 0.

As a result, for $-1 \leq r < 0$ the $(L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R}))$-norm of operator (78) is estimated by $C \epsilon^{r+1}$.

Now, consider

$$
\epsilon \left( A_{\sigma_0}^{-1/2} \sin \left( t \epsilon^{-1} A_{\sigma_0}^{1/2} \right) \Psi^{(+)} - [\varphi_0] (A_0^{\text{hom}})^{-1/2} \sin (t \epsilon^{-1} (A_0^{\text{hom}})^{1/2}) \right) \times \epsilon^r (-d_x^2 + \epsilon^2 I)^{-r/2} F_R.
\vphantom{(A_0^{1/2} - (b_0 k^2))^{-1/2}}
$$

(80)

Similarly to (63), (64), we see that operator (80) equals

$$
\epsilon \left( \Psi^* \left[ (E(k) - \sigma_0)^{-1/2} \sin \left( t \epsilon^{-1} (E(k) - \sigma_0)^{1/2} \right) \right] - [\varphi_0] \Phi^* \left[ (b_0 k^2)^{-1/2} \sin (t \epsilon^{-1} (b_0 k^2)^{1/2}) \right] \right) \left[ \epsilon^r (k^2 + \epsilon^2)^{-r/2} \chi_R(k) \right] \Phi.
\vphantom{(A_0^{1/2} - (b_0 k^2))^{-1/2}}
$$

By (37), (39), and (40),

$$
\left| (E(k) - \sigma_0)^{-1/2} - (b_0 k^2)^{-1/2} \right| \chi_R(k) = \frac{|k|^3 |\tilde{\gamma}_0(k)|}{b_0^{1/2} |k| \left| b_0^{1/2} |k| + |k|^3 \tilde{\gamma}_0(k) \right|} \chi_R(k)
\leq \frac{3 |k| |\tilde{\gamma}_0(0)|}{2 b_0^{1/2} \left| b_0^{1/2} + k^2 \tilde{\gamma}_0(0) \right|} \chi_R(k)
\leq 3 b_0^{-1} |k| |\tilde{\gamma}_0(0)| \chi_R(k),
$$

whence

$$
\left| (E(k) - \sigma_0)^{-1/2} - (b_0 k^2)^{-1/2} \right| \epsilon^{r+1} (k^2 + \epsilon^2)^{-r/2} \chi_R(k)
\leq 3 b_0^{-1} |\tilde{\gamma}_0(0)| \epsilon^{r+1} k (k^2 + \epsilon^2)^{-r/2}.
$$

Here we have taken into account that the function $k (k^2 + \epsilon^2)^{-r/2}$ monotonically increases in $k \geq 0$.

Next, from (42) it directly follows that

$$
\epsilon \| \Psi^* - [\varphi_0] \Phi^* \| (b_0 k^2)^{-1/2} \sin (t \epsilon^{-1} (b_0 k^2)^{1/2}) \epsilon^r (k^2 + \epsilon^2)^{-r/2} \chi_R(k) \| \Phi \| 
\leq C \epsilon^{\min(1, r+1)}, \quad 0 < \epsilon \leq 1.
$$

It remains to estimate

$$
\epsilon \| \Psi^* \left[ (b_0 k^2)^{-1/2} \left( \sin \left( t \epsilon^{-1} (E(k) - \sigma_0)^{1/2} \right) - \sin \left( t \epsilon^{-1} (b_0 k^2)^{1/2} \right) \right) \right] \epsilon^r (k^2 + \epsilon^2)^{-r/2} \chi_R(k) \| \Phi \|_{L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R})}.
$$
We have
\[
\sin(t\varepsilon^{-1}(E(k) - \sigma_0)^{1/2}) - \sin(t\varepsilon^{-1}(b_0k^2)^{1/2})
= 2 \cos\left(\frac{1}{2}t\varepsilon^{-1}((E(k) - \sigma_0)^{1/2} + (b_0k^2)^{1/2})\right) \sin\left(\frac{1}{2}t\varepsilon^{-1}((E(k) - \sigma_0)^{1/2} - (b_0k^2)^{1/2})\right).
\]

Obviously, \(|\cos\left(\frac{1}{2}t\varepsilon^{-1}((E(k) - \sigma_0)^{1/2} + (b_0k^2)^{1/2})\right)| \leq 1\). The application of upper estimate (47) together with (37) completes the proof of inequalities (74), (75). ■

Remark 6.1: In estimate (57) for the Schrödinger equation, there is the factor \(e^{-it\varepsilon^{-2}\sigma_0}\) which corresponds to the spectral shift. To obtain similar result (71), (72) for the hyperbolic equation it is more convenient for us to add the spectral shift \(\varepsilon^{-2}\sigma\) in (68), since \(\cos(tA_{\sigma_0,\varepsilon}^{1/2})\) and \(A_{\sigma_0,\varepsilon}^{-1/2}\sin(tA_{\sigma_0,\varepsilon}^{1/2})\) are the functions of the square root of \(A_{\sigma_0,\varepsilon}\).

### 6.2. The case where Condition 4.3 is fulfilled

Now it is convenient to realize points of the circle in (29) in the following way:

\[\Psi_l: L_2(\mathbb{R}) \to L_2(\Omega_{l-1}^j), \quad l = 1, \ldots, s,\]

where \(\tilde{\Omega}_j = \Omega_j + \pi\).

Consider the Cauchy problem for the nonstationary Schrödinger equation

\[
\begin{cases}
i \frac{\partial}{\partial t}w_\varepsilon(x, t) = (A_\varepsilon w_\varepsilon)(x, t), \\
w_\varepsilon(x, 0) = (\tilde{\Psi}^{(-)}_\varepsilon) f(x),
\end{cases}
\]  

where

\[\tilde{\Psi}^{(-)}_\varepsilon f \in \mathcal{E}_{A_\varepsilon}[0, \varepsilon^{-2}\sigma_{\varepsilon}]L_2(\mathbb{R}),\]

and the corresponding ‘homogenized’ problem

\[
\begin{cases}
i \frac{\partial}{\partial t}w_0(x, t) = - (A^{\text{hom}}_{\varepsilon} w_0)(x, t), \\
w_0(x, 0) = f(x).
\end{cases}
\]

Here \(A^{\text{hom}}_{\varepsilon} = -b_\varepsilon \frac{d^2}{dx^2}, \text{ Dom } A^{\text{hom}}_{\varepsilon} = H^2(\mathbb{R}),\) is the operator acting in \(L_2(\mathbb{R}),\) which is called the effective operator at the right edge \(\sigma_{\varepsilon} = E(\varepsilon)\) of the band \(R(E)\).

The solutions of problems (81) and (82) can be represented as follows:

\[
w_\varepsilon = e^{-itA_\varepsilon}(\tilde{\Psi}^{(-)}_\varepsilon) f, \quad w_0 = e^{-it(A^{\text{hom}}_{\varepsilon})} f,
\]

where \(\tilde{\Psi}^{(-)}_\varepsilon := \sum_{j=1}^s \int_{\varepsilon}^{R^-_{j-1}} \Psi_j s_{j-1} \Phi \delta_{\varepsilon}(\varepsilon x/k)\), and \(R^j\) is the operator of restriction to \(\tilde{\Omega}_j\).

**Theorem 6.3:** Suppose that Conditions 4.3 and (36) are fulfilled. Let \(w_\varepsilon\) be the solution of problem (81), and let \(w_0\) be the solution of problem (82). Let \(t \neq 0, f \in H^q(\mathbb{R}), 0 \leq q \leq 2\). We have

\[
\|w_\varepsilon(\cdot, t) - e^{-it\varepsilon^{-2}\sigma_\varepsilon} \varphi \varepsilon e^{it\varphi_\varepsilon(\cdot/t)} w_0(\cdot, t)\|_{L_2(\mathbb{R})} \leq C(1 + |t|/q) \|f\|_{H^q(\mathbb{R})},
\]

\[0 < \varepsilon \leq 1, \quad 0 < |t|/\varepsilon^{-1/2} \leq \varepsilon,\]

with the constant \(C = C(q, \kappa, \|\varphi_\varepsilon\|_{L_\infty}, \|\varphi_\varepsilon(\varepsilon)\|, \|\theta_\varepsilon\|_{M_\kappa(\varepsilon)}).\)
Proof: By (83), we have to prove the estimate
\[
\left\| e^{-itA} \widetilde{\Upsilon}_\pi(e^{-1}x) e^{-it(-A_\pi^{\text{hom}})} \right\|_{H^q(\mathbb{R}) \to L_2(\mathbb{R})} \leq C(1 + |t|^{q/4})e^{q/2},
\]
where \( t \neq 0, 0 < \varepsilon \leq 1, 0 < \varepsilon |t|^{-1/2} \leq \varepsilon, 0 \leq \varepsilon \leq 2 \). From (59) and the unitarity of the scaling transformation it follows that
\[
\left\| e^{-itA} \widetilde{\Upsilon}_\pi(e^{-1}x) e^{-it(-A_\pi^{\text{hom}})} \right\|_{H^q(\mathbb{R}) \to L_2(\mathbb{R})} = \left\| \left( e^{-itA} - e^{-ite^{-2}\sigma_\pi e^{it(-A_\pi^{\text{hom}})} \right) \varepsilon(q(-d_x^2 + \varepsilon^2 I) - q/2) \right\|_{L_2(\mathbb{R}) \to L_2(\mathbb{R})}.
\]
Here \( \widetilde{\Upsilon}_\pi(e^{-1}x) = \sum_j |\Psi_j R'_s|_{s,j+1} \Phi[e^{itx}] \). Next,
\[
\Phi^* [\varepsilon(q((k - \pi)^2 + \varepsilon^2)) - q/2] \Phi[e^{itx}] = [e^{itx}] \varepsilon(q(-d_x^2 + \varepsilon^2 I) - q/2),
\]
\[
\Phi^* [\varepsilon(q((k - \pi)^2 + \varepsilon^2)) - q/2] \Phi[e^{itx}] = [e^{itx}] \varepsilon(q(-d_x^2 + \varepsilon^2 I) - q/2),
\]
whence
\[
\left\| \left( e^{-itA} \sum_j |\Psi_j R'_s|_{s,j+1} - e^{-ite^{-2}\sigma_\pi \varepsilon \pi} \Phi^* [\varepsilon(q((k - \pi)^2 + \varepsilon^2)) - q/2] \right) \right\|_{L_2(\mathbb{R}) \to L_2(\mathbb{R})}.
\]
Obviously, \( \varepsilon(q((k - \pi)^2 + \varepsilon^2)) - q/2(1 - \chi_p(k - \pi)) \leq \kappa^{-q} \varepsilon q \), and so
\[
\left\| \left( e^{-itA} \sum_j |\Psi_j R'_s|_{s,j+1} - e^{-ite^{-2}\sigma_\pi \varepsilon \pi} \Phi^* [\varepsilon(q((k - \pi)^2 + \varepsilon^2)) - q/2] \right) \right\|_{L_2(\mathbb{R}) \to L_2(\mathbb{R})} \leq (1 + \|\varepsilon\|_{L_\infty}) \kappa^{-q} \varepsilon q.
\]
Taking into account the equality
\[
e^{-ite^{-2}\sigma_\pi \varepsilon \pi} \sum_j |\Psi_j R'_s|_{s,j+1} \varepsilon(q((k - \pi)^2 + \varepsilon^2)) - q/2 \chi_p(k - \pi) \Phi
\]
\[
= \Psi^* [\varepsilon(q((k - \pi)^2 + \varepsilon^2)) - q/2 \chi_p(k - \pi)] \Phi
\]
and the estimate (which follows from (41))
\[
\left\| (\Psi^* - \varepsilon \pi) \Phi^* \varepsilon(q((k - \pi)^2 + \varepsilon^2)) - q/2 \chi_p(k - \pi) \Phi \right\| \leq C \varepsilon^{\min(1,q)}, \quad 0 < \varepsilon \leq 1,
\]
we conclude that it remains to estimate the norm
\[
\left\| \Psi^* [\varepsilon(q((k - \pi)^2 + \varepsilon^2)) - q/2 \chi_p(k - \pi)] \Phi \right\|_{L_2(\mathbb{R}) \to L_2(\mathbb{R})}.
\]
By (34), we have
\[ e^{-it\varepsilon^{-2}E(k)} - e^{-it\varepsilon^{-2}q_{\pi}}e^{i\varepsilon^{-2}b_{\pi}(k-\pi)^2} \]
\[ = -e^{-it\varepsilon^{-2}(\sigma_{\pi} - b_{\pi}(k-\pi)^2)}e^{-\frac{1}{2}it\varepsilon^{-2}(k-\pi)^4\gamma_{\pi}(k)} \cdot 2i \sin \left( \frac{1}{2}t\varepsilon^{-2}(k - \pi)^4\gamma_{\pi}(k) \right). \]

Application of estimate (45) together with
\[ \left| e^{-it\varepsilon^{-2}(\sigma_{\pi} - b_{\pi}(k-\pi)^2)} \right| = 1, \quad \left| e^{-\frac{1}{2}it\varepsilon^{-2}k^4\gamma_{\pi}(k)} \right| = 1 \]
completes the proof.

Let \( f, g \in L_2(\mathbb{R}) \). Consider the Cauchy problem for the hyperbolic equation
\[ \begin{cases}
\frac{\partial^2}{\partial t^2}z_{\varepsilon}(x, t) = (A_{\varepsilon}(x, t) - \varepsilon^{-2}\sigma_{\pi})z_{\varepsilon}(x, t), \\
z_{\varepsilon}(x, 0) = (\tilde{\gamma}_{\varepsilon}(-)\tilde{f})(x), \quad (\partial_{t}z_{\varepsilon})(x, 0) = (\tilde{\gamma}_{\varepsilon}(-)\tilde{g})(x),
\end{cases} \quad (85) \]
and the corresponding 'homogenized' problem
\[ \begin{cases}
\frac{\partial^2}{\partial t^2}z_0(x, t) = -(A_{\pi}^{\text{hom}}z_0)(x, t), \\
z_0(x, 0) = f(x), \quad (\partial_{t}z_0)(x, 0) = g(x).
\end{cases} \quad (86) \]

The solutions of problems (85) and (86) satisfy the following representations:
\[ z_{\varepsilon} = \cos \left( tA_{\pi}^{1/2} \tilde{\gamma}_{\varepsilon}(-) \tilde{f} \right) + A_{\pi}^{-1/2} \sin \left( tA_{\pi}^{1/2} \tilde{\gamma}_{\varepsilon}(-) \tilde{g}, \right) \]
\[ z_0 = \cos \left( tA_{\pi}^{\text{hom}}^{1/2} \tilde{f} \right) + A_{\pi}^{-1/2} \sin \left( tA_{\pi}^{\text{hom}}^{1/2} \tilde{g}, \right), \]
where \( A_{\pi}^{\text{hom}} := (\varepsilon^{-2}\sigma_{\pi} - A_{\pi}) \varepsilon A_{\pi}, [0, \varepsilon^{-2}\sigma_{\pi}] \). The following statement can be proved similarly to the proofs of Theorems 6.2 and 6.3.

**Theorem 6.4:** Suppose that Conditions 4.3 and (36) are fulfilled. Let \( z_{\varepsilon} \) be the solution of problem (85), and let \( z_0 \) be the solution of problem (86). Let \( t \neq 0 \), \( f \in H^q(\mathbb{R}), g \in H^r(\mathbb{R}) \). We have
\[ \|z_{\varepsilon}(\cdot, t) - \varphi_{\pi}^{\varepsilon}e^{it\varepsilon^{-2}\gamma_{\pi}(\cdot)\tilde{f}}(\cdot, t)\|_{L_2(\mathbb{R})} \]
\[ \leq C \left( (1 + |t|)^{q/3}e^{2q/3}\|f\|_{H^q(\mathbb{R})} + (1 + |t|^{(r+1)/3})e^{(2r+2)/3}\|g\|_{H^r(\mathbb{R})} \right), \]
\[ 0 < \varepsilon \leq 1, \quad 0 < \varepsilon |t| \leq \tilde{c}, \quad 0 \leq q \leq 3/2, \quad 0 \leq r \leq 1/2, \]
\[ \|z_{\varepsilon}(\cdot, t) - \varphi_{\pi}^{\varepsilon}e^{it\varepsilon^{-2}\gamma_{\pi}(\cdot)\tilde{f}}(\cdot, t)\|_{L_2(\mathbb{R})} \leq C \left( (1 + |t|^{q/3})e^{2q/3}\|f\|_{H^q(\mathbb{R})} + (1 + |t|^{(r+1)/3})e^{(2r+2)/3}\|g\|_{H^r(\mathbb{R})} \right), \]
\[ 0 < \varepsilon \leq 1, \quad 0 < \varepsilon |t| \leq \tilde{c}, \quad 0 \leq q \leq 3/2, \quad -1 \leq r < 0, \]
with the constants \( C = C(q, r, \kappa, a_0, \beta_0, \beta_1, \sigma_{\pi}, b_{\pi}, \|\varphi_{\pi}\|_{L_\infty(\mathbb{R})}, |\gamma_{\pi}(\pi)|, \|\theta_{\pi}\|_{M_\kappa(\pi)}). \)
6.3. The case where Condition 4.2 is fulfilled

In $L_2(\mathbb{R})$, consider the operator $A_0^{\text{hom}} = -b_0 \frac{d^2}{dx^2}$, $\text{Dom} A_0^{\text{hom}} = H^2(\mathbb{R})$, which is called the effective operator at the right edge $\sigma_0 = E(0)$ of the band $R(E)$. Let $f, g \in L_2(\mathbb{R})$. Consider the following Cauchy problem for the nonstationary Schrödinger equation and the corresponding ‘homogenized’ problem:

$$
\begin{align*}
\begin{cases}
\frac{\partial}{\partial t} \tilde{u}_e(x, t) = (A_e \tilde{u}_e)(x, t), \\
\tilde{u}_e(x, 0) = (\gamma_e^{(-)} f)(x), \\
\end{cases} & \quad \begin{cases}
\frac{\partial}{\partial t} \tilde{u}_0(x, t) = -(A_0^{\text{hom}} \tilde{u}_0)(x, t), \\
\tilde{u}_0(x, 0) = f(x),
\end{cases}
\end{align*}

$$

(89)

where

$$
(\gamma_e^{(-)} f)(x) := (2\pi)^{-1/2} \int \Phi(k) \sum_{j=1}^{s} e^{ikx} \psi_j(x, t) \chi_{\Omega_{e+j+1}}(\epsilon k) \, dk,
$$

and the similar Cauchy problems for the hyperbolic equations:

$$
\begin{align*}
\begin{cases}
\frac{\partial^2}{\partial t^2} \tilde{v}_e(x, t) = (A_e \tilde{v}_e)(x, t) \\
\tilde{v}_e(x, 0) = (\gamma_e^{(-)} f)(x), \\
(\partial_{t} \tilde{v}_e)(x, 0) = (\gamma_e^{(-)} g)(x), \\
\end{cases} & \quad \begin{cases}
\frac{\partial^2}{\partial t^2} \tilde{v}_0(x, t) = -(A_0^{\text{hom}} \tilde{v}_0)(x, t), \\
\tilde{v}_0(x, 0) = f(x), \\
(\partial_{t} \tilde{v}_0)(x, 0) = g(x).
\end{cases}
\end{align*}

$$

(90)

Theorem 6.5: Suppose that Conditions 4.2 and (36) are fulfilled. Let $\tilde{u}_e, \tilde{u}_0$ be the solutions of problems (89), $t \neq 0$, $f \in H^q(\mathbb{R})$, $0 \leq q \leq 2$. We have

$$
\|\tilde{u}_e(\cdot, t) - e^{-i\epsilon t} \sigma_0 \psi_0^{\tilde{u}}(\cdot, t)\|_{L_2(\mathbb{R})} \leq C(1 + |t|^{q/4}) e^{2q/2} \|f\|_{H^q(\mathbb{R})},
$$

$$
0 < \epsilon \leq 1, \quad 0 < \epsilon |t|^{-1/2} \leq \tilde{c},
$$

(91)

with the constant $C = C(q, \kappa, \|\psi_0\|_{L_\infty}, |\gamma_0(0)|, \|\theta_0\|_{M_\epsilon(0)})$.

Theorem 6.6: Suppose that Conditions 4.2 and (36) are fulfilled. Let $\tilde{v}_e, \tilde{v}_0$ be the solutions of problems (90), and let $t \neq 0$, $f \in H^q(\mathbb{R})$, $g \in H^r(\mathbb{R})$. We have

$$
\|\tilde{v}_e(\cdot, t) - \varphi_0^{\tilde{v}}(\cdot, t)\|_{L_2(\mathbb{R})} 
\leq C \left( (1 + |t|^{q/3}) e^{2q/3} \|f\|_{H^q(\mathbb{R})} + (1 + |t|^{(r+1)/3}) e^{(2r+2)/3} \|g\|_{H^r(\mathbb{R})} \right),
$$

$$
0 < \epsilon \leq 1, \quad 0 < \epsilon |t|^{-1} \leq \tilde{c}, \quad 0 \leq q \leq 3/2, \quad 0 \leq r \leq 1/2,
$$

(92)

$$
\|\tilde{v}_e(\cdot, t) - \varphi_0^{\tilde{v}}(\cdot, t)\|_{L_2(\mathbb{R})} \leq C \left( (1 + |t|^{q/3}) e^{2q/3} \|f\|_{H^q(\mathbb{R})} 
+ ((1 + |t|^{(r+1)/3}) e^{(2r+2)/3} + |t|^{1/3} e^{2/3}) \|g\|_{H^r(\mathbb{R})} \right),
$$

$$
0 < \epsilon \leq 1, \quad 0 < \epsilon |t|^{-1} \leq \tilde{c}, \quad 0 \leq q \leq 3/2, \quad -1 \leq r < 0,
$$

(93)

with the constants $C = C(q, r, \kappa, \sigma_0, \beta_0, \beta_1, \sigma_0, b_0, \|\psi_0\|_{L_\infty}, |\gamma_0(0)|, \|\theta_0\|_{M_\epsilon(0)})$. 


6.4. The case where Condition 4.4 is fulfilled

In \( L_2(\mathbb{R}) \), consider the operator \( A_{\pi}^{\text{hom}} = -b_{\pi} \frac{d^2}{d x^2} \), \( \text{Dom} A_{\pi}^{\text{hom}} = H^2(\mathbb{R}) \), which is called the effective operator at the left edge \( \sigma_{\pi} = E(\pi) \) of the band \( R(E) \). Let \( f, g \in L_2(\mathbb{R}) \). Consider the following Cauchy problem for the nonstationary Schrödinger equation and the corresponding 'homogenized' problem:

\[
\begin{aligned}
&\left\{ \begin{array}{ll}
\frac{\partial}{\partial t} \tilde{w}_e(x,t) = (A_e \tilde{w}_e)(x,t), \\
\tilde{w}_e(x,0) = (\tilde{\Upsilon}_e^{(+)} f)(x),
\end{array} \right. \\
&\left\{ \begin{array}{ll}
\frac{\partial}{\partial t} \tilde{w}_0(x,t) = (A_{\pi}^{\text{hom}} \tilde{w}_0)(x,t), \\
\tilde{w}_0(x,0) = f(x),
\end{array} \right.
\end{aligned}
\]

(94)

where

\[
(\tilde{\Upsilon}_e^{(+)} f)(x) := (2\pi)^{-1/2} \int_{\mathbb{R}} (\Phi f)(k - \varepsilon^{-1} \pi) \sum_{j=\pm} e^{i k x} \varphi_j(x/\varepsilon, \varepsilon k) \chi_{\Omega j}(\varepsilon k) \, dk,
\]

\[
\tilde{\Upsilon}_e^{(+)} f \in \mathcal{E}_{A_e}[\varepsilon^{-2} \sigma_{\pi}, \infty)L_2(\mathbb{R}),
\]

and the similar Cauchy problems for the hyperbolic equations:

\[
\begin{aligned}
&\frac{\partial^2}{\partial t^2} \tilde{z}_e(x,t) = -(A_e \tilde{z}_e)(x,t) \\
&\tilde{z}_e(x,0) = (\tilde{\Upsilon}_e^{(+)} f)(x), \\
&\tilde{z}_e(x,0) = (\tilde{\Upsilon}_e^{(+)} g)(x),
\end{aligned}
\]

(95)

**Theorem 6.7:** Suppose that Conditions 4.4 and (36) are fulfilled. Let \( \tilde{w}_e, \tilde{w}_0 \) be the solutions of problems (94), and let \( t \neq 0, f \in H^3(\mathbb{R}), 0 \leq q \leq 2 \). We have

\[
\| \tilde{w}_e(\cdot,t) - e^{-it \varepsilon^{-2} \sigma_{\pi}} e^{i \pi (\cdot)/\varepsilon} \tilde{w}_0(\cdot,t) \|_{L_2(\mathbb{R})} \leq C(1 + |t|^{q/4}) \| f \|_{H^q(\mathbb{R})},
\]

\[
0 < \varepsilon \leq 1, \quad 0 < \varepsilon |t|^{-1/2} \leq \varepsilon,
\]

(96)

with the constant \( C = C(q, \kappa, \| \varphi_{\pi} \|_{L_{\infty}}, |\gamma_{\pi}(\pi)|, \| \theta_{\pi} \|_{M_{e}(\pi)}) \).

**Theorem 6.8:** Suppose that Conditions 4.4 and (36) are fulfilled. Let \( \tilde{z}_e, \tilde{z}_0 \) be the solutions of problems (95), and let \( t \neq 0, f \in H^3(\mathbb{R}), g \in H^r(\mathbb{R}) \). We have

\[
\| \tilde{z}_e(\cdot,t) - e^{-it \varepsilon^{-2} \sigma_{\pi}} e^{i \pi (\cdot)/\varepsilon} \tilde{z}_0(\cdot,t) \|_{L_2(\mathbb{R})} \leq C \left( 1 + |t|^{q/3} \| f \|_{H^q(\mathbb{R})} + (1 + |t|^{(r+1)/3}) e^{(2r+2)/3} \| g \|_{H^r(\mathbb{R})} \right),
\]

\[
0 < \varepsilon \leq 1, \quad 0 < \varepsilon |t|^{-1} \leq \varepsilon, \quad 0 \leq q \leq 3/2, \quad 0 \leq r \leq 1/2,
\]

(97)

\[
\| \tilde{z}_e(\cdot,t) - e^{-it \varepsilon^{-2} \sigma_{\pi}} e^{i \pi (\cdot)/\varepsilon} \tilde{z}_0(\cdot,t) \|_{L_2(\mathbb{R})} \leq C \left( 1 + |t|^{q/3} \| f \|_{H^q(\mathbb{R})} + (1 + |t|^{(r+1)/3}) e^{(2r+2)/3} \| g \|_{H^r(\mathbb{R})} \right),
\]

(98)

with the constants \( C = C(q, r, \kappa, \alpha_0, \beta_0, \beta_1, \sigma_{\pi}, b_{\pi}, \| \varphi_{\pi} \|_{L_{\infty}}, |\gamma_{\pi}(\pi)|, \| \theta_{\pi} \|_{M_{e}(\pi)}) \).
7. Concluding remarks

7.1. Under the assumptions of Theorems 6.1, 6.3, 6.5, 6.7 for \( q = 2 \) and under the assumptions of Theorems 6.2, 6.4, 6.6, 6.8 for \( q = 3/2 \), \( r = 1/2 \) we have the error estimates of order \( O((1 + |t|^{1/2}\epsilon) \). The first power of \( \epsilon \) is order-sharp.

If \( q = 0 \) and \( r = -1 \), then applying the Banach–Steinhaus theorem ([48, Chapter II, Section 1, 18]) one can obtain the strong convergence of the solutions. For example, for \( u_\epsilon \) and \( v_\epsilon \), using Theorems 6.1, 6.2, the \( (L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R}))\)-boundedness of the operators \( e^{-itA_\epsilon} \), \( \cos(tA_{\sigma_0,\epsilon}) \) and the \( (H^{-1}(\mathbb{R}) \rightarrow L_2(\mathbb{R}))\)-boundedness of the operator \( A_{\sigma_0,\epsilon}^{-1/2} \sin(tA_{\sigma_0,\epsilon}) \) (and similar operator-valued functions of 'homogenized' operator), we have

\[
\lim_{\epsilon \to 0} \| u_\epsilon (\cdot, t) - e^{-itA_\epsilon} \varphi_0^e u_0(\cdot, t) \|_{L_2(\mathbb{R})} = 0, \quad \lim_{\epsilon \to 0} \| v_\epsilon (\cdot, t) - \varphi_0^e v_0(\cdot, t) \|_{L_2(\mathbb{R})} = 0.
\]

7.2. Estimates (72), (88), (93), (98) are of interest, when the right-hand sides are small (i.e. when \( \epsilon |t|^{1/2} \) is small). If \( \epsilon |t|^{1/2} \leq 1 \), then the norms in the left-hand sides of (72), (88), (93), (98) are estimated by

\[
C \left( (1 + |t|^{q/3})\epsilon^{2q/3} \| f \|_{H^q(\mathbb{R})} + (1 + |t|^{(r+1)/3})\epsilon^{(2r+2)/3} \| g \|_{H^r(\mathbb{R})} \right)
\]

under the same assumptions.

7.3. With the help of the lower estimates formulated in Lemma 5.2, it can be proved that the results of Theorems 6.1–6.8 are sharp with respect to the norm type as well as with respect to the dependence on \( t \) (for large \( t \)). Let us show that for the example of the result of Theorem 6.1 (for \( q = 2 \)).

**Theorem 7.1:** Suppose that Conditions 4.1 and (36) are fulfilled. Let \( u_\epsilon \) be the solution of problem (54), and let \( u_0 \) be the solution of problem (55). Let \( t \neq 0 \) and \( 0 \leq q' < 2 \). Then there does not exist a constant \( C(t) > 0 \) such that the estimate

\[
\| u_\epsilon (\cdot, t) - e^{-itA_\epsilon} \varphi_0^e u_0(\cdot, t) \|_{L_2(\mathbb{R})} \leq C(t)\epsilon \| f \|_{H^{q'}(\mathbb{R})}
\]

(99) holds for all sufficiently small \( \epsilon > 0 \).

**Proof:** We prove by contradiction. Without loss of generality, it suffices to assume that \( 1 \leq q' < 2 \). Suppose that (99) is valid. By (56), it means that the estimate

\[
\left\| e^{-itA_\epsilon} \Psi^{(+)} - e^{-itA_\epsilon} \varphi_0^e e^{-itA_0^{bom}} \right\|_{H^{q'}(\mathbb{R}) \rightarrow L_2(\mathbb{R})} \leq C(t)\epsilon
\]

holds for all sufficiently small \( \epsilon \). By virtue of (59)–(66) (with \( q \) replaced by \( q' \)), we conclude that

\[
\left\| \Psi^* \left[ 2ie^{-it\epsilon^{-2}(\sigma_0+b_0k^2+(1/2)k^4\gamma_0(k))} \sin \left( \frac{1}{2}te^{-2k^4\gamma_0(k)} \right) \right. \right.
\]

\[
\times e^{q'(k^2 + \epsilon^2)^{-q'/2}} \chi_R(k) \left. \Phi \right|_{L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R})} \leq \tilde{C}(t)\epsilon
\]

with a constant \( \tilde{C}(t) > 0 \) for all sufficiently small \( \epsilon \). Note that the initial space of the operator \( \Psi^* \) coincides with \( \text{Ran} \Psi = L_2(\mathbb{R}) \). Obviously, the range of the operator

\[
\left[ 2ie^{-it\epsilon^{-2}(\sigma_0+b_0k^2+(1/2)k^4\gamma_0(k))} \sin \left( \frac{1}{2}te^{-2k^4\gamma_0(k)} \right) e^{q'(k^2 + \epsilon^2)^{-q'/2}} \chi_R(k) \right] \Phi
\]

is contained in the initial space of \( \Psi^* \). Together with lower estimate (45), this yields

\[
\frac{e^{q'/2}|t|^{q'/4}}{(|\gamma_0(0)|^{-1/2} + \epsilon |t|^{1/2})^{q'/2}} \leq \tilde{C}(t)\epsilon
\]

for all sufficiently small \( \epsilon \). But this is not true if \( q' < 2 \). This contradiction completes the proof. □
Theorem 7.2: Suppose that Conditions 4.1 and (36) are fulfilled. Let $u_{\varepsilon}$ be the solution of problem (54), and let $u_0$ be the solution of problem (55). Let $q' \geq 2$. Then there does not exist a function $C(t) > 0$ such that $t \to C(t)/|t|^{1/2}$ = 0 and estimate (99) holds for all $t \in \mathbb{R}$ and all sufficiently small $\varepsilon > 0$.

Proof: We also prove by contradiction. Repeating the same arguments that were used in the proof of Theorem 7.1, we obtain the estimates

$$\frac{\varepsilon^{q'/4} |t|^{q'/4}}{(|\gamma_0(0)|^{-1/2} + \varepsilon |t|^{1/2})^{q'/2}} \leq \tilde{C}(t) \varepsilon,$$

for $2 \leq q' \leq 4$,

$$|\gamma_0(0)| \varepsilon \leq \tilde{C}(t) \varepsilon,$$

for $q' > 4$,

for $t \neq 0$ and all sufficiently small $\varepsilon$, which can be rewritten as

$$\frac{\varepsilon^{q'/2-1} |t|^{q'/4-1/2}}{(|\gamma_0(0)|^{-1/2} + \varepsilon |t|^{1/2})^{q'/2}} \leq \frac{\tilde{C}(t)}{|t|^{1/2}},$$

for $2 \leq q' \leq 4$,

$$|\gamma_0(0)| \varepsilon \leq \frac{\tilde{C}(t)}{|t|^{1/2}},$$

for $q' > 4$,

where $\tilde{C}(t)$ is a positive function such that $\lim_{t \to \infty} \frac{\tilde{C}(t)}{|t|^{1/2}} = 0$. But this estimate is not true for large $|t|$ and sufficiently small $\varepsilon = |t|^{-1/2}$. This contradiction completes the proof. 

7.4. Throughout this paper, we have assumed that at least one of Conditions 4.1–4.4 is fulfilled. Now consider the case where some bands touch. For definiteness let $s \geq 3$ be an odd number and let $E_{s-1}(0) = E_s(0) =: \sigma_0$. According to the general analytic perturbation theory there exist real-analytic functions $\lambda_l(k), l = 1, 2$, (the branches of eigenvalues) and real-analytic $\mathcal{H}^1(0, 1)$-valued functions $\phi_l(\cdot, k), l = 1, 2$, (the branches of eigenfunctions) of the operator $\Pi$. Moreover, we have the following power series expansions in $k$

$$\lambda_l(k) = \sigma_0 + a_l k + b_l k^2 + \ldots,$$

$$\phi_l(x, k) = \phi_{0, l}(x) + \ldots,$$

convergent in a neighborhood of $k = 0$. Obviously, $E_s(k) = \lambda_n(k), v_s(x, k) = \phi_n(x, k), -\pi < k \leq 0$; and $E_s(k) = \lambda_m(k), v_s(x, k) = \phi_m(x, k), 0 \leq k < \pi$, for some numbers $n, m \in \{1, 2\}$.

Consider the following Cauchy problem for the nonstationary Schrödinger-type equation and the corresponding ‘homogenized’ problems:

$$\begin{cases}
  i \frac{\partial}{\partial t} u_\varepsilon^\infty(x, t) = (A_\varepsilon u_\varepsilon^\infty)(x, t), \\
  u_\varepsilon^\infty(x, 0) = (\Upsilon^\varepsilon(\cdot))(x),
\end{cases}$$

$$\begin{cases}
  i \frac{\partial}{\partial t} u_0(p)(x, t) = (A^\text{hom}(p) u_0(p))(x, t), \\
  u_0(p)(x, 0) = f(p)(x),
\end{cases}$$

where $\Upsilon^\varepsilon(\cdot)$ is the same as in (54); $A^\text{hom}(p) := -b_p \frac{\partial}{\partial x^p}, p \in \{n, m\}; f^{(n)} := \Phi^* [\chi(-\infty, 0)(k)] \Phi f$, $f^{(m)} := \Phi^* [\chi(0, \infty)(k)] \Phi f$. Then the analog of (57) looks as follows:

$$\|u_\varepsilon^\infty(\cdot, t) - e^{-it - \varepsilon^{-2} \sigma_0} \left( \phi^\varepsilon_{0,n} U^{(n)}(\cdot + \varepsilon^{-1} a_n t, t) + \phi^\varepsilon_{0,m} U^{(m)}(\cdot + \varepsilon^{-1} a_m t, t) \right) \|_{L^2(\mathbb{R})} \leq C(1 + |t|^{q/3}) \varepsilon^{q/3} \|f\|_{H^q(\mathbb{R})}, \quad 0 \leq q \leq 3.$$
7.5. Finally, throughout the paper we have assumed that (36) is fulfilled. This is the generic case. Now suppose that

$$\gamma_{k_0}(k) = (k - k_0)^{2m} \gamma_{k_0}(k), \quad \gamma_{k_0}(k_0) \neq 0, \quad m \geq 1.$$  

Then the statements of Theorems 6.1–6.8 can be improved. In particular, for $$f \in H^{q_1}(\mathbb{R}), q_1 = (m + 2)/(m + 1),$$ the norms in the left-hand sides of (57), (84), (91), (96) are estimated by

$$C(1 + |t|^{1/(2m+2)})\varepsilon \|f\|_{H^{q_1}(\mathbb{R})};$$  

and for $$f \in H^{q_2}(\mathbb{R}), g \in H^{q_3}(\mathbb{R}), q_2 = (2m + 3)/(2m + 2), q_3 = 1/(2m + 2),$$ the norms in the left-hand sides of (71), (87), (92), (97) are estimated by

$$C(1 + |t|^{1/(2m+2)})\varepsilon (\|f\|_{H^{q_2}(\mathbb{R})} + \|g\|_{H^{q_3}(\mathbb{R})}).$$

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**References**

[1] Bakhvalov NS, Panasenko GP. Homogenization: averaging processes in periodic media. Mathematical problems in mechanics of composite materials. Dordrecht: Kluwer Academic Publishers Group; 1989. (Mathematics and Its Applications (Soviet Series); Vol. 36).

[2] Bensoussan A, Lions J-L, Papanicolaou G. Asymptotic analysis for periodic structures. Amsterdam, North-Holland: 1978. (Studies in Mathematics and its Applications; Vol. 5).

[3] ZhikovVV,KozlovSM,OlejnikOA.Homogenizationofdifferentialoperators.Berlin:Springer-Verlag; 1994.

[4] BirmanMSh,SuslinaTA.Secondorderperiodicdifferentialoperators.Thresholdpropertiesandhomogenization.AlgebraAnal.2003;15(5):1–108.Englishtransl.,StPetersburgMathJ.2004;15(5):639–714.

[5] AllaireG.Homogenizationandtwo-scaleconvergence.SIAMJMathAnal.1992;23(6):1482–1518.

[6] Sevost’yanova EV. An asymptotic expansion of the solution of a second order elliptic equation with periodic rapidly oscillating coefficients. Mat Sb. 1981;115(2):204–222. English transl., Math USSR-Sb. 1982;43(2):181–198.

[7] ZhikovVV.Spectralapproximatoasymptoticdiffusionproblems.DifferUravn.1989;25(1):44–50.Englishtransl.,DifferEqu.1989;25(1):33–39.

[8] Conca C, Vanninathan M. Homogenization of periodic structures via Bloch decomposition. SIAM J Math Anal. 1992;23(6):1482–1518.

[9] Suslina TA. On homogenization of periodic parabolic systems. Funktsional Anal Prilozhen. 2004;38(4):86–90. English transl., Funct Anal Appl. 2004;38(4):309–312.

[10] Suslina TA. Homogenization of a periodic parabolic Cauchy problem. Amer Math Soc Transl (2). 2007;220:201–233.

[11] Birman MSh, Suslina TA. Homogenization with corrector term for periodic elliptic differential operators. Algebra Anal. 2005;17(6):1–104. English transl., St Petersburg Math J. 2006;17(6):897–973.
[14] Birman MSh, Suslina TA. Homogenization with corrector term for periodic differential operators. Approximation of solutions in the Sobolev class $H^1(\mathbb{R}^d)$. Algebra Anal. 2006;18(6):1–130. English transl., St Petersburg Math J. 2007;18(6):857–955.

[15] Vasilevskaya ES. A periodic parabolic Cauchy problem: homogenization with corrector. Algebra Anal. 2009;21(1):3–60. English transl., St Petersburg Math J. 2010;21(1):1–41.

[16] Suslina TA. Homogenization of a periodic parabolic Cauchy problem in the Sobolev space $H^1(\mathbb{R}^d)$. Math Model Nat Phenom. 2010;5(4):390–447.

[17] Zhikov VV. On some estimates of homogenization theory. Dokl Ros Akad Nauk. 2006;406(5):597–601. English transl., Dokl Math. 2006;73:96–99.

[18] Zhikov VV, Pastukhova SE. On operator estimates for some problems in homogenization theory. Russ J Math Phys. 2005;12(4):515–524.

[19] Zhikov VV, Pastukhova SE. Estimates for operator estimates for a parabolic equation with periodic coefficients. Russ J Math Phys. 2006;13(2):224–237.

[20] Zhikov VV, Pastukhova SE. Operator estimates in homogenization theory. Uspekhi Matem Nauk. 2016;71(3):27–122. English transl., Russ Math Surv. 2016;71(3):417–511.

[21] Birman MSh, Suslina TA. Operator error estimates in the homogenization problem for nonstationary periodic equations. Algebra Anal. 2008;20(6):30–107. English transl., St Petersburg Math J. 2009;20(6):873–928.

[22] Meshkova YM. On operator error estimates for homogenization of hyperbolic systems with periodic coefficients. J Spectr Theory. 2021;11(2):587–660.

[23] Suslina TA. Spectral approach to homogenization of hyperbolic equations with periodic coefficients. Funktsional Anal Prilozhen. 2022;56(3):93–99. English transl.: Funct Anal Appl. 2022;56(3):229–234.

[24] Birman MSh, Suslina TA. Operator error estimates in the homogenization problem for nonstationary periodic equations. Algebra Anal. 2008;20(6):30–107. English transl., St Petersburg Math J. 2009;20(6):873–928.

[25] Suslina TA. Homogenization of the Schrödinger-type equations: operator estimates with correctors. Funktsional Anal Prilozhen. 2022;56(3):93–99. English transl.: Funct Anal Appl. 2022;56(3):229–234.

[26] Barletti L, Ben Abdallah N. Quantum transport in crystals: effective mass theorems and homogenization of wave equations. J Eur Math Soc. 2022;24(9):3031–3053.

[27] Craster RV, Kaplunov J, Pichugin AV. High-frequency homogenization for periodic media. Proc R Soc A. 2010;466(2120):2341–2362.

[28] Harutyunyan D, Milton GW, Craster RV. High-frequency homogenization for travelling waves in periodic media. Proc R Soc A. 2016;472(2191):20160066.

[29] Ceresoli L, Abdeddaim R, Antonakakis T, et al. Dynamic effective anisotropy: simulations, and microwave experiments with dielectric fibers. Phys Rev B. 2015;92(17):174307.

[30] Allaire G. Periodic Homogenization and Effective Mass Theorems for the Schrödinger Equation. In: Abdallah NB, Frosali G, editors. Lecture Notes in Mathematics. Quantum Transport. Springer; 2008. p. 1–44.

[31] Allaire G, Piatnitski A. Homogenization of the Schrödinger equation and effective mass theorems. Comm Math Phys. 2005;258(1):1–22.

[32] Kuchment P, Raich A. Green’s function asymptotics near the internal edges of spectra of periodic elliptic operators. spectral edge case. Math Nachr. 2012;285(14-15):1880–1894.

[33] Kha M, Kuchment P, Raich A. Green’s function asymptotics near the internal edges of spectra of periodic elliptic operators. spectral gap interior. J Spectr Theory. 2017;7(4):1171–1233.

[34] Birman MSh. On homogenization procedure for periodic operators near the edge of an internal gap. Algebra Anal. 2003;15(4):61–71. English transl., St Petersburg Math J. 2004;15(4):507–513.

[35] Suslina TA, Kharin AA. Homogenization with corrector for a periodic elliptic operator near an edge of inner gap. Problemy Mat Analiza. 2009;41:127–141. English transl., J Math Sci. 2009;159(2):264–280.

[36] Mishulovich AA, Slouschch VA, Suslina TA. Homogenization of a one-dimensional periodic elliptic operator at the edge of a spectral gap: operator estimates in the energy norm. Zap Nauchn Sem POMI. 2022;519:114–151. English transl.: J Math Sci. Forthcoming.

[37] Akhmatova AR, Aksenova ES, Slouschch VA, et al. Homogenization of the parabolic equation with periodic coefficients at the edge of a spectral gap. Complex Var Elliptic Equ. 2022;67(3):523–555.
[42] Birman MSh, Suslina TA. Homogenization of a multidimensional periodic elliptic operator in a neighborhood of the edge of an internal gap. Zap Nauchn Sem POMI. 2004;318:60–74. English transl., J Math Sci. 2006;136(2):3682–3690.

[43] Suslina TA, Kharin AA. Homogenization with corrector for a multidimensional periodic elliptic operator near an edge of an inner gap. Problemy Mat Analiza. 2011;59:177–193. English transl., J Math Sci. 2011;177(1):208–227.

[44] Mishulovich AA. Homogenization of the multidimensional parabolic equations with periodic coefficients at the edge of a spectral gap. Zap Nauchn Sem POMI. 2022;516:135–175. English transl.: J Math Sci. Forthcoming.

[45] Dorodnyi MA. High-energy homogenization of a multidimensional nonstationary Schrödinger equation. arXiv:2301.05907 [Preprint]. 2023 [cited 2023 Feb 25]: [26 p.]. Available at https://arxiv.org/abs/2301.05907.

[46] Kirsch W, Simon B. Comparison theorems for the gap of Schrödinger operators. J Funct Anal. 1987;75(2):396–410.

[47] Reed M, Simon B. Methods of modern mathematical physics, vol. 4: analysis of operators. New York: Academic Press; 1978.

[48] Dunford N, Schwartz JT. Linear operators, part 1: general theory. New York: Interscience Publishers; 1958.