Constraints on $f(R_{ijkl} R^{ijkl})$ gravity: evidence against the covariant resolution of the Pioneer anomaly

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Abstract

We consider corrections in the form of $\Delta L(R_{ijkl} R^{ijkl})$ to the Einstein–Hilbert Lagrangian. Then we compute the corrections to the Schwarzschild geometry due to the inclusion of this general term to the Lagrangian. We show that $\Delta L_3 = \alpha_3 (R_{ijkl} R^{ijkl})^3$ gives rise to a constant anomalous acceleration for objects orbiting the Sun, toward the Sun. This leads to the conclusion that $\alpha_3 = (13.91 \pm 2.11) \times 10^{-26} \left(\frac{\text{meter}}{1}\right)^3$ would have covariantly resolved the Pioneer anomaly if this value of $\alpha_3$ had not contradicted other observations.

We note that the experimental bounds on $\Delta L_3$ grow stronger in case we examine the deformation of the spacetime geometry around objects lighter than the sun. We therefore use the high precision measurements around the Earth (LAGEOS and LLR) and obtain a very strong constraint on the corrections in the form of $\Delta L(R_{ijkl} R^{ijkl})$ and in particular $\Delta L = a_n (R_{ijkl} R^{ijkl})^n$. This bound requires $\alpha_3 \leq 6.12 \times 10^{-29} \left(\frac{\text{meter}}{1}\right)^3$. Therefore, it refutes the covariant resolution of the Pioneer anomaly.

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(Some figures in this article are in colour only in the electronic version)

1. Dynamics of the empty spacetime geometry

As we currently understand nature, matters\(^1\) deform their surrounding spacetime geometry, and gravity is a side effect of this deformation.

The simplest secular action capable of describing the deformation around matters’ distribution, or equivalently the dynamics of the spacetime, is the Einstein–Hilbert action:

$$S_{\text{EH}} = \int d^4x \sqrt{-\det g} R,$$

\(^1\) Non-vanishing energy–momentum tensor.
which happens to provide a very good phenomenological description of the gravitational phenomena within the solar system. Let us describe the solar system itself in a covariant language. In doing so we note that the spacetime geometry within the solar system possesses the following covariant characteristics:

1. An almost vanishing Ricci scalar and tensor outside the world lines of the Sun and planets,
   \[ R = 0, \quad R_{\mu\nu} = 0. \]

2. \( R_{\mu\nu\lambda\eta} R^{\mu\nu\lambda\eta} \geq 7.53 \times 10^{-71} \text{ (meters)}^4. \)

It should be highlighted that the first of the above-assigned properties to the solar system is not exact due to the cosmological constant. The presence of a cosmological constant of the order \( \Lambda \approx 10^{-120} \text{ m}^{-4} \) seems to be the most economical description of the expansion of the universe which has been indicated by the distance–redshift relation. Ignoring the cosmological constant, however, seems to be a legitimate approximation when \( R_{ijkl} R^{ijkl} \gg \Lambda^2 = 10^{-104} \text{ m}^{-4} \) holds. Since \( R_{ijkl} R^{ijkl} \gg \Lambda^2 \) is violated beyond 10AU (astronomical units) from the Sun, a Ricci-flat geometry approximation to the geometry in the solar system should be indeed very accurate.

Note that the Newtonian gravitational interaction is the leading term in the effective gravitational interaction assigned to (1). If we had been interested only in the effective gravitational interaction then we could have simply introduced some desired distance-dependent terms into the phenomenological effective gravitational potential. ‘Distance’, however, is not a covariant quantity. We adhere to the standpoint that allows only generally covariant modification to (1). Once we add a modification, we can address what the modified term implies for the geometry around the Sun, Earth and even for time-dependent solutions. This perhaps enables us to employ the high precision measurements around the Earth to test the validity of a covariant correction that is suggested by some accurate observations around the Sun.

From a phenomenological standpoint, the Einstein–Hilbert action is only a model compatible with observational quantities and experimental data. Unborn or approaching finer observations and preciser experiments may lead to some corrections to the Einstein–Hilbert action. These corrections can be any scalar constructed from the Riemann tensor and its covariant derivatives.

From the theoretical standpoint, the simplest example of the corrections to the action we may consider is an arbitrary functional of the Ricci scalar and Ricci tensor:

\[
L = R + \epsilon \Delta L + O(\epsilon^2),
\]

where \( \epsilon \) encodes the perturbative expansion. The Lagrangian given by (2) leads to the following equations of motion:

\[
R_{\mu\nu} + \epsilon k g_{\mu\nu} f(R, R_{ij}) + \epsilon O_{\mu\nu} \left[ \frac{\partial f(R, R_{ij})}{\partial R}, \frac{\partial f(R, R_{ij})}{\partial R_{ijkl}} \right] = O(\epsilon^2),
\]

where \( O_{\mu\nu} \) is a linear operator acting on its arguments, and \( k \) is a constant number. We should solve (3) in a perturbative fashion:

\[
g_{\mu\nu} = g_{\mu\nu}^{(0)} + \epsilon g_{\mu\nu}^{(1)} + O(\epsilon^2),
\]

This standpoint also allows us to address what observed history of the cosmological evolution of the universe implies on the time-independent solutions. This suggests that what causes the cosmological inflationary paradigm provides a lower bound on the mass of black holes [1].

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where $g_{\mu\nu}^{(0)}$ is a Ricci flat metric. Inserting (4) in (3) leads to

$$\Box^{(0)}(g_{\mu\nu}^{(1)}) + kf(R, R_{ij})|_{R_{ij}=0} g_{\mu\nu}^{(0)} + O_{\mu\nu} \left[ \frac{\partial f(R, R_{ij})}{\partial R} \bigg|_{R_{ij}=0}, \frac{\partial f(R, R_{ij})}{\partial R_{ij}} \bigg|_{R_{ij}=0} \right] = 0,$$

where $\Box^{(0)}$ is a linear second-order differential operator. When $f(R, R_{ij})$ has an expansion in terms of its variable around $R_{ij}=0$ then its partial derivative with respect to $R$ or $R_{ij}$ either vanishes or diverges at $R_{ij}=0$. If the partial derivatives vanish then a perturbative solution exists. For such case since $O_{\mu\nu}$ is a linear operator then (5) simplifies to

$$\Box^{(0)}(g_{\mu\nu}^{(1)}) + kf(0, 0)g_{\mu\nu}^{(0)} = 0.$$  

Note that the above equation could have been obtained from the variation of the Einstein–Hilbert action in the presence of a tiny cosmological constant,

$$S = \int d^4x \sqrt{-\det g(R + \epsilon/\Lambda) + O(\epsilon^2)},$$

where $\Lambda \propto f(0, 0)$. The exact solutions of (7) are known. So the inclusion of the corrections in the form of a functional of the Ricci scalar and the Ricci tensor does not give rise to not-yet investigated perturbative corrections to the spacetime geometry around the Sun. The same conclusion holds for any functional of the Ricci scalar, Ricci tensor and their covariant derivatives provided the functional has an expansion in terms of its variables around a Ricci flat geometry.

A correction to the action thus would ‘non-trivially’ perturb the spacetime geometry around a Ricci flat geometry in case the correction involves the Riemann tensor per se. The simplest of such corrections is perhaps an arbitrary functional of the Riemann tensor squared:

$$L = R + \epsilon/\Delta L + O(\epsilon^2),$$

$$\Delta L = \Delta L(R_{ijkl} R_{ijkl}).$$

Examples of this form of correction include

$$\Delta L_1 = \alpha_1 R_{ijkl} R_{ijkl},$$
$$\Delta L_2 = \alpha_2 (R_{ijkl} R_{ijkl})^2,$$
$$\Delta L_3 = \alpha_3 (R_{ijkl} R_{ijkl})^3. $$

Having modified the dynamics of the spacetime in small distances, the heterotic and type I string theories lead to corrections to the Einstein–Hilbert action which includes $\Delta L_1$. Note that $\alpha_2$ is a dimensionless parameter, a constant number. Thus $\alpha_2$ represents a possible structure constant for the spacetime geometry. On the other hand, the dimension of $\alpha_1$ is negative with respect to the dimension of $\alpha_2$. $\Delta L_3$ thus may require a modification of the dynamics of the spacetime in very small Riemann curvatures.

This work aims to identify the best experimental limits, and observational bounds on $\Delta L(R_{ijkl} R_{ijkl})$ and especially $\Delta L = \alpha_n (R_{ijkl} R_{ijkl})^n$ regardless of what theory governs the

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3 Examples: the partial derivative of $f(R) = R^2$ vanishes at $R = 0$ while the partial derivative of $f(R) = R^\frac{3}{2}$ diverges.

4 Besides, the solar and cosmological constraints on $f(R)$ or $R^n$ make them not attractive [21].

5 In the compactification of type II string theories, we also get non-perturbative world-sheet corrections in the form of $L_1$. 

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dynamics of the spacetime in a very high or low but not-yet achieved curvature. The work will be organized in the following order:

In the second section, we will add a general correction in the form of $\Delta L(R_{ijkl} R^{ijkl})^{\frac{1}{2}}$ to the Einstein–Hilbert Lagrangian. After that we will compute the corrections to the Schwarzschild geometry due to the inclusion of this general term in the Lagrangian.

In the third section, we will obtain the effective modified Newtonian gravitational potential for spacecraft in the spherical and static extrema of $R + \epsilon \Delta L + O(\epsilon^2)$. We show that $\Delta L = \alpha_{\frac{1}{2}}(R_{ijkl} R^{ijkl})^{\frac{1}{2}}$ culminates in a constant anomalous acceleration for the objects orbiting the Sun, toward the Sun. This leads us to the conclusion that $\alpha_{\frac{1}{2}} = (13.91 \pm 2.11) \times 10^{-26} \left(\frac{\text{meters}}{\text{meters}}\right)^{\frac{1}{2}}$ would have covariantly resolved the Pioneer anomaly if this value of $\alpha_{\frac{1}{2}}$ had not contradicted other observations. We note that the experimental bounds for such correction grow stronger for the spacetime geometry around objects lighter than the Sun.

In the fourth section, we will utilize the high precision measurements around the Earth (LAGEOS and LLR) to obtain a strong limit on the corrections in the form of $\Delta L = \alpha_n(R_{ijkl} R^{ijkl})^n$. This bound requires $\alpha_{\frac{1}{2}}$ to be smaller than $6.12 \times 10^{-29} \left(\frac{\text{meters}}{\text{meters}}\right)^{\frac{1}{2}}$; therefore, it clearly refutes the covariant resolution of the Pioneer anomaly.

In the fifth section, we will note that $\Delta L_2 = \alpha_{\frac{1}{2}}(R_{ijkl} R^{ijkl})^{\frac{1}{2}}$ gives rise to an effective logarithmic gravitational potential. We will discuss whether $\Delta L_2 = \alpha_{\frac{1}{2}}(R_{ijkl} R^{ijkl})^{\frac{1}{2}}$ can describe the anomalous flat rotational velocity curves of the spiral galaxies. We show that a simple correction in the form of $\Delta L_2 = \alpha_{\frac{1}{2}}(R_{ijkl} R^{ijkl})^{\frac{1}{2}}$ is neither in agreement with the high precision measurements around the Earth, nor fails to describe the flat rotational velocity curves of the spiral galaxies.

In the last section, we will provide a summary of the results.

2. Generic but simplest corrections around the Sun

The Schwarzschild metric is an isotropic and static solution to the Einstein–Hilbert action. In four dimensions, in the standard preferred coordinates, it reads

$$ds^2 = -c^2 \left(1 - \frac{r_h}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_h}{r}} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$

(10)

where $r_h = \frac{2Gm}{c^2}$ in which $G$ is the Newton constant, $m$ represents the mass, and $c$ stands for the speed of light. We are interested in the spacetime geometry around the Sun; we thus set $m = M_\odot = 1.98 \times 10^{30}$ kg for which $r_h \equiv r_\odot \sim 3$ km.

As argued in the first section, since the Einstein tensor vanishes for the Schwarzschild metric, the geometry around the Sun receives corrections in case the corrections to the action involve the Riemann tensor per se. We consider the simplest form of these corrections, those which are generic functional of the Riemann tensor’s square:

$$S \equiv \int d^4x L[g_{\mu\nu}, R_{\mu\nu\rho\sigma}] + O(\epsilon^2),$$

(11a)

$$L[g_{\mu\nu}, R_{\mu\nu\rho\sigma}] = \sqrt{-\det g} (R + \epsilon \mathcal{L}[R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}]),$$

(11b)

where $\epsilon$ is the parameter of the expansion, and $\mathcal{L}$ is a generic functional. Computing the first variation of (11) with respect to the metric, we obtain [4]

$$0 = -\frac{\partial L}{\partial g_{ij}} - \frac{\partial L}{\partial R_{\mu\nu\rho\sigma}} R^j_{\mu\nu\rho\sigma} - 2\nabla_\mu \nabla_\nu \frac{\partial L}{\partial R_{\mu\nuij}},$$

(12)
where partial derivatives are taken assuming that $g_{\mu\nu}$ and $R_{\mu\nu\alpha\beta}$ are independent variables, and the partial derivative coefficients appearing in (12) are uniquely fixed to have precisely the same tensor symmetries as the varied quantities. Note that our conventions are such that $R_{\phi\theta\phi\theta}$ as well as the Ricci curvature scalar is positive for the standard metric on the 2-sphere.

We then insert the explicit form of $\mathcal{L}$ presented in (11) into (12) to obtain

\[ E_{ij}^{(0)} \equiv \text{rhs of } (12) = E_{ij}^{(0)} + \epsilon E_{ij}^{(1)} + O(\epsilon^2), \]  

(13a)

\[ E_{ij}^{(0)} = R_{ij} - \frac{1}{2} g_{ij} R, \]  

(13b)

\[ E_{ij}^{(1)} = 2R^\rho_{i\alpha\beta} R^\rho_{j\alpha\beta} \frac{\partial \mathcal{L}[x]}{\partial x_i \partial x_j} - \frac{1}{2} R_{ij} \mathcal{L}(\mathcal{R}^2) - 4 \nabla^\alpha \nabla^\beta \left( \frac{\partial \mathcal{L}[x]}{\partial x_i} \right) R^\rho_{\alpha\beta} R^\rho_{i\alpha\beta} \mathcal{L}(\mathcal{R}^2), \]  

(13c)

whereafter $\mathcal{R}^2 \equiv R_{\mu\nu\rho\gamma} R^{\mu\nu\rho\gamma}$ is inferred.

In the following lines from the outset, we consider the perturbations around the Sun in the standard preferred coordinate:

\[ ds^2 = -A(r)c^2 dt^2 + B(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]  

(14a)

\[ A(r) = \left( 1 - \frac{r_0}{r} \right) (1 + \epsilon a(r)) + O(\epsilon^2), \]  

(14b)

\[ B(r) = \frac{1}{A(r)} \left( 1 + \epsilon b(r) \right) + O(\epsilon^2). \]  

(14c)

Now let us utilize

\[ \mathcal{R}^2 \simeq \frac{12r_0^2}{r^6} + \ldots \]  

(15)

and define

\[ L(r) \equiv \mathcal{L}[\mathcal{R}^2]. \]  

(16)

\[ \tilde{L}(r) \equiv \frac{\partial \mathcal{L}[x]}{\partial x_i} \bigg|_{x=x^2} = -\frac{r^7}{72r_0^2} L'(r) + O(\epsilon). \]  

(17)

It follows that the non-vanishing and independent components of (13) for (14) are

\[ \frac{1}{c^2} E_{tt}^{(0)} = \frac{\epsilon}{r^2} \left( 1 - \frac{r_0}{r} \right) (r - r_0) (a' + b') - a + b) + O(\epsilon^2), \]  

(18a)

\[ E_{rr}^{(0)} = \frac{\epsilon}{r(r-r_0)} ((r-r_0)a' + a - b) + O(\epsilon^2), \]  

(18b)

\[ E_{\theta\theta}^{(0)} = \frac{\epsilon r}{2} \left( (r-r_0)a'' + 2a' - b' + \frac{b' r_0}{2r} \right) + O(\epsilon^2), \]  

(18c)

and

\[ \frac{E_{tt}^{(1)}}{c^2 (r-r_0)} = -\frac{2r_0}{r^6} (2r (r-r_0) \tilde{L}'' - (2r - 3r_0) \tilde{L}') + \frac{r L' + 6 L}{12r} + O(\epsilon^2), \]  

(19)

\[ (r-r_0) E_{rr}^{(1)} = \frac{r^2 L' + 6r L}{12} - \frac{2r_0 (-3r_0 + 2r)}{r^4} L' + O(\epsilon^2) \]  

(20)
\[ E_{b_{\mu
u}}^{(1)} = -\frac{2}{r^3}(r(r - r_\odot)\dddot{L} - 2(r - 3r_\odot)\ddot{L}), \quad -\frac{r\dddot{L} + 6L}{12} r^2 + O(\epsilon^2), \] (21)

where the dependence of the various functions on \( r \) is understood. Then (13) gives rise to a non-homogeneous second-order differential equation for \( a(r) \) and a non-homogeneous first-order differential equation for \( b(r) \). In accordance with the preceding studies of string world-sheet corrections to various black holes [5–7], we demand that the corrections must not diverge on the possible event horizon at \( r = r_\odot \), provided that the contribution of \( L(R^2) \) remains bounded on the horizon. We find out that this precondition is satisfied by the following solution:

\[ b(r) = -4r_\odot \int_{r_\odot}^{r} \frac{\dddot{L}(x)}{x^2}, \quad \] (22a)

\[ a(r) = -\frac{1}{2(r - r_\odot)} \int_{r_\odot}^{r} \frac{dx}{x^4} \left[ -12r_\odot^2 (x \ddot{L}(x) + \dot{L}(x)) + 8r_\odot x^2 \ddot{L}(x) + x^6 L(x) + 2x^4 b(x) \right], \quad \] (22b)

The above boundary conditions also reproduce the correct results around ordinary stars. In order to illustrate this claim, we note that we are solving the equations of motion in a perturbative fashion. We note that \( R_{ijkl} R^{ijkl} \) inside the Sun is larger than \( R_{ijkl} R^{ijkl} \) outside the Sun. Since the perturbation that we are interested in grows as the Riemann curvature decreases (see the end of third section) then the perturbation holds valid inside the Sun. Let (4) represent the perturbative solution. Inserting this perturbative expansion in the equation of motion leads to a second-order non-homogeneous differential equation for \( g^{(1)}_{\mu\nu} \). Let us separately write the equations of motion of \( g^{(1)}_{\mu\nu} \) for inside and outside the matter’s distribution:

\[ \square^{(0)} \cdot g_{m}^{(1)} = F[g_{m}^{(0)}], \] (23)

\[ \square^{(0)} \cdot g_{out}^{(1)} = F[g_{out}^{(0)}], \] (24)

where indices are understood but not written, and \( F \) stands for the non-homogeneous part. Outside the star we have \( g_{out}^{(0)} = g_{Schwarzschild} \). The consistency of the perturbation requires that \( g_{in}^{(1)} \) remains bounded inside the star. Requiring the existence of the metric and its first derivatives on the surface of the star then provides the physical boundary condition for \( g_{out}^{(1)} \). This boundary condition, however, requires first solving the equations inside the star. Therefore, a method capable of reproducing the physical boundary condition (which does not require solving the equations inside the star) is appreciated.

Since the equation for \( g_{out}^{(1)} \) is linear, \( g_{out}^{(1)} \) for a star reads

\[ g_{out}^{(1)} = c_1 g_{out}^{(1)} + c_2 g_{out}^{(1)} + g_{non\text{-}homogeneous} \] (25)

where \( g_{out}^{(1)} \) and \( g_{out}^{(1)} \) are solutions of the homogeneous part of (24) that respectively diverge or converge when they are extrapolated toward the Schwarzschild radius associated with the central mass. \( c_1 \) and \( c_2 \) should be chosen such that the metrics inside and outside the star match each other. In the Einstein–Hilbert action, and for a star that is formed by collapse of dust, \( c_1 \) and \( c_2 \) can only depend on the total central mass: \( c_1 = c_1(M), \) \( c_2 = c_2(M) \). On the other hand, \( c_1 \) and \( c_2 \) are numbers while \( M \) is parameter with a dimension. Therefore, \( c_1 \) and \( c_2 \) must be independent of the mass of the star. Subsequently, once a physical criterion fixes \( c_1 \) and \( c_2 \) for a spherical mass distribution then they are fixed for every spherical time-independent distribution of mass. Since \( c_1 = 0, \) \( c_2 = 1 \) for a large black hole then it holds \( c_1 = 0, \) \( c_2 = 1 \) for the Sun. We thus conclude that (22) also describes the perturbation around the Sun.
It is worth noting that though the precondition we employed does not fix all the boundary conditions, it fixes the radial-dependent part of $a$ and $b$. The constant parts in $a$ and $b$ do not affect the force exerted onto a spacecraft, the quantity we need in the following sections. In particular, for $\epsilon \mathcal{L}[\mathcal{R}^2] = a_n(R_{\mu\nu\lambda\eta}R^{\mu\nu\lambda\eta})^n$, the large radius behavior of $a(r)$ and $b(r)$ simplifies to

$$a(r) \simeq -4(12)^n n(1 - n)\alpha_n r^{-4n} \left(\frac{r}{r_\odot}\right)^{3 - 6n} - \frac{1}{3 - 6n}, \quad (26a)$$

$$b(r) \simeq -\left(\frac{5}{2} - 3n\right) a(r), \quad (26b)$$

where $n \neq \frac{1}{2}$ is presumed. For $n = \frac{1}{2}$ one finds

$$a(r) \simeq b(r) \simeq -\frac{1}{2} \alpha_\frac{1}{2} \ln \left(\frac{r}{r_\odot}\right). \quad (27)$$

For the sake of simplicity, we will consider only the corrections in the form of $\epsilon \mathcal{L}[\mathcal{R}^2] = a_n(R_{\mu\nu\lambda\eta}R^{\mu\nu\lambda\eta})^n$ but the results can be directly generalized to a general case. We also note that the perturbations increase as we go further away from the sun. We, however, shall address the regime of the validity of the perturbation in the fourth section after having independently obtained the experimental bounds on $a_n$.

### 3. Effective potential and the covariant resolution of the Pioneer anomaly

The effective potential for spacecrafts in the spherical and static geometry of (14) is [8]

$$V_{\text{eff}}(r) = \frac{l^2}{2r^2 B(r)} - \frac{E^2}{2c^2 A(r) B(r)} + \frac{c^2}{2B(r)}, \quad (28)$$

where the equatorial plane is chosen to be orthogonal to the angular momentum, $E$ stands for the energy (per unit rest mass), and $l$ represents the magnitude (per unit rest mass) of the angular momentum.

Inserting the $\epsilon$ perturbative expansion series of $A(r)$ and $B(r)$ into (28) yields

$$V_{\text{eff}}(r) = V_{\text{eff}}^{(0)}(r) + \epsilon V_{\text{eff}}^{(1)}(r) + O(\epsilon^2), \quad (29a)$$

$$V_{\text{eff}}^{(0)}(r) = -\frac{GM_\odot}{r} + \frac{l^2}{2r^2} = \frac{GM_\odot l^2}{r^3c^2}, \quad (29b)$$

$$V_{\text{eff}}^{(1)}(r) = \frac{E^2}{2c^2} - b(r) + \frac{c^2}{2} a(r) + O \left(\frac{l^2}{2r^2}\right), \quad (29c)$$

where a constant term is tacitly recognized in (29c). We note that $V_{\text{eff}}^{(0)}(r)$ is the effective potential for the Einstein–Hilbert action. Thus it is $\epsilon V_{\text{eff}}^{(1)}(r)$ which gives rise to the anomalous accelerations for spacecrafts deployed to explore the outer solar system.

The Pioneers 10 and 11, spacecrafts deployed for exploring the outer solar system, are reported to have experienced a constant anomalous acceleration of magnitude $a_p \equiv (8.74 \pm 1.33) \times 10^{-16} \text{ m s}^{-2}$ in the direction toward the Sun at a distance of 20–70 AU from

6 It is not legitimate to extrapolate (26) toward far infinity since it has been obtained after expanding the equations for regions of the spacetime which meets $R_{\mu\nu\lambda\eta}R^{\mu\nu\lambda\eta} > \frac{7.53 \times 10^{-74} \text{ m}^6}{\text{kg}^2}$. 

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the Sun\textsuperscript{7} [2]. What causes this anomaly might be on-board systematics, but the smoking gun has not been found yet [3]. Let us see if a correction in the form of $\Delta L = L(R_{ijkl}R^{ijkl})$ may resolve the Pioneer anomaly.

The effective potential culminating to the observed constant anomalous acceleration of the Pioneers is

\begin{equation}
\epsilon V^{(i)}_{\text{eff}}(r) = -a_p r. \tag{30}
\end{equation}

From now on we assume that (30) is valid for $r \in 0–70$ AU.

The anomalous acceleration is obtained by analyzing the Doppler shift of the electromagnetic wave that the Pioneers had been sending assuming that the spacetime geometry around the Sun coincides to the Schwarzschild geometry. Deviation from the Schwarzschild geometry affects the gravitational red/blue shifts and subsequently alters the anomalous acceleration assigned to the Pioneers [9–11]\textsuperscript{8}. We first assume that the deviation from the Schwarzschild geometry needed to describe the Pioneer anomaly does not significantly affect the gravitational red/blue shifts of the signals sent by the spacecrafts. We confirm this assumption after having identified the deviation from the Schwarzschild geometry.

If the Pioneer effective potential (30) is due to gravitational effects of the Sun, then (29) and (30) result

\begin{equation}
\epsilon \left( \frac{E^2 - c^4}{2c^2} b(r) + \frac{c^2}{2} a(r) \right) \simeq -a_p r. \tag{31}
\end{equation}

One notes that the Pioneers have classical velocity and $\frac{E^2 - c^4}{2c^2} \simeq \frac{c^2}{4} \ll \frac{c^2}{c^2}$ thereafter $v$ stands for the radial velocity of the spacecrafts with respect to the Sun. Therefore, (31) can be further approximated to

\begin{equation}
\epsilon a(r) = -\frac{2a_p}{c^2} r + O \left( \frac{v^2}{c^2} \right), \tag{32}
\end{equation}

where rearranging the terms is induced. Note that in these distances it holds $\epsilon a(r) \ll \frac{v^2}{c^2}$ and the approximation in (32) is much lesser than the error bar in $a_p$.

Equation (32) identifies the deviation from the Schwarzschild geometry. In the following, we are going to argue that this deviation does not significantly alter the gravitational Doppler shifts to which the anomalous acceleration is assigned. The Pioneer spacecraft\textsuperscript{9} sends a photon at time $t$ in $x = \tilde{r}_{\text{Pioneer}}$ of frequency $\nu_1$ in the local frame that is fixed at $\tilde{r}_{\text{Pioneer}}$. The detector on the Earth receives the photon on time $t + T(t)$ with a frequency of $\nu_2$ in the local from that is fixed at $x = \tilde{r}_{\text{Detector}}$; see figure 1. $T(t)$ can be identified in terms of the initial position of the spacecraft, Earth and the detector.

In the absence of gravity we assign energy of $\hbar \nu_1$ to a photon related to light’s wave of frequency $\nu_1$. In the presence of the weak gravity, therefore, we should assign an energy of $\hbar \nu + V_{\text{eff}} \frac{\hbar \nu}{c^2}$ to the total energy for the photon of frequency $\hbar \nu$, where $V_{\text{eff}}$ is the effective gravitational potential where the photon is localized. Requiring the conservation of energy for the photon sent by the Pioneer yields

\begin{equation}
\nu_1 \left( 1 + \frac{1}{c^2} V_{\text{eff}}[r_{\text{Detector}(t+T)}] \right) = \nu_2 \left( 1 + \frac{1}{c^2} V_{\text{eff}}[r_{\text{Pioneer}(t)}] \right) \implies \frac{\nu_1 - \nu_2}{\nu_2} = \frac{1}{c^2} (V_{\text{eff}}[r_{\text{Pioneer}(t)}] - V_{\text{eff}}[r_{\text{Detector}(t+T)}]). \tag{33}
\end{equation}

\textsuperscript{7} The major part in the error bar of $a_p$ is systematic. The statistical error is only $0.01 \times 10^{-10}$ m s\textsuperscript{-2}.

\textsuperscript{8} I appreciate the comment made by the referee of CQG that led to adding a discussion on this issue.

\textsuperscript{9} The deep space network antennas tracked the Pioneer spacecrafts with a S-band signal at about 2.11 GHZ. The tracking was done by sending a signal from the Earth which the Pioneers were replying. The simple one-way Doppler shift analysis of this section, however, suffices for our conclusions.
Figure 1. The acceleration of the Pioneer spacecrafts is identified by analyzing the Doppler shift of the spacecrafts’ emission. The gravitational red/blue shifts are part of this analysis. Altering the geometry affects the gravitational red/blue shifts. Therefore, the back reaction of the change of geometry to the constant anomalous acceleration should be addressed if the anomaly is due to a covariant correction.

Employing (29) then yields

$$\frac{c^2 \Delta v}{v} = V_{\text{eff}}^{(0)}[r_{\text{Detector}}(t + T)] - V_{\text{eff}}^{(0)}[r_{\text{Pioneer}}(t)] + \frac{\epsilon}{2} [a[r_{\text{Detector}}(t + T)] - a[r_{\text{Pioneer}}(t)]].$$  \hspace{1cm} (34)

We note that the time that a photon needs to travel from the spacecraft to the detector also has an $\epsilon$ expansion:

$$T = T^{(0)} + \epsilon T^{(1)} + O(\epsilon^2),$$ \hspace{1cm} (35)

where $r_D = r_{\text{Detector}}$. Inserting (35) in (33) leads to

$$\left( \frac{\Delta v}{v} \right)_{\text{missing}} = \frac{\epsilon}{c^2} \frac{\partial V_{\text{eff}}^{(0)}(r)}{\partial r} \left|_{r=r_D} \right. \frac{\partial r_D(t)}{\partial t} T^{(1)} + \frac{\epsilon a_p}{c^3} [r_{\text{Pioneer}}(t) - r_{\text{Detector}}(t + T^{(0)})],$$ \hspace{1cm} (36)

where $\left( \frac{\Delta v}{v} \right)_{\text{EH}}$ is the prediction of the Einstein–Hilbert action. This states that ignoring the deviation from the Einstein–Hilbert action leads to a systematic error in $\Delta v/v$ given by

$$\left( \frac{\Delta v}{v} \right)_{\text{missing}} = \epsilon \frac{\partial V_{\text{eff}}^{(0)}(r)}{\partial r} \left|_{r=r_D} \right. \frac{\partial r_D(t)}{\partial t} T^{(1)} + \frac{\epsilon a_p}{c^3} [r_{\text{Pioneer}}(t) - r_{\text{Detector}}(t + T^{(0)})],$$ \hspace{1cm} (37)

which in turn results to a systematic error in determining the acceleration of the spacecraft:

$$\frac{a}{a_{\text{missing}}} = \epsilon \frac{d}{dt} \left( \frac{\Delta v}{v} \right)_{\text{missing}}.$$ \hspace{1cm} (38)

We note that $T^{(0)} \approx \frac{c}{v_{\text{Earth}}}$, $T^{(1)}$ should be proportional to $a_p$. Therefore, $T^{(1)} \approx a_p \left( \frac{c}{v_{\text{Pioneer}}} \right)^2$. This helps us to obtain the order of magnitude of the terms present in $a_{\text{missing}}$:

$$\frac{a}{a_{\text{missing}}} = \epsilon \left( \frac{v_{\text{Earth}}}{c} \right) + O \left( \frac{v_{\text{Pioneer}}}{c} \right) + O \left( \frac{R_{\text{Earth}}}{c} \right).$$ \hspace{1cm} (39)

Therefore, due to the error bar in $a_p = (8.74 \pm 1.33) \times 10^{-10} \text{ m s}^{-2}$, it is legitimate to neglect $a_{\text{missing}}$.

We note that the covariant resolution of the Pioneer anomaly gives rise to periodic terms (with periodicity of one day and one year) in the Doppler shift (36). References [12] and [13] report that the residual of the fit with constant anomalous acceleration contains clear periodic terms. Reference [13] argues that these periodic terms should be assigned to the Earth and its...
atmosphere, while [12] discusses that they are somehow fingerprints of what causes the Pioneer anomaly. The periodic terms reported in [12, 13] are at the order of 
\[ \frac{\Delta\nu}{\nu} \approx 1 \times 10^{-12}. \]
The term of periodicity of one year in (36) is at the order of magnitude
\[ \Delta\nu_{yr} \approx \frac{1}{2} \times 2 \text{ AU} = 10^{-15}. \]
and the term of periodicity of one day in (36) is at the order
\[ \Delta\nu_{day} \approx \frac{1}{2} \times 6400 \text{ km} = 10^{-19}. \]
Therefore, no covariant resolution of the constant Pioneer anomaly is able to account for the residual periodicity of the fit with constant anomalous acceleration (note that this statement was derived only from (32)).

Now let us come back to the main issue of this section: how does (32) help us to identify the correction to the action? We see that (32) beside (22) leads to an integral equation for the correction to the Einstein–Hilbert action. Variation of this integral equation with respect to \( r \) leads to a non-homogeneous linear third-order differential equation for \( L(r) \). Equation (26) then shows that one solution of this differential equation is
\[ \epsilon L[R^2] = \alpha^2 \times (R_{\mu\nu\eta\gamma} R_{\mu\nu\eta\gamma})^\frac{1}{3}, \]
which leads to
\[ \epsilon a_1^2 (r) = -\epsilon a_1^2 \left( \frac{12}{13} \right) \frac{8r_0}{g}, \quad b_1^2 (r) = \frac{3}{2} a_1^2 (r). \]

It is worth noting that (40) is only one solution to the corresponding non-homogeneous linear third-order differential equation for \( L(r) \). Other solutions differ with (40) by terms which do not affect the motion of a spacecraft. Since we are interested in the motion of a spacecraft, we consider only (40). Then comparing (41) to (32) identifies \( \alpha \) to
\[ \alpha^p_{13} = (13.91 \pm 2.11) \times 10^{-26} \left( \frac{1}{\text{meter}} \right)^\frac{2}{3}, \]
which would have covariantly resolved the Pioneer anomaly if it had not been in contradiction with the other observations.

4. Constraints from the Earth and Moon

In the previous section, we have examined a general family of the covariant corrections and found the covariant correction capable of describing the Pioneer anomaly. We note that the corrections to the spacetime geometry given in equation (41) would increase if the mass of the Sun decreases (recall that \( r_0 = \frac{2GM}{c^2} \)). It implies that the accurate measurements of the geometry around the Earth would provide a strong constraint on the covariant corrections in the form of
\[ S = \int d^4x \sqrt{-\det g} \left( R + \alpha_n (R_{\mu\nu\eta\gamma} R_{\mu\nu\eta\gamma})^n \right). \]
Recalling that (43) and (29c) hold a perturbative expansion like that of (22) and (26–27) around the Earth, we find that a probe with a classical velocity experiences an anomalous acceleration of magnitude \( a_v(r) \), given by
\[ a_v(r) = -a_n r^{2-6n}, \]
\[ a_n = 2(12)^n (1-n) c^2 \left( \frac{2GM_{\text{Earth}}}{c^2} \right)^{2n-1} \]

at the distance \( r \) from the center of the Earth toward the center of the Earth in the case that equation (43) governs the dynamics of the spacetime. To put it another way, a satellite in a circular orbit experiences the following gravitational field around the Earth:
\[ F_G = \frac{GM_{\text{Earth}}}{r^2} + a_n r^{2-6n}. \]
where $M_{\text{Earth}}$ is the inertial (effective) mass of the Earth. The effective gravitational mass of the Earth is defined by
\[
GM_{\text{eff}, \text{Earth}}(r) = r^2 F_G = GM_{\text{Earth}} + a_n r^{4-6n}.
\]
(46)

In contrast to the Newtonian dynamic, the effective gravitational mass is not a radius-independent quantity. Analyzing the circular orbit of any satellite or the Moon identifies the effective gravitational mass of the Earth within that orbit.

The accurate value of the mass ratio of the Sun/(Earth + Moon) from the lunar laser ranging can be combined with the solar GM and the lunar GM from lunar orbiting spacecrafts [16] to give the effective gravitational mass of the Earth in an Earth-centered reference frame with a precision of one part in 10^8:
\[
GM_{\text{LLR}, \text{Earth}}(d_{\text{Earth–Moon}}) = 398 600.443 \pm 0.004 \text{ km}^3 \text{s}^{-2},
\]
(47)

where $d_{\text{Earth–Moon}}$ is the distance between the Moon and Earth [17]. The effective gravitational mass of the Earth has also been measured by various artificial Earth satellites [18], including the accurate tracking of the LAGEOS satellites orbiting the Earth in nearly circular orbits with semimajor axes about twice the radius of the Earth:
\[
GM_{\text{LAGEOS}, \text{Earth}}(2r_{\text{Earth}}) = 398 600.4419 \pm 0.002 \text{ km}^3 \text{s}^{-2},
\]
(48)

Using (44b), (50) results
\[
|\alpha_n| \leq 0.004 \text{ km}^3 \text{s}^{-2} \left( \frac{GM_{\text{Earth}}}{c^2} \right)^{1-2n} \frac{2^{2n} n (1-n) c^2 d_{\text{Earth–Moon}}^{1-6n}}{12}.
\]
(51)

The values of $|\alpha_n|$ which meet (51) are illustrated in figure 2 for 0.05 $\leq n \leq 0.65$. The limit on $|\alpha_\frac{1}{2}|$ is $|\alpha_\frac{1}{2}| \leq 6.12 \times 10^{-29} \left( \frac{1}{\text{meter}} \right)^\frac{3}{2}$. Therefore, $\alpha_\frac{1}{2} = (13.91 \pm 2.11) \times 10^{-26} \left( \frac{1}{\text{meter}} \right)^\frac{3}{2}$, which is needed to covariantly resolve the Pioneer anomaly, is clearly not compatible with the accurate measurements around the Earth. So the Pioneer anomaly cannot be covariantly resolved within the general family of corrections that we have considered. This supports the idea that the Pioneer anomaly is on board systematic or due to non-gravitational effects. This idea is in agreement with other independent studies: the precession of the longitudes of perihelia of the solar planets [14] or the trajectories of long-period comets [15] have not been reported to experience an anomalous gravitational field toward the Sun of the magnitude capable of describing the Pioneer anomaly.

Having obtained the experimental bounds on $\alpha_n$ (51), we would like to find the minimum distance from the Sun that the perturbation breaks in. Note that we have assumed that the spacetime geometry has a perturbation around the Schwarzschild geometry:
\[
-g_{rr} = A(r) = \left( 1 - \frac{r_G}{r} \right) \left( 1 + \epsilon a(r) + O(\epsilon^2) \right),
\]
(52)

wherein we have assumed that the Schwarzschild geometry describes the spacetime geometry with a very good approximation. The existence of the perturbation means that $\epsilon a(r) \ll 1$.

10 I thank the comment of the referee of CQG that led to adding the subsequent paragraphs in this section.
Figure 2. The blue region is the range of parameters describing the covariant corrections to gravity which are consistent with the accurate measurements of the spacetime geometry around the Earth (50). The dots represent the values which will be needed to covariantly resolve the Pioneer anomaly or the anomalous velocity curves of the spiral galaxies. The thickness of the dots corresponds to their uncertainties.

The Schwarzschild geometry remains a good approximation if \( \epsilon a(r) \ll \frac{r}{r_{\odot}} \). The perturbation breaks when \( \epsilon a(r) \approx \frac{r}{r_{\odot}} \) or \( \epsilon a(r) \approx 1 \). Using the bounds on \( \alpha_n \) we then obtain

\[
\epsilon a(\mathcal{F}) \approx 1 \rightarrow \xi_{\odot} \approx \left( 1 + \frac{3 - 6n}{12^2 4n(n-1)} \frac{1}{\alpha_n r_{\odot}^{-4n}} \right) \frac{1}{\xi_{\odot}},
\]

\[
\epsilon a(\mathcal{F}) \approx \frac{r_{\odot}}{r} \rightarrow \xi_{\odot} \approx \left( \frac{3 - 6n}{12^2 4n(n-1)} \frac{1}{\alpha_n r_{\odot}^{-4n}} \right) \frac{1}{\alpha_n r_{\odot}^{-4n}},
\]

where \( \alpha_n \) is such that the bound in (51) is saturated, and in the last line it is assumed that \( n \neq \frac{1}{2} \). Figure 3 plots \( \xi_{\odot} \). The first observation is that the perturbation remains valid inside the solar system. It is also interesting that the perturbation around the Schwarzschild metric breaks before perturbation in \( \alpha_n(R_{ijkl}R^{ijkl})^n \). This means that there exist some regions where \( R + \epsilon a_n(R_{ijkl}R^{ijkl})^n + O(\epsilon^2) \) is perturbative in the sense that terms of order \( \epsilon^2 \) can be consistently neglected while the dominant term in \( R + \epsilon a_n(R_{ijkl}R^{ijkl})^n \) is \( \alpha_n(R_{ijkl}R^{ijkl})^n \) not \( R \). It might be interesting to solve the ‘exact’ equations in these regions.

We would like to re-emphasize that the combined LLR and LAGEOS measurements indeed provide a strong constraint on the form of the covariant correction. In order to illustrate further the power of this constraint let us investigate if it is satisfied by the covariant corrections proposed in [23]. Reference [23] considers a family of \( f(R) \) gravity in the presence of the cosmological constant before showing that in the vicinity of the Sun, there exists a set of \( f(R) \) corrections capable of describing the Pioneer anomaly. It shows that the effective gravitational acceleration is

\[
a_{\text{gravity}} = -\frac{GM_{\odot}}{r^2} - a_{\text{constant}},
\]

where the second term is ‘a constant acceleration (while is) independent of the (considered central) mass’. It then sets \( a_{\text{constant}} = a_p = (8.73 \pm 1.33) \times 10^{-10} \text{ m s}^{-2} \). Reference [23] however has not considered the implication of the covariant correction they studied...
to the spacetime geometry around the Earth. Equation (54) implies the following effective gravitational acceleration around the Earth:

\[
a_{\text{gravity}} = -\frac{G M_{\text{Earth}}}{r^2} - a_p, \tag{55}
\]

which leads to the following effective gravitational mass of the Earth,

\[
G M_{\text{eff}}^\text{Earth} = G M_{\text{Earth}} + a_p r^2. \tag{56}
\]

The combined LAGEOS and LLR measurements then require

\[
a_p (d_{\text{Earth–Moon}}) \leq 0.004 \text{ km}^3 \text{s}^{-2}, \tag{57}
\]

while \( a_p (d_{\text{Earth–Moon}}) = (0.129 \pm 0.020) \text{ km}^3 \text{s}^{-2}. \) Therefore, the Earth–Moon system also refutes the covariant resolution of [23] for the Pioneer anomaly.

A lesson we should learn here is that ‘phenomenologically good’ covariant corrections to the action which remain perturbative at the ‘close’ vicinity of a spherical central mass seem to be those whose predicted corrections to the spacetime geometry had decreased if we decreased the central mass. The perturbative studies of \( f(R_{ijkl} R^{ijkl}) \) (this paper) and \( f(R) \) [23] show that this criterion is not satisfied in general. It is interesting to systematically study what kind of corrections meet this criterion. In doing so, perhaps, it is interesting to study

\[
(\pi i)^e_0 \sqrt{3} \alpha y_9 \bar{c}^5 \sin \left( \frac{r}{r_9} \right) + O(\alpha^2), \tag{58}
\]

which is suggested by the covariant resolution of the anomalous flat rotational curves of the spiral galaxies [22].

5. On the anomalous rotational velocity curves of the spiral galaxies

Equations (27) and (28) demonstrate that a correction in the form of \( \Delta L = \alpha_9 (R_{ijkl} R^{ijkl}) \frac{1}{2} \) leads to the following effective perturbative gravitational potential,

\[
V_{\text{eff}}(r) = -\frac{G M}{r} - 2\sqrt{3} \alpha c^2 \ln \left( \frac{r}{r_9} \right) + O(\alpha^2), \tag{59}
\]

around any spherical static distribution of matter of total inertial mass \( M_g \). Equation (58) suggests that \( \Delta L = \alpha_9 (R_{ijkl} R^{ijkl}) \frac{1}{2} \) might have a chance to resolve the anomalous flat rotational velocity curves of the spiral galaxies without considering dark matter. Mathematically speaking, the current work does not approve or reject this suggestion, due to the following reasons:

(1) The anomalous velocity curves of the spiral galaxies occur at the boundaries of the spiral galaxies. The matter’s distribution inside the presumed galaxy is disk-like and not spherical for the stars in the boundary of the spiral galaxies. Equations (27) and (28) are derived for a spherical distribution of matter.

(2) The anomalous velocity curves of the spiral galaxies are no small deviation from what Newtonian gravity predicts. The exact solutions of the modified action might precede a possible resolution of the anomalous velocity curve.

Despite the above obstacles we tend to examine whether a value of \( \alpha_9 \) compatible with (50) has a chance to describe the flat rotational velocity curves of the spiral galaxies. In doing so, let us extrapolate (58) toward the boundary of a typical spiral galaxy of mass \( M_g = 10^{12} M_\odot \). This generalization leads to the following relation for the velocity of the stars moving on a circular orbit around the center of the galaxy:

\[
v^2 = \frac{G M}{r} + 2\sqrt{3} \alpha c^2 + O(\alpha^2), \tag{59}
\]
Figure 3. We have used the bounds on $\alpha_n$, which are presented in figure 1, to find how far from the Sun the perturbation remains valid. The perturbation in terms of $\alpha_n(R_{ijkl} R_{ijkl})$ is valid below the blue continuous line. Below the dashed line, the perturbation around the Schwarzschild geometry is valid. Note that the $y$-axis is both logarithmic and represents $\log_{10}(r_{\text{break}})$, where $r_{\text{break}}$ is the minimum distance from the Sun that the ‘corresponding’ perturbation breaks in.

Figure 4. Rotational curves of spiral galaxies obtained by combining CO data for the central regions, optical for disks, and HI for outer disk and halo [20].

where $v$ stands for the velocity of the star around the center of the presumed galaxy. Examining the rotational curves of the spiral galaxies in figure 4, we see that the constant asymptotic velocity can be approximated by $200 \pm 50$ km s$^{-1}$ in large scales. In these distances, the first term of (59) is small, thus (59) implies that

$$\alpha^g = (13.61 \pm 6.39) \times 10^{-8}$$

is needed to describe the constant velocity of the within the borders of a typical spiral galaxy. The high precision measurement around the Earth (50), however, requires

$$\alpha^g \lesssim 6.67 \times 10^{-20},$$

therefore discards (60) and implies that $\Delta L = (R_{ijkl} R_{ijkl})^{\frac{1}{2}}$ has no chance of describing the flat rotational velocity curves of the spiral galaxies.
It is worth noting that the Riemann scalar curvature in the solar system and around the Earth satisfies
\[
7.53 \times 10^{-71} \text{ meter}^4 \lesssim R_{\mu\nu\eta\gamma} R^{\mu\nu\eta\gamma},
\] (62)
while in the regime where the anomalous rotational curvature of the spiral galaxies happens, it satisfies
\[
10^{-104} \text{ meter}^4 \lesssim R_{\mu\nu\eta\gamma} R^{\mu\nu\eta\gamma} \lesssim 10^{-92} \text{ meter}^4,
\] (63)
where a galaxy with a central mass at the order $10^{12} M_\odot$ with the boundary of about 100 kpc is alleged. If we assume that a simple action in the form of a polynomial in terms of the Riemann tensor dictates the dynamics of the spacetime in both of the regimes given by (62) and (63), we observe that the value of $\alpha_1$ will be needed to describe the anomalous flat rotational curve of the spiral galaxies (60) and is not in agreement with the accurate measurements of the spacetime geometry around the Earth, as illustrated in figure 2 as well. We, however, lack experimental justification or observational data supporting this assumption. Any functional of the Riemann tensor squared, $\Theta[R^2] = \Theta[R_{\mu\nu\eta\gamma} R^{\mu\nu\eta\gamma}]$, which becomes sufficiently small for (62) but constant for (63) can be utilized to suggest the following phenomenological action for gravity:
\[
S = \int d^4x \sqrt{-\det g} \left( R + \alpha_1^2 \Theta[R^2](R_{\mu\nu\eta\gamma} R^{\mu\nu\eta\gamma})^{\frac{1}{2}} \right),
\] (64)
which has a chance not only to be consistent with the solar system’s data but meets (60) as well. The exact solutions of (60) precede reaching a concrete conclusion on the validity of the above suggestion. Addressing the exact solution of (60) or a similar action in which $R_{\mu\nu\eta\gamma} R^{\mu\nu\eta\gamma}$ is replaced with the Gauss–Bonnet Lagrangian lying outside the scope of the current work.

6. Conclusions

We have approximated the spacetime geometry in the solar system by a Ricci flat geometry, a geometry of vanishing Ricci tensor. We have shown that a correction to the Einstein action would ‘non-trivially’ perturb the spacetime geometry around a Ricci flat geometry in case the correction involves the Riemann tensor per se. After that we have considered the simplest family of these corrections; the corrections which are arbitrary functional of the Riemann tensor’s squared:
\[
L = R + \epsilon \Delta L + O(\epsilon^2), \quad \Delta L = L(R_{ijkl} R^{ijkl}).
\] (65a)

Then we have computed the corrections to the Schwarzschild black hole in an asymptotically flat 4D geometry for a general $\Delta L$.

We have observed that $\Delta L_3 = \alpha_1(R_{ijkl} R^{ijkl})^\frac{1}{2}$ gives rise to a constant anomalous acceleration for objects orbiting the Sun, toward the Sun. This led us to the conclusion that $\alpha_1 = (13.91 \pm 2.11) \times 10^{-26}(\text{ meter})^\frac{1}{2}$ would have covariantly resolved the Pioneer anomaly if this value of $\alpha_1$ had not contradicted with other observations.

We have shown that the experimental bounds on $\Delta L_3$ become stronger in the case where we examine the deformation of the spacetime geometry around objects lighter than the Sun. We have, therefore, used the high precision measurements around the Earth (LAGEOS and lunar laser ranging) and obtained a strong constraint on the corrections in the form of $\Delta L(R_{ijkl} R^{ijkl})$. 
and in particular $\Delta L = \alpha_n (R_{ijkl} R^{ijkl})^n$. It is interesting that the high precision measurements around the Earth provide a strong constraint on the possible correction to the Einstein–Hilbert action.

The high precision measurements around the Earth require $\alpha_1^3 \leq 6.12 \times 10^{-29} (\text{meters})^2$; therefore, they refute the covariant resolution of the Pioneer anomaly. So the Pioneer anomaly cannot be covariantly resolved within the general family of corrections that we have considered. This supports the idea that the Pioneer anomaly is on board systematic or due to non-gravitational effects.

We have also noted that $\Delta L_2 = \alpha_2 (R_{ijkl} R^{ijkl})^{1/2}$ gives rise to an effective logarithmic gravitational potential. We have raised the question of whether $\Delta L_2 = \alpha_2 (R_{ijkl} R^{ijkl})^{1/2}$ may be useful in describing the anomalous flat rotational velocity curves of the spiral galaxies, before having proved that a simple correction in the form of $\Delta L_2 = \alpha_2 (R_{ijkl} R^{ijkl})^{1/2}$ is neither in agreement with the high precision measurements around the Earth nor can it describe the flat rotational velocity curves of the spiral galaxies.

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