Trigonometric $osp(1|2)$ Gaudin model

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Abstract

The problems connected with Gaudin models are reviewed by analyzing model related to the trigonometric $osp(1|2)$ classical $r$-matrix. The eigenvectors of the trigonometric $osp(1|2)$ Gaudin Hamiltonians are found using explicitly constructed creation operators. The commutation relations between the creation operators and the generators of the trigonometric loop superalgebra are calculated. The coordinate representation of the Bethe states is presented. The relation between the Bethe vectors and solutions to the Knizhnik-Zamolodchikov equation yields the norm of the eigenvectors. The generalized Knizhnik-Zamolodchikov system is discussed both in the rational and in the trigonometric case.
1 Introduction

Classifying integrable systems solvable in the framework of the quantum inverse scattering method [1, 2, 3] by underlying dynamical symmetry algebras, one could say that the Gaudin models are the simplest ones being based on loop algebras and classical r-matrices. More sophisticated solvable models correspond to more complicated algebras: Yangians, quantum affine algebras, elliptic quantum groups, dynamical quantum groups, etc.

Gaudin models [4, 5] are related to classical r-matrices, and the density of Gaudin Hamiltonians

\[ H^{(a)} = \sum_{b \neq a} r_{ab}(z_a - z_b) \]  

(1.1)

coincides with the r-matrix. Condition of their commutativity \([H^{(a)}, H^{(b)}] = 0\) is nothing else but the classical Yang-Baxter equation (YBE)

\[ [r_{ab}(z_a - z_b), r_{ac}(z_a - z_c) + r_{bc}(z_b - z_c)] + [r_{ac}(z_a - z_c), r_{bc}(z_b - z_c)] = 0 \, , \]  

(1.2)

where \(r\) is antisymmetric and belongs to the tensor product \(g \otimes g\) of a Lie algebra \(g\), or its representations and the indices fix the corresponding factors in the \(N\)-fold tensor product of this algebra (see Section 2).

The Gaudin models (GM) related to classical r-matrices of simple Lie algebras were studied intensively (see [5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and references therein). The spectrum and eigenfunctions were found using different methods (coordinate and algebraic Bethe Ansatz [5, 6], separated variables [6], etc.). The correlation functions were evaluated for \(g = sl(2)\) by the Gauss factorization approach [7]. A relation to the Knizhnik-Zamolodchikov (KZ) equation of conformal filed theory was established [11, 12, 13].

There exists a variety of classical r-matrices with trigonometric dependence on spectral parameter. Although algebraic construction of integrals of motion is straightforward the calculation of the spectrum and the corresponding eigenstates, by means of algebraic Bethe Ansatz, still depends on the underlining Lie algebra and r-matrix. Moreover, many trigonometric r-matrices are invariant under diagonal action of the Cartan subalgebra only

\[ [h_a + h_b, r_{ab}(z_a - z_b)] = 0 \, , \]  

(1.3)

as opposed to the rational case where classical r-matrix is invariant under the action of the whole Lie algebra \(g\). Hence, one can modify the Gaudin Hamiltonian (1.1) by adding a local generator of the Cartan subalgebra

\[ H_a \rightarrow \tilde{H}^{(a)} = g h_a + H^{(a)} \, . \]  

(1.4)

This modification does not change the creation operators, but the Bethe equations and solutions to the KZ system. However, the dependence on the parameter \(g\) (a magnetic field) will be described by a difference equation [15, 16].

The aim of this paper is to review problems connected with Gaudin models by analyzing the model related to the trigonometric osp(1|2) classical r-matrix. Results obtained here are in many respect similar to the ones we obtained in the case of osp(1|2)-invariant rational r-matrix [17]. However, connection of Gaudin model with magnetic field and KZ equations requires modification of the later by adding dynamical difference equation [15, 16].

There are additional peculiarities of Gaudin models related to classical r-matrices based on Lie superalgebras due to \(Z_2\)-grading of representation spaces and operators. The study of
the $osp(1|2)$-invariant Gaudin model corresponding to the simplest non-trivial super-case of the $osp(1|2)$ invariant $r$-matrix \cite{17} started in \cite{19}. The spectrum of the $osp(1|2)$ invariant Gaudin Hamiltonians $H^{(a)}$ was given, antisymmetry property of their eigenstates was claimed, and a two site model was connected with some physically interesting one (a Dicke model). Let us also point out that recently rational and trigonometric $sl(2)$ Gaudin models were used to describe different physical phenomena in metallic grains \cite{20} and a condensate fragmentation of confined bosons \cite{21}. Connection with perturbed WZNW models of conformal field theory was found in \cite{22}.

The creation operators used in the $sl(2)$ Gaudin model (and similarly for $sl(n)$ case) coincide with one of the $L$-matrix entry \cite{5,6}. However, in the $osp(1|2)$ case, as it was shown for rational $r$-matrix \cite{17}, the creation operators are complicated polynomials of the two generators $X^+(\lambda)$ and $v^+(\mu)$ of the loop superalgebra. We introduce $B$-operators belonging to the Borel subalgebra of the trigonometric loop superalgebra $L_t(osp(1|2))$ by a recurrence relation. Acting on the lowest spin vector (bare vacuum) $B_M(\mu_1,\ldots,\mu_M)\Omega^-$ the $B$-operators generate exact eigenstates of the Gaudin Hamiltonians $H^{(a)}$, provided Bethe equations are imposed on parameters $\{\mu_j\}$ of the states. For this reason the $B$-operators are sometimes refereed to as the creation operators and the eigenstates as the Bethe vectors, or simply $B$-vectors. Furthermore, the recurrence relation is solved explicitly and the commutation relations between the $B$-operators and the generators of the loop superalgebra $L_t(osp(1|2))$ as well as the generators of the global superalgebra $osp(1|2) \subset L_t(osp(1|2))$ are calculated. We prove that the constructed states are eigenvectors of the generator of the global Cartan subalgebra $h_{gl}$, but the $B$-vectors are not the lowest spin vectors anymore, as it was the case for the invariant model \cite{23,17}. Analogously to the rational case \cite{17}, a striking coincidence between the spectrum of the $osp(1|2)$ invariant Gaudin Hamiltonians of spin $s$ and the spectrum of the Hamiltonians of the $sl(2)$ Gaudin model of the integer spin $2s$ is also confirmed in the trigonometric case.

A connection between the $B$-states, when the Bethe equations are not imposed on their parameters ("off-shell Bethe states"), of the Gaudin models for simple Lie algebras to the solutions of the Knizhnik-Zamolodchikov equation was established in the papers \cite{7,12}. An explanation of this connection based on Wakimoto modules at critical level of the underlying affine algebra was given in \cite{12}. An explicit form of the Bethe vectors in the coordinate representation was given in both papers \cite{7,12}. The coordinate Bethe Ansatz for the $B$-states of the $osp(1|2)$ Gaudin model is obtained in our paper as well. Using commutation relations between the $B$-operators and the transfer matrix $t(\lambda)$, as well as the Hamiltonians $H^{(a)}$, we give an algebraic proof of the fact that explicitly constructed $B$-states yield a solution to the Knizhnik-Zamolodchikov equation corresponding to a conformal field theory. This connection permits us to calculate the norm of the eigenstates of the Gaudin Hamiltonians. An analogous connection is expected between quantum $osp(1|2)$ spin system related to the graded Yang-Baxter equation \cite{13,24,25,26} and quantum Knizhnik-Zamolodchikov equation following the lines of \cite{27}. We point out possible modifications of the Gaudin Hamiltonians and corresponding modifications of the Knizhnik-Zamolodchikov equation, similar to the case of the $sl(2)$ Gaudin model which was interpreted in \cite{28,29} as a quantization of the Schlesinger system for isomonodromy deformation.

The norm and correlation functions of the $sl(2)$ invariant Gaudin model were evaluated in \cite{7} using Gauss factorization of a group element and Riemann-Hilbert problem. The study of this problem for the trigonometric Gaudin model based on the $osp(1|2)$ Lie superalgebra is in progress. However, we propose a formula for the scalar products of the Bethe states which is analogous to the $sl(2)$ case.

The paper is organized as follows. In Section 2 we review main data of the quantum trigonometric $osp(1|2)$ spin system: the $osp(1|2)$ solution to the graded Yang-Baxter equation ($R$-
its center is spanned by the q-deformed Casimir element which can be written also in the form of the corresponding Gaudin model can be obtained easily as a quasi-classical limit of these data. The trigonometric osp(1|2) Gaudin model and its creation operators $B_M$ are discussed thoroughly in Section 3. Some of the most important properties of these operators are formulated and demonstrated pure algebraically: antisymmetry with respect to their arguments, commutation relations with the trigonometric loop superalgebra generators, commutation relations with the generating function $t(\lambda)$ of the Gaudin Hamiltonians, a differential identity, valid in the case of the Gaudin realization of the loop superalgebra. Using these properties of the $B$-operators we prove in Section 4 that acting on the lowest spin vector $\Omega_-$ these operators generate eigenvectors of the generating function of integrals of motion, provided the Bethe equations are imposed on the arguments of the $B$-operators. Possible modifications of the Gaudin Hamiltonians are pointed out, also. In particular, one of them yields Richard type Hamiltonian. An algebraic proof is given in Section 5 that constructed Bethe vectors are entering into solutions of the Knizhnik-Zamolodchikov equation of conformal field theory. Quasi-classical asymptotic with respect to a parameter of the Knizhnik-Zamolodchikov equation permits us to calculate the norm of the eigenstates of the Gaudin Hamiltonian. We pointed out that modification of Gaudin Hamiltonians, a differential identity, valid in the case rational case. Further development on possible evaluation of correlation functions is discussed in Conclusion. Finally, some definitions of the orthosymplectic Lie superalgebra osp(1|2) are given in the Appendix.

2 Quantum osp(1|2) superalgebra and corresponding spin system

The quantum superalgebra $\mathcal{U}_q(osp(1|2))$ as a deformation of the universal enveloping algebra of the Lie superalgebra $osp(1|2)$ (see Appendix), is generated by three elements $h, v^+, v^-$. The q-deformed commutation relations between the generators are

$$[h, v^\pm] = \pm v^\pm, \quad [v^+, v^-]_+ = -\frac{q^h - q^{-h}}{q - q^{-1}} := -[h]_q. \quad (2.1)$$

Its center is spanned by the q-deformed Casimir element

$$c_2(q) = A(q) ([h]_q)^2 + B(q) - \frac{(q^{1/2} + q^{-1/2})^2}{2} [v^+, v^-]_+ + \frac{(q + q^{-1})}{4} \left( q^h + q^{-h} \right) [v^+, v^-]_-. \quad (2.2)$$

with $2A(q) = q + q^{-1} + \frac{1}{2} (q - q^{-1})^2$ and $B(q) = (q^{1/2} + q^{-1/2})^2$. The q-deformed Casimir element can be written also in the form

$$c_2(q) = ([h - 1/2]_q)^2 - \left( q^{1/2} + q^{-1/2} \right)^2 v^2 v^- + \left( q^{h-1} + q^{-h+1} \right) v^+ v^- \quad (2.3)$$

Like in the case of the Lie superalgebra $osp(1|2)$ (see Appendix), there exists an element

$$c_1(q) = \left( [h - 1/2]_q + \left( q^{1/2} + q^{-1/2} \right) v^+ v^- \right), \quad (c_1(q))^2 = c_2(q), \quad (2.4)$$

with a grading property $[c_1, h] = 0$ and $c_1 v^\pm = -v^\pm c_1$. In the quasi-classical limit $q \to 1$ the relations (2.1-4) are reduced to the Lie superalgebra $osp(1|2)$ ones $[33]$. There is a coproduct
map $\Delta : \mathcal{U}_q \rightarrow \mathcal{U}_q \otimes \mathcal{U}_q$ consistent with the commutation relations (2.1) and a universal R-matrix $R$ as an element of $\mathcal{U}_q \otimes \mathcal{U}_q$ [31, 32]. Let us write its matrix form $R(q) = (\rho \otimes \rho) R$ in the basis $e_1 \otimes e_1, e_1 \otimes e_2, e_1 \otimes e_3, \ldots, e_3 \otimes e_3$ of the tensor product of two copies of the fundamental representation $V^{(1)} \otimes V^{(1)}$, $\rho : \mathcal{U}_q \rightarrow \text{End} (V^{(1)})$,

$$R(q) = \begin{pmatrix} q & 1 & a \\ q^{-1} & b & c \\ 1 & b & a \\ 1 & 1 & 1 \end{pmatrix},$$

(2.5)

here $a = q - q^{-1}$, $b = q^{-3/2} - q^{1/2}$ and $c = (1 + q^{-1})(q - q^{-1})$. Multiplying $R(q)$ by the graded permutation $\mathcal{P}$ of $V^{(1)} \otimes V^{(1)} : (\mathcal{P})_{abcd} = (-1)^{p(a)p(b)} \delta_{ab}\delta_{cd}$, one gets the braid group form of the R-matrix $\tilde{R}(q) = R(q) \mathcal{P}$ which has a spectral decomposition. Using the projectors on the irreducible representation components in the Clebsch-Gordan decomposition $V^{(1)} \otimes V^{(1)} = V^{(2)} \oplus V^{(0)}$ one can represent this R-matrix in the form [31, 32]

$$\tilde{R}(q) = -q^{-2}P_0(q) - q^{-1}P_1(q) + qP_2(q)$$

(2.6)

where projectors are

$$P_0(q) = \frac{q^{1/2} + q^{-1/2}}{q^{3/2} + q^{-3/2}} \begin{pmatrix} 0 & 0 & q^{-1/2} & -1 \\ q^{-1/2} & 0 & -1 & q^{1/2} \\ -q^{-1/2} & -1 & 0 & q \\ -1 & -q^{1/2} & 0 & 0 \end{pmatrix},$$

(2.7)

$$P_1(q) = \frac{1}{q + q^{-1}} \begin{pmatrix} 0 & q^{-1} & -1 & -1 \\ q^{-1} & -1 & -\nu & 1 \\ -1 & -\nu & -1 & q \\ 1 & \nu & \nu^2 & -\nu \\ 1 & \nu & -1 & q \end{pmatrix},$$

(2.8)

here $\nu = q^{1/2} + q^{-1/2}$, and $P_2(q) = I - P_0(q) - P_1(q)$. By the Baxterization procedure, or simply changing functions $(\lambda - a)/(\lambda + a)$ to the trigonometric ones $\sinh(\lambda - a)/\sinh(\lambda + a)$
in the \(osp(1|2)\)-invariant \(R\)-matrix \([18, 17]\) one gets the trigonometric \(R\)-matrix related to the quantum affine algebra \(U_q(osp(1|2))\)

\[
\hat{R}(\lambda, \eta) = P_2 - \frac{\sinh(\lambda - 2\eta)}{\sinh(\lambda + 2\eta)}P_1 - \frac{\sinh(\lambda - 3\eta)}{\sinh(\lambda + 3\eta)}P_0 ,
\]

(2.9)

here \(q = e^{2\eta}\) (normalized to \(\lambda = 0 \hat{R} = 1\)). The \(L\)-operator of the quantum spin system on a one-dimensional lattice with \(N\) sites coincides with \(R\)-matrix acting on a tensor product \(V_0 \otimes V_a\) of auxiliary space \(V_0\) and the space of states at site \(a = 1, 2, \ldots N\)

\[
L_{0a}(\lambda - z_a) = R_{0a}(\lambda - z_a) ,
\]

(2.10)

where \(z_a\) is a parameter of inhomogeneity (site dependence) and \(R_{0a}(\lambda, \eta) = \mathcal{P}\hat{R}(\lambda, \eta)\) is the usual \(R\)-matrix. Corresponding monodromy matrix \(T\) is an ordered product of the \(L\)-operators

\[
T(\lambda; \{z_a\}_1^N) = L_{0N}(\lambda - z_N) \ldots L_{01}(\lambda - z_1) = \prod_{a=1}^{N} L_{0a}(\lambda - z_a) .
\]

(2.11)

The commutation relations of the \(T\)-matrix entries follow form the \((Z_2\)-graded) FRT-relation \([3]\)

\[
R_{12}(\lambda - \mu)T_1(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)R_{12}(\lambda - \mu) .
\]

(2.12)

Multiplying (2.12) by \(R_{12}^{-1}\) and taking the super-trace over \(V_1 \otimes V_2\), one gets commutativity of the transfer matrix

\[
t(\lambda) = \text{str} T(\lambda) = \sum_j (-1)^{j+1} T_{jj}(\lambda; \{z_a\}_1^N) = T_{11} - T_{22} + T_{33}
\]

(2.13)

for different values of the spectral parameter \(t(\lambda)t(\mu) = t(\mu)t(\lambda)\).

The choice of the \(L\)-operators (2.10) corresponds to the following space of states of the \(osp(1|2)\)-spin system

\[
\mathcal{H} = \bigotimes_{a=1}^{N} V_a^{(1)} .
\]

The eigenvalues of the transfer matrix \(t(\lambda)\) in this space are \([18, 24, 25]\)

\[
\Lambda(\lambda; \{\mu_j\}_1^M) = \alpha_1^{(N)}(\lambda; \{z_a\}_1^N) \prod_{j=1}^{M} S_1(\lambda - \mu_j) - \alpha_2^{(N)}(\lambda; \{z_a\}_1^N) \times \
\]

\[
\times \prod_{j=1}^{M} S_1(\lambda - \mu_j + \eta) S_{-1}(\lambda - \mu_j + 2\eta) + \\alpha_3^{(N)}(\lambda; \{z_a\}_1^N) \prod_{j=1}^{M} S_{-1}(\lambda - \mu_j + 3\eta) ,
\]

(2.14)

where \(\alpha_j^{(N)}(\lambda; \{z_a\}_1^N) = \prod_{b=1}^{N} \alpha_j(\lambda - z_b) ; j = 1, 2, 3\),

\[
\alpha_1(\lambda) = \sinh(\lambda + 2\eta) \sinh(\lambda + 3\eta) , \quad \alpha_2(\lambda) = \sinh(\lambda) \sinh(\lambda + 3\eta) , \quad \alpha_3(\lambda) = \sinh(\lambda) \sinh(\lambda + \eta) , \quad S_n(\mu) = \frac{\sinh(\mu - n\eta)}{\sinh(\mu + n\eta)} .
\]

(2.15)
Although according to (2.14) the eigenvalue has formally two sets of poles at \( \lambda = \mu_j - \eta \) and \( \lambda = \mu_j - 2\eta \), the corresponding residues are zero due to the Bethe equations on the parameters \( \{\mu_j\} \) of the eigenstate [13, 24, 25]

\[
\prod_{a=1}^{N} \frac{\sinh(\mu_j - z_a + \eta)}{\sinh(\mu_j - z_a - \eta)} = \prod_{k=1}^{M} S_1(\mu_j - \mu_k)S_{-2}(\mu_j - \mu_k) .
\] (2.16)

If we take different spins \( l_a \) at different sites of the lattice and the following space of states

\[\mathcal{H} = \otimes_{a=1}^{N} V_{l_a}(l_a) ,\]

then the factors on the left hand side of (2.16) will be spin dependent too.

Due to the more complicated structure of the \( R \)-matrix (2.9) (see (2.7), (2.8)) than the \( gl(n) \), or \( gl(m|n) \) trigonometric \( R \)-matrices, the commutation relations of the entries \( T_{ij}(\lambda) \) of the \( T \)-matrix (2.11) have more terms and construction of the eigenstates of the transfer matrix \( t(\lambda) \) by the algebraic Bethe Ansatz can be done only using a complicated recurrence relation expressed in terms of \( T_{ij}(\mu_k) \) [30]. It will be shown below that due to a simplification of this recurrence relation in the quasi-classical limit \( \eta \to 0 \) one can solve it and find the creation operators for the trigonometric \( osp(1|2) \) Gaudin model explicitly. Furthermore, the commutation relations between the creation operators and the generators of the trigonometric loop superalgebra as well as the generating function \( t(\lambda) \) of the Gaudin Hamiltonians will be given explicitly, yielding the solution to the eigenvalue problem.

3 \( osp(1|2) \) trigonometric Gaudin model

As in the case of any simple Lie algebra, the trigonometric classical \( r \)-matrix of the orthosymplectic Lie superalgebra \( osp(1|2) \) can be expressed in a pure algebraic form as an element in the tensor product \( osp(1|2) \otimes osp(1|2) \)

\[
\hat{\tau}(\lambda) = \coth(\lambda) h \otimes h + \frac{2}{\sinh(\lambda)} \left( e^{-\lambda} X^+ \otimes X^- + e^\lambda X^- \otimes X^+ \right) + \frac{1}{\sinh(\lambda)} \left( e^{-\lambda} v^+ \otimes v^- - e^\lambda v^- \otimes v^+ \right) ,
\] (3.1)

and it is a solution of the \( Z_2 \)-graded classical YBE (1.2) [2]. This \( r \)-matrix can be decomposed naturally into positive and negative parts [34]

\[
\hat{\tau}(\lambda) = \frac{1}{\sinh(\lambda)} \left( e^{\lambda\tau^{(-)}} + e^{-\lambda\tau^{(+)}} \right) \]
(3.2)

\[
= \frac{e^\lambda}{\sinh(\lambda)} \left( \frac{1}{2} h \otimes h + 2X^- \otimes X^+ - v^- \otimes v^+ \right) + \frac{e^{-\lambda}}{\sinh(\lambda)} \left( \frac{1}{2} h \otimes h + 2X^+ \otimes X^- + v^+ \otimes v^- \right) .
\]

It can also be represented in another form useful for modifications

\[
\hat{\tau}(\lambda) = \coth(\lambda) e_{2}^{0} + 2 \left( X^- \otimes X^+ - X^+ \otimes X^- \right) - \left( v^- \otimes v^+ + v^+ \otimes v^- \right) ,
\] (3.3)

here

\[
e_{2}^{0} = h \otimes h + 2 \left( X^+ \otimes X^- + X^- \otimes X^+ \right) + \left( v^+ \otimes v^- - v^- \otimes v^+ \right) .
\] (3.4)
The matrix form of \( \hat{r} \) in the fundamental representation of \( osp(1|2) \) follows from (3.1) by substituting appropriate \( 3 \times 3 \) matrices instead of the \( osp(1|2) \) generators and taking into account \( Z_2 \)-graded tensor product of even and odd matrices. One can get it also as the quasi-classical limit \( \eta \to 0 \) from the \( R \)-matrix (2.4). Let us write explicitly the matrix form of \( \hat{r} \) in the basis of the tensor product of two copies of the fundamental representation \( V^{(1)} \otimes V^{(1)} \) (see Appendix)

\[
\begin{pmatrix}
\cosh(\lambda) & 0 & e^{-\lambda} \\
0 & -\cosh(\lambda) & -e^{-\lambda} \\
e^{\lambda} & 0 & 2e^{-\lambda}
\end{pmatrix}
\]

(3.5)

with all the other entries of this \( 9 \times 9 \) matrix being identically equal to zero.

A quasi-classical limit \( \eta \to 0 \) of the FRT-relations (2.12) \( (R(\lambda; \eta) = I + \eta r(\lambda) + O(\eta^2) \) and \( T(\lambda; \eta) = I + \eta L(\lambda) + O(\eta^2) \) results in a matrix form of the loop superalgebra relation, the so-called Sklyanin linear bracket,

\[
\left[ L_1 (\lambda), L_2 (\mu) \right] = - \left[ r_{12}(\lambda - \mu), L_1 (\lambda) + L_2 (\mu) \right].
\]

(3.6)

Both sides of this relation have the usual commutators of even \( 9 \times 9 \) matrices \( L_1 (\lambda) = L(\lambda) \otimes I_3, \) \( L_2 (\mu) = I_3 \otimes L(\mu) \) and \( r_{12}(\lambda - \mu) \), where \( I_3 \) is \( 3 \times 3 \) unit matrix and \( L(\lambda) \) has loop superalgebra valued entries:

\[
L(\lambda) = \begin{pmatrix}
h(\lambda) & -v^- (\lambda) & 2X^- (\lambda) \\
v^+ (\lambda) & 0 & v^- (\lambda) \\
2X^+ (\lambda) & v^+ (\lambda) & -h(\lambda)
\end{pmatrix}.
\]

(3.7)

From the expression (3.2) of the classical \( r \)-matrix it is natural to assume that the \( L \)-operator has a triangular decomposition \( L_\pm \) as \( \lambda \to \pm \infty \)

\[
L_+ = h \otimes h_{gl} + 4X^- \otimes X^+_{gl} - 2v^- \otimes v^+_{gl},
\]

(3.8)

\[
L_- = h \otimes h_{gl} + 4X^+ \otimes X^-_{gl} + 2v^+ \otimes v^-_{gl}.
\]

(3.9)

Here the first factors are generators in the fundamental representation \( V^{(1)} \) (see Appendix) and the second factors are generators of a finite dimensional \( osp(1|2) \) Lie superalgebra.

The relation (3.6) is a compact matrix form of the following commutation relations between the generators \( h(\lambda), v^\pm (\mu), X^\pm (\nu) \) of the trigonometric loop superalgebra \( L_t(osp(1|2)) \)

\[
[h(\lambda), h(\mu)]_\pm = 0,
\]

\[
[h(\lambda), X^\pm (\mu)]_\pm = \pm \frac{2}{\sinh(\lambda - \mu)} \left( \cosh(\lambda - \mu)X^\pm (\mu) - e^{\pm(\lambda - \mu)}X^\pm (\lambda) \right),
\]

\[
[X^+(\lambda), X^-(\mu)]_\pm = \frac{-e^{(\lambda - \mu)}}{\sinh(\lambda - \mu)} (h(\lambda) - h(\mu)),
\]

\[
[2X^+(\lambda), 2X^- (\mu)]_\pm = \frac{4}{\sinh(\lambda - \mu)} [h(\lambda), h(\mu)]_\pm,
\]

\[
\left[ X^+(\lambda), X^-(\mu) \right]_\pm = \frac{2}{\sinh(\lambda - \mu)} [h(\lambda), X^\pm (\mu)]_\pm,
\]

\[
\left[ h(\lambda), X^\pm (\mu) \right]_\pm = \frac{2}{\sinh(\lambda - \mu)} [h(\lambda), X^\pm (\mu)]_\pm.
\]
Due to the (3.6), (3.10) as Poisson brackets [35]. Also, in the explicit

\[ [h(\lambda), v^\pm(\mu)]_+ = \frac{\pm 1}{\sinh(\lambda - \mu)} \left( \cosh(\lambda - \mu) v^\pm(\mu) - e^{\mp(\lambda - \mu)} v^\pm(\lambda) \right), \]

\[ [v^+(\lambda), v^-(\mu)]_+ = \frac{e^{\lambda - \mu}}{\sinh(\lambda - \mu)} (h(\lambda) - h(\mu)), \]

\[ [v^+(\lambda), v^+(\mu)]_+ = \frac{\pm 2}{\sinh(\lambda - \mu)} \left( e^{\pm(\lambda - \mu)} X^\pm(\mu) - e^{\mp(\lambda - \mu)} X^\pm(\lambda) \right), \]

\[ [X^\pm(\lambda), v^\mp(\mu)]_+ = \frac{e^{\pm(\lambda - \mu)}}{\sinh(\lambda - \mu)} (v^\pm(\mu) - v^\pm(\lambda)), \]

\[ [X^\pm(\lambda), v^\pm(\mu)]_+ = [X^\pm(\lambda), X^\pm(\mu)]_+ = 0. \]  

(3.10)

In order to define a dynamical system besides the algebra of observables we need to specify a Hamiltonian. Due to the \( \tau \)-matrix relation (3.6) the elements

\[ t(\lambda) = \frac{1}{2} \text{str} L^2(\lambda) = h^2(\lambda) + 2[X^+(\lambda), X^-(\lambda)]_+ + [v^+(\lambda), v^-]_- \]

\[ = h^2(\lambda) + h'(\lambda) + 4X^+(\lambda)X^-(\lambda) + 2v^+(\lambda)v^-(-\lambda) \]  

(3.11)

commute for different values of the spectral parameter

\[ t(\lambda)t(\mu) = t(\mu)t(\lambda). \]  

(3.12)

Thus, \( t(\lambda) \) can be considered as a generating function of integrals of motion. The supertrace in (3.11) for an even matrix \( \{A_{ij}\} \) means \( \text{str}A = \sum_{i=1}^{3}(-1)^{p(i)}A_{ii} \), and we use the grading \( p(1) = p(3) = 0, p(2) = 1 \) (see Appendix).

One way to show (3.12) is to notice that the commutation relation between \( t(\lambda) \) and \( L(\mu) \) can be written in the form

\[ [t(\lambda), L(\mu)] = [M(\lambda, \mu), L(\mu)], \]  

(3.13)

where

\[ M(\lambda, \mu) = -\text{str} \left( r_{12}(\lambda - \mu) L(\lambda) \right) - \frac{1}{2} \text{str} \left( r_{12}(\lambda - \mu) \right), \]  

(3.14)

and the second term is a quantum correction, which is absent if we consider the left hand side of (3.10) as Poisson brackets [35]. Also, in the \( sl(n) \) case this term does not contribute since it is proportional to the unit matrix. Using (3.15) and (3.17) it is straightforward to calculate \( M(\lambda, \mu) \) explicitly

\[ M(\lambda, \mu) = \frac{-2}{\sinh(\lambda - \mu)} \left( \begin{array}{ccc} \cosh(\lambda - \mu) h(\mu) & -e^{\lambda - \mu} v^-(\mu) & 2e^{\lambda - \mu} X^-(\mu) \\ e^{-(\lambda - \mu)} v^+(\mu) & 0 & e^{\lambda - \mu} v^-(\mu) \\ 2e^{-(\lambda - \mu)} X^+(\mu) & e^{-(\lambda - \mu)} v^+(\mu) & -\cosh(\lambda - \mu) h(\mu) \end{array} \right) \]

\[ - \frac{1}{\sinh^2(\lambda - \mu)} \left( \begin{array}{ccc} \cosh^2(\lambda - \mu) + 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & \cosh^2(\lambda - \mu) + 1 \end{array} \right). \]  

(3.15)

Substituting (3.15) into the equation (3.13) we obtain the commutation relations between \( t(\lambda) \) and the generators of the superalgebra \( \mathfrak{L}_i(osp(1|2)) \). In particular

\[ [t(\lambda), X^+(\mu)]_+ = 4 \coth(\lambda - \mu) X^+(\mu) h(\lambda) - \frac{4e^{-(\lambda - \mu)}}{\sinh(\lambda - \mu)} X^+(\lambda) h(\mu) + 4X^+(\mu) \]

\[ - \frac{2e^{-(\lambda - \mu)}}{\sinh(\lambda - \mu)} (v^+(\lambda) v^+(\mu) - v^+(-\mu) v^+(\lambda)). \]  

(3.16)
\[ t(\lambda), v^+(\mu) \] = \begin{align*}
2 \coth(\lambda - \mu) v^+(\mu) h(\lambda) - \frac{2e^{-(\lambda-\mu)}}{\sinh(\lambda - \mu)} v^+(\lambda) h(\mu) + v^+(\mu) \\
+ \frac{4}{\sinh(\lambda - \mu)} \left( e^{-(\lambda-\mu)} X^+(\lambda)v^-(\mu) - e^{-(\mu-\lambda)} X^+(\mu)v^-(\lambda) \right).
\end{align*}

(3.17)

Preserving some generality we can consider the representation space \( \mathcal{H} \) of the dynamical algebra to be a lowest spin \( \rho(\lambda) \) representation of the loop superalgebra with the lowest spin vector \( \Omega_- \)

\[ h(\lambda) \Omega_- = \rho(\lambda) \Omega_- , \quad v^-(\lambda) \Omega_- = 0 . \]

(3.18)

One can study spectrum and eigenstates of \( t(\lambda) \) in this general representation \( \mathcal{H} \). However, to have a physical interpretation we will use a local realization of the trigonometric superalgebra \( L_t(osp(1|2)) \) with

\[ \mathcal{H} = \bigotimes_{a=1}^N V_a , \]

as a tensor product of \( osp(1|2) \) representations. Then,

\[ h(\lambda) = \sum_{a=1}^N \coth(\lambda - z_a) h_a \]

(3.19)

\[ v^+(\lambda) = \sum_{a=1}^N \frac{e^{\lambda - z_a}}{\sinh(\lambda - z_a)} v^+_a , \quad v^-(\lambda) = \sum_{a=1}^N \frac{e^{-\lambda + z_a}}{\sinh(\lambda - z_a)} v^-_a , \]

(3.20)

\[ X^+(\lambda) = \sum_{a=1}^N \frac{e^{\lambda - z_a}}{\sinh(\lambda - z_a)} X^+_a , \quad X^-(\lambda) = \sum_{a=1}^N \frac{e^{-\lambda + z_a}}{\sinh(\lambda - z_a)} X^-_a , \]

(3.21)

where \( h_a, v^+_a, X^+_a \in \text{End} (V_a) \) are \( osp(1|2) \) generators in a representation \( V_a \) associated with each site \( a \). If in this realization one considers the limits \( \lambda \rightarrow \pm \infty \) then one finds the expressions of the generators of the global Lie superalgebra \( osp(1|2) \subseteq L_t(osp(1|2)) \) in terms of the local generators

\[ Y_{gl} = \sum_{a=1}^N Y_a , \]

(3.22)

where \( Y = (h, X^\pm, v^\pm) \).

In particular, a representation of the Gaudin realization can be obtained by considering at each site \( a \) an irreducible representations \( V_a(l_a) \) of the Lie superalgebra \( osp(1|2) \) defined by a spin \( l_a \) and a lowest spin vector \( \omega_a \) such that \( v^-_a \omega_a = 0 \) and \( h_a \omega_a = -l_a \omega_a \). Thus,

\[ \Omega_- = \bigotimes_{a=1}^N \omega_a , \quad \text{and} \quad \rho(\lambda) = \sum_{a=1}^N (-l_a) \coth(\lambda - z_a) . \]

(3.23)

It is a well-known fact in the theory of Gaudin models [3, 3] that the Gaudin Hamiltonian is related to the classical \( r \)-matrix \([1, 4]\), \([1, 2]\)

\[ H^{(a)} = \sum_{b \neq a} r_{ab}(z_a - z_b) = \sum_{b \neq a} \frac{1}{\sinh(z_a - z_b)} \left( e^{(z_a - z_b) r^{(-)}_{ab}} + e^{(z_b - z_a) r^{(+)}_{ab}} \right) \]
\[
\begin{align*}
&= \sum_{b \neq a} \coth(z_a - z_b)h_a h_b + \frac{2}{\sinh(z_a - z_b)} \left( e^{- (z_a - z_b)} X_a^+ X_b^- + e^{(z_a - z_b)} X_a^- X_b^+ \right) \\
&+ \frac{1}{\sinh(z_a - z_b)} \left( e^{- (z_a - z_b)} v_a^+ v_b^- - e^{(z_a - z_b)} v_a^- v_b^+ \right), \tag{3.24}
\end{align*}
\]

and can be obtained as the residue of the operator \( t(\lambda) \) at the point \( \lambda = z_a \) using the expansion

\[
t(\lambda) = h_{gl}^2 + \sum_{a=1}^{N} \left( \frac{c_2(a)}{\sinh^2(\lambda - z_a)} + 2 \frac{e^{\lambda - z_a}}{\sinh(\lambda - z_a)} H^{(a)} \right), \tag{3.25}
\]

\( c_2(a) = h_a^2 + 2 \left( X_a^+ X_a^- + X_a^- X_a^+ \right) + v_a^+ v_a^- - v_a^- v_a^+ \). As opposed to the rational \( osp(1|2) \)-invariant case [17], the generating function (3.25) commutes only with one generator \( h_{gl} \) of the global superalgebra \( osp(1|2) \)

\[
[t(\lambda), h_{gl}] = 0 . \tag{3.26}
\]

To construct the set of eigenstates of the generating function of integrals of motion \( t(\lambda) \) we have to define appropriate creation operators. The creation operators used in the \( sl(2) \) Gaudin model coincide with one of the \( L \)-matrix entry [5, 13]. However, in the \( osp(1|2) \) case the creation operators are complicated functions of the two generators of the loop superalgebra \( X^+(\lambda) \) and \( v^+(\mu) \).

**Definition 3.1** Let \( B_M(\mu_1, \ldots, \mu_M) \) belong to the Borel subalgebra of the \( osp(1|2) \) loop superalgebra \( \mathcal{L}_l(osp(1|2)) \) such that

\[
B_M(\mu_1, \ldots, \mu_M) = v^+(\mu_1) B_{M-1}(\mu_2, \ldots, \mu_M) + 2 X^+(\mu_1) \sum_{j=2}^{M} (-1)^{j-1} \frac{e^{-(\mu_1 - \mu_j)}}{\sinh(\mu_1 - \mu_j)} B_{M-j}^{(j)}(\mu_2, \ldots, \mu_M) \tag{3.27}
\]

with \( B_0 = 1, B_1(\mu) = v^+(\mu) \) and \( B_M = 0 \) for \( M < 0 \). The notation \( B_{M-j}^{(j)}(\mu_2, \ldots, \mu_M) \) means that the argument \( \mu_j \) is omitted.

As we will show below, the \( B \)-operators are such that the Bethe vectors are generated by their action on the lowest spin vector \( \Omega_- \) [5, 18]. To prove this result we will need some important properties of the \( B \)-operators. All the properties of the creation operators \( B_M(\mu_1, \ldots, \mu_M) \) listed below can be demonstrated by induction method. Since the proofs are lengthy and quite technical we will present only two of them.

**Lemma 3.1** The creation operators \( B_M(\mu_1, \ldots, \mu_M) \) are antisymmetric functions of their arguments

\[
B_M(\mu_1, \ldots, \mu_k, \mu_{k+1}, \ldots, \mu_M) = -B_M(\mu_1, \ldots, \mu_{k+1}, \mu_k, \ldots, \mu_M) , \tag{3.28}
\]

here \( 1 \leq k < M \) and \( M \geq 2 \).
Lemma 3.2 The commutation relations between the creation operator $B_M$ and the generators $v^+(\lambda), h(\lambda), v^- (\lambda)$ of the loop superalgebra are given by

$$v^+(\lambda)B_M = (-1)^M B_M v^+(\lambda) + 2 \sum_{j=1}^M \frac{(-1)^j}{\sinh(\lambda - \mu_j)} \times \left( e^{-(\lambda-\mu_j)} X^+(\lambda) - e^{-\mu_j} X^+(\mu_j) \right) B_{M-1}^{(j)},$$

$$h(\lambda)B_M = B_M \left( h(\lambda) + \sum_{i=1}^M \coth(\lambda - \mu_i) \right) + \sum_{i=1}^M (-1)^i \frac{e^{-(\lambda-\mu_i)}}{\sinh(\lambda - \mu_i)} \times$$
$$\times \left( v^+(\lambda)B_{M-1}^{(i)} + 2X^+(\lambda) \sum_{j \neq i} (-1)^{j+\Theta(i-j)} \frac{e^{-(\mu_i-\mu_j)}}{\sinh(\mu_i - \mu_j)} \right) \right) + v^+(\lambda) \sum_{i<j} (-1)^{i-j-1}$$
$$\times \frac{e^{-(\mu_i-\mu_j)}}{\sinh(\mu_i - \mu_j)} B_{M-2}^{(i,j)} \left( \frac{e^{-(\lambda-\mu_i)}}{\sinh(\lambda - \mu_i)} + \frac{e^{-(\lambda-\mu_j)}}{\sinh(\lambda - \mu_j)} \right).$$

Here the upper index of $B_{M-1}^{(j)}$ means that the argument $\mu_j$ is omitted, the upper index of $B_{M-2}^{(i,j)}$ means that the parameters $\mu_i, \mu_j$ are omitted and $\Theta(j)$ is Heaviside function

$$\Theta(j) = \begin{cases} 1 & \text{if } j > 0 \\ 0 & \text{if } j \leq 0 \end{cases}.$$

It is useful to have explicit formulas for the commutators between the global generators and the $B$-operators.

Remark 3.1 The commutation relations between the generators of the global osp(1|2) and the $B_M$ elements follow from the previous lemma 3.2. To see this we take the appropriate limit $\lambda \to \pm \infty$ in (3.29), (3.30), (3.31). In this way we obtain

$$v^+_gl B_M = (-1)^M B_M v^+_gl - 2 \sum_{j=1}^M (-1)^j X^+(\mu_j) B_{M-1}^{(j)},$$

$$h_{gl} B_M = B_M (h_{gl} + M),$$

$$v^-_{gl} B_M = (-1)^M B_M v^-_{gl} + \sum_{j=1}^M (-1)^j B_{M-1}^{(j)} \left( h_{gl} + h(\mu_j) + \sum_{k \neq j} e^{\mu_j - \mu_k} \right).$$
Lemma 3.3 The generating function of integrals of motion \( t(\lambda) \) (3.11) has the following commutation relation with the creation operator \( B_M(\mu_1, \ldots, \mu_M) \)

\[
t(\lambda) B_M = B_M t(\lambda) + B_M \left( 2h(\lambda) \sum_{i=1}^{M} \coth(\lambda - \mu_i) + 2 \sum_{i<j}^{M} \coth(\lambda - \mu_i) \coth(\lambda - \mu_j) + M \right) + 2 \sum_{i=1}^{M} (-1)^i \frac{e^{-(\lambda - \mu_i)}}{\sinh(\lambda - \mu_i)} \left( v^+ B^{(i)}_{M-1} + 2 X^+ \left( \sum_{j \neq i}^{M} (-1)^{j+i} \frac{e^{-(\mu_i - \mu_j)}}{\sinh(\mu_i - \mu_j)} B_{M-2}^{(i,j)} \right) \right) \times \frac{\beta_M(\mu_i)}{\sinh(\lambda - \mu_i)} \left( e^{-(\lambda - \mu_i)} X^+(\lambda) v^- (\mu_i) - e^{-(\mu_i - \lambda)} X^+(\mu_i) v^- (\lambda) \right). \tag{3.35}
\]

The notation used here for the operator \( \beta_M(\mu_i) \) is \( \beta_M(\mu_i) = h(\mu_i) + \sum_{j \neq i}^{M} \coth(\mu_i - \mu_j) \).

In the trigonometric Gaudin realization (3.17 – 19) the creation operators \( B_M(\mu_1, \ldots, \mu_M) \) have some specific analytical properties.

Lemma 3.4 The \( B \)-operators in the Gaudin realization (3.17 – 19) satisfy an important differential identity

\[
\frac{\partial}{\partial z_a} B_M = \sum_{j=1}^{M} \frac{\partial}{\partial \mu_j} \left( (-1)^j \frac{e^{\mu_j - z_a}}{\sinh(\mu_j - z_a)} \right) \times \left( v^+ B^{(j)}_{M-1} + 2 X^+ \left( \sum_{k \neq j}^{M} (-1)^{k+j} \frac{e^{-(\mu_j - \mu_k)}}{\sinh(\mu_j - \mu_k)} B_{M-2}^{(j,k)} \right) \right). \tag{3.36}
\]

This identity will be a fundamental step in establishing a connection between the Bethe vectors and solutions to the KZ equation.

The proofs of the lemmas are based on the induction method. As illustrations, we prove explicitly lemma 3.1 and the formula (3.29) in lemma 3.2.

Proof of lemma 3.1. Consider \( M = 2 \)

\[
B_2(\mu_1, \mu_2) = v^+(\mu_1)v^+(\mu_2) + \frac{2e^{-(\mu_1 - \mu_2)}}{\sinh(\mu_1 - \mu_2)} X^+(\mu_1).
\]

Using the commutation relations (3.10) it is straightforward to check that \( B_2(\mu_1, \mu_2) \) is antisymmetric

\[
B_2(\mu_1, \mu_2) = -B_2(\mu_2, \mu_1).
\]

Assume \( B_N(\mu_1, \ldots, \mu_N) \) is antisymmetric \( N \geq 2 \) and for \( N < M \). We have to prove that \( B_M(\mu_1, \ldots, \mu_M) \) is antisymmetric also.

Consider \( j \geq 2 \), the antisymmetry of \( B_M(\mu_1, \ldots, \mu_M) \) with respect to \( \mu_j \) and \( \mu_{j+1} \) follows directly form the recurrence relation (3.27) and our assumption. Namely, the terms \( B_{M-2}^{(j)}(\mu_2, \ldots, \mu_M)e^{-(\mu_1 - \mu_j)}/\sinh(\mu_1 - \mu_j) \) and \( B_{M-2}^{(j+1)}(\mu_2, \ldots, \mu_M)e^{-(\mu_1 - \mu_{j+1})}/\sinh(\mu_1 - \mu_{j+1}) \) enter with the opposite sign.
Therefore we only have to show the antisymmetry of \( B_M(\mu_1, \ldots, \mu_M) \) with respect to the interchange of \( \mu_1 \) and \( \mu_2 \). To see this we have to iterate the recurrence relation (3.27) twice and combine the appropriate terms

\[
B_M(\mu_1, \ldots, \mu_M) = \left( v^+(\mu_1)v^+(\mu_2) + \frac{2e^{-(\mu_1-\mu_2)}}{\sinh(\mu_1 - \mu_2)}X^+(\mu_1) \right) B_{M-2}(\mu_3, \ldots, \mu_M)
+ 2v^+(\mu_1)X^+(\mu_2) \sum_{j=3}^{M} (-1)^{j+1} \frac{e^{-(\mu_2-\mu_j)}}{\sinh(\mu_2 - \mu_j)} B_{M-3}^{(j)}(\mu_3, \ldots, \mu_M)
+ 2v^+(\mu_2)X^+(\mu_1) \sum_{j=3}^{M} (-1)^{j} \frac{e^{-(\mu_1-\mu_j)}}{\sinh(\mu_1 - \mu_j)} B_{M-3}^{(j)}(\mu_3, \ldots, \mu_M)
+ 4X^+(\mu_1)X^+(\mu_2) \sum_{j=3}^{M} (-1)^{j} \frac{e^{-(\mu_1-\mu_j)}}{\sinh(\mu_1 - \mu_j)} \sum_{k=3}^{M} (-1)^{k+\Theta(j-k)} \times
\frac{e^{-(\mu_2-\mu_k)}}{\sinh(\mu_2 - \mu_k)} B_{M-4}^{(j,k)}(\mu_3, \ldots, \mu_M)
\]

(3.37)

where \( B_{M-4}^{(j,k)}(\mu_3, \ldots, \mu_M) \) means that the arguments \( \mu_j \) and \( \mu_k \) are omitted. Since \( v^+(\mu) \) commutes with \( X^+(\nu) \), the antisymmetry of the right hand side of (3.37) with respect to \( \mu_1 \) and \( \mu_2 \) follows. Hence we have demonstrated the lemma.

**Proof of lemma 3.2.** Here we prove explicitly only formula (3.29). In particular, when \( M = 1 \) the expression (3.29) is just the anticommutator between \( v^+(\lambda) \) and \( v^+(\mu) \). Using the recurrence relations (3.27) it is straightforward to check that the formula (3.29) holds for \( M = 2 \)

\[
v^+(\lambda)B_2(\mu_1, \mu_2) = B_2(\mu_1, \mu_2)v^+(\lambda)
- \frac{2}{\sinh(\lambda - \mu_1)} \left( e^{-(\lambda-\mu_1)}X^+(\lambda) - e^{\lambda-\mu_1}X^+(\mu_1) \right) v^+(\mu_2)
+ \frac{2}{\sinh(\lambda - \mu_2)} \left( e^{-(\lambda-\mu_2)}X^+(\lambda) - e^{\lambda-\mu_2}X^+(\mu_2) \right) v^+(\mu_1)
\]

(3.38)

Therefore we can proceed to demonstrate the lemma 3.3 by induction. Assume that the relation (3.29) holds for \( B_N, M \geq N \geq 2 \). Then we have to show the formula (3.29) is valid for \( M + 1 \). We use the recurrence relations (3.27) to write

\[
v^+(\lambda)B_{M+1} = v^+(\lambda) \left( v^+(\mu_1)B_M + 2X^+(\mu_1) \sum_{j=2}^{M+1} \frac{e^{-(\mu_1-\mu_j)}}{\sinh(\mu_1 - \mu_j)} B_{M-1}^{(j)} \right)
= -v^+(\mu_1)v^+(\lambda)B_M - \frac{2}{\sinh(\lambda - \mu_1)} \left( e^{-(\lambda-\mu_1)}X^+(\lambda) - e^{\lambda-\mu_1}X^+(\mu_1) \right) B_M
+ 2X^+(\mu_1) \sum_{j=2}^{M+1} (-1)^j \frac{e^{-(\mu_1-\mu_j)}}{\sinh(\mu_1 - \mu_j)} v^+(\lambda)B_{M-1}^{(j)}.
\]

(3.39)

Now we can substitute the expressions for \( v^+(\lambda)B_M \) and \( v^+(\lambda)B_{M-1}^{(j)} \). After rearranging the terms in an appropriate way we have

\[
v^+(\lambda)B_{M+1} = (-1)^{M+1}B_{M+1}v^+(\lambda) + 2 \sum_{j=1}^{M+1} \frac{(-1)^j}{\sinh(\lambda - \mu_j)}
\]
This completes the proof of the lemma.

The proofs of the other lemmas are analogous to the proofs we have illustrated above. They do not contain illuminating insights and are considerably longer than the two we have seen. Thus, we will omit them.

The recurrence relation (3.27) can be solved explicitly. To be able to express the solution in a compact form it is useful to introduce a contraction operator $d$.

**Definition 3.2** Let $d$ be a contraction operator whose action on an ordered product $\prod_{j=1}^{M} v^{+}(\mu_j)$, $M \geq 2$, is given by

$$d \left( v^{+}(\mu_1)v^{+}(\mu_2)\ldots v^{+}(\mu_M) \right) = 2 \sum_{j=1}^{M-1} X^{+}(\mu_j) \sum_{k=j+1}^{M} (-1)^{\sigma(jk)} \frac{e^{-(\mu_j-\mu_k)}}{\sinh(\mu_j-\mu_k)} \prod_{m \neq j,k}^{M} v^{+}(\mu_m)$$

where $\sigma(jk)$ is the parity of the permutation

$$\sigma : (1,2,\ldots,j,j+1,\ldots,k,\ldots,M) \to (1,2,\ldots,j,k,j+1,\ldots,M) \ .$$

The $d$ operator can be applied on an ordered product $\prod_{j=1}^{M} v^{+}(\mu_j)$ consecutively several times, up to $[M/2]$, the integer part of $M/2$.

**Theorem 3.1** Explicit solution to the recurrence relation (3.27) is given by

$$B_{M}(\mu_{1},\ldots,\mu_{M}) = \prod_{j=1}^{M} v^{+}(\mu_{j}) + \frac{[M/2]}{m!} d^{m} \prod_{j=1}^{M} v^{+}(\mu_{j}) = \exp d \prod_{j=1}^{M} v^{+}(\mu_{j}) \ . \quad (3.42)$$

The properties of the creation operators $B_{M}$ studied in the this Section will be fundamental tools in determining characteristics of the trigonometric $osp(1|2)$ Gaudin model. Our primary interest is to obtain the spectrum and the eigenvectors of the generating function of integrals of motion $t(\lambda)$ (3.11).

**4 Spectrum and eigenstates of trigonometric $osp(1|2)$ Gaudin model**

With the help of the creation operators $B_{M}$ it is possible to obtain the eigenvectors as well as the corresponding eigenvalues of the Hamiltonians of the trigonometric Gaudin model. This result is a direct consequence of the following theorem.

**Theorem 4.1** The lowest spin vector $\Omega_{-}$ (3.18) is an eigenvector of the generating function of integrals of motion $t(\lambda)$ (3.11) with the corresponding eigenvalue $\Lambda_{0}(\lambda)$

$$t(\lambda) \Omega_{-} = \Lambda_{0}(\lambda) \Omega_{-} \ , \ \Lambda_{0}(\lambda) = \rho^{2}(\lambda) + \rho'(\lambda) \ . \quad (4.1)$$
Furthermore, the action of the $B$-operators (3.27) on the lowest spin vector $\Omega_-$ yields the eigenvectors
\[ \Psi(\mu_1, \ldots, \mu_M) = B_M(\mu_1, \ldots, \mu_M) \Omega_-, \] (4.2)
of the $t(\lambda)$ operator
\[ t(\lambda)\Psi(\mu_1, \ldots, \mu_M) = \Lambda(\lambda; \{\mu_j\}_{j=1}^M) \Psi(\mu_1, \ldots, \mu_M), \] (4.3)
with the eigenvalues
\[ \Lambda(\lambda; \{\mu_j\}_{j=1}^M) = y^2 + \partial_\lambda y, \] (4.4)
here
\[ y(\lambda; \{\mu_j\}_{j=1}^M) = \rho(\lambda) + \sum_{k=1}^M \coth(\lambda - \mu_k), \] (4.5)
provided that the Bethe equations are imposed on the parameters $\{\mu_j\}_{j=1}^M$ of the state (4.2).
\[ \beta_M(\mu_j) = \rho(\mu_j) + \sum_{k \neq j}^M \coth(\mu_j - \mu_k) = 0. \] (4.6)

Proof. The equation (4.1) can be checked by a direct substitution of the definitions of the operator $t(\lambda)$ and the lowest spin vector $\Omega_-$, the equations (3.11) and (3.18), respectively.

To show the second part of the theorem, we use the equation (4.2) to express the Bethe vectors
\[ t(\lambda)\Psi(\mu_1, \ldots, \mu_M) = t(\lambda) B_M(\mu_1, \ldots, \mu_M) \Omega_- \] (4.7)
Our next step is to use the third property of the $B$-operators, the equation (3.3), and the definition of the lowest spin vector $\Omega_-$ the equation (3.18) in order to calculate the action of the operator $t(\lambda)$ on the Bethe vectors when the Bethe equations (4.6) are imposed
\[ t(\lambda) B_M \Omega_- = B_M t(\lambda) \Omega_- + \left(2\rho(\lambda) \sum_{i=1}^M \coth(\lambda - \mu_i) + 2 \sum_{i<j}^M \coth(\lambda - \mu_i) \coth(\lambda - \mu_j) + M\right) B_M \Omega_-. \] (4.8)

We can express the first term on the right hand side since we know how the operator $t(\lambda)$ acts on the vector $\Omega_-$, the equation (4.1). Thus we have
\[ t(\lambda) B_M \Omega_- = \Lambda(\lambda; \{\mu_j\}_{j=1}^M) B_M \Omega_-, \] (4.9)
with
\[ \Lambda(\lambda; \{\mu_j\}_{j=1}^M) = \Lambda_0(\lambda) + 2\rho(\lambda) \sum_{i=1}^M \coth(\lambda - \mu_i) + 2 \sum_{i<j}^M \coth(\lambda - \mu_i) \coth(\lambda - \mu_j) + M, \]
and we complete the proof by expressing the eigenvalue as
\[ \Lambda(\lambda; \{\mu_j\}_{j=1}^M) = y^2 + \partial_\lambda y, \quad \text{with} \quad y(\lambda; \{\mu_j\}_{j=1}^M) = \rho(\lambda) + \sum_{k=1}^M \coth(\lambda - \mu_k). \]
\[ \square \]
Corollary 4.1 In the trigonometric Gaudin realization given by the equations (3.19), (3.20),
(3.21) and (3.23) the Bethe vectors \( \Psi(\mu_1, \ldots, \mu_M) \) (4.2) are the eigenvectors of the Gaudin
Hamiltonians (3.24) (see also [37])
\[
H^{(a)} \Psi(\mu_1, \ldots, \mu_M) = E_M^{(a)} \Psi(\mu_1, \ldots, \mu_M),
\]
with the eigenvalues
\[
E_M^{(a)} = \sum_{b=1}^{N} l_a l_b \coth(z_a - z_b) + \sum_{j=1}^{M} l_a \coth(\mu_j - z_a),
\]
when the Bethe equations are imposed
\[
\beta_M(\mu_j) = \rho(\mu_j) + \sum_{k \neq j} M \coth(\mu_j - \mu_k) = \sum_{a=1}^{N} (-l_a) \coth(\mu_j - z_a) + \sum_{k \neq j} \coth(\mu_j - \mu_k) = 0.
\]

Proof. The statement of the corollary follows from residue of the equation (3.3) at the point
\( \lambda = z_a \). The residue can be determined using (3.25), (4.4) and (4.1).

Comparing the eigenvalues \( E_M^{(a)} \) (4.11) of the Gaudin Hamiltonians and the Bethe equations
(4.12) with the corresponding quantities of the \( sl(2) \) Gaudin model [5, 6] we arrive to an inter-
esting observation.

Remark 4.1 The spectrum of the \( osp(1|2) \) trigonometric Gaudin model with the spins \( l_a \) co-
icides with the spectrum of the \( sl(2) \) trigonometric Gaudin system for the integer spins (see
an analogous observation for partition functions of corresponding anisotropic vertex models in
[32]).

Remark 4.2 The Bethe vectors are eigenstates of the global generator \( h_{gl} \)
\[
h_{gl} \Psi(\mu_1, \ldots, \mu_M) = \left( -\sum_{a=1}^{N} l_a + M \right) \Psi(\mu_1, \ldots, \mu_M).
\]
As oppose to the \( osp(1|2) \)-invariant model [17], these Bethe vectors are not the lowest spin
vectors of the global \( osp(1|2) \) since they are not annihilated by the generator \( v_{gl}^- \)
\[
v_{gl}^- \Psi(\mu_1, \ldots, \mu_M) \neq 0,
\]
once the Bethe equations are imposed (4.12). These conclusions follow from the remark 3.1, in
particular the equations (3.33) and (3.34), and the definition of the Bethe vectors (4.2).

As was pointed out already in [3] for the \( sl(2) \) case, there are several modifications of the
Hamiltonians (3.24). One of them is the Richardson’s pairing-force Hamiltonian [36, 38, 5].
These modifications can be formulated in the framework of the universal \( L \)-operator and \( r \)-matrix
formalism [36, 8].

Due to invariance of the \( r \)-matrix (3.5) with respect to the Cartan element
\[
[r(\lambda), h \otimes I + I \otimes h] = 0, \quad h \in osp(1|2)
\]
one can add to the $L$-operator the element $h$

$$L(\lambda) \rightarrow \tilde{L}(\lambda) = gh + L(\lambda),$$

(4.16)

preserving commutation relations (3.6). Then

$$\tilde{t}(\lambda) = \frac{1}{2} \text{str} \tilde{L}^2(\lambda) = t(\lambda) + 2gh(\lambda) + g^2,$$

(4.17)

will have the commutativity property, i.e. $\tilde{t}(\lambda)\tilde{t}(\mu) = \tilde{t}(\mu)\tilde{t}(\lambda)$. Hence we can take $\tilde{t}(\lambda)$ to be the generating function of the (modified) integrals of motion

$$\tilde{t}(\lambda) = (hgl - g)^2 + \sum_{a=1}^{N} \left( \frac{c_2(a)}{\sinh^2(\lambda - z_a)} + 2\frac{e^{\lambda - z_a}}{\sinh(\lambda - z_a)} \tilde{H}^{(a)} \right),$$

(4.18)

(4.19)

$$\tilde{H}^{(a)} = \text{res}_{\lambda=z_a} \tilde{t}(\lambda) = gh_a + H^{(a)}.$$  

(4.20)

In this case the eigenstates $\Psi_M$ are generated by the same B-operators. However, corresponding eigenvalues and Bethe equations are now given by

$$\tilde{\Lambda}(\lambda; \{\mu_j\}_{j=1}^{M}) = (y + g)^2 + \partial_\lambda y,$$

(4.21)

here as before $y(\lambda; \{\mu_j\}_{j=1}^{M}) = \sum_{a=1}^{N} (-l_a) \coth(\lambda - z_a) + \sum_{k=1}^{M} \coth(\lambda - \mu_k),$

$$\tilde{E}_M^{(a)} = E^{(a)}_M + g(-l_a),$$

(4.22)

$$\sum_{a=1}^{N} (-l_a) \coth(\mu_j - z_a) + \sum_{k \neq j}^{M} \coth(\mu_j - \mu_k) + g = 0.$$  

(4.23)

The crucial step in the proof of these equations is the observation that the commutation relations between the operator $\tilde{t}(\lambda)$ (4.17) and the creation operators $\tilde{B}_M$ are equal to the commutation relations (3.3) but with modified operator $\beta_M(\mu_j) \rightarrow \beta_M(\mu_j) + g$. To see this notice the similarity between the terms with $\nu^+ (\lambda) B^{(i)}_{M-1}$ operators and with $X^+ (\lambda) B^{(i,j)}_{M-2}$ operators in the lemma 3.2 the equation (3.30) and in the lemma 3.3 the equation (3.3).

Richardson like Hamiltonian $[16, 17, 18, 20, 21]$ can be obtained as a coefficient in the $\lambda \rightarrow +\infty$ expansion $[17]$

$$\tilde{t}(\lambda) = (hgl + g)^2 + 4e^{-2\lambda} \left( (hgl - 1 + g) \left( \sum_a e^{2z_a} h_a \right) \right) + 4X^+_{gl} \left( \sum_a e^{2z_a} X_a^- \right) + 2\nu^+_gl \left( \sum_a e^{2z_a} \nu_a^- \right) + O \left( e^{-4\lambda} \right).$$

(4.24)

Let us denote the coefficient next to the factor $4e^{-2\lambda}$ by $H_+$

$$H_+ = (hgl - 1 + g) \left( \sum_a e^{2z_a} h_a \right) + 4X^+_{gl} \left( \sum_a e^{2z_a} X_a^- \right) + 2\nu^+_gl \left( \sum_a e^{2z_a} \nu_a^- \right).$$  

(4.25)
This Hamiltonian is obviously not symmetric. Similar Hamiltonian can be obtained as a coefficient in the \(\lambda \to -\infty\) expansion

\[
\tilde{t}(\lambda) = (h_{gl} - g)^2 + 4e^{2\lambda} \left( \sum_a e^{-2z_a} h_a \right) (h_{gl} - 1 - g) + 4 \left( \sum_a e^{-2z_a} X_a^+ \right) X_{gl}^{-} + 2 \left( \sum_a e^{-2z_a} v_a^+ \right) v_{gl}^- + O \left( e^{4\lambda} \right) .
\] (4.26)

Let us denote the coefficient next to the factor \(4e^{2\lambda}\) by \(H_{-}\), which is also not symmetric. Thus, we choose the following symmetric combination for a trigonometric generalization of the Richardson Hamiltonian

\[
H_R = \frac{1}{2} (H_+ + H_-) = (h_{gl} - 1) \left( \sum_a \cosh(2z_a) h_a \right) + g \left( \sum_a \sinh(2z_a) h_a \right) + 2 \left( \sum_a e^{2z_a} X_a^+ \right) X_{gl}^{-} + \left( \sum_a e^{-2z_a} v_a^+ \right) v_{gl}^- .
\] (4.27)

The eigenvalues of \(H_R\) have different dependence on the quasi-momenta from the rational case \([36, 17]\)

\[
H_R \Psi_M(\mu_1, \ldots, \mu_M) = E_R(M) \Psi_M(\mu_1, \ldots, \mu_M) ,
\] (4.28)

with

\[
E_R(M) = \left( \sum_{j=1}^M \cosh(2\mu_j) - \sum_{a=1}^N l_a \cosh(2z_a) \right) \left( M - \sum_{a=1}^N l_a - 1 \right) + \left( \sum_{j=1}^M \sinh(2\mu_j) - \sum_{a=1}^N l_a \sinh(2z_a) \right) g .
\] (4.29)

More complicated modifications of Gaudin models can be obtained considering quasi-classical limit of the quantum spin system with non-periodic boundary conditions and corresponding reflection equation \([39, 40]\). The \(L\)-operator can be expressed in terms of the original one \((3.7)\) as

\[
L^{(bGM)}(\lambda; \{z\}) = L(\lambda; \{z\}) - L(-\lambda; \{z\}) ,
\] (4.30)
in the case of the open chain, and it will satisfy more complicated linear brackets, defining a subalgebra of the loop algebra \((3.6)\) (see also \([41]\) and references therein).

Most of the trigonometric Gaudin model relations have their counterparts in the rational \(osp(1|2)\)-invariant case. To show this one takes a scaling limit \(\lambda \to \varepsilon \lambda, z_a \to \varepsilon z_a\),

\[
\lim_{\varepsilon \to 0} \varepsilon L_{\text{trig}}(\varepsilon \lambda; \{\varepsilon z\}) = L_{\text{inv}}(\lambda; \{z\}) ,
\] (4.31)

and in this way one reproduces known results for the \(osp(1|2)\)-invariant model. However, as we shall see in the next Section some relations of the invariant GM have quite complicated analogs in the trigonometric case (a generalization of KZ system to include a “magnetic field” parameter \(g\),

18
requires a difference dynamical equation \((\text{[15 16]}))\). Also the modified \(L\)-operator \((\text{[16]}))\) requires to scale the parameter \(g \rightarrow g/\varepsilon\).

Another modification can be obtained by performing the similarity transformation on the \(r\)-matrix \((\text{[3,1]}))\) by the tensor square of the element \(\exp(tX^+).\) Then the scaling limit \(\lambda \rightarrow \varepsilon \lambda,\) \(t \rightarrow \xi/2\varepsilon\) results in a modified \(r\)-matrix

\[
\hat{r}(\lambda) = \frac{e^{\xi |K|}}{\lambda} + \xi \left( h \otimes X^+ - X^+ \otimes h - v^+ \otimes v^+ \right).
\]

The loop superalgebra will be modified, as well as corresponding Hamiltonians \((\text{[1]}))\). Similarly, the algebraic Bethe Ansatz will require changes, although the Bethe equations and the spectrum will be the same as in the \(osp(1|2)\)-invariant case (see the \(sl(2)\) case in \((\text{[2]}))\).

The expression of the eigenvectors of a solvable model in terms of local variables parameterized by sites of the chain or by space coordinates, is known as coordinate Bethe Ansatz \((\text{[5]}))\). The coordinate representation of the Bethe vectors gives explicitly analytical dependence on the parameters \(\{\mu_i\}_{1}^{M}\) and \(\{z_a\}_{1}^{N}\) useful in a relation to the Knizhnik-Zamolodchikov equation (Section 5). Using the Gaudin realization \((\text{[3,19]}, \text{[3,20]}, \text{[3,21]}))\) of the generators

\[
v^+(\mu) = \sum_{a=1}^{N} \frac{e^{\mu - z_a}}{\sinh(\mu - z_a)} v^+_a, \quad X^+(\mu) = \sum_{a=1}^{N} \frac{e^{\mu - z_a}}{\sinh(\mu - z_a)} X^+_a,
\]

and the definition of the creation operators \((\text{[3,42]}))\), one can get the coordinate representation of the \(B\)-operators:

\[
B_M(\mu_1, \mu_2, \ldots, \mu_M) = \sum_{\pi} \left( v^+_1 \cdots v^+_M \right)_{\pi} \prod_{a=1}^{N} \varphi\left( \{\mu_m(a)\}_{1}^{|K_a|}; z_a \right), \quad (4.33)
\]

where the first sum is taken over ordered partitions \(\pi\) of the set \(\{1, 2, \ldots, M\}\) into subsets \(K_a, a = 1, 2, \ldots, N,\) including empty subsets with the constraints

\[
\bigcup_{a} K_a = \{1, 2, \ldots, M\}, \quad K_a \bigcap K_b = \emptyset \quad \text{for} \quad a \neq b.
\]

The corresponding subset of quasimomenta

\[
\left( \mu_1^{(a)} = \mu_{j_1}^{(a)}, \mu_2^{(a)} = \mu_{j_2}^{(a)}, \ldots, \mu_{|K_a|}^{(a)} = \mu_{j_{|K_a|}}^{(a)}; j_m \in K_a \right),
\]

where \(|K_a|\) is the cardinality of the subset \(K_a\), and \(j_k < j_{k+1}\), entering into the coordinate wave function

\[
\varphi\left( \{\nu_m\}_{1}^{|K|}; z \right) = \sum_{\sigma \in \mathcal{S}_{|K|}} (-1)^{p(\sigma)} \frac{e^{\nu_{(1)} - \nu_{(2)}}}{\sinh(\nu_{(1)} - \nu_{(2)})} \frac{e^{\nu_{(2)} - \nu_{(3)}}}{\sinh(\nu_{(2)} - \nu_{(3)})} \cdots \frac{e^{\nu_{(|K|)} - z}}{\sinh(\nu_{|K|} - z)}.
\]

Due to the alternative sum over permutations \(\sigma \in \mathcal{S}_{|K|}\) this function is antisymmetric with respect to the quasi-momenta. Finally the first factor in \((\text{4.33)}\)

\[
\left( v^+_1 \cdots v^+_M \right)_{\pi}
\]

means that for \(j_m \in K_a\) corresponding indices of \(v^+_a\) are equal to \(a\) so that \(v^+_a = v^+_a\). One can collect these operators into product \(\prod_{a=1}^{N} (v^+_a)^{|K_a|}\), consequently we have an extra sign factor \((-1)^{p(\sigma)}\).
This coordinate representation is similar to the representations obtained in [11, 12, 13] for the Gaudin models related to the simple Lie algebras (see also [43]). The $\mathbb{Z}_2$-grading of superalgebra results in extra signs, while the complicated structure of the $B_M$-operators (for the $\mathfrak{sl}(2)$ Gaudin model they are just products of $B_1$-operators $B_1(\mu_j) = X^+_j(\mu_j)$) is connected with the fact that $(v_j^+)^2 = X^+_j \neq 0$ while for $j \neq k$ $v_j^+$ and $v_k^+$ anticommute.

5 Solutions to the Knizhnik-Zamolodchikov equation

Correlation functions $\psi(z_1, \ldots, z_N)$ of a two dimensional conformal field theory satisfy the Knizhnik-Zamolodchikov equation [44]

$$\kappa \partial_{z_a} \psi(z_1, \ldots, z_N) = H^{(a)} \psi(z_1, \ldots, z_N),$$

where $H^{(a)}$ are the Gaudin Hamiltonians (3.24) and $\psi(z_1, \ldots, z_N)$ is a function of $N$ complex variables with its values in a tensor product $\mathcal{H} = \bigotimes_{a=1}^N V^{(l_a)}_a$.

A relation between the Bethe vectors of the Gaudin model related to simple Lie algebras and the solutions to the Knizhnik-Zamolodchikov equation is well known for sometime [11, 12]. Approach used here to obtain solutions to the Knizhnik-Zamolodchikov equation corresponding to conformal field theory and Lie superalgebra $\mathfrak{osp}(1|2)$ starting from B-vectors (4.2) is based on [11].

A solution in question is represented as a contour integral over the variables $\mu_1, \ldots, \mu_M$

$$\psi(z_1, \ldots, z_N) = \oint \cdots \oint \phi(\vec{\mu}|\vec{z}) \Psi(\vec{\mu}|\vec{z}) \ d\mu_1 \ldots d\mu_M,$$

where an integrating factor $\phi(\vec{\mu}|\vec{z})$ is a scalar function

$$\phi(\vec{\mu}|\vec{z}) = \prod_{i<j} \sinh(\mu_i - \mu_j) \prod_{a<b} \sinh(z_a - z_b)^{l_a l_b} \left( \prod_{k=1}^M \prod_{c=1}^N \sinh(\mu_k - z_c)^{-l_{c+k}} \right),$$

and $\Psi(\vec{\mu}|\vec{z})$ is a Bethe vector (4.2) where the corresponding Bethe equations are not imposed.

As a first step in the proof that $\psi(z_1, \ldots, z_N)$ given by (5.2) is a solution of (5.1) we differentiate the product $\phi\Psi$ with respect to $z_a$ and obtain

$$\partial_{z_a} (\phi\Psi) = \partial_{z_a} (\phi) \Psi + \phi \partial_{z_a} (\Psi).$$

Using (5.3) the first term on the right hand side can be calculated explicitly

$$\kappa \partial_{z_a} \phi = \left( \sum_{b=1}^N l_a l_b \coth(z_a - z_b) - \sum_{j=1}^M l_a \coth(z_a - \mu_j) \right) \phi = E^{(a)}_{M} \phi.$$

Furthermore, taking a residue of (5.3) at $\lambda = z_a$ we have

$$H^{(a)} \Psi = E^{(a)}_{M} \Psi + \sum_{j=1}^M (-1)^j \frac{e^{-z_a - \mu_j}}{\sinh(z_a - \mu_j)^{\beta_M(\mu_j)}} \Psi(j, a),$$

where $\beta_M(\mu_j)$ denotes the $\mu_j$-th highest root of the root system $\mathfrak{g}$ and $E^{(a)}_{M} \Psi$ is the Bethe vector of the $\mu_j$-th component of the Bethe system (4.2) with $\mu_j = z_a$. This completes the proof.
\[ \tilde{\Psi}(j,a) = \left( v_{+}^{+} B_{M-1}^{(j)} + 2 X_{a}^+ \sum_{k \neq j}^{M} (-1)^{k+\Theta(j-k)} \frac{e^{-(\mu_{j}-\mu_{k})}}{\sinh(\mu_{j} - \mu_{k})} B_{M-2}^{(j,k)} \right) \Omega_{-}. \] (5.7)

Hence (5.4) can be written as

\[ \kappa \partial_{z_{a}} (\phi \Psi) = H^{(a)} (\phi \Psi) + \phi \sum_{j=1}^{M} (-1)^{j} \frac{e^{\mu_{j}-z_{a}}}{\sinh(\mu_{j} - z_{a})} \beta_{M}(\mu_{j}) \tilde{\Psi}(j,a) + \kappa \phi \partial_{z_{a}} (\Psi). \] (5.8)

Moreover, from (5.3) we also have

\[ \kappa \partial_{\mu_{j}} \phi = \left( \sum_{a=1}^{N} (-l_{a}) \coth(\mu_{j} - z_{a}) + \sum_{k=1}^{M} \coth(\mu_{j} - \mu_{k}) \right) \phi = \beta_{M}(\mu_{j}) \phi, \] (5.9)

and from the lemma 3.4 follows

\[ \partial_{z_{a}} \Psi = \sum_{j=1}^{M} (-1)^{j} \partial_{\mu_{j}} \left( \frac{e^{\mu_{j}-z_{a}}}{\sinh(\mu_{j} - z_{a})} \tilde{\Psi}(j,a) \right) \] (5.10)

Thus, using (5.9) and (5.10), we can combine the last two terms in (5.8) into a sum of first order derivatives in \( \mu_{j} \)

\[ \kappa \partial_{z_{a}} (\phi \Psi) = H^{(a)} (\phi \Psi) + \kappa \sum_{j=1}^{M} (-1)^{j} \partial_{\mu_{j}} \left( \frac{e^{\mu_{j}-z_{a}}}{\sinh(\mu_{j} - z_{a})} \phi \tilde{\Psi}(j,a) \right) \] . (5.11)

A closed contour integration of \( \phi \Psi \) with respect to \( \mu_{1}, \ldots, \mu_{M} \) will cancel the contribution from the terms under the sum in (5.11) and therefore \( \tilde{\psi}(z_{1}, \ldots, z_{N}) \) given by (5.2) satisfies the Knizhnik-Zamolodchikov equation.

Conjugated Bethe vectors \( (B_{M} \Omega_{-})^{*} \) are entering into the solution \( \tilde{\psi}(z_{1}, \ldots, z_{N}) \) of the dual Knizhnik-Zamolodchikov equation

\[ - \kappa \frac{\partial}{\partial z_{a}} \tilde{\psi}(z_{1}, \ldots, z_{N}) = \tilde{\psi}(z_{1}, \ldots, z_{N}) H^{(a)}. \] (5.12)

The scalar product \( \left( \tilde{\psi}(z_{1}, \ldots, z_{N}) , \psi(z_{1}, \ldots, z_{N}) \right) \) does not depend on \( \{z_{j}\}_{1}^{N} \) and its quasi-classical limit \( \kappa \to 0 \) gives the norm of the Bethe vectors due to the fact that the stationary points of the contour integrals for \( \kappa \to 0 \) are solutions to the Bethe equations [13]

\[ \frac{\partial S}{\partial \mu_{j}} = \sum_{a=1}^{N} (-l_{a}) \coth(\mu_{j} - z_{a}) + \sum_{k=1}^{M} \coth(\mu_{j} - \mu_{k}) = 0, \] (5.13)

\[ S(\vec{\mu}|\vec{z}) = \kappa \ln \phi = \sum_{a<b}^{N} l_{a} l_{b} \ln (\sinh(z_{a} - z_{b})) + \sum_{i<j}^{M} \ln (\sinh(\mu_{i} - \mu_{j})) - \sum_{a=1}^{N} \sum_{j=1}^{M} l_{a} \ln (\sinh(z_{a} - \mu_{j})). \] (5.14)
According to the remark in the end of Section 4 analytical properties of the Bethe vectors of the trigonometric \( \text{osp}(1|2) \) Gaudin model coincide with the analytical properties of the trigonometric \( \text{sl}(2) \) Gaudin model. Thus, the expression for the norm of the Bethe vectors \( \Psi \) (4.2) obtained as the first term in the asymptotic expansion \( \kappa \to 0 \) coincides also

\[
(\Psi, \Psi) = \det \left( \frac{\partial^2 S}{\partial \mu_j \partial \mu_k} \right),
\]

(5.15)

\[
\frac{\partial^2 S}{\partial \mu_j^2} = \sum_{a=1}^{N} \frac{l_a}{\sinh^2(\mu_j - z_a)} - \sum_{k \neq j}^{M} \frac{1}{\sinh^2(\mu_j - \mu_k)}, \quad \frac{\partial^2 S}{\partial \mu_j \partial \mu_k} = \frac{1}{\sinh^2(\mu_j - \mu_k)},
\]

(5.16)

for \( j \neq k \).

Finally we notice that the modification of the Gaudin Hamiltonians we discussed at the end of the previous Section, can be easily transfered to the corresponding modification of the Knizhnik-Zamolodchikov equations. The modification (4.16) for the \( \text{sl}(2) \)-invariant Gaudin model was studied in [29] as a quantization of the Schlesinger system (see also [28]). This modification is related with extra factor in the integrating scalar function (5.3)

\[
\phi_j = \exp \left( \frac{S_j}{\kappa} \right), \quad j = 0, 1,
\]

(5.17)

where \( S_0 = S \) (5.14) and

\[
S_1 = S_0 + g \sum_{j=1}^{M} \mu_j - g \sum_{a=1}^{N} l_a z_a,
\]

(5.18)

correspond to the modification (4.16).

Moreover, following the lines of [29], one can try to extend the connection between the KZ equation and the Guadin model based on the modified \( L \)-operator (4.16) by extending the KZ system to include an equation of the form

\[
\left( \kappa \frac{\partial}{\partial g} - H_{\text{Rich}} \right) \psi = 0.
\]

(5.19)

However, such a straightforward generalization has failed in the trigonometric case (see below).

We can comment on the extension in the rational case [17] as a scaling limit of the trigonometric Gaudin model (4.31). The equations of original KZ system are defined by mutually commuting differential operators (see (5.4))

\[
\nabla_a = \kappa \frac{\partial}{\partial z_a} - \tilde{H}^{(a)} = \kappa \frac{\partial}{\partial z_a} - gh_a - \sum_{b \neq a} \frac{c^g_{ab} z_b}{z_a - z_b}.
\]

(5.20)

The operator

\[
\nabla_g = \kappa \frac{\partial}{\partial g} - H_{\text{Rich}} = \kappa \frac{\partial}{\partial g} - \sum_{a=1}^{N} z_a h_a - \frac{1}{2g} (c_2(gl) - h_{gl}(h_{gl} - 1)),
\]

(5.21)

is commuting pairwise with the operators \( \nabla_a \). Thus in the rational case the KZ system can be generalized to include the operator \( \nabla_g \).
To prove that the solution to the modified KZ system with \( \phi_1 \) given by (5.17) and (5.18) is a solution to the generalized KZ system we have to extend the trigonometric KZ equations (5.1) with modified Hamiltonians \( gh_a + H^{(a)} \) along the lines of [15, 16]. A difference equation must be introduced

\[
K(z_1, \ldots, z_N; g) \psi(z_1, \ldots, z_N; g) = \psi(z_1, \ldots, z_N; g - 2\kappa), \tag{5.22}
\]

instead of (5.21). The operator \( K \) is defined on the space \( \mathcal{H} \)

\[
K(z_1, \ldots, z_N; g) = \exp\left(-2\sum_{a=1}^{N} z_a h_a\right) P(g; h_{gl}, v_{gl}^+, v_{gl}^-), \tag{5.23}
\]

where the operator \( P \) depends on the global generators of the subalgebra \( \mathfrak{osp}(1|2) \subset \mathcal{L}_t \), and is constructed form the extremal projector \( p(h, v^+, v^-) \) by a shift of the Cartan generator (see Appendix). We introduce only one \( K \) operator since the rank of \( \mathfrak{osp}(1|2) \) Lie superalgebra is one. In general case, of simple Lie superalgebra of rank \( r \), one has to consider a set of \( K_k, k = 1, \ldots, r \) (see [15, 16]).

6 Conclusion

By analyzing the model related to the trigonometric \( \mathfrak{osp}(1|2) \) classical \( r \)-matrix the algebraic Bethe Ansatz approach to the Gaudin models is reviewed. The results presented in this paper are in some sense analogous to the ones we obtained for the \( \mathfrak{osp}(1|2) \)-invariant model [17]. In particular, a striking similarity between some of the most fundamental characteristics of this system and the \( sl(2) \) trigonometric Gaudin model was confirmed. Although explicitly constructed creation operators \( B_M \) (3.42) of the Bethe vectors are complicated polynomials of the \( L \)-operator entries \( v^+(\lambda) \) and \( X^+(\lambda) \), the coordinate form of the eigenfunctions differs only in signs from the corresponding states in the case of \( sl(2) \) trigonometric model, being antisymmetric functions of the quasi-momenta. Moreover, the eigenvalues and the Bethe equations coincide, provided that the \( sl(2) \) Gaudin model with integer spins is considered. Analogously, the KZ equations based on both trigonometric models and for the nontrivial magnetic field \( g \) require extension of the system of equations by the dynamical difference equation.

Let us point out that by the method presented in this paper one can construct explicitly creation operators of the Gaudin models related to trigonometric Izergin-Korepin \( r \)-matrix [3, 30] corresponding to the twisted affine algebra \( A_2^{(2)} \). Similarly to the simple Lie algebra case solutions to the Knizhnik-Zamolodchikov equation were constructed from the Bethe vectors using algebraic properties of the creation operators \( B_M \) and the Gaudin realization of the loop superalgebra \( \mathcal{L}_t(\mathfrak{osp}(1|2)) \). This interplay between the Gaudin model and the Knizhnik-Zamolodchikov equation enabled us to determine the norm of eigenfunctions of the Gaudin Hamiltonians

\[
\| \Psi(\mu_1, \ldots \mu_M; \{z_a\}_1^N) \|^2 = \det\left( \frac{\partial^2 S}{\partial \mu_j \partial \mu_k} \right).
\]

The difficult problem of correlation function calculation for general Bethe vectors

\[
C\left( \{\nu_j\}_1^M; \{\mu_i\}_1^M; \{\lambda_k\}_1^K \right) = \left( \Omega_-, B_M^*(\nu_1, \ldots \nu_M) \prod_{k=1}^{K} h(\lambda_k) B_M(\mu_1, \ldots \mu_M) \Omega_- \right)
\]

was solved nicely for the \( sl(2) \)-invariant Gaudin model in [7] using the Gauss factorization of the loop algebra group element and the appropriate Riemann-Hilbert problem. Although the
corresponding factorization is known even for the quantum superalgebra $U_q(osp(1|2))$ [15]. The final expression of the correlation functions is difficult to obtain due to the complicated structure of the creation operators $B_M(\mu_1, \ldots, \mu_M) = \text{Poly}(v^+, X^+)$ (5.42). The study of this problem is in progress and the following expression for the scalar product of the Bethe states is conjectured (see [7])

$$(\Omega_-, B_M^*(\nu_1, \ldots, \nu_M) B_M(\mu_1, \ldots, \mu_M) \Omega_-) = \sum_{\sigma \in S_M} (-1)^{p(\sigma)} \det \mathcal{M}^\sigma,$$

where the sum is over symmetric group $S_M$ and $M \times M$ matrix $\mathcal{M}^\sigma$ is given by

$$\mathcal{M}^\sigma_{jj} = \frac{e^{\mu_j - \nu_{\sigma(j)}}}{\sinh(\mu_j - \nu_{\sigma(j)})} (\rho(\mu_j) - \rho(\nu_{\sigma(j)})) - \sum_{k \neq j} \frac{e^{\mu_j - \mu_k} e^{-(\nu_{\sigma(j)} - \nu_{\sigma(k)})}}{\sinh(\mu_j - \mu_k) \sinh(\nu_{\sigma(j)} - \nu_{\sigma(k)})},$$

$$\mathcal{M}^\sigma_{jk} = \frac{e^{\mu_j - \mu_k} e^{-(\nu_{\sigma(j)} - \nu_{\sigma(k)})}}{\sinh(\mu_j - \mu_k) \sinh(\nu_{\sigma(j)} - \nu_{\sigma(k)})}, \quad \text{for } j, k = 1, 2, \ldots, M.$$

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### A Appendix: Orthosymplectic Lie superalgebra $osp(1|2)$

The rank of the orthosymplectic Lie algebra $osp(1|2)$ is one and its dimension is five [33]. The three even generators are $h, X^+, X^-$ and the two odd generators are $v^+, v^-$. The (graded) commutation relations of the generators are

$$[h, X^\pm] = \pm 2X^\pm, \quad [X^+, X^-] = h, \quad [h, v^\pm] = \pm v^\pm, \quad [v^+, v^-] = -h,$$

$$[X^\pm, v^\pm] = v^\mp, \quad [v^\pm, v^\pm] = \pm 2X^\pm, \quad [X^\pm, v^\mp] = 0.$$

(A.1)

The Casimir element is

$$c_2 = h^2 + 2 \left( X^+X^- + X^-X^+ \right) + (v^+v^- - v^-v^+)$$

$$= h^2 - h + 4X^+X^- + 2v^+v^-.$$

(A.2)

It is interesting to point out the existence of a “square root” of this element

$$c_1 = h + 2v^+v^- - \frac{1}{2}, \quad (c_1)^2 = c_2 + 1,$$

(A.3)

with a grading property $[c_1, X^\pm] = 0, [c_1, h] = 0$ and $c_1 v^\pm = -v^\pm c_1$. The finite dimensional irreducible representations $V(l)$ of the $osp(1|2)$ Lie superalgebra are parameterized by an integer $l$, so that their dimensions $2l + 1$ and the values of the Casimir element (A.2) $c_2 = l(l + 1)$ coincide with the same characteristics of the integer spin $l$ irreducible representations of $sl(2)$. 
The fundamental irreducible representation $V^{(1)}$ of $osp(1|2)$ is three dimensional. We choose a grading of the basis vectors $e_1, e_2, e_3$ to be $(0, 1, 0)$. Explicitly we have

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$v^- = (v^+)^\ast = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

together with $X^{\pm} = \pm(v^{\pm})^2$. The matrix $v^+$ in the representation $V^{(l)}$ has $2l$ non-zero elements on the second upper diagonal only, and these elements are

$$\{(v^+_{jj+1}) = (\sqrt{l}, \sqrt{l-1}, \sqrt{2}, \ldots, \sqrt{l}) \text{, } j = 1, 2, \ldots, 2l \} \quad \text{(A.4)}$$

The external projector $[46]$ for $osp(1|2)$ (on the lowest weight vectors):

$$p(h, v^+, v^-) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( (v^+)^{2k} (v^-)^{2k} + (v^+)^{2k+1} (v^-)^{2k+1} \frac{1}{h-k-1} \right) \prod_{j=1}^{k} \frac{1}{h-j}$$

$$= \left( 1 + v^+ v^- \frac{1}{h-1} \right) \left( \sum_{k=0}^{\infty} (X^+)^k (X^-)^k \prod_{j=1}^{k} \frac{1}{j(h-j-1)} \right)$$

$$= p_0(h, v^+, v^-) p_0(h, X^+, X^-) \quad \text{(A.5)}$$

here $p_0(h, X^+, X^-)$ is the usual $sl(2)$ external projector. There is no such factorization property for the external projector of the quantum superalgebra $U_q(osp(1|2))$ [31].
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