SIMPLE TRACIALLY $\mathcal{Z}$-ABSORBING C*-ALGEBRAS

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Abstract. We define a notion of tracial $\mathcal{Z}$-absorption for simple not necessarily unital C*-algebras, study it systematically, and prove its permanence properties. This extends the notion defined by Hirshberg and Orovitz for unital C*-algebras. The Razak-Jacelon algebra, simple nonelementary C*-algebras with tracial rank zero, and simple purely infinite C*-algebras are tracially $\mathcal{Z}$-absorbing. We obtain the first purely infinite examples of tracially $\mathcal{Z}$-absorbing C*-algebras which are not $\mathcal{Z}$-absorbing. We use techniques from reduced free products of von Neumann algebras to construct these examples. A stably finite example was given by Z. Niu and Q. Wang in 2021. We study the Cuntz semigroup of a simple tracially $\mathcal{Z}$-absorbing C*-algebra and prove that it is almost unperforated and the algebra is weakly almost divisible.

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1. Introduction and Main Results

X. Jiang and H. Su [31] constructed a unital separable simple infinite dimensional nuclear C*-algebra $\mathcal{Z}$ with the same K-theoretic invariant as that of $\mathcal{C}$. The Jiang-Su algebra, which is the stably finite analog of the Cuntz algebra $\mathcal{O}_\infty$, has played a central role in Elliott classification program for nuclear C*-algebras. A unital C*-algebra $A$ is $\mathcal{Z}$-absorbing if $A \cong A \otimes \mathcal{Z}$. Certain $\mathcal{Z}$-absorbing C*-algebras are classified by their ordered K-groups [38, 54] and all classes of unital simple nuclear C*-algebras for which the Elliott conjecture is confirmed are already $\mathcal{Z}$-absorbing. It was conjectured
that the properties of strict comparison, finite nuclear dimension, and $\mathcal{Z}$-absorption are equivalent for unital separable simple infinite-dimensional nuclear C*-algebras (Toms-Winter conjecture) \[52\]. It is now known that the last two conditions are equivalent \[11, 12, 51, 53\] (building upon \[39\]). We also know that $\mathcal{Z}$-absorption implies strict comparison for unital simple exact C*-algebras \[49\] and that for a unital separable simple infinite-dimensional nuclear C*-algebra with finitely many extremal traces, strict comparison is equivalent to $\mathcal{Z}$-absorption \[39\]. Finally, we know that, when there is at least one trace on a unital separable simple infinite-dimensional nuclear C*-algebra, then $\mathcal{Z}$-absorption is equivalent to strict comparison plus the uniform $\Gamma$-property \[13\].

Hirshberg and Orovitz introduced the notion of tracial $\mathcal{Z}$-absorption for unital C*-algebras \[28\]. (See Definition 3.1 below.) They show in \[28, Proposition 2.2 and Theorem 4.1\] that for simple unital separable nuclear C*-algebras, $\mathcal{Z}$-absorption is equivalent to tracial $\mathcal{Z}$-absorption. The main result of \[14\] states this equivalence in the nonunital case. The main advantage of tracial $\mathcal{Z}$-absorption is that it is easier to check in certain cases, particularly for crossed products. (See, e.g., \[32\].) This is due to the fact that tracial $\mathcal{Z}$-absorption is defined via a local property, whereas to verify $\mathcal{Z}$-absorption one needs to verify the existence of an isomorphism which may not be easy.

The aim of this paper is twofold. First, there has recently been considerable interest in the classification and structure of nonunital simple C*-algebras \[25, 26, 27\]. Nonunital tracial $\mathcal{Z}$-absorption may be useful in applying the results of the classification of nonunital C*-algebras (or, rather, stably projectionless C*-algebras), in particular, the classification of crossed products of simple nonunital C*-algebras by actions with Rokhlin type properties. This requires, however, further progress in the recently started classification project for simple nonunital C*-algebras. (See e.g. \[18\].) Second, tracial $\mathcal{Z}$-absorption can substitute for $\mathcal{Z}$-absorption when attempting to prove various regularity properties for simple C*-algebras, but holds more generally and is easier to verify. There are therefore motivations to study tracial $\mathcal{Z}$-absorption in its own right.

In this paper we extend the notion of tracial $\mathcal{Z}$-absorption to the simple nonunital case and study it systematically.

**Definition A.** We say that a simple C*-algebra $A$ is **tracially $\mathcal{Z}$-absorbing** if $A \not\cong \mathbb{C}$ and for every $x, a \in A_+$ with $a \neq 0$, every finite set $F \subseteq A$, every $\varepsilon > 0$, and every $n \in \mathbb{N}$, there is a c.p.c. order zero map $\varphi : M_n \to A$ such that:

1. $(x^2 - x\varphi(1)x - \varepsilon)_+ \lesssim a$.
2. $\|\varphi(z)\| < \varepsilon$ for any $z \in M_n$ with $\|z\| \leq 1$ and any $b \in F$.

Part (1) contains our main idea of “nonunital tracial smallness.” If $A$ is unital, then we can take $x = 1$, and it turns out that this definition extends
the definition of the unital case ([28, Definition 2.1]); see Remark 3.7. This definition motivated the second named author and Forough to introduce a suitable notion of weak tracial Rokhlin property for finite group actions on simple nonunital C*-algebras. See Definition 1.1 of [21] and the discussion after that. Also, see [20, Definition 6.6], [30], and [22, Definition 4.4] in which Definition A is quoted from the unpublished version of the current paper, showing that this idea of nonunital tracial smallness has been motivating. This definition also appeared recently in [14] which has minor overlap in results with ours.

We prove in Section 7 that if $A$ is finite then one can add the following condition to Definition A:

(3) $\|\varphi(1)a\varphi(1)\| > 1 - \varepsilon$.

This is parallel to the definition of the weak tracial Rokhlin property for finite group actions [24, 21], and is needed for the proof of a result in [1] that the permutation action on the minimal tensor product of finitely many copies of a tracially $\mathbb{Z}$-absorbing C*-algebra has the weak tracial Rokhlin property, provided that the tensor product is finite.

Different characterizations of Definition A and its permanence properties are given in Sections 3 and 4. In particular, it is preserved under inductive limits and stable isomorphism, and passes to hereditary subalgebras.

The class of simple tracially $\mathbb{Z}$-absorbing C*-algebras is large.

**Theorem B.** The following classes of C*-algebras are tracially $\mathbb{Z}$-absorbing:

1. simple nonelementary C*-algebras with tracial rank zero;
2. simple purely infinite C*-algebras;
3. simple $\mathbb{Z}$-absorbing C*-algebras.

Part (1) implies that many simple stably finite C*-algebras are tracially $\mathbb{Z}$-absorbing. It is proved in Lemma 3.12 (unital case) and Proposition 4.10 (nonunital case). The special case of Part (1) where $A$ is unital and separable follows from [22, Proposition 6.9 and Theorem 4.11] and [23, Theorem 5.9] in which the authors use different methods from ours. A similar result by Matui and Sato is that every simple separable unital nuclear infinite-dimensional C*-algebra with tracial rank zero $\mathbb{Z}$-absorbing [39, Theorem 5.4]. (The nonunital version of this result follows from Part (1) and [14, Theorem A].) An example of a simple separable unital C*-algebra with tracial rank zero (hence with stable rank one and real rank zero [36, 37]) which is not $\mathbb{Z}$-absorbing is constructed in [40]. It is nonnuclear and exact. As far as we know, this stably finite example and purely infinite examples of Theorem C (below) are the first known tracially $\mathbb{Z}$-absorbing C*-algebras which are not $\mathbb{Z}$-absorbing.

Part (2) follows directly form Definition A and characterizations of pure infiniteness [33, Proposition 5.4] (see Example 3.11). We obtain purely infinite C*-algebras which are not $\mathbb{Z}$-absorbing.
**Theorem C.** There are simple separable unital purely infinite (hence tracially $\mathcal{Z}$-absorbing) C*-algebras which are not $\mathcal{Z}$-absorbing.

The proof of this result is based on techniques from reduced free products of von Neumann algebras [17, 10]. See Proposition 5.5 and Example 5.6. As a concrete example, the reduced C*-algebra free product

$$(M_2 \otimes M_2) \star_r C([0, 1])$$

is simple separable unital purely infinite (hence tracially $\mathcal{Z}$-absorbing) but not $\mathcal{Z}$-absorbing (Example 5.7). By [28, Theorem 4.1], this example is nonnuclear.

We remark that an interesting result of [22] says that every simple separable tracially approximately divisible C*-algebra is purely infinite or has stable rank one [22, Corollary 6.5]. Since tracial approximate divisibility is equivalent to tracial $\mathcal{Z}$-absorption for simple separable C*-algebras [22, Theorem 4.11], we see that every simple separable tracially $\mathcal{Z}$-absorbing C*-algebra is purely infinite or has stable rank one.

Part (3) of Theorem B follows from tracial $\mathcal{Z}$-stability of $\mathcal{Z}$ [28, proposition 2.2] and that tracial $\mathcal{Z}$-absorption passes to minimal tensor products (Theorem 5.1). In particular, the Razak-Jacelon algebra $\mathcal{W}$ is a stably projectionless C*-algebra which is tracially $\mathcal{Z}$-absorbing.

It is shown in [28, Theorem 3.3] that if $A$ is a simple tracially $\mathcal{Z}$-absorbing C*-algebra then the Cuntz semigroup $W(A)$ is almost unperforated and hence $A$ has strict comparison. We extend this to the nonunital case. We define a notion of strict comparison (called weak strict comparison) in a sense suitable for nonunital C*-algebras (Definition 6.8). Purely infinite simple C*-algebras have weak strict comparison, and for simple unital C*-algebras, we show that our definition reduces to strict comparison. The proof of different parts of the following result is given in Section 6.

**Theorem D.** Let $A$ be a simple tracially $\mathcal{Z}$-absorbing C*-algebra. Then

1. $W(A)$ is almost unperforated;
2. $A$ has weak strict comparison;
3. $A$ is weakly almost divisible.

Weak almost divisibility in Part (3) is a weak version of divisibility introduced in [42], and is useful to show that certain crossed products have strict comparison in the absence of tracial $\mathcal{Z}$-stability [42].

The structure of this paper is as follows. In Section 2, we recall some facts and prove some results on Cuntz subequivalence. In Section 3, we define the notion of tracial $\mathcal{Z}$-absorption for simple not necessarily unital C*-algebras (Definition 3.6) and investigate its primary properties. In Section 4, we show that tracial $\mathcal{Z}$-absorption passes to hereditary subalgebras, matrix algebras, and direct limits, and that it is Morita invariant. In Section 5, we compare tracial $\mathcal{Z}$-absorption with $\mathcal{Z}$-absorption. In Section 6, we study the Cuntz
semigroup of tracially $\mathcal{Z}$-absorbing C*-algebras. In Section 7, we verify finite tracially $\mathcal{Z}$-absorbing C*-algebras.

We use the following (standard) notation. For a C*-algebra $A$, $A_+$ denotes the positive cone of $A$. Also, $A^+$ denotes the unitization of $A$ (adding a new identity even if $A$ is unital), while $A^\sim = A$ if $A$ is unital and $A^\sim = A^+$ if $A$ is nonunital. The notation $a \approx b$ means $\|a - b\| < \varepsilon$. We write $\mathcal{K} = K(\ell^2)$ and $M_n = M_n(\mathbb{C})$. We take $\mathbb{Z} = \mathbb{Z}/n\mathbb{Z}$. (The $p$-adic integers will never appear.) We take $\mathbb{N} = \{1, 2, \ldots\}$. We abbreviate “completely positive contractive” to “c.p.c.”. For any C*-algebra $A$, we denote its tracial state space by $T(A)$.

2. Preliminaries on Cuntz subequivalence

In this section we recall some results and provide some lemmas on Cuntz subequivalence needed in the subsequent sections. Some of these are already in the literature. See [2] for an extensive discussion of Cuntz subequivalence.

Let $A$ be a C*-algebra. For $a, b \in A_+$, we write $a \lesssim b$ if $a$ is Cuntz subequivalent to $b$, i.e., there is a sequence $(v_n)_{n \in \mathbb{N}}$ in $A$ such that $\|a - v_nbv^*_n\| \to 0$. We write $a \sim b$ if both $a \lesssim b$ and $b \lesssim a$. The following lemma will be used in many places.

**Lemma 2.1** ([34], Lemma 2.2). Let $A$ be a C*-algebra, let $a, b \in A$ be positive, and let $\varepsilon > 0$. If $\|a - b\| < \varepsilon$ then there is a contraction $d \in A$ such that $(a - \varepsilon)_+ = dbd^*$. In particular, $(a - \varepsilon)_+ \lesssim b$.

Observe that if $a \lesssim b$ in $A$ then, by definition, there is a sequence $(v_n)_{n \in \mathbb{N}}$ in $A$ such that $\|a - v_nbv^*_n\| \to 0$. But it is not the case that there always exists a bounded sequence with this property. However, we have the following lemma. It may be in the literature. We give a proof for the sake of completeness. (There is a similar result in [34, Lemma 2.4(ii)], but there is a gap in the proof, and the statement may well be false [50]. Its proof essentially implies our lemma.)

**Lemma 2.2.** Let $A$ be a C*-algebra, let $a, b \in A_+$, and let $\delta > 0$. If $a \lesssim (b - \delta)_+$ then there exists a sequence $(v_n)_{n \in \mathbb{N}}$ in $A$ such that $\|a - v_nbv^*_n\| \to 0$ and $\|v_n\| \leq \|a\|^{1/2}\delta^{-1/2}$ for every $n \in \mathbb{N}$.

**Proof.** Let $n \in \mathbb{N}$. Since $a \lesssim (b - \delta)_+$, there exists $w_n \in A$ such that $\|a - w_n(b - \delta)_+w^*_n\| < \frac{1}{n}$. By Lemma 2.1, there exists $d_n \in A$ such that $(a - \frac{1}{n})_+ = d_nw_n(b - \delta)_+w^*_nd^*_n$.

By [34, Lemma 2.4(i)], there exists $v_n \in A$ such that

$$(a - \frac{1}{n})_+ = v_nbv^*_n \quad \text{and} \quad \|v_n\| \leq \|(a - \frac{1}{n})_+\|^{1/2}\delta^{-1/2}.$$  

Now $v_nbv^*_n \to a$ and $\|v_n\| \leq \|a\|^{1/2}\delta^{-1/2}$. \hfill $\Box$

**Lemma 2.3.** Let $A$ be a C*-algebra, let $x \in A$ be nonzero, and let $b \in A_+$. Then for any $\varepsilon > 0$,

$$(xbx^* - \varepsilon)_+ \lesssim x(b - \varepsilon/\|x\|^2)_+x^* \lesssim (b - \varepsilon/\|x\|^2)_+.$$
If \( \|x\| \leq 1 \) then \((xbx^* - \varepsilon)_+ \preceq x(b - \varepsilon)_+ x^* \preceq (b - \varepsilon)_+\).

**Proof.** For the first statement, we have

\[
\|xbx^* - x(b - \varepsilon/\|x\|^2)_+ x^*\| \leq \|x\|^2 \|b - (b - \varepsilon/\|x\|^2)_+\|
\]

\[
\leq \|x\|^2 \cdot \left(\frac{\varepsilon}{\|x\|^2}\right) = \varepsilon.
\]

Let \( \rho > 0 \). Using [33, Lemma 2.5(i)] at the first step, and Lemma 2.1 at the second step, we have

\[
((xbx^* - \varepsilon)_+ - \rho)_+ = (xbx^* - \varepsilon - \rho)_+ \preceq x(b - \varepsilon/\|x\|^2)_+ x^*.
\]

Since \( \rho > 0 \) is arbitrary, it follows from [33, Proposition 2.6] that

\[
(xbx^* - \varepsilon)_+ \preceq x(b - \varepsilon/\|x\|^2)_+ x^*.
\]

Clearly \( x(b - \varepsilon/\|x\|^2)_+ x^* \preceq (b - \varepsilon/\|x\|^2)_+ x^* \).

If \( \|x\| \leq 1 \) then \( \varepsilon/\|x\|^2 \geq \varepsilon \) and so

\[
(xbx^* - \varepsilon)_+ \preceq x(b - \varepsilon/\|x\|^2)_+ x^* \leq x(b - \varepsilon)_+ x^* \preceq (b - \varepsilon)_+.
\]

This completes the proof. \( \square \)

In general, the reverse of the Cuntz subequivalence relation in Lemma 2.3 does not hold. However, a weaker version holds. See Lemma 2.5 below. We need the following proposition in its proof. It should be in the literature. We provide a proof for the sake of completeness.

**Proposition 2.4.** Let \( A \) be a C*-algebra, let \( K \subseteq \mathbb{C} \) be a compact set, and let \( f: K \to \mathbb{C} \) be a continuous function. Set

\[
S = \{x \in M(A) : x \text{ is normal and } \text{sp}(x) \subseteq K\}.
\]

Then the functional calculus map \( x \mapsto f(x) \), from \( S \) to \( M(A) \), is strictly continuous. Furthermore, if \((x_i)_{i \in I} \) is a net in \( M(A)_+ \) which converges to \( x \in M(A) \) strictly, then \( x \in M(A)_+ \) and \((x_i - \varepsilon)_+ \to (x - \varepsilon)_+ \) strictly.

**Proof.** First note that \( S \) is norm bounded, by \( \sup\{|\lambda| : \lambda \in K\} \).

Let \((x_i)_{i \in I} \) be a net in \( S \) which tends to \( x \in S \) strictly. We have to show that \( f(x_i) \to f(x) \) strictly. Since \((x_i)_{i \in I} \) is bounded and the product is jointly continuous in the strict topology on bounded subsets of \( M(A) \), and since \( a \mapsto a^* \) is strictly continuous, the statement holds when \( f(\zeta) \) is a polynomial in \( \zeta \) and \( \overline{\zeta} \). Now let \( y \in A \). We need to show that \( f(x_i)y \to f(x)y \) and \( yf(x_i) \to yf(x) \) in norm in \( A \). We do only the first; the second is similar. Let \( \varepsilon > 0 \). Choose \( \delta > 0 \) such that \( \delta \|y\| < \varepsilon/3 \). There is a polynomial \( g(\zeta) \) in \( \zeta \) and \( \overline{\zeta} \) such that \( \sup_{\zeta \in K} |g(\zeta) - f(\zeta)| < \delta \). Since \( g(x_i)y \to g(x)y \), there is \( i_0 \in I \) such that \( \|g(x_i)y - g(x)y\| < \frac{\varepsilon}{3} \) for any \( i \in I \) with \( i \geq i_0 \). Thus for
$i \geq i_0$ we have
\[
\|f(x_i)y - f(x)y\| \\
\leq \|f(x_i)y - g(x_i)y\| + \|g(x_i)y - g(x)y\| + \|g(x)y - f(x)y\| \\
< \delta \|y\| + \frac{\epsilon}{3} + \delta \|y\| < \epsilon.
\]

So $f(x_i) \to f(x)$ strictly.

Since $M(A)_+$ is strictly closed in $M(A)$, the second part of the statement follows from the first. \hfill $\square$

We will use the following lemma in the proof of Proposition 3.18.

**Lemma 2.5.** Let $A$ be a $\sigma$-unital $C^*$-algebra, let $x$ be a strictly positive element, let $b \in M(A)$, and let $\delta > 0$. Then for every $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ we have
\[
(x(b - \delta)_+ x - \epsilon)_+ \preceq x (x^{1/n} bx^{1/n} - \delta)_+ x \preceq (x^{1/n} bx^{1/n} - \delta)_+.
\]

**Proof.** Consider $A \subseteq M(A)$ and note that $x^{1/n} \to 1_{M(A)}$ strictly. So $x^{1/n} bx^{1/n} \to b$ strictly. Thus, by Proposition 2.4, $(x^{1/n} bx^{1/n} - \delta)_+ \to (b - \delta)_+$ strictly. Hence $x(x^{1/n} bx^{1/n} - \delta)_+ x \to x(b - \delta)_+ x$ in $A$, and so there is $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$,
\[
\|x(x^{1/n} bx^{1/n} - \delta)_+ x - x(b - \delta)_+ x\| < \epsilon.
\]

Therefore, using Lemma 2.1 at the first step, we have
\[
(x(b - \delta)_+ x - \epsilon)_+ \preceq x (x^{1/n} bx^{1/n} - \delta)_+ x \preceq (x^{1/n} bx^{1/n} - \delta)_+.
\]

This completes the proof. \hfill $\square$

3. **Tracial $\mathcal{Z}$-absorption**

In this section we define tracial $\mathcal{Z}$-absorption for simple not necessarily unital $C^*$-algebras (Definition 3.6). We show that this definition extends the unital version defined by Hirshberg and Orovitz (see Remark 3.7). As an example, every simple purely infinite $C^*$-algebra is tracially $\mathcal{Z}$-absorbing. In the separable case, we give an equivalent definition for tracial $\mathcal{Z}$-absorption in terms of the central sequence algebra (Proposition 3.14). At the end of this section we give another version of tracial $\mathcal{Z}$-absorption (called strong tracial $\mathcal{Z}$-absorption; see Definition 3.15) and we compare it with Definition 3.6.

We recall the definition of a unital tracially $\mathcal{Z}$-absorbing $C^*$-algebra.

**Definition 3.1 ([28], Definition 2.1).** A unital $C^*$-algebra $A$ is called **tracially $\mathcal{Z}$-absorbing** if $A \not\cong \mathbb{C}$ and for every finite set $F \subseteq A$, every $\epsilon > 0$, every $a \in A_+ \setminus \{0\}$, and every $n \in \mathbb{N}$, there is a c.p.c. order zero map $\varphi: M_n \to A$ such that the following hold:

1. $1 - \varphi(1) \preceq a$.
2. $\|([\varphi(z), b])\| < \epsilon$ for any $z \in M_n$ with $\|z\| \leq 1$ and any $b \in F$.
Remark 3.2. Although Definition 3.1 makes sense for unital C*-algebras, it works well mainly for unital simple C*-algebras. For example, in [28] the assumption of simplicity appears in almost all of the results. Because of this, in this paper we define tracial $\mathcal{Z}$-absorption only for (not necessarily unital) simple C*-algebras. However, we will not use the assumption of simplicity in some results (e.g., Theorems 4.1 and 4.7).

We need the following equivalent version of tracial $\mathcal{Z}$-absorption in order to define this notion for nonunital C*-algebras.

Lemma 3.3. Let $A$ be a unital C*-algebra with $A \not\cong \mathbb{C}$. Then $A$ is tracially $\mathcal{Z}$-absorbing if and only if for every finite set $F \subseteq A$, every $\varepsilon > 0$, every $a \in A_+ \setminus \{0\}$, and every $n \in \mathbb{N}$ there is a c.p.c. order zero map $\varphi: M_n \to A$ such that:

1. $(1 - \varphi(1) - \varepsilon)_+ \preceq a$.
2. $\|\varphi(z), b\| < \varepsilon$ for any $z \in M_n$ with $\|z\| \leq 1$ and any $b \in F$.

Proof. The forward implication is obvious because $(1 - \varphi(1) - \varepsilon)_+ \preceq 1 - \varphi(1)$.

Now suppose that we are given $F$, $\varepsilon$, $a$, and $n$ as in the statement. We have to find a c.p.c. order zero map $\psi: M_n \to A$ such that:

3. $1 - \psi(1) \preceq a$.
4. $\|\psi(z), b\| < \varepsilon$ for any $z \in M_n$ with $\|z\| \leq 1$ and any $b \in F$.

Choose $\delta > 0$ such that

$$\delta < \frac{\varepsilon}{1 + 2 \max\{\|x\| : x \in F\}}.$$  

By assumption there is a c.p.c. order zero map $\varphi: M_n \to A$ such that $(1 - \varphi(1) - \delta)_+ \preceq a$ and $\|\varphi(z), b\| < \delta$ for any $z \in M_n$ with $\|z\| = 1$. Define a continuous function $f: [0, 1] \to [0, 1]$ by

$$f(\lambda) = \begin{cases} \frac{\lambda}{1 - \delta} & 0 \leq \lambda \leq 1 - \delta \\ 1 & 1 - \delta < \lambda \leq 1 \end{cases}.$$  

Using functional calculus for c.p.c. order zero maps ([55, Corollary 4.2]), set $\psi = f(\varphi)$. Thus $\psi: M_n \to A$ is a c.p.c. order zero map, and, similar to the proof of [3, Lemma 2.8], we have

$$\|\varphi(z) - \psi(z)\| \leq \delta \|z\|$$  

for any $z \in M_n$ with $\|z\| \leq 1$, and

$$1 - \psi(1) = \frac{1}{1 - \delta}(1 - \varphi(1) - \delta)_+ \sim (1 - \varphi(1) - \delta)_+ \preceq a.$$  

By (3.2) we get (3). To prove (4), let $z \in M_n$ with $\|z\| \leq 1$ and let $x \in F$.

By (3.1) we get

$$\|\psi(z)x - x\psi(z)\|$$

$$\leq \|\psi(z)x - \varphi(z)x\| + \|\varphi(z)x - x\varphi(z)\| + \|x\varphi(z) - x\psi(z)\|$$

$$\leq \delta \|x\| + \delta + \delta \|x\| = \delta(1 + 2\|x\|) < \varepsilon.$$  

This completes the proof. \qed
One may also work with $\epsilon$-order zero maps to check tracial $\mathcal{Z}$-absorption. Recall that a c.p.c. map $\varphi: M_n \to A$ is $\epsilon$-order zero if whenever $y, z \in M_n$ are orthogonal positive elements of norm at most 1, then $\|\varphi(y)\varphi(z)\| < \epsilon$.

**Proposition 3.4.** Let $A$ be a unital $C^*$-algebra with $A \not\cong \mathbb{C}$. Then $A$ is tracially $\mathcal{Z}$-absorbing if and only if for any finite set $F \subseteq A$, $\epsilon > 0$, $a \in A_+ \setminus \{0\}$, and $n \in \mathbb{N}$, there is a c.p.c. $\epsilon$-order zero map $\varphi: M_n \to A$ such that:

1. $(1 - \varphi(1) - \epsilon)_+ \lesssim a$.
2. $\|[\varphi(z), b]\| < \epsilon$ for any $z \in M_n$ with $\|z\| \leq 1$ and any $b \in F$.

**Proof.** This follows easily from Lemma 3.3 and [35, Proposition 2.5].

The following lemma will be used several times in the sequel.

**Lemma 3.5.** Let $A$ be a $C^*$-algebra and let $a \in A_+ \setminus \{0\}$. Suppose that an element $x \in A_+$ has the following property. For any finite set $F \subseteq A$, any $\epsilon > 0$, and any $n \in \mathbb{N}$, there is a c.p.c. order zero map $\varphi: M_n \to A$ such that:

1. $(x^2 - x\varphi(1)x - \epsilon)_+ \lesssim a$.
2. $\|[\varphi(z), b]\| < \epsilon$ for any $z \in M_n$ with $\|z\| \leq 1$ and any $b \in F$.

Then every positive element $y \in \overline{Ax}$ also has the same property.

**Proof.** Let $F \subseteq A$ be a finite subset, let $\epsilon > 0$, and let $n \in \mathbb{N}$. Let $\delta > 0$ satisfy

$$\delta < \min \left(1, \frac{\epsilon}{4(2\|y\| + 1)}\right).$$

Since $y \in \overline{Ax}$, there is $c \in A \setminus \{0\}$ such that $\|y - cx\| < \delta$. Set

$$\eta = \min \left(\epsilon, \frac{\epsilon}{2\|c\|^2}\right).$$

By assumption there is a c.p.c. order zero map $\varphi: M_n \to A$ such that

1. $(x^2 - x\varphi(1)x - \eta)_+ \lesssim a$.
2. $\|[\varphi(z), b]\| < \eta$ for any $z \in M_n$ with $\|z\| \leq 1$ and any $b \in F$.

By Lemma 2.3 we have

$$\left(cx^2c^* - cx\varphi(1)xc^* - \frac{\epsilon}{2}\right)_+ \lesssim \left(x^2 - x\varphi(1)x - \frac{\epsilon}{2\|c\|^2}\right)_+ \lesssim (x^2 - x\varphi(1)x - \eta)_+ \lesssim a.$$
Also we have
\[\|y^2 - y\varphi(1)y - (cx^2c^* + cx\varphi(1)x^* - \frac{\epsilon}{2})\|\]
\[\leq \|y^2 - y\varphi(1)y - cx^2c^* + cx\varphi(1)x^*\| + \frac{\epsilon}{2}\]
\[\leq \|y^2 - cx^2c^*\| + \|cy - cx^2c^*\| + \|y\varphi(1)y - y\varphi(1)x^*\|\]
\[+ \|y\varphi(1)x^* - cx\varphi(1)x^*\| + \frac{\epsilon}{2}\]
\[\leq \delta\|y\| + \delta\|cx\| + \delta\|y\| + \delta\|cx\| + \frac{\epsilon}{2}\]
\[< 2\delta(2\|y\| + 1) + \frac{\epsilon}{2} < \epsilon.\]

Thus,
\[\left(y^2 - y\varphi(1)y - \epsilon\right) \preceq \left(cx^2c^* - cx\varphi(1)x^* - \frac{\epsilon}{2}\right) \preceq a.\]

Since \(\eta < \epsilon\), for any \(b \in F\) we have \(||[\varphi(z), b]\| < \epsilon\). \(\square\)

**Definition 3.6.** We say that a simple C*-algebra \(A\) is *tracially Z-absorbing* if \(A \not\cong \mathbb{C}\) and for every \(x, a \in A^+\) with \(a \neq 0\), every finite set \(F \subseteq A\), every \(\varepsilon > 0\), and every \(n \in \mathbb{N}\), there is a c.p.c. order zero map \(\varphi: M_n \rightarrow A\) such that:

1. \(\left(x^2 - x\varphi(1)x - \varepsilon\right) \preceq a.\)
2. \(||[\varphi(z), b]\| < \varepsilon\) for any \(z \in M_n\) with \(||z\|| \leq 1\) and any \(b \in F\).

Some definitions of this sort, such as for the weak tracial Rokhlin property (see, [21], for a definition in nonunitl case), have an additional condition, which here would be to assume \(||a|| = 1\) and require \(||\varphi(1)a\varphi(1)|| > 1 - \varepsilon\). We don’t include such a condition here because it doesn’t appear in Definition 3.1 ([28, Definition 2.1]). When \(A\) is finite in a suitable sense, it is automatic that such a condition can be satisfied, as we show in Section 7.

**Remark 3.7.** Let \(A\) be a simple C*-algebra.

1. If \(A\) is unital, Definition 3.6 is equivalent to Definition 3.1, even without assuming simplicity. In fact, by Lemma 3.3, Definition 3.6 implies Definition 3.1. The converse holds by Lemma 3.3 and taking \(x = 1\) in Lemma 3.5.
2. If \(A\) is \(\sigma\)-unital, it is enough that the properties in Definition 3.6 hold for some strictly positive element \(x \in A\). This follows from Lemma 3.5.
3. In Definition 3.6, the “c.p.c. order zero” condition on \(\varphi\) may be replaced by “c.p.c. \(\varepsilon\)-order zero” (cf. Proposition 3.4).
4. Definition 3.6 holds for \(A = \mathbb{C}\) by taking \(\varphi = 0\). To rule this out, the technical condition \(A \not\cong \mathbb{C}\) is added. In fact, every nonzero simple tracially Z-absorbing C*-algebra is infinite dimensional (Lemma 3.9 below) and not type I (Corollary 4.2 below).
(5) In Definition 3.6, it is enough to have the map \( \varphi: M_n \to A \) for some \( l \in \mathbb{N} \) and each \( n > l \). In fact, if Definition 3.6 holds for \( n = km \) with \( k, m \in \mathbb{N} \), then it also holds for \( n = m \).

**Remark 3.8.** It is easy to see that in Definition 3.6 it is enough to take \( x \) in a norm dense subset of \( A_+ \). Moreover, if \( (e_i)_{i \in I} \) is an approximate identity for \( A \), it is enough to take \( x \) from the set \( \{ e_i : i \in I \} \). This follows from Lemma 3.5 because the set \( \{ x \in A_+ : x \in Ae_i \text{ for some } i \in I \} \) is dense in \( A_+ \).

**Lemma 3.9.** Let \( A \) be a nonzero simple tracially \( \mathbb{Z} \)-absorbing \( C^* \)-algebra. Then \( A \) is infinite dimensional.

**Proof.** Suppose \( \dim(A) < \infty \). Choose \( n \in \mathbb{N} \) such that \( n^2 > \dim(A) \). We claim that there is no nonzero c.p.c. order zero map \( \varphi: M_n \to A \). Indeed, if we had such a map \( \varphi \), then [55, Theorem 3.3] would provide a homomorphism \( \pi: M_n \to A \) such that \( \varphi(z) = \pi(z)\varphi(1) \) for all \( z \in M_n \). But \( \dim(A) < n^2 \), so \( \pi = 0 \). Thus \( \varphi = 0 \). This proves the claim.

Now it follows from Definition 3.6 that for every \( x, a \in A_+ \) with \( a \neq 0 \) and every \( \varepsilon > 0 \), we have \( (x^2 - \varepsilon)_+ \preceq a \). So \( x \preceq a \) for every \( x, a \in A_+ \) with \( a \neq 0 \). Since \( \dim(A) < \infty \), this can only happen if \( A \cong \mathbb{C} \), which is excluded in Definition 3.6, or \( A = 0 \). \( \square \)

It may happen that \( \varphi \) in Definition 3.6 is zero. See Example 3.11 below. First we recall (see before Proposition 1.6 of [15]) that a simple \( C^* \)-algebra \( A \) (not necessarily unital) is purely infinite if for every \( a \in A_+ \setminus \{0\} \) there is an infinite projection in \( aAa \). For our purposes, the following characterization is often more useful. It is the simple case of [33, Definition 4.1], according to which a not necessarily simple \( C^* \)-algebra \( A \) is purely infinite if and only if there are no characters on \( A \) and for all \( a,b \in A_+ \) such that \( a \in AbA \), we have \( a \preceq b \).

**Definition 3.10 ([33], Proposition 5.4).** A simple \( C^* \)-algebra \( A \) is purely infinite if and only if \( A \not\cong \mathbb{C} \) and \( a \sim b \) for any \( a,b \in A_+ \setminus \{0\} \).

**Example 3.11.** Let \( A \) be a simple purely infinite \( C^* \)-algebra. Then \( A \) is tracially \( \mathbb{Z} \)-absorbing. In fact, we can take \( \varphi = 0 \) in Definition 3.6.

**Lemma 3.12.** Let \( A \) be an infinite dimensional simple unital \( C^* \)-algebra with tracial rank zero [37, Definition 3.6.2]. Then \( A \) is tracially \( \mathbb{Z} \)-absorbing.

In Proposition 4.10 below, we remove the requirement that \( A \) be unital. A variant of this lemma is stated in [39, Lemma 5.1] with the assumption that \( A \) is separable, and is used to show that every simple separable unital nuclear infinite-dimensional \( C^* \)-algebra with tracial rank zero is approximately divisible, and hence \( \mathbb{Z} \)-absorbing [39, Theorem 5.4].
Proof of Lemma 3.12. We prove the following statement, which clearly implies the conclusion. Let $a \in A \setminus \{0\}$, let $F \subseteq A$ be finite, let $\varepsilon > 0$, and let $n \in \mathbb{N}$. Then there is a homomorphism $\varphi : M_n \to A$ such that:

1. $1 - \varphi(1) \lesssim a$.
2. $||[\varphi(z), x]|| < \varepsilon$ for any $z \in M_n$ with $\|z\| \leq 1$ and any $x \in F$.

Since $A$ has real rank zero (by [37, Theorem 3.6.11]) and is unital, infinite dimensional, and simple, there are nonzero orthogonal projections $q_1, q_2 \in aAa$. Apply the definition of tracial rank zero to find a projection $p \in A$, $m, r(1), r(2), \ldots, r(m) \in N$, and a unital injective homomorphism $\psi$ from $B = \bigoplus_{l=1}^{m} M_{r(l)}$ to $pAp$ such that:

3. $||p, x|| < \frac{\varepsilon}{6}$ for all $x \in F$.
4. $\text{dist}(pxp, \psi(B)) < \frac{\varepsilon}{6}$ for all $x \in F$.
5. $1 - p \lesssim q_1$.

Choose $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \inf_{\tau \in T(A)} \tau(q_2).$$

Choose $\nu \in \mathbb{N}$ so large that $mn > n_0$.

For $l = 1, 2, \ldots, m$, let $(e_{j,k}^{(l)})_{j,k=1,2,\ldots,r(l)}$ be the standard system of matrix units for $M_{r(l)}$. Use [43, Lemma 2.3] to choose projections $f_0^{(l)}, f_1^{(l)}, f_2^{(l)}, \ldots, f_{\nu n}^{(l)} \in A(e_{1,1})A\psi(e_{1,1})$ such that

$$f_1^{(l)} \sim f_2^{(l)} \sim \cdots \sim f_{\nu n}^{(l)}, \quad f_0^{(l)} \lesssim f_1^{(l)}, \quad \text{and} \quad \sum_{j=0}^{\nu n} f_j^{(l)} = \psi(e_{1,1}^{(l)}).$$

There is then a homomorphism $\beta_l : M_{\nu n} \to A$ such that the images of the diagonal matrix units in $M_{\nu n}$ are $f_1^{(l)}, f_2^{(l)}, \ldots, f_{\nu n}^{(l)}$. Define a homomorphism $\beta : M_{\nu n} \to A$ by

$$\beta(z) = \sum_{l=1}^{m} \sum_{k=1}^{r(l)} \psi(e_{k,1}^{(l)}) \beta_l(z) \psi(e_{1,k}^{(l)})$$

for $z \in M_{\nu n}$. Clearly $\beta(z)$ commutes with $\psi(b)$ for all $z \in M_{\nu n}$ and $b \in B$, and $\beta(1) \leq p$.

Let $x \in F$ and $z \in M_n$ satisfy $\|z\| \leq 1$. Apply (4) to choose $b \in B$ such that $\|\psi(b) - pxp\| < \frac{\varepsilon}{6}$. We have $[\beta(z), (1 - p)x(1 - p)] = 0$, and so by (3) at the third step, we get

$$||[\beta(z), x]|| \leq ||[\beta(z), pxp]|| + ||[\beta(z), px(1 - p)]|| + ||[\beta(z), (1 - p)xp]||$$

$$\leq 2||\beta(z)|| \cdot \|pxp - \psi(b)\|$$

$$+ \|\beta(z)|| \cdot \|px(1 - p)\| + ||\beta(z)|| \cdot \|(1 - p)xp\|$$

$$< 2 \left( \frac{\varepsilon}{6} \right) + \left( \frac{\varepsilon}{6} \right) + \left( \frac{\varepsilon}{6} \right) < \varepsilon.$$
For every \( \tau \in T(A) \), for \( l = 1, 2, \ldots, m \) we have
\[
\tau(f_1^{(l)}) = \tau(f_2^{(l)}) = \cdots = \tau(f_m^{(l)}), \quad \tau(f_0^{(l)}) \leq \tau(f_1^{(l)}),
\]
and
\[
\sum_{j=0}^{m} \tau(f_j^{(l)}) = \tau(\psi(e_{1,1}^{(l)})).
\]
Therefore,
\[
\tau(f_0^{(l)}) \leq \frac{1}{\nu n + 1} \tau(\psi(e_{1,1}^{(l)})).
\]
Since
\[
p - \beta(1) = \sum_{l=1}^{m} \sum_{k=1}^{r_{(l)}} \psi(e_{k,1}^{(l)}) f_0^{(l)} \psi(e_{1,k}^{(l)}),
\]
it follows that
\[
\tau(p - \beta(1)) = \sum_{l=1}^{m} r(l) \tau(f_0^{(l)}) \leq \frac{1}{\nu n + 1} \sum_{l=1}^{m} r(l) \tau(\psi(e_{1,1}^{(l)})
\]
\[
= \frac{\tau(p)}{\nu n + 1} \leq \frac{1}{\nu n + 1} < \frac{1}{n_0} < \tau(q_2).
\]
Since \( A \) has strict comparison of projections using traces [37, Theorem 3.7.2], it follows that \( (p - \beta(1)) \lesssim q_2 \). We already have \( 1 - p \lesssim q_1, q_1 q_2 = 0 \), and \( q_1 + q_2 \in aAa \), so \( 1 - \beta(1) \lesssim a \). This is (1) with \( \beta \) in place of \( \varphi \).

To complete the proof, we define \( \varphi: M_n \to A \) by \( \varphi(z) = \beta(1_{M_n} \otimes z) \) for \( z \in M_n \).

Now we give an equivalent definition for the tracial \( \mathbb{Z} \)-absorption in terms of the central sequence algebra.

**Notation 3.13.** If \((A_i)_{i \in I}\) is a family of \( \text{C}^* \)-algebra, we let \( \prod_{i \in I} A_i \) denote that set of all families \( (a_i)_{i \in I} \) in the set theoretic product of the \( A_i \) such that \( \sup_{i \in I} \| a_i \| < \infty \). (This is the product in the category of \( \text{C}^* \)-algebras.)

For a \( \text{C}^* \)-algebra \( A \), we write
\[
A_\infty = \prod_{n \in \mathbb{N}} A / \bigoplus_{n \in \mathbb{N}} A.
\]
We identify \( a \in A \) with the equivalence class in \( A_\infty \) of the corresponding constant sequence \( (a_n)_{n \in \mathbb{N}} \). We write \( A_\infty \cap A' \) for the relative commutant of \( A \) in \( A_\infty \).

The algebra \( A_\infty \cap A' \) is often written \( A^\infty \). We warn that some authors write \( A^\infty \) for our \( A_\infty \).

**Proposition 3.14.** Let \( A \) be a simple separable \( \text{C}^* \)-algebra. The following statements are equivalent:

(1) \( A \) is tracial \( \mathbb{Z} \)-absorbing.
(2) For every $a,x \in A_+$ with $a \neq 0$ and every $n \in \mathbb{N}$, there exists a c.p.c. order zero map $\psi: M_n \to A_\infty \cap A'$ such that $x^2 - x\psi(1)x \precsim a$ in $A_\infty$.

If moreover $A$ is unital, then these statements are equivalent to the following:

(3) For every nonzero $a \in A_+$ and every $n \in \mathbb{N}$, there exists a c.p.c. order zero map $\psi: M_n \to A_\infty \cap A'$ such that $1_{A_\infty} - \psi(1) \precsim a$ in $A_\infty$.

Proof. (1) $\Rightarrow$ (2): Let $\{a_1,a_2,\ldots\}$ be a dense subset of the closed unit ball of $A$. Put $F_m = \{a_1,a_2,\ldots,a_m\}$ for $m \in \mathbb{N}$. Let $a,x \in A_+$ with $a \neq 0$ and let $n \in \mathbb{N}$. We may assume that $\|a\| = 1$. Tracial $\mathcal{Z}$-absorption of $A$ implies that for every $m \in \mathbb{N}$ there exists a c.p.c. order zero map $\varphi_m: M_n \to A$ such that

$$(x^2 - x\varphi_m(1)x - \frac{1}{m})_+ \precsim (a - \frac{1}{m})_+$$

and $\|\varphi_m(z),b\| < \frac{1}{m}$ for every $z \in M_n$ with $\|z\| \leq 1$ and every $b \in F_m$. Now let $\pi: \prod_{m \in \mathbb{N}} A \to A_\infty$ be the quotient map, and define $\psi: M_n \to A_\infty$ by

$$\psi(z) = \pi((\varphi_m(z))_{m \in \mathbb{N}})$$

for $z \in M_n$. Then $\psi(M_n) \subseteq A_\infty \cap A'$. By Lemma 2.2, for every $m \in \mathbb{N}$ there exists $v_m \in A$ such that $\|v_m\| \leq 2\|x\|^2$ and

$$\|x^2 - x\varphi_m(1)x - \frac{1}{m} - v_mav_m^*\| < \frac{1}{m}.$$ 

Hence $\|x^2 - x\varphi_m(1)x - v_mav_m^*\| < \frac{2}{m}$. Put $v = \pi((v_m)_{m \in \mathbb{N}}) \in A_\infty$. Then we have $x^2 - x\psi(1)x = vav^* \precsim a$ in $A_\infty$.

(2) $\Rightarrow$ (1): Suppose that (2) holds. To show that $A$ is tracially $\mathcal{Z}$-absorbing, we verify the condition in Remark 3.7(3) (Definition 3.6 with order zero replaced by $\varepsilon$-order zero). So let $a,x,\varepsilon,F,$ and $n$ be as in Definition 3.6. Choose a c.p.c. order zero map $\psi: M_n \to A_\infty \cap A'$ such that $x^2 - x\psi(1)x \precsim a$ in $A_\infty$. Since $M_n$ is nuclear, there is a sequence $(\psi_m)_{m \in \mathbb{N}}$ of c.p.c. maps from $M_n$ to $A$ such that, if we define

$$\tilde{\psi} = (\psi_m)_{m \in \mathbb{N}}: M_n \to \prod_{m \in \mathbb{N}} A,$$

then $\psi = \pi \circ \tilde{\psi}$. Since $F$ is finite and $M_n$ is finite dimensional, it follows that there is $m \in \mathbb{N}$ such that $\|\psi_m(z),y\| < \varepsilon$ for every $z \in M_n$ with $\|z\| \leq 1$ and every $y \in F$, such that $\|\psi_m(y)\psi_m(z)\| < \varepsilon$ for all $y,z \in M_n$ with $0 \leq y,z \leq 1$ and $yz = 0$, and such that there is $w \in A$ with $\|x^2 - x\psi_m(1)x - waw^*\| < \varepsilon$. Thus, $(x^2 - x\psi_m(1)x - \varepsilon)_+ \precsim a$. Hence, $\psi_m$ satisfies the conditions of Remark 3.7(3), and so $A$ is tracially $\mathcal{Z}$-absorbing. Thus (1) holds.

Suppose that $A$ is unital. The implication (3) $\Rightarrow$ (2) is obvious since $x^2 - x\psi(1)x \precsim 1_{A_\infty} - \psi(1)$. Also, (2) $\Rightarrow$ (3) follows by taking $x = 1$ in (2). □

To conclude this section, we give another version of tracial $\mathcal{Z}$-absorption (called strong tracial $\mathcal{Z}$-absorption, Definition 3.15 below), and we compare
it with Definition 3.6. By Remark 3.7(1) and Remark 3.17, both definitions are equivalent to [28, Definition 2.1] in the simple unital case. Definition 3.15 was our first proposal for the definition of tracial $\mathcal{Z}$-absorption. At first glance, it seems to be a more natural definition extending the unital case. However, it turned out that Definition 3.6 is better. We do not have examples to show that these two definitions are actually different.

**Definition 3.15.** We say that a simple C*-algebra $A$ is strongly tracially $\mathcal{Z}$-absorbing if $A \not\cong \mathbb{C}$ and for every $x, a \in A_+$ with $a \neq 0$, every finite set $F \subseteq A$, every $\varepsilon > 0$, and every $n \in \mathbb{N}$, there is a c.p.c. order zero map $\varphi: M_n \to A$ such that:

1. $x^2 - x\varphi(1)x \preceq a$.
2. $\|\varphi(z), b\| < \varepsilon$ for any $z \in M_n$ with $\|z\| \leq 1$ and any $b \in F$.

As an example, by taking $\varphi = 0$, we see that every simple (not necessarily unital) purely infinite C*-algebra is strongly tracially $\mathcal{Z}$-absorbing. (Compare with Example 3.11.) In particular, Examples 5.6 and 5.7 show that strong tracial $\mathcal{Z}$-absorption does not imply $\mathcal{Z}$-absorption.

**Lemma 3.16.** Let $A$ be a C*-algebra and let $x, y \in A_+$. If $y \in \overline{xA}$ then for any $c \in (A^\times)_+$ we have $ycy \preceq xcx$.

**Proof.** Choose $(b_n)_{n \in \mathbb{N}}$ in $A$ such that $yb_n \to y$. Then $b_n^*xcxb_n \to ycy$ and for any $n \in \mathbb{N}$, we have $b_n^*xcxb_n \preceq xc$. Taking the limit in the latter gives the result.

**Remark 3.17.** By Lemma 3.16, if the condition stated in Definition 3.15 holds for some $x \in A_+$ then it also holds for any positive element $y \in \overline{xA}$. In particular, Definition 3.15 is equivalent to Definition 3.6 in the unital case. Moreover, it follows from Lemma 3.16 that if $A$ is $\sigma$-unital, then $A$ is strongly tracially $\mathcal{Z}$-absorbing if the properties in Definition 3.15 hold for some strictly positive element $x \in A$.

The following result clarifies the relation between Definitions 3.6 and 3.15.

**Proposition 3.18.** Let $A$ be a simple $\sigma$-unital C*-algebra and let $x \in A$ be a strictly positive element. Then $A$ is strongly tracially $\mathcal{Z}$-absorbing if and only if for every $a \in A_+$ with $a \neq 0$, every finite set $F \subseteq A$, every $\varepsilon > 0$, and every $n \in \mathbb{N}$, there is a c.p.c. order zero map $\varphi: M_n \to A$ such that:

1. $(x^{2/m} - x^{1/m}\varphi(1)x^{1/m} - \varepsilon)_+ \preceq a$ for every $m \in \mathbb{N}$.
2. $\|\varphi(z), b\| < \varepsilon$ for any $z \in M_n$ with $\|z\| \leq 1$ and any $b \in F$.

**Proof.** Assume first that $A$ is strongly tracially $\mathcal{Z}$-absorbing. Let $a, F, \varepsilon$, and $n$ be as in the hypotheses, and apply Definition 3.15 with these elements and $x$ as given, obtaining a c.p.c. order zero map $\varphi: M_n \to A$. This map satisfies (2) by construction. For (1), let $m \in \mathbb{N}$. Then $x^{1/m} \in \overline{xA}$, so, using Lemma 3.16 at the second step,

$$(x^{2/m} - x^{1/m}\varphi(1)x^{1/m} - \varepsilon)_+ \preceq x^{2/m} - x^{1/m}\varphi(1)x^{1/m} \preceq x^2 - x\varphi(1)x.$$
For the converse, let \( a, F, \varepsilon, \) and \( n \) be as in Definition 3.15. Choose \( \delta > 0 \) with
\[
\delta < \frac{\varepsilon}{1 + 2 \max \{ \| y \| : y \in F \}}.
\]
By assumption there is a c.p.c. order zero map \( \varphi : M_n \to A \) such that (1) and (2) hold for \( \delta \) instead of \( \varepsilon \). Define a map \( f : [0, 1] \to [0, 1] \) by
\[
f(\lambda) = \begin{cases} 
\frac{\lambda}{1-\delta} & 0 \leq \lambda \leq 1 - \delta \\
1 & 1 - \delta < \lambda \leq 1.
\end{cases}
\]
Using functional calculus for c.p.c. order zero maps ([55, Corollary 4.2]), set \( \psi = f(\varphi) \). As in the proof of Lemma 3.3, in \( A^\sim \) we have
\[(3.3) \quad 1 - \psi(1) = \frac{1}{1-\delta} (1 - \varphi(1) - \delta)_+.
\]
We show that (1) in Definition 3.15 holds, i.e., \( x^2 - x\psi(1)x \preceq a \). Let \( \eta > 0 \). Then by (3.3) we have
\[(1 - \delta)(x^2 - x\psi(1)x) - \eta)_+ = (x(1 - \varphi(1) - \delta)_+x - \eta)_+.
\]
By Lemma 2.5, there is \( m \geq 1 \) such that
\[(3.4) \quad (x(1 - \varphi(1) - \delta)_+x - \eta)_+ \preceq (x^{1/m}(1 - \varphi(1))x^{1/m} - \delta)_+.
\]
The right hand side of (3.4) is Cuntz subequivalent to \( a \) by (1). Since \( \eta > 0 \) is arbitrary, and using (3.3) at the first step, we get
\[x^2 - x\psi(1)x \sim x(1 - \varphi(1) - \delta)_+x \preceq a.
\]
The proof of (2) in Definition 3.15 is similar to the last part of the proof of Lemma 3.3. So \( A \) is strongly tracially \( \mathcal{Z} \)-absorbing.

\[\square\]

4. PERMANENCE PROPERTIES

In this section we show that tracial \( \mathcal{Z} \)-absorption passes to hereditary C*-subalgebras (Theorem 4.1). In particular, no simple tracially \( \mathcal{Z} \)-absorbing C*-algebra is type I. Then we show that tracial \( \mathcal{Z} \)-absorption passes to matrix algebras and direct limits. We also prove that it is preserved under Morita equivalence in the class of \( \sigma \)-unital simple C*-algebras (Corollary 4.12). Moreover, we compare our nonunital definition of tracial \( \mathcal{Z} \)-absorption with the unital definition when the algebra has an approximate identity (not necessarily increasing) consisting of projections (Proposition 4.9).

**Theorem 4.1.** Let \( A \) be a simple tracially \( \mathcal{Z} \)-absorbing C*-algebra and let \( B \) be a hereditary C*-subalgebra of \( A \). Then \( B \) is also tracially \( \mathcal{Z} \)-absorbing.

**Proof.** Let \( x \in B_+ \), let \( a \in B_+ \setminus \{0\} \), let \( \varepsilon > 0 \), let \( F \subseteq B \) be finite, and let \( n \in \mathbb{N} \). We must find a c.p.c. order zero map \( \varphi : M_n \to B \) such that:

(1) \( (x^2 - x\varphi(1)x - \varepsilon)_+ \preceq a \).

(2) \( \| [\varphi(z), b] \| < \varepsilon \) for any \( z \in M_n \) with \( \| z \| \leq 1 \) and any \( b \in F \).
The case \( x = 0 \) is trivial, so, by Lemma 3.5, we may assume that \( \|x\| = 1 \). Also, we may assume that \( \|b\| \leq 1 \) for all \( b \in F \). Let \( y_0 \) be the sum of \( x \) with the positive and negative parts of the real and imaginary parts of all the elements of \( F \), and set \( y = \|y_0\|^{-1}y_0 \). Then \( y \in B_+ \), \( \|y\| = 1 \), and \( F \cup \{x\} \subseteq yBy \). Since \( (y^{1/m})_{m \in \mathbb{N}} \) is an approximate identity for \( yBy \), there is \( m \in \mathbb{N} \) such that

\[
(4.1) \quad \|y^{1/k}b\| < \frac{\varepsilon}{m} \quad \text{and} \quad \|y^{1/k}b - b\| < \frac{\varepsilon}{m}
\]

for \( k \in \{m, 2m, m/2\} \) and \( b \in F \cup \{x\} \). Choose \( \delta > 0 \) for \( \frac{\varepsilon}{m} \) according to [3, Lemma 2.7]. We may assume that \( \delta < \frac{\varepsilon}{m} \). Since \( A \) is tracially \( \mathcal{Z} \)-absorbing, there exists a c.p.c. order zero map \( \psi: M_n \to A \) such that:

1. \( \langle y^{2/m} - y^{1/m}\psi(1)y^{1/m} - \delta \rangle_+ \lesssim a \).
2. \( \|\psi(z), b\| < \delta \) for every \( z \in M_n \) with \( \|z\| \leq 1 \) and every \( b \in F \cup \{x, y^{2/m}, y^{1/m}\} \).

Let us show that the conditions of [3, Lemma 2.7] are satisfied for \( y^{1/m} \), \( \psi \), and \( B \) in place of \( x \), \( \varphi_0 \), and \( B \). Since \( \|y\| \leq 1 \), we have \( 0 \leq y^{1/m} \leq 1 \). Also by (4), \( \|\psi(z), y^{1/m}\| < \delta \) for any \( z \in M_n \) with \( \|z\| \leq 1 \). Finally, we claim that \( \text{dist}(\psi(z)y^{1/m}, B) < \delta \) for any \( z \in M_n \) with \( \|z\| \leq 1 \). In fact, \( y^{1/(2m)}\psi(z)y^{1/(2m)} \in B \) (since \( B \) is hereditary) and by (4) we have

\[
\|\psi(z)y^{1/m} - y^{1/(2m)}\psi(z)y^{1/(2m)}\| \leq \|\psi(z)y^{1/(2m)} - y^{1/(2m)}\psi(z)\| < \delta,
\]

proving the claim. Hence by the choice of \( \delta \) using [3, Lemma 2.7], there is a c.p.c. order zero map \( \varphi: M_n \to B \) such that for every \( z \in M_n \) with \( \|z\| \leq 1 \) we have

\[
(4.2) \quad \|\psi(z)y^{1/m} - \varphi(z)\| < \frac{\varepsilon}{6}.
\]

Now we show that \( \varphi \) has properties (1) and (2) above. For (2), let \( b \in F \) and let \( z \in M_n \) satisfy \( \|z\| \leq 1 \). Using (4.1), (4.2), and (4) at the second step, we have

\[
\|\varphi(z)b - b\varphi(z)\| \leq \|\varphi(z) - \psi(z)y^{1/m}\| \cdot \|b\| + \|\psi(z)\| \cdot \|y^{1/m}b - by^{1/m}\| + \|\psi(z)b - b\psi(z)\| \cdot \|y^{1/m}\| + \|b\| \cdot \|\psi(z)y^{1/m} - \varphi(z)\| < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \delta + \frac{\varepsilon}{6} < \frac{\varepsilon}{3} + \frac{\varepsilon}{6} \leq \varepsilon.
\]

To prove (1), first we use (4.2) and (4) at the third step to get

\[
\|\langle y^{2/m} - y^{1/m}\psi(1)y^{1/m} - \delta \rangle_+ - \langle y^{2/m} - y^{1/(2m)}\varphi(1)y^{1/(2m)} \rangle_+\|
\leq \delta + \|y^{1/m}\psi(1)y^{1/m} - y^{1/(2m)}\varphi(1)y^{1/(2m)}\| \leq \delta + \|y^{1/(2m)}\| \cdot \|y^{1/(2m)}\psi(1) - \psi(1)y^{1/(2m)}\| \cdot \|y^{1/m}\| + \|y^{1/(2m)}\| \cdot \|\psi(1)y^{1/m} - \varphi(1)\| \cdot \|y^{1/(2m)}\| < \delta + \frac{\varepsilon}{6} < \frac{\varepsilon}{3} + \frac{\varepsilon}{6} < \frac{\varepsilon}{2}.
\]

Thus by (3) we have

\[
(4.3) \quad \langle y^{2/m} - y^{1/(2m)}\varphi(1)y^{1/(2m)} - \frac{\varepsilon}{2} \rangle_+ \lesssim \langle y^{2/m} - y^{1/m}\psi(1)y^{1/m} - \delta \rangle_+ \lesssim a.
\]
At the second step using \( \|x - xy^{1/(2m)}\| = \|x - y^{1/(2m)}x\| \) and (4.1) several times, we get
\[
\|\left( x^2 - x\varphi(1)x \right) - \left( xy^{2/m}x - xy^{1/(2m)}\varphi(1)y^{1/(2m)}x - \frac{x}{2} \right) \| \\
\leq \frac{\varepsilon}{2} + \|x\| \cdot \|x - y^{2/m}x\| + \|x - xy^{1/(2m)}\| \cdot \|\varphi(1)x\| \\
+ \|xy^{1/(2m)}\varphi(1)x\| \cdot \|x - y^{1/(2m)}x\| \\
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Using Lemma 2.3 at the second step and (4.3) at the third step, we now have
\[
\left( x^2 - x\varphi(1)x - \varepsilon \right) \preceq \left( xy^{2/m}x - xy^{1/(2m)}\varphi(1)y^{1/(2m)}x - \frac{x}{2} \right) \\
\succeq (y^{2/m} - y^{1/(2m)}\varphi(1)y^{1/(2m)} - \frac{x}{2}) \preceq a.
\]

It remains to prove that \( B \not\cong \mathbb{C} \). So suppose \( B \cong \mathbb{C} \). Since \( B \) is hereditary and \( A \) is simple, it follows that there is a Hilbert space \( H \) such that \( A \cong K(H) \). We have \( A \not\cong \mathbb{C} \) and \( A \neq 0 \), so \( A \) has a hereditary subalgebra \( D \) isomorphic to \( M_2 \). Since \( D \not\cong \mathbb{C} \), by what we have already done, \( D \) is tracially \( \mathbb{Z} \)-absorbing. This contradicts Lemma 3.9.

**Corollary 4.2.** Let \( A \) be a nonzero simple tracially \( \mathbb{Z} \)-absorbing C*-algebra. Then \( A \) is not type I. In particular, \( K \) is not tracially \( \mathbb{Z} \)-absorbing.

**Proof.** Suppose that \( H \) is a Hilbert space and \( K(H) \) is tracially \( \mathbb{Z} \)-absorbing. By Lemma 3.9, \( H \) is infinite dimensional. Let \( p \) a projection in \( K(H) \) with \( \text{rank}(p) = 2 \). Then \( pK(H)p \cong M_2 \) would be tracially \( \mathbb{Z} \)-absorbing by Theorem 4.1. But this is not the case by Lemma 3.9.

**Theorem 4.3.** Let \( A \) be a simple C*-algebra. The following are equivalent:

1. \( A \) is tracially \( \mathbb{Z} \)-absorbing.
2. Every \( \sigma \)-unital hereditary C*-subalgebra of \( A \) is tracially \( \mathbb{Z} \)-absorbing.
3. For every positive element \( x \) in \( A \), \( x\mathbb{A}x \) is tracially \( \mathbb{Z} \)-absorbing.

**Proof.** By Theorem 4.1, (1) implies (2). Clearly, (2) is equivalent to (3) since a hereditary C*-subalgebra \( B \subseteq A \) is \( \sigma \)-unital if and only if there is \( x \in A_+ \) such that \( B = x\mathbb{A}x \). It remains to show that (3) implies (1). Suppose that (3) holds. Thus, in particular, \( A \not\cong \mathbb{C} \). Suppose that we are given \( a, x \in A_+ \setminus \{0\}, \varepsilon > 0 \), a finite set \( F \subseteq A \), and \( n \in \mathbb{N} \). Choose \( y \in A_+ \) such that \( F \cup \{x\} \subseteq y\mathbb{A}y \) (e.g., let \( y \) be the sum of \( x \) with the positive and negative parts of the real and imaginary parts of the elements of \( F \)). We get \( \varphi \) as in Definition 3.6 by applying this definition to \( y\mathbb{A}y \).

**Lemma 4.4** (cf. [28], Lemma 2.3). Let \( A \) be a simple C*-algebra which is not of type I and let \( n \in \mathbb{N} \). For every nonzero positive element \( a \in M_n \otimes A = M_n(A) \), there exists a nonzero positive element \( b \in A \) such that \( 1 \otimes b \preceq a \).

**Proof.** For \( j, k = 1, 2, \ldots, n \), let \( e_{j,k} \in M_n \) be the standard matrix unit. Since \( a \) is a positive and nonzero element of \( M_n(A) \), there exists \( j \) such that
(e_{ij} \otimes 1_{A^*})a(e_{ij} \otimes 1_{A^*}) \neq 0$. By replacing $a$ with $(e_{ij} \otimes 1_{A^*})a(e_{ij} \otimes 1_{A^*})$ we may assume that $a$ is of the form $e_{ij} \otimes c$ for some nonzero positive $c \in A$. Use \cite[Lemma 2.1]{45} to find $b_1, b_2, \ldots, b_n \in A_+ \setminus \{0\}$ such that $b_1 \sim b_2 \sim \cdots \sim b_n$, such that $b_jb_k = 0$ for $j \neq k$, and such that $b_1 + b_2 + \cdots + b_n \in cAc$. Then
\[
1 \otimes b \sim e_{ij} \otimes (b_1 + b_2 + \cdots + b_n) \preceq e_{ij} \otimes c,
\]
as desired. \hfill \Box

The following is a nonunital analog of \cite[Lemma 2.4]{28}.

**Proposition 4.5.** Let $A$ be a simple tracially $\mathcal{Z}$-absorbing $C^*$-algebra and let $n \in \mathbb{N}$. Then $M_n(A)$ is also tracially $\mathcal{Z}$-absorbing.

**Proof.** Let $x, a \in M_n(A)_+$ with $a \neq 0$, let $\varepsilon > 0$, let $m \in \mathbb{N}$, and let $F \subseteq M_n(A)$ be finite.

By Lemma 4.4, it is enough to consider the case where $a = 1 \otimes b \in M_n \otimes A = M_n(A)$ for some nonzero element $b \in A_+$. Also, there exists $y \in A_+$ such that $x \in M_n(A)(1 \otimes y)$. Thus, by Lemma 3.5, it is enough to take $x = 1 \otimes y$ with $y \in A_+$. Let $E$ be the set of all matrix entries of elements of $F$. Put $\delta = \varepsilon/n^2$. Since $A$ is tracially $\mathcal{Z}$-absorbing, there exists a c.p.c. order zero map $\varphi: M_m \to A$ such that:

1. $(y^2 - y\varphi(1)y - \delta)_+ \preceq b$.
2. $||[\varphi(z), d]|| < \delta$ for any $z \in M_n$ with $||z|| \leq 1$ and any $d \in E$.

Define a c.p.c. order zero map $\psi: M_m \to M_n \otimes A$ by $\psi(z) = 1 \otimes \varphi(z)$ for $z \in M_n$. Then
\[
(x^2 - x\varphi(1)x - \varepsilon)_+ = 1 \otimes (y^2 - y\varphi(1)y - \varepsilon)_+ \preceq 1 \otimes (y^2 - y\varphi(1)y - \delta)_+ \preceq 1 \otimes b = a.
\]

Also, for any $z \in M_n$ with $||z|| \leq 1$ and any $d = (d_{j,k})_{1 \leq j, k \leq n} \in F$ we have
\[
||[\psi(z), d]|| = ||[1 \otimes \varphi(z), d]|| = ||([\varphi(z), d_{j,k})_{1 \leq j, k \leq n}|| \leq \sum_{j,k=1}^n ||[\varphi(z), d_{j,k}]|| < n^2\delta = \varepsilon.
\]

We have shown that $M_n(A)$ is tracially $\mathcal{Z}$-absorbing. \hfill \Box

Proposition 4.5 also follows from Theorem 5.1 below.

**Definition 4.6.** We say that a $C^*$-algebra $A$ is locally tracially $\mathcal{Z}$-absorbing if for any $\varepsilon > 0$ and any finite subset $F \subseteq A$ there exists a simple tracially $\mathcal{Z}$-absorbing $C^*$-subalgebra $B$ of $A$ such that $F \subseteq \varepsilon B$, that is, for any $a \in F$ there exists $b \in B$ such that $||a - b|| < \varepsilon$.

**Theorem 4.7.** Let $A$ be a simple locally tracially $\mathcal{Z}$-absorbing $C^*$-algebra. Then $A$ is tracially $\mathcal{Z}$-absorbing.
Proof. It follows from Definition 4.6 that $A \not\cong \mathbb{C}$. Let $x, a, F, \varepsilon$, and $n$ be as in Definition 3.6. Write $F = \{f_1, f_2, \ldots, f_m\}$. We may assume that $\|a\| = 1$, that $\|x\| \leq \frac{1}{\delta}$, and that $\varepsilon < 1$. Set $E = F \cup \{x^{1/2}, a^{1/2}\}$. Choose $\delta > 0$ such that

$$\delta < \frac{\varepsilon}{4}, \quad (2 + \delta)\delta < \frac{\varepsilon}{12}, \quad \text{and} \quad (2\|a^{1/2}\| + \delta)\delta < \frac{\varepsilon}{12}.$$ 

By assumption there is a simple tracially $\mathcal{Z}$-absorbing C*-subalgebra $B$ of $A$ such that $E \subseteq B$.

In particular, there exists $b \in B$ such that $\|a^{1/2} - b\| < \delta$. Then

$$\|b^*b - a\| \leq \|b^*b - b^*a^{1/2}\| + \|b^*a^{1/2} - a\|$$

$$\leq \|b\|\delta + \|a^{1/2}\|\delta \leq (\|a^{1/2}\| + \delta + \|a^{1/2}\|)\delta < \frac{\varepsilon}{12}.$$ 

Set $d = (b^*b - \frac{\varepsilon}{12})_+$. Then the estimate just done implies that

$$d \precsim a. \tag{4.4}$$

Also, $d \neq 0$ since $\|b^*b\| > 1 - \frac{\varepsilon}{12}$ and $\varepsilon < 1$.

Similarly, there exists $y \in B$ such that $\|x^{1/2} - y\| < \delta$, and the element $w = y^*y$ satisfies $\|w - x\| < \frac{\varepsilon}{12}$. Since $\|x\| \leq \frac{1}{\delta}$ and $\frac{\varepsilon}{12} < \frac{1}{12}$, we have $\|w\| < 1$. Therefore

$$\|w^2 - x^2\| < \frac{\varepsilon}{6}. \tag{4.5}$$

Choose $e_1, e_2, \ldots, e_m \in B$ such that $\|e_j - f_j\| < \frac{\varepsilon}{4}$ for $j = 1, 2, \ldots, m$. Since $B$ is tracially $\mathcal{Z}$-absorbing, there exists a c.p.c. order zero map $\varphi: M_n \to B$ such that:

1. $(w^2 - w\varphi(1)w - \frac{\varepsilon}{4})_+ \precsim d.$
2. $\|\varphi(z), e_j\| < \frac{\varepsilon}{4}$ for $j \in \{1, 2, \ldots, m\}$ and $z \in M_n$ with $\|z\| \leq 1$.

Using (4.5), $\|w\| \leq 1$, and $\|x\| \leq \frac{1}{\delta}$ at the third step, we then get

$$\|(w^2 - w\varphi(1)w - \frac{\varepsilon}{4})_+ - (x^2 - x\varphi(1)x)\|$$

$$\leq \|(w^2 - w\varphi(1)w - \frac{\varepsilon}{4})_+ - w^2 - w\varphi(1)w\|$$

$$+ \|(w^2 - w\varphi(1)w) - (x^2 - x\varphi(1)x)\|$$

$$\leq \frac{\varepsilon}{4} + \|w^2 - x^2\| + \|w\varphi(1)x - w\varphi(1)x\| + \|w\varphi(1)x - x\varphi(1)x\|$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{12} < \varepsilon.$$ 

Hence, using (1) at the second step and (4.4) at the third step, we get

$$(x^2 - x\varphi(1)x - \varepsilon)_+ \precsim (w^2 - w\varphi(1)w - \frac{\varepsilon}{4})_+ \precsim d \precsim a.$$ 

Finally, for any $z \in M_n$ with $\|z\| \leq 1$ and for $j \in \{1, 2, \ldots, m\}$, by (2) we have

$$\|\varphi(z), f_j\| \leq \|\varphi(z)f_j - \varphi(z)e_j\| + \|\varphi(z)e_j - e_j\varphi(z)\| + \|e_j\varphi(z) - f_j\varphi(z)\|$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon.$$ 

So $A$ is tracially $\mathcal{Z}$-absorbing. \qed
Corollary 4.8. Let $A$ be a $C^*$-algebra which is the direct limit of a system of simple tracially $\mathbb{Z}$-absorbing $C^*$-algebras. Then $A$ is also simple and tracially $\mathbb{Z}$-absorbing.

Proof. This follows immediately from Theorem 4.7. □

For $C^*$-algebras with an approximate identity (not necessarily increasing) consisting of projections we give a “unital” equivalent statement for our “nonunital” definition of tracial $\mathbb{Z}$-absorption.

Proposition 4.9. Let $A$ be a simple $C^*$-algebra with an approximate identity $(p_i)_{i \in I}$ (not necessarily increasing) consisting of projections. Then $A$ is tracially $\mathbb{Z}$-absorbing if and only if $p_iAp_i$ is tracially $\mathbb{Z}$-absorbing in the unital sense for each $i \in I$.

Proof. The statement follows from Remark 3.7(1), Theorem 4.1, and Theorem 4.7. □

We can now generalize Lemma 3.12 to the nonunital case.

Proposition 4.10. Let $A$ be a nonelementary simple not necessarily $C^*$-algebra with tracial rank zero (which is tracially AF; see [36, Definition 2.1]). Then $A$ is tracially $\mathbb{Z}$-absorbing.

Proof. By [21, Corollary A.22(1)], the algebra $A$ has real rank zero. Therefore $A$ has an approximate identity $(p_i)_{i \in I}$ (not necessarily increasing) consisting of projections. (This also follows from [36, Corollary 2.8].) The algebras $p_iAp_i$ have tracial rank zero by [21, Corollary A.22(3)]. So they are tracially $\mathbb{Z}$-absorbing by Lemma 3.12. Now Proposition 4.9 implies that $A$ is tracially $\mathbb{Z}$-absorbing. □

Proposition 4.11. Let $A$ be a simple $C^*$-algebra. Then $A$ is tracially $\mathbb{Z}$-absorbing if and only if $A \otimes \mathcal{K}$ is tracially $\mathbb{Z}$-absorbing.

Proof. The forward implication follows from Proposition 4.5 and Corollary 4.8. The converse follows from Theorem 4.1 because $A$ is isomorphic to a hereditary subalgebra of $A \otimes \mathcal{K}$. □

Corollary 4.12. Let $A$ and $B$ be simple $C^*$-algebras.

(1) If $A$ and $B$ are stably isomorphic, and $A$ is tracially $\mathbb{Z}$-absorbing, then so is $B$.

(2) If $A$ and $B$ are $\sigma$-unital and Morita equivalent, and $A$ is tracially $\mathbb{Z}$-absorbing, then so is $B$.

Proof. Part (1) follows from Proposition 4.11. For (2), let $A$ and $B$ be $\sigma$-unital and Morita equivalent. By [46, Theorem 5.55], $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$. Now the statement follows from (1). □

It seems plausible that Corollary 4.12(2) holds even if $A$ and $B$ are not $\sigma$-unital, but more work is needed.
5. $\mathcal{Z}$-absorption and tracial $\mathcal{Z}$-absorption

In this section we compare tracial $\mathcal{Z}$-absorption with $\mathcal{Z}$-absorption. We show that $\mathcal{Z}$-absorption implies tracial $\mathcal{Z}$-absorption, that the converse is false, but that the converse is true when the algebra is separable and nuclear \cite{A6} (see Remark 5.3).

First we prove the following general result.

**Theorem 5.1.** Let $A$ be a simple tracially $\mathcal{Z}$-absorbing $C^*$-algebra and let $B$ be a simple $C^*$-algebra. Then $A \otimes_{\min} B$ is tracially $\mathcal{Z}$-absorbing.

**Proof.** Suppose we are given $\varepsilon > 0$, a finite subset $F \subseteq A \otimes_{\min} B$, $v, c \in (A \otimes_{\min} B)_+$ with $c \neq 0$, and $n \in \mathbb{N}$. We have to find a c.p.c. order zero map $\varphi: M_n \to A \otimes_{\min} B$ such that the following hold:

1. $(v^2 - v\varphi(1)v - \varepsilon)_+ \precsim c$.
2. $||\varphi(z), f|| < \varepsilon$ for any $z \in M_n$ with $||z|| \leq 1$ and any $f \in F$.

We may assume that there are $m \in \mathbb{N}$ and

$$r_1, r_2, \ldots, r_m \in A \quad \text{and} \quad s_1, s_2, \ldots, s_m \in B$$

such that $||r_j|| \leq 1$ and $||s_j|| \leq 1$ for $j = 1, 2, \ldots, m$ and such that $F = \{r_j \otimes s_j : j = 1, 2, \ldots, m\}$. By Kirchberg’s Slice Lemma \cite{Kirchberg2}, there are nonzero elements $a \in A_+$ and $b \in B_+$ such that $a \otimes b \precsim c$ in $A \otimes_{\min} B$. Choose $\delta > 0$ such that $\delta < \varepsilon/3$. Since $A \otimes_{\min} B$ has an approximate identity consisting of elementary tensors $x \otimes y$ with $x$ and $y$ positive and of norm at most one, by Remark 3.8 we may assume that $v = x \otimes y$ with $x \in A_+$, $y \in B_+$, and $||x||$, $||y|| \leq 1$. By \cite{Jiang1}, Proposition 2.7(v)], there is $k \in \mathbb{N}$ such that:

3. $(y^2 - \delta)_+ \precsim b \otimes 1_k$ in $M_\infty(B)$.

Since $A$ is simple and tracially $\mathcal{Z}$-absorbing, Corollary 4.2 implies that $A$ is not type I. Now Lemma 4.4 implies that there is $a_0 \in A_+ \setminus \{0\}$ such that:

4. $a_0 \otimes 1_k \precsim a$ in $M_\infty(A)$.

By assumption $A$ is tracially $\mathcal{Z}$-absorbing. Applying Definition 3.6 with $\{r_1, r_2, \ldots, r_m\}$ in place of $F$, with $\delta$ in place of $\varepsilon$, with $n$ as given, and with $a_0$ in place of $a$, we obtain a c.p.c. order zero map $\varphi_0: M_n \to A$ such that the following hold:

5. $(x^2 - x\varphi_0(1) + \delta)_+ \precsim a_0$.
6. $||\varphi_0(z), r_j|| < \delta$ for $j = 1, 2, \ldots, m$ and any $z \in M_n$ with $||z|| \leq 1$.

Choose $e \in B_+$ such that

7. $||e|| \leq 1$, $||yey - y^2|| < \delta$, and $||es_j - s_j|| < \delta$ and $||s_j e - s_j|| < \delta$ for $j = 1, 2, \ldots, m$.

Now define $\varphi: M_n \to A \otimes_{\min} B$ by $\varphi(z) = \varphi_0(z) \otimes e$ for $z \in M_n$. Then $\varphi$ is a c.p.c. order zero map.
We show that (1) and (2) hold. To prove (1), first recall that $v = x \otimes y$ at the first step and use (7) at the last step to get
\[
\| (v^2 - v \varphi(1)v) - (x^2 - x \varphi_0(1)x - \delta) \otimes (y^2 - \delta)_+ \|
\leq \| x^2 \otimes y^2 - x \varphi_0(1)x \otimes yey - (x^2 - x \varphi_0(1)x) \otimes y^2 \| + 2\delta
\]
\[= \| x \varphi_0(1)x \otimes (y^2 - yey) \| + 2\delta \]
\[\leq \| yey - y^2 \| + 2\delta < 3\delta. \]

Then use Lemma 2.1 at the first step, use (3) and (5) at the second step, and use (4) at the fourth step, to get
\[
(v^2 - v \varphi(1)v - 3\delta)_+ \leq (x^2 - x \varphi_0(1)x - \delta)_+ \otimes (y^2 - \delta)_+
\]
\[\leq a_0 \otimes (b \otimes 1_k) \sim (a_0 \otimes 1_k) \otimes b \leq a \otimes b \leq c. \]
Thus
\[
(v^2 - v \varphi(1)v - \varepsilon)_+ \leq (v^2 - v \varphi(1)v - 3\delta)_+ \leq c. \]

For (2), let $f \in F$. Then there is $j \in \{1, 2, \ldots, m\}$ such that $f = r_j \otimes s_j$. Let $z \in M_n$ satisfy $\|z\| \leq 1$. Using (6) and (7) at the fourth step, we get
\[
\| \varphi(z) \| f
\]
\[= \| \varphi_0(z) \otimes e, r_j \otimes s_j \|
\[= \| \varphi_0(z)r_j \otimes es_j - r_j \varphi_0(z) \otimes s_je \|
\[\leq \| \varphi_0(z), r_j \otimes es_j \| + \| r_j \varphi_0(z) \otimes (es_j - s_je) \|
\[< \delta + \delta + \delta < \varepsilon. \]
This completes the proof. \hfill \Box

**Corollary 5.2.** Let $A$ be a simple $\mathcal{Z}$-absorbing $C^*$-algebra. Then $A$ is tracially $\mathcal{Z}$-absorbing.

**Proof.** By definition, $A \cong \mathcal{Z} \otimes A$. By [28, Proposition 2.2], $\mathcal{Z}$ is tracially $\mathcal{Z}$-absorbing. Now Theorem 5.1 implies that $A$ is tracially $\mathcal{Z}$-absorbing. \hfill \Box

For example, the Razak-Jacelon algebra $\mathcal{W}$ (see [29, 41]) is a stably projectionless tracially $\mathcal{Z}$-absorbing $C^*$-algebra.

**Remark 5.3.** In [12], it is proved that every simple separable nuclear tracially $\mathcal{Z}$-absorbing $C^*$-algebra $A$ is $\mathcal{Z}$-absorbing. The special case that $A$ is not stably projectionless follows from the corresponding result for the unital case ([28, Theorem 4.1]), the Morita invariance of $\mathcal{Z}$-absorption in the class of separable $C^*$-algebras ([52, Corollary 3.2]), and Theorems 4.1 and 5.1.

We obtain a different proof of a special case of [52, Corollary 3.4].

**Corollary 5.4.** Let $A$ be a simple separable nuclear $C^*$-algebra which is not stably projectionless and which is the direct limit of a system of simple $\mathcal{Z}$-absorbing $C^*$-algebras. Then $A$ is $\mathcal{Z}$-absorbing.
Proof. The statement follows from Corollaries 5.2 and 4.8 and Remark 5.3. □

We now give examples of tracially $Z$-absorbing $C^*$-algebras which are not $Z$-absorbing. Our examples are unital; no such examples were previously known, even in the unital case. We start with purely infinite examples. By Example 3.11, it is enough to find some separable purely infinite simple unital $C^*$-algebra which is not $Z$-absorbing. The work for this is contained in Proposition 5.5. It is based on the ideas of [17, Corollary 1.2], in which it is shown that an algebra $A$ as in Proposition 5.5 is not approximately divisible. The execution here is much messier because we can’t use projections. A stably finite example is then obtained by applying Lemma 3.12 to an example in [40].

**Proposition 5.5.** For $j = 1, 2$ let $P_j$ be a von Neumann algebra, let $\omega_j: P_j \to \mathbb{C}$ be a faithful normal state, and let $G_j \subseteq P_j$ be a discrete subgroup of the unitary group $U(P_j)$ which is contained in the centralizer of $\omega_j$ and such that whenever $u, v \in G_j$ are distinct then $\omega_j(v^*u) = 0$. Suppose $G_1$ contains an element $a \neq 1$, and $G_2$ contains distinct elements $b, c \neq 1$. Let $(P, \omega)$ be the reduced free product von Neumann algebra $(P, \omega) = (P_1, \omega_1) \star_r (P_2, \omega_2)$, equipped with its free product state $\omega$. Let $A \subset P$ be any $C^*$-subalgebra satisfying $a, b, c \in A$. Then $A$ is not $Z$-stable.

For a von Neumann algebra $N$ and a state $\rho: N \to \mathbb{C}$, a unitary $u \in N$ is in the centralizer of $\rho$ if and only if $\rho \circ \text{Ad}(u) = \rho$.

The algebras are called $(M_j, \varphi_j)$ in [10], but this notation conflicts with notation for matrix algebras.

As discussed at the beginning of [10, Section 2], the state $\omega$ is necessarily faithful and normal. By [10, Theorem 2], $P$ is a factor of type III.

**Proof of Proposition 5.5.** Let $D$ be the dimension drop interval algebra

$$D = \{ g \in C([0, 1], M_2 \otimes M_3) : g(0) \in M_2 \otimes \mathbb{C} \cdot 1 \text{ and } g(1) \in \mathbb{C} \cdot 1 \otimes M_3 \}.$$ 

It suffices to show that there is no approximately central sequence of unital homomorphisms from $D$ to $A$. To do this, we set

$$N = 24 \quad \text{and} \quad \varepsilon = \frac{1}{70}.$$

We let $(e_{j,k})_{j,k=1,2}$ be the standard system of matrix units for $M_2$, and we let $(f_{j,k})_{j,k=1,2,3}$ be the standard system of matrix units for $M_3$. For $n = 1, 2, \ldots, N$, define $h_n: [0, 1] \to [0, 1]$ by

$$h_n(\lambda) = \begin{cases} 
0 & 0 \leq \lambda \leq \frac{n-1}{N} \\
N\lambda - n + 1 & \frac{n-1}{N} < \lambda < \frac{n}{N} \\
1 & \frac{n}{N} \leq \lambda \leq 1.
\end{cases}$$

Then define functions $r_{j,n} \in D$ for $j = 1, 2$ and $s_{j,n} \in D$ for $j = 1, 2, 3$ by

$$r_{j,n}(\lambda) = (1 - h_n(\lambda))(e_{j,j} \otimes 1) \quad \text{and} \quad s_{j,n}(\lambda) = h_n(\lambda)(1 \otimes f_{j,j}).$$
Next, define a compact set $S \subseteq D$ to consist of the functions defined for $\lambda \in [0, 1]$ by

$$\lambda \mapsto (u \otimes 1) r_{1,n}(\lambda) (u \otimes 1)^*$$

for $u \in U(M_2)$ and $n = 1, 2, \ldots, N$, and

$$\lambda \mapsto (1 \otimes v) s_{1,n}(\lambda) (1 \otimes v)^*$$

for $v \in U(M_3)$ and $n = 1, 2, \ldots, N$.

If there were an approximately central sequence of unital homomorphisms from $D$ to $A$, then, in particular, there would be a unital homomorphism $\varphi: D \to A$ such that

$$||[\varphi(g), a]|| < \varepsilon, \quad ||[\varphi(g), b]|| < \varepsilon, \quad \text{and} \quad ||[\varphi(g), c]|| < \varepsilon$$

for all $g \in S$. We will show that this can’t happen.

Suppose therefore that $\varphi: D \to A$ is a unital homomorphism such that (5.1) holds for all $g \in S$. Regard $C([0, 1])$ as a unital subalgebra of $D$ in the obvious way. Let $\mu$ be the Borel probability measure on $[0, 1]$ such that $(\omega \circ \varphi)(g) = \int_{[0,1]} g \, d\mu$ for all $g \in C([0,1])$. We have

$$\sum_{n=1}^N \mu\left(\left[\frac{n-1}{N}, \frac{n}{N}\right]\right) \leq 2\mu([0,1]) = 2,$$

so there exists $n \in \{1, 2, \ldots, N\}$ such that, if we set $I = \left[\frac{n-1}{N}, \frac{n}{N}\right]$, then $\mu(I) \leq \frac{2}{N}$. Set $\alpha = (\omega \circ \varphi)(h_n)$.

We have $r_{1,n} + r_{2,n} = 1 - h_n$, so

$$(\omega \circ \varphi)(r_{1,n}) + (\omega \circ \varphi)(r_{2,n}) = 1 - \alpha.$$ 

Therefore there are $j, k \in 1, 2$ such that

$$(\omega \circ \varphi)(r_{j,n}) \geq \frac{1 - \alpha}{2} \quad \text{and} \quad (\omega \circ \varphi)(r_{k,n}) \leq \frac{1 - \alpha}{2}.$$ 

The set of functions

$$\lambda \mapsto (u \otimes 1) r_{1,n}(\lambda) (u \otimes 1)^*,$$

for $u \in U(M_2)$, is connected and contains both $r_{j,n}$ and $r_{k,n}$. Therefore there is $u \in U(M_2)$ such that the function

$$r(\lambda) = (u \otimes 1) r_{1,n}(\lambda) (u \otimes 1)^*$$

satisfies $(\omega \circ \varphi)(r) = \frac{1}{2}(1 - \alpha)$. A similar argument, starting with

$$(\omega \circ \varphi)(s_{1,n}) + (\omega \circ \varphi)(s_{2,n}) + (\omega \circ \varphi)(s_{3,n}) = \alpha,$$

produces $v \in U(M_3)$ such that the function

$$s(\lambda) = (1 \otimes v) s_{1,n}(\lambda) (1 \otimes v)^*$$

satisfies $(\omega \circ \varphi)(s) = \frac{\alpha}{2}$. Then $(\omega \circ \varphi)(r + s) \in \left[\frac{1}{2}, \frac{3}{2}\right]$. It follows that

$$\omega \circ \varphi)(r + s) - [(\omega \circ \varphi)(r + s)]^2 \geq \frac{1}{6}.$$
Define \( l \in C([0,1]) \) by
\[
l(\lambda) = \|r(\lambda) + s(\lambda) - (r(\lambda) + s(\lambda))^2\|
\]
for \( \lambda \in [0,1] \). We have \( 0 \leq r + s \leq 1 \), so \( 0 \leq (r + s)^2 \leq r + s \), and
\[
0 \leq r + s - (r + s)^2 \leq l \leq 1.
\]
Also \( l(\lambda) = 0 \) for \( \lambda \notin I \). Therefore
\[
0 \leq (\omega \circ \varphi)(r + s) - (\omega \circ \varphi)((r + s)^2)
\]
\[
\leq (\omega \circ \varphi)(l) = \int_{[0,1]} l \, d\mu \leq \mu(I) \leq \frac{2}{N} = \frac{1}{12}.
\]
Combining this inequality with (5.2) gives
\[
(\omega \circ \varphi)((r + s)^2) - [(\omega \circ \varphi)(r + s)]^2 \geq \frac{1}{12}.
\]
Recall that for \( x \in P \), we have \( \|x\|_{\omega} = \omega(x^*x)^{1/2} \). In particular, \( \|x\|_{\omega} \leq \|x\| \). The proof of [10, Theorem 11] shows that for every \( x \in P \) we have
\[
\|x - \omega(x) \cdot 1\|_{\omega} \leq 14 \max \left( \|x, a\|_{\omega}, \|x, b\|_{\omega}, \|x, c\|_{\omega} \right).
\]
If \( 0 \leq x \leq 1 \), a calculation shows that
\[
\|x - \omega(x) \cdot 1\|_{\omega}^2 = \omega(x^2) - \omega(x)^2.
\]
Put \( x = r + s \) and use (5.3) at the first step and (5.1) at the third step to get
\[
\frac{1}{12} \leq 14^2 \max \left( \|x, a\|_{\omega}^2, \|x, b\|_{\omega}^2, \|x, c\|_{\omega}^2 \right)
\]
\[
\leq 14^2 \max \left( \|x, a\|^2, \|x, b\|^2, \|x, c\|^2 \right) < 14^2 \varepsilon^2 = \frac{1}{25},
\]
a contradiction. \( \square \)

**Example 5.6.** There is a simple separable unital purely infinite C*-algebra which is not \( Z \)-stable.

We start with the example at the end of [10], which gives \((P_1, \omega_1)\) and \((P_2, \omega_2)\) satisfying the hypotheses of Proposition 5.5. Let \((P, \omega)\) and \(a, b, c \in P\) be as there.

Since, by [10, Theorem 2], \( P \) is a factor of type III, we can apply [17, Proposition 1.3(i)] (whose method goes at least back to [6, Proposition 2.2])

Example 5.6. There is a simple separable unital purely infinite C*-algebra which is not \( Z \)-stable.

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Since, by [10, Theorem 2], \( P \) is a factor of type III, we can apply [17, Proposition 1.3(i)] (whose method goes at least back to [6, Proposition 2.2]) to find a purely infinite simple separable C*-algebra \( A \subseteq P \) such that \( a, b, c \in A \). Then \( A \) is not \( Z \)-stable by Proposition 5.5.

In fact, one can explicitly write down a purely infinite simple separable reduced free product algebra which is not \( Z \)-stable.

**Example 5.7.** Let \( A \) be the reduced free product \( A = (M_2 \otimes M_2) \ast_r C([0,1]) \), taken with respect to the Lebesgue measure state on \( C([0,1]) \) and the state on \( M_2 \otimes M_2 \) given by tensor product of the usual tracial state \( tr \) with the
state $\rho(x) = \text{tr}(\text{diag}(\frac{1}{2}, \frac{1}{2}) x)$ on $M_2$. We claim that $A$ is purely infinite and simple but not $\mathcal{Z}$-stable.

To prove pure infiniteness, in [16, Example 3.9(iii)] take $A_1 = M_2$ with the state $\text{tr}$, take $F = M_2$ with the state $\rho$, and take $B = C([0, 1])$ with the state given by Lebesgue measure. These choices satisfy the hypotheses there. So $A$ is purely infinite and simple.

To prove failure of $\mathcal{Z}$-stability, in Proposition 5.5 take $P_1 = M_2 \otimes M_2$ with the state $\text{tr} \otimes \rho$, and take $P_2 = L^\infty([0, 1])$ with the state given by Lebesgue measure. Take $G_1 = \left\{ 1, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1 \right\}$, and take $G_2$ to be the set of functions $\lambda \mapsto e^{2\pi i n \lambda}$ for $n \in \mathbb{Z}$. These choices satisfy the hypotheses there. Moreover, $G_2 \subseteq C([0, 1])$. Therefore the reduced free product $A = (M_2 \otimes M_2)^\star C([0, 1])$ is a subalgebra of the algebra $P$ of Proposition 5.5 which contains $a$, $b$, and $c$, so is not $\mathcal{Z}$-stable by Proposition 5.5.

One might hope that if $A$ is a purely infinite simple separable C*-algebra, then the conditions of $O_\infty$-stability, $\mathcal{Z}$-stability, and approximate divisibility are all equivalent. This seems too good to be true.

Example 5.8. There is a simple separable stably finite unital C*-algebra which is tracially $\mathcal{Z}$-absorbing but not $\mathcal{Z}$-absorbing. Start with the example in [40] of a simple separable stably finite unital C*-algebra which has tracial rank zero [37, Definition 3.6.2] but is not $\mathcal{Z}$-absorbing. This algebra is tracially $\mathcal{Z}$-absorbing by lemma 3.12.

6. The Cuntz semigroup

In this section we generalize Theorem 3.3 of [28] to the nonunital case. More precisely, we show that if $A$ is a simple tracially $\mathcal{Z}$-absorbing C*-algebra then $W(A)$ is almost unperforated. This implies that $A$ has strict comparison in a sense suitable for nonunital C*-algebras. (See Definition 6.8 below.) The proof is a modification of the argument given in [28]. Moreover, we show that $A$ is weakly almost divisible, as in Definition 6.11.

Definition 6.1. Let $A$ be a C*-algebra and let $a \in M_\infty(A)_+$. We say that $a$ is weakly purely positive if $0$ is an accumulation point of $\text{sp}(a)$.

Remark 6.2. Recall (see before [2, Corollary 2.24]) that $a \in M_\infty(A)_+$ is called purely positive if there is no projection $p \in M_\infty(A)_+$ such that $\langle p \rangle = \langle a \rangle$ in $W(A)$. By [2, Proposition 2.23], if $A$ is a unital C*-algebra with $\text{tsr}(A) = 1$ then $a \in M_\infty(A)_+$ is weakly purely positive if and only if $a$ is purely positive. In general, pure positivity implies weak pure positivity. We don’t know whether the converse holds.

The following lemma is a nonunital analog of [28, Lemma 3.2].

Lemma 6.3. Let $A$ be a simple tracially $\mathcal{Z}$-absorbing C*-algebra. Let $a, b \in A_+$. Suppose $b$ is weakly purely positive and there is a positive integer $n$ such that $n(a) \leq n(b)$ in $W(A)$. Then $a \lesssim b$. 
Some parts of the proof are similar to the proof of [28, Lemma 3.2]. (See [28, Notation 2.13] for some of the notation used in that proof.) We provide a complete proof because of some differences due to the absence of the identity. (Also, in the proof of [28, Lemma 3.2], we could not follow one step in the proof of the estimate \(|\hat{c}(b - \delta) + \hat{c} - a_1| < \frac{\delta}{3} \|.

Proof of Lemma 6.3. We may assume that \(|a|| = \|b|| = 1\). Let \(\varepsilon > 0\); we prove that \((a - \varepsilon)_+ \precsim b\).

The assumption says \(a \otimes 1_n \precsim b \otimes 1_n\). By [33, Proposition 2.6], there is \(\delta > 0\) such that

\[
(a \otimes 1_n - \frac{\delta}{2})_+ \precsim (b \otimes 1_n - \delta)_+.
\]

Choose \(c_0 \in M_n(A)\) such that

\[
\|c_0[(b \otimes 1_n - \delta)_+]c_0^* - (a \otimes 1_n - \frac{\delta}{2})_+\| < \frac{\delta}{2}.
\]

It follows from Lemma 2.1 that there is \(c_1 \in M_n(A)\) such that

\[
c_1c_0[(b \otimes 1_n - \delta)_+]c_0^* c_1 = ((a \otimes 1_n - \frac{\delta}{2})_+ - \frac{\delta}{2})_+.
\]

Thus,

\[
c_1c_0[(b - \delta)_+ \otimes 1_n](c_1c_0)^* = (a - \varepsilon)_+ \otimes 1_n.
\]

Let \(f, h \in C_0((0,1])\) be nonnegative functions such that \(f = 0\) on \([\frac{\delta}{2}, 1]\), \(f > 0\) on \((0, \frac{\delta}{2})\), and \(\|f\| = 1\), and such that \(h = 0\) on \([0, \frac{\delta}{2}]\) and \(h = 1\) on \([\frac{\delta}{2}, 1]\). Put \(d = f(b)\). Since \(b\) is weakly purely positive, \(d \neq 0\). Put \(c = c_1c_0[h(b) \otimes 1_n]\). Since \(h(\lambda)(\lambda - \delta)_+ = (\lambda - \delta)_+\) for \(\lambda \in [0,1]\) and \(hf = 0\), we get

\[
c[(b - \delta)_+ \otimes 1_n]c^* = (a - \varepsilon)_+ \otimes 1_n
\]

Write \(c = (c_{j,k})_{1 \leq j,k \leq n}\) with \(c_{j,k} \in A\) for \(j,k = 1,2,\ldots,n\). Then

(6.1) \(c_{j,k}d = 0\).

For any \(\mu > 0\) we will find \(z \in A\) such that

(6.2) \(\|z[(b - \delta)_+ + d]z^* - (a - \varepsilon)_+\| < \mu\).

This will prove the first step in the calculation

\((a - \varepsilon)_+ \precsim (b - \delta)_+ + d \precsim b;\)

since the second step is clear, and \(\varepsilon > 0\) is arbitrary, we will have finished the proof.

So fix \(\mu > 0\). We have

(6.3) \[\sum_{l=1}^{n} c_{j,l}(b - \delta)_+c_{k,l}^* = \begin{cases} (a - \varepsilon)_+ & j = k \\ 0 & j \neq k. \end{cases}\]

Similarly to the proof of [28, Lemma 3.1] define \(g,h \in C_0((0,1])\) by

\[
g(\lambda) = \begin{cases} \sqrt{\frac{13\lambda}{\mu}} & \lambda < \mu/13 \\ \frac{1}{\lambda} & \lambda \geq \mu/13 \end{cases} \quad \text{and} \quad h(\lambda) = 1 - \sqrt{1 - \lambda}
\]
Then given, we obtain a c.p.c. order zero map \( \phi \) \((6.10)\)

Put \( M_\lambda \) for all \( C \) \( \phi \) such that \((6.8)\)

\[ \| x \| \leq 1 \]

for any \( k \) \( \phi \). For any \( C \)-algebras \( B \), \( A \), any map \( \varphi \): \( B \to A \), and any subset \( F \subseteq A \), we define

\[ C(\varphi, F) = \sup \{ \| [\varphi(w), y] \| : w \in B \text{ satisfies } \| w \| \leq 1 \text{ and } y \in F \} \]

It follows from [28, Lemma 2.8] that there is \( \eta_0 > 0 \) such that if \( \varphi: M_n \to A \)

a c.p.c. order zero map with \( C(\varphi, F) < \eta_0 \), then

\[ C(\varphi, F) < \frac{\mu}{24n^4}, \quad C(\varphi^{1/2}, F) < \frac{\mu}{24n^4}, \quad \text{and} \quad C(h(\varphi), F) < \frac{\mu}{6} \]

Use [4, Lemma 2.5] to choose \( \eta > 0 \) such that \( \eta < \min (\eta_0, \frac{\mu}{24n^4}) \) and such that whenever \( e, s \in A_+ \) satisfy

\[ \| e \| \leq 1, \quad \| s \| \leq 1, \quad \text{and} \quad \| es - se \| < 9\eta, \]

then \( \| e^{1/2} s - s e^{1/2} \| < \mu/6 \). Choose \( x \in A_+ \) such that

\[ \| x \| \leq 1 \quad \text{and} \quad \| x(a - \varepsilon)_+ - (a - \varepsilon)_+ \| < \eta. \]

Applying Definition 3.6, with \( \eta \) in place of \( \varepsilon \), and with \( n, F, \) and \( x \) as given, we obtain a c.p.c. order zero map \( \varphi: M_n \to A \) such that

\[ C(\varphi, F) < \eta \quad \text{and} \quad (x^2 - x\varphi(1)x - \eta)_+ \leq d. \]

Put

\[ r_0 = 1 - \varphi(1) \quad \text{and} \quad r = (x^2 - x\varphi(1)x - \eta)_+ = (xr_0x - \eta)_+. \]

Then \( r_0 \in A^\sim \) and \( r \in A \). Set

\[ a_1 = \varphi(1)(a - \varepsilon)_+, \quad a_2 = r_0^{1/2}(a - \varepsilon)_+ r_0^{1/2}, \quad \text{and} \quad a_3 = r^{1/2}(a - \varepsilon)_+ r^{1/2}. \]
Since
\[(6.11) \quad \| r_0, (a - \varepsilon)_+ \| = \| [1 - \varphi(1), (a - \varepsilon)_+] \| = \| [\varphi(1), (a - \varepsilon)_+] \| < \eta, \]
the choice of \( \eta \) implies
\[(6.12) \quad \| (a - \varepsilon)_+ - (a_1 + a_2) \| < \frac{\mu}{6}. \]
For \( j, k = 1, 2, \ldots, n \), let \( e_{j,k} \in M_n \) be the standard matrix unit, and set
\[(6.13) \quad g_{j,k} = g(\varphi)(e_{j,k}) \quad \text{and} \quad \tilde{c}_{j,k} = \varphi^{1/2}(1)g_{j,k}c_{j,k}. \]
Then set \( \tilde{c} = \sum_{j,k=1}^n \tilde{c}_{j,k}. \)
We claim that
\[(6.14) \quad \| \tilde{c}(b - \delta)_+ \tilde{c}^* - a_1 \| < \frac{\mu}{6}. \]
We follow the proof of [28, Lemma 3.1]. One checks, following the steps there and using (6.6) and (6.3), that
\[(6.15) \quad \sum_{j,k,l,m=1}^n g_{j,k}g_{l,m}c_{j,k}(b - \delta)_+c^*_{m,l} = g(\varphi)(1)^2 \cdot (a - \varepsilon)_+. \]
Now, using (6.13) at the second step, using (6.7) at the third step, using (6.7) and (6.5) at the fourth step, using (6.15) at the fifth step, and using (6.4) at the sixth step, we have
\[
\begin{align*}
\tilde{c}(b - \delta)_+ \tilde{c}^* &= \sum_{j,k,l,m=1}^n \tilde{c}_{j,k}(b - \delta)_+c^*_{m,l} \\
&= \varphi^{1/2}(1) \left( \sum_{j,k,l,m=1}^n g_{j,k}c_{j,k}(b - \delta)_+c^*_{m,l}g_{l,m} \right) \varphi^{1/2}(1) \\
&\approx \frac{\mu}{24} \varphi^{1/2}(1) \left( \sum_{j,k,l,m=1}^n g_{j,k}g_{l,m}c_{j,k}(b - \delta)_+c^*_{m,l} \right) \varphi^{1/2}(1) \\
&\approx \frac{\mu}{24} \varphi(1) \left( \sum_{j,k,l,m=1}^n g_{j,k}g_{l,m}c_{j,k}(b - \delta)_+c^*_{m,l} \right) \\
&= \varphi(1) \cdot g(\varphi)(1)^2 \cdot (a - \varepsilon)_+ \\
&\approx \frac{\mu}{12} \varphi(1)(a - \varepsilon)_+ \\
&= a_1.
\end{align*}
\]
This proves (6.14).
We claim that
\[(6.16) \quad \| a_2 - a_3 \| < \frac{\mu}{2}. \]
First, using (6.10) at the first step, using (6.8) at the second and fourth steps, and using (6.11) and (6.10) at the third and fifth steps, we have

\[ r(a - \varepsilon)_+ \approx_\eta x r_0 x (a - \varepsilon)_+ \approx_\eta x r_0 (a - \varepsilon)_+. \]

Thus

\[ \| r(a - \varepsilon)_+ - r_0(a - \varepsilon)_+ \| < 5\eta \quad \text{and} \quad \| r(a - \varepsilon)_+ - (a - \varepsilon)_+ r_0 \| < 4\eta. \]

Taking adjoints in the second inequality and combining gives

\[ \| r(a - \varepsilon)_+ - (a - \varepsilon)_+ r \| < 9\eta. \]

By the choice of \( \eta \), and using (6.11) for the first inequality,

\[ \| r_0^{1/2} (a - \varepsilon)_+ - (a - \varepsilon)_+ r_0^{1/2} \| < \frac{\mu}{6} \quad \text{and} \quad \| r^{1/2} (a - \varepsilon)_+ - (a - \varepsilon)_+ r^{1/2} \| < \frac{\mu}{6}. \]

Now

\[ \| a_2 - a_3 \| = \| r_0^{1/2} (a - \varepsilon)_+ r_0^{1/2} - r^{1/2} (a - \varepsilon)_+ r^{1/2} \| \]

\[ \leq \frac{\mu}{6} + \frac{\mu}{6} + \| r_0 (a - \varepsilon)_+ - r (a - \varepsilon)_+ \| \]

\[ < \frac{\mu}{3} + 5\eta < \frac{\mu}{3} + \frac{\mu}{6} = \frac{\mu}{2}. \]

This proves (6.16).

Using (6.9) and (6.10) for the second step, we get \( r \lesssim d \). Choose \( s \in A \) such that \( \| sds^* - a_3 \| < \frac{\mu}{6} \). Since \( d = f(b) \), as in the proof of [28, Lemma 3.1], replacing \( s \) with \( sk(b) \) for a function \( k \in C([0,1]) \) which is 1 on \( [0, \frac{\mu}{2}] \) and vanishes on \( [\delta, 1] \), we may assume that \( s(b - \delta)_+ = 0 \). Set \( z = \tilde{c} + s \). Using (6.1) and (6.13) at the first step, and using (6.14), (6.16), and (6.12) at the third step, we get

\[ \| z[(b - \delta)_+ + d]z^* - (a - \varepsilon)_+ \| \]

\[ = \| z[(b - \delta)_+ \tilde{c}^* + sds^* - (a - \varepsilon)_+ \| \]

\[ \leq \| z[(b - \delta)_+ \tilde{c}^* - a_1] + \| sds^* - a_3 \| + \| a_3 - a_2 \| + \| a_1 + a_2 - (a - \varepsilon)_+ \| \]

\[ < \frac{\mu}{6} + \frac{\mu}{6} + \frac{\mu}{2} + \frac{\mu}{6} = \mu. \]

This proves (6.2) and completes the proof.

The following theorem is the nonunital generalization of [28, Theorem 3.1]. The proof is essentially the same.

**Theorem 6.4.** Let \( A \) be a simple tracially \( \mathcal{Z} \)-absorbing \( C^* \)-algebra. Then \( W(A) \) is almost unperforated.

**Proof.** Let \( a, b \in M_\infty(A)_+ \). Suppose there is \( k \geq 2 \) such that \( k(a) \leq (k - 1)(b) \). We have to show that \( a \lesssim b \). Choose \( n \in \mathbb{N} \) such that \( a, b \in M_n(A)_+ \). Since \( M_n(A) \) is tracially \( \mathcal{Z} \)-absorbing (by Proposition 4.5), we may assume that \( a, b \in A \). If \( b \) is weakly purely positive then Lemma 6.3 implies that \( a \lesssim b \). If \( b \) is not weakly purely positive then \( b \) is not purely positive. So
there exists a projection $p \in A$ such that $\langle p \rangle = \langle b \rangle$. We may assume that $p \neq 0$. Since $A$ is not type I (by Corollary 4.2), we can apply [28, Lemma 3.1] to obtain a weakly purely positive element $c \in A$ such that $(k-1)\langle p \rangle \leq k\langle c \rangle$ and $c \leq p$. Thus $k\langle a \rangle \leq (k-1)\langle p \rangle \leq k\langle c \rangle$. Therefore, by Lemma 6.3, $a \preceq c \leq p \sim b$. \hfill $\square$

For strict comparison, we need dimension functions. We give careful statements since there is conflicting terminology in the literature.

**Definition 6.5.** Let $A$ be a $C^*$-algebra. A dimension function $d$ on $A$ is an additive order preserving function $d : W(A) \to [0, \infty]$. It is equivalent to give a function $d : M_\infty(A)_+ \to [0, \infty]$ such that $d(a \oplus b) = d(a) + d(b)$ and $a \preceq b$ implies $d(a) \leq d(b)$, and we use the same letter for both versions of a dimension function. The set of all dimension functions on $A$ is denoted by $DF(A)$.

This definition is given after [49, Theorem 4.5]. We warn that at the beginning of [47, Section 4], the algebra $A$ is assumed unital and it is required that $d((1_A)) = 1$; a state there is an additive order preserving function from $W(A)$ to $[0, \infty)$. In [8, Definition I.1.2], the algebra $A$ is not assumed unital, but it is required that $\sup_{a \in A_+} d(a) = 1$. (In [8, Definition I.1.2], a dimension function $d$ is actually taken to be defined on $M_\infty(A)$, but to satisfy $d(a) = d(a^*a)$ for all $a \in M_\infty(A)$.)

The following is also from after [49, Theorem 4.5], except that the definition of lower semicontinuity is from the beginning of [47, Section 4].

**Definition 6.6.** Let $A$ be a $C^*$-algebra, and let $d : M_\infty(A)_+ \to [0, \infty]$ be a dimension function. We say that $d$ is lower semicontinuous if whenever $(a_n)_{n \in \mathbb{N}}$ is a sequence in $M_\infty(A)_+$ and $\|a_n - a\| \to 0$, then $d(a) \leq \liminf_{n \to \infty} d(a_n)$.

**Lemma 6.7** (Proposition 4.1 of [47]). Let $A$ be a $C^*$-algebra, and let $d : M_\infty(A)_+ \to [0, \infty]$ be a dimension function. Define $\overline{d} : M_\infty(A)_+ \to [0, \infty]$ by

$$\overline{d}(a) = \lim_{\varepsilon \to 0^+} d((a - \varepsilon)_+)$$

for $a \in M_\infty(A)_+$. Then $\overline{d}$ is lower semicontinuous, and $\overline{d} = d$ if and only if $d$ is lower semicontinuous.

**Proof.** The proof of [47, Proposition 4.1] works just as well in the present situation, without normalization and with values allowed to be in $[0, \infty]$. \hfill $\square$

We have not found the following definition in the literature in the nonunital case. (For example, it isn’t in [9].)

**Definition 6.8.** Let $A$ be a $C^*$-algebra. We say that $A$ has weak strict comparison if whenever $a, b \in M_\infty(A)_+$ satisfy $a \in \overline{AbA}$ and $d(a) < d(b)$ for every dimension function $d$ on $A$ with $d(b) = 1$, then $a \preceq b$. 

Purely infinite simple C*-algebras have weak strict comparison, since then $a \in \mathcal{A}b\mathcal{A}$ implies $a \preceq b$.

For simple unital C*-algebras, we show that our definition reduces to strict comparison.

**Proposition 6.9.** Let $A$ be a simple unital C*-algebra which has a quasitrace. Then $A$ has strict comparison (defined using quasitraces) if and only if $A$ has weak strict comparison.

As usual, we take quasitraces to be normalized 2-quasitraces, and we denote the set of quasitraces on $A$ by $\QT(A)$.

Some authors define strict comparison using only tracial states. Without knowing that every quasitrace is a trace, this might be a different concept.

**Proof of Proposition 6.9.** Observe that if $b \in M_\infty(A)_+ \setminus \{0\}$, then $d(b) > 0$ for every nonzero dimension function $d$ on $A$. Indeed, suppose $d(b) = 0$ and $b \in M_n(A)_+$. Then $\{a \in M_n(A) : d(a^*a) = 0\}$ is a proper ideal in $M_n(A)$ (a priori not necessarily closed). But $M_n(A)$ has no nontrivial ideals, closed or not, so $b = 0$. It follows that $d_\tau(b) > 0$ for all $\tau \in \QT(A)$.

Suppose $A$ has strict comparison. Let $a, b \in M_\infty(A)_+$ satisfy $a \in \mathcal{A}b\mathcal{A}$ and $d(a) < d(b)$ for every dimension function $d$ on $A$ with $d(b) = 1$. If $b = 0$ then $a = 0$, so $a \preceq b$. Otherwise, let $\tau \in \QT(A)$. Then $d = d_\tau(b)^{-1}d_\tau$ is a dimension function on $A$ with $d(b) = 1$. So $d(a) < d(b)$ by hypothesis, and it follows that $d_\tau(a) < d_\tau(b)$. Since $\tau$ is arbitrary, strict comparison implies $a \preceq b$.

Conversely, assume that $A$ has weak strict comparison. We first claim that $W(A)$ is almost unperforated. Let $a, b \in M_\infty(A)_+$, and suppose there are are $m, n \in \mathbb{N}$ such that $n(a) \leq m(b)$ and $n > m$. We want to show that $a \preceq b$.

(This version of almost unperforation is clearly equivalent to the usual, and is [49, Definition 3.1].) If $b = 0$ then $a = 0$, so $a \preceq b$. Otherwise, for any dimension function $d$ on $A$ with $d(b) = 1$, we have $nd(a) \leq md(b) < nd(b)$, so $d(a) < d(b)$. Then $a \preceq b$ by weak strict comparison.

The rest is essentially contained in the proof of [47, Theorem 5.2(a)]. Let $a, b \in M_\infty(A)_+$ satisfy $d_\tau(a) < d_\tau(b)$ for all $\tau \in \QT(A)$. Clearly $b \neq 0$. Fix $\varepsilon > 0$; we claim that for all dimension functions $d$ on $A$ such that $d(b) = 1$, we have $d((a - \varepsilon)_+) < d(b)$.

We prove the claim. Since $A$ is simple and unital, for every $c \in M_\infty(A)_+$ there is $n$ such that $n(c) \leq n(b)$. This shows that the values of $d$ are in $[0, \infty)$, that is, $d$ is a state on $W(A)$ in the sense described at the beginning of [47, Section 3]. So the dimension function $\overline{d}$ of Lemma 6.6 also has values in $[0, \infty)$. It is nonzero because $\overline{d}(1) = d(1)$. Theorem II.2.2 of [8] provides $\tau \in \QT(A)$ such that $d(1)^{-1}\overline{d} = d_\tau$. Now, using the assumption $d_\tau(a) < d_\tau(b)$ at the third step,

$$d((a - \varepsilon)_+) \leq \overline{d}(a) = d(1)d_\tau(a) < d(1)d_\tau(b) = \overline{d}(b) \leq d(b).$$

The claim is proved.
Combining the claim and [47, Proposition 3.1], we get \((a - \varepsilon)_+ \preceq b\). Since this is true for all \(\varepsilon > 0\), we have \(a \preceq b\), as desired. \(\square\)

Definition 6.8 was chosen because it is the conclusion of [49, Corollary 4.7], and gives the same answer as the usual definition in the simple unital case.

More direct analogs of strict comparison seem to be unsuitable. As a heuristic example, consider \(A = C(S^2 \times S^2) \otimes K\). The algebra \(A_0 = C(S^2 \times S^2)\) doesn’t have strict comparison, because there is a rank 2 projection \(q_1 \in M_\infty(A_0)\) which is not Murray-von Neumann subequivalent to the trivial rank 3 projection \(q_2 = 1_{M_3} \otimes 1\). So \(A\) shouldn’t have strict comparison either. It presumably would if one restricted to dimension functions with finite values. Indeed, define

\[x = \text{diag}(1, \frac{1}{2}, \frac{1}{3}, \ldots) \in K \quad \text{and} \quad e = \text{diag}(1, 0, 0, \ldots) \in K.\]

Any dimension function \(d\) on \(A\) has \(d(1 \otimes x) \geq nd(1 \otimes e)\) for every \(n \in \mathbb{N}\). So if \(d(1 \otimes x)\) is finite then \(d(1 \otimes x) = 0\). Similar reasoning shows that in fact if \(d(1 \otimes x)\) is finite then \(d(p) = 0\) for every projection \(p \in A\). So \(d(q_1) = d(q_2) = 0\) for every dimension function with finite values, and the standard example to show lack of strict comparison fails.

It doesn’t help to restrict to lower semicontinuous dimension functions with finite values.

It seems, therefore, that infinite values must be allowed. But now the dimension function given by \(d(0) = 0\) and \(d(a) = \infty\) for all nonzero \(a \in M_\infty(A)_+\) gives \(d(q_1) = d(q_2) = \infty\), so again the standard example to show lack of strict comparison fails. This dimension function is lower semicontinuous, so again it doesn’t help to restrict to lower semicontinuous dimension functions.

This discussion leaves several other possibilities, which we do not address.

(1) In Definition 6.8 use only lower semicontinuous dimension functions.

(2) Use all dimension functions which are finite on the Pedersen ideal of \(A \otimes K\).

(3) In (2) use only lower semicontinuous dimension functions.

Returning to the main development, from Theorem 6.4 we get the following consequence.

Proposition 6.10. Let \(A\) be a simple (not necessarily unital) tracially \(\mathcal{Z}\)-absorbing \(C^*\)-algebra. Then \(A\) has weak strict comparison.

Proof. Let \(a, b \in M_\infty(A)_+\) satisfy \(a \in \overline{AbA}\) and \(d(a) < d(b)\) for every dimension function \(d\) on \(A\) with \(d(b) = 1\). Choose \(n\) such that \(a, b \in M_n(A)\). Then \(M_n(A)\) is tracially \(\mathcal{Z}\)-absorbing by Proposition 4.5. By Theorem 6.4, \(W(M_n(A))\) is almost unperforated. Now apply [49, Corollary 4.7], noting that this result (unlike some other results in [49]) does not require the algebra to be unital. \(\square\)
Definition 6.11 ([42], Definition 3.1). Let $A$ be a C*-algebra. We say that $A$ is weakly almost divisible if for any $a \in M_\infty(A)_+$, any $n \in \mathbb{N}$, and any $\varepsilon > 0$, there is $\eta \in W(A)$ such that $n\eta \leq \langle a \rangle$ and $\langle (a - \varepsilon)_+ \rangle \leq (n + 1)\eta$.

Weak almost divisibility is a weak version of divisibility. It seems to be needed when one wants to prove that the crossed product by an action with the tracial Rokhlin property preserves strict comparison in the absence of tracial $\mathbb{Z}$-stability. See [42].

Proposition 6.12. Let $A$ be a simple C*-algebra. If $A$ is tracially $\mathbb{Z}$-absorbing, then $A$ is weakly almost divisible.

Proof. Let $a \in M_\infty(A)_+$, $n \in \mathbb{N}$, and $\varepsilon > 0$ be as in Definition 6.11. We have to find $b \in M_\infty(A)_+$ such that
\[ n\langle b \rangle \leq \langle a \rangle \quad \text{and} \quad \langle (a - \varepsilon)_+ \rangle \leq (n + 1)\langle b \rangle. \]

We may assume that $(a - \varepsilon)_+ \neq 0$. Since $M_m(A)$ is also tracially $\mathbb{Z}$-absorbing for any $m \in \mathbb{N}$ (Proposition 4.5), we may assume that $a \in A$. Use [45, Lemma 2.1] to find $z \in A_+ \setminus \{0\}$ such that
\[ (n + 2)\langle z \rangle \leq \langle (a - \varepsilon)_+ \rangle. \]

Choose $z_0 \in B_+ \setminus \{0\}$ such that $z_0 \not\lesssim z$ in $A$. By Theorem 4.3, the subalgebra $B = aAa$ of $A$ is tracially $\mathbb{Z}$-absorbing. Thus, there exists a c.p.c. order zero map $\psi: M_n \to B$ such that
\[ (a - a^{1/2}\psi(1)a^{1/2} - \frac{\varepsilon}{2})_+ \not\lesssim z_0. \]

Let $t \in C_0((0, 1])$ denote the identity function, and use [55, Corollary 4.1] to find a homomorphism $\rho: C_0((0, 1]) \otimes M_n \to B$ satisfying $\rho(t \otimes x) = \psi(x)$ for all $x \in M_n$. For $j, k = 1, 2, \ldots, n$, let $e_{j,k} \in M_n$ be the standard matrix unit. Set $b = \psi(e_{1,1})$. We have
\[ \langle 1 \rangle = \sum_{j=1}^n \psi(e_{j,j}) \sim 1_n \otimes \psi(e_{1,1}) = 1_n \otimes b. \]

So $n\langle b \rangle = \langle \psi(1) \rangle$. Since $\psi(1) \in B = aAa$, it follows that $n\langle b \rangle \leq \langle a \rangle$.

Now we show that $\langle (a - \varepsilon)_+ \rangle \leq (n + 1)\langle b \rangle$. Using [45, Lemma 1.5] in the first step, (6.18) in the second step, and (6.19) in the third step, we get
\[ \langle (a - \varepsilon)_+ \rangle \not\lesssim (a^{1/2}\psi(1)a^{1/2} - \frac{\varepsilon}{2})_+ \oplus (a - a^{1/2}\psi(1)a^{1/2} - \frac{\varepsilon}{2})_+ \]
\[ \not\lesssim \psi(1) \oplus z_0 \not\lesssim (1_n \otimes b) \oplus z. \]

Hence
\[ \langle (a - \varepsilon)_+ \rangle \leq n\langle b \rangle + \langle z \rangle. \]

It remains to show that $\langle z \rangle \leq \langle b \rangle$. Since $A$ is simple and $b \neq 0$, we certainly have $z \in A_+A$. Since $A$ has weak strict comparison (by Proposition 6.10), it now suffices to show that $d(z) < d(b)$ for all $d \in DF(A)$ with
If \( d(z) = 0 \), we are done. Otherwise, by (4.1) at the beginning of [47, Section 4], there is \( m \in \mathbb{N} \) such that \( \langle (a - \varepsilon) \rangle \leq m \langle b \rangle \). Then

\[
d(z) \leq d((a - \varepsilon) \rangle \leq nd(b) = m.
\]

That is, \( d(z) \) is finite. By (6.17) and (6.20) we have

\[
d(z) < (n + 2)d(z) - d(z) \leq d((a - \varepsilon) \rangle - d(z) \leq nd(b).
\]

We have shown that \( d(z) < d(b) \) for all \( d \in DF(A) \) with \( d(b) = 1 \), as desired. \( \square \)

7. Tracial \( \mathcal{Z} \)-absorption for finite C*-algebras

The objective of this section is to prove Proposition 7.12, which states that if \( A \) is a finite simple tracially \( \mathcal{Z} \)-absorbing C*-algebra, then in Definition 3.6 we can assume \( \|a\| = 1 \) and require \( \|\varphi(1)a\varphi(1)\| > 1 - \varepsilon \). This is parallel to the definition of the weak tracial Rokhlin property for finite group actions.

We don’t actually use Proposition 7.12. However, almost all the work which goes into it is needed for the proof of a result in [1] that the permutation action on the minimal tensor product of finitely many copies of a tracially \( \mathcal{Z} \)-absorbing C*-algebra has the weak tracial Rokhlin property, provided that the tensor product is finite.

It isn’t entirely clear what the “right” definition of finiteness is for nonunital C*-algebras, even simple ones. We will use the following definition, which seems to be the most common.

**Definition 7.1** (V.2.2.1 in [7]). A nonunital C*-algebra \( A \) is finite if its unitization \( A^+ \) is finite in the usual sense, that is, 1 is a finite projection in \( A^+ \). Otherwise, \( A \) is infinite.

Equivalently, all one sided invertible elements in \( A^+ \) are two sided invertible.

There is further discussion of finiteness in [7, Sections V.2.2 and V.2.3]. For our further discussion, we need the following definition and notation.

**Definition 7.2** (Definition 3.2 in [33]). Let \( A \) be a C*-algebra and let \( a \in A_+ \). Then \( a \) is infinite if there exists \( b \in A_+ \setminus \{0\} \) such that \( a \oplus b \not\leq a \). Otherwise, \( a \) is finite.

**Notation 7.3.** For any C*-algebra \( A \), we let Ped(\( A \)) denote the Pedersen ideal (smallest dense ideal) in \( A \); see [44, Section 5.6].

The following elementary fact will be used often enough that we record it here.

**Lemma 7.4.** Let \( A \) be a C*-algebra, let \( a \in A_+ \), and let \( \varepsilon > 0 \). Then \( (a - \varepsilon)_+ \in \text{Ped}(A) \).

**Proof.** This is immediate from the proof of [44, Theorem 5.6.1]. \( \square \)

For a simple nonunital C*-algebra \( A \), we can consider at least four further conditions, none of which appears in [7]:
(1) Every positive element in Ped($A$) (see Notation 7.3) is finite.
(2) The finite elements of $A_+$ are dense in $A_+$.
(3) For every $a \in A_+$ and $\varepsilon > 0$, the element $(a - \varepsilon)_+$ is finite.
(4) There are a dimension function (see Definition 6.5) $d$ on $A$ and a positive element $a$ in Ped($A$) such that $0 < d(a) < \infty$.

It isn’t reasonable to require that all positive elements of $A$ be finite, since it follows from [33, Proposition 3.7] that every nonzero separable stable C*-algebra contains infinite elements.

Condition (3) is the condition we actually use.

Conditions (1), (2), and (3) are very similar. Apart from the obvious implications, from (1) to (3) (because $(a - \varepsilon)_+ \in$ Ped($A$) by Lemma 7.4) and from (3) to (2), we have not investigated the relations between these three conditions. Proposition 7.8 below shows that finiteness as in Definition 7.1 implies (3). Corollary 7.7 below shows that Condition (4) implies Condition (3). Definition 7.1 has the advantage that important permanence properties are immediate.

Remark 7.5. Finiteness as in Definition 7.1 has the following properties.

(1) All subhomogeneous C*-algebras are finite.
(2) Direct limits of direct systems of finite C*-algebras with injective maps are finite.
(3) Arbitrary direct limits of subhomogeneous C*-algebras are finite.
(4) Subalgebras of finite C*-algebras are finite.

Lemma 7.6. Let $A$ be a simple C*-algebra, and let $d$ be a dimension function on $A$ (Definition 6.5).

(1) If there is $a \in$ Ped($A$)$_+ \setminus \{0\}$ such that $d(a) = 0$, then $d(b) = 0$ for all $b \in$ Ped($A$)$_+$.
(2) If there is $a \in$ Ped($A$)$_+ \setminus \{0\}$ such that $d(a)$ is finite, then $d(b)$ is finite for all $b \in$ Ped($A$)$_+$.

Proof. Both the sets

\[
\{ a \in A : d(a^*a) = 0 \} \quad \text{and} \quad \{ a \in A : d(a^*a) < \infty \}
\]

are easily seen to be (not necessarily closed) ideals in $A$. Therefore each is either equal to $\{0\}$ or contains Ped($A$). \hfill \Box

Corollary 7.7. Let $A$ be a simple C*-algebra. Assume that there is a dimension function (see Definition 6.5) $d$ on $A$ and $a \in$ Ped($A$)$_+$ such that $0 < d(a) < \infty$. Then all positive elements of Ped($A$) are finite.

Proof. Suppose there is an infinite element $a \in$ Ped($A$)$_+$. By definition, there is $b \in A_+ \setminus \{0\}$ such that $a \oplus b \not\leq a$.

Let $d$ be a dimension function on $A$ as in the statement. Then $d(a) + d(b) \leq d(a)$. So $d(b) = 0$ or $d(a) = \infty$. If $d(a) = \infty$ then $d(x) = \infty$ for all $x \in$ Ped($A$)$_+ \setminus \{0\}$ by Lemma 7.6(2), contradicting the hypothesis of the statement. So $d(a) < \infty$ and $d(b) = 0$. Choose $\varepsilon > 0$ such that
\[ \varepsilon < \|b\|. \] Then \((b - \varepsilon)_+ \neq 0\), and is in \(\text{Ped}(A)\) by Lemma 7.4. Therefore, 
\[ d((b - \varepsilon)_+) \leq d(b) = 0, \] which implies that \(d(x) = 0\) for all \(x \in \text{Ped}(A)_+\) by Lemma 7.6(1), contradicting the hypothesis of the statement. \(\Box\)

**Proposition 7.8.** Let \(A\) be a simple \(C^*\)-algebra. Suppose that there are \(a \in A_+\) and \(\varepsilon > 0\) such that \((a - \varepsilon)_+\) is infinite. Then \(A\) is infinite in the sense of Definition 7.1.

Although it wasn’t convenient to write the proof this way, having chosen \(b \in A_+ \setminus \{0\}\) which is orthogonal to \((a - \varepsilon)_+\), heuristically the key point is that \((a - \varepsilon)_+\) is in the algebraic ideal generated by \(b\).

This proposition should also be true in the unital case, but we don’t need it there.

**Proof of Proposition 7.8.** Replacing \(a\) by \(\|a\|^{-1}a\) and \(\varepsilon\) by \(\|a\|^{-1}\varepsilon\), we can assume that \(\|a\| = 1\). Then clearly \(\varepsilon < 1\).

Define continuous functions \(f_1, f_2: [0, 1] \to [0, 1]\) by

\[
f_1(\lambda) = \begin{cases} 
0 & \text{if } 0 \leq \lambda \leq \frac{2\varepsilon}{3} \\
3\varepsilon - 1 - 2\frac{\varepsilon}{3} < \lambda < \varepsilon & \text{and } f_2(\lambda) = \begin{cases} 
0 & \text{if } 0 \leq \lambda \leq \frac{\varepsilon}{3} \\
3\varepsilon - 1 - \lambda & \text{if } \frac{\varepsilon}{3} < \lambda < \frac{2\varepsilon}{3} \\
1 & \text{if } \frac{2\varepsilon}{3} \leq \lambda \leq 1 
\end{cases}
\end{cases}
\]

Define

\[ a_0 = (a - \varepsilon)_+, \quad e_1 = f_1(a), \quad \text{and } e_2 = f_2(a). \]

Then \(e_1a_0 = a_0\). By assumption, there is \(c \in A_+ \setminus \{0\}\) such that \(a_0 \oplus c \precsim a_0\). Since \(A\) is not unital, \((1 - e_2)A(1 - e_2)\) is a nonzero hereditary subalgebra of \(A\). By [45, Lemma 2.4], there exists a positive element \(c_0 \in (1 - e_2)A(1 - e_2)\) such that \(\|c_0\| = 1\) and \(c_0 \precsim c\). Then set \(b = f_2(c_0)\) and \(b_0 = f_1(c_0)\). So \(b_0 \neq 0\) and \(bb_0 = b_0\). Also \(b \precsim c_0\), so \(a_0 + b \precsim a_0\). Note that \(a_0c_0 = 0\), and so \(a_0b = 0\).

Since \(A\) is simple, [45, Lemma 1.13] provides \(n \in \mathbb{N}\) and \(y_1, y_2, \ldots, y_n \in A\) such that \(\|e_2 - \sum_{j=1}^n y_jy_j^*\| < \frac{1}{n}\). In \(W(A)\) we then have

\[
\langle e_1 \rangle \leq \langle (e_2 - \frac{1}{n})_+ \rangle \leq \sum_{j=1}^n \langle y_jy_j^* \rangle \leq \sum_{j=1}^n \langle y_jb_0y_j^* \rangle \leq n\langle b \rangle \leq \langle a_0 \rangle + n\langle b \rangle.
\]

It follows from an induction argument that \(\langle a_0 \rangle + (n + 1)\langle b \rangle \leq \langle a_0 \rangle\), so \(e_1 + b \precsim a_0\). Therefore there is \(v \in A\) such that

\[
\langle a_0 \rangle + (n + 1)\langle b \rangle \leq \langle a_0 \rangle, \quad \text{so } e_1 + b \precsim a_0.
\]

Therefore there is \(v \in A\) such that

\[
\langle a_0 \rangle + (n + 1)\langle b \rangle \leq \langle a_0 \rangle, \quad \text{so } e_1 + b \precsim a_0.
\]

Define

\[
s = va_0^{1/2} + (1 - e_1 - b)^{1/2}.
\]

Note that if \(A\) is unital then \(s \in A\), and if \(A\) is nonunital then \(s \in A^+\).

Since \(e_1a_0 = a_0\) and \(ba_0 = 0\), we have

\[
(1 - e_1 - b)a_0 = a_0(1 - e_1 - b) = 0,
\]
and it follows (for example, by polynomial approximation) that

\[(7.2) \quad (1 - e_1 - b)^{1/2}a_0^{1/2} = a_0^{1/2}(1 - e_1 - b)^{1/2} = 0.\]

We claim that \(s\) is right invertible. To prove this, we show that \(\|1-ss^*\| < \frac{1}{2}\). Now

\[ss^* = va_0^{1/2}a_0^{1/2}v^* + (1 - e_1 - b) + va_0^{1/2}(1 - e_1 - b)^{1/2} + (1 - e_1 - b)^{1/2}a_0^{1/2}v^*.\]

The last two terms are zero by (7.2). The first is equal to \(va_0v^*\). So the relation \(\|1-ss^*\| < \frac{1}{2}\) follows from (7.1). The claim is proved.

Now we claim that \(sb_0 = 0\). Since \(e_1b_0 = 0\) and \(bb_0 = b_0\), we have \((1 - e_1 - b)b_0 = 0\), so \((1 - e_1 - b)^{1/2}b_0 = 0\). Using \(a_0b_0 = 0\), so \(a_0^{1/2}b_0 = 0\), we now get \(sb_0 = 0\), as desired.

It follows from the last claim that \(s\) is not two sided invertible, completing the proof. \(\Box\)

**Lemma 7.9.** For every \(\varepsilon > 0\) there is \(\delta > 0\) such that whenever \(A\) is a \(C^*\)-algebra, and \(x, h \in A_+\) satisfy \(\|x\| \leq 1\), \(\|h\| \leq 1\), and \(\|xhx\| > 1 - \delta\), then \(\|xhx\| > 1 - \varepsilon\).

**Proof.** The statement is trivial if \(\varepsilon > 1\), so assume \(\varepsilon \leq 1\). Set \(\delta = 1 - (1 - \varepsilon)^{1/2} > 0\). Now let \(A, x, h\) be as in the hypotheses. Using selfadjointness of \(xhx\) at the second step and \(x^2 \leq x\) at the last step, we have

\[1 - \varepsilon = (1 - \delta)^2 < \|(xhx)^2\| = \|xhx^2hx\| \leq \|hx^2h\| \leq \|hxh\|.

This completes the proof. \(\Box\)

The following lemma generalizes [45, Lemma 1.8].

**Lemma 7.10.** For every \(\varepsilon > 0\) there is \(\delta > 0\) such that whenever \(A\) is a \(C^*\)-algebra, \(\alpha, \beta \in [0, \infty)\), and \(r, x, h \in A_+\) satisfy

\[\|r\| \leq 1, \quad \|x\| \leq 1, \quad \|h\| \leq 1, \quad \text{and} \quad \|rx - x\| < \delta,

then

\[(r,xr - [\alpha + \beta + \varepsilon])_+ \preceq (h^{1/2}xh^{1/2} - \alpha)_+ \oplus (r^2 - rhr - \beta)_+.

**Proof.** Use [37, Lemma 2.5.11] to choose \(\delta > 0\) such that whenever \(A\) is a \(C^*\)-algebra and \(r, x \in A_+\) satisfy \(\|r\| \leq 1, \|x\| \leq 1, \text{and} \|rx - x\| < \delta\), then \(\|rx^{1/2} - x^{1/2}\| < \frac{\delta}{2}\).

Now let \(A, \alpha, \beta, r, x, h\) be as in the hypotheses. We have

\[\|x^{1/2}rhrx^{1/2} - x^{1/2}hx^{1/2}\| \leq 2\|rx^{1/2} - x^{1/2}\| < \varepsilon,

so, by [45, Lemma 1.6],

\[(7.3) \quad (x^{1/2}rhrx^{1/2} - [\alpha + \varepsilon])_+ \preceq (x^{1/2}hx^{1/2} - \alpha)_+.\]
Now, using [19, Proposition 2.3(ii)] at the first and fourth steps, [45, Lemma 1.5] at the second step, and (7.3) and Lemma 2.3 at the third step, we have
\[
(rx - r^2 x^2)^+ \sim \left(\frac{1}{2}r^2 x^2 \frac{1}{2} - [\alpha + \beta + \varepsilon]\right) + \frac{1}{2}(x - x^2 - \alpha + \varepsilon)^+ \oplus \frac{1}{2}(x^2 - x^2 - \beta)^+.
\]
This completes the proof.

The following lemma is a nonunital version of [45, Lemma 2.9].

**Lemma 7.11.** Let \( A \) be a simple \( C^* \)-algebra which is not of type I, let \( r \in A_+ \) be a finite element, and let \( x \in (xAr)^+ \) satisfy \( \|x\| = 1 \). Then for every \( \varepsilon > 0 \) there are \( \delta > 0 \) and \( y \in (xAr)^+ \setminus \{0\} \) such that whenever \( h \in A_+ \) satisfies \( \|h\| \leq 1 \) and \( (r^2 - x^r - \delta)^+ \lesssim y \), then \( \|x^r x - x^r\| \geq 1 - \varepsilon \).

**Proof.** By Lemma 7.9, and changing the value of \( \varepsilon \), it suffices to prove this with the conclusion \( \|x^r x - x^r\| > 1 - \varepsilon \) instead of \( \|x^r x - x^r\| > 1 - \varepsilon \).

If \( \varepsilon > 1 \), there is nothing to prove. So assume \( \varepsilon \leq 1 \).

Set \( \varepsilon_0 = \frac{\varepsilon}{10} \). Apply Lemma 7.10 with \( \varepsilon_0 \) in place of \( \varepsilon \), getting \( \rho > 0 \) (called \( \delta \) there).

For \( \alpha, \beta \in [0, \infty) \) with \( \alpha < \beta \), define a continuous function \( f_{\alpha, \beta}: [0, \infty) \to [0, 1] \) by
\[
f_{\alpha, \beta}(\lambda) = \begin{cases} 0 & 0 \leq \lambda \leq \alpha \\ (\beta - \alpha)^{-1}(\lambda - \alpha) & \alpha < \lambda < \beta \\ 1 & \beta \leq \lambda. \end{cases}
\]
The elements \( f_{1/n, 2/n}(r) \), for \( n \in \mathbb{N} \), form an approximate identity for \( xAr \).

Choose \( n \in \mathbb{N} \) so large that \( \frac{2}{n} < \varepsilon_0 \) and the element \( r_0 = f_{1/n, 2/n}(r) \) satisfies \( \|r_0 x - x\| < \min\left(\frac{\varepsilon_0}{3}, \varepsilon_0\right) \). Define a continuous function \( g: [0, \infty) \to [0, \infty) \) by
\[
g(\lambda) = \begin{cases} \frac{n^2 + 1/4}{\lambda} & 0 \leq \lambda \leq \frac{2}{n} \\ \frac{1}{\lambda} & \frac{2}{n} < \lambda. \end{cases}
\]
Set \( s = g(r) \). Then \( \|s\| \leq \frac{n}{2} \). We have
\[
\|sr - r_0\| \leq \frac{2}{n} < \varepsilon_0 \quad \text{and} \quad \|sr\| \leq 1.
\]
So
\[
(7.4) \quad \|s(r^2 - x^r) - (r_0^2 - r_0^r)\| < 4\varepsilon_0.
\]

By [45, Lemma 2.8], there are \( a, b \in (xAr)^+ \) such that \( \|a\| = \|b\| = 1 \), \( ab = b \), and whenever \( d \in bAb \) satisfies \( \|d\| \leq 1 \), then \( \|xd - d\| < \min\left(\frac{\varepsilon_0}{3}, \varepsilon_0\right) \).

Use [45, Lemma 2.4] to choose nonzero orthogonal positive elements \( z_1, z_2 \in bAb \) such that \( \|z_1\| = \|z_2\| = 1 \). For \( j = 1, 2 \), set \( b_j = f_{\varepsilon_0, 2\varepsilon_0}(z_j) \). Use [45,
Lemma 2.6] to choose nonzero positive elements $c_j \in f_{2\varepsilon_0,3\varepsilon_0}(z_j)A f_{2\varepsilon_0,3\varepsilon_0}(z_j)$ for $j = 1, 2$ such that $c_1 \sim c_2$. We may clearly assume $\|c_1\| = \|c_2\| = 1$. Then

$$b_1b_2 = 0, \quad 0 \leq c_j \leq b_j \leq 1, \quad ab_j = b_j, \quad \text{and} \quad b_jc_j = c_j.$$ 

Define $y = c_1$ and $\delta = 4\varepsilon_0/n^2$.

Now let $h \in A_+$ satisfy $\|h\| \leq 1$ and $(r^2 - rh\varepsilon - \delta)_+ \lesssim y$. Suppose $\|xhx\| \leq 1 - \varepsilon$. The choices of $a$ and $b$ imply that

$$\|x(b_1+b_2)-(b_1+b_2)\| < \min(\varepsilon_0, \frac{\rho}{3}) \quad \text{and} \quad \|x(b_1+b_2)^{1/2}-(b_1+b_2)^{1/2}\| < \varepsilon_0.$$ 

Therefore

$$\|h^{1/2}(b_1 + b_2)h^{1/2}\| = \|(b_1 + b_2)^{1/2}h(b_1 + b_2)^{1/2}\|$$

$$\leq \|(b_1 + b_2)^{1/2}xhx(b_1 + b_2)^{1/2}\| + \varepsilon_0$$

$$\leq \|xhx\| + \varepsilon_0 \leq 1 - \varepsilon + 2\varepsilon_0 = 1 - 13\varepsilon_0.$$

We have

$$\|r_0(b_1 + b_2) - (b_1 + b_2)\| \leq \|r_0x - x\| \cdot \|b_1 + b_2\| + 2\|x(b_1 + b_2) - (b_1 + b_2)\|$$

$$< \min(\frac{\rho}{3}, \varepsilon_0) + 2\min(\frac{\rho}{3}, \varepsilon_0) = \min(\rho, 3\varepsilon_0),$$

so

$$\|r_0(b_1 + b_2)r_0 - (b_1 + b_2)\| < 6\varepsilon_0.$$ 

Now, using $1 - \varepsilon_0 < 1$ and [45, Lemma 1.10] at the first step; using (7.7) and [45, Corollary 1.6] at the second step; and at the third step using (7.6) and the choice of $\rho$ using Lemma 7.10, with $\alpha = 1 - 13\varepsilon_0$, with $\beta = 5\varepsilon_0$, and with $\varepsilon_0$ in place of $\varepsilon$,

$$c_1 + c_2 \lesssim (b_1 + b_2 - (1 - \varepsilon_0))_+$$

$$\lesssim (r_0(b_1 + b_2)r_0 - (1 - 7\varepsilon_0))_+$$

$$\lesssim (h^{1/2}(b_1 + b_2)h^{1/2} - (1 - 13\varepsilon_0))_+ \oplus (r_0^2 - r_0hr_0 - 5\varepsilon_0)_+.$$ 

It follows from (7.5) that $(h^{1/2}(b_1 + b_2)h^{1/2} - (1 - 13\varepsilon_0))_+ = 0$. So, using (7.4) and [45, Corollary 1.6] at the second step, using Lemma 2.3 at the third step, and using the choice of $\delta$ and $\|s\| \leq \frac{\rho}{2}$ at the fourth step,

$$c_1 + c_2 \lesssim (r_0^2 - r_0hr_0 - 5\varepsilon_0)_+$$

$$\lesssim (s(r^2 - rh\varepsilon)s - \varepsilon_0)_+$$

$$\lesssim (r^2 - rh\varepsilon - \|s\|^2\varepsilon_0)_+ \lesssim (r^2 - rh\varepsilon - \delta)_+ \lesssim y = c_1.$$ 

Since $A$ is simple, $c_1c_2 = 0$, $c_1 \sim c_2$, and $r$ is finite, this contradicts [33, Lemma 3.8]. The proof is complete. \qed
**Proposition 7.12.** Let $A$ be a finite simple tracially $\mathcal{Z}$-absorbing $C^*$-algebra. Then for every $x, a \in A_+$ with $\|a\| = 1$, every finite set $F \subseteq A$, every $\varepsilon > 0$, and every $n \in \mathbb{N}$, there is a c.p.c. order zero map $\varphi : M_n \to A$ such that:

1. $(x^2 - x\varphi(1)x - \varepsilon)_+$ $\preceq a$.
2. $\|\varphi(z)b\| < \varepsilon$ for any $z \in M_n$ with $\|z\| \leq 1$ and any $b \in F$.
3. $\|\varphi(1)a\varphi(1)\| > 1 - \varepsilon$.

By Lemma 7.9, we could equally well use the condition $\|a\varphi(1)a\| > 1 - \varepsilon$.

**Proof of Proposition 7.12.** Let $x, a \in A_+$ with $\|a\| = 1$, let $F \subseteq A$ be finite, let $\varepsilon > 0$, and let $n \in \mathbb{N}$.

We may assume that $x \neq 0$. Define

$$
\eta = \min \left( \frac{1}{8 \varepsilon}, \frac{1}{24 \|x\|}\left(\frac{\varepsilon}{\|x\| + 1}\right) \right).
$$

Choose $x_0 \in A_+$ such that

$$
\|x_0\| = 1, \quad \|x_0x - x\| < \eta, \quad \text{and} \quad \|x_0a - a\| < \eta.
$$

Define $x_1 = (x_0 - \eta)_+$. Then $x_1$ is finite by Proposition 7.8. Since $\|a\| = 1$, we have

$$(7.8) \|x_1ax_1 - a\| \leq 2\|x_1 - x_0\| + \|x_0a - a\| + \|ax_0 - a\| < 4\eta.$$

So $\|x_1ax_1\| > 1 - 4\eta$. Thus $x_1ax_1 \neq 0$, and we can set $a_0 = \|x_1ax_1\|^{-1}x_1ax_1$.

We claim that

$$(7.9) \|a_0 - a\| < \frac{\varepsilon}{2}.$$

To prove the claim, using $1 \geq \|x_1ax_1\| > 1 - 4\eta$ at the first step, using (7.8) at the second step, using $\eta \leq \frac{\varepsilon}{2}$ at the third step, and using $\eta \leq \frac{\varepsilon}{2\|x\|}$ at the fourth step,

$$
\|a_0 - a\| \leq \left(1 - \frac{1}{1 - 4\eta}\right) + \|x_1ax_1 - a\| < \frac{4\eta}{1 - 4\eta} + 4\eta \leq 8\eta + 4\eta \leq \frac{\varepsilon}{2}.
$$

The claim is proved.

Apply Lemma 7.11 with $x_1$ in place of $r$, with $\eta$ in place of $\varepsilon$, and with $a_0$ in place of $x$, getting $\delta > 0$ and $y \in (a_0Aa_0)_+ \setminus \{0\}$ as there. By [45, Lemma 2.4], there is $c \in A_+ \setminus \{0\}$ such that $c \preceq y$ and $c \preceq a$. Set

$$
\rho = \min \left( \delta, \frac{\varepsilon}{2}, \frac{\varepsilon}{2\|x\|^2} \right).
$$

Apply the definition of tracial $\mathcal{Z}$-absorption with $x_1$ in place of $x$, with $\rho$ in place of $\varepsilon$, with $c$ in place of $a$, and with $F$ and $n$ as given, getting a c.p.c. order zero map $\varphi : M_n \to A$ such that:

4. $(x_1^2 - x_1\varphi(1)x_1 - \rho)_+ \preceq c$.
5. $\|\varphi(z)b\| < \rho$ for any $z \in M_n$ with $\|z\| \leq 1$ and any $b \in F$. 


Since \( \rho \leq \varepsilon \), Condition (2) in the conclusion follows from (5).

We prove Condition (1) in the conclusion. We have \( \|x_0x - x\| < \eta \) and \( \|x_1 - x_0\| \leq \eta \), so \( \|x_1x - x\| < (1 + \|x\|)\eta \). Therefore, using the choice of \( \eta \) at the third step,

\[
\| (xx_1x^2 - x x_1 \varphi(1)x_1x) - (x^2 - x \varphi(1)x) \| \leq 4\|x\| \cdot \|x_1x - x\| \\
< 4\|x\|((\|x\| + 1)\eta \leq \varepsilon/2.
\]

Using this and [45, Corollary 1.6] at the first step, using \( \|x\|^2 \rho \leq \varepsilon/2 \) at the second step, using Lemma 2.3 at the third step, using (4) at the fourth step, and using \( c \preceq a \) at the fifth step, we now get

\[
(x^2 - x \varphi(1)x - \varepsilon) + \preceq (xx_1x^2 - x x_1 \varphi(1)x_1x - \varepsilon) + \\
\preceq (xx_1x^2 - x x_1 \varphi(1)x_1x - \|x\|^2 \rho) + \\
\preceq (x^2 - x_1 \varphi(1)x_1x - \rho) + \preceq c \preceq a.
\]

This is (1).

Finally, since \( c \preceq y \), \( \rho \leq \delta \), and \( \eta \leq \varepsilon/2 \), (4) and the choices of \( \delta \) and \( y \) using Lemma 7.11 imply that \( \|\varphi(1)a_0\varphi(1)\| > 1 - \varepsilon/2 \). Combining this estimate with (7.9) gives Condition (3) in the conclusion. □

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While preparing the final draft of this paper, a preprint appeared on arXiv [14] shortly before our first arXiv version. Though there are minor overlap between the results of two papers (mainly some permanence properties and a variant of Part (1) of Theorem D), the main results are different. The main result of [14] is that every separable simple nuclear tracially \( \mathcal{Z} \)-absorbing C*-algebra is \( \mathcal{Z} \)-absorbing, which extends [28, Theorem 4.1] to the nonunital case and answers an open question of our unpublished version, and in a sense complements our work. We had a proof of this result in the special case where the C*-algebra is not stably projectionless (Remark 5.3). Though we posted the first arXiv version of the current paper in September 2021, its ideas and results had been obtained long time ago. For instance, Definition A was quoted from the unpublished version of the current paper in [20, Definition 6.6], [30], and [22, Definition 4.4]. Moreover, this definition and some results of this paper (including the permanence properties and almost unperforated Cuntz semigroup of tracially \( \mathcal{Z} \)-absorbing C*-algebras) were announced by the authors in several conferences, for instance, by the fourth author in his ICM talk in 2018 entitled
“Tracial $\mathcal{Z}$-stability in the nonunital case” (ICM Operator Algebras Satellite Conference, http://mtm.ufsc.br/icmoa/), by the second author in his talk in 2020 entitled “Group Actions on Tracially $\mathcal{Z}$-absorbing C*-algebras” (The 7th Workshop on Operator Algebras and their Applications, IPM, http://math.ipm.ac.ir/conferences/2020/OpeAlg2020/), and by the third author in his talk in 2021 in COSy entitled “Simple Tracially $\mathcal{Z}$-absorbing C*-algebras” (http://www.fields.utoronto.ca/activities/20-21/COSy).

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