$N \times N$ matrix time–band limiting examples

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Abstract
Time– and band–limiting in the context of orthogonal polynomials has been studied since the 1980’s. It involves finding differential or difference operators with special commutative properties. More recently this topic has been generalized to the cases were the orthogonal polynomials are matrix-valued. Leading to differential or difference operators with matrix coefficients. So far the only explicit examples of such operators were $2 \times 2$ matrices. In this paper we give a number of $N \times N$ examples and a counterexample to illustrate the role that strong Pearson equations can play in finding such examples.

1 Introduction
In this paper we examine a number of $N \times N$ matrix-valued orthogonal polynomials (MVOP) which are examples that fit into the noncommutative time–band limiting framework introduced in [13]. So in this introduction we will first mention some of the work that led up to [13].

1.1 Time–Band limiting
In the 1960’s Slepian, Landau and Pollack published a series of seminal papers [18, 19, 24, 26] studying time–limiting and band–limiting in the context of the Fourier expansion. The results they were able to derive depended strongly on a seemingly miraculous commutation between a certain integral operator

$$(K \cdot \phi)(x) = \int_{-1}^{1} \frac{\sin(c(x-y))}{\pi(x-y)} \phi(y) dy$$

and a differential operator,

$$(D \cdot \phi)(x) = (1 - x^2)\phi''(x) - 2x\phi'(x) - c^2 x^2 \phi(x)$$

and in particular on their sharing of eigenfunctions.

This miraculous commutation seemed to be more than just a lucky fluke since various generalizations of this situation resulted in useful commuting differential operators. We recommend [25] for a very nice introduction to the topic.

Next we will briefly discuss bispectrality, what the connection is with time–band limiting and some noncommutative generalizations that have been made.
1.2 Bispectrality

Although bispectrality started out \[6\] and still is a topic within integrable systems, a discussion of these roots is beyond the scope of this introduction.

The notion of bispectrality usually involves two (differential or difference) operators. One we call $D$ acting on a variable $x$ and another will be called $L$ acting on another variables $y$. Lastly we need a so-called bispectral function $\phi(x,y)$ that is an eigenfunction of both

$$(D \cdot \phi)(x,y) = \lambda(y)\phi(x,y), \quad (L \cdot \phi)(x,y) = \mu(x)\phi(x,y).$$

One of the most basic examples of a bispectral function is

$$\phi(x,y) = e^{-2\pi ixy}, \quad L = \partial_y, \quad D = \partial_x.$$  

This is also the kernel of the Fourier transform and the kernel of the integral operator \([\text{I}]\) comes from the expression

$$\frac{\sin(\pi T(x-y))}{\pi (x-y)} = \int_{-T/2}^{T/2} e^{2\pi ixt} e^{-2\pi iyt} dt.$$ 

This may not seem like a strong link between time–band limiting and bispectrality at first, given the ubiquity of the exponential function as well as that of Fourier analysis. However in \([\text{II}]\) certain types of integral kernels of the form

$$\mathcal{K}(x,y) = \int_{\Gamma} \phi(x,z)\phi(y,z)dz,$$

were found to have differential operators commuting with their integral operators, where the $\phi$ are bispectral functions and $\Gamma$ is a contour in $\mathbb{C}$. After that, similar results were found in \([\text{I}]\) of the form

$$\mathcal{K}(x,y) = \int_{\Gamma} \phi(x,z)\phi^\dagger(y,z)dz,$$

were $\phi$ and $\phi^\dagger$ are bispectral functions associated to KP hierarchy\([\text{I}]\). It should be noted though that the link between time–band limiting and bispectrality was suggested already in \([6]\) at bispectrality’s inception.

In an effort to better understand the miraculous existence of a differential operator with nice commutative properties such as the one in \([\text{I}]\), a lot of work was done to generalize this kind of analysis to different settings. See for instance \([\text{III}]\).

A sequence of classical orthogonal polynomials $(p_n(x))_n$ taken as a whole is also a bispectral function if we view the index as a discrete variable $y = n \in \mathbb{N}_0$. We can then define the operator $\delta$ as acting on a sequence $u_n$. We define it as a shift $(\delta^j \cdot u)_n = u_{n+j}$ as long as $n + j \geq 0$ and define any entries with negative

\[1\] The integrable hierarchy associated to the Kadomtsev-Petviashvili equation.
index to equal 0. Then for \( \phi(x, y) = p_n(x) \) the second eigenvalue equation would be the three term recurrence relation

\[
(L \cdot p)_n(x) = xp_n(x), \quad L = a_n \delta + b_n + c_n \delta^{-1},
\]

and the first would be either a differential or difference operator acting on the variable \( x \) for which the polynomials are eigenfunctions. The second operator is guaranteed to exist because we are dealing with \textit{classical} orthogonal polynomials.

Casting classical orthogonal polynomials in the time–band limiting framework started with \([8, 10]\) for continuous orthogonal polynomials, and for the discrete case in \([20–22]\). In these works analogous differential operators (or difference operators in the discrete case) were found that commuted with the analogous integral operator. Most of these papers contained, in addition to the construction of the operators, some analysis of the spectrum. For analysis of the eigenfunctions we refer the reader to \([23]\).

### 1.3 Noncommutative bispectrality and time–band limiting

A noncommutative version of bispectrality was introduced in \([9]\) with the goal of studying noncommutative analogues of integrable lattice equations.

Bispectrality in the noncommutative context involves pairs of operators where one acts from the left and the other from the right. It was also this thinking that eventually led to the formalism of the Fourier algebras which play a crucial role in \([2]\).

This has been a motivation to pose a noncommutative versions of time–band limiting problems in particular those involving matrix-valued orthogonal polynomials \([11,12]\). More work was done in this direction \([3,4,13]\) and it is \([13]\) that we will build on further.

The authors of \([13]\) show how to construct matrix-valued differential operators that commute with a corresponding integral operator with a matrix-valued kernel. Various explicit examples that fit well into this framework have been studied before, but these were always matrices of size \(2 \times 2\). The goal of this paper is to give \(N \times N\) examples and to show how the strong Pearson equations help to satisfy the conditions that are unique to the matrix case. As a contrasting case we also treat an example in which the associated matrix weight does not satisfy strong Pearson equations and we subsequently see that this case only works for \(N = 2\) and not for a number of larger matrix sizes.

### 1.4 Overview

The rest of this paper is structured as follows. Section \([2]\) reviews the bare minimum of the noncommutative time–band limiting framework that we will need for the results in Section \([4]\). In Section \([3]\) we highlight some similarities as well as differences between the different weights and operators that will appear in
the remainder of the paper. We then apply this time–band limiting framework to
MVOP of Hermite–type, Laguerre–type, Gegenbauer–type and Charlier–type in
Section 4. We also consider a slightly different Hermite–type scenario in Section
5, to which the framework is applicable for matrix size $N = 2$ but not for some
larger matrix sizes. Finally the appendix has a double role. We use it to collect
some explicit expressions of matrix weights and related quantities that appear
in the proofs, but also to correct some typos that have appeared in previous
work.

2 Noncommutative Time–Band Limiting for MVOP

In this section we will very briefly recall the parts of [13] that we will need to
treat the examples that are the main focus of this paper.

We consider a $N \times N$ matrix weight $W$ and its monic MVOP $(P_n)_n$ with
squared norms $H_n$. We then assume that the $P_n$ are simultaneous eigenfunctions
of a $W$-symmetric second order differential operator $D$, acting from the right
and with eigenvalue matrix $\Lambda_n$ on the left

$$P_n \cdot D = \Lambda_n P_n.$$  

Definition 1. Given $M \in \mathbb{N}_0$ and a matrix weight $W$, we define the time–
limiting operator $\chi_T^{(M)}$ to act as

$$\left( F \cdot \chi_T^{(M)} \right)(x) = \sum_{n=0}^{M} (F, P_n) H_n^{-1} P_n(x)$$

for matrix functions $F$ for which the matrix inner product $(F, P_n)$ corresponding
to $W$ is well-defined and has finite entries.

Given $\Omega \in \mathbb{R}$ the band–limiting operator $\chi_B^{(\Omega)}$ acts by multiplication of a
characteristic function $(F \cdot \chi_B^{(\Omega)})(x) = F(x)\chi(-\infty, \Omega)(x)$.

Remark The subscripts for $\chi_T$ and $\chi_B$ refer to "time" and "band". The
variable $n$ plays the role of time, $x$ plays the role of the spectral variable and
each $\chi$ limits its respective variable.

Remark Note that the band–limiting operator $\chi_B^{(\Omega)}$ is the same no matter
which MVOP are under consideration, whereas this is not the case for the
time–limiting operator $\chi_T^{(M)}$.

One of the main results from [13] is the construction of a differential operator
$T$ that commutes with both the time– and band–limiting operators separately.
A central requirement given in [13, Equation (9)] is that we must be able to find
a matrix $R$ which does not depend on $x$ or $\Omega$ and satisfies

$$(R - x(\Lambda_M + \Lambda_{M+1})) W(x) = W(x) (R - x(\Lambda_M + \Lambda_{M+1}))^*.$$  (2)
Note that $R$ is allowed to depend on $M$ and any other parameters that the weight $W$ might have.

Finally equation (11a) in [13] gives the construction of $T$ as

$$T = xD + D(x - 2\Omega) - x(\Lambda_M + \Lambda_{M+1}) + R,$$

where $T$ acts from the right. Note that the condition (2) guarantees that $T$ will be $W$-symmetric. This is because $x$ and $D$ are $W$-symmetric and so $xD + Dx$ and $-2\Omega D$ are as well.

**Remark** Strictly speaking the results presented in [13] only apply to $W$-symmetric differential operators. However upon inspection, the proofs apply equally well to $W$-symmetric difference operators. We will therefore apply it to the Charlier–type example in Section 4.2.

Each of the examples we treat in this paper have been introduced in previous works [7,15–17], including the matrix weight and the $W$-symmetric second order differential or difference operator. What is left for us to do is find the matrix $R$ that satisfies (2) in order to construct $T$.

### 3 Weights, Parameters and Operators

The matrix weights discussed in this paper have a number of similarities. They all have a free parameter $\nu$. They also all have some other parameters $(\alpha_j)_{j=1}^N$ and $(t_j^{(\nu)})_{j=1}^N$, which can be required to satisfy a certain set of nonlinear equations. This requirement then implies that the matrix weight satisfies a strong Pearson equation. In the appendix we list some parameter values such that this requirement is met. Only the matrix weight in Section 5 does not need to satisfy this requirement as we do not need a strong Pearson equation, and so we only require the parameters $\alpha_j$ and $t_j^{(\nu)}$ to be positive real numbers.

Each matrix weight is given in its LDU decomposition

$$W^{(\nu)}(x) = L^{(\nu)}(x)T^{(\nu)}(x)L^{(\nu)}(x)^*,$$

where the diagonal matrix entries are $(T^{(\nu)}(x))_{jj} = t_j^{(\nu)}w_j^{(\nu)}(x)$ with each $w_j^{(\nu)}$ a corresponding scalar weight. For example for the Hermite–type case $w_j^{(\nu)}(x) = e^{-x^2}$ does not depend on $j$ or $\nu$ but for the Gegenbauer–type case $w_j^{(\nu)}(x) = (1 - x^2)^{\nu+j-1/2}$.

The lower triangular matrix has nonzero entries of the form $L^{(\nu)}(x)_{jk} = \frac{\alpha_k}{\alpha_j}p_j^{(\nu+k)}(x)$ where $p_n^{(\nu)}$ are the corresponding scalar orthogonal polynomials and where the parameter $\nu + k$ shifts with the column index only when it is appropriate.\footnote{\textsuperscript{2}The scalar Hermite polynomials do not have any such parameter so unsurprisingly in that case $L^{(\nu)}(x) = L(x)$. However, this is also true for the Charlier polynomials who do have a parameter that could have been shifted. This is not done because their scalar ladder relations do not involve shifting this parameter.}
Even without the restriction on the parameters $\alpha_j$ and $t_j^{(\nu)}$, each matrix weight already has at least one $W^{(\nu)}$-symmetric second order differential or difference operator with the MVOP as simultaneous eigenfunctions. When we impose the requirements to obtain strong Pearson equations we get an additional operator with these properties. These additional operators are the ones that appear in Sections 4 and the other kind of operator is studied in Section 5 which, as we will see, does not fit into the framework of [13] as nicely.

Remark  The matrix weights we consider in this paper are all of the form

$$W(x) = w(x)Q(x),$$

where $w$ is a scalar classical weight and $Q$ is a matrix polynomial of degree $2N - 2$. This means that (2) in these cases will always be a matrix polynomial equation. Or since we are looking to solve for the entries of $\mathcal{R}$, (2) is a inhomogeneous linear system. We point this out to note that roughly speaking, as $N$ grows, the number of equations for the entries of $\mathcal{R}$ grows as $N^3$ whereas the number of parameters obviously is just $N^2$. So we conclude that cases where we can find $\mathcal{R}$ for all $N \in \mathbb{N}$ are far from generic.

4 Examples with strong Pearson equations

4.1 Hermite, Laguerre and Gegenbauer

In this section we discuss three examples which correspond to Hermite–, Laguerre– and Gegenbauer–type MVOP introduced in [15], [17] and [18] respectively. We discuss them at the same time due to their close similarity. All three of these cases have a parameter family of weights $W^{(\nu)}$ that satisfy certain requirements we will call strong Pearson equations

$$\begin{align*}
W^{(\nu+1)}(x) &= W^{(\nu)}(x)\Phi^{(\nu)}(x), \\
W^{(\nu+1)'}(x) &= W^{(\nu)}(x)\Psi^{(\nu)}(x),
\end{align*}$$

where $\Phi^{(\nu)}$ and $\Psi^{(\nu)}$ are matrix polynomials of degree $\leq 2$ and exactly equal to 1 respectively

$$\begin{align*}
\Phi^{(\nu)}(x) &= x^2\phi_2^{(\nu)} + x\phi_1^{(\nu)} + \phi_0^{(\nu)}, \\
\Psi^{(\nu)}(x) &= x\psi_1^{(\nu)} + \psi_0^{(\nu)}.
\end{align*}$$

Explicit expressions for these polynomials are listed in the appendix.

The parameter $\nu$ is usually taken to be positive real though it is possible to extend it in some cases.
Another consequence is
\[ W^{(\nu)}(x)\Phi^{(\nu)}(x) = \Phi^{(\nu)}(x)^*W^{(\nu)}(x), \quad W^{(\nu)}(x)\Psi^{(\nu)}(x) = \Psi^{(\nu)}(x)^*W^{(\nu)}(x), \]
which follows from the fact that \( W^{(\nu+1)} \) as well as \( W^{(\nu+1)^*} \) are symmetric matrices. The idea of the proof of the following theorem is that the previous two equations can be combined into an equation of the form of (2) and hence provide us with a matrix \( R^{(\nu)} \).

**Theorem 1.** Let \( T^{(\nu)} \) be the following matrix differential operator
\[ T^{(\nu)} = xD^{(\nu)} + D^{(\nu)}(x - 2\Omega) - x(\Lambda^{(\nu)}_M + \Lambda^{(\nu)}_{M+1}) + R^{(\nu)}, \]
as given in [13, equation (11a)].

- When \( D^{(\nu)} \) is as in the Hermite–type example [15, Corollary 3.11], then \( R^{(\nu)}_H = -(2M + 1)\psi_0^{(\nu)*} \) satisfies (2).
- When \( D^{(\nu)} \) is as in the Laguerre–type example [17, Corollary 6.3], then \( R^{(\nu)}_L = -M^2\phi_1^{(\nu)*} - (2M + 1)\psi_0^{(\nu)*} \) satisfies (2).
- When \( D^{(\nu)} \) is as in the Gegenbauer–type example [16, Corollary 2.5] (denoted \( \Phi^{(\nu)} \)), then \( R^{(\nu)}_G = -\left( \frac{M^2}{2\nu + N} + 2M + 1 \right)\psi_0^{(\nu)*} \) satisfies (2).

**Proof.** For the Hermite–type case \( \Phi^{(\nu)} \) is a degree 1 polynomial, see [13]. So by (3) we have \( \Lambda^{(\nu)}_n = n\psi_1^{(\nu)*} \). This means that when we choose \( R^{(\nu)} = -(2M + 1)\psi_0^{(\nu)*} \) the degree 1 polynomial that appears in (2) is
\[ R^{(\nu)} - x(\Lambda^{(\nu)}_M + \Lambda^{(\nu)}_{M+1}) = -(2M + 1)\Psi^{(\nu)}(x)^*. \]
The second equation in (4) is then equivalent to the desired condition in (2).

For the Laguerre case \( \Phi^{(\nu)} \) is of degree 2 but \( \phi_0^{(\nu)} = 0 \), see [16]. So now \( x^{-1}\phi_1^{(\nu)}(x) = x\phi_2^{(\nu)} + \phi_1^{(\nu)} \) is a degree 1 polynomial. We can leverage this and (4) to obtain
\[ (x\phi_2^{(\nu)*} + \phi_1^{(\nu)*})W^{(\nu)}(x) = W^{(\nu)}(x)(x\phi_2^{(\nu)*} + \phi_1^{(\nu)*})^*, \]
\[ (x\phi_1^{(\nu)*} + \psi_0^{(\nu)*})W^{(\nu)}(x) = W^{(\nu)}(x)(x\phi_1^{(\nu)*} + \psi_0^{(\nu)*})^*. \]
Since in this case by (3) \( \Lambda^{(\nu)}_n = n(n - 1)\phi_2^{(\nu)*} + n\psi_1^{(\nu)*} \), we can use \( R^{(\nu)} = -M^2\phi_1^{(\nu)*} - (2M + 1)\psi_0^{(\nu)*} \) to get
\[ R^{(\nu)} - x(\Lambda^{(\nu)}_M + \Lambda^{(\nu)}_{M+1}) = -M^2\Phi^{(\nu)}(x)^* - (2M + 1)\Psi^{(\nu)}(x)^*. \]
Using both equations in (5) it can be seen that the condition in (2) is satisfied.

For the Gegenbauer case we have the good fortune that the leading coefficients of \( \Phi^{(\nu)} \) and \( \Psi^{(\nu)} \) are equal up to scalar factor \( \phi_1^{(\nu)} = \frac{1}{2\nu + N}\psi_1^{(\nu)} \), see [16].
So then the eigenvalue in (3) simplifies to $\Lambda_n^{(\nu)} = \left(\frac{n(n-1)}{2\nu+N} + n\right)\psi_1^{(\nu)*}$, and in a similar way to the Hermite case, we can set $R^{(\nu)} = -\left(\frac{M^2}{2\nu+N} + 2M + 1\right)\psi_0^{(\nu)*}$.

The degree 1 polynomial that appears in (2) is then $R^{(\nu)} - x(\Lambda_M^{(\nu)} + \Lambda_{M+1}^{(\nu)}) = -\left(\frac{M^2}{2\nu+N} + 2M + 1\right)\Psi^{(\nu)}(x)^*$.

which satisfies (2) again due to the equation in (4) involving $\Psi^{(\nu)}$.

In [13] the reason to construct an operator like $T^{(\nu)}$ is its commutativity with the time– and band–limiting operators.

**Corollary 1.** Given the values $\Omega \in \mathbb{R}$ and $M \in \mathbb{N}_0$ the matrix differential operator $T^{(\nu)}$ in Theorem 1 constructed with the quantities in [13] Corollary 3.11 and with $R^{(\nu)} = R_H^{(\nu)}$, commutes with the band–limiting operator $\chi^{(\Omega)}_B$ and the time–limiting operator $\chi^{(M)}_T$ that corresponds to the weight $W^{(\nu)}$ described in [13] Section 3.3.

**Proof.** The proof follows immediately from the main results in [13].

Analogous results hold for the Laguerre– and Gegenbauer–type examples. In each example the band–limiting operator is always the same but the time–limiting operator is different for each case, because it involves the matrix inner product and the MVOP.

### 4.2 Charlier

In this section we discuss the Charlier–type MVOP that were introduced in [7]. Let us first recall some notation for the forward and backwards finite shift operators

$$(F \cdot \Delta)(x) = F(x+1) - F(x), \quad (F \cdot \nabla)(x) = F(x) - F(x-1).$$

As before we have a family of weights $W^{(\nu)}$ parametrized by $\nu \in \mathbb{N}_0$ that satisfies certain requirements which are discrete analogues to the strong Pearson equations

$$\begin{cases}
W^{(\nu+1)}(x) = W^{(\nu)}(x)\Phi^{(\nu)}(x), \\
(W^{(\nu+1)} \cdot \nabla)(x) = W^{(\nu)}(x)\Psi^{(\nu)}(x),
\end{cases}$$

where $\Phi^{(\nu)}$ and $\Psi^{(\nu)}$ are matrix polynomials

$$\Phi^{(\nu)}(x) = x^2\phi_2^{(\nu)} + x\phi_1^{(\nu)} + \phi_0^{(\nu)}, \quad \Psi^{(\nu)}(x) = x\psi_1^{(\nu)} + \psi_0^{(\nu)}.$$ 

Explicit expressions for these polynomials were given in [7] but are also listed in the appendix.
As in Section 4.1 one of the main consequences of these strong Pearson equations is that the MVOP $P_n^{(\nu)}$ satisfy the eigenvalue equation $P_n^{(\nu)} D^{(\nu)} = \Lambda_n^{(\nu)} P_n^{(\nu)}$ but now with a difference operator (which is denoted $\Delta S^{(\lambda)}$ in [7])

$$D^{(\nu)} = -\Delta \Phi^{(\nu)}(x)^* - \nabla \Psi^{(\nu)}(x)^*, \quad \Lambda_n^{(\nu)} = -n(n-1)\phi_2^{(\nu)^*} - n\psi_1^{(\nu)^*}. \quad (6)$$

Another consequence is

$$W^{(\nu)}(x) \Phi^{(\nu)}(x) = \Phi^{(\nu)}(x)^* W^{(\nu)}(x), \quad W^{(\nu)}(x) \Psi^{(\nu)}(x) = \Psi^{(\nu)}(x)^* W^{(\nu)}(x), \quad (7)$$

which follows from the fact that $W^{(\nu+1)}$ is a symmetric matrix. The idea of the proof of the following theorem is that the previous two equations can be combined into an equation of the form of (2) and hence provide us with a matrix $R^{(\nu)}$.

**Theorem 2.** Let $T^{(\nu)}$ be the following matrix difference operator

$$T^{(\nu)} = xD^{(\nu)} + D^{(\nu)}(x - 2\Omega) - x(\Lambda_M^{(\nu)} + \Lambda_{M+1}^{(\nu)}) + R^{(\nu)}.$$

as given in [1] equation (11a)]. When $D^{(\nu)}$ is as in the Charlier–type example (6), then $R^{(\nu)}_C = M^2(\phi_1^{(\nu)^*} - \psi_1^{(\nu)^*}) + (2M + 1)\psi_0^{(\nu)^*}$ satisfies (2).

**Proof.** We start off using (6) to show that $\Lambda_M^{(\nu)} + \Lambda_{M+1}^{(\nu)} = -M^2\phi_2^{(\nu)^*} - (2M + 1)\psi_1^{(\nu)^*}$. For the Charlier–type case $\Phi^{(\nu)}$ and $\Psi^{(\nu)}$ have the same constant term: $\psi_0^{(\nu)} = \phi_0^{(\nu)}$. This means that $x^{-1}(\Phi^{(\nu)}(x) - \Psi^{(\nu)}(x)) = x\phi_2^{(\nu)} + \phi_1^{(\nu)} - \psi_1^{(\nu)}$ is a degree 1 polynomial. Due to (7) we see that this polynomial has a desired commutation property with $W^{(\nu)}$. If we now choose $R^{(\nu)}_C = M^2(\phi_1^{(\nu)^*} - \psi_1^{(\nu)^*}) + (2M + 1)\psi_0^{(\nu)^*}$ we get that

$$R^{(\nu)}_C - x(\Lambda_M^{(\nu)} + \Lambda_{M+1}^{(\nu)}) = \frac{M^2}{x}(\Phi^{(\nu)}(x)^* - \Psi^{(\nu)}(x)^*) + (2M + 1)\psi_0^{(\nu)^*},$$

which then satisfies (2). \qed

## 5 Counterexample

In Section 4.1 we studied the Hermite–type matrix weight given in [15] Section 3.3] that had certain restrictions on its parameters $\alpha_j$ and $\gamma_j$. In [15] Section 3.2] however the weight is described without these restrictions and so in this case one cannot derive the strong Pearson equations and the differential operator that follows from them. This latter type of weight has also been studied in [2]. Without the Pearson equations the parameter $\nu$ loses meaning so we drop that part of the notation.

Nevertheless there is another $W$-symmetric second order differential operator that has the MVOP as eigenfunctions. We denote this operator with a slightly different normalization than in [15] Proposition 3.5] as

$$D = -\frac{1}{2} \partial_x^2 + \partial_x (xI - A) + J.$$
Here $J = \text{diag}(1, 2, \ldots, N)$ is diagonal and $A$ only has nonzero entries on the first subdiagonal $A_{i,i-1} = \frac{2\alpha_i}{\alpha_{i-1}}$. $D$ has the monic MVOP $P_n$ as eigenfunctions with eigenvalue $\Lambda_n = nI + J$.

**Remark** We note that the parameters $\alpha_j$ and $t_j^{(\nu)}$ are free in this context, though we do take them to be positive in order to guarantee that the weight is irreducible and positive definite.

In this situation (2) looks very different because the $n$-dependent part of $\Lambda_n$ commutes with $W$ and hence does not contribute. This also means that $\mathcal{R}$ will not depend on $M$. What we are left with is

$$\mathcal{R}W(x) - W(x)\mathcal{R}^* = 2x[J,W(x)],$$

where $[,]$ denotes the usual commutator. The above equation is inherently antisymmetric and can be reduced to a matrix polynomial equation. We therefore have a system of $N(N - 1)/2$ scalar polynomial equations.

### 5.1 $N = 2$

Using the expressions in the appendix we can write out the $2 \times 2$ weight as

$$W(x) = e^{-x^2}t_1 \begin{pmatrix} 1 & \frac{2x}{\alpha_2} \\ 2x\frac{\alpha_2}{\alpha_1} & \frac{\alpha_1}{\alpha_2} + 4x^2\frac{\alpha_2}{\alpha_1} \end{pmatrix}.$$  

Then equation (8) amounts to one independent polynomial equation that must hold for all $x \in \mathbb{R}$

$$\mathcal{R}e_{21} - \mathcal{R}_{12} \frac{t_2}{t_1} - 2(\mathcal{R}_{11} - \mathcal{R}_{22}) \frac{\alpha_2}{\alpha_1} x - 4\mathcal{R}_{12} \frac{\alpha_2}{\alpha_1} x^2 = 4\frac{\alpha_2}{\alpha_1} x^2.$$  

This can be easily solved with

$$\mathcal{R} = \begin{pmatrix} \frac{c}{t_2} & -1 \\ \frac{c}{t_1} & c \end{pmatrix}, \quad c \in \mathbb{R}.$$

### 5.2 $N > 2$

For higher matrix size $N$ the system of equations becomes larger quickly, but not necessarily difficult because (8) is just an inhomogeneous linear system of equations in the entries of $\mathcal{R}$, as mentioned in the Remark on page 6.

Computer algebra calculations for $N \in \{3, 4, 5, 6\}$ indicate that (8) has no solution for $\mathcal{R}$.

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Appendix: Glossary of Explicit Expressions

The matrix differential operators that appear in this paper which commute with their corresponding time– and band– limiting operators can be constructed with the ingredients listed in this appendix.

A Hermite–type

The matrix weight introduced in [15] Equation (3.5)] is given in LDU form by (we omit the $\alpha$ superscript)

$$W^{(\nu)}(x) = L(x)T^{(\nu)}(x)L(x)^*,$$

with $T^{(\nu)}(x)_{jj} = t^{(\nu)}_j e^{-x^2}$ and $L(x)_{j \geq k} = \frac{\alpha_j H_{n-k}(x)}{\alpha_k (j-k)!}$, where $H_n(x)$ denotes the $n$-th standard scalar Hermite polynomial.

For the matrix weight in Section 5 we can leave the parameters $t^{(\nu)}_j$ and $\alpha_j$ as free positive real numbers, but for the Hermite–type example in Section 4 we need them to satisfy additional conditions given in [15, equations (3.7) and (3.9)]. Three explicit parameter sets that satisfy these conditions are given in [15] which we list below.

For $k \in \{1, \ldots, N\}$, $\nu > 0$, $\lambda > 0$:

$$\begin{cases}
  d^{(\nu)} = \frac{1}{\nu+1}, & \alpha_k = \frac{2^{1-k}(N-k+1)_{k-1}}{(k-1)!}, \\
  e^{(\nu)} = \frac{\nu}{\nu+1}, & t^{(\nu)}_j = \frac{\nu+1}{(k-1)!}, \\
  d^{(\nu)} = \lambda, & \alpha_k = 2^{1-k} \lambda^{(k-1)!}(N-k+1)_{k-1}, \\
  e^{(\nu)} = \lambda \nu, & t^{(\nu)}_j = 2^{-k} \lambda^{(k-1)!} \Gamma(\nu+k),
\end{cases}$$

and for the last set we additionally require $\rho > 0$ and $C \geq 0$:

$$\begin{cases}
  d^{(\nu)} = \rho, & \alpha_k = 1, \\
  e^{(\nu)} = C + \nu \rho, & t^{(\nu)}_j = \frac{(\nu+1+C/\rho)_{k-1}}{(k-1)!} \Gamma(\nu+1+C/\rho).
\end{cases}$$

For any of these parameter sets we have

$$\Phi^{(\nu)}(x) = x \phi_1^{(\nu)} + \phi_0^{(\nu)} \quad \Psi^{(\nu)}(x) = x \psi_1^{(\nu)} + \psi_0^{(\nu)}$$

with $\phi_1^{(\nu)} = -d^{(\nu)} A^*$, $\phi_0^{(\nu)} = d^{(\nu)} (J + \frac{1}{2} (A^*)^2) + c^{(\nu)} I$ and

$$\begin{align*}
  \psi_1^{(\nu)} &= 2(d^{(\nu)} (J - (N+1)I) - c^{(\nu)} I), \\
  \psi_0^{(\nu)} &= A^*(c^{(\nu)} I + d^{(\nu)} ((N+1)I - J)) + \frac{1}{2} d^{(\nu)} \tilde{A} J (NI - J),
\end{align*}$$
with \( J = \text{diag}(1, \ldots, N) \), \( A_{k,k-1} = 2\frac{\alpha_k}{\alpha_{k-1}} \) and \( \tilde{A}_{k,k-1} = 2\frac{\alpha_k}{\alpha_k} - 1, k \in \{2, \ldots, N\} \).

**B Laguerre–type**

The matrix weight introduced in [17] Equation (3.8) is given in LDU form by (we omit the \( \alpha \) superscript)

\[
W^{(\nu)}(x) = L(x)T^{(\nu)}(x)L(x)^*,
\]

with \( T^{(\nu)}(x)_{jj} = x^{\nu+k}e^{-x}\Delta^{(\nu)}_{jj} = x^{\nu+k}e^{-x}t_j^{(\nu)}(x) \) and \( L(x)_{j \geq k} = \frac{\alpha_j}{\alpha_k} e^{\alpha(k+k)}(x) \), where \( t_n^{(\nu)}(x) \) denotes the \( n \)-th standard scalar Laguerre polynomial. We give three explicit parameter sets that ensure the weight satisfies strong Pearson equations as given in [17], except the first set which contained a typo for the \( t_j^{(\nu)} \). These hold for \( k \in \{1, \ldots, N\}, \nu > 0, \lambda > 0 \):

\[
\begin{align*}
\{d^{(\nu)} = 1, \quad &\alpha_k = \sqrt{(N-k+1)k}, \\
e^{(\nu)} = \nu, \quad &t_k^{(\nu)} = \Gamma(\nu+1) \prod_{s=1}^{k-1} (1 + \frac{\nu}{s}), \\
d^{(\nu)} = \lambda, \quad &\alpha_k = \sqrt{(k-1)!(N-k+1)k^{-1}}, \\
e^{(\nu)} = \lambda \nu, \quad &t_k^{(\nu)} = \lambda^\nu \Gamma(\nu + k),
\end{align*}
\]

and for the last set we additionally require \( \rho > 0 \) and \( C \geq 0 \):

\[
\begin{align*}
\{d^{(\nu)} = \rho, \quad &\alpha_k = 1, \\
e^{(\nu)} = C + \nu \rho, \quad &t_k^{(\nu)} = \frac{(\nu+1+C/\rho)_{k-1}}{(k-1)(N-k+1)_{k-1}} \rho^\nu \Gamma(\nu + 1 + C/\rho). 
\end{align*}
\]

For these parameters we have

\[
\Phi^{(\nu)}(x) = x^2\phi_2^{(\nu)} + x\phi_1^{(\nu)}, \quad \Psi^{(\nu)}(x) = x\psi_1^{(\nu)} + \psi_0^{(\nu)}
\]

with \( \phi_2^{(\nu)} = -d^{(\nu)}(L(0)^*)^{-1}A^*L(0)^* \), \( \phi_1^{(\nu)} = d^{(\nu)}(L(0)^*)^{-1}JL(0)^* + e^{(\nu)}I, \) and

\[
\begin{align*}
\psi_1^{(\nu)} &= d^{(\nu)}(L(0)^*)^{-1}(J - A^*(J + (\nu + 1)I)L(0)^*) + d^{(\nu)}(\nu + 1 + e^{(\nu)}I), \\
\psi_0^{(\nu)} &= (L(0)^*)^{-1}\left( (J + (\nu + 1)I)(d^{(\nu)}J + e^{(\nu)}I) + \Delta^{(\nu)-1}A\Delta^{(\nu+1)} \right) L(0)^* 
\end{align*}
\]

with \( J = \text{diag}(1, \ldots, N) \) and \( A_{k,k-1} = -\frac{\alpha_k}{\alpha_{k-1}} \).

**C Gegenbauer–type**

The matrix weight introduced in [16] Theorem 2.2 is given in LDU form by

\[
W^{(\nu)}(x) = L^{(\nu)}(x)T^{(\nu)}(x)L^{(\nu)}(x)^*,
\]

with \( T^{(\nu)}(x)_{jj} = t_j^{(\nu)}(1 - x^2)^{\nu+j-1/2} \) and \( L(x)_{j \geq k} = \frac{C^{(\nu)}_{j,k}e^{(\nu+k)}(x), \) where \( C^{(\nu)}_n(x) \) denotes the standard scalar Gegenbauer polynomials. Note however
that we have adhered to the index notation used in [16], which is different from all the other cases described in this paper. The matrix size is \( N = 2\ell + 1 \) with \( \ell \in \mathbb{N} \) and the matrix index takes values \( j \in \{0, \ldots, 2\ell\} \).

We give the parameters that appear in the weight \( \beta_{j,k}^{(\nu)} = \frac{j!}{k!(2\nu+2k)_{j-k}} \) and

\[
l_{k}^{(\nu)} = \frac{k!(\nu)_k}{(\nu + 1/2)_k} \frac{(2\nu + 2\ell)_{\ell}(2\nu + \nu)_{\nu}}{(2\ell - k + 1)_{\ell}(2\nu + k - 1)_{\nu}},
\]
as well as another parameter that will allow for more compact expressions

\[
c^{(\nu)} = \frac{(2\nu + 1)(2\ell + \nu + 1)\ell^2}{\nu(2\nu + 2\ell + 1)(2\ell + \nu)(\ell + \nu)}.
\]

To the same end we give the following diagonal matrices

\[ J = \text{diag}(0, \ldots, 2\ell), \quad A_{j,j-1} = 1, \quad K^{(\nu)}_n = \frac{2\ell + 2\nu + n}{-\ell^2}(J + \nu I)((2\ell + \nu)I - J), \]

the first two of which do not appear in [16]. We should also note that our \( \Phi^{(\nu)}(x) = x^2 \phi_2^{(\nu)} + x \phi_1^{(\nu)} + \phi_0^{(\nu)} \) and \( \Psi^{(\nu)}(x) = x \psi_1^{(\nu)} + \psi_0^{(\nu)} \) follow a slightly different convention than in [16, Theorem 2.4] due to a difference in the form of the strong Pearson equations. We use the coefficients

\[
\phi_2^{(\nu)} = \frac{c^{(\nu)}}{2\ell+2\nu+1} K_1^{(\nu)}, \quad \psi_1^{(\nu)} = c^{(\nu)} K_1^{(\nu)} - \psi_1^{(\nu)} - \phi_1^{(\nu)} J_{\nu}^{(\nu)} - 4(2\ell + 1)I - 2J + (2\ell + \nu)I - J
\]

\[
\phi_1^{(\nu)} = c^{(\nu)} \left( ((2\ell + 1)I - 2J)(J - (2\ell + 1)I) + ((2\ell + 1)I - 2J) A^* J \right)
\]

\[
\phi_0^{(\nu)} = c^{(\nu)} \left( 4(\ell + \nu)^2 I + ((2\ell + 2)I - J)((2\ell + 1)I - J) A^2 + 2J^2 - 4\ell J + 2\ell I + (A^* J)^2 \right)
\]

\[
\psi_0^{(\nu)} = c^{(\nu)} \left( A(J - 2\ell I)(J + \nu I) - A^* J ((2\ell + \nu)I - J) \right)
\]

The only difference with [16] is that here we have absorbed the factor \( c^{(\nu)} \) into the coefficients.

Lastly we note that the second term in \( \psi_0^{(\nu)} \) has an errant opposite sign in [16].

## D Charlier–type

The matrix weight introduced in [17] is given in LDU form by

\[ W^{(\nu)}(x) = (I + A)^{x+\nu} T^{(\nu)}(x) (I + A^*)^{x+\nu}, \quad x, \nu \in \mathbb{N}_0, \quad a > 0, \]

with the parameters \( t_j^{(\nu)} > 0 \), \( \alpha_j > 0 \) in the diagonal matrices

\[
A_{j,k} = \begin{cases} \frac{\alpha_j}{\alpha_j - 1} & j = k + 1 \\ 0, & j \neq k + 1 \end{cases}, \quad T^{(\nu)}(x) = \frac{a^x}{x!} \text{diag}(t_1^{(\nu)}, \ldots, t_N^{(\nu)}).
\]
Here the indices take values $j, k \in \{1, \ldots, N\}$ again.

The following parameter values
\[
\left(\frac{a_j}{\alpha_k}\right)^2 = a^{k-j}(N-k)!, \quad \ell_k^{(\nu)} = \left(\frac{a}{2}\right)^\nu (k)\nu,
\]
ensure that the weight satisfies the discrete strong Pearson equations. The matrix polynomials that arise from those equations
\[
\Phi^{(\nu)}(x) = W^{(\nu)}(x)^{-1}W^{(\nu+1)}(x),
\]
\[
\Psi^{(\nu)}(x) = W^{(\nu)}(x)^{-1}(W^{(\nu+1)}(x) - W^{(\nu+1)}(x - 1)),
\]
are then of degree two and one respectively. In particular we have
\[
\Phi^{(\nu)}(x) = x^2 \phi_2^{(\nu)} + x\phi_1^{(\nu)} + \phi_0^{(\nu)}, \quad \Psi^{(\nu)}(x) = x\psi_1^{(\nu)} + \psi_0^{(\nu)},
\]
where
\[
\phi_2^{(\nu)} = -\frac{1}{2}A^*(A^* + I)^{-1},
\]
\[
\phi_1^{(\nu)} = \frac{1}{2} \left(2J - (N + 1)I - aA^* - (2\nu + 1)A^*(A^* + I)^{-1}\right),
\]
\[
\phi_0^{(\nu)} = \psi_0^{(\nu)} = (A^* + I)^{-\nu}(T^{(\nu)}(0))^{-1}(A + I)T^{(\nu+1)}(0)(A^* + I)^{\nu+1},
\]
\[
\psi_1^{(\nu)} = \frac{1}{2}(J - (N + 1 + \nu)I - aA^* - (\nu + 1)A^*(I + A^*)^{-1}).
\]

**Remark** To see that this weight is in fact very similar to the Hermite–, Laguerre– and Gegenbauer–type weights we have also seen in this paper, we can also define
\[
L(x)_{j,k} = (-a)^{j-k}\frac{\alpha_j \ell_{j-k}^{(\nu)}(x)}{\alpha_k (j-k)!}, \quad j \geq k, \quad L(x)_{j,k} = 0, \quad j < k.
\]

Since $L(x)$ satisfies $L(x+1) = L(x)(I+A)$, due to ladder relations of the scalar Charlier polynomials, we have $L(x) = L(0)(I+A)^x = (I+A)^x L(0)$ for $x \in \mathbb{Z}$. So then it is clear that the Charlier–type weight in this appendix is congruent to a weight of the form $L(x)T^{(\nu)}(x)L(x)^* = L(-\nu)W^{(\nu)}(x)L(-\nu)^*$ which looks more similar to the other weights in this chapter.

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