Research Article

On Hopf-Cyclic Cohomology and Cuntz Algebra

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We demonstrate that Hopf cyclic cocycles, that is, cyclic cocycles with coefficients in stable anti-Yetter-Drinfeld modules, arise from invariant traces on certain ideals of Cuntz-type extension of the algebra.

1. Introduction

Let $k$ be a field of characteristic zero and $A$ an algebra over $k$. In [1] the construction of cyclic cocycles over $A$ was related to the construction of traces over some ideals in the Cuntz algebra extension $qA$. Let us briefly remind the basic construction.

Definition 1. Let $qA$ be an algebra generated by $A$ and symbols $q(a)$ subject to the relation

$$q(ab) = q(a) b + a q(b) - q(a) q(b),$$

for all $a, b \in A$. Equivalently one may identify $qA$ with an ideal with in a free product algebra $A \ast A$.

Further, define $J^n$ as an ideal of $qA$ generated by $a_0 q(a_1) \cdots q(a_m)$ and $q(a_1) \cdots q(a_m)$ $m \geq n$. The main result of Connes and Cuntz [1] states as following.

Theorem 2 (see [1, Proposition 3]). If $T$ is a trace on $J^n$, $n$ even, that is a linear functional such that

$$T(xy) = T(yx), \quad \forall x \in J^k, y \in J^l, k + l = n + 1,$$

then

$$\tau(a_0, a_1, \ldots, a_n) = T(q(a_0) q(a_1) \cdots q(a_n))$$

defines an even cyclic cocycle on $A$.

Odd cocycles arise from graded $\rho$-traces on $J^{n+1}$:

$$T(xy) = T(y \rho (x)), \quad \forall x \in J^k, y \in J^l, k + l = n + 1,$$

where $\rho$ is a $\mathbb{Z}_2$ action on $qA$:

$$\rho(q(a_0) \cdots q(a_m)) = (-1)^m (a_0 - q(a_0)) q(a_1) \cdots q(a_m),$$

$$\rho(q(a_1) \cdots q(a_m)) = (-1)^m q(a_1) \cdots q(a_m).$$

In the paper we will extend this result to a version of Hopf-cyclic cohomology (see [2–4]) for review and details) with coefficients in a stable anti-Yetter-Drinfeld module and present, as a particular example, the case of a twisted cyclic cohomology. The latter was already studied in [5], with the view to geometric construction of modular Fredholm modules.

2. $H$-Module and Comodule Algebras and Hopf-Cyclic Cohomology

Let $H$ be a Hopf algebra with an invertible antipode and $A$ a left $H$-module algebra. Throughout the paper we use the Sweedler notation for coproduct:

$$\Delta(h) = h_{(1)} \otimes h_{(2)},$$

and coaction. The action of $h \in H$ on $a \in A$ (from the left) we denote simply by $ha$.
We begin with the basic lemma, which follows directly from the definition of \( qA \).

**Lemma 3.** If \( A \) is a left \( H \)-module algebra then so is \( qA \), with the action of \( H \) extended through:
\[
hq(a) = q(ha).
\]
(7)

Similarly, if \( B \) is a left \( H \)-comodule algebra then so is \( qB \), with the coaction of \( H \) extended through:
\[
\Delta_L q(b) = b_{(-1)} \otimes q(b_{(0)}).
\]
(8)

Let us recall the following.

**Definition 4.** A left-right stable anti-Yetter-Drinfeld module \( M \), over \( H \), is a right \( H \)-module and left \( H \)-comodule, such that
\[
m_{(0)}m_{(-1)} = m, \quad \forall m \in M,
\]
\[
mh_{(-1)} \otimes mh_{(0)} = (Sh_{(3)})m_{(-1)}h_{(1)} \otimes m_{(0)}h_{(2)},
\]
(9)
\[
\forall m \in M, h \in H.
\]

Let \( \Omega(A) \) be a differential graded algebra with an injective map \( i : A \rightarrow \Omega^0(A) \). Let us assume that the \( \Omega(A) \) has an \( H \)-module structure compatible with that of \( A \) and with the exterior derivative \( d \):
\[
h(i(a)) = i(ha), \quad hd(\omega) = d(h\omega),
\]
\[
\forall a \in A, \omega \in \Omega^i(A), h \in H.
\]
(10)

Now we are ready to define the following.

**Definition 5.** We say that \( \int_M \) is an \( H \)-invariant twisted closed graded trace on \( M \otimes \Omega^n(A) \) if
\[
\int_M (mh_{(-1)} \otimes S(h_{(2)}) \omega) = e(h) \int_M (m \otimes \omega),
\]
\[
\forall m \in M, \omega \in \Omega^n(A), h \in H,
\]
\[
\int_M (m \otimes \omega a) = \int_M (m_{(0)} \otimes S^{-1}(m_{(-1)}) a \omega),
\]
\[
\forall m \in M, \omega \in \Omega^n(A), a \in A, h \in H,
\]
\[
\int_M (m \otimes da) = 0, \quad \forall m \in M, \omega \in \Omega^{n-1}(A).
\]
(11)

Then the following is true.

**Proposition 6.** If \( \Omega(A) \) is a differential graded algebra over \( A \), with an action of \( H \), and \( \int_M \) is an \( H \)-invariant closed graded trace as defined above, then the following map:
\[
\phi(m,a_0,a_1,\ldots,a_n) = \int_M (m \otimes a_0 \, da_1 \, da_2 \cdots da_n),
\]
(12)
defines a Hopf cyclic-cocycle.

**Proof.** First, let us check the cyclicity:
\[
\phi(m,a_0,a_1,\ldots,a_n) = \int_M (m \otimes a_0 \, da_1 \, da_2 \cdots da_n),
\]
\[
= (-1)^n \int_M (m \otimes d(a_0 \, da_1 \, da_2 \cdots da_{n-1} \, a_n)
\]
\[
- m \otimes da_0 \, da_1 \, da_2 \cdots da_n)
\]
\[
= \int_M (m \otimes da_0 \, da_1 \, da_2 \cdots da_n)
\]
\[
= \int_M (m_{(0)} \, (S^{-1}m_{(-1)} a_n) \, da_0 \, da_1 \, da_2 \cdots da_{n-1})
\]
\[
= (-1)^n \phi(m_{(0)}, (S^{-1}m_{(-1)} a_n), a_0, a_1, \ldots, a_{n-1}).
\]
(13)

Similarly, one proves that the Hochschild coboundary of \( \phi \) vanishes:
\[
b\phi(m,a_0,a_1,\ldots,a_{n+1})
\]
\[
:= \sum_{i=0}^{n} (-1)^i \phi(m,a_0,\ldots,a_{a_i},\ldots,a_{n+1})
\]
\[
+ (-1)^{n+1} \phi(m_{(0)}, S^{-1}(m_{(-1)}) a_{n+1} a_0, a_1, \ldots, a_n)
\]
\[
= \int_M (m \otimes a_0 a_1 \, da_2 \cdots da_{n+1}),
\]
\[
+ \sum_{i=1}^{n} (-1)^i \int_M (m \otimes a_0 \, da_1 \cdots d(a_{a_i+1}) \cdots da_{n+1}),
\]
\[
+ (-1)^{n+1} \int_M (m_{(0)} \, S^{-1}(m_{(-1)}) a_{n+1} a_0 \, da_1 \cdots da_n)
\]
\[
= (-1)^n \int_M (m \otimes a_0 a_1 \, da_2 \cdots da_{n+1} a_{n+1})
\]
\[
+ (-1)^{n+1} \int_M (m_{(0)} \, S^{-1}(m_{(-1)}) a_{n+1} a_0 \, da_1 \cdots da_n)
\]
\[
= 0.
\]
(14)

In a trivial way we can also prove the inverse of that theorem, by taking as \( \Omega(A) \) the universal differential graded algebra over \( A \) and setting the \( H \)-invariant trace on the bimodule of \( n \)-forms as the given cocycle on all elements \( m \otimes a_0 \, da_1 \cdots da_n \), and as \( 0 \) on all elements \( m \otimes da_1 \cdots da_n \).

In the following, we define an \( H \)-invariant trace on \( M \otimes A \).

**Definition 7.** An \( H \)-invariant trace on \( M \otimes A \) is a bilinear functional \( \phi : M \otimes A \rightarrow k \), which satisfies
\[
\phi(mh_{(0)} \otimes Sh_{(2)}a) = e(h) \phi(m,a),
\]
(15)
\[
(b\phi)(m,a,b) = \phi(m,ab) - \phi(m_{(0)}, (S^{-1}m_{(-1)} b)a) = 0.
\]
(16)
The main result is as follows.

**Proposition 8.** If $\phi$ is an $H$-invariant trace on $M \otimes J^n \subset M \otimes qA$, $n$ even, then

$$
\xi (m, a_0, a_1, \ldots, a_k) = \phi (m, q (a_0) q (a_1) \cdots q (a_k))
$$

defines a Hopf-cyclic cocycle on $M \otimes A$.

**Proof.** Clearly, since $\phi$ is $H$-invariant, so is $\xi$. What remains to be checked is the cyclicity and the condition that $\xi$ is a Hochschild cycle. This, however, will be taken care of by the extension of the map $\eta$ from $[1]$.

Using the result of $[1]$ we know that the maps

$$
\eta(a) = \begin{pmatrix} a & 0 \\ 0 & q(a) \end{pmatrix}, \quad \eta(da) = \begin{pmatrix} 0 & -q(a) \\ q(a) & 0 \end{pmatrix}
$$

(18)

define a morphism of differential graded algebras from $\Omega_k(A)$ to $M_2(qA)$. Hence, the image is a differential graded algebra, which we will call $\Omega(A)$. Observe that the bimodule of $n$-forms is contained in $M_2(J^{n+1})$.

If $\tau$ is an $H$-invariant trace on $M \otimes J^n$ as defined in Definition 7, then the following defines a closed, graded $H$-invariant trace on $M \otimes \Omega^n(A)$:

$$
\int_M (m \otimes \omega) = \tau (m \otimes \eta(\omega)_{11}) - \tau (m \otimes \eta(\omega)_{22}) .
$$

(19)

That $\int_M$ is closed follows immediately from the fact that a product of even number of elements $\eta(da)$ is proportional to identity matrix in $M_2(qA)$. It is clear that the map $\eta$ is $H$-linear. Therefore it remains only to check the $H$-cyclicity of $\int_M$. But again, since $\eta(\omega)$ is diagonal for any even $n$-form $\omega$ this follows directly from the fact that $\eta(a)$ is diagonal and $\tau$ is an $H$-twisted trace.

In a similar way, odd Hopf-cyclic cocycles can be associated with $\rho$-twisted $H$-invariant traces on $M \otimes J^n$. Consider now the space of $H$-invariant (15) linear functionals on $M \otimes J^n$ and let us split them into odd and even, with respect to the action of $\rho$. For any $k \geq n$ and any such functional we define

$$
T^{(k)}_+ (m, a_0, a_1, \ldots, a_k, b_0) = T (m \otimes a_0 q (a_1) \cdots q (a_k) b_0) .
$$

We have the following

**Proposition 9.** An even $H$-invariant functional $T_+$ is a trace if and only if $bT^{(k)}_+ = T^{(k+1)}_+$ and

$$
T^{(k)}_+ (m, a_0, a_1, \ldots, a_k) = -T^{(k)}_+ (m_{(0)}, S^{-1} m (-1)_a a_0, a_1, \ldots, a_k) .
$$

(20)

of all odd $k \geq n$.

An odd $H$-invariant functional $T_-$ is a trace if and only if $T_-(m \otimes a_0 q (a_1) \cdots q (a_k))$ is a Hopf-cyclic cocycle and $bT^{(k)}_- = T^{(k+1)}_-$ and

$$
T^{(k)}_- (m, a_0, a_1, \ldots, a_k) = T^{(k)}_- (m_{(0)}, S^{-1} m (-1)_a a_0, a_1, \ldots, a_k) + 2T^{(k+1)}_- (m_{(0)}, S^{-1} m (-1)_a a_0, a_1, \ldots, a_k) .
$$

(22)

Since the proof is purely algebraic and follows $[1, Proposition 5]$, the only difference being in the application of cyclicity and $H$-invariance, we skip it. In the conclusion we have the following.

**Corollary 10.** For any even $n$ the Hopf-cyclic cohomology $HC^n_H (M \otimes A)$ is isomorphic to the quotient

$$
\frac{Traces \ on \ J^{n-1}}{Traces \ on \ J^n} \ \text{such that} \ T^{(n-2)} = 0 .
$$

(23)

The full quotient

$$
\frac{Traces \ on \ J^{n-1}}{Traces \ on \ J^n}
$$

(24)

is isomorphic with the quotient of the Hopf-cyclic cohomology group $HC^n_H (M \otimes A)$ by the image of $HC^{n-1}_H (M \otimes A)$ through the periodicity operator $S$.

Similar statement for $\rho$-traces gives the correspondence to odd Hopf-cyclic cohomology.

### 3. Example: Twisted Cyclic Cocycles

Twisted cyclic cocycles appeared first in a context of quantum deformations $[6]$, where they appeared to be a good replacement of the usual cyclic cocycles. In particular, for the quantum $SU_q(2)$ and the family of quantum spheres, certain automorphisms lead to a similar behavior of twisted cyclic theory as in the classical nondeformed case, without the dimension drop, that appears in the standard cyclic homology $[7]$. A detailed study of the twisted case, including the geometric realization through modular Fredholm modules, was presented in $[5]$; here we recall the basic facts to illustrate the above general case.

The notation used in this section is as follows: again $A$ is an algebra (not necessarily unital) over $k$ and $\sigma$ is an automorphism of $A$. Consider $H = CZ$, group algebra of $Z$ with the action on $A$ through the automorphism $\sigma$. As an easy corollary of Lemma 3 we have the following.

**Corollary 11.** The automorphism $\sigma$ extends naturally as an automorphism on $qA$ through

$$
\sigma (q(a)) = q (\sigma(a)) .
$$

(25)

Moreover, the ideals $J^n$ are $\sigma$-invariant, $\sigma (J^n) \subset J^n$. 

Consider now stable anti-Yetter-Drinfeld modules over $H$. The simplest example comes from one-dimensional vector space $M_1$ with the right action and left coaction given by
\begin{equation}
\Delta v = e \otimes v, \quad v \triangleleft e = v,
\end{equation}
where $e$ denotes the generator of $Z$ and $v$ a vector from $M_1$.

We have the following.

**Lemma 12.** Let $A$, $\sigma$ be an algebra and its automorphism. Then, any $Z$-invariant, cyclic trace on $M \otimes A$ corresponds to a $\sigma$-twisted trace $T$ on $A$:
\begin{equation}
T(xy) = T(y\sigma(x)), \quad \forall x \in j^k, \; y \in j^l, \; k + l = n + 1
\end{equation}

We skip the proof as it follows directly from the properties of Hopf-cyclic traces applied to this particular example. As a corollary, we obtain the following.

**Proposition 13.** If $T$ is a $\sigma$-twisted trace on $qA$ then the functional
\begin{equation}
\tau_r(a_0, a_1, \ldots, a_n) = T(q(a_0)q(a_1)\cdots q(a_n))
\end{equation}
defines a $\sigma$-twisted $n$-cyclic cocycle on $A$ for even $n$.

Similarly, by composing $\sigma$ with the map $\rho$ (5), we obtain another automorphism of $qA$:
\begin{equation}
\bar{\sigma} = \sigma \circ \rho = \rho \circ \sigma.
\end{equation}

Then, we can define odd $\sigma$-traces on $f^n$, which satisfy
\begin{equation}
T(xy) = T(y\bar{\sigma}(x)), \quad \forall x \in f^k, \; y \in f^l, \; k + l = n + 1.
\end{equation}

The respective functionals, which arise from $\bar{\sigma}$-traces, give $\sigma$-twisted odd cyclic cocycles.

The detailed presentation of the construction of twisted cyclic cocycles from finitely summable modular Fredholm modules is in [5].

### 4. Example: Hopf Algebras

A different set of examples of Hopf-cyclic cohomology originated from studies of Hopf algebras. Let us begin with an example of the Hopf-cyclic homology of an $H$-comodule algebra. In this section, $A$ is a right $H$-comodule algebra and $M$ is a right-right stable anti-Yetter-Drinfeld module.

First, we observe the following.

**Remark 14.** The coaction of $H$ extends to $qA$ through
\begin{equation}
\Delta q(a) = q(a_0) \otimes q(a_1), \quad \forall a \in A.
\end{equation}

An $n$ Hopf-cyclic cocycle with values in $M$ is a multilinear map $\psi$ from $A^{n+1}$ to $M$, which is cyclic:
\begin{equation}
\psi(a_0, a_1, \ldots, a_n) = (-1)^n\psi(a_{n(0)}, a_0, \ldots, a_{n-1})a_{n(1)},
\end{equation}

$H$-colinear:
\begin{equation}
\Delta \psi(a_0, a_1, \ldots, a_n)
= \psi(a_0(0), a_1(0), \ldots, a_n(0)) \otimes a_0(1) a_1(1) \cdots a_n(1),
\end{equation}
and that its coboundary vanishes:
\begin{equation}
\begin{aligned}
& b\psi(a_0, a_1, \ldots, a_n, a_{n+1}) \\
& = \sum_{i=0}^{n} (-1)^i \psi(a_0, \ldots, a_{i+1}, \ldots, a_{n+1}) \\
& \quad + (-1)^{n+1} \psi(a_0(0)a_0(1), \ldots, a_{n-1})a_{n(1)},
\end{aligned}
\end{equation}
The proof follows exactly the same lines as in the previous section and therefore we skip it. What is interesting, however, is the application, which was discussed in [8].

**Lemma 16.** If $A = H$ and one takes the coproduct as the coalgebra structure, and the anti-Yetter-Drinfeld module $M = k$ is determined through a modular pair in involution: $\gamma$ is a grouplike element, $\chi$ is a character of $A$, such that $\chi(\gamma) = 1$ and the right coaction and action are
\begin{equation}
\Delta(v) = v \otimes \gamma, \quad v \triangleleft h = v\chi(h),
\end{equation}
for any $v \in k$ (for details see [9]).

The compatibility condition $\gamma$ and $\chi$ is
\begin{equation}
S_{\gamma, \chi}(h) = \chi(h_{(1)}) S(h_{(2)}), \quad S_{\gamma, \chi}^2 = \text{id}.
\end{equation}

Then, since $qA$ is a comodule algebra over $A$ and $k$ remains an anti-Yetter-Drinfeld module, one can construct even Hopf-cyclic cocycles over $A$ with values in $k$ from $k$-valued linear maps on $f^n$, $n$-even, that satisfy
\begin{equation}
\begin{aligned}
T(x) \otimes y = T(x_{(0)}) \otimes x_{(1)}, \\
T(xy) = T(y_{(0)}) \chi(y_{(1)}),
\end{aligned}
\end{equation}
for each $x \in f^n$, $xy \in f^n$.

Again, the proof is a direct consequence of Proposition 9 and Corollary 10.

### 5. Conclusions

We have shown that the results of [1] extend to the case of Hopf-cyclic cohomology with coefficients. This is, in itself, an anticipated result. Its value, however, is that such presentation offers a possibility for a geometric presentation of Hopf-cyclic cocycles thus opening a new insight in the theory. Similarly as in the standard or twisted case it is conceivable that Hopf-cyclic cocycles might be constructed from certain type of objects like Fredholm modules. While the general theory is still not available yet, the above construction shows a path, which could be followed, at least in some particular cases, like for the modular pair in involution. The work in this direction is already in progress.
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