Quandle theory and optimistic limits of representations of knot groups

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October 17, 2014

Abstract

Quandle of a link diagram is very useful tool to describe the knot group via Wirtinger presentation. Following the conjugation quandle proposed by Inoue and Kabaya, we see the octahedral triangulation is always non-degenerate for any boundary-parabolic representation of the knot group.

On the other hand, the same triangulation was also used in the optimistic limit. In this paper, we show the relationship between the quandle theory and the optimistic limit of a link. Especially, we interpret the optimistic limit in the language of quandle, and generalize it to any boundary-parabolic representation of the knot group. Furthermore, we show that the hyperbolicity equation, which determines the hyperbolic structure of the triangulation, always have a solution. This solution is explicitly constructed by the shadow-coloring of the quandle induced by the representation.

1 Introduction

Quandle of a link diagram is an algebraic object whose axioms are naturally induced by Reidemeister moves of links. Due to the natural definition, it has fruitful structures and various applications to knot theory, including classical knots, surface knots and virtual knots. (A good guide of quandle theory is [1].)

Especially, quandle is a very useful tool to describe the fundamental group of a link via Wirtinger presentation. Inoue and Kabaya, in [7], used it for a boundary-parabolic representation \( \rho : \pi_1(L) \to \text{PSL}(2, \mathbb{C}) \), where \( \pi_1(L) \) is the fundamental group of a link \( L \), which is called the knot group of \( L \), and boundary-parabolic means the image of the peripheral subgroup \( \pi_1(\partial(S^3 \setminus L)) \) is a parabolic subgroup of \( \text{PSL}(2, \mathbb{C}) \). They defined shadow-coloring of the (conjugation) quandle consisting of parabolic elements induced by \( \rho \), and suggested a natural way to determine the geometric shape of the octahedral triangulation of \( S^3 \setminus (L \cup \{\text{two points}\}) \). Especially, they showed that any tetrahedron in the triangulation is non-degenerate.

On the other hand, the same octahedral triangulation appears naturally from the link diagram together with its dual graph (see Section 3 of [14] for detail), and is used in many
The optimistic limit was first appeared in [8] when Kashaev proposed the volume conjecture, which relates certain limits of knot invariants, called Kashaev invariants, with the hyperbolic volumes. The optimistic limit is the value of certain potential function evaluated at saddle point, where the function and the value are expected to be an analytic continuation of the Kashaev invariant and the limit of the invariant, respectively. As a matter of fact, physicists usually consider the optimistic limit as “actual limit” and develop their theories. Due to many works on the optimistic limit in [10], [4] and [15], the volume conjecture will be solved if the coincidence of the optimistic limit and the actual limit is proved.

On the other hand, the original definition of the optimistic limit was so complicated that applying it to other theories was really hard. Therefore, we improved the definition at [3] by using the octahedral triangulation of $S^3 \setminus (L \cup \text{two points})$ without any collapsing of tetrahedra. We claimed the revised definition is more natural because many technical assumptions are removed and the geometry of the triangulation works more naturally. This article will show another example of such naturalness by the harmony with the quandle theory of [7].

The optimistic limit is defined by the potential function $V(z_1, \ldots, z_n, w^i_j, \ldots)$. Previously, in [3], this function was defined purely by the link diagram, but here we modify it using the information of the representation $\rho$. (The definition is in Section 3) We consider a solution of the following set

$$H := \left\{ \exp(z_k \frac{\partial V}{\partial z_k}) = 1, \quad \exp(w^i_k \frac{\partial V}{\partial w^i_k}) = 1 \mid j: \text{degenerate, } k = 1, \ldots, n \right\},$$

which is a saddle-point of the potential function $V$. Then Proposition 3.1 will show that $H$ becomes the hyperbolicity equations, which determines the hyperbolic structure of the octahedral triangulation. We assumed $H$ has a solution in [3] because, if not, we cannot do anything with that potential function. (This is because the chosen link diagram fixes the triangulation and the function together.) When there is no solution, we had to change the function by modifying the link diagram and we conjecturally expected proper change will guarantee the existence of a solution. However, in this article, we will explicitly construct a solution $(z_1^{(0)}, \ldots, z_n^{(0)}, (w^i_k)^{(0)}, \ldots)$ of $H$ from the shadow-coloring induced by the representation $\rho$. (The exact formulas are in Theorem 3.2) Furthermore, after defining $V_0(z_1, \ldots, z_n, (w^i_k), \ldots) := V(z_1, \ldots, z_n, (w^i_k), \ldots)$

$$- \sum_k \left( z_k \frac{\partial V}{\partial z_k} \right) \log z_k - \sum_{j: \text{degenerate}} \left( w^i_k \frac{\partial V}{\partial w^i_k} \right) \log w^i_k,$$
we will show

\[ V_0(z_1^{(0)}, \ldots, z_n^{(0)}, (w_j^{(0)}, \ldots) \equiv i(\text{vol}(\rho) + i\text{cs}(\rho)) \pmod{\pi^2} \]  

(1)

at Theorem 3.3, where vol(\rho) and cs(\rho) are the volume and the Chern-Simons invariant of \( \rho \) defined in [16], respectively. The left-hand side of (1) is called the optimistic limit of \( \rho \), and vol(\rho) + i cs(\rho) in the right-hand side is called the complex volume of \( \rho \).

Note that the optimistic limit gives a very convenient method to calculate the complex volume of a given representation \( \rho \). As a matter of fact, when we consider the following formal series of the asymptotic expansion of the Kashaev invariant in (1-3) of [5],

\[ \mathcal{Z}_M(h) = \exp \left( \frac{1}{\hbar} S_{M,0} - \frac{3}{2} \log \hbar + S_{M,1} + \sum_{n \geq 2} \hbar^{n-1} S_{M,n} \right), \quad \hbar = \frac{2\pi i}{N}, \]

our potential function of the optimistic limit detects the complex volume term \( S_{M,0} \). The author expects there should be another function, which is also induced by the Kashaev invariant, that detects the Ray-Singer torsion term \( S_{M,1} \). Actually, the results of [5] and [12] support this conjecture. One of the advantage of the approach in this article is that we already know the right solution, so what should be done is to find the right function.

This article consists of the following contents. In Section 2, we will summarize some results of [7]. Although the article [7] discussed general theory of quandle homology, we restrict our attention only to the conjugation quandle \((P, *)\), which consists of parabolic elements of PSL(2,\( \mathbb{C} \)) with the operation induced by the conjugation. Also, we will discuss the triangulation of the link complement of [3] and slightly modify it in the viewpoint of [7]. Section 3 will discuss the optimistic limit and Section 4 will contain two simple examples, the figure-eight knot 4_1 and the trefoil knot 3_1 with one degenerate crossing.

2 Quandle

In this section, we will survey the quandle theory of [7], which is an essential tool to new interpretation of the optimistic limit. We remark that all contents of this section come from [7] and series lectures of Ayumu Inoue given at Seoul National University during spring of 2012.

2.1 Conjugation quandle of parabolic elements

**Definition 2.1.** A quandle is a set \( X \) with a binary operation \( * \) satisfying the following three conditions:

1. \( a * a = a \) for any \( a \in X \),

2. the map \( \ast b : X \to X \) (\( a \mapsto a * b \)) is bijective for any \( b \in X \),

3. \( (a * b) * c = (a * c) * (b * c) \) for any \( a, b, c \in X \).
The inverse of \( *b \) is notated by \( *^{-1}b \). In other words, the equation \( a *^{-1}b = c \) is equivalent to \( c * b = a \).

**Definition 2.2.** Let \( G \) be a group and \( X \) be a subset of \( G \) satisfying
\[ gXg^{-1} = X \quad \text{for any } g \in G. \]
Define the binary operation \( * \) on \( X \) by
\[ a * b = bab^{-1} \tag{2} \]
for any \( a, b \in X \). Then \((X, *)\) becomes a quandle and is called the conjugation quandle.

As an example, let \( \mathcal{P} \) be the set of parabolic elements of \( \text{PSL}(2, \mathbb{C}) = \text{Isom}^+(\mathbb{H}^3) \). Then
\[ g\mathcal{P}g^{-1} = \mathcal{P} \]
holds for any \( g \in \text{PSL}(2, \mathbb{C}) \). Therefore, \((\mathcal{P}, *)\) is a conjugation quandle, and this is the only quandle we are using in this article.

To perform concrete calculations, explicit expression of \((\mathcal{P}, *)\) was introduced in [7]. At first, note that
\[
\begin{pmatrix} p & q \\ r & s \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 + rs & s^2 \\ -r^2 & 1 - rs \end{pmatrix},
\]
for \( \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \text{PSL}(2, \mathbb{C}) \). Therefore, we can identify \((\mathbb{C}^2 \setminus \{0\})/\pm\) with \( \mathcal{P} \) by
\[
(\alpha \beta) \leftrightarrow \begin{pmatrix} 1 + \alpha\beta & \beta^2 \\ -\alpha^2 & 1 - \alpha\beta \end{pmatrix}, \tag{3}
\]
where \( \pm \) means the equivalence relation \((\alpha \beta) \sim (-\alpha -\beta)\). To use the left-side action of \( \text{PSL}(2, \mathbb{C}) \) on \((\mathbb{C}^2 \setminus \{0\})/\pm\), we consider the transpose of (3) by
\[
(\alpha \beta) \leftrightarrow \begin{pmatrix} 1 + \alpha\beta & -\alpha^2 \\ \beta^2 & 1 - \alpha\beta \end{pmatrix},
\]
and define the operation \( * \) by
\[
\begin{pmatrix} \alpha \\ \beta \end{pmatrix} * \begin{pmatrix} \gamma \\ \delta \end{pmatrix} := \begin{pmatrix} 1 + \gamma\delta & -\gamma^2 \\ \delta^2 & 1 - \gamma\delta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in (\mathbb{C}^2 \setminus \{0\})/\pm,
\]
where the matrix multiplication on the right-hand side is the standard multiplication. Note that this definition coincides with the operation of the conjugation quandle \((\mathcal{P}, \ast)\) by
\[
\begin{pmatrix} \alpha \\ \beta \end{pmatrix} * \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 1 + \gamma\delta & -\gamma^2 \\ \delta^2 & 1 - \gamma\delta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in (\mathbb{C}^2 \setminus \{0\})/\pm
\]
\[
\leftrightarrow \begin{pmatrix} 1 + \gamma\delta & -\gamma^2 \\ \delta^2 & 1 - \gamma\delta \end{pmatrix} \begin{pmatrix} 1 + \alpha\beta & -\alpha^2 \\ \beta^2 & 1 - \alpha\beta \end{pmatrix} \begin{pmatrix} 1 + \gamma\delta & -\gamma^2 \\ \delta^2 & 1 - \gamma\delta \end{pmatrix}^{-1}
\]
\[
= \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix}^{-1} \in \text{PSL}(2, \mathbb{C}).
\]
The inverse operation is given by
\[
\begin{pmatrix} \alpha \\ \beta \end{pmatrix}^{-1} \star \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 1 - \gamma \delta & \gamma^2 \\ -\delta^2 & 1 + \gamma \delta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.
\]

From now on, we use the notation \( \mathcal{P} \) instead of \((\mathbb{C}^2\setminus\{0\})/\pm\).

### 2.2 Knot group and shadow-coloring

Consider a representation \( \rho : \pi_1(L) \to \text{PSL}(2, \mathbb{C}) \) of a hyperbolic link \( L \). We call \( \rho \) boundary-parabolic when the peripheral subgroup \( \pi_1(\partial(S^3\setminus L)) \) of \( \pi_1(L) \) maps to a subgroup of \( \text{PSL}(2, \mathbb{C}) \) whose elements are all parabolic.

For a fixed oriented link diagram \( D \) of \( L \), Wirtinger presentation gives an algorithmic expression of \( \pi_1(L) \). For each arc \( \alpha_k \) of \( D \), we draw a small arrow labelled \( a_k \) as in Figure 1, which presents a loop. (The details are in [13]. Here we are using the opposite orientation of \( a_k \) to be consistent with the operation of the conjugation quandle.) This loop corresponds to one of the meridian curves of the boundary tori, so \( \rho(a_k) \) is an element in \( \mathcal{P} \). Hence we call \( \{\rho(a_1), \ldots, \rho(a_n)\} \) arc-coloring of \( D \), whereas each \( \rho(a_k) \) is assigned to the corresponding arc \( \alpha_k \).

![Figure 1: The figure-eight knot 4_1](image)

Wirtinger presentation shows that the knot group is presented by
\[
\pi_1(L) = \langle a_1, \ldots, a_n; r_1, \ldots, r_n \rangle,
\]

\(^1\) We always assume the diagram does not contain a trivial knot component which has only over-crossings or under-crossings or no crossing. If it happens, then we change the diagram of the trivial component slightly. For example, applying Reidemeister second move to make different types of crossings or Reidemeister first move to add a kink is good enough. This assumption is necessary to guarantee that the five-term triangulation becomes a topological triangulation of \( S^3 \setminus (L \cup \{\text{two points}\}) \).
where the relation $r_l$ is assigned to each crossing as in Figure 2. Note that $r_l$ coincides with (2), so we can write down relation of the arc-colors as in Figure 3. As a matter of fact, Figure 3 is usually the defining relation of arc-coloring in general quandle. (Refer Section 4 of [7] for this.)

(a) $r_l: a_{l+1} = a_ka_l a_k^{-1}$

(b) $r_l: a_l = a_ka_{l+1} a_k^{-1}$

Figure 2: Relations at crossings

Figure 3: Arc-coloring

From now on, we always assume $\rho: \pi_1(L) \to \text{PSL}(2, \mathbb{C})$ is a given boundary-parabolic representation.

**Definition 2.3.** The *Hopf map* $h: \mathcal{P} \to \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ is defined by

$$
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}
\mapsto
\frac{\alpha}{\beta}
$$

To avoid redundant notations, arc-color is denoted by $\{a_1, \ldots, a_n\}$ without $\rho$ from now on. Choose an element $s \in \mathcal{P}$ corresponding to certain region of the diagram $D$ and determine elements corresponding to the other regions using the relation in Figure 4.

The assignment of elements of $\mathcal{P}$ to all regions using the relation in Figure 4 is called *region-coloring*. This assignment is well-defined because the two curves in Figure 5 which we call the cross-changing pair, determine the same region-coloring, and any pair of curves with the same starting and ending points can be transformed each other by finite sequence of cross-changing pairs.
An arc-coloring together with a region-coloring is called shadow-coloring. The following lemma shows important property of shadow-colorings, which is crucial for showing the existence of solutions of certain equations.

**Lemma 2.4.** Let $L$ be a link and assume an arc-coloring is already given by the boundary-parabolic representation $\rho : \pi_1(L) \to PSL(2, \mathbb{C})$. Then, for any triple $(a_k, s, s * a_k)$ of an arc-color $a_k$ and its surrounding region-colors $s, s * a_k$ as in Figure 4, there exists a region-coloring satisfying

$$ h(a_k) \neq h(s) \neq h(s * a_k) \neq h(a_k). $$

(4)

**Proof.** We follow some part of the proof of Proposition 2 in [7].

For the given arc-colors $a_1, \ldots, a_n$, we choose region-colors $s_1, \ldots, s_m$ so that

$$ \{h(s_1), \ldots, h(s_m)\} \cap \{h(a_1), \ldots, h(a_n)\} = \emptyset. $$

(5)

This is always possible because, the number of $h(s_1)$ satisfying $h(s_1) \in \{h(a_1), \ldots, h(a_n)\}$ is finite, and $h(s_2), \ldots, h(s_m)$ are uniquely determined by $h(s_1)$. Therefore, the number of $h(s_1)$ satisfying

$$ \{h(s_1), \ldots, h(s_m)\} \cap \{h(a_1), \ldots, h(a_n)\} \neq \emptyset $$

is finite, but we have infinite freedom to choose $h(s_1) \in \mathbb{CP}^1$. 
Now consider the case of Figure 4 and assume \( h(s * a_k) = h(s) \). Then we obtain

\[
h(s * a_k) = \hat{a}_k(h(s)) = h(s),
\]

where \( \hat{a}_k : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \) is the Möbius transformation

\[
\hat{a}_k(z) = \frac{(1 + \alpha_k \beta_k)z - \alpha_k^2}{\beta_k^2 z + (1 - \alpha_k \beta_k)}
\]

of \( a_k = \left( \frac{\alpha_k}{\beta_k} \right) \). Then (6) implies \( h(s) \) is the fixed point of \( \hat{a}_k \), which means \( h(a_k) = h(s) \) that contradicts (5).

\[
\square
\]

We remark that the condition (5) is stronger than what we actually need in (4). Even though some region-coloring does not satisfy (5), if it satisfies (4), then all results of this article can be applicable. Examples in Section 4 are this type.

The arc-coloring induced by \( \rho \) together with the region-coloring satisfying Lemma 2.4 is called the *shadow-coloring induced by \( \rho \)*. This shadow-coloring will determine the exact coordinates of points of the octahedral triangulation in the next section.

### 2.3 Octahedral triangulations of link complements

In this section, we describe the ideal triangulation of \( \mathbb{S}^3 \setminus (L \cup \{ \text{two points} \}) \) appeared in [3].

To obtain the triangulation, we consider the crossing \( j \) in Figure 6 and place an octahedron \( A_jB_jC_jD_jE_jF_j \) on each crossing \( j \) as in Figure 7(a). Then we twist the octahedron by identifying edges \( B_jF_j \) to \( D_jF_j \) and \( A_jE_j \) to \( C_jE_j \), respectively. The edges \( A_jB_j, B_jC_j, C_jD_j \) and \( D_jA_j \) are called *horizontal edges* and we sometimes express these edges in the diagram as arcs around the crossing as in Figure 6.

![Crossing j with shadow-coloring](image)

(a) Positive crossing  
(b) Negative crossing

Figure 6: Crossing \( j \) with shadow-coloring
Then we glue faces of the octahedra following the lines of the link diagram. Specifically, there are three gluing patterns as in Figure 8. In each cases (a), (b) and (c), we identify the faces $\triangle A_jB_jE_j \cup \triangle C_jB_jE_j$ to $\triangle C_{j+1}D_{j+1}F_{j+1} \cup \triangle C_{j+1}B_{j+1}F_{j+1}$, $\triangle B_jC_jF_j \cup \triangle D_jC_jF_j$ to $\triangle D_{j+1}C_{j+1}F_{j+1} \cup \triangle B_{j+1}C_{j+1}F_{j+1}$ and $\triangle A_jB_jE_j \cup \triangle C_jB_jE_j$ to $\triangle C_{j+1}B_{j+1}E_{j+1} \cup \triangle A_{j+1}B_{j+1}E_{j+1}$, respectively.

Note that this gluing process identifies vertices $\{A_j, C_j\}$ to one point, denoted by $-\infty$, and $\{B_j, D_j\}$ to another point, denoted by $\infty$, and finally $\{E_j, F_j\}$ to the other points, denoted by $P_t$ where $t = 1, \ldots, c$ and $c$ is the number of the components of the link $L$. The regular neighborhoods of $-\infty$ and $\infty$ are two 3-balls and that of $\cup_{t=1}^c P_t$ is a tubular neighborhood of the link $L$. Therefore, after removing all vertices of the gluing, we obtain an octahedral decomposition of $S^3 \setminus (L \cup \{\pm \infty\})$. The octahedral triangulation is obtained by subdividing all the octahedra of the decomposition using the arc-coloring from $\rho$ as follows.

If $h(a_k) \neq h(a_l)$ in Figure 6, then we subdivide the octahedron into four tetrahedra by adding edge $E_jF_j$ as in Figure 7(b). Also, if $h(a_k) = h(a_l)$, then we subdivide it by adding edge $A_jC_j$ as in Figure 7(c). The result is called the octahedral (ideal) triangulation of $S^3 \setminus (L \cup \{\pm \infty\})$. 

Figure 7: Octahedron on the crossing $j$

Figure 8: Three gluing patterns
Consider a shadow-coloring of a link diagram $D$ induced by $\rho$, and let \{a_1, a_2, \ldots, a_n\} be the arc-colors and \{s_1, s_2, \ldots, s_m\} be the region-colors. The number of these colors is finite, so can choose an element $p \in \mathcal{P}$ satisfying $h(p) \notin \{h(a_1), \ldots, h(a_n), h(s_1), \ldots, h(s_m)\}$.

The geometric shape of the triangulation is determined by the shadow-coloring induced by $\rho$ in the following way. If $h(a_k) \neq h(a_l)$ and the crossing $j$ is positive, then let the signed coordinates of the tetrahedra $E_j F_j C_j D_j$, $E_j F_j A_j D_j$, $E_j F_j A_j B_j$, $E_j F_j C_j B_j$ be

\[(a_l, a_k, s * a_l, p), -(a_l, a_k, a_k, s * a_l, p), -(a_l * a_k, a_k, (s * a_l) * a_k, p),\]

respectively. Here, the minus sign of the coordinate means the orientation of the tetrahedron does not coincide with the one induced by the vertex-ordering. Also, if the crossing $j$ is negative, then let the signed coordinates of the tetrahedra $E_j F_j C_j D_j$, $E_j F_j A_j D_j$, $E_j F_j A_j B_j$, $E_j F_j C_j B_j$ be

\[(a_l, a_k, s, p), -(a_l, a_k, s * a_l, p), (a_l * a_k, a_k, (s * a_l) * a_k, p), -(a_l * a_k, a_k, s * a_l, p),\]

The tetrahedra in Figure 9 shows the signed coordinates of (7) and (8), and Figure 10 shows the relationship between Figure 7(b) and Figure 9. Here, gluing the pairs of faces $E_j F_j D_j$ and $E_j F_j' D_j$, $E_j F_j B_j$ and $E_j' F_j' B_j'$, $E_j F_j C_j$ and $E_j' F_j' C_j'$, $E_j F_j A_j$ and $E_j' F_j A_j'$ of Figure 10 respectively, induces Figure 7(b).

![Positive crossing](image1)

(a) Positive crossing

![Negative crossing](image2)

(b) Negative crossing

Figure 9: Coordinates of tetrahedra when $h(a_k) \neq h(a_l)$

On the other hand, if $h(a_k) = h(a_l)$ and $j$ is positive, then let the signed coordinates of the tetrahedra $F_j A_j C_j D_j$, $E_j A_j C_j D_j$, $E_j A_j C_j B_j$, $F_j A_j C_j B_j$ be

\[-(a_k, s, s * a_l, p), (a_l, s, s * a_l, p), -(a_l * a_k, s * a_k, (s * a_l) * a_k, p), (a_k, s * a_k, (s * a_l) * a_k, p),\]

(9)
Figure 10: Octahedron in Figure 7(b) after gluing pairs of faces

respectively. If \( j \) is negative, then let the signed coordinates be

\[-(a_k, s*a_l, s, p), (a_l, s*a_l, s, p), -(a_l*a_k, (s*a_l)*a_k, s*a_k, p), (a_k, (s*a_l)*a_k, s*a_k, p), \]

(10)

respectively.

The tetrahedra in Figure 11 shows the signed coordinates of (9) and (10). Note that the orientations of (7)–(10) are different from [7] and match with [3].

**Definition 2.5.** Let \( v_0, v_1, v_2, v_3 \in \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\} = \partial \mathbb{H}^3 \). The hyperbolic ideal tetrahedron with signed coordinate \( \sigma(v_0, v_1, v_2, v_3) \) with \( \sigma \in \{\pm 1\} \) is called **degenerate** when some of the vertices \( v_0, v_1, v_2, v_3 \) have the same coordinate, and **non-degenerate** when all the vertices have different coordinates. The **cross-ratio** \([v_0, v_1, v_2, v_3]^{\sigma}\) of the non-degenerate signed coordinate \( \sigma(v_0, v_1, v_2, v_3) \) is defined by

\[ [v_0, v_1, v_2, v_3]^\sigma = \left( \frac{v_3 - v_0}{v_2 - v_0} \frac{v_2 - v_1}{v_3 - v_1} \right)^\sigma \in \mathbb{C}\{0, 1\}. \]

The tetrahedra in (7)–(10) have elements of the coordinates in \( \mathcal{P} \). Therefore, we need to send them to points in the boundary of the hyperbolic 3-space \( \partial \mathbb{H}^3 \) so as to obtain hyperbolic ideal tetrahedra. The Hopf map \( h \), defined in Definition 2.3, plays the role.

**Lemma 2.6.** The images of (7)–(10) under the Hopf map \( h \) are non-degenerate tetrahedra. Specifically, if \( h(a_k) \neq h(a_l) \) and the crossing \( j \) is positive, then

\[
(h(a_l), h(a_k), h(s*a_l), h(p)), -(h(a_l), h(a_k), h(s), h(p)), \]

(11)

\[
(h(a_l*a_k), h(a_k), h(s*a_k), h(p)), -(h(a_l*a_k), h(a_k), h((s*a_l)*a_k), h(p)), \]

\[
(h(a_l*a_k), h(a_k), h(s*a_k), h(p)), -(h(a_l*a_k), h(a_k), h((s*a_l)*a_k), h(p)), \]

\[
(h(a_l*a_k), h(a_k), h(s*a_k), h(p)), -(h(a_l*a_k), h(a_k), h((s*a_l)*a_k), h(p)), \]
and, if \(h(a_k) \neq h(a_l)\) and the crossing \(j\) is negative, then
\[
(h(a_l), h(a_k), h(s), h(p)), -(h(a_l), h(a_k), h(s \ast a_l), h(p)), \\
(h(a_l \ast a_k), h(a_k), h((s \ast a_l) \ast a_k), h(p)), -(h(a_l \ast a_k), h(s \ast a_k), h(p)),
\]
are non-degenerate hyperbolic ideal tetrahedra.

If \(h(a_k) = h(a_l)\) and the crossing \(j\) is positive, then
\[
(h(a_l), h(s), h(s \ast a_l), h(p)), -(h(a_k), h(s), h(s \ast a_l), h(p)), \\
(h(a_k), h(s \ast a_k), h((s \ast a_l) \ast a_k), h(p)), -(h(a_l \ast a_k), h(s \ast a_k), h(p)),
\]
and, if \(h(a_k) = h(a_l)\) and the crossing \(j\) is negative, then
\[
(h(a_l), h(s \ast a_l), h(s), h(p)), -(h(a_k), h(s \ast a_l), h(s), h(p)), \\
(h(a_k), h((s \ast a_l) \ast a_k), h(s \ast a_k), h(p)), -(h(a_l \ast a_k), h((s \ast a_l) \ast a_k), h(s \ast a_k), h(p)),
\]
are non-degenerate hyperbolic ideal tetrahedra.
Proof. The shadow-coloring we are considering satisfies Lemma 2.4, and all endpoints of edges are adjacent, as \( a_k, s, s \ast a_k \) in Figure 4, or one of them is \( p \), expect the edges \((a_l, a_k), (a_l \ast a_k, a_k)\) in the case of \( h(a_k) \neq h(a_l) \). Therefore, it is enough to show that \( h(a_k) \neq h(a_l) \) implies \( h(a_l \ast a_k) \neq h(a_k) \), which is trivial because \( h(a_l \ast a_k) = h(a_k \ast a_k) \) implies \( h(a_l) = h(a_k) \).

Note that, when \( h(a_k) = h(a_l) \), the first two tetrahedra in (13) share the same coordinate with different signs and the other two do the same. Therefore, all tetrahedra cancel each other out and we can remove the octahedron of the crossing. Also, the same holds for (14).

When a geometric shape of a triangulation is given, it determines another representation \( \rho' : \pi_1(L) \to \text{PSL}(2, \mathbb{C}) \) by the Yoshida’s construction in Section 4.5 of [9]. This construction determines the representation \( \rho' \) up to conjugate, and we know \( \rho' \) is conjugate with \( \rho \) by the following observation. The Poincaré polyhedron theorem claims the fundamental group \( \pi_1(L) \) is generated by the face identifications with certain relations. The face identifications in Figure 9, after applying the Hopf map \( h \), are the Möbius transformations \( \hat{a}_1, \ldots, \hat{a}_n \) and the identity map. Note that \( a_1, \ldots, a_n \) are also generators in Wirtinger presentation of \( \pi_1(L) \) in Section 2.2. Hence, two representations \( \rho \) and \( \rho' \) map generators to the same elements, which implies they are the same.

### 2.4 Complex volume of \( \rho \)

Consider an ideal tetrahedron with vertices \( v_0, v_1, v_2, v_3 \), where \( v_k \in \mathbb{C}P^1 \). For each edge \( v_kv_l \), we assign \( g_{kl} \) and \( \hat{g}_{kl} \in \mathbb{C}P^1 \), and call them long-edge parameter and edge parameter, respectively. (See Figure 12) Later, we will distinguish them by considering \( g_{kl} \) is assigned to the edge of a triangulation and \( \hat{g}_{kl} \) to the edge of a tetrahedron.

![Figure 12: Long-edge parameter](image)

**Definition 2.7.** For the (long-)edge parameter \( g_{kl} \) of an ideal tetrahedron, **Ptolemy identity** is the following equation:

\[
g_{02}g_{13} = g_{01}g_{23} + g_{03}g_{12}.
\]

For example, if we define long-edge parameter \( g_{kl} \equiv v_l - v_k \), then direct calculation shows

\[
(v_2 - v_0)(v_3 - v_1) = (v_1 - v_0)(v_3 - v_2) + (v_3 - v_0)(v_2 - v_1),
\]

(15)
which is the Ptolemy identity. Furthermore, these long-edge parameters satisfy

\[ [v_0, v_1, v_2, v_3] = \frac{g_{03}g_{12}}{g_{02}g_{13}}. \]  \hspace{1cm} (16)

The Ptolemy identity and \( 16 \) are the most important properties, so this definition of long-edge parameters looks good enough now. However, to apply the results of \([16]\) and \([6]\), the value of \( g_{kl} \) should depend only on the edge of the triangulation. In other words, if two edges are glued together in the triangulation and they have the long-edge parameters \( g_{kl} \) and \( g_{ab} \) respectively, then we need \( g_{kl} = g_{ab} \). We call this condition the coincidence of long-edge parameter. (We also need extra condition that the orientations of the two glued edges induced by the vertex-orientations of each tetrahedra should coincide. However, the vertex-orientation in \([11]\)–\([14]\) always satisfy it.) Unfortunately, \( g_{kl} := v_l - v_k \) does not satisfy this condition, so we will modify them using \([7]\) as follows.

At first, consider two elements \( a = \left( \alpha_1 \alpha_2 \right) \), \( b = \left( \beta_1 \beta_2 \right) \) in \( \mathcal{P} \). We define determinant \( \det(a, b) \) by

\[ \det(a, b) := \pm \det \left( \frac{\alpha_1 \beta_1}{\alpha_2 \beta_2} \right) = \pm(\alpha_1\beta_2 - \beta_1\alpha_2). \]

Note that the determinant is defined up to sign due to the choice of the representative \( a = \left( \alpha_1 \alpha_2 \right) = \left( -\alpha_1 -\alpha_2 \right) \in \mathcal{P} \). To remove this ambiguity, we fix representatives \( 2 \) of arc-colors in \( \mathbb{C}^2 \setminus \{0\} \) once and for all. Then we fix a representative of one region-color, which uniquely determines the representatives of all the other region-colors by the arc-coloring. (This is due to the fact that \( s \ast (\pm a) = s \ast a \) for any \( s, a \in \mathbb{C}^2 \setminus \{0\} \).)

After fixing all the representatives of shadow-colors, we obtain a well-defined determinant

\[ \det(a, b) = \det \left( \frac{\alpha_1}{\alpha_2} \frac{\beta_1}{\beta_2} \right) = \alpha_1\beta_2 - \beta_1\alpha_2. \]  \hspace{1cm} (17)

**Lemma 2.8.** For \( a, b, c \in \mathbb{C}^2 \setminus \{0\} \), the determinant satisfies

\[ \det(a \ast c, b \ast c) = \det(a, b). \]

**Proof.** Let \( a = \left( \alpha_1 \alpha_2 \right) \), \( b = \left( \beta_1 \beta_2 \right) \), \( c = \left( \gamma_1 \gamma_2 \right) \), and \( C = \left( \begin{array}{cc} 1 + \gamma_1\gamma_2 & -\gamma_2 \\ \gamma_2 & 1 - \gamma_1\gamma_2 \end{array} \right) \). Then

\[ \det(a \ast c, b \ast c) = \det(C \left( \frac{\alpha_1}{\alpha_2} \frac{\beta_1}{\beta_2} \right), C \left( \frac{\beta_1}{\beta_2} \right)) = \det C \cdot \det(a, b) = \det(a, b). \]

\( \square \)

\( \text{The difference with } \[7\] \text{ is that they chose a sign of the determinant once and for all. Their choice is good enough to define long-edge parameter } g_{jk}, \text{ but not for edge parameter } \hat{g}_{jk}. \)
Consider the shadow-coloring and the coordinates of tetrahedra in Figures 9 and 11. We define the edge parameter \( \hat{g}_{kl} \) using those coordinates. Specifically, when the signed coordinate of the tetrahedron \( \mp(a_l, a_k, s, p) \) in the left-hand or the right-hand side of Figures 9 are defined by

\[
\hat{g}_{l1} = \det(a_l, a_k), \quad \hat{g}_{02} = \det(a_1, s), \quad \hat{g}_{03} = \det(a_1, p),
\]

\[
\hat{g}_{12} = \det(a_k, s), \quad \hat{g}_{13} = \det(a_k, p), \quad \hat{g}_{23} = \det(s, p).
\]

**Lemma 2.9.** The edge parameter \( \hat{g}_{kl} \) of the tetrahedron \( \sigma(a_0, a_1, a_2, a_3) \) defined in (18) satisfies the Ptolemy identity and

\[
[h(a_0), h(a_1), h(a_2), h(a_3)] = \frac{\hat{g}_{03}\hat{g}_{12}}{\hat{g}_{02}\hat{g}_{13}}.
\]

**Proof.** From (17), we obtain

\[
h(x) - h(y) = \frac{x_1}{x_2} - \frac{y_1}{y_2} = \frac{\det(x, y)}{x_2y_2},
\]

where \( x = \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \) and \( y = \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) \).

Let \( a_k = \left( \begin{array}{c} \alpha_k \\ \beta_k \end{array} \right) \) for \( k = 0, \ldots, 3 \), and let \( v_k = h(a_k) = \frac{\alpha_k}{\beta_k} \). Then (15) and (20) imply

\[
\frac{\det(a_0, a_2)}{\beta_0\beta_2} \frac{\det(a_1, a_3)}{\beta_1\beta_3} = \frac{\det(a_0, a_1)}{\beta_0\beta_1} \frac{\det(a_2, a_3)}{\beta_2\beta_3} + \frac{\det(a_0, a_3)}{\beta_0\beta_3} \frac{\det(a_1, a_2)}{\beta_1\beta_2},
\]

which is equivalent to the Ptolemy identity \( \hat{g}_{02}\hat{g}_{13} = \hat{g}_{01}\hat{g}_{23} + \hat{g}_{03}\hat{g}_{12} \).

Also, using (20), we obtain

\[
[h(a_0), h(a_1), h(a_2), h(a_3)] = \frac{\det(a_0, a_2)}{\beta_0\beta_2} \frac{\det(a_1, a_3)}{\beta_1\beta_3} = \frac{\hat{g}_{03}\hat{g}_{12}}{\hat{g}_{02}\hat{g}_{13}},
\]

\[
[h(a_0), h(a_3), h(a_1), h(a_2)] = \frac{\hat{g}_{02}\hat{g}_{13}}{\hat{g}_{01}\hat{g}_{23}}, \quad [h(a_0), h(a_2), h(a_3), h(a_1)] = -\frac{\hat{g}_{01}\hat{g}_{23}}{\hat{g}_{03}\hat{g}_{12}}.
\]

Note that, by the same calculation of the proof above, we obtain

\[
[h(a_0), h(a_3), h(a_1), h(a_2)] = \frac{\hat{g}_{02}\hat{g}_{13}}{\hat{g}_{01}\hat{g}_{23}}, \quad [h(a_0), h(a_2), h(a_3), h(a_1)] = -\frac{\hat{g}_{01}\hat{g}_{23}}{\hat{g}_{03}\hat{g}_{12}}.
\]

If we put \( z^\sigma = [h(a_0), h(a_1), h(a_2), h(a_3)] \), using Ptolemy identity, the above equations are expressed by

\[
z^\sigma = \frac{\hat{g}_{03}\hat{g}_{12}}{\hat{g}_{02}\hat{g}_{13}}, \quad 1 - z^\sigma = \frac{\hat{g}_{02}\hat{g}_{13}}{\hat{g}_{01}\hat{g}_{23}}, \quad 1 - \frac{1}{z^\sigma} = -\frac{\hat{g}_{01}\hat{g}_{23}}{\hat{g}_{03}\hat{g}_{12}}.
\]
The edge parameter \( \hat{g}_{jk} \) defined above satisfies all needed properties of the long-edge parameter \( g_{jk} \) except the coincidence of long-edge parameter, which \( \hat{g}_{jk} \) satisfies up to sign. To see this phenomenon, consider the two edges of Figure 9(a) as in Figure 13, which are glued in the triangulation. Assume the chosen representative of \( a_m \) in Figure 13 satisfies \( a_m = -a_l \ast a_k \in \mathbb{C}^2 \setminus \{0\} \). (This actually happens often. For example, the minus signs of (46) and (47) in Section 4 show this situation.) Then the edge parameters satisfy

\[
\hat{g}_{01} = \det(a_l, a_k) = \det(a_l \ast a_k, a_k) = - \det(a_m, a_k) = -\hat{g}'_{01}.
\]

![Diagram](image)

Figure 13: Example of the inconsistency of edge parameter

To define the long-edge parameter \( g_{jk} \), we assign certain signs to the edge parameters

\[ g_{jk} = \pm \hat{g}_{jk}, \]

so that the consistency of the long-edge parameter holds. Due to Lemma 6 of [7], any choice of values of \( g_{jk} \) determines the same complex volume. Actually, in Section 3, we do not consider the exact values of \( g_{jk} \), but use the existence of them.

The relations (21) of the edge parameters become

\[
z^\sigma = \pm \frac{g_{03}g_{12}}{g_{02}g_{13}}, \quad \frac{1}{1 - z^\sigma} = \pm \frac{g_{02}g_{13}}{g_{01}g_{23}}, \quad 1 - \frac{1}{z^\sigma} = \pm \frac{g_{01}g_{23}}{g_{03}g_{12}}.
\]

Using (22), we define integers \( p \) and \( q \) by

\[
\begin{align*}
 p\pi i &= - \log z^\sigma + \log g_{03} + \log g_{12} - \log g_{02} - \log g_{13}, \\
 q\pi i &= \log(1 - z^\sigma) + \log g_{02} + \log g_{13} - \log g_{01} - \log g_{23}.
\end{align*}
\]

Now we consider the tetrahedron with the signed coordinate \( \sigma(a_0, a_1, a_2, a_3) \) and the signed triples \( \sigma[z^\sigma; p, q] \in \hat{P}(\mathbb{C}) \). (The extended pre-Bloch group is denoted by \( \hat{P}(\mathbb{C}) \) here. For the definition, see Definition 1.6 of [16].) To consider all signed triples corresponding to all tetrahedra in the triangulation, we denote the triple by \( \sigma_t[z^\sigma_t; p_t, q_t] \), where \( t \) is the index of tetrahedra. We define a function \( \hat{L} : \hat{P}(\mathbb{C}) \to \mathbb{C}/\pi^2 \mathbb{Z} \) by

\[
[z; p, q] \mapsto \text{Li}_2(z) + \frac{1}{2} \log z \log(1 - z) + \frac{\pi i}{2} (q \log z + p \log(1 - z)) - \frac{\pi^2}{6},
\]
where $\text{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt$ is the dilogarithm function. (Well-definedness of $\hat{L}$ was proved in [11].) Recall that, for a boundary-parabolic representation $\rho$, the volume $\text{vol}(\rho)$ and the Chern-Simons invariant $\text{cs}(\rho)$ was already defined in [16]. We call $\text{vol}(\rho) + i \text{cs}(\rho)$ the complex volume of $\rho$. The following theorem is one of the main result of [7].

**Theorem 2.10** ([16], [7]). For a given boundary-parabolic representation $\rho$ and the shadow-coloring induced by $\rho$, the complex volume of $\rho$ is calculated by

$$\sum_t \sigma_t \hat{L}[z_{ot}; p_t, q_t] \equiv i(\text{vol}(\rho) + i \text{cs}(\rho)) \pmod{\pi^2},$$

where $t$ is over all tetrahedra of the triangulation defined in Section 2.3.

**Proof.** See Theorem 5 of [7]. Note that the removal of the tetrahedra in (13) and (14) does not have any effect on the complex volume. For example, if we put $[z; p, q]$ and $-[z'; p', q']$ the corresponding triples of the tetrahedron $(h(a_l), h(s), h(s*a_l), h(p))$ and $-(h(a_k), h(s), h(s*a_l), h(p))$ in (13), respectively, and put $\{g_{kl}\}$, $\{g'_{kl}\}$ the sets of long-edge parameters of the two tetrahedra, respectively. Then, from $h(a_l) = h(a_k)$, we obtain $z = z'$. Furthermore, we can choose long-edge parameters so that $g_{kl} = g'_{kl}$ holds for all pairs of edges sharing the same coordinate, which induces $p = p'$, $q = q'$ and $\hat{L}[z; p, q] - \hat{L}[z'; p', q'] = 0$.

$\square$

## 3 Optimistic limit

In this Section, we will use the result of Section 2 to redefine the optimistic limit of [3] and prove the existence of solutions of $\mathcal{H}$. At first, we consider a given boundary-parabolic representation $\rho$ and its shadow-coloring of a link diagram $D$. For the diagram, define sides of the diagram by the lines connecting two adjacent crossings. (The word edge is more common than side here. However, we want to keep the word edge for the edges of a triangulation.)

For example, the diagram in Figure 14 has eight sides. We assign $z_1, \ldots, z_n$ to sides of $D$ as in Figure 14 and call them side variables.

For the crossing $j$ in Figure 15, let $z_e, z_f, z_g, z_h$ be side variables and let $a_l, a_k$ be the arc-colors. If $h(a_k) \neq h(a_l)$, then we define the potential function $V_j$ of the crossing $j$ by

$$V_j(z_e, z_f, z_g, z_h) = \text{Li}_2\left(\frac{z_f}{z_e}\right) - \text{Li}_2\left(\frac{z_f}{z_g}\right) + \text{Li}_2\left(\frac{z_h}{z_g}\right) - \text{Li}_2\left(\frac{z_h}{z_e}\right). \quad (25)$$

On the other hand, if $h(a_l) = h(a_k)$ in Figure 15, then we introduce new variables $w^j_e, w^j_f, w^j_g$ of the crossing $j$ and define

$$V_j(z_e, z_f, z_g, z_h, w^j_e, w^j_f, w^j_g) = -\log w^j_e \log z_e + \log w^j_f \log z_f - \log w^j_g \log z_g + \log \frac{w^j_f w^j_g}{w^j_e} \log z_h. \quad (26)$$
Figure 14: Sides of a link diagram

Figure 15: A crossing $j$ with arc-colors and side variables

For notational convenience, we put $w^j_h := w^j_e w^j_g / w^j_f$. (We can choose any three of $w^j_e, w^j_f, w^j_g, w^j_h$ as free variables in (26).) We call the crossing in Figure 15 degenerate when $h(a_l) = h(a_k)$ holds. In particular, when the degenerate crossing forms a kink, as in Figure 16, we put

$$V_j(z_e, z_f, z_g, w^j_e, w^j_f)$$

$$= - \log w^j_e \log z_e + \log w^j_f \log z_f - \log w^j_f \log z_f + \log \frac{w^j_e w^j_f}{w^j_f} \log z_g$$

$$= - \log w^j_e \log z_e + \log w^j_f \log z_g.$$

Consider the crossing $j$ in Figure 15 and place the octahedron $A_j B_j C_j D_j E_j F_j$ as in Figure 7. When the crossing $j$ is non-degenerate, in other words $h(a_k) \neq h(a_l)$, we consider Figure 7(b) and assign shape parameters $\frac{z_e}{z_f}, \frac{z_f}{z_g}, \frac{z_h}{z_e}$ and $\frac{z_e}{z_h}$ to the horizontal edges $A_j B_j, B_j C_j, C_j D_j, D_j A_j$, respectively. On the other hand, if the crossing $j$ is degenerate, in other words $h(a_k) = h(a_l)$, then we consider Figure 7(c) and assign shape parameters $w^j_e, w^j_f, w^j_g$ and $w^j_h$ to the edges $A_j F_j, B_j E_j, C_j F_j$ and $D_j E_j$, respectively.\footnote{Note that, when $h(a_k) = h(a_l)$, by adding one more edge $B_j D_j$ to Figure 7(c), we obtain another...}
The potential function \( V(z_1, \ldots, z_n, w_k^j, \ldots) \) of the link diagram \( D \) is defined by

\[
V(z_1, \ldots, z_n, w_k^j, \ldots) = \sum_j V_j,
\]

where \( j \) is over all crossings. For example, if \( h(a_1) \neq h(a_2) \) in Figure 14, then \( a_4 = a_1 * a_2 \) implies \( h(a_1) \neq h(a_2) \). If \( h(a_3) \neq h(a_1) \), then \( a_2 = a_3 * a_4 \). Furthermore, the shape-parameters assigned to \( D \) are \( 1/w_k \) and \( 1/w_k^j \), respectively.

Figure 16: Kink

Note that, if \( h(a_1) \neq h(a_k) \) for any crossing \( j \) in Figure 15, then the definition of the potential function above coincides with the definition in Section 2 of \([3]\). Therefore, the above definition is a slight modification of the previous one.

On the other hand, if \( h(a_1) = h(a_2) \) in Figure 14, then \( a_1 * a_2 = a_1 \). This equation and subdivision of the octahedron with five tetrahedra. (This subdivision was already used in \([2]\).) Focusing on the middle tetrahedron that contains all horizontal edges, we obtain \( w_{e}^j w_{h}^j = w_{f}^j w_{h}^j \). Furthermore, the shape-parameters assigned to \( D_j F_j \) and \( B_j F_j \) are \( 1 - 1/w_k \) and \( 1 - 1/w_k^j \), respectively.

4 \( h(a_4) = h(a_2) \), then \( h(a_2 * a_2) = h(a_2) \) and \( h(a_4) = h(a_1 * a_2) \) induces \( h(a_2) = h(a_1) \), which is contradiction.

5 \( h(a_3) = h(a_3) \), then \( h(a_3 * a_3) = h(a_3) = h(a_2) = h(a_1 * a_3) \) induces \( h(a_2) = h(a_3) = h(a_1) \), which is contradiction. Likewise, if \( h(a_1) = h(a_3) \), then \( h(a_2) = h(a_1 * a_3) = h(a_1) \) is contradiction.
the relations at crossings induce $a_1 = a_2 = a_3 = a_4$, and the potential function becomes

$$V(z_1, \ldots, z_8, w_1^1, w_1^4, w_1^7, w_1^8, w_2^2, w_3^2, w_3^3, w_5^2, w_5^4, w_6^4)$$

$$= -\log w_8^1 \log z_8 + \log w_1^4 \log z_4 - \log w_1^7 \log z_7 + \log w_1^8 \log z_5$$

$$- \log w_2^2 \log z_4 + \log w_3^3 \log z_3 - \log w_3^2 \log z_5 - \log w_6^4 \log z_6,$$

where $w_5^1 = w_3^1 w_1^1 / w_1^4$, $w_2^2 = w_3^2 w_1^2 / w_8^2$, $w_3^2 = w_3^3 w_5^2 / w_3^3$ and $w_4^1 = w_2^4 w_1^4 / w_2^4$.

For the potential function $V(z_1, \ldots, z_n, w_k^j, \ldots)$, let $\mathcal{H}$ be the set of equations

$$\mathcal{H} := \left\{ \exp(z_k \frac{\partial V}{\partial z_k}) = 1, \exp(w_k^j \frac{\partial V}{\partial w_k^j}) = 1 \right\}, \quad k = 1, \ldots, n, j : \text{degenerate}$$

and $\mathcal{S} = \{(z_1, \ldots, z_n, w_k^j, \ldots)\}$ be the solution set of $\mathcal{H}$. Here, solutions are assumed to satisfy the properties that $z_k \neq 0$ for all $k = 1, \ldots, n$ and $\frac{z_f}{z_e} \neq 1, \frac{z_g}{z_f} \neq 1, \frac{z_h}{z_e} \neq 1, \frac{z_h}{z_g} \neq 1, \frac{z_g}{z_e} \neq 1, \frac{z_g}{z_f} \neq 1$ in Figure 15 for any non-degenerate crossing, and $w_k^j \neq 0$ for any degenerate crossing $j$ and the index $k$. (All these assumptions are essential to avoid singularity of the equations in $\mathcal{H}$ and log 0 in the formula $V_0$ defined in (32). Even though we allow $w_k^j = 1$ here, the value we are interested in always satisfies $w_k^j \neq 1$.)

**Proposition 3.1.** For the arc-coloring of a link diagram $D$ induced by $\rho$ and the potential function $V(z_1, \ldots, z_n, w_k^j, \ldots)$, the set $\mathcal{H}$ induces the whole set of hyperbolicity equations of the octahedral triangulation defined in Section 2.3.

The hyperbolicity equations consist of the Thurston’s gluing equations of edges and the completeness condition.

**Proof of Proposition 3.1.** When no crossing is degenerate, this proposition was already proved in Section 3 of [3]. To see the main idea, check Figures 10–13 and equations (3.1)–(3.3) of [3]. Equation (3.1) is a completeness condition along a meridian of certain annulus, and (3.2)–(3.3) are gluing equations of certain edges. These three types of equations induce all the other gluing equations.

Therefore, we consider the case when the crossing $j$ in Figure 15 is degenerate. Then, the following three equations

$$\exp(w_e^j \frac{\partial V}{\partial w_e^j}) = \frac{z_h}{z_e} = 1, \quad \exp(w_f^j \frac{\partial V}{\partial w_f^j}) = \frac{z_f}{z_e} = 1, \quad \exp(w_g^j \frac{\partial V}{\partial w_g^j}) = \frac{z_h}{z_g} = 1$$

induce $z_e = z_f = z_g = z_h$. This guarantees the gluing equations of horizontal edges trivially by the assigning rule of shape parameters. (Note that the shape parameters assigned to the horizontal edges of the octahedron at a degenerate crossing are always 1.)
There are four possible cases of gluing pattern as in Figure 17, and we assume the crossing \( j \) is degenerate and \( j+1 \) is non-degenerate. (The case when both of \( j \) and \( j+1 \) are degenerate can be proved similarly.)

The part of the potential function \( V \) containing \( z_k \) in Figure 17(a) is
\[
V^{(a)} = \log w_k^j \log z_k + \text{Li}_2 \left( \frac{z_e}{z_k} \right) - \text{Li}_2 \left( \frac{z_f}{z_k} \right),
\]
and
\[
\exp \left( z_k \frac{\partial V}{\partial z_k} \right) = \exp \left( z_k \frac{\partial V^{(a)}}{\partial z_k} \right) = w_k^j \left( 1 - \frac{z_e}{z_k} \right) \left( 1 - \frac{z_f}{z_k} \right)^{-1} = 1
\]
is equivalent with the following completeness condition
\[
\frac{1}{w_k^j} \left( 1 - \frac{z_k}{z_e} \right)^{-1} \left( 1 - \frac{z_k}{z_f} \right) = 1
\]
along a meridian \( m \) in Figure 18(a). (Compare it with Figure 11 of [3].) Here, \( a_j, b_j, c_j, b_{j+1}, c_{j+1}, d_{j+1} \) in Figure 18(a) are the points of the cusp diagram, which lie on the edges \( A_jE_j, B_jE_j, C_jE_j, B_{j+1}F_{j+1}, C_{j+1}F_{j+1}, D_{j+1}F_{j+1} \) of Figure 7(a), respectively.

The part of the potential function \( V \) containing \( z_k \) in Figure 17(b) is
\[
V^{(b)} = -\log w_k^j \log z_k - \text{Li}_2 \left( \frac{z_e}{z_k} \right) + \text{Li}_2 \left( \frac{z_k}{z_f} \right),
\]
and
\[
\exp \left( z_k \frac{\partial V}{\partial z_k} \right) = \exp \left( z_k \frac{\partial V^{(b)}}{\partial z_k} \right) = \frac{1}{w_k^j} \left( 1 - \frac{z_k}{z_e} \right) \left( 1 - \frac{z_k}{z_f} \right)^{-1} = 1
\]
is equivalent with the following completeness condition
\[
\frac{1}{w_k^j} \left( 1 - \frac{z_k}{z_f} \right)^{-1} \left( 1 - \frac{z_k}{z_e} \right) = 1
\]

---

6 The relation \( a_4 = a_1 \ast a_2 \) induces \( a_4 = a_1, a_4 = a_3 \ast a_1 \) does \( a_4 = a_3, \) and \( a_2 = a_3 \ast a_4 \) does \( a_2 = a_4. \)
along a meridian \( m \) in Figure 18(b). Here, \( b_j, c_j, d_j, a_{j+1}, b_{j+1}, c_{j+1} \) in Figure 18(b) are the points of the cusp diagram, which lie on the edges \( B_jF_j, C_jF_j, D_jF_j, A_{j+1}E_{j+1}, B_{j+1}E_{j+1}, C_{j+1}E_{j+1} \) of Figure 7(a), respectively. (To simplify the cusp diagram in Figure 18(b), we subdivided the polygon \( A_jB_jC_jD_jF_j \) in Figure 7(c) into three tetrahedra by adding the edge \( B_jD_j \).)

The part of the potential function \( V \) containing \( z_k \) in Figure 17(c) is

\[
V^{(c)} = - \log w_k^j \log z_k + \text{Li}_2 \left( \frac{z_e}{z_k} \right) - \text{Li}_2 \left( \frac{z_f}{z_k} \right),
\]

and

\[
\exp \left( z_k \frac{\partial V}{\partial z_k} \right) = \exp \left( z_k \frac{\partial V^{(c)}}{\partial z_k} \right) = \frac{1}{w_k^j} \left( 1 - \frac{z_e}{z_k} \right) \left( 1 - \frac{z_f}{z_k} \right)^{-1} = 1
\]
is equivalent with the following gluing equation

\[ w^j_k \left( 1 - \frac{z_e}{z_k} \right)^{-1} \left( 1 - \frac{z_f}{z_k} \right) = 1 \]

of \( c_j = c_{j+1} \) in Figure 18(c). (Compare it with Figure 12 of [3].) Here, \( b_j, c_j, d_j, b_{j+1}, c_{j+1}, d_{j+1} \) in Figure 18(c) are the points of the cusp diagram, which lie on the edges \( B_jF_j, C_jF_j, D_jF_j, B_{j+1}F_{j+1}, C_{j+1}F_{j+1}, D_{j+1}F_{j+1} \) of Figure 7(a), respectively, and the edges \( d_jc_j \) and \( bjc_j \) are identified to \( b_{j+1}c_{j+1} \) and \( d_{j+1}c_{j+1} \), respectively. (To simplify the cusp diagram in Figure 18(c), we subdivided the polygon \( A_jB_jC_jD_jF_j \) in Figure 7(c) into three tetrahedra by adding the edge \( B_jD_j \).)

The part of the potential function \( V \) containing \( z_k \) in Figure 17(d) is

\[ V^{(d)} = \log w^j_k \log z_k - \text{Li}_2 \left( \frac{z_k}{z_e} \right) + \text{Li}_2 \left( \frac{z_k}{z_f} \right), \]

and

\[ \exp \left( z_k \frac{\partial V}{\partial z_k} \right) = \exp \left( z_k \frac{\partial V^{(d)}}{\partial z_k} \right) = w^j_k \left( 1 - \frac{z_k}{z_e} \right) \left( 1 - \frac{z_k}{z_f} \right)^{-1} = 1 \]

is equivalent with the following gluing equation

\[ w^j_k \left( 1 - \frac{z_k}{z_e} \right) \left( 1 - \frac{z_k}{z_f} \right)^{-1} = 1 \]

of \( b_j = b_{j+1} \) in Figure 18(d). (Compare it with Figure 13 of [3].) Here, \( a_j, b_j, c_j, a_{j+1}, b_{j+1}, c_{j+1} \) in Figure 18(d) are the points of the cusp diagram, which lie on the edges \( A_jE_j, B_jE_j, C_jE_j, A_{j+1}E_{j+1}, B_{j+1}E_{j+1}, C_{j+1}E_{j+1} \) of Figure 7(a), respectively, and the edges \( a_jb_j \) and \( c_jb_j \) are identified to \( c_{j+1}b_{j+1} \) and \( a_{j+1}b_{j+1} \), respectively.

Note that the case when both of the crossings \( j \) and \( j+1 \) in Figure 17 are degenerate can be proved by the same way.

On the other hand, it was already shown in [3] that all hyperbolicity equations are induced by these types of equations (see the discussion that follows Lemma 3.1 of [3]), so the proof is done.

\[
\square
\]

In [3], we could not prove the existence of a solution of \( \mathcal{H} \), in other words \( S \neq \emptyset \), so we assumed it. However, the following theorem proves the existence by directly constructing one solution from the given boundary-parabolic representation \( \rho \) together with the shadow-coloring.

**Theorem 3.2.** Consider a shadow-coloring of a link diagram \( D \) induced by \( \rho \) which satisfies Lemma 2.4 and the potential function \( V(z_1, \ldots, z_n, w^j_k, \ldots) \) from \( D \). For each side of \( D \) with the side variable \( z_k \), arc-color \( a_l \) and the region-color \( s \), as in Figure 19, we define

\[ z_k^{(0)} := \frac{\det(a_l, p)}{\det(a_l, s)}. \]

(30)
Also, if the positive crossing $j$ in Figure 20(a) is degenerate, then we define

$$ (w^j_e)(0) := \frac{\det(s, p)}{\det(s * a_k, p)}, \quad (w^j_f)(0) := \frac{\det((s * a_l) * a_k, p)}{\det(s * a_k, p)}, $$

$$ (w^j_g)(0) := \frac{\det((s * a_l) * a_k, p)}{\det(s * a_k, p)}, \quad (w^j_h)(0) := \frac{\det(p, s)}{\det(s * a_k, p)}, $$

and, if the negative crossing $j$ in Figure 20(b) is degenerate, then we define

$$ (w^j_e)(0) := \frac{\det(s * a_l, p)}{\det((s * a_l) * a_k, p)}, \quad (w^j_f)(0) := \frac{\det(s * a_k, p)}{\det((s * a_l) * a_k, p)}, $$

$$ (w^j_g)(0) := \frac{\det(s * a_k, p)}{\det(s * a_l, p)}, \quad (w^j_h)(0) := \frac{\det(s * a_l, p)}{\det(s, p)}. $$

Then $z^j_k(0) \neq 0, 1, \infty, (w^j_k)(0) \neq 0, 1$ for all possible $j, k$, and $(z^j_1(0), \ldots, z^j_n(0), (w^j_k)(0), \ldots) \in S$.

Figure 19: Region-coloring

Figure 20: Crossings with shadow-colors and side-variables

Note that the ± signs in the arc-colors of Figure 20 appears due to the representatives of the colors in $\mathbb{C}^2 \setminus \{0\}$. However, ± does not change the value of $z^j_k(0)$ because

$$ \frac{\det(\pm a_l, p)}{\det(\pm a_l, s)} = \frac{\det(a_l, p)}{\det(a_l, s)} = z^j_k(0). $$

Likewise, the value of $(w^j_k)(0)$ does not depend on the choice of ± because the representatives of region-colors are uniquely determined from the fact $s * (\pm a) = s * a$ for any $s, a \in \mathbb{C}^2 \setminus \{0\}$.
Proof of Theorem 3.2

At first, when the crossing $j$ in Figure 20 is degenerate, we will show

$$z_e^{(0)} = z_f^{(0)} = z_g^{(0)} = z_h^{(0)},$$

which satisfies (29). Using $h(a_k) = h(a_l)$, we put $a_k = \left( \begin{array}{c} \alpha \\ \beta \end{array} \right)$ and $a_l = \left( \begin{array}{c} c \alpha \\ c \beta \end{array} \right) = c a_k$ for some constant $c \in \mathbb{C} \setminus \{0\}$. Then, we obtain $a_l * a_k = a_l$ and, if $j$ is positive crossing, then

$$z_e^{(0)} = \frac{c \det(a_k, p)}{c \det(a_k, s)} = \frac{\det(a_l, p)}{\det(a_l, s)} = z_h^{(0)};$$

$$z_f^{(0)} = \frac{\det(\pm a_l * a_k, p)}{\det(\pm a_l * a_k, s * a_k)} = \frac{\det(a_l * a_k, s * a_k)}{\det(a_l * a_k, s * a_k)} = \frac{\det(a_l, p)}{\det(a_l, s)} = z_h^{(0)};$$

$$z_g^{(0)} = \frac{c \det(a_k, p)}{c \det(a_k, s * a_l)} = \frac{\det(a_l, p)}{\det(a_l, s * a_l)} = z_h^{(0)}.$$

If $j$ is negative crossing, then by exchanging the indices $e \leftrightarrow g$ in the above calculation, we obtain the same result.

Note that Lemma 2.4 and the definition of $p$ in Section 2.3 guarantee $z_k^{(0)} \neq 0, 1, \infty$ and $(w_k^{(0)}) \neq 0, 1$, so we will concentrate on proving $(z_1^{(0)}, \ldots, z_n^{(0)}, (w_k^{(0)}), \ldots) \in S$.

Consider the positive crossing $j$ in Figure 20(a) and assume it is non-degenerate. Also consider the tetrahedra in Figures 9(a) and 10 and assign variables $z_e, z_f, z_g, z_h$ to sides of the link diagram as in Figure 20(a). Then, using (19) and (30), the shape parameters assigned to the horizontal edges $A_j B_j$ and $D_j A_j$ are

$$1 \neq [h(s * a_k), h(p), h(\pm a_l * a_k), h(a_k)] = \frac{\det(s, a_k)}{\det(s * a_k, \pm a_l * a_k)} \frac{\det(p, \pm a_l * a_k)}{\det(p, a_k)} = \frac{z_e^{(0)}}{z e^{(0)}},$$

$$1 \neq [h(s), h(p), h(a_k), h(a_l)] = \frac{\det(s, a_l)}{\det(s, a_k)} \frac{\det(p, a_k)}{\det(p, a_l)} = \frac{z_e^{(0)}}{z h^{(0)}};$$

respectively. Likewise, the shape parameters assigned to $B_j C_j$ and $C_j D_j$ are $\frac{z_e^{(0)}}{z f^{(0)}}$ and $\frac{z h^{(0)}}{z g^{(0)}}$ respectively. Furthermore, for any $a, b \in \mathbb{C} \setminus \{0\}$, we can easily show that $h(a * b - a) = h(b)$. If $\frac{z_e}{z e} = \frac{\det(a_k, s)}{\det(a_k, s * a_l)} = 1$, then $h(a_k) = h(s * a_l - s) = h(a_l)$, which is contradiction. Therefore, we obtain $\frac{z_e}{z e} \neq 1$, and $\frac{z h}{z f} \neq 1$ can be obtained similarly.

We can verify the same holds for non-degenerate negative crossing $j$ by the same way.

Now consider the case when the positive crossing $j$ in Figure 20(a) is degenerate. (See Figures 7(c) and 11(a).) Then, using (19) and (31), the shape parameters assigned to the
edges $F_jA_j$, $E_jB_j$, $F_jC_j$ and $E_jD_j$ in Figure 7(c) are

$$
[h(a_k), h(s), h(p), h(s * a_l)] [h(a_k), h(s * a_k), h((s * a_l) * a_k), h(p)] \\
= \frac{\det(s, p)}{\det(s * a_k, p)} = (w_{c_j}^l)^{(0)},
$$

$$
[h(\pm a_l * a_k), h(p), h((s * a_l) * a_k), h(s * a_k)] \\
= \frac{\det(p, (s * a_l) * a_k)}{\det(p, s * a_k)} = (w_{f_j}^l)^{(0)},
$$

$$
[h(a_k), h((s * a_l) * a_k), h(p), h(s * a_k)] [h(a_k), h(s * a_l), h(s), h(p)] \\
= \frac{\det((s * a_l) * a_k, p)}{\det(s * a_l, p)} = (w_{g_j}^l)^{(0)},
$$

respectively. We can verify the same holds for degenerate negative crossing $j$ by the same way.

Therefore $(z_1^{(0)}, \ldots, z_n^{(0)}, (w_{k_j}^j)^{(0)}, \ldots)$ satisfies the hyperbolicity equations of octahedral triangulation defined in Section 2.3 and, from Proposition 3.1, we obtain $(z_1^{(0)}, \ldots, z_n^{(0)}, (w_{k_j}^j)^{(0)}, \ldots)$ is a solution of $\mathcal{H}$. By the definition of $\mathcal{S}$, we obtain $(z_1^{(0)}, \ldots, z_n^{(0)}, (w_{k_j}^j)^{(0)}, \ldots) \in \mathcal{S}$.

We remark that, in the viewpoint of [3], Theorem 3.2 can be interpreted that a sufficient condition for $\mathcal{S} \neq \emptyset$ of the solution set $\mathcal{S}$ defined in [3] is that the parabolic elements corresponding to meridians of two arcs, in other words $a_l$ and $a_k$ in Figure 20 of any crossing have distinct fixed points.

To obtain the complex volume of $\rho$ from the potential function $V(z_1, \ldots, z_n, (w_{k_j}^j), \ldots)$, we modify it to

$$
V_0(z_1, \ldots, z_n, (w_{k_j}^j), \ldots) := V(z_1, \ldots, z_n, (w_{k_j}^j), \ldots) \\
- \sum_k \left( z_k \frac{\partial V}{\partial z_k} \right) \log z_k - \sum_{j: \text{degenerate}} \left( w_{k_j}^j \frac{\partial V}{\partial w_{k_j}^j} \right) \log w_{k_j}^j.
$$

This modification guarantees the invariance of the value under the choices of branches of $z_k^{(0)}$ and $(w_{k_j}^j)^{(0)}$. (See Lemma 2.1 of [3].) Note that $V_0(z_1^{(0)}, \ldots, z_n^{(0)}, (w_{k_j}^j)^{(0)}, \ldots)$ means evaluation of $V_0(z_1, \ldots, z_n, (w_{k_j}^j), \ldots)$ at $(z_1^{(0)}, \ldots, z_n^{(0)}, (w_{k_j}^j)^{(0)}, \ldots)$.

**Theorem 3.3.** Consider a hyperbolic link $L$, the shadow-coloring induced by $\rho$, the potential function $V(z_1, \ldots, z_n, (w_{k_j}^j), \ldots)$ and the solution $(z_1^{(0)}, \ldots, z_n^{(0)}, (w_{k_j}^j)^{(0)}, \ldots) \in \mathcal{S}$ defined in Theorem 3.2. Then,

$$
V_0(z_1^{(0)}, \ldots, z_n^{(0)}, (w_{k_j}^j)^{(0)}, \ldots) \equiv i(\text{vol}(\rho) + i \text{cs}(\rho)) \pmod{\pi^2}.
$$
Proof. When the crossing $j$ is degenerate, direct calculation shows that the potential function $V_j$ of the crossing in (26) satisfies

$$ (V_j)_0(z, z, z, z, w_1, w_2, w_3) = 0, \quad (34) $$

for any nonzero values of $z, w_1, w_2, w_3$. To simplify the potential function, we rearrange the side variables $z_1, \ldots, z_n$ to $z_1, \ldots, z_r, z_{r+1}, z_{r+2}, \ldots, z_t$ so that the endpoints of sides with variables $z_1, \ldots, z_r$ are non-degenerate crossings and the degenerate crossings induce

$$ z_{r+1}^{(0)} = (z_{r+1}^{(0)})^0 = (z_{r+1}^{(0)})^0, \ldots, z_t^{(0)} = \ldots = (z_t^{(0)})^0. $$

Then we define simplified potential function $\hat{V}$ by

$$ \hat{V}(z_1, \ldots, z_t) := \sum_{j: \text{non-degenerate}} V_j(z_1, \ldots, z_r, z_{r+1}, z_{r+2}, \ldots, z_t, z_t, z_t, z_t). $$

Note that $\hat{V}$ is obtained from $V$ by removing the potential functions (26) of the degenerate crossings and substituting the side variables $z_f, z_g, z_h$ with $z_e$. From (34), we have

$$ \hat{V}_0(z_1^{(0)}, \ldots, z_t^{(0)}) = V_0(z_1^{(0)}, \ldots, z_n^{(0)}, (w_j^{(0)})^0, \ldots), $$

which suggests $\hat{V}$ is just a simplification of $V$ with the same value. Therefore, from now on, we use $\hat{V}$ instead of $V$ and substitute the side variables $z_{r+1}, z_{r+2}, z_{r+3}$ to $z_{r+1}$ and $z_1, \ldots, z_t$ to $z_t$, etc. Also, we remove octahedra (13) of degenerate crossings because they do not have any effect on the complex volume. (See the comment in the proof of Theorem 2.10.)

Now we will follow ideas of the proof of Theorem 1.2 in [3]. However, due to the degenerate crossings, we will improve the proof to cover more general cases. At first, we define $r_k$ by

$$ r_k \pi i = \frac{\partial \hat{V}}{\partial z_k} \bigg|_{z_1 = z_1^{(0)}, \ldots, z_t = z_t^{(0)}}, \quad (35) $$

for $k = 1, \ldots, t$, where $|_{z_1 = z_1^{(0)}, \ldots, z_t = z_t^{(0)}}$ means the evaluation of the equation at $(z_1^{(0)}, \ldots, z_t^{(0)})$. Unlike [3], we cannot guarantee $r_k$ is an even integer yet, so we need the following lemma.

Lemma 3.4. For the value $z_k^{(0)}$ defined in Theorem 3.2, $(z_1^{(0)}, \ldots, z_t^{(0)})$ is a solution of the following set of equations

$$ \hat{H} = \left\{ \exp(z_k \frac{\partial \hat{V}}{\partial z_k}) = 1 \bigg| k = 1, \ldots, t \right\}. $$

Proof. Note that, for the degenerate crossing $j$ in Figure 15, we have

$$ \frac{(w_j^{(0)})^0}{(w_j^{(0)})^0} = 1. $$

Also, for $r+1 \leq k \leq t$, (31) implies

$$ (z_k^{(0)}) = (z_k^{(0)}) = \ldots = (z_k^{(0)}). $$
Therefore, assuming $\frac{w_j w_j}{w_d w_d} = 1$ for any degenerate crossing $j$, we obtain
\[
\exp(z_k \frac{\partial \hat{V}}{\partial z_k}) = \exp(z_k \frac{\partial V}{\partial z_k}) \left| \begin{array}{c}
\ldots \exp(z_k^3 \frac{\partial V}{\partial z_k}) \left| z_k = z_k
\end{array} \right.,
\]
for $r + 1 \leq k \leq t$. Therefore, Theorem 3.2 induces the statement of this lemma.

To avoid redundant complicate indices, we use $z_k$ instead of $z_k^{(0)}$ in this proof from now on. Using the even integer $r_k$, we can denote $V_0(z_1, \ldots, z_t)$ by
\[
\hat{V}_0(z_1, \ldots, z_t) = \hat{V}(z_1, \ldots, z_t) - \sum_{k=1}^{t} r_k \pi i \log z_k.
\]

Now we use variables $\alpha_m, \beta_m, \gamma_l, \delta_j$ for the long-edge parameters in (18). We assign $\alpha_m$ and $\beta_m$ to non-horizontal edges as in Figure 21, where $m = a, b, c, d$. We also assign $\gamma_l$ to horizontal edges, where $l$ is over all regions, and $\delta_j$ to the edge $E_j F_j$ inside the octahedron. Although we have $\alpha_a = \alpha_c$ and $\beta_b = \beta_d$, we use $\alpha_a$ for the tetrahedron $E_j F_j A_j B_j$ and $E_j F_j A_j D_j$, $\alpha_c$ for $E_j F_j C_j B_j$ and $E_j F_j C_j D_j$, $\beta_b$ for $E_j F_j A_j B_j$ and $E_j F_j C_j B_j$, $\beta_d$ for $E_j F_j C_j D_j$ and $E_j F_j A_j D_j$, respectively. Note that the labeling is consistent even when some crossing is degenerate because, when the crossing $j$ in Figure 21 is degenerate, we obtain $z_a = z_b = z_c = z_d$ and, after removing the octahedron of the crossing, the long-edge parameters satisfy $\alpha_a = \alpha_b = \alpha_c = \alpha_d$ and $\beta_a = \beta_b = \beta_c = \beta_d$.

\[\text{Figure 21: Long-edge parameters of non-horizontal edges}\]
Now consider a side with variable $z_k$ and two possible cases in Figure 22. We consider the case when the crossing is non-degenerate, or equivalently, $z_a \neq z_k \neq z_b$. (If it is degenerate, we assume there is no octahedron at the crossing.) For $m = a, b$, let $\sigma_k^m \in \{\pm 1\}$ be the sign of the tetrahedron assigned to the horizontal edge. We put $\tau_k^m = 1$ when $z_k$ is the numerator of $(u_k^m)\sigma_k^m$ and $\tau_k^m = -1$ otherwise. We also define $p_k^m$ and $q_k^m$ by (23) so that $\sigma_k^m (u_k^m)^{\sigma_k^m}; p_k^m, q_k^m$ becomes the element of $\hat{P}(\mathbb{C})$ corresponding to the tetrahedron. Then $\frac{1}{2} \sum_{k,m} \sigma_k^m (u_k^m)^{\sigma_k^m}; p_k^m, q_k^m$ is the element of $\hat{B}(\mathbb{C})$ corresponding to the octahedral triangulation in Section 2.3, and

$$\frac{1}{2} \sum_{k,m} \sigma_k^m \hat{L}(u_k^m)^{\sigma_k^m}; p_k^m, q_k^m \equiv i(\text{vol}(\rho) + i\text{cs}(\rho)) \pmod{\pi^2},$$

from Theorem 2.10.

By definition, we know

$$u_k^a = \frac{z_k}{z_a}, \quad u_k^b = \frac{z_k}{z_b}. \quad (38)$$

In the case of Figure 22(a), we have

$$\sigma_k^a = 1, \quad \sigma_k^b = -1 \quad \text{and} \quad \tau_k^a = \tau_k^b = 1.$$  

Using the equation (23) and Figure 23(a), we decide $p_k^m$ and $q_k^m$ as follows:

$$\begin{align*}
\left\{ \begin{array}{l}
\log \frac{z_k}{z_a} + p_k^a \pi i = \log \alpha_k - \log \beta_k - \log \alpha_a - \log \beta_a, \\
\log \frac{z_k}{z_b} + p_k^b \pi i = \log \alpha_k - \log \beta_k - \log \alpha_b - \log \beta_b,
\end{array} \right. \quad (39)
\end{align*}$$

$$\begin{align*}
\left\{ \begin{array}{l}
- \log (1 - \frac{z_k}{z_a}) + q_k^a \pi i = \log \beta_k - \log \alpha_k - \log \gamma_1 - \log \delta_1, \\
- \log (1 - \frac{z_k}{z_b}) + q_k^b \pi i = \log \beta_k - \log \alpha_b - \log \gamma_2 - \log \delta_1.
\end{array} \right. \quad (40)
\end{align*}$$

In the case of Figure 22(b), we have

$$\sigma_k^a = -1, \quad \sigma_k^b = 1 \quad \text{and} \quad \tau_k^a = \tau_k^b = -1.$$  

7 Sign of a tetrahedron is the sign of the coordinate in (11) or (12).
8 The coefficient $\frac{1}{2}$ appears because the same tetrahedron is counted twice in the summation.
Figure 23: Tetrahedra of Figure 22

Using the equation (23) and Figure 23(b), we decide $p^m_k$ and $q^m_k$ as follows:

\[
\begin{align*}
\log \frac{z^a}{\alpha^a} + p^a_k \pi i &= (\log \alpha - \log \beta_a) - (\log \alpha_k - \log \beta_k), \\
\log \frac{z^b}{\alpha^b} + p^b_k \pi i &= (\log \alpha - \log \beta_b) - (\log \alpha_k - \log \beta_k), \\
\log \frac{1 - \frac{z^a}{\alpha^a}}{\epsilon} + q^a_k \pi i &= \log \beta_a + \log \alpha_k - \log \gamma_1 - \log \delta_1, \\
\log \frac{1 - \frac{z^b}{\alpha^b}}{\epsilon} + q^b_k \pi i &= \log \beta_b + \log \alpha_k - \log \gamma_2 - \log \delta_1.
\end{align*}
\] (41)

The equations (39) and (41) holds for all possible non-degenerate crossings, so we get the following observation.

**Observation 3.5.** We have

\[
\log \alpha_k - \log \beta_k \equiv \log z_k + A \pmod{\pi i},
\]

for all $k = 1, \ldots, t$, where $A$ is a complex constant number independent of $k$.

Recall that the potential function $\hat{V}$ is expressed by

\[
\hat{V}(z_1, \ldots, z_t) = \frac{1}{2} \sum_{k,m} \sigma^m_k \text{Li}_2((u_k^m)^{\sigma^m_k}),
\] (43)

where $k$ is over all sides and the range of $m = a, b, \ldots$ is determined by $k$. From now on, we put the range of $m$ by $m = a, \ldots, d$. Recall that $r_k$ was defined in (35). Direct calculation

\[9\] This does not mean $m = a, b, c, d$ in general. If $z_k$ is connected to a degenerate crossing, then $m$ can have bigger range.
\[
    r_k \pi i = - \sum_{m=a,\ldots,d} \sigma_k^m \tau_k^m \log(1 - (u_k^m)^{\sigma_k^m}).
\]

Combining (40) and (42), we obtain
\[
    \sum_{m=a,b} \sigma_k^m \tau_k^m \{ - \log(1 - (u_k^m)^{\sigma_k^m}) + q_k^m \pi i \} = - \log \gamma_1 + \log \gamma_2,
\]
for both cases in Figure 22 (Note that \(\alpha_a = \alpha_b\) in (40) and \(\beta_a = \beta_b\) in (42).) Therefore, we obtain
\[
    \sum_{m=a,\ldots,d} \sigma_k^m \tau_k^m \{ - \log(1 - (u_k^m)^{\sigma_k^m}) + q_k^m \pi i \} = 0,
\]
and
\[
    r_k \pi i = - \sum_{m=a,\ldots,d} \sigma_k^m \tau_k^m q_k^m \pi i. \tag{44}
\]

Lemma 3.6. For all possible \(k\) and \(m\), we have
\[
    \frac{1}{2} \sum_{k,m} \sigma_k^m q_k^m \pi i \log(u_k^m)^{\sigma_k^m} \equiv - \sum_{k=1}^t r_k \pi i \log z_k \pmod{2\pi^2}. \tag{45}
\]

Proof. Note that, by definition, \(\sigma_k^m = \sigma_m^k\), \(\tau_k^m = -\tau_m^k\) and
\[
    (u_k^m)^{\sigma_k^m} = \left(\frac{z_k}{z_m}\right)^{\tau_k^m} = (z_k)^{\tau_k^m} (z_m)^{\tau_m^k}.
\]
Using the above and (44), we can directly calculate
\[
    \frac{1}{2} \sum_{k=1}^t \sum_{m=a,\ldots,d} \sigma_k^m q_k^m \pi i \log(u_k^m)^{\sigma_k^m} \equiv \sum_{k=1}^t \left( \sum_{m=a,\ldots,d} \sigma_k^m \tau_k^m q_k^m \pi i \right) \log z_k \pmod{2\pi^2}
    = - \sum_{k=1}^t r_k \pi i \log z_k.
\]

Lemma 3.7. For all possible \(k\) and \(m\), we have
\[
    \frac{1}{2} \sum_{k,m} \sigma_k^m \log(1 - (u_k^m)^{\sigma_k^m}) \left(\log(u_k^m)^{\sigma_k^m} + p_k^m \pi i\right) \equiv - \sum_{k=1}^t r_k \pi i \log z_i \pmod{2\pi^2}.
\]

Proof. From (39) and (41), we have
\[
    \log(u_k^m)^{\sigma_k^m} + p_k^m \pi i = \tau_k^m (\log \alpha_k - \log \beta_k) + \tau_m^k (\log \alpha_m - \log \beta_m).
\]
Therefore,
\[
\frac{1}{2} \sum_{k,m} \sigma_k^m \log \left(1 - \left(u_k^m\right)^\sigma_k^m\right) \left(\log (u_k^m)^\sigma_k^m + p_k^m \pi i\right)
\]
\[= \sum_{k=1}^t \left( \sum_{m=a,\ldots,d} \sigma_k^m r_k^m \log \left(1 - \left(u_k^m\right)^\sigma_k^m\right) \left(\log \alpha_k - \log \beta_k\right) \right)
\]
\[= - \sum_{k=1}^t r_k \pi i (\log \alpha_k - \log \beta_k).
\]

Note that
\[
\sum_{k=1}^t r_k \pi i = \sum_{k=1}^t z_k \frac{\partial \hat{V}}{\partial z_k} = 0
\]
because \(\hat{V}\) is expressed by the summation of certain forms of \(\text{Li}_2\left(\frac{z_a}{z_b}\right)\) and
\[
\frac{z_a}{z_a} \frac{\partial \text{Li}_2(z_a/z_b)}{\partial z_a} + z_b \frac{\partial \text{Li}_2(z_a/z_b)}{\partial z_b} = - \log \left(1 - \frac{z_a}{z_b}\right) + \log \left(1 - \frac{z_a}{z_b}\right) = 0.
\]

By using Observation 3.5, the above and the fact that \(r_k\) is even, we have
\[
- \sum_{k=1}^t r_k \pi i (\log \alpha_k - \log \beta_k) \equiv - \sum_{k=1}^t r_k \pi i (\log z_k + A) = - \sum_{k=1}^t r_k \pi i \log z_k \pmod{2\pi^2}.
\]

Combining (37), (43), Lemma 3.6 and Lemma 3.7, we complete the proof of Theorem 3.3 as follows:
\[
i(\text{vol}(\rho) + i \text{cs}(\rho)) \equiv \frac{1}{2} \sum_{k,m} \sigma_k^m \text{Li}_2\left((u_k^m)^\sigma_k^m\right) \left(p_k^m \cdot q_k^m\right)
\]
\[= \frac{1}{2} \sum_{k,m} \sigma_k^m \left(\text{Li}_2\left((u_k^m)^\sigma_k^m\right) - \frac{\pi^2}{6}\right) + \frac{1}{4} \sum_{k,m} \sigma_k^m q_k^m \pi i \log (u_k^m)^\sigma_k^m
\]
\[+ \frac{1}{4} \sum_{k,m} \sigma_k^m \log \left(1 - \left(u_k^m\right)^\sigma_k^m\right) \left(\log (u_k^m)^\sigma_k^m + p_k^m \pi i\right)
\]
\[\equiv \hat{V}(z_1, \ldots, z_n) - \sum_{k=1}^t r_k \pi i \log z_k = \hat{V}_0(z_1, \ldots, z_n) \pmod{\pi^2}.
\]
4 Examples

4.1 Figure-eight knot 4

For the figure-eight knot diagram in Figure 24 let the elements of $\mathcal{P}$ corresponding to the arcs be

$$a_1 = \begin{pmatrix} 0 \\ t \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} -t \\ 1 + t \end{pmatrix}, \quad a_4 = \begin{pmatrix} -t \\ t \end{pmatrix},$$

where $t$ is a solution of $t^2 + t + 1 = 0$. These elements satisfy

$$a_1 \ast a_2 = a_4, \quad a_3 \ast a_4 = a_2, \quad a_1 \ast a_3 = -a_2, \quad a_3 \ast a_1 = a_4,$$

(46)

where the identities are expressed in $\mathbb{C}^2 \setminus \{0\}$, not in $\mathcal{P} = (\mathbb{C}^2 \setminus \{0\})/\pm$. Let $\rho : \pi_1(4_1) \to \text{PSL}(2, \mathbb{C})$ be the boundary-parabolic representation determined by $a_1, \ldots, a_4$. We define the shadow-coloring of Figure 24 induced by $\rho$ by letting

$$s_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad s_3 = \begin{pmatrix} -t - 1 \\ t + 2 \end{pmatrix}, \quad s_4 = \begin{pmatrix} -2t - 1 \\ 2t + 3 \end{pmatrix},$$

$$s_5 = \begin{pmatrix} -2t - 1 \\ t + 4 \end{pmatrix}, \quad s_6 = \begin{pmatrix} 1 \\ t + 2 \end{pmatrix}, \quad p = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Direct calculation shows this shadow-coloring satisfies (4) in Lemma 2.4 (However, this does not satisfy (5).)

All values of $h(a_1), \ldots, h(a_4)$ are different, hence the potential function $V(z_1, \ldots, z_8)$ of
Figure 24 is (27). Applying Theorem 3.2 we obtain

\[
\begin{align*}
    z_1^{(0)} &= \frac{\det(a_1, p)}{\det(a_1, s_6)} = 2, \\
    z_2^{(0)} &= \frac{\det(a_1, p)}{\det(a_1, s_5)} = \frac{-2}{2t + 1}, \\
    z_3^{(0)} &= \frac{\det(a_2, p)}{\det(a_2, s_6)} = \frac{1}{t + 2}, \\
    z_4^{(0)} &= \frac{\det(a_2, p)}{\det(a_2, s_1)} = 1, \\
    z_5^{(0)} &= \frac{\det(a_3, p)}{\det(a_3, s_4)} = -3t - 2, \\
    z_6^{(0)} &= \frac{\det(a_3, p)}{\det(a_3, s_5)} = \frac{3t + 2}{2t}, \\
    z_7^{(0)} &= \frac{\det(a_4, p)}{\det(a_4, s_4)} = \frac{1}{2}, \\
    z_8^{(0)} &= \frac{\det(a_4, p)}{\det(a_4, s_3)} = 3,
\end{align*}
\]

and \((z_1^{(0)}, \ldots, z_8^{(0)})\) becomes a solution of \(H = \{\exp(z_k \frac{\partial V}{\partial z_k}) = 1 \mid k = 1, \ldots, 8\}\). Applying Theorem 3.3 we obtain

\[
V_0(z_1^{(0)}, \ldots, z_8^{(0)}) \equiv i(\text{vol}(\rho) + i \text{cs}(\rho)) \pmod{\pi^2},
\]

and numerical calculation verifies it by

\[
V_0(z_1^{(0)}, \ldots, z_8^{(0)}) = \begin{cases} 
  i(2.0299\ldots + 0i) = i(\text{vol}(4_1) + i \text{cs}(4_1)) & \text{if } t = -\frac{1 - \sqrt{3}i}{2}, \\
  i(-2.0299\ldots + 0i) = i(-\text{vol}(4_1) + i \text{cs}(4_1)) & \text{if } t = \frac{-1 + \sqrt{3}i}{2}.
\end{cases}
\]

### 4.2 Trefoil knot \(3_1\)

Figure 25: Trefoil knot \(3_1\) with parameters

For the trefoil knot diagram in Figure 25 let the elements of \(P\) corresponding to the arcs be

\[
a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad a_3 = a_4 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\]

(Note that crossing 4 is degenerate.) These elements satisfy

\[
a_4 \ast a_2 = -a_1, \quad a_2 \ast a_1 = a_3, \quad a_1 \ast a_4 = a_2, \quad a_4 \ast a_3 = a_3.
\]
where the identities are expressed in $\mathbb{C}^2 \setminus \{0\}$, not in $\mathcal{P} = (\mathbb{C}^2 \setminus \{0\})/\pm$. Let $\rho : \pi_1(3_1) \to \text{PSL}(2, \mathbb{C})$ be the boundary-parabolic representation determined by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$. We define the shadow-coloring of Figure 24 induced by $a_1, a_2, a_3, a_4$.

Direct calculation shows this shadow-coloring satisfies (4) in Lemma 2.4. (However, this does not satisfy (5).)

All values of $h(a_1), h(a_2), h(a_3) = h(a_4)$ are different, hence the potential function $V$ of Figure 25 is

$$V(z_1, \ldots, z_8, w_6^4, w_7^4) = \text{Li}_2\left(\frac{z_2}{z_5}\right) - \text{Li}_2\left(\frac{z_2}{z_4}\right) + \text{Li}_2\left(\frac{z_1}{z_4}\right) - \text{Li}_2\left(\frac{z_1}{z_5}\right) + \text{Li}_2\left(\frac{z_6}{z_3}\right) - \text{Li}_2\left(\frac{z_6}{z_2}\right) + \text{Li}_2\left(\frac{z_5}{z_3}\right) - \text{Li}_2\left(\frac{z_5}{z_2}\right) + \text{Li}_2\left(\frac{z_4}{z_6}\right) - \text{Li}_2\left(\frac{z_4}{z_2}\right) + \text{Li}_2\left(\frac{z_3}{z_6}\right) - \text{Li}_2\left(\frac{z_3}{z_2}\right),$$

and the simplified potential function $\hat{V}$ defined in the proof of Theorem 3.3 is

$$\hat{V}(z_1, \ldots, z_6) = \text{Li}_2\left(\frac{z_2}{z_5}\right) - \text{Li}_2\left(\frac{z_2}{z_4}\right) + \text{Li}_2\left(\frac{z_1}{z_4}\right) - \text{Li}_2\left(\frac{z_1}{z_5}\right) + \text{Li}_2\left(\frac{z_6}{z_3}\right) - \text{Li}_2\left(\frac{z_6}{z_2}\right) + \text{Li}_2\left(\frac{z_5}{z_3}\right) - \text{Li}_2\left(\frac{z_5}{z_2}\right) + \text{Li}_2\left(\frac{z_4}{z_6}\right) - \text{Li}_2\left(\frac{z_4}{z_2}\right) + \text{Li}_2\left(\frac{z_3}{z_6}\right) - \text{Li}_2\left(\frac{z_3}{z_2}\right).$$

Applying Theorem 3.2 we obtain

$$z_1^{(0)} = \frac{\det(a_4, p)}{\det(a_4, s_5)} = \frac{3}{2}, \quad z_2^{(0)} = \frac{\det(a_1, p)}{\det(a_1, s_2)} = \frac{1}{2}, \quad z_3^{(0)} = \frac{\det(a_1, p)}{\det(a_1, s_5)} = 1,$$

$$z_4^{(0)} = \frac{\det(a_2, p)}{\det(a_2, s_3)} = -2, \quad z_6^{(0)} = \frac{\det(a_3, p)}{\det(a_3, s_4)} = 2,$$

$$z_5^{(0)} = \frac{\det(s_1, p)}{\det(s_4, p)} = \frac{5}{2}, \quad (w_6^4)^{(0)} = \frac{\det(s_1, p)}{\det(s_6, p)} = \frac{5}{8}.$$
respectively. Applying Theorem 3.3, we obtain
\[ V_0(z_1^{(0)}, \ldots, z_6^{(0)}) \equiv \tilde{V}_0(z_1^{(0)}, \ldots, z_6^{(0)}) \equiv i(\text{vol}(\rho) + i \text{cs}(\rho)) \pmod{\pi^2}, \]
and numerical calculation verifies it by
\[ \tilde{V}_0(z_1^{(0)}, \ldots, z_6^{(0)}) = i(0 + 1.6449\ldots i), \]
where \( \text{vol}(3_1) = 0 \) holds trivially and \( 1.6449\ldots = \frac{\pi^2}{6} \) holds numerically.

Acknowledgments The author appreciates Yuichi Kabaya and Jun Murakami for suggesting this research and many discussions. Ayumu Inoue gave wonderful lectures on his work [7], which become the framework of Section 2 of this article. Many people including Hyuk Kim, Seonhwa Kim, Roland van der Veen, Hitoshi Murakami, Satoshi Nawata, Stephané Baseilhac, and KIAS topology members, gave many suggestions on my incomplete work, which helped me a lot to complete this article.

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