On gauge invariant regularization of fermion currents. *

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Abstract

We compare Schwinger and complex powers methods to construct regularized fermion currents. We show that although both of them are gauge invariant they not always yield the same result.

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A difficulty specific to quantum field theories is the occurrence of infinities and hence the necessity of regularizing and renormalizing the theory. Whenever a field theory possesses a classical symmetry—and hence a conserved current—it is desirable to have at hand regularization procedures preserving that symmetry.\textsuperscript{1}

The calculation of vacuum expectation values of vector currents involves the evaluation of the Green function for the particle fields at the diagonal, so a regularization is required. In a classical paper J. Schwinger introduced a point splitting method to regularize fermion currents maintaining gauge symmetry on the quantum level \[1\].

More recently, the so called $\zeta$-function method, based on complex powers of pseudodifferential operators \[2\], has proved to be a very valuable gauge invariant regularizing tool (see for example \[3\]). Some time ago, we used it to get fermion currents in 2 and 3 dimensional models \[4\].

It is the aim of this work to compare the results obtained by the above mentioned methods.

Let $\mathcal{D} = i \partial + A$ be an Euclidean Dirac operator coupled with a gauge field $A$ defined on an $n$-dimensional compact boundaryless manifold $M$. The operator $\mathcal{D}$ is elliptic and, since its principal symbol has only real eigenvalues, it fulfills the Agmon cone condition \[2\]. Thus, the complex powers $\mathcal{D}^s$ can be constructed following Seeley \[2\]. For $\text{Re} \ s < 0$ we can write

$$\mathcal{D}^s := \frac{i}{2\pi} \int_{\Gamma} \lambda^s (\mathcal{D} - \lambda)^{-1} d\lambda,$$

where $\Gamma$ is a contour enclosing the spectrum of $\mathcal{D}$, and we define $\mathcal{D}^s$ for $\text{Re} \ s \geq 0$ by using $\mathcal{D}^{s+1} = \mathcal{D}^s \circ \mathcal{D}$.

For each $s \in \mathbb{C}$, $\mathcal{D}^s$ turns out to be a pseudodifferential operator of order $s$ and so, if $\text{Re} \ s < -n$, its Schwartz kernel $K_s(x, y)$ is a continuous function. The evaluation at the diagonal $x = y$ of this kernel, $K_s(x, x)$, admits a meromorphic extension to the whole complex $s$-plane $\mathbb{C}$, with at most simple poles at $s \in \mathbb{Z}^-$. This extension will be also denoted by $K_s(x, x)$.

Since $K_{-1}(x, y)$ coincides with the Green function for $x \neq y$, the finite part of $K_s(x, x)$ at $s = -1$ can be used to obtain gauge invariant regularized

\textsuperscript{1}As it is well known, not always it is possible to preserve all the classical symmetries present simultaneously and anomalies can arise.
fermion currents \[4\]:

\[ J_\mu(x) = -\text{tr} \gamma_\mu \left( \text{FP} K_s(x,x) \right) \]  \hfill (2)

In order to compare this regularizing procedure with Schwinger’s one, it is convenient to consider the kernels \( K_s(x,x) \) in the framework developed in \[5\]. Since we are interested in studying the behaviour of these kernels for \( s \to -1 \), we shall carry out our analysis just for \(-1 \leq \text{Re} \, s < 0\).

By considering the finite expansion (see for instance \[6\])

\[ \sigma(\mathcal{D}^s) = \sum_{\ell=0}^{N} c_{s-\ell}(x,\xi) + r_N(x,\xi,s), \]  \hfill (3)

with \( N = n - 1 \), of the symbol of the operator \( \mathcal{D}^s \), with \( c_{s-\ell}(x,\xi) \) positively homogeneous of degree \( s-\ell \) for \( |\xi| \geq 1 \), we can write, for \( s \neq -1 \) the Schwartz kernel of this operator as

\[ K_s(x,y) = \sum_{\ell=0}^{N} H_{-n-s+\ell}(x,u) + R_N(x,u,s), \]  \hfill (4)

where \( H_{-n-s+\ell}(x,u) \) is the Fourier transform in the variable \( \xi \) of \( \tilde{c}_{s-\ell}(x,\xi) \), the homogeneous extension of \( c_{s-\ell}(x,\xi) \), evaluated at \( u = x - y \), and consequently \( u \)-homogeneous of degree \(-n - s + \ell \) and \( R_N(x,u,s) \) is that of \( r_N(x,\xi,s) - \sum_{\ell=0}^{N} (\tilde{c}_{s-\ell} - c_{s-\ell})(x,\xi) \). Note that \( (\tilde{c}_{s-\ell} - c_{s-\ell})(x,\xi) \equiv 0 \) for \( |\xi| \geq 1 \).

Now, for \( u \neq 0 \), simple poles can arise at \( s = -1 \) in \( H_{-n-s+N} \) and in \( R_N(x,u,s) \). Since \( K_s(x,x-u) \) is holomorphic in the variable \( s \) for \( u \neq 0 \), these poles cancel each other. In fact, they are just due to the singularity of \( \tilde{c}_{s-N}(x,\xi) \) at \( \xi = 0 \) and then

\[ \text{res}_{s=-1} R_N(x,u,s) = -\text{res}_{s=-1} H_{-n-s+N}(x,u). \]  \hfill (5)

Thus, for \( u \neq 0 \), we have for \( G(x,y) \), the Green function of \( \mathcal{D} \),

\[ G(x,y) = \lim_{s \to -1} K_s(x,y) = \sum_{\ell=0}^{N} G_{-n+1+\ell}(x,u) + R_G(x,u), \]  \hfill (6)
with \( G_{-n+1+\ell} (x, u) = \lim_{s \to -1} H_{-n-s+\ell} (x, u) \) for \( \ell < N \), \( G_{-n+1+N} (x, u) = \lim_{s \to -1} H_{-n-s+N} (x, u) \) and \( R_G(x, u) = \lim_{s \to -1} R_N(x, u, s) \).

Then, taking into account that, for \( s \neq -1 \), (see, for instance [5])

\[
K_s(x, x) = R_N(x, 0, s),
\]

we have

\[
\text{FP} K_s(x, x) = R_G(x, 0),
\]

On the other hand, the fermionic currents regularized according to Schwinger’s prescription are given by [1]

\[
J_\mu(x) = -\text{Sch-lim}_{y \to x} \left( \gamma_\mu G(x, y) e^{i \int_x^y A.dz} \right),
\]

where

\[
\int_x^y A.dz = - \int_0^1 A_\mu(x - tu) u_\mu dt.
\]

and Sch-lim (Schwinger limit) is the usual limit when it exists, it vanishes for \( u \)-homogeneous functions of negative degree and for logarithmic ones, and it coincides with the mean value at \(|u| = 1\) for \( u \)-homogeneous functions of zero degree. The exponential factor was introduced by Schwinger [1] in order to maintain gauge invariance.

From (2), (8) and (9) we see that both methods yield the same result for \( J_\mu \) if and only if

\[
\text{Sch-lim}_{y \to x} \left( \gamma_\mu \sum_{\ell=0}^N G_{-n+1+\ell} (x, u) e^{i \int_x^y A.dz} \right) = 0
\]

since, being \( R_G(x, u) \) continuous at \( x = y \),

\[
\text{Sch-lim}_{y \to x} \left( \gamma_\mu R_G(x, u) e^{i \int_x^y A.dz} \right) = \text{Sch-lim}_{y \to x} (\gamma_\mu R_G(x, u))
\]

\[
= \lim_{u \to 0} (\gamma_\mu R_G(x, u)) = \text{tr} \left( \gamma_\mu \text{FP} K_s(x, x) \right).
\]
Now, we shall see how this works in $n = 2, 3$ and $4$. By computing the $G_{-n+1+\ell}(x, u)$'s we shall be able to establish when (11) holds and so, when both methods yield the same regularized currents.

In a local coordinate chart

$$\mathcal{D} = \gamma_\mu D_\mu = \gamma_\mu (i\partial_\mu + A_\mu),$$

where the algebra of the $\gamma$-matrices is

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = \delta_{\mu\nu}. \quad (14)$$

Its symbol, $\sigma(\mathcal{D}; x, \xi)$, is

$$\sigma(\mathcal{D}; x, \xi) = -\xi - A(x). \quad (15)$$

The symbol of the resolvent, $\sigma((\mathcal{D} - \lambda)^{-1}; x, \xi)$, has an asymptotic expansion $\sum_\ell \tilde{C}_{-1-\ell}(x, \xi, \lambda)$, where $\tilde{C}_{-1-\ell}(x, \xi, \lambda)$ is homogeneous in $\xi$ and $\lambda$ of degree $-1 - \ell$. Then

$$(\mathcal{D} - \lambda)^{-1} \varphi(x) \sim \frac{1}{(2\pi)^{n/2}} \int \sum_\ell \tilde{C}_{-1-\ell}(x, \xi, \lambda) e^{i\xi x} \hat{\varphi}(\xi) \, d\lambda \, d\xi, \quad (16)$$

Applying $\mathcal{D} - \lambda$ to Equation (3) we get recursive equations for determining the $\tilde{C}_{-1-\ell}(x, \xi, \lambda)$'s:

$$-(\xi + \lambda) \tilde{C}_{-1}(x, \xi, \lambda) = 1,$$

$$\mathcal{D}_x \tilde{C}_{-1-\ell}(x, \xi, \lambda) - (\xi + \lambda) \tilde{C}_{-1-\ell-1}(x, \xi, \lambda) = 0. \quad (17)$$

Consequently,

$$\tilde{C}_{-1-\ell}(x, \xi, \lambda) = -\frac{(\xi - \lambda)}{\xi^2 - \lambda^2} \left[ \mathcal{D}_x \frac{(\xi - \lambda)}{\xi^2 - \lambda^2} \right]^\ell. \quad (18)$$

Now, from equation (14),

$$H_{-n-s+\ell}(x, u) = \frac{1}{(2\pi)^n} \int \tilde{G}_{s-\ell}(x, \xi) e^{ix u} \, d\lambda \, d\xi$$

$$= \frac{i}{(2\pi)^{n+1}} \int \int_{\Gamma} \tilde{C}_{-1-\ell}(x, \xi, \lambda) \lambda^s e^{i\xi u} \, d\lambda \, d\xi, \quad (19)$$
where the contour $\Gamma$ can be chosen as shown in Figure 1. Therefore,

$$H_{-n-s+\ell}(x, u)$$

$$\begin{align*}
&= \frac{-i}{(2\pi)^{n+1}} \int \int_{\Gamma} \frac{(g - \lambda)}{(\xi^2 - \lambda^2)^{\ell+1}} \left[ \mathcal{D}_x (g - \lambda) \right]^\ell \lambda^s e^{i\xi \cdot u} \, d\lambda \, d\xi \\
&= \frac{-i}{(2\pi)^{n+1}} \int \int_{\Gamma} \frac{-i \, \phi_n - \lambda}{(\xi^2 - \lambda^2)^{\ell+1}} \left[ \mathcal{D}_x (-i \, \phi_n - \lambda) \right]^\ell \lambda^s e^{i\xi \cdot u} \, d\lambda \, d\xi.
\end{align*}$$

(20)

Taking into account that, for any polynomial $P(\lambda)$,

$$\frac{i}{2\pi} \int_{\Gamma} \frac{\lambda^s P(\lambda)}{(\xi^2 - \lambda^2)^{\ell+1}} \, d\lambda$$

$$\begin{align*}
&= \frac{i}{2\pi} \left\{ \int_{\infty}^{0} \frac{(z \, e^{i\frac{\pi}{2}})^s \, P(iz)}{\xi^2 + z^2)^{\ell+1}} \, i \, dz + \int_{0}^{\infty} \frac{(z \, e^{-i\frac{3\pi}{2}})^s \, P(iz)}{(\xi^2 + z^2)^{\ell+1}} \, i \, dz \right\} \\
&= \frac{i}{\pi} \, e^{-iz} \sin(\pi s) \, P(-\partial_\alpha) \left[ \int_{0}^{\infty} \frac{z^s \, e^{-iaz}}{\xi^2 + z^2)^{\ell+1}} \, dz \right]_{a=0},
\end{align*}$$

(21)
we can write
\[
H_{-n-s+\ell}(x, u) = \left( \frac{i}{\pi} \right) e^{-i\Phi s} \sin(\pi s)(-i \, \partial_u + \partial_a) \left[ \mathcal{D}_x (\Phi u + \partial_a) \right]^\ell \\
\times \sum_{k=0}^{\ell+1} \frac{(-ia)^k}{k!} \int_0^\infty z^{s+k} \frac{1}{(2\pi)^n} \int_0^1 \frac{1}{(\zeta^2 + z^2)^{\ell+1}} e^{iz \xi u} \, d\xi \, dz \bigg|_{a=0}.
\]

(22)

Now, the integrals in (22) can be performed using the known identities
\[
\frac{1}{(2\pi)^n} \int (\zeta^2 + z^2)^s e^{iz \xi u} \, d\xi = \frac{2^{1+s}}{(2\pi)^{s+1}} \frac{1}{\Gamma(-s)} \left( \frac{z}{u} \right)^{s+1} K_{s+1}(zu)
\]

(23)

where \( K_{s+1} \) is a Bessel function (see for instance [8]), and
\[
\int_0^\infty z^\mu K_{s+1}(zu) \, dz = 2^{\mu-1} u^{-\mu-1} \Gamma \left( \frac{1+s+\mu}{2} \right) \Gamma \left( \frac{1+s-\mu}{2} \right),
\]

(24)

(see for example [7]).

Finally, we thus get the following expression for \( H_{-n-s+\ell}(x, u) \):
\[
H_{-n-s+\ell}(x, u) = \left( \frac{i}{\pi} \right) e^{-i\Phi s} \sin(\pi s) \sum_{k=0}^{\ell+1} \frac{(-ia)^k}{k!} \int_0^\infty z^{s+k} \frac{1}{(2\pi)^{s+1}} \frac{1}{\Gamma(-s)} \left( \frac{z}{u} \right)^{s+1} K_{s+1}(zu) \bigg|_{a=0}.
\]

(25)

The first four terms \( H_{-n-s+\ell}(x, u) \)'s, obtained from (25) after a straightforward but tedious computation just involving \( \gamma \)-matrices's algebra and derivatives, are shown in Table I. There, as usual, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -i(D_\mu A_\nu - D_\nu A_\mu) \). It is worth noticing that the first terms of the exponential
\[
e^{-i \int_y^x A \cdot dz} = 1 + i(u.A) - \frac{(u.D)(u.A)}{2!} - \frac{i(u.D)(u.D)(u.A)}{3!} + \ldots
\]

(26)
Table 1: The first four $H_{n-s+i}(x,u)$'s.

$$H_{n-s}(x,u) = \frac{2^{s-1}}{\pi^{\frac{s}{2}+1}} e^{-i\frac{\pi}{2} s} \sin(\pi s)$$

$$\times \left[ \Gamma \left( \frac{1+s}{2} \right) \Gamma \left( \frac{n+s+1}{2} \right) u^{-n-s-1} \mu - \Gamma \left( \frac{2+s}{2} \right) \Gamma \left( \frac{n+s}{2} \right) u^{-n-s} \right]$$

$$H_{n-s+1}(x,u) = \frac{2^{s-1}}{\pi^{\frac{s}{2}+1}} e^{-i\frac{\pi}{2} s} \sin(\pi s)$$

$$\times \left[ \Gamma \left( \frac{1+s}{2} \right) \Gamma \left( \frac{n+s+1}{2} \right) u^{-n-s-1} \mu - \Gamma \left( \frac{2+s}{2} \right) \Gamma \left( \frac{n+s}{2} \right) u^{-n-s} \right] i(u.A)$$

$$H_{n-s+2}(x,u) = \frac{2^{s-1}}{\pi^{\frac{s}{2}+1}} e^{-i\frac{\pi}{2} s} \sin(\pi s)$$

$$\times \left\{ \left[ \Gamma \left( \frac{1+s}{2} \right) \Gamma \left( \frac{n+s+1}{2} \right) u^{-n-s-1} \mu - \Gamma \left( \frac{2+s}{2} \right) \Gamma \left( \frac{n+s}{2} \right) u^{-n-s} \right] \left( \frac{(u.D)(u.A)}{2!} \right) \right\}$$

$$+ \frac{i}{8} \left[ \Gamma \left( \frac{1+s}{2} \right) \Gamma \left( \frac{n+s-1}{2} \right) u^{-n-s+1} u_\rho \gamma_\mu \gamma_\rho \gamma_\nu + \Gamma \left( \frac{2+s}{2} \right) \Gamma \left( \frac{n+s-2}{2} \right) u^{-n-s+2} \gamma_\mu \gamma_\nu \right] F_{\mu\nu}$$

$$H_{n-s+3}(x,u) = \frac{2^{s-1}}{\pi^{\frac{s}{2}+1}} e^{-i\frac{\pi}{2} s} \sin(\pi s)$$

$$\times \left\{ \left[ \Gamma \left( \frac{1+s}{2} \right) \Gamma \left( \frac{n+s+1}{2} \right) u^{-n-s-1} \mu - \Gamma \left( \frac{2+s}{2} \right) \Gamma \left( \frac{n+s}{2} \right) u^{-n-s} \right] \left( -\frac{(u.D)(u.D)(u.A)}{6} \right) \right\}$$

$$+ \frac{i}{8} \left[ \Gamma \left( \frac{1+s}{2} \right) \Gamma \left( \frac{n+s-1}{2} \right) u^{-n-s+1} u_\rho \gamma_\mu \gamma_\rho \gamma_\nu + \Gamma \left( \frac{2+s}{2} \right) \Gamma \left( \frac{n+s-2}{2} \right) u^{-n-s+2} \gamma_\mu \gamma_\nu \right] F_{\mu\nu} i(u.A)$$

$$+ \frac{1}{24} \left[ \Gamma \left( \frac{1+s}{2} \right) \Gamma \left( \frac{n+s-1}{2} \right) u^{-n-s+1} \left( -\frac{3}{2} u_\rho u_\sigma \gamma_\mu \gamma_\rho \gamma_\sigma \partial_\sigma F_{\mu\nu} - u_\mu u_\rho \gamma_\rho \partial_\nu F_{\mu\nu} + u_\mu u_\nu \gamma_\rho \partial_\rho F_{\mu\nu} \right) \right.$$

$$+ \left. \Gamma \left( \frac{2+s}{2} \right) \Gamma \left( \frac{n+s-2}{2} \right) u^{-n-s+2} \left( -\frac{3}{2} u_\mu \gamma_\nu \gamma_\rho \partial_\rho F_{\nu\mu} + u_\mu \partial_\nu F_{\nu\mu} \right) \right.$$
start to appear as an overall factor in the sum of the expansion (1) for $K_s(x, y)$.

Now, we shall compute the sum in expression (11) in order to see whether both methods coincide or not. Taking into account that $G_{-n+1+\ell} (x, u) = \lim_{s \to -1} H_{-n-s+\ell} (x, u)$ for $\ell < N$ and $G_{-n+1+N} (x, u) = \text{FP} H_{-n-s+N} (x, u)$, from Table I we get the following relations.

For $n = 2$, we have

$$\sum_{\ell=0}^{1} G_{-2+1+\ell} (x, u) e^{i\int_x^y A \, dz} = -\frac{i}{2\pi} \frac{\gamma\iota}{u^2} (1 + o(u^2)),$$  \hspace{1cm} (27)

so it is clear that (11) holds in this case.

For $n = 3$, we get

$$\sum_{\ell=0}^{2} G_{-3+1+\ell} (x, u) e^{i\int_x^y A \, dz} = -\frac{i}{4\pi} \frac{\gamma\iota}{u^3} (1 + o(u^3))$$

$$+ \frac{1}{16\pi} \left[ \frac{u_\rho}{u} \gamma_\mu \gamma_\rho \gamma_\nu + \gamma_\mu \gamma_\nu \right] F_{\mu\nu},$$  \hspace{1cm} (28)

and so

$$\text{Sch-lim} \, \text{tr} \left( \gamma_\mu \sum_{\ell=0}^{2} G_{-3+1+\ell} (x, u) e^{i\int_x^y A \, dz} \right) = \frac{1}{16\pi} \text{tr} [\gamma_\mu \gamma_\rho \gamma_\nu] F_{\rho\nu},$$  \hspace{1cm} (29)

which vanishes or not depending on the $\gamma$'s representation (it does not vanish if the $2 \times 2$ Pauli matrices are chosen).

Finally, we consider $n = 4$. In this case, a pole is present in $H_{-4-s+3}(x, u)$ at $s = -1$. After computing the finite part in order to get $G_{-4+1+3}(x, u)$ we have

$$\sum_{\ell=0}^{3} G_{-4+1+\ell} (x, u) e^{i\int_x^y A \, dz} = -\frac{i}{2\pi^2} \frac{\gamma\iota}{u^4} (1 + o(u^4))$$

$$+ \frac{1}{16\pi^2} \frac{u_\rho}{u^2} \gamma_\mu \gamma_\rho \gamma_\nu F_{\mu\nu} (1 + o(u^2))$$

$$- \frac{i}{48\pi^2} \frac{u_\rho u_\sigma}{u^2} (\gamma_\mu \gamma_\rho \gamma_\nu \partial_\sigma F_{\mu\nu} - \gamma_\rho \partial_\mu F_{\sigma\nu} + \gamma_\mu \partial_\mu F_{\sigma\nu})$$

$$- \frac{i}{24\pi^2} (\ln 2 - \ln u - \frac{i\pi}{2} + \Gamma'(1)) \gamma_\nu \partial_\mu F_{\mu\nu},$$  \hspace{1cm} (30)
which, in general, clearly yields a nonzero result for expression (11).

So, we see that although Schwinger and complex powers methods are both gauge invariant, they only coincide for the two-dimensional case. In 3 dimensions the coincidence depends on the representation chosen for the \( \gamma \)-matrices’s, while for \( n = 4 \) they in general disagree.

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