Network Models in Class C on Arbitrary Graphs

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Abstract

We consider network models of quantum localisation in which a particle with a two-component wave function propagates through the nodes and along the edges of an arbitrary directed graph, subject to a random SU(2) rotation on each edge it traverses. The propagation through each node is specified by an arbitrary but fixed $S$-matrix. Such networks model localisation problems in class C of the classification of Altland and Zirnbauer, and, on suitable graphs, they model the spin quantum Hall transition. We extend the analyses of Gruzberg, Ludwig and Read and of Beamond, Cardy and Chalker to show that, on an arbitrary graph, the mean density of states and the mean conductance may be calculated in terms of observables of a classical history-dependent random walk on the same graph. The transition weights for this process are explicitly related to the elements of the $S$-matrices. They are correctly normalised but, on graphs with nodes of degree greater than 4, not necessarily non-negative (and therefore interpretable as probabilities) unless a sufficient number of them happen to vanish. Our methods use a supersymmetric path integral formulation of the problem which is completely finite and rigorous.

I. INTRODUCTION

Network models of quantum localisation were first introduced by Chalker and Coddington [1] to model the transition between plateaux in integer quantum Hall systems. Reduced to their essentials, they describe the propagation of a single quantum-mechanical particle along the directed edges and through the nodes of a graph. For the Chalker-Coddington model, this graph is some large but bounded domain of the L-lattice, a square lattice whose edges are directed in such a way that the particle turns through $\pm 90^\circ$ at each node. In propagating along each edge, the single-component wave function is multiplied by random phases, which are i.i.d. random variables with a uniform distribution in $[0, 2\pi)$. The propagation through

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each node is described by a unitary $2 \times 2$ $S$-matrix of amplitudes between the two incoming and two outgoing edges.

The integer quantum Hall plateau transition is only one among several possible universality classes of quantum localisation transitions, which have been classified by Altland and Zirnbauer [2] according to the symmetry properties of the underlying single-particle hamiltonian $\mathcal{H}$. Another, known as class C, corresponds to the existence of a symmetry

$$\sigma_y \mathcal{H} \sigma_y = -\mathcal{H}^*.$$  \hspace{1cm} (1.1)

This gives rise to a pairing between eigenstates with energies $\pm E$, while $E = 0$ is special and may correspond to delocalised eigenstates, even in two dimensions. Class C is supposed to be realised in disordered spin-singlet superconductors in which time-reversal symmetry is broken, but Zeeman splitting is negligible [2]. The fact that spin is still conserved can then lead to a spin quantum Hall effect.

The appropriate network model for this on the L-lattice was formulated and studied numerically by Kagalovsky et al [3]. An equivalent spin-chain hamiltonian was also investigated by Senthil et al [4]. However, in a remarkable paper, Gruzberg, Ludwig and Read [5] showed that the mean single-particle Green function, as well as the mean conductance, may be expressed in terms of classical averages of appropriate observables of the hulls (boundaries) of clusters in classical bond percolation on the square lattice. The critical exponents [5] and some other universal properties [6] of the spin quantum Hall transition in two dimensions are thus exactly known.

The methods of Gruzberg et al [5] used supersymmetry to average over quenched disorder. One of the essential features of this, which will also appear in our analysis, is the reduction of the Hilbert space on each edge to one of finite dimension. They then analysed the transfer matrix for the L-lattice, and demonstrated its equivalence to that for percolation hulls.

One of the interesting features of percolation hulls on the square lattice is that they may be generated, independently of the underlying percolation problem, as history-dependent random walks on the L-lattice. Consider a random walk which begins on some edge, and steps through one node in unit time. At each node it turns to the left or right with probabilities $p$ or $1 - p$ respectively. However, it cannot traverse a given edge more than once, so that whenever it returns to a node it has already visited it is forced to exit along the other (empty) edge. Eventually, on a closed graph, it will return to its initial edge. The statistical properties of such loops are identical to those of a single closed hull in the percolation problem. On the L-lattice in the thermodynamic limit, when $p \neq \frac{1}{2}$, loops close almost surely after a finite number of steps. This corresponds to quantum localisation in the network model. The only delocalised states occur when $p = \frac{1}{2}$, at the bond percolation threshold.

Motivated by this work, Beamond, Cardy and Chalker [7] investigated class C network models on arbitrary graphs. Their methods did not use supersymmetry. Instead they showed that, for quantities like the average Green function $G$ and the mean conductance (related to $|G|^2$), there is a massive cancellation between paths in the Feynman expansion of these quantities which leaves essentially classical paths whose weights correspond to those of a history-dependent random walk. This reproduces the results of Gruzberg et al [5] when specialised to the L-lattice.
However, the proof of Beamond et al [7] was restricted to graphs in which each node has $N = 2$ incoming and outgoing directed edges. It becomes too cumbersome to generalise it to graphs with nodes with $N > 2$. Nevertheless, it seems important to find such a generalisation in order to be able to investigate, for example, the properties of such network models on simple regular lattices embedded in three or more dimensions.

In this paper, we find this generalisation. Like Gruzberg et al [5], we use supersymmetry to perform the quenched average. However, we do this within a path integral, rather than a Hilbert space, formulation of the model, which allows for the treatment of an arbitrary graph, not only those regular lattices which admit a transfer matrix. The result is positive, in the sense that we can prove that the mean of $G$ and of $|G|^2$ may be expressed as a sum of history-dependent classical random walks on the graph. The weights at each node, given that a given set of incoming edges is occupied by the walk, when summed over all possible outcomes, correctly sum to unity. However, in general these weights are not positive, and therefore cannot be interpreted as probabilities. In fact we show that, for $N > 2$, this condition can only be satisfied if a certain number of elements of $S$ vanish. A sufficient condition for this is that the $S$-matrix at each node is a direct product of $S$-matrices for $2 \to 2$ nodes, that is, the node may be decomposed into $2 \to 2$ nodes (in which case, of course, the analysis of Beamond et al [7] applies.) For $N = 3$ and 4 we have shown that this condition is also necessary.

Despite this negative result, the proof of the general theorem sheds further light on the analysis of Gruzberg et al [5], as well as giving a more elegant derivation of Beamond et al [7]. These methods may also be used to determine which combinations of averages of higher-point Green functions in the network model may be related to observables in the classical problem, thus giving a simpler derivation of some of the results of Mirlin, Evers, and Mildenberger [8].

The layout of this paper is as follows. In the next Section, we define the network models and the observables of interest. Then we are able to state our main Theorems (1 and 2). In Sec. III we introduce the path integral machinery necessary to compute these observables, and to perform quenched averages. The supersymmetric path integral involves both bosonic (commuting) and fermionic (anticommuting) variables on each edge, but after the quenched average is taken we show that these reduce to the propagation of only single fermion-fermion or boson-fermion pairs. In Sec. IV we then show that the propagation of a fermion-fermion pair through the lattice obeys the rules of a classical history-dependent random walk, once all the other degrees of freedom are traced out. This proves Theorem 1. We also consider the case of open systems, and conductance measurements. Finally, in Sec. V, we consider the probabilistic interpretation in terms of history-dependent random walks, and prove Theorem 4 which states conditions under which these weights are non-negative.

II. DEFINITION OF THE MODEL AND OBSERVABLES.

Let $G$ be a graph consisting of $N$ directed edges, and nodes. Initially we consider only closed graphs. At each node there is an equal number of incoming and outgoing directed edges. Apart from this, $G$ is arbitrary. We wish to define a network model on this graph with describes the propagation of a quantum-mechanical particle whose hamiltonian $\mathcal{H}$ obeys
the symmetry (1.1). First we note that the single-particle Hilbert space must be even-dimensional in order to be able to define the action of \(\sigma_y\). For simplicity we take this to be two-dimensional (in [7] a method was proposed for reducing any class C network model with an even-dimensional single-particle Hilbert space to this case.) Since the network describes propagation over finite time steps \(\Delta t\), we need to define the unitary evolution operators \(U_e\) and \(U_n\) which evolve the wave function along each edge and node respectively. By (1.1), they must obey \(\sigma_y U \sigma_y = U^*\). Each \(U_e\), therefore, must be an element of \(Sp(2)\sim SU(2)\). We take these to be i.i.d. random variables, each uniformly distributed with respect to the Haar measure of \(SU(2)\). Quenched averages with respect to this measure will be denoted by an overline. \(U_n\) is the unitary \(S\)-matrix for the node \(n\). As will become clear, it suffices to take this to be diagonal in the \(SU(2)\) indices, so, by the above, it must be real and therefore an element of \(O(N)\), where \(N\) is the number of incoming (and outgoing) edges at the node. The \(U_n\) may vary from node to node, and in principle they may also be random variables. However, our theorems apply to a fixed realisation of these \(S\)-matrix elements.

The full unitary evolution operator \(U\) for the whole network is then a direct sum over the edges and nodes of \(1 \otimes \ldots \otimes U_e \otimes \ldots \otimes 1\) and \(1 \otimes \ldots \otimes U_n \otimes \ldots \otimes 1\). Of interest is the Green function, which is the matrix element of the resolvent operator \((1 - zU)^{-1}\) between states localised on two edges:

\[
G(e_2, e_1; z) \equiv \langle e_2 | (1 - zU)^{-1} | e_1 \rangle. \tag{2.1}
\]

This is a \(2 \times 2\) matrix in \(SU(2)\) space. For \(|z| < 1\) this may be expanded as a sum of Feynman paths on \(\mathcal{G}\) beginning at \(e_1\) and ending at \(e_2\): each path is weighted by an ordered product of \(SU(2)\) matrices \(U_e\) along the edges it traverses, and a product of \(S\)-matrix elements according to how it passes through each node, as well as a factor \(z\) raised to the power of its length. However, each edge may be traversed an arbitrary number of times. If we were to formulate the sum over such paths using a transfer matrix formalism (assuming that \(\mathcal{G}\) allows this), the Hilbert space on each edge would be infinite-dimensional.

Alternatively, for \(|z| > 1\), we may write the resolvent as \(-z^{-1}U^\dagger (1 - z^{-1}U\dagger)^{-1}\) and expand \(G\) in powers of \(z^{-1}\), as a similar sum over paths. Each path is now weighted by an ordered product of factors \(z^{-1}U_e^{\dagger}\), as well as an overall factor of \(-z^{-1}\).

The Green function may be used to compute the density of states of \(U\) via its diagonal elements \(G(e, e; z)\). For a closed graph, these are of the form \(\exp(ie_j)\). We define the density of states \(\rho(\epsilon) = \sum_j \delta(\epsilon - \epsilon_j)\). Then

\[
\rho(\epsilon) = \frac{1}{2\pi} \frac{1}{2N} \sum_{e} \lim_{\delta \to 0} \left( \text{Tr} G(e, e; (1 - \delta)e^{-i\epsilon}) - \text{Tr} G(e, e; (1 + \delta)e^{-i\epsilon}) \right). \tag{2.2}
\]

An open system may be defined by cutting open a subset of the edges of \(\mathcal{G}\). We may then perform a conductance measurement by attaching leads to a subset \(\{e_{\text{in}}\}\) of the incoming edges, and to a subset \(\{e_{\text{out}}\}\) of the outgoing edges. The transmission matrix \(t\) between these two leads has elements \(\langle e_{\text{out}} | (1 - U) | e_{\text{in}} \rangle\), and the conductance is then given by the Landauer formula as \(g = \text{Tr} t\dagger t\). In particular, the point conductance between two edges is \(\text{Tr} G(e_{\text{out}}, e_{\text{in}}; 1)\dagger G(e_{\text{out}}, e_{\text{in}}; 1)\).

The above defines the quantum problem which we wish to study. Our main theorems will relate it to a classical problem, defined as follows. For each node in \(\mathcal{G}\), adopt an arbitrary
but fixed labelling of the incoming edges \( j \in \{1, \ldots, N\} \) and outgoing edges \( i \in \{1, \ldots, N\} \), and denote the elements of the corresponding \( S \)-matrix by \( S_{ij} \). Note that \( \det S = \pm 1 \), with the sign being dependent on the choice of labelling.

Define a trail \( \tau \) on \( \mathcal{G} \) as a sequence of distinct edges \((e_1, \ldots, e_{|\tau|})\) such that \( e_k \) and \( e_{k+1} \) are incoming and outgoing edges of the same node, for each \( 1 \leq k \leq |\tau| - 1 \). It is a (rooted) closed trail if \( e_{|\tau|} \) and \( e_1 \) also share the same node. Note that a trail cannot pass along a given edge more than once, but it may pass through a given node any number of times, up to its order.

For a particular trail \( \tau \), and a particular node \( n \), denote the set of incoming edges on \( \tau \) by \( J_{n;\tau} \), and the set of outgoing edges by \( I_{n;\tau} \). These are both ordered subsets of \( \{1, 2, \ldots, N\} \).

A given trail associates an element of \( I_{n;\tau} \) with each element of \( J_{n;\tau} \), and vice versa, and thus may be associated with a permutation \( \pi_{n;\tau} \) of the ordered elements of \( J_{n;\tau} \). Denote the signature of this by \((-1)^{\pi_{n;\tau}}\). Let \( \det S_{I,J} \) denote the minor of \( S \) restricted to the ordered subsets \( I \) and \( J \) of the outgoing and incoming channels respectively.

We are now ready to state

**Theorem 1.** The mean of \( G(e_1, e_2; z) \) vanishes if \( e_1 \neq e_2 \), while in the case of equality it is given by

\[
\text{Tr} \frac{G(e, e; z)}{z} = \begin{cases} 2 - \sum_{\tau(e)} \cos 2|\tau(e)| & : |z| < 1 \\ \sum_{\tau(e)} \cos 2|\tau(e)| & : |z| > 1 \end{cases} \quad (2.3)
\]

where the sums are over all closed trails \( \tau(e) \) rooted at \( e \) and \( w_{\tau(e)} \) is the weight of each, given by the product over all the nodes on \( \tau(e) \) of factors

\[
\Omega(I_{n;\tau}; J_{n;\tau}) \equiv (-1)^{\pi_{n;\tau}} \prod_{j \in J_{n;\tau}} S_{\pi_{n;\tau}(j),j} \quad (\det S_{I_{n;\tau},J_{n;\tau}}) \quad (2.4)
\]

Note that the first two factors are the term in the expansion of \( \det S_{I_{n;\tau},J_{n;\tau}} \) corresponding to the permutation \( \pi_{n;\tau} \): if we were to sum (2.4) over all permutations, we would obtain \((\det S_{I_{n;\tau},J_{n;\tau}})^2\). (2.4) is a generalisation of the main result of [7], which applies to the case \( N = 2 \). In this case, the elements of \( S \) may be taken to be \( S_{11} = S_{22} = \cos \theta \) and \( S_{12} = -S_{21} = \sin \theta \). If \( J = \{1\} \) and \( I = \{1\} \), or if \( J = \{2\} \) and \( I = \{2\} \), the weight is \( \cos^2 \theta \). If \( J = \{1\} \) and \( I = \{2\} \), or if \( J = \{2\} \) and \( I = \{1\} \), it is \( \sin^2 \theta \). But if \( J = \{1,2\} \) and \( I = \{1,2\} \), or \( \{2,1\} \), it is unity.

The next theorem gives the equivalent result for conductance measurements.

**Theorem 2.** The mean point conductance \( \bar{\sigma} \) between two edges \( e_{\text{in}} \) and \( e_{\text{out}} \) is given by twice the sum over all open trails on \( \mathcal{G} \) connecting the two edges, each such path being weighted as for the closed loops in Theorem 1.

Thus, if the weights on the trails can be interpreted as a probability measure, the mean conductance between two contacts is just twice the expected number of open trails which connect them.

The weight for a single trail \( \tau \) is given by a product of weights corresponding to each node thorough which \( \tau \) passes, once the whole of \( \tau \) is given. Alternatively, we may build up these weights as a product of factors incurred each time \( \tau \) passes through a given node. For example, the first time it passes through, entering via edge \( j_1 \) and leaving by edge \( i_1 \), it incurs a weight, according to (2.4), of \( \Omega(i_1; j_1) = S_{i_1,j_1}^2 \). If it passes through the same node
again, this time entering along \( j_2 \) and leaving along \( i_2 \), it then incurs a conditional weight

\[ w(i_1, i_2; j_1, j_2) = \frac{\Omega(i_1, i_2; j_1, j_2)}{\Omega(i_1; j_1)} \]

and so on. In general

\[ w(i_1, \ldots, i_p; j_1, \ldots, j_p) = \frac{\Omega(i_1, \ldots, i_{p-1}, i_p; j_1, \ldots, j_{p-1}, j_p)}{\Omega(i_1, \ldots, i_{p-1}; j_1, \ldots, j_{p-1})} \quad (2.5) \]

The next theorem states that these conditional weights are properly normalised, in the sense that they give unity when summed over all possible outcomes:

**Theorem 3.** The weights \( w(i_1, \ldots, i_p; j_1, \ldots, j_p) \) satisfy

\[ \sum_{i_p \not\in \{i_1, \ldots, i_{p-1}\}} w(i_1, \ldots, i_p; j_1, \ldots, j_p) = 1 \quad (2.6) \]

Thus, as long as they are non-negative, they define a set of transition probabilities for a discrete random process whereby the ensemble of trails may be dynamically generated with the correct weights, the trail growing by one unit at each time step. Since the weights at a given node depend on whether (and how) it has been visited in the past, the process may be thought of as a history-dependent random walk. When all nodes have \( N = 2 \), this is straightforward: the first time \( \tau \) passes through a given node, it incurs a factor \( \cos^2 \theta \) or \( \sin^2 \theta \). If it passes through a second time, this factor is unmodified.

We may ask whether this positivity can extend to nodes with \( N \geq 3 \). To answer this, we need a notion of reducibility. The \( S \)-matrix at a node with \( N \geq 3 \) is said to be reducible if it admits a factorisation of the form \( S = S^{(1)} S^{(2)} \), where (after a possible re-ordering of the incoming and outgoing channel labels) the \( N \times N \) \( S^{(1)} \) and \( S^{(2)} \) matrices have the block diagonal forms

\[ S^{(1)} = \begin{pmatrix} s^{(1)}_p & 0 \\ 0 & 1_{N-p} \end{pmatrix} \quad S^{(2)} = \begin{pmatrix} 1_q & 0 \\ 0 & s^{(2)}_{N-q} \end{pmatrix} \quad (2.7) \]

where \( s^{(1)}_p \) and \( s^{(2)}_{N-q} \) are orthogonal \( p \times p \) and \( (N - q) \times (N - q) \) matrices respectively, and \( p > q \). This is illustrated in Fig. 1. This procedure may be repeated. An \( N \times N \) \( S \)-matrix is said to be completely reducible if it can be factorised in this way into \( 2 \rightarrow 2 \) \( S \)-matrices.

**Theorem 4.** At a node with \( N \geq 3 \), a sufficient condition for the weights (2.4) to be all non-negative is that the \( S \)-matrix is completely reducible. For \( N = 3 \) and \( 4 \) this is also necessary. Thus, in these cases, the network model on \( \mathcal{G} \) could have been described on an equivalent graph with only \( N = 2 \) nodes.

### III. PATH INTEGRAL REPRESENTATION.

In the standard way, \( G \) may be written as a path integral over commuting (bosonic) variables. The notation is a little complicated, but the basic idea is simple. Label each end

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1This assumes \( S_{i_1,j_1} \neq 0 \). If the conditional weight can be interpreted as a probability, \( w(i_1, i_2; j_1, j_2) \leq 1 \) and therefore it has a finite limit as \( S_{i_1,j_1} \to 0 \). Even if this is not the case, the unconditional weight \( w(i_1, i_2; j_1, j_2) w(i_1; j_1) \) vanishes in this limit.
of a given directed edge $e$ by $e_R$ and $e_L$, in the direction of propagation $e_R \to e_L$. Introduce complex integration variables $b_R(e)$ and $b_L(e)$, which are each 2-component column vectors in SU(2) space, their components being labelled $b_{Ra}(e)$ and $b_{La}(e)$ respectively, where $a = 1, 2$. Then $G$ can be written as

$$G(e_2, e_1; z) = \langle b_L(e_2) b_{L}^\dagger(e_1) \rangle = \frac{\int \prod_e [db_L(e)][db_R(e)] b_L(e_2) b_{L}^\dagger(e_1) e^{W_b}}{\int \prod_e [db_L(e)][db_R(e)] e^{W_b}}$$  \hspace{1cm} (3.1)$$

where $W_b = W_{edge} + W_{node}$ with

$$W_{edge} = z \sum_e b_{L}^\dagger(e) U_e b_R(e)$$  \hspace{1cm} (3.2)$$

$$W_{node} = \sum_n \sum_a \sum_{ij} b_{Ra}^\ast(e_i)(S_n)_{ij} b_{La}(e_j)$$  \hspace{1cm} (3.3)$$

and the integration is wrt the usual coherent state measure

$$\int [db] = (1/\pi^2) \int e^{-b_{L}^\dagger b_R} \prod a dRe b_a dIm b_a$$  \hspace{1cm} (3.4)$$

Note that there is a finite number of integrations, if $\mathcal{G}$ is finite, and that no time-ordering necessary: we can imagine writing everything out in terms of components, and all quantities in the path integral are commuting. On a finite graph, only a finite number of integrations is necessary. The exponentiation of $W_b$ correctly takes into account the multiple traversing of edges by Feynman paths.

The next step is to average over the quenched random variables $U_e$. As usual, since these occur in both the numerator and denominator, this is most easily done either by introducing replicas, or by adding an anticommuting (fermionic) copy of the bosonic variables, making it supersymmetric. We opt for the latter. Thus to each pair of complex integration variables $(b^\dagger, b)$ we introduce a pair of Grassmann variables $(\bar{f}, f)$ with corresponding labels, and we add to $W_b$ a term $W_f$ of identical form with bosonic variables replaced by fermionic ones. The Grassmann integration is defined by

$$\int [df] = \int d\bar{f} df e^{-ff}$$  \hspace{1cm} (3.5)$$

so that
\[
\int [df] f = \int [df] \bar{f} = 0; \quad (3.6)
\]
\[
\int [df] 1 = \int [df] f \bar{f} = 1 \quad (3.7)
\]

Integrating over the fermionic variables cancels the denominator in (3.1), so that we may write
\[
G(e_2, e_1) = \prod_e [db_L(e)][db_R(e)][df_L(e)][df_R(e)] b_L(e_2) b_R^\dagger(e_1) e^{W_b + W_f}. \quad (3.8)
\]

However, \( W_b + W_f \) is invariant under global supersymmetry, so \( G \) may equally well be expressed, for example, as \( \langle f_L(e_2) \bar{f}_L(e_1) \rangle \).

### Quenched average.

The average over the SU(2) matrix \( U \) on a given edge has the form
\[
\int dU \exp(z b_R^\dagger U b_R + z \bar{f}_L U f_R) \quad (3.9)
\]
where the integral is with respect to the invariant measure on SU(2), normalised so that \( f dU = 1 \).

**Lemma 1:** The above integral equals \( 1 + \frac{1}{2} z^2 \det M \), where \( M \) is the \( 2 \times 2 \) matrix with components \( M_{ij} = b_{Li}^\dagger b_{Rj} + \bar{f}_{Li} f_{Rj} \).

**Proof:** Because the fermionic variables have only two components, and any such component squares to zero, the expansion in the fermionic part terminates:
\[
\int dU \exp(z b_R^\dagger U b_R + z \bar{f}_L U f_R + \frac{1}{2} z^2 (\bar{f}_L U f_R)^2) \quad (3.10)
\]

The first term, the purely bosonic integral, is identically equal to unity. This follows from the observation that the integral is invariant under the substitutions \( b_R \to \lambda V_R b_R, \ b_L^\dagger \to \lambda^{-1} b_L^\dagger V_L^\dagger \), where \( V_L \) and \( V_R \) are independent SU(2) matrices, and \( \lambda \) is a complex number, and there is no combination of \( b_L^\dagger \) and \( b_R \) which has this property, save a constant. However, an explicit proof is given in Appendix A.

The third, purely fermionic, term is also easy:
\[
(\bar{f}_L U f_R)^2 = (\bar{f}_{L1} U_{11} f_R1 + \bar{f}_{L2} U_{21} f_R1 + \bar{f}_{L1} U_{12} f_R2 + \bar{f}_{L2} U_{22} f_R2)^2 \quad (3.11)
\]
\[
= 2(\bar{f}_{L1} U_{11} f_R1)(\bar{f}_{L2} U_{22} f_R2) + 2(\bar{f}_{L2} U_{21} f_R1)(\bar{f}_{L1} U_{12} f_R2) \quad (3.12)
\]
\[
= 2 \bar{f}_{L1} \bar{f}_{L2} f_R2 f_R1(U_{11} U_{22} - U_{12} U_{21}) \quad (3.13)
\]
\[
= 2 \bar{f}_{L1} \bar{f}_{L2} f_R2 f_R1 \quad (3.14)
\]

where the fourth line follows because \( \det U = 1 \). The expression is therefore independent of \( U \), and the integration is then the same as in the purely bosonic term, which gives a factor 1 as before.
The second term can also be worked out explicitly, but it easier to invoke the supersymmetry, and simply add \( b^*_L b R_j \) to each element \( \tilde{f}_{i1} f_{Rj} \) of the above matrix. Note that the purely bosonic part of the determinant vanishes. 

The result of the quenched average over the SU(2) matrix on a given edge is therefore

\[
1 + \frac{1}{2} z^2 (b^*_L \tilde{f}_{L2} - b^*_L \tilde{f}_{L1})(b_{R1} f_{R2} - b_{R2} f_{R1}) + z^2 (\tilde{f}_{L1} \tilde{f}_{L2})(f_{R2} f_{R1})
\]

(3.16)

The interpretation of this is clear: after averaging over the SU(2) matrices, the only paths which contribute are those in which on each edge the allowed propagation is of either the identity, a pair of fermions \( f_1 f_2 \), or a boson-fermion pair \((1/\sqrt{2})(b_1 f_2 - f_1 b_2)\). Note that in each case the combinations in parentheses above are SU(2) singlets. Note also that, having averaged over the edge variables \( U_e \), the distinction between \( L \) and \( R \) is now immaterial, and we can henceforth drop these labels.

The above result has several important consequences. First, there is now only a finite number \( 3^N \) of possibilities for propagation along the \( N \) edges of a finite graph \( G \). (This is equivalent to the result of Gruzberg et al [5] that the Hilbert space of the transfer matrix is finite-dimensional.) Second, it is clear why the assumption that the scattering at the nodes in diagonal in the SU(2) indices was not crucial: only the singlet invariant amplitude matters. Third, the only non-zero two-point functions with \( e_2 \neq e_1 \) are

\[
\frac{1}{2} \langle (b_1(e_2) f_2(e_2) - b_2(e_2) f_1(e_2))(b^*_1(e_1) \tilde{f}_2(e_1) - b^*_2(e_1) \tilde{f}_1(e_1)) \rangle
\]

(3.17)

\[
= \langle f_1(e_2) f_2(e_2) \tilde{f}_2(e_1) \tilde{f}_1(e_1) \rangle
\]

(3.18)

\[
= \frac{G_{11} G_{22} - G_{12} G_{21}}{\det G(e_2, e_1; z)}
\]

(3.19)

Let us for the moment take \( z \) real. Then \( G(e_2, e_1; z) \), as a sum over Feynman paths, is a linear combination of SU(2) matrices with real coefficients. Any such \( 2 \times 2 \) matrix is itself proportional to an SU(2) matrix, up to a real scalar (see Appendix.) Thus we may write \( G = \lambda G \) where \( \lambda \) is real and \( G \in \text{SU}(2) \). Hence \( \det G = \lambda^2 \), and \( G^\dagger G = \lambda^2 I \), so that \( \text{Tr} G^\dagger G = 2 \det G \). The right hand side is a polynomial in \( z \). For general complex \( z \) we have, therefore,

\[
2 \det G(e_2, e_1; z) = \text{Tr} G(e_2, e_1; z^*) G(e_2, e_1, z)
\]

(3.20)

When \( z = 1 \) this is the mean point conductance, which is therefore given, up to a factor 2, by the two-point functions in (3.17).

Since only SU(2) singlets now propagate, it follows that two-point functions like \( \langle f_a(e_2) \overline{f}_a(e_1) \rangle = G(e_2, e_1) \) vanish if \( e_2 \neq e_1 \). This is because, once the matrices \( U_e \) have been traced out, the supersymmetric path integral possesses a local SU(2) gauge invariance under \((b(e), f(e)) \rightarrow (V_e b(e), V_e f(e))\) with \( V_e \in \text{SU}(2) \).

However, this does not apply if \( e_2 = e_1 \). In fact, because of (3.17), it follows that

\[
G(e, e; z)_{11} = \langle f_1(e) \tilde{f}_1(e) \rangle = \langle f_1(e) f_2(e) \tilde{f}_2(e) \tilde{f}_1(e) \rangle = \overline{\det G(e, e; z)}
\]

(3.21)
IV. PROPAGATION THROUGH THE NODES.

In the last section, we showed that, for a graph $G$ with $N$ edges, the quenched average of the path integral can be written as a sum of $3N$ terms, according to which of the three terms in (3.16) (corresponding to the propagation of a $bf$ pair, an $ff$ pair, or the identity) is chosen on each edge. Let us now consider just one of these terms, and one particular node.

The contribution to the path integral from this node has the form

$$\prod_i A_{\alpha_i}(r_i) S \prod_j \overline{A}_{\alpha_j}(r_j)$$

where $A_1 = 1$, $A_2 = f_1 f_2$ and $A_3 = (1/\sqrt{2})(b_1 f_2 - b_2 f_1)$ and

$$S = \exp\left(\sum_{a=1}^2 \sum_{i,j} (b^*_a S_{ij} b_{ja} + \bar{f}_{ia} S_{ij} f_{ja})\right)$$

In doing this, we have brought together in the path integral all the factors associated with the given node. There is a subtlety, however, because the boson-fermion variables $A_3$ and $\overline{A}_3$ anticommute with each other. At a given node, we may arrange these factors in the standard order determined by the fixed (but arbitrary) labelling of the incoming and outgoing edges. For a given term out of the $3^N$ possibilities, this will introduce an overall factor $\pm 1$.

Define a decomposition of the node as a pairing of each outgoing edge $i$ with a unique incoming edge $j$. This defines a permutation $\pi$ of the edge labels, whereby the outgoing edge $i$ paired with the incoming edge $j$ is $\pi(j)$. Carried through for every node in turn, this decomposes $G$ into a union of disjoint directed closed loops (and open paths if $G$ is open), such that every edge lies on just one loop or open path, and each loop or open path may pass along a given edge no more than once. The following Proposition shows that we are allowed to do this inside the path integral, as long as we weight each decomposition correctly:

**Proposition 1.** The result of performing the integration over the variables $(b_j, f_j)$ and $(b^*_i, \bar{f}_i)$ in (4.1) is the same as if $S$ were replaced by

$$\det S \sum_{\pi} (-1)^\pi \prod_{ij} \delta_{i,\pi(j)} S_{ij} \delta_{\alpha_i, \alpha_j}$$

that is, it is given by a weighted sum over all decompositions $\pi$. In each decomposition, each state on the incoming edge $j$ propagates freely to $\pi(j)$.

**Proof:** Since the numbers of each component of both bosons and fermions are the same in the incoming and outgoing channels, and bosons are always paired with fermions, it follows that the numbers of $ff$ and $fb$ pairs are individually conserved at every node. Let us call the subsets of the $N$ outgoing channels occupied by an $ff$ pair, a $bf$ pair, or empty, $FF$, $FB$, and $E$, respectively, and similarly for the incoming channels, $\overline{FF}$, $\overline{FB}$ and $\overline{E}$. The integrations may now be performed, expanding $S$ to second order in the $S_{ij}$ and using Wick’s theorem. Each fermion (boson) in outgoing channel $i$, when contracted with a fermion (boson) in the incoming channel $j$, gives (up to a sign) a factor $\delta_{ab} S_{ij}$. The bosons in $FB$ may only contract onto the bosons in $\overline{FB}$, but the complication is that some of the fermions in $FF$...
may contract onto fermions in \( \overline{FB} \), and some of those in \( FB \) may contract onto those in \( FF \).

However, every set of possible contractions will involve each outgoing channel in \( FF \cup FB \) and each incoming channel in \( \overline{FF} \cup \overline{FB} \) exactly twice. Thus, if \( \sigma \) denotes a permutation of the channels in \( FF \cup FB \), then, the general form of the result will be

\[
\sum_{\sigma, \sigma'} a_{\sigma, \sigma'} \prod_{i \in FF \cup FB} S_{i, \sigma(i)} \prod_{i' \in FF \cup FB} S_{i', \sigma'(i')} \tag{4.4}
\]

where the \( a_{\sigma, \sigma'} \) are numerical coefficients.

We have already introduced the notation \( \det S_{I,J} \) for the minor of \( S \) restricted to the ordered subsets \( I \) and \( J \) of outgoing and incoming channels. Now define \( \text{perm} S_{I,J} \) to be the corresponding permanent, that is, with all the terms having the same sign +1. Then the claim is that the result of the integration is

\[
\det S_{FF,\overline{FF}} \cdot \text{perm} S_{FB,\overline{FB}} \cdot \det S_{FF \cup FB, \overline{FF} \cup \overline{FB}} \tag{4.5}
\]

This expression has the correct properties in that: (a) each channel index appears exactly twice in each term; (b) \( S_{ij} \) with \( i \in FF \) and \( j \in \overline{FB} \) (and also with \( i \in FB \) and \( j \in \overline{FF} \)) occurs at most once; (c) it is symmetric under permutations of the channels in \( FF \), and separately in \( \overline{FF} \); (d) it is antisymmetric under permutations of the channels in \( FB \), and separately in \( \overline{FB} \); and (e) it has the correct overall numerical coefficient.

In order to prove (4.5), it is helpful first to consider what happens if each \( SU(2) \) singlet \( \frac{1}{\sqrt{2}}(b_1 f_2 - b_2 f_1) \) is replaced by \( b_1 f_2 \) (and similarly for the conjugate variables in the incoming channels.) In that case, the result follows immediately. The \( f_{1i} \) with \( i \in FF \) can contract only onto the \( \overline{f}_{1j} \) with \( j \in \overline{FF} \), giving the first factor in (4.5). Similarly, the \( b_{1i} \) with \( i \in FB \) can contract only onto the \( b_{1j}^{*} \) with \( j \in \overline{FB} \), giving the second factor. Finally the \( f_{2i} \) are free to contract onto any of the \( \overline{f}_{2j} \), leading to the last factor. The reason that this result continues to hold when each \( b_1 f_2 \) is replaced by the singlet combination is the local gauge invariance already alluded to: we could imagine multiplying the whole amplitude by an independent \( SU(2) \) matrix in each channel, and averaging over this. The final result, being gauge invariant, would not change, but it would project \( b_1 f_2 \) onto the singlet combination.

We now need the following property of \( O(N) \) matrices:

**Lemma 2.** If \( S \in O(N) \), and \( \det S' \) and \( \det S'' \) are complementary minors of \( S \), then \( \det S' = \det S'' \cdot \det S \).

**Proof:** This relies on the fact that if \( S' \) has rank \( p \), and \( T_{j_1...j_p} \) is a tensor of rank \( p \), then \( \epsilon_{j_1...j_p j_{p+1}...j_N} T_{j_1...j_p} \) (where \( \epsilon_{...} \) is the Levi-Civita symbol) transforms under proper rotations as a tensor of rank \( N - p \), and changes sign under parity. \( \blacksquare \)

In our case this implies that \( \det S_{FF \cup FB, \overline{FF} \cup \overline{FB}} = \det S_{E, \overline{E}} \cdot \det S \), so that (4.5) reads

\[
\det S_{FF,\overline{FF}} \cdot \text{perm} S_{FB,\overline{FB}} \cdot \det S_{E, \overline{E}} \cdot \det S \tag{4.6}
\]

Now look at (4.3), inserted into the path integral instead of \( S \). The Kronecker deltas which conserve the labels \( \alpha \) restrict the sum over permutations \( \pi \) to those which map \( \overline{FF} \) onto some permutation \( \pi_{FF} \) of \( FF \), \( \overline{FB} \) onto some permutation \( \pi_{FB} \) of \( FB \), and so on. The signature \((-1)^{\pi}\) decomposes into a product of the signatures of the three permutations. Now, since the \( ff \) pairs propagate freely (and they commute among themselves), the integrations over
these variables give unity. The sum over $\pi_{FF}$ is therefore $\sum_{\pi_{FF}} (-1)^{\pi_{FF}} \prod_{i \in FF} S_{i,\pi_{FF}(i)} = \det S_{FF,FF}$, which gives the first factor in (4.6). Although the $bf$ pairs also propagate freely, they are fermionic, which means that their contractions give rise to an extra factor $(-1)^{\pi_{FB}}$. On summing over all $\pi_{FB}$, we get the second factor in (4.6). The remaining factors of $S_{\tau(j),j}$, with $j \in E$, when summed over $\pi_E$, give the last factor.

We have shown the equivalence of the two expressions $S$ and (4.3) at each node, for each of the $3^N$ terms in the expansion of the path integral. We may now restore the anticommuting $bf$ factors to their original ordering in the path integral, thus removing the possible overall sign. This concludes the proof of Prop. 1. $\blacksquare$

**Proof of Theorems 1 and 2.**

First consider the case when $G$ is closed. In Sec. III it was shown that $\overline{G(e,e)}$ is given by the correlation function $\langle f_{L1}(e)f_{L2}(e)f_{L3}(e)f_{L1}(e) \rangle$ in the supersymmetric path integral. By the results of the previous Section, this is given by a sum of terms in which each edge except $e$ is occupied by either an $ff$ pair, a $bf$ pair, or the identity (and $e$ is occupied only by an $ff$ pair.) Moreover, the path integral is given by a sum of terms, each corresponding to a decomposition of $G$ into closed loops. Along all but one of the closed loops can freely propagate an $ff$ pair, giving an overall factor +1, a $bf$ pair, giving -1, or the identity, giving +1. The first two contributions cancel, leaving a factor +1 for each of these closed loops. The exception is the unique loop which contains the edge $e$, which can be thought of as a closed trail $\tau(e)$, rooted at $e$. Along this only an $ff$ pair is allowed to propagate.

Now sum over all decompositions of $G$ which contain the specified trail $\tau(e)$. At a given node $n$, $\tau(e)$ occupies the incoming edges $J_{n,\tau}$ and the outgoing edges $I_{n,\tau}$. The sum in (4.3) includes only those permutations $\pi$ for which $\pi(J_{n,\tau})$ is some permutation of $I_{n,\tau}$. This implies that $\pi$ acting on the complementary subset $J_{n,\tau}$ is some permutation $\bar{\pi}$ of the complement $I_{n,\tau}$. If we now sum the factors of $S_{ij}$ in (4.3) with $i \in J_{n,\tau}$ and $j \in I_{n,\tau}$ over the permutations $\bar{\pi}$, weighted by $(-1)^{\bar{\pi}}$, we get $\det S_{\tau,\tau}$. Using Lemma 2 again, this equals $\det S_{I_{n,\tau},J_{n,\tau}} \cdot \det S$. The latter factor of $\det S$ combines with explicit one in (4.3) to give unity. The remaining factors then give the weight (2.4) of the node $n$ on the trail $\tau(e)$. This proves Theorem 1. $\blacksquare$

Theorem 2 follows similarly. For an open graph, $\text{Tr} \overline{G^T(e_2,e_1)G(e_2,e_1)}$ is given by a sum of decompositions of $G$ as before, into closed loops as well as open paths which connect the incoming and outgoing external edges. Along these propagate either $ff$ pairs, $bf$ pairs, or the identity, with weights at each node given by (4.3). In each decomposition, there is a unique open trail $\tau$ from $e_1$ to $e_2$, carrying an $ff$ pair. The other open paths must carry the identity, otherwise the path integration over the free bosonic and fermionic variables at their ends would give zero. They therefore contribute a factor 1. All the other closed loops also contribute a factor 1 after the cancellation between the $ff$ and $bf$ pairs which propagate around each of them. We are left with a single $ff$ pair propagating along $\tau$. The summation over all the decompositions of $G$ containing a given open trail $\tau$ then gives a factor $\det S_{I_{n,\tau},J_{n,\tau}} \cdot \det S$ at each node as above. This proves Theorem 2. $\blacksquare$
V. PROBABILISTIC INTERPRETATION.

Normalisation.

We now prove Theorem 3, which states that the weights \( \Omega(I, J) \) in Theorem 1 lead, if non-negative, through (2.5) to correctly normalised transition probabilities \( w(i_1, \ldots, i_p; j_1, \ldots, j_p) \) for the trail \( \tau(e) \) interpreted as a classical random walk on the edges of \( \mathcal{G} \). A necessary and sufficient condition for this is

\[
\sum_{i_p \notin \{i_1, \ldots, i_{p-1}\}} \Omega(i_1, \ldots, i_{p-1}, i_p; j_1, \ldots, j_{p-1}, j_p) = \Omega(i_1, \ldots, i_{p-1}; j_1, \ldots, j_{p-1}) \tag{5.1}
\]

Without loss of generality, we may relabel the rows and columns of \( S \) so that \( i_k = k \) for \( 1 \leq k \leq p - 1 \), and \( j_k = k \) for \( 1 \leq k \leq p \). Notice that we can remove the restriction on the sum over \( i_p \) because the summand formally vanishes whenever \( 1 \leq s_p \leq p - 1 \). The index \( i_p \) occurs on the left hand side of (5.1) in the factor \( S_{i_p, p} \) as well as in each term of the expansion of the minor \( \det S_{\{1, \ldots, i_p\}; \{1, \ldots, p\}} \), where it occurs as \( S_{i_p, k} \) with \( 1 \leq k \leq p \). Thus the sum over \( i_p \), in each term in the expansion of the determinant, has the form \( \sum_{i_p} S_{i_p, p} S_{i_p, k} = \delta_{pk} \), from the orthonormality of the rows of \( S \). The coefficient of this term is just the subminor \( \det S_{\{1, \ldots, p-1\}; \{1, \ldots, p-1\}} \) which occurs on the right hand side of (5.1). All the remaining factors \( \prod_{1 \leq k \leq p-1} S_{k, k} \) are the same on both sides. This demonstrates the validity of (5.1) and thus Theorem 3.

Positivity of the weights.

Although we have argued that the weights \( \Omega \) appearing in Theorem 1 are normalised, they may only be interpreted as probabilities if they are all non-negative. This places strong constraints on the \( S \)-matrix at each node.

Taking first the case when the sets \( I \) and \( J \) comprise all the outgoing and incoming edges of the node, we see that the weights are all non-negative if and only if every term in the expansion of \( \det S \) has the same sign, or vanishes. In fact, this is also a sufficient condition for all the weights to be non-negative when \( I \) and \( J \) are proper subsets. This is because, by Lemma 2, \( \det S_{I, J} \) is, up to a factor \( \det S = \pm 1 \), the same as its conjugate minor, and therefore each term in (2.4) is, up to an overall sign, a sum of a subset of terms in the expansion of \( \det S \). They therefore all have the same sign, or vanish, if this is true of the individual terms in the expansion.

For \( N = 2 \), this is always the case. If \( \det S = 1 \), we can write \( S = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \), so that the terms in the expansion are \((\cos^2 \theta, \sin^2 \theta)\); or if \( \det S = -1 \) we can write \( S = \begin{pmatrix} \sin \theta & \cos \theta \\ \cos \theta & -\sin \theta \end{pmatrix} \), in which case they are \((-\sin^2 \theta, -\cos^2 \theta)\). However, for an orthogonal matrix with \( N > 2 \), this constraint becomes nontrivial.
Consider first the case $N = 3$. It is elementary to show that if the $3!$ terms in the expansion of the determinant of any $3 \times 3$ matrix all have the sign (or vanish) then there must be at least one vanishing element. For consider the product of all these terms. This contains each element $S_{ij}$ exactly twice. There are six terms in all, and three of these, corresponding to the odd permutations, occur with minus signs. Hence the product of all the terms is $-\prod_{i=1}^{3} \prod_{j=1}^{3} S_{ij}^2 \leq 0$. This would be impossible if all the $S_{ij}$ were non-vanishing.

Now any O(3) rotation can be composed of three suitable O(2) rotations about different axes, for example through the Euler angles. This composition may, in general, be pictured using a diagram like that in Fig. 2. Each intersection of lines labelled by $i$ and $j$ corresponds to an O(2) rotation in the $ij$ plane, represented by an O(2) matrix $s^{(a)}$ with $a = 1, 2, 3$. The element $S_{ij}$ of the full O(3) matrix is given by a sum over directed paths from $j$ to $i$ in the diagram, each path being weighted by a product of the appropriate O(2) matrix elements. For example,

$$S_{13} = s_{13}^{(2)}$$

$$S_{31} = s_{32}^{(3)} s_{21}^{(1)} + s_{33}^{(3)} s_{31}^{(1)}$$

Each topologically distinct way of drawing and labelling Fig. 2 corresponds to a different but equivalent Euler angle parametrisation.

We can always draw the diagram so that the matrix element which vanishes by the above argument (in this example $S_{13}$) is given by a simple form like (5.2). This implies that $s_{13}^{(2)} = s_{31}^{(2)} = 0$, and therefore that $s_{11}^{(2)} = s_{33}^{(2)} = 1$ (note that $s^{(2)}$ can always be chosen as a proper rotation.) This means that we can picture the lines 1 and 3 simply crossing at the vertex (2), and that the full O(3) rotation reduces into a product of just two O(2) rotations, as in the definition (2.7) of reducibility.
2. $N > 3$.

An $O(N)$ matrix has several distinct but equivalent Euler angle representations as a composition of $\frac{1}{2}N(N - 1)$ $O(2)$ rotations, which may be pictured using a generalisation of Fig.2. An example for $N = 4$ is shown in Fig. 3. In such a diagram any given line intersects each of the others exactly once. If the matrix is completely reducible there is at least one representation which has a tree structure, that is, contains no cycles. An example is shown in Fig. 3. In this case, many elements of $S$ must vanish, and those which do not are each given by a single term which is a product of $O(2)$ matrix elements along a single possible path through the diagram.

Complete reducibility is a sufficient condition for the weights in (2.4) all to be non-negative. One way to see this is to note that we can in this case decompose the node into a tree of $2 \to 2$ nodes. The internal edges of this tree can be made to carry an arbitrary $SU(2)$ matrix, which can however always be set equal to 1 by making a gauge transformation on the $SU(2)$ matrices on the incoming and outgoing edges of the node (this is not always possible if there are cycles.) We may therefore introduce such matrices on each internal edge of the tree and integrate over them without changing the problem. Thus the weights for the node are products of weights for $2 \to 2$ nodes, which we have already argued are always non-negative.

Next we consider whether this condition is necessary. Consider the terms in the expansion of $\det S$ which contain a factor $S_{11}S_{22} \ldots S_{N-3,N-3}$. The coefficient of this term is the $3 \times 3$ minor $\det S_{I,J}$ with $I = J = \{N - 2, N - 1, N\}$. The above Lemma about $3 \times 3$ matrices then shows that either this submatrix has at least one vanishing element, or the product $S_{11}S_{22} \ldots S_{N-3,N-3}$ vanishes. In general, every $3 \times 3$ submatrix of $S$ must have at least one vanishing element, or every term in the expansion of its complementary minor must vanish.
For \( N = 4 \), this implies that there must be at least 3 vanishing elements, not all in the same row or column. By considering the different cases, together with a suitably chosen Euler angle representation, it is possible to show that in each case a sufficient number of the \( \text{O}(2) \) matrix elements must vanish that the diagram breaks up into a tree. This shows that, for \( N = 4 \), the condition of complete reducibility is also necessary for non-negative weights. However, we have not found a general argument for all \( N \) and indeed there may be exceptions. What can be shown straightforwardly is that \( S \) must have at least \( N - 1 \) vanishing elements.

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**APPENDIX: SOME PROPERTIES OF SU(2) MATRICES**

We show explicitly that the integral
\[
I \equiv \int dU \exp(z b_L^i U b_R^i) = 1 \tag{A1}
\]

Any SU(2) matrix may be parametrised as \( U = \exp(i \alpha \sigma \cdot n) = \cos \alpha + i \sigma \cdot n \sin \alpha \). The Haar measure is then
\[
\int dU = (2 \pi^2)^{-1} \int_{-1}^1 d \cos \theta \int_0^\pi \sin^2 \alpha d \alpha \int d \Omega_n \tag{A2}
\]
The exponent in (A1) has the form \( A \cos \alpha + i n \cdot B \) where \( A = z b_L^i b_R^i \) and \( B = z b_L^i \sigma b_R^i \). Note that \( B^2 = A^2 \). Although these are in general complex, since \( I \) is an analytic function of each of their components, we can first assume they are real. Then, without loss of generality, we can assume that \( B \) is real and points in the \( z \)-direction. Then
\[
I = (1/\pi) \int_0^\pi \sin^2 \alpha d \alpha \int_{-1}^1 d (\cos \theta) \exp(A(\cos \alpha + i \cos \theta \sin \alpha)) \tag{A3}
\]
The integral over \( \cos \theta \) is simple, and the result may be expanded in a power series in \( A \). All terms, save that \( O(A^0) \), then vanish on integration over \( \alpha \).

We show that any real linear combination of SU(2) matrices is itself, up to a real constant, an SU(2) matrix. From the above representation, it may be written as
\[
G = \sum_j a_j \cos \alpha_j + i \sum_j a_j n_j \cdot \sigma \sin \alpha_j \tag{A4}
\]
which has the form \( A + i B N \cdot \sigma \) where \( A \) and \( B \) are real, and \( N \) is another unit 3-vector. Writing \( A = \rho \cos \alpha \) and \( B = \rho \sin \alpha \) then gives the required result.
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