Polynomial Representations of the Lie Superalgebra $\mathfrak{osp}(1|2n)$

A.K. Bisbo\textsuperscript{1\dagger}, H. De Bie\textsuperscript{2\ddagger} and J. Van der Jeugt\textsuperscript{3\dagger}

\textsuperscript{1}Department of Applied Mathematics, Computer Science and Statistics, Faculty of Sciences, Ghent University, Krijgslaan 281-S9, B-9000 Gent, Belgium.

\textsuperscript{2}Department of Electronics and Information Systems, Faculty of Engineering and Architecture, Ghent University, Krijgslaan 281-S8, B-9000 Gent, Belgium.

\textsuperscript{3}E-mail: Asmus.Bisbo@UGent.be

\textsuperscript{1} \textsuperscript{2} \textsuperscript{3} \textsuperscript{\dagger} \textsuperscript{\ddagger}

Abstract

We study a particular class of infinite-dimensional representations of $\mathfrak{osp}(1|2n)$. These representations $W_n(p)$ are characterized by a positive integer $p$, and are the lowest component in the $p$-fold tensor product of the metaplectic representation of $\mathfrak{osp}(1|2n)$. We construct a new polynomial basis for $W_n(p)$ arising from the embedding $\mathfrak{osp}(1|2np) \supset \mathfrak{osp}(1|2n)$. The basis vectors of $W_n(p)$ are labelled by semi-standard Young tableaux, and are expressed as Clifford algebra valued polynomials with integer coefficients in $np$ variables. Using combinatorial properties of these tableau vectors it is deduced that they form indeed a basis. The computation of matrix elements of a set of generators of $\mathfrak{osp}(1|2n)$ on these basis vectors requires further combinatorics, such as the action of a Young subgroup on the horizontal strips of the tableau.

1 Introduction

In representation theory of Lie algebras, Lie superalgebras or their deformations, there are often three problems to be tackled. The first is the existence or the classification of representations. The second is obtaining character formulas for representations. And the third is the construction of (a class of) representations. Concretely, this third step consists of finding an explicit basis for the representation space and the explicit action of a set of algebra generators in this basis (i.e. find all matrix elements). Mathematicians are primarily interested in the first two problems and often ignore the third one. This goes together with the impression that the third problem is computationally quite hard and not necessarily leads to interesting mathematical structures. For applications in physics however, the third step is often indispensable, as one needs to compute physical quantities such as energy spectra coming from eigenvalues of Hamiltonians, or transition matrix elements coming from explicit actions, e.g. [14]. A typical example of the third problem is the construction of the Gelfand-Zetlin basis for finite-dimensional irreducible representations of the Lie algebra $\mathfrak{gl}(n)$ or $\mathfrak{sl}(n)$, see e.g. [23].

In this paper we are dealing with a class of representations of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$. The representations considered here are infinite-dimensional irreducible lowest weight representations, with lowest weight coordinates $(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2})$, for $p$ a positive integer [1, 8, 7]. As we shall see in the paper, the construction of basis vectors and generator actions leads to many interesting mathematical and combinatorial concepts.

The motivation for studying these representations comes from physics: this class of representations of $\mathfrak{osp}(1|2n)$ corresponds to the so-called paraboson Fock spaces. Parabosons were introduced by Green [11] in 1953, as generalizations of bosons. Parabosons have been of interest in quantum field theory [25], in generalizations of quantum statistics [13, 2] and in Wigner quantum systems [17, 19]. Whereas creation and annihilation operators of bosons satisfy simple commutation relations, those of parabosons satisfy more complicated triple relations. Moreover where
there is only one boson Fock space, there are an infinite number of paraboson Fock spaces, each of them characterized by a positive integer \( p \) (called the order of statistics). Many years after their introduction, it was shown that the triple relations for \( n \) pairs of paraboson operators are in fact defining relations for the Lie superalgebra \( \mathfrak{osp}(1|2n) \) [10] and that the paraboson Fock space of order \( p \) coincides with the unitary irreducible lowest weight representation \( W_n(p) \) of \( \mathfrak{osp}(1|2n) \) with lowest weight \((\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2})\) in the natural basis of the weight space.

The construction of a convenient basis for \( W_n(p) \), with the explicit action of the paraboson operators, turns out to be a difficult problem. In principle, one can follow Green’s approach [11, 16] and identify \( W_n(p) \) as an irreducible component in the \( p \)-fold tensor product of the boson Fock space of \( \mathfrak{osp}(1|2n) \) (which is \( W_n(1) \)). However, there are computational difficulties to find a proper basis this way.

Some years ago [20], a solution was found for the construction of a basis for \( W_n(p) \), using in particular the embedding \( \mathfrak{osp}(1|2n) \supseteq \mathfrak{gl}(n) \). This allowed the construction of a proper Gelfand-Zetlin basis for \( W_n(p) \), and using further group theoretical techniques the actions of the paraboson operators in this basis could be computed [20]. This offered a solution to a long-standing problem.

A disadvantage of this solution and the Gelfand-Zetlin basis is the rather complicated expressions for the generator matrix elements.

In the present paper, we construct a new polynomial basis for these representations \( W_n(p) \). Starting from the embedding \( \mathfrak{osp}(1|2np) \supseteq \mathfrak{osp}(1|2n) \), \( W_n(p) \) can be identified with a submodule of the decomposition of the \( \mathfrak{osp}(1|2np) \) Fock space \( W_{np}(1) \). Equivalently, the Howe dual pair \((\mathfrak{osp}(1|2n), Pin(p))\) for this Fock space can be employed. Using furthermore the character for \( W_n(p) \), this yields a polynomial basis consisting of vectors \( \omega_A(p) \), labelled by semi-standard Young tableaux of length at most \( p \) with entries \( 1, 2, \ldots, n \). The construction of this basis is carefully developed in sections 2–5. Each basis vector \( \omega_A(p) \) is a specific Clifford algebra valued polynomial with integer coefficients in \( np \) variables \( x_{i,\alpha} \) \((i = 1, \ldots, n; \alpha = 1, \ldots, p)\), thus involving \( p \) Clifford elements \( e_\alpha \). The \( \mathfrak{osp}(1|2n) \) generators \( X_i \) and \( D_i \) are realized as (Clifford algebra valued) multiplication operators or differentiation operators with respect to the \( x_{i,\alpha} \)'s. These are given in Section 2, where the definition of \( \mathfrak{osp}(1|2n) \) is recalled and \( W_n(p) \) is defined. The irreducibility of \( W_n(p) \) is discussed in Section 3, where its equivalence with other definitions of this \( \mathfrak{osp}(1|2n) \) module are settled. In Section 4 we introduce the above mentioned tableau vectors \( \omega_A(p) \) as candidate basis vectors for \( W_n(p) \). The same is done for tableau vectors \( v_A(p) \) of a module \( V_n(p) \), which is a natural induced \( \mathfrak{osp}(1|2n) \) module of which \( W_n(p) \) is a quotient module. In this section, the knowledge of the characters of \( W_n(p) \) and \( V_n(p) \) in terms of Schur polynomials plays an essential role. In order to show that the tableau vectors \( \omega_A(p) \) constitute indeed a basis for \( W_n(p) \), their linear independence must be shown, and this is established in Section 5. The proof is non-trivial and depends on a total ordering for the set of semi-standard Young tableaux with entries from \( \{1, 2, \ldots, n\} \). This allows the identification of a unique “leading term” in \( \omega_A(p) \) which does not appear in \( \omega_B(p) \) if \( B < A \). In the last section (Sec. 6) we compute the action of the \( \mathfrak{osp}(1|2n) \) generators \( X_i \) and \( D_i \) on the tableau vectors \( \omega_A(p) \) of \( W_n(p) \). This is rather technical: rewriting \( X_i\omega_A(p) \) or \( D_i\omega_A(p) \) as linear combinations of tableau vectors \( \omega_B(p) \) is not trivial. Fortunately, also here the identification of “leading terms” is very helpful to solve the problem. We also give some examples, making the technical parts comprehensible.

To improve the readability of the paper, we make a list of the commonly used symbols. Here \( n, p \in \mathbb{N} \) and \( i \in \{1, \ldots, n\} \).
2 The Polynomial Paraboson Fock Space \( W_n(p) \)

The Lie superalgebra \( \mathfrak{osp}(1|2n) \) is usually defined as a matrix Lie superalgebra \([15, 9]\). It can also be defined as a symbolic algebra with generators and relations \([10]\). Adopting the latter definition \( \mathfrak{osp}(1|2n) \) is generated by \( 2n \) odd elements \( B^+_i \) and \( B^-_i \), for \( i \in \{1, \ldots, n\} \), satisfying the structural relations

\[
\{B^+_i, B^+_j\}, B^-_i] = (\epsilon - \xi)\delta_{i,l}B^+_j + (\epsilon - \eta)\delta_{j,l}B^-_i,
\]

for \( i, j, l \in \{1, \ldots, n\} \) and \( \eta, \epsilon, \xi \in \{+, -\} \), to be interpreted as \( \pm 1 \) in the algebraic relations. The brackets \( [\cdot, \cdot] \) and \( \{\cdot, \cdot\} \) in (2.1) denote commutators and anti-commutators respectively.

The Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{osp}(1|2n) \) has a basis consisting of the \( n \) commuting elements

\[
h_i = \frac{1}{2}\{B^+_i, B^-_i\},
\]

for \( i \in \{1, \ldots, n\} \). Letting \( \epsilon_i \), for \( i \in \{1, \ldots, n\} \), be the corresponding dual basis for \( \mathfrak{h}^* \), we are able to define the lowest weight modules we are interested in. Fixing \( \{\epsilon_1, \ldots, \epsilon_n\} \) as basis for \( \mathfrak{h}^* \), the notation \( (p_1, \ldots, p_n) \) will from now on be used for the weight \( \sum_{i=1}^n p_i\epsilon_i \in \mathfrak{h}^* \).
Definition 2.1. Given \( p \in \mathbb{N} \), let \( \mathcal{H}_n(p) \) be the irreducible lowest weight module of \( \mathfrak{osp}(1|2n) \) of lowest weight \((\frac{p}{2}, \ldots, \frac{p}{2})\) and with lowest weight vector \( |0\rangle \). The actions of the generators \( B_i^+ \) and \( B_i^- \), for \( i \in \{1, \ldots, n\} \), of \( \mathfrak{osp}(1|2n) \) on \( \mathcal{H}_n(p) \) are defined by relations
\[
B_i^- |0\rangle = 0, \quad \text{and} \quad \{ B_i^+, B_j^- \} |0\rangle = p \delta_{i,j} |0\rangle, \tag{2.3}
\]
for all \( i, j \in \{1, \ldots, n\} \).

These modules were originally introduced in the context of parastatistics with \( \mathcal{H}_n(p) \) as the Fock spaces of \( n \) parabosonic particles of order \( p \). Here they are usually referred to as paraboson Fock spaces \([11, 13]\). When \( p = 1 \), the parabosonic particles revert to usual bosonic particles, and \( \mathcal{H}_n(1) \) becomes the usual boson Fock space. From a different point of view \( \mathcal{H}_n(1) \) is the Hilbert space of the quantum harmonic oscillator and is for that reason also referred to as the oscillator representation \([24]\).

From now on we will consider \( n \) and \( p \) to be fixed positive integers. The treatment of the representation \( \mathcal{H}_n(p) \) will be carried out through a certain polynomial realization, the construction of which will take up the rest of this section.

The Clifford algebra \( \mathcal{C}_p \) is generated by \( p \) elements \( e_\alpha \), for \( \alpha \in \{1, \ldots, p\} \), satisfying
\[
\{ e_\alpha, e_\beta \} = 2 \delta_{\alpha,\beta}, \tag{2.4}
\]
for all \( \alpha, \beta \in \{1, \ldots, p\} \). Let \( \mathbb{C}[\mathbb{R}^{np}] \) denote the space of polynomials in \( np \) variables \( x_{i,\alpha} \), for \( i \in \{1, \ldots, n\} \) and \( \alpha \in \{1, \ldots, p\} \), with complex coefficients, and define
\[
\mathcal{A} := \mathbb{C}[\mathbb{R}^{np}] \otimes \mathcal{C}_p \tag{2.5}
\]
to be its Clifford algebra valued counterpart. Denoting the \( 2np \) odd generators of \( \mathfrak{osp}(1|2np) \) by \( B_{i,\alpha}^\pm \) we can consider \( \mathbb{C}[\mathbb{R}^{np}] \) as an \( \mathfrak{osp}(1|2np) \)-module with action
\[
B_{i,\alpha}^+ q(x) = x_{i,\alpha} q(x) \quad \text{and} \quad B_{i,\alpha}^- q(x) = \partial_{i,\alpha} q(x), \tag{2.6}
\]
for all \( q(x) \in \mathbb{C}[\mathbb{R}^{np}] \), where \( \partial_{i,\alpha} := \frac{\partial}{\partial x_{i,\alpha}} \). The relations
\[
[\partial_{i,\alpha}, x_{j,\beta}] = \delta_{i,j} \delta_{\alpha,\beta}, \tag{2.7}
\]
for \( i, j \in \{1, \ldots, n\} \) and \( \alpha, \beta \in \{1, \ldots, p\} \), imply the structural relations \((2.1)\) of \( \mathfrak{osp}(1|2np) \). In fact \( \mathbb{C}[\mathbb{R}^{np}] \sim \mathcal{H}_n(1) \) as an \( \mathfrak{osp}(1|2np) \) module. Similarly we can consider \( \mathcal{A} \) as an \( \mathfrak{osp}(1|2n) \)-module by defining operators
\[
X_i := \sum_{\alpha=1}^p x_{i,\alpha} e_\alpha \quad \text{and} \quad D_i := \sum_{\alpha=1}^p \partial_{i,\alpha} e_\alpha, \tag{2.8}
\]
and letting
\[
B_i^+(q(x) \otimes f) = X_i(q(x) \otimes f) = \sum_{\alpha=1}^p x_{i,\alpha} q(x) \otimes e_\alpha f, \tag{2.9}
\]
\[
B_i^-(q(x) \otimes f) = D_i(q(x) \otimes f) = \sum_{\alpha=1}^p \partial_{i,\alpha} q(x) \otimes e_\alpha f,
\]
for all \( i \in \{1, \ldots, n\} \), \( \alpha \in \{1, \ldots, p\} \), \( q(x) \in \mathbb{C}[\mathbb{R}^{np}] \) and \( f \in \mathcal{C}_p \). It is easily verified that \( \mathcal{A} \) is an \( \mathfrak{osp}(1|2n) \)-module by checking that the actions \((2.9)\) satisfy the relations \((2.1)\). The subsequent result, Proposition 2.3, tells us that \( \mathcal{A} \) has an irreducible submodule isomorphic to \( \mathcal{H}_n(p) \). The
shape of the operators in (2.8) is that of the Green’s ansatz [11] with the internal components here being realized using Clifford algebra elements. This type of realization of the Green’s ansatz dates back to [13, 12].

The module $A$ admits the Howe dual pair [5] ($\mathfrak{osp}(1|2n), G$), where $G = Pin(p)$ when $p$ is even and $G = Spin(p)$ when $p$ is odd. This gives a multiplicity free decomposition of $A$ into a direct sum of irreducible modules of $\mathfrak{osp}(1|2n) \times G$ of the form $H_{\mathfrak{osp}(\mu)} \otimes H_G(\nu)$ [3, 4, 26]. Here $H_{\mathfrak{osp}(\mu)}$ denotes an irreducible lowest weight module of $\mathfrak{osp}(1|2n)$ of lowest weight $\mu$, and $H_G(\nu)$ denotes an irreducible highest weight module of $G$ of highest weight $\nu$.

We are interested in exactly one of the irreducible modules in this decomposition, namely the one with the constant polynomial $1 \otimes I \in A$ as lowest weight vector. Here $I \in \mathcal{C}_p$ is the identity element in the Clifford algebra.

We shall from now on use the notation $1 := 1 \otimes I \in A$ and in general suppress the tensor products when dealing with elements of $A$.

\textbf{Definition 2.2.} Let $W_n(p)$ be the submodule of $A$ generated by the action of $\mathfrak{osp}(1|2n)$ on $1 \in A$.

\textbf{Proposition 2.3.} The module $W_n(p)$ is equivalent to the irreducible lowest weight $\mathfrak{osp}(1|2n)$-module $H_n(p)$.

\textit{Proof.} In the module $A$ the basis for the Cartan subalgebra of $\mathfrak{osp}(1|2n)$, as described in (2.2), is represented by operators

$$h_i \mapsto \frac{1}{2}\{X_i, D_i\} = \frac{p}{2} + \sum_{\alpha=1}^{p} x_{i,\alpha} \partial_{i,\alpha},$$

for all $i \in \{1, \ldots, p\}$. Since

$$D_i(1) = 0 \text{ and } \frac{1}{2}\{X_i, D_j\}(1) = \frac{p}{2} \delta_{i,j}(1),$$

for all $i, j \in \{1, \ldots, n\}$, it follows that $1$ is a lowest weight vector of weight $(\frac{p}{2}, \ldots, \frac{p}{2})$. As mentioned above, the module $A$ decomposes into a direct sum of irreducible lowest weight modules of $\mathfrak{osp}(1|2n)$ as a result of admitting the Howe dual pair $(\mathfrak{osp}(1|2n), G)$. It follows that $W_n(p)$ must be one of the components in the decomposition and hence be irreducible. It can thus be concluded that $W_n(p)$ is equivalent to $H_n(p)$. \hfill \square

In the following sections we will construct a basis for $W_n(p)$ together with formulas for the action of $\mathfrak{osp}(1|2n)$ on the basis.

\section{Irreducibility of $W_n(p)$}

In this section a different path to proving the irreducibility of $W_n(p)$ is presented. The benefit of this approach is that it more clearly motivates the basis we will construct for $W_n(p)$. To do so yet another $\mathfrak{osp}(1|2n)$ module must be constructed following the procedure described in the paper [20]. Consider first the parabolic subalgebra of $\mathfrak{osp}(1|2n)$ given by

$$\mathfrak{P} = \text{span}\{\{B_i^+, B_j^\pm\}, B_i^-, \{B_i^-, B_j^-\} : i, j \in \{1, \ldots, n\}\}. \quad (3.1)$$

Let $|0\rangle$ describe the lowest weight vector of the module $H_n(p)$, then the action (2.3) of $\mathfrak{P}$ on $|0\rangle$ generates a one dimensional module, which we will denote $\mathbb{C}[0]$. From this module we can induce a module of $\mathfrak{osp}(1|2n)$ by the following procedure. Let $U(\mathfrak{osp}(1|2n))$ be the universal enveloping algebra of $\mathfrak{osp}(1|2n)$. Then

$$\mathcal{B} = U(\mathfrak{osp}(1|2n)) \otimes \mathbb{C}[0], \quad (3.2)$$
becomes a module of \( \mathfrak{osp}(1|2n) \) with the action given by
\[
B_i^+(g \otimes |0\rangle) = (B_i^+ g) \otimes |0\rangle,
\]
(3.3)
for all \( i \in \{1, \ldots, n\} \) and \( g \in U(\mathfrak{osp}(1|2n)) \). Let \( \mathcal{M} \) denote the submodule of \( \mathcal{B} \) generated by the vectors
\[
(gh) \otimes |0\rangle - g \otimes h|0\rangle,
\]
(3.4)
for all \( g \in U(\mathfrak{osp}(1|2n)) \) and \( h \in \mathfrak{P} \). We can now construct the induced module of \( \mathfrak{osp}(1|2n) \).

**Definition 3.1.** The induced module of \( \mathfrak{osp}(1|2n) \) relative to \( \mathfrak{P} \) is the quotient module
\[
\overline{V}_n(p) = \text{Ind}_{\mathfrak{P}}^{\mathfrak{osp}(1|2n)} \mathbb{C}|0\rangle := \mathcal{B}/\mathcal{M}.
\]
(3.5)

**Lemma 3.2.** Let \( M_n(p) \) be the maximal non-trivial submodule of \( \overline{V}_n(p) \), then the quotient module
\[
V_n(p) := \overline{V}_n(p)/M_n(p)
\]
(3.6)
is equivalent to \( \mathcal{H}_n(p) \).

**Proof.** First, by observing that \( \overline{V}_n(p) \) is generated by the action of \( \mathfrak{osp}(1|2n) \) on \( |0\rangle \), it follows that the quotient module \( V_n(p) \) is generated by the corresponding action of \( \mathfrak{osp}(1|2n) \) on the equivalence class of \( |0\rangle \), also denoted by \( |0\rangle \). Second, choosing \( M_n(p) \) maximal insures that \( V_n(p) \) is irreducible. Finally, \( |0\rangle \in V_n(p) \) is a lowest weight vector of weight \( (\frac{p}{2}, \ldots, \frac{p}{2}) \), so it follows that \( V_n(p) \) is equivalent to \( \mathcal{H}_n(p) \).

The module \( \overline{V}_n(p) \) allows for the construction of a canonical basis through use of the Poincaré-Birkhoff-Witt theorem for Lie superalgebras [15, 9].

**Proposition 3.3.** The vectors
\[
(B_i^+)_{k_1} \cdots (B_n^+)_{k_n} \prod_{1 \leq i \neq j \leq n} \{B_i^+, B_j^+\}_{k_{i,j}} |0\rangle,
\]
(3.7)
for \( k_1, \ldots, k_n, k_{1,2}, k_{1,3}, \ldots, k_{n-1,n} \in \mathbb{N} \), form a basis for the module \( \overline{V}_n(p) \).

Let \( \mathfrak{osp}(1|2n)^+ \) denote the subalgebra of \( \mathfrak{osp}(1|2n) \) generated by \( B_i^+ \), for \( i \in \{1, \ldots, n\} \),
\[
\mathfrak{osp}(1|2n)^+ = \text{span} \left\{ B_i^+, \{B_i^+, B_j^+\} : i, j \in \{1, \ldots, n\} \right\}.
\]
(3.8)
As vector spaces \( \mathfrak{osp}(1|2n) = \mathfrak{P} \oplus \mathfrak{osp}(1|2n)^+ \). From Proposition 3.3 we get the following corollary which will be useful later on.

**Corollary 3.4.** The map
\[
\Phi_p : U(\mathfrak{osp}(1|2n)^+) \rightarrow \overline{V}_n(p), \quad B \mapsto B|0\rangle,
\]
(3.9)
is an isomorphism of vector spaces.

The basis described in Proposition 3.3 does unfortunately not contain a subbasis for the module \( M_n(p) \subset \overline{V}_n(p) \), unless \( M_n(p) = \{0\} \). Therefore this basis does not translate directly to bases of \( V_n(p) \) and \( W_n(p) \). The basis is still useful though, as it allows us to define the following surjective intertwining operator. Let
\[
\Psi_p : \overline{V}_n(p) \rightarrow W_n(p),
\]
(3.10)
be the operator acting on the basis for Proposition 3.3 as follows

\[
\Psi_p \left( \left( B_1^+ \right)^{k_1} \cdots \left( B_n^+ \right)^{k_n} \prod_{1 \leq i < j \leq n} \{ B_i^+, B_j^+ \}^{k_{i,j}} \{ 0 \} \right)
\]

\[
:= X_1^{k_1} \cdots X_n^{k_n} \prod_{1 \leq i < j \leq n} \{ X_i, X_j \}^{k_{i,j}} \langle 0 \rangle
\]

for all \( k_1, \ldots, k_n, k_{1,2}, k_{1,3}, \ldots, k_{n-1,n} \in \mathbb{N} \).

A clear path to proving the irreducibility of \( W_n(p) \) can now be seen. By showing that \( M_n(p) \) coincides with the kernel of \( \Psi_p \), a module isomorphism \( \tilde{\Psi}_p : V_n(p) \to W_n(p) \) can be induced from \( \Psi_p \).

We shall now introduce some notation that helps to characterize the submodule \( M_n(p) \) of \( V_n(p) \).

For each \( k \in \mathbb{N} \), consider the index set

\[
\mathcal{I}(k) := \left\{ (i_1, \ldots, i_k) \in \{ 1, \ldots, n \}^k : i_1 < \cdots < i_k \right\}.
\]

(3.12)

To each \( I = (i_1, \ldots, i_k) \in \mathcal{I}(k) \) we assign an operator defined as

\[
X_I := \sum_{\sigma \in S_k} \sgn(\sigma) X_{i_{\sigma(1)}} \cdots X_{i_{\sigma(k)}},
\]

(3.13)

which is an antisymmetric sum over the permutations \( \sigma \) of the symmetric group \( S_k \). A short calculation allows us to rewrite this as a sum of monomials,

\[
X_I = k! \sum_{\alpha_1, \ldots, \alpha_k \in \{ 1, \ldots, p \}, \alpha_i \neq \alpha_j \text{ with } i \neq j} x_{i_1, \alpha_1} \cdots x_{i_k, \alpha_k} e_{\alpha_1} \cdots e_{\alpha_k},
\]

(3.14)

with the sum being over all sets of \( k \) mutually distinct integers in \( \{ 1, \ldots, p \} \). Clearly, if \( k > p \), then \( X_I = 0 \), for each \( I \in \mathcal{I}(k) \).

We are now almost ready to give our alternative proof for the irreducibility of \( W_n(p) \), needing only the following two lemmas.

**Lemma 3.5.** The module \( M_n(p) \) is generated by the action of \( \mathfrak{osp}(1|2n) \) on \( \binom{n}{p+1} \) vectors \( w_I \), for \( I \in \mathcal{I}(p+1) \), with corresponding weights

\[
\epsilon_I := \frac{p}{2} \sum_{i=1}^n \epsilon_i + \sum_{k=1}^{p+1} \epsilon_{i_k},
\]

(3.15)

for all \( I \in \mathcal{I}(p+1) \).

**Proof.** In the paper [20] a Gelfand-Zetlin basis for the module \( \overline{V}_n(p) \) is constructed. That basis contains a subset forming a basis for the submodule \( M_n(p) \subset \overline{V}_n(p) \). From the way these basis vectors are constructed it follows that \( M_n(p) \) is generated by the action of \( \mathfrak{osp}(1|2n) \) on the Gelfand-Zetlin basis vectors with top row

\[
(1, 1, \ldots, 1, 0, 0, \ldots, 0),
\]

(3.16)

in their Gelfand-Zetlin labelling. There are exactly \( \binom{n}{p+1} \) such basis vectors \( w_I \) with weights \( \epsilon_I \), for \( I \in \mathcal{I}(p+1) \). \( \square \)
It is clear from Lemma 3.5 that $M_n(p) = \{0\}$ if and only if $p \geq n$. Thus the vectors

$$X_1^{k_1} \cdots X_n^{k_n} \prod_{1 \leq i < j \leq n} \{X_i, X_j\}^{k_{i,j}}(1),$$

for all $k_1, \ldots, k_n, k_{1,2}, k_{1,3}, \ldots, k_{n-1,n} \in \mathbb{N}$, form a basis for $W_n(p)$ if and only if $p \geq n$. This is in fact closely related to a more general result [6],[18, Theorem 1.1]. In the context of the present paper the result can be stated as follows. If $p \geq 2n$ and $W$ is any irreducible factor in the decomposition of $A$, with lowest weight vectors $v_1 \ldots v_r$, then the following vectors

$$X_1^{k_1} \cdots X_n^{k_n} \prod_{1 \leq i < j \leq n} \{X_i, X_j\}^{k_{i,j}}(v_s),$$

for all $k_1, \ldots, k_n, k_{1,2}, k_{1,3}, \ldots, k_{n-1,n} \in \mathbb{N}$ and $s \in \{1, \ldots, r\}$, form a basis for $W$.

**Lemma 3.6.** The map $\Psi_p$ is a surjective $\mathfrak{osp}(1|2n)$-module homomorphism, with

$$\ker \Psi_p = M_n(p).$$

*Proof.* By construction $\Psi_p$ is a surjective $\mathfrak{osp}(1|2n)$-module homomorphism, so it suffices to prove (3.19). The maximality of $M_n(p)$ implies that $\ker \Psi_p \subset M_n(p)$. Let

$$\nu_I = \sum_{\sigma \in S_{p+1}} \text{sgn}(\sigma)B_{s_{\sigma(1)}}^+ \cdots B_{s_{\sigma(p+1)}}^+ |0\rangle,$$

for all $I = (i_1, \ldots, i_{p+1}) \in I(p+1)$. For each $I \in I(p+1)$,

$$\Psi_p(\nu_I) = \sum_{\sigma \in S_{p+1}} \text{sgn}(\sigma)X_{s_{\sigma(1)}} \cdots X_{s_{\sigma(p+1)}}(1) = 0,$$

so $\nu_I \in \ker \Psi_p$, for all $I \in I(p+1)$. Hence it is sufficient to prove that $\{\nu_I : I \in I(p+1)\}$ is a generating set for $M_n(p)$.

For any $I \in I(p+1)$, simple calculations show that in the expansion of $\nu_I$ in the basis of Proposition 3.3 the vector

$$B_{i_1}^+ \cdots B_{p+1}^+ |0\rangle,$$

has coefficient $(p+1)!$. Hence $\nu_I \neq 0$, for all $I \in I(p+1)$. Furthermore, the $\binom{n}{p+1}$ vectors $\nu_I$, for $I \in I(p+1)$, satisfy

$$\frac{1}{2}\{B_{i}^+ B_{i}^-\} \nu_I = \begin{cases} (1 + \frac{n}{2})\nu_I, & \text{if } i \in \{i_1, \ldots, i_{p+1}\}, \\ \frac{n}{2}\nu_I, & \text{if } i \notin \{i_1, \ldots, i_{p+1}\}, \end{cases}$$

for all $i \in \{1, \ldots, n\}$. The weight of $\nu_I$ therefore coincides with the weight $\epsilon_I$ defined in Lemma 3.5, for all $I \in I(p+1)$. It follows that, for each $I \in I(p+1)$, the vector $\nu_I$ must be proportional to the generator $w_I$ of $M_n(p)$ described in Lemma 3.5, thus proving that

$$\ker \Psi_p = M_n(p).$$

□

We can now give an alternative proof of the irreducibility of $W_n(p)$, or equivalently an alternative proof of Proposition 2.3.

*Alternative proof of Proposition 2.3.* It follows from Lemma 3.6 that $\Psi_p$ induces a module isomorphism

$$\tilde{\Psi}_p : V_n(p) \to W_n(p),$$

thus proving that $W_n(p)$ is equivalent to $\mathcal{H}_n(p)$, when applying the result of Lemma 3.2. □
4 Tableau Vectors in $W_n(p)$ and $\nabla_n(p)$

The main result of this paper is the explicit construction of bases for the modules $W_n(p)$ and $\nabla_n(p)$ which are related by the map $\Psi_p$. In this section we will specify exactly which properties we expect of bases we are looking for, see Proposition 4.1. We will furthermore construct the sets of vectors that will be our candidates for the desired bases, see Definition 4.2. Proving that they are indeed bases for the modules is postponed to the next section. As touched upon in the proof of Lemma 3.5, bases for the modules $V_n(p)$ and $\nabla_n(p)$, which are related by the quotient map

$$\nabla_n(p) \rightarrow \nabla_n(p)/M_n(p) = V_n(p),$$

are found and studied in the paper [20]. These basis vectors are labelled by a Gelfand-Zetlin pattern relative to the $\mathfrak{gl}(n)$ subalgebra of $\mathfrak{osp}(1|2n)$. The new basis constructed in this section, on the contrary, will consist of explicit polynomials in $W_n(p)$.

Partitions, Young tableaux and symmetric functions will be used throughout this paper. In the interest of consistency, we will be using the notation established in Macdonald [22]. Let the set of all partitions be denoted by $\mathcal{P}$, and let $\ell(\lambda)$ be the length of the partition $\lambda \in \mathcal{P}$.

From [20, Theorem 7] we have

$$\text{char } V_n(p) = (t_1 \cdots t_n)^{\frac{p}{2}} \sum_{\lambda \in \mathcal{P}} s_\lambda(t_1, \ldots, t_n),$$

with $s_\lambda$ denoting the Schur function indexed by $\lambda$, and $t_i$ denoting the formal exponential $e^{t_i}$. Introducing the Kostka numbers enumerating semistandard (s.s.) Young tableaux as

$$K_{\lambda,\mu} := \#\{\text{Semistandard Young tableaux of shape } \lambda \text{ and weight } \mu\},$$

for $\lambda \in \mathcal{P}$ and $\mu \in \mathbb{N}_0^n$, we can expand the Schur functions in terms of monomials

$$\text{char } V_n(p) = (t_1 \cdots t_n)^{\frac{p}{2}} \sum_{\lambda \in \mathcal{P}} \sum_{\mu \in \mathbb{N}_0^n} K_{\lambda,\mu} t_1^{\mu_1} \cdots t_n^{\mu_n}.$$ 

The weight lattice of $V_n(p)$ is

$$\left\{ \mu + \frac{p}{2} := \mu + \left( \frac{p}{2}, \ldots, \frac{p}{2} \right) : \mu \in \mathbb{N}_0^n \right\}.$$ 

This implies that $V_n(p)$ can be written as direct sum of weight spaces corresponding to weights $\mu + \frac{p}{2}$, for $\mu \in \mathbb{N}_0^n$. Denote the weight space corresponding to the weight $\mu + \frac{p}{2}$ by

$$V_n(p)_{\mu + \frac{p}{2}} = \{ v \in V_n(p) : h_i v = (\mu_i + \frac{p}{2}) v, \text{ for } i \in \{1, \ldots, n\} \}.$$ 

The defining formula for characters is,

$$\text{char } V_n(p) = (t_1 \cdots t_n)^{\frac{p}{2}} \sum_{\mu \in \mathbb{N}_0^n} \dim V_n(p)_{\mu + \frac{p}{2}} t_1^{\mu_1} \cdots t_n^{\mu_n}.$$ 

The formulas (4.4) and (4.7) imply the following dimension formula for the weight spaces of $V_n(p)$

$$\dim V_n(p)_{\mu + \frac{p}{2}} = \sum_{\lambda \in \mathcal{P}} K_{\lambda,\mu},$$ 

for all $\mu \in \mathbb{N}_0^n$. We note here that this gives the dimensions of all weight spaces of $V_n(p)$.
The character formula for \( W_n(p) \) obtained in the papers \([21],[20, \text{Theorem 7}]\) has the form

\[
\text{char} \ W_n(p) = \text{char} \ V_n(p) = (t_1 \cdots t_n)^\frac{p}{2} \sum_{\lambda \in \mathcal{P}, \ell(\lambda) \leq p} s_\lambda(t_1, \ldots, t_n). \tag{4.9}
\]

Denote the weight spaces of \( W_n(p) \) by

\[
W_{n}(p)_{\mu + \frac{p}{2}} = \left\{ w \in W_n(p) : h_i w = (\mu_i + \frac{p}{2})w, \text{ for } i \in \{1, \ldots, n\} \right\}, \tag{4.10}
\]

for all \( \mu \in \mathbb{N}_0^n \). Following the same arguments, which lead to (4.8), the dimensions of these weight spaces are

\[
\dim W_{n}(p)_{\mu + \frac{p}{2}} = \sum_{\lambda \in \mathcal{P}, \ell(\lambda) \leq p} K_{\lambda, \mu}, \tag{4.11}
\]

for all \( \mu \in \mathbb{N}_0^n \). Let \( \mathbb{Y}_n \) denote the set of semistandard Young tableaux with weights in \( \mathbb{N}_0^n \), that is with numbers 1, \ldots, \( n \) as entries. Let \( \lambda_A \in \mathcal{P} \) and \( \mu_A \in \mathbb{N}_0^n \) denote the shape and weight of \( A \in \mathbb{Y}_n \) respectively. Finally, let

\[
\mathbb{Y}_n(p) = \{ A \in \mathbb{Y}_n : \ell(\lambda_A) \leq p \} \tag{4.12}
\]

be the set of s.s. Young tableaux in \( \mathbb{Y}_n \) with at most \( p \) rows. The dimension formulas (4.8) and (4.11) together with Lemma 3.6 imply the following result on the existence of related bases for \( \overline{V}_n(p) \) and \( W_n(p) \).

**Proposition 4.1.** There exist bases for \( \overline{V}_n(p) \) and \( W_n(p) \) denoted by

\[
\{ \bar{u}_A(p) \in \overline{V}_n(p) : A \in \mathbb{Y}_n \} \tag{4.13}
\]

and

\[
\{ u_A(p) \in W_n(p) : A \in \mathbb{Y}_n(p) \} \tag{4.14}
\]

with the following properties. Given \( A \in \mathbb{Y}_n \),

1. \( \Psi_p(\bar{u}_A(p)) = \begin{cases} u_A(p), & \text{if } A \in \mathbb{Y}_n(p), \\ 0, & \text{if } A \notin \mathbb{Y}_n(p). \end{cases} \)

2. The weights of \( \bar{u}_A(p) \) and \( u_A(p) \) are both \( \mu_A + \frac{p}{2} \).

The existence of such bases is not significant in itself unless we can find a relatively simple description of the vectors in terms of the corresponding tableaux. The construction of such a basis will begin with some introductory notation inspired by (3.13). Given \( k \in \mathbb{N} \) and \( I = (i_1, \ldots, i_k) \in \mathcal{I}(k) \) we define

\[
B_I^+ := \sum_{\sigma \in S_k} \text{sgn}(\sigma)B_{i_{\sigma(1)}}^+ \cdots B_{i_{\sigma(k)}}^+ \in U(\mathfrak{osp}(1|2n)). \tag{4.15}
\]

For the construction of the basis vectors for \( W_n(p) \) and \( \overline{V}_n(p) \) the idea is to interpret each column of a given s.s. Young tableau \( A \in \mathbb{Y}_n \) as an operator of the form (4.15), with \( k \) varying over the heights of the columns, and then let each column-operator act on the lowest weight vector with the leftmost column acting first.

To formalize this we need some notation. Given a partition \( \lambda \in \mathcal{P} \), we denote the conjugate partition by \( \lambda' \). Let \( a_l \) denote the \( l \)'th column, counted from left to right, of \( A \in \mathbb{Y}_n \), for all \( l \in \{1, \ldots, (\lambda_A)_1\} \). Since the columns of \( A \) are strictly increasing downwards we can naturally consider \( a_l \) as an element of \( \mathcal{I}((\lambda_A)_l') \), for all \( l \in \{1, \ldots, (\lambda_A)_1\} \), noting here that the amount of columns in \( A \) is indeed \( (\lambda_A)_1 \).
Definition 4.2. Given $A \in \mathbb{Y}_n$ we let $m = (\lambda_A)_1$ and define

$$v_A(p) := B_{a_1}^+ \cdots B_{a_1}^+(0)$$

and

$$\omega_A(p) := \Psi_p(v_A(p)) = X_{a_1} \cdots X_{a_1}(1).$$

The next section is dedicated to proving that these sets of vectors are bases with the properties described in Proposition 4.1. In order to illustrate the above-mentioned concepts, consider an example $n = 5$, $\lambda_A = (4,3,1)$, $\mu_A = (2,2,3,1,0)$ and

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 3 & 4 \end{bmatrix}.$$  \hspace{1cm} (4.18)

The columns of $A$ are then $a_1 = (1,2,4)$, $a_2 = (1,3)$, $a_3 = (2,3)$ and $a_4 = (3)$. In that case

$$v_A(p) = B_{2}^+ B_{(2,3)}^+ B_{(1,3)}^+ B_{(1,2,4)}^+(0)$$

and

$$\omega_A(p) = X_3 X_{(2,3)} X_{(1,3)} X_{(1,2,4)}(1),$$

where $B_{a_l}^+$ and $X_{a_l}$, for $l \in \{1, \ldots , 4\}$, are defined in (4.15) and (3.13) respectively. In the case $p \geq 3$, $v_A(p) \neq 0$, $\omega_A(p) \neq 0$. On the other hand, if $p < 3$, then $v_A(p) \neq 0$ and $\omega_A(p) = 0$.

5 Bases for $W_n(p)$ and $\mathcal{V}_n(p)$

This section is dedicated to proving that the sets $\{v_A(p) : A \in \mathbb{Y}_n\}$ and $\{\omega_A(p) : A \in \mathbb{Y}_n(p)\}$, defined in Definition 4.2, indeed form bases for $\mathcal{V}_n(p)$ and $W_n(p)$ satisfying the conditions of Proposition 4.1. This is the content of Theorem 5.8.

The overarching strategy will be to construct a total ordering of $\mathbb{Y}_n(p)$ with the following property

$$\omega_A(p) \notin \text{span}\{\omega_B(p) : B \in \mathbb{Y}_n(p), B < A\}. \hspace{1cm} (5.1)$$

By (4.11) the weight spaces of $W_n(p)$ are finite dimensional, so the existence of such an ordering will prove the linear independence of the vectors

$$\{\omega_A(p) : A \in \mathbb{Y}_n(p)\} \hspace{1cm} (5.2)$$

and that they form a basis for $W_n(p)$.

Definition 5.1. Given $A \in \mathbb{Y}_n$ and $k \in \{1, \ldots , n\}$ the $k$th subtableau of $A$ is defined to be the tableau $A^k \in \mathbb{Y}_n$ obtained by truncating $A$ to only the entries containing the numbers $1, \ldots , k$.

For example, we have

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & 3 & 4 \end{bmatrix} \Rightarrow A^4 = A, \hspace{0.5cm} A^3 = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 \end{bmatrix}, \hspace{0.5cm} A^2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \text{ and } A^1 = \begin{bmatrix} 1 \end{bmatrix}. \hspace{1cm} (5.3)$$

To define the total ordering of $\mathbb{Y}_n$ we endow both $\mathbb{N}_0$ and $\mathcal{P}$ with the graded lexicographic ordering, both denoted by $<$. The definition of these orderings are given in the Appendix A.

Definition 5.2. Given $A, B \in \mathbb{Y}_n$, then we write $A < B$ if $\mu_A < \mu_B$ in $\mathbb{N}_0$ or if $\mu_A = \mu_B$ and there exists $k \in \{1, \ldots , n\}$ such that, for all $l < k$,

$$\lambda_{A^l} = \lambda_{B^l} \text{ and } \lambda_{A^l} < \lambda_{B^l} \text{ in } \mathcal{P}. \hspace{1cm} (5.4)$$
The relation $<$ on $\mathbb{V}_n$ defined above is in fact a total order. This follows from the fact that the graded lexicographic order gives a total ordering of both $\mathbb{N}_0^n$ and $\mathcal{P}$. The total ordering of $\mathbb{V}_n$ defined above is inherited by $\mathbb{V}_n(p)$.

To get a feel for this ordering of $\mathbb{V}_n$ consider the following example. Let $n = 5$, and consider the ascending chain of all 13 s.s. Young tableaux of weight $\mu = (2, 1, 1, 1, 0)$:

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 5 & 4 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 4 & 3 & 5 \\
1 & 2 & 4 & 5 & 3 \\
1 & 2 & 3 & 5 & 4 \\
1 & 2 & 4 & 3 & 5 \\
1 & 2 & 4 & 5 & 3 \\
1 & 2 & 5 & 4 & 3 \\
1 & 2 & 3 & 5 & 4 \\
1 & 2 & 4 & 5 & 3 \\
1 & 2 & 4 & 3 & 5 \\
1 & 2 & 5 & 3 & 4 \\
\end{array}
\]

To see the connection between this ordering and the prospective basis for $W_n(p)$ we need to consider the expansion of the vectors $\omega_A(p)$ into monomials. To do so a new class of tableaux are needed. Let

\[
\mathbb{T}(\lambda, p) := \left\{ \text{Fillings of the Young diagram of shape } \lambda \in \mathcal{P} \text{ with numbers } 1, \ldots, p \text{ occuring at most once in each column.} \right\}. \quad (5.5)
\]

We will refer to the tableaux in $\mathbb{T}(\lambda, p)$ as column distinct (c.d.) Young tableaux. Some examples of these are:

\[
\begin{array}{ccc}
2 & 1 & 2 \\
3 & 3 & 1 \\
1 & 4 & 3 \\
\end{array}, \quad \begin{array}{ccc}
3 & 1 & 2 \\
2 & 4 & 3 \\
1 & 3 & 2 \\
\end{array}, \quad \begin{array}{ccc}
3 & 1 & 3 \\
2 & 4 & 1 \\
1 & 3 & 2 \\
\end{array}. \quad (5.6)
\]

From now on we shall adopt the slightly abusive notation of writing $\lambda$ both for a partition in $\mathcal{P}$ and for the set of coordinates in the corresponding Young diagram. That is,

\[
\lambda = \left\{ (k, l) : l \in \{1, \ldots, \lambda_1\} \text{ and } k \in \{1, \ldots, \lambda'_1\} \right\}, \quad (5.7)
\]

for all $\lambda \in \mathcal{P}$. With the notion of c.d. Young tableaux the following monomials in $\mathbb{C}[\mathbb{R}^{np}]$ and $\mathcal{C}_p$ can be defined. Given a tableau $C \in \mathbb{T}(\lambda, p)$ and $A \in \mathbb{V}_n(p)$ of shape $\lambda = \lambda_A \in \mathcal{P}$. We denote the entries of $A$ and $C$ by $a_{k,l}$ and $c_{k,l}$ respectively, for all $(k,l) \in \lambda$. Letting $m = \lambda_1$ be the number of columns in $A$ and $C$ we define monomials

\[
e_C := (e_{c_{1,m}} \cdots e_{c_{\lambda_m,m}}) \cdots (e_{c_{1,1}} \cdots e_{c_{\lambda'_1,1}}) \in \mathcal{C}_p \quad (5.8)
\]

and

\[
x_{A,C} := (x_{a_{1,m},c_{1,m}} \cdots x_{a_{\lambda_m,m},c_{\lambda_m,m}}) \cdots (x_{a_{1,1},c_{1,1}} \cdots x_{a_{\lambda'_1,1},c_{\lambda'_1,1}}) = \prod_{(k,l) \in \lambda} x_{a_{k,l},c_{k,l}} \in \mathbb{C}[\mathbb{R}^{np}] \quad (5.9)
\]

For example, consider the case $n = 4$, $p = 4$, and

\[
A = \begin{array}{cc}
1 & 2 \\
2 & 3 \\
3 & 4 \\
\end{array} \quad \text{and} \quad C = \begin{array}{cc}
2 & 3 \\
1 & 4 \\
\end{array}. \quad (5.10)
\]

The monomials defined in (5.8) and (5.9) will then take the form

\[
e_C = e_3e_5e_4e_2e_3e_1 = e_2e_4 \quad (5.11)
\]

and

\[
x_{A,C} = x_2x_3x_4x_1x_2x_3x_4 = x_1x_2x_3x_4. \quad (5.12)
\]

To each partition $\lambda \in \mathcal{P}$ we associate the following factorial:

\[
\lambda! := \lambda'_1! \cdots \lambda'_\lambda! \in \mathbb{N}. \quad (5.13)
\]
Proposition 5.3. For all $A \in \mathbb{Y}_n(p)$, we have
\[
\omega_A(p) = \lambda_A! \sum_{C \in \mathbb{T}(\lambda_A, p)} x_{A,C} e_C. \tag{5.14}
\]

Proof. Let $\lambda = \lambda_A \in \mathcal{P}$ be the shape of $A \in \mathbb{Y}_n(p)$, and write $m = \lambda_1$. Denoting the columns of $A$ by $a_1, \ldots, a_m$ and the entries of $A$ by $a_{k,l}$, for all $(k,l) \in \lambda$, the statement is proved by the following calculation using Definition 4.2 together with (3.14), (5.8) and (5.9):
\[
\begin{align*}
\omega_A(p) &= X_{a_m} \cdots X_{a_1}(1) \\
&= \left(\lambda'_m! \sum_{c_{1,m}, \ldots, c_{\lambda'_m,m} \in \{1, \ldots, p\}, \ c_{i,m} \neq e_{j,m} \text{ with } i \neq j} x_{a_1,m,c_{1,m}} \cdots x_{a_{\lambda'_m,m},c_{\lambda'_m,m}} e_{c_{1,m}} \cdots e_{c_{\lambda'_m,m}} \right) \times \\
&\quad \times \left(\lambda'_1! \sum_{c_{1,1}, \ldots, c_{\lambda'_1,1} \in \{1, \ldots, p\}, \ c_{i,1} \neq e_{j,1} \text{ with } i \neq j} x_{a_{1,1},c_{1,1}} \cdots x_{a_{\lambda'_1,1},c_{\lambda'_1,1}} e_{c_{1,1}} \cdots e_{c_{\lambda'_1,1}} \right) \\
&= \lambda! \sum_{c_{1,m}, \ldots, c_{\lambda'_m,m} \in \{1, \ldots, p\}, \ c_{i,m} \neq e_{j,m} \text{ with } i \neq j} \cdots \sum_{c_{1,1}, \ldots, c_{\lambda'_1,1} \in \{1, \ldots, p\}, \ c_{i,1} \neq e_{j,1} \text{ with } i \neq j} (x_{a_1,m,c_{1,m}} \cdots x_{a_{\lambda'_m,m},c_{\lambda'_m,m}}) \cdots (x_{a_{1,1},c_{1,1}} \cdots x_{a_{\lambda'_1,1},c_{\lambda'_1,1}}) (e_{c_{1,m}} \cdots e_{c_{\lambda'_m,m}}) \cdots (e_{c_{1,1}} \cdots e_{c_{\lambda'_1,1}}) \\
&= \lambda! \sum_{C \in \mathbb{T}(\lambda_A, p)} x_{A,C} e_C. \tag{5.15}
\end{align*}
\]
Here $C \in \mathbb{T}(\lambda, p)$ is the c.d. Young tableau with entries $c_{k,l}$, for all $(k,l) \in \lambda$. \hfill \Box

Consider as an example $n = 2, p = 2$ and
\[
A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}. \tag{5.16}
\]

Definition 4.2 gives
\[
\omega_A(p) = X_2X_1X_{1,2} = 2!(x_{2,1}e_1 + x_{2,2}e_2)(x_{1,1}e_1 + x_{1,2}e_2)(x_{1,1}x_{2,2}e_1e_2 + x_{1,2}x_{2,1}e_2e_1). \tag{5.17}
\]

On the other hand Proposition 5.3 gives
\[
\begin{align*}
\omega_A(p) &= 2x \left( \begin{array}{c} \varepsilon_1, \varepsilon_1 \\ \varepsilon_2, \varepsilon_2 \end{array} \right) e_1e_2 + 2x \left( \begin{array}{c} \varepsilon_1, \varepsilon_2 \\ \varepsilon_1, \varepsilon_1 \end{array} \right) e_1e_2 + 2x \left( \begin{array}{c} \varepsilon_2, \varepsilon_1 \\ \varepsilon_2, \varepsilon_2 \end{array} \right) e_1e_2 \\
&\quad + 2x \left( \begin{array}{c} \varepsilon_1, \varepsilon_2 \\ \varepsilon_2, \varepsilon_1 \end{array} \right) e_2e_1 + 2x \left( \begin{array}{c} \varepsilon_1, \varepsilon_1 \\ \varepsilon_2, \varepsilon_2 \end{array} \right) e_2e_1 + 2x \left( \begin{array}{c} \varepsilon_2, \varepsilon_2 \\ \varepsilon_1, \varepsilon_1 \end{array} \right) e_2e_1 \\
&\quad + 2x \left( \begin{array}{c} \varepsilon_2, \varepsilon_1 \\ \varepsilon_1, \varepsilon_2 \end{array} \right) e_2e_1 \tag{5.18}
\end{align*}
\]

Both formulas give the expansion
\[
\omega_A(p) = -4x_{1,1}x_{1,2}x_{2,1}x_{2,2} - 2x_{1,1}x_{1,2}x_{2,1}x_{2,2}e_1e_2 - 2x_{1,2}^2x_{2,1}x_{2,2}e_1e_2 - 2x_{1,1}^2x_{2,1}x_{2,2}e_1e_2 \\
+ 2x_{1,1}x_{1,2}x_{2,2}e_1e_2 + 2x_{1,2}^2x_{2,2} + 2x_{1,1}^2x_{2,2}. \tag{5.19}
\]

When looking at an expansion such as the one in Proposition 5.3, it is natural to ask the following questions. Given $A \in \mathbb{Y}_n(p)$ and $C \in \mathbb{T}(\lambda_A, p)$, does there exist $C' \in \mathbb{T}(\lambda_A, p)$ with $C \neq C'$ such that
\[
x_{A,C} e_C = \pm x_{A,C'} e_{C'}. \tag{5.20}
\]
If so, which \( C' \in T(\lambda_A,p) \) has this property. As is demonstrated by the following example, the answer to the first question is in some instances yes, though we will shortly describe a class of c.d. Young tableaux for which the answer is always no.

\[
x(\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}) \delta_{12} = x_{12} x_{21} x_{11} x_{22} = x(\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}) \delta_{12} = x(\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}) \delta_{12}
\]

(5.21)

The second question will be dealt with in more detail in Section 6. However we will make the following preliminary considerations here. For each non-negative integer \( n \times p \) matrix \( \gamma \in M_{n,p}(\mathbb{N}_0) \) we denote the column and row sums as follows

\[
\mu_\gamma := \left( \sum_{\alpha=1}^{p} \gamma_{1,\alpha}, \ldots, \sum_{\alpha=1}^{p} \gamma_{n,\alpha} \right) \quad \text{and} \quad \eta_\gamma := \left( \sum_{i=1}^{n} \gamma_{1,i}, \ldots, \sum_{i=1}^{n} \gamma_{i,p} \right).
\]

(5.22)

Define also the following monomials

\[
x^\gamma := \prod_{i=1}^{n} \prod_{\alpha=1}^{p} x_{i,\alpha}^{\gamma_{i,\alpha}} \in \mathbb{C}[\mathbb{R}^{np}] \quad \text{and} \quad e^{\eta_\gamma} := (\epsilon_1^{(1)} \cdot \ldots \cdot \epsilon_p^{(p)}) \in \mathbb{C} \ell_p.
\]

(5.23)

The vector space \( \mathcal{A} \) admits a canonical inner product given by

\[
\langle x^\gamma e_J, x'^\gamma e_{J'} \rangle := \delta_{\mu_\gamma \mu_{\gamma'}} \delta_{J,J'},
\]

(5.24)

for all \( \gamma, \gamma' \in M_{n,p}(\mathbb{N}_0), k, k' \leq p, J \in \mathcal{I}(k) \) and \( J' \in \mathcal{I}(k') \). Note here that the Clifford algebra valued polynomials \( x^\gamma e_J \) form a basis for \( \mathcal{A} \). Denote the normalized coefficient of \( x^\gamma e^{\eta_\gamma} \) in \( \omega_A(p) \) by

\[
c_A(\gamma) := \frac{1}{\lambda_A!} \langle x^\gamma e^{\eta_\gamma}, \omega_A(p) \rangle,
\]

(5.25)

for all \( A \in \mathbb{Y}_n(p) \). For each pair of tableaux \( A \in \mathbb{Y}_n(p) \) and \( C \in T(\lambda_A,p) \) there exists a positive integer \( N(C) \in \mathbb{N} \) and a unique matrix \( \gamma_{A,C} \in M_{n,p}(\mathbb{N}_0) \) with \( \mu_{\gamma_{A,C}} = \mu_A \) such that

\[
x_{A,C} e_C = (-1)^{N(C)} x^{\gamma_{A,C}} e^{\eta_{\gamma_{A,C}}}.
\]

(5.26)

The expansion of \( \omega_A(p) \) presented in Proposition 5.3 implies the following result.

**Lemma 5.4.** Let \( A \in \mathbb{Y}_n(p) \), then

\[
\omega_A(p) = \lambda_A! \sum_{\gamma \in M_{n,p}(\mathbb{N}_0)} c_A(\gamma) x^\gamma e^{\eta_\gamma},
\]

(5.27)

where

\[
c_A(\gamma) = \sum_{C \in T(\lambda_A,p)} (-1)^{N(C)},
\]

(5.28)

for all \( \gamma \in M_{n,p}(\mathbb{N}_0) \).

While it is true, as illustrated by (5.21), that some terms appear multiple times, possibly with different signs, in the expansion from Proposition 5.3 other terms are more special. Specifically, it will be proven in Proposition 5.5 that the terms corresponding to c.d. Young tableaux of the following type are linearly independent of the rest of the terms in the expansion.

Given \( A \in \mathbb{Y}_n(p) \), let

\[
D_A \in T(\lambda_A,p)
\]

(5.29)
be the c.d. Young tableau with all 1’s in the 1st row, 2’s in the 2nd row and \( k \)'s in the \( k \)th row. For example,

\[
\begin{array}{cccc}
1 & 1 & 2 & 3 \\
2 & 2 & 3 & 4 \\
3 & 4 & & \\
\end{array}
\]

If \( A = \begin{array}{cccc}
1 & 1 & 2 & 3 \\
2 & 2 & 3 & 4 \\
3 & 4 & & \\
\end{array} \), then \( D_A = \begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
3 & 3 & & \\
\end{array} \) \hspace{1cm} (5.30)

**Proposition 5.5.** Let \( A, B \in \mathcal{Y}_n(p) \) and \( C \in \mathcal{T}(\lambda_A, p) \).

a) If \( x_{A,C} = x_{A,D_A} \) then \( C = D_A \).

b) If \( A < B \), then \( x_{B,D_B} \notin \{x_{A,C} : C \in \mathcal{T}(\lambda_A, p)\} \).

In other words, the expansion of \( \omega_A(p) \) described in Proposition 5.3 contains a unique leading term \( x_{A,D_A}e^{D_A} \) which is linearly independent of all the other terms in the expansion. Furthermore, if \( A < B \), the leading term \( x_{B,D_B}e^{D_B} \) of \( \omega_B(p) \) does not appear in the expansion of \( \omega_A(p) \).

**Proof.** We begin with the proof of statement (a). Given that \( A, C \) and \( D_A \) are all fillings of a Young diagram of shape \( \lambda_A \), their entries can all be indexed by \( \lambda_A \). Write \( a_{k,l} \) and \( c_{k,l} \) for the \((k,l)\)th entries of \( A \) and \( C \) respectively, and recall that by definition the \((k,l)\)th entry of \( D_A \) is \( k \), for all \((k,l) \in \lambda_A \). Combining the assumption \( x_{A,C} = x_{A,D_A} \) with (5.9) gives

\[
\prod_{(k,l) \in \lambda_A} x_{a_{k,l},c_{k,l}} = \prod_{(k,l) \in \lambda_A} x_{a_{k,l},k}. \hspace{1cm} (5.31)
\]

Being a s.s. Young tableau the columns of \( A \) are strictly increasing downwards, which means that \( a_{k,l} \geq k \), for all \((k,l) \in \lambda_A \), which together with (5.31) leads to

\[
a_{k,l} \geq c_{k,l}, \hspace{1.5cm} (5.32)
\]

for all \((k,l) \in \lambda_A \). The entries of the exponent matrix \( \gamma := \gamma_{A,D_A} \in M_{n,p}(\mathbb{N}_0) \) of \( x_{A,D_A} \), as defined in (5.26), are given as follows

\[
\gamma_{i,\alpha} := \# \{ t \in \{1, \ldots, (\lambda_A)_\alpha \} : a_{\alpha,t} = i \} = \text{the number of } i \text{'s in the } \alpha \text{'th row of } A. \hspace{1cm} (5.33)
\]

The observations (5.32) and (5.33) allows us to prove the statement.

Suppose by contradiction that \( C \neq D_A \). From (5.31) this is equivalent to the existence of \((k,l) \in \lambda_A \) such that \( c_{k,l} \neq k \). Let \((k,l)\) be such a coordinate chosen such that it maximizes \( k \) and minimizes the number \( s \) for which \((k,l) \in \lambda_{A^s} \). Explained in words, \((k,l)\) is the coordinate for the box in the smallest possible subdiagram of \( \lambda_A \) and the nethermost row such that \( c_{k,l} \neq k \).

If \( s = 1 \), then \( k = 1 \) and \( a_{k,l} = 1 \). This leads to the contradiction \( c_{k,l} = k \) by use of (5.32). Thus we may assume \( s > 1 \). The minimality condition put on \((k,l)\) leads to the observation that \( c_{k',l'} \neq k' \) if \((k',l') \in \lambda_{A^{s-1}} \). Together with (5.32) and the fact that entries in the columns of \( C \) are distinct this observation gives us the following restriction on \( c_{k,l} \):

\[
c_{k,l} \in \{k, \ldots, s\}. \hspace{1cm} (5.34)
\]

If \( k = s \) this already leads to a contradiction. Now assuming that \( k < s \), we can find \( k'' \in \{k + 1, \ldots, s\} \) such that \( c_{k,l} = k'' \). The maximality condition on \((k,l)\) leads to the following observation: If \( l' \in \{1, \ldots, (\lambda_A)_{k''} \} \) and \((k'',l') \in \lambda_{A^s} \), then \( c_{k'',l'} \neq k'' \). Letting \( \zeta = \gamma_{A,C} \in M_{n,p}(\mathbb{N}_0) \) be the exponent matrix of \( x_{A,C} \) it follows that

\[
\zeta_{s,k''} \geq \# \{ (k,l) \cup \{ l \in \{1, \ldots, (\lambda_A)_{k''} \} : (k'',l) \in \lambda_{A^s} \} \} = 1 + \gamma_{s,k''}. \hspace{1cm} (5.35)
\]
Similarly by using the property that each column of $k, l$ the $(k, l)′$th entry of $D$ is $k$. Therefore, it follows that

$$\sum_{i,\alpha \leq t} \zeta_{i,\alpha} = \sum_{\alpha \leq t} (\lambda_{B^*})_\alpha.$$  \hspace{1cm} (5.39)

Similarly by using the property that each column of $C$ has distinct entries we get

$$\sum_{i,\alpha \leq t} \gamma_{i,\alpha} \leq \sum_{\alpha \leq t} (\lambda_{A^*})_\alpha.$$  \hspace{1cm} (5.40)

By use of (5.37) it is clear that $\sum_{i,\alpha \leq t} \gamma_{i,\alpha} < \sum_{i,\alpha \leq t} \zeta_{i,\alpha}$, proving that $x_{B, D_B} \neq x_{A, C}$. 

In terms of the coefficients $c_A(\gamma)$ the result can be stated as

**Corollary 5.6.** Let $A, B \in \mathbb{Y}_n(p)$ with $A < B$ and $\lambda = \lambda_A$, then

a) $c_A(\gamma_{A, D_A}) = (-1)^{\sum_{j=1}^{\lambda_1} \frac{(j-1)\lambda_j' (\lambda_j' - 1)}{2}} \neq 0$,

b) $c_A(\gamma_{B, D_B}) = 0$.

**Proof.** For each $j \in \{1, \ldots, \lambda_1\}$, define the $p$-tuple $r(j)$ to be

$$r_\alpha(j) = \begin{cases} 1, & \text{for } \alpha \leq \lambda_j', \\ 0, & \text{for } \alpha > \lambda_j', \end{cases}$$  \hspace{1cm} (5.41)

and $e(r(j)) = e_1^{r_1(j)} \cdots e_p^{r_p(j)}$. Then

$$e_{D_A} = e^{r(1)} \cdots e^{r(p)} = (-1)^s e_1^{\lambda_1} \cdots e_p^{\lambda_p},$$  \hspace{1cm} (5.42)

where

$$s = \sum_{i, j=1}^{\lambda_1} \sum_{\alpha, \beta=1}^{\lambda_j'} r_\alpha(i) r_\beta(j) = \sum_{j=1}^{\lambda_1} \frac{(j-1)\lambda_j'(\lambda_j' - 1)}{2}. $$  \hspace{1cm} (5.43)

This proves statement (a). Statement (b) follows directly from the Proposition 5.5. 

One should note that $x_{A,D,A}e_{D,A}$ is not the only term in the decomposition (5.14) of $\omega_A(p)$ with these properties. Proposition 5.5 is satisfied by any term $x_{A,C,e}C$, for which $C \in \mathbb{T}(\lambda_A, p)$ is a c.d. Young tableau for which all entries of any single row are equal.

**Corollary 5.7.** For any $B \in \mathcal{Y}_n(p)$, $\omega_B(p) \neq 0$ and $\omega_B(p) \notin \text{span}_C\{\omega_A(p) : A < B\}$.

**Theorem 5.8.** The sets of vectors

$$\{v_A(p) : A \in \mathcal{Y}_n\} \text{ and } \{\omega_A(p) : A \in \mathcal{Y}_n(p)\}, \quad (5.44)$$

defined in Definition 4.2, form bases for the $\mathfrak{osp}(1|2n)$-modules $\mathcal{V}_n(p)$ and $W_n(p)$ satisfying the properties of Proposition 4.1.

**Proof.** We begin by proving that the two properties of Proposition 4.1 are satisfied. That is, we first prove that, for all $A \in \mathcal{Y}_n$,

1. $\Psi_p(v_A(p)) = \begin{cases} \omega_A(p), & \text{if } A \in \mathcal{Y}_n(p), \\ 0, & \text{if } A \notin \mathcal{Y}_n(p). \end{cases}$

2. The weights of $v_A(p)$ and $\omega_A(p)$ are both $\mu_A + \frac{p}{2}$.

Statement (1) is a direct consequence of $X_I = 0$, for all $I \in \mathcal{I}(k)$ with $k > p$, and the definition of the vectors. Statement (2) follows from $\Psi_p$ being a module homomorphism and the fact that the operators $B_i^+$, for $i \in \{1, \ldots, n\}$, raise the weight of a weight vector $v \in \mathcal{V}_n(p)$ by $\epsilon_i$, which in turn implies that given $A \in \mathcal{Y}_n$ the operator $B_{a_1}^+ \cdots B_{a_n}^+$ raises the weight of $v$ by $\sum_{i=1}^n (\mu_A)_i \epsilon_i$.

It now only remains to prove that both sets actually form bases for the relevant modules. We begin with $\{\omega_A(p) : A \in \mathcal{Y}_n(p)\}$. It is enough to prove that we have bases for each of the weight spaces of $W_n(p)$. That is, it is enough to prove that, for any $\mu \in \mathbb{N}_0^n$, the set

$$\{\omega_A(p) : A \in \mathcal{Y}_n(p), \mu_A = \mu\} \subset W_n(p)_{\mu + \frac{p}{2}}, \quad (5.45)$$

forms a basis for $W_n(p)_{\mu + \frac{p}{2}}$. Since the set is finite, linear independence follows from Corollary 5.7. Equations (4.11) and (4.3) tells us that the cardinality of the set (4.3) agrees with the dimension of the weight space proving that we indeed have a basis.

To prove that $\{v_A(p) : A \in \mathcal{Y}_n\}$ is a basis for $\mathcal{V}_n(p)$ we first note that Lemma 3.5 implies $M_q(q) = \{0\}$, for all $q \geq n$. Thus taking $q \geq n$, $\Psi_q : \mathcal{V}_n(q) \to W_n(q)$ becomes an isomorphism of $\mathfrak{osp}(1|2n)$-modules by Lemma 3.6. Together with (3.4) the composition of the maps $\Psi_q$ and $\Phi_q$ yields the following isomorphism of vector spaces,

$$\Psi_q \circ \Phi_q : U(\mathfrak{osp}(1|2n)^+) \to W_n(q). \quad (5.46)$$

In the notation of (4.15), we define an element

$$B_A^+ := B_{a(\lambda_A)}^+ \cdots B_{a_1}^+ \in U(\mathfrak{osp}(1|2n)^+), \quad (5.47)$$

for each $A \in \mathcal{Y}_n$. Noting that $\Psi_q \circ \Phi_q(B_A^+) = \omega_A(q)$, for all $A \in \mathcal{Y}_n$, it follows that

$$\{B_A^+ \in W_n(q) : A \in \mathcal{Y}_n\} \quad (5.48)$$

is a basis for $U(\mathfrak{osp}(1|2n)^+)$ and thus by Corollary 3.4 that

$$\{v_A(p) = B_A^+ |0\} = \Phi_p(B_A^+) : A \in \mathcal{Y}_n\}, \quad (5.49)$$

is a basis for $\mathcal{V}_n(p)$ regardless of the value of $p$. \hfill \Box

The proof of this Theorem yields the following corollary.

**Corollary 5.9.** The set

$$\{B_A^+ := B_{a(\lambda_A)}^+ \cdots B_{a_1}^+ : A \in \mathcal{Y}_n\} \quad (5.50)$$

forms a basis for $U(\mathfrak{osp}(1|2n)^+)$.
6  Action of $\mathfrak{osp}(1|2n)$ on Tableau Vectors $\omega_A(p)$

In this section we will obtain formulas for the action of the $\mathfrak{osp}(1|2n)$ generators $X_i$ and $D_i$ on the basis for $W_n(p)$ constructed in the previous sections. More precisely, considering the normalization

$$\tilde{\omega}_A(p) := \frac{1}{\lambda_A} \omega_A(p), \quad (6.1)$$

for all $A \in \mathcal{Y}_n(p)$, we want to obtain coefficients $\tilde{c}_B(i, p, A)$ and $\tilde{c}_B(i, p, A)$, for all $i \in \{1, \ldots, n\}$ and $B \in \mathcal{Y}_n(p)$, such that

$$X_i \tilde{\omega}_A(p) = \sum_{B \in \mathcal{Y}_n(p)} \tilde{c}_B(i, p, A) \tilde{\omega}_B(p), \quad D_i \tilde{\omega}_A(p) = \sum_{B \in \mathcal{Y}_n(p)} \hat{c}_B(i, p, A) \tilde{\omega}_B(p). \quad (6.2)$$

Regarding the actions of $B_i^+$ and $B_i^-$ on the basis vectors $\tilde{v}_A(p) = \frac{1}{\lambda_A} v_A(p)$ of $\mathcal{V}(p)$, the corresponding expansion coefficients can be calculated from the coefficients in (6.2). The details of this will be discussed at the end of this section after we obtain formulas for the calculation of the coefficients $\tilde{c}_B(i, p, A)$ and $\hat{c}_B(i, p, A)$, see Proposition 6.6.

Before producing general formulas for obtaining the expansions in (6.2), we take a look at a few cases where the actions of $X_i$ and $D_i$ on $\tilde{\omega}_A(p)$ are particularly simple. First, if $(\mu_A)_i = 0$ then $D_i \tilde{\omega}_A(p) = 0$. Second, if $i$ is greater than or equal to the topmost entry of the rightmost column of $A$, then $X_i \tilde{\omega}_A(p) = \tilde{\omega}_B(p)$, where $B$ is the tableau obtained by adding a single column containing only the box $\blacksquare$ to the right side of $A$. For example, if $A = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$, then

$$X_i \tilde{\omega}_A(p) = \tilde{\omega}_B(p), \quad (6.3)$$

for all $i \geq 3$.

The vectors $X_i \tilde{\omega}_A(p)$ and $D_i \tilde{\omega}_A(p)$ are weight vectors, for all $i \in \{1, \ldots, n\}$ and $A \in \mathcal{Y}_n(p)$, specifically

$$X_i \tilde{\omega}_A(p) \in W_n(p)_{\mu_A + \epsilon_i + \frac{\lambda}{2}} \quad \text{and} \quad D_i \tilde{\omega}_A(p) \in W_n(p)_{\mu_A - \epsilon_i + \frac{\lambda}{2}}. \quad (6.4)$$

This fact leads to the following observation

**Remark 6.1.** For all $i \in \{1, \ldots, n\}$ and $A, B \in \mathcal{Y}_n(p)$,

$$\tilde{c}_B(i, p, A) = 0 \text{ if } \mu_B \neq \mu_A + \epsilon_i \quad (6.5)$$

and

$$\hat{c}_B(i, p, A) = 0 \text{ if } \mu_B \neq \mu_A - \epsilon_i. \quad (6.6)$$

Given a weight $\mu \in \mathbb{N}^n_0$ we let $d_\mu$ be the dimension of the weight space $W_n(p)_{\mu + \frac{\lambda}{2}}$, see (4.11),

$$d_\mu := \dim W_n(p)_{\mu + \frac{\lambda}{2}}. \quad (6.7)$$

Remark 6.1 then tells us that $X_i \tilde{\omega}_A(p)$ and $D_i \tilde{\omega}_A(p)$ are linear combinations of at most $d_{\mu_A + \epsilon_i}$ and $d_{\mu_A - \epsilon_i}$ tableau vectors respectively.

The remaining coefficients in (6.2) can be obtained by solving the system of linear equations that comes from comparing the monomial expansions, Lemma 5.4, of $X_i \tilde{\omega}_A$ and $D_i \tilde{\omega}_A$ with those of $\tilde{\omega}_B(p)$ for $\mu_B = \mu_A + \epsilon_i$ and $\mu_B = \mu_A - \epsilon_i$ respectively. This process is very inefficient as it entails calculating all the coefficients in the monomial expansion of the involved vectors. Our approach
will be to reduce the number of coefficients we need to determine to only the necessary ones and subsequently to find formulas for calculating them.

Given $A \in \mathcal{Y}_n(p)$ and $C \in \mathcal{T}(\lambda_A, p)$ the coefficient of $x_{A,C} e_C$ in the monomial expansion of a given vector $v \in W_n(p)$ is equal to
\begin{equation}
\langle x_{A,C} e_C, v \rangle.
\end{equation}
Given $\mu \in \mathbb{N}_0^n$ we denote the $d_\mu$ s.s. Young tableaux $A_1, \ldots, A_{d_\mu}$ of weight $\mu$ in such a way that
\begin{equation}
A_1 < \cdots < A_{d_\mu}.
\end{equation}
Define the $d_\mu \times d_\mu$ matrix $U_\mu$ and the vectors $f_\mu(v) \in \mathbb{C}^{d_\mu}$, for all $v \in W_n(p)_{\mu+\frac{\varepsilon}{2}}$, as follows
\begin{equation}
(U_\mu)_{k,l} := \langle x_{A_k, D_{A_k}} e_{D_{A_k}}, \tilde{\omega}_{A_l}(p) \rangle
\end{equation}
and
\begin{equation}
(f_\mu)_k(v) := \langle x_{A_k, D_{A_k}} e_{D_{A_k}}, v \rangle,
\end{equation}
for all $k, l \in \{1, \ldots, d_\mu\}$. The $(k, l)$'th entry of the matrix $U_\mu$ is thus the coefficient of the leading term $x_{A_k, D_{A_k}} e_{D_{A_k}}$ of $\tilde{\omega}_{A_l}(p)$ as it appears in $\tilde{\omega}_{A_l}(p)$.

**Proposition 6.2.** For any $\mu \in \mathbb{N}_0^n$, the matrix $U_\mu$ is $M_{d_\mu, d_\mu}(\mathbb{Z})$ is integer and upper unitriangular. Furthermore, for any $v \in W_n(p)_{\mu+\frac{\varepsilon}{2}}$, we have
\begin{equation}
v = \sum_{k=1}^{d_\mu} \tilde{c}_{A_k}(v) \tilde{\omega}_{A_k}(p),
\end{equation}
with
\begin{equation}
\tilde{c}_{A_k}(v) = (U_\mu^{-1} : f_\mu(v))_k
= \langle x_{A_k, D_{A_k}} e_{D_{A_k}}, v \rangle - \sum_{l=k+1}^{d_\mu} \langle x_{A_k, D_{A_k}} e_{D_{A_k}}, \tilde{\omega}_{A_l}(p) \rangle \tilde{c}_{A_l}(v).
\end{equation}

In the context of (6.2), Proposition 6.2 lets us calculate the non-zero coefficients
\begin{equation}
\tilde{c}_B(i, p, A) = \begin{cases} 
\tilde{c}_B(X_i \tilde{\omega}_A(p)), & \text{if } \mu_B = \mu_A + \epsilon_i,
0, & \text{otherwise},
\end{cases}
\end{equation}
and
\begin{equation}
\tilde{c}_B(i, p, A) = \begin{cases} 
\tilde{c}_B(D_i \tilde{\omega}_A(p)), & \text{if } \mu_B = \mu_A - \epsilon_i,
0, & \text{otherwise}.
\end{cases}
\end{equation}

**Proof.** The expansion in Proposition 5.3 implies that the matrix $U_\mu$ is integer, and Proposition 5.5 implies that it is upper unitriangular.

Since $v \in W_n(p)_{\mu+\frac{\varepsilon}{2}}$ it is a linear combination of the vectors $\tilde{\omega}_{A_k}(p)$, for $k \in \{1, \ldots, d_\mu\}$. By statement (b) of Proposition 5.5 the monomial $x_{A_{d_\mu}, D_{A_{d_\mu}}} e_{D_{A_{d_\mu}}}$ only appears in the vector $\tilde{\omega}_{A_{d_\mu}}(p)$, so taking Proposition 5.3 into account we get
\begin{equation}
\tilde{c}_{A_{d_\mu}}(v) = \langle x_{A_{d_\mu}, D_{A_{d_\mu}}} e_{D_{A_{d_\mu}}}, v \rangle.
\end{equation}
By a similar line of reasoning we may conclude that, for any $k \in \{1, \ldots, d_\mu\}$, the monomial $x_{A_k, D_{A_k}} e_{D_{A_k}}$ can only appear in the vectors $\tilde{\omega}_{A_l}(p)$, for $k \leq l \leq d_\mu$, hence
\begin{equation}
\tilde{c}_{A_k}(v) = \langle x_{A_k, D_{A_k}} e_{D_{A_k}}, v \rangle - \sum_{l=k+1}^{d_\mu} \langle x_{A_k, D_{A_k}} e_{D_{A_k}}, \tilde{\omega}_{A_l}(p) \rangle \tilde{c}_{A_l}(v).
\end{equation}
This is the back-substitution algorithm for solving linear systems of linear equations of the form $Tx = b$, where $T$ is an upper unitriangular matrix. This implies that $\tilde{c}_{A_k}(v) = (U^{-1}_{\mu} \cdot f_\mu(v))_k$, for all $k \in \{1, \ldots, d_\mu\}$. \hfill $\square$

To apply Proposition 6.2 and use it to calculate the expansion coefficients introduced in (6.2), it is necessary to determine the relevant coefficients from the monomial expansions of $X_i\tilde{\omega}_A$, $D_i\tilde{\omega}_A$ and $\tilde{\omega}_B(p)$ with $\mu_B = \mu_A + \epsilon_i$ or $\mu_B = \mu_A - \epsilon_i$ respectively. This can be done, however inefficiently, by performing the monomial expansions of the vectors and then extracting the relevant coefficients. Fortunately a better method exists in the form of a formula for calculating any individual monomial coefficient of $\tilde{\omega}_A(p)$. After the following example, this formula will be constructed.

The following is a sketch of how to determine the expansion of $X_i\tilde{\omega}_A(p)$ using Proposition 6.2. We do not include calculations of the entries of $U_\mu$ and $f_\mu(X_i\tilde{\omega}_A(p))$. Consider the case $n = 4$, $p = 2$, $i = 1$ and
\[
A = \begin{pmatrix} 2 & 3 \\ 4 \end{pmatrix}.
\] (6.18)
Then $X_1\tilde{\omega}_p^b(2)$ is a linear combination of the vectors $\tilde{\omega}_B(2)$ of weights $\mu = \mu_A + \epsilon_1 = (1, 1, 1, 1)$. Since $d_\mu = 6$, there are 6 s.s. Young tableaux of weight $(1, 1, 1, 1)$. These are, in increasing order,

$A_1 = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $A_4 = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, $A_5 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $A_6 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Had we instead chosen $p \geq 4$, then we would also have to take into account the 4 remaining s.s. Young tableaux of weight $(1, 1, 1, 1)$: \(\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\), \(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\) and \(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\). When $p = 2$ no basis vectors correspond to these tableaux as they have strictly more than 2 rows.

Calculating the coefficients of the relevant terms we get
\[
U_\mu = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\] and
\[
f_\mu(X_1\tilde{\omega}_p^b(2)) = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.
\] (6.19)

Note that in general the entries of $U_\mu$ take values in all of $\mathbb{Z}$ and not just in $\{\pm 1, 0\}$. Calculating the inverse of $U_\mu$ we get
\[
U^{-1}_\mu = \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\] (6.20)

and
\[
U^{-1}_\mu \cdot f_\mu(X_1\tilde{\omega}_p^b(2)) = (-2, -1, 2, 0, 1, 0).
\] (6.21)

Using Proposition 6.2 this tells us that
\[
X_1\tilde{\omega}_p^b(2) = -2\tilde{\omega}_p^b(2) - \tilde{\omega}_p^b(2) + 2\tilde{\omega}_p^b(2) + \tilde{\omega}_p^b(2).
\] (6.22)

To determine general formulas for the entries of $U_\mu$, $f_\mu(X_i\tilde{\omega}_A(p))$ and $f_\mu(D_i\tilde{\omega}_A(p))$ we first write them as functions of the coefficients $c_B(\gamma)$ introduced in Lemma 5.4. Simple calculations and applications of definitions yield the following formulas.
Lemma 6.3. Let $A, B \in \mathbb{Y}_n(p)$ and $i \in \{1, \ldots, n\}$. Then
\begin{equation}
\langle x_{B,D_B}e_{D_B}, \bar{w}_A(p) \rangle = c_B(\gamma_{B,D_B})c_A(\gamma_{B,D_B}),
\end{equation}
\begin{equation}
\langle x_{B,D_B}e_{D_B}, X_i \bar{w}_A(p) \rangle = c_B(\gamma_{B,D_B}) \sum_{\alpha=1, \beta=1}^{p} (-1)^{\langle \lambda_{B} \rangle_{\beta}} c_A(\gamma_{B,D_B} - \epsilon_{i,\alpha}),
\end{equation}
\begin{equation}
\langle x_{B,D_B}e_{D_B}, D_i \bar{w}_A(p) \rangle = c_B(\gamma_{B,D_B}) \sum_{\alpha=1, \beta=1}^{p} (-1)^{\langle \lambda_{B} \rangle_{\beta}} (\epsilon_{i,\alpha} + 1) c_A(\gamma_{B,D_B} + \epsilon_{i,\alpha}),
\end{equation}

where $c_A(\gamma_{B,D_B} - \epsilon_{i,\alpha}) := 0$ if $\gamma_{B,D_B} = 0$, and where $\epsilon_{i,\alpha} \in M_{n,p}(\mathbb{N}_0)$ for which
\begin{equation}
(\epsilon_{i,\alpha})_{j,\beta} := \delta_{i,j} \delta_{\alpha,\beta},
\end{equation}
for all $i, j \in \{1, \ldots, n\}$ and $\alpha, \beta \in \{1, \ldots, p\}$.

Using Corollary 5.6 we already have a formula for $c_B(\gamma_{B,D_B})$. The rest of this section will be dedicated to obtaining concrete formulas for the calculation of $c_A(\gamma)$ given any $A \in \mathbb{Y}_n(p)$ and $\gamma \in M_{n,p}(\mathbb{N}_0)$. Consider first the definition as given in Lemma 5.4
\begin{equation}
c_A(\gamma) = \sum_{C \in \mathbb{T}(\lambda_A,p)} (-1)^{N(C)},
\end{equation}
where $N(C) \in \mathbb{N}_0$ is a number such that
\begin{equation}
x_{A,C}e_C = (-1)^{N(C)}x^{\gamma_{A,C}}e^{\eta_{A,C}}.
\end{equation}
The general idea behind our approach to the calculation of $c_A(\gamma)$ is to determine the set
\begin{equation}
\{C \in \mathbb{T}(\lambda_A,p) : \gamma_{A,C} = \gamma\},
\end{equation}
and for each element in this set calculate the number $(-1)^{N(C)}$. Consider the following more general class of Young tableaux,
\begin{equation}
\mathbb{E}(\lambda,p) := \{ \text{Fillings of the Young diagram of shape } \lambda \in \mathcal{P} \text{ by numbers } 1, \ldots, p \}.
\end{equation}
We shall refer to the elements of $\mathbb{E}(\lambda,p)$ as Young tableaux. Note that $\mathbb{T}(\lambda,p) \subset \mathbb{E}(\lambda,p)$. The definitions (5.8) and (5.9) can naturally be extended to hold, for any $A \in \mathbb{Y}_n(p)$ with $\lambda = \lambda_A$ and $T \in \mathbb{E}(\lambda,p)$, by letting $m = (\lambda_A)_1$ and defining
\begin{equation}
e_T := (e_{t_{1,1}} \cdots e_{t_{\lambda_1,m}}) \cdots (e_{t_{1,1}} \cdots e_{t_{\lambda_1,1}}) \in \mathcal{C}_p
\end{equation}
and
\begin{equation}
x_{A,T} := (x_{a_{1,1},t_{1,1}} \cdots x_{a_{\lambda_1,m},t_{\lambda_1,m}}) \cdots (x_{a_{1,1},t_{1,1}} \cdots x_{a_{\lambda_1,1},t_{\lambda_1,1}}) \in \mathbb{C}[\mathbb{R}^{np}].
\end{equation}
Let furthermore $\gamma_{A,T} \in M_{n,p}(\mathbb{N}_0)$ and $N(T)$ such that $x_{A,T}e_T = (-1)^{N(T)}x^{\gamma_{A,T}}e^{\eta_{A,T}}$. Define now, for each $\mu \in \mathbb{N}_0^n$, the following permutation group,
\begin{equation}
S_{\mu} := S_{\mu_1} \times \cdots \times S_{\mu_n}.
\end{equation}
The reason for introducing these objects is that the set $\mathbb{E}(\lambda,p)$ carries a useful action of the permutation group $S_{\mu}$, for each $A \in \mathbb{Y}_n(p)$ with $\mu_A = \mu$ and $\lambda_A = \lambda$. To define this action, let
\begin{equation}
y_A(i,s) = (k_A(i,s), l_A(i,s)) \in \lambda_A,
\end{equation}
for \(i \in \{1, \ldots, n\}\) and \(s \in \{1, \ldots, (\mu_A)_i\}\), be the coordinates of \(\lambda_A\) such that
\[
\lambda_{A^t} - \lambda_{A^{t-1}} = \{y_A(i, s) : s \in \{1, \ldots, (\mu_A)_i\}\},
\]
and \(l_A(i, 1) > \cdots > l_A(i, (\mu_A)_i)\). Here \(\lambda_{A^t} - \lambda_{A^{t-1}}\) is the set of coordinates \(y \in \lambda_A\) whose corresponding entries \(a_y\) in \(A\) are \(i\). Specifically
\[
l_A(i, s) = \max\{l \in \{1, \ldots, (\lambda_{A^t})_i\} : s = \sum_{r=l+1}^{(\lambda_{A^t})_i} (\lambda_{A^t})_r - (\lambda_{A^{t-1}})_r\},
\]
\[
k_A(i, s) = (\lambda_{A^t})_l - (\lambda_{A^{t-1}})_l.
\]
Note that these coordinates cover all entries of \(\lambda_A\)
\[
\lambda_A = \{y_A(i, s) : i \in \{1, \ldots, n\}, s \in \{1, \ldots, (\mu_A)_i\}\}.
\]
In the terminology of Macdonald \[22\], \(\{y_A(i, s) : s \in \{1, \ldots, (\mu_A)_i\}\}\) is the \(i\)'th horizontal strip of \(A\) consisting of the boxes \(\parallel\) and indexed from right to left by \(s\). As an example consider
\[
A = \begin{array}{cccc}
1 & 1 & 1 & 2 \\
2 & 2 & 2 & 3 \\
3 & & & \\
\end{array}.
\]
Then \(\mu_1 = 3, \mu_2 = 4\) and \(\mu_3 = 2\). For \(i = 2\), the coordinates \(y_A(2, s) \in \lambda_A\) for \(s \in \{1, 2, 3, 4\}\) are given by
\[
y_A(2, 1) = (1, 5), \ y_A(2, 2) = (1, 4), \ y_A(2, 3) = (2, 2) \text{ and } y_A(2, 4) = (2, 1).
\]
Visually these coordinates refer to the gray boxes in the Young diagram \(\lambda_A\):
\[
(6.40)
\]
The action
\[
\pi_A : S_{\mu_A} \times \mathbb{E}(\lambda_A, p) \to \mathbb{E}(\lambda_A, p),
\]
is defined by letting \(T^{\sigma, A} := \pi_A(\sigma, T)\) be the Young tableau with entries \(t_{y_A}^{\sigma, A}\), for \(y \in \lambda_A\), defined by
\[
t_{y_A}^{\sigma, A}(i, s) := t_{y_A(i, \sigma_i^{-1}(s))},
\]
for all \(i \in \{1, \ldots, n\}\) and \(s \in \{1, \ldots, (\mu_A)_i\}\). Consider \(A\) as in the previous example, and let \(p = 4\) and
\[
T = \begin{array}{cccc}
2 & 1 & 1 & 2 \\
4 & 3 & 3 & \\
3 & & & \\
\end{array} \in \mathbb{T}((5, 3, 1), 4).
\]
Let \(I\) denote the identity permutation, and let \(\sigma, \tau \in S_{\mu_A} = S_3 \times S_4 \times S_2\) with
\[
\sigma = (I, (13), I) \quad \text{and} \quad \tau = (I, (2413), I).
\]
The \(\pi_A\) action of these permutations only permutes the entries of \(T\) corresponding to the coordinates of the 2nd horizontal strip of \(A\). The boxes corresponding to these coordinates are marked with gray below.
\[
T^{\sigma, A} = \begin{array}{cccc}
2 & 1 & 1 & 2 \\
4 & 2 & 3 & \\
3 & & & \\
\end{array} \quad \text{and} \quad T^{\tau, A} = \begin{array}{cccc}
2 & 1 & 1 & 3 \\
2 & 2 & 3 & \\
& & & \\
\end{array}.
\]
Note here that \(T^{\sigma, A} \in \mathbb{T}((5, 3, 1), 4),\) whereas \(T^{\tau, A} \notin \mathbb{T}((5, 3, 1), 4).\)
Lemma 6.4. Let \( A \in \mathcal{Y}_n(p) \) and \( T, T' \in \mathcal{E}(\lambda_A, p) \), then \( x_{A,T} = x_{A,T'} \) if and only if there exists \( \sigma \in S_{\mu_A} \) such that \( T' = T^{\sigma,A} \).

Equivalently this lemma states that
\[
\{ T' \in \mathcal{E}(\lambda_A, p) : \gamma_{A,T'} = \gamma_{A,T} \} = \{ T^{\sigma,A} : \sigma \in S_{\mu_A} \}. \tag{6.46}
\]

Proof. Let \( a_y, t_y \) and \( t_y' \) for \( y \in \lambda_A \), denote the entries of \( A, T \) and \( T' \) respectively. Note first that, for all \( \sigma = (\sigma_1, \ldots, \sigma_n) \in S_{\mu_A} \),
\[
x_{A,T} = \prod_{i=1}^n \prod_{s=1}^{(\mu_A)_i} x^{a_{y_A(i,s)}, t_{y_A(i,s)}} = \prod_{i=1}^n \prod_{s=1}^{(\mu_A)_i} x^{a_{y_A(1,s)}, t'_{y_A(1,s)}} = x_{A,T^{\sigma,A}}. \tag{6.47}
\]

This yields one of the implications stated in the lemma. If \( x_{A,C} = x_{A,C'} \) then
\[
\prod_{s=1}^{(\mu_A)_i} x^{a_{y_A(i,s)}, t_{y_A(i,s)}} = \prod_{s=1}^{(\mu_A)_i} x^{a_{y_A(i,s)}, t'_{y_A(i,s)}}, \tag{6.48}
\]
for all \( i \in \{1, \ldots, n\} \). This implies that the vectors \( (t_{y_A(i,1)}, \ldots, t_{y_A(i,(\mu_A)_i)}) \) and \( (t'_{y_A(i,1)}, \ldots, t'_{y_A(i,(\mu_A)_i)}) \) contain the same amount of \( \alpha \)-entries, for all \( \alpha \in \{1, \ldots, p\} \), letting us construct a permutation \( \sigma_i \in S_{\mu_A} \) such that
\[
t'_{y_A(i,s)} = t_{y_A(i,\sigma_i^{-1}(s))}, \tag{6.49}
\]
for all \( s \in \{1, \ldots, (\mu_A)_i\} \). Repeating this process for all \( i \in \{1, \ldots, n\} \) and defining \( \sigma := (\sigma_1, \ldots, \sigma_n) \), it is clear that \( \sigma \in S_{\mu_A} \) and \( T' = T^{\sigma,A} \).

A Young tableau \( T_{\gamma_A} \in \mathcal{E}(\lambda_A, p) \) for which \( \gamma_{A,T_{\gamma_A}} = \gamma \) can be constructed in the following way. For all \( \alpha \in \{1, \ldots, p\}, i \in \{1, \ldots, n\} \) and \( s \) with \( \sum_{\beta=1}^{\alpha-1} \gamma_{i,\beta} < s \leq \sum_{\beta=1}^{\alpha} \gamma_{i,\beta} \) we define the entry of \( T_{\gamma_A} \) at the coordinate \( y_A(i,s) \) to be
\[
(t_{\gamma_A})_{y_A(i,s)} := \alpha. \tag{6.50}
\]
That is, for any given \( i \in \{1, \ldots, n\} \) we let the boxes of the \( i \)'th horizontal strip be filled, from right to left, by \( \gamma_{i,1} \) '1's, then \( \gamma_{i,2} \) '2's, followed by \( \gamma_{i,3} \) '3's and so on.

As an example consider \( n = 3, p = 4 \),
\[
A = \begin{bmatrix}
1 & 1 & 1 & 2 & 2 \\
2 & 2 & 3
\end{bmatrix}
\]
and \( \gamma = \begin{pmatrix} 2 & 1 & 0 & 0 \\
0 & 2 & 1 & 1 \\
0 & 0 & 2 & 0 \end{pmatrix} \). \tag{6.51}

In that case
\[
T_{\gamma_A} = \begin{bmatrix}
2 & 1 & 1 & 2 & 2 \\
4 & 3 & 3 \\
3
\end{bmatrix}. \tag{6.52}
\]
The gray boxes correspond to the 2nd horizontal strip. A notable feature of this construction is that \( T_{\gamma_A,D_A} = D_A \).

By applying Lemma 6.4 we can now calculate \( c_A(\gamma) \). This is the content of Theorem 6.5. For any \( A \in \mathcal{Y}_n(p) \) and \( \gamma \in M_n(p) \) with \( \mu_\gamma = \mu_A \), let
\[
f_{\gamma_A}(\sigma) := \begin{cases} (-1)^{N(T^{\sigma,A}_{\gamma_A})}, & \text{if } T^{\sigma,A}_{\gamma_A} \in \mathcal{P}(\lambda_A, p), \\ 0, & \text{otherwise,} \end{cases} \tag{6.53}
\]
for all \( \sigma \in S_{\mu_A} \). Here \( T^{\sigma,A}_{\gamma_A} \) is the permutation of \( T_{\gamma_A} \), defined in (6.50), by \( \sigma \) via the action defined in (6.42).
Theorem 6.5. For any \( A \in \mathcal{Y}_n(p) \) and \( \gamma \in M_n(p) \) with \( \mu_\gamma = \mu_A \),

\[
c_A(\gamma) = \prod_{1 \leq i \leq n} \frac{1}{\gamma_{i,\alpha_1}} \sum_{\sigma \in S_{\mu_A}} f_{\gamma,A}(\sigma),
\]

where

\[
\#\{\sigma' \in S_{\mu_A} : T_{\gamma,A}^{\sigma',A} = T_{\gamma,A}^{\sigma,A}\} = \prod_{1 \leq i \leq n} \gamma_{i,\alpha_1}!,
\]

is an over-counting factor. 

To obtain a simple formula for the signs \((-1)^{N(T)}\) we consider the following total order on the coordinates \((k, l)\) of \( \lambda \in \mathcal{P} \)

\[
(k, l) < (k', l') \text{ if and only if } l > l', \text{ or } l = l' \text{ and } k < k',
\]

that is

\[
(1, \lambda_1) < \cdots < (\lambda'_1, \lambda_1) < \cdots < (1, 1) < \cdots < (\lambda'_1, 1).
\]

This ordering is motivated by the definition

\[
e_T = e_{t(1, \lambda_1)} \cdots e_{t(\lambda'_1, \lambda_1)} \cdots e_{t(1, 1)} \cdots e_{t(\lambda'_1, 1)},
\]

for all \( T \in \mathcal{E}(\lambda, p) \). With it we can write

\[
e_T = (-1)^\#\{(k, l), (k', l') \in \lambda : (k, l) > (k', l') \text{ and } t(k, l) < t(k', l')\} e_1^{(\eta_{\gamma_A,T})_{1}} \cdots e_p^{(\eta_{\gamma_A,T})_{p}},
\]

which implies that

\[
(-1)^{N(T)} = (-1)^\#\{(k, l), (k', l') \in \lambda : (k, l) > (k', l') \text{ and } t(k, l) < t(k', l')\}.
\]

We have now reached a point where any coefficient of the form \( c_A(\gamma) \) can be explicitly calculated. To show how such a calculation is done consider \( A \) and \( \gamma \) as in (6.51), with \( n = 3 \) and \( p = 4 \). Here \( \mu_A = (3, 4, 2) \) meaning that \( S_{\mu_A} \) contains \( |S_{\mu_A}| = 3!4!2! = 288 \) different permutations. Taking the over-counting factor (6.55) into account the total number of Young tableaux generated by the permutation action \( \pi_A \) is

\[
\#\{T_{\gamma,A}^{\sigma,A} : \sigma \in S_{\mu_A}\} = \prod_{1 \leq i \leq n} \frac{|S_{\mu_A}|}{\gamma_{i,\alpha_1}!} = \frac{3!4!2!}{2^3} = 36.
\]

Out of these 36 tableaux 19 are not column distinct and thus contribute with factors of 0 to \( c_A(\gamma) \), see (6.53). An example of this is obtained by letting \( \tau = (I, (2413), I) \). The corresponding tableau

\[
T_{\gamma,A}^{\tau,A} = \begin{array}{cccc}
2 & 1 & 1 & 3 \\
2 & 2 & 3 & 1 \\
3 & & & \\
\end{array}
\]

has two entries containing a 2 in its first column and is thus not column distinct. The remaining 17 tableaux are column distinct and each contribute with a factor of \( \pm 1 \) to \( c_A(\gamma) \). Examples from these 17 tableaux include the ones corresponding to permutations

\[
\sigma = (I, (13), I) \quad \text{and} \quad \kappa = ((13), (2413), I).
\]
Those are
\[ T_{\gamma, A}^{\sigma} = \begin{bmatrix} 2 & 1 & 1 & 2 & 3 \\ 4 & 2 & 3 \\ 3 \end{bmatrix} \quad \text{and} \quad T_{\gamma, A}^V = \begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 3 \\ 3 \end{bmatrix}. \] (6.64)

Calculated with (6.60) the contributions of these tableaux are
\[ (-1)^{N(T_{\gamma, A}^{\sigma})} = (-1)^1 = -1 \quad \text{and} \quad (-1)^{N(T_{\gamma, A}^V)} = (-1)^20 = 1. \] (6.65)

Out of all 17 column distinct tableaux one will find that 9 have positive sign and 8 have negative meaning that
\[ c_A(\gamma) = 9 - 8 = 1. \] (6.66)

Obtaining the coefficients \( c_A(\gamma) \) in this way requires the construction of all the relevant Young tableaux. Formulas for doing these calculations without explicitly constructing the tableaux become rather complicated. For the interested readers such formulas are included in Appendix B.

In this section, we have so far explained how to calculate all the coefficients in the expansions from (6.2). Such calculations are done by combining Proposition 6.2 with Theorem 6.5 and Lemma 6.3. Proposition 6.6 shows how, using the coefficients from (6.2), we can obtain the expansions for the actions of \( B_i^+ \) and \( B_i^- \) on the basis vectors \( \tilde{v}_A(p) = \frac{1}{\lambda_A} v_A(p) \) of \( \nabla_n(p) \).

**Proposition 6.6.** Let \( i \in \{1, \ldots, n\} \) and \( A, B \in \nabla_n \), then
\[
B_i^+ \tilde{v}_A(p) = \sum_{B \in \nabla_n} \tilde{c}_B(i, n, A) \tilde{v}_B(p), \\
B_i^- \tilde{v}_A(p) = \sum_{B \in \nabla_n} (n + 1 - p) \tilde{c}_B(i, n, A) - (n - p) \tilde{c}_B(i, n + 1, A) \tilde{v}_B(p). \] (6.67)

By Theorem 5.8, formulas corresponding to Equation (6.67) also hold for the actions of \( X_i \) and \( D_i \) on \( W_n(p) \), though with the sums only being over \( B \in \nabla_n(p) \). Proposition 6.6 tells us that by using the tools presented in this section to determine matrix coefficients of the \( X_i \) and \( D_i \) actions on \( W_n(n) \) and \( W_n(n + 1) \) we can determine the matrix coefficients of any other action, be it on \( W_n(p) \) or \( \nabla_n(p) \), by simple linear combination of coefficients.

**Proof.** Let \( |0\rangle \) be the lowest weight vector of \( \nabla_n(p) \). Corollary 5.9 tells us that there exist coefficients \( d_B(i, A) \), independent of \( p \), such that
\[
B_i^+ \tilde{v}_A(p) = B_i^+ \left( \frac{1}{\lambda_A} B_A^+ \right) |0\rangle = \sum_{B \in \nabla_n} d_B(i, A) \frac{1}{\lambda_B} B_B^+ |0\rangle = \sum_{B \in \nabla_n} d_B(i, A) \tilde{v}_B(p). \] (6.68)

Since \( \Psi_n \) is an isomorphism of vector spaces with \( \Psi_n(\tilde{\omega}_B(n)) = \tilde{v}_B(n) \), it follows that \( d_B(i, A) = \tilde{c}_B(i, n, A) \), for all \( B \in \nabla_n = \nabla_n(n) \), when we compare with the coefficients in (6.2).

Using Equation (2.1) we can find \( B_1^{1, +}(i, j, A), B_2^{2, +}(i, A), B_3^{3, +}(i, A) \in U(\mathfrak{osp}(1|2n)^+) \) such that
\[
B_i^- B_A^+ = \left( \sum_{1 \leq i, j \leq n} B_1^{1, +}(i, j, A) \{ B_i^-, B_j^+ \} \right) + B_2^{2, +}(i, A) + B_3^{3, +}(i, A) B_i^- . \] (6.69)

Corollary 5.9 together with (2.3) then tells us that there exists coefficients \( d_B^1(i, A) \) and \( d_B^2(i, A) \),
The fact that $\Psi$ and $\Psi_{n+1}$ are isomorphisms of vector spaces with $\Psi_n(\tilde{\omega}_B(n)) = \tilde{v}_B(n)$ and $\Psi_{n+1}(\tilde{\omega}_B(n+1)) = \tilde{v}_B(n+1)$ lets us compare coefficients with (6.2) and obtain the equations

$$d^1_B(i, A)n + d^2_B(i, A) = \hat{c}_B(i, n, A),$$

and

$$d^1_B(i, A)(n+1) + d^2_B(i, A) = \hat{c}_B(i, n+1, A).$$

Solving these equations for $d^1_B(i, A)$ and $d^2_B(i, A)$ and inserting the results into (6.70) gives the statement of the proposition.

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\appendix

\section{Graded Lexicographic Order}

The definitions of the graded lexicographic ordering on $\mathbb{N}_0^n$ and $\mathcal{P}$ were omitted in the main text. For the readers unfamiliar with these orderings the definitions are included here.

\begin{definition}
Given $\mu, \nu \in \mathbb{N}_0^n$ we say that $\mu < \nu$ with respect to the graded lexicographic ordering if $|\mu| = \sum_{i=1}^n \mu_i < |\nu| = \sum_{i=1}^n \nu_i$, or if $|\mu| = |\nu|$ and $\mu_j < \nu_j$ for the first $j$ where $\mu_j$ and $\nu_j$ differ. For $n = 4$, the $\mu \in \mathbb{N}_0^4$ with $|\mu| \leq 2$ are ordered as follows,

$$(0, 0, 0, 0) < (0, 0, 0, 1) < (0, 0, 1, 0) < (0, 1, 0, 0) < (1, 0, 0, 0) <$$

$$(0, 0, 0, 2) < (0, 0, 1, 1) < (0, 0, 2, 0) < (0, 1, 0, 1) < (0, 1, 1, 0)$$

$$(0, 2, 0, 0) < (1, 0, 0, 1) < (1, 0, 1, 0) < (1, 1, 0, 0) < (2, 0, 0, 0)$$

\end{definition}

\begin{definition}
Given $\lambda, \kappa \in \mathcal{P}$ we say that $\lambda < \kappa$ with respect to the graded lexicographic ordering if $|\lambda| < |\kappa|$, or if $|\lambda| = |\kappa|$ and $\lambda_j < \kappa_j$ for the first $j$ where $\lambda_j$ and $\kappa_j$ differ. The $\lambda \in \mathcal{P} \in \mathbb{N}_0^n$ with $|\lambda| \leq 4$ are ordered as follows,

$$0 < (1) < (1, 1) < (2) < (1, 1, 1) < (2, 1) < (3) < (1, 1, 1, 1) < (2, 1, 1) < (2, 2) < (3, 1) < (4)$$

\end{definition}
B Alternative Formula for $c_A(\gamma)$

We present here a formula for calculating the coefficients $c_A(\gamma)$ of $\hat{\omega}_A(p)$ that is more explicit than the one produced at the end of Section 6. What follows are two technical lemmas in which constituents of the final formula are obtained. Following those lemmas the final formula is presented, marking the end of this appendix. Due to them being technical and not very enlightening the proofs of the following results are omitted.

Consider the function

$$G(l, k, l') = \begin{cases} 
  l' - l, & \text{if } l \leq l' \leq k, \\
  l' - k, & \text{if } l \leq k < l', \\
  0, & \text{otherwise}, 
\end{cases} \quad (B.1)$$

for all $l, k, l' \in \mathbb{N}_0$.

**Lemma B.1.** Given $A \in \mathbb{V}_n(p)$ and $\gamma \in M_n(p)$ with $\mu_\gamma = \mu_A$, then

$$e_{T_{\gamma, A}} = (-1)^{N_1(\gamma, A) + N_2(\gamma, A)} e^{\eta_\gamma}, \quad (B.2)$$

where

$$N_1(\gamma, A) = \sum_{1 \leq \alpha' < \alpha' \leq p} \sum_{1 \leq i < i' \leq n} \sum_{s=1}^{\sum_{1 \leq i < i' \leq n} \gamma_i, \beta} G \left( \alpha' - 1, \prod_{\beta=1}^{\alpha'} \gamma_i', \beta, \prod_{l=A(i,s)+1}^{\lambda_{A'}-1} \gamma_l', \beta \right) \quad (B.3)$$

and

$$N_2(\gamma, A) = \sum_{1 \leq \alpha < \alpha' \leq p} \sum_{1 \leq i < i' \leq n} \sum_{s=1}^{\sum_{1 \leq i < i' \leq n} \gamma_i, \beta} G \left( \alpha' - 1, \prod_{\beta=1}^{\alpha'} \gamma_i', \beta, \prod_{l=A(i,s)+1}^{\lambda_{A'}-1} \gamma_l', \beta \right) \quad (B.4)$$

Let $k \in \mathbb{N}$ and consider a $k$-tuple of positive integers $L = (L_1, \ldots, L_k) \in \mathbb{N}_k$. If $L_m \neq L_{m'}$, for all $m \neq m'$, then we define $\sigma_L \in S_k$ to be the permutation such that

$$L_{\sigma(1)} < \cdots < L_{\sigma(k)}. \quad (B.5)$$

With this we can define the sign of $L$ to be

$$\text{sgn}(L) = \begin{cases} 
  \text{sgn} \sigma_L, & \text{if } L_m \neq L_{m'}, \text{ for all } m \neq m', \\
  0, & \text{otherwise}. \quad (B.6) 
\end{cases}$$

**Lemma B.2.** Given $A \in \mathbb{V}_n(p)$, $\sigma \in S_{\mu_A}$ and $\gamma \in M_{n,p}(\mathbb{N}_0)$ with $\mu_\gamma = \mu_A$, then

$$\text{sgn}(\sigma) \prod_{\alpha=1}^{p} \text{sgn}(L_{\gamma, A}(\sigma, \alpha))(-1)^{N_0(\gamma, A)} e_{T_{\gamma, A}} = \begin{cases} 
  e_{T_{\gamma, A}}^{\sigma(\gamma, A)}, & \text{if } T_{\gamma, A}^{\sigma(\gamma, A)} \in T(\lambda_A, p), \\
  0, & \text{otherwise}. \quad (B.7) 
\end{cases}$$

where

$$N_0(\gamma, A) = \sum_{1 \leq \alpha < \alpha' \leq p} \sum_{1 \leq i < i' \leq n} \sum_{s=1}^{\sum_{1 \leq i < i' \leq n} \gamma_i, \beta} G \left( \alpha - 1, \prod_{\beta=1}^{\alpha} \gamma_i', \beta, \prod_{l=A(i,s)+1}^{\lambda_{A'}-1} \gamma_l', \beta \right) \quad (B.8)$$
and \( L_{\gamma, A}(\sigma, \alpha) \in \mathbb{N}^{(\eta_\gamma)}_\alpha \) defined such that, for all \( \alpha \in \{1, \ldots, p\}, \ i \in \{1, \ldots, n\} \) and \( t \) with 
\[
\sum_{j=1}^{i-1} \gamma_{j, \alpha} < t \leq \sum_{j=1}^{i} \gamma_{j, \alpha},
\]
\[
(L_{\gamma, A}(\sigma, \alpha))_t = l_A \left( i, \sigma^{-1} \left( t - \sum_{j=1}^{i-1} \gamma_{j, \alpha} + \sum_{\beta=1}^{\alpha-1} \gamma_{i, \beta} \right) \right).
\]

(B.9)

If \( (\eta_\gamma)_\alpha = 0 \), then \( \text{sgn}(L_{\gamma, A}(\sigma, \alpha)) := 1 \).

Proposition B.3. Let \( A \in \mathbb{Y}_n(p) \) and \( \gamma \in M_{n,p}(\mathbb{N}_0) \) with \( \mu_\gamma = \mu_A \), then
\[
c_A(\gamma) = \sum_{\sigma \in S_{\mu_A}} \prod_{1 \leq \alpha \leq p} \frac{1}{\gamma_{i, \alpha}!} \text{sgn}(\sigma)(-1)^{N(\gamma,A)} \text{sgn}(L_A(\sigma, \alpha)),
\]

(B.10)

where \( N(\gamma, A) = N_0(\gamma, A) + N_1(\gamma, A) + N_2(\gamma, A) \).

References

[1] Blank, J., and Havlček, M. “Irreducible *-representations of Lie superalgebras \( B(0, n) \) with finite-degenerated vacuum.” Journal of Mathematical Physics 27, no. 12 (1986): 2823–2831.

[2] Chaturvedi, S. “Canonical partition functions for parastatistical systems of any order.” Physical Review E 54, no. 2 (1996): 1378-1382.

[3] Cheng, S.-J., and Kwon, J.-H., and Wang, W. “Kostant homology formulas for oscillator modules of Lie superalgebras.” Advances in Mathematics 224, no. 4 (2010): 1548–1588.

[4] Cheng, S.-J., and Wang. W. Dualities and Representations of Lie Superalgebras, American Mathematical Society, 2012.

[5] Cheng, S.-J., and Zhang, R. B. “Howe duality and combinatorial character formula for orthosymplectic Lie superalgebras.” Advances in Mathematics 182, no. 1 (2004): 124–172.

[6] Colombo, F., and Sabadini, I., and Sommen, F., and Struppa, D. Analysis of Dirac systems and computational algebra, Boston: Birkhuser, 2004.

[7] Dobrev, V. K., and Salom, I. “Positive Energy Unitary Irreducible Representations of the Superalgebras \( osp(1|2n, \mathbb{R}) \) and Character Formulae.” Journal of Physics: Conference Series 804, no. 1 (2017): 012015.

[8] Dobrev, V. K., and Zhang, R. B. “Positive energy unitary irreducible representations of the superalgebras \( osp(1|2n, \mathbb{R}) \)” Physics of Atomic Nuclei 68, no. 10 (2005): 1660–1669.

[9] Frappat, L., and Sorba, P., and Sciarrino, A. Dictionary on Lie algebras and superalgebras, London: Academic Press, 2000

[10] Ganchev, A. Ch., and Palev, T. D. “A Lie superalgebraic interpretation of the para-Bose statistics” Journal of Mathematical Physics 21, no. 4 (1980): 797–799.

[11] Green, H. S. “A Generalized Method of Field Quantization” Physical Review 90, no. 2 (1953): 270–273.

[12] Greenberg, O.W., and Macrae, K.I. “Locally gauge-invariant formulation of parastatistics” Nuclear Physics B 219, no. 2 (1983): 358–366.
[13] Greenberg, O.W., and Messiah, A. M. L. “Selection Rules for Parafields and the Absence of Paraparticles in Nature” Physical Review 138, no. 5B (1965): 1155–1167.

[14] Iachello, F., and Van Isacker, P. The interacting boson-fermion model, Cambridge: Cambridge University Press, 1991

[15] Kac, V. C. “Lie superalgebras” Advances in Mathematics 26, no. 1 (1977): 8–96.

[16] Kanakoglou, K., and Daskaloyannis, C. “A braided look at Green ansatz for parabosons” Journal of Mathematical Physics 48, no. 11 (2007): 113516.

[17] King, R. C., and Palev, T. D., and Stoilova, N. I., and Van der Jeugt, J. “The non-commutative and discrete spatial structure of a 3D Wigner quantum oscillator” Journal of Physics: A. Mathematical and General 36, no. 15 (2003): 4337–4362.

[18] Lvička, R., and Souček, V. “Fischer decomposition for spinor valued polynomials in several variables”, 2017, arXiv: 1708.01426.

[19] Lievens, S., and Stoilova, N. I., and Van der Jeugt, J. “Harmonic oscillators coupled by springs: discrete solutions as a Wigner quantum system” Journal of Mathematical Physics 47, no. 11 (2006): 113504.

[20] Lievens, S., and Stoilova, N. I., and Van der Jeugt, J. “The Paraboson Fock Space and Unitary Irreducible Representations of the Lie Superalgebra \( \mathfrak{osp}(1|2n) \)” Communications in Mathematical Physics 281, no. 3 (2008): 805–826.

[21] Loday, J.-L., and Popov, T. “Parastatistics Algebra, Young Tableaux and the Super Plactic Monoid” International Journal of Geometric Methods in Modern Physics 5, no. 8 (2008): 1295-1314.

[22] Macdonald, I. G. Symmetric Functions and Hall Polynomials, 2nd ed. Oxford: Oxford University Press, 1995.

[23] Molev, A. I. “Gelfand-Tsetlin bases for classical Lie algebras” Handbook of algebra 4 (2006): 109–170.

[24] Nishiyama, K. “Oscillator representations for orthosymplectic algebras” Journal of Algebra 129, no. 1 (1990): 231–262.

[25] Ohnuki, Y., and Kamefuchi, S. Quantum field theory and parastatistics Berlin: Springer, 1982.

[26] Salom, I. “Role of the orthogonal group in construction of \( \mathfrak{osp}(1|2n) \) representations”, 2013, arXiv: 1307.1452.