SHARP LIPSCHITZ ESTIMATES FOR OPERATOR $\bar{\partial}_M$
ON A $q$-CONCAVE CR MANIFOLD

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ABSTRACT. We prove that the integral operators $R_r$ and $H_r$ constructed in $\overline{\mathbb{F}}$ and such that
\[ f = \bar{\partial}_M R_r(f) + R_{r+1}(\bar{\partial}_M f) + H_r(f), \]
for a differential form $f \in C_{0,\alpha}^\infty(M)$ on a regular $q$-concave CR manifold $M$ admit sharp estimates in the Lipschitz scale.

1. Introduction.

Let $M$ be a CR submanifold in a complex $n$-dimensional manifold $G$ such that for any $z \in M$ there exist a neighborhood $V \ni z$ in $G$ and smooth real valued functions $\{\rho_k\}$ $k = 1, \ldots, m$ ($1 < m < n - 1$) on $V$ such that
\[ M \cap V = \{z \in G \cap V : \rho_1(z) = \cdots = \rho_m(z) = 0\}, \]
\[ \partial \rho_1 \wedge \cdots \wedge \partial \rho_m \neq 0 \text{ on } M \cap V. \] (1)

In this paper we continue the study of regularity of the operator $\bar{\partial}_M$ on a submanifold $M$ satisfying special concavity condition. In $[P]$ we proved sharp estimates for solutions of the $\bar{\partial}_M$ equation with an $L^\infty$ right hand side. Here we push sharp estimates higher on the Lipschitz scale.

Before formulating the main result we will introduce the necessary notations and definitions. The CR structure on $M$ is induced from $G$ and is defined by the subbundles
\[ T''(M) = T''(G)|_M \cap CT(M) \quad \text{and} \quad T'(M) = T'(G)|_M \cap CT(M), \]
where $CT(M)$ is the complexified tangent bundle of $M$ and the subbundles $T''(G)$ and $T'(G) = \overline{T''(G)}$ of the complexified tangent bundle $CT(G)$ define the complex structure on $G$.

We will denote by $T'(M)$ the subbundle $T(M) \cap [T'(M) \oplus T''(M)]$. If we fix a hermitian scalar product on $G$ then we can choose a subbundle $N \subset T(M)$ of real dimension $m$ such that $T^c(M) \perp N$ and for a complex subbundle $N = CN$ of $CT(M)$ we have
\[ CT(M) = T'(M) \oplus T''(M) \oplus N, \quad T'(M) \perp N \text{ and } T''(M) \perp N. \]

We define the Levi form of $M$ as the hermitian form on $T'(M)$ with values in $N$
\[ L_z(L(z)) = \sqrt{-1} \cdot \pi \left( [\overline{T}, L] \right)(z) \quad (L(z) \in T'_z(M)), \]
where $[\overline{T}, L] = \overline{T}L - L\overline{T}$ and $\pi$ is the projection of $CT(M)$ along $T'(M) \oplus T''(M)$ onto $N$.

If the functions $\{\rho_k\}$ are chosen so that the vectors $\{\text{grad } \rho_k\}$ are orthonormal then the Levi form of $M$ may be defined as
\[ L_z(M) = - \sum_{k=1}^m (L_z \rho_k(\zeta)) : \text{grad } \rho_k(z), \]
where $\zeta$. 

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where $L_z\rho(\zeta)$ is the Levi form of the real valued function $\rho \in C^4(D)$ at the point $z$:

$$L_z\rho(\zeta) = \sum_{i,j} \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j} (z) \zeta_i \cdot \zeta_j.$$ 

Analogously by the Hessian form of $M$ at the point $z \in M$ we call the hermitian form on the complex tangent space $T^c_z(M)$ of $M$ at $z$ with values in $N_z$, defined by the formula:

$$\mathcal{H}_z(M) = - \sum_{k=1}^{m} (\mathcal{H}_z \rho_k(\zeta)) \cdot \text{grad} \rho_k(z),$$

where $\mathcal{H}_z \rho(\zeta)$ is the Hessian form of the real valued function $\rho \in C^4(D)$ at the point $z$:

$$\mathcal{H}_z \rho(\zeta) = \sum_{i,j} \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j} (z) \zeta_i \cdot \zeta_j + \text{Re} \left\{ \sum_{i,j} \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j} (z) \zeta_i \cdot \zeta_j \right\}.$$ 

For a pair of vectors $\mu = (\mu_1, \ldots, \mu_n)$ and $\nu = (\nu_1, \ldots, \nu_n)$ we will denote $\langle \mu, \nu \rangle = \sum_{i=1}^{n} \mu_i \cdot \nu_i$. 

For a unit vector $\theta = (\theta_1, \ldots, \theta_m) \in \mathbb{R}N_z$ we define the Levi form of $M$ at the point $z \in M$ in the direction $\theta$ as the scalar hermitian form on $T^c_z(M)$

$$\langle \theta, L_z(M) \rangle = - L_z \rho(\zeta),$$

and the Hessian form of $M$ at the point $z \in M$ in the direction $\theta$ as the hermitian form

$$\langle \theta, \mathcal{H}_z(M) \rangle = - \mathcal{H}_z \rho(\zeta),$$

where $\rho(\zeta) = \sum_{k=1}^{m} \theta_k \rho_k(\zeta)$.

Following [12] we introduce the notions of q-pseudoconcave and q-concave CR manifolds.

Namely, we call $M$ q-pseudoconcave (weakly q-pseudoconcave) at $z \in M$ in the direction $\theta$ if the Levi form of $M$ at $z$ in this direction $\langle \theta, L_z(M) \rangle$ has at least $q$ negative (q nonpositive) eigenvalues on $T^c_z(M)$ and we call $M$ q-concave (weakly q-concave) at $z \in M$ in the direction $\theta$ if the Hessian form of $M$ at $z$ in this direction $\langle \theta, \mathcal{H}_z(M) \rangle$ has at least $q$ negative (q nonpositive) eigenvalues on $T^c_z(M)$.

We call $M$ q-pseudoconcave (weakly q-pseudoconcave) at $z \in M$ if it is q-pseudoconcave (weakly q-pseudoconcave) in all directions. Analogously, we call $M$ q-concave (weakly q-concave) at $z \in M$ if it is q-concave (weakly q-concave) in all directions.

We call a q-concave CR manifold $M$ by a regular q-concave CR manifold (cf. [P]) if for any $z \in M$ there exist an open neighborhood $U \ni z$ in $M$ and a family $E_q(\theta, z)$ of q-dimensional complex linear subspaces in $T^c_z(M)$ smoothly depending on $(\theta, z) \in S^{m-1} \times U$ and such that the Hessian form $(\theta, \mathcal{H}_z(M))$ is strictly negative on $E_q(\theta, z)$.

Following [S] we define spaces $\Gamma^{p,\alpha}(M)$ for nonnegative $p \in \mathbb{Z}$ and $0 < \alpha < 2$. Namely, we say that function $h \in \Gamma^{p,\alpha}(M)$ if for any set of tangent vector fields $D_1, \ldots, D_p$ on $M$ such that $\|D_i\|_{C^{p+2}(M)} \leq 1$

\[ \|h\|_{\Gamma^{p,\alpha}(M)} = \|D_1 \circ \cdots \circ D_p h\|_{\Lambda^p(M)} + \sup \left\{ \|D_1 \circ \cdots \circ D_p h(x(\cdot))\|_{\Lambda^\alpha([0,1])} \right\} < \infty, \]

where the sup is taken over all curves $x : [0,1] \to M$ such that

(i) $|x'(t)|, |x''(t)| \leq 1$,

(ii) $x'(t) \in T^c(M)$.

For a differential form $g = \sum_{I,J} g_{I,J}(z) dz^I \wedge d\bar{z}^J$ with $|I| = k$ and $|J| = r$ we say that $g \in \Gamma_{k,r}^{p,\alpha}(M)$ if $g_{I,J} \in \Gamma^{p,\alpha}(M)$.

The following theorem represents the main result of the paper.
Theorem 1. Let $0 < \alpha < 1$ and let a $C^\infty$ submanifold $M \subset G$ of the form (1) be regular $q$-concave with $q \geq 2$. Then for any $r = 1, \ldots, q - 1$ there exist linear operators
\[ R_r : \Gamma_{0,r}^{0,\alpha}(M) \to \Gamma_{0,r-1}^{0,\alpha+1}(M) \quad \text{and} \quad H_r : \Gamma_{0,r}^{p,\alpha}(M) \to \Gamma_{0,r}^{p,\alpha}(M) \]
such that $R_r$ is bounded and $H_r$ is compact and such that for any differential form $f \in C_{\infty,r}^{0}(M)$ the equality:
\[ f = \tilde{\partial}_M R_r(f) + R_{r+1}(\tilde{\partial}_M f) + H_r(f) \tag{2} \]
holds.

The study of the $\tilde{\partial}_M$ complex on a real submanifold $M \subset \mathbb{C}^n$ was initiated by J. J. Kohn and H. Rossi in [K], [KR]. For a closed strongly pseudoconvex hypersurface $M \subset \mathbb{C}^n$ J. J. Kohn [K], A. Andreotti, C. D. Hill [AnH], I. Naruki [Na] proved the solvability of the nonhomogeneous $\tilde{\partial}_M$ equation:
\[ \tilde{\partial}_M g^{0,r-1} = g^{0,r} \]
for $r < n - 1$ and any $g^{0,r} \in C_{\infty,(0,r)}^{0}(M)$ satisfying the solvability condition
\[ \tilde{\partial}_M g^{0,r} = 0. \]

A version of the main theorem for $q$-pseudoconvex hypersurfaces $(m = 1)$ and different spaces $\Gamma^\alpha(M)$ with $\alpha \geq 0$ was proved by G. B. Folland and E. M. Stein in [FS] (cf. also [H1]).

Generalizing notion of $q$-concavity to the manifolds of higher codimension I. Naruki in [Na] using Kohn-Hormander’s method constructed bounded operators $R_r : L^2_{(0,r)}(M) \to L^2_{(0,r-1)}(M')$ for the $(0, r)$ forms with $r > n - m - q$.

Then in [H2] and [AH] with the use of explicit integral formulas bounded operators $R_r : L^\infty_{(0,r)}(M) \to \Gamma^{0,1-%}{r-1}(M)$ were constructed on a $q$-pseudoconcave CR manifold of higher codimension for the forms of type $(0, r)$ with $r < q$ or $r > n - m - q$.

Existence of solution $f \in C_{\infty,(0,r-1)}^{0}(M)$ of equation $\tilde{\partial}_M f = g$ for $g \in L^\infty_{(0,r)}(M)$ for $q$-pseudoconcave CR manifold was obtained by M. Y. Barkatou in [Ba] and existence of solution $f \in \Gamma^{0,\beta+1}_{(0,r-1)}(M)$ for $g \in \Gamma^{\beta}_{(0,r)}(M)$, $0 < \beta < 1$ and $M$ - quadratic $q$-pseudoconcave CR manifold was obtained in the paper [BGG] by R. Beals, B. Gaveau and P.C. Greiner.

The problem considered here is closely related to the problem of obtaining sharp Lipschitz estimates for solutions of $\tilde{\partial}$ equation on complex manifolds with boundary. Paper by Y.-T. Siu [Siu] devoted to this problem has motivated some of the constructions in our paper.

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2. Construction of $R_r$ and $H_r$.

The local versions of operators $R_r$ and $H_r$ from the theorem [1] were constructed in [P]. Before describing these local operators we will introduce necessary definitions and notations.

For a vector-valued function $\eta = (\eta_1, \ldots, \eta_n)$ we will use the notation:
\[ \omega'(\eta) = \sum_{k=1}^n (-1)^{k-1} \eta_k \land_{j \neq k} d\eta_j, \quad \omega(\eta) = \land_{j=1}^n d\eta_j. \]

If $\eta = \eta(\zeta, z, t)$ is a smooth function of $\zeta \in \mathbb{C}^n$, $z \in \mathbb{C}^n$ and a real parameter $t \in \mathbb{R}^n$ satisfying the condition
\[ \sum_{k=1}^n \eta_k(\zeta, z, t) \cdot (\zeta_k - z_k) = 1 \tag{3} \]
\[ d\omega'(\eta) \wedge \omega(\zeta) \wedge \omega(z) = 0 \]

or, separating differentials,
\[ d_t\omega'(\eta) + \tilde{\partial}_\zeta \omega'(\eta) + \tilde{\partial}_z \omega'(\eta) = 0. \tag{4} \]

Also, if \( \eta(\zeta, z, t) \) satisfies (3) then the differential form \( \omega'(\eta) \wedge \omega(\zeta) \wedge \omega(z) \) can be represented as:
\[ \sum_{r=0}^{n-1} \omega'_r(\eta) \wedge \omega(\zeta) \wedge \omega(z), \tag{5} \]
where \( \omega'_r(\eta) \) is a differential form of the order \( r \) in \( d\tilde{z} \) and respectively of the order \( n-r-1 \) in \( d\zeta \) and \( dt \). From (4) and (5) follow equalities:
\[ d_t\omega'_r(\eta) + \tilde{\partial}_\zeta \omega'_r(\eta) + \tilde{\partial}_z \omega'_{r-1}(\eta) = 0 \quad (r = 1, \ldots, n), \tag{6} \]
and
\[ \omega'_r(\eta) = \frac{1}{(n-r-1)!r!} \text{Det} \left[ \eta, \tilde{\partial}_z \eta, \tilde{\partial}_\zeta \eta, \omega'_1(\eta), \ldots, \omega'_{n-r-1}(\eta) \right], \tag{7} \]
where the determinant is calculated by the usual rules but with external products of elements and the position of the element in the external product is defined by the number of its column.

Let \( \bar{U} \) be an open neighborhood in \( \mathbf{G} \) and \( U = \bar{U} \cap \mathbf{M} \). We call a vector function
\[ P(\zeta, z) = (P_1(\zeta, z), \ldots, P_n(\zeta, z)) \quad \text{for} \quad (\zeta, z) \in (\bar{U} \setminus U) \times \bar{U} \]
by strong \( \mathbf{M} \)-barrier for \( \bar{U} \) if there exist \( C, c > 0 \) such that the inequality:
\[ |\Phi(\zeta, z)| > C \cdot \left( \rho(\zeta) + |\zeta - z|^2 \right)^2, \tag{8} \]
holds for \( (\zeta, z) \in (\bar{U} \setminus U) \times U \), where
\[ \Phi(\zeta, z) = \langle P(\zeta, z), \zeta - z \rangle = \sum_{i=1}^{n} P_i(\zeta, z) \cdot (\zeta_i - z_i). \]

In \( \mathbb{F} \) special strong \( \mathbf{M} \)-barriers for a q-concave CR submanifold of the form (1) were constructed. Below we describe these barriers.

According to (1) we may assume that \( U = \bar{U} \cap \mathbf{M} \) is a set of common zeros of smooth functions \( \{\rho_k, \quad k = 1, \ldots, m\} \). Then, using the q-concavity of \( \mathbf{M} \) and applying Kohn’s lemma to the set of functions \( \{\rho_k\} \) we can construct a new set of functions \( \tilde{\rho}_1, \ldots, \tilde{\rho}_m \) of the form:
\[ \tilde{\rho}_k(z) = \rho_k(z) + A \cdot \left( \sum_{i=1}^{m} \rho_i^2(z) \right), \]
with large enough constant \( A > 0 \) and such that for any \( z \in \mathbf{M} \) there exist an open neighborhood \( U \ni z \) and a family \( E_{q+m}(\theta, z) \) of \( q + m \) dimensional complex linear subspaces in \( \mathbb{C}^n \) smoothly depending on \( (\theta, z) \in \mathbb{S}^{m-1} \times U \) and such that \( -\mathcal{H}_z \tilde{\rho}_\theta \) is strictly negative on \( E_{q+m}(\theta, z) \) with all negative eigenvalues not exceeding some \( c < 0 \).

To simplify notations we will assume that the functions \( \rho_1, \ldots, \rho_m \) already satisfy this condition.

Let \( E_{n-q-m}^\perp(\theta, z) \) be the family of \( n - q - m \) dimensional subspaces in \( T(\mathbf{G}) \) orthogonal to \( E_{q+m}(\theta, z) \) and let
\[ a_j(\theta, z) = (a_{j1}(\theta, z), \ldots, a_{jn}(\theta, z)) \quad \text{for} \quad j = 1, \ldots, n - q - m \]
be a set of $C^2$ smooth vector functions representing an orthonormal basis in $E_{n-q-m}^\perp(\theta, z)$.

Defining for $(\theta, z, w) \in S^{m-1} \times U \times \mathbb{C}^n$

$$A_j(\theta, z, w) = \sum_{i=1}^n a_{ji}(\theta, z) \cdot w_i, \quad (j = 1, \ldots, n-q-m)$$

we construct the form

$$\mathcal{A}(\theta, z, w) = \sum_{j=1}^{n-q-m} A_j(\theta, z, w) \cdot \bar{A}_j(\theta, z, w)$$

such that the hermitian form

$$\mathcal{H}_x \rho_\theta(w) + \mathcal{A}(\theta, z, w)$$

is strictly positive definite in $w$ for $(\theta, z) \in S^{m-1} \times U$.

Then we define for $\zeta, \zeta \in U$:

$$Q_i^{(k)}(\zeta) = -\frac{\partial_{\zeta_k} \rho_\theta(z)}{\rho\theta(z)}(\zeta),$$

$$F^{(k)}(\zeta, z) = \langle Q^{(k)}(\zeta), \zeta - z \rangle,$$

$$P_i(\zeta, z) = \sum_{k=1}^m Q_i^{(k)}(\zeta) \cdot F^{(k)}(\zeta, z) + \sum_{j=1}^{n-q-m} a_{ji}(\theta(\zeta), z) \cdot \bar{A}_j(\theta(\zeta), z, \zeta - z) \cdot A(\theta(\zeta), z, \zeta - z),$$

$$\Phi(\zeta, z) = \langle P(\zeta, z), \zeta - z \rangle$$

$$= \sum_{k=1}^m F^{(k)}(\zeta, z) \cdot \bar{F}^{(k)}(\zeta, z) + \frac{1}{4} \mathcal{A}^2(\theta(\zeta), z, \zeta - z)$$

with

$$\theta_k(\zeta) = -\frac{\rho_\theta(\zeta)}{\rho(\zeta)} \quad \text{for} \quad k = 1, \ldots, m.$$ 

From the definitions of $Q_i^{(s)}(\zeta, z)$, $F^{(s)}(\zeta, z)$ and $\mathcal{A}(\zeta, z)$ follow the equalities that will be used in the further estimates

$$\sum_{i=1}^n Q_i^{(s)}(\zeta) d\zeta_i = d\zeta F^{(s)}(\zeta, z), \quad \bar{\partial}_{\bar{\zeta}}(\mathcal{A} \bar{A}_j) = \mu^{(j)}(\zeta, z) + \mu^{(j)}(\zeta, z)$$

where

$$\mu^{(j)}(\zeta, z) = \mathcal{A}(\zeta, z) \cdot \sum_{i=1}^n \bar{a}_{ji}(\theta(\zeta), z) d\zeta_i +$$

$$+ \bar{A}_j(\zeta, z) \cdot \sum_{i=1}^n \sum_{k=1}^{n-q-m} A_k(\zeta, z) \bar{a}_{ki}(\theta(\zeta), z) d\zeta_i,$$

$$\mu^{(j)}(\zeta, z) = \mathcal{A}(\zeta, z) \cdot \sum_{i=1}^n (\zeta_i - \bar{z}_i) \bar{\partial}_{\bar{\zeta}}(\bar{a}_{ji}(\theta(\zeta), z) +$$

$$+ \bar{A}_j(\zeta, z) \cdot \sum_{i=1}^n (\zeta_i - \bar{z}_i) \sum_{k=1}^{n-q-m} A_k(\zeta, z) \bar{\partial}_{\bar{\zeta}}(\bar{a}_{ki}(\theta(\zeta), z) +$$

$$+ \bar{A}_j(\zeta, z) \cdot \sum_{i=1}^n (\zeta_i - \bar{z}_i) \sum_{k=1}^{n-q-m} \bar{A}_k(\zeta, z) \bar{\partial}_{\bar{\zeta}}(a_{ki}(\theta(\zeta), z),$$
where $\chi_s(\zeta, z) = \sum_{i=1}^{n} (\bar{\zeta}_i - \bar{z}_i) \cdot \chi_{is}(\zeta, z)$

are differential forms in $\zeta$ with smooth coefficients and

$$\kappa_s(\zeta, z) = \sum_{i=1}^{n} (\bar{\zeta}_i - \bar{z}_i) \cdot \kappa_{is}(\zeta, z)$$

are differential forms in $z$ with smooth coefficients.

In our construction of the global integral formula on $\mathcal{M}$ we will use local formula from [P]. To describe this formula we introduce the following notations.

We define the tubular neighborhood $\mathbf{G}_\epsilon$ of $\mathcal{M}$ in $\mathbf{G}$ as follows:

$$\mathbf{G}_\epsilon = \{ z \in \mathbf{G} : \rho(z) < \epsilon \},$$

where $\rho(z) = \left( \sum_{k=1}^{m} \rho_k^2(z) \right)^{1/2}$. The boundary of $\mathbf{G}_\epsilon - \mathcal{M}_\epsilon$ is defined by the condition

$$\mathcal{M}_\epsilon = \{ z \in \mathbf{G} : \rho(z) = \epsilon \}.$$

For a sufficiently small neighborhood $\bar{U} \subset \mathbf{G}$ we may assume that functions

$$\rho_k(\zeta), \text{ Im}F^{(k)}(\zeta, z) \{ k = 1, \ldots, m \}$$

have a nonzero jacobian in Re$\zeta_1, \ldots, \text{Re}\zeta_m$, Im$\zeta_1, \ldots, \text{Im}\zeta_m$ for $z, \zeta \in \bar{U}$. Therefore, for any fixed $z \in \bar{U}$ these functions may be chosen as local $C^\infty$ coordinates in $\zeta$. We may also complement the functions above by holomorphic functions $w_j(\zeta) = u_j(\zeta) + iv_j(\zeta)$ with $j = 1, \ldots, n - m$ so that the functions

$$\rho_k(\zeta), \text{ Im}F^{(k)}(\zeta, z) \{ k = 1, \ldots, m \},$$

$$u_j(\zeta), v_j(\zeta) \{ j = 1, \ldots, n - m \},$$

represent a complete system of local coordinates in $\zeta \in \bar{U}$ for any fixed $z \in \bar{U}$.

The following complex valued vector fields on $U$ for any fixed $z \in U$

$$Y_{i,\zeta}(z) = \frac{\partial}{\partial \text{Im}F^{(i)}(\zeta, z)} \text{ for } i = 1, \ldots, m,$$

$$\frac{\partial}{\partial w_i}, \frac{\partial}{\partial \bar{w}_i} \text{ for } i = 1, \ldots, n - m,$$

represent a basis in $\mathbf{CT}(\mathcal{M})$ at any $z \in U$.

We need also to consider local extensions of functions and forms from $U = \bar{U} \cap \mathcal{M}$ to $\bar{U}$. Defining appropriate functional spaces on these neighborhoods we say that a function $h$ on $\bar{U}(\epsilon) = \mathbf{G}_\epsilon \cap \bar{U}$ is in $\Gamma^{p,\alpha} \left( \{ \rho \}, \bar{U}(\epsilon) \right)$ if

$$\|h\|_{\Gamma^{p,\alpha} \left( \{ \rho \}, \bar{U}(\epsilon) \right)} = \|h\|_{A^{p+\alpha/2}(\bar{U}(\epsilon))} + \sup_{(\sum_{k=1}^{m} \delta_k^2)^{1/2} < \epsilon} \|h\|_{\Gamma^{p,\alpha}(U(\delta_1, \ldots, \delta_m))} < \infty$$

where

$$U(\delta_1, \ldots, \delta_m) = \{ z \in \bar{U} : \rho_1(z) = \delta_1, \ldots, \rho_m(z) = \delta_m \}.$$
For a differential form \( f = \sum_{I,J} f_{I,J}(z) dz^I \wedge d\bar{z}^J \) with \(|I| = l\) and \(|J| = r\) we say that \( f \in \Gamma_{(l,r)}^p(\{\rho\}, \bar{U}(\epsilon)) \) if \( f_{I,J} \in \Gamma_{(l,r)}^p(\{\rho\}, \bar{U}(\epsilon)) \).

We introduce a local extension operator for \( \bar{U}(\epsilon) \) and \( U = \bar{U}(\epsilon) \cap M \)

\[
E_U : \Gamma_{(0,r)}^p(U) \to \Gamma_{(0,r)}^p(\{\rho\}, \bar{U}(\epsilon))
\]

which we define by extending all the coefficients of the differential form identically with respect to coordinates, complementary to \( \rho_1, \ldots, \rho_m \) in \( \bar{U} \). From the construction it follows that \( E_U \) satisfies the following estimate

\[
\|E_U(g)\|_{\Gamma_{(0,r)}^p(\{\rho\}, \bar{U}(\epsilon))} \leq C \cdot \|g\|_{\Gamma_{(0,r)}^p(U)}.
\]

The following proposition provides local integral formula for \( \bar{\partial}_M \).

**Proposition 2.1.** Let \( M \subset G \) be a \( C^\infty \) regular \( q \)-concave CR submanifold of the form (1) and \( \bar{U} \) an open neighborhood in \( G \) with analytic coordinates \( z_1, \ldots, z_n \).

Then for \( r = 1, \ldots, q-1 \) and any differential form \( g \in \mathcal{C}_{(0,r)}^\omega(M) \) with compact support in \( U \) the following equality

\[
g = \bar{\partial}_MR_r(g) + R_{r+1}(\bar{\partial}_Mg) + H_r(g),
\]

holds, where

\[
R_r(g)(z) = (-1)^r \frac{(n-1)!}{(2\pi i)^n} \cdot pr_M \circ \lim_{\epsilon \to 0} \int_{M_r \times [0,1]} \bar{g}(\zeta) \wedge \omega_{r-1} \left((1-t) - \frac{\bar{z} - z}{|\zeta - z|^2} + \frac{t}{\Phi(\zeta, z)}\right) \wedge \omega(\zeta),
\]

\[
H_r(g)(z) = (-1)^r \frac{(n-1)!}{(2\pi i)^n} \cdot pr_M \circ \lim_{\epsilon \to 0} \int_{M_r} \bar{g}(\zeta) \wedge \omega_{r} \left(\frac{P(\zeta, z)}{\Phi(\zeta, z)}\right) \wedge \omega(\zeta),
\]

\( \bar{g} = E_U(g) \) is the extension of \( g \), \( \Phi(\zeta, z) \) is a local barrier for \( \bar{U} \) constructed in (9) and \( pr_M \) denotes the operator of projection to the space of tangential differential forms on \( M \).

**Remark.** As follows from the definitions above spaces \( \Gamma_{(0,r)}^p(\{\rho\}, \bar{U}(\epsilon)) \) as well as the extension operator \( E_U \) depend on the choice of functions \( \rho_1, \ldots, \rho_m \). But we notice (cf. [P]) that the operators \( R_r \) and \( H_r \) are independent of the choice of functions \( \rho_1, \ldots, \rho_m \) and of extension operator \( E_U \).

To construct now global formula on \( M \) we consider two finite coverings \( \{\bar{U}_i \subset \bar{U}_i'\} \) of \( G \) and two partitions of unity \( \{\vartheta_i\} \) and \( \{\vartheta_i'\} \) subordinate to these coverings and such that \( \vartheta_i'(z) = 1 \) for \( z \in \text{supp}(\vartheta_i) \).

Applying proposition 2.1 to the form \( \vartheta_i g \) in \( U_i' \) we obtain

\[
\vartheta_i(z)g(z) = \bar{\partial}_MR_i(g)(z) + R_{i+1}(\bar{\partial}_M\vartheta_i g)(z) + H_i(\vartheta_i g)(z).
\]

Multiplying the equality above by \( \vartheta_i'(z) \) and using equalities

\[
\vartheta_i'(z) \cdot \bar{\partial}_MR_i(g)(z) = \bar{\partial}_M[\vartheta_i'(z) \cdot R_i(g)(z)] - \bar{\partial}_M\vartheta_i'(z) \wedge R_i(g)(z)
\]

and

\[
R_{i+1}(\bar{\partial}_M\vartheta_i g)(z) = R_{i+1}(\bar{\partial}_M\vartheta_i \wedge g)(z) + R_{i+1}(\vartheta_i' \bar{\partial}_Mg)(z)
\]

we obtain

\[
\vartheta_i(z)g(z) = \bar{\partial}_MR_i(g)(z) + R_{i+1}(\bar{\partial}_Mg)(z) + H_i(\vartheta_i g)(z)
\]

with

\[
R_i(g)(z) = \vartheta_i'(z) \cdot R_i(\vartheta_i g)(z)
\]

and

\[
H_i(\vartheta_i g)(z) = -\bar{\partial}_M\vartheta_i'(z) \wedge R_i(\vartheta_i g)(z) + \vartheta_i'(z) \cdot R_{i+1}(\bar{\partial}_M\vartheta_i \wedge g)(z) + \vartheta_i(\vartheta_i g)(z).
\]

\[
= -\bar{\partial}_M\vartheta_i'(z) \wedge R_i(\vartheta_i g)(z) + \vartheta_i'(z) \cdot R_{i+1}(\bar{\partial}_M\vartheta_i \wedge g)(z) + \vartheta_i(\vartheta_i g)(z).
\]

\[
= -\bar{\partial}_M\vartheta_i'(z) \wedge R_i(\vartheta_i g)(z) + \vartheta_i'(z) \cdot R_{i+1}(\bar{\partial}_M\vartheta_i \wedge g)(z) + \vartheta_i(\vartheta_i g)(z).
\]
Adding equalities (14) for all $t$ we obtain

**Proposition 2.2.** Let $M \subset G$ be a $C^\infty$ regular $q$-concave CR submanifold of the form (1).

Then for $r = 1, ..., q - 1$ and any differential form $g \in C^{(0,r)}(M)$ the following equality

$$g = \partial_M R_r(g) + R_{r+1}(\partial_M g) + H_r(g),$$

holds, where

$$R_r(g)(z) = \sum_i \vartheta_i^r(z) \cdot R^i_r(\vartheta_i g)(z)$$

and

$$H_r(g)(z) = \sum_i [-\partial_M \vartheta_i^r(z) \wedge R^i_r(\vartheta_i g)(z) + R^i_{r+1}(\partial_M \vartheta_i \wedge g)(z) + \vartheta_i^r(z) \cdot H^i_r(\vartheta_i g)(z)].$$

### 3. Boundedness of $R_r$.

From the construction of operator $R_r$ we conclude that in order to prove necessary estimates for operator $R_r$ it suffices to prove these estimates for operator $R_r$. In the proposition below we state necessary estimates for operator $R_r$.

**Proposition 3.1.** Let $0 < \alpha < 1$, $M \subset G$ be a $C^\infty$ regular $q$-concave CR submanifold of the form (1) and let $g \in \Gamma^{p,\alpha}_{(0,r-1)}(M)$ be a form with compact support in $U = \overline{U} \cap M$.

Then operator $R_r$, defined in (13) satisfies the following estimate

$$\| R_r(g) \|_{\Gamma^{p,\alpha+1}_{(0,r-1)}(U)} \leq C \cdot \| g \|_{\Gamma^{p,\alpha}_{(0,r)}(U)}$$

with a constant $C$ independent of $g$.

In our proof of proposition 3.1 we will use the approximation of $R_r$ by the operators

$$R_r(\epsilon)(f)(z) = (-1)^r \cdot \text{pr}_M \circ \frac{(n-1)!}{(2\pi i)^n}$$

$$\times \sum_i \int_{M_r \times [0,1]} \vartheta_i(\zeta) \check{f}(\zeta) \wedge \omega_{r-1} \left( (1-t) \frac{\zeta - \bar{z}}{|\zeta - z|^2} + t \frac{P^i(\zeta, z)}{\Phi^i(\zeta, z)} \right) \wedge \omega(\zeta)$$

when $\epsilon$ goes to 0.

Using equality (12) we obtain the following representation of kernels of these integrals on $\overline{U} \times [0,1] \times M$:

$$\vartheta_i(\zeta) \cdot \omega_{r-1} \left( (1-t) \frac{\zeta - \bar{z}}{|\zeta - z|^2} + t \frac{P^i(\zeta, z)}{\Phi^i(\zeta, z)} \right) \wedge \omega(\zeta)$$

$$= \sum_{i,J} a_{i,J}(t, \zeta, z) dt \wedge \lambda^{i,J}_{r-1}(\zeta, z) + \sum_{i,J} b_{i,J}(t, \zeta, z) dt \wedge \gamma^{i,J}_{r-1}(\zeta, z),$$

where $i$ is an index, $J = \bigcup_{i=1}^{11} J_i$ is a multiindex such that $i \notin J$, $a_{i,J}(t, \zeta, z)$ and $b_{i,J}(t, \zeta, z)$ are polynomials in $t$ with coefficients that are smooth functions of $z$, $\zeta$ and $\theta(\zeta)$, and $\lambda^{i,J}_{r-1}(\zeta, z)$ and $\gamma^{i,J}_{r-1}(\zeta, z)$ are defined as follows:

$$\lambda^{i,J}_{r-1}(\zeta, z) = \frac{1}{|\zeta - z|^{2(|J|_1 + |J|_2 + 1)} \cdot \Phi(\zeta, z)^{n-|J|_1 - |J|_2 - 1}}$$

$$\times \sum_{\bar{\zeta}, \bar{z}, Q^{(i)}} \text{Det} \left[ \zeta - \bar{z}, Q^{(i)} F^{(i)}, \bar{d}_{\zeta}, \bar{Q} dp, \bar{Q} \cdot \bar{\chi}, \bar{A} \cdot \partial Q a, \right]$$

$$\text{Det} \left[ \zeta - \bar{z}, Q^{(i)} F^{(i)}, \bar{d}_{\zeta}, \bar{Q} dp, \bar{Q} \cdot \bar{\chi}, \bar{A} \cdot \partial Q a, \right].$$
where

\[ \rho(I) = \begin{cases} \sum_{j \in J_0} \frac{1}{a \cdot \mu_\nu, a \cdot \mu_\tau, d\zeta, A\partial_a, a \cdot \partial_\zeta(A\partial_\zeta, F \cdot \partial_\zeta Q, \nabla \cdot Q) \wedge \omega(\zeta),} \\
\end{cases} \]

and

\[ \gamma_{r-1}^{i,j}(\zeta, z) = \frac{1}{|\zeta - z|^{|J_1|+|J_2|+1}} \cdot \Phi(\zeta, z)^n - |J_1| - |J_2| - 1 
\]

\[ \times \sum_{j \in J_0} \text{Det} \left[ \frac{1}{a \cdot \mu_\nu, a \cdot \mu_\tau, d\zeta, A\partial_a, a \cdot \partial_\zeta(A\partial_\zeta, F \cdot \partial_\zeta Q, \nabla \cdot Q) \wedge \omega(\zeta).} \right] \]

In the proof of boundedness of operators \( R_r \) we will need to know the differentiability properties of integrals with kernels \( \lambda_{r-1}^{i,j}(\zeta, z) \) and \( \gamma_{r-1}^{i,j}(\zeta, z) \). In the lemmas below we prepare necessary tools for the proof of boundedness of \( R_r \).

At first we consider the differentiability property of slightly different kernels

\[ \kappa_{d,j}^I(\zeta, z) = \left\{ \frac{\rho(\zeta)}{|I_1|} (\zeta - z)^{I_2} (\bar{\zeta} - \bar{z})^{I_3} a \cup \partial \partial F^{(i)} \wedge dp_k \wedge d\partial(\zeta) \wedge d\sigma_{2n-m}(\zeta), \right\} \]

where \( I = \bigcup_{j=1}^6 I_j \) and \( I_j \) for \( j = 1, \ldots, 6 \) are multiindices such that \( I_1 \) contains \( m \) indices, \( I_2, I_3 \) contain \( n \) indices, \( I_4 \cup I_5 \cup I_6 \) contains \( m - 1 \) indices, \( |I_4| + |I_5| + |I_6| = m - 1 \), and \( \{\rho(\zeta)\} I_1 = \Pi_{i \in I_2} \rho s(\zeta)^s, (\zeta - z)^I = \Pi_{i \in I_2} (\zeta - \bar{z})^s, (\bar{\zeta} - \bar{z})^{I_3} = \Pi_{i \in I_2} (\bar{\zeta} - \bar{z})^s. \)

For kernels \( \kappa_{d,j}^I \) we introduce also the following notation

\[ k \left( \kappa_{d,j}^I \right) = d - |I_2| - |I_3|, \]

\[ h \left( \kappa_{d,j}^I \right) = 2j, \]

\[ l \left( \kappa_{d,j}^I \right) = |I_1| + |I_5|, \]

\[ s \left( \kappa_{d,j}^I \right) = |I_4|. \]

**Lemma 3.2.** Let \( U = \bar{U} \cap M \) be a neighborhood with \( \Phi(\zeta, z) \) constructed in (9), \( U(\epsilon) = \bar{U} \cap M_\epsilon \) and let \( g(\zeta, z, \theta, t) \) be a smooth form with compact support in \( \bar{U}_\zeta \times \bar{U}_z \times S^{n-1} \times [0,1] \).

Then for \( g(\zeta, z, t) = g(\zeta, z, \theta(\zeta), t) \) and a vector field

\[ D = \sum_{j=1}^n a_j(z) \frac{\partial}{\partial z_j} + \sum_{j=1}^n b_j(z) \frac{\partial}{\partial \zeta_j} \in CT(M) \]

the following equality holds

\[ D \left( \int_{U(\epsilon) \times [0,1]} g(\zeta, z, t) \cdot \kappa_{d,j}^I(\zeta, z) dt \right) = \int_{U(\epsilon) \times [0,1]} [Dg(\zeta, z, t)] \cdot \kappa_{d,j}^I(\zeta, z) dt \]

\[ + \int_{U(\epsilon) \times [0,1]} [D_\zeta g(\zeta, z, t)] \cdot \kappa_{d,j}^I(\zeta, z) dt + \sum_{S,a,b} \int_{U(\epsilon) \times [0,1]} c_{(S,a,b)}(\zeta, z, t) \cdot g(\zeta, z, t) \cdot \kappa_{d,j}^S(\zeta, z) dt \]

\[ + \sum_{k=1}^m \sum_S \int_{U(\epsilon) \times [0,1]} c_{(k,S)}(\zeta, z, t) \cdot [Y_k, \zeta(\zeta) g(\zeta, z, t)] \cdot \kappa_{d,j}^S(\zeta, z) dt \]
where
\[ g(\zeta, z, t) = g(\zeta, z, \theta(\zeta), t), \]
\[ c_{\{S,a,b\}}(\zeta, z, t), c_{\{i,S\}}(\zeta, z, t) \] are \( C^\infty \) functions of \( \zeta, z, \theta(\zeta), t, \)
vector field \( D_\zeta \) defined as
\[ D_\zeta = \sum_{j=1}^{n} a_j(z) \frac{\partial}{\partial \zeta_j} + \sum_{j=1}^{n} b_j(z) \frac{\partial}{\partial \zeta_j} \]
and indices \( a, b \) and multiindices \( S \) satisfy the following conditions
\[ k \left( \mathcal{K}_{a,b}^S \right) + h \left( \mathcal{K}_{a,b}^S \right) - l \left( \mathcal{K}_{a,b}^S \right) - s \left( \mathcal{K}_{a,b}^S \right) \]
\[ \leq k \left( \mathcal{K}_{d,j}^l \right) + h \left( \mathcal{K}_{d,j}^l \right) - l \left( \mathcal{K}_{d,j}^l \right) - s \left( \mathcal{K}_{d,j}^l \right), \]
\[ k \left( \mathcal{K}_{a,b}^S \right) + 2h \left( \mathcal{K}_{a,b}^S \right) - 2l \left( \mathcal{K}_{a,b}^S \right) - 2s \left( \mathcal{K}_{a,b}^S \right) \]
\[ \leq k \left( \mathcal{K}_{d,j}^l \right) + 2h \left( \mathcal{K}_{d,j}^l \right) - 2l \left( \mathcal{K}_{d,j}^l \right) - 2s \left( \mathcal{K}_{d,j}^l \right). \]

**Proof.**
To prove the lemma we represent integral from the left hand side of (21) as
\[ D \left( \int_{U(\epsilon) \times [0,1]} g(\zeta, z, t) \cdot \mathcal{K}_{d,j}^l(\zeta, z) dt \right) = \int_{U(\epsilon) \times [0,1]} [Dg(\zeta, z, t)] \cdot \mathcal{K}_{d,j}^l(\zeta, z) dt \]
\[ - \int_{U(\epsilon) \times [0,1]} g(\zeta, z, t) \left[ D_\zeta \mathcal{K}_{d,j}^l(\zeta, z) \right] dt + \int_{U(\epsilon) \times [0,1]} g(\zeta, z, t) \left[ \left( D + D_\zeta \right) \mathcal{K}_{d,j}^l(\zeta, z) \right] dt. \]

To transform the second term of the right hand side of (23) we apply integration by parts and obtain
\[ \int_{U(\epsilon) \times [0,1]} g(\zeta, z, t) \left[ D_\zeta \mathcal{K}_{d,j}^l(\zeta, z) \right] dt = - \int_{U(\epsilon) \times [0,1]} [D_\zeta g(\zeta, z, t)] \mathcal{K}_{d,j}^l(\zeta, z) dt. \]

To transform the third term of the right hand side of (23) we will use the estimates below that follow from the definitions of \( F^{(k)}(\zeta, z) \) and \( \mathcal{A}(\zeta, z) \) and from the fact that \( D \in \mathcal{CT}(\mathcal{M}) \)
\[ (D + D_\zeta) \mathcal{A}^2(\zeta, z) = \mathcal{O} \left( \mathcal{A}^2(\zeta, z) \right), \]
\[ (D + D_\zeta) \text{Re} F^{(k)}(\zeta, z) = \mathcal{O} \left( |\zeta - z|^2 \right), \]
\[ (D + D_\zeta) \text{Im} F^{(k)}(\zeta, z) = \mathcal{O} \left( |\zeta - z| \right). \]

Applying the operators \( D \) and \( D_\zeta \) to \( \mathcal{K}_{d,j}^l(\zeta, z) \) we obtain
\[ (D + D_\zeta) \mathcal{K}_{d,j}^l(\zeta, z) \]
\[ = (-j) \frac{\rho(\zeta)}{|\zeta - z|^d} \cdot \Phi(\zeta, z)^{d+1} \cdot \left\{ \left[ (D + D_\zeta) \mathcal{A}^2 \right] d_\zeta \text{Re} F^{(i)} \wedge d\theta_1(\zeta) \wedge d\sigma_{2n-m}(\zeta) \right. \]
\[ + \sum_{k=1}^{m} 2\text{Re} F^{(k)} \cdot \left[ (D + D_\zeta) \text{Re} F^{(k)} \right] d_\zeta \text{Re} F^{(i)} \wedge d\theta_1(\zeta) \wedge d\sigma_{2n-m}(\zeta) \]
\[ + \sum_{k=1}^{m} 2\text{Im} F^{(k)} \cdot \left[ (D + D_\zeta) \text{Im} F^{(k)} \right] d_\zeta \text{Re} F^{(i)} \wedge d\theta_1(\zeta) \wedge d\sigma_{2n-m}(\zeta) \right\} \]
From estimates (24) we conclude that the first two terms of the right hand side of (25) can be represented as linear combinations with $C^\infty \left( \hat{U}_\zeta \times \hat{U}_z \times \mathbb{S}^{n-1} \times [0,1] \right)$ coefficients of kernels $K_{d,j+1}^S$ with either

$$|S_2| + |S_3| = |I_2| + |I_3| + 4$$

or

$$|S_1| = |I_1| + 1 \text{ and } |S_2| + |S_3| = |I_2| + |I_3| + 2.$$ 

To handle the last term of the right hand side of (25) we use the third estimate from (24) and represent corresponding integrals as

$$\int_{U(\zeta) \times [0,1]} \left( -j \right) \frac{\{\rho(\zeta)\} I_1(\zeta - z) I_2(\zeta - \bar{z}) I_3 2\text{Im}\left(\frac{1}{\zeta - z} F^{(k)}(\cdot) \right)}{\zeta - z + d} \cdot \Phi(\zeta, z) J^{+1} \times \left( (D + D_\zeta) \text{Im}\left(\frac{1}{\zeta - z} F^{(k)}(\cdot) \right) \right) \frac{d\zeta \Phi(\zeta, z) J^{+1}}{I_2(\zeta - \bar{z}) S_3}$$

$$\times \left( \int_{U(\zeta) \times [0,1]} c(S) (\zeta, z, t) \cdot g(\zeta, z, t) \cdot \frac{\{\rho(\zeta)\} I_1(\zeta - z) S_2 (\zeta - \bar{z}) S_3}{|\zeta - z|^d} \right) \right) dt.$$

Kernels of the second term of the right hand side of (26) may be represented as linear combinations with $C^\infty \left( \hat{U}_\zeta \times \hat{U}_z \times \mathbb{S}^{n-1} \times [0,1] \right)$ coefficients of kernels $K_{d,j+1}^S$ with either

$$|S_2| + |S_3| = |I_2| + |I_3| + 4$$

or

$$|S_1| = |I_1| + 1 \text{ and } |S_2| + |S_3| = |I_2| + |I_3| + 2.$$ 

The first term of the right hand side of (26) we transform applying integration by parts by

$$\int_{U(\zeta) \times [0,1]} c(S) (\zeta, z, t) \cdot g(\zeta, z, t) \cdot \frac{\{\rho(\zeta)\} I_1(\zeta - z) S_2 (\zeta - \bar{z}) S_3}{|\zeta - z|^d} \right) dt.$$

From (25), (26) and (27) follows statement of the lemma for the third term of the right hand side of (23).
In the next two lemmas we prove transformation and differentiation formulas for the kernels
\[ B^{T,T}_{d,j}(\zeta, z) = \{\text{Im}F(\zeta, z)\}^T \cdot K_{d,j}(\zeta, z), \]
and
\[ B^{T,0}_{d,0}(\zeta, z) = \{\text{Im}F(\zeta, z)\}^T \cdot K_{d,0}(\zeta, z) \cdot \log \Phi(\zeta, z), \]
where \( \{\text{Im}F(\zeta, z)\}^T = \prod_{s \in T} \left(\text{Im}F^{(s)}(\zeta, z)\right)^{t_s} \) for a multiindex \( T = (t_1, \ldots, t_m) \).

Generalizing corresponding notations for \( K_{d,j} \) we denote
\[
\begin{align*}
  k \left( B^{T,T}_{d,j} \right) &= d - |I_2| - |I_3|, \\
  h \left( B^{T,T}_{d,j} \right) &= 2j - |T|, \\
  l \left( B^{T,T}_{d,j} \right) &= |I_1| + |I_5|, \\
  s \left( B^{T,T}_{d,j} \right) &= |I_4|.
\end{align*}
\]

**Lemma 3.3.** Let \( U = \tilde{U} \cap M \) be a neighborhood of \( z_0 \in M \) with \( \Phi(\zeta, z) \) constructed in (9), \( U(\epsilon) = \tilde{U} \cap M_\epsilon \), \( g(\zeta, z, \theta, t) \) be a smooth form with compact support and let \( T \) be a multiindex such that \( t_k \) is a first nonzero index in \( T \).

Then the following equality holds for \( g(\zeta, z, t) = g(\zeta, z, \theta(\zeta), t) \) and \( j > 0 \)
\[
\int_{U(\epsilon) \times [0,1]} g(\zeta, z, t) \cdot B^{T,T}_{d,j}(\zeta, z) dt
\]
\[
= c \cdot \int_{U(\epsilon) \times [0,1]} \{Y_{k,\zeta}(z)g(\zeta, z, t)\} \cdot B^{T,T}_{d,j-1}(\zeta, z) dt
\]
\[
+ \sum_{\{L,S,a,b\}} \int_{U(\epsilon) \times [0,1]} c_{(L,S,a,b)}(\zeta, z, t) \cdot g(\zeta, z, t) \cdot B^{L,S}_{a,b}(\zeta, z) dt,
\]
where \( \hat{T} = (t_1, \cdots, t_k - 1, \cdots, t_m) \), \( c \) is a constant, \( c_{(L,S,a,b)}(\zeta, z, t) = c_{(L,S,a,b)}(\zeta, z, \theta(\zeta), t) \) with \( c_{(L,S,a,b)}(\zeta, z, \theta, t) \in C^\infty \left( \tilde{U}_\zeta \times \tilde{U}_z \times S^{n-1} \times [0,1] \right) \)
and indices \( a, b \) and multiindices \( L, S \) are such that
\[
|L| < |T|,
\]
\[
k \left( B^{L,S}_{a,b} \right) + h \left( B^{L,S}_{a,b} \right) - l \left( B^{L,S}_{a,b} \right) - s \left( B^{L,S}_{a,b} \right)
\]
\[
\leq k \left( B^{T,T}_{d,j} \right) + h \left( B^{T,T}_{d,j} \right) - l \left( B^{T,T}_{d,j} \right) - s \left( B^{T,T}_{d,j} \right),
\]
\[
k \left( B^{L,S}_{a,b} \right) + 2h \left( B^{L,S}_{a,b} \right) - 2l \left( B^{L,S}_{a,b} \right) - 2s \left( B^{L,S}_{a,b} \right)
\]
\[
\leq k \left( B^{T,T}_{d,j} \right) + 2h \left( B^{T,T}_{d,j} \right) - 2l \left( B^{T,T}_{d,j} \right) - 2s \left( B^{T,T}_{d,j} \right).
\]
For $j = 1$ and $|T| = 1$ stronger inequalities hold for the terms $\mathcal{B}_{a,0}^{0,0}$ of the right hand side of (28)

$$
k \left( \mathcal{B}_{a,0}^{0,0} \right) - l \left( \mathcal{B}_{a,0}^{0,0} \right) - s \left( \mathcal{B}_{a,0}^{0,0} \right) + 1$$

$$
\leq k \left( \mathcal{B}_{d,1}^{T,I} \right) + h \left( \mathcal{B}_{d,1}^{T,I} \right) - l \left( \mathcal{B}_{d,1}^{T,I} \right) - s \left( \mathcal{B}_{d,1}^{T,I} \right),
$$

$$
k \left( \mathcal{B}_{a,0}^{0,0} \right) - 2l \left( \mathcal{B}_{a,0}^{0,0} \right) - 2s \left( \mathcal{B}_{a,0}^{0,0} \right) + 2$$

$$
\leq k \left( \mathcal{B}_{d,1}^{T,I} \right) + 2h \left( \mathcal{B}_{d,1}^{T,I} \right) - 2l \left( \mathcal{B}_{d,1}^{T,I} \right) - 2s \left( \mathcal{B}_{d,1}^{T,I} \right).
$$

**Proof.**

The integral in the left hand side of (28) may be represented as a sum of two integrals

$$
\int_{U(\varepsilon) \times [0,1]} g(\zeta, z, t) \cdot \{\text{Im} F(\zeta, z)\}^{T} K_{d,j}^{I}(\zeta, z) dt
$$

$$
= -\frac{1}{2(j - 1)} \int_{U(\varepsilon) \times [0,1]} g(\zeta, z, t) \{\text{Im} F(\zeta, z)\}^{T} \left\{ \rho(\zeta) \right\}^{T} \left( \zeta - z \right)^{I_{3}} 1 \left| \zeta - z \right|^{d}
$$

$$
\times d\zeta \left( \frac{1}{\Phi(\zeta, z)} \right)^{j} \frac{i \in I_{4}}{d\zeta \text{Re} F(i)} \wedge d\theta_{1}(\zeta) \wedge \left( d\zeta \text{Im} F^{(k)} \vee d\sigma_{2n-m}(\zeta) \right) dt
$$

$$
- \int_{U(\varepsilon) \times [0,1]} g(\zeta, z, t) \{\text{Im} F(\zeta, z)\}^{T} \left\{ \rho(\zeta) \right\}^{T} \left( \zeta - z \right)^{I_{3}} 1 \left| \zeta - z \right|^{d}
$$

$$
\times \sum_{i=1}^{m} \text{Re} F(i) \frac{d\zeta \text{Re} F(i)}{\Phi(\zeta, z)} \frac{1}{d\zeta A^{2}} \frac{i \in I_{4}}{d\zeta \text{Re} F(i)} \wedge d\theta_{1}(\zeta) \wedge \left( d\zeta \text{Im} F^{(k)} \vee d\sigma_{2n-m}(\zeta) \right) dt.
$$

As follows from (24) the second integral admits necessary representation with indices $a, b$ and multiindices $L, S$ satisfying conditions (29). To transform the first integral from the right hand side of (31) we apply integration by parts and obtain

$$
= \frac{1}{2(j - 1)} \int_{U(\varepsilon) \times [0,1]} g(\zeta, z, t) \{\text{Im} F(\zeta, z)\}^{T} \left\{ \rho(\zeta) \right\}^{T} \left( \zeta - z \right)^{I_{3}} 1 \left| \zeta - z \right|^{d}
$$

$$
\times d\zeta \left( \frac{1}{\Phi(\zeta, z)} \right)^{j} \frac{i \in I_{4}}{d\zeta \text{Re} F(i)} \wedge d\theta_{1}(\zeta) \wedge \left( d\zeta \text{Im} F^{(k)} \vee d\sigma_{2n-m}(\zeta) \right) dt
$$

$$
= \frac{1}{2(j - 1)} \left[ \int_{U(\varepsilon) \times [0,1]} [Y_{k,\zeta}(\zeta) g(\zeta, z, t)] \cdot \{\text{Im} F(\zeta, z)\}^{T} K_{d,j-1}^{I}(\zeta, z) dt
$$

$$
+ \sum_{\{L, |T| = 1\}} \int_{U(\varepsilon) \times [0,1]} c_{(L)}(\zeta, z, t) \cdot g(\zeta, z, t) \cdot \{\text{Im} F(\zeta, z)\}^{L} K_{d,j-1}^{I}(\zeta, z) dt
$$

$$
+ \sum_{\{S_{a} + S_{b} = |T| + |T| + 1\}} \int_{U(\varepsilon) \times [0,1]} c_{(S)}(\zeta, z, t) \cdot g(\zeta, z, t) \cdot \{\text{Im} F(\zeta, z)\}^{T} K_{d,j-1}^{S}(\zeta, z) dt
$$

with kernels satisfying (29).
Lemma 3.4. Let $U = \tilde{U} \cap M$ be a neighborhood of $z_0 \in M$ with $\Phi(\zeta, z)$ constructed in (9), $U(\epsilon) = \tilde{U} \cap M_\epsilon$, and let $g(\zeta, z, \theta, t)$ be a smooth form with compact support in $\tilde{U}_\zeta \times \tilde{U}_z \times \mathbb{S}^{n-1} \times [0, 1]$. 

Then for $g(\zeta, z, t) = g(\zeta, z, \theta(\zeta), t)$ and a vector field

$$ D = \sum_{j=1}^{n} a_j(z) \frac{\partial}{\partial z_j} + \sum_{j=1}^{n} b_j(z) \frac{\partial}{\partial z_j} \in CT(M) $$

the following equality holds

$$ D \int_{U(\epsilon) \times [0,1]} g(\zeta, z, t) \cdot B_{d,0}^{0,1}(\zeta, z) dt = \int_{U(\epsilon) \times [0,1]} Dg(\zeta, z, t) \cdot B_{d,0}^{0,1}(\zeta, z) dt $$

(32)

$$ + \int_{U(\epsilon) \times [0,1]} [D_\zeta g(\zeta, z, t)] \cdot B_{d,0}^{0,1}(\zeta, z) dt + \sum_{\{S, a\}} \int_{U(\epsilon) \times [0,1]} c_{\{S, a\}}(\zeta, z, t) g(\zeta, z, t) \cdot K_{a,1}(\zeta, z) dt $$

(33)

with $a$ and $S$ satisfying (22).

Proof. 

Proof of the lemma is analogous to the proof of lemma 3.2.

We represent the integral from the left hand side of (32) as

$$ D \left( \int_{U(\epsilon) \times [0,1]} g(\zeta, z, t) \cdot B_{d,0}^{0,1}(\zeta, z) dt \right) = \int_{U(\epsilon) \times [0,1]} [Dg(\zeta, z, t)] \cdot B_{d,0}^{0,1}(\zeta, z) dt $$

(33)

$$ - \int_{U(\epsilon) \times [0,1]} g(\zeta, z, t) \left[ D_\zeta B_{d,0}^{0,1}(\zeta, z) \right] dt + \int_{U(\epsilon) \times [0,1]} g(\zeta, z, t) \left[ (D + D_\zeta) B_{d,0}^{0,1}(\zeta, z) \right] dt. $$

To transform the second term of the right hand side of (33) we apply integration by parts and obtain

$$ \int_{U(\epsilon) \times [0,1]} g(\zeta, z, t) \left[ D_\zeta B_{d,0}^{0,1}(\zeta, z) \right] dt = - \int_{U(\epsilon) \times [0,1]} [Dg(\zeta, z, t)] B_{d,0}^{0,1}(\zeta, z) dt. $$

To transform the third term of the right hand side of (33) we use estimates (24) and obtain

$$ (D + D_\zeta) B_{d,0}^{0,1}(\zeta, z) = \sum_{\{S, a\}} c_{\{S, a\}}(\zeta, z, t) \cdot K_{a,1}(\zeta, z) $$

(34)

$$ + \sum_{k=1}^{m} \frac{\{p(\zeta)\}^{I_1}(\zeta - z)^{I_2}(\bar{\zeta} - \bar{z})^{I_3}}{|\zeta - z|^d \cdot \Phi(\zeta, z)} $$

$$ \times 2\text{Im}F^{(k)} \cdot \left[ (D + D_\zeta) \text{Im}F^{(k)} \right] d_\zeta \text{Re}F^{(i)} \wedge d_\theta(\zeta) \wedge d\sigma_{2n-m}(\zeta). $$

Again using estimates (24) and applying integration by parts as in lemma 3.2 we obtain (32).

The following two simple lemmas (cf. [4]) will be used in the further estimates.
Lemma 3.5. Let $M$ be a generic CR submanifold in the unit ball $B^n$ in $\mathbb{C}^n$ of the form:

$$M = \{ z \in B^n : \rho_1(z) = \cdots = \rho_m(z) = 0 \},$$

where $\{ \rho_k \}$, $k = 1, \ldots, m$ ($m < n$) are real valued functions of the class $C^\infty$ satisfying

$$\partial \rho_1 \wedge \cdots \wedge \partial \rho_m \neq 0 \text{ on } M.$$

Then for any point $\zeta_0 \in M$ there exists a neighborhood $V_\varepsilon(\zeta_0) = \{ \zeta : |\zeta - \zeta_0| < \varepsilon \}$ such that for any $n \geq s > n - m$ and $p > 2n - s - m$ the following representation holds in $V_\varepsilon$:

$$d\zeta_{i_1} \wedge \cdots d\zeta_{i_p} \wedge d\xi_{s_1} \wedge \cdots d\xi_{s_k} = \sum d\rho_{j_1} \wedge \cdots d\rho_{j_p} \wedge g_{j_1 \cdots j_p}^{1 \cdots s} (\zeta),$$

with $g_{j_1 \cdots j_p}^{1 \cdots s}$ of the class $C^\infty(V_\varepsilon)$.

Lemma 3.6. Let

$$B(1) = \{ (\rho, \eta, w) \in \mathbb{R}^s \times \mathbb{R}^m \times \mathbb{C}^{n-m} : \sum_{i=1}^s \rho_i^2 + \sum_{i=1}^m \eta_i^2 + \sum_{i=1}^{n-m} |w_i|^2 < 1 \},$$

$$V(\delta) = \{ (\rho, \eta, w) \in \mathbb{R}^s \times \mathbb{R}^m \times \mathbb{C}^{n-m} : \sum_{i=1}^s |\rho_i| + \sum_{i=1}^m |\eta_i| + \sum_{i=1}^{n-m} |w_i|^2 < \delta^2 \},$$

$$K \{ \alpha, k, h, s \} (\rho, \eta, w, \varepsilon) = \frac{1}{(\varepsilon + \sum_{i=1}^s |\rho_i| + \sum_{i=1}^m |\eta_i| + \sum_{i=1}^{n-m} |w_i|)^k} \wedge \sum_{i=1}^s d\rho_i \wedge \sum_{i=1}^m d\eta_i \wedge \sum_{i=1}^{n-m} (dw_i \wedge dw_i) / (\sqrt{\varepsilon + \sum_{i=1}^s \sqrt{|\rho_i|} + \sum_{i=1}^m \sqrt{|\eta_i|} + \sum_{i=1}^{n-m} |w_i|)^{2k}},$$

with $0 \leq \alpha < 1$ and $k, h, s \in \mathbb{Z}$.

Let

$$I_1 \{ \alpha, k, h, s \} (\eta, \varepsilon, \delta) = \int_{V(\delta)} K \{ \alpha, k, h, s \} (\eta, w, \varepsilon),$$

and

$$I_2 \{ \alpha, k, h, s \} (\eta, \varepsilon, \delta) = \int_{B(1) \setminus V(\delta)} K \{ \alpha, k, h, s \} (\eta, w, \varepsilon).$$

Then

$$\begin{cases}
O \left( \varepsilon^{2n-m-k-h-s} \cdot (\log \varepsilon)^2 \right) & \text{if } k \geq 2n - 2m \text{ and } k + h - s \geq 2n - m, \\
O (\varepsilon^k) & \text{if } k \geq 2n - 2m \text{ and } k + h - s \leq 2n - m - 1,
\end{cases}
$$

$$= O (\varepsilon^0) \text{ if } \alpha > 0, \text{ and }$$

$$I_1 \{ \alpha, k, h, s \} (\eta, \varepsilon, \delta)
$$

$$\begin{cases}
O \left( \varepsilon^{2n-k-2(h-s)} / 2 \cdot \log \varepsilon \right) & \text{if } k \leq 2n - 2m - 1 \text{ and } k + 2(h - s) \geq 2n, \\
O (\varepsilon^k) & \text{if } k \leq 2n - 2m - 1 \text{ and } k + 2(h - s) \leq 2n - 1,
\end{cases}
$$

$$= O (\delta^0) \text{ if } \alpha > 0, \text{ and }$$

$$I_2 \{ \alpha, k, h, s \} (\eta, \varepsilon, \delta)
$$

$$\begin{cases}
O \left( \varepsilon^{2n-2m-k-h-s} \cdot (\log \varepsilon)^2 \right) & \text{if } k \geq 2n - 2m \text{ and } k + h - s \geq 2n - m, \\
O (\varepsilon^k) & \text{if } k \geq 2n - 2m \text{ and } k + h - s \leq 2n - m - 1,
\end{cases}
$$

$$= O (\delta^0) \text{ if } \alpha > 0, \text{ and }$$

$$I_1 \{ \alpha, k, h, s \} (\eta, \varepsilon, \delta)
$$

$$\begin{cases}
O \left( \varepsilon^{2n-2m-k-h-s} \cdot (\log \varepsilon)^2 \right) & \text{if } k \geq 2n - 2m \text{ and } k + h - s \geq 2n - m, \\
O (\varepsilon^k) & \text{if } k \geq 2n - 2m - 1 \text{ and } k + 2(h - s) \geq 2n - 1,
\end{cases}
$$

$$= O (\delta^{-1}) \text{ if } \alpha > 0, \text{ and }$$

$$I_2 \{ \alpha, k, h, s \} (\eta, \varepsilon, \delta)
$$

$$\begin{cases}
O \left( \varepsilon^{2n-2m-k-h-s} \cdot (\log \varepsilon)^2 \right) & \text{if } k \geq 2n - 2m \text{ and } k + h - s \geq 2n - m, \\
O (\varepsilon^k) & \text{if } k \geq 2n - 2m - 1 \text{ and } k + 2(h - s) \geq 2n + 1,
\end{cases}
$$

and

$$\begin{cases}
O \left( \varepsilon^{2n-2m-k-h-s} \cdot (\log \varepsilon)^2 \right) & \text{if } k \geq 2n - 2m \text{ and } k + h - s \geq 2n - m, \\
O (\varepsilon^k) & \text{if } k \geq 2n - 2m - 1 \text{ and } k + 2(h - s) \geq 2n + 2.
\end{cases}$$
Proof of proposition 3.1.

According to (17) in order to prove statement of the proposition it suffices to prove the estimates

$$\left\| \int_{\mathcal{U}(\epsilon) \times [0,1]} a(i,J)(t,\zeta,z) dt \wedge \tilde{g}(\zeta) \wedge \lambda_{r-1}^{i,J}(\zeta,z) \right\|_{\Gamma^{r,\alpha+1}(\mathcal{U})} \leq C \cdot \| g \|_{\Gamma^{r,\alpha}(\mathcal{U})},$$

(36)

and

$$\left\| \int_{\mathcal{U}(\epsilon) \times [0,1]} b(i,J)(t,\zeta,z) dt \wedge \tilde{g}(\zeta) \wedge \gamma_{r-1}^{i,J}(\zeta,z) \right\|_{\Gamma^{r,\alpha+1}(\mathcal{U})} \leq C \cdot \| g \|_{\Gamma^{r,\alpha}(\mathcal{U})}$$

with constant $C$ independent of $g$ and $\epsilon$.

Using the estimates

$$|A^2 \cdot \tilde{\partial} a| = O(|\zeta - z|^3), \quad |\mu_r| = O(|\zeta - z|^3), \quad |\mu_r| = O(|\zeta - z|^2),$$

$$|A^2 \cdot \tilde{\partial} a| = O(|\zeta - z|^3), \quad |a \cdot \tilde{\partial} (\mathcal{A} \mathcal{A})| = O(|\zeta - z|^2),$$

$$|\kappa(\zeta,z)| = O(|\zeta - z|), \quad |\chi(\zeta,z)| = O(|\zeta - z|),$$

$$F^{(k)}(\zeta,z) = \frac{1}{T} (\rho_k(z) - \rho_k(\zeta)) + O(|\zeta - z|^2) + \sqrt{\text{Im} F^{(k)}(\zeta,z)}$$

for the terms of the determinants in (19) and (20) and applying lemma 3.3 to the differential form

$$\int_{T+|E| \leq |J_0|} d\zeta \wedge \chi \wedge \mu_r \wedge \omega(\zeta)$$

we obtain representations

$$a(i,J)(t,\zeta,z) dt \wedge \tilde{g}(\zeta) \wedge \lambda_{r-1}^{i,J}(\zeta,z) = \sum_{|T|+|E| \leq |J_0|+1} c_{I,d,j}(\zeta,z,t) \tilde{g}(\zeta) \{\text{Im} F(\zeta,z)\}^T \mathcal{K}_{d,j}^{I,J,S,T,E}(\zeta,z),$$

(38)

and

$$b(i,J)(t,\zeta,z) dt \wedge \tilde{g}(\zeta) \wedge \gamma_{r-1}^{i,J}(\zeta,z) = \sum_{|T|+|E| \leq |J_0|} c_{I,d,j}(\zeta,z,t) \tilde{g}(\zeta) \{\text{Im} F(\zeta,z)\}^T \mathcal{K}_{d,j}^{I,J,S,T,E}(\zeta,z).$$

(39)

Multiindices $T$ and $E$ in (38) are obtained from the decomposition

$$\{\mathcal{T}(\zeta,z)\}^{J_0,J}$$

$$= \sum_{|T|+|E| \leq |J_0|} c_{I,T,E,G,H}(\zeta,z) \{\text{Im} F(\zeta,z)\}^T \{\rho(\zeta)\}^E (\zeta - z)^G (\zeta - z)^H$$

and multiindices $I_i$ for $i = 1, \ldots, 6$ and indices $d, j$ in (38) satisfy the conditions below

$$d = 2(|J_1| + |J_7| + 1),$$

$$j = n - |J_1| - |J_7| - 1,$$

$$|I_1| = |E|,$$

$$|I_2| + |I_3| = 1 + |J_3| + 3|J_4| + 3|J_5| + 2|J_6| + 3|J_8| + 2|J_9| + |J_{11}|$$

+2(|J_{10}| + 1 - |T| - |E|),

$$|I_4| = 0,$$

$$|I_5| = |J_1| + |J_2| + |J_3| + |J_6| + r + m - n.$$
Multiindices $T$ and $E$ in (39) are obtained from the decomposition
\[
\{ T(\zeta, z) \}^{J_{10}} = \sum_{|T| + |E| + \frac{1}{2}(|G| + |H|) = |J_{10}|} c_{\{T,E,G,H\}}(\zeta, z) \{ \text{Im}F(\zeta, z) \}^T \{ \rho(\zeta) \}^E (\zeta - z)^G (\bar{\zeta} - \bar{z})^H
\]
and multiindices $I_i$ for $i = 1, \ldots, 6$ and indices $d, j$ in (39) satisfy the conditions
\[
\begin{align*}
&d = 2(|J_1| + |J_7| + 1), \\
j &= n - |J_1| - |J_7| - 1, \\
&|I_1| = |E|, \\
&|I_2| + |I_3| = 4 + |J_3| + 3|J_4| + 3|J_5| + 2|J_6| + 3|J_8| + 2|J_9| + |J_{11}| + 2(|J_{10}| - |T| - |E|), \\
&|I_4| = 0, \\
&|I_5| = |J_1| + |J_2| + |J_3| + |J_6| + r + m - n.
\end{align*}
\]
Using representations (38) and (39) we reduce the problem of proving (36) to each term
\[
\bar{g}(\zeta) B_{d,j}^{T,I}(\zeta, z)
\]
of the right hand side of these representations.

We further reduce the problem using transformation from lemma 3.3 to obtain a representation
\[
\int_{U(\zeta) \times [0, 1]} \bar{g}(\zeta) B_{d,j}^{T,I}(\zeta, z) dt = \sum_{i=0}^{p} \sum_{|H| = i} \int_{U(\zeta) \times [0, 1]} c_{\{H,M\}}(\zeta, z, t) \left[ \{ Y_\zeta(z) \}^H \bar{g}(\zeta) \right] B_{a,b}^{P(H),M}(\zeta, z) dt,
\]
where \( c_{\{H,M\}}(\zeta, z, t) = c_{\{H,M\}}(\zeta, z, \theta(\zeta), t) \) with
\[
c_{\{H,M\}}(\zeta, z, \theta, t) \in C^\infty \left( \overline{U_\zeta} \times \overline{U_z} \times S^{n-1} \times [0, 1] \right)
\]
and indices $a, b$ and multiindices $H, P(H), M$ are such that
\[
P(H) = \emptyset \text{ if } |H| < p,
\]
\[
\begin{align*}
&k \left( B_{a,b}^{P(H),M} \right) + h \left( B_{a,b}^{P(H),M} \right) - l \left( B_{a,b}^{P(H),M} \right) - s \left( B_{a,b}^{P(H),M} \right) \\
&\leq k \left( B_{d,j}^{T,I} \right) + h \left( B_{d,j}^{T,I} \right) - l \left( B_{d,j}^{T,I} \right) - s \left( B_{d,j}^{T,I} \right) - |H|,
\end{align*}
\]
\[
\begin{align*}
&k \left( B_{a,b}^{P(H),M} \right) + 2h \left( B_{a,b}^{P(H),M} \right) - 2l \left( B_{a,b}^{P(H),M} \right) - 2s \left( B_{a,b}^{P(H),M} \right) \\
&\leq k \left( B_{d,j}^{T,I} \right) + 2h \left( B_{d,j}^{T,I} \right) - 2l \left( B_{d,j}^{T,I} \right) - 2s \left( B_{d,j}^{T,I} \right) - 2|H|.
\end{align*}
\]
To obtain such a representation we repeatedly apply transformation from lemma 3.3 to a selected term of the right hand side of (38) or (39) and all the resulting terms until either $P = \emptyset$ or $|H| = p$. The procedure will stop in at most $|J_{10}| + 1$ steps for $\lambda_{i,j,1}$ and $|J_{10}|$ steps for $\gamma_{P,1}$ because $|P| \leq |J_{10}| + 1$ for $\lambda_{i,j,1}$ and $|P| \leq |J_{10}|$ for $\gamma_{P,1}$ and on every step $|P|$ decreases at least by one. Also, inequality $|T| \leq j$ is satisfied for the kernels of terms of representations (38) and (39) and is preserved under transformation from lemma 3.3, therefore the kernels of
terms in the right hand side of representation (42) with \( P = \emptyset \) will be \( K_{a,b}^M \) or \( B_{a,0}^0 \).

Conditions (43) will be satisfied because according to lemma 3.3 after every application of \( Y_{i,\zeta}(z) \) to \( \tilde{g}(\zeta) \) numbers \( b \) and \( |P| \) decrease by one with all other multiindices unchanged, therefore decreasing \( h\left(B_{a,b}^P(H), M\right) \) by one.

For each term of representation (42) with \( |H| = i \) we repeatedly apply lemma 3.2 or lemma 3.4 \( p - i \) times to obtain the following representation

\[
D_1 \circ \cdots \circ D_{p-i} \left( \int_{U(\epsilon) \times [0,1]} \left[ \{Y_{\zeta}(z)\}^H \tilde{g}(\zeta) \right] B_{a,b}^{P(H), M}(\zeta, z)dt \right)
\]

(44)

\[
= \sum_{r=1}^{\frac{p}{T,I}} \sum \int_{U(\epsilon) \times [0,1]} c_{\{T,I\}}(\zeta, z, t) \left[ \{Y_{i,\zeta}(z), D_{\zeta}\}^r \tilde{g}(\zeta) \right] B_{d,j}^{T,I}(\zeta, z)dt,
\]

where \( \{Y_{i,\zeta}(z), D_{\zeta}\}^r \) denotes a composition of differentiations \( Y_{j,\zeta}(z) \) and \( D_{\zeta} \) applied \( r \) times and indices \( d, j \) and multiindices \( T, I \) of representation (44) satisfy

\[
k\left(B_{d,j}^{T,I}\right) + h\left(B_{d,j}^{T,I}\right) - l\left(B_{d,j}^{T,I}\right) - s\left(B_{d,j}^{T,I}\right) - i,
\]

(45)

Finally, we apply operator \( D_1 \circ \cdots \circ D_i \) to each term of the representation (44) with \( r \leq p \) by differentiating the kernel and obtain

\[
D_1 \circ \cdots \circ D_i \left( \int_{U(\epsilon) \times [0,1]} \left[ \{Y_{i,\zeta}(z), D_{\zeta}\}^T \tilde{g}(\zeta) \right] B_{d,j}^{T,I}(\zeta, z)dt \right)
\]

(46)

\[
= \sum_{r \leq p} \sum_{P, M} \int_{U(\epsilon) \times [0,1]} c_{\{P, M\}}(\zeta, z, t) \left[ \{Y_{i,\zeta}(z), D_{\zeta}\}^r \tilde{g}(\zeta) \right] B_{a,b}^{P,M}(\zeta, z)dt,
\]

with indices \( a, b \) and multiindices \( P, M \) satisfying

\[
k\left(B_{a,b}^{P,M}\right) + h\left(B_{a,b}^{P,M}\right) - l\left(B_{a,b}^{P,M}\right) - s\left(B_{a,b}^{P,M}\right) - i,
\]

(47)

Conditions (47) are satisfied because after each application of \( D_i \) to \( B_{d,j}^{T,I}(\zeta, z) \) either \( k\left(B_{d,j}^{T,I}\right) \) increases by one or \( h\left(B_{d,j}^{T,I}\right) \) increases by one or \( h\left(B_{d,j}^{T,I}\right) \) increases by two and \( k\left(B_{d,j}^{T,I}\right) \) decreases by three.

From (42), (44) and (46) we conclude that in order to prove the statement of the proposition it suffices to prove that

\[
\left\| \int_{U(\epsilon) \times [0,1]} c(\zeta, z, t) \left[ \{Y_{i,\zeta}(z), D_{\zeta}\}^P \tilde{g}(\zeta) \right] B_{a,b}^{P,M}(\zeta, z)dt \right\|_{L^p, \alpha + 1}(U) \leq C \cdot \|g\|_{L^p, \alpha}(U)
\]

(48)
for the kernel $\mathcal{B}_{a,b}^{P,M}$ obtained from $\lambda_{i-j}^{\gamma}$ after described above application of lemmas 3.2, 3.3 and 3.4.

We will prove (48) as a corollary of the lemma below.

**Lemma 3.7.** Let $0 < \alpha < 1$, $g \in \Gamma^{\alpha}\left(\{\rho\}, \tilde{U}(\emptyset)\right)$ be a function with compact support and let indices $a, b$ and multiindices $P, M$ satisfy conditions

\[
k \left(\mathcal{B}_{a,b}^{P,M}\right) + h \left(\mathcal{B}_{a,b}^{P,M}\right) - l \left(\mathcal{B}_{a,b}^{P,M}\right) - s \left(\mathcal{B}_{a,b}^{P,M}\right) \leq 2n - m - 2,
\]

for $b \geq 1$ and

\[
k \left(\mathcal{B}_{a,b}^{0,M}\right) - l \left(\mathcal{B}_{a,b}^{0,M}\right) - s \left(\mathcal{B}_{a,b}^{0,M}\right) \leq 2n - m - 3
\]

for $|P| = 0$ and $b = 0$

Then

\[
f_{\epsilon}(z) := \left(\int_{U(\epsilon) \times [0,1]} c(\zeta, z, t) g(\zeta) \mathcal{B}_{a,b}^{P,M}(\zeta, z) dt \right) \in \Gamma^{\alpha+1}(U)
\]

for $\epsilon < \epsilon_{0}$ and

\[
\|f_{\epsilon}\|_{\Gamma^{\alpha+1}(U)} \leq C \cdot \|g\|_{\Gamma^{\alpha}(U)}
\]

with $C$ independent of $g$ and $\epsilon$.

**Proof.**

At first we will prove the inclusion $f_{\epsilon} \in \Lambda^{\alpha+1}(U)$. For a fixed point $w \in U$ and arbitrary $z \in U$ we denote $\delta = |z - w|$ and $c = 1 + \max_{i,j,z \in U} \left|\left|Q_{j}^{(k)}(z)\right|\right|$ and introduce the following neighborhood

\[
V(w, z) = \left\{ \zeta \in U : |\zeta - w|^{2} + \sum_{i=1}^{m} |P_{i}(\zeta)| + \sum_{i=1}^{m} \left|\text{Im} F^{(i)}(\zeta, w)\right| \leq 4cn^{2}\delta \right\},
\]

containing $z$ and $w$ and such that the estimates

\[
|\Phi(\zeta, w)|, |\Phi(\zeta, z)| \geq C \cdot \delta^{2}
\]

hold for $\zeta \notin V(w, z)$ with constant $C > 0$ independent of $\delta$ for $\delta$ small enough.

Denoting $V(\epsilon) = V(w, z) \cap U(\epsilon)$ we represent $f_{\epsilon}(z)$ as follows

\[
f_{\epsilon}(z) = g(w) \cdot \int_{U(\epsilon) \times [0,1]} c(\zeta, z, t) \mathcal{B}_{a,b}^{P,M}(\zeta, z) dt
\]

\[
+ \int_{V(\epsilon) \times [0,1]} (g(\zeta) - g(w)) c(\zeta, z, t) \mathcal{B}_{a,b}^{P,M}(\zeta, z) dt
\]

\[
+ \int_{(U(\epsilon) \setminus V(\epsilon)) \times [0,1]} (g(\zeta) - g(w)) c(\zeta, z, t) \mathcal{B}_{a,b}^{P,M}(\zeta, z) dt.
\]

Applying lemmas 3.2, 3.3 and 3.4 as earlier in the proof of proposition 3.1 to the first term of the right hand side of (51) we obtain

\[
D \left( g(w) \int_{U(\epsilon) \times [0,1]} c(\zeta, z, t) \cdot \mathcal{B}_{a,b}^{P,M}(\zeta, z) dt \right)
\]

\[
= g(w) \sum_{T, I, d, j} \int_{U(\epsilon) \times [0,1]} c_{\{T, I, d, j\}}(\zeta, z, t) \cdot \mathcal{B}_{d,j}^{T,I}(\zeta, z) dt
\]

with $c_{\{T, I, d, j\}}(\zeta, z, t) = c_{\{T, I, d, j\}}(\zeta, z, \theta(\zeta), t)$ and

\[
c_{\{T, I, d, j\}}(\zeta, z, \theta, t) \in C^{\infty}(\tilde{U}_{\zeta} \times \tilde{U}_{z} \times \mathbb{S}^{n-1} \times [0, 1])
\]
and indices $d, j$ and multiindices $T, I_1, I_2, I_3$ satisfying (49). Then applying formula

$$d\rho_t|_{U(\epsilon)} = \epsilon d\theta_i$$

and lemma 3.6, we obtain

$$\left| g(w) \cdot D \left( \int_{U(\epsilon) \times [0,1]} c(\zeta, z, t) B^{P,M}_{a,b}(\zeta, z) dt \right) \right|$$

$$= \|g\|_{\Lambda^0(U)} \cdot O \left( \int_{U(\epsilon) \times [0,1]} \frac{s(B) - m - s(B)}{\lambda_i d\theta_i(\zeta) \wedge d\sigma_{2n-m}(\zeta)} \right)$$

$$\leq \|g\|_{\Lambda^0(U)} \cdot O \left( \int_{U(\epsilon) \times [0,1]} |\zeta - z| k(B) \cdot |\Phi(\zeta, z)| \right)$$

$$= \|g\|_{\Lambda^0(U)} \cdot O \left( I_1 \{0, k(B), h(B) - l(B), s(B)\} \{\epsilon, 1\} \right) = \|g\|_{\Lambda^0(U)} \cdot O(1),$$

which shows that the first term of the right hand side of (51) is in $\Lambda^1(U)$.

For the second term of the right hand side of (51) we have

$$\left| \int_{V(\epsilon) \times [0,1]} (g(\zeta) - g(w)) c(\zeta, z, t) B^{P,M}_{a,b}(\zeta, z) dt \right|$$

$$= \|g\|_{\Gamma^0(U)} \cdot \delta^{\frac{\epsilon}{4}} \cdot O \left( \int_{V(\epsilon) \times [0,1]} \frac{s(B) - m - s(B)}{\lambda_i d\theta_i(\zeta) \wedge d\sigma_{2n-m}(\zeta)} \right)$$

$$= \|g\|_{\Gamma^0(U)} \cdot \delta^{\frac{\epsilon}{4}} \cdot O \left( I_1 \{0, k(B), h(B) - l(B), s(B)\} \{\epsilon, \sqrt{\delta}\} \right) = \|g\|_{\Gamma^0(U)} \cdot O \left( \delta^{\frac{\epsilon}{8}} \right),$$

where we used lemma 3.6 and the estimate

$$|g(\zeta) - g(w)| = \|g\|_{\Gamma^0(U)} \cdot O \left( \delta^{\frac{\epsilon}{2}} \right)$$

for $\zeta \in V(w, z)$.

For the third term of the right hand side of (51) we have

$$\left| \int_{(U(\epsilon) \setminus V(\epsilon)) \times [0,1]} (g(\zeta) - g(w)) c(\zeta, z, t) B^{P,M}_{a,b}(\zeta, z) dt \right|$$

$$- \int_{(U(\epsilon) \setminus V(\epsilon)) \times [0,1]} (g(\zeta) - g(w)) c(\zeta, z, t) B^{P,M}_{a,b}(\zeta, z) dt$$

$$\leq \|g\|_{\Gamma^0(U)} \cdot \delta \cdot O \left[ I_2 \{\alpha, k(B) + 1, h(B) - l(B), s(B)\} \{\epsilon, \sqrt{\delta}\} \right]$$

$$+ \|g\|_{\Gamma^0(U)} \cdot \delta \cdot O \left[ I_2 \{\alpha, k(B), h(B) - l(B) + 1, s(B)\} \{\epsilon, \sqrt{\delta}\} \right]$$

$$+ \|g\|_{\Gamma^0(U)} \cdot \delta \cdot O \left[ I_2 \{\alpha, k(B), h(B) - l(B), s(B)\} \{\epsilon, \sqrt{\delta}\} \right] = \|g\|_{\Gamma^0(U)} \cdot O \left( \delta^{\frac{\epsilon + 1}{2}} \right),$$

where we again used lemma 3.6.

Representation (51) together with the estimates above show that

$$\|f_\epsilon\|_{\Lambda^0(U)} \leq C \cdot \|g\|_{\Gamma^0(U)}$$
uniformly with respect to $\epsilon$.

To complete the proof of the lemma we have to prove that

$$\|D^c f_\epsilon\|_{\Lambda^c(U)} \leq C \cdot \|g\|_{\Gamma^a(U)}$$

where differentiation $D^c \in T^c(M)$ and

$$\|h\|_{\Lambda^c(U)} = \sup \left\{ |h(x(\cdot))|_{\Lambda^c(\{0,1\})} \right\}$$

with the sup taken over all curves $x: [0,1] \to M$ such that

1. $|x'(t)|, |x''(t)| \leq 1,$
2. $x'(t) \in T^c(M).$

To prove this estimate we use the following representation

$$D^c f_\epsilon(z) = g(z) \cdot D^c \left( \int_{U(\epsilon)} c(\zeta, z, t) B_{a,b}^{P,M}(\zeta, z) dt \right)$$

(52)

+ \left( \int_{U(\epsilon) \times [0,1]} (g(\zeta) - g(z)) D^c \left[ c(\zeta, z, t) B_{a,b}^{P,M}(\zeta, z) \right] dt \right).

Applying as for the first term of (51) lemmas 3.3, 3.2 and 3.4 and using lemma 3.6 we conclude that

$$D^c \left( \int_{U(\epsilon) \times [0,1]} c(\zeta, z, t) B_{a,b}^{P,M}(\zeta, z) dt \right) \in \Lambda^1(U)$$

and therefore the first term of the right hand side of (52) is in $\Lambda^a(U)$.

To estimate the second term of the right hand side of (52) we consider $z, w \in U$ and introduce the neighborhoods

$$W(w, z) = \left\{ \zeta \in U : |\zeta - w|^2 + \sum_{i=1}^{m} |\rho_i(\zeta)| + \sum_{i=1}^{m} |\text{Im} F^{(i)}(\zeta, w)| \leq 4cn^2 \delta^2 \right\},$$

$$W'(w, z) = \left\{ \zeta \in U : |\zeta - w|^2 + \sum_{i=1}^{m} |\rho_i(\zeta)| + \sum_{i=1}^{m} |\text{Im} F^{(i)}(\zeta, w)| \leq 16cn^2 \delta^2 \right\},$$

$$W(\epsilon) = W(w, z) \cap U(\epsilon),$$

$$W'(\epsilon) = W'(w, z) \cap U(\epsilon)$$

and a function $\phi(\zeta) \in C^\infty(\bar{U})$ such that $0 \leq \phi(\zeta) \leq 1$, $\phi \equiv 1$ on $W(w, z)$, $\phi \equiv 0$ on $\bar{U} \setminus W'(w, z)$, $|\text{grad} \phi(\zeta)| \leq 2/\delta^2$ and $|\text{grad}_\perp \phi(\zeta)| \leq 2/\delta$.

Then we consider the following representation of the second term of the right hand side of (52)

$$\int_{U(\epsilon) \times [0,1]} (g(\zeta) - g(z)) D^c \left[ c(\zeta, z, t) B_{a,b}^{P,M}(\zeta, z) \right] dt$$

(53)

$$- \int_{U(\epsilon) \times [0,1]} (g(\zeta) - g(w)) D^c \left[ c(\zeta, w, t) B_{a,b}^{P,M}(\zeta, w) \right] dt$$

$$+ \int_{U(\epsilon) \times [0,1]} (g(\zeta) - g(w)) \phi(\zeta) D^c \left[ c(\zeta, z, t) B_{a,b}^{P,M}(\zeta, z) \right] dt$$

$$- \int_{U(\epsilon) \times [0,1]} (g(\zeta) - g(w)) \phi(\zeta) D^c \left[ c(\zeta, w, t) B_{a,b}^{P,M}(\zeta, w) \right] dt$$

$$+ \int_{U(\epsilon) \times [0,1]} (g(\zeta) - g(z)) (1 - \phi(\zeta)) D^c \left[ c(\zeta, z, t) B_{a,b}^{P,M}(\zeta, z) \right]$$

$$- (g(\zeta) - g(w)) (1 - \phi(\zeta)) D^c \left[ c(\zeta, w, t) B_{a,b}^{P,M}(\zeta, w) \right] dt.$$
To estimate integrals on the right hand side of (53) we use the estimate

\[ D^c \left[ F^{(k)}(\zeta, z) \right] = O \left( |\zeta - z| \right) \]  

(54)

for \( k = 1, \ldots, m \) and obtain a representation

\[ D^c \left[ c(\zeta, z, t) \mathcal{B}_{a,b}^{P,M}(\zeta, z) \right] = \sum_{T,I,d,j} c_{(T,I,d,j)}(\zeta, z, t) \cdot \mathcal{B}_{d,j}^{T,I}(\zeta, z) \]  

(55)

with \( c_{(T,I,d,j)}(\zeta, z, t) = c_{(T,I,d,j)}(\zeta, z, \theta(\zeta), t) \) and

\[ c_{(T,I,d,j)}(\zeta, z, \theta(\zeta), t) \in C^\infty \left( \tilde{U}_\zeta \times \tilde{U}_z \times \mathbb{S}^{n-1} \times [0,1] \right), \]

and indices and multiindices in the right hand side of (55) satisfying

\[ k \left( \mathcal{B}_{d,j}^{T,I} \right) + h \left( \mathcal{B}_{d,j}^{T,I} \right) - l \left( \mathcal{B}_{d,j}^{T,I} \right) - s \left( \mathcal{B}_{d,j}^{T,I} \right) \leq 2n - m - 1, \]

\[ k \left( \mathcal{B}_{d,j}^{T,I} \right) + 2h \left( \mathcal{B}_{d,j}^{T,I} \right) - 2l \left( \mathcal{B}_{d,j}^{T,I} \right) - 2s \left( \mathcal{B}_{d,j}^{T,I} \right) \leq 2n. \]  

(56)

Then for the first two integrals of the right hand side of (53) using representation (55) with conditions (56) and lemma \[ \overline{3} \] we obtain

\[
\left| \int_{U(\epsilon) \times [0,1]} (g(\zeta) - g(z)) \phi(\zeta)c(\zeta, z, t)\mathcal{B}_{d,j}^{T,I}(\zeta, z) dt \right| \\
= O \left( \int_{W(\epsilon) \times [0,1]} (g(\zeta) - g(z)) c(\zeta, z, t)\mathcal{B}_{d,j}^{T,I}(\zeta, z) dt \right) \\
= \|g\|_{\Gamma^\alpha(U)} \cdot O \left( \int_{W(\epsilon)} \left| \zeta - \zeta \right|^{(k)(B)} \cdot |\Phi(\zeta, z)| \frac{h(B) - l(B)}{2} \cdot \frac{|\sigma_{2n-m}(\zeta)|}{2} \right)
\]

= \|g\|_{\Gamma^\alpha(U)} \cdot O \left( I_1 \{ \alpha, k(B), h(B) - l(B), s(B) \} (\epsilon, \delta) \right) = \|g\|_{\Gamma^\alpha(U)} \cdot O (\delta^\alpha).

To estimate the third integral of the right hand side of (53) we represent it as

\[ \int_{U(\epsilon) \times [0,1]} (g(\zeta) - g(z)) \left( 1 - \phi(\zeta) \right) D^c \left[ c(\zeta, z, t)\mathcal{B}_{a,b}^{P,M}(\zeta, z) \right] dt \]  

(57)

\[ - \int_{U(\epsilon) \times [0,1]} (g(\zeta) - g(w)) \left( 1 - \phi(\zeta) \right) D^c \left[ c(\zeta, w, t)\mathcal{B}_{a,b}^{P,M}(\zeta, w) \right] dt \]

\[ = \int_{U(\epsilon) \times [0,1]} (g(\zeta) - g(z)) \left( 1 - \phi(\zeta) \right) \]

\[ \times \left( D^c \left[ c(\zeta, z, t)\mathcal{B}_{a,b}^{P,M}(\zeta, z) \right] - D^c \left[ c(\zeta, w, t)\mathcal{B}_{a,b}^{P,M}(\zeta, w) \right] \right) dt \]

\[ + |g(w) - g(z)| \cdot \int_{U(\epsilon) \times [0,1]} (1 - \phi(\zeta)) D^c \left[ c(\zeta, w, t)\mathcal{B}_{a,b}^{P,M}(\zeta, w) \right] dt. \]

To estimate the first integral of the right hand side of (57) we use representation (55) with kernels satisfying (56) and estimate

\[ |F^{(k)}(\zeta, z) - F^{(k)}(\zeta, w)| = O \left( \delta \cdot |\zeta - z| + \delta^2 \right) \]  

(58)

for \( k = 1, \ldots, m \) and \( \zeta \in U \setminus W(\epsilon) \).

Then we obtain

\[
\left| \int_{U(\epsilon) \times [0,1]} (g(\zeta) - g(z)) \left( 1 - \phi(\zeta) \right) \right. \\
\left. \times \left( D^c \left[ c(\zeta, z, t)\mathcal{B}_{a,b}^{P,M}(\zeta, z) \right] - D^c \left[ c(\zeta, w, t)\mathcal{B}_{a,b}^{P,M}(\zeta, w) \right] \right) dt \right|
\]
To obtain necessary estimate for the second integral of the right hand side of (57) it suffices to prove the following estimate

\[ \left| \int_{U(\varepsilon) \times [0,1]} (1 - \phi(\zeta)) D^c [c(\zeta, w, t) B_{a,b}^{P,M}(\zeta, w)] dt \right| = O(1). \]  

(59)

To obtain this estimate for kernels \( B_{a,b}^{P,M} \) with \(|P| = 0\) and \( b \geq 1 \) (i.e. for \( K_{a,b}^{M} \)) we use representation from lemma 3.2

\[ D^c \int_{U(\varepsilon) \times [0,1]} (1 - \phi(\zeta)) c(\zeta, w, t) K_{a,b}^{M}(\zeta, w) dt \]

(60)

\[ = \int_{U(\varepsilon) \times [0,1]} (1 - \phi(\zeta)) \left[ D^c c(\zeta, w, t) \right] \cdot K_{a,b}^{M}(\zeta, w) dt \]

\[ + \sum_{S,d,j} \int_{U(\varepsilon) \times [0,1]} c_{S,d,j}(\zeta, w, t) \cdot (1 - \phi(\zeta)) \cdot K_{a,b}^{M}(\zeta, w) dt \]

\[ + \int_{U(\varepsilon) \times [0,1]} D^c (1 - \phi(\zeta)) \cdot c(\zeta, w, t) K_{a,b}^{M}(\zeta, w) dt \]

\[ + \sum_{k=1}^{m} \sum_{L} \int_{U(\varepsilon) \times [0,1]} [Y_{k,L}(w) (1 - \phi(\zeta))] \cdot c_{k,L}(\zeta, w, t) \cdot K_{a,b}^{M}(\zeta, w) dt \]

with indices of the kernels of all the terms except the last term satisfying (49). For the kernels of the last term of the right hand side of (60) according to (26) and (27) we have

\[ k \left( K_{a,b}^{L} \right) = k \left( K_{a,b}^{M} \right) - 1. \]  

(61)

For the first two terms of the right hand side of (60) we obtain estimate (59) using lemma 3.6 and conditions (49). For the third term of the right hand side of (60) we obtain the following estimate

\[ \left| \int_{U(\varepsilon) \times [0,1]} D^c (1 - \phi(\zeta)) \cdot c(\zeta, w, t) K_{a,b}^{M}(\zeta, w) dt \right| \]

\[ = \delta^{-1} \cdot O \left( I_2 \{ 0, k(\mathcal{K}), h(\mathcal{K}) - l(\mathcal{K}), s(\mathcal{K}) \} (\varepsilon, \delta) \right) = O(1), \]

where we used properties of the function \( \phi \).

We obtain the same estimate for the integrals of the fourth term of the right hand side of (60) if we use the inequality

\[ |\zeta - w| = O(\delta) \quad \text{for} \quad \zeta \in W'(w, z), \]

property (61) and the estimate \(|\text{grad } \phi(\zeta)| \leq 2/\delta^2\).

Proof of the same estimates for kernels \( B_{a,0}^{h,M} \) is analogous to the proof for \( K_{a,b}^{M} \) but uses condition (50) instead of conditions (49).

For a general kernel \( B_{a,b}^{P,M} \) with \(|P| \neq 0\) we use representation from lemma 3.3 and obtain

\[ D^c \int_{U(\varepsilon) \times [0,1]} (1 - \phi(\zeta)) c(\zeta, w, t) \cdot B_{a,b}^{P,M}(\zeta, w) dt \]

(62)

\[ = \int_{U(\varepsilon) \times [0,1]} [Y_{k,L}(z) (1 - \phi(\zeta))] \cdot D^c \left[ c(\zeta, w, t) B_{a,b-1}^{T,I}(\zeta, w) \right] dt \]
\[ + \sum_{\{L,S,d,j\}} D^c \int_{U(\epsilon) \times [0,1]} (1 - \phi(\zeta)) c_{\{L,S,d,j\}}(\zeta, w, t) \cdot \mathcal{B}^{L,S}_{d,j}(\zeta, w) dt, \]

with \(|T| = |P| - 1\), indices of the kernels \(\mathcal{B}^{L,S}_{d,j}\) satisfying (49) and \(|L| < |P|\). Therefore the problem of estimating integral in (59) will be reduced to estimating the same integral for a kernel with smaller \(|P|\) if we prove estimate (59) for the second term of the right hand side of (62). We obtain this estimate analogously to the estimate of the fourth term of the right hand side of (60) with the use of estimate (54).

In order to prove applicability of lemma 3.7 to the kernels obtained from \(\lambda^{i,J}_{r-1}\) and \(\gamma^{i,J}_{r-1}\) after applications of lemmas 3.2, 3.3 and 3.4 we have to prove relations (49) and (50) for these kernels. But according to lemmas 3.2, 3.3 and 3.4 expressions in the left hand sides of these relations don’t increase under transformations from these lemmas. Also, as follows from lemma 3.3, relation (50) for kernels \(\mathcal{B}^{\emptyset,M}_{a,0}\) is a corollary of the first relation from (49) for kernels \(\mathcal{B}^{T,I}_{d,j}(\zeta, z)\) satisfying conditions (40) and (41).

We notice that since \(s\left(\mathcal{B}^{T,I}_{d,j}\right) = 0\) for the original kernels we will omit it in our calculations.

Second condition from (49) is always satisfied for the indices satisfying (40) as can be seen from the inequality

\[ l(\mathcal{B}) + \frac{1}{2} \left(2n - k(\mathcal{B}) - 2h(\mathcal{B})\right) \geq \frac{1}{2} \left(|J_8| + m - |J_2| - |J_3| - |J_{11}|\right) \geq \frac{1}{2}, \]

where we used relations

\[ |J_1| + |J_3| + |J_6| + r + m - n \geq 0, \]

\[ \sum_{i=1}^6 |J_i| = n - r - 1, \]

\[ |J_2| + |J_3| + |J_{11}| \leq m - 1 \]

for the multiindices of \(\lambda^{i,J}_{r-1}\).

Condition (63) is also satisfied for the indices satisfying (41) which follows from the inequality

\[ l(\mathcal{B}) + \frac{1}{2} \left(2n - k(\mathcal{B}) - 2h(\mathcal{B})\right) \geq \frac{1}{2} \left(1 + |J_8| + m - |J_2| - |J_3| - |J_{11}|\right) \geq \frac{1}{2}, \]

where we used relations

\[ |J_1| + |J_3| + |J_6| + r + m - n \geq 0, \]

\[ \sum_{i=1}^6 |J_i| = n - r - 1, \]

\[ |J_2| + |J_3| + |J_{11}| \leq m \]

for the multiindices of \(\gamma^{i,J}_{r-1}\).

First condition from (49) is not satisfied for all kernels \(\mathcal{B}^{T,I}_{d,j}(\zeta, z)\). But in the lemma below we prove that if it is not satisfied then the corresponding terms of the integral formula for \(R_r(\epsilon)\) do not survive under the limit when \(\epsilon \to 0\). This lemma is a simplified version of the lemma 4 in [P].
Lemma 3.8. If
\[ k(\mathcal{B}) + h(\mathcal{B}) - l(\mathcal{B}) \geq 2n - m - 1 \quad \text{and} \quad s(\mathcal{B}) = 0 \]
then
\[ \left\| \int_{U(\varepsilon) \times [0,1]} \tilde{g}(\zeta) c(\zeta, z, t) B_{d,j}^{T,1}(\zeta, z) dt \right\|_{L^\infty(M)} = O(\sqrt{\varepsilon} \cdot \log \varepsilon) \cdot \|g\|_{L^\infty(M)}. \]

Proof.
We use the inequality
\[ 2n - m + l(\mathcal{B}) - k(\mathcal{B}) - h(\mathcal{B}) \geq n - |J_1| - |J_2| - |J_7| - 1 \geq 1, \tag{64} \]
which is a corollary of the definitions of $k(\mathcal{B})$, $h(\mathcal{B})$ and $l(\mathcal{B})$ and equality
\[ \sum_{i=1}^{6} |J_i| = n - r - 1. \]
From the condition of the lemma and the inequality (64) we obtain
\[ k(\mathcal{B}) + h(\mathcal{B}) - l(\mathcal{B}) = 2n - m - 1 \]
and
\[ n - |J_1| - |J_2| - |J_7| - 1 = 1, \]
which leads to
\[ |J_1| + |J_2| = n - r - 1, \quad |J_3| = 0, \quad |J_6| = 0, \quad |J_7| = r - 1, \]
and hence to
\[ l(\mathcal{B}) \geq |J_1| + |J_2| + |J_3| + |J_6| + r + m - n = m - 1 > 1. \]
Using lemma 3.6 in the estimate of the integral in lemma we obtain
\[ \left\| \int_{U(\varepsilon) \times [0,1]} \tilde{g}(\zeta) c(\zeta, z, t) B_{d,j}^{T,1}(\zeta, z) dt \right\|_{L^\infty(M)} = \|g\|_{L^\infty(M)} \cdot e^{l(\mathcal{B})} \cdot O \left( \left\| I \left( \{ 0, k(\mathcal{B}), h(\mathcal{B}), 0 \} \right( \varepsilon, 1) \right) \right) \]
\[ = \|g\|_{L^\infty(M)} \cdot \begin{cases} e^{l(\mathcal{B})} \cdot O \left( e^{2n - m - k(\mathcal{B}) - h(\mathcal{B})} \cdot (\log \varepsilon)^2 \right) & \text{if } k(\mathcal{B}) \geq 2n - 2m, \\ e^{l(\mathcal{B})} \cdot O \left( e^{(2n - k(\mathcal{B}) - 2h(\mathcal{B}))/2} \cdot \log \varepsilon \right) & \text{if } k(\mathcal{B}) \leq 2n - 2m - 1. \end{cases} \]
In the first subcase of the above we have the necessary estimate because of the inequality (64) and condition $l(\mathcal{B}) \geq 1$. In the second subcase we have the necessary estimate from the inequality (63) and again condition $l(\mathcal{B}) \geq 1$. \qed

This completes the proof of proposition 3.4.

4. Compactness of $H_r$.

From the definition of operator $H_r$ we conclude that in order to prove its compactness it suffices to prove compactness of each of the terms below
\[ \tilde{\partial}_M \varphi_i(z) \wedge R_r(\varphi_i g)(z) , \quad \varphi_i(z) \cdot R^r_{i+1}(\tilde{\partial}_M \varphi_i \wedge g)(z) \quad \text{and} \quad \varphi_i(z) \cdot H^r_i(\varphi_i g)(z). \]
Compactness of the first two of these terms follows from the boundedness of operators $R_r$ proved in proposition 3.3 and compactness of the embedding
\[ \Gamma^{p,\alpha}(U) \to \Gamma^{p,\beta}(U) \]
for $\alpha > \beta$ [Ad]. Another application of compactness of this embedding shows that compactness of the third term follows from the proposition below.
Proposition 4.1. Let \( 0 < \alpha < 1, \ M \subset \mathbb{C} \) be a \( C^\infty \) regular \( q \)-concave CR submanifold of the form (1) and let \( g \in \Gamma^{p,\alpha}_{(0,r)}(M) \) be a form with compact support in \( U = \bar{U} \cap M \).

Then for \( r < q \) the operator \( H_r \), defined in (13) satisfies the following estimate

\[
\| H_r(g) \|_{\Gamma^{p,\alpha+1}(U)} < C \cdot \| g \|_{\Gamma^{p,\alpha}(U)}
\]

with a constant \( C \) independent of \( g \).

Proof.

In our proof of proposition [12] we will use the approximation of \( H_r \) by the operators

\[
H_r(\epsilon)(g)(z) = (-1)^r \frac{(n-1)!}{(2\pi i)^n} \cdot \text{pf}_M \circ \int_{U_r} \partial(\zeta) \bar{g}(\zeta) \wedge \omega_r \left( \frac{P(\zeta,z)}{\Phi(\zeta,z)} \right) \wedge \omega(\zeta)
\]

when \( \epsilon \) goes to 0.

Kernel of the operator \( H_r(\epsilon) \) with the use of equalities (10) and (12) may be represented on \( \bar{U} \times U \) as

\[
\bar{\phi}_r^{i,J}(\zeta,z) = \frac{1}{\Phi(\zeta,z)^n} \times \text{Det} \left[ Q^{(i)} \bar{F}^{(i)}, Q^{(j)} d_\zeta \bar{F}^{(j)}, \bar{A} \cdot \partial_\zeta a, \right.
\]

\[
\left. \sum_{j \in J_1} \sum_{j \in J_2} \left( \sum_{j \in J_3} \sum_{j \in J_4} \sum_{j \in J_5} \sum_{j \in J_6} \sum_{j \in J_7} \sum_{j \in J_8} \right) \right] \wedge \omega(\zeta),
\]

and

\[
\bar{\psi}_r^{i,J}(\zeta,z) = \frac{1}{\Phi(\zeta,z)^n} \times \text{Det} \left[ a_i \bar{A} \bar{A}_i, Q^{(j)} d_\zeta \bar{F}^{(j)}, \bar{A} \cdot \partial_\zeta a, \right.
\]

\[
\left. \sum_{j \in J_1} \sum_{j \in J_2} \left( \sum_{j \in J_3} \sum_{j \in J_4} \sum_{j \in J_5} \sum_{j \in J_6} \sum_{j \in J_7} \sum_{j \in J_8} \right) \right] \wedge \omega(\zeta).
\]

Applying then (10) we conclude that \( \bar{\phi}_r^{i,J}(\zeta,z) \) can be represented as a finite sum of the following terms with smooth coefficients in \( \zeta, z \) and \( \theta(\zeta) \):

\[
\phi_r^{i,J}(\zeta,z) = \frac{\bar{F}^{(i)}}{\Phi(\zeta,z)^n} \cdot d_\zeta \bar{F}^{(j)} \wedge d_\zeta \bar{F}^{(j)} \wedge \wedge \bar{A} \cdot \partial_\zeta a \wedge a \cdot \partial_\mu \wedge a \cdot \partial_\mu
\]

\[
\left( \sum_{j \in J_1} \sum_{j \in J_2} \sum_{j \in J_3} \sum_{j \in J_4} \sum_{j \in J_5} \sum_{j \in J_6} \sum_{j \in J_7} \sum_{j \in J_8} \right) \wedge \bar{A} \cdot \partial_\zeta a \wedge a \cdot \partial_\zeta (\bar{A} \bar{A}) \wedge F \cdot \partial_\zeta Q \wedge \wedge \left( d_\zeta \right).
\]
Similarly, \( \psi_t^{i,J}(\zeta, z) \) can be represented as a finite sum of the following terms with smooth coefficients in \( \zeta, z \) and \( \theta(\zeta) \):

\[
\psi_t^{i,J}(\zeta, z) = \frac{a_i A A_i}{\Phi(\zeta, z)} \cdot d_\zeta F^{(j)} \wedge d_\zeta F^{(j)} \wedge \Lambda A \cdot \partial_\zeta a \wedge a \cdot \mu_\nu \wedge a \cdot \mu_\tau \tag{70}
\]

\[
\Lambda A \cdot \partial_\zeta a \wedge a \cdot \partial_\zeta (A A) \wedge F \cdot \partial_\zeta Q \wedge \kappa \wedge d_\zeta .
\]

As in the proof of proposition 3.1 we use estimates (37) for the terms of the determinants in (67) and (68) and obtain the representations

\[
a_{(i, J)}(\zeta, z) \wedge \overline{g}(\zeta) \wedge \phi_{\xi}^{i,J}(\zeta, z) = \sum_{|T|+|E| \leq |J_0|+1} c_{(I, d, j)}(\zeta, z) \overline{g}(\zeta) \{\text{Im} F(\zeta, z)\}^T K_{d, j}^{I(J, J, T, E)}(\zeta, z), \tag{71}
\]

and

\[
b_{(i, J)}(\zeta, z) \wedge \overline{g}(\zeta) \wedge \psi_t^{i,J}(\zeta, z) = \sum_{|T|+|E| \leq |J_0|} c_{(I, d, j)}(\zeta, z) \overline{g}(\zeta) \{\text{Im} F(\zeta, z)\}^T K_{d, j}^{(J, J, T, E)}(\zeta, z). \tag{72}
\]

Multiindices \( T \) and \( E \) in (71) are obtained from the decomposition

\[
\{F(\zeta, z)\}^{[J_0, J_1]}
\]

\[
= \sum_{|T|+|E| \leq |J_0|+1} c_{(I, d, j)}(\zeta, z) \{\text{Im} F(\zeta, z)\}^T \{\rho(\zeta)\}^E (\zeta - z)^{G(\zeta - \bar{z})^H}
\]

and multiindices \( I_i \) for \( i = 1, \ldots, 5 \) and indices \( d, j \) in (71) satisfy the conditions below

\[
d = 0,
\]

\[
j = n,
\]

\[
|I_1| = |E|,
\]

\[
|I_2| + |I_3| = 3|J_2| + 3|J_3| + 2|J_4| + 3|J_5| + 2|J_6| + |J_8|
\]

\[
+ 2(|J_T| + 1 - |T| - |E|),
\]

\[
|I_4| = |J_1|,
\]

\[
|I_5| = 0.
\]

Multiindices \( T \) and \( E \) in (72) are obtained from the decomposition

\[
\{F(\zeta, z)\}^{[J_T]}
\]

\[
= \sum_{|T|+|E| \leq |J_0|+1} c_{(I, d, j)}(\zeta, z) \{\text{Im} F(\zeta, z)\}^T \{\rho(\zeta)\}^E (\zeta - z)^{G(\zeta - \bar{z})^H}
\]
and multiindices $I_i$ for $i = 1, \ldots, 5$ and indices $d, j$ in (72) satisfy the conditions

\begin{align*}
    d &= 0, \\
    j &= n, \\
    |I_1| &= |E|, \\
    |I_2| + |I_3| &= 3 + 3|J_2| + 3|J_3| + 2|J_4| + 3|J_5| + 2|J_6| + |J_8| \\
    &+ 2(|J_7| - |T| - |E|), \\
    |I_4| &= |J_1|, \\
    |I_5| &= 0.
\end{align*}

Using representations (71) and (72) we reduce the statement of proposition 4.1 to the same statement for each term $\tilde{g}(\zeta)B^{T,I}_{d,j}(\zeta, z)$ of the right hand side of these representations.

Proceeding then as in the proof of proposition 3.1 we reduce the problem to lemma 3.7. To check applicability of lemma 3.7 we have to prove only the second condition of (49) for the kernels obtained from $\phi^{i,J}_r(\zeta, z)$ and $\psi^{i,J}_r(\zeta, z)$ and satisfying (73) and (74) respectively. The reason for that is that for these kernels we have $k(B) \leq 0 \leq 2n - 2m - 1$ and therefore only the second condition of (49) is needed for application of lemma 3.6.

For multiindices satisfying (73) we have

\begin{align*}
    2n - k(B) - 2h(B) + 2l(B) + 2s(B) \\
    &= 2n + |I_2| + |I_3| - 4n + 2|T| + 2|I_1| + 2|I_4| \\
    &= |J_2| + |J_3| + |J_5| - |J_8| \geq |J_1| + |J_2| + |J_3| - m + 1,
\end{align*}

where we used the inequality $|J_1| + |J_8| \leq m - 1$.

Therefore, if

\begin{align*}
    2n - k(B) - 2h(B) + 2l(B) + 2s(B) \leq 0
\end{align*}

then

\begin{align*}
    |J_1| + |J_2| + |J_3| \leq m - 1
\end{align*}

and hence

\begin{align*}
    |J_5| \geq n - r - 1 - |J_1| - |J_2| - |J_3| \geq n - r - m > n - q - m,
\end{align*}

which is impossible.

Analogously, for multiindices satisfying (74) we have

\begin{align*}
    2n - k(B) - 2h(B) + 2l(B) + 2s(B) \\
    &= 2n + |I_2| + |I_3| - 4n + 2|T| + 2|I_1| + 2|I_4| \\
    &= |J_2| + |J_3| + |J_5| - |J_8| + 1 \geq |J_1| + |J_2| + |J_3| - m + 1,
\end{align*}

where we used the inequality $|J_1| + |J_8| \leq m$.

The same arguments as above again show that

\begin{align*}
    2n - k(B) - 2h(B) + 2l(B) + 2s(B) \geq 1.
\end{align*}
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