Research Article

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Some estimates for commutators of Littlewood-Paley $g$-functions

Abstract: The aim of this paper is to establish the boundedness of commutator $[b, \dot{g}]$ generated by Littlewood-Paley $g$-functions $\dot{g}$, and $b \in \text{RBMO}(\mu)$ on non-homogeneous metric measure space. Under assumption that $\lambda$ satisfies $\varepsilon$-weak reverse doubling condition, the author proves that $[b, \dot{g}]$ is bounded from Lebesgue spaces $L^p(\mu)$ into Lebesgue spaces $L^q(\mu)$ for $p \in (1, \infty)$ and also bounded from spaces $L^1(\mu)$ into spaces $L^{1,\infty}(\mu)$. Furthermore, the boundedness of $[b, \dot{g}]$ on Morrey space $M^p_1(\mu)$ and on generalized Morrey $L^{p,\Phi}(\mu)$ is obtained.

Keywords: non-homogeneous metric measure space, Littlewood-Paley $g$-function, commutator, space RBMO($\mu$), generalized Morrey space

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1 Introduction

Although the metric measure spaces equipped with the polynomial growth conditions (see [1–5]) and spaces of homogeneous type in the sense of Coifman and Weiss [6,7] are two important classes of function spaces in harmonic analysis, there exist no relations between the non-doubling measure spaces and spaces of homogeneous type. To solve this problem, Hytönen [8] first introduced the non-homogeneous metric measure spaces satisfying the so-called geometrically doubling and upper doubling conditions. From then on, many papers focus on the properties of function spaces and operators over non-homogeneous metric measure spaces. For example, Cao and Zhou [9] obtained the definition of Morrey space on non-homogeneous metric measure space and also proved that Hardy-Littlewood maximal operator, Calderón-Zygmund operator and fractional integral are bounded on Morrey space. Fu and Zhao [10] proved that generalized homogeneous Littlewood-Paley $g$-function is bounded from atomic Hardy space $H_{\text{atom}}(\mu)$ into space $L^1(\mu)$ and also bounded from space RBMO($\mu$) into space RBLO($\mu$). For more development of harmonic analysis on non-homogeneous metric measure space the readers can see [11–19] and references therein.

Let $(X, d, \mu)$ be a non-homogeneous metric measure space in the sense of Hytönen [8]. In this setting, the author proves that the commutator $[b, \dot{g}]$ generated by $b \in \text{RBMO}(\mu)$ and generalized homogeneous Littlewood-Paley $g$-function $\dot{g}$, is bounded on Lebesgue spaces $L^p(\mu)$ for $p \in (1, \infty)$ and also bounded from spaces $L^1(\mu)$ into spaces $L^{1,\infty}(\mu)$. Furthermore, the boundedness of $[b, \dot{g}]$ on Morrey space $M^p_1(\mu)$ and on generalized Morrey space $L^{p,\Phi}(\mu)$ is also established in this paper. In 2016, Fu and Zhao obtained the boundedness of $\dot{g}$ on atomic Hardy space $H_{\text{atom}}(\mu)$ and on space RBMO($\mu$) (see [10]). In 2021, Lu and Tao [11] proved that the $\dot{g}$ is bounded on Lipschitz space $\text{Lip}_\beta(\mu)$ and on generalized Morrey space $L^{p,\Phi}(\mu)$.

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Before stating the main results of this paper, we first recall some necessary notions. The following definitions of geometrically doubling and upper doubling conditions are from [8].

**Definition 1.1.** A metric space $(X, d)$ is said to be geometrically doubling if there exists some $N_0 \in \mathbb{N}$ such that, for any ball $B(x, r) \subset X$ with $x \in X$ and $r \in (0, \infty)$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most $N_0$.

**Remark 1.2.** Let $(X, d)$ be a metric measure. Hytönen [8] showed that the geometrically doubling $(X, d)$ is equivalent to the following statement: for any $\epsilon \in (0, 1)$ and any ball $B(x, r) \subset X$ with $x \in X$ and $r \in (0, \infty)$, there exists a finite ball covering $\{B(x_i, \epsilon r)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most $e^{-\epsilon n_0}$, where $n_0 = \log_2 N_0$.

**Definition 1.3.** A metric measure space $(X, d, \mu)$ is said to be upper doubling if $\mu$ is a Borel measure on $X$ and there exist a dominating function $\lambda : X \times (0, \infty) \to (0, \infty)$ and constant $C_\lambda > 0$, depending on $\lambda$, such that, for each $x \in X$, $r \to \lambda(x, r)$ is non-decreasing and for all $x \in X$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \leq \lambda(x, r) \leq C_\lambda \lambda(x, r/2).$$

Moreover, Hytönen et al. [12] showed that there exists another dominating function $\tilde{\lambda}$ such that $\tilde{\lambda} \leq \lambda$, $C_{\tilde{\lambda}} \leq C_\lambda$, and, for all $x, y \in X$ with $d(x, y) \leq r$,

$$\tilde{\lambda}(x, r) \leq C_\lambda \tilde{\lambda}(y, r).$$

Here and in what follows, we always assume that $\tilde{\lambda}$ satisfies (1.2).

Although the doubling measure condition is not assumed uniformly for all balls on $(X, d, \mu)$, Hytönen [8] showed that there exist many balls satisfying the $(\alpha, \beta)$-doubling condition, i.e., let $\alpha, \beta \in (1, \infty)$, a ball $B \subset X$ is said to be $(\alpha, \beta)$-doubling if $\mu(aB) \leq \beta \mu(B)$. Throughout this paper, for any $\alpha \in (1, \infty)$ and ball $B$, the smallest $(\alpha, \beta_\alpha)$-doubling ball of the form $aB$ with $a \in \mathbb{R}$ is denoted by $B\alpha^a$, where

$$\beta_\alpha := a^{\lambda_{\max}(n, v)} + 30^a + 30^v.$$

We always denote by $\tilde{B}$ the smallest $(6, \beta_6)$-doubling ball of the form $B\tilde{\alpha}$ in this paper.

We now recall the definition of coefficient $K_{B,S}$ introduced in [8], which is very close to the quantity $K_{\Omega,R}$ introduced by Tolsa [3], that is, for any two balls $B \subset S$, define

$$K_{B,S} = 1 + \int_{(2\Omega)^1} \frac{1}{\lambda(c(B), d(x, c(B)))} \, d\mu(x),$$

where $c(B)$ represents the center of ball $B$. For more properties of the $K_{B,S}$, we can see [12, Lemma 2.1].

The following regularized bounded mean oscillation (RBMO) space was from [8].

**Definition 1.4.** Let $\rho > 1$. A function $f \in L^1_{\text{loc}}(\mu)$ is said to be in the space $\text{RBMO}(\mu)$ if there exist a positive constant $C$ and, for any ball $B$, a number $f_B$ such that

$$\frac{1}{\mu(\rho B)} \int_B |f(y) - f_B| \, d\mu(y) \leq C$$

and, for any two balls $B$ and $S$ such that $B \subset S$,

$$|f_B - f_S| \leq CK_{B,S}. \tag{1.5}$$

The infimum of the positive constants $C$ satisfying both (1.4) and (1.5) is defined to be the RBMO$(\mu)$ norm of $f$ and denoted by $\|f\|_{\text{RBMO}(\mu)}$. Moreover, Hytönen [8] also showed that the space RBMO$(\mu)$ is independent of the choice of the constant $\rho \in (1, \infty)$. 

In 2015, Tan and Li [13] gave an approximation of the identity \( S = \{S_k\}_{k \in \mathbb{Z}} \) associated with \((2, 2(C_\lambda + 1))-\)doubling balls \( \{Q_{x,k}\}_{x \in \text{supp} \mu} \) on \((X, d, \mu)\), which are integral operators associated with kernels \( S_k(x, y) \) on \( X \times X \) satisfying the following conditions:

(A-1) \( S_k(x, y) = S_k(y, x) \) for all \( x, y \in \text{supp} \mu \).

(A-2) For any \( k \in \mathbb{Z} \) and \( x \in \text{supp} \mu \), \( S_1(x, 1) = 1 = S_k^*1(x) \), where \( S_k^* \) is the adjoint operator of \( S_k \).

(A-3) For each \( k \in \mathbb{Z} \) and \( x \in \text{supp} \mu \), \( \text{supp}(S_k(x, \cdot)) \subset Q_{x,k-1} \).

(A-4) For all \( x, y \in X \) and \( k \in \mathbb{Z} \), if \( x \neq y \) or \( Q_{x,k} \neq |x| \), then there exists a non-negative constant \( C \) such that

\[
0 \leq S_k(x, y) \leq \frac{C}{\lambda(x, r(Q_{x,k}) + r(Q_{y,k}) + d(x, y))}.
\]

(A-5) For all \( x, \tilde{x}, y \in X \) and \( k \in \mathbb{Z} \), if \( x, \tilde{x} \in Q_{x_0,k} \) for some \( x_0 \in \text{supp} \mu \) and \( x \neq \tilde{x} \), then there exists a positive constant \( C \) such that

\[
|S_k(x, y) - S_k(\tilde{x}, y)| \leq \left( \frac{d(x, \tilde{x})}{r(Q_{x_0,k})} \right)^\varepsilon \frac{C}{\lambda(x, r(Q_{x,k}) + r(Q_{y,k}) + d(x, y))},
\]

where \( \varepsilon \in (0, \infty) \) and \( Q_{x,k} \) represents a fixed doubling ball center as \( x \) of generation \( k \).

Moreover, Tan and Li [13] showed that the aforementioned results through (A-1) to (A-5) are still correct if the \((2, 2(C_\lambda + 1))-\)doubling balls are replaced by \((6, \beta))-\)doubling balls.

We now recall the definition of generalized homogeneous Littlewood-Paley \( g \)-function introduced in [10].

**Definition 1.5.** Let \( k \in \mathbb{Z} \), \( r \in [2, \infty) \), \( D_k(x, y) = S_k(x, y) - S_k_{-1}(x, y) \) for all \( x, y \in X \), and \( D_k \) be the corresponding integral operator associated with the kernel \( D_k(x, y) \). Then, the generalized homogeneous Littlewood-Paley \( g \)-function \( \tilde{g}_r \) is defined by

\[
\tilde{g}_r(f)(x) = \left( \sum_{k \in \mathbb{Z}} |D_k(x, \cdot) f(x)|^r \right)^{\frac{1}{r}}, \quad \text{for any } x \in X.
\]

Given a function \( b \in \text{RBMO}(\mu) \), the commutator \([b, \tilde{g}_r] \), which is generated by \( b \) and \( \tilde{g}_r \) as in (1.6), is defined by

\[
[b, \tilde{g}_r](x) = \left( \sum_{k \in \mathbb{Z}} |D_k(x, \cdot) (b(x) - b(\cdot)) f(x)|^r \right)^{\frac{1}{r}},
\]

where \( x \in X \) and \( f \in L^\infty(\mu) \) being the space of all \( L^\infty(\mu) \) functions with bounded support.

The following definition of generalized Morrey space is from [14].

**Definition 1.6.** Let \( k > 1 \) and \( 1 < p < \infty \). Suppose that \( \Phi \) is an increasing function on \((0, \infty)\). Then, the generalized Morrey space \( L^{p, \Phi}(\mu) \) is defined by

\[
L^{p, \Phi}(\mu) = \{ f \in L^p_{\text{loc}}(\mu) : \|f\|_{L^{p, \Phi}(\mu)} < \infty \},
\]

where

\[
\|f\|_{L^{p, \Phi}(\mu)} = \sup_B \left[ \Phi(\mu(B)) \right]^{\frac{1}{p}} \left( \int_B |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}}.
\]

**Remark 1.7.**

1. From [14], Lu and Tao showed that the norm \( \|f\|_{L^{p, \Phi}(\mu)} \) is independent of the choice of parameter \( k \) for \( k > 1 \).
(2) If we take \( \phi(t) = t^{1-q} \) with \( t > 0 \) and \( 1 < q \leq p < \infty \), then \( L^{p,\phi}(\mu) \) defined as in (1.8) is just the Morrey space \( M^p_\phi(\mu) \), which was introduced by Cao and Zhou [9], that is, let \( k > 1 \) and \( 1 < q \leq p < \infty \), then Morrey space \( M^p_\phi(\mu) \) is defined by

\[
M^p_\phi(\mu) = \{ f \in L^p_{\text{loc}}(\mu) : \| f \|_{M^p_\phi(\mu)} < \infty \},
\]

where

\[
\| f \|_{M^p_\phi(\mu)} := \sup_B [\mu(kB)]^{\frac{1}{p}} \left( \int_B |f(y)|^q \text{d}\mu(y) \right)^{\frac{1}{q}}.
\] (1.9)

We now recall the following \( \varepsilon \)-weak reverse doubling condition introduced in [16].

**Definition 1.8.** Let \( \epsilon \in (0, \infty) \). A dominating function \( \lambda \) is said to satisfy the \( \epsilon \)-weak reverse doubling condition if, for all \( r \in (0, 2\text{diam}(\mathcal{X})) \) and \( a \in (1, 2\text{diam}(\mathcal{X})/r) \), there exists a number \( C(a) \in [1, \infty) \), depending only on \( a \) and \( \mathcal{X} \), such that, for all \( x \in \mathcal{X} \),

\[
\lambda(x, ar) \geq C(a) \lambda(x, r)
\]

and, moreover,

\[
\sum_{k=1}^{\infty} \frac{1}{(C(a^k))^2} < \infty.
\] (1.10)

The main theorems of this paper are stated as follows:

**Theorem 1.9.** Let \( b \in \text{RBMO}(\mu) \) and \( r \in [2, \infty) \). Suppose that \( \lambda \) satisfies the \( \epsilon \)-weak reverse doubling condition defined as in Definition 1.8. Then, there exists a constant \( C > 0 \) such that, for all \( f \in L^p(\mu) \) with \( p \in (1, \infty) \),

\[
\| [b, \tilde{g}_r](f) \|_{L^p(\mu)} \leq C\| b \|_{\text{RBMO}(\mu)} \| f \|_{L^p(\mu)}.
\]

**Theorem 1.10.** Let \( b \in \text{RBMO}(\mu) \) and \( r \in [2, \infty) \). Suppose that \( \lambda \) satisfies the \( \epsilon \)-weak reverse doubling condition defined as in Definition 1.8. Then, there exists a constant \( C > 0 \) such that, for all \( f \in L^1(\mu) \),

\[
\| [b, \tilde{g}_r](f) \|_{L^{1,\infty}(\mu)} \leq C\| b \|_{\text{RBMO}(\mu)} \| f \|_{L^1(\mu)}.
\]

**Theorem 1.11.** Let \( b \in \text{RBMO}(\mu) \), \( r \in [2, \infty) \) and \( 1 < q \leq p < \infty \). Suppose that \( \lambda \) satisfies the weak reverse doubling condition defined as in Definition 1.8. Then, there exists a constant \( C > 0 \) such that, for all \( f \in M^p_\phi(\mu) \),

\[
\| [b, \tilde{g}_r](f) \|_{M^p_\phi(\mu)} \leq C\| b \|_{\text{RBMO}(\mu)} \| f \|_{M^p_\phi(\mu)}.
\]

**Theorem 1.12.** Let \( b \in \text{RBMO}(\mu) \), \( r \in [2, \infty) \) and \( p \in (1, \infty) \). Suppose that \( \phi : (0, \infty) \rightarrow (0, \infty) \) is an increasing function and the mapping \( s \mapsto \frac{\phi(s)}{s} \) is almost decreasing, namely, there is a constant \( C \in (0, \infty) \) such that, the following inequality

\[
\frac{\phi(w)}{w} \leq C\frac{\phi(s)}{s}
\] (1.11)

holds for \( s \geq w \). Then, there exists a positive constant \( C \) such that, for all \( f \in L^{p,\phi}(\mu) \),

\[
\| [b, \tilde{g}_r](f) \|_{L^{p,\phi}(\mu)} \leq C\| b \|_{\text{RBMO}(\mu)} \| f \|_{L^{p,\phi}(\mu)}.
\]

Finally, we make some conventions on notation. Throughout the paper, \( C \) represents a positive constant, which is independent of the main parameters. For any subset \( E \) of \( \mathcal{X} \), we use \( \chi_E \) denote its characteristic function. Given any \( q \in (1, \infty) \), let \( q' = q/(q - 1) \) denote its conjugate index. For any ball \( B, c(B) \) and \( r(B) \) represent the center and radius of ball \( B \), respectively. Furthermore, \( m_B(f) \) denotes the mean value of function \( f \) over ball \( B \), that is, \( m_B(f) = \frac{1}{\mu(B)} \int_B f(y) \text{d}\mu(y) \).
2 Preliminaries

To prove the main theorems of this paper, we should recall some necessary results in this section (see [10, 14, 16, 17]).

Lemma 2.1. Let \((X, d, \mu)\) be a non-homogeneous metric measure space satisfying the weak reverse doubling condition, \(r \in [2, \infty) \) and \(p \in (1, \infty)\). Then, there exists a constant \(C > 0\) such that, for all \(f \in L^p(\mu)\),

\[
\|\hat{f}_r(f)\|_{L^{p}(\mu)} \leq C\|f\|_{L^{p}(\mu)}.
\] (2.1)

Corollary 2.2. If \(f \in \text{RBMO}(\mu)\), then there exists a constant \(C > 0\) such that, for any ball \(B, \tau \in (1, \infty)\) and \(s \in [1, \infty)\),

\[
\left(\frac{1}{\mu(\tau B)} \int_B |f(y) - f_B|^s \, d\mu(y)\right)^{\frac{1}{s}} \leq C\|f\|_{\text{RBMO}(\mu)}.
\] (2.2)

Lemma 2.3.

(1) Let \(p \in (1, \infty), t \in (1, p)\) and \(\rho \in [5, \infty)\). The following maximal operators, respectively, defined for all \(f \in L^1_{\text{loc}}(\mu)\) and \(x \in X\),

\[
M_{x, \rho}f(x) := \sup_{B \ni x} \left(\frac{1}{\mu(\rho B)} \int_B |f(y)|^t \, d\mu(y)\right)^{\frac{1}{t}}
\] (2.3)

and

\[
Nf(x) := \sup_{B \text{ doubling}} \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y)
\]

are bounded on \(L^p(\mu)\) and also bounded from \(L^1(\mu)\) into \(L^{1, \infty}(\mu)\).

(2) For all \(f \in L^1_{\text{loc}}(\mu)\), it holds true that \(|f(x)| \leq Nf(x)\) for \(\mu\)-a.e \(x \in X\).

Lemma 2.4. Let \(f \in L^1_{\text{loc}}(\mu)\) satisfy \(\int_X f(x) \, d\mu(x) = 0\) when \(\|f\| = \mu(x) < \infty\). Assume that, for some \(p, q\) satisfying \(1 < q \leq p < \infty\), \(\inf_{1 \leq q \leq p} \frac{p}{q} \in M(\mu)\). Then, there exists a constant \(C > 0\),

\[
\|Nf\|_{M^p(\mu)} \leq C\|M^qf\|_{M^p(\mu)},
\] (2.4)

where the sharp maximal operator \(M^q\) is defined by, for all \(f \in L^1_{\text{loc}}(\mu)\) and \(x \in X\),

\[
M^qf(x) := \sup_{B \ni x} \frac{1}{\mu(6B)} \int_B |f(y) - m_B(f)| \, d\mu(y) + \sup_{\substack{x \in B \setminus S \text{ doubling} \quad \|m_B(f) - m_s(f)\|}} \frac{K_{B, S}}{K_{B, S}}.
\] (2.5)

Lemma 2.5. Let \(\rho > 1, 1 < t < p < \infty\) and \(\phi : (0, \infty) \to (0, \infty)\) be an increasing function. Suppose that \(M_{x, \rho}\) is defined as in (2.3) and the mapping \(s \mapsto \phi(s)\) satisfies the condition (1.11). Then, there exists a positive constant \(C > 0\) such that, for all \(f \in L^p(\mu)\),

\[
\|M_{x, \rho}f\|_{L^{p, \phi}(\mu)} \leq C\|f\|_{L^{p, \phi}(\mu)}.
\]
3 Proof of Theorems 1.9–1.12

Proof of Theorem 1.9. For any $r \in [2, \infty)$, $p \in (1, \infty)$, $f \in L^p(\mu)$ and $x \in X$, we first claim that

$$M^r([b, \dot{g}_r](f))(x) \leq C\|b\|_{\text{RBMO}(\mu)} \{M_{r, \delta}^r(f)(x) + M_{r, \delta}^r(f)(x) + M_{r, \delta}^r(f)(x)\}. \tag{3.1}$$

Once (3.1) is proved, by applying Lemma 2.3 and Theorem 4.2 in [15], we have

$$\|\{b, \dot{g}_r\}(f)\|_{L^p(\mu)} \leq C\|b\|_{\text{RBMO}(\mu)} \{M_{r, \delta}^r(f)(x) + M_{r, \delta}^r(f)(x) + M_{r, \delta}^r(f)(x)\},$$

which is just the desired conclusion.

To show (3.1), by the definition of sharp maximal function $M^r$, there exists a family of numbers $\{b_{\delta}\}_{\delta}$ such that, for all $x \in X$ and $B \ni x$,

$$\frac{1}{\mu(6B)} \int_B |g_{r, \delta}(f)(x) - b_{\delta}| \, d\mu(x) \leq C\|b\|_{\text{RBMO}(\mu)} \{M_{r, \delta}^r(f)(x) + M_{r, \delta}^r(f)(x) + M_{r, \delta}^r(f)(x)\}, \tag{3.2}$$

and, for all balls $B$, $S$ satisfying $B \subset S$ and $B \ni x$,

$$|h_B - h_S| \leq CK_{B, S} \|b\|_{\text{RBMO}(\mu)} \{M_{r, \delta}^r(g_r(f))(x) + M_{r, \delta}^r(f)(x) + M_{r, \delta}^r(f)(x)\}, \tag{3.3}$$

where

$$B = m_B(g_r((b - b_B)\eta_{X \setminus 6B})) \quad \text{and} \quad h_S = m_S((b - b_S)\eta_{X \setminus 6S}).$$

To prove (3.2), for a fixed ball $B$ and $x \in B$, write

$$[b, \dot{g}_r](f)(x) \leq C\frac{1}{\mu(6B)} \left| \int_B \frac{\sum_{k \in Z} |D_{k}f(x)|}{|x|} \right|^\frac{1}{2} + C\left( \frac{1}{\mu(6B)} \int_B \frac{\sum_{k \in Z} |D_{k}(b(\cdot) - b_B)f_1(x)|}{|x|} \right)^\frac{1}{2} + C\left( \frac{1}{\mu(6B)} \int_B \frac{\sum_{k \in Z} |D_{k}(b(\cdot) - b_B)f_2(x)|}{|x|} \right)^\frac{1}{2} \leq C\|b\|_{\text{RBMO}(\mu)} \{M_{r, \delta}^r(g_r(f))(x) + M_{r, \delta}^r(f)(x) + M_{r, \delta}^r(f)(x)\},$$

where $f_1 = f_{6B}$ and $f_2 = f - f_1$.

By applying Hölder inequality, (2.1) and (2.2), we obtain that

$$\frac{1}{\mu(6B)} \int_B |b(x) - b_B| \dot{g}_r(f)(x) |d\mu(x) \leq \frac{1}{\mu(6B)} \int_B |\dot{g}_r(f)(x)| d\mu(x) \frac{1}{\mu(6B)} \int_B |b(x) - b_B| \dot{g}_r(f)(x) |d\mu(x) \leq C\|b\|_{\text{RBMO}(\mu)} \{M_{r, \delta}^r(g_r(f))(x)\}. \tag{3.4}$$

To estimate $\dot{g}_r((b(\cdot) - b_B)f_1)$, take $s = \sqrt{r}$. From Hölder inequality, (2.1) and (2.2), it follows that

$$\frac{1}{\mu(6B)} \int_B |\dot{g}_r((b(\cdot) - b_B)f_1)(x)| d\mu(x) \leq \frac{1}{\mu(6B)^\frac{1}{2}} \left( \int_B \frac{1}{\mu(6B)^\frac{1}{2}} \left( \int_B |\dot{g}_r((b(\cdot) - b_B)f_1)(x)|^2 d\mu(x) \right)^\frac{1}{2} \right) \leq C \|b\|_{\text{RBMO}(\mu)} \|f_1\|_{L^\infty(\mu)} \tag{3.5}$$

$$\leq C \left( \frac{1}{\mu(6B)} \int_{6B} |b(y) - b_B|^s \, d\mu(y) \right)^{\frac{1}{s}} \left( \frac{1}{\mu(6B)} \int_{6B} |f(y)|^s \, d\mu(y) \right)^{\frac{1}{s}} \leq C\|b\|_{\text{RBMO}(\mu)} \{M_{r, \delta}^r(f)(x)\}.$$
Together with the aforementioned estimates, to get (3.2), we still need to estimate the difference $\hat{g}_t((b(\cdot) - b_B) f_2) - h_B$. For all $y_1, y_2 \in B$, write

$$|\hat{g}_t((b(\cdot) - b_B) f_2)(y_1) - h_B| \leq \frac{1}{\mu(B)} \int_B \left| \sum_{k \in \mathbb{Z}} \left( \int_X D_k(y_1, z)(b(z) - b_B) f_2(z) \, d\mu(z) \right) \right|^\frac{p}{2} \, d\mu(y_2)$$

$$- \left( \sum_{k \in \mathbb{Z}} \int_X D_k(y_2, z)(b(z) - b_B) f_2(z) \, d\mu(z) \right) \, d\mu(y_2)$$

$$\leq \frac{1}{\mu(B)} \int_B \left( \sum_{k \in \mathbb{Z}} \left( \int_X (D_k(y_1, z) - D_k(y_2, z))(b(z) - b_B) f_2(z) \, d\mu(z) \right) \right) \, d\mu(y_2),$$

therefore, we only consider

$$\left( \sum_{k \in \mathbb{Z}} \left( \int_X (D_k(y_1, z) - D_k(y_2, z))(b(z) - b_B) f_2(z) \, d\mu(z) \right) \right)^\frac{1}{p}.$$ 

From Fubini-Tonelli theorem, Hölder inequality, (1.1), (1.6) and (2.2), it follows that

$$\left( \sum_{k \in \mathbb{Z}} \left( \int_X (D_k(y_1, z) - D_k(y_2, z))(b(z) - b_B) f_2(z) \, d\mu(z) \right) \right)^\frac{1}{p}$$

$$\leq \int_X \left( \sum_{k \in \mathbb{Z}} |D_k(y_1, z) - D_k(y_2, z)| \right)^\frac{p}{2} |b(z) - b_B| |f(z)|^\frac{p}{2} \, d\mu(z)$$

$$\leq C \sum_{i=0}^{H-3} \int_{Q_{y_1} \setminus Q_{y_1+i}} \left[ \frac{d(y_1, y_2)}{r(Q_{y_1})} \right]^r |b(z) - b_B| |f(z)|^r \, d\mu(z)$$

$$+ C \left( \int_{Q_{y_1} \setminus Q_{y_1+i}} \frac{|b(z) - b_B| |f(z)|^r}{|\lambda(y_1, d(y_1, z))|^r} \, d\mu(z) \right)^\frac{1}{r}$$

$$\leq C \sum_{i=0}^{H-3} \int_{Q_{y_1} \setminus Q_{y_1+i}} \left[ \frac{r(B)}{r(Q_{y_1} r^{-i})} \right]^r \frac{|b(z) - b_B| |f(z)|^r}{|\lambda(y_1, d(y_1, z))|^r} \, d\mu(z)$$

$$+ C \left( \int_{Q_{y_1} \setminus Q_{y_1+i}} \frac{|b(z) - b_B| |f(z)|^r}{|\lambda(y_1, d(y_1, z))|^r} \, d\mu(z) \right)^\frac{1}{r}$$

$$\leq C \sum_{i=0}^{H-3} \left[ \frac{r(B)}{r(Q_{y_1} r^{-i})} \right]^r \frac{1}{|\lambda(y_1, r(Q_{y_1} r^{-i-1}))|^r} \int_{Q_{y_1} \setminus Q_{y_1+i}} |b(z) - b_B| |f(z)|^r \, d\mu(z)$$

$$+ \frac{C}{\lambda(y_1, r(6B))} \int_{Q_{y_1} \setminus Q_{y_1+i}} |b(z) - b_B| |f(z)|^r \, d\mu(z)$$

$$\leq C \left( \sum_{i=0}^{H-3} \left[ \frac{r(B)}{r(Q_{y_1} r^{-i})} \right]^r \frac{1}{|\lambda(y_1, r(Q_{y_1} r^{-i-1}))|^r} \int_{Q_{y_1} \setminus Q_{y_1+i}} |b(z) - b_B| |f(z)|^r \, d\mu(z) \right)^\frac{1}{r}$$

$$+ \frac{C}{\lambda(y_1, r(6B))} \int_{Q_{y_1} \setminus Q_{y_1+i}} |b(z) - b_B| |f(z)|^r \, d\mu(z)$$

From Fubini-Tonelli theorem, Hölder inequality, (1.1), (1.6) and (2.2), it follows that
\[ C \sum_{l=1}^{\infty} \frac{r(B)}{r(Q_{y_1, r+1})} \geq \frac{1}{[l(y_1, r(Q_{y_1, r+1})]^l} \]

\[ \times \int_{Q_{y_1, r+1}} |f(z) - b_{Q_{y_1, r+1}} f(z)|^l \, d\mu(z) + |b_B - b_{Q_{y_1, r+1}}|^l \int_{Q_{y_1, r+1}} |f(z)|^l \, d\mu(z) \]

\[ \leq C \frac{1}{[l(y_1, r(Q_{y_1, r+1})]^l} \int \left\| b_{\RBMO(\mu)} [\mu(6Q_{y_1, r+1})] M_{r, \delta}(f)(x) \right\| \]

\[ + C \int \left\| b_{\RBMO(\mu)} [\mu(6Q_{y_1, r+1})] M_{r, \delta}(f)(x) \right\| \]

\[ \leq C \int \left\| b_{\RBMO(\mu)} [\mu(6Q_{y_1, r+1})] M_{r, \delta}(f)(x) \right\| \]

where \( H_{y_1}^n \) represents the largest integer \( k \) satisfying \( B \subset B_{y_1,k} \) for any ball \( B \) and \( y_1 \in B \cap \text{supp}(\mu) \). Moreover, we also need some known facts proved in [10].

\[ \lambda \setminus 6B = \bigcup_{i=3}^{\infty} \left( Q_{y_1, r+1-i} \setminus Q_{y_1, r+1-i} \right) \cup \left( Q_{y_1, r+1-i} \setminus 6B \right) \]

and, for all \( y_1, y_2 \in B \) and \( z \in \lambda \setminus Q_{y_1, r+1-2} \),

\[ \sum_{k \in Z} |D_k(y_1, z) - D_k(y_2, z)|^l \leq C \frac{d(y_1, y_2)}{r(Q_{y_1, r+1-i})} \geq \frac{1}{[l(y_1, d(y_1, z))^l} \]

Combining the estimates for (3.4), (3.5) and (3.6), we get (3.2).

Now, we show the condition (3.3). Consider two balls \( B, S \subset \lambda \) satisfying \( B \subset S \) and \( x \in B \), and let \( N = N_{\lambda, \delta} + 1 \). Write

\[ |h_B - h_S| \leq |m_{\delta} (\hat{g}_B((b-B)_S f(x, y^B_{y_1, y_B}))\rangle + m_{\delta} (\hat{g}_B((b-B)_S f(x, y^B_{y_1, y_B}))\rangle + |m_{\delta} (\hat{g}_B((b-B)_S f(x, y^B_{y_1, y_B}))\rangle + m_{\delta} (\hat{g}_B((b-B)_S f(x, y^B_{y_1, y_B}))\rangle\]

\[ = E_1 + E_2 + E_3 + E_4. \]

With an argument similar to that used in the estimate of (3.6), it is easy to see that

\[ E_3 \leq C \int \left\| b_{\RBMO(\mu)} [M_{r, \delta}(f)(x) + M_{r, \delta}(f)(x)] \right\|. \]

For \( y \in B \), by applying Fubini-Tonelli Theorem, Minkowski inequality, Hölder inequality and (2.2), we can deduce that

\[ E_1 \leq \frac{1}{\mu(B)} \int_B \left\| \hat{g}_B((b_B) f(x, y^B_{y_1, y_B})) (y) \right\| d\mu(y) \]

\[ \leq \frac{1}{\mu(B)} \int_B \left\{ \int_{x^B_{y_1, y_B}} \left[ \sum_{k \in Z} \left| D(y, z) \right|^l \right] \left( b(z) - b_{B_S} \right)^l |f(z)|^l d\mu(z) \right\}^l d\mu(y) \]
\[
\frac{1}{\mu(B)} \int_{\mathcal{B}(B) \setminus 6\mathcal{B}_1(B)} \left( \int_{\mathcal{B}} \left( \sum_{L \in \mathcal{L}} |D(y, z)|^{\frac{1}{n}} \right)^{\frac{n}{2}} \|b(z) - b_0\|f(z)\|d\mu(z) \right)
\leq \frac{1}{\mu(B)} \int_{\mathcal{B}(B) \setminus 6\mathcal{B}_1(B)} \left( \int_{\mathcal{B}} \left( \sum_{L \in \mathcal{L}} |D(y, z)|^{\frac{1}{n}} \right)^{\frac{n}{2}} \|b(z) - b_0\|f(z)\|d\mu(z) \right)
\leq C \sum_{j=1}^{N-1} \int_{\mathcal{B}(B) \setminus 6\mathcal{B}_1(B)} \frac{|b(z) - b_0|}{\lambda(c(B), d(c(B), z))} |f(z)| d\mu(z)
\leq C \sum_{j=1}^{N-1} \frac{1}{\lambda(c(B), 6\mathcal{B}_1(B))} \int_{6\mathcal{B}_1(B)} |b(z) - b_0| |f(z)| d\mu(z) + |b_0| \int |f(z)| d\mu(z)
\leq CK_{B,S} \|b\|_{RBMO(B)} M_{r,\delta}(f)(x) \sum_{j=1}^{N-1} \frac{\mu(6j+1\mathcal{B})}{\lambda(c(B), 6\mathcal{B}_1(B))}
\leq CK_{B,S} \|b\|_{RBMO(B)} M_{r,\delta}(f)(x)
\]
Similarly, it follows that \(E_2 \leq CK_{B,S} \|b\|_{RBMO(B)} M_{r,\delta}(f)(x)\).

Now, let us estimate \(E_4\). For any \(y \in B\), from (1.5), Fubini-Tonelli Theorem, Minkowski inequality and Hölder inequality, we have
\[
\frac{1}{\mu(B)} \int_{\mathcal{B}} |\mu_B g_B(y) - b_0| |f(x)| d\mu(y) \leq CK_{B,S} \|b\|_{RBMO(B)} \left( M_{r,\delta}(g_B(f))(x) + M_{r,\delta}(f)(x) \right)
\]

**Proof of Theorem 1.10.** By applying Lemma 2.3 and (3.1), it is obvious to see that
\[
\|b, g_B\|_{L^{\infty}(\mu)} \leq \|N((b, g_B)(f))\|_{L^{\infty}(\mu)}
\leq C \|M^1((b, g_B)(f))\|_{L^{\infty}(\mu)}
\leq C \|b\|_{RBMO(\mu)} \left\{ \|M_{r,\delta}(g_B(f))\|_{L^{\infty}(\mu)} + \|M_{r,\delta}(f)\|_{L^{\infty}(\mu)} \right\}
\leq C \|b\|_{RBMO(\mu)} \|f\|_{L^p(\mu)}
\]
which is the desired result.

**Proof of Theorem 1.11.** By applying (2.4), (3.1) and Theorem 14 (see [9]), we have
\[
\|b, g_B\|_{L^{p^*}(\mu)} \leq \|N((b, g_B)(f))\|_{M^{p^*}(\mu)}
\leq C \|M^1((b, g_B)(f))\|_{M^{p^*}(\mu)}
\leq C \|b\|_{RBMO(\mu)} \left\{ \|M_{r,\delta}(g_B(f))\|_{M^{p^*}(\mu)} + \|M_{r,\delta}(f)\|_{M^{p^*}(\mu)} \right\}
\leq C \|b\|_{RBMO(\mu)} \|f\|_{L^{p^*}(\mu)}
\]

**Proof of Theorem 1.12.** From Remark 1.7 (1), we assume \(k = 6\) in (1.8). By applying Definition 1.6, Lemma 2.3 (2), Theorem 4.2 in [15] and (3.1), we can deduce that
\[
\|b, g_B\|_{L^{p^*}(\mu)} \leq \sup_B \frac{C}{\|\mu(6\mathcal{B})\|^{\frac{1}{n}}} \|N((b, g_B)(f))\|_{L^{p^*}(\mu,B)}
\leq \sup_B \frac{C}{\|\mu(6\mathcal{B})\|^{\frac{1}{n}}} \|M^1((b, g_B)(f))\|_{L^{p^*}(\mu,B)}
\leq C \|b\|_{RBMO(\mu)} \left\{ \|M_{r,\delta}(g_B(f))\|_{L^{p^*}(\mu)} + \|M_{r,\delta}(f)\|_{L^{p^*}(\mu)} \right\}
\leq C \|b\|_{RBMO(\mu)} \|f\|_{L^{p^*}(\mu)}
\]
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