Abstract—The continuum deformation leader-follower cooperative control strategy models vehicles in a multi-agent system as particles of a deformable body. A desired continuum deformation is defined based on leaders’ trajectories and acquired by followers in real-time through local communication. The existing continuum deformation theory requires followers to be placed inside the convex simplex defined by leaders. This constraint is relaxed in this paper. We prove that under suitable assumptions any \( n + 1 \) \((n = 1, 2, 3)\) vehicles forming an \( n \)-D simplex can be selected as leaders while followers, arbitrarily positioned inside or outside the leading simplex, can acquire a desired continuum deformation in a decentralized fashion. The paper’s second contribution is to assign a one-to-one mapping between leaders’ smooth trajectories and homogeneous deformation features obtained by continuum deformation eigen-decomposition. Therefore, a safe and smooth continuum deformation coordination can be planned either by shaping homogeneous transformation features or by choosing appropriate leader trajectories. This is beneficial to efficiently plan and guarantee collision avoidance in a large-scale group. A simulation case study is reported in which a virtual convex simplex contains a quadcopter vehicle team at any time \( t \); \( A^* \) search is applied to optimize quadcopter team continuum deformation in an obstacle-laden environment.

Keywords—Path Planning, Collision Avoidance, Multi-vehicle System (MVS), Eigen Decomposition, Local Communication

I. INTRODUCTION

Formation and cooperative control algorithms [1] have been applied to problems in biology [2], computer science [3], aerospace engineering [4], [5], and elsewhere. Virtual structure [6], [7], consensus [8], [17], containment [18]–[25], and continuum deformation [26]–[28] are available multi-agent system (MAS) coordination methods. While the virtual structure (VS) method is commonly exploited for centralized coordination, the other three methods provide decentralized solutions. The VS method treats MAS as particles of a virtual rigid body; rigid body translation and rotation prescribes agents’ trajectories in a 3-D motion space. Consensus algorithm stability has been analyzed under fixed and switching communication topologies [9], [29] and in the presence of fixed and time-varying delays [10], [11], [17], [30]. Finite-time consensus under fixed and switching communication topologies is developed in Refs. [12], [13], while leader-follower consensus is investigated in Refs. [14], [15].

In containment control, leaders independently guide collective motion, and followers acquire the desired coordination via local communication. Containment control stability and convergence [31] with fixed [18] and switching [19], [22], [25] communication topologies have been analyzed in the existing literature. Retarded containment with fixed [18] and time-varying [20], [24] time-delays and finite-time containment control and coordination [21], [23] have also been investigated.

Defining agent coordination by continuum deformation [26] was first introduced in Ref. [28]. Leader-follower continuum deformation proposed in [26], [33] treats leader and follower agents as particles of a deformable body. Leaders form an \( n \)-D simplex containing follower agents during MAS evolution \((n = 1, 2, 3)\). A desired formation is defined by a homogeneous transformation (deformation) uniquely related to the trajectories of \( n + 1 \) leaders. Followers acquire the desired homogeneous transformation in real-time through local communication and apply communication weights consistent with each agent’s position in the reference configuration. Also, Refs. [34], [35] offer a leader-follower affine transformation method for multi-agent coordination where graph rigidity is explained to specify followers’ communication weights based on agents’ reference configuration. Continuum deformation supports fixed [26] and switching [27] communication topologies. Ref. [26] analyzes continuum deformation stability in the presence of communication delays. Alignment and polyhedral communication topologies are analyzed in [26], [36], and continuum deformation with more than \( n + 1 \) moving leaders is studied in [37]. Containment control and continuum deformation are both leader-follower methods. Continuum deformation extends containment control by assuring inter-agent collision avoidance as well as containment.

This paper offers a novel eigen-decomposition method for continuum deformation coordination of a multi-vehicle system (MVS) in a 3-D motion space. This eigen-decomposition leads to a computationally-efficient and scalable continuum deformation coordination approach in an obstacle-free environment and a less conservative safety condition for inter-agent collision avoidance. By relaxing limitations considered in the previous work, we advance the theory of continuum deformation acquisition in obstacle-laden and obstacle-free environments. Furthermore, we study continuum deformation planning and optimization in a cluttered environment. Compared to the available literature and the authors’ prior work [26], [27], [33], [37], this article offers the following novel contributions:

- This paper advances continuum deformation coordination theory [26], [27], [36], [37] by relaxing the containment requirement considered previously. Specifically, we

1 A continuum is defined as a continuous domain containing an infinite number of particles with infinitesimal size [32].
show in this paper that any \( n + 1 \) \( (n = 1, 2, 3) \) vehicles forming an \( n\)-D simplex can be selected as leaders. Followers, placed either inside or outside of the leading simplex, infer the desired continuum deformation in real time through local communication.

- We advance the theory of continuum deformation for integrator agents toward continuum deformation of vehicles with input-output linearizable dynamics. Assuming each vehicle has minimum-phase dynamics, this paper guarantees inter-agent collision avoidance in a motion governed by the continuum deformation algorithm with significant rotation and deformation possible.

- The existing continuum deformation coordination method ensures inter-agent collision avoidance by assigning a single lower-limit for all deformation eigenvalues. This could make continuum deformation coordination restrictive specifically when agents are non-uniformly distributed in the reference configuration as agent minimum separation constraints are related to deformation matrix eigenvalues. This paper guarantees inter-agent collision avoidance by assigning a lower-limit on one of the eigenvalues of the pure deformation matrix that characterizes the minimum separation distance in the reference configuration of the vehicles, while the other two eigenvalues only need to be positive to maintain the requirement of the continuum deformation coordination (see Theorem 5 below). This paper also relaxes agent spacing requirements in regions where the single minimum eigenvalue was unnecessarily restrictive. This new less conservative strategy is advantageous because more aggressive continuum deformation maneuvers are possible with distinct lower limits for the deformation eigenvalues.

In this paper, MVS desired homogeneous transformation is uniquely represented by the following features: (i) A rotation matrix, (ii) A positive definite deformation matrix defining principle deformations (eigenvalues) and their orientations (eigenvectors) along with a rigid-body displacement vector. A one-to-one mapping is obtained to relate leader trajectories defining an \( n\)-D homogeneous transformation to homogeneous transformation features. Safe continuum deformations will be planned either by shaping homogeneous transformation features or by choosing desired leader trajectories. In an obstacle-free environment, a large-scale continuum deformation coordination is planned strictly by shaping homogeneous transformation features. This is beneficial because safe leaders’ trajectories, ensuring inter-agent collision avoidance, can be determined at low computational cost. Alternatively, desired homogeneous transformations through an obstacle-laden environment can be planned by optimizing leaders’ trajectories such that the prescribed lower limit on homogeneous deformation eigenvalues required for obstacle collision avoidance is satisfied. Case studies are presented illustrating how safety constraints can be incorporated into optimizing continuum deformation given initial and target MAS formations.

This paper is organized as follows. Preliminaries presented in Section II are followed by inter-agent communication topology and graph theory definitions in Section III. Section IV presents the formulations and statements of the problems considered in this paper. MVS collective dynamics is obtained in Section V. Safety requirements of MVS continuum deformation coordination are obtained in Section VI. Continuum deformation planning is formulated in Section VII. Case study results in Section VIII are followed by a conclusion in Section IX. 

II. PRELIMINARIES

A. Position Notations

Agent positions are expressed with respect to a Cartesian frame with unit basis vectors \( \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \). Expressing \( \dot{\mathbf{e}}_1 = [1 \ 0 \ 0]^T \), \( \dot{\mathbf{e}}_2 = [0 \ 1 \ 0]^T \), \( \dot{\mathbf{e}}_3 = [0 \ 0 \ 1]^T \), the paper defines the following position notations for every agent \( i \in \mathcal{V} \):

- **Actual Position** vector of vehicle \( i \in \mathcal{V} \) denoted by \( \mathbf{r}_i = [x_i, y_i, z_i]^T \) is considered as the output of the control system of every vehicle \( i \in \mathcal{V} \).

- **Initial Position** vector of vehicle \( i \in \mathcal{V} \) is denoted by \( \mathbf{r}_{i,x} = [x_{i,x}, y_{i,x}, z_{i,x}]^T \in \mathbb{R}_x \), where \( \mathbb{R}_x \subset \mathbb{R}^3 \) is a finite set.

- **Reference Position** vector of vehicle \( i \in \mathcal{V} \) is denoted by \( \mathbf{r}_{i,0} = [x_{i,0}, y_{i,0}, z_{i,0}]^T \in \mathbb{R}_0 \), where \( \mathbb{R}_0 \subset \mathbb{R}^3 \) is a finite set.

- **Global Desired Position** vector of vehicle \( i \in \mathcal{V} \) denoted by \( \mathbf{r}_{i,HT} = [x_{i,HT}, y_{i,HT}, z_{i,HT}]^T \) is defined by homogeneous deformation that is presented in Section II-D.

- **Local Desired Position** vector of vehicle \( i \in \mathcal{V} \) denoted by \( \mathbf{r}_{d,i} = [x_{d,i}, y_{d,i}, z_{d,i}]^T \) is defined in Section V.

B. Motion Space Discretization

Let \( \mathbf{p}_1 \in \mathbb{R}^{n+1}, \ldots, \mathbf{p}_{n+1} \in \mathbb{R}^{n+1} \), and \( \mathbf{c} \) be position vectors of \( n+2 \) points in an \( n\)-D hyperplane. Defining a scalar function

\[
\Psi_n(\mathbf{p}_1, \ldots, \mathbf{p}_{n+1}) = \text{rank} \left( \left[ \mathbf{p}_2 - \mathbf{p}_1 \quad \cdots \quad \mathbf{p}_{n+1} - \mathbf{p}_1 \right] \right),
\]

(1)

vectors \( \mathbf{p}_1 \in \mathbb{R}^{n+1}, \ldots, \mathbf{p}_{n+1} \in \mathbb{R}^{n+1} \) assign positions of vertices of an \( n\)-D simplex, if \( \Psi_n(\mathbf{p}_1, \ldots, \mathbf{p}_{n+1}) = n \). If \( \Psi_n(\mathbf{p}_1, \ldots, \mathbf{p}_{n+1}) = n \), we can define a vector function

\[
\Theta_n(\mathbf{p}_1, \ldots, \mathbf{p}_{n+1}, \mathbf{c}) = \left[ \begin{array}{c} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \vdots \\ \mathbf{p}_{n+1} \\ \mathbf{c} \end{array} \right].
\]

(2)

The following properties hold for the vector \( \Theta_n \in \mathbb{R}^{(n+1)\times1} : 

1) The sum of the entries of \( \Theta_n \) is 1, i.e., \( \mathbf{1}_{(n+1)} \Theta_n = \mathbf{1} \)

2) If \( \Theta_n(\mathbf{p}_1, \ldots, \mathbf{p}_{n+1}, \mathbf{c}) > 0 \), then, point \( \mathbf{c} \) is inside the simplex by \( \mathbf{p}_1, \ldots, \mathbf{p}_{n+1} \). Otherwise \( \mathbf{c} \) is outside.

C. Rotation Matrix

Angles \( \beta_1, \beta_2, \) and \( \beta_3 \) define rotation matrix

\[
\mathbf{R}(\beta_1, \beta_2, \beta_3) = \begin{bmatrix}
S_{\beta_1}C_{\beta_2} & C_{\beta_1}S_{\beta_2} & -S_{\beta_1}S_{\beta_2} \\
S_{\beta_1}S_{\beta_3}C_{\beta_2} - C_{\beta_1}S_{\beta_3} & C_{\beta_1}S_{\beta_3}C_{\beta_2} + S_{\beta_1}C_{\beta_3} & S_{\beta_1}S_{\beta_3}S_{\beta_2} - C_{\beta_1}C_{\beta_3} \\
S_{\beta_1}S_{\beta_3}C_{\beta_2} + C_{\beta_1}S_{\beta_3} & C_{\beta_1}S_{\beta_3}C_{\beta_2} - S_{\beta_1}C_{\beta_3} & S_{\beta_1}S_{\beta_3}S_{\beta_2} + C_{\beta_1}C_{\beta_3}
\end{bmatrix},
\]

(3)

where \( C_{(\cdot)} \) and \( S_{(\cdot)} \) abbreviate \( \cos(\cdot) \) and \( \sin(\cdot) \), respectively. Rotation matrix \( \mathbf{R}(\beta_1, \beta_2, \beta_3) \) has the following properties:
D. MVS Homogeneous Deformation Coordination

We consider collective motion of an MVS consisting of \( N \) vehicles that are identified by index numbers defined by set \( \mathcal{V}_R \). We treat the vehicles constituting the MVS as particles of a deformable body, where the global desired position of vehicle \( i \in \mathcal{V}_R \), denoted \( \mathbf{r}_{i,HT} \), is defined by homogeneous transformation.

Assumption 1. Vehicles’ initial positions are given by

\[
\mathbf{r}_{i,s} = \mathbf{Q}_i \mathbf{r}_{i,0} + \mathbf{d}_i,
\]

where \( \mathbf{Q}_i = \mathbf{Q}(t_s) \) and \( \mathbf{d}_i = \mathbf{d}(t_s) \) denote Jacobian matrix and rigid-body displacement vector, respectively at the initial time instant \( t = t_s \). This paper assumes that \( \mathbf{Q}_i \) is an orthogonal matrix.

1) Leader-Follower Homogeneous Deformation Coordination: If \( \mathbf{Q} \) and \( \mathbf{d} \) are known at a time \( t \), leaders’ trajectories can be assigned using the homogeneous transformation given in (4). However, obstacle collision avoidance may not be necessarily guaranteed when Eq. (4) is directly used to define a continuum deformation coordination. This issue can be handled by defining a large-scale continuum deformation coordination by choosing appropriate leaders’ trajectories that avoid obstacles rather than shaping \( \mathbf{Q} \) and \( \mathbf{d} \) at any time \( t \).

Assumption 2. It is assumed that leader vehicles 1, 2, \( \cdots, n+1 \) form an \( n \)-D simplex in the reference configuration. Therefore,

\[
\Psi_n(\mathbf{r}_{1,0}, \cdots, \mathbf{r}_{n+1,0}) = n.
\]

Because leaders’ reference positions satisfy Assumption 2, the global desired position of vehicle \( i \in \mathcal{V}_R \) at \( t = t_0 \) is expressed as a linear combination of leader positions [26]:

\[
i \in \mathcal{V}_R, \; t \geq t_s, \quad \mathbf{r}_{i,HT} = \sum_{j=1}^{n+1} \alpha_{i,j} \mathbf{r}_{j,HT}(t),
\]

where \( \alpha_{i,1}, \alpha_{i,2}, \cdots, \alpha_{i,n+1} \) are called reference \( \alpha \)-parameters and obtained by

\[
[\alpha_{i,1} \cdots \alpha_{i,n+1}]^T = \Theta_n(\mathbf{r}_{1,0}, \cdots, \mathbf{r}_{n+1,0}, \mathbf{r}_{i,0}).
\]

2) Homogeneous Deformation Decomposition: Matrix \( \mathbf{Q}(t) \) in (3) can be decomposed as

\[
\mathbf{Q}(t) = \mathbf{R}_D(t) \mathbf{U}_D(t),
\]

where

\[
\mathbf{R}_D(t) = \mathbf{R}(\phi_u(t), \theta_u(t), \psi_u(t)),
\]

\[
\mathbf{U}_D(t) = \sum_{i=1}^{3} \lambda_i \mathbf{u}_i(\phi_u(t), \theta_u(t), \psi_u(t)) \hat{\mathbf{u}}_i^T(\phi_u(t), \theta_u(t), \psi_u(t)),
\]

\[
i = 1, 2, 3, \quad \hat{\mathbf{u}}_i = \mathbf{R}^T(\phi_u(t), \theta_u(t), \psi_u(t)) \hat{\mathbf{e}}_i.
\]

Note that \( \mathbf{u}_1, \mathbf{u}_2, \) and \( \mathbf{u}_3 \) are the eigenvectors of \( \mathbf{U}_D \) while \( \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \) and \( \hat{\mathbf{e}}_3 \) are the base vectors of the inertial coordinate system defined in Section II-A. A desired homogeneous transformation (4) can thus be uniquely expressed by the following features: (i) Rotation angles \( \phi_u(t), \theta_u(t), \psi_u(t) \), (ii) Deformation eigenvalues \( \lambda_1(t), \lambda_2(t), \lambda_3(t) \), (iii) Deformation angles \( \phi_u(t), \theta_u(t) \), and (iv) Rigid body displacement components \( d_1(t), d_2(t), \) and \( d_3(t) \). The decomposition of \( n \)-D homogeneous deformation coordination into such features is detailed in Appendix A where \( n = 1, 2, 3 \). The features are all real-valued signals with ranges given in Table II.

III. INTER-AGENT COMMUNICATION

Suppose an MVS consists of \( N \) vehicles moving in a 3-D motion space. The set \( \mathcal{V}_R = \{1, 2, \cdots, N\} \) defining identification (index) numbers of the vehicles is expressed as \( \mathcal{V}_R = \mathcal{V}_L \cup \mathcal{V}_F \) where \( \mathcal{V}_L \) and \( \mathcal{V}_F \) define index numbers of leaders and followers, respectively. The paper considers cases in which vehicles are distributed on an \( n \)-D \( (n = 1, 2, 3) \) Euclidean space in \( \mathbb{R}^3 \). The MVS is guided by \( n+1 \) leaders with index numbers \( \mathcal{V}_L = \{1, 2, \cdots, n+1\} \). Followers’ index numbers are defined by the set \( \mathcal{V}_F = \{n+2, \cdots, N\} \).

Let \( \Omega_e \) be an arbitrary closed domain enclosing all real vehicles at a reference configuration, where \( \Omega_e \) auxiliary nodes are arbitrarily distributed on the boundary \( \partial \Omega_e \) and identified by the set \( \mathcal{V}_aux = \{N+1, N+2, \cdots, N+N_a\} \).
TABLE I: n-dimensional Homogeneous Deformation Parameters

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| d1 | > 0 | > 0 | > 0 | > 0 | > 0 | > 0 | > 0 | > 0 | > 0 | > 0 |
| d2 | = 1 | = 1 | = 1 | = 1 | = 1 | = 1 | = 1 | = 1 | = 1 | = 1 |
| d3 | = 0 | = 0 | = 0 | = 0 | = 0 | = 0 | = 0 | = 0 | = 0 | = 0 |
| θφ | ∈ [−π/2, π/2] | ∈ [0, 2π] | ∈ [0, π] | ∈ [0, 2π] | ∈ [0, 2π] | ∈ [0, 2π] | ∈ [0, 2π] | ∈ [0, 2π] | ∈ [0, 2π] | ∈ [0, 2π] |
| φr | ∈ [−π/2, π/2] | ∈ [0, 2π] | ∈ [0, π] | ∈ [0, 2π] | ∈ [0, 2π] | ∈ [0, 2π] | ∈ [0, 2π] | ∈ [0, 2π] | ∈ [0, 2π] | ∈ [0, 2π] |
| φr | ∈ [−π/2, π/2] | ∈ [0, 2π] | ∈ [0, π] | ∈ [0, 2π] | ∈ [0, 2π] | ∈ [0, 2π] | ∈ [0, 2π] | ∈ [0, 2π] | ∈ [0, 2π] | ∈ [0, 2π] |

Assumption 5. It is assumed that in-neighbor vehicles \(i_1, i_2, \ldots, i_n+1\) form an n-D simplex enclosing follower \(i \in V_F\) in the reference configuration. Therefore,

\[
\forall i \in V_R, \quad \Psi_n(r_{i,0}, \ldots, r_{i,n+1}) = n, \quad (13a)
\]

\[
\forall i \in V_R, \quad \Theta_n(r_{i,0}, \ldots, r_{i,n+1}, r_{i,0}) > 0. \quad (13b)
\]

Defining reference in-neighbor set of follower \(i \in V_F\), communication weights \(w_{i,q}\) through \(w_{i,q+1}\) are all positive. This is because the communication simplex defined by \(i_1, i_2, \ldots, i_{n+1}\) encloses follower \(i \in V_F\).

We define the weight matrix \(W = [W_{ij}] \in \mathbb{R}^{(N+N_u)\times(N+N_u)}\) as follows:

\[
W_{ij} = \begin{cases} 
-1 & i = j \\
1 & i \in (V_F \cup V_{aux}), \ j \in N_i \\
0 & \text{otherwise}.
\end{cases} \quad (15)
\]

The matrix \(W\) can be partitioned as follows:

\[
W = \begin{bmatrix} 
-I_{n+1} & 0_{(n+1)\times(N-n-1)} & 0_{(n+1)\times N_u} \\
-W_{f,l} & A & W_{f,a} \\
-W_{a,l} & 0_{N_u\times(N-n-1)} & -I_{N_u}
\end{bmatrix}, \quad (16)
\]

where \(I_{n+1} \in \mathbb{R}^{(n+1)\times(n+1)}\) and \(I_{N_u} \in \mathbb{R}^{N_u\times N_u}\) are the identity matrices, \(0_{(n+1)\times(N-n-1)} \in \mathbb{R}^{(n+1)\times(N-n-1)}\), \(0_{(n+1)\times N_u} \in \mathbb{R}^{(n+1)\times N_u}\), and \(0_{N_u\times(N-n-1)} \in \mathbb{R}^{N_u\times(N-n-1)}\) are the zero-entry matrices, \(W_{f,l} \in \mathbb{R}^{(N-n-1)\times(n+1)}\) and \(W_{f,a} \in \mathbb{R}^{(N-n-1)\times N_u}\) are non-negative matrices, and \(A \in \mathbb{R}^{(N-n-1)\times(N-n-1)}\). Also,

\[
W_{a,l} = \begin{bmatrix} 
\Theta_n^T(r_{1,0}, \ldots, r_{n,0}, r_{N+1,0}) \\
\vdots \\
\Theta_n^T(r_{1,0}, \ldots, r_{n,0}, r_{N+1,0}) \\
\end{bmatrix} \in \mathbb{R}^{N_u\times 3}, \quad (17)
\]

is one-sum row, where \(\alpha_{i,k} (i \in V_{aux}, k \in V_L)\) is assigned by Eq. (9).

Proposition 1. Let \(z_{q,l,0} = [q_{1,0} \cdots q_{n+1,0}]^T\), \(z_{q,f,0} = [q_{n+2,0} \cdots q_{N,0}]^T\), and \(z_{q,aux,0} = [q_{N+1,0} \cdots q_{N+N_u,0}]^T\) define...
component \( q \in \{x, y, z\} \) of leaders, followers, and auxiliary nodes, respectively. Then, \( z_{q,f,0} \) is related to \( z_{q,l,0} \) by

\[
L \begin{bmatrix} z_{q,l,0} \\ z_{q,f,0} \end{bmatrix} = \begin{bmatrix} -z_{q,l,0} \\ 0 \end{bmatrix}
\]

where

\[
L = \begin{bmatrix} -I_{n+1} & 0_{(n+1)\times(N-n-1)} \\ B & A \end{bmatrix},
\]

and

\[
B = W_{f,j} + W_{f,a} W_{a,l}.
\]

Proof: When reference communication weights and \( \alpha \) parameters are consistent with agents’ reference positions and assigned using Eqs. (14) and (9),

\[
W^T \begin{bmatrix} z_{q,l,0}^T \\ z_{q,f,0}^T \\ z_{q,aux,0}^T \end{bmatrix} = 0.
\]

Substituting \( z_{q,aux,0} = W_{aux}(z_{q,l,0}) \), Eq. (21) simplifies to relation (18).

**Theorem 1.** If Assumptions 1, 2, and 3 are satisfied, then the following properties hold:

1. Matrix \( A \in \mathbb{R}^{(N-n-1)\times(N-n-1)} \) is Hurwitz.
2. For an arbitrary placement of the auxiliary agents on \( \partial \Omega_c \),

\[
W_L = A^{-1} B \in \mathbb{R}^{(N-n-1)\times(n+1)}
\]

is i.e., the sum of the row-elements is one for every row of matrix \( W_L \), where \( \Omega_c \) is an arbitrary closed domain enclosing MVS reference configuration (\( \Omega_c \subset \Omega_w \)).

Proof: By Assumption 1 the position information can be transmitted from a leader to every follower. Consequently, \( W \in \mathbb{R}^{(N-N_w}\times(N-N_w)} \) is not reducible. By Assumption 3 and due to how communication weights are defined by Eq. (14), it follows that: (i) The sum of the elements is for rows \( n+2 \) through \( N+N_{w} \) of \( W \). (ii) Except for diagonal elements of \( W \) that are all \( -1 \), off-diagonal elements are non-negative at rows \( n+2 \) through \( N \). (ii) Because \( W \) is not reducible, the sum of the row elements are negative in at least \( n+1 \) rows of matrix \( A \in \mathbb{R}^{(N-n-1)\times(N-n-1)} \). Hence, the matrix \( A \) can be expressed as \( A = -I_{N-n-1} + H \) where \( H \in \mathbb{R}^{(N-n-1)\times(N-n-1)} \) has a spectral radius \( \rho < 1 \), i.e. eigenvalues of matrix \( H \) are all located inside a disk with radius \( \rho < 1 \) centered at the origin on a complex plane. Therefore, \( -A \) is a nonsingular M-matrix [38] and matrix \( A \) is Hurwitz.

Because Assumption 2 is satisfied, the position of follower \( i \in \mathcal{V}_f \) can be expressed with respect to leaders as given in Eq. (8) where \( \alpha_{1,1}, \ldots, \alpha_{n,1} \) are unique and assigned by Eq. (9). In other words, component \( q \) of the followers’ and leaders’ reference positions can be related by \( z_{q,f,0} = W_L z_{q,l,0} \), where

\[
W_L = \begin{bmatrix} \Theta_n^T (r_{1,0}, \ldots, r_{n,1,0}, r_{n+2,0}) \\ \vdots \\ \Theta_n^T (r_{1,0}, \ldots, r_{n,1,0}, r_{N,0}) \end{bmatrix} = \begin{bmatrix} \alpha_{n,1} & \ldots & \alpha_{n+1,1} \\ \vdots & \vdots & \vdots \\ \alpha_{N,1} & \ldots & \alpha_{N,n+1} \end{bmatrix}.
\]

Because \( \alpha \)-parameters \( \alpha_{1,1}, \ldots, \alpha_{n+1,n+1} \) are assigned by Eq. (2). \( W_L \) is one-sum row. On the other hand, \( z_{q,f,0} \) and \( z_{q,l,0} \) can be also related with Eq. (18):

\[
z_{q,f,0} = A^{-1} B z_{q,l,0}.
\]

By equating right-hand sides of Eqs. (23) and (24), the following properties hold: (i) \( W_L \) is obtained as given in Eq. (22), and (ii) \( W_{Lij} = \alpha_{i+n+1,j} \) \( (i+n+1) \in \mathcal{V}_f, j \in \mathcal{V}_L \). ■

**Remark 3.** Let \( z_{q,f,HT}^{i,H}(t) = [q_{1,HT} \cdots q_{n+1,HT}]^T \in \mathbb{R}^{n+1} \) and \( z_{q,f,HT}(t)(t) = [q_{n+2,HT} \cdots q_{N,HT}]^T \in \mathbb{R}^{N-n-1} \) denote the vector formed from component \( q \in \{x, y, z\} \) of the global desired positions of leaders and followers, e.g. \( r_i,HT = x_i,HT \hat{e}_1 + y_i,HT \hat{e}_2 + z_i,HT \hat{e}_3 \) is the global desired position of vehicle \( i \in (\mathcal{V}_L \cup \mathcal{V}_F) \). Then, \( z_{q,f,HT} \) and \( z_{q,l,HT} \) are related by

\[
t \geq t_s, \quad z_{q,f,HT}(t) = W_L z_{q,l,HT}(t).
\]

Therefore,

\[
j = 0, 1, \ldots, \rho, \quad \frac{d^j z_{q,HT}}{dt^j} = \begin{bmatrix} I_{n+1} \\ W_L \end{bmatrix} \begin{bmatrix} \frac{d^j q_{1,HT}}{dt^j} \\ \vdots \\ \frac{d^j q_{n+1,HT}}{dt^j} \end{bmatrix},
\]

where \( z_{q,HT}(t) = [q_{1,HT}(t) \cdots q_{N,HT}(t)]^T \).

**Assumption 6.** It is assumed that leader vehicles 1, 2, \ldots, \( n+1 \) form an \( n-D \) simplex at any time \( t \in [t_s,t_f] \), where \( t_s \) and \( t_f \) denote the initial and final times, respectively. Therefore,

\[
\Psi_n (r_{1,HT}(t), \ldots, r_{n+1,HT}(t)) = n.
\]

**B. Coordination Graph and Real Communication Weights**

As mentioned in Section III, auxiliary nodes do not move and they are just defined in the reference configuration in order to obtain communication weights and ensure stability of collective motion. When the MVS is moving, follower vehicles only communicate with real in-neighbor where inter-agent communication is defined by coordination graph \( G_c \) \((\mathcal{V}_R, E_c)\) with edge set \( E_c \subset \mathcal{V}_R \times \mathcal{V}_R \). Real in-neighbors of follower \( i \in \mathcal{V}_F \) are defined by time-invariant set \( I_i = i \in \mathcal{V}_F \) which is called real in-neighbor set. We define the local desired position of vehicle \( i \in \mathcal{V}_R \) by

\[
r_{d,i} = \begin{cases} r_{i,HT} \\ \sum_{j \in I_i} \sigma_{i,j} r_j \end{cases} \quad i \in \mathcal{V}_L
\]

where \( \sigma_{i,j} > 0 \) is the real communication weight between follower \( i \in \mathcal{V}_F \) and vehicle \( j \in I_i \subset \mathcal{V}_R \) and

\[
\sum_{j \in I_i} \sigma_{i,j} = 1.
\]

By considering Proposition 1 and Theorem 1, the following properties hold about the real in-neighbor set \( I_i \) and real communication weight \( \sigma_{i,j} \):
1) If $N_i \cap V_{aux} = \emptyset$, then, $I_i = N_i$ and $\sigma_{i,j} = w_{i,j}$ for every vehicle $i \in V_F$ with every in-neighbor vehicle $j \in I_i = N_i$.

2) If $N_i \cap V_{aux} \neq \emptyset$, then, $I_i \neq N_i$ and $\sigma_{i,j}$ is defined as follows:

$$\sigma_{i,j} = \begin{cases} w_{i,j} & \text{if } j \in (I_i \cap N_i) \\ \sum_{k \in (N_i \cap V_{aux})} w_{i,k} a_{k,j} & \text{otherwise} \end{cases}$$

IV. PROBLEM FORMULATION

This paper studies the properties of our homogeneous deformation approach for an $N$-vehicle MVS where vehicles are treated as particles of an $n$-D deformable body ($n = 1, 2, 3$). The desired MVS vehicle positions are guided by $n+1$ leaders and acquired by the remaining followers through local communication, where followers are arbitrarily distributed in a 3D motion space. Dynamics of the vehicle $i \in V_R$ is represented by a nonlinear model

$$\begin{align*}
\dot{x}_i &= f_i(x_i) + g_i(x_i)u_i \\
\dot{r}_i &= r_i(x_i),
\end{align*}$$

where $u_i \in U_i \subset \mathbb{R}^{n_u}$ and $U_i$ is compact. In Eq. (28), $x_i \in \mathbb{R}^{n_x \times 1}$ and $u_i \in \mathbb{R}^{n_u \times 1}$ are state and control input, respectively, the actual position $r_i \in \mathbb{R}^{3 \times 1}$ is the output of vehicle dynamics (28), and $f_i : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x}$ and $g_i : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x \times n_u}$ are smooth. Defining $F_L = [f_1^T \cdots f_{N_i}^T]^T$, $F_F = [f_{N+1}^T \cdots f_N^T]^T$, $G_L = [g_1^T \cdots g_{N_i}^T]^T$, $G_F = [g_{N+1}^T \cdots g_N^T]^T$, the leader state vector by $X_L = [x_1^T \cdots x_{N_i}^T]^T$, the follower state vector by $X_F = [x_{N+1}^T \cdots x_N^T]^T$, the leader input vector by $U_L = [u_1^T \cdots u_{N_i}^T]^T$, the follower input vector $U_F = [u_{N+1}^T \cdots u_N^T]^T$, the MVS collective dynamics can be expressed by

$$\begin{align*}
\dot{X}_L &= F_L(X_L) + G_L(X_L)U_L \\
\dot{X}_F &= F_F(X_F) + G_F(X_F)U_F.
\end{align*}$$

We also define

$$\begin{align*}
P_L &= \text{vec} \left( \begin{bmatrix} r_1 & \cdots & r_{n+1} \\
\frac{dr_1}{dt} & \cdots & \frac{dr_{n+1}}{dt} \end{bmatrix}^T \right) \\
P_F &= \text{vec} \left( \begin{bmatrix} r_{N+2} & \cdots & r_N \\
\frac{dr_{N+2}}{dt} & \cdots & \frac{dr_N}{dt} \end{bmatrix}^T \right)
\end{align*}$$

are the outputs of follower and leader MVS dynamics, respectively. “vec” is the matrix vectorization operator. Also,

$$P_{L,d} = \text{vec} \left( \begin{bmatrix} r_{1,HT} & \cdots & r_{n+1,HT} \\
\frac{dr_{1,HT}}{dt} & \cdots & \frac{dr_{n+1,HT}}{dt} \end{bmatrix}^T \right)$$

in the reference input of the MVS control system, and

$$P_{F,d}(t) = D_{MVS}P_F(t) + B_{MVS}P_L(t)$$

is the reference input of the follower MVS dynamics, where $D_{MVS} = I_{3\Xi} \otimes \mathbf{D}$, $B_{MVS} = I_{3\Xi} \otimes \mathbf{B}$, $\mathbf{D} = [I_{N-1} \otimes \mathbf{A}]$, $\otimes$ is the Kronecker product symbol, $\Xi \subseteq \mathbb{R}_+$ is known, and $I_{3\Xi} \in \mathbb{R}^{3\Xi \times 3\Xi}$ and $I_{N-1} \in \mathbb{R}^{(N-1) \times (N-1)}$ are the identity matrices.

This paper studies the following problems:

**Problem 1** (MVS Continuum Deformation Coordinated Control): The first problem to be addressed is to determine $U_L(t)$ and $U_F(t)$ such that position deviation of every vehicle $i \in V_L \cup V_F$ is less than or equal to $\delta > 0$ at any time $t$ ($\|r_i(t) - r_{i,HT}\| \leq \delta$, $\forall i \in V$, $t \in [t_1, t_f]$). Note that the local desired position $r_{i,HT}$, defined in Section II-A is the reference input for every vehicle $i \in V_L \cup V_F$. Therefore, leader $i \in V_L$ knows the global desired position $r_{i,HT}$ while follower $i \in V_F$ does not know $r_{i,HT}$ but acquires it through local communication. The block diagram of the MVS collective dynamics is shown in Fig. (3).

**Assumption 7.** The following assumption is made: Dynamics of every vehicle $i \in V_L \cup V_F$ is input-output linearizable.

Stability and convergence of the MVS collective dynamics are investigated in Section VI.

**Problem 2** (Guaranteeing Continuum Deformation Safety Specification): The objective is to mathematically specify safety conditions for large-scale MVS coordination in obstacle-free and obstacle-laden motion spaces. In particular, a large-scale coordination is labelled safe if inter-agent collision is avoided, obstacle collision avoidance is avoided, and vehicle input constraints are satisfied. This paper also advances the existing condition for assurance of inter-agent collision avoidance in a large-scale continuum deformation coordination by proposing a new and less restrictive inter-agent collision avoidance condition. Specifically, we show that how inter-agent collision can be avoided only by constraining one of the eigenvalues of the deformation matrix, if the shear-deformation angles remain constant at all times $t$. This can significantly improve the flexibility and maneuverability of the large-scale continuum deformation coordination.

**Problem 3** (MVS Continuum Deformation Coordination Planning): For the planning of a continuum deformation coordination, the global desired trajectories of leaders are assigned in obstacle-free and obstacle-laden environments such that the initial conditions at time $t_1$, final conditions at time $t_f$, and continuum deformation safety condition at any time $t \in [t_1, t_f]$ are satisfied. We define the desired trajectory of every leader $i \in V_L$ by

$$\begin{align*}
r_{i,HT} &= r_i(t) \left( s_{i,k}(t) \right),
\end{align*}$$

Fig. 3: Continuum deformation coordination block diagram.
where

\[ s^n_{\varrho}(t) = \left[ s^n_{\varrho,1}(t) \cdots s^n_{\varrho,n_{\varrho},1}(t) \right]^T \in \mathbb{R}^{n_{\varrho} \times 1} \]  

is the generalized coordinate vector, \( n \in \{1,2,3\} \) is the dimension of the homogeneous deformation coordination and

\[ g_{n,\varrho} = \begin{cases} n(n+1) & \varrho = \text{OL}, \ n = 1,2,3 \\ 6 & \varrho = \text{OF}, \ n = 1 \\ 8 & \varrho = \text{OF}, \ n = 2 \\ 9 & \varrho = \text{OF}, \ n = 3 \end{cases} \]  

is the number of generalized coordinates. Also, \( s^n_{\varrho,\varpi} : [t_k,t_f] \rightarrow \mathbb{R} \) is the \( h \)-th generalized coordinate (\( h = 1,\ldots,n(n+1) \)), \( \varrho \in \{\text{OF},\text{OL}\} \) is a discrete variable, and “OF” and “OL” stand for “obstacle free” and “obstacle laden”, respectively. Note that \( s^n_{\varrho,\varpi} \) is defined based on leaders’ position components while \( s^n_{\varrho,\varphi} \) specifies the homogeneous transformation features. The generalized coordinate vector \( s^n_{\varrho}(t) \) is given by

\[
s^n_{\varrho}(t) = s^n_{\varrho,1}(1-\beta(t,T_k)) + \beta(t,T_k) s^n_{\varrho,k+1,t} \]  

where \( k = 1,\ldots,n_{\varrho} \), \( \varrho \in \{\text{OL,OF}\} \), \( t_1 = t_k \), \( t_{n_{\varrho}+1} = t_f \), \( s^n_{\varrho,1,t} \), \( s^n_{\varrho,2,t} \), \( \ldots \), \( s^n_{\varrho,n_{\varrho}+1,t} \), determine intermediate configurations for the generalized coordinate, \( T_k = t_{k+1} - t_k \), and \( \beta(t,T_k) \) is defined by the following fifth order polynomial:

\[
\beta(t,T_k) = \sum_{j=0}^{5} \zeta_{j,k} \left( \frac{t-t_k}{T_k} \right)^j ,
\]

where \( \beta(t,T_k) \in [0,1] \) is an increasing function of \( t \) over \( [t_k,t_{k+1}] \), \( \beta(t,T_k) = 0 \), \( \beta(t_{k+1},T_k) = 1 \), \( \zeta_{j,k} \) through \( \zeta_{5,k} \) are constant and determined based on boundary conditions at times \( t_k \) and \( t_{k+1} \). Note that \( T_1 \) through \( T_n \) are the design parameters determined such that the continuum deformation safety conditions are all satisfied.

For continuum deformation planning in obstacle-free environments, \( n_{\varrho} = 1 \) and Eq. (33) is used to assign the global desired trajectories of the leaders by shaping the homogeneous deformation features. For motion planning in obstacle-laden environments, \( s^n_{\varrho,1,\text{OL}}, \ldots, s^n_{\varrho,n+1,\text{OL}} \) are determined using A* search such that the leaders’ travel distance is minimized. This paper determines the leaders’ paths through connecting intermediate way-points assigned by the A* planner. Therefore, the additional conditions \( \beta(t_k,T_k) = 0 \), \( \beta(t_f,T_k) = 0 \), \( \beta(t_{n_{\varrho}+1},T_k) = 0 \), \( \beta(t_{k+1},T_k) = 0 \) are all satisfied in order to ensure the continuity of the desired leaders’ velocity and acceleration at all times \( t \).

V. PROBLEM 1: MVS CONTINUUM DEFORMATION
COORDINATED CONTROL

By Assumption [7], we can find a state transformation \( x_i \rightarrow (r_i,\varpi_i) \), such that \( \varpi_i \) is decomposed into external dynamics

\[
q \in \{x,y,z\}, \quad \frac{d^{\rho_q}q_i}{dt^{\rho_q}} = v_{q,i} \]  

and internal dynamics

\[
\omega_i = f_{\text{int},i} \left( \omega_i, x_i, \cdots, L_f^{\rho_q-1} x_i, y_i, \cdots, L_f^{\rho_q-1} y_i, z_i, \cdots, L_f^{\rho_q-1} z_i \right),
\]

where \( v_{q,i} \) is assigned as

\[
v_{q,i} = -\sum_{k=1}^{\rho_q} \gamma_{q,k} L_f^{\rho_q-k} q_i + \sum_{k=\rho_q}^{\rho_q} \gamma_{i,k} L_f^{\rho_q-k} (q_{d,i} - q_i),
\]

and where \( L_f G = (\nabla G)^T F \) is the Lie derivative of a smooth function \( G \) with respect to a vector field \( F \), \( q_{d,i} \) is the component \( q \in \{x,y,z\} \) of local desired position \( r_{d,i} \) defined by Eq. (27). Control gains \( \gamma_{q,i} \) through \( \gamma_{i,\rho_q} \) are assigned such that the MVS collective dynamics is stable. In Eq. (35), \( \rho_q \) (\( q \in \{x,y,z\} \)) is the relative degree, \( \rho = \rho_x + \rho_y + \rho_z \leq n_x \) is the total relative degree, \( \Xi < \rho_q \) is constant, and

\[
\begin{align*}
\psi_{x,i} & = M_i u_i + n_i, \\
\psi_{y,i} & = M_i u_i + n_i,
\end{align*}
\]

the following:

Assumption 8. For the system \( \Sigma \) and \( \Sigma_{\text{SYS}} \), (i) \( n_u \geq 3 \), (ii) \( \text{Rank}(M_i) = 3 \), and (iii) zero dynamics of vehicle \( i \), \( \omega = f_{\text{int},i}(\omega_i,0,0,0) \), is locally asymptotically stable at the origin.

Define

\[
X_{\text{SYS},q} = \begin{bmatrix} q_1 & \cdots & q_{n-1} & L_{T_{k_1}} q_1 & \cdots & L_{T_{k_{n_q}}} q_{n_q} \end{bmatrix} \in \mathbb{R}^{(n_q+1) \times 1},
\]

\[
X_{\text{SYS}} = \begin{bmatrix} X_{\text{SYS},q}^T \ X_{\text{SYS,y}}^T \ X_{\text{SYS,z}}^T \end{bmatrix}^T \in \mathbb{R}^{(n+1) \times 1},
\]

\[
U_{\text{SYS},q} = \begin{bmatrix} L_{T_{k_1}}^{-1} q_{1,HT} & \cdots & L_{T_{k_{n_q}}}^{-1} q_{n_q,HT} \end{bmatrix},
\]

\[
U_{\text{SYS}} = \begin{bmatrix} U_{\text{SYS},q}^T \ U_{\text{SYS,y}}^T \ U_{\text{SYS,z}}^T \end{bmatrix}^T \in \mathbb{R}^{(n+1) \times 1},
\]

where \( q_{i,HT} \) denotes component \( q \in \{x,y,z\} \) of actual and global desired positions of vehicle \( i \). The MVS collective dynamics can be expressed by the following normal form:

External Dynamics : \( \dot{X}_{\text{SYS}} = A_{\text{SYS}} X_{\text{SYS}} + B_{\text{SYS}} U_{\text{SYS}}, \)  

Internal Dynamics : \( \frac{d\omega}{dt} = F_{\text{INT}}(\omega, X_{\text{SYS}}) \),

where \( \omega = [\omega_1 \cdots \omega_N]^T \), \( F_{\text{INT}} = \begin{bmatrix} f_{\text{int},1} & \cdots & f_{\text{int},N} \end{bmatrix}^T \),

\[
A_{\text{SYS}} = \begin{bmatrix} A_{\text{SYS},x} & 0 & 0 \\
0 & A_{\text{SYS},y} & 0 \\
0 & 0 & A_{\text{SYS},z} \end{bmatrix}, \quad B_{\text{SYS}} = \begin{bmatrix} B_{\text{SYS},x} & 0 & 0 \\
0 & B_{\text{SYS},y} & 0 \\
0 & 0 & B_{\text{SYS},z} \end{bmatrix}
\]

\[
\begin{align*}
\ w_i &= \int_{t_k}^{t_{k+1}} \dot{\omega}_i dt, \\
\ x_i &= \int_{t_k}^{t_{k+1}} \dot{x}_i dt, \\
\ y_i &= \int_{t_k}^{t_{k+1}} \dot{y}_i dt, \\
\ z_i &= \int_{t_k}^{t_{k+1}} \dot{z}_i dt,
\end{align*}
\]

and \( j = \rho_q - \Xi + 1, \cdots, \rho_q \), \( \Gamma_{j,i} = \text{diag}(\gamma_{j,i}, \cdots, \gamma_{j,n+1}) \in \mathbb{R}^{(n+1) \times (n+1)}, \)

\[
\begin{align*}
\dot{\omega}_i &= f_{\text{int},i} \left( \omega_i, x_i, \cdots, L_f^{\rho_q-1} x_i, y_i, \cdots, L_f^{\rho_q-1} y_i, z_i, \cdots, L_f^{\rho_q-1} z_i \right),
\end{align*}
\]
\( q \in \{x, y, z\}, \quad B_{\text{SYS},q} = \begin{bmatrix} 0_{(n+1) \times (\rho_q-1)N} & I_{n+1} & 0_{(n+1) \times (N-n-1)} \end{bmatrix}^T. \)

\( q \in \{x, y, z\}, \quad A_{\text{SYS},q} = \begin{bmatrix} 0_N & I_N & \cdots & 0_N \\ \vdots & \ddots & \ddots & \vdots \\ 0_N & 0_N & \cdots & I_N \\ \Gamma_{p_{\text{q}}} H_{\text{S},q} & \Gamma_{p_{\text{q}}} - \Gamma_{p_{\text{q}}} H_{\text{S},q} & \cdots & \Gamma_{p_{\text{q}}} H_{\text{S},q} \end{bmatrix}. \)

Note that \( I_N \in \mathbb{R}^{N \times N} \) and \( 0_N \in \mathbb{R}^{N \times N} \) are the identity and zero-entry matrices. \( H_{\text{S},q} \in \mathbb{R}^{n \times (n+1)} \) is the system matrix.

The proof is inspired by Theorem 6.3 in Ref. [39].

A. Error Dynamics

Defining

\[
X_{\text{SYS},HT,q} = \begin{bmatrix} q_{\text{HT}} & \cdots & q_N \end{bmatrix}^T, \quad X_{\text{SYS},HT} = \begin{bmatrix} X_{\text{SYS},HT,x}^T & X_{\text{SYS},HT,y}^T & X_{\text{SYS},HT,z}^T \end{bmatrix}^T.
\]

where

\[
X_{\text{SYS},HT} = \begin{bmatrix} X_{\text{SYS},HT,x}^T & X_{\text{SYS},HT,y}^T & X_{\text{SYS},HT,z}^T \end{bmatrix}^T.
\]

and \( q \in \{x, y, z\} \). The transient error \( E_{\text{SYS}} = X_{\text{SYS}} - X_{\text{SYS},HT} \) assigning deviation vehicles' actual position components from their global desired position components are updated by

\[
E_{\text{SYS}} = A_{\text{SYS}} E_{\text{SYS}} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} V_{\text{SYS}}, \quad (39)
\]

where

\[
V_{\text{SYS}} = \begin{bmatrix} V_{\text{SYS},x}^T & V_{\text{SYS},y}^T & V_{\text{SYS},z}^T \end{bmatrix}^T \in \mathbb{R}^{3N \times 1}, \quad (AA)
\]

The transient error \( V_{\text{SYS},q,i} = \sum_{j=0}^{\rho_q-\Xi} \Gamma_j \frac{d}{dt} q_{\text{HT},j} \) of deviation of every vehicle is bounded at any time \( t \in \{t_s, t_f\} \). This condition can be expressed as follows:

\[
\forall t \in \{t_s, t_f\}, \forall i \in \mathcal{V}, \quad \|r_i(t) - r_{\text{HT},i}(t)\|_2 \leq \delta, \quad (41)
\]

where \( \delta > 0 \) is constant.

Safety Condition 2 (Inter-Agent Collision Avoidance): Inter-agent collision must be avoided in an MVS continuum deformation coordination by satisfying the condition

\[
\forall t \in \{t_s, t_f\}, \forall i, j \in \mathcal{V}, \ i \neq j, \quad \|r_i(t) - r_j(t)\|_2 \geq 2\epsilon, \quad (42)
\]

where \( \epsilon > 0 \) is the radius of the ball enclosing each vehicle.

Safety Condition 3 (Vehicle Containment Condition):

For continuum deformation coordination in an obstacle-laden environment, the MVS needs to be contained by a virtual n-D simplex, called \textit{virtual containment simplex} (VCS), at any time \( t \). By ensuring MVS containment, obstacle collision avoidance is guaranteed if the VCS boundary surfaces do not hit obstacles in the motion space.

Lemma 3 is used to define the error zero dynamics, obtained by substituting \( V_{\text{SYS},q,i} = 0 \) in Eq. (39), is a locally asymptotically stable at the origin.

Proof: The proof is inspired by theorem 6.3 in Ref. [39].

\[
\begin{bmatrix} \dot{E}_{\text{SYS}} \\ \frac{d\omega}{dt} \end{bmatrix} = F_{\text{LE}} E_{\text{SYS},y} + F_{\text{LF}} E_{\text{SYS},z} + F_{\text{Le},\omega} \omega + H.O.T.
\]

A. Safety Conditions for MVS Continuum Deformation Coordination

This paper considers the following four safety requirements for a valid continuum deformation coordination:

Safety Condition 1 (Bounded Deviation): Error bounded at any time \( t \) and \( t_s \leq t \leq t_f \). Required safety conditions are specified in Section VI-A. Also, sufficient conditions for inter-agent collision avoidance and vehicle containment are provided in Section VI-B.

A. Problem 2: Continuum Deformation Safety Specification

A continuum deformation coordination is called valid over time interval \( [t_s, t_f] \) if the safety requirements are all satisfied at any time \( t \). Required safety conditions are specified in Section VI-A. Also, sufficient conditions for inter-agent collision avoidance and vehicle containment are provided in Section VI-B.
time $t$ per Section [II-D].  The MVS containment condition is mathematically specified by
\[ t \in [t_s, t_f], \forall i \in \mathcal{V}, \quad \Theta_n(h_1(t), \ldots, h_{n+1}(t), r_i(t)) > 0, \] (44)
where $\Theta_n$ is an appropriately defined function defined by Eq. (2).

Safety Condition 4 (Feasibility of Vehicular Inputs): A desired continuum deformation is planned by leaders and must be followed by every vehicle in the team. In other words, the generalized coordinate $s_0^i(t)$ ($q \in \{OF, OL\}$, $n = 1, 2, 3$) must be planned such that every vehicle can follow the desired coordination by employing control inputs which satisfy control constraints
\[ t \in [t_s, t_f], \forall i \in \mathcal{V}, \quad u_i(t) \in \mathcal{U}_i. \] (45)

B. Sufficient conditions for Collision Avoidance

It is computationally expensive to ensure inter-agent collision avoidance by satisfying the safety condition (41) for every vehicle pair $i$ and $j$. Because MVS desired coordination is defined by an affine transformation, inter-agent collision avoidance and follower containment conditions can be enforced by assigning lower limits on eigenvalues of the matrix $U_D = (Q^TQ)^{-\frac{1}{2}}$. Section VI-B.1 provides a conservative condition for follower containment and inter-agent collision avoidance in an obstacle-laden environment. Furthermore, an aggressive condition for inter-agent collision avoidance in obstacle-free environments is provided in Section VI-B.2.

1) Conservative Collision Avoidance Condition: We can ensure collision avoidance and vehicle containment by assigning a single lower-limit for all eigenvalues of matrix $U_D(t)$, denoted by $\lambda_1(t)$, $\lambda_2(t)$, and $\lambda_3(t)$, at any time $t$.

Proposition 2. Assume agents’ desired positions are defined by the homogeneous transformation given in Eq. (4). Suppose that inequality (41) is satisfied at any time $t$. Inter-agent collision avoidance, defined by Eq. (42), is guaranteed, if
\[ t \in [t_s, t_f], \forall i, j \in \mathcal{V}, i \neq j, \quad \|r_i,HT(t) - r_j,HT(t)\|_2 \geq 2(\delta + \epsilon). \] (46)

Proof: Note that
\[ (r_i - r_j) = (r_i,HT - r_j,HT) - (r_i,HT - r_j) - (r_j - r_j,HT). \]

Hence,
\[ \|r_i - r_j\|_2 \geq \|r_i,HT - r_j,HT\|_2 - \|r_i,HT - r_j\|_2 - \|r_j - r_j,HT\|_2 \geq 2(\delta + \epsilon) - \delta - \delta = 2\epsilon. \]

Proposition 2 implies that, assuming (41) holds, Eq. (46) can be used instead of Eq. (42) to ensure inter-agent collision avoidance. Substituting $r_i,HT = Qr_i,0 + d$ and $r_j,HT = Qr_j,0 + d$, (46) holds if and only if
\[ \forall i, j \in \mathcal{V}, i \neq j, \quad 4(\delta + \epsilon)^2 \leq (r_i,0 - r_j,0)^T U_D^2(t)(r_i,0 - r_j,0). \] (47)

for all $t$ in $[t_s, t_f]$ where $U_D^2(t) = Q^T(t)Q(t)$.

Theorem 3. Define
\[ d_s = \min_{i, j \in \mathcal{V}_k, i \neq j} \|r_i,0 - r_j,0\| \] (48)
as the minimum separation distance between any two agents in the reference configuration. Inter-agent collision avoidance is guaranteed if
\[ \lambda_i \geq \lambda_{\text{min}}, \quad i = 1, 2, 3, \] (49)
where $\lambda_1$, $\lambda_2$, and $\lambda_3$ are the eigenvalues of $U_D(t)$ and
\[ \lambda_{\text{min}} = 2 \frac{\delta + \epsilon}{d_s}. \] (50)

Proof: The right-hand side of Eq. (47) satisfies the following inequality:
\[ \forall i, j \in \mathcal{V}, i \neq j, \quad \lambda_{\text{min}}^2 d_s^2 \leq (r_i,0 - r_j,0)^T U_D^2(t)(r_i,0 - r_j,0). \] (51)

Inter-agent collision avoidance is guaranteed when
\[ 4(\delta + \epsilon)^2 \leq \lambda_i^2 d_s^2, \quad i = 1, 2, 3. \]
Thus, collision avoidance is ensured if Eq. (49) is satisfied.

Theorem 4. Assume each vehicle is enclosed by a ball with radius $\epsilon$. Given deviation upper-bound $\delta$ (Eq. (41)), we define

\[ \delta_{\text{max}} = \min \{ (d_b - \epsilon), \frac{1}{2}(d_s - 2\epsilon) \}, \] (52)

where $d_b$ is the minimum distance from the boundary of the leading simplex at in the reference configuration. The vehicle containment and inter-agent collision avoidance are guaranteed at time instant $t$ if eigenvalues $\lambda_1$, $\lambda_2$, $\lambda_3$ of $U_D(t)$ satisfy the following inequality constraint (53):

\[ j = 1, 2, 3, \quad \lambda_j \geq \lambda_{\text{CD,min}} \] (53)

where
\[ \lambda_{\text{CD,min}} = \frac{\delta + \epsilon}{\delta_{\text{max}} + \epsilon}. \] (54)

Proof: See the proof of Theorem 4 in Ref. [37].
2) Relaxation of the Collision Avoidance Condition: The issue in Remark 4 can be dealt with, if we only constrain the eigenvalue \( \lambda_1 \) of matrix \( U_D(t) \) and direct eigenvector \( \hat{u}_1 \) along the vector connecting a pair of agents with the minimum separation distance. Given MVS reference positions, reference eigenvector \( \hat{u}_{1,0} \) is defined as follows:

\[
\hat{u}_{1,0} = \min_{i,j \in V_R, i \neq j} \left\{ \frac{r_{i,0} - r_{j,0}}{\|r_{i,0} - r_{j,0}\|} \right\}
\]

(55)

Because we use the 3-2-1 Euler angles to define a rigid-body rotation, the eigenvector \( \hat{u}_i \) of \( U_D(t) \) only depends on \( \theta_u \) and \( \psi_u \) at any time \( t \) (see Eq. (11c)). Thus, the reference deformation angles \( \theta_{u,0} \) and \( \psi_{u,0} \) are obtained by

\[
\theta_{u,0} = -\sin^{-1} \left( \hat{u}_{1,0} \cdot \hat{e}_3 \right),
\]

(56a)

\[
\psi_{u,0} = \tan^{-1} \left( \frac{\hat{u}_{1,0} \cdot \hat{e}_2}{\hat{u}_{1,0} \cdot \hat{e}_1} \right).
\]

(56b)

Suppose now that the desired continuum deformation is designed such that deformation angles \( \phi_u(t) = \phi_{u,0} \), \( \theta_u(t) = \theta_{u,0} \) and \( \psi_u(t) = \psi_{u,0} \) remain constant for \( t \in [t_s, t_f] \), where \( \phi_{u,0} \) takes an arbitrary value between 0 and \( 2\pi \).

**Lemma 1.** If \( \phi_u(t) = \phi_{u,0} \), \( \theta_u(t) = \theta_{u,0} \) and \( \psi_u(t) = \psi_{u,0} \) are constant for \( t \in [t_s, t_f] \) and \( r_{i,HT} \) and \( r_{j,HT} \) are defined by the homogeneous transformation given in (4) (\( \forall i, j \in V_R \)), then, the following relation holds:

\[
\forall i, j \in V_R, \quad (r_{i,HT}(t) - r_{j,HT}(t))^T (R_D(t) \hat{u}_{1,0}) = \lambda_1(t) (r_{i,0} - r_{j,0})^T \hat{u}_{1,0}
\]

(57)

**Proof:** Given global desired positions of vehicles \( i, j \in V_R \), defined by Eq. (4), we can write

\[
(r_{i,HT}(t) - r_{j,HT}(t))^T = Q(t) (r_{i,0}(t) - r_{j,0})
\]

\[
= R_D(t) \left( \sum_{h=1}^{3} \lambda_h(t) \hat{u}_{h,0}^T (r_{i,0} - r_{j,0}) \right) U_D(t) = R_D(t) \left( \sum_{h=1}^{3} \lambda_h(t) (r_{i,0} - r_{j,0})^T \hat{u}_{h,0} \right)
\]

Note that \( \lambda_h(t) (r_{i,0} - r_{j,0})^T \hat{u}_{h,0} \in \mathbb{R} \) at any time \( t \) \((h = 1, 2, 3) \). Thus, pre-multiplying both sides of Eq. (55) by \( \hat{u}_{i,0}^T R_D(t) \) and substituting

\[
\hat{u}_{i,0}^T R_D(t) \hat{u}_{i,0} = \begin{bmatrix} 1 & 0 \\ 0 & h \neq l \end{bmatrix}
\]

Eq. (57) follows.

**Theorem 5.** Assume every vehicle is enclosed by a ball of radius \( \epsilon \). MVS inter-agent collision avoidance is guaranteed if

\[
\forall t \in [t_s, t_f], \quad \lambda_1(t) \geq \lambda_{\min},
\]

(59)

where \( \lambda_1(t) \) is the first eigenvalues of \( U_D(t) \), \( d_s \) is the minimum separation distance defined in (48) and

\[
\lambda_{\min} = 2 \frac{\delta + \epsilon}{d_s}.
\]

(60)

**Proof:** Vectors \( R_D(t) \hat{u}_{1,0} \), \( R_D(t) \hat{u}_{2,0} \), \( R_D(t) \hat{u}_{3,0} \) are parallel to \( \hat{u}_{1}(t) \), \( \hat{u}_{2}(t) \), \( \hat{u}_{3}(t) \), respectively. Because the lowest minimum separation distance is along the unit vector \( \hat{u}_{1}(t) = R_D(t) \hat{u}_{1,0} \), inter-agent collision is avoided if

\[
\forall i, j \in V_R, \quad (r_{i,HT}(t) - r_{j,HT}(t))^T (R_D(t) \hat{u}_{1,0}) \geq 2(\delta + \epsilon).
\]

(61)

By Lemma 1, it follows that

\[
\min_{i, j \in V_R} \left\{ (r_{i,HT}(t) - r_{j,HT}(t))^T (R_D(t) \hat{u}_{1,0}) \right\} = \lambda_1(t) \min_{i, j \in V_R} \left\{ (r_{i,0} - r_{j,0})^T \hat{u}_{1,0} \right\}
\]

Thus the inter-agent collision is avoided if

\[
\forall t \in [t_s, t_f], \quad \lambda_1(t) d_s \geq 2(\delta + \epsilon).
\]

(62)

This is ensured if (59) and (60) are satisfied.

**Remark 5.** Eigenvalues \( \lambda_2(t) \) and \( \lambda_3(t) \) of matrix \( U_D \) must be positive at any time \( t \) as is required for continuum deformation coordination. Furthermore, shear deformation angle \( \phi_{u,0} \) can be arbitrarily selected. Without loss of generality, this paper chooses \( \phi_{u}(t) = \phi_{u,0} = 0 \) at any time \( t \).
 VII. PROBLEM 3: CONTINUUM DEFORMATION PLANNING

The paper offers two distinct strategies for continuum deformation planning in obstacle-free and obstacle-laden environments. In an obstacle-free environment, safe continuum deformation coordination is planned by shaping homogeneous transformation features. In an obstacle-laden environment, A* search is used to plan a safe continuum deformation by choosing optimal leaders’ paths given MVS initial and target formations.

A. Obstacle-Free Environment

For continuum deformation coordination in an obstacle-free environment, generalized coordinate vector $s^*_{OF}$ is defined as follows:

$$s^*_{OF} = \begin{bmatrix} 1 & \theta_r & \psi_r & d_1 & d_2 & d_3 \end{bmatrix}^T,$$

$$s^2_{OF} = \begin{bmatrix} 1 & \lambda_2 & \phi_r & \theta_r & \psi_r & d_1 & d_2 & d_3 \end{bmatrix}^T,$$

$$s^3_{OF} = \begin{bmatrix} 1 & \lambda_2 & \lambda_3 & \phi_r & \theta_r & \psi_r & d_1 & d_2 & d_3 \end{bmatrix}^T,$$

where $s^*_{OF}$ is defined by quintic polynomial $(63)$ with $t_1 = t_s$ and $t_2 = t_f$. The global desired position of vehicle $i$ is given by $(61)$, where $d = [d_1(t) d_2(t) d_3(t)]^T$ for $n = 1, 2, 3$, and

$$Q(\lambda_1, \theta_r, \psi_r) \quad n = 1,$$

$$Q(\lambda_1, \lambda_2, \phi_r, \theta_r, \psi_r) \quad n = 2,$$

$$Q(\lambda_1, \lambda_2, \lambda_3, \phi_r, \theta_r, \psi_r) \quad n = 3.$$  

Remark 6. For coordination in an obstacle-free environment, shear deformation angles $\theta_r$, $\psi_r$, and $\phi_r$ are either 0 or constant at any time $t \in [t_s, t_f]$ (See Section VII-B 2). Therefore, shear deformation angles are not included as generalized coordinates.

B. Obstacle-Laden Environment

For coordination in an obstacle-laden environment,

$$s^*_{OL}(t) = \text{vec} \left( \mathbf{h}_1(t) \cdots \mathbf{h}_{n+1}(t) \right)^T,$$

is the generalized coordinate vector which is defined based on the $n(n+1)$ leader position components. Note that “vec” is the matrix vectorization symbol. Also,

$$r_i, HT(t) = \left( I_3 \otimes \mathbf{O}_n \left( \mathbf{h}_{1,0}, \cdots, \mathbf{h}_{n+1,0}, \mathbf{r}_{i,0} \right) \right) s^*_{OL},$$

where $n \in \{1, 2, 3\}$ is the dimension of the homogeneous deformation, $\mathbf{r}_{1,0}$ is the reference position of vehicle $i \in \mathcal{V}_R$, $\mathbf{h}_{1,0}$ through $\mathbf{h}_{n+1,0}$ denote reference positions of VCS vertices.

This paper applies A* search to optimally determine intermediate generalized coordinate vectors $s^*_{OL}, s^2_{OL}, \cdots, s^3_{n+1,OL}$, given initial and final generalized coordinate vectors $s^0_{OL}$ and $s^0_{n+1,OL}$, respectively. We define the following:

$$\Omega^s = (\mathbf{h}_{1,s}, \cdots, \mathbf{h}_{n+1,s}) := \text{Initial VCS},$$

$$\Omega^f = (\mathbf{h}_{1,f}, \cdots, \mathbf{h}_{n+1,f}) := \text{Final VCS},$$

$$\Omega^c = (\mathbf{h}_{1,c}, \cdots, \mathbf{h}_{n+1,c}) := \text{Current VCS},$$

$$\Omega^\text{next} = (\mathbf{h}_{1,\text{next}}, \cdots, \mathbf{h}_{n+1,\text{next}}) := \text{Next VCS},$$

where $\mathbf{h}_{i,0}$, $\mathbf{h}_{i,f}$, $\mathbf{h}_{i,c}$, and $\mathbf{h}_{i,n}$ denote position of vertex $i$ of the VCS at initial time $t_s$, final time $t_f$, current time, and next time, respectively. Leaders’ optimal paths are determined by minimizing continuum deformation cost, given by

$$n = 1, 2, 3, \quad F(\Omega^\text{next}) = G(\Omega^\text{next}) + H(\Omega^\text{next}),$$

where $\Omega^\text{next}$ is a valid continuum deformation from $\Omega^c$ and

$$n = 1, 2, 3, \quad H(\Omega^\text{next}) = \sum_{i=1}^{n+1} \left\| \mathbf{h}_{i,\text{next}} - \mathbf{h}_i \right\|_2^2,$$

is the heuristic cost. Furthermore,

$$n = 1, 2, 3, \quad G(\Omega^\text{next}) = \min \left\{ G(\Omega^c) + C^s_{\text{next}} \right\},$$

is the minimum estimated cost from $\Omega^c$ to $\Omega^\text{next}$, where

$$C^s_{\text{next}} = \sum_{i=1}^{n+1} \left\| \mathbf{h}_{i,\text{next}} - \mathbf{h}_{i,c} \right\|_2$$

$(n = 1, 2, 3)$.

C. Planning of travel time

From $(39)$

$$E_{\text{SYS}}(t) = e^{A_{\text{SYS}}(t-t_s)} E_{\text{SYS}}(t_s) + \int_{t_s}^{t_f} e^{A_{\text{SYS}}(t'-t_s)} \mathbf{V}_{\text{SYS}}(t') dt'.$$

Define $C_i = [C_i(t)] \in \mathbb{R}^{(\rho_q + \rho_s + \rho_t) N}$, with elements

$$C_i(t) = \begin{cases} \rho_q & (t \in (t_s, t_f) \lor t = \bar{t}_q) \lor (t = t_k), \\ \rho_t & (t_s < t < t_{k+1}), \\ 0 & \text{else} \end{cases}$$

Assume consecutive generalized coordinate vectors $s^0_{OL}$ and $s^0_{n+1,OL}$ are known for $q \in \{0, 0, 0\}$, then.

$$C_i(t) \in \mathcal{U}_i, \quad \forall t \in [t_s, t_{k+1}], \forall i \in \mathcal{V},$$

$$\beta(t, T_k) = \sum_{j=0}^{5} \xi_j k \left( \frac{T_k - t_k}{T_k} \right)^j,$$

subject to $(68)$ and

$$s^0_{OL}(t) = s^0_{OL}(1 - \beta(t, T_k)) + \beta(t, T_k) s^0_{OL}, \quad t \in [t_k, t_{k+1}],$$

$$E^T_{\text{SYS}}(t) C_i E_{\text{SYS}}(t) \leq \xi_0, \quad \forall t \in [t_k, t_{k+1}], \forall i \in \mathcal{V},$$

$$u_i(t) \in \mathcal{U}_i, \quad \forall t \in [t_k, t_{k+1}], \forall i \in \mathcal{V},$$

where $t_s = t_1$, $t_f = t_{n+1}$, $t_{k+1} = t_k + T_k$, and $\xi_0$ through $\xi_5$ are specified coefficients satisfying assumptions discussed after $(33)$. Note that typically $E_{\text{SYS}}(t_s) = 0$. Because $\beta(t, T_k)$ is assigned by Eq. $(54)$ for $t \in [t_k, t_{k+1}]$, the following properties hold:

1) $\frac{d \beta(t, T_k)}{dt}$ is a decreasing function with respect to $T_k$ at any time $t \in [t_k, t_{k+1}]$ for $j \in \{\rho_q - \Xi, \cdots, \rho_q \}$ (where $q \in \{x, y, z\}$).

2) $\mathbf{V}_{\text{SYS}}(t)$, defined by $(68)$, and $E_{\text{SYS}}(t)$, defined by $(68)$, tend to $0 \in \mathbb{R}^{3N \times 1}$ and $0 \in \mathbb{R}^{(\rho_q + \rho_t + \rho_s) N \times 1}$, respectively, at any time $t \in [t_k, t_{k+1}]$ as $T_k \rightarrow \infty$.

Consequently, there exists a minimum travel time $T_k^*$ such that the safety inequality constraints given in $(69)$ are all satisfied.
VIII. SIMULATION RESULTS

Case studies of 2-D, and 3-D continuum deformation are presented below with and without obstacles. We assume each vehicle is a quadcopter. The variables \( \phi_i \), \( \theta_i \), and \( \psi_i \) denote the \( i \)th quadcopter Euler angles, the variables \( x_i \), \( y_i \), \( z_i \) denote its position coordinates, \( v_{x_i} \), \( v_{y_i} \), \( v_{z_i} \) denote its velocity components, \( m_i \) is mass, \( F_{T,i} \) is the total thrust magnitude, \( \bar{T} \) is the thrust force per unit mass, and \( g = 981 m/s^2 \) is the gravity acceleration. Quadcopter \( i \) dynamics is then given by Eq. (28)

\[ \dot{X_i} = [x_i \, y_i \, z_i \, v_{x,i} \, v_{y,i} \, v_{z,i} \, F_{T,i} \, \dot{\phi}_i \, \dot{\theta}_i \, \dot{\psi}_i]^T \]

where \( \rho_k \) is the gravity acceleration. Quadcopter components, \( \psi \)

\[ f_e = [v_{x,i} \, v_{y,i} \, v_{z,i} \, f_{k,i} \, f_{s,i} \, f_{u,i} \, \dot{F}_{T,i} \, \dot{\phi}_i \, \dot{\theta}_i \, \dot{\psi}_i]^T, \]

\[ g = \begin{bmatrix} 0_{6 \times 9} \quad I_3 \quad 0_{3 \times 1} \end{bmatrix}^T. \]

Using feedback linearization, quadcopter dynamics can be expressed as

\[ \frac{d^4 r_i}{dt^4} = \mathbf{v}_i \quad (70a) \]

\[ \frac{d^2 \psi_i}{dt^2} = \ddot{\psi}_{d,HT} + k_{\psi} (\psi_{d,i} - \psi_i) + k_{\psi} (\psi_{d,i} - \psi_i), \quad (70b) \]

where \( \rho_s = \rho_0 = \rho_2 = 4 \) and (70b) represents the internal dynamics of the quadcopter system. Parameters \( k_{\psi} \) and \( k_{\psi} \) are positive constants and we choose \( \psi_{d,i} = 0 \), \( \psi_{d,i} = 0 \), and \( \ddot{\psi}_{d,i} = 0 \). Therefore, the quadcopter zero-dynamics is stable. For the quadcopter model [37], \( \mathbf{v}_i = \mathbf{M}_i \mathbf{u}_i + \mathbf{n}_i, \) \( \mathbf{M}_i = [A_{i,1} \, A_{i,2} \, A_{i,3}] \), \( \mathbf{n}_i = [A_{i,1} \, A_{i,2} \, A_{i,3}] \), \( \lambda_{0,i} = \dot{\psi}_i \)

\[ \lambda_{1,i} = \begin{bmatrix} -C_{\phi_i} S_{\phi_i} C_{\psi_i} + S_{\phi_i} S_{\theta_i} & C_{\phi_i} S_{\phi_i} C_{\psi_i} + C_{\phi_i} S_{\theta_i} & C_{\phi_i} C_{\theta_i} \\ -C_{\phi_i} S_{\phi_i} S_{\psi_i} + C_{\phi_i} C_{\theta_i} & C_{\phi_i} S_{\phi_i} C_{\psi_i} - C_{\phi_i} S_{\theta_i} & -S_{\phi_i} C_{\theta_i} \\ -S_{\phi_i} S_{\phi_i} C_{\psi_i} + S_{\phi_i} C_{\theta_i} & -S_{\phi_i} S_{\phi_i} S_{\psi_i} - C_{\phi_i} C_{\theta_i} & 0 \end{bmatrix} \]

\[ \lambda_{2,i} = \begin{bmatrix} -C_{\phi_i} C_{\psi_i} & C_{\phi_i} \phi_i & C_{\phi_i} \phi_i \\ -C_{\phi_i} C_{\psi_i} & C_{\phi_i} \phi_i & -S_{\phi_i} C_{\theta_i} \end{bmatrix} \]

A. 2-D Continuum Deformation Coordination

Consider an MVS with reference formation shown in Fig. 5(a). The MVS consists of \( N = 27 \) quadcopters; quadcopters 1, 2, and 3 are leaders and the remaining quadcopters 4, 5, \ldots, 27 are followers, e.g. \( \mathcal{V}_L = \{1, 2, 3\} \) and \( \mathcal{V}_F = \{4, 5, \ldots, 27\} \). Boundary quadcopters are defined by the set \( \mathcal{V}_B = \{1, 4, 5, 10, 12, 14, 17, 24, 25, 26, 27\} \) and \( N_a = 11 \) auxiliary nodes are defined by the set \( \mathcal{V}_{aux} = \{28, 29, \ldots, 38\} \). Without loss of generality, reference positions of auxiliary nodes and boundary quadcopters are the same (Fig. 5(a)). Unidirectional and bidirectional communications are shown by one-sided green arrows and double-sided blue arrows, respectively. Leaders move independently. Therefore, no edge is incident to a node \( i \in \mathcal{V}_L \). Every follower communicates with three in-neighbor nodes where followers’ reference communication weights are determined by using Eq. (14) given quadcopters’ reference positions shown in Fig. 5(a). Every auxiliary node communicates with leader agents 1, 2, and 3, e.g. \( \mathcal{N}_i = \{1, 2, 3\} \) if \( i \in \mathcal{V}_{aux} \). Note that communication links between auxiliary nodes and leaders are not shown in Fig. 5(a). As shown in Fig. 5(a), reference VCS is a containing triangle with vertices positioned at \( \mathbf{h}_{1,0} = (0.40, 0), \mathbf{h}_{2,0} = (60.100), \) and \( \mathbf{h}_{3,0} = (0.140) \). Given reference VCS configuration and the MVS formation, \( d_s = 5.5875 \) is the minimum separation distance between two quadcopters and \( d_b = 4.5358 \) is the minimum distance from the VCS sides. Assuming every quadcopter is enclosed by a ball with radius \( \epsilon = 0.5m \), \( \delta_{max} = 2.2938m \) is obtained by using Eq. (52). We consider MVS collective motion in an obstacle.
laden environment shown in Fig. 5. Because the \( z \) component of the MVS initial formation is 0, reference and initial MVS formations are considered the same; \( t_0 = t_r = 0 \), \( \mathbf{R}_L = \mathbf{I}_3 \in \mathbb{R}^{3 \times 3} \), and \( \mathbf{d}_L = \mathbf{0} \in \mathbb{R}^{3 \times 1} \). Optimal leaders’ paths, minimizing travel distance between initial and target MVS formations, are determined using A* search and shown in Figs. 5 (a-h) (See Section VII-B). Given leaders’ trajectories, followers use the communication graph shown in Fig. 5(a) to acquire the desired continuum deformation through local communication. MVS formations at sample times 0s, 100s, 500s, and 812s are shown in Figs. 5 (a-d). Blue and red triangles show the containing VCS and leading triangle configurations in Figs. 5(a-d). Given leaders’ positions at reference time \( t = 0 \) and current time time \( t \), \( \mathbf{U}_D \) eigenvalues, \( \lambda_1 \) and \( \lambda_2 \), are plotted versus time in Fig. 5 (e), e.g. \( \lambda_3(t) = 1 \) at any time \( t \). As shown \( \lambda_{CD, min} = 0.32 \) is a lower limit for the \( \mathbf{U}_D \) eigenvalues. Substituting \( \lambda_{CD, min} = 0.32 \), quadcopter size \( \epsilon = 0.5m \), \( \delta_{max} = 2.2938m \) into Eq. (54), \( \delta \) is obtained from:

\[
\delta = \lambda_{CD, min} (\delta_{max} + \epsilon) - \epsilon = 0.3940m.
\]

Transient error \( ||\mathbf{r}_i - \mathbf{r}_{i,HT}|| \) is plotted versus time for every follower quadcopter \( i \in \mathcal{V}_F \) in Fig. 5(f). Note that the deviation of every follower remains less than \( \delta = 0.3940m \). Therefore, inequality constraints (41) and (53) are satisfied, and inter-agent and obstacle collision avoidance, as well as quadcopter containment constraints are satisfied.

B. 3-D Continuum Deformation Coordination

We simulate collective takeoff with \( N = 16 \) quadcopter MVS. Leaders are indexed by 1 through 4 \( (\mathcal{V}_L = \{1, 2, 3, 4\}) \), followers are defined by set \( \mathcal{V}_F = \{5, \ldots, 16 \} \) where \( \mathcal{V}_B = \{1, 2, 3, 4, 14, 15, 16 \} \) specifies the boundary quadcopters. \( \mathcal{N}_a = 3 \) auxiliary nodes are defined by \( \mathcal{V}_{aux} = \{17, 18, 19 \} \) where \( \mathbf{r}_{14,0} = \mathbf{r}_{17,0} \), \( \mathbf{r}_{15,0} = \mathbf{r}_{18,0} \), and \( \mathbf{r}_{16,0} = \mathbf{r}_{19,0} \). Reference positions of auxiliary nodes 17, 18, and 19 and boundary quadcopters 14, 15, and 16 are the same. The MVS initial formation is shown in Fig. 6 (a) where leaders are illustrated by red and followers 5 through 13 are shown by black. In addition, follower quadcopters 14, 15, and 16 (auxiliary nodes 17, 18, and 19) are shown in green in Fig. 6 (b). Initial and reference formations are considered the same in a 3-D continuum deformation coordination. Follower communication weights are assigned using Eq. (14). Table II lists initial positions and reference communication weights per Eq. (14).

Given quadcopter initial reference positions, the shear deformation angles \( \theta_{a,0} = -0.1721 \) rad and \( \phi_{a,0} = 0.7130 \) rad are obtained using Eq. (50). Furthermore, we choose \( \phi_{a,0} = 0 \) at any time \( t \) (see Remark 5). We consider MVS collective motion over the time interval [0,250] \( (t_s = 0s \text{ and } t_f = 250s) \) where \( \theta_s(t_f) = 0 \text{ rad, } \theta_r(t_f) = 0.0713 \text{ rad, } \phi_s(t_f) = \frac{\pi}{2} \text{ rad, } d_1(t_f) = 100m, \text{ and } d_3(t_f) = 200m \). Given \( s_{OF}^3(0) \) and \( s_{OF}^3(250) \), a homogeneous transformation is defined by Eq. (4) and acquired by followers through local communication. MVS formations at sample times 0s, 10s, 50s, 100s, and 250s are shown in Fig. 6 (b).

Inter-Agent Collision Avoidance: Given the MVS initial formation, \( \delta_s = 4.6607m \) is the minimum separation distance.

![MVS Initial Formation](image1)

![MVS Evolution](image2)

![Transient Error](image3)

Fig. 6: (a) MVS initial formation and inter-agent communication in case-study 4. (b) MVS at times 0s, 90s, 150s, 250s. Red and black nodes show leaders and followers, respectively. Leader 1, 2, 3, 4 paths are shown by black, green, red, and pink curves. (c) Deviation of each follower \( i \in \mathcal{V}_F \) versus time in case study 4. Note that sup \( ||\mathbf{r}_i - \mathbf{r}_{i,HT}|| \leq \delta = 0.6652m \), \( \forall i \in \mathcal{V}_F \).

Given \( \lambda_{min} = \lambda_1(t_f) = 0.5 \), deviation upper-bound \( \delta \) becomes \( \delta = \lambda_{CD, min} (\delta_{max} + \epsilon) - \epsilon = 0.6458m \). Deviation of every follower from the desired position, defined by the continuum deformation, is plotted versus time in Fig. 6 (c). Because sup \( ||\mathbf{r}_i(t) - \mathbf{r}_{i,HT}(t)|| \leq 0.6652 \) (\( \forall i \in \mathcal{V}_F \)) inter-agent collision avoidance is guaranteed.

IX. CONCLUSION

This paper advanced continuum deformation coordination by relaxing existing containment constraints. We showed that any \( n + 1 \) agents forming an \( n \)-D simplex can be considered as leaders; followers can be placed inside or outside the leading
simplex in an n-D homogeneous transformation (n = 1, 2, 3). This paper also formulated continuum deformation coordination eigen-decomposition to determine a nonsingular mapping between leader position components and homogeneous transformation features assigned by continuum deformation eigen-decomposition. With this approach, leader trajectories ensuring collision avoidance and quadcopter containment can be safely planned. Furthermore, this paper advances the existing condition for inter-agent collision avoidance in a large-scale continuum deformation. This new safety condition is much less restrictive and significantly advances the maneuverability and flexibility of the continuum deformation coordination.

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| t | x_1 | x_2 | x_3 | x_4 | x_5 | x_6 | x_7 |
|---|---|---|---|---|---|---|---|
| 1 | -30 | -40 | 0 | 0 | 0 | 0 | 0 |
| 2 | -30 | 40 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

TABLE II: Quadcopters’ initial positions and reference communication weights of followers in case study 4
APPENDIX A

n-D HOMOGENEOUS DEFORMATION DECOMPOSITION

By decomposition, homogeneous transformation features are uniquely related to leaders’ position components at any time $t \geq t_0$ given leaders’ reference positions. Decomposition is straightforward for a 3-D homogeneous transformation ($n = 3$) when four leaders define a desired homogeneous transformation, e.g. 12 leaders position components can be simply related to the homogeneous transformation features. However, homogeneous transformation decomposition will not be straightforward for 1-D and 2-D continuum deformation coordination, defined by 2 and 3 leaders, in a 3-D motion space because the number of leader position components differs from the number of homogeneous transformation features.

1) 1-D Homogeneous Transformation Decomposition: Assume that vehicles are treated as particles of a 1-D deformable body distributed along a deformable line segment in a 3D motion space at a time $t$. The MVS transformation is guided by two leaders (agents 1 and 2) and the remaining agents are followers. For 1-D homogeneous transformation, $\lambda_1(t) > 0$, $\lambda_2(t) = \lambda_1(t) = 1$ and $\phi_\theta(t) = \theta_o(t) = \psi_r(t) = \theta_r = 0$ at all times $t$, see the first row of Table 3.

Proposition 3. If agents are distributed along $U_D$ eigenvector $\hat{u}_1$ at time $t \geq t_0$, they were distributed along $\hat{e}_1 = [1\ 0\ 0]^T$ at reference time $t_0$.

Proof: Substituting $\hat{\phi}_u = \theta_o = \phi_r = \theta_r = \psi_r = 0$ in Eq. (10), $\hat{u}_1 = \hat{e}_1 = [1\ 0\ 0]^T$, $\hat{u}_2 = \hat{e}_2 = [0\ 1\ 0]^T$, and $\hat{u}_3 = \hat{e}_3 = [0\ 0\ 1]^T$.

Theorem 6 describes how to determine $(\lambda_1, \phi_\theta, \theta_r, d_1, d_2, d_3)$ leader position components $(\lambda_1, HT, \phi_\theta, \theta_r, d_1, d_2, d_3)$ at any time $t$.

Theorem 6. If leaders’ reference positions $(r_{1,0}$ and $r_{1,0})$ and leaders’ global desired positions $r_{1,HT}(t)$ and $r_{2,HT}(t)$ at current time $t \geq t_0$ are given, then,

$$\lambda_1(t) = \frac{||r_{2,HT}(t) - r_{1,HT}(t)||}{||r_{2,0} - r_{1,0}||} = \frac{||r_{2,HT}(t) - r_{1,HT}(t)||}{x_{2,0} - x_{1,0}}$$

Also,

$$\hat{u}_1(t) = \frac{r_{2,HT}(t) - r_{1,HT}(t)}{r_{2,0} - r_{1,0}},$$

$$\theta_r = -\sin^{-1}(\hat{u}_1 \cdot \hat{e}_3),$$

$$\psi_r = \tan^{-1}\left(\frac{\hat{u}_1 \cdot \hat{e}_2}{\hat{u}_1 \cdot \hat{e}_1}\right),$$

where $\hat{e}_1 = [1\ 0\ 0]^T$, $\hat{e}_2 = [0\ 1\ 0]^T$, and $\hat{e}_3 = [0\ 0\ 1]^T$.

Additionally, $d_1$, $d_2$, and $d_3$ are assigned by Eq. (4): $d(t) = [d_1(t)\ d_2(t)\ d_3(t)]^T = r_{1,HT}(t) - Q_1(t)$.

Proof: Leaders’ desired positions satisfy Eq. (4), thus, $\forall t, r_{2,HT}(t) - r_{1,HT}(t) = \hat{Q} [r_{2,0} - r_{1,0}]$ and

$$(r_{2,HT} - r_{1,HT})^T (r_{2,HT} - r_{1,HT}) = (r_{2,0} - r_{1,0})^T R_D U_D^T R_D (r_{2,0} - r_{1,0}),$$

at any time $t \in [t_1, t_f]$, where $r_{2,0} - r_{1,0} = [1\ x_{2,0} - x_{1,0} \ 0\ 0\ 0]$. Because $R_D$ is orthogonal, $R_D^T R_D = I$ and $\lambda_1$ is obtained as given in Eq. (7). Because vehicles’ desired positions are distributed along $\hat{u}_1(t)$ at any time $t$, $\hat{u}_1(t) = r_{2,HT}(t) - r_{1,HT}(t)$. Considering Proposition 3 vehicles were distributed along the $\hat{e}_1$ axis at reference time $t_0$. Therefore,

$$r_{2,0} - r_{1,0} = \left[\begin{array}{c} \|r_{2,0} - r_{1,0}\| \ 0 \ 0 \end{array}\right]^T.$$

Because $r_{i,0}$ and $r_{i,HT}(t) (i = 1, 2)$ satisfy Eq. (4), we can write:

$$r_{2,HT} - r_{1,HT} = ||r_{2,HT} - r_{1,HT}|| \hat{u}_1 = \lambda_1 ||r_{2,0} - r_{1,0}|| \begin{bmatrix} \cos \theta_r \cos \psi_r \\ \cos \theta_r \sin \psi_r \\ -\sin \theta_r \end{bmatrix}.$$

Therefore, $\hat{u}_1 = [\cos \theta_r \cos \psi_r \cos \theta_r \sin \psi_r - \sin \theta_r]^T$ and $\theta_r$ and $\psi_r$ are per Eqs. (73b) and (73c). Given $\lambda_1(t)$, $\theta_r(t)$, and $\psi_r(t)$ at time $t$, $U_D$ and $R_D$ are determined using Eq. (11). Substituting $r_1(t)$ and $r_1(t)$ into Eq. (4),

$$d(t) = [d_1(t)\ d_2(t)\ d_3(t)]^T = r_{1,HT}(t) - Q_1(t).$$

2) 2-D Homogeneous Transformation Decomposition: Assume that vehicles’ global desired positions are on a plane in a 3-D motion space at current time $t$. The MVS transformation is guided by three leaders (vehicles 1, 2, and 3) and the remaining vehicles are followers. In a 2-D homogeneous transformation, $\lambda_3(t) = 1$ and $\phi_\theta(t) = \psi_r(t) = 0$ at any time $t$. Therefore, $U_D$ simplifies to

$$U_D = \begin{bmatrix} \lambda_1 \cos^2 \psi_r + \lambda_2 \sin^2 \psi_r & (\lambda_1 - \lambda_2) \sin \psi_r \cos \psi_r & 0 \\ (\lambda_1 - \lambda_2) \sin \psi_r \cos \psi_r & \lambda_1 \sin^2 \psi_r + \lambda_2 \cos^2 \psi_r & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

(74)
eigenvectors of $\mathbf{U}_D$), they were on this plane at reference time to.

**Proof:** Substituting $\phi_0 = \theta_0 = \psi_0 = \phi_r = \theta_r = \psi_r = 0$ in Eq. (11), $\mathbf{u}_1 = \mathbf{e}_1 = [1 \ 0 \ 0]^T$, $\mathbf{u}_2 = \mathbf{e}_2 = [0 \ 1 \ 0]^T$, and $\mathbf{u}_3 = \mathbf{e}_3 = [0 \ 0 \ 1]^T$. ■

**Remark.**  Given leaders' reference positions $\mathbf{r}_i(0) = [x_{i,0} \ y_{i,0} \ 0]^T$ and $\mathbf{r}_j(0) = [x_{j,0} \ y_{j,0} \ 0]^T$ and leaders' global desired positions $\mathbf{r}_{i,HT}(t) = [x_{i,HT}(t) \ y_{i,HT}(t) \ z_{i,HT}(t)]^T$ and $\mathbf{r}_{j,HT}(t) = [x_{j,HT}(t) \ y_{j,HT}(t) \ z_{j,HT}(t)]^T$, satisfying homogeneous transformation condition [4], the following relation holds:

$$(\mathbf{r}_{d,i} - \mathbf{r}_{j,HT})^T (\mathbf{r}_{d,i} - \mathbf{r}_{j,HT}) = (\mathbf{r}_{i,0} - \mathbf{r}_{j,HT})^T \mathbf{U}_D^2 (\mathbf{r}_{d,i} - \mathbf{r}_{j,HT}),$$

where

$$\mathbf{U}_D^2 = \begin{bmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$a = \lambda_1^2 \cos^2 \psi_u + \lambda_2^2 \sin^2 \psi_u,$$

$$b = (\lambda_1^2 - \lambda_2^2) \sin \psi_u \cos \psi_u,$$

$$c = \lambda_1^2 \sin^2 \psi_u + \lambda_2^2 \cos^2 \psi_u.$$

(79a)

(79b)

(79c)

(79d)

Considering Proposition [4], it is concluded that $z$ components of leaders' reference positions are zero ($\mathbf{r}_{i,0} = [x_{i,0} \ y_{i,0} \ 0]^T$ $(i = 1, 2, 3, i \in \mathcal{V}_L$). Substituting $\mathbf{r}_{i,0}$ and $\mathbf{r}_{i,HT}(t)$ $(i = 1, 2, 3)$ into Eqs. (78a), (78b), and (78c), $a$, $b$, and $c$ are obtained as given in Eq. (77). Given $a$, $b$, and $c$, $\lambda_1$, $\lambda_2$, and $\lambda_3$ are obtained as given in Eq. (76).

**Rotation Matrix $\mathbf{R}_D$:** Given leaders' reference positions $\mathbf{r}_{1,0}$, $\mathbf{r}_{2,0}$, and $\mathbf{r}_{3,0}$, leaders desired positions $\mathbf{r}_{1,HT}(t)$, $\mathbf{r}_{2,HT}(t)$, and $\mathbf{r}_{3,HT}(t)$, the pure deformation matrix $\mathbf{U}_D$ is computed using Eq. (74). Under a homogeneous transformation, $\mathbf{v}_{1,0} = \mathbf{r}_{2,0} - \mathbf{r}_{1,0}$, $\mathbf{v}_{2,0} = \mathbf{r}_{3,0} - \mathbf{r}_{1,0}$, and $\mathbf{v}_{3,0} = \mathbf{r}_{3,0} - \mathbf{r}_{2,0}$ are transferred to $\mathbf{v}_{1,HT} = \mathbf{r}_{2,HT} - \mathbf{r}_{1,HT}$, $\mathbf{v}_{2,HT} = \mathbf{r}_{3,HT} - \mathbf{r}_{2,HT}$, and $\mathbf{v}_{3,HT} = \mathbf{r}_{3,HT} - \mathbf{r}_{2,HT}$, where $\mathbf{v}_{i,HT} = \mathbf{Q} \mathbf{v}_{i,0}$. Define

$$\mathbf{I}_0 = [\mathbf{v}_{1,0} \ \mathbf{v}_{2,0} \ \mathbf{v}_{3,0}]$$

and $\mathbf{I}_d = [\mathbf{v}_{1,HT} \ \mathbf{v}_{2,HT} \ \mathbf{v}_{3,HT}]$.

If leaders 1, 2, and 3 form a triangle at time $t$, the elements of $\mathbf{Q}$ are obtained by

$$\mathbf{vec}(\mathbf{Q}^T) = (\mathbf{I}_3 \otimes \mathbf{I}_0)^{-1} \mathbf{vec}(\mathbf{I}_d).$$

(80)

Given $\mathbf{Q}$ and $\mathbf{U}_D$, $\mathbf{R}_D = \mathbf{Q} \mathbf{U}_D^{-1}$ can be obtained.

**Rigid-Body Displacement Vector $\mathbf{d}$:** Given $\mathbf{Q}$, $\mathbf{r}_{i,0}$, and $\mathbf{r}_{d,i}$, $\mathbf{d} = \mathbf{r}_{d,i} - \mathbf{Q} \mathbf{r}_{i,0}$.

3-D Homogeneous Transformation Decomposition:

When an MVS is guided by four leaders and leaders form a tetrahedron at any time $t$, leaders' global desired positions satisfy the following rank condition:

$$\forall t \geq t_0.$$ 

$$\text{Rank} \left( [\mathbf{r}_{2,HT} - \mathbf{r}_{1,HT} \ \mathbf{r}_{3,HT} - \mathbf{r}_{1,HT} \ \mathbf{r}_{d,4} - \mathbf{r}_{1,HT}] \right) = 3.$$ 

(81)

Define

$$\mathbf{P}_0 = [x_{1,0} \ y_{1,0} \ z_{1,0}] \text{ and } \mathbf{P}_d = [x_{1,HT} \ y_{1,HT} \ z_{1,HT} \ x_{2,HT} \ y_{2,HT} \ z_{2,HT} \ x_{3,HT} \ y_{3,HT} \ z_{3,HT} \ x_{4,HT} \ y_{4,HT} \ z_{4,HT}]$$

Elements of $\mathbf{Q}$ and $\mathbf{d}$ are uniquely related to leaders' position components by [40]:

$$\mathbf{vec}(\mathbf{Q}^T(t)) = \left[ \mathbf{I}_3 \otimes \mathbf{P}_0 \ \mathbf{I}_3 \otimes \mathbf{I}_4 \right]^{-1} \mathbf{vec}(\mathbf{P}_d(t)).$$

(82)

where $\mathbf{I}_3 \in \mathbb{R}^{3 \times 3}$ is an identity matrix and $\mathbf{I}_4 \in \mathbb{R}^{4 \times 1}$ is a ones vector.
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