Bound state studies in light-front QCD of mesons containing at least one heavy quark.

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We present the first numerical QCD bound-state calculation based on a renormalization group-improved light-front Hamiltonian formalism. The QCD Hamiltonian is determined to second order in the coupling, and it includes two-body confining interactions. We make a momentum expansion, obtaining an equal-time-like Schrödinger equation. This is solved for quark-antiquark constituent states, and we obtain a set of self-consistent parameters by fitting B mesons. Applying the approach to charmonium spectra leads to a prediction of the hyperfine splitting between $J/\psi(1S)$ and $\eta_c(1S)$ which is in good agreement with experiment.

1 Introduction

Recently, a new approach to QCD bound states has been proposed \cite{1,2}. The goal is to build a bridge between QCD and a constituent quark model (CQM). It has been argued that it is convenient to use a light-front formulation of the theory, because on the light-front it is possible to make the vacuum trivial simply by implementing a small longitudinal cutoff. As a result, all partons in a hadronic state are connected to the hadron, instead of being disconnected excitations in a complicated medium. The price to pay is a considerably more complicated renormalization problem.

The new approach consists of two (major) steps. The first step is to find an effective Hamiltonian, starting from the canonical light-front Hamiltonian regulated by a large cutoff (for details on the cutoff scheme we refer the reader to refs. \cite{1,2}). Unitary transformations are used to find counterterms which remove the dependence on the large cutoff, and to bring the Hamiltonian towards band-diagonal form with respect to free light-front energies. These transformations form a renormalization (semi)group. This is repeated until the Hamiltonian is band-diagonal with respect to a typical hadronic scale. At the end of this first step the Hamiltonian is still a complicated field theory Hamiltonian, but it does not couple states which differ in their free light-front energies by more than the hadronic scale. Instead, it contains effective potentials. It was shown that the effective potentials contain a Coulomb and a logarithmic confining potential already at order $g^2$ \cite{2}.

In the second step we want to solve this Hamiltonian and find its spectrum. We divide the effective Hamiltonian into a part $H_0$ which is solved nonperturbatively, and a part $V$ which is then calculated in bound-state perturbation theory. We want to choose $H_0$ so that it is manageable (i.e. something that we can solve) and we want it to contain the essential physics. Taking hints from the constituent quark model, we include two body potentials and use constituent masses in the $H_0$, but do not include
emission and absorption. Any approximations can always be done by adding a term to $H_0$ and subtracting it from $V$ which is treated in bound state perturbation theory. Not including the emission and absorption in the $H_0$ has an important consequence: different Fock states decouple. We can thus solve few body problems. The consistency of this procedure has to be checked in bound state perturbation theory.

In this talk I present one of the simplest QCD bound state calculation based on similarity transformations [3] and using coupling coherence [4]. We find the effective Hamiltonian to order $g^2$, and then solve for $q\bar{q}$ bound states. The calculation is carried through for quarks of arbitrary but nonzero masses. At the end, we concentrate on mesons containing at least one heavy quark.

2 The effective Hamiltonian to order $g^2$

The effective Hamiltonian, which generated by the similarity transformation to order $g^2$, is band-diagonal in light-front energy with respect to a hadronic scale $\Lambda^2_{P^+}$, and it can be written as:

$$H_{\text{eff}} = H_{\text{free}} + V_1 + V_2 + V_{2 \text{ eff}} ,$$

where $H_{\text{free}}$ is the light-front kinetic energy (we remind the reader that the light-front kinetic energy of a particle with transverse momentum $\vec{p}^\perp$ and longitudinal momentum $p^+$ is $\frac{p^\perp + m^2}{p^+}$), $V_1$ is $O(g)$ emission and absorption with nonzero matrix elements only between states with energy difference smaller than the hadronic scale $\frac{\Lambda^2}{P^+}$, $V_2$ is $O(g^2)$ instantaneous interaction, and $V_{2 \text{ eff}}$ includes the effective interactions generated by similarity, also $O(g^2)$. The effective interactions generated to this order contain one-body and two-body operators. In particular, the effective one-body operator is:

$$\frac{g}{2\pi P^+} \left\{ \alpha_s C_F \left( 2 \frac{P^+}{P^+} \Lambda^2 \log \left( \frac{P^+}{\epsilon P^+} \right) + 2 \frac{P^+}{P^+} \Lambda^2 \log \left( \frac{x_a \frac{P^+}{P^+} \Lambda^2}{x_a \frac{P^+}{P^+} \Lambda^2 + m_a^2} \right) \right) + \frac{3}{2} \frac{P^+}{P^+} \Lambda^2 + \frac{1}{2} \frac{m_a^2}{x_a \frac{P^+}{P^+} \Lambda^2 + m_a^2} \right\} ,$$

where $x_a = \frac{P^+}{P^+}$ is the longitudinal fraction of the momentum carried by the constituent under consideration, $m_a$ is its mass, $P^+$ is the total longitudinal momentum of the state, $P^+$ is the longitudinal scale required in the cutoff by dimensional arguments, and $\epsilon$ is an infrared cutoff which is to be taken to zero. The divergence in the effective one-body operator exactly cancels against the divergence in the effective two-body operator if the state is a color singlet.

The effective two-body operators have the following matrix elements between states containing a quark of momentum $\vec{p}_i$ and an antiquark of momentum $\vec{k}_i$, $i = 1, 2$ referring
to the initial and final state, respectively:

\[-g_\Lambda^2 \bar{u}(p_2, \sigma_2) \gamma^\mu u(p_1, \sigma_1) \bar{v}(k_2, \lambda_2) \gamma^\nu v(k_1, \lambda_1) \langle T_a T_b \rangle \times \left[ \frac{1}{q^2} D_{\mu\nu}(q) \left( \frac{\theta([D_1]-\frac{m_a^2}{D_1})}{m_a} + \frac{\theta([D_2]-\frac{m_b^2}{D_2})}{m_b} \right) \right] \]  

(3)

where \( \sigma_i \) and \( \lambda_i \) are light-front helicities of the quark and antiquark, respectively, \( u(p, \sigma) \) and \( v(k, \lambda) \) are their spinors, \( D_{\mu\nu}(q) = \frac{q_+^2}{q^2} \eta_\mu \eta_\nu + \frac{1}{q^2} (\eta_\mu q^+ \eta_\nu + \eta_\nu q^+ \eta_\mu) - g^\perp_{\mu\nu} \) is the gluon propagator in light-front gauge, \( \eta_\mu = (0, \eta^+, 0, 0) \), \( q = q^+ - q^\perp \) is the exchanged momentum and \( q^- = \frac{q^2}{q^2} \). \( D_1, D_2 \) are energy denominators: \( D_1 = p^+_1 - p^-_2 - q^- \) and \( D_2 = k^+_2 - k^-_1 - q^- \).

### 3 Bound state calculation

For the purpose of the bound-state calculation we divide the effective Hamiltonian into two parts: \( H_0 \) which is solved nonperturbatively, and \( V \equiv H_{\text{eff}} - H_0 \) which is solved in bound-state perturbation theory. In \( H_0 \), we include nonrelativistic limit of the kinetic energy, the self-energies, and the rotationally symmetric part of the \( \eta_\mu \eta_\nu \) term of the two-body interactions (both instantaneous and generated by similarity) [5]. Let the masses of the constituents be \( m_a \) and \( m_b \), and

\[ M_{ab} \equiv m_a + m_b. \]  

(4)

In the nonrelativistic limit, the light-front scale \( \Lambda^2 \) is naturally replaced by \( \mathcal{L} \equiv \frac{\Lambda^2}{p^+ M_{\text{as}}} \), which carries dimension of mass [6].

The Hamiltonian \( H_0 \) is:

\[ H_0 = 2M_{ab} \left[ -\frac{1}{2m} \nabla^2 + \bar{\Sigma} - \frac{C_F}{r} \right] - \frac{C_F}{\pi} V_0(Lr) \],  

(5)

where \( m \) is the reduced mass and

\[ V_0(Lr) = 2 \log R - 2Ci(R) + \frac{4Si(R)}{R} - 2 \left( \frac{1 - \cos R}{R^2} \right) + 2 \frac{\sin R}{R} - 5 + 2\gamma, \]  

(6)

where \( \gamma \) is Euler constant. \( \bar{\Sigma} \) contains the finite shift produced by the self-energies after subtracting terms needed to make the confining potential vanish at the origin:

\[ \bar{\Sigma} = \frac{\alpha C_F \mathcal{L}}{2\pi} \left[ \left( 1 + \frac{3m_a}{4\mathcal{L}} \right) \log \left( \frac{m_a}{\mathcal{L} + m_a} \right) + \left( 1 + \frac{3m_b}{4\mathcal{L}} \right) \log \left( \frac{m_b}{\mathcal{L} + m_b} \right) ight] + \frac{1}{4} \frac{m_a}{\mathcal{L} + m_a} + \frac{1}{4} \frac{m_b}{\mathcal{L} + m_b} + \frac{5}{2}. \]  

(7)
3.1 Schrödinger equation.

We now want to find the mass of a $q \bar{q}$ bound state and its wave function $\psi(\kappa^-, x)$:

$$|P\rangle = \int \frac{d^2 \kappa^- dx}{2(2\pi)^3 \sqrt{x(1-x)}} \psi(\kappa^-, x) b^d d^l |0\rangle .$$

(8)

We use a Lorentz-invariant normalization for the states:

$$\langle P'|P\rangle = 2(2\pi)^3 P^+ \delta^3(\vec{P} - \vec{P}') ,$$

and the wave function is normalized to one:

$$\int \frac{d^2 \kappa^- dx}{2(2\pi)^3} |\psi(\kappa^-, x)|^2 = 1 .$$

The bound state satisfies:

$$H_0 |P\rangle = M^2 |P\rangle ,$$

(9)

where $M^2$ is the invariant mass of the bound state. Let the mass of the bound state be

$$M^2 = (m_a + m_b)^2 + 2(m_a + m_b)E ,$$

(10)

which defines $E$.

After substituting for $H_0$ and $M^2$ in equation (9), some straightforward algebra leads to a bound-state equation for the wave function $\psi$:

$$M_{ab} \left( E - \tilde{\Sigma} + \frac{1}{2m} \frac{d^2}{d\vec{r}^2} \right) \psi(\vec{r}) = M_{ab} \left[ -\frac{\alpha C_F}{r} + \frac{\alpha C_F L}{\pi} V_{\text{conf}}(Lr) \right] \psi(\vec{r}) .$$

(11)

It is convenient to use a dimensionless separation $\mathcal{R} = Lr$ that naturally arises in the confining piece of the potential, and to absorb $-\tilde{\Sigma}$ into a definition of the eigenvalue $\tilde{E}$ of the Schrödinger equation. When extracting the bound state mass, $-\tilde{\Sigma}$ has to be subtracted. The bound-state equation in dimensionless form is:

$$\left[ -\frac{\mathcal{L}^2}{2m} \frac{d^2}{d\mathcal{R}^2} + \mathcal{L} \alpha C_F \left( \frac{1}{\pi} V_{\text{conf}}(\mathcal{R}) + V_{\text{coul}}(\mathcal{R}) \right) \right] \psi(\mathcal{R}) = \tilde{E} \psi(\mathcal{R}) .$$

(12)

Multiplying both sides of the equation by $2m/\mathcal{L}^2$ and introducing a dimensionless coupling and eigenvalue:

$$c \equiv \frac{2m \alpha C_F}{\mathcal{L}} ,$$

(13)

$$e \equiv \frac{2m \tilde{E}}{\mathcal{L}^2} ,$$

(14)

we obtain a Schrödinger equation, which depends only on dimensionless variables:

$$\left[ -\frac{d^2}{d\mathcal{R}^2} + c \left( \frac{1}{\pi} V_{\text{conf}}(\mathcal{R}) + V_{\text{coul}}(\mathcal{R}) \right) \right] \psi(\mathcal{R}) = e \psi(\mathcal{R}) .$$

(15)
This form is advantageous for numerical study, but moreover, it is quite general - one obtains an equation of this form for any quark-antiquark systems and any choice of the confining potential in the nonrelativistic limit, regardless of the masses, providing they are nonzero. For different systems $L$, $c$, $e$ would differ, but the resulting dimensionless Schrödinger equation will be the same. Thus in the leading order, qualitative characteristics of spectra depend only on one particular combination of the masses and the coupling, as seen from equations (13) and (14) [5].

3.2 Results and conclusion

We refer the reader to ref. [5] for more details on what one can learn from this dimensionless Schrödinger equation, and the dimensionless results. We applied the approach to B mesons [5], and we obtained a set of self-consistent parameters. Applying the approach to charmonium, we used 1S, 1P and 2S levels to fit our parameters ($m_c = 1.5$ GeV, $\Lambda = 1.7$ GeV, $\alpha = 0.5$) and then predicted the hyperfine splitting in the charmonium ground state (the splitting between $J/\psi(1S)$ and $\eta_c(1S)$) to be about 0.13 GeV [6].

Our study shows that the new approach proposed by Wilson et al. is promising.

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