Inherent Frequency and Spatial Decomposition of the Lorenz Chaotic Attractor

Gonzalo Álvarez,1 Shujun Li,2 Jinhu Lü,3 and Guanrong Chen2

1Instituto de Física Aplicada, Consejo Superior de Investigaciones Científicas, Serrano 144—28006 Madrid, Spain
2Department of Electronic Engineering, City University of Hong Kong, Kowloon Toon, Hong Kong, China
3Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Science, Beijing 100080, China

Abstract

This letter suggests a new way to investigate 3-D chaos in spatial and frequency domains simultaneously. After spatially decomposing the Lorenz attractor into two separate scrolls with peaked spectra and a 1-D discrete-time zero-crossing series with a wide-band spectrum, it is found that the Lorenz chaotic attractor has an inherent frequency uniquely determined by the three system parameters. This result implies that chaos in the Lorenz attractor is mainly exhibited when the trajectory crosses from one scroll to another, not within the two scrolls. This is also true for some other double-scroll Lorenz-like chaotic attractors, such as Chua’s attractor. Some possible applications of the inherent frequency and the spatial decomposition are also discussed.

PACS numbers: 05.45.Ac, 05.45.Gg, 05.45.Pq, 05.45.Vx

Keywords: chaotic attractor, spectrum, spatial decomposition, zero-crossing, Lorenz, Chua, phase coherence
Since the discovery of the first chaotic attractor by Lorenz [1], 3-D chaotic attractors have been extensively studied to clarify how chaos occurs in the real world and to help better understand the essence of chaos [2]. This letter suggests a novel way to investigate chaotic attractors in frequency domain via a spatial decomposition method. It is found that an inherent frequency, i.e., a prominent spectrum peak, exists in all of the three variables of the Lorenz attractor (but is particularly distinguished in the $z$-variable), and that the Lorenz attractor can be spatially decomposed into two single-scroll sub-attractors with peaked spectra at the inherent frequency and a 1-D zero-crossing time series with wide-band spectrum. This result strongly implies that, from the spectral point of view, chaos in the Lorenz attractor is mainly exhibited at the boundary of the two scrolls, not within them. Further numerical experiments show that the inherent frequency is uniquely determined by the three system parameters. The inherent frequency and the spatial decomposition also exist for some other Lorenz-like chaotic attractors with double scrolls, such as Chua’s attractor [3]. Compared with previous work on spectral analysis of “phase incoherent chaos” [4], this letter reveals much subtler features inherently-existing in Lorenz-like chaotic attractors.

Consider the following Lorenz system [1, 2]:

$$\begin{align*}
\dot{x} &= \sigma (y - x), \\
\dot{y} &= rx - y - xz, \\
\dot{z} &= xy - bz,
\end{align*}$$

where $\sigma, r, b$ are three positive parameters. When $\sigma = 10, r = 28, b = 8/3$, the Lorenz system has a double-scroll chaotic attractor shown in Fig. 1. Because the Lorenz system is invariant under the coordinate transformation $(x, y, z) \to (-x, -y, z)$, the chaotic attractor is symmetric with respect to the two planes $x = 0$ and $y = 0$. When $r > 1$, the centers of the two scrolls, which are two equilibrium points of the Lorenz system, can be calculated as $\left( \pm \sqrt{b(r-1)}, \pm \sqrt{b(r-1)}, r-1 \right)$. Roughly speaking, the double-scroll attractor is developed by the chaotic trajectory spirally rotating around the two points in a chaotic fashion.

Although the Lorenz chaotic attractor has been extensively studied in the past three decades, the relationship between the chaotic behaviors and the spectra of $x(t), y(t), z(t)$ has not yet been clarified [4]. This letter tries to give a clearer description of this problem via a novel spatial decomposition method. In the following, without loss of generality, the
system parameters are always set as the typical values $\sigma = 10, r = 28, b = 8/3$.

In Fig. 2 the power spectra of $x(t)$, $y(t)$ and $z(t)$ are shown (calculated by DFT with 4-term Blackman-Harris window, the same hereinafter). It can be seen that the spectra of $x(t)$ and $y(t)$ are locally wide-band in the low frequency range and gradually approach to zero as the frequency increases, which shows the existence of chaos in $x(t)$ and $y(t)$. In comparison, the power spectrum of $z(t)$ is much simpler: it has a significant spectral peak at the frequency $f_z \approx 1.3$ Hz, which implies that $z(t)$ is nearly periodic, i.e., weakly chaotic. Observing the time evolution of $z(t)$ shown in Fig. 3 one can see that the spectrum peak of $z(t)$ is a natural reflection of the nearly-invariant short-time period of $z(t)$.

Considering the spatial symmetry of the Lorenz attractor with respect to the two planes...
FIG. 3: The nearly-periodic evolution of $z(t)$

$x = 0$ and $y = 0$, the spectrum peak of $z(t)$ implies that the rotation frequency of the chaotic trajectory around each scroll is nearly invariant and the mean frequency is $f_z$. As a natural result, when the chaotic trajectory rotates within each scroll, the local (i.e., short-time) spectra of $x(t)$ and $y(t)$ should also have a peak at the frequency $f_z$. Is it really true? If so, then it can be expected that the chaoticity of the Lorenz attractor is mainly exhibited when the trajectory crosses the boundary between the two scrolls.

In the following, a simple spatial decomposition method is introduced as a tool to give an answer. The basic idea of the spatial decomposition method is to divide the chaotic trajectory into three parts: two separate single-scroll sub-attractors and a 1-D zero-crossing time series. The method is described as follows:

- **Two separate single-scroll sub-attractors:**
  1) connect all disjoint positive parts of $x(t)$ together to make an embedded signal $x_+(t) > 0$, and connect all its disjoint negative parts to make an embedded signal $x_-(t) < 0$ (see Fig. 4 for a schematic illustration on how the two embedded signals are extracted from the original signal $x(t)$);
  2) determine the sub-signals of $y(t)$, $z(t)$ corresponding to $x_+(t)$ and $x_-(t)$ in the time axis: $y_+(t), y_-(t)$ and $z_+(t), z_-(t)$.

As a result, $(x_+(t), y_+(t), z_+(t))$ constitutes a positive sub-attractor, and $(x_-(t), y_-(t), z_-(t))$ constitutes a negative sub-attractor (see Fig. 5). Both sub-attractors have a single scroll.

- **A 1-D zero-crossing time series $\{ZC(i)\}_{i=1}^{\infty}$:**
  1) use the zero-crossing times, i.e., the times when the chaotic trajectory crosses the boundary $x = 0$ from one scroll to another, to create a discrete-time series $\{t_{zc}(i)\}_{i=1}^{\infty}$;
2) $ZC(i)$ is defined as the elapsed time between $t_{zc}(i + 1)$ and $t_{zc}(i)$: $ZC(i) = t_{zc}(i + 1) - t_{zc}(i)$.

![Graph of $x(t)$ and $y(t)$](image1)

**FIG. 4:** The extraction of $x_+(t)$ and $x_-(t)$ from $x(t)$

![Graph of $z(t)$ and $y(t)$](image2)

**FIG. 5:** The positive (solid) and the negative (dashed) sub-attractors of the Lorenz attractor

Note that the above spatial decomposition can also be made with respect to the plane $y = 0$ to get two sub-attractors in the same way. After the above spatial decomposition of the Lorenz attractor, the three time series are analyzed via the spectral analysis method. It can be expected that the spectra of the two sub-attractors have a peak at the frequency of $f_z$ and the spectrum of $ZC(i)$ is wide-band. The experimental results well support this prediction. Observing the power spectra of $x_+(t)$, $y_+(t)$, $x_-(t)$ and $y_-(t)$ shown in Fig. 6, it can be clearly seen that a spectrum peak occurs at the same frequency $f_z \approx 1.3$Hz in
each spectrum. Note that the frequency $f_z$ of $z(t)$ is actually a reflection of the nearly-periodic trajectory within the two sub-attractors, since $z(t)$ is independent of the chaotic zero-crossing behaviors.

![Graph showing power spectra](image)

**FIG. 6:** The relative power spectra of $x_+(t)$, $y_+(t)$, $x_-(t)$ and $y_-(t)$

Now, let us consider the power spectrum of $ZC(i)$ shown in Fig. 7a, which keeps wide-band in the whole frequency range and has many spectral peaks without decaying to zero. Comparing all the five spectra, it is obvious that the local wide-band spectra of $x(t)$ and $y(t)$ stems from the wide-band spectrum of $ZC(i)$. Because the elapsed time between two consecutive zero-crossing events is at least equal to the time of one cycle of $x(t)$, i.e., the mean period of the Lorenz attractor $T_z = 1/f_z$, the occurrence frequency of the zero-crossing event will not be greater than $f_z$. It is the reason why the original spectra of $x(t)$ and $y(t)$ gradually decay to zero as the frequency increases beyond $f_z$.

Here, note the following fact: since the two sub-attractors are both approximately periodic, $ZC(i)$ will always be about $n$ times of $T_z$, where $n$ is an integer. In other words, $ZC(i)/T_z$ approximates the number of rotational cycles of the chaotic trajectory within the current sub-attractor. This can be clearly seen from Fig. 8a. Then, one can define a new time series $ZC_n(i) = \text{round}(ZC(i)/T_z)$, where round(·) denotes the function rounding a real number to a nearest integer. In Fig. 8b, $ZC_n(i)$ is displayed to show its approximation to $ZC(i)/T_z$. Apparently, $ZC_n(i)$ has a clearer physical meaning than $ZC(i)$, and has a similar spectrum to $ZC(i)$. Figure 7 gives a comparison of the power spectra of $ZC(i)$ and $ZC_n(i)$, which are almost identical.
The above spatial decomposition of the Lorenz attractor shows that the chaotic trajectory’s short-time behavior within each scroll is approximately periodic (i.e., weakly chaotic) and that the strongly chaotic behaviors mainly occur instantaneously when the trajectory crosses the boundary of the two scrolls with a mean invariant frequency not greater than $f_z$. Another obvious physical meaning of the spatial decomposition is the collapse of 3-D chaos into 1-D space. Considering that any 3-D chaotic system has only one positive Lyapunov exponent, such a collapse seems apprehensible.

A plenty of experiments were performed to verify the existence of this nearly-invariant spectral peak for other valid parameters of the Lorenz chaotic attractor. It is found that such an inherent frequency $f_z$ always exists and the spatial decomposition works well. Then, what is the relation between $f_z$ and the three parameters $r, \sigma, b$? When $r = 28$, the surface of $f_z = F(\sigma, b)$ is plotted in Fig. 9. As $r$ increases, the height of the surface will rise but the basic shape of the surface remains. At present, an explicit mathematical formula has not been found to describe such a relationship.
FIG. 9: The relation between $f_z$ and $(\sigma, b)$ when $r = 28$ (some fluctuations may be induced by numerical errors)

Is it possible to extend the above results on the Lorenz attractor to other 3-D chaotic attractors? For Chua’s chaotic attractor generated with the following equations:

\[
\begin{align*}
\dot{x} &= p(-x + y - f(x)), \\
\dot{y} &= x - y + z, \\
\dot{z} &= -qy, 
\end{align*}
\]

where $f(x) = m_0 x + 0.5(m_1 - m_0)(|x + 1| - |x - 1|)$, our experiments show that its inherent frequency also exists and the above spatial decomposition works as well. Further experiments have shown the existence of the inherent frequencies in some other 3-D Lorenz-like chaotic attractors with double scrolls. Considering prominent spectrum peaks also exist in many other chaotic attractors, such as the well-known phase-coherent Rössler attractors, it is an interesting question to ask if the spatial decomposition-based inherent frequencies exist in all 3-D chaotic attractors and how to explain this delicate phenomenon theoretically. In other words, the question raised is: does the only one positive Lyapunov exponent mean “3-D chaos = 1-D chaos + 2-D near periodicity”? No definite answer is given at this time.

Finally, some possible applications of the inherent frequency and the spatial decomposition are discussed.

**System Identification** The deterministic relationship between the inherent frequency $f_z$ and the system parameters reveals a new way of realizing system identification from the short-time waveform of any one of the three variables $x(t)$, $y(t)$ or $z(t)$. This is useful for adaptive synchronization of chaos.
Chaotic Cryptanalysis  The identification of system parameters from the inherent frequency actually means the breaking of some chaos-based secure communication systems, if the system parameters serve as the secret key for encryption. In addition, another possibility of using the inherent frequency in cryptanalysis is direct extracting the plain-signal from the cipher-signal. For example, in chaotic modulation systems, the plain-signal is used to change the system parameters dynamically, which will change the short-time periods of the transmitted cipher-signal. By distinguishing the change of the short-time period, it is possible to directly estimate the plain-signal without identifying the system parameters [7].

Pseudo-Random Numbers Generation  The wide-band spectrum of $ZC(i)$ means that good pseudo-random numbers may be generated using $ZC(i)$. A possible algorithm is to generate a symbolic 0-1 bit sequence according to which scroll the chaotic trajectory stays in [2, Chap. 6]. Initial experiments show that the generated pseudo-random bits can pass some statistical tests.

Suppressing Chaos  Due to the symmetry of the two sub-attractors, the Lorenz attractor can merge into an approximately periodic single-scroll attractor with the following transform: $x(t) \rightarrow G_x(x(t))$ and $y(t) \rightarrow G_y(y(t))$, where $G_x(\cdot)$ and $G_y(\cdot)$ are even symmetric functions. Such a transform can fold the two scrolls from two different quadrants into the same quadrant and merge them into a single scroll. As a natural result, the strongly chaotic zero-crossing actions are suppressed and the whole attractor becomes approximately periodic at the inherent frequency $f_z$. Apparently, the simplest transformation functions are $G_x(x(t)) = |x(t)|$ and $G_y(y(t)) = |y(t)|$, which correspond to the merging of the positive and the negative sub-attractors obtained via the spatial decomposition.

This research was supported by the Applied R&D Center, City University of Hong Kong, Hong Kong SAR, China, under Grants no. 9410011 and no. 9620004, and by the Ministerio de Ciencia y Tecnología of Spain, research grant TIC2001-0586 and SEG2004-02418.

* The first two authors, Gonzalo Álvarez and Shujun Li, are equivalent contributors of this letter.
† E-mail: gonzalo@iec.csic.es
‡ E-mail: hooklee@mail.com; URL: www.hooklee.com

[1] E. N. Lorenz, J. Atmosph. Sc. 20, 130 (1963).
[2] C. Sparrow, *The Lorenz Equations: Bifurcations, Chaos and Strange Attractors* (Springer-Verlag, New York, 1982).

[3] L. O. Chua, J. Circuits, Systems and Computers 4, 117 (1994).

[4] D. Farmer, J. Crutchfield, H. Froehling, N. Packard, and R. Shaw, Annals of New York Academy of Science 357, 453 (1980).

[5] T. E. Vadivasova and V. S. Anishchenoko, J. Communications Technology and Electronics 49, 69 (2004).

[6] G. Chen and X. Dong, *From Chaos to Order* (World Scientific, Singapore, 1998).

[7] G. Álvarez and S. Li, *Estimating short-time period to break different types of chaotic modulation based secure communications*, in preparation (2004).