EXACT BOSONIC AND SUPERSYMMETRIC STRING BLACK HOLE SOLUTIONS

I. Jack, D. R. T. Jones and J. Panvel
DAMTP, University of Liverpool, Liverpool L69 3BX, UK

Abstract

We show that Witten’s two-dimensional string black hole metric is exactly conformally invariant in the supersymmetric case. We also demonstrate that this metric, together with a recently proposed exact metric for the bosonic case, are respectively consistent with the supersymmetric and bosonic σ-model conformal invariance conditions up to four-loop order.
1. Introduction

Lately, intense activity has been devoted to the construction of conformally invariant theories representing strings propagating in black-hole type backgrounds, mostly in two dimensions but with some higher dimensional examples\cite{1,2,3}. This endeavour is not new\cite{2}, but the recent spate of interest was stimulated by Witten’s observation\cite{1} that the $SL(2, R)/U(1)$ gauged Wess-Zumino-Witten (WZW) model\cite{4,5} has an interpretation as a string in a two-dimensional black hole. The essential point of Witten’s insight is to provide us with an exactly conformally invariant theory with a black hole interpretation, whereas previous solutions were only valid perturbatively. This opens up the possibility of investigating the nature of spacetime singularities in string theory using the techniques of conformal field theory.

In Witten’s original analysis, upon gauge-fixing and integrating out the gauge fields in the $SL(2, R)/U(1)$ gauged WZW model, a non-linear $\sigma$-model is obtained with metric and dilaton background fields. The metric exhibits explicitly the properties associated with a black hole, such as an event horizon enclosing a spacetime singularity. However, the metric and dilaton fields only satisfy the $\sigma$-model conformal invariance conditions (which are obtained from the $\beta$-functions for the $\sigma$-model) to lowest order, despite the fact that the original gauged WZW model was exactly conformally invariant (since it represents a coset model\cite{6}). This discrepancy was ascribed to additional contributions arising from the functional measure when integrating out the gauge field. In a later analysis\cite{7}, the Virasoro operators $L_0, \bar{L}_0$ for the gauged $SL(2, R)/U(1)$ model were expressed as differential operators. By identifying these operators with the Laplacian obtained from the string effective action, proposed exact solutions for the metric and dilaton background were obtained. This derivation is somewhat indirect and moreover leaves out of account higher order perturbative corrections to the string effective action, and hence it is of interest to check whether this metric/dilaton background does indeed yield a solution to the conformal invariance conditions beyond one loop. It has recently been shown\cite{8} that the purported exact solution is in fact consistent with conformal invariance up to three loops in $\sigma$-model perturbation theory. At three loops, issues emerge concerned with scheme dependence of the conformal invariance conditions, which may be further illuminated by considering higher orders. Hence one purpose of this paper is to show that the exact metric and dilaton background given in Ref. [7] is indeed conformally invariant up to four loops, within renormalisation scheme ambiguities.

The four-loop calculation is also of interest in the supersymmetric context. Witten’s original string black hole solution was derived for the bosonic gauged $SL(2, R)/U(1)$ gauged WZW model. However, it is also possible to construct the supersymmetric extension of this model\cite{9}. By repeating the analysis of Ref. [7], we shall argue that Witten’s original metric/dilaton background should be exactly conformally invariant in the supersymmetric case. This is manifestly consistent with the fact that the $\beta$-function for the $N = 1$ supersymmetric $\sigma$-model is equal to that for the bosonic $\sigma$-model at one loop and vanishes at the two\cite{10} and three\cite{11} loop level. However, there is in general a non-vanishing contribution at four loops\cite{12}. We shall show that although the four-loop $N = 1$ supersymmetric $\beta$-function computed using minimal subtraction fails to vanish for Witten’s metric/dilaton
background, nevertheless there exists a renormalisation scheme in which it does.

2. The string black hole

In this section we briefly recapitulate previous results [1,7] on string black hole solutions. The Wess-Zumino-Witten (WZW) model[4] may be regarded as a special case of a nonlinear $\sigma$-model with an antisymmetric tensor field background in addition to a metric background. In general we may gauge a subgroup of the isometry group of the $\sigma$-model provided certain conditions are satisfied[13]. For the WZW model based on the group $G$, the isometry group is $G \times G$, and for a subgroup $H \subset G$ we can gauge the diagonal subgroup $H \times H$, providing a Lagrangian realisation of the coset model $G/H$. In the case at hand, we consider the $SL(2, R)$ WZW model and gauge a $U(1)$ subgroup. We can gauge either a vector or an axial realisation of the $U(1)$ subgroup, corresponding to the symmetry under $g \to hgh^{-1}$ or $g \to hgh$, respectively. We choose to gauge the axial subgroup here (gauging the vector subgroup leads to the “dual” $\sigma$-model[7, 14]). We parametrise the group by “Euler” angles, writing a generic group element $g$ as

$$g = e^{i/2 \phi_L e^{1/2 r} \sigma_1 e^{i/2 \phi_R \sigma_2}}$$

We will consider gauging the group generated by $\sigma_2$, so that the local gauge transformations correspond to $\phi_{L,R} \to \phi_{L,R} + \alpha$. This will yield a 2-dimensional black hole of Euclidean signature. The gauged WZW action takes the form

$$S_{GWZW} = S_{ZW} + \frac{k}{2\pi} \int d^2z [A(\bar{\partial}_L \phi_L + \cosh r \bar{\partial}_R \phi_R)$$

$$+ (\bar{\partial}_R \phi_R + \cosh r \partial_\phi_L) \bar{A} - A\bar{A} (\cosh r + 1)]$$

with the ungauged WZW action $S_{ZW}$ given by

$$S[r, \phi_L, \phi_R] = \frac{k}{4\pi} \int d^2z (\partial r \bar{\partial} r - \partial \phi_L \bar{\partial} \phi_L - \partial \phi_R \bar{\partial} \phi_R - 2 \cosh r \partial \phi_L \bar{\partial} \phi_R)$$

At this point, following Witten one can pick a gauge by setting $\phi_L = -\phi_R = \phi$ and integrate out the gauge fields using their equations of motion. The result has the form of the 2-dimensional non-linear $\sigma$-model action,

$$S = \frac{1}{4\pi\alpha'} \int d^2x \sqrt{\gamma} \gamma_{\mu\nu} \partial_\mu \phi i^j \partial_{\nu} \phi j^i (\phi) + \frac{1}{4\pi\alpha'} \int d^2x \sqrt{\gamma} D(\phi) R(2)$$

where $\gamma_{\mu\nu}$ is a metric on the 2-dimensional worldsheet and $R(2)$ is the worldsheet Ricci scalar. $\phi i(\lambda), i = 1, ..., N$, can be regarded as co-ordinates on an N-dimensional manifold.
with metric $g_{ij}$ and dilaton $D(\phi)$. In the case at hand we have $\{\phi_i\} = \{r, \phi\}$ with a metric and dilaton given by

$$ds^2 = \frac{e' \kappa}{4} (dr^2 + e^{2\lambda(r)}d\phi^2)$$

$$\lambda(r) = \ln \left(2 \tanh \frac{r}{2}\right)$$

$$D(r) = -\ln \cosh \frac{r}{2} + \text{const.}$$

(2.5)

In fact the dilaton field does not appear upon naively integrating over the gauge fields, and its existence is inferred somewhat indirectly. Moreover, although the metric and dilaton given by Eq.(2.5) satisfy the $\sigma$-model conformal invariance conditions at one loop, they fail to do so at higher order. Since the original gauged $SL(2, R)/U(1)$ WZW model was exactly conformally invariant, this is a somewhat unsatisfactory outcome. This situation can be remedied by following the analysis of Ref.[7]. The gauge field is parametrised as

$$A = \partial \pi_L$$

$$\bar{A} = \bar{\partial} \pi_R$$

(2.6)

with $\pi_L = \pi_R^*$. Upon making the shift

$$\phi_L \to \phi_L + \pi_L, \quad \phi_R \to \phi_R + \pi_R$$

(2.7)

the action Eq.(2.2) takes the form

$$S_{GWZW} = S_{WZW} + S[\pi] + S[b, c]$$

(2.8)

where

$$S[\pi] = \frac{k}{4\pi} \int d^2z \partial \pi \bar{\partial} \pi$$

$$\pi = \pi_L - \pi_R$$

(2.9)

and $S[b, c]$ is a ghost action derived from the Jacobian of the change of field variables Eq.(2.6), given by

$$S[b, c] = \int d^2z (b \partial c + \bar{b} \partial \bar{c})$$

(2.10)

If we also impose the gauge condition

$$\bar{\partial} A = \partial \bar{A}$$

(2.11)

then we can write

$$\pi(z, \bar{z}) = \varphi(z) + \bar{\varphi}(\bar{z})$$

(2.12)

The holomorphic conserved current is given by

$$2kg - 1\partial g = i\sigma_2 J_2 + i\sigma_3 \frac{1}{2}(J_+ - J_-) + \sigma_1 \frac{1}{2}(J_+ + J_-)$$

(2.13)
with
\[
J_2 = (\partial \phi_R + \cosh r \partial \phi_L)
\]
\[
J_{\pm} = e^{\pm i \phi_R} (\partial r \mp \sinh r \partial \phi_L)
\] (2.14)

The holomorphic part of the energy momentum tensor is given in terms of the currents through the Sugawara construction [15] as
\[
T(z) = \frac{1}{k-2} (J_2 - J_1 J_2) - \frac{k}{4} (\partial \pi) (\partial \pi) + b \partial c
\] (2.15)

The physical operators of the theory are defined by requiring that they commute with the BRST charge
\[
Q_{BRST} = \int dz c (J_2 + \frac{1}{2} k \partial \pi) + c.c.
\] (2.16)

It is argued in [7] that there is only one propagating degree of freedom in the 2-dimensional string theory represented by the \(SL(2,R)/U(1)\) models and that this may be taken to be the tachyon field. The primary fields of the coset conformal field theory represent the vertex operators for the tachyon field. Writing the tachyon vertex operator, \(V(z, \bar{z})\), as
\[
V(z, \bar{z}) = T(r(z, \bar{z}), \phi_L(z, \bar{z}), \phi_R(z, \bar{z})) e^{i q_R \varphi(z)} + i q_L \bar{\varphi}(\bar{z})
\] (2.17)

we find that the zero modes \(J_{0a}\) of the \(SL(2,R)\) currents in Eq.(2.12) may be represented as differential operators acting on \(T(r, \phi_L, \phi_R)\) of the form
\[
J_{02} = \frac{\partial}{\partial \phi_R}
\]
\[
J_{0\pm} = e^{\pm i \phi_R} \left[ \frac{\partial}{\partial r} \mp \frac{i}{\sinh r} \left( \frac{\partial}{\partial \phi_L} - \cosh r \frac{\partial}{\partial \phi_R} \right) \right]
\] (2.18)

Using the condition that \(V(z, \bar{z})\) in Eq.(2.17) commutes with \(Q_{BRST}\) in Eq.(2.16), we find from Eqs.(2.15), (2.17) and (2.18) that the Virasoro operators \(L_0, \bar{L}_0\) are represented by the operators
\[
L_0 = -\frac{1}{k-2} \Delta_0 - \frac{1}{k} \frac{\partial^2}{\partial \phi^2_R}
\]
\[
\bar{L}_0 = -\frac{1}{k-2} \Delta_0 - \frac{1}{k} \frac{\partial^2}{\partial \phi^2_L}
\] (2.19)

where
\[
\Delta_0(r, \phi_L, \phi_R) = \frac{\partial}{\partial r^2} + \coth r \frac{\partial}{\partial r} + \frac{1}{\sinh 2r} \left( \frac{\partial^2}{\partial \phi^2_L} - 2 \cosh r \frac{\partial}{\partial \phi_L} \frac{\partial}{\partial \phi_R} + \frac{\partial^2}{\partial \phi^2_R} \right)
\] (2.20)

Using the physical state condition
\[
(L_0 - \bar{L}_0) T(r, \phi_L, \phi_R) = 0
\] (2.21)
we can decompose \( T(r, \phi_L, \phi_R) \) as

\[
T(r, \phi_L, \phi_R) = T(r, \phi) + \tilde{T}(r, \tilde{\phi})
\]

\[
\phi = \frac{1}{2}(\phi_L - \phi_R), \quad \tilde{\phi} = \frac{1}{2}(\phi_L + \phi_R)
\]

so that acting on \( T(r, \phi) \), \( L_0 \) has the form

\[
L_0 = -\frac{1}{k+2} \left( \frac{\partial^2}{\partial r^2} + \coth r \frac{\partial}{\partial r} + \frac{1}{4} \sinh \frac{2r}{k} \frac{\partial^2}{\partial \phi^2} \right) - \frac{1}{4k} \frac{\partial^2}{\partial \phi^2} (2.23)
\]

In the \( \sigma \)-model approach to string theory, on the other hand, the tachyon is described to lowest order by a target-space effective action of the form

\[
S = \int d^2\phi e^{-2D\sqrt{g} \left( \frac{1}{2} \alpha' g_{ij} \partial_i T \partial_j T - 2T^2 \right)} (2.24)
\]

The equations of motion derived from this effective action give the lowest order conformal invariance conditions for the tachyon background field. The conformal invariance conditions should be equivalent to the physical state conditions expressed in terms of the Virasoro operators, and hence we identify \( L_0 \) in Eq.(2.23) with the Laplacian derived from Eq.(2.24), namely

\[
L_0 = -\frac{\alpha'}{4e^{-2D\sqrt{g}}} \partial_i e^{-2D\sqrt{g}g_{ij} \partial_j} (2.25)
\]

This identification leads to a solution for \( g_{ij} \) and \( D \) given by

\[
ds^2 = \frac{a'}{4} (k-2) (dr^2 + \beta^2(r)d\phi^2)
\]

\[
D = -\frac{1}{2} \ln \frac{\sinh r}{\beta(r)} (2.26)
\]

\[
\beta(r) = 2 \left( \coth \frac{r}{2} - \frac{2}{k} \right) - \frac{1}{2}
\]

This form for the metric and dilaton is claimed to be an exact solution of the conformal invariance conditions. Before investigating this claim in detail, we shall repeat the above analysis for the supersymmetric case.
3. The Supersymmetric Gauged SL(2,R) WZW model

In this section we consider the supersymmetric extension of the gauged \( SL(2,R) \) WZW model. The general supersymmetric WZW model was considered in Ref. [9] and the gauged \( SL(2,R)/U(1) \) version was discussed in Ref. [16]. The analysis of the previous section may readily be repeated for this case. The field element in Eq(2.1) is replaced by a superfield \( G(z,\bar{z},\theta,\bar{\theta}) \) where \( \theta, \bar{\theta} \) are superspace Grassman coordinates. \( G \) has an expansion in terms of components

\[
G = g + \theta \gamma + \bar{\theta} \bar{\gamma} + \theta \bar{\theta} f
\]  

and is parametrised as

\[
G(z,\theta) = e^{i \frac{1}{2} \Phi_L \sigma_2} e^{i \frac{1}{2} R \sigma_1} e^{i \frac{1}{2} \Phi_R \sigma_2}
\]  

for superfields \( \Phi_{L,R}, R \). The supersymmetric gauged WZW action is now of the form Eqs. (2.2), (2.3), but with

\[
\int d^2z \rightarrow \int d^2zd^2\theta, \quad \partial \rightarrow D, \quad \bar{\partial} \rightarrow \bar{D}, \quad r \rightarrow R, \quad \phi_{L,R} \rightarrow \Phi_{L,R}
\]  

where the superspace covariant derivatives are given by

\[
D = \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial z}, \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} - \bar{\theta} \frac{\partial}{\partial \bar{z}}
\]  

Following the analysis of Section 2, if we pick the gauge

\[
\bar{D}A = D\bar{A}
\]  

we can parametrise the gauge fields as

\[
A = D\Pi_L, \quad \bar{A} = \bar{D}\Pi_R
\]  

and then the action reduces to the form

\[
S_{GWZW} = S_{WZW} + S[\Pi] + S[B,C]
\]  

with

\[
\Pi = \Pi_L - \Pi_R
\]

\[
S[\Pi] = \frac{-k}{4\pi} \int d^2z d^2\theta D\Pi \bar{D}\Pi
\]

\[
S[B,C] = \int d^2z d^2\theta (B\bar{D}C + \bar{B}DC)
\]  

where \( B \) and \( C \) are anti-ghost and ghost superfields corresponding to the redefinition Eq. (3.6). If we impose the gauge condition

\[
\bar{D}A = -D\bar{A}
\]  

6
then we can write
\[ \Pi = \Psi + \bar{\Psi} \quad (3.10) \]
where
\[ \bar{D}\Psi = D\bar{\Psi} = 0. \quad (3.11) \]
The holomorphic conserved currents are given by
\[ 2kG - 1DG = i\sigma_2 J_2 + J_1 \sigma_1 + iJ_3 \sigma_3 \quad (3.12) \]
with
\[ J_1 = \frac{1}{2}(J_+ + J_-), \quad J_3 = \frac{1}{2}(J_+ - J_-) \quad (3.13) \]
and
\[ J_2 = D\bar{\Phi}_R + \cosh R D\Phi_L \]
\[ J_\pm = e^{\pm i\Phi_R}(DR \mp \sinh R D\Phi_L) \quad (3.14) \]
The holomorphic part of the superconformal tensor, [9,17], \( T(z, \theta) \), is then given in terms of the currents by
\[ T(z, \theta) = \frac{1}{k} \left( J_2 D J_2 - \frac{1}{2} J_+ D J_- + \frac{1}{2} J_- D J_+ \right) - \frac{2i}{3k^2} f_{abc} J_a (J_b J_c) - \frac{k}{4\pi} D\Pi D\Pi + BDC \quad (3.15) \]
where \( f_{abc} \) are the structure constants for \( SL(2, R) \). We now introduce a vertex operator for the tachyon field which we write as
\[ V(z, \bar{z}, \theta, \bar{\theta}) = T(R, \Phi_L, \Phi_R) e^{i q R} \Psi + i q L \bar{\Psi} \quad (3.16) \]
and which is required to commute with the BRST charge given by an expression analogous to Eq.(2.10),
\[ Q_{BRST} = \int dz C \left( D\Pi + \frac{1}{2k} J_2 \right) + \int d\bar{z} C \left( \bar{D}\bar{\Pi} + \frac{1}{2k} \bar{J}_2 \right) \quad (3.17) \]
The zero modes of the \( SL(2, R) \) currents may be represented as differential operators acting on \( T(R, \Phi_L, \Phi_R) \) of the form
\[ J_{02} = \theta \frac{\partial}{\partial \Phi_L} \]
\[ J_{0\pm} = \theta e^{\pm i\Phi_R} \left[ \frac{\partial}{\partial R} \mp \frac{i}{\sinh R} \left( \frac{\partial}{\partial \Phi_L} - \cosh R \frac{\partial}{\partial \Phi_R} \right) \right] \quad (3.18) \]
Upon requiring \( V \) in Eq.(3.16) to commute with \( Q_{BRST} \) in Eq.(3.17), we find using Eqs.(3.15), (3.18) that the Virasoro operators \( L_0, \bar{L}_0 \) are represented acting on \( T(R, \Phi_L, \Phi_R) \) by the operators
\[ L_0 = -\frac{1}{k} \Delta_0 (R, \Phi_L, \Phi_R) - \frac{1}{k} \frac{\partial^2}{\partial \Phi_R^2} \]
\[ \bar{L}_0 = -\frac{1}{k} \Delta_0 (R, \Phi_L, \Phi_R) - \frac{1}{k} \frac{\partial^2}{\partial \Phi_L^2} \quad (3.19) \]
with $\Delta_0$ as given by Eq.(2.20). As in Section 2, we use the physical state condition analogous to Eq.(2.21) to decompose $T(R, \Phi_L, \Phi_R)$ as

$$T(R, \Phi_L, \Phi_R) = T(R, \Phi) + \tilde{T}(R, \tilde{\Phi})$$

$$\Phi = \frac{1}{2}(\Phi_L - \Phi_R)$$

$$\tilde{\Phi} = \frac{1}{2}(\Phi_L + \Phi_R)$$

so that finally we find that acting on $T(R, \Phi)$, $L_0$ has the form

$$L_0 = -\frac{1}{k} \left( \frac{\partial^2}{\partial R^2} + \coth R \frac{\partial}{\partial R} + \frac{1}{4} \coth^2 \frac{R}{2} \frac{\partial^2}{\partial \Phi^2} \right)$$

Identifying $L_0$ with the Laplacian derived from the tachyon effective action, Eq.(2.18), according to Eq.(2.19), we find $g_{ij}, D$ given by

$$ds^2 = \frac{a'k}{4} \left( dR^2 + 4 \tanh \frac{R}{2} d\Phi^2 \right)$$

$$D = -\ln \cosh \frac{R}{2}$$

which we recognise as the solution Eq.(2.5) obtained by Witten in the bosonic case by simple integration over the gauge fields. In other words, Witten’s original black hole solution is exact in the supersymmetric case.

4. Checking the exact solutions

In this section we shall check the results of the previous two sections by showing that the proposed exact solutions in the bosonic and supersymmetric cases are consistent with perturbative results for the conformal invariance conditions up to four loop order. In the supersymmetric case, it is known that there is no contribution to the conformal invariance conditions at two [10] or three [11] loop order, and hence four-loop calculations [12] furnish the first non-trivial check beyond one loop. In the bosonic case, the exact solution of Eq.(2.26) has already been checked up to three-loop order [8], but nevertheless we feel it is worthwhile to pursue verification to the limit of available perturbative results.

The conformal invariance conditions for the $\sigma$-model, as given in Eq.(2.4), may be expressed as [18,19]

$$B_{g_{ij}} = \beta g_{ij} + 2\nabla(iS_j) + 2a'\nabla_i \partial_j D = 0$$

$$BD = \beta D + Si \partial_i D + a'\partial_i D \partial_i D = 0$$
where $\beta_{g_{ij}}, \beta D$ are the renormalisation group $\beta$-functions for the metric and dilaton respectively, and together with $S_i$ may be calculated perturbatively as a power series in $\alpha'$. $\beta_{g_{ij}}, \beta D$ and $S_i$ are known up to four-loop order for both the bosonic [20,21,22] and the $N = 1$ supersymmetric [10-12] $\sigma$-models (and up to 5 loops for the $N = 2$ supersymmetric $\sigma$-model [23]). In fact, if $\beta_{g_{ij}} = 0$ then $\beta D$ is guaranteed to be a constant [24], and in our case this constant is determined by the lowest order results. Hence we shall only need to focus our attention on Eq.(4.1a). If we postulate a solution to Eq.(4.1) of the form

$$ds^2 = dx^2 + e^{2\lambda(x)d\phi^2}, \quad D \equiv D(x), \quad (4.2)$$

then we may simply solve Eq.(4.1a) for $\lambda(x)$ and $D(x)$ as a power series in $\alpha'$. The Christoffel symbols for the metric in Eq.(4.2) are

$$\Gamma_{r\phi\phi} = -\lambda', \quad \Gamma_{\phi r\phi} = e^{2\lambda'} \quad (4.3)$$

and the Riemann tensor is given by

$$R_{klmn} = \frac{1}{2} R(g_{km}g_{ln} - g_{kn}g_{lm}) \quad (4.4)$$

with

$$R = -2(\lambda'' + \lambda'2). \quad (4.5)$$

Using the bosonic results for $\beta_{g_{ij}}$ and $S_i$ up to four-loop order collected in Ref. [19], we find

$$\lambda = \lambda_0 + \ln t + at^2 + a2(-3t^2 + 2t4)$$
$$+ a3 \left\{ \left[ -\frac{1}{6}\zeta(3) + \frac{64}{9} \right] t^2 + \left[ \frac{4}{15}\zeta(3) - \frac{518}{45} \right] t4 + \left[ -\frac{1}{10}\zeta(3) + \frac{86}{15} \right] t6 \right\} + \ldots$$

$$D = D_0 - \ln \cosh br + \frac{1}{2} at^2 + a2 \left[ -\frac{1}{2}t^2 + \frac{1}{2}t4 \right]$$
$$+ a3 \left\{ -3t^2 + \left[ \frac{1}{24}\zeta(3) + \frac{89}{36} \right] t4 + \left[ -\frac{1}{60}\zeta(3) - \frac{13}{30} \right] t6 \right\} + \ldots \quad (4.6)$$

where $\lambda_0$ and $D_0$ are constants, and where

$$t \equiv \tanh bx$$
$$a \equiv \alpha' b^2 \quad (4.7)$$

with $b$ a constant. On the other hand, the proposed exact solution given in Eq.(2.26) may be cast in the form of Eq.(4.2) by taking

$$bx \equiv \frac{1}{2} r, \quad a = \frac{1}{k-2} \quad (4.8)$$
and then it corresponds to taking
\[
\lambda = -\ln b + \ln t - \frac{1}{2} \ln \left(1 - \frac{2a}{1 + 2at^2}\right) \\
D = -\frac{1}{2} \ln 2b - \frac{1}{2} \ln \sinh 2bx + \frac{1}{2} \lambda(x)
\] (4.9)

which has the expansion
\[
\lambda = -\ln b + \ln t + at^2 + a2(-2t^2 + t4) + a3(4t^2 - 2t^4 - \frac{2}{3}t^6) + \ldots \\
D = -\frac{1}{2} \ln 4b - \ln \cosh bx + \frac{1}{2} at^2 + a2(-t^2 + \frac{1}{2}t^4) + a3(2t^2 - t^4 - \frac{1}{3}t^6) + \ldots
\] (4.10)

Comparing Eqs.(4.6) and (4.10) we find agreement at \(O(\alpha'0)\) and \(O(\alpha)\) (with appropriate choice of \(\lambda_0\) and \(D_0\) in Eq. (4.6)), but not at \(O(\alpha'2)\) or \(O(\alpha'3)\). However, in the \(\sigma\)-model approach we have the freedom to change the renormalisation scheme used to calculate the \(\beta\)-functions, which corresponds to making local covariant redefinitions of the metric and dilaton fields. Hence we should explore the possibility that Eqs. (4.6) and (4.10) are in fact related by such a legitimate field redefinition. When considering the effects of a field redefinition one can adopt two different (but equivalent) viewpoints; in the first case, which is the more usual in the \(\sigma\)-model context, one considers the conformal invariance conditions involving the \(\beta\)-functions modified by a field redefinition, and one can then seek a solution of these new conformal invariance conditions of the form given by Eq.(4.2). In the second case we start from a metric and dilaton given by Eqs. (4.2), (4.6) and simply modify them by a local covariant redefinition. The two procedures are manifestly equivalent, since if we redefine \(g = g(\bar{g})\), for instance, we have
\[
\bar{\beta}(\bar{g}) = \mu \frac{d}{d\mu} \bar{g} = \beta(g) \frac{\partial}{\partial g} \bar{g}
\] (4.11)

and hence \(\bar{\beta}(\bar{g}) = 0 \iff \beta(g) = 0\). The second approach is simpler and it is the one we shall adopt. There is one subtlety, however; a general local covariant redefinition of the metric will not leave the metric in the form Eq.(4.2) and so we have to make a co-ordinate change to restore the form Eq.(4.2). If we make a redefinition
\[
\bar{g}_{ij} = g_{ij} + T_{ij} \\
\bar{D} = D + K
\] (4.12)

(where \(T_{ij}\) and \(K\) are restricted to be local covariant quantities) then to recover the form Eq.(4.2) we also need to make a co-ordinate transformation
\[
\bar{x} = x + q(x)
\] (4.13)

where
\[
q' = \frac{1}{2} T_{xx}
\] (4.14)
so that the resulting combined field redefinition and co-ordinate transformation of $\lambda$, $D$ is

\[
\tilde{\lambda} = \lambda - q\lambda' + \frac{1}{2}e^{-2\lambda T_{\phi\phi}}
\]

\[
\tilde{D} = D - qD' + K
\]  

(4.15)

From the other point of view, in the case of a $\sigma$-model with a target space of arbitrary dimensions, at three and higher loops there are tensor structures in the $\beta$-functions which are “scheme-independent” and cannot be modified by field redefinitions [25]. However in the present case of a two dimensional target space, many curvature invariants which are in general distinct become related, and one might expect that there would no longer be any invariant structures. Nevertheless, a remnant of scheme independence persists, since it is not possible to make an arbitrary redefinition of $\lambda$ via Eq.(4.15). In fact, an $O(\alpha')$ covariant metric redefinition of the form Eq.(4.12) makes a change in $\lambda$ of the form

\[
\delta \lambda = \sum_{i=1}^{n} n\lambda_i t2i
\]  

with the constraint

\[
\sum \lambda_i = 0
\]  

(4.17)

Comparing Eqs.(4.6) and (4.10), we see that the difference between the respective solutions for $\lambda$ and $D$ is given by

\[
\delta \lambda = a2(-t2 + t4) + a3
\]

\[
+ \left\{ \left( -\frac{1}{6}\zeta(3) + \frac{28}{9} \right) t2 + \left( \frac{4}{15}\zeta(3) - \frac{428}{45} \right) t4 + \left( -\frac{1}{10}\zeta(3) + \frac{32}{5} \right) t6 \right\} + \ldots
\]

\[
\delta D = a2\left(\frac{1}{2}t2\right) + a3\left\{ -5t2 + \left( \frac{1}{24}\zeta(3) + \frac{125}{36} \right) t4 + \left( -\frac{1}{60}\zeta(3) - \frac{1}{10} \right) t6 \right\} + \ldots
\]

(4.18)

and it is clear that the constraint Eq.(4.17) is satisfied at $O(\alpha'^2)$ and $O(\alpha'^3)$. This analysis was first carried out at $O(\alpha'^2)$ in Ref. [8] where an explicit local covariant field redefinition was displayed which yields the correct $\delta \lambda$ and $\delta D$ at this order. At $O(\alpha'^3)$, the most general local field redefinition is obtained by taking in Eq. (4.12)

\[
T_{ij} = a_1R3g_{ij} + a_2\partial_iR\partial_jR + a_3\nabla_i\partial_jR + a_4\partial R.\partial R
\]

\[
+ a_5R\nabla^2Rg_{ij} + a_6\nabla^2\nabla^2Rg_{ij} + a_7\nabla_i\nabla_j\nabla^2R
\]

\[
K = b_1R3 + b_2\partial R.\partial R + b_3\nabla^2R + b_4\nabla^2\nabla^2R.
\]  

(4.19)

There is a certain amount of freedom in selecting a field redefinition which effects the required transformation; one possibility is

\[
a_1 = \frac{1}{256}\zeta(3) + \frac{5}{48}, \quad a_2 = a_3 = 0,
\]

\[
a_4 = \frac{17}{48}, \quad a_5 = a_6 = a_7 = 0, \quad b_1 = \frac{1}{1536}(\zeta(3) - 205),
\]

\[
b_2 = -\frac{77}{768}, \quad b_3 = b_4 = 0.
\]  

(4.20)
An interesting phenomenon is that the coefficients of the terms in Eq. (4.19) which involve total derivatives, namely \(a_6, a_7\) and \(b_4\), turn out to be related amongst themselves by a homogeneous equation

\[
4b_4 = 2a_6 - a_7,
\]

which means that \(a_6, a_7\) and \(b_4\) can be chosen to be all zero, independently of the values assigned to the other coefficients. The same effect can also be observed at the three-loop order, although its significance is still unclear.

Hence we have shown that up to \(O(\alpha' \Lambda)^3\), corresponding to four-loop order, the exact black hole solution of Eq.(2.26) is a solution of the \(\sigma\)-model conformal invariance conditions in some renormalisation scheme. In fact, the validity at this order of Witten’s original solution Eq.(2.5) in the \(N = 1\) supersymmetric case follows as an immediate corollary. The \(\beta\)-function for the \(N = 1\) supersymmetric \(\sigma\)-model is the same as in the bosonic case at one loop, zero at two and three loops, and at four loops can be obtained from the four-loop bosonic \(\sigma\)-model \(\beta\)-function by retaining only the \(\zeta(3)\) terms [21]. The \(\zeta(3)\) terms in Eq.(4.6), and hence Eq.(4.18), are therefore unchanged in passing from the bosonic to the supersymmetric case, and the analogue of Eq.(4.6) in the supersymmetric case consists of the zeroth order term and the \(\zeta(3)\) terms. Since the zeroth order term in Eq.(4.6) reproduces Witten’s original solution Eq.(2.5), and since the \(\zeta(3)\) terms in Eq.(4.6) have the property Eq.(4.17), it follows that the Witten solution Eq.(2.5) is valid up to four loops in the \(N = 1\) supersymmetric case. In fact, since the \(N = 1\) supersymmetric WZW model is known to possess \(N = 2\) supersymmetry [26], and since the five-loop contribution to the \(\beta\)-function for the \(N = 2\) supersymmetric \(\beta\)-function can be field-redefined to zero [23], we can deduce that the Witten solution is valid up to five-loop order.

Conclusions

We have proved that Witten’s original black-hole metric, Eq.(2.5), is exactly conformally invariant in the supersymmetric case. We have demonstrated this by explicit comparison with the \(\sigma\)-model conformal invariance conditions up to five-loop order. We have also shown that the exact solution for the bosonic case, proposed in Ref. [7] and given in Eq. (2.26), is consistent with the \(\sigma\)-model conformal invariance conditions up to four-loop order.

Finally, a comment is in order on the derivation of the exact solution in [7]. This derivation relies on identifying the Virasoro operators \(L_0, L_0\) with the one-loop string effective action for the tachyon field. However, the tachyon string effective action gets contributions at three and higher orders in perturbation theory in minimal subtraction. Nevertheless, if one included these contributions the effect would simply be a further local covariant redefinition of the metric and hence could not be detected by the methods used here. To this extent, there is an intrinsic ambiguity in the identification of the metric and
dilaton fields starting from the gauged WZW model.

Acknowledgements

We are indebted to Hugh Osborn and Arkady Tseytlin for useful and illuminating discussions. I. J. and J. P. thank the S. E. R. C. for support.

References

1. E. Witten, Phys. Rev. D44 (1991) 314.
2. C. G. Callan, R. C. Myers and M. J. Perry, Nucl. Phys. B311 (1988) 673; G. W. Gibbons and K. Maeda, Nucl. Phys. B298 (1988) 741; D. Garfinkle, G. T. Horowitz and A. Strominger, Phys. Rev. D43 (1991) 3140; G. T. Horowitz and A. Strominger, Nucl. Phys. B360 (1991) 197.
G. Mandal, A. Sengupta and S. Wadia, Mod. Phys. Lett. A6 (1991) 1685.
3. N. Ishibashi, M. Li and A. R. Steif, “Two-dimensional charged black holes in string theory”, Santa Barbara preprint UCSBTH-91-28; J. H. Horne and G. T. Horowitz, “Exact black string solutions in three dimensions”, Santa Barbara preprint UCSBTH-91-39;
P. Horava, “Some exact solutions of string theory in four and five dimensions”, Enrico Fermi Institute preprint EFI-91-57;
S. P. Khastgir and A. Kumar, “String effective action and two-dimensional charged black hole”, Bhubaneswar preprint IP/BBSR/91-31;
M. D. McGuigan, C. R. Nappi and S. A. Yost, “Charged black holes in two-dimensional string theory”, preprint IASSNS-HEP-91/57;
M. Gasperini, J. Maharana and G. Veneziano, “From trivial to non-trivial conformal string backgrounds via O(d, d) transformations”, preprint CERN-TH-6214/91.
4. E. Witten, Comm. Math. Phys. 92 (1984) 455.
5. P. di Vecchia and P. Rossi, Phys. Lett. B140 (1984) 334; P. di Vecchia, B. Durhuus and J. L. Petersen, Phys. Lett. B144 (1984) 245; D. Gonzales and A. N. Redlich, Phys. Lett. B147 (1984) 150; A. N. Redlich and H. J. Schnitzer, Phys. Lett. B193 (1987) 471.
6. P. Goddard, A. Kent and D. Olive, Phys. Lett. B152 (1985) 88; K. Gawędski and A. Kupiainen, Phys. Lett. B215 (1988) 119;
7. R. Dijkgraaf, E. Verlinde and H. Verlinde, “String propagation in a black hole geometry”, preprint IASSNS-HEP-91/22.
8. A. A. Tseytlin, Phys. Lett. B268 (1991) 175.
9. P. di Vecchia, V. G. Knizhnik, J. L. Petersen and P. Rossi, Nucl. Phys. B253 (1985) 701.
10. L. Alvarez-Gaumé, D. Z. Freedman and S. Mukhi, Ann. Phys. (N. Y.) 134 (1981) 85.
11. L. Alvarez-Gaumé, Nucl. Phys. B184 (1981) 180.
12. M. T. Grisaru, A. E. M. van de Ven and D. Zanon, Nucl. Phys. B277 (1986) 409.
13. C. M. Hull and B. Spence, Phys. Lett. B232 (1989) 204; Nucl. Phys. B353 (1991) 379; Mod. Phys. Lett. A6 (1991) 969; Nucl. Phys. B363 (1991) 593; I. Jack, D. R. T. Jones, N. Mohammedi and H. Osborn, Nucl. Phys. B332 (1990) 359.
14. A. Giveon, Mod. Phys. Lett. A6 (1991) 2843; E. B. Kiritsis, Mod. Phys. Lett. A6 (1991) 2871; M. Roček and E. Verlinde, “Duality, quotients and currents”, preprint IASSNS-HEP-91/68.
15. See P. Goddard and D. Olive, Int. J. Mod. Phys. A1 (1986) 303 for a review and further references.
16. S. Nojiri, “Superstring in two-dimensional black hole”, preprint FERMILAB-PUB-91/230-T.
17. J. Fuchs, Nucl. Phys. B286 (1986) 455; C. M. Hull and B. Spence, Phys. Lett. B241 (1990) 357.
18. G. M. Shore, Nucl. Phys. B286 (1987) 349.
19. A. A. Tseytlin, Phys. Lett. B178 (1986) 34; Nucl Phys. B294 (1987) 383.
20. D. Friedan, Phys. Rev. Lett. 51 (1980) 334; Ann. Phys. (N. Y.) 163 (1985) 316; A. A. Tseytlin, Nucl. Phys. B276 (1986) 391; S. J. Graham, Phys. Lett. B197 (1987) 543; A. P. Foakes and N. Mohammedi, Phys. Lett. B198 (1987) 359; Nucl. Phys. B306 (1988) 343; I. Jack, D. R. T. Jones and D. A. Ross, Nucl. Phys. B307 (1988) 531.
21. I. Jack, D. R. T. Jones and N. Mohammedi, Nucl. Phys. B322 (1989) 431.
22. I. Jack, D. R. T. Jones and N. Mohammedi, Nucl. Phys. B332 (1990) 330.
23. M. T. Grisaru, D. I. Kazakov and D Zanon, Nucl. Phys. B287 (1987) 189.
24. G. Curci and G. Paffuti, Nucl. Phys. B286 (1987) 399.
25. R. R. Metsaev and A. A. Tseytlin, Phys. Lett. B191 (1987) 354.
26. Y. Kazama and H. Suzuki, Nucl. Phys. B321 (1989) 232.