FINITE TREES INSIDE THIN SUBSETS OF $\mathbb{R}^d$

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Abstract. Bennett, Iosevich and Taylor proved that compact subsets of $\mathbb{R}^d$, $d \geq 2$, of Hausdorff dimensions greater than $\frac{d+1}{2}$ contain chains of arbitrary length with gaps in a non-trivial interval. In this paper we generalize this result to arbitrary tree configurations.

1. Introduction

We begin with a seminal result due to Tamar Ziegler, [12], which generalizes an earlier result due to Furstenberg, Katznelson and Weiss [6]. See also [3].

**Theorem 1.1.** Let $E \subset \mathbb{R}^d$, of positive upper Lebesgue density in the sense that

$$\limsup_{R \to \infty} \frac{\mathcal{L}^d(E \cap [-R, R]^d)}{(2R)^d} > 0,$$

where $\mathcal{L}^d$ denotes the $d$-dimensional Lebesgue measure. Let $E_\delta$ denote the $\delta$-neighborhood of $E$. Let $V = \{0, v^1, v^2, \ldots, v^{k-1}\} \subset \mathbb{R}^d$, where $k \geq 2$ is a positive integer. Then there exists $l_0 > 0$ such that for any $l > l_0$ and any $\delta > 0$ there exists \{x^1, \ldots, x^k\} $\subset$ $E_\delta$ congruent to $lV = \{0, lv^1, \ldots, lv^{k-1}\}$.

In particular, this result shows that we can recover every simplex similarity type and sufficiently large scaling inside a subset of $\mathbb{R}^d$ of positive upper Lebesgue density. It is reasonable to wonder whether the assumptions of Theorem 1.1 can be weakened, but the following result due to Falconer ([4]) (see also Maga [11]) shows that conclusion may fail even if we replace the upper Lebesgue density condition with the assumption that the set is of dimension $d$.

**Theorem 1.2.** ([11]) For any $d \geq 2$ there exists a full dimensional compact set $A \subset \mathbb{R}^d$ such that $A$ does not contain the vertices of any parallelogram. If $d = 2$, then given any triple of points $x^1, x^2, x^3, x^3 \in A$, there exists a full dimensional compact set $A \subset \mathbb{R}^2$ such that $A$ does not contain the vertices of any triangle similar to $\Delta x^1 x^2 x^3$.

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The general question is to study the distance graph with vertices in a compact set of a given Hausdorff dimension. (For more on graph theory, see [2].) More precisely, let $E$ be a compact subset of $\mathbb{R}^d$, $d \geq 2$, and view its points as vertices of a graph where two vertices $x, y$ are connected by an edge if $|x - y| = t$, with $| \cdot |$ denoting the Euclidean distance and $t$ a positive real number. Denote the resulting graph by $G_t(E)$. Theorem 1.2 says that if $d = 2$ and the Hausdorff dimension of $E$ is equal to 2, then $G_t(E)$ does not in general contain a triangle. The situation changes in higher dimensions, as was demonstrated by the first listed author of this paper and Bochen Liu in [8].

**Theorem 1.3.** ([8]) For every $d \geq 4$ there exists $s_0 < d$ such that if the Hausdorff dimension of $E$ is $> s_0$, then $E$ contains vertices of an equilateral triangle.

**Definition 1.4.** A path in a graph is a finite or infinite sequence of edges that connect a sequence of distinct vertices. A path of length $k$ connects a sequence of $(k+1)$-vertices, and we refer to this sequence of vertices as a $k$-chain.

Bennett and the two authors of this paper proved in [1] that if the Hausdorff dimension of $E \subset \mathbb{R}^d$, $d \geq 2$, is greater than $\frac{d+1}{2}$, then $G_t(E)$ contains an arbitrarily long path. More generally, they proved the following.

**Theorem 1.5.** (Theorem 1.7 in [1]) Suppose that the Hausdorff dimension of a compact set $E \subset \mathbb{R}^d$, $d \geq 2$, is greater than $\frac{d+1}{2}$. Then for any $k \geq 1$, there exists an open interval $\tilde{I}$ such that for any $\{t_i\}_{i=1}^k \subset \tilde{I}$ there exists a non-degenerate $k$-chain in $E$ with gaps $\{t_i\}_{i=1}^k$.

One of the key aspects of the proof of Theorem 1.5 is the following estimate.

**Theorem 1.6.** (Theorem 1.8 in [1]) Suppose that $\mu$ is a compactly supported non-negative Borel measure such that

\begin{equation}
\mu(B(x,r)) \leq Cr^s_\mu,
\end{equation}

where $B(x,r)$ is the ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$, for some $s_\mu \in (\frac{d+1}{2}, d]$. Then for any $t_1, \ldots, t_k > 0$ and $\epsilon > 0$,

\begin{equation}
\mu \times \mu \times \cdots \times \mu\{(x^1, x^2, \ldots, x^{k+1}) : t_i - \epsilon \leq |x^{i+1} - x^i| \leq t_i + \epsilon; \ i = 1, 2, \ldots, k\} \leq C\epsilon^k.
\end{equation}

For the purposes of this paper we are interested in the special case of Theorem 1.5 where all the $t_i$’s are equal. Our goal is to extend Theorem 1.5 to more general configurations.

**Definition 1.7.** A tree is an (undirected) graph in which any two vertices are connected by exactly one path.

Our main result is the following.
**Theorem 1.8.** Let $E \subset \mathbb{R}^d$, compact of Hausdorff dimension greater than $\frac{d+1}{2}$ and let $T$ be a tree on $k+1$ vertices. Then there exists a non-empty interval $I$ such that for all $t \in I$, $G_t(E)$ contains $T$ as a subgraph.

**Remark 1.9.** For an analogous result in sets of positive upper Lebesgue density, see a result due to Lyall and Magyar in [10].

2. **Proof of Theorem 1.8**

The proof of Theorem 1.8 is obtained by streamlining and extending the proof of Theorem 1.5 in a direct and simple way.

Let $T$ be a graph on $k+1$ vertices. Enumerate the vertices of $T$ and let $\mathcal{E}(T)$ denote the set of pairs $(i, j)$, $i < j$, such that the $i$th vertex is connected with the $j$th vertex by an edge. Let $\mu$ be a Borel measure supported on $E$ and define

$$T^\epsilon_{T,t}(\mu) = \int \cdots \int \left\{ \prod_{(i, j) \in \mathcal{E}(T)} \sigma^\epsilon_i(x^i - x^j) \right\} d\mu(x^1) \cdots d\mu(x^{k+1}).$$

It is not difficult to see that Theorem 1.8 would follow from the following estimates:

$$T^\epsilon_{T,t}(\mu) \leq C_k,$$

and

$$\liminf_{\epsilon \to 0} T^\epsilon_{T,t}(\mu) \geq c_k > 0,$$

where $t \in I$, a non-empty interval, $\epsilon > 0$ is taken sufficiently small, and both the upper and lower bounds of $c_k$ and $C_k$ respectively hold independently of $t \in I$ and $\epsilon < \epsilon_0$. We will prove that these estimates hold when the measure $\mu$ is replaced in each variable by the restriction of $\mu$ to an appropriate subset of $E$ of positive $\mu$-measure.

In the proof of Theorem 1.5 in [1], the upper bound was established using the observation that if $T f = \lambda * f$, where $\lambda$ is a compactly supported measure satisfying $|\hat{\lambda}(\xi)| \leq C|\xi|^{-\alpha}$ for some $\alpha > 0$ and $\mu$ is a compactly supported Borel measure satisfying $\mu(B(x, r)) \leq C r^s$ for some $s > d - \alpha$, then $T$ is a bounded operator from $L^2(\mu)$ to $L^2(\mu)$. The lower bound was established using an inductive procedure generalizing the argument due to the authors of this paper and Mihalis Mourgoglou in [9]. In this paper we streamline the procedure by proving the upper bound and the lower bound at the same time.
The key feature of our argument is the following calculation.

**Lemma 2.1.** Set

\[ G = G_{t, \epsilon}(1) = \{ x \in E : c < \sigma^*_t \ast \mu(x) < 2^{m(1)} \}, \]

where \( c > 0 \) and \( m(1) \in \mathbb{N} \). There exists a non-empty open interval \( I \), an \( \epsilon_0 > 0 \), and a choice of \( c, m(1), \) and \( \delta > 0 \) so that

\[ \mu(G_{t, \epsilon}) > \delta > 0 \]

whenever \( t \in I \) and \( 0 < \epsilon < \epsilon_0 \).

To prove this result, let \( f_t^{\epsilon} = \sigma^*_t \ast \mu(x) \). It was proved in [1] that there exists a non-empty open interval \( I \) and an \( \epsilon_0 > 0 \) so that simultaneously the \( L^1(\mu) \) norm of \( f_t^{\epsilon} \) is uniformly bounded below and the \( L^2(\mu) \) norm is bounded above for all \( t \in I \) and \( 0 < \epsilon < \epsilon_0 \). Denote these uniform lower and upper bounds by \( 0 < \tilde{c} \) and \( C \) respectively. Let \( \epsilon, \) and \( t \) be such, and set \( f = f_t^{\epsilon} \).

Set \( G = G_{t, \epsilon}(1) = \{ x : c < f(x) < 2^{m(1)} \} \), where \( m(1) \in \mathbb{N} \) is to be determined. Now,

\[ c < \int f d\mu(x) = \int_{\{f \leq \tilde{c}\}} f d\mu(x) + \int_{G} f d\mu(x) + \sum_{l=m(1)}^{\infty} \int_{\{2^l \leq f \leq 2^{l+1}\}} f d\mu(x). \]

(2.4) It is a straight-forward consequence of Chebyshev’s inequality and Cauchy-Schwarz that \( \mu(\{2^l \leq f \leq 2^{l+1}\}) < C 2^{-2l} \). Plugging this into (2.4) and taking \( m(1) \) sufficiently large, it quickly follows that \( \mu(G) \) is bounded below away from zero with constants independent of \( \epsilon > 0 \) and \( t \).

By induction, using an identical argument to the one above, one can find the following nested sequence of sets of positive \( \mu \)-measure (where the lower bound on the measure is independent of \( t \in I \) and \( \epsilon \) small).

**Lemma 2.2.** For \( j \in \mathbb{N} \), set

\[ G_{t, \epsilon}(j + 1) = \{ x \in G_{t, \epsilon}(j) : c(j + 1) < \sigma^*_t \ast \mu_j(x) < 2^{m(j)} \}, \]

where \( \mu_j(x) \) denotes restriction of the measure \( \mu \) to the set \( G_{t, \epsilon}(j) \) and \( c(j + 1) > 0 \). Then there exists numbers \( m(j + 1) \in \mathbb{N}, c(j + 1) > 0, \) and \( \delta_{j+1} > 0, \) so that if \( t \in I \) and \( 0 < \epsilon < \epsilon_0, \) then

\[ \mu(G_{t, \epsilon}(j + 1)) > \delta_{j+1} > 0. \]

We now demonstrate the pigeon-holing argument that allows us to deduce (2.2) and (2.3) when \( \mu \) in each variable is appropriately restricted.
Fix \( k \in \mathbb{N} \), and let \( T \) be a tree on \( k + 1 \) vertices. We say that a vertex is isolated if it is connected to only one other vertex. Let \( V(1) \) denote the set of isolated vertices of \( T \), and let \( x^1, \ldots, x^{N(1)} \) denote the collection of vertices who are connected to at least on vertex in \( V(1) \). Let \( k_1, \ldots, k_{N(1)} \) denote the number of isolated vertices connected to \( x^1, \ldots, x^{N(1)} \) respectively.

Consider the expression in (2.1). Integrating in each \( v^j \in V(1) \), we replace each of the expressions

\[
\sigma^\epsilon_t (x^i - v^j) d\mu(v^j)
\]

whenever \( x^i \) is connected to \( v^j \). So, if \( v^{j_1}, \ldots, v^{j_{k_i}} \in V(1) \) are all connected to \( x^i \), then we get an expression of the form

\[
(\sigma^\epsilon_t * \mu(x^i))^{k_i}
\]

in the integrand.

The next step is to restrict the vertices \( x^1, \ldots, x^{N(1)} \) to the set \( G(1) \) as in Lemma 2.1. In this way, for each \( x^i \), the expression in (2.5) is bounded above and below by positive constants independent of \( t \in I \) and \( 0 < \epsilon < \epsilon_0 \). Due to the positivity of the integrand, we can now consider the expression in (2.1) with terms of the form in (2.5) removed. Finally, let \( T(2) \) denote the tree with all of the vertices in \( V(1) \) removed, and let \( T^\epsilon_{T,t}(2) \) denote the expression in (2.5) with the above mentioned modifications (so any evidence of the vertices in \( V(1) \) has been removed).

We repeat this process. For \( j \in \mathbb{N} \), let \( T(j + 1) \) denote the tree that is obtained after repeating this process \( j \)-times. Let \( V(j + 1) \) denote the set of isolated vertices of \( T(j + 1) \), and let \( y^1, \ldots, y^{N(j)} \) denote the collection of vertices who are connected to at least on vertex in \( V(j + 1) \). Let \( K_1, \ldots, K_{N(j)} \) denote the number of isolated vertices connected to \( y^1, \ldots, y^{N(j)} \) respectively.

Consider the expression in \( T^\epsilon_{T,t}(j+1) \). Integrating in each \( v^j \in V(j+1) \), we replace each of the expressions

\[
\sigma^\epsilon_t (y^i - v^j) d\mu_j(v^j)
\]

whenever \( y^i \) is connected to \( v^j \). So, if \( v^{j_1}, \ldots, v^{j_{k_i}} \in V(j + 1) \) are all connected to \( y^i \), then we get an expression of the form

\[
(\sigma^\epsilon_t * \mu_j(x^i))^{k_i}
\]

in the integrand.
The next step is to restrict the vertices $y^1, \ldots, y^N(j)$ to the set $G(j + 1)$ as in Lemma 2.2. In this way, for each $y^i$, the expression in $T_{j,t}^c(j + 1)$ is bounded above and below by positive constants independent of $t \in T$ and $0 < \epsilon < \epsilon_0$. Due to the positivity of the integrand, we can now consider the expression in $T_{j,t}^c(j + 1)$ with terms of the form in (2.6) removed. Finally, let $T(j + 2)$ denote the tree with all of the vertices in $V(j + 1)$ removed.

This procedure terminates after a finite number of steps, and we are left with an expression of the form

$$\int \int \sigma^t_\epsilon(z^1 - z^2) d\mu_{J_1}(z^1) d\mu_{J_2}(z^2) = \int \sigma^t_\epsilon \ast \mu_{J_1}(z^2) d\mu_{J_2}(z^2),$$

where we assume without loss of generality that $J_2 \geq J_1$ (so that $G(J_2) \subset G(J_1)$). If $J_2 > J_1$, then this expression is bounded above and below by positive constants independent of $0 < \epsilon < \epsilon_0$ and independent of $t \in T$. We obtain the same conclusion when $J_2 = J_1$ by simply restricting the variable $z^2$ to the set $G(J_3)$ defined above in Lemma 2.2.

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