Smallify: Learning Network Size while Training

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Abstract

As neural networks become widely deployed in different applications and on different hardware, it has become increasingly important to optimize inference time and model size along with model accuracy. Most current techniques optimize model size, model accuracy and inference time in different stages, resulting in suboptimal results and computational inefficiency. In this work, we propose a new technique called Smallify that optimizes all three of these metrics at the same time. Specifically we present a new method to simultaneously optimize network size and model performance by neuron-level pruning during training. Neuron-level pruning not only produces much smaller networks but also produces dense weight matrices that are amenable to efficient inference. By applying our technique to convolutional as well as fully connected models, we show that Smallify can reduce network size by 35X with a 6X improvement in inference time with similar accuracy as models found by traditional training techniques.

1 Introduction

Neural networks are used in an increasingly wide variety of applications on a diverse set of hardware architectures, ranging from laptops to phones to embedded sensors. This wide variety of deployment settings means that inference time and model size are becoming as important as prediction accuracy when assessing model quality. However, currently these three dimensions, prediction accuracy, inference time, and model size, are optimized independently, often with sub-optimal results.

Our approach to optimize the three dimensions also stands in contrast to existing techniques, which can be categorized into two general approaches: (1) quantization [1] and code compilation, techniques that can be applied to any network, and (2) techniques which analyze the structure of the network and systematically prune connections or neurons [2, 3]. While the first category is useful, it has limited impact on the network size. The second category can reduce the model size much more but has several drawbacks: first, those techniques often negatively impact model quality. Second, they can also (surprisingly) negatively impact inference time as they transform dense matrix operations into sparse ones, which can be substantially slower to execute on GPUs which do not support efficiently sparse linear algebra [2]. Third, these techniques generally start by optimizing a particular architecture for prediction performance, and then, as a post-processing step, applying compression to generate a smaller model that meets the resource constraints of the deployment setting. Because the network architecture is essentially fixed during this post-processing, model architectures that work better in small settings may be missed – this is especially true in large networks like many-layered CNNs, where it is infeasible to try explore even a small fraction of possible network configurations.

In contrast, in this paper we present a new and surprisingly simple method to simultaneously optimize network size and model performance. The key idea is to learn the right network size at the same time that we optimize for prediction performance. Our approach, called Smallify, starts with an over-sized network, and dynamically shrinks it by eliminating unimportant neurons—those that do not contribute

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to prediction performance—during training time. We achieve this, by introducing a new layer, called **SwitchLayer**, which can switch neurons on and off, and is co-optimized while training the neural net. Furthermore, the layer-based approach makes it not only easy to implement **Smallify** in various neural net frameworks, but also to use it as part of existing network architectures. **Smallify** has two main benefits. First, it explores the architecture of models that are both small and perform well, rather than starting with a high-performing model and making it small. **Smallify** accomplishes this goal by using a single new hyperparameter that effectively models the target network size. Second, in contrast to existing neural network compression techniques \[2, 4\], our approach results in models that are not only small, but where the weight matrices are dense, leading to better inference time.

In summary, our contributions are as follows:

1. We propose a novel technique based on dynamically switching on and off neurons, which allows us to optimize the network size while the network is trained.
2. We extend our technique to remove entire neurons, leading not only to smaller networks, but also dense matrices, which yield improved inference times as networks shrink. Furthermore, our switching layers used during training can be safely removed before the model is used for inference, meaning they add no additional overhead at inference time.
3. We show that our technique is a relaxation of group LASSO \[5\] and prove that our problem admits many global minima.
4. We evaluate **Smallify** with both fully-connected as well as convolutional neural networks. For CIFAR10, we achieve the same accuracy as a traditionally trained network while reducing the network size by a factor of 2.2X. Further, while sacrificing just 1% of performance, **Smallify** finds networks that are 35X smaller. All in all, this leads to speedups in inference time of up to 6X.

## 2 Related Work

There are several lines of work related to optimizing network structure.

**Hyperparameter optimization techniques:** One way to optimize network architecture is to use hyperparameter optimization. Although many methods have been proposed for hyperparameter optimization, simple techniques such as randomized search have been shown to work surprisingly well in practice \[6, 7\]. Alternative more advanced techniques include Bayesian techniques and/or various bandit algorithms (e.g. \[8, 9\]). Although these methods can be used to tune the size of each layer in a network, in practice, related work presents limited experimental evidence regarding this, likely because treating each layer as a hyperparameter would lead to an excessively large search space. In contrast, with **Smallify**, the size of the network can be tuned with a single parameter. Recently, methods based on reinforcement learning have been proposed (\[10, 11\]) and shown to generate very accurate networks (NAS-Net). However as stated in \[10\], they still used the popular heuristic that doubles the number of channels every time the dimension of features is reduced without challenging it.

**Model Compression:** Model compression techniques focus on reducing the model size after training, in contrast to **Smallify**, which reduces it while training. Optimal brain damage \[3\] identifies connections in a network that are unimportant and then prunes these connections. DeepCompression \[2\] takes this one step further and in addition to pruning connections, it quantizes weights to make inference extremely efficient. A different vein of work such as \[12, 13\] proposes techniques for distilling a network into a simpler network or a different model. Because these techniques work after training, they are orthogonal and complementary to **Smallify**. Further, some of these techniques, e.g., \[14, 3\], produce sparse matrices that are not likely to improve inference times even though they reduce network size.

**Dynamically Sizing Networks**. The techniques closest to our proposed method are those based on group sparsity such as \[15, 16\], nuclear norm \[17\], low-rank constraints \[18\], exclusive sparsity \[19\], and even physics-inspired methods \[20\]. In \[21\], authors look beyond removing channels and experiment with shape and depth. In \[22\], the authors propose a method called Adaptive Radial-Angular Gradient Descent that adds and removes neurons on the fly via an \(l_2\) penalty. This approach requires a new optimizer and takes longer to converge compared to **Smallify**. \[23\] is similar to **Smallify** in that they both scale each channel/neuron by a scalar. Our approach is more general since
it can be used with any architecture, does not depend on batch normalization layers, and in contrast to [23] we propose some implementation details to make the framework more practical in section 4. Most of these methods train for sparsity and deactivate neurons at the end of the training process except [17] that do a single step of garbage collection at epoch 15. Our pipeline allows early detection of the least important neurons/channels and take advantage from it to speed up training.

3 The Smallify Approach

In this section we describe the Smallify approach. We discuss first the new SwitchLayers which are used to deactivate neurons, followed by a description of how we adapt the training loss function.

3.1 Overview

At a high-level, our approach consists of two interconnected stages. The first one identifies neurons that do not improve the prediction accuracy of the network and deactivates them. The second stage then removes neurons from the network (explicitly shrinking weight matrices and updating optimizer state) thus leading to smaller networks and faster inference.

Deactivating Neurons On-The-Fly: During the first stage, Smallify applies an on/off switch to every neuron of an initially over-sized network. We model the on/off switches by multiplying each input (or output) of each layer by a parameter $\beta \in \{0, 1\}$. A value of 0 will deactivate the neuron, while 1 will let the signal go through. These switches are part of a new layer, called the SwitchLayer; this layer applies to fully connected as well as convolutional layers.

Our objective is to minimize the number of “on” switches to reduce the model size as much as possible while preserving prediction accuracy. This can be achieved by jointly minimizing the training loss of the network and applying an $l_0$ norm to the $\beta$ parameters of the SwitchLayer. Since minimizing the $l_0$ norm is an NP-Hard problem, we instead relax the constraint to an $l_1$ norm by constraining $\beta$ to be a real number instead of a binary value.

Neuron Removal: During this stage, the neurons that are deactivated by the switch layers are actually removed from the network, effectively shrinking the network size. This step improves inference times. We choose to remove neurons at training time because we have observed that this allows the remaining active neurons to adapt to the new network architecture and we can avoid a post-training step to prune deactivated neurons.

Next we describe in detail the switch layer as well as and the training process for Smallify, and then describe the removal process in Section 4.

3.2 The Switch Layer

Let $L$ be a layer in a neural network that takes an input tensor $x$ and produces an output tensor $y$ of shape $(c \times d_1 \times \cdots \times d_n)$ where $c$ is the number of neurons in that layer. For instance, for fully connected layers, $n=0$ and the output is single dimensional vector of size $c$ (ignoring batch size for now) while for a 2-D convolutional layer, $n=2$ and $c$ is the number of output channels or feature maps.

We want to tune the size of $L$ by applying a SwitchLayer, $S$, containing $c$ switches. The SwitchLayer is parametrized by a vector $\beta \in \mathbb{R}^c$ such that the result of applying $S$ to $L(x)$ is a also a tensor size $(c \times d_1 \times \cdots \times d_n)$ such that:

$$S_\beta(L(x))_{i,...} = \beta_i L(x)_{i,...} \forall i \in [1 \ldots c]$$

(1)

Once passed through the switch layer, each output channel $i$ produced by $L$ is scaled by the corresponding $\beta_i$. Note that when $\beta_i = 0$, the $i^{th}$ channel is multiplied by zero and will not contribute to any computation after the switch layer. If this happens, we say the switch layer has deactivated the neuron corresponding to channel $i$ of layer $L$.

We place SwitchLayer after each layer whose size we wish to tune; these are typically fully connected and convolutional layers. We discuss next how to train Smallify.
We can prove that under the condition:

\[ \forall \lambda \text{ with } A \text{ where } \lambda \in \mathbb{R} \]

which we will discuss in the following subsection.

This expression augments the regular training loss with a regularization term for the switch parameters and another on the network weights. Interestingly, there exists a connection between Smallify and group sparsity regularization (LASSO) which we will discuss in the following subsection.

3.3 Training Smallify

For training, we need to account for the effect of the SwitchLayers on the loss function. The effect of SwitchLayers can be expressed in terms of a sparsity constraint that pushes values in the \( \beta \) vector to 0. In this way, given a neural network parameterized by weights \( \theta \) and switch layer parameters \( \theta \), we optimize Smallify loss as:

\[
L_{SN}(x, y; \theta, \beta) = L(x, y; \beta) + \lambda \| \beta \|_1 + \lambda_2 \| \theta \|_p^p
\]

This expression augments the regular training loss with a regularization term for the switch parameters and another on the network weights.

3.4 Relation to Group Sparsity (LASSO)

Smallify removes neurons, i.e., inputs/outputs of layers. For a fully connected layer defined as:

\[
f_{A,b}(x) = a(Ax + b)
\]

where \( A \) represents the connections and \( b \) the bias, removing an input neuron \( j \) is equivalent to having \( (A^T)_{j} = 0 \). Removing an output neuron \( i \) is the same as setting \( A_i = 0 \) and \( b_i = 0 \). Solving optimization problems while trying to set entire group of parameters to zero is the goal of group sparsity regularization [15]. In any partitioning of the set of parameters \( \theta \) defining a model in \( p \) groups: \( \theta = \bigcup_{i=1}^{p} \theta_i \), group sparsity penalty is defined as:

\[
\Omega_{\lambda}^{gp} = \lambda \sum_{i=1}^{p} \sqrt{\text{card(} \theta_i \text{)}} \| \theta_i \|_2
\]

with \( \lambda \) being the regularization parameter. In fully-connected layers, the groups are either columns of \( A \) if we want to remove inputs, or rows of \( A \) and the corresponding entry in \( b \) if we want to remove outputs. For simplicity, we focus our analysis on the simple one-layer case. As filtering outputs does not make sense in this case, we only consider removing inputs. The group sparsity regularization then becomes (when \( \sqrt{n} \) is folded into the \( \lambda \))

\[
\Omega_{\lambda}^{gp} = \lambda \sum_{j=1}^{p} \| (A^T)_{j} \|_2
\]

Interestingly, group sparsity and Smallify try to achieve the same goal and are closely related. First let’s recall the two problems. In the context of approximating \( y \) with a linear regression from features \( x \), the two problems are:

**Smallify:** \( \min_{A, \beta} \| y - A \text{diag(} \beta \text{)} x \|_2^2 + \lambda \| \beta \|_1 \)  \hspace{1cm}  **Group sparsity:** \( \min_{A} \| y - Ax \|_2^2 + \Omega_{\lambda}^{gp} \)

We can prove that under the condition: \( \forall j \in [1, p], \| (A^T)_{j} \|_2 = 1 \) the two problems are equivalent by taking \( \beta_j = \| (A^T)_{j} \|_2^2 \), and replacing \( A \) by \( A \text{diag(} \beta \text{)}^{-1} \). However, if we relax this constraint then Smallify becomes non-convex and has no global minimum. The latter is true because one can divide \( \beta \) by an arbitrarily large constant and multiplying \( A \) by the same value. Fortunately, by adding an extra term to the Smallify regularization term we can avoid that problem and prove that:

\[
\min_{A, \beta} \| y - A \text{diag(} \beta \text{)} x \|_2^2 + \Omega_{\lambda}^{gp} + \lambda_2 \| A \|_p^p
\]

has global minimums for all \( p > 0 \). More specifically there are at least \( 2^k \), where \( k \) is the total number of components in \( \beta \). Indeed, for any solution, one can obtain the same output by flipping any sign in \( \beta \) and the corresponding entries in \( A \). This is the reason we defined the regularized Smallify penalty above in Eq. (3). In practice, we observed that \( p = 2 \) or \( p = 1 \) are good a choice; note that the latter will also introduce additional sparsity into the parameters because the \( l_1 \) is, thlest best convex approximation of the \( l_0 \) norm.
4 Smallify in Practice

In this section we discuss practical aspects of Smallify, including neuron removal and several optimizations.

On-The-Fly Neuron Removal. Switch layers are initialized with weights sampled from $\mathcal{N}(0, 1)$; their values change as part of the training process so as to switch on or off neurons. Using gradient descent, it is very unlikely that the unimportant components of $\beta$ will ever be exactly 0. In most cases, irrelevant neurons will see their SwitchLayer oscillate close to 0, while never reaching 0, influenced solely by the $L_1$ penalty. Our goal is to detect this situation and effectively force them to 0 to deactivate them. We evaluated multiple screening strategies but the most efficient and flexible one was the Sign variance strategy: At each update we measure the sign of each component of $\beta$ ($-1$ or $1$). We maintain two metrics: the exponential moving average (EMA) of its mean and variance. When the variance exceeds a predefined threshold, we assume that the neuron does not contribute significantly to the output, so we effectively deactivate it. This strategy is parametrized by two hyper-parameters, the threshold but also the momentum of the statistics we keep.

Preparing for Inference. With Smallify we obtain reduced-sized networks during training, which is the first steps towards faster inference. This networks are readily available for inference. However, because they include switch layers—and therefore more parameters—they introduce unnecessary overhead at inference time. To avoid this overhead, we reduce the network parameters by combining each switch layer with its respective network layer by multiplying the respective parameters before emitting the final trained network. As a result, the final network is a dense network without any switching layers.

Neural Garbage Collection. Smallify decides on-the-fly which neurons to deactivate. Since Smallify deactivate a large fraction of neurons, we must dynamically remove these neurons at runtime to not unnecessarily impact network training time. We implemented a neural garbage collection method as part of our library which takes care of updating the necessary network layers as well as updating optimizer state to reflect the neuron removal.

5 Evaluation

The goal of our evaluation is to explore (1) whether, by varying $\lambda$, Smallify can efficiently explore (in terms of number of training runs) the spectrum of high-accuracy models from small to large, on both CNNs and fully connected networks. Our results show that, for each network size, we obtain models that perform as well or better than Static Networks, trained via traditional hyperparameter optimization; (2) whether, because these smaller networks are dense, they result in improved inference times on both CPUs and GPUs; and (3) whether the Smallify approach results in network architectures that are substantially different than the best network architectures (in terms of relative number of neurons per layer) identified in the literature.

We implemented SwitchLayers and the associated training procedure as a library in pytorch [24]. The layer can be freely mixed with other popular layers such as convolutional layers, batchnorm layers, fully connected layers, and used with all the traditional optimizers. We use our implementation to evaluate Smallify throughout the evaluation section.

5.1 Can Smallify achieve good accuracy?

To answer this question we compare Smallify with a traditional network. In both cases, we need to perform hyperparameter optimization to explore different network sizes. We perform random search, which is an effective technique for this purpose [6]. We evaluate Smallify on two architectures. One for which it is not possible to explore the entire space of network architectures (VGG) and one for which it is possible to do so (3 layers perceptron).

We assume no prior knowledge on the optimal batch size, learning rate, $\lambda$ or weight decay ($\lambda_2$). Instead, we trained a number of models, randomly and independently selecting the values of these parameters from a range of values commonly used in practice. Training is done using the Adam optimizer [25]. We start with randomly sampled learning rate; we divide the learning rate by 10 every 5 consecutive epochs without improvement. We stop when the learning rate is under $10^{-7}$. We pick the epoch with the best validation accuracy after the size of network converged and report the
corresponding testing accuracy. We also measure the total size, in terms of number of floating point parameters, excluding the \textbf{SwitchLayers} because as described in Section 4 these are eliminated after training.

\subsection{Large Network Setting: CIFAR10}

\textit{CIFAR10} is an image classification dataset containing 60000 color images \((3 \times 32 \times 32)\), belonging to 10 different classes. We use it with the \textit{VGG16 network} \cite{vgg16}. We applied \textbf{Smallify} to the \textit{VGG16 network} by adding \textbf{SwitchLayers} after each \textit{BatchNorm} and each fully connected layer (except for the last layer). Recall that \textbf{Smallify} assume that the starting size of the network is an upper bound on the optimal size. Thus, we started with a network with \(2\times\) the original size for each layer.

As the baseline we use a fixed-sized network, which architecture is configured by a total of 13 parameters for the convolutional layers and \(2\) for the fully connected layers. \textbf{Smallify} effectively fuse all these parameters in a single \(\lambda\). However, for traditional conventional architectures where all of these parameters are free, it is infeasible to obtain a reasonable sample for such a large search space. To obtain a baseline, we therefore use the same conventional heuristic that the original VGG architecture and many other CNNs use, which doubles the number of channels after every \textit{MaxPool} layer. For \textit{Static Networks} we sample the size between \(0.1\) and \(2\times\) the size original one, designed for \textit{ImageNet}. We report the same numbers as we did for \textbf{Smallify} and we compare the two distributions.

The results are shown in the top figure of Fig. 1, with blue dots indicating models produced by \textbf{Smallify} and orange dots indicating static networks. model, we plot its accuracy and model size. The lines show the Pareto frontier of models in each of the two optimization settings. \textbf{Smallify} explore the trade-off between model size and accuracy more effectively. Note that the best performing \textbf{Smallify} model has 92.07\% accuracy which is identical to the accuracy of the static network, while the \textbf{Smallify} model is 2.22 times smaller. In addition, if we give up just 1\% error, \textbf{Smallify} find a model that is 35.5 times smaller than any static network that performs as good.

\subsection{Small Network Setting: COVERTYPE}

The \textit{COVERTYPE} \cite{cov} dataset contains 581012 descriptions of geographical area (elevation, inclination, etc...) and the goal is to predict the type of forest growing in each area. We picked this dataset for two reasons. First it is simple, such that we can reach good accuracy with only a few fully-connected layers. This is important because we want to show that \textbf{Smallify} find sizes as good as \textit{Static Networks}, even if we are sampling the entire space of possible network sizes. Second, Scardapane et al \cite{scardapane} perform their evaluation on this dataset, which allows us to compare the results obtained by our method with the method in \cite{scardapane}. We compare \textbf{Smallify} against the same architecture used in \cite{scardapane}, i.e., a three fully-connected layers network with no \textit{Dropout} \cite{dropout} and no \textit{BatchNorm}. In this case, for the \textit{Static Networks}, we independently sample the sizes of the three different layers to explore all possible architectures.

The results are shown in the top figure of Fig. 2. Here, \textit{Static} method finds models that perform well at a variety of sizes, because it is able to explore the entire parameter space. This is as expected; the fact that \textbf{Smallify} perform as well as the Static indicates that \textbf{Smallify} are doing an effective job of exploring the parameter space using just the single \(\lambda\) parameter. Note that the best performing \textbf{Smallify} models has 96.91\% accuracy while the best static model is only 96.66\% accurate, while the \textbf{Smallify} shrink model is 2.51 times smaller. In addition, if we give up just 0.5\% error, \textbf{Smallify} find a model that is 38.6X smaller than any static network with equivalent accuracy.

\subsection{Can \textbf{Smallify} speed up inference?}

The previous experiment showed that \textbf{Smallify} find networks of similar or better accuracy than static networks that are much smaller. As noted in the introduction, for some applications, compact models that offer fast inference times are as important as absolute accuracy. In this section, we study the relationship between accuracy, network size and inference time. To do this, we select the smallest model that achieves a given accuracy for the both \textbf{Smallify} and Static approach. For each model, we measure the time to run inference with the model. We then compute the ratio of the network size and inference time between \textbf{Smallify} and Static at each accuracy level, and plot them on the bottom of Figure 1 and 2. We limit our plots to the models with \(80\% - 100\%\) accuracy range because those are the ones that we consider to be practically useful.
The middle plot in each figure shows the ratio of model size between Smallify and Static (values >1 mean Smallify are smaller) at different accuracy levels. These figures show that size improvements are particularly significant for CIFAR10. In the range of accuracies we are interested in, improvements in size go from 4x to 40x. The fact that the COVERTYPE networks are not dramatically smaller is expected: as the distribution at the top of Figure 2 shows, the static method is able to explore most of the parameter search space.

For speedup, we experimented with both CPUs and GPUs. For each data set/GPU/CPU combination, we show results with batch size 1, as well as with a batch size large enough to fully utilize the hardware on each dataset and hardware configuration. Note that when using a batch size of 1 on GPU, we do not expect to (and do not) observe any improvement because inference times are very small (typically about 10 µs), such that setup time dominates overall runtime.

The bottom four graphs in each figure show the results. Again, the CIFAR10 results show the benefit of the Smallify approach most dramatically. On CPU, speedups range up to 6x depending on the batch size, with many models exceeding 3x speedup. In general, speedups are less than compression ratios, due to overheads in problem setup, invocation, and result generation in Python/PyTorch. On GPU, the speedups are less substantial because the CUDA benchmarking utility that we use for
evaluation can choose better algorithms for larger matrices which masks some of our benefit, although they are still often 1.5x–2x faster for large batch sizes.

A key takeaway of these speedup results is that, unlike local sparsity compression methods, our methods’ improvement on size translates directly to higher throughput at inference time [14].

5.3 Architectures obtained after convergence

Smallify effectively explore the frontier of model size and accuracy. For a given target accuracy, the size needed is significantly smaller than when we use the "channel doubling" heuristic commonly used to size convolutional neural networks. This suggests that this conventional heuristic may not in fact be optimal, especially when looking for smaller models. Empirically we observed this to often be the case. For example, during our experimentations on the MNIST [28] and FashionMNIST [29] datasets (not reported here due to space constraints), we observed that even though these datasets have the same number of classes, input features, and output distributions, for a fixed \( \lambda \) Smallify converged to considerably bigger networks in the case of FashionMNIST. This evidence shows that optimal architecture not only depends on the output distribution or shape of the data but actually reflects the dataset. This makes sense, as MNIST is a much easier problem than FashionMNIST.

To illustrate this point on a larger dataset, we show two examples of architectures learned by Smallify in Figure 3. In the plot, the dashed line shows the number of neurons in each layer of the original VGG net, and the shaded regions show the size of the Smallify as it converges (with the darkest region representing the fully converged network). Observe that the final network that is trained looks quite different in the two cases, with the optimal performing network appearing similar to the original VGG net, whereas the shrunken network allocates many fewer neurons to the middle layers, and then additional neurons to the final fewer layers.

![Figure 3: Evolution of the size of each layer over time (lighter: beginning, darker: end). On the left a simpler model with 90.5% accuracy, on the right a very large network performing 92.07%.

6 Conclusion

We presented Smallify, an approach to learn deep network sizes while training. Smallify employs a SwitchLayer, which deactivates neurons, as well as of a method to remove them, which reduces network sizes, leading to faster inference times. We demonstrated these claims on two well-known datasets, on which we achieved networks of the same accuracy as traditional neural networks, but up to 35X smaller, with inference speedups of up to 6X.

7 Appendix

Proposition 7.1. \( \forall (n, p) \in \mathbb{N}^2, \ y \in \mathbb{R}^n, \ x \in \mathbb{R}^p \lambda \in \mathbb{R}: \)

\[
\min_{A, \beta} ||y - Ax||_2^2 + \lambda \sum_{j=1}^p \left( (A^T)_j \right)_{1} = \begin{cases} 
\min_{A', \beta} ||y - A' \text{diag}(\beta)x||_2^2 + \lambda ||\beta||_1 \\
\text{s.t.} \forall j, 1 \leq j \leq p, \left( (A'^T)_j \right)_2 = 1
\end{cases}
\]

Proof. First, we prove that there is at least one global minimum. Then, we how to construct \( 2^k \) distinct solutions from a single global minimum. In order to prove this second statement, we first
show that for any solution \( A \) to the first problem, there exists a solution in the second with the exact same value, and vice-versa.

**Part 1** Assume we have a potential solution \( A \) for the first problem. We define \( \beta \) such that \( \beta_j = \| (A^T)_j \|_2^2 \), and \( A' = A (\text{diag} (\beta))^{-1} \). It is easy to see that the constraint on \( A' \) is satisfied by construction. Now:

\[
\| y - Ax \|_2^2 + \lambda \sum_{j=1}^{p} \left\| (A^T)_j \right\|_2 = \| y - A' \text{diag} (\beta) x \|_2^2 + \lambda \sum_{j=1}^{p} \left\| (A'^T)_j \beta_j \right\|_2
\]

\[
= \| y - A' \text{diag} (\beta) x \|_2^2 + \lambda \sum_{j=1}^{p} |\beta_j| \cdot 1 = \| y - A' \text{diag} (\beta) x \|_2^2 + \lambda \| \beta \|_1
\]

**Part 2** Assuming we take an \( A' \) that satisfies the constraint and a \( \beta \), we can define \( A = A' \text{diag} (\beta) \). We can apply the same operations in reverse order and obtain an instance of the first problem with the same value.

**Conclusion** There is no way these two problems have different minima, because we are able to construct a solution to a problem from the solution of the other while preserving the value of the objective.

**Proposition 7.2.** \( \| y - A \text{diag} (\beta) x \|_2^2 \) is not convex in \( A \) and \( \beta \).

**Proof.** To prove this we will take the simplest instance of the problem: where everything is a scalar. We have \( f(a, \beta) = (y - a\beta x)^2 \). For simplicity we’ll take \( y = 0 \) and \( x > 0 \). If we consider two candidates \( s_1 = (0, 2) \) and \( s_2 = (2, 0) \), we have \( f(s_1) = f(s_2) = 0 \). However \( f\left(\frac{2}{3}, \frac{2}{3}\right) = x > \frac{1}{2} f(0, 2) + \frac{1}{2} f(2, 0) \), which break the convexity property. Since we showed that a particular case of the problem is non-convex then necessarily the general case cannot be convex.

**Proposition 7.3.** \( \min_{A, \beta} \| y - A \text{diag} (\beta) x \|_2^2 + \lambda \| \beta \|_1 \) has no solution if \( \lambda > 0 \).

**Proof.** Let’s assume this problem has a minimum \( A^*, \beta^* \). Let’s consider \( 2A^* \cdot \frac{1}{2} \beta^* \). Trivially the first component of the sum is identical for the two solutions, however \( \lambda \| \frac{1}{2} \beta^* \| < \lambda \| \beta^* \| \). Therefore \( A^*, \beta^* \) cannot be the minimum. We conclude that this problem has no solution.

**Proposition 7.4.** For this proposition we will not restrict ourselves to single layer but the composition of an an arbitrary large (\( n \)) layers as defined individually as \( f_{A_k, \beta_k, b_k}(x) = a_k A_k \text{diag} (\beta_k) x + b_k \). Suppose the entire network is denoted by the function \( N(x) \). For \( \lambda > 0 \), \( \lambda_k > 0 \) and \( p \) we have that \( \min \| y - N(x) \|_2^2 + \Omega_{\lambda, \lambda_k, p} \) has at least \( 2^k \) global minimum where \( k = \sum_{i=1}^{n} \text{card}(\beta_i) \)

**Proof.** We split this proof into two parts. First we show that there is at least one global minimum, then we show how to construct \( 2^n - 1 \) other distinct solutions with the same objective.

**Part 1:** The two components of the expression are always positive so we know that this problem is bounded by below by 0. \( \Omega_{\lambda_k, \lambda_k, p} \) is trivially coercive. Since we have a sum of terms, all bounded by below by 0 and one of them is coercive, so the entire function admits at least one global minimum.

**Part 2:** Let’s consider one global minimum. For each component \( k \) of \( \beta_i \) for some \( i \). Negating it and negating the \( k^{th} \) column of \( A_k \) does not change the the first part of the objective because the two factors cancel each other. The two norms do not change either because by definition the norm is independent of the sign. As a result these two sets of parameters have the same value and by extension also a global minimum. It is easy to see that going from this global minimum, we can decide to negate or not each element in each \( \beta_i \). We have a binary choice for each parameter, there are \( k = \sum_{i=1}^{n} \text{card}(\beta_i) \) parameters, so we have at least \( 2^k \) global minima.

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