EXTENSIONS OF $\infty$-GROUP SHEAVES

PÁL ZSÁMBOKI

ABSTRACT. Let $\mathcal{X}$ be an $\infty$-topos, for example the $\infty$-category of simplicial sheaves on a Grothendieck site. Then $\infty$-group sheaves are group objects in $\mathcal{X}$. Let $A \in \text{Grp } \mathcal{X}$ be such a group object. Then as $\mathcal{X}$ is an $\infty$-topos, there exists a universal $BA$-fiber bundle $BA \sslash \text{Aut } A$. We make $q$ pointed, and show that as a pointed map, via the looping-delooping equivalence, it is a universal extension of group objects by $A$. In particular, semidirect products of group objects by $A$ are classified by $BA \sslash \text{Aut } A$.

CONTENTS

Introduction 1
1. Groups and principal bundles in an $\infty$-topos 2
1.1. Small object classifiers in an $\infty$-topos 2
1.2. Principal bundles and fiber bundles 4
2. Classifying group extensions and semidirect products 6
2.1. Preliminaries about pointed $\infty$-categories 7
2.2. Classifying group extensions and semidirect products 8
Conclusion 9
References 9

INTRODUCTION

The construction of Lie algebras on group sheaves as described in [SGA3, Exposé II] naturally involves extensions of group sheaves. We are interested in generalizing this construction to $\infty$-group sheaves and so were motivated to first examine extensions of $\infty$-group sheaves. Let us first review the Lie algebra of group sheaves construction.

Let $\mathcal{X}$ be a big site of a scheme $S$, and let $G \in \text{Grp } \mathcal{X}$ be a group sheaf. Then the tangent sheaf $TG = \text{Hom}_S(\text{Spec}(\mathcal{O}_S \oplus \varepsilon \mathcal{O}_S), G)$ fits into a semi-direct product of group sheaves $0 \to \text{Lie } G \to TG \to G \to 1.$ This corresponds to the adjoint action map $G \xrightarrow{\text{ad}} \text{Aut}_{\text{Grp}}(\text{Lie } G)$. One can show that this action is $\mathcal{O}_S$-linear. Applying the Lie functor to this action map, we get an $\mathcal{O}_S$-linear morphism

$$\text{Lie } G \to \text{Lie}(\text{Aut}_{\mathcal{O}_S} \text{Lie } G) \cong \text{End}_{\mathcal{O}_S}(\text{Lie } G),$$

which in turn gives the Lie bracket map.
One can see that in order to generalize this construction to the case of $\infty$-group sheaves, we need to get a handle on semidirect products of $\infty$-group sheaves. In particular, we need to have an analogue of the adjoint action map classifying a semidirect product. In this article, we classify semidirect products of $\infty$-group sheaves.

In section 1.1, we review small object classifiers in $\infty$-topoi [Lur09, §6.1.6]. In section 1.2, we review how one can use small object classifiers to classify fiber bundles [NSS14, §4.1]. Let $A \in \text{Grp}_X$ be a group object. Then there exists a universal $B A$-fiber bundle $B A \sslash Aut A$ in $\mathcal{X}$. After establishing some preliminary results about pointed $\infty$-categories in section 2.1, we prove our classification statement in section 2.2. We make $q$ pointed, and show that as a pointed map, via the looping-delooping equivalence, it is a universal extension of group objects by $A$. In particular, semidirect products of group objects are classified by $B A \sslash Aut A$.

Our interest in Lie algebras of $\infty$-group sheaves comes from the following. Let $G$ be an algebraic group on a scheme $S$, and let $X \to S$ be a morphism of schemes. Via the adjoint action map $G \to Aut(Lie G)$, we can get a twisting map $f_\ast B G \to f_\ast Form(Lie G)$, where the codomain is the stack of families of Lie algebras locally isomorphic to $Lie G$. It might be possible to compactify $f_\ast Form(Lie G)$ by letting the Lie algebras degenerate to Lie algebras on perfect complexes, which in turn could give information about limits in the stack of families of $G$-torsors $f_\ast B G$. But for this, we need to handle $Lie G$, in the case where $G$ is the automorphism $\infty$-group of forms of a perfect complex.

An analogous result to ours has been obtained in [Jar15, Theorem 9.66], but there only extensions of groupoid sheaves by group sheaves are classified, and not $\infty$-groups, which we need so that we can encode higher homotopies of perfect complexes.

We would like to thank Ajneet Dhillon, Nicole Lemire and Chris Kapulkin for the fruitful conversations on the topic of the article.

1. Groups and principal bundles in an $\infty$-topos

1.1. Small object classifiers in an $\infty$-topos.

**Notation 1.1.1.** Let $\mathcal{X}$ be an $\infty$-category. Then we let $\text{Arr} \mathcal{X} = \mathcal{X}^{\Delta^1}$ denote the codomain fibration. Let $S$ be a collection of morphisms in $\mathcal{X}$. Then let $\text{Arr}^S \mathcal{X} \subseteq \text{Arr} \mathcal{X}$ denote the full subcategory spanned by $S$. Suppose that $\mathcal{X}$ has pullbacks, and that $S$ is closed under pullbacks. Then we denote by $\text{Cart} \mathcal{X} \subseteq \text{Arr} \mathcal{X}$ the 2-full subcategory with morphisms the pullback squares, and by $\text{Cart}^S \mathcal{X} = \text{Cart} \mathcal{X} \cap \text{Arr}^S \mathcal{X}$.

**Remark 1.1.1.** The target map $\text{Arr} \mathcal{X} \to \mathcal{X}$ is a coCartesian fibration classified by the map $\mathcal{X} \to \mathcal{X}^{\Delta^1} \xrightarrow{\text{res}_1} \mathcal{X}$ denote the codomain fibration. Let $S$ be a collection of morphisms in $\mathcal{X}$. Then let $\text{Arr}^S \mathcal{X} \subseteq \text{Arr} \mathcal{X}$ denote the full subcategory spanned by $S$. Suppose that $\mathcal{X}$ has pullbacks, and that $S$ is closed under pullbacks. Then we denote by $\text{Cart} \mathcal{X} \subseteq \text{Arr} \mathcal{X}$ the 2-full subcategory with morphisms the pullback squares, and by $\text{Cart}^S \mathcal{X} = \text{Cart} \mathcal{X} \cap \text{Arr}^S \mathcal{X}$.

**Definition 1.1.2.** Let $\mathcal{X}$ be an $\infty$-category which admits pullbacks, and let $S$ be a collection of morphisms of $\mathcal{X}$ which is closed under pullbacks. Then we say that a morphism $X \to Y$ in $\mathcal{X}$
classifies \( S \), if it is a final object of the category \( \text{Cart}^S \mathcal{X} \). In this case, we also say that \( Y \in \mathcal{X} \) is a classifying object for \( S \), and that \( \pi \) is a universal map of property \( S \).

**Definition 1.1.3.** Suppose that \( \pi \) classifies \( S \). Consider the following zigzag.

\[
\text{Cart}^S \mathcal{X} \leftarrow (\text{Cart}^S \mathcal{X})_{/\pi} \overset{\text{cod}_{/\pi}}{\longrightarrow} \mathcal{X}_{/Y}
\]

The left-hand arrow is a trivial fibration by definition. Since the map \( \text{Cart}^S \mathcal{X} \overset{\text{cod}}{\longrightarrow} \mathcal{X} \) is the restriction to the full subcategory of cod-Cartesian edges of the Cartesian fibration \( \text{Arr}^S(\mathcal{X}) \overset{\text{cod}}{\longrightarrow} \mathcal{X} \), the right-hand arrow is a trivial fibration by Lemma 1.1.3.1. This explains why \( \pi \) is called a universal map of property \( S \): every morphism \( X' \overset{f}{\rightarrow} Y' \) of property \( S \) fits into an essentially unique Cartesian diagram of the form

\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow^f & \downarrow & \downarrow^\pi \\
Y' & \longrightarrow & Y.
\end{array}
\]

In this situation, the map \( c_f \) is called the map classifying \( f \).

**Lemma 1.1.3.1.** Let \( K \overset{p}{\longrightarrow} K' \) be a right fibration of simplicial sets, \( x \in K \) a vertex, and \( K_{/x} \overset{p_{/x}}{\longrightarrow} K'_{/p(x)} \) the induced map on overcategories. Then the map \( p_{/x} \) is a trivial fibration.

**Proof.** Let \( n \geq 0 \). Then a lifting problem

\[
\begin{array}{ccc}
\partial\Delta^n & \longrightarrow & K_{/x} \\
\downarrow & & \downarrow^{p_{/x}} \\
\Delta^n & \longrightarrow & K'_{/p(x)}
\end{array}
\]

corresponds to a lifting problem

\[
\begin{array}{ccc}
\Lambda^{n+1} & \longrightarrow & K \\
\downarrow & & \downarrow^p \\
\Delta^{n+1} & \longrightarrow & K',
\end{array}
\]

which has a solution, since \( p \) is a right fibration. \( \square \)

**Theorem 1.1.4.** [[Lur09, Theorem 6.1.6.8]]

Let \( \mathcal{X} \) be a presentable \( \infty \)-category. Then \( \mathcal{X} \) is an \( \infty \)-topos if and only if the following conditions are satisfied.

(1) Colimits in \( \mathcal{X} \) are universal.

(2) For all sufficiently large cardinals \( \kappa \), there exists a universal relatively \( \kappa \)-compact morphism in \( \mathcal{X} \).

**Notation 1.1.5.** Let \( \kappa \) be a regular cardinal, and let \( S \) denote the collection of relatively \( \kappa \)-compact morphisms. Then we let \( \text{Cart}^\kappa \mathcal{X} = \text{Cart}^S \mathcal{X} \). We denote a universal relatively \( \kappa \)-compact morphism by \( \text{Obj}_{\kappa} \overset{\pi}{\rightarrow} \text{Obj}_{\kappa}' \).
1.2. **Principal bundles and fiber bundles.** In this subsection, let us fix an infinity-topos $\mathcal{X}$.

**Definition 1.2.1.** Let $G \in \text{Grp}(\mathcal{X})$ be a group object and $P \in \mathcal{X}$ an object. Then a (right) $G$-action on $P$ is a morphism $P_* \to G$ in $\text{Grpd}(\mathcal{X})$ such that for all morphisms $[m] \to [n]$ in $\Delta$, the square

$$
\begin{array}{ccc}
P_n & \longrightarrow & P_m \\
\downarrow & & \downarrow \\
G_n & \longrightarrow & G_m
\end{array}
$$

is Cartesian, and we have $P_0 \simeq P$. Let $\text{Action-}G \subseteq \text{Grpd}(\mathcal{X})/G$ denote the full subcategory of right $G$-actions.

**Remark 1.2.1.1.** Let $G$ be a discrete group acting on a set $P$ from the right via the action map $P \times G \xrightarrow{\rho} P$. Then the square

$$
\begin{array}{ccc}
P \times G & \xrightarrow{\rho} & P \\
\downarrow & & \downarrow \\
G & \longrightarrow & *
\end{array}
$$

is Cartesian.

**Remark 1.2.1.2.** [NSS14, Proposition 3.15] shows that it is enough to assume less, and the proof needs even less than assumed.

**Theorem 1.2.2.** [NSS14, Theorem 3.19]

Let $G \in \text{Grp}(\mathcal{X})$ be a group object, and $X \in \mathcal{X}$ be an object. Then principal bundles are classified by morphisms $X \to B\!G$. That is, we have an equivalence

$$(\text{Action-}G) \times \mathcal{X} \{X\} \simeq \text{Map}(X, B\!G)$$

where the map $\text{Action-}G \to \mathcal{X}$ is given by taking a map of groupoids $P \to G$ to $\lim P$. The equivalence is given by taking geometrical realizations and restricting to $-1 \in \Delta_+$.

**Definition 1.2.3.** Let $V, X \in \mathcal{X}$ be objects. Then a $V$-fiber bundle over $X$ is a map $E \xrightarrow{p} X$ such that there exists an effective epimorphism $U \twoheadrightarrow X$, and a Cartesian square

$$
\begin{array}{ccc}
U \times V & \longrightarrow & E \\
\downarrow^{pr_U} & & \downarrow^{p} \\
U & \longrightarrow & X.
\end{array}
$$

The space of $V$-fiber bundles is the full subcategory $\text{Bun}_V \subseteq \text{Cart}_{\mathcal{X}}$ of $V$-fiber bundles.

**Proposition 1.2.4.** Let $V \in \mathcal{X}$ be a small object, ie. a $\kappa$-compact object for some regular cardinal $\kappa$. Then every $V$-fiber bundle is relatively $\kappa$-compact.

**Proof.** This is a special case of [Lur09, Lemma 6.1.6.5].
Corollary 1.2.4.1. We have \( \text{Bun}_V \subseteq \text{Cart}_\kappa^X \).

Definition 1.2.5. Let \( V \in \mathcal{X} \) be a \( \kappa \)-compact object. Then its inner automorphism group \( \text{Aut} V \) with an action \( \rho_V : \text{Aut} V \curvearrowright V \) is defined via the following diagram of Cartesian squares.

\[
\begin{array}{ccc}
V & \rightarrow & V \parallel \text{Aut} V \rightarrow \text{Obj}_k \\
\downarrow & & \downarrow \pi \\
* & \rightarrow & \mathbb{B} \text{Aut} V \leftarrow \text{Obj}_k \\
\end{array}
\]

Proposition 1.2.6. [NSS14, Proposition 4.10]

Let \( E \xrightarrow{p} X \) be a \( V \)-fiber bundle. Then its classifying map \( X \xrightarrow{c_p} \text{Obj}_\kappa \) factors through the monomorphism \( \mathbb{B} \text{Aut} V \hookrightarrow \text{Obj}_\kappa \), resulting in the following pasting diagram.

\[
\begin{array}{ccc}
E & \rightarrow & V \parallel \text{Aut} V \rightarrow \text{Obj}_k \\
\downarrow & & \downarrow \pi \\
X & \rightarrow & \mathbb{B} \text{Aut} V \leftarrow \text{Obj}_k \\
\end{array}
\]

Theorem 1.2.7. We have a zigzag of trivial fibrations

\[ \text{Bun}_V \leftrightarrow (\text{Bun}_V)_q \rightarrow \mathcal{X} \mathcal{B} \text{Aut} V, \]

that is \( q \) is a universal \( V \)-fiber bundle.

Proof. Let us denote the right-hand Cartesian square of Definition 1.2.5 by \( \alpha \). Then the map \( \mathbb{B} \text{Aut} V \xrightarrow{i} \text{Obj}_\kappa \) is a monomorphism if and only if \( i \in \mathcal{X} \parallel \text{Obj}_\kappa \) is \((-1)\)-truncated, which in turn is equivalent to \( \alpha \in (\text{Cart}_\kappa^X)/\pi \) being \((-1)\)-truncated, that is \( \alpha \) is a monomorphism. Therefore, the induced map \( (\text{Cart}_\kappa^X)/q \rightarrow (\text{Cart}_\kappa^X)/\pi \) is fully faithful by Lemma 1.2.7.1. Thus, the commutative square

\[
\begin{array}{ccc}
(\text{Bun}_V)_q & \xrightarrow{\cong} & (\text{Cart}_\kappa^X)/q \\
\downarrow & & \downarrow \approx \\
\text{Bun}_V & \leftarrow & \text{Cart}_\kappa^X \end{array}
\]

is Cartesian, which proves our claim.

\[ \square \]

Lemma 1.2.7.1. Let \( X \xrightarrow{f} Y \) be a morphism in an \( \infty \)-category \( \mathcal{C} \). Suppose that \( f \) is a monomorphism. Then the postcomposition map \( \mathcal{C}_{/X} \xrightarrow{f^*} \mathcal{C}_{/Y} \) is a monomorphism, ie. fully faithful.

Proof. It will be enough to show that the unit map \( \text{id}_{\mathcal{C}_{/X}} \xrightarrow{\eta} f^* f_! \) is an equivalence. Consider a bundle \( P \xrightarrow{p} X \). It gives the following diagram.

5
Since \( f \) is a monomorphism, so is its pullback \((fp)^*f\). Since \(((fp)^*f)\eta_p = \text{id}_p\), by the uniqueness of the epi-mono factorization, the map \( \eta_p \) is an equivalence, as required.

\[ \square \]

### 2. Classifying group extensions and semidirect products

**Definition 2.0.8.** Let \( \mathcal{X} \) be an \( \infty \)-topos, and \( G, A \) two group objects in it. A group extension of \( G \) by \( A \) is a fibration sequence

\[ A \rightarrow \tilde{G} \rightarrow G \]

in \( \text{Grp} \mathcal{X} \). That is, group extensions of \( G \) by \( A \) are equivalent to pointed fibration sequences of the form

\[ B A \rightarrow B \tilde{G} \rightarrow B G \]

If we could interpret the latter as \( BA \)-fiber bundles in \( \mathcal{X} \), then they could be classified as such. Unfortunately, this is not possible, due to the fact that if \( \mathcal{X} \) is nontrivial, then \( \mathcal{X} \) is not an \( \infty \)-topos.

**Proposition 2.0.9.** Let \( \mathcal{X} \) be a pointed \( \infty \)-topos. Then \( \mathcal{X} \) is contractible.

**Proof.** Let \( X \in \mathcal{X} \) be an object. Then we have a canonical map \( X \rightarrow * \). Since \( * \) is also an initial object, \( X \) needs to be initial too [Lur09] Lemma 6.1.3.6]. This shows that every object in \( \mathcal{X} \) is initial, and thus \( \mathcal{X} \) is contractible.

\[ \square \]

In Section 2.1, we show that the forgetful functor \( \mathcal{X} \xrightarrow{U} \mathcal{X} \) has as left adjoint the adjoining a disjoint point functor \( \mathcal{X} \xrightarrow{\sqcup} \mathcal{X} \), and that a pointed square \((\Lambda^2_2)^\circ \xrightarrow{d} \mathcal{X} \) is Cartesian if and only if its image \((\Lambda^2_2)^\circ \xrightarrow{d} \mathcal{X} \) is so. Therefore, we can define the \( \infty \)-category of group extensions by \( A \) as follows.

**Definition 2.0.10.** The \( \infty \)-category of group extensions by \( A \) is the full subcategory \( \text{Ext}_A \subseteq \text{Cart} \mathcal{X} \) on maps with codomain of the form \( B G \) for some \( G \in \text{Grp} \mathcal{X} \), and with fiber \( BA \). Note that by construction the domain is also the delooping of a group object. Let \( G \in \text{Grp} \mathcal{X} \) be a group object. Then \( \text{Ext}(G, A) = \text{Ext}_A(BG) \) is the classifying space of extensions of \( G \) by \( A \).

We can make the universal \( BA \)-fiber bundle \( BA \xrightarrow{q} B \text{Aut}BA \) pointed in such a way that the Cartesian diagram

\[
\begin{array}{ccc}
  BA & \xrightarrow{q} & B \text{Aut}BA \\
  \downarrow & & \downarrow \\
  * & \xrightarrow{e_BA} & B \text{Aut}BA
\end{array}
\]

is pointed. Then in section 2.2, we show that \( q \) as a pointed map is a universal group extension by \( A \).
2.1. Preliminaries about pointed $\infty$-categories.

**Notation 2.1.1.** Let $\mathcal{X}$ be an $\infty$-category with a final object. Then let $\mathcal{X} \to \mathcal{X}$ denote the forgetful map, and let $\mathcal{X} \to \mathcal{X}^*$ denote the map adding a disjoint point.

**Proposition 2.1.2.** Let $\mathcal{X}$ be an $\infty$-category with a final object. Then the pair $(F, U)$ is an adjunction.

*Proof.* Let $X \in \mathcal{X}$ and $(Y, y) \in \mathcal{X}^*$. We want the inclusion map $X \to X \sqcup^* Y$ to serve as a unit map. Then we need to show that the composite

$$\text{Map}_{\mathcal{X}}(X \sqcup^* Y) \xrightarrow{U} \text{Map}_{\mathcal{X}}(X \sqcup^* Y) \xrightarrow{\eta_X} \text{Map}_{\mathcal{X}}(X, Y)$$

is an equivalence. By [Lur09, Lemma 5.5.5.12 and Theorem 4.2.4.1], we have a pasting diagram

$$\begin{array}{ccc}
\text{Map}_{\mathcal{X}}(X \sqcup^* Y) & \xrightarrow{U} & \text{Map}_{\mathcal{X}}(X \sqcup^* Y) \\
\downarrow & & \downarrow \\
\ast & \xrightarrow{(y)} & \text{Map}_{\mathcal{X}}(\ast, Y) \xrightarrow{\eta_X} \ast,
\end{array}$$

which proves our claim. \[\square\]

**Proposition 2.1.3.** Let $* \sqcup \Lambda^2_2 \xrightarrow{\partial_*} \mathcal{X}$ be a pointed square. Then it is Cartesian precisely when its image $* \sqcup \Lambda^2_2 \xrightarrow{U \partial_*} \mathcal{X}$ is Cartesian.

*Proof.* If $\partial_*$ is Cartesian, then so is $U \partial_*$, since $U$ is a right adjoint, which shows necessity. To prove sufficiency, let us suppose that $U \partial_*$ is Cartesian. Let us denote by $\sigma_*$ the restriction $\partial_*|\Lambda^2_2$. We need to show that for any $n \geq 0$, any lifting problem of the form

$$\begin{array}{ccc}
\partial \Delta^n & \to & (\mathcal{X})/_{\partial_*} \\
\downarrow & & \downarrow \\
\Delta^n & \to & (\mathcal{X})/_{\partial_*}
\end{array}$$

has a solution. But such a lifting problem is the same as a lifting problem of the form

$$\begin{array}{ccc}
* \sqcup \partial \Delta^n & \to & (\mathcal{X})/_{U \partial_*} \\
\downarrow & & \downarrow \\
* \sqcup \Delta^n & \to & (\mathcal{X})/_{U \partial_*}
\end{array}$$

which has a solution, since $U \partial_*$ is Cartesian. \[\square\]
2.2. Classifying group extensions and semidirect products. Now we are ready to classify group extensions. Let us fix a group object \( A \in \text{Grp} \).

**Theorem 2.2.1.** Let \( B A \mapsto \text{Aut}(BA) \) denote the universal \( BA \)-fiber bundle in \( \mathcal{X} \). Let us make \( q \) a pointed map in such a way that the Cartesian diagram

\[
\begin{array}{ccc}
BA & \rightarrow & BA \mapsto \text{Aut}(BA) \\
\downarrow & & \downarrow q \\
* & \rightarrow & B \text{Aut}(BA)
\end{array}
\]

is pointed. Then \( q \) is a universal group extension by \( A \).

**Corollary 2.2.1.1.** Semidirect products of \( G \) and \( A \) are classified by pointed morphisms \( BG \rightarrow BA \mapsto \text{Aut}A \).

**Remark 2.2.1.2.** Suppose that \( A \) is an \( E_2 \)-group. Then an \( \infty \)-group extension \( A \rightarrow \hat{G} \rightarrow G \) is called central, when the map \( BA \rightarrow \hat{B} G \) is deloopable. The fibration sequence \( BA \rightarrow * \rightarrow B^2A \) is classified by a map \( B^2A \rightarrow \text{Aut}BA \). Therefore, a group extension of \( G \) by \( A \) is central precisely when its classifying map factors through \( Z \). This is why the classifying space of central extensions is defined as \( \text{Map}(BG, B^2A) \) in [NSS14]. Note also that the map classifying a semidirect product which is central factors through \( * \), and is thus trivial.

**Proof of Theorem 2.2.1.** We claim that \( q \in \text{Ext}_A \) is a final object, that is the restriction map \( (\text{Ext}_A)_q \rightarrow \text{Ext}_A \) is a trivial fibration. Let \( n \geq 0 \), and suppose given a lifting problem of the following form

\[
\begin{array}{ccc}
\partial \Delta^n & \rightarrow & (\text{Ext}_A)_q \\
\downarrow & & \downarrow \\
\Delta^n & \rightarrow & \text{Ext}_A.
\end{array}
\]

This gives rise to the following lifting problem

\[
\begin{array}{ccc}
\Delta^{[1,n+1]} \cup_{\partial \Delta^{[1,n+1]}} \Lambda_n^{[1,n+1]+} & \rightarrow & \text{Cart}_{\mathcal{X}} \\
\downarrow & & \downarrow \\
\Delta^{[0,n+1]} \cup_{\Lambda_0^{[0,n+1]}} \Lambda^{[1,n+1]} & \rightarrow & \mathcal{X},
\end{array}
\]

where we denote by \( \Delta^{[1,n+1]} \subseteq \Delta^{n+2} \) the sub-simplicial set with all subsets of \([0, n+2]\) not containing \([1, n+1]\). One can check that the inclusion \( \Delta^{[1,n+1]} \cup_{\partial \Delta^{[1,n+1]}} \Lambda_n^{[1,n+2]} \subset \Delta^{[0,n+1]} \cup_{\Lambda_0^{[0,n+1]}} \Delta^{[1,n+1]} \) is actually \( \partial \Delta^{[1,n+2]} \subset \Lambda_n^{n+2} \). This inclusion is right anodyne, because a lifting problem along it can be understood as a lifting problem along \( \partial \Delta^{[1,n+1]} \times [1] \subset \partial \Delta^{[1,n+1]} \times 1 \) by extending \( \Delta^0 \) to \( \partial \Delta^{[1,n+1]} \times [0] \) in a degenerate way, and that is right anodyne [Lur09 Proposition 2.1.2.6]. Let \( \Lambda_0^{n+2} \rightarrow \text{Cart}_{\mathcal{X}} \) be a lift. Then by Proposition 2.1.3, the restriction \( U_{\tau BA} \) gives a lifting problem of the form...
Lemma 2.2.1.3. Let $n \geq 0$. Then the inclusion $(\partial d_{n+3}) \cup d_0 \subset \Lambda_{n+3}^{n+4}$ is inner anodyne.

Proof. It will be enough to show that the intersections with $d_j$ for $j \in \{0, \ldots, n+3, n+4\}$ are inner anodyne. For $j = 0$, we have $d_0 \subseteq d_0$. For $0 < j < n+3$, we have $d_0 \cup d_{j(n+3)} \subseteq d_j$, which is of the form $\Delta^{[0,j-1]} \star \Delta^{[j+1,n+3,n+4]} \cup \Delta^{[1,j-1]} \star \Delta^{[1,n+4]} \subset \Delta^{[0,j-1]} \star \Delta^{[1,n+4]}$, and that is inner anodyne, because $\Delta^{[1,j-1]} \subset \Delta^{[0,j-1]}$ is right anodyne by Lemma 2.2.1.4. For $j = n+4$, we have $d_0 \cup d_{(n+4)(n+4)} \subset d_{n+4}$, which is of the form $\Delta^{[0,1]} \star \Delta^{[2,n+4]} \cup \Delta^{[1]} \star \Delta^{[2,n+4]} \subset \Delta^{[0,1]} \star \Delta^{[2,n+4]}$, and that is inner anodyne, because $\Delta^{[1]} \subset \Delta^{[0,1]}$ is right anodyne by Lemma 2.2.1.4.

Lemma 2.2.1.4. Let $\ell > 0$. Then for all $0 < k \leq \ell$, the inclusions $\Delta^{[k,\ell]} \hookrightarrow \Delta^{\ell}$ are right anodyne.

Proof. Let us prove this by induction on $\ell$. This follows by definition for $\ell = 1$. Let $\ell > 1$ and $0 < k \leq \ell$. Since the inclusion $\Delta^{[k,\ell]} \hookrightarrow \Delta^{[1,\ell]}$ is right anodyne by hypothesis, we can assume that $k = 1$. Then we can see that the inclusion $\Delta^{[1,\ell]} \hookrightarrow \Delta^{\ell}$ is right anodyne by hypothesis, thus we are done.

Conclusion

As stated in Corollary 2.2.1.1, a semi-direct product of $\infty$-group sheaves $1 \to A \to \hat{G} \to G \to 1$ is classified by a pointed map $BG \to BA \amalg \text{Aut} BA$, which we can view as the $\infty$-categorical version of the adjoint action $G \to \text{Aut} A$.

Towards constructing Lie algebras of $\infty$-group sheaves, the next step will be to specialize to the case of the semidirect product $0 \to \text{Lie} G \to TG \to G \to 1$, which can be constructed analogously to the classical way in Homotopical Algebraic Geometry [TV08 §1.4.1]. Following [SGA3, Exposé II], we will first have to show that the adjoint action $BG \to B(\text{Lie} G) \amalg \text{Aut} B(\text{Lie} G)$ is $\partial S$-linear.

References

[SGA3] Michael Artin, Jean-Étienne Bertin, Michel Demazure, Pierre Gabriel, Alexander Grothendieck, Michel Raynaud, and Jean-Pierre Serre, Schémas en groupes (SGA 3) (Philippe Gille and Patrick Polo, eds.), Société Mathématique de France, 2011. ↑1, 9
[Jar15] John F. Jardine, *Local homotopy theory*, Springer Monographs in Mathematics, Springer, New York, 2015. MR3309296

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR2522659 (2010j:18001)

[NSS14] Thomas Nikolaus, Urs Schreiber, and Danny Stevenson, *Principal ∞-bundles: general theory*, Journal of Homotopy and Related Structures, posted on 2014, 1–53, DOI 10.1007/s40062-014-0083-6, (to appear in print).

[TV08] Bertrand Toën and Gabriele Vezzosi, *Homotopical algebraic geometry. II. Geometric stacks and applications*, Mem. Amer. Math. Soc. 193 (2008), no. 902, x+224, DOI 10.1090/memo/0902, available at http://perso.math.univ-toulouse.fr/btoen/files/2012/04/HAGII.pdf MR2394633 (2009h:14004)

E-mail address: pzsambok@uwo.ca

Department of Mathematics, Middlesex College, The University of Western Ontario, London, Ontario, Canada, N6A 5B7