Sound beam through a circular aperture and the far-field nonparaxial regime

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Propagation of sound beam diffracted from a circular aperture in far-field region has been studied in this paper by the method of angular spectrum representation and stationary phase method. This nonparaxial theory is useful when beam angle is not very small and the wavelength $\lambda$ is comparable to aperture diameter $a$ unlike the situation in paraxial approximation. Here we have studied two cases, one for a Gaussian source and other for a plane piston source.

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I. INTRODUCTION

Pressure field radiation and diffraction from a source with a circular aperture and its propagation through elastic medium is quite well known and a well studied subject in acoustics. Paraxial approximation has been successfully applied to study such phenomena. This approximation neglects $\frac{\partial^2 P}{\partial z^2}$ term (the second derivative of pressure with respect to propagation axial co-ordinate) in the Helmholtz equation, as the term is of higher order in beam angle $\theta$ which is very small ($\sin \theta \approx \theta$). But when $\theta$ is not so small, paraxial approximation is no longer valid. Some good works has already been done in optics in such nonparaxial correction. But, in acoustics, such work is still missing. This paper concentrates on the propagation of the acoustic beam transmitted from a monochromatic source from a circular aperture in nonparaxial far-field zone, through a comparatively general approach employing the angular spectrum distribution and the stationary phase method. Gaussian and plane piston are taken as the initial source. The paper begins with a discussion of an analytic nonparaxial formalism with analytical results. This is followed by a presentation of numerical results. Discussions with conclusions are the final feature of this paper.

II. NONPARAXIAL THEORY

A. Helmholtz equation and angular spectrum representation

The governing equation for spatial part of pressure field is given by the Helmholtz equation

$$[\nabla^2 + k^2]P(x, y, z) = 0$$

In any plane $z = constant$, let us assume the field can be represented by a Fourier integral as

$$P(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(u, v, z)e^{i(k(ux+vy) - \omega t)}du dv$$

which when substituted in Eq.1 yields

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\nabla^2 + k^2]P(u, v, z)e^{i(k(ux+vy) - \omega t)}du dv = 0$$

and then differentiating under integral sign, we find that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[(-u^2 - v^2 + k^2)P(u, v, z) + \frac{\partial^2 P(u, v, z)}{\partial z^2}\right] e^{i(k(ux+vy) - \omega t)}du dv = 0$$

For any values of $x$ and $y$, the term under bracket will go to zero. Hence the function $P(u, v, z)$ will satisfy the following differential equation

$$\frac{\partial^2 P(u, v, z)}{\partial z^2} + u^2 P(u, v, z) = 0$$

Where $w^2 = 1 - u^2 - v^2$. When $u^2 + v^2 \leq 1$, $w = +\sqrt{1 - u^2 - v^2}$. Again, when $u^2 + v^2 > 1$, then $w = +i\sqrt{u^2 + v^2 - 1}$.

The general solution of the partial differential equation (5) is

$$P(u, v, z) = A(u, v)e^{ikwz} + B(u, v)e^{-ikwz},$$

where $A(u, v)$ and $B(u, v)$ are arbitrary functions of $u$ and $v$. On substitution of equation (2), we obtained

$$P(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(u, v)e^{i(k(ux+vy+wz) - \omega t)}du dv + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B(u, v)e^{-i(k(ux+vy+wz) - \omega t)}du dv$$

If we don’t consider reflection mode, then $B(u, v) = 0$ (soft baffle case). Otherwise for rigid baffle, the pressure will be doubled due to reflection. So

$$P(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(u, v)e^{i(k(ux+vy+wz) - \omega t)}du dv, \quad (8)$$
here $A(u,v)$ is defined as:

$$A(u,v) = \left(\frac{k}{2\pi}\right)^{2} \iint_{x^{2}+y^{2}\leq a^{2}} P(x,y,0) e^{-ik(ux+vy)} dxdy$$

(9)

Let, the source function be defined as

$$P(x,y,0) = P_{0}(x,y)T(x,y)$$

(10)

where $T(x,y)$ is the window function for circular aperture defined as

$$T(x,y) = \begin{cases} 1 & \text{if } x^{2}+y^{2} \leq a^{2} \\ 0 & \text{otherwise} \end{cases}$$

(11)

Therefore $A(u,v)$ can be determined by

$$A(u,v) = \left(\frac{k}{2\pi}\right)^{2} \int_{0}^{a} \int_{0}^{2\pi} P_{0}(\rho,0) e^{-ik(u\rho \cos \theta + v\rho \sin \theta)} \rho d\rho d\theta$$

(12)

Transforming from Cartesian coordinate to cylindrical coordinate system i.e. $x = \rho \cos \theta$ and $y = \rho \sin \theta$, and assuming the source is axially symmetric (Fig.1), the integral becomes

$$A(u,v) = \left(\frac{k^{2}}{2\pi}\right) \int_{0}^{a} P_{0}(\rho,0) J_{0}(k \rho \sqrt{u^{2}+v^{2}}) \rho d\rho$$

(13)

using the identity

$$\int_{0}^{2\pi} e^{-ik\rho \cos(\psi-\xi)} d\psi = 2\pi J_{0}(k \rho)$$

(14)

Thus $P(x,y,z)$ is

$$P(x,y,z) = \left(\frac{k^{2}}{2\pi}\right) \int_{0}^{a} \int_{u^{2}+v^{2}\leq a^{2}} P_{0}(\rho,0) J_{0}(k \rho \sqrt{u^{2}+v^{2}}) \rho d\rho dudv$$

(15)

(16)

B. Method of stationary phase

The integral over $u$ and $v$ on the right hand side of Eq.(16) may be approximated by method of stationary phase. Here $w = +(1-u^{2}-v^{2})^{\frac{1}{2}}$ as $u^{2}+v^{2} \leq 1$, $k = \frac{\omega}{c}$. As the evanescent or exponentially decaying wave does not give any significant contribution in far field zone, we have avoided another condition $m = i(u^{2} + v^{2} - 1)$ in our problem. Now let us specify some other variables $s_{x} = x/r$, $s_{y} = y/r$, $s_{z} = z/r$.

$$I = \int \int_{u^{2}+v^{2}\leq a^{2}} J_{0}(k \rho \sqrt{u^{2}+v^{2}}) \times e^{ikr(u_{s}x_{s}+v_{s}y_{s}+w_{s})} dudv,$$

(18)

$$= \int \int_{u^{2}+v^{2}\leq a^{2}} a(u,v) e^{ikrg(u,v;s_{x},s_{y})} dudv$$

(19)

where $r = \sqrt{x^{2}+y^{2}+z^{2}}$, $a(u,v) = J_{0}(k \rho \sqrt{u^{2}+v^{2}})$ and $g(u,v;s_{x},s_{y}) = s_{x}u + s_{y}v + s_{z}w$. The critical stationary points for this integral will be $u_{1} = s_{x}$, $v_{1} = s_{y}$ and $w_{1} = s_{z}$. Hence in this approximation one obtains

$$I \sim -\frac{2\pi i}{kr\sqrt{\Delta}} a(u_{1},v_{1}) e^{ikrg(u_{1},v_{1};s_{x},s_{y})}$$

(20)

$$\sim -\frac{2\pi i}{k} \left(\frac{z}{r}\right) J_{0}(k \rho \sqrt{x^{2}+y^{2}}) e^{ikr},$$

(21)

where

$$\Delta = (g_{xx}g_{yy} - g_{xy}^{2})_{u_{1},v_{1}}$$

Becomes

$$P(x,y,z) = \frac{k}{i} \left(\frac{z}{r}\right) e^{ikr} \int_{0}^{a} P_{0}(\rho,0) J_{0}(k \rho \sqrt{x^{2}+y^{2}}) \rho d\rho,$$

(22)

III. PRESSURE FIELDS FOR DIFFERENT AXISSYMMETRIC SOURCES

A. Gaussian source

Sources with Gaussian amplitude is often been used because of its functional form and invariant nature in different integral transforms. Let, the source function be

$$P_{0}(\rho,0) = P_{0} e^{-\frac{\rho^{2}}{2\sigma^{2}}}$$

(23)

where $P_{0}$ = peak amplitude of the source. So, Eq.(22) becomes

$$P(x,y,z) = \frac{P_{0}k}{i} \left(\frac{z}{r}\right) e^{ikr} \int_{0}^{a} e^{-\frac{\rho^{2}}{2\sigma^{2}}} J_{0}(k \rho \sqrt{x^{2}+y^{2}}) \rho d\rho$$

(24)

$J_{0}(x)$ can be expanded as

$$J_{0}(x) = \sum_{m=0}^{\infty} (-1)^{m} \frac{\left(\frac{x}{2}\right)^{2m}}{(m!)^{2}}$$

(25)
Thus, we can get [3]:

\[ P(x, y, z) = P_0 \frac{i}{2k} \left( \frac{z}{r} \right) \left( \frac{e^{ikr}}{r} \right) \times \sum_{n=0}^{\infty} (-1)^n (ka)^{2n+2} \left( \frac{x^2 + y^2}{4r^2} \right)^n \times [\gamma(1+n, 1) - n!] \left( \frac{1}{n!} \right)^2 \]  

(26)

where \( \gamma(\alpha, \beta) \) is incomplete Gamma function defined as:

\[ \gamma(\alpha, \beta) = \int_{0}^{\beta} t^{\alpha-1} e^{-t} dt = \alpha^{-1} \beta^{\alpha} e^{-\beta} \, _1F_1(1; 1 + \alpha; \beta) \]

where \(_1F_1(1; 1 + \alpha; \beta)\) is the confluent hypergeometric function of the first kind. Eq.(26) is the basic analytical result for far-field behavior of nonparaxial Gaussian beam with circular aperture. Paraxial result can be obtained from this result by expanding \( r \) and retaining the first term in amplitude part, and retaining up-to the second term in phase part,

\[ r \approx z + \frac{x^2 + y^2}{2z} \]  

(27)

So that, Eq.(26) simplifies to

\[ P_{para}(x, y, z) = P_0 \frac{i}{2k} \frac{e^{ikz}}{z} e^{\frac{ik}{r}(x^2+y^2)} \sum_{n=0}^{\infty} (-1)^n (ka)^{2n} \times \left( \frac{x^2 + y^2}{4z^2} \right)^m \left[ \frac{1}{n!} \right] \frac{\gamma(1+n, 1) - n!}{(n!)^2} \]  

(28)

Eq.(28) is the Fraunhofer diffraction formula for apertured Gaussian beam with a circular aperture in paraxial regime.

B. Piston source

In piston sources, the amplitude distribution can be described as

\[ P(x, y, 0) = P_0 T(x, y) \]  

(29)

Here the window function is same as Eq.(11). For this case, the resulting amplitude takes the integral form

\[ P(x, y, z) = -P_0 ik \left( \frac{z}{r} \right) \frac{e^{ikr}}{r} \int_{0}^{a} J_0(kp\sqrt{x^2 + y^2})dp \]

\[ = -iP_0 ka^2 \left( \frac{z}{r} \right) \frac{e^{ikr}}{r} \left( \frac{r}{ak\sqrt{x^2 + y^2}} \right) \times J_1 \left( \frac{ak\sqrt{x^2 + y^2}}{r} \right) \]  

(30)

This is another result derived in this paper for pressure field of nonparaxial circular aperture in far-field region. Again, if we expand \( r \) in series as given in Eq.(27), we can arrive at paraxially approximated result.
and Eq.(31). We have taken two different source frequencies, these are \( f_1 = 5 \times 10^5 \text{Hz} \) and \( f_2 = 5 \times 10^6 \text{Hz} \). Aperture diameter has been taken as 3.175mm. Velocity of sound \( c \) has been taken as 1500 m/sec. The axial distances that has been taken to compute the field is 10\( Z_R \), where \( Z_R \) is Rayleigh distance defined as \( Z_R = \frac{\lambda^2}{\alpha^2} \). The axial distances are 105.5641mm and 1055.6mm for \( f_1 \) and \( f_2 \) respectively. Results has been shown in Fig.2 and Fig.3.

V. DISCUSSION

An analytical method has been proposed to investigate sound beams beyond paraxial regime. Angular spectrum representation and method of stationary phase are the two mathematical tools used. The analytical study and final results are for a general situation in far-field zone. Paraxial results can be obtained as the special cases of those results. Eq.(26) is the general nonparaxial solution for Gaussian source whereas Eq.(28) is its paraxial form. Similarly for the case of plane piston type source, Eq.(30) is the general nonparaxial solution and Eq.(31) is its paraxial form. Fig.(2) and Fig.(3) are the plots of the comparative numerical results for Gaussian and plane piston sources respectively. In Fig.2(a), nonparaxial and paraxial results differ. Although the maxima is same for nonparaxial and paraxial values, but the spread is different. But as \( \frac{f}{\lambda} \) increases [Fig.2(b)], this difference diminishes. Similar result has been shown in Fig(3), i.e. the results for plane piston source. We conclude that the analytical results presented here represents a generalization of propagation of sound beam diffracted from circular aperture in far-field zone. Not only generalization, but a significant correction over paraxial approximation too.

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\[ P_{parax} = -iP_0ka^2 \left( e^{ikz} \frac{\cos(kx^2+y^2)}{z} \right) \left( J_1 \left( \frac{ak\sqrt{x^2+y^2}}{z} \right) \right) \]

\[ \text{Matlab 7.2 co} \]

IV. NUMERICAL RESULTS

des has been written to solve Eq.(26), Eq.(28), Eq.(30) and Eq.(31). We have taken two different source frequen-

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