ROTA-BAXTER OPERATORS ON WITT AND VIRASORO ALGEBRAS

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Abstract. The homogeneous Rota-Baxter operators on Witt and Virasoro algebras are classified. As applications the induced solutions of the classical Yang-Baxter equation and pre-Lie and PostLie algebra structures are obtained respectively.

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1. Introduction

Rota-Baxter operators were originally defined on associative algebras by G. Baxter to solve an analytic formula in probability [4] and then developed by the school of Rota [14].

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They have been closely related to many fields in mathematics and mathematical physics such as number theory, combinatorics, operads and quantum field theory (see [10, 11] and the references therein).

On the other hand, Rota-Baxter operators in the context of Lie algebras were developed with their own motivation. In fact Semenov-Tian-Shansky’s fundamental work [15] shows that a Rota-Baxter operator of weight 0 on a Lie algebra is exactly the operator form of the classical Yang-Baxter equation (CYBE), which was regarded as a “classical limit” of the quantum Yang-Baxter equation [5]. Whereas the latter is also an important topic in many fields such as symplectic geometry, integrable systems, quantum groups and quantum field theory (see [4] and the references therein).

The study of Rota-Baxter operators on Lie algebras has practical meanings. First, both Rota-Baxter operators of weight 0 and 1 on a Lie algebra $g$ give rise to solutions of CYBE in the double Lie algebra $g \ltimes_{\text{ad}^*} g^*$ over the direct sum $g \oplus g^*$ of the Lie algebra $g$ and its dual space $g^*$. Note that such a relationship holds for any Lie algebra, which is different from the correspondence given by Semenov-Tian-Shansky with a strict constraint on the Lie algebra itself. Secondly, there are certain interesting algebraic structures coming out of the Rota-Baxter operators, notably the pre-Lie algebras from Rota-Baxter operators of weight 0 on Lie algebras and the PostLie algebras from Rota-Baxter operators of weight 1 on Lie algebras. Pre-Lie algebras are a class of non-associative algebras emerged from the study of convex homogeneous cones, affine manifolds and deformations of associative algebras [12, 8, 17]. PostLie algebras were introduced in the context of operads [16]. These two algebraic structures have appeared in many other fields in mathematics and mathematical physics (see [6, 3] and the references therein).

Most of the study on Rota-Baxter operators has been focused on finite dimension cases. For example, a detailed study of Rota-Baxter operators of weight 0 on $sl_2(\mathbb{C})$ is available [13]. It is natural to consider the infinite dimensional case. As a guide for further study, we consider Rota-Baxter operators on two important infinite dimensional Lie algebras: Witt algebra and its central extension Virasoro algebra. These two algebras have played a crucial role in many areas of mathematics and physics. The following three issues are studied in this paper:

1. Classify the homogeneous Rota-Baxter operators of weight 0 and 1 on the Witt algebra $W$ and the Virasoro algebra $V$ respectively.
2. Give the induced solutions of the CYBE in the Lie algebras $W \ltimes_{\text{ad}^*} W^*$ and $V \ltimes_{\text{ad}^*} V^*$ respectively.
3. Give the induced pre-Lie and PostLie algebra structures respectively.

We note that the study in (2) and (3) can be regarded as applications of the classification results given in (1).

Our results can be briefly summarized as follows. In Section 2, we classify the homogeneous Rota-Baxter operators of weight 0 and 1 on the Witt algebra $W$. In Section 3, we classify the homogeneous Rota-Baxter operators of weight 0 and 1 on the Virasoro algebra $V$. In Section 4, we give the induced solutions of the CYBE in the Lie algebras $W \ltimes_{\text{ad}^*} W^*$ and $V \ltimes_{\text{ad}^*} V^*$ respectively. In Section 5, we give the induced pre-Lie algebras from the Rota-Baxter operators of weight 0 on the Witt algebra $W$ and Virasoro algebra $V$ respectively. In Section 6, we give the induced PostLie algebras from the Rota-Baxter operators of weight 1 on $W$ and $V$ respectively.
2. Homogeneous Rota-Baxter operators on the Witt algebra $W$

**Definition 2.1.** A Rota-Baxter operator of weight $\lambda \in \mathbb{F}$ on a Lie algebra $\mathfrak{g}$ is a linear map $R : \mathfrak{g} \to \mathfrak{g}$ satisfying

\[(2.1) \quad [R(x), R(y)] = R([R(x), y] + [x, R(y)]) + \lambda R([x, y]), \quad \forall x, y \in \mathfrak{g}.\]

Note that if $R$ is a Rota-Baxter operator of weight $\lambda \neq 0$, then $\lambda^{-1}R$ is a Rota-Baxter operator $R$ of weight 1. Therefore, we only consider Rota-Baxter operators of weights 0 and 1 in this paper. We also assume that $\mathbb{F} = \mathbb{C}$, the complex field, since both the Witt and Virasoro algebras are defined over $\mathbb{C}$.

**Definition 2.2.** The Witt algebra $W$ is the Lie algebra with a basis $\{L_n | n \in \mathbb{Z}\}$ subject to the following relations:

\[(2.2) \quad [L_m, L_n] = (m - n)L_{m+n}, \quad \forall m, n \in \mathbb{Z}.\]

There is a natural $\mathbb{Z}$-grading on the Witt algebra $W$, namely

$$W = \bigoplus_{n \in \mathbb{Z}} W_n,$$

where $W_n = \mathbb{C}L_n$ for any $n \in \mathbb{Z}$.

**Definition 2.3.** A homogeneous Rota-Baxter operator $R_k$ with degree $k$ on $W$ is a Rota-Baxter operator on the Witt algebra $W$ of the following form

\[(2.3) \quad R_k(L_m) = f(m+k)L_{m+k}, \quad \forall m \in \mathbb{Z},\]

where $f$ is a $\mathbb{C}$-valued function defined on $\mathbb{Z}$.

### 2.1. Homogeneous Rota-Baxter operators of weight 0 on the Witt algebra $W$.

Let $R_k$ be a homogeneous Rota-Baxter operator of weight 0 with degree $k$ on the Witt algebra $W$ satisfying Eq. (2.3). Then by Eqs. (2.1) and (2.2), we see that the function $f$ satisfies the following equation:

\[(2.4) \quad f(m)f(n)(m-n) = f(m+n)(f(m)(m-n+k) + f(n)(m-n-k)), \quad \forall m, n \in \mathbb{Z}.\]

**Proposition 2.4.** With the notations as above, degree 0 Rota-Baxter operator $f$ of weight 0 is given by

$$f(m) = af_{m,0}, \forall m \in \mathbb{Z},$$

for $a \in \mathbb{C}$.

**Proof.** When $k = 0$, Eq. (2.4) becomes

$$(m-n)f(m)f(n) = (m-n)f(m+n)(f(m) + f(n)), \forall m, n \in \mathbb{Z}.$$

Plugging $n = 0$ in the equation, we have

$$mf(m)^2 = 0.$$

Thus $f(m) = a\delta_{m,0}$ for some $a \in \mathbb{C}$. \hfill $\square$

When $k \neq 0$, taking $n = 0$ in Eq. (2.4), we have

\[(2.5) \quad f(m)((m+k)f(m) - kf(0)) = 0, \forall m \in \mathbb{Z}.\]
Proposition 2.5. With the notations as above, when the degree \( k \neq 0 \) and \( f(0) = 0 \), we have
\[
f(m) = a\delta_{m,-k}, \forall m \in \mathbb{Z},
\]
where \( a \in \mathbb{C} \).

Proof. If \( f(0) = 0 \), then by Eq. (2.5), we have
\[
(m + k)(f(m))^2 = 0, \forall m \in \mathbb{Z}.
\]
Thus, the function \( f \) satisfies
\[
f(m) = a\delta_{m,-k}, \forall m \in \mathbb{Z},
\]
where \( a \in \mathbb{C} \).

When \( f(0) \neq 0 \), if follows from Eq. (2.5) that \( f(-k) = 0 \). Moreover, substituting this into Eq. (2.4) with \( m = k \) and \( n = -k \), we have \( f(k) = 0 \). For such an \( f \) satisfying Eq. (2.4) so that \( kf(0) \neq 0 \), we set
\[
I = \{ m \in \mathbb{Z} | f(m) = 0 \}, \quad J = \{ m \in \mathbb{Z} | (m + k)f(m) - kf(0) = 0 \}.
\]
Thus \( -k, k \in I \) and \( I \cap J = \emptyset, I \cup J = \mathbb{Z} \).

Lemma 2.6. Let \( f \) be a \( \mathbb{C} \)-valued function defined on \( \mathbb{Z} \) satisfying Eq. (2.4). Suppose that \( f(0) \neq 0 \) and \( k \neq 0 \). If \( n \in J \) and \( m \neq n, n + k \), then \( m \in I \) if and only if \( m + n \in I \).

Proof. If \( m \in I \), \( m \neq n + k \) and \( n \in J \), then by Eq. (2.4), we have
\[
(m - n - k)f(n)f(n + m) = 0.
\]
Since \( n \in J \), we have \( f(n + m) = 0 \). Conversely, if \( m + n \in I \), \( m \neq n \) and \( n \in J \), then by Eq. (2.4), we have
\[
(m - n)f(m) = 0.
\]
Hence \( m \in I \).

For an integer \( m \in \mathbb{Z} \), set
\[
J_m = \{ n \in J | mn \in J \}, \quad I_m = \{ n \in J | mn + k \in I \}.
\]

Proposition 2.7. With the conditions as above, we have

1. \( J_0 = J_1 = J \).
2. \( J \setminus \{ -\frac{k}{2m} \} \cap J_m \subset J_{-m} \) for every \( m \neq 0 \). In particular, \( J \setminus \{ -\frac{k}{2} \} \subset J_{-1} \).
3. \( J \setminus \{ -\frac{k}{2}, \frac{k}{m+1} \} \cap J_{m-1} \subset J_m \), \( J \setminus \{ -\frac{k}{m+1} \} \cap J_{1-m} \subset J_{-m} \) for \( m \geq 2 \).
4. \( J \setminus \{ -\frac{k}{2m-1} \} \cap J_{1-m} \subset J_m \), \( J \setminus \{ -\frac{k}{2}, -\frac{k}{2m-1} \} \cap J_{m-1} \subset J_{-m} \) for \( m \geq 2 \).

Proof. (1) follows immediately from definition. We only give a proof for (2) as the proofs of (3) and (4) are similar.

In fact, it is straightforward to check that \( 0 \in (J \setminus \{ -\frac{k}{2m} \} \cap J_m \) and \( 0 \in J_{-m} \) for \( m \neq 0 \). Let \( n \neq 0 \) and \( n \in (J \setminus \{ -\frac{k}{2m} \}) \cap J_m \). To prove (2), we only need to show that \(-nm \in J \). Otherwise, \(-nm \in I \). Then by Lemma 2.6 we have \( nm - nm = 0 \in I \), which is a contradiction with the assumption that \( f(0) \neq 0 \).
Corollary 2.8. With the conditions as above, we have

\[ J \setminus \left\{ \frac{-k}{2} \right\} \subset \bigcap_{m \in \mathbb{Z}} J_m. \]

Proof. We only need to show that \( J \setminus \left\{ \frac{-k}{2} \right\} \subset J_m \cap J_{-m} \) for every \( m \geq 1 \).

By Proposition 2.7 we show that \( J \setminus \left\{ \frac{-k}{2} \right\} \subset J_0 \). Moreover, since \( J_1 = J_0 = J \), we have \( J \setminus \left\{ \frac{-k}{2} \right\} \subset J_1 \cap J_{-1} \).

By Proposition 2.7 again, we have

\[ J \setminus \left\{ \frac{-k}{2}, \frac{k}{3} \right\} \subset J_2, \; J \setminus \left\{ \frac{-k}{2}, \frac{k}{4} \right\} \subset J_2, \; J \setminus \left\{ \frac{k}{4} \right\} \subset J_2, \; J \setminus \left\{ \frac{-k}{2}, \frac{k}{4}, \frac{k}{3} \right\} \subset J_2. \]

Therefore

\[ J \setminus \left\{ \frac{-k}{2}, \frac{k}{3}, \frac{k}{4} \right\} = (J \setminus \left\{ \frac{k}{4} \right\}) \cap (J \setminus \left\{ \frac{-k}{2}, \frac{k}{3} \right\}) \subset (J \setminus \left\{ \frac{k}{4} \right\}) \cap J_2 \subset J_2. \]

Hence

\[ J \setminus \left\{ \frac{-k}{2} \right\} = (J \setminus \left\{ \frac{-k}{2}, \frac{k}{3}, \frac{k}{4} \right\}) \cup (J \setminus \left\{ \frac{-k}{2}, \frac{k}{3}, \frac{k}{4} \right\}) \subset J_2. \]

Similarly, we show that \( J \setminus \left\{ \frac{-k}{2} \right\} \subset J_2 \).

Now assume that \( J \setminus \left\{ \frac{-k}{2} \right\} \subset J_{m-1} \cap J_{1-m} \) holds for \( m > 2 \). By Proposition 2.7 we have that

\[ J \setminus \left\{ \frac{-k}{2}, \frac{k}{m+1} \right\} = (J \setminus \left\{ \frac{-k}{2}, \frac{k}{m+1} \right\}) \cap (J \setminus \left\{ \frac{-k}{2}, \frac{k}{m+1} \right\}) \subset (J \setminus \left\{ \frac{-k}{2}, \frac{k}{m+1} \right\}) \cap J_{m-1} \subset J_m, \]

and

\[ J \setminus \left\{ \frac{-k}{2}, \frac{k}{2m-1} \right\} = (J \setminus \left\{ \frac{-k}{2}, \frac{k}{2m-1} \right\}) \cap (J \setminus \left\{ \frac{k}{2m-1} \right\}) \subset J \setminus \left\{ \frac{-k}{2}, \frac{k}{2m-1} \right\} \cap J_{1-m} \subset J_m. \]

Since \( \frac{k}{m+1} \neq \frac{k}{2m-1} \) for \( m > 2 \), we have

\[ J \setminus \left\{ \frac{-k}{2} \right\} = (J \setminus \left\{ \frac{-k}{2}, \frac{k}{m+1} \right\}) \cup (J \setminus \left\{ \frac{-k}{2}, \frac{k}{2m-1} \right\}) \subset J_m. \]

Similarly we show that \( J \setminus \left\{ \frac{-k}{2} \right\} \subset J_{-m} \) for \( m > 2 \). \( \square \)

Proposition 2.9. With the conditions as above, we have

1. \( I_0 = J \).
2. \( J \setminus \left\{ \frac{-k}{2}, \frac{k}{m} \right\} \subset I_m \) for every \( m \neq 0 \).
3. \( J \setminus \left\{ \frac{-k}{2}, \frac{k}{2m} \right\} \subset I_m \) for every \( m \neq 0 \).

Proof. (1) follows from the fact that \( k \in I \). We only give an explicit proof of (3) and the proof of (2) is similar.

Let \( m \) be a fixed non-zero integer. Since \( 0 \notin J \) and \( k \notin I \), we show that \( 0 \notin I_m \). Let \( n_0 \) be an arbitrary nonzero integer in \( J \setminus \left\{ \frac{-k}{2}, \frac{k}{2m} \right\} \). Then we have \( k + mn_0 \neq -mn_0 \) and \( k + mn_0 \neq -mn_0 + k \). By Corollary 2.8 we have \(-mn_0 \in J \). Hence by Lemma 2.6 and since \( mn_0 + k - mn_0 = k \in I \), we have \( mn_0 + k \in I \). \( \square \)

By Proposition 2.9 we get the following result.
Corollary 2.10. With the conditions as above, we have
\[ J \setminus \{ -\frac{k}{2} \} \subset \bigcap_{m \in \mathbb{Z}} I_m. \]

Proposition 2.11. With the conditions as above, let \( n \in J \setminus \{ -\frac{k}{2} \} \) and \( n \neq 0 \). Then we have \( n \nmid k \), and for any \( m \in \mathbb{Z} \),
\begin{enumerate}
  \item if \( m \in I \), then \( m + n\mathbb{Z} \in I \);
  \item if \( m \in J \), then \( m + n\mathbb{Z} \in J \).
\end{enumerate}

Proof. If \( m \) is neither in \( n\mathbb{Z} \) nor in \( k + n\mathbb{Z} \), the conclusion holds due to Lemma 2.6. On the other hand, by Corollary 2.8, we show that \( \gcd(n) \neq 0 \). By Corollary 2.10 and the fact that \( k \in I \), we have \( n \in \bigcap_{l \in \mathbb{Z}} J_l \). Hence \( n \mathbb{Z} \subset J \). Furthermore, \( n \mathbb{Z} \notin J \), and the fact that \( k \in I \), we have \( n \in \bigcap_{l \in \mathbb{Z}} I_l \). Thus \( k + n\mathbb{Z} \subset I \). Therefore for any \( m \in n\mathbb{Z} \) or \( m \in k + n\mathbb{Z} \), if \( m \in I \), then \( m \in k + n\mathbb{Z} \) and hence \( m + n\mathbb{Z} \in I \), and if \( m \in J \), then \( m \in n\mathbb{Z} \) and hence \( m + n\mathbb{Z} \in J \). Moreover \( n \nmid k \). Otherwise we have \( n\mathbb{Z} = k + n\mathbb{Z} \subset I \cap J \), which is a contradiction. \( \square \)

For any two \( m, n \in \mathbb{Z} \), let \( \gcd(m, n) \) denote the greatest common divisor of \( m \) and \( n \).

Corollary 2.12. With the conditions as above, if \( n_1 \in J \), \( n_2 \in J \setminus \{ 0, -\frac{k}{2} \} \), then \( \gcd(n_1, n_2) \mathbb{Z} \subset J \).

Proof. If \( n_1 \neq -\frac{k}{2} \), then by Proposition 2.11, we show that for every \( m_1, m_2 \in \mathbb{Z} \), \( n_1 m_1 + n_2 m_2 \in J \). Furthermore, we have \( \gcd(n_1, n_2) \mathbb{Z} = n_1 \mathbb{Z} + n_2 \mathbb{Z} \). Thus \( \gcd(n_1, n_2) \mathbb{Z} \subset J \).

If \( n_1 = -\frac{k}{2} \in J \), then by Proposition 2.11, we show that \( n_1 + n_2 \in J \). On the other hand, we have \( n_2, n_1 + n_2 \in J \setminus \{ -\frac{k}{2} \} \). Hence \( \gcd(n_1 + n_2, n_2) \mathbb{Z} \subset J \). Since \( \gcd(n_1 + n_2, n_2) \mathbb{Z} = \gcd(n_1, n_2) \mathbb{Z} \), we have \( \gcd(n_1, n_2) \mathbb{Z} \subset J \).

Proposition 2.13. Let \( f \) be a \( \mathbb{C} \)-valued function defined on \( \mathbb{Z} \) satisfying Eq. (2.4). Suppose that \( f(0) \neq 0 \) and \( k \neq 0 \). If \( -\frac{k}{2} \in \mathbb{Z} \) and \( -\frac{k}{2} 
 J \), then \( J = \{ 0, -\frac{k}{2} \} \), and in this case,
\[
(2.6) \quad f(m) = \delta_{m,0} f(0) + 2\delta_{m,-\frac{k}{2}} f(0), \forall m \in \mathbb{Z}.
\]

Proof. It is obvious that \( \{ 0, -\frac{k}{2} \} \subset J \). Conversely, if there exists an \( n_0 \in J \) such that \( n_0 \neq 0, -\frac{k}{2} \), then by Corollary 2.12, we have \( \gcd(n_0, -\frac{k}{2}) \mathbb{Z} \subset J \). Since \( J \neq \mathbb{Z} \), we have \( \gcd(n_0, -\frac{k}{2}) \neq 1 \). Set \( d = \gcd(n_0, -\frac{k}{2}) \). Then \( d|\frac{k}{2} \). Hence \( d|k \). By Proposition 2.11, we show that \( d = \frac{k}{2} \). Thus \( n_0 = k m_0 \) for some \( m_0 \neq 0, -1 \). However, by Lemma 2.6, we can show that \( \frac{k}{2} \in \mathbb{Z} \) for every \( m \neq 0, -1 \). It is a contradiction. Hence \( J = \{ 0, -\frac{k}{2} \} \). \( \square \)

Proposition 2.14. Let \( f \) be a \( \mathbb{C} \)-valued function defined on \( \mathbb{Z} \) satisfying Eq. (2.4). Suppose that \( f(0) \neq 0 \) and \( k \neq 0 \). If \( -\frac{k}{2} \notin J \), and \( \{ 0 \} \subset \mathbb{Z} \), then there exists a non-zero integer \( n_0 \in J \), \( n_0 \nmid k \) such that \( \left| n_0 \right| \) is minimal. In this case, we have
\[
J = n_0 \mathbb{Z},
\]
and thus
\[
(2.7) \quad f(m) = \begin{cases} 
\frac{k}{m+k} f(0) & m \in n_0 \mathbb{Z}; \\
0 & m \notin n_0 \mathbb{Z}.
\end{cases}
\]
Proof. Since \( \{0\} \subseteq J \), there exists an integer \( n_0 \in J \) such that \( n_0 \neq 0 \), \( |n_0| \) is minimal and \( n_0 \nmid k \). By Proposition 2.11 and the minimality of \( |n_0| \), we have \( m \in I \) for any \( m \notin n_0\mathbb{Z} \). On the other hand, since \( 0 \in J \) and by Proposition 2.11 again, we have \( n_0\mathbb{Z} \subset J \). Hence \( J = n_0\mathbb{Z} \) and thus the conclusion holds. \( \square \)

Summarizing Propositions 2.4, 2.5, 2.13 and 2.14, we get the following result:

**Theorem 2.15.** A homogeneous Rota-Baxter operator \( R_k \) of weight 0 with degree \( k \) on the Witt algebra \( W \) should satisfy one of the following equations:

1. \( R_k(L_m) = \begin{cases} \alpha L_{-k} & m = -2k; \\ 0 & m \neq -2k, \end{cases} \)
   where \( \alpha \in \mathbb{C} \).

2. \( R_k(L_m) = \begin{cases} \alpha L_0 & m = -k; \\ 2\alpha L_{-\frac{k}{2}} & m = -\frac{3k}{2}; \\ 0 & m \neq -k, -\frac{3k}{2}, \end{cases} \)
   where \( k \) is a nonzero even number and \( 0 \neq \alpha \in \mathbb{C} \).

3. \( R_k(L_m) = \begin{cases} \frac{k}{m+2k} L_{m+k} & m \in -k + n_0\mathbb{Z}; \\ 0 & m \notin -k + n_0\mathbb{Z}, \end{cases} \)
   where \( k \neq 0 \), \( n_0 \in \mathbb{Z} \) satisfying \( n_0 \nmid k \) and \( 0 \neq \alpha \in \mathbb{C} \).

**Remark 2.16.** It is known that \( R \) is a Rota-Baxter operator of weight 0 on a Lie algebra \( g \) if and only if \( \alpha R \) is a Rota-Baxter operator of weight 0 on \( g \) for \( 0 \neq \alpha \in \mathbb{C} \). So the set of Rota-Baxter operators of weight 0 on any Lie algebra carries an action of \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) by scalar multiplication. In this sense, the above theorem can be rewritten as follows. A complete set of representatives of the set of homogeneous Rota-Baxter operators of weight 0 with degree \( k \) on the Witt algebra \( W \) under the action of \( \mathbb{C}^* \) by scalar multiplication consists of the following operators:

1. \( R_k(L_m) = 0 \), for any \( m \in \mathbb{Z} \).

2. \( R_k(L_m) = \begin{cases} L_{-k} & m = -2k; \\ 0 & m \neq -2k, \end{cases} \)

3. \( R_k(L_m) = \begin{cases} L_0 & m = -k; \\ 2L_{-\frac{k}{2}} & m = -\frac{3k}{2}; \\ 0 & m \neq -k, -\frac{3k}{2}, \end{cases} \)
   where \( k \) is a nonzero even number.

4. \( R_k(L_m) = \begin{cases} \frac{k}{m+2k} L_{m+k} & m \in -k + n_0\mathbb{Z}; \\ 0 & m \notin -k + n_0\mathbb{Z}, \end{cases} \)
   where \( k \neq 0 \), \( n_0 \in \mathbb{Z} \) satisfying \( n_0 \nmid k \).

2.2. Homogeneous Rota-Baxter operators of weight 1 on the Witt algebra \( W \).

It is straightforward to show by definition that there does not exist a homogeneous Rota-Baxter operator of weight 1 with a nonzero degree \( k \) on the Witt algebra \( W \).

Let \( R_0 \) be a homogeneous Rota-Baxter operator of weight 1 with degree 0 on the Witt algebra \( W \) satisfying Eq. (2.3), that is,

\[
R_0(L_m) = f(m)L_m, \quad \forall m \in \mathbb{Z}.
\]
Then by Eqs. (2.1) and (2.2), we show that the function $f$ exactly satisfies the following equation:

$$(2.9) \quad f(m)f(n)(m - n) = f(m + n)(f(m) + f(n) + 1)(m - n), \; \forall m, n \in \mathbb{Z}.$$

Let $n = 0$ in Eq. (2.9), then we have

$$m f(m)(f(m) + 1) = 0, \forall m \in \mathbb{Z}.$$

Set

$$\mathcal{J}_1 = \{ m \in \mathbb{Z} | f(m) = 0 \}, \quad \mathcal{J}_2 = \{ m \in \mathbb{Z} | f(m) = -1 \}.$$

**Lemma 2.17.** Let $f$ be a $\mathbb{C}$-valued function defined on $\mathbb{Z}$ satisfying Eq. (2.9). If $m, n \in \mathbb{Z}$ such that $m \neq n$ and $m, n \in \mathcal{J}_i$, then $m + n \in \mathcal{J}_i$ ($i = 1, 2$).

*Proof.* If $m \neq n$ and $m, n \in \mathcal{J}_1$, then Eq. (2.9) implies $m + n \in \mathcal{J}_1$. Similarly, if $m \neq n$ and $m, n \in \mathcal{J}_2$, then $m + n \in \mathcal{J}_2$. □

**Proposition 2.18.** With the notations as above, if there exists a nonzero integer $m_0$ such that $m_0, -m_0 \in \mathcal{J}_1$, then $\mathcal{J}_1, \mathcal{J}_2$ belong to one of the following cases:

1. $\mathcal{J}_1 = \{ m \mid m \leq 1 \}, \mathcal{J}_2 = \{ m \mid m \geq 2 \}$;
2. $\mathcal{J}_1 = \{ m \mid m \geq -1 \}, \mathcal{J}_2 = \{ m \mid m \leq -2 \}$;
3. $\mathcal{J}_1 = \emptyset, \mathcal{J}_2 = \emptyset$.

*Proof.* By Lemma 2.17, $0 = m_0 + (-m_0) \in \mathcal{J}_1$. Hence Eq. (2.9) with $n = -m \neq 0$ implies $f(m)f(-m) = 0, \forall m \neq 0$.

Therefore if $m \neq 0$ and $m \in \mathcal{J}_2$, then $-m \notin \mathcal{J}_1$.

Let $l$ be the minimal positive integer such that $l \in \mathcal{J}_1$. Then $l = 1$. Otherwise, $l \geq 2$. So $1 \in \mathcal{J}_2$. Hence $-1 \in \mathcal{J}_1$. By Lemma 2.17, we show that $l - 1 = -1 + l \in \mathcal{J}_1$ which contradicts with the minimality of $l$. Similarly, $-1 \notin \mathcal{J}_1$.

1. If $2 \in \mathcal{J}_2$, then $-2 \notin \mathcal{J}_1$. Since $-1, -2 \notin \mathcal{J}_1$, by Lemma 2.17 and induction, $\mathcal{J}_1$ contains all negative integers. Thus for any $m \geq 2, 2 - m \in \mathcal{J}_1$. It implies $m \notin \mathcal{J}_2$.

Otherwise, by Lemma 2.17, $2 = (2 - m) + m \in \mathcal{J}_1$ which contradicts with the assumption that $2 \notin \mathcal{J}_2$. In this case, $\mathcal{J}_1 = \{ m \mid m \leq 1 \}, \mathcal{J}_2 = \{ m \mid m \geq 2 \}$.

2. Similarly, if $-2 \notin \mathcal{J}_2$, then $\mathcal{J}_1 = \{ m \mid m \geq -1 \}, \mathcal{J}_2 = \{ m \mid m \leq -2 \}$.

3. If $2, -2 \notin \mathcal{J}_1$, then $\mathcal{J}_1 = \mathbb{Z}, \mathcal{J}_2 = \emptyset$.

Hence the conclusion holds. □

Similarly, we have the following conclusion.

**Proposition 2.19.** With the notations as above, if there exists a nonzero integer $m_0$ such that $m_0, -m_0 \in \mathcal{J}_2$, then $\mathcal{J}_1, \mathcal{J}_2$ belong to one of the following cases:

1. $\mathcal{J}_1 = \{ m \mid m \geq 2 \}, \mathcal{J}_2 = \{ m \mid m \leq 1 \}$;
2. $\mathcal{J}_1 = \{ m \mid m \leq -2 \}, \mathcal{J}_2 = \{ m \mid m \geq -1 \}$;
3. $\mathcal{J}_1 = \emptyset, \mathcal{J}_2 = \mathbb{Z}$.

**Proposition 2.20.** With the notations as above, if there does not exist a nonzero integer $m$ such that $m, -m \in \mathcal{J}_i$, $i = 1, 2$, then either

$$\mathbb{Z}_+ \subset \mathcal{J}_1, \mathbb{Z}_- \subset \mathcal{J}_2,$$
or

\[ Z_- \subset I_1, Z_+ \subset I_2, \]

where \( Z_+ \) denotes the set of all positive integers and \( Z_- \) denotes the set of all negative integers. Moreover, \( f(0) \in \mathbb{C} \) is arbitrary.

Proof. In this case, \( f(m) \neq f(-m) \) for every \( m \neq 0 \). Thus \( m \in I_1 \) if and only if \( -m \in I_2 \). So the conclusion about \( I_1, I_2 \) holds. Moreover, Eq. (2.9) holds automatically when we set \( n = 0 \) or \( m + n = 0 \), that is, \( f(0) \in \mathbb{C} \) is arbitrary. \( \square \]

Summarizing Propositions 2.18, 2.19 and 2.20, we obtain the following conclusion:

**Theorem 2.21.** A homogeneous Rota-Baxter operator \( R_0 \) of weight 1 with degree zero on the Witt algebra \( W \) should satisfy one of the following equations:

1. \( R_0(L_m) = \begin{cases} -L_m & m \geq 2; \\ 0 & m \leq 1. \end{cases} \)
2. \( R_0(L_m) = \begin{cases} -L_m & m \leq -2; \\ 0 & m \geq -1. \end{cases} \)
3. \( R_0(L_m) = 0, \) for any \( m \in \mathbb{Z}. \)
4. \( R_0(L_m) = \begin{cases} -L_m & m \leq 1; \\ 0 & m \geq 2. \end{cases} \)
5. \( R_0(L_m) = \begin{cases} -L_m & m \geq -1; \\ 0 & m \leq -2. \end{cases} \)
6. \( R_0(L_m) = -L_m, \) for any \( m \in \mathbb{Z}. \)
7. \( R_0(L_m) = \begin{cases} -L_m & m < 0; \\ \alpha L_0 & m = 0; \\ 0 & m > 0, \end{cases} \)
where \( \alpha \in \mathbb{C}. \)
8. \( R_0(L_m) = \begin{cases} -L_m & m > 0; \\ \alpha L_0 & m = 0; \\ 0 & m < 0, \end{cases} \)
where \( \alpha \in \mathbb{C}. \)

**Remark 2.22.** It is known that \( R \) is a Rota-Baxter operator of weight 1 on a Lie algebra \( g \) if and only if \( -R \circ \text{Id} \) is also a Rota-Baxter operator of weight 1 on \( g \), where \( \text{Id} \) is the identity map on \( g \). In this sense, we have the following correspondences for the Rota-Baxter operators listed in Theorem 2.21:

(1) \( \iff \) (4), (2) \( \iff \) (5), (3) \( \iff \) (6), (7) with \( \alpha \iff \) (8) with \( -\alpha - 1. \)

3. **Homogeneous Rota-Baxter operators on the Virasoro algebra \( V \)**

**Definition 3.1.** The *Virasoro algebra* \( V \) is a Lie algebra with a basis \( \{L_m, c| m \in \mathbb{Z}\} \) satisfying the following relations:

\[ [L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c, \forall m, n \in \mathbb{Z}. \]
The Virasoro algebra \( V \) is a central extension of the Witt algebra \( W \), and has a natural \( \mathbb{Z} \)-grading as well:

\[
V = \bigoplus_{n \in \mathbb{Z}} V_n,
\]

where \( V_n = \mathbb{C}L_n \) for \( n \in \mathbb{Z} \setminus \{0\} \) and \( V_0 = \mathbb{C}L_0 \oplus \mathbb{C}c \).

**Definition 3.2.** A homogeneous Rota-Baxter operator \( R_k \) with degree \( k \) on the Virasoro algebra \( V \) is a Rota-Baxter operator on \( V \) such that

\[
R_k(V_m) \subseteq V_{m+k}, \quad \forall m \in \mathbb{Z}.
\]

Hence \( R_k \) has the following form:

\[
(3.3) \quad R_k(L_m) = f(m + k)L_{m+k} + \mu \delta_{m+k,0}c, \quad \forall m \in \mathbb{Z};
\]

\[
(3.4) \quad R_k(c) = \theta L_k + \nu \delta_{k,0}c,
\]

where \( f \) is a \( \mathbb{C} \)-valued function defined on \( \mathbb{Z} \) and \( \mu, \theta, \nu \in \mathbb{C} \).

### 3.1. Homogeneous Rota-Baxter operators of weight 0 on the Virasoro algebra \( V \).

**Theorem 3.3.** A homogeneous Rota-Baxter operator \( R_0 \) of weight 0 with degree 0 on the Virasoro algebra \( V \) satisfies

\[
R_0(L_m) = \delta_{m,0}(\alpha L_m + \mu c), \forall m \in \mathbb{Z};
\]

\[
R_0(c) = \theta L_0 + \nu c,
\]

where \( \alpha, \theta, \mu, \nu \in \mathbb{C} \) are arbitrary.

**Proof.** Let \( R_0 \) be a homogeneous Rota-Baxter operator of weight 0 with degree 0 on \( V \) satisfying Eqs. (3.3) and (3.4). By Eqs. (2.1), (3.1) and (3.2), we have the following equations:

\[
(3.5) \quad f(m)f(n)(m - n) = (f(m) + f(n))(m - n)f(m + n)
\]

for any \( m + n \neq 0 \),

\[
(3.6) \quad 2mf(m)f(-m) = (f(m) + f(-m))\left(2mf(0) + \frac{m^3 - m}{12}\theta\right), \forall m \in \mathbb{Z},
\]

and

\[
(3.7) \quad f(m)f(-m)\frac{m^3 - m}{12} = (f(m) + f(-m))(2m\mu + \frac{m^3 - m}{12}\nu), \forall m \in \mathbb{Z}.
\]

Let \( n = 0 \) in Eq. (3.5). Then \( f(m) = 0 \) for any \( m \neq 0 \). Therefore, all the above equations hold automatically. Hence \( f(0) = \alpha, \theta, \mu, \nu \in \mathbb{C} \) are arbitrary.

**Remark 3.4.** In the sense of Remark 2.16, a complete set of representatives of the set of homogeneous Rota-Baxter operators of weight 0 with degree 0 on \( V \) under the action of \( \mathbb{C}^* \) by scalar multiplication consists of the following operators:
(1) \( R_0(L_m) = \mu \delta_{m,0} c \), for any \( m \in \mathbb{Z} \), and \( R_0(c) = \theta L_0 + \nu c \), where \( \theta, \mu, \nu \in \mathbb{C} \) are arbitrary.

(2) \( R_0(L_m) = \delta_{m,0}(L_m + \mu c) \), for any \( m \in \mathbb{Z} \), and \( R_0(c) = \theta L_0 + \nu c \), where \( \theta, \mu, \nu \in \mathbb{C} \) are arbitrary.

Let \( R_k \) be a homogeneous Rota-Baxter operator of weight 0 with a nonzero degree \( k \) on \( V \) satisfying Eqs. (3.3) and (3.4). In this case, it is obvious that \( \nu = 0 \), that is,
\[
R_k(L_n) = f(n + k)L_{n+k} + \mu \delta_{n+k,0} c, \quad \forall n \in \mathbb{Z};
\]
\[
R_k(c) = \theta L_k.
\]

By Eqs. (2.1), (3.1) and (3.2), we have the following equations:
\[
f(m + k)\theta \left( mL_{m+2k} + \frac{(m + k)^3 - (m + k)}{12} \delta_{m+2k,0} c \right)
= \theta(m - k)(f(m + 2k)L_{m+2k} + \mu \delta_{m+2k,0} c) + \theta^2 \frac{m^3 - m}{12} \delta_{m+k,0} L_k, \forall m \in \mathbb{Z};
\]
\[
f(m)f(n)\left( (m - n)L_{m+n} + \frac{m^3 - m}{12} \delta_{m+n,0} c \right)
= (f(m)(m - n + k) + f(n)(m - n - k))(f(m + n)L_{m+n} + \mu \delta_{m+n,0} c)
+ \frac{m^3 - m}{12} f(m) - \frac{n^3 - n}{12} f(n) \delta_{m+n,k} \theta L_k, \forall m, n \in \mathbb{Z}.
\]

**Proposition 3.5.** With the notations as above, if \( \theta = 0 \), then \( f \) and \( \mu \) belong to one of the following cases:

1. \( f(m) = 0 \) and \( \mu \in \mathbb{C} \).
2. \( f(m) = \alpha \delta_{m,-k} \), where \( \alpha \in \mathbb{C} \), and \( \mu = 0 \).
3. \( f(m) = \alpha(\delta_{m,0} + 2\delta_{m,-\frac{k}{2}}) \), where \( \alpha \in \mathbb{C} \), and \( \mu \in \mathbb{C} \).

**Proof.** If \( \theta = 0 \), then Eq. (3.8) holds automatically and Eq. (3.9) becomes
\[
f(m)f(n)(m - n) = f(m + n)(f(m)(m - n + k) + f(n)(m - n - k)), \forall m, n \in \mathbb{Z},
\]
and
\[
\frac{m^3 - m}{12} f(m)f(-m) = \mu(f(m)(2m + k) + f(-m)(2m - k)), \forall m \in \mathbb{Z}.
\]

Note that Eq. (3.10) is exactly Eq. (2.4). By the discussion in the previous section, \( f \) satisfies one of the following equations:

(i) \( f(m) = \delta_{m,-k} f(-k), \forall m \in \mathbb{Z} \).

(ii) \( f(m) = (\delta_{m,0} + 2\delta_{m,-\frac{k}{2}}) f(0), \forall m \in \mathbb{Z} \).

(iii) \( f(m) = \begin{cases} \frac{k}{m+k} f(0) & m \in n_0\mathbb{Z}; \\ 0 & m \notin n_0\mathbb{Z}. \end{cases} \)

For (i), Eq. (3.11) implies that either \( \mu = 0 \) or \( f(-k) = 0 \), which corresponds to the cases (2) and (1) respectively.

For (ii), Eq. (3.11) holds automatically. Thus \( \mu \in \mathbb{C} \) is arbitrary. It corresponds to the case (3).

For (iii), it does not satisfy Eq. (3.11). \( \square \)
Proposition 3.6. With the notations as above, if \( \theta \neq 0 \), then
\[
f(m) = -\frac{k^2 - 1}{24}\theta \delta_{m,k}, \quad \forall m \in \mathbb{Z},
\]
and \( \mu = 0 \).

Proof. In this case, Eq. (3.8) implies the following equations:
\[
(m-k)f(m) = (m-2k)f(m+k), \quad \forall m \neq 0,
\]
\[
\frac{1}{2}f(0) = f(k) + \frac{k^2 - 1}{24}\theta
\]
and
\[
\frac{k^2 - 1}{12}f(-k) = 3\mu.
\]
Eq. (3.9) implies the following equations:
\[
f(m)f(n)(m-n) = f(m+n)(f(m)(m-n+k) + f(n)(m-n-k))
\]
for \( m+n \neq k \),
\[
f(m)f(n)(m-n) = f(k)(2mf(m) - 2nf(n)) + \theta \left( f(m)\frac{m^3 - m}{12} - f(n)\frac{n^3 - n}{12} \right)
\]
for \( m+n = k \) and
\[
\frac{m^3 - m}{12}f(m)f(-m) = \mu(f(m)(2m+k) + f(-m)(2m-k)), \quad \forall m \in \mathbb{Z}.
\]

Let \( n = 0 \) in Eq. (3.16). Then we have
\[
f(m)((m+k)f(m) - kf(0)) = 0, \quad \forall m \neq k.
\]

Hence for any \( m \neq k, -k \), either \( f(m) = 0 \) or
\[
f(m) = \frac{k}{m+k}f(0).
\]

In addition, \( f(-k)f(0) = 0 \).

On the other hand, let \( m = -k \) in Eq. (3.13). Then we have
\[
2f(-k) = 3f(0).
\]
Hence \( f(0) = 0 \). Therefore \( f(m) = 0 \) for any \( m \neq k \). Thus Eqs. (3.13), (3.14), (3.15), (3.16), (3.17) and (3.18) become
\[
f(k) = -\frac{k^2 - 1}{24}\theta, \quad \mu = 0.
\]

Therefore the conclusion holds. \( \Box \)

Summarizing Propositions 3.5 and 3.6, we have the following result:

Theorem 3.7. A homogeneous Rota-Baxter operator \( R_k \) of weight 0 with a nonzero degree \( k \) on the Virasoro algebra \( \mathcal{V} \) should satisfy one of the following equations:

(1) For any \( m \in \mathbb{Z} \), \( R_k(L_m) = \alpha \delta_{m,-k}c \) for \( \alpha \in \mathbb{C} \), and \( R_k(c) = 0 \).

(2) For any \( m \in \mathbb{Z} \), \( R_k(L_m) = \alpha \delta_{m,-2k}L_{m+k} \) for some \( \alpha \in \mathbb{C}^* \), and \( R_k(c) = 0 \).
(3) For any \( m \in \mathbb{Z} \), \( R_k(L_m) = \alpha(\delta_{m,-k} + 2\delta_{m,-2k})L_{m+k} + \mu\delta_{m,-kc}, \) for some \( \alpha \in C^* \), \( \mu \in \mathbb{C} \), \( k \) a nonzero even number. Moreover \( R_k(c) = 0 \).

(4) For any \( m \in \mathbb{Z} \) \( R_k(L_m) = -\alpha\delta_{m,0} \frac{k^2 - 1}{24}L_{m+k} \) for some \( \alpha \in C^* \), and \( R_k(c) = \alpha L_k \).

**Remark 3.8.** In the sense of Remark 2.16, a complete set of representatives of the set of homogeneous Rota-Baxter operators of weight 0 with a nonzero degree \( \nu \) and \( \theta \) is a complete set of representatives of the set of homogeneous Rota-Baxter operators of weight 0 with a nonzero degree \( \nu \) and \( \theta \).

3.2. **Homogeneous Rota-Baxter operators of weight 1 on the Virasoro algebra** \( V \). Let \( R_k \) be a homogeneous Rota-Baxter operator of weight 1 with degree \( k \) on \( V \) satisfying Eqs. \( 3.3 \) and \( 3.4 \). By Eqs. \( 2.1 \), \( 3.1 \) and \( 3.2 \), we have the following equations:

\[
(3.19) \quad f(m)f(n) \left( (m - n)L_{m+n} + \frac{m^3 - m}{12} \delta_{m+n,0}c \right)
\]

\[
= f(m)(m - n + k)(f(m + n)L_{m+n} + \mu\delta_{m+n,0}c)
\]

\[
+ f(m)\frac{m^3 - m}{12}\delta_{m+n,k}(\theta L_k + \nu\delta_{k,0}c)
\]

\[
+ f(n)(m - n - k)(f(m + n)L_{m+n} + \mu\delta_{m+n,0}c)
\]

\[
- f(n)\frac{n^3 - n}{12}\delta_{m+n,k}(\theta L_k + \nu\delta_{k,0}c)
\]

\[
+ (m - n)(f(m + n - k)L_{m+n-k} + \mu\delta_{m+n,k}c)
\]

\[
+ \frac{(m - k)^3 - (m - k)}{12}\delta_{m+n,2k}(\theta L_k + \nu\delta_{k,0}c), \forall m, n \in \mathbb{Z}.
\]

If \( k \neq 0 \), then by Eq. \( 3.19 \), we have

\[
(m - n)f(m + n - k) = 0, \forall m, n \in \mathbb{Z}.
\]

Thus \( f(m) = 0 \) for any \( m \in \mathbb{Z} \). In this case, by Eq. \( 3.19 \) again, we show that \( \mu = 0 \), \( \nu = 0 \) and \( \theta = 0 \). Hence, any homogeneous Rota-Baxter operator of weight 1 with a nonzero degree \( k \) on \( V \) is zero.

Next let \( R_0 \) be a homogeneous Rota-Baxter operator of weight 1 with degree 0 on \( V \). Then Eq. \( 3.19 \) becomes

\[
(3.20) \quad f(m)f(n) \left( (m - n)L_{m+n} + \frac{m^3 - m}{12} \delta_{m+n,0}c \right)
\]

\[
= (m - n)(f(m + n)L_{m+n} + \mu\delta_{m+n,0}c)(f(m) + f(n) + 1)
\]

\[
+ \frac{m^3 - m}{12}\delta_{m+n,0}(\theta L_0 + \nu c)(f(m) + f(n) + 1), \forall m, n \in \mathbb{Z}.
\]
By this equation, we have the following equations:

\[(3.21)\]
\[f(m)f(n) = (f(m) + f(n) + 1)f(m + n), \forall m \neq n, m + n \neq 0;\]

\[(3.22)\]
\[mf(m)f(-m) = (f(m) + f(-m) + 1)\left( mf(0) + \frac{m^3 - m}{24} \theta \right), \forall m \in \mathbb{Z};\]

\[(3.23)\]
\[\frac{m^3 - m}{24}f(m)f(-m) = (m\mu + \frac{m^3 - m}{24} \nu)(f(m) + f(-m) + 1), \forall m \in \mathbb{Z}.\]

Let \( n = 0 \) in Eq. (3.21). Then we have

\[(f(m) + 1)f(m) = 0, \forall m \neq 0.\]

Hence \( f(m) = 0 \) or \(-1\) for \( m \neq 0 \).

Set

\[\mathcal{J}_1 = \{m \mid f(m) = 0\}, \mathcal{J}_2 = \{m \mid f(m) = -1\}.\]

By Lemma 2.17, we have the following conclusion.

**Lemma 3.9.** Let \( f \) be a \( \mathbb{C} \)-valued function defined on \( \mathbb{Z} \) satisfying Eq. (3.21). If \( m, n \in \mathbb{Z} \) such that \( m \neq n, m + n \neq 0 \) and for \( i = 1, 2, m, n \in \mathcal{J}_i \), then \( m + n \in \mathcal{J}_i \).

Let \( m = 1 \) in Eqs. (3.22) and (3.23). Then we have

\[(3.24)\]
\[f(1)f(-1) = (f(1) + f(-1) + 1)f(0);\]

\[(3.25)\]
\[(f(1) + f(-1) + 1)\mu = 0.\]

Therefore we can divide the situation into four cases:

(i) \( 1, -1 \in \mathcal{J}_1; \)
(ii) \( 1, -1 \in \mathcal{J}_2; \)
(iii) \( 1 \in \mathcal{J}_1, -1 \in \mathcal{J}_2; \)
(iv) \( 1 \in \mathcal{J}_2, -1 \in \mathcal{J}_1.\)

**Proposition 3.10.** If \( 1, -1 \in \mathcal{J}_1 \), then \( \mu = 0 \) and \( f, \theta, \nu \) belong to one of the following cases:

(1) \( \mathcal{J}_1 = \{m \mid m \leq 1\}, \mathcal{J}_2 = \{m \mid m \geq 2\}, \) and \( \theta, \nu \in \mathbb{C} \) are arbitrary.
(2) \( \mathcal{J}_1 = \{m \mid m \geq -1\}, \mathcal{J}_2 = \{m \mid m \leq -2\}, \) and \( \theta, \nu \in \mathbb{C} \) are arbitrary.
(3) \( \mathcal{J}_1 = \mathbb{Z}, \mathcal{J}_2 = \emptyset \) and \( \theta = \nu = 0. \) In this case, \( R_0 = 0.\)

**Proof.** When \( 1, -1 \in \mathcal{J}_1 \), Eqs. (3.24) and (3.25) become

\[f(0) = 0, \mu = 0.\]

Thus \( 0 \in \mathcal{J}_1.\) Moreover, there are following three cases:

(1) If \( 2 \in \mathcal{J}_2 \), then \(-2 \in \mathcal{J}_1.\) Otherwise, if \(-2 \in \mathcal{J}_2 \), then Eq. (3.23) implies \( \nu = -1.\) Thus Eq. (3.23) becomes

\[\frac{m^3 - m}{24}f(m)f(-m) = -\frac{m^3 - m}{24}(f(m) + f(-m) + 1), \forall m \in \mathbb{Z},\]

which contradicts with the assumption that \( 1, -1 \in \mathcal{J}_1.\) Since \(-1, -2 \in \mathcal{J}_1,\) by Lemma 3.9 and induction, \( \mathcal{J}_1 \) contains all negative integers. Therefore, for
any $m \geq 2$, $2 - m \in J_1$. Hence $m \in J_2$. Otherwise, by Lemma 3.9 we have $2 = (2 - m) + m \in J_1$ which contradicts with the assumption that $2 \in J_2$. Hence

$$J_1 = \{m \mid m \leq 1\}, J_2 = \{m \mid m \geq 2\}.$$  

In this case, Eqs. (3.22) and (3.23) hold automatically for $m \neq 0, \pm 1$. Thus $\theta$ and $\nu$ are arbitrary.

(2) Similarly, if $-2 \in J_2$, then $J_1 = \{m \mid m \geq -1\}, J_2 = \{m \mid m \leq -2\}$ and $\theta$ and $\nu$ are arbitrary.

(3) If $2, -2 \in J_1$, then $J_1 = \mathbb{Z}, J_2 = \emptyset$. In this case, Eqs. (3.22) and (3.23) become

$$\frac{m^3 - m}{24} \theta = 0, \forall m \neq 0, \pm 1; \nu = 0.$$  

Thus $\theta = 0$ and hence $R_0 = 0$.

Therefore the conclusion holds.

Similarly, for the case (ii) that $-1, 1 \in J_2$, we have the following conclusion:

**Proposition 3.11.** If $1, -1 \in J_2$, then $\mu = 0$ and $f, \theta, \nu$ belong to one of the following cases:

1. $J_1 = \{m \mid m \geq 2\}, J_2 = \{m \mid m \leq 1\}$, and $\theta, \nu \in \mathbb{C}$ are arbitrary.
2. $J_1 = \{m \mid m \leq -2\}, J_2 = \{m \mid m \geq -1\}$, and $\theta, \nu \in \mathbb{C}$ are arbitrary.
3. $J_1 = \emptyset, J_2 = \mathbb{Z}$ and $\theta = 0, \nu = -1$. In this case, $R_0 = -\text{Id}$.

For the cases (iii) and (iv), it is straightforward to get the following conclusions.

**Proposition 3.12.** If $1 \in J_2, -1 \in J_1$, then $\mathbb{Z}_+ \subset J_1, \mathbb{Z}_- \subset J_2$, and $f(0), \theta, \mu, \nu \in \mathbb{C}$ are arbitrary.

**Proposition 3.13.** If $1 \in J_2, -1 \in J_1$, then $\mathbb{Z}_+ \subset J_2, \mathbb{Z}_- \subset J_1$, and $f(0), \theta, \mu, \nu \in \mathbb{C}$ are arbitrary.

Summarizing Propositions 3.10, 3.11, 3.12 and 3.13 we have the following conclusion.

**Theorem 3.14.** A homogeneous Rota-Baxter operator $R_0$ of weight 1 with degree 0 on the Virasoro algebra $V$ should satisfy one of the following equations:

1. $R_0(L_m) = \begin{cases} -L_m & m \geq 2; \\ 0 & m \leq 1, \end{cases}$
   
   $R_0(c) = \theta L_0 + \nu c$ for some $\theta, \nu \in \mathbb{C}$.

2. $R_0(L_m) = \begin{cases} -L_m & m \leq -2; \\ 0 & m \geq -1, \end{cases}$
   
   $R_0(c) = \theta L_0 + \nu c$ for some $\theta, \nu \in \mathbb{C}$.

3. $R_0 = 0$.

4. $R_0(L_m) = \begin{cases} -L_m & m \leq 1; \\ 0 & m \geq 2, \end{cases}$
   
   $R_0(c) = \theta L_0 + \nu c$ for some $\theta, \nu \in \mathbb{C}$.

5. $R_0(L_m) = \begin{cases} -L_m & m \geq -1; \\ 0 & m \leq -2, \end{cases}$
   
   $R_0(c) = \theta L_0 + \nu c$ for some $\theta, \nu \in \mathbb{C}$.
In the sense of Remark 2.22, we have the following correspondence between $R$ and $-R - \text{Id}$ for the Rota-Baxter operators listed in Theorem 3.14:

1. with $\theta, \nu \iff (4)$ with $-\theta, -\nu - 1$,
2. with $\theta, \nu \iff (5)$ with $-\theta, -\nu - 1$,
3. with $\theta, \nu \iff (6)$ with $-\theta, -\nu - 1$,
4. with $\alpha, \theta, \mu, \nu \iff (8)$ with $-\alpha - 1, -\theta, -\mu, -\nu - 1$.

4. Solutions of the CYBE in $W \ltimes_{\text{ad}^*} W^*$ and $V \ltimes_{\text{ad}^*} V^*$

First we give some notations. Let $\mathfrak{g}$ be a Lie algebra. An element $r = \sum a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$ is called a solution of the classical Yang-Baxter equation (CYBE) in $\mathfrak{g}$ if $r$ satisfies

\[ [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \quad \text{in} \quad U(\mathfrak{g}), \]

where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ and

\[ r_{12} = \sum_i a_i \otimes b_i \otimes 1, \quad r_{13} = \sum_i a_i \otimes 1 \otimes b_i, \quad r_{23} = \sum_i 1 \otimes a_i \otimes b_i. \]

Set

\[ r^{21} = \sum_i b_i \otimes a_i. \]

It is obvious that $r$ is skew-symmetric if and only if $r = -r^{21}$.

Let $\text{ad}: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ be the adjoint representation of $\mathfrak{g}$ defined by $\text{ad}(x)(y) = [x, y]$ for any $x, y \in \mathfrak{g}$. Let $\text{ad}^*: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}^*)$ be the dual representation of the adjoint representation of $\mathfrak{g}$. On the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$, there is a natural Lie algebra structure (denoted by $\mathfrak{g} \ltimes_{\text{ad}^*} \mathfrak{g}^*$) given by

\[ [x_1 + f_1, x_2 + f_2] = [x_1, x_2] + \text{ad}^*(x_1)f_2 - \text{ad}^*(x_2)f_1, \quad \forall x_1, x_2 \in \mathfrak{g}, f_1, f_2 \in \mathfrak{g}^*. \]  

(4.1)

A linear map is said to be of finite rank if its image has finite dimension. A linear map $R: \mathfrak{g} \rightarrow \mathfrak{g}$ of finite rank can be identified as an element in $\mathfrak{g} \otimes \mathfrak{g}^* \subset (\mathfrak{g} \ltimes_{\text{ad}^*} \mathfrak{g}^*) \otimes (\mathfrak{g} \ltimes_{\text{ad}^*} \mathfrak{g}^*)$ as follows. Let $\{e_i\}_{i \in I}$ be a basis of $\text{Im} R$, then for each $x \in \mathfrak{g}$, $R(x)$ can be written as a linear combination of the basis. In other words, for each $i \in I$ there exists a unique linear functional $R_i \in \mathfrak{g}^*$ such that

\[ R(x) = \sum_{i \in I} R_i(x)e_i, \quad \forall x \in \mathfrak{g}. \]
Note that $I$ is finite since $R$ is of finite rank. Then we have
\begin{equation}
R = \sum_{i \in I} e_i \otimes R_i \in \mathfrak{g} \otimes \mathfrak{g}^* \subset (\mathfrak{g} \ltimes_{\text{ad}^*} \mathfrak{g}^*) \otimes (\mathfrak{g} \ltimes_{\text{ad}^*} \mathfrak{g}^*).
\end{equation}

\textbf{Lemma 4.1.} \([1]\) Let $\mathfrak{g}$ be a Lie algebra. A linear map $R : \mathfrak{g} \to \mathfrak{g}$ of finite rank is a Rota-Baxter operator of weight 0 on $\mathfrak{g}$ if and only if $r = R - R^{21}$ is a skew-symmetric solution of the CYBE in $\mathfrak{g} \ltimes_{\text{ad}^*} \mathfrak{g}^*$.

\textbf{Remark 4.2.} Note that the above conclusion was originally proved for the finite dimensional case and it is easily extended to the infinite dimensional case for linear maps of finite rank.

For the Witt algebra $W$, let $\{L_n^*\}_{n \in \mathbb{Z}}$ be the dual basis of $\{L_n\}_{n \in \mathbb{Z}}$. Then the Lie algebra structure on $W \ltimes_{\text{ad}^*} W^*$ is given by
\begin{equation}
[L_m, L_n] = (m-n)L_{m+n}, \quad [L_m, L_n^*] = (n-2m)L_{n-m}^*, \quad [L_m^*, L_n] = 0, \forall m, n \in \mathbb{Z}.
\end{equation}

Note that the Rota-Baxter operators of weight 0 on $W$ given in Theorem \([2, 15]\) are of finite rank except those of type (3). By Lemma \([1]\) we obtain the following skew-symmetric solutions of the CYBE in $W \ltimes_{\text{ad}^*} W^*$.

1. $r = \alpha(L_{-k} \otimes L_{-2k}^* - L_{-2k}^* \otimes L_{-k})$, where $k \in \mathbb{Z}, \alpha \in \mathbb{C}$;
2. $r = \alpha(L_0 \otimes L_{-k}^* - L_{-k}^* \otimes L_0 + 2(L_{-k/2} \otimes L_{-3k/2}^* - L_{-3k/2}^* \otimes L_{-k/2}))$, where $k$ is a nonzero even integer and $\alpha \in \mathbb{C}$.

For the Virasoro algebra $V$, let $\{L_n^*\}_{n \in \mathbb{Z}} \cup \{c^*\}$ be the dual basis of $\{L_n\}_{n \in \mathbb{Z}} \cup \{c\}$. Then the Lie algebra structure on $V \ltimes_{\text{ad}^*} V^*$ is given by
\begin{equation}
[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12} \delta_{m+n,0} c, \quad [L_m, L_n^*] = (n-2m)L_{n-m}^*,
\end{equation}
\[ [L_m, c^*] = -\frac{m^3 - m}{12} L_{m}^*, \quad [L_m^*, L_n] = [L_m^*, c] = [L_m, c] = [c^*, c] = 0, \forall m, n \in \mathbb{Z}. \]

Note that the Rota-Baxter operators of weight 0 on $V$ given in Theorem \([3, 3]\) and \([3, 7]\) are all of finite rank. By Lemma \([4, 5]\) we obtain the following skew-symmetric solutions of the CYBE in $V \ltimes_{\text{ad}^*} V^*$.

1. $r = (\alpha L_0 + \mu c) \otimes L_{-k}^* + (\theta L_0 + \nu c) \otimes c^* - L_0^* \otimes (\alpha L_0 + \mu c) - c^* \otimes (\theta L_0 + \nu c)$, where $\alpha, \theta, \mu, \nu \in \mathbb{C}$;
2. $r = \alpha(c \otimes L_{-k}^* - L_{-k}^* \otimes c)$, where $\alpha \in \mathbb{C}, k \in \mathbb{Z} \setminus \{0\}$;
3. $r = \alpha(L_{-k} \otimes L_{-2k}^* - L_{-2k}^* \otimes L_{-k})$, where $\alpha \in \mathbb{C}^*, k \in \mathbb{Z} \setminus \{0\}$;
4. $r = \alpha(L_0 \otimes L_{-k}^* - L_{-k}^* \otimes L_0) + 2\alpha(L_{-k/2} \otimes L_{-3k/2}^* - L_{-3k/2}^* \otimes L_{-k/2}) + \mu(c \otimes L_{-k}^* - L_{-k}^* \otimes c)$, where $\alpha \in \mathbb{C}^*, \mu \in \mathbb{C}, k \in \mathbb{Z} \setminus \{0\}$;
5. $r = -\frac{k^2 - 1}{24} \alpha(L_0 \otimes L_{-k}^* - L_{-k}^* \otimes L_0) + \alpha(L_k \otimes c^* - c^* \otimes L_k)$, where $\alpha \in \mathbb{C}^*, k \in \mathbb{Z} \setminus \{0\}$.

\textbf{Lemma 4.3.} \([3]\) Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $R : \mathfrak{g} \to \mathfrak{g}$ be a linear map. Then $R$ is a Rota-Baxter operator of weight 1 on $\mathfrak{g}$ if and only if both $(R - R^{21}) + \text{Id}$ and $(R - R^{21}) - \text{Id}^{21}$ are solutions of the CYBE in $\mathfrak{g} \ltimes_{\text{ad}^*} \mathfrak{g}^*$.

\textbf{Remark 4.4.} Since $R$ is a Rota-Baxter operator of weight 1 on a Lie algebra $\mathfrak{g}$ if and only if $-R - \text{Id}$ is also a Rota-Baxter operator of weight 1 on $\mathfrak{g}$ and
\[ (-R - \text{Id}) - (-R - \text{Id})^{21} + \text{Id} = -((R - R^{21}) - \text{Id}^{21}), \]
we only list the solutions of the CYBE obtained from \( (R - R^{21}) + \text{Id} \).

For infinite dimensional vector spaces \( V_1, V_2 \), we define the formal tensor product \( V_1 \hat{\otimes} V_2 \) to be the space of formal series on the basis of \( V_1 \otimes V_2 \). Its elements are called formal tensors. A formal tensor can also be identified as an infinite matrix with the basis of \( V_1 \) as its row-index set and the basis of \( V_2 \) as its column-index set. We will not distinguish these two presentations in this paper.

For a Lie algebra \( \mathfrak{g} \), the CYBE in \( \mathfrak{g} \) is an equation of tensors in \( \mathfrak{g} \otimes \mathfrak{g} \). We need to generalize the notion of CYBE to formal tensors with suitable conditions.

Note that for \( r = \sum a_{ij}e_i \otimes e_j \in \mathfrak{g} \hat{\otimes} \mathfrak{g} \), the CYBE equals to the following equations:

\[
[r](e_i, e_j, e_k) := \sum_{s,t \in I}(C^i_{st}a_{sj}a_{tk} + a_{is}C^j_{st}a_{tk} + a_{is}a_{jt}C^k_{st}) = 0, \quad \forall i, j, k \in I,
\]

where \( C^i_{st} \) are the structural coefficients of \( \mathfrak{g} \). The summation is finite since only finitely many coefficients of \( r \) are nonzero.

An infinite matrix \( (a_{ij})_{i,j \in J} \) is said to be row-finite if each of its rows contains only finitely many nonzero entries. An infinite matrix is said to be column-finite if each of its columns contains only finitely many nonzero entries. For example, a linear map, viewed as an infinite matrix, is column-finite and vice versa. An infinite matrix which is both row-finite and column-finite is said to be row-and-column-finite.

For a formal tensor \( r = \sum a_{ij}e_i \otimes e_j \in \mathfrak{g} \hat{\otimes} \mathfrak{g} \), to ensure the summation in Eq. (4.5) is finite, we need \( (a_{ij})_{i,j \in I} \) to be a row-and-column-finite matrix.

Therefore, a formal tensor \( r = \sum a_{ij}e_i \otimes e_j \in \mathfrak{g} \hat{\otimes} \mathfrak{g} \) is called a solution of the formal CYBE if it is row-and-column-finite and satisfies Eq. (4.5).

A linear map \( R: \mathfrak{g} \rightarrow \mathfrak{g} \) can be identified as an element in \( \mathfrak{g} \hat{\otimes} \mathfrak{g}^* \subset (\mathfrak{g} \ltimes \text{ad}^* \mathfrak{g}^*) \hat{\otimes} (\mathfrak{g} \ltimes \text{ad}^* \mathfrak{g}^*) \) as follows. Let \( \{e_i\}_{i \in I} \) be a basis of \( \mathfrak{g} \) and \( \{e_i^*\}_{i \in I} \) be its dual defined by \( e_i^*(e_j) = \delta_{ij}, \quad \forall i, j \in I \).

By Zorn’s lemma, \( \{e_i^*\}_{i \in I} \) can be extended to a basis of \( \mathfrak{g}^* \), say \( \{e_i^*\}_{i \in I} \cup \{f_j^*\}_{j \in J} \). Then we have

\[
R = \sum_{i \in I} R(e_i) \otimes e_i^* + \sum_{j \in J} 0 \otimes f_j^* \in \mathfrak{g} \hat{\otimes} \mathfrak{g}^* \subset (\mathfrak{g} \ltimes \text{ad}^* \mathfrak{g}^*) \hat{\otimes} (\mathfrak{g} \ltimes \text{ad}^* \mathfrak{g}^*).
\]

If \( R^{21} \) is also column-finite, then we say \( R \) is balanced. Both Lemma 4.1 and 4.3 can be easily extended to the infinite dimensional case for balanced Rota-Baxter operators. Therefore we have the following conclusion.

**Lemma 4.5.** Let \( \mathfrak{g} \) be a Lie algebra and \( R: \mathfrak{g} \rightarrow \mathfrak{g} \) be a balanced linear map. Then we have the following results.

1. \( R \) is a Rota-Baxter operator of weight 0 on \( \mathfrak{g} \) if and only if \( r = R - R^{21} \) is a skew-symmetric solution of the formal CYBE in \( \mathfrak{g} \ltimes \text{ad}^* \mathfrak{g}^* \). In particular, when \( R \) is of finite rank, it coincides with the conclusion in Lemma 4.1, that is, in this case, the two corresponding solutions of (formal) CYBE coincide.

2. \( R \) is a Rota-Baxter operator of weight 1 on \( \mathfrak{g} \) if and only if both \( (R - R^{21}) + \text{Id} \) and \( (R - R^{21}) - \text{Id}^{21} \) are solutions of the formal CYBE in \( \mathfrak{g} \ltimes \text{ad}^* \mathfrak{g}^* \).
Indeed, by Eq. (4.1), one sees that Eq. (4.5) is trivial except for the cases: $[[r]](e_i^+, e_j^+, e_k^+)$, $[[r]](e_i^+, e_j, e_k^+)$ and $[[r]](e_i, e_j^+, e_k^+)$ for $i, j, k \in I$. However, for $r = (R - R^{21}) + \text{Id}$ (resp. $r = R - R^{(21)}$), these equations are nothing but Eq. (2.1) with $\lambda = 1$ (resp. $\lambda = 0$) and $x = e_i, y = e_j; x = e_j, y = e_k; x = e_j, y = e_k$ respectively.

Therefore, except for the solutions of CYBE obtained from the Rota-Baxter operators of weight 0 on $W$ which are of finite rank, Lemma 4.5 (1) gives the following solutions of the formal CYBE in $W_{\kappa_{ad}^*}W^*$ from the Rota-Baxter operators of weight 0 on $W$ which are of type (3) given in Theorem 2.15:

$$r = \sum_{n_0 | m + k} \frac{k}{m + 2k} \alpha (L_{m+k} \otimes L_m^* - L_m^* \otimes L_{m+k}),$$

where $k, n_0 \in \mathbb{Z}$, $k \neq 0$, $n_0 \nmid k$ and $\alpha \in \mathbb{C}^*$. Moreover, Lemma 4.5 (2) gives the following solutions of the formal CYBE in $W_{\kappa_{ad}^*}W^*$ from the Rota-Baxter operators of weight 1 on $W$ given in Theorem 2.21. Note that here $\text{Id} = \sum_{m \in \mathbb{Z}} L_m \otimes L_m^*$:

1. $r = \sum_{m \leq 1} L_m \otimes L_m^* + \sum_{m > 1} L_m^* \otimes L_m$;
2. $r = \sum_{m \geq -1} L_m \otimes L_m^* + \sum_{m < -1} L_m^* \otimes L_m$;
3. $r = \sum_{m} L_m \otimes L_m^*$;
4. $r = \sum_{m > 1} L_m \otimes L_m^* + \sum_{m \leq 1} L_m^* \otimes L_m$;
5. $r = \sum_{m > 1} L_m \otimes L_m^* + \sum_{m < 1} L_m^* \otimes L_m$;
6. $r = \sum_{m} L_m^* \otimes L_m$;
7. $r = \sum_{m < 0} L_m^* \otimes L_m + \sum_{m > 0} L_m \otimes L_m^* + (\alpha + 1) L_0 \otimes L_0^* - \alpha L_0^* \otimes L_0$, where $\alpha \in \mathbb{C}$;
8. $r = \sum_{m > 0} L_m^* \otimes L_m + \sum_{m < 0} L_m \otimes L_m^* + (\alpha + 1) L_0 \otimes L_0^* - \alpha L_0^* \otimes L_0$, where $\alpha \in \mathbb{C}$.

We remark that although the summation is infinite, the solution of formal CYBE on any highest irreducible representation of $W$ will be finite.

Lemma 4.5 (2) also gives the following solutions of the formal CYBE in $V_{\kappa_{ad}^*}V^*$ from the Rota-Baxter operators of weight 1 on $V$ given in Theorem 3.14. Note that here $\text{Id} = \sum_{m \in \mathbb{Z}} L_m \otimes L_m^* + c \otimes c^*$:

1. $r = \sum_{m \leq 1} L_m \otimes L_m^* + \sum_{m > 1} L_m^* \otimes L_m + (\theta L_0 + \nu c) \otimes c^* - c^* \otimes (\theta L_0 + \nu c) + c \otimes c^*$ where $\theta, \nu \in \mathbb{C}$;
2. $r = \sum_{m \geq -1} L_m \otimes L_m^* + \sum_{m < -1} L_m^* \otimes L_m + (\theta L_0 + \nu c) \otimes c^* - c^* \otimes (\theta L_0 + \nu c) + c \otimes c^*$ where $\theta, \nu \in \mathbb{C}$;
3. $r = \sum_{m} L_m \otimes L_m^* + c \otimes c^*$;
4. $r = \sum_{m > 1} L_m \otimes L_m^* + \sum_{m \leq 1} L_m^* \otimes L_m + (\theta L_0 + \nu c) \otimes c^* - c^* \otimes (\theta L_0 + \nu c) + c \otimes c^*$ where $\theta, \nu \in \mathbb{C}$;
5. $r = \sum_{m \leq 1} L_m \otimes L_m^* + \sum_{m \geq -1} L_m^* \otimes L_m + (\theta L_0 + \nu c) \otimes c^* - c^* \otimes (\theta L_0 + \nu c) + c \otimes c^*$ where $\theta, \nu \in \mathbb{C}$;
6. $r = \sum_{m} L_m^* \otimes L_m + c \otimes c^*$;
7. $r = \sum_{m < 0} L_m^* \otimes L_m + \sum_{m > 0} L_m \otimes L_m^* + ((\alpha + 1) L_0 + \mu c) \otimes L_0^* - \alpha L_0^* \otimes (\alpha L_0 + \mu c) + (\theta L_0 + \nu c) \otimes c^* - c^* \otimes (\theta L_0 + \nu c) + c \otimes c^*$, where $\alpha, \theta, \mu, \nu \in \mathbb{C}$;
8. $r = \sum_{m > 0} L_m^* \otimes L_m + \sum_{m < 0} L_m \otimes L_m^* + ((\alpha + 1) L_0 + \mu c) \otimes L_0^* - \alpha L_0^* \otimes (\alpha L_0 + \mu c) + (\theta L_0 + \nu c) \otimes c^* - c^* \otimes (\theta L_0 + \nu c) + c \otimes c^*$, where $\alpha, \theta, \mu, \nu \in \mathbb{C}$.
We remark that although the summation is infinite, the solution of the formal CYBE on any highest irreducible representation of $V$ will be a finite expression.

5. **Induced pre-Lie algebras from Rota-Baxter operators of weight 0 on Witt and Virasoro algebras**

**Definition 5.1.** A pre-Lie algebra $A$ is a vector space $A$ with a bilinear product $\ast$ satisfying

\[(x \ast y) \ast z - x \ast (y \ast z) = (y \ast x) \ast z - y \ast (x \ast z), \quad \forall x, y, z \in A.\]

**Proposition 5.2.** Let $(g, [\cdot, \cdot])$ be a Lie algebra and $R : g \rightarrow g$ be a Rota-Baxter operator of weight 0. Define a new operation $x \ast y = [R(x), y]$ for any $x, y \in A$. Then $(g, \ast)$ is a pre-Lie algebra.

Let $(A, \ast)$ be a pre-Lie algebra. Then the commutator

\[[x, y] = x \ast y - y \ast x, \forall x, y \in A\]

defines a Lie algebra $g(A)$ called the sub-adjacent Lie algebra of $A$ and $A$ is also called a compatible pre-Lie algebra on the Lie algebra $g(A)$.

5.1. **Induced pre-Lie algebras from Rota-Baxter operators of weight 0 on Witt algebra $W$.**

**Theorem 5.3.** In the sense of Proposition 5.2, the homogeneous Rota-Baxter operators $R_k$ of weight 0 with degree $k$ on the Witt algebra $W$ obtained in Theorem 2.15 give the following pre-Lie algebras:

1. $L_m \ast L_n = 0$, for any $m, n \in \mathbb{Z}$.
2. $L_m \ast L_n = \begin{cases} (k-n)L_{n-k} & m = 0; \\ 0 & m \neq 0, \end{cases}$
   where $k \in \mathbb{Z}$.
3. $L_m \ast L_n = \begin{cases} (k-n)L_n & m = 0; \\ (k-2n)L_{-\frac{k}{2}+n} & m = -\frac{k}{2}; \\ 0 & m \neq 0, -\frac{k}{2}; \end{cases}$
   where $k$ is a nonzero even number.
4. $L_m \ast L_n = \begin{cases} \frac{(m+k-n)k}{m+k}L_{m+n} & m \in n_0\mathbb{Z}; \\ 0 & m \notin n_0\mathbb{Z}, \end{cases}$
   where $k \neq 0$, $n_0 \in \mathbb{Z}$ satisfying $n_0 \nmid k$.

Moreover, these pre-Lie algebras are not mutually isomorphic.

**Proof.** The conclusion follows from Proposition 3.2 by a direct computation. Note that we use a linear transformation defined by $L_m \rightarrow \frac{1}{\alpha}L_m$ for any $\alpha \in \mathbb{C}^*$ (also see Remark 2.16). Moreover, we also use a degree shifting by for $L_m \rightarrow L_{m+2k}$ in the above (2), $L_m \rightarrow L_{m+k}$ in the above (3) and $L_m \rightarrow L_{m+k}$ in the above (4) respectively. It is also straightforward to show that these pre-Lie algebras are not mutually isomorphic. \qed

The following conclusion is an immediate consequence.

**Proposition 5.4.** The sub-adjacent Lie algebras of the pre-Lie algebras in 5.3 are given as follows respectively:
(1) \([L_m, L_n] = 0\), for any \(m, n \in \mathbb{Z}\).

(2) \([L_m, L_n] = \begin{cases} (k - n)L_{n-k} & m = 0, n \neq 0; \\ 0 & m, n \neq 0; \end{cases}\)

where \(k \in \mathbb{Z}\):

(3) \([L_m, L_n] = \begin{cases} (k - n)L_n & m = 0, n \neq 0, -\frac{k}{2}; \\ (k - 2n)L_{-\frac{k}{2}+n} & m = -\frac{k}{2}, n \neq 0, -\frac{k}{2}; \\ \frac{k}{2}L_{-\frac{k}{2}} & m = 0, n = -\frac{k}{2}; \\ 0 & m, n \neq 0, -\frac{k}{2}, \end{cases}\)

where \(k\) is a nonzero even number;

(4) \([L_m, L_n] = \begin{cases} (m + k - n)kL_{m+n} & m \in n_0\mathbb{Z}, n \notin n_0\mathbb{Z}; \\ (m - n)(m + n + k)kL_{m+n} & m, n \in n_0\mathbb{Z}; \\ 0 & m, n \notin n_0\mathbb{Z}, \end{cases}\)

where \(k \neq 0\), \(n_0 \in \mathbb{Z}\) satisfying \(n_0 \nmid k\).

5.2. Induced pre-Lie algebras from Rota-Baxter operators of weight 0 on Virasoro algebra \(V\).

Theorem 5.5. In the sense of Proposition 5.2, the homogeneous Rota-Baxter operators \(R_0\) of weight 0 with degree 0 on the Virasoro algebra \(V\) obtained in Theorem 3.3 give the following pre-Lie algebras:

(1) \(L_m \ast L_n = c \ast L_m = L_m \ast c = 0\) for any \(m, n \in \mathbb{Z}\).

(2) \(L_m \ast L_n = -n\delta_{m,0}L_n, c \ast L_n = -nL_n, L_n \ast c = 0\), for any \(m, n \in \mathbb{Z}\).

(3) \(L_m \ast L_n = -n\delta_{m,0}L_n, c \ast L_n = L_n \ast c = 0\), for any \(m, n \in \mathbb{Z}\).

(4) \(L_m \ast L_n = 0, c \ast L_n = -nL_n, L_n \ast c = 0\), for any \(m, n \in \mathbb{Z}\).

Moreover, these pre-Lie algebras are not mutually isomorphic.

Proof. By Proposition 5.2 we show that the induced pre-Lie algebra from \(R_0\) is given by

\(L_m \ast L_n = -n\alpha\delta_{m,0}L_n, c \ast L_n = -n\theta L_n, L_n \ast c = 0, \forall m, n \in \mathbb{Z}\),

where \(\alpha, \theta \in \mathbb{C}\). For \(\alpha \neq 0\) or \(\theta \neq 0\), we use the linear transformation by \(L_m \rightarrow \frac{1}{\alpha}L_m\) for any \(m \in \mathbb{Z}\) or \(c \rightarrow \frac{1}{\theta}c\). Then the conclusion follows. \(\square\)

The following conclusion is obtained immediately.

Proposition 5.6. The sub-adjacent Lie algebras of the pre-Lie algebras in Theorem 5.5 are given as follows respectively:

(1) \([L_m, L_n] = [c, L_m] = 0\) for any \(m, n \in \mathbb{Z}\).

(2) \([L_m, L_n] = -n\delta_{m,0}L_n + m\delta_{n,0}L_m, [c, L_n] = -nL_n\), for any \(m, n \in \mathbb{Z}\).

(3) \([L_m, L_n] = -n\delta_{m,0}L_n + m\delta_{n,0}L_m, [c, L_n] = 0\), for any \(m, n \in \mathbb{Z}\).

(4) \([L_m, L_n] = 0, [c, L_n] = -nL_n\), for any \(m, n \in \mathbb{Z}\).

Theorem 5.7. In the sense of Proposition 5.2, the homogeneous Rota-Baxter operators \(R_k\) of weight 0 with degree \(k \neq 0\) on the Virasoro algebra \(V\) obtained in Theorem 3.4 give the following pre-Lie algebras:

(1) \(L_m \ast L_n = c \ast L_m = L_m \ast c = 0\) for any \(m, n \in \mathbb{Z}\).
(2) \( L_m \ast L_n = \delta_{m,0} \left((k - n)L_{n-k} + \frac{k-3}{12} \delta_{n,3k}c\right), \quad c \ast L_n = L_n \ast c = 0, \) for any \( m, n \in \mathbb{Z}, \)

where \( k \in \mathbb{Z} \setminus \{0\}. \)

(3) \( L_m \ast L_n = \delta_{m,0}((k-n)L_n + \delta_{m,-\frac{k}{2}} \left((k-2n)L_{-\frac{k}{2}+n} + \frac{k-3}{6} \delta_{n,3k}c\right)), \quad c \ast L_n = L_n \ast c = 0, \) for any \( m, n \in \mathbb{Z}, \) where \( k \in 2\mathbb{Z} \setminus \{0\}. \)

(4) \( L_m \ast L_n = -\frac{k^2}{24} \delta_{m,0} c \ast L_n, \quad c \ast L_m = (k-n)L_{n+k} + \frac{k^3-k}{12} \delta_{n,-k}c, \quad L_n \ast c = 0, \) for any \( m, n \in \mathbb{Z}, \) where \( k \in \mathbb{Z} \setminus \{0\}. \)

Moreover, these pre-Lie algebras are not mutually isomorphic.

\textbf{Proof.} The conclusion follows from Proposition 5.2 by a direct computation. Note that we use a linear transformation given by \( L_m \rightarrow \frac{1}{\alpha}L_m \) and \( c \rightarrow \frac{1}{\alpha}c \) for any \( \alpha \in \mathbb{C}^* \) (also see Remark 3.8). Moreover, we also use a degree shifting: \( L_m \rightarrow L_{m+2k} \) for (2) and \( L_m \rightarrow L_{m+k} \) for (3) respectively. It is easy to check that these pre-Lie algebras are not mutually isomorphic. \( \square \)

\textbf{Proposition 5.8.} The sub-adjacent Lie algebras of the pre-Lie algebras in Theorem 5.7 are given as follows respectively:

\( (1) \quad [L_m, L_n] = [c, L_m] = 0 \) for any \( m, n \in \mathbb{Z}. \)

\( (2) \quad [L_m, L_n] = \delta_{m,0}((k-n)L_{n-k} + \frac{k-3}{12} \delta_{n,3k}c) - \delta_{n,0}((k-m)L_{m-k} + \frac{k-3}{12} \delta_{m,3k}c), \)

\( [c, L_n] = 0, \) for any \( m, n \in \mathbb{Z}, \) where \( k \in \mathbb{Z} \setminus \{0\}. \)

\( (3) \quad [L_m, L_n] = \delta_{m,0}((k-n)L_n + \frac{1}{2} \delta_{m,-\frac{k}{2}} \left((k-2n)L_{-\frac{k}{2}+n} + \frac{k-3}{6} \delta_{n,3k}c\right)) - \delta_{n,0}(k-m)L_{m-k} - \delta_{n,-\frac{k}{2}} \left((k-2m)L_{-\frac{k}{2}+m} + \frac{k-3}{6} \delta_{m,3k}c\right), \)

\( [c, L_n] = 0, \) for any \( m, n \in \mathbb{Z}, \) where \( k \in 2\mathbb{Z} \setminus \{0\}. \)

\( (4) \quad [L_m, L_n] = -\frac{k^2}{24} \delta_{m,0} [c, L_n] + \frac{k^2}{24} \delta_{n,0} [c, L_m], \quad [c, L_n] = (k-n)L_{n+k} + \frac{k^3-k}{12} \delta_{n,-k}c, \)

for any \( m, n \in \mathbb{Z}, \) where \( k \in \mathbb{Z} \setminus \{0\}. \)

6. \textbf{Induced PostLie algebras from Rota-Baxter operators of weight 1 on the Witt and Virasoro algebras}

\textbf{Definition 6.1.} (\textit{[16]}) A \textit{PostLie algebra} is a Lie algebra \((\mathfrak{g}, [\cdot, \cdot])\) with a bilinear product \( \circ \) satisfying the following equations:

\( (6.1) \quad ((x \circ y) \circ z - x \circ (y \circ z)) - ((y \circ x) \circ z - y \circ (x \circ z)) + [x, y] \circ z = 0, \)

\( (6.2) \quad z \circ [x, y] - [z \circ x, y] - [x, z \circ y] = 0, \)

for all \( x, y, z \in \mathfrak{g}. \) We denote it by \((\mathfrak{g}, [\cdot, \cdot], \circ)\).

\textbf{Lemma 6.2.} (\textit{[2]}) Let \((\mathfrak{g}, [\cdot, \cdot])\) be a Lie algebra and \( R : \mathfrak{g} \rightarrow \mathfrak{g} \) be a Rota-Baxter operator of weight 1. Define a new operation \( x \circ y = [R(x), y]. \) Then \((\mathfrak{g}, [\cdot, \cdot], \circ)\) is a PostLie algebra.

Let \((\mathfrak{g}, [\cdot, \cdot], \circ)\) be a PostLie algebra. Then the following operation (cf. \textit{[2]})

\( (6.3) \quad \{x, y\} = x \circ y - y \circ x + [x, y], \quad \forall x, y \in \mathfrak{g}, \)

defines a Lie algebra structure on \( \mathfrak{g}. \)
6.1. Induced PostLie algebras from Rota-Baxter operators of weight 1 on the Witt algebra $W$.

**Theorem 6.3.** In the sense of Lemma 6.2, the homogeneous Rota-Baxter operators $R_0$ of weight 1 with degree zero on the Witt algebra $W$ provided in Theorem 2.21 give the following PostLie algebras $(W, [\cdot, \cdot], \circ)$, where $(W, [\cdot, \cdot])$ is the Witt algebra:

- (1) $L_m \circ L_n = \begin{cases} -(m-n)L_{m+n} & m \geq 2; \\ 0 & m \leq 1. \end{cases}$
- (2) $L_m \circ L_n = 0$, for any $m, n \in \mathbb{Z}$.
- (3) $L_m \circ L_n = \begin{cases} -(m-n)L_{m+n} & m \leq 1; \\ 0 & m \geq 2. \end{cases}$
- (4) $L_m \circ L_n = -(m-n)L_{m+n}$, for any $m, n \in \mathbb{Z}$.
- (5) $L_m \circ L_n = \begin{cases} -\alpha nL_n & m = 0; \\ 0 & m > 0, \end{cases}$ where $\alpha \in \mathbb{C}$.

Moreover, these PostLie algebras are not mutually isomorphic.

**Proof.** The conclusion follows from Lemma 6.2 by a direct computation. Note that the Rota-Baxter operators of type (1) and (2) in Theorem 2.21 give the PostLie algebra (1); the Rota-Baxter operators of type (3) in Theorem 2.21 give the PostLie algebra (2); the Rota-Baxter operators of type (4) and (5) in Theorem 2.21 give the PostLie algebra (3); the Rota-Baxter operators of type (6) in Theorem 2.21 give the PostLie algebra (4); the Rota-Baxter operators of type (7) and (8) in Theorem 2.21 give the PostLie algebras (5). In fact, the PostLie algebras obtained by (2), (5) and (8) in Theorem 2.21 are isomorphic to the PostLie algebras obtained by (1), (4) and (7) respectively through the linear transformation $L_m \rightarrow -L_{-m}$. Moreover, it is also straightforward to show that these PostLie algebras are not mutually isomorphic. \hfill \Box

**Proposition 6.4.** The PostLie algebras in Theorem 6.3 give the following Lie algebras $\{\cdot, \cdot\}$ in the sense of Eq. (6.5):

- (1) $\{L_m, L_n\} = \begin{cases} -2(m-n)L_{m+n} & m, n \geq 2; \\ -(m-n)L_{m+n} & m \geq 2, n \leq 1; \\ 0 & m, n \leq 1. \end{cases}$
- (2) $\{L_m, L_n\} = 0$, for any $m, n \in \mathbb{Z}$.
- (3) $\{L_m, L_n\} = \begin{cases} -2(m-n)L_{m+n} & m, n \leq 1; \\ -(m-n)L_{m+n} & m \leq 1, n \geq 2; \\ 0 & m, n \geq 2. \end{cases}$
- (4) $\{L_m, L_n\} = -2(m-n)L_{m+n}$, for any $m, n \in \mathbb{Z}$.
\[
\begin{aligned}
(5) \quad \{L_m, L_n\} &= \begin{cases}
-2(m-n)L_{m+n} & m, n < 0; \\
-(m-n)L_{m+n} & m < 0, n > 0; \\
-(1-\alpha)mL_m & m < 0, n = 0; \\
-\alpha nL_n & m = 0, n > 0; \\
0 & m > 0, n > 0,
\end{cases}
\end{aligned}
\]

where \(\alpha \in \mathbb{C}\).

**Remark 6.5.** For the first Lie algebra, the subalgebra spanned by \(\{L_m|m, n \geq 2\}\) is a Lie subalgebra isomorphic to the subalgebra of \(W\) spanned by the same set.

For the third Lie algebra, the subalgebra spanned by \(\{L_m|m, n < 1\}\) is a Lie subalgebra isomorphic to the subalgebra of \(W\) spanned by the same set.

For the fourth Lie algebra, it is isomorphic to \(W\).

For every Lie algebra in the fifth class, the subalgebra spanned by \(\{L_m|m, n < 0\}\) is a Lie subalgebra isomorphic to the subalgebra of \(W\) spanned by the same set.

### 6.2. Induced PostLie algebras from Rota-Baxter operators of weight 1 on the Virasoro algebra \(V\).

**Theorem 6.6.** In the sense of Lemma \[6.2\], the homogeneous Rota-Baxter operators \(R_0\) of weight 1 with degree zero on the Virasoro algebra \(V\) given in Theorem \[3.14\] give rise to the following PostLie algebras \((V, [,], \circ)\), where \((V, [,])\) is the Virasoro algebra:

\[
(1) \quad L_m \circ L_n = \begin{cases}
-(m-n)L_{m+n} - \frac{m^3-m}{12}\delta_{m+n,0}c & m \geq 2; \\
0 & m \leq 1,
\end{cases}
\]

\[
(2) \quad L_m \circ L_n = \theta n L_n, \quad L_m \circ c = 0, \quad \text{where } \theta \in \mathbb{C}, \text{ for any } m, n \in \mathbb{Z}.
\]

\[
(3) \quad L_m \circ L_n = \begin{cases}
-(m-n)L_{m+n} - \frac{m^3-m}{12}\delta_{m+n,0}c & m \leq 1; \\
0 & m \geq 2,
\end{cases}
\]

\[
(4) \quad L_m \circ L_n = -(m-n)L_{m+n} - \frac{m^3-m}{12}\delta_{m+n,0}c, \quad c \circ L_n = L_m \circ c = 0, \quad \text{for any } m, n \in \mathbb{Z}.
\]

\[
(5) \quad L_m \circ L_n = \begin{cases}
-(m-n)L_{m+n} - \frac{m^3-m}{12}\delta_{m+n,0}c & m < 0; \\
-\alpha nL_n & m = 0; \\
0 & m > 0,
\end{cases}
\]

\[
c \circ L_n = -\theta n L_n, \quad L_m \circ c = 0, \quad \text{where } \alpha, \theta \in \mathbb{C}, \text{ for any } m, n \in \mathbb{Z}.
\]

**Proof.** The conclusion follows from Lemma \[6.2\] by a direct computation. Note that the Rota-Baxter operators of type (1) and (2) in Theorem \[3.14\] give the PostLie algebras (1); the Rota-Baxter operator of type (3) in Theorem \[3.14\] gives the PostLie algebra (2); the Rota-Baxter operators of type (4) and (5) in Theorem \[3.14\] give the PostLie algebras (3); the Rota-Baxter operator of type (6) in Theorem \[3.14\] gives the PostLie algebra in (4); the Rota-Baxter operators of type (7) and (8) in Theorem \[3.14\] give the PostLie algebras (5). In fact, the PostLie algebras obtained by Rota-Baxter operators of type (2), (5) and (8) in Theorem \[3.14\] are isomorphic to the PostLie algebras obtained by Rota-Baxter operators of type (1), (4) and (7) respectively through the linear transformation of basis \(L_m \to -L_{-m}, c \to -c\). Moreover, it is also straightforward to show that these PostLie algebras are not mutually isomorphic. \(\Box\)
Proposition 6.7. The PostLie algebras in Theorem 6.6 give rise to the following Lie algebras \{c, L_n\} in the sense of Eq. (6.3):

(1) \( \{L_m, L_n\} = \begin{cases} -2(m - n)L_{m+n} - \frac{m^3-m}{6}\delta_{m+n,0}c & m, n \geq 0; \\ 0 & m, n \leq 1; \end{cases} \)

(2) \( \{L_m, L_n\} = \{c, L_n\} = 0, \text{ for any } m, n \in \mathbb{Z}. \)

(3) \( \{L_m, L_n\} = \begin{cases} -2(m - n)L_{m+n} - \frac{m^3-m}{12}\delta_{m+n,0}c & m, n \leq 0; \\ 0 & m, n \geq 1; \end{cases} \)

(4) \( \{L_m, L_n\} = \begin{cases} -2(m - n)L_{m+n} - \frac{m^3-m}{6}\delta_{m+n,0}c & m, n \geq 0; \\ 0 & m, n \leq 1; \end{cases} \)

(5) \( \{L_m, L_n\} = \begin{cases} -2(m - n)L_{m+n} - \frac{m^3-m}{12}\delta_{m+n,0}c & m, n \geq 0; \\ 0 & m, n \leq 1; \end{cases} \)

Remark 6.8. For the Lie algebra in the first class with \( \theta = 0 \), the subalgebra spanned by \( \{L_m|m, n \geq 2\} \cup \{c\} \) is a Lie subalgebra isomorphic to the subalgebra of \( V \) spanned by the same set.

For the Lie algebra in the third class with \( \theta = 0 \), the subalgebra spanned by \( \{L_m|m, n < 1\} \cup \{c\} \) is a Lie subalgebra isomorphic to the subalgebra of \( V \) spanned by the same set.

For the fourth Lie algebra, it is isomorphic to \( V \).

For every Lie algebra in the fifth class with \( \theta = 0 \), the subalgebra spanned by \( \{L_m|m, n < 0\} \cup \{c\} \) is a Lie subalgebra isomorphic to the subalgebra of \( V \) spanned by the same set.

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