Research Article

A Normalized Transfer Matrix Method for the Free Vibration of Stepped Beams: Comparison with Experimental and FE(3D) Methods

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Received 2 June 2017; Revised 22 August 2017; Accepted 15 October 2017; Published 28 November 2017

1. Introduction

Stepped beam-like structure plays an important role in the construction of mechanical and civil engineering systems. Flexural vibrations were first investigated by Euler-Bernoulli in the eighteenth century. The rotary inertia effect was considered by Rayleigh [1]. Almost 95 years ago, Timoshenko introduced a correction for the beam theory to include the shear deformation effect [2]. The effect of rotary inertia and shear deformations on the beam natural frequencies is small at the lower normal modes and large at the higher normal modes. Cowper [3] derived formulae for the precise evaluation of the shear coefficient for rectangular and round cross sections as a function of Poisson’s ratio υ.

The free vibrations of beams with discontinuities can be solved using either exact or approximate solution. The exact methods include the derivation of the transcendental eigenvalue equations in order to evaluate the beam natural frequencies. In the case of relatively simple problems closed form solutions for the eigenvalue problem were obtained [4, 5].

For complicated problems, the beam eigenvalues are obtained by the decomposition of the domain. Numerical assembly technique (NAT) is one of the common methods used for the evaluation of the eigenvalue problem for beams with multiple discontinuity [6–8]. In this method, the size of the frequency equation determinant is $4n \times 4n$ for beam with $n$ segments. Dynamic stiffness matrix is one of methods that is similar to the finite element method in assembling of the elements but with exact element rather than approximate element [9, 10]. The frequency equation determinant size for $n$ segments is $(2n + 2) \times (2n + 2)$. The Laplace transformation method is used to obtain a solution for a Timoshenko beam mounted on elastic foundation with several combinations of discrete in-span attachments and with several combinations of attachments at the boundaries [11].
The exact free vibration of a mechanical system composed of two elastic Timoshenko segments carried on an intermediate eccentric rigid body or on elastic supports was introduced by Farghaly and El-Sayed [34]. Their analysis was based on both analytical and experimental methods. They claimed a good agreement between the analytical and experimental results. Experimental setup using electromagnetic-acoustic transducer (EMAT) was introduced by Díaz-De-Anda et al. [35]. They compared the experimental results with those obtained theoretically using Timoshenko beam theory (TBT) with one and two shear coefficients. The flexural frequencies and amplitudes for cylindrical and rectangular Timoshenko beams were examined experimentally [35]. They found that the experimental results coincide very well with theoretical predictions. The transverse vibration of Bernoulli-Euler beams with discontinuous geometry and elastic support was investigated experimentally and analytically [36].

During the last decades, many literatures were focused on the problem of free and forced vibration analysis of Timoshenko beam and the accuracy of the natural frequency predictions. To the authors’ knowledge, there is not enough research that has tackled the experimental modal frequencies of stepped thick beams, computationally and experimentally. Therefore, the main aim of this work is to investigate the results of the modal frequencies for such beams using analytical, experimental, and the three-dimensional finite element FE(3D). An analytical analysis is proposed which is based on the derivation of a set of fundamental solutions that suits the analysis of Timoshenko beams. This set of solutions is used to modify the TMM to include no inverse matrix procedure which may be called normalized transfer matrix method (NTM). The comparison between the experimental NTM and FE(3D) is done for selected single-step and two-step application models. The percentage deviations between NTM and FE(3D) are investigated. The results show that the finite element results are very close to the experimental results. The study includes the effect of increasing the step ratio \( \hat{d} \), step location parameter \( \mu \), and the length ratio. Finally, the capability of the present analysis to solve the free vibration of tapered beams has been investigated.

2. Mathematical Model

The mathematical model for beam with multiple-stepped sections is shown in Figure 1. The total length of the beam is \( L \). The beam model is divided into \( n \) segments. The beam has \( (n + 1) \) stations as shown in Figure 1. The station numbering corresponding to the start, intermediate, and end location is represented by \( (1, i + 1, n + 1) \), respectively. At each station, there are linear and rotational elastic supports and concentrated mass with mass moment of inertia. As shown in Figure 1, the beam segments are described by their material and cross-sectional properties \( \rho_s, E_s, G_s, A_s \) and \( I_i \ (i = 1, 2, \ldots, n) \), which are the density, Young’s modulus of elasticity, rigidity modulus, cross-sectional area, and second moment of inertia, respectively. In this section, the frequency equation of the model is driven using the proposed normalized transfer matrix method. Since the current analysis is based on Timoshenko beam theory, the rotary inertia and
shear deformations effects are considered. In the current analysis, the analytical solution is subject to the assumptions that the shear strain is assumed constant over the cross-section; therefore, a shear coefficient $k_i$ is used to compensate this assumption. In addition, the effect of stress concentration at the beam steps is neglected.

2.1. Analytical Method and Frequency Equation. The objective of this section is to derive the system frequency equation which represents the model shown in Figure 1. Timoshenko differential coupled equations of motion may be written here for $i$th span as follows:

$$
(k'GA_i)(y_i'' - \psi_i')(x_i, t) - \rho_i A_i \ddot{y}_i (x_i, t) = 0,
$$

$$
(EI)_i \psi_i''(x_i, t) - \rho_i I \ddot{\psi}_i (x_i, t) + (k'GA_i)(y_i'' - \psi_i')(x_i, t) = 0.
$$

(1)

Let

$$
y_i = Y_i(\xi_i) e^{j\omega t},
$$

$$
\psi_i = \Psi_i(\xi_i) e^{j\omega t},
$$

$$
\xi_i = \frac{x_i}{L_i},
$$

where $Y_i$ is the normal function of $y_i$, $\Psi_i$ is the normal function of $\psi_i$, $\xi_i$ is nondimensional length of each beam span $i$, and $f = \sqrt{-1}$.

Substituting (2) into (1) and omitting the factor $e^{j\omega t}$, the following equations can be derived:

$$
y_i''(\xi_i) + \lambda_i^4 \xi_i^2 Y_i(\xi_i) - L_i \Psi_i'(\xi_i) = 0,
$$

$$
s_i^2 L_i \Psi_i''(\xi_i) + \left( \lambda_i^4 r_i^2 s_i^2 - 1 \right) L_i \Psi_i'(\xi_i) + Y_i'(\xi_i) = 0,
$$

(3)

(4)

where

$$
\lambda_i^4 = \frac{\rho_i A_i L_i^4 \omega^2}{E_i I_i},
$$

$$
s_i^2 = \frac{2 r_i^2 (1 + \nu_i)}{k_i'},
$$

(5a)

(5b)

After decoupling the functions $Y_i(\xi_i)$ and $L_i \Psi_i(\xi_i)$ in (3)-(4), the decoupled fourth-order differential equations in the nondimensional form can be written as

$$
Y_i'''(\xi_i) + \alpha_i Y_i''(\xi_i) + \beta_i^2 Y_i(\xi_i) = 0,
$$

$$
L_i \Psi_i'''(\xi_i) + \alpha_i L_i \Psi_i''(\xi_i) + \beta_i^2 L_i \Psi_i(\xi_i) = 0,
$$

(6)

(7)

where

$$
\alpha_i = \lambda_i^4 \left( r_i^2 + s_i^2 \right),
$$

$$
\beta_i^2 = \lambda_i^4 \left( \lambda_i^4 r_i^2 s_i^2 - 1 \right).
$$

(7a)

(7b)

The general solution of (6) and (7) can be written, respectively, in the form

$$
Y_i(\xi_i) = C_{11} \sin(\alpha \xi_i) + C_{21} \cos(\alpha \xi_i) + C_{31} \sinh(\beta \xi_i) + C_{41} \cosh(\beta \xi_i),
$$

$$
L_i \Psi_i(\xi_i) = - \left( \frac{\delta_{11}}{a_i} \right) C_{12} \cos(\alpha \xi_i) + \left( \frac{\delta_{21}}{a_i} \right) C_{22} \sin(\alpha \xi_i) + \left( \frac{\delta_{31}}{b_i} \right) C_{32} \sinh(\beta \xi_i) + \left( \frac{\delta_{41}}{b_i} \right) C_{42} \cosh(\beta \xi_i).
$$

(8)

Here,

$$
a_i^2 = \left( \frac{\alpha_i}{2} \right) + \sqrt{\left( \frac{\alpha_i}{2} \right)^2 - \beta_i^2},
$$

$$
b_i^2 = \left( \frac{\alpha_i}{2} \right) + \sqrt{\left( \frac{\alpha_i}{2} \right)^2 - \beta_i^2}.
$$

(9a)

(9b)
One can derive the expressions of $\delta_{ii}$ and $\delta_{ij}$ using (8), together with (3) or (4) in the form

$$\delta_{ii} = s_i^2 \lambda_i^2 - a_i^2, \quad \delta_{ij} = s_i^2 \lambda_i^2 + b_i^2.$$  \hfill (10a)

Here $i$ denotes the $i$th span, $i = 1, 2, \ldots, n$ in the case of multispans.

In order to introduce the current analysis, the linearly independent fundamental solutions $SY_i(\xi), SY_{ii}(\xi), SY_{ij}(\xi), SY_{ij}(\xi)$ and the corresponding $SY_{ii}(\xi), SY_{ii}(\xi), SY_{ij}(\xi), SY_{ij}(\xi)$ are derived. In order to simplify the solution of Timoshenko beam, the following dependent functions are defined:

$$\Gamma_1(\xi) = Y_1'(\xi) - L_1Y_1(\xi),$$  \hfill (11a)

$$\Gamma_{ii}(\xi) = SY_{ii}'(\xi) - SY_{ii}(\xi),$$  \hfill (11b)

$$\Gamma_{ij}(\xi) = SY_{ij}'(\xi) - SY_{ij}(\xi),$$  \hfill (11c)

$$\Gamma_{ij}(\xi) = SY_{ij}'(\xi) - SY_{ij}(\xi).$$  \hfill (11d)

The Timoshenko solution will be normalized at the origin of coordinates as follows:

$$[SY_{ii}(0) \ \ SY_{ii}'(0) \ \ SY_{ij}'(0) \ \ SY_{ij}(0)]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$  \hfill (12)

Substituting the general solution of (8) in each raw in (12) we get the following set of fundamental solutions:

$$SY_{ii}(\xi) = \frac{1}{\delta_{ii} - \delta_{ij}} (-\delta_{ij} \cos (a_i \xi)$$

$$+ \delta_{ii} \cosh (b_i \xi)), \quad \xi \in (-a_i, a_i)$$  \hfill (13a)

$$SY_{ij}(\xi) = \frac{1}{\delta_{ij}^2 + \delta_{ii}^2 } (a_i (\delta_{ij} - b_i^2) \sin (a_i \xi)$$

$$+ b_i (\delta_{ij} + a_i^2) \sinh (b_i \xi)), \quad \xi \in (-a_i, a_i)$$  \hfill (13b)

$$SY_{ij}(\xi) = \frac{1}{\delta_{ij} - \delta_{ii}} (\cos (a_i \xi) - \cosh (b_i \xi)), \quad \xi \in (-a_i, a_i).$$  \hfill (13c)
\[ \Phi_1 = \frac{\phi_1 L_1}{E_1 I_1}, \quad \text{(16d)} \]

\[ \overline{m}_1 = \frac{m_1}{\rho_1 A_1 L_1}, \quad \text{(16e)} \]

where \( J_1 \) is the mass moment of inertia at station 1, \( k_1 \) and \( \phi_1 \) are the linear and rotational elastic supports at station 1, respectively, and \( m_1 \) is the concentrated mass at station 1; see Figure 1 for details.

Substituting the solutions presented in (13a), (13b), (13c), (13d), (13e), (13f), (13g), (13h)-(14) into (15), the following equations are obtained:

\[ \left( \overline{T}_1 \lambda^4 - \Phi_1 \right) \psi_{10} + \psi'_{10} = 0, \]

\[ s_1^2 (\overline{m}_1 \lambda^4 - Z_1) \psi_{10} + y_{10} = 0. \quad \text{(17)} \]

The start boundary conditions in (17) can be presented in matrix form as

\[ \begin{bmatrix} 0 & (\overline{T}_1 \lambda^4 - \Phi_1) & 1 & 0 \\ s_1^2 (\overline{m}_1 \lambda^4 - Z_1) & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{10} \\ \psi_{10} \\ \psi'_{10} \\ y_{10} \end{bmatrix} = 0. \quad \text{(18)} \]

This equation can be simply written as

\[ [U^T] [\Delta]_1 = 0, \quad \text{(19)} \]

where

\[ [\Delta]_1 = [y_{10} \quad \psi_{10} \quad \psi'_{10} \quad y_{10}]^T, \quad \text{(20)} \]

where the superscript \( ^T \) indicates vector transpose.

At station \((n+1)\), the beam end boundary conditions can be written in the nondimensional form as

\[ \psi'_{n} (1) + \left( \Phi_{n+1} - \lambda^4_{n+1} \right) \psi_{n} (1) = 0, \]

\[ \Gamma_n (1) + \left( Z_{n+1} - \overline{m}_{n+1} \lambda^4_n \right) s_n^2 \psi_{n} (1) = 0. \quad \text{(21)} \]

Equation (23) can be written in the matrix form as

\[ \begin{bmatrix} (\psi'_{n} (1) + (\Phi_{n+1} - \lambda^4_{n+1}) \psi_{n} (1) ) & (\psi'_{n} (1) + (\Phi_{n+1} - \lambda^4_{n+1}) \psi_{n} (1) ) & (\psi'_{n} (1) + (\Phi_{n+1} - \lambda^4_{n+1}) \psi_{n} (1) ) & (\psi'_{n} (1) + (\Phi_{n+1} - \lambda^4_{n+1}) \psi_{n} (1) ) \\
(\psi_{n} (1) + s_n^2 (Z_{n+1} - \overline{m}_{n+1} \lambda^4_n) \psi_{n} (1) ) & (\psi_{n} (1) + s_n^2 (Z_{n+1} - \overline{m}_{n+1} \lambda^4_n) \psi_{n} (1) ) & (\psi_{n} (1) + s_n^2 (Z_{n+1} - \overline{m}_{n+1} \lambda^4_n) \psi_{n} (1) ) & (\psi_{n} (1) + s_n^2 (Z_{n+1} - \overline{m}_{n+1} \lambda^4_n) \psi_{n} (1) ) \end{bmatrix} \begin{bmatrix} \psi_{n} (1) \\ \psi'_{n} (1) \\ \psi_{n+1} (1) \\ \psi'_{n+1} (1) \end{bmatrix} = 0. \quad \text{(24)} \]

This equation can be simply written as

\[ [U^E]_{2 \times 4} [\Delta]_n = 0, \quad \text{(25)} \]

where

\[ [\Delta]_n = [\psi_{n0} \quad \psi'_{n0} \quad \psi_{n0} \quad \psi'_{n0}]^T. \quad \text{(26)} \]

The beam intermediate continuity conditions can be presented in nondimensional form as

\[ Y_1 (1) = Y_{i+1} (0), \]

\[ L_{(i+1)j} L_j Y_1 (1) = L_{(i+1)j} Y_{i+1} (0), \]
\begin{align*}
(EL)_{i+1} + L^2_{i+1}(\psi_{i+1}^0 + (\phi_{i+1} - J_{i+1}a_i^2)\psi_{i+1})
&= L_{i+1}\psi_{i+1}(0), \\
\left(k'\Gamma A\right)_{i+1} + L_{i+1}(\Gamma_{i+1} + s_i^2(Z_{i+1} - m_{i+1}a_i^2)Y_i(1))
&= \Gamma_{i+1}(0),
\end{align*}

where

\begin{align*}
J_{i+1} &= \frac{J_{i+1}}{E_{i+1}}, \\
\lambda_i &= \frac{k_{i+1}L_i^2}{E_{i+1}}, \\
Z_{i+1} &= \frac{k_{i+1}L_i^2}{E_{i+1}}, \\
\phi_{i+1} &= \frac{\phi_{i+1}L_i}{E_{i+1}}, \\
m_{i+1} &= \frac{m_{i+1}}{\rho_iA_iL_i},
\end{align*}

where $J_{i+1}$ is the mass moment of inertia at station $(i + 1)$, $k_{i+1}$ and $\phi_{i+1}$ are the linear and rotational stiffness at station $(i + 1)$, respectively, and $m_{i+1}$ is the concentrated mass at station $(i + 1)$.

Substituting the solutions in (13a), (13b), (13c), (13d), (13e), (13f), (13g), (13h)-(14) into (27), we get the following equations:

\begin{align*}
\begin{bmatrix}
SY_{1i}(1) \\
L_{i-1}\psi_{i+1}(1) \\
W_1(\psi_{i+1}(1) + (\phi_{i+1} - J_{i+1}a_i^2)\psi_{i+1}(1)) \\
W_1(\psi_{i+1}(1) + (\phi_{i+1} - J_{i+1}a_i^2)\psi_{i+1}(1)) \\
W_1(SY_{1i}(1) + \frac{\gamma_i}{\Delta}S\Psi_{1i}(1)) \\
W_1(SY_{1i}(1) + \frac{\gamma_i}{\Delta}S\Psi_{1i}(1))
\end{bmatrix}
&= \begin{bmatrix}
Y_{i+10} \\
Y_{i+10} \\
\psi_{i+10} \\
\psi_{i+10}
\end{bmatrix},
\end{align*}

where

\begin{align*}
W_1 &= (EI)_{i+1} + L^2_{i+1}, \\
W_2 &= \left(k'\Gamma A\right)_{i+1} + L_{i+1}.
\end{align*}

Equation (30) can be presented as

\begin{align*}
[T]_{4\times4} \ast [\Delta]_1 &= [\Delta]_{i+1},
\end{align*}

where

\begin{align*}
[\Delta]_i &= \begin{bmatrix} Y_{i+10} & \psi_{i+10} & \psi_{i+10} & Y_{i+10} \end{bmatrix},
[\Delta]_{i+1} &= \begin{bmatrix} Y_{i+10} & \psi_{i+10} & \psi_{i+10} & Y_{i+10} \end{bmatrix}.
\end{align*}

From (32), one can find that

\begin{align*}
[\Delta]_n &= \left[T\right]_{n-1} \ast [\Delta]_{n-1} = \left[T\right]_{n-1} \ast \left[T\right]_{n-2} \cdots \ast \left[T\right]_2 \ast [\Delta]_1.
\end{align*}

The intermediate spans transfer matrix can be presented as

\begin{align*}
\left[T\right]_{4\times4} &= \left[T\right]_{n-1} \ast \left[T\right]_{n-2} \cdots \ast \left[T\right]_2;
\end{align*}

then (34) can be presented as

\begin{align*}
[\Delta]_n &= \left[T\right]_{4\times4} \ast [\Delta]_1.
\end{align*}
Substituting (36) into the end condition of (25) results in the following equations:

\[
\begin{bmatrix} U^E \end{bmatrix}_{2\times4} \begin{bmatrix} T \end{bmatrix}_{4\times4} \{\Delta\}_1 = 0, \tag{37}
\]

\[
\begin{bmatrix} U^{IE} \end{bmatrix}_{2\times4} = \begin{bmatrix} U^E \end{bmatrix}_{2\times4} \begin{bmatrix} T \end{bmatrix}_{4\times4}, \tag{38}
\]

\[
\begin{bmatrix} U^{IE} \end{bmatrix}_{2\times4} \{\Delta\}_1 = 0. \tag{39}
\]

The general beam equation can be presented using the start boundary condition in (19) and the beam intermediate and end condition in (39) as shown below

\[
\begin{bmatrix} U^{tot} \end{bmatrix}_{4\times4} \{\Delta\}_1 = 0, \tag{40}
\]

where

\[
\begin{bmatrix} U^{tot} \end{bmatrix}_{4\times4} = \begin{bmatrix} U^S \end{bmatrix}_{2\times4} + \begin{bmatrix} U^{IE} \end{bmatrix}_{2\times4}. \tag{41}
\]

Equating the determinant of [\(U^{tot}\)] by zero results in the system frequency equation. In general, the TMM has advantages over the traditional methods in that the final frequency equation is \(4 \times 4\) for any number of beam segments. The advantage of the current method NTM over the TMM is significant in the using of tailored solution that is normalized at the origin of coordinates. This type of solution enables the formulation of the system equations without the need to any inverse matrix procedures as shown previously. This reduces the computational time comparing with the TMM.

2.2. Finite Element Method. Among the numerical tools, finite element method is considered one of most efficient methods to perform the vibration analysis of mechanical and structural components. In this section, finite element is used to obtain the natural frequencies and mode shapes of uniform and stepped beams. ANSYS finite element commercial package is used to perform the finite element analysis. The analysis is done using three-dimensional (3D) solid element models and SOLID95 elements are used for meshing. Since all the experimentally investigated samples in the current work are round and stepped. The beam cross-section is free meshed using 87 SOLID95 elements for smaller cross-section and 171 elements for the larger cross-section. This mesh is then extruded using 40 elements along the length of the beam. The total number of the element is ranging from 3480 (40 \(\times\) 87) to 6840 (40 \(\times\) 171) elements based on the location of the step; see Figure 2. Modal analysis module is used in this analysis and Block Lanczos method is used for the mode extraction method. The finite element model FE(3D) results are compared with those obtained experimentally and analytically.

3. Results and Discussion

3.1. Verification and Validation of NTM Results

3.1.1. Verification Example 1. In this example, the first five nondimensional natural frequencies (\(\lambda_i^* = \rho_i A_i L^4 \omega_i^2 / E_i I_i\)) of stepped beam are compared with the exact solution presented by Gutierrez et al. [18]; see Figure 3. The model is solved at two different step locations and several values of \(b_{21}\) and \(h_{21}\) as shown in Table 1. Three different values of rotary inertia \(r_{21}^2 = vs, 0.0036\) and \(0.01\) are considered in order to validate the current model in case of Bernoulli-Euler and Timoshenko beams. The value of \(s_{21}^2 = 3.12r_{21}^2\) is considered in order to evaluate the shear deformation [18]. The values of nondimensional linear and rotational elastic supports stiffness at start and the end are \((Z_1 = 10, \Phi_1 = vl),\) \((Z_3 = vs, \Phi_3 = vl),\) respectively. The results of Table 1 show that the present NTM results are in good agreement with the exact solution presented by Gutierrez et al. [18].

3.1.2. Verification Example 2. In this example, Timoshenko beam with three-step round cross-section presented in [19] is investigated; see Figure 4. An intermediate lumped mass of \(m_i^* = m_i / \rho_i A_i L = 1\) is located at a distance of 750 mm from point 1. The input data for this example is listed in the caption of Figure 4. Table 2 shows the results of the first five natural frequencies \(\omega_i\) in (rad/sec) for pinned-pinned, free-clamped, clamped-free, clamped-pinned, and clamped-clamped configurations. The results of reference [19] are
Table 1: The first five natural frequencies $\lambda_i^{*2}$ for stepped Timoshenko beam in the case where the beam is rigidly restrained against rotation $\phi_1^* = nf$ and elastically restrained in translation $Z_1^* = 10$ in comparison with [18].

| $r_i^2$ | $\mu$ | $b_1$ | $h_1$ | $\lambda_1^{*2}$ | $\lambda_2^{*2}$ | $\lambda_3^{*2}$ | $\lambda_4^{*2}$ | $\lambda_5^{*2}$ |
|---------|-------|-------|-------|-------------------|-------------------|-------------------|-------------------|-------------------|
| $10^{-7}$ |       |       |       |                   |                   |                   |                   |                   |
| [18]    | 1.0   | 0.8   |       | 3.010            | 9.696             | 34.010            | 74.430            | 132.341           |
| Present NTM | 0.25 | 0.8   | 0.8   | 3.249            | 9.664             | 34.315            | 74.645            | 131.803           |
| [18]    | 0.8   | 0.6   |       | 3.385            | 8.117             | 28.630            | 60.997            | 103.439           |
| Present NTM | 1.0   | 0.8   |       | 3.851            | 8.171             | 28.632            | 60.9943           | 103.4075          |
| [18]    | 0.50  | 0.8   | 0.8   | 3.124            | 10.165            | 34.688            | 80.145            | 139.666           |
| Present NTM | 0.8   | 0.6   |       | 3.249            | 10.1655           | 34.6882           | 80.570            | 139.1753          |
| [18]    | 0.8   | 0.6   |       | 3.284            | 9.499             | 28.501            | 70.123            | 118.482           |
| Present NTM | 0.0036| 0.8   | 0.8   | 3.284            | 9.4989            | 28.5006           | 70.1232           | 118.4830          |
| [18]    | 1.0   | 0.8   |       | 3.007            | 9.668             | 33.571            | 72.393            | 126.111           |
| Present NTM | 0.25 | 0.8   | 0.8   | 3.245            | 9.635             | 33.862            | 72.607            | 125.617           |
| [18]    | 0.8   | 0.6   |       | 3.381            | 8.103             | 28.364            | 59.839            | 100.245           |
| Present NTM | 1.0   | 0.8   |       | 3.812            | 10.103            | 38.643            | 59.8385           | 100.2447          |
| [18]    | 0.50  | 0.8   | 0.8   | 3.120            | 10.0304           | 32.926            | 71.741            | 116.725           |
| Present NTM | 0.8   | 0.6   |       | 3.274            | 9.391             | 27.414            | 64.020            | 102.697           |
| [18]    | 0.01  | 0.8   | 0.8   | 3.2721           | 9.3908            | 27.4490           | 64.0200           | 102.6966          |
| Present NTM | 1.0   | 0.8   |       | 3.001            | 9.619             | 32.841            | 69.229            | 117.238           |
| [18]    | 0.25  | 0.8   | 0.8   | 3.239            | 9.586             | 33.109            | 69.444            | 116.798           |
| Present NTM | 0.8   | 0.6   |       | 3.374            | 8.078             | 27.914            | 57.970            | 95.370            |
| [18]    | 1.0   | 0.8   |       | 2.926            | 9.710             | 30.711            | 61.768            | 95.623            |
| Present NTM | 0.50  | 0.8   | 0.8   | 2.9255           | 9.7104            | 30.7111           | 61.7680           | 95.6230           |
| [18]    | 0.8   | 0.6   |       | 3.0908           | 9.8086            | 30.4643           | 61.9117           | 95.3257           |
| Present NTM | 0.01  | 0.8   | 0.8   | 3.251            | 9.211             | 25.893            | 56.578            | 86.647            |

Figure 3: Verification Example 1 [18], two-span stepped beam.
Table 2: Comparison of the first five natural frequencies, in rad/sec, using the present NTM results with those obtained in [19] for the example shown in Figure 4.

| Method          | \( \omega \) (rad/sec) | Bernoulli-Euler \( r_1^2 = 0 \) | Timoshenko \( r_1^2 \neq 0 \) |
|-----------------|------------------------|---------------------------------|-------------------------------|
| Pinned-pinned   | \( \omega_1 \)        | NAT [19] 319.4341               | 316.5288                      |
| (P-P)           | \( \omega_2 \)        | NTM 319.4340                    | 316.4855                      |
|                 | \( \omega_3 \)        | NAT [19] 1853.3864              | 1789.4207                     |
|                 | \( \omega_4 \)        | NTM 1853.3864                   | 1784.8723                     |
|                 | \( \omega_5 \)        | NAT [19] 4110.1341              | 3825.8438                     |
|                 |                        | NTM 4110.1341                   | 3836.5347                     |
| Free-clamped    | \( \omega_1 \)        | NAT [19] 371.5354               | 367.9440                      |
| (F-C)           | \( \omega_2 \)        | NTM 371.5354                    | 367.1487                      |
|                 | \( \omega_3 \)        | NTM 1243.9063                   | 1216.6220                     |
|                 | \( \omega_4 \)        | NTM 1243.9065                   | 1211.5331                     |
|                 | \( \omega_5 \)        | NTM 3082.0846                   | 2827.0352                     |
|                 |                        | NTM 3082.0845                   | 2849.1465                     |
| Clamped-free    | \( \omega_1 \)        | NAT [19] 55.1257                | 55.0130                       |
| (C-F)           | \( \omega_2 \)        | NTM 616.6018                    | 598.5917                      |
|                 | \( \omega_3 \)        | NTM 2702.3238                   | 2532.1483                     |
|                 | \( \omega_4 \)        | NTM 5483.2379                   | 4969.1659                     |
|                 | \( \omega_5 \)        | NTM 9570.9586                   | 7956.5299                     |
| Clamped-pinned  | \( \omega_1 \)        | NTM 480.8915                    | 469.8444                      |
| (C-P)           | \( \omega_2 \)        | NTM 2098.0556                   | 1991.9611                     |
|                 | \( \omega_3 \)        | NTM 5007.3315                   | 4589.6576                     |
|                 | \( \omega_4 \)        | NTM 8366.8073                   | 7049.5393                     |
|                 | \( \omega_5 \)        | NTM 13678.7112                  | 11215.0349                    |
| Clamped-clamped | \( \omega_1 \)        | NTM 836.3527                    | 8070.9977                     |
| (C-C)           | \( \omega_2 \)        | NTM 3019.5886                   | 2772.2732                     |
|                 | \( \omega_3 \)        | NTM 5583.6469                   | 4980.5538                     |
|                 | \( \omega_4 \)        | NTM 9625.8943                   | 7734.8319                     |
|                 | \( \omega_5 \)        | NTM 14403.3163                  | 11538.1581                    |

Figure 4: Verification Example 2 [19], three-step beam having \( E = 206.9 \) GPa, \( \rho = 7836.9 \) kg/m\(^3\), \( G = 79.5769 \) GPa, and \( k' = 0.75 \).
Table 3: Comparison of the present NTM results with those obtained in [5, 20], for F-F beam shown in Figure 5.

| Mode number | DQEM [20] I | Exp. [5] II | NTM (40) III | FE(3D) IV | % error |
|-------------|-------------|-------------|--------------|----------|---------|
| 1           | 292.440     | 291         | 290.316      | 290.120  | 0.302   |
| 2           | 1181.300    | 1165        | 1162.158     | 1167.100 | -0.180  |
| 3           | 1804.100    | 1771        | 1767.321     | 1772.200 | -0.068  |

Figure 5: Verification Example 3 [5, 20], single-step (F-F) aluminum beam having, $E = 71.7$ GPa, $\rho = 2830$ Kg/m$^3$, and $v = 0.33$.

3.1.3. Verification Example 3. The third verification example is shown in Figure 5 with identical dimensions to that used in [5, 20]. The rotary inertia and shear deformation are considered. The shear coefficient $k'$ is calculated based on [3], keeping Poisson's ratio $\nu = 0.33$. Table 3 shows the first three nonzero free-free (F-F) eigenvalues in Hz and in comparison with [5, 20]. The present results are computed using both numerically (40) and FE(3D) methods. The results of Table 3 show that the present analysis results are very close to the experimental results. The percentage error between the present FE(3D) results and the experimental results of [5] is less than 0.302%. This represents the importance of including the effect of rotary inertia and shear deformation.

3.1.4. Verification Example 4. The fourth example is shown in Figure 6. It is for a twenty-span uniform beam carrying 19 equally spaced concentrated masses with $m_i^* = 0.1$. Several beam start and end conditions are investigated as listed in Table 4. The results are evaluated using the present NTM method and previously published numerical assembly technique NAT [6] method. The computational time required to obtain the first three frequency parameters $\lambda_1^*, \lambda_2^*$, and $\lambda_3^*$ using both NAT and NTM methods is calculated and listed in Table 4. Considerable reduction in computational time is observed in all the investigated cases as shown in Table 4.

3.2. Test Samples and Experimental Procedures. In order to measure the natural frequencies of the system under study, the free-free test samples were put in free oscillations by using an instrumented hammer model B and K 8202. An accelerometer model B and K 4366 is fixed to the shaft in order to capture the vibration signal. The output of the charge amplifier B and K 4365 is connected to NI 6216 data acquisition card. This card is connected to the PC and managed by Lab VIEW software. Figure 10 shows a photo of the current experimental setup. The used card settings are
Table 4: Validation of the present NTM using 20-span uniform beam with 19 equally spaced concentrated masses \( m_i^* = 0.1 \), \( r_i^2 = s_i^2 = v_s \) in comparison with [6].

| Method         | \( Z_1^*, \Phi_1^* \) | \( Z_2^*, \Phi_2^* \) | \( \lambda_1^* \) | \( \lambda_2^* \) | \( \lambda_3^* \) | Computational time (sec) |
|----------------|----------------------|----------------------|------------------|------------------|------------------|-------------------------|
| NAT [6]        | \( v_s, v_s \)       | \( v_s, v_s \)       | 3.7177           | 6.1716           | 8.6392           | 48.17                   |
| Present NTM    | \( v_s, v_s \)       | \( v_s, v_s \)       | 3.7177           | 6.1716           | 8.6392           | 13.53                   |
| NAT [6]        | \( v_l, v_s \)       | \( v_l, v_s \)       | 2.3871           | 4.7742           | 7.1612           | 34.67                   |
| Present NTM    | \( v_l, v_s \)       | \( v_l, v_s \)       | 2.3871           | 4.7742           | 7.1612           | 12.14                   |
| NAT [6]        | \( v_l, v_l \)       | \( v_l, v_l \)       | 3.5941           | 5.9671           | 8.35471          | 40.40                   |
| Present NTM    | \( v_l, v_l \)       | \( v_l, v_l \)       | 3.5941           | 5.9671           | 8.35471          | 13.47                   |
| NAT [6]        | 10, 10               | 10, 10               | 1.5989           | 2.5329           | 4.5969           | 22.13                   |
| Present NTM    | 10, 10               | 10, 10               | 1.5989           | 2.5329           | 4.5969           | 10.23                   |

Table 5: Typical samples dimension of group S2 with \( \mu = 0.75 \) shown in Figure 7.

| Group S2   | \( L \) | \( L_1 \) | \( L_2 \) | \( \mu \) | \( d_1 \) | \( d_2 \) | \( m_i \) (kg) | \( r_i^2 \) |
|------------|--------|--------|--------|--------|--------|--------|-------------|--------|
| S2-40/40   | 200    | 40     | 160    | 0.2    | 40     | 30     | 1.282       | 0.002  |
| S2-40/100  | 200    | 100    | 100    | 0.5    | 40     | 30     | 1.375       | 0.002  |
| S2-40/140  | 200    | 140    | 60     | 0.7    | 40     | 30     | 1.541       | 0.002  |

Figure 7: Schematic drawings for the experimental samples Group S2.

Sample frequency of 20 kHz, sampling time of 3, 5, 8 sec, and the size of samples block to read is 1k. The time domain data is captured and transformed into frequency domain. The resonant frequencies were obtained by the average of the results of 10 impacts applied in three different locations of the sample. Figures 7–9 show two groups of different stepped test samples, S2 and S3, respectively. All the test samples are manufactured from steel rods of dimensions Ø 80 and Ø 40 mm. The common data for the steel test samples are \( \rho = 7850 \text{ kg/m}^3 \), \( E = 206.8 \text{ GPa} \), and \( k' = 0.845 \).

Two-span twelve test samples are shown in group S2, namely, S2-40/40, S2-40/100, and S2-40/140, shown in Figure 7. The dimensions of these samples are presented in Table 5. The natural frequency results of these samples are presented in Table 7 using experimental, analytical, and FE(3D) methods. The captured signal for the free vibration response for selected case from Table 7 is shown in Figure 11. This case is italicized in the table. The results of Table 7 show that the three-dimensional FE solution is closer to the experimental results than the analytical results. The maximum percentage error between the experimental and the FE results is less than 0.5%.

Group S3 consists of four different samples, namely, S3-80/250, S3-80/200, S3-80/150, and S3-80/100. More details about the geometrical dimensions and material properties for these samples are shown in Table 6 and Figure 8. The results of Table 8 show that the deviation between the FE(3D) and the experimental is less than 1.16%. The free vibration
Table 6: The sample dimensions of group S3 as shown in Figure 8.

| Group S3 | $L$ | $L_1$ | $L_2$ | $d_1$ | $d_2$ | $m_i$ (kg) | $L_*^*$ | $r_1^*$ |
|----------|-----|-------|-------|-------|-------|------------|--------|--------|
| S3-80/250 | 500 | 200   | 250   | 20    | 80    | 10.480     | 0.5    | 0.000100 |
| S3-80/200 | 500 | 225   | 200   | 20    | 80    | 7.398      | 0.4    | 0.000100 |
| S3-80/150 | 500 | 250   | 150   | 20    | 80    | 6.781      | 0.3    | 0.000100 |
| S3-80/100 | 500 | 275   | 100   | 20    | 80    | 4.932      | 0.2    | 0.000100 |

Figure 8: Schematic drawings for the experimental samples Group S3.

Figure 9: Real image for typical experimental test samples.
signals of selected case italicized in Table 8 are plotted in Figure 12. In general, the results of the three-span samples reveal the same conclusion drawn from the two-span samples that the FE(3D) results are closer to the experimental results than the analytical results. The conclusion driven from the investigated two- and three-span samples is that the three-dimensional finite element can be trusted in the prediction of the natural frequency of stepped beam.

3.3. Percentage Modal Deviation between FE(3D) and NTM Method. The results of the previous section show the accuracy of the FE(3D) model in evaluating the natural frequencies of stepped beam. Therefore, in this section, a free-free two-span model is deeply investigated using (40) and FE(3D) methods to justify the validity of the analytical solution in predicting the stepped beam results. Two categories of samples are considered as shown in Figure 13. The first category includes 600 mm length samples with \( r^2 = 2.78 \times 10^{-4} \) and the second category includes 200 mm length samples with \( r^2 = 205 \times 10^{-3} \). The effects of changing \( \mu \) and \( \overline{d} \) are investigated, \( \mu \) varies from 0 to 1, and meanwhile, only three values are listed in Table 9. Four different values of \( \overline{d} \) are investigated 0.5, 0.625, 0.75, and 0.875. The study in
Table 7: Percentage error between the computational and experimental results for single-step test samples (Group S2). Lowest four nonzero free-free modes.

| $d$  | $L_1$ | $\mu$ | Method          | 1     | 2     | 3     | 4     |
|------|-------|-------|-----------------|-------|-------|-------|-------|
| 40   | 0.2   |       | (A) EXP         | 1848.940 | 5028.140 | 9854 | 16303 |
|      |       |       | (B) NTM         | 1873.981 | 5194.229 | 10211.79 | 16776.400 |
|      |       |       | (C) FE(3D)      | 1859.490 | 5056.749 | 9893.345 | 16380.170 |
|      |       |       | (D) % error (A, B) | 1.354 | 3.303 | 3.630 | 2.903 |
|      |       |       | (E) % error (A, C) | 0.570 | 0.568 | 0.399 | 0.473 |
|      |       |       | (A) EXP         | 2153.900 | 7713.530 | 13701.800 | 19845.900 |
|      |       |       | (B) NTM         | 2244.601 | 7923.127 | 13661.990 | 20230.700 |
| 20/40| 100   | 0.5   | (C) FE(3D)      | 2142.012 | 7700.169 | 13663.580 | 20634.360 |
|      |       |       | (D) % error (A, B) | 4.211 | −0.290 | 1.938 | 1.938 |
|      |       |       | (E) % error (A, C) | −0.551 | −0.173 | −0.278 | −0.264 |
|      |       |       | (A) EXP         | 3596.820 | 8103.960 | 15773.100 | 20733.800 |
|      |       |       | (B) NTM         | 3813.426 | 8160.003 | 15739.160 | 21174.340 |
|      |       |       | (C) FE(3D)      | 3581.521 | 8090.723 | 15705.070 | 20634.360 |
|      |       |       | (D) % error (A, B) | 6.022 | 0.691 | −0.215 | 2.124 |
|      |       |       | (E) % error (A, C) | −0.425 | −0.163 | −0.431 | −0.479 |
|      |       |       | (A) EXP         | 2430.240 | 6510.630 | 12186.700 | 19359.300 |
|      |       |       | (B) NTM         | 2444.111 | 6638.069 | 12492.520 | 19588.580 |
|      |       |       | (C) FE(3D)      | 2430.512 | 6518.094 | 12223.080 | 19314.540 |
|      |       |       | (D) % error (A, B) | 4.158 | 1.957 | 2.509 | 1.184 |
|      |       |       | (E) % error (A, C) | 0.011 | 0.298 | −0.231 | 0.173 |
|      |       |       | (A) EXP         | 2796.050 | 8693.500 | 14542.200 | 22182.430 |
|      |       |       | (B) NTM         | 2912.321 | 8781.311 | 14633.780 | 22182.430 |
|      |       |       | (C) FE(3D)      | 2806.782 | 8698.963 | 14551.070 | 22095.920 |
|      |       |       | (D) % error (A, B) | 4.158 | 1.010 | 0.629 | 0.218 |
|      |       |       | (E) % error (A, C) | 0.383 | 0.060 | −0.172 | −0.172 |
|      |       |       | (A) EXP         | 3023.690 | 7906   | 14381.400 | 21856.600 |
|      |       |       | (B) NTM         | 3023.321 | 7946.457 | 14429.850 | 21806.140 |
|      |       |       | (C) FE(3D)      | 3014.442 | 7878.443 | 14307.960 | 21781.750 |
|      |       |       | (D) % error (A, B) | −0.012 | 0.511 | 0.336 | −0.230 |
|      |       |       | (E) % error (A, C) | −0.305 | −0.348 | −0.510 | −0.342 |
|      |       |       | (A) EXP         | 3401.940 | 9329.910 | 15681.400 | 23607.700 |
|      |       |       | (B) NTM         | 3478.703 | 9293.555 | 15691.770 | 23326.460 |
|      |       |       | (C) FE(3D)      | 3405.324 | 9313.349 | 15645.690 | 23513.650 |
|      |       |       | (D) % error (A, B) | 2.256 | −0.389 | 0.066 | −1.191 |
|      |       |       | (E) % error (A, C) | 0.999 | −0.177 | −0.227 | −0.398 |
|      |       |       | (A) EXP         | 4201.680 | 9322.630 | 16596.400 | 23967.700 |
|      |       |       | (B) NTM         | 4268.272 | 9431.643 | 16437.830 | 23729.240 |
|      |       |       | (C) FE(3D)      | 4211.100 | 9322.286 | 16530.200 | 23881.400 |
|      |       |       | (D) % error (A, B) | 1.584 | 1.169 | −0.955 | −0.994 |
|      |       |       | (E) % error (A, C) | 0.224 | −0.003 | −0.398 | −0.360 |
Table 7: Continued.

| \(d\) | \(L_1\) | \(\mu\) | Method       | \(f_1\) | \(f_2\) | \(f_3\) | \(f_4\)    |
|-----|-----|-----|-------------|------|------|------|-------|
| 40  | 0.2 |     | (A) EXP     | 3608.280 | 9126.360 | 16128.600 | 23817.3 |
|     |     |     | (B) NTM     | 3607.895 | 9104.530 | 16004.810 | 23500.960 |
|     |     |     | (C) FE(3D)  | 3607.465 | 9106.544 | 16054.460 | 23720.110 |
|     |     |     | (D) % error (A, B) | \(-0.010\) | \(-0.239\) | \(-0.792\) | \(-1.328\) |
|     |     |     | (E) % error (A, C) | \(-0.022\) | \(-0.217\) | \(-0.459\) | \(-0.408\) |
| 35/40 | 100 | 0.5 | (A) EXP     | 3885.030 | 9778.800 | 16719.900 | 24538.800 |
|     |     |     | (B) NTM     | 3906.518 | 9705.047 | 16577.640 | 24125.780 |
|     |     |     | (C) FE(3D)  | 3882.419 | 9759.382 | 16658.180 | 24417.410 |
|     |     |     | (D) % error (A, B) | \(0.553\) | \(-0.754\) | \(-0.850\) | \(-1.683\) |
|     |     |     | (E) % error (A, C) | \(-0.067\) | \(-0.198\) | \(-0.369\) | \(-0.495\) |
| 140 | 0.7 |     | (A) EXP     | 4248.240 | 9888.540 | 17028.000 | 24738.600 |
|     |     |     | (B) NTM     | 4253.660 | 9878.822 | 16849.850 | 24311.230 |
|     |     |     | (C) FE(3D)  | 4247.271 | 9870.286 | 16963.900 | 24616.760 |
|     |     |     | (D) % error (A, B) | \(0.127\) | \(-0.098\) | \(-1.046\) | \(-1.726\) |
|     |     |     | (E) % error (A, C) | \(-0.022\) | \(-0.184\) | \(-0.376\) | \(-0.491\) |

Table 8: Percentage error between the computational and experimental results for two-step samples (Group S3). Lowest four nonzero free-free modes.

| Method       | \(f_1\) | \(f_2\) | \(f_3\) | \(f_4\) |
|-------------|------|------|------|-------|
| S3-80/250   |      |      |      |       |
| (A) EXP     | 403.148 | 2126.460 | 4139.830 | 5153.500 |
| (B) NTM     | 420.212 | 2203.984 | 4334.192 | 5605.556 |
| (C) FE(3D)  | 404.180 | 2126.900 | 4153.700 | 5169.800 |
| (D) % error (A, B) | \(-3.966\) | \(-3.517\) | \(-4.484\) | \(-8.428\) |
| (E) % error (A, C) | \(-0.255\) | \(-0.020\) | \(-0.333\) | \(-0.315\) |
| S3-80/200   |      |      |      |       |
| (A) EXP (Figure 12) | 354.790 | 1727.900 | 2291.490 | 4531.380 |
| (B) NTM     | 367.381 | 1788.472 | 2551.555 | 4660.983 |
| (C) FE(3D)  | 355.230 | 1728.300 | 2318.400 | 4532.200 |
| (D) % error (A, B) | \(-3.549\) | \(-3.386\) | \(-10.192\) | \(-2.860\) |
| (E) % error (A, C) | \(0.124\) | \(0.023\) | \(1.160\) | \(0.018\) |
| S3-80/150   |      |      |      |       |
| (A) EXP     | 335.859 | 1360.050 | 1545.130 | 3778.200 |
| (B) NTM     | 346.243 | 1434.113 | 1638.174 | 3889.033 |
| (C) FE(3D)  | 336.440 | 1364.200 | 1548.500 | 3781.900 |
| (D) % error (A, B) | \(-3.091\) | \(-5.445\) | \(-6.021\) | \(-2.933\) |
| (E) % error (A, C) | \(0.172\) | \(0.305\) | \(0.218\) | \(0.097\) |
| S3-80/100   |      |      |      |       |
| (A) EXP     | 337.400 | 1018.240 | 1371.070 | 3221.200 |
| (B) NTM     | 345.942 | 1074.574 | 1424.992 | 3303.837 |
| (C) FE(3D)  | 337.480 | 1024.000 | 1372.900 | 3215.000 |
| (D) % error (A, B) | \(-2.531\) | \(-5.532\) | \(-3.932\) | \(-2.565\) |
| (E) % error (A, C) | \(0.023\) | \(0.565\) | \(0.133\) | \(-0.192\) |
this section focuses only on the first three nonzero free-free modes. The percentage modal deviation in analytical NTM solution prediction in reference to the FE(3D) solution is presented in Figure 13 for long samples and in Figure 14 for short samples. This percentage deviation is calculated using the following formula:

\[
\text{Dev}_i = \frac{f_{(\text{NTM})i} - f_{(\text{FE})i}}{f_{(\text{FE})i}} \times 100, \tag{42}
\]

where

- \(\text{Dev}_i\) is the percentage deviation in the \(i\)th modal frequency,
- \(f_{(\text{FE})i}\) is the \(i\)th mode natural frequency using three-dimensional finite element,
- \(f_{(\text{NTM})i}\) is the \(i\)th mode natural frequency using analytical NTM method.

The results of Table 9 show that, for the investigated examples with step ratio smaller than one, the increase in \(\mu\) and/or \(d\) increases the modal frequencies. The percentage deviations in the analytical NTM results for the short samples are higher than those for the longer samples. The percentage deviations in the analytical natural frequency prediction are plotted for the long and short examples in Figures 14 and 15, respectively. Figures 14(a) and 15(a) present a plot for \(\text{Dev}_1\), Figures 14(b) and 15(b) present a plot for \(\text{Dev}_2\), and Figures 14(c) and 15(c) present a plot for \(\text{Dev}_3\). Figure 16(a) presents the first mode shape at the conditions of the peak point in Figure 15(a), Figure 16(b) presents the second mode shape at the conditions of the peak point in Figure 15(b), and Figure 16(c) presents the third mode shape at the conditions of the peak point in Figure 15(c).

Figures 14(a) and 15(a) show that, for long and short stepped samples, the \(\text{Dev}_1\) attains the peak when \(\mu\) lies
Table 9: Typical relative deviation between present NTM and finite element (3D) results. Lowest three nonzero modes for two categories of single-step F-F beam; see Figure 13.

| \( \bar{a} \) | Method | \( \mu \) | \( L = 600 \text{ mm} \) | Modal frequencies in Hz | \( L = 200 \text{ mm} \) |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 1 | 2 | 3 |
| 0.5 | NTM | 0.1 | 217.034 | 635.604 | 1267.590 | 1878.302 | 5163.860 | 9645.650 |
| | | 0.4 | 232.036 | 797.543 | 1834.110 | 1995.497 | 6510.740 | 13831.40 |
| | | 0.7 | 447.629 | 1037.270 | 2253.940 | 3813.426 | 8160 | 15739.200 |
| | FE(3D) | 0.1 | 216.987 | 634.850 | 1263.259 | 1876.267 | 5123.790 | 9448.138 |
| | | 0.4 | 229.554 | 785.722 | 1819.305 | 1930.166 | 6271.645 | 13642.314 |
| | | 0.7 | 437.591 | 1033.915 | 2245.125 | 3813.426 | 8091.330 | 15706.439 |
| | % dev. | 0.1 | −0.022 | −0.119 | −0.343 | −0.108 | −0.782 | −2.090 |
| | | 0.4 | −1.082 | −1.504 | −0.814 | −3.385 | −3.812 | −1.386 |
| | | 0.7 | −2.294 | −0.324 | −0.392 | −6.431 | −0.849 | −0.208 |
| 0.625 | NTM | 0.1 | 283.930 | 812.357 | 1608.220 | 2446.347 | 6532.940 | 11979.500 |
| | | 0.4 | 308.616 | 991.955 | 2074.890 | 2639.638 | 7837.580 | 15025.700 |
| | | 0.7 | 495.792 | 1128.820 | 2333.290 | 4167.385 | 8797.250 | 16074.300 |
| | FE(3D) | 0.1 | 283.898 | 811.795 | 1604.956 | 2445.458 | 6509.821 | 11855.709 |
| | | 0.4 | 305.955 | 981.347 | 2072.554 | 2570.408 | 7647.979 | 15037.839 |
| | | 0.7 | 489.234 | 1121.452 | 2336.177 | 4021.301 | 8655.042 | 16159.248 |
| | % dev. | 0.1 | −0.011 | −0.069 | −0.203 | −0.036 | −0.355 | −1.044 |
| | | 0.4 | −0.870 | −1.081 | −0.113 | −3.693 | −2.479 | 0.080 |
| | | 0.7 | −1.341 | −0.657 | 0.123 | −3.633 | −1.643 | 0.526 |
| 0.750 | NTM | 0.1 | 355.027 | 993.279 | 1944.090 | 3030.222 | 7818.400 | 14004 |
| | | 0.4 | 381.639 | 1149.720 | 2248.190 | 4268.272 | 9431.640 | 16074.300 |
| | | 0.7 | 511.227 | 1230.890 | 2406.190 | 4201.301 | 8759.634 | 16159.248 |
| | FE(3D) | 0.1 | 355.036 | 993.179 | 1943.016 | 3031.657 | 7819.722 | 13978.716 |
| | | 0.4 | 379.583 | 1144.159 | 2251.351 | 4253.660 | 9324.425 | 16529.197 |
| | | 0.7 | 508.439 | 1224.396 | 2410.039 | 4205.930 | 9312.425 | 16529.197 |
| | % dev. | 0.1 | 0.002 | −0.010 | −0.055 | 0.047 | 0.017 | 0.181 |
| | | 0.4 | −0.542 | −0.486 | 0.144 | −1.731 | −0.936 | 0.576 |
| | | 0.7 | −0.548 | −0.530 | 0.160 | −1.482 | −1.280 | 0.553 |
| 0.875 | NTM | 0.1 | 429.273 | 1177.820 | 2274.860 | 3616.212 | 9013.570 | 15741.600 |
| | | 0.4 | 448.134 | 1269.840 | 2419.690 | 3759.295 | 9566.420 | 16569.700 |
| | | 0.7 | 511.380 | 1312.660 | 2498.460 | 4253.660 | 9878.820 | 16849.800 |
| | FE(3D) | 0.1 | 429.347 | 1178.446 | 2277.013 | 4268.272 | 9431.640 | 16337.800 |
| | | 0.4 | 447.258 | 1269.273 | 2423.695 | 3793.834 | 9595.934 | 16695.492 |
| | | 0.7 | 510.778 | 1310.738 | 2502.153 | 4205.930 | 9870.045 | 16963.843 |
| | % dev. | 0.1 | −0.016 | −0.045 | 0.165 | −0.520 | 0.308 | 0.753 |
| | | 0.4 | −0.118 | −0.146 | 0.148 | −0.153 | −0.089 | 0.672 |

between 0.5 and 0.7; that is, the step in the beam is located around the peak of the first free-free mode shape; see Figure 16(a). Figures 14(b) and 15(b) show that there are two peaks in Dev prediction. The location of these peaks is found to be near the position of the peaks of the second free-free mode shape; see Figure 16(b). In addition, the value of Dev, approaches zero when the step location lies in the semistraight line between the two peaks in the second F-F mode shape. The same trend is repeated in the third mode shape as shown in Figures 14(c), 15(c), and 16(c). In addition, when the location of diametric step in shaft coincides with a peak point in \( i \) mode shape, the value of Dev, is increased. Meanwhile, when the step lies in a straight portion of the mode shape, the Dev, approaches zero.

3.4. Tapered Beam Approach. Due to the importance of tapered or conical beams in many engineering applications, the current section is devoted to show how to use the present analysis to solve the problem of taper beam. The current analysis is based on uniform beams, while the partial differential equation which represents the lateral vibration of tapered or conical beams is fourth-order Bessel equation [30, ...
31, 37, 38]. To simulate nonuniform beam using the current analysis, the beam is divided into multiple equal length spans as shown in Figure 17. The height and/or width of these spans are varying linearly between the start and the end to simulate the tapered or conical beam. The height and/or width ratio of \( i \) span can be calculated from the following formula:

\[
\begin{align*}
\hat{h}_i &= h_{n1} + (1 - h_{n1}) \left( -\frac{x_i}{L} + \frac{L_{i1}}{2n} \right), \\
\hat{b}_i &= b_{n1} + (1 - b_{n1}) \left( -\frac{x_i}{L} + \frac{L_{i1}}{2n} \right),
\end{align*}
\]

where

\[
\begin{align*}
h_{i1} &= \frac{h_i}{h_1}, \\
h_{n1} &= \frac{h_n}{h_1}, \\
b_{i1} &= \frac{b_i}{b_1}, \\
b_{n1} &= \frac{b_n}{b_1},
\end{align*}
\]

\[
L_{i1} = \frac{L_i}{L_1}.
\]

\[
h_i \quad \text{is the height of} \quad i \quad \text{beam segment and} \quad b_i \quad \text{is the width of} \quad i \quad \text{beam segment.}
\]

To verify the suitability of the current model to represent conical beams, the results of the current model are compared with the exact solution for cantilevered (C-F) conical beam with variable taper ratio \( h_{n1} = b_{n1} = 0.2, 0.5, \) and 0.7 as shown in Table 10. The model was investigated using the present NTM and using several number of spans \( n = 20, 100, \) and 1000. The first three eigenvalues are evaluated for each taper ratio and number of spans. The time used for computing the first three natural frequencies is evaluated. The model results are also evaluated using NAT previously published in [6] at \( n = 20 \) in order to compare the time saving when using the present method.

The results of Table 10 show that increasing the number of spans results in increasing the accuracy of the evaluated beam eigenvalues in comparison with the exact solution in [30]. On the other hand, the results show that increasing the number of spans results in increasing the computational time. The comparison between the computational time using the present NTM and previously published NAT shows that the
Figure 15: Percentage modal deviation between FE(3D) and NTM as a function of $\mu$ and $d$, for free-free short samples $L = 200\text{ mm}$: (a) 1st mode, (b) 2nd mode, and (c) 3rd mode.

Figure 16: Typical first three modal shapes at the peak points shown in Figure 15 for free-free short sample and $d = 0.5$. 
Table 10: The first three eigenvalues for C-F conical beam using NTM in comparison with those of the exact solution presented in [30] and NAT results using the program of [6].

| $h_{01} = h_{01}$ | [30] $n = 20$ | NAT [6] $n = 20$ | NTM $n = 20$ | NTM $n = 100$ | NTM $n = 200$ | NTM $n = 1000$ |
|-------------------|----------------|------------------|--------------|---------------|---------------|----------------|
|                   | Time (s)       | Time (s)         | Time (s)     | Time (s)      | Time (s)      | Time (s)       |
| 0.2               |                |                  |              |               |               |                |
| $\lambda_{1}^{2}$ | 6.1964         | 6.1683           | 6.1706       | 6.1954        | 6.1962        | 6.1963         |
| $\lambda_{2}^{2}$ | 18.3855        | 18.2513          | 18.2513      | 18.3801       | 18.3840       | 18.3853        |
| $\lambda_{3}^{2}$ | 39.8336        | 39.4809          | 39.4809      | 39.8194       | 39.8300       | 39.8336        |
| 0.5               |                |                  |              |               |               |                |
| $\lambda_{1}^{2}$ | 4.6252         | 4.6155           | 4.6175       | 4.6249        | 4.6250        | 4.6251         |
| $\lambda_{2}^{2}$ | 19.5476        | 19.5074          | 19.5074      | 19.5460       | 19.5472       | 19.5477        |
| $\lambda_{3}^{2}$ | 48.5789        | 48.4725          | 48.4725      | 48.5746       | 48.5779       | 48.5788        |
| 0.7               |                |                  |              |               |               |                |
| $\lambda_{1}^{2}$ | 4.0669         | 4.0615           | 4.0635       | 4.0668        | 4.0669        | 4.0670         |
| $\lambda_{2}^{2}$ | 20.5554        | 20.5360          | 20.5361      | 20.5547       | 20.5553       | 20.5554        |
| $\lambda_{3}^{2}$ | —              | 53.9625          | 53.9625      | 54.0131       | 54.0147       | 54.0152        |
present NTM method is quicker than the NAT method [6] for the same number of spans.

To investigate the capability of the present model to evaluate the natural frequencies of taper or conical beam with variable boundary conditions, some cases of nonuniform beams are selected form [31]. The analysis in [31] is numerical and based on solving the taper beam partial differential equation using Runge-Kutta method. The boundary conditions and the taper ratio are listed in Table 11. The number of spans used in evaluating the current example is $n = 1000$. The eigenvalue results using the present NTM method are in good agreement with numerical results of [31].

### 4. Conclusion

A new proposed normalized transfer matrix NTM uses new set of fundamental solution in combination with the transfer matrix method. This method has the advantage of the TMM in that the determinant of the frequency equation is $4 \times 4$ for $n$ number of spans. In addition, the formation of the system frequency equation determinant is not included in any inverse matrix steps which reduces the computational time.

The current work introduces a comparison between the experimental and analytical NTM and three-dimensional finite element analyses for stepped thick beams. Different system parameters such as the step diameters ratio $\overline{d}$, the step location parameter $\mu$, and the elastic segment length ratio $L^*_2$ are considered. The following conclusions can be drawn from the present work:

1. An excellent agreement between the present experimental results, using several test samples, and those of FE(3D) predictions has been recorded.
2. The results show that the three-dimensional finite element can be trusted in the prediction of the modal frequencies of stepped thick beams in structural and mechanical system.
3. An interesting percentage of modal deviations between the FE(3D) results and those obtained using the present frequency equation (40) has been illustrated. An increase in the step down location parameter ($\mu$) and/or ($\overline{d}$) increases the modal deviations. The percentage deviations are higher for shorter beam than for longer one. The maximum deviation value in a specific normal mode is obtained when the step location is adjacent to a peak location in this normal mode. Moreover, the minimum percentage deviation in a specific mode shape is obtained when the step location lies in a straight portion in the mode shape. This may be explained by the smaller effect of stress concentration at the step under such conditions.
Nomenclature

A: Cross-sectional area of the beam
a, b: Polynomial roots
d: Segment diameter
d: Segment diameter ratio
E: Young’s modulus of elasticity
f: Frequency (Hz)
G: Shear modulus of rigidity
I: Moment of inertia of the beam cross-section about the neutral axis
J: Rotational moment of inertia of the station

\[ \tilde{I}_{n+1} = I_{n+1}/(\rho AL^3) \]

\( k \): Shear deformation shape coefficient
\( k, \phi \): Elastic stiffness
\( k_1, k_{n+1} \): End translational spring stiffness
\( L \): Length of the beam (between points 1 and \( n + 1 \))
\( L_j \): Ratio \( L_j/L \)
\( m_i \): Concentrated mass at \( i \) point
\( \bar{m}_{i+1} \): Total mass of beam
\( r_i \): Rotary inertia parameter \( I_1/A_i L^2 \)
\( s_i \): Shear deformation parameter \( E_1/\gamma k_1 \)
\( Y \): Nondimensional lateral deflection
x, y: System coordinate of the beam
\( Z_1, Z_{n+1} \): Nondimensional stiffness parameters defined as \( k_1 L^3/E_1 I_1 \) and \( k_{n+1} L^3/E_{n+1} I_n \) respectively
\( Z_1^*, Z_{n+1}^* \): Nondimensional stiffness parameters defined as \( k_1 L^3/E_1 I_1 \) and \( k_{n+1} L^3/E_{n+1} I_n \) respectively
\( \Gamma_i \): Nondimensional length term \( (Y^1 - L \Psi) \)
\( \delta_{1}, \delta_{2} \): Set of nondimensional terms defined as (10a) and (10b)
\( \lambda_i, \lambda_i^* \): Frequency parameters \( \rho_i A_i L^2 \omega^2 /E_i I_1 \)
\( \xi \): Poisson’s ratio
\( \rho \): Mass density of the beam material (kg/m^3)
\( \phi_i, \phi_{n+1} \): End rotational spring stiffness
\( \Phi_1, \Phi_{n+1} \): Nondimensional rotational spring parameters defined as \( \phi_1 L/E_1 I_1 \) and \( \phi_{n+1} L/E_{n+1} I_n \) respectively
\( \psi \): Slope due to bending
\( (\cdot) \): 1st derivative with respect to \( x \) or \( \xi \)
\( (\cdot)'' \): 2nd derivative with respect to \( x \) or \( \xi \)
\( (\cdot)''' \): 3rd derivative with respect to \( x \) or \( \xi \)
\( (\cdot)'''' \): 4th derivative with respect to \( x \) or \( \xi \)
C: Clamped (fixed) end
F: Free end
P: Pinned (hinged) end
S1: Two-span with single-step sample
S2: Two-span with single-step sample
S3: Two-span with single-step sample
v1: 10E + 12
vs: 10E – 12.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors gratefully thank the assistants, Eng. M. Zain, workshop technician, and office secretary for their technical support during all stages of this work.

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