Compressive Sensing Recovery Algorithms and Applications- A Survey

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Abstract. Compressive sensing is an efficient method of acquiring signals or images with minimum number of samples, assuming that the signal is sparse in a certain transform domain. Conventional technique for signal acquisition follows the Shannon’s sampling theorem, which requires signals to be sampled at a rate at least twice the maximum frequency (i.e. $f_s \geq 2f_m$). As compared with this traditional acquisition technique, compressive sensing technique captures wide range of signals at a rate significantly lower than Nyquist rate without losing the imperative information, so this technique can be widely used in MRI. Suitable reconstruction algorithms are needed for recovering the original signals from compressed sampled signal. This paper introduces a survey of the various reconstruction algorithms which might enable the use of this technology for wide spread hardware compatible implementation in the near future.

1. Introduction
The basic concept behind compressive sensing (CS) is to directly collect the signal in compressed form Figure 1. According to the Shannon/Nyquist sampling theorem, analogue signal can be recovered perfectly, if it can be sampled at a rate at least twice the maximum frequency component present in the signal (Nyquist 1928; Shannon 1949). CS builds on the work of Candes, Romberg, and Tao [1] and Donoho [2], were possibly the first set of researchers who had mathematically shown that a sparse signal can be recovered from a few set of measurements. Since there were only a few set of measurements, these researchers termed it as compressive sensing. It offers the possibility to capture less data then is commonly done, but still to be able to reconstruct the entire information. This can also appear as a consequence of omitting samples that are exposed to different kinds of noise or losing some parts of the signal during the transmission. These omitting samples recovered by using suitable CS reconstruction algorithms.
2. Compressive Sensing Essentials

Nearly all the digital camera records every pixel in an image, which is immediately discarded for reducing the storage space for saving an image. A natural question asks why we need to acquire this abundance of data, just to throw most of it away immediately. This notation sparked the theory of compressive sensing. As an alternative to the traditional sampling theory, compressive sensing approach provides grate quality to the signal without increasing the amount of data required for reconstruction. Thus CS specifically important in applications related to medical signal processing. The important feature of Compressed Sensing is to avoid compression after acquisition and to directly acquire data in the compressed form. To make this possible we should know about two terms, Sparsity and Incoherence.

2.1. Sparsity

Sparsity is an important term used in compressed sensing. In particular most of the signals are sparse i.e most values are zero and only a few contain essential data when represented in certain domain. Figure 2, shows sparsity in time domain. Figure 2 (a) the actual signal that has 256 samples with only 5 non zeros, which is very sparse. Figure 2 (b) is the number of measurements to be taken, here we take only 64 random measurements. Figure 2 (c) is the recovery result by taking only 64 random measurements, which specify that the actual signal can be recovered from few set of measurements. In the frequency domain, it is completely shown by two peaks. In, Figure 2, the time domain signals was already in its sparse domain, hence representation basis $\psi$ was simply the identity matrix $I_n$. However in Figure 3, $\psi$ transforms a time domain signal in to the frequency domain, i.e the DFT matrix Figure 3 (b). Figure 3 (c) shows the number of measurements to be taken, here we take only 64 random measurements. The original time domain signal had 256 values, by taking only 64 random measurements from it we can recover all 256 original values Figure 3 (c). This differ from the example sparse in time domain, is that the signal of interest was not in the time domain, but in the frequency domain.

2.2. Incoherence

Compressed sensing is based on a choice of a sensing basis $\phi$ relative to representation basis $\psi$, which use an incoherence property $\mu$ for measure the correlation between vectors from each basis. The minimum correlation between any two elements of two different matrices is called coherence, it is denoted by $\mu(\phi,\psi)$. We can reconstruct signals with fewer measurements if the value of incoherence is to small [3].

3. Reconstruction Algorithms

The important feature of compressive sensing is that it requires an efficient reconstruction algorithms. The reconstruction of compressed sampled signal involves solution of an
underdetermined system of linear equations and therefore has infinitely many solutions. The basic sensing equation is given by

\[ y = Ax \]  \hspace{1cm} (1)

where, \( y \) is the measured values, \( A \) is the sensing matrix, \( x \) is the original Signal. Reconstruction requires the solution of equation for \( x \). Direct solution of the equation never yields a unique solution even with the prior knowledge of the sparseness of the solution. The inverse equation is given by

\[ \hat{x} = (A^T A)^{-1} A^T \]  \hspace{1cm} (2)

The signal reconstruction process is essentially to choose the best estimate of the original signal from all the possible solutions obtained from the above inverse equation. This may be achieved by the convex optimization algorithm. Various reconstruction algorithms used for reconstruction of compressed sampled signal may be classified as shown in Figure 4.

3.1. Convex Relaxation
Convex Relaxation is the class of algorithms which can solve reconstruction issues through linear programming. Basis Pursuit, Basis Pursuit De-Noising, Least Absolute Shrinkage and Selection Operator (LASSO) are some examples of such algorithms.

Basis pursuit is the mathematical optimization of a problem in the form of

\[ \min ||x||_1 \hspace{1cm} \text{subject to} \hspace{0.5cm} y = Ax \]  \hspace{1cm} (3)
These types are mainly used in cases where there is an underdetermined signal of linear equation of \( y = Ax \). This algorithm is more complex and time consuming, so it cannot be applicable in time critical reconstruction applications. Figure 5, represents reconstruction of a input signal of length 256 by using basis pursuit algorithm. In this experiment we take 64 samples, for reconstructing signal of length 256 with sparsity 8. From the experiment we can conclude that when sparsity level increases accuracy decreases and will get approximately accurate signal. The signal reconstruction quality is analysed via the objective test method using some methods including Signal-to-Noise Ratio (SNR), Relative Error (RE) etc., as shown in Table 1 and Table 2. From Table 2, \( M > K \log(N/K) \) we can see that greater the quality of signal other wise signal quality will be decreased.

In 1996, Tibshirani [4] introduced the method Least Absolute Shrinkage And Selection Operator (LASSO). It is a commonly used technique in sparse modeling, originally developed in the field of statistics. As in the case for compressive sensing, LASSO also utilizes \( l_1 \) norm, but the solution \( x \) is obtained as

\[
x_{\text{LASSO}} = \arg\min ||y - Ax||_2^2 + \mu ||X||_1 \tag{4}
\]

Where, \( \mu > 0 \) is a fixed regularization parameter. The first term in the argument of the right-hand side describes goodness of fit, which is commonly used in least-squares fitting, and the second term is the penalty term based on the \( l_1 \)-norm. The regularization parameter \( \mu \) adjusts degree of sparsity by changing the weight of the \( l_1 \)-norm penalty.

For instance, a large \( \mu \) prefers a solution with very few non-zero components, yet \( \mu = 0 \) provide no sparsity. Optimization of \( \mu \) in LASSO is a major problem. Information criteria or cross

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**Figure 4.** Reconstruction algorithms

**Figure 5.** Reconstructed output data using BP
Table 1. Sparsity level Vs Error rate for reconstruction using Compressive Sensing

| Sr. No | Sparsity | Error       |
|--------|----------|-------------|
| 1      | 10       | 6.2780e-14  |
| 2      | 20       | 8.1502e-14  |
| 3      | 30       | 8.7132e-14  |
| 4      | 40       | 3.3142      |
| 5      | 50       | 21.7598     |

Table 2. Number of measurements Vs SNR for reconstruction using Compressive Sensing
N=256, Sparsity(K)=10

| Sr. No | Measurement Number(M) | SNR     |
|--------|------------------------|---------|
| 1      | 512                    | 6.0206  |
| 2      | 64                     | 6.0206  |
| 3      | 20                     | 5.6047  |
| 4      | 15                     | -2.9872 |
| 5      | 10                     | -0.4331 |

Validation methods can be used for this optimization. Another method is to adjust \( \mu \), based on the estimate of the sparsity, which might be obtained from some a priori information or theoretical predictions. Optimized value of \( \mu \) is dependent of characteristics of problems to be solved (i.e., the shapes of observation matrix \( A \) and also image \( x \)). LASSO yields a sparse solution of which the parameter vector \( \mu \) has nearly few nonzero coefficients. In contrast, the \( l_2 \) regularised least squares generally has all coefficients nonzero. Because it actively reduces the number of variables, the lasso is useful in some situation. The basic theory behind this algorithm is to shrinks some coefficients and hence tries to retain the good feature of both subset selection and ridge regression. Amusing work on the lasso is being carried out in many fields, including statistics, engineering, mathematics and computer science.

3.2. Non Convex Minimization Algorithm
Replacing \( l^1 \) norm in equation (1) with the \( l^p \) quasi-norm \((0 < p < 1)\), for recover \( x \) from fewer linear measurements. ie, \( ||x||_p \) subject to \( y=Ax \). In practical case convex optimization problem can be solvable using Non convex minimization algorithms [5]. Focal Underdetermined System Solution (FOCUSS), Sparse Baysian Learning algorithms are some examples of such algorithms. In medical imaging tomography Non-Convex optimization used. Focal underdetermined system solution(FOCUSS) is an algorithm for finding localised energy solution from limited data. The inverse problem with sparse solutions.

\[
y = Ax + \nu
\]  

Where \( y \) is the observed data, \( A \in \mathbb{R}^{M \times N} \) is the generating matrix, \( x \) is the sparse solution and \( \nu \) is the additive noise. Sparse vector selection and a dictionary learning step are the main steps involved in this algorithm. FOCUSS also used to solve for sparse solution of linear inverse problems when \( A \) is known and perform the vector selection step of the algorithm. The dictionary learning step is performed by gradient descent. Sparse Bayesian Learning (SBL) is an efficient technique to perform sparse processing, which is build by a Bayesian framework,
approximately solves a non-convex optimization problem using fixed point updates. Sparse Bayesian Learning algorithms was introduced for regression and classification problems in the context of machine learning it has been used since for signal processing with various modifications and extensions. Since SBL does not impose explicitly any sparsity constraints but determines sparsity automatically.

3.3. Greedy Iterative Algorithm

The greedy algorithms are based on finding the elements of the transform matrix called dictionary that best matches the signal through iterations. These algorithms are less computationally complex and therefore much faster compared to the $l_1$-norm based optimization techniques, but are also less precise. Matching pursuit (MP), Orthogonal Matching Pursuit (OMP) and Compressive Sampling Matching Pursuit (CoSaMP) are the commonly used greedy iterative algorithms [9]. The matching pursuit (MP) is a sort of exhaustive search that attempts to select in a greedy fashion the subset columns (the support-set that identified the locations of the non-zero elements in a sparse signal-vector) of the dictionary that match the measurements the most. The columns with the biggest inner product with the signal are selected first. The method then optimises the selection by least squares in every iteration, subtracting from the signal an approximation of the selected columns, until the signal is well decomposed (the residual is less than a defined threshold).

MP was extended to orthogonal MP (OMP) of higher computational cost and more accurate results. OMP calculates in every iteration the projection of the signal on all dictionary columns selected, and updates the coefficients. OMP works by finding a basis vector in $\phi$ that maximizes the correlation with the residual (initialized to $y$), and then recomputing the residual and coefficients by projecting the residual on all atoms in the dictionary using existing coefficients. OMP has the major advantage that, each column vector of the sensing matrix picked up once wont be picked up in the next iteration. This is achieved by maintaining an atom index. which is updated on each iteration. As, a result OMP produces an estimate of the sparse signal in m iterations. The case $Ax = b$ is considered. Both A and b are created with random values. Using OMP, a sparse x is found. Figure 6. shows the result.

Regularised OMP is an iterative algorithm which is also based on OMP with some differences. The algorithm selects many vectors at each iteration not like OMP, which at each step only selects one vector. CoSaMP is the latest algorithm developed based on OMP. The performance of CoSaMP is not accurate as OMP algorithms, but using CoSaMP recovery is extremely fast.

Figure 6. Reconstruction of signal using OMP Algorithm
Without the noise, the recovery results by OMP are much more accurate than the results by CoSaMP. The advantage with CoSaMP is that it works well when the samples are disrupted with noise.

3.4. Combinatorial / Sublinear Algorithms
By this method, acquire highly structured samples of the signal and reconstruction by group testing [8]. This class include Fourier sampling uses random (but structured) time samples to recover signals that are compressible with respect to the discrete Fourier basis. Fourier Sampling Algorithm, Chaining Pursuit Heavy Hitters on Steroids (HHS) are the commonly used combinatorial algorithms. These are fast and efficient as compared to convex relaxation or greedy algorithms.

3.5. Iterative Thresholding Algorithms
For this class of algorithms, correct measurements are recovered by soft or hard thresholding. Iterative Hard Thresholding is a class of low computational algorithms, which has been proposed for reconstruction of sparse signals, starting from \( x^0 = 0 \), IHT iteratively finds the sparse solution of \( y = Ax \) using the following iteration,

\[
x^{n+1} = H_k(x^n + \phi^T(y - \phi x^n))
\]

Where \( H_k \) is a nonlinear operator that sets all but the \( k \) largest (in magnitude) coefficients of its input vector to Zero. It can be shown that if \( ||\phi||_2 < 1 \) the guaranteed to coverage to the local minimum of the cost function \( ||y - \phi x||_2 \) under the constraint that \( x \) is \( k \) sparse.

Variations of the IHT includes Message Passing[11], Expander Matching Pursuit, Sparse Matching Pursuits, Sequential Sparse Matching Pursuits, and Belief Propagation. Variations of the IST include fast iterative soft thresholding (FISTA) [10].

4. APPLICATIONS
4.1. Medical Imaging
CS reduces the sampling rate without losing the useful information, so this technique can be used in MRI. MRI uses magnetic fields to excite hydrogen atoms in the body, which result in measurements of the Fourier transform of the image. The number of measurements to be captured is proportional to the scan duration. Longer scan are unpleasant for patients and also more prone to motion artifacts and impossible for certain parts of the body (example, those affected by breathing). However, the introduction of CS based techniques provide great quality to the image and reduce the scan duration through reduction in the number of collected measurements[11].

4.2. Radar
The main function of Radar system is to target detection and range determination, also it’s function extends to air and traffic control, air defence systems, antimissile systems etc. In radar signal processing in order to determine the target location, large bandwidth signals needs to be launched, so it requires a very high signal sampling rate and system calculations. Radar echo signal is highly sparse, so compressed sensing theory can be used to reduce the sampling rate and in the lower sampling rate, target parameter can be estimated from radar echo. Compressed sensing enables the achievement of better target resolution with minimum number of samples.

4.3. CS in Cameras
The single-pixel camera developed at Rice University is a one of the most important application of compressed sensing [12]. Single pixel camera mainly consists of a Digital Micro mirror Device
(DMD), two lenses, a photon detector to sense data and an electronic switching device to flip the mirrors at high speed. Each mirror in Digital Micromirror Device (DMD) array performs one of these two tasks: either reflect light towards the sensor or reflect light away from it. Therefore, light received at sensor (photodiode) end is weighted average of many different pixels, whose combination gives a single pixel. A stranded digital camera uses a massive CCD arrays to collect millions of pixels, which are typically compressed to reduce their cost for storage and transmission. For example, a 20 mega pixel camera measure 20,000,000 pixels using a CCD or CMOS light sensors. On the other hand, the single pixel camera, uses one light sensor to measure the entire image, compress the image data before the pixels are recorded. As a result it’s able to capture an image with minimum number of samples.

5. Conclusions
A review of the compressed sensing is discussed, which indicates the proper selection of sparse representation of the signals, selection of measurement matrix for acquisition, algorithms for reconstruction and also discussed compressive sensing applications. Here we can see that, algorithms used for the signal reconstruction focus to find the solution of an underdetermined system of linear equations using sparseness constraints. Generally most of the natural signals are sparse or sparse under some domain. If we can find the location of those non-zero entries, we can reconstruct the signal uniquely. The traditional process of signal compression is quite costly, because this acquisition system acquire all samples of the original signal and a significant portion of which is immediately discarded. The new idea of signal compression combines signal acquisition and compression as one step, which reduces the overall cost significantly. As compared with the traditional Nyquist-Shannon sampling theory, CS afford grate quality to the signals without increasing the quantity of required data, which means that the original signal can be reconstructed with minimum number of samples ($f_s < 2f_m$). This paper broadly speaking the survey of various algorithms for reconstructing compressed sampled signals and compares their complexity. In future work, test various algorithms to reconstruct compressed sensed images can be done.

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