Concentration of measure 
and generalized product of random vectors 
with an application to Hanson-Wright-like inequalities.

Cosme Louart*  Romain Couillet†

Abstract
Starting from concentration of measure hypotheses on $m$ random vectors $Z_1, \ldots, Z_m$, this article provides an expression for the concentration of functionals $\phi(Z_1, \ldots, Z_m)$, where the variations of $\phi$ on each variable depend on the product of the norms (or semi-norms) of the other variables (as if $\phi$ were a product). We illustrate the importance of this result through various generalisations of the Hanson-Wright concentration inequality and through a study of the random matrix $XDX^T$ and its resolvent $Q = (I_p - \frac{1}{n} XDX^T)^{-1}$, where $X$ and $D$ are random, which are of fundamental interest in statistical machine learning applications.

Keywords: Concentration of Measure; Hanson Wright inequality; Random Matrix Theory.

MSC2020 subject classifications: 00-08, 60B20, 62J07.

Introduction

Among the various assumptions one could pose on random vectors $Z_1, \ldots, Z_m$ to study the concentration of a functional $\phi(Z_1, \ldots, Z_m)$ of limited variations (on $Z_1, \ldots, Z_m$), Concentration of measure hypotheses provide flexible properties that allow one to (i) characterize a wide range of settings where, in particular, the independent entries hypothesis is relaxed, and (ii) to obtain rich concentration inequalities with precise convergence bounds. The historical result of concentration of measure theory was first obtained on the uniform distribution on the sphere by Lévy [Lé51] and later Milman and Gromov extended the approach to other families of distributions, in particular involving isoperimetric inequalities and the Ricci curvature in [GM83]. An important part of the theory was developed by Talagrand, whose results are not discussed here, and a full description of the various results can be found in the monographs [Led05, BLM13].

To present the simplest picture possible, we admit for the moment that what we call “concentrated vectors” (or “Lipschitz concentrated vectors”) are transformations $X = F(Z) \in \mathbb{R}^p$ of a Gaussian vector $Z \sim \mathcal{N}(0, I_d)$ for a given 1-Lipschitz (for the Euclidean norm) mapping $F : \mathbb{R}^d \rightarrow \mathbb{R}^p$. This class of random vectors derives from a core result of concentration of measure theory [Led05, Corollary 2.6], which states

*GIPSA-lab. E-mail: [cosmelouart@gmail.com]
†LIG-lab, GIPSA-lab. E-mail: romain.couillet@gipsa-lab.grenoble-inp.fr
Concentration of product of random vectors

that for any $\lambda$-Lipschitz mapping $f : \mathbb{R}^d \to \mathbb{R}$ (where $\mathbb{R}^d$ and $\mathbb{R}$ are endowed with the Euclidean norm $\| \cdot \|$ and the absolute value $| \cdot |$, respectively),

$$\forall t > 0 : \mathbb{P} (|f(Z) - \mathbb{E}[f(Z)]| \geq t) \leq C e^{-(t/c)^2}, \quad (0.1)$$

where $C = 2$ and $c = \sqrt{2}$ (these constants do not depend on the dimensions $d$!). Note that the concentration rate is proportional to the Lipschitz parameter of $f$. In particular, this implies that the standard deviation of the random variable $f(Z)$ — called the “$\lambda$-Lipschitz observation of $Z$” — does not depend on the dimension $d$ (if $\lambda$ remains constant as $d$ tends to $\infty$). We briefly denote this property as $Z \propto C \mathcal{E}_2(c)$ or, if we are in the quasi-asymptotic regime where the dimension $d$ (or $p$) is large, we do not pay attention to the constants appearing in the exponential bound (as long as $C, c \leq d \to \infty O(1)$, the result would not be much different) and instead write $Z \propto \mathcal{E}_2$.

We can then derive a variety of concentration inequalities for any observation $g(F(Z))$ for $g : \mathbb{R}^p \to \mathbb{R}$ Lipschitz. If $F$ is, say, $\sigma$-Lipschitz, with $\sigma$ possibly depending on the dimension, we have the concentration $X = F(Z) \propto \mathcal{E}_2(\sigma)$. This succinct notation ($X \propto \mathcal{E}_2(\sigma)$ is analogous to (0.1) with $\sigma$ replacing $c$) shows only the central quantity describing the concentration of $X$, namely the “observable diameter of $X$”: $O(\sigma)$. In fact, the implicit concentration inequalities constrain the standard deviations of any $\nu$-Lipschitz observations of $X$ to be of the order of $O(\nu \sigma)$.

The goal of this article is to go beyond the Lipschitz case and express, using our shorthand notation, the concentration of products of concentrated random vectors. As an illustrative example, let $Y = X_1 \circ \cdots \circ X_m \in \mathbb{R}^p$, for a given product “$\circ$” satisfying:

$$\forall x_1, \ldots, x_m \in \mathbb{R}^p, \quad \|x_1 \circ \cdots \circ x_m\| \leq \|x_1\| \cdots \|x_m\|. \quad (0.2)$$

(For example, $\circ$ could be the entry-wise product, and the norm would be the infinite norm). In particular, if $\mathbb{E}[X] = 0$, we will see that

$$Y \propto \mathcal{E}_2(p \overbrace{\cdots \sigma^m}^{m \text{ times}}) + \mathcal{E}_2(\sigma^m), \quad (0.3)$$

where $\mu$ satisfies $\mathbb{E}[\|X\|] \leq O(\mu)$, which in our framework means that there exist two constants $C, c > 0$ (independent of $p$) such that for any 1 Lipschitz mapping $f : \mathbb{R}^p \to \mathbb{R}$, $\forall t > 0$,

$$\mathbb{P} (|f(Y) - \mathbb{E}[f(Y)]| \geq t) \leq C \exp \left( - \frac{t}{c p \overbrace{\cdots \sigma^m}^{m \text{ times}}} \right) + C \exp \left( - \frac{t}{c \sigma^m} \right)^{2/m}. \quad (0.2)$$

We see here that the term $\mathcal{E}_2(\sigma^m)$ in (0.3) controls the tail of the distribution of $f(Y)$, but its first moments are controlled by the term $\mathcal{E}_2(p \overbrace{\cdots \sigma^m}^{m \text{ times}})$. Specifically, its standard deviation is of the order of $O(p \overbrace{\cdots \sigma^m}^{m \text{ times}})$, the observable diameter of $Y$. In a sense, the result is quite intuitive, looking back at the algebraic inequality (0.2): here $\mathbb{E}[\|X\|] \leq O(\sqrt{p})$ and the variations of $Y$ are bounded by $\mathbb{E}[\|X\|] \overbrace{\cdots \sigma^m}^{m \text{ times}}$ times the variation of $X$. This simple scheme generalizes to more complex products of random vectors $X_1, \ldots, X_m$ belonging to different normed vector spaces, where $Y$ may not be a multilinear mapping of $(X_1, \ldots, X_m)$ but still satisfy an inequality similar to (0.2) (with semi-norms possibly replacing some of the norms). The complete description of these possible settings is the central result of this paper: Theorem 5.1.

Quite similar results with this multiple exponential regime can be found in [Lat06] in the Gaussian case and later in [AW15], which extends Latala’s result to more general hypotheses of concentration. In several aspects, the approach of [AW15] may seem
more "structural"—but also somehow less flexible—than ours, in particular because
the lower bound given in the case of polynomials of Gaussian variables meets (up to
a constant) the upper bound given in the case of d-differentiable functions of more
general variables. However, it does not seem that their result can recover ours—even if
one does not consider the specific case of nondifferentiable functionals and operations
on general algebras (and not only on random variables), which we are the only ones
to treat, and the hypotheses of concentration, which are quite similar. We give an
expression of their result and compare it with ours after Theorem 5.1.

As a simple but fundamental application of our main result is given by the Hanson-
Wright inequality expressing the concentration of $X^TAX$, where $A \in M_m$ and $X$ is
either a random vector of $R^p$ or a random matrix of $M_{p,n}$. The historical result (see
[BLM13] or [Ver17]) was given on random vectors $X = (X_1, \ldots , X_p) \in R^p$ with indepen-
dent sub-Gaussian entries, satisfying some sub-Gaussian concentration inequality, say
$X_i \sim \mathcal{E}_2(K)$, then one gets the concentration:

$$X^TAX \prec \mathcal{E}_2(K^2 ||A||_F) + \mathcal{E}_1(K^2 ||A||)$$ (0.4)

Note that the standard deviation of $X^TAX$ is of order $||A||_F$ (where $||.||_F$ is the Frobenius
norm). Based on concentration of measure hypotheses (allowing for dependence be-
tween entries), good concentration inequalities were already obtained in [VW14] (with
a term $\mathcal{E}_2(\log n ||A||_F)$ replacing $\mathcal{E}_2(||A||_F)$) and then improved in [Ada15] to exactly the
same result as (0.4).

Although we extend this concentration result to the case of random matrices
$X, Y \in M_{p,n}$, which is not a big improvement, unlike [VW14] and [Ada15], we do not
take convex concentration hypotheses (derived from a well-known result of Talagrand),
because Theorem 5.1 could not be proven in this setting.1

To illustrate our central result with more general products (when $m \geq 3$), we con-
cider the concentration of $X^TDY$, where $D \in M_n$ is a diagonal random matrix and
$X, Y \in M_{p,n}$ are two random matrices, all satisfying $X, D, Y \prec \mathcal{E}_2$. In a last step, in
the same setting, to go beyond the multilinear case, we consider the concentration of the
resolvent $Q = (I_p - \frac{1}{n} XDXT)^{-1}$ studied in [PP09, GLPP14], but with a diagonal ma-
trix $D$ possibly depending on $X, Y$. This setting appears in robust regression problems
[EKBB13, MLC19, SLTC21]. With the possibly complex dependencies between the en-
tries of $x_i$, they allow, the concentration of measure hypotheses are very light compared
to the classical Gaussian hypotheses adopted in large dimensional statistics and statisti-
cal learning [Hua17, DKT20]. To obtain a good concentration of $Q$, one must assume
that the columns $x_1, \ldots, x_n$ of $X$ are all independent and that for all $i \in [n]$ there exists
a diagonal random matrix $D(i) \in M_n$, not too far from $D$ and independent with $x_i$.

The remainder of the article is organized as follows. After presenting Lipschitz con-
centration of measure hypotheses and basic probabilistic inferences (I), we introduce
the class of linearly concentrated random vectors (II) and explain how their norm can
be controlled in generic normed vector spaces (III). We then briefly discuss the fact
that the random vector $(X_1, \ldots, X_m)$ (as a whole) is not always concentrated if one only
assumes that each of the $X_i$/’s, $i \in [m]$, is concentrated (IV). This provides us with the
ingredients to establish the concentration of $\phi(X_1, \ldots, X_m)$ in Theorem 5.1, the core
result of the article, and to provide a first set of elementary consequences (V). As an
application of Theorem 5.1 we next provide a generalization of the Hanson-Wright The-
orem (VI). Then we end the article with a study of the concentration of $XDX^T$ (VII)
and the resolvent $Q = (I_p - \frac{1}{n} XDXT)^{-1}$ (VIII).

1A result analogous to Theorem 5.1 can be proven in the convex concentration setting and for the entry
wise product in $R^p$ or the matrix product in $M_{p,n}$, but this is not the purpose of this article.
Concentration of product of random vectors

1 Basics and notations of the concentration of measure framework

To discuss concentration of measure, we choose here to adopt the viewpoint of Levy families where the goal is to track the influence of the vector dimension over the concentration. Specifically, given a sequence of random vectors \((Z_p)_{p \in \mathbb{N}}\) where each \(Z_p\) belongs to a space of dimension \(p\) (typically \(\mathbb{R}^p\)), we wish to obtain inequalities of the form:

\[
\forall p \in \mathbb{N}, \forall t > 0 : P(\|f_p(Z_p) - a_p\| \geq t) \leq \alpha_p(t),
\]

where, for every \(p \in \mathbb{N}\), \(\alpha_p : \mathbb{R}^+ \to [0, 1]\) is called a concentration function: it is left-continuous, decreasing, and tends to 0 at infinity; \(f_p : \mathbb{R}^p \to \mathbb{R}\) is a 1-Lipschitz function; and \(a_p\) is either a deterministic variable (typically \(E(f_p(Z_p))\)) or a random variable (for instance \(f_p(Z'_p)\)) with \(Z'_p\) an independent copy of \(Z_p\). The sequences of random vectors \((Z_p)_{p \geq 0}\) satisfying inequality (1.1) for all sequences of 1-Lipschitz functions \((f_p)_{p \geq 0}\) are called Levy families or more simply concentrated vectors (with this denomination, we implicitly omit the dependence on \(p\) and abusively call “vectors” the sequences of random vectors of growing dimension).

Concentrated vectors admitting an exponentially decreasing concentration function \(\alpha_p\) are extremely flexible objects. We dedicate the next two subsections to further definitions of the fundamental notions involved under this setting. These are of central interest to the present article – this approach is primarily inspired by the Gaussian fundamental example satisfying (0.1).

Our main interest is in two classes of concentrated vectors, characterized by the regularity of the class of admissible sequences of functions \((f_p)_{p \in \mathbb{R}}\) satisfying (1.1). When (1.1) holds for all the 1-Lipschitz mappings \(f_p, Z_p\), it is said to be Lipschitz concentrated; when true for all 1-Lipschitz linear mappings \(f_p, Z_p\), it is said to be linearly concentrated (the convex concentration, not studied here, occurs when (1.1) is satisfied for all 1-Lipschitz convex mappings \(f_p\) \([VW14]\)). As such, the concentration of a random vector \(Z_p\) is only defined through the concentration of what we refer to as its “observations” \(f_p(Z_p)\) for all \(f_p\) in a specific class of functions.

We will work with normed (or semi-normed) vector spaces, although concentration of measure theory is classically developed in metric spaces. The presence of a norm (or a semi-norm\(^3\)) on the vector space is particularly important when establishing the concentration of a product of random vectors.

**Definition/Proposition 1.1.** Given a sequence of normed or semi-normed vector spaces \((E_p, \|\cdot\|_p)_{p \geq 0}\), a sequence of random vectors \((Z_p)_{p \geq 0} \in \prod_{p \geq 0} E_p\), a sequence of positive reals \((\sigma_p)_{p \geq 0} \in \mathbb{R}_+^\mathbb{N}\) and a parameter \(q > 0\), we say that \(Z_p\) is \(q\)-exponentially concentrated with observable diameter of order \(O(\sigma_p)\) iff one of the following three equivalent assertions is satisfied:\(^4\)

\(^2\)A semi-norm becomes a norm when it satisfies the implication \(\|x\| = 0 \Rightarrow x = 0\).
\(^3\)A random vector \(Z\) of \(E\) is a measurable function from a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) to the normed vector space \((E, \|\cdot\|)\) endowed with the Borel \(\sigma\)-algebra; one should indeed write \(Z : \Omega \to E\), but we abusively simply denote \(Z \in E\).
\(^4\)Aside from the fact that they all give interesting interpretation of the concentration of a random vector, all three characterizations can be relevant, depending on the needs:

- the characterization with the independent copy is employed in Remark 1.3 and in the proof of Theorem 5.1;
- the characterization with the median is employed in the proof of Lemma 1.4;
- the characterization with the expectation, likely the most intuitive, is used to establish Proposition 7.4, Theorem 6.1 and Lemma 8.3.
Concentration of product of random vectors

- \exists C, c > 0 \mid \forall p \in \mathbb{N}, \forall 1\text{-Lipschitz } f : E_p \to \mathbb{R}, \forall t > 0 : 
  \Pr (|f(Z_p) - f(Z'_p)| \geq t) \leq Ce^{-(t/c\sigma_p)^y},

- \exists C, c > 0 \mid \forall p \in \mathbb{N}, \forall 1\text{-Lipschitz } f : E_p \to \mathbb{R}, \forall t > 0 : 
  \Pr (|f(Z_p) - mf| \geq t) \leq Ce^{-(t/c\sigma_p)^y},

- \exists C, c > 0 \mid \forall p \in \mathbb{N}, \forall 1\text{-Lipschitz } f : E_p \to \mathbb{R}, \forall t > 0 : 
  \Pr (|f(Z_p) - E[f(Z_p)]| \geq t) \leq Ce^{-(t/c\sigma_p)^y},

where \( Z'_p \) is an independent copy of \( Z_p \) and \( mf \) is a median of \( f(Z_p) \); the mappings \( f \) are 1-Lipschitz for the norm (or semi-norm) \( \| \cdot \|_p \). We denote in this case \( Z_p \sim \mathcal{E}_q(\sigma_p) \) (or more simply \( Z \sim \mathcal{E}_q(\sigma) \)).

The equivalence between the three definitions is proven in [Led05] (full details are given in [Led13] Propositions 1.2, 1.18 Corollary 1.24), also the existence of the expectation in the third point is guaranteed if one of the two first item is valid thanks to Fubini Theorem that transforms concentration inequalities into bounds on the moments ([Led05] Proposition 1.7)).

**Remark 1.2** (Quasi-asymptotic regime). Most of our results will be expressed under the quasi-asymptotic regime where \( p \) is large. Sometimes, it will be natural to index the sequences of random vectors with two (or more) indices (e.g., the numbers of rows and columns for random matrices): in these cases, the quasi-asymptotic regime is not well defined since the different indices could have different convergence speed. This issue is overcome with the extensive use of the notation \( O(\sigma_t) \), where \( t \in \Theta \) designates the (possibly multivariate) index. Given two sequences \( (a_t)_{t \in \Theta} \), \( (b_t)_{t \in \Theta} \in \mathbb{R}_+^\Theta \), we will denote \( a_t \leq O(b_t) \) if there exists a constant \( C > 0 \) such that \( \forall t \in \Theta, a_t \leq Cb_t \) and \( a_t \geq O(b_t) \) if \( \forall t \in \Theta, a_t \geq Cb_t \). A “constant” \( K > 0 \) is a quantity that does not depend on our asymptotic variables, it satisfies therefore \( K \leq O(1) \) and \( K \geq O(1) \). For a concentrated random vector \( Z_t \sim \mathcal{E}_q(\sigma_t) \), any sequence \( (\nu_t)_{t \in \Theta} \in \mathbb{R}_+^\Theta \) such that \( \sigma_t \leq O(\nu_t) \) is also an observable diameter of \( Z_t \). When \( \sigma_t \leq O(1) \), we simply write \( Z_t \sim \mathcal{E}_q \).

**Remark 1.3** (Metric versus normed spaces). It is more natural, as done in [Led05], to introduce the notion of concentration in metric spaces, as one only needs to resort to Lipschitz mappings which merely require a metric structure on \( E \). However, to exploit Theorem 5.1, we will need to control the amplitude of concentrated vectors which is easily conducted when the metric is a norm, under linear concentration assumptions.

When a concentrated vector \( Z_p \sim \mathcal{E}_q(\sigma_p) \) takes values only on some subset \( A_p \equiv Z_p(\Omega) \subset E_p \) (where \( \Omega \) is the universe), it might be useful to be able to establish the concentration of observations \( f_p(Z_p) \) where \( f_p \) is only 1-Lipschitz on \( A_p \) (and possibly non Lipschitz on \( E_p \setminus A_p \)). This would be an immediate consequence of Definition 1.1 if one were able to extend \( f_p|_{A_p} \) into a mapping \( f_p \) Lipschitz on the whole vector space \( E_p \); but this is rarely possible. Yet, the observation \( f_p(Z_p) \) does concentrate under the hypotheses of Definition 1.1

**Lemma 1.4** (Concentration of locally Lipschitz observations). Given a sequence of random vectors \( Z_p : \Omega \to E_p \), satisfying \( Z_p \sim \mathcal{E}_q(\sigma_p) \), for any sequence of mappings \( f_p : E_p \to \mathbb{R} \), which are 1-Lipschitz on \( Z_p(\Omega) \), we have the concentration \( f_p(Z_p) \sim \mathcal{E}_q(\sigma_p) \).

\[ \Pr (f(Z_p) \geq mf) \geq \frac{1}{2} \text{ and } \Pr (f(Z_p) \leq mf) \geq \frac{1}{2}. \]
Concentration of product of random vectors

Proof. considering a sequence of median $m_{f,p}(Z_p)$ and the (sequence of) sets $S_p = \{f_p \leq m_{f,p}\} \subset E_p$, if we note for any $z \in E_p$ and $U \subset E_p$, $U \not= \emptyset$, $d(z, U) = \inf\{\|z - y\|, y \in U\}$, then we have for any $z \in A$ and $t > 0$:

$$f_p(z) \geq m_{f,p} + t \quad \Rightarrow \quad d(z, S_p) \geq t,$$
$$f_p(z) \leq m_{f,p} - t \quad \Rightarrow \quad d(z, S_p^c) \geq t,$$

since $f_p$ is 1-Lipschitz on $A$. Therefore, since $z \mapsto d(z, S_{p})$ and $z \mapsto d(z, S_{p}^{c})$ are both 1-Lipschitz on $E$ and both admit 0 as a median ($P(d(Z_p, S_p) = 0) = 1 \geq \frac{1}{2}$ and $P(d(Z_p, S_p) \leq 0) \geq P(f_p(Z_p) \leq m_{f,p}) \geq \frac{1}{2}$),

$$P\left(\left| f_p(Z_p) - m_{f,p}\right| \geq t\right) \leq P\left(d(Z_p, S_p) \geq t\right) + P\left(d(Z_p, S_p^c) \geq t\right) \leq 2Ce^{-(t/c_\sigma)}.$$

One could argue that, instead of Definition 1.1 we could have posed hypotheses on the concentration of $Z_p$ on $Z_p(\Omega)$ only; however, we considered the present definition of concentration already quite complex as it stands. This locality aspect must be kept in mind: it will be exploited to obtain the concentration of products of random vectors.

Lemma 1.4 is particularly interesting when working with conditioned variables.

Remark 1.5 (Concentration of conditioned vectors). Given a (sequence of) random vectors $Z \propto \mathcal{E}_q(\sigma)$ and a (sequence of) events $A$ such that $P(A) \geq O(1)$, it is straightforward to show that $(Z \mid A) \propto \mathcal{E}_q(\sigma)$, since there exist two constants $C, c > 0$ such that for any $p \in \mathbb{N}$ and any 1-Lipschitz mapping $f : E_p \rightarrow \mathbb{R}$:

$$\forall t > 0 : P\left(\left| f(Z_p) - f(Z'_p)\right| \geq t \mid A\right) \leq \frac{1}{P(A)}P\left(\left| f(Z_p) - f(Z'_p)\right| \geq t\right) \leq Ce^{-(t/c_\sigma)}.$$ 

This being said, Lemma 1.4 allows us to obtain the same concentration inequality for any mapping $f : E_p \rightarrow \mathbb{R}$ 1-Lipschitz on $Z_p(A)$ (that will be abusively denoted $A$ later on).

A simple but fundamental consequence of Definition 1.1 is that, as announced in the introduction, any Lipschitz transformation of a concentrated vector is also a concentrated vector. The Lipschitz coefficient of the transformation controls the concentration.

Proposition 1.6 (Stability through Lipschitz mappings). In the setting of Definition 1.1 given a sequence $(\lambda_p)_{p \geq 0} \in \mathbb{R}^+_\mathbb{N}$, a supplementary sequence of normed vector spaces $(E_p', \| \|_p')_{p \geq 0}$ and a sequence of $\lambda_p$-Lipschitz transformations $F_p : (E_p', \| \|_p') \rightarrow (E_p', \| \|_p')$, we have

$$Z_p \propto \mathcal{E}_q(\sigma_p) \quad \Rightarrow \quad F_p(Z_p) \propto \mathcal{E}_q(\lambda_p \sigma_p).$$

There exists a range of elemental concentrated random vectors, which may be found for instance in the monograph [Led05]. We recall below some of the major examples. In the following theorems, we only consider sequences of random vectors of the normed vector spaces $(\mathbb{R}^p, \| \|)$. For readability of the results, we will omit the index $p$.

Theorem 1.7 (Fundamental examples of concentrated vectors). The following sequences of random vectors are concentrated and satisfy $Z \propto \mathcal{E}_2$:

\[\]
Concentration of product of random vectors

- $Z$ is uniformly distributed on the sphere $\sqrt{p}S^{p-1}$.
- $Z \sim N(0, I_p)$ has independent standard Gaussian entries.
- $Z$ is uniformly distributed on the ball $\sqrt{p}B = \{x \in \mathbb{R}^p, \|x\| \leq \sqrt{p}\}$.
- $Z$ is uniformly distributed on $[0, \sqrt{p}]^p$.
- $Z$ has the density $dP_Z(z) = e^{-U(z)}d\lambda_p(z)$ where $U : \mathbb{R}^p \to \mathbb{R}$ is a positive functional with Hessian bounded from below by, say, $cI_p$ with $c \geq O(1)$ and $d\lambda_p$ is the Lebesgue measure on $\mathbb{R}^p$.

Some fundamental results also give concentrations $Z \propto \mathcal{E}_1$ (when $Z \in \mathbb{R}^p$ has independent entries with density $\frac{1}{p}e^{-\|z\|^2}d\lambda_p$, [Tal95]) or $Z \propto \mathcal{E}_q\left(p^{-\frac{q}{2}}\right)$ (when $Z \in \mathbb{R}^p$ is uniformly distributed on the unit ball of the norm $\|\cdot\|_p$, [Led05]).

A very explicit characterization of exponential concentration is given by an upper bound on the different centered moments that lets appear as expected a dependence on the observable diameter.

**Proposition 1.8** (Characterization with the centered moments), [Led05, Proposition 1.10] A random vector $Z \in E$ is $q$-exponentially concentrated with an observable diameter of order $\sigma$ (i.e., $Z \propto \mathcal{E}_q(\sigma)$) if and only if there exist two constants $C, c > 0$ such that for all $p \in \mathbb{N}$, any (sequence of) 1-Lipschitz functions $f : E_p \to \mathbb{R}$:

$$\forall r > 0 : \mathbb{E}\left[\left\|f(Z_p) - f(Z'_p)\right\|^r\right] \leq C \left(\frac{r}{q}\right)^\frac{r}{q} (c\sigma_p)^r,$$

(1.2)

where $Z'_p$ is an independent copy of $Z_p$. Inequality (1.2) also holds if we replace $f(Z'_p)$ with $\mathbb{E}[f(Z_p)]$ (of course the constants $C$ and $c$ might be slightly different).

The Lipschitz-concentrated vectors described in Definition 1.1 belong to the larger class of linearly concentrated random vectors that only requires the linear observations to concentrate. This “linear concentration” presents less stability properties than those described by Proposition 1.6 but is still a relevant notion because:

1. although it must be clear that a concentrated vector $Z$ is generally far from its expectation (for instance Gaussian vectors lie on an ellipse), it can still be useful to have some control on $\|Z - \mathbb{E}[Z]\|$ to express the concentration of product of vectors; linear concentration is a sufficient assumption for this control,
2. there are some examples (Proposition 6.1 and 7.2) where we can only derive linear concentration inequalities from a Lipschitz concentration hypothesis. In that case, we say that the Lipschitz concentration “degenerates” into linear concentration that appears as a “residual” concentration property.

2 Linear concentration and control on high order statistics

**Definition 2.1** (Linearly concentrated vectors). Given a sequence of normed vector spaces $(E_p, \|\cdot\|_p)_{p \geq 0}$, a sequence of random vectors $(Z_p)_{p \geq 0} \in \prod_{p \geq 0} E_p$, a sequence of deterministic vectors $(\tilde{Z}_p)_{p \geq 0} \in \prod_{p \geq 0} E_p$, a sequence of positive reals $(\sigma_p)_{p \geq 0} \in \mathbb{R}^\mathbb{N}$ and a parameter $q > 0$, $Z_p$ is said to be $q$-exponentially linearly concentrated around the deterministic equivalent $\tilde{Z}_p$ with an observable diameter of order $O(\sigma_p)$ iff there exist two constants $c, C > 0$ such that $\forall p \in \mathbb{N}$ and for any unit-normed linear form $f \in E'_p$ ($\forall p \in \mathbb{N}, \forall x \in E : |f(x)| \leq \|x\|$):

$$\forall t > 0 : \mathbb{P}\left(\left\|f(Z_p) - f(\tilde{Z}_p)\right\| \geq t\right) \leq C e^{(t/c\sigma_p)^q}.$$
When the property holds, we write \( Z \in \tilde{Z} + \mathcal{E}_q(\sigma) \). If it is unnecessary to mention the deterministic equivalent, we will simply write \( Z \in \mathcal{E}_q(\sigma) \). If we just need to control its amplitude, we can write \( Z \in O(\theta) \pm \mathcal{E}_q(\sigma) \) when \( \| \tilde{Z}_p \| \leq O(\theta_p) \).

When \( q = 2 \), we retrieve the well known class of sub-Gaussian random vectors. We need this definition with generic \( q \) to prove Proposition 5.1 which involves a weaker tail decay.

Of course linear concentration is stable through affine transformations.

**Proposition 2.2** (Stability through affine mappings). Given two (sequences of) normed vector spaces \((E, \| \cdot \|_E)\) and \((F, \| \cdot \|_F)\), a (sequence of) random vectors \( Z \in E \), a (sequence of) deterministic vectors \( \tilde{Z} \in E \) and a (sequence of) affine mappings \( \phi : E \to F \) such that \( \forall x \in E : \| \phi(x) - \phi(0) \|_F \leq \lambda \| x \|_E \):

\[
Z \in \tilde{Z} + \mathcal{E}_q(\sigma) \quad \implies \quad \phi(Z) \in \phi(\tilde{Z}) \pm \mathcal{E}_q(\lambda\sigma).
\]

When the expectation can be defined, there exists an implication link between Lipschitz concentration (Definitions 1.1) and linear concentration (Definition 2.1).

**Lemma 2.3.** Given a normed space \((E, \| \cdot \|)\) and a random vector \( Z \in E \) admitting an expectation, we have the implication:

\[
Z \propto \mathcal{E}_q(\sigma) \quad \implies \quad Z \in E[Z] \pm \mathcal{E}_q(\sigma).
\]

This implication becomes an equivalence in law dimensional spaces (i.e. when the sequence index “\( p \)” is not linked to the dimension of the vector spaces \( E_p \)); then the distinction between linear concentration and Lipschitz concentration is not relevant anymore.

The next lemma is a formal expression of the assessment that “any deterministic vector located at a distance smaller than the observable diameter to a deterministic equivalent is also a deterministic equivalent”, we omit the proof that straightforward.

**Lemma 2.4.** Given a random vector \( Z \in E \), a deterministic vector \( \tilde{Z} \in E \) such that \( Z \in \tilde{Z} + \mathcal{E}_q(\sigma) \), we have the equivalence:

\[
Z \in \tilde{Z}' + \mathcal{E}_q(\sigma) \quad \iff \quad \| \tilde{Z} - \tilde{Z}' \| \leq O(\sigma)
\]

**Definition 2.5** (Centered moments of random vectors). Given a random vector \( X \in \mathbb{R}^p \) and an integer \( r \in \mathbb{N} \), we call the “\( r \)th centered moment of \( X \)” the symmetric \( r \)-linear form \( \mathcal{C}_r^X : (\mathbb{R}^p)^r \to \mathbb{R} \) defined for any \( u_1, \ldots, u_r \in \mathbb{R}^p \) by

\[
\mathcal{C}_r^X(u_1, \ldots, u_r) = \mathbb{E} \left[ \prod_{i=1}^r (u_i^T X - \mathbb{E}[u_i^T X]) \right].
\]

When \( r = 2 \), the centered moment is the covariance matrix.

We define the operator norm of an \( r \)-linear form \( S \) of \( \mathbb{R}^p \) as

\[
\| S \| \equiv \sup_{\| u_1 \|, \ldots, \| u_r \| \leq 1} S(u_1, \ldots, u_r).
\]

When \( S \) is symmetric, we employ the simpler formula \( \| S \| = \sup_{\| u \| \leq 1} S(u, \ldots, u) \). We then have the following characterization, similar to Proposition 1.3 (refer to [LC18 Proposition 1.21, Lemma 1.21] for the technical arguments required to go from a bound on \( r \in \mathbb{N} \) to a bound on \( r > 0 \)).
Concentration of product of random vectors

Proposition 2.6 (Moment characterization of linear concentration). Given \( q > 0 \), a sequence of random vectors \( X_p \in \mathbb{R}^p \), and a sequence of positive numbers \( \sigma_p > 0 \), we have the following equivalence:

\[
X \in \mathcal{E}_q(\sigma) \iff \exists C, c > 0, \forall p \in \mathbb{N}, \forall r \geq q : \|C^r X_p\| \leq C \left(\frac{r}{q}\right)^{\frac{q}{2}} (c \sigma_p)^r
\]

In particular, if we note \( C = E[X X^T] - E[X]E[X]^T \), the covariance of \( X \in \mathcal{E}_q(\sigma) \), we see that \( \|C\| \leq O(\sigma^2) \), if in addition \( X \in O(\sigma) \cdot \mathcal{E}_q(\sigma) \) (which means that \( \|E[X]\| \leq O(\sigma) \)), then \( \|E[X X^T]\| \leq O(\sigma^2) \)

With these results at hand, we are in position to explain how a control on the norm can be deduced from a linear concentration hypothesis.

3 Control of the norm of linearly concentrated random vectors

Given a random vector \( Z \in (E, \| \cdot \|) \), if \( Z \in \tilde{Z} \pm \mathcal{E}_q(\sigma) \), the control of \( \|Z - \tilde{Z}\| \) can be done easily when the norm \( \| \cdot \| \) can be defined as the supremum on a set of linear forms; for instance when \( (E, \| \cdot \|) = (\mathbb{R}^p, \| \cdot \|_\infty) \): \( \|x\|_\infty = \sup_{1 \leq i \leq p} e_i^T x \) (where \( e_1, \ldots, e_p \) is the canonical basis of \( \mathbb{R}^p \)). We can then bound:

\[
\mathbb{P}\left(\|Z - \tilde{Z}\|_\infty \geq t\right) = \mathbb{P}\left(\sup_{1 \leq i \leq p} e_i^T (Z - \tilde{Z}) \geq t\right)
\leq \min\left(1, p \sup_{1 \leq i \leq p} \mathbb{P}\left(e_i^T (Z - \tilde{Z}) \geq t\right)\right)
\leq \min\left(1, p C e^{-ct/\sigma^2}\right) \leq \max(C, e) \exp\left(-\frac{ct^q}{2\sigma^q \log(p)}\right),
\]

for some constants \( c, C > 0 \) (\( C \leq O(1) \), \( c \geq O(1) \)).

To manage the infinity norm, the supremum is taken on a finite set \( \{e_1, \ldots, e_p\} \). Problems arise when this supremum must be taken on an infinite set. For instance, for the Euclidean norm, the supremum is taken over the whole unit ball \( B_{\mathbb{R}^p} \equiv \{u \in \mathbb{R}^p, \|u\| \leq 1\} \) since for any \( x \in \mathbb{R}^p \), \( \|x\| = \sup\{u^T x, \|u\| \leq 1\} \). This loss of cardinality control can be overcome if one introduces so-called \( \varepsilon \)-nets to discretize the ball with a net \( \{u_i\}_{i \in I} \) (with \( I \) finite - \( |I| < \infty \)) in order to simultaneously:

1. approach sufficiently the norm to ensure

\[
\mathbb{P}\left(\|Z - \tilde{Z}\|_\infty \geq t\right) \approx \mathbb{P}\left(\sup_{i \in I} u_i^T (Z - \tilde{Z}) \geq t\right),
\]

2. control the cardinality \( |I| \) for the inequality

\[
\mathbb{P}\left(\sup_{i \in I} u_i^T (Z - \tilde{Z}) \geq t\right) \leq |I| \mathbb{P}\left(\sup_{1 \leq i \leq p} u_i^T (Z - \tilde{Z}) \geq t\right)
\]

not to be too loose.

One can then show that there exist two constants \( C, e > 0 \) such that:

\[
\mathbb{P}(\|Z - \tilde{Z}\| \geq t) \leq \max(C, e) \exp\left(-\frac{ct^q}{\sigma^q \log(p)}\right).
\]

(3.1)

The approach with \( \varepsilon \)-nets in \( (\mathbb{R}^p, \| \cdot \|) \) can be generalized to any normed vector space \( (E, \| \cdot \|) \) when the norm can be written as a supremum through an identity of the kind:

\[
\forall x \in E : \|x\| = \sup_{f \in E', \|f\|_1 \leq 1} f(x), \quad \text{with } H \subset E' \text{ and } \dim(\text{Vect}(H)) < \infty,
\]

(3.2)
for a given $H \subset E'$ (for $E'$, the dual space of $H$) and with $\text{Vect}(H)$ the subspace of $E'$ generated by $H$. Such a $H \subset E'$ exists in particular when $(E, \| \cdot \|)$ is a reflexive space \cite{jam57}.

When $(E, \| \cdot \|)$ is of infinite dimension, it is possible to establish \eqref{eq:3.2} for some $H \subset E$ when $E$ is reflexive thanks to a result from \cite{jam57}, or for some choice of semi-norms $\| \cdot \|$. Without going into details, we introduce the notion of norm degree which will help us adapt the concentration rate $\rho$ appearing in the exponential term of concentration inequality \eqref{eq:3.1} (concerning $(\mathbb{R}^p, \| \cdot \|)$) to other normed vector spaces.

**Definition 3.1 (Norm degree).** Given a normed (or semi-normed) vector space $(E, \| \cdot \|)$, and a subset $H \subset E'$, the degree $\eta_H$ of $H$ is defined as:

- $\eta_H \equiv \log(|H|)$ if $H$ is finite,
- $\eta_H \equiv \dim(\text{Vect}(H))$ if $H$ is infinite.

If there exists a subset $H \subset E'$ such that \eqref{eq:3.2} is satisfied, we denote $\eta(E, \| \cdot \|)$, or more simply $\eta_{\| \cdot \|}$, the degree of $\| \cdot \|$, defined as:

$$
\eta_{\| \cdot \|} = \eta(E, \| \cdot \|) \equiv \inf \left\{ \eta_H, H \subset E' \mid \forall x \in E, \|x\| = \sup_{f \in H} f(x) \right\}.
$$

**Example 3.2.** We can give some examples of norm degrees:

- $\eta(\mathbb{R}^p, \| \cdot \|_\infty) = \log(p)$ ($H = \{ x \mapsto e_i^T x, 1 \leq i \leq p \}$),
- $\eta(\mathbb{R}^p, \| \cdot \|_1) = p$ ($H = \{ x \mapsto u^T x, u \in \mathbb{B}_{\mathbb{R}^p} \}$),
- $\eta(M_{p,n}, \| \cdot \|_1) = n + p$ ($H = \{ M \mapsto u^T M v, (u, v) \in \mathbb{B}_{\mathbb{R}^p} \times \mathbb{B}_{\mathbb{R}^n} \}$),
- $\eta(M_{p,n}, \| \cdot \|_F) = np$ ($H = \{ M \mapsto \text{Tr}(AM), A \in M_{n,p}, \|A\|_F \leq 1 \}$),
- $\eta(M_{p,n}, \| \cdot \|_\infty) = np$ ($H = \{ M \mapsto \text{Tr}(AM), A \in M_{n,p}, \|A\| \leq 1 \}$).

Just to give some justification, if $E = \mathbb{R}^p$ or $E = M_{p,n}$, the dual space $E'$ can be identified with $E$ through the representation with the scalar product. Given a subset $H' \subset E$ such that:

$$
\forall x \in \mathbb{R}^p, \|x\|_\infty = \sup_{u \in H'} u^T x,
$$

we can set that all $u \in H'$ satisfy $\|u\|_1 = \sum_{i=1}^p |u_i| \leq 1$ because if we note $u' = (\text{sign}(u_i))_{i \in [p]}$, we can bound $\|u\|_1 = u^T u' \leq \sup_{v \in H'} v^T u' \leq \|u\|_\infty \leq 1$. Then, noting $H = \{e_1, \ldots, e_p\}$, we know that $H \subset H'$, otherwise, if, say $e_i \notin H'$, then one could bound $\|e_i\|_\infty = \sup_{u \in H'} u^T e_i < 1$ (because if $\|u\|_1 \leq 1$ and $u \neq e_i$, then $u_i < 1$). Therefore $H \subset H'$ and it consequently reaches the minimum of $\eta_{H'}$. The value of the other norm indexes is justified with the same arguments.

Depending on the ambient vector space, one can employ one of these examples along with the following proposition borrowed from \cite{LC18} Proposition 2.9.2.11, Corollary 2.13 to establish the concentration of the norm of a random vector.

**Proposition 3.3.** Given a reflexive vector space $(E, \| \cdot \|)$ and a concentrated vector $Z \in E$ satisfying $Z \in \tilde{Z} \pm \mathcal{E}_q(\sigma)$:

$$
\|Z - \tilde{Z}\| \sim \mathcal{E}_q\left(\eta_{\| \cdot \|}^{1/q} \sigma\right) \quad \text{and} \quad \mathbb{E}\left[\|Z - \tilde{Z}\|\right] \leq O\left(\eta_{\| \cdot \|}^{1/q} \sigma\right).
$$

\text{\footnote{Introducing the mapping $J : E \to E''$ (where $E''$ is the bidual of $E$) satisfying $\forall x \in E$ and $\phi \in E'$: $J(x)(\phi) = \phi(x)$, the normed vector space $E$ is said to be “reflexive” if $J$ is onto.}}

\text{\footnote{\| \cdot \|$_*$ is the nuclear norm defined for any $M \in M_{p,n}$ by $\|M\|_* = \text{Tr}(\sqrt{MM^*})$: it is the dual norm of $\| \cdot \|$, which means that for any $A, B \in M_{p,n}$, $\text{Tr}(AB^*) \leq \|A\| \|B\|_*$. One must be careful that Proposition 3.3 is rarely useful to bound the nuclear norm as explained in footnote 19.}}
Concentration of product of random vectors

**Remark 3.4.** In Proposition 3.3, if \( Z \sim \mathcal{E}_q(\sigma) \), the norm satisfies the same concentration as it is a Lipschitz observation, and one gets
\[
\| Z - \bar{Z} \| \in O\left( \eta^{1/q} \| \sigma \| \right) + \mathcal{E}_q(\sigma).
\]

**Example 3.5.** Given two random vectors \( Z \in \mathbb{R}^p \) and \( X \in \mathcal{M}_{p,n} \):

- if \( Z \sim \mathcal{E}_2 \) in \( \mathbb{R}^p \), then \( \mathbb{E} \| Z \| \leq \| \mathbb{E}[Z] \| + O(\sqrt{p}) \),
- if \( X \sim \mathcal{E}_2 \) in \( \mathcal{M}_{p,n} \), then \( \mathbb{E} \| X \| \leq \| \mathbb{E}[X] \| + O(\sqrt{p} + n) \),
- if \( X \sim \mathcal{E}_2 \) in \( \mathcal{M}_{p,n} \), then \( \mathbb{E} \| X \| \leq \| \mathbb{E}[X] \|_F + O(\sqrt{p}n) \).

Let us consider the semi norm \( \| \cdot \|_d \) that will be useful later and that satisfies:

**Definition 3.6.** Given \( M \in \mathcal{M}_n \), we define the diagonal norm of \( M \) as:
\[
\| M \|_d = \left( \sum_{i=1}^{n} M_{i,i}^2 \right)^{1/2} = \sup_{\| D \|_F \leq 1} \text{Tr}(DM).
\]
where \( D \in \mathcal{D}_{p,n} \) is the set of diagonal matrices of \( \mathcal{M}_{p,n} \) defined as:
\[
D \in \mathcal{D}_{p,n} \iff (i \neq j \implies D_{i,j} = 0).
\]

For simplicity, we note the (non zero) diagonal terms of any \( D \in \mathcal{D}_{n,p} : D_1, \ldots, D_{\min(n,p)} \).

**Example 3.7.** We see directly that \( \eta_{\mathcal{M}_n} \| \cdot \|_d \) = \#\mathcal{D}_n = n \) and therefore for a given \( X \in \mathcal{M}_n \) such that \( X \sim \mathcal{E}_2 \), we can bound \( \mathbb{E} \| X \|_d \leq \| \mathbb{E}[X] \| + O(\sqrt{n}) \).

Proposition 3.3 is not always the optimal way to bound norms. For instance, given a vector \( Z \in \mathbb{R}^p \) and a deterministic matrix \( A \in \mathcal{M}_p \), if \( Z \sim \mathcal{E}_q \), one is tempted to bound naively thanks to Proposition 3.3:

- if \( \| \mathbb{E}[Z] \| \leq O(p^{1/2}) \), \( \mathbb{E}[\| AZ \|] \leq \| A \| \mathbb{E}[\| Z \|] \leq O(\| A \| p^{1/2}) \);  
- if \( \| \mathbb{E}[Z] \| \leq O(1) \), decomposing \( A = P^T \Lambda Q \), where \( P, Q \in \mathcal{O}_p \), \( \Lambda = \text{Diag}(\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p \) and setting \( \tilde{Z} = (\tilde{Z}_1, \ldots, \tilde{Z}_p) \equiv QZ \):
\[
\mathbb{E}[\| AZ \|] = \mathbb{E}[\| \Lambda QZ \|] = \mathbb{E} \left[ \sum_{1}^{p} \lambda_i^2 \tilde{Z}_i^2 \right] \leq \| \lambda \|_\infty \mathbb{E} \left[ \| \tilde{Z} \|_\infty \right] \leq \| A \|_F O \left( (\log p)^{1/2} \right).
\]

Note indeed that \( \tilde{Z} \sim \mathcal{E}_2 \) and therefore \( \mathbb{E}[\| \tilde{Z} \|_\infty] \leq \| A \| \| \tilde{Z} \|_\infty + O \left( (\log p)^{1/2} \right) \) \leq \| \mathbb{E}[Z] \| + O \left( (\log p)^{1/2} \right) .

However, here, Proposition 3.3 is suboptimal: one can reach a better bound thanks to the following lemma that was taken from the proof of [Ada15, Theorem 2.5]. We give a result for random vectors and random matrices, they are actually equivalent.

---

\(^{9}\) The notation \( Z \in O(\theta) \pm \mathcal{E}_q(\sigma) \) was presented in Definition 2.1 for linearly concentrated vectors, it can be extended to concentrated random variables.

\(^{10}\) One must be careful here that Theorem 1.7 just provides concentration in the Euclidean spaces \( (\mathbb{R}^p, \| \cdot \|) \) or \( (\mathcal{M}_{p,n}, \| \cdot \|_F) \) from which one can deduce concentration in \( (\mathbb{R}^p, \| \cdot \|_\infty) \) or \( (\mathcal{M}_{p,n}, \| \cdot \|_\infty) \) since for all \( x \in \mathbb{R}^p, \| x \|_\infty \leq \| x \| \) and for all \( M \in \mathcal{M}_{p,n} \), \( \| M \| \leq \| M \|_F \). However one cannot obtain a better bound than \( \| M \|_\infty \leq \sqrt{\min(n,p) \| M \|_F} \) for instance implies that a random matrix \( X = (x_1, \ldots, x_n) \) with \( x_1, \ldots, x_n \) i.i.d. satisfying \( \forall i \in [n], x_i \sim \mathcal{N}(0, I_p) \) follows the concentration \( X \sim \mathcal{E}_2(\sqrt{\min(p,n)}) \) in \( (\mathcal{M}_{p,n}, \| \cdot \|_\infty) \).
Concentration of product of random vectors

**Lemma 3.8.** Given a random vector in \((\mathbb{R}^p, \| \cdot \|)\) such that \(\|E[ZZ^T]\| \leq O(1)\) and a deterministic matrix \(A \in \mathcal{M}_p\):

\[
E[\|AZ\|] \leq O(\|A\|_F).
\]

and given a random matrix \(X \in \mathcal{E}_2\) in \((\mathcal{M}_{p,n}, \| \cdot \|_F)\) such that \(\|E[X]\|_F \leq O(1)\) and a supplementary deterministic matrix \(B \in \mathcal{M}_n\):

\[
E[\|AXB\|_F] \leq O(\|A\|_F \|B\|_F).
\]

Thanks to [2.6] we know that the condition \(\|E[ZZ^T]\| \leq O(1)\) can be obtained if one assumes that \(Z \in \mathcal{E}_q\) and \(\|E[Z]\| \leq O(1)\) thanks to the inequality \(E[ZZ^T] = C_Z^2 + E[Z]E[Z]^T\), where \(C_Z\) is the covariance of \(Z\). The same way.

**Proof.** One can bound with Jensen’s inequality:

\[
E[\|AZ\|] \leq \sqrt{E[ZZ^T AZ]} = \sqrt{E[\text{Tr}(\Sigma A^T A)]} \leq \sqrt{\|\Sigma\|_F \|A\|_F} \leq O(\|A\|_F).
\]

The second result is basically the same. If we introduce \(\tilde{X} \in \mathbb{R}^{mn}\) satisfying \(\tilde{X}_{(j-1)+j} = X_{1,j}\), we know that \(\tilde{X} \sim \mathcal{E}_2\) like \(X\) (since \(\|\tilde{X}\| = \|X\|_F\)) and thanks to the previous result we can bound:

\[
E[\|AXB\|_F] = E[\|A \otimes B \tilde{X}\|] \leq O(\|A \otimes B\|_F) = O(\|A\|_F \|B\|_F).
\]

Returning to Lipschitz concentration, in order to control the concentration of the sum \(X + Y\) or the product \(XY\) of two random vectors \(X\) and \(Y\), a first step is to express the concentration of the concatenation \((X, Y)\). This last result is easily obtained for the class of linearly concentrated random vectors but a tight concentration of the product with good observable diameter is in general not accessible. In the class of Lipschitz concentrated vectors, the concentration of \((X, Y)\) is far more involved, and assumptions of independence here play a central role (unlike for linear concentration).

The sum being a 2-Lipschitz operation (for the norm \(\| \cdot \|_{\infty}\)), the concentration of \(X + Y\) is easily handled with Proposition 1.6 and directly follows from the concentration of \((X, Y)\). For products of vectors, more work is required.

### 4 Multi regime concentration of expression with concentrated variations

Given four parameters \(\sigma_1, \sigma_2, q_1, q_2 > 0\), such that \(\sigma_1 \leq \sigma_2\) and \(q_1 \leq q_2\), we have the equivalence:

\[
e^{-c(t/\sigma_1)^{q_1}} \geq e^{-c(t/\sigma_2)^{q_2}} \iff t \leq (\frac{\sigma_1}{\sigma_2})^{\frac{1}{q_1/q_2}}, \quad (4.1)
\]

one then introduce naturally the threshold \(t^{\sigma_2, q_2}_{\sigma_1, q_1}\) that will allow us to distinguish the different regimes of exponential concentration and see is some of them can be removed when they are irrelevant for all \(t > 0\).

**Definition 4.1.** Given \(m \in \mathbb{N}_+\), a set of \(m\) couples of parameters \((q_1, \sigma_1), \ldots, (q_m, \sigma_m) \in (\mathbb{R}^2)^m\) is called a family of multi-regime parameters if:

- \(q_1 > \cdots > q_m > 0\)
Concentration of product of random vectors

- \( \sigma_1 > \cdots > \sigma_m > 0 \).

This family is said to be thrifty if \( m \leq 2 \) or \( \forall 1 \leq k < l \leq m \):

\[
t_{\sigma_k,q_k}^{q_k+1} \leq t_{\sigma_k,q_k}^{q_k} \leq t_{\sigma_{k-1},q_{k-1}}^{q_{k-1}},
\]

where we noted:

\[
t_{\sigma_k,q_k}^{q_k} = \left( \frac{\sigma_k^{q_k}}{\sigma_k^{q_k}} \right)^{\frac{1}{q_k-q_k}}.
\]

It is called a free family of multi-regime parameters if in addition for all \( 1 \leq k < l \leq m-1, k < l \):

\[
t_{\sigma_k,q_k}^{q_k+1} \leq t_{\sigma_k,q_k}^{q_k} < t_{\sigma_{k+1},q_{k+1}}^{q_{k+1}}.
\]

This definition is justified by the following proposition that explains how one can remove some regimes of concentration when they are always under other regimes. One first need a preliminary lemma that we provide without proof since it is trivial.

**Lemma 4.2.** Given three couples \( (\sigma_1,q_1), (\sigma_2,q_2), (\sigma_3,q_3) \in \mathbb{R}^2 \), such that \( \sigma_1 \leq \sigma_2 \leq \sigma_3 \) and \( q_1 \leq q_2 \leq q_3 \), if we note for any \( l, k \in [3], l \neq k, t_{l,k} \equiv t_{\sigma_l,q_l}^{q_l} = (\sigma_k^{q_k}/\sigma_l^{q_l})^{1/q_k-q_l} \), then, we have the relation:

\[
t_{1,2}^{q_1-q_2} t_{2,3}^{q_2-q_3} = t_{1,3}^{q_1-q_3}.
\]

Now, we can set the following proposition:

**Proposition 4.3.** In the setting and with the notations of Lemma 4.2, one has the equivalence:

\[
t_{1,2} \geq t_{1,3} \quad \iff \quad t_{1,3} \geq t_{2,3}
\]

and if one assumes the two assertions of this equivalence, one can bound:

\[
\forall t > 0 : e^{-(t/\sigma_2)^{q_2}} \leq \sup (e^{-(t/\sigma_1)^{q_1}}, e^{-(t/\sigma_3)^{q_3}}).
\]

In other words, \( E_{q_2}(\sigma_2) \leq E_{q_1}(\sigma_1) + E_{q_3}(\sigma_3) \).

**Proof.** The equivalence is just a consequence of Lemma 4.2. If we now assume that the two assertion of the equivalence are true, one can bound when \( t \leq t_{1,3} \) thanks to (4.1) and the fact that \( t_{1,3} \leq t_{1,2} \):

\[
e^{-c(t/\sigma_2)^{q_2}} \leq e^{-c(t/\sigma_1)^{q_1}}.
\]

The same way, when \( t \geq t_{1,3} \geq t_{2,3} \):

\[
e^{-c(t/\sigma_2)^{q_2}} \leq e^{-c(t/\sigma_3)^{q_3}},
\]

we thus exactly showed that \( E_{q_2}(\sigma_2) \leq E_{q_1}(\sigma_1) + E_{q_3}(\sigma_3) \).

An important local characterization of thrifty multi-regime families is given in next proposition.

**Proposition 4.4.** Given \( m \geq 3 \), a family of multi-regime parameters \( (q_1,\sigma_1), \ldots, (q_m,\sigma_m) \in (\mathbb{R}^2)^m \) is thrifty if and only if one of the following property is satisfied:

- for any tuple of integers \( (i,j,k,l) \in [m]^4 \) such that \( i \leq j \leq k \leq l \), we have the inequality:

\[
\left( \frac{\sigma_i}{\sigma_j} \right)^{\frac{q_j}{q_i}} \leq \left( \frac{\sigma_k}{\sigma_l} \right)^{\frac{q_k}{q_l}}.
\]

(4.2)
When we then assume that relation (4.4) is true when we bound:

\[ \frac{\sigma_i}{\sigma_{i+1}} \frac{q_i q_{i+1}}{q_{i+1} q_i} \leq \frac{\sigma_{i+1}}{\sigma_{i+2}} \frac{q_{i+1} q_{i+2}}{q_{i+2} q_{i+1}}, \]

(4.3)

Proof. Let us start with the following equivalence:

\[ t_{\sigma_i, q_i} \leq t_{\sigma_k, q_k} \iff \left( \frac{\sigma_i}{\sigma_j} \right)^{q_i q_j} \leq \left( \frac{\sigma_k}{\sigma_j} \right)^{q_k q_j}, \]

\[ \iff \frac{\sigma_i}{\sigma_j} \frac{q_i q_j}{q_k q_j} \leq \frac{\sigma_k}{\sigma_j}, \]

(4.4)

(the last equivalence is obtained putting the inequality to the power \( \frac{q_i q_j}{q_k q_j} \geq 0 \), we have indeed \( \frac{\sigma_i}{\sigma_j} \geq 1 \)). Now, given a supplementary integer \( l \geq k \), assuming \( t_{\sigma_i, q_i} \leq t_{\sigma_k, q_k} \leq t_{\sigma_l, q_l} \), we can conclude thanks to the previous equivalence:

\[ \left( \frac{\sigma_i}{\sigma_j} \right)^{q_i q_j} \leq \left( \frac{\sigma_k}{\sigma_j} \right)^{q_k q_j} \leq \left( \frac{\sigma_l}{\sigma_j} \right)^{q_l q_j}. \]

The reverse implication is simply obtained taking \( k = j \).

Since relation (4.3) is a particular case of relation (4.2), we just show the reverse implication. The proof is done iteratively on \( k-i \) with the characterisation given in (4.4). When \( k-i \geq 2 \), we are in the case of relation (4.3), given an integer \( n \in \{2, \ldots, m-1\} \), we then assume that relation (4.4) is true when \( k-i \leq n \), let us then consider a case where \( k-i = n+1 \) and, say, \( k-j \geq 2 \), employing \( \sigma_j \) and \( \sigma_{j+1} \) as a pivot \( (j+1-i, k-j \leq n) \) we bound:

\[ \frac{\sigma_i}{\sigma_k} \frac{q_i q_j}{q_j q_i} \leq \frac{\sigma_{j+1}}{\sigma_j} \frac{q_j q_{j+1}}{q_{j+1} q_j} \]

\[ \leq \frac{\sigma_{j+1}}{\sigma_j} \frac{q_j q_{j+1}}{q_{j+1} q_j} \frac{q_j q_{j+1}}{q_{j+1} q_j} \frac{q_j q_{j+1}}{q_{j+1} q_j} \]

\[ \leq \left( \frac{\sigma_i}{\sigma_j} \frac{q_i q_j}{q_j q_i} \right) \left( \frac{q_i q_j}{q_j q_i} \right) \left( \frac{q_i q_j}{q_j q_i} \right) \left( \frac{q_i q_j}{q_j q_i} \right) \]

One can then conclude thanks to the identity:

\[ \left( \frac{1}{q_i+1} \right) - \left( \frac{1}{q_i} \right) = \left( \frac{1}{q_i} - \frac{1}{q_i+1} \right) \left( \frac{1}{q_i} - \frac{1}{q_i+1} \right) = \left( \frac{1}{q_i+1} - \frac{1}{q_i} \right) \left( \frac{1}{q_i} - \frac{1}{q_i+1} \right) \]

This last proposition allows to extract thrifty multi-regime families of multi-regime parameters from non thrifty ones.
Concentration of product of random vectors

**Proposition 4.5.** Given \( m \geq 3 \) and a family of multi-regime parameters \( (q_1, \sigma_1), \ldots, (q_m, \sigma_m) \in (\mathbb{R}^2)^m \), there exists a unique free thrifty extraction of this family \( (q_1, \sigma_1), \ldots, (q_k, \sigma_k) \in (\mathbb{R}^2)^k \) for \( k \leq m \) such that \( 1 \leq i_1 < \cdots < i_k = m \) and for any \( t > 0 \):

\[
\sup_{i \in [m]} \mathcal{E}_{q_i}(\sigma_i) = \sup_{j \in [k]} \mathcal{E}_{q_{i_j}}(\sigma_{i_j}). \tag{4.5}
\]

**Proof.** For all \( k, l \in [m] \), we employ again the short notation \( t_{k,l} \equiv t_{\sigma_k, \sigma_l}^{q_k, q_l} \). The extraction can be done iteratively. The first component of our iterative extraction is \( (\sigma_1, q_1) = (\sigma, q) \). Now, given \( 2 \leq l \leq m - 1 \) and assuming that \( (q_1, \sigma_1), \ldots, (q_i, \sigma_i) \) is thrifty, we introduce the set:

\[
A_{i_l} \equiv \{ i \in \{ l+1, \ldots, m \} \text{ s.t. } t_{u, i_l} < t_{u, i_l+1} \},
\]

and if \( A_{i_l} \) is not empty, we denote \( i_{l+1} = \inf A_{i_l} \). We know from Proposition 4.4 that \( (q_1, \sigma_1), \ldots, (q_{i_l}, \sigma_{i_l}) \) is thrifty, we can thus repeat the procedure until we found \( i_n \in [m] \) such that \( A_{i_n} \) is empty (we know that \( A_{i_2} \supset \cdots \supset A_{i_l} \supset A_1 \supset \cdots \supset A_m = \emptyset \)), then, noting \( k = l_0 + 1 \), and \( i_k = m \), by construction, \( (q_1, \sigma_1) \cdots (q_k, \sigma_k) \) is a thrifty family.

To prove (4.5), we consider \( l \in [k - 1] \), and \( i \in [m] \) such that \( i_l \leq i \leq i_{l+1} \) we know from the definition of \( A_{i_l} = \emptyset \) that \( t_{u,i} \geq t_{u,i_{l}+1} \), therefore we know from Proposition 4.3 that we also have the inequality \( t_{u,i_{l}+1} \leq t_{u,i_{l}+1} \) and one can bound:

\[
\mathcal{E}_{q_{i_l}}(\sigma_{i_l}) \leq \mathcal{E}_{q_{i_l}}(\sigma_{i_l}) + \mathcal{E}_{q_{i_{l+1}}}(\sigma_{i_{l+1}}),
\]

which ends our proof. \( \Box \)

Let us now provide We can then adapt Proposition 4.4 to the case where \( \forall i \in [m], q_i = \frac{q}{2} \), for some \( q > 0 \), to obtain the following characterization.

**Theorem 4.6.** Given two (sequence of) normed or (semi normed) vector spaces \( (E, \| \cdot \|) \) and \( (F, \| \cdot \|) \), a (sequence of) random vectors \( Z \in (E, \| \cdot \|) \) such that \( Z \sim \mathcal{E}_q(\sigma) \) and a mapping \( \Phi : E \to F \) satisfying on the set of drawings of \( Z \):

\[
\| \phi(Z) - \phi(Z') \| \leq \mathbb{E} \| Z - Z' \|,
\]

for \( Z' \sim E \), an independent copy of \( Z \) and \( V \) a random variable satisfying:

\[
V \in O(\sigma_0) \pm \sum_{i=1}^{m} \mathcal{E}_{q_i}(\sigma_i),
\]

for some multi-regime family \( (\sigma_1, q_1), \ldots, (\sigma_m, q_m) \) and \( \sigma_0 \in (0, \sigma_1] \). Then one can deduce the concentration:

\[
\phi(Z) \sim \mathcal{E}_q(\sigma_0 \sigma) + \sum_{i=1}^{m} \mathcal{E}_{q_i \frac{q_i}{\sigma_i + q_i}}(\sigma_i \sigma)
\]

**Proof.** Let us prove the theorem iteratively on the number of regimes describing the concentration of \( V \). In a first time, we consider that the multi-regime family \( (\sigma_1, q_1), \ldots, (\sigma_m, q_m) \) is thrifty. To prove the initializing point we assume that \( V \in O(\sigma_0) \pm \mathcal{E}_{q_1}(\sigma_1) \). Then introducing:

\[
K_l = \max \left( \sigma_0, \sigma_1 \left( \frac{l}{\sigma_1} \right)^{\frac{q}{\sigma_1 + q}} \right),
\]

\[
K = \max_{l} \left( \sigma_0, \sigma_1 \left( \frac{l}{\sigma_1} \right)^{\frac{q}{\sigma_1 + q}} \right)
\]

Page 15/41
Concentration of product of random vectors

we know that \( K_t \geq \sigma_0 \) and we can first bound:

\[
\mathbb{P}(V \geq 2K_t) \leq \mathbb{P}(V \geq 2K_t) \leq \mathbb{P}(|V - \sigma_0| \geq K_t) \leq Ce^{-c(t/\sigma_M)^{\frac{\gamma t}{1+\gamma}}}. 
\]

Besides, under \( \{V \leq K_t\} \), \( \phi \) is \( K_t \)-Lipschitz, and one can bound:

\[
\mathbb{P}(|\Phi(Z) - \Phi(Z')| \geq t, V \leq 2K_t) \leq Ce^{-c(t/\sigma_0)^{\gamma}} \leq \max \left( Ce^{-c(t/\sigma_0)^{\gamma}}, Ce^{-c(t/\sigma_1)^{\gamma}} \right).
\]

Finally we are able to bound:

\[
\mathbb{P}(|\Phi(Z) - \Phi(Z')| \geq t) \leq \max \left( Ce^{-c(t/\sigma_0)^{\gamma}}, Ce^{-c(t/\sigma_1)^{\gamma}} \right).
\]

Let us now assume that our theorem is true for any \( m \leq M - 1 \) and that we have the concentration:

\[
V \in O(\sigma_0) \pm \sum_{l \in [M]} \mathcal{E}_l(\sigma_l).
\]

With the notation \( t_{l-1} \equiv t_{l-1}^\sigma = \frac{(\frac{\sigma_l}{\sigma_{l-1}})^{\frac{1}{\gamma_l-1}}}{\gamma_l-1} \), we can bound (since the family is thrifty):

\[
\forall t > 0 : \quad \mathbb{P}(V \geq t | V \leq t_{M-1}) \leq \sup_{l \in [M-1]} Ce^{-c(t/\sigma_l)^{\gamma}}.
\]

and our iteration hypothesis allows us to write:

\[
\forall t > 0 : \quad \mathbb{P}(|\Phi(Z) - \Phi(Z')| \geq t | V \leq t_{M-1}) \leq \sup_{l \in [M-1]} Ce^{-c(t/\sigma_l)^{\gamma}}.
\]

When \( V \geq t_{M-1} \), we introduce the parameter

\[
K_t = \max \left( \frac{\sigma_0}{\sigma_M} \left( \frac{t}{\sigma_M} \right)^{\frac{1}{\gamma_M+1}} \right),
\]

it satisfies:

\[
\mathbb{P}(V \geq K_t | V \geq t_{M-1}) \leq Ce^{-c(t/\sigma_M)^{\gamma_M}}
\]

and:

\[
\mathbb{P}(|\Phi(Z) - \Phi(Z')| \geq t, V \leq K_t) \leq \max \left( Ce^{-c(t/\sigma_0)^{\gamma}}, Ce^{-c(t/m^\gamma/\sigma_M)^{\gamma_M}} \right).
\]

When the family \( (\sigma_1, q_1), \ldots, (\sigma_m, q_m) \) is not thrifty, one can still consider its free thrifty extraction \( (\sigma_1', q_1), \ldots, (\sigma_k', q_k) \) thanks to Proposition 4.6 and then show:

\[
\Phi(Z) \propto \mathcal{E}_q(\sigma_0) + \sum_{l=1}^{k} \mathcal{E}_q(\sigma_{l+1}) \propto \mathcal{E}_q(\sigma_0) + \sum_{l=1}^{m} \mathcal{E}_q(\sigma_l)
\]

\[\square\]

It is possible to optimize the expression of the concentration in Theorem 4.6 if one uses some stability properties provided by the following proposition. some stability properties of the class of thrifty multi-regime families.
Proposition 4.7. Given a positive parameter $\alpha > 0$, a sequence of decreasing positive parameters $q_1 > \cdots > q_m > 0$, given $\sigma_1 > \cdots > \sigma_m > 0$ and $\sigma'_1 > \cdots > \sigma'_m > 0$ such that $(\sigma_1, q_1), \ldots, (\sigma_m, q_m)$ and $(\sigma'_1, q'_1), \ldots, (\sigma'_m, q'_m)$ are two thrifty multi-regime family, the family $((\sigma_1, \sigma'_1)^\alpha, q_1), \ldots, ((\sigma_m, \sigma'_m)^\alpha, q_m)$ is also thrifty.

The same way, given a sequence of decreasing positive parameters $\sigma_1 > \cdots > \sigma_m > 0$, given $q_1 > \cdots > q_m > 0$ and $q'_1 > \cdots > q'_m > 0$ such that $(\sigma_1, q_1), \ldots, (\sigma_m, q_m)$ and $(\sigma_1, q'_1), \ldots, (\sigma_m, q'_m)$ are two thrifty multi-regime family, the family $((\sigma_1, q_1)^\alpha, \sigma_1), \ldots, ((\sigma_m, q_m)^\alpha, \sigma_m)$ is also thrifty.

Remark 4.8. To provide a geometric interpretation of the last proposition, note that the set of families of positive parameters $\mu_1 > \cdots > \mu_m$ such that $(e^{\mu_1}, q_1), \ldots, (e^{\mu_m}, q_m)$ is a thrifty multi-regime family is a cone of $\mathbb{R}^m$. The same way, the set of families of positive parameters $s_1 > \cdots > s_m > 0$ such that $(\sigma_1, 1), \ldots, (\sigma_m, 1)$ is a thrifty multi-regime family is a cone of $\mathbb{R}^m$.

Proof. For all $i \in [m]$, we note:

$$\mu_1^{(i)} = \log(\sigma_1^{(i)}) \quad \text{and} \quad s_1^{(i)} = \frac{1}{q_i^{(i)}}.$$  

It is then easy to see that for any $1 < i < m$:

$$\left(\frac{\sigma_i}{\sigma_i + 1}\right)^{\frac{q_i - q_{i+1}}{q_i}} \leq \left(\frac{\sigma_i}{\sigma_i + 1}\right)^{\frac{q_i - q_{i+1}}{q_i}} \iff \frac{\mu_i - \mu_i'}{s_i - s_{i+1}} \leq \frac{\mu_i - \mu_i'}{s_i - s_{i+1}},$$

and the same identity hold for $\mu_1^{(i)} - \mu_1^{(i+1)}$, $\mu_1^{(i+1)}$, $s_1^{(i)}, s_1^{(i+1)}$, $s_i^{(i)}$, $s_i^{(i+1)}$. One can then conclude with the implications:

$$\begin{align*}
\begin{cases}
\frac{\mu_i - \mu_i}{s_i - s_{i+1}} \\
\frac{\mu_i' - \mu_i'}{s_i - s_{i+1}}
\end{cases} \leq \begin{cases}
\frac{\mu_i - \mu_i}{s_i - s_{i+1}} \\
\frac{\mu_i' - \mu_i'}{s_i - s_{i+1}}
\end{cases} \Rightarrow \frac{\mu_i - \mu_i}{s_i - s_{i+1}} \leq \frac{\mu_i - \mu_i'}{s_i - s_{i+1}} \leq \frac{\mu_i - \mu_i'}{s_i - s_{i+1}} \leq \frac{\mu_i - \mu_i'}{s_i - s_{i+1}}.
\end{align*}$$

Taking into account those stability properties of the class of thrifty multi-regime families, one can obtain the following offshoot of Theorem 4.6.

Corollary 4.9. In the setting of Theorem 4.6 if we further assume that $(q_1, \sigma_1), \ldots, (q_m, \sigma_m)$ is thrifty, and we note $l_0 = \inf\{l \in [m] \mid \sigma_0 \leq \sigma'_l\}$, then $(q, \sigma_0), (\frac{q}{q_0}, \sigma_0), \ldots, (\frac{q}{q_l}, \sigma_0)$ is also thrifty and one has the concentration:

$$\Phi(Z) \propto E_q(\sigma_0) + \sum_{l=0}^{m} E_{\frac{q}{q+l}}(\sigma_{l-1} \sigma)$$

Page 17/41
Concentration of product of random vectors

Proof. For simplicity, we introduce for all $i \in [m]$ the notation:

$$
\sigma'_i = \sigma \sigma_i,
$$

and

$$
q'_i = \frac{qq_i}{q + q_i}.
$$

We already know that:

$$
\Phi(Z) \propto \mathcal{E}_q(\sigma \sigma_0) + \sum_{i=1}^{m} \mathcal{E}_{q'_i}(\sigma'_i),
$$

Considering $l \in \{1, \ldots, l_0 - 1\}$ the characterization given by Proposition 4.4 allows us to set the equivalence:

$$
\ell_{\sigma \sigma_0, q} \leq \ell_{\sigma'_0, q'_0} \iff \left( \frac{\sigma_0}{\sigma_1} \right)^{\sigma'_0 q'_0} \leq \left( \frac{\sigma_1}{\sigma_{l_0}} \right)^{\sigma' q'_0} \leq \sigma_0 \iff \frac{\sigma_0 q}{\sigma_1} \geq \frac{1}{\sigma_{l_0}} \equiv \ell_{\sigma_0, q_0} \quad (4.6)
$$

(no primes in this last inequality). Therefore, since $\sigma_0 > \ell_{\sigma_0-1, q_0-1} \geq \ell_{\sigma'_0, q'_0}$, the upper equivalence allows us to set:

$$
\ell_{\sigma \sigma_0, q} \geq \ell_{\sigma'_0, q'_0},
$$

which implies, thanks to Proposition 4.3 $\mathcal{E}_{q'_i}(\sigma'_i) \leq \mathcal{E}_q(\sigma \sigma_0) + \mathcal{E}_{q'_i}(\sigma'_i)$ and therefore:

$$
\Phi(Z) \propto \mathcal{E}_q(\sigma \sigma_0) + \sum_{i=1}^{m} \mathcal{E}_{q'_i}(\sigma'_i).
$$

To show that $(q, \sigma \sigma_0), (q_{\sigma_0}, \sigma_{\sigma_0}), \ldots, (q_{\sigma_m}, \sigma_{\sigma_m})$ is thrifty, let us first recall from Proposition 4.7 that $(\sigma_1, q_1), \ldots, (\sigma_m, q_m)$ is thrifty since $(\sigma_1, q'_1), \ldots, (\sigma_m, q'_m)$ are both thrifty and $(\sigma'_0, q'_0), \ldots, (\sigma'_0, q'_m)$ are thrifty since $(\sigma_1, q'_1), \ldots, (\sigma_m, q'_m)$ and $(\sigma'_0, q'_0), \ldots, (\sigma'_0, q'_m)$ are both thrifty. We then know from Proposition 4.4 that we are just left to show that $\ell_{\sigma'_0, q'_0} \leq \ell_{\sigma'_0, q'_0}$, but this is immediate thanks to equivalence (4.6) and the hypothesis $\sigma_0 \leq \ell_{\sigma_0-1, q_0-1}$. We end with the characterization of thrifty family in the case where $\forall i, q_i = \frac{q}{q_i}$, for some $q > 0$. This says $\sigma_0 > \ell_{\sigma_0-1, q_0-1}$, so it comes naturally when one uses several times Theorem 4.6. Indeed, when $Z \propto \mathcal{E}_2(\sigma)$ and $V \in O(\sigma_0) \pm \mathcal{E}_2(\sigma_1)$, $\Phi(Z) \propto \mathcal{E}_2(\sigma \sigma_0) + \mathcal{E}_1(\sigma_1)$ and if $\forall i \in [p]$, $V \in O(\sigma_0) \pm \mathcal{E}_2(\sigma_1) + \mathcal{E}_1(\sigma_2)$, $\Phi(Z) \propto \mathcal{E}_2(\sigma \sigma_0) + \mathcal{E}_1(\sigma_1) + \mathcal{E}_1(\sigma_2)$. We will see it appear in some practical examples.

**Proposition 4.10.** Given $q > 0$ and $m$ parameters $\sigma_1, \ldots, \sigma_k > 0$, the family of multi regime parameters $(q, \sigma_1), \ldots, (\frac{q}{m}, \sigma_m)$ is thrifty if and only if one of the following properties is satisfied:

- $\forall i \in [m - 2]$ : $\sigma_i \sigma_{i+2} \leq \sigma_{i+1}^2$
- $\forall k, l \in [m - 1], k \leq l$ : $\sigma_k \sigma_{k+1} \leq \sigma_{k+1} \sigma_l$.

**Proof.** Thanks to Proposition 4.4, the only difficulty is to show that the first point implies the second point. It is obtained straightforwardly by multiplying the two inequalities:

$$
\sigma_k \sigma_{l+1} \leq \sigma_{k+1} \sigma_{l+1-k}
$$

and

$$
\sigma_k \sigma_{l+1} \leq \sigma_{l+1-k} \sigma_{l+1}.
$$
Concentration of product of random vectors

Proposition 1.8, which provides a control of the centered moments of a concentrated vector, cannot be directly applied when the concentration follows differing exponential regimes as in Theorem 5.1. We give here a generalization of this result.

Proposition 4.11 (Moment characterization of multi-regime concentration). Given an integer \( m \in \mathbb{N} \), and a sensible family of \( 2m \) multi-regime parameters \( \sigma_1, \ldots, \sigma_m > 0 \), \( q_1, \ldots, q_m > 0 \), a random variable \( Z \in \mathbb{R} \) satisfies the concentration:

\[
Z \propto \sum_{l=1}^{m} E_q (\sigma_l)
\]

for some constants \( C, c > 0 \) if and only if there exist two constants \( C', c' > 0 \) depending only on \( C, c \) such that for all \( r > 0 \), we have the bound:

\[
E[\|Z - E[Z]\|^r] \leq C' \max_{l \in [m]} \left( \frac{r}{qc'} \right)^{\frac{q}{ql}} \sigma_l^{r}.
\]  \( (4.8) \)

Proof. This proof is mainly a rewriting of [Led05, Proposition 1.10] with a fine study of the different concentration regimes. We start with the direct implication which is easier to prove. Assume that there exists two constants \( C, c > 0 \) such that:

\[
\forall t > 0, \quad \mathbb{P}\left( |Z - E[Z]| \geq t \right) \leq C \max_{l \in [m]} e^{-c(t/\sigma_l)^{q_l}}.
\]

Given \( r > 0 \):

\[
E[\|Z - E[Z]\|^r] = \int_{0}^{\infty} \mathbb{P}\left( \|Z - E[Z]\|^r \geq t \right) dt
\]

\[
= \int_{0}^{\infty} \frac{r}{t} t^{r-1} \mathbb{P}\left( |Z - E[Z]| \geq t \right) dt
\]

\[
\leq \max_{l \in [m]} Cr \int_{0}^{\infty} t^{r-1} e^{-(t/\sigma_l)^{q_l}} dt
\]

\[
= \max_{l \in [m]} C \left( \frac{\sigma_l}{c_l^{1/q_l}} \right)^{r} \int_{0}^{\infty} t^{r-1} e^{-c^{q_l} dt},
\]

and, if we assume that \( r \geq q_l \) (\( \geq q_l \) for all \( l \in [m] \)):

\[
\frac{r}{q} \int_{0}^{\infty} t^{r-1} e^{-tc^{q_l} dt} = \frac{r}{q} \Gamma \left( \frac{r}{q}, \frac{r}{q} \right) \leq \left( \frac{r}{q} \right)^{q_l - 1} \Gamma \left( q_l, \frac{r}{q} \right),
\]

when \( r < q_l \), one can still bound with Jensen’s inequality (since \( \frac{r}{q_l} \leq 1 \)):

\[
E[\|Z - E[Z]\|^r] \leq E[\|Z - E[Z]\|^q_l] \leq C \max_{l \in [m]} \left( \frac{r}{q_l} \right)^{q_l} \leq C \max_{l \in [m]} \left( \frac{r}{q_l} \right)^{q_l} \leq C \max_{l \in [m]} \left( \frac{r}{q_l} \right)^{q_l}.
\]

Since \( m^m \), \( \max(C, 1) \leq o(1) \), we can choose cleverly our constants to set the first implication of the proposition.

Let us now assume \( (4.8) \). We deduce from Markov inequality and basic integration calculus that \( \forall r > 0 \):

\[
\mathbb{P}\left( |Z - E[Z]| \geq t \right) \leq \frac{E[\|Z - E[Z]\|^r]}{t^r} \leq C \max_{l \in [m]} \left( \frac{r}{qc} \right)^{q_l} \sigma_l^{r}.
\]  \( (4.9) \)

Given any \( k, l \in [m] \), noting:

\[
t_{k,l} \equiv \left( \frac{\sigma_k^{q_k}}{\sigma_l^{q_l}} \right)^{\frac{1}{q_k + q_l}}
\]
and we note $t_i \equiv t_{i-1},t$. We know in particular that $0 = t_0 \leq t_1 \leq \cdots \leq t_{m+1} = \infty$. Given $l \in [m]$ and $t \in [t_l,t_{l+1}]$ if we chose $r = \frac{\pi}{e} \left(\frac{\sigma}{\nu}\right)^{q_l}$, then, for all $k \in [m]$, we want to bound with a $E_l$ decay the quantity:

$$c_k(t) = C \left(\frac{r}{e^{\nu k}}(\sigma_k/t)^{q_k}\right)^{\frac{\omega_k}{q_k}} \leq C \left(\frac{\nu_k}{e^{\nu k}}(\sigma_k/t)^{q_k}\right)^{\frac{\omega_k}{q_k}}$$

to be able to bound the concentration inequality (4.9).

If $k = l$, we have directly:

$$c_k(t) = c_l(t) = C \left(\frac{\nu_k}{e^{\nu k}}(\sigma_k/t)^{q_k}\right)^{\frac{\omega_k}{q_k}} \leq C e^{-\frac{\nu_k}{e^{\nu k}}(\sigma_k/t)^{q_k}}.$$  

If $k \leq l - 1$, $q_k \geq q_l$ then $t_{k,l} \leq t \leq t_l$, which implies $1/t^{q_k-q_l} \leq 1/t^{q_k-q_l}$ and:

$$c_k(t) \leq \left(\frac{\nu_k}{e^{\nu k}}(\sigma_k/t)^{q_k}\right)^{\frac{\omega_k}{q_k}} \leq C e^{-\frac{\nu_k}{e^{\nu k}}(\sigma_k/t)^{q_k}}.$$  

And the same way, when $k \geq l + 1$, $q_k \leq q_l$, then $t \leq t_{l+1} \leq t_{k,l}$, and we can bound $1/t^{q_k-q_l} = 1/t^{q_k-q_{l+1}} \leq t_{l+1}^{q_k-q_{l+1}}$, which allows us to conclude again that $c_k(t) \leq C e^{-\frac{\nu_k}{e^{\nu k}}(\sigma_k/t)^{q_k}}$. When $t \in (0,t_l]$, choosing $r = \frac{\nu_k}{e^{\nu k}}(\frac{\sigma_k}{\nu_k})^{q_k}$, we show the same way that $\forall k \in [m]$, $c_k(t) \leq C e^{-\frac{\nu_k}{e^{\nu k}}(\sigma_k/t)^{q_k}}$. We eventually obtain for all $t \in [0 \leq t_i \leq t_{i+1}] \supset \mathbb{R}^*$:

$$P (\|Z - E(Z)\| \geq t) \leq \max_{l \in [m]} C e^{-c'(\frac{\sigma_k}{\nu_k})^{q_k}},$$

with $c' = \frac{\omega_k}{q_k}$. This is the looked for concentration. \hfill \square

**Remark 4.12.** Proposition 4.11 is generally employed to bound the first centered moments of an observation. In this case, $\left(\frac{\omega_k}{q_k}\right)^{\frac{\omega_k}{q_k}} \leq O(1)$, and when $\sigma \leq O(\mu(1))$ (which is generally the case), there exists a constant $C > 0$ such that we can bound for any constant $r > 0$ ($r \leq O(1)$):

$$E [\|f(Z) - E(f(Z))\|^r] \leq C(\sigma_1)^r,$$

since $\sigma_1 \geq \sigma_l, \forall l \in [m]$. We then see that the first exponential regime $E_{q_l}(\sigma)$ controls the first statistics of the observations and we then say that the observable diameter of $Z$ is of order $O(\sigma_l)$.

## 5 Concentration of generalized products of random vectors

To treat the product of vectors, we provide a general result of concentration of what could be called "multilinearly $m$-Lipschitz mappings" on normed vector spaces. Instead of properly defining this class of mappings we present it directly in the hypotheses of the theorem. Briefly, these mappings are multivariate functions which are Lipschitz on each variable, with a Lipschitz parameter depending on the product of the norms (or semi-norms) of the other variables and/or constants. To express the observable diameter of such an observation, one needs a supplementary notation.

Given a vector of parameters $\nu \in \mathbb{R}_+^m$, we denote for any $k \in [m]$:  

$$\nu^{(k)} \equiv \max_{1 \leq i_1 < \cdots < i_k \leq m} \nu_{i_1} \cdots \nu_{i_k} = \nu^{(m-k+1)} \cdots \nu^{(m)},$$
Concentration of product of random vectors

where \( \{ \nu(t) \}_{t \in [m]} = \{ \nu(t) \}_{t \in [m]} \) and \( \nu(t) \leq \cdots \leq \nu(m) \). This Theorem is the iterative consequence of Theorem 4.6 provided at the end of the section and that has the particularity to control the variations of \( \phi \) with concentrated variables instead of norms (or semi-norms) \((\| \cdot \|_j)_{j \in [m]}(t)\).

**Theorem 5.1 (Concentration of generalized product).** Given a constant \( m (m \leq O(1)) \), let us consider:

- \( m \) sequences of normed vector spaces \( (E_1, \| \cdot \|_1), \ldots, (E_m, \| \cdot \|_m) \).
- \( m \) sequences of norms (or semi-norms) \( \| \cdot \|_1, \ldots, \| \cdot \|_m \), respectively defined on \( E_1, \ldots, E_m \).
- \( m \) sequences of random vectors \( Z_1 \in E_1, \ldots, Z_m \in E_m \) satisfying

\[
Z \equiv (Z_1, \ldots, Z_m) \propto \mathcal{E}_q(\sigma)
\]

for some (sequence of) positive numbers \( \sigma \in \mathbb{R}_+ \), and for both norms\(^\text{11}\):

\[
\| z_1, \ldots, z_m \|_\infty = \sup_{i=1}^{m} \| z_i \|_i \quad \text{and} \quad \| (z_1, \ldots, z_m) \|_\infty = \sup_{i=1}^{m} \| z_i \|_i \text{ defined on } E = E_1 \times \cdots \times E_m.
\]

- a (sequence of) normed vector spaces \( (F, \| \cdot \|) \), a (sequence of) mappings \( \phi : E_1, \ldots, E_m \to F \), such that \( \forall (z_1, \ldots, z_m) \in E_1 \times \cdots \times E_m \) and \( z'_i \in E_i \):

\[
\| \phi(z_1, \ldots, z_m) - \phi(z_1, \ldots, z_i, z'_i, \ldots, z_m) \| \leq \frac{\prod_{j=1}^{m} \max(\| z'_j \|_j, \mu_j)}{\max(\| z_i \|_i, \mu_i)} \| z_i - z'_i \|_i.
\]

where \( \mu_i > 0 \) is a (sequence of) positive reals such that \( \mu_i \geq O(\sigma)_\text{12} \).

Then we have the concentration\(^\text{13}\)

\[
\phi(Z) \propto \max_{l \in [m]} \mathcal{E}_{q/l} \left( \sigma^l \mu^{(m-l)} \right) \quad (5.1)
\]

It is explained in Remark 4.12 that, in the setting of Theorem 5.1, the standard deviation (resp. the \( r \)th centered moment with \( r \leq O(1) \)) of any 1-Lipschitz observation of \( \phi(Z) \) is of order \( O(\sigma^{l-m-1}) \) (resp. \( O((\sigma^{l-m-1})^r) \)) thus the observable diameter, is given by the first exponential decay, \( \mathcal{E}_q(\sigma^{m-l-1}) \), which represent the guiding term of \( \mathcal{E}_q \).

Note that the multi-regime family \( (\sigma^l \mu^{(m-l), \frac{1}{l}})_{l \in [m]} \) is clearly thrifty thanks to Proposition 4.10.

The proof is simply a consequence of Theorem 4.6.

**Proof.** We only prove the result for \( \sigma = 1 \) since it is easy to get back to this setting from a general cases (replacing \( Z \) by \( Z/\sigma \) and \( \mu \) by \( \mu/\sigma \)). Let us assume Theorem 5.1 up to \( m = m_0 - 1 \) and let us try to show its validity for \( m = m_0 \) thanks to Theorem 4.6. We can show with the iteration hypothesis for \( m = m_0 - 1 \) that for all \( i \in [m] \):

\[
\| Z_1 \|_1 \cdots \| Z_{i-1} \|_{i-1} \| Z_{i+1} \|_{i+1} \cdots \| Z_m \|_m \leq O(\mu_i^{m-l}) \sup_{l \in [m-1]} \mathcal{E}_l \left( \mu_i^{m-l-1} \right).
\]

\( ^\text{11} \) One just needs to assume the concentration \( \| Z_i \|_i \in \mu_i \pm \mathcal{E}_q(\sigma) \); the global concentration of \( Z_i \) for the norm (or seminorm) \( \| \cdot \|_i \) is not required.

\( ^\text{12} \) This is a very light assumption: it is hard to find any practical example where \( \mu_i \ll \sigma \).

\( ^\text{13} \) Which means that there exist two constants \( C, c > 0 \) such that for all indexes and for all 1-Lipschitz mapping \( f : F \to \mathbb{R} \), and \( \forall t > 0, 5.3 \) is satisfied. Here since \( m \leq O(1) \), taking the maximum over \( l \in [m] \) is equivalent to taking the sum, up to a small change of the constants; we will thus indifferently write \( \phi(Z) \propto \max_{l \in [m]} \mathcal{E}_{q/m} \left( \sigma^{(l-m)} \right) \) or \( \phi(Z) \propto \sum_{l=1}^{m} \mathcal{E}_{q/m} \left( \sigma^{(l-m)} \right) \).
Concentration of product of random vectors

where \( \mu_{-i} \equiv \mu_1 \cdots \mu_{i-1} \mu_{i+1} \cdots \mu_m \). Now, since for all \( k \in [m-1] \), \( \mu_{-i}^{(k)} \leq O(\mu^{(i)}) \), we retrieve the hypotheses of Theorem 4.6 with \( \forall i \in [m] : \sigma_i \equiv \mu_{-i}^{(m-i)} \) and we can prove Theorem 5.1 for \( m = m_0 \).

Let us now provide the result of Adamczac and Wolff in [AW15] to compare it with Theorem 5.1. It relies on some notations originally introduced by Latala in [Lat06]. Let us denote \( \mathcal{P}_m \), the sets of partition of \([m]\) into nonempty, pairwise disjoint sets. Given a tensor \( A = (\alpha_x)_{x \in [n]^m} \) and a partition under ordered sets, \( \mathcal{J} = \{J_1, \ldots, J_l\} \), define:

\[
\|A\|_{\mathcal{J}} \equiv \sup \left\{ \sum_{\alpha \in [m]^m} a_{\alpha} \prod_{k=1}^{l} x_{\alpha_{J_k}}^k \mid x \in R^J, \|x\| = \sqrt{\sum_{i \in J_k} x_i^2} \right\},
\]

where for any multiindex \( \alpha \in [n]^m \), and any \( k \in [l] \), we noted \( \alpha_{J_k} = (\alpha_i)_{i \in J_k} \) and for any \( x \in R^J \), \( \|x\| \equiv \sqrt{\sum_{i \in J_k} x_i^2} \).

\textbf{Theorem 5.2 (}[AW15], Theorem 1.2), Given a (sequence of) random vector \( X \in \mathbb{R}^n \), satisfying for any mapping \( h : \mathbb{R}^n \rightarrow \mathbb{R} \) and any \( r \geq 2 \):

\[
E \left[ \|h(X) - \mathbb{E}[h(X)]\|^r \right] \leq \sigma^r \mathbb{E}[\|\nabla h(X)\|^r]
\]

and a \( C^m \) mapping \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) such that \( x \mapsto d^m f|_x \) is uniformly bounded on \( \mathbb{R}^n \), we have the concentration:

\[
f(X) \propto \max_{J \in \mathcal{P}_m} \left( \mathbb{E}_X \left( \sigma^m \sup_{x \in \mathbb{R}^n} \|d^m f|_x\|_{\mathcal{J}} \right) \right) + \max_{J \in \mathcal{P}_m} \left( \mathbb{E}_X \left( \sigma^r \mathbb{E}[\|d^m f|_x\|_{\mathcal{J}}] \right) \right)
\]

\textbf{Remark 5.3 (Comparison between Theorem 5.1 and Theorem 5.2)). The hypotheses of Theorem 5.2 imply that for any 1-Lipschitz mapping \( f : \mathbb{R}^p \rightarrow \mathbb{R} \):

\[
\forall r > 0 : E \left[ \|f(Z_p) - f(Z_p')\|^r \right] \leq \left( \frac{r}{2} \right)^{\frac{r}{2}} \sigma^r,
\]

which then allows us, thanks to Proposition 1.3, to set \( X \propto \mathcal{E}_2 \). Therefore, the hypotheses look slightly weaker than those of Theorem 5.1 but not so much because there is a way to connect their hypotheses to log-Sobolev inequalities as seen in [ABW17, AS94] which can be connected to our hypotheses (see [Led05, Theorem 5.3].)

To simplify the picture, one could observe that our result concerns only the first-order variations of the functionals on each variable, and manage the variability of the coefficient of variation (which in practice is some product of semi-norms of random vectors) with some truncation methods. One of the main limitations is then that one has to treat each random term appearing in the variation coefficient independently, thus producing an observable diameter depending on the mean of the norm of the variation term (and not the norm of the mean of the variations). The strength of [AW15] is to postpone the computation of the norm to the higher order derivative of the functionals thanks to the iterative invocation of its hypothesis of a generalized Poincaré inequality (you can bound the differences of \( f \) to its expectation with the expectation of the norm of the derivative, which you can then bound with the norm of its expectation plus the expectation of the norm of the second derivative thanks to the triangular inequality and the generalized Poincaré inequality). This approach becomes very powerful in the case of polynomial functionals, since a certain order of the derivative will vanish, the bound then being composed only of norms of expectations, allowing some cancellation of terms

\footnote{This ordering is not specified in [AW15], but otherwise, we do not see how to define properly the notation \( \alpha_{J_{k}} \) for \( \alpha \in [n]^m \) and \( k \in [l] \).}
Concentration of product of random vectors

(our result only gives the same bound for the concentration of the monomial). However, outside the special case of polynomials, their theorem requires the computation of many complex norms. Moreover, if no large order derivative cancels, they will still keep a term like $\sup_\ell \|d^m f\|_\ell$, without any hint of resolution if it is not bounded (imagine for instance a functional $f(X_1 \ldots X_n)$ with the derivatives of $f$ not bounded). In these rather common cases, our approach seems to be the only relevant one.

**Remark 5.4 (Regime decomposition).** Let us rewrite the concentration inequality (5.1) to let appear the implicit parameter $t$. There exist two constants $C, c > 0$ (in particular, $C, c \leq O(1)$) such that for any 1-Lipschitz mapping $f : F \to \mathbb{R}$, for any $t > 0$:

$$
P \left( |f(\Phi(Z)) - E[f(\Phi(Z))]| \geq t \right) \leq C \max_{i \in [m]} \exp \left( - \left( \frac{t}{\mu^{(m-i)}} \right)^\frac{m}{m-1} \right), \quad (5.3)
$$

This expression displays $m$ regimes of concentration, depending on $t \in [m]$: the first one, $C e^{-c(t/\mu^{(m-i)})^{\frac{m}{m-1}}}$ ($l = 1$), controls the probability for the small values of $t$, and the last one, $C e^{-c t^m / \sigma}$ ($l = m$), controls the tail. Let us define:

$$
t_1 = 0; \quad \forall i \in [m] \setminus \{1\} : t_i \equiv \mu^{(m-i)} \mu_i^i = \frac{\mu^{(m-i+1)}}{\mu^{(m-i)}}; \quad t_{m+1} = \infty.
$$

Recalling that $\mu(1) \leq \cdots \leq \mu(m)$, we see that $t_1 \leq \cdots \leq t_m$. One can then show that for any $i \in [m]$, we have the equivalence:

$$
t \in [t_i, t_{i+1}] \iff \forall j \in [m] \setminus \{i\} : \exp \left( - \left( \frac{t}{\mu^{(m-j)}} \right)^\frac{m}{m-1} \right) \leq \exp \left( - \left( \frac{t}{\mu^{(m-i)}} \right)^\frac{m}{m-1} \right).
$$

Now, if, for a given $i \in [m]$, $i \geq 2$, $\mu_i = \mu_i(i+1)$, then $t_i = t_{i+1}$, therefore the term $C \exp \left( - \left( \frac{t}{\mu^{(m-i)}} \right)^\frac{m}{m-1} \right)$ can be removed from the expression of the concentration inequality since it never reaches the maximum.

In particular, when $\mu(1) = \cdots = \mu(m) \equiv \mu_0$ \footnote{To be precise, it is sufficient to assume $\mu(2) = \cdots = \mu(m)$, since $\mu(1)$ never appears in the definition of the $t_i$ for $i \in [m]$.} $\forall i \in [m], t_i = \mu_0^n$. In this case, there are only two regimes and we can more simply write:

$$
\Phi(Z) \propto \mathcal{E}_q(\sigma \mu_0^{m-1}) + \mathcal{E}_\infty(\sigma^m).
$$

**Corollary 5.5.** In the setting of Theorem 5.1 when $\forall i \in [m], \|E[Z]_i\|_i \leq O(\sigma \eta_i^{1/4})$, we have the simpler concentration:

$$
\phi(Z) \propto \max_{i \in [m]} \mathcal{E}_q / \left( \sigma^m \eta(i-m) \right),
$$

where we denoted $\eta = \left( \eta_i^{1/4}, \ldots, \eta_i^{1/4} \right)_{i \in [m]}$.

**Proof.** Proposition 3.3 allows us to choose $\mu_i = C' \sigma \eta_i^{1/4}$ for some constant $C' > 0$; we thus retrieve the result thanks to Theorem 5.1.

Let us give examples of “multilineary Lipschitz mappings” that would satisfy the hypotheses of Theorem 5.1.
which provides an observable diameter of order $O(1)$ for all matrices $M$. 

### 6 Generalized Hanson-Wright theorems

**Example 5.6** (Entry-wise product). Letting $\odot$ be the entry-wise product in $\mathbb{R}^p$ defined as $[x \odot y]_i = x_i y_i$ (it is the Hadamard product for matrices), $\phi : (\mathbb{R}^p)^m \ni (x_1, \ldots, x_m) \mapsto x_1 \odot \cdots \odot x_m \in \mathbb{R}^p$ is multilinearly Lipschitz since we have for all $i \in [m]$:

$$
\|x_1 \odot \cdots \odot x_{i-1} \odot (x_i - x'_i) \odot x_{i+1} \odot \cdots \odot x_m\| \leq \left( \prod_{j \neq i} \|x_j\| \right) \|x_i - x'_i\|,
$$

for all vectors $x_1, \ldots, x_m, x'_1, \ldots, x'_m \in \mathbb{R}^p$. As a practical case, if $(Z_1, \ldots, Z_m) \propto E_2$ and $\forall i \in [m]$ $\|E[Z_i]\|_\infty \leq O(1)$, then Corollary 5.5 and Remark 5.4 imply:

$$
Z_1 \odot \cdots \odot Z_m \propto E_2 \left( \log(p) \frac{m}{p} \right) + E_{\frac{p}{m}}.
$$

It is explained in Remark 3.12 that in this case, the observable diameter of $Z_1 \odot \cdots \odot Z_m$ is provided by $E_2 \left( \log(p) \frac{m}{p} \right)$: as such, under this very common setting, the entry-wise product has almost no impact on the rate of concentration.

**Example 5.7** (Matrix product). The mapping $\phi : (M_p)^q \ni (M_1, \ldots, M_q) \mapsto M_1 \cdots M_q \in M_p$ is multilinearly Lipschitz since for all $i \in [m]$ $\mathbb{E}[X_i]$:

$$
\|M_1 \cdots M_{i-1}(M_i - M'_i)M_{i+1} \cdots M_m\|_F \leq \left( \prod_{j \neq i} \|M_j\| \right) \|M_i - M'_i\|_F,
$$

for all matrices $M_1, \ldots, M_m, M'_1, \ldots, M'_m \in M_p$. Given $m$ random matrices $X_1, \ldots, X_m \in M_p$ such that $(X_1, \ldots, X_m) \propto E_2$ and $\forall i \in [m]$ $\|E[X_i]\|_\infty \leq O(\sqrt{p})$, Corollary 5.5 implies:

$$
X_1 \cdots X_m \propto E_2 \left( p \frac{m-1}{m} \right) + E_{\frac{p}{m}}.
$$

In particular, for a “data” matrix\footnote{One could have equivalently considered, for even $m \in \mathbb{N}$, the mapping $\phi : M_{p,n}^m \ni M \mapsto M$ satisfying $\forall M_1, \ldots, M_m \in M_{p,n}$, $\phi(M_1, \ldots, M_m) = M_1 M_2 \cdots M_{m-1} M_n$.} $X = (x_1, \ldots, x_n) \in M_{p,n}$ satisfying $X \propto E_2$ and $\|X\|_\infty \leq O(\sqrt{n})$, the sample covariance matrix satisfies the concentration:

$$
\frac{1}{n} XX^T \propto E_2 \left( \sqrt{p + n} \frac{1}{n} \right) + E_1 \left( \frac{1}{n} \right),
$$

which provides an observable diameter of order $O(1/\sqrt{n})$ when $p \leq O(n)$.

### 6 Generalized Hanson-Wright theorems

To give some more elaborate consequences of Theorems 4.6 and 5.1 let us first provide a matricial version of the popular Hanson-Wright concentration inequality. \cite{HansonWright1971}.

**Proposition 6.1** (Hanson-Wright). Given two random matrices $X, Y \in M_{p,n}$, assume that $(X, Y) \propto E_2$ and $\|E[X]\|_F, \|E[Y]\|_F \leq O(1)$ (as $n,p \to \infty$). Then, for any deterministic matrix $A \in M_p$, we have the linear concentration in $(M_{p,n}, \|\cdot\|_F)$:

$$
Y^T A X \propto E_2 (\|A\|_F) + E_1 (\|A\|).
$$

This proposition, which provides a result in terms of linear concentration, points out an instability of the class of Lipschitz concentrated vectors which (here through products) degenerates into a mere linear concentration. This phenomenon fully justifies the introduction of the notion of linear concentration: it will occur again in Proposition 7.2 and Lemma 8.3. We present the proof directly here as it is a short and convincing application of Theorem 5.1.
Concentration of product of random vectors

Proof. Considering a deterministic matrix $B \in \mathcal{M}_n$ such that $\|B\|_F \leq 1$ we introduce the semi-norm $\|\cdot\|_{A,B}$ defined on $\mathcal{M}_{p,n}$ and satisfying for all $M \in \mathcal{M}_{p,n}$, $\|M\|_{A,B} \equiv \|AMB\|_F$. Note that for any $M, P \in \mathcal{M}_{p,n}$:

$$\text{Tr}(BMAP^T) \leq \left\{ \frac{\|M\|_{A,B}\|P\|_F}{\|M\|_F \|P\|_{A,B}} \right\}.$$

Thanks to Lemma 3.8 $E[\|X\|_{A,B}], E[\|Y\|_{A,B}] \leq O(\|A\|_F)$, therefore:

$$\sup(\|X\|_{A,B}, \|Y\|_{A,B}) \in O(\|A\|_F) \pm E_2(\|A\|).$$

Therefore, the hypotheses of Theorem 4.6 are satisfied and we can deduce the result.

\[\square\]

Remark 6.2. For the reader information, we mention that in [Ada15], the concentration is even expressed on the random variable $\sup_{A \in A} X^TAX$ where $A$ is a bounded set of matrices (and $X \in \mathbb{R}^p$).

In [VW14] and [Ada15], the result is even obtained assuming convex concentration for $X = Y \in \mathbb{R}^p$, i.e., the inequalities of Definition 1.1 are satisfied for all 1-Lipschitz convex functionals. This definition is less constrained, thus the class of convexly concentrated random vector is larger\(^{19}\) than the class of Lipschitz concentrated random vectors. A well-known theorem of [Tal95] provides the concentration of the Lipschitz and convex observations of any random vector $X$ built as an affine transformation of a random vector with bounded (with respect to $p$) and independent entries.

These looser hypotheses are not very hard to handle in this particular case of quadratic functionals since these observations exhibit convex properties. The main issue is to find a result analogous to Lemma 1.4 to show the convex concentration of $X^TAX$ on events $\{\|X\| \leq K\}$ for $K > 0$ (note that these events are associated to convex subsets of $\mathbb{R}^p$). These details go beyond the scope of the article: we have shown in the ongoing work [Lou22] that Theorem 5.1 can extend to entry-wise products of $m$ convexly random vectors and to matrix products of $m$ convexly concentrated random matrices (for the latter operations, the concentration is not as good as in the Lipschitz case).

Let us end this section with a useful consequence of Proposition 6.1

Corollary 6.3. Given a deterministic matrix $A \in \mathcal{M}_p$ satisfying $\|A\|_F \leq 1$ and two random matrices $X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_n) \in \mathcal{M}_{p,n}$ satisfying $(X,Y) \propto E_2$ and $\sup_{i \in [n]} \|E[x_i]\|, \|E[y_i]\| \leq O(1)$ such that we have the concentration:

$$\|X^TAX\|_d \in E_2\left(\sqrt{\log(np)}\right) + E_1.$$

\(\text{If}^{20} \text{ in addition, } \|A\|_* \leq O(1) \text{ or } \sup_{i \in [n]} \|E[y_i x_i^T]\|_F \leq O(1)\) then $E[\|X^TAY\|_d] \leq O(\sqrt{n})$.

\(^{18}\)It is even strictly larger as it was shown in [Gib99] that the uniform distribution on $\{0,1\}^p$ is convexly concentrated but not Lipschitz concentrated (with interesting concentration speed).

\(^{19}\)Actually, to bound the expectation we just need the concentration of each of the couples $(x_i, y_i)$ but not of the matrix couple $(X,Y)$. Note that the concentration of each the $x_1, \ldots, x_n$ does not imply the concentration of the whole matrix $X$, even if the columns are independent. To tackle this issue, some authors [PP09] require a logconcave distribution for all the columns because the product of logconcave distribution is also logconcave. However our assumptions are more general because they allow to take for $x_i$ any $O(1)$-Lipschitz transformation of a Gaussian vector which represents a far larger class of random vectors.

\(^{20}\)To be precise, one just needs $\sup_{i \in [n]} |E[x_i^T A y_i]| \leq O(1)$. 

Page 25/41
Concentration of product of random vectors

Remark 6.4. Recall from Proposition 2.6 that if \((x_i, y_i) \in \mathcal{E}_2\) and \(||E[(x_i, y_i)]||_F \leq O(1)\) (as under the hypotheses of Corollary 6.3), \(||E[y_i x_i^T]||_F \leq O(1)\) and therefore, \(||E[y_i x_i^T]||_F \leq \sqrt{p}||E[y_i x_i^T]||_F \leq O(\sqrt{p})\). In particular, when \(x_i\) is independent with \(y_i\) (and \(||E[x_i]||, ||E[y_i]|| \leq O(1)\)), we can bound \(||E[y_i x_i^T]||_F \leq ||E[y_i]||||E[x_i]|| \leq O(1)\).

Proof of Corollary [6.3]. Decomposing \(A = U^T AV\) with \(A = \text{Diag}(\lambda) \in \mathcal{D}_n\) and \(U, V \in \mathcal{O}_p\), noting \(X \equiv VX\) and \(Y \equiv UY\) we have the identity:

\[
||Y^T AX||_d \leq \sup_{\rho \in \mathcal{D}_n} \text{Tr}(DY^T AX) \leq \sup_{\rho \in \mathcal{D}_n} \sum \rho_i \parallel D \parallel_1 \leq \parallel X \parallel_2 \parallel Y \parallel_2,
\]

and the same, \(||Y^T AX||_d \leq \parallel X \parallel_2 \parallel Y \parallel_2\). Now we can bound thanks to Proposition 3.

\[
E[\parallel X \parallel_2] \leq E[\parallel X \parallel_2] + O(\sqrt{\log(p)}) \leq E[\parallel X \parallel] + O(\sqrt{\log(p)}) \leq O(\sqrt{\log(p)}),
\]

and the holds for \(E[\parallel Y \parallel_2]\). Therefore, applying Theorem 4.6 with the concentrations \(\tilde{X}, \tilde{Y} \sim \mathcal{E}_2\) and the variations concentrations \(\parallel X \parallel_2, \parallel Y \parallel_2 \in O(\sqrt{\log(p)})\) ± \(\mathcal{E}_2\), we obtain the looked for concentration.

To bound the expectation, we start with the identity \(||X^T AY||_d = \sqrt{\sum_{i=1}^{n} (x_i A y_i)^2}\), and we note that the hypotheses of Proposition 5.1 are satisfied and therefore \(x_i A y_i \in \mathcal{E}_1\). Now, if \(\parallel A \parallel_* \leq 1\), we can bound:

\[
E[\parallel X^T AY||_d] = \parallel E[y_i x_i^T] \parallel \parallel A \parallel_* \leq O(1),
\]

thanks to Proposition 2.6 \(((x_i, y_i) \sim \mathcal{E}_2\) and \(||E[x_i]||, ||E[y_i]||^T \leq O(1))\). The same bound is true when \(||E[y_i x_i^T]||_F \leq O(1)\) because \(||E[y_i x_i^T]||_F \leq \parallel E[y_i x_i^T] \parallel \parallel A \parallel F\). As a consequence, \(x_i A y_i \in O(1) \pm \mathcal{E}_1\) (with the same concentration constants for all \(i \in [n]\)), and we can bound:

\[
E[\parallel X^T AY\parallel_2] \leq \sqrt{\sum_{i=1}^{n} E[(x_i A y_i)^2]} \leq O(\sqrt{n}).
\]

Let us now give an example of application of Theorem 5.1 when \(m \geq 3\).

7 Concentration of \(XDX^T\)

Considering three random matrices \(X, Y \in M_{p,n}\) and \(D \in \mathcal{D}_n\) such that \((X, Y, D) \sim \mathcal{E}_2\) and \(||E[D]||, ||E[X]||_F, ||E[Y]||_F \leq O(1)\) we wish to study the concentration of \(XDX^T\). Theorem 5.1 just allows us to obtain the concentration \(XDX^T \sim \mathcal{E}_2(n) + \mathcal{E}_2(\sqrt{n}) + \mathcal{E}_2/2\) since we cannot get a better bound than \(||XDX^T\parallel_2 \leq ||X\parallel_2 ||D\parallel_2 ||Y\parallel_2\). However, considering some particular observations on \(XDX^T\), it appears that the observable diameter can be smaller than \(n\). Next Propositions reveal indeed that for any deterministic \(u \in \mathbb{R}^p\) and \(A \in M_{p}\):

1. \(XDX^T u\) is Lipschitz concentrated with an observable diameter of order \(O(||u||_F \sqrt{\log(n)})\)
2. \(\text{Tr}(AXDY^T)\) is concentrated with a standard deviation of order \(O(||A||_F \sqrt{(n+p) \log(np)})\) if \(\sup_{i \in [n]} ||E[x_i y_i^T]||_F \leq O(1)\) and \(O(||A||_* \sqrt{(n+p) \log(np)})\) otherwise.

Page 26/41
Proposition 7.1. Given three random matrices $X, Y = (y_1, \ldots, y_n) \in \mathcal{M}_{p,n}$ and $D \in \mathcal{D}_n$ diagonal such that $(X, Y, D) \propto \mathcal{E}_2$, $\|E[X]\| \leq O(\sqrt{p + n})$ and $\|E[D]\|, \sup_{i \in [n]} \|E[y_i]\| \leq O(1)$, for any deterministic vector $u \in \mathbb{R}^p$ such that $\|u\| \leq 1$:

$$X D Y^T u \propto \mathcal{E}_2 \left( \sqrt{(p + n) \log(np)} \right) + \mathcal{E}_1 \left( \sqrt{p + n} \right) + \mathcal{E}_{2/3} \quad \text{in} \ (\mathbb{R}^p, \| \cdot \|).$$

Proof. The Lipschitz concentration of $X D Y^T u$ is obtained thanks to the inequalities:

$$\| X D Y^T u \| \leq \begin{cases} \| X \| \| D \| \| Y^T u \|, \\ \| X \| \| D \|_F \| Y^T u \|_\infty. \end{cases}$$

Thanks to the bounds already presented in Example 3.5 (the spectral norm $\| \cdot \|$ on $\mathcal{D}_n$ is like the infinity norm $\| \cdot \|_\infty$ on $\mathbb{R}^n$), we know that:

- $\mathbb{E}[\| D \|] \leq O(\sqrt{\log n}) + O(\| E[D] \|) \leq O(\sqrt{\log n})$,
- $\mathbb{E}[\| Y^T u \|_\infty] \leq O(\sqrt{\log n})$, $\| E[Y^T u] \|_\infty \leq O(\sqrt{\log n}) + \sup_{i \in [n]} \| E[y_i] \| \leq O(\sqrt{\log n})$,
- $\mathbb{E}[\| X \|] \leq O(\sqrt{\log n})$,
- $\mathbb{E}[\| Y^T u \|] \leq O(\sqrt{\log n}) + \| E[Y^T u] \| \leq O(\sqrt{\log n}) + \sqrt{n} \sup_{i \in [n]} \| E[y_i] \| \leq O(\sqrt{n})$.

One then obtains the concentrations:

- $\| D \| \| Y^T u \| \in O(\sqrt{\log(n)n}) + \mathcal{E}_2(\sqrt{n}) + \mathcal{E}_1$,
- $\| X \| \| Y^T u \|_\infty \in O(\sqrt{\log(n)(n + p)}) + \mathcal{E}_2(\sqrt{n + p}) + \mathcal{E}_1$,

from which Theorem 4.6 allows us to conclude.

The concentration of $\text{Tr}(AXD^T)$ signifies a linear concentration of $X D Y^T$, demonstrating as in Proposition 4.1 the relevance of the notation of linear concentration. Note besides that this result can be seen as a weak offshot of Hanson-Wright concentration inequality if one takes $D = \sqrt{n}E_{1,1}$, where $|E_{1,1}|_{i,j} = 0$ for all $(i, j) \neq (1, 1)$ and $|E_{1,1}|_{1,1} = 1$.

Proposition 7.2. Given three random matrices $X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_n) \in \mathcal{M}_{p,n}$ and $D \in \mathcal{D}_n$ such that $(X, Y, D) \propto \mathcal{E}_2$, $\|E[D]\|_F \leq O(\sqrt{n})$, $\|E[X]\|_F, \|E[Y]\|_F \leq O(1)$, we have the linear concentration\(^{21}\):

$$X D Y^T \in \mathcal{E}_2 \left( \sqrt{n} \right) + \mathcal{E}_1 \left( \sqrt{\log np} \right) + \mathcal{E}_{2/3} \quad \text{in} \ (\mathcal{M}_p, \| \cdot \|).$$

If, in addition, $\sup_{i \in [n]} \|E[y_i x_i^T]\|_F \leq O(1)$:

$$X D Y^T \in \mathcal{E}_2 \left( \sqrt{n} \right) + \mathcal{E}_1 \left( \sqrt{\log np} \right) + \mathcal{E}_{2/3} \quad \text{in} \ (\mathcal{M}_p, \| \cdot \|_F).$$

Before proving this corollary let us give a preliminary lemma of independent interest.

Lemma 7.3. Given two random matrices $X \in \mathcal{M}_{p,n}$ and $D \in \mathcal{D}_n$ such that $(X, D) \propto \mathcal{E}_2$, $\|E[X]\| \leq O(1)$ and $\|E[D]\| \leq O(\sqrt{n})$ and a deterministic matrix $A \in \mathcal{M}_p$, such that $\|A\|_F \leq 1$, we have the concentration:

$$AXD \propto \mathcal{E}_2(\sqrt{\log(np)}) + \mathcal{E}_1 \quad \text{in} \ (\mathcal{M}_{p,n}, \| \cdot \|_F).$$

in addition, one can bound: $\mathbb{E}[\|AXD\|_F] \leq O(\sqrt{n})$.

---

\(^{21}\)The estimation of $\mathbb{E}[X D Y^T]$ is done in Proposition 7.4.
Concentration of product of random vectors

Proof. With the same decomposition \( A = U^T \Lambda V \) and notation \( \hat{X} \equiv VX \) as in the proof of Corollary 6.3 we have the identity:

\[
\|AXD\|_F = \|AX\|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^p \lambda_i^2 \hat{x}_{ij}^2 D_j^2} \leq \|\hat{X}\|_\infty \|D\|_F,
\]

and besides, \( \|AXD\|_F \leq \|\hat{X}\|_F \|D\| \), we can thus employ Theorem 4.6 with the concentrations \( \bar{X}, D \propto E \) and:

\[
\|\bar{X}\|_\infty \in O(\sqrt{\log \rho n}) + \varepsilon_2 \quad \text{and} \quad \|D\| \in O(\sqrt{\log n}) + \varepsilon_2,
\]
to obtain \( AXD \propto \varepsilon_2(\sqrt{\log \rho n}) + \varepsilon_1 \). To bound the expectation, let us note that for all \( i \in [n], j \in [p], \bar{X}_{i,j} \in O(1) \pm \varepsilon_2 \) and \( D_j \in O(1) \pm \varepsilon_2 \); therefore, \( (\bar{X}_{i,j} D_j)^2 \in O(1) \pm O(\varepsilon_2^2) \), with concentration constants independent of \( i, j \). Finally, we can bound:

\[
\mathbb{E}[\|AXD\|_F] \leq \sqrt{\sum_{i=1}^n \sum_{j=1}^p \lambda_i^2 \mathbb{E}[\bar{x}_{ij}^2 D_j^2]} \leq \left( \sum_{i=1}^n O(1) \right) \left( \sum_{j=1}^p \lambda_j^2 \right) \leq O(\sqrt{n}).
\]

\( \square \)

Proof of Proposition 7.2. Considering a deterministic matrix \( A \in \mathcal{M}_{p,n} \), we will assume that \( \|A\|_F \leq 1 \) if \( \sup_{i \in [n]} \|E[y_i x_i^T]\|_F \leq O(1) \) (to show a concentration in \( (\mathcal{M}_{p,n}, \| \cdot \|_F) \)) and that \( \|A\|_* \leq 1 \) otherwise (to show a concentration in \( (\mathcal{M}_{p,n}, \| \cdot \|) \)). In both cases, Corollary 6.3 and Lemma 7.3 allows us to set:

\[
\|Y^T A X\|_{\text{tr}}, \|AXD\|_F, \|DY^T A\|_F \in O(\sqrt{n}) + \varepsilon_2(\sqrt{\log \rho n}) + \varepsilon_1.
\]

Besides, we can bound:

\[
\text{Tr}(AXDY^T) \leq \left\{ \begin{array}{ll} \|AXD\|_F \|Y\|_F \\ \|DY^T A\|_F \|X\|_F \\ \|Y^T AX\|_{\text{tr}} \|D\|_{\text{tr}}, \end{array} \right.
\]

which allows us to conclude thanks to Theorem 4.6. \( \square \)

In the setting of Proposition 7.1 once one knows that \( XDY \) is concentrated it is natural to look for a simple deterministic equivalent. The next proposition help us for such a design. Note that the hypotheses are far lighter, in particular, we just need the linear concentration of \( D \).

**Proposition 7.4.** Given three random matrices \( D \in \mathcal{D}_n \), \( X = (x_1, \ldots, x_n) \), \( Y = (y_1, \ldots, y_n) \) in \( \mathcal{M}_{p,n} \) and a deterministic matrix \( \bar{D} \in \mathcal{D}_n \), such that \( D \in \bar{D} \pm \varepsilon_2 \) in \( (\mathcal{D}_n, \| \cdot \|) \) and for all \( i \in [n] \), \( (x_i, y_i) \propto \varepsilon_2 \) and \( \sup_{i \in [n]} \|E[x_i]\|, \|E[y_i]\| \leq O(1) \), we have the estimate:

\[
\|E[XYD^T] - E[XE[D]Y^T]\|_F \leq O(n).
\]

We can precisce the estimation with supplementary assumptions:

\[\text{If we adopt the stronger assumptions } (X, Y) \propto \varepsilon_2 \text{ in } (\mathcal{M}_{p,n}, \| \cdot \|) \text{ and } \|E[X]\|_F, \|E[X]\|_F \leq O(1), \text{ we can show more directly thanks to Propositions 7.1, 6.3 and 3.3:} \]

\[
\left| E[\text{Tr}(AXDY^T)] - E[\text{Tr}(AXE[D]Y^T)] \right| \leq \left| E \left[ \text{Tr} \left( \left( Y^T AX - E[Y^T AX] \right) (D - E[D]) \right) \right] \right| \leq \left[ E \|Y^T AX - E[Y^T AX]\|_F^2 \right] E \|D - E[D]\|_F^2 \leq O(n). \]

Page 28/41
Concentration of product of random vectors

- If $\|\bar{D} - E[D]\|_F \leq O(1)$ then $\|E[X(D - \bar{D})Y^T]\|_F \leq O\left(\sqrt{n \max(p,n)}\right)$
- if $\sup_{i \in [n]} \|E[x_i y_i^T]\|_F \leq O(1)$ then $\|E[XY^T] - E[X\bar{D}Y^T]\|_F \leq O(n)$.

**Proof.** Considering a deterministic matrix $A \in \mathcal{M}_p$, such that $\|A\|_F \leq 1$:

$$
\left|\left|E[\mathrm{Tr}(AXDY^T)] - E[\mathrm{Tr}(AXE[D]Y^T)]\right|\right| \leq \sum_{i=1}^{n} \left|E\left[D_i x_i^T A y_i - E[D_i] x_i^T A y_i\right]\right|
$$

$$
= \sum_{i=1}^{n} \left|E\left[(x_i^T A y_i - E[x_i^T A y_i]) (D_i - E[D_i])\right]\right|
$$

$$
\leq \sum_{i=1}^{n} \sqrt{E\left[|x_i^T A y_i - E[x_i^T A y_i]|^2\right]} \left|E[D_i - E[D_i]]\right| \leq O(n)
$$

thanks to H"older’s inequality applied to the bounds given by Proposition 4.11 (we know that $D_i \in E[D_i] + \mathcal{E}_2$ and from Proposition 6.1 that $x_i^T A y_i \in E[x_i^T A y_i] + \mathcal{E}_2 + \mathcal{E}_1$; note that the concentration constants are the same for all $i \in [n]$).

Now, $\|E[x_i^T A y_i]\|_F \leq \|A\|_F \|E[y_i x_i^T]\|_F \leq O(\sqrt{p})$ thanks to Remark 5.4 and if $\|E[D] - \bar{D}\|_F \leq O(1)$, we can bound:

$$
\left|\left|E\left[\mathrm{Tr}\left(AX\left(E[D] - \bar{D}\right)Y^T\right)\right]\right|\right| \leq \sum_{i=1}^{n} \left|E\left[x_i^T A y_i\right]\right| \left|\left(E[D_i] - \bar{D_i}\right)\right|
$$

$$
\leq \sup_{i \in [n]} \|E[x_i^T A y_i]\| \sqrt{n} \|E[D] - \bar{D}\|_F \leq O(\sqrt{np}).
$$

If, $\|E[D] - \bar{D}\|_F$ is possibly of order far bigger than $O(1)$, but $\sup_{i \in [n]} \|E[y_i x_i^T]\|_F \leq O(1)$, then $\sup_{i \in [n]} \|E[x_i^T A y_i]\| \leq O(1)$, and we can still bound:

$$
\left|\left|E\left[\mathrm{Tr}\left(AX\left(E[D] - \bar{D}\right)Y^T\right)\right]\right|\right|_F \leq n \sup_{i \in [n]} \|E[x_i^T A y_i]\| \left|\left|E[D] - \bar{D}\right|\right| \leq O(n).
$$

Let us end this article with a non multi-linear application of Theorem 5.1.

8 Concentration of the resolvent $(I_p - \frac{1}{n} XDY^T)^{-1}$

We study here the concentration of a resolvent $Q = (I_p - \frac{1}{n} XDY^T)^{-1}$ with the assumption of Proposition 7.2 for $X$, $Y$ and $D$ (in particular $D$ is random). Among other use, this object appears when studying robust regression [EKB13, MLC19]. In several settings, robust regression can be expressed by the following fixed point equation:

$$
\beta = \frac{1}{n} \sum_{i=1}^{n} f(x_i^T \beta) x_i, \quad \beta \in \mathbb{R}^p, \quad (8.1)
$$

where $\beta$ is the weight vector performing the regression (to classify data, for instance). It was then shown in [SLTC21] that the estimation of the expectation and covariance of $\beta$ (and therefore, of the performances of the algorithm) rely on an estimation of $Q$, with $D = \text{Diag}(f(x_i^T \beta))$. To obtain a sharp concentration on $Q$ (as it is done in Theorem 8.1 below), one has to understand the dependence between $Q$ and $x_i$, for all $i \in [n]$. This is performed with the notation, given for any $M = (m_1, \ldots, m_n) \in \mathcal{M}_{p,n}$ or any $\Delta = \text{Diag}_{i \in [n]}(\Delta_i) \in \mathcal{D}_n$.
Given a random diagonal matrix $D \in \mathcal{D}_n$ and a random matrices $X = (x_1, \ldots, x_n)$, in the regime $p \leq O(n)$ and under the assumptions:

- $(X, D) \propto \mathcal{E}_2$, 
- all the couples $(x_i, y_i)$ are independent,
- $O(1) \leq \sup_{i \in [n]} \|E[x_i]\|, \sup_{i \in [n]} \|E[y_i]\| \leq O(1)$.
- for all $i \in [n]$, there exists a random diagonal matrix $D^{(i)}$, independent of $x_i$, such that $\sup_{i \in [n]} \|D_{-i} - D^{(i)}_{-i}\|_F \leq O(1)$,
- there exist constants $\kappa_X, \kappa_D, \varepsilon > 0 (\varepsilon \geq O(1))$ and $\kappa_X, \kappa_D \leq O(1)$, such that $\|X\| \leq \sqrt{n} \kappa_X$, $\|D\| \leq \kappa_D$ and $\kappa_X^2 \kappa_D \leq 1 - \varepsilon$.

The resolvent $Q \equiv (I_p - \frac{1}{n} XDX^T)^{-1}$ follows the linear concentration

\[ Q \in \mathcal{E}_2 \left( \frac{\log^2 n}{\sqrt{n}} \right) \pm \mathcal{E}_2 \left( \frac{\log^2 n}{n} \right) \quad \text{in} \quad (\mathcal{M}_p, \|\cdot\|). \]

Inspiring from the identities provided in [SB95, PP09, LC21], one can further estimate this random matrix thanks to the following notation, given $\delta, D \in \mathcal{M}_n$:

\[ \tilde{Q}^{\delta}(D) \equiv \left( I_p - \frac{1}{n} \sum_{i=1}^{n} E \left[ \frac{D_{-i}}{1 + \delta D_{-i}} \right] \Sigma_i \right)^{-1}, \]

where we noted for all $i \in [n], \Sigma_i \equiv E[x_i y_i^T]$.

**Theorem 8.2.** Given a random diagonal matrix $D \in \mathcal{D}_n$, the fixed point equation:

\[ \delta = \text{Diag}_{i \in [n]} \left( \Sigma_i, \tilde{Q}^{\delta}(D) \right) \]

admits a unique solution $\delta \in \mathcal{D}_n$ and, under the hypotheses of Theorem 8.1 one can estimate:

\[ \|E[Q] - \tilde{Q}^{\delta}(D)\|_F \leq O(\log^2 n). \]

**Remark 8.3.** Let us give two examples of the matrices $D^{(i)}$ that one could choose, depending on the cases:

- For all $i \in [n]$, $D_i = f(x_i)$ for $f : \mathbb{R}^{2p} \to \mathbb{R}$, bounded, then, $D_i$ just depends on $(x_i, y_i)$ so one can merely take $D^{(i)} = D_{-i}$ for all $i \in [n]$.

---

$^{23}$It is not necessary to assume that $p \leq O(n)$ but it simplifies the concentration result (if $p \gg n$, the concentration is not as good, but it can still be expressed).

$^{24}$The assumptions $\|X\|/\sqrt{n}$ and $\|Y\|/\sqrt{n}$ bounded and $\kappa^2 \kappa_D \leq 1 - \varepsilon$ might look a bit strong (since it is not true for matrices with i.i.d. Gaussian entries) and it is indeed enough to assume that $E[\|X\|] \leq O(\sqrt{n})$ introduce a parameter $\varepsilon > 0$ and study the behavior of $(zI_p - \frac{1}{n} XDX^T)^{-1}$ when $z$ is far from the spectrum of $\frac{1}{n} XDX^T$ - as it is done in [LC21]. We however preferred here to make a relatively strong hypothesis not to have supplementary notations and proof precautions, that might have blurred the message.
Concentration of product of random vectors

• For the robust regression described by Equation 8.1 as in [SLTC21], we can assume for simplicity that \( \|f\|_\infty, \|f'\|_\infty, \|f''\|_\infty \leq O(1) \). If we choose \( D = \text{Diag}(f'(x_i^T \beta)) \), then it is convenient to assume \( \frac{1}{n} \|f''\|_\infty \|X_i\|^2 \leq 1 - \varepsilon \) (which implies in particular \( \frac{1}{n} \|X\| \|D\|Y\| \leq 1 - \varepsilon \)) so that \( \beta \) is well defined, being solution of a contractive fixed point equation. One can further introduce \( \beta(i) \in \mathbb{R}^p \), the unique solution to

\[
\beta(i) = \frac{1}{n} \sum_{j \neq i}^n f(x_j^T \beta(i)) x_j.
\]

By construction, \( \beta(i) \) is independent of \( x_i \) and so:

\[
D^{(i)} \equiv \text{Diag} \left( f'(x_i^T \beta(i)), \ldots, f'(x_{i-1}^T \beta(i)), 0, f'(x_{i+1}^T \beta(i)), \ldots, f'(x_n^T \beta(i)) \right).
\]

Besides \( \|D_{-i} - D_{-i}^{(i)}\|_F \leq \|f''\|_\infty \|X_{-i}^T(\beta - \beta(i))\|_F \). Now, the identities:

\[
X_{-i}^T \beta = \frac{1}{n} X_{-i}^T X f(X^T \beta) \quad \text{and} \quad X_{-i}^T \beta(i) = \frac{1}{n} X_{-i}^T X_{-i} f(X_{-i}^T \beta(i))
\]

(where \( f \) is applied entry-wise) imply:

\[
\|X_{-i}^T(\beta - \beta(i))\|_F \leq \frac{1}{n} \|f''\|_\infty \|X_{-i}\|_2^2 \|X_{-i}^T(\beta - \beta(i))\|_F + \frac{1}{n} \|f(x_i^T \beta)X_{-i} x_i\|.
\]

We can then deduce (since \( \frac{1}{n} \|f''\|_\infty \|X_{-i}\|_2^2 \leq 1 - \varepsilon \) by hypothesis):

\[
\|D_{-i} - D_{-i}^{(i)}\|_F \leq \|f''\|_\infty \|X_{-i}^T(\beta - \beta(i))\|_F \leq \frac{\|f''\|_\infty}{n\varepsilon} f(x_i^T \beta)X_{-i} x_i \leq O(1).
\]

The preliminary lemmas to the proof of Theorem 8.1 are here to prove a similar result to Corollary 6.3, namely the concentration of \( \|X^T Q A Q X\|_d \), given in Lemma B.9. The proof of Theorem 8.2 requires a finer approach presented in Appendix B.4.

Conclusion

With the complexity of nowadays machine learning algorithms, it becomes crucial to devise simple and efficient notations to comprehend their structural logic. For that purpose, the present work provides a systematic approach to comprehend the probabilistic issues involving concentrated vectors, as a model for real data, and their use in statistical learning methods. Indeed, on the one hand, as justified in [STC19], the very realistic artificial images created by generative adversarial networks are concentrated random vectors by construction: this strongly suggests that most commonly studied databases satisfy our hypotheses. On the other hand, the flexibility of the hypotheses of Theorem 5.1 and of Theorem 4.6 ensure that a wide range of real functionals involved in machine learning problems are concerned by those results.

As such, in essence, the article provides a catalogue of ready-to-use results for a probabilistic approach of machine learning. To summarize, establishing a concentration inequality on a given random quantity \( Y \) generally follows the steps:

1. Identify the random vectors \( X_1, \ldots, X_m \) (independent or not) upon which \( Y \) is built, and verify that \( (X_1, \ldots, X_m) \sim \mathcal{E}_2 \);
2. Bound the variations of \( Y \) with a functional \( \delta_i \) when \( X_i \) varies, \( \forall i \in [m] \);
3. Express the concentration of \( \delta_i \), for all \( i \in [m] \) and deduce the concentration of \( Y \) from Theorem 4.6 or from Theorem 5.1 depending on \( \delta_i \).

25The bound \( \|f\|_\infty \leq O(1) \) is not necessary to set the concentration of \( Q \), but it avoids a lot of complications.
Concentration of product of random vectors

Our work strongly relates to general log-concave settings (very similar to the setting we proposed here: in [Ada11] for Wigner matrices and in [PP09] for Wishart matrices) for which the asymptotic behavior of the spectral distribution of random matrices was shown only to depend on the first moments of the entries. In these probabilistic contexts, the random objects behave as if the initial data were Gaussian because the only relevant statistics of the asymptotic behavior is composed of the means and covariances of the data. The laborious Gaussian calculus (with the Stein method as in [Pas05], possibly combined with Poincaré inequalities as in [Cha17]) then appears as superfluous and can be replaced by concentration of measure arguments in more general settings. This being said, a result from [Kla07, FGP07] establishing a central limit theorem (CLT) for deterministic projections of concentrated random vectors allowed us in a parallel contribution to employ Gaussian inference for the estimation of quantities depending on such projections [SLTC21]. Nonetheless, in this case, a small number of projections do not satisfy the CLT\(^{26}\), which restricts the application of the argument.

A Proof of the concentration of generalized products

B Proofs of resolvent concentration properties

B.1 Lipschitz concentration of Q

**Lemma B.1.** Under the assumptions of Theorem 8.1, \( \|Q\| \leq \frac{1}{\varepsilon} \leq O(1) \).

Then we can show a Lipschitz concentration of \( Q \) but with looser observable diameter that the one given by Theorem 8.1 (as for \( XD^TX^T \), we get better concentration speed in the linear concentration framework).

**Lemma B.2.** Under the hypotheses of Theorem 8.1:

\[
\left( Q, \frac{1}{\sqrt{n}}QX \right) \preceq \mathcal{E}_2 \quad \text{in} \quad (\mathcal{M}_{p,n}, \| \cdot \|_F).
\]

**Proof.** Let us just show the concentration of the resolvent, the concentration of \( \frac{1}{\sqrt{n}}QX \) is treated the same way thanks to the bound \( \|X\| \leq O(\sqrt{n}) \). If we note \( \phi(X, D) = Q \) and we introduce \( X' \in \mathcal{M}_{p,n} \) and \( D' \in \mathcal{D}_n \), satisfying \( \|X'\| \leq \kappa_X \sqrt{n} \) and \( \|D'\| \leq \kappa_D \) as \( X, D \), we can bound:

\[
\|\phi(X, D) - \phi(X', D)\|_F \leq \frac{1}{n} \left( \|\phi(X, D)(X - X')D^T \phi(X', D)\|_F \right.
\]

\[
+ \frac{1}{n} \left( \|\phi(X, D)X'D(X - X')^T \phi(X', D)\|_F \right) \leq \frac{2\kappa_X \kappa_D}{\varepsilon^2 \sqrt{n}} \|X - X'\|_F,
\]

thanks to the hypotheses and Lemma [B.1] given above. The same way, we can bound:

\[
\|\phi(X, D) - \phi(X, D')\|_F \leq \frac{\kappa^2}{\varepsilon^2} \|D - D'\|_F.
\]

Therefore, as a \( O(1) \)-Lipschitz transformation of \( (X, D) \), \( Q \preceq \mathcal{E}_2. \)

\(^{26}\)Given a random variable \( z_1 \sim \text{Unif}([0, 1]) \) and \( p - 1 \) i.i.d. random variables \( z_2, \ldots, z_p \sim \mathcal{N}(0, 1) \), we know that \( Z = (z_1, \ldots, z_p) \) is not Gaussian. It is stated in [Kla07] that for most \( u \in \mathbb{S}^{p-1} \), \( u^T Z \) is quasi-Gaussian (the measure of the complementary set to such \( u \) is exponentially decreasing with the maximal distance in infinity norm between the Gaussian CDF and the CDFs of \( u^T Z \)).
B.2 Control on the dependency on \((x_i, y_i)\)

The dependence between \(Q\) and \(x_i\) prevent us from bounding straightforwardly \(\|Q_i\|\) with Lemma B.1 and the hypotheses on \(x_i\). We can still disentangle this dependence thanks to some notations and classical random matrix identities. Let us denote:

\[
Q_{-i} = \left( I_p - \frac{1}{n} X_{-i}^T D X_{-i}^T \right)^{-1} \quad \text{and} \quad Q_{-i}^{(i)} = \left( I_p - \frac{1}{n} X_{-i}^T D^{(i)} X_{-i}^T \right)^{-1}.
\]

We can indeed bound:

\[
\|E[Q_{-i,x_i}]\| \leq \|E[Q_{-i}^{(i)}]\| E[|x_i|] \leq O(1), \tag{B.1}
\]

and we even have interesting concentration properties that will be important later:

**Lemma B.3.** Under the assumptions of Theorem 6.1

\[
Q_{-i}^{(i)}, x_i, 1/n X_{-i}^T Q_{-i}^{(i)} x_i \in O(1) \pm \mathcal{E}_2 \quad \text{and} \quad \frac{1}{n} x_i^T Q_{-i}^{(i)} x_i \in O(1) \pm \mathcal{E}_2 \left( \frac{1}{\sqrt{n}} \right) + \mathcal{E}_1 \left( \frac{1}{n} \right).
\]

Besides, for any deterministic \(A \in \mathcal{M}_p:\)

\[
x_i^T Q_{-i}^{(i)} A Q_{-i}^{(i)} x_i \in O(\|A\|_*) + \mathcal{E}_2(\|A\|_F) + \mathcal{E}_1(\|A\|).
\]

**Proof.** Considering \(u \in \mathbb{R}^p\), deterministic such that \(\|u\| \leq 1\), we can bound thanks to the independence between \(Q_{-i}^{(i)}\) and \(x_i\):

\[
\left| u^T Q_{-i}^{(i)} x_i - E[u^T Q_{-i}^{(i)} x_i] \right| \leq \left| u^T Q_{-i}^{(i)} (x_i - E[x_i]) \right| + \left| u^T \left( Q_{-i}^{(i)} - E[Q_{-i}^{(i)}] \right) E[x_i] \right|.
\]

Therefore, the concentrations \(x_i \propto \mathcal{E}_2\) and \(Q_{-i}^{(i)} \propto \mathcal{E}_2\) given in Lemma B.2 imply that there exist two constants \(C, c > 0\) such that \(\forall t > 0\) such that if we note \(\mathcal{A}_{-i}\), the sigma algebra generated by \(X_{-i}\) and \(x_{-i}\) (it is independent with \(x_i\)):

\[
P \left( \left| u^T Q_{-i}^{(i)} x_i - E[u^T Q_{-i}^{(i)} x_i] \right| \geq t \right) \\
\leq E \left[ P \left( \left| u^T Q_{-i}^{(i)} (x_i - E[x_i]) \right| \geq \frac{t}{2} \mid \mathcal{A}_{-i} \right) \right] + P \left( \left| u^T \left( Q_{-i}^{(i)} - E[Q_{-i}^{(i)}] \right) E[x_i] \right| \geq \frac{t}{2} \right) \\
\leq E \left[ C e^{(t/c\|Q_{-i}^{(i)}\|)^2} \right] + C e^{(t/c\|E[x_i]\|)^2} \leq C' e^{-t^2/c'},
\]

for some constants \(C', c' > 0\), thanks to the bounds \(\|E[x_i]\| \leq O(1)\) in the assumptions and \(\|Q_{-i}^{(i)}\| \leq O(1)\) given by Lemma B.1.

The linear concentration of \(X_{-i}^T Q_{-i}^{(i)} x_i / \sqrt{n}\) is proven thanks to the concentration \(X_{-i}^T Q_{-i}^{(i)}/\sqrt{n} \propto \mathcal{E}_2\) given in Lemma B.2.

The concentration of \(\frac{1}{\sqrt{n}} x_i^T Q_{-i}^{(i)} x_i\) is proven similarly with the bound:

\[
\left| \frac{1}{\sqrt{n}} x_i^T Q_{-i}^{(i)} x_i - E \left[ \frac{1}{\sqrt{n}} x_i^T Q_{-i}^{(i)} x_i \right] \right| \leq \frac{1}{\sqrt{n}} x_i^T Q_{-i}^{(i)} x_i - \frac{1}{\sqrt{n}} \text{Tr} \left( \Sigma_i Q_{-i}^{(i)} \right) \\
+ \frac{1}{\sqrt{n}} \text{Tr} \left( \Sigma_i Q_{-i}^{(i)} \right) - E \left[ \frac{1}{\sqrt{n}} \text{Tr} \left( \Sigma_i Q_{-i}^{(i)} \right) \right],
\]

and the concentrations \(Q_{-i}^{(i)} \propto \mathcal{E}_2\) in \((\mathcal{M}_p, \|\cdot\|_F)\) and \(x_i^T A x_i \in \text{Tr}(\Sigma_i A) \pm \mathcal{E}_2(\sqrt{n}\|A\|) \pm \mathcal{E}_1(\|A\|)\) for any deterministic \(A \in \mathcal{M}_p\).

Finally, the concentration of \(x_i^T Q_{-i}^{(i)} A Q_{-i}^{(i)} x_i\) is also shown the same way, knowing that \(M \mapsto \text{Tr}(\Sigma, M A M)\) is \(O(\|M\|_F)\)-Lipschitz on \(\{|M| \leq \frac{1}{2}\}\) and for any \(B \in \mathcal{M}_p\), \(x_i^T B x_i \in O(\|B\|_* \pm \mathcal{E}_2(\|B\|_F) + E_i(\|B\|)\) thanks to Theorem 6.1 and the fact that \(E[x_i^T B x_i] = \text{Tr}(\Sigma_i B) \leq \|\Sigma\| \|B\|_* \leq \|B\|_*\) thanks to the hypotheses on \(X\).
The link between $Qx_i$ and $Q_{-i}x_i$ is made possible thanks to classical Schur identities:

$$Q = Q_{-i} - \frac{1}{n} D_i Q_{-i} x_i^T Q_{-i} \quad \text{and} \quad Qx_i = \frac{Q_{-i} x_i}{1 + D_i \Delta_i},$$  \hfill (B.2)

where we noted $\Delta_i \equiv \frac{1}{n} x_i^T Q_{-i} x_i$. The link between $Q_{-i}x_i$ and $Q^{(i)}_{-i}x_i$ is made thanks to:

**Lemma B.4.** Under the hypotheses of Theorem [8.1] for all $i \in [n]$:

$$\|Q_{-i}x_i - Q^{(i)}_{-i}x_i\| \leq O \left( \sqrt{\log n} \right) \pm \mathcal{E}_2 \left( \sqrt{\log n} \right).$$

Let us first prove a Lemma of independent interest:

**Lemma B.5.** Under the hypotheses of Theorem [8.1] for all $i \in [n]$

$$\|Q^{(i)}_{-i}x_i\|_\infty \leq O(\sqrt{\log p}) \pm \mathcal{E}_2(\sqrt{\log p}) \quad \text{and} \quad \left\| \frac{1}{\sqrt{n}} X^T Q^{(i)}_{-i} x_i \right\|_\infty \leq O(\sqrt{\log n}) \pm \mathcal{E}_2(\sqrt{\log n}).$$

**Proof.** The control on the variation is given by Lemma B.3 (\| \cdot \|_\infty \leq \| \cdot \|_F \) and the bound on the expectation is a consequence of Proposition B.3 and the bound:

$$\frac{1}{\sqrt{n}} \left\| E \left[ X^T Q^{(i)}_{-i} x_i \right] \right\|_\infty \leq \frac{1}{\sqrt{n}} \left\| E \left[ X^T Q^{(i)}_{-i} \right] E[x_i] \right\| \leq O(1). \hfill (B.3)$$

One can show the same way that $\left\| E[Q^{(i)}_{-i} x_i] \right\|_\infty \leq O(1)$.

**Proof of Lemma B.5.** Let us bound directly:

$$\left\| (Q_{-i} - Q^{(i)}_{-i}) x_i \right\| \leq \frac{1}{n} \left\| Q_{-i} X_{-i} (D_{-i}^{(i)} - D_{-i}) X^T Q^{(i)}_{-i} x_i \right\|$$

$$\leq \frac{1}{n} \left\| Q_{-i} X_{-i} \right\| \left\| D^{(i)}_{-i} - D_{-i} \right\|_F \left\| X^T Q^{(i)}_{-i} x_i \right\|_\infty \leq O \left( \frac{1}{\sqrt{n}} \left\| X^T Q^{(i)}_{-i} x_i \right\|_\infty \right).$$

We can then conclude thanks to Lemma B.3.

We end this subsection with two fast consequences to Lemma B.4 that will find some use.

**Lemma B.6.** Under the hypotheses of Theorem [8.1]

$$\Delta_i \equiv \frac{1}{n} x_i^T Q_{-i} x_i \in \Delta_i \pm \mathcal{E}_2 \left( \frac{\sqrt{\log n}}{n} \right) + \mathcal{E}_1 \left( \frac{\sqrt{\log n}}{n} \right),$$

where we noted $\bar{\Delta}_i \equiv \frac{1}{n} E[x_i^T Q_{-i} x_i]$ (recall that $\forall i \in [n] : |\bar{\Delta}_i| \leq \frac{\delta}{\varepsilon} n$).

**Proof.** We know from Lemma B.3 that $\frac{1}{n} x_i^T Q^{(i)}_{-i} x_i \in O(1) \pm \mathcal{E}_2 \left( \frac{\sqrt{\log n}}{n} \right) + \mathcal{E}_1 \left( \frac{1}{n} \right)$. One can then conclude with Lemma B.4.

**Lemma B.7.** Under the hypotheses of Theorem [8.1] and given $A \in \mathcal{M}_p$, such that $\|A\|_F \leq 1$:

$$\|AQ_{-i}x_i\| \in O \left( \sqrt{\log n} \|A\|_F \right) \pm \mathcal{E}_2 \left( \sqrt{\log n} \|A\| \right)$$

**Proof.** The concentration of $\|AQ_{-i}x_i\|$ is a consequence of Lemma B.4 and the fact that:

$$E[\|AQ^{(i)}_{-i}x_i\|] \leq \sqrt{E[x_i^T Q^{(i)}_{-i} A Q^{(i)}_{-i} x_i]} \leq \frac{1}{\varepsilon} \|E[x_i x_i^T]\|^{1/2} \|A\|_F \leq O(\|A\|_F).$$
Concentration of product of random vectors

B.3 Proof of Theorem B.1

Let us first show the concentration of the pseudo-quadratic forms $x_i^TQAQx_i$ for some $A \in \mathcal{M}_p$.

**Lemma B.8.** Under the hypotheses of Theorem B.1 given a deterministic matrix $A \in \mathcal{M}_{p,n}$ such that $\|A\|_F \leq 1$:

$$x_i^TQ_{-i}AQ_{-i}x_i \in O(\|A\|_+ + \|A\|_F \log n) \pm \mathcal{E}_2(\|A\|_F \log n) + \mathcal{E}_1(\|A\| \log n)$$

**Proof.** We already know from Lemma B.3 that:

$$x_i^TQ_{-i}AQ_{-i}x_i \in O(\|A\|_+ + \|A\|_F) + \mathcal{E}_1(\|A\|)$$

we then use the identity:

$$x_i^TQ_{-i}AQ_{-i}x_i = x_i^TQ_{-i}AQ_{-i}x_i + 2x_i^T(Q_{-i} - Q_{-i}^o)AQ_{-i}x_i + x_i^T(Q_{-i} - Q_{-i}^o)A(Q_{-i} - Q_{-i}^o)x_i,$$

the bounds:

$$\begin{cases}
2x_i^T(Q_{-i} - Q_{-i}^o)AQ_{-i}x_i \leq \|x_i^T(Q_{-i} - Q_{-i}^o)\|\|AQ_{-i}x_i\| \\
|x_i^T(Q_{-i} - Q_{-i}^o)A(Q_{-i} - Q_{-i}^o)x_i| \leq \|x_i^T(Q_{-i} - Q_{-i}^o)\|^2\|A\|,
\end{cases}$$

and the concentrations:

- $\|AQ_{-i}x_i\| \in O(\sqrt{\log n}\|A\|_F) \pm \mathcal{E}_2(\sqrt{\log n}\|A\|)$ thanks to Lemmas B.7
- $\|\frac{1}{\sqrt{n}}x_i^TQ_{-i}X_{-i}\|_\infty \in O(\sqrt{\log n}) \pm \mathcal{E}_2(\sqrt{\log n})$ thanks to Lemma B.5

to show the concentration of $x_i^TQ_{-i}AQ_{-i}x_i$ with Theorem B.6.

**Lemma B.9.** Under the hypotheses of Theorem B.1 given a deterministic matrix $A \in \mathcal{M}_{p,n}$, $B \in \mathcal{M}_{p,n}$ such that $\|A\|_+ \leq 1$, $\|B\|_+ \leq 1$:

$$X^TQAQX \succeq \mathcal{E}_2\left(\|A\|_+ \log^2 n\right) + \mathcal{E}_1\left(\|A\|_F \log^2 n\right) + \mathcal{E}_2\left(\|A\| \log^2 n\right) \quad \text{in } (\mathcal{M}_n, \cdot \|.\|).$$

and $\mathbb{E}\|X^TQAQX\|_d \leq O\left(\sqrt{\log^2 n}\|A\|_+\right)$.

To control the variation of the upper quantities, one first needs the following lemma.

**Lemma B.10.** Under the hypotheses of Theorem B.1 for any deterministic matrix $U \in \mathcal{M}_p$ such that $\|U\| \leq 1$:

$$\begin{cases}
\|U^TQX\|_\infty \in O(\sqrt{\log n}) \pm \mathcal{E}_2(\sqrt{\log n}) \\
\|X^TQ'_{-i}AQ_{-i}X\|_\infty \in O\left(\|A\|_+ \log^2 n\right) + \mathcal{E}_2\left(\|A\|_F \log^2 n\right) + \mathcal{E}_1\left(\|A\| \log^2 n\right)
\end{cases}$$

**Proof.** If $p \leq n$, one can replace $U$ with a matrix $U' \in \mathcal{M}_{p,n}$ that satisfies $\|UQX\|_\infty = \|U'QX\|_\infty$, we thus assume from now on that $U \in \mathcal{M}_{p,n}$. Noting the columns of $U$: $u_1, \ldots, u_n$, we can bound:

$$\|U^TQx_i\| = \sup_{i,j} u_j^TQ_{-i}x_i \leq \sup_{i,j} \frac{u_j^TQ_{-i}x_i}{1 + D_i\delta_i} \leq \sup_{i,j} |u_j^TQ_{-i}^{(i)}x_i| + \sup_{r \in [n]} \|Q_{-i}x_i - Q_{-i}^{(i)}x_i\|$$

and we deduce the concentration of $\|UQX\|_\infty$ since we know:
• from Lemma B.3 and Proposition 3.3 that sup_{i,j \in [p]} |u_i^T Q(0) x_i| \in O(\sqrt{\log(n^2)}) ± \mathcal{E}_2(\sqrt{\log(n^2)}), and of course, log n^2 ≤ O(\log n).
• from Lemma B.4 that \|Q_{\cdot i} x_i - Q(0)_{\cdot i} x_i\| \in O(\sqrt{\log n}) ± \mathcal{E}_2(\sqrt{\log n}).

For the second result, we recall first that for any x, y ∈ \mathbb{R}^p and A ∈ \mathcal{M}_p nonnegative symmetric:

\[ 2 \|x^T Ay\| ≤ \|x^T Ax\| + \|y^T Ay\|. \]

Therefore, we rather bound:

\[ \sup_{i \in [n]} \|x_i^T QAQ x_i\| ≤ \sup_{i \in [n]} \|x_i^T Q_{\cdot i} AQ_{\cdot i} x_i\|, \]

(we know that \(x_i^T Q = \frac{1}{\sqrt{n}}x_i^T \Lambda_{\cdot i} x_i\) and \(D_i \Delta_i \geq 0\). We can then conclude thanks to Proposition 3.3 and Lemma B.3 that states that \(x_i^T Q_{\cdot i} AQ_{\cdot i} x_i \in O(\|A\|_* + \|A\|_F \log n) ± \mathcal{E}_2(\|A\|_F \log n) + \mathcal{E}_1(\|A\| \log n)\).

**Proof of Lemma B.3.** Let us introduce \(X', D'\), respectively an independent copy of \(X\) and \(D\) note \(\Phi(X, D) = \|X^T QAX\|_d\), \(Q' = (I_p - X'DX')^{-1}\) and \(Q'' = (I_p - XD'X')^{-1}\). One can first bound:

\[
\Phi(X, D) - \Phi(X', D') \leq \|(X - X')^T QAX\|_d + \frac{1}{n} \|X'^T Q(X - X') DX'Q'AX\|_d + \frac{1}{n} \|X'^T Q' A(QX - X') DX'Q'X\|_d + \frac{1}{n} \|X'^T Q' A(X - X') DX'Q'X\|_d.
\]

Some of the term are treated similarly, we will therefore just bound the first and the second one. Inspiring from the proof of Corollary 3.3 we decompose again \(A = \Lambda D V^T\) with \(\Lambda = \text{Diag}(\lambda) \in \mathcal{D}_n\) and \(U, V \in \mathcal{O}_p\), noting \(\hat{X} \equiv V^TQX\), \(\hat{X}' \equiv V^TQX'\) and \(\hat{Y} \equiv U^TQX\) we have the identity:

\[
\|(X - X')^T QAX\|_d \leq \sup_{D \in \mathcal{D}_n} \|D^T ((\hat{X} - \hat{X}') \odot \hat{Y}) \|_F \|A\|_F \|X - X'\|_F \|\hat{Y}\|_\infty.
\]

and we know from Lemma B.10 that \(\|\hat{Y}\|_\infty \in O(\sqrt{\log(pn)}) ± \mathcal{E}_2(\sqrt{\log(pn)})\).

The second terms bounds similarly, noting this time \(\hat{X} \equiv \frac{1}{n} V^T Q'XDQX'\), \(\hat{X}' \equiv \frac{1}{n} V^T Q'XDQX'\) and \(\hat{Y} \equiv U^TQX\). Indeed, one can bound:

\[
\frac{1}{n} \|X^T Q(X - X') DX'Q'AX\|_d \leq \|((\hat{X} - \hat{X}') \odot \hat{Y})\|_F \|X - X'\|_F \|\hat{Y}\|_\infty.
\]

Let us now bound the variations on \(D\):

\[
|\Phi(X, D) - \Phi(X, D')| \leq \frac{1}{n} \|X^T Q(X - D') DX'Q'AX\|_d + \frac{1}{n} \|X^T Q' A(X - D') DX'Q'X\|_d.
\]

The two terms are similar, we therefore just bound:

\[
\frac{1}{n} \|X^T Q(X - D') DX'Q'AX\|_d \leq \|D - D'\|_F \sup_{i,j} \|x_i^T Q' A x_j\|.
\]
Concentration of product of random vectors

But we know from Lemma B.10 that:

\[ \|X^T Q_{\cdot -} A Q_{\cdot -} X\|_\infty \in O \left( \|A\|_* \log^2 n \right) \pm \mathcal{E}_2 \left( \|A\| F \log^2 n \right) + \mathcal{E}_1 \left( \|A\| \log^{3/2} n \right) \]

One can then conclude on the concentration with Theorem 4.6.

For the control on the diagonal norm of the expectation, one can merely bound:

\[
\mathbb{E}[\|X^T A Q X\|_d] \leq \sqrt{\mathbb{E} \left[ \sum_{i=1}^n x_i^T A Q x_i \right]} = \sqrt{\mathbb{E} \left[ \sum_{i=1}^n x_i^T Q_{\cdot i} A Q_{\cdot i} x_i \right] (1 + D_1 \Delta_i)^2} \\
\leq \sqrt{\sum_{i=1}^n \mathbb{E} \left[ x_i^T Q_{\cdot i} A Q_{\cdot i} x_i \right]} \leq O \left( \sqrt{n} \log^2 \|A\|_* \right).
\]

Proof of Theorem 8.1 Let us consider \( A \in M_{p,n} \) such that \( \|A\|_* \leq 1 \) and let us note \( \phi(X, D) = \text{Tr}(AQ) \). We abusively work with \( X, D \) and independent copies \( X', D' \) satisfying \( \|X\|, \|X'\| \leq \sqrt{n} \) and \( \|D\|, \|D'\| \leq \kappa_D \) as if they were deterministic variables, and we note \( Q_X' \equiv \phi(X, D), Q_{D'} \equiv \phi(X, D') \). Let us bound the variations

\[ |\phi(X, D) - \phi(X, D')| = \frac{1}{n} |\text{Tr}(AQ(X - X')DXQ_X)| \leq \frac{\kappa_D}{\varepsilon^2 \sqrt{n}} \|X - X'\|_F. \]

We can also bound as in the proof of Proposition 7.2

\[ |\phi(X, D) - \phi(X, D')| \leq \frac{1}{n} \|Q_{D'} AQ X\|_d \|D - D'\|_F. \]

and we know from Lemma B.9 that:

\[ \|Y Q_{D'} AQ X\|_d \in O \left( \sqrt{n} \log^2 n \right) \pm \mathcal{E}_4 \left( \log^{3/2} n \right) \]

(although Lemma B.9 gives the concentration of \( \|Y Q_{D'} AQ X\|_d \), but the proof remains the same if one replaces one of the \( Q \) with \( Q_{D'} \), for a diagonal matrix \( D' \), independent with \( D \)). We can then conclude on the concentration of \( Q \) applying Theorem 4.6.

\[ \square \]

B.4 Proof of Theorem 8.2

The existence of the deterministic parameters \( \delta \in \mathcal{D}_n \) such that:

\[ \delta = \frac{1}{n} \text{Diag}_{i \in [n]} \text{Tr}(\Sigma_i \hat{Q}^{\delta}(D)) \quad \left( \text{recall that } \hat{Q}^{\delta}(D) \equiv \left( I_p - \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ D_1 \frac{D}{1 + D_1 \delta_i} \right] \Sigma_i \right)^{-1} \right) \]

is proved in a similar way as in [LC21, Theorem 1] thanks to a semimetric \( d_s \) defined for any \( \delta, \delta' \in \mathcal{D}_n(\mathbb{R}^+) \) as \( d_s(\delta, \delta') = \frac{\| \delta - \delta' \|_F}{\sqrt{\delta^2}} \). To prove Theorem 8.2 we are going to invoke a result also taken from [LC21] providing a deterministic equivalent for resolvent of the form \( Q = (I_p - \frac{1}{n} X X^T)^{-1} \) where the columns of \( X \) are independent but possibly not identically distributed (that concerns in particular the case of matrices \( (I_p - \frac{1}{n} X D X^T)^{-1} \) for deterministic diagonal matrices \( D \) as stated below).

**Theorem B.11** ([LC21], Theorem 4, Corollary 1). In the regime \( p \leq O(n) \), given two random matrices \( X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_n) \in M_{p,n} \) such that \( X, Y \sim \mathcal{E}_2, O(1) \leq \sup_{i \in [n]} \|E[x_i]\|, \sup_{i \in [n]} \|E[y_i]\| \leq O(1) \) and all the couples \( (x_i, y_i) \) are independent, for
any deterministic diagonal matrix $\tilde{D}$ satisfying $\|\tilde{D}\| \leq O(1)$ and $\|X \tilde{D}X^T\| \leq 1 - \varepsilon$, the equation $\delta = \frac{1}{n} \text{Tr} (\Sigma_i \tilde{Q}^3 (\tilde{D}))$ admits a unique solution $\delta$ and we can bound:

$$\left\| \left( I_p + \frac{1}{n} X \tilde{D} X^T \right)^{-1} - \tilde{Q}^3 (\tilde{D}) \right\|_F \left( \frac{1}{\sqrt{n}} \right).$$

Let us first explain why the resolvent $\tilde{Q}' \equiv (I_p - \frac{1}{n} X E[D]X^T)^{-1}$ is not a relevant a deterministic equivalent of $Q$.

**Remark B.12.** Considering a deterministic vector $u \in \mathbb{R}^p$ such that $\|u\| \leq 1$:

$$|E[u^T Q u] - E[u^T Q' u]| = \frac{1}{n} |E [u^T Q X (D - E[D]) X^T Q' u]|$$

and the concentrations $D_i \propto O(1) \pm \varepsilon_2$ given by our hypotheses, $x_i^T Q_{-i} A Q' x_i \in O (\log n) \pm \varepsilon_1 (\log n)$ given by Lemma [B.3] and $\Delta_i \in O(1) \pm \varepsilon_1 \left(\sqrt{\log n/n}\right)$ given by Lemma [B.6] that:

$$E \left[ (x_i^T Q u u^T Q x_i - E[x_i^T Q u u^T Q' x_i])^2 \right] \leq (\log n)^2.$$

The Hölder inequality then allows us to conclude ($E[D^2_i] \leq O(1)$):

$$\left\| E[Q] - \tilde{Q}^3 (E[D]) (E[D]) \right\| \leq O (\log^2 n).$$

This result has no interest because we already knew that $\|E[Q]\| \leq O(1)$.

What happens is that the concentration of $x_i^T Q A Q x_i$ for general $A$ is not good and can not be improved from the concentration of $x_i Q_{-i} A Q' x_i$ and the identity (B.4). Indeed $D_i$ only has an observable diameter of order $O(1)$ and $x_i Q_{-i} A Q' x_i$ can possibly be of size $O(\log n / \sqrt{n})$ when one only assumes that $\|A\| \leq 1$.

To get a relevant estimation of $E[Q]$, one needs the following lemma.

**Lemma B.13.** Under the hypotheses of Theorem [B.1] given a deterministic matrix $A \in \mathcal{M}_{p,n}$ such that $\|A\| \leq 1$:

$$x_i^T Q A Q x_i (1 + D_i \Delta_i) \propto \mathcal{E}_1 (\log n) + \mathcal{E}_2 \left( \frac{\log^2 n}{\sqrt{n}} \right).$$

be careful that the mean is taken on $\Delta_i$ and not on $D_i$, all the subtlety is here since it does not seem possible to show such a tight concentration for $x_i^T Q A Q x_i (1 + E[D_i] \Delta_i)$ as explained above.

**Proof.** The concentration of:

$$x_i^T Q A Q x_i (1 + D_i \Delta_i) = x_i^T Q_{-i} A Q x_i = x_i^T Q A Q_{-i} x_i \in O \left( \sqrt{n \log n} \right) \pm \mathcal{E}_2 (\log n) + \mathcal{E}_1 (\|A\| \log n)$$
Concentration of product of random vectors

is proven the same way as in the proof of Lemma [B.8] taking advantage of the quasi
independence between \( x_i \) and \( Q_{-i} \).

To show the concentration of \( x_i^T QAQ x_i (1 + D_i \Delta_i) \), note that:

\[
|x_i^T QAQ x_i (1 + D_i \Delta_i) - x_i^T QAQ x_i (1 + D_i \Delta_i)| \leq \kappa_D |x_i^T QAQ x_i| |\Delta_i - \Delta_i|,
\]

one can then conclude thanks to the concentration of \( x_i^T QAQ x_i \) and \( \Delta_i \in \Delta_i \pm E_1(\sqrt{\log n/n}) \) given in Lemma [B.6]

We have now all the elements to estimate \( \mathbb{E}[Q] = \mathbb{E}[(I_p - \frac{1}{n} XDXT)^{-1}] \).

**Proof of Theorem [B.2]** Let us introduce the resolvent \( \tilde{Q} \equiv (I_p - \frac{1}{n} XDXT)^{-1} \) where we defined

\[
\tilde{D} \equiv \left( \tilde{\Delta} - \left( \mathbb{E} \left[ \left( I_p + D\tilde{\Delta} \right)^{-1} \right] \right)^{-1} \right).
\]

As will be seen later, this elaborated definition is taken for \( \tilde{D} \) to satisfy the following relation:

\[
\frac{\tilde{D}}{1 + D\tilde{\Delta}} = \mathbb{E} \left[ \frac{D}{I_p + D\tilde{\Delta}} \right],
\]

it implies in particular that \( \tilde{Q}^\delta(D) = \tilde{Q}^\delta(\tilde{D}) \) for any \( \delta \in \mathcal{D}_n \). Let us then consider a
deterministic matrix \( A \in \mathcal{M}_p \), such that \( \|A\|_F \leq 1 \) and bound:

\[
|\mathbb{E}[\text{Tr}(AQ)] - \mathbb{E}[\text{Tr}(AQ)]| = \frac{1}{n} \sum_{i=1}^n |\mathbb{E} [x_i^T QAQ x_i \tilde{\Delta}_i^{-1} (\tilde{\Delta}_i D_i + 1 - (\tilde{\Delta}_i \tilde{D}_i + 1))]| = \frac{1}{n} \sum_{i=1}^n |\mathbb{E} [x_i^T QAQ x_i \tilde{\Delta}_i^{-1} (\tilde{\Delta}_i D_i + 1) (\tilde{\Delta}_i \tilde{D}_i + 1) \left( \frac{1}{\Delta_i \tilde{D}_i + 1} \right)]| = \frac{1}{n} \sum_{i=1}^n |\mathbb{E} [x_i^T QAQ x_i (\Delta_i D_i + 1) (\tilde{\Delta}_i \tilde{D}_i + 1) \left( \frac{D_i}{\Delta_i \tilde{D}_i + 1} \right)]| = \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ (x_i^T QAQ x_i (\Delta_i D_i + 1) - \mathbb{E} [x_i^T QAQ x_i (\Delta_i D_i + 1)]) \cdot \frac{D_i (\Delta_i \tilde{D}_i + 1)}{\Delta_i D_i + 1} \right]| = \frac{\kappa_D}{\epsilon} \left( \frac{\kappa_D^2}{\epsilon} + 1 \right) \sup \mathbb{E} \left[ (x_i^T QAQ x_i (\Delta_i D_i + 1) - \mathbb{E} [x_i^T QAQ x_i (\Delta_i D_i + 1)])^2 \right] \leq O \left( \log^{3/2} n \right),
\]

thanks to Lemma [B.13].

Theorem [B.11] then allows us to state that given \( \delta \in \mathcal{D}_n \) solution to:

\[
\delta = \frac{1}{n} \text{Tr} \left( \Sigma_i \tilde{Q}^\delta(D) \right) = \frac{1}{n} \text{Tr} \left( \Sigma_i \tilde{Q}^\delta(D) \right),
\]

one has the estimation:

\[
\left\| \mathbb{E}[Q] - \tilde{Q}^\delta(D) \right\|_F \leq \left\| \mathbb{E}[Q] - \mathbb{E}[\tilde{Q}] \right\|_F + \left\| \mathbb{E}[Q] - \tilde{Q}^\delta(D) \right\|_F \leq O \left( \log^{3/2} n + \frac{1}{\sqrt{n}} \right) \leq O \left( \log^{3/2} n \right).
\]
Concentration of product of random vectors

References

[ABW17] Radosław Adamczak, Witold Bednorz, and Paweł Wolff. Moment estimates implied by modified log-sobolev inequalities. ESAIM: Probability and Statistics, 21:467–494, 2017.

[Ada11] Radosław Adamczak. On the marchenko-pastur and circular laws for some classes of random matrices with dependent entries. Electronic Journal of Probability, 16:1065–1095, 2011.

[Ada15] Radosław Adamczak. A note on the hanson-wright inequality for random vectors with dependencies. Electronic Communications in Probability, 20(72):1–13, 2015.

[AS94] Shigeki Aida and Daniel Stroock. Moment estimates derived from poincaré and logarithmic sobolev inequalities. Mathematical Research Letters, 1(1):75–86, 1994.

[AW15] Radosław Adamczak and Paweł Wolff. Concentration inequalities for non-lipschitz functions with bounded derivatives of higher order. Probability Theory and Related Fields, 162(3):531–586, 2015.

[BLM13] Stéphane Boucheron, Gabor Lugosi, and Pascal Massart. Concentration Inequalities: a Nonasymptotic Theory of Independence. Oxford University Press, 2013.

[Cha17] Sourav Chatterjee. Fluctuations of eigenvalues and second order poincaré inequalities. Probability Theory and Related Fields, 143:1–40, 2017.

[DKT20] Zeyu Deng, Abla Kammoun, and Christos Thrampoulidis. A model of double descent for high-dimensional binary linear classification. ICASSP’2020 IEEE International Conference on Acoustics, Speech and Signal Processing, 2020.

[EKBB+13] Noureddine El Karoui, Derek Bean, Peter J Bickel, Chinghway Lim, and Bin Yu. On robust regression with high-dimensional predictors. Proceedings of the National Academy of Sciences, 110(36):14557–14562, 2013.

[FGP07] B. Fleury, O. Guédon, and G. Paouris. A stability result for mean width of l p-centroid bodies. Advances in Mathematics, 214:865–877, 2007.

[GLPP14] Olivier Guédon, Anna Lytova, Alain Pajor, and Leonid Pastur. The central limit theorem for linear eigenvalue statistics of the sum of independent matrices of rank one. American Mathematical Society Translations: Series 2, 233(1):145–164, 2014.

[GM83] Mikhael Gromov and Vitali D Milman. A topological application of the isoperimetric inequality. American Journal of Mathematics, 105(4):843–854, 1983.

[Hua17] Hanwen Huang. Asymptotic behavior of support vector machine for spiked population model. Journal of Machine Learning Research, 18(45):1–21, 2017.

[HW71] D. L. Hanson and F. T. Wright. A bound on tail probabilities for quadratic forms in independent random variables. Annals of Mathematical Statistics, 42(3):1079–1083, 1971.

[Jam57] Robert C. James. Reflexivity and the supremum of linear functionals. Annals of Mathematics, 66(1):159 – 169, 1957.

[Kla07] B. Klartag. A central limit theorem for convex sets. Inventiones mathematicae, 168:91–131, 2007.

[Lat06] Rafał Latała. Estimates of moments and tails of gaussian chaos. The Annals of Probability, 34(6):2315–2331, 2006.

[LC18] Cosme Louart and Romain Couillet. Concentration of measure and large random matrices with an application to sample covariance matrices. arXiv preprint arXiv:1805.08295, 2018.

[LC21] Cosme Louart and Romain Couillet. Spectral properties of sample covariance matrices arising from random matrices with independent non identically distributed columns. arXiv preprint arXiv:2109.02644, 2021.

[Led05] Michel Ledoux. The concentration of measure phenomenon. Number 89. American Mathematical Soc., 2005.

[Lou22] Cosme Louart. Sharp bounds for the concentration of the resolvent in convex concentration settings. arXiv preprint arXiv:2201.00284, 2022.
Concentration of product of random vectors

[Le51] Paul Lévy. *Problèmes concrets d’analyse fonctionnelle*. Gauthier-Villars, 1951.

[MLC19] Xiaoyi Mai, Zhenyu Liao, and Romain Couillet. A large scale analysis of logistic regression: Asymptotic performance and new insights. *ICASSP 2019 - 2019 IEEE International Conference on Acoustics, Speech and Signal Processing*, 2019.

[Pas05] Leonid Pastur. A simple approach to the global regime of gaussian ensembles of random matrices. *Ukrainian Mathematical Journal*, 57(6):936–966, 2005.

[PP09] Alain Pajor and Leonid Pastur. On the limiting empirical measure of the sum of rank one matrices with log-concave distribution. *Studia Mathematica, Institut Matematyczny Polska Akademia nauk*, 195:11–29, 2009.

[SB95] J. W. Silverstein and Z. D. Bai. On the empirical distribution of eigenvalues of a class of large dimensional random matrices. *Journal of Multivariate Analysis*, 54(2):175–192, 1995.

[SLTC21] Mohamed El Amine Seddik, Cosme Louart, Mohamed Tamaazousti, and Romain Couillet. The unexpected deterministic and universal behavior of large softmax classifiers. *AISTATS*, 2021.

[STC19] Mohamed El Amine Seddik, Mohamed Tamaazousti, and Romain Couillet. Kernel random matrices of large concentrated data: the example of gan-generated images. *ICASSP’19*, 2019.

[Tal88] Michel Talagrand. An isoperimetric theorem on the cube and the kintchine-kahane inequalities. *Proceedings of the American Mathematical Society*, 81(3):73—205, 1988.

[Tal95] Michel Talagrand. Concentration of measure and isoperimetric inequalities in product spaces. *Publications mathématiques de l’IHÉS*, 104:905–909, 1995.

[Ver17] R. Vershynin. *High dimensional probability*. Cambridge University Press, 2017.

[VW14] Van Vu and Ke Wang. Random weighted projections, random quadratic forms and random eigen vectors. *Random Structures and Algorithms*, 2014.