UNBRANCHED RIEMANN DOMAINS OVER STEIN SPACES

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Abstract. In this article, we show that if $\Pi : X \to \Omega$ is an unbranched Riemann domain with $\Omega$ Stein and $\Pi$ a locally 1-complete morphism, then $X$ is Stein. This gives in particular a positive answer to the local Steinness problem, namely if $X$ is a Stein space and, if $\Omega$ is a locally Stein open set in $X$, then $\Omega$ is Stein.

1. Introduction

A holomorphic map $\Pi : X \to Y$ of complex spaces is said to be a locally $r$-complete morphism if for every $x \in Y$, there exists an open neighborhood $U$ of $x$ such that $\Pi^{-1}(U)$ is $r$-complete. When $r = 1$, $\Pi$ is called locally 1-complete or locally Stein morphism.

In [3] Coltoiu and Diederich proved the following

Theorem 1. Let $X$ and $Y$ be complex spaces with isolated singularities and $\Pi : X \to Y$ an unbranched Riemann domain such that $Y$ is Stein and $\Pi$ is a locally Stein morphism. Then $X$ is Stein.

In this article, we prove that the same result follows if we assume only that $Y$ is an arbitrary Stein space.

An immediate consequence of this result is the

Corollary 1. Let $X$ be a Stein space, and let $\Omega \subset X$ be an open subset which is locally Stein in the sense that every point $x \in \partial \Omega$ has an open neighborhood $U$ in $X$ such that $U \cap \Omega$ is Stein. Then $\Omega$ is itself Stein.

2. Preliminaries

We start by recalling some definitions which are important for our purposes.

Let $\Omega$ be an open set in $\mathbb{C}^n$ with complex coordinates $z_1, \ldots, z_n$. Then it is known that a function $\phi \in C^\infty(\Omega)$ is $q$-convex if for every point $z \in \Omega$, there exists a complex vector subspace $E(z)$ of $\mathbb{C}^n$ of dimension at least $n - q + 1$ such that the

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levi form \( L_z(\phi, \xi) = \sum_{i,j} \frac{\partial^2 \phi(z)}{\partial z_i \partial z_j} \xi_i \xi_j \) is positive definite at each point \( \xi \in E(z) \).

A smooth real valued function \( \phi \) on a complex space \( X \) is called \( q \)-convex if every point \( x \in X \) has an open neighborhood \( U \) isomorphic to a closed analytic set in a domain \( D \subset \mathbb{C}^n \) such that the restriction \( \phi|_U \) has an extension \( \tilde{\phi} \in C^\infty(D) \) which is \( q \)-convex on \( D \).

We say that \( X \) is \( q \)-complete if there exists a \( q \)-convex function \( \phi \in C^\infty(X, \mathbb{R}) \) which is exhaustive on \( X \) i.e. \( \{ x \in X : \phi(x) < c \} \) is relatively compact for any \( c \in \mathbb{R} \).

The space \( X \) is said to be cohomologically \( q \)-complete if for every coherent analytic sheaf \( \mathcal{F} \) on \( X \) the cohomology groups \( H^p(X, \mathcal{F}) \) vanish for all \( p \geq q \).

An open subset \( D \) of \( \Omega \) is called \( q \)-Runge if for every compact set \( K \subset D \), there is a \( q \)-convex exhaustion function \( \phi \in C^\infty(\Omega) \) such that

\[
K \subset \{ x \in \Omega : \phi(x) < 0 \} \subset \subset D
\]

It is shown in [2] that if \( D \) is \( q \)-Runge in \( \Omega \), then for every \( \mathcal{F} \in \text{coh}(\Omega) \) the cohomology groups \( H^p(D, \mathcal{F}) \) vanish for \( p \geq q \) and, the restriction map

\[
H^p(\Omega, \mathcal{F}) \rightarrow H^p(D, \mathcal{F})
\]

has dense image for all \( p \geq q - 1 \).

3. Main result

**Lemma 1.** Let \( X \) and \( Y \) be complex spaces and \( \Pi : X \rightarrow Y \) an unbranched Riemann domain. Assume that there exists a smooth \( q \)-convex function \( \phi \) on \( Y \). Then, for any real number \( c \) and every coherent analytic sheaf \( \mathcal{F} \) on \( X \) if \( \text{dih}(\mathcal{F}) > q \) and \( X'_c = \{ x \in X : \phi \Pi(x) > c \} \), the restriction map \( H^p(X, \mathcal{F}) \rightarrow H^p(X'_c, \mathcal{F}) \) is an isomorphism for \( p \leq \text{dih}(\mathcal{F}) - q - 1 \).

Here \( \text{dih}(\mathcal{F}) \) denotes the homological dimension of \( \mathcal{F} \).

Let \( V \) be a closed analytic set in a domain \( D \subset \mathbb{C}^n \) and \( \phi \in C^\infty(V) \) a \( q \)-convex function. Let \( \xi \in V \) and suppose we can find a \( q \)-convex function \( \tilde{\phi} \in C^\infty(D) \) with \( \tilde{\phi}|_V = \phi \) and that \( n \) is equal to the dimension of the Zariski tangent space at \( \xi \).

Then in order to prove lemma 1 we shall need the following result due to Andreotti-Grauert [2].
Theorem 2. For any coherent analytic sheaf \( F \) on \( V \) with \( \text{dih}(F) > q \), there exists a fundamental system of Stein neighborhoods \( U \subset D \) of \( \xi \) such that if \( Y = \{ z \in V : \phi(z) > 0 \} \), then \( H^p(Y \cap U, F) = 0 \) for \( 0 < p < \text{dih}(F) - q \) and \( H^0(U \cap V, F) \to H^0(U \cap Y, F) \) is an isomorphism.

Proof. Let \( \xi \in X \) such that \( \phi|\pi(\xi) = c \), and let \( V \subset X \) be a hyperconvex open neighborhood of \( \xi \), biholomorphic by \( \Pi \) to the open subset \( W = \Pi(V) \subset Y \). We may take \( V \) so that \( W \) is biholomorphic to a closed analytic subset of a domain \( D \) in \( \mathbb{C}^n \) of minimal dimension and \( \phi|W \) extends to a smooth \( q \)-convex function in a neighborhood \( W_1 \subset D \) of \( W \). Let \( \psi : V \to ]-\infty, 0[ \) be a continuous strictly plurisubharmonic function. Then it is clear that \( \psi_k = \frac{k}{k^2} \psi + \phi \| \Pi \) \( k \geq 1 \), is an increasing sequence of \( q \)-convex functions on \( V \). If we put \( V_k = \{ x \in V : \psi_k(x) > c \} \), then \( \bigcup_{k \geq 1} V_k = V \cap X' \). Moreover, for any \( x \in V \), there exists, by theorem 2, a fundamental system of connected Stein neighborhoods \( U \subset V \) such that \( H^r(U \cap V_k, F) = 0 \) for \( 1 \leq r < \text{dih}(F) - q \) and \( H^r(U \cap V_k, F) \to H^r(U \cap V_k, F) \) is an isomorphism, or equivalently (see [4] or [1]) \( H^r_{S_k} (F) = 0 \) for \( r \leq \text{dih}(F) - q \), where \( H^r_{S_k} (F) \) is the cohomology sheaf with support in \( S_k = \{ x \in V : \psi_k(x) \leq c \} \) and coefficients in \( F \). Furthermore, there exists a spectral sequence

\[
H^p_{S_k} (V, F) \leftrightarrow E^p,q_2 = H^p (V, H^q_{S_k} (F))
\]

Since \( H^p_{S_k} (F) = 0 \) for \( p \leq \text{dih}(F) - q \), then for any \( p \leq \text{dih}(F) - q \) the cohomology groups \( H^p_{S_k} (V, F) \) vanish and, the exact sequence of local cohomology

\[
\cdots \to H^p_{S_k} (V, F) \to H^p (V, F) \to H^p (V_k, F) \to H^{p+1}_{S_k} (V, F) \to \cdots
\]

implies that \( H^p (V_k, F) \cong \tilde{H}^0 (V, F) \) for all \( p \leq \text{dih}(F) - q - 1 \). Hence

\[
H^p (V_k, F) = 0 \text{ for } 1 \leq p \leq \text{dih}(F) - q - 1 \text{ and } H^p (V_k, F) \cong H^p (V, F) \text{ for every integer } k.
\]

Since \( V \cap X'_c \) is an increasing union of \( V_k \), \( k \in \mathbb{N} \), then, by (2), lemma, p.250), we deduce that \( H^p (V \cap X'_c, F) = 0 \) for \( 1 \leq p \leq n - q - 1 \) and \( H^p (V, F) \to H^p (V \cap X'_c, F) \) is an isomorphism. Since each point of \( X \) has a fundamental system of hyperconvex neighborhoods, then, if \( S = \{ x \in X : \phi \Pi (x) \leq c \} \), the cohomology sheaf \( H^p_S (F) \) vanishes for all \( p \leq \text{dih}(F) - q \). Therefore the spectral sequence

\[
H^p_S (X, F) \leftrightarrow E^p,q_2 = H^p (X, H^q_S (F))
\]

shows that \( H^p_S (X, F) = 0 \) for any \( p \leq \text{dih}(F) - q \), and from the exact sequence

\[
\cdots \to H^p_S (X, F) \to H^p (X, F) \to H^p (X'_c, F) \to H^{p+1}_S (X, F) \cdots
\]

we see that \( H^r (X, F) \cong H^r (X'_c, F) \) for any \( c \in \mathbb{R} \) and all \( r \leq n - q - 1 \). \( \square \)
Lemma 2. Let $Y$ be a Stein space and $\Pi : X \to Y$ an unbranched Riemann domain and locally $r$-complete morphism. Then for any coherent analytic sheaf $\mathcal{F}$ on $X$ with $\text{dih}(\mathcal{F}) \geq r + 2$, the cohomology group $H^p(X, \mathcal{F}) = 0$ for all $p \geq r$.

Proof. Since $\Pi : X \to Y$ is locally $r$-complete, it follows from [7] that $H^p(X, \mathcal{F}) = 0$ for all $p \geq r + 1$. It is therefore enough to prove that $H^r(X, \mathcal{F}) = 0$.

We consider a covering $\mathcal{V} = (V_i)_{i \in \mathbb{N}}$ of $Y$ by open sets $V_i \subset \Omega$ such that $\Pi^{-1}(V_i)$ is $r$-complete for all $i \in \mathbb{N}$. By the Stein covering lemma of Stshelé [6], there exists a locally finite covering $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$ of $Y$ by Stein open subsets $U_i \subset \subset Y$ such that $\mathcal{U}$ is a refinement of $\mathcal{V}$, $\bigcup_{i \leq j} U_i$ is Stein for all $i$ and Stein for all $j$. Moreover, there exists for all $j \in \mathbb{N}$ a continuous strictly plurisubharmonic function $\phi_{j+1}$ on $\bigcup_{i \leq j} U_i$ such that

$$\bigcup_{i \leq j} U_i \cap U_{j+1} = \{x \in U_{j+1} : \phi_{j+1}(x) < 0\}$$

Note also that $\Pi^{-1}(U_i)$ is $r$-complete for all $i \in \mathbb{N}$ and, if $X_j = \Pi^{-1}(\bigcup_{i \leq j} U_i)$ and $X_j' = \Pi^{-1}(U_{j+1})$, then $X_j \cap X_j' = \{x \in X_{j+1} : \phi_{j+1} \circ \Pi(x) < 0\}$ is clearly $r$-Runge in $X'_j$.

We shall first prove by induction on $j$ that $H^r(X_j, \mathcal{F}) = 0$. For $j = 0$, this is clear, since $\Pi^{-1}(U_0)$ is $r$-complete. Assume that $j \geq 1$, $H^r(X_j, \mathcal{F}) = 0$ and put $Y_j = \{x \in X_j : \phi_{j+1} \circ \Pi(x) > 0\}$ and $Y_j' = \{x \in X_j' : \phi_{j+1} \circ \Pi(x) > 0\}$. Then, by lemma 1, $H^p(Y_j, \mathcal{F}) \cong H^p(X_j, \mathcal{F})$ and $H^p(Y_j', \mathcal{F}) \cong H^p(X_j', \mathcal{F})$ for $p \leq r$.

Since $Y_{j+1} = \{x \in X_{j+1} : \phi_{j+1} \circ \Pi(x) > 0\} = Y_j \cup Y_j'$ and $Y_j \cap Y_j' = \emptyset$, then we have

$$H^p(X_{j+1}, \mathcal{F}) \cong H^p(Y_{j+1}, \mathcal{F}) \cong H^p(Y_j, \mathcal{F}) \oplus H^p(Y_j', \mathcal{F}) \text{ for all } p \leq r$$

This proves in particular that $H^r(X_j, \mathcal{F}) = 0$ for all $j \in \mathbb{N}$.

Moreover, since $X$ is an increasing union of $(X_j)_{j \geq 0}$ and $H^{r-1}(X_{j+1}, \mathcal{F}) \cong H^{r-1}(X_j, \mathcal{F}) \oplus H^{r-1}(X_j', \mathcal{F})$, then, by [2, lemma, p. 250], the restriction map $H^r(X, \mathcal{F}) \to H^r(X_0, \mathcal{F})$ is an isomorphism, which implies that $H^r(X, \mathcal{F}) = 0$. 

Theorem 3. Let $\Pi : X \to Y$ be an unbranched Riemann domain with $Y$ a Stein space of dimension $n$ and $\Pi$ a locally Stein morphism. Then $X$ is Stein.

Proof. The proof is by induction on the dimension of $Y$.

In order to prove theorem 3 we have only to verify that $H^1(X, \mathcal{O}_X) = 0$. (See [5].)

Suppose that $n = 2$, and let $\xi : \tilde{Y} \to Y$ be a normalization of $Y$. If $\tilde{X}$ denotes the fiber product of $\Pi : X \to Y$ and the normalization $\xi : \tilde{Y} \to Y$, then $\tilde{X} = \text{fiber product of } \Pi \text{ and } \xi$. 


\{(x, \tilde{y}) \in X \times \tilde{Y} : \Pi(x) = \xi(\tilde{y})\} \) and, it is clear that the projection \( \Pi_2 : \tilde{X} \to \tilde{Y} \) is an unbranched Riemann domain over the 2-dimensional Stein normal space \( \tilde{Y} \).
Moreover, since \( \Pi_2 \) is obviously a locally Stein morphism, it follows from [3] that \( \tilde{X} \) is Stein. On the other hand, it is easy to verify that the projection \( \Pi_1 : \tilde{X} \to X \) is a finite holomorphic surjection, which implies that \( X \) is Stein. (See e.g. [8]).
We now suppose that \( n \geq 3 \) and that the theorem has already proved if \( \text{dim}(Y) \leq n - 1 \).
Since a complex space \( X \) is Stein if and only if each irreducible component \( X_i \) of \( X \) is Stein, then we may assume that \( Y \) is irreducible.
Let \( f \) be a holomorphic function on \( Y \), \( f \neq 0 \), but \( Z = \{f = 0\} \neq \emptyset \). Then \( \Pi|_{Z'} : Z' = \Pi^{-1}(Z) \to Z \) is an unbranched Riemann domain and locally Stein. Therefore \( Z' \) is Stein by the induction hypothesis. Furthermore, if \( \mathcal{I}(Z') \) denotes the ideal sheaf of \( Z' \), it follows from [2] that \( \text{dih}(\mathcal{I}(Z')) = \text{dih}(\mathcal{O}_{Z'}) + 1 \geq 3 \) and, by lemma 2, we obtain \( H^1(X, \mathcal{I}(Z')) = 0 \). Consider now the exact sequence of sheaves

\[ 0 \to \mathcal{I}(Z') \to \mathcal{O}_X \to \mathcal{O}_{Z'} \to 0 \]

Since \( H^1(X, \mathcal{I}(Z')) = 0 \) and \( Z' \) is Stein, we deduce from the long exact sequence of cohomology that \( H^1(X, \mathcal{O}_X) = 0 \).

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