Chiral random matrix theory for two-color QCD at high density

Takuya Kanazawa\textsuperscript{1}, Tilo Wettig\textsuperscript{2}, and Naoki Yamamoto\textsuperscript{1}
\textsuperscript{1}Department of Physics, The University of Tokyo, Tokyo 113-0033, Japan
\textsuperscript{2}Department of Physics, University of Regensburg, 93040 Regensburg, Germany

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We identify a non-Hermitian chiral random matrix theory that corresponds to two-color QCD at high density. We show that the partition function of the random matrix theory coincides with the partition function of the finite-volume effective theory at high density, and that the Leutwyler-Smilga-type spectral sum rules of the random matrix theory are identical to those derived from the effective theory. The microscopic Dirac spectrum of the theory is governed by the BCS gap, rather than the conventional chiral condensate. We also show that with a different choice of a parameter the random matrix theory yields the effective partition function at low density.

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Random matrix theory (RMT) has found numerous applications within and outside of physics, e.g., in nuclear physics [1], mesoscopic condensed matter physics [2], quantum chromodynamics (QCD) and QCD-like theories [3], quantum gravity [4], number theory [5], econophysics [6], wireless communications [7], and many more. Most of these applications are described by Hermitian random matrix models. While the first paper on non-Hermitian RMT by Ginibre appeared already in 1965 [8], there was essentially no activity in this area for a long time. However, in the past 10 years or so numerous new applications have been found in many fields of science that can be described by non-Hermitian random matrix models [9]. Prominent examples are QCD and QCD-like theories at nonzero density which, depending on their anti-unitary symmetries, are described by chiral versions of the Ginibre ensembles [10]. So far, the applications of non-Hermitian RMT in QCD have been restricted to moderate densities, roughly up to the critical density for the chiral phase transition. However, in recent years there was quite some activity in the study of QCD and QCD-like theories at very high densities, by means of effective theories and renormalization group methods, and a number of interesting new phenomena such as color superconductivity [11] and color-flavor locking [12] have been elucidated. The present paper is the first application of non-Hermitian RMT at high density and is expected to open up many more applications. We shall see that the situation is quite different compared to the one at low density. We will focus on the case of QCD with two colors which can in principle be tested in lattice simulations because the fermion sign problem is absent with two colors which can in principle be tested in lattice simulations because the fermion sign problem is absent.

Recently we have constructed the low-energy effective theory for high-density QCD with two colors and an even number of fermions in the fundamental representation [13]. At sufficiently large quark chemical potential $\mu$ ($\mu \gg \Lambda_{\text{SU}(2)}$), chiral symmetry is broken spontaneously by a diquark condensate as

$$\text{SU}(N_f)\text{L} \times \text{SU}(N_f)\text{R} \times \text{U}(1)\text{B} \times \text{U}(1)\text{A} \rightarrow \text{Sp}(N_f)\text{L} \times \text{Sp}(N_f)\text{R},$$

and the degrees of freedom of the effective theory are the Nambu-Goldstone (NG) bosons associated with (1).

We identify a non-Hermitian chiral random matrix theory that corresponds to two-color QCD at high density. We show that the partition function of the random matrix theory coincides with the partition function of the finite-volume effective theory at high density, and that the Leutwyler-Smilga-type spectral sum rules of the random matrix theory are identical to those derived from the effective theory. The microscopic Dirac spectrum of the theory is governed by the BCS gap, rather than the conventional chiral condensate. We also show that with a different choice of a parameter the random matrix theory yields the effective partition function at low density.

The sum rules we obtained earlier plus a universal argument hinted at the existence of a corresponding RMT. Here, we propose that the non-Hermitian chiral RMT describing two-color QCD at large $\mu$ for an even number of fundamental fermions is given by

$$Z(\{m_f\}) = \int dA dB \ e^{-N \text{tr}(AA^T + BB^T)} \times \prod_{f=1}^{N_f} \text{det} \left( \begin{array}{cc} m_f 1 & A \\ B & m_f 1 \end{array} \right),$$

where $A$ and $B$ are real $N \times N$ matrices, the integration measure is the flat Cartesian measure, the $m_f$ are dimensionless quantities corresponding to the quark masses, and $N_f$ is assumed to be even. At this stage physical scales have not been introduced yet. (The matching of random-matrix quantities to physical quantities is given in Eq. (9) below. There is no need to introduce a Gaussian width parameter for $A$ and $B$, as this parameter can be absorbed in the $m_f$.) Note that $A$ and $B$ are square matrices. In principle, the model (2) can be extended to rectangular matrices $A$ and $B$ [16], see Eq. (4) below, giving rise to topological zero modes. However, at high density the topological susceptibility is strongly suppressed [17] so that topological zero modes are irrelevant in the physical situation we are studying.

We stress that our model (2) differs from earlier RMT approaches to QCD at $\mu \neq 0$ [18–20]. In these approaches, the corresponding low-energy effective theory
is formulated in terms of NG bosons parametrizing the coset space \( SU(2N_f)/Sp(2N_f) \) of the symmetry breaking pattern by a chiral condensate at \( \mu = 0 \) [21], i.e., \( \mu \) is considered as a small perturbation. On the other hand, the low-energy effective theory corresponding to (2) is formulated in terms of NG bosons parametrizing the coset space \( SU(N_f) \times SU(N_f)/[Sp(N_f) \times Sp(N_f)] \) (see [13] and Eq. (8) below), i.e., it realizes the pattern of spontaneous chiral symmetry breaking induced by BCS-type diquark condensation at high density, with a vanishing chiral condensate. To see this explicitly, we rewrite (2) in the chiral limit as

\[
Z(\{0\}) = \int dA \det^{N_f/2} \begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix} e^{-N \text{ tr} AA^*} \times \int dB \det^{N_f/2} \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} e^{-N \text{ tr} BB^*}.
\]

(3)

Each of the two factors on the RHS of this equation corresponds to a chiral orthogonal ensemble at \( \mu = 0 \), for which the symmetry breaking pattern is known to be \( SU(N_f) \rightarrow Sp(N_f) \) [22]. Thus, we can immediately conclude that Eq. (3) exhibits the symmetry breaking pattern \( SU(N_f) \times SU(N_f) \rightarrow Sp(N_f) \times Sp(N_f) \), which agrees with Eq. (1). There are also NG bosons associated with the breaking of \( U(1)_B \) (baryon) and \( U(1)_A \) (axial). The former decouples completely and does not affect the quark mass dependence of the partition function [13]. The latter decouples in the chiral limit in which (3) was considered. At nonzero quark masses, see (2), we have a nontrivial \( U(1)_A \) integral, see Eq. (8) below.

The model (2) is not new. It corresponds to class \( 2P \) in Magnen’s classification of non-Hermitian ensembles, see Table 2 of [23]. What is new is the realization that Eq. (2) describes dense two-color QCD. Below we show explicitly that the effective theory resulting from Eq. (2) at large \( N \) exactly reproduces the finite-volume partition function of dense two-color QCD in the microscopic limit, obtained in [13] based on symmetry principles and weak-coupling calculations at large \( \mu \). This strongly suggests that Eq. (2) is the correct RMT for dense two-color QCD, paving the way towards a quantitative understanding of the Dirac spectrum at large \( \mu \).

A full proof of the equivalence, which would also establish the equality of all spectral correlation functions, requires the study of the partially quenched theory, see [24], which we leave to future work. However, we will also see that the Leutwyler-Smilga-type sum rules derived from Eq. (2) coincide with those obtained in [13]. These sum rules are moments of the spectral correlation functions and thus provide a nontrivial piece of evidence for the equivalence.

Before going into details, let us list some basic properties of the model (2). First, the matrix \( D \equiv \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \) is neither Hermitian nor anti-Hermitian. This implies that the eigenvalues of \( D \) are distributed over the complex plane. Second, \( D \) preserves chiral symmetry (i.e., it anticommutes with \( \gamma_5 \)). Third, it follows from chiral symmetry and from the fact that \( A \) and \( B \) are real that the eigenvalues of \( D \) appear either in pairs \((\lambda, -\lambda)\) with \( \lambda \in \mathbb{R} \cup i\mathbb{R} \) or in quartets \((\lambda, -\lambda, \lambda^*, -\lambda^*)\) with \( \lambda \in \mathbb{C} \) [25]. All of the above features are in common with dense two-color QCD [13].

Let us also elucidate the relation between our ensemble (2) and related ensembles in the literature. The first two-matrix model for \( \mu \neq 0 \), in which the matrix multiplying \( \mu \) is not a unit matrix but an independent random matrix, was introduced by Osborn [26]. His original analysis was restricted to Dyson index \( \beta = 2 \), corresponding to three-color QCD. Subsequently, extensions to \( \beta = 4 \) [27] and \( \beta = 1 \) [16] were constructed. Two-color QCD corresponds to \( \beta = 1 \), and the two-matrix model, which was studied mathematically in [28] for \( N_f = 0 \), reads

\[
Z(\hat{\mu}, \{m_f\}) = \int dC \int dD \ e^{-2N \text{ tr}(CC^T + DD^T)} \times \prod_{f=1}^{N_f} \det \left(-C^T + \hat{\mu}D^T \frac{C + \hat{\mu}D}{m_f} \right).
\]

(4)

Here, \( C \) and \( D \) are \( N \times (N + \nu) \) real matrices and \( \hat{\mu} \) is a parameter corresponding to the physical chemical potential \( \mu \). For \( \nu = 0 \) and \( \hat{\mu} = 1 \), (4) reduces to (2).

We shall now try to derive the effective theory corresponding to (4) at large \( N \) for \( 0 \leq \hat{\mu} \leq 1 \), following a similar analysis at \( \hat{\mu} = 0 \) [22]. The mass terms in (4) are generalized to a generic mass term \( \mathbf{m}^T P_R + \mathbf{m} P_L \) as in the QCD Lagrangian, where \( \dim(\mathbf{m}) = N_f \) and \( P_R/L = \tfrac{1}{2}(1 \pm \gamma_5) \). We introduce Grassmann variables to write the determinants as exponentials, integrate out the Gaussian matrix elements, and then introduce auxiliary matrices for a Hubbard-Stratonovich transformation. After integrating out the Grassmann variables we obtain

\[
Z(\hat{\mu}, \mathbf{m}) = \int dK \int dL dP \ e^{-8N \text{ tr}((KK^T + LL^T) + 2PP^T)} \times \text{ Pf}^N \left( \begin{pmatrix} \sqrt{1 + \mu^2} & \sqrt{1 - \mu^2} \mathbf{m}^T \frac{1}{\sqrt{1 + \mu^2}} \\ \sqrt{1 - \mu^2} P & \frac{1}{\sqrt{1 + \mu^2}} \mathbf{m}^* \end{pmatrix} \right) \times \text{ Pf}^{N+\nu} \left( \begin{pmatrix} \sqrt{1 + \mu^2} K \frac{1}{\sqrt{1 + \mu^2} L} & \frac{1}{\sqrt{1 + \mu^2} \mathbf{m}} \\ \sqrt{1 - \mu^2} P \frac{1}{\sqrt{1 + \mu^2} L} & \frac{1}{\sqrt{1 + \mu^2} \mathbf{m}^T} \end{pmatrix} \right),
\]

(5)

where \( K, L \) and \( P \) are \( N_f \times N_f \) complex-valued matrices, with \( K \) and \( L \) antisymmetric. Pf denotes the Pfaffian of the matrix. So far no approximation has been made.

The so-called strong non-Hermiticity limit is defined by \( N \gg 1 \) with \( \mu \) fixed. We now focus on the case relevant for us, i.e., \( \hat{\mu} = 1 \) and \( \nu = 0 \). In this case \( P \) drops out and we find

\[
Z_0(1, \mathbf{m}) = \int dK dL \ e^{-N \text{ tr}(KK^T + LL^T)} \times \text{ Pf}^N \left( \begin{pmatrix} \mathbf{m}^T & -\mathbf{m}^* \end{pmatrix} \right) \text{ Pf}^N \left( \begin{pmatrix} K^T -\mathbf{m}^T L \end{pmatrix} \right),
\]

(6)

where the normalization has been changed slightly.

Below we assume that \( N_f \) is even. Since \( K \) and \( L \) are antisymmetric they can be brought to the standard form
\( K = U A U^T \) with \( U \in U(N_f)/[Sp(2)]^{N_f/2} \) and \( \Lambda \) a real antisymmetric matrix with \( \Lambda_{k,k+1} = -\Lambda_{k+1,k} \geq 0 \) and all other matrix elements zero (likewise for \( L \)). For \( \|m\| \ll 1 \) the integration over \( \Lambda \) can be estimated by a saddle-point approximation at \( m = 0 \), while the integration over \( U \) is soft and has to be done exactly at \( m \neq 0 \). Elementary calculation shows that the saddle point is given by \( \Lambda = \Lambda_0 I/\sqrt{2} \) with \( I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). Since \( U \) enters \( Z_0 \) only in the form \( U U^T \), the integration manifold can be confined to \( U(N_f) \). Thus

\[
Z_0(1,m) \sim \int_{U(N_f)} dU dV \, \text{Pr}^N \left( \frac{1}{\sqrt{2}} (V I V^T)^t \frac{-m^*}{\sqrt{2} U U^T} \right) \times \text{Pr}^N \left( \frac{1}{\sqrt{2}} (U I U^T)^t \frac{-m^*}{\sqrt{2} V V^T} \right) \\
\sim \int_{U(N_f)} dU dV \, \exp \left[ -2N \text{Re} \text{tr}(m U I U^T m^T V^* V^t) \right],
\]

(7)

where we expanded the exponent to the first nontrivial order in \( m \). Redefining the variables as \( U = \tilde{U} e^{i(\theta+i\varphi)} \) and \( V = \tilde{V} e^{i(\theta-i\varphi)} \) with \( \tilde{U}, \tilde{V} \in SU(N_f) \) and \( 0 \leq \varphi \leq \pi \), and setting \( A \equiv e^{2i\varphi} \), we obtain

\[
Z_0(1,m) \sim \int_{U(1)} dA \int_{SU(N_f)} d\tilde{U} d\tilde{V} \times \exp \left[ -2N \text{Re} \text{tr}(A^2 \tilde{U} I \tilde{U}^t m^T V^* V^+ \tilde{V}^+) \right],
\]

(8)

which exactly coincides with the finite-volume partition function at large \( \mu \) \cite[Eq. (4.2), Eq. (4.24)]{13} if we identify

\[
N m^2 \equiv \frac{3}{4\pi^2} V_4 \Delta^2 M^2,
\]

(9)

where \( M \) is the dimensionful physical mass matrix and \( V_4 \) is the space-time volume. Therefore, our model \cite{2} depends on the chemical potential \( \mu \) only implicitly through \( \Delta \). (Note that the gap \( \Delta \) depends on \( \mu \) via the relation \( \Delta \sim \frac{\mu}{e^{-1/y}} \) \cite{29} and that \( \Delta \) is related to the diquark condensate via the relation \( \langle qq \rangle \sim \mu^2 \Delta/y \) \cite{13, 30}.) The absence of terms linear in \( m \) in the exponent of \( (8) \) is a consequence of the \( \mathbb{Z}(2)_L \times \mathbb{Z}(2)_R \) symmetry (flipping left-handed and right-handed quarks independently, i.e., \( q_L \rightarrow \pm q_L \) and \( q_R \rightarrow \pm q_R \)) of the diquark pairing \( (q_L q_L) \) and \( (q_R q_R) \), which distinguishes the model \cite{2} from conventional chiral RMTs. The physical limits in which the model \cite{2} describes two-color QCD are given by the inequalities defining the microscopic domain (or lowest order of the \( \varepsilon \)-regime) at high density \cite{13},

\[
\frac{1}{\Delta} \ll L \ll \frac{1}{m_{\mu,\pi,n'}},
\]

(10)

where \( L \) is the linear extent of the Euclidean box and \( m_{\mu,\pi,n'} \) is the mass of the NG bosons at high density.

In addition to the mass term, we could also add a diquark source term \( \propto j \bar{q} g q \) to the model \cite{2}. It can be shown that the partition function at \( N \gg 1 \) then contains a term linear in \( j \), resulting in a nonvanishing diquark condensate in this model. The addition of the diquark source term makes the model somewhat more complicated and will be addressed in future work \cite{31}.

Since in the large-\( N \) limit the partition function \( (2) \) has the same mass dependence as the partition function \( (8) \) of the effective theory, it follows immediately from the calculations performed in \cite{13} that the Leutwyler-Smilga-type sum rules for inverse Dirac eigenvalues of the model \cite{2} agree with those obtained in \cite{13}, as stated earlier.

For \( N_f = 2 \) we can also calculate the partition function at finite \( N \). In \cite{16} the correlation function of two characteristic polynomials was studied in the quenched version of the model \cite{4}, which corresponds exactly to \( Z_0 \) for \( N_f = 2 \) in \cite{4}. The result for \( \bar{\mu} = 1 \) (after a change of notation and correcting typos) reads \cite[Eq. (34)]{16}

\[
Z_\nu(1, \{m_1, m_2\}) \propto \sum_{\ell=0}^{N} \frac{1}{\sqrt{(\ell + \nu)!}} (2N m_1 m_2)^{2\ell + \nu} \\
\times I_\nu(4N m_1 m_2) \text{ as } N \rightarrow \infty.
\]

(11)

The final expression \( I_\nu(4N m_1 m_2) \) can be obtained from \( (8) \) as well, thus verifying our saddle-point result.

We now comment on the case of odd \( N_f \). With \( \bar{\mu} = 1 \), assuming \( m = m_1 \) for simplicity, and allowing for \( \nu \neq 0 \) again, we find for the Pfaffians in the integrand of \( (6) \)

\[
\text{Pr}^N \left( K \right) = \text{Pr}^{N-\nu} \left( \frac{K}{m} \right) \text{Pr}^{\nu} \left( \frac{-m^T}{m} \right)
\]

(12)

\[
\propto \det^{N/2}(m^{2 \nu} + L^T K) \cdot \det^{N \nu / 2}(m^{2 \nu} + L K^+).
\]

Since \( K \) (or \( L \)) is an antisymmetric matrix of odd size, at least one of its eigenvalues must be zero, and evidently the same holds for \( L^+ K \). Thus \( Z_\nu(1, m) \propto |m|^{2N \nu} \), and hence the model with odd \( N_f \) seems to have an ill-defined chiral limit, with a divergent chiral condensate. Further analysis is left to future work.

While so far we have mainly been concerned with the \( \bar{\mu} = 1 \) case, there exists another region of \( \bar{\mu} \) which is nontrivial, yet analytically tractable: the weak non-Hermiticity limit \( (N \gg 1 \text{ with } \bar{\mu}^2 \text{ fixed}) \) \cite{32}, to which we now turn for the sake of completeness. First, define

\[
\mathcal{D} \equiv \begin{pmatrix} K & -P \nonumber \\
L & -L^+ & - \mu \end{pmatrix} \quad \text{and} \quad \mathcal{M} \equiv \begin{pmatrix} 0 & m^T \\
m & 0 \nonumber \end{pmatrix}.
\]

(13)

In the following discussion \( N_f \) is not restricted to be even.

Since both \( \bar{\mu}^2 \) and \( m \) are treated as infinitesimal parameters, the saddle point is estimated at \( \bar{\mu} = 0 \) and \( m = 0 \), i.e., \( \mathcal{D} = U I U^T / 4 \) with \( U \in U(2N_f) \). Taylor expansion in \( \bar{\mu}^2 \) yields

\[
\text{Pr} \left( \sqrt{1 + \bar{\mu}^2} K^+ \\
1 - \bar{\mu}^2 \right) \left( \sqrt{1 + \bar{\mu}^2} K^+ \\
1 - \bar{\mu}^2 \right) \left( \sqrt{1 + \bar{\mu}^2} L \right) \left( \sqrt{1 + \bar{\mu}^2} L \right)
\]

(14)

\[
\sim \text{det} U \exp \left[ 4 \bar{\mu}^2 \text{tr}(\mathcal{D}^T \mathcal{D}^+ \mathcal{M}) + 4 \text{tr}(\mathcal{M}^T \mathcal{M}) \right],
\]

where \( \mathcal{D} \) and \( \mathcal{M} \) are matrices of size \( 2N_f \) and \( 2N_f \times 1 \), respectively.
where we used $\mathcal{D}^\dagger \mathcal{D} = 1/16$ and introduced the $2N_f \times 2N_f$ baryon charge matrix $\mathcal{B} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Isolating the U(1) part by $U \to U e^{i\theta/2N_f}$ we find

$$Z_\nu(\hat{\mu}, \mathbf{m}) \sim \int d\theta \exp \int dU \quad \text{SU}(2N_f)$$

$$\times \exp \left\{ \frac{N}{2} \mu^2 \text{tr} \left( U U^T \mathcal{B}^T (U U^T)^\dagger \mathcal{B} \right) + 2N \text{Re} \text{tr} \left( e^{i\theta/N} (U U^T)^\dagger \mathcal{M} \right) \right\}. \quad (15)$$

The exponent in this partition function exactly reproduces the static part of the chiral Lagrangian [21, Eq. (42)] obtained from symmetry principles, with the production of the usual low-energy constant, $m_\pi$ is the pion mass, and in the last equality the Gell-Mann–Oakes–Renner relation was used. (A similar analysis is given in [20, Eq. (4.15) of 2nd reference] based on the one-matrix formulation.) It is intriguing that a single chiral RMT, Eq. (4), can describe two extreme cases, $\mu \ll \Lambda_{\text{SU}(2)}$ and $\mu \gg \Lambda_{\text{SU}(2)}$, that have totally distinct patterns of spontaneous symmetry breaking, by two different choices of the parameter ($\mu \sim O(1/\sqrt{N})$ and $\mu = 1$, respectively) and two different mappings of the random-matrix quark masses to the physical quark masses (rescaling by $\langle |\langle \psi \psi^\dagger \rangle| \rangle$ and $\Delta$, respectively). We expect that this paper will stimulate work in several directions. (i) While we do not doubt that (2) yields the correct microscopic spectral correlation functions for two-color QCD at high density, for completeness it would be useful to prove the equivalence rigorously using the partially quenched theory. (ii) We are currently investigating the calculation of the microscopic spectral correlation functions using the methods developed in [28]. (iii) It would be stimulating to extend our model to two-color QCD at intermediate densities, the latter being relevant to the phase structure of two-color QCD [20, 33]. The issue of continuity between the hadronic phase and the BCS superfluid phase is of particular interest, in view of the formal similarity of the partition functions at small and large $\mu$ for degenerate masses [13]. (iv) Since two-color QCD with even $N_f$ and pairwise degenerate quark masses is free from the sign problem even at $\mu \neq 0$, it is in principle possible to test the predictions of our model by lattice simulations. (v) Finally, the most important extension from a phenomenological viewpoint is to the color-superconducting phase of three-color QCD [15]. Work in most of these directions is in progress.

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