THE VALUES OF ZETA FUNCTIONS COMPOSED BY THE HURWITZ AND PERIODIC ZETA FUNCTIONS AT INTEGERS

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ABSTRACT. For \( s \in \mathbb{C} \) and \( 0 < a < 1 \), let \( \zeta(s, a) \) and \( \text{Li}_s(e^{2\pi ia}) \) be the Hurwitz and periodic zeta functions, respectively. For \( 0 < a < 1/2 \), put \( Z(s, a) := \zeta(s, a) + \zeta(s, 1-a) \), \( P(s, a) := \text{Li}_s(e^{2\pi ia}) + \text{Li}_s(e^{2\pi i(1-a)}) \), \( Y(s, a) := \zeta(s, a) - \zeta(s, 1-a) \) and \( O(s, a) := -i(\text{Li}_s(e^{2\pi ia}) - \text{Li}_s(e^{2\pi i(1-a)}) \).

Let \( n \geq 0 \) be an integer and \( b := r/q \), where \( q > r > 0 \) are coprime integers. In this paper, we prove that the values \( Z(-n, b) \), \( \pi^{-2n-2}P(2n+2, b) \), \( Y(-n, b) \) and \( O(-n, b) \) are rational numbers, in addition, \( \pi^{-2n-2}Z(2n+2, b) \), \( P(-n, b) \), \( \pi^{-2n-1}Y(2n+1, b) \) and \( O(-n, b) \) are polynomials of \( \cos(2\pi/q) \) and \( \sin(2\pi/q) \) with rational coefficients. Furthermore, we show that \( Z(-n, a) \), \( \pi^{-2n-2}P(2n+2, a) \), \( Y(-n, a) \) and \( \pi^{-2n-1}O(2n+1, a) \) are polynomials of \( 0 < a < 1 \) with rational coefficient, in addition, \( \pi^{-2n-2}Z(2n+2, a) \), \( P(-n, a) \), \( \pi^{-2n-1}Y(2n+1, a) \) and \( O(-n, a) \) are rational functions of \( \exp(2\pi ia) \) with rational coefficients. Note that the rational numbers, polynomials and rational functions mentioned above are given explicitly.

Moreover, we show that \( P(s, a) \equiv 0 \) for all \( 0 < a < 1/2 \) if and only if \( s \) is a negative even integer. We also prove similar assertions for \( Z(s, a) \), \( Y(s, a) \), \( O(s, a) \) and so on. In addition, we prove that the function \( Z(s, |a|) \) appears as the spectral density of some stationary self-similar Gaussian distributions.

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1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

1.1. Special values of the Riemann zeta function. For a complex variable \( s = \sigma + it \), where \( \sigma, t \in \mathbb{R} \), the Riemann zeta function is defined by

\[
\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \sigma > 1.
\]

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According to the integral representation
\[
\zeta(s) = \frac{e^{-i\pi s}\Gamma(1-s)}{2\pi i} \int_C \frac{z^{s-1}}{e^z - 1} \, dz,
\]
where the contour \( C \) starts at infinity on the positive real axis, encircles the origin once in the positive direction, excluding the points \( \pm 2\pi i, \pm 4\pi i, \ldots \), and returns to the positive infinity, we can see that \( \zeta(s) \) is a meromorphic function with a simple pole at \( s = 1 \) with residue 1. Moreover, the function \( \zeta(s) \) satisfies the functional equation
\[
\zeta(1-s) = 2\Gamma(s) \cos \left( \frac{\pi s}{2} \right) \zeta(s)
\]
from the integral representation above. The functional equation and integral representation derive the following (see for example [2, Section 12.12] and [12, Section 2.4]). Note that the \( n \)-th Bernoulli number \( B_n \) is defined in Section 2.1.

**Theorem A.** For \( n \in \mathbb{N} \), we have
\[
\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{(2n)!} B_{2n}.
\]
For every integer \( n \geq 0 \), it holds that
\[
\zeta(-n) = -\frac{B_{n+1}}{n+1}.
\]

Let \( \chi \) be a Dirichlet character and \( L(s, \chi) \) be the Dirichlet \( L \)-function associated to the character \( \chi \). And let \( \chi \) be a primitive character and \( n \) be a natural number. Then it is known that \( \pi^{-n} L(n, \chi) \) is written by a Gauss sum and the generalized Bernoulli number if \( \chi(-1) = (-1)^n \) (see for example [1, Theorem 9.6]). Furthermore, it is also known that \( L(-n, \chi) \) is written by a generalized Bernoulli number when \( n \) is a non-negative integer (see [1, Theorem 9.10]). Obviously, these facts are an analogue of Theorem A. In general, there are no such explicit evaluation formulas at integers for automorphic \( L \)-function, Epstein zeta functions, the prime zeta function and so on. Hence, there are few \( L \)-functions of which the values at both positive and negative integers are expressed by \( \pi \) and (generalized) Bernoulli numbers.

1.2. The Hurwitz and periodic zeta functions. The Hurwitz zeta function \( \zeta(s, a) \) is defined by the series
\[
\zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad \sigma > 1, \quad 0 < a < 1.
\]
The function \( \zeta(s, a) \) is meromorphic and has a simple pole at \( s = 1 \) whose residue is 1 (see for instance [2, Section 12]). Next, we define the periodic zeta function by
\[
\text{Li}_s(e^{2\pi i a}) := \sum_{n=1}^{\infty} \frac{e^{2\pi i a}}{n^s}, \quad \sigma > 1, \quad 0 < a < 1
\]
(see for example [2, Exercise 12.2]). The periodic zeta function \( \text{Li}_s(e^{2\pi i a}) \) with \( 0 < a < 1 \) is analytically continuable to the whole complex plane since the Dirichlet series of \( \text{Li}_s(e^{2\pi i a}) \) converges uniformly in each compact subset of the half-plane \( \sigma > 0 \) when \( 0 < a < 1 \) (see for example [7, p. 20]). Note that \( \zeta(-n, a) \) and \( \text{Li}_{-n}(e^{2\pi i a}) \), where \( n \) is a non-negative integer, are written by the Bernoulli number and Stirling number of the second kind,
are elements of the polynomial ring \( \mathbb{Q}(e^{2\pi i a}) \), where \( n \) is an integer greater than 1.

For \( 0 < a \leq 1/2 \), let

\[
Z(s,a) := \zeta(s,a) + \zeta(s,1-a), \quad P(s,a) := \text{Li}_s(e^{2\pi i a}) + \text{Li}_s(e^{2\pi i (1-a)}),
\]

\[
2Q(s,a) := Z(s,a) + P(s,a) = \zeta(s,a) + \zeta(s,1-a) + \text{Li}_s(e^{2\pi i a}) + \text{Li}_s(e^{2\pi i (1-a)}),
\]

\[
Y(s,a) := \zeta(s,a) - \zeta(s,1-a), \quad O(s,a) := -i(\text{Li}_s(e^{2\pi i a}) - \text{Li}_s(e^{2\pi i (1-a)})),
\]

\[
2X(s,a) := Y(s,a) + O(s,a) = \zeta(s,a) - \zeta(s,1-a) - i(\text{Li}_s(e^{2\pi i a}) - \text{Li}_s(e^{2\pi i (1-a)})).
\]

It should be noted that the functions \( Y(s,a), O(s,a) \) and \( X(s,a) \) are entire (see [8, Section 3.3]). We remark that one has \( Y(s,1/2) \equiv O(s,1/2) \equiv X(s,1/2) \equiv 0 \).

In [8, Section 1.2] and [9, Section 1.1], the following are shown.

**Theorem B.** All real zeros of the function \( Z(s,a) \) are simple and only at the non-positive even integers if and only if \( 1/4 \leq a \leq 1/2 \).

Moreover, all real zeros of the function \( P(s,a) \) are simple and only at the negative even integers if and only if \( 1/4 \leq a \leq 1/2 \).

**Theorem C.** All real zeros of the quadrilateral zeta function \( Q(s,a) \) are simple and only at the negative even integers if and only if \( a_0 < a \leq 1/2 \), where \( a_0 = 0.1183751396... \) satisfies \( Z(1/2,a_0) = P(1/2,a_0) = Q(1/2,a_0) = 0 \).

**Theorem D.** All real zeros of the functions \( Y(s,a), O(s,a) \) or \( X(s,a) \) with \( 0 < a < 1/2 \) are simple and only at the negative odd integers.

It should be emphasised that from the theorems above, the gap between consecutive real zeros of \( Y(s,a), O(s,a), X(s,a) \) with \( 0 < a < 1/2, Z(s,a) \) and \( P(s,a) \) with \( 1/4 \leq a \leq 1/2 \) and \( Q(s,a) \) with \( a_0 < a \leq 1/2 \) is always 2, in other words, the the gaps do not depend on \( a \) just like the Riemann zeta function \( \zeta(s) \).

### 1.3. Main results

In the present paper, we investigate the values of zeta functions \( Z(s,a), P(s,a), Q(s,a), Y(s,a), O(s,a) \) and \( X(s,a) \) at integers (see Theorems 1.1, 1.2, 1.3 and 1.4). Moreover, we show that the zeta function \( Z(s,a) \) is related to some stationary self-similar Gaussian distributions in Proposition 1.5.

When \( a = r/q \) is a rational number, we have the following as an analogue of the fact \( \zeta(2n) \in \mathbb{Q}\pi^{2n} \) proved by Theorem A. Note that the explicit evaluation formulas for the special vales below are given in Section 2.2.

**Theorem 1.1.** Let \( n \) be a non-negative integer and \( q > r > 0 \) be coprime integers. Then

\[
Z(-n,r/q), \quad Y(-n,r/q), \quad \pi^{-2n-2}P(2n+2,r/q), \quad \pi^{-2n-1}O(2n+1,r/q)
\]

are rational numbers. Moreover,

\[
\pi^{-2n-2}Z(2n+2,r/q), \quad P(-n,r/q), \quad \pi^{-2n-1}Y(2n+1,r/q), \quad O(-n,r/q),
\]

\[
\pi^{-2n-2}Q(2n+2,r/q), \quad Q(-n,r/q), \quad \pi^{-2n-1}X(2n+1,r/q), \quad X(-n,r/q)
\]

are elements of the polynomial ring \( \mathbb{Q}[\cos(2\pi/q), \sin(2\pi/q)] \).

Next we prove the following when \( 0 < a < 1/2 \) is irrational. It should be emphasised that the polynomials and rational functions in the theorem below are given explicitly in Section 3.2.
**Theorem 1.2.** Let \( n \) be a non-negative integer. Then
\[
Z(-n, a), \quad \pi^{-2n-2}P(2n + 2, a), \quad Y(-n, a), \quad \pi^{-2n-1}O(2n + 1, a)
\]
are polynomials with rational coefficients of \( 0 < a < 1/2 \). Furthermore,
\[
\pi^{-2n-2}Z(2n + 2, a), \quad P(-n, a), \quad \pi^{-2n-1}Y(2n + 1, a) \quad O(-n, a)
\]
\[
\pi^{-2n-2}Q(2n + 2, a), \quad Q(-n, a), \quad \pi^{-2n-1}X(2n + 1, a), \quad X(-n, a)
\]
are rational functions with rational coefficients of \( \exp(2\pi i a) \).

We can see that \( P(s, a) \) identically vanishes for all \( 0 < a < 1/2 \) if \( s \) is a negative even integer by the functional equation of \( P(s, a) \) (see Theorem B and Lemma 2.3). The next theorem implies that \( P(s, a) \equiv 0 \) for all \( 0 < a < 1/2 \) only if \( s \) is a negative even integer.

**Theorem 1.3.** Let \( s \neq 1 \). Then we have
\[
Z(s, a) \equiv 0 \quad \text{for all} \quad 0 < a < 1/2
\]
if and only if \( s \) is a non-positive even integer. Furthermore it holds that
\[
Q(s, a) \equiv 0 \quad \text{for all} \quad 0 < a < 1/2
\]
if and only if \( s \) is a negative even integer.

Next let \( s \in \mathbb{C} \). Then one has
\[
P(s, a) \equiv 0 \quad \text{for all} \quad 0 < a < 1/2
\]
if and only if \( s \) is a negative even integer. Moreover, we have
\[
Y(s, a) \equiv 0 \quad \text{for all} \quad 0 < a < 1/2
\]
if and only if \( s \) is an odd negative integer. The same statement holds for the zeta functions \( O(s, a) \) and \( X(s, a) \).

On the other hand, we have the following for \( \zeta(s, a) \) and \( \text{Li}_s(e^{2\pi i a}) \).

**Theorem 1.4.** For any \( 1 \neq s \in \mathbb{C} \), there exists \( 0 < a < 1/2 \) such that
\[
\zeta(s, a) \neq 0.
\]
For any \( s \in \mathbb{C} \), there is \( 0 < a < 1/2 \) such that
\[
\text{Li}_s(e^{2\pi i a}) \neq 0.
\]
Moreover, we have the following proposition which implies that \( Z(s, |a|) \) appears as the spectral density of some stationary self-similar Gaussian distributions (for details see Appendix or [11, Section 1]).

**Proposition 1.5.** Let \( P \) be a one-dimensional stationary Gaussian distribution on \( X \) with \( \mathbb{E}x_t = 0 \). Then the distribution \( P \) is an s.s.d, if and only if its spectral density \( \rho_{\lambda}(\alpha) \) has the form
\[
\rho_{\lambda}(\alpha) := C|e^{2\pi i \alpha} - 1|^2Z(\lambda + 1, |\alpha|), \quad -1/2 \leq \alpha \leq 1/2,
\]
where \( C > 0 \) is a constant.

In Section 2 and 3, we prove Theorems 1.1 and 1.2, respectively. We prove Theorems 1.3 and 1.4 in Section 4. In Section 5, we prove Proposition 1.5.
2. Proof of Theorem 1.1

2.1. Bernoulli polynomials and functional equations. We denote by \( B_n(t) \) the Bernoulli polynomial of order \( n \) defined as

\[
\frac{z e^{tz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{z^n}{n!}.
\]

The first few are:

\[
B_0(t) = 1, \quad B_1(t) = t - \frac{1}{2}, \quad B_2(t) = t^2 - t + \frac{1}{6},
\]

\[
B_3(t) = t^3 - \frac{3}{2} t^2 + \frac{1}{2} t, \quad B_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30}.
\]

The following equation is well-known (see for example [2, Exercise 12.11]).

\[
B_n(1-a) = (-1)^n B_n(a), \quad n \geq 0. \tag{2.1}
\]

And we define the \( n \)-th Bernoulli number \( B_n \) by

\[
B_n := B_n(1).
\]

The following are well-known (see for instance [2, Theorems 12.19 and 12.13]).

Lemma 2.1. If \( k \in \mathbb{N} \) and \( 0 < a < 1 \), one has

\[
B_{2k}(a) = (-1)^{k+1} \frac{2(2k)!}{(2\pi)^{2k}} \sum_{m=1}^{\infty} \frac{\cos 2\pi ma}{m^{2k}}, \quad B_{2k-1}(a) = (-1)^{k} \frac{2(2k-1)!}{(2\pi)^{2k-1}} \sum_{m=1}^{\infty} \frac{\sin 2\pi ma}{m^{2k-1}}.
\]

Lemma 2.2. For every integer \( n \geq 0 \), it holds that

\[
\zeta(-n, a) = -\frac{B_{n+1}(a)}{n+1}.
\]

Next we quote the functional equations for \( \zeta(s, a) \) and \( \text{Li}_s(e^{2\pi i a}) \) (see [2, Theorem 12.6 and Exercise 12.2]) and \( Z(s, a) \), \( P(s, a) \), \( Y(s, a) \), \( Q(s, a) \) and \( X(s, a) \) (see [8, Sections 3.3 and 4.2]).

Lemma 2.3. It holds that

\[
\zeta(1 - s, a) = \frac{\Gamma(s)}{(2\pi)^s} \left( e^{-\pi is/2} \text{Li}_s(e^{2\pi i a}) + e^{\pi is/2} \text{Li}_s(e^{2\pi i(1-a)}) \right),
\]

\[
\text{Li}_{1-s}(e^{2\pi i a}) = \frac{\Gamma(s)}{(2\pi)^s} \left( e^{\pi is/2} \zeta(s, a) + e^{-\pi is/2} \zeta(s, 1-a) \right),
\]

\[
Z(1 - s, a) = \frac{2\Gamma(s)}{(2\pi)^s} \cos \left( \frac{\pi s}{2} \right) P(s, a), \quad P(1 - s, a) = \frac{2\Gamma(s)}{(2\pi)^s} \cos \left( \frac{\pi s}{2} \right) Z(s, a),
\]

\[
Y(1 - s, a) = \frac{2\Gamma(s)}{(2\pi)^s} \sin \left( \frac{\pi s}{2} \right) O(s, a), \quad O(1 - s, a) = \frac{2\Gamma(s)}{(2\pi)^s} \sin \left( \frac{\pi s}{2} \right) Y(s, a),
\]

\[
Q(1 - s, a) = \frac{2\Gamma(s)}{(2\pi)^s} \cos \left( \frac{\pi s}{2} \right) Q(s, a), \quad X(1 - s, a) = \frac{2\Gamma(s)}{(2\pi)^s} \sin \left( \frac{\pi s}{2} \right) X(s, a).
\]
2.2. Proof of Theorem 1.1. In this subsection, we give explicit evaluation formulas for $Z(-n, b), P(2n + 2, b), Y(-n, b), O(2n + 1, b), Z(2n + 2, b), P(-n, b), Y(2n + 1, b)$ and $O(-n, b)$, where $n$ is a non-positive integer, $q > r > 0$ are coprime integers and $b := r/q,$ which prove Theorem 1.1.

The next well-known formula plays an important role in the proof of Theorem 1.1.

Lemma 2.4. Let $r, q \in \mathbb{N}$ be coprime and $q > r > 0$. The one has

$$\text{Li}_s(e^{2\pi i rm/q}) = q^{-s} \sum_{m=1}^{q} e^{2\pi i rm/q} \zeta(s, m/q), \quad s \in \mathbb{C}. \quad (2.2)$$

Proof. For readers convenience, we write the proof. Let $|z| = 1$ and $\sigma > 1$. Then it is easy to see that

$$\sum_{l=0}^{\infty} z^{l} = \sum_{m=1}^{q} \sum_{l=0}^{\infty} (q l + m)^{s} = q^{-s} \sum_{m=1}^{q} z^{m} \sum_{l=0}^{\infty} (l + m/q)^{s}. \quad (2.2)$$

By putting $z = e^{2\pi i r/q}$, we have (2.2). □

Proposition 2.5. Let $r, q \in \mathbb{N}$ be coprime and $q > r > 0$. Then, for $n \in \mathbb{N}$, we have

$$Z(-n, r/q) = \frac{(-1)^{n}-1}{n+1} B_{n+1}(r/q), \quad Y(-n, r/q) = \frac{(-1)^{n+1}-1}{n+1} B_{n+1}(r/q),$$

$$P(-n, r/q) = -\frac{2q^n}{n+1} \sum_{m=1}^{q} \cos(2\pi rm/q) B_{n+1}(m/q),$$

$$O(-n, r/q) = -\frac{2q^n}{n+1} \sum_{m=1}^{q} \sin(2\pi rm/q) B_{n+1}(m/q).$$

Proof. By using Lemma 2.2, we have

$$Z(-n, r/q) = \zeta(-n, r/q) + \zeta(-n, 1 - r/q) = -\frac{B_{n+1}(r/q)}{n+1} - \frac{B_{n+1}(1 - r/q)}{n+1}. \quad (2.1)$$

Hence we obtain the first equation of Proposition 2.5 from (2.1). Similarly, we have

$$Y(-n, r/q) = \zeta(-n, r/q) - \zeta(-n, 1 - r/q) = -\frac{B_{n+1}(r/q)}{n+1} + \frac{B_{n+1}(1 - r/q)}{n+1},$$

which implies the second equation of Proposition 2.5. From (2.2), it holds that

$$q^{s} P(s, r/q) = q^{s} \left( \text{Li}_s(e^{2\pi i rm/q}) + \text{Li}_s(e^{2\pi i (q-r)m/q}) \right)$$

$$= \sum_{m=1}^{q} e^{2\pi i rm/q} \zeta(s, m/q) + \sum_{m=1}^{q} e^{-2\pi i rm/q} \zeta(s, m/q) = 2 \sum_{m=1}^{q} \cos(2\pi rm/q) \zeta(s, m/q).$$

By (2.2), we similarly obtain

$$i q^{s} O(s, r/q) = i q^{s} \left( \text{Li}_s(e^{2\pi i rm/q}) - \text{Li}_s(e^{2\pi i (q-r)m/q}) \right)$$

$$= \sum_{m=1}^{q} e^{2\pi i rm/q} \zeta(s, m/q) - \sum_{m=1}^{q} e^{-2\pi i rm/q} \zeta(s, m/q) = 2i \sum_{m=1}^{q} \sin(2\pi rm/q) \zeta(s, m/q).$$

Hence we have the third and fourth formulas of Proposition 2.5 from Lemma 2.2. □
The next proposition is proved by Proposition 2.5 above and the functional equations in Lemma 2.3.

**Proposition 2.6.** Let \( r, q \in \mathbb{N} \) be coprime. Then, for \( n \in \mathbb{N} \), we have

\[
Z(2n, r/q) = (-1)^{n+1} q^{2n-1} \frac{(2\pi)^{2n}}{(2n)!} \sum_{m=1}^{q} \cos(2\pi rm/q) B_{2n}(m/q),
\]

\[
Y(2n - 1, r/q) = (-1)^{n} q^{2n-2} \frac{(2\pi)^{2n-1}}{(2n - 1)!} \sum_{m=1}^{q} \sin(2\pi rm/q) B_{2n-1}(m/q),
\]

\[
P(2n, r/q) = (-1)^{n+1} \frac{(2\pi)^{2n}}{(2n)!} B_{2n}(r/q), \quad O(2n - 1, r/q) = (-1)^{n} \frac{(2\pi)^{2n-1}}{(2n - 1)!} B_{2n-1}(r/q).
\]

**Proof.** From the functional equation of \( P(1 - s, a) \) in Lemma 2.3, we have

\[
P(1 - 2n, a) = \frac{2(2n - 1)!}{(2\pi)^{2n}} \cos(\pi n) Z(2n, a) = (-1)^{n} \frac{2(2n - 1)!}{(2\pi)^{2n}} Z(2n, a).
\]

Thus we obtain the first formula of this proposition from

\[
P(1 - 2n, r/q) = -\frac{2q^{2n-1}}{2n} \sum_{m=1}^{q} \cos(2\pi rm/q) B_{2n}(m/q)
\]

which is proved by Proposition 2.5. Similarly, one has

\[
O(2 - 2n, a) = \frac{2(2n - 2)!}{(2\pi)^{n-1}} \sin\left(\frac{2n - 1}{2} \pi\right) Y(2n - 1, a)
\]

by the functional equation of \( O(1 - s, a) \) in Lemma 2.3. Hence, we have the second equation of Proposition 2.6 and

\[
O(2 - 2n, a) = -\frac{2q^{2n-2}}{2n - 1} \sum_{m=1}^{q} \sin(2\pi rm/q) B_{2n-1}(m/q)
\]

derived from Proposition 2.5.

By the definition of \( P(s, a) \), it holds that

\[
P(2n, a) = \text{Li}_{2n}(e^{2\pi ia}) + \text{Li}_{2n}(e^{2\pi(1-a)}) = 2 \sum_{m=1}^{\infty} \frac{\cos 2\pi ma}{m^{2n}}. \quad (2.3)
\]

In addition, we have

\[
O(2n - 1, a) = \frac{1}{i} \left( \text{Li}_{2n-1}(e^{2\pi ia}) - \text{Li}_{2n-1}(e^{2\pi(1-a)}) \right) = 2 \sum_{m=1}^{\infty} \frac{\sin 2\pi ma}{m^{2n-1}}. \quad (2.4)
\]

from the definition of \( O(s, a) \). Hence, the third and fourth equations in this proposition are prove by (2.3), (2.4) and Lemma 2.1. \( \square \)

We can immediately show the following by the propositions above and the definitions of \( Q(s, a) \) and \( X(s, a) \).
Corollary 2.7. Let \( r, q \in \mathbb{N} \) be coprime. Then, for \( n \in \mathbb{N} \), one has
\[
2Q(-n, r/q) = \frac{(-1)^n - 1}{n+1} B_{n+1}(r/q) - \frac{2q^n}{n+1} \sum_{m=1}^{q} \cos(2\pi rm/q) B_{n+1}(m/q),
\]
\[
2X(-n, r/q) = \frac{(-1)^{n+1} - 1}{n+1} B_{n+1}(r/q) - \frac{2q^n}{n+1} \sum_{m=1}^{q} \sin(2\pi rm/q) B_{n+1}(m/q).
\]

Corollary 2.8. Let \( r, q \in \mathbb{N} \) be coprime. Then, for \( n \in \mathbb{N} \), one has
\[
2Q(2n, r/q) = (-1)^n \frac{(2\pi)^2}{(2n)!} \left( B_{2n}(r/q) + q^{2n-1} \sum_{m=1}^{q} \cos(2\pi rm/q) B_{2n}(m/q) \right),
\]
\[
2X(2n-1, r/q) = (-1)^n \frac{(2\pi)^2}{(2n-1)!} \left( B_{2n-1}(r/q) + q^{2n-2} \sum_{m=1}^{q} \sin(2\pi rm/q) B_{2n-1}(m/q) \right).
\]

Proof of Theorem 1.1. We can prove Theorem 1.1 from Propositions 2.5 and 2.6, Corollary 2.7 and 2.8 and de Moivre’s identity
\[
\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n, \quad n \in \mathbb{N}, \quad \theta \in \mathbb{R},
\]
and fact that Bernoulli polynomials are polynomials with rational coefficients. \(\square\)

3. Proof of Theorem 1.2

3.1. Generalized Euler polynomials. For \( 0 < a < 1 \), we define the generalized Euler polynomial \( E_{c,n}(t) \) by
\[
\frac{(1 + c)e^{tz}}{e^z + c} = \sum_{n=0}^{\infty} E_{c,n}(t) \frac{z^n}{n!}, \quad c := -\exp(2\pi ia).
\]
The polynomial \( E_{c,n}(t) \) above is introduced in [10, Section 4.1]. Note that similar polynomials are defined by Apostol [3] and Frobenius [5]. For simplicity, we put \( b := -(1+c)^{-1} \). Then we have (see [10, Section 4.1])
\[
E_{c,n}(t) = t^n + b \sum_{k=0}^{n-1} \binom{n}{k} E_{c,n}(t), \quad \frac{d}{dt} E_{c,n}(t) = nE_{c,n-1}(t), \quad n > 0,
\]
\[
E_{c,n}(t+1) + cE_{c,n}(t) = (1+c)t^n, \quad E_{1,2n+1}(1/2) = 0,
\]
\[
E_{c,n}(1-t) = (-1)^n E_{c-1,n}(t), \quad E_{c-1,n}(0) = (-1)^{n+1} c E_{c,n}(0).
\]
For instance, one has
\[
E_{c,0}(t) = 1, \quad E_{c,1}(t) = t + b, \quad E_{c,2} = t^2 + 2bt + 2b^2 + b,
\]
\[
E_{c,3}(t) = t^3 + 3bt^2 + (6b^2 + 3b)t + 6b^3 + 6b^2 + b, \quad b := -(1+c)^{-1}.
\]
When \( n \in \mathbb{N} \) and \( 0 < a < 1 \), we define \( F_n(a) \) by
\[
F_n(a) := \sum_{l \in \mathbb{Z}} \frac{1}{(l+a)^{n+1}} = \sum_{l=0}^{\infty} \frac{1}{(l+a)^{n+1}} + (-1)^{n+1} \sum_{l=0}^{\infty} \frac{1}{(l+1-a)^{n+1}}.
\]
We have the following by \( E_{c,n}(0) = (1+c^{-1})n!(2\pi i)^{-n-1}F_n(a) \) proved in [10, Theorem 4.2].
Lemma 3.1. For $n \in \mathbb{N}$, it holds that
\[
F_n(a) = \frac{(2\pi i)^{n+1} E_{c,n}(0)}{n!(1+c^{-1})}, \quad c := -\exp(2\pi i a). \tag{3.1}
\]

By using Yamamoto’s formula (see [13, Proposition 3.2] or [6, p. 17]) and the functional equation of $\text{Li}_{-s}(e^{2\pi ia})$ (see Lemma 2.3), we have the following.

Lemma 3.2. For every integer $n \geq 0$, it holds that
\[
\text{Li}_{-n}(e^{2\pi ia}) = \sum_{r=0}^{n} \frac{r!(n-r)^{n+1} \zeta(n+1,a) + (-i)^{n+1} \zeta(n+1,1-a)}{(1+c)^{r+1}} = \frac{n!}{(2\pi i)^{n+1}} F_n(a) = \frac{n!}{(2\pi i)^{n+1}} F_n(1-a)
\]
which implies the second equal sign of (3.2). We obtain the third equal sign of (3.2) from Lemma 3.1.

3.2. Proof of Theorem 1.2. In this subsection, we give explicit evaluation formulas for $Z(-n,a), P(2n+2,a), Y(-n,a), O(2n+1,a), Z(2n+2,a), P(-n,a), Y(2n+1,a), O(-n,a), Q(2n+2,a), Q(-n,a), X(2n+1,a), X(-n,a)$, where $n$ is a non-positive integer, which prove Theorem 1.2.

Proposition 3.3. For $n \in \mathbb{N}$, we have
\[
Z(2n,a) = \frac{(2\pi i)^{2n} E_{c,2n-1}(0)}{(1+c^{-1})(2n-1)!} = \frac{c(2\pi i)^{2n}}{(2n-1)!} \sum_{r=0}^{2n-1} \frac{(-1)^r r! S(2n-1,r)}{(1+c)^{r+1}}.
\]

For every integer $n \geq 0$, it holds that
\[
Z(-n,a) = -\frac{B_{n+1}(a) + B_{n+1}(1-a)}{n+1} = \frac{(-1)^n - 1}{n+1} B_{n+1}(a).
\]

Proof. Obviously, we have
\[
Z(2n,a) = F_{2n-1}(a).
\]
Hence, the first formula is proved by (3.1), (3.2) and
\[
\frac{E_{c,0}(0)}{1+c^{-1}} = \sum_{r=0}^{n} \frac{(-c^{-1})^r S(n,r)}{(1+c^{-1})^{r+1}} = c \sum_{r=0}^{n} \frac{r!(-1)^r S(n,r)}{(c+1)^{r+1}}. \tag{3.3}
\]
We obtain the second formula from (2.1) and Lemma 2.2. □
Proposition 3.4. For \( n \in \mathbb{N} \), we have
\[
P(2n, a) = (-1)^{n+1} \frac{(2\pi)^{2n}}{(2n)!} B_{2n}(a).
\]
For every integer \( n \geq 0 \), it holds that
\[
P(-n, a) = \frac{1 - (-1)^n}{1 + c^{-1}} E_{c,n}(0) = \frac{1 - (-1)^n}{1 + c^{-1}} \sum_{r=0}^{n} \frac{(-1)^r r! S(n, r)}{(1 + c)^{r+1}}.
\]

Proof. We have the first formula from (2.3) and Lemma 2.1. The second formula is shown by (3.3), the definition of \( P(s, a) \), Lemma 3.2, the formula
\[
P(-n, a) = E_{c-1,n}(0) \frac{1}{1 + c^{-1}}
\]
and the equation \( E_{c-1,n}(0) = (-1)^{n+1} c E_{c,n}(0) \) (see Section 2.3 or [10, (4.3g)]). \( \Box \)

Proposition 3.5. For \( n \in \mathbb{N} \), we have
\[
Y(2n - 1, a) = \frac{(2\pi i)^{2n-1} E_{c,2n-2}(0)}{(1 + c^{-1})(2n - 2)!} = \frac{c(2\pi i)^{2n-1}}{(1 + c^{-1})(2n - 2)!} \sum_{r=0}^{2n-2} \frac{(-1)^r r! S(2n - 2, r)}{(1 + c)^{r+1}}.
\]
For every integer \( n \geq 0 \), it holds that
\[
Y(-n, a) = -\frac{B_{n+1}(a) - B_{n+1}(1 - a)}{n + 1} = \frac{(-1)^{n-1} - 1}{n + 1} B_{n+1}(a).
\]

Proof. The first formula is shown by (3.3), Lemma 3.1 and
\[
Y(2n - 1, a) = F_{2n-2}(a)
\]
if \( n \geq 2 \). The case \( n = 1 \) is shown by
\[
limit_{s \to 1} Y(s, a) = \sum_{n=0}^{\infty} \left( \frac{1}{n + a} - \frac{1}{n + 1 - a} \right) = \psi(1 - a) - \psi(a) = \pi \cot \pi a,
\]
where \( \psi(a) \) the digamma function. We have the second formula of this proposition from (2.1) and Lemma 2.2 again. \( \Box \)

Proposition 3.6. For \( n \in \mathbb{N} \), we have
\[
O(2n - 1, a) = (-1)^{n} \frac{(2\pi)^{2n-1}}{(2n - 1)!} B_{2n-1}(a).
\]
For every integer \( n \geq 0 \), it holds that
\[
O(-n, a) = \frac{1 + (-1)^n}{i(1 + c^{-1})} E_{c,n}(0) = \frac{1 + (-1)^n}{i(c^{-1} + c^{-2})} \sum_{r=0}^{n} \frac{(-1)^r r! S(n, r)}{(1 + c)^{r+1}}.
\]

Proof. We have the first formula of this proposition from (2.4) and Lemma 2.1. We obtain the second formula by Lemma 3.2, the equations
\[
O(-n, a) = \frac{1}{i} \left( \frac{E_{c-1,n}(0)}{1 + c} - \frac{E_{c,n}(0)}{1 + c^{-1}} \right)
\]
and \( E_{c-1,n}(0) = (-1)^{n+1} c E_{c,n}(0) \) again. \( \Box \)

By the propositions above and definitions of \( Q(s, a) \) and \( X(s, a) \), we have the following.
Corollary 3.7. For \( n \in \mathbb{N} \), we have

\[
2Q(2n, a) = \frac{(2\pi i)^{2n}E_{c,2n-1}(0)}{(1 + c^{-1})(2n - 1)!} - (-1)^n \frac{(2\pi)^{2n}}{(2n)!} B_{2n}(a).
\]

For every integer \( n \geq 0 \), it holds that

\[
2Q(-n, a) = \frac{1 - (-1)^n}{1 + c^{-1}} E_{c,n}(0) - \frac{1 - (-1)^n}{n + 1} B_{n+1}(a).
\]

Corollary 3.8. For \( n \in \mathbb{N} \), we have

\[
2X(2n - 1, a) = \frac{(2\pi i)^{2n-1}E_{c,2n-2}(0)}{(1 + c^{-1})(2n - 2)!} + (-1)^n \frac{(2\pi)^{2n-1}}{(2n - 1)!} B_{2n-1}(a).
\]

For every integer \( n \geq 0 \), it holds that

\[
2X(-n, a) = \frac{1 + (-1)^n}{i(1 + c^{-1})} E_{c,n}(0) - \frac{1 + (-1)^n}{n + 1} B_{n+1}(a).
\]

4. Proofs of Theorems 1.3 and 1.4

4.1. Proof of Theorem 1.3. Recall the Hermite formula

\[
\zeta(s, a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + 2 \int_0^\infty \frac{\sin(s \arctan(x/a))}{(x^2 + a^2)s/2(e^{2\pi x} - 1)} dx,
\]

where the integral involved in the formula above converges for all \( s \in \mathbb{C} \) (see for example [14, Section 13.2]). On the other hand, the following equation is well-known:

\[\zeta(s, a) = a^{-s} + \zeta(s, 1 + a). \tag{4.1}\]

Proof of Theorem 1.3 for \( Z(s, a) \) and \( P(s, a) \). From (4.1), for all \( 1 \neq s \in \mathbb{C} \) with \( \sigma > 0 \), it holds that

\[|Z(s, a)| \geq a^{-\sigma} - (1 - a)^{-\sigma} - |\zeta(s, 1 + a)| - |\zeta(s, 2 - a)|.
\]

This inequality and the Hermite formula imply

\[|Z(s, a)| \to \infty, \quad a \to +0. \tag{4.2}\]

Hence for any \( 1 \neq s \in \mathbb{C} \) with \( \sigma > 0 \), there is \( 0 < a < 1/2 \) such that \( Z(s, a) \neq 0 \). Next let \( \sigma > 1 \). Then we have

\[
\int_0^1 Z(1-s, a)^2 da = \left( \frac{2\Gamma(s)}{(2\pi)^s} \cos \left( \frac{\pi s}{2} \right) \right)^2 \int_0^1 P(s, a)^2 da
\]

\[= \left( \frac{2\Gamma(s)}{(2\pi)^s} \cos \left( \frac{\pi s}{2} \right) \right)^2 \int_0^1 \sum_{m,n=1}^\infty \cos 2\pi ma \cos 2\pi na \frac{1}{m^s n^s} da = \left( \frac{2\Gamma(s)}{(2\pi)^s} \cos \left( \frac{\pi s}{2} \right) \right)^2 \frac{\zeta(2s)}{2}.
\]

from the functional equation of \( Z(1-s, a) \) in Lemma 2.3 and the formula

\[2 \cos \alpha \cos \beta = \cos(\alpha + \beta) + \cos(\alpha - \beta), \quad \alpha, \beta \in \mathbb{R}.
\]

According to (4.1) and the Hermite formula, the integral \( \int_0^1 Z(1-s, a)^2 da \) converges when \( \sigma > 1/2 \) since one has

\[
\int_0^1 Z(1-s, a)^2 da \ll_s \int_0^1 a^{2s-2} da = O_s(1).
\]
It is well-known that \( \zeta(2s) \) does not vanish when \( \sigma > 1/2 \) by the Euler product of the Riemann zeta function. Therefore, for any \( s \in \mathbb{C} \) with \( \sigma < 1/2 \) and \( -s \not\in 2\mathbb{N} \cup \{0\} \), there exists \( 0 < a < 1/2 \) such that \( Z(s,a) \neq 0 \) from
\[
0 \neq \int_{0}^{1} Z(1-s,a)^2 da = \left( \int_{0}^{1/2} + \int_{1/2}^{1} \right) Z(1-s,a)^2 da
\]
\[
= \int_{0}^{1/2} Z(1-s,a)^2 da + \int_{1/2}^{1} Z(1-s,1-a)^2 da = 2 \int_{0}^{1/2} Z(1-s,a)^2 da
\]
which is shown by \( Z(s,a) = Z(s,1-a) = \zeta(s,a) + \zeta(s,1-a) \). Thus, we have \( Z(s,a) \equiv 0 \) for all \( 0 < a < 1/2 \) if and only if \( s \) is a non-positive even integer.

When \( \sigma > 1 \), we have
\[
2 \int_{0}^{1/2} P(s,a)^2 da = \int_{0}^{1} P(s,a)^2 da = \int_{0}^{1} \sum_{m,n=1}^{\infty} \frac{\cos 2\pi ma \cos 2\pi na}{m^s n^s} da = \frac{\zeta(2s)}{2}
\]
from \( P(s,a) = P(s,1-a) \). Hence, for any \( s \in \mathbb{C} \) with \( \sigma > 1 \), there is \( 0 < a < 1/2 \) such that \( P(s,a) \neq 0 \). By using the functional equation of \( P(1-s,a) \) in Lemma 2.3 and fact proved above that for any \( 0,1 \neq s \in \mathbb{C} \) with \( \sigma > -1 \), there is \( 0 < a < 1/2 \) such that \( Z(s,a) \) does not vanish, we can see that for any \( s \in \mathbb{C} \) with \( \sigma < 2 \) and \( -s \not\in \{-1,0\} \cup 2\mathbb{N} \), there exists \( 0 < a < 1/2 \) such that \( P(s,a) \neq 0 \). From [8, (4.12)], one has the following equations
\[
P(1,a) = -2 \log(2 \sin \pi a), \quad P(0,a) = -1.
\]
Thus, we have \( P(s,a) \equiv 0 \) for all \( 0 < a < 1/2 \) if and only if \( s \) is a negative even integer. □

Proof of Theorem 1.3 for \( Y(s,a) \) and \( O(s,a) \). When \( \sigma > 0 \), we can show \( |Y(s,a)| \to \infty \) as \( a \to +0 \) by modifying the proof of (4.2). For \( \sigma > 1 \), we have
\[
\int_{0}^{1} Y(1-s,a)^2 da = \left( \frac{2\Gamma(s)}{(2\pi)^s} \sin \left( \frac{\pi s}{2} \right) \right)^2 \int_{0}^{1} O(s,a)^2 da
\]
\[
= \left( \frac{2\Gamma(s)}{(2\pi)^s} \sin \left( \frac{\pi s}{2} \right) \right)^2 \int_{0}^{1} \sum_{m,n=1}^{\infty} \frac{\sin 2\pi ma \sin 2\pi na}{m^s n^s} da = \left( \frac{2\Gamma(s)}{(2\pi)^s} \sin \left( \frac{\pi s}{2} \right) \right)^2 \frac{\zeta(2s)}{2}
\]
by the functional equation of \( Y(1-s,a) \) in Lemma 2.3 and the equation \( 2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta) \). It should be mentioned that the integral \( \int_{0}^{1} Y(1-s,a)^2 da \) converges when \( \sigma > 1/2 \) by the Hermite formula and
\[
\int_{0}^{1} Y(1-s,a)^2 da \ll \int_{0}^{1} a^{2s-2} da = O_s(1).
\]
Furthermore, it holds that
\[
0 \neq \int_{0}^{1} Y(1-s,a)^2 da = 2 \int_{0}^{1/2} Y(1-s,a)^2 da
\]
by \( Y(s,a) = Y(s,1-a) \). Hence we can prove that for any \( -s \not\in 2\mathbb{N}-1 \), there is \( 0 < a < 1/2 \) such that \( Y(s,a) \neq 0 \).

When \( \sigma > 1 \), it holds that
\[
2 \int_{0}^{1/2} O(s,a)^2 da = \int_{0}^{1} O(s,a)^2 da = \int_{0}^{1} \sum_{m,n=1}^{\infty} \frac{\sin 2\pi ma \sin 2\pi na}{m^s n^s} da = \frac{\zeta(2s)}{2}.
\]
Thus, for any \( s \in \mathbb{C} \) with \( \sigma > 1 \), there is \( 0 < a < 1/2 \) such that \( O(s, a) \neq 0 \). From the functional equation of \( O(1-s, a) \) in Lemma 2.3 and the fact proved above that for any \( s \in \mathbb{C} \) with \( \sigma > -1 \), there is \( 0 < a < 1/2 \) such that \( Y(s, a) \) does not vanish, we can see that for any \( s \in \mathbb{C} \) with \( \sigma < 2 \) and \(-s \not\equiv 2\mathbb{N} - 1\), there exists \( 0 < a < 1/2 \) such that \( O(s, a) \neq 0 \). Therefore, we have \( O(s, a) \equiv 0 \) for all \( 0 < a < 1/2 \) if and only if \( s \) is a negative odd integer. \( \square \)

**Proof of Theorem 1.3 for \( Q(s, a) \) and \( X(s, a) \).** According to the functional equation of \( P(s, a) \) in Lemma 2.3, one has

\[
2Q(s, a) = Z(s, a) + P(s, a) = Z(s, a) + \frac{(2\pi)^s}{2\Gamma(s)\cos(\pi s/2)}Z(1-s, a)
\]

\[= Z(s, a) + 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)Z(1-s, a). \quad (4.3)
\]

From (4.1) and the Hermite formula, we have

\[Z(s, a) = a^{-s} + O_s(1), \quad Z(1-s, a) = a^{s-1} + O_s(1)\]

when \( a \to +0 \), \( s \neq 1 \) and \( \sigma > 1/2 \). Hence, for any \( 1 \neq s \in \mathbb{C} \) with \( \sigma > 1/2 \), there exists \( 0 < a < 1/2 \) such that \( Q(s, a) \neq 0 \) by (4.3). From (4.3) and the equation \( Z(1/2 - it, a) = Z(1/2 + it, a) \), one has

\[2Q(1/2 + it, a)\]

\[= Z(1/2 + it, a) + 2(2\pi)^{-1/2-it}\Gamma(1/2 - it)\sin\left(\frac{\pi (1 + 2it)}{4}\right)Z(1/2 + it, a)\]

\[= a^{-1/2-it} + 2(2\pi)^{-1/2-it}\Gamma(1/2 - it)\sin\left(\frac{\pi (1 + 2it)}{4}\right)a^{-1/2+it} + O_t(1).\]

Therefore, for any \( t \in \mathbb{R} \), there exist \( 0 < a < 1/2 \) such that \( Q(1/2 + it, a) \neq 0 \). Hence, for any \( 1 \neq s \in \mathbb{C} \) with \( \sigma \geq 1/2 \), there exists \( 0 < a < 1/2 \) such that \( Q(s, a) \) does not vanish. According to [9, (2.4)], that we have

\[Q(0, a) = -1/2 = \zeta(0) \neq 0.\]

Thus, by using the functional equation of \( Q(s, a) \) in Lemma 2.3, we have that for any \( 1 \neq s \in \mathbb{C} \) with \( -\sigma \not\equiv 2\mathbb{N} \), there exists \( 0 < a < 1/2 \) such that \( Q(s, a) \neq 0 \). We can similarly show that for any \( 1 \neq s \in \mathbb{C} \) with \( -\sigma \not\equiv 2\mathbb{N} - 1 \), there exists \( 0 < a < 1/2 \) such that \( X(s, a) \) does not vanish. \( \square \)

**Proof of Theorem 1.4.** Let \( 1 \neq s \in \mathbb{C} \) with \( \sigma > 0 \). Then, there exists \( 0 < a < 1/2 \) such that \( \zeta(s, a) \neq 0 \) since we have \( |\zeta(s, a)| \to \infty \) as \( a \to +0 \) by modifying the proof of (4.2).

When \( \sigma > 1 \), one has

\[\int_0^1 \zeta(1-s, a)^2 da = \frac{2\Gamma(s)^2}{(2\pi)^{2s}}\zeta(2s)\]

according to the functional equation of \( \zeta(1-s, a) \) in Lemma 2.3. The integral converges absolutely when \( \sigma > 1/2 \) from (4.1) and the Hermite formula. Hence for any \( s \in \mathbb{C} \) with \( \sigma < 1/2 \), there is \( 0 < a < 1 \) such that \( \zeta(s, a) \neq 0 \). In addition, we have

\[\int_0^{1/2} \zeta(1-s, a) da + \int_{1/2}^1 \zeta(1-s, a) da = \int_0^1 \zeta(1-s, a) da = 0\]
by the functional equation of $\zeta(1 - s, a)$ and $\int_0^1 \zeta(1 - s, a) \, da = 0$ for $\sigma > 1$. The integral $\int_0^1 \zeta(1 - s, a) \, da$ converges absolutely when $\sigma > 0$ from (4.1) and the Hermite formula. Thus, for any $s \in \mathbb{C}$ with $\sigma < 1/2$, there is $0 < a < 1/2$ such that $\zeta(s, a)$ does not vanish. Suppose $\sigma > 1$. Then we have

$$
2 \int_0^{1/2} \mathrm{Li}_s(e^{2\pi a}) \mathrm{Li}_s(e^{2\pi i(1 - a)}) \, da
= \int_0^{1/2} \mathrm{Li}_s(e^{2\pi a}) \mathrm{Li}_s(e^{2\pi i(1 - a)}) \, da + \int_{1/2}^1 \mathrm{Li}_s(e^{2\pi a}) \mathrm{Li}_s(e^{2\pi i(1 - a)}) \, da
= \int_0^1 \mathrm{Li}_s(e^{2\pi i a}) \mathrm{Li}_s(e^{2\pi i(1 - a)}) \, da = \zeta(2s).
$$

Hence, for any $s \in \mathbb{C}$ with $\sigma > 1$, there is $0 < a < 1/2$ such that $\mathrm{Li}_s(e^{2\pi a}) \neq 0$. By using (4.1), the Hermite formula and functional equation of $\mathrm{Li}_{1-s}(e^{2\pi i a})$ in Lemma 2.3, we have

$$
\mathrm{Li}_{1-s}(e^{2\pi i a}) = \frac{\Gamma(s)}{(2\pi)^s} \left( e^{\pi i s/2} a^{-s} + O_s(1) \right), \quad a \to +0
$$

when $\sigma > 0$. Hence, for any $s \in \mathbb{C}$ with $\sigma < 1$, there is $0 < a < 1/2$ such that $\mathrm{Li}_s(e^{2\pi i a})$ does not vanish. Furthermore, it holds that

$$
\frac{\partial}{\partial a} \mathrm{Li}_s(e^{2\pi i a}) = 2\pi i \mathrm{Li}_{s-1}(e^{2\pi i a}), \quad 0 < a < 1
$$

which implies

$$
\mathrm{Li}_{1+it}(e^{2\pi i a}) = \frac{1}{2\pi i} \frac{\partial}{\partial a} \mathrm{Li}_{2+it}(e^{2\pi i a}), \quad 0 < a < 1.
$$

On the other hand, one has $\mathrm{Li}_{2+it}(1) = \zeta(2 + it) \neq 0$ and

$$
\mathrm{Li}_{2+it}(e^{\pi i}) = (2^{-1-it} - 1) \zeta(2 + it) \neq \mathrm{Li}_{2+it}(1)
$$

which is proved by

$$
\mathrm{Li}_s(e^{\pi i}) = \sum_{n=1}^{\infty} \frac{(-1)^n}{ns^s} = -\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots = -\zeta(s) + 2 \cdot 2^{-s} \zeta(s), \quad \sigma > 1.
$$

Hence, there is $0 < a < 1/2$ such that $(\partial/\partial a) \mathrm{Li}_{1+it}(e^{2\pi i a}) \neq 0$ by $\mathrm{Li}_{2+it}(1) \neq \mathrm{Li}_{2+it}(e^{\pi i})$. Therefore, for any $t \in \mathbb{R}$, there exists $0 < a < 1/2$ such that $\mathrm{Li}_{1+it}(e^{2\pi i a}) \neq 0$. \hfill \Box

**Remark.** The condition $a \in (0, 1/2)$ in Theorems 1.3 and 1.4 can be replaced by $a \in I$, where $I \subset (0, 1/2)$ is an open interval by the identity theorem and fact that the functions $(s - 1)\zeta(s, a)$ and $\mathrm{Li}_s(e^{2\pi i a})$ are real analytic with respect to $a \in (0, 1/2)$.

5. $Z(s, a)$ and Stationary Self-Similar Distribution

We first define one-dimensional stationary self-similar distributions (see [11, Section 1]). Let $X$ be the space of realizations of a one-dimensional random field $x := \{x_l : l \in \mathbb{Z}\}$. Note that each random variable $x$ takes on real values, and the space $X$ is a vector space. There is a group $\{T_l : l \in \mathbb{Z}\}$ of translations acting naturally on the space $X$. The symbols $\mathfrak{M}$ and $\mathfrak{M}^\mathrm{st}$ denote the space of all probability distributions on $X$ and all stationary distributions on $X$ (namely, distributions invariant with respect to the group $\{T^*_l : l \in \mathbb{Z}\}$ of translations, where $\{T^*_l : l \in \mathbb{Z}\}$ is the group adjoint to $\{T_l : l \in \mathbb{Z}\}$ which acts on $\mathfrak{M}$), respectively.
For each $1 < \lambda < 2$, we introduce the multiplicative semigroup $A_k(\lambda) = A_k$, where $k \in \mathbb{N}$, of linear endomorphisms of $X$ whose action is given by the formula
\[
\tilde{x}_l = (A_k x)_l := \frac{1}{k^{\lambda/2}} \sum_{tk \leq r < (t+1)k} x_r, \quad l \in \mathbb{Z}.
\]

Let $\{A_k : k \in \mathbb{N}\}$ denote the adjoint semigroup acting on the space $\mathcal{M}$, namely,
\[
(A_k^* P)(C) = P(A_k^{-1} C), \quad C \subset X, \quad P \in \mathcal{M}.
\]

**Definition A.** A probability distribution $P \in \mathcal{M}$ is called a self-similar distribution (s.d.) if one has
\[
A_k^* P = P \quad \text{for all } k \in \mathbb{N}.
\]

In other words, an s.d. is a fixed point of the semigroup $\{A_k^* : k \in \mathbb{N}\}$ acting on the space $\mathcal{M}$. On the other hand, it follows from the definition of $A_k$ that $A_k T_{lk} = T_{l}A_k$. Hence, if $P \in \mathcal{M}^{\text{st}}$, then $A_k^* P \in \mathcal{M}^{\text{st}}$ for any $k \in \mathbb{N}$.

**Definition B.** An s.d. distribution $P \in \mathcal{M}$ is called a stationary self-similar distribution (s.s.d.) if $P \in \mathcal{M}^{\text{st}}$.

Now let $P$ be a one-dimensional stationary Gaussian distribution on $X$ with $\mathbb{E}x_l = 0$, where $\mathbb{E}x_l$ is the expected value of $x_l$. Then we have the following.

**Theorem E** ([11, Theorem 2.1]). The distribution $P$ is an s.s.d. if and only if its spectral density $\rho_\lambda(\alpha)$ has the form
\[
\rho_\lambda(\alpha) := C|e^{2\pi i \alpha} - 1|^2 \sum_{n \in \mathbb{Z}} \frac{1}{|n + \alpha|^\lambda + 1}, \quad -1/2 \leq \alpha \leq 1/2,
\]
where $C > 0$ is a constant.

By the next proposition, we can easily see that the spectral density $\rho_\lambda(\alpha)$ above is written by $Z(\lambda, |\alpha|)$.

**Proposition 5.1.** When $\sigma > 1$ and $\alpha \neq 0$, one has
\[
\sum_{n \in \mathbb{Z}} \frac{1}{|n + \alpha|^s} = Z(s, |\alpha|), \quad -1/2 \leq \alpha \leq 1/2.
\]

**Proof.** For $0 < \alpha \leq 1/2$, we have
\[
\sum_{n \in \mathbb{Z}} \frac{1}{|n + \alpha|^s} = \sum_{n=0}^{\infty} \frac{1}{|n + \alpha|^s} + \sum_{n=-\infty}^{\infty} \frac{1}{|n + \alpha|^s} = \sum_{n=0}^{\infty} \frac{1}{(n + \alpha)^s} + \sum_{n=0}^{\infty} \frac{1}{(n + 1 - \alpha)^s} \quad = \zeta(s, \alpha) + \zeta(s, 1 - \alpha) = Z(s, \alpha) = Z(s, |\alpha|).
\]

When $-1/2 \leq \alpha < 0$, it holds that
\[
\sum_{n \in \mathbb{Z}} \frac{1}{|n + \alpha|^s} = \sum_{n=1}^{\infty} \frac{1}{|n + \alpha|^s} + \sum_{n=0}^{\infty} \frac{1}{|n + \alpha|^s} = \sum_{n=0}^{\infty} \frac{1}{(n + 1 - |\alpha|)^s} + \sum_{n=0}^{\infty} \frac{1}{(n + |\alpha|)^s} \quad = \zeta(s, 1 - |\alpha|) + \zeta(s, |\alpha|) = Z(s, |\alpha|).
\]

The equations above imply Proposition 5.1. \qed

**Proof of Proposition 1.5.** This is easily proved by Theorem E and Proposition 5.1. \qed
Remark. Fukasawa and Takabatake [4, p. 1877] considered a sequence of $n$-dimensional centered Gaussian random vectors which covariance functions are characterized by the following spectral density:

$$
\rho^2 \delta_n^2 \Gamma(2H + 1) \sin(\pi H) \left(\frac{2 - 2 \cos(2\pi \alpha)}{2\pi} \right)^{\psi+1} \sum_{n \in \mathbb{Z}} \frac{1}{|n + \alpha|^{1+2H+2\psi}},
$$

where $\rho, \psi > 0$, $0 < H \leq 1$, $n$ is the sample size and $\delta_n$ is the length of sampling intervals. Note that the infinite series above coincides with $Z(1 + 2H + 2\psi, |\alpha|)$ by Proposition 5.1. Moreover, the function $Z(1 + 2H + 2\psi, |\alpha|)$ can be expressed as a rational function with rational coefficients of $\exp(2\pi i |\alpha|)$ from Proposition 3.3 (see also Section 3.1).

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