Global buckling of composite laminated box beam

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Abstract. Based on the theory of first-order shear deformation beam, the equivalent constitutive Eq. of composite laminated box beam is established. The analytical formula of equivalent bending stiffness and equivalent torsional stiffness characterized by the stiffness coefficient of laminated plates are derived. The critical loads of bending instability and torsional instability are obtained by solving the integral buckling equilibrium differential Eq. of composite laminated box beam under axial load. The variation of critical load under different ply layers and different ply angles is analyzed.

1. Introduction
The resin-based fiber reinforced composite material has the advantages of high specific strength, large specific modulus, good corrosion resistance and good fatigue resistance. As a typical structural member subjected to tensile/compression, bending and torsional loads, composite laminated box beam is widely used in bridge engineering [1], aerospace [2, 3], wind power generation [4] and other fields. The composite laminated box beam may undergo bending instability or torsional instability under the axial load of the end, resulting in the momentary loss of bearing capacity of the structure [5]. Therefore, the overall bending and torsion buckling analysis of composite laminated box beam has important engineering significance.

2. Equivalent constitutive Eq.

2.1. Kinematics analysis
The following hypothesis is made using the shear-deformed beam theory that comprehensively considers transverse shear deformation, primary and secondary warping, non-uniform torsion and three-dimensional strain effect [6]:
(1) The cross section does not undergo in-plane deformation;
(2) The transverse shear strains $\gamma_{xy}$ and $\gamma_{xz}$ are evenly distributed in the cross section;
(3) Using a constrained torsion model, the torsion angle $\phi$ is a function of the longitudinal coordinate;
(4) In addition to considering the main warpage displacement along the contour of the centerline, the secondary warping displacement deviating from the contour of the centerline is also considered;
(5) Only small deformations in the linear elastic range occur.

The Cartesian coordinate system (x,y,z) is used as the overall coordinate, the origin of the coordinate is located at the centroid of the cross section; the curvilinear coordinate system (z,s,n) is used as the local coordinate, and the origin of the coordinate is located in the midline of the cross section, n is The normal line direction of the middle line, s is the tangential direction of the middle line, as shown in Fig. 2.
According to the above assumptions, the displacement field at any point on the beam section can be obtained as [7, 8]:

\[
\begin{align*}
    &u(x, y, z) = u_0(z) - y\phi(z) \\
    &v(x, y, z) = v_0(z) + x\phi(z) \\
    &w(x, y, z) = w_0(z) + \theta_x(z)[y(s) - n \frac{dx}{ds}] \\
    &+ \theta_y(z)[x(s) + n \frac{dy}{ds}] - \phi'(z)[F_w(s) + na(s)]
\end{align*}
\]  

(1)

Where

\[
\begin{align*}
    &\theta_x(z) = \gamma_{yz}(z) - v_0'(x) \\
    &\theta_y(z) = \gamma_{zx}(z) - u_0'(x) \\
    &a(s) = -y(s) \frac{dy}{ds} - x(s) \frac{dx}{ds} \\
    &F_w(s) = \int_0^s \left[ r_n(s) - \psi(s) \right] ds = \int_0^s \left[ r_n(s) - \frac{F(s)}{t(s)} \right] ds
\end{align*}
\]  

(2)

\(u, v, w\) are the displacements along the \(x, y,\) and \(z\) directions of the coordinate axis. \(\theta_x(z), \theta_y(z)\) and \(\varphi(z)\) are the rotation angles around the coordinate axes \(x, y,\) and \(z,\) respectively. \(u_0(z), v_0(z)\) and \(w_0(z)\) are rigid body displacements in three directions, and the direction is as shown in Fig. 2. \(\gamma_{yz}(z)\) and \(\gamma_{zx}(z)\) are two transverse shear strains. When neglecting transverse shear and out-of-plane warping, \(\gamma_{yz} = \gamma_{zx} = 0,\) the model is simplified to the Euler-Bernoulli beam model without considering shear deformation. \(a(s)\) is the height of the right triangle in the geometric relationship, \(F_w(s)\) is the generalized fan coordinate, \(\psi(s)\) is the warping function, \(F(s)\) is the St. Venant torsional shear flow, For a box section with a wall thickness of \(T,\) a section height of \(h,\) and a section width of \(b:\)

\[
\psi(s) = \frac{F(s)}{t(s)} = \frac{b_1 b_2}{b_1 + b_2}, \quad b_1 = h - T, \quad b_2 = b - T
\]  

(3)
2.2. Geometric Eq.
From hypothesis (1) and (5), the strain field is obtained.

Axial strain:

\[ \varepsilon_z(z, s, n) = \varepsilon_0(z, s) + n\varepsilon_1(z, s) \]  \hspace{1cm} (4)

\( \varepsilon_0(z, s) \) and \( \varepsilon_1(z, s) \) are the strain caused by primary and secondary warping, respectively, which is calculated by the following formula:

\[ \begin{align*}
\varepsilon_0(z, s) &= \nu_0'(z) + \theta_0'(z)x(s) + \theta_0'(z)y(s) - \phi(z)F_z(s) \\
\varepsilon_1(z, s) &= \theta_1'(z)\frac{dy}{ds} + \phi(z)\alpha(s)
\end{align*} \]  \hspace{1cm} (5)

Circumferential shear strain:

\[ \gamma_{sz}(x, s, n) = \gamma_{sz}^0 dx + \gamma_{sz}^1 ds + \gamma_{sz}^2 \left[ u_0 + \theta_0(z) \right] + \gamma_{sz}^3 \left[ v_0 + \theta_0(z) \right] + \psi(s)\phi(z) \]  \hspace{1cm} (6)

Transverse shear strain:

\[ \gamma_{nz}(z, s, n) = \gamma_{nz}^0 dy - \gamma_{nz}^1 dz + \gamma_{nz}^2 \left[ u_0 + \theta_0(z) \right] - \gamma_{nz}^3 \left[ v_0 + \theta_0(z) \right] \]  \hspace{1cm} (7)

2.3. Equivalent stiffness calculation
The constitutive Eq. of the composite single layer is [9]:

\[ \begin{bmatrix} \sigma_z \\ \tau_{sz} \\ \tau_{nz} \end{bmatrix} = \begin{bmatrix} C_{11}^* & C_{12}^* & 0 \\ C_{12}^* & C_{22}^* & 0 \\ 0 & 0 & C_{33}^* \end{bmatrix} \begin{bmatrix} \varepsilon_z \\ \gamma_{sz} \\ \gamma_{nz} \end{bmatrix} \]  \hspace{1cm} (8)

Where

\[ C_{ij}^* = \frac{Q_{ij}}{Q_{zz}}, \quad C_{12}^* = \frac{Q_{12}}{Q_{zz}}, \quad C_{22}^* = \frac{Q_{22}}{Q_{zz}}, \quad C_{33}^* = \frac{Q_{33}}{Q_{zz}} \]  \hspace{1cm} (9)

\( \overline{Q_{ij}} \) is the reduced modulus component in the classical laminate theory, which is calculated by:

\[ \overline{Q_{ij}} = C_{ij}^* - \frac{C_{13}^* C_{3j}^*}{C_{33}^*} (i, j = 1, 2, 6) \]  \hspace{1cm} (10)

\( \overline{C_{ij}} \) is the stiffness coefficient of the single-layer three-dimensional constitutive relation of composite materials, a more detailed explanation can be found in Ref. [9]. It can be seen that the stiffness coefficient of the composite single-layer constitutive Eq. can be expressed by the reduced modulus component, which can simplify the following Integral operation. The geometric relationship between \((x, y, z)\) and \((z, s, n)\) at any point on the midline is:

\[ dx = ds\cos\theta, dy = ds\sin\theta \]  \hspace{1cm} (11)
Integrate the stress in the cross-section of the beam along the section to obtain:

\[ N_z = \int (\tau_z \cos \theta + \tau_{nz} \sin \theta) dsdn, \]
\[ V_x = \int (\tau_x \cos \theta + \tau_{nx} \sin \theta) dsdn, \]
\[ V_y = \int (\tau_y \sin \theta - \tau_{ny} \cos \theta) dsdn, \]
\[ M_x = \int \sigma_x (y - nz \sin \theta) dsdn, \]
\[ M_y = \int \sigma_x (x + nz \cos \theta) dsdn, \]
\[ M_z = \int \tau_x y(s) dsdn \]
\[ M_{o} = \int \sigma_z[F_w(s) + na(s)] dsdn \]  

(12)

where \( N_z \) is the axial force. \( V_x \) and \( V_y \) are the shear forces in the \( x \) and \( y \) directions respectively. \( M_x, M_y, \) and \( M_z \) are the bending moments around the \( x, y, \) and \( z \) axes, respectively. \( M_{o} \) is the double moment generated by the constrained torsional normal stress. The relationship between internal force and displacement is

\[
\begin{bmatrix}
N_z \\
M_x \\
M_y \\
V_x \\
V_y \\
M_{o}
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17}
\end{bmatrix}
\begin{bmatrix}
w_0 \\
\theta_x \\
\theta_y \\
w_0' + \theta_x \\
w_0' + \theta_y \\
\phi'
\end{bmatrix}
\]

(13)

\( a_{ij} \) is called the equivalent stiffness coefficient, which can be derived from the integral of Eq.(12).

Circumferentially uniform stiffness (CUS) is a typical composite pipe fitting method with consistent hoop stiffness [10]. The composite laminated box beam can be formed by CUS layering by lamination or winding method, which can reduce the number of times of prepreg cutting, the initial defects are small, and the overall mechanical properties are superior. Therefore, this paper focuses on the stiffness performance of CUS composite laminated box beam. According to the symmetry of \( \theta(y)=\theta(-y) \) of the CUS ply configuration, the internal force-strain relationship can be simplified as:

\[
\begin{bmatrix}
N_z \\
M_x \\
M_{y}
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{17} & \left[\begin{array}{c}
w_0' \\
\phi'
\end{array}\right]
\end{bmatrix}
\]

(14)

\[
\begin{bmatrix}
M_x \\
V_x \\
M_{o}
\end{bmatrix} =
\begin{bmatrix}
a_{22} & 0 & 0 & a_{25} & 0 \\
0 & a_{33} & a_{34} & 0 & 0 \\
0 & a_{34} & a_{44} & 0 & 0 \\
a_{25} & 0 & 0 & a_{55} & 0 \\
0 & 0 & 0 & 0 & a_{66}
\end{bmatrix}
\begin{bmatrix}
\theta_x' \\
\theta_y' \\
w_0' + \theta_x \\
w_0' + \theta_y \\
\phi'
\end{bmatrix}
\]

(15)

It can be found that for the composite laminated box beam of CUS ply, the stiffness coefficient \( a_{17} \) characterizing the tensile-torsional coupling effect and the stiffness coefficient \( a_{25}, a_{34} \) characterizing the bending-shear coupling effect are not zero. Assuming that the box beam works under pure bending, the equivalent bending stiffness of the \( x \)-axis and \( y \)-axis is obtained by inverting Eq. (15) [10,11]

\[ [EI] = a_{33} \frac{a_{44}}{a_{34}} [EI] = a_{22} \frac{d_{33}}{d_{55}} \]

(16)
Assuming that working under pure torsion, the equivalent torsional stiffness can be obtained by inverting Eq. (14).

\[
[GJ] = a_{11} - a_{11}^2
\]  

(17)

3. Global buckling analysis

3.1. Governing differential Eq.

According to the classical thin-walled structural mechanics, the composite laminated box beam will undergo bending instability or torsional instability under axial load. The differential Eq. of buckling equilibrium is [12]:

\[
\begin{bmatrix}
-\{EI\}_x & 0 & 0 & -\{EI\}_x \\
0 & \{EI\}_y & 0 & \{EI\}_y \\
0 & 0 & \{GJ\} & \{Pz/\pi\} \\
-\{EI\}_x & \{EI\}_y & \{GJ\} & \{Pz/\pi\}
\end{bmatrix}
\begin{bmatrix}
\phi_x''(z) \\
\phi_y''(z) \\
\varphi''(z) \\
\varphi''(z)
\end{bmatrix} = 0
\]  

(18)

where \(P^0\) is the critical buckling load; \([EI]_x, [EI]_y\) are the equivalent bending stiffnesses for the x-axis and the y-axis, respectively, calculated by Eq. (16); \([GJ]\) is the equivalent torsional stiffness, Eq. (17) is calculated; \([EI]_w = a_{06},\) which is the equivalent warpage stiffness; \(I_p\) is the polar moment of inertia of the section; the corresponding boundary condition is substituted to obtain the solution of Eq. (18):

\[
P_x = \frac{\pi^2 [EI]_x}{(\mu_x l)^2}, \quad P_y = \frac{\pi^2 [EI]_y}{(\mu_y l)^2}, \quad P_0 = A \frac{\pi^2 [EI]_w}{I_p} + [GJ]
\]  

(19)

Where \(P_x\) and \(P_y\) are the critical load of bending instability around the x-axis and y-axis respectively; \(P_0\) is the critical load of torsional instability; \(\mu_x, \mu_y\) and \(\mu_0\) are length factors, for the simply supported beam, \(\mu_x = \mu_y = \mu_0 = 1\).

3.2. Numerical examples analysis

In order to compare the size and variation of \(P_x, P_y\) and \(P_0\), a composite laminated box beam is introduced as a numerical example. The paving parameters and geometric parameters are shown in Table 1. The boundary conditions are hinged at both ends. The mechanical properties of the material are: \(E_1 = 135\text{ GPa}, E_2 = E_3 = 8.8\text{ GPa}, G_{12} = G_{13} = 4.47\text{ GPa}, G_{23} = 3.0\text{ GPa},\) \(v_{12} = v_{13} = v_{23} = 0.33.\)

| No.  | Lay-ups | \(b\) | \(h\) | \(T\) | \(t\) |
|------|---------|-------|-------|------|------|
| CRL1 | \([\theta]_{50}\) | 164   | 230   | 10   | 0.2  |
| CRL2 | \([\pm \theta]_{25}\) |       |       |      |      |

Firstly, the length \(l=2600\text{ mm}\) is fixed, the global buckling critical load of the composite laminated box beam with the ply angle \(\theta\) is calculated by Eq. (19), as shown in Fig. 3. It can be seen that the bending instability critical load \(P_x, P_y\) decreases with the increase of \(\theta\), and takes the maximum value at \(\theta=0^\circ\); the torsional instability load \(P_0\) is much larger than the bending instability critical load, \(P_0\) takes maximum value at \(\theta=45^\circ\).
Let $\theta=30^\circ$, $15^\circ$, obtain the curve of the overall buckling load of the composite laminated box beam with the slenderness ratio $\lambda_x$, as shown in Fig. 4. It can be seen that the bending instability critical load $P_x$, $P_y$ decreases with the increase of the slenderness ratio $\lambda_x$; $\lambda_x \geq 30$, $P_x$ and $P_y$ change less; the torsional instability load $P_0$ does not change with the slenderness ratio $\lambda_x$; $\lambda_x \geq 15$, $P_0$ is much larger than the bending instability critical load; compared with the CRL1 layup, the overall buckling critical load of the CRL2 laminated composite laminated box beam is larger.
4. Conclusion
(1) By combining the shear-deformed beam theory with the classical laminated plate theory, the analytical formula of the equivalent bending stiffness and the equivalent torsional stiffness of the composite laminated box beam can be obtained. The analytical formula is composed of the stiffness coefficient of the laminated plate and the geometrical dimensions of the box beam section.
(2) Bending instability or torsional instability of composite laminated box beam under axial load, increasing the 0° ply ratio can increase the critical load of bending instability, and increasing the 45° ply ratio can improve the torsion loss. Stable critical load.
(3) The critical load of bending instability decreases with the increase of slenderness ratio, and its value is much smaller than the critical load of torsional instability, indicating that the composite laminated box beam will first bend and destabilize under the axial compression load.

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