VARIATIONAL REGULARIZATION THEORY BASED ON IMAGE SPACE APPROXIMATION RATES

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Abstract. We present a new approach to convergence rate results for variational regularization. Avoiding Bregman distances and using image space approximation rates as source conditions we prove a nearly minimax theorem showing that the modulus of continuity is an upper bound on the reconstruction error up to a constant. Applied to Besov space regularization we obtain convergence rate results for $0, 2, q$- and $0, p, p$-penalties without restrictions on $p, q \in (1, \infty)$. Finally we prove equivalence of Hölder-type variational source conditions, bounds on the defect of the Tikhonov functional, and image space approximation rates.

Key words. regularization, convergence rates, real interpolation, source conditions, converse results

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1. Introduction. The subject of this paper are ill-posed equations $Ax = g$ with $A$ a bounded linear operator mapping from a Banach space $X$ to a Hilbert space $Y$. We analyze approximations of an unknown $x \in X$ given noisy, indirect observations $g^\delta$ satisfying $\|g^\delta - Ax\|_Y \leq \delta$ with a fixed noise level $\delta > 0$. In this context ill-posedness means that the unknown $x$ does not depend continuously on the observations $g^\delta$. As a naive application of the inverse of $A$ may therefore amplify the noise indefinitely regularization is needed to compute stable approximations of the unknown. Here, we study variational regularization with a convex penalty $\mathcal{R}$ defined on $X$. More precisely, we consider the Tikhonov functional given by

$$T_\alpha(x, g) := \frac{1}{2\alpha}\|g - Ax\|_Y^2 + \mathcal{R}(x) \quad \text{for } \alpha > 0, x \in \text{dom}(\mathcal{R}) \text{ and } g \in Y$$

and denote its set of minimizers by

$$R_\alpha(g) := \text{argmin}_{x \in \text{dom}(\mathcal{R})} T_\alpha(x, g) \subseteq \text{dom}(\mathcal{R}).$$

A central aim of regularization theory are upper bounds on the distance $L(x, \hat{x}_\alpha)$ between $x$ and estimators $\hat{x}_\alpha \in R_\alpha(g^\delta)$ with respect to some loss function $L$. For ill-posed problems the convergence of $\hat{x}_\alpha$ to $x$ for $\delta \to 0$ can be arbitrarily slow in general. Therefore, upper bounds on the error require regularity conditions on the true solution $x$, which are referred to as source conditions in regularization theory. The name comes from the first such conditions in a Hilbert space setting, $x = (A^*A)^{\nu/2}\omega, \nu > 0$, where $\omega$ is referred to as source generating $x$. This condition implies the convergence rate $\|x - \hat{x}_\alpha\|_X = O(\delta^{\frac{\nu}{\nu+1}})$ in the Hilbert space norm that defines the penalty. In [21] convergence rates in Hilbert scales are proven under source conditions of the form $x = \varphi(A^*A)\omega$ for more general functions $\varphi$. Nevertheless, we restrict our attention to Hölder-type convergence rates in this paper. A generalization of the above source condition for $\nu = 1$ to convex or Banach space penalties is given by source-wise representations

$$A^*\omega \in \partial\mathcal{R}(x) \quad \text{for some } \omega \in Y$$

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leading to the convergence rate $O(\delta)$ in the Bregman divergence of $\mathcal{R}$ (see [3]). Slower rates of convergence in Banach space settings can be shown under variational source conditions [23, 24] or under approximate source conditions [13, 14]. We refer to [6] for a comparison of the latter two concepts. Recently in [11] convergence rates are shown under the condition $(A^*A)^\nu \omega \in \partial \mathcal{R}(x)$ for convex penalties defined on Hilbert spaces. In [16] upper bounds on $T_\alpha(x,Ax) - T_\alpha(x_\alpha,Ax)$ (defect of the Tikhonov functional) in terms of $\alpha$ are used as a source condition.

In this work we consider Hölder-type image space approximation rates, i.e. bounds of the form

\begin{equation}
\|Ax - Ax_\alpha\|_Y \leq c\alpha^\nu \quad \text{for all } \alpha > 0, x_\alpha \in R_\alpha(Ax)
\end{equation}

for some $\nu \in [0, \infty)$ and $c \geq 0$. These will play the role of source conditions. In many situations these kind of bounds can be proven under source conditions. (see e.g. [18, Thm. 2.3], [16, Prop. 6]).

We first prove a bound on $L(x, \tilde{x}_\alpha)$ uniformly on the set of all $x$ satisfying (1.2) in terms of the modulus of continuity. One main advantage of our analysis is the flexibility in the choice of the loss function $L$. Then for penalties given by Banach space norm powers we work out a characterization of condition (1.2) in terms of real interpolation spaces. This leads to convergence rate results with regularity conditions given by real interpolation spaces. As examples we consider weighted $\ell^p$-regularization, Besov space 0, $p,p$ and 0, 2, $q$-regularization. Our approach seems to allow for the first time to obtain minimax optimal rates for all $p$ resp. $q$ in $(1, \infty)$. Finally we compare condition (1.2) to source conditions used in the literature. We prove equivalence of (1.2), Hölder-type variational source conditions (used e.g. in [19, 7]) and Hölder-type bounds on the defect of the Tikhonov functional. In particular, this equivalence yields a characterization of (1.2) that does not directly depend on the minimizers $x_\alpha$.

The structure of the paper is as follows: In Section 2 we present our main results. Section 3, 4 and 5 are devoted for the proofs of the main results and establish some new techniques which may be of some independent interest in variational regularization theory. We finish with an outlook where we also discuss limitations of the present work.

2. Main results. To give an overview over the main results of this paper, we present and discuss the theorems in their precise mathematical form and refer to the proofs given in the sections 3, 4 and 5.

2.1. Minimax convergence rates.

Assumption 2.1. Let $\tau$ be a topology such that $(X, \tau)$ is a locally convex Hausdorff space and $\mathcal{R}: X \to (-\infty, \infty]$ a proper, convex function. We assume that the sublevel set $\{x \in X: \mathcal{R}(x) \leq \lambda\}$ is $\tau$-compact for all $\lambda \in \mathbb{R}$. Note that this implies that $\mathcal{R}$ is lower semi-continuous on $(X, \tau)$.

Let $Y$ be a Hilbert space and $A: X \to Y$ a linear, $\tau$-to-weak continuous operator.

Assumption 2.1 implies $\tau$-compactness of the sublevel sets of the Tikhonov functional. Using the finite intersection property of these sets one can show that $R_\alpha(g)$ is nonempty for all $g \in Y$. Furthermore, for every $g \in \text{im}(A)$ there exist a (possibly not unique) $\mathcal{R}$-minimal $x \in X$ with $Ax = g$, i.e. $\mathcal{R}(x) \leq \mathcal{R}(z)$ for all $z \in X$ with $Az = g$.

Let $\nu \in [0, \infty)$ and $g > 0$. We define

\begin{equation}
\begin{aligned}
g_\nu: X &\to [0, \infty] \quad \text{by} \quad g_\nu(x) = \sup \{\alpha^{-\nu}\|Ax - Ax_\alpha\|_Y : \alpha > 0, x_\alpha \in R_\alpha(Ax)\}, \\
K_\nu^g = \{x \in X : g_\nu(x) \leq g\} \quad \text{and} \quad K_\nu := \{x \in X : g_\nu(x) < \infty\}.
\end{aligned}
\end{equation}
Note that \( x \in K_\nu \) if and only if a bound (1.2) holds true and \( \varrho_\nu(x) \) is the smallest possible constant \( c > 0 \).

Let \( L : \mathcal{X} \times \mathcal{X} \to [0, \infty] \) satisfy the triangle inequality. We use \( L \) to measure the reconstruction error.

The first and central result is a uniform bound in \( K_\nu \) on \( L(x, \hat{x}_\alpha) \) with \( \hat{x}_\alpha \in R_\alpha(g^\delta) \) in terms of the modulus of continuity. Recall that the latter is given by

\[
(2.2) \quad \Omega(\delta, K) := \sup \{ L(x_1, x_2) : x_1, x_2 \in K \text{ with } \|Ax_1 - Ax_2\|_Y \leq \delta \}
\]

for a subset \( K \subset \mathcal{X} \).

We consider two parameter choice rules for the regularization parameter \( \alpha \). An apriori rule requiring prior knowledge of the parameter \( \nu \) in (1.2) characterizing the regularity of the unknown \( x \), and the discrepancy principle as most well-known a-posteriori rule.

**Theorem 1.** Let \( \nu \in (0, 1] \) and \( \nu, \alpha > 0 \). Suppose \( x \in K_\nu \). Let \( \alpha > 0 \) and \( \hat{x}_\alpha \in R_\alpha(g^\delta) \).

1. (apriori rule) Let \( c_r \geq c_l > 0 \). If \( c_l \varrho^{-\frac{1}{\nu}} \delta^\frac{1}{\nu} \leq \alpha \leq c_r \varrho^{-\frac{1}{\nu}} \delta^\frac{1}{\nu} \), then

\[
L(x, \hat{x}_\alpha) \leq \Omega(c_1 \delta, K_{\nu^2}^2)
\]

with \( c_1 := 1 + c_r^{\nu} \) and \( c_2 := 2 + c_l^{-\nu} \).

2. (discrepancy principle) Let \( C_D > c_D > 1 \). If \( c_D \delta \leq \|g^\delta - Ax\|_Y \leq C_D \delta \), then

\[
L(x, \hat{x}_\alpha) \leq \Omega(d_1 \delta, K_{\nu^2}^2)
\]

with \( d_1 := 1 + C_D \) and \( d_2 := 2 + (c_D - 1)^{-1} \).

The proof of Theorem 1 can be found in Subsection 3.5. Under mild assumptions Theorem 1 gives rise to an almost minimax result in the following manner. Recall that the worst case error of a reconstruction map \( R : \mathcal{Y} \to \mathcal{X} \) on a set \( K \subset \mathcal{X} \) is given by

\[
\Delta_R(\delta, K) := \sup \{ L(x, R(g^\delta)) : x \in K, g^\delta \in \mathcal{Y} \text{ with } \|g^\delta - Ax\|_Y \leq \delta \}
\]

and satisfies the lower bound

\[
(2.3) \quad \Delta_R(\delta, K) \geq \frac{1}{2} \Omega(2\delta, K)
\]

(see [5, Rem. 3.12], [28, Lemma 3.11] or [4, 4.3.1. Prop. 1]). Let \( \overline{R}_\alpha : \mathcal{Y} \to \mathcal{X} \) satisfy \( \overline{R}_\alpha(g^\delta) \in R_\alpha(g^\delta) \) for all \( g^\delta \in \mathcal{Y} \) with either \( \alpha = \alpha(\delta) \) satisfying the apriori parameter choice given in Theorem 1.1. or \( \alpha = \alpha(\delta, g^\delta) \) satisfying the discrepancy principle in Theorem 1.2. In the case \( \Omega(\delta, K_\nu^\delta) \sim \delta^e \delta^f \) for some exponents \( e, f > 0 \) this yields a minimax result

\[
\Delta_{\overline{R}_\alpha}(\delta, K_\nu^\delta) \leq C \inf_R \Delta_R(\delta, K_\nu^\delta).
\]

This shows that up to a constant \( C \) no method can achieve a better approximation uniformly on \( K_\nu^\delta \).

Moreover, we would like to highlight the flexibility in the choice of the loss function \( L \). Many recent works in Banach space or convex regularization theory are restricted to error bounds in the Bregman divergence (see e.g. [19], [16], [7], [29]). In some situations the meaning of the Bregman divergence is unclear and lower bounds on the Bregman distance are required to obtain more tangible statements. In [28] these lower bounds cause a restriction on the parameters \( s, p, q \) of the Besov scale. By applying Theorem 1 to Besov space regularization we can overcome these restrictions.
2.2. Convergence rate theory for Banach space regularization. Here we consider \( \mathcal{R}: X \rightarrow [0, \infty) \) given by \( \mathcal{R}(x) = \frac{1}{\theta} \| x \|_X^\theta \) for fixed \( u \in [1, \infty) \). We assume \( X_A \) to be a Banach space with a continuous, dense embedding \( X \subset X_A \) such that \( A \) extends to a norm isomorphism \( A: X_A \rightarrow Y \), i.e. there exists a constant \( M \geq 1 \) such that

\[
\frac{1}{M} \| x \|_{X_A} \leq \| Ax \|_Y \leq M \| x \|_{X_A} \quad \text{for all } \ x \in X_A.
\]

(2.4) Note that injectivity is necessary for (2.4). On the other hand injectivity of \( A: X \rightarrow Y \) suffices for the existence of a space \( X_A \) such that (2.4) holds with \( M = 1 \). (Take the Banach completion of \( X \) in the norm \( x \mapsto \| Ax \|_Y \).

For example, in Besov space settings we will assume \( X_A \) a space with negative smoothness index, and we consider spaces \( X \) with smoothness index 0. Moreover we need the following assumption on \( K_1 \) and \( q_1 \) defined in (2.1). Recall that a quasi-norm satisfies the properties of norm except that the triangle inequality is replaced by \( \| x + y \| \leq c (\| x \| + \| y \|) \) for a constant \( c > 0 \). A complete and quasi-normed vector space is called a quasi-Banach space.

**Assumption 2.2.** Let \( u \in (0, \infty) \). Suppose \( K_1 \) is a vector space and there is a quasi-norm \( \| \cdot \|_{\text{lin}} \) on \( K_1 \) such that \( (K_1, \| \cdot \|_{\text{lin}}) \) is a quasi-Banach space. Moreover assume

\[
\frac{1}{M} q_1(x) \leq \| x \|_{\text{lin}} - 1 \leq M q_1(x) \quad \text{for all } x \in K_1.
\]

This assumption is motivated by the computation of \( K_1 \) for the examples below. Recall that for a quasi-Banach space \( X_S \) with a continuous embedding \( X_S \subset X_A \) and \( \theta \in (0, 1) \) the real interpolation space \( (X_A, X_S)_{\theta, \infty} \) consists of all \( x \in X_A \) such that

\[
\| x \|_{(X_A, X_S)_{\theta, \infty}} := \sup_{t>0} t^{-\theta} K(x, t) < \infty.
\]

Here the \( K \)-functional is given by

\[
K(x, t) := \inf_{z \in X_S} \left( \| x - z \|_{X_A} + t \| z \|_{X_S} \right).
\]

For the definition of the real interpolation spaces \( (X_A, X_S)_{\theta, q} \) for \( q \in (0, \infty) \) we refer to [2].

**Theorem 2** (error bounds). Suppose (2.4) and Assumption 2.2 hold true. If \( X \) is not reflexive, suppose Assumption 2.1 holds true.

Let \( X_L \) be a Banach space with a continuous embedding \( X_L \subset X_A \). Let \( 0 < \xi < \theta < 1 \) and \( \delta, q, \alpha > 0 \) and \( c_\tau \geq c_l > 0 \), \( C_D > c_D > 1 \). Suppose there is a continuous embedding \( (X_A, K_1)_{\xi, 1} \subset X_L \). Assume

\[
x \in (X_A, K_1)_{\theta, \infty} \quad \text{with} \quad \| x \|_{(X_A, K_1)_{\theta, \infty}} \leq \varrho.
\]

Let \( \hat{x}_\alpha \in R_{\alpha}(g^\delta) \). There exists a constant \( C > 0 \) independent of \( x, \delta \) and \( \varrho \) such that whenever \( \alpha \) satisfies either

\[
c_l q \frac{\alpha^\frac{1}{\tau} \delta^{(1-\theta)(\alpha-1)+\delta}}{\theta^\frac{1}{\theta} \alpha^\frac{1}{\tau} \delta^{(1-\theta)(\alpha-1)+\delta}} \leq \alpha \leq c_\tau q \frac{\alpha^\frac{1}{\tau} \delta^{(1-\theta)(\alpha-1)+\delta}}{\theta^\frac{1}{\theta} \alpha^\frac{1}{\tau} \delta^{(1-\theta)(\alpha-1)+\delta}} \quad \text{or} \quad c_D \delta \leq \| g^\delta - Ax_\alpha \|_Y \leq C_D \delta
\]

the bound

\[
\| x - \hat{x}_\alpha \|_{X_L} \leq C q^\delta \delta^{\frac{1}{\delta} - \frac{1}{\delta}}
\]

holds true.
We refer to Subsection 4.4 for the proof of Theorem 2.

Remark 2.3. The statement of the theorem remains valid in the limiting case \( \theta = 1 \) where the source condition in terms of \((\mathcal{X}_A, K_1)_{\theta, \infty}\) has to be replaced by simply \( x \in K_1 \) with \( \|x\|_{\text{lin}} \leq \varrho \). Here the apriori rule is \( \alpha \sim \varrho^{-(\nu-1)\delta} \).

We illustrate the impact of this result by applying it to three more concrete Banach space regularization setups.

Example 1: Weighted \( p \)-Norm Penalization. Let \( A \) be a countable index set, \( p \in (0, \infty) \) and \( \omega = (\omega_j)_{j \in A} \) a sequence of positive reals. We consider weighted sequence spaces \( \ell^p_\omega \) defined by

\[
\ell^p_\omega = \{ x \in \mathbb{R}^A : \|x\|_{\ell^p_\omega} < \infty \} \quad \text{with} \quad \|x\|_{\ell^p_\omega} = \sum_{j \in A} \omega_j |x_j|^p.
\]

We assume that the forward operator maps a weighted \( \ell^2 \)-space isomorphically to the image space \( Y \). More precisely, we suppose that (2.4) holds true with \( X_A = \ell^2_\omega \) for \( \omega = (\omega_j)_{j \in A} \) a sequence of positive real numbers.

Moreover let \( p \in (1,2) \) and \( \mathfrak{r} = (\mathfrak{r}_j)_{j \in A} \) a sequence of weights such that \( \mathfrak{r}^{-1} \) is bounded. We consider \( X = \ell^2_\mathfrak{r} \subset \ell^2_\omega \) (see [20, Prop. A.1]) with \( \mathcal{R}(x) = \frac{1}{p} \|x\|_{\ell^p_\omega}^p \).

Furthermore we introduce weighted weak \( \ell^p \)-spaces. For \( \mu = (\mu_j)_{j \in A} \) and \( \nu = (\nu_j)_{j \in A} \) sequences of positive reals and \( t \in (0, \infty) \) those are defined by the following quasi-norms

\[
\ell^t_{\mu,\nu} = \{ x \in \mathbb{R}^A : \|x\|_{\ell^t_{\mu,\nu}} < \infty \} \quad \text{with} \quad \|x\|_{\ell^t_{\mu,\nu}} = \sup_{\tau > 0} \left( \tau^t \sum_{j \in A} \nu_j \mathbb{I}(\mu_j |x_j| > \tau) \right).
\]

We apply Theorem 2 and obtain the following result.

Corollary 2.4 (error bounds for weighted \( p \)-norm penalties). Let \( p \in (1,2) \), \( t \in (2p-2, p) \) and \( \delta, \varrho, \alpha > 0 \) and \( c_\varrho \geq c_1 > 0 \), \( C_D > c_D > 1 \) and \( \mu := (\omega \mathfrak{r}^{-p})^{\frac{1}{p}} \), \( \nu := (\omega^{-1} \mathfrak{r}^{-1})^{\frac{1}{p}} \). Assume \( x \in \ell^t_{\mu,\nu} \) with \( \|x\|_{\ell^t_{\mu,\nu}} \leq \varrho \) and \( \hat{x}_\alpha \in R_\alpha(g^\delta) \). There is a constant \( C > 0 \) independent of \( x, \delta \) and \( \varrho \) such that whenever \( \alpha \) satisfies either

\[
c_1 \varrho^{\frac{(2p-2)}{2p}} \delta^{\frac{2p-2}{2p}} \leq \alpha \leq c_\varrho \varrho^{\frac{(2p-2)}{2p}} \delta^{\frac{2p-2}{2p}} \quad \text{or} \quad C_D \delta \leq \|g^\delta - A\hat{x}_\alpha\|_Y \leq C_D \delta
\]

the bound

\[
\|x - \hat{x}_\alpha\|_{\ell^p_\omega} \leq C \varrho^{\frac{(2p-2)}{2p}} \delta^{\frac{2p-2}{2p}}.
\]

holds true.

The proof of Corollary 2.4 can be found in Subsection 4.4.

Remark 2.5. In the limiting case \( t = 2p-2 \) the statement remains valid if one replaces \( \ell^t_{\mu,\nu} \) by \( K_1 = \ell^{2p-2}_\omega \) with \( \mathfrak{r} = \mathfrak{r}^{-1} \mathfrak{r}^{-p} \). Here we obtain the rate

\[
\|x - \hat{x}_\alpha\|_{\ell^p_\omega} \leq C \varrho^{\frac{1}{2p-2}} \delta^{\frac{1}{2p-2}}.
\]

In [10] the rate \( O(\delta^\hat{p}) \) is already proven under a condition similar to (1.1). Here we obtain intermediate convergences rates between \( O(\delta^0) \) and \( O(\delta^\hat{p}) \). This has the advantage that we obtain statements on the speed of convergences on larger sets.

Remark 2.6. Corollary 2.4 remains valid word by word in the case \( p = 1 \) (see [20, Thm. 4.4]).
**Example 2: Besov $0, p, p$-Penalties.** We introduce a scale of sequence spaces that allows to characterize Besov function spaces by decay properties of coefficients in wavelet expansions (see [26]).

Let $(\Lambda_j)_{j \in \mathbb{N}_0}$ be a family of sets such that $2^j \leq |\Lambda_j| \leq C_A 2^j$ for some constant $C_A \geq 1$ and all $j \in \mathbb{N}_0$. We consider the index set $\Lambda := \{(j, k) : j \in \mathbb{N}_0, k \in \Lambda_j\}$.

For $p, q \in (0, \infty)$ and $s \in \mathbb{R}$ we set $b^s_{p,q} = \{x \in \mathbb{R}^A : \|x\|_{s,p,q} < \infty\}$ with

$$
\|x\|_{s,p,q} := \sum_{j \in \mathbb{N}_0} 2^{jq(s + \frac{d}{p} - \frac{d}{q})} \left(\sum_{k \in \Lambda_j} |x_{j,k}|^p\right)^{q/p}.
$$

with the usual replacements for $p = \infty$ or $q = \infty$.

Let $a > 0$ and assume that the forward operator $A : b^{-a}_{2,2} \to Y$ satisfies (2.4) with $X_A = b^{-a}_{2,2}$. Let $p \in (1, \infty)$ (for $p = 1$ we refer to [20] again) with $\frac{d}{p} - \frac{d}{2} \leq a$. Then we have a continuous embedding $b^0_{p,p} \subset b^{-a}_{2,2}$ (see [27, 3.3.1.(6), 3.2.4.(1)]).

We use $X = b^0_{p,p}$ with

$$
R(x) = \frac{1}{p} \|x\|_{0,p,p} = \frac{1}{p} \sum_{(j,k) \in \Lambda} 2^{jp\left(\frac{d}{p} - \frac{d}{q}\right)} |x_{j,k}|^p
$$

for $x \in b^0_{p,p}$.

Note that we have

$$
b^s_{p,p} = \ell^p_{s,p} \text{ with equal norm for } (\mathbb{N}_{s,p})(j,k) = 2^{j\left(s + \frac{d}{p} - \frac{d}{q}\right)}.
$$

Hence for $p < 2$, this example is a special case of Example 1.

Let $\tilde{s} = \frac{a}{p-1}$ and $\tilde{t} = 2p - 2$. For $0 < s < \tilde{s}$ we set

$$
k_s := \left(b^{-a}_{2,2}, b^s_{\tilde{t},\tilde{t}}\right)_{\theta,\infty} \text{ with } \theta = \frac{p - 1 + s}{p}.
$$

Here the application of Theorem 2 yields the following error bound.

**Corollary 2.7** (error bounds for $0, p, p$-penalties). Let $0 < s < \tilde{s}$ and $\delta, \varrho, \alpha > 0$, $\epsilon \geq \epsilon_0 > 0$, $C_D > c_D > 1$. Assume $x \in k_s$ with $\|x\|_{k_s} \leq \varrho$ and $\hat{x}_\alpha \in \mathcal{R}_\alpha(g^0)$. There is a constant $C > 0$ independent of $x$, $\delta$ and $\varrho$ such that whenever $\alpha$ satisfies either

$$
c_D \varrho \frac{\delta}{\epsilon_0} \left(\frac{2^{(p-1)a}}{\epsilon_0}\right)^{\frac{d}{p}} \leq \alpha \leq c_D \varrho \frac{\delta}{\epsilon_0} \left(\frac{2^{(p-1)a}}{\epsilon_0}\right)^{\frac{d}{p}} \quad \text{or} \quad c_D \delta \leq \|g^0 - A\hat{x}_\alpha\|_Y \leq C_D \delta
$$

the bound

$$
\|x - \hat{x}_\alpha\|_{0,p,p} \leq C \varrho \frac{\delta}{\epsilon_0} \delta^{1/p}.
$$

holds true.

The proof of Corollary 2.7 can be found in Subsection 4.4.

**Remark 2.8.** In the limiting case $s = \tilde{s}$ the result remains valid if one replaces $k_s$ by $K_1 = b^s_{\tilde{t},\tilde{t}}$ and we obtain the bound $\|x - \hat{x}_\alpha\|_{0,p,p} \leq C \varrho \frac{\delta^{1/p}}{\epsilon_0} \delta^{1/p}$.

For $p = 2$ we have $k_s = b^s_{2,2}$ (see [27, 3.3.6.(9)]). The following proposition provides a nesting of $k_s$ for $p \neq 2$ by Besov sequence spaces.

**Proposition 2.9.** Let $0 < s < \tilde{s}$ and $t = \frac{2pn}{(2-p)s + 2a}$.

1. For $p < 2$ we have continuous embeddings

$$
b^s_{t,t} \subset k_s \subset b^{s-\varepsilon}_{t-\varepsilon} \quad \text{for all } 0 < \varepsilon < t.
$$
For $p > 2$ we have continuous embeddings $b^s_{1,t} \subset k_s \subset b^s_{2,\infty}$.

We refer to Subsection 5.4 for a proof of Proposition 2.9. For $p < 2$ the same argument as in [20, Ex.6.7.] shows that describing the regularity of functions with jumps or kinks via their wavelet expansion in terms of $k_s$ allows for a higher value of $s$ then using $B^p_{s,\infty}(\Omega)$ as in [28]. Therefore we obtain a faster convergence rate for this class of functions.

For $p > 2$ we measure the error in a stronger norm than the $\ell^2$-norm. On the other hand the set on which we obtain convergence rates is smaller than $b^s_{2,\infty}$.

**Example 3: Besov $0,2,q$-Penalties.** Again we consider $a > 0$ and $X_{A} = b^{-a}_{2,2}$ with $A$ satisfying (2.4). Let $q \in (1,\infty)$. Then there is a continuous embedding $\mathcal{K} := b^0_{2,q} \subset b^{-a}_{2,2}$ (see [27, 3.3.1.(7)]) and we choose

$$\mathcal{R}(x) = \frac{1}{q} \|x\|_{0,2,q} = \frac{1}{q} \sum_{j \in \mathbb{N}_0} \left( \sum_{k \in \Lambda_j} |x_{j,k}|^2 \right)^{q/2}.$$  

For a convergence analysis in the case $q = 1$ we refer to [17]. The application of Theorem 2 provides:

**Corollary 2.10** (error bounds for $0,2,q$-penalties). Let $0 < s < \frac{a}{q-1}$ and $\delta, q, \alpha > 0$, $c_r \geq c_l > 0$, $C_D > c_D > 1$. Assume $x \in b^s_{2,\infty}$ with $\|x\|_{s,2,\infty} \leq \tilde{q}$ and $\hat{x}_\alpha \in R_{\alpha}(g^\delta)$. There is a constant $C > 0$ independent of $x$, $\delta$ and $q$ such that whenever $\alpha$ satisfies either

$$c_D \delta^{-\alpha} \leq \alpha \leq c_r \delta^{\frac{2-\alpha+2a}{\alpha}} \quad \text{or} \quad c_D \delta \leq \|g^\delta - A\hat{x}_\alpha\|_Y \leq C_D \delta$$

the bound

$$\|x - \hat{x}_\alpha\|_{0,2,2} \leq C \tilde{q} \delta^{\frac{2-a}{q-1}} \delta^{-\alpha}$$

holds true.

The proof of Corollary 2.10 can by found in Subsection 4.4.

**Remark 2.11.** In the limiting case $s = \frac{a}{q-1}$ the result remains valid if one replaces $b^s_{2,\infty}$ by $K_1 = b^0_{2,\tilde{q}}$ with $\tilde{q} = 2q - 2$. Here we obtain $\|x - \hat{x}_\alpha\|_{0,2,2} \leq C \tilde{q} \delta^{\frac{2-a}{q-1}} \delta^{-\alpha}.$

In contrast to the analysis in [17] we measure the error in the $\ell^2$-norm independent of the value of $q$, i.e. the error norm is not dictated by the penalty term. The smaller $q$ the larger is the region $0 < s < \frac{a}{q-1}$ of regularity parameters for which we guarantee upper bounds. Furthermore we see that changing the fine index $q$ while keeping $p = 2$ does not change the set where convergence rates are guaranteed, but it influences the parameter choice rule.

**Example 4: Radon Transform.** To give a more concrete example we discuss the Radon transform which appears as forward operator in computed tomography (CT) and positron emission tomography (PET). This example also shows how our results apply to operators initially defined on function spaces.

Let $d \in \mathbb{N}$ with $d \geq 2$, $\Omega := \{x \in \mathbb{R}^d : |x| \leq 1\}$, $S^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ and $\mathbb{U} := L^2(S^{d-1} \times [-1,1])$. Then the Radon transform $R : L^2(\Omega) \to L^2(S^{d-1} \times \mathbb{R})$ is given by

$$(Rf)(\theta,t) := \int_{x \cdot \theta = t} f(x) \, dx, \quad \theta \in S^{d-1}, t \in \mathbb{R}.$$
With \( a = \frac{d}{dY} \) it follows from [15, Thm. 3.1] that \( R \) is a norm isomorphism from \( B_{2,2}^{-s}(\Omega) \) to \( Y \). Here \( B_{2,2}^{-s}(\Omega) \) denotes a Besov function space. We refer to the book [9] for an introduction to this scale of function spaces.

Furthermore, with \( \Lambda \) and the scale of spaces \( b_{p,q}^s \) as introduced in Example 2 and \( s_{\text{max}} > a \) we consider a \( s_{\text{max}} \)-regular wavelet system \( (\psi_\lambda)_{\lambda \in \Lambda} \) on \( \Omega \) such that the synthesis operator

\[
S : b_{p,q}^s \to B_{p,q}^s(\Omega) \quad \text{given by} \ x \mapsto \sum_{\lambda \in \Lambda} x_\lambda \psi_\lambda
\]

is well defined and a norm isomorphism for all \( s \in \mathbb{R} \) and \( p,q \in (0,\infty] \) satisfying \( s \in (\sigma_p - s_{\text{max}}, s_{\text{max}}) \) with \( \sigma_p = \max \left\{ d\left( \frac{1}{p} - 1 \right), 0 \right\} \) (see [26]). Now for \( R \) as in Example 2 or Example 3 we consider

\[
(2.7) \quad S_\alpha(g) = S \hat{x}_\alpha \quad \text{with} \ \hat{x}_\alpha \in \text{argmin}_{x \in \text{dom}(R)} \left( \frac{1}{2\alpha} \| g^{\text{obs}} - RSx \|^2_Y + R(x) \right)
\]

and obtain the following convergence rate results.

**Corollary 2.12** (Convergence rates for wavelet regularization of the Radon transform).

1. Let \( p \in (1, \infty) \). With the notation of Example 2 suppose \( 0 < s < \min\{s_{\text{max}}, s_{\text{max}}\} \) and \( \delta, \varrho, \alpha > 0 \), \( c_r \geq c_l > 0 \), \( C_D > c_D > 1 \). Assume \( f \in B_{1,1}^s(\Omega) \) with \( \| f \|_{B_{1,1}^s(\Omega)} \leq \varrho \) and \( \hat{f}_\alpha \in S_\alpha(g^\varrho) \) with \( R \) as in Example 2. Then there is a constant \( C > 0 \) independent of \( f, \delta \) and \( \varrho \) such that whenever \( \alpha \) satisfies either

\[
c_\delta \left( \frac{s}{s + a} \right)^{\frac{(2-p) + 2a}{2a}} \leq \alpha \leq c_r \left( \frac{s}{s + a} \right)^{\frac{(2-p) + 2a}{2a}} \quad \text{or} \quad c_D \delta \leq \| g^\varrho - R\hat{f}_\alpha \|_Y \leq C_D \delta
\]

the bound

\[
\| f - \hat{f}_\alpha \|_{B_{p,p}^s(\Omega)} \leq C \varrho \left( \frac{s}{s + a} \right)^{\frac{(2-p) + 2a}{2a}}
\]

holds true. If \( p \leq 2 \) we also obtain the bound

\[
\| f - \hat{f}_\alpha \|_{L^p(\Omega)} \leq C \varrho \left( \frac{s}{s + a} \right)^{\frac{(2-p) + 2a}{2a}}
\]

2. Let \( q \in (0, \infty) \). Suppose \( 0 < s < \min\{\varphi_{\frac{s}{q}}, s_{\text{max}}\} \) and \( \delta, \varrho, \alpha > 0 \), \( c_r \geq c_l > 0 \), \( C_D > c_D > 1 \). Assume \( f \in B_{2,\infty}^s(\Omega) \) with \( \| f \|_{B_{2,\infty}^s(\Omega)} \leq \varrho \) and \( \hat{f}_\alpha \in S_\alpha(g^\varrho) \) with \( R \) as in Example 3. Then there is a constant \( C > 0 \) independent of \( f, \delta \) and \( \varrho \) such that whenever \( \alpha \) satisfies either

\[
c_\delta \left( \frac{s}{s + a} \right)^{\frac{(2-p) + 2a}{2a}} \leq \alpha \leq c_r \left( \frac{s}{s + a} \right)^{\frac{(2-p) + 2a}{2a}} \quad \text{or} \quad c_D \delta \leq \| g^\varrho - R\hat{f}_\alpha \|_Y \leq C_D \delta
\]

the bound

\[
\| f - \hat{f}_\alpha \|_{L^2(\Omega)} \leq C \varrho \left( \frac{s}{s + a} \right)^{\frac{(2-p) + 2a}{2a}}
\]

holds true.

The proof of Corollary 2.11 can be found in Subsection 4.4. In Corollary 2.12.1. it would also we sufficient to require \( f \in \mathcal{S}_k \) instead of \( f \in B_{1,1}^s(\Omega) \). Transferring the interpolation identity in (2.6) to function spaces shows that \( \mathcal{S}_k \) is independant of the chosen wavelet system (see [20, Sec. 6.2] for a similar discussion).

In the same manner the presented theory can be applied to other linear, finitely smoothening forward operators as inverses of elliptic differential operators with smooth, periodic coefficients or specific periodic convolution operators (see [17, Ex. 2.5] for more details).
2.3. Connections to source conditions. Assuming only Assumption 2.1 we compare (1.2) to source conditions used in the literature. For a concave and upper semi-continuous function \( \phi : [0, \infty) \to [0, \infty) \) we consider variational source conditions of the form

\[
\mathcal{R}(x) - \mathcal{R}(z) \leq \phi(\|Ax - Az\|^2) \quad \text{for all } z \in X.
\]

(2.8)

In [19] this condition is used to prove convergence rates with respect to the twisted Bregman distance of \( \mathcal{R} \) and it is shown that the source condition (1.1) implies (2.8) with \( \phi \sim \sqrt{\cdot} \). In [7] necessity of (2.8) for convergence rates with respect to the twisted Bregman distance under a fixed parameter choice rule is proven.

Inspired by [16] we also study the defect of the Tikhonov functional

\[
\sigma_x(\alpha) := T_\alpha(x, Ax) - T_\alpha(x_\alpha, Ax).
\]

The following result shows that Hölder-type variational source conditions, Hölder-type bounds on the defect of the Tikhonov functional and Hölder type image space approximation rates are equivalent.

**Theorem 3.** Let \( \nu \in (\frac{1}{2}, 1] \). Assume \( x \in \text{dom}(\mathcal{R}) \) is \( \mathcal{R} \)-minimal in \( A^{-1}(\{Ax\}) \) and \( x_\alpha \in R_\alpha(Ax) \) for \( \alpha > 0 \) is any selection of a minimizers for exact data. The following statements are equivalent:

(i) There exists a constant \( c_1 > 0 \) with \( \|Ax - Ax_\alpha\|_Y \leq c_1 \alpha^\nu \) for all \( \alpha > 0 \).

(ii) There exists a constant \( c_2 > 0 \) such that \( \sigma_x(\alpha) \leq c_2 \alpha^{2\nu - 1} \).

(iii) There exists a constant \( c_3 > 0 \) with (2.8) holds true for \( \phi(t) = c_3 t^{\frac{2\nu - 1}{\nu}} \).

More precisely (i) implies (ii) with \( c_2 = \frac{c_1^2}{4\nu - 2} \), (ii) implies (iii) with \( c_3 = 2c_2^{\frac{\nu}{\nu - 1}} \) and (iii) implies (i) with \( c_1 = c_3^\nu \).

We provide a proof of Theorem 3 in Subsection 4.4. The result allows the following representation of \( K_\nu \) in terms of variational source conditions:

\[
K_\nu = \left\{ x \in X : \text{There exists } c > 0 \text{ such that (2.8) with } \phi(t) = ct^{\frac{2\nu - 1}{\nu}} \text{ holds true.} \right\}
\]

(2.9)

for all \( \nu \in \nu \in (\frac{1}{2}, 1] \). Note that since the map \((\frac{1}{2}, 1] \to (0, \frac{1}{2}] \) given by \( \nu \mapsto \frac{2\nu - 1}{2\nu} \) is bijective this characterization grasps all Hölder type functions \( \phi(t) = O(t^\mu) \) for \( \mu \in (0, \frac{1}{2}] \). Due to [16, Prop. 3] the largest meaningful exponent is \( \mu = \frac{1}{2} \). Furthermore, (2.8) implies \( x \in \text{dom}(\mathcal{R}) \) and this in turn yields (i) with \( \nu = \frac{1}{2} \). Therefore, we cannot expect a characterization of (i) for \( \nu < \frac{1}{2} \) by variational source conditions. Hence all meaningful Hölder type variational source conditions of the form (2.8) are covered in Theorem 3 and (2.9). In other words it is not possible to extend Theorem 3 to a larger set of exponents.

Together with Theorem 1 we see that Hölder-type variational source conditions imply upper bounds on the reconstruction error for any loss function given by the modulus of continuity. In contrast as far as the author knows all upper bounds in the literature derived from (2.8) are restricted to the twisted Bregman distance.

3. Minimax convergence rates on \( K_\nu \). The aim of this section is to prove Theorem 1. Here we only assume the topological assumptions given in Assumption 2.1. We will follow an idea presented in the seminal paper [4]: Any feasible procedure is nearly minimax (see [4, 4.3.1.]). In our context feasibility means

1. image space bounds: \( \|Ax - Ax_\alpha\|_Y \leq c \delta \),
2. regularity of the minimizers: \( \rho_v(\hat{x}_\alpha) \leq c \rho_v(x) \) for some constant \( c > 0 \).

After proving feasibility we use the same argument as in [4, 4.3.1. Prop. 2] to obtain a nearly minimax result.

### 3.1. Characterization of \( A \circ R_\alpha \) as proximity mapping.

This subsection provides an important preliminary that we use in several places throughout the paper. We introduce a convex function \( Q \) on \( Y \) that can be seen as a push forward of \( R \) through the linear operator \( A \). We show that the proximity mapping of \( \alpha Q \) equals \( A \circ R_\alpha \). Recall that for a convex, proper and lower semi-continuous function \( Q : Y \to (-\infty, \infty] \) and \( g \in Y \) there is a unique minimizer \( \text{Prox}_Q(g) \) of the function \( y \mapsto \frac{1}{2}\|g - y\|_Y^2 + Q(y) \). The single-valued mapping

\[
\text{Prox}_Q : Y \to Y \quad \text{given by} \quad g \mapsto \text{Prox}_Q(g) := \arg\min_{y \in Y} \left( \frac{1}{2}\|g - y\|^2_Y + Q(y) \right)
\]

is called proximity mapping of \( Q \) (see [1, 11.4, Def. 12.23]).

**Lemma 3.1.** We define

\[
Q : Y \to (-\infty, \infty] \quad \text{by} \quad Q(g) := \inf\{R(x) : x \in X \text{ with } Ax = g\}
\]

with \( \inf \emptyset = \infty \). Then \( Q \) is convex, proper and lower semi-continuous, and we have \( \text{dom}(Q) = A(\text{dom}(R)) \).

**Proof.** Let \( \lambda \in \mathbb{R} \). First we prove that \( L_\lambda := \{g \in Y : Q(g) \leq \lambda\} \) satisfies

\[
L_\lambda = A(\{x \in X : R(x) \leq \lambda\})
\]

To this end let \( g \in L_\lambda \). There exists \( x \in X \) with \( Ax = g \) and \( R(x) \leq \lambda \). Then \( R(x) = Q(g) \leq \lambda \). On the other hand if \( x \in X \) with \( R(x) \leq \lambda \) then \( Q(Ax) \leq R(x) \leq \lambda \). Taking union over \( \lambda \in \mathbb{R} \) yields \( \text{dom}(Q) = A(\text{dom}(R)) \). Hence \( Q \) is proper as \( R \) is proper. The sublevel sets \( L_\lambda \) are convex as the image of a convex set under a linear map and closed as the image of a \( \tau \)-compact set under a \( \tau \)-to-weak continuous map. Hence \( Q \) is convex and lower semi-continuous.

**Remark 3.2.** Note that in the case of an injective forward operator \( A \), the map \( Q \) is given by \( Q(g) = R(A^{-1}g) \) if \( g \in \text{im}(A) \) and \( Q(g) = \infty \) if \( g \in Y \setminus \text{im}(A) \) where \( A^{-1} : \text{im}(A) \to X \) denotes the inverse map of \( A \).

**Proposition 3.3.** Let \( g \in Y \) and \( \alpha > 0 \). Then

\[
A\hat{x}_\alpha = \text{Prox}_{\alpha Q}(g) \quad \text{and} \quad R(\hat{x}_\alpha) = Q(\text{Prox}_{\alpha Q}(g)) \quad \text{for all} \quad \hat{x}_\alpha \in R_\alpha(g).
\]

In particular \( A \circ R_\alpha = \text{Prox}_{\alpha Q} \) is single-valued. Hence \( A\hat{x}_\alpha \) and \( R(\hat{x}_\alpha) \) do not depend on the particular choice of \( \hat{x}_\alpha \in R_\alpha(g) \).

**Proof.** Let \( v \in \text{dom}(Q) \). By Lemma 3.1 we have \( v \in \text{im}(A) \). There exists \( z \in X \) with \( Az = v \) and \( R(z) \leq R(y) \) for all \( y \in X \) with \( Ay = v \). By definition of \( Q \) that is \( R(z) = Q(v) \). The first identity follows from

\[
\frac{1}{2\alpha}\|g - A\hat{x}_\alpha\|^2_Y + Q(A\hat{x}_\alpha) \leq \frac{1}{2\alpha}\|g - A\hat{x}_\alpha\|^2_Y + R(\hat{x}_\alpha) \leq \frac{1}{2\alpha}\|g - Az\|^2_Y + Q(z) = \frac{1}{2\alpha}\|g - v\|^2_Y + Q(v).
\]

Inserting \( v = A\hat{x}_\alpha \) yields \( R(\hat{x}_\alpha) = Q(A\hat{x}_\alpha) = Q(\text{Prox}_{\alpha Q}(g)) \).
The statement in Proposition 3.3 can be read as follows: the function Q on Y stores all relevant information on R and A to recover the mapping A ° R_α in one object. Note that the definition of K_ν can be rephrased only in terms of Q.

Remark 3.4. Suppose x ∈ dom(R), α > 0 and x_α ∈ R_α(Ax). In [16] the authors study upper bounds on |R(x) − R(x_α)| (defect for penalty) and on σ_x(α) (defect for Tikhonov functional) in terms of α. The first quantity bounds the second and it is bounded by the double of the second (see [16, Prop. 2.4]). In [16, Rem. 2.5] the authors rely on this nesting to argue that changing the selection of minimizers changes the defect for penalty at most by a factor of 2. Proposition 3.3 actually shows that the defect for penalty is independent of the choice of x_α ∈ R_α(Ax).

Exploiting firm non-expansiveness (see [1, Def. 4.1]) of proximal operators we draw a further conclusion of Proposition 3.3.

Corollary 3.5 (Firm non-expansiveness). Let g, h ∈ Y, α > 0, x_α ∈ R_α(g) and x̂_α ∈ R_α(h). Then

\[ \| (g - A\hat{x}_\alpha) - (h - A\hat{x}_\alpha) \|_Y^2 + \| A\hat{x}_\alpha - A\hat{x}_\alpha \|_Y^2 \leq \| g - h \|_Y^2. \]

Proof. By [1, Prop. 12.27] the proximity operator Prox_αQ satisfies

\[ \| (g - \text{Prox}_Q(g)) - (h - \text{Prox}_Q(h)) \|_Y^2 + \| \text{Prox}_Q(g) - \text{Prox}_Q(h) \|_Y^2 \leq \| g - h \|_Y^2 \]

for all g, h ∈ Y. Inserting the first identity in Proposition 3.3 yields the claim.

3.2. Properties of the sets K_ν. The following proposition captures properties of the sets K_ν. In particular, we show that K_ν is nontrivial for ν ∈ (0, 1).

Lemma 3.6. We have

1. K_0 = X.
2. K_ν_1 ⊂ K_ν_2 for 0 ≤ ν_1 ≤ ν_2.
3. K_ν = \arg\min_{x∈X} R(z) + ker(A) for all ν > 1.
4. dom(R) + ker(A) ⊂ K_1/2.

Proof. Let x ∈ K_χ_ν. Then

\[ \frac{1}{2\alpha} \| Ax - Ax_0 \|_Y^2 + R(x_0) \leq \frac{1}{2\alpha} \| Ax - Ay \|_Y^2 + R(y). \]

As R(y) ≤ R(x_0) this implies \| Ax - Ax_0 \|_Y ≤ \| Ax - Ay \|_Y. Hence \vartheta_0(x) ≤ D_x < ∞.

2. Suppose x ∈ K_χ_ν. Then

\[ \| Ax - Ax_0 \|_Y = \| Ax - Ax_0 \|_Y^{\alpha_{\nu_1}} \| Ax - Ax_0 \|_Y^{1 - \alpha_{\nu_1}} < \vartheta_{\nu_1}(x)^{\alpha_{\nu_1}} \vartheta_0(x)^{1 - \alpha_{\nu_1}}. \]

implies \vartheta_{\nu_1}(x) ≤ \vartheta_{\nu_0}(x)^{\alpha_{\nu_1}} \vartheta_0(x)^{1 - \alpha_{\nu_1}}.

3. Let ν > 1. Suppose x ∈ K_ν. From [1, Prop. 16.34] and Proposition 3.3 we obtain

\[ \eta_\alpha := \frac{1}{\alpha} (Ax - Ax_0) = \frac{1}{\alpha} (Ax - \text{Prox}_\alpha Q(Ax)) ∈ \partial Q(Ax). \]

Since \eta_\alpha → 0 and Ax_0 → Ax for α → 0 in the norm topology of Y this implies 0 ∈ \partial Q(Ax). Hence Ax ∈ arg\min_{y∈Y} Q(y). Let y ∈ X be R-minimal with Ay = Ax. Then

\[ R(y) = Q(Ax) ≤ Q(Ax) ≤ R(z) \text{ for all } z ∈ X. \]
Hence

\[ x = y + x - y \in \arg\min_{z \in \mathbb{X}} \mathcal{R}(z) + \ker(A). \]

On the other hand assume \( x = y + k \in \arg\min_{z \in \mathbb{X}} \mathcal{R}(z) + \ker(A) \). Then

\[ \frac{1}{2\alpha} \| Ax - Ax_\alpha \|_Y^2 + \mathcal{R}(x_\alpha) \leq T_\alpha(y, Ax) = \mathcal{R}(y) \leq \mathcal{R}(x_\alpha) \]

yields \( \| Ax - Ax_\alpha \|_Y = 0 \). Hence \( x \in K_\nu \).

4. Let \( x = y + k \in \text{dom}(\mathcal{R}) + \ker(A) \). From

\[ \frac{1}{2\alpha} \| Ax - Ax_\alpha \|_Y^2 \leq T_\alpha(x_\alpha, Ax) \leq T_\alpha(y, Ax) = \mathcal{R}(y) \]

we obtain \( g_{1/2}(x) \leq \sqrt{2\mathcal{R}(y)}. \)

The set \( K_\nu \) does not change for \( \nu > 1 \). As announced in Subsection 2.2 we will see that \( K_1 \) is the set of elements satisfying source condition (1.1).

Moreover note that the last inequality in the proof of Lemma 3.6.2 resembles an interpolation inequality. This gives a first hint to a connection to interpolation theory in the case of Banach space regularization.

3.3. Image space bounds. This subsection is devoted to error bounds in the image space \( \mathbb{Y} \) in terms of the deterministic noise level and the image space approximation error for exact data. Let \( \delta \geq 0, x \in \mathbb{X} \) and \( g^\delta \in \mathbb{Y} \) with \( \| g^\delta - Ax \|_\mathbb{Y} \leq \delta \).

**Lemma 3.7.** The following inequalities

\begin{align}
(3.1) & \quad \| Ax - A\hat{x}_\alpha \|_\mathbb{Y} \leq \delta + \| Ax - Ax_\alpha \|_\mathbb{Y}, \\
(3.2) & \quad \| g^\delta - A\hat{x}_\alpha \|_\mathbb{Y} \leq \delta + \| Ax - Ax_\alpha \|_\mathbb{Y}
\end{align}

hold true for all \( \alpha > 0, \hat{x}_\alpha \in R_\alpha(g^\delta), x_\alpha \in R_\alpha(Ax) \).

**Proof.** Corollary 3.5 with \( g = g^\delta \) and \( h = Ax \) yields

\[ \|(Ax - Ax_\alpha) - (g^\delta - A\hat{x}_\alpha)\|_\mathbb{Y}^2 + \| Ax_\alpha - A\hat{x}_\alpha \|_\mathbb{Y}^2 \leq \delta^2. \]

We neglect the first summand on the left hand side and obtain

\[ \| Ax - A\hat{x}_\alpha \|_\mathbb{Y} \leq \| Ax - Ax_\alpha \|_\mathbb{Y} + \| Ax_\alpha - A\hat{x}_\alpha \|_\mathbb{Y} \leq \delta + \| Ax - Ax_\alpha \|_\mathbb{Y} \]

and the second for

\[ \| g^\delta - A\hat{x}_\alpha \|_\mathbb{Y} \leq \|(Ax - Ax_\alpha) - (g^\delta - A\hat{x}_\alpha)\|_\mathbb{Y} + \| Ax - Ax_\alpha \|_\mathbb{Y} \leq \delta + \| Ax - Ax_\alpha \|_\mathbb{Y}. \]

**Proposition 3.8.** Let \( \nu \in (0, 1] \) and \( \alpha, \rho > 0 \). Suppose \( x \in K_\rho^\nu \).

1. Let \( c_r > 0 \). If \( \alpha \leq c_r \rho^{-\frac{1}{2}} \delta^{\frac{1}{2}} \) then

\[ \| Ax - A\hat{x}_\alpha \|_\mathbb{Y} \leq (1 + c_r^\nu) \delta \quad \text{for all} \quad \hat{x}_\alpha \in R_\alpha(g^\delta). \]

2. Let \( c_D > 1 \). If \( \hat{x}_\alpha \in R_\alpha(g^\delta) \) satisfies \( c_D \delta \leq \| g^\delta - A\hat{x}_\alpha \| \), then

\[ (c_D - 1)^{\frac{1}{2}} \rho^{-\frac{1}{2}} \delta^{\frac{1}{2}} \leq \alpha. \]

**Proof.** Let \( x_\alpha \in R_\alpha(Ax) \).
1. By (3.1) and the definition of \( g_\nu \) we obtain
\[
\| Ax - A\hat{x}_\alpha \|_Y \leq \delta + \varrho\alpha^\nu \leq (1 + \varrho^\nu)\delta.
\]

2. The bound (3.2) implies
\[
c_d\delta \leq \delta + \| Ax - Ax_\alpha \|_Y \leq \delta + \varrho\alpha^\nu.
\]

Subtracting \( \delta \) and rearranging yields the claim. \( \square \)

#### 3.4. Regularity of the minimizers.

First we recall the well-known fact that the source condition (1.1) implies a linear convergence rate in the image space (see e.g. [12, Lem. 3.5]).

**Lemma 3.9.** Let \( z \in \mathcal{X} \) and assume \( \omega \in \mathcal{Y} \) with \( A^*\omega \in \partial \mathcal{R}(z) \). Then
\[
\| Az - Az_\alpha \|_Y \leq \| \omega \|_Y \alpha \quad \text{for all } \alpha > 0 \text{ and } z_\alpha \in R_\alpha(Az).
\]

**Proof.** The first order optimality condition yields \( \xi_\alpha := \frac{1}{\alpha}A^*A(z - z_\alpha) \in \partial \mathcal{R}(z_\alpha) \).

Solving the inequality
\[
\frac{1}{\alpha}\| Az - Az_\alpha \|^2 = \langle \xi_\alpha, z - z_\alpha \rangle \leq R(z) - R(z_\alpha) \leq \langle A^*\omega, z - z_\alpha \rangle \leq \| \omega \|_Y \| Az - Az_\alpha \|_Y
\]

for \( \| Az - Az_\alpha \|_Y \) proves the claim. \( \square \)

**Lemma 3.10.** Let \( \alpha > 0 \), \( \hat{x}_\alpha \in R_\alpha(g^\delta) \). Furthermore let \( \beta > 0 \), \( (\hat{x}_\alpha)_\beta \in R_\beta(A\hat{x}_\alpha) \) and \( x_\beta \in R_\beta(Ax) \).

1. If \( \beta \in (0, \alpha] \) then
\[
\| A\hat{x}_\alpha - A(\hat{x}_\alpha)_\beta \|_Y \leq \frac{\beta\delta}{\alpha} + \| Ax - Ax_\beta \|_Y.
\]

2. If \( \beta \in [\alpha, \infty) \) then
\[
\| A\hat{x}_\alpha - A(\hat{x}_\alpha)_\beta \|_Y \leq \delta + 2\| Ax - Ax_\beta \|_Y.
\]

**Proof.**
1. By the first order optimality condition the element \( \hat{x}_\alpha \) satisfies the prerequisite \( A^*\omega \in \partial \mathcal{R}(\hat{x}_\alpha) \) of Lemma 3.9 with \( \omega = \frac{\delta}{\alpha}(g^\delta - A\hat{x}_\alpha) \).

By Lemma A.1 the map \( \alpha \mapsto \frac{1}{\alpha}\| Ax - Ax_\alpha \|_Y \) is non increasing. Together with (3.2) we obtain
\[
\| \omega \|_Y = \frac{1}{\alpha}\| g^\delta - A\hat{x}_\alpha \|_Y \leq \frac{\delta}{\alpha} + \frac{1}{\alpha}\| Ax - Ax_\alpha \|_Y \leq \frac{\delta}{\alpha} + \frac{1}{\beta}\| Ax - Ax_\beta \|_Y.
\]

Hence Lemma 3.9 implies the claim.

2. We use first Corollary 3.5 with \( g = Ax \) and \( h = A\hat{x}_\alpha \) then (3.1) and finally non decreasingness of \( \alpha \mapsto \| Ax - Ax_\alpha \|_Y \) (see Lemma A.1) to estimate
\[
\|( Ax - Ax_\beta) - (A\hat{x}_\alpha - A(\hat{x}_\alpha)_\beta) \|_Y \leq \| Ax - A\hat{x}_\alpha \|_Y
\]
\[
\leq \delta + \| Ax - Ax_\beta \|_Y
\]
\[
\leq \delta + \| Ax - Ax_\beta \|_Y
\]

The triangle inequality finishes the proof. \( \square \)
**Proposition 3.11.** Let \( \nu \in (0,1] \) and \( \varrho, c_1, \alpha > 0 \). Suppose \( x \in K_\nu^\varrho \) and \( \hat{x}_\alpha \in R_\alpha(g^\delta) \). If \( c_1 \varrho^{-\frac{1}{2}} \delta^{\frac{1}{2}} \leq \alpha \), then \( \varrho_\nu(\hat{x}_\alpha) \leq (2 + c_1^{-\nu}) \varrho \).

**Proof.** Let \( \beta \leq \alpha \). With \( \delta \leq c_1^{-\nu} \varrho \alpha^\nu \) we estimate

\[
\frac{\delta \beta}{\alpha} \leq c_1^{-\nu} \varrho \alpha^{\nu-1} \beta \leq c_1^{-\nu} \varrho \beta^\nu.
\]

Furthermore

\[
\delta \leq c_1^{-\nu} \varrho \alpha^\nu \leq c_1^{-\nu} \varrho \beta^\nu \text{ for all } \beta \geq \alpha.
\]

Together with \( \|Ax - Ax_\beta\|_Y \leq \varrho \beta^\nu \) for all \( \beta > 0 \) and \( x_\beta \in R_\beta(Ax) \) the result follows from \( \text{Lemma 3.10} \).

#### 3.5. Almost minimaxity on the sets \( K_\nu \)

Now we are in position to give the proof of Theorem 1.

**Proof of Theorem 1.**

1. By Proposition 3.8 we have \( \|Ax - A\hat{x}_\alpha\|_Y \leq c_1 \delta \) and Proposition 3.11 yields \( x, \hat{x}_\alpha \in K_{c_1 \delta \nu}^\varrho \).

2. Using the triangle inequality we obtain

\[
\|Ax - A\hat{x}_\alpha\|_Y \leq \delta + \|\varrho^\delta - A\hat{x}_\alpha\|_Y \leq d_1 \delta.
\]

**Proposition 3.8** provides \( (c_D - 1)^\frac{1}{2} \varrho^{-\frac{1}{2}} \delta^{\frac{1}{2}} \leq \alpha \). Therefore Proposition 3.11 yields \( x, \hat{x}_\alpha \in K_{c_1 \delta \nu}^\varrho \).

In both cases the claim follows from the definition of the modulus \( \Omega \).

#### 4. Convergence rates theory for Banach space regularization.

**4.1. Source-wise representations and linear image space approximation.** We start with a converse to Lemma 3.9: A linear bound \( \|Ax - Ax_\alpha\|_Y = \mathcal{O}(\alpha) \) implies the source condition (1.1) and the minimal \( \mathcal{O} \)-constant \( \varrho_1(x) \) agrees with the minimal norm \( \|\omega\|_Y \) attended by a source element \( \omega \). Similar results can be found in [12, Lem. 4.1] and [22, Prop. 4.1]. For sake of self-containedness we include a proof.

**Proposition 4.1.** Let \( x \in X \) with \( \mathcal{R}(x) = \inf\{\mathcal{R}(z) : z \in X \text{ with } Az = Ax\} \). Then

\[
\varrho_1(x) = \inf \{\|\omega\|_Y : A^*\omega \in \partial\mathcal{R}(x)\}.
\]

If this quantity is finite and \( x_\alpha \in R_\alpha(Ax) \), \( \alpha > 0 \) is any selection, then the net \( (\frac{1}{\alpha}(Ax - Ax_\alpha))_{\alpha > 0} \) converges weakly for \( \alpha \searrow 0 \) to the unique \( \omega \in Y \) with \( A^*\omega \in \partial\mathcal{R}(x) \) and \( \|\omega\|_Y = \varrho_1(x) \).

**Proof.** Taking the infimum over \( \omega \) in Lemma 3.9 yields

\[
\varrho_1(x) = \inf \{\|\omega\|_Y : A^*\omega \in \partial\mathcal{R}(x)\}.
\]

To prove the remaining inequality let \( x \in X \) with \( \varrho_1(x) < \infty \). Then the net \( (\frac{1}{\alpha}(Ax - Ax_\alpha))_{\alpha > 0} \) is norm bounded in the Hilbert space \( Y \). By the Banach–Alaoglu theorem every null sequence of positive numbers has a subsequence \( \alpha_n > 0 \) such that \( \frac{1}{\alpha_n}(Ax - Ax_{\alpha_n}) \) converges weakly to some \( \omega \in Y \) with \( \|\omega\|_Y \leq \varrho_1(x) \). **Lemma A.2** and the minimality assumption yield

\[
\frac{1}{2\alpha_n} \|Ax - Ax_{\alpha_n}\|_Y^2 + \mathcal{R}(x_{\alpha_n}) \to \mathcal{R}(x).
\]
Together with \( \|Ax - Ax_{\alpha_n}\|_Y \leq \rho_1(x)\alpha_n \) we obtain \( R(x_{\alpha_n}) \to R(x) \). The first order optimality condition yields \( \lim_{n \to \infty} A^*A(x - x_{\alpha_n}) \in \partial R(x_{\alpha_n}) \). Hence for \( z \in X \) we obtain

\[
\begin{align*}
R(x) + (A^*\omega, z - x) &= R(x) + \langle \omega, A(z - x) \rangle \\
&= \lim_{n \to \infty} R(x_{\alpha_n}) + \left( \frac{1}{\alpha_n} A(x - x_{\alpha_n}), A(z - x_{\alpha_n}) \right) \\
&= \lim_{n \to \infty} R(x_{\alpha_n}) + \left( \frac{1}{\alpha_n} A^*A(x - x_{\alpha_n}), z - x_{\alpha_n} \right) \leq R(z).
\end{align*}
\]

This shows \( A^*\omega \in \partial R(x) \). Therefore the stated identity is proven.

Being the the preimage of the convex set \( \partial R(x) \) under the linear map \( A^* \), the set \( \{ \omega \in Y : A^*\omega \in \partial R(x) \} \) is convex. Strict convexity of \( \| \cdot \|_Y \) yields uniqueness of \( \omega \). In particular this implies the convergence of the net.

**Corollary 4.2.** We have \( \rho_1(x) = 0 \) if and only if \( x \in \text{argmin}_{z \in X} R(z) \).

**Proof.** By the second statement in Proposition 4.1 we have \( \rho_1(x) = 0 \) if and only if \( 0 \in \partial R(x) \). Hence the first order optimality condition \( x \in \text{argmin}_{z \in X} R(z) \) if and only if \( 0 \in \partial R(x) \) yields the claim. \( \square \)

**Example 4.3.** Let \( p \in [1, 2] \), \( X = \ell^p := \ell^p(\mathbb{N}), \ Y = \ell^2 := \ell^2(\mathbb{N}), \ A : \ell^p \to \ell^2 \) the embedding operator given by \( x \mapsto x \) and \( R \) given by \( R(x) = \frac{1}{p} \|x\|_p^p \). Let \( x \in \ell^p \).

If \( p > 1 \) then \( \partial R(x) = \{ \xi \} \) with \( |\xi_j| = |x_j|^{p-1} \). The adjoint \( A^* \) identifies with the embedding operator \( \ell^2 \to \ell^p \) with \( p' \) the Hölder conjugate of \( p \). Hence \( x \in K_1 \) if and only if \( \|\xi\|_{\ell^2} < \infty \), and we have

\( \rho_1(x) = \|\xi\|_{\ell^2} = \left( \sum_{j \in \mathbb{N}} |x_j|^{2p-2} \right)^{1/2} = \|x\|_{2p-2}^{p-1} \).

Therefore Assumption 2.2 is satisfied in this case.

For \( p = 1 \) we have \( \xi \in \partial R(x) \) if and only if \( \xi_j = 1 \) for \( x_j > 0 \), \( \xi_j = -1 \) for \( x_j < 0 \) and \( |\xi_j| \leq 1 \) for \( x_j = 0 \). Hence \( K_1 \) consists of all elements with finitely many non vanishing coefficients. We have \( \rho_1(x) = \# \{ j \in \mathbb{N} : x_j \neq 0 \}^{1/2} \) and Assumption 2.2 is not fulfilled.

### 4.2. Computation of \( K_1 \) for Banach space regularization

In this subsection we assume \( X_A \) is a Banach space with a dense, continuous embedding \( X \subset X_A \) and that \( A \) extends to \( X_A \) such that (2.4) is satisfied. Let \( u \in [1, \infty) \) and consider the penalty given by \( R(x) = \|x\|_u^u \).

If \( X \) is reflexive we choose \( \tau \) to be the weak topology on \( X \). Then the sublevel sets of \( R \) are \( \tau \)-compact by the Banach-Alaoglu theorem. Moreover \( A \) is weak-to-weak continuous as it is bounded. Therefore, Assumption 2.1 is automatically satisfied in this case.

In this subsection we compute \( K_1 \) for the three penalties covered in the examples in Subsection 2.2. We start with a tool that helps computing the function \( \rho_1 \) up to equivalence. Note that the density \( X \subset X_A \) allows us to view the adjoint of the embedding as an embedding \( X'_A \subset X' \).

**Proposition 4.4.** We have \( x \in K_1 \) if and only if \( \partial R(x) \cap X'_A \neq \emptyset \). The function

\[
\overline{\rho}_1 : X_A \to [0, \infty) \quad \text{given by} \quad \overline{\rho}_1(x) = \inf \{ \|\xi\|_{X'_A} : \xi \in \partial R(x) \cap X'_A \}
\]

satisfies

\[
\frac{1}{M} \rho_1(x) \leq \overline{\rho}_1(x) \leq M \rho_1(x) \quad \text{for all } x \in X.
\]
Proof. Suppose $\xi \in \partial R(x) \cap X'\Lambda$. Let $z \in X\Lambda$, then
\[ \langle \xi, z \rangle \leq \|\xi\|_{X'\Lambda} \|z\|_{X\Lambda} \leq M\|\xi\|_{X'\Lambda} \|Az\|_{Y}. \]

Proposition B.2 provides $\omega \in Y$ with $\|\omega\|_{Y} \leq M\|\xi\|_{X'\Lambda}$ and $A^{*}\omega = \xi \in \partial R(x)$. Together with Proposition 4.1 this yields the first inequality. Let $\omega \in Y$, such that $A^{*}\omega \in \partial R(x)$. Then
\[ \langle A^{*}\omega, z \rangle = \langle \omega, Az \rangle \leq \|\omega\|_{Y} \|Az\|_{Y} \leq M\|\omega\|_{Y} \|z\|_{X\Lambda} \]
for all $z \in X\Lambda$. Hence $\|A^{*}\omega\|_{X'\Lambda} \leq M\|\omega\|_{Y}$. This proves the second inequality. \quad \square

Computation of $K_{1}$ for weighted $p$-norm penalization. We revisit the first example in Subsection 2.2. Recall $X\Lambda = \ell_{p}^{\omega}$ and $X = \ell_{p}^{\omega}$ with $p \in (1, 2)$.

PROPOSITION 4.5. Let $\pi = (\frac{1}{p}, \frac{p}{p-1})$. Then $K_{1} = \ell_{\pi}^{2p-2}$ with
\[ \frac{1}{M}q_{1}(x) \leq \|x\|_{\ell_{\pi}^{2p-2}} \leq Mq_{1}(x) \quad \text{for all} \quad x \in \ell_{\pi}^{2}. \]

Proof. Let $x \in \ell_{\pi}^{2}$. Then $\partial R(x) = \{ \xi \} \text{ with } |\xi| = \nu_{j} = \nu_{j}^{p}|x_{j}|^{p-1}$. With $\overline{\nu}_{1}$ as in Proposition 4.4 and in view of Proposition B.1 we obtain
\[ \overline{\nu}_{1}(x) = \|\xi\|_{a^{-1},2} = \left( \sum_{j \in \Lambda} a_{j}^{-2} \nu_{j}^{2p} |x_{j}|^{2p-2} \right)^{1/2} = \|x\|_{\ell_{\pi}^{2p-2}}^{p-1}. \]

Proposition 4.4 yields the result. \quad \square

Computation of $K_{1}$ for Besov $0,p,p$-penalties. Next we characterize $K_{1}$ for Example 2. Recall $X\Lambda = b_{2,\alpha}^{\alpha}$ and $X = b_{p,q}^{0}$ with $p \in (1, \infty)$.

PROPOSITION 4.6. Let $\check{s} = \frac{\alpha}{p-1}$ and $\check{l} = 2p - 2$. Then $K_{1} = b_{\check{s},\check{l}}^{\check{a}}$ with
\[ \frac{1}{M}q_{1}(x) \leq \|x\|_{\ell_{\check{s},\check{l}}^{\check{a}}-1} \leq Mq_{1}(x) \quad \text{for all} \quad x \in b_{2,\alpha}^{\alpha}. \]

Proof. The proof works along the lines of the proof of Proposition 4.5 by identifying the expression for $\|\xi\|_{a,2,2}$ with $\|x\|_{\check{s},\check{l},\check{a}}$. \quad \square

Computation of $K_{1}$ for Besov $0,2,q$-penalties. Finally we compute $K_{1}$ for Example 3. with $X\Lambda = b_{2,\alpha}^{\alpha}$ and $X = b_{2,q}^{0}$ with $q \in (1, \infty)$.

PROPOSITION 4.7. Let $\check{s} = \frac{\alpha}{q-1}$ and $\check{q} = 2q - 2$. Then $K_{1} = b_{2,\check{q}}^{\check{a}}$ with
\[ \frac{1}{M}q_{1}(x) \leq \|x\|_{\ell_{\check{s},\check{q}}^{\check{a}}-1} \leq Mq_{1}(x) \quad \text{for all} \quad x \in b_{2,\alpha}^{\alpha}. \]

Proof. If $x \in b_{2,q}^{0}$, then $\partial R(x) = \{ \xi \} \text{ with } \xi_{j,k} = \left( \sum_{k'} |x_{j,k}|^{2} \right)^{1/2} \nu_{j}^{q-1} |x_{j,k}|$. With $\overline{\nu}_{1}$ as in Proposition 4.4 and using Proposition B.1 we obtain $\overline{\nu}_{1}(x) = \|\xi\|_{a,2,2} = \|x\|_{\ell_{\check{s},\check{q}}^{\check{a}}-1}^{q-1}$. Proposition 4.4 yields the result. \quad \square

Note that Assumption 2.2 holds true for all three examples.

4.3. Characterizations of $K_{\nu}$. 

K_\nu via approximation by elements of K_1. In [3, Prop. 1] the authors point out that the set of elements satisfying the source condition (1.1) is the set of possible minimizers of the Tikhonov functional. Therefore one might suggest that the approximation error of x \in X by x_\alpha \in R_\alpha(Ax) is determined by the best approximation from the family of sets

\[ B_r := \{ x \in X : g_1(x) \leq r \} \quad \text{with} \quad r \geq 0. \]

We consider the best approximation error

\[ \gamma_x : [0, \infty) \to [0, \infty) \quad \text{given by} \quad \gamma_x(r) = \inf_{z \in B_r} \| Ax - Az \|_Y. \]

The function \( \gamma_x \) is well defined as Corollary 4.2 yields \( \emptyset \neq \text{argmin}_{z \in X} R(z) \subseteq B_r \) for all \( r \geq 0 \). Moreover it is non increasing as \( B_{r_1} \subseteq B_{r_2} \) for \( r_1 \leq r_2 \).

The following proposition is the starting point to prove equivalence of Hölder-type bounds on \( \gamma_x \) and on \( \| Ax - Ax_\alpha \|_Y \).

**Proposition 4.8.** Let \( x \in X, \alpha > 0 \) and \( x_\alpha \in R_\alpha(Ax) \). Then

\[ \gamma_x \left( \frac{1}{\alpha} \| Ax - Ax_\alpha \|_Y \right) \leq \| Ax - Ax_\alpha \|_Y \leq 4 \gamma_x \left( \frac{1}{4\alpha} \| Ax - Ax_\alpha \|_Y \right). \]

**Proof.** Proposition 4.1 and the first order optimality condition \( \frac{1}{\alpha} A^* A(x - x_\alpha) \in \partial R(x_\alpha) \) provide \( g_1(x_\alpha) \leq \frac{1}{\alpha} \| Ax - Ax_\alpha \|_Y \). This proves the first inequality by definition of \( \gamma_x \).

To show the second inequality let \( z \in B_r \). By Proposition 4.1 there is \( \omega \in Y \) with \( \| \omega \|_Y \leq r \) and \( A^* \omega \in \partial R(z) \) hence

\[ R(z) - R(x_\alpha) \leq \langle A^* \omega, z - x_\alpha \rangle \leq r \| Az - Ax_\alpha \|_Y. \]

From \( 2\alpha T_\alpha(x_\alpha, Ax) \leq 2\alpha T_\alpha(z, Ax) \) and the last inequality we deduce

\[ \| Ax - Ax_\alpha \|_Y^2 \leq \| Ax - Az \|_Y^2 + 2\alpha r \| Ax - Ax_\alpha \|_Y \leq \| Ax - Az \|_Y^2 + 2\alpha r \| Ax - Ax_\alpha \|_Y + 2\alpha r \| Ax - Az \|_Y. \]

Taking the infimum over \( z \in B_r \) and estimating the third summand using \( ab \leq \frac{1}{2} a^2 + \frac{1}{2} b^2 \) we obtain

\[ \| Ax - Ax_\alpha \|_Y^2 \leq \gamma_x(r)^2 + 2\alpha r \gamma_x(r) + 2\alpha^2 r^2 + \frac{1}{2} \| Ax - Ax_\alpha \|_Y^2 \]

\[ \leq 2(\gamma_x(r) + \alpha r)^2 + \frac{1}{2} \| Ax - Ax_\alpha \|_Y^2. \]

Hence \( \| Ax - Ax_\alpha \|_Y \leq 2 \gamma_x(r) + 2\alpha r \) and the choice \( r = \frac{1}{4\alpha} \| Ax - Ax_\alpha \|_Y \) yields the second inequality.

As announced we see equivalence of Hölder-type bounds on \( \gamma_x \) and on \( \| Ax - Ax_\alpha \|_Y \) as a consequence.

**Proposition 4.9.** Let \( \nu \in (0, \infty) \) and \( x \in X \). The following statements are equivalent:

(i) There exists a constant \( c_1 > 0 \) such that \( \gamma_x(r) \leq c_1 r^{-\nu} \) for all \( r > 0 \).

(ii) There exists a constant \( c_2 > 0 \) such that \( \| Ax - Ax_\alpha \|_Y \leq c_2 \alpha^{1+\nu} \) for all \( \alpha > 0 \) and \( x_\alpha \in R_\alpha(Ax) \).
More precisely (i) implies (ii) with $c_2 = 4 c_1 \frac{1}{1 - \nu}$ and (ii) implies (i) with $c_1 = c_2^{1 + \nu}$.

**Proof.**

(i)⇒(ii): The second inequality in Proposition 4.8 yields

$$\|Ax - Ax_\alpha\|_Y \leq 4^{1+\nu} c_1 \alpha^\nu \|Ax - Ax_\alpha\|_Y^{\nu}$$

Multiplying by $\|Ax - Ax_\alpha\|_Y^{\nu}$ and taking the power $\frac{1}{1+\nu}$ yields

$$\|Ax - Ax_\alpha\|_Y \leq 4 c_1 \frac{1}{\alpha} \alpha^{\frac{1}{1+\nu}}$$

(ii)⇒(i): Let $r > 0$. For $\alpha = c_2^{1+\nu} r^{-(1+\nu)}$ we obtain $\frac{1}{\alpha} \|Ax - Ax_\alpha\|_Y \leq c_2 \alpha^{\frac{1}{1+\nu}} = r$.

Hence the first inequality in Proposition 4.8 yields

$$\gamma_x(r) \leq \gamma_x \left( \frac{1}{\alpha} \|Ax - Ax_\alpha\|_Y \right) \leq \|Ax - Ax_\alpha\|_Y \leq c_2 \alpha^{\frac{1}{1+\nu}} = c_2^{1+\nu} r^{-\nu}. \quad \square$$

**$K_\nu$ via real interpolation.** Again we assume $X_A$ is a Banach space such that (2.4) holds true. The next lemma shows that under Assumption 2.2 the spaces $(X_A, K_1)_{\theta, \infty}$ classify the image space approximation precision.

**Proposition 4.10 ($K_\nu$ as a real interpolation space).** Suppose Assumption 2.2 holds true. Let $\theta \in (0, 1)$ and $\nu := \frac{\theta}{1-\theta(u-1)+\nu}$. We have $K_\nu = (X_A, K_1)_{\theta, \infty}$ with

$$C_1 \|x\|_{(X_A, K_1)_{\theta, \infty}} \leq \varrho_\nu(x) \leq C_2 \|x\|_{(X_A, K_1)_{\theta, \infty}}$$

for all $x \in (X_A, K_1)_{\theta, \infty}$ with constants $C_1, C_2 > 0$ depending only on $u, \theta$ and $M$.

**Proof.** Assume $\varrho := \varrho_\nu(x) < \infty$. Proposition 4.9 provides the bound

$$\gamma_x(r) \leq \varrho \frac{1}{(1-\theta)(u-1)+\theta} r^{-\frac{\nu}{(1-\theta)(u-1)}}$$

Let $t > 0$. We choose $r := \varrho^{\frac{1}{(1-\theta)(u-1)+\theta}} t^{-\frac{1}{(1-\theta)(u-1)}}$. If $\varepsilon > 0$ then there exists $z \in K_1$ with $\varrho_1(z) \leq r$ and $\|Ax - Az\|_Y \leq \gamma_x(r) + \varepsilon$. Therefore we obtain

$$K(x, t) \leq \|x - z\|_{X_A} + t \|z\|_{\text{lin}} \leq M(\gamma_x(r) + \varepsilon) + t M^{-\frac{1}{1-\nu}} r^{\frac{1}{1-\nu}}.$$ 

For $\varepsilon \to 0$ we obtain

$$K(x, t) \leq M \varrho \frac{1}{(1-\theta)(u-1)+\theta} r^{-\frac{\nu}{(1-\theta)(u-1)}} + t M^{-\frac{1}{1-\nu}} r^{\frac{1}{1-\nu}} = \left(M + M^{-\frac{1}{1-\nu}}\right) \varrho \frac{1}{(1-\theta)(u-1)+\theta} t^{\theta}.$$ 

This proves the first inequality. Assume $n := \|x\|_{(X_A, K_1)_{\theta, \infty}} < \infty$. We prove a bound on $\gamma_x$ and apply Proposition 4.9.

Let $r > 0$. We choose $t := 2 M \frac{1}{(1-\theta)(u-1)+\theta} n \frac{1}{1-\nu} r^{-\frac{1}{(1-\theta)(u-1)}}$. Since $2^{1-\theta} > 1$ there exists $z \in X_A$ such that

$$M^{-1} \|Ax - Az\| + t M^{-\frac{1}{1-\nu}} \varrho_1(z) \frac{1}{n^{\frac{1}{1-\nu}}} \leq \|x - z\|_{X_A} + t \|z\|_{\text{lin}} \leq 2^{1-\theta} K(x, t) \leq 2^{1-\theta} n t^{\theta} = t M^{-\frac{1}{1-\nu}} r^{\frac{1}{1-\nu}}.$$ 

Neglecting the first summand on the left hand side we obtain $\varrho_1(z) \leq r$. Therefore

$$\gamma_x(r) \leq \|Ax - Az\|_Y \leq M 2^{1-\theta} n t^{\theta} = 2 M \frac{1}{(1-\theta)(u-1)+\theta} \frac{1}{n^{\frac{1}{1-\nu}}} r^{-\frac{1}{(1-\theta)(u-1)}}.$$ 

Proposition 4.9 yields $\varrho_\nu(x) \leq 8 M \frac{1}{n^{\frac{1}{1-\nu}}}.$ \quad \square
Remark 4.11. As already exposed in Example 4.3 we cannot expect Assumption 2.2 to hold true for $\ell^1$-type norms like Besov 0, 1, 1 or 0, 2, 1-norms. Nevertheless one may use Proposition 4.9 directly to characterize the sets $K_\nu$ in this case. Applying Theorem 1 then reproduces the convergence rates results for the 0, 2, 1-penalty in [17] and for weighted $\ell^1$-penalties in [20] in the case of linear operators.

4.4. Error bounds. We apply Theorem 1 to obtain error bounds measured in the norm of certain Banach spaces $X_L$ with a continuous embedding $X_L \subset X_A$.

To this end we consider the loss function $L: X \times X \rightarrow [0, \infty]$ given by $L(x_1, x_2) = \|x_1 - x_2\|_{X_L}$ if $x_1 - x_2 \in X_L$ and $L(x_1, x_2) = \infty$ if $x_1 - x_2 \notin X_L$. Before we prove Theorem 2 we state a proposition that characterizes for which spaces $X_L$ Hölder-type bounds on the modulus of continuity on balls of a given quasi-Banach space $X_S \subset X$ are satisfied.

**Proposition 4.12** (bound on the modulus). Let $X_S \subset X$ be a quasi-Banach space and $X_L$ a Banach space with continuous embeddings $X_S \subset X_L \subset X_A$ and $e \in (0, 1)$. For $\varrho > 0$ we denote

$$K_{X_S}^\varrho := \{x \in X_S: \|x\|_{X_S} \leq \varrho\}.$$

The following statements are equivalent:

(i) There is a continuous embedding $(X_A, X_S)_{e, 1} \subset X_L$.

(ii) There exists a constant $c > 0$ with $\Omega(\delta, K_{X_S}^\varrho) \leq c \varrho^e \delta^{1-e}$ for all $\delta, \varrho > 0$.

**Proof.** By [2, Sec. 3.5, Thm. 3.11.4] statement (i) is equivalent to an interpolation inequality

$$\|z\|_{X_L} \leq C \|z\|_{X_A}^{1-e} \|z\|_{X_S}^e \quad \text{for all } z \in X_S. \tag{4.1}$$

Let $x_1, x_2 \in K_{X_S}^\varrho$ with $\|Ax_1 - Ax_2\|_Y \leq \delta$. The quasi-triangle inequality yields $\|x_1 - x_2\|_{X_S} \leq 2c\varrho$ and from (2.4) we obtain $\|x_1 - x_2\|_{X_A} \leq M\delta$. Hence (4.1) with $z = x_1 - x_2$ yields $\|x_1 - x_2\|_{X_L} \leq CM^{1-e}(2c)^e \varrho^e \delta^{1-e}$. Taking the supremum over $x_1, x_2$ yields (ii).

Assuming a bound on the modulus we obtain (4.1) from

$$\|z\|_{X_L} \leq \Omega \left(M\|z\|_{X_A}, K_{X_S}^\varrho\right). \quad \square$$

Next we give the proof of Theorem 2.

**Proof of Theorem 2.** For $\nu$ as in Proposition 4.10 the second inequality therein yields

$$x \in K_{\nu}^0_{(X_A, K_1)_{e, \infty}} \subset K_{\nu}^{\overline{\nu}}$$

with $\overline{\nu} = (C_2 \varrho)^{1/(1-e)/(1-e) + \vartheta}$. In view of Theorem 1 it remains to prove an upper bound on $\Omega(c_1 \delta, K_{\nu}^{\overline{\nu}}) \leq C_2 \delta^{1/2}$ for constants $c_1, c_2 > 0$ given therein. The first inequality in Proposition 4.10 provides

$$K_{\nu}^{\overline{\nu}} \subset K_{(X_A, K_1)_{e, \infty}}^{c_3 \varrho} \quad \text{with} \quad c_3 = C_1^{-1} C_2 c_2^{1/2 - \theta/(1-e) + \vartheta/(1-e) + \vartheta}.$$

The reiteration theorem (see [2, Thm. 3.11.5]) yields

$$\tag{4.2} (X_A, K_1)_{e, 1} = \left((X_A, (X_A, K_1)_{e, \infty})_{e, 1}\right)^{1/2}.$$
Proposition 4.12 with \( \mathcal{X}_S = (\mathcal{X}_A, K_1)_{\theta, \infty} \subset \mathcal{X}_L \subset \mathcal{X}_A \). Hence Proposition 4.12 with \( \mathcal{X}_S = (\mathcal{X}_A, K_1)_{\theta, \infty} \) yields a constant \( c_4 \) with

\[ \Omega \left( c_1 \delta, K_{\nu, \infty} \right) \leq \Omega \left( c_1 \delta, K^c_{(\mathcal{X}_A, K_1)_{\theta, \infty}} \right) \leq C \theta^d \delta^{1 - \frac{d}{p}} \]

with \( C = c_4 \theta^d c_1^{-\frac{d}{p}} \).

For the discrepancy principle the bound \( \Omega(d_1 \delta, K^c_{\nu, \infty}) \leq C \theta^d \delta^{1 - \frac{d}{p}} \) follows by replacing \( c_1 \) by \( d_1 \) and \( c_2 \) by \( d_2 \).

Remark 4.13. The statement in Remark 2.3 for the limiting case \( \theta = 1 \) follows along the same lines leaving out the step involving the reiteration theorem.

Remark 4.14. The relation \( (\mathcal{X}_A, K_1)_{\theta, 1} \subset \mathcal{X}_L \) is necessary to obtain error bounds as in Theorem 2 in the following sense: Assuming \( \mathcal{X}_L \) satisfies an error bound

\[ \| x - \hat{x}_\alpha \|_{\mathcal{X}_L} \leq C \theta^d \delta^{1 - \epsilon} \]

for some \( \epsilon < 1 \) and all \( x \in K^e_{(\mathcal{X}_A, K_1)_{\theta, \infty}} \) under some apriori parameter choice \( \alpha = \alpha(\delta) \), then the lower bound (2.3) yields

\[ \frac{1}{2} \Omega(2\delta, K^e_{(\mathcal{X}_A, K_1)_{\theta, \infty}}) \leq \Delta R_{\alpha(\delta)}(\delta, K^e_{(\mathcal{X}_A, K_1)_{\theta, \infty}}) \leq C \theta^d \delta^{1 - \epsilon}. \]

Thus the converse implication in Proposition 4.12 and the identity (4.2) provides

\[ (\mathcal{X}_A, K_1)_{\theta, 1} = (\mathcal{X}_A, (\mathcal{X}_A, K_1)_{\theta, \infty})_{\epsilon, 1} \subset \mathcal{X}_L. \]

Error bounds for weighted \( p \)-norm penalization. To prove Corollary 2.4 we return to the setting of Example 1.

Proof of Corollary 2.4. First note that Assumption 2.2 holds true by Proposition 4.5. By [8, Thm. 2, Rem.] we have

\[ \ell^p_{\tau} = \left( \ell_{\tau}^2, \ell_{\tau}^{2p-2} \right)_{\xi, p} = (\mathcal{X}_A, K_1)_{\xi, p} \quad \text{with} \quad \xi := \frac{p-1}{p}. \]

Hence by [2, Thm. 3.4.1 (b); Sec. 3.11] there is a continuous embedding \( \left( \ell^2_{\tau}, \ell_{\tau}^{2p-2} \right)_{\xi, 1} \subset \ell^p_{\tau} \). Hence the choice \( \mathcal{X}_L = \ell^p_{\tau} \) satisfies the assumption of Theorem 2. The interpolation spaces \( (\mathcal{X}_A, K_1)_{\theta, \infty} = \left( \ell^2_{\tau}, \ell_{\tau}^{2p-2} \right)_{\theta, \infty} \) are characterized by weighted weak \( p \)-spaces \( \ell_{\mu, \nu}^{\ell, \infty} \) in the following manner:

\[ \ell_{\mu, \nu}^{\ell, \infty} = \left( \ell^2_{\tau}, \ell_{\tau}^{2p-2} \right)_{\theta, \infty} \quad \text{with} \quad \frac{1}{\mu} = \frac{1}{2} - \frac{\theta}{2p-2}, \mu := (\tau^{2p-2})^{\frac{1}{2p-2}}, \nu := (\tau^{-1})^{\frac{2p}{2p-2}} \]

with equivalent quasi-norms (see [8, Thm. 2]).

The application of Theorem 2 yields Corollary 2.4 and Remark 2.5 follows from Remark 2.3.

Error bounds for Besov \( 0, p, p \)-penalties. Next we revisit Example 2.

Proof of Corollary 2.7. Here Assumption 2.2 holds true by Proposition 4.6. The identification [8, Thm. 2, Rem.] for \( p \neq 2 \) and [27, 3.3.6.9] for \( p = 2 \) yield

\[ b^0_{p, p} = \left( b^0_{\alpha, 2}, b^0_{2, \xi} \right)_{\xi, p} = (\mathcal{X}_A, K_1)_{\xi, p} \quad \text{with} \quad \xi := \frac{p-1}{p}. \]
Hence the choice $X_L = b^0_{p_0, p}$ satisfies the assumption in Theorem 2. We apply Theorem 2 to obtain Corollary 2.7. Remark 2.8 follows from Remark 2.3.

Furthermore we prove the nestings given in Proposition 2.9.

Proof of Proposition 2.9. Let $\theta = \frac{p - 1 + a}{a}$. Then $\frac{a}{\theta} = \frac{1 - \delta}{2} + \frac{\theta}{t}$. With [8, Thm. 2, Rem.] and [2, Thm. 3.4.1 (b)] we obtain

$$b^\theta_{t, t} = \left( b^{-a}_{2, 2}, b^\frac{a}{\theta}_{t, t} \right)_{\theta, t} \subset \left( b^{-a}_{2, 2}, b^\frac{a}{\theta}_{t, t} \right)_{\theta, \infty} = k_s$$

in both cases. Suppose $p < 2$. Then $\tilde{t} < 2$ and $t \in (\tilde{t}, 2)$. Let $\varepsilon > 0$ such that $t - \varepsilon \in (\tilde{t}, 2)$. There are $s < s' < s$ and $\theta < \theta' < 1$ such that $b^\theta_{t - \varepsilon, t - \varepsilon} = \left( b^{-a}_{2, 2}, b^{\theta'}_{t, t} \right)_{\theta', t - \varepsilon}$. The reiteration theorem (see [2, Thm. 3.11.5]) yields $k_s = \left( b^{-a}_{2, 2}, b^{\theta'}_{t, t} \right)_{\theta', t - \varepsilon}$. From $t - \varepsilon < 2$ we obtain the continuous embeddings $b^{-a}_{2, 2} \subset b^{\theta'}_{\infty, \infty} \subset b^{a}_{\infty, \infty}$ (see [27, 3.2.4(1), 3.3.1(9)]). Together with the interpolation result $b^{-a}_{t - \varepsilon, \infty} = \left( b^{-a}_{t - \varepsilon, \infty}, b^{\theta'}_{t, \infty} \right)_{\infty}$ (see [27, 3.3.6(9)]) we obtain the second inclusion using [27, 2.4.1 Rem. 4]. By [27, 3.3.1(9)] therefore obtain the second inclusion for all $0 < \epsilon < t$. For $p > 2$ we have $b^{\theta}_{t, t} \subset b^{\frac{a}{\theta}}_{t, t}$ (see [27, 3.3.1(9)]). Hence [27, 3.3.6(9)] and [27, 2.4.1 Rem. 4] yield $k_s \subset \left( b^{-a}_{2, 2}, b^{\frac{a}{\theta}}_{t, t} \right)_{\theta, \infty} = b^{\frac{a}{\theta}}_{2, \infty}$. 

Error bounds for Besov 0, 2, $q$-penalties. Next we treat Example 3.

Proof of Corollary 2.10. Due to Proposition 4.7, the Assumption 2.2 is satisfied. By [27, 3.3.6(9)] we have

$$b^0_{2, 2} = \left( b^{-a}_{2, 2}, b^\frac{a}{\theta}_{2, 2} \right)_{\xi, 2} \text{ with } \xi = \frac{q - 1}{q}.$$

Therefore the choice $X_L = b^0_{2, 2}$ satisfies the assumption on $X_L$ in Theorem 2. Moreover for $0 < s < \frac{q}{q - 1}$ we have

$$b^{\theta}_{2, \infty} = \left( b^{-a}_{2, 2}, b^\frac{a}{\theta}_{2, \infty} \right)_{\theta, 2} \text{ with } \theta = \frac{q - 1 + a}{q}.$$

Hence the application of Theorem 2 yields Corollary 2.10 and Remark 2.3 yields Remark 2.11.

Error bounds for the Radon transform. Finally we turn to proof of the convergence rate result with the Radon transform as forward operator.

Proof of Corollary 2.12. 1. Since $a < s_{\text{max}}$ the synthesis operator $S$ is a norm isomorphism $b^{-a}_{2, 2} \rightarrow B^{s}_{2, 2}(\Omega)$. Hence the operator $R \circ S$ satisfies (2.4) with $X_A = b^{-a}_{2, 2}$. The inequality $\frac{a}{p} - \frac{d}{q} \leq a$ implies $s_t - s_{\text{max}} \leq s_t \leq s$. Hence $S: b^t_{t, t} \rightarrow B^{s}_{t, t}(\Omega)$ is a norm isomorphism. Let $c_1$ be the operator norm of the inverse of $S$. Then $f = Sx$ with $x \in b^t_{t, t}$ and $\|x\|_{s, t} \leq c_1 \theta$. Let $c_2$ be the embedding constant of $b^{s}_{t, t}$ (see Proposition 2.9). Then we obtain $x \in k_s$ with $\|x\|_{k_s} \leq c_1 c_2 \theta$. With $\tilde{x}_\alpha$ given by Corollary 2.7 we obtain the bound

$$\|x - \tilde{x}_\alpha\|_{0, p, p} \leq \tilde{C} \theta^{\frac{a}{q} - \frac{d}{q}} \frac{\alpha}{\delta}.$$
with a constant $C > 0$ independent of $f, \delta$ and $g$. Hence the first bound in Corollary 2.12 implies
\[
\|f - \hat{f}_\alpha\|_{B_{p,p}(\Omega)} = \|S(x - \hat{x}_\alpha)\|_{B_{p,p}(\Omega)} \leq c_3\|x - \hat{x}_\alpha\|_{0,p,p}
\]
with $c_3$ the operator norm of $S: B_{p,p}^0(\Omega) \to B_{p,p}^0(\Omega)$. The bound in the $L^p$-norm for $p \leq 2$ follows from the continuity of the embedding $B_{p,p}^0(\Omega) \subset L^p(\Omega)$ (see [27]).

2. This follows along the lines of the proof of 1. using Corollary 2.10 instead of Corollary 2.7.

5. Connection to other source conditions. In this section we return to the setting of Subsection 2.1 and assume only the assumptions in the first lines of Subsection 2.1. The aim of this section is the proof Theorem 3.

5.1. A preliminary: differentiability of the minimal value function.

Definition 5.1 (minimal value function). For $g \in Y$ we define
\[
\vartheta_g : (0, \infty) \to \mathbb{R} \quad \text{by} \quad \vartheta_g(\alpha) = \inf_{x \in \text{dom}(R)} T_\alpha(x, g) = \frac{1}{2\alpha}\|g - A\hat{x}_\alpha\|^2_\Omega + R(\hat{x}_\alpha).
\]

independent of the choice $\hat{x}_\alpha \in R_\alpha(g)$.

The main result of this subsection is the differentiability of the minimal value function. The approximation error $\|g - A\hat{x}_\alpha\|_Y$ is represented by calculus rules of $\vartheta_g$.

Recall that the Moreau envelope function of some function $Q: Y \to (-\infty, \infty]$ for $\alpha > 0$ is given by
\[
Q_\alpha(g) = \inf_{y \in Y} \left( \frac{1}{2\alpha}\|g - y\|^2_Y + Q(y) \right)
\]

and the infimum is uniquely attained at $\text{Prox}_{\alpha Q}(g) \in Y$. The key ingredient is the following result by T. Strömberg:

Lemma 5.2. (see [25, Prop. 3(iii)]) Let $Q: Y \to (-\infty, \infty]$ be convex, proper and lower semi-continuous. The family of Moreau envelope functions $Q_\alpha : Y \to \mathbb{R}, \alpha > 0$ satisfies
\[
\frac{\partial}{\partial \alpha} Q_\alpha(g) = -\frac{1}{2\alpha}(\nabla Q_\alpha(g)|\|^2_Y.
\]

We apply Lemma 5.2 to the function $Q$ defined in Lemma 3.1. Note that due to Proposition 3.3 we have
\[
Q_\alpha(g) = \frac{1}{2\alpha}\|g - \text{Prox}_{\alpha Q}(g)\|^2_Y + Q(\text{Prox}_{\alpha Q}(g)) = \vartheta_g(\alpha).
\]

Proposition 5.3. Let $g \in Y$ and $\hat{x}_\alpha \in R_\alpha(g), \alpha > 0$ any selection. The function $\vartheta_g$ is convex, non-increasing and continuously differentiable with
\[
\vartheta_g'(\alpha) = -\frac{1}{2\alpha^2}\|g - A\hat{x}_\alpha\|^2_Y.
\]

Proof. The Moreau envelope function $Q_\alpha$ is convex, real valued and continuous with the Fenchel conjugate $(Q_\alpha)^* = Q^* + \frac{\alpha}{2}\|\cdot\|^2_Y$ (see [1, Prop. 12.15; Prop. 13.21]).

The biconjugation theorem implies
\[
\vartheta_g(\alpha) = Q_\alpha(g) = \left(Q^* + \frac{\alpha}{2}\|\cdot\|^2_Y\right)^*(g) = \sup_{v \in Y} \left(\langle g, v \rangle - Q^*(v) - \frac{\alpha}{2}\|v\|^2_Y\right).
\]
Hence, \( \vartheta_g \) is convex and non-increasing being the supremum of affine non-increasing functions.

By [1, Prop. 12.29] \( Q_\alpha \) is Fréchet differentiable with \( \nabla Q_\alpha = \frac{1}{\alpha}(\text{Id}_Y - \text{Prox}_\alpha Q) \).

Lemma 5.2 yields differentiability of \( \alpha \mapsto Q_\alpha(g) \) with derivative \( -\frac{1}{2\alpha^2}\|g - \text{Prox}_\alpha Q(g)\|_Y \) for all \( g \in Y \). Therefore, \( \vartheta_g \) is differentiable and we conclude with Proposition 3.3

\[
\vartheta'_g(\alpha) = -\frac{1}{2}\|Q_\alpha(g)\|^2 = -\frac{1}{2\alpha^2}\|g - \text{Prox}_\alpha Q(g)\|_Y = -\frac{1}{2\alpha^2}\|g - A\hat{\alpha}\|_Y.
\]

Finally, \( \vartheta'_g \) is continuous as \( \vartheta_g \) is convex and differentiable.

\[\] 5.2. Defect function and its link to variational source conditions. For the rest of this paper we always assume \( x \in \text{dom}(R) \) is \( R \)-minimal in \( A^{-1}(\{Ax\}) \) and \( x_\alpha \in R_\alpha(Ax) \) for \( \alpha > 0 \) is any selection of a minimizer for exact data.

If \( A \) is injective then the minimality is trivially satisfied for all \( x \in \text{dom}(R) \). As already mentioned we consider the defect of the Tikhonov functional \( \sigma_x : (0, \infty) \to (0, \infty) \) given by

\[
\sigma_x(\alpha) = T_\alpha(x, Ax) - T_\alpha(x_\alpha, Ax) = R(x) - R(x_\alpha) - \frac{1}{2\alpha}\|Ax - Ax_\alpha\|_Y^2.
\]

The next proposition collects properties of the defect function.

**Lemma 5.4.**
1. \( \sigma_x \) is concave, non-decreasing and continuously differentiable with \( \sigma'_x(\alpha) = \frac{1}{2\alpha^2}\|Ax - Ax_\alpha\|_Y^2 \).
2. We have \( \lim_{\alpha \to 0} \sigma_x(\alpha) = 0 \).
3. The function \( (0, \infty) \to (0, \infty) \) given by \( \alpha \mapsto \sigma_x(\frac{1}{\alpha}) \) is convex and continuous.

**Proof.** We have \( \sigma_x(\alpha) = R(x) - \vartheta_{Ax}(\alpha) \) with the minimal value function \( \vartheta_{Ax} \) from Definition 5.1. Hence 1. follows from Proposition 5.3. Lemma A.2 yields 2. because of the \( R \)-minimality assumption on \( x \). Let \( h \) be the function given in 3. Then \( h \) is differentiable and 1. yields

\[
h'(\alpha) = -\frac{1}{\alpha} \sigma'_x(\frac{1}{\alpha}) = -\frac{1}{2}\|Ax - Ax_\frac{1}{\alpha}\|_Y.
\]

By A.1.2. the function \( \alpha \mapsto \|Ax - Ax_\alpha\|_Y \) is non-decreasing. Hence \( h' \) is non-decreasing. Therefore \( h \) is convex. Continuity follows from the first statement.

Let \( \alpha > 0 \). We write

\[
\sigma_x(\alpha) = \sup_{z \in X} \left( R(x) - R(z) - \frac{1}{2\alpha}\|Az - Ax\|^2 \right)
\]

to note a similarity to the distance function in [7, (3.1)] and [6, Chapter 12] and [16, Chapter 3] used to derive variational source conditions of the form (2.8). In [16, Prop. 4] its shown that a variational source condition (2.8) implies bounds on the defect function \( \sigma_x \). The next result provides a sharp connection between bounds on the defect function and variational source conditions. We introduce two partially ordered sets of functions

\[
\Sigma = \{ \sigma : (0, \infty) \to [0, \infty] : \sigma \text{ is proper, non-decreasing and } \sigma(1/\cdot) \text{ is convex l.s.c.} \}
\]
\[
\Phi = \{ \phi : [0, \infty) \to [0, \infty] : \phi \text{ concave and upper semi-continuous} \}
\]
with pointwise ordering. Here l.s.c. is an abbreviation for lower semi-continuous. Moreover, we consider the map $F: \Sigma \to \Phi$ given by

$$(5.2) \quad (F(\sigma))(t) := \inf_{\alpha > 0} \left( \sigma(\alpha) + \frac{1}{2\alpha} t \right) \text{ for } t \geq 0.$$ 

In Lemma C.2 we prove that $F$ is well-defined, order preserving and bijective. The order preserving inverse $F^{-1}: \Phi \to \Sigma$ is given by

$$(5.3) \quad (F^{-1}(\phi))(\alpha) = \sup_{t \geq 0} \left( \phi(t) - \frac{1}{2\alpha} t \right) \text{ for } \alpha > 0.$$ 

By Lemma 5.4, we have $\sigma_x \in \Sigma$. It turns out that $\phi = F(\sigma_x)$ is the minimal function in $\Phi$ satisfying (2.8).

**Lemma 5.5.** Let $\phi \in \Phi$. Then the following statements are equivalent:

(i) $F(\sigma_x) \leq \phi$

(ii) $\sigma_x \leq F^{-1}(\phi)$

(iii) $R(x) - R(z) \leq \phi(\|Ax - Az\|_Y^2)$ for all $z \in X$.

In particular, we always have

$$R(x) - R(z) \leq (F(\sigma_x))(\|Ax - Az\|_Y^2) \text{ for all } z \in X \quad (5.4)$$

**Proof.** The equivalence of (i) and (ii) is immediate by Lemma C.2. Next we prove (5.4). To this end let $z \in X$ and $\alpha > 0$. Then

$$T_\alpha(x, Ax) \leq \frac{1}{2\alpha} \|Ax - Az\|_Y^2 + R(z)$$

and $R(x) = T_\alpha(x, Ax)$. We obtain

$$R(x) - R(z) = T_\alpha(x, Ax) - T_\alpha(x, Ax) + T_\alpha(x, Ax) - R(z)$$

$$\leq \sigma_x(\alpha) + \frac{1}{2\alpha} \|Ax - Az\|_Y^2.$$

Taking the infimum over $\alpha$ on the right hand side yields (5.4).

Hence (i) implies (iii). Assuming (iii) we estimate

$$\sigma_x(\alpha) \leq \phi(\|Ax - Ax_\alpha\|_Y^2) - \frac{1}{2\alpha} \|Ax - Ax_\alpha\|_Y^2$$

$$\leq \sup_{t \geq 0} \left( \phi(t) - \frac{1}{2\alpha} t \right) = (F^{-1}(\phi))(\alpha).$$

Hence $\sigma_x \leq F^{-1}(\phi)$. This yields (i) as $F$ is order preserving.

**Remark 5.6.** Inequality (5.4) is sharp for $z = x_\alpha \in R_\alpha(g)$ for all $\alpha > 0$. To see this note that by definition $(F(\sigma_x))(t) \leq \sigma_x(\alpha) + \frac{1}{2\alpha} t$ for all $t \geq 0$ and $\alpha > 0$. By (5.4) we have

$$R(x) - R(x_\alpha) \leq (F(\sigma_x))(\|Ax - Ax_\alpha\|_Y^2)$$

$$\leq \sigma_x(\alpha) + \frac{1}{2\alpha} \|Ax - Ax_\alpha\|_Y^2$$

$$= R(x) - R(x_\alpha).$$
5.3. Link between defect function and image space approximation. The result of this subsection is that \( \sigma_x \) and hence also the smallest index function \( \phi \) allowing for a variational source condition \((2.8)\) depends only on the net \((\|Ax - Ax_\alpha\|_Y)_\alpha > 0\). Further we will exploit a condition when a bound \(\|Ax - Ax_\alpha\|_Y \leq \psi(\alpha)\) implies a bound on the defect function \(\sigma_x\).

**Lemma 5.7.** We have
\[
(5.5) \quad \sigma_x(\alpha) = \int_0^\alpha \frac{1}{2\beta^2} \|Ax - Ax_\beta\|_Y^2 \, d\beta \quad \text{for all } \alpha > 0.
\]

*Proof.* Let \(0 < \varepsilon < \alpha\). Lemma 5.4.1 yields
\[
\sigma_x(\alpha) - \sigma_x(\varepsilon) = \int_\varepsilon^\alpha \sigma'_x(\beta) \, d\beta = \int_\varepsilon^\alpha \frac{1}{2\beta^2} \|Ax - Ax_\beta\|_Y^2 \, d\beta.
\]
In view of Lemma 5.4.2, the expression for \(\sigma_x\) follows by taking the limit \(\varepsilon \to 0\).

**Proposition 5.8 (Image space approximation).**
1. We have
\[
\|Ax - Ax_\alpha\|_Y \leq \sqrt{2\alpha \sigma_x(\alpha)} \quad \text{for all } \alpha > 0.
\]
2. Let \(\psi: [0, \infty) \to [0, \infty)\) be continuous. Assume that there is a constant \(C_\psi > 0\) with
\[
(5.6) \quad \int_0^\alpha \frac{1}{\beta} \psi(\beta) \, d\beta \leq C_\psi \psi(\alpha) \quad \text{for all } \alpha > 0.
\]
Then a bound \(\|Ax - Ax_\alpha\|_Y \leq \sqrt{2\alpha \psi(\alpha)}\) for all \(\alpha > 0\) implies \(\sigma_x(\alpha) \leq C_\psi \psi(\alpha)\) for all \(\alpha > 0\).

*Proof.* 1. By Lemma 5.4 the continuous extension of \(\sigma_x\) to \([0, \infty)\) is concave. Hence the claim follows from
\[
\frac{1}{2\alpha^2} \|Ax - Ax_\alpha\|_Y^2 = \sigma'_x(\alpha) \leq \frac{1}{\alpha} \sigma_x(\alpha).
\]
2. Using \((5.5)\) and \((5.6)\) we obtain
\[
\sigma_x(\alpha) = \int_0^\alpha \frac{1}{2\beta^2} \|Ax - Ax_\beta\|_Y^2 \, d\beta \leq \int_0^\alpha \frac{1}{\beta} \psi(\beta) \, d\beta \leq C_\psi \psi(\alpha).
\]

5.4. Equivalence theorem for Hölder-type bounds.

*Proof of Theorem 3.*
(i)\(\Rightarrow\)(ii): Consider the continuous function
\[
\psi: [0, \infty) \to [0, \infty) \quad \text{given by } \psi(\alpha) = \frac{1}{2} c_1^2 \alpha^{2\nu - 1}.
\]
Then \(c_1 \alpha^\nu = \sqrt{2\alpha \psi(\alpha)}\) for all \(\alpha > 0\). We have
\[
\int_0^\alpha \frac{1}{\beta} \psi(\beta) \, d\beta = \frac{1}{2} c_1^2 \int_0^\alpha \beta^{2\nu - 2} \, d\beta = \frac{1}{2\nu - 1} \psi(\alpha).
\]
Hence \((5.6)\) is satisfied with \(C_\psi = \frac{1}{2\nu - 1}\). Proposition 5.8. implies \(\sigma_x(\alpha) \leq \frac{c_1^2}{4\nu - 2} \alpha^{2\nu - 1}\).
Corollary 3.5

Lemma 5.5

Remark 2.5

Proposition 5.3

Lemma A.1

Proposition 5.3

Lemma 3.7

Another direction is the application to further concrete settings as in the three presented examples. An idea is to formulate a weaker version of Assumption 2.2 by require a nesting $X_{1a} \subseteq K_1 \subseteq X_{1b}$ with quasi-Banach spaces $X_{1a}, X_{1b}$ and try to prove a generalized version of Theorem 2. The author believes that this approach would cover e.g. Besov norm penalties with mixed indices $p, q$ with $p \neq 2$.

Appendix A. Elementary facts from regularization theory.

Lemma A.1. Let $g \in \mathcal{Y}$ and $\hat{x}_\alpha \in R_\alpha(g)$, $\alpha > 0$ any selection.

1. The function $(0, \infty) \rightarrow \mathbb{R}$ given by $\alpha \mapsto R(\hat{x}_\alpha)$ is non increasing.

2. The function $(0, \infty) \rightarrow [0, \infty)$ given by $\alpha \mapsto \|g - A \hat{x}_\alpha\|_\mathcal{Y}$ is non decreasing.

3. The function $(0, \infty) \rightarrow (0, \infty)$ given by $\alpha \mapsto \|g - A \hat{x}_\alpha\|_\mathcal{Y}$ is non increasing.

Proof. To prove 1.2. let $\alpha < \beta$. Set $m = \frac{1}{2}\|g - A \hat{x}_\alpha\|_\mathcal{Y}^2 - \frac{1}{2}\|g - A \hat{x}_\beta\|_\mathcal{Y}^2$. From $T_\alpha(\hat{x}_\alpha, g) \leq T_\alpha(\hat{x}_\beta, g)$ and $T_\beta(\hat{x}_\beta, g) \leq T_\beta(\hat{x}_\alpha, g)$ we obtain

$$m \leq \alpha (R(\hat{x}_\beta) - R(\hat{x}_\alpha)) \leq \frac{\alpha}{\beta} m.$$ 

Hence $m \leq 0$. 3. follows from Proposition 5.3.  

6. Discussion and Outlook. We close this paper by addressing some open questions and possible extensions.

The identification of $A \in R_\alpha$ as a proximity mapping (see Subsection 3.1) seems to be a new structural insight in convex regularization theory. It allows to apply convex analysis tools leading to interesting statements and new simple proofs (see e.g. Corollary 3.5, Proposition 5.3, Lemma 3.7, Lemma A.1). So far the presented theory is limited to Hilbert space data fidelity terms. It would be interesting to generalize the arguments in Section 3 to Banach spaces $\mathcal{Y}$. A generalization to nonlinear operators seems even more challenging.

So far the presented theory is restricted to Hölder-type convergence rates. To also cover exponentially ill-posed problems it is of interest to investigate logarithmic convergence rates and source conditions. At first sight condition (5.6) seems to fail for index functions not of Hölder-type. Thus it remains open whether an equivalence between image space approximation rates and variational source conditions remains valid for more general upper bounds.

As for approaches using variational source conditions the fastest convergence rate we are able to prove for a $p$-homogeneous penalty term is $O(\frac{1}{\sqrt{t}})$ (see Remark 2.5, Remark 2.8 and Remark 2.11). It seems to be an interesting question to extend the presented approach to higher order convergence rates.

Another direction is the application to further concrete settings as in the three presented examples. An idea is to formulate a weaker version of Assumption 2.2 by require a nesting $X_{1a} \subseteq K_1 \subseteq X_{1b}$ with quasi-Banach spaces $X_{1a}, X_{1b}$ and try to prove a generalized version of Theorem 2. The author believes that this approach would cover e.g. Besov norm penalties with mixed indices $p, q$ with $p \neq 2$. 

(ii)$\Rightarrow$(iii): For $\sigma(\alpha) := c_2 \alpha^{2\nu - 1}$ inserting $\alpha = \left(\frac{1}{2}\right)^{\frac{1}{\mu}}$ yields

$$(F(\sigma))(t) = \inf_{\alpha > 0} \left( c_2 \alpha^{2\nu - 1} + \frac{1}{2\alpha} t \right) \leq 2^{\frac{1}{\mu}} c_2^{\frac{1}{\mu}} t^{\frac{2\nu}{\mu}} \leq 2 c_2^{\frac{1}{\mu}} t^{\frac{2\nu}{\mu}}.$$ 

Lemma 5.5 with $\phi = F(\sigma)$ yields the claim.

(iii)$\Rightarrow$(i): (see also [16, proof of Prop. 6]) The first order condition

$$\xi_\alpha := \frac{1}{\alpha} A^* A(x - x_\alpha) \in \partial R(x_\alpha)$$

provides

$$\frac{1}{\alpha} \|Ax - Ax_\alpha\|_\mathcal{Y}^2 = \langle \xi_\alpha, x - x_\alpha \rangle \leq R(x) - R(x_\alpha) \leq c_3 \|Ax - Ax_\alpha\|_\mathcal{Y}^{2\nu - 1}.$$ 

Solving for $\|Ax - Ax_\alpha\|_\mathcal{Y}$ yields the claim. 

Proof. To prove 1.2. let $\alpha < \beta$. Set $m = \frac{1}{2}\|g - A \hat{x}_\alpha\|_\mathcal{Y}^2 - \frac{1}{2}\|g - A \hat{x}_\beta\|_\mathcal{Y}^2$. From $T_\alpha(\hat{x}_\alpha, g) \leq T_\alpha(\hat{x}_\beta, g)$ and $T_\beta(\hat{x}_\beta, g) \leq T_\beta(\hat{x}_\alpha, g)$ we obtain

$$m \leq \alpha (R(\hat{x}_\beta) - R(\hat{x}_\alpha)) \leq \frac{\alpha}{\beta} m.$$ 

Hence $m \leq 0$. 3. follows from Proposition 5.3.
Lemma A.2. Let \( x \in \mathcal{X}_A \) and \( x_n \in R_\alpha(Ax), \alpha > 0 \) any selection. Then
\[
\lim_{\alpha \searrow 0} \left( \frac{1}{2\alpha} \|Ax - Ax_\alpha\|_2^2 + \mathcal{R}(x_\alpha) \right) = \inf\{\mathcal{R}(z) : z \in \mathcal{X}_A \text{ with } Az = Ax\}.
\]

Proof. Due to (5.1) and [1, Prop. 12.32] we have
\[
\frac{1}{2\alpha}\|Ax - Ax_\alpha\|_2^2 + \mathcal{R}(x_\alpha) = \mathcal{Q}_\alpha(Ax) \to \mathcal{Q}(Ax) \quad \text{for } \alpha \searrow 0
\]
with \( \mathcal{Q} \) defined in Lemma 3.1 and \( \mathcal{Q}_\alpha \) its Moreau envelope (see Subsection 5.1).

**Appendix B. Properties of Banach spaces.**

Proposition B.1. 1. Let \( p \in [1, \infty) \) and \( \omega = (\omega_j)_{j \in \Lambda} \) a sequence of positive reals. Let \( p' \in (1, \infty] \) with \( \frac{1}{p} + \frac{1}{p'} = 1 \). Then the pairing
\[
\langle \cdot, \cdot \rangle : \ell^p_{\omega-1} \times \ell^p_{\omega} \to \mathbb{R} \quad \text{given by } (\xi, x) = \sum_{j \in \Lambda} \xi_j x_j
\]
is well defined and gives rise to an isometric isomorphism \( (\ell^p_{\omega})' \cong \ell^{p'}_{\omega-1} \).

2. Let \( p, q \in [1, \infty) \) and \( s \in \mathbb{R} \). Then the pairing
\[
\langle \cdot, \cdot \rangle : b^s_{p,q} \times b^{-s}_{p,q} \to \mathbb{R} \quad \text{given by } (\xi, x) = \sum_{(j,k) \in \Lambda} \xi_{j,k} x_{j,k}
\]
is well defined and gives rise to an isometric isomorphism \( (b^s_{p,q})' \cong b^{-s}_{p',q'} \). (see [27, 2.11.2 (1)])

Proposition B.2. [23, Lem. 8.21.] Let \( A \colon \mathcal{X} \to \mathcal{Y} \) be a bounded linear operator between Banach spaces and \( \xi \in \mathcal{X}' \). The following statements are equivalent:

(i) There exists a constant \( c \geq 0 \) such that \( \langle \xi, x \rangle \leq c \|Ax\|_{\mathcal{Y}} \) for all \( x \in \mathcal{X} \).

(ii) There exists \( \omega \in \mathcal{Y}' \) with \( \|\omega\|_{\mathcal{Y}'} \leq c \) and \( A^* \omega = \xi \).

**Appendix C. Index function calculus.**

Let \( \Gamma := \{f \colon \mathbb{R} \to (-\infty, \infty] \colon f \) is proper, convex and lower semi-continuous\}.

Lemma C.1. Suppose \( f \in \Gamma \). Then
1. \( f \) is positive with \( \text{dom}(f) \subseteq (-\infty, 0] \) if and only if \( f^*|_{[0, \infty)} \leq 0 \).
2. \( f \) is non-decreasing if and only if \( \text{dom}(f^*) \subseteq [0, \infty) \).

Proof. 1. \( f \) is positive with \( \text{dom}(f) \subseteq (-\infty, 0] \) if and only if \( \chi_{(-\infty,0]} \leq f \).
\( f^*|_{[0, \infty)} \leq 0 \) if and only if \( f^* \leq \chi_{[0, \infty)} \). Hence the claim follows from \( \chi_{[0, \infty)} = \chi_{(-\infty,0]} \).

2. Suppose \( f \) is non-decreasing and let \( t < 0 \). Let \( \beta_0 \in \text{dom}(f) \). Then
\[
\beta t - f(\beta) \geq \beta t - f(\beta_0) \quad \text{for all } \beta \leq \beta_0.
\]
As \( \beta t - f^*(\beta_0) \to \infty \) for \( \beta \to -\infty \) this shows \( f^*(t) = \sup_{\beta \in \mathbb{R}} \beta t - f(\beta) = \infty \). Hence \( \text{dom}(f^*) \subseteq [0, \infty) \).

Vice versa assume \( \text{dom}(f^*) \subseteq [0, \infty) \). Then \( f(\beta) = \sup_{t \geq 0} t \beta - f^*(t) \) is non-decreasing as a supremum over non-decreasing functions. \( \square \)

Lemma C.2. The map \( F \) defined in (5.2) is well-defined, order preserving and bijective. The expression (5.3) holds true.

Proof. We define the following sets
\[
\Gamma_1 = \{ f \in \Gamma \colon \text{f is non-decreasing with } \text{dom}(f) \subseteq (-\infty, 0] \}
\]
\[
\Gamma_2 = \{ f \in \Gamma \colon \text{dom}(f) \subseteq [0, \infty) \text{ and } f|_{[0, \infty)} \leq 0 \}
\]
By Lemma C.1 the Fenchel conjugation $\ast : \Gamma_1 \to \Gamma_2$ is an order reversing bijection and its inverse is given by the Fenchel conjugation $\ast : \Gamma_2 \to \Gamma_1$. We will construct bijections $G_1 : \Sigma \to \Gamma_1$ and $G_2 : \Gamma_2 \to \Phi$, such that $F = G_2 \circ \ast \circ G_1$.

Let $\sigma \in \Sigma$. Then we define $f_\sigma : \mathbb{R} \to [0, \infty]$ by

$$f_\sigma(\beta) = \begin{cases} \sigma \left( \frac{-1}{2\beta} \right) & \text{if } \beta < 0 \\ \lim_{\alpha \to \infty} \sigma(\alpha) & \text{if } \beta = 0 \\ \infty & \text{if } \beta > 0 \end{cases}$$

Then $f_\sigma$ is proper, non-decreasing and $\text{dom}(f_\sigma) \subset (-\infty, 0]$. Convexity and lower semi-continuity of $\sigma(\frac{1}{t})$ yields convexity and lower semi-continuity of $f_\sigma$ on $(-\infty, 0)$. We have

$$f_\sigma(0) = \lim_{\beta \nearrow 0} \sigma \left( \frac{-1}{2\beta} \right) = \lim_{\beta \to 0} f_\sigma(\beta).$$

Hence $f_\sigma$ is convex and lower semi-continuous.

It is easy to see that $G_1 : \Sigma \to \Gamma_1$ given by $\sigma \mapsto f_\sigma$ is an order preserving bijection. Its inverse is given by $(G_1^{-1}(f))(\alpha) = f \left( \frac{-1}{2\alpha} \right)$.

Moreover, the map $\Gamma_2 \to \Phi$ given by $g \mapsto -(g|_{[0, \infty)})$ is well defined, bijective and order reversing. Its inverse is given by $\phi \mapsto g\phi$ with

$$g\phi : \mathbb{R} \to (-\infty, \infty] \text{ given by } g\phi(t) = \begin{cases} -\phi(t) & \text{if } t \geq 0 \\ \infty & \text{if } t < 0 \end{cases}.$$ 

If $\sigma \in \Sigma$ and $t \geq 0$ then

$$\lim_{\beta \nearrow 0} \beta t - f_\sigma(\beta) = -\lim_{\beta \nearrow 0} f_\sigma(\beta) = -f_\sigma(0).$$

Hence

$$(F(\sigma))(t) = \inf_{\alpha > 0} \left( \sigma(\alpha) + \frac{1}{2\alpha} t \right) = \inf_{\beta < 0} f_\sigma(\beta) - \beta t = \inf_{\beta \leq 0} f_\sigma(\beta) - \beta t = -f_\sigma^*(t) = ((G_2 \circ \ast \circ G_1)(\sigma))(t)$$

This shows $F = G_2 \circ \ast \circ G_1$. Therefore $F$ is an order preserving bijection. It remains to compute $F^{-1} = G_1^{-1} \circ \ast \circ G_2^{-1}$. If $\phi \in \Phi$ and $\alpha > 0$, then

$$(F^{-1}(\phi))(\alpha) = g_\phi^* \left( \frac{-1}{2\alpha} \right) = \sup_{t \geq 0} \left( -g_\phi(t) - \frac{1}{2\alpha} t \right) = \sup_{t \geq 0} \left( \phi(t) - \frac{1}{2\alpha} t \right).$$

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