Dynamics of quantum anisotropies in Taub Universe in the WKB approximation

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We analyze the dynamics of a Taub cosmological model in the presence of a massless minimally coupled scalar field and a cosmological constant, in the limit when both the Universe volume and the scalar field live in a quasi-classical approximation. By other words, we study the dynamics of a quantum small anisotropy evolving on a de Sitter background and in the presence of a kinetic term of the inflaton field.

We demonstrate that the quantum anisotropy exponentially decays during the Universe expansion and an isotropic limit for the Universe is recovered. We also compare the obtained quantum decay (behaviour of the anisotropy variance) and that one when the system is treated as a fully classical problem. We see that, on a quantum level, the anisotropy decay is slower by a factor $e^{5/2\tau}$ than the classical suppression, $\tau$ being the inflation $e$-folding. This result suggests that the quantum isotropization of the Universe during a de Sitter phase is much weaker than the corresponding classical evolution, favouring the survival of certain degree of anisotropy to the de Sitter phase. Finally we analyze the case when also the scalar field is considered as quantum variable, by showing how its variance naturally spreads because of no potential term significantly affects its dynamics. This behaviour results to be different from the anisotropy which is subjected to the potential coming from the spatial curvature.

I. INTRODUCTION

One of the most interesting open questions in theoretical cosmology concerns how a primordial quantum Universe (whose cosmological singularity is intended regularized by a cut-off effect into a Big-Bounce \cite{1}) reaches a classical isotropic limit \cite{2}. The reason to hypothesize, near the singularity, a very general morphology of the Universe, relies on the request to address the quantum cosmological problem on a general ground, at least within the framework of the Bianchi homogeneous models \cite{9} (we recall that the Bianchi types VIII and IX are prototype for the generic inhomogeneous cosmological problem) \cite{4,10}. In fact, the isotropic Robertson-Walker model is highly symmetric and it does not contain real gravitational degrees of freedom (actually in cosmology, the two gravitational degrees of freedom are identified in the anisotropies of space). Furthermore, the implementation of symmetry restrictions and the canonical quantization procedure do not commute in general.

In \cite{7} it was argued that, starting with a generic quantum inhomogeneous Universe, it can reaches a classical limit only after it has also become essentially isotropic, otherwise no stable averaged background can emerge.

While the question concerning how the quantum anisotropies can be reduced to small effects is still fully open, in \cite{8} it was shown how such small anisotropies can be naturally damped on a quantum level. This conclusion was mainly based on the features of the basic modes of the associated quantum dynamics.

Here, we focus our attention to the Taub cosmological model \cite{9} in order to deepen and complete the previous study, by analyzing in detail the evolution of wave packets and the comparison of the classical and quantum decaying of the anisotropies during a de Sitter phase.

More specifically, we consider a Taub cosmology, in the presence of a cosmological constant and a free minimally coupled scalar field (these two last ingredients well mimic the slow-rolling phase of an inflationary scenario \cite{2,10}). The analysis of the dynamics is performed in the semi-classical picture developed in \cite{11} to interpret the anisotropy wave function. By other words, we consider the Universe volume and the scalar field as quasi-classical variables while the anisotropy variable contained in the model is fully quantized. The potential term appearing in the Taub Hamiltonian is then expanded for small value of the anisotropy, according to the ideal proposed in \cite{11} that the quantum subsystem must be ”small”, for a better characterization of this hypothesis see also \cite{12}.

Validity of the small anisotropy approximation across the system dynamics is then ensured by analyzing the time dependence of the surviving harmonic potential and of the decaying of a tunnelling process probability toward large values of the anisotropy variable.

The resulting system is a quantum harmonic oscillator in the anisotropy variable, having a frequency rapidly increasing with time. The behaviour of Gaussian packets is investigated both via an expansion of the initial condition in terms of the basic modes of the time-dependent harmonic oscillator, as well as by an exact Gaussian solution (taken in the spirit of \cite{13}). Both these studies unambiguously demonstrate that the variance of the anisotropy rapidly decreases, reducing essentially to zero the probability for an anisotropic space.

The observed quantum decay of the anisotropy var-
The line element of the space-time reads as
\[ ds^2 = N^2(t)dt^2 - e^{2\alpha}(e^{2\beta})_{ab}\omega^a \omega^b, \]
where \( \omega^a = \omega^a_i dx^i \) are the left-invariant one-forms. The variable \( \alpha(t) \) describes the isotropic expansion of the model and the gravitational degrees of freedom of the Universe are associated to \( \beta_+ \), the anisotropy. It is determined in the following traceless symmetric matrix
\[ \beta_{ab} = \text{diag}(\beta_+, \beta_+, -2\beta_+). \]

In addition, Taub model is a particular case of Bianchi IX model once \( \beta_- \equiv 0 \). We introduce the cosmological constant \( \Lambda \) because we want to describe the de Sitter phase and a scalar field \( \phi \). The behaviour of a massless scalar field well approximates that one of an inflaton field during the slow-rolling dynamics, \( \dot{\phi}^2 \ll |V(\phi)| \), when the potential term is essentially constant and it provides the cosmological term.

Finally, we consider separately the case when the free massless scalar field is quantum too, in order to outline that its behaviour is intrinsically different to that of one of the anisotropies. In fact, the scalar field is essentially potential-free during the slow-rolling phase and we see that its quantum variance spreads, suggesting that it is not suppressed by the exponential expansion of the Universe, but it remains a pure quantum degree of freedom. Not suppressed by the exponential expansion of the Universe remains a pure quantum degree of freedom. It is just such a behaviour that allows the scalar field inhomogeneities (not addressed here) to be the natural origin of the actual Universe clumpiness, while the scalar curvature acts on the anisotropy degrees of freedom so that their evolution is strongly damped (although in a much softer way on a quantum setting).

The paper is structured as follows. In Sec. II we give a detailed description of the Taub cosmological model, showing the metric morphology and the associated dynamics. In Sec. III we discuss the WKB approach to a small quantum subsystem, expressing a necessary condition for a possible division of the phase-space into a classical and quantum one. In Sec. IV we derive the basic equations and solutions to describe the anisotropy dynamical evolution, showing its behaviour during a de Sitter phase. Sec. V is devoted to analyze the behaviour of probability density of the anisotropy variable as the Universe expands from the singularity, building a complete wave function and studying the wave packets with a Gaussian Ansatz. In Sec. VI we calculate to which extent the quantum anisotropy is suppressed and, for a clearer understanding, the ratio between quantum and classical one. In Sec. VII we investigate quantum scalar field fluctuations showing how they can survive to the de Sitter phase producing seeds. Finally in Sec. VIII conclusions are drawn.

II. THE TAUB MODEL

The Taub cosmological model is an homogeneous Universe. The presence of a different evolution of a scale factor from the other two makes this model anisotropic. For this reason, Taub Universe is invariant around rotation about one axis of three-dimensional space.

The line element of the space-time reads as
The dynamical picture is completed by taking into account the choice $\dot{a} = 1$ which fixes the temporal gauge. So that $N$ reads as

$$2N = \frac{a}{k} p_a,$$  (9)

where $p_a$ is considered to be negative to have compatibility between the time gauge and a positive lapse function.

Since $a(t)$ plays the role of a time, the cosmological singularity appears as $a \to 0$. Far from the singularity, the cosmological constant term dominates over the scalar fields kinetic energy and it is necessary for the development of the inflationary scenario.

### III. Vilenkin Approach to the Small Quantum Subsystem

In quantum cosmology, the wave function of the Universe is a functional defined on the minisuperspace metric $h_{ab}(x)$, i.e.

$$\psi(h_{ab}(x)).$$  (10)

We stress that an external time definition is absent because of the null scalar constraint $\mathcal{H} = 0$. In this perspective, we can consider a small quantum subsystem of the semiclassical Universe. Hence, the Hamiltonian reads as

$$\mathcal{H} = H_0 + H_q.$$  (11)

We also assume that the quantum variables $q^\nu$ with $\nu = (1, \ldots, n - m)$ does not effect the dynamics of the classical ones $h^\alpha$ with $\alpha = (1, \ldots, m)$ which is a Wentzel-Kramers-Brillouin approach similar to the Born-Oppenheimer approximation.

The Wheeler-DeWitt equation corresponding from the action (3) can be written as follows

$$(\nabla_0^2 - U_0 - H_q)\psi = 0,$$  (12)

in which the operator $H_0 = \nabla_0^2 - U_0(h)$ represents the classical Hamiltonian obtained by neglecting the quantum variables and the respective momenta $p_i = -i\partial/\partial q^i$. To justify the smallness of the quantum subspace, Vilenkin [11] imposed the following reasonable assumption

$$\frac{H_q \psi}{H_0 \psi} = O(h),$$  (13)

so that

$$\nabla_0^2 = O(h^{-1}).$$  (14)

The necessary condition for a possible division of the full phase-space into a classical and a quantum subsystem is that the minisuperspace metric tensor $g^{(0)}_{ab} \sim 1$ and $g_{ab} = O(h)$.

The wave function of the Universe can be written as

$$\psi(h, q) = \psi(h)\chi(h, q),$$  (15)

where $\psi(h) = A(h)e^{iH(h)}$.

In such a way, equation (12) can be decomposed in three equations in order of $h$. In the lowest order we obtain the Hamilton-Jacobi equation for the classical action $I$ and, in the next order, an equation for the amplitude $A$ which takes the form of a continuity equation. They respectively are

$$g^{ab}(\nabla_a I)(\nabla_b I) + U = 0$$  (16)

$$2\nabla A \cdot \nabla I + A\nabla^2 I = 0.$$  (17)

The equation for the wave function $\chi(h, q)$ of the quantum subspace has the form

$$2i(\nabla_0 I)\nabla_0 \chi = H_q \chi.$$  (18)

Using the Hamilton-Jacobi equation, (18) can be rewritten as

$$i\frac{\partial \chi}{\partial \tau} = H_q \chi,$$  (19)

with $d\tau = N(t)dt$. Hence, the Schrödinger equation we find for the subsystem in the background defined by $h^\alpha(t)$ allows to define a dynamical evolution for the quantum subsystem. In both follows, the minisuperspace variables division between $h^\alpha$ and $q^\nu$ corresponds to the following: the volume of the Universe i.e. the scale factor $a$ and the scalar field are taken as classical variables while the anisotropy $\beta_+$ is regarded as the quantum one.

The total probability density is defined by the wave function $\psi = A(h)e^{iH(h)}\chi(h, q)$ and corresponds to a conserved current. It is the product of the classical and the quantum one

$$\rho(h, q, t) = \rho_0(h, t)|\chi(q, h(t), t)|^2$$  (20)

in which $\rho_0(h, t)$ is normalized by

$$\int \rho_0 d\Sigma_0 = 1,$$  (21)

where $d\Sigma_0$ is the surface element in the subsystem defined by $h^\alpha$ and $\chi(q, h, t)$ can be normalized by

$$\int |\chi|^2 d\Omega_q = 1$$  (22)

in which $d\Omega_q = |det g_{\mu\nu}|^{1/2} d^m q$. This is the standard interpretation of the wave function for a small subspace of the Universe.
IV. BASIC EQUATIONS AND SOLUTIONS FOR THE TAUB MODEL

To describe the dynamical evolution of the Taub model, we need to analyze the three equations derived by Vilenkin approach. Equation (16) and (17) become

\[-(\partial_\alpha I)^2 + (\partial_\phi I)^2 + \Lambda e^{6\alpha} = 0,\]  

\[\partial_\alpha (A^2 \partial_\alpha I) + \partial_\phi (A^2 \partial_\phi I) = 0.\]  

(23) (24)

It has been used the notation \(e^{\alpha}\) to simplify the analytical integration. From the first equation (16) it is possible to derive the action \(I(\alpha, \phi)\) by variables separation

\[\partial_\phi I = p_\phi = \text{const}\]

\[I(\alpha, \phi) = p_\phi \phi + \bar{I}(\alpha),\]  

(25)

where

\[\bar{I}(\alpha) = -\frac{1}{3} \sqrt{\Lambda e^{6\alpha} + p_\phi^2} \left[ \frac{\sqrt{\Lambda e^{6\alpha} + p_\phi^2}}{p_\phi} \right].\]  

(26)

From the continuity equation (17) it has been found the amplitude \(A\) also by variables separation

\[A(\alpha, \phi) = A_1(\alpha) A_2(\phi),\]  

so \(A\) reads as

\[A(\alpha, \phi) = e^{-\frac{1}{4} \text{arctanh}(\Lambda e^{6\alpha} + p_\phi^2)} \sqrt{\Lambda e^{6\alpha} + p_\phi^2} \phi.\]  

(27) (28)

The functions \(I\) and \(A\) provide a complete characterization of the quasi-classical system.

A. Time-dependent harmonic oscillator

To describe the behaviour of the anisotropy \(\beta_+\) we now study the second equation of the same order in \(\hbar\) in (17), that is a pure Schrödinger-like equation

\[2i \frac{e^{3\alpha}}{2N(M)^{1/2}} \left( \alpha \frac{\partial \chi}{\partial \alpha} + \phi \frac{\partial \chi}{\partial \phi} \right) = H_\chi \chi,\]  

\[\chi = \chi(\alpha(t), \phi(t), \beta_+).\]  

(29)

where \(\chi = \chi(\alpha(t), \phi(t), \beta_+).\) Hence, using the previously introduced change of variables \(e^{\alpha} = a\) and a new time variable defined by the relation \(d\tau = (N\hbar/\alpha^3) da,\) (29) can be rewritten as

\[i \partial_\tau \chi = H_\chi \chi.\]  

(30)

The quantum Hamiltonian \(H_\chi\) reads as

\[H_\chi = -\alpha^2 + \frac{a^4}{4\hbar^2} U(\beta_+).\]  

(31)

Moreover, according to the Vilenkin idea of a small quantum system (see also \([12]\)), we consider the quasi-isotropic regime \(|\beta_+| \ll 1\) so that the potential term gets a quadratic form

\[U(\beta_+) = -3 + 24\beta_+^2,\]  

(32)

in which the zero order of the approximate potential, substituted in WDW (i.e. \(-3e^{-4\alpha} \equiv -3a^4\)) would provide a contribution to the Hamilton-Jacobi equation (16) and becomes negligible when the cosmological constant dominates. Instead for the equation (30) we get

\[i \partial_\tau \chi = -\frac{\partial^2}{\partial \beta_+^2} \omega^2 (\tau) \beta_+^2 \chi,\]  

(33)

in which the frequency term is \(\omega^2(\tau) = 6 - \frac{4}{\hbar^2} + \frac{24}{\hbar^2} e^{-4\alpha}\).

Harmonic oscillator quantum theory with time dependent frequency is known and the solution to (30) can be obtained analytically by using the exact invariant method and some transformations \([14,15]\). The solution is connected to the J-eigenfunctions \(\psi_n\) by the relation \(\chi_n(\beta_+, \tau) = e^{i\alpha_n(\tau)} \psi_n(\beta_+, \tau)\) but the general one is a linear combination \(\chi(\beta_+, \tau) = \sum c_n \chi_n(\beta_+, \tau),\) in which \(c_n\) are real or complex coefficients that weight the different wave functions. \(\chi_n\) reads as

\[\chi_n(\beta_+, \tau) = \frac{e^{i\alpha_n(\tau)} N(M)^{1/2}}{\sqrt{\pi} n! \hbar^2 \rho} h_n \left( \frac{\beta_+}{\rho} \right) e^{\left[ i \left( \frac{3}{2} + i \frac{3}{\rho} \right) \beta_+^2 \right].\]  

(34)

In \(34\) \(h_n\) are Hermite polynomials and \(\rho = \rho(\tau)\) is the function satisfying the auxiliary differential equation

\[\dot{\rho} + \omega^2 \rho - \rho^{-3} = 0,\]  

(35)

which gives the solution

\[\rho(\tau) = \sqrt{\frac{\tau}{\sqrt{\tau^2 + 1} + \frac{3}{9}}}.\]  

(36)

The phase \(\alpha(\tau)\) is given by

\[\alpha_n = -\left( n + \frac{1}{2} \right) \int \frac{d\tau}{\rho^2(\tau)} = \frac{3\sqrt{\pi}}{2} \left( n + \frac{1}{2} \right) \left[ \ln(1 + 9C\tau^2/3) \right].\]  

(37)

The above scheme allows us to analyze the evolution of the wave function once assigned a generic initial condition.

V. ANALYSIS OF THE WAVE PACKETS

In this section, we analyze the evolution of small quantum subsystem in correspondence to a Gaussian initial
condition for the probability distribution of $\beta_+$. The probability density for a generic expansion takes the form

$$|\chi|^2 \propto \left| \sum_n c_n h_n(\beta_+/\rho)e^{-\beta^2_+}/\rho^2 \right|^2.$$  \hspace{1cm} (38)

We consider the following initial condition

$$\psi(\beta_+, \tau) = Ae^{-\beta^2_+/(2\sigma^2)},$$  \hspace{1cm} (39)

where $A$ is a normalization constant (i.e. $A = 1/\sqrt{2\pi\sigma^2}$).

By doing this, we can calculate $c_n(\tau)$ and their evolution

$$c_n = \int d\beta_+ \chi_n(\beta_+, \tau) \psi(\beta_+, \tau).$$  \hspace{1cm} (40)

To build the complete wave function we calculate $|\chi_n(\beta_+, \tau)|^2$ at a time close to the singularity with $n = (0, \ldots, 20)$ terms of Hermite polynomials, in order to it evolve up.

A. Gaussian Ansatz for the Schrödinger equation

Since it is evident from the harmonic oscillator eigenfunction that the simplest way to locate the Universe is a Gaussian shape, we search for an exact solution of the time-dependent Schrödinger equation [13] as

$$\psi(\beta_+, \tau) = N(\tau)e^{-1/2\Omega(\tau)\beta^2_+}.$$  \hspace{1cm} (42)

which leads to the following two equations

$$iN'(\tau) = \frac{1}{2} N(\tau)\Omega(\tau),$$  \hspace{1cm} (43)

$$i\Omega'(\tau) = (\Omega(\tau))^2 - \omega^2(\tau).$$  \hspace{1cm} (44)

It is enough to solve (44) for the inverse Gaussian width, to get the physically information on the behaviour of the anisotropy. In fact, we get

$$\Omega = \frac{3}{\tau^{1/3}} + 9\tau.$$  \hspace{1cm} (45)

Again we find that the quantum anisotropy is suppressed (Fig.2) when the Universe expands in the de Sitter phase, i.e. $\tau \to 0$.

B. Isotropic model as an attractor

As we see from (31), the time-dependent frequency term is multiplied in the exact Hamiltonian by the Taub potential. Hence, we get a potential term which changes its shape with the Universe expansion

$$U(\tau, \beta_+) = \frac{1}{\tau^{1/3}} U(\beta_+),$$  \hspace{1cm} (46)

becoming an attractor with a remarkable restoring force. We obtain an increase in depth and width of the potential.
well as time goes to zero. We retained the only large contribution for $\beta_+ \to \infty$, when
\[
-\frac{1}{\tau^{1/3}} e^{-2\beta_+} \ll 1,
\]
to validate the theory of small oscillations approximation of the potential.

In term of $\beta_+$ the condition above reads as
\[
\beta_+ \gg \frac{2}{3} \ln \tau.
\]

Since this condition holds for any values of $\beta_+$ as far as $\tau$ approaches zero, we see that the small oscillation model is a very reliable paradigm for the present analysis, see Fig. (3).

VI. QUANTUM TO CLASSICAL RATIO OF ANISOTROPY VALUES

For a clearer understanding of quantum anisotropy suppression, we want to calculate how it decays. To do this, it has been calculated the classical one by the Hamilton equation
\[
\partial_{\tau} P_+ = \frac{2 p_+}{3(8\pi)^2} d\tau,
\]
so it reads as
\[
\beta_c(\tau) = \frac{2 p_+}{3(8\pi)^2} \tau + \beta_0
\]
where the integration constant $\beta_0$ can be set equal to zero by redefinition of the space coordinates. During the inflation phase, for high values of the scale factor Friedmann equation becomes of the form
\[
H^2 = \left(\frac{\dot{a}}{a}\right) = \frac{c^2 k \Lambda}{9} = \dot{H}^2
\]
where $\Lambda/3 = \rho_\Lambda$ is the density parameter of the cosmological constant. Solving for $a$, we get
\[
a(t) = e^{\dot{H}(t_f-t_i)},
\]
If we want to measure the amount of inflation expansion, we need to use the logarithmic growth of the scale factor named also number of $e$-folding
\[
N = \ln \left(\frac{a_f}{a_i}\right),
\]
where $a_i \equiv a(t_i)$ and $t_i$ is the instant when the De Sitter phase starts.

We now analyze the ratio of quantum to classical anisotropy compared with the scale factor before and after the inflation. Hence, we get
\[
\frac{\beta_q}{\beta_c} = \frac{\tau^{1/6}}{\tau} = a^{5/2},
\]
where $\beta_q$ is estimated by the Gaussian standard deviation. In such a way, it’s clear that the Universe isotropizes both for a classical and a quantum dynamics of the anisotropy variable but in the latter case the anisotropy decays slower. We can write the ratio as
\[
\frac{\beta_q}{\beta_c} = \frac{1}{2} \dot{H}(t_f-t_i)
\]
and this means that they differ from each other by $5/2$ $e$-folding, passing the de Sitter.

VII. BEHAVIOUR OF THE QUANTUM SCALAR FIELD

The cosmological classical field, responsible for the inflation has also small inhomogeneous quantum fluctuations and the generation of density inhomogeneities relies just in considering such small quantum corrections during the slow rolling phase as the sources of the perturbations observed today. In fact, our analysis demonstrated that also a quantum anisotropy degree of freedom would be strongly damped during the de Sitter phase, including its homogeneous component, if it is present.

Since the addressed model is intrinsically homogeneous, we can not consider here the spatial dependence of the quantum field, none the less we can study the case in which also the scalar field is a quantum degree of freedom and compare its behaviour with that one of the anisotropy. Thus, now the classical Hamiltonian contains only the time variable $a$ while the quantum subsystem Hamiltonian reads as
\[
H_q = \left[-\left(\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial \beta^2} + U(\beta_+)\right)\right]
\]
leading to the Schrödinger equation
\[
i \partial_\tau \chi = \left(-\frac{\partial^2}{\partial \phi^2} + H_q \phi^2\right) \chi,
\]
in which $H^A_q$ refers to (31). To solve (57), we take the following wave function

$$\chi(\beta_+, \tau) = e^{-i\phi_0 \tau} \xi(\beta_+)$$

(58)

in which a phase factor in $\phi$ is added. It is easy to check that the function $\chi(\beta_+, \tau)$ still satisfies (30). Moving to the general solution, we get

$$\Psi(\beta_+, \phi, \tau) = \int \frac{dp}{2\pi} e^{-i\phi_0 \tau} e^{i\phi_0 \phi} \xi(\beta_+),$$

(59)

which represents a spreading wave packets in $\phi$ (times the wave function of the small anisotropy) due to the absence of any potential, since during the slow-rolling phase the Universe is on a potential plateau. This can be considered as a starting point to understand that, by adding the dependence on space, its fluctuations can survive to the De Sitter phase producing the seeds for structure formation.

VIII. CONCLUDING REMARKS

In this paper we analyzed the isotropization process of a Taub Universe in which the volume is quasi-classical and exponentially expands during a de Sitter phase, while the anisotropy degree of freedom is treated on a pure quantum level. We included into the dynamics also a massless and minimally coupled scalar field, analyzed first as a classical field which contributes through its energy to the volume dynamics and then on a quantum level, like the anisotropy variable. This field mimics here the contribution of the kinetic term of the inflaton field during a slow-rolling phase, when its potential energy is well-summarized by the cosmological constant term [2] [10].

The main merit of the present analysis consists of a detailed characterization of the quantum anisotropy decaying, as an effect of the exponential expansion of the Universe volume, here behaving as an external clock [11] [17]. Actually, we solved a time-dependent Schrödinger equation for the anisotropy quantum degree of freedom, analyzing the behaviour of Gaussian packets, both as expanded in the basic problem eigenfunctions, as well as exact states of the quantum dynamics. We see that the variance of anisotropy variable decreases to zero as the expansion goes by. However, a crucial point in our study is to demonstrate that the decaying of the quantum anisotropy of the Universe is much smaller (of a factor $e^{5/2\tau}$) with respect to the classical value, as described by pure classical Hamilton equations and in which the spatial curvature is negligible with respect to the cosmological constant term.

Furthermore, when we consider the scalar field as a quantum degree of freedom we see that its variance has a very different behaviour with respect to that one of the anisotropy. In fact, such a quantity spreads as the expansion goes by and this reflects the absence of a significant potential governing its dynamics during the slow-rolling phase.

Here, we are considering a pure homogeneous field but its dynamics could be easily extended to the presence of inhomogeneous quantum corrections and it is clear that just the non-suppression of the scalar mode by the exponential expansion is the reason why it can generate seeds for later structure formation across the Universe.

On the contrary, we identified in the spatial curvature the ingredient responsible for the anisotropy quantum suppression. By other words, when the Universe can be characterized by small quantum anisotropies, in the sense discussed in [11] and in [12], the scalar potential takes the form of a harmonic oscillator, which frequency increases with time as the Universe volume expands. This potential term is then responsible for the damping of the anisotropy, providing a valuable paradigm for the Universe isotropization.

The validity of this picture has to be regarded as viable on a rather general setting also in the presence of local inhomogeneities in the Universe. This offers an intriguing paradigm for the emergence of a classical and homogeneous Universe from a primordial quantum age.

In this respect, a crucial question calls now attention to be investigated: how the full quantum Universe can spontaneously evolves to the proposed picture a la Vilenkin, when its volume is a quasi-classical variable and the anisotropies are small. An answer to this highly non-trivial question probably requires to account for the presence of a Universe radiation component, able to alter the mixmaster dynamics in such a way that central regions of the Bianchi IX potential (though also in a local inhomogeneous scenario) are favoured, see [18] and reference therein.

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