Iitaka conjecture for anticanonical divisors in positive characteristics

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Abstract

Given a separable fibration over an algebraically closed field of positive characteristic, we study an Iitaka-type inequality for the anticanonical divisors. We conclude that it holds when the source of the fibration is a threefold or when the target is a curve. We then give counterexamples in characteristics 2 and 3 for fibrations with “very singular” fibres, constructed from Tango-Raynaud surfaces.

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Introduction

The aim of the minimal model program is to classify varieties $X$ up to birational equivalence. One of the main invariants used in this classification is the Kodaira dimension, $\kappa(X, K_X)$. The Iitaka conjecture addresses the problem of relating the Kodaira dimensions in a fibration.

**Conjecture 0.1** (Iitaka, $C_{n,m}$). Let $f : X \to Z$ be a separable fibration between normal projective varieties, then

$$\kappa(X, K_X) \geq \kappa(F, K_F) + \kappa(Z, K_Z),$$

where $F$ is a general fibre of $f$.

In characteristic 0, this inequality has been proven in many important situations. It is then natural to ask whether the same holds true in positive characteristics. In the
paper [CZ15] the conjecture is proven when the general fibre of \( f \) is a curve. The series of papers [BCZ18], [EZ18], [Zha19], [Eji17] shows that it does hold when \( \dim(X) = 3 \) under mild assumptions on the base field or on the geometric generic fibre. However, it is known that in positive characteristics the Iitaka conjecture does not hold in general. The paper [CEKZ21] gives some counterexamples using the construction of Tango–Raynaud surfaces.

Recently, a similar statement for the anticanonical divisor has been proven in characteristic 0 by C. Chang in the paper [Cha22]. There, it is proven that

\[
\kappa(X, -K_X) \leq \kappa(F, -K_F) + \kappa(Z, -K_Z),
\]

when \( X \) and \( Z \) are \( \mathbb{Q} \)-Gorenstein normal projective varieties, \( X \) has at worst klt singularities, the anticanonical divisor \(-K_X\) is effective and its stable base locus does not dominate \( Z \) (see [Cha22, theorem 1.1]). The result is generalized also for pairs \((X, \Delta)\) with similar assumptions (see [Cha22, theorem 4.1]). We call \( C_{n,m} \) this version of the Iitaka conjecture for the anticanonical divisors.

We may wonder then, does the same inequality also hold in positive characteristics? In the paper [EG19], S. Ejiri and Y. Gongyo studied positivity properties of the relative anticanonical divisor in a fibration \( f : X \to Z \), also in positive characteristics. The paper [Cha22], that proves \( C_{n,m} \) in characteristic 0, combines the same tools with techniques involving canonical bundle formula results as studied in [Amb05]. However, in positive characteristics, it is not possible to follow exactly the same steps to prove the inequality in such generality.

Nonetheless, if we work in low dimensions, the same proofs go through with little modifications. In particular, the results in [EG19] permit to prove \( C_{n,1} \).

**Theorem 0.2 (\( C_{n,1} \), see theorem 2.1).** Let \( f : X \to Z \) be a separable fibration from a normal projective variety onto a smooth curve over an algebraically closed field of characteristic \( p > 0 \). Let \( \Delta \) be an effective \( \mathbb{Q} \)-divisor on \( X \) such that \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier. Suppose that the general fibre \((F, \Delta|_F)\) is strongly \( F \)-regular and the stable base locus of \(-(K_X + \Delta)\) does not dominate \( Z \). Then:

\[
\kappa(X, -(K_X + \Delta)) \leq \kappa(F, -(K_F + \Delta|_F)) + \kappa(Z, -K_Z).
\]

On the other hand, combining the work in [Cha22] with the results in [Wit21] instead of those in [Amb05], allows us to prove \( C_{n,n-1} \) when the fibres of \( f \) are smooth curves and \( \kappa(Z, -K_Z) = 0 \).

At this point, we need to restrict our attention to threefolds. On one hand, it is because in this way we always have that either the base of the fibration or the general fibre is a curve, so that we can use the previous results. On the other hand, when \( \kappa(Z, -K_Z) \neq 0 \), C. Chang, in [Cha22], exploits the Iitaka fibration induced by \(-K_Z\) to reduce to the case when the base has trivial anticanonical Iitaka dimension. However, in positive characteristics, we do not have the same control over the Iitaka fibration and we may lose the properties we need. When \( X \) is a threefold, then \( Z \) is at most a surface, therefore the Iitaka fibration induced by the anticanonical divisor is nice enough to be able to apply the previous results. In this way, \( C_{3,m} \) follows.
Let $f: X \to Z$ be a fibration from a normal projective threefold $X$ to a normal projective variety $Z$ with at most canonical singularities, over an algebraically closed field of characteristic $p \geq 5$. Let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Assume, moreover, that the general fibre $(F, \Delta|_F)$ is strongly $F$-regular and the stable base locus of $-(K_X + \Delta)$ does not dominate $Z$. Then,

$$\kappa(X, -(K_X + \Delta)) \leq \kappa(F, -(K_F + \Delta|_F)) + \kappa(Z, -K_Z).$$

In the last section, we study some counterexamples to $C_{n,m}^-$. The assumptions on the stable base locus and on the normality of the fibres cannot be removed. In fact, it is easy to construct counterexamples to $C_{n,m}$ already in dimension 2 when the stable base locus of $-K_X$ does surject onto $Z$. They are given by some ruled surfaces and they work also in characteristic 0.

Instead, the construction used in [CEKZ21] to prove that $C_{n,m}$ does not hold in positive characteristics, gives counterexamples also to $C_{n,m}^-$ in low characteristics if we remove the assumption on the singularities of the fibres. It is based on the fact that Kodaira vanishing does not hold in positive characteristics. In particular, on a Tango–Raynaud curve $C$, there exists a non-zero effective divisor $D$ such that $H^1(C, \mathcal{O}_C(-D)) \neq 0$. A wise choice of the divisor $D$ and a vector bundle of rank 2 corresponding to a non-split extension in $H^1(C, \mathcal{O}_C(-D))$, allow us to construct Tango–Raynaud surfaces. We can build our counterexamples using them.

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**1. Preliminaries**

**Notations**

By variety we mean a noetherian integral separated scheme of finite type over an algebraically closed field $k$.

Let $\mathcal{F}$ be a reflexive sheaf of rank 1 on a normal variety $X$, with $\mathcal{F}^{[a]}$ we denote the $a$th reflexive power of $\mathcal{F}$, for $a \in \mathbb{N}$.

A fibration $f: X \to Z$ is a projective surjective morphism of normal varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Z$.

The canonical divisor of a variety $X$ will be denoted by $K_X$.  

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To say that two divisors $D_1$ and $D_2$ are linearly equivalent, we write $D_1 \sim D_2$.

Instead, if $D_1$ and $D_2$ are $\mathbb{Q}$-divisors, $D_1 \sim_{\mathbb{Q}} D_2$ means that there is a non-zero integer $m$ such that $mD_1 \sim mD_2$.

To denote that two divisors $D_1$ and $D_2$ are numerically equivalent, we write $D_1 \equiv D_2$.

By $\lceil D \rceil$ (resp. $\lfloor D \rfloor$) we mean the divisor obtained by taking the round-up (resp. round-down) of every coefficient of the components of $D$.

Given a $\mathbb{Q}$-Cartier divisor $D$ on a normal variety $X$, we denote by $\kappa(X, D)$ the Iitaka dimension defined by $D$.

Finally, the stable base locus of a $\mathbb{Q}$-divisor $D$ is denoted by $B(D)$.

### 1.1. Singularities

First, we recall the definitions of the main classes of singularities considered in birational geometry.

**Definition 1.1.** Let $(X, \Delta)$ be a pair such that $X$ is a normal variety and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then, given a birational morphism from a normal variety: $f : Y \to X$, we can write:

$$K_Y + (f^{-1})_* \Delta \sim_{\mathbb{Q}} f^*(K_X + \Delta) + \sum_{i \in I} a(E_i, X, \Delta) E_i,$$

where the $E_i$’s are all the prime exceptional divisors of $f$. The quantity $a(E, X, \Delta)$ is called the **discrepancy** of $E$ with respect to $(X, \Delta)$. We say $(X, \Delta)$ is:

- **terminal** if $a(E, X, \Delta) > 0$ for all possible exceptional divisors $E$ over $X$;
- **canonical** if $a(E, X, \Delta) \geq 0$ for all possible exceptional divisors $E$ over $X$;
- **Kawamata log terminal** or **klt** if $a(E, X, \Delta) > -1$ for all possible exceptional divisors $E$ over $X$ and $\lfloor \Delta \rfloor \leq 0$;
- **log canonical** or **lc** if $a(E, X, \Delta) \geq -1$ for all possible exceptional divisors $E$ over $X$.

One of the peculiarities of fields of positive characteristics is the presence of the Frobenius morphism which gives rise to inseparable morphisms. When working over these fields, it is more natural to define other types of singularities, rather than the “classical” ones we just mentioned. Anyway, they turn out to be strictly related to the latter. These new classes of singularities can be identified according to the behaviour of the Frobenius morphism around them.

**Definition 1.2.** Let $X$ be a scheme of finite type over an algebraically closed field $k$ of characteristic $p > 0$. Define the **absolute Frobenius morphism** $F : X \to X$ to be the identity on the topological space of the scheme and the $p^{th}$-power on the structure sheaf. We will consider also its $e^{th}$-powers, $F^e$ defined likewise by taking the identity on the topological space and the $(p^e)^{th}$-power on the structure sheaf.
**Definition 1.3.** Let $X$ be a normal variety and $\Delta$ an effective $\mathbb{Q}$-divisor on it. The log pair $(X, \Delta)$ is called **globally $F$-pure** if there is an integer $e > 0$ such that the natural map induced by the $e^{th}$ power of the Frobenius morphism

$$O_X \to F_e^*O_X \to F_e^*O_X([\lceil (p^e - 1)\Delta \rceil])$$

splits.

We say, moreover, that it is **globally $F$-regular** if, for every effective Weil divisor $D$, there is an integer $e > 0$ such that the map

$$O_X \to F_e^*O_X \to F_e^*O_X([\lceil (p^e - 1)\Delta \rceil] + D)$$

splits.

We call $(X, \Delta)$ **sharply $F$-pure** (resp. **strongly $F$-regular** or **SFR** for short) if it is covered by a finite number of open subsets $U_i$ such that the pairs $(U_i, \Delta|_{U_i})$ are globally $F$-pure (resp. globally $F$-regular) for all $i$.

**Remark 1.4.** It is possible to prove that if $(X, \Delta)$ is SFR (resp. sharply $F$-pure), then it is klt (resp. log canonical) (see [HW02]).

### 1.2. Easy additivity theorems

Let $f : X \to Z$ be a fibration between normal projective varieties. As we saw in the introduction, the Iitaka conjecture states a relation between the Kodaira dimensions of $X$, $Z$ and the general fibre $F$. Actually, some inequalities dealing with a similar relation have been known for a long time in any characteristic and they hold more in general for the Iitaka dimension with respect to some line bundles. They are the so-called “easy additivity” theorems.

**Theorem 1.5 (Easy additivity).** [Uen75, theorem 6.12] or [Fuj20, lemma 2.3.31] Let $f : X \to Z$ be a fibration between normal projective varieties over an algebraically closed field. Let $D$ be a $\mathbb{Q}$-Cartier divisor on $X$, then:

$$\kappa(X, D) \leq \kappa(F, D|_F) + \dim(Z),$$

where $F$ is a general fibre of $f$.

**Theorem 1.6 (Easy additivity 2).** [BCZ18, lemma 2.20] Let $f : X \to Z$ be a fibration between normal projective varieties over an algebraically closed field. Let $D$ be an effective $\mathbb{Q}$-Cartier divisor on $X$ and $H$ a big $\mathbb{Q}$-Cartier divisor on $Z$. Then,

$$\kappa(X, D + f^*H) \geq \kappa(F, D|_F) + \dim(Z),$$

where $F$ is a general fibre of $f$. 

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1.3. Weak positivity

The main technical property studied in the paper [EG19] is weak positivity of sheaves. We recall here what we need to use in the following.

**Definition 1.7.** Let \( X \) be a normal quasi-projective variety and \( \mathcal{G} \) a coherent sheaf on it. We say that \( \mathcal{G} \) is **generically globally generated** if the map
\[
H^0(X, \mathcal{G}) \otimes O_X \to \mathcal{G}
\]
is surjective over the generic point of \( X \).

The sheaf \( \mathcal{G} \) is called **weakly positive** if, given any ample divisor \( A \) on \( X \) and any natural number \( n \), there exists \( m > 0 \) such that \((\text{Sym}^m(\mathcal{G}))^{**} \otimes O_X(mA)\) is generically globally generated, where the double star indicates the double dual.

**Lemma 1.8.** In the setting of the definition above, if \( \mathcal{G} = O_X(D) \) is an invertible sheaf, then it is generically globally generated if and only if \( D \) is linearly equivalent to an effective divisor. Moreover, it is weakly positive if and only if \( D \) is pseudoeffective.

**Proof.** The first claim follows directly from the definition. Let us then prove the second. The condition of being weakly positive translates to the fact that, for any \( A \) ample and any \( n \in \mathbb{N}_{>0} \), \( D + \frac{1}{n} A \) is \( \mathbb{Q} \)-effective. But then, \( D = \lim_{n \to \infty} (D + \frac{1}{n} A) \) is a limit of \( \mathbb{Q} \)-effective divisors, thus it is pseudoeffective. On the other hand, if \( D \) is pseudoeffective, then it is in the closure of the \( \mathbb{Q} \)-effective cone. Consider the line defined by \( D + tA \). For \( t \in \mathbb{Q}_{>0} \), each of these divisors is \( \mathbb{Q} \)-effective, thus \( D \) is weakly positive. \( \text{qed} \)

2. Proof of \( C_{-1}^{-} \)

In this section, we use the paper [EG19] to prove \( C_{-1}^{-} \) in positive characteristic, when the target of the fibration \( f : X \to Z \) is a curve. In this paper, the authors assume that the relative anticanonical divisor is nef. We weaken a bit the assumption asking for its stable base locus not to surject onto \( Z \). The proofs carry on in the same way as in [EG19, section 3 and section 4], thus we only write down the proof of theorem 2.5. The main result of the section is, then, \( C_{-1}^{-} \).

**Theorem 2.1** \( (C_{-1}^{-}) \). Let \( f : X \to Z \) be a separable fibration from a normal projective variety onto a smooth curve over an algebraically closed field of characteristic \( p > 0 \). Let \( \Delta \) be an effective \( \mathbb{Q} \)-divisor on \( X \) such that \(- (K_X + \Delta) \) is \( \mathbb{Q} \)-Cartier. Suppose that the general fibre \((F, \Delta|_F)\) is strongly \( F \)-regular and the stable base locus of \(- (K_X + \Delta) \) does not dominate \( Z \). Then:

\[
\kappa(X, -(K_X + \Delta)) \leq \kappa(F, -(K_F + \Delta|_F)) + \kappa(Z, -K_Z).
\]

\( \text{(**) Set-up} \)

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We write here the assumptions we make for the first result. Note that they differ from the ones used by [EG19] only for condition (iii): they assume nefness of $L$ instead.

Let $f : X \to Z$ be a separable fibration between normal projective varieties over an algebraically closed field of characteristic $p > 0$. Consider $\mathbb{Q}$-divisors $\Delta$ and $D$ on $X$ and $Z$ respectively such that $K_X + \Delta$ and $D$ are $\mathbb{Q}$-Cartier. Let $\Delta = \Delta^+ - \Delta^-$ be the decomposition by effective $\mathbb{Q}$-divisors with no common components and let

$$L := -(K_X + \Delta) - f^*D.$$ 

Suppose that:

(i) the general fibre $(F, \Delta^+|_F)$ is sharply F-pure;

(ii) $\text{Supp}(\Delta^-)$ does not dominate $Z$;

(iii) $\mathbb{B}(L)$ does not dominate $Z$;

(iv) there exist positive integers $a, b$ not divisible by $p$, such that $a\Delta^+$ and $bL$ are integral.

**Lemma 2.2.** In the above setting, let $D$ be an effective, $\mathbb{Q}$-Cartier divisor on $X$ such that its restriction to the general fibre $F$ is ample. Then, there exists a $\mathbb{Q}$-divisor $\Gamma \geq 0$ on $X$ such that $\Gamma \sim \mathbb{Q}D$ and $(F, \Delta^+|_F + \Gamma|_F)$ is still sharply F-pure.

**Proof.** Pick a general fibre $F$ and choose $m \gg 0$ so that $mD$ is very ample on $F$. Write $|mD| = |M| + B$, where $|M|$ is the movable part of the linear system and $B$ is its fixed divisor. Consider the morphism $\varphi$ to $\mathbb{P}^N_k$ induced by $|M|$. Then, by [SZ13, corollary 6.10], the pair $(F, \Delta^+|_F + \varphi^{-1}_F(H))$ is sharply F-pure for a general hyperplane $H$ of $\mathbb{P}^N_k$. Let

$$\Gamma := \frac{1}{m}(\varphi^{-1}_*(H) + B) \sim \mathbb{Q}D.$$ 

Since $\Delta^+|_F + m\Gamma|_F \geq \Delta^+|_F + \Gamma|_F$, $(F, \Delta^+|_F + \Gamma|_F)$ is sharply F-pure as well. qed

We state the next results in the assumptions in which we need to use them later. In the original papers they are proven in greater generality.

**Theorem 2.3.** [Eji17, theorem 5.1 and example 3.11] Let $f : X \to Z$ be a separable surjective morphism between normal projective varieties over an algebraically closed field of characteristic $p > 0$. Let $\Delta$ be an effective $\mathbb{Q}$-Weil divisor on $X$ such that there is a positive integer $a$ not divisible by $p$ such that $a\Delta$ is integral. Let $\bar{\eta}$ be the geometric generic point of $Z$. Suppose that:

(i) the geometric generic fibre $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ is sharply F-pure;

(ii) the divisor $K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}$ is ample.

Then,

$$(f_*\mathcal{O}_X(am(K_X + \Delta))) \otimes \omega^{-am}_Y$$

is weakly positive for every $m \gg 0$. 

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Theorem 2.4. \[PSZ18\] corollary 3.31] In the setting \(\bullet\) let \(\Delta'\) be an effective \(\mathbb{Q}\)-Weil divisor on \(X\) such that there is a positive integer \(a\) not divisible by \(p\), with \(a\Delta'\) integral. Suppose that \(K_X + \Delta'\) is \(\mathbb{Q}\)-Cartier and \((X_s, \Delta'_s|_{X_s})\) is sharply F-pure for some perfect point \(s \in Z\). Then, there exists a non-empty open subset \(U \subset Y\) such that \((X_s, \Delta'_s|_{X_s})\) is sharply F-pure for all \(s \in U\) and all \(e > 0\). In particular, if we denote by \(\bar{X}_s\) the geometric generic fibre, \((\bar{X}_s, \Delta'_s|_{\bar{X}_s})\) is sharply F-pure.

Theorem 2.5. In the setting above \(\bullet\), fix a positive integer \(l\) such that \(l(K_X + \Delta)\) and \(l(K_Z + D)\) are Cartier and \(l\Delta^-\) is integral. Then, there exists an effective \(f\)-exceptional divisor \(B\) on \(X\) such that

\[O_X(l(-f^*(K_Z + D) + \Delta^- + B))\]

is weakly positive.

Furthermore, \(B\) can be chosen to be 0 if \(Z\) has only canonical singularities.

Proof. We follow almost verbatim the proof given in \[EG19\] theorem 3.1, highlighting the small differences. First, we prove the statement when \(f\) is equi-dimensional. Set \(\mathcal{F} := O_X(l(-f^*(K_Z + D) + \Delta^-))\) and let \(A\) be an ample Cartier divisor on \(X\). We need to show weak positivity of \(\mathcal{F}\), in particular it is enough to see that for any \(\alpha \in \mathbb{Z}_{>0} \setminus p\mathbb{Z}\), there is some \(\beta \in \mathbb{N}_{>0}\) such that \(\mathcal{F}(\alpha \beta)(\beta A)\) is weakly positive.

Since \(B(L)\) does not dominate \(Z\), the restriction of \(L\) to the general fibre \(F\) is semiample. Therefore, \((L + \alpha^{-1}A)|_F\) is ample. By lemma \[2.2\] there exists an effective \(\mathbb{Q}\)-divisor \(\Gamma \sim \mathbb{Q} (L+\alpha^{-1}A)\), such that \((F, \Delta^+|_F + \Gamma|_F)\) is still sharply F-pure. By theorem \[2.4\] this implies that \((X_{\bar{s}}, (\Delta^+ + \Gamma)|_{X_{\bar{s}}})\) is sharply F-pure. Note that \((K_X + \Delta^+ + \Gamma)|_{X_{\bar{s}}} \sim \mathbb{Q} \alpha^{-1}A|_{X_{\bar{s}}}\) is ample. Thus, theorem \[2.3\] proves weak positivity of the sheaf

\[(f_*O_X(lm(K_X + \Delta^+ + \Gamma))) \otimes \omega_Y^{\otimes -lm}.
\]

Then, as shown in \[EG19\] theorem 3.1, for \(\beta \gg 0\), the generically surjective morphism

\[f^*((f_*O_X(\alpha \beta (K_X + \Delta^+ + \Gamma))) \otimes \omega_Y^{\otimes -\alpha \beta l}) \rightarrow \mathcal{F}(\alpha \beta) \otimes O_X(\beta A)
\]

shows that the latter sheaf is weakly positive as well.

In the general case, by the flattening lemma \[AO00\] section 3.3], we have a commutative diagram:

\[
\begin{array}{ccc}
X' & \overset{\sigma}{\longrightarrow} & X \\
\downarrow f' & & \downarrow f \\
Z' & \underset{\alpha}{\longrightarrow} & Z
\end{array}
\]

with \(\sigma\) and \(\rho\) projective birational, \(f'\) equi-dimensional. Let \(E \geq 0\) be a \(\sigma\)-exceptional divisor such that \(-K_{X'} \leq -\sigma^*K_E + E\) and let \(\Delta'\) be a \(\mathbb{Q}\)-Weil divisor on \(X'\) such that \(K_{X'} + \Delta' = \rho^*(K_X + \Delta)\). Then, the stable base locus of \(-(K_{X'} + \Delta') - f'^*\sigma^*D\) does not dominate \(Z'\). From the previous step, we can conclude weak positivity of the sheaf

\[O_{X'}(l(-f'^*(K_{Z'} + \sigma^*D) + \Delta'^-))\],

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where $\Delta' \geq 0$ is defined as the negative component in the decomposition by effective $\mathbb{Q}$-divisors of $\Delta' = \Delta' + \Delta''$. As $-f^*(K_Z + \sigma^*D) \leq -\rho f^*(K_Z + D) + f^*E$, the sheaf

$$\rho_*(O_X((l(-\rho f^*(K_Z + D) + \Delta' + f^*E))))$$

is also weakly positive. Thus, if we set $B := \rho_*f^*E$ we get the claim. Remark that, if $Z$ has at worst canonical singularities, than we can choose $E$ and consequently $B$ to be 0.

$qed$

(*) Set-up

Let $f : X \to Z$ be a separable fibration between normal projective varieties over an algebraically closed field of characteristic $p > 0$, and let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$. Let $L := -(K_X + \Delta) + f^*K_Z$. Suppose that:

(i) $Z$ has at worst canonical singularities;

(ii) $K_X + \Delta$ is $\mathbb{Q}$-Cartier;

(iii) $B(L)$ does not dominate $Z$;

(iv) the general fibre $(F, \Delta|_F)$ is strongly $F$-regular.

Remark 2.6. We can assume that there exists a positive integer $a$, not divisible by $p$, such that $a\Delta$ and $-a(K_X + \Delta)$ are integral divisors. Indeed, we can perturb $\Delta$ with a sufficiently small multiple of an effective divisor $\mathbb{Q}$-equivalent to $-a(K_X + \Delta)$, so that condition $(iv)$ still holds.

All results in [EG19, section 4] hold with exactly the same proofs in this setting using theorem 2.5 instead of [EG19, theorem 3.1]. In particular, we conclude the following.

Corollary 2.7. [EG19, corollary 4.7] In the setting above the restriction map

$$\alpha : \bigoplus_{m \in \mathbb{N}} H^0(X, [mL]) \to \bigoplus_{m \in \mathbb{N}} H^0(F, [mL|_F])$$

is injective.

With this last result it is straightforward to prove $C_{n,1}$.

Proof of [2.1] First of all, we can assume $\kappa(X, -(K_X + \Delta)) \geq 0$. Moreover, as in remark 2.6, by perturbing $\Delta$ with a sufficiently small multiple of $-(K_X + \Delta)$, we can suppose that there exists a positive integer $a$, not divisible by $p$, such that $a\Delta$ and $-a(K_X + \Delta)$ are integral divisors. We distinguish three cases according to the genus of $Z$.

- Let $Z$ be a curve of genus 0.
  
  The result follows immediately from easy additivity theorem 1.5.
• Let $Z$ be a curve of genus $1$. Then, $K_Z \sim 0$, so corollary 2.7 gives the desired inequality.

• Let $Z$ be a curve of genus $\geq 2$. Since $K_Z$ is ample, $B(-(K_X + \Delta) + f^*K_Z) \subseteq B(-(K_X + \Delta))$ does not dominate $Z$. Thus, we can apply corollary 2.7 and easy additivity 2 theorem 1.6 to get:

$$\kappa(F, -(K_F + \Delta|_F)) + \dim(Z) \leq \kappa(X, -(K_X + \Delta) + f^*K_Z),$$

$$\kappa(X, -(K_X + \Delta) + f^*K_Z) \leq \kappa(F, -(K_F + \Delta|_F)).$$

Contradiction. Such case never happens.

3. Partial results on $C_{n,n-1}^-$

The goal of this section is the proof of $C_{n,m}^-$ in positive characteristics when the relative dimension of the fibration is one and the target has zero anticanonical Iitaka dimension. The results follow [Cha22, section 3 and section 4].

\(\star\) Set-up

In this section, we work in the following setting. Let $f : X \to Z$ be a fibration between normal projective varieties over an algebraically closed field $k$ of characteristic $p > 0$. Suppose that $f$ is of relative dimension one with smooth generic fibre. Let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$ and $D$ be any $\mathbb{Q}$-Cartier divisor on $Z$, such that:

• $K_X + \Delta$ is $\mathbb{Q}$-Cartier;

• the pair restricted to the general fibre $(F, \Delta|_F)$ is strongly F-regular;

• $L := -(K_X + \Delta) - f^*D$ is $\mathbb{Q}$-effective and its stable base locus does not surject onto $Z$.

**Theorem 3.1.** In the above setting\(\star\), assume that $\kappa(Z, -K_Z - D) = 0$. Then, the map defined by restriction on a general fibre $F$,

$$\alpha : \bigoplus_{m \in \mathbb{N}} H^0(X, \lfloor mL \rfloor) \to \bigoplus_{m \in \mathbb{N}} H^0(F, \lfloor mL|_F \rfloor)$$

is injective. In particular, when $D = 0$, we get the inequality:

$$\kappa(X, -(K_X + \Delta)) \leq \kappa(F, -(K_F + \Delta|_F)) + \kappa(Z, -K_Z),$$

for a general fibre $F$. 
This result is an analogue of [Cha22, theorem 3.8]. In order to prove it, the author uses techniques involving canonical bundle formula results as in [Amb05]. In positive characteristics, we do not have the same results in such generality, but only when the general fibre of the fibration we are considering is a curve.

**Proposition 3.2.** [Wit21, proposition 3.2] Let $(X, \Delta)$ be a quasi-projective log pair over an algebraically closed field of positive characteristic with $\Delta$ an effective $\mathbb{Q}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : X \to Z$ be a fibration onto a normal variety. Assume that the geometric generic fibre $X_{\eta}$ is a smooth curve, $(X_{\eta}, (\Delta + \Gamma)_{|X_{\eta}})$ is log canonical and $K_X + \Delta = f^* L_Z$ for some $\mathbb{Q}$-Cartier divisor $L_Z$ on $Z$. Then,

$$L_Z \sim_{\mathbb{Q}} K_Z + \Delta_Z,$$

where $\Delta_Z$ is an effective $\mathbb{Q}$-divisor.

**Proposition 3.3.** In the setting $\bigstar -K_Z - D$ is $\mathbb{Q}$-effective.

**Proof.** Pick a general fibre $F$. Since the stable base locus of $L$ does not surject onto $Z$, $L|_F$ is semiample. By [Tan17, proposition 3], there exists an effective $\mathbb{Q}$-divisor $\Gamma \sim_{\mathbb{Q}} L$ on $X$ such that $(F, (\Delta + \Gamma)|_F)$ is sharply F-pure. Then, $K_X + \Delta + \Gamma \sim_{\mathbb{Q}} -f^* D$. Let $X_{\bar{\eta}}$ be the geometric generic fibre of $f$. By theorem 2.3, $(X_{\bar{\eta}}, (\Delta + \Gamma)|_{X_{\bar{\eta}}})$ is sharply F-pure as well. Therefore, we can apply proposition 3.2 to find an effective $\mathbb{Q}$-divisor $\Delta_Z$ on $Z$, such that $-K_Z - D \sim_{\mathbb{Q}} \Delta_Z$. qed

**Remark 3.4.** Applying the above proposition 3.3 to the case $D = 0$, we conclude that, in the assumptions of setting $\bigstar$, $-K_Z$ is $\mathbb{Q}$-effective whenever $-(K_X + \Delta)$ is.

Following verbatim the proofs of [Cha22, proposition 4.2 and proposition 4.3], we have the same results in the more restrictive setting $\bigstar$.

**Proposition 3.5.** Consider the setting $\bigstar$. Let $E$ be a $\mathbb{Q}$-Cartier divisor on $Z$. Moreover, assume there exists an effective $\mathbb{Q}$-divisor on $X$, $\Gamma$, such that $L - f^* E \sim_{\mathbb{Q}} \Gamma$. Then, for $0 < \varepsilon \ll 1$, $-K_Z - D - \varepsilon E$ is $\mathbb{Q}$-effective.

**Proof.** The proof is the same as the one of [Cha22, proposition 4.2]. Note that, as for the klt case, for any $0 < \varepsilon \ll 1$, $(F, (\Delta + \varepsilon \Gamma)|_F)$ is still SFR. qed

**Proof of 3.1.** If the theorem did not hold, there would exist $s$, non-zero section, in the kernel of $\alpha$. Then, up to taking powers, $s$ defines an effective $\mathbb{Q}$-divisor $N \sim_{\mathbb{Q}} -(K_X + \Delta) - f^* D$ containing $F$ in its support. Since $\kappa(Z, -K_Z - D) = 0$, there exists a unique effective $\mathbb{Q}$-divisor $M \sim_{\mathbb{Q}} -K_Z - D$. We can suppose that $f(F) = z \in Z$ is such that:

(i) $f$ is flat in a neighbourhood of $z$;

(ii) $z$ is a regular point of $Z$;

(iii) $z \notin \text{Supp}(M)$. 

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Consider the diagram:

\[ G \subseteq X' \xrightarrow{\pi} X \supseteq F \]
\[ E \subseteq Z' \xrightarrow{p} Z \ni z \]

where
- \( p : Z' \to Z \) is the blow-up at \( z \);
- \( \pi : X' \to X \) is the blow-up at \( F := f^{-1}(z) \). Note that, by flatness of \( f \) near \( z \) and normality of the fibres, \( X' \) is normal and \( X' = X \times_Z Z' \);
- \( G := \text{Exc}(\pi) \cong F \times E = f^* E \), with \( E := \text{Exc}(p) \).

Let \( D' := p^* D \) and \( \Delta' \) be the strict transform of \( \Delta \), then:

\[ -K_{Z'} = p^*(-K_Z) - aE, \quad a = \dim(Z) - 1; \]
\[ -(K_{X'} + \Delta') = \pi^*(-(K_X + \Delta)) - bG, \quad b \leq \text{codim}(F) - 1 = a. \]

By assumption, \( \pi^* L \) is \( \mathbb{Q} \)-effective, its stable base locus does not surject onto \( Z' \) and

\[ \pi^* L = \pi^*(-(K_X + \Delta) - f^* D) = -(K_{X'} + \Delta') + bf^* E - f^* D'. \]

Let \( \beta \) be the coefficient of \( G = f^* E \) in \( \pi^* N \). Since \( F \) is in the support of \( N \), \( \beta > 0 \). The \( \mathbb{Q} \)-divisor \( \pi^* N - \beta f^* E \geq 0 \) is effective, thus, by proposition 3.5 there exists an effective \( \mathbb{Q} \)-divisor \( \Gamma \) which is \( \mathbb{Q} \)-linearly equivalent to \( -K_{Z'} + bE - D' - \varepsilon \beta E \), for some \( \varepsilon > 0 \). But then we would have:

\[ \Gamma + (a - b + \varepsilon \beta)E \sim_{\mathbb{Q}} p^*(-K_Z - D) \sim_{\mathbb{Q}} p^* M. \]

Both sides of the above equation are effective, the LHS has \( E \) in its support, while the RHS does not. However, \( \kappa(Z', p^* M) = 0 \), contradiction. \qed

4. Proof of \( C_{3,m}^- \)

Here, we use the results of the previous sections to prove \( C_{n,m}^- \) in positive characteristics when the source of the fibration is a threefold. The proof follows the ideas in [Cha22, theorem 4.1].

**Theorem 4.1** \((C_{3,m}^-)\). Let \( f : X \to Z \) be a fibration from a normal projective threefold \( X \) to a normal projective variety \( Z \) with at worst canonical singularities, over an algebraically closed field of characteristic \( p \geq 5 \). Let \( \Delta \) be an effective \( \mathbb{Q} \)-divisor on \( X \) such that \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier. Suppose that the general fibre \((F, \Delta|_F)\) is strongly \( F \)-regular and the stable base locus of \(- (K_X + \Delta) \) does not dominate \( Z \). Then,

\[ \kappa(X, -(K_X + \Delta)) \leq \kappa(F, -(K_F + \Delta|_F)) + \kappa(Z, -K_Z). \]
Proof. First of all, we can assume \( \kappa(X, -(K_X + \Delta)) \geq 0 \). Moreover, as in remark 2.6 by perturbing \( \Delta \) with a sufficiently small multiple of \( -(K_X + \Delta) \), we can suppose that there exists a positive integer \( a \), not divisible by \( p \), such that \( a\Delta \) and \( -a(K_X + \Delta) \) are integral divisors. If \( Z \) is a curve, then the result holds by theorem 2.1. Thus, we only need to consider what happens when \( Z \) is a surface. By proposition 3.3, \( -K_Z \) is \( \mathbb{Q} \)-effective. If \( \kappa(Z, -K_Z) = 0 \), we conclude by theorem 3.1 in the previous section. If \( \kappa(Z, -K_Z) = 2 \), easy additivity theorem 1.5 gives the conclusion. We are therefore only left with the case \( \kappa(Z, -K_Z) = 1 \). In order to achieve the result here, we want to get to a situation where we can apply theorem 3.1. To do this, we pass through the Iitaka fibration of \( Z \) following the ideas of [Cha22, theorem 4.1]. However, in positive characteristics if a property holds for a very general fibre, it may happen that it does not hold at all, as the ground field might be countable. Moreover, the generic fibre of a fibration can be very singular even if the source and the target are smooth. Therefore, we may lose the nice properties we need. Nonetheless, since \( Z \) is a surface, these problems do not arise, in the next lemma we see why.

Lemma 4.2. Let \( S \) be a normal projective \( \mathbb{Q} \)-factorial surface such that \( \kappa(S, -K_S) = 1 \) and let \( g : S \to C \) be the rational map induced by the linear system \( | -mK_S| \) for \( m \gg 0 \). Write \( | -mK_S| = |M| + B \), where \( |M| \) is the movable part (i.e. its base locus has codimension \( \geq 2 \)) and \( B \) is the fixed divisor. Then, \( M \sim_{\mathbb{Q}} g^*A \) for an ample \( \mathbb{Q} \)-divisor \( A \) on \( C \) and the support of \( B \) does not dominate \( C \). In particular, the Iitaka fibration relative to \( -K_S \) is induced by \( |M| \) and is a (quasi-)elliptic fibration.

Proof. First of all, note that the rational map induced by \( -mK_S \) coincides with the one induced by \( M \). Since the base locus of \( M \) has dimension \( 0 \), \( M \) is semiample by Zariski–Fujita’s theorem [Fuj81, theorem 2.8]. By possibly substituting it with a multiple, it is then linearly equivalent to \( g^*A \), for an ample divisor \( A \) on \( C \) by construction of the Iitaka fibration. Let \( B^h \) and \( B^v \) be effective divisors decomposing \( B \) into its horizontal and vertical components, respectively. We need to show that \( B^h = 0 \). Suppose this was not true. Then write, for \( 0 < \varepsilon \ll 1 \),

\[
-mK_S \sim (g^*A + \varepsilon B^h) + (1 - \varepsilon)B^h + B^v.
\]

The divisor \( B^h \) is relatively ample. By Nakai–Moishezon criterion, \( g^*A + \varepsilon B^h \) is ample and \( (1 - \varepsilon)B^h + B^v \) is effective, showing that \( -K_S \) is big. Contradiction. Therefore, \( g \) is well-defined everywhere and it coincides with the Iitaka fibration. Moreover, the general fibre of \( g \) has arithmetic genus \( 1 \).

By [KM98, proposition 4.11], since \( Z \) is a surface with canonical singularities, it is in particular \( \mathbb{Q} \)-factorial, thus we can apply the above lemma 4.2. Fix an \( m \gg 0 \) and let \( g : Z \to C \) be the rational map induced by \( | -mK_Z| \). Then, we can write \( -mK_Z \sim g^*A + B \), where \( B \) is the fixed divisor and \( A \) is ample on \( C \). Call \( W \) the
general fibre of $h := g \circ f$, using easy additivity theorem \[1.5\] we get the inequality:

$$\kappa(X, -(K_X + \Delta)) \leq \kappa(W, -(K_W + \Delta|_W)) + \dim(C)$$

To conclude, it is enough to prove that $\kappa(W, -(K_W + \Delta|_W)) \leq \kappa(F, -(K_F + \Delta|_F))$, for $F$ general fibre of $f$. Again by the lemma above \[4.2\] the arithmetic genus of the general fibre is 1, so, if the characteristic of the base field is $\geq 5$, it is smooth.

Let us study the stable base locus of $-(K_W + \Delta|_W)$. Note that a general fibre of $g$ is not contained in $f(B(-(K_X + \Delta)))$. Fix one of these fibres, say $G$, and let $W$ be the fibre of $h$ over $G$. There exist effective $\mathbb{Q}$-divisors $D_i \sim_{\mathbb{Q}} -(K_X + \Delta)$, for $i = 1, 2$, such that $G \not\subseteq f(\text{Supp}(D_1) \cap \text{Supp}(D_2))$. Since $0 \leq D_i|_W \sim_{\mathbb{Q}} -(K_W + \Delta|_W)$, we conclude that the stable base locus of $-(K_W + \Delta|_W)$ does not dominate $G$. Thus, the fibration $f|_W : W \to G$ satisfies:

- $G$ is smooth and $\kappa(G, -K_G) = 0$;
- the general fibre $F$ is normal and $(F, \Delta|_F)$ is SFR;
- the stable base locus of $-(K_W + \Delta|_W)$ does not surject onto $G$.

We are in the right setting to apply theorem \[3.1\] whence:

$$\kappa(W, -(K_W + \Delta|_W)) \leq \kappa(F, -(K_F + \Delta|_F))$$

qed

5. Counterexamples

Counterexamples for $C_{n,m}$ removing the hypothesis on the stable base locus are easy to obtain just looking at ruled surfaces (see [Cha22, example 1.7]). Note that they work in any characteristic.

The next example shows that the assumption on the normality of the fibres is essential. It relies on the construction of Tango–Raynaud surfaces. These surfaces are a good source of counterexamples in positive characteristics. Analysing them, in the paper [CEKZ21], the authors found counterexamples to $C_{n,m}$ in characteristic $p$, for any $p > 0$. The same construction gives counterexamples to $C_{n,m}$ in characteristics 2 and 3.

**Definition 5.1.** Let $C$ be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field of characteristic $p > 0$. Define the **Tango invariant**:

$$n(C) := \max \left\{ \deg \left( \left\lfloor \frac{(df)}{p} \right\rfloor \right) \mid f \in K(C) \right\},$$

where $(df)$ denotes the divisor of zeroes and poles of the differential $df$. Note that $pn(C) \leq 2g - 2$. We say that $C$ is a **Tango curve** if $n(C) > 0$ and that it is a **Tango–Raynaud curve** if, moreover, $pn(C) = 2g - 2$. 

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Example 5.2. Let $e \in \mathbb{N}$ and let $C$ be the plane curve defined by the equation $Y^{pe} - YX^{pe-1} = Z^{pe-1}X$ in $\mathbb{P}^2_{\overline{F}_p}$ with coordinates $[X : Y : Z]$. It is smooth and, by adjunction, if $g$ is its genus, $2g - 2 = pe(pe - 3)$. Consider the differential $d(\frac{Z}{X})$ and denote $\infty$ the point $[0 : 0 : 1] \in C$. Then, $(d(\frac{Z}{X})) = pe(pe - 3)(\infty) = (2g - 2)(\infty)$, showing that $C$ is a Tango–Raynaud curve.

Consider the Frobenius map $F : C \to C$. Denote by $\mathcal{B}$ the cokernel of the induced map, $\mathcal{O}_C \to F_*\mathcal{O}_C$. Thus, for any Cartier divisor $D$, we have the exact sequence:

$$0 \to \mathcal{O}_C(-D) \to F_*\mathcal{O}_C(-pD) \to \mathcal{B}(-D) \to 0.$$  

Lemma 5.3. [Xie10, lemma 2.5] With the same notations as above,

$$H^0(C, \mathcal{B}(-D)) = \{df | f \in K(C), (df) \geq pD\}.$$  

Moreover, if $D$ is effective, $H^0(C, \mathcal{B}(-D))$ is the kernel of $F^* : H^1(C, \mathcal{O}_C(-D)) \to H^1(C, \mathcal{O}_C(-pD))$. Now, if $C$ is a Tango curve, by definition, we can find an effective divisor $D$ on it and a non-zero element $df$ in $H^0(C, \mathcal{B}(-D))$. This determines a non-zero element of $H^1(C, \mathcal{O}_C(-D))$ which is mapped to zero by the Frobenius morphism. Hence $df$ determines a (non-split) short exact sequence

$$0 \to \mathcal{O}_C(-D) \to \mathcal{E} \to \mathcal{O}_C \to 0,$$

which becomes a split exact sequence after applying the Frobenius morphism:

$$0 \to \mathcal{O}_C(-pD) \to F^*\mathcal{E} \to \mathcal{O}_C \to 0.$$

Thus, we get also:

$$0 \to \mathcal{O}_C \to F^*\mathcal{E} \to \mathcal{O}_C(-pD) \to 0.$$

Let $C$ be a Tango–Raynaud curve, let $f$ be a rational map on $C$ such that $(df) = K_C = pD$, with $D$ effective divisor. Consider the rank 2 vector bundle $\mathcal{E}$ on $C$ defined by the extension determined by $df$ as above. Let $g : P := \mathbb{P}(\mathcal{E}) \to C$ and $g_1 : P_1 := \mathbb{P}(F^*\mathcal{E}) \to C$. Thus, we have a commutative diagram:

$$\begin{align*}
\begin{array}{c}
P \ar[d]_{F_{P/C}} \ar[dr]^{F_P} & \\
F_P & \\
P_1 \ar[r]_{W} & P \ar[dl]_{g} & \\
g_1 & \\
C \ar[r]_{g} & C
\end{array}
\end{align*}$$

where the lower square is a fibre product diagram and $F_{P/C}$ is the relative Frobenius.

Let $T$ be a divisor on $P$ such that $\mathcal{O}_P(T) \simeq \mathcal{O}_P(1)$. The short exact sequence \(\boxtimes\) defines a section of $g$, $F \sim T + g^*D$. On the other hand, the short exact sequence \(\boxtimes\) defines
a section of \( g_1, G_1 \sim W^*F - pg_1^*D \). Let \( G := F_{P/C}^\circ G_1 \), then \( G \sim pF - pg^*D \sim pT \). Computing the arithmetic genus of \( G \) and \( F \) using adjunction formula, we see that the curves \( G \) and \( F \) are smooth of genus \( g \). Moreover, they are disjoint.

If there exists \( l > 0 \) such that \( l \) divides \( p+1 \) and \( D = l D' \) for an effective divisor \( D' \), then:

\[
G + F \sim (p + 1)T + g^*D = l(rT + g^*D') = lM
\]

where \( r = \frac{p+1}{l} \) and \( M := rT + g^*D' \). Such a situation happens for example if we take an appropriate \( e \) in the curve of example 5.2. Since the support of the divisor \( G + F \) is smooth, the \( l \)-cyclic cover defined by the above equivalence yields a smooth surface \( S \).

Call \( \pi : S \to P \) the cover and \( f = g \circ \pi : S \to C \).

The last step in this construction consists in taking \( m \)-times the fibre product of \( S \) over \( C \): \( X^{(m)} := S \times_C S \cdots \times_C S \). Let \( p_i : X^{(m)} \to S \) be the projection to the \( i \)th factor and \( f^{(m)} : X^{(m)} \to C \) the composition of \( f \) with any of these projections.

Studying local equations, it is easy to see that the varieties \( X^{(m)} \) are normal and their singular locus is the union of \( \supp(T_i) \cap \supp(T_j) \) for \( i \neq j \); where \( T_i := (\pi \circ p_i)^*T \). Moreover, the same computations show that the fibres of \( X^{(m)} \to C \) are not normal. Indeed, let \( s \) be a local parameter on the fibres of \( g \) such that locally \( G = \{ s^p = 0 \} \) and \( F = \{ s = \infty \} \). First we look at what happens away from \( F \). Let \( (x_1, ..., x_r) = \mathbb{A}^r \) be local affine coordinates on \( C \), then, as showed in [Muk13, section 2] there is \( f(x) \) such that \( S \) affine locally has equation \( t^i = s^p - f(x) \) inside \( \mathbb{A}^1 \times \mathbb{A}^1 \times C \). Then, affine locally, we can see \( X^{(m)} \subseteq \mathbb{A}^m \times \mathbb{A}^m \times C \) with coordinates \( t_1, ..., t_m, s_1, ..., s_m, x \) and equations \( t_i^s = s_i^p - f(x) \) for \( i = 1, ..., m \). Applying Jacobi criterion we get the result.

On the other hand, if \( \sigma = 1/s \) is a local parameter defining \( F \) on \( P \), an affine local equation for \( S \) away from \( G \) is \( t^i = \sigma \). We then find that \( X^{(m)} \) is smooth in this open subset.

**Theorem 5.4.** If we take \( \{p, l\} = \{2, 3\} \), then the fibrations:

\[
\eta : X^{(\mu)} \to X^{(m)}
\]

give counterexamples to \( C_{\mu+1, m+1}^- \) for \( \mu \geq 6 \) and \( 0 \leq m < 6 \) (with the convention that \( X^{(0)} = C \)). Moreover, the stable base locus of the anticanonical divisor of \( X^{(\mu)} \) is empty, for \( \mu \geq 6 \).

**Proof.** The relative anticanonical divisors of \( g \) and \( f \), for our choices of \( p \) and \( l \) are:

\[
K_{P/C} = -2T - pg^*D;
\]

\[
K_{S/C} = \pi^*(K_{P/C} + (l-1)M) = -f^*D'.
\]

Then, the anticanonical sheaf of \( X^{(m)} \) is:

\[
\omega_{X^{(m)}}^{-1} = (\prod_{i=1}^m p_i^*\omega_{S/C}^{-1}) \otimes \omega_C^{-1}.
\]
Therefore, for any positive integer $n$:

$$-nK_{X(m)} = f^{(m)*}(nm - 6n)D'. $$

This sheaf is ample on $C$ for $m > 6$, trivial for $m = 6$ and antiample on $C$ for $m < 6$, so that

$$\kappa(X^{(m)}, -K_{X(m)}) = \begin{cases} 
1 & \text{for } m > 6 \\
0 & \text{for } m = 6 \\
-\infty & \text{for } m < 6 
\end{cases}. $$

Note that the base locus is always empty for $m \geq 6$ as the divisor is vertical. \hfill \text{qed}

References

[Amb05] Florin Ambro. The moduli b-divisor of an lc-trivial fibration. *Compos. Math.*, 141(2):385–403, 2005.

[AO00] Dan Abramovich and Frans Oort. Alterations and resolution of singularities. In *Resolution of singularities (Obergurgl, 1997)*, volume 181 of *Progr. Math.*, pages 39–108. Birkhäuser, Basel, 2000.

[BCZ18] Caucher Birkar, Yifei Chen, and Lei Zhang. Iitaka $C_{n,m}$ conjecture for 3-folds over finite fields. *Nagoya Math. J.*, 229:21–51, 2018.

[CEKZ21] Paolo Cascini, Sho Ejiri, János Kollár, and Lei Zhang. Subadditivity of Kodaira dimension does not hold in positive characteristic. *Comment. Math. Helv.*, 96(3):465–481, 2021.

[Cha22] Chi-Kang Chang. Positivity of anticanonical divisors in algebraic fibre spaces. *Mathematische Annalen*, pages 1–23, 2022.

[CZ15] Yifei Chen and Lei Zhang. The subadditivity of the Kodaira dimension for fibrations of relative dimension one in positive characteristics. *Math. Res. Lett.*, 22(3):675–696, 2015.

[EG19] Sho Ejiri and Yoshinori Gongyo. Nef anti-canonical divisors and rationally connected fibrations. *Compos. Math.*, 155(7):1444–1456, 2019.

[Eji17] Sho Ejiri. Weak positivity theorem and Frobenius stable canonical rings of geometric generic fibers. *J. Algebraic Geom.*, 26(4):691–734, 2017.

[EZ18] Sho Ejiri and Lei Zhang. Iitaka’s $C_{n,m}$ conjecture for 3-folds in positive characteristic. *Math. Res. Lett.*, 25(3):783–802, 2018.

[Fuj80] Takao Fujita. On semipositive line bundles. *Proc. Japan Acad. Ser. A Math. Sci.*, 56(8):393–396, 1980.
Osamu Fujino. *Itaka conjecture—an introduction*. SpringerBriefs in Mathematics. Springer, Singapore, [2020] ©2020.

Nobuo Hara and Kei-Ichi Watanabe. F-regular and F-pure rings vs. log terminal and log canonical singularities. *J. Algebraic Geom.*, 11(2):363–392, 2002.

János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

Shigeru Mukai. Counterexamples to Kodaira’s vanishing and Yau’s inequality in positive characteristics. *Kyoto J. Math.*, 53(2):515–532, 2013.

Zsolt Patakfalvi, Karl Schwede, and Wenliang Zhang. F-singularities in families. *Algebr. Geom.*, 5(3):264–327, 2018.

Karl Schwede and Wenliang Zhang. Bertini theorems for F-singularities. *Proc. Lond. Math. Soc. (3)*, 107(4):851–874, 2013.

Hiromu Tanaka. Semiample perturbations for log canonical varieties over an F-finite field containing an infinite perfect field. *Internat. J. Math.*, 28(5):1750030, 13, 2017.

Kenji Ueno. *Classification theory of algebraic varieties and compact complex spaces*. Lecture Notes in Mathematics, Vol. 439. Springer-Verlag, Berlin-New York, 1975. Notes written in collaboration with P. Cherenack.

Jakub Witaszek. On the canonical bundle formula and log abundance in positive characteristic. *Math. Ann.*, 381(3-4):1309–1344, 2021.

Qihong Xie. Counterexamples to the Kawamata-Viehweg vanishing on ruled surfaces in positive characteristic. *J. Algebra*, 324(12):3494–3506, 2010.

Lei Zhang. Subadditivity of Kodaira dimensions for fibrations of three-folds in positive characteristics. *Adv. Math.*, 354:106741, 29, 2019.