Classical and quantum kinetics of the Zakharov system

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Abstract

A kinetic theory for quantum Langmuir waves interacting nonlinearly with quantum ion-acoustic waves is derived. The formulation allows for a statistical analysis of the quantum correction to the Zakharov system. The influence of a background random phase on the modulational instability is given. In the coherent case, the effect of the quantum correction is to reduce the growth rate. Moreover, in the classical limit, a bifurcation develops in the dispersion curves due to the presence of partial coherence. However, the combined effect of partial coherence and a quantum correction may give rise to an increased modulational instability growth rate, as compared to the classical case. The results may be of significance in dense astrophysical plasmas and laboratory laser–plasma systems.

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I. INTRODUCTION

The effects from the quantum domain have intriguing consequences for the way we view the world and the way we interpret physical models, and these effects can be seen in single particle phenomena. Plasma physics, on the other hand, deals with the collective interaction of charged particles, and processes related to such. However, there are physical parameter domains where quantum mechanics and plasma physics need to be taken into account simultaneously, e.g. dense and/or hot astrophysical plasmas. Indeed, the creation of pair plasmas surrounding neutron stars can be viewed as a collective quantum plasma effect. Thus, there is interest and application for a model taking both collective charged particle effects and quantum phenomena into account (see, e.g. Refs. 1 and 2).

One of the most prominent models in plasma physics is described by the Zakharov equations, in which high frequency Langmuir waves are coupled nonlinearly to low frequency ion-acoustic waves. The statistical properties of this system has been analyzed in Ref. 3, where a Landau-like damping was found. Recently, a generalization of the Zakharov system was derived, taking quantum effects into account. The effect of the quantum correction was to introduce higher order dispersion into the system of equations, thus altering the behavior of wave evolution. It was argued in Ref. 3 that these contributions could be important in astrophysical plasmas, as the plasma densities may become significant.

In this paper, we will introduce a kinetic description of the quantum Zakharov equation, by applying the Wigner transform to the Langmuir propagation equation. The resulting system of equations may be useful for understanding the properties of partially coherent Langmuir waves interacting with quantum ion-acoustic waves. We derive the general dispersion relation, and analyze the stability of the system. A comparison between monoenergetic Langmuir waves and random-phase Langmuir waves is given, and it is found that the interplay between quantum corrections and spectral broadening may alter the instability properties in novel ways. In particular, it is found that the growth rate for a short wavelength partially coherent quantum Langmuir wave is larger than the corresponding growth rate for the classical Langmuir wave. Thus, the interplay between incoherence and quantum effects gives rise to modified modulational instability growth rates, a result that may be relevant to astrophysical and intense laboratory laser-plasmas, for which the quantum parameter may take on significant values.
II. BASIC EQUATIONS

The dynamics of the nonlinearily coupled quantum Langmuir and ion-acoustic waves is given by the Zakharov-like equations\(^5\)

\[
\begin{align}
&i \partial_t E(t, x) + \partial_x^2 E(t, x) - H^2 \partial_x^4 E(t, x) = n(t, x) E(t, x), \quad (1a) \\
&(\partial_t^2 - \partial_x^2) n(t, x) + H^2 \partial_x^4 n(t, x) = \partial_x^2 |E(t, x)|^2, \quad (1b)
\end{align}
\]

where \(H \equiv \hbar \omega_{pi} / k_B T_e\) is the quantum parameter due to a quantum pressure emanating from the underlying hydrodynamic model\(^5\). Here \(\hbar\) is Planck’s constant divided by \(2\pi\), \(\omega_{pi} = (n_0 e^2 / m_i \epsilon_0)^{1/2}\) is the ion plasma frequency, \(k_B\) is Boltzmann’s constant, \(T_e\) is the electron temperature, \(n_0\) is the constant background density, and \(m_i\) is the ion rest mass. The electric field \(E\) has been normalized according to \(E \rightarrow (\epsilon_0 m_i / 16 m_e n_0 k_B T_e)^{1/2} E\), while the density \(n\) is normalized by \(n \rightarrow (m_i / 4m_e n_0) n\), where \(m_e\) is the electron mass. The coordinates have been rescaled using \(t \rightarrow (2m_e / m_i) \omega_{pe} t\) and \(x \rightarrow 2(m_e / m_i)^{1/2} x / \lambda_e\), where \(\omega_{pe} = (n_0 e^2 / m_e \epsilon_0)^{1/2}\) is the electron plasma frequency and \(\lambda_e\) is the electron Debye length. As \(H \rightarrow 0\), we regain the classical Zakharov equations from (1). However, in some astrophysical plasmas, the quantum parameter \(H\) may approach unity, since in such environments, high densities are not uncommon (see, e.g. Ref. 6). We see that the effect of the quantum parameter is to introduce higher order dispersion.

III. QUANTUM KINETICS

The Fourier transform of the two-point correlation function, as given for the electric field by

\[
\rho(t, x, p) = \frac{1}{2\pi} \int d\xi e^{ip\xi} \langle E^* (t, x + \xi/2) E(t, x - \xi/2) \rangle \tag{2}
\]

was introduced by Wigner\(^7\) in quantum statistical mechanics. Here the angular brackets denotes the ensemble average, and the asterisk denotes the complex conjugation operation. The Wigner function \(\rho\) is a generalized distribution function, which satisfies

\[
\langle |E(t, x)|^2 \rangle = \int dp \rho(t, x, p). \tag{3}
\]
Applying the transformation (2) to Eq. (1a) gives the kinetic equation
\[ \partial_t \rho(t, x, p) + (2p \partial_x + 4H^2 p^3 \partial_x - H^2 p \partial_x^3) \rho(t, x, p) - 2n(t, x) \sin \left( \frac{1}{2} \partial_x \partial_p \right) \rho(t, x, p) = 0, \] (4)
which is coupled to the ion-acoustic equation (1b) via Eq. (3). Here the sin-operator is defined by its Taylor expansion, and arrows denote direction of operation. Keeping the lowest order derivative in this Taylor expansion, corresponding to the long wavelength limit, gives a modified Vlasov equation
\[ \partial_t \rho(t, x, p) + p D_x \rho(t, x, p) - (\partial_x n(t, x))(\partial_p \rho(t, x, p)) = 0, \] (5)
for the quantum Langmuir wave, driven by the ion-acoustic ponderomotive force. Here \(D_x \equiv (2 + 4H^2 p^2 - H^2 \partial_x^3)\partial_x\). Thus, in the classical limit \(H \rightarrow 0\), \(D_x \rightarrow 2\partial_x\), and we obtain a Vlasov-like equation for the long wavelength Langmuir waves.

IV. THE MODULATIONAL INSTABILITY

In order to analyze Eqs. (1b), (3), and (4), we perform a perturbative expansion. Letting \(\rho(t, x, p) = \rho_0(p) + \rho_1 \exp(ikx - i\omega t)\), where \(|\rho_1| \ll \rho_0\), and \(n(t, x) = n_0 + n_1 \exp(ikx - i\omega t)\), we linearize with respect to the perturbation variables. We then obtain the dispersion relation
\[ -\omega^2 + (1 + H^2 k^2)k^2 = k^2 \int dp \frac{\rho_0(p + k/2) - \rho_0(p - k/2)}{\omega - kp(2 + 4H^2 p^2 + H^2 k^2)}. \] (6)
The dispersion relation (6) generalizes the results in Refs. 4 and 5, and is valid for partially coherent quantum Langmuir waves interacting nonlinearly with quantum ion-acoustic waves.

A. Monoenergetic Langmuir waves

In the case of a monoenergetic Langmuir wave, we have \(\rho_0(p) = I_0 \delta(p)\), where \(I_0 = |E_0|^2\) is the background intensity. Then the dispersion relation (6) becomes
\[ \left[ \omega^2 - (1 + H^2 k^2)k^2 \right] \left[ \omega^2 - (1 + H^2 k^2)^2k^4 \right] = 2I_0(1 + H^2 k^2)k^4, \] (7)
such that
\[ \omega^2 = \frac{1}{2} \left[ H^2 k^2 + H^4 k^4 \pm H k^2 \sqrt{H^2 + 8I_0 - 2H^4 k^2 + H^6 k^4} \right], \] (8)
where $H \equiv 1 + H^2 k^2$. Letting $\omega = i\gamma$, the instability growth rate is given by
\[
\gamma = \frac{1}{\sqrt{2}} \left[ H^2 \sqrt{H^2 + 8I_0 - 2H^4k^2 + H^6k^4} - H^2k^2 - H^4k^4 \right]^{1/2}. 
\] (9)

Starting from $H = 0$, successively higher values of $H$ tend to suppress the instability, giving lower growth rates with a cut-off at a lower wavenumber, see Fig. 1.

B. Partial coherence

The coherent monoenergetic background distribution gives important information on wave instabilities. However, in many applications the background field is not fully coherent, but rather displays partial decoherence due to, e.g. noise. The noise, either classical or quantum, may stem from different sources, such as thermal effects, weak turbulence, or quantum fluctuations. Such sources of noise may lead to a background field $E_0$ with a random phase $\varphi(x)$ such that
\[
\langle e^{-i[\varphi(x+\xi/2)-\varphi(x-\xi/2)]} \rangle = e^{-p_W|\xi|}, 
\] (10)

with the corresponding distribution function $\rho_0$ is given by the Lorentzian
\[
\rho_0(p) = \frac{I_0}{\pi} \frac{p_W}{p^2 + p_W^2}, 
\] (11)

where $p_W$ is the width of the distribution. The integrand of (6) has three poles, one real and two complex, where the real pole is given by
\[
p_0 = -\frac{1}{A^{1/3}(\omega, k)} + \frac{A^{1/3}(\omega, k)}{3b(k)}. 
\] (12)

Here $A(\omega, k) = 3[9ab^2 + \sqrt{3}(4b^3 + 27a^2b^4)^{1/2}]/2$, $a(\omega, k) = \omega/(2 + H^2k^2)$, and $b(k) = 4H^2/(2 + H^2k^2)$. As the quantum parameter $H$ approaches zero, the complex poles approach complex infinity. Thus, in the integration of Eq. (6) we will neglect these poles, only taking the real pole $p_0$ into account, since the modes corresponding to the complex poles are quickly damped. Thus, we have
\[
-\omega^2 + (1 + H^2 k^2)k^2 = 2I_0k^4 \left\{ \frac{g(k) - h(k)}{[\omega + 2ip_Wh(k)k]^2 - k^4[g(k) - h(k)]^2} + \frac{ip_Wp_0}{k^2(2 + H^2k^2)((p_0 + k/2)^2 + p_W^2)[(p_0 - k/2)^2 + p_W^2]} \right\} 
\] (13)
where we have defined the real and positive functions \( g(k) = H^2(k^2 + 8p_W^2) \) and \( h(k) = 1 + 2H^2(k^2 + p_W^2) \). The dispersion relation (13) describes the effects of partial coherence for the quantum Zakharov system (1). The damping character due to the finite width of the background distribution can clearly be seen, as well as the Landau damping due to the real pole. We note that as \( p_W \to 0 \), we regain the monoenergetic dispersion relation (17).

C. The classical limit

If \( H \to 0 \), we obtain the classical limit of the dispersion relation (13), when we use a kinetic photon description for the Langmuir waves. The effects of statistical broadening on the Zakharov system was also analyzed in Ref. 4. We note that the two complex poles approaches infinity, and only the real pole remains with the value \( p_0 = \omega/2k \), as it should. The dispersion relation then reads

\[
\left(\omega^2 - k^2 \right) \left[ \omega^4 + k^8 + 8p_W^2 k^6 + 8p_W^2 k^2 \omega^2 + 2k^4 (8p_W^4 - \omega^2) \right] = 2I_0 k^4 (\omega^2 - k^2 - 8ip_W k\omega - 4p_W^2 k^2),
\]

and although the quantum effects have been neglected, the behavior of the function \( \omega(k; p_W) \) is still rather complicated. The dispersion relation (13) with \( H = 0 \) was analyzed analytically in the long wavelength limit, i.e. \( \omega/k \gg 1 \), in Ref. 4, and the growth rate was found for a Gaussian background spectrum. Here we will solve the equation (14) for all wave lengths. Letting \( \omega = \text{Re } \omega + i\gamma \), we may solve Eq. (13) numerically for the growth rate \( \gamma \), using different values of the widths \( p_W \). In the Figs. 2 and 3 we have plotted the solutions for \( H = 0 \) and a number of values of \( p_W \). The results show on a more complicated dispersion structure than in the coherent case. The asymptotic behavior of the growth rate for short wavelengths has been depicted in Fig. 4, using a number of different values on the decoherence width \( p_W \). For large \( k \), the growth rate has a linear dependence on the wavenumber, the slope being determined by the values of the width \( p_W \).

D. Quantum effects on the instability growth rate

When \( H \) is nonzero, the combined effects of quantum correction and decoherence make themselves apparent in the dispersion relation (13) through new and novel wave modes. In Fig. 5 we display the growth rate \( \gamma \) as a function of the wave number \( k \). As compared to
the classical case, the combined effect of partial coherence and quantum effects, i.e. finite $p_W$ and $H$ respectively, is to make the modulational instability growth rate smaller for long wavelength perturbations. However, the interesting effect is for short wavelengths, where the modes introduced due the finite spectral width is amplified by the quantum corrections. Thus, we may expect much stronger growth rates for short wavelength perturbations, making these dominant in quantum plasmas. In Fig. 5 the dispersion curves for a value $H = 0.25$ of the quantum parameter has been plotted. The strong growth rate for short wavelengths can clearly be seen.

V. CONCLUSIONS

The effects of partial coherence in quantum plasmas, such as in the form of a random phase, is of interest in certain plasmas, such as astrophysical plasmas\textsuperscript{5,6}, and the next generation laser plasma systems\textsuperscript{11,12}. Moreover, in such system, the density may even reach values of $10^{23} - 10^{31}$ m$^{-3}$ for temperatures of the order $10^5 - 10^7$ K, giving $H = \hbar \omega_{pi}/k_B T_e \sim 10^{-7} - 1$. Thus, the quantum parameter $H$ may attain appreciable values, such that the higher order dispersive terms in Eqs. become important. Even in the cases of small a quantum correction, this effect combined with Langmuir wave decoherence will yield a strongly growing mode for short wavelengths, and could lead to significant changes in extreme astrophysical and laboratory plasmas. Thus, the combination of incoherence and quantum effects may yield rich and interesting dynamics of Langmuir wave propagation in such plasmas. However, a detailed analysis of possible applications is left for future research.

Here we have analyzed the statistical properties of the quantum Zakharov system, giving the dynamics of high frequency Langmuir waves in terms of a kinetic equation. This enabled the investigation into the effects of partial coherence of the quantum Langmuir wave, in particular the implications due to a random phase, and it was found that such a system exhibits an interesting dispersion structure. In particular, the combined effect of decoherence and quantum corrections gives rise to new dispersion curves as well as increased modulational instability growth rates, as compared to the case of a classical coherent and partial coherent Langmuir wave.
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Fig. 1: The growth rate $\gamma$ as given by Eq. (9) plotted as a function of the wavenumber $k$. Seen from the right, the values $H = 0, 0.5$ and 1 have been used together with the intensity $I_0 = 0.5$. The effect of the quantum parameter is thus to suppress the instability.

Fig. 2: The classical limit of the dispersion relation (13) for different values of the background distribution width $p_W$. We see that the upper curve in panel (a) closely resembles the uppermost curve in Fig. 4 but is slightly damped. The effects of the spectral width can clearly be seen through the damping of the mode present in the coherent case, as well as the presence of a completely new mode. In the different panels, we have the following widths: (a) $p_W = 0.025$, (b) $p_W = 0.05$, (c) $p_W = 0.0667$, (d) $p_W = 0.0909$, and (e) $p_W = 0.267$. The intensity in all the panels is $I_0 = 0.50$.

Fig. 3: The classical limit of the dispersion relation (13) for a select set of values of the background distribution width $p_W$, taken to larger wavenumbers. In the different panels, we have the following widths: (a) $p_W = 0.0714$, (b) $p_W = 0.111$, and (c) $p_W = 0.333$. The intensity in all the panels is $I_0 = 0.50$.

Fig. 4: The inverse of the classical growth rate, i.e. $H = 0$, plotted as a function of the wavelength $\lambda = 2\pi/k$, giving the asymptotic behavior of $\gamma$ for short wavelengths. Starting from the top of the panel, we have used the values $p_W = 0.025$, $p_W = 0.05$, and $p_W = 0.091$ on the respective curve. We note the generic linear behavior for short wavelengths.

Fig. 5 (Color online): The combined effects of a quantum correction and partial coherence on the growth rate (black curve), obtained from Eq. (13) using $H = 0.25$, as compared to the classical case for the same spectral width (red curve). In the different panels, we have the following widths: (a) $p_W = 0.05$, (b) $p_W = 0.111$, and (c) $p_W = 0.2$. We note the mode due to the quantum corrections combined with the partial coherence gives rise to a larger growth rate for short wavelengths, while the long wavelength modes are damped by the quantum corrections (see Fig. 1).
FIG. 1:
FIG. 2:
(a)

(b)
FIG. 3:

FIG. 4:
FIG. 5: