Self-Adjointness in Klein-Gordon Theory on Globally Hyperbolic Spacetimes

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Abstract

We show that the spatial part of the linear Klein-Gordon operator for a massive scalar field with external potential in a globally hyperbolic spacetime is a strictly positive essentially self-adjoint operator. The proof is conducted by a fusion of new results concerning globally hyperbolic manifolds, the theory of weighted Hilbert spaces and operational analytic advances in the aforementioned area.

1 Introduction

Quantum field theory (QFT) in curved spacetime studies the behavior of quantum fields that propagate in the presence of a classical gravitational field, where the quantum behavior of the gravitational field is neglected. It can be seen as an intermediate (and mostly rigorous) step towards a complete theory of quantum gravity (see [1][2][3][4][5] for excellent reviews).

One particularly fruitful context arises from the restriction to globally hyperbolic spacetimes. The advantage of this class of spacetimes is the existence of a choice of time, or equivalently the existence of a global Cauchy surface. This guarantees that the linear wave equations arising as the equations of motion of classical field theory before quantization, provide well posed initial value problems [2, Theorem 4.1.2][1].

A well established path towards constructing a free QFT starts with the phase space of initial data of a classical field theory on a spacelike hypersurface. This comes naturally equipped with a symplectic structure. Adding a complex structure converts this space into a “one particle structure” which is then second quantized into a Fock space [6],[7]. For the case of stationary spacetimes this construction has been done in [8] and (more rigorously) in [9]. More recently, the question of finding a complex structure for non-stationary spacetimes has been addressed e.g. in [10], [11] (and references therein).

In the present article we shall focus on the Klein-Gordon theory. An important ingredient in the construction of the QFT is then the operator \( w^2 \) encoding the spatial part of the Klein Gordon equation (see Equation 2.4). In particular, one requires
essential self-adjointness and positivity of this operator. The positivity requirement is imposed in order to take the (unique) square-root of the operator which is used in the construction of complex structures. Proving positivity and essential self-adjointness in a rather general setting is the goal of the present paper. Kay was able to show essential self-adjointness for the important class of stationary spacetime under certain additional boundedness requirements [9, Theorem 7.2]. Our results considerably extend this by showing essential self-adjointness for general globally hyperbolic spacetimes while dropping at the same time the additional boundedness requirements. To this end we use the functional analytic theory of weighted Hilbert spaces (see [12],[13]).

An additional reason for the significance of essential self-adjointness of the spatial part of the Klein-Gordon equation is its role in the construction of the Hamiltonian operator, see e.g. [14, Equation 1.16]. In particular, in order for the quantum evolution to be unitary the quantum Hamiltonian has to be self-adjoint which in turn requires (among other things) the spatial operator to be essentially self-adjoint. Recent results [15] have also shown an important connection between the existence of an essentially self-adjoint quantum Hamiltonian (that guarantees unitary evolution) and the requirement for those operators to be bounded from below. Hence, in addition to essential self-adjointness, positivity plays again an important role.

Throughout this work we use Greek letters $\mu, \nu = 0, \ldots, 3$ for spacetime indices and we use Latin letters $i, j, k, \ldots$ for spatial components which run from $1, \ldots, 3$.

2 Klein-Gordon Theory on Globally Hyperbolic Spacetimes

We start with basic definitions that are needed for the subsequent results.

**Definition 2.1 (Globally hyperbolic spacetime).** We denote a spacetime by $(M, g)$ where $M$ is a smooth, four-dimensional manifold and $g$ is a Lorentzian metric on $M$ with signature $(-1, +1, +1, +1)$. In addition we assume orientability and time-orientability of the manifold. **Time-orientability** means that there exists a $C^\infty$-vector-field $u$ on $M$ that is everywhere timelike, i.e. $g(u, u) < 0$. A smooth curve $\gamma : I \rightarrow M$, $I$ being a connected subset of $\mathbb{R}$, is called **causal** if $g(\dot{\gamma}, \dot{\gamma}) \leq 0$ where $\dot{\gamma}$ denotes the tangent vector of $\gamma$. A causal curve is called **future directed** if $g(\dot{\gamma}, u) < 0$ and **past directed** if $g(\dot{\gamma}, u) > 0$ all along $\gamma$ and for a global timelike vector-field $u$. For any point $x \in M$, $J^\pm(x)$ denotes the set of all points in $M$ which can be connected to $x$ by a future($+$)/past ($-$)-directed causal curve. An orientable and time-orientable spacetime is called **globally hyperbolic** if for each pair of points $x, y \in M$ the set $J^-(x) \cap J^+(y)$ is compact whenever it is non-empty. This definition is equivalent to the existence of a smooth foliation of $M$ in Cauchy surfaces, where a smooth hypersurface of $M$ is called a **Cauchy surface** if it is intersected exactly once by each inextendible causal curve.

The advantages of requiring a spacetime to be globally hyperbolic are best displayed by the following theorem [14].

**Theorem 2.1.** Given a spacetime $(M, g)$ the following statements are equivalent:

- $(M, g)$ is globally hyperbolic
- There exists a (global) Cauchy surface in $(M, g)$
- There exists a choice of time on $(M, g)$

Hence, an effective way of thinking of a globally hyperbolic spacetimes is to think that these spacetimes admit a choice of time. One can show that the manifold of any global hyperbolic spacetime is topologically equivalent to $\mathbb{R} \times \Sigma$ for any Cauchy surface $\Sigma$ [16]. The authors in [17] were able to solve a long-standing conjecture by proving
that any globally hyperbolic spacetime can be smoothly foliated into Cauchy surfaces [17, Theorem 1.1].

**Theorem 2.2.** Let \((M, g)\) be a globally hyperbolic spacetime. Then, it is isometric to the smooth product manifold

\[
\mathbb{R} \times \Sigma, \quad \text{with metric} \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + h_{ij} dx^i dx^j,
\]

where \(\Sigma\) is a smooth 3-manifold, \(t : \mathbb{R} \times \Sigma \mapsto \mathbb{R}\) is the natural projection, \(N : \mathbb{R} \times \Sigma \mapsto (0, \infty)\) a smooth function, and \(h\) a 2-covariant symmetric tensor field on \(\mathbb{R} \times \Sigma\), satisfying the following condition: Each hypersurface \(\Sigma_t\) at constant \(t\) is a Cauchy surface, and the restriction \(h(t)\) of \(h\) to such a \(\Sigma_t\) is a Riemannian metric (i.e. \(\Sigma_t\) is spacelike).

In general \(N\) and \(h\) depend on the time and space coordinates. We proceed to consider the Klein-Gordon equation in \((M, g)\) with an external potential \(V\). A solution \(\phi\) satisfies,

\[
(\Box g - m^2 - V)\phi = 0,
\]

where \(\Box g\) is the Laplace-Beltrami operator with respect to the metric \(g\). That is, \(\Box g = (\sqrt{|g|})^{-1} \partial_\mu (\sqrt{|g|} g^{\mu\nu} \partial_\nu)\) with \(|g|\) denoting the absolute value of the determinant of the metric \(g\). As in [14] we denote for each Cauchy surface \(\Sigma\) the space of smooth Cauchy data of compact support by

\[
\mathcal{S}_\Sigma := C_0^\infty(\Sigma) \oplus C_0^\infty(\Sigma).
\]

Moreover, due to Leray’s Theorem [18], the Cauchy data \(\Phi \in \mathcal{S}_\Sigma\) given by

\[
\Phi = \left( \begin{array}{c} \varphi \\ \varphi_1 \\ \varphi_2 \\ \pi \\ \pi_1 \\ \pi_2 \\ \phi \\ \phi_1 \end{array} \right) = \left( \begin{array}{c} \phi \\ n^\mu \nabla_\mu \phi \left|_{\Sigma} \right. \\ N^{-1} \partial_t \phi \left|_{\Sigma} \right. \\ \partial_\mu (\sqrt{|h|} h^{\mu\nu} \partial_\nu) \sqrt{|h|} d^3x \\ \partial_\mu (\sqrt{|h|} h^{\mu\nu} \partial_\nu) \sqrt{|h|} d^3x \end{array} \right),
\]

define a unique solution \(\phi\) in \(C^\infty(M)\), where \(N^{-1} \partial_t = n^\mu \nabla_\mu\). Let us further define the corresponding symplectic form \(\Omega_\Sigma : \mathcal{S}_\Sigma \times \mathcal{S}_\Sigma \mapsto \mathbb{R}\) on \(\mathcal{S}_\Sigma\) as

\[
\Omega_\Sigma(\Phi_1, \Phi_2) = \int_{\Omega_\Sigma} (\pi_1 \varphi_2 - \pi_2 \varphi_1) \sqrt{|h|} d^3x
\]

where the former integral exists since the functions \(\Phi_1\) and \(\Phi_2\) are compactly supported on \(\Sigma\). Moreover, for given global solutions of the Klein-Gordon equation (2.2) the integral is independent of the choice of the Cauchy surface. The symplectic structure \(\Omega_\Sigma\) makes \(\mathcal{S}_\Sigma\) into a symplectic vector space (a fact that relies on the requirement that the spacetime is globally hyperbolic, see [19]). Next, let the operator \(w^2\) be given by

\[
w^2 = -\frac{N}{\sqrt{|h|}} \partial_t (\sqrt{|h|} Nh^{ij} \partial_j) + N^2 m^2 + N^2 V
\]

\[
= -N^2 (\Delta_h - m^2 - V) - Nh^{ij} \partial_i N \partial_j,
\]

where \(\Delta_h\) is the Laplace-Beltrami operator with respect to the associated spatial metric \(h\). The operator \(w^2\) is defined such that the Klein-Gordon equation takes the form,

\[
(\partial_t^2 + f(t, x) \partial_t + w^2) \phi = 0,
\]

where \(f(t, x) = -N^{-1} \partial_t N + (\sqrt{|h|})^{-1} \partial_t \sqrt{|h|}\).

### 3 Weighted Manifolds and Essential Self-Adjointness

We proceed to introduce the notion of weighted Manifolds and weighted Hilbert spaces. For further details we direct the reader to the excellent reference [12]. We begin this section with the following definition, [12, Chapter 3.6, Definition 3.17].
**Definition 3.1.** A triple $(\Sigma, h, \mu)$ is called a weighted manifold, if $(\Sigma, h)$ is a Riemannian manifold and $\mu$ is a measure on $\Sigma$ with a smooth and everywhere positive density function. Hence, if the corresponding measure (volume element on $\Sigma$) is given by $d\Sigma$ and $\rho$ is a positive (smooth) density function then the corresponding measure $\mu$ is given by $d\mu = \rho \, d\Sigma$. Equipped with the measure $\mu$ the weighted Hilbert space, denoted as $L^2(\Sigma, \mu)$, is given as the space of all square-integrable functions on the manifold $\Sigma$ with respect to the measure $\mu$. The corresponding weighted Laplace-Beltrami operator (also called the Dirichlet-Laplace operator), denoted by $\Delta_\mu$ is given by

$$\Delta_\mu = \frac{1}{\rho \sqrt{|h|}} \partial_i (\rho \sqrt{|h|} h^{ij} \partial_j).$$

The interesting fact about weighted Hilbert spaces is that this notion allows to extend the essential self-adjointness of the Laplace-Beltrami operator to the class of weighted Laplace-Beltrami operators. In order to give the domain of essential self-adjointness we need to define Sobolev spaces with respect to the weighted Hilbert spaces first [12, Chapter 4.1]. The first Sobolev space is given as

$$W^1(\Sigma, h, \mu) := \{ u \in L^2(\Sigma, \mu) : \nabla u \in L^2(\Sigma, \mu) \}.$$  

As usual the Sobolev space $W^1$ is a space of functions and not merely a space of equivalence classes of functions. It is easy to prove that $W^1(\Sigma, h, \mu)$ is a Hilbert subspace. Next, we define the functional space $W^2_0(\Sigma, h, \mu)$ as the closure of $C_0^\infty(\Sigma, h, \mu)$ (smooth functions of compact support with scalar product with respect to the measure $\mu$) in $W^1(\Sigma, h, \mu)$ and

$$W^2_0(\Sigma, h, \mu) = \{ u \in W^1_0(\Sigma, h, \mu) : \Delta_\mu u \in L^2(\Sigma, \mu) \}.$$  

The space $W^2_0(\Sigma, h, \mu)$ has the same inner product as $W^1(\Sigma, h, \mu)$ and is a Hilbert space as a closed subspace of $W^1(\Sigma, h, \mu)$ and obviously we have the following relation $C_0^\infty(\Sigma, h, \mu) \subset W^2_0(\Sigma, h, \mu)$. It can be easily seen that the weighted Laplace-Beltrami operator is a symmetric operator on $C_0^\infty(\Sigma, h, \mu)$ and the next theorem states that the densely defined operator $\Delta_\mu|_{W^2_0}$ that is (clearly) an extension of the operator $\Delta_\mu|_{C_0^\infty}$ is essentially self-adjoint [12, Chapter 4, Theorem 4.6].

**Theorem 3.1.** On any weighted manifold, the weighted Laplace-Beltrami operator $\Delta_\mu|_{W^2_0}$ is a self-adjoint positive operator in $L^2(\Sigma, \mu)$. Furthermore, $\Delta_\mu|_{W^2_0}$ is a unique self-adjoint extension of $\Delta_\mu|_{C_0^\infty}$ whose domain is contained in $W^2_0$.

Another useful fact about weighted manifolds that we shall give as a proposition is [12, Chapter 3, Exercise 3.11].

**Proposition 3.1.** Let $a, b$ be smooth and everywhere positive functions on a weighted manifold $(\Sigma, h, \mu)$ and define a new metric $\tilde{h}$ and measure $\tilde{\mu}$ by

$$\tilde{h} = a \, h, \quad \text{and} \quad d\tilde{\mu} = b \, d\mu.$$  

Then, the weighted Laplace-Beltrami operator $\tilde{\Delta}_{\tilde{\mu}}$ of the weighted manifold $(\Sigma, \tilde{h}, \tilde{\mu})$ is given by

$$\tilde{\Delta}_{\tilde{\mu}} = \frac{1}{b} \text{div}_\mu \left( \frac{b}{a} \nabla \right),$$

where in local coordinates the divergence of a vector field $v$ is given by

$$\text{div}_\mu v = \frac{1}{\rho} \frac{\partial}{\partial x^i} (\rho v^i).$$

In particular, if $a = b$ then

$$\tilde{\Delta}_{\tilde{\mu}} = \frac{1}{a} \Delta_\mu.$$
Proof. The proof is done by using functions of compact support (on the weighted manifold \((\Sigma, \tilde{h}, \tilde{\mu})\), and the Green formula [12, Chapter 3, Theorem 3.16],

\[
\int_{\Sigma} u(\tilde{\Delta}\tilde{v}) \, d\tilde{\mu} = -\int_{\Sigma} (\partial_{i} u) \tilde{h}^{ij} (\partial_{j} \tilde{v}) \, d\tilde{\mu} = \int_{\Sigma} \frac{b}{a} \text{div}_{\mu}(\nabla \tilde{v}) \, d\mu = \int_{\Sigma} u \cdot \frac{1}{b} \text{div}_{\mu}(\nabla \tilde{v}) \, d\tilde{\mu}.
\]

For \(b = a\) we have,

\[
\int_{\Sigma} u(\tilde{\Delta}\tilde{v}) \, d\tilde{\mu} = \int_{\Sigma} u \cdot \frac{1}{a} \text{div}_{\mu}(\nabla \tilde{v}) \, d\tilde{\mu}.
\]

Remark 3.1. Note that the weighted Laplace-Beltrami operator is negative, i.e. \(-\Delta_{\mu}\) is a positive operator. The proof can be done analogously as for the case \(\rho = 1\) where in the weighted case the measure is simply multiplied with a positive smooth function \(\rho\) that does not interfere with the common proof of positivity. In particular, consider the eigenvalue equation

\[-\Delta_{\mu} u = \lambda u,
\]

where \(u\) is an eigenfunction associated to the eigenvalue \(\lambda\) (due to the self-adjointness, see Theorem 3.1, all eigenvalues are real) and write

\[-\int_{\Sigma} u(\Delta_{\mu} u) \, d\mu = \lambda \int_{\Sigma} u^{2} \, d\mu.
\]

Performing integration by parts we have

\[-\int_{\Sigma} u(\Delta_{\mu} u) \, d\mu = \int_{\Sigma} (\partial_{i} u) h^{ij} (\partial_{j} u) \, d\mu = \int_{\Sigma} |\nabla u|^{2} \, d\mu
\]

and by putting the last two equations together we arrive at

\[\int_{\Sigma} |\nabla u|^{2} \, d\mu = \lambda \int_{\Sigma} u^{2} \, d\mu,
\]

from which we conclude that the eigenvalues are positive, i.e. \(\lambda \geq 0\) and therefore the operator \(-\Delta_{\mu}\) is positive.

Before proceeding to our general result we need to mention another theorem that we will make use of. First, define a local \(L^{2}(\Sigma, \nu)\) function \(f\) as a function that is square integrable (with respect to the scalar product of the weighted Hilbert space \(L^{2}(\Sigma, \nu)\)) on every compact subset of the manifold \(\Sigma\) and we write \(f \in L^{2}_{\text{loc}}(\Sigma, \nu)\). Then, the theorem of Shubin states the following [13, Theorem 1.1] (see also [20]).

**Theorem 3.2.** Let the Riemannian manifold \((\Sigma, h')\) be complete and let the potential \(V \in L^{2}_{\text{loc}}(\Sigma, \nu)\) be such that we can write \(V = V_{+} + V_{-}\), where \(V_{+} \in L^{2}_{\text{loc}}(\Sigma, \nu) \geq 0\) and \(V_{-} \in L^{2}_{\text{loc}}(\Sigma, \nu) \leq 0\) point-wise and the corresponding operator \(-\Delta_{\mu} + V\) be semi-bounded from below. Then, the operator \(-\Delta_{\mu} + V\) is an essentially self-adjoint operator on \(C^{\infty}_{0}(\Sigma, \nu)\).
4 Main Result on Self-adjointness

In this section we present our main result, i.e. proving essential self-adjointness and positivity of the operator $w^2$ that was given by formula (2.4). In contrast to previous works, our result covers the case of general \textit{globally hyperbolic spacetimes} with unbounded $N$.

Let $\Sigma$ be a complete Cauchy surface, $h$ the induced spatial metric and $\mu$ the corresponding measure with the smooth positive density function $N$, i.e. $d\mu = N\,d\Sigma$, from Theorem 2.2 (as well as Equation 2.1). We consider the weighted manifolds $(\Sigma, h, \mu)$ and $(\Sigma, h, \tilde{\mu})$, where $d\tilde{\mu} = N^{-2}d\mu$ and $\tilde{h} = N^{-2}h$. The measure $d\tilde{\mu}$ is $N^{-1}d\Sigma$ represents the measure one usually uses in field theory in curved spacetimes to define the symplectic structure or a real inner product (see [2, Equation 4.2.6] or [8]). Hence, one is in general interested in proving the essential self-adjointness of the operator $w^2$ with respect to the measure $d\tilde{\mu}$. By using the former results and theorems we are able to obtain the following general result.

\textbf{Theorem 4.1.} Assume that the potential $V \in L^2_{\text{loc}}(\Sigma, \tilde{\mu})$, with the property $V > -m^2 + \epsilon$ everywhere for some $\epsilon > 0$, that in addition can be written as $V = V_+ + V_-$, where $V_+ \in L^2_{\text{loc}}(\Sigma, \tilde{\mu}) \geq 0$ and $V_- \in L^2_{\text{loc}}(\Sigma, \tilde{\mu}) \leq 0$ point-wise. Then, the operator $w^2$ (from Equation 2.4) is an essentially self-adjoint operator on $C^\infty_0(\Sigma, \tilde{\mu}) \subset L^2(\Sigma, \tilde{\mu})$-the closure being positive and invertible and the square root, i.e. $w$, is given as a unique self-adjoint operator.

\textit{Proof.} The plan of the proof is as follows. First, let us look at the operator $N^{-2}w^2 - (m^2 + V)$ which can be written as a weighted Laplace-Beltrami operator with respect to the weighted manifold $(\Sigma, h, \mu)$ with $d\mu = N\,d\Sigma$, i.e. where the positive smooth density function is given by $N$, where positivity and smoothness of the function $N$ follow from global hyperbolicity (see Theorem 2.2),

$$N^{-2}w^2 - (m^2 + V) = -\Delta_\mu = -\frac{1}{N\sqrt{|h|}}\partial_i(N\sqrt{|h|}h^{ij}\partial_j).$$

The domain of this operator is given by $C^\infty_0(\Sigma, \mu)$. Next, we use Proposition 3.1 by making the following transformation

$$\tilde{h} = N^{-2}h, \quad d\tilde{\mu} = N^{-2}d\mu = N^{-1}d\Sigma.$$

Note that the operator $N^{-2}$ satisfies the conditions of Proposition 3.1 (the conditions on $a$) as a multiplication operator since global hyperbolicity demands from the operator $N$ to be smooth, positive and invertible. After applying the aforementioned transformations we obtain the weighted Laplace-Beltrami operator of the weighted manifold $(\Sigma, \tilde{h}, \tilde{\mu})$,

$$\tilde{\Delta}_{\tilde{\mu}} = N^2\Delta_\mu.$$

By the use of Theorem 3.1 we conclude the first part of the proof, i.e. that the Laplace-Beltrami part of $w^2$ is essentially self-adjoint on $C^\infty_0(\Sigma, \tilde{\mu})$. The proof that the whole operator $w^2$ is essentially self-adjoint follows from the fact that the Laplace-Beltrami part of the operator is written in a manner that we can use Theorem 3.2, where the condition $V > -m^2 + \epsilon$ for some $\epsilon > 0$ guarantees the semi-boundedness from below, i.e.

$$\langle \Phi, w^2\Phi \rangle = \langle \Phi, (-\tilde{\Delta}_{\tilde{\mu}} + N^2(m^2 + V))\Phi \rangle \geq c||\Phi||^2,$$

which holds $\forall \Phi \in C^\infty_0(\Sigma, \tilde{\mu})$ and some constant $c > 0$. Moreover, because $N^2$ is strictly positive and smooth the signs of the respective parts of the potential do not change and the local integrability is preserved by the fact (due to the smoothness) that $N^2$ is locally integrable, i.e.

$$||N^2V||^2_{L^2_{\text{loc}}} \leq ||N^2||_{L^2_{\text{loc}}}||V||_{L^2_{\text{loc}}} < \infty.$$
Since the operator \( w^2 \) is a semi-bounded essentially self-adjoint operator, it has only one semi-bounded self-adjoint extension which is the Friedrich extension ([21, Theorem X.23, Theorem X.26]) that is bounded by the same constant \( c \) (see Equation 4.1). The proof that the square root is given as a unique self-adjoint operator can be done by the use of the spectral theorem ([22, Chapter V.III.3]) or by using [23, Theorem 3.35], [24] or for a new shorter proof see [25].

Remark 4.1. Friedrich’s extension theorem tells us that if an operator \( T \) is densely defined symmetric and positive, then there is an operator in the Hilbert space that is self-adjoint and positive and whose restriction to \( \mathcal{D}(T) \) (dense domain of the operator \( T \)) is equal to \( T \). Since the operator \( w^2 \) is a positive symmetric operator (with the additional requirement on the potential) we could have used the Friedrich extension (as suggested in [8]) to prove that there exists a self-adjoint extension. However, the problem with this approach is lack of uniqueness. In particular the extension given by the Friedrich theorem is, although self-adjoint, not necessarily unique. Proving essential self-adjointness on the other hand implies the uniqueness of the Friedrich extension.

5 Conclusion and Outlook

Essential self-adjointness of the spatial part \( w^2 \) of the Klein Gordon operator has been proven for the case that the Cauchy surface under consideration is complete and under the assumption that the external potential is positive (i.e., \( V > -m^2 + \epsilon \) everywhere for some \( \epsilon > 0 \)). This generalizes a corresponding result [9, Theorem 7.2] of Kay, who had to assume in addition that the spacetime is stationary and that \( N \) is bounded as a multiplication operator.

Although we require global hyperbolicity of spacetime, a possible application of our theorem to QFT on certain spacetimes that are not globally hyperbolic might be possible. The reason therefore lies in the manner how QFT can be studied on such spacetimes. In particular, Kay in [26] proposed a generalization of the algebraic approach to quantum field theory to the case of non-globally hyperbolic spacetimes by defining a property called \( F \)-locality. This property requires that every point around a given spacetime \( M \) should have a globally hyperbolic neighborhood \( N \) such that the restricted algebra of the spacetime \( M \) to \( N \) should be equal to the algebra obtained by regarding the neighborhood \( N \) as a globally hyperbolic spacetime on its own right (with some choice of time). With regards to our result this means that we can define a Klein-Gordon equation on each such globally hyperbolic neighborhood of an \( F \)-local algebra and use Theorem 4.1 to prove the essential self-adjointness of the spatial part of the Klein-Gordon operator on that neighborhood.

A different proposal for QFT on non-globally hyperbolic spacetimes is given in [19] (with the difference to [26] being the construction of algebras that in general fail to fulfill \( F \)-locality, see [27]). This uses the fact that every spacetime admits an open covering by locally causal sets, i.e. subsets \( U \subset (M, g) \) s.t. \( U \) is open, connected and the spacetime \( (U, g|_U) \) is globally hyperbolic. Hence, our result can be used for the spatial part of the Klein Gordon operator on each locally causal set.

Another approach for doing QFT on not necessarily globally hyperbolic spacetimes is general boundary quantum field theory [28][29]. There, Hilbert spaces of states are associated to hypersurfaces that need not even be spacelike. However, in many contexts the restriction to spacelike hypersurfaces is sufficient and our result could be of use there.
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