Effect of Symmetry Breaking Perturbations in the One-Dimensional SU(4) Spin-Orbital Model

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We study the effect of symmetry breaking perturbations in the one-dimensional SU(4) spin-orbital model. We allow the exchange in spin ($J_1$) and orbital ($J_2$) channel to be different and thus reduce the symmetry to SU(2) $\otimes$ SU(2). A magnetic field $h$ along the $S^z$ direction is also applied. Using the formalism developed by Azaria et al., we extend their analysis of the isotropic $J_1 = J_2$, $h = 0$ case and obtain the low-energy effective theory near the SU(4) point in the generic case $J_1 \neq J_2$, $h \neq 0$. In zero magnetic field, we retrieve the same qualitative low-energy physics than in the isotropic case. In particular, the isotropic massless behavior found on the line $J_1 = J_2 < K/4$ extends in a large anisotropic region. We discover however that the anisotropy plays its trick in allowing non trivial scaling behaviors of the physical quantities. For example the mass gap $M$ has two different scaling behaviors depending on the anisotropy. In addition, we show that in some regions, the anisotropy is responsible for anomalous finite size effects and may change qualitatively the shape of the computed critical line in a finite system. When a magnetic field is present the effect of the anisotropy is striking. In addition to the usual commensurate-incommensurate phase transition that occurs in the spin sector of the theory, we find that the field may induce a second transition of the KT type in the remaining degrees of freedom to which it does not couple directly. In this sector, we find that the effective theory is that of an SO(4) Gross-Neveu model with an $h$-dependent coupling that may change its sign as $h$ varies.

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I. INTRODUCTION

In the very past years, there has been an intense interest devoted to one dimensional spin-orbital models (1). The main reason stems from the recent discovery of the new quasi one dimensional spin gapped materials Na\textsubscript{2}Ti\textsubscript{2}Sb\textsubscript{2}O \textsuperscript{2} and NaV\textsubscript{2}O\textsubscript{3} \textsuperscript{3}. It is believed that the unusual magnetic properties observed in these compounds should be explained by a simple two-band Hubbard models at quarter filling. At this filling, and in the large Coulomb repulsion the effective Hamiltonian simplifies greatly and is equivalent to a model of two interacting spin one-half Heisenberg models \textsuperscript{4,5}:

$$
\mathcal{H} = \sum_i J_1 \vec{S}_i \cdot \vec{S}_{i+1} + J_2 \vec{T}_i \cdot \vec{T}_{i+1} + K \left( \vec{S}_i \cdot \vec{S}_{i+1} \right) \left( \vec{T}_i \cdot \vec{T}_{i+1} \right) (1)
$$

where $\vec{S}_i$ and $\vec{T}_i$ are spin-1/2 operators that represent spin and orbital degrees of freedom at each site $i$ and $J_{1(2)}$ and $K$ are positive. It is important to notice that (4) is not only SU(2) invariant in $\vec{S}$ space but also in $\vec{T}$ space. For generic couplings $J_{1(2)}$ and $K$, the Hamiltonian (4) is SU(2)$_x$ $\otimes$ SU(2)$_y$ symmetric. Two particular cases are of interest. First, when $J_1 = J_2$ there is an additional $Z_2$ symmetry in the exchange between $\vec{S}_i$ and $\vec{T}_i$. The other important case is when $J_1 = J_2 = K/4$ in which case the Hamiltonian is SU(4) invariant (5). In fact, the Hamiltonian (4) is a simplified version of the most general case. Indeed, depending on the microscopic couplings, there can be other terms that breaks the SU(2) symmetry in both spin and orbital sectors. The choice of studying (4) is dictated by the fact that since it retains some symmetries it is a simple starting point from which one can expect to gain some insight before tackling with the more general case. In this respect, the model (4) describes the simplest physically relevant symmetry breaking pattern SU(4) $\rightarrow$ SU(2)$_x$ $\otimes$ SU(2)$_y$.

The Hamiltonian (4) can be interpreted in terms of a two-leg spin ladder coupled by a four spins interaction. Such an interaction can be generated either by phonons or, in the doped state, by conventional Coulomb repulsion between the holes (6). Some of the properties of (4) are well established in the weak coupling limit. In the limit $K \ll J_{1(2)}$, the Hamiltonian (4) describes a non-Haldane spin liquid where magnon excitations are incoherent (7).

The strong coupling regime, $K \sim J_{1(2)}$, has just begin to be investigated (8,9,10). From the theoretical point of view the situation is awkward. Indeed, as stated above, at the special point $J_1 = J_2 = K/4$, the Hamiltonian (4) has an enlarged SU(4) symmetry (11) and is exactly solvable by the Bethe-ansatz (12). The model is critical with three gapless bosonic modes and flows in the infrared towards the Wess-Zumino-Novikov-Witten (WZNW) model SU(4)$_1$ (13). It is conformally invariant with the central charge $c = 3$. Therefore, there should be
a qualitative change in the physical behavior of $\Phi$ when going from small to large $K$. From the theoretical point of view this situation is striking since this means that one cannot go continuously from weak to strong coupling. This is a manifestation of the Zamolodchikov c-theorem [13] that states that starting at $K = 0$ with two decoupled $S = 1/2$ chains with the central charge $c = 2$ (two gapless bosonic modes) one cannot flow, in the Renormalization Group (RG) sense, towards the SU(4) point which has $c = 3$ (three gapless bosonic modes). Very recently, a new approach to tackle with the strong coupling regime has been developed by Azaria et al [14]. The idea is to start from the strong coupling fixed point SU(4)$_1$ and to perturb around it.

This strategy has been applied to the symmetric line $J = J_1 = J_2$. It was shown that when $J < K/4$ a small deviation from the SU(4) point is irrelevant and thus the low energy physics is governed by the SU(4)$_1$ fixed point. In contrast, when $J > K/4$ the interaction is relevant and a gap opens in the spectrum. The system dimerizes with alternating spin and orbital singlets. In addition and a gap opens in the spectrum. The system dimerizes with wavevector near $\mathbf{q}$.

We obtain the phase diagram in the plane $(J, K)$ and discuss the asymptotic behaviors of the mass gap $M$.

To this end, let us introduce the SU(4) Hubbard model with $U > 0$:

\[ \mathcal{H}_U = \sum_{i\sigma} \left( -t \, c_{i+1\sigma}^\dagger c_{i\sigma} + h.c. \right) + \frac{U}{2} \sum_{i\sigma\sigma'} n_{i\sigma} n_{i\sigma'} (1 - \delta_{\sigma\sigma'}) . \] (2)

Here $c_{i\sigma}$ denotes electron creation with channel (orbital) $\alpha = 1, 2$ and spin $\sigma = \uparrow, \downarrow$ at the $i$-th site. The occupation number is defined by: $n_{i\sigma} = c_{i\sigma}^\dagger c_{i\sigma}$. In the limit of large positive $U$, and at quarter filling, it is not difficult to show that (3) reduce to (1) with the identification:

\[ \tilde{\sigma}_i = \frac{1}{2} \sum_{\alpha} c_{i\alpha\sigma}^\dagger \bar{\sigma}_{\alpha\beta} c_{i\alpha\beta} \]

\[ \bar{T}_i = \frac{1}{2} \sum_{\alpha} c_{i\alpha\sigma}^\dagger \bar{\tau}_{ab} c_{i\alpha\beta} \] (3)

where $\bar{\sigma}$ (resp. $\bar{\tau}$) are the Pauli matrices acting in the spin (resp. orbital) space. The low energy physics can be described in terms of right movers $R_{\sigma \sigma'}$ and left movers $L_{\sigma \sigma'}$ fermions which correspond to the lattice fermion $c_{i\alpha\sigma}$ in the continuum limit:

\[ c_{i\alpha\sigma} \sim R_{\sigma \sigma'} (x) \exp (ik_F x) + L_{\sigma \sigma'} (x) \exp (-ik_F x), \quad x = ia_0 \] (4)

where $a_0$ is the lattice spacing and the Fermi wavevector is defined by: $k_F = \pi/4a_0$. At this point we bosonize and introduce four chiral bosonic fields $\Phi_{\sigma \sigma', R, L}$ using the Abelian bosonization of Dirac fermions [15]:

\[ R_{\sigma \sigma'} = \frac{\kappa_{\sigma \sigma'}}{\sqrt{2\pi a_0}} \exp \left( i\sqrt{4\pi} \Phi_{\sigma \sigma'} \right) \]

\[ L_{\sigma \sigma'} = \frac{\kappa_{\sigma \sigma'}}{\sqrt{2\pi a_0}} \exp \left( -i\sqrt{4\pi} \Phi_{\sigma \sigma'} \right) \] (5)

where the bosonic fields satisfy the following commutation relation:

\[ [\Phi_{\sigma \sigma', R}, \Phi_{\delta \delta', L}] = \frac{i}{4} \delta_{\sigma\delta} \delta_{\sigma', \delta'} \] (6)

so that $\{R_{\sigma \sigma'} (x), L_{\delta \delta'} (y)\} = 0$. The anticommutation between fermions with different spin-channel indexes is insured by the presence of Klein factors (here Majorana fermions) $\kappa_{\sigma \sigma'}$ with the following anticommutation rule:

\[ \{\kappa_{\sigma \sigma'}, \kappa_{\delta \delta'}\} = 2\delta_{\sigma\delta} \delta_{\sigma', \delta'} \] (7)

As in the the solution of the two-channel Kondo effect by Abelian bosonization [16] it is suitable to introduce the physically transparent basis:

\[ \Phi_c = \frac{1}{2} (\Phi_{1\uparrow} + \Phi_{1\downarrow} + \Phi_{2\uparrow} + \Phi_{2\downarrow}) \]

\[ \Phi_s = \frac{1}{2} (\Phi_{1\uparrow} - \Phi_{1\downarrow} + \Phi_{2\uparrow} - \Phi_{2\downarrow}) \]

\[ \Phi_f = \frac{1}{2} (\Phi_{1\uparrow} + \Phi_{1\downarrow} - \Phi_{2\uparrow} - \Phi_{2\downarrow}) \]

\[ \Phi_{sf} = \frac{1}{2} (\Phi_{1\uparrow} - \Phi_{1\downarrow} - \Phi_{2\uparrow} + \Phi_{2\downarrow}) \] (8)
In this new basis, the total charge degree of freedom is described by the bosonic field \( \Phi_c \), while the other “spin-orbital” degrees of freedom, are faithfully bosonized by the three bosonic fields \( \Phi_s, \Phi_f, \Phi_{sf} \). It is now straightforward to obtain the continuum limit of the Hubbard Hamiltonian \( \mathcal{H}_U \):  
\[ \mathcal{H}_U = \mathcal{H}_c + \mathcal{H}_{sf} \]  
(9)
where  
\[ \mathcal{H}_c = \frac{v_F}{2} ( (\partial_x \Phi_c)^2 + (\partial_x \Theta_c)^2 ) + \frac{3U a_0}{2\pi} (\partial_x \Phi_c)^2, \]  
(10)
and  
\[ \mathcal{H}_{sf} = \sum_{a=s,f,sf} \left( \frac{v_F}{2} ( (\partial_x \Phi_a)^2 + (\partial_x \Theta_a)^2 ) \right) - \frac{U a_0}{2\pi} (\partial_x \Phi_a)^2 + \frac{U}{\pi^2 a_0} \left( \cos \sqrt{4\pi} \Phi_s \cos \sqrt{4\pi} \Phi_f + \cos \sqrt{4\pi} \Phi_{sf} \cos \sqrt{4\pi} \Phi_{sf} \right). \]  
(11)
As in the SU(2) Heisenberg chain, spin and charge degrees of freedom separate. Notice however that at this order in \( U \) there are no umklapp terms in the charge sector since we are at quarter filling and the 4\( k_F \) contribution to the effective Hamiltonian oscillates. Umklapp terms will arise at higher order in perturbation theory and will be responsible of a Mott transition at a finite value of \( U = U_c \). Assuming that \( U >> U_c \), we focus now on the spin-orbital sector.

The interaction term in \( \mathcal{H}_{sf} \) has scaling dimension 2 and is therefore marginal. This term is nothing but the SU(4) current-current interaction. The 15 SU(4) currents \( \mathcal{J}_R^a \) and \( \mathcal{J}_L^a \) can be expressed in terms of the three bosonic fields \( \Phi_s, \Phi_f \) and \( \Phi_{sf} \) and the spin-orbital part of the Hamiltonian \( \mathcal{H}_{sf} \) takes the form:
\[ \mathcal{H}_{sf} = \frac{2\pi v_s}{5} \sum_{a=1,15} (\mathcal{J}_R^a \mathcal{J}_R^a + \mathcal{J}_L^a \mathcal{J}_L^a) + 2g_s \sum_{a=1,15} \mathcal{J}_R^a \mathcal{J}_L^a \]  
(12)
where  
\[ g_s = -U a_0, \]  
\[ v_s = v_F - U a_0 / 2\pi. \]  
(13)
The first term in Eq. (12) is just the Sugawara form of the SU(4)_1 WZNW model while the second term is the marginal current-current interaction. When \( U > 0 \), \( g_s < 0 \), the current-current interaction \( \mathcal{J}_R^a \mathcal{J}_L^a \) is thus irrelevant, as a consequence the spin orbital sector is described by the SU(4)_1 WZNW model [14].

B. Majorana representation and the SO(6) Gross-Neveu model

As first emphasized by Shelton et al. [18] in their study of the two-leg spin 1/2 ladders it is very convenient to formulate spin ladders problem in terms of real (Majorana) fermions. This can be done by refermionising the three bosonic fields \( \Phi_s, \Phi_f \) and \( \Phi_{sf} \). Let us introduce the six Majorana fermions \( \xi_a^s, a = 1...6 \) as follows:
\[ (\xi^1 + i\xi^2)_{R(L)} = \frac{\eta_i}{\sqrt{\pi a_0}} \exp \left( \pm i \sqrt{4\pi} \Phi_{R(L)} \right) \]  
(14)
\[ (\xi^3 + i\xi^4)_{R(L)} = \frac{\eta_2}{\sqrt{\pi a_0}} \exp \left( \pm i \sqrt{4\pi} \Phi_{fR(L)} \right) \]  
(15)
\[ (\xi^5 + i\xi^6)_{R(L)} = \frac{\eta_3}{\sqrt{\pi a_0}} \exp \left( \pm i \sqrt{4\pi} \Phi_{sfR(L)} \right) \]  
where \( \eta_i \) are Klein factors. With all these relations at hand, one can rewrite the Hamiltonian [11] in terms of six Majorana fermions:
\[ \mathcal{H}_{sf} = -\frac{i v_s}{2} \sum_{a=1}^{6} (\xi_R^a \partial_s \xi_R^a - \xi_L^a \partial_s \xi_L^a) + g_s \sum_{i<j} \kappa_i \kappa_j \]  
(16)
where we have introduced the energy density of the different Ising models: \( i \varepsilon_i = \kappa_a = \kappa_{R(L)}^a \). The Hamiltonian (13) is nothing but that of the SO(6) GN model with a marginally irrelevant interaction when \( U > 0 \). Thus in the far infrared the six Majorana fermions decouple and remain massless. This is the SO(6)_1 fixed point.

The above result assumes that \( U \) is small and the question that naturally arises is whether it can be extended to large values of \( U \) where (2) reduces to (1). For exactly the same reasons as for the SU(2) Heisenberg chain the answer is positive. We know from the exact solution that the SU(4) model is critical with three massless bosonic modes or equivalently six massless Majorana fermions. We know from Conformal Field Theory that the fixed point Hamiltonian can only be the SU(4)_1 ~ SO(6)_1 WZNW model. The marginal interaction \( \sum_{i<j} \kappa_i \kappa_j \) is the only one that respects both the SO(6) symmetry as well as translation invariance. Therefore in the vicinity of the fixed point the SU(4) Heisenberg model will be given by:
\[ \mathcal{H} = -\frac{i u_s}{2} \sum_{a=1}^{6} (\xi_R^a \partial_s \xi_R^a - \xi_L^a \partial_s \xi_L^a) + 2G_3 \sum_{i<j} \kappa_i \kappa_j \]  
(17)
\[ \tilde{\xi}_{R(L)} = (\xi^2, \xi^1, \xi^0)_{R(L)} \]
\[ \tilde{\xi}_{tR(L)} = (\xi^4, \xi^3, \xi^5)_{R(L)}. \]  (17)

These quantities transform like a vector under spin SO(3), and orbital SO(3) rotations. These correspond to the SU(2) s and SU(2) t transformations acting on the operators \( \vec{S} \) and \( \vec{T} \) respectively.

To get a complete description of the SO(6) fixed point one needs the continuum expressions for the effective spin and orbital densities in terms of the Majorana fermions:

\[ \vec{S} = \vec{J}_R + \vec{J}_L + \exp(\pm i \pi x/2a_0) \vec{N}_s + \text{H.c.} + (-1)^{k/a_0} \vec{n}_s(x) \]
\[ \vec{T} = \vec{J}_R + \vec{J}_L + \exp(\pm i \pi x/2a_0) \vec{N}_t + \text{H.c.} + (-1)^{k/a_0} \vec{n}_t(x) \]  (18)

where \( \vec{J}_{s,t} \) is the uniform, \( k = 0 \) part and \( \vec{N}_{s,t} \) and \( \vec{n}_{s,t} \) are the \( 2k_F = \pi/2a_0 \) and \( 4k_F = \pi/a_0 \) contributions. Notice that in contrast with the SU(2) Heisenberg chain, the spin density has three oscillating components. The reason for this comes from Conformal Field Theory. Indeed the different oscillating components of the spin density are primary fields of the SU(4) 4 WZNW model. There are three of them with scaling dimensions \((3/4, 1/3, 1/4)\). They all belong to the representations (building blocks) of SU(4) with Young tableau consisting of \( a \) \((a=1,2,3)\) boxes and one column. In particular the staggered components at \( 4k_F = \pi, \vec{n}_{s,t} \), are components of an antisymmetric tensor of rank 2.

The uniform components express in terms of the Majorana fermions as follows:

\[ \vec{J}_{sR(L)} = -\frac{i}{2} \tilde{\xi}_{sR(L)} \wedge \tilde{\xi}_{sR(L)} \]
\[ \vec{J}_{tR(L)} = -\frac{i}{2} \tilde{\xi}_{tR(L)} \wedge \tilde{\xi}_{tR(L)} \]  (19)

Notice that in contrast with the SU(2) Heisenberg chain, the uniform part of the spin densities are \( SU(2) \) currents. The expressions for the \( 4k_F = \pi/a_0 \) densities are given by

\[ \vec{n}_s = iB \tilde{\xi}_{sR} \wedge \tilde{\xi}_{sL} \]
\[ \vec{n}_t = iB \tilde{\xi}_{tR} \wedge \tilde{\xi}_{tL} \]  (20)

where \( B \) is a non universal constant. Their scaling dimension at the SO(6) fixed point is \( \Delta_s = 1 \). Both densities \((13)\) and \((20)\) are rather simple when expressed in terms of the Majorana fermions. This is not the case with the \( 2k_F = \pi/2a_0 \) densities \( \vec{N}_{s,t} \) that are non local in the Majorana fermions \( \xi^a \). Indeed they involve order and disorder operators \( \sigma^a \) and \( \mu_0 \), of the six Ising models that are associated with the six Majorana fermions. The expressions of both \( \vec{N}_{s,t} \) are lengthy and we shall give here only the \( z \) component that will be sufficient for our purpose:

\[ \vec{N}^z_s = A (i\mu_1 \mu_2 \sigma_4 \sigma_5 \sigma_6 + \sigma_1 \sigma_2 \sigma_4 \mu_3 \sigma_4 \mu_5 \sigma_6) \]
\[ \vec{N}^z_t = A (i\sigma_1 \sigma_2 \sigma_4 \sigma_5 \sigma_6 + \mu_1 \mu_2 \sigma_3 \sigma_4 \mu_5 \sigma_6) \]  (21)

where \( A \) is also a non universal constant. Since at the free fermion point, the order and disorder operators have scaling dimension \( 1/8 \), the \( 2k_F \) densities \( \vec{N}_{s,t} \) have the scaling dimension \( \Delta_{s,t} = 3/4 \).

This completes the continuum description at the SU(4) point. The theory is critical and flows to the SO(6) fixed point. There is a marginally irrelevant correction which magnitude is nonuniversal and is given by the unknown coupling \( G_3 < 0 \). This is in complete agreement with the non-Abelian bosonization approach of Affleck [12]. With the continuum expressions of the spin and orbital operators at the SU(4) point in terms of the six Majorana fermions and the associated Ising models we shall be able us to investigate the properties of \((1)\) close to the SU(4) symmetric model. But before moving to this point, let us stress that the effective theory depends on three unknown parameters \( A \) and \( B \) and \( G_3 \) which can be in principle extracted from numerical studies. In the following, we shall assume that the above description still holds for small deviations of the SU(4) point. The only modification being a renormalization of the non universal constants \( A, B \) and \( G_3 \).

C. The SU(2)\( \otimes \)SU(2) symmetry breaking perturbation

We shall now derive the effective low energy theory associated with \((1)\) for arbitrary values of \( J_1 \) and \( J_2 \) close to the SU(4) invariant point given by \( J_1 = J_2 = K/4 \). To this end, let us parametrize the couplings as follows:

\[ J_1 = \frac{K}{4} + G_1 \]
\[ J_2 = \frac{K}{4} + G_2 \]  (22)

where both \( G_1 \) and \( G_2 \) are much smaller than \( K \). The Hamiltonian \((1)\) can then be written as:

\[ \mathcal{H} = \mathcal{H}_{SU(4)} + G_1 \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + G_2 \sum_i \vec{T}_i \cdot \vec{T}_{i+1}. \]  (23)

Using the low energy description of the spin-orbital operators \((14), (21) \) and \((21)\), one can expand \((23)\) around the SO(6) fixed point:

\[ \mathcal{H} = -\frac{i u_s}{2} (\tilde{\xi}_{sR} \cdot \partial_x \tilde{\xi}_{sR} - \tilde{\xi}_{sL} \cdot \partial_x \tilde{\xi}_{sL}) + g_1 (\kappa_1 + \kappa_2 + \kappa_6)^2 \]
\[ -\frac{i u_t}{2} (\tilde{\xi}_{tR} \cdot \partial_x \tilde{\xi}_{tR} - \tilde{\xi}_{tL} \cdot \partial_x \tilde{\xi}_{tL}) + g_2 (\kappa_3 + \kappa_4 + \kappa_5)^2 + 2G_3 (\kappa_1 + \kappa_2 + \kappa_6) (\kappa_3 + \kappa_4 + \kappa_5) \]  (24)

where

\[ g_1 = G_1 + G_3 \]
\[ g_2 = G_2 + G_3 \]  (25)

and the two renormalized velocities \( u_s \) and \( u_t \) are given by:

4
\[ u_s = v_s + 2G_1/\pi \]
\[ u_t = v_s + 2G_2/\pi. \]

In the above equations and in the remaining of this paper, we include the effect of the 4k_F components of the spin and orbital densities in a redefinition of the couplings: \( G_{1,2} \rightarrow (1 + B^2)G_{1,2} \). The Hamiltonian \([24]\) describes two marginally coupled \( \text{SO}(3) \) Gross-Neveu models: one in the spin channel described by the three Majorana \( \xi_s \) and one in the orbital channel described by the three Majorana \( \xi_c \). The situation at hand is to be contrasted another time with the one encountered in the study of spin ladders. In the latter models deviation from criticality leads, in general, to relevant perturbation and a gap always opens independently of the sign of the couplings. There are however notable exceptions where frustration plays its trick. This is the case of some three-leg frustrated ladders where for some particular values of the couplings only marginal interaction remains in the effective action. As a result a non trivial critical state, the so-called "chirally stabilized" liquid, shows up \([24]\). In the model the situation is even more striking since we find only marginal interactions in a finite region of the couplings. This is mainly due to the fact that the frustration is maximal in the strong coupling region. A direct consequence of the marginality of all the interaction is that we expect the phase diagram to result from a delicate balance between the different terms entering in \([24]\). This is why it is now worth discussing the effect of the three different terms in \([24]\).

Consider first the case where \( G_3 = 0 \). Then we are left with the two decoupled \( \text{SO}(3)_s \) and \( \text{SO}(3)_t \) GN models which are exactly solvable. At issue is the sign of \( G_1 \) and \( G_2 \). When \( G_1 > 0 \) and \( G_2 > 0 \) a gap opens in both spin and orbital channels. The spectrum consists of kinks and antikinks (there are no fermions) \([24]\). When \( G_1 < 0 \) and \( G_2 < 0 \), the interaction is irrelevant and the model flows towards the isotropic \( \text{SO}(6)_1 \) fixed point with the central charge \( c = 3 \). When \( G_1G_2 < 0 \) one of the \( \text{SO}(3) \) GN model will become massive while the other one will flow towards the \( \text{SO}(3)_1 \) fixed point. The Hamiltonian will remain critical with three massless Majorana fermions leaving the whole system with the central charge \( c = 3/2 \).

Now the physically relevant question is whether or not this scheme survives a small negative \( G_3 \). Indeed, it would be not correct to neglect the last term in \([24]\). First of all, from the point of view of the lattice Hamiltonian \([23]\), the Hamiltonian \([24]\) has to be thought as the effective Hamiltonian obtained by integrating out high energy modes up to a scale where one sits close enough to the \( \text{SO}(6)_1 \) fixed point where the continuum limit can be taken. Thus, generically \( G_3 \) is not zero. The second reason is that all other interactions are marginal. In such a case, it is well known that operators that are naively irrelevant may become dangerous and strongly modify the physics in the infrared. Therefore, even though \( G_3 < 0 \), one has to keep it and analyze with care \([24]\) with all couplings different from zero. As we shall see, the strong tendency to the \( \text{SO}(3)_1 \) criticality in the regions \( G_1G_2 < 0 \) will be spoiled in most of the parameter space. There will be though still a finite region where the Hamiltonian will be critical but with an approximate \( \text{SO}(6) \) symmetry.

### III. Renormalization Group Analysis

#### A. The renormalization group flow

The RG equations for the couplings entering in \([24]\) are given at leading order by:

\[
\dot{g}_1 = \frac{1}{\pi u_s} g_1^2 + \frac{3}{\pi u_t} G_3^2 \\
\dot{g}_2 = \frac{1}{\pi u_t} g_2^2 + \frac{3}{\pi u_s} G_3^2 \\
\dot{G}_3 = \frac{2G_3}{\pi} \left( \frac{g_1}{u_s} + \frac{g_2}{u_t} \right)
\]

In the above equation, \( \dot{G} \) means \( \partial G/\partial t \) where \( t = \log(\lambda) \) is the RG scale. It is more convenient to express the set of Eqs. \([27]\) in terms of the couplings that enter in the lattice Hamiltonian \([23]\):

\[
\dot{G}_1 = G_1^2 - 2G_2G_3 \\
\dot{G}_2 = G_2^2 - 2G_1G_3 \\
\dot{G}_3 = 2G_3(G_1 + G_2 + 2G_3),
\]

where, for simplicity we have made the following redefinition:

\[
G_1 \rightarrow 1/\pi \sqrt{u_s u_t} (\alpha G_1 - (1 - \alpha)G_3) \\
G_2 \rightarrow 1/\pi \sqrt{u_s u_t} (1/\alpha G_2 - (1 - 1/\alpha)G_3) \\
G_3 \rightarrow 1/\pi \sqrt{u_s u_t} G_3
\]

with \( \alpha = \sqrt{u_t/u_s} \). At the leading order, the velocities \( u_s \) and \( u_t \) also renormalize:

\[
\dot{u}_s = -6u_s G_3^2 \mathcal{I}(\frac{u_t}{u_s}) \\
\dot{u}_t = -6u_t G_3^2 \mathcal{I}(\frac{u_s}{u_t})
\]

where

\[
\mathcal{I}(x) = \frac{1 - x}{1 + x}.
\]

The RG equations \([28]\) and \([30]\) can be exactly solved and reduce to a single differential equation. To show this let us introduce the following variables:

\[
\mu = G_1G_2 + G_1G_3 + G_2G_3 \\
d = G_1 - G_2 \]

\[
s = G_1 + G_2 + 2G_3
\]
Eqs. (28) and (30) greatly simplify to
\[ \begin{align*}
\dot{\mu} &= \mu s \\
\dot{d} &= ds \\
\dot{G}_3 &= 2G_3s
\end{align*} \] (33)

and
\[ u_s - u_t = -6G_3^2 (u_s - u_t) \]
\[ u_s u_t = \text{constant.} \] (34)

While the meaning of both \( s \) and \( d \) is clear, the physical interpretation of the variable \( \mu \) is not straightforward. As we shall see in the following, \( \mu \) measures the departure from criticality.

The solution of both (33) and (34) are then easily obtained:
\[ \begin{align*}
\mu(t) &= \mu(0)X(t) \\
d(t) &= d(0)X(t) \\
G_3(t) &= G_3(0)X^2(t) \\
(u_s - u_t)(t) &= (u_s - u_t)(0) \exp\left(-6G_3^2(0) \int_0^t X^4(\tau)d\tau\right) \\
(u_s u_t)(t) &= (u_s u_t)(0)
\end{align*} \] (35)

where
\[ X(t) = \exp\int_0^t s(\tau)d\tau \] (36)

is the solution of the differential equation:
\[ \left(\frac{X}{X}\right)^2 = P(X). \] (37)

In the latter equation \( P(X) \) is a fourth order polynomial that depends on the initial conditions of the flow:
\[ P(X) = 4G_3^2(0)X^4 + d^2(0)X^2 + 4\mu(0)X. \] (38)

Once the solution \( X(t) \) of (37) is known, the behavior of the RG flow is completely determined. In particular the time evolution of couplings \( G_1(t) \) and \( G_2(t) \) is given by:
\[ G_{(1,2)}(t) = \frac{1}{2} \left( \epsilon(s) \sqrt{P(X(t))} - 2G_3(0)X^2(t) \pm d(0)X(t) \right) \] (39)

where \( \epsilon(s) = \text{sign}(s) \).

As in the XXZ model we distinguish three different phases A, B and C that are separated by the two surfaces defined by \( \mu = 0 \).

- The A Phase. There one has \( \mu(0) > 0 \) and \( s(0) > 0 \) and \( X(t) \to -\infty \) as \( t \to \infty \) (see Fig. 1(a)). All couplings are relevant and a gap opens in the spectrum. In addition, the velocities \( u_s(t) \) and \( u_t(t) \) goes to a fixed value \( u_s^* = u_t^* \). Looking in more details in the large \( t \) behavior of the couplings, one finds that \( G_1(t) \sim G_2(t) \sim G_3(t) \) so that the effective interaction in the far infrared takes the suggestive form: \( \sim ((\kappa_1 + \kappa_2 + \kappa_3) - (\kappa_3 + \kappa_4 + \kappa_5))^2 \) which after the transformation \( \xi_R \to \xi_R; \xi_L \to -\xi_L \) acquires a manifestly SO(6) invariant form \( \sim (\sum_i \kappa_i)^2 \). One may therefore be tempted to conclude that the SO(6) symmetry is restored in the far infrared and that the low energy spectrum of the Hamiltonian would be approximatively that of the SO(6) GN model. However, as pointed out by Azaria, Lecheminant and Tsvelik [22], one cannot conclude on the basis of perturbation theory alone. In fact looking at the flow in the infrared is not sufficient and can be misleading. The point is that in the A phase, since \( X(t) \to 0 \) as \( t \to -\infty \) all couplings goes to zero in the ultraviolet and the theory is asymptotically free. From the field theoretical point of view this means that a well defined renormalized theory with three different renormalized couplings \( G_{1R}, G_{2R} \) and \( G_{3R} \) exists [23]. Therefore, it is most likely that the SO(6) symmetry is not restored in the A phase. The situation here is very similar to the XXZ model where in the A phase (the Ising region), though there is an apparent SU(2) symmetry restoration in the far infrared, the exact solution tells us that this is not the case [3].

- The B Phase. There one has \( \mu(0) > 0 \) and \( s(0) < 0 \) and the flow in the X variable is reversed (Fig. 1(b)). All couplings goes to zero in the infrared and the interaction is irrelevant. The six Majorana fermions are massless and the model is critical with the central charge \( c = 6 \times 1/2 = 3 \). However, for generic values of the initial conditions \( (d(0) \neq 0) \), the fixed point Hamiltonian does not have the SO(6) symmetry. Indeed, the velocities in both spin and orbital sector have different fixed point values \( u_s^* \neq u_t^* \) so that the symmetry at the fixed point is SO(3) \( \otimes \) SO(3)\_z.
It is only on the symmetric line \((G_1 = G_2)\) that the fixed point symmetry is SO(6).

- The C Phase. There one has \(\mu(0) < 0\) and \(s\) can have both signs. As seen in Fig. 1(c), \(X(t) \to \infty\) when both \(t \to \pm \infty\). The theory is not asymptotically free neither in the infrared nor in the ultraviolet. Though one certainly expects that a gap opens in the spectrum, perturbation theory is pathological \([24]\). Indeed, from the field theoretical point of view, the lack of asymptotic freedom in the theory is pathological \([24]\). Indeed, from the field theoretical point of view, the lack of asymptotic freedom in the ultraviolet implies that a well defined renormalized theory with \(3\) different couplings \(G_{1R}, G_{2R}\) and \(G_{3R}\) does not exist. The low energy physics in such a situation is a highly non trivial and essentially non perturbative problem and one is left to make a sensible conjecture.

As in the A phase discussed above, in the far infrared \(G_1(t) \sim G_2(t) \sim -2G_3(t)\) as well as \(u_1^* = u_2^*\) so one may wonder again whether the SO(6) symmetry is approximatively restored or not. We stress that the situation at hand here is very different to what happens in the A phase since the theory is not asymptotically free in the ultraviolet. This may have important consequences for the low energy physics. Indeed it is well known that the divergency of some couplings at high energy is reminiscent of the fact that some \(part\) of the interaction is \(irrelevant\) at low energy. Of course, perturbation theory alone cannot tell us \(which\) part of the interaction is irrelevant and at present all what we have at hand to make a reasonable hypothesis is our perturbative results. The simplest scenario is that the SO(6) symmetry is approximatively restored provided the initial conditions are not too anisotropic (see however subsection C). As a support of this conjecture let us mention that this is what happens in the C phase of the XXZ model where the Bethe ansatz solution tells us that the SU(2) symmetry is restored up to exponentially small corrections \([23]\). We are of course aware that our hypothesis is highly questionable but it is the simplest one and could in principle be tested either numerically or experimentally. Indeed, the immediate consequence of the possible SO(6) restoration is that the effective low energy effective theory of the Hamiltonian \((40)\) in the C phase is approximatively that of the SO(6) GN model:

\[
\mathcal{H} = -\frac{iu^*}{2} \sum_{a=1}^{6} (\xi_R^a \partial_x \xi_R^a - \xi_L^a \partial_x \xi_L^a) + G \sum_{i<j} \kappa_i \kappa_j
\]

(40)

where the \(effective\) coupling \(G\) is positive. The model \((40)\) is integrable. Its spectrum is known and consists of the fundamental fermion, with mass \(M\), together with a kink of mass \(m = M/\sqrt{2}\) \([23]\). At this point the question that naturally arises is how this enlarged SO(6) symmetry reflects in the spin and orbital correlation functions.

The answer to this important question requires the computation of the exact dynamical correlation functions in the SO(6) restored massive phase. This could be accomplished in principle by the form factors approach as in the frustrated two leg ladder considered in Ref. \((45)\). However this task is even more for the SO(6) case and goes beyond the scope of this paper.

B. Phase diagram

Let us now sum up our results and present the phase diagram associated with Hamiltonian \((33)\). As even at the SU(4) symmetric point the coupling \(G_3\) is not zero, the best way to visualize the phase diagram is to fix the value of \(G_3\) and to look in the plane \((G_1, G_2)\). As one can see from Fig. 2, there are two curves separating the three regions A, B and C which are given by the equation:

\[
\mu = G_1 G_2 + G_1 G_3 + G_2 G_3 = 0.
\]

(41)

In the region B all models are critical with approximate SO(6) symmetry. The spectrum consists into six massless Majorana fermions with different velocities in both spin and orbital sectors. As one crosses the critical line \(\Sigma\) one enters in a fully massive phase (C phase) with an approximate SO(6) symmetry. For large positive values of the couplings \(G_1\) and \(G_2\) one may finally enter the region A. Notice that there is no phase transition between the A phase and the C phase since both phases are massive but rather a smooth cross over.

![FIG. 2. Phase diagram for anisotropic couplings in the plane \((G_1, G_2)\) at a fixed value of \(G_3\). The massless phase B is separated from the massive C phase by the critical line \(\Sigma\). The special point S on the symmetric line which is at the border between the C and the A phases is SO(6) symmetric with \(G_1 = G_2 = -2G_3\). We plot also the two trajectories labelled \(\gamma_1\) and \(\gamma_2\) for which the scaling of the mass gap have two different qualitative behaviors.](image-url)
The most important feature of our phase diagram is that the two regions with \( G_1 G_2 < 0 \) which would have been massless in the absence of \( G_3 \) do not survive an arbitrarily small value of \( G_1 \). Therefore the \( c = 3/2 \) phases discussed in the previous section are not stable. Though the whole region is essentially massive there are still room for criticality since the B phase extends in both quadrants \( G_1 G_2 < 0 \). From Eq. (41), the width of the critical region is of the order \([G_3]\). Therefore the main effect of \( G_3 \) is to drive the system either to a fully massive phase (phase C) or to a fully massless phase (phase B), both with an approximate SO(6) symmetry. Of course, the phase diagram as depicted in the Fig. 2 is strictly valid in the small \( G \) limit. In particular higher order corrections may modify the shape of the critical curve \( \Sigma \). We stress however, that its behavior in the vicinity of the SU(4) point at \( G_1 = G_2 = 0 \) is given by our one loop result. In particular the fact that \( \Sigma \) crosses the symmetric line with a right angle will not change as one includes higher order in perturbation theory. However, as one goes to large deviations from the SU(4) point our effective theory will not apply since for large enough positive (respectively negative) \( G_1 \) and negative (respectively positive) \( G_2 \) the orbital (respectively spin) degrees of freedom order ferromagnetically \([10]\).

C. Effect of the anisotropy

At this point one may wonder whether the anisotropy has no effect at all in the physical quantities. Apart from the velocity anisotropy in the B phase, one may still expect some non trivial effect of the anisotropy in the C phase. Indeed, after all eventhough both \( c = 3/2 \) phases are unstable there should be some significant signature of the presence of the SO(3)\(_1\) fixed point in the scaling of the physical quantities and in the finite size scaling. The very reason for this is that the SO(6) symmetry is restored dynamically with help of a marginal operator.

\textit{Cross-over and finite size scaling}. Let us look at the RG flow in more details. The regions of interest are those with \( G_1 G_2 < 0 \) in which, as discussed in the previous section, if not for \( G_3 \) one of the coupling would have been irrelevant. Then either the spin or the orbital degrees of freedom would have remained massless. In the following we shall concentrate on the orbital degrees of freedom. All the results that will be given can be straightforwardly extended to the spin degrees of freedom.

Consider first initial conditions slightly above the phase B in the lower right quadran with \( G_1 G_2 < 0 \) (see Fig. 2). We are then in the phase C with \( G_1(0) > 0 \) and \( G_2(0) < 0 \). As seen in the Fig. 3(a), the flow can be splitted into two "time" slices. At first \( |G_3| \) strongly decreases and remains almost constant. Then, both spin and orbital degrees of freedom weakly interact since \( |G_{1,2}| >> |G_3| \). In the mean time, the coupling \( G_2 \) increases and changes its sign at a time \( t_{02} \) where it vanishes. At "times" \( t < t_{02} \) the orbital degrees of freedom do not know yet they shall enter in a massive phase and the system is on the influence of the SO(3)\(_1\) fixed point. Finally, as \( t \rightarrow t_M \) all the couplings blow up and one enters in the strong coupling region. The physical interpretation of this phenomenon is clear: it is the spin degrees of freedom that drive the orbital degrees of freedom away from criticality and it takes roughly \( \exp(t_{02}) \) RG iterations for the orbital sector to escape from the basin of attraction of the SO(3)\(_1\) fixed point.

The other interesting region is the portion of the B phase delimited by the critical curve \( \Sigma \) and the \( G_2 \)-axis in upper left quadran of Fig. 2. There \( G_1(0) < 0 \) and \( G_2(0) > 0 \). As seen in Fig. 3(b), it is the spin degrees of freedom that drive the orbital ones to criticality. The coupling \( G_2 \) changes its sign and vanishes at a time \( t_{02} \) and goes to zero in the limit \( t \rightarrow \infty \). In both cases, the cross-over "time" \( t_{02} \) is given by the implicit equation:

\[
X(t_{02}) = \sqrt{\frac{\mu(0)}{d(0) G_3(0)}}.
\]  

(42)

Of course a similar discussion holds for the coupling \( G_1 \) and defines another cross-over time \( t_{01} \) which is defined by the implicit equation:

\[
X(t_{01}) = \sqrt{-\frac{\mu(0)}{d(0) G_3(0)}}.
\]  

(43)

We plot in Fig. 4 the iso-\( t_{02} \) and iso-\( t_{01} \) curves in the plane \((G_1,G_2)\).
We stress that this cross-over phenomenon may have important practical consequences in numerical simulations. Indeed, in a finite system of size $L$, the critical region will seem to extend towards the iso-$t_{01}$ with $t_{01} = \ln L$ in the region $G_1 < 0$, $G_2 > 0$ and towards the iso-$t_{02}$ with $t_{02} = \ln L$ in the region $G_1 > 0$, $G_2 < 0$. The low lying spectrum will look like if either the spin or the orbital degrees of freedom would have been massless. It is important to notice that these two pseudo-critical lines bend upwards in contrast with the critical line $\Sigma$.

**The mass gap.** The anisotropy has also non trivial effect on the mass gap of the system. To see this we have computed the mass gap $M$ with help of the RG equations \[ (28) \] As well known, the gap $M$ is defined as the scale where perturbation theory breaks down. More precisely it is given by the scale $t_M = \ln(1/Ma_0)$ at which the couplings blow up:

$$X(t_M) = \infty.$$ \[ (44) \]

The above implicit equation can be solved but the resulting analytical expression is too cumbersome to be quoted here. Details are given in the Appendix and we shall content ourselves by its asymptotic behaviors as one approaches the critical surface $\Sigma$.

We find two qualitatively different behaviors depending on the way one approaches $\Sigma$. When one approaches $\Sigma$ at the SU(4) point $(G_1 = G_2 = 0)$ we find:

$$M \sim \Lambda \exp \left(-C(\gamma_1)/\Delta^{2/3}\right)$$ \[ (45) \]

where $\Delta = \sqrt{G_1^2 + G_2^2} \to 0$ and the constant $C(\gamma_1)$ depends on the trajectory labelled $\gamma_1$ in the Fig. 2. On the other hand, as one approaches $\Sigma$ at any other point the asymptotic behavior of the gap is different:

$$M \sim \Lambda \exp \left(-C(\gamma_2)/\Delta\right)$$ \[ (46) \]

where now $\Delta$ is the Euclidean distance of the critical surface. As above $C(\gamma_2)$ depends on the curve labelled $\gamma_2$ in Fig. 2. There are two other marginal cases as one approaches $\Sigma$ tangentially: the exponents of $\Delta$ in both (45) and (46) are doubled. We see that the gap is generically **larger** in the regions with small anisotropy. To visualize this phenomenon we present in the Fig. 5 the isogap curves in the plane $(G_1, G_2)$. We observe that at a given distance $\Delta$ from the critical line $\Sigma$ the gap increases as one moves towards the symmetric region $(G_1 \sim G_2)$ and is maximum on the symmetric line $(G_1 = G_2)$. Similarly, to keep the value of the gap to some constant $M$ one has to move away from $\Sigma$ as one leaves the symmetric region.

**Effect of anisotropy on the SO(6) symmetry restoration.** We conclude this section by looking at the effect of the anisotropy on the SO(6) symmetry restoration in the C phase. As discussed above, in the C phase the couplings $G_1$ and $G_2$ tend to $-2G_3$ in the far infrared (i.e. as $t \to t_M$) so that the effective Hamiltonian has an apparent SO(6) symmetry. Looking in more details at the asymptotic behavior of the couplings $G_1(t)$ and $G_2(t)$ we find that:

$$G_{(1,2)}(t) \sim -2G_3(0)X^2(t) \pm \frac{d(0)}{2} X(t).$$ \[ (47) \]
Therefore, as when \( t \rightarrow t_M, X(t) \rightarrow \infty \), there remains a subdominant infrared singularity when \( d(0) \neq 0 \). We thus expect some corrections to the SO(6) behavior. To get an estimate of these corrections we make the reasonable hypothesis that, when the anisotropy is small enough, the effective theory will be given by an SO(6) GN model with a small symmetry-breaking perturbation proportional to:

\[
\mathcal{H} \sim -i\mu^a \sum_{a=1}^6 \left( \xi_R^a \partial_x \xi_R^a - \xi_L^a \partial_x \xi_L^a \right) + G \sum_{i<j} \kappa_i \kappa_j \\
+ \lambda \left( (\kappa_3 + \kappa_4 + \kappa_5)^2 - (\kappa_1 + \kappa_2 + \kappa_6)^2 \right),
\]

where \( \lambda \ll G \). Of course we do not know the value of the effective coupling \( \lambda \), but it is sufficient for our purpose that it is small enough which will be the case if \( d(0) \) is small. From Eq. (48) one can get an estimate of the mass splitting in both the spin and orbital sectors:

\[
\frac{M_s}{M_t} \sim 1 + \frac{\lambda}{\ln(1/\lambda_0)}.
\]

Therefore, we expect the SO(6) symmetry will hold approximately only for small enough gap and anisotropy. The above argument is of course far from being rigorous but it helps to get an idea of the effect of the anisotropy in the symmetry restoration process. This should serve as a warning that, once again, one should be careful with the conclusions drawn from perturbation theory.

IV. EFFECT OF A MAGNETIC FIELD

In contrast with the SU(2) Heisenberg chain there is no unique way to apply a magnetic field in the SU(4) model. Indeed, while in the SU(2) case there is, apart from the total spin, only one conserved charges \( S^z \) in the SU(4) model there are three of them. These are the three Cartan generators of the SU(4) Lie algebra. In this work we have chosen for the three commuting generators:

\[
H^1 = S^z, \quad H^2 = T^z \quad \text{and} \quad H^3 = 2S^zT^z.
\]

Other choices are possible but \( \xi_1 \) is the one that is physically relevant to our problem. We thus see that one is at liberty to apply a magnetic field in any “direction” in the Cartan basis \( \xi_1 \). However, if one relies on the physical interpretation of both operators \( \vec{S} \) and \( \vec{T} \) as spin and orbital operators, a magnetic field only couples to \( H^1 = S^z \). The resulting Hamiltonian thus writes:

\[
\mathcal{H}_h = \sum_i J_1 \vec{S}_i \cdot \vec{S}_{i+1} + J_2 \vec{T}_i \cdot \vec{T}_{i+1} \\
+ K \left( \vec{S}_i \cdot \vec{S}_{i+1} \right) \left( \vec{T}_i \cdot \vec{T}_{i+1} \right) + h \sum_i S_i^z.
\]

The magnetic field breaks the SU(4) symmetry down to SU(2) \( \otimes U(1) \). For a sufficiently large value of \( h > h_0 \), the spin degrees of freedom align parallel to the field and the remaining degrees of freedom will decouple. More interesting is the situation when \( h \) is small. In this case, one may expand \( \xi_1 \) around the SU(4) fixed point, as in previous section, and study its effects in perturbation. For small values of \( h \), the effective low energy Hamiltonian associated with \( \xi_1 \) can be easily derived using the expression of the spin operator \( S^z \) in terms of the Majorana fermions as given by Eqs. (14) and (15). Dropping all oscillating terms, the only contribution comes from the uniform part of the spin density and the effect of the magnetic field is just to add to the Hamiltonian (24) the interaction \( \mathcal{H}_Z = i\hbar (\xi_R^1 \xi_R^2 + \xi_L^1 \xi_L^2) \) and the low-energy Hamiltonian thus writes:

\[
\mathcal{H}_h = - \frac{i\mu_s}{2} \left( \xi_R^1 \partial_z \xi_R^1 - \xi_L^1 \partial_z \xi_L^1 + g_1 (\kappa_1 + \kappa_2 + \kappa_6)^2 \right) \\
- \frac{i\mu_s}{2} \left( \xi_R^2 \partial_z \xi_R^2 - \xi_L^2 \partial_z \xi_L^2 + g_2 (\kappa_3 + \kappa_4 + \kappa_5)^2 \right) \\
+ 2G_1 (\kappa_1 + \kappa_2 + \kappa_6) (\kappa_3 + \kappa_4 + \kappa_5) \\
+ i\hbar (\xi_R^1 \xi_R^2 + \xi_L^1 \xi_L^2)
\]

where we recall that

\[
g_1 = G_1 + G_3 \\
g_2 = G_2 + G_3.
\]

A. Renormalization group analysis

The situation at hand is similar to that of the XXZ model in a field \( \xi_1 \). There exists a magnetic length \( \lambda_h \) below which the magnetic field has no substantial effect on the physics. However as the scale \( \lambda \gg \lambda_h \), the field strongly modifies the low energy behavior. Indeed, in this regime, as we shall see below, the two Majorana fermions \( \xi_1 \) and \( \xi_2 \) completely decouple from the four others. The best way to see this is to bosonize the “1-2” sector of the Hamiltonian (52) with help of Eq. (14). The resulting Hamiltonian becomes:

\[
\mathcal{H}_h = \frac{\nu_0}{2} \left( \frac{1}{\Lambda} (\partial_x \Phi_s)^2 + K (\partial_x \Theta_s)^2 \right) \\
- \frac{i\nu_6}{2} (\xi_R^6 \partial_x \xi_R^6 - \xi_L^6 \partial_x \xi_L^6) \\
- \frac{i\nu_1}{2} (\xi_R^1 \partial_z \xi_R^1 - \xi_L^1 \partial_z \xi_L^1) \\
+ 2f_2 \kappa_6 (\kappa_3 + \kappa_4 + \kappa_5) \\
+ 2f_3 \kappa_6 (\kappa_3 + \kappa_4 + \kappa_5) \\
- 2if_4 \cos \left( \sqrt{4\pi} \Phi_s + 2\theta_s \right) (\kappa_3 + \kappa_4 + \kappa_5) \\
- 2if_5 \cos \left( \sqrt{4\pi} \Phi_s + 2\theta_s \right) \kappa_6
\]

where
\[ K = 1 - \frac{g_1}{\pi u_s}, \]
\[ v_\phi = u_s + \frac{g_1}{\pi}, \]
\[ q = -\frac{K h}{v_\phi}. \] (55)

In the above Hamiltonian, we have introduced new velocities and coupling constants with bare values: \( v_\phi = u_s, \) \( v_t = u_s, f_2 = g_2, \) \( f_3 = f_4 = G_3 \) and \( f_5 = g_1 \) corresponding to the operators that are stable under renormalization. As readily seen, the cosine terms involving \( \Phi_s \) will start to oscillate with wavevector \( q \sim h \). Therefore, at sufficiently large magnetic field the two last terms in (54) can be dropped and the bosonic field \( \Phi_s \) completely decouples. When a gap is present, i.e. when one sits in the B phase at \( h = 0 \), there exists a critical field \( h_c \sim M \) below which the field has little effects on the low energy physics. As \( h \) increases and becomes greater than \( h_c \), the spin sector undergoes a commensurate-incommensurate transition of the NJT type [2;] to an incommensurate massless phase with central charge \( c_s = 1 \). In contrast, in the absence of a gap, i.e. when one sits in the B phase at \( h = 0 \), the spin sector always decouples at sufficiently low energies. Of course, this decoupling procedure has to be understood in the framework of the renormalization group. To this end we have computed the one loop recursion relation associated with the Hamiltonian (54). Neglecting the velocity renormalization we obtain:

\[ \dot{K} = -(3f_2^2 + f_3^2)J_0(2q), \]
\[ \dot{q} = q - (3f_2^2 + f_3^2)J_1(2q), \]
\[ \dot{f}_2 = f_2^2 + f_3^2 + 2f_2^2J_0(2q), \]
\[ \dot{f}_3 = 2f_2f_3 + 2f_4f_5J_0(2q), \]
\[ \dot{f}_4 = (1 - K)f_4 + (2f_2f_4 + f_3f_5)(1 + J_0(2q))/2, \]
\[ \dot{f}_5 = (1 - K)f_5 + 3f_3f_4(1 + J_0(2q))/2 \] (56)

where the couplings have been rescaled as \( f \to f/\pi v^* \), \( v^* \) being a velocity scale, and \( J_{0,1}(x) \) are the Bessel functions of the first and second kinds. Apart from the inherent anisotropy in our problem, the above equations resemble to that of the XXZ model in a field [2]. We see on Eqs. (50) that when \( q(t) \sim \pi \) the two Bessel functions start to oscillate and the renormalization conming from both the \( f_4 \) and \( f_5 \)-terms is stopped. This defines the magnetic length \( \lambda_h \) as \( \dot{q}(t_h) = \pi (t_h = \ln \lambda_h) \) where the spin degrees of freedom decouple from the rest of the interaction (as well known, there is no unique way to define the magnetic length but we have checked that other reasonable choices do not affect qualitatively the physics). At this point, it is worth stressing that in the massive phase, \( t_h \) must be much smaller than \( t_M \), the scale at which perturbation theory breaks down. This means, of course, that \( h > M \) as discussed above.

At times \( t > t_h \), the couplings \( f_4 \) and \( f_5 \) do not participate in the RG equations of the couplings \( K, q, f_2 \) and \( f_3 \) and Eqs.(56) reduce to:

\[ \dot{K} = 0 \]
\[ \dot{q} = q \] (57)

and

\[ \dot{f}_2 = f_2^2 + f_3^2 \]
\[ \dot{f}_3 = 2f_2f_3 \] (58)

This means that in the regime \( t > t_h \) the Hamiltonian decouples:

\[ \mathcal{H}_h = \mathcal{H}_s + \mathcal{H}_z \] (59)

where

\[ \mathcal{H}_s = \frac{v_\phi}{2} \left( \frac{1}{K} \left( \partial_x \Phi_s \right)^2 + K \left( \partial_x \Theta_s \right)^2 \right) \] (60)

and

\[ \mathcal{H}_z = -\frac{iv_0}{2} \left( \xi_R^* \partial_x \xi_R - \xi_L^* \partial_x \xi_L \right) \]
\[ -\frac{iv_t}{2} \left( \xi_R^* \partial_x \xi_R - \xi_L^* \partial_x \xi_L \right) + 2f_2 \left( \kappa_3 \kappa_4 + \kappa_3 \kappa_5 + \kappa_4 \kappa_5 \right) + 2f_3 \kappa_6 \left( \kappa_3 + \kappa_4 + \kappa_5 \right) \] (61)

where \( K, v_s, v_t, f_2 \) and \( f_3 \) are the effective couplings at the magnetic length \( \lambda_h \).

Let us first concentrate on the spin sector as given by [4]. The Hamiltonian (54) is that of a Luttinger Liquid with stiffness \( K \). The spin sector is thus massless and contribute to a central charge \( c_s = 1 \). In addition, the correlation functions involving the field \( \Phi_s \) will be incommensurable with \( h \)-dependent wavevector \( q \). From Eqs. (21) one sees that the incommensurability will reflect in both the spin and orbital correlation functions.

We focus now on the remaining part of the interaction. The Hamiltonian \( \mathcal{H}_z \) describes the interaction between the orbital sector with the remaining spin-orbital \( Z_2 \) degree of freedom (\( z^a \)) of the spin sector. The low energy physics in this sector is non trivial and at issue is the behavior of the RG flow associated with Eq. (58) where the initial conditions have to be taken at the magnetic length with \( f_2(t_h) \) and \( f_3(t_h) \) at \( t_h = \ln(\lambda_h) \).

These equations are trivially solved. Indeed, upon introducing the new variables:

\[ f_\pm = f_2 \pm f_3 \] (62)

Eqs.(58) decouple:

\[ \dot{f}_\pm = f_\pm^2 \] (63)

As in the previous section, we distinguish between three phases A, B and C depending on the initial conditions of the flow:

- **The A Phase.** This is when \( f_+(t_h) > 0 \) and \( f_-(t_h) > 0 \). Both couplings are relevant and a gap opens in the
proximatively restored. In the far infrared the effective
ion, one may conjecture that the SO(4) symmetry is ap-
to zero in the infrared. Therefore, as in the previous sec-
become massless leaving the theory with the central
ues of the effective couplings at the magnetic length are
incommensurate transition in the spin sector depends
in zero magnetic field the time evolution of
Fig. 3(b)). The presence of a field
orbital sector and could change its sign at a time
$G_1 < 0$ is thus unstable. The physical in-
interpretation of this result is clear. In zero field, it was the
spins degrees of freedom that drove the orbital degrees of
freedom to criticality. When the field is large enough, its
effect is to decouple a part of the spin degrees of freedom
before the remaining fluctuations had a chance to enter
the bassin of attraction of the SO(4) fixed point. Thus,
the effect of the magnetic field in this region is to reduce
the extension of the phase B.

- The other interesting region, is when one sits in the
C phase just above the critical surface $\Sigma$ in the lower
right quadrant of Fig. 2, i.e. when $G_1 > 0$ and $G_2 < 0$.
There $G_2$ is negative but is driven to positive values by
the spin degrees of freedom. It changes sign at a time
$t_{02}(h)$ where it vanishes. Now if $t_h < t_{02}(h)$, $f_-(t_h)$
will be still negative while if $t_h > t_{02}(h)$, $f_-(t_h)$ will be
positive. Therefore there exists a critical field $h_{c0}$
below which the orbital and spin-orbital sectors will be
still massive. Above $h_{c0}$, the gap will close. Again, the
physical reason why the gap vanishes above $h_{c0}$ is that,
the spin degrees of freedom did not have enough time to
drive the orbital sector to strong coupling. We therefore
conclude that the B phase has a tendency to extend in the
region $G_1 > 0$, $G_2 < 0$ when a field is present.

To summarize, we expect two kinds of transition as one
varies the magnetic field. When the theory is massive at
$h = 0$, as one increases $h$ there will be a first transition
in the spin sector to an incommensurate phase with the
central charge $c_s = 1$. The remaining degrees of freedom
will be still massive but are described by the SO(4) GN
model. The coherent fermionic excitations of the SO(6)
GN model disappear from the spectrum and the only mas-
sive excitations that remain are the kinks of the SO(4)
GN model. Consequently all excitations will be incoher-
ent. What happens as one increases $h$ further strongly
depends on the anisotropy. If $G_2 > 0$ the magnetic field
just renormalizes the mass of the SO(4) kinks, the spec-
trum is still massive. The total central charge of the
model is thus $c = c_s + c_L = 1$. However, when $G_2 < 0$,
there is a second phase transition at a field $h_{c0}$ of the
Kosterlitz-Thouless (KT) type \[28\] to a commensurate
massless phase with $c_s = 2$. Both spin and orbital de-
grees of freedom are massless and the total central charge
is $c = c_s + c_L = 3$.

Notice that there will be three different velocities: $v_0^* \neq v_0^* \neq v_0^*$ so that the symmetry at the fixed point is not
SO(6) but rather SO(3) $\otimes$ U(1) $\otimes$ Z_2.

When there is no gap at zero field the spin sector is al-
ways critical with incommensurate correlation functions.

**B. Phase diagram**

It is clear from the discussion given above that the val-
ues of the effective couplings at the magnetic length are
crucial. Of course, the existence of the commensurate-
incommensurate transition in the spin sector depends
only on the mass gap $M$ of the zero field Hamiltonian.
What is more interesting, is what happens in the remain-
ning orbital and spin-orbital sectors described by the Ma-
jorana fermions $\xi^3, \xi^4, \xi^5, \xi^6$. As we shall see now the
anisotropy will play its tricks. Indeed, what is into ques-
tion is the sign of the coupling $f_-(t_h)$ at the magnetic
length. Returning to the original variables, one finds
that $f_0 = G_2$. We saw in the preceding section that in
zero magnetic field the time evolution of $G_2$ was very
sensitive to the presence of the SO(3)$_1$ fixed point in the
orbital sector and could change its sign at a time $t_{02}$ de-
pending on the the initial conditions (see Fig. 3(a) and
Fig. 3(b)). The presence of a field $h$ does not affect qual-
itatively this feature and provides for a renormalization of
$t_{02}$ which becomes $h$-dependent: $t_{02} \rightarrow t_{02}(h)$. We
consider now two cases.

- First is when one sits in the B phase between the $G_1$
axis and the critical surface $\Sigma$, with $G_1 < 0$ (see Fig. 
2). At zero magnetic field the system is critical with

the central charge $c = 3$. There $f_-$ is positive and de-
creases as $t$ increases vanishes at some RG time $t_{02}(h)$
and then changes sign. Now if $t_h < t_{02}(h)$, $f_-(t_h)$ will
be still positive and a gap will open in the orbital and
spin-orbital sector according to (64). On the other hand,
if $t_h > t_{02}(h)$ then $f_-(t_h) < 0$ and there is no gap. This
means that there exists a critical value of the field $h_{c0}$
above which a gap opens. The portion of critical surface
$\Sigma$ in the region $G_1 < 0$ is thus unstable. The physical in-
interpretation of this result is clear. In zero field, it was the
spins degrees of freedom that drove the orbital degrees of
freedom to criticality. When the field is large enough, its
effect is to decouple a part of the spin degrees of freedom
before the remaining fluctuations had a chance to enter
the bassin of attraction of the SO(4) fixed point. Thus,
the effect of the magnetic field in this region is to reduce
the extension of the phase B.

- The other interesting region, is when one sits in the
C phase just above the critical surface $\Sigma$ in the lower
right quadrant of Fig. 2, i.e. when $G_1 > 0$ and $G_2 < 0$.
There $G_2$ is negative but is driven to positive values by
the spin degrees of freedom. It changes sign at a time
$t_{02}(h)$ where it vanishes. Now if $t_h < t_{02}(h)$, $f_-(t_h)$
will be still negative while if $t_h > t_{02}(h)$, $f_-(t_h)$ will be
positive. Therefore there exists a critical field $h_{c0}$
below which the orbital and spin-orbital sectors will be
still massive. Above $h_{c0}$, the gap will close. Again, the
physical reason why the gap vanishes above $h_{c0}$ is that,
the spin degrees of freedom did not have enough time to
drive the orbital sector to strong coupling. We therefore
conclude that the B phase has a tendency to extend in the
region $G_1 > 0$, $G_2 < 0$ when a field is present.

To summarize, we expect two kinds of transition as one
varies the magnetic field. When the theory is massive at
$h = 0$, as one increases $h$ there will be a first transition
in the spin sector to an incommensurate phase with the
central charge $c_s = 1$. The remaining degrees of freedom
will be still massive but are described by the SO(4) GN
model. The coherent fermionic excitations of the SO(6)
GN model disappear from the spectrum and the only mas-
sive excitations that remain are the kinks of the SO(4)
GN model. Consequently all excitations will be incoher-
ent. What happens as one increases $h$ further strongly
depends on the anisotropy. If $G_2 > 0$ the magnetic field
just renormalizes the mass of the SO(4) kinks, the spec-
trum is still massive. The total central charge of the
model is thus $c = c_s + c_L = 1$. However, when $G_2 < 0$,
there is a second phase transition at a field $h_{c0}$ of the
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massless phase with $c_s = 2$. Both spin and orbital de-
grees of freedom are massless and the total central charge
is $c = c_s + c_L = 3$.

Notice that there will be three different velocities: $v_0^* \neq v_0^* \neq v_0^*$ so that the symmetry at the fixed point is not
SO(6) but rather SO(3) $\otimes$ U(1) $\otimes$ Z_2.

When there is no gap at zero field the spin sector is al-
ways critical with incommensurate correlation functions.
What happens for the spin-orbital and orbital degrees of freedom depends again strongly on the anisotropy. When $G_2 < 0$ they remain massless and the total central charge is thus $c = 3$. However, if $G_2 > 0$, there will be a KT type phase transition at a critical field $h_{c0}$ to a massive phase with approximate SO(4) symmetry.

We stress that the mechanism that leads to the KT type phase transition at the magnetic field $h_{c0}$ is highly non trivial since the magnetic field does not couple directly to both the orbital and the spin-orbital degrees of freedom.

V. CONCLUSIONS

In the present work we have studied the effect of symmetry breaking perturbations in the one dimensional SU(4) spin-orbital model. Using the low energy effective field theory developed in Ref. \[14\], we investigated the phase diagram of the SU(2)$\otimes$SU(2) model where the exchange in both the spin ($J_1$) and the orbital ($J_2$) sectors are different. We found that the different phases of the symmetric $J_1 = J_2$ line extend to the case $J_1 \neq J_2$. In particular the massless phase, governed by the SO(6)$_1$ fixed point, extends to a finite region in the plane ($J_1$, $J_2$) around the SU(4) point ($J_1 = K/4$, $J_2 = K/4$). Similarly, in the vicinity of the critical surface, the massive phase also has an approximate SO(6) symmetry provided the anisotropy is not too large. In this phase, as in the isotropic case, the system spontaneously breaks translational invariance and dimerizes with alternate spin and orbital singlets $[14]$. Both spin and orbital excitation are coherent at wavevector $k = \pi/2$. All these results remain valid in the vicinity of the SU(4) point. The question that naturally arises is what happens when $K$ decreases. Indeed, in the limit $K \ll J_{(1,2)}$ one enters in the weak coupling limit where magnon excitations are incoherent at wavevector $k = \pi |L|$. In the simplest scenario, as discussed in Ref. $[14]$, one expects that the coherent peak at $k = \pi/2$ in the dynamical susceptibility will disappear at a critical value of $K = K_D$. Such a special point where an oscillating component of the correlation function disappears is a disorder point $[24]$ and therefore, we do not expect a phase transition at $K_D$ but rather a smooth cross over.

Though these results could have been anticipated on the basis of the previous study of the symmetric case $[14]$, since the interactions are marginal, the anisotropy reveals itself in a non trivial scaling of the physical quantities. Indeed, we have shown that the anisotropy plays its tricks in two particular regions of the phase diagram with $G_1G_2 < 0$, where $G_{(1,2)} = J_{(1,2)} - K/4$ measures the departure from the SU(4) point. In these regions, both spin and orbital degrees of freedom compete. For instance, when $G_1 > 0$ and $G_2 < 0$, the spin sector tends to open a gap while the orbital one wants to remain massless. Since both sectors interact marginally, at issue is a delicate balance between the strength of the interactions: it is one kind of degrees of freedom that drives the other to its favorite behavior. In particular, this is the reason why the massless phase extends in the region with either $G_1 > 0$ or $G_2 > 0$. In any cases, the whole system ultimately becomes either fully massless or fully massive. The crucial point is that since the interactions are only marginal, it may take a very long time, in the renormalization group sense, to reach the asymptotic low energy regime. This has important consequences.

First of all is the non trivial behavior of the mass gap of the system. We found that the gap $M$ is generically smaller in the regions with large anisotropy, i.e. in the two quadrants $G_1G_2 < 0$ above the critical curve $\Sigma$. This is due precisely to the strong tendency to massless behavior in these regions. As a consequence the gap $M$ has two qualitatively different scaling behaviors as one approaches $\Sigma$ either from the symmetric region or the asymmetric one (the trajectories labelled $\gamma_1$ and $\gamma_2$ in Fig. 2): $M \sim \exp(-c_1/\Delta^{2/3})$ and $M \sim \exp(-c_2/\Delta)$.

Second is the finite size scaling. Since the gap opens exponentially it is very difficult to localize accurately the critical line $\Sigma$ in a finite system. In the current model the situation is even more awkward in the regions $G_1G_2 < 0$. Indeed, in a finite system of size $L$, the critical region will seem to extend and the pseudo-critical lines will be given by the two iso-$t_{01}$ and iso-$t_{02}$ curves, with $t_{0(1,2)} = \ln L$, that have the opposite curvature than the true critical line $\Sigma$ (see Fig. 4). In this pseudo-critical region, either the spin or the orbital degrees of freedom will look massless. The phase diagram in zero magnetic field as obtained by very recent DMRG calculations $[30]$ is in qualitative agreement with our RG analysis. However, the critical line obtained in these numerical computations has the opposite curvature that the one loop result $\Sigma$. Our interpretation of this fact is that what has been observed are the two iso-$t_{0(1,2)}$ curves. This reflects once again the non trivial finite size scaling induced by the anisotropy.

Finally, is the effect of a magnetic field. The magnetic field affects the spin degrees of freedom in the usual way. In the massless phase it leads to incommensuration in the spin sector while when a gap is present, a commensurate-incommensurate transition can occur at a critical field. However, what happens to the remaining degrees of freedom strongly depends on the anisotropy. In the region, where both degrees of freedom do not compete, i.e. when $G_1G_2 > 0$, the remaining orbital and spin-orbital sector remains either massless or massive with an approximate SO(4) symmetry. On the other hand, the most striking effect occurs when the spin and orbital fluctuations compete i.e. for $G_1G_2 < 0$. In this region the field reinforces the effect of the orbital degrees of freedom and can induce a second phase transition, of the KT type, for a sufficiently large field, from massive to massless approximate SO(4) behavior. The origin of this non trivial effect of the magnetic field stems from the interplay of the presence of orbital degeneracy and anisotropy. We hope that
this transition will be observed in further experiments on quasi one dimensional spin gapped materials with orbital degeneracy.

Note added. After this work was completed, we became aware of a work by Itoi et al.\textsuperscript{[30]} who also predict the extension of the massless phase in the anisotropic region.

VI. APPENDIX

In this Appendix, we shall compute the mass gap $M$ in the phase C, and obtain the asymptotics of $M$ in the vicinity of the critical surface $\Sigma$. As well known, within perturbation theory the gap defines the scale $t_M = \ln(1/Ma_0)$ where all the couplings blow up. Clearly, $t_M$ is given by the equation:

$$X(t_M) = \infty.$$  \hfill (65)

Integrating \textsuperscript{[37]} and recalling the dynamic of $X$ in the phase C we find:

$$t_M = \left( \int_{X^*}^{\infty} + \epsilon \int_{1}^{X^*} \right) \frac{dX}{X \sqrt{P(X)}} \hfill (66)$$

where $\epsilon = \text{sign}(s)$; $P(X)$ is given by Eq.(38) and has only two reals roots 0 and $X^* \leq 1$. In the following $s, \mu$ and $G_3$ have to be understood as initial conditions.

Performing the integrals we obtain:

$$t_M = \sqrt{\frac{p}{|\mu|}} [E(\alpha(0), k) - \epsilon E(\alpha(1), k)]$$

$$+ \frac{1}{2} \sqrt{\frac{p}{|\mu|}} \left( \frac{u^*_p}{p} - 1 \right) [F(\alpha(0), k) - \epsilon F(\alpha(1), k)]$$

$$- \frac{G_3}{\mu} \left[ \frac{1}{p + u^*} - \frac{s}{2G_3(p + u^* - 1)} \right] \hfill (67)$$

where $F$ and $E$ the elliptic functions of the first and the second kind respectively, with parameters:

$$\alpha(u) = 2 \arctan \sqrt{\frac{u^* - u}{p}}$$

$$k = \sqrt{\frac{p + u^* - G_3^2/2\mu u^2}{2p}} \hfill (68)$$

$u^*$ and $p^2$ being given by:

$$u^* = 1/X^*$$

$$p^2 = u^*_p - \frac{2G_3^2}{\mu u^*}. \hfill (69)$$

Developing $t_M$ around a point $(G_{1c}, G_{2c}, G_{3c})$ belonging to the critical surface $\Sigma$ between the phases B and C (see Fig. 2), we obtain the asymptotics

$$M \sim \Lambda \exp \left( -C(\gamma_1)/\Delta^{2/3} \right) \quad \text{if} \quad G_{1c}^c = G_{2c}^c = 0$$

$$M \sim \Lambda \exp \left( -C(\gamma_2)/\Delta \right) \quad \text{if} \quad G_{1c}^c \neq G_{2c}^c \hfill (70)$$

where $\Delta$ is the Euclidian distance from $\Sigma$. The two constants $C(\gamma_1)$ and $C(\gamma_2)$ are given by:

$$C(\gamma_1) = 0.6845 \quad G_{3c}^{-1/3} (\cos \theta)^{-2/3}$$

$$C(\gamma_2) = \left( (1 + 2G_{3c}^2/(G_{1c} - G_{2c})^2 \cos^2 \theta \right)^{-1/2} \hfill (71)$$

where $\theta$ is the angle to the normal of $\Sigma$.  


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[28] Notice that the transition does not have the usual SO(2) symmetry for the standard KT transition but rather an SO(4) symmetry. This generalization of the KT transi-

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