Integrability of $N=3$ super Yang-Mills equations *

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Abstract
We describe the harmonic superspace formulation of the Witten-Manin supertwistor correspondence for $N=3$ extended super Yang-Mills theories. The essence is that on being sufficiently supersymmetrised (up to the $N=3$ extension), the Yang-Mills equations of motion can be recast in the form of Cauchy-Riemann-like holomorphicity conditions for a pair of prepotentials in the appropriate harmonic superspace. This formulation makes the explicit construction of solutions a rather more tractable proposition than previous attempts.

1. Introduction

Alik Berezin was enthusiastic about the possibility of solving the Yang-Mills equations and he frequently discussed this intriguing problem. Recalling these discussions, we feel that he would have enjoyed knowing about our recent work on this theme, which we shall describe here as our contribution to this volume dedicated to his memory. Today the above possibility certainly shows much promise and many existence proofs exist (e.g. [1]). An older promise based on the twistor transform [2,3,4] has also been renewed recently [5]; and this will be the subject of the present paper. The latter approach was based on the observation that one could approach non-self-dual Yang-Mills fields by combining self-dual (SD) and anti-self-dual (ASD) fields in some way. Specifically, the twistor transform for

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self-dual Yang-Mills [6] establishes a correspondence between self-dual fields and certain holomorphic fields, which effectively linearises the (nonlinear) self-duality equations, and the observation of [2,3] concerned the intermingling of SD and ASD holomorphic data to extend to non-self-dual data. In general, the procedure works in a certain formal neighbourhood of the self-dual solution, but Witten [2] observed that for the N=3 supersymmetric equations the SD and ASD data actually interlock to give an exact non-self-dual solution. This observation was given a global formulation by Manin [7], who also discussed some solutions for rather complicated gauge groups.

Restricting ourselves entirely to local considerations, we shall show that this supertwistorial construction enjoys a very elegant formulation in the language of ‘harmonic superspace’, which promises to be very effective for the explicit construction of local solutions. The crux of the formulation is that the \( N = 3 \) extended supersymmetric Yang-Mills equations can be rewritten as Cauchy-Riemann-like conditions for a pair of prepotentials in an appropriate harmonic superspace. ‘Holomorphic’ prepotentials therefore encode local N=3 super Yang-Mills solutions; leaving only the decodification as the remaining technical problem. The formulation involves a crucial modification of the harmonic (super)space formulation of (anti-)self-duality equations (see, e.g. [8, 9] and previous references therein). The latter involve the harmonisation of “half” the Lorentz group and only allow one to deal with base spaces of signatures (4,0), (2,2) or with complexified space. For non-self-dual \( N = 3 \), however, we need to harmonise the whole Lorentz group, allowing us to consider a base space of any signature, including (3,1), the Minkowski one.

We should mention that harmonics were originally introduced [10] in order to construct the first unconstrained off-shell \( N = 2 \) and \( N = 3 \) supersymmetric gauge theories. In that case the internal \( SU(2) \) and \( SU(3) \) groups were harmonised instead of the Lorentz group which is harmonised here. Harmonisation of the three-dimensional Lorentz group was discussed for \( N = 6 \ d = 3 \) gauge theories by Zupnik [11].

2. \( N = 3 \) super Yang-Mills equations

The thrice-extended Yang-Mills multiplet contains the following fields [12]: the gauge vector field represented by its self- and anti-dual field-strengths \( f_{\dot{\alpha}\dot{\beta}} \) and \( f_{\alpha\beta} \), defined by

\[
[\nabla_{\alpha\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] \equiv \epsilon_{\alpha\beta} f_{\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}} f_{\alpha\beta},
\]

spinor singlet and triplet fields \( \{\lambda_\alpha, \dot{\lambda}_{\dot{\alpha}}, \chi_\dot{i}, \chi_{i\alpha}\} \) and two triplets of scalar fields \( W^i, W_i \) where \( \alpha \) and \( \dot{\alpha} \) are undotted and dotted Lorentz spinor indices while \( i = 1, 2, 3 \) is the \( SU(3) \) index. All fields are in the adjoint representation of gauge group and are Lie-algebra-valued.
The dynamical equations for this supermultiplet are [13]

\[
\begin{align*}
\nabla_{\beta}^{\alpha} f_{\beta\alpha} + \nabla_{\beta}^{\dot{\alpha}} f_{\dot{\beta}\alpha} &= \{\chi_{\beta}, \chi_{k\dot{\beta}}\} + \{\lambda_{\beta}, \lambda_{\dot{\beta}}\} + [W^{i}, \nabla_{\beta} W_{i}] + [W_{i}, \nabla_{\beta} W^{i}] \\
\nabla_{\dot{\alpha}}^{\beta} \lambda_{\dot{\beta}} &= [\chi_{i\dot{\beta}}, W_{k}] \\
\nabla_{\dot{\alpha}}^{\alpha} \lambda_{j\dot{\beta}} &= [\chi_{i\dot{\beta}}, W_{k}] \epsilon_{ijk} - [\lambda_{\dot{\beta}}, W^{j}] \\
\nabla_{\dot{\alpha}}^{\alpha} \chi_{j\dot{\alpha}} &= [\chi_{i\dot{\beta}}, W_{k}] \epsilon_{ijk} - \frac{1}{2} [\lambda^{i}_{\dot{\alpha}}, W^{k}] \\
\nabla_{\dot{\alpha}}^{\alpha} \nabla_{\dot{\alpha}} W_{j} &= -2[[W^{i}, W_{j}], W_{i}] + [[W^{i}, W_{j}], W_{j}] - \{\chi_{j\dot{\alpha}}, \lambda^{i}_{\dot{\beta}}\} + \frac{1}{2} \epsilon_{ijk} \{\chi_{j\dot{\alpha}}, \chi_{k\dot{\beta}}\} \\
\nabla_{\dot{\alpha}}^{\alpha} \nabla_{\dot{\alpha}} W^{j} &= -2[[W_{i}, W^{j}], W_{i}] + [[W_{i}, W^{j}], W_{j}] - \{\chi_{j\dot{\alpha}}, \lambda^{i}_{\dot{\beta}}\} + \frac{1}{2} \epsilon_{ijk} \{\chi_{i\dot{\alpha}}, \chi_{k\dot{\beta}}\}.
\end{align*}
\]

To describe this theory invariantly in the customary superspace with coordinates

\[
\{x^{\dot{\alpha}}, \theta^{i\alpha}, \bar{\theta}^{i\dot{\alpha}}\},
\]

one introduces gauge-covariant derivatives \(D_{A} \equiv \partial_{A} A_{A} = (\nabla_{\dot{\alpha}}^{\alpha}, D_{i\alpha}, \bar{D}_{j\dot{\alpha}}), i, j = 1, 2, 3\). The super-connections \(A_{A}\) contain the above supermultiplet consistently only if the gauge covariant derivatives are constrained as follows [14, 15]:

\[
\begin{align*}
\{D_{(i\alpha}, D_{j)\dot{\beta}}\} &= 0 \\
\{\bar{D}^{(i}_{\dot{\alpha}, \bar{D}^{j)}_{\dot{\beta}}\} &= 0 \\
\{D_{i\alpha}, \bar{D}^{j}_{\dot{\beta}}\} &= 2 \delta^{j}_{i} \nabla_{\dot{\alpha}}^{\dot{\beta}},
\end{align*}
\]

The crucial message [13] is that these constraints turn out to be equivalent to the equations of motion (1). Moreover, they take the form of a Cauchy-Riemann system in an appropriately enlarged base space, as we now describe.

### 3. Harmonic superspace

Let us begin with Euclidean superspace. The coordinates (2) parametrise the coset of the super Poincare group by its Lorentz subgroup. In this case the Lorentz group is SO(4) = SU(2) × SU(2), with independent SU(2) groups. Factoring, instead, by a subgroup of the Lorentz group, yields a correspondingly larger space, a construction which turns out to be very useful. In particular, factoring by the \(U(1) \times U(1)\) subgroup of the Lorentz group, yields an enlargement of superspace by a direct product of two 2-spheres, \(S^{2} = \frac{SU(2)}{U(1)}\). As coordinates for these spheres we shall use harmonics \(u^{+\dot{\alpha}}, u^{-\dot{\alpha}}\) and \(v^{\oplus\alpha}, v^{\ominus\alpha}\) [8], defined up to the respective \(U(1)\) phases, with \(\oplus, \ominus\) and \(\ominus, \oplus\) being the respective \(U(1)\) charges, and obeying the constraints:

\[
u^{+\dot{\alpha}} v^{-\dot{\alpha}} = 1, \quad v^{\oplus\alpha} v^{\ominus\alpha} = 1\]
(or, equivalently, satisfying the completeness relations
\[ u^+{}^\alpha u^-{}^\beta - u^-{}^\alpha u^+{}^\beta = \delta^\alpha{}^\beta, \quad v^\oplus{}^\alpha v^\ominus{}^\beta - v^\ominus{}^\alpha v^\oplus{}^\beta = \delta^\alpha{}^\beta \].

This enlarged space, with spinor harmonics as additional coordinates, is our harmonic superspace.

For the signature (3,1) Minkowski space, the Lorentz group is the simple group \( SL(2, C) \). In this case the analogous construction involves an enlargement by a coset \( \frac{SL(2, C)}{SL(1, C)} \), so that the charge of the harmonics becomes complex: the \( u \) and \( v \) harmonics; and the \((+, -)\) and \((\oplus, \ominus)\) charges becoming complex conjugates of each other. On the other hand, for a signature (2,2) space, the Lorentz group is \( SL(2, R) \times SL(2, R) \), a direct product of two real groups. In these noncompact cases there appear richer structures and peculiarities. We shall not go into subtleties in the present paper. Suffice to say that in these cases we understand the construction in the sense of a Wick rotated version of the Euclidean one.

The upshot is that we have additional coordinates and are able to pass to a basis of harmonic superspace with coordinates
\[ \{x^\pm{}^\ominus, x^\pm{}^\oplus, \vartheta^i{}^\oplus, \vartheta^i{}^\ominus, \bar{\vartheta}^\pm{}^i, u^\pm{}^\alpha, v^\oplus{}^\alpha, v^\ominus{}^\alpha\} \]

where the \( x \)'s and \( \vartheta \)'s are related to the usual superspace coordinates (2) by
\[ x^\pm{}^\ominus = x^{\alpha\dot{\alpha}} u^\pm{}^\alpha v^\ominus{}^\alpha, \quad x^\pm{}^\oplus = x^{\alpha\dot{\alpha}} u^\pm{}^\alpha v^\oplus{}^\alpha, \]
\[ \vartheta^i{}^\oplus = \vartheta^{i\alpha} v^\oplus{}^\alpha, \quad \vartheta^i{}^\ominus = \vartheta^{i\alpha} v^\ominus{}^\alpha, \quad \bar{\vartheta}^\pm{}^i = \bar{\vartheta}^{i\alpha} u^\pm{}^\alpha. \]

In virtue of these relations we may recover customary superspace fields as coefficients in the double harmonic expansions (in both \( u \) and \( v \)) of harmonic superspace fields.

4. The harmonic superconnections

Now we come to the crucial point. In the coordinates (4) the constraints (3) are radically simplified: They turn out to be equivalent to the following set of commutation relations
\[ \{\bar{D}^+{}^i, D^+{}^j\} = 0 = \{D^\oplus{}^i, D^\oplus{}^j\} \]
\[ \{\bar{D}^+{}^i, D^\ominus{}^i\} = 2\nabla^\ominus{}^\oplus, \]

where \( \bar{D}^+{}^i, D^\ominus{}^i \) are gauge-covariant spinorial derivatives and \( \nabla^\ominus{}^\oplus = \frac{\partial}{\partial x^\ominus{}^\oplus} + A^+{}^\ominus \), together with the conditions
\[ [D^{++}, \bar{D}^+{}^j] = 0 = [D^{++}, D^\ominus{}^i] \]
\[ [D^{++}, \nabla^\ominus{}^\oplus] = 0 = [D^\ominus{}^\oplus, \nabla^\ominus{}^\oplus] \]
\[ [D^\ominus{}^\oplus, \bar{D}^+{}^j] = 0 = [D^\ominus{}^\oplus, D^\ominus{}^i] \]

and the important consistency relation
\[ [D^{++}, D^\ominus{}^\oplus] = 0, \]
where $D^{\oplus \oplus}, D^{++}$ are harmonic space derivatives which act on the respective negatively-charged harmonic space coordinates to yield their positively-charged counterparts, i.e.

\[ D^{++} x^{-\Theta} = x^{+\Theta}, \quad D^{++} x^{-\Theta} = x^{+\Theta}, \quad D^{++} u_{\dot{\alpha}}^{-} = u_{\dot{\alpha}}^{+}, \quad D^{++} \bar{\vartheta}^{-} = \bar{\vartheta}^{+} \]

and

\[ D^{\ominus \ominus} x^{\pm \Theta} = x^{\pm \Theta}, \quad D^{\ominus \ominus} v_{\dot{\alpha}}^{\ominus} = v_{\dot{\alpha}}^{\ominus}, \quad D^{\ominus \ominus} \vartheta^{\ominus} = \vartheta^{\ominus}, \]

while giving zero when applied to the respective positively charged coordinates:

\[ D^{++} x^{+\Theta} = 0, \quad D^{++} x^{+\Theta} = 0, \quad D^{++} u_{\dot{\alpha}}^{+} = 0, \quad D^{++} \bar{\vartheta}^{+} = 0 \]

and

\[ D^{\ominus \ominus} x^{\pm \Theta} = 0, \quad D^{\ominus \ominus} v_{\dot{\alpha}}^{\ominus} = 0, \quad D^{\ominus \ominus} \vartheta^{\ominus} = 0. \]

Correspondingly, the action on derivatives is given by

\[
\begin{align*}
[D^{++}, \partial^{-\Theta}] &= \partial^{+\Theta}, & [D^{++}, \partial^{-\Theta}] &= \partial^{+\Theta}, & [D^{++}, \partial^{-i}] &= \partial^{+i}, \\
[D^{\ominus \ominus}, \partial^{\pm \Theta}] &= \partial^{\pm \Theta}, & [D^{\ominus \ominus}, \partial^{-i}] &= \partial^{+i}, \\
[D^{++}, \partial^{+\Theta}] &= [D^{++}, \partial^{+\Theta}] = [D^{++}, \partial^{+i}] = 0, \\
[D^{\ominus \ominus}, \partial^{\pm \Theta}] &= [D^{\ominus \ominus}, \partial^{-i}] = 0.
\end{align*}
\]

In virtue of these properties of $D^{++}, D^{\ominus \ominus}$ the conditions (6) ensure that in this basis, the covariant derivatives \{\(\bar{D}^{+i}, D^{\oplus i}, \nabla^{+\ominus}\)\} are homogeneous of degree one in the correspondingly charged harmonics. The relations (5) mean that these covariant derivatives take the pure-gauge forms

\[
\begin{align*}
\bar{D}^{+i} &= D^{+i} - D^{+i} \varphi^{-1} \\
D^{\oplus i} &= D^{\oplus i} - D^{\oplus i} \varphi^{-1} \\
\nabla^{+\ominus} &= \partial^{+\ominus} - \partial^{+\ominus} \varphi^{-1},
\end{align*}
\]

in other words, equations (5) are integrability conditions of the system

\[
\begin{align*}
\bar{D}^{+i} \varphi &= 0 \\
D^{\oplus i} \varphi &= 0 \\
\nabla^{+\ominus} \varphi &= 0.
\end{align*}
\]

The matrix function $\varphi$ takes values in the gauge group and is defined up to the gauge freedom

\[
\varphi \mapsto e^{-\tau} \varphi e^{\lambda}, \quad D^{++} \tau = 0 = D^{\ominus \ominus} \tau, \quad D^{+i} \lambda = 0 = D^{\oplus i} \lambda = \partial^{+\ominus} \lambda,
\]

where $\tau$ and $\lambda$ are matrix functions in the gauge algebra.

In contrast to the covariant derivatives (8), the harmonic derivatives $D^{++}, D^{\ominus \ominus}$ are ‘short’ (i.e. have no connection) in this frame. This choice of frame (the ‘central frame’) is
actually inherited from the four-dimensional superspace and is not the most natural one for harmonic superspace. However, we may pass to another frame, what we call the ‘analytic frame’, in which the derivatives \{D^+, D_+^\oplus, \nabla^+\oplus\} are ‘short’ and \(D^{++}, D^{\oplus\oplus}\) are ‘long’ (i.e. acquire Lie-algebra-valued connections) instead. Namely,

\[
D^{++} \mapsto D^{++} = \varphi^{-1}[D^{++}]\varphi = D^{++} + V^{++}
\]

\[
D^{\oplus\oplus} \mapsto D^{\oplus\oplus} = \varphi^{-1}[D^{\oplus\oplus}]\varphi = D^{\oplus\oplus} + V^{\oplus\oplus},
\]

with the thus acquired harmonic superconnections given by

\[
V^{++} = \varphi^{-1}D^{++}\varphi
\]

\[
V^{\oplus\oplus} = \varphi^{-1}D^{\oplus\oplus}\varphi,
\]

and the covariant derivatives \(\{D^+, D_+^\oplus, \nabla^+\oplus\}\) lose their connections, i.e. instead of (8) we have, in this analytic frame,

\[
D^+ = D^+
\]

\[
D_+^\oplus = D_+^\oplus
\]

\[
\nabla^+\oplus = \partial^+\oplus.
\]

5. The Cauchy-Riemann equivalence

In this analytic frame it is natural to use an analytic basis which manifestly distinguishes the analytic subspace, analogous to the well known chiral basis of ordinary superspace which distinguishes the chiral subspace. Such a basis is defined by the change of coordinates to:

\[
x_A^+ = x^+ + \bar{\vartheta}_i^+ \vartheta^i, \quad x_A^- = x^- - \bar{\vartheta}_i^+ \vartheta^i
\]

\[
x_A^\ominus = x^{-\ominus}, \quad x_A^{+\ominus} = x^{+\ominus},
\]

with all other coordinates remaining unchanged. In such an analytic basis the derivatives occurring the system (5) take the form

\[
D_+^\oplus = - \frac{\partial}{\partial \vartheta_i^\ominus} + \bar{\vartheta}_i^+ \partial_A^+\ominus
\]

\[
D^+ = \frac{\partial}{\partial \vartheta^i} - \vartheta^i \partial_A^+\oplus
\]

\[
\partial_A^+\oplus = \frac{\partial}{\partial x_A^{-\ominus}}
\]

\[
D^{++} = u^+ \frac{\partial}{\partial u^{-\alpha}} + x_A^+(\partial_A^+\ominus) + \bar{\vartheta}_i^+ \frac{\partial}{\partial \vartheta^i} - \bar{\vartheta}_i^+ \vartheta^i \partial_A^+(\partial_A^+\oplus)
\]

\[
D^{\oplus\oplus} = v^{\oplus\alpha} \frac{\partial}{\partial v^{\ominus\alpha}} + x_A^+(\partial_A^-\ominus) + \vartheta^i \frac{\partial}{\partial \vartheta^i} - \vartheta^i \bar{\vartheta}_i^+ \partial_A^-(\partial_A^{-}\oplus),
\]
where brackets denote symmetrisations in the corresponding charges. In the analytic frame the equations (5) become identities for the derivatives (10) whereas the equations (6) take the form of generalised Cauchy-Riemann conditions

\begin{align*}
\frac{\partial}{\partial \bar{\vartheta}} V^{++} &= 0 = \frac{\partial}{\partial \bar{\vartheta}} V^{\ominus \ominus} \\
\frac{\partial}{\partial \vartheta} V^{++} &= 0 = \frac{\partial}{\partial \vartheta} V^{\ominus \ominus} \\
\frac{\partial}{\partial x^{-\ominus}} V^{++} &= 0 = \frac{\partial}{\partial x^{-\ominus}} V^{\ominus \ominus}
\end{align*}

(11)

with the consistency relation (7) taking the form of a zero curvature relation

\[ D^{++} V^{\ominus \ominus} - D^{\ominus \ominus} V^{++} + [V^{++}, V^{\ominus \ominus}] = 0. \]  

(12)

These equations are merely analytic frame versions of the central frame equations (5), (6) and (7); and are therefore equivalent to the constraints (3). The complete dynamical information of the N=3 Yang-Mills system (1) is therefore coded into harmonic connections \{V^{++}, V^{\ominus \ominus}\} solving the first order linear differential equations (11,12) in harmonic superspace. The self-dual (resp. anti-self-dual ) subsets of solutions simply correspond to v- (resp. u-) independent chiral (i.e. \(\bar{\vartheta}\)- (resp. \(\vartheta\)-) independent) solutions of (11). The condition of v (resp. u) independence being tantamount to the vanishing of one of the harmonic connections, viz. \(V^{\ominus \ominus}\) (resp. \(V^{++}\)); and chirality implying the independence of an additional x-variable: \(x^{-\ominus}\) (resp. \(x^{++}\)). This formulation of the super self-duality equations is described in [9]. The interlocking of SD and ASD data mentioned at the beginning is clearly manifest in (11,12).

In order to construct the superconnection satisfying (3), we first need to recover the ‘bridge’ \(\varphi\) connecting the analytic frame to the central frame by solving the linear system of equations (9)

\begin{align*}
D^{++} \varphi &= \varphi V^{++} \\
D^{\ominus \ominus} \varphi &= \varphi V^{\ominus \ominus}
\end{align*}

(13)

for arbitrary holomorphic (i.e. independent of \(\{x^{-\ominus}, \vartheta^{\ominus i}, \bar{\vartheta}^{-\ominus}\}\)) superfields \(\{V^{++}, V^{\ominus \ominus}\}\), which enjoy (12) as an integrability condition. For semi-simple gauge groups it follows from (13) that the determinant of the bridge obeys the equations

\[ D^{++} \det \varphi = D^{\ominus \ominus} \det \varphi = 0. \]

Using a consistent solution \(\varphi\) of this system, we may return to the central basis in which solutions of the constraints (3) take the form

\begin{align*}
A^{++} &= - D^{++} \varphi \varphi^{-1} \\
A^{\ominus \ominus} &= - D^{\ominus \ominus} \varphi \varphi^{-1} \\
A^{+\ominus} &= - \partial^{+\ominus} \varphi \varphi^{-1}.
\end{align*}
These \( \{A^t, A^\oplus\} \) (resp. \( \{A^t, A^\oplus\} \)) are guaranteed by (5) to be linear in \( u \) (resp. \( v \)) in the central frame, so the superconnections satisfying (1) afford immediate extraction from the harmonic expansions

\[
\begin{align*}
A^t &= u^\alpha A^t_\alpha \\
A^\oplus &= v^\alpha A^\oplus_\alpha \\
A^\oplus &= u^\alpha v^\alpha A^\alpha \dot{\alpha}.
\end{align*}
\]

6. The static limit

The explicit construction is somewhat simpler for the static case, which is similar to the three-dimensional \( N = 6 \) Yang-Mills theory considered in [11]. In these cases the spinor indices \( \alpha \) and \( \dot{\alpha} \) as well as two sets of harmonics get identified and we have the relations to three-dimensional superspace coordinates:

\[
\begin{align*}
\vartheta_i^\pm &= \theta_i^\alpha u_\alpha^\pm, & \bar{\vartheta}_i^\pm &= \bar{\theta}_i^\alpha u_\alpha^\pm,
\end{align*}
\]

which supersymmetrise the three dimensional twistor relations of [16]. In this reduced harmonic superspace, the harmonic connections \( V^{++} \) and \( V^{\oplus\oplus} \) become identified and the somewhat difficult consistency relation (12) disappears, leaving the simplified CR system

\[
\begin{align*}
\frac{\partial}{\partial \vartheta_i^-} V^{++} &= 0 \\
\frac{\partial}{\partial \vartheta_i^-} V^{++} &= 0 \\
\frac{\partial}{\partial x^-} V^{++} &= 0,
\end{align*}
\]

in which the imposition of chirality (i.e. independence of \( \vartheta_i^+ \) as well, which implies \( x^{+-} \) independence) corresponds to the Bogomolny reduction for self-dual monopoles (supersymmetrisuing the construction of [17]). We shall present explicit solutions elsewhere [5].

7. Conclusion

We hope to have convinced the reader that the harmonic superspace approach is an effective framework for discussing integrability properties of the \( N = 3 \) Yang Mills equations. We find it remarkable that after sufficient supersymmetrisation non-integrable equations become integrable and the corresponding quantum theory becomes ultraviolet finite; and we expect an analogous phenomenon for the higher extended supergravity theories, which we shall discuss elsewhere.

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