Quantum Parameterized Complexity

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Parameterized complexity theory was developed in the 1990s to enrich the complexity-theoretic analysis of problems that depend on a range of parameters. In this paper we establish a quantum equivalent of classical parameterized complexity theory, motivated by the need for new tools for the classifications of the complexity of real-world problems. We introduce the quantum analogues of a range of parameterized complexity classes and examine the relationship between these classes, their classical counterparts, and well-studied problems. This framework exposes a rich classification of the complexity of parameterized versions of QMA-hard problems, demonstrating, for example, a clear separation between the Quantum Circuit Satisfiability problem and the Local Hamiltonian problem.

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I. INTRODUCTION

Quantum and classical complexity theory provide an essential tool for establishing potential benchmarks by which we can classify the cost of computing a given problem. It allows us to group applications by complexity classes that determine the asymptotic tractability of a problem. However, standard complexity-theoretic classification only considers a single parameter—usually the size of the input to the problem. In practice there may be considerably more structure to a problem than can be quantified by a single variable. This limitation can lead to characterisations that are too broad, such as grouping problems as typical for notoriously difficult complexity classes, e.g., NP (non-deterministic polynomial time), QMA (quantum Merlin Arthur), and QCMA (quantum classical Merlin Arthur), and subsequently might suggest problems are intractable for a given set of parameters, when in realistic scenarios these problems could be practically solvable. For example the $k$-COLOURING problem is trivial on trees and 3-SAT is polynomial-time solvable when the number of variables or the number of clauses is fixed. By varying the parameters considered, we can gain a more nuanced understanding of the conditions that contribute to the tractability of problems.

Parameterized complexity theory was developed in the 1990s to enrich the complexity-theoretic analysis of problems that depend on a range of parameters [1–4]. The key motivation was to develop tools that more closely mirror the real-world use cases of heuristics. Central to the theory is an examination of how the complexity of a problem might change if a certain parameter $k$ varies independently of the instance size $n$. The parameterized analogue of the complexity class $P$ (polynomial time) is the class $FPT$ (fixed-parameter tractable). Informally, it is the class of problems that can be solved in time $f(k) \cdot n^{O(1)}$ for some computable function $f$. A classic example of this is VERTEX COVER, which is NP-complete in its usual form but has an $O(n)$ time algorithm if we consider constant-size covers [5] and hence is in $FPT$. Another important application of parameterized complexity is its use in determining the potential intractability of a problem. Parameterized intractability classes include para-NP (parameterized non-deterministic polynomial time), XP (slice-wise polynomial time), $W[P]$, and the $W$ hierarchy (weft hierarchy).

Parameterized complexity has been applied to the classical simulation of quantum systems. In particular, fixed-parameter tractable algorithms have been established for simulating quantum systems when parameterized by the treewidth [6] and by the number of non-Clifford gates [7–9]. This was further extended to quantum Merlin Arthur problems when parameterized by the number of non-Clifford gates in the verification circuit [10].

A. Quantum Parameterized Complexity Framework

We establish a quantum equivalent of classical parameterized complexity theory, motivated by the need to establish new tools for the complexity-theoretic classification of real-world problems. We introduce the quantum analogues of a range of parameterized complexity classes (see Table I), and examine the relationship between these classes, their classical counterparts, and well-studied problems (see Fig. 1). In some cases, the quantum generalisations and their relationships follow almost directly from the equivalent classical definitions and their relationships. However, in other cases, the quantum generalisations are much less straightforward, for example, in the case of the quantum generalisation of the $W$ hierarchy.

For quantum parameterized tractability, we introduce the complexity class $FPQT$ (fixed-parameter quantum tractable; Definition 5). Informally, $FPQT$ is the class of parameterized decision problems that are tractable on a quantum computer. We discuss the relationship of $FPQT$ with other well-studied complexity classes. While the generalisation of $FPT$ to $FPQT$ is relatively straightforward,
the quantum generalisations of parameterized intractability classes is more complicated. As is standard in quantum complexity theory, we consider QMA and QCMA as the natural generalisations of the class NP. This gives the parameterized generalisations of para-NP as para-QMA (parameterized quantum Merlin Arthur; Definition 15) and para-QCMA (parameterized quantum classical Merlin Arthur; Definition 23). The parameterized class XP also readily generalises to the quantum case as XQP (slice-wise quantum polynomial time; Definition 33). In classical parameterized complexity theory the W hierarchy, its finite levels W[t], and their asymptotic limit W[P] play an important role in bounding FPT. We introduce quantum generalisations of the W hierarchy via the WEIGHT-k QUANTUM CIRCUIT SATISFIABILITY problem to establish QW[P] (Definition 42) and the QW hierarchy (quantum weft hierarchy; Definition 52) and the CLASSICAL WEIGHT-k QUANTUM CIRCUIT SATISFIABILITY problem to establish QCW[P] (quantum classical weft hierarchy; Definition 56) and the QCW hierarchy (Definition 63). Further, we introduce a variation of the QCW hierarchy via the HAMMING WEIGHT-k QUANTUM CIRCUIT SATISFIABILITY problem to establish the QCW_c hierarchy (Definition 66).

We establish several structural results concerning quantum and classical parameterized complexity classes, for example, we show that FPT = FPQT if and only if P = BQP (Proposition 9). We also...
establish key technical components of quantum parameterized complexity such as FPQT reductions (Section II B). Further, we apply the notion of fixed-parameter quantum tractability to the problem of approximate counting (Section IV A) and approximating quantum circuit probability amplitudes (Theorem 68 and Corollary 69).

One of our most important observations concerns the complexity of weighted quantum Merlin Arthur problems, i.e., where the witness state is constrained to be a superposition of $n$-bit strings of Hamming weight $k$. We show that the Weight-$k$ Quantum Circuit Satisfiability problem is $\text{QW}[P]$-complete under FPQT reductions (Proposition 48) and the Weight-$k$ $l$-Local Hamiltonian problem is in $\text{XP}$ (Proposition 49). Since Weight-$k$ Quantum Circuit Satisfiability cannot be in $\text{XP}$ unless $\text{P} = \text{BQP}$, this demonstrates a clear separation between the two problems.

B. Discussion and Open Problems

There are several important open problems that still remain. The most compelling of these is the question of how to saturate the power of FPQT in a natural way. While it is possible to construct problems for FPT-sized quantum circuits, this does not seem to naturally capture the role of the parameter. Further, it seems that many known QMA-complete problems do not have parameterizations that are known to be in FPQT. This is in contrast to the classical case, where there are several NP-complete problems whose parameterizations are in FPT.

Another important open problem is to identify natural complete problems for finite levels of the QW hierarchy. While we establish natural complete problems for para-QMA ($l$-Local Hamiltonian; Corollary 22) and para-QCMA (Ground State $k$-connectivity; Corollary 31), we have been unable to establish natural complete problems for finite levels of the QW hierarchy. This is contrast to the $W$ hierarchy, where there are several examples of natural complete problems, e.g., $k$-Independent Set for $W[1]$ and $k$-Dominating Set for $W[2]$ [3].

II. QUANTUM PARAMETERIZED TRACTABILITY

We begin by introducing the theory encompassing tractability for parameterized problems. A parameterized language is a language equipped with an additional specified input, the parameter. Historically, parameterized problems have been defined using either an explicit parameter [1, 3], or by defining a parameterization [2]. While there are subtle differences between these two approaches, for practical purposes they are equivalent, and we use the explicit parameterization.

Definition 1 (Parameterization). A parameterization of a finite alphabet $\Sigma$ is a mapping $\kappa : \Sigma^* \rightarrow \mathbb{Z}^+$ that is polynomial-time computable. The trivial parameterization $\kappa_{\text{trivial}}$ is the parameterization with $\kappa_{\text{trivial}}(x) = 1$ for all $x \in \Sigma^*$.

We now define a parameterized problem.

Definition 2 (Parameterized problem). A parameterized problem over a finite alphabet $\Sigma$ is a pair $(L, \kappa)$ where $L \subseteq \Sigma^*$ is a set of strings over $\Sigma$ and $\kappa$ is a parameterization of $\Sigma$. We say that a parameterized problem $(L, \kappa)$ over the alphabet $\Sigma$ is trivial if either $L = \emptyset$ or $L = \Sigma^*$.

It is sometimes useful to consider parameterized problems for fixed values of the parameter. To do so the formal definition of a slice of a parameterized problem is used.

Definition 3 (Slice). Let $(L, \kappa)$ be a parameterized problem over the finite alphabet $\Sigma$ and let $l \in \mathbb{Z}^+$ be a positive integer. The $l$th slice of $(L, \kappa)$ is the standard problem

$$(L, \kappa)_l := \{ x \in L \mid \kappa(x) = l \}.$$
We now describe the foundational definitions for tractability of quantum parameterized problems.

### A. Fixed-Parameter Quantum Tractable

The central complexity class for establishing tractability in the quantum parameterized framework is Fixed-Parameter Quantum Tractable (FPQT). Informally, an FPQT algorithm is a quantum algorithm that, for a parameterized problem \((L, \kappa)\), decides if \(x\) is a member of \(L\) with error probability at most \(1/3\) in time \(f(\kappa(x)) \cdot |x|^{O(1)}\) for some computable function \(f\).

**Definition 4** (FPQT algorithm). Let \((L, \kappa)\) be a parameterized problem over the alphabet \(\Sigma\). An algorithm \(A\) is a **FPQT algorithm** for \((L, \kappa)\) if the following conditions are satisfied.

1. There is a computable function \(f : \mathbb{Z}^+ \to \mathbb{Z}^+\) and a polynomial \(p \in \mathbb{N}[X]\), such that, for every \(x \in \Sigma^*\), the size of an FPT-uniform quantum circuit that computes \(A\) on input \(x\) is at most \(f(\kappa(x)) \cdot p(|x|)\).

2. For every \(x \in \Sigma^*\),
   - If \(x \in L\), then \(\Pr[A(x) \text{ accepts}] \geq \frac{2}{3}\).
   - If \(x \notin L\), then \(\Pr[A(x) \text{ accepts}] \leq \frac{1}{3}\).

The error probability of \(1/3\) is completely arbitrary and can be replaced by any constant non-zero probability less than \(1/2\). Note that we have adopted a uniform notion of an FPQT algorithm, however, it is also possible to adopt a non-uniform notion. We shall now introduce the complexity class \(FPQT\), which consists of all parameterized problems with an FPQT algorithm.

**Definition 5** (FPQT). The class **FPQT** consists of all parameterized problems that have an FPQT algorithm.

FPQT is the quantum analogue of the classical complexity class Fixed-Parameter Polynomial time (FPT) [1] and the parameterized version of the quantum complexity class Bounded-error Quantum Polynomial time (BQP) [11]. It is easy to see that the slices of any problem in \(FPQT\) are in \(BQP\).

**Proposition 6.** Let \((L, \kappa)\) be a parameterized problem and let \(l \in \mathbb{Z}^+\) be a positive integer. If \((L, \kappa)\) is in \(FPQT\), then \((L, \kappa)_l\) is in \(BQP\).

**Proof.** The proof follows from the fact that \(\kappa\) is polynomial-time computable. 

We shall now present some alternative characterisations of \(FPQT\).

**Theorem 7.** Let \((L, \kappa)\) be a parameterized problem over the alphabet \(\Sigma\). Then the following statements are equivalent.

1. \((L, \kappa)\) is in \(FPQT\).

2. \((L, \kappa)\) is in \(BQP\) after a precomputation on the parameter. That is, there exists an alphabet \(\Pi\), a computable function \(\pi : \mathbb{Z}^+ \to \Pi^*\), and a problem \(X \subseteq \Sigma^* \times \Pi^*\) such that \(X\) is in \(BQP\) and, for all instances \(x\) of \(L\), we have \(x \in L\) if and only if \((x, \pi(\kappa(x))) \in X\).

3. \(L\) is decidable and \((L, \kappa)\) is eventually in \(BQP\). That is, there exists a computable function \(\rho : \mathbb{Z}^+ \to \mathbb{Z}^+\) and a polynomial-time quantum algorithm that on input \(x \in \Sigma^*\) with \(|x| \geq \rho(\kappa(x))\), decides if \(x\) is a member of \(L\) with error probability at most \(1/3\).
Proof. The proof follows similarly to that of the equivalent classical theorem [2, Theorem 1.37]. ■

The following containment is straightforward.

**Proposition 8.** $\text{FPT} \subseteq \text{FPQT}$.

We conjecture that this containment is strict, i.e., $\text{FPT} \neq \text{FPQT}$. However, proving a separation between $\text{FPT}$ and $\text{FPQT}$ is as difficult as proving a separation between $\text{P}$ and $\text{BQP}$. Therefore, resolving this conjecture is a hard open problem.

**Proposition 9.** $\text{FPT} = \text{FPQT}$ if and only if $\text{P} = \text{BQP}$.

Proof. Suppose that $\text{FPT} = \text{FPQT}$. For every problem $L \subseteq \Sigma^*$ in $\text{BQP}$, we have that $(L, \kappa_{\text{trivial}})$ is in $\text{FPQT}$. It then follows that $(L, \kappa_{\text{trivial}})$ is in $\text{FPT}$ and therefore $L$ is in $\text{P}$. Hence $\text{P} = \text{BQP}$.

Now suppose instead that $\text{P} = \text{BQP}$, then the characterisation of $\text{FPQT}$ given by Theorem 7 (2.) is equivalent to a characterisation of $\text{FPT}$ [2, Theorem 1.37 (2.)]. Hence $\text{FPT} = \text{FPQT}$. ■

We now prove some results that characterise $\text{FPQT}$.

**Proposition 10.** $\text{FPT}^{\text{FPQT}} = \text{FPQT}$.

Proof. The inclusions $\text{FPQT} \subseteq \text{FPT}^{\text{FPQT}}$ and $\text{FPT}^{\text{FPQT}} \subseteq \text{FPQT}^{\text{FPQT}}$ are trivial. We have that $\text{FPQT}^{\text{FPQT}} = \text{FPQT}$ by a similar proof to that showing $\text{BQP}^{\text{BQP}} = \text{BQP}$ [11]. Hence, $\text{FPT}^{\text{FPQT}} = \text{FPQT}$, completing the proof. ■

The next result gives an alternative characterisation for $\text{FPQT}$ in terms of $\text{FPT}^{\text{BQP}}$. As a consequence of this result, we can use $\text{BQP}$-complete problems to define $\text{FPQT}$-complete problems.

**Proposition 11.** $\text{FPT}^{\text{BQP}} = \text{FPT}^{\text{FPQT}}$.

Proof. The inclusion $\text{FPT}^{\text{BQP}} \subseteq \text{FPT}^{\text{FPQT}}$ is trivial. To show that $\text{FPT}^{\text{FPQT}} \subseteq \text{FPT}^{\text{BQP}}$, we apply Theorem 7, which states that $(L, \kappa) \in \text{FPQT}$ if and only if $L \in \text{BQP}$ after a precomputation of the parameter. This precomputation can be performed by an $\text{FPT}$ machine, completing the proof. ■

We present some application of fixed-parameter quantum tractability in Section IV.

### B. Fixed-Parameter Quantum Tractable Reductions

We shall now introduce the notion of a reduction in quantum parameterized complexity that we use throughout the remainder of the paper.

**Definition 12.** Let $(L, \kappa)$ and $(L', \kappa')$ be parameterized problems over the alphabets $\Sigma$ and $\Sigma'$ respectively. A *FPQT reduction* from $(L, \kappa)$ to $(L', \kappa')$ is a mapping $R : \Sigma^* \rightarrow (\Sigma')^*$ such that the following conditions are satisfied.

1. For all $x \in \Sigma^*$, $x \in L \iff R(x) \in L'$.
2. $R$ is computable by an FPQT algorithm with respect to the parameter $\kappa$, (i.e. $R(x)$ is computable using an FPT-uniform collection of circuits of size $f(\kappa(x)) \cdot p(|x|)$ with high probability).
3. There is a computable function $g : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ such that $\kappa'(R(x)) \leq g(\kappa(x))$ for all $x \in \Sigma^*$.

This definition gives the crucial property that $\text{FPQT}$ is closed under $\text{FPQT}$ reductions.
Proposition 13. FPQT is closed under FPQT reductions. That is, if \((L', \kappa')\) is in FPQT and there is an FPQT reduction from \((L, \kappa)\) to \((L', \kappa')\), then \((L, \kappa)\) is in FPQT.

Proof. Let \((L', \kappa')\) be a parameterized problem in FPQT and let \(R\) be an FPQT reduction from \((L, \kappa)\) to \((L', \kappa')\) computable in time \(g(\kappa(x)) \cdot q(|x|)\) with \(\kappa(x') \leq g(\kappa(x))\), where \(g\) and \(h\) are computable functions and \(q \in \mathbb{N}[X]\) is a polynomial. Let \(A\) be an FPQT algorithm for deciding \((L', \kappa')\) in time \(f(\kappa(x)) \cdot p(|x|)\) with error probability at most 1/3. Then we can decide if \(x\) is a member of \(L\) by firstly computing \(R(x)\) and then deciding if \(R(x)\) is a member of \(L'\). This requires time at most \(g(\kappa(x)) \cdot q(|x|) + f(\kappa(x)) \cdot p(g(\kappa(x))) \cdot p(q(|x|))\). Then by applying a simple error gap argument, we obtain an FPQT algorithm for \((L, \kappa)\). Hence, \((L, \kappa)\) is in FPQT. \(\blacksquare\)

III. QUANTUM PARAMETERIZED INTRACTABILITY

Classical parameterized complexity has a very rich theory of intractability with a series of fine-grained complexity hierarchies. This richness is reflected in the quantum case and perhaps even more so. In the following section we introduce several quantum analogues to the classical classes \(\text{para-NP}, \text{XP}, \text{W}[P]\) and the \(W\) hierarchy. In contrast to the apparently straightforward comparison between \(FPT\) and \(FPQT\), the quantum intractability classes reveal interesting aspects particular to quantum computation.

A. Parameterized Quantum Merlin Arthur

The most immediate intractable classical parameterized class is \(\text{para-NP}\) — a direct parameterized analogue of \(\text{NP}\).

Definition 14 (para-NP). A parameterized problem \((L, \kappa)\) over the alphabet \(\Sigma\) is in para-NP if there is a verification procedure \(\{V_{n, \kappa}\}_{n, \kappa \in \mathbb{Z}^+}\) such that the following conditions are satisfied.

1. There is a computable function \(f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+\) and a polynomial \(p \in \mathbb{N}[X]\), such that, for every \(x \in \Sigma^*\), \(V_{|x|, \kappa(x)}\) on input \(x\) runs in time at most \(f(\kappa(x)) \cdot p(|x|)\) on a deterministic Turing machine.

2. For every \(x \in \Sigma^*\),
   - If \(x \in L\), then there exists a bit string \(y\), such that \(V_{|x|, \kappa(x)}(x, y)\) accepts.
   - If \(x \notin L\), then for every bit string \(y\), \(V_{|x|, \kappa(x)}(x, y)\) rejects.

The equivalent notion to \(\text{NP}\)-completeness is obtained naturally using \(FPT\)-reductions, however there is also an immediate theorem [2, Theorem 2.14] demonstrating the intractability of para-NP-complete problems where problems that are \(\text{NP}\)-complete for any finite set of values of the parameter are para-NP-complete. Thus several standard \(\text{NP}\)-complete problems are para-NP-complete with their obvious parameterizations. For example, \(k\)-\text{COLOURING} is para-NP-complete when parameterized by the number of colours \(k\), \(k\)-\text{SAT} is para-NP-complete when parameterized by the size of the clauses \(k\). We give the analogous theorem for QMA-complete problems in Theorem 21.

In the context of quantum complexity classes, Quantum Merlin Arthur (QMA) [12] occupies a place congruent to \(\text{NP}\) in the classical case. We introduce the class parameterized Quantum Merlin Arthur (para-QMA), as the quantum analogue of para-NP [2] and the parameterized version of QMA.
Definition 15 (para-QMA). A parameterized problem $(L, \kappa)$ over the alphabet $\Sigma$ is in para-QMA($c, s$) if there is a quantum verification procedure $\{V_{n,k}\}_{n,k \in \mathbb{Z}^+}$ such that the following conditions are satisfied.

1. There is a computable function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ and a polynomial $p \in \mathbb{N}[X]$, such that, for every $x \in \Sigma^*$, the size of an FPT-uniform quantum circuit that computes $V_{|x|, \kappa(x)}$ on input $x$ is at most $f(\kappa(x)) \cdot p(|x|)$.

2. For every $x \in \Sigma^*$,
   - If $x \in L$, then there exists a quantum state $|\psi\rangle$, such that $\Pr[V_{|x|, \kappa(x)}(x, |\psi\rangle) \text{ accepts}] \geq c$.
   - If $x \notin L$, then for every quantum state $|\psi\rangle$, $\Pr[V_{|x|, \kappa(x)}(x, |\psi\rangle) \text{ accepts}] \leq s$.

The class para-QMA is defined to be para-QMA($\frac{2}{3}, \frac{1}{3}$).

We shall now present some alternative characterisations of para-QMA.

Proposition 16. Let $(L, \kappa)$ be a parameterized problem over the alphabet $\Sigma$. Then the following statements are equivalent.

1. $(L, \kappa)$ is in para-QMA.

2. $(L, \kappa)$ is in QMA after a precomputation on the parameter. That is, there exists an alphabet $\Pi$, a computable function $\pi : \mathbb{Z}^+ \to \Pi^*$, and a problem $X \subseteq \Sigma^* \times \Pi^*$ such that $X$ is in QMA and, for all instances $x$ of $L$, we have $x \in L$ if and only if $(x, \pi(\kappa(x))) \in X$.

3. $L$ is decidable and $(L, \kappa)$ is eventually in QMA. That is, there exists a computable function $\rho : \mathbb{Z}^+ \to \mathbb{Z}^+$ and a QMA algorithm that on input $x \in \Sigma^*$ with $|x| \geq \rho(\kappa(x))$, decides if $x$ is a member of $L$ with error probability at most $1/3$.

Proof. The proof follows similarly to that of Theorem 7. □

The following containments are straightforward.

Proposition 17. para-NP $\subseteq$ para-QMA and FPQT $\subseteq$ para-QMA.

We conjecture that these containments are strict, i.e., para-NP $\neq$ para-QMA and FPQT $\neq$ para-QMA. However, proving a separation between para-NP and para-QMA is as difficult as proving a separation between NP and QMA, and proving a separation between FPQT and para-QMA is as difficult as proving a separation between BQP and QMA. Therefore, resolving these conjectures is a hard open problem.

Proposition 18. para-NP = para-QMA if and only if NP = QMA.

Proof. Suppose that para-NP = para-QMA. For every problem $L \subseteq \Sigma^*$ in QMA, we have that $(L, \kappa_{\text{trivial}})$ is in para-QMA. It then follows that $(L, \kappa_{\text{trivial}})$ is in para-NP and therefore $L$ is in NP. Hence NP = QMA.

Now suppose instead that NP = QMA, then the characterisation of para-QMA given by Proposition 16 (2.) is equivalent to a characterisation of para-NP [2, Proposition 2.12 (2.)]. Hence para-NP = para-QMA. □

Proposition 19. FPQT = para-QMA if and only if BQP = QMA.
Proof. Suppose that \( \text{FPQT} = \text{para-QMA} \). For every problem \( L \subseteq \Sigma^* \) in \( \text{QMA} \), we have that \((L, \kappa_{\text{trivial}})\) is in \( \text{para-QMA} \). It then follows that \((L, \kappa_{\text{trivial}})\) is in \( \text{FPQT} \) and therefore \( L \) is in \( \text{BQP} \). Hence \( \text{BQP} = \text{QMA} \).

Now suppose instead that \( \text{BQP} = \text{QMA} \), then the characterisation of \( \text{FPQT} \) given by Theorem 7 (2.) and the characterisation of \( \text{para-QMA} \) given by Proposition 16 (2.) are equivalent. Hence \( \text{FPQT} = \text{para-QMA} \). \( \blacksquare \)

We shall now develop the theory of \( \text{para-QMA} \)-completeness. Firstly, we show that \( \text{para-QMA} \) is closed under \( \text{FPQT} \) reductions.

**Proposition 20.** \( \text{para-QMA} \) is closed under \( \text{FPQT} \) reductions.

*Proof.* The proof follows similarly to that of Proposition 13. \( \blacksquare \)

The following theorem allows us to establish the \( \text{para-QMA} \)-completeness of a wide range of problems.

**Theorem 21.** Let \((L, \kappa)\) be a non-trivial parameterized problem in \( \text{para-QMA} \). Then the following statements are equivalent.

1. \((L, \kappa)\) is \( \text{para-QMA} \)-complete under \( \text{FPQT} \) reductions.

2. The union of a finite number of slices of \((L, \kappa)\) is \( \text{QMA} \)-complete.

*Proof.* The proof follows similarly to that of the equivalent classical theorem [2, Theorem 2.14]. \( \blacksquare \)

We now proceed to show that the \( l \)-Local Hamiltonian problem is \( \text{para-QMA} \)-complete.

**l-Local Hamiltonian:**

- **Instance:** An \( l \)-local Hamiltonian \( H := \sum_i H_i \) on \( n \) qubits that comprises at most a polynomial in \( n \) many terms \( \{H_i\} \), which each act non-trivially on at most \( l \) qubits and have operator norm \( \|H_i\| \) bounded from above by a polynomial in \( n \). Two positive numbers \( a, b \in (0, 1) \), such that \( b - a > \frac{1}{\text{poly}(n)} \).

- **Parameter:** A natural number \( l \geq 2 \).

- **Problem:** Decide whether \( H \) has an eigenvalue less than or equal to \( a \) or all of the eigenvalues of \( H \) are greater than or equal to \( b \), given the promise that one of these is the case.

**Corollary 22.** \( l \)-Local Hamiltonian is \( \text{para-QMA} \)-complete.

*Proof.* The proof follows from Theorem 21 and the fact that \( l \)-Local Hamiltonian is \( \text{QMA} \)-complete for constant \( l \) [13, 14]. \( \blacksquare \)

**B. Parameterized Quantum Classical Merlin Arthur**

We shall now introduce the complexity class parameterized Quantum Classical Merlin Arthur (\( \text{para-QCMA} \)), which is the subclass of \( \text{para-QMA} \) restricted to classical proofs. Alternatively, it is the parameterized version of the quantum complexity class Quantum Classical Merlin Arthur (\( \text{QCMA} \)) [15].

**Definition 23** (\( \text{para-QCMA} \)). A parameterized problem \((L, \kappa)\) over the alphabet \( \Sigma \) is in \( \text{para-QCMA}(c, s) \) if there is a quantum verification procedure \( \{\mathcal{V}_{n,k}\}_{n,k \in \mathbb{Z}^+} \) such that the following conditions are satisfied.
1. There is a computable function \( f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \) and a polynomial \( p \in \mathbb{N}[X] \), such that, for every \( x \in \Sigma^* \), the size of an FPT-uniform quantum circuit that computes \( V_{[x],\kappa(x)} \) on input \( x \) is at most \( f(\kappa(x)) \cdot p(|x|) \).

2. For every \( x \in \Sigma^* \),
   - If \( x \in L \), then there exists a bit string \( y \), such that \( \Pr[V_{[x],\kappa(x)}(x, y) \text{ accepts}] \geq c \).
   - If \( x \notin L \), then for every bit string \( y \), \( \Pr[V_{[x],\kappa(x)}(x, y) \text{ accepts}] \leq s \).

The class para-QCMA is defined to be para-QCMA\((\frac{2}{3}, \frac{1}{3})\).

We shall now present some alternative characterisations of para-QCMA.

**Proposition 24.** Let \((L, \kappa)\) be a parameterized problem over the alphabet \(\Sigma\). Then the following statements are equivalent.

1. \((L, \kappa)\) is in para-QCMA.

2. \((L, \kappa)\) is in QCMA after a precomputation on the parameter. That is, there exists an alphabet \(\Pi\), a computable function \(\pi : \mathbb{Z}^+ \rightarrow \Pi^*\), and a problem \(X \subseteq \Sigma^* \times \Pi^*\) such that \(X\) is in QCMA and, for all instances \(x\) of \(L\), we have \(x \in L\) if and only if \((x, \pi(\kappa(x))) \in X\).

3. \(L\) is decidable and \((L, \kappa)\) is eventually in QCMA. That is, there exists a computable function \(\rho : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+\) and a QCMA algorithm that on input \(x \in \Sigma^*\) with \(|x| \geq \rho(\kappa(x))\), decides if \(x\) is a member of \(L\) with error probability at most \(1/3\).

**Proof.** The proof follows similarly to that of Theorem 7. \(\Box\)

The following containments are straightforward.

**Proposition 25.** para-NP \(\subseteq\) para-QCMA, FPQT \(\subseteq\) QCMA, and para-QCMA \(\subseteq\) para-QMA.

We conjecture that these containments are strict, i.e., para-NP \(\neq\) para-QCMA, FPQT \(\neq\) para-QCMA, and para-QCMA \(\neq\) para-QMA. However, proving a separation between para-NP and para-QCMA is as difficult as proving a separation between NP and QCMA, proving a separation between FPQT and para-QCMA is as difficult as proving a separation between BQP and QCMA, and proving a separation between para-QCMA and para-QMA is as difficult as proving a separation between QCMA and QMA. Therefore, resolving these conjectures is a hard open problem.

**Proposition 26.** para-NP = para-QCMA if and only if NP = QCMA.

**Proof.** The proof follows similarly to that of Proposition 18. \(\Box\)

**Proposition 27.** FPQT = para-QCMA if and only if BQP = QCMA.

**Proof.** The proof follows similarly to that of Proposition 19. \(\Box\)

**Proposition 28.** para-QCMA = para-QMA if and only if QCMA = QMA.

**Proof.** Suppose that para-QCMA = para-QMA. For every problem \(L \subseteq \Sigma^*\) in QCMA, we have that \((L, \kappa_{\text{trivial}})\) is in para-QMA. It then follows that \((L, \kappa_{\text{trivial}})\) is in para-QCMA and therefore \(L\) is in QCMA. Hence QCMA = QMA.

Now suppose instead that QCMA = QMA, then the characterisation of para-QCMA given by Proposition 24 (2.) and the characterisation of para-QMA given by Proposition 16 (2.) are equivalent. Hence para-QCMA = para-QMA. \(\Box\)
We shall now develop the theory of para-QCMA-completeness. Firstly, we show that para-QCMA is closed under FPQT reductions.

**Proposition 29.** para-QCMA is closed under FPQT reductions.

**Proof.** The proof follows similarly to that of Proposition 13. ■

The following theorem allows us to establish the para-QCMA-completeness of a wide range of problems.

**Theorem 30.** Let \((L, \kappa)\) be a non-trivial parameterized problem in para-QCMA. Then the following statements are equivalent.

1. \((L, \kappa)\) is para-QCMA-complete under FPQT reductions.

2. The union of a finite number of slices of \((L, \kappa)\) is QCMA-complete.

**Proof.** The proof follows similarly to that of Theorem 21. ■

We now proceed to show that the **Ground State \(k\)-Connectivity** problem is para-QCMA-complete.

**Ground State \(k\)-Connectivity:**

**Instance:** A local Hamiltonian \(H := \sum_i H_i\) on \(n\) qubits, where each term \(H_i\) has infinity norm \(||H_i||_\infty \leq 1\). A polynomial \(p \in \mathbb{N}[X]\). Two positive numbers \(a, b \in (0, 1)\), such that \(b - a > \frac{1}{\text{poly}(n)}\). Two polynomial-size quantum circuits \(U_\psi\) and \(U_\phi\) generating states \(|\psi\rangle = U_\psi |0^n\rangle\) and \(|\phi\rangle = U_\phi |0^n\rangle\) such that \(\langle \psi | H | \psi \rangle \leq a\) and \(\langle \phi | H | \phi \rangle \leq a\).

**Parameter:** A natural number \(k\).

**Problem:** Decide whether there exists a sequence of \(k\)-local unitary matrices \((U_i)_{i=1}^{p(n)}\), such that:

1. For all \(m \in [p(n)]\), the intermediate states \(|\psi_m\rangle := \prod_{i=1}^{m} U_i |\psi\rangle\) satisfy \(\langle \psi_i | H | \psi_i \rangle \leq a\).

2. The final state \(|\psi_{p(n)}\rangle := \prod_{i=1}^{p(n)} U_i |\phi\rangle\) satisfies \(|||\psi_{p(n)}\rangle - |\phi\rangle||_2 \leq a\).

Otherwise, if for all sequences of \(k\)-local unitary matrices \((U_i)_{i=1}^{p(n)}\), either:

1. There exists an \(m \in [p(n)]\) and an intermediate state \(|\psi_m\rangle := \prod_{i=1}^{m} U_i |\psi\rangle\) such that \(\langle \psi_i | H | \psi_i \rangle \geq b\).

2. The final state \(|\psi_{p(n)}\rangle := \prod_{i=1}^{p(n)} U_i |\phi\rangle\) satisfies \(|||\psi_{p(n)}\rangle - |\phi\rangle||_2 \geq b\).

Given the promise that one of these is the case.

**Corollary 31.** **Ground State \(k\)-Connectivity** is para-QCMA-complete.

**Proof.** The proof follows from Theorem 30 and the fact that **Ground State \(k\)-Connectivity** is QCMA-complete for constant \(k\) [16]. ■
C. Slice-Wise Quantum Polynomial Time

The class $XP$ is often used in a similar way to $EXP$ (exponential time) as there is both a strict separation from FPT under the time hierarchy, and the parameterizations of several $EXP$-complete problems are $XP$-complete. This includes the Peg Game when parameterized by the number of rings [1] and the Pebble Game when parameterized by size of the start set [1]. The $n^k$-Step Halting Problem for deterministic Turing machines is also $XP$-complete when parameterized by $k$ [2].

Definition 32 ($XP$). The class $XP$ consists of all parameterized problems $(L, \kappa)$ whose slices $(L, \kappa)_l$ for $l \geq 1$ are all in $P$.

This class provides a direct means for establishing that a problem is unlikely to be para-$NP$-complete, as $para-NP \subseteq XP$ implies $P = NP$ [2, Proposition 2.20]. Thus membership in $XP$ is a useful tool for demonstrating the possibility of tractability. A similar theorem holds in the quantum case, providing a similar tool.

We shall now briefly introduce the complexity class Slice-wise Quantum Polynomial time ($XQP$), which is the quantum analogue of the classical complexity class Slice-wise Polynomial time ($XP$) [1].

Definition 33 ($XQP$). The class $XQP$ consists of all parameterized problems $(L, \kappa)$ whose slices $(L, \kappa)_l$ for $l \geq 1$ are all in $BQP$.

The following containments are straightforward.

Proposition 34. $XP \subseteq XQP$ and $FPQT \subseteq XQP$.

We conjecture that these containments are strict, i.e., $XP \neq XQP$ and $FPQT \neq XQP$. We also prove the following.

Proposition 35. If $FPQT \subseteq XP$ then $P = BQP$.

Proof. If $FPQT \subseteq XP$ then any $BQP$-complete problem with trivial parameterization is contained in $XP$. Implying that $P = BQP$.

We shall now study the relationship between $XQP$, para-$QMA$, and para-$QCMA$. We have the following propositions.

Proposition 36. If $BQP \neq QMA$ then $para-QMA \not\subseteq XQP$.

Proof. If $para-QMA \subseteq XQP$ then $\kappa$-LOCAL HAMILTONIAN is in $XQP$. Hence 2-LOCAL HAMILTONIAN is in $BQP$, which implies $BQP = QMA$. Here we use the fact that 2-LOCAL HAMILTONIAN is QMA-complete [14].

Proposition 37. If $BQP \neq QCMA$ then $para-QCMA \not\subseteq XQP$.

Proof. If $para-QCMA \subseteq XQP$ then GROUND STATE $\kappa$-CONNECTIVITY is in $XQP$. Hence GROUND STATE 2-CONNECTIVITY is in $BQP$, which implies $BQP = QCMA$. Here we use the fact that GROUND STATE 2-CONNECTIVITY is QCMA-complete [16].

The class $XQP$ is a non-uniform class and, in fact, contains problems that are undecidable. It is easy to see this because the class $XP$ contains problems that are undecidable [2]. We define the following uniform version of $XQP$.

Definition 38 ($XQP_{unif}$). The class $XQP_{unif}$ consists of all parameterized problems $(L, \kappa)$ over the alphabet $\Sigma$ for which there is a computable function $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ and a quantum algorithm that, given $x \in \Sigma^*$, decides if $x$ is a member of $L$ with error probability at most $1/3$ and runs in time less than $|x| f(\kappa(x)) + f(\kappa(x))$. 
D. The Quantum Weft Hierarchy

The Weft hierarchy \((W)\) and its related class \(W[P]\) are the central tools for demonstrating intractability in the classical parameterized setting. The \(W\) hierarchy consists of an infinite hierarchy of classes \(W[t]\) for \(t \in \mathbb{N}\) and is contained in the class \(W[P]\). The class \(W[P]\) may be intuitively thought of as the subclass of para-NP with sufficiently limited non-determinism to also be a subclass of \(XP\). The problem \(\text{Weight-}\text{k Circuit Satisfiability}\) is complete for \(W[P]\), \(k\)-INDEPENDENT SET is complete for \(W[1]\), and \(k\)-DOMINATING SET is complete for \(W[2]\) \cite{3}. Before introducing the quantum analogues of these classes, we review their classical definitions.

**Definition 39 \((W[P])\).** A parameterized problem \((L, \kappa)\) over the alphabet \(\Sigma\) is in \(W[P]\) if there is a verification procedure \(\{V_{n,\kappa}\}_{n,\kappa \in \mathbb{Z}^+}\) such that the following conditions are satisfied.

1. There is a computable function \(f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+\) and a polynomial \(p \in \mathbb{N}[X]\), such that, for every \(x \in \Sigma^*\), \(V_{|x|,\kappa(x)}\) on input \(x\) runs in time at most \(f(\kappa(x)) \cdot p(|x|)\) on a deterministic Turing machine.
2. For every \(x \in \Sigma^*\),
   - If \(x \in L\), then there exists a bit string \(y\) comprising at most \(f(\kappa(x)) \cdot \log |x|\) bits, such that \(V_{|x|,\kappa(x)}(x, y)\) accepts.
   - If \(x \notin L\), then for every bit string \(y\) comprising at most \(f(\kappa(x)) \cdot \log |x|\) bits, \(V_{|x|,\kappa(x)}(x, y)\) rejects.

To define the complexity class \(W[t]\), we require the notion of circuit weft.

**Definition 40 \((\text{Circuit weft})\).** Given a Boolean circuit \(C\) comprising generalised Toffoli gates and one and two bit fan-in gates. The weft of \(C\) is the maximum number of Toffoli gates that act on any path from input bit to output bit.

**Weight-\(k\) Weft-\(t\) Depth-\(d\) Circuit Satisfiability:**

*Instance:* A weft-\(t\) depth-\(d\) Boolean circuit \(C\) on \(n\) input bits.

*Parameter:* A natural number \(k\).

*Problem:* Decide whether there exists an \(n\)-bit Hamming weight-\(k\) string \(y\), such that \(C(y)\) accepts.

**Definition 41 \((W[t])\).** For \(t \in \mathbb{N}\), the class \(W[t]\) consists of all parameterized problems that are FPT reducible to \(\text{Weight-}\text{k Weft-}\text{t Depth-}\text{d Circuit Satisfiability}\) for some \(d \geq t\).

We shall now introduce the Quantum Weft hierarchy \((QW)\), which is the quantum version of the Weft hierarchy \((W)\). We begin by defining the complexity class \(QW[P]\)— the quantum version of the complexity class \(W[P]\).

**Definition 42 \((QW[P])\).** A parameterized problem \((L, \kappa)\) over the alphabet \(\Sigma\) is in \(QW[P]\) if there is a quantum verification procedure \(\{V_{n,\kappa}\}_{n,\kappa \in \mathbb{Z}^+}\) such that the following conditions are satisfied.

1. There is a computable function \(f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+\) and a polynomial \(p \in \mathbb{N}[X]\), such that, for every \(x \in \Sigma^*\), the size of an FPT-uniform quantum circuit that computes \(V_{|x|,\kappa(x)}\) on input \(x\) is at most \(f(\kappa(x)) \cdot p(|x|)\).
2. For every \(x \in \Sigma^*\),
• If \( x \in L \), then there exists a quantum state \( |\psi\rangle \) comprising at most \( f(\kappa(x)) \cdot \log |x| \) qubits, such that \( \Pr[\mathcal{V}_{|x,\kappa(x)}(x, |\psi\rangle) \text{ accepts}] \geq c. \)

• If \( x \not\in L \), then for every quantum state \( |\psi\rangle \) comprising at most \( f(\kappa(x)) \cdot \log |x| \) qubits, \( \Pr[\mathcal{V}_{|x,\kappa(x)}(x, |\psi\rangle) \text{ accepts}] \leq s. \)

The class \( \text{QW}[\mathcal{P}] \) is defined to be \( \text{QW}[\mathcal{P}](\frac{3}{2}, \frac{1}{3}) \).

The following containments are straightforward.

**Proposition 43.** \( \text{W}[\mathcal{P}] \subseteq \text{QW}[\mathcal{P}] \) and \( \text{FPQT} \subseteq \text{QW}[\mathcal{P}] \).

We conjecture that these containments are strict, i.e., \( \text{W}[\mathcal{P}] \neq \text{QW}[\mathcal{P}] \) and \( \text{FPQT} \neq \text{QW}[\mathcal{P}] \). However, proving a separation between \( \text{FPQT} \) and \( \text{QW}[\mathcal{P}] \) is as difficult as proving a separation between \( \text{BQP} \) and \( \text{QMA} \).

**Proposition 44.** If \( \text{FPQT} \neq \text{QW}[\mathcal{P}] \) then \( \text{BQP} \neq \text{QMA} \).

**Proof.** Since \( \text{FPQT} \subseteq \text{QW}[\mathcal{P}] \subseteq \text{para-QMA} \), we have that \( \text{FPQT} \neq \text{QW}[\mathcal{P}] \) implies \( \text{FPQT} \neq \text{para-QMA} \). It then follows from Proposition 19, that \( \text{BQP} \neq \text{QMA} \).

We also have the following containment.

**Proposition 45.** \( \text{QW}[\mathcal{P}] \subseteq \text{XQP} \cap \text{para-QMA} \).

**Proof.** It is straightforward to prove that \( \text{QW}[\mathcal{P}] \subseteq \text{para-QMA} \). To prove that \( \text{QW}[\mathcal{P}] \subseteq \text{XQP} \), observe that any problem in \( \text{QW}[\mathcal{P}] \) can be solved in quantum time \( O \left( 2^{f(\kappa(x)) \cdot \log |x|} \right) = |x|^{O(f(\kappa(x)))} \), see, for example, Section IV B.

We shall now develop the theory of \( \text{QW}[\mathcal{P}] \)-completeness. Firstly, we show that \( \text{QW}[\mathcal{P}] \) is closed under \( \text{FPQT} \) reductions.

**Proposition 46.** \( \text{QW}[\mathcal{P}] \) is closed under \( \text{FPQT} \) reductions.

**Proof.** The proof follows similarly to that of Proposition 13.

We now introduce the \textbf{Weight-}\( k \) \textbf{Quantum Circuit Satisfiability} problem and show that this problem is \( \text{QW}[\mathcal{P}] \)-complete. This requires the notion of the \textit{weight} of a quantum state.

**Definition 47** (Weight of a quantum state). A quantum state \( |\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle \) on \( n \) qubits is said to have \textit{weight} \( k \) if \( \alpha_x = 0 \) for all \( x \) not of Hamming weight \( k \).

**Weight-}\( k \) \textbf{Quantum Circuit Satisfiability:}

**Instance:** A quantum circuit \( C \) on \( n \) witness qubits and \( \text{poly}(n) \) ancilla qubits. Two positive numbers \( a, b \in (0, 1) \), such that \( b - a > \frac{1}{\text{poly}(n)} \).

**Parameter:** A natural number \( k \).

**Problem:** Decide whether there exists an \( n \)-qubit weight-\( k \) quantum state \( |\psi\rangle \), such that \( \Pr[C(|\psi\rangle) \text{ accepts}] \geq b. \) Otherwise, if for every \( n \)-qubit weight-\( k \) quantum state \( |\psi\rangle \), \( \Pr[C(|\psi\rangle) \text{ accepts}] \leq a. \) Given the promise that one of these is the case.

We now establish our completeness result.

**Proposition 48.** \textbf{Weight-}\( k \) \textbf{Quantum Circuit Satisfiability} is \( \text{QW}[\mathcal{P}] \)-complete under \( \text{FPQT} \) reductions.
Proof. Firstly, we show that \textsc{Weight-k Quantum Circuit Satisfiability} is in \textsc{QP}. Let \( \mathcal{C} \) be a quantum circuit on \( n \) qubits, \( k \) a natural number, and \( f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \) a computable function. Further let \( S_{n,k} \) denote the set of all \( n \)-bit strings with Hamming weight \( k \) and let \( \varepsilon \) be a binary enumeration of the elements of \( S_{n,k} \). An \( n \)-qubit weight-\( k \) quantum state \( |\psi\rangle = \sum_{x \in S_{n,k}} \alpha_x |x\rangle \) can be described using \( f(k) \cdot \log(n) \) qubits by the quantum state \( |\psi_x\rangle = \sum_{x \in S_{n,k}} \alpha_x |\varepsilon(x)\rangle \). Let \( \mathcal{M}_{n,k} \) be a verification procedure for deciding whether the weight of an \( n \)-qubit quantum state is \( k \). The verification procedure \( \mathcal{V}_{n,k} \) constructs the state \( |\psi_x\rangle \) and accepts if and only if \( \mathcal{C}(\mathcal{M}_{n,k} |\psi_x\rangle) \) accepts. Applying the gap amplification scheme of Marriott and Watrous [17] to this procedure completes the claim.

We now prove that \textsc{Weight-k Quantum Circuit Satisfiability} is \textsc{QP}-complete. Let \((L, \kappa)\) be a problem in \textsc{QP} with verification procedure \( \{\mathcal{V}_{n,k}\}_{n,k \in \mathbb{Z}^+} \). Further let \( f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \) be a computable function and define \( k_x := \kappa(x) \). For input \( x \in \Sigma^* \), we shall construct a quantum circuit \( \mathcal{C}_x \) that is satisfiable by a weight-\( k_x \) quantum state if and only if \( \mathcal{V}_{|x|,\kappa(x)}(x) \) is satisfiable. The circuit \( \mathcal{C}_x \) takes as input \( n \) qubits and firstly decides whether the input state has weight \( k_x \) using the verification procedure \( \mathcal{M}_{n,k_x} \). Finally, the circuit inputs the quantum state into the verifier \( \mathcal{V}_{|x|,\kappa(x)} \). Therefore, \( \mathcal{C}_x \) is satisfiable by a weight-\( k_x \) quantum state if and only if \( \mathcal{V}_{|x|,\kappa(x)}(x) \) is satisfiable. This completes the proof.

It is natural to ask whether the \textsc{l-Local Hamiltonian} variant of this problem is \textsc{QP}-complete. However, as we shall see this problem is in \textsc{XP}. Note that the slices of the \textsc{Weight-k Quantum Circuit Satisfiability} problem are \textsc{BQP}-complete and so it cannot be in \textsc{XP} unless \( P = \text{BQP} \).

\textbf{Weight-}\( k \) \textsc{l-Local Hamiltonian}:

\textbf{Instance}: An \( l \)-local Hamiltonian \( H := \sum_i H_i \) on \( n \) qubits that comprises at most a polynomial in \( n \) many terms \( \{H_i\} \), which each act non-trivially on at most \( l \) qubits and have operator norm \( \|H_i\| \) bounded from above by a polynomial in \( n \). Two positive numbers \( a, b \in (0, 1), \) such that \( b - a > \frac{1}{\text{poly}(n)} \).

\textbf{Parameter}: A natural number \( k \).

\textbf{Problem}: Decide whether there exists an \( n \)-qubit weight-\( k \) quantum state \( |\psi\rangle \), such that \( \langle \psi | H | \psi \rangle \leq a \). Otherwise, if for every \( n \)-qubit weight-\( k \) quantum state \( |\psi\rangle \), \( \langle \psi | H | \psi \rangle \geq b \). Given the promise that one of these is the case.

\textbf{Proposition 49}. \textbf{Weight-}\( k \) \textsc{l-Local Hamiltonian} is in \textsc{XP}.

\textbf{Proof}. Let \( S_{n,k} \) denote the set of all \( n \)-bit strings with Hamming weight \( k \) and let \( \varepsilon \) be an enumeration of the elements of \( S_{n,k} \). We define the matrix \( H_\varepsilon \) such that \( \langle \varepsilon(x) | H_\varepsilon | \varepsilon(y) \rangle := \langle x | H | y \rangle \) for all \( x, y \in S_{n,k} \), and for an \( n \)-qubit weight-\( k \) quantum state \( |\psi\rangle := \sum_{x \in S_{n,k}} \alpha_x |x\rangle \), we define the quantum state \( |\psi_\varepsilon\rangle := \sum_{x \in S_{n,k}} \alpha_x |\varepsilon(x)\rangle \). Then, for any \( n \)-qubit weight-\( k \) quantum state \( |\psi\rangle \), we have \( \langle \psi | H | \psi \rangle = \langle \psi_\varepsilon | H_\varepsilon | \psi_\varepsilon \rangle \). Therefore, it is sufficient to compute the smallest eigenvalue \( \lambda_{\min}(H_\varepsilon) \) of \( H_\varepsilon \). However, since the dimension of \( H_\varepsilon \) is \( n^{O(k)} \) and each of its entries can be computed in time \( n^{O(1)} \), we can compute \( \lambda_{\min}(H_\varepsilon) \) in time \( n^{O(k)} \). Hence, \textsc{Weight-}\( k \) \textsc{l-Local Hamiltonian} is in \textsc{XP}. This completes the proof.

We prove the following.

\textbf{Proposition 50}. If \( \text{QP} \subseteq \text{XP} \) then \( P = \text{BQP} \).

\textbf{Proof}. If \( \text{QP} \subseteq \text{XP} \) then \( \text{FPQT} \subseteq \text{XP} \), and so \( P = \text{BQP} \) by Proposition 35. 

We shall now define the complexity class \textsc{QP}[\( t \)] — the quantum version of the complexity class \textsc{W}[\( t \)]. This requires the notion of \textit{quantum circuit weft}. 

We prove the following.

**Definition 51** (Quantum circuit weft). Given a quantum circuit $C$ comprising generalised Toffoli gates, one and two-qubit gates, and unbounded classical fan-out. The weft of $C$ is the maximum number of Toffoli gates that act on any path from input qubit to output qubit.

**Weight-\(k\) Weft-\(t\) Depth-\(d\) Quantum Circuit Satisfiability:**

**Instance:** A weight-\(t\) depth-\(d\) quantum circuit $C$ on $n$ witness qubits and $\text{poly}(n)$ ancilla qubits. Three positive numbers $a, b, c \in (0, 1)$, such that $b - a > c$.

**Parameter:** A natural number $k$.

**Problem:** Decide whether there exists an $n$-qubit weight-\(k\) quantum state $|\psi\rangle$, such that $\Pr[C(|\psi\rangle)\text{ accepts}] \geq b$. Otherwise, if for every $n$-qubit weight-\(k\) quantum state $|\psi\rangle$, $\Pr[C(|\psi\rangle)\text{ accepts}] \leq a$. Given the promise that one of these is the case.

**Definition 52** ($\text{QW}[t]$). For $t \in \mathbb{N}$, the class $\text{QW}[t]$ consists of all parameterized problems that are FPQT reducible to Weight-\(k\) Weft-\(t\) Depth-\(d\) Quantum Circuit Satisfiability for some $d \geq t$.

The following containments are straightforward.

**Proposition 53.** For any $t \in \mathbb{N}$, $W[t] \subseteq \text{QW}[t]$, $\text{QW}[t] \subseteq \text{QW}[t + 1]$, and $\text{QW}[t] \subseteq \text{QW}[\mathbb{P}]$.

The complexity classes $\text{QW}[t]$, for $t \geq 1$, define the $\text{QW}$ hierarchy, while note that $\text{QW}[0] = \text{FPQT}$. We prove the following.

**Proposition 54.** For any $t \in \mathbb{N}$, if $W[t] = \text{QW}[t]$ then $P = \text{BQP}$.

**Proof.** If $W[t] = \text{QW}[t]$ then FPQT $\subseteq \text{XP}$, and so $P = \text{BQP}$ by Proposition 35.

**Proposition 55.** For any $t \in \mathbb{N}$, if $\text{QW}[t] \subseteq \text{XP}$ then $P = \text{BQP}$.

**Proof.** If $\text{QW}[t] \subseteq \text{XP}$ then FPQT $\subseteq \text{XP}$, and so $P = \text{BQP}$ by Proposition 35.

**E. The Quantum Classical Weft Hierarchy**

We shall now introduce the Quantum Classical Weft hierarchy ($\text{QCW}$). We begin by define the complexity class $\text{QCW}[\mathbb{P}]$, which is the subclass of $\text{QW}[\mathbb{P}]$ restricted to classical proofs.

**Definition 56** ($\text{QCW}[\mathbb{P}]$). A parameterized problem $(L, \kappa)$ over the alphabet $\Sigma$ is in $\text{QCW}[\mathbb{P}]|(c, s)$ if there is a quantum verification procedure $\{V_{n,k}\}_{n,k \in \mathbb{Z}^{+}}$ such that the following conditions are satisfied:

1. There is a computable function $f : \mathbb{Z}^{+} \rightarrow \mathbb{Z}^{+}$ and a polynomial $p \in \mathbb{N}[X]$, such that, for every $x \in \Sigma^{*}$, the size of an FPT-uniform quantum circuit that computes $V_{|x|,\kappa(x)}$ on input $x$ is at most $f(\kappa(x)) \cdot p(|x|)$.

2. For every $x \in \Sigma^{*}$,
   - If $x \in L$, then there exists a bit string $y$ comprising at most $f(\kappa(x)) \cdot \log |x|$ bits, such that $\Pr[V_{|x|,\kappa(x)}(x, y)\text{ accepts}] \geq c$.
   - If $x \notin L$, then for every bit string $y$ comprising at most $f(\kappa(x)) \cdot \log |x|$ bits, $\Pr[V_{|x|,\kappa(x)}(x, y)\text{ accepts}] \leq s$.

The class $\text{QCW}[\mathbb{P}]$ is defined to be $\text{QCW}[\mathbb{P}]|(\frac{2}{3}, \frac{1}{3})$. 
The following containments are straightforward.

**Proposition 57.** \( W[P] \subseteq QCW[P], \) FPQT \( \subseteq QCW[P], \) and \( QCW[P] \subseteq QW[P]. \)

We conjecture that these containments are strict, i.e., \( W[P] \neq QCW[P], \) FPQT \( \neq QCW[P], \) and \( QCW[P] \neq QW[P]. \) Similar to Proposition 44, proving a separation between FPQT and QCW[P] is as difficult as proving a separation between BQP and QCMA.

**Proposition 58.** If \( FPQT \neq QCW[P] \) then \( BQP \neq QCMA. \)

**Proof.** Since \( FPQT \subseteq QCW[P] \subseteq para-QCMA, \) we have that \( FPQT \neq QCW[P] \) implies \( FPQT \neq para-QCMA. \) It then follows from Proposition 27, that \( BQP \neq QCMA. \)

We also have the following containment.

**Proposition 59.** \( QCW[P] \subseteq XQP \cap para-QCMA. \)

**Proof.** The proof follows similarly to that of Proposition 45.

We shall now develop the theory of QCW[P]-completeness. Firstly, we show that QCW[P] is closed under FPQT reductions.

**Proposition 60.** QCW[P] is closed under FPQT reductions.

**Proof.** The proof follows similarly to that of Proposition 13.

We now introduce the Weight-k Quantum Circuit Satisfiability problem and show that this problem is QW[P]-complete. This requires the notion of the weight of a quantum state.

**Definition 61** (Classical weight of a quantum state). A quantum state \( |\psi\rangle \) on \( n \) qubits is said to have classical weight \( k \) if there exists a quantum state \( |\phi\rangle \) on \( k \) qubits and an \( n \)-qubit swap network \( S_n \), such that \( |\psi\rangle = S_n(|\phi\rangle |0^{n-k}\rangle) \).

**Classical Weight-k Quantum Circuit Satisfiability:**

*Instance:* A quantum circuit \( C \) on \( n \) witness qubits and poly\((n)\) ancilla qubits. Two positive numbers \( a, b \in (0, 1) \), such that \( b - a > \frac{1}{\text{poly}(n)} \).

*Parameter:* A natural number \( k \).

*Problem:* Decide whether there exists an \( n \)-qubit classical weight-\( k \) quantum state \( |\psi\rangle \), such that \( \Pr[C(|\psi\rangle) \text{ accepts}] \geq b. \) Otherwise, if for every \( n \)-qubit classical weight-\( k \) quantum state \( |\psi\rangle \), \( \Pr[C(|\psi\rangle) \text{ accepts}] \leq a. \) Given the promise that one of these is the case.

**Proposition 62.** Classical Weight-k Quantum Circuit Satisfiability is QCW[P]-complete.

**Proof.** Firstly, we show that Classical Weight-k Quantum Circuit Satisfiability is in QCW[P]. Let \( C \) be a quantum circuit on \( n \) qubits, \( k \) a natural number, and \( f: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \) a computable function. An \( n \)-qubit classical weight-\( k \) quantum state \( |\psi\rangle \) can be described to an arbitrary constant precision \( \epsilon > 0 \) using \( f(k) \cdot \log(n) \) classical bits by specifying a quantum circuit on \( k \) qubits \( D_k \) and an \( n \)-qubit swap network \( S_n \), such that \( \| |\psi\rangle - S_n D_k |0^n\rangle \|_2 \leq \epsilon. \) The verification procedure \( V_{n,k} \) constructs the state \( S_n D_k |0^n\rangle \) and accepts if and only if \( C(S_n D_k |0^n\rangle) \) accepts. Applying the gap amplification scheme of Marriott and Watrous [17] to this procedure completes the claim.

We now prove that Classical Weight-k Quantum Circuit Satisfiability is QCW[P]-hard. Let \( (L, \kappa) \) be a problem in QCW[P] with verification procedure \( \{V_{n,k}\}_{n,k \in \mathbb{Z}^+} \). Further let
Proposition 64. For any $t \in \mathbb{N}$, $W[t] \subseteq QCW[t]$, $QCW[t] \subseteq QCW[t + 1]$, $QCW[t] \subseteq QW[t]$, and $QCW[t] \subseteq QCW[P]$.

The complexity classes $QCW[t]$, for $t \geq 1$, define the QCW hierarchy, while note that, for $t = 0$, we have $QCW[0] = FPQT$. We now establish an alternate $QCW[P]$-complete problem, which leads to an variation of the QCW hierarchy.

**Hamming Weight-$k$ Quantum Circuit Satisfiability:**
Instance: A quantum circuit $C$ on $n$ witness qubits and $\text{poly}(n)$ ancilla qubits. Two positive numbers $a, b \in (0, 1)$, such that $b - a > \frac{1}{\text{poly}(n)}$.
Parameter: A natural number $k$.
Problem: Decide whether there exists an $n$-bit Hamming weight-$k$ string $y$, such that $\text{Pr}[C(y) \text{ accepts}] \geq b$. Otherwise, if for every $n$-bit Hamming weight-$k$ string $y$, $\text{Pr}[C(y) \text{ accepts}] \leq a$. Given the promise that one of these is the case.

Proposition 65. Hamming Weight-$k$ Quantum Circuit Satisfiability is $QCW[P]$-complete.

Proof. The proof follows similarly to that of Proposition 62. □
Definition 66 (QCW$_c^t$). For $t \in \mathbb{N}$, the class QCW$_c^t$ consists of all parameterized problems that are FPQT reducible to Hamming Weight-$k$ Weft-$t$ Depth-$d$ Quantum Circuit Satisfiability for some $d \geq t$.

The following containments are straightforward.

Proposition 67. For any $t \in \mathbb{N}$, $W[t] \subseteq$ QCW$_c^t$, QCW$_c^t \subseteq$ QCW$_c^{t+1}$, QCW$_c^t \subseteq$ QCW$_c^P$, and QCW$_c^t \subseteq$ QCW$_c^{P}$.

The complexity classes QCW$_c^t$, for $t \geq 1$, define the QCW$_c$ hierarchy, while note that, for $t = 0$, we have QCW$_c^0 =$ QCW$_c^P =$ FPQT.

IV. APPLICATIONS

In this section we shall explore some applications of fixed-parameter quantum tractability.

A. Approximate Counting

In this section we shall apply some standard arguments in quantum computation and approximate counting to the parameterized setting. We proceed by establishing an FPQT algorithm for approximating quantum probability amplitudes.

Theorem 68. Fix $\epsilon > 0$. Let $\{C_n\}_{n \in \mathbb{Z}^+}$ be a polynomial-time uniform family of quantum circuits each acting on $p(n)$ qubits and define $q(x) := \langle 0^{|x|} | C_{|x|} (x) | 0^{|x|} \rangle$. Further let $\kappa$ be a parameterization and $f: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ a computable function. Then there is an FPQT algorithm that, for any input $x$, outputs an approximation to $q(x)$ to within an additive error of $\frac{\epsilon}{f(\kappa(x)) \cdot \text{poly}(|x|)}$.

Proof. We apply the Hadamard test to $C_{|x|}(x)$ to sample from random variables with expectation values equal to $\text{Re}(q(x))$ and $\text{Im}(q(x))$. It follows from the Chernoff-Hoeffding bound that repeating this procedure $f(\kappa(x))^2 \cdot \text{poly}(|x|) \cdot \epsilon^{-2}$ times allows us to obtain an approximation $\tilde{q}(x)$ to $q(x)$, such that

$$\Pr \left[ |q(x) - \tilde{q}(x)| \geq \frac{\epsilon}{f(\kappa(x)) \cdot \text{poly}(|x|)} \right] \leq e^{-\text{poly}(|x|)}.$$

Then, with high probability, we have

$$|q(x) - \tilde{q}(x)| \leq \frac{\epsilon}{f(\kappa(x)) \cdot \text{poly}(|x|)}.$$

This completes the proof. ■

We obtain the immediate corollary.

Corollary 69. In the notation of Theorem 68. Let $\kappa$ be a parameterization such that, for any input $x$, $|q(x)| \geq \frac{1}{f(\kappa(x)) \cdot \text{poly}(|x|)}$. Then there is an FPQT algorithm that, for any input $x$, outputs a multiplicative $\epsilon$-approximation to $q(x)$.

It is well known that quantum probability amplitudes encode the evaluation of Jones polynomials at principal non-lattice roots of unity, i.e., $t = \exp(2\pi i/k)$ for $k = 5$ or $k \geq 7$ [18, 19]. This is the key observation used to establish the quantum algorithm of Aharonov, Jones, and Landau [19] for approximating the evaluation of such Jones polynomials. We shall apply Theorem 68 and
Corollary 69 to extend this algorithm to the parameterized setting. Recall that a braid is a collection of strands that may cross over and under each other, and must always move from left to right. The plat closure of a 2n-strand braid b is the link formed by connecting pairs of adjacent strands on the left and the right of the braid. We have the following corollary.

**Corollary 70.** Fix $\epsilon > 0$. Let $k = 5$ or $k \geq 7$ be an integer, and $t = \exp(2\pi/k)$ its corresponding root of unity. Let $b$ be a braid on 2n strands with at most a polynomial in $n$ number of crossings, and let $b^{pl}$ denote its plat closure. Further let $\kappa$ be a parameterization and $f: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ a computable function. Then there is an FPQT algorithm that outputs an approximation to the evaluation of the Jones polynomial $V_{b^{pl}}(t)$ to within an additive error of $e^{\epsilon(2\cos(\pi/k)\kappa n)}/f(\kappa(t)) \cdot \text{poly}(|x|)$. Furthermore, if $\kappa$ is a parameterization such that $|V_{b^{pl}}(t)| \geq \epsilon(2\cos(\pi/k)\kappa n)/f(\kappa(t)) \cdot \text{poly}(|x|)$, then the FPQT algorithm outputs a multiplicative $\epsilon$-approximation to $V_{b^{pl}}(t)$.

**Proof.** By a result of Aharonov, Jones, and Landau [19] there exists a polynomial-time uniform family of quantum circuits $\{C_{n,k}\}_{n,k \in \mathbb{Z}^+}$ and a polynomial $p \in \mathbb{N}[X]$ such that, for any braid $b$ on 2n strands with at most a polynomial in $n$ number of crossings,

$$\langle 0^{p(|x|)} | C_{n,k}(b) | 0^{p(|x|)} \rangle = \frac{e^{3\pi/2k^2}}{(2\cos(\pi/k))^n} V_{b^{pl}}(t),$$

where $w(b^{pl})$ is the writhe of $b^{pl}$, which can be computed in polynomial time. By applying Theorem 68, we have an FPQT algorithm that outputs an approximation to $V_{b^{pl}}(t)$ to within an additive error of $e^{\epsilon(2\cos(\pi/k)\kappa n)}/f(\kappa(t)) \cdot \text{poly}(|x|)$. Then by Corollary 69, if $\kappa$ is a parameterization such that $|V_{b^{pl}}(t)| \geq \epsilon(2\cos(\pi/k)\kappa n)/f(\kappa(t)) \cdot \text{poly}(|x|)$, then the FPQT algorithm outputs a multiplicative $\epsilon$-approximation to $V_{b^{pl}}(t)$. This completes the proof.

Fenner et al. [20] showed that the solution to arbitrary problems in GapP can be efficiently encoded in quantum probability amplitudes. Recall that GapP is the closure of $\#P$ under subtraction. This allows us to establish an FPQT algorithm for approximating the solution to arbitrary problems in GapP. However, as we shall see later, this can be achieved by a classical FPT algorithm.

**Corollary 71.** Fix $\epsilon > 0$. Let $g$ be a function in GapP, and $p \in \mathbb{N}[X]$ a polynomial such that, for any input $x$, $g(x)$ takes values in the range $[-2^p(|x|), 2^p(|x|)]$. Further let $\kappa$ be a parameterization and $f: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ a computable function. Then there is an FPQT algorithm that, for any input $x$, outputs an approximation to $g(x)$ to within an additive error of $e^{\epsilon g(x)}/f(\kappa(x)) \cdot \text{poly}(|x|)$. Furthermore, if $\kappa$ is a parameterization such that $|g(x)| \geq 2^p(|x|)/f(\kappa(x)) \cdot \text{poly}(|x|)$, then the FPQT algorithm outputs a multiplicative $\epsilon$-approximation to $g(x)$.

**Proof.** By a result of Fenner et al. [20, Theorem 3.2] there exists a polynomial-time uniform family of quantum circuits $\{C_n\}_{n \in \mathbb{Z}^+}$ and a polynomial $q \in \mathbb{N}[X]$ such that, for all $x$ of length $n$,

$$\langle 0^{q(|x|)} | C_x(x) | 0^{q(|x|)} \rangle = \frac{g(x)}{2^p(|x|)}.$$

By applying Theorem 68, we have an FPQT algorithm that outputs an approximation to $g(x)$ to within an additive error of $e^{\epsilon g(x)}/f(\kappa(b)) \cdot \text{poly}(|x|)$. Then by Corollary 69, if $\kappa$ is a parameterization such that $|g(x)| \geq 2^p(|x|)/f(\kappa(x)) \cdot \text{poly}(|x|)$, then the FPQT algorithm outputs a multiplicative $\epsilon$-approximation to $g(x)$. This completes the proof.

The following theorem is a folklore result in parameterized counting.
Theorem 72. Fix $\epsilon > 0$. Let $g$ be a function in $\text{GapP}$, and $p \in \mathbb{N}[X]$ a polynomial such that, for any input $x$, $g(x)$ takes values in the range $[-2^p(|x|), 2^p(|x|)]$. Further let $\kappa$ be a parameterization and $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ a computable function. Then there is an $\text{FPT}$ algorithm that, for any input $x$, outputs an approximation to $g(x)$ to within an additive error of $\frac{\epsilon 2^p(|x|)}{f(\kappa(x)) \cdot \text{poly}(|x|)}$. Furthermore, if $\kappa$ is a parameterization such that $|g(x)| \geq \frac{2^p(|x|)}{f(\kappa(x)) \cdot \text{poly}(|x|)}$, then the $\text{FPT}$ algorithm outputs a multiplicative $\epsilon$-approximation to $g(x)$.

Proof. For any instance $x$ of $g$, we evaluate $m$ computational paths uniformly at random from the $2^p(|x|)$ possible. Let $\{X_i\}_{i=1}^m$ be the set of random variables such that $X_i$ takes the value $+1$ if the $i$th computational path accepts instance $x$ and $-1$ otherwise. We then approximate $g(x)$ by $\tilde{g}(x) = \frac{1}{m} \sum_{i=1}^m X_i$, which has expectation value $\mathbb{E}[\tilde{g}(x)] = g(x)$. By taking $m = f(\kappa(x))^2 \cdot \text{poly}(|x|) \cdot \epsilon^{-2}$, it follows from the Chernoff-Hoeffding bound that

$$\Pr \left[ |g(x) - \tilde{g}(x)| \geq \frac{\epsilon \cdot 2^p(|x|)}{f(\kappa(x)) \cdot \text{poly}(|x|)} \right] \leq e^{-\text{poly}(|x|)}.$$  

Then, with high probability, we have

$$|g(x) - \tilde{g}(x)| \leq \frac{\epsilon \cdot 2^p(|x|)}{f(\kappa(x)) \cdot \text{poly}(|x|)}.$$  

Furthermore, if $\kappa$ is a parameterization such that $|g(x)| \geq \frac{2^p(|x|)}{f(\kappa(x)) \cdot \text{poly}(|x|)}$, then we have

$$|g(x) - \tilde{g}(x)| \leq \epsilon |g(x)|.$$  

This completes the proof. ■

The results of this section concern approximately counting the number of accepting paths when there are a large number of them. However, it is also possible to approximately count the number of accepting paths when there are a small number of them provided we have access to an oracle that solves the decision problem $[21, 22]$. We remark that our results also apply to combinatorial structures of fixed-parameter tractable size. In particular, we can approximate the evaluation of Jones polynomials of the plat closure of tractable number of crossings.

B. Quantum Merlin Arthur Proofs

In this section we study the complexity class $\text{QMA}_k$ and its connection to quantum parameterized complexity. Recall that $\text{QMA}_k$ is the complexity class consisting of all languages for there exists a $\text{QMA}$ verification procedure on $k$ witness qubits. Marriott and Watrous [17] proved that $\text{QMA}_{\log} = \text{BQP}$, we follow their analysis to establish an $\text{FPQT}$ algorithm for any problem in $\text{QMA}_k$ when parameterized by the witness length $k$.

Theorem 73. Let $L \subseteq \Sigma^*$ be a language in $\text{QMA}_k$ and let $\kappa : \Sigma^* \rightarrow \mathbb{Z}^+$ be the parameterization with $\kappa(x) = k$ for all $x \in L$, then $(L, \kappa)$ is in $\text{FPQT}$.

Proof. Since $L$ is in $\text{QMA}_{k,\left(\frac{7}{3}, \frac{11}{3}\right)}$, by Ref. [17, Theorem 3.3], we have that $L$ is in $\text{QMA}_{k}(c, s)$ with $c = 1 - \frac{1}{3}2^{-k}$ and $s = \frac{2}{3}2^{-k}$. Let $A$ be a $k$-qubit verification procedure for $L$ with $n$ workspace qubits and let $\{\Pi_0, \Pi_1\}$ be a measurement defined by the projectors

$$\Pi_0 := |0\rangle \langle 0| \otimes I_{k+n-1} \quad \text{and} \quad \Pi_1 := |1\rangle \langle 1| \otimes I_{k+l-1},$$
which decides whether \( \mathcal{A} \) accepts or rejects. For each \( x \in \Sigma^* \), we define an operator \( \mathcal{Q}_x \) by

\[
\mathcal{Q}_x := (I_k \otimes \langle 0^n \mid) \mathcal{A}^\dagger(x) \Pi_1 \mathcal{A}(x) (I_k \otimes |0^n\rangle).
\]

Since \( \mathcal{Q}_x \) is positive semidefinite, then there exists a quantum state \( |\psi\rangle \), such that, if \( x \in L \), then

\[
\text{Tr}(\mathcal{Q}_x) \geq \langle \psi| \mathcal{Q}_x |\psi\rangle = \Pr[\mathcal{A}(x, |\psi\rangle) \text{ accepts}] \geq 1 - \frac{1}{3} 2^{-k} \geq \frac{2}{3}.
\]

Similarly, if \( x \notin L \), then

\[
\text{Tr}(\mathcal{Q}_x) \leq 2^k \langle \psi| \mathcal{Q}_x |\psi\rangle = 2^k \Pr[\mathcal{A}(x, |\psi\rangle) \text{ accepts}] \leq \frac{1}{3}.
\]

We now establish an FPQT algorithm \( \mathcal{B} \) for deciding \((L, \kappa)\). The algorithm \( \mathcal{B} \) constructs the maximally mixed state \( 2^{-k} I_k \) on \( k \) qubits and then runs the verification procedure \( \mathcal{A} \) with input \( 2^{-k} I_k \). We then have

\[
\Pr[\mathcal{B}(x) \text{ accepts}] = 2^{-k} \text{Tr}(\mathcal{Q}_x).
\]

Hence, if \( x \in L \), then

\[
\Pr[\mathcal{B}(x) \text{ accepts}] \geq \frac{2}{3} 2^{-k}.
\]

Otherwise, if \( x \notin L \), then

\[
\Pr[\mathcal{B}(x) \text{ accepts}] \leq \frac{1}{3} 2^{-k}.
\]

Since these probabilities are bounded away from one another by an inverse exponential in \( k \), by a simple gap amplification argument, we obtain an FPQT algorithm for \((L, \kappa)\). Hence, \((L, \kappa)\) is in \text{FPQT}.

\[\blacksquare\]

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[1] R. G. Downey and M. R. Fellows, \textit{Parameterized Complexity} (Springer New York, 1999).
