ASYMPTOTICS OF THE COLEMAN-GURTIN MODEL

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Abstract. This paper is concerned with the integrodifferential equation
\[ \partial_t u - \Delta u - \int_0^\infty \kappa(s) \Delta u(t-s) \, ds + \varphi(u) = f, \]
arising in the Coleman-Gurtin’s theory of heat conduction with hereditary memory, in presence of a nonlinearity \( \varphi \) of critical growth. Rephrasing the equation within the history space framework, we prove the existence of global and exponential attractors of optimal regularity and finite fractal dimension for the related solution semigroup, acting both on the basic weak-energy space and on a more regular phase space.

1. Introduction

1.1. The model equation. Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with a sufficiently smooth boundary \( \partial\Omega \). For \( t > 0 \), we consider the integrodifferential equation in the variable \( u = u(x,t) : \Omega \times \mathbb{R} \to \mathbb{R} \)
\[ \partial_t u - \Delta u - \int_0^\infty \kappa(s) \Delta u(t-s) \, ds + \varphi(u) = f, \]
subject to the Dirichlet boundary condition
\[ u(x,t)|_{x \in \partial\Omega} = 0. \]
The function \( u \) is supposed to be known for all \( t \leq 0 \). Accordingly, the boundary-value problem (1.1)-(1.2) is supplemented with the “initial condition”
\[ u(x,t) = \hat{u}(x,t), \quad \forall t \leq 0, \]
where \( \hat{u} : \Omega \times (-\infty,0] \to \mathbb{R} \) is a given function accounting for the initial past history of \( u \). In the sequel, we agree to omit the dependence on \( x \in \Omega \).

Among many other diffusive phenomena, equation (1.1) models heat propagation in a homogeneous isotropic heat conductor with hereditary memory. Here, the classical Fourier law ruling...
the heat flux is replaced by the more physical constitutive relation devised in the seminal paper of B.D. Coleman and M.E. Gurtin [6], based on the key assumption that the heat flux evolution is influenced by the past history of the temperature gradient (see also [8, 15, 16, 17, 22, 24, 26]). In that case, $u$ represents the temperature variation field relative to the equilibrium reference value, $f$ is a time-independent external heat supply, and the nonlinear term $\varphi(u)$ has to comply with some dissipativity assumptions, although it can exhibit an antidiissipative behavior at low temperatures. Such a nonlinearity is apt to describe, for instance, temperature-dependent radiative phenomena (cf. [23]).

1.2. Basic assumptions. We take $f \in L^2(\Omega)$ and $\varphi \in C^2(\mathbb{R})$, with $\varphi(0) = 0$, satisfying the growth and the dissipation conditions

$$|\varphi''(u)| \leq c(1 + |u|^p), \quad p \in [0, 3],$$

$$\liminf_{|u| \to \infty} \varphi'(u) > -\lambda_1,$$

where $\lambda_1 > 0$ is the first eigenvalue of the Laplace-Dirichlet operator on $L^2(\Omega)$. Concerning the memory kernel, we assume

$$\kappa(s) = \kappa_0 - \int_0^s \mu(\sigma) \, d\sigma, \quad \kappa_0 > 0,$$

for some (nonnegative) nonincreasing summable function $\mu$ on $\mathbb{R}^+ = (0, \infty)$ of total mass

$$\int_0^\infty \mu(s) \, ds = \kappa_0.$$

Consequently, $\kappa$ is nonincreasing and nonnegative. Moreover, we require the inequality (cf. [14])

$$\kappa(s) \leq \Theta \mu(s)$$

(1.6)

to hold for every $s > 0$ and some $\Theta > 0$. Observe that (1.6) implies the exponential decay

$$\kappa(s) \leq \kappa_0 e^{-s/\Theta}.$$

As a byproduct, $\kappa$ is summable on $\mathbb{R}^+$. To avoid the presence of unnecessary constants, we agree to put

$$\int_0^\infty \kappa(s) \, ds = \int_0^\infty s \mu(s) \, ds = 1,$$

where the first equality follows from an integration by parts.

1.3. Asymptotic behavior. The present paper is focused on the asymptotic properties of the solutions to (1.1)-(1.3). Setting the problem in the so-called history space framework [9] (see the next Section 3), in order to have a solution semigroup, our goal is to obtain global and exponential attractors of optimal regularity and finite fractal dimension. We address the reader to the books [1, 5, 19, 20, 21, 25, 29] for a detailed discussion on the theory of attractors.

The existence of the global attractor in the weak-energy space $\mathcal{H}^0$ (where $u \in L^2(\Omega)$) has been proved in [8], generalizing some earlier results from [16]. However, both its finite fractal dimension and the existence of exponential attractors are established only for $p < 3$. It should be noted that, without growth conditions on $\varphi$ other than (1.4) (e.g. the same polynomial control rate from above and below), the case $p = 3$ is critical, which explains the difficulties faced by [8].

In this work, we are mainly interested to solutions in the higher regularity space $\mathcal{H}^1$ (where $u \in H^1_0(\Omega)$). Here, the treatment of the case $p = 3$ is even more delicate, since the same problems encountered in [8] arise from the very beginning. Besides, our assumptions on the memory kernel...
\( \mu \) are more general (as shown in [3], the most general within the class of decreasing kernels). The strategy to deal with the critical case leans on an instantaneous regularization of \( u \), obtained by means of estimates of “hyperbolic” flavor, demanding in turn a skillful treatment of the memory terms. The effect of such a regularization is to render the nonlinearity subcritical in all respects, allowing to construct regular exponentially attracting sets in \( \mathcal{H}^1 \). Incidentally, once the existence of global and exponential attractors in \( \mathcal{H}^1 \) is established, it is standard matter to recover analogous results in the less regular space \( \mathcal{H}^0 \), extending the analysis of [8] to the critical case \( p = 3 \).

1.4. Plan of the paper. The functional setting is introduced in the next Section 2. In Section 3, we recall some known facts on the solution semigroup. The main result are then stated in Section 4. The rest of the paper is devoted to the proofs: in Section 5, we study an auxiliary problem, which will be used in the subsequent Section 6, in order to draw the existence of a strongly continuous semigroup in a more regular space; in Section 7, we demonstrate the existence of a regular exponentially attracting set, while the final Section 8 contains the conclusions of the proofs.

2. Functional Setting and Notation

Throughout this work, \( \mathcal{I}(\cdot) \) will stand for a generic increasing positive function.

Given a Hilbert space \( \mathcal{H} \), we denote by \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) and \( \| \cdot \|_{\mathcal{H}} \) its inner product and norm, and we call \( \mathcal{L}(\mathcal{H}) \) the Banach space of bounded linear operators on \( \mathcal{H} \). For \( R > 0 \), we put

\[
B_{\mathcal{H}}(R) = \{ z \in \mathcal{H} : \| z \|_{\mathcal{H}} \leq R \}.
\]

The Hausdorff semidistance between two sets \( \mathcal{X}, \mathcal{Y} \subset \mathcal{H} \) is defined as

\[
dist_{\mathcal{H}}(\mathcal{X}, \mathcal{Y}) = \sup_{x \in \mathcal{X}} \inf_{y \in \mathcal{Y}} \| x - y \|_{\mathcal{H}},
\]

while the fractal dimension of a (relatively) compact set \( K \subset \mathcal{H} \) is

\[
dim_{\mathcal{H}}(K) = \limsup_{\varepsilon \to 0^+} \frac{\ln \mathcal{N}_\varepsilon(K)}{\ln(1/\varepsilon)},
\]

\( \mathcal{N}_\varepsilon(K) \) being the smallest number of \( \varepsilon \)-balls of \( \mathcal{H} \) necessary to cover \( K \).

We consider the strictly positive Laplace-Dirichlet operator on \( L^2(\Omega) \)

\[
A = -\Delta, \quad \text{dom}(A) = H^2(\Omega) \cap H^1_0(\Omega),
\]

generating, for \( r \in \mathbb{R} \), the scale of Hilbert spaces (we omit the index \( r \) when \( r = 0 \))

\[
H^r = \text{dom}(A^{r/2}), \quad \langle u, v \rangle_r = \langle A^{r/2}u, A^{r/2}v \rangle_{L^2(\Omega)}.
\]

In particular, \( H = L^2(\Omega) \), \( H^1 = H^1_0(\Omega) \), \( H^2 = H^2(\Omega) \cap H^1_0(\Omega) \). Whenever \( r_1 > r_2 \), the embedding \( H^{r_1} \subset H^{r_2} \) is compact and

\[
\| u \|_{r_1} \geq \lambda^{(r_1-r_2)/2} \| u \|_{r_2}, \quad \forall u \in H^{r_1}.
\]

Next, we introduce the memory spaces

\[
\mathcal{M}^r = L^2_{\mu}(\mathbb{R}^+; H^r), \quad \langle \eta, \psi \rangle_{\mathcal{M}^r} = \int_0^\infty \mu(s) \langle \eta(s), \psi(s) \rangle_r \, ds,
\]

along with the infinitesimal generator of the right-translation semigroup on \( \mathcal{M}^r \), i.e. the linear operator

\[
T\eta = -\eta', \quad \text{dom}_r(T) = \{ \eta \in \mathcal{M}^r : \eta' \in \mathcal{M}^r, \, \eta(0) = 0 \},
\]
where the prime stands for the distributional derivative, and \( \eta(0) = \lim_{s \to 0} \eta(s) \) in \( \mathcal{M}^r \). For every \( \eta \in \text{dom}_r(T) \), we have the basic inequality (see [18])

\[
\langle T\eta, \eta \rangle_{\mathcal{M}^r} \leq 0.
\]

Finally, we define the phase spaces

\[
\mathcal{H}^r = \mathcal{H}^r \times \mathcal{M}^{r+1}, \quad \mathcal{V} = \mathcal{H}^2 \times \mathcal{M}^2.
\]

**A word of warning.** Without explicit mention, we will perform several formal estimates, to be in a position to exploit (2.1), for instance. As usual, the estimates are justified within a proper Galerkin approximation scheme.

### 3. The Semigroup

Introducing the auxiliary variable \( \eta = \eta^t(s) : [0, \infty) \times \mathbb{R}^+ \to \mathbb{R} \), accounting for the integrated past history of \( u \), and formally defined as (see [9, 18])

\[
\eta^t(s) = \int_0^s u(t - y)dy,
\]

we recast (1.1)-(1.3) in the history space framework. This amounts to considering the Cauchy problem in the unknowns \( u = u(t) \) and \( \eta = \eta^t \).

\[
\begin{aligned}
\partial_t u + Au + \int_0^\infty \mu(s)A\eta(s)ds + \varphi(u) &= f, \\
\partial_t \eta = T\eta + u, \\
(u(0), \eta^0) &= z,
\end{aligned}
\]

where \( z = (u_0, \eta_0) \) and \( f \in \mathcal{H} \) is independent of time.

**Remark 3.1.** The original problem (1.1)-(1.3) is recovered by choosing

\[
u_0 = \hat{u}(0), \quad \eta_0(s) = \int_0^s \hat{u}(-y)dy.
\]

We address the reader to [2, 18] for more details on the equivalence between the two formulations, which, within the proper functional setting, is not merely formal.

Problem (3.2) generates a (strongly continuous) semigroup of solutions \( S(t) \), on both the phase spaces \( \mathcal{H}^0 \) and \( \mathcal{H}^1 \) (see, e.g. [8]). Thus,

\[
(u(t), \eta^t) = S(t)z.
\]

In particular, \( \eta^t \) has the explicit representation formula [18]

\[
\eta^t(s) = \begin{cases}
\int_0^s u(t - y)dy & s \leq t, \\
\eta_0(s - t) + \int_0^t u(t - y)dy & s > t.
\end{cases}
\]

**Remark 3.2.** As shown in [2], the linear homogeneous version of (3.2), namely,

\[
\begin{aligned}
\partial_t u + Au + \int_0^\infty \mu(s)A\eta(s)ds &= 0, \\
\partial_t \eta = T\eta + u,
\end{aligned}
\]

generates a strongly continuous semigroup \( L(t) \) of (linear) contractions on every space \( \mathcal{H}^r \), satisfying, for some \( M \geq 1, \varepsilon > 0 \) independent of \( r \), the exponential decay

\[
\|L(t)\|_{\mathcal{L}(\mathcal{H}^r)} \leq Me^{-\varepsilon t}.
\]
Let us briefly recall some known facts from [8].

- For \( i = 0, 1 \), there exists \( R_i > 0 \) such that
  \[ B_i := B_{H^i}(R_i) \]
  is an absorbing set for \( S(t) \) in \( H^i \).
- Bounded sets \( B \subset H^0 \) are exponentially attracted by \( B_1 \) in the norm of \( H^0 \):
  \[ \text{dist}_{H^0}(S(t)B, B_1) \leq \mathcal{I} \left( \|B\|_{H^0} \right) e^{-\varepsilon_0 t}, \]
  for some \( \varepsilon_0 > 0 \).
- For every \( z \in B_{H^1}(R) \),
  \[ \|S(t)z\|_{H^1} + \int_t^{t+1} \|u(\tau)\|_2^2 d\tau \leq \mathcal{I}(R). \]

**Remark 3.3.** These results have been obtained under the commonly adopted assumption
\[ \mu'(s) + \delta \mu(s) \leq 0, \quad \text{a.e. } s > 0, \]
for some \( \delta > 0 \). On the other hand, it is not hard to show that (3.6) can be equivalently written as
\[ \mu(s + \sigma) \leq Ce^{-\delta \sigma} \mu(s), \quad \text{a.e. } s > 0, \ \forall \sigma \geq 0, \]
for some \( \delta > 0 \) and \( C \geq 1 \), which is easily seen to coincide with (3.8) when \( C = 1 \). However, if \( C > 1 \), the gap between (3.8) and (3.9) is quite relevant (see [3] for a detailed discussion).

4. **Main Results**

Defining the vector
\[ z_f = (u_f, \eta_f) \in H^2 \times \text{dom}_2(T) \subset \mathcal{V}, \]
with \( u_f = \frac{1}{2} A^{-1} f \) and \( \eta_f(s) = u_f s \), the main result of the paper reads as follows.

**Theorem 4.1.** There exists a compact set \( \mathcal{E} \subset \mathcal{V} \) with \( \dim_\mathcal{V}(\mathcal{E}) < \infty \), and positively invariant under the action of \( S(t) \), satisfying the exponential attraction property
\[ \text{dist}_\mathcal{V}(S(t)B, \mathcal{E}) \leq \mathcal{I} \left( \|B\|_{H^1} \right) e^{-\omega_1 t}, \quad \forall t > 0, \]
for some \( \omega_1 > 0 \) and every bounded set \( B \subset H^1 \). Moreover,
\[ \mathcal{E} = z_f + \mathcal{E}_*, \]
where \( \mathcal{E}_* \) is a bounded subset of \( H^3 \), whose second component belongs to \( \text{dom}_2(T) \).

**Remark 4.2.** The theorem implicitly makes a quite interesting assertion: whenever \( t > 0 \) and \( B \subset H^1 \) is bounded, \( S(t)B \) is a bounded subset of \( \mathcal{V} \) (cf. Proposition 6.1 below).

Such a set \( \mathcal{E} \) is called an *exponential attractor*. It is worth noting that, as \( \eta_f \in \text{dom}_2(T) \), the second component of \( \mathcal{E} \) belongs to \( \text{dom}_2(T) \) as well. Besides, if \( f \in H^1 \), it is immediate to deduce the boundedness of \( \mathcal{E} \) in \( H^3 \).
Corollary 4.3. With respect to the Hausdorff semidistance in $H^1$, the attraction property improves to

$$\text{dist}_{H^1}(S(t)B, \mathcal{E}) \leq \mathcal{I}(\|B\|_{H^1})e^{-\omega_1 t}.$$ 

As a byproduct, we establish the existence of the $(H^1, \mathcal{V})$-global attractor.

Theorem 4.4. There exists a compact set $\mathfrak{A} \subset \mathcal{E}$ with $\dim_{\mathcal{V}}(\mathfrak{A}) < \infty$, and strictly invariant under the action of $S(t)$, such that

$$\lim_{t \to \infty} \left[ \text{dist}_{\mathcal{V}}(S(t)B, \mathfrak{A}) \right] = 0,$$

for every bounded set $B \subset H^1$.

As observed in [8], the semigroup $S(t)$ fulfills the backward uniqueness property on the attractor (in fact, on the whole space $H^0$), a typical feature of equations with memory. A straightforward consequence is

Corollary 4.5. The restriction of $S(t)$ on $\mathfrak{A}$ is a group of operators.

The next result provides the link between the two components of the solutions on the attractor.

Proposition 4.6. Any solution $(u(t), \eta^t)$ lying on $\mathfrak{A}$ satisfies (3.1) for all $t \in \mathbb{R}$.

Remark 4.7. In particular, we obtain the uniform estimates

$$\sup_{t \in \mathbb{R}} \|\eta^t(s)\|_2 \leq c_0 s \quad \text{and} \quad \sup_{t \in \mathbb{R}} \sup_{s > 0} \|\eta^t(s)\|_2 \leq c_0,$$

for every $(u(t), \eta^t)$ lying on the attractor, with $c_0 = \sup\{\|u_0\|_2 : (u_0, \eta_0) \in \mathfrak{A}\}$.

We now focus our attention on $S(t)$ as a semigroup on the phase space $H^0$. Indeed, the set $\mathcal{E}$ of Theorem 4.1 turns out to be an exponential attractor on $H^0$ as well.

Corollary 4.8. We have

$$\text{dist}_{H^0}(S(t)B, \mathcal{E}) \leq \mathcal{I}(\|B\|_{H^0})e^{-\omega_0 t},$$

for some $\omega_0 > 0$ and every bounded set $B \subset H^0$.

Corollary 4.9. The set $\mathfrak{A}$ is also the global attractor for the semigroup $S(t)$ on $H^0$; namely,

$$\lim_{t \to \infty} \left[ \text{dist}_{H^0}(S(t)B, \mathfrak{A}) \right] = 0,$$

whenever $B$ is a bounded subset of $H^0$.

The remaining of the paper is devoted to the proofs of the results.

5. An Auxiliary Problem

This section deals with the analysis of the Cauchy problem in the variable $Z(t) = (u(t), \eta^t)$

$$\begin{aligned}
\partial_t u + Au + \int_0^\infty \mu(s)A\eta(s) \, ds &= f + g, \\
\partial_t \eta &= T\eta + u, \\
Z(0) &= z,
\end{aligned}$$

(5.1)

where $z = (u_0, \eta_0) \in H^1$, $f \in H$ is independent of time and $g \in L^2_{\text{loc}}(\mathbb{R}^+; H^1)$.

We need a definition and a preliminary lemma.
Definition 5.1. A nonnegative function Λ on \( \mathbb{R}^+ \) is said to be translation bounded if
\[
T(\Lambda) := \sup_{t \geq 0} \int_t^{t+1} \Lambda(\tau) \, d\tau < \infty.
\]

Lemma 5.2. For \( i = 0, 1, 2 \), let \( \Lambda_i \) be nonnegative functions such that \( T(\Lambda_i) \leq m_i \). Assuming \( \Lambda_0 \) absolutely continuous, let the differential inequality
\[
\frac{d}{dt} \Lambda_0 \leq \Lambda_0 \Lambda_1 + \Lambda_2
\]
hold almost everywhere in \( \mathbb{R}^+ \). Then, for every \( t \geq 0 \),
\[
\Lambda_0(t) \leq e^{m_1} \Lambda_0(0) e^{-t} + \frac{e^{m_1}(m_0 + m_0 m_1 + m_2)}{1-e^{-t}}.
\]

Proof. Setting
\[
\tilde{\Lambda}_1(t) = \Lambda_1(t) - 1 - m_1, \quad \tilde{\Lambda}_2(t) = (1 + m_1) \Lambda_0(t) + \Lambda_2(t),
\]
we rewrite the differential inequality as
\[
\frac{d}{dt} \Lambda_0 \leq \Lambda_0 \tilde{\Lambda}_1 + \tilde{\Lambda}_2.
\]
Observe that
\[
\int_\tau^t \tilde{\Lambda}_1(s) \, ds \leq -(t - \tau) + m_1, \quad \forall t \geq \tau,
\]
and
\[
T(\tilde{\Lambda}_2) \leq m_0 + m_0 m_1 + m_2.
\]
Hence, an application of the Gronwall lemma entails
\[
\Lambda_0(t) \leq e^{m_1} \Lambda_0(0) e^{-t} + e^{m_1} \int_0^t e^{-(t-\tau)} \tilde{\Lambda}_2(\tau) \, d\tau,
\]
and the inequality (cf. [4])
\[
\int_0^t e^{-(t-\tau)} \tilde{\Lambda}_2(\tau) \, d\tau \leq \frac{1}{1-e^{-t}} \, T(\tilde{\Lambda}_2)
\]
yields the desired result. \( \square \)

Given \( Z = (u, \eta) \in V \) and \( f \in H \), we define the functional
\[
\Lambda[Z,f] = \|u\|_2^2 + \alpha \|Z\|_H^2 + 2\langle \eta, u \rangle_{M^2} - 2\langle f, Au \rangle + 8\|f\|^2;
\]
with \( \alpha > 0 \) large enough such that
\[
\frac{1}{2} \|Z\|_V^2 \leq \Lambda[Z,f] \leq 2\alpha \|Z\|_V^2 + \alpha \|f\|^2.
\]
Moreover, given \( u \in L^2_{\text{loc}}(\mathbb{R}^+; H^2) \), we set
\[
F(t; u) = \int_0^t \mu(t-s) \|u(s)\|_2^2 \, ds.
\]

Remark 5.3. Exchanging the order of integration, we have
\[
T(F(\cdot, u)) \leq \kappa_0 T(\|u\|_2^2).
\]

We now state and prove several results on the solution \( Z(t) \) to problem (5.1).
Lemma 5.4. There is a structural constant $\alpha > 0$, large enough to comply with (5.2), such that the functional

$$\Lambda(t) = \Lambda[Z(t), f]$$

satisfies (within the approximation scheme) the differential inequality

\begin{equation}
\frac{d}{dt}\Lambda(t) \leq \mu(t)\Lambda(t) + \vartheta F(t; u) + \vartheta \|f\|^2 + \vartheta \|g(t)\|^2,
\end{equation}

for some positive constant $\vartheta = \vartheta(\alpha)$.

Proof. Multiplying the first equation of (5.1) by $A\partial_t u$, and using the second equation, we obtain the differential equality

\begin{equation}
\frac{d}{dt}\left\{\|u\|^2 + 2\langle \eta, u \rangle_{M^2} - 2\langle f, Au \rangle\right\} + 2\|\partial_t u\|^2 = 2\kappa_0 \|u\|^2 + 2\langle T\eta, u \rangle_{M^2} + 2\langle g, \partial_t u \rangle_1.
\end{equation}

Arguing exactly as in [11, Lemma 4.3], we find $\alpha > 0$, depending only on the total mass $\kappa_0$ of $\mu$, such that

\begin{equation}
2\langle T\eta, u \rangle_{M^2} \leq \|u\|^2 + \mu \|u\|^2 + \alpha F - 2\alpha \langle T\eta, \eta \rangle_{M^2}.
\end{equation}

Clearly, due to (2.1), the estimate is still valid for a larger $\alpha$. Thus, controlling the last term as

$$2\langle g, \partial_t u \rangle_1 \leq \|g\|^2 + \|\partial_t u\|^2,$$

we end up with

\begin{equation}
\frac{d}{dt}\left\{\|u\|^2 + 2\langle \eta, u \rangle_{M^2} - 2\langle f, Au \rangle\right\} 
\leq (1 + 2\kappa_0) \|u\|^2 + \mu \|u\|^2 + \alpha F - 2\alpha \langle T\eta, \eta \rangle_{M^2} + \|g\|^2.
\end{equation}

A further multiplication of (5.1) by $Z$ in $H^1$ entails

\begin{equation}
\frac{d}{dt}\|Z\|^2_{H^1} + 2\|u\|^2 = 2\langle T\eta, u \rangle_{M^2} + 2\langle f, Au \rangle + 2\langle g, u \rangle_1.
\end{equation}

Exploiting the straightforward relation

$$2\langle f, Au \rangle + 2\langle g, u \rangle_1 \leq \|u\|^2 + 2\|f\|^2 + 2\lambda_1^{-1}\|g\|^2,$$

we are led to the inequality

\begin{equation}
\frac{d}{dt}\|Z\|^2_{H^1} + \|u\|^2 \leq 2\langle T\eta, \eta \rangle_{M^2} + 2\|f\|^2 + 2\lambda_1^{-1}\|g\|^2.
\end{equation}

We now choose $\alpha \geq 1 + 2\kappa_0$ such that (5.2) and (5.5) hold. Adding (5.6) and $\alpha$-times (5.7), we finally get (5.4). $\Box$

Lemma 5.5. Assume that

$$\mathbb{T}(\|Z\|^2_\beta) \leq \beta \quad \text{and} \quad \|g(t)\|_1 \leq \gamma(1 + \|Z(t)\|^2_\beta),$$

for some $\beta, \gamma > 0$. Then, there exists $D = D(\beta, \gamma, \|f\|) > 0$ such that

$$\|Z(t)\|_V \leq D\left(\frac{1}{\sqrt{t}} + 1\right).$$

If $z \in V$, the estimate improves to

$$\|Z(t)\|_V \leq D\|z\|_V e^{-t} + D.$$
Proof. From (5.2), it is readily seen that
\[ \|g\|_1 \leq \gamma + 2\gamma \Lambda. \]
Thus, defining
\[ \Lambda_1(t) = \mu(t) + 2\partial \gamma \|g(t)\|_1, \]
\[ \Lambda_2(t) = \partial F(u; t) + \vartheta \|f\|^2 + 2\partial \gamma \|g(t)\|_1, \]
iequation (5.4) turns into
\[ \frac{d}{dt}\Lambda \leq \Lambda \Lambda_1 + \Lambda_2. \]
Using again (5.2), and recalling (5.3), we learn that
\[ T(\Lambda) + T(\Lambda_1) + T(\Lambda_2) \leq C, \]
for some $C > 0$ depending (besides on $\kappa_0$) only on $\beta, \gamma, \|f\|$. Hence, Lemma 5.2 together with a further application of (5.2) entail the second assertion of the lemma. If $z \notin \mathcal{V}$, we apply a standard trick: we set
\[ \tilde{\Lambda}(t) = \frac{t}{1+t} \Lambda(t), \]
which satisfies
\[ \frac{d}{dt}\tilde{\Lambda} \leq \tilde{\Lambda} \Lambda_1 + \tilde{\Lambda}_2, \]
where $\tilde{\Lambda}_2(t) = \Lambda(t) + \Lambda_2(t)$. Note that
\[ T(\tilde{\Lambda}) + T(\tilde{\Lambda}_2) \leq 2C. \]
As in the previous case, the first assertion follows from Lemma 5.2 and (5.2). \qed

Lemma 5.6. Suppose that $z \in \mathcal{V}$, $f = 0$ and
\[ \|g(t)\|_1 \leq k \|Z(t)\|_{\mathcal{V}}, \]
for some $k \geq 0$. Then, there are $D_1 > 0$ and $D_2 = D_2(k) > 0$ such that
\[ \|Z(t)\|_{\mathcal{V}} \leq D_1 \|z\|_{\mathcal{V}} e^{D_2 t}. \]
Proof. Under these assumptions, (5.2) and (5.4) become
\[ \frac{1}{2}\|Z\|^2_{\mathcal{V}} \leq \Lambda \leq 2\alpha \|Z\|^2_{\mathcal{V}} \]
and
\[ \frac{d}{dt}\Lambda \leq (\mu + 2\partial \alpha k^2)\Lambda + \partial F. \]
Moreover, exchanging the order of integration,
\[ \int_0^t F(\tau) \, d\tau \leq \kappa_0 \int_0^t \|u(\tau)\|^2_2 \, d\tau \leq 2\kappa_0 \int_0^t \Lambda(\tau) \, d\tau. \]
Hence, integrating the differential inequality on $(0, t)$, we arrive at
\[ \Lambda(t) \leq \Lambda(0) + \int_0^t \left[ \mu(\tau) + 2\partial(\alpha k^2 + \kappa_0)\Lambda(\tau) \right] \, d\tau. \]
Making use of the integral Gronwall lemma,
\[ \|Z(t)\|^2_{\mathcal{V}} \leq 2 \Lambda(t) \leq 2\Lambda(0)e^{\kappa_0}e^{2\partial(\alpha k^2 + \kappa_0)t} \leq 4\alpha e^{\kappa_0}\|z\|^2_{\mathcal{V}}e^{2\partial(\alpha k^2 + \kappa_0)t}, \]
and the result follows by choosing $D_1 = 4\alpha e^{\kappa_0}$ and $D_2 = 2\partial(\alpha k^2 + \kappa_0)$. \qed
6. The Semigroup on $\mathcal{V}$

We begin with a suitable regularization property for the solutions departing from $H^1$.

**Proposition 6.1.** Let $z \in B_{H^1}(R)$. Then, for every $t > 0$, $S(t)z \in \mathcal{V}$ and the estimate

$$
\|S(t)z\|_\mathcal{V} \leq J(R)\left(\frac{1}{\sqrt{t}} + 1\right)
$$

holds. If in addition $z \in \mathcal{V}$,

$$
\|S(t)z\|_\mathcal{V} \leq J(R)\|z\|_\mathcal{V} e^{-t} + J(R).
$$

**Proof.** We know from (3.7) that the solution $Z(t) = S(t)z$ fulfills

$$
\Theta(\|Z\|^2_\mathcal{V}) \leq J(R),
$$

whereas (1.4), (3.7) and the Agmon inequality

$$(6.1) \quad \|u\|_{L^\infty(\Omega)}^2 \leq c_\Omega \|u\|_1 \|u\|_2$$

entail

$$
\|\varphi(u)\|_1 = \|\varphi'(u) \nabla u\| \leq \|\varphi'(u)\|_{L^\infty(\Omega)} \|u\|_1 \leq J(R)(1 + \|z(t)\|_\mathcal{V}^2).
$$

Hence, Lemma 5.5 with $g = -\varphi(u)$ applies. \hfill \Box

**Corollary 6.2.** There exists $R_\mathcal{V} > 0$ such that the set

$$
\mathfrak{B}_\mathcal{V} := B_{\mathcal{V}}(R_\mathcal{V})
$$

has the following property: for every $R > 0$ there is a time $t_\mathcal{V} = t_\mathcal{V}(R) > 0$ such that

$$
S(t)B_{H^1}(R) \subset \mathfrak{B}_\mathcal{V}, \quad \forall t \geq t_\mathcal{V}.
$$

**Proof.** Let $z \in \mathfrak{B}_1$. According to Proposition 6.1

$$
\|S(t)z\|_\mathcal{V} \leq J(R_1)\left(\frac{1}{\sqrt{t}} + 1\right).
$$

Thus, setting $R_\mathcal{V} = 2J(R_1)$, the inclusion $S(t)\mathfrak{B}_1 \subset \mathfrak{B}_\mathcal{V}$ holds for every $t \geq 1$. Since $\mathfrak{B}_1$ is absorbing in $H^1$, for every $R > 0$ there exists $t_1 = t_1(R)$ such that $S(t)B_{H^1}(R) \subset \mathfrak{B}_1$ whenever $t \geq t_1$. We conclude that

$$
S(t)B_{H^1}(R) \subset S(t-t_1)\mathfrak{B}_1 \subset \mathfrak{B}_\mathcal{V}, \quad \forall t \geq t_\mathcal{V},
$$

with $t_\mathcal{V} = t_1 + 1$. \hfill \Box

In particular, Proposition 6.1 tells that $S(t)$ is a semigroup on $\mathcal{V}$, which, by Corollary 6.2, possesses the absorbing set $\mathfrak{B}_\mathcal{V}$. In fact, $S(t)$ is a strongly continuous semigroup, as the next proposition shows.

**Proposition 6.3.** For $i = 1, 2$, let $z_i \in B_{\mathcal{V}}(R)$. Then, we have the continuous dependence estimate

$$
\|S(t)z_1 - S(t)z_2\|_\mathcal{V} \leq D_1 \|z_1 - z_2\|_\mathcal{V} e^{3(R)t}.
$$

**Proof.** Calling $(u_i(t), \eta_i^t) = S(t)z_i$, the difference $(\bar{u}(t), \bar{\eta}^t) = S(t)z_1 - S(t)z_2$ fulfills the problem

$$
\begin{cases}
\partial_t \bar{u} + A\bar{u} + \int_0^\infty \mu(s)A\eta(s) \, ds = \varphi(u_2) - \varphi(u_1), \\
\partial_t \bar{\eta} = T\eta + \bar{u}, \\
\bar{z}(0) = z_1 - z_2.
\end{cases}
$$
Due to Proposition 6.1, \( \|u_i\|_2 \leq \mathcal{I}(R) \). Exploiting (1.4) and the Agmon inequality (6.1), it is then immediate to see that
\[
\|\varphi(u_2) - \varphi(u_1)\|_1 \leq \mathcal{I}(R) \|\bar{u}\|_2,
\]
and the claim is a consequence of Lemma 5.6 with \( f = 0 \) and \( g = \varphi(u_2) - \varphi(u_1) \).

\[\Box\]

**Proposition 6.4.** For every fixed \( z \in \mathcal{V} \),
\[ t \mapsto S(t)z \in \mathcal{C}([0, \infty), \mathcal{V}). \]

**Proof.** Let \( \tau > 0 \) be fixed. Given \( z \in \mathcal{V} \), choose a regular sequence \( z_n \to z \) in \( \mathcal{V} \), such that \( t \mapsto S(t)z_n \in \mathcal{C}([0, \infty), \mathcal{V}) \). For every \( n, m \in \mathbb{N} \), Proposition 6.3 provides the estimate
\[
\sup_{t \in [0, \tau]} \|S(t)z_n - S(t)z_m\|_V \leq C \|z_n - z_m\|_V,
\]
for some \( C > 0 \) depending on \( \tau \) and on the \( \mathcal{V} \)-bound of \( z_n \). Therefore, \( t \mapsto S(t)z_n \) is a Cauchy sequence in \( \mathcal{C}([0, \tau], \mathcal{V}) \). Accordingly, its limit \( t \mapsto S(t)z \) belongs to \( \mathcal{C}([0, \tau], \mathcal{V}) \). Since \( \tau > 0 \) is arbitrary, we are done.

\[\Box\]

Finally, we dwell on the linear homogeneous case, that is, system (3.4). From the previous results, we know that \( L(t) \) is a strongly continuous semigroup of linear operators on \( \mathcal{V} \). We prove that \( L(t) \) is exponentially stable as well.

**Proposition 6.5.** The semigroup \( L(t) \) satisfies the exponential decay property
\[
(6.2) \quad \|L(t)\|_{\mathcal{L}(\mathcal{V})} \leq M_1 e^{-\varepsilon_1 t},
\]
for some \( M_1 \geq 1 \) and \( \varepsilon_1 > 0 \).

**Proof.** Let \( z \in \mathcal{V} \). By virtue of (3.5) we have that
\[
\|\eta^t\|_{\mathcal{M}^2} \leq \|L(t)z\|_{\mathcal{H}^1} \leq M e^{-\varepsilon t} \|z\|_{\mathcal{H}^1}.
\]
Thus,
\[
\int_0^\infty \|\eta^\tau\|^2_{\mathcal{M}^2} d\tau < \infty.
\]
On the other hand, multiplying (3.3) times \( (u, \eta) \) in \( \mathcal{H}^1 \), and using (2.1), we get
\[
\frac{d}{dt}\|S(t)z\|_{\mathcal{H}^1}^2 + 2\|u(t)\|_{\mathcal{H}^1}^2 \leq 0.
\]
Integrating the inequality, we obtain
\[
\int_0^\infty \|u(\tau)\|_{\mathcal{H}^1}^2 d\tau \leq \frac{1}{2} \|z\|_{\mathcal{H}^1}^2 < \infty.
\]
We conclude that
\[
\int_0^\infty \|L(\tau)z\|_{\mathcal{V}}^2 d\tau < \infty, \quad \forall z \in \mathcal{V},
\]
and the result follows from the celebrated theorem of R. Datko [10] (see also [28]).

\[\Box\]
7. Regular Exponentially Attracting Sets

7.1. The result. We show the existence of a compact subset of \( \mathcal{V} \) which exponentially attracts \( \mathcal{B}_V \), with respect to the Hausdorff semidistance in \( \mathcal{V} \). To this end, we introduce the further space

\[
\mathcal{W} = \{ \eta \in \mathcal{M}^4 \cap \text{dom}_2(T) : \Xi[\eta] < \infty \},
\]
where

\[
\Xi[\eta] = \|T\eta\|_{\mathcal{M}^2}^2 + \sup_{x \geq 1} \left[ \int_{(0,1/x) \cup (x,\infty)} \mu(s)\|\eta(s)\|_{\mathcal{M}^2}^2 \, ds \right].
\]

This is a Banach space endowed with the norm

\[
\|\eta\|_{\mathcal{W}} = \|\eta\|_{\mathcal{M}^4} + \Xi[\eta].
\]

Finally, we define the product space

\[
\mathcal{Z} = H^3 \times \mathcal{W} \subset H^3.
\]

Remark 7.1. By means of a slight generalization of [27, Lemma 5.5], the embedding \( \mathcal{Z} \subset \mathcal{V} \) is compact (this is the reason why \( \mathcal{Z} \) is needed), contrary to the embedding \( H^3 \subset \mathcal{V} \), which is clearly continuous, but never compact. Moreover, closed balls of \( \mathcal{Z} \) are compact in \( \mathcal{V} \) (see [7]).

Theorem 7.2. Let \( z_f \) be given by (4.1). There exists \( R_* > 0 \) such that

\[
\mathcal{B} := z_f + B_{\mathcal{Z}}(R_*)
\]
fulfills the following properties:

(i) There is \( t_* = t_*(R_*) > 0 \) such that \( S(t)\mathcal{B} \subset \mathcal{B} \) for every \( t \geq t_* \).

(ii) The inequality

\[
\text{dist}_V(S(t)\mathcal{B}, \mathcal{B}) \leq C_1 e^{-\varepsilon_1 t}
\]
holds for some \( C_1 > 0 \), with \( \varepsilon_1 \) as in (6.2).

Theorem 7.2 is a consequence of the next lemma, proved in Subsection 7.2.

Lemma 7.3. Let \( \mathcal{J}_i(\cdot) \) denote generic increasing positive functions. For every \( z \in B_V(R) \), the semigroup \( S(t)z \) admits the decomposition

\[
S(t)z = z_f + \ell_1(t;z) + \ell_2(t;z),
\]
where

\[
\|\ell_1(t;z)\|_V \leq \mathcal{J}_1(R)e^{-\varepsilon_1 t},
\]
\[
\|\ell_2(t;z)\|_{\mathcal{Z}} \leq \mathcal{J}_2(R).
\]

If in addition \( z \in z_f + B_{\mathcal{Z}}(\varrho) \), we have the further estimate

\[
\|\ell_1(t;z)\|_{\mathcal{Z}} \leq \mathcal{J}_3(\varrho)e^{-\varepsilon_2 t} + \mathcal{J}_4(R),
\]
for some \( \varepsilon_2 > 0 \).

Proof of Theorem 7.2. For any given \( R, \varrho > 0 \) and

\[
z \in B_V(R) \cap [z_f + B_{\mathcal{Z}}(\varrho)],
\]
it is readily seen from (7.2)-(7.3) that

\[
\|S(t)z - z_f\|_{\mathcal{Z}} \leq \mathcal{J}_3(\varrho)e^{-\varepsilon_2 t} + \mathcal{J}_2(R) + \mathcal{J}_4(R).
\]
We fix then \( \mathcal{B} \) by selecting

\[
R_* = 2\mathcal{J}_2(R_V) + 2\mathcal{J}_4(R_V),
\]
with \( R_V \) as in Corollary \( 6.2 \). In particular, defining
\[
g_* = \mathcal{I}_3(R_*) + \mathcal{I}_4(\|B\|_V) + \mathcal{I}_4(\|B\|_V),
\]
inequality (7.4) provides the inclusion
\[
S(t)B \subset z_f + B_Z(g_*), \quad \forall t \geq 0.
\]
On the other hand, by Corollary \( 6.2 \) there is a time \( t_e \geq 0 \) (the entering time of \( B_V \) into itself) for which
\[
S(t_e)B \subset S(t_e)B_V \subset B_V = B_V(R_V).
\]
In conclusion,
\[
S(t_e)B \in B_V(R_V) \cap [z_f + B_Z(g_*)],
\]
and a further application of (7.4) for \( t \geq t_e \) leads to
\[
\|S(t)z - z_f\|_V \leq \mathcal{I}_3(g_*)e^{-\varepsilon_2(t-t_e)} + \frac{1}{2}R_*, \quad \forall z \in B.
\]
Accordingly, (i) holds true by taking a sufficiently large \( t_* = t_*(R_*) \geq t_e \). Finally, since \( \mathcal{I}_2(R_V) < R_* \), relations (7.1)-(7.2) immediately entail the estimate
\[
\text{dist}_V(S(t)B_V, B) \leq \mathcal{I}_1(R_V)e^{-\varepsilon_1 t},
\]
establishing (ii). \( \square \)

7.2. Proof of Lemma 7.3. We will make use of the following technical lemma (see [8] for a proof).

**Lemma 7.4.** Given \( \eta_0 \in \mathcal{W} \) and \( u \in L_\text{loc}^\infty(\mathbb{R}^+; H^2) \), let \( \eta = \eta^t(s) \) be the unique solution to the Cauchy problem in \( \mathcal{M}^2 \)
\[
\begin{aligned}
\partial_t \eta^t &= T\eta^t + u(t), \\
\eta^t(0) &= \eta_0.
\end{aligned}
\]
Then, \( \eta^t \in \text{dom}_2(T) \) for every \( t > 0 \), and
\[
\Xi[\eta^t] \leq Q^2 \Xi[\eta_0]e^{-2\nu t} + Q^2 \|u\|_{L_\text{loc}^\infty(0,t; H^2)}^2,
\]
for some \( Q \geq 1 \) and some \( \nu > 0 \), both independent of \( \eta_0 \) and \( u \).

In the sequel, \( C > 0 \) will denote a generic constant, which may depend (increasingly) only on \( R \). Given \( z \in B_V(R) \), we put
\[
\ell_1(t; z) = L(t)(z - z_f), \quad \ell_2(t; z) = W(t),
\]
where, by comparison, the function \( W(t) = (w(t), \xi^t) \) solves the problem
\[
\begin{aligned}
\partial_t w + Aw + \int_0^\infty \mu(s)A\xi(s)ds + \varphi(u) &= 0, \\
\partial_t \xi &= T\xi + w, \\
W(0) &= 0.
\end{aligned}
\]
In light of Proposition 6.5, we get at once (7.7). Indeed,
\[
\|L(t)(z - z_f)\|_V \leq M_1\|z - z_f\|_V e^{-\varepsilon_1 t} \leq Ce^{-\varepsilon_1 t}.
\]
If \( z \in z_f + B_Z(g) \), the decay property (5.5) provides the estimate
\[
\|L(t)(z - z_f)\|_{H^3} \leq M\|z - z_f\|_{H^3}e^{-et} \leq M_0 e^{-et}.
\]
The second component of $L(t)(z - z_f) = (v(t), \psi^t)$ fulfills the problem

$$\begin{cases}
\partial_t \psi^t = T \psi^t + v, \\
\psi^0 = \eta_0 - \eta_f,
\end{cases}$$

and the $V$-estimate above ensures the uniform bound

$$\|v(t)\|_2 \leq C.$$ 

Therefore, by Lemma 7.4,

$$\Xi[\psi^t] \leq Q^2 \Xi[\eta_0 - \eta_f]e^{-2\nu t} + C \leq Q^2 \varrho e^{-2\varrho t} + C.$$ 

Putting $\varepsilon_2 = \min\{\varepsilon, \nu\}$, we obtain

$$\|L(t)(z - z_f)\|_2^2 = \|L(t)(z - z_f)\|_{\mathcal{H}^3}^2 \leq (M^2 + Q^2)\varrho e^{-2\varrho t}.$$ 

This proves (7.3). We now turn to system (7.5). Thanks to Proposition 6.1,

$$\|u(t)\|_2 \leq C.$$ 

By (1.4), it is then standard matter to verify that

$$\|v(u(t))\|_2 \leq C.$$ 

Multiplying (7.5) by $W$ in $\mathcal{H}^3$, and using (2.1), we arrive at

$$\frac{d}{dt}\|W\|_{\mathcal{H}^3}^2 + 2\|w\|_4^2 \leq \|\varphi(u)\|_2 \|w\|_4 \leq \|w\|_4^2 + C.$$ 

In order to reconstruct the energy, following [2], we introduce the functional

$$\Upsilon(t) = \int_0^\infty k(s)\|\xi^t(s)\|_4^2 ds,$$

which, in light of (1.6), satisfies the bound

$$\Upsilon \leq \Theta \|\xi\|^2_{\mathcal{M}^4},$$

and the differential inequality

$$\frac{d}{dt}\Upsilon = -\|\xi\|^2_{\mathcal{M}^4} + 2\int_0^\infty k(s)\langle \xi(s), w\rangle_4 \, ds$$

$$\leq -\|\xi\|^2_{\mathcal{M}^4} + 2\Theta \|w\|_4 \int_0^\infty \mu(s)\|\xi(s)\|_4 \, ds \leq -\frac{1}{2}\|\xi\|^2_{\mathcal{M}^4} + \kappa_0 \Theta^2 \|w\|_4^2.$$ 

Defining then

$$\Psi(t) = \Theta_0\|W(t)\|_{\mathcal{H}^3}^2 + \Upsilon(t),$$

for $\Theta_0 > \max\{\Theta, \kappa_0 \Theta^2\}$ (so that, in particular, $\Psi$ and $\|W\|_{\mathcal{H}^3}^2$ control each other) the differential inequality

$$\frac{d}{dt}\Psi + \varpi \Psi \leq C,$$

holds for some $\varpi = \varpi(\Theta_0, \Theta, \lambda_1) > 0$. Hence, the Gronwall lemma gives the uniform bound

$$\|W(t)\|_{\mathcal{H}^3} \leq C.$$ 

Finally, applying Lemma 7.4 to the second equation of (7.5), we get

$$\Xi[\xi^t] \leq C.$$ 

Summarizing,

$$\|W(t)\|_Z \leq C.$$ 

This establishes (7.2) and completes the proof of the lemma.
8. Exponential Attractors

The next step is to demonstrate the existence of a regular set $E$ which exponentially attracts $B_V$.

**Theorem 8.1.** There exists a compact set $E \subset V$ with $\dim(E) < \infty$, and positively invariant for $S(t)$, such that

$$\text{dist}_V(S(t)B_V, E) \leq C_* e^{-\omega_1 t},$$

for some $C_* > 0$ and some $\omega_1 > 0$.

We preliminary observe that, thanks to the exponential decay property of Theorem 7.2

$$\text{dist}_V(S(t)B_V, B) \leq C_1 e^{-\varepsilon_1 t},$$

and the continuous dependence estimate provided by Proposition 6.3, the transitivity of the exponential attraction, devised in [13], applies. Hence, it suffices to prove the existence of a set $E$ complying with the statement of the theorem, but satisfying only the weaker exponential decay estimate

$$(8.1) \quad \text{dist}_V(S(t)B, E) \leq C_0 e^{-\omega t},$$

for some $C_0 > 0$ and some $\omega > 0$. Thus, in light of the abstract result from [12] on the existence of exponential attractors for discrete semigroups in Banach spaces, and thereafter constructing the attractor for the continuous case in a standard way, Theorem 8.1 applies provided that we show the following facts:

(i) There exist positive functions $\gamma(\cdot)$ and $\Gamma(\cdot)$, with $\gamma$ vanishing at infinity, such that the decomposition

$$S(t)z_1 - S(t)z_2 = \ell_1(t; z_1, z_2) + \ell_2(t; z_1, z_2),$$

holds for every $z_1, z_2 \in B$, where

$$\|\ell_1(t; z_1, z_2)\|_V \leq \gamma(t)\|z_1 - z_2\|_V,$$

$$\|\ell_2(t; z_1, z_2)\|_Z \leq \Gamma(t)\|z_1 - z_2\|_V.$$

(ii) There exists $K \geq 0$ such that

$$\sup_{z \in B} \|S(t)z - S(\tau)z\|_V \leq K|t - \tau|, \quad \forall t, \tau \in [t_*, 2t_*].$$

Indeed, recalling that $B$ is closed in $V$, by means of (i) we obtain the existence of an exponential attractor $E_d \subset B$ for the discrete semigroup $S_n := S(nt_*) : B \to B$. Then, we define

$$E = \bigcup_{t \in [t_*, 2t_*]} S(t)E_d.$$
where $W(t) = (\bar{w}(t), \bar{\xi}(t))$ solves the problem

$$
\begin{cases}
\partial_t \bar{w} + A \bar{w} + \int_0^\infty \mu(s) A \bar{\xi}(s) ds = \varphi(u_2) - \varphi(u_1), \\
\partial_t \bar{\xi} = T \bar{\xi} + \bar{w}, \\
W(0) = 0.
\end{cases}
$$

(8.2)

By Proposition 6.5

$$
\|L(t)\bar{z}\|_V \leq M_1 \|\bar{z}\|_V e^{-\varepsilon t}.
$$

Taking advantage of (1.4) and Proposition 6.3

$$
\|\varphi(u_2(t)) - \varphi(u_1(t))\|_2 \leq C \|u_2(t) - u_1(t)\|_2 \leq C \|\bar{z}\|_V e^{Ct}.
$$

Hence, multiplying (8.2) by $W$ in $H^3$, and using (2.1), we obtain

$$
\frac{d}{dt}\|W\|_{H^3}^2 \leq C \|\bar{z}\|_V^2 e^{Ct},
$$

and an integration in time readily gives

$$
\|W(t)\|_{H^3}^2 \leq C \|\bar{z}\|_V^2 e^{Ct}.
$$

Accordingly, from Lemma 7.4 applied to the second equation of (8.2),

$$
\Xi[\bar{\xi}^t] \leq C \|\bar{z}\|_V^2 e^{Ct}.
$$

Consequently, we learn that

$$
\|W(t)\|_Z \leq C \|\bar{z}\|_V e^{Ct}.
$$

Therefore, (i) holds with the choice $\ell_1(t; z_1, z_2) = L(t)\bar{z}$ and $\ell_2(t; z_1, z_2) = W(t)$. □

**Proof of (ii).** We will show that

$$
\sup_{t \in [t, 2t]} \|\partial_t S(t)z\|_V \leq C,
$$

which clearly implies (ii). For $z = (u_0, \eta_0) \in \mathcal{B}$, the function $(\tilde{u}(t), \tilde{\eta}^t) = \partial_t S(t)z$ fulfills the Cauchy problem

$$
\begin{cases}
\partial_t \tilde{u} + A \tilde{u} + \int_0^\infty \mu(s) A \tilde{\eta}(s) ds + \varphi'(u)\tilde{u} = 0, \\
\partial_t \tilde{\eta} = T \tilde{\eta} + \tilde{u}, \\
(\tilde{u}(0), \tilde{\eta}^0) = \tilde{z},
\end{cases}
$$

where

$$
\tilde{z} = (-Au_0 - \int_0^\infty \mu(s) A\eta_0(s) ds - \varphi(u_0) + f, T\eta_0 + u_0).
$$

Observe that

$$
\|\varphi'(u)\tilde{u}\|_1 \leq C \|\tilde{u}\|_2.
$$

Thus, applying Lemma 5.6 with $f = 0$ and $g = -\varphi'(u)\tilde{u}$, and noting that $\|\tilde{z}\|_V \leq C$, the claim follows. □
9. PROOFS OF THE MAIN RESULTS

We have now all the ingredients to carry out the proofs of the results stated in Section 4.

Proofs of Theorem 4.1 and Corollary 4.3. Let \( B \subset B_{H^1}(R) \), for some \( R > 0 \). According to Corollary 6.2, there is a positive time \( t_V = t_V(R) \) such that

\[
S(t)B \subset B_{V}, \quad \forall t \geq t_V.
\]

Therefore, by Theorem 8.1,

\[
\text{dist}_V(S(t)B, E) \leq I(R)e^{-\omega_1 t}, \quad \forall t \geq t_V.
\]

On the other hand, by virtue of Proposition 6.1,

\[
\text{dist}_V(S(t)B, E) \leq I(R)\sqrt{t}, \quad \forall t \in (0, t_V).
\]

Collecting the two inequalities we obtain

\[
\text{dist}_V(S(t)B, E) \leq I(R)e^{-\omega_1 t}\sqrt{t}, \quad \forall t > 0.
\]

The remaining properties of \( E \) are ensured by Theorem 8.1. With respect to the Hausdorff semidistance in \( H^1 \), we have

\[
\text{dist}_{H^1}(S(t)B, E) \leq (\lambda_1^{-1/2} + 1)\text{dist}_V(S(t)B, E) \leq I(R)e^{-\omega_1 t}, \quad \forall t \geq 1,
\]

and, due to (3.7),

\[
\text{dist}_{H^1}(S(t)B, E) \leq I(R), \quad \forall t < 1.
\]

Hence, Corollary 4.3 follows. \( \square \)

Theorem 4.4 is a direct consequence of Theorem 4.1 (cf. [1, 29]).

Proof of Proposition 4.6. Let \( Z(t) = (u(t), \eta^t) \) be a solution lying on \( A \). Assume first \( t > 0 \). Fixed an arbitrary \( \tau > 0 \), denote \( z_\tau = S(-\tau)Z(0) \) and set

\[
(u_\tau(t), \eta^t_\tau) = S(t)z_\tau.
\]

Observing that

\[
(u_\tau(t + \tau), \eta^{t+\tau}_\tau) = (u(t), \eta^t),
\]

the representation formula (3.3) for \( \eta^{t+\tau} \) gives

\[
\eta^t(s) = \eta^{t+\tau}_\tau(s) = \int_0^s u_\tau(t + \tau - y)dy = \int_0^s u(t - y)dy,
\]

whenever \( 0 < s \leq t + \tau \). From the arbitrariness of \( \tau > 0 \), we conclude that (3.1) is valid for all \( t > 0 \). If \( t \leq 0 \), the argument is similar, and left to the reader. \( \square \)

Proof of Corollary 4.8. Let \( B \subset B_{H^0}(R) \), for some \( R > 0 \). From (3.6),

\[
\text{dist}_{H^0}(S(t)B, B_1) \leq I(R)e^{-\omega_1 t},
\]

whereas Corollary 4.3 implies, in particular, that

\[
\text{dist}_{H^0}(S(t)B_1, E) \leq I(R_1)e^{-\omega_1 t}.
\]

Besides, exploiting (1.5), the continuous dependence estimate

\[
\|S(t)z_1 - S(t)z_2\|_{H^0} \leq e^{ct}\|z_1 - z_2\|_{H^0}
\]
is easily seen to hold for some $c > 0$ and every $z_1, z_2 \in \mathcal{H}^0$. Once again, we take advantage of the transitivity of the exponential attraction\,[13], and we obtain the required exponential attraction property. \hfill \Box

Similarly to the case of Theorem 4.5, Corollary 4.9 is a byproduct of Corollary 4.8 and of the $\nu$-regularity of the (exponentially) attracting set.

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