CP-violating asymmetries in $B \to \pi\pi$ and $B \to \rho\rho$ decays can help specify the weak phase $\phi_2 = \alpha$ of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. We discuss the impact of improved measurements of these processes such as will be available in the near future, finding special value in better measurement of the time-dependent CP violation parameter $S_{00}$ in $B^0 \to \pi^0\pi^0$ and $B^0 \to \rho^0\rho^0$. Reducing the errors on $B \to \rho\rho$ measurements by a factor of two can potentially lead to an error in $\phi_2 = \alpha$ just above $2^\circ$, at which level the $\rho$ width and isospin-breaking corrections must be considered.

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Precision measurements of the phases of weak charge-changing transitions, as encoded in the Cabibbo-Kobayashi-Maskawa (CKM) matrix, are a potential window to new physics if inconsistencies are uncovered. The unitarity of the CKM matrix may be expressed in terms of a triangle in the complex plane, expressing the relation

$$V^*_{ub}V_{ud} + V^*_{cb}V_{cd} + V^*_{tb}V_{td} = 0.$$  \hspace{1cm} (1)

In present fits to data the angles of the triangle add up to $\pi$ within a few degrees, as illustrated in Table 1. Small differences between the fits of Refs. [1] and [2] may be ascribed to differing inputs and statistical methods, and are indicative of present systematic uncertainties.

The weak phase $\beta = \phi_1$ is measured with fractional-degree accuracy by CP asymmetries in such processes as $B^0(\bar{B}^0) \to J/\psi K_S$. Individual measurements of the CKM phases $\alpha$ and $\gamma$ carry considerably larger uncertainties. The phase $\alpha$ can be extracted from isospin analyses of $B \to \pi\pi$ and $B \to \rho\rho$ decays [3]. For instance, the Babar collaboration [4] has used $B \to \pi\pi$ to constrain this phase to a range $71^\circ < \alpha < 109^\circ$ at a $1\sigma$ level, while Belle [5] obtained a weaker constraint. More precise determinations of $\alpha$ have been obtained from analyses of longitudinally polarized $B \to \rho\rho$, for which Babar [6] and Belle [7] find values of $\alpha$, $(92.4^{+6.9}_{-6.3})^\circ$ and $(93.7 \pm 10.6)^\circ$, respectively. A smaller uncertainty can be obtained from $B \to \rho\rho$ analyses relying on the approximate validity of SU(3) [8]. Studying $B \to \pi\rho$ decays is more complicated as a result of the non-identity of the final-state particles [9,10].
Table 1: Fits to angles of the unitarity triangle expressing the sum rule (1) as quoted by CKMfitter [1] and UTfit [2].

| Fit       | $\alpha = \phi_2 =$ | $\beta = \phi_1 =$ | $\gamma = \phi_3 =$ |
|-----------|----------------------|----------------------|----------------------|
|           | Arg($V_{tb}^*V_{td}/V_{ub}^*V_{ud}$) | Arg($-V_{cb}^*V_{td}/V_{tb}^*V_{td}$) | Arg($-V_{ub}^*V_{ud}/V_{cb}^*V_{cd}$) |
| CKMfitter | $90.4^{+2.0}_{-1.0}$ | $22.6^{+0.44}_{-0.22}$ | $67.01^{+0.88}_{-1.99}$ |
| UTfit     | $88.6 \pm 3.3$          | $22.03 \pm 0.86$         | $69.2 \pm 3.4$          |

Isospin analyses usually neglect a higher-order electroweak penguin amplitude [11] and isospin-breaking effects. Inclusion of the former amplitude decreases the value of $\alpha$ determined in $B \rightarrow \pi\pi, \rho\rho$ by a calculable amount of 1.8$^\circ$ [12][13]. Uncertainties at this same small level are introduced by isospin-breaking corrections [14][15] and by a finite $\rho$ width effect [16].

In this note we concentrate on ways to improve the determination of $\alpha$ from $B \rightarrow \pi\pi$ and $B \rightarrow \rho\rho$ decays using isospin, by identifying the major sources of statistical and systematic error. We identify one uncertainty as the large statistical error in the difference between time-integrated rates for $B^0 \rightarrow \pi^0\pi^0$ and $\bar{B}^0 \rightarrow \pi^0\pi^0$, encoded in the parameter $C_{00}$, and another in the parameter $S_{00}$ measured in time-dependent studies. The uncertainty in the branching fraction for $B^+ \rightarrow \pi^+\pi^0$ could stand some improvement as well. As has been noted [17], measurement of time-dependent CP violation in $B^0(\bar{B}^0) \rightarrow \pi^0\pi^0$ can help to reduce discrete ambiguities in the determination of $\alpha$. We find that $B \rightarrow \rho\rho$ decays are subject to the same discrete ambiguity arising in the extraction of $\alpha$ from $B \rightarrow \pi\pi$ decay. The error in $B \rightarrow \rho\rho$ decays can be reduced by improving measurements of the longitudinal branching ratios for $B^0 \rightarrow \rho^+\rho^-$ and $B^+ \rightarrow \rho^+\rho^0$, and especially by improving measurement of the parameter $S_{00}$ describing time-dependent CP violation in $B^0 \rightarrow \rho^0\rho^0$.

We begin by identifying the algebraic source of information on $\alpha$ based on known $B \rightarrow (\pi\pi, \rho\rho)$ rates and CP asymmetries. The formalism for obtaining $\alpha$ from $B \rightarrow \pi\pi$ decays was proposed in Ref. [3] and is reviewed in Ref. [18]. One may define phases of amplitudes such that

$$A(B^0 \rightarrow \pi^+\pi^-) = |T|e^{i\gamma} + |P|e^{i\delta},$$

(2)

where $|T|$ is the magnitude of a tree amplitude with weak phase $\gamma$, while $|P|$ is the magnitude of a penguin amplitude with strong phase $\delta$. The unitarity relation (1) has been used to express $V_{tb}^*V_{td} = -V_{ub}^*V_{ud} - V_{cb}^*V_{cd}$, and the resulting first term with a phase $\gamma$ incorporated into $T$. An initial $B^0$ or $\bar{B}^0$, defined by tagging the production vertex, evolves as [10][20]

$$\Gamma(B^0(t)/\bar{B}^0(t)) \sim e^{-\Gamma t}[1 \pm C_{+-} \cos \Delta mt \mp S_{+-} \sin \Delta mt]$$

(3)

with

$$C_{+-} \equiv \frac{1 - |\lambda_{\pi\pi}|^2}{1 + |\lambda_{\pi\pi}|^2}, \quad S_{+-} \equiv \frac{2 \text{Im}(\lambda_{\pi\pi})}{1 + |\lambda_{\pi\pi}|^2}, \quad \lambda_{\pi\pi} \equiv e^{-2i\beta} \frac{A(\bar{B}^0 \rightarrow \pi^+\pi^-)}{A(B^0 \rightarrow \pi^+\pi^-)},$$

(4)

The tree transition $b \rightarrow u\bar{u}d$ carries isospin 1/2 and 3/2, while the penguin transition $b \rightarrow d$ carries only isospin 1/2. The spinless two-pion state can only have isospin 0 and 2, so the
Table 2: Inputs to the determination of $\alpha$ from an isospin analysis of $B \to \pi\pi$ \cite{21,22}.

| Quantity | Value ($\times 10^{-6}$) | Quantity | Value |
|----------|--------------------------|----------|-------|
| $B_{av}(B^+ \to \pi^+\pi^0)$ | $5.11 \pm 0.37^a$ | $C_{+-}$ | $-0.31 \pm 0.05$ |
| $B_{av}(B^0 \to \pi^+\pi^0)$ | $5.12 \pm 0.19$ | $C_{00}$ | $-0.43 \pm 0.24$ |
| $B_{av}(B^0 \to \pi^0\pi^0)$ | $1.17 \pm 0.13$ | $S_{+-}$ | $-0.66 \pm 0.06$ |

$^a$Branching ratio corrected by factor \cite{22} $\tau(B^0)/\tau(B^+) = 0.929$.

Table 3: Current (preliminary) status of $B_{av}(B^0 \to \pi^0\pi^0)$.

| Source | Value ($10^{-6}$) |
|--------|-------------------|
| Belle \cite{17} | $0.90 \pm 0.12 \pm 0.10$ |
| BaBar \cite{4} | $1.83 \pm 0.21 \pm 0.13$ |
| Average | $1.165 \pm 0.132$ |

$B \to \pi\pi$ amplitudes obey the relation

$$A(B^0 \to \pi^+\pi^-)/\sqrt{2} + A(B^0 \to \pi^0\pi^0) = A(B^+ \to \pi^+\pi^0),$$

with a corresponding relation for $\bar{B}$. The amplitude $A(B^+ \to \pi^+\pi^0)$ has no penguin contribution and thus has the weak phase $\gamma$, while $A(B^- \to \pi^-\pi^0)$ has weak phase $-\gamma$. Thus if we multiply all $\bar{B}$ amplitudes by $e^{2i\gamma}$, defining them with a tilde, we can express the triangle relations as

$$A_{+-}/\sqrt{2} + A_{00} = A_{+0}, \quad \bar{A}_{+-}/\sqrt{2} + \bar{A}_{00} = \bar{A}_{-0},$$

where the triangles have the same base: $A_{+0} = \bar{A}_{-0}$. They would be identical in the absence of the penguin amplitude, and recalling that $\gamma + \beta = \pi - \alpha$, one would have $\sin(2\alpha) = S_{+-}/(1 - C_{+-}^2)^{1/2}$. The deviation from this value depends on the shapes of both triangles, governed by the separate rates of $B$ and $\bar{B}$ decays.

The measurements used in our determination of $\alpha$ are summarized in Table 2. They are taken from Ref. \cite{21} except for $B_{av}(B^0 \to \pi^0\pi^0)$, which is based on averaging a new preliminary Belle measurement \cite{17} with an earlier BaBar one (see Table 3), and $C_{00}$, which is taken from Ref. \cite{22}. The subscript “av” denotes the average for the process and its CP conjugate. We assume no CP violation in $B^+ \to \pi^+\pi^0$.

We obtain separate branching ratios for $B^0$ decays and their CP conjugates using the relations

$$B(B^0 \to f) = (1 + C_f)B_{av}(B^0 \to f), \quad B(\bar{B}^0 \to f) = (1 - C_f)B_{av}(B^0 \to f).$$

The sides of the triangles are then specified, and the angle $\theta_f$ between the $B^0(\bar{B}^0) \to f^0$ and $B^\pm \to f^\pm$ sides is calculated using the law of cosines. For $B \to \pi^+\pi^-$ this yields $\theta_{+-} = \text{Arg}(A_{+-}/A_{+0})$ for $B$ decays and $\theta_{+-} = \text{Arg}(\bar{A}_{+-}/\bar{A}_{-0})$ for $\bar{B}$ decays. The difference...
Figure 1: Isospin triangles for the decays $B \rightarrow \pi\pi$. Amplitudes for $B$ decays are those without a tilde, while amplitudes with a tilde correspond to those for $\bar{B}$ decays, multiplied by the phase $e^{2i\gamma}$ so that the bases of the two triangles coincide.

between these two angles, $\Delta \theta_{+-} = \tilde{\theta}_{+-} - \theta_{+-}$, then may be used in the determination of $\alpha$ via the relation

$$\sin(2\alpha + \Delta \theta_{+-}) = \frac{S_{+-}}{\sqrt{1 - (C_{+-})^2}}.$$  \hspace{1cm} (8)

The triangles for a typical set of decays are shown in Fig. 1. We shall also need the angles $\theta_{00}$ and $\tilde{\theta}_{00}$:

$$\theta_{00} \equiv \text{Arg}(A_{00}/A_{+0}) , \quad \tilde{\theta}_{00} \equiv \text{Arg}(\tilde{A}_{00}/\tilde{A}_{-0}) ; \quad \Delta \theta_{00} \equiv \tilde{\theta}_{00} - \theta_{00} ,$$  \hspace{1cm} (9)

determining the CP-violation parameter

$$S_{00} = \sqrt{1 - (C_{00})^2} \sin(2\alpha + \Delta \theta_{00}) ,$$  \hspace{1cm} (10)

By definition, the $+-$ and $00$ angles have opposite signs. Either triangle can be flipped about its base, giving a four-fold ambiguity in $\Delta \theta_{+-}$ and hence $\alpha$. Furthermore, each value of $\sin(2\alpha + \Delta \theta_{+-})$ corresponds to two values of $2\alpha + \Delta \theta_{+-}$. In practice $[1,2]$ all but one or two solutions for $\alpha$ are incompatible with the unitarity relation (1).

We find solutions for $\alpha$ using a Monte Carlo program which generates the six observables of Table 2 assuming they obey Gaussian distributions. One first generates the five observables $B_{+-} \equiv B(B^+ \rightarrow \pi^+\pi^0)$, $B_{+-}^{av} \equiv B_{av}(B^0 \rightarrow \pi^+\pi^-)$, $B_{00} \equiv B_{av}(B^0 \rightarrow \pi^0\pi^0)$, $C_{+-}$, and $C_{00}$. For the central values in Table 2, the $B$ triangle does not close, so the points of minimum $\chi^2 > 0$ are those in which it just barely closes, and hence lies flat with $\theta_{+-} = 0$. The contribution of the sixth observable $S_{+-}$ to $\chi^2$ depends on the orientation of the isospin triangles through the quantity $\Delta \theta_{+-}$, and the orientation giving the lowest $\chi^2$ is chosen. (As $\theta_{+-} = 0$ for the $B \rightarrow \pi\pi$ solutions with lowest $\chi^2$, only the sign of $\tilde{\theta}_{+-}$ matters.) The
Table 4: Results of a fit to parameters determining \( \alpha \) from an isospin analysis of \( B \to \pi \pi \).

| Quantity | Value \( \times 10^{-6} \) | \( \chi^2 \) | Quantity | Value | \( \chi^2 \) |
|----------|----------------|---------|----------|-------|---------|
| \( B_{av}(B^+ \to \pi^+\pi^0) \) | 5.019\(^a\) | 0.061 | \( C_{\pi\pi}^- \) | -0.303 | 0.021 |
| \( B_{av}(B^0 \to \pi^+\pi^-) \) | 5.134 | 0.006 | \( C_{00} \) | -0.316 | 0.227 |
| \( B_{av}(B^0 \to \pi^0\pi^0) \) | 1.190 | 0.023 | \( S_{\pi\pi}^+ \) | -0.66 ± 0.06\(^b\) | |
| \( \alpha \) (degrees) | 95.0, 141.1 | | \( \bar{\theta}_{\pi\pi}^- = \theta_{00} = 0 \) | | \( \chi^2_{total} = 0.338 \) |
| Other solutions | 128.9, 175.0 | \( \bar{\theta}_{\pi\pi}^- = 33.9^\circ \), \( \theta_{00} = -54.6^\circ \) | |

\(^a\) Branching ratio corrected by factor \( \tau(B^0)/\tau(B^+)=0.929 \).

\(^b\) Retained as input to determine \( \alpha \).

predicted observables are updated each time a Monte Carlo event gives a lower \( \chi^2 \) than found previously. Typically one obtains sufficient accuracy with 3 million generated events, though one must smooth out fluctuations when isospin triangles are close to flat. The values obtained are summarized in Table 4, with individual \( \chi^2 \) contributions and their sum. The flatness of the \( B \) isospin triangle in the favored fit means that the eightfold ambiguity is reduced to a fourfold one, as only the \( B \) triangle can be flipped. A fit to the observables in Table 4 results in \( \chi^2 \) values shown in Fig. 2. [Fluctuations due to limited Monte Carlo statistics have been smoothed out with piecewise parabolic fits to regions near \( \chi^2 \) minima.] Minimum values of \( \chi^2 \) = 0.338 occur at \( \alpha = (95, 128.9, 141.1, 175)^\circ \). \( \Delta \chi^2 \leq 1 \) is satisfied for \( \alpha \) in the range ([87,104],[120,150],[166,183])\(^\circ \). These results are in accord with those found by the CKMfitter Collaboration \( \cite{1} \). Note that for every solution \( \alpha \), there is another solution at \( 270^\circ - \alpha \), with both isospin triangles flipped so that \( \Delta \theta_{\pi\pi}^- \to -\Delta \theta_{\pi\pi}^- \).

In order to gauge the dependence of \( \alpha \) on the input parameters, we display their fitted values for the range \( 87^\circ \leq \alpha \leq 104^\circ \) in Fig. 3.

We note several features of the determination of \( \alpha \) using only \( B \to \pi \pi \) decays.

- The greatest dependence of \( \alpha \) on the parameters in Table 2 normalized by their experimental uncertainty, is on \( C_{00} \). Indeed, the full \( \pm 1 \sigma \) variation of \( C_{00} \) is not permitted. If \( C_{00} \) is too negative, the \( B \) isospin triangle cannot close. The requirement that the isospin triangles close was used in Ref. \( \cite{23} \) to place bounds on \( B_{av}(B^0 \to \pi^0\pi^0) \) and on \( \Delta \theta_{\pi\pi}^- \).
- The uncertainty on \( B(B^+ \to \pi^+\pi^0) \) has greater effect on \( \alpha \) than the experimental errors of either \( B^0 \) decay mode.
- Reduction of \( B_{av}(B^0 \to \pi^0\pi^0) \) reduces the allowable parameter range for \( C_{00} \), as it prevents the \( B \) isospin triangle from closing for a wider range of \( C_{00} \).

The interplay of \( C_{00} \) and \( B_{av}(B^0 \to \pi^0\pi^0) \) is keenly illustrated by the recent preliminary Belle value for the latter quantity \( \cite{17} \). The significant reduction in \( B_{av}(B^0 \to \pi^0\pi^0) \) from the previous PDG average of \( (1.91 \pm 0.22) \times 10^{-6} \) is what has prevented the \( B \) isospin triangle from closing when all other parameters are taken at their central values. As stated
Figure 2: Values of $\chi^2$ as a function of $\alpha = \phi_2$ as derived from an isospin analysis of $B \to \pi\pi$. The horizontal dashed line denotes a value of $\chi^2$ one unit above the minimum.
Figure 3: Dependence of fitted input parameters describing $B \rightarrow \pi \pi$ decays on $\alpha$ in the range $[87,104]^\circ$. Fluctuations are due to limited Monte Carlo statistics.
in Ref. [17], any remeasurement of $B_{av}(B^0 \rightarrow \pi^0\pi^0)$ must be regarded as preliminary until accompanied by a remeasurement of $C_{00}$.

If a subsequent measurement finds $C_{00} = -0.316 \pm 0.12$, corresponding to the fitted central value in Table 4 with half the present error while other inputs remain as in Table 2 the minimum $\chi^2$ is reduced to near zero, while the shape of the curve in Fig. 2 is essentially preserved. Thus, the values of $\alpha$ at the minimum, and the range for which $\Delta \chi^2 < 1$, remain unchanged.

Now take central values of all parameters in Table 4 with errors as in Table 2 except for $\delta C_{00} = 0.12$. The resulting plot of $\chi^2$ vs. $\alpha$ is shown in the left-hand panel of Fig. 4. The $\chi^2$ curves are somewhat flattened at their minima, but the values of $\alpha$ are not greatly affected. If the central value of $C_{00}$ is raised to $-0.2$, other parameters being kept fixed, the resulting plot is shown in the right-hand panel of Fig. 4. Here neither isospin triangle is flattened, so the full eight-fold degeneracy of solutions occurs. The $\chi^2$ minima are near 89.5, 102, 121.5, 134, 136, 148.5, 168, and 180.5 degrees (note the symmetry under $\alpha \leftrightarrow 270^\circ - \alpha$). The ranges allowed for $\Delta \chi^2 \leq 1$ are ([84,107],[117,153],[163,186])°.

Other parameters in Fig. 3 which show some $\alpha$ dependence are $B_{+0} \equiv B(B^+ \rightarrow \pi^+\pi^0)$ and $S_{+-}$. We have studied the effect of taking each parameter with half its present experimental error. The reduction of the error on $B_{+0}$ by a factor of two increases the overall $\chi^2$ by less than 0.1. Halving the $S_{+-}$ error reduces the $\alpha$ range to ([88,103],[120,150],[167,182])°. Finally, the effect of reducing all experimental errors in Table 2 by a factor of two leads to an allowed $\alpha$ range of ([91,100],[124,146], [170,179])°. Thus the error on $\alpha$ scales roughly as the error on all six variables, while reducing the error on any individual variable does not significantly affect the error on $\alpha$. 

Figure 4: Left: $\chi^2$ vs. $\alpha$ for central values of all parameters in Table 4 with errors as in Table 2 except for $\delta C_{00} = 0.12$. Right: Same except $C_{00} = -0.20 \pm 0.12$. 

8
We next discuss the potential impact of a measurement of the time-dependent CP-violation parameter $S_{00}$, given by Eq. (10). We may calculate $S_{00}$ for each orientation of the isospin triangles and for each pair of $\alpha$ values resulting from the value of $\sin(2\alpha + \Delta\theta_{00})$. The results are shown in Table 5 where the $B$ triangle has been taken to be flat.

Table 5: Values of $\alpha$ consistent with the measurements in Table 2 and their corresponding values of $S_{00}$. Angles are given in degrees. We are using $C_{00}$ from Table 4.

| $\Delta\theta_{00}$ | $\alpha$ | $S_{00}$ |
|---------------------|----------|---------|
| $< 0$               | 95.0°    | 0.67    |
|                     | or 141.1° | −0.70  |
| $> 0$               | 128.9°   | −0.70   |
|                     | or 175.0° | 0.67   |

Future measurements of $S_{00}$ at the Belle II $B$ factory using external photon conversion on a data sample of $50 \times 10^9 B\bar{B}$ pairs [24] may be able to favor one of the two predicted values of $S_{00}$ over the other. As an example, we compare in Fig. 5 the $\chi^2$ dependence on $\alpha$ when $S_{00} = 0.67 \pm 0.25$ (left) or $−0.70 \pm 0.25$ (right).

![Figure 5: Dependence of $\chi^2$ on $\alpha$ as extracted from an isospin analysis of $B \rightarrow \pi\pi$ in the presence of a measurement of $S_{00}$.
Left: $S_{00} = 0.67 \pm 0.25$; right: $S_{00} = −0.70 \pm 0.25$. We show only the range $80° \leq \alpha \leq 135°$ because there exist solutions with $\alpha \leftrightarrow 270° − \alpha$ and the sign of $\Delta\theta_{+-}$ changed.

A distinction between solutions with $\alpha = (95,175)^°$ and $(129,141)^°$ is possible. The allowed ranges of $\alpha$ within these solutions are reduced slightly (e.g., to the interval [90,99].]
are those which make the leading to a degeneracy of solutions. The \( \chi \) inputs analogous to the six multiplied by the fraction \( f \) of \( S \) of same central values with errors divided by two are shown in Fig. 6. The first five parameters yields the values in Table 8. As in the case of listed in Table 6. The inputs leading to the first three entries are summarized in Table 7.

...experimental uncertainty, originates in \( f \) consistent with other CKM constraints, the second solution near 1.80

Table 6: Inputs to the determination of \( \alpha \) from an isospin analysis of \( B \to \rho \rho \). Observed branching fractions are multiplied by observed longitudinal \( \rho \) polarization fractions \[21][22].

| Quantity                      | Value \((\times 10^{-6})\) | Quantity | Value |
|-------------------------------|-----------------------------|----------|-------|
| \( f_L B_{\text{av}}(B^+ \to \rho^+ \rho^0) \) | 21.18 \( \pm \) 1.71 \(^a\) | \( C_{+-} \) | 0.00 \( \pm \) 0.09 |
| \( f_L B_{\text{av}}(B^0 \to \rho^+ \rho^-) \) | 27.42 \( \pm \) 1.95 | \( C_{00} \) | 0.20 \( \pm \) 0.85 |
| \( f_L B_{\text{av}}(B^0 \to \rho^0 \rho^0) \) | 0.67 \( \pm \) 0.12 \(^b\) | \( S_{+-} \) | -0.14 \( \pm \) 0.13 |

\(^a\)Branching ratio corrected by factor \[22\] \( \tau(B^0)/\tau(B^+) = 0.929 \).

\(^b\)Averaged values of branching ratio and longitudinal fraction using also Ref. \[25\].

Table 7: Individual measurements used to calculate longitudinal branching fractions (first three entries of Table 6). We denote \( B^{ij} \equiv B(B \to \rho^i \rho^j) \) given in units of \( 10^{-6} \), \( f_L^{ij} \equiv f_L(B \to \rho^i \rho^j) \).

| Quantity | Belle \[7][27][28] | Babar \[6][26][29] | LHCb \[25\] | Average |
|----------|---------------------|---------------------|--------------|---------|
| \( B^{+0} \) | 31.7 \( \pm \) 7.1\(^{+2.3}_{-6.7} \) | 23.7 \( \pm \) 1.4\(\pm 1.4 \) | – | 24.0\( \pm 1.9 \) |
| \( f_L^{+0} \) | 0.95 \( \pm \) 0.11 \( \pm 0.02 \) | 0.950 \( \pm \) 0.015 \( \pm 0.006 \) | – | 0.950\( \pm 0.016 \) |
| \( B^{+-} \) | 28.3 \( \pm \) 1.5 \( \pm 1.5 \) | 25.5 \( \pm 2.1^{+3.6}_{-3.9} \) | – | 27.7\( \pm 1.9 \) |
| \( f_L^{+-} \) | 0.988 \( \pm \) 0.012 \( \pm 0.023 \) | 0.992\( \pm 0.024^{+0.026}_{-0.013} \) | – | 0.990\( \pm 0.019 \) |
| \( B^{00} \) | 1.02 \( \pm \) 0.30 \( \pm 0.15 \) | 0.92 \( \pm \) 0.32 \( \pm 0.14 \) | 0.94\( \pm 0.17 \pm 0.09 \pm 0.06 \) | 0.95\( \pm 0.15 \) |
| \( f_L^{00} \) | 0.21\( ^{+0.18}_{-0.22} \pm 0.15 \) | 0.75\( ^{+0.11}_{-0.14} \pm 0.05 \) | 0.745\( ^{+0.048}_{-0.058} \pm 0.034 \) | 0.71\( \pm 0.06 \) |

degrees). There still remains a two-fold ambiguity in \( \alpha \). Anticipating a value of \( \alpha \) near 90\( ^\circ \) consistent with other CKM constraints, the second solution near 180\( ^\circ \) with the same value of \( S_{00} \) can be then easily excluded.

We now perform similar analyses for \( B \to \rho \rho \) decays. We use branching fractions multiplied by the fraction \( f_L \) of decays leading to longitudinal \( \rho \) polarization. We first examine inputs analogous to the six \( B \to \pi \pi \) observables: three \( B \)'s, two \( C \)'s, and \( S_{+-} \). They are listed in Table 6. The inputs leading to the first three entries are summarized in Table 7.

Here, both the \( B \) and \( \bar{B} \) triangles fail to close for the listed central values. A \( \chi^2 \) fit to the first five parameters yields the values in Table 8. As in the case of \( B \to \pi \pi \), these parameters are those which make the \( B \) triangle exactly flat. In this case the \( \bar{B} \) triangle is also flat, leading to a degeneracy of solutions. The \( \chi^2 \) distributions for nominal variables and for the same central values with errors divided by two are shown in Fig. 6.

The greatest sensitivity of \( \alpha \) to the measurements in Table 6 normalized by their experimental uncertainty, originates in \( f_L B_{\text{av}}(B^+ \to \rho^+ \rho^0) \) and \( f_L B_{\text{av}}(B^0 \to \rho^+ \rho^-) \). More precise information on branching fractions would be helpful. Significant improvement is expected in thirteen-year-old Belle results for \( B^+ \to \rho^+ \rho^0 \) \[27\], based on only about ten percent of the final Belle \( \Upsilon(4S) \) sample.
Table 8: Results of a fit to the six parameters in Table 6

| Quantity                      | Value ($\times 10^{-6}$) | $\chi^2$ | Quantity | Value | $\chi^2$ |
|-------------------------------|---------------------------|-----------|----------|-------|----------|
| $f_L B_{av}(B^+ \rightarrow \rho^+ \rho^0)$ | 20.73$^a$                  | 0.070     | $C_{+-}$ | -0.008 | 0.008    |
| $f_L B_{av}(B^0 \rightarrow \rho^+ \rho^-)$ | 27.78                     | 0.034     | $C_{00}$ | 0.036  | 0.037    |
| $f_L B_{av}(B^0 \rightarrow \rho^- \rho^0)$ | 0.68                      | 0.011     | $S_{+-}$ | $-0.14 \pm 0.13^b$ |          |
| $\alpha$ (degrees)           | 94, 176                   |           | $\chi^2_{total}$ | 0.160 |          |

$^a$ Branching ratio corrected by factor [22] $\tau(B^0)/\tau(B^+) = 0.929$.

$^b$ Retained as input to determine $\alpha$.

Figure 6: Values of $\chi^2$ as a function of $\alpha = \phi_2$ from an isospin analysis of $B \rightarrow \rho \rho$ based on the six parameters of Table 6. Left: present experimental errors, with $\Delta \chi^2 \leq 1$ corresponding to $\alpha = (94 \pm 8)^\circ$ or $(176 \pm 8)^\circ$. Right: present errors divided by two, leading to $\alpha = (94 \pm 5)^\circ$ or $(176 \pm 5)^\circ$. 
Figure 7: Isospin triangle fits to $B \to \rho\rho$ observables in Table 6 when the measurement (11) is included. (a) nominal experimental errors. (b) Same as (a) but with present error on $S_{00}$ divided by two. (c) Same as (a) but with all experimental errors divided by two.

An additional piece of experimental information is available in the case of $B \to \rho\rho$. The BaBar Collaboration [26] has measured

$$S_{00} = 0.3 \pm 0.7 \pm 0.2 = 0.3 \pm 0.73.$$  (11)

Despite its large uncertainty, this measurement has a significant effect on $\alpha$. There are now two quantities, $S_{+-}$ and $S_{00}$, which depend on $\alpha$. With $S_{+-}$ alone, a $\chi^2$ fit is governed solely by the geometry of the isospin triangles. When both $S_{+-}$ and $S_{00}$ are specified, some tension can arise between their favored values of $\alpha$, and the geometry of the isospin triangles can be adjusted to minimize this tension.

We show in Fig. 7(a) the effect of adding the observable (11), related to $\alpha$ through Eq. (10), to those in Table 6 (We show only the solution consistent with other observables.) The value of $\alpha$ corresponding to $\Delta \chi^2 \leq 1$ is now $\alpha = (92.0^{+4.7}_{-5.0})^\circ$. In Fig. 7(b) we show the $\chi^2$ distribution when the error on $S_{00}$ is divided by two, leading to $\alpha = (91.7^{+3.8}_{-3.7})^\circ$. We also checked that a substantial reduction of the error on $f_L B_{av}(B^0 \to \rho^0 \rho^0)$, potentially achievable at the LHCb upgrade, would have an insignificant effect on improving the precision in $\alpha$. Finally, in Fig. 7(c) we show the $\chi^2$ distribution when all errors are divided by two, in which case one finds $\Delta \chi^2 \leq 1$ for $\alpha = (92.0 \pm 2.5)^\circ$.

We have discussed ways to narrow the uncertainty in the CKM phase $\alpha = \phi_2$ as derived from isospin analyses of $B \to \pi\pi$ and $B \to \rho\rho$. No single variable in $B \to \pi\pi$ dominates the present error of $9^\circ$ in $\alpha$. Reduction of that error by a factor of two is achieved if the errors in all six inputs of Table 2 are cut in half. The time-dependent CP violation parameter $S_{00}$ will help to distinguish solutions near $\alpha = 129^\circ$ and $141^\circ$, yielding $S_{00} \simeq -0.70$, from those near $95^\circ$ and $175^\circ$, yielding $S_{00} \simeq 0.67$.

For the $B \to \rho\rho$ analysis, improving measurements of longitudinal branching fractions of $B^+ \to \rho^+ \rho^0$ and $B^0 \to \rho^+ \rho^-$ would reduce the $5^\circ$ current error in $\alpha$ as determined in these processes. The measurement of $S_{00}$ in $B^0 \to \rho^0 \rho^0$ with an error reduced by a factor of two (or more) also would have a significant effect on the accuracy of determining $\alpha$. However, reduction by a factor of two of all experimental errors (including that of $S_{00}$) would reduce
the error on \( \alpha \) to 2.5\(^\circ\), a point at which one should begin to take into account the \( \rho \) width and isospin-breaking corrections.

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