Homology of infinite loop spaces

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Contents

1 Recollection on the Segal machine. ........................................ 3
2 Homology of Γ-spaces. ..................................................... 6
3 Stabilization. .................................................................. 9

Introduction.

A spectrum \( X \) is a sequence of pointed topological spaces \( X_0, X_1, \ldots \) and homotopy equivalences \( X_n \cong \Omega X_{n+1}, n \geq 0 \) (we tacitly assume that all the topological spaces in consideration are nice enough, e.g. having homotopy type of a CW complex). A spectrum \( X \) is connected if all its components \( X_n, n \geq 0 \) are connected. Homology \( H_*(X_\ast, \mathbb{Z}) \) of a spectrum \( X \), with coefficients in a ring \( R \) is given by

\[
H_*(X_\ast, R) = \lim_{\to} \tilde{H}_{*+n}(X_n, R),
\]

where \( \tilde{H}_*(-, R) \) denotes reduced homology of a pointed topological space, and the limit is taken with respect to maps \( \Sigma X_n \to X_{n+1} \) adjoint to the structure maps \( X_n \to \Omega X_{n+1} \). For any \( n, i \geq 0 \), we then have a natural map

\[
\tilde{H}_{i+n}(X_n, R) \to H_i(X_\ast, R).
\]

If the spectrum \( X \) is connected, this map is an isomorphism for \( i < n \).

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1
The forgetful functor sending a spectrum $X_q$ to its component $X_0$ is conservative on the category of connected spectra, so that up to a homotopy equivalence, a connected spectrum $X_q$ can be reconstructed from a pointed topological space $X_0$ equipped with an additional structure. This structure is usually called an “infinite loop space structure”, and it can be described in several ultimately equivalent ways, mostly discovered in the early 1970es and sometimes called “machines” (see [A] for an all-time great overview of the subject). One of these machines is that of G. Segal [S], where a connected spectrum is constructed from a so-called special $\Gamma$-space. This turned out to be very useful, since e.g. in algebraic $K$-theory, the relevant $\Gamma$-space often can be obtained almost for free.

The goal of this short note is to give a simple expression for the homology of a connected spectrum $X_q$ in terms of the associated special $\Gamma$-space. We state right away that the expression is not new, and it is due to T. Pirashvili – namely, it can be deduced rather directly from [P2, Proposition 2.2], and for Eilenberg-Maclane spectra, the results goes back at least to [JP] (see the end of Section 3 for more details). All the basic ideas behind the proof are also definitely due to Pirashvili. However, the result itself is never stated explicitly in the general corpus of Pirashvili’s work, and while well-known to experts, is not universally known. So, a short and self-contained independent proof might be useful. This is what the present paper aims to provide.

The paper consists of three parts: in Section 1, we recall the details of the Segal machine in a convenient form, in Section 2, we build a homological counterpart of the theory, and finally in Section 3, we state and prove our results, and sketch an alternative approach using [P2, Proposition 2.2].

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A note on notation. For the convenience of the reader, here is a brief comparison between our notation and that of Pirashvili. In [P2] and elsewhere, our $\Gamma_+$ is $\Gamma$. Our functor $T$ is denoted $L$ in [PB]. Our $t$ is $t$. 

2
1 Recollection on the Segal machine.

We start by briefly recalling Segal’s approach to infinite loop spaces and rephrasing it in a language that suits our goal.

Denote by $\Gamma_+$ the category of finite pointed sets. For any integer $n \geq 0$, denote by $[n]_+ \in \Gamma_+$ the set with $n$ unmarked elements (plus one distinguished element $o \in [n]_+$). Alternatively, $\Gamma_+$ is equivalent to the category $\Gamma'_+$ of finite sets and partially defined maps between them – that is, a map from $S_1$ to $S_2$ is given by a diagram

$$
\begin{array}{ccc}
S_1 & \xrightarrow{\iota} & S \\
\downarrow & & \downarrow \\
S_2 \\
\end{array}
$$

with injective $\iota$. The equivalence $\gamma : \Gamma'_+ \to \Gamma_+$ adds a distinguished element $o$ to a set $S \in \Gamma'_+$, and for any $f : S_1 \to S_2$ represented by a diagram (1.1), $\gamma(f)$ sends $S_1 \setminus \iota(S)$ to this added distinguished element in $S_2$.

For any injective map $\iota : S_1 \to S_2$ of finite sets, let $\iota^# : S_2 \to S_1$ be the map in $\Gamma'_+$ represented by the diagram

$$
\begin{array}{ccc}
S_2 & \xleftarrow{\iota} & S_1 \\
\downarrow & & \downarrow \\
S_1 \\
\end{array}
$$

Definition 1.1. (i) A $\Gamma$-space is a functor $X : \Gamma_+ \to \text{Top}_+$ from $\Gamma_+$ to the category of compactly generated pointed topological spaces. Say that a $\Gamma$-space $X$ is normalized if $X([0]_+)$ is the one-point set $\text{pt}$.

(ii) A $\Gamma$-space is special if it is normalized, and for any $S_1, S_2 \in \Gamma'_+$ with the natural embeddings $\iota_1 : S_1 \to S_1 \coprod S_2$, $\iota_2 : S_2 \to S_1 \coprod S_2$, the map

$$
X(\gamma(S_1 \coprod S_2)) \xrightarrow{X(\gamma(\iota_1^#)) \times X(\gamma(\iota_2^#))} X(\gamma(S_1)) \times X(\gamma(S_2))
$$

is a homotopy equivalence.

Remark 1.2. Sometimes it is convenient to relax the normalization condition on special $\Gamma$-spaces by only requiring that $X([0]_+)$ is contractible. However, the stronger condition is harmless: replacing $X([n]_+)$, $n \geq 1$, with the homotopy fiber of the map $X([n]_+) \to X([0]_+)$ corresponding to the unique map $[n]_+ \to [0]_+$, one can always achieve $X([0]_+) = \text{pt}$.

The category of $\Gamma$-spaces is denoted $\Gamma_+ \text{ Top}_+$. For any two $\Gamma$-spaces $X_1, X_2 \in \Gamma_+ \text{ Top}_+$, we define $X_1 \vee X_2$, $X_1 \times X_2$ and $X_1 \wedge X_2$ pointwise.

Let $\widetilde{\Gamma_+} \text{ Top}_+ \subset \Gamma_+ \text{ Top}_+$ be the full subcategory spanned by normalized $\Gamma$-spaces. Then the forgetful functor

$$
U : \widetilde{\Gamma_+} \text{ Top}_+ \to \text{ Top}_+, \quad U(X) = X([1]_+)
$$

3
has a left-adjoint
\[ T : \text{Top}_+ \to \widetilde\Gamma_+ \text{Top}_+. \]

Explicitly, for a pointed topological space \( X \), the \( \Gamma \)-space \( T(X) \) is given by
\[
T(X)([n]_+) = \bigvee_{s \in [n]_+ \setminus \{o\}} X = X \wedge [n]_+. \tag{1.2}
\]

The adjunction map \( \text{id} \to U \circ T \) is an isomorphism, so that \( T \) is a full embedding, and the adjunction map \( \tau : T \circ U \to \text{id} \) can be described as follows: for any \( X \in \Gamma_+ \text{Top}_+ \), \( [n]_+ \in \Gamma_+ \), we have
\[
\tau = \bigvee_{s \in [n]_+ \setminus \{o\}} X(i_s) : T(U(X))([n]_+) = \bigvee_{s \in [n]_+ \setminus \{o\}} X([1]_+) \to X([n]_+),
\]
where \( i_s : [1]_+ \to [n]_+ \) is the embedding onto the subset \( \{s, o\} \subset [n]_+ \).

Let \( \Delta \) be, as usual, the category of finite non-empty totally ordered sets, with \( [n] \in \Delta \) denoting the set of integers from 0 to \( n \), and let \( S : \Delta^\text{opp} \to \text{Sets} \) be the standard simplicial circle – that is, the simplicial set obtained by gluing together the two ends of the standard 1-simplex. The glued ends give a natural distinguished element in \( S([n]), [n] \in \Delta \), so that \( S \) is actually a pointed simplicial set. Moreover, \( S([n]) \cong [n]_+ \) is a finite set for any \( [n] \in \Delta^\text{opp} \), so that \( S \) can be interpreted as a functor \( \sigma : \Delta^\text{opp} \to \Gamma_+ \).

Recall that for any simplicial topological space \( X : \Delta^\text{opp} \to \text{Top}_+ \), we have its geometric realization \( \text{Real}(X) \in \text{Top}_+ \), and this construction is functorial in \( X \) and compatible with products and colimits. For any simplicial abelian group \( M \), denote by \( N_\idot(M) \) the corresponding standard complex with terms \( N_n(M) = M([n]) \) and differential given by the alternating sum of the face maps. Then for any ring \( R \), the reduced singular chain complex \( \widetilde{C}_\idot(\text{Real}(X), R) \) is naturally quasiisomorphic to the total complex of a bicomplex
\[
N_\idot(\widetilde{C}_\idot(\text{Real}(X), R)). \tag{1.3}
\]

The geometric realization \( \text{Real}(S) \) of the simplicial circle is homeomorphic to the 1-sphere \( S^1 \). For any simplicial pointed topological space \( X : \Delta^\text{opp} \to \text{Top}_+ \), the realization
\[
\text{Real}(X \wedge S)
\]
of the pointwise smash product \( X \wedge S \) is homeomorphic to the suspension \( \Sigma X = S^1 \wedge X \).
Definition 1.3. Geometric realization $\text{Real}(X)$ of a $\Gamma$-space $X$ is given by

$$\text{Real}(X) = \text{Real}(\sigma^* X).$$

In particular, for any $X \in \text{Top}_+$, we have $\sigma^* T(X) \cong S \wedge X$, so that $\text{Real}(T(X)) \cong \Sigma X$.

Now consider the product $\Gamma_+^2 = \Gamma_+ \times \Gamma_+$. Let $\pi_1, \pi_2 : \Gamma_+^2 \to \Gamma_+$ be the projections onto the first resp. second factor, and let $\beta : \Gamma_+^2 \to \Gamma_+$ be the smash-product functor, $\beta([n_1]_+ \times [n_2]_+) = [n_1]_+ \wedge [n_2]_+ \cong [n_1 n_2]_+$. Denote by $\Gamma_+^2 \text{Top}_+$ the category of functors from $\Gamma_+^2$ to $\text{Top}_+$, and let $U_1, \text{Real}_1 : \Gamma_+^2 \text{Top}_+ \to \Gamma_+ \text{Top}_+$, $T_1 : \Gamma_+ \text{Top}_+ \to \Gamma_+^2 \text{Top}_+$ be the functors obtained by applying $U$ resp. $\text{Real}$ resp. $T$ fiberwise over fibers of the projection $\pi_1 : \Gamma_+^2 \to \Gamma_+$ (in particular, $U_1 \cong i_1^*$, where $i_1 : \Gamma_+ \to \Gamma_+^2$ is the embedding onto $\Gamma_+ \times [1]_+ \subset \Gamma_+^2$). For any normalized $\Gamma$-space $X : \Gamma_+ \to \text{Top}_+$, let

$$B X = \text{Real}_1(\beta^* X),$$

and let

$$\Sigma(X) = \text{Real}_1(T_1(X)).$$

Note that for any $[n]_+$, we have

$$\Sigma(X)([n]_+) = \text{Real}_1(T_1(X))(n)_+ = \text{Real}(T(X([n]_+))) \cong \Sigma X([n]_+),$$

and in particular, $U(\Sigma(X)) \cong \Sigma U(X)$. Moreover, since $\beta \circ i_1 \cong \text{id}$, we have $U_1(\beta^* X) \cong X$, so that we obtain a natural adjunction map

(1.4) $$\tau : T_1(X) \cong T_1(U_1(\beta^* X)) \to \beta^* X$$

and its realization

(1.5) $$\rho_X = \text{Real}_1(\tau) : \Sigma(X) \to B X.$$ 

Segal, then, proved the following.

Proposition 1.4. Assume given a special $\Gamma$-space $X$. Then

(i) the $\Gamma$-space $B X$ is also special, and
(ii) the natural map

\[ U(X) \to \Omega U(BX) \]

adjoint to the map

\[ U(\rho_X) : \Sigma U(X) \cong U(\Sigma(X)) \to U(BX) \]

is a homotopy equivalence. \qed

By (i), the functor \( B \) can be iterated, so that every special \( \Gamma \)-space \( X \) gives rise to a sequence of special \( \Gamma \)-spaces \( B^nX \); by (ii), the sequence \( U(B^nX) \) with the maps \( U(\rho_{B^nX}) \) then naturally forms a spectrum. We will denote this spectrum by \( EX \).

2 Homology of\( \Gamma \)-spaces.

Fix once and for all a commutative ring \( R \), and consider the category \( \text{Fun}(\Gamma_+, R) \) of functors from \( \Gamma_+ \) to the category \( R\)-mod of \( R \)-modules. This is an abelian category with enough injectives and projectives. We equip it with pointwise tensor product, and we denote by \( D(\Gamma_+, R) \) its derived category. An obvious set of projective generators is given by representable functors \( T_n \),

\[ T_n([m]_+) = R(\Gamma_+([n]_+, [m]_+)), \]

since by Yoneda, we have \( \text{Hom}(T_n, E) \cong E([n]_+) \) for any \( E \in \text{Fun}(\Gamma_+, R) \).

Let \( T \in \text{Fun}(\Gamma_+, R) \) be the functor given by

\[ T([n]_+) = \bigoplus_{s \in [n]_+ \setminus \{o\}} R, \]

that is, the reduced span functor. We have an obvious direct sum decomposition \( T_1 \cong T_0 \oplus T \).

Consider the functor \( T : R\text{-mod} \to \text{Fun}(\Gamma_+, R) \) given by

\[ T(M) = T \otimes_R M \]

for any \( R \)-module \( M \). This is consistent with previous notation, in the sense that for any \( X \in \text{Top}_+ \) with reduced singular chain complex \( \tilde{C}_*(X, R) \), (1.2) immediately gives a canonical isomorphism

\[ T(\tilde{C}_*(X, R)) \cong \tilde{C}_*(T(X), R). \]

The functor \( T : R\text{-mod} \to \text{Fun}(\Gamma_+, R) \) is exact, and it has a right and a left-adjoint \( R, Q : \text{Fun}(\Gamma_+, R) \to R\text{-mod} \).
Lemma 2.1. For any $E \in \text{Fun}(\Gamma_+, R)$, we have a canonical decomposition $E([1]+) \cong R(E) \oplus E([0]+)$. The functor $R$ is exact, the functor $T$ is fully faithful, and its extension $T : \mathcal{D}(R\text{-mod}) \to \mathcal{D}(\Gamma_+, R)$ is also fully faithful.

Proof. The decomposition is induced by the decomposition $T_1 \cong T \oplus T_0$. Exactness of $R$ follows; to see that the embedding $T$ is fully faithful, note that $R \circ T \cong \text{id}$ both on the abelian and on the derived category level. □

Definition 2.2. The homology $H^\Gamma_*(E)$ of a functor $E \in \text{Fun}(\Gamma_+, R)$ is given by
$$H^\Gamma_*(E) = L^*\mathcal{Q}(E),$$
the derived functors of the functor $\mathcal{Q}$ left-adjoint to the full embedding $T : R\text{-mod} \to \text{Fun}(\Gamma_+, R)$.

Explicitly, homology can be expressed as
$$H^\Gamma_*(E) = \text{Tor}^\Gamma_*(t, E),$$
where $t : \Gamma_+^{\text{opp}} \to R\text{-mod}$ is given by $t([n]+) = \text{Hom}_R(T([n]+), R)$, and $\text{Tor}_*$ is taken over the small category $\Gamma_+$ in the usual way, see e.g. [K, Secton 1.1]. To compute it, it suffices to find a projective resolution of the functor $t$. One very elegant way to do it was discovered by Pirashvili and Jibladze, and it leads to the so-called cube construction of MacLane (see [LP], or a slightly less computational exposition in [K, Section 3.3]). Whatever resolution $Q$ one fixes, one immediately obtains a canonical way to generalize homology to complexes: for any complex $E_*$ in $\text{Fun}(\Gamma_+, R)$, we obtain a complex
$$Q_*(E_*) = Q_* \otimes^{\Gamma_+} E_*.$$
(2.2)
of $R$-modules whose homology we denote by $H^\Gamma_*(E_*)$. If the complex $E_* = E$ is concentrated in degree 0, we have $H^\Gamma_*(E_*) \cong H^\Gamma_*(E)$

The following Lemma is the crucial result of the theory.

Lemma 2.3. Assume given $E_1, E_2 \in \text{Fun}(\Gamma_+, R)$ such that $E_1([0]+) = E_2([0]+) = 0$. Then $H^\Gamma_*(E_1 \otimes_R E_2) = 0$.

Proof. This is [PB, Lemma 2]; I give a proof for the convenience of the reader.

Consider the product $\Gamma_+ \times \Gamma_+$, let $\pi_1, \pi_2 : \Gamma_+ \times \Gamma_+ \to \Gamma_+$ be the projections onto the first resp. second factor, and let $\iota_1, \iota_2 : \Gamma_+ \to \Gamma_+ \times \Gamma_+$ be the embeddings sending $[n]+$ to $[n]+ \times [0]+$ resp. $[0]+ \times [n]+$. Moreover, let
m : Γ_+ × Γ_+ → Γ_+ be the coproduct functor, and let δ : Γ_+ → Γ_+ × Γ_+ be the diagonal embedding. Then π_i is right-adjoint to i_i, i = 1, 2, and m is right-adjoint to δ. We obviously have

\[ m^* T \cong \pi_1^* T \oplus \pi_2^* T, \]

and this decomposition induces an isomorphism

\[ m^* \circ T \cong (\pi_1^* \circ T) \oplus (\pi_2^* \circ T). \]

By adjunction, we obtain a functorial isomorphism

\[ L^* Q(\delta^* E) \cong L^* Q(\iota_1^* E) \oplus L^* Q(\iota_2^* E) \]

for any functor \( E : \Gamma_+ \times \Gamma_+ \to R\text{-mod} \). Take \( E = E_1 \boxtimes_R E_2 \), and note that by assumption, \( \iota_1^* E = \iota_2^* E = 0 \), while \( \delta^* E \cong E_1 \otimes_R E_2 \). \( \square \)

**Definition 2.4.** The homology \( H^\Gamma_*(X, R) \) of a \( \Gamma \)-space \( X \) is given by

\[ H^\Gamma_*(X, R) = H^\Gamma_*(\widetilde{C}_*(X, R)), \]

where \( \widetilde{C}_*(X, R) \) is a complex in \( \text{Fun}(\Gamma_+, R) \) obtained by taking pointwise the reduced singular chain complex \( \widetilde{C}_*(-, R) \).

**Lemma 2.5.** For any \( X \in \text{Top}_+ \), we have a canonical isomorphism

\[ \widetilde{H}_*(X, R) \cong H^\Gamma_*(T(X), R). \]

**Proof.** By virtue of the quasiisomorphism (2.1), this immediately follows from the last claim of Lemma 2.1: we have

\[ \text{Tor}^\Gamma_+(t, T) \cong \text{Hom}_R(\text{Ext}^\Gamma_+(T, T), R) \cong \text{Hom}_R(\text{Ext}^\Gamma_*(R, R)), \]

and the right-hand side is \( R \) in degree 0 and 0 in higher degrees, so that for any complex \( C_* \) of \( R \)-modules, the groups

\[ H^\Gamma_*(T(C_*)) \cong \text{Tor}^\Gamma_+(t, C_* \otimes T) \]

coincide with the homology groups of the complex \( C_* \) itself. \( \square \)

Lemma 2.3 has the following implication for the homology of \( \Gamma \)-spaces. For any \( n \geq 0 \), let \( \mu_n : \Gamma_+ \to \Gamma_+ \) be the functor given by \( \mu_n([m]_+) = [n]_+ \wedge [m]_+ \). Assume given two pointed finite sets \([n]_+, [n']_+ \in \Gamma_+ \), identify
and consider the natural maps \([n]_+ ∨ [n']_+ \cong [n+n']_+\), and \([n+n']_+ ∨ [n']_+ \cong [n+n']_+\). These maps then induce maps
\[
\iota : \mu_n \to \mu_{n+n'}, \quad \iota' : \mu_{n'} \to \mu_{n+n'},
\]
and for any \(\Gamma\)-space \(X\), we obtain natural maps
\[
\iota : \mu_n^* X \to \mu_{n+n'}^* X, \quad \iota' : \mu_{n'}^* X \to \mu_{n+n'}^* X
\]
and
\[
\iota^\# : \mu_{n+n'}^* X \to \mu_n^* X, \quad \iota'^\# : \mu_{n+n'}^* X \to \mu_{n'}^* X.
\]

**Corollary 2.6.** Assume that the \(\Gamma\)-space \(X\) is special, Then the natural map
\[
\iota ∨ \iota' : \mu_n^* X ∨ \mu_{n'}^* X \to \mu_{n+n'}^* X
\]
of \(\Gamma\)-spaces induces an isomorphism of homology \(H^\Gamma_\ast (-, R)\).

**Proof.** For any two pointed topological spaces \(X_1, X_2\), we have a cofiber sequence
\[
X_1 ∨ X_2 \to X_1 \times X_2 \to X_1 \wedge X_2.
\]
Since \(X\) is special, the natural map
\[
\mu_{n+n'}^* X \xrightarrow{\iota^\# \times \iota'^\#} \mu_n^* X \times \mu_{n'}^* X
\]
is a pointwise homotopy equivalence. Therefore the sequence
\[
\mu_n^* X ∨ \mu_{n'}^* X \xrightarrow{\iota ∨ \iota'} \mu_{n+n'}^* X \xrightarrow{\iota^\# \wedge \iota'^\#} \mu_n^* X \wedge \mu_{n'}^* X
\]
is a pointwise cofiber sequence, and it suffices to prove that
\[
H^\Gamma_\ast (\mu_n^* X \wedge \mu_{n'}^* X, R) = 0.
\]
This immediately follows from Lemma 2.3 and the Künneth formula. \(\square\)

### 3 Stabilization.

We can now formulate and prove the main result of the paper. For any special \(\Gamma\)-space \(X\), let
\[
\tau_X : T(U(X)) \to X
\]
be the adjunction map, and let \(\rho_X : \Sigma(X) \to BX\) be as in (1.5).
Lemma 3.1. For any special $\Gamma$-space $X$, the diagram

$$
\begin{array}{ccc}
\Sigma(T(U(X))) & \cong & T(U(\Sigma(X))) \\
\downarrow_{T(U(\rho_X))} & & \downarrow_{\rho_X} \\
T(U(BX)) & \longrightarrow & \Sigma(X)
\end{array}
\quad \begin{array}{cc}
\Sigma(\tau X) & \longrightarrow \\
\downarrow & \downarrow \\
\tau BX & \longrightarrow
\end{array}
BX
$$

is commutative.

Proof. By (1.2), we have

$$
T_1(T(Y))(\lfloor n_1 \rfloor_+ \times \lfloor n_2 \rfloor_+) \cong \lfloor n_2 \rfloor_+ \wedge T(Y)(\lfloor n_1 \rfloor_+) \cong \lfloor n_2 \rfloor_+ \wedge \lfloor n_1 \rfloor_+ \wedge Y \\
\cong \lfloor n_1 n_2 \rfloor_+ \wedge Y \cong \beta^*T(Y)(\lfloor n_1 \rfloor_+ \times \lfloor n_2 \rfloor_+)
$$

for any $Y \in \text{Top}_+$, $[n_1]_+, [n_2]_+ \in \Gamma_+$, so that $T_1(T(Y)) \cong \beta^*T(Y)$. Taking $Y = U(X)$, we obtain a natural commutative diagram

$$
\begin{array}{ccc}
T_1(T(U(X))) & \longrightarrow & T_1(X) \\
\downarrow & & \downarrow \\
\beta^*T(U(X)) & \longrightarrow & \beta^*X.
\end{array}
$$

Applying Real$_1$, we get the claim. \hfill \Box

Taking homology $H^\Gamma_U(-, R)$ and using Lemma 2.5, we obtain a commutative diagram

$$
\begin{array}{ccc}
\tilde{H}_*(U(X), R) & \longrightarrow & H^\Gamma_{U}(X, R) \\
\downarrow & & \downarrow \\
\tilde{H}_{*+1}(U(BX), R) & \longrightarrow & H^\Gamma_{*+1}(BX, R),
\end{array}
$$

and passing to the limit, we get a natural map

$$
(3.1) \quad H_*(EX_*, R) \to \lim_{\leftarrow n} H^\Gamma_{*+n}(U(B^n X), R).
$$

Here is, then, our main result.

Theorem 3.2. Assume given a special $\Gamma$-space $X$, and let $EX_*$ be the corresponding spectrum. Then the natural map (3.1) factors through an isomorphism

$$
H_*(EX_*, R) \cong H^\Gamma_{*}(X, R),
$$

where the right-hand side is as in Definition 2.4.
The proof is a combination of the following two results.

**Lemma 3.3.** Assume given a special $\Gamma$-space $X$. Then the map $\rho_X$ of (1.5) induces an isomorphism

$$H^\Gamma_i(\Sigma(X), R) \cong H^\Gamma_i(BX, R).$$

*Proof.* Combining (1.3) and (2.2), we see that for any $X \in \Gamma^2_+$, the homology $H^\Gamma_i(\text{Real}_1(X), R)$ can be computed by the total complex of the triple complex $Q_*(N_*(C_*(X), R))$. This gives rise to a convergent spectral sequence

$$H^\Gamma_i(i^*n T_1(X), R) \Rightarrow H^\Gamma_i(i^*\beta^* X, R).$$

induced by the map $\tau$ of (1.4) is an isomorphism. By (1.2), we have

$$i^*n T_1(X) \cong [n]_+ \wedge X \cong \bigvee_{s \in [n]_+ \setminus \{o\}} X$$

and by definition,

$$i^*_n \beta^* X \cong \mu^*_n X.$$

so that the statement immediately follows by induction on $n$ from Corollary 2.6. \hfill $\square$

**Lemma 3.4.** Assume given a special $\Gamma$-space $X$, and assume that $U(X)$ is $n$-connected for some $n \geq 1$. Then the natural map

$$\tilde{H}^\Gamma_i(U(X), R) \cong H^\Gamma_i(T(U(X)), R) \to H^\Gamma_i(X, R)$$

induces by the map $\tau_X$ is an isomorphism for $i < 2n$.

*Proof.* Since $U(X)$ is $n$-connected, $H_0(U(X), R) \cong R$ and $H_i(U(X), R) = 0$ when $0 < i < n$. Then by Definition 1.1 (ii), $X([n]_+)$ is homotopy-equivalent to $X([1]_+)^n = U(X)^n$ for any $n \geq 1$, and then the Künneth formula immediately implies that the map

$$\tilde{C}_*(T(U(X)), R) \to \tilde{C}_*(X, R)$$

11
induced by $\tau_X$ is a quasiisomorphism in homological degrees less than $2n$. Applying $L^*Q$, we get the claim. □

Proof of Theorem 3.2. By Lemma 3.3 the natural map

$$H^\Gamma_q(X, R) \to \lim_{\to n} H^\Gamma_{q+n}(U(B^nX), R)$$

is an isomorphism, and by Lemma 3.4, (3.1) is also an isomorphism – in fact, in each homological degree, it becomes an isomorphism at some finite step in the inductive sequence. □

To finish the paper, let us explain how Theorem 3.2 can be deduced from the work of T. Pirashvili mentioned in the introduction. Note that any abelian group can be treated as a pointed set, by taking 0 as the distinguished point and forgetting the rest of the group structure. Thus a functor $F : \Gamma_+ \to \Delta_{opp} \text{Ab}$ from $\Gamma_+$ to the category of simplicial abelian groups can be treated as a pointed simplicial $\Gamma$-set. Then even if such $F$ is not special in the sense of Definition 1.1 (ii), the map (1.6) is still well-defined, so that the sequence $B^nF$, $n \geq 0$ forms a pre-spectrum. One denotes by $\pi^{st}_*(F)$ the homotopy groups of the corresponding spectrum. Then [P2, Proposition 2.2] claims that there exists a natural isomorphism

$$\pi^{st}_*(F) \cong \text{Tor}^\Gamma_+(t, F)$$

(to be precise, [P2, Proposition 2.2] is stated only for functors to constant simplicial groups, but generalization to arbitrary ones is immediate). Pirashvili’s proof of this fact also uses Lemma 2.3, but it is in fact simpler since working with an arbitrary $F$ gives more latitude. Then to deduce Theorem 3.2 one has to take $F = C\cdot(X, R)$, and show that

$$H_*(EX\cdot, R) \cong \pi^{st}_*(F).$$

This is also rather straightforward. So in a nutshell, Pirashvili’s proof is ultimately simpler but relies on some context, while our proof is longer but elementary and self-contained.

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