Linear instability of nonlinear Dirac equation in 1D with higher order nonlinearity

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Abstract

We consider the nonlinear Dirac equation in one dimension, also known as the Soler model in (1+1) dimensions, or the massive Gross-Neveu model:

\[
\begin{align*}
    i\partial_t \psi &= -i\alpha \partial_x \psi + m\beta \psi - f(\psi^* \beta \psi) \beta \psi, \\
    \psi(x,t) &\in \mathbb{C}^2, \quad x \in \mathbb{R}; \\
    g &\in \mathbb{C}^\infty(\mathbb{R}), \quad m > 0,
\end{align*}
\]

where \( \alpha, \beta \) are \( 2 \times 2 \) hermitian matrices which satisfy \( \alpha^2 = \beta^2 = 1, \alpha \beta + \beta \alpha = 0 \). We study the spectral stability of solitary wave solutions \( \phi_\omega(x)e^{-i\omega t} \). More precisely, we study the presence of point eigenvalues in the spectra of linearizations at solitary waves of arbitrarily small amplitude, in the limit \( \omega \to m \). We prove that if \( f(s) = s^k + O(s^{k+1}), \quad k \in \mathbb{N}, \quad k \geq 3 \), then one positive and one negative eigenvalue are present in the spectrum of linearizations at all solitary waves with \( \omega \) sufficiently close to \( \omega = m \). This shows that all solitary waves of sufficiently small amplitude are linearly unstable. The approach is based on applying the Rayleigh-Schrödinger perturbation theory to the nonrelativistic limit of the equation.

The results are in formal agreement with the Vakhitov-Kolokolov stability criterion.

Let us mention a similar independent result \[GG\] on linear instability for the nonlinear Dirac equation in three dimensions, with cubic nonlinearity (this result is also in formal agreement with the Vakhitov-Kolokolov stability criterion).

1 Introduction

A natural simplification of the Dirac-Maxwell system \[Gro66\] is the nonlinear Dirac equation, such as the massive Thirring model \[Thi58\] with vector-vector self-interaction and the Soler model \[Sol70\] with scalar-scalar self-interaction (known in dimension \( n = 1 \) as the massive Gross-Neveu model \[GN74\], \[LG75\]). There was an enormous body of research devoted to the nonlinear Dirac equation, which we can not cover comprehensively in this short note. The existence of standing waves in the nonlinear Dirac equation was studied in \[Sol70\], \[CV86\], \[Mer88\], and \[ES95\]. Numerical confirmation of spectral stability of solitary waves of small amplitude is contained in \[Chu07\]. The overview of the well-posedness of the nonlinear Dirac equation in 1D is contained in \[Pel10\]. The asymptotic stability of small amplitude solitary waves in the external potential has been studied in \[Bou08\], \[PS10\].

These models with self-interaction of local type have been receiving a lot of attention in the particle physics, as well as in the theory of Bose-Einstein condensates \[HC09\], \[MJZ+10\]. The question of stability of solitary waves is of utmost importance: perturbations ensure that we only ever encounter stable configurations. Recent attempts at asymptotic stability \[PS10\], \[BCT11\] rely on the fundamental question of spectral stability:

Consider the Ansatz \( \psi(x,t) = (\phi_\omega(x) + \rho(x,t))e^{-i\omega t} \), with \( \phi_\omega(x)e^{-i\omega t} \) a solitary wave solution. Let \( \partial_t \rho = A_\omega \rho \) be the linearized equation on \( \rho \). Does \( A_\omega \) have eigenvalues in the right half-plane?
In spite of a very clear picture of the spectral stability of nonlinear Schrödinger and Klein-Gordon equations [VK73, Sha83, Wei85, SS85, GSS87] and many attempts at the spectral stability in the context of the nonlinear Dirac (let us mention [AS83, AS86, SV86, Bou08, CKMS10]), in the latter case the question of spectral stability is still completely open. Our numerical results [BC09] show that in the 1D Soler model (cubic nonlinearity) all solitary waves are spectrally stable. Let us mention the related results [CP06, Chu07].

Our previous result [Com11] shows that the Vakhitov-Kolokolov criterion in the case of the nonlinear Dirac equation gives a less definite answer about the spectral stability than in the case of the nonlinear Schrödinger equation. All we know is that when \( \partial_\omega Q(\omega) = 0 \), with \( Q(\omega) \) being the charge of the solitary wave \( \psi_\omega e^{-i\omega t} \), two eigenvalues collide at \( \lambda = 0 \), but we do not know where these eigenvalues are located when \( \partial_\omega Q(\omega) \neq 0 \). On the other hand, for the solutions to the nonlinear Schrödinger equation, the condition \( \partial_\omega Q(\omega) > 0 \) is enough to conclude that one positive and one negative eigenvalue move out of \( \lambda = 0 \) along the real axis; see [VK73, GSS87, CP03].

In the present paper, we show that if the Vakhitov-Kolokolov guarantees linear instability for the system obtained in the nonrelativistic limit, then the same result is true for the small amplitude solitary waves in the original relativistic system. We use our approach to show that the small amplitude solitary wave solutions to the nonlinear Dirac equation in 1D with higher order nonlinearities are linearly unstable.

According to our results, the spectrum of the linearization at small amplitude solitary waves in 1D nonlinear Dirac equation with the cubic and quintic nonlinearities has no real eigenvalues; instead, one can prove the existence of two purely imaginary eigenvalues. To prove that these solitary waves are spectrally stable, one also needs to prove that there are no complex eigenvalues with \( \text{Re} \lambda > 0 \); we already have partial results which we will publish elsewhere.

Our approach is based on the idea that the family of real eigenvalues of the linearization of the nonlinear Dirac equation bifurcating from \( \lambda = 0 \) is a deformed family of eigenvalues of the linearization of the nonlinear Schrödinger equation. The model and the main results are described in Section 2. The necessary constructions in the context of the nonlinear Schrödinger equation are presented in Section 3. The asymptotics of solitary waves of the nonlinear Dirac equation and the linearization is covered in Section 4. The statement of Theorem 2.1 for \( \omega \lesssim m \) follows from Proposition 5.2 which we prove using the Rayleigh-Schrödinger perturbation theory.

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## 2 Main result

We consider the nonlinear Dirac equation in one dimension,

\[
  i \partial_t \psi = D_m \psi - f(\psi^* \beta \psi) \beta \psi, \quad x \in \mathbb{R}, \quad \psi \in \mathbb{C}^2,
\]

where \( D_m \) is the Dirac operator:

\[
  D_m = -i\alpha \partial_x + m\beta, \quad m > 0.
\]

Above, \( \psi^* \) being the Hermitian conjugate of \( \psi \). We assume that the nonlinearity \( f(s) \) is smooth, real-valued, and satisfies \( f(0) = 0 \). The Hermitian matrices \( \alpha \) and \( \beta \) are chosen so that

\[
  D_m^2 = (-i\alpha \partial_x + \beta m)^2 = (-\partial_x^2 + m^2) I_2,
\]

where \( I_2 \) is the \( 2 \times 2 \) unit matrix. That is, \( \alpha \) and \( \beta \) are to satisfy

\[
  \alpha^2 = I_2, \quad \beta^2 = I_2; \quad \alpha \beta + \beta \alpha = 0.
\]

The generalized massive Gross-Neveu model, or, in the terminology of [CKMS10], the scalar-scalar case with \( k \in \mathbb{N} \), corresponds to the nonlinearity \( f(s) = s^k \).
According to the Dirac-Pauli theorem (cf. [Dir28 vdW32 Pau36] and [Tha92 Lemma 2.25]), the particular choice of \( \alpha \) and \( \beta \) matrices does not matter. We choose

\[
\alpha = -\sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \beta = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

For a large class of nonlinearities \( f(s) \), there are solitary wave solutions of the form

\[
\psi(x, t) = \phi_\omega(x)e^{-i\omega t}, \quad \phi_\omega = \begin{bmatrix} \psi(x, \omega) \\ u(x, \omega) \end{bmatrix} \in H^1(\mathbb{R}, \mathbb{C}^2), \quad |\omega| < m,
\]

with \( \psi(x, \omega) \) and \( u(x, \omega) \) real-valued and with finite \( H^1 \) norms. For example, this is the case for the Soler model with the nonlinearity \( f(s) = s \). For details, see Section 4.

Due to the \( U(1) \)-invariance, for solutions to (2.1) the value of the charge functional

\[
Q(\psi) = \int_{\mathbb{R}} |\psi(x, t)|^2 \, dx
\]

is formally conserved. For brevity, we also denote by \( Q(\omega) \) the charge of the solitary wave \( \phi_\omega(x)e^{-i\omega t} \):

\[
Q(\omega) = \int_{\mathbb{R}} |\phi_\omega(x)|^2 \, dx.
\]

We are interested in the spectral stability of linearization of (2.1) at a solitary wave solution (2.3).

**Theorem 2.1.** Let

\[
f(s) = as^k + O(s^{k+1}), \quad a > 0.
\]

If \( k \geq 3 \), then the solitary wave solutions \( \phi_\omega(x)e^{-i\omega t} \) to (2.1) are linearly unstable for \( \omega \in I \), where \( I \subset \Omega \) is the largest interval with \( \text{supp} I = m \) such that \( \partial_\omega Q(\omega) \) does not vanish on \( I \). More precisely, let \( A_\omega \) be the linearization of the nonlinear Dirac equation at a solitary wave \( \phi_\omega(x)e^{-i\omega t} \). Then there are eigenvalues \( \pm \lambda_\omega \in \sigma_p(A_\omega) \), with \( \lambda_\omega > 0 \) for \( \omega \in I \), with \( \lambda_\omega = O(m - \omega) \).

See Figure 1.

**Remark 2.2.** We only need to prove the linear instability for \( \omega \lesssim m \). Then, by [Com11], the positive and negative eigenvalues remain trapped on the real axis, not being able to collide at \( \lambda = 0 \) for \( \omega \in I = (\omega_0, m) \) as long as \( \partial_\omega Q(\omega) \) does not vanish on \( I \) (this nonvanishing is a sufficient condition for the dimension of the generalized null space to remain equal to two). These eigenvalues can not leave into the complex plane, either, since they are simple, while the spectrum of the operator is symmetric with respect to the real and imaginary axes.

**Remark 2.3.** We do not know what happens at \( \omega = \inf I \); as \( \omega \) drops below \( \inf I \), it could be that either the pair of real eigenvalues, having collided at \( \lambda = 0 \), turn into a pair of purely imaginary eigenvalues (linear instability disappears), or instead two purely imaginary eigenvalues, having met at \( \lambda = 0 \), turn into the second pair of real eigenvalues (linear instability persists).

**Remark 2.4.** Theorem 2.1 is in the formal agreement with the Vakhitov-Kolokolov stability criterion [VK73], since for \( \omega \lesssim m \) one has \( Q'(\omega) < 0 \) for \( k = 1 \) and \( Q'(\omega) > 0 \) for \( k \geq 3 \). Let us mention that the sign of the instability criterion, \( Q'(\omega) > 0 \) differs from [VK73] because of their writing the solitary waves in the form \( \varphi(x)e^{+i\omega t} \).

**Remark 2.5.** We expect that in the 1D case with \( k = 1 \) the small amplitude solitary waves are spectrally stable; we will prove this elsewhere.

We also expect that in the 1D case with \( k = 2 \) ("quintic nonlinearity", \( f(\psi^5 \beta \psi)\beta \psi = O(|\psi|^5) \)) the small solitary wave solutions of the nonlinear Dirac equation in 1D are spectrally stable. For the corresponding nonlinear Schrödinger equation (quintic nonlinearity in 1D), the charge is constant, thus the zero eigenvalue of a linearized operator is always of higher algebraic multiplicity. For the Dirac equation, this degeneracy is "resolved": the charge is now a decaying function for \( \omega \lesssim m \) (with nonzero limit as \( \omega \to m \)), suggesting that there are two purely imaginary eigenvalues \( \pm \lambda_\omega \) in the spectrum of \( A_\omega \), with \( \lambda_\omega = o(m - \omega) \), but no eigenvalues with nonzero real part.
i(m + \omega)

i(m - \omega)

-\lambda_\omega
\lambda_\omega

Figure 1: Main result: The point spectrum of the linearization of the nonlinear Dirac equation with the nonlinearity

\( f(s) = s^k + O(k^{k+1}), \ k \geq 3, \) at a solitary wave with \( \omega \leq m \) contains two nonzero real eigenvalues, \( \pm \lambda_\omega, \) with \( \lambda_\omega = O(m - \omega). \) See Theorem 2.1. Also plotted on this picture is the essential spectrum, with the edges at \( \lambda = \pm i(m - \omega) \) and with the embedded threshold points (branch points of the dispersive relation) at \( \lambda = \pm i(m + \omega). \)

Remark 2.6. The same approach can be used to show instability of small amplitude solitary wave solutions to the nonlinear Dirac equation in \( n \geq 3, \) with \( f(s) = as^k + O(s^{k+1}), \ a > 0, \) with any \( k \in \mathbb{N}. \) These results in 3D have been independently obtained in [GG]. Let us notice that in the 3D case for the cubic nonlinearity \( f(s) = s \) (this is the original Soler model from [Sol70]), based on the numerical evidence from [Sol70, ASS83], one expects that the charge \( Q(\omega) \) has a local minimum at \( \omega = 0.936m, \) suggesting that the solitary waves with \( 0.936m < \omega < m \) are linearly unstable, but then at \( \omega = 0.936m \) the real eigenvalues collide at \( \lambda = 0, \) and there are no nonzero real eigenvalues in the spectrum for \( \omega \lesssim 0.936m. \)

Remark 2.7. We can not rule out the possibility that the eigenvalues with nonzero real part could bifurcate directly from the imaginary axis into the complex plane. Such a mechanism is absent for the nonlinear Schrödinger equation linearized at a solitary wave, for which the point eigenvalues always remain on the real or imaginary axes. At present, though, we do not have examples of such bifurcations in the context of the nonlinear Dirac equation linearized at a solitary wave.

Remark 2.8. The existence of solitary waves stated in the theorem follows from [CV86]. We will reproduce their argument in order to have the asymptotics of profiles of solitary waves for \( \omega \) near \( m. \)

3 Nonlinear Schrödinger equation

We are going to use the fact that the nonlinear Dirac equation in the nonrelativistic limit coincides with the nonlinear Schrödinger equation,

\[
i\partial_t \psi = -\frac{1}{2} \partial_x^2 \psi + m\psi - f(|\psi|^2)\psi, \quad \psi(x,t) \in \mathbb{C}, \quad x \in \mathbb{R}.
\]

We will assume that

\( f(s) = s^k, \ k \in \mathbb{N}. \)

Solitary waves

The solitary wave solutions

\[ \psi(x,t) = \phi_\omega(x)e^{-i\omega t}, \quad \phi_\omega \in H^1(\mathbb{R}), \]
exist for \( \omega \in (-\infty, m) \). These solitary waves and their asymptotics are well-known, to the extent that we do not know which of the multiple references would be most appropriate. Often we will not indicate explicitly the dependence of the amplitude of the solitary wave on \( \omega \), writing \( \phi \) instead of \( \phi_\omega \). This amplitude satisfies the equation
\[
\partial_x^2 \phi(x) = 2(m - \omega)\phi - 2\phi^{2k+1}, \quad x \in \mathbb{R},
\]
which could be integrated to the relation
\[
\partial_x \phi = -\phi \sqrt{2(m - \omega) - \frac{2\phi^{2k}}{k+1}}.
\]
Introducing \( \mathcal{X}(x, \omega) = \phi_\omega^2(x) \), we get
\[
\partial_x \mathcal{X} = -2\mathcal{X} \sqrt{2(m - \omega) - \frac{2\mathcal{X}^{k}}{k+1}}, \quad (3.2)
\]
We will perform the scaling in terms of
\[
\varepsilon = \sqrt{2(m - \omega)}. \quad (3.3)
\]
Then, as can be seen from (3.2),
\[
\mathcal{X}(x, \omega) = \varepsilon^{2/k} U(\varepsilon x), \quad (3.4)
\]
with \( U(y) \) positive spherically symmetric (even) solution to the equation
\[
\partial_y U(y) = -2U \sqrt{1 - \frac{2U^k}{k+1}}, \quad \lim_{y \to \pm\infty} U(y) = 0.
\]
Such a solution exists and is unique; it is explicitly given by
\[
U(y) = \left( \frac{k+1}{2\cosh^2 ky} \right)^{1/k}. \quad (3.5)
\]

### Linearization at a solitary wave

To derive the linearization of the nonlinear Schrödinger equation (3.1) at a solitary wave \( \psi(x, t) = \phi_\omega(x)e^{-i\omega t} \), we use the Ansatz
\[
\psi(x, t) = (\phi_\omega(x) + \rho(x, t))e^{-i\omega t}, \quad \rho(x, t) \in \mathbb{C}, \quad x \in \mathbb{R},
\]
and arrive at the linearized equation
\[
\partial_t \rho = jL(\omega)\rho, \quad \rho(x, t) = \begin{bmatrix} \text{Re } \rho(x, t) \\ \text{Im } \rho(x, t) \end{bmatrix}, \quad (3.6)
\]
with
\[
j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} L_- & 0 \\ 0 & L_+ \end{bmatrix}, \quad (3.7)
\]
\[
L_-(\omega) = -\frac{1}{2} \partial_x^2 + m - \omega - f(\phi_\omega^2), \quad L_+(\omega) = L_- - 2f'(\phi_\omega^2)\phi_\omega^2. \quad (3.8)
\]
In the case of the nonlinearity \( f(s) = s^k \), taking into account the explicit form of \( \mathcal{X}(x) \) given by (3.4), one obtains
\[
L_-(\omega) = -\frac{1}{2} \partial_x^2 + m - \omega - \varepsilon^2 \frac{k+1}{2\cosh^2(\varepsilon kx)}, \quad L_+(\omega) = -\frac{1}{2} \partial_x^2 + m - \omega - \varepsilon^2 \frac{(2k+1)(k+1)}{2\cosh^2(\varepsilon kx)}. \quad (3.9)
\]

**Lemma 3.1** (Vakhitov-Kolokolov stability criterion). *For the linearization (3.6) at a particular solitary wave \( \phi_\omega(x)e^{-i\omega t} \), there are real nonzero eigenvalues \( \pm \lambda \in \sigma_d(jL) \), \( \lambda > 0 \), if and only if \( \frac{d}{d\omega} \|\phi_\omega\|^2_{L^2} > 0 \) at this value of \( \omega \).*
Let us consider the spectrum of the linearized equation (3.10) for the nonlinearity \( f(s) = s^k \). Using (3.14), one derives for the corresponding charge:

\[
Q(\omega) = \int_R \phi^2(x) \, dx = \int_R \mathcal{X}(x, \omega) \, dx = \varepsilon^{\frac{\omega}{2} - 1} C = (2(m - \omega))^{\frac{3}{2}} C, \quad \omega < m, \tag{3.10}
\]

where \( C = \int_R U(y) \, dy > 0 \). We see from (3.10) that one has \( Q'(\omega) < 0 \) for \( k = 1 \), \( Q'(\omega) \equiv 0 \) for \( k = 2 \), and \( Q'(\omega) > 0 \) for \( k \geq 3 \).

**Lemma 3.2.** Let \( f(s) = s^k, \, k \in \mathbb{N} \). If \( k = 1 \) or \( k = 2 \), then \( \sigma_p(jL) \subset i\mathbb{R} \). If \( k \geq 3 \), then \( \sigma_p(jL) \ni \{ \pm \varepsilon^2 \Lambda \} \), for some \( \Lambda > 0 \) and \( \varepsilon = \sqrt{2/(m - \omega)} \).

**Proof.** In the case \( k = 1 \), since \( Q'(\omega) < 0 \) by (3.10), Lemma 3.4 guarantees that there are no nonzero real eigenvalues: \( \sigma_p(jL(\omega)) \cap (\mathbb{R} \setminus 0) = \emptyset \). In the case \( k = 2 \), \( Q'(\omega) \equiv 0 \), and the eigenvalue \( \lambda = 0 \) is always of increased algebraic multiplicity 6 (generically, the algebraic multiplicity of \( \lambda = 0 \) is 4, jumping to 6 when \( Q'(\omega) = 0 \)). In the case \( k \geq 3 \), since \( Q'(\omega) > 0 \) by (3.10), Lemma 3.4 states that there are two real eigenvalues \( \pm \lambda_\omega \in \sigma_p(jL(\omega)) \), with \( \lambda_\omega > 0 \). The rescaling \( y = \varepsilon x \) shows that \( \lambda_\omega = \varepsilon^2 \Lambda \), with some \( \Lambda > 0 \).

For \( k \in \mathbb{N} \), consider the Schrödinger operators

\[
\hat{L} = \begin{bmatrix} \hat{L}_- & 0 \\ 0 & \hat{L}_+ \end{bmatrix}, \quad \hat{L}_- = -\frac{1}{2} \partial_y^2 + \frac{1}{2} - \frac{k + 1}{2 \cosh^2 ky}, \quad \hat{L}_+ = -\frac{1}{2} \partial_y^2 + \frac{1}{2} - \frac{(2k + 1)(k + 1)}{2 \cosh^2 ky}. \tag{3.11}
\]

Let \( \phi(y) = \frac{1}{\cosh^{\frac{1}{2}} ky} \); then

\[
\hat{L}_- \phi = 0, \quad \hat{L}_+ \partial_y \phi = 0. \tag{3.12}
\]

**Proposition 3.3.** If \( k \geq 3 \), the discrete spectrum of the operator \( j\hat{L} = \begin{bmatrix} 0 & \hat{L}_- \\ -\hat{L}_+ & 0 \end{bmatrix} \) contains two real nonzero eigenvalues \( \pm \Lambda, \, \Lambda > 0 \). If \( k \leq 2 \), \( \sigma_d(j\hat{L}) \subset i\mathbb{R} \).

**Proof.** We start with constructing several important relations. By (3.12),

\[
\hat{L}_+ \phi = \hat{L}_- \phi + (\hat{L}_- - \hat{L}_+) \phi = -\frac{k(k + 1)}{\cosh^2 ky} \phi. \tag{3.13}
\]

For \( \mu > 0 \), let \( \phi_\mu(y) = \phi(\mu y) \) and \( \hat{L}_\mu = -\frac{1}{2} \partial_y^2 + \mu^2 \left( \hat{L}_- + \frac{k + 1}{2 \cosh^2 \mu ky} \right) \); then \( \hat{L}_\mu \phi_\mu = 0 \). Taking the derivative of this relation with respect to \( \mu \) and evaluating it at \( \mu = 1 \), we get:

\[
\hat{L}_+ \theta = \left( -1 + \frac{k + 1}{\cosh^2 ky} \right) \phi, \quad \text{where} \quad \theta := \frac{\partial}{\partial \mu} \bigg|_{\mu = 1} \frac{1}{\cosh^2 \mu ky} = -\frac{y \sinh ky}{\cosh^2 ky}. \tag{3.14}
\]

By (3.11) and (3.14),

\[
\hat{L}_+ (-\theta - \frac{1}{k} \phi) = \phi. \tag{3.15}
\]

Now we follow the argument from [27].

**Lemma 3.4.** Let \( k \geq 3 \). The minimum of \( \langle r, \hat{L}_+ r \rangle \) under the constraints \( \langle r, r \rangle = 1 \) and \( \langle \phi, r \rangle = 0 \) is negative.

**Proof.** The vector \( r \) corresponding to the minimum of \( \langle r, \hat{L}_+ r \rangle \) under the constraints \( \langle r, r \rangle = 1 \) and \( \langle \phi, r \rangle = 0 \) satisfies the equation \( \hat{L}_+ r = \alpha r + \beta \phi \), where \( \alpha, \beta \in \mathbb{R} \) are Lagrange multipliers. Since \( \langle r, \hat{L}_+ r \rangle = \langle r, \alpha r + \beta \phi \rangle = \alpha \), we need to know the sign of \( \alpha \). Denote

\[
f(z) = \langle \phi, (\hat{L}_+ - z)^{-1} \phi \rangle, \quad z \in \rho(\hat{L}_+).
\]
We note that \( f(z) \) has a removable singularity at \( z = 0 \) since \( \phi \in L^2(\mathbb{R}) \), while the restriction of \( \hat{L}_+ \) onto the space of spherically symmetric (that is, even) functions, \( \hat{L}_+ : L^2_s(\mathbb{R}) \to L^2_s(\mathbb{R}) \), has a bounded inverse. (being even) Using (3.15), we have:

\[
f(0) = \langle \phi, \hat{L}_+^{-1}\phi \rangle = \langle \phi, (-\theta - \frac{1}{k}\phi) \rangle = \int_\mathbb{R} \frac{1}{\cosh^{1+\frac{1}{k}} y} \left( \frac{y \sinh ky}{\cosh^{1+\frac{1}{k}} ky} \right) dy
\]

\[
= \int_\mathbb{R} \left( \frac{z \sinh z}{\cosh^{1+\frac{1}{k}} z} - \frac{dz}{k} \right) dz = \int_\mathbb{R} \left( -z \cdot \frac{d}{dz} \left( \frac{1}{2 \cosh^{1+\frac{1}{k}} z} \right) \right) dz = \left( \frac{1}{2} - \frac{1}{k} \right) \int_\mathbb{R} \frac{dz}{\cosh^{1+\frac{1}{k}} z} > 0,
\]

where we took into account that \( k \geq 3 \). Since \( f(0) > 0 \) and \( \lim_{z \to \pm \lambda_0} f(z) = -\infty \) (it is \(-\infty \) since \( f'(z) > 0 \)), where \( \lambda_0 < 0 \) is the smallest eigenvalue of \( \hat{L}_+ \), there is \( \mu \in (\lambda_0, 0) \) such that \( f(\mu) = 0 \).

Since \( \hat{L}_- \) is positive-definite and \( r \) in Lemma 3.4 is orthogonal to \( \ker \hat{L}_- \) (which is spanned by \( \phi \)), we may define \( R = \hat{L}_-^{-1/2} r \); then

\[
\langle R, \hat{L}_-^{1/2} \hat{L}_+^{1/2} R \rangle = \langle r, \hat{L}_+ r \rangle < 0.
\]

Therefore, \( \sigma_d(\hat{L}_- \hat{L}_+) = \sigma_d(\hat{L}_-^{1/2} \hat{L}_+^{1/2}) \) contains a negative eigenvalue \(-\Lambda^2\), where \( \Lambda > 0 \). Let \( \xi \) be a corresponding eigenvector, so that \( \hat{L}_- \hat{L}_+ \xi = -\Lambda^2 \xi \). Then

\[
\begin{bmatrix} 0 & \hat{L}_- \\ -\hat{L}_+ & 0 \end{bmatrix} \begin{bmatrix} \Lambda \xi \\ \hat{L}_+ \xi \end{bmatrix} = \pm \Lambda \begin{bmatrix} \Lambda \xi \\ \hat{L}_+ \xi \end{bmatrix}, \text{ hence } \pm \Lambda \in \sigma_d(\hat{jL}).
\]

\[\square\]

Remark 3.5. Since the spectrum of the linearization at zero solitary wave is purely imaginary, one has \( \lim_{\omega \to m^-} \lambda_\omega = 0 \).

Remark 3.6. Comparing \( L \) (see (3.7), (3.9)) to the operator \( \hat{L} \) introduced in (3.11),

\[
\hat{L} = \begin{bmatrix} \hat{L}_- & 0 \\ 0 & \hat{L}_+ \end{bmatrix}, \quad \hat{L}_- = -\frac{1}{2} \partial^2_y + \frac{1}{2} - \frac{k + 1}{2 \cosh^2 ky}, \quad \hat{L}_+ = -\frac{1}{2} \partial^2_y + \frac{1}{2} - \frac{(2k + 1)(k + 1)}{2 \cosh^2 ky},
\]

one concludes that

\[
\sigma(L_-) = \varepsilon^2 \sigma(\hat{L}_-), \quad \sigma(L_+) = \varepsilon^2 \sigma(\hat{L}_+), \quad \sigma(jL) = \varepsilon^2 \sigma(j\hat{L}).
\]

Therefore, \( \lambda_\omega = \varepsilon^2 \Lambda \), where \( \Lambda \) is the positive eigenvalue of \( j\hat{L} \).

4 Nonlinear Dirac equation

In this section, we will use the notation

\[
g(s) = ms - f(s).
\]

In terms of the components, \( \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \psi_1, \psi_2 \in \mathbb{C} \), we write the nonlinear Dirac equation (2.1) as a system

\[
\begin{cases}
i \partial_t \psi_1 = \partial_x \psi_2 + g(|\psi_1|^2 - |\psi_2|^2) \psi_1, \\
i \partial_t \psi_2 = -\partial_x \psi_1 - g(|\psi_1|^2 - |\psi_2|^2) \psi_2.
\end{cases}
\]

(4.1)
Solitary waves

**Definition 4.1.** The solitary waves are solutions to (2.1) of the form

\[ S = \{ \psi(x,t) = \phi_\omega(x)e^{-i\omega t}; \phi_\omega \in H^1(\mathbb{R}, \mathbb{C}^2), \ \omega \in \mathbb{R} \}. \]

We start by demonstrating the existence of solitary wave solutions and exploring their properties. The following result follows from [CV86]; we follow our article [BC09].

**Lemma 4.2.** Let \( G \) be the antiderivative of \( g \) such that \( G(0) = 0 \). Assume that there is \( \omega_0 < m \) such that for given \( \omega \in (\omega_0, m) \) there exists \( \Gamma_\omega > 0 \) such that

\[ \omega \Gamma_\omega = G(\Gamma_\omega), \quad \omega \neq g(\Gamma_\omega), \quad \text{and} \quad \omega s < G(s) \quad \text{for} \quad s \in (0, \Gamma_\omega). \tag{4.2} \]

Then there is a solitary wave solution \( \psi(x,t) = \phi_\omega(x)e^{-i\omega t} \) to (2.1), where

\[ \phi_\omega(x) = \begin{bmatrix} v(x,\omega) \\ u(x,\omega) \end{bmatrix}, \tag{4.3} \]

with both \( v \) and \( u \) real-valued, belonging to \( H^1(\mathbb{R}) \) as functions of \( x \), \( v \) being even and \( u \) odd.

More precisely, for \( x \in \mathbb{R} \) and \( \omega \in (\omega_0, m) \), let us define \( \mathcal{X}(x,\omega) \) and \( \mathcal{Y}(x,\omega) \) by

\[ \mathcal{X} = v^2 - u^2, \quad \mathcal{Y} = vu. \tag{4.4} \]

Then \( \mathcal{X}(x,\omega) \) is a unique positive symmetric solution to

\[ \partial_x^2 \mathcal{X} = -\partial_x ( -2G(\mathcal{X})^2 + 2\omega^2 \mathcal{X}^2), \quad \lim_{x \to \pm \infty} \mathcal{X}(x,\omega) = 0, \tag{4.5} \]

and \( \mathcal{Y}(x,\omega) = -\frac{1}{\omega} \partial_x \mathcal{X}(x,\omega) \). This solution satisfies \( \mathcal{X}(0,\omega) = \Gamma_\omega \).

**Proof.** From (4.1), we obtain:

\[ \begin{cases} \omega v = \partial_x u + g(|v|^2 - |u|^2)v, \\ \omega u = -\partial_x v - g(|v|^2 - |u|^2)u. \end{cases} \tag{4.6} \]

Assuming that both \( v \) and \( u \) are real-valued (this will be justified once we found real-valued \( v \) and \( u \)), we can rewrite (4.6) as the following Hamiltonian system:

\[ \begin{cases} \partial_x u = \omega v - g(v^2 - u^2)v = \partial_x h(v,u), \\ -\partial_x v = \omega u + g(v^2 - u^2)u = \partial_u h(v,u), \end{cases} \tag{4.7} \]

where the Hamiltonian \( h(v,u) \) is given by

\[ h(v,u) = \frac{\omega}{2}(v^2 + u^2) - \frac{1}{\omega}G(v^2 - u^2). \tag{4.8} \]

The solitary wave with a particular \( \omega \in (\omega_0, m) \) corresponds to a trajectory of this Hamiltonian system such that

\[ \lim_{x \to \pm \infty} v(x,\omega) = \lim_{x \to \pm \infty} u(x,\omega) = 0, \]

hence \( \lim_{x \to \pm \infty} \mathcal{X} = 0 \). Since \( G(s) \) satisfies \( G(0) = 0 \), we conclude that \( h(v(x),u(x)) \equiv 0 \), which leads to

\[ \omega(v^2 + u^2) = G(v^2 - u^2). \tag{4.9} \]

We conclude from (4.9) that solitary waves may only correspond to \(|\omega|<m, \omega \neq 0 \).

The functions \( \mathcal{X}(x,\omega) \) and \( \mathcal{Y}(x,\omega) \) introduced in (4.1) are to solve

\[ \begin{cases} \partial_x \mathcal{X} = -4\omega \mathcal{Y}, \\ \partial_x \mathcal{Y} = -(v^2 + u^2)g(\mathcal{X}) + \omega \mathcal{X} = -\frac{1}{\omega}G(\mathcal{X})g(\mathcal{X}) + \omega \mathcal{X}, \end{cases} \tag{4.10} \]

and to have the asymptotic behavior \( \lim_{|x| \to \infty} \mathcal{X}(x) = 0, \lim_{|x| \to \infty} \mathcal{Y}(x) = 0 \). In the second equation in (4.10), we used the relation (4.9). The system (4.10) can be written as the following equation on \( \mathcal{X} \):

\[ \partial_x^2 \mathcal{X} = -\partial_x ( -2G(\mathcal{X})^2 + 2\omega^2 \mathcal{X}^2) = 4(G(\mathcal{X})g(\mathcal{X}) - \omega^2 \mathcal{X}). \tag{4.11} \]

This equation describes a particle in the potential \(-2G(s)^2 + 2\omega^2 s^2\). The condition (4.2) is needed for the existence of the turning point of the zero energy trajectory in this potential, at \( s = \Gamma_\omega \). \qed
Solitary waves in the nonrelativistic limit

From now on, without loss of generality, we will assume that the nonlinearity \( f \in C^\infty(\mathbb{R}) \) is such that

\[
f(s) = s^k + o(s^{k+1}), \quad k \in \mathbb{N},
\]

and consider \( k \) fixed throughout the rest of the article (this is to avoid writing subscripts). By Lemma 4.2 there is \( \omega_0 < m \) such that there are solitary wave solutions for \( \omega \in (\omega_0, m) \).

We are going to determine the asymptotics of solitary waves in the nonrelativistic limit \( \omega \to m \).

We consider the function \( g(s) := m - f(s) \) and its antiderivative,

\[
g(s) = m - s^k + O(s^{k+1}), \quad G(s) = ms - \frac{s^{k+1}}{k+1} + O(s^{k+2}).
\]

Then the expression in right-hand side of (4.11) takes the following form:

\[
G(s)g(s) - \omega^2 s = (ms - \frac{s^{k+1}}{k+1} + O(s^{k+2}))(m - s^k + O(s^{k+1})) - \omega^2 s = (m^2 - \omega^2)s - \frac{k+2}{k+1}s^{k+1} + O(s^{k+2}).
\]

Let \( \omega \in (\omega_0, m) \), so that there is a solitary wave \( \phi_\omega(x)e^{-i\omega t} \) with this particular value of \( \omega \). Denote

\[
\epsilon = \sqrt{m^2 - \omega^2}.
\]

Let \( \mathcal{X}(x, \omega) \) be a positive symmetric solution to (4.11); we rewrite (4.11) as

\[
-\frac{1}{4}(\partial_x^2 \mathcal{X} - \frac{k+2}{k+1} \mathcal{X}^{k+1} + \epsilon^2 \mathcal{X}) = O(\mathcal{X}^{k+2}).
\]

(4.13)

Let \( U(y) \) be a positive symmetric solution to

\[
-\frac{1}{4} \partial_y^2 U - \frac{k+2}{k+1} U^{k+1} + U = 0, \quad \lim_{y \to \infty} U(y) = 0.
\]

(4.14)

Such a solution \( U(y) \) exists and is unique, and is explicitly given by (3.5).

Lemma 4.3. For \( \omega \in (\omega_0, m) \), there are the relations

\[
\mathcal{X}(x, \omega) = \epsilon \hat{\mathcal{X}} U(\epsilon x) + O(\epsilon^{1+}), \quad \text{(4.15)}
\]

\[
v(x, \omega) = \epsilon \hat{v} (U(\epsilon x))^{1\over 2} + O(\epsilon^{1+}), \quad u(x, \omega) = O(\epsilon^{1+}). \quad \text{(4.16)}
\]

Remark 4.4. Similar asymptotics are established in [Gua08] for the nonlinear Dirac equation in 3D. These asymptotics are used in [GG] for the proof of linear instability of solitary waves with \( \omega \lesssim m \) to the nonlinear Dirac equation with cubic nonlinearity.

Proof. Since \( U(y) > 0 \) for \( y \in \mathbb{R} \), one concludes from (4.11) that \( \lambda = 0 \) is the lowest eigenvalue of the operator

\[
H_{-} = -\frac{1}{4} \partial_y^2 - \frac{k+2}{k+1} U^k + 1.
\]

Taking the derivative of (4.14), we find:

\[
H_{+} \partial_y U := -\frac{1}{4} \partial_y^3 U - (k+2) U^k \partial_y U + \partial_y U = 0. \quad \text{(4.17)}
\]

The operator \( H_{+} \) in this relation is the same as in (3.11). With \( \partial_y U \) having one node, we conclude that \( \lambda = 0 \) is the second lowest eigenvalue of \( H_{+} \).
We define the function $X(y, \epsilon)$ by the relation $\mathcal{X}(x, \omega) = e^{2/k}X(\epsilon x, \epsilon)$, where $\epsilon$ and $\omega$ are related by (4.12). Then (4.13) takes the form

$$-rac{1}{4} \partial_y^2 X + X - \frac{k+2}{k+1} X^{k+1} = \epsilon^{-2} \frac{2}{k} O(\mathcal{X}^{k+2}) = \epsilon^{2/k} O(X^{k+2}).$$

(4.18)

Subtracting (4.14) from (4.18), we find:

$$-rac{1}{4} \partial_y^2 (X - U) + (X - U) - \frac{k+2}{k+1} (X^{k+1} - U^{k+1}) = \epsilon^{2/k} O(X^{k+2}),$$

which we rewrite as

$$-rac{1}{4} \partial_y^2 (X - U) + (X - U) - (k+2)U^k (X - U) = \epsilon^{2/k} O(X^{k+2}) + O((X - U)^2).$$

Thus, denoting $Z(y, \epsilon) = X(y, \epsilon) - U(y)$, one has

$$Z = (H_+|_{L^2})^{-1} \left( \epsilon^{2/k} O(U^{k+2}) + \epsilon^{2/k} O(Z) + O(Z^2) \right).$$

(4.19)

Above, $H_+|_{L^2} : L^2_r(\mathbb{R}) \to L^2_r(\mathbb{R})$ is the restriction of $H_+$ to the space of spherically symmetric (even) functions; its inverse is bounded (from $L^2_\omega$ to $L^2_\omega$) since by (4.17) we know that $\ker H_+ = \text{span}(\partial_\theta U)$, which is skew-symmetric (let us mention that $\lambda = 0$ is a simple eigenvalue of $H_+$, which is straightforward in one dimensional case). The right-hand side of (4.19) is well-defined since both $U$ and $Z = X - U$ are spherically symmetric.

For given $\epsilon > 0$, let $B_{\epsilon, \omega} = \{ Z \in H^1(\omega) ; Z(x) = Z(-x), \| Z \|_{H^1} < \epsilon^{1/k} \} \subset H^1(\omega)$ be the space of symmetric functions with $\| Z \|_{H^1} \leq \epsilon^{1/k}$. Consider the map

$$B_{\epsilon, \omega} \to H^1(\omega), \quad Z \mapsto (H_+|_{L^2})^{-1} \left( \epsilon^{2/k} O(U^{k+2}) + \epsilon^{2/k} O(Z) + O(Z^2) \right).$$

(4.20)

We can choose $\omega_1 \in (\omega_0, m)$ so close to $m$, that for $\omega \in (\omega_1, m)$ the value of $\epsilon = \sqrt{m^2 - \omega^2}$ is small enough for the map (4.20) to be an endomorphism and a contraction on $B_{\epsilon, \omega}$; by the contraction mapping theorem, there is a unique stationary point $Z \in B_{\epsilon, \omega}$, and, by (4.19), $Z$ satisfies $\| Z \|_{H^1} = O(\epsilon^{2/k})$. It follows that

$$\| \mathcal{X}(x, \omega) - \epsilon^{2/k} U(\epsilon x) \|_{L^\infty} = \epsilon^{2/k} \| Z \|_{L^\infty(\omega)} = O(\epsilon^{4/k}).$$

Now the asymptotics of $\mathcal{X}(x, \omega)$ could be determined from (4.10) and then $v(x, \omega), u(x, \omega)$ are determined by (4.4).

5 Linear instability of small solitary waves

Linearization at a solitary wave

To derive the linearization of equation (2.1) at a solitary wave (2.3), we consider the solution in the form of the Ansatz

$$\psi(x, t) = (\phi_\omega(x) + \rho(x, t)) e^{-i\omega t}, \quad \phi_\omega(x) = \begin{bmatrix} v(x, \omega) \\ u(x, \omega) \end{bmatrix}, \quad \phi_\omega \in H^1(\mathbb{R}, \mathbb{C}^2),$$

(5.1)

where $\rho(x, t) \in \mathbb{C}^2$. Note that $v(x, \omega)$ and $u(x, \omega)$ are real-valued by Lemma 4.2. Then, by (2.1), the linearized equation on $\rho$ takes the following form:

$$i \dot{\rho} = D_m \rho - \omega \rho - f(\phi_\omega^* \beta \phi_\omega) \beta \rho - (\phi_\omega^* \beta \rho + \rho^* \beta \phi_\omega) f'(\phi_\omega^* \beta \phi_\omega) \beta \phi_\omega, \quad D_m = -i a \partial_x + m \beta.$$

(5.2)

We note that the above equation is $\mathbb{R}$-linear but not $\mathbb{C}$-linear, due to the presence of the $\rho^*$ term. We denote $\rho = \begin{bmatrix} R_1 + i S_1 \\ R_2 + i S_2 \end{bmatrix}$, with $R_j, S_j$ real-valued. Since $v$ and $u$ are real-valued,

$$\phi_\omega^* \beta \rho + \rho^* \beta \phi_\omega = 2 \text{Re}(\phi_\omega^* \beta \rho) = 2(vR_1 - uR_2),$$

(5.3)
we rewrite equation (5.2) in the following form in terms of \( R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}, \ S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \):

\[
\partial_t \begin{bmatrix} R \\ S \end{bmatrix} = J \left\{ \begin{bmatrix} D_m - \omega - f \beta \\ 0 \\ 0 \\ D_m - \omega - f \beta \end{bmatrix} \begin{bmatrix} R \\ S \\ 0 \\ 0 \end{bmatrix} - 2 \text{Re}(\phi_\omega^* \beta \rho)' \frac{\beta \phi_\omega}{0} \right\},
\]

(5.3)

where

\[
f = f(\phi_\omega^* \beta \phi_\omega), \quad f' = f'(\phi_\omega^* \beta \phi_\omega),
\]

and \( J \) corresponds to \( 1/i \):

\[
J = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix},
\]

where \( I_2 \) is the \( 2 \times 2 \) unit matrix. We define \( L_+ (\omega) \) and \( L_-(\omega) \) by the following:

\[
L_-(\omega) = \begin{bmatrix} m - \omega - f & \partial_x \\ -\partial_x & -m - \omega + f \end{bmatrix}, \quad L_+(\omega) = \begin{bmatrix} m - \omega - f - 2f'v^2 & \partial_x + 2f'vu \\ -\partial_x + 2f'vu & -m - \omega + f - 2f'u^2 \end{bmatrix};
\]

(5.4)

\[
L(\omega) = \begin{bmatrix} L_+(\omega) & 0 \\ 0 & L_-(\omega) \end{bmatrix}.
\]

Let us remind the reader that both \( v \) and \( u \) in (5.4) depend on \( \omega \). Then equation (5.3) which describes the linearization at the solitary wave \( \phi_\omega e^{i \omega t} \) takes the form

\[
\partial_t \begin{bmatrix} R \\ S \end{bmatrix} = J L(\omega) \begin{bmatrix} R \\ S \end{bmatrix} = \begin{bmatrix} 0 & L_-(\omega) \\ -L_+(\omega) & 0 \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix}.
\]

Lemma 5.1. For any nonlinearity \( f(s) \) in (2.1), the spectrum of the linearization at a solitary wave \( \phi_\omega e^{i \omega t} \) has the following properties:

1. \( \sigma_{\text{ess}}(JL(\omega)) = i \mathbb{R} \setminus (-i(m - \omega), i(m - \omega)) \);

2. \( \sigma_d(JL(\omega)) \supset \{ \pm 2\omega i; 0 \} \).

See [Com11].

Unstable eigenvalue of \( JL \) for \( \omega \lesssim m \)

Proposition 5.2. Let \( k \geq 3 \). There is \( \omega_0 \leq m \) such that for \( \omega \in (\omega_0, m) \) there are two families of eigenvalues

\[
\pm \lambda_\omega \in \sigma_p(JL(\omega)), \quad \lambda_\omega > 0, \quad \lambda_\omega = O(\epsilon^2).
\]

Proof. In the explicit form, the relation

\[
\begin{bmatrix} 0 \\ -L_+ \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix} = \lambda \begin{bmatrix} R \\ S \end{bmatrix}
\]

can be written as follows:

\[
\begin{bmatrix} -\lambda & 0 & g - \omega & \partial_x \\ 0 & -\lambda & -\partial_x & g - \omega \\ -m + \omega + f + 2f'v^2 & -\partial_x - 2f'vu & -\lambda & 0 \\ \partial_x - 2f'vu & m + \omega - f - 2f'u^2 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ S_1 \\ S_2 \end{bmatrix} = 0.
\]

(5.5)

Dividing the first and the third rows by \( \epsilon^2 \), the second and the fourth rows by \( \epsilon \), and substituting \( y = \epsilon x \), \( R_2 = \epsilon R_2 \), \( S_2 = \epsilon S_2 \), we get

\[
\begin{bmatrix} -\lambda \epsilon^2 \\ -m + \omega + f + 2f'v^2 \\ \partial_y - 2f'vu \end{bmatrix} = \frac{1}{\epsilon} \begin{bmatrix} m - \omega - f \\ \partial_y - 2f'vu \\ \partial_y - 2f'vu \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ S_1 \end{bmatrix} = 0,
\]

(5.6)
which simplifies to

\[
\begin{bmatrix}
-\frac{\lambda}{\epsilon} & 0 & \frac{m-\omega-f}{\epsilon^2} & -\partial_y \\
0 & -\lambda & -\partial_y & -m - \omega + f \\
-m + \omega + f + 2f'u^2 & -\partial_y - \frac{2f'u}{\epsilon} & -\frac{\lambda}{\epsilon^2} & 0 \\
\partial_y - \frac{2f'u}{\epsilon} & m + \omega - f + 2f'u^2 & 0 & -\lambda
\end{bmatrix}
\begin{bmatrix}
R_1 \\
\hat{R}_2 \\
S_1 \\
\hat{S}_2
\end{bmatrix} = 0. \quad (5.7)
\]

Let \( \Lambda = \lim_{\epsilon \to 0} \frac{\lambda}{\epsilon^2} \). We introduce the matrices

\[
A_\Lambda = \begin{bmatrix}
-\Lambda & 0 & \frac{1}{2} - U^k & \partial_y \\
0 & 0 & -\partial_y & -2 \\
-\frac{1}{2} + (2k + 1)U^k & -\partial_y & -\Lambda & 0 \\
\partial_y & 0 & 0 & 0
\end{bmatrix}, \quad K_1 = \text{diag}[1, 0, 1, 0], \quad K_2 = \text{diag}[0, 1, 0, 1]. \quad (5.8)
\]

Above,

\[
U(y) = \left( \frac{k + 1}{2 \cosh^2(ky)} \right)^{1/k}; \quad (5.9)
\]

cf. (3.5). Noting that one has \( \mathcal{X}(x) = \epsilon \hat{U}(\epsilon x) + O(\epsilon^2) \) by Lemma 4.3, we can write (5.7) in the form

\[
A_\Lambda \eta = \left( \frac{\lambda}{\epsilon^2} - \Lambda \right) K_1 \eta + \lambda K_2 \eta + W \eta, \quad (5.10)
\]

where \( \eta = \begin{bmatrix} R_1 \\ \hat{R}_2 \\ S_1 \\ \hat{S}_2 \end{bmatrix} \in \mathbb{C}^4 \) and

\[
W(y, \epsilon) = A_\Lambda - \begin{bmatrix}
-\frac{\lambda}{\epsilon^2} & 0 & \frac{m-\omega-f}{\epsilon^2} & -\partial_y \\
0 & -\lambda & -\partial_y & -m - \omega + f \\
-m + \omega + f + 2f'u^2 & -\partial_y - \frac{2f'u}{\epsilon} & -\frac{\lambda}{\epsilon^2} & 0 \\
\partial_y - \frac{2f'u}{\epsilon} & m + \omega - f + 2f'u^2 & 0 & -\lambda
\end{bmatrix} - \left( \frac{\lambda}{\epsilon^2} - \Lambda \right) K_1 - \lambda K_2. \quad (5.11)
\]

is the zero order differential operator with \( L^\infty \) coefficients.

**Lemma 5.3.** \( \|W(\cdot, \epsilon)\|_{L^\infty(\mathbb{R}, \mathbb{C}^4 \to \mathbb{C}^4)} \leq O(\epsilon^{2/k}) \).

**Proof.** By Lemma 4.3, one has

\[
v(x, \omega) = \epsilon \hat{U}(\epsilon x) + O(\epsilon^2), \quad \|u(\cdot, \omega)\|_{L^\infty} = O(\epsilon^{1+\frac{2}{k}}).
\]

Then

\[
f(v^2 - u^2) = v^{2k} + O(\epsilon^{2+\frac{2}{k}}) = \epsilon^2 U(\epsilon x)^k + O(\epsilon^{2+\frac{2}{k}}),
\]

\[
f'(v^2 - u^2) = kv^{2k-2} + O(\epsilon^2) = O(\epsilon^{2-\frac{2}{k}}),
\]

\[
f''(v^2 - u^2)v = \epsilon^2 k U(\epsilon x)^k + O(\epsilon^{2+\frac{2}{k}}),
\]

\[
f''(v^2 - u^2)u = O(\epsilon^3).
\]

Now the proof follows from the definition of \( A_\Lambda \) and \( K_1, K_2 \) in (5.8). \( \square \)

**Lemma 5.4.** \( \dim \ker A_\Lambda > 0 \) if and only if \( \Lambda \) is an eigenvalue of the operator

\[
jH = \begin{bmatrix}
\frac{1}{2} \partial_y^2 - \frac{1}{2} & 0 & -\frac{1}{2} \partial_y - \frac{1}{2} \\
0 & \frac{1}{2} & -\frac{1}{2} \partial_y^2 + \frac{1}{2} - U^k
\end{bmatrix}.
\]

When \( k \geq 3 \), there is \( \Lambda > 0 \) such that \( \pm \Lambda \in \sigma_d(jH) \).

12
Proof. The first statement of the Lemma follows from the structure of the second and the fourth rows of $A_A$. The second statement follows from Proposition 5.3.

By Lemma 5.4, if $k \geq 3$, then there is $\Lambda > 0$ such that $\pm \Lambda \in \sigma_d(j)$. Now we will use the Rayleigh-Schrödinger perturbation theory to show that there are $\lambda \in \sigma_d(jL)$ with $\lambda \approx \pm \epsilon^2 \Lambda$. Let $\Phi_A \in \ker A_A$ with $\|\Phi_A\|_{L^2} = 1$, and let $P_A : L^2(\mathbb{R}, \mathbb{C}^4) \rightarrow H^\infty(\mathbb{R}, \mathbb{C}^4)$ be the spectral projector onto the corresponding eigenspace. Coupling (5.10) with $j\Phi_A$ and projecting (5.10) onto Range $(1 - P_A)$, one has:

$$0 = \langle j\Phi_A, A_A\eta \rangle = \left(\frac{\lambda}{\epsilon^2} - \Lambda\right) \langle j\Phi_A, K_1 \eta \rangle + \lambda \langle j\Phi_A, K_2 \eta \rangle + \langle j\Phi_A, W \eta \rangle,$$

$$\quad (1 - P_A)\eta = A_A^{-1}(1 - P_A) \left(\frac{\lambda}{\epsilon^2} - \Lambda\right) K_1 + \lambda K_2 + W \right)\eta. \quad (5.13)$$

We will find $\eta$ in the form $\eta = \Phi_A + \zeta$, with $\zeta \in \text{Range} (1 - P_A)$. Denote $\mu = \frac{\lambda}{\epsilon^2} - \Lambda$. Then the relations (5.12) and (5.13) can be written as

$$0 = \mu \langle j\Phi_A, K_1 \eta \rangle + \mu \langle j\Phi_A, K_2 \eta \rangle + \epsilon^2(\Lambda + \mu) \langle j\Phi_A, K_2(\Phi_A + \zeta) \rangle + \langle j\Phi_A, W(\Phi_A + \zeta) \rangle, \quad (5.14)$$

$$\zeta = A_A^{-1}(1 - P_A) \left(\mu K_1 + \epsilon^2(\Lambda + \mu) K_2 + W \right)\Phi_A + \zeta. \quad (5.15)$$

Let us note that $\langle j\Phi_A, K_1 \eta \rangle \neq 0$. The equations (5.14), (5.15) could be written as $\mu = M(\mu, \zeta), \zeta = Z(\mu, \zeta)$, with functions $M : \mathbb{R} \times L^2(\mathbb{R}, \mathbb{C}^4) \rightarrow \mathbb{R}, Z : \mathbb{R} \times L^2(\mathbb{R}, \mathbb{C}^4) \rightarrow L^2(\mathbb{R}, \mathbb{C}^4)$ given by

$$M(\mu, \zeta) = \frac{1}{\langle j\Phi_A, K_1 \Phi_A \rangle} \left[\mu \langle j\Phi_A, K_1 \eta \rangle + \epsilon^2(\Lambda + \mu) \langle j\Phi_A, K_2(\Phi_A + \zeta) \rangle + \langle j\Phi_A, W(\Phi_A + \zeta) \rangle\right], \quad (5.16)$$

$$Z(\mu, \zeta) = A_A^{-1}(1 - P_A) \left(\mu K_1 + \epsilon^2(\Lambda + \mu) K_2 + W \right)\Phi_A + \zeta. \quad (5.17)$$

Pick $\Gamma \geq 1$ such that

$$\Gamma \geq 2\|A_A^{-1}(1 - P_A)K_1 \Phi_A\|_{L^2}. \quad (5.18)$$

Lemma 5.5. Consider $\mathbb{R} \times L^2(\mathbb{R}, \mathbb{C}^4)$ endowed with the metric

$$\| (\mu, \zeta) \|_\Gamma = |\Gamma| |\mu| + \| \zeta \|_{L^2(\mathbb{R}, \mathbb{C}^4)}. \quad$$

There is $\omega_1 \in (\omega_0, \omega)$ such that for $\omega \in (\omega_1, \omega)$ the map

$$M \times Z : \mathbb{R} \times L^2(\mathbb{R}, \mathbb{C}^4) \rightarrow \mathbb{R} \times L^2(\mathbb{R}, \mathbb{C}^4), \quad (\mu, \zeta) \mapsto (M(\mu, \zeta), Z(\mu, \zeta)), \quad (5.19)$$

restricted onto the set

$$B_{\epsilon^2/k} = \{ (\mu, \zeta) \in \mathbb{R} \times L^2(\mathbb{R}, \mathbb{C}^4) ; \| (\mu, \zeta) \|_\Gamma \leq \epsilon^{1/k} \} \subset \mathbb{R} \times L^2(\mathbb{R}, \mathbb{C}^4)$$

is an endomorphism and a contraction with respect to $\| \cdot \|_\Gamma$.

Proof. The proof follows from the definitions (5.16), (5.17). We needed to introduce the factor $\Gamma$ into the definition of the metric to make sure that the contribution of the term $A_A^{-1}(1 - P_A)\mu K_1 \Phi_A$ from (5.17) into $\| (M(\mu, \zeta), Z(\mu, \zeta)) \|_\Gamma$ is bounded by $\frac{1}{\epsilon^2} |\mu| \leq \frac{1}{\epsilon^2} |\mu|$. The contribution of all other terms is bounded by $\frac{1}{\epsilon^2} |\mu| |\zeta| + O(\epsilon^{2/k})$, placing $(M(\mu, \zeta), Z(\mu, \zeta))$ inside $B_{\epsilon^2/k}$ whenever $(\mu, \zeta) \in B_{\epsilon^2/k}$, as long as $\omega_1$ is sufficiently close to $\omega$, so that $\epsilon = \sqrt{\omega^2 - \omega^2}$ is sufficiently small. The contribution $O(\epsilon^{2/k})$ comes from the terms with $W \Phi_A$, due to the bound $\| W (\cdot, \epsilon) \|_{L^\infty(\mathbb{R}, \mathbb{C}^4)} = O(\epsilon^{2/k})$ established in Lemma 5.3.

The contribution of the term $A_A^{-1}(1 - P_A)\mu K_1 \Phi_A$ into the norm $\| Z(\mu, \zeta) - Z(\mu', \zeta') \|_{L^2}$ is bounded by $\frac{1}{\Gamma} |\mu - \mu'| \leq \frac{1}{\Gamma} |(\mu - \mu', \zeta - \zeta')|$. The contribution of all other terms from (5.16), (5.17) into $\| (M(\mu, \zeta) - M(\mu', \zeta'), Z(\mu, \zeta) - Z(\mu', \zeta')) \|_\Gamma$ could be made smaller than $\frac{1}{\Gamma} |(\mu - \mu', \zeta - \zeta')|_\Gamma$ by choosing $\omega_1$ sufficiently close to $\omega$, so that $\epsilon = \sqrt{\omega^2 - \omega^2}$ is sufficiently small. It follows that $(M \times Z)|_{B_{\epsilon^2/k}}$ is a contraction in the metric $\| \cdot \|_\Gamma$. 

According to Lemma 5.5 by the contraction mapping theorem, for $\omega \in (\omega_1, m)$, the map (5.19) has a fixed point 
$$(\mu_0(\omega), \zeta_0(\omega)) \in B_{\epsilon/k} \subset \mathbb{R} \times L^2(\mathbb{R}, \mathbb{C}^4).$$
Thus, we have
$$\pm \epsilon^2(\Lambda + \mu_0(\omega)) \in \sigma_p(JL(\omega)), \quad \omega \in (\omega_1, m),$$
with $|\mu_0(\omega)| \leq \epsilon^{1/k}$, finishing the proof of the proposition. \hfill \Box

By Remark 2.2 Proposition 5.2 finishes the proof of Theorem 2.1.

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