REGULARITY ESTIMATES FOR GREEN OPERATORS OF DIRICHLET AND NEUMANN PROBLEMS ON WEIGHTED HARDY SPACES

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Abstract. In this paper we first study the generalized weighted Hardy spaces $H^{p}_{L,w}(X)$ for $0 < p \leq 1$ associated to nonnegative self-adjoint operators $L$ satisfying Gaussian upper bounds on the space of homogeneous type $X$ in both cases of finite and infinite measure. We show that the weighted Hardy spaces defined via maximal functions and atomic decompositions coincide. Then we prove weighted regularity estimates for the Green operators of the inhomogeneous Dirichlet and Neumann problems in suitable bounded or unbounded domains including bounded semiconvex domains, convex regions above a Lipschitz graph and upper half-spaces. Our estimates are in terms of weighted $L^p$ spaces for the range $1 < p < \infty$ and in terms of the new weighted Hardy spaces for the range $0 < p \leq 1$. Our regularity estimates for the Green operators under the weak smoothness assumptions on the boundaries of the domains are new, especially the estimates on Hardy spaces for the full range $0 < p \leq 1$ and the case of unbounded domains.

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1. Introduction

Let $\Omega$ be an open connected domain in $\mathbb{R}^n$. Denote by $W^{1,2}(\Omega)$ the Sobolev space on $\Omega$ with the norm

$$\|f\|_{W^{1,2}(\Omega)} = \|f\|_{L^2(\Omega)} + \|\nabla f\|_{L^2(\Omega)}.$$ 

The closure of $C_c^\infty(\Omega)$ in $W^{1,2}(\Omega)$ will be denoted by $W^{1,2}_0(\Omega)$.

Consider the inhomogeneous Dirichlet problem for the Laplacian

$$\begin{cases}
\Delta u = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}$$

(1)

Denote by $G_D$ the Green operator for Dirichlet problem (1), i.e. the solution operator which maps each $f \in C^\infty(\Omega)$ to the unique solution $u := G_D(f) \in W^{1,2}_0(\Omega)$ of the problem (1).

We also consider the inhomogeneous Neumann problem for the Laplacian

$$\begin{cases}
\Delta u = f & \text{in } \Omega \\
\partial_\nu u = 0 & \text{on } \partial \Omega
\end{cases}$$

(2)

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for a suitable domain $\Omega$. Denote by $G_N$ the Green operator for Neumann problem (2), i.e. the solution operator which maps each $f \in C^\infty(\overline{\Omega})$ with $\int_\Omega f = 0$ to the unique solution $u := G_N(f) \in W^{1,2}(\Omega)$ of the problem (2).

One of the most interesting and important problems concerning problems (1) and (2) is the regularity estimate for the Green operators $G_D$ and $G_N$. We would like to give a shortlist of known results in this research direction for the $L^p$-boundedness with $1 < p < \infty$ (see for example [27]).

(i) The $L^p$-boundedness for $\nabla^2 G_D$ and $\nabla^2 G_N$ with $1 < p < \infty$ on a bounded $C^\infty$ domain was obtained in [4] and [40].

(ii) In [37], it was proved that $\nabla^2 G_D$ is well-defined and bounded on $L^2(\Omega)$ provided that $\Omega$ is a bounded convex domain.

(iii) Under the assumption that $\Omega$ is a bounded and convex domain, the weak type (1,1) for $\nabla^2 G_D$ was proved in [25, 31], meanwhile the boundedness for $\nabla^2 G_D$ on the suitable Hardy space was obtained in [2].

(iv) The $L^2$-boundedness for $\nabla^2 G_N$ appeared first in [32]. Then it was proved that it is bounded from some Hardy space into $L^1(\Omega)$, hence by interpolation it is bounded on $L^p(\Omega)$ for $1 < p < 2$. See for example [3].

(v) It is important to note that the $L^p$-boundedness for $\nabla^2 G_D$ and $\nabla^2 G_N$ may fail in the class of Lipschitz domains for any $p \in (1, \infty)$ and in the class of convex domains for any $p \in (2, \infty)$. For the further details, see [2, 3, 23, 35, 44] and the references therein.

The following brief summary gives an overview of the progress concerning the boundedness for $\nabla^2 G_D$ and $\nabla^2 G_N$ for $0 < p \leq 1$.

(i) In [16, 17, 18] the authors studied the theory of Hardy spaces on domains. In [16], they obtained the boundedness of $\nabla^2 G_D$ and $\nabla^2 G_N$ on these Hardy spaces with the range $0 < p \leq 1$ when the domains are bounded $C^\infty$ domains. The boundedness on the Hardy spaces with the range $\frac{n}{n+1} < p \leq 1$ for $\nabla^2 G_D$ and $\nabla^2 G_N$ on either bounded Lipschitz domains or the upper half-spaces was proved in [17, 18] (see also [42]).

(ii) In the case when $\Omega$ is a bounded Lipschitz domain satisfying a uniform exterior ball condition, the estimates of $\nabla^2 G_D$ on Besov and Triebel–Lizorkin spaces were proved in [43]. These results include the boundedness of $\nabla^2 G_D$ on local Hardy spaces for $\frac{n}{n+1} < p \leq 1$.

(iii) Recently, the authors in [27] developed the theory of Hardy spaces associated to Dirichlet Laplacians on bounded semiconvex domains and Neumann Laplacians on bounded convex domains. Then they gave a new approach to obtain the boundedness of $\nabla^2 G_D$ and $\nabla^2 G_N$ on local Hardy spaces for $\frac{n}{n+1} < p \leq 1$. These results were extended to weighted Orlicz-Hardy spaces in [15].

Although the regularity estimates for $\nabla^2 G_D$ and $\nabla^2 G_N$ have been investigated intensively, there are still a number of interesting open problems.

**Problem 1:** The Hardy space estimates for $\nabla^2 G_D$ and $\nabla^2 G_N$ for a full range $0 < p \leq 1$ are only known when $\Omega$ is a bounded $C^\infty$ domain. Under weaker smoothness assumptions such as Lipschitz domains, the range $\frac{n}{n+1} < p \leq 1$ is known but the range $0 < p \leq \frac{n}{n+1}$ is still open.

**Problem 2:** There are a number of results for the Hardy space estimates for $\nabla^2 G_D$ and $\nabla^2 G_N$ on bounded domains, while to the best of our knowledge similar results on unbounded domains are still open. See [18] for the boundedness of $\nabla^2 G_D$ and $\nabla^2 G_N$ on Hardy spaces with $\frac{n}{n+1} < p \leq 1$ on the upper half-spaces. See [1, 3, 32] for the boundedness of $\nabla^2 G_D$ and $\nabla^2 G_N$ from the Hardy spaces ($p = 1$) to $L^1$ on certain unbounded Lipschitz domains. In the case of general unbounded domains, the Hardy space estimates for both $\nabla^2 G_D$ and $\nabla^2 G_N$ are still unknown even for $p = 1$.

**Problem 3:** Concerning the weighted estimates, recently in [15] the authors introduced the local weighted Orlicz-Hardy spaces in a bounded semiconvex/convex domain and they obtained the boundedness of $\nabla^2 G_D$ and $\nabla^2 G_N$ on these spaces with a limited range of $p$. However, it seems that the class of weights in [15] is not optimal. It is natural to raise the question on
finding better class of weights and weighted $L^p$ estimates for $1 < p < \infty$ not only for bounded
domains but also for unbounded domains.

The aim of this paper is to address Problems 1, 2 and 3 for different types of domains. Our
main results are for bounded domains in Theorem 1.5 and for unbounded domains in Theorem
1.7. In the specific case of upper half spaces, in addition to estimates from Theorem 1.7, we
give further results in Theorem 1.9. Our approach in this paper might be applicable to other
problems in different settings since we state our assumptions on certain heat kernel estimates;
see the theorems in Sections 3.1 and 3.2.

In order to state the main results precisely, we first give definitions of weighted Hardy spaces.
For the weighted Hardy spaces on $\mathbb{R}^n$, we recall the definitions in [50]. Assume that $p \in (0, 1]$,
the weight $w$ belongs to the Muckenhoupt class $A_{\infty}(\mathbb{R}^n)$ (see Section 2 for the Muckenhoupt
weights) and $q \in (q_w, \infty]$ where $q_w$ is defined in (7). A bounded, measurable function $a$ is called
a $(p, q, w)$-atom if

(i) $a$ is supported in a ball $B \subset \mathbb{R}^n$;

(ii) $\|a\|_{L^p_w} \leq w(B)^{1/q-1/p}$;

(iii) $\int_B x^\alpha a(x)dx = 0$ for all multi–indices $\alpha$ with $|\alpha| \leq \lfloor n(q_w/p - 1)\rfloor$.

The Hardy space $H^{p,q}_w(\Omega)$ is defined as the set of all distributions $f \in \mathcal{F}'$ such that

$$f = \sum_{j=1}^{\infty} \lambda_j a_j$$

where $a_j$ are $(p, q, w)$-atoms and $\lambda_j$ are scalars with $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$. We also set

$$\|f\|_{H^{p,q}_w(\mathbb{R}^n)} = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j|^p : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\}$$

where the infimum is taken over all such decompositions.

It is well known that for $p \in (0, 1]$, $w \in A_{\infty}(\mathbb{R}^n)$ and $q \in (q_w, \infty]$ we have

$$H^{p,q}_w(\mathbb{R}^n) \equiv H^{p,\infty}_w(\mathbb{R}^n).$$

Hence, for any $p \in (0, 1]$ and $w \in A_{\infty}(\mathbb{R}^n)$ we define $H^p_w(\mathbb{R}^n)$ as any space $H^{p,q}_w(\mathbb{R}^n)$ with
$q \in (q_w, \infty]$.

We next recall the weighted Hardy spaces on domains of Miyachi [46].

**Definition 1.1.** Let $\Omega$ is an open set in $\mathbb{R}^n$. Let $p \in (0, 1]$, $w \in A_{\infty}(\mathbb{R}^n)$ and $q \in (q_w, \infty]$. A
bounded, measurable function $a$ from $\Omega$ to $\mathbb{R}$ is called a $(p, q, w)_{M_i}$-atom if

(i) $a$ is supported in a ball $B \subset \Omega$;

(ii) $\|a\|_{L^p_w(\Omega)} \leq w(B)^{1/q-1/p}$;

(iii) either $2B \subset \Omega$ and $4B \cap \partial\Omega \neq \emptyset$, or $4B \subset \Omega$ and

$$\int_B x^\alpha a(x)dx = 0$$

for all multi–indices $\alpha$ with $|\alpha| \leq \lfloor n(q_w/p - 1)\rfloor$.

The Hardy space $H^{p,q}_{M_i,w}(\Omega)$ is defined as the set of all $f \in \mathcal{F}'$ such that

$$f = \sum_{j=1}^{\infty} \lambda_j a_j$$

where $a_j$ are $(p, q, w)_{M_i}$-atoms and $\lambda_j$ are scalars with $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$. We also set

$$\|f\|_{H^{p,q}_{M_i,w}(\Omega)} = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j|^p : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\}$$

where the infimum is taken over all such decompositions.
Let $\phi \in C_c^\infty(B(0,1))$ be a non-negative radial function such that $\int \phi(x)dx = 1$. It was proved in [45] that the Hardy spaces $H_{M_1,w}^{p,q}(\Omega)$ can be characterized in terms of maximal functions of the form
\[
 f^+_M(x) = \max_{0 < t < \delta(x)/2} |\phi_t \ast f(x)|
\]
where $\delta(x) = d(x, \Omega^c)$ and $\phi_t(x) = t^{-n} \phi(x/t)$. More precisely, we have the following theorem from [46]:

**Theorem 1.2.** Let $p \in (0,1]$, $w \in A_\infty(\mathbb{R}^n)$ and $q \in (q_w, \infty]$. Then we have $f \in H_{M_1,w}^{p,q}(\Omega)$ if and only if $f^+_M \in L_w^p(\Omega)$; moreover,
\[
 \|f\|_{H_{M_1,w}^{p,q}(\Omega)} \sim \|f^+_M\|_{L_w^p(\Omega)}.
\]

From Theorem 1.2 for $w \in A_\infty(\mathbb{R}^n)$ and $p \in (0,1]$ we will write the Hardy spaces $H_{M_1,w}^p(\Omega)$ for any space $H_{M_1,w}^{p,q}(\Omega)$ with $q \in (q_w, \infty]$. The Hardy space $H_{M_1,w}^p(\Omega)$ is closely related to the Hardy space $H_{r,w}^p(\Omega)$ defined by
\[
 H_{r,w}^p(\Omega) = \{f \in S' : \text{there exists } F \in H_w^p(\mathbb{R}^n) \text{ so that } F|_{\Omega} = f\}
\]
with the norm
\[
 \|f\|_{H_{r,w}^p(\Omega)} = \inf\{\|F\|_{H_w^p(\mathbb{R}^n)} : F \in H_w^p(\mathbb{R}^n), F|_{\Omega} = f\}.
\]

Arguing similarly to [18] we can prove that if $\Omega$ is a bounded Lipschitz domain or a convex domain above a Lipschitz graph, then $H_{M_1,w}^p(\Omega) \equiv H_{r,w}^p(\Omega)$ for $w \in A_\infty(\mathbb{R}^n)$ and $\frac{mp}{n+1} < p \leq 1$.

Let $p \in (0,1]$, $w \in A_\infty(\mathbb{R}^n)$ and $q \in (q_w, \infty]$. A bounded, measurable function $a$ is called a local $(p,q,w)$-atom if

(i) $a$ is supported in a ball $B \subset \mathbb{R}^n$;
(ii) $\|a\|_{L_w^p(\Omega)} \leq w(B)^{1/q-1/p}$;
(iii) $\int_B x^\alpha a(x)dx = 0$ for all multi–indices $\alpha$ with $|\alpha| \leq \lfloor n(q_w/p - 1) \rfloor$, if $r_B < 1$.

Similarly to the weighted Hardy space $H_w^p(\mathbb{R}^n)$ we can define the weighted local Hardy spaces via local atomic decompositions for $p \in (0,1]$ and $w \in A_\infty(\mathbb{R}^n)$, and we denote these local weighted Hardy spaces by $h_w^p(\mathbb{R}^n)$. See for example [11].

**Definition 1.3.** Let $\Omega$ be an open set in $\mathbb{R}^n$. Let $p \in (0,1]$ and $w \in A_\infty(\mathbb{R}^n)$. The weighted Hardy space $H_{r,w}^p(\Omega)$ can be defined as follows:

\[
 H_{r,w}^p(\Omega) := \left\{ f \in H_w^p(\mathbb{R}^n) : f \equiv 0 \text{ on } \Omega^c \right\}, \quad \text{if } \Omega \text{ is unbounded}
\]

\[
 H_{r,w}^p(\Omega) := \left\{ f \in h_w^p(\mathbb{R}^n) : f \equiv 0 \text{ on } \Omega^c \right\}, \quad \text{if } \Omega \text{ is bounded},
\]

with the norm defined by
\[
 \|f\|_{H_{r,w}^p(\Omega)} := \left\{ \begin{array}{ll}
 \|f\|_{H_w^p(\Omega)}, & \text{if } \Omega \text{ is unbounded} \\
 \|f\|_{h_w^p(\Omega)}, & \text{if } \Omega \text{ is bounded},
\end{array} \right.
\]

We remark that if $\Omega$ is bounded domain and $w \equiv 1$, the Hardy spaces $H_{r,w}^p(\Omega)$ coincides with the local Hardy spaces of extension $h_w^p(\Omega)$ defined in [18]. We also note that in the case of bounded domain, the local Hardy spaces $h_w^p(\Omega)$ defined in [18] and the Hardy spaces $H_w^p(\Omega)$ defined by Coifmann and Weiss in [21] are the equivalent. For this reason we use the same notation $H_{r,w}^p(\Omega)$ for both cases of bounded and unbounded domains.

**Definition 1.4.** Let $p \in (0,1]$, $w \in A_\infty(\mathbb{R}^n)$, $q \in (q_w, \infty]$. A bounded, measurable function $a : \Omega \to \mathbb{R}$ is called an $(p,q,w)_\Omega$-atom if

(i) $a$ is supported in a ball $B \subset \mathbb{R}^n$ and $a \equiv 0$ on $\Omega^c$;
(ii) $\|a\|_{L_w^p(\Omega)} \leq w(B)^{1/q-1/p}$;
(iii) \( \int_B x^\alpha a(x) dx = 0 \) for all multi–indices \( \alpha \) with \(|\alpha| \leq \lfloor n(q_w/p - 1) \rfloor \).

In the case where \( \Omega \) is bounded, a function \( a \) can be viewed as an atom if
\[
\|a\|_{L^p_w(\Omega)} \leq w(\Omega)^{1/q - 1/p}.
\]

The Hardy space \( H_{at,w}^p(\Omega) \) is defined as the set of all distributions \( f \in \mathcal{S}' \) such that
\[
f = \sum_{j=1}^{\infty} \lambda_j a_j
\]
where \( a_j \) are \((p,q,w)\)-atoms and \( \lambda_j \) are scalars with \( \sum_j |\lambda_j|^p < \infty \). We also set
\[
\|f\|_{H_{at,w}^p(\Omega)}^p = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j|^p : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\}
\]
where the infimum is taken over all such decompositions.

It is easy to see that for \( p \in (0,1] \), \( w \in A_p(\mathbb{R}^n) \) and \( q \in (q_w,\infty] \) we have
\[
H_{at,w}^p(\Omega) = H_{\mathbb{S}^1,w}^p(\Omega)
\]
for either \( \Omega \) is bounded or \( \Omega \) is unbounded.

In what follows, denote by \( \Delta_D \) and \( \Delta_N \) the Dirichlet Laplacian and the Neumann Laplacian, respectively. For \( w \in A_\infty(\mathbb{R}^n) \) and \( 0 < p \leq 1 \), we denote by \( H_{\Delta_D,w}^p(\Omega) \) and \( H_{\Delta_N,w}^p(\Omega) \) the weighted Hardy spaces associated to \( \Delta_D \) and \( \Delta_N \), respectively. See the definitions of these function spaces in Section 2. Our first main result concerning the weighted estimates for \( \nabla^2 G_D \) and \( \nabla^2 G_N \) on bounded domains.

**Theorem 1.5.** Assume that \( \Omega \subset \mathbb{R}^n \) is a bounded, simply connected, semiconvex domain (see [27, Definition 4.6]) for the Dirichlet problem and \( \Omega \) is a bounded convex set for the Neumann problem. Then we have

(i) The operators \( \nabla^2 G_D \) and \( \nabla^2 G_N \) extend as bounded operators on \( L^p_w(\Omega) \) for all \( 1 < p < 2 \) and \( w \in A_p(\mathbb{R}^n) \cap RH(2/p)'(\mathbb{R}^n) \), and as bounded operators from \( L^p_w(\Omega) \) into \( L^1_{w,\infty}(\Omega) \) for \( w \in A_1(\mathbb{R}^n) \cap RH_2(\mathbb{R}^n) \).

(ii) The operator \( \nabla^2 G_D \) extends as a bounded operator from \( H_{\Delta_D,w}^p(\Omega) \) into \( H_{\Delta_D,w}^p(\Omega) \) for all \( 0 < p \leq 1 \) and \( w \in \bigcup_{1 < r < 2} A_r(\mathbb{R}^n) \cap RH(2/r)'(\mathbb{R}^n) \). As a consequence, the operator \( \nabla^2 G_D \) extends as a bounded operator on \( H_{\mathbb{S}^1,w}^p(\Omega) \) for all \( \frac{n}{n+1} < p \leq 1 \) and \( w \in \bigcup_{1 < r < r_0} A_r(\mathbb{R}^n) \cap RH(2/r)'(\mathbb{R}^n) \) where \( r_0 = \frac{2p}{n+1} \).

(iii) The operator \( \nabla^2 G_N \) extends as a bounded operator from \( H_{\Delta_N,w}^p(\Omega) \) into \( H_{\Delta_N,w}^p(\Omega) \) for all \( 0 < p \leq 1 \) and \( w \in \bigcup_{1 < r < 2} A_r(\mathbb{R}^n) \cap RH(2/r)'(\mathbb{R}^n) \). As a consequence, the operator \( \nabla^2 G_D \) extends as a bounded operator from \( H_{\mathbb{S}^1,w}^p(\Omega) \) into \( H_{\mathbb{S}^1,w}^p(\Omega) \) for all \( \frac{n}{n+1} < p \leq 1 \) and \( w \in \bigcup_{1 < r < r_0} A_r(\mathbb{R}^n) \cap RH(2/r)'(\mathbb{R}^n) \) where \( r_0 = \frac{2p}{n+1} \).

**Remark 1.6.** (a) The main parts of Theorem 1.5 are (ii) and (iii).

(b) The \( L^p \)-weighted estimates in part (i) was obtained in [8] for \( w \in A_1(\Omega) \cap RH_2(\Omega) \). Here we prove the results (i) in terms of \( w \in A_1(\mathbb{R}^n) \cap RH_2(\mathbb{R}^n) \). While the class \( A_1(\mathbb{R}^n) \) is smaller than \( A_1(\Omega) \), it is not clear about the two classes \( RH_2(\mathbb{R}^n) \) and \( RH_2(\Omega) \). We note that the our proof in this paper can be used to reproduce the result for \( w \in A_1(\Omega) \cap RH_2(\Omega) \) as well.

(c) The boundedness of \( \nabla^2 G_D \) and \( \nabla^2 G_N \) from \( H_{\Delta_D,w}^p(\Omega) \) into \( H_{\Delta_D,w}^p(\Omega) \) and from \( H_{\Delta_N,w}^p(\Omega) \) into \( H_{\Delta_N,w}^p(\Omega) \) for \( 0 < p \leq 1 \) are new even for \( w \equiv 1 \). This answers the open question in [33] for the case \( 0 < p \leq \frac{n}{n+1} \) under weak smoothness condition on the boundary of the domain.

(d) The weighted estimates for \( \frac{n}{n+1} < p \leq 1 \) was obtained in [15, Theorems 1.8–1.9] for the local weighted Orlicz-Hardy spaces. However, our estimates are sharper than those in [15,
Theorems 1.8–1.9] since the class of weights \( w \in \bigcup_{1 < r < r_0} A_r(\mathbb{R}^n) \cap RH_{2/r}(\mathbb{R}^n) \) where \( r_0 = \frac{np}{n+1} \) is strictly larger than those in [15, Theorems 1.8–1.9] which is \( w \in A_q(\mathbb{R}^n) \cap RH_r(\mathbb{R}^n) \) where \( q < \frac{np}{n+1}, \, r > \frac{2(2-q)}{2-q} \) and \( \frac{2q}{p} < \frac{n+1}{n} + \frac{r-1}{pr} \).

Our next main result is the following.

**Theorem 1.7.** Let \( \Omega \subset \mathbb{R}^n \) be a convex domain above a Lipschitz graph. Then we have

(i) The operators \( \nabla^2 G_D \) and \( \nabla^2 G_N \) extend as bounded operators on \( L^p_w(\Omega) \) for all \( 1 < p < 2 \) and \( w \in A_p(\mathbb{R}^n)(\mathbb{R}^n) \cap RH_{2/p}(\mathbb{R}^n) \), and as bounded operators from \( L^1_w(\Omega) \) into \( L^{1,\infty}(\Omega) \) for \( w \in A_1(\mathbb{R}^n) \cap RH_2(\mathbb{R}^n) \).

(ii) The operator \( \nabla^2 G_D \) extends as a bounded operator from \( H^p_{\Delta D,w}(\Omega) \) into \( H^p_{M_1,w}(\Omega) \) for all \( 0 < p \leq 1 \) and \( w \in \bigcup_{1 < r < 2} A_r(\mathbb{R}^n) \cap RH_{2/r}(\mathbb{R}^n) \). As a consequence, the operator \( \nabla^2 G_D \) extends as a bounded operator on \( H^p_{r,w}(\Omega) \) for all \( \frac{n}{n+1} < p \leq 1 \) and \( w \in \bigcup_{1 < r < r_0} A_r(\mathbb{R}^n) \cap RH_{2/r}(\mathbb{R}^n) \) where \( r_0 = \frac{np}{n+1} \).

(iii) The operator \( \nabla^2 G_N \) extends as a bounded operator from \( H^p_{\Delta N,w}(\Omega) \) into \( H^p_{M_1,w}(\Omega) \) for all \( 0 < p \leq 1 \) and \( w \in \bigcup_{1 < r < 2} A_r(\mathbb{R}^n) \cap RH_{2/r}(\mathbb{R}^n) \). As a consequence, the operator \( \nabla^2 G_N \) extends as a bounded operator from \( H^p_{r,w}(\Omega) \) into \( H^p_{r,w}(\Omega) \) for all \( \frac{n}{n+1} < p \leq 1 \) and \( w \in \bigcup_{1 < r < r_0} A_r(\mathbb{R}^n) \cap RH_{2/r}(\mathbb{R}^n) \) where \( r_0 = \frac{np}{n+1} \).

**Remark 1.8.** The \( L^p \)-boundedness from a suitable Hardy space into \( \nabla^2 G_D \) and \( \nabla^2 G_N \) for \( 1 < p < 2 \) when \( \Omega \subset \mathbb{R}^n \) is a convex domain above a Lipschitz graph was obtained in [2, 3]. The results in Theorem 1.7 are new; moreover the results in (ii) and (iii) are new even for unweighted cases.

In the case of half spaces, in addition to Theorem 1.7, we can have further estimates as follows.

**Theorem 1.9.** Let \( \Omega = \mathbb{R}^n_+ \) be the upper half-space. Then we have

(i) The operators \( \nabla^2 G_D \) and \( \nabla^2 G_N \) extend as bounded operators on \( L^p_w(\Omega) \) for all \( 1 < p < \infty \) and \( w \in A_p(\mathbb{R}^n) \), and as bounded operators from \( L^1_w(\Omega) \) into \( L^{1,\infty}(\Omega) \) for \( w \in A_1(\mathbb{R}^n) \).

(ii) The operator \( \nabla^2 G_D \) extends as a bounded operator from \( H^p_{\Delta D,w}(\Omega) \) into \( H^p_{M_1,w}(\Omega) \) for all \( 0 < p \leq 1 \) and \( w \in A_\infty(\mathbb{R}^n) \). As a consequence, the operator \( \nabla^2 G_D \) extends as a bounded operator on \( H^p_{r,w}(\Omega) \) for all \( \frac{n}{n+1} < p \leq 1 \) and \( w \in \bigcup_{1 < r < r_0} A_r(\mathbb{R}^n) \cap RH_{2/r}(\mathbb{R}^n) \) where \( r_0 = \frac{np}{n+1} \).

(iii) The operator \( \nabla^2 G_D \) extends as a bounded operator from \( H^p_{\Delta D,w}(\Omega) \) into \( H^p_{M_1,w}(\Omega) \) for all \( 0 < p \leq 1 \) and \( w \in A_\infty(\mathbb{R}^n) \).

(iv) The operator \( \nabla^2 G_N \) extends as a bounded operator from \( H^p_{\Delta N,w}(\Omega) \) into \( H^p_{M_1,w}(\Omega) \) for all \( 0 < p \leq 1 \) and \( w \in A_\infty(\mathbb{R}^n) \). As a consequence, the operator \( \nabla^2 G_D \) extends as a bounded operator from \( H^p_{\Delta N,w}(\Omega) \) into \( H^p_{r,w}(\Omega) \) for all \( 0 < p \leq 1 \) and \( w \in A_\infty(\mathbb{R}^n) \).

**Remark 1.10.** In [18], it was proved that \( \nabla^2 G_D \) is bounded on \( H^p_{r}(\Omega) \) for all \( \frac{n}{n+1} < p \leq 1 \), and \( \nabla^2 G_D \) and \( \nabla^2 G_N \) are bounded from \( H^p_{r}(\Omega) \) into \( H^p_{r}(\Omega) \) for all \( 0 < p \leq 1 \). Hence, our results can be viewed as an extension of those in [18] to the weighted estimates, meanwhile the boundedness from \( H^p_{\Delta D,w}(\Omega) \) into \( H^p_{M_1,w}(\Omega) \) for \( \nabla^2 G_D \) are new even for unweighted case.

Our approach relies on the theory of Hardy spaces associated to operators which was initially developed in [6] and has been studied intensively by many authors. See for example [29, 34, 33] and the references therein. We first prove that the Hardy spaces defined via atomic decompositions and maximal functions are equivalent. See Theorem 2.4. This plays a crucial role in
the proof of our main results and is interesting in its own right. We note that [48, 49] showed
unweighted estimates when the underlying space has infinite measure.

We also remark that the atomic decompositions for the Hardy spaces were obtained in [27, 15]
by using the existing atomic decomposition results for the tent spaces. However, it seems that
the approach in [27, 15] is not applicable to our setting when the domain is bounded since the
atomic decomposition results for the tent spaces might not be true for the bounded domains.
To overcome this trouble, we adopt some ideas in [10] which makes use of kernel estimates for
functional calculus, estimates for maximal functions and the Whitney covering lemma. Note
that our approach can be easily applied to study the problems in the Musielak–Hardy spaces
and this might be done elsewhere.

The organization of our paper is as follows. In Section 2, we prove the equivalence between the
atomic Hardy spaces and maximal Hardy spaces in the general setting of spaces of homogeneous
type. This result is interesting in its own right. The proofs of the main results will be addressed
in Section 3.

Notation. As usual we use $C$ and $c$ to denote positive constants that are independent of the
main parameters involved but may differ from line to line. The notation $A \lesssim B$ means $A \leq CB,$
and $A \sim B$ means that both $A \lesssim B$ and $B \lesssim A$ hold. We use $\int_{E} f d\mu = \frac{1}{|E|} \int_{E} f d\mu$ to denote
the average of $f$ over $E.$ We write $B(x, r)$ to denote the ball centred at $x$ with radius $r.$ By
a ‘ball’ $B$ we mean the ball $B(x_{B}, r_{B})$ with some fixed centre $x_{B}$ and radius $r_{B}.$ The annuli
around a given ball $B$ will be denoted by $S_{j}(B) = 2^{j+1}B \setminus 2^{j}B$ for $j \geq 1$ and $S_{0}(B) = 2B$ for
$j = 0.$

2. Weighted Hardy spaces associated to operators

In this section, we study weighted Hardy spaces on a general space of homogeneous type $X$
which is of interest in its own right and has a doubling domain $\Omega \subset \mathbb{R}^{n}$ as a special case.

Let $(X, d, \mu)$ be a metric space endowed with a nonnegative Borel measure $\mu$ satisfying the
doubling condition: there exists a constant $C_{1} > 0$ such that
\begin{equation}
\mu(B(x, 2r)) \leq C_{1} \mu(B(x, r))
\end{equation}
for all $x \in X,$ $r > 0$ and all balls $B(x, r) := \{ y \in X : d(x, y) < r \}.$ For the moment $\mu(X)$ may
be finite or infinite.

It is not difficult to see that the condition (4) implies that there exists a “dimensional”
constant $n \geq 0$ so that
\begin{equation}
\mu(B(x, \lambda r)) \leq C_{2} \lambda^{n} \mu(B(x, r))
\end{equation}
for all $x \in X, r > 0$ and $\lambda \geq 1,$ and
\begin{equation}
\mu(B(x, r)) \leq C_{3} \mu(B(y, r)) \left( 1 + \frac{d(x, y)}{r} \right)^{n}
\end{equation}
for all $x, y \in X, r > 0.$

A weight $w$ is a non-negative measurable and locally integrable function on $X.$ We say that
$w$ belongs to the Muckenhoupt class $A_{p}(X)$ for $1 < p < \infty,$ if there exists a constant $C$
such that for every ball $B \subset X,$
\[
\left( \int_{B} w(x) d\mu(x) \right) \left( \int_{B} w^{-1/(p-1)}(x) d\mu(x) \right)^{p-1} \leq C.
\]
For $p = 1,$ we say that $w \in A_{1}(X)$ if there is a constant $C$ such that for every ball $B \subset X,$
\[
\int_{B} w(y) d\mu(y) \leq C w(x) \text{ for a.e. } x \in B.
\]
We define $A_{\infty}(X) = \cup_{p \geq 1} A_{p}(X).$
The reverse H"older classes of weights $RH_q$ are defined in the following way: $w \in RH_q$, $1 < q < \infty$, if there is a constant $C$ such that for any ball $B \subset X$,

$$\left( \int_B w^q(y) d\mu(y) \right)^{1/q} \leq C \int_B w(x) d\mu(x).$$

The endpoint $q = \infty$ is given by the condition: $w \in RH_\infty$ whenever, there is a constant $C$ such that for any ball $B \subset X$,

$$w(x) \leq C \int_B w(y) d\mu(y) \text{ for a.e. } x \in B.$$

Let $w \in A_\infty(\mathbb{R}^n)$, for $1 \leq p < \infty$, the weighted spaces $L^p_w(X)$ can be defined by

$$\left\{ f : \int_X |f(x)|^p w(x) d\mu(x) < \infty \right\}$$

with the norm

$$\|f\|_{L^p_w(X)} = \left( \int_X |f(x)|^p w(x) d\mu(x) \right)^{1/p}.$$

We sum up some of the standard properties of classes of weights in the following lemma. For the proofs, see for example [50].

**Lemma 2.1.** The following properties hold:

(i) $A_1(X) \subset A_p(X) \subset A_q(X)$ for $1 < p \leq q < \infty$.

(ii) $RH_\infty(X) \subset RH_q(X) \subset RH_p(X)$ for $1 < p \leq q < \infty$.

(iii) If $w \in A_p(X), 1 < p < \infty$, then there exists $1 < r < p < \infty$ such that $w \in A_r$.

(iv) If $w \in RH_q(X), 1 < q < \infty$, then there exists $q < p < \infty$ such that $w \in RH_p(X)$.

(v) $A_\infty(X) = \bigcup_{1 \leq p < \infty} A_p(X) \subset \bigcup_{1 < q \leq \infty} RH_q(X)$

For $w \in A_\infty(\mathbb{R}^n)$ we define

$$(7) \quad q_w = \sup\{ p \in (1, \infty) : w \in A_p(X) \}.$$

In this paper, we will also assume the existence of an operator $L$ that satisfies the following two conditions:

(A1) $L$ is a nonnegative self-adjoint operator on $L^2(X)$;

(A2) $L$ generates a semigroup $\{e^{-tL}\}_{t>0}$ whose kernel $p_t(x,y)$ admits a Gaussian upper bound.

That is, there exist two positive constants $C$ and $c$ so that for all $x, y \in X$ and $t > 0$,

$$(GE) \quad |p_t(x,y)| \leq \frac{C}{\mu(B(x,\sqrt{t}))} \exp\left(-\frac{d(x,y)^2}{ct}\right).$$

Then for $0 < p \leq 1$ one can define three types of Hardy spaces related to $L$. The first type is through linear combinations of atoms that appropriately encode the cancellation inherent in $L$. The second and third types are $H^p_{\text{max}}$ and $H^p_{\text{rad}}$, which are defined via the non-tangential maximal function and the radial maximal function respectively. For the reader’s convenience, we recall these Hardy spaces below.

**Definition 2.2** (Atoms for $L$). Let $p \in (0,1]$, $q \in (1, \infty]$, $M \in \mathbb{N}$ and $w \in A_\infty(X)$. A function $a$ supported in a ball $B$ is called an $(L,p,q,w,M)$-atom if there exists a function $b \in \mathcal{D}(L^M)$ such that

(i) $a = L^M b$;

(ii) $\supp L^k b \subset B$, $k = 0,1, \ldots, M$;

(iii) $\|L^k b\|_{L^q_w(X)} \leq r_B^{2(M-k)} w(B)^{\frac{1}{q} - \frac{k}{q}}$, $k = 0,1, \ldots, M$.

In the particular case where $\mu(X) < \infty$, the constant function $\left[w(X)\right]^{-1/p}$ is also considered as an atom.
In contrast to the concept of atoms in [15] in which the atoms are defined via $L^q$-norms, in Definition 2.2 our atoms are defined via the weighted $L^q$-norms. This plays an essential role in proving the weighted Hardy estimates for the Green operators.

Given $p \in (0, 1]$, $q \in (1, \infty]$, $M \in \mathbb{N}$ and $w \in A_{\infty}(X)$, we say that $f = \sum_{j=1}^{\infty} \lambda_j a_j$ is an atomic $(L, p, q, w, M)$-representation if $\{\lambda_j\}_{j=0}^{\infty} \in \ell^p$, each $a_j$ is a $(L, p, q, w, M)$-atom, and the sum converges in $L^2(X)$. The space $H_{L,w,at,M}^p(X)$ is then defined as the completion of

$$\{f \in L^2(X) : f \text{ has an atomic } (L, p, q, w, M)\text{-representation}\},$$

with the norm given by

$$\|f\|_{H_{L,w,at,M}^p(X)} = \inf \left\{ \left( \sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} : f = \sum_{j=1}^{\infty} \lambda_j a_j \text{ is an atomic } (L, p, q, w, M)\text{-representation} \right\}.$$

**Definition 2.3** (Maximal Hardy spaces for $L$). For $f \in L^2(X)$, we define the non-tangential maximal function associated to $L$ of $f$ by

$$f_L^t(x) = \sup_{0 < t \leq d_X^2} \sup_{d(x,y) < t} |e^{-tL}f(y)|$$

and the radial maximal function by

$$f_L^+(x) = \sup_{0 < t \leq d_X^2} |e^{-tL}f(x)|$$

where $d_X = \text{diam } X$.

Given $p \in (0, 1]$ and $w \in A_{\infty}(X)$, the Hardy space $H_{L,w,\max}^p(X)$ is defined as the completion of

$$\{f \in L^2(X) : f_L^+ \in L_w^p(X)\},$$

with the norm given by

$$\|f\|_{H_{L,w,\max}^p(X)} = \|f_L^+\|_{L_w^p(X)}.$$

Similarly, the Hardy space $H_{L,w,\rad}^p(X)$ is defined as the completion of

$$\{f \in L^2(X) : f_L^+ \in L_w^p(X)\},$$

with the norm given by

$$\|f\|_{H_{L,w,\rad}^p(X)} = \|f_L^+\|_{L_w^p(X)}.$$

We will show that the three types of Hardy space are equivalent as in the following result.

**Theorem 2.4.** Let $X$ be a space of homogeneous type with finite or infinite measure and let $L$ be an operator satisfying (A1) and (A2). Let $p \in (0, 1]$, $w \in A_{\infty}(X)$, $q \in (\eta_w, \infty]$, and $M > \frac{q}{p} (\frac{q}{p} - 1)$. Then the Hardy spaces $H_{L,w,at,M}^p(X)$, $H_{L,w,\max}^p(X)$ and $H_{L,w,\rad}^p(X)$ coincide with equivalent norms.

We note that the unweighted case of the theorem was proved in [10]. The proof of Theorem 2.4 will be given at the end of this section.

2.1. **Some maximal function estimates.** Let $L$ satisfy (A1) and (A2). Denote by $E_L(\lambda)$ the spectral decomposition of $L$. Then by spectral theory, for any bounded Borel function $F : [0, \infty) \to \mathbb{C}$ we can define

$$F(L) = \int_0^{\infty} F(\lambda)dE_L(\lambda)$$

as a bounded operator on $L^2(X)$. It is well-known that the kernel $K_{\cos(t\sqrt{L})}$ of $\cos(t\sqrt{L})$ satisfies

$$\text{supp } K_{\cos(t\sqrt{L})} \subset \{(x,y) \in X \times X : d(x,y) \leq t\}.$$ 

See for example [22]. We have the following useful lemmas.
Let \( \varphi, \psi \) be even functions. Then for every \( k \in \mathbb{N} \), the kernel \( K_{(t \sqrt{L})^k \Phi(t \sqrt{L})}(\cdot, \cdot) \) of \( (t \sqrt{L})^k \Phi(t \sqrt{L}) \) satisfies
\[
\text{supp} K_{(t \sqrt{L})^k \Phi(t \sqrt{L})}(x, y) \subset \{(x, y) \in X \times X : d(x, y) \leq t \},
\]
and
\[
|K_{(t \sqrt{L})^k \Phi(t \sqrt{L})}(x, y)| \leq \frac{C}{\mu(B(x, t))}. \tag{9}
\]

**Lemma 2.6.** (a) Let \( \varphi \in \mathcal{S}(\mathbb{R}) \) be an even function. Then for any \( N > 0 \) there exists \( C \) such that
\[
|K_{\varphi(t \sqrt{L})}(x, y)| \leq \frac{C}{\mu(B(x, t)) + \mu(B(y, t))} \left(1 + \frac{d(x, y)}{t}\right)^{-n-N},
\]
for all \( t > 0 \) and \( x, y \in X \).

(b) Let \( \varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}) \) be even functions. Then for any \( N > 0 \) there exists \( C \) such that
\[
|K_{\varphi_1(t \sqrt{L}) \varphi_2(s \sqrt{L})}(x, y)| \leq \frac{C}{\mu(B(x, t)) + \mu(B(y, t))} \left(1 + \frac{d(x, y)}{t}\right)^{-n-N},
\]
for all \( t \leq s < 2t \) and \( x, y \in X \).

(c) Let \( \varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}) \) be even functions with \( \varphi_2^{(0)}(0) = 0 \) for \( \nu = 0, 1, \ldots, 2\ell \) for some \( \ell \in \mathbb{Z}^+ \). Then for any \( N > 0 \) there exists \( C \) such that
\[
|K_{\varphi_1(t \sqrt{L}) \varphi_2(s \sqrt{L})}(x, y)| \leq C \left(\frac{1}{s}\right)^{2\ell} \frac{1}{\mu(B(x, t)) + \mu(B(y, t))} \left(1 + \frac{d(x, y)}{t}\right)^{-n-N},
\]
for all \( t \geq s > 0 \) and \( x, y \in X \).

**Proof.** (a) The estimate (11) was proved in [12, Lemma 2.3] in the particular case \( X = \mathbb{R}^n \) but the proof is still valid in the spaces of homogeneous type. For the items (b) and (c) we refer to [9].

We record the following result in [26].

**Lemma 2.7.** Let \( \varphi \in \mathcal{S}(\mathbb{R}) \) be an even function with \( \varphi(0) = 1 \) and let \( N > 0 \). Then there exist even functions \( \phi, \psi \in \mathcal{S}(\mathbb{R}) \) with \( \phi(0) = 1 \) and \( \psi^{(0)}(0) = 0, \nu = 0, 1, \ldots, N \) so that for every \( f \in L^2(X) \) and every \( j \in \mathbb{Z} \) we have
\[
f = \phi(2^{-j} \sqrt{L}) \varphi(2^{-j} \sqrt{L}) f + \sum_{k \geq j} \psi(2^{-k} \sqrt{L}) (\varphi(2^{-k} \sqrt{L}) - \varphi(2^{-k+1} \sqrt{L})) f \in L^2(X).
\]

The following elementary estimate will be used frequently. Its proof is simple and we omit it.

**Lemma 2.8.** Let \( \epsilon > 0 \). We have
\[
\int_X \frac{1}{\mu(B(x, s)) \wedge \mu(B(y, s))} \left(1 + \frac{d(x, y)}{s}\right)^{-n-\epsilon} |f(y)| d\mu(y) \lesssim \mathcal{M} f(x).
\]
for all \( x \in X, s > 0 \) where \( \mathcal{M} f(x) \) is the Hardy-Littlewood maximal function of \( f \).

Let \( F \) be a measurable function on \( X \times (0, \infty) \). For \( \alpha > 0 \) we set
\[
F^*_\alpha(x) = \sup_{0 < t < d \alpha} \sup_{d(x, y) < d \alpha} |F(y, t)|.
\]
In the particular case \( \alpha = 1 \), we write \( F^* \) instead of \( F^*_1 \).

We have the following result:

**Lemma 2.9.** For any \( p > 0, w \in A_q \) and \( 0 < \alpha_2 \leq \alpha_1 \), there exists \( C \) depending on \( n \) and \( p \) so that
\[
\|F^*_\alpha \|_{L^p_w(X)} \leq C \left(1 + \frac{\alpha_1}{\alpha_2}\right)^{nq/p} \|F^*_\alpha \|_{L^p_w(X)}.
\]
Proof. The proof of this lemma is similar to that of [14, Theorem 2.3], hence we just sketch the main ideas.

Set
\[ E_1 = \{ x \in X : F^*_{\alpha_1}(x) > \lambda \}, \quad \text{and} \quad E_2 = \{ x \in X : F^*_{\alpha_2}(x) > \lambda \}. \]

Then if \( x_0 \in E_1 \), arguing similarly to the proof of [14, Theorem 2.3] we can find \( 0 < t_0 < d_X \) so that
\[ \frac{\mu(B(x_0, (\alpha_1 + \alpha_2)t_0) \cap E_2)}{V(x_0, (\alpha_1 + \alpha_2)t_0)} \geq C \left( 1 + \frac{2\alpha_1}{\alpha_2} \right)^{-n}. \]

This implies
\[ \frac{w(B(x_0, (\alpha_1 + \alpha_2)t_0) \cap E_2)}{w(B(x_0, (\alpha_1 + \alpha_2)t_0))} \geq C_0 \left( 1 + \frac{2\alpha_1}{\alpha_2} \right)^{-nq}. \]

As a consequence, we have
\[ E_1 \subset \left\{ x \in X : \mathcal{M}_w(\chi_{E_2})(x) > C_0 \left( 1 + \frac{2\alpha_1}{\alpha_2} \right)^{-nq} \right\}. \]

This, along with the weak type \((1,1)\) of the maximal function \( \mathcal{M}_w \), yields
\[ w(E_1) \leq C_1 \left( 1 + \frac{2\alpha_1}{\alpha_2} \right)^{nq} w(E_2), \]
or equivalently,
\[ w\{ x \in X : F^*_{\alpha_1}(x) > \lambda \} \leq C_1 \left( 1 + \frac{2\alpha_1}{\alpha_2} \right)^{nq} w\{ x \in X : F^*_{\alpha_2}(x) > \lambda \}. \]

Therefore,
\[
\|F^*_{\alpha_1}\|_{L^p_w(X)}^p = p \int_0^p \lambda^{p-1} w\{ x \in X : F^*_{\alpha_1}(x) > \lambda \} \, d\lambda \\
\leq pC_1 \left( 1 + \frac{2\alpha_1}{\alpha_2} \right)^{nq} \int_0^p \lambda^{p-1} w\{ x \in X : F^*_{\alpha_2}(x) > \lambda \} \, d\lambda \\
\leq pC_1 \left( 1 + \frac{2\alpha_1}{\alpha_2} \right)^{nq} \|F^*_{\alpha_2}\|_{L^p_w(X)}^p.
\]

This completes our proof. \( \square \)

From the lemmas above we obtain the following result.

Lemma 2.10. For any \( p \in (0,1] \), \( w \in A_p(X) \) and \( \lambda > nq/p \), there exists \( C \) depending on \( n \) and \( p \) so that
\[
\left\| \sup_{0<t<d_X} \sup_y F(y,t) \left( 1 + \frac{d(x,y)}{t} \right)^{-\lambda} \right\|_{L^p_w(X)} \leq C \|F^*\|_{L^p_w(X)}.
\]

For any even function \( \varphi \in \mathcal{S}(\mathbb{R}) \), \( \alpha > 0 \) and \( f \in L^2(X) \) we define
\[ \varphi^\perp_{L,\alpha}(f)(x) = \sup_{0<t<d_X} \sup_{d(x,y)<\alpha t} |\varphi(t\sqrt{L})f(y)|, \]
and
\[ \varphi^\perp_{L,\alpha}(f)(x) = \sup_{0<t<d_X} |\varphi(t\sqrt{L})f(x)|. \]

As usual, we drop the index \( \alpha \) when \( \alpha = 1 \).

We now are in position to prove the following estimate.

Proposition 2.11. Let \( p \in (0,1] \) and \( w \in A_{\infty}(X) \). Let \( \varphi_1, \varphi_2 \in \mathbb{R} \) be even functions with \( \varphi_1(0) = 1 \) and \( \varphi_2(0) = 0 \) and \( \alpha_1, \alpha_2 > 0 \). Then for every \( f \in L^2(X) \) we have
\[ \|\varphi_2\|_{L,\alpha_2} \|_{L^p_w(X)} \leq \|\varphi_1\|_{L,\alpha_1} \|_{L^p_w(X)}. \]

As a consequence, for every even function \( \varphi \) with \( \varphi(0) = 1 \) and \( \alpha > 0 \) we have
\[ \|\varphi^\perp_{L,\alpha} f\|_{L^p_w(X)} \sim \|f^\perp_{L} \|_{L^p_w(X)}. \]
Proof. From Lemma 2.9 it suffices to prove the proposition with \( \alpha_1 = \alpha_2 = 1 \).

Fix \( N > n \) and \( \lambda > nq \omega/p \) and \( M > \lambda/2 \). Fix \( t \in (0, d_X) \) and let \( j_0 \in \mathbb{Z}^+ \) so that \( 2^{-j_0 + 1} \leq t < 2^{-j_0 + 2} \). According to Lemma 2.7 there exist even functions \( \phi, \psi \in \mathbb{R} \) with \( \phi(0) = 1 \) and \( \psi(0) = 0 \) for \( \nu = 0, 1, \ldots, 2M \) so that

\[
f = \phi(2^{-j_0} \sqrt{L}) \varphi_1(2^{-j_0} \sqrt{L}) f + \sum_{k \geq j_0} \psi(2^{-k} \sqrt{L}) [\varphi_1(2^{-k} \sqrt{L}) - \varphi_1(2^{-k+1} \sqrt{L})]
\]

which implies

\[
\varphi_2(t \sqrt{L}) f(y) = \varphi_2(t \sqrt{L}) \phi(2^{-j_0} \sqrt{L}) \varphi_1(2^{-j_0} \sqrt{L}) f(y) + \sum_{k \geq j_0} \varphi_2(t \sqrt{L}) \psi(2^{-k} \sqrt{L}) [\varphi_1(2^{-k} \sqrt{L}) - \varphi_1(2^{-k+1} \sqrt{L})] f(y)
= : I_1(y, t) + I_2(y, t).
\]

Since \( 2^{-j_0} \sim t \), by Lemma 2.6 we have

\[
|I_1(y, t)| \lesssim \int_X \left( 1 + \frac{d(y, z)}{t} \right)^{-\lambda-N} \varphi_1(t \sqrt{L}) f(z) |d\mu(z)|.
\]

This implies that for a fixed \( x \in X \) and \( 0 < t < d_X \) we have

\[
\sup_{d(x,y) < t} |I_1(y, t)| \lesssim \sup_{z \in X} \left( 1 + \frac{d(x, z)}{t} \right)^{-\lambda} \varphi_1(t \sqrt{L}) f(z).
\]

Applying the estimate in Lemma 2.8 we have

\[
\sup_{d(x,y) < t} |I_1(y, t)| \lesssim \sup_{z \in X} \left( 1 + \frac{d(x, z)}{t} \right)^{-\lambda} \varphi_1(t \sqrt{L}) f(z).
\]

For the second term \( I_2 \), using (13) and the fact that \( t \sim 2^{-j_0} \) we have

\[
|I_2(y, t)| \lesssim \sum_{k \geq j_0} \int_X 2^{-2M(k-j_0)} \frac{1}{V(z, 2^{-j_0})} \left( 1 + \frac{d(y, z)}{2^{-j_0}} \right)^{-\lambda-N} \varphi_1(2^{-k} \sqrt{L}) f(z) |d\mu(z)|
+ \sum_{k \geq j_0} \int_X 2^{-2M(k-j_0)} \frac{1}{V(z, 2^{-j_0})} \left( 1 + \frac{d(y, z)}{2^{-j_0}} \right)^{-\lambda-N} \varphi_1(2^{-k+1} \sqrt{L}) f(z) |d\mu(z)|
\]

Hence, for a fixed \( x \in X \) and \( 0 < t < d_X \) we have

\[
\sup_{d(x,y) < t} |I_2(y, t)| \lesssim \sum_{k \geq j_0} \int_X 2^{-2M(k-j_0)} \frac{1}{V(z, 2^{-j_0})} \left( 1 + \frac{d(x, z)}{2^{-j_0}} \right)^{-\lambda-N} \varphi_1(2^{-k} \sqrt{L}) f(z) |d\mu(z)|
+ \sum_{k \geq j_0} \int_X 2^{-2M(k-j_0)} \frac{1}{V(z, 2^{-j_0})} \left( 1 + \frac{d(x, z)}{2^{-j_0}} \right)^{-\lambda-N} \varphi_1(2^{-k+1} \sqrt{L}) f(z) |d\mu(z)|.
\]

Note that for \( k \geq j_0 \) we have \( 2^{-k+1} \leq t < d_X \). This, along with Lemma 2.8 and above inequality, implies

\[
\sup_{d(x,y) < t} |I_2(y, t)| \lesssim \sup_{0 < s < d_X} \sup_{z \in X} \left( 1 + \frac{d(x, z)}{s} \right)^{-\lambda} \varphi_1(s \sqrt{L}) f(z).
\]

Taking this, (17) and (16) into account we conclude that

\[
\sup_{0 < t < d_X} \sup_{d(x,y) < t} |\varphi_2(t \sqrt{L}) f(y)| \lesssim \sup_{0 < s < d_X} \sup_{z \in X} \left( 1 + \frac{d(x, z)}{s} \right)^{-\lambda} \varphi_1(s \sqrt{L}) f(z).
\]

Then applying Lemma 2.10, (14) follows directly.
To prove (15), we apply (14) for $\varphi_1(\lambda) = \varphi(\lambda) - e^{-\lambda^2}$, $\varphi_2(\lambda) = e^{-\lambda^2}$, $\alpha_1 = \alpha$ and $\alpha_2 = 1$ to obtain

$$\left\| \sup_{t > 0} \sup_{d(x,y) < \alpha t} \left| \varphi(t\sqrt{L})f(y) - e^{-t^2L}f(y) \right| \right\|_{L^p_c(X)} \lesssim \| f \|_{L^p_c(X)}.$$  

This, along with Lemma 2.9, yields

$$\| \varphi_{L,\alpha}f \|_{L^p_c(X)} \lesssim \| f \|_{L^p_c(X)}.$$  

Similarly, we obtain

$$\| f \|_{L^p_c(X)} \lesssim \| \varphi_{L,\alpha}^*f \|_{L^p_c(X)}.$$  

This proves (15). \hfill \Box

For each $\lambda > 0$ and each even function $\varphi \in \mathcal{S}(\mathbb{R})$ we define

$$M_{L,\varphi,\lambda}^* f(x) = \sup_{t > 0} \sup_{y \in X} \left( \frac{|\varphi(t\sqrt{L})f(y)|}{1 + \frac{d(x,y)}{t}} \right)^\lambda,$$

for each $f \in L^2(X)$.

Obviously, we have $\varphi_{L}^* f(x) \leq M_{L,\varphi,\lambda}^* f(x)$ for all $x \in X$, $\lambda > 0$ and even functions $\varphi \in \mathcal{S}(\mathbb{R})$.

**Proposition 2.12.** Let $p \in (0,1]$ and $w \in A_w(X)$. Let $\varphi \in \mathcal{S}(\mathbb{R})$ be an even function with $\varphi(0) = 1$. Then we have, for every $f \in L^2(X)$,

$$\left\| M_{L,\varphi,\lambda}^* f \right\|_{L^p_c(X)} \lesssim \| \varphi_{L}^* f \|_{L^p_c(X)},$$

provided $\lambda > nq_w/p$.

As a consequence, we have

$$\left\| \varphi_{L}^* f \right\|_{L^p_c(X)} \lesssim \| \varphi_{L}^* f \|_{L^p_c(X)}.$$  

**Proof.** Fix $N > n$ and $\lambda > nq_w/p$ and $M > \lambda/2$. Fix $\theta \in (0,p)$ so that $p/\theta > q_w$ and $\lambda > n/\theta$. We now rewrite (16):

$$\varphi_2(t\sqrt{L})f(y) = \varphi_2(t\sqrt{L})\phi(2^{-j_0}\sqrt{L})\varphi_1(2^{-j_0}\sqrt{L})f(y) + \sum_{k \geq j_0} \varphi_2(t\sqrt{L})\psi(2^{-k}\sqrt{L})(\varphi_1(2^{-k}\sqrt{L}) - \varphi_1(2^{-k+1}\sqrt{L}))f(y)$$

$$=: I_1(y,t) + I_2(y,t).$$

Arguing similarly to (17) we have, for a fixed $x \in X$, all $y \in X$ and $t > 0$,

$$(1 + \frac{d(x,y)}{t})^{-\lambda} |I_1(y,t)| \lesssim \int_X \frac{1}{V(z,t)} \left( 1 + \frac{d(x,z)}{t} \right)^{-\lambda} |\varphi(t\sqrt{L})f(z)| d\mu(z)$$

$$\lesssim \int_X \frac{1}{V(z,t)} \left( 1 + \frac{d(x,z)}{t} \right)^{-\lambda} |\varphi(t\sqrt{L})f(z)| d\mu(z)$$

(20)

\begin{align*}
&\lesssim \int_X \frac{1}{V(z,t)} \left( 1 + \frac{d(x,z)}{s} \right)^{-\lambda} |\varphi(t\sqrt{L})f(z)| d\mu(z) \\
&\lesssim \int_X \frac{1}{V(z,t)} \left( 1 + \frac{d(x,z)}{t} \right)^{-\lambda} |\varphi(t\sqrt{L})f(z)| d\mu(z) \\
&\lesssim [M_{L,\varphi,\lambda}^* f(x)]^{-\theta} \int_X \frac{1}{V(z,t)} \left( 1 + \frac{d(x,z)}{t} \right)^{-\theta\lambda} |\varphi(t\sqrt{L})f(z)|^\theta d\mu(z). \end{align*}

Applying Lemma 2.8 we have

$$\left( 1 + \frac{d(x,y)}{t} \right)^{-\lambda} |I_1(y,t)| \lesssim [M_{L,\varphi,\lambda}^* f(x)]^{-\theta} \mathcal{M}(|\varphi_{L}^* f|^\theta)(x).$$
Likewise, we have
\begin{equation*}
(1 + \frac{d(x, y)}{t})^{-\lambda} |I_2(y, t)| \lesssim |M^*_{L, \varphi, \lambda}f(x)|^{1-\theta} \mathcal{M}(\varphi^+L f^\theta)(x).
\end{equation*}
Therefore, for all $y \in X$ and $0 < t < d_X$ we have
\begin{equation*}
(1 + \frac{d(x, y)}{t})^{-\lambda} |\varphi(t\sqrt{L})f(y)| \lesssim |M^*_{L, \varphi, \lambda}f(x)|^{1-\theta} \mathcal{M}(\varphi^+L f^\theta)(x)
\end{equation*}
which implies that
\begin{equation*}
M^*_{L, \varphi, \lambda}f(x) \lesssim |M^*_{L, \varphi, \lambda}f(x)|^{1-\theta} \mathcal{M}(\varphi^+L f^\theta)(x).
\end{equation*}
As a result, we come up with
\begin{equation*}
M^*_{L, \varphi, \lambda}f(x) \lesssim \left[ \mathcal{M}(\varphi^+L f^\theta)(x) \right]^{\frac{1}{\theta}}.
\end{equation*}
Since $p/\theta > q_w$ we have $w \in A_{p/\theta}$. Hence,
\begin{equation*}
\|M^*_{L, \varphi, \lambda}f\|_{L^p_w(X)} \lesssim \left\| \mathcal{M}(\varphi^+L f^\theta)(x) \right\|_{L^p_w(X)} \lesssim \|\varphi^+L f\|_{L^p_w(X)}.
\end{equation*}
This completes our proof. 

\[ \square \]

2.2. Maximal function characterizations. We are ready to give the proof of Theorem 2.4.

Proof of Theorem 2.4: The unweighted case of the theorem was proved in [10], we now adapt this argument to our present situation with some modifications due to the presence of the weight $w \in A_\infty(X)$. We will give the proof for the case $\mu(X) < \infty$, since the case $\mu(X) = \infty$ is similar and even easier.

We now divide the proof into 2 steps.

**Step 1.** Let $p \in (0, 1]$, $w \in A_\infty(X)$ and $M > \frac{p}{2} \left( \frac{2p}{p} - 1 \right)$. We now claim that $H^p_{L,w,\text{max}}(X) \hookrightarrow H^p_{L,w,\text{at},M}(X)$. To do this, fix $f \in H^p_{L,w,\text{max}}(X) \cap L^2(X)$. We shall show that $f$ has an $(L, p, \infty, w, M)$-representation $\sum_j \lambda_j a_j$ with $\langle \sum_j |\lambda_j|^p \rangle^{1/p} \lesssim \|f\|_{H^p_{L,w,\text{max}}}$.

Let $\Phi$ be a function from Lemma 2.5. For $M \in \mathbb{N}$, $M > \frac{p}{2} \left( \frac{2p}{p} - 1 \right)$ we have
\begin{equation}
\tag{21}
f = c_{\Phi, M} \int_0^\infty (t^2 L)^M \Phi(t\sqrt{L})\Phi(t\sqrt{L}) f \frac{dt}{t}
\end{equation}
in $L^2(X)$, where $c_{\Phi, M} = \left[ \int_0^\infty x^{2M} \Phi(x) \frac{dx}{x} \right]^{-1}$.

Then we have
\begin{equation}
\tag{22}
f = c_{\Phi, M} \int_0^{T_0} (t^2 L)^M \Phi(t\sqrt{L})\Phi(t\sqrt{L}) f \frac{dt}{t} + c_{\Phi, M} \int_{T_0}^\infty (t^2 L)^M \Phi(t\sqrt{L})\Phi(t\sqrt{L}) f \frac{dt}{t} \quad =: f_1 + f_2
\end{equation}
in $L^2(X)$ where $T_0 = d_X/2$ and
\begin{equation}
\tag{23}
\psi(x) = c_{\Phi, M} \int_x^\infty 2^M \Phi^2(t) \frac{dt}{t} = c_{\Phi, M} \int_1^\infty (tx)^{2M} \Phi^2(tx) \frac{dt}{t}.
\end{equation}
It is easy to check that $\psi \in \mathcal{C}(\mathbb{R})$ and is an even function with $\psi(0) = 1$. We now define the maximal operator
\begin{equation*}
M^*_L f(x) = \sup_{0 < t < d_X} \sup_{d(x, y) < 8t} \left[ |\psi(t\sqrt{L})f(y)| + |\Phi(t\sqrt{L})f(y)| \right].
\end{equation*}
Then Proposition 2.12 yields
\[(24) \quad \|M_{L,f}\|_{L^p_v(X)} \lesssim \|f\|_{H^p_{L,\omega,\max}(X)}.\]

Since \(T_0 = \text{diam } X/2\), we have, for any \(x \in X\),
\[|\psi(T_0 \sqrt{L})f(x)| \leq \sup_{d(z,y) < 8T_0} |\psi(T_0 \sqrt{L})f(z)| \leq \inf_{y \in X} M_{L,f}(y).\]

This, along with (24), implies that
\[\|f_2\|_{L^\infty(X)} \leq w(X)^{-1/p}\|M_{L,f}\|_{L^p_v(X)} \lesssim w(X)^{-1/p}\|f\|_{H^p_{L,\omega,\max}(X)}.\]

Therefore, \(f_2\) is an \((L, p, \infty, w, M)\) atom (with a harmless multiple constant).

We now take care of the component \(f_1\). For each \(k \in \mathbb{Z}\) we set
\[\Omega_i := \{x \in X : M_{L,f}(x) > 2^k\}.\]

Since \(M_{L,f}\) is lower–continuous and \(X\) is bounded, there exists \(i_0\) so that \(\Omega_{i_0} = X\) and \(\Omega_{i_0+1} \neq X\).

Without loss of generality we may assume that \(i_0 = 0\). Then for each \(t > 0\) we define
\[(25) \quad \Omega_i^t = \begin{cases} \Omega_0, & i = 0, \\ \{x : d(x, \Omega_i^0) > 4t\}, & i > 0, \end{cases}\]

and \(\hat{\Omega}_i^t = \Omega_i^t \setminus \Omega_i^{t+1}\).

It is clear that \(X = \bigcup_{i=0}^{\infty} \hat{\Omega}_i^t\) for each \(t > 0\). Hence,
\[(26) \quad f_1 = \sum_{i=0}^{\infty} c_{\Phi,M} \int_0^{T_0} (t^2L)^M \Phi(t\sqrt{L}) \left[ \Phi(t\sqrt{L})f \cdot \chi_{\hat{\Omega}_i^t} \right] \frac{dt}{t} = \sum_{i=0}^{\infty} f_1^i.\]

Arguing similarly to [10] we obtain
\[|f_1^0(x)| \lesssim 1, \quad \forall x \in X\]

which implies that
\[|f_1^0(x)| \lesssim w(X)^{-1/p}w(\Omega_0)^{1/p}\]
\[\lesssim w(X)^{-1/p} \sum_{i=0}^{\infty} 2^i w(\Omega_i^0)^{1/p} \sim w(X)^{-1/p}\|M_{L,f}\|_{L^p_v(X)}\]
\[\lesssim w(X)^{-1/p}\|f\|_{H^p_{L,\omega,\max}(X)}.\]

Hence, \(f_1^0\) is an \((L, p, \infty, w, M)\) atom (with a harmless multiple constant).

We now take care of the term \(f_1^i\) with \(i > 0\). To do this, for each \(i > 0\) we apply a covering lemma in [21] (see also [26, Lemma 5.5]) to obtain a collection of balls \(\{B_{i,k} := B(x_{B_{i,k}}, r_{B_{i,k}}) : x_{B_{i,k}} \in \Omega_i, r_{B_{i,k}} = d(x_{B_{i,k}}, \Omega_i^0)/2, k = 1, \ldots\}\) so that
(i) \(\Omega_i = \bigcup_k B(x_{B_{i,k}}, r_{B_{i,k}});\)
(ii) \(\{B(x_{B_{i,k}}, r_{B_{i,k}}/5)\}_{k=1}^{\infty}\) are disjoint.

For each \(i, k \in \mathbb{N}^+\) and \(t \geq 0\) we set \(B_{i,k}^t = B(x_{i,k}, r_{B_{i,k}} + 2t)\) which is a ball having the same center as \(B_{i,k}\) with radius being \(2t\) greater than the radius of \(B_{i,k}\). Then, for each \(i, k \in \mathbb{N}^+\) and \(t > 0\), we set
\[R_{i,k}^t = \begin{cases} \hat{\Omega}_i^t \cap B_{i,k}^t, & \text{if } \hat{\Omega}_i^t \cap B_{i,k} \neq \emptyset, \\ 0, & \text{if } \hat{\Omega}_i^t \cap B_{i,k} = \emptyset, \end{cases}\]
and
\[(27) \quad E_{i,k}^t = R_{i,k}^t \setminus \bigcup_{\ell > k} R_{i,k}^\ell.\]
It is easy to see that for each $i \in \mathbb{N}^+$ and $t > 0$ we have
\[
\hat{\Omega}^t_i = \bigcup_{k \in \mathbb{N}^+} E^t_{i,k}.
\]
Hence, from (26) we have, for $i \in \mathbb{N}^+$,
\[
f^i_1 = \sum_{k \in \mathbb{N}^+} c_{i,k} \int_0^{T_0} (t^2 L)^M \Phi(t \sqrt{L}) \left[ \Phi(t \sqrt{L}) f \cdot \chi_{E^t_{i,k}} \right] \frac{dt}{t}
\]
and set $a_i = 0$ if $E^t_{i,k} = \emptyset$.

We now define $\lambda_{i,k} = 2^i w(B_{i,k})^{1/p}$ and $a_i = L^M b_i$ where
\[
(28)
\begin{align*}
b_{i,k} &= \frac{c_{i,k}}{\lambda_{i,k}} \int_0^{T_0} t^{2M} \Phi(t \sqrt{L}) \left[ \Phi(t \sqrt{L}) f \cdot \chi_{E^t_{i,k}} \right] \frac{dt}{t}.
\end{align*}
\]
Then it can be seen that
\[
f_1 = \sum_{i \in \mathbb{N}^+} f^i_1 = \sum_{i,k \in \mathbb{N}^+} \lambda_{i,k}^i a_{i,k}
\]
in $L^2(X)$; moreover,
\[
\sum_{i,k \in \mathbb{N}^+} |\lambda_{i,k}|^p = \sum_{i,k \in \mathbb{N}^+} 2^{ip} w(B_{i,k}) \lesssim \sum_{i \in \mathbb{N}^+} 2^{ip} w(\Omega_i) \lesssim \|M_L f\|_{L^p(X)} \lesssim \|f\|_{H^{p \wedge 1}_p}^{p}.
\]
Therefore, it suffices to prove that each $a_{i,k} \neq 0$ is an $(L, p, \infty, w, M)$ atom associated to the ball $B_{i,k}^* : = 8B_{i,k}$. Indeed, if $r_{B_{i,k}} < t/2$, then we have $d(x_{B_{i,k}}, \Omega_i^t) = 2r_{B_{i,k}} < t$. Therefore,
\[
B^t_{i,k} = B(x_{B_{i,k}}, r_{B_{i,k}} + 2t) \subset \{ x \colon d(x, \Omega^t_i) < 4t \}.
\]
This implies that $R^t_{i,k} := \hat{\Omega}^t_i \cap B^t_{i,k} = \emptyset$. Hence, if $a_{i,k} \neq 0$, then $r_{B_{i,k}} \geq t/2$. This, along with (28) and Lemma 2.5, implies that
\[
\text{supp} L^m b_{i,k} \subset B^t_{i,k}, \quad \forall m = 0, 1, \ldots, M.
\]
By a similar argument to that in [10] we can show that
\[
\|L^m b_{i,k}\|_{L^\infty(X)} \lesssim 2^M(2^{M-m}) \mu(B_{i,k})^{-1/p}, \quad \forall m = 0, 1, \ldots, M.
\]
This implies that each $a_{i,k} \neq 0$ is an $(L, p, \infty, w, M)$ atom and hence this completes the proof of Step 1.

**Step 2.** Let $p \in (0, 1]$, $w \in A_\infty(X)$, $q \in (q_w, \infty]$, and $M > \frac{q}{2}(q_w - 1)$. We now claim that $H^{p,q}_{L,w,\max}(X) \hookrightarrow H^p_{L,w,\max}(X)$. It suffices to show that there exists $C > 0$ so that
\[
(29) \quad \sup_{0 < t < d^2_X} \|e^{-tL}a\|_{L^p_c(X)} \leq C
\]
for all $(L, p, q, w, M)$ atoms.

The proof of (29) is quite standard and is not difficult, hence we omit the details.

From the results in Steps 1 and 2 we conclude that for all $p \in (0, 1]$, $w \in A_\infty(X)$, $q \in (q_w, \infty]$, and $M > \frac{q}{2}(q_w - 1)$ we have
\[
H^{p,q}_{L,w,\max}(X) \equiv H^p_{L,w,\max}(X)
\]
which, together with Proposition 2.12, yields that
\[
H^{p,q}_{L,w,\max}(X) \equiv H^p_{L,w,\max}(X) \equiv H^p_{L,w,\rad}(X).
\]
This completes our proof. \qed
3. Weighted regularity estimates for inhomogeneous Dirichlet and Neumann problems

This section is dedicated to the proofs of Theorems 1.5-1.9. To do this, we first prove a number of estimates for inhomogeneous Dirichlet problems and Neumann problems (see Subsections 3.1 and 3.2). These results are of independent interest and should have applications in other settings apart from those in this paper.

3.1. Dirichlet Laplacian problems. Let Ω be a open connected domain in \( \mathbb{R}^n \). Denote by \( \Delta_D \) the Dirichlet Laplacian define on \( \mathcal{D}(\Delta_D):= \{ u \in W_0^{1,2}(\Omega) : \Delta u \in L^2(\Omega) \} \) such that

\[
\langle \Delta_D f, g \rangle = \int_{\Omega} \nabla f \cdot \nabla g, \quad \forall f \in \mathcal{D}(\Delta_D), \ g \in W_0^{1,2}(\Omega).
\]

Denote by \( p_{t,\Delta_D}(x,y) \) the kernel of \( e^{-t\Delta_D} \). It is well-known that

\[
0 \leq p_{t,\Delta_D}(x,y) \leq \frac{1}{(4\pi t)^{n/2}} \exp \left( -\frac{|x-y|^2}{4t} \right)
\]

for all \( t > 0 \) and \( x,y \in \Omega \).

We now consider the following condition: There exist a constant \( \beta > 0 \) and \( C > 0 \) so that

\[
\left( \int_{\Omega} |\nabla p_{t,\Delta_D}(x,y)|^2 e^{\beta|x-y|^2} \frac{dt}{t} \right)^{1/2} \leq C \beta t^{-2} |B_\Omega(y, \sqrt{t})|^{-1}, \ \forall y \in \Omega, \ t > 0.
\]

It is obviously that if \( \nabla^2 p_{t,\Delta_D} \) has a Gaussian upper bound, then (22) is satisfied. In our applications, we will show that (22) is satisfied if \( \Omega \) is one of the following domains:

(i) a bounded, simply connected, semiconvex domain;
(ii) a convex domain above a Lipschitz graph;
(iii) the upper-half space.

Theorem 3.1. Suppose that the second derivative of the Green function \( \nabla^2 \Delta_D^{-1} \) is bounded on \( L^{p_0}(X) \) for some \( p_0 \geq 2 \). Suppose that the Dirichlet Laplacian \( \Delta_D \) satisfies (22). Then we have

(i) The second derivative of the Green function \( \nabla^2 \Delta_D^{-1} \) is bounded on \( L^{p_0}_w(\Omega) \) for all \( 1 < p < p_0 \) and \( w \in A_p(\mathbb{R}^n) \cap RH_{(p_0/p)'}(\mathbb{R}^n) \), and is bounded from \( L^1_w(\Omega) \) into \( L^1_w(\Omega) \) for \( w \in A_1(\mathbb{R}^n) \cap RH_{p_0}(\mathbb{R}^n) \).
(ii) The second derivative of the Green function \( \nabla^2 \Delta_D^{-1} \) is bounded from \( H^p_{\Delta_D,w}(\Omega) \) into \( H^p_{\Delta_D,w}(\Omega) \) for all \( 0 < p \leq 1 \) and \( w \in \bigcup_{1 < p < p_0} A_p(\mathbb{R}^n) \cap RH_{(p_0/p)'}(\mathbb{R}^n) \).

In order to prove the item (i) in Theorem 3.1 we need the following criterion for the weighted estimates for singular integrals whose proof is similar to that of [8, Theorem 3.1], and hence we omit the details.

Theorem 3.2. Let \( T \) be a bounded linear operator on \( L^{p_0}(\Omega) \) with \( 1 < p_0 < \infty \). Also assume that there exist \( m \in \mathbb{N}, \ \delta > 0 \) and \( 1 < p_2 < \infty \) such that for any ball \( B \subset \Omega \), the operator \( T(I - e^{-\frac{t}{2}L})^m \) has a kernel \( K_{m,r_B}(x,y) \) satisfying

\[
\left( \int_{S_1(B_\Omega)} |K_{m,r_B}(y,z)|^{p_0} dy \right)^{1/p_0} \leq C 2^{-j\delta} |2^j B_\Omega|^{1/p_0-1}
\]

for all \( z \in B \) and all \( j \geq 2 \).

Then, we have:

(a) If \( \delta > 0 \), then for any \( 1 < p < p_0 \) and \( w \in A_p(\mathbb{R}^n) \cap RH_{(p_0/p)'}(\mathbb{R}^n) \), the operator \( T \) is bounded on \( L^p(\Omega, w) \).
(b) For any \( w \in A_1(\mathbb{R}^n) \cap RH_{p_0}(\mathbb{R}^n) \), if \( \delta > n \), then \( T \) is bounded from \( L^1(\Omega, w) \) into \( L^{1,\infty}(\Omega, w) \).
Proof of Theorem 3.1: (i) Fix \( w \in A_p(\mathbb{R}^n) \cap RH_{(p_0/p)^+} \). Then we can find \( q_0 \in (1, p_0) \) so that \( w \in A_q(\mathbb{R}^n) \cap RH_{(q_0/p)^+}(\mathbb{R}^n) \). Fix a ball \( B \) with radius \( r_B \). For \( m > n/2 \), we observe that

\[
\Delta_D^{-1} = \int_0^\infty e^{-t\Delta_D} dt
\]

so that

\[
\nabla^2 \Delta_D^{-1}(I - e^{-r_B^2\Delta_D})^m = \int_0^\infty \nabla^2 e^{-t\Delta_D}(I - e^{-r_B^2\Delta_D})^m dt \frac{dt}{\sqrt{t}} = \int_0^\infty g_{r_B,m}(t) \nabla^2 e^{-t\Delta_D} dt
\]

where \( g_{r_B,m} : \mathbb{R}^+ \to \mathbb{R} \) is a function such that

\[
\int_0^\infty |g_{r_B,m}(t)|e^{-\alpha t} dt \leq C_{m,\alpha} t^{-j_m},
\]

for any \( \alpha > 0 \). See for example [5, p.932].

It follows that

\[
K_{m,r_B}(y,z) = \int_0^\infty g_{r_B,m}(t) \nabla^2 p_{t,\Delta_D}(y,z) dt.
\]

Hence, for \( z \in B \) and \( j \geq 2 \) we have

\[
\left( \int_{S_j(B)} |K_{m,r_B}(y,z)|^{q_0} dy \right)^{1/q_0} \leq \int_0^\infty g_{r_B,m}(t) \left( \int_{S_j(B)} |\nabla^2 p_{t,\Delta_D}(y,z)|^{q_0} dy \right)^{1/q_0} dt.
\]

On the other hand, since \( \nabla^2 \Delta_D^{-1} \) is bounded on \( L^{p_0} \)

\[
\left( \int_{S_j(B)} |\nabla^2 p_{t,\Delta_D}(y,z)|^{q_0} dy \right)^{1/q_0} \leq \left( \int_{S_j(B)} |\nabla^2 p_{t,\Delta_D}(y,z)|^{p_0} dy \right)^{1/p_0} \leq \left( \int_{S_j(B)} |\Delta_D p_{t,\Delta_D}(y,z)|^{p_0} dy \right)^{1/p_0} \leq t^{\frac{n}{2} \frac{n}{p_0} - 1} \lesssim |2^j B_\Omega|^{1/p_0 - 1} \left( \frac{4^j r_B^2}{t} \right)^{\frac{n}{2} - \frac{n}{p_0}}
\]

for all \( z \in B_\Omega \) and \( t > 0 \). From (32), we have

\[
\left( \int_{S_j(B_\Omega)} |\nabla^2 p_{t,\Delta_D}(y,z)|^{q_0} dy \right)^{1/2} \lesssim |B_\Omega(z, \sqrt{t})|^{-1/2} t^{-1} e^{-\beta \frac{4^j r_B^2}{t}},
\]

Interpolating this and (36) we obtain

\[
\left( \int_{S_j(B_\Omega)} |K_{m,r_B}(y,z)|^{q_0} dy \right)^{1/q_0} \lesssim |2^j B_\Omega|^{1/q_0 - 1} t^{-1} e^{-\frac{4^j r_B^2}{t}}.
\]

Therefore,

\[
\left( \int_{S_j(B_\Omega)} |K_{m,r_B}(y,z)|^{q_0} dy \right)^{1/q_0} \lesssim |2^j B_\Omega|^{1/q_0 - 1} t^{-1} e^{-\beta \frac{4^j r_B^2}{t}}.
\]

which along with (35) implies that

\[
\left( \int_{S_j(B_\Omega)} |K_{m,r_B}(y,z)|^{q_0} dy \right)^{1/q_0} \lesssim 2^{-m_j} |2^j B_\Omega|^{1/q_0 - 1}, \quad j \geq 2.
\]

Applying Theorem 3.2, we get (i).

(ii) Fix \( w \in \bigcup_{1 < r < p_0} A_r(\mathbb{R}^n) \cap RH_{(p_0/r)^+}(\mathbb{R}^n) \). Then there exists \( q \in (1, p_0) \) so that \( w \in A_q \cap RH_{(q_0/p)^+} \). Let \( a = \Delta_D^{-1} b \) be an \((\Delta_D, p, q, w, M)\) atom associated to a ball \( B \) where \( M \in \)
we need to claim that $\nabla^2 \Delta_D^{-1} a$ is a $(p, q, w)_{M_i}$ atom. Indeed, we consider three cases.

**Case 1:** $4B \subset \Omega$. We first observe that $\nabla^2 \Delta_D^{-1} a = \nabla^2 \Delta_D^{-1} b$. Since $\text{supp} \Delta_D^{-1} b \subset B$, we have $\text{supp} \nabla^2 \Delta_D^{-1} a \subset B$. Moreover, from (i) we have 
\[
\|\nabla^2 \Delta_D^{-1} a\|_{L^2_{\Omega}(\Omega)} \lesssim \|a\|_{L^2_{\Omega}(\Omega)} \lesssim w(B)^{1/q-1/p}.
\]
We now verify that
\[
(37) \quad \int_B x^\alpha \nabla^2 \Delta_D^{-1} a(x)dx = 0
\]
for every multi-index $\alpha$ with $|\alpha| \leq \lfloor n(q_w/p - 1) \rfloor$.

Let $\psi \in C^\infty(\Omega)$ so that $\text{supp} \psi \subset 2B$ and $\psi = 1$ in $\frac{3}{2}B$. We then have $\text{supp} \partial^\beta [x^\alpha \psi(x)] \subset 2B$ and $\partial^\beta [x^\alpha \psi(x)] = 0$ on $(2B)^c$ for every multi-index $\beta$. Therefore, by integration by part we have
\[
\int_B x^\alpha \nabla^2 \Delta_D^{-1} a(x)dx = \int_B x^\alpha \psi(x) \nabla^2 \Delta_D^{-1} b(x)dx
\]
\[
= \int_B \nabla^2 [x^\alpha \psi(x)] \Delta_D^{-1} b(x)dx.
\]
Since $\nabla^2 [x^\alpha \psi(x)] \in W^{1,2}_0(\Omega)$, we have
\[
\int_B x^\alpha \nabla^2 \Delta_D^{-1} a(x)dx = \int_B \nabla^2 [x^\alpha \psi(x)] \Delta_D^{-2} b(x)dx.
\]
Using integration by part again, we have
\[
\int_B x^\alpha \nabla^2 \Delta_D^{-1} a(x)dx = \int_B \nabla^4 [x^\alpha \psi(x)] \Delta_D^{-2} b(x)dx.
\]
Repeating this process $(M - 2)$ times we come up with
\[
\int_B x^\alpha \nabla^2 \Delta_D^{-1} a(x)dx = \int_B \nabla^{2M} [x^\alpha \psi(x)] b(x)dx.
\]
Since $\psi = 1$ on $B$ and $M > \frac{1}{2} [n(q_w/p - 1)]$, we have $\nabla^{2M} [x^\alpha \psi(x)] = 0$ for $x \in B$. As a consequence,
\[
\int_B x^\alpha \nabla^2 \Delta_D^{-1} a(x)dx = 0
\]
which proves (37).

Therefore, $\nabla^2 \Delta_D^{-1} a$ is a $(p, q, w)_{M_i}$ atom associated to $B$.

**Case 2:** $4B \cap \Omega^c \neq \emptyset$. Set $\tilde{a} = \nabla^2 \Delta_D^{-1} a$. Arguing similar to Case 1, we have $\text{supp} \tilde{a} \subset B_\Omega$ and
\[
\|\tilde{a}\|_{L^p_{\Omega}(\Omega)} \lesssim w(B_\Omega)^{1/q-1/p} \sim w(B)^{1/q-1/p}.
\]
From Theorem 1.2 we need to claim that
\[
\|\tilde{a}\|_{L^p_{\Omega}(\Omega)} \lesssim 1.
\]
Indeed, let $\phi$ be a function as in (3). We have
\[
\|\tilde{a}\|_{L^p_{\Omega}(\Omega)} \lesssim \|\tilde{a}\|_{L^p_{(8B_\Omega)^c}(8B_\Omega)} + \|\tilde{a}\|_{L^p_{(8B_\Omega)^c}(8B_\Omega)^c}}.
\]
We claim that $\|\tilde{a}\|_{L^p_{(8B_\Omega)^c}(8B_\Omega)^c}} = 0$. Indeed, for $x \in (8B_\Omega)^c$ we have
\[
\tilde{a}(x) = \sup_{0 < r < \delta(x)/2} \left| \int_{B_\Omega} \frac{1}{t^n} \phi \left( \frac{x-y}{t} \right) \tilde{a}(y)dy \right|
\]
\[
\leq \sup_{0 < r \leq 7r_B} \left| \int_{B_\Omega} \frac{1}{t^n} \phi \left( \frac{x-y}{t} \right) \tilde{a}(y)dy \right| + \sup_{7r_B \leq r < \delta(x)/2} \left| \int_{B_\Omega} \frac{1}{t^n} \phi \left( \frac{x-y}{t} \right) \tilde{a}(y)dy \right|
\]
\[
=: E_1 + E_2.
\]
Since \(|x - y| > 7r_B\) for \(y \in B_\Omega\) and \(x \in (8B_\Omega)^c\), \(E_1 = 0\). Note that the term \(E_2\) is valid if \(\delta(x) > 14r_B\). In this situation, for \(y \in B_\Omega\) and \(x \in (8B_\Omega)^c\) we have
\[
|x - y| > \delta(x) - \delta(y).
\]
Since \(4B \cap \Omega^c \neq \emptyset\), we have \(\delta(y) < 4r_B\) for each \(y \in B_\Omega\). Hence, for \(y \in B_\Omega\) and \(x \in (8B_\Omega)^c\) we have
\[
|x - y| > \delta(x) - \delta(y) > \delta(x) - 4r_B > \delta(x) - \delta(x)/2 = \delta(x)/2 > t.
\]
As a consequence,
\[
\phi \left( \frac{x - y}{t} \right) = 0
\]
for \(y \in B_\Omega\) and \(x \in (8B_\Omega)^c\) and \(7r_B \leq t < \delta(x)/2\). Hence, \(E_2 = 0\).

The estimates of \(E_1\) and \(E_2\) yield \(\|\tilde{\alpha}^+_{\Omega}\|_{L^p((8B_\Omega)^c)} = 0\).

Therefore,
\[
\|\tilde{\alpha}^+_{\Omega}\|_{L^p(\Omega)} \lesssim \|\tilde{\alpha}^+_{\Omega}\|_{L^p((8B_\Omega))}.
\]

Applying H"older’s inequality we obtain
\[
\|\tilde{\alpha}^+_{\Omega}\|_{L^p(\Omega)} \lesssim \|\tilde{\alpha}^+_{\Omega}\|_{L^p((8B_\Omega))} w(8B_\Omega)^{1/p-1/q} \\
\lesssim w(8B_\Omega)^{1/q-1/p} w(8B_\Omega)^{1/p-1/q} \\
\lesssim 1
\]
where in the second inequality we used the fact that \(\tilde{\alpha}^+_{\Omega} \lesssim \mathcal{M} \tilde{a}\) and \(\mathcal{M}\) is bounded on \(L^p_w\) for \(w \in A_q\).

**Case 3:** \(\Omega\) is bounded. In this case, apart from the atoms considered in two cases above, it remains to consider the case \(a = w(\Omega)^{-1/p} \chi_\Omega\). This case can be done similarly to that of the case 2.

This completes our proof. \(\square\)

The next result gives regularity estimates for the Dirichlet Green operator.

**Theorem 3.3.** Suppose that the second derivative of the Green function \(\nabla^2 \Delta_D^{-1}\) is bounded on \(L^{p_0}(X)\) for some \(p_0 \geq 2\). Assume that the Dirichlet Laplacian \(\Delta_D\) satisfies (32), and there exist \(\gamma \in (0, 1]\) and \(C, c > 0\) so that
\[
\|p_t(\Delta_D)(x, y) - p_t(\Delta_D)(x, y')\| \leq \left( \frac{|y - y'|}{\sqrt{t}} \right)^\gamma \frac{C}{\mu(B_\Omega(x, \sqrt{t}))} \exp \left( - \frac{|x - y|^2}{c t} \right)
\]
for all \(0 < t < d_\Omega := \text{diam} \Omega\) and \(x, y, y' \in \Omega\) so that \(|y - y'| < \sqrt{t}/2\).

Then the second derivative of the Green function \(\nabla^2 \Delta_D^{-1}\) is bounded on \(H^p_{M_i, w}(\Omega)\) for all \(\frac{n}{n+\gamma} < p \leq 1\) and \(w \in \bigcup_{1 < r < r_0} A_r(\mathbb{R}^n) \cap RH_{(p_0/r)'}(\mathbb{R}^n)\) where \(r_0 = \frac{p(n+\gamma)}{n}\).

Hence, \(\nabla^2 \Delta_D^{-1}\) is bounded from \(H^p_{z,w}(\Omega)\) into \(H^p_{M_i, w}(\Omega)\) for all \(\frac{n}{n+\gamma} < p \leq 1\) and \(w \in \bigcup_{1 < r < r_0} A_r(\mathbb{R}^n) \cap RH_{(p_0/r)'}(\mathbb{R}^n)\).

**Proof.** From Theorem 3.1 in order to prove that \(\nabla^2 \Delta_D^{-1}\) is bounded on \(H^p_{M_i, w}(\Omega)\), it suffices to show that
\[
H^p_{M_i, w}(\Omega) \hookrightarrow H^p_{\Delta_D, w}(\Omega)
\]
for all \(\frac{n}{n+\gamma} < p \leq 1\) and \(w \in \bigcup_{1 < r < p_0} A_r(\mathbb{R}^n) \cap RH_{(p_0/r)'}(\mathbb{R}^n)\).

By Theorem 2.4 it suffices to prove that there exists \(C > 0\) so that
\[
\|\mathcal{M}_{\Delta_D} a\|_{L^p_w(\Omega)} \leq C
\]
for every \((p, \infty, w)\)-atom \(a\) associated to a ball \(B\), where
\[
\mathcal{M}_{\Delta_D} a(x) = \sup_{0 < t < d_\Omega^2} |e^{-t\Delta_D} a(x)|.
\]
Indeed, we have
\[ \| \mathcal{M}_{\Delta_D} a \|_{L^p_c(\Omega)} \lesssim \| \mathcal{M}_{\Delta_D} a \|_{L^p_c(4B)} + \| \mathcal{M}_{\Delta_D} a \|_{L^p_c(\Omega \setminus 4B)}. \]

Since \( \mathcal{M}_{\Delta_D} a \lesssim M_a \), we have
\[ \| \mathcal{M}_{\Delta_D} a \|_{L^p_c(4B)} \lesssim w(4B)^{1/p} \| \mathcal{M}_{\Delta_D} a \|_{L^\infty} \lesssim w(4B)^{1/p} \| a \|_{L^\infty} \lesssim 1. \]

For the second term we remark that \( \mathcal{M}_{\Delta_D} a(x) \leq \sup_{0 < t < 4r_B^2} |e^{-t\Delta_D} a(x)| + \sup_{4r_B^2 \leq t < d_B^2} |e^{-t\Delta_D} a(x)| =: I_1(x) + I_2(x). \)

By the Gaussian upper bound (31) we have
\[ I_1(x) \lesssim \sup_{0 < t < 4r_B^2} \int_B 1 \cdot \exp \left( - \frac{1}{ct^2} \right) |a(y)| dy \lesssim \sup_{0 < t < 4r_B^2} \int_B \frac{1}{ct^2} \exp \left( - \frac{|x - y|^2}{ct^2} \right) |a(y)| dy \lesssim \| a \|_{L^1} \frac{r_B^\gamma}{|x - x_B|^{n+\gamma}}. \]

For the term \( I_2(x) \) we consider two cases. If \( 4B \subset \Omega \), then by the cancellation property \( \int a = 0 \) and (38) we will come up with
\[ I_2(x) \lesssim \| a \|_{L^1} \frac{r_B^\gamma}{|x - x_B|^{n+\gamma}}. \]

If \( 2B \subset \Omega \) and \( 4B \cap \Omega^c \neq \emptyset \), then we have
\[ I_2(x) = \sup_{0 < t < 4r_B^2} \left| \int_B (p_t\Delta_D(x, y) - p_t\Delta_D(x, y_0)) a(y) dy \right| \]
where \( y_0 \) is any point in \( 4B \cap \partial \Omega \).

Using (38) we also obtain
\[ I_2(x) \lesssim \| a \|_{L^1} \frac{r_B^\gamma}{|x - x_B|^{n+\gamma}}. \]

Taking the estimates of \( I_1 \) and \( I_2 \) into account we find that
\[ \mathcal{M}_{\Delta_D} a(x) \lesssim \| a \|_{L^1} \frac{r_B^\gamma}{|x - x_B|^{n+\gamma}} \lesssim \frac{r_B^\gamma}{|x - x_B|^{n+\gamma}} |B| w(4B)^{-1/p}. \]

This implies
\[ \| \mathcal{M}_{\Delta_D} a \|_{L^p_c(\Omega \setminus 4B)} \lesssim 1 \]
as long as \( w \in \bigcup_{1 < r < r_0} A_r(\mathbb{R}^n) \cap RH_{(p_0/r')}(\mathbb{R}^n) \).

We can obtain regularity estimates for the Dirichlet Green operator for a larger range of \( p \) if we have stronger assumptions on the derivatives of heat kernels.

**Theorem 3.4.** Suppose that the second derivative of the Green function \( \nabla^2 \Delta_D^{-1} \) is bounded on \( L^{p_0}(X) \) for some \( p_0 \geq 2 \). Assume that the Dirichlet Laplacian \( \Delta_D \) satisfies (32), and assume that for any multi-index \( \alpha \), there exist \( C, c > 0 \) so that

\[ |\partial_x^\alpha p_t\Delta_D(x, y)| \leq \frac{C}{t^{\alpha|\beta_1(x, \sqrt{t})|}} \exp \left( - \frac{|x - y|^2}{ct} \right) \]

for all \( 0 < t < \text{diam} \Omega \) and \( x, y \in \Omega \).

Then the second derivative of the Green function \( \nabla^2 \Delta_D^{-1} \) is bounded \( H^p_{\varepsilon, w}(\Omega) \) into \( H^{p_0}_{\varepsilon, w}(\Omega) \) for all \( 0 < p \leq 1 \) and \( w \in \bigcup_{1 < r < r_0} A_r(\mathbb{R}^n) \cap RH_{(p_0/r')}(\mathbb{R}^n) \).
Proof. By a similar argument to that of the proof of Theorem 3.3 with minor modifications, we can show that for all \(0 < p \leq 1\) and \(w \in \bigcup_{1 < r < p_0} A_r(\mathbb{R}^n) \cap RH_{(p_0/r)'}(\mathbb{R}^n)\) there exists \(C > 0\) such that
\[
\|M_{\Delta_D}u\|_{L^p_w(\Omega)} \leq C
\]
for every \((p, \infty, w)_{Mi}\)-atom \(a\) associated to a ball \(B\).

This, along with Theorem 2.4, implies that \(H^p_{\Delta_D,w}(\Omega) \subset H^p_{\Delta_D,w}(\Omega)\). Hence, Theorem 3.4 follows immediately from Theorem 3.1. \(\square\)

Note that in Theorems 3.1, 3.3 and 3.4, condition (32) is only used to obtain the weighted estimates. Hence, in the unweighted case when \(w \equiv 1\), condition (32) can be removed. More precisely, we have the following result.

**Theorem 3.5.** Suppose the second derivative of the Green function \(\nabla^2 \Delta_D^{-1}\) is bounded on \(L^p(X)\) for some \(p_0 \geq 2\). Then we have:

(i) The operator \(\nabla^2 \Delta_D^{-1}\) is bounded from \(H^p_{\Delta_D}(\Omega)\) into \(H^p_{\Delta_D}(\Omega)\) for all \(0 < p \leq 1\).

(ii) If (38) is satisfied, then \(\nabla^2 \Delta_D^{-1}\) is bounded on \(H^p_{\Delta_D}(\Omega)\) for all \(\frac{n}{n+\gamma} < p \leq 1\). Hence, \(\nabla^2 \Delta_D^{-1}\) is bounded from \(L^p_{\Delta_D}(\Omega)\) into \(H^p_{\Delta_D}(\Omega)\) for all \(\frac{n}{n+\gamma} < p \leq 1\).

(iii) If (39) is satisfied, then \(\nabla^2 \Delta_D^{-1}\) is bounded from \(H^p_{\Delta_D}(\Omega)\) into \(H^p_{\Delta_D}(\Omega)\) for all \(0 < p \leq 1\).

### 3.2. Neumann Laplacian problems

Let \(\Omega\) be an open connected domain in \(\mathbb{R}^n\). Denote by \(\Delta_N\) the Neumann Laplacian defined on \(D(\Delta_N) := \{u \in W^{1,2}(\Omega) : \Delta u \in L^2(\Omega)\}\) such that
\[
(\Delta_N f, g) = \int_{\Omega} \nabla f \cdot \nabla g, \quad \forall f \in D(\Delta_N), \quad g \in W^{1,2}(\Omega).
\]

Denote by \(p_t,\Delta_N(x,y)\) the kernel of \(e^{-t\Delta_N}\). We assume that there exist \(C, c > 0\) so that
\[
0 \leq p_t,\Delta_N(x,y) \leq \frac{C}{|B_1(x, \sqrt{t})|} \exp\left(-\frac{|x-y|^2}{ct}\right)
\]
for all \(t > 0\) and \(x, y \in \Omega\), where \(B_1(x, \sqrt{t}) = B(x, \sqrt{t}) \cap \Omega\). It was proved in [24] that the Gaussian upper bound (41) is satisfied if \(\Omega\) has the extension property.

We now consider the following condition: There exist a constant \(\beta > 0\) and \(C > 0\) so that
\[
(\int_{x \in \Omega} |\nabla_x^2 p_t,\Delta_N(x,y)|^2 e^{\frac{|x-y|^2}{\beta t^2}} dx)^{1/2} \leq C\beta t^{-2} |B_\Omega(y, \sqrt{t})|^{-1}, \quad \forall y \in \Omega, t > 0.
\]

Similarly to the condition (32), it can be seen that the estimates (42) is satisfied if \(\nabla^2_x p_t,\Delta_N\) has a Gaussian upper bound. In our applications, we will show that the estimate (42) is satisfied if \(\Omega\) is one of the following domains:

(i) a bounded convex domain;
(ii) a convex domain above a Lipschitz graph;
(iii) the upper-half space.

Arguing similarly to the proof of Theorem 3.1 we have:

**Theorem 3.6.** Suppose that the second derivative of the Green function \(\nabla^2 \Delta_N^{-1}\) is bounded on \(L^p(X)\) for some \(p_0 \geq 2\). Suppose that the Neumann Laplacian \(\Delta_N\) satisfies (42). Then we have

(i) The second derivative of the Green function \(\nabla^2 \Delta_N^{-1}\) is bounded on \(L^p_w(\Omega)\) for all \(1 < p < p_0\) and \(w \in A_p(\mathbb{R}^n) \cap RH_{(p_0/p)'},\) and is bounded from \(L^1_w(\Omega)\) into \(L^1_{w,\infty}(\Omega)\) for \(w \in A_1 \cap RH_{p_0} \).

(ii) The second derivative of the Green function \(\nabla^2 \Delta_N^{-1}\) is bounded from \(H^p_{\Delta_N,w}(\Omega)\) into \(H^p_{\Delta_N,w}(\Omega)\) for all \(0 < p \leq 1\) and \(w \in \bigcup_{1 < r < p_0} A_r(\mathbb{R}^n) \cap RH_{(p_0/r)'}(\mathbb{R}^n)\).

In the particular case when the kernel \(p_t,\Delta_N(x,y)\) satisfies the Hölder continuity (43) below, Theorem 3.6 deduces to the boundedness of \(\nabla^2 \Delta_N^{-1}\) from \(H^p_{\Delta_N,w}(\Omega)\) into \(H^p_{\Delta_N,w}(\Omega)\) for certain \(p \leq 1\). We have:
Theorem 3.7. Suppose that the second derivative of the Green function $\nabla^2 \Delta_N^{-1}$ is bounded on $L^{p_0}(X)$ for some $p_0 \geq 2$. Assume that the Neumann Laplacian $\Delta_N$ satisfies (32), and there exist $\gamma \in (0, 1)$ and $C, c > 0$ so that
\begin{equation}
|p_t,\Delta_N(x, y) - p_t,\Delta_N(x, y')| \leq \left( \frac{|y - y'|}{\sqrt{t}} \right)^\gamma \frac{C}{\mu(B_t(x, \sqrt{t}))} \exp \left( - \frac{|x - y|^2}{ct} \right)
\end{equation}
for all $0 < t < d_\Omega := \text{diam} \Omega$ and $x, y, y' \in \Omega$ so that $|y - y'| < \sqrt{t}/2$.

Then the second derivative of the Green function $\nabla^2 \Delta_N^{-1}$ is bounded from $H^p_{z,w}(\Omega)$ into $H^p_{M_t,w}(\Omega)$ for all $\frac{n}{n+\gamma} < p \leq 1$ and $w \in \bigcup_{1 < r < r_0} A_r(\mathbb{R}^n) \cap RH_{(p_0/r)^2}(\mathbb{R}^n)$ where $r_0 = \frac{p(n+\gamma)}{n}$.

Proof. From Theorem 3.6 in order to prove that $\nabla^2 \Delta_N^{-1}$ is bounded from $H^p_{z,w}(\Omega)$ into $H^p_{M_t,w}(\Omega)$, it suffices to show that for any multi-index $\alpha$
\begin{equation}
\left\| \mathcal{M}_{\Delta_N} a \right\|_{L^p_w(\Omega)} \leq C
\end{equation}
for every $(p, q, w)$-atom $a$ associated to a ball $B$, where
\begin{equation}
\mathcal{M}_{\Delta_N} a(x) = \sup_{0 < t < d_\Omega^2} |e^{-\Delta_N t} a(x)|.
\end{equation}
It follows that
\begin{equation}
H^p_{z,w}(\Omega) \hookrightarrow H^p_{\Delta_N,w}(\Omega).
\end{equation}
This, in combination with Theorem 2.4, yields
\begin{equation}
H^p_{z,w}(\Omega) \hookrightarrow H^p_{\Delta_N,w}(\Omega),
\end{equation}
which completes our proof. \hfill \Box

Theorem 3.8. Suppose that the second derivative of the Green function $\nabla^2 \Delta_N^{-1}$ is bounded on $L^{p_0}(X)$ for some $p_0 \geq 2$. Assume that the Neumann Laplacian $\Delta_N$ satisfies (32), and assume that for any multi-index $\alpha$, there exist $C, c > 0$ so that
\begin{equation}
|\partial^\alpha_x p_t,\Delta_N(x, y)| \leq \frac{C}{t^{\alpha/2}} \frac{1}{\mu(B_t(x, \sqrt{t}))} \exp \left( - \frac{|x - y|^2}{ct} \right)
\end{equation}
for all $0 < t < \text{diam} \Omega$ and $x, y \in \Omega$.

Then the second derivative of the Green function $\nabla^2 \Delta_N^{-1}$ is bounded from $H^p_{z,w}(\Omega)$ into $H^p_{M_t,w}(\Omega)$ for all $0 < p \leq 1$ and $w \in \bigcup_{1 < r < r_0} A_r(\mathbb{R}^n) \cap RH_{(p_0/r)^2}(\mathbb{R}^n)$.

Proof. From Theorem 3.6 it suffices to verify that
\begin{equation}
H^p_{z,w}(\Omega) \hookrightarrow H^p_{\Delta_N,w}(\Omega).
\end{equation}
To do this, we need to show that for all $0 < p \leq 1$ and $w \in \bigcup_{1 < r < r_0} A_r(\mathbb{R}^n) \cap RH_{(p_0/r)^2}(\mathbb{R}^n)$ there exists $C > 0$ so that
\begin{equation}
\left\| \mathcal{M}_{\Delta_N} a \right\|_{L^p_w(\Omega)} \leq C
\end{equation}
for every $(p, q, w)$-atom $a$.

The proof of (45) is standard and we omit the details. This completes our proof. \hfill \Box

It is worth noticing that in the unweighted case when $w \equiv 1$, condition (42) can be removed. We have the following result.

Theorem 3.9. Suppose the second derivative of the Green function $\nabla^2 \Delta_N^{-1}$ is bounded on $L^{p_0}(X)$ for some $p_0 \geq 2$. Then we have:
(i) The operator $\nabla^2 \Delta_N^{-1}$ is bounded from $H^p_{\Delta_N}(\Omega)$ into $H^p_{M_t}(\Omega)$ for all $0 < p \leq 1$. 

(ii) If (43) is satisfied, $\nabla^2 \Delta^{-1}_D$ is bounded from $H^p_\Omega$ into $H^p_{M1}(\Omega)$ for all $\frac{n}{n+\gamma} < p \leq 1$.

(iii) If (44) is satisfied, then $\nabla^2 \Delta^{-1}_N$ is bounded from $H^p_\Omega$ into $H^p_{M1}(\Omega)$ for all $0 < p \leq 1$.

3.3. Proof of main results. We now ready to give the proofs of Theorems 1.5–1.9.

Proof of Theorem 1.5: Note that the $L^2$-boundedness of $\nabla^2 \Delta^{-1}_D$ and $\nabla^2 \Delta^{-1}_N$ can be found in [27, Theorem 4.8]. The condition (38) for any $\gamma \in (0, 1)$ and condition 32 for the inhomogeneous Dirichlet problems were verified in Lemma 2.7 and Proposition 4.15 in [27]. Meanwhile, the Gaussian upper bound (41), the Hölder continuity condition 43 for $\gamma = 1$ and the estimate (42) can be found in Lemma 2.8 and Proposition 4.15 in [27]. Therefore, Theorem 1.5 follows directly from Theorem 3.1, Theorem 3.3, Theorem 3.6 and Theorem 3.7.

To prove Theorem 1.7, we need the following technical result:

Lemma 3.10. (a) Let $\Omega$ be a convex domain above a Lipschitz graphs in $\mathbb{R}^n$, and assume that $f \in L^2(\Omega)$. Then the unique solution $u \in W^{1,2}(\Omega)$ to the Neumann problem (2) has the property that for any $\psi \in C^\infty(\mathbb{R}^n)$,

$$\int_\Omega \psi^2 |\nabla^2 u|^2 dx \leq C \int_\Omega |\nabla \psi|^2 |\nabla u|^2 dx + C \int_\Omega \psi^2 f^2 dx,$$

for some finite constant $C > 0$ independent of $f$.

(b) The result of part (a) still holds if the function $f \in L^2(\Omega)$ with $\int_\Omega f dx = 0$ and the Neumann problem (2) is replaced by the Dirichlet problem (2).

Proof. The statement (a) is essential taken from Theorem 2.1 in [3]. The proof of (b) can be done similarly and hence we leave it to interested readers.

Proof of Theorem 1.7: The $L^2$-boundedness of $\nabla^2 \Delta^{-1}_D$ and $\nabla^2 \Delta^{-1}_N$ was proved in [2] and [3], respectively. The Gaussian upper bound (41) follows from Theorem 3.2.9 in [24]. The Hölder continuity conditions (38) and (43) for any $\gamma \in (0, 1)$ can be found in [7]. We now verify conditions (32) and (42). Let us take care of (32) first. By using (b) in Lemma 3.10 and arguing similarly to the proof of Proposition 4.16 in [27] we conclude that there exist a constant $\beta > 0$ and $C > 0$ so that

$$\left( \int_{x \in \Omega} |\nabla^2 p_t \Delta_D(x,y)|^2 e^{\beta \frac{|y|^2}{t}} dx \right)^{1/2} \leq C_\beta t^{-2} |B_\Omega(y, \sqrt{t})|^{-1}, \quad \forall y \in \Omega, t > 0$$

which proves (32).

Similarly, we obtain that (42) is satisfied.

Hence, Theorem 1.5 follows directly from Theorem 3.1, Theorem 3.3, Theorem 3.6 and Theorem 3.7.

Proof of Theorem 1.9: The boundedness of $\nabla^2 \Delta^{-1}_D$ and $\nabla^2 \Delta^{-1}_N$ on $L^p(\Omega)$ for $1 < p < \infty$ is classical. See for example [18]. The Gaussian upper bound (41), the Hölder continuity conditions (38) and (43), and conditions (32) and (42) follow directly from the explicit expression for the kernels of $p_t \Delta_D(x,y)$ and $p_t \Delta_N(x,y)$.

Therefore, Theorem 1.9 follows immediately from Theorem 3.4 and Theorem 3.8.

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