Normalized ground states for nonlinear Schrödinger equations with general Sobolev critical nonlinearities

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Abstract

In this paper, we study the existence of normalized solutions to the following nonlinear Schrödinger equation

\[
\begin{align*}
-\Delta u &= f(u) + \lambda u \quad \text{in } \mathbb{R}^N, \\
u &\in H^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u|^2 \, dx = c,
\end{align*}
\]

where \( N \geq 3 \), \( c > 0 \), \( \lambda \in \mathbb{R} \) and \( f \) has a Sobolev critical growth at infinity but does not satisfy the Ambrosetti-Rabinowitz condition. By analysing the monotonicity of the ground state energy with respect to \( c \), we develop a constrained minimization approach to establish the existence of normalized ground state solutions for all \( c > 0 \).

I Introduction and main results

In this paper, we deal with the following nonlinear Schrödinger equation

\[
-\Delta u = f(u) + \lambda u \quad \text{in } \mathbb{R}^N
\]

under the constraint

\[
\int_{\mathbb{R}^N} |u|^2 \, dx = c,
\]

where \( N \geq 3 \), \( c > 0 \), \( f \in C^1(\mathbb{R}, \mathbb{R}) \) and \( \lambda \in \mathbb{R} \) is a Lagrange multiplier. A function \( u \in H^1(\mathbb{R}^N) \) satisfying (1.1)-(1.2) is usually referred as a normalized solution of (1.1). The study of normalized solutions for (1.1) is motivated by the search of standing waves solutions with form \( \Psi(t,x) = e^{-i\lambda t} u(x) \) of prescribed mass \( c \) for the following time-dependent Schrödinger equation

\[
\partial_t \Psi(t,x) + \Delta_x \Psi(t,x) + g \left( |\Psi(t,x)|^2 \right) \Psi(t,x) = 0, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.
\]

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Consider the associated energy functional \( J : H^1(\mathbb{R}^N) \to \mathbb{R} \) given by

\[
J(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} F(u) dx,
\]

where \( F(u) := \int_0^u f(s) ds \). Set \( S_c := \{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 dx = c \} \). Clearly, \( J \) is of \( C^1 \) under suitable conditions on \( f \), and any critical point of \( J \) on \( S_c \) corresponds to a normalized solution of (1.1). Furthermore, any normalized solution of (1.1) stays in the following Nehari-Pohozaev type set

\[
\mathcal{M} := \left\{ u \in H^1(\mathbb{R}^N) \setminus \{0\} : \mathcal{P}(u) = 0 \right\}.
\]

Here \( \mathcal{P} : H^1(\mathbb{R}^N) \to \mathbb{R} \) is the Nehari-Pohozaev functional defined by

\[
\mathcal{P}(u) := \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{N}{2} \int_{\mathbb{R}^N} H(u) dx, \quad \forall u \in H^1(\mathbb{R}^N),
\]

where \( H(u) := f(u)u - 2F(u) \).

If \( f \) admits a \( L^2 \) subcritical growth at infinity, i.e., \( f(s) \) has a growth \(|s|^{p-1}\) with \( p < 2 + \frac{4}{N} \) as \( |s| \to +\infty \), then \( J|_{S_c} \) is bounded below and one can use the minimization method to get a global minimizer, see for example Refs. [1–3].

If \( f \) admits a \( L^2 \) supercritical growth at infinity, i.e., \( p > 2 + \frac{4}{N} \), then \( J|_{S_c} \) is unbounded below and the direct minimization does not work. The first breakthrough in the case of \( L^2 \) supercritical but Sobolev subcritical was made by Jeanjean [4], where a mountain-pass type argument for the scaled functional \( \tilde{J}(u,t) := J(tu) \) with \( t \neq 0 \) was introduced. Subsequently, Bartsch and de Valeriola [5] applied the genus theory to obtain infinitely many normalized solutions of (1.1). In Ref. [6], Ikoma and Tanaka established a deformation result on \( S_c \), and gave an alternative proof of the results in Refs. [4, 5]. Bartsch and Soave [7, 8] introduced a minimax approach on \( S_c \cap \mathcal{M} \) to study the existence and multiplicity of normalized solutions to (1.1). In these studies, the following Ambrosetti-Rabinowitz (AR) condition

\[
\text{there exists } \alpha > 2 + \frac{4}{N} \text{ such that } f(s)s \geq \alpha F(s) > 0, \quad s \neq 0
\]

plays an essential role in obtaining the bounded constrained Palais-Smale sequences of \( J \).

In Ref. [9], Jeanjean and Lu studied the normalized solutions of (1.1) when \( f \) satisfies a monotonicity condition (see (f3) below), but not the (AR) condition. By analysing the behavior of the ground state energy as the prescribed mass varies, they developed a minimax argument on \( S_c \cap \mathcal{M} \) to prove the existence of normalized ground state solutions. Here a normalized solution \( u_c \in H^1(\mathbb{R}^N) \) of (1.1) is called a ground state solution means it has minimal energy among all the solutions belonging to \( S_c \), that is,

\[
J(u_c) := \inf \left\{ J(u) : u \in S_c, dJ|_{S_c}(u) = 0 \right\},
\]

where \( dJ|_{S_c}(u) \) denotes the constrained derivative of \( J \) on \( S_c \). Bieganowski and Mederski [10] presented a constrained minimization method for the \( L^2 \) supercritical problem without imposing the (AR) condition. In order to get the existence of normalized ground state solutions, they first transformed the search of minimizers for \( J \) on \( S_c \cap \mathcal{M} \) into looking for minimizers for \( J \) on \( \mathcal{D}_c \cap \mathcal{M} \), where

\[
\mathcal{D}_c := \left\{ u \in H^1(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} |u|^2 dx \leq c \right\}.
\]

Then the proof was performed by the analysis of Lagrange multipliers for the constraints \( S_c \) and \( \mathcal{M} \) respectively.
When \( f \) admits a Sobolev critical growth, there have been many works about the study of normalized solutions to (1.1). In Ref. [11], Sovaje studied the existence of normalized ground state solutions of the following nonlinear Schrödinger equation with combined power nonlinearities

\[
-\Delta u = \lambda u + \mu |u|^{q-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, \ N \geq 3,
\]

where \( 2 < q < 2^* := \frac{2N}{N-2} \) and the condition \( \mu^{(1-\gamma_0)}q < \alpha(N, q) \) was imposed. Here \( \gamma_q = \frac{N(q-2)}{2q} \), \( \alpha(N, q) = +\infty \) if \( N = 3, 4 \) and \( \alpha(N, q) \) is finite for \( N \geq 5 \). In Ref. [11], the ground state solution was shown to be a local minimizer when \( 2 < q < 2 + \frac{4}{N} \), while it is a critical point of mountain pass type when \( 2 + \frac{4}{N} \leq q < 2^* \).

Afterwards, under the \( L^2 \)-subcritical \( (2 < q < 2 + \frac{4}{N}) \), \( L^2 \)-critical \( (q = 2 + \frac{4}{N}) \), and \( L^2 \)-supercritical \( (2 + \frac{4}{N} < q < 2^*) \) perturbation \( \mu |u|^{q-2}u \) respectively, there have been a series of progress made towards a complete comprehension of (1.3), one is referred to see Refs. [12–16] and their references.

In this paper, we develop a direct minimization approach on \( S_c \cap \mathcal{M} \) to study the existence of normalized ground state solutions of (1.1) when \( f \) satisfies a general Sobolev critical growth but not the (AR) type condition. By analysing the monotonicity of the ground state energy with respect to the prescribed mass \( c \), we shall establish the existence of normalized ground state solutions of (1.1) for all \( c > 0 \).

Our assumptions are formulated as follows:

\begin{enumerate}
\item[(f_1)] \( f \in C^1(\mathbb{R}, \mathbb{R}) \) and \( \lim_{|x| \to 0} \frac{F(x)}{|x|^q} = 0 \);
\item[(f_2)] there exists \( \eta > 0 \) such that \( \eta := \lim_{|x| \to \infty} \frac{F(x)}{|x|^{q-2}} \);
\item[(f_3)] \( \frac{H(s)}{|s|^{q-1}} \) is strictly decreasing on \((-\infty, 0)\) and strictly increasing on \((0, +\infty)\);
\item[(f_4)] there exist constants \( p > 2 + \frac{4}{N} \) and \( \mu > 0 \) such that \( sgn(s)f(s) \geq \mu |s|^{p-2}s \),
\end{enumerate}

where \( sgn(s) = \begin{cases} 1, & s > 0, \\ -1, & s < 0 \end{cases} \)

\begin{enumerate}
\item[(f_5)] \( f(s)s < 2^*F(s) \), \( \forall s \in \mathbb{R} \setminus \{0\} \).
\end{enumerate}

The main result of this paper is stated as follows.

**Theorem 1.1.** Assume that (f_1)-(f_5) hold. Then, for any \( c > 0 \), there exists \( \mu_0 > 0 \) such that, for all \( \mu > \mu_0 \), problem (1.1) admits a normalized ground states solution \( u \in H^1(\mathbb{R}^N) \) for some \( \lambda < 0 \).

For any \( c > 0 \), we define

\[ m(c) := \inf_{u \in S_c \cap \mathcal{M}} J(u). \]

The monotonicity of the ground state energy \( m(c) \) with respect to \( c \) plays a crucial role in establishing the existence of normalized ground state solutions, see for example Refs. [9, 17] for the Sobolev subcritical case. In this paper, since the nonlinearity has a Sobolev critical growth and the (AR) condition is not assumed, the problem becomes more complicated. In addition, we remark that problem (1.1) with general Sobolev critical growth was also studied recently in Ref. [18] by following the strategy in Ref. [10], where the ground state normalized solutions were obtained without using the the monotonicity of the ground state energy \( m(c) \). Furthermore, one can easily verify that, for \( \mu > 0 \) large enough, our results can not be covered by the results in Ref. [18].
To prove Theorem 1.1, the main obstacle is to recover the compactness of the minimizing sequence \( \{u_n\} \) of \( J \) on \( S_c \cap M \). We shall overcome this difficulty by proving that the function \( c \mapsto m(c) \) is strictly decreasing, which is based on a delicate energy estimate and the concentration compactness arguments. Then, by showing that \( M \) is a natural constraint, we prove that the minimizer \( u_0 \) of \( J \) on \( S_c \cap M \) is a normalized ground state solution of (1.1).

The paper is organized as follows. In Section 2, we introduce some preliminary results. In Section 3, the monotonicity of \( m(c) \) with respect to \( c > 0 \) will be analyzed. In Section 4, we give the proof of Theorem 1.1.

Regarding the notations, for \( p \geq 1 \), the (standard) \( L^p \) norm of \( u \in L^p(\mathbb{R}^N) \) is denoted by \( ||u||_p \) and \( || \cdot || \) denotes the norm in \( H^1(\mathbb{R}^N) \).

II Preliminaries

In this section, we prove some preliminary results. We recall that, for \( N \geq 2 \) and \( q \in (2, 2^*) \), there exists \( C_{N,q} > 0 \) depending on \( N \) and \( q \) such that the following Gagliardo-Nirenberg inequality holds:

\[
||u||_q \leq C_{N,q}||u||_2^{1 - \frac{q}{2}}||\nabla u||_2, \quad \forall u \in H^1(\mathbb{R}^N),
\]

where \( \gamma_q := N(\frac{1}{2} - \frac{1}{q}) \). See Ref. [19]. If \( N \geq 3 \), by Ref. [20], there exists an optimal constant \( S > 0 \) depending only on \( N \) such that

\[
S||u||_2^2 \leq ||\nabla u||_2^2, \quad \forall u \in H^1(\mathbb{R}^N).
\]

**Lemma 2.1.** Assume that \((f_1) - (f_2)\) and \((f_5)\) hold. Then \( \inf_{S \cap M} ||\nabla u||_2 > 0 \).

**Proof.** We assume by contradiction that there exists \( \{u_n\} \subset S_c \cap M \) such that

\[
||\nabla u_n||_2 \to 0.
\]

By \((f_1) - (f_2)\), there exist \( p < p_1 < 2^* \) and \( C_{\eta, p_1} > 0 \) such that

\[
F(s) \leq C_{\eta, p_1}|s|^{p_1} + \eta|s|^{2^*}, \quad \forall s \in \mathbb{R}.
\]

Then, by (2.1)-(2.4), \((f_5)\) and \( \gamma_{p_1, p_1} > 2 \), we get

\[
\int_{\mathbb{R}^N} f(u_n)u_n dx \leq 2^* \int_{\mathbb{R}^N} F(u_n) dx \\
\leq 2^* \int_{\mathbb{R}^N} \left( C_{\eta, p_1}|u_n|^{p_1} + \eta|u_n|^{2^*} \right) dx \\
\leq \left( 2^* C_{\eta, p_1} C_{p_1}^{\frac{q}{p_1}} \frac{\gamma_{p_1, p_1}^{1 - \frac{q}{2}}}{p_1 - 1} \right) \int_{\mathbb{R}^N} ||\nabla u_n||_2^2 dx \\
\leq \frac{1}{4} \int_{\mathbb{R}^N} ||\nabla u_n||_2^2 dx
\]

for \( n \) large enough. Hence, using \((f_5)\) again,

\[
\int_{\mathbb{R}^N} H(u_n) dx = \int_{\mathbb{R}^N} \left[ f(u_n)u_n - 2F(u_n) \right] dx \leq \frac{2}{N} \int_{\mathbb{R}^N} f(u_n) u_n dx \leq \frac{1}{N} \int_{\mathbb{R}^N} ||\nabla u_n||_2^2 dx,
\]

which together with \( \{u_n\} \subset M \) implies that

\[
0 = \mathcal{P}(u_n) = \int_{\mathbb{R}^N} ||\nabla u_n||_2^2 dx - \frac{N}{2} \int_{\mathbb{R}^N} H(u_n) dx \geq \frac{1}{2} \int_{\mathbb{R}^N} ||\nabla u_n||_2^2 dx.
\]

This is a contradiction with \( \{u_n\} \subset S_c \). Therefore, \( \inf_{S \cap M} ||u||_2 > 0 \). \( \square \)
For any \( u \in H^1(\mathbb{R}^N) \setminus \{0\} \) and \( t > 0 \), set
\[
(t \ast u)(x) := t^2 u(tx), \quad \forall x \in \mathbb{R}^N.
\]
Clearly, \( t \ast u \in \mathcal{S}_c \) if \( u \in \mathcal{S}_c \).

**Lemma 2.2.** Suppose that \((f_1) - (f_4)\) hold. Then, for any \( u \in H^1(\mathbb{R}^N) \setminus \{0\} \), there exists a unique number \( t_u > 0 \) such that \( t_u \ast u \in \mathcal{M} \). Moreover, \( J(t_u \ast u) > J(t \ast u) \) for all \( t > 0 \) with \( t \neq t_u \).

**Proof.** By (2.4) and (f_4), we get
\[
\frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq J(t \ast u) \geq \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - C_{\eta,p,t} \frac{N}{p(p-2)} \int_{\mathbb{R}^N} |u|^{p_1} \, dx - \eta t^{2r} \int_{\mathbb{R}^N} |u|^{2r} \, dx
\]
and
\[
J(t \ast u) \leq \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{\mu}{p} \frac{N}{p(p-2)} \int_{\mathbb{R}^N} |u|^p \, dx.
\]
Then, by \( p_1 > p > 2 + \frac{4}{N} \), we have
\[
J(t \ast u) \to 0^+ \quad \text{as} \quad t \to 0^+,
\]
\[
J(t \ast u) \to -\infty \quad \text{as} \quad t \to +\infty,
\]
which implies that \( \Phi_u(t) := J(t \ast u) \) admits a global maximum point \( t_u > 0 \) and thus \( \frac{d}{dt} \Phi_u(t)|_{t=t_u} = 0. \) Since
\[
\frac{d}{dt} J(t \ast u) = t^{-1} \mathcal{P}(t \ast u),
\]
we deduce \( t_u \ast u \in \mathcal{M} \).

In what follows, we prove the uniqueness. Assume by contradiction that there exist \( 0 < t_u < t_u \) such that \( t_u \ast u, t_u \ast u \in \mathcal{M} \). Then
\[
\frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{N}{2} \mathcal{P}_{t_u} \int_{\mathbb{R}^N} H(t_u \ast u) \, dx = 0, \quad \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{N}{2} \mathcal{P}_{t_u} \int_{\mathbb{R}^N} H(t_u \ast u) \, dx = 0.
\]
By similar arguments as in Remark 2.2 in Ref. [9], we may assume that \( \frac{H(s)}{|s|^{2+\frac{4}{N}}} \) is continuous on \( \mathbb{R} \), \( \frac{H(s)}{|s|^{2+\frac{4}{N}}} = 0 \) at \( s = 0 \). Furthermore, it is strictly decreasing on \((-\infty, 0]\) and strictly increasing on \([0, +\infty)\). Then by (2.5) we get
\[
\int_{\mathbb{R}^N} \left( \frac{H(t_u \ast u)}{|t_u \ast u|^{2+\frac{4}{N}}} - \frac{H(t_u \ast u)}{|t_u \ast u|^{2+\frac{4}{N}}} \right) |u|^{2+\frac{4}{N}} \, dx = 0.
\]
However, by \((f_3)\) we can deduce
\[
\int_{\mathbb{R}^N} \left( \frac{H(t_u \ast u)}{|t_u \ast u|^{2+\frac{4}{N}}} - \frac{H(t_u \ast u)}{|t_u \ast u|^{2+\frac{4}{N}}} \right) |u|^{2+\frac{4}{N}} \, dx > 0,
\]
which provides a contradiction. Hence \( t_u = t_u \), which implies that \( t_u \ast u \in \mathcal{M} \) is the unique global maximum point of \( \Phi_u \) and \( J(t_u \ast u) > J(t \ast u) \) for all \( t \neq t_u \).

**Lemma 2.3.** Assume that \((f_1) - (f_5)\) hold. \( \inf_{\mathcal{S} \subseteq \mathcal{M}} J(u) = \inf_{u \in \mathcal{S}} \max_{t > 0} J(t \ast u) > 0. \)
Proof. By Lemma 2.2, it is easily seen that \( \inf_{S \subseteq M} J(u) = \inf_{u \in S} \max_{t > 0} J(t \ast u) \). Then, by (2.4), for any \( t > 0 \) and \( u \in S \subseteq M \), we have

\[
J(u) \geq \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} \left( C_{n,p} t^{\frac{N}{2}(p-2)} |u|^p + \eta \frac{|\nabla u|^{2^*}}{2^*} \right) \, dx
\]

\[
\geq \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \left( C_{n,p} C_{p,p} t^{\frac{N}{2}(p-2)} c_{\cdot}^{1-\gamma \delta_1} + \eta \delta \frac{|\nabla u|^{2^*}}{2^*} \right).
\]

Hence, in view of \( p_1 > 2 + \frac{2}{N} \), taking \( t = \delta \| \nabla u \|_2 \) with \( \delta > 0 \) small enough, we deduce

\[
J(u) \geq \frac{\delta^2}{2} - \left( C_{n,p} C_{p,p} c_{\cdot}^{1-\gamma \delta_1} + \eta \delta \| \nabla u \|_2 \right) \geq \frac{1}{4} \delta^2 > 0.
\]

\[\square\]

Lemma 2.4. Assume that \((f_1) - (f_4)\) hold. Then, for any \( u \in H^1(\mathbb{R}^N) \setminus \{0\} \), we have

(i) The map \( u \mapsto t_u \) is continuous.

(ii) \( t_{u(\cdot + y)} = t_u \) for any \( y \in \mathbb{R}^N \).

Proof. For (i), by Lemma 2.2, the mapping \( u \mapsto t_u \) is well defined. Let \( \{u_n\} \subset H^1(\mathbb{R}^N) \setminus \{0\} \) be any sequence such that \( u_n \to u \) in \( H^1(\mathbb{R}^N) \). We first show that \( \{t_{u_n}\} \) is bounded. If not, \( t_{u_n} \to +\infty \). Then, by \((f_4)\) it follows that

\[
0 \leq t_{u_n}^{-2} J(t_{u_n} \ast u_n) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n|^{2} dx - t_{u_n}^{-2(N+1)} \int_{\mathbb{R}^N} F(t_{u_n}^\delta u_n) \, dx
\]

\[
\leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n|^{2} dx - \frac{\mu}{p} \left( t_{u_n}^{-2(N+1)} \right) \int_{\mathbb{R}^N} |u_n|^p \, dx \to -\infty \quad \text{as} \quad t \to +\infty,
\]

which is a contradiction. Hence, the sequence \( \{t_{u_n}\} \) is bounded, which implies that there exists \( t^* \geq 0 \) such that \( t_{u_n} \to t^* \). Due to \( u_n \to u \) in \( H^1(\mathbb{R}^N) \) and \( P(t_{u_n} \ast u_n) = 0 \), we get \( t_{u_n} \ast u_n \to t^* \ast u \) in \( H^1(\mathbb{R}^N) \) and \( P(t^* \ast u) = 0 \). By Lemma 2.2, we have \( t^* = t_u \) and thus (i) is proved. By the definition of \( t_{u(\cdot + y)} \) and direct computations, we get (ii). \[\square\]

III Monotonicity of the ground state energy

In this section, we shall study the behavior of the ground state energy \( m(c) \).

Lemma 3.1. Assume that \((f_1) - (f_5)\) hold. Then the function \( c \mapsto m(c) \) is continuous.

Proof. For any \( c > 0 \), we take \( \{c_n\} \) be such that \( c_n \to c \) as \( n \to \infty \). For any \( u \in S \subseteq M \), set \( u_n = \sqrt{c_n} u, \) \( n \in \mathbb{N}^+ \). Clearly, \( u_n \in S_c \) and \( u_n \to u \) in \( H^1(\mathbb{R}^N) \). Then, by Lemma 2.4(i), we have

\[
t_{u_n} \to t_{u} = 1, \quad t_{u_n} \ast u_n \to t_u \ast u = u \quad \text{in} \quad H^1(\mathbb{R}^N),
\]

which implies that

\[
\limsup_{n \to \infty} m(c_n) \leq \limsup_{n \to \infty} J(t_{u_n} \ast u_n) = J(u).
\]

Since \( u \in S \subseteq M \) is arbitrary, we get

\[
\limsup_{n \to \infty} m(c_n) \leq m(c). \quad (3.1)
\]
In what follows, we prove

$$m(c) \leq \liminf_{n \to \infty} m(c_n).$$  \hspace{1cm} (3.2)

For any $\varepsilon > 0$ small enough, there exists $v_n \in S_{c_n} \cap \mathcal{M}$ such that

$$J(v_n) \leq m(c_n) + \varepsilon. \hspace{1cm} (3.3)$$

We claim $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. If not, due to $\|v_n\|^2 = c_n \to c$, we deduce $\|\nabla v_n\|_2 \to \infty$. Set $w_n = \frac{1}{s_n^*} v_n$, where $s_n = \|\nabla v_n\|_2$. Clearly,

$$s_n \to \infty, \quad s_n^* \in S_{c_n} \cap \mathcal{M} \quad \text{and} \quad \|\nabla w_n\|_2 = 1,$$

which implies that $\{w_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Set

$$\hat{\delta} := \limsup_{n \to \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B(y;1)} |w_n|^2 \, dx \right).$$

We distinguish the following two cases:

Case 1. $\hat{\delta} > 0$, i.e., non-vanishing occurs. Then, up to a subsequence, there exists $\{z_n\} \subset \mathbb{R}^N$ such that

$$w_n(\cdot + z_n) \to w \neq 0 \quad \text{in} \quad H^1(\mathbb{R}^N) \quad \text{and} \quad w_n(x + z_n) \to w(x) \quad \text{for a.e.} \ x \in \mathbb{R}^N.$$ 

By $(f_4)$ and Lemma 2.3, we have

$$0 \leq s_n^{-2} J(v_n) = s_n^{-2} J(s_n^* w_n) \leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w_n|^2 \, dx - s_n^{-2} \int_{\mathbb{R}^N} F(s_n^* w_n) \, dx \leq \frac{1}{2} \frac{\mu (\nu (p-2+\frac{4}{p}))}{p} \int_{\mathbb{R}^N} |w_n(x + z_n)|^p \, dx \to -\infty,$$

which is a contradiction.

Case 2. $\hat{\delta} = 0$, i.e., $\{w_n\}$ is vanishing. By the Lions lemma (see Ref. [21]), we deduce

$$\int_{\mathbb{R}^N} |w_n|^p \, dx \to 0. \hspace{1cm} (3.4)$$

Moreover, by (2.2), (2.4), (3.3), Lemma 2.2 and $\|\nabla w_n\|_2 = 1$, for any $s > 0$, we have

$$m(c_n) + \varepsilon \geq J(v_n) = J(s_n^* w_n) \geq J(s \star w_n) \geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w_n|^2 \, dx - s^{-N} \int_{\mathbb{R}^N} F(s \nabla w_n) \, dx \geq \frac{1}{2} s^2 - s^\nu (p-1-\nu) C_{\eta,p_1} \int_{\mathbb{R}^N} |w_n|^{p_1} \, dx - s^{-\eta} \int_{\mathbb{R}^N} |w_n|^{2 \eta} \, dx \geq \frac{1}{2} s^2 - s^\nu (p-1-\nu) C_{\eta,p_1} \int_{\mathbb{R}^N} |w_n|^{p_1} \, dx - s^{-\eta} \int_{\mathbb{R}^N} |w_n|^{2 \eta} \, dx.$$

Hence, by (3.1), (3.4) and $\varepsilon > 0$ small enough, we obtain $m(c) \geq \frac{1}{2} s^2 - s^\nu \eta \int_{\mathbb{R}^N} |w_n|^{2 \eta} \, dx, \forall s \in \mathbb{R} \setminus \{0\}$. Taking $s = (2^\nu) \frac{\int_{\mathbb{R}^N} |w_n|^{2 \eta}}{s^\nu} \in \mathbb{R}$, we get

$$m(c) \geq \frac{1}{N} (2^\nu) \frac{\int_{\mathbb{R}^N} |w_n|^{2 \eta}}{s^\nu} \mathbb{S}^N. \hspace{1cm} (3.5)$$
However, by $(f_\delta)$, for any $u \in \mathcal{S}$,
\[
m(c) \leq \max_{t \geq 0} J(t \ast u) \leq \max_{t \geq 0} \left\{ \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\mu}{p} \left( \frac{N(p-2)}{2} + 1 \right) \int_{\mathbb{R}^N} |u|^p dx \right\}
\leq C \left( \frac{1}{\mu} \right)^{\frac{4}{N(p-2)-4}} \to 0 \quad \text{as } \mu \to +\infty,
\] (3.6)
which contradicts with (3.5). Hence, $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$.

Take $\tilde{v}_n := v_n(\cdot / \tau_n)$ with $\tau_n = (\frac{1}{\tau_n})^{\frac{1}{p'}}$. Clearly, $\tilde{v}_n \in \mathcal{S}$, $t_n \ast \tilde{v}_n \in \mathcal{M}$ and $\{\tilde{v}_n\}$ is bounded in $H^1(\mathbb{R}^N)$. We claim that
\[
\lim_{n \to \infty} \sup \tau_n < +\infty.
\] (3.7)
We first show $\{\tilde{v}_n\}$ is non-vanishing, i.e.,
\[
\tilde{\delta} := \lim_{n \to \infty} \sup \left( \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |\tilde{v}_n|^2 dx \right) > 0.
\]

If not, $\tilde{\delta} = 0$, then by the Lions lemma [11], $\tilde{v}_n \to 0$ in $L^q(\mathbb{R}^N)$ for $q \in (2, 2^*)$. Clearly, $\int_{\mathbb{R}^N} |v_n|^{p_1} dx = \tau_n^{-N} \int_{\mathbb{R}^N} |v_n|^{p_1} dx \to 0$. Then, by $\mathcal{P}(v_n) = 0$, $(f_\delta)$, (2.2) and (2.4), we get
\[
\int_{\mathbb{R}^N} |\nabla v_n|^2 dx = \frac{N}{2} \int_{\mathbb{R}^N} H(v_n) dx
\leq \frac{N}{2} (2^* - 2) \int_{\mathbb{R}^N} F(v_n) dx
\leq \frac{N}{2} (2^* - 2) \int_{\mathbb{R}^N} \left( C_{N,p_1} |v_n|^{p_1} + \eta |v_n|^2 \right) dx
\leq 2^* C_{N,p_1} \int_{\mathbb{R}^N} (|v_n|^{p_1} dx + 2^* \eta \sup_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \right)^{\frac{N}{p_1}}
\leq 2^* \eta \sup_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \right)^{\frac{N}{p_1}} + o_n(1).
\] (3.8)
Since $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$, in view of Lemma 2.1, we may assume that, up to a subsequence, $\|\nabla v_n\|_2^2 \to I > 0$. Combining with (3.8), we deduce
\[
I \geq (2^* \eta) \frac{1}{2} \frac{N}{p_1} S^{\frac{N}{p_1}}.
\] (3.9)
By (2.2), (2.4), (3.3) and Lemma 2.2, for any $t > 0$, we have
\[
m(c_n) + \varepsilon \geq J(t \ast v_n) = J(0 \ast v_n)
\geq J(t \ast v_n)
= \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - t^{-N} \int_{\mathbb{R}^N} F(t \ast v_n) dx
\geq \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - t^{2^* (p_1 - 2)} \eta \int_{\mathbb{R}^N} |v_n|^{p_1} dx - t^{2^*} \eta \sup_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \right)^{\frac{N}{2}}
\geq \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - t^{2^* (p_1 - 2)} \eta \int_{\mathbb{R}^N} |v_n|^{p_1} dx - t^{2^*} \eta \sup_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \right)^{\frac{N}{2}}
\geq \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - t^{2^*} \eta \sup_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \right)^{\frac{N}{2}} + o_n(1).
Then, taking $t = (2^+ \eta)^{\frac{2-N}{2N}} S^{\frac{2}{N}} / \frac{1}{2}$, together with (3.1) and (3.9) it follows that
\[
m(\varepsilon) \geq \frac{1}{N} (2^+ \eta)^{\frac{2-N}{2N}} S^{\frac{2}{N}},
\]
which produces a contradiction with (3.6). Hence, $\{\varphi_n\}$ is non-vanishing, which implies that there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ and $\nu \in H^1(\mathbb{R}^N)$ such that, up to a subsequence, $\varphi_n(\cdot + y_n) \to \nu \neq 0$ almost everywhere in $\mathbb{R}^N$.

Now, we prove (3.7). By contradiction, up to a subsequence $\tau_n \to +\infty$ as $n \to \infty$. By Lemma 2.4(ii), we have $\tau_n(\varepsilon + y_n) = \tau_n \to +\infty$. Using (f4), we deduce
\[
0 \leq \int_{\mathbb{R}^N} \nabla \varphi_n(\cdot + y_n) \cdot d\sigma_n(\cdot + y_n)
\]
which is a contradiction. Then (3.7) holds and using the fact that $\{\varphi_n\}$ is bounded in $H^1(\mathbb{R}^N)$ it follows that $\{\tau_n \ast \nu_n\}$ is bounded in $H^1(\mathbb{R}^N)$. By (3.3) and Lemma 2.2, we obtain
\[
m(\varepsilon) \leq J(\tau_n \ast \nu_n)
\]
\[
\leq J(\tau_n \ast \nu_n) + \left| J((\tau_n \ast \nu_n) - J(\tau_n \ast \nu_n) \right|
\leq J(\nu_n) + \frac{\|\nabla \nu_n\|^2}{2} - \tau_n \int_{\mathbb{R}^N} F(\nabla \nu_n) dx - \frac{\|\nabla \nu_n\|^2}{2} + \tau_n \int_{\mathbb{R}^N} |\nabla \nu_n|^2 dx + \tau_n \int_{\mathbb{R}^N} F(\nabla \nu_n) dx
\leq m(\varepsilon) + \|\nu_n\|^2 - 1 \cdot \int_{\mathbb{R}^N} |\nabla \varphi_n(\cdot + \nu_n)|^2 dx + |\nu_n|^2 - 1 \cdot \int_{\mathbb{R}^N} |F(\tau_n \ast \nu_n)| dx.
\]
Hence, since $\tau_n \to 1$ and $\varepsilon$ is arbitrary, we get (3.2). Together with (3.1), it follows that $c \mapsto m(\varepsilon)$ is continuous.

**Lemma 3.2.** Assume that (f1) – (f3) hold. Then $m(\varepsilon)$ is non-increasing with respect to $c > 0$.

**Proof.** For any $u_1 \in S_{c_1}$, set $u_2(x) = \theta^{-\frac{\sigma}{2N}} u_1(\theta^{-\frac{\sigma}{2}} x)$, where $\theta = \frac{\nu}{\sigma} > 1$. Clearly, $\|\nabla u_2\|_2^2 = \|\nabla u_1\|_2^2$ and $\|u_2\|_2^2 = c_2$. For any $s \in \mathbb{R} \setminus \{0\}$, we define $G_s(\sigma) := F(s - \sigma \frac{\nu}{\sigma} s), \forall \sigma \geq 1$. Clearly, $G_s(1) = 0$. By (f3), we get
\[
G'_s(\sigma) = -\frac{N}{2} \sigma^{\frac{\nu}{2}} \left( F(\sigma^{\frac{\nu}{2}} s) - \frac{N-2}{2N} f(\sigma^{\frac{\nu}{2}} s) \sigma^{\frac{\nu}{2}} s \right) < 0.
\]
Then we deduce $G_s(\theta) < 0$, which implies that
\[
\int_{\mathbb{R}^N} F(t_2 \ast u) dx - \theta^{-\frac{\sigma}{2}} \int_{\mathbb{R}^N} F \left( \theta^{-\frac{\sigma}{2}} (t_2 \ast u) \right) dx < 0. \tag{3.10}
\]
Take \( t_2 > 0 \) such that \( t_2 * u_2 \in \mathcal{S}_c \cap \mathcal{M} \). Then by Lemma 2.2, Lemma 2.3 and (3.10), we get
\[
m(c_2) \leq \max_{t > 0} J(t * u_2)
\]
\[
= J \left( \frac{t}{t_2} u_2(t_2x) \right)
\]
\[
= J \left( \frac{t}{t_2} \theta^{2-N} u_1(\theta - \frac{1}{2} t_2 x) \right)
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla (\frac{t}{t_2} \theta^{2-N} u_1(\theta - \frac{1}{2} t_2 x))|^2 \, dx - \int_{\mathbb{R}^N} F \left( \frac{t}{t_2} \theta^{2-N} u_1(\theta - \frac{1}{2} t_2 x) \right) \, dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla (t_2 * u_1)|^2 \, dx - \theta \frac{N}{2} \int_{\mathbb{R}^N} F \left( \theta^{2-N} (t_2 * u_1) \right) \, dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla (t_2 * u_1)|^2 \, dx - \int_{\mathbb{R}^N} F(t_2 * u_1) \, dx + \int_{\mathbb{R}^N} F(t_2 * u_1) \, dx - \theta \frac{N}{2} \int_{\mathbb{R}^N} F \left( \theta^{2-N} (t_2 * u_1) \right) \, dx
\]
\[
\leq J(t_2 * u_1),
\]
which implies that
\[
m(c_2) \leq \inf_{u \in \mathcal{S}_c} \max_{t > 0} J(t * u) = m(c_1).
\]

To show the function \( c \mapsto m(c) \) is strictly decreasing, we need the following result.

**Lemma 3.3.** Suppose that \((f_1) - (f_5)\) hold. Assume that there exists \( u \in \mathcal{S}_c \cap \mathcal{M} \) such that \( J(u) = m(c) \). Then \( m(c) > m(c') \) for any \( c' > c \) close enough to \( c \).

**Proof.** For any \( s > 0 \), set \( \alpha(s) := J(t_s * (sw)) \), where \( w \in \mathcal{S}_c \). Clearly,
\[
\alpha(s) = \frac{1}{2} s^2 t_s \int_{\mathbb{R}^N} |\nabla w|^2 \, dx - t_s^N \int_{\mathbb{R}^N} F(sw) dx.
\]
Furthermore, we have
\[
\frac{d}{ds} \alpha(s) = st_s^2 \int_{\mathbb{R}^N} |\nabla w|^2 \, dx - t_s^N \int_{\mathbb{R}^N} f(sw) w dx = \frac{1}{s} J'(t_s * (sw)) (t_s * (sw)),
\]
where \( J'(u) \) denotes the unconstrained derivative of \( J \). By \( t_s * w \in \mathcal{M} \) and \((f_5)\), we get
\[
J'(t_s * w) (t_s * w) = \frac{N - 2}{2} \int_{\mathbb{R}^N} \left[ f(t_s * w) (t_s * w) - 2F(t_s * w) \right] \, dx < 0.
\]
Then by (3.11), we deduce for a fixed \( \delta_1 > 0 \) small enough, \( \frac{d}{ds} \alpha(s) < 0, \forall s \in [1 - \delta_1, 1) \). Then by the mean value theorem, we obtain
\[
\alpha(1) = \alpha(s) + (1 - s) \frac{d}{ds} \alpha(s) \big|_{s = \xi} < \alpha(s),
\]
where \( 1 - \delta_1 \leq s < \xi < 1 \). For any \( c' > c \) close enough to \( c \), set \( s_1 = \sqrt{\frac{1}{c'}} \). Clearly, \( s_1 \in [1 - \delta_1, 1) \) and there exists \( w \in \mathcal{S}_c \) such that \( u = s_1 w \). Hence, by Lemma 2.2, we have
\[
m(c') \leq \alpha(1) < \alpha(s_1) = J(t_{s_1} * (s_1 w)) = J(t_{s_1} * u) \leq J(t_s * u) = J(u) = m(c).
\]

By Lemma 3.2 and Lemma 3.3, we directly obtain

**Lemma 3.4.** Assume that \((f_1) - (f_5)\) hold. If there exists \( u \in \mathcal{S}_c \cap \mathcal{M} \) such that \( J(u) = m(c) \), then \( m(c) > m(c') \) for any \( c' > c \).
IV Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. Firstly, we show the minimizer of \( J(u) \) constrained on \( \mathcal{S} \cap M \) is attained.

**Lemma 4.1.** Assume that \((f_1) - (f_3)\) hold. Then there exists \( u_0 \in \mathcal{S} \cap M \) such that \( J(u_0) = m(c) \).

**Proof.** By Lemma 2.3 and the Ekeland variational principle, there exists a minimizing sequence \( \{u_n\} \subset \mathcal{S} \cap M \) such that

\[
J(u_n) \to m(c) \quad \text{as} \quad n \to +\infty. \tag{4.1}
\]

By similar arguments as in the proof of Lemma 3.1, we can get that \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^N) \). We claim that \( \{u_n\} \) is non-vanishing. We assume by contradiction that \( \{u_n\} \) is vanishing. Then by the Lions lemma [21], we deduce

\[
\int_{\mathbb{R}^N} |u_n|^p dx \to 0. \tag{4.2}
\]

Then, by \( \mathcal{P}(u_n) = 0, (f_5), (2.2) \) and (2.4), we have

\[
\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \leq \frac{N}{2} (2^* - 2) \int_{\mathbb{R}^N} F(u_n) dx \\
\leq 2^* C_{\eta, p_1} \int_{\mathbb{R}^N} |u_n|^{p_1} dx + 2^* \eta \int_{\mathbb{R}^N} |u_n|^2 dx \\
\leq 2^* \eta S^{\frac{N}{2^*}} \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \right)^{\frac{2^*}{2}} + o_n(1). \tag{4.3}
\]

Since \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^N) \), by Lemma 2.1, we may assume that, up to a subsequence, \( \|\nabla u_n\|_2^2 \to l^* > 0 \). By (4.3), we obtain \( l^* \geq (2^* \eta)^{\frac{2}{2^*}} S^{\frac{N}{2}} \). Moreover, by (4.1) and Lemma 2.2, for any \( t > 0 \), we have

\[
m(c) + o_n(1) = J(u_n) = J(t \ast u_n) \\
\geq J(t \ast u_n) \\
= t^2 \int_{\mathbb{R}^N} |\nabla u_n|^2 dx - t^2 \int_{\mathbb{R}^N} F(\frac{N}{2} u_n) dx \\
\geq t^2 \int_{\mathbb{R}^N} |\nabla u_n|^2 dx - t^2 \eta \int_{\mathbb{R}^N} |u_n|^{p_1} dx - t^2 \eta \int_{\mathbb{R}^N} |u_n|^2 dx \\
\geq t^2 \int_{\mathbb{R}^N} |\nabla u_n|^2 dx - t^2 \eta \int_{\mathbb{R}^N} |u_n|^{p_1} dx - t^2 \eta S^{\frac{N}{2^*}} \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \right)^{\frac{2^*}{2}}.
\]

Hence, taking \( t = (2^* \eta)^{\frac{N}{2^*}} S^{\frac{N}{2}} (l^*)^{-\frac{1}{2}} \) and \( n \to +\infty \), by (4.2), we obtain \( m(c) \geq \frac{1}{4} (2^* \eta)^{\frac{N}{2^*}} S^{\frac{N}{2}} \), which is contrary to (3.6). Therefore the claim holds. Thus there exist a sequence \( \{y_n\} \subset H^1(\mathbb{R}^N) \), and \( u_0 \in H^1(\mathbb{R}^N) \backslash \{0\} \) such that up to a subsequence \( u_n(\cdot + y_n) \rightharpoonup u_0 in H^1(\mathbb{R}^N) \). Denote \( u_{n,0} = u_n - u_0 \). By Brezis-Lieb lemma, we have

\[
\|u_n\|_2^2 = \|u_0\|_2^2 + \|u_{n,0}\|_2^2 + o_n(1) \quad \text{and} \quad \|\nabla u_n\|_2^2 = \|\nabla u_0\|_2^2 + \|\nabla u_{n,0}\|_2^2 + o_n(1). \tag{4.4}
\]

Using the similar arguments as Lemma 3.2 in Ref. [22], we deduce

\[
\int_{\mathbb{R}^N} (F(u_n) - F(u_0) - F(u_0)) dx = o_n(1) \quad \text{and} \quad J(u_n) = J(u_0) + J(u_{n,0}) + o_n(1). \tag{4.5}
\]

Furthermore, by Lemma 2.9 in Ref. [18], we get

\[
\int_{\mathbb{R}^N} H(u_n) dx = \int_{\mathbb{R}^N} H(u_0) dx + \int_{\mathbb{R}^N} H(u_{n,0}) dx + o_n(1). \tag{4.6}
\]
Then by the second equality in (4.4) and $u_n \in \mathcal{M}$, we obtain
\[ 0 = P(u_n) = P(u_0) + P(u_n,0) + o_n(1). \] (4.7)

We claim that $\mathcal{P}(u_0) \leq 0$. In fact, up to a subsequence if necessary, taking $\alpha_n := \int_{\mathbb{R}^N} |\nabla u_{n,0}|^2 \, dx \to \alpha_0 \geq 0$ as $n \to +\infty$, we divide into the following two cases:

Case 1. $\alpha_0 = 0$. By (2.2), we deduce
\[ \int_{\mathbb{R}^N} |u_{n,0}|^q \, dx \to 0 \quad \text{for } q \in (2, 2^*]. \] (4.8)

Furthermore, by (2.4), $(f_3)$ and $(f_5)$, there exist $p_1 \in (2 + \frac{4}{N}, 2^*)$ and $C_{\eta, p_1} > 0$ such that
\[ \int_{\mathbb{R}^N} F(u_{n,0}) \, dx \leq \int_{\mathbb{R}^N} \left( C_{\eta, p_1} |u_{n,0}|^{p_1} + \eta |u_{n,0}|^{2^*} \right) \, dx. \] (4.9)

Combining (4.8) and (4.9) gives
\[ 0 \leq \int_{\mathbb{R}^N} H(u_{n,0}) \, dx \leq (2^* - 2) \int_{\mathbb{R}^N} F(u_{n,0}) \, dx \to 0, \]
which implies that $\int_{\mathbb{R}^N} H(u_{n,0}) \, dx \to 0$ as $n \to +\infty$. Together with $\int_{\mathbb{R}^N} |\nabla u_{n,0}|^2 \, dx \to 0$, we get $P(u_{n,0}) \to 0$ as $n \to +\infty$. Hence, by (4.7), we obtain $\mathcal{P}(u_0) = 0$ and the claim holds.

Case 2. $\alpha_0 > 0$. By contradiction, we assume that $\mathcal{P}(u_0) > 0$. By $(f_1) - (f_5)$, we have
\[ f(s) \geq (2 + \frac{4}{N}) F(s), \quad \forall s \in \mathbb{R}. \] (4.10)

Then
\[ J(u_0) = \frac{1}{2} \mathcal{P}(u_0) + \frac{N}{4} \int_{\mathbb{R}^N} \left( f(u_0) u_0 - (2 + \frac{4}{N}) F(u_0) \right) \, dx > 0. \] (4.11)

In view of (4.7), we get $\mathcal{P}(u_{n,0}) \leq 0$. Then by Lemma 2.2 and $(f_3)$, there exists $t_{u_{n,0}} \in (0, 1]$ such that $\mathcal{P}(t_{u_{n,0}} u_{n,0}) = 0$. Furthermore, we have
\[
J(u_{n,0}) - J(t_{u_{n,0}} u_{n,0}) = \frac{1 - t_{u_{n,0}}^2}{2} \int_{\mathbb{R}^N} |\nabla u_{n,0}|^2 \, dx + t_{u_{n,0}}^{-N} \int_{\mathbb{R}^N} F(t_{u_{n,0}}^N u_{n,0}) \, dx - \int_{\mathbb{R}^N} F(u_{n,0}) \, dx
\]
\[
= \frac{1 - t_{u_{n,0}}^2}{2} \mathcal{P}(u_{n,0}) + \frac{N}{4} \int_{\mathbb{R}^N} \left( 1 - t_{u_{n,0}}^2 \right) f(u_{n,0}) u_{n,0} - \left( 1 + \frac{N}{2} \right) F(u_{n,0}) + t_{u_{n,0}}^{-N} F(t_{u_{n,0}}^N u_{n,0}) \, dx
\]
\[
\geq \frac{1 - t_{u_{n,0}}^2}{2} \mathcal{P}(u_{n,0}) + \int_{\mathbb{R}^N} t_{u_{n,0}}^{-N} \left( \frac{N}{2} |t_{u_{n,0}}|^{2^*} \frac{H(u_{n,0})}{|u_{n,0}|^{2^*}} \right) dt \, dx
\]
\[
\geq \frac{1 - t_{u_{n,0}}^2}{2} \mathcal{P}(u_{n,0}).
\]
Denote $c_{n,0} := \|u_{n,0}\|_2^2$. Clearly, $c_{n,0} \leq c$. Then by (4.5), (4.6), (4.10) and Lemma 3.2, we obtain

$$m(c) = \lim_{n \rightarrow +\infty} \left[ J(u_n) - \frac{1}{2} \mathcal{P}(u_n) \right]$$

$$= \lim_{n \rightarrow +\infty} \left[ \frac{N}{4} \int_{\mathbb{R}^N} H(u_n)dx - \int_{\mathbb{R}^N} F(u_n)dx \right]$$

$$= \frac{N}{4} \int_{\mathbb{R}^N} (H(u_0) + H(u_{n,0})) dx - \int_{\mathbb{R}^N} (F(u_0) + F(u_{n,0})) dx$$

$$= \frac{N}{4} \int_{\mathbb{R}^N} \left( f(u_0)u_0 - \left( 2 + \frac{4}{N} \right) F(u_0) \right) dx + J(u_{n,0}) - \frac{1}{2} \mathcal{P}(u_{n,0})$$

$$\geq J(u_{n,0}) - \frac{1}{2} \mathcal{P}(u_{n,0})$$

$$\geq J(t_{u_{n,0}} \ast u_0) - \frac{t_{u_{n,0}}^2}{2} \mathcal{P}(u_{n,0})$$

$$\geq J(t_{u_{n,0}} \ast u_0) \geq m(c_{n,0}) \geq m(c),$$

which implies that $\mathcal{P}(u_{n,0}) = 0$ and

$$J(u_{n,0}) = m(c_{n,0}) = m(c). \tag{4.12}$$

Therefore, by (4.5), (4.11) and (4.12), it follows that

$$m(c) = J(u_n) + o_n(1) = J(u_0) + J(u_{n,0}) + o_n(1) > J(u_{n,0}) = m(c),$$

which is a contradiction and then claim holds.

Since $\mathcal{P}(u_0) \leq 0$, as before, one can see that there exists $t_0 \in (0, 1]$ such that $t_0 \ast u_0 \in \mathcal{S}_0 \cap \mathcal{M}$ and

$$J(u_0) - J(t_0 \ast u_0) \geq \frac{1 - t_0^2}{2} \mathcal{P}(u_0). \tag{4.13}$$

Denote $c_0 = \int_{\mathbb{R}^N} |u_0|^2 dx$. Clearly, $c_0 \in (0, c]$. Therefore, by (4.5), (4.6), (4.10), (4.13) and Lemma 3.2, we obtain

$$m(c) = \lim_{n \rightarrow +\infty} \left[ J(u_n) - \frac{1}{2} \mathcal{P}(u_n) \right]$$

$$= \lim_{n \rightarrow +\infty} \left[ \frac{N}{4} \int_{\mathbb{R}^N} H(u_n)dx - \int_{\mathbb{R}^N} F(u_n)dx \right]$$

$$= \frac{N}{4} \int_{\mathbb{R}^N} (H(u_0) + H(u_{n,0})) dx - \int_{\mathbb{R}^N} (F(u_0) + F(u_{n,0})) dx$$

$$= J(u_0) - \frac{1}{2} \mathcal{P}(u_0) + \frac{N}{4} \int_{\mathbb{R}^N} \left( f(u_{n,0})u_{n,0} - \left( 2 + \frac{4}{N} \right) F(u_{n,0}) \right) dx$$

$$\geq J(u_0) - \frac{1}{2} \mathcal{P}(u_0)$$

$$\geq J(t_0 \ast u_0) - \frac{t_0^2}{2} \mathcal{P}(u_0)$$

$$\geq J(t_0 \ast u_0) \geq m(c_0) \geq m(c),$$

which implies $m(c_0) = m(c)$ and $\mathcal{P}(u_0) = 0$, that is $t_{u_0} = 1$. Then we obtain $u_0 \in \mathcal{S}_c \cap \mathcal{M}$ and $J(u_0) = m(c_0)$.

Using Lemma 3.4 at $c_0$ and $m(c_0) = m(c)$, we deduce $c_0 = c$ and thus $J(u_0) = m(c)$.
Proof. Theorem 1.1. Set $E(u) := J(t_u * u)$, where $u \in S_c$ and $t_u * u \in M$. We claim that for any $u \in S_c$ and $\psi \in T_u S_c$, $E \in C^1(S_c, \mathbb{R})$ and

$$J'(t_u * u)(t_u * \psi) = E'(u)\psi.$$  \hfill (4.14)

For $|s|$ small enough, we estimate $E(u + s\psi) - E(u)$. By Lemma 2.2, for any $u \in S_c$, there exists $t_u$ such that $t_u * u \in M$ and $J(t_u * u) > J(t * u)$ for any $t \neq t_u$. Then, for any $\psi \in T_u S_c$, by the mean value theorem we obtain

$$E(u + s\psi) - E(u) = J(t_{u+s\psi} * (u + s\psi)) - J(t_u * u)$$

$$\leq J(t_{u+s\psi} * (u + s\psi)) - J(t_u + s\psi * u)$$

$$= \frac{1}{2} \int_{\mathbb{R}^N} (2s \nabla u \cdot \nabla \psi + s^2 |\nabla \psi|^2) dx - t_u^{-N} \int_{\mathbb{R}^N} f \left( \frac{s}{t_u + s\psi} (u + s \psi) \right) t_u^{-N} s\psi dx,$$

where $s \in [0, 1]$ and $\xi_s \in (0, 1)$. Similarly,

$$E(u + s\psi) - E(u) = J(t_{u+s\psi} * (u + s\psi)) - J(t_u * u)$$

$$\geq J(t_{u} * (u + s\psi)) - J(t_u * u)$$

$$= \frac{1}{2} t_u^{-N} \int_{\mathbb{R}^N} (2s \nabla u \cdot \nabla \psi + s^2 |\nabla \psi|^2) dx - t_u^{-N} \int_{\mathbb{R}^N} f(t_u * u)(t_u * \psi) dx.$$

By Lemma 2.4(i), we get $\lim_{s \to 0} t_{u+s\psi} = t_u$. Hence, we have

$$\lim_{s \to 0} \frac{E(u + s\psi) - E(u)}{s} = t_u^2 \int_{\mathbb{R}^N} \nabla u \cdot \nabla \psi dx - t_u^{-N} \int_{\mathbb{R}^N} f(t_u * u) t_u^{-N} \psi dx.$$

By $(f_1)$, the Hölder inequality and Gagliardo-Nirenberg inequality, it follows that the Gâteaux derivative of $E$ is bounded linear in $\psi$. In view of Lemma 2.4(i), it is continuous in $u$. Therefore, by Proposition 1.3 in Ref. [20], we deduce $E : S_c \to \mathbb{R}$ is of class $C^1$. Furthermore, by directly computations,

$$E'(u)(\psi) = \int_{\mathbb{R}^N} \nabla (t_u * u) \cdot \nabla (t_u * \psi) dx - \int_{\mathbb{R}^N} f(t_u * u)(t_u * \psi) dx = J'(t_u * u)(t_u * \psi).$$

Hence the claim holds.

By Lemma 2.2 and Lemma 2.3, we get

$$m(c) = \inf_{u \in S_c \cap M} J(u) = \inf_{u \in S_c} \max_{t > 0} J(u) = \inf_{u \in S_c} J(t_u * u) = \inf_{u \in S_c} E(u).$$

Using Lemma 4.1, we obtain $t_0 \in S_c \cap M$ such that $J(t_0) = m(c)$. Then there exists $v_0 \in S_c$ such that $t_0 * v_0 = u_0$ and $E(v_0) = J(t_0 * v_0) = J(t_0) = m(c)$, which implies $v_0$ is a minimizer of $E$ constrained on $S_c$. Together with (4.14), we obtain

$$\|dJ(t_0)\|_{(T_{u_0} S_c)'} = \sup_{\phi \in T_{t_0} S_c, \|\phi\| \leq 1} \left| dJ(t_0) \phi \right|$$

$$= \sup_{\phi \in T_{t_0} S_c, \|\phi\| \leq 1} \left| dJ(t_0 \ast v_0) \left[ t_0 \ast (t_0^{-1} \ast \phi) \right] \right|$$

$$= \sup_{\phi \in T_{t_0} S_c, \|\phi\| \leq 1} \left| dE(v_0) \left[ t_0^{-1} \ast \phi \right] \right|$$

$$\leq \|dE(v_0)\|_{(T_{u_0} S_c)'} \cdot \sup_{\phi \in T_{t_0} S_c, \|\phi\| \leq 1} \|t_0^{-1} \ast \phi\|$$

$$= \|dE(v_0)\|_{(T_{u_0} S_c)'} \cdot \left( t_0^{-1} \|\nabla \phi\|_{2} + \|\phi\|_{2} \right)$$

$$\leq \max \left\{ t_0^{-1}, 1 \right\} \|dE(v_0)\|_{(T_{u_0} S_c)'} = 0.$$
It follows that \( u_0 \) is a critical point of \( J \) constrained on \( S_c \). Thus there exists \( \lambda \in \mathbb{R} \) such that \( u_0 \) is a weak solution of

\[
-\Delta u = f(u) + \lambda u.
\]

Furthermore, \( J(u_0) = \inf_{u \in S_c} J(u) = m(c) \). Therefore \( u_0 \) is a normalized ground state solution for problem (1.1). Moreover, by \( u_0 \in S_c \cap M \) and \( (f_5) \), we have \( \lambda < 0 \). This completes the proof.

\[\square\]

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Author Declarations

Conflict of interest

The authors have no conflicts to disclose.

Author Contributions

Xiaojun Chang: Methodology (equal); Project administration (equal); Writing – review & editing (equal).

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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