The consensus problem for opinion dynamics with local average random interactions

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Abstract

We study the consensus formation for an agents based model, generalizing that originally proposed by Krause [Kr], by allowing the communication channels between any couple of agents to be switched on or off randomly, at each time step, with a probability law depending on the proximity of the agents’ opinions. Namely, we consider a system of agents sharing their opinions according to the following updating protocol. At time $t+1$ the opinion $X_i(t+1) \in [0, 1]$ of agent $i$ is updated at the weighted average of the opinions of the agents communicating with it at time $t$. The weights model the confidence level an agent assign to the opinions of the other agents and are kept fixed by the system dynamics, but the set of agents communicating with any agent $i$ at time $t+1$ is randomly updated in such a way that the agent $j$ can be chosen to belong to this set independently of the other agents with a probability that is a non increasing function of $|X_i(t) - X_j(t)|$. This condition models the fact that a communication among the agents is more likely to happen if their opinions are close. We prove that the system reaches consensus,
i.e. as the time tends to infinity the agents’ opinions will reach the same value exponentially fast.

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1 Introduction, notations and results

Opinion dynamics is a topic in applied mathematics which has witnessed a growing interest in the last decades. This is due to the possibility to describe the emergence of collective phenomena such as the reaching of consensus in a community of peers by means of simple models of interacting agents [CFL]. The literature on the subject concentrates mainly on two families of models: those cast in the framework of interacting particle
systems (IPS) e.g. the voter model and the majority-vote process \[\text{Li}\], the Axelrod model and its generalizations \[\text{La}\], the Deffuant-Weisbuch model \[\text{DNAW}, \text{Ha}, \text{HH}\], and those belonging to the family of coupled dynamical systems (CDS) e.g. \[\text{Kr}\] (see \[\text{FF}\] for linear models).

The common feature of these models is that the interactions among the agents are designed in such a way that an agent adjust its opinion to that of its neighbours and that agents are more likely to interact with those sharing similar opinions. On the other hand, the main difference between IPS type models and CDS type models, aside from the fact that IPS type models typically evolve in continuous time while CDS type models evolve in discrete time, is that in CDS type models all the agents can change the value of their opinion at each time step while, in IPS type models, at a given time (usually at the tick of a Poisson clock), only the elements of a randomly chosen subset of agents are allowed to change the value of their opinion, and the rule under which the opinion of the selected agents are updated, being stochastic in general (see \[\text{CF}\] for a stochastic version of the Deffuant model), can be deterministic too \[\text{DNAW}, \text{HH}\].

In this paper we present a model of consensus formation which, although it represents a modification of that originally proposed by Krause, it is defined by an updating rule of the agents opinions that recalls those characterizing IPS type models. In particular, at each time step, firstly the set of neighboring peers of any agent \(i\) is selected at random in such a way that the events that any two distinct agents belong to the neighbourhood of \(i\) are independent. Then, each agent update the value of its opinion to the average value of the opinions of its neighbours.

More precisely, we consider a collection of agents which form the set of vertices of a directed graph \(G := (V, E)\), \(E \subseteq V \times V\). We assume that agent \(u\) communicate with agent \(v\) if the directed edge \((u, v)\) is in \(E\). Each agent hold an opinion (belief) represented by a variable taking values in \([0, 1]\). Agent’s beliefs evolve in time in such a way that the opinion \(X_u(t + 1)\) of the agent labelled by the graph vertex \(u\) at time \(t + 1\) is updated at the weighted average of the beliefs \(X_v(t)\) of the agents communicating with \(u\) at time \(t\). The weights appearing in the just mentioned average represent the quality of the information exchange among the agents and do not change in time. On the other hand, the set of the agents communicating with agent \(u\) at time \(t\) is randomly chosen according to a probability distribution which gives more chance to a communication exchange between agent \(u\) and agent \(v\) to happen if the value of their opinion are close. It is also natural to assume that the information exchange an agent has with itself is always maximal.

In the rest of the section we introduce the notation used throughout the paper, formally display the definition of the model and present the results obtained in the following sections about the emergence of consensus for the finite size system as well as the extension of these results when the size of the system is very large.

We stress that, as in \[\text{Kr}\], in this work, the dynamics of the state of the edges of \(G\), representing the communication channels amid the agents, is synchronous, i.e.
they are all updated at each time step. On the other hand, one may wish to modify the system’s dynamics by letting the state of the edges of $G$ to evolve under an asynchronous dynamics. In the case of a finite size system, this can be realized, for example, by updating at each time step the state of just one edge sampled uniformly at random among the elements of $E$ and leaving unchanged the states of the other edges. In fact, it seems that the techniques used to analyse the emergence of consensus in the synchronous case do not apply to this particular case. Therefore, the discussion about the possibility to reach consensus for the opinion exchange model characterized by the same updating rule for the values of the agents’ opinions presented in this paper, but subject to the asynchronous evolution of the communication exchange among the agents just described, are deferred to a forthcoming paper.

We also remark that the differences between the way, synchronous or asynchronous, one chooses to update the state of the communication links shared by the agents reflects in the large system limit analysis. As a matter of fact, for the model where the state of just one communication channel is updated at a time, as the one just described, with a suitable modification of the interaction among the agents, the evolution of the system, in the large system limit, can be analysed looking at the evolution of the empirical distribution of the agents’ opinions, in the spirit of the kinetic limit for models of flocking (see e.g. [GO], [CCH]), as it has already been discussed for the stochastic version of Deffuant model in [CP] and more recently for other IPS models [AM]. On the contrary, this is not possible in models where the state of all the communication channels are update at once as for the one studied in this paper. However, if the size of the neighborhood of the agents is finite it is possible to define directly the evolution of the system on very large, possibly infinite, graphs as in [La] and [HH].

1.1 Notations

If $A$ is a set and $B \subseteq A$, $1_B$ denotes the indicator function of $A$ and $B^c := A \setminus B$. Let $\mathcal{P}(A)$ the set of the subsets of $A$. For any $k \geq 1$, we set $\mathcal{P}_k(A) := \{ B \in \mathcal{P}(A) : |B| = k \}$ and denote by $\mathcal{P}_0(A) := \bigvee_{k \geq 1} \mathcal{P}_k(A)$ the set of finite subsets of $A$.

If $A$ is a metric space, $\mathcal{B}(A)$ denotes its Borel $\sigma$-algebra. We also denote by $BM(A)$ the Banach space of bounded measurable functions on $A$, by $C(A)$ the Banach space of real-valued continuous functions on $A$ and by $\|\cdot\|$ the sup-norm. Moreover, if $\text{Lip}(A)$ is the Banach space of real-valued bounded Lipschitz functions on $A$, for any $\varphi \in \text{Lip}(A)$, we denote its norm by $\|\varphi\|_L$.

For $A, A'$ metric spaces, $BL(A, A')$ denotes the space of bounded linear operators on $A$ with values in $A'$. If $A' = A$ we set $BL(A, A) := BL(A)$. In particular, if $\mathcal{V}$ is a finite set, we denote by $St(\mathcal{V})$ the convex subset of $BL(\mathbb{R}^\mathcal{V})$ of stochastic matrices.

We denote by $\mathbb{E}$ the expected value of a random element when there is no need to specify the probability space on which it is defined and consequently write $\mathbb{P}\{B\}$.
for the expected value of the indicator function $1_B$ of an event $B \subseteq A$. The same notation will be also kept when considering conditional expectations and conditional probabilities. Besides, given a $\sigma$-algebra $\mathcal{A}$ of subsets of $A$, we denote by $\mathfrak{B} (A, \mathcal{A})$ the set of probability measures on $(A, \mathcal{A})$. If $\mu \in \mathfrak{B} (A, \mathcal{A})$, $\text{spt} \mu$ denotes the support of $\mu$, and, for $\mu, \nu \in \mathfrak{B} (A, \mathcal{A}), \| \mu - \nu \|$ denotes the total variation distance between the two measures.

Let $A := A^V$ and denote by $a := \{a_v\}_{v \in V}$. If $A$ is a poset w.r.t. the partial order: $a \leq a'$ if $a_v \leq a'_v$, for any $v \in V$, we say that a real-valued function $\varphi$ on $A$ is non-decreasing if $\varphi (a) \leq \varphi (a')$ whenever $a \leq a'$. Given two probability measures $P, P'$ on $(A, A^{\otimes V})$ we say that $P$ is stochastically dominated by $P'$, and denote this property by $P \preceq P'$, if for any bounded non-decreasing function $\varphi, \mathbb{E} [\varphi] = \int dP (a) \varphi (a) \leq \int dP' (a) \varphi (a) = \mathbb{E}' [\varphi]$. Moreover, if $A$ is finite, a probability measure $P$ on $(A, \mathcal{P} (A))$ is called irreducible if starting from any element of $A$ with positive $P$-probability one can reach any other element with positive $P$-probability via successive coordinate changes without passing through elements with zero $P$-probability [GHM], [Ge].

If $A := A^N$, for any $N \in \mathbb{N}$, we set $a_N := (a_1, ..., a_N)$ and denote by $\mathcal{C} (A)$ the cylinder $\sigma$-algebra that is the $\sigma$-algebra generated by the cylinder subsets

$$C_N (B) := \{ a \in A : a_N \in B \} ,$$

with $B \subseteq A^N$ if $A$ is a discrete set, while $B \in \mathcal{B} (A^N) = \mathcal{B} (A)^{\otimes N}$ if $A$ is a metric space.

If $\mathcal{L} (A)$ denotes the algebra of real-valued bounded local (cylinder) functions on $A$ we denote by $\mathcal{L} (A)$ the space of real-valued bounded quasi-local functions on $A$ that is the closure in topology of uniform convergence of the algebra of cylinder functions [Ge].

If $V$ is denumerable we denote by $\mathcal{E}_A$ the product $\sigma$-algebra $A^{\otimes V}$ on $A := A^V$.

1.1.1 Graphs

We recall some basic definition of graph theory useful to give a mathematical definition of consensus for the system. The connection of graph theory with Markov chains will be exploited in the next section. We refer the reader to basic textbooks such as [Bo] and [St] for an account on this subject.

A directed graph $G$ is a ordered pair of sets $(V, E)$ where $V$ is a finite set called set of vertices and $E \subseteq V \times V$ is called set of edges or bonds. $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq (V' \times V') \cap E$ is said to be a subgraph of $G$ and this property is denoted by $G' \subseteq G$. If $G' \subseteq G$, we denote by $V (G')$ and $E (G')$ respectively the set of vertices and the collection of the edges of $G'$. $| V (G') |$ is called the order of $G'$ while $| E (G') |$ is called its size. Given $G_1, G_2 \subseteq G$, we denote by $G_1 \cup G_2 := (V (G_1) \cup V (G_2), E (G_1) \cup E (G_2)) \subseteq G$ the graph union of $G_1$ and $G_2$. Moreover, we say that $G_1, G_2 \subseteq G$ are disjoint if $V (G_1) \cap V (G_2) = \emptyset$. For any $E' \subseteq E$, we denote by $G (E') := (V, E')$ the spanning
graph of $E'$. We also define

$$V(E') := \left( \bigcup_{e \in E'} e \right) \subset V .$$

(2)

Given $V' \subseteq V$, we set

$$E(V') := \{ e \in E : e \subset V' \}$$

and denote by $G[V'] := (V', E(V'))$ that is called the subgraph of $G$ induced or spanned by $V'$.

Two vertices $u, v$ are said to be adjacents if belong to the same bond i.e. if $V(e) = \{u, v\}$. If $e = (u, v)$, $e$ is said to be outgoing from $u$ and ingoing in $v$. Let

$$E_v^- := \{ e \in E : e = (u, v), u \in V \} , E_v^+ := \{ e \in E : e = (v, u), u \in V \}$$

be the set of edges respectively ingoing, outgoing from $v$. We denote by $N^-(v) := \left( \bigcup_{e \in E_v^-} V(e) \right) \subseteq V$ the closed ingoing neighborhood of $v$ and by $N^+(v) := \left( \bigcup_{e \in E_v^+} V(e) \right) \subseteq V$ the closed outgoing neighborhood of $v$. Moreover, for any $W \subseteq V$, we set $N^+(W) := \bigcup_{e \in W} N^+(v)$ to be the closed outgoing neighborhood of $W$. Given $v \in V$, we set $N^+_k(v) := N^+(v)$ and, for $k \geq 2$, $N^+_k(v) := N^+(N^+_k(v))$ to be the outgoing $k$-neighborhood of $v$. Given two vertices $u$ and $v$, $v$ is said to communicate with $u$ if there exists $k \geq 1$ such that $u \in N^+_k(v)$. Therefore, $u, v \in V$ are said to be connected if one communicates with the other. Indeed, since if $u \in N^+_k(v)$ for some $k \geq 1$, then $u \in N^+_l(v)$, $\forall l > k$, for $u$ and $v$ to be connected there must be $k_1, k_2, k \geq 1$ such that $u \in N^+_k(v)$ and $v \in N^+_k(u)$, that is $u \in N^+_k(v), v \in N^+_k(u)$. $G$ is then said to be strongly connected if any two distinct vertices are connected. The maximal connected subgraphs of $G$ are called components of $G$ and to denote that $G' \subset G$ is a component of $G$ we write $G' \sqsubseteq G$.

An example of directed graph is the one which can be associated to a Markov chain. In this case, $V$ coincides with the set of states of the chain and, denoting by $P$ the transition matrix associated to the chain, $E = E(P) := \{(u, v) \in V \times V : P_{u,v} > 0\}$. Then, the directed graph associated to the Markov chain with transition matrix $P$ is denoted by $G(P)$. Hence, the Markov chain and therefore $P$ are said to be irreducible if and only if $G(P)$ is strongly connected.

In the following, if $e = (u, v)$ is an edge of a directed graph $(V, E)$, we will occasionally note $\tilde{e}$ for the edge $(v, u) \in E$.

### 1.2 Description of the model and results

In the following, unless differently specified, we will be concerned only with graphs $G$ being subgraphs of the complete directed graph $G = (V, E)$ of finite order where $E := (V \times V)$.
A bond (or edge) configuration is a map $E \ni e \mapsto \omega_e \in \{0, 1\}$ so that a bond $e$ is said to be open if $\omega_e = 1$. Setting $\forall u, v \in V, \omega_{u,v} := \omega_e \delta_{e,(u,v)}$ and defining
\[
\Omega := \left\{ \omega \in \{0, 1\}^E : \forall u \in V, \omega_{u,u} = 1 \right\},
\]
we define
\[
\Omega \ni \omega \mapsto E(\omega) := \{ e \in E : \omega_e = 1 \} \in P(E)
\]
and consequently
\[
G(\omega) := G(E(\omega)) \subseteq G.
\]
We also set, $\forall v \in V$,
\[
E_v^-(\omega) := \{ e \in E(\omega) : e = (u,v), \ u \in V \},
\]
\[
E_v^+(\omega) := \{ e \in E(t) : e = (v,u), \ u \in V \},
\]
\[
N_\pm (v,\omega) := \left( \bigcup_{e \in E_v^\pm(\omega)} V(e) \right),
\]
\[
N_1^+(v,\omega) := N_\pm (v,\omega) \cap N_{k-1}^+(v,\omega), \ k \geq 2.
\]
Moreover, given a $\Omega$-valued random sequence $\{\omega(t)\}_{t \geq 0}$ we set $E(t) := E(\omega(t))$, $G(t) := G(\omega(t))$ as well as, $\forall v \in V, E_v^+(t) := E_v^+(\omega(t))$ and $\forall k \geq 1, N_k^+(v,\omega(t))$.

A belief configuration is a map $V \ni v \mapsto X_v \in [0, 1]$. We set $\Xi := [0, 1]^V$ and consider the sequence $\{X(t)\}_{t \geq 0}$ representing the beliefs evolution in time.

### 1.2.1 Beliefs dynamics

The beliefs evolution is given by the system of equations
\[
\begin{cases}
X_v(t+1) := \frac{\sum_{u \in N^-((v,t),\omega)} r_{u,v} X_u(t)}{\sum_{u \in N^-((v,t),\omega)} r_{u,v}} = \frac{\sum_{u \in V} r_{u,v} \omega_{u,v} X_u(t)}{\sum_{u \in V} r_{u,v} \omega_{u,v}} = X_v(t) + \frac{\sum_{u \in V} r_{u,v} \omega_{u,v} X_u(t)}{\sum_{u \in V} r_{u,v} \omega_{u,v}(t)} - \frac{\sum_{u \in V} r_{u,v} \omega_{u,v} X_u(t)}{\sum_{u \in V} r_{u,v} \omega_{u,v}(t)} - X_v(t),

X_v(0) = X_v^0,
\end{cases}
\]
which, by (\(\square\)), can be rewritten as
\[
\begin{cases}
X_v(t+1) = \frac{\sum_{e \in E_v^-} r_e \sum_{u \in V} \delta_{e,(u,v)} X_u(t)}{\sum_{e \in E_v^-} r_e} = X_v(t) + \frac{\sum_{e \in E_v^-} r_e \omega_e (t) \Delta_e X(t)}{\sum_{e \in E_v^-} r_e \omega_e (t)} - \frac{\sum_{e \in E_v^-} r_e \omega_e (t) \Delta_e X(t)}{\sum_{e \in E_v^-} r_e \omega_e (t)},

X_v(0) = X_v^0,
\end{cases}
\]
where, $\forall e \in E,$
\[
\Delta_e X(t) := (X_u(t) - X_v(t)) \mathbf{1}_{(v,u)}(e)
\]
and $r_e \in [0, 1]$ is the communication rate between the agents labelled by the the vertices incident in $e$, namely $r_{u,v} := r_e \delta_{e,(u,v)}$ represents the confidence level assigned by the agent $v$ to the belief of the agent $u$.
1.2.2 Communication channels dynamics

For any $v \in V$, the $\mathcal{P}(E)$-valued sequence $\{E_{v}^{(t)}\}_{t \geq 0}$, as well as the $G$-valued sequence $\{G^{(t)}\}_{t \geq 0}$, are constructed by $\{\omega^{(t)}\}_{t \geq 0}$ through the random evolution described by the collection of regular conditional probabilities

$$
P \{\omega_{e}(t + 1) = \omega_{e}'|X(t)\} = \delta_{\omega_{e},1} p(|\Delta_{e}X(t)|) + \delta_{\omega_{e},0} (1 - p(|\Delta_{e}X(t)|))$$

$$= \omega_{e}' |\Delta_{e}X(t)| + (1 - \omega_{e}') (1 - p(|\Delta_{e}X(t)|)) \ , \ e \in E \ ,$$

where $X(t) \in \Xi$ is the belief configuration at time $t \geq 0$ and $p : [0, 1] \to$ is a nonincreasing function such that $p(0) = 1$.

Notice that, for any $t \geq 0$, given $e, f \in E$ such that $\Delta_{e}X(t) = \Delta_{f}X(t)$, the r.v.'s $\omega_{e}(t + 1)$ and $\omega_{f}(t + 1)$ have the same conditional probabilities w.r.t. $X(t)$. In particular this holds for $e = (u, v)$ and $f = \bar{e} = (v, u)$, although the edge configurations $\omega_{e}$ and $\omega_{f}$ are different in general.

In the following we will consider $\forall e \in E, r_{e} > 0$. As a matter of fact, since the $r_{e}$'s are fixed, we can restrict ourselves to consider instead of $G$ each component of its spanning subgraph $G_{r} := G(E_{r})$, where

$$E_{r} := \{e \in E : r_{e} > 0\} \ ,$$

because, by (12), if $G_{1}, G_{2} \subseteq G_{r}$ the evolution of the beliefs labeled by the vertices of $G_{1}$ is never affected by those labeled by the vertices of $G_{2}$.

Moreover, since it is reasonable to assume that the agents put maximal confidence on their own beliefs, we can set $r_{(u, u)} = 1$, for any $u \in V$.

1.2.3 Results for the finite system

Let $V$ be of finite order. If $X^{0} \in \Xi$ is such that $\forall v \in V, X^{0}_{v} = x \in [0, 1]$, then by (12) $X(t) = X^{0}, \forall t \geq 0$. Hence these configuration, called consensus configurations, are stationary for the system evolution.

In the next section we will prove the following result.

**Theorem 1** The agents system reaches consensus for any realization of the initial value of the noise $\omega_{0} \in \Omega$ and any initial configuration $X^{0} \in \{X \in \Xi : \Gamma(X) > 0\}$.

Where $\Gamma : \Xi \to [0, 1]$ is defined in (11).

Moreover, we will also prove that, the random sequence $\{(X^{(t)}, \omega^{(t)})\}_{t \geq 0}$ started at $(X^{0}, \omega^{0}) \in \{X \in \Xi : \Gamma(X) > 0\} \times \Omega$ in the limit as $t$ tends to infinity weakly converges at geometric rate to $(X^{\infty}, \bar{1})$, where $X^{\infty} \in \Xi$ is such that, for any $v \in V, X_{v}^{\infty} = x$, for some $x \in [0, 1]$, and $\bar{1}$ is the element of $\Omega$ such that all its entries are equal to 1. As a byproduct of this result we will obtain that the random sequence $\{(X^{(2t)}, X^{(2t + 1)})\}_{t \geq 0}$ started at $(X^{0}, X^{1}) \in \{X \in \Xi : \Gamma(X) > 0\} \times \Xi$ will also weakly converge, in the limit as $t$ tends to infinity, to $(X^{\infty}, X^{\infty})$ at geometric rate.
1.2.4 Results for the large system

Let \( V := \mathbb{N}, E := \{(u, v) \in \mathbb{N} \times \mathbb{N}\} \) and set \( \Xi := [0, 1]^\mathbb{N} \) and \( \Omega \) as in \([5]\). Given \( N \in \mathbb{N} \), let \( V_N := \{1, \ldots, N\} \subset \mathbb{N} \) and \( E_N := \{(u, v) \in V_N \times V_N\} \). We denote by \( X_N := (X_1, \ldots, X_N) \) the element of \( \Xi_N := [0, 1]^N \) representing the restriction of the belief configuration \( X \in \Xi \) to \( V_N \) and by \( \omega_N \) the restriction of the configuration \( \omega \in \Omega \) to \( \Omega_N := \{0, 1\}^{E_N} \) and, by \([8]\), if \( E := E(\omega) \), we set \( E_N := E \cap E_N \).

Assuming that \( R := \{(u, v) \in V \times V : r_{uv} > 0\} \) is finite, in the last section we prove that the random sequence \( \{(X(2t), X(2t + 1))\}_{t \geq 0} \) started at \((X^0, X^1) \in \{X' \in \Xi : \inf_{N \in \mathbb{N}} \Gamma(X'_N) > 0\} \times \Xi\), in the limit as \( t \) tends to infinity, weakly converges at geometric rate to \((X^\infty, X^\infty)\).

2 Finite system evolution

Therefore, the evolution of the system is given by the following algorithm:

**Algorithm 2**

1. Label the elements of \( V \) from 1 to \( N \) in such a way that \( V := \{1, \ldots, N\} \) and consequently label \((i, j)\) the elements of \( E := V \times V \), then go to the next step.

2. Set \( t := 0, X(0) = (X_1(0), \ldots, X_N(0)) := (X^0_1, \ldots, X^0_N) \in [0, 1]^N, \omega(0) := \{\omega^0_{i,j}\}_{(i,j) \in E} \in \{0, 1\}^E \) such that \( \forall i = 1, \ldots, N, \omega^0_{i,i} = 1 \), and go to the next step.

3. Set \( i := 1 \) and go to the next step.
   
   (a) Set \( j := 1 \) and go to the next step.
   
   (b) Compute \( p_{i,j}(t) := p(|X_i(t) - X_j(t)|) \) and form the vector
       \[
       p(t) := (p_{1,1}(t), \ldots, p_{1,N}(t), p_{2,1}(t), \ldots, p_{2,N}(t), \ldots, p_{i,1}(t), \ldots, p_{i,j}(t)) \quad (17)
       \]
       and go to the step.
   
   (c) Set \( j := j+1 \). If \( j + 1 \leq N \) go back to step 3.b, otherwise go to the next step.
   
   (d) Set \( i := i + 1 \). If \( i + 1 \leq N \) go back to step 3.a, otherwise go to the next step.

4. Set \( i := 1 \) and go to the next step.
   
   (a) Compute \( X_i(t + 1) \) according to \((12)\) and form the vector \( X(t + 1) := (X_1(t + 1), \ldots, X_i(t + 1)) \), then go to the next step.
   
   (b) Set \( i := i + 1 \). If \( i + 1 \leq N \) go back to step 4.a, otherwise go to the next step.

5. Read \( X(t) = (X_1(t), \ldots, X_N(t)) \). If \( X(t + 1) = X(t) \) stop, otherwise go to the next step.
6. Set $i := 1$ and go to the next step.

(a) Set $j := 1$ and go to the next step.

(b) Read the $p_{i,j}(t)$ entry of the vector $p(t)$. Sample a random variable $U$ uniformly distributed on $[0,1]$. If $U \leq p_{i,j}(t)$ then set $\omega_{i,j}(t+1) := \omega_{i,j}(t)$ if $\omega_{i,j}(t) = 1$, otherwise set $\omega_{i,j}(t+1) := 1 - \omega_{i,j}(t)$. If $U > p_i(t)$ then set $\omega_{i,j}(t+1) := 1 - \omega_{i,j}(t)$ if $\omega_{i,j}(t) = 1$, otherwise set $\omega_{i,j}(t+1) := \omega_{i,j}(t)$.

Form the vector $\omega(t+1) := (\omega_{1,1}(t+1), \ldots, \omega_{1,N}(t+1), \ldots, \omega_{i,j}(t+1), \ldots, \omega_{i,j}(t+1))$ (18) and go to the next step.

(c) Set $j := j+1$. If $j+1 \leq N$ go back to step 6.b, otherwise go to the next step.

(d) Set $i := i+1$. If $i+1 \leq N$ go back to step 6.a, otherwise go to the next step.

7. Set $t := t+1$, $X(0) := X(t+1)$, $\omega(0) := \omega(t+1)$ and go to step 3.

In terms of stochastic process the system evolution can be described as follows.

Let $\Omega \ni \omega \mapsto P(\omega) \in St(\mathbf{V})$ the stochastic matrix-valued function on $\mathbb{R}^V$ such that, for any $\omega \in \Omega$,

$$P_{v,u}(\omega) := \frac{\sum_{e \in E^-\omega(v)} \delta_e(u,v) r_e}{\sum_{e \in E^-\omega(v)} r_e} = \frac{r_{u,v} 1_{\mathcal{N}^-\omega(v)}(u)}{\sum_{u \in \mathcal{V}} r_{u,v}} = \frac{r_{u,v} \omega_{u,v}}{\sum_{u \in \mathcal{V}} r_{u,v} \omega_{u,v}}, \quad u, v \in \mathbf{V}. \quad (19)$$

**Remark 3** We remark that, given $\omega \in \Omega$, $v \in \mathbf{V}$, by (19) $u \in \mathcal{N}^-\omega(v)$ iff $P_{v,u}(\omega) > 0$.

Therefore, denoting by $G(\omega) := G(P(\omega))$ the graph associated to $P(\omega)$, this is the spanning graph of $\mathcal{E}(\omega) := \{ e \in \mathcal{E} : e = (u,v) \text{ if } (v,u) \in E(\omega) \}$.

Considering $\Xi := [0,1]^V \subset \mathbb{R}^V$ endowed with the norm $\|X\| := \|X\|_{\infty} = \sup_{v \in \mathbf{V}} |X_v|$, let $(\Xi, \mathcal{F})$ be the measurable space such that $\mathfrak{X} := \Xi \times \Omega$ and, if $\mathbf{V}$ is a finite set, $\mathcal{F} := \mathcal{B}(\Xi) \otimes \mathcal{P}(\Omega)$.

For any $\omega \in \Omega$, $P(\omega) \in BL(\Xi)$, therefore we set

$$\mathfrak{X} \ni (X, \omega) \mapsto \mathcal{T}_\omega(X, \omega) := \sum_{u \in \mathbf{V}} P_{v,u}(\omega) X_u \in [0,1], \quad (20)$$

and consider the measurable map

$$\mathfrak{X} \ni (X, \omega) \mapsto \mathcal{T}(X, \omega) := \{ \mathcal{T}_\omega(X, \omega) \}_{\omega \in \mathbf{V}} \in \Xi. \quad (21)$$
Defining, by \((15)\), the probability kernel from \((\Xi, \mathcal{B}(\Xi))\) to \((\Omega, \mathcal{P}(\Omega))\)

\[
\mathcal{X} \ni (X, \omega) \mapsto \Pi(\omega|X) := \prod_{e \in E} [\delta_{x_e,1}p(\Delta_e X) + \delta_{x_e,0}(1 - p(\Delta_e X))] \tag{22}
\]

\[
= \prod_{e \in E} [\omega_e p(\Delta_e X) + (1 - \omega_e)(1 - p(\Delta_e X))] \in [0, 1],
\]

we introduce the positive linear operator on \(BM(\mathcal{X})\) such that

\[
BM(\mathcal{X}) \ni \varphi \mapsto \mathcal{X} \varphi (X, \omega) := \sum_{\omega \in \Omega} \varphi(T(X, \omega), \omega') \Pi(\omega'|X) \in BM(\mathcal{X}). \tag{23}
\]

Let \(\mathbb{P}_0\) be the probability distribution on \((\mathcal{X}^{\mathbb{Z}_{+}}, \mathcal{C})\), where \(\mathcal{C} := \mathcal{C}(\Xi) \otimes \mathcal{C}(\Omega)\), describing the homogeneous discrete time Markov process started at \((X^0, \omega^0)\) by the one-step transition probability kernel associated to \(\mathcal{F}\). We denote by \(\{\chi_t\}_{t \geq 0}\) the random process on \((\mathcal{X}^{\mathbb{Z}_{+}}, \mathcal{C}, \mathbb{P}_0)\) such that, \(\forall t \geq 0\),

\[
\mathcal{X}^{\mathbb{Z}_{+}} \ni x \mapsto \chi_t(x) = (X(t), \omega(t)) \in \mathcal{X} \tag{24}
\]

and by \(\{\mathcal{F}_t\}_{t \geq 0}\), with \(\mathcal{F}_t := \bigvee_{s=0}^{t} \chi_s^{-1}(\mathcal{B}(\Xi) \otimes \mathcal{P}(\Omega))\), the associated natural filtration.

Therefore, denoting by \(\mathbb{E}_0\) the expectation value w.r.t. \(\mathbb{P}_0\), for any bounded measurable function \(\varphi\) on \(\mathcal{X}\),

\[
\mathbb{E}_0 [\varphi \circ \chi_{t+1} | \mathcal{F}_t] = \mathbb{E}_0 [\varphi \circ \chi_{t+1} | \chi_t] = (\mathcal{X} \varphi)(\chi_t) \mathbb{P}_0 - a.s. . \tag{25}
\]

Notice that, by \((20)\), \(\mathcal{X} : C(\mathcal{X}, \mathbb{R}) \subset\), that is \(\{\chi_t\}_{t \geq 0}\) is a Feller process.

Setting \(\pi_\omega : \mathcal{X} \mapsto \Omega\), \(\pi_X : \mathcal{X} \mapsto \Xi\), we denote by \(\{\mathbf{w}_t\}_{t \geq 0}, \{\mathbf{r}_t\}_{t \geq 0}\) the random processes on \((\mathcal{X}^{\mathbb{Z}_{+}}, \mathcal{C}, \mathbb{P}_0)\) such that, \(\forall t \geq 0\),

\[
\mathcal{X}^{\mathbb{Z}_{+}} \ni x \mapsto \mathbf{w}_t(x) := \pi_\omega \circ \chi_t(x) = \omega(t) \in \Omega \tag{26}
\]

and

\[
\mathcal{X}^{\mathbb{Z}_{+}} \ni x \mapsto \mathbf{r}_t(x) := \pi_X \circ \chi_t(x) = X(t) \in \Xi. \tag{27}
\]

Hence, \(\{\chi_t\}_{t \geq 0}\) can be represented as \(\{(\mathbf{r}_t, \mathbf{w}_t)\}_{t \geq 0}\). We also set \(\{\mathcal{F}_t^\omega\}_{t \geq 0}\), with \(\mathcal{F}_t^\omega := \bigvee_{s=0}^{t} \mathbf{w}_s^{-1}(\mathcal{P}(\Omega))\), and \(\{\mathcal{F}_t^X\}_{t \geq 0}\), with \(\mathcal{F}_t^X := \bigvee_{s=0}^{t} \mathbf{r}_s^{-1}(\mathcal{B}(\Xi))\).

\textbf{Remark 4} Notice that neither \(\{\mathbf{w}_t\}_{t \geq 0}\) nor \(\{\mathbf{r}_t\}_{t \geq 0}\) are Markov processes. Indeed, by \((13)\), for any \(t \geq 0\), \(\mathbf{w}_{t+1}\) is independent of \(\mathbf{w}_t\). Moreover, since \(\forall t \geq 0, \mathcal{F}_t^X\) and \(\mathcal{F}_t^\omega\) are
In particular, by (29), we get that
\[ \phi \]
for any bounded measurable function \( \phi \) of \( F \),
while, by (15), (23) and (25), 
\( \forall \phi \in BM(\Xi, \mathbb{R}) \), since for any \( t \geq 0 \), \( \mathcal{F}_t^X \) is a subalgebra of \( \mathcal{F}_t \),
\[
E_0 [\phi \circ \pi_{t+1}|\mathcal{F}_t^X] = E_0 [\phi \circ \pi_X \circ \chi_{t+1}|\mathcal{F}_t^X] = E_0 [E_0 [\phi \circ \pi_X \circ \chi_{t+1}|\mathcal{F}_t]|\mathcal{F}_t^X] \tag{29}
\]
\[
\quad = E_0 \left[ \sum_{\omega' \in \Omega} \phi (T (\xi_t, \omega')) \Pi (\omega'|\xi_t) \right] \tag{30}
\]
\[
\quad = \sum_{\omega' \in \Omega} \sum_{\omega \in \Omega} \phi (T (\xi_t, \omega)) \Pi (\omega|\xi_t) \Pi (\omega'\xi_t) \tag{31}
\]
\[
\quad = \sum_{\omega \in \Omega} \phi (T (\xi_t, \omega)) \Pi (\omega|\xi_t) = E_0 [\phi \circ \pi_{t+1}|\xi_t, \xi_{t-1}] \quad P_0 \text{ - a.s.}
\]

In particular, by (29), we get that \( \{\eta_t\}_{t \geq 0} \) such that \( \forall t \geq 0 \), \( \eta_t := (\xi_{2t}, \xi_{2t+1}) \) is a homogeneous Markov process on \( (\Xi^{2}, \mathcal{F}, \mathbb{P}) \). Indeed, denoting by \( \mathcal{F}^\eta_{t+1} \) the filtration generated by \( \{\eta_t\}_{t \geq 0} \), since \( \forall t \geq 0 \), \( \mathcal{F}_t^X = \mathcal{F}^\eta_{2t+1} \), for any bounded measurable function \( \phi \) on \( \mathbb{R} \),
\[
E [\phi \circ \eta_{t+1}|\mathcal{F}_t^X] = E [\phi (\eta_{2t+2}, \eta_{2t+3})|\mathcal{F}_{2t+1}^X] = E [\phi (\xi_{2t+2}, \xi_{2t+3})|\eta_{2t+1}, \eta_{2t}] \tag{32}
\]
\[
\quad = E [\phi \circ \eta_{t+1}|\eta_t] .
\]

Therefore, the transition operator associated to \( \{\eta_t\}_{t \geq 0} \) is
\[
(T \phi) (X_1, X_2) := \sum_{\omega, \omega' \in \Omega} \phi (T (X_2, \omega), T (T (X_2, \omega), \omega')) \Pi (\omega|X_1) \Pi (\omega'|X_2) \tag{33}
\]
and, setting
\[
\Xi^2 \ni (X_1, X_2) \rightarrow \pi_i (X_1, X_2) := X_1 \delta_{i, 1} + X_2 \delta_{i, 2} \in \Xi , \quad i = 1, 2 ,
\]
for any bounded measurable function \( \phi \) on \( \Xi \), we have
\[
E_0 [\phi \circ \pi_{t+1}|\pi_t, \pi_{t-1}] = E_0 [\phi \circ \pi_1 (\eta_{t+1})|\eta_t] \tag{34}
\]
\[
\quad = E_0 [T (\phi \circ \pi_1) (\eta_t)] \quad P_0 \text{ - a.s.}
\]
2.1 Consensus

If $X^0 \in \Xi$ is such that $\forall v \in V, X^0_v = x \in [0,1]$, then by (12) $X(t) = X^0, \forall t \geq 0$. Hence these configurations, called consensus configurations, are stationary for the system evolution.

We denote by

$$I := \bigcup_{x \in [0,1]} I_x$$

(34)

where

$$I_x := \{ X \in \Xi : X_v = x, \forall v \in V \}$$

(35)

and by $M : \Xi \rightarrow \Xi$ the consensus projection map, that is the map associating to each belief configuration $X$ the consensus configuration $M X$ such that $\forall v \in V, (M X)_v := \frac{1}{|V|} \sum_{u \in V} X_u$. It is easy to see that $M$ is a projection operator on $I$, moreover an orthogonal projection if $\Xi$ is endowed with the Euclidean structure $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^V$. Indeed, $\forall X \in \Xi$, $M^2 X = M X$.

Therefore, $\forall X \in \Xi$, we set

$$\text{dist} (I, X) := \inf_{Y \in I} \| X - Y \| \leq [I - M] X,$$

(36)

where we denote by $I$ the identity operator on $\mathbb{R}^V$.

Consequently, since if $X = M X, \forall (u, v) \in E, X_u - X_v = 0$, we can modify the algorithm erasing the line 5 and adding the line

3.e If $\sum_{i=1}^{N} \sum_{j=1}^{N} (1 - \delta_{i,j}) p_{i,j} (t) = N (N - 1)$ stop, otherwise proceed to the next step.

2.2 Invariant measures for $\Xi$ and $T$

Setting $X := (I - M) \Xi$, we can represent $\Xi$ as $I \oplus X$. Moreover, for any $\omega \in \Omega, I$ is invariant under $T (\cdot, \omega)$, since $X \in \Xi$, by (21) we get $T (M X, \omega) = M X$. Therefore

$$T (X, \omega) = T (M X + (I - M) X, \omega) = M X + T ((I - M) X, \omega).$$

(37)

Moreover, by (22), for any $\omega \in \Omega, \Pi (\omega | X) = \Pi (\omega | (I - M) X)$. Hence, denoting by $\delta_{\bar{1}}$ the Dirac measure at $\bar{1}$, by the definition of $p$ and by (15), given $X \in I, \forall \omega \in \Omega, \Pi (\omega | X) = \prod_{e \in E} \delta_{\omega_e, 1} = \delta_{\bar{1}}$.

Denoting by $\delta^X$ the probability measure on $(\Xi, \mathcal{B} (\Xi))$ concentrated on the beliefs configuration $X \in \Xi$, let $\delta^X$ be the probability measure on $(\Xi, \mathcal{B} (\Xi))$ putting mass 1 on the configuration $X \in I$. It is easy to see that the probability measure $\delta^X \otimes \delta_{\bar{1}}$ on $(X, F)$ is invariant for the evolution given by $X$. Indeed, if $X \in I$, by (34) and (35), there exists $x \in [0,1]$ such that $X \in I_x$. Hence, by (21), for any $\omega \in \Omega, T (X, \omega) = X$. Therefore, given
any bounded measurable function \( \varphi \) on \( \mathcal{X} \), by (21), \( \forall \omega, \omega' \in \Omega, \delta^X_T [\varphi (T (\cdot, \omega), \omega')] = \delta^X_T [\varphi (\cdot, \omega')] \). Thus, by (23),

\[
\delta^X_T \otimes \delta^\Omega_T \left[ \mathfrak{F} \varphi \right] = \delta^\Omega_T \left[ \delta^X_T \left[ \sum_{\omega' \in \Omega} \varphi (T (\cdot, \omega), \omega') \Pi (\omega' | \cdot) \right] \right]
\]

\[
= \delta^\Omega_T \left[ \sum_{\omega' \in \Omega} \delta^X_T \left[ \varphi (T (\cdot, \omega), \omega') \Pi (\omega' | \cdot) \right] \right]
\]

\[
= \delta^\Omega_T \left[ \sum_{\omega' \in \Omega} \delta^X_T \left[ \varphi (\cdot, \omega') \prod_{e \in E} \delta_{\omega'_e, 1} \right] \right]
\]

\[
= \delta^X_T \otimes \delta^\Omega_T \left[ \sum_{\omega' \in \Omega} \varphi (\cdot, \omega') \prod_{e \in E} \delta_{\omega'_e, 1} \right]
\]

\[
= \delta^X_T \otimes \delta^\Omega_T \left[ \varphi (\cdot, \cdot) \right] = \delta^X_T \otimes \delta^\Omega_T \left[ \varphi \right].
\]

Thus the set \( \mathcal{I}_T \) of invariant probability measures under \( \mathfrak{F} \) is the weak limit of convex combinations of elements of the set \( \{ \delta^\omega \otimes \delta^X_T \}_{X \in \mathcal{I}} \subset \mathfrak{P} (\mathcal{X}, \mathcal{F}) \).

Since for any \( X_2 \in \mathcal{I} \) and any \( \mu \in \mathfrak{P} (\Xi, \mathcal{B} (\Xi)) \), from (31) it follows that

\[
\mu \otimes \delta^X_T \left[ T \varphi \right] = \int \mu (dX_1) \delta^X_T \left[ \sum_{\omega, \omega' \in \Omega} \varphi (T (\cdot, \omega), T (T (\cdot, \omega), \omega')) \Pi (\omega | X_1) \Pi (\omega' | \cdot) \right]
\]

\[
= \int \mu (dX_1) \sum_{\omega, \omega' \in \Omega} \varphi (X_2, T (X_2, \omega')) \Pi (\omega | X_1) \prod_{e \in E} \delta_{\omega'_e, 1}
\]

\[
= \int \mu (dX_1) \sum_{\omega \in \Omega} \varphi (X_2, T (X_2, \bar{1})) \Pi (\omega | X_1)
\]

\[
= \int \mu (dX_1) \sum_{\omega \in \Omega} \varphi (X_2, X_2) \Pi (\omega | X_1) = \varphi (X_2, X_2)
\]

we have that the set \( \mathcal{I}_T \) of invariant probability measures under \( T \) is the weak limit of convex combinations of elements of the set \( \left\{ \delta^{(X,X)}_T \right\}_{X \in \mathcal{I}} \subset \mathfrak{P} (\Xi^2, \mathcal{B} (\Xi^2)) \), where \( \delta^{(X,X)}_T := \delta^X_T \otimes \delta^X_T \).

### 2.3 Emergence of consensus

**Definition 5** Given \( E \subseteq \mathcal{E} \), consider the spanning graph \( G (E) = (\mathcal{V}, E) \). We call pivots the elements \( w \) of \( \mathcal{V} \) such that \( \mathcal{N}^+ (w) = \mathcal{V} \) and denote their collection by \( \mathcal{P} (E) \). Moreover, for any \( \omega \in \Omega, \) we set \( \mathcal{P} (\omega) := \mathcal{P} (E (\omega)) \) and define \( \Omega_F := \{ \omega \in \Omega : \mathcal{P} (\omega) \neq \emptyset \} \).
Let us denote by $\gamma$ the r.v.\footnote{Notice that $1 - \gamma$ is the coefficient of ergodicity of the transition probability matrix of the Markov chain on $V$ whose components are two independent versions of the Markov chain defined by the transition probability matrix $\{P_{u,v}(\omega)\}_{u,v\in V}$.}

$$\Omega \ni \omega \mapsto \gamma(\omega) := \min_{u,v \in V : u \neq v} \sum_{w,z \in V} P_{u,w}(\omega) P_{v,z}(\omega) \land P_{u,z}(\omega) P_{v,w}(\omega) \in [0, 1)$$ \hfill (40)

and by $\Gamma$ the r.v.

$$\Xi \ni X \mapsto \Gamma(X) := E[\gamma|X] = \sum_{\omega \in \Omega} \Pi(\omega|X) \gamma(\omega) \in [0, 1).$$ \hfill (41)

**Lemma 6** Given $\omega \in \Omega$, $\gamma(\omega) > 0$ if and only if $P(\omega)$ is not empty.

**Proof.** Let $\omega \in \Omega$ be such that $P(\omega) \neq \emptyset$. Denoting by $u = u(\omega), v = v(\omega)$ the elements of $V$ such that

$$\sum_{w,z \in V} P_{u,w}(\omega) P_{v,z}(\omega) \land P_{u,z}(\omega) P_{v,w}(\omega) = \min_{u',v' \in V} \sum_{w,z \in V} P_{u',w}(\omega) P_{v',z}(\omega) \land P_{u',z}(\omega) P_{v',w}(\omega),$$

for any $\tilde{w} \in P(\omega)$, we have

$$\gamma(\omega) = \sum_{w,z \in V} P_{u,w}(\omega) P_{v,z}(\omega) \land P_{u,z}(\omega) P_{v,w}(\omega) = \sum_{w \in V} P_{u,w}(\omega) P_{v,w}(\omega) + \sum_{w,z \in V : w \neq z} P_{u,w}(\omega) P_{v,z}(\omega) \land P_{u,z}(\omega) P_{v,w}(\omega) \geq (P_{u,\tilde{w}}(\omega) \land P_{v,\tilde{w}}(\omega))^2 > 0.$$

Conversely by (19), $\gamma(\omega) > 0$ iff, for any $u, v \in V$ such that $u \neq v, N^-(u, \omega) \cap N^-(v, \omega) \neq \emptyset$, which is equivalent to say that $\gamma(\omega) > 0$ implies that there exists at least one $\tilde{w} = \tilde{w}(\omega)$ in $V$ such that, by (50), $N^+(\tilde{w}, \omega) = V$, or, in other words, by Definition\footnote{1}{5} that $P(\omega)$ is not empty. $\blacksquare$

In the next section we will prove the following result.

**Theorem 7** The agents system reaches consensus for any realization of the initial value of the noise $\omega_0 \in \Omega$ and any initial configuration $X^0 \in \{X \in \Xi : \Gamma(X) > 0\}$.

More precisely, for any $X^0 \in \{X \in \Xi : \Gamma(X) > 0\}$ and any $\omega_0 \in \Omega$, the sequence of probability measures $\{\mu_t^{\omega_0}\}_{t \geq 0}$ on $(\Xi, B(\Xi))$ such that

$$B(\Xi) \ni A \mapsto \mu_t^{\omega_0}(A) := \mathbb{P}_0 \{x \in X^{Z^+: \pi_X \circ \chi_t(x) \in A} \in [0, 1],$$

converges to a probability measure $\mu_0^{\omega_0}$ supported on $\mathcal{I}$.\hspace{1cm} (44)
Remark 8 We stress that this result give no information on the common value of the beliefs when consensus is reached.

Given $X \in \Xi$, let
\[ W(X) := \max_{u,v \in V} |X_u - X_v|. \]  
(45)

Since
\[ W(\|I - M\|X) = W(X), \]  
\[ W \] is a seminorm on $\mathbb{R}^V$ and therefore induces a norm on $W := \mathbb{R}^V/\text{Ran} M$.

Hence, because $M I = I$, for any $Y \in \mathcal{I}$, we have
\[ \|X - Y\| = W(X - Y) = W((I - M) (X - Y)) = W(X), \]  
\[ \] which implies
\[ \text{dist} (\mathcal{I}, X) = W(X). \]  
(48)

For any $t \geq 0$, let $W(t) := W(X(t))$. In the following we will prove that the random sequence $\{W(t)\}_{t \geq 0}$ converges to zero w.p.1 w.r.t. the noise, hence proving Theorem 7.

**Proposition 9** The sequence $\{W(t)\}_{t \geq 0}$ is non-increasing hence bounded. Moreover, $\{W(t)\}_{t \geq 0}$ is a non-negative $L^1$-supermartingale w.r.t. $\{\mathcal{F}_t\}_{t \geq 0}$, therefore $\mathbb{P}_0$-a.s. convergent to a $L^1(X, \mathcal{F}, \mathbb{P}_0)$ r.v. which we denote by $W$.

**Proof.** By (12), given $u, v \in V$ such that $u \neq v$, for $t \geq 0$,
\[ X_u(t+1) - X_v(t+1) = (X_u(t+1) - X_u(t)) - (X_v(t+1) - X_v(t)) + X_u(t) - X_v(t) \]  
\[ = X_u(t) - X_v(t) + \sum_{e \in E_u^-(t)} r_e 1_{(u,w)}(e) \frac{\left[X_w(t) - X_u(t)\right]}{\sum_{e \in E_u^-(t)} r_e}, \]  
\[ - \sum_{e' \in E_v^-(t)} r_{e'} 1_{(v,z)}(e') \frac{\left[X_z(t) - X_v(t)\right]}{\sum_{e' \in E_v^-(t)} r_{e'}}, \]  
\[ = \sum_{e \in E_u^-(t)} r_e 1_{(u,w)}(e) X_w(t) - \sum_{e' \in E_v^-(t)} r_{e'} 1_{(v,z)}(e') X_z(t) \].  
(49)

By (12), setting
\[ P_{u,v}(t) := P_{u,v}(\omega(t)) = \frac{\sum_{e \in E_u^-(t)} \delta_{e,(v,u)} r_e}{\sum_{e \in E_u^+(t)} r_e} = \frac{r_{v,u} 1_{X^-(u,v)}(v)}{\sum_{v \in X^-(u,v)} r_{v,u}}, \]  
(50)
we can rewrite the previous expression as

\[
X_u(t + 1) - X_v(t + 1) = \sum_{w \in V} P_{u,w}(t) X_w(t) - \sum_{z \in V} P_{v,z}(t) X_z(t) .
\] (51)

Since, \( \forall t \geq 0, \sum_{v \in V} P_{u,v}(t) = 1 \), we have

\[
X_u(t + 1) - X_v(t + 1) = \sum_{u,z \in V} P_{u,w}(t) P_{v,z}(t) [X_w(t) - X_z(t)]
\] (52)

and, since \([X_w(t) - X_z(t)] = -[X_z(t) - X_w(t)]\) , we obtain

\[
X_u(t + 1) - X_v(t + 1) = \frac{1}{2} \sum_{u,z \in V} \left\{ P_{u,w}(t) P_{v,z}(t) - P_{u,z}(t) P_{v,w}(t) \right\} [X_w(t) - X_z(t)] .
\] (53)

Hence

\[
|X_u(t + 1) - X_v(t + 1)| \leq \frac{1}{2} \sum_{u,z \in V} |P_{u,w}(t) P_{v,z}(t) - P_{u,z}(t) P_{v,w}(t)| |X_w(t) - X_z(t)| .
\] (54)

Since \( \forall a, b \in \mathbb{R}, a \land b = \frac{a + b - |a - b|}{2} \),

\[
|X_u(t + 1) - X_v(t + 1)| \leq \sum_{u,z \in V} \left\{ \frac{P_{u,w}(t) P_{u,z}(t) + P_{u,z}(t) P_{v,w}(t)}{2} \right\} |X_w(t) - X_z(t)|
\] (55)

\[
- P_{u,w}(t) P_{v,z}(t) \land P_{u,z}(t) P_{v,w}(t) \right\} |X_w(t) - X_z(t)|
\]

\[
\leq \sum_{u,z \in V} \left\{ \frac{P_{u,w}(t) P_{u,z}(t) + P_{u,z}(t) P_{v,w}(t)}{2} \right\}
\]

\[
- P_{u,w}(t) P_{v,z}(t) \land P_{u,z}(t) P_{v,w}(t) \right\} \max_{w,z \in V} |X_w(t) - X_z(t)|
\]

\[
\leq \left\{ 1 - \sum_{u,z \in V} P_{u,w}(t) P_{v,z}(t) \land P_{u,z}(t) P_{v,w}(t) \right\} \max_{w,z \in V} |X_w(t) - X_z(t)| .
\]

Therefore, choosing \( u, v \in V \) such that

\[
|X_u(t + 1) - X_v(t + 1)| = \max_{w,z \in V} |X_w(t + 1) - X_z(t + 1)| ,
\] (56)

by (40) we get

\[
W(t + 1) \leq \left\{ 1 - \min_{u,v \in V : u \neq v} \sum_{w,z \in V} P_{u,w}(t) P_{v,z}(t) \land P_{u,z}(t) P_{v,w}(t) \right\} W(t) \quad (57)
\]

\[
= (1 - \gamma(\omega(t))) W(t) ;
\]
Proof. For any $t \geq 0$, $W(t) \leq W(0) \leq 1$.

Thus, representing the random sequence $\{W(t)\}_{t \geq 0}$ as $\{W \circ \gamma_t\}_{t \geq 0}$, from (57) we get

$$
\mathbb{E}_0 \left[ W(t+1) | \mathcal{F}_X^t \right] = \mathbb{E}_0 \left[ W \circ \gamma_{t+1} | \mathcal{F}_X^t \right] = \mathbb{E}_0 \left[ \mathbb{E}_0 \left[ W \circ \pi_X \circ \chi_{t+1} | \mathcal{F}_X^t \right] | \mathcal{F}_X^t \right] 
\leq \mathbb{E}_0 \left[ \mathbb{E}_0 \left[ \left\{ 1 - \gamma \circ \pi_{\omega} \right\} W \circ \pi_X \circ \chi_t | \mathcal{F}_X^t \right] | \mathcal{F}_X^t \right] 
= \mathbb{E}_0 \left[ \mathbb{E}_0 \left[ 1 - \gamma \left( \pi_{\omega} \circ \chi_t \right) \right] W \circ \gamma_t \right] 
= \left\{ 1 - \mathbb{E}_0 \left[ \gamma \left( \pi_{\omega} \circ \chi_t \right) \right] \right\} W(t) \leq W(t), 
$$

that is $\{W(t)\}_{t \geq 0}$ is a $L^1$-supermartingale w.r.t. $\{\mathcal{F}_X^t\}_{t \geq 0}$.

2.3.1 Asymptotic estimate of $\mathbb{E}_0 \left[ W(t) \right]$

Lemma 10 The sequence $\left\{ \mathbb{E}_0 \left[ \gamma | \mathcal{F}_X^t \right] \right\}_{t \geq 0}$ is predictable w.r.t. the filtration $\{\mathcal{F}_X^t\}_{t \geq 0}$.

Proof. For any $t \geq 1$, by (19),

$$
\mathbb{E}_0 \left[ \gamma | \mathcal{F}_X^t \right] = \mathbb{E}_0 \left[ \gamma \left( \pi_{\omega} \circ \chi_t \right) | \mathcal{F}_X^t \right] = \mathbb{E}_0 \left[ \gamma \left( \pi_{\omega} \circ \chi_{t-1} \right) | \mathcal{F}_X^{t-1} \right] = 
\sum_{\omega \in \Omega} \Pi \left( \omega | \chi_{t-1} \right) \min_{v \neq u} \sum_{w,z \in V} \delta_{\omega,v} \left( \delta_{\omega,u} \delta_{\omega,z} \right) P_{\omega,w} (\omega) P_{\omega,z} (\omega) = \Gamma \left( \chi_{t-1} \right). 
$$

Let us set

$$
\mathcal{X} \ni (X, \omega) \mapsto \Pi (\omega | W(X)) := \prod_{e \in E} \left[ \delta_{\omega,e} p (W(X)) + \delta_{\omega,e} (1 - p (W(X))) \right] 
= \prod_{e \in E} \left[ \omega_e p (W(X)) + (1 - \omega_e) (1 - p (W(X))) \right] \in [0,1].
$$

Since $\Omega$ is a poset w.r.t. the partial order relation: $\omega \leq \omega'$ if, $\forall e \in E, \omega_e \leq \omega'_e$, we have

Lemma 11 For any $X \in \Xi, \Pi (\cdot | X) \overset{st}{\geq} \Pi (\cdot | W(X))$. Moreover, for any $t \geq 0, \Pi (\cdot | W(t+1)) \overset{st}{\geq} \Pi (\cdot | W(t))$.

Proof. Let us consider first the statement $\Pi (\cdot | X) \overset{st}{\geq} \Pi (\cdot | W(X))$. For $X \in \mathcal{X}$, by (22) and (60) $\Pi (\cdot | X)$ and $\Pi (\cdot | W(X))$ coincide. Let now $X \in \mathcal{X}$. By (22) and (61) $\Pi (\cdot | \cdot)$ is irreducible, then to prove $\Pi (\cdot | X) \overset{st}{\geq} \Pi (\cdot | W(X))$ is enough to prove that the Holley inequality is satisfied, namely

$$
\Pi (\omega \cup \omega' | X) \Pi (\omega \cap \omega' | W(X)) \geq \Pi (\omega | X) \Pi (\omega' | W(X)), \quad \omega, \omega' \in \Omega. 
$$
where $\omega \vee \omega' \in \Omega$ is such that $\forall e \in E, (\omega \vee \omega')_e = \omega_e \vee \omega'_e$ and $\omega \wedge \omega' \in \Omega$ is such that $\forall e \in E, (\omega \wedge \omega')_e = \omega_e \wedge \omega'_e$. This is equivalent to prove that, for any $e, f \in E$,

$$\Pi (\omega^{(e)}|X) \Pi (\omega^{(f)}|W (X)) \geq \Pi (\omega^{(e)}|W (X)) \Pi (\omega^{(f)}|X)$$

where, for any $e \in E$, $\Pi (\omega^{(e)}|X) = \frac{\omega^{(e)}_X}{\omega^{(e)}}$ and $\Pi (\omega^{(e)}|W (X)) = \frac{\omega^{(e)}_W}{\omega^{(e)}}$ (62).

(19)

which can be rewritten as

$$\Pi (\omega^{(e)}|X) \Pi (\omega^{(f)}|W (X)) \geq \Pi (\omega^{(e)}|W (X)) \Pi (\omega^{(f)}|X),$$

and

$$\Pi (\omega^{(e)}|X) \Pi (\omega^{(f)}|W (X)) \geq \Pi (\omega^{(e)}|W (X)) \Pi (\omega^{(f)}|X),$$

where, for any $E \subset E, \omega^E \in \Omega$ is such that $\forall e \in E, \omega^E_e := \omega_e 1_{E^c} (e) + 1_E (e)$ and $\omega_E \in \Omega$ is such that $\forall e \in E, (\omega^E)_e := \omega_e 1_{E^c} (e)$ (see e.g. [Gr] Theorem 2.3). But, by (22) and (60), $\Pi (\cdot|X)$ and $\Pi (\cdot|W (X))$ are product measures, then (62) becomes

$$\Pi (\omega = 1|X) \Pi (\omega = 0|W (X)) \geq \Pi (\omega = 1|W (X)) \Pi (\omega = 0|X),$$

which can be rewritten as

$$p (|\Delta_e X|) (1 - p (W (X))) \geq p (W (X)) (1 - p (|\Delta_e X|))$$

and (63) becomes

$$\Pi (\omega = 1|X) \Pi (\omega = 0|W (X)) \Pi (\omega = 0|W (X)) \geq \Pi (\omega = 1|W (X)) \Pi (\omega = 0|X) \Pi (\omega = 1|X)$$

which is again (64). Since by (15), for any $e \in E, W (X) \geq \Delta_e X$ and since $p : [0, 1] \cap$ is non increasing, we have, for any $e \in E, p (|\Delta_e X|) \geq p (W (X))$ and consequently

$$(1 - p (W (X))) \geq (1 - p (|\Delta_e X|))$$

which proves (64).

The proof of the statement $\Pi (\cdot|W (t + 1)) \geq \Pi (\cdot|W (t)), t \geq 0$, follow the same lines of the proof of $\Pi (\cdot|X) \geq \Pi (\cdot|W (X))$, since, by (57), $W (t + 1) \leq W (t)$, which implies $p (W (t + 1)) \geq p (W (t))$. ■

**Proof of Theorem 7** Since $\gamma$ is an non-decreasing function, by (41) and by the previous lemma we have that

$$\Gamma (W (t)) \geq \sum_{e \in \Omega} \mathbb{P} (W |W (X)) \gamma (\omega) = \Gamma (W (X))$$

and, for any $t \geq 0$, $\Gamma (W (t + 1)) \geq \Gamma (W (t))$. Then, by (58) and Lemma (10) we get

$$E_0 [W (t)] = E_0 [E_0 [W \circ \chi_t \circ \chi_{t-1}^X]] \leq E_0 [(1 - E_0 [\gamma |\chi_{t-1}^X]) W \circ \chi_t \circ \chi_{t-1}]$$

$$= E_0 [(1 - \Gamma (X (t - 2))) W (t - 1)] \leq E_0 [(1 - \Gamma (W (t - 2))) W (t - 1)]$$

$$= E_0 [(1 - \Gamma (W \circ \chi_{t-2})) E_0 [W \circ \chi_t \circ \chi_{t-1}|\chi_{t-2}^X]]$$

$$\leq E_0 [(1 - \Gamma (W \circ \chi_{t-2}) (1 - E_0 [\gamma |\chi_{t-2}^X]) W \circ \chi_t \circ \chi_{t-2}]$$

$$= E_0 [(1 - \Gamma (W (t - 2))) (1 - \Gamma (X (t - 2))) W (t - 2)]$$

$$\leq E_0 [(1 - \Gamma (W (t - 2)))^2 W (t - 2)] \leq E_0 [(1 - \Gamma (W (t - 3)))^2 W (t - 2)].$$
Iterating this inequality, after \( k \) steps, with \( k \leq t \), we obtain

\[
\mathbb{E}_0 [W(t)] \leq \mathbb{E}_0 \left[ (1 - \Gamma (W(t - k)))^{k-1} W(t - k + 1) \right]
\]

which, by (67) implies

\[
\mathbb{E}_0 [W(t)] \leq \mathbb{E}_0 [W(1)] (1 - \Gamma (W(X^0)))^{t-1} \leq W(X^0) (1 - \Gamma (W(X^0)))^t .
\]

Therefore, for any \( X^0 \in \{ X \in \Xi : \Gamma (X) > 0 \} \), since \( W(X^0) \) and for any \( \varepsilon > 0 \) the Markov inequality implies

\[
\mathbb{P}_0 \{ W(t) > \varepsilon \} \leq \frac{\mathbb{E}_0 [W(t)]}{\varepsilon} \leq \varepsilon^{-1} (1 - \Gamma (W(X^0)))^t ,
\]

by the Borel-Cantelli Lemma \( \{ W(t) \}_{t \geq 0} \) converges to zero \( \mathbb{P}_0 \)-a.s., that is \( \mu^\infty := \lim_{t \to \infty} \mathbb{P}_0 \{ X(t) \in \cdot \} \) is supported on \( \mathcal{I} \).

2.4 Convergence to the stationary measure of \( \{ \chi_t \}_{t \in \mathbb{Z}^+} \) and \( \{ u_t \}_{t \in \mathbb{Z}^+} \)

We can rephrase (69) and therefore the content of Theorem 7 in terms of exponential (more precisely geometric since \( t \in \mathbb{Z}^+ \)) convergence to an element of the set of the invariant measures of the Markov chains defined by the transition operators \( \mathcal{I} \) and \( \mathcal{T} \). More precisely, for any \( \varepsilon > 0 \) and \( t > 0 \), given \( \chi^0 = (X^0, \omega^0) \in \{ X \in \Xi : \Gamma (X) > 0 \} \times \Omega \), by (36) \( \{ \|(I - M) \right)_t \| > \varepsilon \} \subseteq \{ W(t) > \varepsilon \} \). Hence, by Theorem 7, \( \{ u_t \}_{t \in \mathbb{Z}^+} \) converges to zero \( \mathbb{P}_0 \)-a.s. and, by (71), \( \{ u_t \}_{t \in \mathbb{Z}^+} \) converges to \( \overline{\mathbb{P}}_0 \)-a.s.. But, since, by (23), for any \( Y \in \mathcal{I}, \omega \in \Omega, \exists \varphi (Y, \omega) = \varphi (Y, 1), \{ u_t \}_{t \in \mathbb{Z}^+} \) converges \( \mathbb{P}_0 - a.s. \) to an element of \( \mathcal{I} \) which we denote by \( X^\infty \).

Given \( X \in \Xi \), let us set \( X = (U, V) \) such that \( U := M U, V := (I - M) X \) and consider the random processes \( \{ u_t \}_{t \in \mathbb{Z}^+} \) and \( \{ u_t \}_{t \in \mathbb{Z}^+} \) such that \( \forall t \geq 0, u_t := M u_t \) and \( u_t := (I - M) u_t \). From (22), (23) and (37), for any bounded measurable function \( \varphi \) on \( \Xi^2 \times \Omega \),

\[
\mathcal{I} \varphi (U, V, \omega) = \sum_{\omega' \in \Omega} \varphi (U + M T (V, \omega), (I - M) T (V, \omega), \omega') \Pi (\omega' | V) .
\]

Hence, \( \{ u_t \}_{t \in \mathbb{Z}^+} \), such that, \( \forall t \geq 0, u_t := (v_t, w_t) \), is an homogeneous Markov process.

Let us introduce on \( \Omega \) the metric

\[
\Omega \times \Omega \ni (\omega, \omega') \mapsto d(\omega, \omega') := \frac{1}{|E|} \sum_{e \in E} (1 - \delta_{\omega, \omega'}) \in [0, 1] .
\]

Lemma 12 From (43), for any \( (X^0, \omega^0) \in \Xi \) and \( t \geq 1 \), we have

\[
\mathbb{E} [d(v_t, 1) | (X^0, \omega^0)] \leq \mathbb{E} [(1 - p(W \circ u_{t-1})) | (X^0, \omega^0)] .
\]
Proof. For any $\omega \in \Omega$, we get
\[
d(\omega, \bar{1}) = \frac{1}{|E|} \sum_{e \in E} (1 - \delta_{\omega, 1}) = \frac{1}{|E|} \sum_{e \in E} (1 - \omega_e) .\tag{74}
\]
Hence, by the Markov property, from (23) and (22) we have
\[
E_0 [d(m_t, \bar{1})] = E_0 [E [d(m_t, \bar{1}) | (\xi_{t-1}, m_{t-1})]]
= E_0 \left[ \sum_{\omega' \in \Omega} \frac{1}{|E|} \sum_{e \in E} (1 - \omega'_e) \Pi (\omega'_t | \xi_{t-1}) \right]
= \frac{1}{|E|} \sum_{e \in E} E_0 \left[ \sum_{\omega' \in \Omega} (1 - \omega'_e) (1 - p (|\Delta_e \xi_{t-1}|)) \times (1 - \omega'_e) (1 - p (|\Delta_e m_{t-1}|)) \right]
= \frac{1}{|E|} \sum_{e \in E} E_0 \left[ (1 - p (|\Delta_e \xi_{t-1}|)) \right]
\leq \frac{1}{|E|} \sum_{e \in E} E_0 \left[ (1 - p (W(t - 1))) \right]
\leq E_0 \left[ (1 - p (W(t - 1))) \right].
\tag{75}
\]

Let $\Xi := \Xi \setminus \mathcal{I}$, Given a bounded measurable function $\varphi$ on $\Xi \times \Omega \subset \mathcal{X}$, let
\[
\|\nabla V \varphi\|_1 := \sup_{(V', \omega) \in \Xi \times \Omega} \sum_{V \in V} \left| \frac{\partial}{\partial V_v} \varphi (V', \omega) \right| ,
\tag{76}
\]
\[
\|\varphi\|_\Omega := \sup_{V \in \Xi} \sup_{\omega, \omega' \in \Omega : \omega \neq \omega'} \frac{|\varphi (V, \omega) - \varphi (V, \omega')|}{d(\omega, \omega')} ,
\tag{77}
\]
and consider the Banach space $L$ of measurable functions $\varphi$ on $\Xi \times \Omega$, with norm
\[
\|\varphi\|_L := \sup_{(V, \omega) \in \Xi \times \Omega} |\varphi (V, \omega)| + \|\nabla V \varphi\|_1 + \|\varphi\|_\Omega .
\tag{78}
\]
Since for any $X \in \Xi$,
\[
\|(I - \Pi) X\| = \sup_{v \in \mathbb{V}} |X_v - (\Pi X)_v| = \sup_{v \in \mathbb{V}} \left| X_v - \frac{1}{|\mathbb{V}|} \sum_{u \in \mathbb{V}} X_u \right| \quad (79)
\]

\[
\begin{align*}
&= \sup_{v \in \mathbb{V}} \left| \left(1 - \frac{1}{|\mathbb{V}|}\right) X_v - \frac{1}{|\mathbb{V}|} \sum_{u \in \mathbb{V} : u \neq v} X_u \right| \\
&= \sup_{v \in \mathbb{V}} \left| \frac{|\mathbb{V}| - 1}{|\mathbb{V}|} X_v - \frac{1}{|\mathbb{V}|} \sum_{u \in \mathbb{V} : u \neq v} X_u \right| \\
&= \sup_{v \in \mathbb{V}} \left| \frac{1}{|\mathbb{V}|} \sum_{u \in \mathbb{V} : u \neq v} (X_v - X_u) \right| \leq W(X),
\end{align*}
\]

then, for $\varphi \in \mathbb{L}$, by (79), we have
\[
|\varphi((I - \Pi) X, \omega) - \varphi(0, \omega)| = \left| \int_0^1 ds \sum_{v \in \mathbb{V}} \left( \frac{\partial}{\partial V_v} \varphi \right) (s (I - \Pi) X, \omega) ((I - \Pi) X)_v \right| \quad (80)
\]
\[
= \left| \int_0^1 ds \sum_{v \in \mathbb{V}} \left( \frac{\partial}{\partial V_v} \varphi \right) (s X - (\Pi X), \omega) \frac{1}{|\mathbb{V}|} \sum_{u \in \mathbb{V} : u \neq v} (X_v - X_u) \right| \leq \|\nabla V \varphi\|_1 W(X).
\]

**Proposition 13** Starting from an initial state $X^0 = (X^0, \omega^0) \in \{X \in \Xi : \Gamma(X) > 0\} \times \Omega \subset \mathcal{X}$, the Markov chain $\{X_t\}_{t \geq 0}$ weakly converges to the degenerate random vector $(X^\infty, \bar{1}) \in \mathcal{I} \times \Omega$, where $X^\infty$ is the $\mathbb{P}_0$-a.s. limit of the random process $\{u_t\}_{t \in \mathbb{Z}^+}$. Moreover, if $p$ is concave function and $\lim_{x \downarrow 0} \frac{1-p(x)}{x}$ exists, the rate of convergence is exponential.

**Proof.** Given $\varphi \in \mathbb{L}$, by (80) and (75) we have
\[
|\mathbb{E}_0[\varphi(v_t, w_t)] - \varphi(0, \bar{1})| \leq \mathbb{E}_0[|\varphi(v_t, w_t) - \varphi(0, \bar{1})|] \leq \mathbb{E}_0[|\varphi(v_t, w_t) - \varphi(0, \bar{1})|] + \mathbb{E}_0[|\varphi(0, \bar{1}) - \varphi(0, \bar{1})|] \leq \|\nabla \chi \varphi\|_1 \mathbb{E}_0[W(t)] + \|\varphi\|_\Omega \mathbb{E}_0[(1 - p(W(t - 1)))]
\]
\[
\leq \|\varphi\|_L (\mathbb{E}_0[W(t)] \vee \mathbb{E}_0[(1 - p(W(t - 1)))]
\]

which tends to zero in the limit $t \to \infty$ by Theorem [7]. Clearly, if $p$ is concave, $1 - p$ is convex, hence, by (69),
\[
\mathbb{E}_0[(1 - p(W(t - 1)))] \leq (1 - p(W(t - 1))) \leq 1 - p \left(W(X^0) \left(1 - \Gamma(X^0)\right)^{t-1}\right),
\]

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Therefore, if \( \lim_{x \to 0} \frac{1-p(x)}{x} \) is finite the rate of convergence is exponential. 

Similar conclusions hold for the Markov chain \( \{\eta_t\}_{t \geq 0} \). Indeed, by (31) and (37), setting \( X_1 = (U_1, V_1) \), \( X_2 = (U_2, V_2) \), for any bounded measurable \( \varphi : \Xi^4 \times \Omega \to \mathbb{R} \),

\[
(T\varphi)(U_1, V_1, U_2, V_2) := \sum_{\omega, \omega' \in \Omega} \varphi(U_2 + MT(V_2, \omega), (I - M)T(V_2, \omega), \omega', (I - M)T(V_2, \omega), \omega')) \times \\
\times \Pi(\omega | V_1) \Pi(\omega' | V_2)
\]

Hence, \( \{3_t\}_{t \geq 0} \) such that \( \forall t \geq 0, 3_t := (v_{2t}, v_{2t+1}) \) is an homogeneous Markov process.

Moreover, from (57), for any \( t \geq 0 \) we get

\[
\mathbb{E}_0[W \circ \tau_{t+1}3^X_{t+1}] = \mathbb{E}_0[\mathbb{E}_0[W \circ \tau_{t+1}3^X_{t+1}] | 3^X_t] = \mathbb{E}_0[(1 - \Gamma \circ \tau_{t-1}) W \circ \tau_t | 3^X_t] = \mathbb{E}_0[W \circ \tau_t | 3^X_t] (1 - \Gamma \circ \tau_{t-1}) \leq (1 - \Gamma \circ \tau_{t-1}) (1 - \Gamma \circ \tau_{t-2}) W \circ \tau_{t-1} \leq (1 - \Gamma \circ W \circ \tau_{t-1}) (1 - \Gamma \circ W \circ \tau_{t-2}) W \circ \tau_{t-1} \leq (1 - \Gamma \circ W \circ \tau_{t-2})^2 W \circ \tau_{t-1}.
\]

Hence, by (32), defining

\[
\Xi^2 \ni (X_1, X_2) \mapsto \tilde{\Gamma}(X_1, X_2) := (\Gamma \circ \pi_1)(X_1, X_2) \in (0, 1) \quad (85)
\]

\[
\Xi^2 \ni (X_1, X_2) \mapsto \tilde{W}(X_1, X_2) := (W \circ \pi_1)(X_1, X_2) \in [0, 1] \quad (86)
\]

from (84) we have

\[
\mathbb{E}_0[W \circ \eta_{t+1}3^\eta_t] \leq (1 - \Gamma \circ \eta_t)^2 W \circ \eta_t, \quad t \geq 0. \quad (87)
\]

Therefore, proceeding as in (67), by (87) we get

\[
\mathbb{E}_0[W \circ \eta_t] \leq \mathbb{E}_0[(1 - \tilde{\Gamma} \circ \eta_{t-1})^2 \tilde{W} \circ \eta_{t-1}] \leq \mathbb{E}_0[(1 - \tilde{\Gamma} \circ \tilde{W} \circ \eta_{t-1})^2 \tilde{W} \circ \eta_{t-1}] \leq \mathbb{E}_0[(1 - \tilde{\Gamma} \circ \tilde{W} \circ \eta_{t-2})^2 \tilde{W} \circ \eta_{t-1}].
\]

Thus, iterating,

\[
\mathbb{E}_0[T^t\tilde{W}] \leq (1 - \tilde{\Gamma} \circ \eta_0)^{2t} \tilde{W} \circ \eta_0. \quad (89)
\]

Let us denote by \( \mathcal{L} \) be the Banach space of bounded measurable functions \( \varphi \) on \( \Xi^2 \) with norm

\[
\|\varphi\|_{\mathcal{L}} := \sup_{(V_1, V_2) \in \Xi^2} |\varphi(V_1, V_2)| + \|\nabla \varphi\|_1, \quad (90)
\]

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where
\[
\|\nabla \varphi\|_1 := \sup_{(V', V'') \in \Xi} \sum_{v \in V} \left[ \left| \frac{\partial}{\partial (V_1)} \varphi (V', V'') \right| + \left| \frac{\partial}{\partial (V_2)} \varphi (V', V'') \right| \right].
\]

**Corollary 14** The Markov chain \( \{ \eta_t \}_{t \geq 0} \) started at \((X^0, X^1) \in \{ X \in \Xi : \Gamma (X) > 0 \} \times \Xi \) converges weakly to the degenerate random vector \((X^\infty, X^\infty) \in \Xi^2 \) with exponential rate.

**Proof.** Given \( \varphi \in \mathcal{L} \), proceeding as in (80), by (58) and (89), we have
\[
|E_0 [\varphi (3t)] - \varphi (0, 0)| \leq E_0 \| \varphi (3t) - \varphi (0, 0) \| \leq \| \varphi \|_L E_0 [W \circ \eta_t].
\]

\( \blacksquare \)

## 3 Large system evolution

Given \( N \in \mathbb{N} \), let \( \mathbf{V}_N := \{1, \ldots, N\} \subset \mathbb{N} \) and denote by \( \mathbf{E}_N \) the subset of \( \mathbf{E} := \{(u, v) \in \mathbb{N} \times \mathbb{N}\} \) such that \( \mathbf{E}_N := \{(u, v) \in \mathbf{V}_N \times \mathbf{V}_N\} \).

In this section we set \( \Xi := [0, 1]^N \) and, denoting by \( X := \{X_1, \ldots, X_N\} \) the element of \( \Xi_N := [0, 1]^N \) representing the restriction of the beliefs configuration \( X \in \Xi \) to \( \mathbf{V}_N \), we endow \( \Xi \) with the metric induced by the norm \( \|X\| := \sum_{N \in \mathbb{N}} 2^{-N} \|X_N\|_\infty \).

Setting \( \Omega := \{0, 1\}^\mathbf{E} \), we denote by \( \omega_N \) the restriction of the configuration \( \omega \in \Omega \) to \( \Omega_N := \{0, 1\}^{\mathbf{E}_N} \) and, by (90), if \( E := E (\omega) \), we set \( E_N := E \cap \mathbf{E}_N \).

Then, for any \( X \in \Xi, \bar{\Pi} (\cdot|X) \) denotes the random field on \((\Omega, \mathcal{C} (\Omega))\) such that, for any cylinder event \( C_N (\omega') = \{\omega \in \Omega : \omega_N = \omega'\} \), \( N \geq 1, \omega' \in \Omega_N \),
\[
\sum_{\omega \in \Omega} \bar{\Pi} (\omega|X) 1_{C_N (\omega')} (\omega) = \bar{\Pi} (\omega'|X_N)
\]
where \( \bar{\Pi} (\omega'|X_N) \) is given by (22). Moreover, for any \( M \geq N \),
\[
\sum_{\omega \in \Omega_M} \bar{\Pi} (\omega|X_M) 1_{\{\omega \in \Omega_M : \omega_N = \omega'\}} (\omega) = \sum_{\omega \in \Omega} \bar{\Pi} (\omega|X) 1_{C_N (\omega')} (\omega) = \bar{\Pi} (\omega'|X_N)
\]

Setting \( \mathbf{V} := \mathbb{N} \) for notational convenience, let \( \mathcal{K} (\mathbf{V}) \) be the set of the transition probability kernels on \((\mathbf{V}, \mathcal{P} (\mathbf{V}))\). From (111), given the \( \mathcal{C} (\Omega)\)-measurable function \( \Omega \ni \omega \mapsto P (\omega) \in \mathcal{K} (\mathbf{V}) \) such that \( \forall \omega \in \Omega, \)
\[
P_{v,u} (\omega) = \frac{\sum_{e \in E \cap E (\omega)} r_{\delta_e (u,v)}}{\sum_{e \in E \cap E (\omega)} r_e} = p_{v,u} (E (\omega)) \in \{0, 1\}; \ v, u \in \mathbf{V},
\]

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we denote as in (21) and (20) $X \ni (X, \omega) \mapsto \mathcal{T}_v (X, \omega) \in \Xi$ such that, for any $v \in V$ and any $(X, \omega) \in X, \mathcal{T}_v (X, \omega) : = \sum_{u \in V} p_{v,u} (E(\omega)) X_u$. Then, the operator

$$BM (X) \ni \varphi \mapsto \mathcal{T}_v (X, \omega) : = \sum_{\omega' \in \Omega} \varphi (\mathcal{T} (X, \omega), \omega') \Pi (\omega' | X) \in BM (X)$$

(96)

represents the transition probability kernel of the homogeneous Markov chain $\{X_t\}_{t \geq 0}$ on $(\mathcal{X}^\mathbb{Z}_{\geq}, \mathcal{F}, \mathbb{P}_0)$ with initial condition $(X^0, \omega^0) \in X$ such that, by (23), for any $\omega' \in \Omega_N, B \in \mathcal{B} (\{0, 1\}^N)$,

$$\mathbb{P}_0 (\{X_{t+1} \in \mathcal{C}_N (B) \times \mathcal{C}_N (\omega') \} | \chi_t) = (\mathcal{T}_1 \mathcal{C}_N (B) \times \mathcal{C}_N (\omega')) (\chi_t)
$$

(97)

$$= \Pi (\omega' | X_N (t)) 1_B (\mathcal{T}_N (X (t), \omega (t))) .$$

Consequently, the operator

$$\mathcal{L} (\Xi^2) \ni \varphi \mapsto (\mathcal{T}_v \varphi) (X, Y) : = \sum_{\omega, \omega' \in \Omega} \varphi (\mathcal{T} (Y, \omega), \mathcal{T} (\mathcal{T} (Y, \omega), \omega')) \Pi (\omega | X) \Pi (\omega' | Y) \in \mathcal{L} (\Xi^2)$$

(98)

defined as in (31), represents the transition probability kernel of the homogeneous Markov chain $\{\eta_t\}_{t \geq 0}$ on $(\mathcal{X}^\mathbb{Z}_{\geq}, \mathcal{F}, \mathbb{P}_0)$ such that, for any $B \in \mathcal{B} (\{0, 1\}^N)$,

$$\mathbb{P}_0 (\{\eta_{t+1} \in \mathcal{C}_N (B) \} \cup \eta_t) = (\mathcal{T}_1 \mathcal{C}_N (B)) (\eta_t) = \sum_{\omega, \omega' \in \Omega} 1_B (\mathcal{T}_N (X (2t+1), \omega), \mathcal{T}_N (\mathcal{T} (X (2t+1), \omega), \omega')) \Pi (\omega | X (2t)) \Pi (\omega' | X (2t+1)) = \sum_{\omega, \omega' \in \Omega} 1_B (\mathcal{T}_N (X (2t+1), \omega), \mathcal{T}_N (\mathcal{T} (X (2t+1), \omega), \omega')) \Pi (\omega | X (2t)) \Pi (\omega' | X (2t+1)) = \sum_{\omega, \omega' \in \Omega} 1_B (\mathcal{T} (X_N (2t+1), \omega), \mathcal{T} (\mathcal{T} (X_N (2t+1), \omega), \omega')) \Pi (\omega | X (2t)) \Pi (\omega' | X (2t+1)) = \sum_{\omega, \omega' \in \Omega} 1_B (\mathcal{T} (X_N (2t+1), \omega), \mathcal{T} (\mathcal{T} (X_N (2t+1), \omega), \omega')) \Pi (\omega | X (2t)) \Pi (\omega' | X (2t+1)) .$$

(99)

Remark 15 Notice that if the cardinality of the set $R : = \{(u, v) \in V \times V : r_{u,v} > 0\}$ is finite, there exists $M > N$ such that

$$\mathbb{P}_0 (\{X_{t+1} \in \mathcal{C}_N (B) \times \mathcal{C}_N (\omega') \} | \chi_t) = \Pi (\omega' | X_N (t)) 1_B (\mathcal{T} (X_N (t), \omega_M (t)))$$

(100)
\[ P_0 \left( \{ n_{t+1} \in C_N(B) \} \mid n_t \right) = \sum_{\omega_M, \omega'_M \in \Omega_M} 1_B \left( \mathcal{T} (X_N(2t+1), \omega), \mathcal{T} (\mathcal{T} (X_N(2t+1), \omega), \omega') \right) \times \Pi (\omega_M | X_M(2t)) \Pi (\omega'_M | X_M(2t+1)) = \left( T_1 \left\{ (X^{(1)}, X^{(2)}) \in \Xi_M : (X^{(1)}_N, X^{(2)}_N) \in B \right\} \right) (n_t). \]

In the following we make this assumption.

**Proposition 16** Let the initial datum \( X^0 \in \Xi \) be such that \( \alpha := \inf_{N \in \mathbb{N}} \Gamma (X^0_N) > 0 \). Then, for any \( \varphi \in C(\Xi) \) and any \( X^1 \in \Xi \),
\[
\lim_{t \to \infty} \left| E \left[ \varphi \circ Z_t | Z_0 = (X^0, X^1) \right] - \varphi (0,0) \right| = 0.
\]

**Proof.** For any \( \varphi \in C(\Xi) \) there exists a sequence \( \{ \varphi_N \}_{N \in \mathbb{N}} \) such that, \( \forall N \geq 1, \varphi_N \) is a continuous \( \mathcal{B}(\Xi_N) \) measurable function uniformly convergent to \( \varphi \). Hence, given \( \varepsilon > 0 \), there exists \( N_\varepsilon \geq 1 \) such that for any \( N > N_\varepsilon, \| \varphi - \varphi_N \|_\infty < \varepsilon \). Moreover, denoting by \( I_N := \bigcup_{x \in [0,1]} \{ X_N \in \Xi_N : (X_N)_v = x, \forall v \in V_N \} \) and \( \Xi_N := \Xi \setminus I_N \), by the Stone-Weierstass theorem there exists \( \phi \in \mathcal{L}_N \) (with \( \mathcal{L}_N \) defined as \( \mathcal{L} \) in the previous section) such that \( \| \phi_N - \varphi_N \|_\infty < \varepsilon \). Then, the thesis follows from Corollary 14. Indeed,
\[
\left| E \left[ \varphi \circ n_t | n_0 = (X^0, X^1) \right] - \varphi (0,0) \right| \leq 2\varepsilon + \left| E \left[ \varphi_N \circ n_t | n_0 = (X^0, X^1) \right] - \varphi_N (0,0) \right|
\]
\[
\leq 4\varepsilon + \left| E \left[ \phi_N \circ n_t | n_0 = (X^0, X^1) \right] - \phi_N (0,0) \right|
\]
\[
= 4\varepsilon + \left| d(X^0_M, X^1_M)^T (\phi_N - \phi_N (0,0)) \right|,
\]
where \( M \geq N \). But,
\[
d(X^0_M, X^1_M)^T | \phi_N - \phi_N (0,0)| \leq \| \phi_N \|_{\mathcal{L}_N} (1 - \alpha)^2t.
\]

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