GENERALIZED $\zeta$-PRODUCT FOR HIGHT ORDER TENSORS WITH APPLICATIONS USING GPU COMPUTATIONS.

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Abstract. In this paper, we will present a generalization of the $\zeta$-tensor product ($\zeta$-product) including generalization of the well known tensor cosine and $T$-products that were defined for third-order tensors and based on fast Fourier transform and discrete cosine transform (DCT). We will give some applications on tensor completion. To solve some optimization problems linked with the problem of tensor completion, we will use the Proximal Gradient Algorithm (PGA) to solve some derived optimization problems. Numerical tests are given to show the effectiveness of the proposed methods and also present some tests using GPU computation.

Key words. Tensor completion, tensor nuclear norm, tensor $\zeta$-product, tensor $\zeta$-svd, GPU.

1. Introduction. In the last decade, tensors become an important multilinear algebra tool involved in many modern problems such completion [9, 18, 24], principal component analysis [13], image processing [20, 12, 4] and others. The classical $n$-mode product leads to many concepts and developements when working with multidimensional data. The CP and the Tucker compressions were introduced as natural generalization of the classical singular value decomposition (SVD) for matrices; see [15, 4, 12, 13, 24]. In the last years, new tensor-tensor products such as cosine-product ($c$-product), using discrete cosine or $T$-product, using Fast Fourier Transform (FFT), were introduced for third-order tensors, studied and applied to image processing and other fields; see [19, 1, 25, 13, 20]. In the present paper, we generalize those tensor-tensor products for high-order tensors. Using those new products, we will propose new completion models. We give some theoretical results and some numerical examples in color video processing.

The outline of this paper will as follows In Section 2 we will give some definitions and remind some known results of the third-order tensor-tensor product based on Fast Fourier Transform and cosine transform. In Section 3 we present our new generalized $c$-product for any order of tensors and give some important results. Section 4 presents a novel models of tensor completion for tensors of any order by using the PGA. Section 5 will be devoted to some numerical experiments with someexperiments using GPU.

2. Definitions and notations. In this subsection we will define some notions that will help us in the paper. We denote tensors by Euler script letters, e.g., $X$, matrices will be denoted by boldface capital letters, e.g., $X$, vectors by boldface lowercase letters, e.g., $x$ and scalars by lowercase letters, e.g., $x$. Also we will denote the $(i_1, i_2, \ldots, i_K)^{th}$ element of a $K$-th order tensor $X$ by $X_{i_1, i_2, \ldots, i_K}$. Also we will denote $\mathbb{C}^{I_1 \times I_2 \times \ldots \times I_K}$ by $\mathbb{C}^{I_1 I_2 \cdots I_K}$, and the $K^{th}$-order tensor-scalar space as $\mathbb{K}^{I_1 \times I_2 \times \ldots \times I_K} = \mathbb{C}^{1 \times I_2 \times \ldots \times I_K}$.

Let $A$ and $B$ be two $K^{th}$-order tensors in $\mathbb{K}^{I_1 I_2 \cdots I_K}$, we will define the inner product between $A$ and $B$ by

$$\langle A, B \rangle = \sum_{i_1, i_2, \ldots, i_K=1}^{I_1, I_2, \ldots, I_K} A_{i_1, i_2, \ldots, i_K} B_{i_1, i_2, \ldots, i_K}. \quad (2.1)$$

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The associated norm is defined by
\[
|\mathcal{A}|_F^2 = \sum_{i_1, i_2, \ldots, i_K=1}^{I_1, I_2, \ldots, I_K} A_{i_1, i_2, \ldots, i_K}^2.
\] (2.2)

The notion of columns and rows of matrices are extended to the case of tensors, where in this case we talk about n-mode fiber instead of columns and rows, with the n-mode fiber of a Kth-order tensor \(\mathcal{X}\) is defined by fixing all the indexes except the nth one.

There is some ways to transform a tensor to a matrix which consist to make the operations on tensors easier, there is for example the n-mode matricization [4] [24] defined as follows:

**Definition 2.1.** Let \(\mathcal{X} \in \mathbb{K}^{I_1 \times I_2 \times \cdots \times I_N}\), then the n-mode matricization of \(\mathcal{X}\) denoted by \(X(n) \in \mathbb{K}^{I_n \times \prod_{k=1, k \neq n}^{I_k}}\) and it is defined by making the n-mode fibers as columns of \(X(n)\), i.e., the \((i_1, i_2, \ldots, i_N)^{th}\) element of \(\mathcal{X}\) maps to a matrix element \((i_n, J)\) satisfying
\[
j = 1 + \sum_{k=1, k \neq n}^{N} (i_k - 1)J_k \text{ for } J_k = \prod_{m=1, m \neq n}^{N} I_m.
\]

For third-order tensors \(\mathcal{X} \in \mathbb{K}^{I_1 \times I_2}\), \(X(1)\), \(X(2)\) and \(X(3)\) are given by
\[
X(1) = [X^{(1)}, X^{(2)}, \ldots, X^{(I_1)}],
\]
\[
X(2) = [X^{(1)}T, (X^{(2)})T, \ldots, (X^{(I_3)})T],
\]
\[
X(3) = [\text{sq}(X(1, 1))^T, \text{sq}(X(1, 2))^T, \ldots, \text{sq}(X(1, I_2))^T],
\]
where \(X^{(i)}\) denotes the i\(^{th}\) frontal slice \((X^{(i)} = X(:, i))\) and \(\text{sq}\) transforms the tensor \(\mathcal{X} \in \mathbb{K}^{I_1 \times 1}\) to a matrix \(X \in \mathbb{K}^{I_1 \times I_1}\), i.e., \(X = \text{sq}(\mathcal{X})\).

The n-mode product, which is a product between a tensor and a matrix in the n-mode [4] is defined in the following definition:

**Definition 2.2.** Let \(\mathcal{X} \in \mathbb{K}^{I_1 \times I_2} \times \mathbb{K}^{J \times I_1}\) and \(U \in \mathbb{K}^{J \times I_1}\) where \(J\) is a positive nonzero integer. Then the n-mode product \(\mathcal{X} \times_n U\) is the tensor in \(\mathbb{K}^{I_1 \times I_2 \times \cdots \times I_N \times J \times I_{n+1} \times \cdots \times I_K}\), where its \((i_1, \ldots, i_{n-1}, j, i_{n+1}, \ldots, i_K)^{th}\) element is defined by
\[
(\mathcal{X} \times_n U)_{i_1, \ldots, i_{n-1}, j, i_{n+1}, \ldots, i_K} = \sum_{i_n=1}^{I_n} \mathcal{X}_{i_1, i_2, \ldots, i_K} U_{j, i_n}.
\] (2.3)

Some useful properties of the n-mode product are given as follows. Let the tensor \(\mathcal{X}\), and the matrices \(U\) and \(V\) of appropriate sizes, then
\[
\mathcal{Y} = \mathcal{X} \times_n U \iff Y_{(n)} = U X_{(n)}, \text{ and}
\]
\[
\mathcal{X} \times_n U \times_m V = \mathcal{X} \times_m V \times_n U.
\]

We will also use the notion of tensor face-wise product defined next:

**Definition 2.3.** Let \(\mathcal{X} \in \mathbb{K}^{I_1 \times I_2}\) and \(\mathcal{Y} \in \mathbb{K}^{I_1 \times I_2}\) two third-order tensors, then the face-wise product between \(\mathcal{X}\) and \(\mathcal{Y}\) is given by the tensor of size \(I_1 \times I_2 \times I_3\) where its \(i^{th}\) frontal slice is given from the product between the \(i^{th}\) frontal slices of \(\mathcal{X}\) and \(\mathcal{Y}\), i.e.,
\[
(\mathcal{X} \triangle \mathcal{Y})^{(i)} = \mathcal{X}^{(i)} \triangle \mathcal{Y}^{(i)}.
\] (2.4)
Classical tensor decompositions such as CP decomposition [4], Tucker decomposition [24], block term decomposition [16] give nice results in many tensor applications. However, those decompositions suffer from the high computational cost for large problems. In the recent years new tensor decompositions of the third-order case and based on tensor-tensor product using the Fourier domain such as the t-product [12] and cosine-product (c-product) [2], have been defined and used for many image processing applications; see [1, 3, 9, 13].

In this section we will try to remind the most important results of those types of tensor-tensor product.

The main idea of this type of tensor products is to transform the tensors to another domain which is called the transform domain, like Fourier domain, cosine domain. Then all the operations are done in the transformed domain using for example FFT on each tube to speed-up the executing time. This kind of transformation could be defined in the following.

**Definition 2.4.** Let $M$ be an invertible matrix of size $I_3 \times I_3$, we define the operator $\mathcal{L}$ as

$$\mathcal{L} : \mathbb{K}^{I_1 \times I_2} \rightarrow \mathbb{K}^{I_3 \times I_2}$$

$$\mathcal{A} \rightarrow \mathcal{A} \times M$$

and its inverse is defined as

$$\mathcal{L}^{-1} : \mathbb{K}^{I_1 \times I_2} \rightarrow \mathbb{K}^{I_3 \times I_2}$$

$$\mathcal{A} \rightarrow \mathcal{A} \times M^{-1}$$

Now we can define the tensor-tensor product of two third-order tensors.

**Definition 2.5.** Let $\mathcal{L}$ be an invertible operator, then the tensor-tensor product between two third-order tensors associated with the operator $\mathcal{L}$, is denoted by $\ast_{\mathcal{L}}$ and is given by

$$A \ast_{\mathcal{L}} B = \mathcal{L}^{-1} (\mathcal{L}(A) \triangle \mathcal{L}(B)).$$

(2.5)

where the tensors $\mathcal{A}$ in $\mathbb{K}^{I_1 \times I}$ and $B$ in $\mathbb{K}^{I \times I_2}$.

The matrix $M$ in Definition 2.4 depends on the type of the product, for example if we use the t-product [12], the matrix $M$ is the matrix of discrete Fourier transform $F_{I_3}$ where the Fourier matrix $F_n \in \mathbb{C}^{n \times n}$ is given by

$$F_n = [\omega_n^{(i-1)(j-1)}] ; i,j = 1, \ldots, n-1; \text{ and } \omega_n = e^{-\frac{2\pi i}{n}}.$$  

(2.6)

for $n \in \mathbb{N}^*$. Notice that $\frac{F_n}{\sqrt{n}}$ is unitary, i.e., $F_n F_n^* = nI_n$.

In the case of c-product [2], the matrix $M$ is defined as

$$M = W_{I_3}^{-1} C_{I_3} (I_{I_3} + Z_{I_3}),$$

(2.7)

where $W_{I_3} = \text{diag}(C_{I_3}(:,1))$, the matrix $Z_{I_3}$ is the circulant upshift matrix defined by $Z_{I_3} = \text{diag(ones}(I_3 - 1,1,1))$ and $C_{I_3}$ is the matrix of discrete cosine transform of size $I_3 \times I_3$ and its $(i,j)$th element is defined as

$$ (C_{I_3})_{i,j} = \sqrt{\frac{2-\delta_{i,j}}{I_3}} \cos \left( \frac{(i-1)(2j-1)\pi}{2I_3} \right) \text{ with } 1 \leq i,j \leq I_3,$$

(2.8)

where $\delta_{i,j}$ is the Kronecker symbol. Notice that the matrix $C_n$ is orthogonal for all $n \in \mathbb{N}^*$. We have also to mention that, in this case also, the matrix $M$ is invertible and $M_{I_3}^{-1} = \cdot \cdot \cdot $
\((I_k + Z_k)^{-1} C_k^T W_k\). Using those tensor-tensor products, all the classical matrix decomposition, such as svd, QR and Shur decompositions have been generalized to the tensor case; see \cite{2, 7}. Many applications of tensor-tensor product use some optimization algorithms and in our present work, we will use the Proximal Gradient Algorithm \cite{10} in tensor completion. The method consists in solving the optimization problem

\[
\min_{X \in H} g(X) \quad \text{s.t.} \quad A(X) = B,
\]

where \(H\) is an Hilbert space equipped with a norm \(\|\cdot\|\), \(g\) is a continuous function, \(A\) is a linear map and \(B\) is an observation. By referring to \cite{5, 14}, this optimization problem can be solved by solving the following one

\[
\min_{X \in H} \{F(X) = \mu g(X) + f(X)\},
\]

where \(f(X) = \frac{1}{2} \|A(X) - B\|^2\) and \(\mu > 0\) is the relaxation parameter. The penalty function \(f\) is convex and smooth with Lipshitz constant \(l_f\). To solve (2.10), we minimize the quadratic function \(Q(X, Y)\), where \(Y\) is chosen and \(Q\) is defined as follows

\[
Q(X, Y) = \mu g(X) + f(Y) + \langle \nabla f(Y), X - Y \rangle + \frac{l_f}{2} \|X - Y\|^2.
\]

Getting a solution \(X\) satisfying (2.11) is equivalent to solve the following minimization problem

\[
\min_{X \in H} Q(X, Y) = \min_{X \in H} \mu g(X) + \frac{l_f}{2} \|X - G\|^2,
\]

where \(G = Y - \frac{1}{l_f} \nabla f(Y)\). The problem is solved iteratively by computing \(X^{p+1}\) such that

\[
X^{p+1} = \arg \min_{X \in H} Q(X, Y^p).
\]

In \cite{14}, \(Y^p\) was computed by \(Y^p = X^p + \frac{t_{p+1}^{-1} - 1}{t_p} (X^p - X^{p-1})\) instead of \(Y^p = X^p\) for computationally reasons and \(t_{p+1} = \frac{1 + \sqrt{4t_p^2 + 1}}{2}\). The steeps of this algorithm can be summarized in the following algorithm

\textbf{Algorithm 1} Proximal Gradient Algorithm (PGA).

1: while not converged do
2: \(Y^p = X^p + \frac{t_{p+1}^{-1} - 1}{t_p} (X^p - X^{p-1})\).
3: \(G^p = Y^p - \frac{1}{l_f} \nabla f(Y^p)\).
4: \(X^{p+1} = \arg \min_{X \in H} \mu g(X) + \frac{l_f}{2} \|X - G^p\|^2\).
5: \(t_{p+1} = \frac{1 + \sqrt{4t_p^2 + 1}}{2}\).
6: \(p = p + 1\).
7: end while
3. Generalized tensor-tensor cosine product. The main inconvenient of tensor-tensor products above is the fact that they could be used only for third-order tensors. In [27], the authors proposed a generalization of the t-product and in the present work we propose the generalization of the c-product with some applications. We will first recall some important results linked with the c-product for third-order tensors described in [2]. For two third-order tensors $A \in \mathbb{K}_{I_3 \times I_3}$ and $B \in \mathbb{K}_{I_3 \times I_3}$, the c-product $A \times_c B$ is defined by

$$A \times_c B = \text{ten}(\text{btph}(A) \text{btph}(B)) \in \mathbb{K}_{I_3 \times I_2},$$  \hspace{1cm} (3.1)

where $\text{btph}$ represents the block-Toeplitz-plus-Hankel matrix defined as

$$\text{btph}(A) = \begin{pmatrix} A^{(1)} & \cdots & A^{(I_3)} \\ A^{(2)} & \cdots & A^{(I_3-1)} \\ \vdots & \ddots & \vdots \\ A^{(I_3)} & \cdots & A^{(1)} \end{pmatrix} + \begin{pmatrix} A^{(I_3)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A^{(I_3)} \end{pmatrix} \in \mathbb{K}_{I_1 I_3 \times I_2 I_3} \hspace{1cm} (3.2)$$

and the operator $\text{ten}$ is the operator to get back a tensor from its $\text{btph}$, where $\text{ten}(\text{btph}(A)) = A$. Computing this product by the above formula can be expensive, because the matrix $\text{btph}$ may be very large. Some properties of $\text{btph}$ are given in [2]. Among them, the fact that for a tensor $A \in \mathbb{K}_{I_3 \times I_2}$, the matrix $\text{btph}$ is block diagonalizable by $(C_{I_3} \otimes I_{I_2})$ and we have

$$(C_{I_3} \otimes I_{I_2}) \text{btph}(A) (C_{I_3}^* \otimes I_{I_2}) = \text{bdiag}(\hat{A}),$$ \hspace{1cm} (3.3)

where $\hat{A} = A \times_3 M$ with $M$ is defined by $\text{btph}$ and

$$\text{bdiag}(\hat{A}) = \begin{pmatrix} \hat{A}^{(1)} \\ \hat{A}^{(2)} \\ \vdots \\ \hat{A}^{(I_3)} \end{pmatrix} \in \mathbb{K}_{I_1 I_3 \times I_2 I_3}. \hspace{1cm} (3.4)$$

From this last result, we can define the c-product between two third-order tensors $A$ and $B$ of appropriate sizes with $I_3$ frontal slices. Before giving the generalized version of the high-order c-product, we give some definitions and notations. First, we will call a tensor in the scalar space ($\mathbb{K}_{I_3 \times \cdots \times I_N}$) a scalar-tensor, which will replace the notion of tubes in the case of third-order tensors. For an $N^{th}$-order tensor $\mathcal{X} \in \mathbb{K}_{I_3 \times \cdots \times I_N}$, we will define the operator $\text{Vec}$ that transforms the tensor into a matrix as follows

$$\text{Vec}(\mathcal{X}) = [X^1, X^2, \ldots, X^{n_3}, X^{n_3+1}, \ldots, X^P] \hspace{1cm} (3.5)$$

where $P = I_3 I_4 \ldots I_N$ and $X^P = \mathcal{X}(i_3, k_3, k_4, \ldots, k_N) \in \mathbb{K}_{I_1 \times I_2}$ with $p = k_3 + \sum_{i=1}^{N-1} \frac{(k_i - 1) P}{\prod_{s=3}^{N} I_s}$. This generalizes the notion of frontal slices in the case of third-order tensors. Notice that in the case of scalar-tensors $\mathcal{X} \in \mathbb{K}_{I_3 \times \cdots \times I_N}$, the $X^P$'s are scalars and $\text{Vec}(\mathcal{X})$ is a row vector. In [2], the authors defined the block-Toeplitz-plus-Hankel matrix for third-order tensor and here we will define the block-Toeplitz-plus-Hankel matrix for an $N^{th}$-order tensor for $N \geq 3$ by using the block Toepplitz plus Hankel for an $(N-1)^{th}$-order tensor. To explain this we will present this procedure only for a fourth-order tensor $\mathcal{X} \in \mathbb{K}_{I_3 \times I_2}$. We define the matrix $\text{btph} \in \mathbb{R}^{n_1 P \times n_2 P}$ as the matrix block-Toeplitz-plus-Hankel where each block of the matrices
Toepplitz and Hankel is a block-Toeplitz-plus-Hankel matrix of third-order tensors, respectively. Then the block-Toeplitz-plus-Hankel matrix of $\mathcal{X}$ is defined as follows

$$\text{btph}(\mathcal{X}) = \begin{pmatrix} \text{btph}(\mathcal{X}(:,:,1)) & \text{btph}(\mathcal{X}(:,:,2)) & \ldots & \text{btph}(\mathcal{X}(:,:,I_3)) \\ \text{btph}(\mathcal{X}(:,:,2)) & \text{btph}(\mathcal{X}(:,:,1)) & \ldots & \text{btph}(\mathcal{X}(:,:,I_3 - 1)) \\ \vdots & \vdots & \ddots & \vdots \\ \text{btph}(\mathcal{X}(:,:,I_3)) & \text{btph}(\mathcal{X}(:,:,I_3 - 1)) & \ldots & \text{btph}(\mathcal{X}(:,:,1)) \\ \end{pmatrix}$$

$\text{ten}(\text{btph}(\mathcal{X})) = \mathcal{X}$, this operator allows to reconstruct the original tensor from its associate btph matrix. The block diagonal matrix of the tensor $\mathcal{X}$ is given by

$$\text{bdiag}(\mathcal{X}) = \begin{pmatrix} X^1 & 0 & \ldots & 0 \\ 0 & X^2 & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & X^P \\ \end{pmatrix} \in \mathbb{R}^{I_3 \times \cdots \times I_3 \times P}, \quad (3.6)$$

where the $X^i$'s matrices of size $I_1 \times I_2$ are given in (3.5) as the representative matrices of $\mathcal{X}$. Notice that in the case of scalar-tensors, the blocks $X^i$ are scalars in $\mathbb{C}$ and in this case, the block diagonal matrix $\text{bdiag}(\mathcal{X})$ is just a diagonal matrix and $\text{bdiag}(\mathcal{X}) = \text{diag}(\mathcal{X})$.

### 3.1. Generalized cosine product c-product

In this subsection, we will introduce a generalized version of the c-product for high-order tensors. To this end, we first need some theoretical results. First, remind that in the third-order case, the Toeplitz-plus-Hankel matrix of a tube $a \in \mathbb{K}_{I_3}$ is diagonalizable using the DCT matrix of order $I_3$, i.e.,

$$C_{I_3} \text{tph}(a)C_{I_3}^* = \text{diag}(d); \quad \text{where } d = W_{I_3}^{-1}C_{I_3}(I + Z_{I_3}) \text{vec}(a), \quad (3.7)$$

with $W_{I_3}$ and $Z_{I_3}$ as defined earlier and vec$(a)$ is the vector of size $I_3$ whose elements are the coefficients of the tube $a$. This result is extended to the high-order case by considering scalar-tensors instead of tubes.

**Theorem 3.1.** Let $\mathcal{X} \in \mathbb{K}_{I_{3\times\cdots\times I_N}}$, a scalar-tensor, then its block-Toeplitz-plus-Hankel matrix is diagonalizable, and we have

$$(C_{I_N} \otimes C_{I_{N-1}} \otimes \cdots \otimes C_{I_3}) \text{btph}(\mathcal{X}) (C_{I_N}^* \otimes C_{I_{N-1}}^* \otimes \cdots \otimes C_{I_3}^*) = \text{diag}(\mathcal{X}) \quad (3.8)$$

where $\mathcal{X} = \mathcal{X} \times_3 M_{I_3} \times_4 M_{I_4} \times_5 \cdots \times_N M_{I_N}$ and $C_{I_N}$ are DCT matrices.

**Proof.** For simplicity we consider only the case of fourth-order scalar-tensors $\mathcal{X}$ in $\mathbb{K}_{I_3 \times I_4}^{1 \times 1}$. Using the fact that the matrix block-Toeplitz-plus-Hankel for a third-order tensor is block-diagonalizable using the discrete cosine matrix, we get

$$(I_{I_3} \otimes C_{I_3}) \text{btph}(\mathcal{X}) (I_{I_3} \otimes C_{I_3}^*) = \text{btph}(\mathcal{X}^c), \quad (3.9)$$
Next, we define the new operator
\[
\text{btph}(\mathcal{X}') = \begin{pmatrix}
\text{diag}(\mathcal{X}'_1) & \cdots & \text{diag}(\mathcal{X}'_{i-1}) \\
\text{diag}(\mathcal{X}'_2) & \text{diag}(\mathcal{X}'_1) & \cdots & \text{diag}(\mathcal{X}'_{i-1}) \\
\vdots & \ddots & \ddots & \vdots \\
\text{diag}(\mathcal{X}'_i) & \cdots & \text{diag}(\mathcal{X}'_{i-1}) & \text{diag}(\mathcal{X}'_1)
\end{pmatrix}
\]
\[+
\begin{pmatrix}
\text{diag}(\mathcal{X}'_2) & \text{diag}(\mathcal{X}'_3) & \cdots & \text{diag}(\mathcal{X}'_i) & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \vdots & \ddots & \text{diag}(\mathcal{X}'_{i-1}) & \text{diag}(\mathcal{X}'_1)
\end{pmatrix}
\]
where \(\mathcal{X}'_i = \mathcal{X}'(i;:,:)\) for \(i = 1, 2, \ldots, I_4\). Using a similar result as (3.3), we get
\[
(C_{I_4} \otimes I_{I_3}) \text{btph}(\mathcal{X}') (C_{I_4}^* \otimes I_{I_3}) = \text{diag}(d) \in \mathbb{K}^{I_4 \times I_3 \times I_4}
\] (3.10)

Then, from (3.9) and (3.10) we obtain
\[
(C_{I_4} \otimes I_{I_3}) (I_{I_4} \otimes C_{I_3}) \text{btph}(\mathcal{X}) (I_{I_4} \otimes C_{I_3}^*) (C_{I_4}^* \otimes I_{I_3}) = \text{diag}(d).
\] (3.11)

Therefore, we can deduce that \(\text{diag}(d) = \text{bdiag}(\hat{\mathcal{X}}) = \text{diag}(\hat{\mathcal{X}})\), with \(\hat{\mathcal{X}} = \mathcal{X}'_{1 \times 3} M_{I_5} \times M_{I_4}\) is a scalar-tensor, with \(M_{I_i} = W_{I_i}^{-1} C_{I_i} (I_i + Z_{I_i})\) for \(i = 3, 4\). Then, from (3.11), we obtain
\[
(C_{I_4} \otimes C_{I_3}) \text{btph}(\mathcal{X}) (C_{I_4}^* \otimes C_{I_3}^*) = \text{bdiag}(\hat{\mathcal{X}}).
\]

The result for fifth-order tensor could be obtained from the one for the fourth-order case and so on, the result for an \(N\)th-order scalar-tensor, can be found recursively from the one for an \((N - 1)\)th-order scalar-tensor. This is due to the fact that the \text{btph} matrix of an \(N\)th-order tensor is computed by using the \text{btph} matrix of an \((N - 1)\)th-order tensor. Therefore, for a general scalar-tensor order \(\mathcal{X} \in \mathbb{K}^{I_3 \times I_4 \times I_N}\), we obtain
\[
(C_{I_N} \otimes I_{I_{N-1}} \otimes \cdots \otimes C_{I_3}) \text{btph}(\mathcal{X}) (C_{I_N}^* \otimes C_{I_{N-1}}^* \otimes \cdots \otimes C_{I_3}^*) = \text{diag}(\hat{\mathcal{X}}),
\]
where \(\hat{\mathcal{X}} = \mathcal{X}'_{1 \times 3} M_{I_5} \times 4 M_{I_4} \times 5 \cdots \times M_{I_N}\). \(\square\)

Next, we define the new operator \(\mathcal{L}\) as follows.

**Definition 3.2.** Let \(\mathcal{L} : \mathbb{K}^{I_3 \times I_4 \times \cdots \times I_N} \rightarrow \mathbb{K}^{I_3 \times I_4 \times \cdots \times I_N}\) be the operator defined by
\[
\mathcal{L}(\mathcal{A}) = \hat{\mathcal{A}} = \mathcal{A} \times_3 M_{I_5} \times 4 M_{I_4} \times 5 \cdots \times N M_{I_N},
\]
and its inverse
\[
\mathcal{A} = \mathcal{L}^{-1}(\hat{\mathcal{A}}) = \hat{\mathcal{A}} \times N M_{I_N}^{-1} \times N-1 M_{I_{N-1}}^{-1} \times \cdots \times 3 M_{I_3}^{-1},
\]
where \(M_{I_i} = W_{I_i}^{-1} C_{I_i} (I_i + Z_{I_i})\) with \(W_{I_i} = \text{diag}(C_{I_i})(1, 1)\) and \(C_{I_i}\) is the matrix of discrete cosine and \(Z_{I_i}\), \(i = 3, 4, \ldots, N\) was already defined.

In the last theorem we proved that a block-Toeplitz-plus-Hankel matrix of a scalar-tensor is diagonalizable and in the next theorem we will prove that a block-Toeplitz-plus-Hankel matrix of an \(N\)th-order tensor is block diagonalizable.
Theorem 3.3. Let $X \in \mathbb{K}_{l_1 \times \ldots \times l_K}^I$, then its block Toeplitz-plus-Hankel matrix is block diagonalizable and
\[
\text{btph}(X) = (C^*_I \otimes \cdots \otimes C^*_I) \text{bdiag}(\mathcal{L}(X)) (C_I \otimes \cdots \otimes C_I) \quad (3.12)
\]

Proof. For a tensor $X \in \mathbb{K}_{l_1 \times \ldots \times l_K}$ we have
\[
\text{btph}(X) = \sum_{i_1=1}^{l_1} \sum_{i_2=1}^{l_2} \text{btph}(X(i_1, i_2, \ldots, :)) \otimes e_{i_1} e_{i_2}^T
\]
where $e_i$ is the $i$-th canonical vector of size $I_i$ for $i = 1, 2$. Therefore, using Theorem 3.1 we get
\[
\text{btph}(X) = \sum_{i_1=1}^{l_1} \sum_{i_2=1}^{l_2} (C^*_I \otimes \cdots \otimes C^*_I) \text{bdiag}(\mathcal{L}(X(i_1, i_2, \ldots, :))) (C_I \otimes \cdots \otimes C_I) \otimes e_{i_1} e_{i_2}^T
\]
\[= (C^*_I \otimes \cdots \otimes C^*_I) \text{bdiag}(\mathcal{L}(X)) (C_I \otimes \cdots \otimes C_I) \otimes (e_{i_1} e_{i_2}^T).
\]

Next, we define the generalized c-product.

Definition 3.4. Let $A \in \mathbb{K}_{l_1 \times l_2}$ and $B \in \mathbb{K}_{l_3 \times \ldots \times l_N}$. The generalized c-product $A \ast_c B$ is the tensor of $\mathbb{K}_{l_1 \times \ldots \times l_N}$ defined as follows
\[
A \ast_c B = \text{ten}(\text{btph}(A) \text{btph}(B)). \quad (3.13)
\]

In the next definition, we generalize the face-wise product that was already defined for the third-order tensors.

Definition 3.5. Generalized face-wise product
Let $A \in \mathbb{K}_{l_1 \times \ldots \times l_N}$ and $B \in \mathbb{K}_{l_1 \times \ldots \times l_N}$, we define the face-wise product between $A$ and $B$ by the tensor $C = A \triangle B \in \mathbb{K}_{l_1 \times \ldots \times l_N}$, where the $p^{th}$ representative matrix of $C$, is computed by the product of the $p^{th}$ representative matrices of $A$ and $B$, respectively, i.e.,
\[
C^p = A^p B^p, \quad p = 1, 2, \ldots, P, \quad (3.14)
\]
where the matrices $A^p$, $B^p$ and $C^p$ for $p = 1, 2, \ldots, P$ are the representative matrices given by (3.5) of $A$, $B$ and $C$, respectively.

Lemma 1. The generalized c-product of two $N^{th}$-order tensors $A \in \mathbb{K}_{l_1 \times \ldots \times l_N}$ and $B \in \mathbb{K}_{l_1 \times \ldots \times l_N}^I$ can be also computed in the cosine domain by
\[
A \ast_c B = \mathcal{L}^{-1} (\mathcal{L}(A) \triangle \mathcal{L}(B)). \quad (3.15)
\]

Proof. The generalized c-product of the two tensors $A \in \mathbb{K}_{l_1 \times \ldots \times l_N}$ and $B \in \mathbb{K}_{l_1 \times \ldots \times l_N}$ is given by
\[
A \ast_c B = \text{ten}(\text{btph}(A) \text{btph}(B))
\]
Therefore, using the notation $C_N^p = (C_{I_N} \otimes \cdots \otimes C_{I_p} \otimes I_p)$ for $p \geq 1$ and $N \geq 3$, we get
\[
btph(\mathcal{A} \ast_c \mathcal{B}) = btph(\mathcal{A}) \cdot btph(\mathcal{B}) = (C_N^{I_1})^* C_N^{I_1} \cdot btph(\mathcal{A}) (C_N^{I_N})^* C_N^{I_N} \cdot btph(\mathcal{B}) (C_N^{I_2})^* C_N^{I_2} = (C_N^{I_1})^* \cdot bdiag(\mathcal{L}(\mathcal{A})) \cdot bdiag(\mathcal{L}(\mathcal{B})) \cdot C_N^{I_2} = btph(\mathcal{L}^{-1}(\mathcal{L}(\mathcal{A}) \triangle \mathcal{L}(\mathcal{B}))).
\]

It follows that
\[
\mathcal{A} \ast_c \mathcal{B} = \mathcal{L}^{-1}(\mathcal{L}(\mathcal{A}) \triangle \mathcal{L}(\mathcal{B})).
\]

\[\square\]

3.2. Generalized tensor-tensor product. In this subsection, we define a general tensor-tensor product for high-order tensor. We first define $\mathcal{L}$ as the following operator
\[
\mathcal{L} : \mathbb{K}^{I_1 \times I_2}_{I_3 \times \cdots \times I_N} \rightarrow \mathbb{K}^{I_1 \times I_2}_{I_3 \times \cdots \times I_N},
\]
\[
\mathcal{A} \mapsto \mathcal{L}(\mathcal{A}) = \mathcal{A} \ast \mathcal{L} = \mathcal{A} \otimes M_3 \cdots \otimes M_N
\]

with $M_i \in \mathbb{K}^{I_1 \times I_1}_I$ such that $M_i = \alpha_i R_i$ for $i = 3, 4, \ldots, N$, where $\alpha_i > 0$ and $R_i$ is an unitary matrix. The inverse operator of $\mathcal{L}$ is defined as $\mathcal{L}^{-1}(\mathcal{A}) = \mathcal{A} \ast 3 \cdot M_3^{-1} \times 4 \cdots \times N \cdot M_N^{-1}$. We will denote $\alpha = \alpha_3 \alpha_4 \ldots \alpha_N$. Next, we need the relation between the norm of a tensor and its norm in the transformed domain (for example Fourier or cosine) given by
\[
|\mathcal{A}|_F = \frac{1}{\sqrt{\alpha}} |\mathcal{L}(\mathcal{A})|_F, \quad (3.16)
\]
and we also have
\[
\langle \mathcal{A}, \mathcal{B} \rangle = \frac{1}{\alpha} \langle \mathcal{L}(\mathcal{A}), \mathcal{L}(\mathcal{B}) \rangle, \quad (3.17)
\]

In the next and for any $N^{th}$-order tensor $\mathcal{X}$, we will denote the $p^{th}$ representative matrix of $\mathcal{L}(\mathcal{X})$ by $\mathcal{L}(\mathcal{X})^p$. Now we can define the generalized $\ast_\mathcal{L}$-product

**Definition 3.6.** Let $\mathcal{L}$ be the operator defined above, then the generalized $\ast_\mathcal{L}$-product of two $N^{th}$-order tensors $\mathcal{A} \in \mathbb{K}^{I_1 \times \cdots \times I_N}$ and $\mathcal{B} \in \mathbb{K}^{I_1 \times \cdots \times I_N}$ is given by
\[
\mathcal{A} \ast_\mathcal{L} \mathcal{B} = \mathcal{L}^{-1}(\mathcal{L}(\mathcal{A}) \triangle \mathcal{L}(\mathcal{B})) \in \mathbb{K}^{I_1 \times I_2}_{I_3 \times \cdots \times I_N} \quad (3.18)
\]

The whole steps are summarized in the following algorithm.

**Algorithm 2** The $\ast_\mathcal{L}$-product.

1. **Inputs:** $\mathcal{A} \in \mathbb{K}^{n_1 \times n_N}$ and $\mathcal{B} \in \mathbb{K}^{n_2 \times n_N}$.
2. **Output:** $\mathcal{C} = \mathcal{A} \ast_\mathcal{L} \mathcal{B} \in \mathbb{K}^{n_1 \times n_2}$.
3. Compute $\hat{\mathcal{A}} = \mathcal{L}(\mathcal{A})$ and $\hat{\mathcal{B}} = \mathcal{L}(\mathcal{B})$.
4. for $i = 1, \ldots, P$ do
   5. $\hat{\mathcal{C}}^i = \hat{\mathcal{A}}^i \hat{\mathcal{B}}^i$
6. end for
7. $\mathcal{C} = \mathcal{L}^{-1}(\hat{\mathcal{C}})$.
Proposition 1. Let $\mathcal{X} \in \mathbb{K}^{I_1 \times I_N}_{l_1 \times \cdots \times l_N}$ and $\mathcal{Y} \in \mathbb{K}^{I_2 \times I_N}_{l_1 \times \cdots \times l_N}$ two $K$-order tensors. Then we have

\[
(\mathcal{X} \ast_{\mathcal{L}} \mathcal{Y})_{i,j} = \sum_{k=1}^{l} \mathcal{X}_{i,k} \ast_{\mathcal{L}} \mathcal{Y}_{k,j}, \quad \text{for } 1 \leq i \leq I_1 \text{ and } 1 \leq j \leq I_2,
\]

(3.19)

where $\mathcal{X}_{i,j} = \mathcal{X}(i_1, i_2, \ldots, i_N)$ is the $(i,j)^{th}$ scalar-tensor of $\mathcal{X}$. 

Proof. For or $1 \leq i \leq I_1$ and $1 \leq j \leq I_2$, we have

\[
(\mathcal{X} \ast_{\mathcal{L}} \mathcal{Y})_{i,j} = \mathcal{L}^{-1} (\mathcal{X} \ast_{\mathcal{L}} \mathcal{Y})_{i,j}
\]

which shows the result. □

Proposition 2. Let $\mathcal{A} \in \mathbb{K}^{I_1 \times I_N}_{l_1 \times \cdots \times l_N}$ and $\mathcal{B} \in \mathbb{K}^{I_2 \times I_N}_{l_1 \times \cdots \times l_N}$ be two $K$-order tensors, then we can express the $\ast_{\mathcal{L}}$-product of $\mathcal{A}$ and $\mathcal{B}$ as

\[
\mathcal{A} \ast_{\mathcal{L}} \mathcal{B} = [\mathcal{A} \ast_{\mathcal{L}} \mathcal{B}_{1}, \mathcal{A} \ast_{\mathcal{L}} \mathcal{B}_{2}, \ldots, \mathcal{A} \ast_{\mathcal{L}} \mathcal{B}_{I_2}],
\]

(3.20)

where $\mathcal{B}_{i_2} = \mathcal{B}(i_1, i_2, \ldots, i_N) \in \mathbb{K}^{I_1 \times I_N}_{l_1 \times \cdots \times l_N}$; $i_2 = 1, \ldots, I_2$.

Proof. From the definition of the $\ast_{\mathcal{L}}$-product of $\mathcal{A}$ and $\mathcal{B}$ and for $1 \leq i_2 \leq I_2$, we get

\[
(\mathcal{A} \ast_{\mathcal{L}} \mathcal{B})_{i_2} = \mathcal{L}^{-1} (\mathcal{A} \ast_{\mathcal{L}} \mathcal{B})_{i_2}
\]

(3.22)

which gives the desired result. □

Related to the generalized $\ast_{\mathcal{L}}$-product, we give the definitions of the identity, transpose and orthogonal tensors.

Definition 3.7. (The identity tensor)

The tensor identity tensor $\mathcal{I} \in \mathbb{K}^{I_1 \times I_N}_{l_1 \times \cdots \times l_N}$ is such that $(\mathcal{L}(\mathcal{I}))^p = I$ for $p = 1, 2, \ldots, P$ where $(\mathcal{L}(\mathcal{I}))^p$ is the $p^{th}$ representative matrix of $\mathcal{L}(\mathcal{I})$.

From the previous definition, we can conclude that if $\mathcal{A} \in \mathbb{K}^{I_1 \times I_N}_{l_1 \times \cdots \times l_N}$, then $\mathcal{A} \ast_{\mathcal{L}} \mathcal{I} = \mathcal{I} \ast_{\mathcal{L}} \mathcal{A} = \mathcal{A}$; because $\mathcal{A} \ast_{\mathcal{L}} \mathcal{I} = \mathcal{L}^{-1} (\mathcal{L}(\mathcal{A}) \ast_{\mathcal{L}} \mathcal{L}(\mathcal{I})) = \mathcal{L}^{-1} (\mathcal{L}(\mathcal{A}))$, and the same for $\mathcal{I} \ast_{\mathcal{L}} \mathcal{A}$.

Definition 3.8. (The transpose)

Let $\mathcal{A} \in \mathbb{K}^{I_1 \times I_N}_{l_1 \times \cdots \times l_N}$, the transpose $\mathcal{A}^T$ of the tensor $\mathcal{A}$ is such that $\mathcal{L}(\mathcal{A}^T)^p = (\mathcal{L}(\mathcal{A})^p)^T$ for $p = 1, 2, \ldots, P$. 

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This definition ensure the multiplication reversal property for the transpose under the $\ast_{\mathcal{L}}$-product, i.e., for $\mathcal{A}$ and $\mathcal{B}$ two $N$th-order tensors of appropriate sizes, we get $(\mathcal{A} \ast_{\mathcal{L}} \mathcal{B})^T = B^T \ast_{\mathcal{L}} A^T$; for explanation we have

$$\mathcal{L} \left( (\mathcal{A} \ast_{\mathcal{L}} \mathcal{B})^T \right)^p = (\mathcal{L}(\mathcal{A} \ast_{\mathcal{L}} \mathcal{B})^T)^p = (\mathcal{L}(\mathcal{A})^p \mathcal{L}(\mathcal{B})^p)^T$$

$$= (\mathcal{L}(\mathcal{B})^p)^T (\mathcal{L}(\mathcal{A})^p)^T$$

$$= (\mathcal{L}(\mathcal{B}^T))^p (\mathcal{L}(\mathcal{A}^T))^p$$

$$= \mathcal{L}(B^T \ast_{\mathcal{L}} A^T)^p$$

**Definition 3.9. (Orthogonal tensor)**

The tensor $\mathcal{Q} \in \mathbb{K}_{I_1 \times I_2}^{I_3 \times \times I_N}$ is orthogonal under the $\ast_{\mathcal{L}}$-product, iff $\mathcal{Q} \ast_{\mathcal{L}} \mathcal{Q}^T = \mathcal{Q}^T \ast_{\mathcal{L}} \mathcal{Q} = \mathcal{I}$ which means that for each $i \in \{1, 2, \ldots, P\}$, $\mathcal{L}(\mathcal{Q})^i$ is an orthogonal matrix.

Notice that if $\mathcal{Q} \in \mathbb{K}_{I_1 \times I_2}^{I_3 \times \times I_N}$ is orthogonal, then for an $N$th-order tensor $\mathcal{A}$ of an appropriate size, we have

$$|\mathcal{A} \ast_{\mathcal{L}} \mathcal{Q}|_F^2 = \frac{1}{c} |\mathcal{L}(\mathcal{A}) \triangle \mathcal{L}(\mathcal{Q})|_F^2 = \frac{1}{c} \sum_{i=1}^P \|\mathcal{L}(\mathcal{A})^i \mathcal{L}(\mathcal{Q})^i\|_F^2$$

(3.25)

$$= \frac{1}{c} \sum_{i=1}^P \|\mathcal{L}(\mathcal{A})^i\|_F^2$$

(3.26)

$$= |\mathcal{A}|_F^2$$

(3.27)

**Definition 3.10. f-diagonal tensor**

An $N$th-order tensor $\mathcal{X}$ is f-diagonal, if each $\mathcal{L}(\mathcal{X})^p$ is a diagonal matrix for all $p$ in $\{1, 2, \ldots, P\}$. [1]

**Theorem 3.11.** The set $(\mathbb{K}_{I_1 \times I_2}^{I_3 \times \times I_N}, +, \ast_{\mathcal{L}})$ is a commutative ring.

*Proof.* It is easy to prove that $(\mathbb{K}_{I_1 \times I_2}^{I_3 \times \times I_N}, +)$ is an abelian group with $0 \in \mathbb{K}_{I_1 \times I_2}^{I_3 \times \times I_N}$ as a neutral element. On the other hand, we have

- Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are scalar-tensors in $\mathbb{K}_{I_1 \times I_2}^{I_3 \times \times I_N}$, then we will get

$$\mathcal{A} \ast_{\mathcal{L}} (\mathcal{B} \ast_{\mathcal{L}} \mathcal{C}) = \mathcal{L}^{-1} \left( \mathcal{L}(\mathcal{A}) \triangle \mathcal{L} \left( \mathcal{L}^{-1} \left( \mathcal{L}(\mathcal{B}) \triangle \mathcal{L}(\mathcal{C}) \right) \right) \right)$$

$$= \mathcal{L}^{-1} \left( \mathcal{L}(\mathcal{A}) \triangle \mathcal{L}(\mathcal{B}) \triangle \mathcal{L}(\mathcal{C}) \right)$$

$$= \mathcal{L}^{-1} \left( \mathcal{L} \left( \mathcal{L}^{-1} \left( \mathcal{L}(\mathcal{A}) \triangle \mathcal{L}(\mathcal{B}) \right) \right) \right) \triangle \mathcal{L}(\mathcal{C})$$

$$= (\mathcal{A} \ast_{\mathcal{L}} \mathcal{B}) \ast_{\mathcal{L}} \mathcal{C}.$$  

- We also have

$$\mathcal{A} \ast_{\mathcal{L}} (\mathcal{B} + \mathcal{C}) = \mathcal{L}^{-1} \left( \mathcal{L}(\mathcal{A}) \triangle \mathcal{L}(\mathcal{B} + \mathcal{C}) \right)$$

$$= \mathcal{L}^{-1} \left( \mathcal{L}(\mathcal{A}) \triangle \left( \mathcal{L}(\mathcal{B}) + \mathcal{L}(\mathcal{C}) \right) \right)$$

$$= \mathcal{L}^{-1} \left( \mathcal{L}(\mathcal{A}) \triangle \mathcal{L}(\mathcal{B}) + \mathcal{L}(\mathcal{A}) \triangle \mathcal{L}(\mathcal{C}) \right)$$

$$= \mathcal{L}^{-1} \left( \mathcal{L}(\mathcal{A}) \triangle \mathcal{L}(\mathcal{B}) \right) + \mathcal{L}^{-1} \left( \mathcal{L}(\mathcal{A}) \triangle \mathcal{L}(\mathcal{C}) \right)$$

$$= \mathcal{A} \ast_{\mathcal{L}} \mathcal{B} + \mathcal{A} \ast_{\mathcal{L}} \mathcal{C}.$$  

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Then we obtain
\[ L = U^{-1} (L(A) \triangle L(B)) = L^{-1} (L(B) \triangle L(A)) = B *_L A. \]

**THEOREM 3.12.** (The tensor \(*_L\)-SVD)

Let \( A \in \mathbb{K}_{I_1 \times \cdots \times I_N}^{I_1 \times I_2} \) be an \( N\)-th order tensor, then \( A \) can be decomposed as
\[
A = U *_L S *_L V^T = \sum_{k=1}^{r} U(:,k,:),\ldots,:) *_L S(k,k,:),\ldots,:) *_L V(:,k,:),\ldots,:)^T, \tag{3.28}
\]
where \( U \in \mathbb{K}_{I_1 \times I_1}^{I_1 \times I_1} \) and \( V \in \mathbb{K}_{I_2 \times I_2}^{I_2 \times I_2} \) are orthogonal tensors, \( S \in \mathbb{K}_{I_1 \times I_2}^{I_1 \times I_2} \) is an f-diagonal and \( r \) is the tubal rank of \( A \) which will be defined in the next.

**Proof.** For a \( K\)-th order tensor \( A \) and \( p = 1, 2, \ldots, P \), consider SVD decomposition of the matrices
\[ L(A)^p = U_L^p S_L^p (V_L^p)^T. \]
Then we obtain
\[ L(A) = U_L \triangle S_L \triangle (V_L)^T, \]
with \( L(U) = U_L \), \( L(S) = S_L \) and \( L(V) = V_L \), where those tensors are well defined since the operator \( L \) is invertible. Therefore
\[ L(A) = L(U) \triangle L(S) \triangle L(V)^T \iff A = L^{-1} \left( L(U) \triangle L(S) \triangle L(V)^T \right) \]
which gives
\[ A = U *_L S *_L V^T. \]

The following algorithm summarises the different steps for computing the tensor \(*_L\)-SVD of an \( N\)-th order tensor.

**Algorithm 3 The \(*_L\)-svd.**

1: **Inputs:** \( A \in \mathbb{K}_{I_1 \times I_2}^{I_1 \times I_2} \).
2: **Output:** \( U \in \mathbb{K}_{I_1 \times I_1}^{I_1 \times I_1} \), \( S \in \mathbb{K}_{I_1 \times I_2}^{I_1 \times I_2} \), \( V \in \mathbb{K}_{I_2 \times I_2}^{I_2 \times I_2} \).
3: Compute \( \hat{A} = L(A) \).
4: for \( i = 1, \ldots, P \) do
5: \[ [U^i, S^i, V^i] = \text{svd}(\hat{A}^i) \]
6: end for
7: \( U = L^{-1}(\hat{U}), S = L^{-1}(\hat{S}) \) and \( V = L^{-1}(\hat{V}) \)

**COROLLARY 1.** Let \( A \) be an \( N\)-th order tensor in \( \mathbb{K}_{I_1 \times \cdots \times I_N}^{I_1 \times I_2} \), and let \( S_i = S(i,i,:,\ldots,:) \), where \( S \) is given by \( 3.28 \). Then
\[ \|A\|^2_F = \|S\|^2_F = \sum_{i=1}^{\min(I_1,I_2)} \|S_i\|^2_F. \]
Furthermore

\[ |S_1|_F \geq |S_2|_F \geq \cdots \geq 0. \quad (3.30) \]

Proof. Since the tensors \( U \) and \( V \) given from the \( \ast_L \)-svd are orthogonal, then

\[
\begin{align*}
|A|_F^2 &= |U \ast_L S \ast_L V^T|_F^2 = |S|_F^2 \\
&= \frac{1}{\alpha} \sum_{p=1}^{P} \| \mathcal{L}(S)^p \|_F^2 \\
&= \frac{1}{\alpha} \sum_{i=1}^{P} \min(i_1, i_2) \sum_{i=1}^{P} \left( \mathcal{L}(S)^p(i,i) \right)^2 \\
&= \sum_{i=1}^{\min(i_1, i_2)} |S_i|_F^2.
\end{align*}
\]

On the other hand,

\[
|S|_F^2 = \frac{1}{\alpha} \| \mathcal{L}(S) \|_F^2 = \frac{1}{\alpha} \sum_{p=1}^{P} \left( \mathcal{L}(S)^p(i,i) \right)^2 \geq \frac{1}{\alpha} \sum_{p=1}^{P} \left( \mathcal{L}(S)^p(i+1,i+1) \right)^2 = |S_{i+1}|_F^2.
\]

\( \square \)

Next, we give different definitions of a rank of a high-order tensor.

**Definition 3.13.** (The \( \ast_L \)-tubal rank)

Let \( A \in K_{I_1 \times I_2 \times \cdots \times I_N} \), then the tensor \( \ast_L \)-tubal rank is defined as

\[
\text{rank}_L(A) = \text{card}\{i \mid S(i,i,:) \neq 0\},
\]

where \( S \) is the \( f \)-diagonal tensor given from the \( \ast_L \)-svd of \( A \) \((3.28)\).

**Lemma 2.** For a \( K \)-th-order tensor, the \( \ast_L \)-tubal rank can be written as

\[
\text{rank}_L(A) = \text{card}\{i \mid \exists p \in \{1, 2, \ldots, P\}; \mathcal{L}(S)^p(i,i) \neq 0\}. \quad (3.32)
\]

Proof. The key idea for proving the above result is the following equivalence,

\[
S(i,i,:,:) = 0 \iff \forall p \in \{1, 2, \ldots, P\}; \mathcal{L}(S)^p(i,i) = 0, i = 1, 2, \ldots, \min(I_1, I_2),
\]

which is equivalent to

\[
S(i,i,:,:) \neq 0 \iff \exists p \in \{1, 2, \ldots, P\}; \mathcal{L}(S)^p(i,i) \neq 0, i = 1, 2, \ldots, \min(I_1, I_2).
\]

Therefore,

\[
S(i,i,:,:) \neq 0 \iff \max_{1 \leq p \leq P} \mathcal{L}(S)^p(i,i) \neq 0, i = 1, 2, \ldots, \min(I_1, I_2).
\]

\( \square \)
Definition 3.14. (Multirank and average rank)

For an \( N \)-th order tensor \( A \) of size \( I_1 \times I_2 \times \cdots \times I_N \), its multirank under the \( \ast_L \)-product is defined as the vector \( \rho \) of size \( P \), where its \( i \)-th element is the rank of \( L(A)^i \), i.e.,

\[
\rho_i = \text{rank}(L(A)^i) \quad (3.34)
\]

The average rank of \( A \) is defined as the mean of the vector \( \rho \), i.e.,

\[
\text{rank}_a(A) = \frac{\sum_{i=1}^{P} \rho_i}{P} \quad (3.35)
\]

Remark 1. We notice that the average rank of an \( N \)-th order tensor is defined as the rank of the block-diagonal matrix of \( L(A) \) divided by \( P \),

\[
\text{rank}_a(A) = \frac{\text{bdiag}(L(A))}{P} \quad (3.36)
\]

Next, we give a generalized version of the well known Eckart Young using the \( \ast_L \)-product.

Theorem 3.15. (Eckart Young)

Let \( A \in \mathbb{K}^{I_1 \times \cdots \times I_N} \) and \( A_k = U(:,1:k) \ast_L S(1:k,1:k) \ast_L V(1:k,:)^T \), where \( k \) is a nonzero positive integer. Let \( C \) be the set defined by

\[
C = \{ \mathcal{X} \ast_L \mathcal{Y} / \mathcal{X} \in \mathbb{K}^{I_1 \times \cdots \times I_N} , \mathcal{Y} \in \mathbb{K}^{k \times I_2} \}.
\]

Then the tensor \( A_k \) solves the minimisation problem

\[
\min_{\mathcal{X} \in C} \| A - \mathcal{X} \|_F,
\]

and the error-norm is given by

\[
\| A - A_k \|_F^2 = \sum_{i=k+1}^{P} \| S_i \|_F^2.
\]

Proof. Let \( B = \mathcal{X} \ast_L \mathcal{Y} \in C \), then we have

\[
\| A - B \|_F^2 = \frac{1}{\alpha} \sum_{i=1}^{P} \| L(A)^i - L(B)^i \|_F^2.
\]

Now, since \( L(B)^i = L(\mathcal{X})^i L(\mathcal{Y})^i \) and by using the matrix Eckart Young theorem we obtain the result showing that the \( k \)-best approximation of the matrix \( L(A) \) is given by \( L(\mathcal{U})^i (:,1:k)L(S)^i (1:k,1:k)L(\mathcal{V})^i (1:k,:)^T \), where \( \mathcal{U} \), \( S \) and \( \mathcal{V} \) are given by (3.28). \( \square \)

Proposition 3. The tensor spectral norm of an \( N \)-th order tensor \( A \) satisfies the following equation

\[
|A| = |\text{bdiag}(L(A))| \quad (3.37)
\]
 Proof. The spectral norm of a tensor $A$, is defined by

$$
\|A\| = \sup_{\|V\|_F = 1} \|A \ast_{\mathcal{L}} V\|_F = \frac{1}{\sqrt{\alpha}} \sup_{\|V\|_F = 1} \|\mathcal{L}(A) \triangle \mathcal{L}(V)\|_F
$$

(3.38)

$$
= \frac{1}{\sqrt{\alpha}} \sup_{\|V\|_F = 1} \|\text{bdiag}(\mathcal{L}(A)) \text{bdiag}(\mathcal{L}(V))\|_F
$$

(3.39)

$$
= \|\text{bdiag}(\mathcal{L}(A))\|_\alpha
$$

(3.40)

\[\square\]

**Definition 3.16. Tensor nuclear norm**

The tensor nuclear norm of a tensor $A \in \mathbb{K}^{I_1 \times \ldots \times I_N}$ is defined as its dual norm, i.e.,

$$
\|A\|_\alpha = \sup_{\|B\|_\alpha \leq 1} \|\langle A, B \rangle\|_\alpha = \frac{1}{\alpha} \sup_{\|B\|_\alpha \leq 1} \|\text{bdiag}(\mathcal{L}(A))\|_\alpha
$$

(3.41)

**Theorem 3.17.** The nuclear norm of a tensor $A \in \mathbb{K}^{I_1 \times \ldots \times I_N}$ verifies the relation

$$
\|A\|_\alpha = \frac{1}{\alpha} \|\text{bdiag}(\mathcal{L}(A))\|_\alpha
$$

(3.42)

**Proof.** Starting by the definition of the nuclear norm in (3.41), we will get

$$
\|A\|_\alpha = \sup_{\|B\|_\alpha \leq 1} \|\langle A, B \rangle\| = \frac{1}{\alpha} \sup_{\|B\|_\alpha \leq 1} \|\langle \text{bdiag}(\mathcal{L}(A)), \text{bdiag}(\mathcal{L}(B)) \rangle\|
$$

$$
= \frac{1}{\alpha} \|\text{bdiag}(\mathcal{L}(A))\|_\alpha
$$

Which gives the first equality of (3.42), and the last equality is trivial since $\text{bdiag}(\mathcal{L}(A)) = \text{bdiag}(\mathcal{L}(U)) \text{bdiag}(\mathcal{L}(S)) \text{bdiag}(\mathcal{L}(V^T))$ where $U$, $S$ and $V$ are given from the $\ast_{\mathcal{L}}$-svd of $A$.

\[\square\]

We can also express the tensor nuclear norm of a $K^{th}$-order tensor as

$$
\|A\|_\alpha = \frac{1}{\alpha} \sum_{p=1}^{P} \left( \min(I_j, I_j) \sum_{i=1}^{\min(I_j, I_j)} \mathcal{L}(S)^p(i, i) \right)
$$

(3.43)

**Theorem 3.18.** The envelope convex of the function tensor average rank on the set $S = \{A \in \mathbb{K}^{I_1 \times \ldots \times I_N}, \|A\| \leq 1\}$ is the tensor nuclear norm.

**Proof.** Let $A \in S$, then $\|A\| \leq 1$, which means that for $i = 1, 2, \ldots, P$, we have

$$
\mathcal{L}(A)^p \leq 1
$$

and by using the fact that the tensor average rank is the average of the tensor multirank, it follows (see [3]) that for each $i = 1, 2, \ldots, P$, the envelope convex of $\rho_i$ is $\|\mathcal{L}(A)^p\|_\alpha$.

Consequently, the envelope convex of the function average rank is the tensor nuclear norm.\[\square\]
Next, we define the tensor singular value thresholding under the $\ast_L$-product ($\ast_L$-svt).

**Definition 3.19.** Let $A \in \mathbb{K}_{I_1 \times I_2}^{I_1 \times I_2 \times \cdots \times I_N}$ an $N^{th}$-order tensor and $\tau > 0$, then we call its tensor singular value thresholding the following tensor

$$
\mathcal{D}_\tau(A) = U \ast \mathcal{S}_\tau \ast V^T
$$

(3.44)

where $U$, $S$ and $V$ are given from the generalize t-svd (3.28), and $\mathcal{S}_\tau = L^{-1}((L(S) - \tau)_+)$. We also have the following important result that links the tensor nuclear norm and the tensor singular value thresholding.

**Theorem 3.20.** Let $A \in \mathbb{K}_{I_1 \times I_2}^{I_1 \times I_2 \times \cdots \times I_N}$ and $\tau > 0$, then we have

$$
\mathcal{D}_\tau(A) = \arg \min_{Y \in \mathbb{K}_{I_1 \times I_2}^{I_1 \times I_2 \times \cdots \times I_N}} \tau \|Y\|_* + \frac{1}{2} \|Y - A\|_F^2
$$

(3.45)

**Proof.** Solving the optimization problem (3.45) is equivalent to solve the following one

$$
\arg \min_{Y \in \mathbb{K}_{I_1 \times I_2}^{I_1 \times I_2 \times \cdots \times I_N}} \tau \|\text{bdiag}(L(Y))\|_* + \frac{1}{2} \|\text{bdiag}(L(Y)) - \text{bdiag}(L(A))\|_F^2.
$$

(3.46)

The optimization problem given in (3.46) can be solved by solving $P$ subproblems independently, i.e., for each $i \in \{1, 2, \ldots, P\}$ we will try to find the solution of

$$
\arg \min_{\mathcal{L}(Y) \in \mathbb{K}_{I_1 \times I_2}^{I_1 \times I_2 \times \cdots \times I_N}} \left\{ \tau \|\mathcal{L}(Y)^i\|_* + \frac{1}{2} \|\mathcal{L}(Y)^i - \mathcal{L}(A)^i\|_F^2 \right\}.
$$

(3.47)

Using [11], the solution of each subproblem in (3.47) is $\mathcal{L}(D_\tau(A))^i$. Therefore, $\mathcal{D}_\tau(A)$ solves (3.45). $\square$

In the following algorithm we give the different steps of computing the tensor $\ast_L$-svt

**Algorithm 4** The $\ast_L$-svt.

1. **Inputs:** $A \in \mathbb{K}_{I_1 \times I_2}^{I_1 \times I_2 \times \cdots \times I_N}$ and $\tau > 0$.
2. **Output:** $\mathcal{D}_\tau(A) \in \mathbb{K}_{I_1 \times I_2}^{I_1 \times I_2 \times \cdots \times I_N}$.
3. **Compute:** $\hat{A} = \mathcal{L}(A)$
4. **for** $i = 1, \ldots, P$ **do**
5. **end for**
6. **for** $i = 1, \ldots, P$ **do**
7. **end for**
8. **Output:** $\hat{A} = \mathcal{L}^{-1}(\hat{A})$.

4. **Tensor completion using $\ast_L$-product.** Tensor completion is the problem that consists in finding some unknown pixels of the data from an observed data that contains some known pixels. Many algorithms have been developed the last years; see [8, 22, 26]. Some of those methods use regularization techniques such as the total variation regularization [24, 18]. Those algorithms suffer from the computationally costs and the slowness. In [11, 28] the problem
of tensor completion using the t-product and the c-product with regularized total variation gave good results. The problem of these methods is the fact that they are applied only to third-order tensors. Next, we propose to extend those methods to high order tensors using the $L$ product we defined in the preceding section for tensors of order greater than three.

The main optimization problem that solves the problem of tensor completion consists in finding a low-rank tensor that contains the main information (the known pixels), which depends on the definition of the rank that we will consider. In our proposed method, we will consider the average rank of a tensor. Thus our main optimization problem is given as follows

$$\arg \min_{\mathcal{X} \in \mathbb{K}^{I_1 \times I_2 \times \ldots \times I_N}} \rank_a(\mathcal{X})$$

s.t. $\mathcal{P}_\Omega(\mathcal{X}) = \mathcal{P}_\Omega(\mathcal{M})$, \hspace{1cm} (4.1)

where $\mathcal{X}$ is the underlying tensor, $\mathcal{M} \in \mathbb{K}^{I_1 \times I_2 \times \ldots \times I_N}$ is the observed tensor, $\Omega$ is the set of the known pixels and $\mathcal{P}_\Omega$ is the projection operator that copy the values of the pixels onto $\Omega$. However, the optimization problem (4.1) is NP-hard [23]. It is known that convex optimization problems are the easiest optimization problems to solve and for this reason we will use the approximation of the function average rank given in Theorem 3.18. Our optimization problem is transformed to the following one

$$\min_{\mathcal{X} \in \mathbb{K}^{I_1 \times I_2 \times \ldots \times I_N}} \|\mathcal{X}\|_*$

s.t. $\mathcal{P}_\Omega(\mathcal{X}) = \mathcal{P}_\Omega(\mathcal{M})$, \hspace{1cm} (4.2)

where $\| \cdot \|_*$ is the tensor nuclear norm in Theorem 3.17. The main techniques for solving (4.2) is the Proximal Gradient Algorithm (PGA). The problem (4.2) will be solved iteratively as

$$\mathcal{X}^{k+1} = \arg \min_{\mathcal{X} \in \mathbb{K}^{I_1 \times I_2 \times \ldots \times I_N}} \mu \|\mathcal{X}\|_* + \frac{1}{2} \|\mathcal{X} - \mathcal{G}^k\|_F^2,$$ \hspace{1cm} (4.3)

where

$$\mathcal{G}^k = \mathcal{Y}^k - (\mathcal{P}_\Omega(\mathcal{Y}^k) - \mathcal{P}_\Omega(\mathcal{M})) = \mathcal{P}_\Omega(\mathcal{Y}^k) - \mathcal{P}_\Omega(\mathcal{M}), \hspace{1cm} \text{and}$$

$$\mathcal{Y}^k = \mathcal{X}^k + \frac{t_{k-1} - 1}{t_k} (\mathcal{X}^k - \mathcal{X}^{k-1}).$$ \hspace{1cm} (4.4)

Notice that in this case, the function $f$ involved in (2.10) is given by $f(\mathcal{X}) = \frac{1}{2} \|\mathcal{P}_\Omega(\mathcal{X}) - \mathcal{P}_\Omega(\mathcal{M})\|_F^2$ with $\nabla f(\mathcal{X}) = \mathcal{P}_\Omega(\mathcal{X}) - \mathcal{P}_\Omega(\mathcal{M})$. Therefore, the Lipshitz constant $l_f$ of $\nabla f$ is equal to 1. It is clear from Theorem 3.20 that the solution of the problem (4.3) is the t-svt of the tensor $\mathcal{G}^k$, i.e.,

$$\mathcal{X}^{k+1} = \mathcal{D}_\mu(\mathcal{G}^k).$$ \hspace{1cm} (4.6)

The following algorithm summarizes all the steps of the proposed method
Algorithm 5: Tensor completion using tensor nuclear norm by PGA.

1: **Inputs:** $\mathcal{M} \in \mathbb{K}_{I_1 \times \cdots \times I_N}$, $t_0$, $\nu$, $\mu_0$.
2: **Initialize:** $\mathcal{X}^0 = \mathcal{X}^1 = 0$, $t_0 = t_1 = 0$, $\bar{\mu} = \nu \mu_0$, $itermax = 100$, $k=1$.
3: **while** not converged **do**
4: Update $\mathcal{Y}^k$ from (4.5).
5: Update $\mathcal{G}^k$ from (4.4).
6: Update $\mathcal{X}^{k+1}$ from (4.6).
7: $t_{k+1} = 1 + \sqrt{4t_k^2 + 1}$.
8: $\mu_{k+1} = \max(\nu \mu_k, \bar{\mu})$.
9: **end while**

In Table 4.1, we give the cost of the different tensor operations (\(*\mathcal{L}\)-product, \(*\mathcal{L}\)-svd and \(*\mathcal{L}\)-svt) when using FFT or DCT.

| \(*\mathcal{L}\)-product of $\mathcal{X} \in \mathbb{K}_{I_1 \times \cdots \times I_N}$ and $\mathcal{Y} \in \mathbb{K}_{I_1 \times \cdots \times I_N}$ | FFT | DCT |
| --- | --- | --- |
| $O\left((PI_{1n} + PI{1n} + P) \sum_{i=3}^{N} \log(I_i)\right)$ | $O\left((PI_{1n} + PI{1n} + P) \sum_{i=3}^{N} \log(I_i)\right)$ |
| $+ 4O\left(P I_{1n}^2 I_2\right)$ | $+ O\left(P I_{1n}^2 I_2\right)$ |

| \(*\mathcal{L}\)-svd of $\mathcal{X} \in \mathbb{K}_{I_1 \times \cdots \times I_N}$ | FFT | DCT |
| --- | --- | --- |
| $O\left((2PI_1 I_2 + P I_2^2 + P I_1^2) \sum_{i=3}^{N} \log(I_i)\right)$ | $O\left((2PI_1 I_2 + P I_2^2 + P I_1^2) \sum_{i=3}^{N} \log(I_i)\right)$ |
| $+ 2O\left(P \min(I_1^2 I_2, I_1 I_2)\right)$ | $+ O\left(P \min(I_1^2 I_2, I_1 I_2)\right)$ |

| \(*\mathcal{L}\)-svt of $\mathcal{X} \in \mathbb{K}_{I_1 \times \cdots \times I_N}$ | FFT | DCT |
| --- | --- | --- |
| $O\left((2PI_1 I_2 + P I_2^2 + P I_1^2) \sum_{i=3}^{N} \log(I_i)\right)$ | $O\left((2PI_1 I_2 + P I_2^2 + P I_1^2) \sum_{i=3}^{N} \log(I_i)\right)$ |
| $+ 2O\left(P \min(I_1^2 I_2, I_1 I_2)\right)$ | $+ 8O\left(P I_{1n}^2 I_2\right)$ |
| $+ O\left(P \min(I_1^2 I_2, I_1 I_2)\right)$ | $+ 2O\left(P I_{1n}^2 I_2\right)$ |

Table 4.1: The cost of computing the \(*\mathcal{L}\)-product, \(*\mathcal{L}\)-svd and the \(*\mathcal{L}\)-svt by using Fourier and cosine transforms.
5. Numerical experiments. In this section we test the performance of our algorithms for high order tensor completion using the Fourier and Cosine for the operator $L$ and we will compare the obtained results with those attained by some existing known algorithms on color videos. In Subsection 5.1 the tests were performed with Matlab 2018a, on an Intel i5 laptop with 16 Go of memory, and in Subsection 5.2 we use codes with Python on a machine that uses a CPU of type Intel Xeon Gold 6152 with a frequency from 2.1 Ghz to 3.7 GHz and a GPU of type NVIDIA Tesla Pascal 40. All the tests are computed using a single core. The quality of the obtained data can be computed by the peak signal-to-noise-ration (PSNR) defined by

$$\text{PSNR} = 10 \log_{10} \frac{\text{Max}^{2}_{A_{\text{obt}}}}{\|A_{\text{obt}} - A_{\text{ori}}\|_{F}^{2}}, \quad (5.1)$$

and the relative squared error (RSE) given by

$$\text{RSE} = \frac{\|A_{\text{ori}} - A_{\text{obt}}\|_{F}^{2}}{\|A_{\text{obt}}\|_{F}^{2}}, \quad (5.2)$$

where $A_{\text{ori}}$ is the original tensor, $A_{\text{obt}}$ is the obtained recovered tensor and $\text{Max}_{A_{\text{obt}}}$ is the maximum pixel of the recovered tensor. The quality of the recovered data is good when the value of RSE is small and the value of PSNR is high. In our experiments of tensor completion, we use $M_{\text{f}} = F_{i}$ for Fourier transform and $M_{\text{c}} = C_{i}$ for Cosine transform, where the transformed data $\mathcal{L}(\mathcal{X})$ for an $N^{th}$-order tensor is given by $\text{fft}(\mathcal{X}, [], 3)$ and $\text{dct}(\mathcal{X}, [], 3)$, respectively. The parameters $\mu_{0}$ and $\nu$ stated the algorithm of completion have to be fixed. We set $\nu = 0.9$ and $\mu_{0} = \nu \|\mathcal{M}\|_{F}$, where the stopping criterion convergence of this algorithm is as follows

$$\frac{\|y^{k} - X^{k+1}\|_{F}}{\|X^{k+1}\|_{F}} \leq 10^{-4}. \quad (5.3)$$

Figure 5.1 shows the data tests used in our experiments. The video of xylophone is available from Matlab, and the videos Akiyo and News are available from [1]. In Table 5.1 we give the size of the different used color videos tests.

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1urlhttp://trace.eas.asu.edu/yuv/
Table 5.1: The name of the test data (color videos) and their sizes.

| name    | size       |
|---------|------------|
| xylophone | 240 × 320 × 3 × 141 |
| Akiyo   | 144 × 176 × 3 × 300 |
| news    | 144 × 176 × 3 × 300 |

5.1. Tensor completion for color videos. In this part we show the obtained results of our algorithms TNN-PGA-F and TNN-PGA-C on fourth-order tensors (color videos) and compare them with other ones such as the methods named Tmac [22] and HaLRTC [26]. The comparison will be in terms of the efficiency and executing times by comparing the values of RSE and PSNR, the number of iterations the required cpu-time. In Figure 5.2 we show the 20th bound of the color videos xylophone, Akiyo and news with only 5% of the original data. In Figure 5.3 we show the 20th bound of the recovered data for each color video obtained by the algorithms HaLRTC, Tmac, TNN-PGA-F and TNN-PGA-C.

Figure 5.2: The 20th bound of color videos of the test data with only 5% of the original data, i.e., sr = 0.05.

In Table 5.2, we report the values of RSE, PSNR, the number of iterations and the CPU time required by the algorithms HaLRTC, Tmac, TNN-PGA-F and TNN-PGA-C. Figure 5.4 shows the curves representing the evolution of the RSE and PSNR versus the iteration number for Akiyo-video with sr = 0.05. In Figure 5.5 we give the values of the first 1000 pixels of the recovered data of Akiyo-video for sr = 0.05. Figure 5.6 shows the values of RSE and PSNR for each bound of the xylophone-video for sr = 0.1.
Figure 5.3: The 20th bound of the recovered color videos obtained from the algorithms HaLRTC, Tmac, TNN-PGA-F and TNN-PGA-C for sr = 0.05.

| Video  | Algorithm | RSE | PSNR | Iteration | time  | RSE | PSNR | Iteration | time  |
|--------|-----------|-----|------|-----------|-------|-----|------|-----------|-------|
|        |           |     |      |           |       |     |      |           |       |
| Akiyo  | HaLRTC    | 0.3741 | 15.49 | 163 | 2464.8 | 0.2409 | 19.31 | 94 | 1632.4 |
|        | Tmac      | 0.0879 | 30.25 | 343 | 991.0  | 0.0824 | 30.91 | 210 | 587.6  |
|        | TNN-PGA-F | 0.0497 | 33.84 | 73  | 619.8  | 0.0356 | 36.65 | 70  | 640.8  |
|        | TNN-PGA-C | 0.0496 | 33.91 | 99  | 571.2  | 0.0350 | 36.80 | 76  | 475.1  |
| xylophone | HaLRTC  | 0.2732 | 17.49 | 163 | 3247.3 | 0.1910 | 20.60 | 96  | 2012.0 |
|        | Tmac      | 0.1148 | 26.96 | 376 | 3354.7 | 0.1066 | 27.43 | 347 | 1703.7 |
|        | TNN-PGA-F | 0.0888 | 28.87 | 73  | 1054.2 | 0.0680 | 30.77 | 73  | 1041.4 |
|        | TNN-PGA-C | 0.0889 | 28.38 | 90  | 906.0  | 0.0660 | 30.90 | 79  | 760.9  |
| news   | HaLRTC    | 0.4951 | 14.46 | 142 | 1876.3 | 0.3694 | 17.01 | 89  | 1038.2 |
|        | Tmac      | 0.1344 | 27.95 | 1157 | 4480.0 | 0.1171 | 29.36 | 415 | 1752.6 |
|        | TNN-PGA-F | 0.1058 | 29.07 | 71  | 621.9  | 0.0796 | 31.53 | 69  | 573.4  |
|        | TNN-PGA-C | 0.1156 | 28.20 | 99  | 614.0  | 0.0838 | 31.15 | 77  | 475.5  |

Table 5.2: The values of RSE, PSNR, the number of iteration and the time required by the algorithms HaLRTC, Tmac, TNN-PGA-F and TNN-PGA-C for sr = 0.05 and sr = 0.1.

From Figures 5.3 we can see that our algorithms return very good results. Table 5.2 confirms this fact showing an advantage for the cosine transform as compared to the Fourier transform. Figure 5.4 shows that the curves obtained by our algorithms decreases (for RSE) and increases (for PSNR) quickly towards the minimum and the maximum value, respectively. Figure 5.5 shows that the values of the recovered data obtained by the proposed two algorithms are very
Figure 5.4: The evolution of the RSE and the PSNR values on each iteration of the algorithms HaLRTC, Tmac, TNN-PGA-F and TNN-PGA-C for the video of Akiyo with $sr = 0.05$.

Figure 5.5: Comparison of the first 1000th pixels of the video Akiyo obtained by the Algorithms TNN-PGA-F and TNN-PGA-F with $sr = 0.05$ by the original data of the video.

Figure 5.6: The values of RSE and PSNR ones each bound of the recovered video xylophone for $sr = 0.1$.

close to the original data. Figure 5.6 shows the efficiency of our algorithms as compared to other ones for each bound of the used video.
5.2. Porting the python code for information completion to GPU using CuPy.

Parallel computation can be very important in high performance computing due to the limit of the use of a single core. As a consequence, the latest CPU manufacturers compete to have the most cores on a single CPU. For highly parallel problems that can benefit from more cores using a GPU (Graphics Processing Unit), it is the best approach as GPU sacrifices memory for more cores per unit and this leads to having a massively parallel capabilities over CPU. The mainly use of a GPU is to rapidly manipulate and alter memory to accelerate the creation of images.

In our tests, all the GPU accelerated libraries utilize CUDA toolkit libraries which is a parallel computing platform and an API that allows interaction with a GPU in order to perform general purpose processing. That goes beyond just image data manipulation allowing a more general approach called GP-GPU which stands for general purpose computing on graphics processing units. We used Cupy to accelerate some part of the CPU code. CuPy is an open-source library with NumPy syntax that increases speed by doing matrix operations on NVIDIA GPUs. It is accelerated with the CUDA platform from NVIDIA and also uses CUDA-related libraries, including cuBLAS, cuDNN, cuRAND, cuSOLVER, cuSPARSE, and NCCL, to make full use of the GPU architecture. CuPy’s interface is highly compatible with NumPy and in most cases it can be used as a drop-in replacement that can easily integrated in already existing CPU code to boost the performance without much code changes.

5.2.1. Porting the python code to GPU using CuPy.

Cupy provides an easy way to port a Python code using Numpy and Scipy by accelerating them using GPU. The porting process can be simple by replacing some Numpy/Scipy functions by their equivalent in Cupy.

Code of tsvt by fft and dct:

- Before using CuPy

```python
import numpy as np
from scipy.fftpack import fft, ifftn

def tsvt_fourier_1(tensor, tau):
    dim = tensor.shape
    X = np.zeros(dim, dtype=complex)
    tensor = fftn(tensor, axes = (2, ), axes = (3, ))
    for j in range(dim[3]):
        for t in range(dim[2]):
            v = np.linalg.svd(tensor[:, :, j, t], full_matrices = False)
            u = 1 - tau
            s = np.eye(1)
            v = np.diag(s) @ v
            X[:, :, t, j] = X @ np.diag(s) @ v
    return np.real(ifftn(X, axes = (2, ), axes = (3, )))
```
• After using Cupy

```python
import cupy as cp
from cupyx.scipy.fft import fftn, ifftn

def tsvt_fourier_1(tensor, tau):
    dim = np.shape(tensor)
    X = cp.zeros((mx, dy, complex)
    tensor = fftn(tensor, axes = (2, 3), axes = (3, 3))
    for j in range(dim[1]):
        for t in range(dim[2]):
            u, s, v = cp.linalg.svd(tensor[:, :, t, j], full_matrices = False)
            s = s - tau
            s[s < 0] = 0
            X[:, :, t, j] = u @ cp.diag(s) @ v
    cp.cuda.Stream.null.synchronize()
    return cp.real(fft(X, axes = (2, 3), axes = (3, 3)))
```

For the case of tsvt-cosine by CPU we will use the same steps by changing `fftn` and `ifftn` by `dctn` and `idctn`, respectively. This code provides a good example on how we can introduce Cupy to an already existing piece of code by identifying the heavy work functions and see if they have a Cupy equivalent. The most time consuming functions are svd, fftn and ifftn. We can see in the code below that cupy provides a GPU-accelerated implementation of those functions. For the algorithm that uses the DCT function instead of the FFT, the problem was more difficult since Cupy don’t support the parallelizable version.

5.2.2. Numerical experiments of the problem of completion. In Table 5.3 we give the size of all the data test used in our experiments (color videos: fourth-order tensors).

| Name   | Size           |
|--------|----------------|
| xylophone | 240 × 320 × 3 × 30 |
| car         | 1920 × 1080 × 3 × 30 |
| Mgrass      | 2160 × 4096 × 3 × 30 |
| notes       | 3840 × 2160 × 3 × 30 |

Table 5.3: The size of all the data used in the experiments.

In this part we give the results of our codes of completion (TNN-PGA-F and TNN-PGA-C) by
using CPU and GPU computation. In the next we denote by PGA-F and PGA-C the codes using CPU and by PGA-F-GPU and PGA-C-GPU those using GPU.

In Figures 5.7 and 5.8 we compare the evolution of the RSE and the error during the execution of the codes by CPU and GPU for two different videos ‘mglass’ and ‘notes’ with two values of \( sr, sr = 0.05 \) and \( sr = 0.1 \). In Figure 5.9 we give an histogramme representing the required time of PGA-F-CPU, PGA-F-GPU, PGA-C-CPU and PGA-C-GPU for \( sr = 0.1 \).

![Figure 5.7](image)

**Figure 5.7**: The evolution of RSE and the error on each iteration for mglass starting with 5% of the original data

Figures 5.7 and 5.8 show that the RSE and the error does not change when using the Cupy function as they are almost identical to the Numpy and Scipy ones.

In Table 5.4 we show the speed-up between the sequential and parallel computations. We reported the average \( \tau \) defined by using the following formula

\[
\tau = \frac{\text{cpu\_time}}{\text{gpu\_time}}.
\]

(5.4)

As shown in Table 5.4 the obtained speed-up values show how much we can boost the performance of our codes by using the GPU. When the data set is small, as in the case for xylophone, there is no need to use GPU because in that case the returned cpu-time is smaller than the one obtained by GPU. This performance can be explained by the fact that for small problems, the transfer of the data in parallel computation, requires a significant time compared to the classical computation for which no need of transfer data is needed. When the data becomes larger and the targeted accelerated function is taking a significant time from the total runtime we can see a big speed up using the GPU accelerated functions as kernels.
Figure 5.8: The evolution of RSE and the error on each iteration for notes starting with 10% of the original data.

Figure 5.9: Time comparison for data starting with 10% of the original data.
|       | xylophone | car | mglass | notes |
|-------|-----------|-----|--------|-------|
| PGA-C | 0.77      | 1.68| 1.5    | 1.80  |
| PGA-F | 0.86      | 2.11| 2.47   | 2.89  |

Table 5.4: Speed up percentage for the data set that started from 10% from the original data.

6. Conclusion. In this paper we presented a new tensor-tensor product for high orders. Using this product, we defined a new high-order SVD and some related properties. We gave some theoretical results for the tensor product. We used this tensor product for tensor completion using the proximal gradient algorithm. In the numerical section, we showed some test on color-videos and used GPU computation to get fast computation. The presented numerical experiments show the efficiency of our proposed algorithms.

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