Robust matrices in the interval max-plus algebra

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Abstract. With $\varepsilon = -\infty$, the set $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$ where $\mathbb{R}$ denotes the set of all real numbers, is called max-plus algebra when it is equipped with the operations of maximum and addition. The matrices in the size of $m \times n$ whose elements belong to $\mathbb{R}_\varepsilon$ can be formed and so called matrices over max-plus algebra. Robust matrices in the max-plus algebra have been discussed. Suppose $I(\mathbb{R})_\varepsilon = \{x = \begin{bmatrix} x & \varepsilon \end{bmatrix} | x, \varepsilon \in \mathbb{R}, \varepsilon < x \leq \varepsilon \} \cup \{\varepsilon\}$ and $\varepsilon = [\varepsilon, \varepsilon]$. The set $I(\mathbb{R})_\varepsilon$ which is equipped with maximum and addition operations is called interval max-plus algebra. Matrices of size $m \times n$, whose entries belong to $I(\mathbb{R})_\varepsilon$ are called matrices over interval max-plus algebra. In this paper, we discuss robust matrices in the interval max-plus algebra and obtain some of its properties.

1. Introduction

Suppose $\mathbb{R}$ is the set of all real numbers. Max-plus algebra is the set $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$, $\varepsilon = -\infty$, that is equipped with two operations $\oplus$ (maximum) and $\otimes$ (addition). Max-plus algebra has been used to model and analyze problems in planning, communication, production system, queueing system with finite capacity, parallel computation, and traffic ([1]). From the set $\mathbb{R}_\varepsilon$, the set of matrices in the size $m \times n$ with the elements in $\mathbb{R}_\varepsilon$ can be formed, and so called the set of matrices over max-plus algebra and denoted by $\mathbb{R}_\varepsilon^{m \times n}$ ([2], [3]). [4] has discussed about the applications of max-plus algebra in the production system.

Let $A = (A_{ij}) \in \mathbb{R}_\varepsilon^{n \times n}$ be a production matrix where $A_{ij}$ is the time of production process from machine $j$ to machine $i$, while vector $x(k) = (x_i(k)) \in \mathbb{R}_\varepsilon^n$ where $x_i(k)$ is the starting time of machine $i$ at the $k^{th}$ stage. In this production process is obtained the equation $x(k + 1) = A \otimes x(k)$. One of the criteria is used by the manufacturers is that the production process is expected to take place periodically with fixed period $\lambda$ such that the equation $x(k + 1) = \lambda \otimes x(k)$ is obtained. From $x(k + 1) = A \otimes x(k)$ and $x(k + 1) = \lambda \otimes x(k)$, we get $A \otimes x(k) = \lambda \otimes x(k)$ which is known as the problem of eigenvalues and eigenvectors or the problem of eigen (eigen problem). In the eigen problem, we can determine the value of $\lambda$ and vector $x(k)$ which are called eigenvalue and eigenvector of matrix $A$ respectively. [2], [4] and [5] have discussed about the eigen problem. Furthermore, [6] have discussed the eigen problem for powers of irreducible matrices and robust matrices in the max-plus algebra.

To solve the network problems with scheduling activity time as fuzzy numbers such as fuzzy scheduling system and fuzzy queuing system, max-plus algebra has been generalized into the interval max-plus algebra and fuzzy numbers max-plus algebra. Interval max-plus algebra, is the set $I(\mathbb{R})_\varepsilon$ equipped with operations $\oplus$ and $\otimes$, while fuzzy number max-plus algebra is the set $F(\mathbb{R})_\varepsilon$ equipped with operations $\oplus$ and $\otimes$. Let $A = (A_{ij}) \in F(\mathbb{R})_\varepsilon^{n \times n}$ be a fuzzy production matrix where $A_{ij}$ is the time of production process from machine $j$ to machine $i$, while vector $x(k) = (x_i(k)) \in F(\mathbb{R})_\varepsilon^n$ where $x_i(k)$ is the starting time of machine $i$ at the $k^{th}$ stage. In this fuzzy production process is obtained the equation $x(k + 1) = A \otimes x(k)$.
with operations $\oplus$ dan $\otimes$ ([7]). [8] have discussed about the eigen spaces matrix over interval max-plus algebra. In addition, [9] has discussed the problem of eigenvalues for powers of irreducible matrices in the interval max-plus algebra. Based on [8] and [9] as well as in line with [5], in this research will be discussed about the robust matrices in the interval max-plus algebra and its properties.

### 2. Preliminaries

We present the concepts needed in the paper. The definitions and theorems about interval max-plus algebra, matrices over interval max-plus algebra, and graph can be found in [7]. The closed interval $x$ in $\mathbb{R}_e$ is a subset of $\mathbb{R}_e$ which has form $x = [\overline{x}, \overline{x}] = \{ x \in \mathbb{R}_e | \overline{x} \leq x \leq \overline{x} \}$. Interval $x$ in $\mathbb{R}_e$ is called interval max-plus. A number $x \in \mathbb{R}_e$ can be expressed as the interval $[x, x]$.

**Definition 2.1.** Let $I(R)_e = \{ x = [\overline{x}, \overline{x}] | \overline{x}, \overline{x} \in \mathbb{R}, \overline{x} \leq x \leq \overline{x} \} \cup \{ \varepsilon \}$ where $\varepsilon = [\varepsilon, \varepsilon]$. For any $x, y \in I(R)_e$, the maximum $(\oplus)$ and plus $(\otimes)$ operations are defined,

1. $x \oplus y = [\overline{x} \oplus \overline{y}, \overline{x} \oplus \overline{y}]$.
2. $x \otimes y = [\overline{x} \otimes \overline{y}, \overline{x} \otimes \overline{y}]$.

The set $I(R)_e$ equipped with two operations $\oplus$ and $\otimes$ are commutative idempotent semiring with neutral element $\varepsilon = [\varepsilon, \varepsilon]$ and unit element $\overline{0} = [0, 0]$. Furthermore, it is called the interval max-plus algebra and is denoted by $I(\mathbb{R})_{max} = (I(\mathbb{R})_e; \oplus, \otimes)$.

**Definition 2.2.** The set of matrices in the size $m \times n$ elements in $I(R)_e$ is denoted by $I(R)_{e}^{m \times n}$ and is defined by $I(R)_{e}^{m \times n} = \{ A = (A_{ij}) | A_{ij} \in I(R)_e ; i = 1, 2, ..., m, j = 1, 2, ..., n \}$. The element of $I(R)_{e}^{m \times n}$ is called max-plus interval matrix. Max-plus interval matrix is called the interval matrix.

**Definition 2.3.** For $A \in I(R)_{e}^{m \times n}$, two matrices defined $\underline{A} = (A_{ij}) \in \mathbb{R}^{m \times n}_e$ and $\overline{A} = (A_{ij}) \in \mathbb{R}^{m \times n}_e$ are called lower bound and upper bound of $A$, respectively.

**Definition 2.4.** Given the interval matrix $A \in I(R)_{e}^{m \times n}$ where $\underline{A}$ is the lower bound and $\overline{A}$ is the upper bound of matrix $A$. Matrix interval of $A$ is defined by $[\underline{A}, \overline{A}] = \{ A = (A_{ij}) | \underline{A} \leq A \leq \overline{A} \}$ and the set of matrix intervals of $A$ is defined by $I(R)_{e}^{m \times n}_b = \{ [\underline{A}, \overline{A}] : A \in I(R)_{e}^{m \times n} \}$.

**Definition 2.5.**

1. For $\alpha \in I(R)_e$, $[\underline{A}, \overline{A}], [\underline{B}, \overline{B}] \in I(R)_{e}^{m \times n}_b$ is defined
   i. $\alpha \otimes [\underline{A}, \overline{A}] = [\underline{A} \otimes \alpha, \alpha \otimes \overline{A}]$.
   ii. $[\underline{A}, \overline{A}] \oplus [\underline{B}, \overline{B}] = [\underline{A} \oplus \underline{B}, \overline{A} \oplus \overline{B}]$.

2. For $[\underline{A}, \overline{A}] \in I(R)_{e}^{m \times n}_b$, $[\underline{B}, \overline{B}] \in I(R)_{e}^{n \times k}_b$ is defined $[\underline{A}, \overline{A}] \otimes [\underline{B}, \overline{B}] = [\underline{A} \otimes \underline{B}, \alpha \otimes \overline{B}]$.

**Theorem 2.6.** The algebraic structure of $I(R)_{e}^{n \times n}_b$ that is equipped with two operations $\oplus$ and $\otimes$ is denoted by $I(R)_{e}^{n \times n}_{\oplus, \otimes}$ is dioid (idempotent semiring), while $I(R)_{e}^{n \times n}_b$ is semimodule over $I(R)_e$. Semimodule $I(R)_{e}^{n \times n}_b$ over $I(R)_e$ is isomorphic with semimodule $I(R)_{e}^{n \times n}_b$ over $I(R)_e$, with mapping $f : I(R)_{e}^{n \times n}_b \rightarrow I(R)_{e}^{n \times n}_b$.

The interval of matrix $[\underline{A}, \overline{A}] \in I(R)_{e}^{n \times n}_b$ is called corresponding to matrix of interval $A \in I(R)_{e}^{n \times n}$ and denoted by $A \approx [\underline{A}, \overline{A}]$.

**Definition 2.7.** Defined $I(R)_{e}^{n} = \{ x = [x_1, x_2, ..., x_n]^T | x_i \in I(R)_e ; i = 1, 2, ..., n \}$. The set $I(R)_{e}^{n}$ can be considered as the set of $I(R)_{e}^{1 \times n}$. The element of $I(R)_{e}^{n}$ are called vector of interval over $I(R)_e$. Vector of interval $x$ corresponding to interval of vector $[\underline{x}, \overline{x}]$ that is $x \approx [\underline{x}, \overline{x}]$.

Furthermore, the concept of interval weighted directed graph is presented. Suppose a directed graph $D = (N, E)$ where $N = \{1, 2, ..., n\}$ is called interval weighted if every edge $(j, i)$ in $E$ assigned to an closed interval of real $A_{ij} \in I(R)_e - \{[\varepsilon, \varepsilon] \}$. 


The interval of real number $A_j$ is called interval weighted of edge $(j, i)$, is denoted by $w(i, j)$. A precedence graph (communication graph) of the matrix $A \in I(\mathbb{R})_A^{n \times n}$ is defined as an interval weighted directed graph $D_A = (N, E)$ with $N = \{1, 2, ..., n\}$ and $E = \{(j, i) | w(i, j) = A_{ij} \neq [\epsilon, \epsilon]\}$. Conversely, for every the interval weighted directed graph $D_A = (N, E)$ always can be defined a matrix $A \in I(\mathbb{R})_A^{n \times n}$ and so called interval weighted directed graph $D$, that is

$$A_{ij} = \begin{cases} w(j, i), & \text{ jika } (i, j) \in E \\ [\epsilon, \epsilon], & \text{ jika } (i, j) \notin E \end{cases}$$

Some of the results of previous research which support this research are maximum path interval, the weighted average cycle interval ([7]), eigenvector space and basis of a eigenvector space and its dimension which is called the dimension of matrix over interval max-plus algebra ([8]).

**Definition 2.8.** Given $A \in I(\mathbb{R})_A^{n \times n}$ and communication graph $D_A$ for $A$. Suppose $\pi = (i_1, i_2, ..., i_p)$ is a path in $D_A$, weighted $\pi$ is $||A, \pi|| = A_{i_1i_2} + A_{i_2i_3} + ... + A_{i_{p-1}i_p}$, if $p \neq 1$ and $||A, \pi|| = \epsilon$ if $p = 1$.

**Definition 2.9.** Given $A \in I(\mathbb{R})_A^{n \times n}$ and graph communication $D_A$ for $A$. Suppose $\sigma = (i_1, i_2, ..., i_k, i_1)$ a cycle in $D_A$, cycle average (mean cycle) for $\sigma$ is denoted by $\mu(A, \sigma)$ and defined by $\mu(A, \sigma) = \frac{|\sigma(A, \sigma)|}{k}$. Maximum cycle average $\sigma$ of all cycle is denoted by $\lambda(A)$ and defined by $\lambda(A) = \max_k \mu(A, \sigma)$.

**Definition 2.10.** Given $A \in I(\mathbb{R})_A^{n \times n}$ with $A \approx [\tilde{A}, \bar{A}] \in I(\mathbb{R})_b^{n \times n}$, $E \approx [\tilde{E}, \bar{E}] \in I(\mathbb{R})_b^{n \times n}$. $\lambda = [\tilde{\lambda}, \bar{\lambda}] \in I(\mathbb{R})_b$, defined

1. $\Lambda(A) = \{ \tilde{\lambda} \in I(\mathbb{R})_b | V(A, \tilde{\lambda}) \neq \{\epsilon\}; V(\bar{A}, \tilde{\lambda}) \neq \{\epsilon\} \},$
2. $V(A) = \bigcup_{\lambda \in \Lambda(A)} V(A, \lambda),$
3. $V^+(A, \lambda) = V(A, \lambda) \cap I(\mathbb{R})^n,$
4. $V^+(A) = V(A) \cap I(\mathbb{R})^n.$

**Definition 2.11.** Given $A \in I(\mathbb{R})_A^{n \times n}$ with $A \approx [\tilde{A}, \bar{A}] \in I(\mathbb{R})_b^{n \times n}$ and $N = \{1, 2, ..., n\}$ defined $E(A) = E(\tilde{A}) \cap E(\bar{A})$ are eigen points or critical points of weighted interval directed graf corresponding to $A$.

The elements of $E(A)$ are called eigen points or critical points of weighted directed graf corresponding to $\tilde{A}$, the elements of $E(\bar{A})$ are called eigen points or critical points of weighted directed graf corresponding to $\bar{A}$. Cycle $\sigma$ is called as critical cycle if $\mu(A, \sigma) = \lambda(A)$. From the critical points and edge all cycle critical, the directed graph $C(A)$ can be made and so called the critical directed graph for $A$.

**Lemma 2.12.** Suppose $A \in I(\mathbb{R})_A^{n \times n}$. If $C(A)$ is critical directed graph for $A$ then all cycle in $C(A)$ are critical cycle.

Two points $i$ and $j$ in $C(A)$ are said to be equivalent if both $i$ and $j$ are contained in the same critical cycle from $A$ and denoted by $i \sim j$. It can be proved that $\sim$ is the equivalence relation in $E(A)$. The number of maximum set of equivalent eigen vector basis or the number of strongly connected components in $C(A)$ is called the dimension of the eigen space and denoted by $\dim(A)$.

**Definition 2.13.** Suppose $\bar{g}_k$ and $\tilde{g}_k$, $k = 1, 2, ..., n$ each are the columns of the matrix $I(\tilde{A}_k)$ and $I(\bar{A}_k)$. Formed matrix $\Gamma(A_k)$ where its columns is determined as follows:

1. If pair $\bar{g}_k$ and $\tilde{g}_k$ such that $\bar{g}_k \leq \tilde{g}_k$, $\forall k=1,2,3, ..., n$ then obtained one column i.e.

$$\text{interval vector } \bar{g}_k \approx [\bar{g}_k, \tilde{g}_k].$$
2. If pair $g_k$ and $\overline{g}_k$ does not satisfy $g_k \leq \overline{g}_k$, $\forall k = 1, 2, 3, \ldots, n$ then can be formed $\overline{g}_k^* = \delta \otimes \overline{g}_k$, where $\delta = \max_i ( (g_k^i) - (\overline{g}_k^i) )$, $i = 1, 2, 3, ..., n$ and $g_k \approx \begin{bmatrix} g_k \vline \overline{g}_k \end{bmatrix}$.

3. Results and Discussion

In this section, the research results of robust matrices properties in the interval max-plus algebra will be discussed. The following definition of the orbit of a matrix and the set whose elements are the $x$ where the intersection of the orbit matrix $A$ with initial vector $x$ and its set of eigen vector is not empty.

**Definition 3.1.** Suppose $A = (A_{ij}) \in I(\mathbb{R})_{\mathbb{R}}^{n \times n}$, $x \in I(\mathbb{R})_\mathbb{R}^n$. The orbit of $A$ with initial vector $x$ is $O(A, x) = \{A^r \otimes x | r = 0, 1, 2, \ldots \}$.

Let $T(A) = \{x \in I(\mathbb{R})_\mathbb{R}^n | O(A, x) \cap V(A) \neq \emptyset \}$. The theorem which discuss about the necessary and sufficient conditions of the irreducible matrix so that $T(A) = V(A)$ is given.

**Theorem 3.2.** Suppose $A = (A_{ij}) \in I(\mathbb{R})_{\mathbb{R}}^{n \times n}$ irreducible matrix. $T(A) = V(A)$ if only if for every $x \in I(\mathbb{R})_{\mathbb{R}}^n - \{\varepsilon\}$, $A \otimes x \in V(A) \iff x \in V(A)$.

**Proof.** Suppose $A = (A_{ij}) \in I(\mathbb{R})_{\mathbb{R}}^{n \times n}$ irreducible $T(A) = V(A)$. Let $A \approx \begin{bmatrix} A & \overline{A} \end{bmatrix}$ where $A = (A_{ij}) \in \mathbb{R}_{\mathbb{R}}^{n \times n}$ and $\overline{A} = (\overline{A}_{ij}) \in \mathbb{R}_{\mathbb{R}}^{n \times n}$ irreducible. According to the theorem holds in the max-plus algebra, $T(A) = V(A)$ and $T(\overline{A}) = V(\overline{A})$. Let $x \in I(\mathbb{R})_{\mathbb{R}}^n - \{\varepsilon\}$, suppose $x \approx \{x, \overline{x}\}$ that is $x \in \mathbb{R}_{\mathbb{R}}^n - \{\varepsilon\}$ and $\overline{x} \in \mathbb{R}_{\mathbb{R}}^n - \{\varepsilon\}$. According to the theorem holds in the max-plus algebra, $A \otimes x \in V(A) \iff x \in V(A)$ and $\overline{A} \otimes \overline{x} \in V(\overline{A})$. Therefore $T(A) = V(A)$ and $T(\overline{A}) = V(\overline{A})$. As a result, $T(A) = V(A)$.

Furthermore, the definition of robust matrix and its characteristics are presented in the following definition and theorem.

**Definition 3.3.** Suppose $A \in I(\mathbb{R})_{\mathbb{R}}^{n \times n}$. If $T(A) = I(\mathbb{R})_\mathbb{R}^n - \{\varepsilon\}$ then $A$ is called robust.

**Theorem 3.4.** Suppose $A \in I(\mathbb{R})_{\mathbb{R}}^{n \times n}$ where $A \approx \begin{bmatrix} A & \overline{A} \end{bmatrix}$. Matrix $A$ is robust if only if $A$ and $\overline{A}$ robust.

**Proof.** If it is known that a robust matrix, so that $T(A) = I(\mathbb{R})_\mathbb{R}^n - \{\varepsilon\}$. Suppose $x \in T(A)$ implies $x \in I(\mathbb{R})_\mathbb{R}^n$ so that $O(A, x) \cap V(A) \neq \emptyset$. Suppose $x \approx \{x, \overline{x}\}$, implies $\overline{x} \in \mathbb{R}_{\mathbb{R}}^n$ so that $O(A, \overline{x}) \cap V(A) \neq \emptyset$. Since $T(A) = I(\mathbb{R})_\mathbb{R}^n - \{\varepsilon\}$ implies $x \neq \varepsilon$. Thus $x \neq \varepsilon$ and $x \neq \varepsilon$. As a result $x \in T(A)$ and $\overline{x} \in T(\overline{A})$ so that $T(A) = I(\mathbb{R})_\mathbb{R}^n - \{\varepsilon\}$ and $T(\overline{A}) = I(\mathbb{R})_\mathbb{R}^n - \{\varepsilon\}$. Therefore both $A$ and $\overline{A}$ robust. Conversely, $A$ and $\overline{A}$ robust implies $T(A) = I(\mathbb{R})_\mathbb{R}^n - \{\varepsilon\}$ and $T(\overline{A}) = I(\mathbb{R})_\mathbb{R}^n - \{\varepsilon\}$ implies $T(A) = I(\mathbb{R})_\mathbb{R}^n - \{\varepsilon\}$.

The definition of ultimately periodic matrix and its relation to robust matrix are presented in the following definition and theorem.

**Definition 3.5.** Matrix $A \in I(\mathbb{R})_{\mathbb{R}}^{n \times n}$ is called ultimately periodic if there are positif integer $p$ and positif integer $k_0$ for some $\lambda \in I(\mathbb{R})$ such that $A^{k+p} = \lambda \otimes A^k$ for every $k \geq k_0$.

The smallest positif integer $p$ so that $A^{k+p} = \lambda \otimes A^k$ for every $k \geq k_0$ is called period of $A$ and is denoted by $\text{per}(A)$. If $A$ is not ultimately periodic then $\text{per}(A) = \infty$.

**Theorem 3.6.** If $A \in I(\mathbb{R})_{\mathbb{R}}^{n \times n}$ irreducible then $A$ is robust if only if $\text{per}(A) = 1$.

**Proof.** Suppose $A \in I(\mathbb{R})_{\mathbb{R}}^{n \times n}$ robust irreducible matrix and $A \approx \begin{bmatrix} A & \overline{A} \end{bmatrix}$, where $A = (A_{ij}) \in \mathbb{R}_{\mathbb{R}}^{n \times n}$ and $\overline{A} = (\overline{A}_{ij}) \in \mathbb{R}_{\mathbb{R}}^{n \times n}$ irreducible. Likewise $A$ and $\overline{A}$ robust. According to the theorem that applies in the max-plus algebra, $\text{per}(A) = \text{per}(\overline{A}) = 1$. As a result, $\text{per}(A) = 1$. Conversely, it is known that $\text{per}(A) = 1$ then $\overline{A} = \overline{A} = 1$. As a result, $\overline{A}$ and $\overline{A}$ robust matrices implies that $A$ robust matrix.

By using the definition of a robust matrix, ultimately periodic matrix, and Theorem 3.6, following results are obtained.
Corollary 3.7. If \( A \in I(\mathbb{R})_{e}^{n \times n} \) irreducible, per(\( A \)) = 1 and \( x \in I(\mathbb{R})_{e}^{n} - \{\varepsilon\} \) then there is positive integer \( k_{0} \) such that \( A^{k} \otimes x \) finite for any \( k \geq k_{0} \).

Proof. Suppose \( A \in I(\mathbb{R})_{e}^{n \times n} \) irreducible. Suppose \( A \approx [A, \bar{A}] \) where \( A = (A_{ij}) \in \mathbb{R}^{n \times n}_{e} \) and \( \bar{A} = (\bar{A}_{ij}) \in \mathbb{R}^{n \times n}_{e} \) irreducible. Since per(\( A \)) = 1 then per(\( A \)) = per(\( \bar{A} \)) = 1. Let \( x \in I(\mathbb{R})_{e}^{n} - \{\varepsilon\} \). Suppose \( x \approx [\bar{x}, \bar{x}] \) implies \( x \in \mathbb{R}^{n}_{e} - \{\varepsilon\} \) and \( \bar{x} \in \mathbb{R}^{n}_{e} - \{\varepsilon\} \). According to the theorem that applies in the max-plus algebra there are positive integer \( k_{0} \) and \( \bar{k}_{0} \) such that \( A^{k} \otimes x \) finite for any \( k \geq k_{0} \) and \( \bar{A}^{\bar{k}} \otimes \bar{x} \) finite for some \( \bar{k} \geq \bar{k}_{0} \). Therefore, there is positive integer \( k_{0} \) such that \( A^{k} \otimes x \) finite for any \( k \geq k_{0} \).

Furthermore, the theorem about period of an irreducible matrix is presented.

Theorem 3.8. Suppose \( A = (A_{ij}) \in I(\mathbb{R})_{e}^{n \times n} \) irreducible matrix and \( g_{s} = \left[ g_{s1}, g_{s2} \right] \) be called the greatest common divisor (gcd) of the lengths of critical cycles in the \( s^{th} \) strongly connected component of \( C(A) \) where \( g_{s1} \) and \( g_{s2} \) are gcd of the lengths of critical cycles in the \( s^{th} \) strongly connected component of \( C(A) \) and \( C(\bar{A}) \) respectively, then per(\( A \)) = \( \left[ \text{per}(A), \text{per}(\bar{A}) \right] \) is called the least common multiple (lcm) of \( (g_{1}, g_{2}, \ldots) \).

Proof. Suppose \( A \in I(\mathbb{R})_{e}^{n \times n} \) irreducible matrix and \( A \approx [A, \bar{A}] \) where \( A = (A_{ij}) \in \mathbb{R}^{n \times n}_{e} \) and \( \bar{A} = (\bar{A}_{ij}) \in \mathbb{R}^{n \times n}_{e} \) irreducible. Suppose \( g_{s} = \left[ g_{s1}, g_{s2} \right] \) be the greatest common divisor (gcd) of the lengths of critical cycles in the \( s^{th} \) strongly connected component of \( C(A) \) where \( g_{s1} \) and \( g_{s2} \) are gcd of the lengths of critical cycles in the \( s^{th} \) strongly connected component of \( C(A) \) and \( C(\bar{A}) \) respectively. According to the theorem that applies in the max-plus algebra, per(\( A \)) = lcm(\( g_{1}, g_{2}, \ldots \)) and per(\( \bar{A} \)) = lcm(\( g_{3}, g_{4}, \ldots \)). Therefore, per(\( A \)) = \( \left[ \text{per}(A), \text{per}(\bar{A}) \right] \) is called the least common multiple of \( (g_{1}, g_{2}, \ldots) \).

By using the definition Theorem 3.4 and 3.6, the following theorem is obtained.

Theorem 3.9. The irreducible matrix \( A \in I(\mathbb{R})_{e}^{n \times n} \) robust if only if in every strongly connected component of \( C(A) \) the lengths of all critical cycles are co-prime.

Proof. Suppose \( A \in I(\mathbb{R})_{e}^{n \times n} \) irreducible and \( A \approx [A, \bar{A}] \), where \( A = (A_{ij}) \in \mathbb{R}^{n \times n}_{e} \) and \( \bar{A} = (\bar{A}_{ij}) \in \mathbb{R}^{n \times n}_{e} \) irreducible. Suppose \( A \) robust implies that \( A \) and \( \bar{A} \) robust. According to the theorem holds in the max-plus algebra every strongly connected component of \( C(A) \) and \( C(\bar{A}) \) the lengths of all critical cycles are co-prime. As result in strongly connected component of \( C(A) \) are co-prime. Conversely every strongly connected component of \( C(A) \) the lengths of all critical cycles are co-prime. Since \( A \approx [A, \bar{A}] \) so every strongly connected component of \( C(A) \) and \( C(\bar{A}) \) the lengths of all critical cycles are co-prime. According to the theorem that applies in the max-plus algebra than \( A \) and \( \bar{A} \) robust. As a result, \( A \) robust.

The characteristic of robust matrix concerns with eigen space of \( A, A^{2}, \ldots, A^{n} \). Previously definition strongly irreducible matrix is given.

Definition 3.10. The matrix \( A = (A_{ij}) \in I(\mathbb{R})_{e}^{n \times n} \) is called strongly irreducible if \( A^{k} \) is irreducible for every \( k = 1, 2, \ldots \).

Theorem 3.11. The strongly irreducible matrix \( A = (A_{ij}) \in I(\mathbb{R})_{e}^{n \times n} \) robust if only if eigen space of \( A, A^{2}, \ldots, A^{n} \) have same dimension.

Proof. Suppose \( A \in I(\mathbb{R})_{e}^{n \times n} \) strongly irreducible and robust. Suppose \( A \approx [A, \bar{A}] \) where \( A = (A_{ij}) \in \mathbb{R}^{n \times n}_{e} \) and \( \bar{A} = (\bar{A}_{ij}) \in \mathbb{R}^{n \times n}_{e} \) strongly irreducible and implies that both \( A \) and \( \bar{A} \) robust. According to the theorem that implies in the max-plus algebra, eigen space of \( A, A^{2}, \ldots, A^{n} \) and eigen space \( \bar{A}, \bar{A}^{2}, \ldots, \bar{A}^{n} \) have same dimension. Therefore, eigen space of \( A, A^{2}, \ldots, A^{n} \) have same dimension. Conversely eigen space of \( A, A^{2}, \ldots, A^{n} \) have same dimension. Since \( A \approx [A, \bar{A}] \) so eigen space of...
\( A, A^2, \ldots, A^n \) and eigen space \( \overline{A}, \overline{A^2}, \ldots, \overline{A^n} \) have same dimension. According to the theorem that applies in the max-plus algebra, \( A \) and \( \overline{A} \) robust, so \( A \) robust. ■

**Theorem 3.12.** For every strongly irreducible matrix \( A = (A_{ij}) \in I(\mathbb{R})_e^{n \times n} \) robust if only if eigen space of \( A, A^2, \ldots, A^n \) are coincide.

**Proof.** Suppose \( A \in I(\mathbb{R})_e^{n \times n} \) strongly irreducible and robust. Suppose \( A \approx [A, \overline{A}] \) where \( A = (A_{ij}) \in \mathbb{R}_e^{n \times n} \) and \( \overline{A} = (\overline{A}_{ij}) \in \mathbb{R}_e^{n \times n} \) strongly irreducible, and \( A, \overline{A} \) robust. According to the theorem that applies in the max-plus algebra such that eigen space of \( A, A^2, \ldots, A^n \) coincide and also eigen space of \( \overline{A}, \overline{A^2}, \ldots, \overline{A^n} \) similar. As a result, eigen space of \( A, A^2, \ldots, A^n \) coincide. Conversely if every eigen space of \( A, A^2, \ldots, A^n \) coincide implies that eigen space of \( A, A^2, \ldots, \overline{A^n} \) coincide and also eigen space of \( \overline{A}, \overline{A^2}, \ldots, \overline{A^n} \) coincide. According to the theorem that applies in the max-plus algebra, both \( A \) and \( \overline{A} \) robust. As a result, \( A \) robust. ■

By using Theorems 3.11 and 3.12, the following corollaries are obtained.

**Corollary 3.13.** Every strongly irreducible matrix \( A = (A_{ij}) \in I(\mathbb{R})_e^{n \times n} \) robust if only if eigen space of all power matrices of \( A \) coincide.

**Proof.** Suppose \( A \in I(\mathbb{R})_e^{n \times n} \) strongly irreducible and robust. Suppose \( A \approx [A, \overline{A}] \) where \( A = (A_{ij}) \in \mathbb{R}_e^{n \times n} \) and \( \overline{A} = (\overline{A}_{ij}) \in \mathbb{R}_e^{n \times n} \) irreducible. Similarly \( A \) and \( \overline{A} \) robust. According to the theorem that applies in the max-plus algebra, eigen space all power matrices of \( A \) dan \( \overline{A} \). As a result eigen space of all power matrices \( A \) coincide. Conversely, eigen space of all power matrices \( A \) coincide, therefore eigen space of all powers matrices \( A \) dan \( \overline{A} \) coincide. According to the theorem that applies in the max-plus algebra, \( A \) and \( \overline{A} \) robust. As a result, \( A \) robust. ■

**Corollary 3.14.** Every irreducible matrix \( A = (A_{ij}) \in I(\mathbb{R})_e^{n \times n} \) robust if only if \( A_{ii} = \lambda(A) \) for every \( i \in E(A) \).

**Proof.** Let \( A \in I(\mathbb{R})_e^{n \times n} \) irreducible and robust matrix. Suppose \( A \approx [A, \overline{A}] \) where \( A = (A_{ij}) \in \mathbb{R}_e^{n \times n} \) and \( \overline{A} = (\overline{A}_{ij}) \in \mathbb{R}_e^{n \times n} \) irreducible and robust. According to the theorem that applies in the max-plus algebra, \( A_{ii} = \lambda(A) \) and \( \overline{A}_{ii} = \lambda(\overline{A}) \) for every \( i \in E(A) \) and \( i \in E(\overline{A}) \). As a result, \( A_{ii} = \lambda(A) \) for every \( i \in E(A) \). Conversely \( A_{ii} = \lambda(A) \) for every \( i \in E(A) \). Therefore, \( A_{ii} = \lambda(A) \) and \( \overline{A}_{ii} = \lambda(\overline{A}) \) for every \( i \in E(A) \) and \( i \in E(\overline{A}) \). According to the theorem that applies max-plus algebra \( A = (A_{ij}) \in \mathbb{R}_e^{n \times n} \) and \( \overline{A} = (\overline{A}_{ij}) \in \mathbb{R}_e^{n \times n} \) robust. As a result, \( A \) robust. ■

4. Conclusion

In this paper, we obtain some results concerning to characteristic of robust matrix in interval max-plus algebra, namely lower bound and upper bound, period, strongly connected component of all critical cycle and dimension of eigenspace for powers of matrices and its eigenvalue.

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