On a Problem of Harary and Schwenk on Graphs with Distinct Eigenvalues

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Abstract

Harary and Schwenk posed the problem forty years ago: Which graphs have distinct adjacency eigenvalues? In this paper, we obtain a necessary and sufficient condition for an Hermitian matrix with simple spectral radius and distinct eigenvalues. As its application, we give an algebraic characterization to the Harary-Schwenk’s problem. As an extension of their problem, we also obtain a necessary and sufficient condition for a positive semidefinite matrix with simple least eigenvalue and distinct eigenvalues, which can provide an algebraic characterization to their problem with respect to the (normalized) Laplacian matrix.

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\section{Introduction}

Let $\mathcal{M}_n(\mathbb{F})$ be the set of $n$-by-$n$ matrices with entries from a field $\mathbb{F}$, where $\mathbb{F}$ is usually the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$ in this paper. By $\mathbb{C}^n$ and $\mathbb{R}^n$ we denote the $n$ dimensional \textit{real vector space} and the \textit{complex vector space}, respectively.
matrix \( H = [h_{ij}] \in \mathcal{M}_n(\mathbb{C}) \) is said to be Hermitian if \( H = H^* \), where \( H^* = \overline{H}^T = [\overline{h_{ji}}] \).

It is well-known that all the eigenvalues of \( H \) are real. So its eigenvalues can be ordered as \( \lambda_1 \geq \lambda_2 \cdots \geq \lambda_n \), where the set \( \sigma(H) \) of them and the largest one \( \lambda_1 \) of them are, respectively, called the spectrum and the spectral radius of \( H \). Let \( \text{alg}(\lambda) \) and \( \text{geo}(\lambda) \) be, respectively, the algebraic multiplicity and geometric multiplicity of an eigenvalue \( \lambda \). A well-known fact is that \( \text{geo}(\lambda) \leq \text{alg}(\lambda) \). An eigenvalue \( \lambda \) is said to be simple if \( \text{alg}(\lambda) = 1 \).

All graphs considered here are undirected and simple (i.e., loops and multiple edges are not allowed). Let \( G = (V(G), E(G)) \) be a graph with order \( n = |V(G)| \) and size \( m = |E(G)| \). Let \( M = M(G) \) be a corresponding graph matrix defined in a prescribed way. The \( M \)-eigenvalues of \( G \) are the eigenvalues of \( M(G) \). The \( M \)-spectral radius of \( G \) is the largest \( M \)-eigenvalue of \( G \). In the literature there are several graph matrices, including the adjacency matrix \( A \), the degree matrix \( D \), the Laplacian matrix \( L = D - A \), the signless Laplacian matrix \( Q = D + A \) and so on.

Generally, most of the \( A \)-eigenvalues of a graph are distinct. If a graph has only few distinct \( A \)-eigenvalues, then it appears that the graph has a special structure. As an easy example, a connected graph \( G \) has one or two \( A \)-eigenvalues if and only if \( G \) is, respectively, an isolated vertex or a complete graph of order at least two. It has been shown that the strong regular graphs has three distinct \( A \)-eigenvalues. This field is perhaps originally studied by Doob [6]. Subsequently, van Dam makes much important contributions to this topic [4, 5].

On the other side, a graph whose all \( A \)-eigenvalues are distinct is a long standing problem that was proposed by Harary and Schwenk forty years ago (\[7\], see also \[2\], pp. 266):

**Harary-Schwenk Problem:** Which graphs have distinct \( A \)-eigenvalues?

As far as we know, for forty years more, there have been only two results on this problem. The first one is due to Mowshowitz [10].

**Proposition 1.1.** [10]. Let \( G \) be a finite, simple and undirected graph and \( G(X) \) be its group of automorphisms. If \( G \) has distinct \( A \)-eigenvalues, then every nonidentity element in \( G(X) \) is of order 2 (which implies \( G(X) \) is Abelian).

The other one is a generalization of the above theorem, due to Chao [1].

**Proposition 1.2.** [1]. Let \( G \) be a finite and simple graph (directed or undirected, with or without loops) and \( G(X) \) be its group of automorphisms. If the \( A \)-eigenvalues of \( G \) (in the complex number field) are distinct, then \( G(X) \) is Abelian.
Both of the above two results are related to group theory. In this paper, we will, however, use the theory of Hermitian matrices to investigate the Harary-Schwenk Problem. The organization of the paper is as follows: In Section 2 we study the Hermitian matrices with simple spectral radius and distinct eigenvalues. In Section 3 we give a complete algebraic characterization for the Harary-Schwenk Problem. In Section 4 we discuss the positive semidefinite matrix with simple least eigenvalue and distinct eigenvalues, and extend the Harary-Schwenk Problem to the (normalized) Laplacian matrix.

2 Hermitian matrices with simple spectral radius

First of all, let us recall some important properties of Hermitian matrices. To start with, the following one is simple but helpful (see [8], Theorem 1.1.6).

**Proposition 2.1.** Let $B \in \mathcal{M}_n(\mathbb{F})$ and $g(\cdot)$ be a given polynomial. If $\alpha$ is an eigenvector of $B$ associated with $\lambda$, then $\alpha$ is an eigenvector of $g(B)$ associated with $g(\lambda)$.

As pointed out in [8], provided that $H$ and $N$ are Hermitian, $H^k$ ($k = 1, 2, 3, \cdots$) and $aH + bN$ ($a, b \in \mathbb{R}$) are also Hermitian, then the result below is implied.

**Proposition 2.2.** Let $H \in \mathcal{M}_n(\mathbb{C})$ be an Hermitian matrix and $g(\cdot)$ be a real polynomial. Then $g(H)$ is Hermitian.

The next one is the spectral theorem for Hermitian matrices [8].

**Proposition 2.3.** Let $H \in \mathcal{M}_n(\mathbb{C})$ be given. Then $H$ is Hermitian if and only if there is a unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$ and a real diagonal matrix $\Lambda \in \mathcal{M}_n(\mathbb{C})$ such that $H = U \Lambda U^*$. Moreover, $H$ is real Hermitian (i.e., real symmetric) if and only if there is a real orthogonal matrix $B \in \mathcal{M}_n$ and a real diagonal matrix $\Lambda \in \mathcal{M}_n(\mathbb{C})$ such that $A = BAB^*$.

The above proposition shows that if $H$ is Hermitian or real symmetric, then $H$ is diagonalizable. For the diagonalization matrices, we have the following judgements from the matrix theory (see, for example, [8]).

**Proposition 2.4.** Let $B \in \mathcal{M}_n(\mathbb{C})$ and $g(\cdot)$ be a polynomial.

(i) If $B$ is diagonalizable, then $g(B)$ is also diagonalizable;

(ii) $B$ is diagonalizable if and only if the eigenvalues of $B$ are in $\mathbb{C}$, and for each eigenvalue $\lambda$ of $B$, $\text{alg}(\lambda) = \text{geo}(\lambda)$. 

(iii) Let all the distinct eigenvalues of $B$ be $\lambda_1, \lambda_2, \cdots, \lambda_k$. Then $B$ is diagonalizable if and only if its minimal polynomial $m(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_k)$.

As we know, the rank of an Hermitian $H$ is the number of non-zero eigenvalues of $H$. While if the rank is one, the matrix can be expressed as the following form \cite{8}.

**Proposition 2.5.** Let $B \in M_n(\mathbb{C})$. Then $B$ has rank one if and only if there exist two non-zero $n$-vectors $x, y \in \mathbb{C}^n$ such that $B = xy^*$. Moreover, $Bx = (y^*x)x$, where $y^* = y^T$.

Let $I$ and $O$ be the identity matrix and the zero matrix in $M_n(\mathbb{F})$, respectively. For a vector $\alpha \in \mathbb{C}^n$, let $\|\alpha\|_2$ be the Euclidean norm, that is, $\|\alpha\|_2^2 = \alpha^*\alpha$. We are now in the stage to show the following main result of this section.

**Theorem 2.6.** Let $H \in M_n(\mathbb{C})$ be a Hermitian matrix with simple spectral radius. Then $H$ has exactly $k$ ($2 \leq k \leq n$) distinct eigenvalues if and only if there are $k$ distinct real numbers $\lambda_1, \lambda_2, \cdots, \lambda_k$ satisfying

(i) $H - \lambda_i I$ is a singular matrix for $2 \leq i \leq k$;

(ii) $\prod_{i=2}^{k}(H - \lambda_i I) = byy^*$ and $Hy = \lambda_1 y$, where $b \in \mathbb{C}\{0\}$ and $y \in \mathbb{C}^n\{0\}$.

Moreover, $\lambda_1, \lambda_2, \cdots, \lambda_k$ are exactly the $k$ distinct eigenvalues of $G$.

**Proof.** We first show the necessity. Let $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ be the $k$ distinct eigenvalues of $H$. Then, 0 is an eigenvalue of $H - \lambda_i I$ ($2 \leq i \leq k$) and thus (i) follows. For (ii), since $\lambda_1$ is simple, then alg($\lambda_i$) = $m_i$ ($i = 2, 3, \cdots, k$) verifies that

$$m_2 + m_3 + \cdots + m_k = n - 1. \quad (1)$$

Let $f(x) = \prod_{i=2}^{k}(x - \lambda_i)$. Clearly, $f(x)$ is a real polynomial. So by Proposition 2.2

$$f(H) = \prod_{i=2}^{k}(H - \lambda_i I)$$

is Hermitian. From Proposition 2.1 and (1), it follows that the eigenvalues of $f(H)$ are $f(\lambda_1)$ with alg($f(\lambda_1)$) = 1 and $f(\lambda_i) = 0$ ($2 \leq i \leq k$) with alg($0$) = $n - 1$. Hence, the rank of $f(H)$ is one. In line with Proposition 2.5 there exist two non-zero $n$-vectors $x, y \in \mathbb{C}^n$ such that

$$f(H) = xy^* \quad \text{and} \quad f(H)x = (y^*x)x. \quad (2)$$

The second one of (2) indicates that $y^*x$ is just the only one non-zero eigenvalue of $f(H)$. Actually, $y^*x = f(\lambda_1)$. Due to the first one of (2), we get $y^*f(H) = f(\lambda_1)y^*$ which leads to

$$f(H)y = f(H)^*y = f(\lambda_1)y. \quad (3)$$
Assume that $H$ is a real symmetric matrix with simple spectral radius. Then $H$ has exactly $k$ distinct eigenvalues if and only if there are $k$ distinct real numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ satisfying

(i) $H - \lambda_i I$ is a singular matrix for $2 \leq i \leq k$;

(ii) $\prod_{i=2}^{k} (H - \lambda_i I) = b y y^T$, where $b \in \mathbb{R}^+$, $H y = \lambda_1 y$ and $y \in \mathbb{R}^n \setminus \{0\}$.

Moreover, $\lambda_1, \lambda_2, \ldots, \lambda_k$ are exactly the $k$ distinct $A$-eigenvalues of $G$.

Proof. By Theorem 2.6, we only need to prove $b \in \mathbb{R}^+$ and $y \in \mathbb{R}^n$. Employing Theorem 2.6 (i), we get $|H - \lambda_1 I| = 0$, and thus $(H - \lambda_i I)x = 0$ has non-zero real solutions, because $H$ is real symmetric. Hence, $y \in \mathbb{R}^n$. As shown in Corollary 2.7, $x = by$ is the eigenvector of $f(A)$ associated with eigenvalue $f(\lambda_1) = y^T x$. Since $f(H)$
is also real symmetric, then \(x, y \in \mathbb{R}^n\) and thus \(b \in \mathbb{R}\setminus\{0\}\). Assume that \(b\) is negative. Then

\[
f(H) = byy^T = -(\sqrt{-b}y)(\sqrt{-b}y^T) = -\alpha\alpha^T \quad \text{with} \quad \alpha = \sqrt{-b}y.
\]

Let \(\alpha = (a_1, a_2, \cdots, a_n)\) with \(a_i \in \mathbb{R} \ (1 \leq i \leq n)\). Hence, by (7) we get

\[
\text{tr}(f(H)) = -\sum_{i=1}^{n} a_i^2 < 0.
\]

On the other hand, recall that the eigenvalues of \(f(H)\) are \(f(\lambda_1) = \prod_{i=1}^{2}(\lambda_1 - \lambda_i) > 0\) with \(\text{alg}(f(\lambda_1)) = 1\) and \(0\) with \(\text{alg}(0) = n - 1\). Hence, \(\text{tr}(f(H)) = f(\lambda_1) > 0\) contradicting to (8). Therefore, \(b \in \mathbb{R}^+\).

**Remark 2.8.** Alternatively, we can adopt a rather different method to prove the necessity of Theorem 2.6, which will be used in Lemma 4.1 in Section 4. But, the reader will see that the method to show the necessity of Theorem 2.6 is not suitable for Lemma 4.1 and that it can provide more information (see the corollary below).

We now look back on Proposition 2.1 again. Most notably, its converse is not generally true. As an example, if we set \(f(x) = x^3 + x^2 + 6\) and

\[
B = \begin{pmatrix}
1 & 1 & 2 \\
0 & -1 & 0 \\
2 & 0 & -1
\end{pmatrix}, \quad \text{then} \quad f(B) = \begin{pmatrix}
16 & 5 & 10 \\
0 & 6 & 0 \\
10 & 0 & 6
\end{pmatrix}.
\]

Easily to obtain \(\sigma(B) = \{\pm\sqrt{5}, -1\}\) and \(\sigma(f(B)) = \{11 \pm \sqrt{5}, 6\}\). Obviously, \(f(0) = 6 \in \sigma(f(B))\), but \(0 \notin \sigma(B)\). Even so, the derivation from (3) to (6) of Theorem 2.6 provides a special case which makes the converse true.

**Corollary 2.9.** Let \(H \in M_n(\mathbb{C})\) be an Hermitian matrix with simple spectral radius \(\lambda\) and minimal polynomial \(m(x)\). Set \(f(x) = \frac{m(x)}{x^2 - \lambda}\). If \(\alpha\) is an eigenvector of \(f(H)\) associated with eigenvalue \(f(\lambda)\), then \(\alpha\) is an eigenvector of \(H\) associated with \(\lambda\).

### 3 Applications to the Harary-Schwenk Problem

We now apply Corollary 2.7 to tackle the Harary-Schwenk Problem. Clearly, the adjacency matrix \(A\) is real symmetric. If \(G\) is a connected graph, then \(A\) is an irreducible nonnegative matrix. By Perron-Frobenius Theorem we know that the \(A\)-spectral radius is simple and its associated eigenvector is positive. Setting \(\alpha = \sqrt{\beta}y\) in Corollary 2.7 we get the following result.

**Theorem 3.1.** Let \(G\) be connected graph of order \(n \geq 2\). Then \(G\) has exactly \(k\) \((2 \leq k \leq n)\) distinct \(A\)-eigenvalues if and only if there are \(k\) distinct real numbers \(\lambda_1, \lambda_2, \cdots, \lambda_k\) satisfying
(i) \( A - \lambda_i I \) is a singular matrix for \( 2 \leq i \leq k \);

(ii) \[ \prod_{i=2}^{k} (A - \lambda_i I) = \alpha \alpha^T \text{ and } A \alpha = \lambda_1 \alpha, \] where \( \alpha \in \mathbb{R}^n \setminus \{0\} \).

Moreover, \( \lambda_1, \lambda_2, \cdots, \lambda_k \) are exactly the \( k \) distinct \( A \)-eigenvalues of \( G \).

van Dam [5] used the following equality

\[ (A - \lambda_2 I)(A - \lambda_2) = \alpha \alpha^T, \quad \text{with } A \alpha = \lambda_1 \alpha, \]

to show that a regular graph with three distinct \( A \)-eigenvalues must be strongly regular. While, it is just the case \( k = 3 \) in Theorem 3.1.

\textbf{Remark 3.2.} Noteworthily, the case \( k = n \) in Theorem 3.1 is exactly the answer to the Harary-Schwenk Problem, which offers an algebraic characterization. Further investigation is how to determine the structures of such graphs by the theorem, however, this will not be as easy as it is seen.

\textbf{Remark 3.3.} When \( G \) is connected, the signless Laplacian matrix \( Q \) has the same properties as above those of adjacency matrix. So, Theorem 3.1 also holds for the signless Laplacian matrix.

\begin{proof}
In the end of this section, we give a new proof for the relation between the diameter and the number of distinct \( A \)-eigenvalues of a graph; see [2] for example.

\textbf{Corollary 3.4.} Let \( G \) be a connected graph with \( k \) distinct \( A \)-eigenvalues. Then the diameter \( \text{diam}(G) \) of \( G \) is at most \( k - 1 \).

\textbf{Proof.} By Theorem 3.1 (ii) we get

\[ \prod_{i=2}^{k} (A - \lambda_i I) = A^{k-1} + a_1 A^{k-2} + a_2 A^{k-3} + \cdots + a_{k-2} A + a_{k-1} I = \alpha \alpha^T = (b_{ij})_{n \times n}. \quad (9) \]

Since \( \alpha \) is a positive vector, then \( b_{ij} > 0 \). Assume that \( \text{diam}(G) > k - 1 \). By the definition of diameter, for some \( v_i \) and \( v_j \) the elements \( a_{ij}^{(s)} \) from the \( i \)-th row and from the \( j \)-column of the matrices \( A^{(s)} \) (\( 1 \leq s \leq k - 1 \)) satisfy

\[ a_{ij}^{(k-1)} = a_{ij}^{(k-2)} = \cdots = a_{ij} = 0, \]

which together with (9) results in

\[ b_{ij} = a_{ij}^{(k-1)} + \alpha_1 a_{ij}^{(k-2)} + \alpha_2 a_{ij}^{(k-3)} + \cdots + \alpha_{k-2} a_{ij} = 0, \]

a contradiction. Hence, \( \text{diam}(G) \leq k - 1 \). \qed
4 Extensions of Harary-Schwenk Problem

In this section we extend the Harary-Schwenk Problem to other graph matrices. As is known to all, the positive semidefinite matrices must be Hermitian, and the eigenvalues of such matrices are nonnegative real numbers.

Lemma 4.1. Let $H \in \mathcal{M}_n(\mathbb{C})$ be a positive semidefinite matrix with simple least eigenvalue. Then $H$ has exactly $k$ ($2 \leq k \leq n$) distinct eigenvalues if and only if there are $k$ distinct real numbers $\mu_1, \mu_2, \ldots, \mu_k$ satisfying

(i) $H - \mu_i I$ is a singular matrix for $2 \leq i \leq k$;

(ii) $\prod_{i=2}^{k} (H - \mu_i I) = \frac{\prod_{i=2}^{k-1} (\mu_k - \mu_i)}{\|\alpha\|_2^2} \alpha \alpha^*$ and $H \alpha = \mu_k \alpha$, where $\alpha \in \mathbb{C}^n \setminus \{0\}$.

Moreover, $\mu_1, \mu_2, \ldots, \mu_k$ are exactly the $k$ distinct eigenvalues of $H$.

Proof. Let $\mu_1 > \mu_2 > \cdots > \mu_{k-1} > \mu_k$ be the $k$ distinct eigenvalues of $H$. Since $H$ is diagonalizable, by Proposition 2.4 (iii) we get that the minimal polynomial of $H$ is

$$m(x) = (x - \mu_k)(x - \mu_1) \cdots (x - \mu_{k-1}),$$

which leads to

$$(H - \mu_k I) \prod_{i=1}^{k-1} (H - \mu_i I) = 0.$$ 

Let $\alpha = (a_1, a_2, \ldots, a_n)$ be the eigenvector of $H$ associated to eigenvalue $\mu_k$. Since $\text{alg}(\mu_k) = 1$, then by Proposition 2.4 (ii) we have $\text{geo}(\mu_k) = 1$, which indicates that any eigenvector of $H$ associated to the eigenvalue $\mu_k$ is a scalar multiple of $\alpha$. Hence, each column of matrix $\prod_{i=1}^{k-1} (H - \mu_i I)$ can be written in the form $b_i \alpha$ with $b_i \in \mathbb{C}$ ($i = 1, 2, \cdots, n$), and so

$$\prod_{i=1}^{k-1} (H - \mu_i I) = \alpha (b_1, b_2, \cdots, b_n). \quad (10)$$

Since $\alpha^* (H - \mu_i I) = \alpha^* H - \mu_i \alpha = (\mu_k - \mu_i) \alpha^*$, multiplying $\alpha^*$ to both sides of (10), we obtain

$$\prod_{i=1}^{k-1} (\mu_k - \mu_i) \alpha^* = \alpha^* (b_1, b_2, \cdots, b_n) = \|\alpha\|_2^2 (b_1, b_2, \cdots, b_n).$$

Thereby,

$$b_i = \frac{\prod_{i=1}^{k-1} (\mu_k - \mu_i)}{\|\alpha\|_2^2} \alpha_i, \quad i = 1, 2, \cdots, n.$$

Hence, the necessity follows.
The proof of the sufficiency is similar to that of Theorem 2.6. From (i) it follows that the system of homogeneous linear equations \((H - \mu_i I)x = 0\) has a non-zero solution, say \(\alpha_i\), and thus \(H\alpha_i = \mu_i\alpha_i\) which indicates that \(\mu_i\) is an eigenvalue of matrix \(H\) \((2 \leq i \leq k)\). From (ii) we get that 0 is an eigenvalue of \(H\). Therefore, we have shown that \(H\) has \(k\) distinct eigenvalues \(\mu_1, \mu_2, \cdots, \mu_{k-1}, 0\). Assume that \(H\) has an extra eigenvalue \(\mu_{k+1}\). Recall that \(f(x) = \prod_{i=1}^{k-1} (x - \lambda_i)\) and that \(f(\mu_i)\) \((1 \leq i \leq k+1)\) is the eigenvalue of \(f(H)\). Obviously, \(f(\lambda_i) = 0\) \((1 \leq i \leq k-1)\), \(f(0) \neq 0\) and \(f(\mu_{k+1}) \neq 0\). By (ii) and Proposition 2.5, the rank of \(f(H)\) is one, and so \(f(H)\) has only one none-zero simple eigenvalue, a contradiction.

This finishes the proof.

\[\Box\]

**Remark 4.2.** Go back to Remark 2.8. Applying the method used in Lemma 4.1 to the necessity of Theorem 2.6, we get \(b = \frac{\prod_{i=2}^{k-1}(\lambda_i - \lambda_1)}{\|\alpha\|^2} > 0\).

The following corollary immediately follows from Lemma 4.1.

**Corollary 4.3.** Let \(H \in \mathcal{M}_n(\mathbb{R})\) be a positive semidefinite matrix with simple least eigenvalue. Then \(H\) has exactly \(k\) \((2 \leq k \leq n)\) distinct eigenvalues if and only if there are \(k\) distinct real numbers \(\mu_1, \mu_2, \cdots, \mu_k\) satisfying

(i) \(H - \mu_i I\) is a singular matrix for \(2 \leq i \leq k\);

(ii) \(\prod_{i=2}^{k}(H - \mu_i I) = \frac{\prod_{i=2}^{k-1}(\mu_i - \mu_{i-1})}{\|\alpha\|^2} \alpha^T\) and \(H\alpha = \mu_k\alpha\), where \(\alpha \in \mathbb{R}^n \setminus \{0\}\).

Moreover, \(\mu_1, \mu_2, \cdots, \mu_k\) are exactly the \(k\) distinct eigenvalues of \(H\).

It is generally known that the Laplacian matrix \(L\) of a graph \(G\) is real symmetric and positive semidefinite. Moreover, 0 is the least \(L\)-eigenvalue with eigenvector \(\alpha^T = (1, 1, \cdots, 1) \in \mathbb{R}^n\), and \(\text{alg}(0)\) is equal to the number of the connected components of \(G\). So, if \(G\) is connected, then 0 is a simple \(L\)-eigenvalue. Consequently, substituting \(\mu_k = 0\), \(\alpha\alpha^T = J\) (the all-ones matrix) and \(\|\alpha\|^2 = n\) into Corollary 4.3 we get the following result.

**Theorem 4.4.** Let \(G\) be a connected graph with order \(n\). Then \(G\) has \(k\) \((2 \leq k \leq n)\) distinct \(L\)-eigenvalues if and only if there are \(k-1\) distinct non-zero real numbers \(\mu_1, \mu_2, \cdots, \mu_{k-1}\) satisfying

(i) \(L - \mu_i I\) is a singular matrix for \(2 \leq i \leq k\);

(ii) \(\prod_{i=2}^{k}(L - \mu_i I) = (-1)^{k-1}\frac{\prod_{i=2}^{k-1}\mu_i}{n} J\), where \(J\) is the all-ones matrix.

Moreover, \(\mu_1, \mu_2, \cdots, \mu_{k-1}, 0\) are exactly the \(k\) distinct \(L\)-eigenvalues of \(G\).
We finally turn to the normalized Laplacian matrix of a graph $G$, which is introduced by Chung [3] and defined as
\[ \mathcal{L} = D^{-\frac{1}{2}}LD^{\frac{1}{2}}. \]
Mohar [9] calls this matrix the transition Laplacian. As a fact, $\mathcal{L}$ is also real symmetric and positive semidefinite. Moreover, if $G$ is connected, 0 is a simple least $\mathcal{L}$-eigenvalue with eigenvector $\alpha^T = (\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n})$, where $d_i$ is the degree of vertex $v_i$ ($1 \leq i \leq n$). Substituting $\mu_k = 0$ and $\|\alpha\|^2 = \sum_{i=1}^{n} d_i$ into Corollary 4.3 we obtain the following results.

**Theorem 4.5.** Let $G$ be a connected graph with order $n$ and size $m$. Then $G$ has $k$ distinct $\mathcal{L}$-eigenvalues if and only if there are $k - 1$ ($2 \leq k \leq n$) distinct non-zero real numbers $\mu_1, \mu_2, \cdots, \mu_{k-1}$ satisfying

(i) $\mathcal{L} - \mu_i I$ is a singular matrix for $2 \leq i \leq k$;

(ii) $\prod_{i=2}^{k} (\mathcal{L} - \mu_i I) = (-1)^{k-1} \frac{\prod_{i=1}^{k-1} \mu_i}{2m} \alpha \alpha^T$, where $\alpha^T = (\sqrt{d_1}, \sqrt{d_2}, \cdots, \sqrt{d_n})$.

Moreover, $\mu_1, \mu_2, \cdots, \mu_{k-1}, 0$ are exactly the $k$ distinct $\mathcal{L}$-eigenvalues of $G$.

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