800 CONICS IN A SMOOTH QUARTIC SURFACE

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Abstract. We construct an example of a smooth spatial quartic surface that contains 800 irreducible conics.

1. Introduction

This short note was motivated by Barth, Bauer [1], Bauer [2], and my recent paper [5]. Generalizing [2], define $N_{2n}(d)$ as the maximal number of smooth rational curves of degree $d$ that can lie in a smooth degree $2n$ $K3$-surface $X \subset \mathbb{P}^{n+1}$. (All algebraic varieties considered in this note are over $\mathbb{C}$.) The bounds $N_{2n}(1)$ have a long history and currently are well known, whereas for $d = 2$ the only known value is $N_0(2) = 285$ (see [5]). In the most classical case $2n = 4$ (spatial quartics), the best known examples have 352 or 432 conics (see [1, 2]), whereas the best known upper bound is 5016 (see [2], with a reference to S. A. Strømme).

For $d = 1$, the extremal configurations (for various values of $n$) tend to exhibit similar behaviour. Hence, contemplating the findings of [5], one may speculate that

- it is easier to count all conics, both irreducible and reducible, but
- nevertheless, in extremal configurations all conics are irreducible.

On the other hand, famous Schur’s quartic (the one on which the maximum $N_4(1)$ is attained) has 720 conics (mostly reducible), suggesting that 432 should be far from the maximum $N_4(2)$. Therefore, in this note I suggest a very simple (although also implicit) construction of a smooth quartic with 800 irreducible conics.

Theorem 1.1 (see §3.3). There exists a smooth quartic surface $X_4 \subset \mathbb{P}^3$ containing 800 irreducible conics.

The quartic $X_4$ is Kummer in the sense of [1, 2]: it contains 16 disjoint conics. I conjecture that $N_4(2) = 800$ and, moreover, 800 is the sharp upper bound on the total number of conics (irreducible or reducible) in a smooth spatial quartic.

There has been a considerable development after this note appeared in the arXiv. X. Rouleau observed that, computing the projective automorphism group and using [3], $X_4$ in Theorem 1.1 must be given by the Mukai polynomial

$$z_0^4 + z_1^4 + z_2^4 + z_3^4 + 12z_0z_1z_2z_3 = 0;$$

even though only 320 conics were found in [3]. Then, B. Naskręcki found explicit equations of all 800 conics.

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2. The Leech lattice (see [4])

2.1. The Golay code. The (extended binary) Golay code is the only binary code of length 24, dimension 12, and minimal Hamming distance 8. We regard codewords as subsets of \( \Omega := \{1, \ldots, 24\} \) and denote this collection of subsets by \( \mathcal{C} \); clearly, \( |\mathcal{C}| = 2^{12} \). The code \( \mathcal{C} \) is invariant under the complement \( o \mapsto \Omega \setminus o \). Apart from \( \emptyset \) and \( \Omega \) itself, it consists of 759 octads (codewords of length 8), 759 complements thereof, and 2576 dodecads (codewords of length 12).

The setwise stabilizer of \( \mathcal{C} \) in the full symmetric group \( S(\Omega) \) is the Mathieu group \( M_{24} \) of order 244823040; the actions of this group on \( \Omega \) and \( \mathcal{C} \) are described in detail in § 2 of [4, Chapter 10].

2.2. The square 4 vectors. The Leech lattice is the only root-free unimodular even positive definite lattice of rank 24. For the construction, consider the standard Euclidean lattice \( E := \bigoplus \mathbb{Z} e_i, \ i \in \Omega \), and divide the form by 8, so that \( e_i^2 = 1/8 \). (Thus, we avoid the factor \( 8^{-1/2} \) appearing throughout in [4].) Then, the Leech lattice is the sublattice \( \Lambda \subset E \) spanned over \( \mathbb{Z} \) by the square 4 vectors of the form

\[
(\mp 3, \pm 1^{23}) \quad \text{(the upper signs are taken on a codeword} \ o \in \mathcal{C}).
\]

(We use the notation of [4]: \( a^m, b^n, \ldots \) means that there are \( m \) coordinates equal to \( a \), \( n \) coordinates equal to \( b \), etc.) Apart from (2.1), the square 4 vectors in \( \Lambda \) are

\[
(\pm 2^8, 0^{16}) \quad \text{(±2 are taken on an octad, the number of + is even)}, \quad \text{or}
\]

\[
(\pm 4^2, 0^{22}) \quad \text{(no further restrictions)}.
\]

Altogether, there are 196560 square 4 vectors: \( 24 \cdot |\mathcal{C}| = 98304 \) vectors as in (2.1), \( 2^7 \cdot 759 = 97152 \) vectors as in (2.2), and \( 2^2 \cdot C(24, 2) = 1104 \) vectors as in (2.3).

3. The construction

In this section, we prove Theorem 1.1.

3.1. The lattice \( S \). Consider the lattice \( V := \mathbb{Z} h + \mathbb{Z} a + \mathbb{Z} u_1 + \mathbb{Z} u_2 + \mathbb{Z} u_3 \) with the Gram matrix

\[
\begin{bmatrix}
4 & 2 & 0 & 0 & 0 \\
2 & 4 & 2 & 0 & 1 \\
0 & 2 & 4 & 2 & -1 \\
0 & 0 & 2 & 4 & 0 \\
0 & 1 & -1 & 0 & 4
\end{bmatrix}.
\]

It can be shown that, up to \( O(\Lambda) \), there is a unique primitive isometric embedding \( V \rightarrow \Lambda \); however, for our example, we merely choose a particular model. Fix an ordered quintuple \( Q := (1, 2, 3, 4, 5) \subset \Omega \) and choose one of the four octads \( O \) such that \( O \cap Q = \{1, 2, 4, 5\} \) (cf. sextets in § 2.5 of [4, Chapter 10]); upon reordering \( \Omega \), we can assume that \( O = \{1, 2, 4, 5, 6, 7, 8, 9\} \) (the underlined positions in the top row of Table 1). Then, the generators of \( V \) can be chosen as shown in the upper part of Table 1. (For better readability, we represent zeros by dots; all components beyond \( O := Q \cup O \) are zeros.)

The choice of \( Q \) and \( O \) is unique up to \( M_{24} \); furthermore, the subgroup \( G \subset M_{24} \) stabilising \( Q \) pointwise and \( O \) as a set can be identified with the alternating group \( A(\Omega \setminus Q) \); in particular, it acts simply transitively on the set of ordered pairs

\[
(p, q) : \ p, q \in O \setminus Q = \{6, 7, 8, 9\}, \quad p \neq q.
\]
Table 1. The lattice $V$ and the conics

| # | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbb{H}$ | 4 | 4 | · | · | · | · | · | · | · |
| $a$ | · | 4 | 4 | · | · | · | · | · | · |
| $u_1$ | · | · | 4 | 4 | · | · | · | · | · |
| $u_2$ | · | · | 4 | 4 | · | · | · | · | · |
| $u_3$ | −2 | 2 | · | −2 | 2 | 2 | 2 | 2 | 2 |

1: 13 $-1$ $1$ $-1$ 1 1 $-1^*$ $-1^*$ $\pm 1^{15}$
2: 3 1 $1$ $-1$ 1 1 $-1^*$ $-1^*$ $\pm 1^{15}$
3: 2 2 · · · · · · · · $\pm 2^6, 0^9$
4: 2 2 · · · · · $2^* -2^*$ $\pm 2^4, 0^{11}$

Table 2. The number of conics in $S$

1: $C(4,2) \cdot 16 = 96$ (codewords $o \in C$ such that $o \cap \bar{O} = \{2, 3, 5, p, q\}$),
2: $C(4,2) \cdot 16 = 96$ (codewords $o \in C$ such that $o \cap \bar{O} = \{1, 4, p, q\}$),
3: $2^5 \cdot 10 = 320$ (octads $o \in C$ such that $o \cap \bar{O} = \{1, 2\}$),
4: $2^3 \cdot P(4,2) \cdot 3 = 288$ (octads $o \in C$ such that $o \cap \bar{O} = \{1, 2, p, q\}$).

Define a conic as a square 4 vector $l \in \Lambda$ such that

\[ l \cdot h = 2, \quad l \cdot a = 1, \quad l \cdot u_1 = l \cdot u_2 = l \cdot u_3 = 0. \]

This strange condition can be recast as follows: $l \cdot h = 2$ and $l$ (as well as $h$) lies in the rank 20 lattice

\[ S := \bar{V}^\perp \subset \Lambda, \quad \text{where } \bar{V} := h^\perp. \]

Using §2.2, we conclude that each conic fits one of the four patterns shown at the bottom of Table 1: there are two for (2.1) and two for (2.2). (If $l$ is as in (2.3), we have $l \cdot a = 0 \mod 2$.) The number of conics within each pattern is computed as shown in Table 2, where

- the ordered or unordered pair $(p, q)$ as in (3.1) designates the two variable special positions marked with a $^*$ in Table 1,
- the underlined factor counts certain codewords $o \in C$; the restrictions given by (2.1) or (2.2) are described in the parentheses, and
- the other factors account for the choice of $(p, q)$ and/or signs in $\pm 2$.

These counts sum up to 800.

3.2. The Néron–Severi lattice. Observe that $h \in 2S^\vee$: indeed, $h - 2a \in \bar{V}$ and we have $x \cdot h = 2x \cdot a = 0 \mod 2$ for any $x \in S$. Thus, we can apply to $S \ni h$ the construction of [5], i.e., consider the orthogonal complement $h_S^\perp = V^\perp \subset \Lambda$, reverse the sign of the form, and pass to the index 2 extension

\[ N := (-(h_S^\perp) \oplus \mathbb{Z}h)^\perp_2, \quad h^2 = 4, \]

containing the vector $c := c(l) := l - \frac{1}{2}h + \frac{1}{2}h$ for some (equivalently, any) conic $l \in S$. These 800 new vectors $c \in N$ are also called conics; one obviously has

\[ c^2 = -2, \quad c \cdot h = 2. \]
Starting from

\[ \text{discr } V \cong \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \oplus \left[ \frac{1}{8} \right] \oplus \left[ \frac{2}{7} \right] \]

(see Nikulin [7] for the concept of discriminant form \( \text{discr } V := V^\vee/V \) and related techniques), we easily compute

\[ \mathcal{N} := \text{discr } N \cong \left[ \frac{1}{8} \right] \oplus \left[ \frac{1}{2} \right] \oplus \left[ \frac{2}{7} \right] \cong \left[ \frac{1}{4} \right] \oplus \left[ \frac{-1}{8} \right] \oplus \left[ \frac{2}{7} \right]. \]

Therefore, \(-\mathcal{N} \cong \text{discr } T\), where \( T := \mathbb{Z}b \oplus \mathbb{Z}c \), \( b^2 = 4 \), \( c^2 = 40 \). Then, it follows from [7] that there is a primitive isometric embedding of the hyperbolic lattice \( N \) to the intersection lattice \( H_2 \) of a \( K3 \)-surface, so that \( T \cong N^\perp \) plays the role of the transcendental lattice. Finally, by the surjectivity of the period map [6], we conclude that there exists a \( K3 \)-surface \( X \) with \( \text{NS}(X) \cong N \).

3.3. Proof of Theorem 1.1. The Néron–Severi lattice \( \text{NS}(X) \cong N \) constructed in the previous section is equipped with a distinguished polarisation \( h \in N \), \( h^2 = 4 \). Since the original lattice \( S \subset \Lambda \) is root free, \( N \) does not contain any of the following “bad” vectors:

- \( e \in N \) such that \( e^2 = -2 \) and \( e \cdot h = 0 \) (exceptional divisors) or
- \( e \in N \) such that \( e^2 = 0 \) and \( e \cdot h = 2 \) (2-isotropic vectors)

(see [5] for details). Hence, by Nikulin [8] and Saint-Donat [9], the linear system \( |h| \) is fixed point free and maps \( X \) onto a smooth quartic surface \( X_4 \subset \mathbb{P}^4 \).

The lattice \( N \) contains 800 conics \( c \) as in (3.2). By the Riemann–Roch theorem, each class \( c \) is effective, i.e., represented by a curve \( C \subset X_4 \) of projective degree 2. Since \( X \) is smooth and contains no lines (or other curves of odd degree, as we have \( h \in 2N^\vee \) by the construction), each of these curves \( C \) is irreducible. This concludes the proof of Theorem 1.1. \( \square \)

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