DEFORMATIONS OF MAXIMAL REPRESENTATIONS IN \( \text{Sp}(4, \mathbb{R}) \)

STEVEN B. BRADLOW, OSCAR GARCÍA-PRADA, AND PETER B. GOTHEN

ABSTRACT. We use Higgs bundles to answer the following question: When can a maximal \( \text{Sp}(4, \mathbb{R}) \)-representation of a surface group be deformed to a representation which factors through a proper reductive subgroup of \( \text{Sp}(4, \mathbb{R}) \)?

1. Introduction

A good way to understand an object of study, as Richard Feynman famously remarked\(^1\), is to “just look at the thing”. In this paper we apply Feynman’s method to answer the following question: given a surface group representation in \( \text{Sp}(4, \mathbb{R}) \), under what conditions can it be deformed to a representation which factors through a proper reductive subgroup of \( \text{Sp}(4, \mathbb{R}) \)?

A surface group representation in a group \( G \) is a homomorphism from the fundamental group of the surface into \( G \). For a surface of genus \( g \geq 2 \), the moduli space of reductive surface group representations into \( G = \text{Sp}(4, \mathbb{R}) \), denoted by \( \mathcal{R}(\text{Sp}(4, \mathbb{R})) \), has \( 3 \cdot 2^{2g+1} + 8g - 13 \) connected components (see [17, 22]). The components are partially labeled by an integer, known as the Toledo invariant, which ranges between \( 2 - 2g \) and \( 2g - 2 \). If \( \mathcal{R}_d \) denotes the component with Toledo invariant \( d \), then there is a homeomorphism \( \mathcal{R}_d \simeq \mathcal{R}_{-d} \) and except for the extremal cases (i.e. \( |d| = 2g - 2 \)) each \( \mathcal{R}_d \) is connected. In contrast, the subspace of maximal representations \( \mathcal{R}^{\text{max}} = \mathcal{R}_{2g-2} \) have \( 3 \cdot 2^{2g} + 2g - 4 \) components. These are our objects of study. The precise question we answer is thus: which maximal components contain representations that factor through reductive subgroups of \( \text{Sp}(4, \mathbb{R}) \)?

One motivation for this question stems from the fundamental work of Goldman [18, 20] and Hitchin [20]. Goldman showed that, in the case of \( \text{PSL}(2, \mathbb{R}) \), the space of maximal representations coincides with Teichmüller space, i.e., the space of Fuchsian representations. Using Higgs bundles, Hitchin constructed distinguished components in the moduli space of reductive representations in the split real form of any complex reductive group. These components, known as Hitchin components, have been the subject of much interest, see for example Burger–Iozzi–Labourie–Wienhard, [4], Fock–Goncharov [14], Guichard-Wienhard [23] and Labourie [30, 31].

Moreover, the representations in these components factor through homomorphisms from \( \text{SL}(2, \mathbb{R}) \) into the split real form. In the case of \( \text{Sp}(4, \mathbb{R}) \) there are \( 2^{2g} \) Hitchin components,
all of which are maximal and contain representations which factor through the irreducible representation of $\text{SL}(2, \mathbb{R})$ in $\text{Sp}(4, \mathbb{R})$. One is thus led to ask whether the other $2^{2g+1} + 2g - 4$ components have similar factorization properties.

In the case of $\text{Sp}(4, \mathbb{R})$ there are $2^{2g}$ Hitchin components. They are projectively equivalent, in the sense that they project to a unique Hitchin component in the moduli space for the projective symplectic group $\text{PSp}(4, \mathbb{R})$. The $\text{Sp}(4, \mathbb{R})$ Hitchin components are all maximal and all contain representations which factor through the irreducible representation of $\text{SL}(2, \mathbb{R})$ in $\text{Sp}(4, \mathbb{R})$. One is thus led to ask whether the other $2^{2g+1} + 2g - 4$ maximal components have similar factorization properties.

To answer our question we need a microscope with which we can “just look at” the components of $\mathcal{R}^{max}$. Higgs bundles provide the tool we need. A Higgs bundle is a holomorphic bundle together with a Higgs field, i.e. a section of a particular associated vector bundle. Such objects appear in the context of surface group representations as follows. Given a real orientable surface, say $S$, and any real reductive Lie group, say $G$, representations of $\pi_1(S)$ in $G$ depend only on the topology of $S$, i.e. on its genus. Fixing a conformal structure, or equivalently a complex structure, transforms $S$ into a Riemann surface (denoted by $X$). This opens the way for holomorphic techniques and brings in Higgs bundles. The group $G$ appears as the structure group of the Higgs bundles, which are hence called $G$-Higgs bundles. By the non-abelian Hodge theory correspondence ([25, 11, 39, 9, 15]), reductive representations of $\pi_1(X)$ in $G$ correspond to polystable $G$-Higgs bundles, and the representation variety, i.e. the space of conjugacy classes of reductive representations, corresponds to the moduli space of polystable Higgs bundles.

Taking $G = \text{Sp}(4, \mathbb{R})$ we denote the moduli space of polystable $\text{Sp}(4, \mathbb{R})$-Higgs bundles by $\mathcal{M}(\text{Sp}(4, \mathbb{R}))$ (or simply $\mathcal{M}$). The non-abelian Hodge theory correspondence then gives a homeomorphism $\mathcal{M} \simeq \mathcal{R}(\text{Sp}(4, \mathbb{R}))$. Let $\mathcal{M}^{max} \subset \mathcal{M}$ be subspace corresponding to $\mathcal{R}^{max}$ under this homeomorphism. If a representation in $\text{Sp}(4, \mathbb{R})$ factors through a subgroup, say $G_\ast \subset \text{Sp}(4, \mathbb{R})$, then the structure group of the corresponding $\text{Sp}(4, \mathbb{R})$-Higgs bundle reduces to $G_\ast$. Through the lens of our Higgs bundle microscope, the question we examine thus becomes: which components of $\mathcal{M}^{max}$ contain polystable $\text{Sp}(4, \mathbb{R})$-Higgs bundles for which the structure group reduces to a subgroup $G_\ast$? This is the question we answer.

The geometry of the hermitean symmetric space $\text{Sp}(4, \mathbb{R})/U(2)$, together with results of Burger, Iozzi and Wienhard [5, 6] (see Section 4) constrain $G_\ast$ to be one of the following three subgroups

- $G_\ast = \text{SL}(2, \mathbb{R})$, embedded via the irreducible representation of $\text{SL}(2, \mathbb{R})$ in $\text{Sp}(4, \mathbb{R})$,
- $G_p$, the normalizer of the product representation
  $$\rho_p : \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \longrightarrow \text{Sp}(4, \mathbb{R}),$$
- $G_\Delta$, the normalizer of the composition of $\rho_p$ with the diagonal embedding of $\text{SL}(2, \mathbb{R})$ in $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$.

For each possible $G_\ast$ we analyze what $G_\ast$-Higgs bundles look like and then, following Feynman’s dictum, we simply check to see which components of $\mathcal{M}^{max}$ contain Higgs bundles of the required type. In practice this means that we carefully describe the structure of maximal $\text{Sp}(4, \mathbb{R})$-Higgs bundles and compare it to that of the $G_\ast$-Higgs bundles.

Our results for each of the possible subgroups are given by Theorems 6.17, 7.12, and 8.16. These lead to our main result, Theorem 5.3, whose essential point is the following.
**Theorem 1.1.** Of the $3 \cdot 2^g + 2g - 4$ components of $M^{\text{max}}$

(1) $2^g$ are Hitchin components in which the corresponding Higgs bundles deform to maximal $\text{SL}(2, \mathbb{R})$-Higgs bundles,

(2) $2 \cdot 2^g - 1$ components have the property that the corresponding Higgs bundles deform to Higgs bundles which admit a reduction of structure group to $G_p$, and also deform to ones which admit a reduction of structure group to $G_{\Delta}$, and

(3) $2g - 3$ components have the property that the corresponding Higgs bundles do not admit a reduction of structure group to a proper reductive subgroup of $\text{Sp}(4, \mathbb{R})$.

The corresponding result for surface group representations, given in Theorem 5.4, says the following:

**Theorem 1.2.** Of the $3 \cdot 2^g + 2g - 4$ components of $R^{\text{max}}$

(1) $2^g$ are Hitchin components, i.e. the corresponding representations deform to ones which factor through (Fuchsian) representations into $\text{SL}(2, \mathbb{R})$,

(2) $2 \cdot 2^g - 1$ components have the property that the corresponding representations deform to ones which factor through $G_p$, and also deform to ones which factor through $G_{\Delta}$, and

(3) $2g - 3$ components have the property that the corresponding representations do not factor through any proper reductive subgroup of $\text{Sp}(4, \mathbb{R})$.

In fact part (1) of Theorems 1.1 and 1.2 follows from Hitchin’s general construction in [26]. It is nevertheless instructive to see the explicit details of the construction in our particular case, namely $G = \text{Sp}(4, \mathbb{R})$, and to view the results from a new perspective. The results about the other maximal components and the other possible subgroups are new. They raise the interesting problem of gaining a better understanding of the representations which do not deform to representations which factor through a proper reductive subgroup of $\text{Sp}(4, \mathbb{R})$.

In addition to the main results in Theorems 1.1 and 1.2 we also give (in Section 3.7) explicit descriptions of some of the components. Together with the main theorems, these have consequences whose import goes beyond the specific case of $G = \text{Sp}(4, \mathbb{R})$. In particular the $2g - 3$ components where representations do not factor through any reductive subgroup are remarkable for the following reasons:

- the representations in these components all have Zariski dense image in $\text{Sp}(4, \mathbb{R})$.
- the components are smooth but, unlike the Hitchin components, topologically non-trivial.

The group $G = \text{Sp}(4, \mathbb{R})$ is thus an example of a Lie group with rank greater than 1 for which the moduli space of surface group representations into $G$ has components with these properties. To the best of our knowledge this is the first such example. Furthermore, by results of Labourie ([30]) and Wienhard ([46]), the mapping class group is known to act properly discontinuously on $R^{\text{max}}$. The components we describe thus give examples of non-trivial manifolds which carry such actions of the mapping class group.

We note, finally, that the case $G = \text{Sp}(4, \mathbb{R})$ has features not shared by $\text{Sp}(2n, \mathbb{R})$ for $n > 2$. In particular, the moduli space of representations (or Higgs bundles) has anomalously large number of connected components when $n = 2$, compared to the case $n \geq 3$. Moreover, we prove in Corollary 9.4 that, when $n \geq 3$, there are no components of $R^{\text{max}}$ in which all $2^g - 3$ components have the property that the corresponding representations do not factor through any proper reductive subgroup of $\text{Sp}(2n, \mathbb{R})$.

\[\text{2 The recent preprint [24] takes interesting steps in this direction.}\]

\[\text{3We thank an anonymous referee for articulating some of these comments}\]
representations have Zariski dense image. The case $n = 2$ thus demands treatment as a special case.

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2. Basic background on Higgs bundles and representations

2.1. Higgs bundles. Our main tool for exploring surface group representations is the relation between such representations and Higgs bundles. We are interested primarily in representations in $\text{Sp}(4, \mathbb{R})$, but it is useful to state the general definition.

Let $G$ be a real reductive Lie group. Following Knapp [27, p. 384], by this we mean that we are given the data $(G, H, \theta, B)$, where $H \subset G$ is a maximal compact subgroup (cf. [27, Proposition 7.19(a)]), $\theta: \mathfrak{g} \to \mathfrak{g}$ is a Cartan involution and $B$ is a non-degenerate bilinear form on $\mathfrak{g}$, which is Ad($G$)-invariant and $\theta$-invariant. The data $(G, H, \theta, B)$ has to satisfy in addition that

- the Lie algebra $\mathfrak{g}$ of $G$ is reductive
- $\theta$ gives a decomposition (the Cartan decomposition)
  $$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$
  into its $\pm 1$-eigenspaces, where $\mathfrak{h}$ is the Lie algebras of $H$,
- $\mathfrak{h}$ and $\mathfrak{m}$ are orthogonal under $B$, and $B$ is positive definite on $\mathfrak{m}$ and negative definite on $\mathfrak{h}$,
- multiplication as a map from $H \times \exp \mathfrak{m}$ into $G$ is an onto diffeomorphism.

We will refer sometimes to the data $(G, H, \theta, B)$, as the Cartan data.

Remark 2.1. If $G$ is semisimple, the data $(G, H, \theta, B)$ can be recovered\footnote{To be precise, the quadratic form $B$ can only recovered up to a scalar but this will be sufficient for everything we do in this paper.} from the choice of a maximal compact subgroup $H \subset G$. There are other situations where less information does the job, e.g. for certain linear groups (see [27, p. 385]).

Remark 2.2. The bilinear form $B$ does not play any role in the definition of $G$-Higgs bundle that follows but it is essential for defining the stability condition and for making sense of the Hitchin–Kobayashi correspondence.

Remark 2.3. Note that the compactness of $H$ together with the last property above say that $G$ has only finitely many components. Note also that we are not assuming, like Knapp, that every automorphism Ad($g$) of $\mathfrak{g}^\mathbb{C}$ is inner for every $g \in G$.

Let $\mathfrak{g}^\mathbb{C}$ and $\mathfrak{h}^\mathbb{C}$ be the complexifications of $\mathfrak{g}$ and $\mathfrak{h}$ respectively, and let $H^\mathbb{C}$ be the complexification of $H$. Let

$$\mathfrak{g}^\mathbb{C} = \mathfrak{h}^\mathbb{C} \oplus \mathfrak{m}^\mathbb{C}$$

(2.1)
be the complexification of the Cartan decomposition. The adjoint action of $G$ on $\mathfrak{g}$ restricts to give a representation – the isotropy representation – of $H$ on $\mathfrak{m}$. Since any two Cartan decompositions of $G$ are related by a conjugation, the isotropy representation is independent of the choice of Cartan decomposition. The same is true of the complexification of this representation, allowing us to define:

**Definition 2.4.** A G-Higgs bundle over $X$ is a pair $(E, \varphi)$ where

- $E$ is a principal holomorphic $H^C$-bundle $E$ over $X$ and
- $\varphi$ is a holomorphic section of $E(\mathfrak{m}^C) \otimes K$, where $E(\mathfrak{m}^C)$ is the bundle associated to $E$ via the isotropy representation of $H^C$ in $\mathfrak{m}^C$ and $K$ is the canonical bundle on $X$.

**Remark 2.5.** If $G = \text{Sp}(4, \mathbb{R})$ then $H = \text{U}(2)$ and $H^C = \text{GL}(2, \mathbb{C})$. It is often convenient to replace the principal $\text{GL}(2, \mathbb{C})$-bundle in Definition 2.4 with the vector bundle associated to it by the standard representation. In the next sections we denote this vector bundle by $V$.

In order to define a moduli space of G-Higgs bundles we need a notion of stability. We briefly recall here the main definitions (see [15, 16] for details). Let $\mathfrak{h}^C$ be the semisimple part of $\mathfrak{h}^\mathbb{C}$, that is, $\mathfrak{h}^\mathbb{C} = [\mathfrak{h}^\mathbb{C}, \mathfrak{h}^\mathbb{C}]$. Choosing a Cartan subalgebra, let $\Delta$ be a fundamental system of roots of $\mathfrak{h}^\mathbb{C}$. For every subset $A \subseteq \Delta$ there is a corresponding parabolic subalgebra $\mathfrak{p}_A$ of $\mathfrak{h}^\mathbb{C}$ and all parabolic subalgebras can be obtained in this way. Denote by $P_A$ the corresponding parabolic subgroup of $H^C$. Let $\chi$ be an antidominant character of $\mathfrak{p}_A$. Using the invariant form on $\mathfrak{h}$ defined by $B$, $\chi$ defines an element $s_\chi \in i\mathfrak{h}$. Now for $s \in i\mathfrak{h}$, define the sets

- $\mathfrak{p}_s = \{ x \in \mathfrak{h}^\mathbb{C} : \text{Ad}(e^{ts})x \text{ is bounded as } t \to \infty \}$
- $P_s = \{ g \in H^C : e^{ts}ge^{-ts} \text{ is bounded as } t \to \infty \}$
- $L_s = \{ x \in \mathfrak{h}^\mathbb{C} : [x, s] = 0 \}$
- $L_s = \{ g \in H^C : \text{Ad}(g)(s) = s \}$.

One has (see [16]) that for $s \in i\mathfrak{h}$, $\mathfrak{p}_s$ is a parabolic subalgebra of $\mathfrak{h}^\mathbb{C}$, $P_s$ is a parabolic subgroup of $H^C$ and the Lie algebra of $P_s$ is $\mathfrak{p}_s$, $L_s$ is a Levi subalgebra of $\mathfrak{p}_s$ and $L_s$ is a Levi subgroup of $P_s$ with Lie algebra $L_s$. Moreover, if $\chi$ is an antidominant character of $\mathfrak{p}_A$, then $\mathfrak{p}_A \subseteq \mathfrak{p}_{s_\chi}$ and $L_A \subseteq L_{s_\chi}$ and, if $\chi$ is strictly antidominant, $\mathfrak{p}_A = \mathfrak{p}_s$ and $L_A = L_s$.

Let $\iota : H^C \to \text{GL}(\mathfrak{m}^C)$ be the isotropy representation. We define

- $\mathfrak{m}_\chi^- = \{ v \in \mathfrak{m}^\mathbb{C} : \iota(e^{t\chi})v \text{ is bounded as } t \to \infty \}$
- $\mathfrak{m}_\chi^0 = \{ v \in \mathfrak{m}^\mathbb{C} : \iota(e^{t\chi})v = v \text{ for every } t \}$.

One has that $\mathfrak{m}_\chi^-$ is invariant under the action of $P_{s_\chi}$ and $\mathfrak{m}_\chi^0$ is invariant under the action of $L_{s_\chi}$.

Let $E$ be a principal $H^C$-bundle and $A \subseteq \Delta$. Let $\sigma$ denote a reduction of the structure group of $E$ to a standard parabolic subgroup $P_A$ and let $\chi$ be an antidominant character of $\mathfrak{p}_A$. Associated to this, there is a number called the degree of $E$ with respect to $\sigma$ and $\chi$ that we denote by $\text{deg}(E)(\sigma, \chi)$. If $\chi$ lifts to a character of $P_A$, $\text{deg}(E)(\sigma, \chi)$ is the degree of the line bundle associated to $E_\sigma$ via the lift.

A G-Higgs bundle $(E, \varphi)$ is called semistable if for any choice of $P_A, \chi, \sigma$ as above such that $\varphi \in H^0(X, E_\sigma(\mathfrak{m}_\chi^-) \otimes K)$, we have

$$\text{deg } E(\sigma, \chi) \geq 0.$$
The Higgs bundle \((E, \varphi)\) is called **stable** if it is semistable and for any \(P_A, \chi\) and \(\sigma\) as above such that \(\varphi \in H^0(X, E_\sigma(m_\chi^{-1}) \otimes K)\) and \(A \neq \emptyset\),

\[
\deg E(\sigma, \chi) > 0.
\]

The Higgs bundle \((E, \varphi)\) is called **polystable** if it is semistable and for each \(P_A, \sigma\) and \(\chi\) as in the definition of semistable \(G\)-Higgs bundle such that \(\deg E(\sigma, \chi) = 0\), there exists a holomorphic reduction of the structure group of \(E_\sigma\) to the Levi subgroup \(L_A\) of \(P_A\), \(\sigma_L \in \Gamma(E_\sigma(P_A/L_A))\). Moreover, in this case, we require \(\varphi \in H^0(X, E(m_\chi^0) \otimes K)\).

We define the **moduli space of polystable \(G\)-Higgs bundles** \(M(G)\) as the set of isomorphism classes of polystable \(G\)-Higgs bundles. The moduli space \(M(G)\) has the structure of a complex analytic variety. This can be seen by the standard slice method (see, e.g., Kobayashi [28]). Geometric Invariant Theory constructions are available in the literature for \(G\) compact algebraic (Ramanathan [35, 36]) and for \(G\) complex reductive algebraic (Simpson [41, 42]). The case of a real form of a complex reductive algebraic Lie group follows from the general constructions of Schmitt [38]. We thus have that \(M(G)\) is a complex analytic variety, which is algebraic when \(G\) is algebraic.

### 2.2. Relation to surface group representations

Let \(G\) be a reductive real Lie group. By a **representation** of \(\pi_1(X)\) in \(G\) we understand a homomorphism \(\rho: \pi_1(X) \to G\). The set of all such homomorphisms, \(\text{Hom}(\pi_1(X), G)\), is a real analytic variety, which is algebraic if \(G\) is algebraic. The group \(G\) acts on \(\text{Hom}(\pi_1(X), G)\) by conjugation:

\[
(g \cdot \rho)(\gamma) = g \rho(\gamma) g^{-1}
\]

for \(g \in G\), \(\rho \in \text{Hom}(\pi_1(X), G)\) and \(\gamma \in \pi_1(X)\). If we restrict the action to the subspace \(\text{Hom}^+(\pi_1(X), G)\) consisting of reductive representations, the orbit space is Hausdorff. By a **reductive representation** we mean one that, composed with the adjoint representation in the Lie algebra of \(G\), decomposes as a sum of irreducible representations. If \(G\) is algebraic this is equivalent to the Zariski closure of the image of \(\pi_1(X)\) in \(G\) being a reductive group. (When \(G\) is compact every representation is reductive.) The **moduli space of representations** of \(\pi_1(X)\) in \(G\) is defined to be the orbit space

\[
\mathcal{R}(G) = \text{Hom}^+(\pi_1(X), G)/G.
\]

It has the structure of a real analytic variety (see e.g. [19]) which is algebraic if \(G\) is algebraic and is a complex variety if \(G\) is complex.

To see the relation between Higgs bundles and representations of \(\pi_1(X)\), let \(h\) be a reduction of structure group of \(E_{H^c}\) from \(H^c\) to \(H\), and let \(E_H\) be the principal \(H\)-bundle defined by \(h\). Let \(d_h\) denote the unique connection on \(E_{H^c}\) compatible with \(h\) and let \(F_h\) be its curvature. If \(\tau\) denotes the compact conjugation of \(g^c\) we can formulate the Hitchin equation

\[
F_h - [\varphi, \tau(\varphi)] = 0.
\]

A fundamental result of Higgs bundle theory (see [25, 39, 14]) is that a \(G\)-Higgs bundle admits a solution to Hitchin’s equation if and only if the Higgs bundle is polystable.

Now if the Hitchin equation is satisfied then

\[
D = d_h + \varphi - \tau(\varphi)
\]

defines a flat connection on the principal \(G\)-bundle \(E_G = E_H \times_H G\). The holonomy of this connection thus defines a representation of \(\pi_1(X)\) in \(G\). A fundamental theorem of Corlette
(and Donaldson [11] for $G = \text{SL}(2, \mathbb{C})$; see also Labourie [29] for a more general set-up) says that this representations is reductive, and that all reductive representations of $\pi_1(X)$ in $G$ arise in this way.

For semisimple groups the above results establish a homeomorphism between isomorphism classes of polystable $G$-Higgs bundles and conjugacy classes of reductive surface group representations in $G$, i.e.

$$\mathcal{M}(G) \simeq \mathcal{R}(G).$$  \hspace{1cm} (2.2)

It is this homeomorphism that allows us to use Higgs bundles to study surface group representations. If $G$ is reductive (but not semisimple) there is a similar correspondence involving representations of a universal central extension of the fundamental group.

2.3. Reduction of structure group. Let $G$ be a real reductive Lie group as defined in Section 2.1. Our main concern is to understand when a surface group representation in $G$ factors through a subgroup of $G$. In this section we reformulate in terms of Higgs bundles what it means for the representation to factor through a subgroup.

A reductive subgroup of $G$ is a reductive group, say $(G', H', \theta', B')$, such that the Cartan data is compatible in the obvious sense with the Cartan data of $(G, H, \theta, B)$ under the inclusion map $G' \hookrightarrow G$. In particular this implies that $H' \subset H$ and we have a commutative diagram

$$g^C = h^C \oplus m^C$$

Moreover, the embedding of isotropy representations $m'^C \hookrightarrow m^C$ is equivariant with respect to the embedding $H'^C \hookrightarrow H^C$.

**Definition 2.6.** Let $G$ be a real reductive Lie group and let $G' \subset G$ be a reductive subgroup. Let $(E, \varphi)$ be a $G$-Higgs bundle. A reduction of $(E, \varphi)$ to a $G'$-Higgs bundle $(E', \varphi')$ is given by the following data:

1. A holomorphic reduction of structure group of $E$ to a principal $H'^C$-bundle $E' \hookrightarrow E$ (equivalently, this is given by a holomorphic section $\sigma$ of $E/H'^C \rightarrow X$).

2. A holomorphic section $\varphi'$ of $E'(m'^C) \otimes K$ which maps to $\varphi$ under the embedding $E'(m'^C) \otimes K \hookrightarrow E(m^C) \otimes K$.

We have the following.

**Proposition 2.7.** Let $G$ be a real reductive Lie group and let $G' \subset G$ be a reductive subgroup. Let $(E_{H^C}, \varphi)$ be a $G$-Higgs bundle whose structure group reduces to $G'$. Let $(E_{H'^C}, \varphi')$ be the corresponding $G'$-Higgs bundle. If $(E_{H^C}, \varphi)$ is polystable as a $G$-Higgs bundle, then $(E_{H'^C}, \varphi')$ is polystable as a $G'$-Higgs bundle.

The key fact in the proof of Proposition 2.7 is that every parabolic subgroup of $H'^C$ and a character of its Lie algebra extend to a parabolic subgroup of $H^C$ and a character of its corresponding Lie algebra. Moreover, the corresponding degrees for parabolic reductions of structure group of the bundles coincide. This can be seen using filtrations of the vector bundles associated to $E_{H^C}$ and $E_{H'^C}$ via an auxiliary representations of $H^C$ (see [16]).
In the situation of Proposition 2.7, the non-abelian Hodge theory correspondence implies that the polystable $G'$-Higgs bundles obtained from polystable $G$-Higgs bundles correspond to $G'$-representations of $\pi_1(X)$. Conversely, let $\rho$ be a reductive surface group representation in $G$ which factors through a reductive subgroup $G'$. Then it is clear that the corresponding polystable $G'$-Higgs bundle is a $G'$-reduction of the $G$-Higgs bundle corresponding to $\rho$. Thus Proposition 2.7 has the following immediate corollary.

**Proposition 2.8.** Let $G$ be a real reductive Lie group and let $G' \subset G$ be a reductive subgroup. 

1. A reductive $\pi_1(X)$-representation in $G$ factors through a reductive representation in $G'$ if and only if the corresponding polystable $G$-Higgs bundle admits a reduction of structure group to $G'$.

2. Let $\rho : \pi_1(X) \to G$ be a reductive representation and let $(E_{\mathfrak{H}}^c, \varphi)$ be the corresponding polystable $G$-Higgs bundle. Suppose that $(E_{\mathfrak{H}}^c, \varphi)$ defines a point in a connected component $M_c(G) \subset M(G)$. The representation $\rho$ deforms to a representation which factors through $G'$ if and only if $M_c(G)$ contains a point represented by a $G$-Higgs bundle that admits a reduction of structure group to $G'$.

Let $G$ be a real reductive Lie group and let $G' \hookrightarrow G$ be an embedding of the Lie group $G'$ as a closed subgroup. One may ask whether the Cartan data of $G'$ induces Cartan data on $G'$. In the following we answer this question.

**Definition 2.9.** An embedding of Lie algebras $\mathfrak{g}' \subset \mathfrak{g}$ is **canonical** with respect to a Cartan involution, $\theta$, on $\mathfrak{g}$ if $\theta(\mathfrak{g}') = \mathfrak{g}'$.

**Lemma 2.10.** Let $G' \subset G$ be a closed Lie subgroup such that $\mathfrak{g}' \subset \mathfrak{g}$ is canonically embedded. Then $H' = H \cap G'$ is a maximal compact subgroup of $G'$. Moreover, if we let $\theta'$ and $B'$ be the restrictions of $\theta$ and $B$, respectively, to $\mathfrak{g}'$, then $(G', H', \theta', B')$ is a reductive subgroup of $(G, H, \theta, B)$.

In view of this Lemma, we make the following convention.

**Convention.** Whenever $G' \subset G$ is a closed subgroup whose Lie algebra is canonically embedded, we consider $G'$ as a reductive subgroup of $G$ with the induced Cartan data.

**Remark 2.11.** If, in the situation of Lemma 2.10, $G'$ is semisimple, the structure of reductive subgroup induced from $G$ must coincide with the one coming from the choice of the maximal compact subgroup $H' = H \cap G'$ (cf. Remark 2.1).

Thus, if we are given a semisimple closed subgroup $G' \subset G$ with an a priori choice of maximal compact $H' \subset G'$, then in order to check that the corresponding Cartan data coincides with the Cartan data induced from $G$, it suffices to check that $H' = H \cap G$ and that $\mathfrak{g}' \hookrightarrow \mathfrak{g}$ is canonically embedded.

### 3. $\text{Sp}(4, \mathbb{R})$-Higgs bundles

**3.1. Definition of $\text{Sp}(4, \mathbb{R})$ and choice of Cartan data.** The Lie group $\text{Sp}(4, \mathbb{R})$ is the subgroup of $\text{SL}(4, \mathbb{R})$ which preserves a symplectic form on $\mathbb{R}^4$. The description of the group depends on the choice of symplectic form. We use the following conventions.

**Definition 3.1.** Let

$$J_{13} = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$
where $I_2$ is the $2 \times 2$ identity matrix. This defines the symplectic form $\omega_{13}(a, b) = a^t J_{13} b$ where $a$ and $b$ are vectors in $\mathbb{R}^4$, i.e.

$$\omega_{13} = x_1 \wedge x_3 + x_2 \wedge x_4. \quad (3.2)$$

The symplectic group in dimension four, defined using $J_{13}$, is thus

$$\text{Sp}(4, \mathbb{R}) = \{ g \in \text{SL}(4, \mathbb{R}) \mid g^t J_{13} g = J_{13} \}. \quad (3.3)$$

Remark 3.2. Later on (see Sections 4.1, 8.1) it will be convenient to consider other choices of symplectic form (denoted by $J_{12}$ and $J_0$). The resulting changes in description will be pointed out as needed.

The maximal compact subgroups of $\text{Sp}(4, \mathbb{R})$ are isomorphic to $U(2)$, i.e. in the notation of the previous section, if $G = \text{Sp}(4, \mathbb{R})$ then $H = U(2)$. Using symplectic form $J_{13}$, we fix the subgroup $U(2) \subset \text{Sp}(4, \mathbb{R})$ given by

$$U(2) = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A^t A + B^t B = I, \ A^t B - B^t A = 0 \right\}, \quad (3.4)$$

i.e. given by the embedding

$$A + iB \mapsto \begin{pmatrix} A \\ -B \\ A \end{pmatrix}. \quad (3.5)$$

It follows from (3.3) and (3.5) that the Cartan decomposition corresponding to our choice of $U(2)$ is defined by the involution

$$\theta(X) = -X^t \quad (3.6)$$

on

$$\mathfrak{sp}(4, \mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid A, B, C \in \text{Mat}_2(\mathbb{R}) ; \ B^t = B, \ C^t = C \right\}. \quad (3.7)$$

This gives

$$\mathfrak{sp}(4, \mathbb{R}) = \mathfrak{u}(2) \oplus \mathfrak{m} \quad (3.8)$$

with

$$\mathfrak{u}(2) = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A, B \in \text{Mat}_2(\mathbb{R}) ; \ A^t = -A, \ B^t = B \right\}, \quad (3.9)$$

$$\mathfrak{m} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mid A, B \in \text{Mat}_2(\mathbb{R}) ; \ A^t = A, \ B^t = B \right\}. \quad (3.10)$$

The complexification of (3.8),

$$\mathfrak{sp}(4, \mathbb{C}) = \mathfrak{gl}(2, \mathbb{C}) \oplus \mathfrak{m}^\mathbb{C} \quad (3.11)$$

is obtained by replacing $\text{Mat}_2(\mathbb{R})$ with $\text{Mat}_2(\mathbb{C})$. In particular, we identify $\mathfrak{gl}(2, \mathbb{C})$ via\footnote{This corresponds to mapping $Z \mapsto \begin{pmatrix} Z - Z^t \\ -z^t + \xi + \eta^t \\ Z + Z^t \\ -z^t + \xi + \eta^t \end{pmatrix}$.}

$$\mathfrak{gl}(2, \mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A, B \in \text{Mat}_2(\mathbb{C}) ; \ A^t = -A, \ B^t = B \right\}. \quad (3.12)$$
Notice that after conjugation by $T = \begin{pmatrix} I & iI \\ I & -iI \end{pmatrix}$, i.e. after the change of basis (on $\mathbb{C}^4$) effected by $T$, we identify the summands in the Cartan decomposition of $\mathfrak{sp}(4, \mathbb{C}) \subset \mathfrak{sl}(4, \mathbb{C})$ as

$$
\mathfrak{gl}(2, \mathbb{C}) = \left\{ \begin{pmatrix} Z & 0 \\ 0 & -Z^t \end{pmatrix} \mid Z \in \text{Mat}_2(\mathbb{C}) \right\},
$$

$$
\mathfrak{m}^C = \left\{ \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \mid \beta, \gamma \in \text{Mat}_2(\mathbb{C}), \beta^t = \beta, \gamma^t = \gamma \right\} = \text{Sym}^2(\mathbb{C}^2) \oplus \text{Sym}^2((\mathbb{C}^2)^*) .
$$

(3.13)

This corresponds to an embedding of $U(2)$ (the maximal compact subgroup of $\text{Sp}(4, \mathbb{R})$) in $\text{SU}(4)$ (the maximal compact subgroup in $\text{SL}(4, \mathbb{C})$) given by

$$
U \mapsto \begin{pmatrix} U & 0 \\ 0 & (U^t)^{-1} \end{pmatrix} \quad \text{where} \quad U^*U = I .
$$

(3.14)

3.2. Definition of $\text{Sp}(4, \mathbb{R})$-Higgs bundles. We fix $G = \text{Sp}(4, \mathbb{R})$ and $H = U(2)$ as in Section 3.1. Given a holomorphic principal $\text{GL}(2, \mathbb{C})$-bundle on $X$, say $E$, let $V$ denote the rank 2 vector bundle associated to $E$ by the standard representation. The Cartan decomposition described in Section 3.1 shows (see (3.13)) that we can identify

$$
E(\mathfrak{m}^C) = \text{Sym}^2(V) \oplus \text{Sym}^2(V^*).
$$

(3.15)

Definition (2.4) thus specializes to the following:

**Definition 3.3.** With $G = \text{Sp}(4, \mathbb{R})$ and $H = U(2)$ as in Section 3.1, an $\text{Sp}(4, \mathbb{R})$-Higgs bundle over $X$ is defined by a triple $(V, \beta, \gamma)$ consisting of a rank 2 holomorphic vector bundles $V$ and symmetric homomorphisms

$$
\beta : V^* \longrightarrow V \otimes K \quad \text{and} \quad \gamma : V \longrightarrow V^* \otimes K.
$$

Except when it is important to keep track of the maximal compact subgroup, we will refer to these objects as $\text{Sp}(4, \mathbb{R})$-Higgs bundles. The composite embedding

$$
\text{Sp}(4, \mathbb{R}) \hookrightarrow \text{Sp}(4, \mathbb{C}) \hookrightarrow \text{SL}(4, \mathbb{C})
$$

(3.16)

allows us to reinterpret the defining data for $\text{Sp}(4, \mathbb{R})$-Higgs bundles as data for special $\text{SL}(4, \mathbb{C})$-Higgs bundles (in the original sense of [26]). Indeed, the embeddings (3.13) show that the triple $(V, \beta, \gamma)$ in Definition 3.3 is equivalent to the pair $(\mathcal{E}, \varphi)$, where

(1) $\mathcal{E}$ is the rank 4 holomorphic bundle $\mathcal{E} = V \oplus V^*$, and

(2) $\varphi$ is a Higgs field $\varphi : \mathcal{E} \longrightarrow \mathcal{E} \otimes K$ given by $\varphi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$.

**Remark 3.4.** The definition of $\text{Sp}(2n, \mathbb{R})$-Higgs bundles for general $n$ is of course entirely analogous and later we shall need the special case $n = 1$, corresponding to $G = \text{Sp}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R})$. Thus an $\text{SL}(2, \mathbb{R})$-Higgs bundle is given by the data $(L, \beta, \gamma)$, where $L$ is a line bundle, $\beta \in H^0(L^2K)$ and $\gamma \in H^0(L^{-2}K)$. 

3.3. Stability. The general definition of (semi-)stability for $G$-Higgs bundles given in Section 2.1 simplifies in the case $G = \text{Sp}(2n, \mathbb{R})$ (see [15, Section 3] or [38]). To state the simplified stability condition, we use the following notation. For any line subbundle $L \subset V$ we denote by $L^{\perp}$ the subbundle of $V^*$ in the kernel of the projection onto $L^*$, i.e.

$$0 \longrightarrow L^{\perp} \longrightarrow V^* \longrightarrow L^* \longrightarrow 0.$$ \hfill (3.17)

Moreover, for subbundles $L_1$ and $L_2$ of a vector bundle $V$, we denote by $L_1 \otimes_S L_2$ the symmetrized tensor product, i.e. the symmetric part of $L_1 \otimes L_2$ inside the symmetric product $S^2 V$ (these bundles can be constructed in standard fashion from the corresponding representations, using principal bundles). For $n = 2$, i.e. for $G = \text{Sp}(4, \mathbb{R})$, the stability condition then takes the following form.

**Proposition 3.5.** An $\text{Sp}(4, \mathbb{R})$-Higgs bundle $(V, \beta, \gamma)$ is semistable if and only if all the following conditions hold

1. If $\beta = 0$ then $\deg(V) \geq 0$.
2. If $\gamma = 0$ then $\deg(V) \leq 0$.
3. Let $L \subset V$ be a line subbundle.
   a. If $\beta \in H^0(L \otimes_S V \otimes K)$ and $\gamma \in H^0(L^{\perp} \otimes_S V^* \otimes K)$ then $\deg(L) \leq \frac{\deg(V)}{2}$.
   b. If $\gamma \in H^0((L^{\perp})^2 \otimes K)$ then $\deg(L) \leq 0$.
   c. If $\beta \in H^0(L^2 \otimes K)$ then $\deg(L) \leq \deg(V)$.

If, additionally, strict inequalities hold in (3), then $(V, \beta, \gamma)$ is stable.

Similarly, the notion of polystability simplifies as follows.

**Proposition 3.6.** Let $(V, \beta, \gamma)$ be an $\text{Sp}(4, \mathbb{R})$-Higgs bundle with $\deg(V) \neq 0$. Then $(V, \beta, \gamma)$ is polystable if it is either stable, or if there is a decomposition $V = L_1 \oplus L_2$ of $V$ as a direct sum of line bundles, such that one of the following conditions is satisfied:

1. The Higgs fields satisfy $\beta = \beta_1 + \beta_2$ and $\gamma = \gamma_1 + \gamma_2$, where

   $$\beta_i \in H^0(L_i \otimes K) \quad \text{and} \quad \gamma_i \in H^0(L_i^{-2} \otimes K)$$

   for $i = 1, 2$. Furthermore, the $\text{Sp}(2, \mathbb{R})$-Higgs bundles $(L_i, \beta_i, \gamma_i)$ are stable for $i = 1, 2$ and there is an isomorphism of $\text{Sp}(2, \mathbb{R})$-Higgs bundles $(L_1, \beta_1, \gamma_1) \simeq (L_2, \beta_2, \gamma_2)$.

2. The Higgs fields satisfy

   $$\begin{cases}
   \beta \in H^0((L_1 L_2 + L_2 L_1) \otimes K) \\
   \gamma \in H^0((L_1^{-1} L_2^{-1} + L_2^{-1} L_1^{-1}) \otimes K)
   \end{cases}$$

   Furthermore, $\deg(L_1) = \deg(L_2) = \deg(V)/2$ and the rank 2 Higgs bundle $(L_1 \oplus L_2^*, (0, \beta))$ is stable.

**Remark 3.7.** If $\deg V = 0$ then there are other possible decompositions for a polystable $\text{Sp}(4, \mathbb{R})$-Higgs bundle; and if $(V, \beta, \gamma)$ is as in (1) of Proposition 3.5 but with $(L_1, \beta_1, \gamma_1)$ and $(L_2, \beta_2, \gamma_2)$ non isomorphic then it is a stable $\text{Sp}(4, \mathbb{R})$-Higgs bundle which is not simple (see Theorem 3.40 in [15] for details).

The following result [15] relating polystability of $\text{Sp}(4, \mathbb{R})$-Higgs bundles to polystability of $\text{GL}(4, \mathbb{C})$-Higgs bundles is useful. It is important to point out that, though the polystability conditions coincide, the stability condition for a $\text{Sp}(4, \mathbb{R})$-Higgs bundle is weaker than the stability condition for the corresponding $\text{GL}(4, \mathbb{C})$-Higgs bundle.
Proposition 3.8 ([15, Theorem 5.13]). An Sp(4, R)-Higgs bundle \((V, \beta, \gamma)\) is polystable if and only if the GL(4, C)-Higgs bundle \((V \oplus V^*, \varphi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix})\) is polystable.

Recall that a GL(4, C)-Higgs bundle \((E, \varphi)\) is stable if, for any proper non-zero \(\varphi\)-invariant subbundle \(F \subseteq E\) satisfies \(\mu(F) < \mu(E)\), where \(\mu(F) = \text{deg}(F)/\text{rk}(F)\) is the slope of the subbundle. The Higgs bundle \((E, \varphi)\) is polystable if it is the direct sum of stable Higgs bundles, all of the same slope. Moreover, to check that the GL(4, C)-Higgs bundle \((E = V \oplus V^*, \varphi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix})\) is stable, it suffices to consider \(\varphi\)-invariant subbundles which respect the decomposition \(E = V \oplus V^*\) (see [2]).

Remark 3.9. Similarly, the stability condition for an SL(2, R)-Higgs bundle \((L, \beta, \gamma)\) simplifies as follows.

(1) If \(\text{deg}(L) > 0\) then \((L, \beta, \gamma)\) is stable if and only if \(\gamma \neq 0\).

(2) If \(\text{deg}(L) < 0\) then \((L, \beta, \gamma)\) is stable if and only if \(\beta \neq 0\).

(3) If \(\text{deg}(L) = 0\) then \((L, \beta, \gamma)\) is polystable if and only if either \(\beta = 0 = \gamma\) or both \(\beta\) and \(\gamma\) are nonzero.

Moreover, if \(\text{deg}(L) \neq 0\), then stability, polystability and semistability are equivalent conditions. Notice that from the semistability condition if \(\text{deg}(L) > 0\), since \(\gamma \neq 0\), we must have that \(\text{deg}(L) \leq g - 1\); and similarly, if \(\text{deg}(L) < 0\), since \(\beta \neq 0\), we must have that \(\text{deg}(L) \geq 1 - g\). We thus have the Milnor–Wood inequality for SL(2, R)-Higgs bundles (see [32, 20, 25]).

Finally, in a manner analogous to Proposition 3.8, we have that \((L, \beta, \gamma)\) is a polystable SL(2, R)-Higgs bundle if and only if

\[ (L \oplus L^{-1}, \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}) \]

is a polystable SL(2, C)-Higgs bundle.

3.4. Toledo invariant and moduli spaces. The basic topological invariant of an Sp(4, R)-Higgs bundle is the degree of \(V\).

Definition 3.10. The Toledo invariant of the Sp(4, R)-Higgs bundle \((V, \gamma, \beta)\) is the integer

\[ d = \text{deg}(V). \]

From the point of view of representations of the fundamental group, the Toledo invariant is defined for representations into any group \(G\) of hermitean type. This justifies the terminology used in the definition.

The following inequality for the Toledo invariant has a long history, going back to Milnor [32], Wood [47], Dupont [12], Turaev [43], Domic–Toledo [10] and Clerc–Ørsted [8]. It is usually known as the Milnor–Wood inequality.

Proposition 3.11. Let \((V, \beta, \gamma)\) be a semistable Sp(4, R)-Higgs bundle. Then

\[ |d| \leq 2g - 2. \]

The sharp bound for \(G = \text{Sp}(4, \mathbb{R})\) was given by Turaev. In its most general form the Milnor–Wood inequality has been proved by Burger, Iozzi and Wienhard. For a proof in the present context of Higgs bundle theory, see [22].
We call $\text{Sp}(4, \mathbb{R})$-Higgs bundles with Toledo invariant $d = 2g - 2$ maximal, and define maximal representations $\rho: \pi_1(X) \to \text{Sp}(4, \mathbb{R})$ similarly.

For simplicity, we shall henceforth use the notation
$$\mathcal{M}_d = \mathcal{M}_d(\text{Sp}(4, \mathbb{R}))$$
for the moduli space parametrizing isomorphism classes of polystable $\text{Sp}(4, \mathbb{R})$-Higgs bundles $(V, \beta, \gamma)$ with $\deg(V) = d$. We will denote the components with maximal positive Toledo invariant by $\mathcal{M}^{\max}$, i.e.
$$\mathcal{M}^{\max} = \mathcal{M}_{2g-2}.$$

We remark (cf. \cite{15}) that there is an isomorphism $\mathcal{M}_d \simeq \mathcal{M}_{-d}$, given by the map $(V, \beta, \gamma) \to (V^*, \gamma, \beta)$. This justifies restricting attention to the case $d \geq 0$ of positive Toledo invariant.

### 3.5. Maximal $\text{Sp}(4, \mathbb{R})$-Higgs bundles and Cayley partners

The Higgs bundle proof \cite{22} of Proposition 3.11 has the following important consequence.

**Proposition 3.12.** Let $(V, \beta, \gamma)$ be a polystable $\text{Sp}(4, \mathbb{R})$-Higgs bundle. If $\deg(V) = 2g - 2$, i.e. if $d$ is maximal and positive, then
$$\gamma: V \to V^* \otimes K$$
is an isomorphism.

If $\gamma: V \to V^* \otimes K$ is an isomorphism, then some of the conditions in Proposition 3.5 cannot occur. The stability condition then reduces to:

**Proposition 3.13.** Let $(V, \beta, \gamma)$ be an $\text{Sp}(4, \mathbb{R})$-Higgs bundle and assume that $\gamma: V \to V^* \otimes K$ is an isomorphism. Set
$$\tilde{\beta} = (\beta \otimes 1) \circ \gamma: V \to V \otimes K^2. \quad (3.18)$$
Then $(V, \beta, \gamma)$ is semi-stable if and only if for any line subbundle $L \subset V$ isotropic with respect to $\gamma$ and such that $\tilde{\beta}(L) \subset L \otimes K^2$, the following condition is satisfied
$$\mu(L) \leq \mu(V).$$
If strict inequality holds then $(V, \beta, \gamma)$ is stable.

If we fix a square root of $K$, i.e. if we pick a line bundle $L_0$ such that $L_0^2 = K$, and define
$$W = V^* \otimes L_0 \quad (3.19)$$
then it follows from Proposition 3.12 that the map
$$q_W := \gamma \otimes I_{L_0^{-1}}: W^* \to W \quad (3.20)$$
defines a symmetric, non-degenerate form on $W$, i.e. $(W, q_W)$ is an $O(2, \mathbb{C})$-holomorphic bundle. The remaining part of the Higgs field, i.e. the map $\beta$ defines a $K^2$-twisted endomorphism
$$\theta = (\gamma \otimes I_{K \otimes L_0}) \circ (\beta \otimes I_{L_0}): W \to W \otimes K^2. \quad (3.21)$$
The map $\theta$ is $q_W$-symmetric, i.e. it takes values in the isotropy representation for $\text{GL}(2, \mathbb{R})$. The pair $(W, q_W, \theta)$ thus satisfies the definition of a $G$-Higgs bundle with $G = \text{GL}(2, \mathbb{R})$, except for the fact that the Higgs field $\theta$ takes values in $E(\mathfrak{m}^C) \otimes K^2$ instead of in $E(\mathfrak{m}^C) \otimes K$. We say that $(W, \theta)$ defines a $K^2$-twisted Higgs pair with structure group $\text{GL}(2, \mathbb{R})$ (see \cite{15} for more details).
Proposition 3.15. Let of the general framework which justifies our terminology. Occasionally, when the section Cayley partner. We refer to [3] for more details on this construction, including an exposition (is not directly relevant for our considerations, we shall also refer to the orthogonal bundle Toledo invariant, i.e. with deg(V) = 2g − 2. Then V can be written as

\[ V = W \otimes L_0, \]

where W is an O(2, C)-bundle and \( L_0 \) is a line bundle such that

\[ L_0^2 = K. \]

Also, the isomorphism \( \gamma \) is given by

\[ \gamma = q \otimes I_{L_0} : W \otimes L_0 \rightarrow W^* \otimes L_0, \]

where \( q \) defines the orthogonal structure on \( W \) and \( I_{L_0} \) is the identity map on \( L_0 \), and

\[ \det(V)^2 = K^2. \]

Definition 3.14. We call \((W, q_W, \theta)\) the Cayley partner of the \( \text{Sp}(4, \mathbb{R})\)-Higgs bundle \((V, \beta, \gamma)\).

The original \( \text{Sp}(4, \mathbb{R})\)-Higgs bundle can clearly be recovered from the defining data for its Cayley partner. We refer to [3] for more details on this construction, including an exposition of the general framework which justifies our terminology. Occasionally, when the section \( \theta \) is not directly relevant for our considerations, we shall also refer to the orthogonal bundle \((W, q_W)\) as the Cayley partner of \((V, \beta, \gamma)\).

The following Proposition sums up the essential point of the constructions of this section.

Proposition 3.15. Let \((V, \beta, \gamma)\) be a polystable \( \text{Sp}(4, \mathbb{R})\)-Higgs bundle with maximal positive Toledo invariant, i.e. with \( \deg(V) = 2g - 2 \). Then \( V \) can be written as

\[ V = W \otimes L_0, \]

where \( W \) is an \( O(2, \mathbb{C}) \)-bundle and \( L_0 \) is a line bundle such that

\[ L_0^2 = K. \]

Also, the isomorphism \( \gamma \) is given by

\[ \gamma = q \otimes I_{L_0} : W \otimes L_0 \rightarrow W^* \otimes L_0, \]

where \( q \) defines the orthogonal structure on \( W \) and \( I_{L_0} \) is the identity map on \( L_0 \), and

\[ \det(V)^2 = K^2. \]

3.6. Connected components of the moduli space. The moduli space \( \mathcal{M}^{\text{max}} \) of maximal \( \text{Sp}(4, \mathbb{R})\)-Higgs bundles is not connected. Its connected components of \( \mathcal{M}^{\text{max}} \) were determined in [22]. In contrast, each moduli space \( \mathcal{M}_d \) for \(|d| < 2g - 2\) is connected (see [17]). In this section we explain the count of components of \( \mathcal{M}^{\text{max}} \) and identify the Higgs bundles appearing in each component.

The key to the count of the components of \( \mathcal{M}^{\text{max}} \) is Proposition 3.15. The fact that the orthogonal bundle \((W, q_W)\) underlying the Cayley partner is an \( O(2, \mathbb{C}) \)-bundle reveals new topological invariants, namely the first and second Stiefel–Whitney classes

\[ w_1(W, q_W) \in H^1(X; \mathbb{Z}/2) \cong \mathbb{Z}/2^{2g} \]

\[ w_2(W, q_W) \in H^2(X; \mathbb{Z}/2) \cong \mathbb{Z}/2. \]

Rank 2 orthogonal bundles were classified by Mumford in [33] (though the reducible case (3) was omitted there):

Proposition 3.16. A rank 2 orthogonal bundle \((W, q_W)\) is one of the following:

1. \( W = L \oplus L^{-1} \), where \( L \) is a line bundle on \( X \), and \( q_W = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) \). In this case \( w_1(W, q_W) = 0 \).
2. \( W = \pi_* (\bar{L} \otimes \iota^* L^{-1}) \) where \( \pi : \bar{X} \rightarrow X \) is a connected double cover, \( \bar{L} \) is a line bundle on \( \bar{X} \), and \( \iota : \bar{X} \rightarrow \bar{X} \) is the covering involution. The quadratic form is locally of the form \( q_W = (\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}) \). In this case \( w_1(W, q_W) \in H^1(X; \mathbb{Z}/2) \) is the non-zero element defining the double cover.
3. \( W = L_1 \oplus L_2 \) where \( L_1 \) and \( L_2 \) are line bundles on \( X \) satisfying \( L_i^2 = \mathcal{O}_X \), and \( q_W = q_1 + q_2 \) where \( q_i \) defines the isomorphism \( L_i \cong L_i^{-1} \). In this case \( w_1(W, q_W) = w_1(L_1, q_1) + w_1(L_2, q_2) \).
Note that cases (1) and (3) above are not mutually exclusive: they coincide when \( V = L \oplus L \) with \( L^2 = 0 \) and \( q_W = (\frac{1}{0} \frac{0}{1}) \).

**Remark 3.17.** In case (2) above, the line bundles of the form \( M = \tilde{L} \otimes \iota^* \tilde{L}^{-1} \) constitute the kernel of \( 1 + \iota^*: \text{Jac}(\tilde{X}) \to \text{Jac}(X) \). Moreover, this kernel consists of two components \( P^+ \) and \( P^- \) (distinguished by the degree of \( \tilde{L} \) modulo 2), each one of them a translate of the Prym variety of the cover (cf. [34]). It can be shown that the value of \( \nu_2(W, q_W) \) is 0 or 1 depending on whether \( M \) belongs to \( P^+ \) or \( P^- \) (see [22, Proposition 5.14]).

Recall that the first Stiefel–Whitney class is the obstruction to the existence of a reduction of structure group to \( \text{SO}(2, \mathbb{C}) \). It can be shown that the value of \( \nu_2(W, q_W) \) is 0 or 1 depending on whether \( M \) belongs to \( P^+ \) or \( P^- \) (see [22, Proposition 5.14]).

**Proposition 3.18.** Let \( (W, q_W) \) be an \( \text{O}(2, \mathbb{C}) \)-bundle. Then \( w_1(W, q_W) \) equals zero if and only if \( (W, q_W) \) is of the kind described in (1) of Proposition 3.16. In this case the second Stiefel–Whitney class, \( w_2(W, q_W) \), lifts to the integer class \( c_1(L) \in H^2(X; \mathbb{Z}) \).

Let \( (V, \beta, \gamma) \) be a maximal semistable \( \text{Sp}(4, \mathbb{R}) \)-Higgs bundle and let \( (W, q_W) \) be defined by \( (3.19) \) and \( (3.20) \). We define topological invariants of \( (V, \beta, \gamma) \) as follows:

\[
w_i(V, \beta, \gamma) = w_i(W, q_W), \quad i = 1, 2.
\]

Note that these invariants are well defined because the Stiefel–Whitney classes are independent of the choice of the square root \( L_0 \) of the canonical bundle used to define the Cayley partner \( (W, q_W) \). When \( w_1(V, \beta, \gamma) = 0 \), the class \( w_2(V, \beta, \gamma) \) lifts to the integer invariant \( \text{deg}(L) \), where \( W = L \oplus L^{-1} = V \otimes L_0^{-1} \) is the vector bundle underlying the Cayley partner of \( (V, \beta, \gamma) \).

**Proposition 3.19.** Let \( (V, \beta, \gamma) \) be a maximal semistable \( \text{Sp}(4, \mathbb{R}) \)-Higgs bundle with \( w_1(V, \beta, \gamma) = 0 \) and let \( (W = L \oplus L^{-1}, q_W = (\frac{0}{1} \frac{1}{0})) \) be its Cayley partner. Then there is a line bundle \( N \) such that

\[
V = N \oplus N^{-1} K,
\]

and, with respect to this decomposition,

\[
\gamma = (\frac{1}{0} \frac{1}{0}) \in H^0(S^2 V^* \otimes K) \quad \text{and} \quad \beta = (\frac{\beta_1}{\beta_2} \frac{\beta_2}{\beta_1}) \in H^0(S^2 V \otimes K).
\]

The degree of \( N \) is given by

\[
\text{deg}(N) = \text{deg}(L) + g - 1.
\]

Moreover,

\[
0 \leq \text{deg}(L) \leq 2g - 2
\]

and, for \( \text{deg}(L) > 0 \),

\[
\beta_2 \neq 0.
\]

When \( \text{deg}(L) = 2g - 2 \) the line bundle \( N \) satisfies

\[
N^2 = K^3. \quad (3.28)
\]

**Proof.** The statement about the shape of \( (V, \beta, \gamma) \) follows by applying Propositions 3.16 and 3.18 to the Cayley partner, letting \( N = LL_0 \).

Assuming without loss of generality that \( \text{deg}(L) \geq 0 \), the fact that \( 0 \neq \beta_2 \in H^0(X, N^{-2} K^3) \) follows easily from semistability (cf. [22]). The rest now follows from \( \text{deg}(N^{-2} K^3) \geq 0 \).
It follows from (3.28) that $N$ is determined by a choice of a square root of the canonical bundle $K$, thus revealing a new discrete invariant of a maximal semistable $\text{Sp}(4, \mathbb{R})$-Higgs bundle with $w_1 = 0$ and $\deg(L) = 2g - 2$. We introduce subspaces of $\mathcal{M}^{\text{max}}$ as follows:

**Definition 3.20.**

1. For $(w_1, w_2) \in H^1(X, \mathbb{Z}/2) \times H^2(X, \mathbb{Z}/2) \setminus (0, 0) \simeq ((\mathbb{Z}/2)^{2g} - \{0\}) \times \mathbb{Z}/2$, define
   
   $$\mathcal{M}_{w_1, w_2} = \{ (V, \beta, \gamma) \mid w_1(V, \beta, \gamma) = w_1, \quad w_2(V, \beta, \gamma) = w_2 \} / \simeq,$$

   where the notation indicates isomorphism classes of $\text{Sp}(4, \mathbb{R})$-Higgs bundles $(V, \beta, \gamma)$.

2. For $c \in H^2(X, \mathbb{Z}) \simeq \mathbb{Z}$ with $0 \leq c \leq 2g - 2$, define
   
   $$\mathcal{M}_c^0 = \{ (V, \beta, \gamma) \mid w_1(V, \beta, \gamma) = 0, \quad \deg(L) = c \} / \simeq,$$

   where $(W = L \oplus L^{-1}, q_W = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}))$ is the Cayley partner of $(V, \beta, \gamma)$.

3. For a square root $K^{1/2}$ of the canonical bundle, define the following subspace of $\mathcal{M}_{2g-2}^0$
   
   $$\mathcal{M}_{K^{1/2}}^T = \{ (V = N \oplus N^{-1}K, \beta, \gamma) \mid N = (K^{1/2})^3 \} / \simeq.$$

In particular, we can therefore write

$$\mathcal{M}_{2g-2}^0 = \bigcup_{K^{1/2}} \mathcal{M}_{K^{1/2}}^T,$$

where $K^{1/2}$ ranges over the $2^{2g}$ square roots of the canonical bundle.

**Remark 3.21.** For the adjoint form of a split real reductive group $G$, Hitchin showed in [26] the existence of a distinguished component of $\mathcal{M}(G)$, isomorphic to a vector space and containing Teichmüller space. This component is known as the Hitchin (or Teichmüller) component. In the case of $\text{Sp}(4, \mathbb{R})$, there are $2^{2g}$ such components, which are exactly the components $\mathcal{M}_{K^{1/2}}^T$. These components are all projectively equivalent, in the sense that the restriction to each of them of the projection to the moduli space for the adjoint group $SO_0(2, 3) \simeq \text{PSp}(4, \mathbb{R})$ is an isomorphism onto the unique Hitchin component in this moduli space (cf. [3]).

**Theorem 3.22 ([22]).** The subspaces $\mathcal{M}_{w_1, w_2}$, $\mathcal{M}_c^0$ with $0 \leq c < 2g - 2$ and $\mathcal{M}_{K^{1/2}}^T$ are connected. Hence the decomposition of $\mathcal{M}^{\text{max}}$ in its connected components is

$$\mathcal{M}^{\text{max}} = ( \bigcup_{w_1, w_2} \mathcal{M}_{w_1, w_2} ) \cup ( \bigcup_{0 \leq c < 2g - 2} \mathcal{M}_c^0 ) \cup ( \bigcup_{K^{1/2}} \mathcal{M}_{K^{1/2}}^T )$$

and the total number of connected components is

$$2(2^{2g} - 1) + (2g - 2) + 2^{2g} = 3 \cdot 2^{2g} + 2g - 4.$$

The proof of the Theorem uses Hitchin’s strategy [25, 26] of considering the Hitchin function, a positive proper function on the moduli space defined by the $L^2$-norm of the Higgs field. Properness of the function means that, in order to show that a given subspace $\mathcal{N}$ of the moduli space is connected, it suffices to prove connectedness of the non-empty subspace of local minima of the Hitchin function restricted to $\mathcal{N}$.

\[ \text{hence the superscript } T \text{ in the notation} \]
3.7. Description of the maximal Higgs bundles. The purpose of this section is to describe the Higgs bundles in each connected component of $\mathcal{M}^{\text{max}}$.

**Proposition 3.23.** Let $(V, \beta, \gamma)$ be an $\text{Sp}(4, \mathbb{R})$-Higgs bundle with $\deg(V) = 2g - 2$.

1. Suppose that $V = N \oplus N^{-1}K$ and that with respect to this decomposition, $\gamma = \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$ is

   \begin{itemize}
   \item[(a)] If $g - 1 < \deg(N) \leq 3g - 3$ then:
   \begin{itemize}
   \item[(i)] $(V, \beta, \gamma)$ is a stable $\text{Sp}(4, \mathbb{R})$-Higgs bundle if and only if $\beta_2 \neq 0$.
   \item[(ii)] If $\beta_2 = 0$ then $(V, \beta, \gamma)$ is not semistable.
   \end{itemize}
   \end{itemize}

2. If $V = W \otimes K^{1/2}$ where $W$ is as in (2) of Proposition 3.16 and $\gamma = q_W \otimes 1_{K^{1/2}}$ then $(V, \beta, \gamma)$ is a stable $\text{Sp}(4, \mathbb{R})$-Higgs bundle.

3. If $V = (L_1 \oplus L_2) \otimes K^{1/2}$ where $L_1$ and $L_2$ are line bundles satisfying $L_2^2 = \mathcal{O}$, $\gamma = \left( \begin{smallmatrix} n \otimes K^{1/2} & 0 \\ 0 & q_2 \otimes K^{1/2} \end{smallmatrix} \right)$ where $q_i$ gives the isomorphism $L_i \simeq L_i^{-1}$ and $1_{K^{1/2}}$ denotes the identity map on $K^{1/2}$, and $\beta = \left( \begin{smallmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{smallmatrix} \right)$ then

   \begin{itemize}
   \item[(a)] $(V, \beta, \gamma)$ is a polystable $\text{Sp}(4, \mathbb{R})$-Higgs bundle.
   \item[(b)] $(V, \beta, \gamma)$ is stable if and only if $L_1 \neq L_2$.
   \end{itemize}

Moreover, if the $\text{Sp}(4, \mathbb{R})$-Higgs bundle $(V, \beta, \gamma)$ is stable then it is simple, unless it is of the form described in Case (3).

**Proof.** Part (1a) follows immediately from Proposition 3.13 and the bounds on $\deg(N)$. Part (1b) follows from Proposition 3.6. Part (2) follows from the fact that in this case $W$ is a stable $O(2)$-bundle. Part (3) follows from Proposition 3.6 and Remark 3.7. \qed

**Remark 3.24.** Proposition 3.23 (1a) says that for $0 < c \leq 2g - 2$ all points in the components $\mathcal{M}_c^0$ are represented by stable $\text{Sp}(4, \mathbb{R})$-Higgs bundles. These Higgs bundles are, moreover, simple and hence represent smooth points of the moduli space (see [16]). It follows that the components $\mathcal{M}_c^0$ are smooth for all $c$ in the range $(0, 2g - 2]$.

The following Proposition gives a description of the $\text{Sp}(4, \mathbb{R})$-Higgs bundles in each component of $\mathcal{M}^{\text{max}}$. Everything in the Proposition follows immediately from what we have said so far, except for the identification of the minima of the Hitchin function (which, though not essential, has been included for completeness; see [22] for the proofs).

**Proposition 3.25.** Let $[V, \beta, \gamma]$ denote an isomorphism class of $\text{Sp}(4, \mathbb{R})$-Higgs bundles in $\mathcal{M}^{\text{max}}$. Then

1. $[V, \beta, \gamma] \in \mathcal{M}^{\ast}_{K^{1/2}}$ if and only if we can take $V = K^{3/2} \oplus K^{-1/2}$, $\gamma = \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$, and $\beta = \left( \begin{smallmatrix} \beta_1 & \beta_3 \\ \beta_3 & 1_{K^{1/2}} \end{smallmatrix} \right)$. It represents a local minimum of the Hitchin function if and only if $\beta_1 = 0$ and $\beta_3 = 0$.

2. $[V, \beta, \gamma] \in \mathcal{M}^0_c$ with $0 < c < 2g - 2$ if and only if we can take $V = N \oplus N^{-1}K$ where $N$ is a line bundle of degree $c + g - 1$, $\gamma = \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$, and $\beta = \left( \begin{smallmatrix} \beta_1 & \beta_2 \\ \beta_2 & \beta_3 \end{smallmatrix} \right)$ with $\beta_2 \neq 0$. It represents a local minimum of the Hitchin function if and only if $\beta_1 = 0$ and $\beta_3 = 0$.
(3) \([V, \beta, \gamma] \in \mathcal{M}^0_c\) if and only if we can take \(V = N \oplus N^{-1}K\) where \(N\) is a line bundle of degree \(g - 1\) and \(\gamma = (0 I)\). It represents a local minimum of the Hitchin function if and only if \(\beta = 0\).

(4) \([V, \beta, \gamma] \in \mathcal{M}_{w_1,w_2}\) if and only if we can take either

(a) \(V = W \otimes K^{1/2}\) where \(W\) is as in (2) of Proposition 3.16, or
(b) \(V = L_1K^{1/2} \oplus L_2K^{1/2}\) where

(i) \(L_1\) and \(L_2\) are line bundles satisfying \(L_i^2 = O\),
(ii) \(w_1(L_1) + w_1(L_2) = w_1\), \(w_1(L_1)w_1(L_2) = w_2\), and
(iii) \(\gamma = (q_1 \otimes \frac{1}{0} \otimes \frac{1}{q_2})\) where \(I\) denotes the identity map on \(K^{1/2}\) and \(q_i\) gives the isomorphism \(L_i \simeq L_i^{-1}\).

It represents a local minimum of the Hitchin function if and only if \(\beta = 0\).

Remark 3.26. The \(Sp(4, \mathbb{R})\)-Higgs bundles of the type described in case (b) of item (4) in Proposition 3.25 have \(L_1 \neq L_2\) since \(w_1(L_1) + w_1(L_2) = w_1 \neq 0\). We point out that \(Sp(4, \mathbb{R})\)-Higgs bundles of this form but with \(L_1 = L_2\) (\(\iff w_1(L_1) + w_1(L_2) = 0\)) are isomorphic to those described in item (3) of the Proposition.

3.8. Description of maximal components. We can use the information in Section 3.7 to completely describe some components of \(\mathcal{M}^\text{max}\). Points in the moduli space correspond to isomorphism classes of Higgs bundles, while Proposition 3.25 describes representatives of the isomorphism classes. We thus need to understand when two representatives belong to the same isomorphism class.

For \(c\) in the range \(0 \leq c \leq 2g-2\), representatives of points in the components \(\mathcal{M}^0_c\) are specified, according to Proposition 3.25, by tuples \((N, \beta_1, \beta_2, \beta_3)\) where \(N\) is a line bundle of degree \(c + g - 1\), \(\beta_1 \in H^0(N^2K), \beta_2 \in H^0(N^{-2}K^3), \beta_3 \in H^0(K^2)\). In the case \(c = 2g - 2\) we require further that \(N^2 = K\) and that \(\beta_2 = 1_{K^{1/2}}\).

Proposition 3.27. Fix \(c\) in the range \(0 < c \leq 2g-2\). Tuples \((N, \beta_1, \beta_2, \beta_3)\) and \((N', \beta_1', \beta_2', \beta_3')\) define the same isomorphism class in \(\mathcal{M}^0_c\) if and only if \(N = N'\) and

1. when \(0 < c < 2g-2\)

\[
(\beta_1', \beta_2', \beta_3') = (t^2 \beta_1, t^{-2} \beta_2, \beta_3)
\]

for some non-zero \(t \in \mathbb{C}^*\), while

2. when \(c = 2g-2\)

\[
(\beta_1', 1_{K^{1/2}}, \beta_3') = (\beta_1, 1_{K^{1/2}}, \beta_3).
\]

Proof. Higgs bundles \((V, \beta, \gamma)\) and \((V', \beta', \gamma')\) are isomorphic if and only if there is a bundle isomorphism \(g : V \longrightarrow V'\) such that

\[
\beta' = g \otimes I_K \circ \beta \circ g^*
\]
\[
\gamma = g^* \otimes I_K \circ \gamma' \circ g
\]

If the bundles are of the form \(N \oplus N^{-1}K\) and \(N' \oplus N'^{-1}K\), and if \(\gamma = \gamma' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\), then above constraints imply that there are two possibilities for \(g\): either \(g = \begin{pmatrix} 0 & g_2 \\ g_3 & 0 \end{pmatrix}\) with \(g_2 \in H^0(N'NK^{-1}), g_3 \in H^0((N'N)^{-1}K)\) and \(g_2g_3 = 1\), or \(g = \begin{pmatrix} g_1 & 0 \\ 0 & g_4 \end{pmatrix}\) with \(g_1 \in H^0(N'N^{-1})\),

\[\text{18}\]
\(g_4 \in H^0(N'^{-1}N)\) and \(g_1 g_4 = 1\). The first possibility can occur only if \(NN' = K\), i.e. if \(c = 0\). Thus when \(0 < c \leq 2g - 2\), the only possibility is that \(N^{-1}N' = \mathcal{O}\), i.e. \(N = N'\), and that

\[
g = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}
\]

(3.35)

where \(t\) is any non-zero complex number. The result follows immediately from this. \(\square\)

Let \(\text{Jac}^d\) be the Jacobian of degree \(d\) line bundles on \(X\) and let

\[
\mathcal{U}_d \longrightarrow \text{Jac}^d(X) \times X
\]

(3.36)

be the universal bundle. Denote the projections from \(\text{Jac}^d(X) \times X\) onto its factors \(\text{Jac}^d(X)\) and \(X\) by \(\pi_J\) and \(\pi_X\) respectively. Define

\[
\mathcal{E}_{d}^{(1)} = \pi_{J*}(\mathcal{U}_{d}^2 \otimes \pi_{X}^*(K))
\]

(3.37)

\[
\mathcal{E}_{d}^{(2)} = \pi_{J*}(\mathcal{U}_{d}^{-2} \otimes \pi_{X}^*(K^3))
\]

(3.38)

\[
\mathcal{E}_d = \mathcal{E}_{d}^{(1)} \oplus \mathcal{E}_{d}^{(2)} \oplus \pi_{J*}(\pi_{X}^*(K^2))
\]

(3.39)

Then \(\mathcal{E}_d\) is a coherent sheaf over \(\text{Jac}^d\). Moreover, for fixed \(c\) in the range

\[
0 < c < g - 1
\]

(3.40)

both \(h^1(N^2K)\) and \(h^1(N^{-2}K^3)\) vanish and thus, by the Riemann-Roch theorem, \(h^0(N^2K)\) and \(h^0(N^{-2}K^3)\) are independent of \(N\). It follows that \(\mathcal{E}_d\) is locally free with fiber over the point represented by the line bundle \(N\) given by

\[
\mathcal{E}_{d,N} = H^0(N^2K) \oplus H^0(N^{-2}K^3) \oplus H^0(K^2)
\]

(3.41)

**Definition 3.28.** Define a \(C^*\)-action on \(\mathcal{E}_d\) by the fiberwise action

\[
C^* \times \mathcal{E}_{d,N} \longrightarrow \mathcal{E}_{d,N}
\]

(3.42)

\[
(t, (\beta_1, \beta_2, \beta_3)) \mapsto (t^2 \beta_1, t^{-2} \beta_2, \beta_3)
\]

(3.43)

**Proposition 3.29.**

(1) For each \(c\) in the range \(0 < c < g - 1\) the component \(\mathcal{M}_c^0\) is the total space of the quotient \(\mathcal{E}_d/C^*\) where

- \(d = c + g - 1\),
- \(\mathcal{E}_d\) denotes \(\mathcal{E}_d\) minus the zero section of \(\mathcal{E}_d^{(2)}\), and
- the \(C^*\) action is as in Definition (3.28).

The fibers of \(\mathcal{M}_c^0\) as a fibration over \(\text{Jac}^d\) are given by \(\mathcal{O}_{P^r}(1)^{\oplus r} \times C^{3g-3}\) where

\[
r = h^0(N^2K) = 2c + 3g - 3
\]

(3.44)

\[
s = h^0(N^{-2}K^3) - 1 = 3g - 4 - 2c
\]

(3.45)

(2) For each choice of a square root \(K^{1/2}\) of the canonical bundle, the component \(\mathcal{M}^T_{K^{1/2}}\) is isomorphic to the vector space \(H^0(K^2) \oplus H^0(K^4)\).

**Remark 3.30.** Part (2) of this proposition is equivalent to Hitchin’s parametrization [20] of his Teichmüller component (cf. Remark 3.21).
Proof. Everything except the description of the fibers of \( M_c \) follows immediately from Propositions \[ 3.25 \text{ and } 3.27 \]. The description of the fibers follows from the claim that

\[
(C^r \oplus (C^*)^{s+1})/C^* = O_{\mathbb{P}^r}(1)^{\oplus r},
\]

where the \( C^* \)-action is given by \( t(\bar{z}, \bar{w}) = (t^2 \bar{z}, t^{-2} \bar{w}) \). But the total space of \( O_{\mathbb{P}^r}(1)^{\oplus r} \) can be identified with the variety

\[
\mathcal{T} = \{ (l, \bar{x}_1, \ldots, \bar{x}_r) \mid l \text{ defines a line in } C^{s+1} \text{ and } \bar{x}_i \in C^{s+1} \text{ lies on } l \}
\]

The map

\[
\frac{(C^r \oplus (C^*)^{s+1})}{C^*} \to \mathbb{P}^s \times (C^{s+1} \oplus \cdots \oplus C^{s+1})
\]

\[(z_1, \ldots, z_r), \bar{w}] \mapsto [\bar{w}], (z_1 \bar{w}, \ldots, z_r \bar{w})
\]

is well defined with a well defined inverse on the subvariety \( \mathcal{T} \), and thus proves our claim. The factor \( C^{3g-3} \) comes from \( H^0(K^2) \). \( \square \)

Remark 3.31. For \( c \) in the range \([g-1, 2g-2]\), there is a map \( f : M_c^0 \to \text{Jac}^{c+g-1} \) but it is not surjective and the fiber dimension is not necessarily constant. Nevertheless, by remark \[ 3.24 \] these components are smooth for all \( c \) in the range \([0, 2g-2]\).

4. Subgroups for maximal representations

4.1. Identification of possible subgroups. The main result of this subsection, Proposition \[ 4.9 \] identifies the possible subgroups of \( \text{Sp}(4, \mathbb{R}) \) through which a maximal representation can factor. The argument leading to this Proposition is due to Wienhard [44]. The basis is the following result of Burger, Iozzi and Wienhard [5, 6].

**Theorem 4.1.** Let \( G \) be of hermitean type. Let \( \rho : \pi_1(X) \to G \) be maximal and let \( \tilde{G} = (\rho(\pi_1(X)))^\circ \) (the identity component of the real part of the Zariski closure). Then

1. \( \tilde{G} \) is hermitean of tube type;
2. the embedding \( \tilde{G} \hookrightarrow G \) is tight.

By classification of tube type domains ([37]) one has the following.

**Lemma 4.2.** The only tube type domains of dimension less than or equal to 3 and rank less than or equal to 2 are \( \mathbb{D}, \mathbb{D} \times \mathbb{D} \) and \( \text{Sp}(4, \mathbb{R})/\text{U}(2) \).

We identify three natural subgroups in \( \text{Sp}(4, \mathbb{R}) \) and then show that, as a result of Lemma \[ 4.2 \] these are essentially the only possibilities. For two of them it is convenient to define \( \text{Sp}(4, \mathbb{R}) \) with respect to the symplectic form

\[
J_{12} = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \quad \text{where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

**Remark 4.3.** The relation between \( J_{12} \) and \( J_{13} \) — and hence between the resulting descriptions of \( \text{Sp}(4, \mathbb{R}) \) — is described in Appendix [A]

The subgroups come from the following three representations:

- The irreducible 4-dimensional representation of \( \text{SL}(2, \mathbb{R}) \) in \( \text{Sp}(4, \mathbb{R}) \),

\[
\rho_1 : \text{SL}(2, \mathbb{R}) \hookrightarrow \text{Sp}(4, \mathbb{R}).
\]

See Section [8] for a full description.
• The representation of $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ given with respect to $J_{12}$ by

$$\rho_2: \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \hookrightarrow \text{Sp}(4, \mathbb{R}) \quad (4.3)$$

$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

• The representation of $\text{SL}(2, \mathbb{R})$ given by,

$$\rho_3 = \rho_2 \circ \Delta: \text{SL}(2, \mathbb{R}) \hookrightarrow \text{Sp}(4, \mathbb{R}), \quad (4.4)$$

where $\Delta$ is the diagonal embedding

$$\text{SL}(2, \mathbb{R}) \hookrightarrow \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}).$$

Remark 4.4. Using the Kronecker product\(^7\), the diagonal embedding $\rho_3$ is given by

$$\rho_3: A \mapsto \begin{cases} I \otimes A \text{ with respect to } J_{12} \\ A \otimes I \text{ with respect to } J_{13} \end{cases} \quad (4.5)$$

Definition 4.5. Let

$$\mathcal{D}_p = \rho_p(\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}))/\rho_p(\text{SO}(2) \times \text{SO}(2));$$

$$\mathcal{D}_\Delta = \rho_\Delta(\text{SL}(2, \mathbb{R}))/\rho_\Delta(\text{SO}(2));$$

$$\mathcal{D}_i = \rho_i(\text{SL}(2, \mathbb{R}))/\rho_i(\text{SO}(2)).$$

With this notation, Lemma 4.2 together with the results of Wienhard et al. on tight embeddings (see \[7, 45\]) implies the following.

Proposition 4.6. Up to isometry of $\text{Sp}(4, \mathbb{R})/\text{U}(2)$, the only proper tube type domains tightly embedded in $\text{Sp}(4, \mathbb{R})/\text{U}(2)$ are $\mathcal{D}_p \simeq \mathcal{D} \times \mathcal{D}$, $\mathcal{D}_\Delta \simeq \mathcal{D}$ and $\mathcal{D}_i \simeq \mathcal{D}$.

Remark 4.7. Note that $\mathcal{D}_i \simeq \mathcal{D}$ is not holomorphically embedded, while the other two are.

Proposition 4.6 is not quite sufficient for identifying the possible embedded subgroups since the subdomains do not uniquely determine the subgroups. Suppose that subgroups $G_1 \subset G_2 \subset \text{Sp}(4, \mathbb{R})$, with maximal compact subgroups $H_1 \subset H_2$, give rise to the same subdomain, i.e. are such that $G_1/H_1 = G_2/H_2$. Then it is straightforward to see that

- $H_1$ is a normal subgroup of $H_2$, and
- if the Cartan decompositions for the subgroups are $g_i = h_i + m_i$, then $m_1 = m_2$.

It follows that $G_1$ is a normal subgroup of $G_2$. The next proposition is thus immediate.

Proposition 4.8. The following subgroups are the largest that give rise to the embedded domains $\mathcal{D}_i, \mathcal{D}_p,$ and $\mathcal{D}_\Delta$ respectively:

$$G_i = N_{\text{Sp}(4, \mathbb{R})}(\rho_1(\text{SL}(2, \mathbb{R}))),$$

$$G_p = N_{\text{Sp}(4, \mathbb{R})}(\rho_2(\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}))),$$

$$G_\Delta = N_{\text{Sp}(4, \mathbb{R})}(\rho_3(\text{SL}(2, \mathbb{R}))).$$

Hence Theorem 4.1 implies the following result.

---

\(^7\)see Appendix A
Proposition 4.9. Let \( \rho: \pi_1(X) \to \text{Sp}(4, \mathbb{R}) \) be maximal and assume that \( \rho \) factors through a proper reductive subgroup \( \tilde{G} \subset G \). Then, up to conjugation, \( \tilde{G} \) is contained in one of the subgroups \( G_i, G_\Delta \) and \( G_p \).

**Note:** We will sometimes use \( G_* \) to denote \( G_i, G_p \) or \( G_\Delta \).

Explicit calculations show that:

Proposition 4.10. We compute that

(1) \( G_p \) is the group generated by \( \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \) and \( \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \). That is, with respect to \( J_{12} \), \( G_p \subset \text{Sp}(4, \mathbb{R}) \) is
\[
G_p = \{ \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} \in \text{Sp}(4, \mathbb{R}) \mid \text{either } Y = Z = 0 \text{ or } X = T = 0 \}.
\]

(2) \( G_\Delta = \text{O}(2) \otimes \text{SL}(2, \mathbb{R}) \) with respect to \( J_{12} \). That is, with respect to \( J_{12} \), \( G_\Delta \subset \text{Sp}(4, \mathbb{R}) \) is
\[
G_\Delta = \{ \begin{pmatrix} xA & yA \\ zA & tA \end{pmatrix} \mid X = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in \text{O}(2) \text{ and } A \in \text{SL}(2, \mathbb{R}) \}.
\]

We defer the calculation of \( G_i \) to Section 5 where the necessary details of the irreducible representation are given. The result we obtain (see Proposition 8.15) is:

Proposition 4.11. \( G_i = \text{SL}(2, \mathbb{R}) \), i.e.
\[
N_{\text{Sp}(4, \mathbb{R})}(\rho_1(\text{SL}(2, \mathbb{R}))) = \rho_1(\text{SL}(2, \mathbb{R})). \tag{4.6}
\]

5. Deformations of representations – main results

5.1. Invariants of representations. Let \( \rho: \pi_1(X) \to \text{Sp}(4, \mathbb{R}) \) be a representation and let \( E_\rho \) be the associated flat \( \text{Sp}(4, \mathbb{R}) \)-bundle. Then the Toledo invariant \( d(\rho) \) of \( \rho \) is simply the first Chern class of the (non-flat) \( \text{U}(2) \)-bundle obtained by a reduction of the structure group of \( E_\rho \) to the maximal compact \( \text{U}(2) \subset \text{Sp}(4, \mathbb{R}) \). In terms of the \( \text{Sp}(4, \mathbb{R}) \)-Higgs bundle \((V, \beta, \gamma)\) associated to \( \rho \) via the non-abelian Hodge theory correspondence, we have \( d(\rho) = \text{deg}(V) \). A representation \( \rho \) is said to be maximal if \( d(\rho) = 2g - 2 \) (cf. Proposition 3.11). Denote the subspace of maximal representations of \( \mathcal{R}(\text{Sp}(4, \mathbb{R})) \) by \( \mathcal{R}^{\text{max}} \). Then the non-abelian Hodge theory correspondence \((2.2)\) gives a homeomorphism
\[
\mathcal{R}^{\text{max}} \cong \mathcal{M}^{\text{max}}. \tag{5.1}
\]

We point out that, by the results of Burger, Iozzi and Wienhard \([5, 6]\), any maximal representation is reductive. Hence the space \( \mathcal{R}^{\text{max}} \) consists of all (isomorphism classes of) maximal representations.

**Definition 5.1.** We denote by \( \mathcal{R}_{w_1, w_2}^0 \), \( \mathcal{R}_c^T \) and \( \mathcal{R}_{K^{1/2}}^0 \) the subspaces of \( \mathcal{R}^{\text{max}} \) corresponding under \((5.7)\) to the subspaces \( \mathcal{M}_{w_1, w_2} \), \( \mathcal{M}_c^0 \) and \( \mathcal{M}_{K^{1/2}}^0 \), respectively, of \( \mathcal{M}^{\text{max}} \) (cf. \((3.29)\), \((3.30)\) and \((3.32)\)).

**Remark 5.2.** Though apparently of a holomorphic nature, the choice of a square root \( K^{1/2} \) of the canonical bundle of \( X \) is in fact purely topological: each such choice corresponds to the choice of a spin structure on the oriented topological surface \( S \) underlying \( X \).
5.2. **Main Theorem.** With these preliminaries in place, we can state our main result. The proof is based on a careful analysis of $G_\ast$-Higgs bundles carried out in Sections 6, 7 and 8 below.

We shall say that a $\text{Sp}(4, \mathbb{R})$-Higgs bundle $(V, \beta, \gamma)$ **deforms** to a $\text{Sp}(4, \mathbb{R})$-Higgs bundle $(V', \beta', \gamma')$, if they belong to the same connected component of the moduli space. In other words, we mean continuous deformation through polystable $\text{Sp}(4, \mathbb{R})$-Higgs bundles. In the setting of representations, we use the analogous notion of deformation.

**Theorem 5.3.** Let $X$ be a closed Riemann surface of genus $g \geq 2$ and let $(V, \beta, \gamma)$ be a maximal polystable $\text{Sp}(4, \mathbb{R})$-Higgs bundle. Then:

1. $(V, \beta, \gamma)$ deforms to a polystable $G_\Delta$-Higgs bundle if and only if it belongs to one of the subspaces $M_{w_1,w_2}$ or $M^0_0$ of $M^{\text{max}}$.

2. $(V, \beta, \gamma)$ deforms to a polystable $G_p$-Higgs bundle if and only if it belongs to one of the subspaces $M_{w_1,w_2}$ or $M^0_0$ of $M^{\text{max}}$.

3. $(V, \beta, \gamma)$ deforms to a polystable $G_i$-Higgs bundle if and only if it belongs to one of the subspaces $M^1_{K^{1/2}}$.

4. There is no proper reductive subgroup $G_\ast \subset \text{Sp}(4, \mathbb{R})$ such that $(V, \beta, \gamma)$ can be deformed to a $G_\ast$-Higgs bundle if and only if $(V, \beta, \gamma)$ belongs to one of the components $M^0_c$ with $0 < c < 2g - 2$.

The corresponding result for surface group representations is:

**Theorem 5.4.** Let $S$ be a closed oriented surface of genus $g \geq 2$ and let $\rho: \pi_1(S) \to \text{Sp}(4, \mathbb{R})$ be a maximal representation. Then:

1. The representation $\rho$ deforms to a representation which factors through the subgroup $G_\Delta \subset \text{Sp}(4, \mathbb{R})$ if and only if it belongs to one of the subspaces $R_{w_1,w_2}$ or $R^0_0$.

2. The representation $\rho$ deforms to a representation which factors through the subgroup $G_p \subset \text{Sp}(4, \mathbb{R})$ if and only if it belongs to one of the subspaces $R_{w_1,w_2}$ or $R^0_0$.

3. The representation $\rho$ deforms to a representation which factors through the subgroup $G_i \subset \text{Sp}(4, \mathbb{R})$ if and only if it belongs to one of the subspaces $R^1_{K^{1/2}}$.

4. There is no proper reductive subgroup $G_\ast \subset \text{Sp}(4, \mathbb{R})$ such that $\rho$ can be deformed to a representation which factors through $G_\ast$ if and only if $\rho$ belongs to a component $R^0_c$ for some $0 < c < 2g - 2$.

**Proof of Theorems 5.3 and 5.4.** Statements (1)–(3) of Theorem 5.3 follow from the results for $G_\ast$-Higgs bundles given in Theorem 6.17 for $G_\ast = G_\Delta$, Theorem 7.12 for $G_\ast = G_p$ and Theorem 8.16 for $G_\ast = G_i$.

Statements (1)–(3) of Theorem 5.4 now follow immediately through the non-abelian Hodge theory correspondence [5.1]. Moreover, by Proposition 4.9, a maximal representation which factors through a proper reductive subgroup must in fact factor through one of the groups $G_\Delta$, $G_p$ and $G_i$. Hence statements (1)–(3) of Theorem 5.3 imply statement (4) of the same Theorem.

Finally, by the non-abelian Hodge theory correspondence [5.1], statement (4) of Theorem 5.3 follows from statement (4) of Theorem 5.4. \[ \square \]

**Remark 5.5.** Part (4) of this theorem says that for any representation, say $\rho: \pi_1(X) \to \text{Sp}(4, \mathbb{R})$, represented by a point in one of the components $R^0_c$, the image $\rho(\pi_1(X))$ in $\text{Sp}(4, \mathbb{R})$
is Zariski dense. Parts (1)–(3) of the theorem say that any other representation can be deformed to one whose image is not Zariski dense, and describe in which subgroups the image \(\rho(\pi_1(X))\) may lie.

Remark 5.6. Though (4) of Theorem 5.3 is a result about \(\text{Sp}(4, \mathbb{R})\)-Higgs bundles our proof depends on the correspondence with representations, since it uses Proposition 4.9. We expect, though, that a pure Higgs bundle proof can be given by applying the Cayley correspondence of [3] (cf. Section 3.5).

6. Analysis of \(G^*_\text{b}-\text{Higgs bundles I: } G_{\Delta}-\text{Higgs bundles}\)

In this section we identify the \(\text{Sp}(4, \mathbb{R})\)-Higgs bundles which admit a reduction of structure group to \(G_{\Delta}\), in the sense of Definition 2.6.

6.1. The embedding of \(G_{\Delta}\) in \(\text{Sp}(4, \mathbb{R})\). Proposition 4.10 describes \(G_{\Delta}\) as an embedded subgroup of \(\text{Sp}(4, \mathbb{R})\) (with respect to \(J_{12}\)). As an abstract group we can identify \(G_{\Delta}\) as the group

\[
G_{\Delta} \simeq (\text{SL}(2, \mathbb{R}) \times O(2))/\mathbb{Z}/2.
\]

(6.1)

This has a maximal compact subgroup

\[
H_{\Delta} \simeq (\text{SO}(2) \times O(2))/\mathbb{Z}/2
\]

(6.2)

and a Cartan decomposition of its Lie algebra

\[
\text{Lie}(G_{\Delta}) \simeq (\mathfrak{so}(2) \oplus \mathfrak{o}(2)) \oplus \mathfrak{m}(\text{SL}(2, \mathbb{R}))
\]

(6.3)

where

\[
\mathfrak{m}(\text{SL}(2, \mathbb{R})) = \{ \begin{pmatrix} x & y \\ y & -x \end{pmatrix} \in \mathfrak{gl}(2, \mathbb{R}) \}
\]

(6.4)

Since we prefer to use \(J_{13}\) when describing \(\text{Sp}(4, \mathbb{R})\)-Higgs bundles, we need to adjust the embedding given in Proposition 4.10. Conjugation by the matrix \(h\) given in Appendix A shows that with respect to \(J_{13}\) the images of \(G_{\Delta}\) and \(H_{\Delta}\) are

\[
G_{\Delta} = \text{SL}(2, \mathbb{R}) \otimes O(2)
\]

(6.5)

\[
H_{\Delta} = \text{SO}(2) \otimes O(2)
\]

(6.6)

\[
\begin{aligned}
&= \left\{ \begin{pmatrix} aX & bX \\ cX & dX \end{pmatrix} \mid X^tX = I \text{ and } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}) \right\} \\
&= \left\{ A \otimes X \in \text{SL}(2, \mathbb{R}) \otimes O(2) \mid A^tA = I, \det(A) = 1 \right\}
\end{aligned}
\]

Lemma 6.1.

1. The Lie algebra of \(G_{\Delta}\) is invariant under the Cartan involution on \(\text{Sp}(4, \mathbb{R})\).
2. \(\text{Lie}(G_{\Delta}) \cap \mathfrak{u}(2) = \text{Lie}(H_{\Delta}) \) where \(\mathfrak{u}(2) \subset \mathfrak{sp}(4, \mathbb{R})\) is as in (3.9).
Remark 6.2. It follows that the Cartan involution on $\text{Sp}(4, \mathbb{R})$ restricts to define a Cartan involution on $G_\Delta$. In fact it is the Cartan involution$^{10}$ corresponding to the decomposition (6.3) and we see that $G_\Delta$ is a reductive subgroup of $\text{Sp}(4, \mathbb{R})$ (see Section 2.3). In particular, $H_\Delta$ lies in the $U(2)$ subgroup embedded in $\text{Sp}(4, \mathbb{R})$ as in (3.4).

The following computations are needed to identify the $\text{Sp}(4, \mathbb{R})$-Higgs bundles whose structure group reduces to $G_\Delta$.

**Proposition 6.3.** (1) The complexification of $G_\Delta$ is
\[ G_\Delta^C \simeq (\text{SL}(2, \mathbb{C}) \times \text{O}(2, \mathbb{C}))/\langle \mathbb{Z}/2 \rangle. \] (6.7)

(2) The complexification of $H_\Delta$ is isomorphic to the complex conformal group, i.e.
\[ H_\Delta^C \simeq (\text{SO}(2, \mathbb{C}) \times \text{O}(2, \mathbb{C}))/\langle \mathbb{Z}/2 \rangle \simeq \text{CO}(2, \mathbb{C}) \] (6.8)

where
\[ \text{CO}(2, \mathbb{C}) = \{ A \in \text{GL}(2, \mathbb{C}) \mid A^tA = \frac{\text{tr}(A^tA)}{2} I \} \] (6.9)

*Proof.* (1) Clear. For (2) identify $^{11} \text{SO}(2, \mathbb{C})$ with $\mathbb{C}^*$ and use the homomorphism
\[ \mathbb{C}^* \times \text{O}(2, \mathbb{C}) \longrightarrow \text{CO}(2, \mathbb{C}) \] defined by $(\lambda, A) \mapsto \lambda A$. This is surjective with kernel $\{ \pm I \}$. □

It follows from (6.7) and (6.8) that the complexification of the Cartan decomposition (6.3) is
\[ \text{Lie}(G_\Delta^C) = \text{Lie}(H_\Delta^C) \oplus m^C_\Delta \] (6.11)

where
\[ m^C(\text{SL}(2, \mathbb{R})) = \left\{ \begin{pmatrix} x & y \\ y & -x \end{pmatrix} \in \mathfrak{gl}(2, \mathbb{C}) \right\}. \] (6.12)

The proof of Proposition 4.10 "complexifies" to show:

**Proposition 6.4.** The embedding of $G_\Delta^C$ in $\text{Sp}(4, \mathbb{C})$ is given by
\[ (A, X) \mapsto \begin{cases} X \otimes A = \begin{pmatrix} xA & yA \\ zA & tA \end{pmatrix} \text{ with respect to } J_{12}, \\ A \otimes X = \begin{pmatrix} aX & bX \\ cX & dX \end{pmatrix} \text{ with respect to } J_{13} \end{cases} \] (6.13)

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $\text{SL}(2, \mathbb{C})$ and $X = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ is in $\text{O}(2, \mathbb{C})$.

$^{10}$We cannot apply Remark 2.11 directly to conclude this, because $G_\Delta$ is not semisimple. However, the explicit verification below of (6.15) justifies our claim.

$^{11}$ via
\[ \lambda \mapsto \begin{pmatrix} \lambda + \lambda^{-1} & -\lambda - \lambda^{-1} \\ \lambda - \lambda^{-1} & \lambda + \lambda^{-1} \end{pmatrix} \]
These embeddings induce embeddings of $\text{Lie}(G^C_\Delta)$ in $\mathfrak{sp}(4, \mathbb{C})$. Let $m^C_\Delta$ denote the image of $m^C(SL(2, \mathbb{R}))$ under the embedding with respect to $J_{13}$. It follows that we can identify $m^C_\Delta \subset \mathfrak{sp}(4, \mathbb{C})$ as

$$m^C_\Delta = \left\{ \begin{pmatrix} aI & bI \\ bI & -aI \end{pmatrix} \mid a, b \in \mathbb{C} \right\}. \quad (6.14)$$

**Remark 6.5.** Comparison with the Cartan decomposition for $\text{Sp}(4, \mathbb{R})$ (see (3.11) and (3.13)) confirms that, as required (cf. (2.3) and Remark 6.2), we get

$$\mathfrak{sp}(4, \mathbb{C}) = \mathfrak{gl}(2, \mathbb{C}) \oplus m^C_\Delta \quad (6.15)$$

where $g^C_\Delta = \text{Lie}(G^C_\Delta)$ and $h^C_\Delta = \text{Lie}(H^C_\Delta)$.

A change of basis via $T = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$ transforms $m^C_\Delta$ into

$$m^C_\Delta = \left\{ \begin{pmatrix} 0 & \bar{\beta}I \\ \bar{\gamma}I & 0 \end{pmatrix} \mid \bar{\beta}, \bar{\gamma} \in \mathbb{C} \right\}, \quad (6.16)$$

where the descriptions in (6.14) and (6.16) are related by

$$\bar{\beta} = 2(a + ib), \quad (6.17)$$
$$\bar{\gamma} = 2(a - ib). \quad (6.18)$$

### 6.2. The principal bundle.

**Lemma 6.6.** Let $V$ be a rank 2 vector bundle associated to a principal $\text{CO}(2, \mathbb{C})$-bundle over $X$. Fix a good cover $\mathcal{U} = \{U_\alpha\}$ for $X$ and suppose that $V$ is defined by transition functions $\{g_{\alpha \beta}\}$ with respect to $\mathcal{U}$. Pick $\{l_{\alpha \beta} \in \mathbb{C}^*\}$ and $\{h_{\alpha \beta} \in \text{O}(2, \mathbb{C})\}$ such that

$$g_{\alpha \beta} = l_{\alpha \beta} h_{\alpha \beta}. \quad (6.19)$$

Then

1. the functions $\{l_{\alpha \beta}^2\}$ define a line bundle, say $L$, and
2. $L^2 = \text{det}^2(V)$

**Proof.** Consider the cocycles $g_{\alpha \beta \gamma}$ defined by

$$g_{\alpha \beta \gamma} = g_{\alpha \beta} g_{\beta \gamma} g_{\gamma \alpha} = (l_{\alpha \beta} l_{\beta \gamma} l_{\gamma \alpha})(h_{\alpha \beta} h_{\beta \gamma} h_{\gamma \alpha}). \quad (6.20)$$

Since $g_{\alpha \beta \gamma} = I$ and the $h_{\alpha \beta}$ are orthogonal, taking $g_{\alpha \beta \gamma}^t g_{\alpha \beta \gamma}$ yields

$$I = (l_{\alpha \beta}^2 l_{\beta \gamma}^2 l_{\gamma \alpha}^2) I. \quad (6.21)$$

This proves (1). Part (2) now follows directly from (6.19). \qed
Remark 6.7. Using the description $\text{CO}(2, \mathbb{C}) = (\text{O}(2, \mathbb{C}) \times \mathbb{C}^*)/(\mathbb{Z}/2)$, we can define a homomorphism

$$\sigma : \text{CO}(2, \mathbb{C}) \longrightarrow \mathbb{C}^*$$

$$[A, \lambda] \mapsto \lambda^2. \quad (6.22)$$

The bundle $L$ is the line bundle associated to $V$ by the representation $\sigma$, i.e. if $E$ is the principal $\text{CO}(2, \mathbb{C})$-bundle underlying $V$ then

$$L = E \times_{\sigma} \mathbb{C}. \quad (6.24)$$

The locally defined transition data $\{l_{\alpha\beta}\}$ or $\{h_{\alpha\beta}\}$ do not in general define $\mathbb{C}^*$ or $\text{O}(2, \mathbb{C})$ bundles. However, if $V$ has even degree, then we get the following decomposition.

Lemma 6.8. Suppose $V$ and $L$ are as in Lemma 6.6 and that $V$ has even degree. Then $\deg(L)$ is even and we can pick a line bundle $L_0$ such that

$$L_0^2 = L. \quad (6.25)$$

We can then decompose $V$ as

$$V = U \otimes L_0, \quad (6.26)$$

where $U$ is an $\text{O}(2, \mathbb{C})$ bundle.

Proof. Using the same notation as in the proof of the previous lemma, let $L_0$ be defined by transition functions $\{n_{\alpha\beta}\}$. By construction we have

$$n_{\alpha\beta}^2 = l_{\alpha\beta}^2. \quad (6.27)$$

Moreover, the bundle $V \otimes L_0^{-1}$ is defined by transition functions

$$v_{\alpha\beta} = (\frac{l_{\alpha\beta}}{n_{\alpha\beta}})h_{\alpha\beta}. \quad (6.28)$$

But then, since $h_{\alpha\beta} \in \text{O}(2, \mathbb{C})$,

$$v_{\alpha\beta}^t v_{\alpha\beta} = (\frac{l_{\alpha\beta}^2}{n_{\alpha\beta}^2})h_{\alpha\beta}^t h_{\alpha\beta} = 1. \quad (6.29)$$

Thus $U = V \otimes L_0^{-1}$ is an $\text{O}(2, \mathbb{C})$ bundle. □

Conversely, we have the following.

Proposition 6.9. If a rank 2 vector bundle $V$ is of the form

$$V = U \otimes L_0, \quad (6.30)$$

where $U$ is an $\text{O}(2, \mathbb{C})$-bundle and $L_0$ is a line bundle, then the structure group of $V$ reduces to $\text{CO}(2, \mathbb{C})$.

Proof. The proof follows immediately from the projection (6.10). □

Remark 6.10. It follows from (6.30) that the line bundle $L_0$ must satisfy

$$L_0^4 = \det(V)^2. \quad (6.31)$$

Corollary 6.11. Let $(V, \beta, \gamma)$ be a polystable $\text{Sp}(4, \mathbb{R})$-Higgs bundle with maximal Toledo invariant, i.e. with $\deg(V) = 2g - 2$. Then the structure group of $V$ (or, equivalently, of the underlying principal $\text{GL}(2, \mathbb{C})$-bundle) reduces to $\text{CO}(2, \mathbb{C})$, i.e. to $H^C_\Delta$. 27
Proof. By Proposition 3.15 we can write $V = U \otimes L_0$, as required by Proposition 6.9. □

6.3. The Higgs field. By Lemma 6.6 we can always give a ‘virtual’ decomposition of a $\text{CO}(2, \mathbb{C})$ bundle $V$ as $V = U^v \otimes L_0^v$, where $U^v$ and $L_0^v$ are ‘virtual’ bundles. This is an honest decomposition into actual bundles if $\text{deg}(V)$ is even, and in all cases there is a line bundle $L$ such that $L = (L_0^v)^2$.

**Proposition 6.12.** Let $V = U^v \otimes L_0^v$ be the vector bundle in a $G_\Delta$-Higgs bundle. The Higgs field is then a pair $(\tilde{\beta}, \tilde{\gamma})$ where

$$\tilde{\beta} \in H^0((L_0^v)^2 K)) \text{ and } \tilde{\gamma} \in H^0((L_0^v)^{-2} K).$$

(6.32)

Proof. The Cartan decomposition of $G_\Delta^C$ (see (6.11)) shows that the isotropy representation of $H_\Delta^C$ is given by

$$H_\Delta^C = \mathbb{C}^* \times \mathbb{Z} \times \text{O}(2, \mathbb{C}) \to \mathbb{C}^* \times \mathbb{C}^*$$

$$[\lambda, g] \mapsto (\lambda^2, \lambda^{-2}).$$

Let $E_{H_\Delta^C}$ be the principal $\text{CO}(2, \mathbb{C})$ bundle underlying $V$. It follows from the above observations that the bundle associated to $E_{H_\Delta^C}$ by the isotropy representation, i.e. $E_{H_\Delta^C}(m_\Delta^C) = E_{H_\Delta^C} \times \text{Ad } m_\Delta^C$, is

$$E_{H_\Delta^C}(m_\Delta^C) = (L_0^v)^2 \oplus (L_0^v)^{-2}. \quad (6.33)$$

The result follows from this. □

**Proposition 6.13.** Let $(V, \beta, \gamma)$ be an $\text{Sp}(4, \mathbb{R})$-Higgs bundle which admits a reduction of structure group to $G_\Delta$. Then Higgs fields $\beta$ and $\gamma$ have to be of the form

$$\beta = \tilde{\beta} I, \quad (6.34)$$

$$\gamma = \tilde{\gamma} I. \quad (6.35)$$

Proof. This is a direct consequence of (6.16). □

We can rephrase Proposition 6.13 in a frame-independent way:

**Corollary 6.14.** Let $(V, \beta, \gamma)$ be a semistable $\text{Sp}(4, \mathbb{R})$-Higgs bundle for which the structure group reduces to $G_\Delta$. Suppose that $V$ has a decomposition as $V = U \otimes L$ where $(U, q_U)$ is an orthogonal bundle and $L$ is a line bundle. Then, using $S^2 V = (S^2 U) \otimes L^2$ and $S^2 V^* = (S^2 U^*) \otimes L^{-2}$, the components of the Higgs field are given by

$$\gamma = q_U \otimes \tilde{\gamma}, \quad \beta = q_U^t \otimes \tilde{\beta}$$

where

$$\tilde{\beta} \in H^0(L^2 K), \quad \tilde{\gamma} \in H^0(L^{-2} K).$$

Remark 6.15. Notice that the section $\tilde{\gamma} \in H^0(L^{-2} K)$ must be non-zero, since otherwise $\gamma = \tilde{\gamma} q_U$ would be zero, contradicting semistability. If $\text{deg}(V) = 2g - 2$ then $\text{deg}(L) = g - 1$ and $\text{deg}(L^{-2} K) = 0$. It follows that in this case $L^2 = K$, i.e. $L$ is a square root of $K$. 28
6.4. **Identifying components with $G_\Delta$-Higgs bundles.** Having characterized $G_\Delta$-Higgs bundles, we now identify which connected components of $\mathcal{M}^{\text{max}}$ contain the $G_\Delta$-Higgs bundles.

**Theorem 6.16.** Let $(V, \beta, \gamma)$ be a polystable $\text{Sp}(4, \mathbb{R})$-Higgs bundle with maximal (positive) Toledo invariant. If $(V, \beta, \gamma)$ represents a point in one of the components $\mathcal{M}_c^0$ with $0 < c < 2g - 2$ or in one of the components $\mathcal{M}_K^T_{K/2}$, then the structure group of $(V, \beta, \gamma)$ does not reduce to $G_\Delta$.

**Proof.** Let $(V, \beta, \gamma)$ be a polystable $\text{Sp}(4, \mathbb{R})$-Higgs bundle for which $\deg(V) = 2g - 2$. Then $\gamma$ is an isomorphism and $V = W \otimes L_0$ where $W$ is an $O(2, \mathbb{C})$-bundle and $L_0^2 = K$ (see Section 3.5). Suppose that the structure group reduces to $G_\Delta$. Then by Corollary 6.14 and the remark following it, $V$ has a second decomposition $V = U \otimes L$ with $L^2 = K$. Since the bundles in this decomposition are determined only up to a twist by a square root of the trivial line bundle, we can assume that $L = L_0$, and hence that $U = W$. It follows, again by Corollary 6.14, that $\beta = q^* \otimes \tilde{\beta}$ where $q$ is the quadratic form on $W$ and $\tilde{\beta} \in H^0(L^2K)$.

If $w_1 = 0$ then $V$ decomposes as

$$V = (L \oplus L^{-1}) \otimes K^{1/2} = N \oplus N^{-1}K$$

and the quadratic form on $W = L \oplus L^{-1}$ is given by

$$q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

It follows that

$$\beta = \begin{pmatrix} 0 & \tilde{\beta} \\ \tilde{\beta} & 0 \end{pmatrix}$$

with respect to the decomposition $V = N \oplus N^{-1}K$. A comparison with the form of $\beta$ given in (1) and (2) of Proposition 3.25 shows that this is not possible if $(V, \beta, \gamma)$ represents a point in $\mathcal{M}_c^0$ with $0 < c < 2g - 2$ or $\mathcal{M}_K^T_{K/2}$. $\square$

Furthermore, by comparing our description of $G_\Delta$-Higgs bundles with the descriptions of minima of the Hitchin function on $\mathcal{M}^{\text{max}}$, and hence with the list of connected components (see Section 3.7), we get:

**Theorem 6.17.** The following components of $\mathcal{M}^{\text{max}}$ contain $G_\Delta$-Higgs bundles:

1. any component in which $w_1 \neq 0$, i.e.

$$\mathcal{M}_{w_1, w_2}$$

for any $(w_1, w_2) \in ((\mathbb{Z}/2)^{2g} - \{0\}) \times \mathbb{Z}/2$,

2. the component in which $w_1 = 0$ and $c_1 = 0$, i.e. $\mathcal{M}_0^0$.

**Proof.** We construct $\text{Sp}(4, \mathbb{R})$-Higgs bundles whose structure group reduces to $G_\Delta$ and show explicitly that they lie in the requisite components of $\mathcal{M}^{\text{max}}$. Let $U$ be a stable $O(2, \mathbb{C})$-bundle over $X$ and let $L$ be a square root of $K$. Let $(w_1, w_2)$ be the first and second Stiefel-Whitney classes of $U$ and let $q_U : U \to U^*$ be the (symmetric) isomorphism which defines the orthogonal structure on $U$. Consider the data $(V, \beta, \gamma)$, in which

- $V = U \otimes L$,
- $\beta : V^* \to VK$ is the zero map, and
- $\gamma : V \to V^*K$ is given by $q_U \otimes I_L$, where $I_L$ is the identity map on $L$, and
By construction, the structure group of $V$ reduces to CO(2, C) and the Higgs fields $\beta$ and $\gamma$ take values in $m^\Delta$. Thus $(V, \beta, \gamma)$ defines a $G^\Delta$-Higgs bundle. It is polystable because the bundle $V$ is stable as an CO(2, C) bundle.

If $(V, \beta, \gamma)$ is polystable as a $G^\Delta$-Higgs bundle then it is polystable as an Sp(4, R)-Higgs bundle. Since $\text{deg}(V) = 2 \text{deg}(L) = 2g - 2$, it follows $(V, \beta, \gamma)$ lies in one of the connected components of $M\max$. As described in Section 3.6, the component containing $(V, \beta, \gamma)$ is labeled by invariants which classify the Cayley partner of $(V, \beta, \gamma)$. Since $L^2 = K$ we may identify $U$ as the Cayley partner. The invariants of $(V, \beta, \gamma)$ are thus $(w_1, w_2)$ if $w_1 \neq 0$. If $w_1 = 0$ then $U$ decomposes as $U = M \oplus M^{-1}$ with $\text{deg}(M) \geq 0$. The invariants of $U$ are then $(0, \text{deg}(M))$. We observe, finally, that $\text{deg}(M) = 0$ if $U$ is polystable. □

7. Analysis of $G^*$-Higgs bundles II: $G_p^*$-Higgs bundles

7.1. Generalities. Recall the abstract description of $G_p^*$ as an extension

$$\{1\} \to \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \to G_p^* \to \mathbb{Z}/2 \to \{0\},$$

in fact, a semi-direct product

$$G_p^* = (\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})) \rtimes \mathbb{Z}/2.$$ (7.2)

Also,

**Proposition 7.1.** The maximal compact subgroups, $H_p^c \subset G_p$, and their complexifications $H_p^c$ are conjugate to

$$H_p = (\text{SO}(2) \times \text{SO}(2)) \rtimes \mathbb{Z}/2,$$ (7.3)

$$H_p^c = (\text{SO}(2, \mathbb{C}) \times \text{SO}(2, \mathbb{C})) \rtimes \mathbb{Z}/2.$$ (7.4)

With respect to $J_{13}$ the embedding (4.3) becomes

$$(A, B) \mapsto (1 0 0 0) + B \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$ (7.5)

showing that $\text{SO}(2) \times \text{SO}(2)$ (a maximal compact subgroup of $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$) embeds in the choice of maximal compact subgroup of $\text{Sp}(4, \mathbb{R})$ (i.e. U(2)) defined by (3.4). After conjugation by $T = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \otimes I$ this yields an embedding of $\text{SO}(2, \mathbb{C}) \times \text{SO}(2, \mathbb{C})$ in $\text{SL}(4, \mathbb{C})$ given by

$$\begin{pmatrix} u & -v \\ v & u \end{pmatrix}, \begin{pmatrix} z & -w \\ w & z \end{pmatrix} \mapsto \begin{pmatrix} u + iv & 0 & 0 & 0 \\ 0 & z + iw & 0 & 0 \\ 0 & 0 & u - iv & 0 \\ 0 & 0 & 0 & z - iw \end{pmatrix}.$$ (7.6)

Either way, since $G_p^*$ is semisimple, it follows from Remark 2.11 that

**Proposition 7.2.** The embedding defined in (1) of Proposition 4.10 makes $G_p^*$ into a reductive subgroup of $\text{Sp}(4, \mathbb{R})$. 30
Proof. For (1), apply (7.6) to the transition functions for the SO(2) bundle.

As for (2), if the structure group of the Higgs bundle reduces to a subgroup $H$, then the Higgs field takes values in $\mathfrak{m}$.

**Proposition 7.4.** If $(V, \beta, \gamma)$ is an $\text{Sp}(4, \mathbb{R})$-Higgs bundle for which the structure group reduces to $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$, then:

1. The bundle $V$ has the form
   \[ V = L_1 \oplus L_2. \] (7.7)
2. The components of the Higgs field are diagonal with respect to this decomposition, i.e.
   \[ \beta = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}, \quad \gamma = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \] (7.8)

with $\beta_i \in H^0(L_i^2 K)$ and $\gamma_i \in H^0(L_i^{-2} K)$.

**Proof.** For (1), apply (7.6) to the transition functions for the $\text{SO}(2, \mathbb{C}) \times \text{SO}(2, \mathbb{C})$ bundle. As for (2), if the structure group of the Higgs bundle reduces to a subgroup $G$, then the Higgs field takes values in $\mathfrak{m} \subset \mathfrak{m} = \mathfrak{g}^C / \mathfrak{h}^C$, with the usual meanings for $\mathfrak{g}^C, \mathfrak{h}^C$, etc. In our case, i.e. $G = \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$, expressed in global terms this means that $\beta$ must lie in
   \[ (L_1^2 \oplus L_2^2)K \subset \text{Sym}^2(L_1 \oplus L_2)K \] (7.9)
and $\gamma$ must lie in
   \[ (L_1^{-2} \oplus L_2^{-2})K \subset \text{Sym}^2(L_1^{-1} \oplus L_2^{-1})K \] (7.10)

**Remark 7.5.** Proposition 7.4 says simply that if the structure group of $(V, \beta, \gamma)$ reduces to $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$, then $(V, \beta, \gamma)$ is a direct sum of $\text{SL}(2, \mathbb{R})$-Higgs bundles, i.e.
   \[ (V, \beta, \gamma) = (L_1, \beta_1, \gamma_2) \oplus (L_2, \beta_2, \gamma_2). \] (7.11)

Of course for $(V, \beta, \gamma)$ to be polystable as an $\text{Sp}(4, \mathbb{R})$-Higgs bundle, each $(L_i, \beta_i, \gamma_i)$ must be (poly)stable as an $\text{SL}(2, \mathbb{R})$-Higgs bundle (cf. Remark 3.9).

7.2. $G_p$-Higgs versus $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$. Let $(V, \beta, \gamma)$ be a $G_p$-Higgs bundle. The obstruction to reducing the structure group to $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \subseteq G_p$ defines an invariant (depending, by Proposition 7.3 only on $V$)
   \[ \xi(V, \beta, \gamma) \in H^1(X, \mathbb{Z}/2). \] (7.12)

Let \( \{t_{\alpha, \beta}\} \) be a Čech $\mathbb{Z}/2$-cocycle representing the class $\xi(V, \beta, \gamma)$ and let
   \[ p : X' \longrightarrow X \] (7.13)
be an unramified double cover defined by $\{t_{\alpha, \beta}\}$. Note that if $\xi(V, \beta, \gamma)$ is non-zero then
   \[ g' = g(X') = 2g - 1. \] (7.14)
Proposition 7.6. Let $V' = p^* V$ be the pull-back of $V$ and let $\beta' = p^* \beta$ and $\gamma' = p^* \gamma$ be the pull-backs of the Higgs fields.

1. The bundle $V'$ admits a reduction of structure group to $\mathbb{C}^* \times \mathbb{C}^*$, i.e. we can write $V'$ as a sum of line bundles $L'_1 \oplus L'_2$.
2. If $\iota : X' \to X'$ is the involution covering the projection onto $X$ then $\iota^*(V') = V'$.
3. Both $\beta'$ and $\gamma'$ decompose, as $(\beta'_1 \oplus \beta'_2)$ and $(\gamma'_1 \oplus \gamma'_2)$ respectively, with respect to the splitting $V' = L'_1 \oplus L'_2$.
4. The pull-back of $G_p$-Higgs bundle $(V, \beta, \gamma)$ defines an $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$-Higgs bundle, namely
   $$p^*(V, \beta, \gamma) = (L'_1, \beta'_1, \gamma'_1) \oplus (L'_2, \beta'_2, \gamma'_2).$$
5. If $(V, \beta, \gamma)$ is polystable and $\deg(V) = 2g - 2$, i.e. if $(V, \beta, \gamma)$ represents a point in $\mathcal{M}^{\text{max}}$, then in $(V', \beta', \gamma')$ we have
   $$\deg(L_1) = \deg(L_2) = g' - 1 = 2g - 2.$$

Proof. Parts (1)–(4) follow by construction. It follows from (2) that $\deg(L_1) = \deg(L_2) = \frac{1}{2} \deg(V')$. Part (5) thus follows from (7.14) and

$$\deg(V') = \deg(p^*(V)) = \int_{X'} c_1(p^*(V)) = \int_{\pi^*(X')} c_1(V) = 2 \int_X c_1(V) = 2 \deg(V).$$

7.3. Identifying components with $G_p$-Higgs bundles. We now determine which components of $\mathcal{M}^{\text{max}}$ contain Higgs bundles for which the structure group reduces to $G_p$ or to $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$. In the next section we consider components for which the invariant $w_1 = 0$, and in section 7.3.2 we consider the case $w_1 \neq 0$.

7.3.1. The case $w_1 = 0$. The invariant $w_1$ is the first Stiefel-Whitney class of the Cayley partner of a maximal $\text{Sp}(4, \mathbb{R})$-Higgs bundle. Using the notation of Section 3.6 the connected components of $\mathcal{M}^{\text{max}}$ in which $w_1 = 0$ are the components $\mathcal{M}_c^0$ (with $0 \leq c < 2g - 2$) and the connected components of $\mathcal{M}_{2g-2}^0$ (i.e. the components $\mathcal{M}_{K_{1/2}}^T$).

Proposition 7.7.

1. For all $c$ in the range $0 < c \leq 2g - 2$ the connected components of $\mathcal{M}_c^0$ do not contain $\text{Sp}(4, \mathbb{R})$-Higgs bundles which admit a reduction of structure group to $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$.
2. The component $\mathcal{M}_c^0$ does contain $\text{Sp}(4, \mathbb{R})$-Higgs bundles which admit a reduction of structure group to $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ — and hence to $G_p$. In fact the structure group can be reduced to the diagonally embedded $\text{SL}(2, \mathbb{R}) \hookrightarrow \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$.

Proof. Let $(V, \beta, \gamma)$ be a maximal $\text{Sp}(4, \mathbb{R})$-Higgs bundle. Recall that $w_1 = 0$ means that

$$V = N \oplus N^{-1} K, \quad \gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \det(V) = K.$$
Suppose furthermore that \((V, \beta, \gamma)\) admits a reduction to \(\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})\). Then by Proposition 7.4 together with the fact that it has maximal Toledo invariant, this means that

\[
V = L_1 \oplus L_2, \quad L_1^2 = L_2^2 = K, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\] (7.17)

For (7.16) and (7.17) to be compatible there must be diagonal embeddings

\[L_{\nu} \hookrightarrow N \oplus N^{-1}K, \quad \nu = 1, 2.\]

This is equivalent to

\[L_1 = L_2 = N = N^{-1}K\]

and hence

\[K = L_1^2 = L_2^2 = N^2.\]

In particular, \(\text{deg}(N) = g - 1\), i.e.

\[c = \text{deg}(N) - (g - 1) = 0.\] (7.18)

This proves (1). To prove (2), pick any \(L\) such that \(L^2 = K\) and construct the \(\text{SL}(2, \mathbb{R})\)-Higgs bundle \((L, 0, \gamma)\) with \(\gamma = 1_L\). Then the polystable Higgs bundle

\[(L, 0, \gamma) \oplus (L, 0, \gamma)\]

proves part (2).

Proposition 7.7 leaves open the possibility that there are \(G_p\)-Higgs bundles with \(w_1 = 0\) but in which the structure group does not reduce to \(\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})\). The next results rules out this possibility.

**Proposition 7.8.** Let \((V, \beta, \gamma)\) be a maximal \(G_p\)-Higgs bundle which does not reduce to an \(\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})\)-Higgs bundle. Then, on the connected double cover

\[X' \xrightarrow{p} X\]

defined by the class \(\xi(V, \beta, \gamma)\), there exist line bundles \(L'_1\) and \(L'_2\) on \(X'\) such that

\[p^*V = L'_1 \oplus L'_2, \quad L_1^2 = L_2^2 = K_{X'}, \quad p^*(\gamma) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.\]

In other words, \(p^*(V, \beta, \gamma)\) is a (maximal) Higgs bundle on \(X'\) with structure group \(\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})\).

**Proof.** Clear. \(\square\)

**Proposition 7.9.** Let \((V, \beta, \gamma)\) be a maximal \(G_p\)-Higgs bundle for which the structure group does not reduce to \(\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})\). Assume that \(w_1(V, \beta, \gamma) = 0\), in other words, that \((V, \beta, \gamma)\) is of the form (7.16). Then \(\text{deg}(N) = g - 1\).

**Proof.** Combining Propositions 7.7 and 7.8 we get that

\[(p^*N)^2 = K_{X'}.\]

Recall, moreover, that \(g(X') = 2g(X) - 1\) and that \(\text{deg}(p^*N) = 2\text{deg}(N)\). The result now follows. \(\square\)
Corollary 7.10. None of the components $\mathcal{M}_c$ with $c > 0$ contains $\text{Sp}(4, \mathbb{R})$-Higgs bundles which admit a reduction of structure group to $G_p$.

7.3.2. The case $w_1 \neq 0$. In this section we prove the following.

Proposition 7.11. For all $(w_1, w_2) \in (H^1(X, \mathbb{Z}/2) - \{0\}) \times H^2(X, \mathbb{Z}/2)$ the component $\mathcal{M}_{w_1, w_2}$ contains $\text{Sp}(4, \mathbb{R})$-Higgs bundles which admit a reduction of structure group to $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \subset G_p$.

Proof. Let $(V, \beta, \gamma)$ be a $\text{Sp}(4, \mathbb{R})$-Higgs bundle of the form

$$V = L_1 \oplus L_2, \quad L_1^2 = L_2^2 = K, \quad \beta = 0, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$  

If we fix a square-root of $K$, i.e. if we pick $L_0$ such that $L_0^2 = K$, and define the Cayley partner $W = V^* \otimes L_0$, then we get

$$W = M_1 \oplus M_2$$  

with $M_i^2 = \mathcal{O}$. Moreover, $\gamma$ defines isomorphisms

$$\tilde{\gamma}_i : M_i \to M_i^*,$$  

that is, $M_1$ and $M_2$ are $\text{O}(1, \mathbb{C})$ bundles. As such, they are determined by their first Stiefel–Whitney classes

$$w_1(M_1), w_1(M_2) \in H^1(X, \mathbb{Z}/2).$$

To determine the invariants of $W$, we need to calculate the total Stiefel–Whitney class

$$w(M_1 \oplus M_2) = 1 + w_1(M_1 \oplus M_2) + w_2(M_1 \oplus M_2)$$  

$$= 1 + w_1(M_1) + w_1(M_2) + w_1(M_1)w_1(M_2).$$

(7.21)

(7.22)

In other words, we need to analyze the map

$$H^1(X, \mathbb{Z}/2) \times H^1(X, \mathbb{Z}/2) \to H^1(X, \mathbb{Z}/2) \times H^2(X, \mathbb{Z}/2)$$  

$$(w_1, w_1') \mapsto (w_1 + w_1', w_1w_1').$$

Using standard coordinates on $H^1(X, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^g$ we write an element in this space as $(a, b) = ((a_1, \ldots, a_g), (b_1, \ldots, b_g))$. The map is then given as follows:

$$(\mathbb{Z}/2)^g \times (\mathbb{Z}/2)^g \to (\mathbb{Z}/2)^2 \times \mathbb{Z}/2,$$  

$$( (a, b), (a', b') ) \mapsto ( (a + a', b + b'), \sum_{i=1}^g (a_ib_i + a'_ib_i) )$$  

(7.23)

One easily sees that $a_i + a'_i = 0$ and $b_i + b'_i = 0$ imply that $a_ib_i + a'_ib_i = 0$. Moreover, one has that

$$( (a, b), (0, 0) ) \mapsto ( (a, b), 0 ).$$

Hence it only remains to show that any element of the form $((\bar{a}, \bar{b}), 1)$ with $(\bar{a}_j, \bar{b}_j) \neq (0, 0)$ for some $j$ is in the image of the map. It is a simple exercise to show that there exists $((a, b), (a', b')) \in (\mathbb{Z}/2)^2 \times (\mathbb{Z}/2)^2$ such that $ab' + a'b = 1$ and $(a + a', b + b') = (\bar{a}_j, \bar{b}_j)$. Now let $(\bar{a}, \bar{b})$ be the element obtained from $((\bar{a}, \bar{b}), 1)$ by substituting $a$ for $\bar{a}_j$ and $b$ for $\bar{b}_j$. Moreover,
Theorem 7.12. The following components of $\mathcal{M}^{\text{max}}$ contain $\text{Sp}(4, \mathbb{R})$-Higgs bundles which admit a reduction of structure group to the subgroup $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ in $G_p$:

- $\mathcal{M}_{w_1, w_2}$, for all $(w_1, w_2) \in H^1(X, \mathbb{Z}/2) - \{0\} \times H^2(X, \mathbb{Z}/2)$,
- $\mathcal{M}_c^0$ for $0 < c < 2g - 2$ and in $\mathcal{M}_{K^{1/2}}^T$ for all choices of $K^{1/2}$, none of the Higgs bundles admit a reduction of structure group to $G_p$. □

7.4. The final tally. Combining Corollary 7.10 and Proposition 7.11 we get, finally, that

Theorem 7.12. The following components of $\mathcal{M}^{\text{max}}$ contain $\text{Sp}(4, \mathbb{R})$-Higgs bundles which admit a reduction of structure group to the subgroup $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ in $G_p$:

- $\mathcal{M}_{w_1, w_2}$, for all $(w_1, w_2) \in H^1(X, \mathbb{Z}/2) - \{0\} \times H^2(X, \mathbb{Z}/2)$,
- $\mathcal{M}_c^0$ for $0 < c < 2g - 2$ and in $\mathcal{M}_{K^{1/2}}^T$ for all choices of $K^{1/2}$, none of the Higgs bundles admit a reduction of structure group to $G_p$. □

8. Analysis of $G_s$-Higgs bundles III: $G_1$-Higgs bundles

8.1. The irreducible representation. The irreducible representation of $\text{SL}(2, \mathbb{R})$ in $\mathbb{R}^4$ comes from its representation on $S^3\mathbb{R}^2$, the third symmetric tensor power of $\mathbb{R}^2$. If we identify $S^3\mathbb{R}^2$ with the space of degree three homogeneous polynomials in two variables, then the representation is defined by

$$\rho_1\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(P)(x, y) = P(ax + cy, bx + dy), \quad (8.1)$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $\text{SL}(2, \mathbb{R})$ and $P$ is a degree three homogeneous polynomial in $(x, y)$.

We get a matrix representation (denoted by $\rho_1$) if we fix a basis for $S^3\mathbb{R}^2$. Taking \(\{x^3, 3x^2y, y^3, 3xy^2\}\) as our basis for $S^3\mathbb{R}^2$ (thought of as the space of degree three homogeneous polynomials in two variables) we get

$$\rho_1\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a^3 & 3a^2b & b^3 & 3ab^2 \\ a^2c & a^2d + 2abc & b^2d & b^2c + 2abd \\ c^3 & 3c^2d & d^3 & 3cd^2 \\ ac^2 & bc^2 + 2acd & bd^2 & ad^2 + 2bcd \end{pmatrix}.$$ 

The standard symplectic form $\omega = dx_1 \wedge dx_2$ on $\mathbb{R}^2$ induces a bilinear form on all tensor powers $(\mathbb{R}^2)^{\otimes n}$, as follows:

$$\Omega((v_1, \ldots, v_n), (w_1, \ldots, w_n)) = \omega(v_1, w_1) \cdots \omega(v_n, w_n),$$

and therefore there is also an induced bilinear form on the symmetric powers of $\mathbb{R}^2$, viewed as subspaces $S^n\mathbb{R}^2 \subset (\mathbb{R}^2)^{\otimes n}$. This form is symmetric when $n$ is even and antisymmetric when $n$ is odd so, in particular, gives us a symplectic form $\Omega$ on $S^3\mathbb{R}^2$. Non-degeneracy follows from the fact that the kernel of the form is an $\text{SL}(2, \mathbb{R})$-submodule of an irreducible representation (and can of course also be seen from the explicit calculation below).

Take the standard basis $\{e_1, e_2\}$ of $\mathbb{R}^2$ and the basis

$$\{e_{ijk} = e_i \otimes e_j \otimes e_k \mid i, j, k = 1, 2\}$$
of \((\mathbb{R}^2)^3\). Then the basis \(\{E_1, E_2, E_3, E_4\}\) for \(S^3\mathbb{R}^2\), where
\[
E_1 = e_{111}, \\
E_2 = e_{112} + e_{121} + e_{211}, \\
E_3 = e_{222}, \\
E_4 = e_{122} + e_{212} + e_{221}
\]
corresponds to the basis \(\{x^3, 3x^2y, y^3, 3xy^2\}\) for \(S^3\mathbb{R}^2\) thought of as the space of degree three homogeneous polynomials of degree in two variables. Calculating the matrix \(J_0\) of the symplectic form \(\Omega\) on \(S^3\mathbb{R}^2\) with respect to this basis one obtains:
\[
J_0 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -3 \\
-1 & 0 & 0 & 0 \\
0 & 3 & 0 & 0
\end{pmatrix}.
\]
If \(ad - bc = 1\), i.e. if \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) is symplectic, then \(\rho_1(A)\) is a symplectic transformation of \((S^3, \mathbb{R}^2, \Omega)\), i.e.
\[
\rho_1(A)^t J_0 \rho_1(A) = J_0. \tag{8.2}
\]
Notice that with
\[
h = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{3}} \\
0 & 0 & 1 & 0 \\
0 & \frac{1}{\sqrt{3}} & 0 & 0
\end{pmatrix}
\]
we get
\[
h^t J_0 h = J_{13}.
\]
Thus using \(J_{13}\) to define \(\text{Sp}(4, \mathbb{R})\), the irreducible representation is given by
\[
\rho_{13}(A) = h^{-1} \rho_1(A) h = \begin{pmatrix}
a^3 & \sqrt{3}ab^2 & \sqrt{3}b^3 & \sqrt{3}a^2b \\
\sqrt{3}ac^2 & ad^2 + 2bcd & \sqrt{3}cd^2 & bc^2 + 2acd \\
\sqrt{3}a^2c & b^2c + 2abd & \sqrt{3}b^2d & a^2d + 2abc \\
\sqrt{3}c^2d & & & \\
\end{pmatrix} \tag{8.3}
\]
If \(A \in \text{SO}(2)\), i.e. if \(d = a, b = -c\) and \(a^2 + c^2 = 1\), then \(\rho_{13} \begin{pmatrix} a & -c \\ c & a \end{pmatrix}\) lies in the copy of \(\text{U}(2)\) embedded in \(\text{Sp}(4, \mathbb{R})\) as in (3.4). Moreover, with the Cartan involution as in (3.6) the image of the induced embedding
\[
\rho_{13*} : \mathfrak{sl}(2, \mathbb{R}) \longrightarrow \mathfrak{sp}(4, \mathbb{R}). \tag{8.4}
\]
is \(\theta\)-invariant, so Remark 2.11 gives us the following.

**Proposition 8.1.** With \(G_i = \rho_{13}(\text{SL}(2, \mathbb{R}))\) defined as above and with the choices for \(\text{Sp}(4, \mathbb{R})\) as in Section 3.7, the subgroup \(G_i\) is a reductive subgroup of \(\text{Sp}(4, \mathbb{R})\).

**Remark 8.2.** This embedding extends to an embedding of \(\text{SL}(2, \mathbb{C})\) in \(\text{Sp}(4, \mathbb{C}) \subset \text{SL}(4, \mathbb{C})\). The restriction to \(\text{SO}(2, \mathbb{C})\) takes values in the copy of \(\text{GL}(2, \mathbb{C})\) embedded in \(\text{SL}(4, \mathbb{C})\) via
\[ Z \mapsto \begin{pmatrix} \frac{Z+Z_{t^{-1}}}{Z_{t^2}+Z} & \frac{Z-Z_{t^{-1}}}{Z_{t^2}-Z} \\ \frac{Z_{t^2}^2-Z}{Z+Z_{t^{-1}}} & \frac{Z_{t^2}^2}{Z+Z_{t^{-1}}} \end{pmatrix}. \tag{8.5} \]

If we conjugate by \( T = \begin{pmatrix} I & iI \\ I & -iI \end{pmatrix} \), that is if we make a complex change of frame from \( \mathbb{R}^4 \otimes \mathbb{C} \) to \( \mathbb{C}^2 \oplus (\mathbb{C}^2)^* \), the embedding of SO(2) becomes (with \( A = \begin{pmatrix} a & -c \\ c & a \end{pmatrix} \))

\[ T \circ \rho_{13}(A) \circ T^{-1} = \begin{pmatrix} \Lambda & 0_2 \\ 0_2 & (\Lambda^t)^{-1} \end{pmatrix} \]

where \( 0_2 \) denotes the \( 2 \times 2 \) zero matrix and

\[ \Lambda = \begin{pmatrix} a^3 + ic^3 & \sqrt{3ac}(ia + c) \\ \sqrt{3ac}(ia + c) & a^3 - 2ac^2 + i(c^3 - 2a^2c) \end{pmatrix}. \]

A further conjugation by

\[ \tilde{H} = \begin{pmatrix} 0 & 0 & \frac{\sqrt{3} - 1}{8}u & \frac{\sqrt{3} - 3}{8}u \\ 0 & 0 & -\frac{\sqrt{3} + 3}{8}v & -\frac{\sqrt{3} + 1}{8}v \\ \frac{\sqrt{3} + 1}{8}u & \frac{\sqrt{3} + 3}{8}v & 0 & 0 \\ \frac{\sqrt{3} - 3}{8}u & -\frac{\sqrt{3} - 1}{8}v & 0 & 0 \end{pmatrix}, \tag{8.6} \]

where \( u = -4\sqrt{6 + 3\sqrt{3}} \) and \( v = 2/\sqrt{2 + \sqrt{3}} \), yields

\[ \tilde{H} \circ T \circ \rho_{13}(A) \circ (\tilde{H} \circ T)^{-1} = \begin{pmatrix} \lambda^3 & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & \lambda^{-3} & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \lambda = a + ic. \]

Remark 8.3. Direct computation shows that with \( \text{Sp}(4, \mathbb{C}) \) defined by \( J_{13} \), conjugation by \( T \) or \( \tilde{H} \) preserves \( \text{Sp}(4, \mathbb{C}) \subset \text{SL}(4, \mathbb{C}) \).

**Definition 8.4.** Let \( \varphi : \text{SL}(2, \mathbb{C}) \longrightarrow \text{Sp}(4, \mathbb{C}) \) be the composite

\[ \varphi(A) = (\tilde{H} \circ T) \circ \rho_{13}(A) \circ (\tilde{H} \circ T)^{-1}. \tag{8.7} \]

We then have a commutative diagram

\[
\begin{array}{ccc}
\text{SL}(2, \mathbb{C}) & \xrightarrow{\varphi} & \text{Sp}(4, \mathbb{C}) \\
\uparrow & & \uparrow \\
\text{GL}(1, \mathbb{C}) & \xrightarrow{\varphi|_{\text{GL}(1, \mathbb{C})}} & \text{GL}(2, \mathbb{C})
\end{array}
\]

where the vertical arrow on the left is given by the identification

\[ \text{GL}(1, \mathbb{C}) \simeq \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{C}^* \right\}. \tag{8.9} \]

and the one on the right is given by (8.5).
8.2. **The embedding of Higgs bundles.** We can compute the infinitesimal version of the embedding (8.3) to find the embedding of $\mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{sp}(4, \mathbb{R})$ (using $J = J_{13}$). With

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and with $\tilde{H}$ and $T$ as above, we compute

**Lemma 8.5.**

$$(\tilde{H}T)_{\rho_{13}}(e - f)(\tilde{H}T)^{-1} = i \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$(\tilde{H}T)_{\rho_{13}}(e + f)(\tilde{H}T)^{-1} = i \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & -1 \\ 0 & -1 & 0 & 0 \\ -1 & 4 & 0 & 0 \end{pmatrix}$$

$$(\tilde{H}T)_{\rho_{13}}(h_0)(\tilde{H}T)^{-1} = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \end{pmatrix}$$

**Proof.** Calculation (Mathematica).

It follows that the restriction of $\varphi$ to $\mathfrak{m}^C(\text{SL}(2, \mathbb{C}))$, where

$$\mathfrak{m}^C(\text{SL}(2, \mathbb{C})) = \{ \begin{pmatrix} x & y \\ y & -x \end{pmatrix} \mid x, y \in \mathbb{C} \} ,$$

gives

$$(\tilde{H}T)_{\rho_{13}}(\begin{pmatrix} x & y \\ y & -x \end{pmatrix})(\tilde{H}T)^{-1} = \begin{pmatrix} 0 & 0 & 0 & 3\beta \\ 0 & 0 & 3\beta & \gamma \\ 0 & \gamma & 0 & 0 \\ \gamma & 4\beta & 0 & 0 \end{pmatrix} \text{ with } \begin{cases} \beta = x + iy \\ \gamma = x - iy \end{cases} .$$

We can make a further transformation so that the bottom left corner is a multiple of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

**Lemma 8.6.** Let

$$S = \begin{pmatrix} 1 & 2(\frac{\beta}{\gamma}) & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2(\frac{\beta}{\gamma}) & 1 \end{pmatrix} \quad (8.10)$$

Then

$$(S\tilde{H}T)_{\rho_{13}}(\begin{pmatrix} x & y \\ y & -x \end{pmatrix})(S\tilde{H}T)^{-1} = \gamma \begin{pmatrix} 0 & 0 & 16(\frac{\beta}{\gamma})^2 & \frac{5(\beta)}{\gamma} \\ 0 & 0 & \frac{5(\beta)}{\gamma} & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$
Next, we recall from Remark 3.4 (cf. [25]) that an SL(2, \(\mathbb{R}\))-Higgs bundles is defined by a triple \((L, \tilde{\beta}, \tilde{\gamma})\) where \(L\) is a holomorphic line bundle, \(\tilde{\beta} \neq 0 \in H^0(L^2K)\), and \(\tilde{\gamma} \in H^0(L^{-2}K)\). Let \(E\) be the principal GL(1, \(\mathbb{C}\))-bundle which defines \(L\). Using the identification of GL(1, \(\mathbb{C}\)) with SO(2, \(\mathbb{C}\)) given by (8.9), \(E\) defines a rank two bundle \(L \oplus L^{-1}\). The Higgs fields \((\tilde{\beta}, \tilde{\gamma})\) then define a bundle map
\[
\begin{pmatrix}
0 & \tilde{\beta} \\
\tilde{\gamma} & 0
\end{pmatrix} : L \oplus L^{-1} \rightarrow (L \oplus L^{-1}) \otimes K.
\]

Theorem 8.7. Let
\[
\rho_{13} : \text{SL}(2, \mathbb{R}) \hookrightarrow \text{Sp}(4, \mathbb{R})
\]
be the irreducible representation as in (8.3), and let
\[
\varphi : \text{SL}(2, \mathbb{C}) \hookrightarrow \text{Sp}(4, \mathbb{C})
\]
be the resulting representation as in (8.7). Use \(\varphi|_{\text{GL}(1, \mathbb{C})}\) to extend the structure group of \(E\) to \(\text{GL}(2, \mathbb{C})\) and use \(\varphi\) to embed \(\mathfrak{m}^\ast(\text{SL}(2, \mathbb{R}))\) in \(\mathfrak{m}^\ast(\text{Sp}(4, \mathbb{R}))\) (cf. (8.8)) . Let
\[
\rho^P_{ir} : \mathcal{M}(\text{SL}(2, \mathbb{R})) \hookrightarrow \mathcal{M}(\text{Sp}(4, \mathbb{R}))
\]
be the induced map from the moduli space of \(\text{SL}(2, \mathbb{R})\)-Higgs bundles to the moduli space of \(\text{Sp}(4, \mathbb{R})\)-Higgs bundles . Let \((L, \tilde{\beta}, \tilde{\gamma})\) be a polystable \(\text{SL}(2, \mathbb{R})\)-Higgs bundle. Then:
(a) If \(0 \leq \text{deg}(L) \leq g - 1\) then
\[
\rho^P_{ir}([L, \tilde{\beta}, \tilde{\gamma}]) = ([L^3 \oplus L^{-1}, \beta, \gamma])
\]
where
\[
\beta = \begin{pmatrix} 0 & 3\tilde{\beta} \\ 3\tilde{\beta} & \tilde{\gamma} \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & \tilde{\gamma} \\ \tilde{\gamma} & 4\tilde{\beta} \end{pmatrix}
\]
(b) If \(\text{deg}(L) = g - 1\) then \(L^2 = K\) and \(\beta\) and \(\gamma\) can be put in the form
\[
\gamma = \tilde{\gamma} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta = \tilde{\gamma} \begin{pmatrix} \beta_1 & \beta_3 \\ \beta_3 & 1 \end{pmatrix}
\]
with \(\beta_3 = 5(\tilde{\gamma})\) and \(\beta_1 = (\frac{16}{25})\beta^2\).

Remark 8.8. The fact that the Higgs bundles obtained in (a) of Theorem 8.7 are not of the standard form given in Proposition 3.25 is due to the fact that unless \(\text{deg}(L) = g - 1\) the Higgs bundles are not maximal, i.e. do not lie in \(\mathcal{M}^{\text{max}}\).

Proof. We use local trivializations and transition functions to describe all bundle data. Fix an open cover \(\{U_i\}\) for \(X\) and local trivializations for \(L\) and \(K\), with transition functions
\[
l_{ij}, k_{ij} : U_i \cap U_j \rightarrow \text{GL}(1, \mathbb{C})
\]
on non-empty intersections \(U_i \cap U_j\). Let the local descriptions of \(\tilde{\beta}\) and \(\tilde{\gamma}\) over \(U_i\) be \(\tilde{\beta}_i\) and \(\tilde{\gamma}_i\) respectively. Then on non-empty intersections \(U_i \cap U_j\)
\[
l_{ij}^2 k_{ij} \tilde{\beta}_j = \tilde{\beta}_j
\]
Similarly
\[
l_{ij}^{-2} k_{ij} \tilde{\gamma}_j = \tilde{\gamma}_j
\]
Observe that if \(L^2 = K\), so that \(l_{ij}^2 = k_{ij}\), this implies
\[
\tilde{\gamma}_j = \tilde{\gamma}_j
\]
The embedding of the SL(2, R)-Higgs bundle \((L, \tilde{\beta}, \tilde{\gamma})\) in the space of Sp(4, R)-Higgs bundles\(^{12}\) is obtained by applying \(\varphi\) to \(T^{-1} \begin{pmatrix} l_{ij} & 0 \\ 0 & l_{ij}^{-1} \end{pmatrix} T\) and \(T^{-1} \begin{pmatrix} 0 & \tilde{\beta}_i \\ \tilde{\gamma}_i & 0 \end{pmatrix} T\), where \(T = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\). We find
\[
\begin{pmatrix} l_{ij} & 0 \\ 0 & l_{ij}^{-1} \end{pmatrix} \mapsto \begin{pmatrix} l_{ij}^3 & 0 & 0 & 0 \\ 0 & l_{ij}^{-1} & 0 & 0 \\ 0 & 0 & l_{ij}^{-3} & 0 \\ 0 & 0 & 0 & l_{ij} \end{pmatrix} = g_{ij} ,
\]
\[
\begin{pmatrix} 0 & \tilde{\beta}_i \\ \tilde{\gamma}_i & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 & 3\tilde{\beta}_i \\ 0 & 0 & 3\tilde{\beta}_i & \tilde{\gamma}_i \\ \tilde{\gamma}_i & 0 & 0 \\ 0 & 4\tilde{\beta}_i & 0 & 0 \end{pmatrix} = \Phi_i .
\]

It follows from this that \(\{g_{ij}\}\) define a bundle \(V \oplus V^*\) with \(V = L^3 \oplus L^{-1}\) and that with respect to this decomposition \(\{\Phi_i\}\) define a Higgs field \(\Phi\) with \(\beta\) and \(\gamma\) as in (8.14). It remains to show that the resulting Sp(4, R)-Higgs bundle, i.e. \((L^3 \oplus L^{-1}, \beta, \gamma)\), is polystable and thus defines a point in \(\mathcal{M}(\text{Sp}(4, \mathbb{R}))\).

Notice that if \(\text{deg}(L) > 0\) and \((L, \tilde{\beta}, \tilde{\gamma})\) is a polystable SL(2, R)-Higgs bundle, then \(\tilde{\gamma} \neq 0\) (cf. Remark 3.9). Thus both \(\beta\) and \(\gamma\) are non-zero. It follows that \((L^3 \oplus L^{-1}, \beta, \gamma)\) is stable if and only if the strict versions of the conditions (3a-c) of Proposition 3.5 are satisfied by line subbundles \(L' \subset L^3 \oplus L^{-1}\). But for any such line subbundle, either \(L' = L^3\) or \(\text{deg}(L') \leq \text{deg}(L^{-1}) < 0\). Conditions (3a-c) are thus clearly satisfied if \(L' \neq L^3\). If \(L' = L^3\) and \(\beta, \gamma\) are as in (8.14), then \(\beta\) fails to satisfy the hypotheses in (a) and (c). Moreover, \(\gamma\) satisfies the hypothesis in (b) only if \(\tilde{\gamma} = 0\), which is not possible if \((L, \tilde{\beta}, \tilde{\gamma})\) is polystable. Thus \(L^3\) is not a destabilizing subbundle and we conclude that \((L^3 \oplus L^{-1}, \beta, \gamma)\) is stable.

Finally, if \(\text{deg}(L) = 0\) then (see Remark 3.9) either \(\tilde{\beta} = \tilde{\gamma} = 0\) or both \(\tilde{\beta}\) and \(\tilde{\gamma}\) are non-zero. In the former case, clearly \((L^3 \oplus L^{-1}, \beta, \gamma)\) is polystable. In the latter case, clearly the conditions on \(\beta\) and \(\gamma\) in (3b-c) of Proposition 3.5 are never satisfied by line subbundles \(L' \subset L^3 \oplus L^{-1}\). The only \(L' \subset L^3 \oplus L^{-1}\) for which the condition on \(\gamma\) in (3a) of Proposition 3.5 is satisfied is \(L' = L^3\). But then the condition on \(\beta\) in (3a) of Proposition 3.5 is not satisfied and we conclude that \((L^3 \oplus L^{-1}, \beta, \gamma)\) is stable. This completes the proof of part (a).

Suppose now that \(\text{deg}(L) = g - 1\). It follows from the definition of polystability for SL(2, R)-Higgs bundles that \(L^2 = K\) and \(\tilde{\gamma} \neq 0\). By (8.18) we can then assume that the \(\tilde{\gamma}_i\) are nowhere zero. We exploit this to define an automorphism of \(V\) which puts \(\gamma\) in a more...
standard form. In the local trivialization over $U_i$, define

$$S_i = \begin{pmatrix}
1 & 2\tilde{\gamma}_i & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -2\tilde{\gamma}_i & 1
\end{pmatrix}$$

(8.19)

Observe that, because of (8.16) and (8.18) we get $g_{ij} S_j g_{ij}^{-1} = S_i$, which verifies that the $\{S_i\}$ define a bundle automorphism. But

$$S_i \Phi_i S_i^{-1} = \begin{pmatrix}
0 & 0 & 16\tilde{\beta}_i^2 & 5\tilde{\beta}_i \\
0 & 0 & 5\tilde{\beta}_i & \tilde{\gamma}_i \\
0 & \tilde{\gamma}_i & 0 & 0 \\
\tilde{\gamma}_i & 0 & 0 & 0
\end{pmatrix}$$

(8.20)

Thus the Sp(4, $\mathbb{R}$)-Higgs bundle defined by $(V, \beta, \gamma)$ is isomorphic to the Sp(4, $\mathbb{R}$)-Higgs bundle defined by $(V, \beta', \gamma')$ where $\beta'$ and $\gamma'$ are as in the statement of the theorem. □

**Corollary 8.9.** Let $(V, \beta, \gamma)$ be the image of $(L, \tilde{\beta}, \tilde{\gamma})$ under $\varphi$.

1. The degree of $V$ is $\deg(V) = 2\deg(L)$.
2. If $L^2 = K$ then $(V, \beta, \gamma)$ lies in the component $\mathcal{M}_T^L$ of $\mathcal{M}^{\text{max}}$.

*Proof.* Part (1) follows immediately from the fact that $V = L^3 \oplus L^{-1}$. For (2), defining $N = L^3$ yields $V = N \oplus N^{-1} K$ with $\deg(N) = 3g-3$. This, together with the characterization of $\mathcal{M}_T^{K_1}$ in Proposition 3.25, yields the result. □

**Corollary 8.10.** Let $(V, \beta, \gamma)$ represent a Sp(4, $\mathbb{R}$) Higgs bundles in $\mathcal{M}_T^{K_1/2}$ and suppose that it admits a reduction of structure group to SL(2, $\mathbb{R}$). Then $(V, \beta, \gamma)$ is isomorphic to a Sp(4, $\mathbb{R}$)-Higgs bundle with $V = K_1^{3/2} \oplus K^{-1/2}$ and $\beta$ and $\gamma$ as in Theorem 8.7.

8.3. The normalizer of SL(2, $\mathbb{R}$). Next we calculate the normalizer of SL(2, $\mathbb{R}$) embedded in Sp(4, $\mathbb{R}$) via the irreducible representation $\rho_1$. We shall need the following standard fact.

**Proposition 8.11.** The outer automorphism group of SL(2, $\mathbb{R}$) is $\mathbb{Z}/2$, generated by conjugation by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Consider the extension of the irreducible representation $\rho_1$ to a representation in SL(4, $\mathbb{R}$). Note that the domain of $\rho_1$ can be extended to SL$\pm$(2, $\mathbb{R}$) = $\{ A \mid \det(A) = \pm 1 \}$: in fact, substituting $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in (8.1) we obtain

$$\rho_1\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

(8.21)

which has determinant 1.

---

13We are grateful to Bill Goldman for explaining this to us.
Next we make a general observation. Let $\tilde{G} \subset G$ be a Lie subgroup. We have the following diagram of exact sequences of groups:

$$
\begin{array}{cccccc}
1 & \rightarrow & Z(\tilde{G}) & \rightarrow & \tilde{G} & \rightarrow & \text{Inn}(\tilde{G}) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & C_G(\tilde{G}) & \rightarrow & N_G(\tilde{G}) & \rightarrow & \text{Aut}(\tilde{G}) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & C_G(\tilde{G})/Z(\tilde{G}) & \rightarrow & N_G(\tilde{G})/\tilde{G} & \rightarrow & \text{Out}(\tilde{G}) & \rightarrow & 1
\end{array}
$$

(8.22)

**Proposition 8.12.** Let $\tilde{G} = \rho_1(\text{SL}(2, \mathbb{R})) \subset G = \text{SL}(4, \mathbb{R})$. Then we have a short exact sequence of groups:

$$1 \rightarrow C_G(\tilde{G})/Z(\tilde{G}) \rightarrow N_G(\tilde{G})/\tilde{G} \rightarrow \mathbb{Z}/2 \rightarrow 1,$$

where the quotient $\mathbb{Z}/2$ is generated by the image of $\rho_1((\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})) \in N_G(\tilde{G})$.

**Proof.** As observed above, $\rho_1((\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}))$ is an element of $G$. Now Proposition 8.11 implies that this element belongs to $N_G(\tilde{G})$ and that the map on the right in the bottom row of (8.22) is surjective. $\Box$

**Proposition 8.13.** Let $\tilde{G} = \rho_1(\text{SL}(2, \mathbb{R})) \subset G = \text{SL}(4, \mathbb{R})$. The centralizer of $\tilde{G}$ in $G$ equals the centre $\{\pm I\}$ of $\tilde{G}$.

**Proof.** Any element in the centralizer of $\tilde{G}$ is also in the centralizer of its complexification. Since this complexification is just the 4-dimensional irreducible representation of $\text{SL}(2, \mathbb{C})$, Schur’s Lemma implies that any element centralizing $\tilde{G}$ is a complex multiple of the identity. But the only multiples of the identity in $\text{SL}(4, \mathbb{R})$ are $\pm I$. $\Box$

**Corollary 8.14.** The normalizer of $\tilde{G} = \rho_1(\text{SL}(2, \mathbb{R}))$ in $\text{SL}(4, \mathbb{R})$ fits in the short exact sequence of groups

$$1 \rightarrow \tilde{G} \rightarrow N_{\text{SL}(4, \mathbb{R})}(\tilde{G}) \rightarrow \mathbb{Z}/2 \rightarrow 1,$$

where the quotient $\mathbb{Z}/2$ is generated by the image $\rho_1((\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})) \in N_{\text{SL}(4, \mathbb{R})}(\tilde{G})$.

**Proof.** Immediate from Propositions 8.12 and 8.13. $\Box$

**Proposition 8.15.** Let $\tilde{G} = \rho_1(\text{SL}(2, \mathbb{R})) \subset \text{Sp}(4, \mathbb{R})$. Then the normalizer of $\tilde{G}$ in $\text{Sp}(4, \mathbb{R})$, i.e. $G_i$, coincides with $\tilde{G}$:

$$G_i = N_{\text{Sp}(4, \mathbb{R})}(\tilde{G}) = \tilde{G}.$$
Proof. Consider $N_{\text{Sp}(4,\mathbb{R})(G)} \subset \text{Sp}(4,\mathbb{R}) \subset \text{SL}(4,\mathbb{R})$ as a subgroup of $\text{SL}(4,\mathbb{R})$. Clearly,
\[ \bar{G} \subset N_{\text{Sp}(4,\mathbb{R})(G)} \subset N_{\text{SL}(4,\mathbb{R})(G)}. \]

We conclude from Corollary 8.14 that either $N_{\text{Sp}(4,\mathbb{R})(\bar{G})}$ coincides with the index 2 subgroup $\bar{G} \subset N_{\text{SL}(4,\mathbb{R})(\bar{G})}$ or it equals $N_{\text{SL}(4,\mathbb{R})(\bar{G})}$. In the latter case, we must have $\rho_1((\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})) \in N_{\text{Sp}(4,\mathbb{R})(\bar{G})}$. But from (8.22) one easily checks that $\rho_1((\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}))$ does not satisfy (8.2) and hence does not belong to $\text{Sp}(4,\mathbb{R})$. This concludes the proof. \hfill $\Box$

8.4. Summary. Putting together Theorem 8.16, Corollary 8.9 and the fact that $G_i = \text{SL}(2,\mathbb{R})$, we finally obtain:

**Theorem 8.16.** A maximal polystable $\text{Sp}(4,\mathbb{R})$-Higgs bundle deforms to a polystable $G_i$-Higgs bundle if and only if it belongs to one of the Hitchin components $\mathcal{M}_{K_{1/2}}^T$.

9. The case $n \geq 3$

In this section we make a digression to the case of $n \geq 3$, showing that in this case any maximal polystable $\text{Sp}(2n,\mathbb{R})$-Higgs bundle can be deformed to a $G$-Higgs bundle for some proper reductive Zariski closed subgroup $G \subset \text{Sp}(2n,\mathbb{R})$.

9.1. The moduli space of $\text{Sp}(2n,\mathbb{R})$-Higgs bundles. An $\text{Sp}(2n,\mathbb{R})$-Higgs bundle on $X$ (cf. Remark 3.4) is a triple $(V, \beta, \gamma)$, where $V$ is a rank $n$ holomorphic vector bundle on $X$, $\beta \in H^0(X, S^2V \otimes K)$ and $\gamma \in H^0(X, S^2V^* \otimes K)$. The moduli space of polystable $\text{Sp}(2n,\mathbb{R})$-Higgs bundles is denoted by $\mathcal{M}(\text{Sp}(2n,\mathbb{R}))$ and is homeomorphic to the moduli space $\mathcal{R}(\text{Sp}(2n,\mathbb{R}))$ of reductive representations of $\pi_1(X)$ in $\text{Sp}(2n,\mathbb{R})$.

The Milnor–Wood inequality for a $\text{Sp}(2n,\mathbb{R})$-Higgs bundle says that $|\text{deg}(V)| \leq n(g-1)$. The moduli space of maximal $\text{Sp}(2n,\mathbb{R})$-Higgs bundles is
\[ \mathcal{M}_{\text{max}}(\text{Sp}(2n,\mathbb{R})) = \{ [V, \beta, \gamma] \in \mathcal{M}(\text{Sp}(2n,\mathbb{R})) \mid \text{deg}(V) = n \}. \]

The space $\mathcal{M}_{\text{max}}(\text{Sp}(2n,\mathbb{R}))$ is homeomorphic to the moduli space of maximal representations of $\pi_1(X)$ in $\text{Sp}(2n,\mathbb{R})$.

For any maximal $\text{Sp}(2n,\mathbb{R})$-Higgs bundle $(V, \beta, \gamma)$, the map $\gamma : V \to V^* \otimes K$ is an isomorphism and $(V, \beta, \gamma)$ has a Cayley partner $(W, q_W, \theta)$ defined as in (3.19)–(3.21). This leads to the existence of invariants $w_1(V, \beta, \gamma) \in H^1(X, \mathbb{Z}/2)$ and $w_2(V, \beta, \gamma) \in H^2(X, \mathbb{Z}/2)$ defined by the Stiefel–Whitney classes of $(W, q_W)$ (cf. [15]).

The count of connected components of $\mathcal{M}_{\text{max}}(\text{Sp}(2n,\mathbb{R}))$ was carried out in [15], where the following theorem is proved.

**Theorem 9.1.** The moduli space of maximal $\text{Sp}(2n,\mathbb{R})$-Higgs bundles has $3 \cdot 2^{2g}$ connected components:

1. For each $(w_1, w_2) \in H^1(X, \mathbb{Z}/2) \times H^2(X, \mathbb{Z}/2)$ there is a component $\mathcal{M}_{w_1,w_2}$. Any $\text{Sp}(2n,\mathbb{R})$-Higgs bundle $(V, \beta, \gamma)$ in such a component has invariants $(w_1, w_2)$ and can be deformed to one with $\beta = 0$.

2. For each choice of a square root $K^{1/2}$ of the canonical bundle of $X$, there is a Hitchin component $\mathcal{M}_{K^{1/2}}^T$. Any $\text{Sp}(2n,\mathbb{R})$-Higgs bundle $(V, \beta, \gamma)$ in such a component has $\beta \neq 0$ and can be deformed to a $\rho_i(\text{SL}(2,\mathbb{R}))$-Higgs bundle, where $\rho_i : \text{SL}(2,\mathbb{R}) \to \text{Sp}(2n,\mathbb{R})$ is the irreducible representation.
Remark 9.2. An $\text{Sp}(2n, \mathbb{R})$-Higgs bundle in $\mathcal{M}_{w_1, w_2}$ has invariants $(w_1, w_2)$, and an $\text{Sp}(2n, \mathbb{R})$-Higgs bundle in $\mathcal{M}_{R}^{w_1/2}$ has $w_2 = 0$. See [15, Proposition 8.2] for the value of $w_1$.

Maximal $\text{Sp}(2n, \mathbb{R})$-Higgs bundles can be constructed as follows (cf. Proposition 7.4). Let $(V_i, \beta_i, \gamma_i)$ be maximal polystable $\text{Sp}(2n_i, \mathbb{R})$-Higgs bundles for $i = 1, 2$ and let $n = n_1 + n_2$. Then the polystable $\text{Sp}(2n, \mathbb{R})$-Higgs bundle $(V, \beta, \gamma)$ defined by

$$V = V_1 \oplus V_2, \quad \beta = \beta_1 + \beta_2, \quad \text{and} \quad \gamma = \gamma_1 + \gamma_2$$

is maximal. Of course, such an $\text{Sp}(2n, \mathbb{R})$-Higgs bundle admits a reduction of structure group to $\text{Sp}(2n_1, \mathbb{R}) \times \text{Sp}(2n_2, \mathbb{R})$.

**Proposition 9.3.** Let $n \geq 3$.

1. Let $(w_1, w_2) \in H^1(X, \mathbb{Z}/2) \times H^2(X, \mathbb{Z}/2)$ be different from $(0, 1)$. Then there is a maximal $\text{Sp}(2n, \mathbb{R})$-Higgs bundle which represents a point in $\mathcal{M}_{w_1, w_2}$ and admits a reduction of structure group to $\text{SL}(2, \mathbb{R}) \times \ldots \times \text{SL}(2, \mathbb{R})$ ($n$ copies).

2. There is a maximal $\text{Sp}(2n, \mathbb{R})$-Higgs bundle which represents a point in $\mathcal{M}_{0, 1}$ and admits a reduction of structure group to $\text{Sp}(4, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \times \ldots \times \text{SL}(2, \mathbb{R})$ ($n - 2$ copies of $\text{SL}(2, \mathbb{R})$).

**Proof.** (1) This follows by first using the construction in the proof of Proposition 7.11 to obtain a maximal polystable $\text{SL}(2, \mathbb{R}) \times \ldots \times \text{SL}(2, \mathbb{R})$-Higgs bundle with the required $(w_1, w_2)$ and then taking direct sums with $n - 2$ copies of a maximal polystable $\text{SL}(2, \mathbb{R})$-Higgs bundle with $w_1 = 0$. The $\text{Sp}(2n, \mathbb{R})$-Higgs bundle $(V, \beta, \gamma)$ thus obtained is maximal and has invariants $(w_1, w_2)$. Moreover $(V, \beta, \gamma)$ is strictly polystable. Since any $\text{Sp}(2n, \mathbb{R})$-Higgs bundle in a Hitchin component is strictly stable [16, 26], $(V, \beta, \gamma)$ does not lie in such a component and it follows that $(V, \beta, \gamma)$ represents a point in $\mathcal{M}_{w_1, w_2}$ as required.

(2) Take a maximal polystable $\text{Sp}(4, \mathbb{R})$-Higgs bundle with invariants $(w_1, w_2) = (0, 1)$ (existence follows from Proposition 7.29) and take direct sums of this with $n - 2$ copies of a maximal polystable $\text{SL}(2, \mathbb{R})$-Higgs bundle with $w_1 = 0$. As in the proof of (1), we see that this yields a maximal $\text{Sp}(2n, \mathbb{R})$-Higgs bundle with the required properties. □

We already knew that maximal $\text{Sp}(2n, \mathbb{R})$-Higgs bundles in the Hitchin components can always be deformed to $G'$-Higgs bundles for some proper Zariski closed subgroup $G' \subset \text{Sp}(2n, \mathbb{R})$ (namely $G' = \text{SL}(2, \mathbb{R})$, embedded via the irreducible representation); Proposition 9.3 tells us that the same is true for $\text{Sp}(2n, \mathbb{R})$-Higgs bundles in all other maximal components. Thus the non-abelian Hodge theory correspondence gives the following.

**Corollary 9.4.** Let $n \geq 3$. Then any maximal representation of $\pi_1(S)$ in $\text{Sp}(2n, \mathbb{R})$ can be deformed to one which factors through a proper reductive Zariski closed subgroup of $\text{Sp}(2n, \mathbb{R})$.

**Appendix A. The Kronecker product**

If $A$ is an $m \times m$ matrix with entries $a_{ij}$ and $B$ is an $n \times n$ matrix with entries $b_{ij}$, then the Kronecker product $A \otimes B$ is defined to be the $mn \times mn$ matrix with block entries $a_{ij}B$. Thus if $A$ and $B$ are both $2 \times 2$ matrices, then

$$A \otimes B = \begin{pmatrix}
    a_{11}B & a_{12}B \\
    a_{21}B & a_{22}B
  \end{pmatrix}.$$  

(A.1)
Several formulae in the main body of this paper have convenient forms when expressed in terms of this product. In particular the symplectic forms used to define $\text{Sp}(4, \mathbb{R})$ are given by

$$J_{13} = J \otimes I,$$  \hspace{1cm} (A.2)

$$J_{12} = I \otimes J.$$  \hspace{1cm} (A.3)

We record some elementary but useful properties of the Kronecker product.

**Lemma A.1.** Let $A, C$ be $m \times m$ matrices and $B, D$ be $n \times n$ matrices. Then

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

$$(A \otimes B)^t = A^t \otimes B^t$$

$$\exp(A \otimes I_n + I_m \otimes B) = \exp(A) \otimes \exp(B)$$

If $A$ and $B$ are both $2 \times 2$ matrices and

$$h = h^t = h^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$  \hspace{1cm} (A.5)

then

$$A \otimes B = h^t (B \otimes A) h.$$  \hspace{1cm} (A.6)

Applying (A.6) to $J_{13}$ we see that

$$h J_{12} = J_{13} h.$$  \hspace{1cm} (A.7)

It follows that $g \in \text{SL}(4, \mathbb{R})$ satisfies $g^t J_{12} g = J_{12}$ if and only if $g^t = hgh$ satisfies $g'^t J_{13} g' = J_{13}$. Thus the descriptions of $\text{Sp}(4, \mathbb{R})$ with respect to $J_{12}$ and with respect to $J_{13}$ are related by conjugation with $h$.

**APPENDIX B. Tables**
| Component | Higgs bundle \((V, \beta, \gamma)\) | \(w_1\) | \(\text{deg}(NK^{-1/2})\) | \(w_2\) | \(G_*\) | Number |
|-----------|-------------------------------|---|----------------|---|---|---|
| \(\mathcal{M}_{K^{1/2}}^T\) | \(V = K^{3/2} \oplus K^{-1/2}\) | 0 | 2\(g - 2\) | 0 | \(G_i\) | \(2^{2g}\) |
| \(\mathcal{M}_c^0\) \((c = \text{deg}(NK^{-1/2}))\) | \(V = N \oplus N^{-1}K\), \(g - 1 < \text{deg}(N) < 3g - 3\) | 0 | 2\(g - 3\) | : | \(c\) | \(c \mod 2\) | - | \((2g - 3)\) |
| \(\mathcal{M}_0\) | \(V = N \oplus N^{-1}K\), \(\text{deg}(N) = g - 1\) | 0 | 0 | 0 | \(G_\Delta, G_p\) | \(1\) |
| \(\mathcal{M}_{w_1,w_2}\) \(w_1 \in H^1(X, \mathbb{Z}/2) - \{0\}\), \(w_2 \in H^2(X, \mathbb{Z}/2) = \mathbb{Z}/2\) | \(V = W \otimes L_0\), \(L_0^2 = K\) | \(w_1\) | - | 0 or 1 | \(G_\Delta, G_p\) | \(2.(2^{2g} - 1)\) |

**Table 1.** Higgs bundles in the components of \(\mathcal{M}^\text{max}\). The columns show the form of the Higgs bundles, their topological invariants (when applicable), the subgroups to which the structure group of the Higgs bundles can reduce, and the number of connected components of each type.
| $G_*$ | $V$ | $\beta$ | $\gamma$ |
|------|------|--------|--------|
| $G_i$ | $K^{3/2} \oplus K^{-1/2}$ | $\left( \begin{array}{cc} \beta_1 & \beta_3 \\ \beta_3 & 1 \end{array} \right)$, $\left\{ \begin{array}{l} \beta_3 \in H^0(K^2) \\ \beta_1 = \text{const.}(\beta_3)^2 \end{array} \right.$ | $\left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$ |
| $G_\Delta$ | $U \otimes L$ | $\tilde{q}_U \otimes \tilde{\beta}$ | $\tilde{q}_U \otimes \tilde{\gamma}$ |
| | $U$ orthogonal | $\tilde{\beta} \in H^0(L^2K)$ | $\tilde{\gamma} \in H^0(L^{-2}K)$ |
| $\text{SL}(2,\mathbb{R}) \times \text{SL}(2,\mathbb{R})$ | $L_1 \oplus L_2$ | $\left( \begin{array}{cc} \beta_1 & 0 \\ 0 & \beta_2 \end{array} \right)$ | $\left( \begin{array}{cc} \gamma_1 & 0 \\ 0 & \gamma_2 \end{array} \right)$ |
| $G_p$ | $p^*(V) = L_1 \oplus L_2$ | $p^*(\beta) = \left( \begin{array}{cc} \beta_1 & 0 \\ 0 & \beta_2 \end{array} \right)$ | $p^*(\gamma) = \left( \begin{array}{cc} \gamma_1 & 0 \\ 0 & \gamma_2 \end{array} \right)$ |

Table 2. $G_*$-Higgs bundles in $\mathcal{M}^{\text{max}}$, showing the special form of the defining data $(V, \beta, \gamma)$ for a $\text{Sp}(4,\mathbb{R})$-Higgs bundle which admits a reduction of structure group to the indicated subgroup.
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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, IL 61801, USA

E-mail address: bradlow@math.uiuc.edu

INSTITUTO DE CIENCIAS MATEMÁTICAS CSIC-UAM-UC3M-UCM, SERRANO 121, 28006 MADRID, SPAIN

E-mail address: oscar.garcia-prada@uam.es

DEPARTAMENTO DE MATEMÁTICA PURA, FACULDADE DE CIÊNCIAS, UNIVERSIDADE DO PORTO, RUA DO CAMPO ALEGRE 687, 4169-007 PORTO, PORTUGAL

E-mail address: pbgothen@fc.up.pt

49