NOTES ON ALGEBRAIC CYCLES AND HOMOTOPY THEORY

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Abstract. We show that a conjecture by Lawson holds, that is, the inclusion from the Chow variety $C_{p,d}(\mathbb{P}^n)$ of all effective algebraic $p$-cycles of degree $d$ in $n$-dimensional projective space $\mathbb{P}^n$ to the space $C_p(\mathbb{P}^n)$ of effective algebraic $p$-cycles in $\mathbb{P}^n$ is $2d$-connected. As a result, the homotopy and homology groups of $C_{p,d}(\mathbb{P}^n)$ are calculated up to $2d$. We also show an analogous statement for Chow variety $C_{p,d}(\mathbb{P}^n)$ over algebraically closed field $K$ of arbitrary characteristic and compute their etale homotopy groups up to $2d$.

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1. Introduction

Let $\mathbb{P}^n$ be the complex projective space of dimension $n$ and let $C_{p,d}(\mathbb{P}^n)$ be the space of effective algebraic $p$-cycles of degree $d$ on $\mathbb{P}^n$. A fact proved by Chow and Van der Waerden is that $C_{p,d}(\mathbb{P}^n)$ carries the structure of a closed complex algebraic set. Hence it carries the structure of a compact Hausdorff space.

Let $l_0 \subset \mathbb{P}^n$ be a fixed $p$-dimensional linear subspace. For each $d \geq 1$, we consider the analytic embedding

$$i : C_{p,d}(\mathbb{P}^n) \hookrightarrow C_{p,d+1}(\mathbb{P}^n)$$

defined by $c \mapsto c + l_0$. From this sequence of embeddings we can form the union

$$C_p(\mathbb{P}^n) = \lim_{d \to \infty} C_{p,d}(\mathbb{P}^n).$$

The topology on $C_p(\mathbb{P}^n)$ is the weak topology for $\{C_{p,d}(\mathbb{P}^n)\}_{d=1}^\infty$, that is, a set $C \subset C_p(\mathbb{P}^n)$ is closed if and only if $C \cap C_{p,d}(\mathbb{P}^n)$ is closed in $C_{p,d}(\mathbb{P}^n)$ for all $d \geq 1$. For details, the reader is referred to the book [W]. The background on homotopy theory is referred to the book [L1].

In this note, we will prove the following main result.

Theorem 1. For all $n, p$ and $d$, the inclusion $i : C_{p,d}(\mathbb{P}^n) \hookrightarrow C_p(\mathbb{P}^n)$ induced by Equation (1) is $2d$-connected.

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This fact was conjectured by Lawson [L1], who had proved that the inclusion \( i : C_{p,d}(\mathbb{P}^n) \hookrightarrow C_p(\mathbb{P}^n) \) has a right homotopy inverse through dimension \( 2d \).

The method in the proof of Theorem [L1] comes from Lawson in his proof of the Complex Suspension Theorem [L1], i.e., the complex suspension to the space of \( p \)-homology groups of \( C_{p,d}(\mathbb{P}^n) \) is obtained (cf. Theorem 10).

As applications of the main result, we calculate the first \( 2d + 1 \) homotopy and homology groups of \( C_{p,d}(\mathbb{P}^n) \).

The analogous result of Theorem [L1] for Chow varieties over algebraically closed fields is obtained (cf. Theorem 10).

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2. **The Method in the Proof of the Complex Suspension Theorem**

Now we briefly review Lawson’s method in the proof of the Complex Suspension Theorem. The materials in this section can be found in [L1], [F] and [L2].

Fix a hyperplane \( \mathbb{P}^n \subset \mathbb{P}^{n+1} \) and a point \( \mathbb{P}^0 \in \mathbb{P}^{n+1} - \mathbb{P}^n \). For any non-negative integer \( p \) and \( d \), set

\[
T_{p+1,d}(\mathbb{P}^{n+1}) := \{ c = \sum n_i V_i \in C_{p+1,d}(\mathbb{P}^{n+1}) | \dim(V_i \cap \mathbb{P}^n) = p, \forall i \},
\]

when \( d = 0 \), \( C_{p,0}(\mathbb{P}^n) \) is defined to be the empty cycle.

The following result was proved by Lawson and a generalized algebraic version was proved by Friedlander [F].

**Proposition 2** ([L1]). The set \( T_{p+1,d}(\mathbb{P}^{n+1}) \) is Zariski open in \( C_{p+1,d}(\mathbb{P}^{n+1}) \). Moreover, \( T_{p+1,d}(\mathbb{P}^{n+1}) \) is homotopy equivalent to \( C_{p,d}(\mathbb{P}^n) \). In particular, their corresponding homotopy groups are isomorphic, i.e.,

\[
\pi_*(T_{p+1,d}(\mathbb{P}^{n+1})) \cong \pi_*(C_{p,d}(\mathbb{P}^n)).
\]

Fix linear embedding \( \mathbb{P}^{n+1} \subset \mathbb{P}^{n+2} \) and two points \( x_0, x_1 \in \mathbb{P}^{n+2} - \mathbb{P}^{n+1} \). Each projection \( p_i : \mathbb{P}^{n+2} - \{ x_0 \} \rightarrow \mathbb{P}^{n+1} \) gives us a holomorphic line bundle over \( \mathbb{P}^{n+1} \).

Let \( D \subset C_{n+1,c}(\mathbb{P}^{n+2}) \) be an effective divisor of degree \( c \) in \( \mathbb{P}^{n+2} \) such that \( x_0, x_1 \) are not in \( D \). Denote by \( \widetilde{Div}_c(\mathbb{P}^{n+2}) \subset C_{n+1,c}(\mathbb{P}^{n+2}) \) the subset of all such \( D \).

Any effective cycle \( c \in C_{p+1,d}(\mathbb{P}^{n+1}) \) can be lifted to a cycle with support in \( D \), defined as follows:

\[
\Psi_D(c) = (\Sigma c) \cdot D.
\]

The map \( \Psi(c,D) := \Psi_D(c) \) is a continuous map with variables \( c \) and \( D \). Hence we have a continuous map \( \Psi_D : C_{p+1,d}(\mathbb{P}^{n+1}) \rightarrow C_{p+1,d}(\mathbb{P}^{n+2} - \{ x_0, x_1 \}) \). The composition of \( \Psi_D \) with the projection \( (p_i)_* \) is \( (p_0)_* \circ \Psi_D = e \) (where \( e \cdot c = c + \cdots + c \) \( e \) times). The composition of \( \Psi_D \) with the projection \( (p_1)_* \) gives us a transformation of cycles in \( \mathbb{P}^{n+1} \) which makes most of them intersecting properly to \( \mathbb{P}^n \). To see this, we consider the family of divisors \( tD, 0 \leq t \leq 1 \), given by scalar multiplication by \( t \) in the line bundle \( p_0 : \mathbb{P}^{n+2} - \{ x_0 \} \rightarrow \mathbb{P}^{n+1} \).

Assume \( x_1 \) is not in \( tD \) for all \( t \). Then the above construction gives us a family transformation

\[
F_{tD} := (p_1)_* \circ \Psi_{tD} : C_{p+1,d}(\mathbb{P}^{n+1}) \rightarrow C_{p+1,d}(\mathbb{P}^{n+1})
\]
for $0 \leq t \leq 1$. Note that $F_{0D} \equiv e$ (multiplication by $e$).

The question is that for a fixed $c$, which divisors $D \in C_{n+1,e}(\mathbb{P}^{n+2})$ ($x_0$ is not in $D$ and $x_1$ is not in $\bigcup_{0 \leq t \leq 1} tD$) have the property that

$$F_{1D}(c) \in T_{p+1,de}(\mathbb{P}^{n+1})$$

for all $0 < t \leq 1$.

Set $B_c := \{D \in C_{n+1,e}(\mathbb{P}^{n+2}) \mid F_{1D}(c) \text{ is not in } T_{p+1,de}(\mathbb{P}^{n+1}) \text{ for some } 0 < t \leq 1\}$, i.e., all degree $e$ divisors on $\mathbb{P}^{n+2}$ such that some component of $(p_1)_* \circ \Psi_{tD}(c) \subseteq \mathbb{P}^n$

for some $t > 0$.

An important calculation we will use later is the following result.

**Proposition 3** (L1). For $c \in C_{p+1,d}(\mathbb{P}^{n+1})$, $\text{codim}_{\mathbb{C}} B_c \geq \frac{(p^2+1)}{e}$.

3. PROOF OF THE FIRST MAIN RESULT

In the construction of the last section, $F_{1D}$ maps $C_{p+1,d}(\mathbb{P}^{n+1})$ to $C_{p+1,de}(\mathbb{P}^{n+1})$, i.e.,

$$F_{1D} := (p_1)_* \circ \Psi_{1D} : C_{p+1,d}(\mathbb{P}^{n+1}) \rightarrow C_{p+1,de}(\mathbb{P}^{n+1}).$$

Moreover, the image of $F_{1D}$ is in the Zariski open subset $T_{p+1,d}(\mathbb{P}^{n+1})$ if $D$ is not $B_c$. We can find such a $D$ if $\text{codim}_{\mathbb{C}} B_c \geq \frac{(p^2+1)}{e}$ is positive.

Suppose now that $f : S^k \rightarrow C_{p+1,d}(\mathbb{P}^{n+1})$ is a continuous map for $0 < k \leq 2d$. We may assume that $f$ is piecewise linear up to homotopy. Then the map $e \cdot f = F_{0D} \circ f$ is homotopic to a map $S^k \rightarrow T_{p+1,de}(\mathbb{P}^{n+1})$. To see this, we consider the family

$$F_{1D} \circ f : S^k \rightarrow C_{p+1,de}(\mathbb{P}^{n+1}), \quad 0 \leq t \leq 1,$$

where $D$ lies outside the union $\bigcup_{x \in S_k} B_f(x)$. This is a set of real codimension bigger than or equal to $2\left(\frac{p^2+1}{e}\right) - (k + 1)$. Therefore, if $2\left(\frac{p^2+1}{e}\right) - (k + 1) \geq 1$, i.e., $k \leq 2\left(\frac{p^2+1}{e}\right) - 2$, then such a $D$ exists once we choose a large $e$ such that $2d \leq 2\left(\frac{p^2+1}{e}\right) - 2$. Therefore we have the following commutative diagram

$$\begin{array}{ccc}
T_{p+1,d}(\mathbb{P}^{n+1}) & \longrightarrow & T_{p+1,de}(\mathbb{P}^{n+1}) \\
\downarrow F_D & & \downarrow F_D \\
S^k & \longrightarrow & C_{p+1,de}(\mathbb{P}^{n+1})
\end{array}$$

where $F_D := F_{1D}$.

**Proposition 4.** For any integer $e \geq 1$, the map

$$\phi_e : C_{0,d}(\mathbb{P}^n) \rightarrow C_{0,de}(\mathbb{P}^n), \quad \phi_e(c) = e \cdot c$$

induces injections $\phi_{ke} : \pi_k(C_{0,d}(\mathbb{P}^n)) \rightarrow \pi_k(C_{0,de}(\mathbb{P}^n))$ for $k \leq 2d$.

**Proof.** First note that $C_{0,d}(\mathbb{P}^n) \cong \text{SP}^d(\mathbb{P}^n)$, where $\text{SP}^d(\mathbb{P}^n)$ denotes the $d$-th symmetric product of $\mathbb{P}^n$. Denote by $\Delta : (\mathbb{P}^n)^d \rightarrow (\mathbb{P}^n)^d e = (\mathbb{P}^n)^de$ the diagonal map $\Delta(x) = (x, \ldots, x)$ for $e$ copies of $x$ and and $p_1 : ((\mathbb{P}^n)^d e \rightarrow (\mathbb{P}^n)^d$ the projection on the first component. Hence we have $p_1 \circ \Delta = id : (\mathbb{P}^n)^d \rightarrow (\mathbb{P}^n)^d$ and $p_1 \circ \Delta_\ast = id_\ast : H_k((\mathbb{P}^n)^d) \rightarrow H_k((\mathbb{P}^n)^d)$ for any integer $k \geq 0$. This implies the injectivity of $\Delta_\ast$. 
From the commutative diagram of continuous maps of complex varieties

\[
\begin{array}{ccc}
(P^n)^d & \xrightarrow{\Delta} & (P^n)^{de} \\
\pi & \downarrow & \pi \\
SP^d(P^n) & \xrightarrow{\phi_*} & SP^{de}(P^n),
\end{array}
\]

where \(\pi : X^n \to SP^m X\) is the natural projection, we have the induced commutative diagram on homology groups

\[
(4)\begin{array}{ccc}
H_k((P^n)^d, \mathbb{Q}) & \xrightarrow{\Delta_*} & H_k((P^n)^{de}, \mathbb{Q}) \\
\pi_* & \downarrow & \pi_* \\
H_k(SP^d(P^n), \mathbb{Q}) & \xrightarrow{\phi'_* \otimes \mathbb{Q}} & H_k(SP^{de}(P^n), \mathbb{Q}),
\end{array}
\]

for any \(k \geq 0\).

Now we show that \(\phi'_* \otimes \mathbb{Q}\) is injective for all \(e \geq 1\). Let \(\alpha \in H_k(SP^d(P^n), \mathbb{Q})\) be an element such that \(\phi'_* \otimes \mathbb{Q}(\alpha) = 0\). Since \(H_k((P^n)^d, \mathbb{Q})^{S_d} \simeq H_k(SP^d(P^n), \mathbb{Q})\) for any \(d \geq 1\), there is an element \(\tilde{\alpha} \in H_k((P^n)^d, \mathbb{Q})\) such that \(\pi_*(\tilde{\alpha}) = \alpha\), where \(S_d\) is the \(d\)-th symmetric group and \(H_k((P^n)^d, \mathbb{Q})^{S_d}\) is the \(S_d\)-invariant subgroup of \(H_k((P^n)^d, \mathbb{Q})\). The element \(\tilde{\alpha}\) is \(S_d\)-invariant. Set \(\beta := \Delta_*(\tilde{\alpha})\). From the commutative diagram (4), we have \(\pi_*(\beta) = 0\). Since \(\beta\) is \(S_{de}\)-invariant and \(\pi_*(\beta) = 0\), we get \(\beta = 0\) since \(H_k((P^n)^{de}, \mathbb{Q})^{S_{de}} \simeq H_k(SP^{de}(P^n), \mathbb{Q})\) (cf. e.g. [ES]). This implies that \(\tilde{\alpha} = 0\) since \(\Delta_*\) is injective and \(\Delta_*(\tilde{\alpha}) = \beta = 0\). Therefore \(\alpha = \pi_*(\tilde{\alpha}) = 0\), i.e., \(\phi'_* \otimes \mathbb{Q}\) is injective on rational homology groups.

Note that the map \(\phi_e : SP^d(P^n) \to SP^{de}(P^n)\) induces a commutative diagram

\[
(5)\begin{array}{ccc}
\pi_k(SP^d(P^n)) & \xrightarrow{\phi'_*} & \pi_k(SP^{de}(P^n)) \\
\rho & \downarrow & \rho \\
H_k(SP^d(P^n)) & \xrightarrow{\phi'_*} & H_k(SP^{de}(P^n)),
\end{array}
\]

where \(\rho\) are Hurewicz maps.

We claim that if \(k \leq 2d\), then \(\phi'_* : \pi_k(SP^d(P^n)) \to \pi_k(SP^{de}(P^n))\) is injective. First we will show that \(\phi'_* \otimes \mathbb{Q} : \pi_k(SP^d(P^n)) \otimes \mathbb{Q} \to \pi_k(SP^{de}(P^n)) \otimes \mathbb{Q}\) is injective.

Fix \(x_0 \in P^n\). Let \(i : P^n \to SP^d(P^n)\) be the map given by \(i(x) = x + (d - 1)x_0\) and let \(j : SP^d(P^n) \to SP^{de+1}(P^n)\) be induced by the sequence of maps \(SP^d(P^n) \to SP^{d+m}(P^n), y \mapsto y + mx_0\). Then \(i\) induces a commutative diagram

\[
(6)\begin{array}{ccc}
\pi_k(P^n) & \xrightarrow{i_*} & \pi_k(SP^d(P^n)) \\
\rho & \downarrow & \rho \\
H_k(P^n) & \xrightarrow{i_*} & H_k(SP^d(P^n)),
\end{array}
\]

\[
\begin{array}{ccc}
\pi_k(SP^d(P^n)) & \xrightarrow{j_*} & \pi_k(SP^{de}(P^n)) \\
\rho & \downarrow & \rho \\
\pi_k(SP^{de}(P^n)) & \xrightarrow{DT} & H_k(X),
\end{array}
\]

where the dotted map \(DT : \pi_k(SP^d(P^n)) \to H_k(P^n)\) is the Dold-Thom isomorphism. This follows from the fact that, for any connected finite CW complex \(X\) and a fixed point \(x_0 \in X\), the composed map \(\pi_k(X) \xrightarrow{i_*} \pi_k(SP^d(X)) \xrightarrow{DT} H_k(X)\) of the
induced map $j_*$ by the inclusion $j : X = \text{SP}^1(X) \subset \text{SP}^\infty(X)$ and the Dold-Thom map $DT : \pi_k(\text{SP}^\infty(X)) \to H_k(X)$ is the Hurewicz map (cf. [DT]).

Note that $j_* : \pi_k(\text{SP}^d(P^n)) \to \pi_k(\text{SP}^\infty(P^n))$ is an isomorphism for $k \leq 2d$ (cf. [D] or [Mi]). From equation (6), we obtain the injectivity of the Hurewicz map $\rho : \pi_k(\text{SP}^d(P^n)) \to H_k(\text{SP}^d(P^n))$ for $k \leq 2d$ since $i_* \circ DT \circ j_* = \rho$ and the injectivity of $i_*$. Now the injectivity of $\phi_{e*} \otimes Q : \pi_k(\text{SP}^d(P^n)) \otimes Q \to \pi_k(\text{SP}^{de}(P^n)) \otimes Q$ follows from equation (5) as well as the injectivity of $\rho \otimes Q$ and $\phi_{e*} \otimes Q$. Since

$$\pi_k(\text{SP}^d(P^n)) \cong \pi_k(\text{SP}^{de}(P^n)) \cong \left\{ \begin{array}{ll} \mathbb{Z}, & 0 < k \leq 2d \text{ and } k \text{ even} , \\ 0, & k = 0 \text{ or } k \leq 2d \text{ and } k \text{ odd} \end{array} \right.$$

(cf. [D] or [Mi]), the injectivity of $\phi_{e*} \otimes Q$ implies the injectivity of $\phi_{e*}$. □

**Remark 5.** From the proof of Proposition 4 we obtain that $\phi_{e*} \otimes Q : \pi_k(\text{C}_{0,d}(P^n)) \otimes Q \cong \pi_k(\text{C}_{0,de}(P^n)) \otimes Q$ for all $k \leq 2d$. To see this we note that, for $k \leq 2d$, both $\pi_k(\text{C}_{0,d}(P^n))$ and $\pi_k(\text{C}_{0,de}(P^n))$ are isomorphic to $\pi_k(\text{C}_0(P^n)) \cong H_k(P^n)$. So the injectivity of $\phi_{e*}$ implies an isomorphism for $\phi_{e*} \otimes Q$.

**Lemma 6.** There is a commutative diagram

$$\begin{array}{c}
C_{p,d}(P^n) \xrightarrow{\Phi_{p,d,n,e}} C_{p,de}(P^n) \\
\downarrow \Sigma \quad \downarrow \Sigma \\
T_{p+1,d}(P^{n+1}) \xrightarrow{F_{1D}} T_{p+1,de}(P^{n+1}),
\end{array}$$

where $\Phi_{p,d,n,e}$ ($\Phi_{0,d,n,e} = \phi_e$ in Proposition 4) is the composed map

$$C_{p,d}(P^n) \xrightarrow{c} C_{p,d}(P^n) \times \cdots \times C_{p,d}(P^n) \xrightarrow{(c,\ldots,c)} C_{p,de}(P^n),$$

and $D \in \text{Div}_{e}(P^{n+2})$, $F_{1D}$ is the restriction of $F_{1D} : C_{p+1,d}(P^{n+1}) \to C_{p+1,de}(P^{n+1})$.

**Proof.** Note that the image of the restriction of

$$F_{1D} : C_{p+1,d}(P^{n+1}) \to C_{p+1,de}(P^{n+1})$$

on $T_{p+1,d}(P^{n+1})$ is in $T_{p+1,de}(P^{n+1})$. The remaining part follows from Lemma 5.5 in [L1]. □

**Proposition 7.** For integers $p, d, n$, there is an integer $e_{p,d,n} \geq 1$ such that if $e \geq e_{p,d,n}$, then the map $\Phi_{p+1,d,n+1,e} : C_{p+1,d}(P^n) \to C_{p+1,de}(P^{n+1})$ given by $c \mapsto e \cdot c$ induces injections

$$(\Phi_{p+1,d,n+1,e})_* : \pi_k(C_{p+1,d}(P^{n+1})) \to \pi_k(C_{p+1,de}(P^{n+1}))$$

for $k \leq 2d$.

**Proof.** We prove it by induction. The case that $p = -1$ follows from Proposition 4. We assume that $\Phi_{p,d,n,e} : C_{p,d}(P^n) \to C_{p,de}(P^{n+1})$ defined by $\Phi_{p,d,n,e}(c) = e \cdot c$ induces injections $(\Phi_{p,d,n,e})_* : \pi_k(C_{p,d}(P^n)) \to \pi_k(C_{p,de}(P^n))$ for $k \leq 2d$ and $e \geq e_{p,d,n}$.

Let $\alpha \in \pi_k(C_{p+1,d}(P^{n+1}))$ be an element such that $(\Phi_{p+1,d,n+1,e})_*(\alpha) = 0$, that is, $(F_{1D})_*(\alpha) = 0$. Let $f : S^k \to C_{p+1,d}(P^{n+1})$ be piecewise linear up to homotopy such that $[f] = \alpha$. By assumption, $[F_{1D} \circ f] = 0$. 

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By Lemma 3 and Equation 3, we have

\[
\begin{array}{c}
S^k \xrightarrow{g} C_{p,d}(\mathbb{P}^n) \xrightarrow{\Phi_{p,d,n,e}} C_{p,de}(\mathbb{P}^n) \\
\downarrow \Sigma \quad F_D \downarrow \Sigma \\
T_{p+1,d}(\mathbb{P}^{n+1}) \xrightarrow{T_{p+1,de}(\mathbb{P}^{n+1})} \\
S^k \xrightarrow{f} C_{p+1,d}(\mathbb{P}^{n+1}) \xrightarrow{F_{0D}} C_{p+1,de}(\mathbb{P}^{n+1}).
\end{array}
\]

Since \( F_{0D} \) is homotopy to \( F_D : C_{p+1,d}(\mathbb{P}^{n+1}) \to T_{p+1,de}(\mathbb{P}^{n+1}) \) for \( e \geq e_{p+1,d,n} \) (cf. [L1]), we have \( [F_D \circ f] = 0 \in \pi_k(T_{p+1,de}(\mathbb{P}^{n+1})) \). By Proposition 2, \( \Sigma^e_1([F_D \circ f]) = 0 \). From the above commutative diagram and the injectivity of \( (\Phi_{p,d,n,e})_* \), the map \( f : S^k \to C_{p+1,d}(\mathbb{P}^{n+1}) \) can be lifted to a null homotopy map \( g : S^k \to C_{p,d}(\mathbb{P}^n) \) such that \( \Phi_{p,d,n,e}^*([g]) = \Sigma^e_1([F_D \circ f]) \) and \( [f] = [\Sigma g] \). Hence \( \alpha = [f] = 0 \). That is, \((F_{0D})_* = (\Phi_{p+1,d,n+1,e})_* \) is injective for \( k \leq 2d \).

The proof of Theorem 1. The case that \( p = -1 \) has been proved in [L3] and [M1]. By taking limit \( e \to \infty \) in Proposition 3, we get injections \( \pi_k(C_{p+1,d}(\mathbb{P}^{n+1})) \to \pi_k(C_{p+1}(\mathbb{P}^{n+1})) \) for \( k \leq 2d \). On one hand, from the fact that \( \pi_k(C_{p+1}(\mathbb{P}^{n+1})) \) is isomorphic to either \( \mathbb{Z} \) or \( 0 \) (cf. [L1]) and the injections above, we obtain \( \pi_k(C_{p+1,d}(\mathbb{P}^{n+1})) \) is isomorphic to either \( \mathbb{Z} \) or \( 0 \) for \( k \leq 2d \). On the other hand, the induced map \( i_* : \pi_k(C_{p+1,d}(\mathbb{P}^{n+1})) \to \pi_k(C_{p+1}(\mathbb{P}^{n+1})) \) by equation (4) is surjective for \( k \leq 2d \) (cf. [L1], Theorem 2). Hence \( \pi_k(C_{p+1,d}(\mathbb{P}^{n+1})) \) is isomorphic to \( \pi_k(C_{p+1}(\mathbb{P}^{n+1})) \) for \( k \leq 2d \).

From the fact that the inclusion map \( i : C_{p,d}(\mathbb{P}^n) \subset C_{p}(\mathbb{P}^n) \) in equation (4) induces surjections \( i_* : \pi_k(C_{p,d}(\mathbb{P}^n)) \to \pi_k(C_{p}(\mathbb{P}^n)) \) for \( k \leq 2d \) (cf. [L1]) and

\[
\pi_k(C_{p,(\mathbb{P}^n)}) \cong \pi_k(C_{p}(\mathbb{P}^n)) \cong \begin{cases} \mathbb{Z}, & 0 < k \leq \min\{2d, 2(n-p)\} \\ 0, & \text{all other } k < 2d \end{cases}
\]

we obtain also the injectivity of \( i_* \) for \( k \leq 2d \) since a surjective homomorphism to \( \mathbb{Z} \) is an isomorphism. This completes the proof of Theorem 1.

As applications of Theorem 1 and Lawson’s Complex Suspension Theorem [L1], we get the homotopy and homology groups of \( C_{p,d}(\mathbb{P}^n) \) up to \( 2d \).

**Corollary 8.** The first \( 2d+1 \) homotopy groups of \( C_{p,d}(\mathbb{P}^n) \) is given by the formula

\[
\pi_k(C_{p,d}(\mathbb{P}^n)) \cong \begin{cases} \mathbb{Z}, & \text{if } k \leq \min\{2d, 2(n-p)\} \text{ and even,} \\ 0, & \text{all other } k < 2d \end{cases}
\]

**Proof.** From the proof to Theorem 1 we have \( \pi_k(C_{p,d}(\mathbb{P}^n)) \to \pi_k(C_{p}(\mathbb{P}^n)) \) for \( k \leq 2d \). Recall the fact that \( C_{p}(\mathbb{P}^n) \) is homotopy equivalent to the product \( K(\mathbb{Z},2) \times \cdots \times K(\mathbb{Z},2(n-p)) \) (cf. [L1]), in particular,

\[
\pi_k(C_{p}(\mathbb{P}^n)) \cong \begin{cases} \mathbb{Z}, & \text{if } k \leq 2(n-p) \text{ and even,} \\ 0, & \text{otherwise.} \end{cases}
\]

**Corollary 9.** The first \( 2d+1 \) homology groups of \( C_{p,d}(\mathbb{P}^n) \) is given by the formula

\[
H_k(C_{p,d}(\mathbb{P}^n)) \cong H_k(K(\mathbb{Z},2) \times \cdots \times K(\mathbb{Z},2(n-p)))
\]
for $0 \leq k \leq 2d$ where the right hand side can be computed by using the Künneth formula.

Proof. It also follows from the proof to Theorem 4 and the fact that $C_p(\mathbb{P}^n)$ is homotopy equivalent to the product $K(\mathbb{Z}, 2) \times \cdots \times K(\mathbb{Z}, 2(n-p))$. \hfill $\square$

4. Etale homotopy for Chow varieties over algebraically closed fields

In this section, we will compute the Etale homotopy groups of Chow varieties over algebraically closed fields. Let $\mathbb{P}^n_K$ be the projective space of dimension $n$ over $K$, where $K$ is an algebraic closed field of characteristic $\text{char}(K)$. Let $l$ be a prime number which is different from $\text{char}(K)$. Let $C_{p,d}(\mathbb{P}^n)_K$ be the space of effective $p$-cycles of degree $d$ in $\mathbb{P}^n_K$.

The notations we use in this section can be found in [F]. Recall that the etale topological type functor $(-)_et$ is a functor from simplicial schemes to pro-simplicial sets; the Bousfield-Kan homotopy inverse limit functor $(\mathbb{Z}/l)_\infty$ maps simplicial sets to simplicial sets; the Bousfield-Kan homotopy inverse limit functor $\text{holim}(\quad)$ maps simplicial sets to topological spaces; and the geometric realization functor $\text{Re}(\quad)$ maps simplicial sets to topological spaces.

Definition 1. Let $|\quad| : (\text{algebraic sets}) \rightarrow (\text{topological spaces})$ be the functor as the composition $\text{Re}(\quad) \circ \text{holim}(\quad) \circ (-)_et$. The $k$-th etale homotopy group of $X_K$ is defined to be $\pi_k(|X_K|_et)$.

Let $l_0 \subset \mathbb{P}^n_K$ be a fixed $p$-dimensional linear subspace. For each $d \geq 1$, we consider the closed immersions

$\tilde{i} : C_{p,d}(\mathbb{P}^n_K) \hookrightarrow C_{p,d+1}(\mathbb{P}^n_K)$

defined by $c \mapsto c + l_0$. These immersions induce topological embeddings

$\tilde{i} : |(C_{p,d}(\mathbb{P}^n))_et| \hookrightarrow |(C_{p,d+1}(\mathbb{P}^n))_et|$

(cf. [F], Prop. 2.1).

From this sequence of embeddings we can form the union

$|C_p(\mathbb{P}^n)_et| := \lim_{d \rightarrow \infty} |(C_{p,d}(\mathbb{P}^n))_et|$.

The topology on $|C_p(\mathbb{P}^n)_et|$ is the weak topology for $\{(C_{p,d}(\mathbb{P}^n))_et\}_d^{\infty-1}$. For more general discussion on etale homotopy on spaces of algebraic cycles, the reader is referred to the paper [F].

Our second main result is the following theorem.

Theorem 10. For all $n, p$ and $d$, the inclusion $\tilde{i} : |(C_{p,d}(\mathbb{P}^n)_et| \hookrightarrow |(C_p(\mathbb{P}^n))_et|$ induced by Equation 7 is $2d$-connected.

Lemma 11. For $k \leq 2d$, we have

$\pi_k(|\text{Sp}^d(\mathbb{P}^n)_et|) = \begin{cases} \mathbb{Z}_l, & \text{if } k \leq 2n \text{ and even}, \\ 0, & \text{if } k = 0 \text{ or } k \geq 2n \text{ or odd.} \end{cases}$

Proof. We need to show that $\pi_k(|\text{Sp}^d(\mathbb{P}^n)_et|) \cong H_k(X, \mathbb{Z}_l)$ for $0 < k \leq 2d$, where $H_k(X, \mathbb{Z}_l)$ is the $l$-adic homology group of $X$. First we have $|\text{Sp}^d(\mathbb{P}^n)_et|$ is simply connected for any integer $d \geq 1$. To see this, note that $|\text{Sp}^d(\mathbb{P}^n)_et|$ is homotopy equivalent to $|\text{Sp}^d((\mathbb{P}^n)_et)|$ (cf. the proof of Theorem 4.3 in [F]) and the latter
is simply connected since $|\mathbb{P}_K^n|_\text{et}$ is. Since the inclusion map $\tilde{\varphi} : \mathbb{P}_K^n \to \mathbb{P}_K^n$ is homologically 2d-connected (cf. [F]) and $\pi_1(\mathbb{P}_K^n) = 0$, we obtain the 2d-connectivity of the inclusion map $i$. That is, $\pi_k(\mathbb{P}_K^n) \cong \pi_k(\mathbb{P}_K^n)$. Now the theorem follows from the fact (cf. [F], Corollary 4.4) that
\[
\pi_k(\mathbb{P}_K^n) = \begin{cases} 
\mathbb{Z}_l, & \text{if } 0 < k \leq 2n \text{ and even,} \\
0, & \text{otherwise.}
\end{cases}
\]
\[
\square
\]

**Lemma 12.** For any integer $e \geq 1$, the map
\[
\tilde{\varphi}_e : C_{0,d}(\mathbb{P}_K^n) \to C_{0,d}(\mathbb{P}_K^n), \quad \tilde{\varphi}_e(c) = e \cdot c
\]
induces injections $\tilde{\varphi}_{e*} : \pi_k([C_{0,d}(\mathbb{P}_K^n)]) \to \pi_k([C_{0,d}(\mathbb{P}_K^n)])$ for $k \leq 2d$.

**Proof.** Note that there is a bi-continuous algebraic morphism from $C_{0,d}(\mathbb{P}_K^n)$ to $\mathbb{P}_K^n$ (cf. [F]), we need to show $\tilde{\varphi}_{e*} : \pi_k([\mathbb{P}_K^n]) \to \pi_k([\mathbb{P}_K^n])$ is for $k \leq 2d$. Now the proof is word for word from Proposition 4 except that the homotopy groups (resp. singular homology groups) are replaced by the étale homotopy groups (resp. $l$-adic homology groups), $\mathbb{Z}$ (resp. $\mathbb{Q}$) are replaced by $\mathbb{Z}_l$ (resp. $\mathbb{Q}_l$) and the Dold-Thom theorem is replaced by the $l$-adic analogous version proved by Friedlander [F].

\[
\square
\]

**Lemma 13.** For integers $p, d, n$, there is an integer $e_{p,d,n} \geq 1$ such that if $e \geq e_{p,d,n}$, then the map $\tilde{\Phi}_{p+1,d,n+1,e} : C_{p+1,d}(\mathbb{P}_K^n) \to C_{p+1,d}(\mathbb{P}_K^n)$ given by $c \mapsto e \cdot c$ induces injections
\[
(\tilde{\Phi}_{p+1,d,n+1,e})_* : \pi_k([C_{p+1,d}(\mathbb{P}_K^n)_{\text{et}}]) \to \pi_k([C_{p+1,d}(\mathbb{P}_K^n)_{\text{et}}])
\]
for $k \leq 2d$.

**Proof.** The case that $K = \mathbb{C}$ has been proved in Proposition 7. The analogous argument is given below.

Note that the map $\tilde{\Phi}_{p+1,d,n+1,e} : C_{p+1,d}(\mathbb{P}_K^n) \to C_{p+1,d}(\mathbb{P}_K^n)$ induced a continuous map (also denote by $\tilde{\Phi}_{p+1,d,n+1,e}$) $\tilde{\Phi}_{p+1,d,n+1,e} : [C_{p+1,d}(\mathbb{P}_K^n)_{\text{et}}] \to [C_{p+1,d}(\mathbb{P}_K^n)_{\text{et}}]$ between topological spaces (cf. [F], Prop. 2.1).

We also prove it by induction. The case that $p = -1$ follows from Lemma 12. We assume that $\Phi_{p,d,n,e} : C_{p,d}(\mathbb{P}_K^n) \to C_{p,d}(\mathbb{P}_K^n)$ defined by $\Phi_{p,d,n,e}(c) = e \cdot c$ induces injections $\Phi_{p,d,n,e} : \pi_k([C_{p,d}(\mathbb{P}_K^n)_{\text{et}}]) \to \pi_k([C_{p,d}(\mathbb{P}_K^n)_{\text{et}}])$ for $k \leq 2d$ and $e \geq e_{p,d,n}$.

Let $\alpha \in \pi_k([C_{p+1,d}(\mathbb{P}_K^n)_{\text{et}}])$ be an element such that $\Phi_{p+1,d,n+1,e}((\alpha)) = 0$, that is, $(F_{p,d,n+1,e})_\ast(\alpha) = 0$. Let $\gamma : C_{p+1,d}(\mathbb{P}_K^n)_{\text{et}} \to \pi_k([C_{p+1,d}(\mathbb{P}_K^n)_{\text{et}}])$ be piecewise linear up to homotopy such that $[\gamma] = \alpha$. By assumption, $[F_{p,d,n+1,e} \circ \gamma] = 0$.

By Proposition 3.5 in [F] and the algebraic version of Equation (3), we have
\[
\begin{align*}
S^k & \xrightarrow{g} [C_{p,d}(\mathbb{P}_K^n)_{\text{et}}] \xrightarrow{\Phi_{p,d,n,e}} [C_{p,d}(\mathbb{P}_K^n)_{\text{et}}] \xrightarrow{\Sigma} [T_{p+1,d}(\mathbb{P}_K^n)_{\text{et}}] \\
& \xrightarrow{F_{p+1,d,n+1,e}} [T_{p+1,d}(\mathbb{P}_K^n)_{\text{et}}] \xrightarrow{\Sigma} [C_{p+1,d}(\mathbb{P}_K^n)_{\text{et}}] \\
S^k & \xrightarrow{f} [C_{p+1,d}(\mathbb{P}_K^n)_{\text{et}}] \xrightarrow{F_{p+1,d,n+1,e}} [C_{p+1,d}(\mathbb{P}_K^n)_{\text{et}}].
\end{align*}
\]
Since $F_{0D}$ is homotopy to $F_D : |(C_{p+1,d}(\mathbb{P}^{n+1})_K)_{et}| \to |(T_{p+1,d,c}(\mathbb{P}^{n+1})_K)_{et}|$ for $e \geq \bar{e}_{p+1,d,n}$ (cf. [F]), we have $[F_D \circ f] = 0 \in \pi_k((T_{p+1,d,c}(\mathbb{P}^{n+1})_K)_{et})$. By the algebraic version of Proposition 2 (cf. [F], Prop. 3.2), $\Sigma^{-1}_e((F_D \circ f)) = 0$. From the above commutative diagram and the injectivity of $(\Phi_{p,d,n,e})_*$, the map $f : S^k \to |(C_{p+1,d}(\mathbb{P}^{n+1})_K)_{et}|$ can be lifted to a null homotopy map $g : S^k \to |(C_{p,d}(\mathbb{P}^{n})_K)_{et}|$ such that $(\Phi_{p,d,n,e})_*([g]) = \Sigma^{-1}_e((F_D \circ f))$ and $[f] = [\Sigma \circ g]$. Hence $\alpha = [f] = 0$. That is, $(F_{0D})_* = (\Phi_{p+1,d,n+1,e})_*$ is injective for $k \leq 2d$.

The proof of Theorem 11. The case that $p = -1$ has been proved in [13], [14] and [F]. By taking limit $e \to \infty$ in Lemma 13 we get injections $\pi_k((C_{p+1,d}(\mathbb{P}^{n+1})_K)_{et}) \to |(C_{p+1}(\mathbb{P}^{n+1})_K)_{et}|$ for $k \leq 2d$. On one hand, from the fact that $\pi_k((C_{p+1}(\mathbb{P}^{n+1})_K)_{et})$ is isomorphic to either $\mathbb{Z}_l$ or 0 (cf. [F]) and the injections above, we obtain $\pi_k((C_{p+1}(\mathbb{P}^{n+1})_K)_{et})$ is isomorphic to either $\mathbb{Z}_l$ or 0 for $k \leq 2d$. On the other hand, the induced map $\tilde{i}_* : \pi_k((C_{p+1,d}(\mathbb{P}^{n+1})_K)_{et}) \to \pi_k((C_{p+1}(\mathbb{P}^{n+1})_K)_{et})$ by equation (2) is surjective for $k \leq 2d$. To see this, consider the commutative diagram

\[
\begin{array}{ccc}
|(C_{0,d}(\mathbb{P}^{n})_K)_{et}| & \xrightarrow{\Sigma^p} & |(C_{0,d}(\mathbb{P}^{n+p})_K)_{et}| \\
\downarrow & & \downarrow \tilde{i} \\
|(C_{0}(\mathbb{P}^{n})_K)_{et}| & \xrightarrow{\Sigma^p} & |(C_{p}(\mathbb{P}^{n+p})_K)_{et}| \\
\end{array}
\]

where, $\Sigma^p := \Sigma \circ \Sigma \circ \cdots \circ \Sigma$ is the suspension for $p$ times. By the argument in the proof of Lemma 13 we know that the left vertical arrow in equation (8) is a $2d$-connected mapping, and by Theorem 4.2 in [F] the lower horizontal arrow is a homotopy equivalence. This implies the map induced by $\tilde{i}$ is surjective on homotopy groups for $k \leq 2d$. Hence $\pi_k((C_{p+1,d}(\mathbb{P}^{n+1})_K)_{et})$ is isomorphic to $\pi_k((C_{p+1}(\mathbb{P}^{n+1})_K)_{et})$ for $k \leq 2d$.

By this fact that the inclusion map $\tilde{i} : |(C_{p,d}(\mathbb{P}^{n})_K)_{et}| \subset |(C_{p}(\mathbb{P}^{n})_K)_{et}|$ in equation (2) induces surjections $i_* : \pi_k((C_{p,d}(\mathbb{P}^{n})_K)_{et}) \to \pi_k((C_{p}(\mathbb{P}^{n})_K)_{et})$ for $k \leq 2d$ and $\pi_k((C_{p,d}(\mathbb{P}^{n})_K)_{et}) \cong \pi_k((C_{p}(\mathbb{P}^{n})_K)_{et}) \cong \left\{ \begin{array}{ll} \mathbb{Z}_l, & \text{for } 0 < k \leq \min\{2d, 2(n-p)\} \\
0, & \text{all other } k \leq 2d, \end{array} \right.$

we obtain also the injectivity of $i_*$ for $k \leq 2d$ since a surjective homomorphism to from $\mathbb{Z}_l$ to $\mathbb{Z}_l$ is an isomorphism. This completes the proof of Theorem 13. □

As an application of Theorem 11 and the Algebraic Suspension Theorem [F], we get the homotopy groups of $|(C_{p,d}(\mathbb{P}^{n})_K)_{et}|$ up to $2d$.

**Corollary 14.** The first $2d + 1$ etale homotopy groups of $C_{p,d}(\mathbb{P}^{n})_K$ are given by the formula

$\pi_k((C_{p,d}(\mathbb{P}^{n})_K)_{et}) \cong \left\{ \begin{array}{ll} \mathbb{Z}_l, & \text{if } k \leq \min\{2d, 2(n-p)\} \text{ and even}, \\
0, & \text{all other } k \leq 2d. \end{array} \right.$
Proof. From the proof to Theorem 10, we have 
\[ \pi_k(\langle C_{p,d}^\ast(\mathbb{P}^n)_K \rangle_{et}) \to \pi_k(\langle C_p^\ast(\mathbb{P}^n)_K \rangle_{et}) \]
for \( k \leq 2d \). Recall the fact that \( |C_p^\ast(\mathbb{P}^n)_K \rangle_{et} \) is homotopy equivalent to the product
\[ K(\mathbb{Z}_l, 2) \times \cdots \times K(\mathbb{Z}_l, 2(n-p)) \] (cf. [F]), in particular,
\[ \pi_k(\langle C_p^\ast(\mathbb{P}^n)_K \rangle_{et}) \cong \begin{cases} 
\mathbb{Z}_l, & \text{if } k \leq 2(n-p) \text{ and even,} \\
0, & \text{otherwise.}
\end{cases} \]

\[ \square \]

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