COMPUTING CLASSICAL MODULAR FORMS FOR ARBITRARY
CONGRUENCE SUBGROUPS

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Abstract. In this paper, we prove the existence of an efficient algorithm for the com-
putation of $q$-expansions of modular forms of weight $k$ and level $\Gamma$, where $\Gamma \subseteq SL_2(\mathbb{Z})$ is
an arbitrary congruence subgroup. We also discuss some practical aspects and provide the
necessary theoretical background.

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1. INTRODUCTION

1.1. Motivation. The absolute Galois group of the rational numbers $G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is an
object of central importance in number theory. It is studied through its action on geometric
objects, among which elliptic curves and their torsion points have been playing a dominant
role. Let $E$ be an elliptic curve defined over $\mathbb{Q}$, and let $p$ be a prime. Then $G_{\mathbb{Q}}$ acts on its
$p$-torsion points $E[p]$. Therefore, we obtain a representation $\bar{\rho}_{E,p} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_p)$.

In [25], Serre proved that the index $[GL_2(\mathbb{F}_p) : \bar{\rho}_{E,p}(G_{\mathbb{Q}})]$ is bounded by a constant $C_E$
depending only on $E$. Serre went on to ask the following question, which remains open to
this day, after half a century.

Conjecture 1.1.1 (Serre’s uniformity problem over $\mathbb{Q}$, [25]). Is there a constant $C > 0$ such
that for any prime $p > C$ and any elliptic curve $E$ over $\mathbb{Q}$ without complex multiplication,
the mod $p$ Galois representation $\bar{\rho}_{E,p} : G_{\mathbb{Q}} \to PGL_2(\mathbb{F}_p)$ is surjective?

There has been much progress in working on the conjecture. The remaining and most
difficult case is to exclude the possibility that $\bar{\rho}_{E,p}$ has image contained in the normalizer
of a non-split Cartan subgroup of $GL_2(\mathbb{F}_p)$ for large $p$. These elliptic curves are classified by a
modular curve $X_{ns}^+(p)$. Equivalently, one has to show that for all large enough primes $p$, the only rational points of the modular curve $X_{ns}^+(p)$ are CM points.

Therefore, there is great interest in finding explicit equations for $X_{ns}^+(p)$ over $\mathbb{Q}$.

More generally, Mazur’s Program B [17] suggests, given an open subgroup $G \subseteq GL_2(\hat{\mathbb{Z}})$ to classify all elliptic curves $E$ such that the image of $\rho_E$ is contained in $G$. To this end, one would like to have explicit equations for the modular curve describing those, $X_G$, over $\mathbb{Q}$, in order to search for rational points.

The general method to find explicit equations for the modular curves $X_G$ is to compute $q$-expansions of a basis of cusp forms in $S_k(\Gamma)$ where $\Gamma \subseteq SL_2(\mathbb{Z})$ is the preimage of $G$ in $SL_2(\mathbb{Z})$, and find polynomial relations between them. The complexity of this computation is dominated by the complexity of computing the Hecke operators $\{T_n\}$ on the space of modular forms $S_k(\Gamma)$.

Thus, a main cornerstone of many efforts related to Serre’s uniformity conjecture and Mazur’s Program B is the computation of Hecke operators and $q$-expansions of modular forms with arbitrary level $\Gamma$.

1.2. Main Results. Let $G \subseteq GL_2(\mathbb{Z}/NZ)$ be a subgroup, and let $k \geq 2$ be an integer. Then $G$ pulls back to a congruence subgroup $\Gamma = \Gamma_G \subseteq SL_2(\mathbb{Z})$ of level $N$. This paper is concerned mainly with the computation of the space of modular forms of weight $k$ and level $\Gamma_G$, which we denote by $M_k(\Gamma_G)$.

Our main goal is to construct an explicit model for $M_k(\Gamma_G)$, which includes a finite dimensional vector space equipped with a subspace of cusp forms $S_k(\Gamma_G)$, and a family of Hecke operators $\{T_n\}$ acting on it.

After constructing such a model, we proceed to evaluate $q$-expansions of the eigenforms in this space, thus bridging the gap in the computational aspect of the efforts.

This is achieved most efficiently, under some mild assumptions, using Merel’s results [20]. Let us presently describe these assumptions.

We shall make use of $\eta := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Z})$, and denote by $\lambda_N : GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}/NZ)$ the natural reduction map.

**Definition 1.2.1.** We say that $G \subseteq GL_2(\mathbb{Z}/NZ)$ is of real type if $\lambda_N(\eta)G\lambda_N(\eta)^{-1} = G$. We say that a group $\Gamma \subseteq SL_2(\mathbb{Z})$ is of real type if $\eta\Gamma\eta^{-1} = \Gamma$.

**Remark 1.2.2.** The assumption that $G$ is of real type is essential for the Hecke operators to commute with the star involution (see Lemma 4.1.8). The slightly weaker assumption, that $\Gamma_G$ is of real type, is necessary for $Y_\Gamma := \Gamma \backslash \mathcal{H}$ to be defined over $\mathbb{Q}$.

In the description of complexities below, we denote by $I_G := [SL_2(\mathbb{Z}) : \Gamma_G]$ the index of $\Gamma_G$, and by $d := \dim S_k(\Gamma_G)$ the dimension of the space of cusp forms. We shall describe the complexity of our algorithms in terms of field operations.

**Remark 1.2.3.** Since the size of our matrix entries is bounded linearly by the level $N$, these are essentially logarithmic in $N$. However, when computing $q$-expansions, some of the arithmetic is done over number fields, and the complexity depends on the algorithms used to implement the arithmetic over those fields.
In what follows, we assume the existence of a function \text{CosetIndex}(x, R, \Gamma) that for an element \( x \in SL_2(\mathbb{Z}) \), a group \( \Gamma \) and a list of coset representatives \( R \) for \( \Gamma \setminus SL_2(\mathbb{Z}) \), returns the unique index \( i \), such that \( x \in \Gamma \cdot R[i] \). We will denote by \( C \) the complexity of \text{CosetIndex}.

\textbf{Remark 1.2.4.} The function \text{CosetIndex} could be implemented in general, by noting that \( \Gamma \setminus SL_2(\mathbb{Z}) \cong G_0 \setminus SL_2(\mathbb{Z}/N\mathbb{Z}) \), where \( G_0 := G \cap SL_2(\mathbb{Z}/N\mathbb{Z}) \), and using the Todd-Coxeter algorithm \cite{32}, with complexity of \( O(I_G^3) \) field operations. However, for certain groups \( \Gamma \), there are more efficient implementations. For example, if \( \Gamma = \Gamma_0(N) \), this can be done in \( O(1) \) field operations using the fact that \( \Gamma \setminus SL_2(\mathbb{Z}) \cong \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \).

Moreover, by allowing pre-computation of \( O(N^3) \) field operations and memory of \( O(N^3) \) field elements, we can use a table and access it using \( O(\log N) \) field operations.

\textbf{Theorem 1.2.5 (Corollary 4.5.11).} There exists an algorithm that given a group of real type \( G \subseteq GL_2(\mathbb{Z}/N\mathbb{Z}) \), a prime \( p \nmid N \) such that \( p \mod N \in \det(G) \) and an integer \( k \geq 2 \), computes the Hecke operator \( T_p \) on the space of modular forms \( S_k(\Gamma_G) \) using \( O(C \cdot d \cdot p \log p) \) field operations.

We also provide an algorithm for computing Hecke operators corresponding to arbitrary double cosets. For a matrix \( \alpha \in GL_2^+(\mathbb{Q}) \), we let \( D(\alpha) = \det(\alpha)/d_1(\alpha)^2 \in \mathbb{Z} \), with \( d_1(\alpha) \) the greatest common divisor of all the entries of \( \alpha \). Denote also \( I_{\alpha,\Gamma} = [\Gamma : \alpha^{-1}\Gamma\alpha] \).

\textbf{Theorem 1.2.6 (Corollary 4.3.3).} There exists an algorithm that given a group of real type \( G \subseteq GL_2(\mathbb{Z}/N\mathbb{Z}) \), an element \( \alpha \in GL_2^+(\mathbb{Q}) \) such that \( \eta^{-1}\alpha\eta \in \Gamma_G \alpha \Gamma_G \) and \( \gcd(D(\alpha), N) = 1 \), and an integer \( k \geq 2 \), computes the Hecke operator \( T_\alpha \), corresponding to the double coset \( \Gamma_G \alpha \Gamma_G \), on the space of modular forms \( S_k(\Gamma_G) \), using \( O(C \cdot d \cdot I_{\alpha,\Gamma} \log(N^2 \cdot D(\alpha))) \) field operations.

We describe a general algorithm to compute the Hecke operators, which works for the case \( \gcd(D(\alpha), N) > 1 \) as well. To describe its complexity, denote the complexity of membership testing in \( G \) by \text{In}.

\textbf{Theorem 1.2.7 (Theorem 4.2.10).} There exists an algorithm that given a group of real type \( G \subseteq GL_2(\mathbb{Z}/N\mathbb{Z}) \), an element \( \alpha \in GL_2^+(\mathbb{Q}) \) such that \( \eta^{-1}\alpha\eta \in \Gamma_G \alpha \Gamma_G \) and an integer \( k \geq 2 \), computes the Hecke operator \( T_\alpha \) corresponding to the double coset \( \Gamma_G \alpha \Gamma_G \), on the space of modular forms \( S_k(\Gamma_G) \), using \( O(d \cdot (C \cdot I_{\alpha,\Gamma} \log(N^2 \cdot D(\alpha)) + I_G^2 \cdot \text{In} )) \) field operations.

These algorithms allow us to compute \( q \)-expansions efficiently. For a complete \( q \)-expansion, we need also the values of the Hecke operators at primes dividing the level. Therefore, we need to assume an additional hypothesis.

\textbf{Definition 1.2.8.} Let \( G \subseteq GL_2(\mathbb{Z}/N\mathbb{Z}) \), and let \( p \mid N \) be a prime. Let \( \{\alpha_{p,i}\}_{i=1}^{r_p} \) be elements in \( GL_2^+(\mathbb{Q}) \) such that \( T_p \) is a linear combination of the \( T_{\alpha_{p,i}} \). We say that the Hecke operator \( T_p \) on \( S_k(\Gamma_G) \) is \textit{effectively computable} if there exists an algorithm to compute a set of such elements \( \{\alpha_{p,i}\} \) using \( O(C \cdot d \cdot (p \log p + k I_G \log(k I_G)) + d^3) \) field operations.

\textbf{Example 1.2.9.} (1) When \( \Gamma = \Gamma_0(N) \), \( T_p \) is effectively computable for all \( p \mid N \), as \( \alpha_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \) can be computed in \( O(1) \) field operations.
(2) When the order generated by $\Gamma$ contains no elements of determinant $p$, $T_p = 0$ is effectively computable.

We further denote by $\text{In}$ the complexity, in field operations, of membership test in $G$.

**Corollary 1.2.10** (Corollary [6.3.3]). There exists an algorithm that given a group of real type $G \subseteq GL_2(\mathbb{Z}/N\mathbb{Z})$ with surjective determinant such that for all $p \mid N$, $T_p$ is effectively computable, an integer $k \geq 2$, and a positive integer $L$, returns the $q$-expansions of a basis of eigenforms for $S_k(\Gamma_G)$ up to precision $q^L$ using

$$O(d \cdot (C \cdot (L \log L + N \log N) + NI_G^2 \cdot \text{In} + kI_G \log(kI_G)) + d^3)$$

field operations.

The method illustrated in this paper has been implemented in the MAGMA computational algebra system, by building on the existing package of modular symbols, implemented in MAGMA [7] by William Stein, with contributions by Steve Donnelly and Mark Watkins. The package is publicly available in the github repository [1].

As an application of our algorithm we recover the equation for the canonical embedding of $X_{S_4}(13)$ in $\mathbb{P}^2_{\mathbb{Q}}$, as was done in [3]. In the original paper, the entire section 4, consisting of 7 steps, was devoted to finding the cuspsforms, while we do it in a few seconds, writing several lines of code in Magma.

**Corollary 1.2.11** ([3, Theorem 1.8]). The modular curve $X_{S_4}(13)$ is a genus 3 curve whose canonical embedding in $\mathbb{P}^2_{\mathbb{Q}}$ has the model

$$4x^3y - 3x^2y^2 + 3xy^3 - x^3z + 16x^2yz - 11xy^2z +
+5y^3z + 3x^2z^2 + 9xyz^2 + y^2z^2 + xz^2 + 2y^3z^3 = 0.$$  

Another application recovers the main result of [5], namely explicit equations for the canonical embeddings of $X_{s}^+(13)$ and $X_{s}(13)$ in $\mathbb{P}^2_{\mathbb{Q}}$. Again, using our methods we are able to produce the result in a matter of seconds.

**Corollary 1.2.12** ([5]). The modular curves $X_{s}^+(13)$ and $X_{s}^+(13)$ are defined by the equation

$$(-y - z)x^3 + (2y^2 + zy)x^2 + (-y^3 + zy^2 - 2z^2y + z^3)x + (2z^2y^2 - 3z^3y) = 0.$$  

Similarly, we are also able to reproduce the results of [18] for $X_{s}^+(17)$, $X_{s}^+(19)$ and $X_{s}^+(23)$.

Another interesting problem that requires the computation of such Hecke operators is the question of decomposition of Jacobians of modular curves. The factors in these decompositions should have interesting arithmetic. We could perform the following using our code in 31 minutes.

**Corollary 1.2.13.** The Jacobian of the modular curve $X_{s}^+(97)$ decomposes as the direct sum of 13 Hecke irreducible subspaces, of dimensions $3, 4, 4, 6, 7, 7, 12, 14, 24, 24, 24, 56, 168$. In particular, it has no elliptic curve factor.

**Remark 1.2.14.** Any method of computing modular forms of weight $k = 1$ relies on computation of modular forms of higher integral weights (e.g. [3], [24]), so that our results apply equally well.
1.3. Related Literature. The theory of modular forms for $GL_2$ has been established for quite some time. However, the theory usually restricts to the Iwahori level subgroups, $\Gamma_0(N)$ and $\Gamma_1(N)$. There are a few exceptions: Shimura, in [26], explores a mild generalization, and much work has been devoted to some special cases, such as those groups induced by maximal subgroups of $GL_2(\mathbb{Z}/N\mathbb{Z})$. In this vein, computation has also been restricted to the subgroups $\Gamma_0(N)$ and $\Gamma_1(N)$, as in [29].

Nevertheless, many new applications arise ([3], [4], [13], [31]) in which description of the space of modular forms $M_k(\Gamma)$ is needed, when $\Gamma \subseteq SL_2(\mathbb{Z})$ is an arbitrary congruence subgroup.

1.4. Organization. The paper is structured as follows.

In Section 2, we introduce the definitions and the statement of the problem. In Section 3, we explain the construction of an explicit model for $M_k(\Gamma)$ using modular symbols, how to construct a boundary map $\partial$, and through it the subspace of cusp forms $S_k(\Gamma)$. This description of $S_k(\Gamma)$ was already known to Merel, see [20], but we present a general algorithm for the computation of $\partial$. So far, the literature has only discussed the computation of $\partial$ for Iwahori level subgroup. (see [9], [29], [34]).

Section 4 contains the core of this work, the computation of the Hecke operators. In subsection 4.1, we begin by defining the Hecke operators corresponding to double cosets. Then, in subsection 4.2 we present a general algorithm for computing them, thus proving theorem 1.2.7. We then present a variant of this algorithm which is more efficient, but applicable only to Hecke operators away from the level, thus yielding 1.2.6. In order to define the Hecke operators $T_n$ for integers $n$ that are coprime to the level of $\Gamma$, i.e. $(n, N) = 1$, in subsection 4.4 we need to view the modular curve adelically. We start from the adelic Hecke operators and show that they correspond to certain double coset operators that were already defined. In subsection 4.5 we use the techniques of Merel from [20] to prove theorem 1.2.5.

Section 5 describes the degeneracy maps, and the old and new subspaces, which will be useful when decomposing our space of cusp forms. Section 6 contains miscellaneous algorithms needed to compute the $q$-expansions of eigenforms given the Hecke operators. In section 7 we describe several applications of our result to contemporary research.

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2. Setup and Notation

In this section, we present the basic setup and introduce some notations that will be used throughout the paper.

2.1. Congruence Subgroups. Let $N$ be a positive integer, $G \subseteq GL_2(\mathbb{Z}/N\mathbb{Z})$ a subgroup, and $G_0 = G \cap SL_2(\mathbb{Z}/N\mathbb{Z})$. Let $\lambda_N : M_2(\mathbb{Z}) \to M_2(\mathbb{Z}/N\mathbb{Z})$ be the natural reduction map, and let $\lambda_{N,0} : SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})$ be its restriction to $SL_2(\mathbb{Z})$. 

Then $\Gamma_G := \lambda_{N,0}^{-1}(G_0) \subseteq SL_2(\mathbb{Z})$ contains $\Gamma(N) := \ker(\lambda_{N,0})$, hence it is a congruence subgroup. Moreover, every congruence subgroup arises this way. In general, we will denote by $\Gamma$ a congruence subgroup of $SL_2(\mathbb{Z})$.

We denote by $\mathcal{H}$ the complex upper half plane $\mathcal{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$. Then $\mathcal{H}$ admits a natural action of $GL_2^+(\mathbb{R})$ via Möbius transformations and we let $Y_\Gamma := \Gamma \backslash \mathcal{H}$ be the affine modular curve of level $\Gamma$.

**Definition 2.1.1.** Denote $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$. The group $GL_2^+(\mathbb{Q})$ also acts on $\mathbb{P}^1(\mathbb{Q})$ via Möbius transformations, and we let $X_\Gamma := \Gamma \backslash \mathcal{H}^*$ be the modular curve of level $\Gamma$. The points of $\Gamma \backslash \mathcal{P}^1(\mathbb{Q}) = X_\Gamma - Y_\Gamma$ are called the cusps of $X_\Gamma$.

**Definition 2.1.2.** We say that $\Gamma_G$ is the congruence subgroup induced by $G$.

Here are several useful examples.

**Example 2.1.3.**

1. Let $G = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mid a \in (\mathbb{Z}/N\mathbb{Z})^\times \right\}$. Then $\Gamma_G = \Gamma(N)$.

2. Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in (\mathbb{Z}/N\mathbb{Z})^\times, b \in \mathbb{Z}/N\mathbb{Z} \right\}$. Then $\Gamma_G = \Gamma_1(N)$.

3. Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, c \in (\mathbb{Z}/N\mathbb{Z})^\times, b \in \mathbb{Z}/N\mathbb{Z} \right\}$. Then $\Gamma_G = \Gamma_0(N)$.

Before presenting a non-trivial interesting example, we define the notion of a non-split Cartan subgroup.

**Definition 2.1.4.** Let $A$ be a finite free commutative $\mathbb{Z}/N\mathbb{Z}$-algebra of rank 2 with unit discriminant, such that for every prime $p$ dividing $N$, the $\mathbb{F}_p$-algebra $A/pA$ is isomorphic to $\mathbb{F}_p^2$. Then the unit group $A^\times$ acts on $A$ by multiplication, and a choice of basis for $A$ induces an embedding $A^\times \hookrightarrow GL_2(\mathbb{Z}/N\mathbb{Z})$. The image of $A^\times$ is called a non-split Cartan subgroup of $GL_2(\mathbb{Z}/N\mathbb{Z})$.

**Example 2.1.5.** Let $G$ be a non-split Cartan subgroup of $GL_2(\mathbb{Z}/N\mathbb{Z})$. Then $\Gamma_G = \Gamma_{ns}(N)$. Similarly, if $\mathfrak{N}(G)$ is its normalizer in $GL_2(\mathbb{Z}/N\mathbb{Z})$ then $\Gamma_{\mathfrak{N}(G)} = \Gamma^+_{ns}(N)$.

### 2.2. Modular Forms.

**Definition 2.2.1.** Let $k \geq 2$ be an integer. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$. Let $f : \mathcal{H} \to \mathbb{C}$, where $\mathcal{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$. We define for any $z \in \mathcal{H}$

$$f|_g(z) = (\det g)^{k-1} \cdot (cz + d)^{-k} f(gz)$$

Denote by $M_k(\Gamma)$ the space of holomorphic modular forms of weight $k$ with respect to $\Gamma$. (i.e. holomorphic functions $f : \mathcal{H} \to \mathbb{C}$ such that $f|_\gamma = f$ for all $\gamma \in \Gamma$ and $f$ is holomorphic at the cusps of $\Gamma \backslash \mathcal{H}$). Denote by $S_k(\Gamma)$ the subspace of cusp forms. (those vanishing at the cusps).

Since $\Gamma(N) \subseteq \Gamma$, we know that $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma$, hence $f(z + N) = f(z)$ for all $f \in M_k(\Gamma)$.

It follows that $f$ admits a Fourier expansion $f(z) = \sum_{n=0}^{\infty} a_n q^n$, with $a_n \in \mathbb{C}$ and $q := e^{2\pi i z}$.

This is now enough to state a computational problem:
Problem 2.2.3. Given a group \( G \subseteq GL_2(\mathbb{Z}/N\mathbb{Z}) \), an integer \( k \geq 2 \) and a positive integer \( L \), find the \( q \)-expansion of a basis for \( S_k(\Gamma_G) \) up to precision \( q^L \).

This problem has been answered extensively in the literature for \( \Gamma_G = \Gamma_1(N) \) and \( \Gamma_G = \Gamma_0(N) \) (see [28], [29], [14]). We will address the general case in what follows.

The solution to Problem 2.2.3 will be through the following steps:

1. Construct an explicit vector space representing \( S_k(\Gamma_G) \).
2. Compute the action of Hecke operators on this space.
3. Decompose the action into irreducible subspaces.
4. Write down the \( q \)-expansion from the systems of eigenvalues in the above decomposition.

Among these steps, step (3) and (4) follow from (2) by linear algebra techniques. Moreover, the Hecke algebra of \( S_k(\Gamma_G) \) is generated by the double coset Hecke operators \( \{T_\alpha\} \) (see Corollary 4.1.4). Therefore, we may concentrate our efforts on the following problem.

Problem 2.2.4. Given a group \( G \subseteq GL_2(\mathbb{Z}/N\mathbb{Z}) \), an integer \( k \geq 2 \) and an element \( \alpha \in GL_2^+(\mathbb{Q}) \) compute the matrix of the Hecke operator \( T_\alpha \) with respect to a basis of \( S_k(\Gamma_G) \).

We address this problem in full in section 4, and present several solutions - one general solution with no additional assumptions, and under some mild assumptions, we have a more efficient solution.

3. Explicit Computation of \( S_k(\Gamma) \)

First, we have to construct a model for the vector space \( S_k(\Gamma) \). We will do so through the means of modular symbols. For the results of this section, it suffices to assume that \( \Gamma \) is a finite index subgroup of \( SL_2(\mathbb{Z}) \). Therefore, in this section, we will suppress the notation of the group \( G \) in \( \Gamma_G \).

3.1. Modular Symbols. Let \( M_2 \) be the free abelian group generated by the expressions \( \{\alpha, \beta\} \), for \((\alpha, \beta) \in \mathbb{P}^1(\mathbb{Q})^2\), subject to the relations

\[
\{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\} = 0 \quad \forall \alpha, \beta, \gamma \in \mathbb{P}^1(\mathbb{Q}).
\]

and modulo any torsion. Let \( M_k := \text{Sym}^{k-2} \mathbb{Z}^2 \otimes M_2 \), where \( \text{Sym}^{k-2} \) is the representation of \( GL_2 \) of highest weight \( k - 2 \).

There is a natural left action of \( GL_2(\mathbb{Q}) \) on \( M_k \) by

\[
g(v \otimes \{\alpha, \beta\}) = gv \otimes \{g\alpha, g\beta\}
\]

Definition 3.1.1. Let \( M_k(\Gamma) = (M_k)_{\Gamma} := M_k/\langle \gamma x - x \rangle \) be the space of \( \Gamma \)-coinvariants, the space of modular symbols of weight \( k \) for \( \Gamma \) (over \( \mathbb{Z} \)). The space of modular symbols of weight \( k \) for \( \Gamma \) over a ring \( R \) is

\[
M_k(\Gamma; R) := M_k(\Gamma) \otimes_\mathbb{Z} R.
\]

The reason we are interested in the space of modular symbols \( M_k(\Gamma) \) is the following theorem proved by Manin:

Theorem 3.1.2. ([16] Theorem 1.9) The natural homomorphism \( \varphi : M_2(\Gamma) \to H_1(X_\Gamma, \text{cusps}, \mathbb{Z}) \), sending the symbol \( \{\alpha, \beta\} \) to the geodesic path in \( X_\Gamma \) between \( \Gamma\alpha \) and \( \Gamma\beta \), is an isomorphism.
This will give us the connection to modular forms. We now turn to the reason that the modular symbols are useful for computations.

### 3.2. Manin Symbols.

**Definition 3.2.1.** Let \( g \in SL_2(\mathbb{Z}) \). The **Manin symbol** \([v, g] \in \mathbb{M}_k(\Gamma)\) is defined as

\[
[v, g] := g(v \otimes \{0, \infty\})
\]

Let

\[
\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\]

We have a right action of \( SL_2(\mathbb{Z}) \) on Manin symbols as follows

\[
[v, g]h = [h^{-1}v, gh]
\]

The following theorem gives an explicit recipe for constructing a concrete realization of \( \mathbb{M}_k(\Gamma) \) as a free module of finite rank.

**Theorem 3.2.2.** ([20] Propositions 1,3) The Manin symbol \([v, g]\) depends only on the class \( \Gamma g \in \Gamma \setminus SL_2(\mathbb{Z}) \) and on \( v \). The Manin symbols \( \{[v, g]\}_{g \in SL_2(\mathbb{Z)}, v \in Sym^{k-2}\mathbb{Z}^2} \) generate \( \mathbb{M}_k(\Gamma) \). Furthermore, if \( x \) is a Manin symbol, then

\[
x + x\sigma = 0
\]

\[
x + x\tau + x\tau^2 = 0
\]

\[
x - xJ = 0
\]

Moreover, these are all the relations between Manin symbols, i.e. if \( B \) is a basis for \( Sym^{k-2}\mathbb{Z}^2 \) then

\[
\mathbb{M}_k(\Gamma) \cong \bigoplus_{b \in B, g \in \Gamma \setminus SL_2(\mathbb{Z})} \mathbb{Z} \cdot [b, g] / R
\]

where \( R \) are the above relations, and torsion.

This allows one to compute a model for \( \mathbb{M}_k(\Gamma) \) efficiently.

**Corollary 3.2.3.** There exists an algorithm that given a finite index subgroup \( \Gamma \subseteq SL_2(\mathbb{Z}) \) and an integer \( k \geq 2 \), computes a basis of \( \mathbb{M}_k(\Gamma) \) in \( O([SL_2(\mathbb{Z}) : \Gamma]) \) basic CosetIndex operations.

**Proof.** This was implemented by Stein for \( \Gamma = \Gamma_0(N) \) in [29], and the general case is similar. \( \square \)

**Definition 3.2.4.** \( Sym^{k-2}\mathbb{Z}^2 \) has a basis consisting of homogeneous elements, namely \( \{x^wy^{k-2-w}\}_{w=0}^{k-2} \). It follows that \( \mathbb{M}_k(\Gamma) \) has a basis consisting of elements of the form \([x^wy^{k-2-w}, g] \). From now on, we assume all given bases to be of this form, and say that the **weight** of a basis element \([x^wy^{k-2-w}, g] \) is \( w \).
3.3. Modular Symbols with Character. Suppose \( \Gamma \subseteq \Gamma' \subseteq SL_2(\mathbb{Z}) \) are finite index subgroups such that \( \Gamma \) is a normal subgroup of \( \Gamma' \) and let \( Q := \Gamma'/\Gamma \). Then \( Q \) acts on \( \mathcal{M}_k(\Gamma) \) via

\[
(\gamma'\Gamma) \cdot [v, x] = [v, \gamma'x]
\]

Let \( \varepsilon : Q \to \mathbb{Q}(\zeta)^\times \) be a character, where \( \zeta = \zeta_n \) is an \( n \)-th root of unity, and \( n \) is the order of \( \varepsilon \). Abusing notation, we will denote the induced character on \( \Gamma' \) by \( \varepsilon \) as well.

Let \( \mathcal{M}_k(\Gamma, \varepsilon) \) be the quotient of \( \mathcal{M}_k(\Gamma; \mathbb{Z}[\zeta]) \) by the relations

\[
\gamma' \cdot x - \varepsilon(\gamma') \cdot x = 0
\]

for all \( x \in \mathcal{M}_k(\Gamma; \mathbb{Z}[\zeta]), \gamma' \in Q \), and by any \( \mathbb{Z}\)-torsion.

Note that \( \mathcal{M}_k(\Gamma, \varepsilon) \) has a basis consisting solely of elements of the form \( [v, g] \) where \( g \in \Gamma' \setminus SL_2(\mathbb{Z}) \). Then if \( Q \) is abelian, this action factorizes to the isotypic components:

\[
\mathcal{M}_k(\Gamma) \cong \bigoplus_{\varepsilon \in \hat{Q}} \mathcal{M}_k(\Gamma, \varepsilon)
\]

This decomposition allows for faster computation by implementing \( \mathcal{M}_k(\Gamma) \) as the direct sum of these smaller spaces.

3.4. Cuspidal Modular Symbols. Let \( \mathbb{B}_2 \) be the abelian group generated by the elements \( \{\alpha\}, \alpha \in \mathbb{P}^1(\mathbb{Q}) \). Let \( \mathbb{B}_k = \text{Sym}^{k-2}\mathbb{Z}^2 \otimes \mathbb{B}_2 \). We have a left action of \( GL_2(\mathbb{Q}) \) on \( \mathbb{B}_k \) via

\[
g(v \otimes \{\alpha\}) = gv \otimes \{g\alpha\}
\]

Let \( \mathbb{B}_k(\Gamma) := (\mathbb{B}_k)_\Gamma \). Again we have a theorem that allows us to find a nice description of this free module of finite rank, by “boundary Manin symbols”.

**Theorem 3.4.1.** (\[20\] Propositions 4,5) Let \( \mathcal{R} \) be the equivalence relation on \( \Gamma\setminus\mathbb{Q}^2 \) given by

\[
[\Gamma(\lambda u, \lambda v)] \sim \text{sign}(\lambda)^k[\Gamma(u, v)]
\]

for any \( \lambda \in \mathbb{Q}^\times \). Let \( \mu : \mathbb{B}_k(\Gamma) \to \mathbb{Q}[(\Gamma\setminus\mathbb{Q}^2)/\mathcal{R}] \) be the natural map given by

\[
\mu\left(P \otimes \left\{ \frac{u}{v} \right\}\right) = P(u, v) \cdot [\Gamma(u, v)]
\]

where \( P \in \text{Sym}^{k-2}\mathbb{Z}^2 \) is considered as a homogeneous polynomial in two variables of degree \( k - 2 \) and \( u, v \) are coprime.

Then \( \mu \) is well defined and injective. Moreover,

\[
\mu \circ \partial([P, g]) = P(1, 0)[\Gamma g(1, 0)] - P(0, 1)[\Gamma g(0, 1)]
\]

**Definition 3.4.2.** The boundary map \( \partial : \mathcal{M}_k(\Gamma) \to \mathbb{B}_k(\Gamma) \) is the natural map extending linearly the map

\[
\partial(v \otimes \{\alpha, \beta\}) = v \otimes \alpha - v \otimes \beta
\]

The space \( S_k(\Gamma) = \ker \partial \) is called the space of cuspidal modular symbols. As in subsection 3.3, we can define \( \mathbb{B}_k(\Gamma, \varepsilon) \), a boundary map \( \partial : \mathcal{M}_k(\Gamma, \varepsilon) \to \mathbb{B}_k(\Gamma, \varepsilon) \) and its kernel will be denoted by \( S_k(\Gamma, \varepsilon) \).

In order to have a concrete realization of \( S_k(\Gamma, \varepsilon) \), it remains to efficiently compute the boundary map.
3.5. Efficient computation of the boundary map. In this subsection, we show how to compute the boundary map effectively. For finite index subgroups \( \Gamma \subseteq \Gamma' \subseteq SL_2(\mathbb{Z}) \) such that \( \Gamma \) is a normal subgroup of \( \Gamma' \), and a character \( \varepsilon : \Gamma'/\Gamma \to \mathbb{Q}(\zeta)^\times \), we denote \( c_\varepsilon(\Gamma) = \dim \mathbb{B}_2(\Gamma, \varepsilon) \). We will proceed to prove the following result.

**Theorem 3.5.1.** There exists an algorithm that given groups \( \Gamma \subseteq \Gamma' \subseteq SL_2(\mathbb{Z}) \) such that \( \Gamma \) is a normal subgroup of \( \Gamma' \), a character \( \varepsilon : \Gamma'/\Gamma \to \mathbb{Q}(\zeta)^\times \), and an integer \( k \geq 2 \), computes a basis for \( S_k(\Gamma, \varepsilon) \) in \( O([SL_2(\mathbb{Z}) : \Gamma'] \cdot c_\varepsilon(\Gamma) + [SL_2(\mathbb{Z}) : \Gamma]) \) basic CosetIndex operations.

The proof of Theorem 3.5.1 will occupy this entire subsection. We will obtain the basis by computing the kernel of the boundary map. Thus, given \( M(\Gamma, \varepsilon) \), we would like to compute the boundary map \( \mu \circ \partial : M_k(\Gamma, \varepsilon) \to \mathbb{B}_k(\Gamma, \varepsilon) \).

The idea follows Cremona and Stein [29 Prop. 2.2.3, Lemma 3.2, Prop. 8.13] in that we do not have to compute the space of cusps for \( \Gamma \) apriori. Instead, for every cusp in \( \mathbb{P}^1(\mathbb{Q}) \) that we encounter, we check whether it is equivalent to the ones previously encountered.

Following that route, we should be able to check whether a cusp is equivalent to 0 or to a previously encountered cusp. We will begin by describing the algorithms for testing cusp equivalence and cusp vanishing, and conclude by presenting the algorithm for computing the boundary map.

These algorithms are implemented in [29] and in [3] for \( \Gamma = \Gamma_0(N), \Gamma_1(N) \).

In order to treat the general case, we first prove a criterion for the equivalence.

**Proposition 3.5.2.** Let \( a, b \in \mathbb{P}^1(\mathbb{Q}) \). Let \( \Gamma \subseteq SL_2(\mathbb{Z}) \) be a subgroup. Let \( g_a, g_b \in SL_2(\mathbb{Z}) \) be such that \( g_a(\infty) = a \) and \( g_b(\infty) = b \). Let \( T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). Then \( \Gamma \backslash SL_2(\mathbb{Z}) \) admits a right action by \( T \), via multiplication, and \( \Gamma a = \Gamma b \) if and only if \( \Gamma g_a, \Gamma g_b \) lie in the same \( T \)-orbit of \( \Gamma \backslash SL_2(\mathbb{Z}) \).

**Proof.** If there is some \( \gamma \in \Gamma \) such that \( \gamma a = b \), then \( \gamma g_a(\infty) = g_b(\infty) \), and so

\[ g_b^{-1} \gamma g_a \in \text{Stab}_{SL_2(\mathbb{Z})}(\infty) = \langle T \rangle. \]

Therefore we get \( g_b^{-1} \gamma g_a = T^n \) for some integer \( n \in \mathbb{Z} \). Thus, there exists \( n \in \mathbb{Z} \) for which \( g_b T^n = g_a \). The converse follows similarly. \( \square \)

Thus, in order to test for equivalence, we have to compute the \( T \) action. This can be done apriori for all cosets, as we proceed to show.

**Definition 3.5.3.** Let \( \Gamma \subseteq SL_2(\mathbb{Z}) \) be a finite index subgroup, and let \( R \) be an array of coset representatives for \( \Gamma \backslash SL_2(\mathbb{Z}) \). We say that an array \( S \) is a \( T \)-orbit table of \( \Gamma \) with respect to \( R \) if \( S \) consists of pairs of integers \((o, e)\) such that:

1. The cosets represented by \( R[i] \) and \( R[j] \) lie in the same \( T \)-orbit iff \( S[i].o = S[j].o \).
2. In that case, \( R[j] \cdot T^{S[i].e-S[j].e} \cdot R[i]^{-1} \in \Gamma \).

**Lemma 3.5.4.** There exists an algorithm that given a finite index subgroup \( \Gamma \subseteq SL_2(\mathbb{Z}) \), and a list of coset representatives for \( \Gamma \backslash SL_2(\mathbb{Z}) \), \( R \), computes a \( T \)-orbit table of \( \Gamma \) with respect to \( R \) in \( O([SL_2(\mathbb{Z}) : \Gamma]) \) basic CosetIndex operations.
Proof. Apply Algorithm 3.5.5. It just goes over the cycles, hence has linear running time. □

Algorithm 3.5.5. OrbitTable(Γ, R). Constructing the T-orbit table.

Input:
• a finite index subgroup Γ ⊆ SL₂(Z).
• an array of coset representatives R for Γ\SL₂(Z).

Output: an array S, which is a T-orbit table of Γ with respect to R.
(1) T_map := [CosetIndex(R[i] · T, R, Γ) : i = 1, 2, ..., #R];
(2) idx := orbit := 1; exponent := 0;
(3) while idx ≤ #R do
   (a) S[idx] := (orbit, exponent);
   (b) idx := T_map[idx];
   (c) if initialized(S[idx]) then
      (i) orbit := orbit + 1;
      (ii) exponent := 0;
      (iii) while idx ≤ #R and initialized(S[idx]) do
         (A) idx := idx + 1;
   (d) else
      (i) exponent := exponent + 1;
(4) return S;

And then our algorithm for cusp equivalence testing is rather simple.

Algorithm 3.5.6. CuspEquiv(Γ, R, S, a, b). Test for cusp equivalence.

Input:
• a finite index subgroup Γ ⊊ SL₂(Z).
• an array of coset representatives R for Γ\SL₂(Z).
• an orbit table S (the output of Algorithm 3.5.5).
• a, b ∈ P¹(Q).

Output: If there exists γ ∈ Γ such that γa = b, then (true, γ), else (false, 1).
(1) Find g_a, g_b such that g_a(∞) = a, g_b(∞) = b.
(2) i_a := CosetIndex(g_a, R, Γ); i_b := CosetIndex(g_b, R, Γ);
(3) o_a := S[i_a].o; o_b := S[i_b].o;
(4) if o_a ≠ o_b then return (false, 1);
(5) return (true, g_b · T^{S[i_a].e−S[i_b].e} · g_a⁻¹)

Corollary 3.5.7. There exists an algorithm that given a finite index subgroup Γ ⊊ SL₂(Z), an array of coset representatives R for Γ\SL₂(Z), the T-orbit table of Γ with respect to R, and a, b ∈ P¹(Q), returns whether Γa = Γb, in which case it returns also an element γ ∈ Γ such that γa = b, in O(1) basic CosetIndex operations.

Proof. Apply Algorithm 3.5.6. By Proposition 3.5.2 it returns the correct answer. □

Finally, to test whether a cusp vanishes, we use the following proposition (an analog of [29, Prop. 8.16]). Recall that if Γ ⊊ Γ′ is a normal subgroup, and Q = Γ′/Γ, then Q acts on \( \mathbb{M}_k(\Gamma) \) as in (3.3.1).
**Proposition 3.5.8.** Let $\Gamma \subseteq \Gamma' \subseteq SL_2(\mathbb{Z})$ be finite index subgroups such that $\Gamma$ is a normal subgroup of $\Gamma'$. Let $Q := \Gamma'/\Gamma$, and let $\varepsilon : Q \rightarrow \mathbb{Q}(\zeta)^{\times}$ be a character. Suppose $a \in \mathbb{P}^1(Q)$ is a cusp. For $x \in \mathbb{P}^1(Q)$, write $[x]$ for the equivalence class of $x$ in $\Gamma \backslash \mathbb{P}^1(Q)$. Then $a$ vanishes modulo the relations

$$[\gamma' a] = \varepsilon(\gamma') \cdot [a] \quad \forall \gamma' \in \Gamma'$$

if and only if there exists $q \in Q$ with $\varepsilon(q) \neq 1$ such that $[q \cdot a] = [a]$.

**Proof.** First suppose that such a $q$ exists. Then

$$[a] = [q \cdot a] = \varepsilon(q) \cdot [a]$$

But $\varepsilon(q) \neq 1$, hence $[a] = 0$.

Conversely, suppose that $[a] = 0$. Because all relations are two-term relations and the $\Gamma$-relations identify $\Gamma$-orbits, there must exist $\alpha, \beta \in \Gamma'$ such that

$$[\alpha \cdot a] = [\beta \cdot a]$$

and $\varepsilon(\alpha) \neq \varepsilon(\beta)$. Indeed, if this did not occur, we could mod out by the $\varepsilon$ relations by writing each $[\alpha \cdot a]$ in terms of $[a]$ and there would be no further relations to kill $[a]$. Next observe that

$$[\beta^{-1} \alpha \cdot a] = \varepsilon(\beta^{-1}) \cdot [\alpha \cdot a] = \varepsilon(\beta)^{-1}[\beta \cdot a] = [a]$$

so that if $q \in Q$ is the image of $\beta^{-1} \alpha \in \Gamma'$, then $\varepsilon(q) = \varepsilon(\beta)^{-1} \varepsilon(\alpha) \neq 1$. $\square$

This gives rise to an algorithm for checking whether a cusp vanishes.

**Algorithm 3.5.9.** CuspVanishing($\Gamma, \varepsilon, s, R, S, a$). Test for cusp vanishing.

**Input :**
- a finite index subgroup $\Gamma \subset SL_2(\mathbb{Z})$.
- a character $\varepsilon : Q = \Gamma'/\Gamma \rightarrow \mathbb{Q}(\zeta)^{\times}$.
- a section $s : Q \rightarrow \Gamma'$ of the quotient map.
- an array of coset representatives $R$ for $\Gamma \backslash SL_2(\mathbb{Z})$.
- an orbit table $S$ (the output of Algorithm 3.5.5).
- $a \in \mathbb{P}^1(Q)$.

**Output :** If $[a] = 0$ in $\mathbb{B}_k(\Gamma, \varepsilon)$ return true, return false.

1. For each $q \in Q$ such that $\varepsilon(q) \neq 1$ do:
   a) equiv, $\gamma :=$ CuspEquiv($\Gamma, R, S, s(q) \cdot a, a$).
   b) If equiv then return true.
2. return false.

**Corollary 3.5.10.** There exists an algorithm that given finite index subgroup $\Gamma \subseteq \Gamma' \subseteq SL_2(\mathbb{Z})$ such that $\Gamma$ is normal in $\Gamma'$, a character $\varepsilon : \Gamma'/\Gamma \rightarrow \mathbb{Q}(\zeta)^{\times}$, an array of coset representatives $R$ for $\Gamma \backslash SL_2(\mathbb{Z})$, a $T$-orbit table of $\Gamma$ with respect to $R$, and a cusp $a \in \mathbb{P}^1(Q)$, returns whether $[a] = 0$ in $\mathbb{B}_k(\Gamma, \varepsilon)$ in $O([\Gamma' : \Gamma])$ basic CosetIndex operations.

**Proof.** Apply algorithm 3.5.9. The correctness follows from proposition 3.5.8 $\square$

Relying on 3.5.6 and 3.5.9, we may now formulate an algorithm to compute the boundary map $M_k(\Gamma, \varepsilon) \rightarrow \mathbb{B}_k(\Gamma, \varepsilon)$. 

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Algorithm 3.5.11. BoundaryMap(\(\Gamma, \Gamma', \varepsilon, k, B, R, R'\)). Computation of the boundary map.

Input:
- finite index subgroups \(\Gamma \subseteq \Gamma' \subseteq SL_2(\mathbb{Z})\) such that \(\Gamma\) is normal in \(\Gamma'\).
- a character \(\varepsilon : \Gamma'/\Gamma \rightarrow \mathbb{Q}(\zeta)^\times\).
- an integer \(k \geq 2\).
- \(B\), a basis of Manin symbols for \(M_k(\Gamma, \varepsilon)\).
- \(R\), a list of coset representatives for \(\Gamma \setminus SL_2(\mathbb{Z})\).
- \(R'\), a list of coset representatives for \(\Gamma' \setminus SL_2(\mathbb{Z})\).

Output:
- a matrix \(A\).
- a list of boundary Manin symbols \(C\).
  such that \(A\) represents the boundary map \(\mu \circ \partial\) with respect to the bases \(B, C\).

\(1\) \(C := \{\}\)
\(2\) \(A := [0 \text{ for } b \in B]\)
\(3\) \(S := \text{OrbitTable}(\Gamma, R)\)
\(4\) \(S' := \text{OrbitTable}(\Gamma', R')\)
\(5\) for each \(b = [P, g] \in B\):
  (a) \(w := \text{weight}(b)\)
  (b) if \(w = k - 2\):
    (i) \(a := g(1 : 0)\)
    (ii) if not \(\text{CuspVanishing}(\Gamma, \varepsilon, s, R, S, a)\):
      (A) \(i, \gamma := \text{CuspIndex}(\Gamma', R', S', a, C)\)
      (B) \(A[b] := A[b] + \varepsilon^{-1}(\gamma) \cdot e_i\)
  (c) if \(w = 0\):
    (i) \(a := g(0 : 1)\)
    (ii) if not \(\text{CuspVanishing}(\Gamma, \varepsilon, s, R, S, a)\):
      (A) \(i, \gamma := \text{CuspIndex}(\Gamma', R', S', a, C)\)
      (B) \(A[b] := A[b] - \varepsilon^{-1}(\gamma) \cdot e_i\)
\(6\) return \(A, C\)

Here \(\text{CuspIndex}\) is a function that given a finite index subgroup \(\Gamma \subseteq SL_2(\mathbb{Z})\), a list of coset representatives \(R\) for \(\Gamma \setminus SL_2(\mathbb{Z})\), a \(T\)-orbit table of \(\Gamma\) with respect to \(R\), a cusp \(a\) and a list of cusps \(C\), returns an index \(i\) such that \(a\) is equivalent to \(C[i]\) in \(\mathbb{B}_k(\Gamma)\) and an element \(\gamma \in \Gamma\) such that \(a = \gamma \cdot C[i]\). If \(a \notin C\), it appends \(a\) to \(C\). This function is obtained by using Algorithm 3.5.6 repeatedly.

We can now proceed to prove Theorem 3.5.1.

Proof. (of Theorem 3.5.1). The computation of \(S_k(\Gamma, \varepsilon)\) is performed by using Corollary 3.2.3 to construct a basis \(B\) of modular symbols for \(M_k(\Gamma, \varepsilon)\), and then applying Algorithm 3.5.11.

The computation of the \(T\)-orbit tables for both \(\Gamma\) and \(\Gamma'\) using Algorithm 3.5.5 costs \(O([SL_2(\mathbb{Z}) : \Gamma])\) basic \(\text{CosetIndex}\) operations by Lemma 3.5.4.
The function CuspIndex is implemented by applying Algorithm 3.5.6. The check whether a cusp vanishes is done by applying Algorithm 3.5.9. By Corollary 3.5.7 and Corollary 3.5.10, these algorithms return the required output in $O(1)$ basic CosetIndex operations.

Since there are $2 \cdot [SL_2(\mathbb{Z}) : \Gamma']$ Manin symbols of either maximal or minimal weight, and we have to check equivalence of at most each of these against each of the cusps in the list, which has at most $c_\varepsilon(\Gamma)$ elements in it, it follows that the running time for Algorithm 3.5.11 is $O([SL_2(\mathbb{Z}) : \Gamma'] \cdot c_\varepsilon(\Gamma) + [SL_2(\mathbb{Z}) : \Gamma])$. Thus, we have an algorithm for computing $S_k(\Gamma, \varepsilon)$ in $O([SL_2(\mathbb{Z}) : \Gamma'] \cdot c_\varepsilon(\Gamma) + [SL_2(\mathbb{Z}) : \Gamma])$ basic CosetIndex operations. □

3.6. Pairing Modular Symbols and Modular Forms. Let $\overline{S}_k(\Gamma) = \{ f \in S_k(\Gamma) \}$ be the space of antiholomorphic cusp forms. We then have a natural pairing

$$\langle \cdot, \cdot \rangle : (S_k(\Gamma) \oplus \overline{S}_k(\Gamma)) \times \mathbb{M}_k(\Gamma) \to \mathbb{C}$$

given by

$$\langle (f_1, f_2), P \otimes \{\alpha, \beta\} \rangle = \int_{\alpha} f_1(z) P(z, 1) dz + \int_{\alpha} f_2(z) P(\overline{z}, 1) d\overline{z}$$

The following theorem is the final ingredient needed to get a handle on the actual spaces of modular forms.

**Theorem 3.6.2.** ([27, Theorem 0.2], and [20, Theorem 3]) The pairing

$$\langle \cdot, \cdot \rangle : (S_k(\Gamma) \oplus \overline{S}_k(\Gamma)) \times S_k(\Gamma, \mathbb{C}) \to \mathbb{C}$$

is a nondegenerate pairing of complex vector spaces.

This is still not completely satisfactory, as we would like to separate the holomorphic forms from their antiholomorphic counterparts.

### 3.6.1. The Action of Complex Conjugation.

**Definition 3.6.3.** Let $\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Z})$. If $\eta$ normalizes $\Gamma$, i.e. $\eta\Gamma\eta = \Gamma$, we say that $\Gamma$ is of real type.

We then have the following result.

**Proposition 3.6.4.** (Merel, [20] Proposition 7) Let $\Gamma \subseteq SL_2(\mathbb{Z})$ be a subgroup of real type. Then there is a complex linear involution $\iota : S_k(\Gamma) \oplus \overline{S}_k(\Gamma) \to S_k(\Gamma) \oplus \overline{S}_k(\Gamma)$ which exchanges $S_k(\Gamma)$ and $\overline{S}_k(\Gamma)$, given by $\iota(f)(z) = f(-\overline{z})$. The involution $\iota^* : \mathbb{M}_k(\Gamma) \to \mathbb{M}_k(\Gamma)$ defined by

$$\iota^*(v \otimes \{\alpha, \beta\}) = -\eta v \otimes \{\eta\alpha, \eta\beta\}$$

is adjoint to $\iota$ with respect to the pairing (3.6.1). Moreover, $\iota^*$ acts on Manin symbols via

$$\iota^*([v, g]) = -[\eta v, \eta g\eta^{-1}]$$

We may now state the final result about the pairing, which explains how modular symbols and modular forms are related.

**Theorem 3.6.6.** (Merel, [20] Proposition 8) Let $S_k(\Gamma)^+ \subset S_k(\Gamma)$ be the $+1$ eigenspace for $\iota^*$ on $S_k(\Gamma)$. The pairing (3.6.1) induces a nondegenerate bilinear pairing

$$S_k(\Gamma)^+ \times S_k(\Gamma) \to \mathbb{C}$$
We denote by \( c(\Gamma) = \dim \mathbb{B}_2(\Gamma) \) the number of cusps of \( \Gamma \). Then it follows that

**Corollary 3.6.7.** There exists an algorithm that given a finite index subgroup \( \Gamma \subseteq SL_2(\mathbb{Z}) \) and an integer \( k \geq 2 \), computes a basis for \( S_k(\Gamma) \) in \( O((SL_2(\mathbb{Z}) : \Gamma) \cdot c(\Gamma)) \) basic CosetIndex operations.

**Proof.** In order to compute \( S_k(\Gamma)^+ \), it is necessary to replace \( \mathbb{B}_k(\Gamma) \) by its quotient modulo the additional relation \([(-u, v)] = [(u, v)]\) for all cusps \((u, v)\). Algorithm 3.5.11 can be modified, as in [29], to treat that case, yielding the result by Theorem 3.5.1. \( \square \)

4. Hecke Operators

Now that we have a realization of \( S_k(\Gamma G) \), we would like to be able to compute Hecke Operators on this space. First, let us recall a few basic facts about Hecke operators in general.

4.1. Hecke Operators on \( M_k(\Gamma) \).

**Definition 4.1.1.** Let \( \Gamma \subseteq SL_2(\mathbb{Z}) \) and suppose \( \Delta \subseteq GL_2^+(\mathbb{Q}) \) is a set such that \( \Gamma\Delta = \Delta \Gamma = \Delta \), and \( \Gamma \setminus \Delta \) is finite. Let \( R \) be a set of representatives for \( \Gamma \setminus \Delta \). Let

\[
T_\Delta : M_k(\Gamma) \to M_k(\Gamma)
\]

\[
T_\Delta(f) = \sum_{\alpha \in R} f|_\alpha
\]

where \( f|_\alpha \) is the usual right action of \( GL_2^+(\mathbb{Q}) \) given by \((2.2.2)\). The operator \( T_\Delta \) is called the Hecke operator associated to \( \Delta \).

It is a standard result that this operator is well-defined and independent of \( R \).

**Proposition 4.1.2.** ([11, Chapter 5.1]) The image of \( T_\Delta \) lies in \( M_k(\Gamma) \), and \( T_\Delta \) does not depend on the choice of \( R \).

**Definition 4.1.3.** The algebra generated by the \( \{T_\Delta\}_\Delta \) when \( \Delta \) runs over all subsets of \( GL_2^+(\mathbb{Q}) \) such that \( \Gamma\Delta = \Delta \Gamma = \Delta \) is the Hecke algebra of \( M_k(\Gamma) \).

For \( \alpha \in GL_2^+(\mathbb{Q}) \), we denote \( T_\alpha = T_{\Gamma \alpha \Gamma} \).

**Corollary 4.1.4.** The Hecke algebra of \( M_k(\Gamma) \) is generated by the operators \( \{T_\alpha\}_{\alpha \in GL_2^+(\mathbb{Q})} \).

**Proof.** It is a consequence of the definition that \( \Delta \) must be a union of double cosets \( \Gamma \alpha \Gamma \), where \( \alpha \in GL_2^+(\mathbb{Q}) \). In particular, the Hecke operators are linearly spanned by the \( \{T_\alpha\} \). \( \square \)

**Example 4.1.5.** (1) If \( \Gamma = \Gamma_1(\mathbb{N}) \), one usually considers for any \( n \in \mathbb{Z} \), the set

\[
\Delta_n^1 := \left\{ g \in M_2(\mathbb{Z}) \mid \det(g) = n \text{ and } g \equiv \begin{pmatrix} 1 & \ast \\ 0 & n \end{pmatrix} \mod N \right\}
\]

One usually denotes \( T_n = T_{\Delta_n^1} \). We note that \( \Delta_n^1 = \Gamma \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \Gamma \), so that \( T_n = T_\alpha \), with \( \alpha = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \).
(2) Let \( \Gamma = \Gamma(N) \). We may take for \( n \in \mathbb{Z} \), the set
\[
\Delta_n := \left\{ g \in M_2(\mathbb{Z}) \mid \det(g) = n \text{ and } g \equiv \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \mod N \right\}
\]
Note that again we are taking \( \Delta_n = \Gamma \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \Gamma \).

As our implementation of modular forms is by using the dual space of modular symbols, we need the dual notion.

**Definition 4.1.6.** Let \( \Delta \subseteq GL_2(\mathbb{Q}) \) be such that \( \Gamma \Delta = \Delta \Gamma \) and such that \( \Gamma \setminus \Delta \) is finite. Let \( R \) be a set of representatives of \( \Gamma \setminus \Delta \). Let
\[
T^\vee_\Delta : \mathbb{M}_k(\Gamma) \to \mathbb{M}_k(\Gamma)
\]
\[
v \otimes \{\alpha, \beta\} \mapsto \sum_{\delta \in R} (\delta v) \otimes \{\delta \alpha, \delta \beta\}
\]
Again, this map does not depend on the choice of \( R \). We then have the following result.

**Proposition 4.1.7.** (Merel, [20, Proposition 10]) The operators \( T_\Delta \) and \( T^\vee_\Delta \) are adjoint with respect to the bilinear pairing \( \langle \cdot, \cdot \rangle \) defined in (3.6.1).

Therefore, by Theorem 3.6.2, in order to compute the restriction of the Hecke operators \( T_\Delta \) to \( S_k(\Gamma) \oplus \overline{S}_k(\Gamma) \), it will suffice to compute the Hecke operators \( T^\vee_\Delta \) on \( S_k(\Gamma) \). Note that in order to compute the restriction to \( S_k(\Gamma) \) using Theorem 3.6.6, we would need \( T_\Delta \) to commute with \( \iota \).

**Lemma 4.1.8.** Let \( \Gamma \subseteq SL_2(\mathbb{Z}) \) be a subgroup of real type. Let \( \alpha \in GL_2(\mathbb{Q}) \) be such that \( \eta^{-1} \alpha \eta \in \Gamma \alpha \Gamma \). Then \( T^\vee_\alpha \circ \iota^\vee = \iota^\vee \circ T^\vee_\alpha \).

**Proof.** Let \( R \) be a set of representatives for \( \Gamma \setminus \Gamma \alpha \Gamma \). For each \( \delta \in \Gamma \alpha \Gamma \), we have by the assumptions on \( \Gamma \) and \( \alpha \) that
\[
\eta^{-1} \delta \eta \in \eta^{-1} \Gamma \alpha \eta \eta^{-1} \alpha \eta (\eta^{-1} \Gamma \eta) = \Gamma \eta^{-1} \alpha \eta \Gamma = \Gamma \alpha \Gamma
\]
Therefore, we see that
\[
\Gamma \alpha \Gamma = \bigsqcup_{\delta \in R} \Gamma \delta = \eta^{-1} \left( \bigsqcup_{\delta \in R} \Gamma \delta \right) \eta = \bigsqcup_{\delta \in R} \eta^{-1} \Gamma \delta \eta = \bigsqcup_{\delta \in R} (\eta^{-1} \Gamma \eta) \eta^{-1} \delta \eta = \bigsqcup_{\delta \in R} \eta^{-1} \delta \eta
\]
so that \( R' = \{\eta^{-1} \delta \eta \mid \delta \in R\} \) is also a set of representatives for \( \Gamma \setminus \Gamma \alpha \Gamma \). Now
\[
T^\vee_\alpha (\iota^\vee (v \otimes \{\alpha, \beta\})) = T^\vee_\alpha (\eta v \otimes \{\eta \alpha, \eta \beta\}) =
\]
\[
= \sum_{\delta \in R} (-\eta \delta \eta v \otimes \{\delta \alpha, \delta \beta\}) =
\]
\[
= \sum_{\delta \in R} (-\eta (\eta^{-1} \delta \eta) v \otimes \{\eta (\eta^{-1} \delta \eta) \alpha, \eta (\eta^{-1} \delta \eta) \beta\}) =
\]
\[
= \sum_{\delta' \in R'} (-\eta \delta' \eta v \otimes \{\eta \delta' \alpha, \eta \delta' \beta\}) = \iota^\vee (T^\vee_\alpha (v \otimes \{\alpha, \beta\})). \quad \square
\]
Corollary 4.1.9. Let $G \subseteq GL_2(\mathbb{Z}/\mathbb{N}\mathbb{Z})$ be a subgroup of real type. Let $\alpha \in M_2(\mathbb{Z})$ be such that $\lambda_N(\alpha) \in G$. Then $T^\vee_\alpha \circ \iota^\vee = \iota^\vee \circ T^\vee_\alpha$.

Proof. Since $G$ is of real type, it follows that $\Gamma_G$ is of real type. Also, as $\lambda_N(\alpha) \in G$, we see that $\lambda_N(\eta^{-1} \alpha \eta) \in \lambda_N(\eta)^{-1} G \lambda_N(\eta) = G$.

Moreover, as $\eta$ normalizes $SL_2(\mathbb{Z})$, we see that $SL_2(\mathbb{Z}) \cdot \alpha \cdot SL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) \cdot \eta^{-1} \alpha \eta \cdot SL_2(\mathbb{Z})$.

From [26, Lemma 3.29] we deduce that

$$\Gamma \alpha \Gamma = \{ g \in M_2(\mathbb{Z}) \mid SL_2(\mathbb{Z}) \cdot \alpha \cdot SL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) \cdot g \cdot SL_2(\mathbb{Z}) \text{ and } \lambda_N(g) \in G \}.$$

It then follows that $\eta^{-1} \alpha \eta \in \Gamma \alpha \Gamma$. The result follows from Lemma 4.1.8. \hfill \square

4.2. Naive Computation of $T^\vee_\alpha$. We will now present a naive algorithm to compute the Hecke operator $T^\vee_\alpha$ for an arbitrary $\alpha \in GL_2^+(\mathbb{Q})$.

Algorithm 4.2.1. HeckeOperator($\alpha$, $x$, $B$). Hecke Operator $T^\vee_\alpha$ on $\mathbb{M}_k(\Gamma)$.

**Input :**
- $\alpha \in GL_2^+(\mathbb{Q})$.
- an element $x = v \otimes \{\alpha, \beta\} \in \mathbb{M}_k(\Gamma)$.
- a basis $B$ for the space of modular symbols $\mathbb{M}_k(\Gamma)$.

**Output :** a vector representing $T^\vee_\alpha(x)$ w.r.t. the basis $B$.

1. Let $H := \Gamma \cap \alpha^{-1} \Gamma \alpha$.
2. Let $S$ be a set of representatives for $H \backslash \Gamma$.
3. Let $R := \alpha \cdot S = \{ \alpha \cdot x : x \in S \}$.
4. Return $\sum_{r \in R} [\pi(r \cdot x)]_B$, where $[\pi(r \cdot x)]_B$ is the vector representing the modular symbol $\Gamma \cdot (rv) \otimes \{ r\alpha, r\beta \}$ w.r.t. $B$.

Remark 4.2.2. Even though this is unclear at first sight, the functions $(\Gamma_1, \Gamma_2) \mapsto \Gamma_1 \cap \Gamma_2$ and $(\Gamma, x) \mapsto x^{-1} \Gamma x$ also require a nontrivial implementation.

Note also that $\Gamma \subseteq SL_2(\mathbb{Z})$ is defined using $G \subseteq GL_2(\mathbb{Z}/\mathbb{N}\mathbb{Z})$. Therefore, if $\alpha \in GL_2^+(\mathbb{Q})$ is such that $\alpha^{-1} \Gamma \alpha \not\subseteq SL_2(\mathbb{Z})$, a straightforward computation could be expensive (either find $N$ such that $N \cdot \alpha^{-1} \Gamma \alpha \subseteq SL_2(\mathbb{Z})$ and change computations to fit, or work in $GL_2^+(\mathbb{Q})$ with generators - that requires finding the generators of $\Gamma$). Moreover, as $GL_2^+(\mathbb{Q})$ is not finitely generated, we do not have ready made tools for computing the intersection $\Gamma \cap \alpha^{-1} \Gamma \alpha$.

This has brought us to consider the following alternate path:

**Strategy for computation of Algorithm 4.2.1.**

(a) Implement the function $(\Gamma, x) \mapsto x^{-1} \Gamma x$ for $x \in GL_2^+(\mathbb{Q})$ when $x^{-1} \Gamma x \subseteq SL_2(\mathbb{Z})$.

(b) Implement an intersection function $(\Gamma_1, \Gamma_2) \mapsto \Gamma_1 \cap \Gamma_2$ for $\Gamma_1, \Gamma_2 \subseteq SL_2(\mathbb{Z})$.

(c) Write the set of representatives $\Gamma \backslash \Gamma_0 \Gamma = \alpha \cdot \langle ((\alpha^{-1} \Gamma_0) \cap \Gamma) \backslash \Gamma \rangle$ as a sequence of operations of type (a), (b).

Here is an implementation of the intersection in step 1, using (a) and (b).

Algorithm 4.2.3. ConjInter($\Gamma$, $\alpha$). Conjugation and Intersection.
Input:

- \( \Gamma \subseteq SL_2(\mathbb{Z}) \) a congruence subgroup.
- \( \alpha \in GL_2^+(\mathbb{Q}) \) such that \( \alpha^{-1}\Gamma\alpha \subseteq SL_2(\mathbb{Z}) \).

Output: \( \Gamma \cap \alpha^{-1}\Gamma\alpha \).

(1) Find matrices \( x, \gamma \in SL_2(\mathbb{Z}) \) such that \( x \cdot \alpha^{-1} \cdot \gamma \) is of the form \( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \) with \( a^{-1}b \in \mathbb{Z} \). (Smith Normal Form)
(2) Let \( H := \Gamma \cap (\alpha^{-1} \cdot (\Gamma \cap (\gamma \cdot \Gamma^0(a^{-1}b) \cdot \gamma^{-1})) \cdot \alpha) \).

First, let us prove that Algorithm 4.2.3 indeed returns the correct answer.

**Proposition 4.2.4.** Algorithm 4.2.3 returns the group \( \Gamma \cap \alpha^{-1}\Gamma\alpha \).

**Proof.** First, note that if \( a^{-1}b \in \mathbb{Z} \), then
\[
\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{-1} \cdot SL_2(\mathbb{Z}) \cdot \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \cap SL_2(\mathbb{Z}) = \Gamma^0(a^{-1}b)
\]

Therefore
\[
\gamma^{-1} \cdot \alpha \cdot x^{-1} \cdot SL_2(\mathbb{Z}) \cdot x \cdot \alpha^{-1} \cdot \gamma \cap SL_2(\mathbb{Z}) = \Gamma^0(a^{-1}b)
\]

But \( x \in SL_2(\mathbb{Z}) \), so that \( x^{-1}SL_2(\mathbb{Z})x = SL_2(\mathbb{Z}) \). It follows that
\[
\alpha \cdot SL_2(\mathbb{Z}) \cdot \alpha^{-1} \cap \gamma \cdot SL_2(\mathbb{Z}) \cdot \gamma^{-1} = \gamma \cdot \Gamma^0(a^{-1}b) \cdot \gamma^{-1}
\]

But \( \gamma \in SL_2(\mathbb{Z}) \), so \( \gamma \cdot SL_2(\mathbb{Z}) \cdot \gamma^{-1} = SL_2(\mathbb{Z}) \). It follows that
\[
\alpha \cdot SL_2(\mathbb{Z}) \cdot \alpha^{-1} \cap SL_2(\mathbb{Z}) = \gamma \cdot \Gamma^0(a^{-1}b) \cdot \gamma^{-1}
\]

Therefore
\[
\alpha \cdot SL_2(\mathbb{Z}) \cdot \alpha^{-1} \cap \Gamma = \alpha \cdot SL_2(\mathbb{Z}) \cdot \alpha^{-1} \cap SL_2(\mathbb{Z}) \cap \Gamma = \Gamma \cap \gamma \cdot \Gamma^0(a^{-1}b) \cdot \gamma^{-1}
\]

so that
\[
(4.2.5) \quad SL_2(\mathbb{Z}) \cap \alpha^{-1}\Gamma\alpha = \alpha^{-1} \cdot (\Gamma \cap \gamma \cdot \Gamma^0(a^{-1}b) \cdot \gamma^{-1}) \cdot \alpha
\]

Finally, intersecting with \( \Gamma \), one obtains
\[
\alpha^{-1}\Gamma\alpha \cap \Gamma = \Gamma \cap (\alpha^{-1} \cdot (\Gamma \cap \gamma \cdot \Gamma^0(a^{-1}b) \cdot \gamma^{-1}) \cdot \alpha) = H.
\]

In order to be able to use the strategy 4.2 we should also show that the conjugations we are computing are in \( SL_2(\mathbb{Z}) \).

**Lemma 4.2.6.** \( \gamma \cdot \Gamma^0(a^{-1}b) \cdot \gamma^{-1} \subseteq SL_2(\mathbb{Z}) \) and \( \alpha^{-1} \cdot (\Gamma \cap \gamma \cdot \Gamma^0(a^{-1}b) \cdot \gamma^{-1}) \cdot \alpha \subseteq SL_2(\mathbb{Z}) \).

**Proof.** The first is trivial, as \( \gamma \in SL_2(\mathbb{Z}) \). The second follows from (4.2.5). \( \square \)

**Remark 4.2.7.** Algorithm 4.2.1 can be used to compute \([T^\gamma_\alpha]_B\) by applying it to each of the basis vectors.

It remains to show how to compute the conjugation in (a), and the intersection in (b). For the conjugation in (a), we compute the generators of \( \Gamma \) using Farey symbols (see []). We then conjugate them to obtain the generators of the conjugated subgroup, and can compute the level and the reduction of this conjugated subgroup. For the intersection, we present the following algorithm.
Proposition 4.2.9. Algorithm 4.2.8 returns a group $K$ such that $\Gamma_K = \Gamma_G \cap \Gamma_H$.

Proof. First note that for any $p \nmid N$, and any $e$, $\lambda_{p^e}(\Gamma_G \cap \Gamma_H) = GL_2(\mathbb{Z}/p^e \mathbb{Z})$. Then note that

$$\lambda_N(\Gamma_G \cap \Gamma_H) = \{x \in GL_2(\mathbb{Z}/N\mathbb{Z}) \mid \lambda_{N_G}(x) \in G, \lambda_{N_H}(x) \in H\}$$

We will show that $K$ is the entire subset above, hence the result. Let $x \in \lambda_N(\Gamma_G \cap \Gamma_H)$. Then $\lambda_n(x) \in K$. Then $\lambda_n(x) = \prod z_i^{e_i}$ for some $\{z_i\} \subseteq \text{Generators}(K)$. Write $a_i := a_{z_i}$, and consider $a = \prod a_i^{e_i}$. It satisfies $\lambda_n(a) = \lambda_n(x)$, hence $\lambda_n(a^{-1}x) = 1$.

In particular, $\lambda_{N_G}(a^{-1}x) \in G_d$ and $\lambda_{N_H}(a^{-1}x) \in H_d$. Then there are $\{x_j\} \subseteq \text{Generators}(G_d)$ and $\{y_k\} \subseteq \text{Generators}(H_d)$ such that $\lambda_{N_G}(a^{-1}x) = \prod x_j^{f_j}$, $\lambda_{N_H}(a^{-1}x) = \prod y_k^{g_k}$. Let $\{b_j\}$ be the lift of the $\{x_j\}$ in $A_G$ and $\{c_k\}$ the lifts of the $\{y_k\}$ in $A_H$. Then

$$\lambda_{N_G} \left( \prod b_j^{f_j} \cdot \prod c_k^{g_k} \right) = \lambda_{N_G}(a^{-1}x)$$

and

$$\lambda_{N_H} \left( \prod b_j^{f_j} \cdot \prod c_k^{g_k} \right) = \lambda_{N_H}(a^{-1}x)$$

hence

$$x = \prod a_i^{e_i} \cdot \prod b_j^{f_j} \cdot \prod c_k^{g_k} \in K. \qed$$

In order to measure the complexity of Algorithm 4.2.8, in addition to the CosetIndex operation, we use In to denote group membership test in $G$. We will also need some notation, which we now introduce.

For $\alpha \in GL_2(\mathbb{Q})$, let $d_1(\alpha) \in \mathbb{Q}_{>0}$ be maximal such that $\alpha \in d_1(\alpha) \cdot M_2(\mathbb{Z})$.

Let $D(\alpha) := \det(\alpha)/d_1(\alpha)^2 \in \mathbb{Z}$, and let $I_{\alpha, \Gamma} := [\Gamma : \alpha^{-1}\Gamma \alpha \cap \Gamma]$.

This leads us to the main result of this subsection.
Theorem 4.2.10. There exists an algorithm that given a congruence subgroup of real type \( \Gamma \subseteq SL_2(\mathbb{Z}) \) of level \( N \), an element \( \alpha \in GL_2^+(\mathbb{Q}) \) such that \( \eta^{-1}\alpha \eta \in \Gamma \alpha \Gamma \) and an integer \( k \geq 2 \), computes the Hecke operator \( T_\alpha \) corresponding to the double coset \( \Gamma \alpha \Gamma \), on the space of cusp forms \( S_k(\Gamma) \), in complexity

\[
O(C \cdot I_{\alpha,\Gamma} \log(N^2 \cdot D(\alpha)) + [SL_2(\mathbb{Z}) : \Gamma]^2 \cdot \text{In}).
\]

Proof. We apply Algorithm \texttt{4.2.1} using Algorithm \texttt{4.2.3} to perform step (1). The conjugation function is computed by computing a set of generators for \( \Gamma \), using Farey symbols (see [15, Theorem 6.1]), and conjugating them. Intersection is computed using Algorithm \texttt{4.2.8}. This computes the dual Hecke operator \( T_\alpha^\vee \) on \( M_k(\Gamma) \). Since \( \Gamma \) is of real type, and \( \eta^{-1}\alpha \eta \in \Gamma \alpha \Gamma \), by Corollary \texttt{4.1.9} it follows that \( T_\alpha^\vee \) commutes with \( \iota^\vee \). This shows that it induces an operator on \( S_k(\Gamma)^+ \). Therefore, by Theorem \texttt{3.6.6} and Proposition \texttt{4.1.7} we obtain \( T_\alpha \).

The cost of step (1) is dominated by computing the Farey symbols, which yields the cost \( O([SL_2(\mathbb{Z}) : \Gamma]^2) \).

In step (4), for each element in the basis \([v, g] \in B\), we compute \( g\{0, \infty\} \), to get the modular symbol \( v \otimes \{a, b\} \). Then, for every \( r \in R \) apply \( r \) to get \( rv \otimes \{ra, rb\} \), and then use continued fractions to convert back to Manin symbols. This is done \( I_{\alpha,\Gamma} \) times.

Since \( g \in \Gamma \backslash SL_2(\mathbb{Z}) \), its entries are bounded by \( N \). Further, we can choose the representatives \( r \in R \) up to a multiplication by a scalar, hence we can bound its entries by \( N \cdot D(\alpha) \).

It follows that the entries of \( r \cdot g \) are bounded by \( N^2 \cdot \det \alpha \), and so computing continued fraction expansion (Euclid’s algorithm) has \( O(\log(N^2 \cdot D(\alpha))) \) steps, for each of which one has to perform a CosetIndex operation to find the corresponding element in the vector space.

Thus, the algorithm has complexity

\[
O(C \cdot I_{\alpha,\Gamma} \log(N^2 \cdot D(\alpha)) + [SL_2(\mathbb{Z}) : \Gamma]^2 \cdot \text{In}). \tag*{\Box}
\]

4.3. Faster Implementation of the Hecke Operator \( T_\alpha^\vee \). Algorithm \texttt{4.2.1} is still quite slow in general, since the conjugation function might be very slow, as it depends quadratically on the index. However, under some simplifying assumptions we may use the following algorithm for the conjugation of an induced congruence subgroups.

Here, we assume that \((D(\alpha), N) = 1\).

Algorithm 4.3.1. \texttt{ConjInd}(G, \alpha). Conjugation of an induced congruence subgroup.

Input :

- \( G \subseteq GL_2(\mathbb{Z}/N\mathbb{Z}) \).
- \( \alpha \in GL_2^+(\mathbb{Q}) \) such that \((D(\alpha), N) = 1\). Write \( n := D(\alpha) \).

Output : a group \( H \subseteq GL_2(\mathbb{Z}/nN\mathbb{Z}) \) such that \( \Gamma_H = \alpha^{-1}\Gamma G \alpha \cap SL_2(\mathbb{Z}) \).

1. Find \( x, y \in SL_2(\mathbb{Z}) \) such that \( x \cdot \alpha \cdot y = \begin{pmatrix} d_1(\alpha) & 0 \\ 0 & n \cdot d_1(\alpha) \end{pmatrix} \).
2. Let \( H_N := \lambda_N(d_1(\alpha)^{-1} \alpha)^{-1} \cdot G \cdot \lambda_N(d_1(\alpha)^{-1} \alpha) \subseteq GL_2(\mathbb{Z}/N\mathbb{Z}) \).
3. Let \( H_n := \lambda_n(y) \cdot \lambda_n(\Gamma^0(b/a)) \cdot \lambda_n(y)^{-1} \subseteq GL_2(\mathbb{Z}/n\mathbb{Z}) \).
4. Let \( A_N \subseteq GL_2(\mathbb{Z}/nN\mathbb{Z}) \) be a set of matrices \( a_i \) lifting \( \text{Generators}(H_N) \) such that \( a_i \equiv 1 \mod n \). (Chinese Remainder Theorem)
(5) Let $B_n \subseteq GL_2(\mathbb{Z}/nN\mathbb{Z})$ be a set of matrices $b_j$ lifting $\text{Generators}(H_n)$ such that $b_j \equiv 1 \text{ mod } N$. (Chinese Remainder Theorem)

(6) Return $H := \langle A_n \cup B_n \rangle$.

**Proposition 4.3.2.** Algorithm 4.3.1 returns a group $H$ such that $\Gamma_H = \alpha^{-1}\Gamma_G\alpha \cap SL_2(\mathbb{Z})$.

**Proof.** First note that replacing $\alpha$ by $d_1(\alpha)^{-1}\alpha$, we may assume $\alpha \in M_2(\mathbb{Z})$ and $d_1(\alpha) = 1$. Let $\Gamma' := \alpha^{-1}\Gamma_G\alpha \cap SL_2(\mathbb{Z})$. If $p \mid n \cdot N$, then $\lambda_{p^e}(\Gamma') = GL_2(\mathbb{Z}/p^e\mathbb{Z})$ (for all $e$). We also have

$$\lambda_N(\Gamma') = \lambda_N(\alpha)^{-1} \cdot G \cdot \lambda_N(\alpha)$$

and if $x \cdot \alpha \cdot y = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$, then

$$\lambda_n(\Gamma') = \lambda_n(y) \cdot \lambda_n(\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix})^{-1} \cdot \lambda_n(x^{-1}\Gamma_Gx) \cdot \lambda_n(\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}) \cdot \lambda_n(y)^{-1}$$

But $\lambda_n(x^{-1}\Gamma_Gx) = GL_2(\mathbb{Z}/n\mathbb{Z})$, and, and so we see that $\Gamma' = \{g \in SL_2(\mathbb{Z}) \mid \lambda_n(g) \in H_n, \lambda_N(g) \in H_N\}$, with $H_n$ and $H_N$ as in Algorithm 4.3.1. Since $(n, N) = 1$, we claim that $\Gamma' = \Gamma_H$, or equivalently that $\lambda_{nN}(\Gamma') = H$.

Indeed, if $g \in \lambda_{nN}(\Gamma')$, then $\lambda_n(g) \in H_n$ and $\lambda_N(g) \in H_N$. Therefore, there exist $\{x_i\} \subseteq \text{Generators}(H_n)$ and $\{y_j\} \subseteq \text{Generators}(H_N)$ such that $\lambda_n(g) = \prod x_i^{e_i}$ and $\lambda_N(g) = \prod y_j^{f_j}$. Let $\{a_i\}$ be the matrices in $A_n$ lifting the $\{x_i\}$ and $\{b_j\}$ the matrices in $B_N$ lifting the $\{y_j\}$. Then by the Chinese Remainder Theorem, we have $g = \prod a_i^{e_i} \cdot \prod b_j^{f_j}$. 

Next, we would like to understand the running time of Algorithm 4.2.1 with this improvement.

**Corollary 4.3.3.** Let $\alpha \in GL_2^+(\mathbb{Q})$ be such that $(D(\alpha), N) = 1$. Then Algorithm 4.2.1 has complexity of $O(I_{\alpha, \Gamma} \cdot \log(N^2 \cdot D(\alpha)))$ basic CosetIndex operations.

**Proof.** Using algorithms 4.3.1 and 4.2.8 in step (1), we see that the main contribution to the complexity of algorithm 4.2.1 comes from step (4), and this was computed in the proof of Theorem 4.2.10.

4.4. The Hecke Operators $T_n$. The question of computing the Hecke operators $T_n$, when $(n, N) > 1$, is a difficult one. We have not been able to find in the literature a good reference even for the definition of these operators. Therefore, we restrict ourselves, for the time being, to $n$ such that $(n, N) = 1$.

4.4.1. Adelic Definition. The only definition given in the literature of the Hecke operator for completely general level structure is in adelic terms.

Therefore, before we state the definition, we need to set up some notation.
Definition 4.4.1. Let $G \subseteq GL_2(\mathbb{Z}/NZ)$. We introduce
\[
S_G := \left\{ r \cdot g \cdot s \mid r \in \mathbb{Q}^\times, g \in GL_2(\hat{\mathbb{Z}}), s \in GL_2^+(\mathbb{R}), g \mod N \in G \right\} \subseteq GL_2(\mathbb{A})
\]
where we use the identification $\hat{\mathbb{Z}}/N\hat{\mathbb{Z}} \simeq \mathbb{Z}/NZ$.

Now we may define the Hecke correspondences on the corresponding Shimura variety.

Definition 4.4.2. ([26, Section 7.3]) Let $G \subseteq GL_2(\mathbb{Z}/NZ)$. Let $X_G := GL_2^+(\mathbb{Q}) \setminus (\mathcal{H} \times GL_2(\hat{\mathbb{Q}))) / S_G$ be the Shimura variety of level $S_G$. Let $w \in GL_2(\hat{\mathbb{Q}})$. Let $W = S_G \cap wS_Gw^{-1}$. Then $w$ induces a natural correspondence $X_G \to X_G$ via
\[
[z, g] \mapsto \sum_{\alpha \in S_G/W} [z, g\alpha w]
\]
We denote this correspondence by $X(w)$.

Lemma 4.4.3. The correspondence $X(w)$, for $w \in GL_2(\hat{\mathbb{Q}})$, is well defined.

Proof. First, the choice of representatives for $S_G/W$ does not matter: if we replace each $\alpha$ by $\alpha w_\alpha$, we would get
\[
\sum_{\alpha \in S_G/W} [z, g\alpha w_\alpha w] = \sum_{\alpha \in S_G/W} [z, g\alpha w \cdot w^{-1}w_\alpha w] = \sum_{\alpha \in S_G/W} [z, g\alpha w]
\]
because $w^{-1}Ww \subseteq S_G$. If $\gamma \in GL_2^+(\mathbb{Q})$, then $X(w)[z, g] = X(w)[\gamma z, \gamma g]$. Also, if we consider an element $[z, gs]$, with $s \in S_G$, it will map to
\[
\sum_{\alpha \in S_G/W} [z, gs\alpha w] = \sum_{\alpha \in S_G/W} [z, g\alpha w]
\]
since so $\alpha$ still runs through a set of representatives of $S_G/W$. \qed

We now connect it to our previous definitions of moduli spaces via the following Lemma.

Lemma 4.4.4. ([21, Lemma 5.13, Theorem 5.17]) Let $G \subseteq GL_2(\mathbb{Z}/NZ)$. Let $\mathcal{C}$ be a set of representatives for
\[
GL_2^+(\mathbb{Q}) \setminus GL_2(\hat{\mathbb{Q}}) / S_G \simeq \hat{\mathbb{Z}} / \det(S_G) \simeq (\mathbb{Z}/NZ)^\times / \det(G)
\]
Then
\[
GL_2^+(\mathbb{Q}) \setminus \left( \mathcal{H} \times G(\hat{\mathbb{Q}}) \right) / S_G \simeq \bigcup_{g \in \mathcal{C}} \Gamma_g \setminus \mathcal{H}
\]
where $\Gamma_g = gS_Gg^{-1} \cap GL_2^+(\mathbb{Q})$.

Under this identification, if we denote by $\omega$ the canonical line bundle on $X_G$, then we may identify $M_k(\Gamma_G)$ with the space of global sections $H^0(X_G, \omega^\otimes k)$. The correspondence $X(w)$ induces (by pullback) a map on global sections of line bundles $T(w) : H^0(X_G, \omega^\otimes k) \to H^0(X_G, \omega^\otimes k)$ given by $f \mapsto \sum_{\alpha \in S_G/W} (\alpha w)^*f$. We then get the following corollary.

Corollary 4.4.5. The correspondences $X(w)$ may be identified via this isomorphism (explicitly given as $z \mapsto [z, 1]$) as correspondences $\Gamma_g \setminus \mathcal{H} \to \Gamma_gw \setminus \mathcal{H}$. In particular, if $\det(w) \in \det(S_G)$, then we may identify the correspondence $X(w)$ as correspondences on $\Gamma_G \setminus \mathcal{H}$, and the operator $T(w)$ as an operator on $M_k(\Gamma_G)$. \footnote{22}
Theorem 4.4.6. (Shimura, [26, Theorem 7.9]) Let $w \in GL_2(\hat{\mathcal{Q}})$ be such that $w_l = 1$ for all $l \neq p$, and $w_p \in M_2(\mathbb{Z}_p)$ is an element such that $\det(w_p) = p$. Then if $p \nmid N$, the correspondence $X(w)$ on $X_G$ is an element such that $\det(w_p) = p$. Hence have common eigenvalues. These are called eigenforms.

Let $Z_p(s; X_G/\mathcal{Q})$ be such that $Z_p(s; X_G/\mathcal{Q}) = \frac{1}{Np} \cdot \frac{1}{p^{-s} + T(\sigma) \cdot p^{1-2s}}$. Then similarily to [26], Corollary 7.10 and Theorem 7.11, one obtains that the local factor of the zeta function of $X_G$ at $p$ is precisely

\begin{equation}
Z_p(s; X_G/\mathcal{Q}) = (1 - T(w) \cdot p^{-s} + T(\sigma) \cdot p^{1-2s})^{-1}
\end{equation}

Therefore, if $f = \sum_{n=1}^{\infty} a_n q^n$ be a normalized eigenform for the Hecke algebra, then by (4.4.8) and the standard machinery (see e.g. [26, Section 3.5]) we see that $T(w)f = a_p f$ for all $p$.

Proof. Let $\sigma \in SL_2(\mathbb{Z})$ be such that $\lambda_N(\sigma) = p \cdot \lambda_N(w)^{-2}$. Then similarly to [26], Corollary 7.10 and Theorem 7.11, one obtains that the local factor of the zeta function of $X_G$ at $p$ is precisely

\begin{equation}
Z_p(s; X_G/\mathcal{Q}) = (1 - T(w) \cdot p^{-s} + T(\sigma) \cdot p^{1-2s})^{-1}
\end{equation}

Yet as another consequence, we get the following fact, which could also be shown also directly by double coset computation.

Corollary 4.4.9. If $p \nmid N$, and $w \in GL_2(\hat{\mathcal{Q}})$ is such that $w_l = 1$ for all $l \neq p$, and $w_p \in M_2(\mathbb{Z}_p)$ is an element such that $\det(w_p) = p$, then the operator $T(w)$ is independent of the choice of $w$.

This motivates the following definition.

Definition 4.4.10. If $p \nmid N$, and $w \in GL_2(\hat{\mathcal{Q}})$ is such that $w_l = 1$ for all $l \neq p$, and $w_p \in M_2(\mathbb{Z}_p)$ is an element such that $\det(w_p) = p$, we define $T_p := T(w)$.
Proof. First, let us show that
\[ p \cdot M_2(\mathbb{Z}_p) \subseteq M_2(\mathbb{Z}_p) \cap w_p M_2(\mathbb{Z}_p) w_p^{-1} \]
Indeed, it is enough to prove that \( p \cdot M_2(\mathbb{Z}_p) \subseteq w_p M_2(\mathbb{Z}_p) w_p^{-1} \), which is the equivalent to
\[ w_p^{-1} p \cdot M_2(\mathbb{Z}_p) \cdot w_p \subseteq M_2(\mathbb{Z}_p) \]
However, as \( \det(w_p) = p \), we see that \( w_p^{-1} \cdot p = \tilde{w}_p \in M_2(\mathbb{Z}_p) \), thus the claim is trivial.

Now, let \( x \in W \). Then, as \( x \in S_G \), we may write \( x = r \cdot g \cdot s \) with \( r \in \mathbb{Q}^\times \), \( g \in GL_2(\hat{\mathbb{Z}}) \), \( s \in GL_2^+(\mathbb{R}) \), and \( g \mod N \in G \).

Write \( g = g^p g_p \), with \( g^p \in \prod_{l \neq p} GL_2(\mathbb{Z}_l) \) and \( g_p \in GL_2(\mathbb{Z}_p) \). Since \( x \in w_S G w^{-1} \), we have
\[ r \cdot g^p \cdot w_p^{-1} g_p w_p \cdot s = w^{-1} x w \in S_G \]
Thus, there exists \( r' \in \mathbb{Q}^\times \) such that
\[ r' \cdot g^p \cdot w_p^{-1} g_p w_p \in GL_2(\hat{\mathbb{Z}}) \]
In particular, for all \( l \neq p \), we have \( r' \cdot g_l \in GL_2(\mathbb{Z}_l) \), and as \( g_l \in GL_2(\mathbb{Z}_l) \), we see that \( r' \in \mathbb{Z}_l^\times \). Also, as
\[ r' \cdot w_p^{-1} g_p w_p \in GL_2(\mathbb{Z}_p) \]
and \( g_p \in GL_2(\mathbb{Z}_p) \), we see that \( r' \in \{\pm 1\} \), and \( w_p^{-1} g_p w_p \in GL_2(\mathbb{Z}_p) \).

Therefore, we obtain
\[ g_p \in w_p GL_2(\mathbb{Z}_p) w_p^{-1} \subseteq w_p M_2(\mathbb{Z}_p) w_p^{-1} \]
As we already know \( g_p \in M_2(\mathbb{Z}_p) \), we see that \( g_p \mod p \in R_p \).

Since \( g_p \in GL_2(\mathbb{Z}_p) \), it follows that \( g \mod p = g_p \mod p \in G_p \), and from \( g \mod N \in G \), it follows that \( g \mod Np \in G \times G_p = G_w \). Thus we have shown that \( x \in S_{G_w} \), so that \( W \subseteq S_{G_w} \).

Conversely, assume \( x \in S_{G_w} \). Then \( x = r \cdot g \cdot s \) with \( r \in \mathbb{Q}^\times \), \( g \in GL_2(\hat{\mathbb{Z}}) \), \( s \in GL_2^+(\mathbb{R}) \), and \( g \mod Np \in G_w \).

It follows that \( g \mod N \in G \) and \( g \mod p \in G_p \). In particular, we see immediately that \( x \in S_G \). Moreover,
\[ w^{-1} x w = r \cdot g^p \cdot w_p^{-1} g_p w_p \cdot s \]
Since \( g_p \mod p = g \mod p \in G_p \subseteq R_p \), we have
\[ g_p \in M_2(\mathbb{Z}_p) \cap w_p M_2(\mathbb{Z}_p) w_p^{-1} \]
hence \( w_p^{-1} g_p w_p \in M_2(\mathbb{Z}_p) \). Looking at the determinant, we see that it actually lies in \( GL_2(\mathbb{Z}_p) \). In particular,
\[ g' = g^p \cdot w_p^{-1} g_p w_p \in GL_2(\hat{\mathbb{Z}}) \]
and \( g' \mod N \in G \). Therefore, \( w^{-1} x w \in S_G \), showing equality. \( \square \)

We now show how to use strong approximation to rewrite the operator \( T(w) \) in the classical description of modular curves.
Lemma 4.4.12. Let $G \subseteq GL_2(\mathbb{Z}/NZ)$ be a subgroup. Let $p \in \text{det}(G)$ be a prime. Let $w \in GL_2(\mathbb{Q})$ be such that $w_l = 1$ for all $l \neq p$, and $w_p \in M_2(\mathbb{Z}_p)$ is such that $\text{det}(w_p) = p$. Then the correspondence $X(w)$ can be realized as

$$\Gamma_G z \mapsto \sum_{\alpha \in \Gamma_G/\Gamma_{G_w}} \Gamma_G \cdot q(w) \cdot \alpha^{-1}z$$

for some $q(w) \in GL_2^+(\mathbb{Q})$, explicitly constructed as a function of $w$. Therefore $T_p$ is realized as $f \mapsto \sum_{\alpha \in \Gamma_G/\Gamma_{G_w}} f|_{q(w)-1} \cdot \alpha^{-1}$.

Proof. Let $\delta_p \in G$ be an element such that $\text{det}(\delta_p) = p$. Let $w_p \in M_2(\mathbb{Z})$ be such that $\text{det}(w_p) = p$, and let $\alpha \in \Gamma_G$. Then

$$\text{det}(\delta_p^{-1} \cdot \lambda_N(w_p)) = \text{det}(\delta(p))^{-1} \cdot \text{det}(w_p) = p^{-1} \cdot p = 1$$

so that $\delta_p^{-1} \cdot \lambda_N(w_p) \in SL_2(\mathbb{Z}/NZ)$. Since $\lambda_N : SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/NZ)$ is surjective, there exists $\beta_p \in SL_2(\mathbb{Z})$ such that

$$\lambda_N(\beta_p) = \delta_p^{-1} \cdot \lambda_N(w_p)$$

Let $q(w) = \beta_p \cdot pw_p^{-1} \in GL_2^+(\mathbb{Q})$. Then we have

$$\alpha \cdot q(w)^{-1} = p^{-1} \cdot \alpha w_p \beta_p^{-1} = p^{-1} \cdot \alpha w \cdot w^{-1} w_p \beta_p^{-1}$$

Let $s = w^{-1} w_p \beta_p^{-1}$. Then $s_p = w_p^{-1} w_p \beta_p^{-1} = \beta_p^{-1} \in SL_2(\mathbb{Z}) \subseteq GL_2(\mathbb{Z}_p)$. Also, for all $l \neq p$, one has $s_l = w_p \cdot \beta_p$, so that $\text{det}(s_l) = \text{det}(w_p) \cdot \text{det}(\beta_p)^{-1} = p$ and $s_p \in M_2(\mathbb{Z}) \subseteq M_2(\mathbb{Z}_l)$, showing that $s_l \in GL_2(\mathbb{Z}_l)$. Moreover,

$$\lambda_N(s) = \lambda_N(w_p) \cdot \lambda_N(\beta_p)^{-1} = \delta_p \in G$$

showing that $s \in S_G$. Therefore $\alpha w = \alpha \cdot q(w)^{-1} \cdot s^{-1}$, so that (recall that $p \in \mathbb{Q}^\times$ acts trivially)

$$[z, \alpha w] = [z, p^{-1} \cdot \alpha w \cdot s] = [z, \alpha \cdot q(w)^{-1}] = [q(w) \cdot \alpha^{-1} \cdot z, 1]$$

By Lemma 4.4.11 $W = S_{G_w}$, hence

$$S_G/W \cong G \times GL_2(\mathbb{Z}/p\mathbb{Z})/G_w \cong \Gamma_G/\Gamma_{G_w}$$

It follows that under the isomorphism $[z] \mapsto [z, 1]$, the correspondence $X(w)$ above is interpreted as

$$\Gamma_G z \mapsto \sum_{\alpha \in \Gamma_G/\Gamma_{G_w}} \Gamma_G \cdot q(w) \cdot \alpha^{-1}z.$$ 

\[\square\]

4.4.2. Classical Definition. We can now use Lemma 4.4.12 to find an equivalent definition in classical terms of the Hecke operator $T_p$.

Lemma 4.4.13. Let $\alpha \in M_2(\mathbb{Z})$ be such that $\text{det}(\alpha) = p$ and $\lambda_N(\alpha) \in G$. Then $T_\alpha = T_p$.

Proof. Let $w \in GL_2(\mathbb{Q})$ be such that $w_l = 1$ for all $l \neq p$, $w_p \in M_2(\mathbb{Z}_p)$ and $\text{det}(w_p) = p$. By Lemma 4.4.12, the operator $T_p = T(w)$ is given by $f \mapsto \sum_{\beta \in \Gamma_G/\Gamma_{G_w}} f|_{q(w)-1} \cdot \alpha^{-1}$, where $q(w) \in GL_2^+(\mathbb{Q})$. By Lemma 4.4.11 we know that $G_w = G \times G_p$, with $G_p = R^\times_p$ where $R_p$ is the image in $M_2(\mathbb{Z}/p\mathbb{Z})$ of $M_2(\mathbb{Z}_p) \cap w_p M_2(\mathbb{Z}_p) w_p^{-1}$, therefore

$$\Gamma_G/\Gamma_{G_w} \cong (G \times GL_2(\mathbb{Z}/p\mathbb{Z}))/G_w \cong GL_2(\mathbb{Z}/p\mathbb{Z})/G_p$$

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Recall also that \( \det(w_p) = p \), hence \( w_p \in SL_2(\mathbb{Z}_p) \cdot \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \cdot SL_2(\mathbb{Z}_p) \).

Write \( w_p = \gamma_1 \cdot \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \cdot \gamma_2 \), for some \( \gamma_1, \gamma_2 \in SL_2(\mathbb{Z}_p) \). Then

\[
\begin{align*}
w_p M_2(\mathbb{Z}_p) w_p^{-1} &= \gamma_1 \cdot \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \cdot \gamma_2 M_2(\mathbb{Z}_p) \gamma_2^{-1} \cdot \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \cdot \gamma_1^{-1} \\
&= \gamma_1 \cdot \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \cdot M_2(\mathbb{Z}_p) \cdot \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \cdot \gamma_1^{-1}
\end{align*}
\]

hence

\[
M_2(\mathbb{Z}_p) \cap w_p M_2(\mathbb{Z}_p) w_p^{-1} = \gamma_1 \cdot \Delta^0(p) \cdot \gamma_1^{-1}
\]

where \( \Delta^0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) \mid b \in p\mathbb{Z}_p \right\} \), and passing to the image in \( M_2(\mathbb{Z}/p\mathbb{Z}) \), we get

\[
G_p = R_p^\times = \overline{\gamma}_1 \cdot \Gamma^0(\mathbb{Z}/p\mathbb{Z}) \cdot \overline{\gamma}_1^{-1}
\]

where \( \Gamma^0(\mathbb{Z}/p\mathbb{Z}) \) is the Borel subgroup of lower triangular matrices, and \( \overline{\gamma}_1 \in SL_2(\mathbb{Z}/p\mathbb{Z}) \).

Also, if \( \beta_p \in SL_2(\mathbb{Z}) \) is such that \( \lambda_N(\beta_p) = \lambda_N(p)^{-1} \cdot \lambda_N(\alpha) \cdot \lambda_N(w_p) \), then \( \lambda_N(\beta_p \cdot pw_p^{-1}) = \lambda_N(\alpha) \) and \( \alpha_p^{-1} \beta_p \cdot {}_p w_p^{-1} \in \Gamma(N) \subseteq \Gamma \), so that \( \Gamma \beta_p \cdot pw_p^{-1} = \Gamma \alpha \), thus \( \Gamma \beta_p \cdot pw_p^{-1} \Gamma = \Gamma \alpha \Gamma \), and so (as \( T_\alpha(f) = \sum_{\delta \in \Gamma \cap \alpha \Gamma} f|_\delta \)) we may assume that \( \beta_p \cdot pw_p^{-1} = \alpha \), hence \( q(w) = \alpha \).

Note then that, as \( p \cdot \alpha_p^{-1} = \gamma_1 \cdot \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \cdot \gamma_2 \beta_p^{-1} \), we see that

\[
\lambda_p(\alpha^{-1} \Gamma \alpha) = \lambda_p(\alpha^{-1} \cdot M_2(\mathbb{Z}) \cdot \alpha) = \lambda_p(\gamma_1 \cdot \Delta^0(p) \cdot \gamma_1^{-1})^\times = \lambda_p(\gamma_1 \cdot \Gamma^0(p) \cdot \gamma_1^{-1})^\times = \overline{\gamma}_1 \cdot \Gamma^0(\mathbb{Z}/p\mathbb{Z}) \cdot \overline{\gamma}_1^{-1} = G_p
\]

Therefore

\[
\Gamma/(\Gamma \cap \alpha^{-1} \Gamma \alpha) \cong GL_2(\mathbb{Z}/p\mathbb{Z})/G_p \cong \Gamma_G/\Gamma_{G_w}
\]

But we have bijection

\[
\Gamma/(\Gamma \cap \alpha^{-1} \Gamma \alpha) \rightarrow (\Gamma \cap \alpha^{-1} \Gamma \alpha) \backslash \Gamma \rightarrow \Gamma \backslash \Gamma \alpha \Gamma \\
\beta \mapsto \beta^{-1} \mapsto \alpha \cdot \beta^{-1}
\]

Thus, the map \( \beta \mapsto \alpha \cdot \beta^{-1} : \Gamma_G/\Gamma_{G_w} \rightarrow \Gamma \backslash \Gamma \alpha \Gamma \) is a bijection, so that

\[
T_\alpha(f) = \sum_{\delta \in \Gamma \backslash \Gamma \alpha \Gamma} f|_\delta = \sum_{\beta \in \Gamma_G/\Gamma_{G_w}} f|_{\alpha \cdot \beta^{-1}} = \sum_{\beta \in \Gamma_G/\Gamma_{G_w}} f|_{q(w) \cdot \beta^{-1}} = T(w)(f) = T_p(f)
\]

establishing the result. \( \square \)

**Corollary 4.4.14.** Assume \( p \nmid N \). Let \( \alpha \in M_2(\mathbb{Z}) \) be such that \( \det(\alpha) = p \) and \( \lambda_N(\alpha) \in G \). Then \( T_\alpha \) is independent of \( \alpha \), and if \( f = \sum_{n=1}^{\infty} a_n q^n \) is an eigenform of the Hecke algebra then \( T_\alpha f = a_p f \).

This allows us to define the Hecke operators at primes not dividing the level (and by multiplicativity to all \( n \) such that \( (n, N) = 1 \)).
Definition 4.4.15. Let $n$ be such that $(n, N) = 1$ and $n \in \det(G)$. Let $\alpha \in M_2(\mathbb{Z})$ be such that $\det(\alpha) = n$ and $\lambda_N(\alpha) \in G$. Let $T_n := T_\alpha$.

Therefore, Algorithm [4.2.1] can be used to compute the Hecke operators $T_n$, when $n$ is coprime to $N$.

Remark 4.4.16. Note that the definition of the Hecke operators $\{T_n\}$, although independent of the choice of representatives, does depend on $G$, and not only on $\Gamma_G$. This is not just an artifact of the proof, as the following example shows.

Example 4.4.17. Let

$$G_1 = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \in GL_2(\mathbb{Z}/7\mathbb{Z}) \mid d \in (\mathbb{Z}/7\mathbb{Z})^\times \right\}.$$  

Let $G_2 \subseteq GL_2(\mathbb{Z}/7\mathbb{Z})$ be the subgroup

$$G_2 := \left\{ \pm \begin{pmatrix} 1 & 4 \\ 2 & 6 \end{pmatrix} \right\} \subseteq GL_2(\mathbb{Z}/7\mathbb{Z})$$

which is an abelian group isomorphic to $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Then $\Gamma_{G_1} = \Gamma_{G_2} = \Gamma(7) \cdot \{\pm 1\}$. But while $T^G_{G_1}$ on $S_2(\Gamma(7))$ is the familiar Hecke operator $T_\alpha$ for $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $T^G_{G_2}$ on $S_2(\Gamma(7))$ is simply the identity, since $\alpha \notin G_2$.

Remark 4.4.18. As in [20, Proposition 3.31], one can show that the map $\Gamma \alpha \Gamma \mapsto SL_2(\mathbb{Z}) \alpha SL_2(\mathbb{Z})$, for all $\alpha \in M_2(\mathbb{Z})$ such that $\lambda_N(\alpha) \in \mathfrak{N}(G)$, defines a homomorphism on the Hecke algebras.

However, as the above example shows, this homomorphism is not injective, and elements $\alpha$ in different cosets of $G \setminus \mathfrak{N}(G)$ give rise to different operators.

4.5. Efficient Implementation of the Hecke Operators $T^\gamma_n$, $n \in \det(G)$. Algorithm [4.2.1] is not as efficient as we would have liked. Specifically, the logarithmic factor obtained from the passage to modular symbols is, in practice, due to the constant, a particularly high cost. We therefore use the ideas of Merel in [20] to calculate, at least the operators $\{T_n\}$ more efficiently. Following Section 2.1 in [20], we introduce the definition of a Merel pair.

Definition 4.5.1. Let $\Delta \subseteq GL_2(\mathbb{Q})$ be such that $\Gamma\Delta = \Delta\Gamma$ and such that $\Gamma \setminus \Delta$ is finite. Let $\tilde{\Delta} = \{g \in GL_2(\mathbb{Q}) \mid \tilde{g} := g^{-1}\det(g) \in \Delta\}$. Let $\phi : \tilde{\Delta} \cdot SL_2(\mathbb{Z}) \rightarrow SL_2(\mathbb{Z})$ be a map such that

1. For all $\gamma \in \tilde{\Delta} \cdot SL_2(\mathbb{Z})$ and $g \in SL_2(\mathbb{Z})$ we have $\Gamma \cdot \phi(\gamma g) = \Gamma \cdot \phi(\gamma) \cdot g$.
2. For all $\gamma \in \tilde{\Delta} \cdot SL_2(\mathbb{Z})$, we have $\gamma \cdot \phi(\gamma)^{-1} \in \tilde{\Delta}$ (or equivalently $\phi(\gamma) \cdot \tilde{\gamma} \in \Delta$).
3. The map $\Gamma \setminus \Delta \rightarrow \tilde{\Delta} \cdot SL_2(\mathbb{Z})/SL_2(\mathbb{Z})$ which associates to $\Gamma\delta$ the element $\tilde{\delta}SL_2(\mathbb{Z})$ is injective. (it is necessarily surjective).

We say that the pair $(\Delta, \phi)$ is a Merel pair for $\Gamma$.

We can now state Merel’s condition $(C_{\Delta})$: 27
**Definition 4.5.2.** Let $\sum u_M M \in \mathbb{C}[M_2(\mathbb{Z})]$, and let $(\Delta, \phi)$ be a Merel pair for $\Gamma$. We will say that $\sum u_M M$ satisfies the condition $(C_\Delta)$ if and only if for all $K \in \Delta SL_2(\mathbb{Z})/SL_2(\mathbb{Z})$ we have the following equality in $\mathbb{C}[\mathbb{P}^1(\mathbb{Q})]$:

$$\sum_{M \in K} u_M([M \infty] - [M0]) = [\infty] - [0].$$

Finally, we recall the following extremely useful theorem.

**Theorem 4.5.3 (\cite{20} Theorem 4).** Let $P \in \mathbb{C}_{k-2}[X, Y]$ and $g \in SL_2(\mathbb{Z})$. Let $(\Delta, \phi)$ be a Merel pair for $\Gamma$. Let $\sum u_M M \in \mathbb{C}[M_2(\mathbb{Z})]$ satisfy the condition $(C_\Delta)$. We have in $M_k(\Gamma)$

$$T^\vee_\Delta([P, g]) = \sum_{M, g M \in \Delta SL_2(\mathbb{Z})} u_M[P|_M, \phi(gM)].$$

We also recall Merel’s condition $(C_n)$.

**Definition 4.5.4.** Denote by $M_2(\mathbb{Z})_n$ the set of matrices of $M_2(\mathbb{Z})$ of determinant $n$. We say that an element $\sum_M u_M M \in \mathbb{C}[M_2(\mathbb{Z})_n]$ satisfies the condition $(C_n)$ if for all $K \in M_2(\mathbb{Z})_n/SL_2(\mathbb{Z})$, we have in $\mathbb{C}[\mathbb{P}^1(\mathbb{Q})]$

$$\sum_{M \in K} u_M([M \infty] - [M0]) = [\infty] - [0].$$

**Corollary 4.5.5.** Let $(\Delta_n, \phi_n)$ be a Merel pair for $\Gamma$ with $\Delta_n \subseteq M_2(\mathbb{Z})_n$. Let $\sum_M u_M M$ satisfy the condition $(C_n)$, then in $M_k(\Gamma)$

$$T^\vee_\Delta([P, g]) = \sum_M u_M[P|_M, \phi_n(gM)].$$

where the sum is restricted to matrices $M$ such that $gM \in \Delta_n SL_2(\mathbb{Z})$.

**Example 4.5.6.**

1. Let

$$\Delta_n := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \mid \det(g) = n, N \mid c, N \mid a - 1 \right\}$$

and $\phi_n : \Delta_n SL_2(\mathbb{Z}) \rightarrow SL_2(\mathbb{Z})$ is a map such that $\pi(\phi_n(g)) = (0 : 1) \cdot \lambda_N(g) \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$, where $\pi : SL_2(\mathbb{Z}) \rightarrow \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ is the natural surjection. Then in \cite{20} Lemma 1, Merel shows that $(\Delta_n, \phi_n)$ is a Merel pair for $\Gamma_1(N)$, which is key to modern efficient implementation of Hecke operators.

2. Let $p > 2$ be a prime, and let $u \in \mathbb{F}_p^\times$ be a non-square. Let

$$\Delta_n := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \det(g) = n, p \mid a - d, p \mid b - u \right\}$$

and $\phi_n : \Delta_n SL_2(\mathbb{Z}) \rightarrow SL_2(\mathbb{Z})$ is a map such that $\phi_n(g)^{-1} \cdot \sqrt{u} = \lambda_p(g)^{-1} \cdot \sqrt{u}$, where the action of $\Delta_n SL_2(\mathbb{Z})$ on $\mathbb{F}_p^2 - \mathbb{F}_p$ is via Mobius transformations. Then similarly $(\Delta_n, \phi_n)$ is a Merel pair for $\Gamma_{ns}(p)$ - the nonsplit Cartan subgroup of level $p$.

Let us write $G_0 := G \cap SL_2(\mathbb{Z}/N\mathbb{Z})$, and let $\pi : SL_2(\mathbb{Z}) \rightarrow G_0/SL_2(\mathbb{Z}/N\mathbb{Z})$ be the natural map $\pi(g) = G_0 \cdot \lambda_N(g)$, inducing an isomorphism $\Gamma \backslash SL_2(\mathbb{Z}) \cong G_0/SL_2(\mathbb{Z}/N\mathbb{Z})$. Let $s : G_0/SL_2(\mathbb{Z}/N\mathbb{Z}) \rightarrow SL_2(\mathbb{Z})$ be a section of $\pi$. 

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Let $\delta_n \in G$ be an element such that $\det(\delta_n) = n$, and let
\[
\Delta_n := \{ \alpha \in M_2(\mathbb{Z}) \mid \det(\alpha) = n, \quad \lambda_N(\alpha) \in G \}
\]

Let $\phi_n : M_2(\mathbb{Z}) \to SL_2(\mathbb{Z})$ be the map defined by
\[
\phi_n(\alpha) = s(G_0 \cdot n^{-1} \delta_n \cdot \lambda_N(\alpha))
\]

**Proposition 4.5.8.** $(\Delta_n, \phi_n)$ is a Merel pair.

**Proof.** First, note that
\[
\tilde{\Delta}_n = \{ \alpha \in M_2(\mathbb{Z}) \mid \det(\alpha) = n, \lambda_N(\tilde{\alpha}) \in G \} = \{ \alpha \in M_2(\mathbb{Z}) \mid \det(\alpha) = n, n \cdot \lambda_N(\alpha)^{-1} \in G \} = \{ \alpha \in M_2(\mathbb{Z}) \mid \det(\alpha) = n, n^{-1} \lambda_N(\alpha) \in G \}
\]
Indeed, $\det(\alpha) = \det(\tilde{\alpha})$.

Next, we verify the three conditions in Definition 4.5.1:

1. Let $\gamma \in \tilde{\Delta}_n SL_2(\mathbb{Z})$ and $g \in SL_2(\mathbb{Z})$. Then
\[
\pi(\phi_n(\gamma g)) = G_0 \cdot n^{-1} \delta_n \cdot \lambda_N(\gamma g) = G_0 \cdot n^{-1} \delta_n \cdot \lambda_N(\gamma) \cdot \lambda_N(g) = \pi(\phi_n(\gamma)) \cdot \lambda_N(g) = G_0 \cdot \lambda_N(\phi_n(\gamma) g) = \pi(\phi_n(\gamma) g)
\]
hence $\Gamma \cdot \phi_n(\gamma g) = \Gamma \cdot \phi_n(\gamma) g$.

2. Let $g \in \tilde{\Delta}_n SL_2(\mathbb{Z})$. Assume that $n^{-1} \delta_n \cdot \lambda_N(g) \in G_0$. Then $n^{-1} \lambda_N(g) \in G$, hence $g \in \tilde{\Delta}_n$. Thus $g \in \tilde{\Delta}_n$ iff $n^{-1} \delta_n \cdot \lambda_N(g) \in G_0$. Now, note that
\[
G_0 \cdot n^{-1} \delta_n \cdot \lambda_N(g) = \pi(\phi_n(g)) = G_0 \cdot \lambda_N(\phi_n(g))
\]
hence
\[
n^{-1} \delta_n \cdot \lambda_N(g \phi_n(g)^{-1}) \in G_0
\]
showing that $g \phi_n(g)^{-1} \in \tilde{\Delta}_n$.

3. Let $\delta_1, \delta_2 \in \Delta_n$ be such that $\delta_1 \delta_2^{-1} \in SL_2(\mathbb{Z})$. By definition $\lambda_N(\delta_1), \lambda_N(\delta_2) \in G$, hence
\[
\lambda_N(\delta_1 \delta_2^{-1}) = \lambda_N(\delta_1^{-1} \cdot n \cdot n^{-1} \cdot \delta_2) = \lambda_N(\delta_1^{-1} \delta_2) = \lambda_N(\delta_1)^{-1} \cdot \lambda_N(\delta_2) \in G
\]
hence $\delta_1 \delta_2^{-1} \in \Gamma$. \hfill \Box

**Corollary 4.5.9.** Assume $n \in \det(G)$. Let $\sum_M u_M M$ satisfy the condition $(C_n)$. Then, in $\mathbb{M}_k(\Gamma_G)$
\[
T_n^\vee([P, g]) = \sum_M u_M [P]_{\tilde{\Delta}_n} \cdot \phi_n(g M)
\]

**Proof.** By Definition 4.4.15 and (4.5.7), we see that $T_n^\vee = T_\alpha^\vee$ for any $\alpha \in \Delta_n$, i.e. $T_n^\vee = T_{\Delta_n}^\vee$. The corollary now follows from Corollary 4.5.5 once we realize that $M_2(\mathbb{Z}) \cong \tilde{\Delta}_n \cdot SL_2(\mathbb{Z})$.

Indeed, for $M \in M_2(\mathbb{Z})$, consider
\[
g = M \cdot \phi_n(M)^{-1} = M \cdot s(G_0 \cdot n^{-1} \delta_n \cdot \lambda_N(M))^{-1}
\]
Then
\[ \lambda_N(g)^{-1} = \lambda_N(\phi_n(M)) \cdot \lambda_N(M)^{-1} \in G_0 \cdot n^{-1} \delta(n) \cdot \lambda_N(M) \cdot \lambda_N(M)^{-1} = G_0 \cdot n^{-1} \delta_n \]
hence
\[ n^{-1} \lambda_N(g) \in \delta_n \cdot G_0 \subseteq G \]
showing that \( g \in \tilde{\Delta}_n \).

**Corollary 4.5.11.** Computation of the Hecke operator \( T_p \) on \( S_k(\Gamma_G) \), for \( p \in \text{det}(G) \), can be done in \( O(p \log p) \) basic CosetIndex operations.

**Proof.** In [19] Proposition 8], Merel shows that one can compute a certain set \( S_n \) of \( O(\sigma_1(n) \cdot \log n) \) matrices, which in [20] Proposition 18], he proves to satisfy condition \((C_n)\) (Definition 4.5.4). Therefore, by Corollary 4.5.10 it is enough to compute, for a vector \([P, g] \in M_k(\Gamma)\), the sum (4.5.10), implying the resulting complexity for computation of the operator \( T_p^\vee \) on \( M_k(\Gamma_G) \).

Let \( \alpha \in M_2(\mathbb{Z}) \) be such that \( \text{det}(\alpha) = p \) and \( \lambda_N(\alpha) \in G \). Then by definition \( T_\alpha = T_p \). Since \( \lambda_N(\alpha) \in G \) and \( G \) is of real type, by Corollary 4.1.9 one can then compute \( T_p^\vee \) on \( S_k(\Gamma_G) \). Finally, from Proposition 4.1.7 and Theorem 3.6.6 the result follows.

**Remark 4.5.12.** In general, the computation of the Hecke operators \( \{T_n\} \) for \((n, N) = 1\), can be done by using their multiplicativity property, so Corollary 4.5.11 is all that’s needed, and gives a slightly better complexity of \( O(p_{\text{max}} \cdot \log n) \) where \( p_{\text{max}} \) is the largest prime factor of \( n \).

Note also that this gives an improvement by a factor of \( \log(N) \) over Algorithm 4.2.1 which is very significant in practice.

We would also like to mention that under some mild assumptions, the decomposition 3.3.2 is Hecke-equivariant, hence one can compute the Hecke operator on each of the isotypic subspaces separately.

**Lemma 4.5.13.** Let \( G \subseteq G' \subseteq GL_2(\mathbb{Z}/N\mathbb{Z}) \) be such that \( G \) is normal in \( G' \). Let \( \varepsilon : G'/G \to \mathbb{Q}(\zeta)^\times \) be a character, and let \( \alpha \in M_2(\mathbb{Z}) \) be such that \( \text{det}(\alpha) = p \) is a prime and \( \lambda_N(\alpha) \in G \). Then the subspace \( S_k(\Gamma_G, \varepsilon) \) is invariant under \( T_\alpha \).

**Proof.** Let \( f \in S_k(\Gamma_G, \varepsilon) \), and let \( \gamma \in \Gamma_G \). Let \( R \) be a set of coset representatives for \( \Gamma_G \backslash \Gamma_G \alpha \Gamma_G \). Then
\[ T_\alpha(f)|_\gamma = \sum_{\delta \in R} f|_{\delta \gamma} = \sum_{\delta \in R} (f|_{\gamma})|_{\gamma^{-1} \delta \gamma} = \]
\[ = \sum_{\delta \in R} (\varepsilon(\gamma) \cdot f)|_{\gamma^{-1} \delta \gamma} = \]
\[ = \varepsilon(\gamma) \sum_{\delta \in R} f|_{\gamma^{-1} \delta \gamma} \]
Since \( \gamma \in \Gamma_G' \) and \( G \) is normal in \( G' \), for any \( x \in \Gamma_G \) we have \( \lambda_N(\gamma^{-1} x \gamma) \in G \), hence \( \gamma^{-1} x \gamma \in \Gamma_G \). Thus \( \gamma^{-1} \Gamma_G \gamma = \Gamma_G \).
As $\Gamma_G \alpha \Gamma_G = \bigsqcup_{\delta \in R} \Gamma_G \delta$, it follows that
\[
\Gamma_G \gamma^{-1} \alpha \gamma \Gamma_G = \gamma^{-1} \Gamma_G \alpha \Gamma_G \gamma = \bigsqcup_{\delta \in R} \gamma^{-1} \Gamma_G \delta \gamma = \bigsqcup_{\delta \in R} \Gamma_G \gamma^{-1} \delta \gamma
\]
However, since $\gamma \in \Gamma_G$ and $\lambda_N(\alpha) \in G$, it follows that $\lambda_N(\gamma^{-1} \alpha \gamma) \in G$, and so by [20, Lemma 3.29], we see that
\[
\Gamma_G \gamma^{-1} \alpha \gamma \Gamma_G = \{ g \in M_2(\mathbb{Z}) \mid \det(g) = p \text{ and } \lambda_N(g) \in G \} = \Gamma_G \alpha \Gamma_G.
\]
It follows that $T_\alpha(f) = \sum_{\delta \in R} f|_{\gamma^{-1} \delta \gamma}$, so that $T_\alpha(f)|_\gamma = \varepsilon(\gamma) \cdot T_\alpha(f)$, hence the result. \hfill $\square$

Remark 4.5.14. Note that it is not enough to demand that $\Gamma_G$ will be normal in $\Gamma_G$. For example, if $t^2 \mid N$ and $t \mid 24$, then, as described in [22], there is an element $u_t \in \mathcal{N} (\Gamma_0(N))$ defined over $\mathbb{Q}(\zeta_t)$ which commutes only with the Hecke operators $T_p$ such that $p \equiv 1 \mod t$ (equivalently, $p$ splits in $\mathbb{Q}(\zeta_t)$). Indeed, $u_t$ does not normalize the group $G \subseteq GL_2(\mathbb{Z}/N\mathbb{Z})$ appearing in Example 2.1.3 (3).

5. Degeneracy Maps

Our next step would be to decompose our spaces of modular forms to irreducible modules for the Hecke algebra. This should be possible once we apply our algorithms from Section 4. The only obstacle in our path to obtain the eigenvectors lie in the fact that we still do not have a multiplicity one result due to the existence of oldforms. We therefore make a small detour, and describe the generalization to our situation of the degeneracy maps, and the decomposition to spaces of newforms.

5.1. Petersson Inner Product. Before we describe the degeneracy maps themselves, in order to obtain a good notion of oldforms, we briefly recall the definitions and properties of the Petersson inner product, so that we will consequently be able to define newforms.

Definition 5.1.1. Let $\Gamma \subseteq SL_2(\mathbb{Z})$ be a congruence subgroup. The Petersson inner product,
\[
(\cdot)_{\Gamma} : S_k(\Gamma) \times S_k(\Gamma) \to \mathbb{C},
\]
is given by
\[
(f,g)_{\Gamma} = \frac{1}{V_\Gamma} \int_{X_\Gamma} f(\tau) \overline{g(\tau)} \left( \Im(\tau) \right)^k \, d\mu(\tau)
\]
where $d\mu = \frac{dx dy}{y^2}$ is the hyperbolic measure on $\mathcal{H}$ and $V_\Gamma = \int_{X_\Gamma} d\mu \tau$ is the volume of $X_\Gamma$.

The following standard results then help us to see that it is indeed a well defined inner product, and to compute adjoints of Hecke operators with respect to this inner product.

Proposition 5.1.3 ([11, Section 5.4 and Proposition 5.5.2]). Let $\Gamma \subseteq SL_2(\mathbb{Z})$ be a congruence subgroup. The integral in the Petersson inner product (5.1.2) is well defined and convergent. The pairing is linear in $f$, conjugate linear in $g$, Hermitian-symmetric and positive definite. Let $\alpha \in GL^+_2(\mathbb{Q})$, and set $\alpha' := \det(\alpha) \cdot \alpha^{-1}$. Then

1. If $\alpha^{-1} \Gamma \alpha \subseteq SL_2(\mathbb{Z})$ then for all $f \in S_k(\Gamma)$ and $g \in S_k(\alpha^{-1} \Gamma \alpha)$
\[
(f|_\alpha, g)_{\alpha^{-1} \Gamma \alpha} = (f, g|_{\alpha'})_{\Gamma}.
\]
(2) For all \( f, g \in S_k(\Gamma) \),
\[
\langle T_\alpha f, g \rangle = \langle f, T_{\alpha'} g \rangle.
\]

In particular, if \( \alpha^{-1}\Gamma \alpha = \Gamma \) then the adjoint of \( f \mapsto f|_\alpha \) is \( g \mapsto g|_{\alpha'} \) and in any case \( T_{\alpha}^* = T_{\alpha'}^* \).

The correspondence between \( \alpha \) and \( \alpha' \) forces us to consider some more operators on this space.

**Definition 5.1.4.** Let \( G \subseteq GL_2(\mathbb{Z}/N\mathbb{Z}) \), and let \( \mathcal{N}(G) \) be its normalizer in \( GL_2(\mathbb{Z}/N\mathbb{Z}) \). Then the quotient \( \bar{Q} := \Gamma_{\mathcal{N}(G)}/\Gamma_G \) acts naturally on \( M_k(\Gamma_G) \) via
\[
\langle q \rangle : M_k(\Gamma_G) \to M_k(\Gamma_G)
\]
\[
f \mapsto f|_\alpha \text{ for any } \alpha \in \Gamma_{\mathcal{N}(G)} \text{ with } \alpha \equiv q \text{ mod } \Gamma_G.
\]

This operator is called a **diamond operator**.

For any irreducible representation \( \rho : Q \to \mathbb{C} \), we let
\[
M_k(\Gamma_G, \rho) := \{ f \in M_k(\Gamma) \mid \langle q \rangle f = \rho(q) f \text{ for all } q \in Q \}
\]
Then \( M_k(\Gamma_G) = \bigoplus_\rho M_k(\Gamma_G, \rho) \) where \( \rho \) runs over all irreducible representations of \( Q \).

**Lemma 5.1.5.** Let \( \alpha \in M_2(\mathbb{Z}) \) be such that \( \lambda_N(\alpha) \in G \), and let \( q \in \Gamma_{\mathcal{N}(G)}/\Gamma_G \). Then \( T_\alpha \circ \langle q \rangle = \langle q \rangle \circ T_\alpha \).

**Proof.** Denote by \( \bar{q} \in \Gamma_{\mathcal{N}(G)} \) an element such that \( \bar{q} \equiv q \text{ mod } \Gamma_G \). Since \( \lambda_N(\alpha) \in G \) and \( \lambda_N(\bar{q}) \in \mathcal{N}(G) \), we see that
\[
\lambda_N(\bar{q}^{-1}\alpha \bar{q}) = \lambda_N(\bar{q})^{-1}\lambda_N(\alpha)\lambda_N(\bar{q}) \in \lambda_N(\bar{q})^{-1}G\lambda_N(\bar{q}) = G
\]
Moreover, \( \det(\bar{q}^{-1}\alpha \bar{q}) = \det(\alpha) \), so that \( \lambda_N(\bar{q}^{-1}\alpha \bar{q}) \in G_0\lambda_N(\alpha) \). By [26, Lemma 3.29 (a)], it follows that \( \bar{q}^{-1}\alpha \bar{q} \in \Gamma_G\alpha\Gamma_G \) hence
\[
\bar{q}^{-1}\Gamma_G\alpha\Gamma_G \bar{q} = \Gamma_G \bar{q}^{-1}\alpha \bar{q} \Gamma_G = \Gamma_G \alpha \Gamma_G
\]
so that if \( \Gamma_G\alpha\Gamma_G = \bigsqcup_i \Gamma_G\alpha_i \) is a disjoint union, then
\[
\Gamma_G \alpha \Gamma_G = \bar{q}^{-1}\Gamma_G \alpha \Gamma_G \bar{q} = \bar{q}^{-1} \left( \bigsqcup_i \Gamma_G \alpha_i \right) \bar{q} = \bigsqcup_i \bar{q}^{-1} \Gamma_G \alpha_i \bar{q} = \bigsqcup_i \Gamma_G \bar{q}^{-1} \alpha \bar{q}.
\]
Therefore, for any \( f \in M_k(\Gamma_G) \) we have
\[
\langle q \rangle T_\alpha f = \sum_i f|_{\alpha_i \bar{q}} = \sum_i f|_{\bar{q}^{-1} \alpha_i \bar{q}} = \sum_i \langle q \rangle f|_{\bar{q}^{-1} \alpha_i \bar{q}} = T_\alpha \langle q \rangle f. \quad \square
\]

Before we continue, we need a simple Lemma, which is a variant of [26, Lemma 3.29].

**Lemma 5.1.6.** Let \( \Gamma_1 \subseteq \Gamma_2 \) be congruence subgroups of levels \( N_1, N_2 \) respectively. Let
\[
\Phi_i = \{ \alpha \in M_2(\mathbb{Z}) \mid \det \alpha > 0, (\det(\alpha), N_i) = 1, \lambda_N_i(\Gamma_i \alpha) = \lambda_N(\alpha \Gamma_i) \}, \quad i \in \{1, 2\}
\]
Then the following assertions hold.

1. \( \Gamma_1 \alpha \Gamma_1 = \{ \xi \in \Gamma_2 \alpha \Gamma_2 \mid \lambda_{N_1}(\xi) \in \lambda_{N_1}(\Gamma_1 \cdot \alpha) \} \) if \( \alpha \in \Phi_1 \).
2. \( \Gamma(N_1) \alpha \Gamma(N_1) = \Gamma(N_1) \beta \Gamma(N_1) \) if and only if \( \Gamma_2 \alpha \Gamma_2 = \Gamma_2 \beta \Gamma_2 \) and \( \alpha \equiv \beta \text{ mod } N_1 \).
   (for \( \alpha, \beta \) such that \( (\det(\alpha, N_1) = (\det(\beta, N_1) = 1) \)
(3) $\Gamma_2\alpha\Gamma_2 = \Gamma_2\alpha\Gamma_1 = \Gamma_1\alpha\Gamma_2$ if $\alpha \in \Phi_2$ and $(\det \alpha, N_1) = 1$.

(4) $\Gamma_i\alpha\Gamma_i = \Gamma_i\alpha\Gamma(N_i) = \Gamma(N_i)\alpha\Gamma_i$ if $\alpha \in \Phi_i$ for $i = 1, 2$.

(5) If $\alpha \in \Phi_1$ and $\Gamma_1\alpha\Gamma_1 = \bigcup_i \Gamma_1\alpha_i$, then $\Gamma_2\alpha\Gamma_2 = \bigcup_i \Gamma_2\alpha_i$.

**Proof.** First, (4) is simply a restatement of [26] Lemma 3.29 (4). To show (3), put $a = \det \alpha$. By [26] Lemma 3.28 and [26] Lemma 3.9 (note that $\gcd(N_H, a \cdot N_G) = N_G$), we have

$$\Gamma(N_2) = \Gamma(N_2 \cdot a) \cdot \Gamma(N_1) \subseteq \alpha^{-1}\Gamma(N_2)\alpha\Gamma(N_1)$$

so that $\alpha^{-1}\Gamma(N_2)\alpha\Gamma(N_2) \subseteq \alpha^{-1}\Gamma(N_2)\alpha\Gamma(N_1)$. Hence $\Gamma(N_2)\alpha\Gamma(N_2) \subseteq \Gamma(N_2)\alpha\Gamma(N_1) \subseteq \Gamma(N_2)\alpha\Gamma_1$. Therefore $\Gamma_2\alpha\Gamma(N_2) \subseteq \Gamma_2\alpha\Gamma_1 \subseteq \Gamma_2\alpha\Gamma_2$. But from (4), as $\alpha \in \Phi_2$, we know that there is an equality $\Gamma_2\alpha\Gamma(N_2) = \Gamma_2\alpha\Gamma_2$. This shows (3). Next, to see (1) note that by [26] Lemma 3.29 (1), we have $\Gamma_1\alpha\Gamma_1 = \{\xi \in SL_2(\mathbb{Z})\alpha SL_2(\mathbb{Z}) \mid \lambda_{N_1}(\xi) \in \lambda_{N_1}(\Gamma_1 \cdot \alpha)\}$. Since we also know $\Gamma_1\alpha\Gamma_1 \subseteq \Gamma_2\alpha\Gamma_2$, (1) follows. (2) is a special case of (1). Finally, let $\alpha \in \Phi_1$, and $\Gamma_1\alpha\Gamma_1 = \bigcup_i \Gamma_1\alpha_i$. Then $\Gamma_2\alpha\Gamma_2 = \Gamma_2\alpha\Gamma_1 = \bigcup_i \Gamma_2\alpha_i$ by (3). Assume $\Gamma_2\alpha_i = \Gamma_2\alpha_j$. Then $\alpha_i = \gamma\alpha_j$ for some $\gamma \in \Gamma_2$. By (1)

$$\lambda_{N_1}(\alpha_i) \in \lambda_{N_1}(\Gamma_1) \cdot \lambda_{N_1}(\alpha) = \lambda_{N_1}(\Gamma_1) \cdot \lambda_{N_1}(\alpha_j)$$

Thus, there exists some $\delta \in \Gamma_1$ such that $\alpha_i \equiv \delta \alpha_j \mod N_1$. It follows that $\gamma \equiv \delta \mod N_1$. Since $\Gamma(N_1) \subseteq \Gamma_1$, we have $\gamma \in \Gamma_1$. This proves (5). \[ \square \]

We may now state the adjoints of the Hecke operators with respect to the Petersson inner product.

**Theorem 5.1.7.** In the inner product spaces $S_k(\Gamma_G)$, the diamond operators $\langle \sigma \rangle$ and the Hecke operator $T_p$ for $p \in \det(G)$ a prime, have adjoints

$$\langle \sigma \rangle^* = \langle \sigma \rangle^{-1} \quad \text{and} \quad T_p^* = \langle \sigma_p \rangle \cdot T_p$$

where $\sigma_p \in \Gamma_{\mathfrak{M}(G)}$ is an element such that $\lambda_N(\sigma_p) = p \cdot \lambda_N(\alpha)^{-2}$, and $\alpha \in M_2(\mathbb{Z})$ is any element with $\det(\alpha) = p$ such that $\lambda_N(\alpha) \in G$. Thus the Hecke operators $\langle \sigma \rangle$ and $T_n$ for $n \in \det(G)$ are normal.

**Proof.** The first assertion is just a restatement of Proposition 5.1.3 (1). For the second assertion, we note that if $\alpha \in M_2(\mathbb{Z})$ is such that $\lambda_N(\alpha) \in G$ and $\det(\alpha) = p$, then $\alpha \in \Phi_{\mathfrak{M}(G)}$. Indeed, if $\nu \in \mathfrak{M}(G)$ then for any $g \in G$ we have

$$(\lambda_N(\alpha)\nu\lambda_N(\alpha)^{-1})g(\lambda_N(\alpha)\nu^{-1}\lambda_N(\alpha)^{-1}) \in \lambda_N(\alpha)\nu G \nu^{-1}\lambda_N(\alpha)^{-1} \subseteq G$$

so that $\lambda_N(\alpha)\nu\lambda_N(\alpha)^{-1} \in \mathfrak{M}(G)$.

Therefore, by Lemma 5.1.6 (3), we see that $\Gamma_{\mathfrak{M}(G)}\alpha \Gamma_{\mathfrak{M}(G)} = \Gamma_G\alpha \Gamma_{\mathfrak{M}(G)}$. Moreover, we have

$$\lambda_N(\alpha') = \det(\alpha)\lambda_N(\alpha)^{-1} \in \mathfrak{M}(G)$$

hence $\Gamma_{\mathfrak{M}(G)}\alpha' \Gamma_{\mathfrak{M}(G)} = \Delta_p^{\gamma_{\mathfrak{M}(G)}} = \Gamma_{\mathfrak{M}(G)}\alpha \Gamma_{\mathfrak{M}(G)} = \Gamma_G\alpha \Gamma_{\mathfrak{M}(G)}$.

Let $\gamma \in \Gamma_{\mathfrak{M}(G)}$ be such that $\alpha' \in \Gamma_G\alpha' \gamma$. Now, as $\gamma$ normalizes $\Gamma_G$, we see that $\Gamma_G\alpha' \Gamma_G = \Gamma_G\alpha' \Gamma_G \gamma$ and hence if $\Gamma_G\alpha' \Gamma_G = \bigcup_i \Gamma_G\alpha_i$, then $\Gamma_G\alpha' \Gamma_G = \bigcup_i \Gamma_G\alpha_i \gamma$. It then follows from Proposition 5.1.3 (2) that $T_p^* = T_{\alpha'}^* = T_{\alpha'} = \langle \gamma \rangle T_{\alpha'}$. Finally, we note that

$$\lambda_N(\gamma) \in \lambda_N(\Gamma_G)\lambda_N(\alpha)^{-1} \lambda_N(\alpha') \subseteq G \cdot p\lambda_N(\alpha)^{-2} = G \cdot \lambda_N(\sigma_p)$$

so that $\gamma \in \Gamma_G \cdot \sigma_p$, showing that $\langle \gamma \rangle = \langle \sigma_p \rangle$. \[ \square \]
Remark 5.1.8. The $\sigma_p$ actually acts via a character. Indeed, if $A \subseteq \mathfrak{N}(G)$ is such that $A/G$ is abelian, and $A$ contains the center of diagonal matrices, then $\sigma_p \in A$ and all the irreducible representations of $A/G$ are characters.

This leads to the following Theorem.

Corollary 5.1.9. The space $S_k(\Gamma_G)$ has an orthogonal basis of simultaneous eigenvectors for the Hecke operators $\{T_n \mid n \in \det(G)\}$ and the diamond operators $\{\langle q \rangle \mid q \in \mathfrak{N}(G)/\Gamma_G\}$.

Proof. This is a corollary of the Spectral Theorem for a family of commuting normal operators. \hfill \Box

Definition 5.1.10. We call a vector as in Corollary 5.1.9 an eigenform.

5.2. Degeneracy Maps. We begin by defining degeneracy maps in general, on our space of modular symbols.

Definition 5.2.1. Let $\Gamma_1 \leq \Gamma'_1$, $\Gamma_2 \leq \Gamma'_2$, be two pairs of finite index subgroups in $SL_2(\mathbb{Z})$. Let $\varepsilon_i : \Gamma'_i/\Gamma_i \to \mathbb{Q}(\zeta_i) \times$ for $i = 1, 2$ be characters. Let $t \in GL_2^+(\mathbb{Q})$ be such that $t^{-1}\Gamma'_1 t \subseteq \Gamma_2'$, and $\varepsilon_2 \circ \text{Inn}(t)^{-1} \mid_{\Gamma'_1} = \varepsilon_1$. Fix a choice $R_t$ of coset representatives for $\Gamma'_1 \setminus t \cdot \Gamma_2'$. Let

$$
\alpha_t : M_k(\Gamma_1, \varepsilon_1) \to M_k(\Gamma_2, \varepsilon_2), \quad \beta_t : M_k(\Gamma_2, \varepsilon_2) \to M_k(\Gamma_1, \varepsilon_1)
$$

$$
\alpha_t(x) = t^{-1} x, \quad \beta_t(x) = \sum_{t \gamma_2 \in R_t} \varepsilon_2(t \gamma_2)^{-1} t \gamma_2 \cdot x
$$

We show that these operators are well defined.

Lemma 5.2.2. The operators $\alpha_t$ and $\beta_t$ are well defined, and moreover the composition $\alpha_t \circ \beta_t$ is multiplication by $[\Gamma'_2 : t^{-1}\Gamma_1 t]$.

Proof. First, note that as $t^{-1}\Gamma'_1 t \subseteq \Gamma_2'$, for any $\gamma_1 \in \Gamma'_1$, $\gamma_2 \in \Gamma'_2$ one has

$$
\gamma_1 \cdot t \gamma_2 = t \cdot t^{-1} \gamma_1 t \cdot \gamma_2 \in t \cdot \Gamma_2'
$$

so that $\Gamma'_1$ indeed acts on $t \cdot \Gamma_2'$. To show that $\alpha_t$ is well defined, we must show that for each $x \in M_k(\Gamma_1, \varepsilon_1)$ and $\gamma_1 \in \Gamma'_1$, we have

$$
\alpha_t(\gamma_1 x - \varepsilon_1(t \gamma_1) \cdot x) = 0 \in M_k(\Gamma_2, \varepsilon_2)
$$

We have, by assumption

$$
\alpha_t(\gamma_1 x) = t^{-1} \gamma_1 \cdot x = t^{-1} \gamma_1 t \cdot t^{-1} x = \varepsilon_2(t^{-1} \gamma_1 t) \cdot t^{-1} x = \varepsilon_1(\gamma_1) \cdot \alpha_t(x)
$$

hence the result.

We next verify that $\beta_t$ is well defined.

Suppose that $\gamma_1 \in \Gamma'_1$ and $\gamma_2 \in \Gamma'_2$, then

$$
\gamma_1 \cdot t \gamma_2 = t \cdot (t^{-1} \gamma_1 t) \cdot \gamma_2
$$

Moreover, for any $x \in M_k(\Gamma_2, \varepsilon_2)$, we have

$$
\varepsilon_2(t^{-1} \gamma_1 t) \cdot \gamma_2)^{-1} \cdot \gamma_1 \cdot t \gamma_2 \cdot x = \varepsilon_1(\gamma_1)^{-1} \cdot \varepsilon_2(\gamma_2)^{-1} \cdot \gamma_1 \cdot \gamma_2 \cdot x = \varepsilon_1(\gamma_1)^{-1} \cdot \gamma_1 \cdot (\varepsilon_2(\gamma_2)^{-1} \cdot t \gamma_2 \cdot x)
$$
But as $\varepsilon_2(\gamma_2)^{-1} \cdot t \gamma_2 \cdot x \in M_k(\Gamma_1, \varepsilon_1)$, this is simply $\varepsilon_2(\gamma_2)^{-1} \cdot t \gamma_2 \cdot x$.

Thus, replacing the representative $t \cdot \gamma_2$ by $\gamma_1 \cdot t \gamma_2$ does not change the result, so that $\beta_i^\gamma$ is independent of the choice of the representatives $R_t$.

Next, we must show that for any $\gamma \in \Gamma_2$ and any $x \in M_k(\Gamma_2, \varepsilon_2)$, $\beta_i^\gamma(\gamma x) = \varepsilon_2(\gamma) \cdot \beta_i^\gamma(x)$. However, for $\gamma \in \Gamma_2'$, using the fact that $\beta_i$ is independent of the choice of representatives, and that $\Gamma_2'$ acts on $\Gamma_1' \setminus t \cdot \Gamma_2'$ by right translations, we get

$$
\beta_i^\gamma(\gamma x) = \sum_{t \gamma_2 \in R_t} \varepsilon_2(\gamma_2)^{-1} \cdot t \gamma_2 \cdot \gamma x = \\
= \sum_{t \gamma_2^{-1} \in R_t} \varepsilon_2(\gamma_2^{-1})^{-1} t \gamma_2^{-1} \cdot \gamma x = \\
= \varepsilon_2(\gamma) \cdot \sum_{t \gamma_2 \in R_t} \varepsilon_2(\gamma_2)^{-1} \cdot t \gamma_2 \cdot x = \varepsilon_2(\gamma) \cdot \beta_i(x)
$$

To compute $\alpha_i^\gamma \circ \beta_i^\gamma$, we use that $\# R_t = [\Gamma_2' : \Gamma_1']$:

$$
\alpha_i^\gamma(\beta_i^\gamma(x)) = \alpha_i^\gamma \left( \sum_{t \gamma_2 \in R_t} \varepsilon_2(\gamma_2)^{-1} \cdot t \gamma_2 \cdot x \right) = \\
= \sum_{t \gamma_2 \in R_t} \varepsilon_2(\gamma_2)^{-1} \cdot t \gamma_2 \cdot x = \sum_{t \gamma_2 \in R_t} x = [\Gamma_2' : \Gamma_1'] \cdot x.
$$

We also note that by using the pairing (3.6.1) we can get maps on spaces of cusp forms.

**Definition 5.2.3.** Let $\Gamma_1 \subseteq \Gamma_1'$, $\Gamma_2 \subseteq \Gamma_2'$, be two pairs of finite index subgroups in $SL_2(\mathbb{Z})$. Let $\varepsilon_i : \Gamma_i'/\Gamma_i \to \mathbb{Q}(\zeta_i)\times$ for $i = 1, 2$ be characters. Let $t \in GL_2^+ (\mathbb{Q})$ be such that $t^{-1} \Gamma_1' t \subseteq \Gamma_2'$, and $\varepsilon_2 \circ \text{Im}(t)^{-1} |_{\Gamma_1'} = \varepsilon_1$. Fix a choice $R_t$ of coset representatives for $\Gamma_1' \setminus t \cdot \Gamma_2'$. Let

$$
\alpha_t : S_k(\Gamma_2, \varepsilon_2) \to S_k(\Gamma_1, \varepsilon_1), \\
\beta_t : S_k(\Gamma_1, \varepsilon_1) \to S_k(\Gamma_2, \varepsilon_2)
$$

$$
f \mapsto f|_{t^{-1}} \\
f \mapsto \sum_{t \gamma_2 \in R_t} \varepsilon_2(\gamma_2)^{-1} f|_{t \gamma_2}
$$

5.3. **Degeneracy maps and Hecke operators.** Next, as in the classical theory for Iwahori level, we would like to show that the degeneracy maps commute with the Hecke operators. Alas, this is not always the case as the following example shows.

**Example 5.3.1.** Let $\Gamma_1 = \Gamma(7)$ and let $\Gamma_2 = \Gamma_{ns}(7)$ be the non-split Cartan at level 7. Then there is a natural inclusion map $\alpha_1^\gamma : M_k(\Gamma_1) \to M_k(\Gamma_2)$, but $\alpha_1$ does not commute with the standard Hecke operators for these spaces.

In the next proposition we describe the conditions under which the degeneracy maps commute with the Hecke operators.

**Proposition 5.3.2.** Let $N_G \mid N_H$ be positive integers. Let $H \subseteq GL_2(\mathbb{Z}/N_H\mathbb{Z})$ and $G \subseteq GL_2(\mathbb{Z}/N_G\mathbb{Z})$ be subgroups such that $\lambda_{N_G}(H) \subseteq G$. Let $p$ be a prime number such that $p \in \det(H)$, and let $t \in M_2(\mathbb{Z})$ be such that $N_G \cdot \det(t) \mid N_H$ and $t^{-1} \Gamma_H t \subseteq \Gamma_G$. Let $T_{p,?}^\gamma$ be the Hecke operator at $p$ on the space $M_k(\Gamma_?)$ for $? \in \{G, H\}$. Then

$$
T_{p,G}^\gamma \circ \alpha_t^\gamma = \alpha_t^\gamma \circ T_{p,H}^\gamma.
$$
Proof. Denote \( d := \det(t) \), so that \( d \cdot t^{-1} \in M_2(\mathbb{Z}) \). Since \( t^{-1} \Gamma_H t \subseteq \Gamma_G \), we have also \( d \cdot t^{-1} \Gamma_H t \subseteq d \cdot \Gamma_G \) so that

\[
\lambda_{N_G \cdot d}(dt^{-1}) \cdot \lambda_{N_G \cdot d}(\Gamma_H) \cdot \lambda_{N_G \cdot d}(t) = \lambda_{N_G \cdot d}(dt^{-1} \Gamma_H t) \subseteq \lambda_{N_G \cdot d}(d \cdot \Gamma_G) = d \cdot \lambda_{N_G \cdot d}(\Gamma_G).
\]

However, we also assumed \( N_G \cdot d \mid N_H \), hence

\[
\lambda_{N_G \cdot d}(\Gamma_H) = \lambda_{N_G \cdot d}(\lambda_{N_H}(\Gamma_H)) = \lambda_{N_G \cdot d}(H).
\]

Let \( \Delta^\vee_p := \{ \alpha \in M_2(\mathbb{Z}) \mid \det(\alpha) = p \text{ and } \lambda_{N_H}(\alpha) \in ? \} \), \( ? \in \{ G, H \} \) and let \( \alpha \in \Delta^H_p \). Then combining (5.3.3) and (5.3.4) we see that

\[
\lambda_{N_G \cdot d}(dt^{-1} \alpha t) = \lambda_{N_G \cdot d}(dt^{-1}) \lambda_{N_G \cdot d}(\lambda_{N_H}(\alpha)) \lambda_{N_G \cdot d}(t) \in \lambda_{N_G \cdot d}(dt^{-1}) \lambda_{N_G \cdot d}(H) \lambda_{N_G \cdot d}(t) \subseteq d \cdot \lambda_{N_G \cdot d}(\Gamma_G)
\]

Therefore \( t^{-1} \alpha t \in M_2(\mathbb{Z}) \) and \( \lambda_{N_G}(t^{-1} \alpha t) \in \lambda_{N_G}(\Gamma_G) = G \). It follows that \( t^{-1} \alpha t \in \Delta^G_p \).

Since \( p \in \det(H) \), we also get \( p \in \det(\lambda_{N_G}(H)) \subseteq \det(G) \). Therefore, we have by [26, Lemma 3.29] that \( \Delta^H_p = \Gamma_H \alpha \Gamma_H \) and \( \Delta^G_p = \Gamma_G \cdot t^{-1} \alpha t \cdot \Gamma_G \).

Let \( \Delta^H_p = \bigsqcup_i \Gamma_H \alpha_i \) be its coset decomposition. Then

\[
t^{-1} \Gamma_H t \cdot t^{-1} \alpha t \cdot t^{-1} \Gamma_H t = t^{-1} \Gamma_H \alpha \Gamma_H t = t^{-1} \left( \bigsqcup_i \Gamma_H \alpha_i \right) t = \bigsqcup_i t^{-1} \Gamma_H \alpha_i t = \bigsqcup_i t^{-1} \Gamma_H t \cdot t^{-1} \alpha_i t.
\]

Now, applying Lemma [5.1.6] (5) to \( t^{-1} \Gamma_H t \subseteq \Gamma_G \) and \( t^{-1} \alpha t \), we obtain that

\[
\Delta^G_p = \Gamma_G \cdot t^{-1} \alpha t \cdot \Gamma_G = \bigsqcup_i \Gamma_G \cdot t^{-1} \alpha_i t.
\]

Finally, we use (5.3.5) to compute

\[
T_{p,G}^\vee \circ \alpha^\vee_t (x) = \sum_i t^{-1} \alpha_i t \cdot t^{-1} x = t^{-1} \sum_i \alpha_i x = \alpha^\vee_t \circ T_{p,H}^\vee (x). \quad \square
\]

We also note that by using the pairing (3.6.1) we have the same for cusp forms.

**Corollary 5.3.6.** *Under the assumptions of Corollary 5.3.2, \( \alpha_t \circ T_{p,G} = T_{p,H} \circ \alpha_t \).*

This motivates the following definition.

**Definition 5.3.7.** Let \( N \) be a positive integers. Let \( H \subseteq GL_2(\mathbb{Z}/N\mathbb{Z}) \) be a subgroup. For each \( G \subseteq GL_2(\mathbb{Z}/N\mathbb{Z}) \) such that \( H \subseteq G \), we denote by \( N_G \) the level of \( G \). Let \( T_G \) be a set of representatives for the orbits of \( \Gamma_H \) on the following set

\[
T_G := \Gamma_H \setminus \{ t \in M_2(\mathbb{Z}) \mid \det(t) \mid N/N_G \text{ and } t^{-1} \Gamma_H t \subseteq \Gamma_G \}.
\]

Let \( i_G \) be the map

\[
i_G : (S_k(\Gamma_G))^T_G \to S_k(\Gamma_H)
\]

\[
(f_t)_{t \in T_G} \mapsto \sum_{t \in T_G} \alpha_t(f_t)
\]

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Let Over($H$) be the set of minimal overgroups of $H$ in $GL_2(\mathbb{Z}/N\mathbb{Z})$. The subspace of oldforms at level $H$ is

$$S_k(\Gamma_H)^{\text{old}} = \sum_{G \in \text{Over}(H)} i_G \left( (S_k(\Gamma_G))^T \right)$$

and the subspace of newforms at level $H$ is the orthogonal complement with respect to the Petersson inner product,

$$S_k(\Gamma_H)^{\text{new}} = \left( S_k(\Gamma_H)^{\text{old}} \right)^\perp$$

We have to show that the sum in this definition is well defined.

**Lemma 5.3.9.** Let $N$ be a positive integers. Let $H \subseteq G \subseteq GL_2(\mathbb{Z}/N\mathbb{Z})$ be subgroups. Denote by $N_G$ the level of $G$. Let $T_G$ be the set given in (5.3.8). Then $T_G$ is finite.

**Proof.** We first note that by the condition $\det(t) \mid N/N_G$, the determinant of $t$ assumes finitely many values, hence it is enough to show that the subset $T_{G,d}$ of such $t$’s with $\det(t) = d$ is finite.

Next, we recall that if $\Delta_d$ is the set of matrices in $M_2(\mathbb{Z})$ of determinant $d$, then $SL_2(\mathbb{Z}) \setminus \Delta_d$ is finite. Since $\Gamma_H$ is of finite index in $SL_2(\mathbb{Z})$, the result follows. $\square$

The Hecke operators respect the decomposition of $S_k(\Gamma_G)$ into old and new.

**Proposition 5.3.10.** The subspaces $S_k(\Gamma_G)^{\text{old}}$ and $S_k(\Gamma_G)^{\text{new}}$ are stable under the Hecke operators $T_n$ for $n \in \det(G)$ and the diamond operators $\langle\sigma_p\rangle$, for $p \in \det(G)$.

**Proof.** By Corollary 5.3.6 the Hecke operators $T_p$ for $p \in \det(G)$ commute with the degeneracy maps $\alpha_t$. By multiplicativity, it extends to $T_n$ such that $n \in \det(G)$.

For the diamond operators, let $G_i \subseteq GL_2(\mathbb{Z}/N_i\mathbb{Z})$ for $i = 1, 2$ be such that $\lambda_{N_2}(G_1) \subseteq G_2$, and let $t \in M_2(\mathbb{Z})$ be such that $\det(t) \mid N_1/N_2$ and $t^{-1}\Gamma_1t \subseteq \Gamma_2$, where $\Gamma_i$ is the congruence subgroup induced by $G_i$. Let $d = \det(t)$. Then $dt^{-1} \in M_2(\mathbb{Z})$ and so if $\sigma$ is such that $\lambda_{N_2}(\sigma) \subseteq G$ and $\det(\sigma) = p$, we have

$$\lambda_{N_2,d}(dt^{-1}\sigma p) = \lambda_{N_2,d}(dt^{-1})\lambda_{N_2,d}(p \cdot \lambda_{N_2,d}(\sigma^{-1}) \cdot \lambda_{N_2,d}(t) \in p \cdot \lambda_{N_2,d}(dt^{-1})\lambda_{N_2,d}(G_1)\lambda_{N_2,d}(t)$$

However, since $t^{-1}\Gamma_1t \subseteq \Gamma_2$, we also have

$$\lambda_{N_2,d}(dt^{-1})\lambda_{N_2,d}(G_1)\lambda_{N_2,d}(t) \subseteq d \cdot \lambda_{N_2,d}(\Gamma_2)$$

hence $t^{-1}\sigma p t \in M_2(\mathbb{Z})$ and moreover $\lambda_{N_2}(t^{-1}\sigma p t) \in p \cdot \lambda_{N_2}(\Gamma_2) = p \cdot G_2 \subseteq \mathcal{H}(G_2)$. It follows that $t^{-1}\sigma p t \in \Gamma_{\mathcal{H}(G_2)}$, and so for any $f \in S_k(\Gamma_2)$ we have

$$\langle (\sigma p) \circ \alpha_t \rangle (f) = f|_{t^{-1}\sigma p} = f|_{t^{-1}\sigma p t^{-1}} = \left( \alpha_t \circ \langle t^{-1}\sigma p t \rangle \right) (f).$$

Finally, we note that $\lambda_{N_2}(t^{-1}\sigma p t) = p \cdot \lambda_{N_2}(t^{-1}\alpha t)^{-2}$, and that by the proof of Corollary 5.3.2 $t^{-1}\alpha t \in M_2(\mathbb{Z})$ is an element such that $\lambda_{N_2}(t^{-1}\alpha t) \in G_2$. It follows that $\langle t^{-1}\sigma p t \rangle$ is the diamond operator at $p$ for $G_2$. This concludes the proof. $\square$
Using the results of section 4, we may now compute the $q$-expansions of eigenforms in $S_k(\Gamma_G)$, thus answering Problem 2.2.3.

All the ideas and methods used in this section are the same as in the case of Iwahori level, and are known (see [30], [29]).

The process for doing so is divided into three steps:

1. Compute a subspace of $M_k(\Gamma)^\vee$ that is isomorphic to $S_k(\Gamma)$ as a Hecke module, denoted by $S_k(\Gamma)^\vee$.
2. Decompose the space $S_k(\Gamma)^\vee$ to irreducible Hecke modules.
3. For each irreducible Hecke module in the decomposition, find the system of Hecke eigenvalues, hence a $q$-expansion.

6.1. Constructing Dual Vector Spaces. Since our interest lies in $S_k(\Gamma)$, we would like to have a vector space which is isomorphic to $S_k(\Gamma)^\oplus S_k(\Gamma)$ as a Hecke module. We could do it by using the pairing (3.6.1), but that would involve computation of integrals and approximation if done directly.

Instead, we can use the fact that $S_k(\Gamma)$ and $S_k(\Gamma)$ are isomorphic as Hecke modules, thus leading to the following algorithm.

**Algorithm 6.1.1.** DualVectorSpace($M$). Compute the dual vector space.

**Input:** $M \subseteq M_k(\Gamma)$ a subspace.

**Output:** $M^\vee \subseteq M_k(\Gamma)^\vee$ which is isomorphic to $M$ as a Hecke module.

1. $V := M_k(\Gamma)^\vee$, $p := 2$
2. While ($\dim V > \dim M$) do
   a. $T_p := $ HeckeOperator($V, p$).
   b. $T_{p,M} := $ HeckeOperator($M_k(\Gamma), p$)$|M$.
   c. $c_p := $ CharPoly($T_{p,M}$).
   d. $V := $ ker($c_p(T_p)$).
   e. $p := $ NextPrime($p$).
3. Return $V$.

Here, we assume the existence of a function CharPoly, that given an operator on a vector space, computes its characteristic polynomial.

**Remark 6.1.2.** We have not defined the Hecke operators $T_n$ for $(n, N) > 1$, and these cannot be ignored (see [6, Example 12.2.11]). It is possible to circumvent this problem, as is done in [6, Section 8.10]. Practically when $p \mid N$, we use instead of $T_p$ the double coset Hecke operators $T_\alpha$ with det($\alpha$) = $p$. There are finitely many such operators, and the operator $T_p$ is a linear combination of them. Thus, for the purpose of describing the dual vector space, and later on, decomposition of the space to irreducible Hecke modules, these suffice.

6.2. Decomposition of $S_k(\Gamma)$. The following theorem is a generalization of [29] Theorem 9.23. It is an application of Sturm’s Theorem, which will enable us to decompose the Hecke module $S_k(\Gamma)$ in a finite (effectively bounded) amount of steps.
**Theorem 6.2.1.** Suppose \( \Gamma \) is a congruence subgroup of level \( N \), and let

\[
(6.2.2) \quad r = \text{Sturm}(k, \Gamma) := \left\lfloor \frac{km}{12} - \frac{m-1}{N} \right\rfloor
\]

where \( m = [SL_2(\mathbb{Z}) : \Gamma] \). Then the Hecke algebra

\[
\mathcal{T} = \mathbb{Z}[\ldots, T_n, \ldots] \subseteq \text{End}(S_k(\Gamma))
\]

is generated as a \( \mathbb{Z} \)-module by the Hecke operators \( T_n \) for \( n \leq r \).

**Proof.** Same as in [29, Theorem 9.23]. \( \square \)

This allows us to decompose the space \( S_k(\Gamma)^\vee = S_k(\Gamma) \oplus \overline{S}_k(\Gamma) \) to irreducible Hecke modules, as follows.

**Algorithm 6.2.3.** Decomposition \((M, p)\). Decomposition to irreducible Hecke modules.

**Input :**
- \( M \subseteq M_k(\Gamma) \) - a vector subspace stable under the Hecke action.
- \( p \) - a prime.

**Output :** \( D \), a list of subspaces of \( M \), such that \( M = \bigoplus_{V_d \in D} V_d \) and \( V_d^\vee \) is an irreducible Hecke module.

1. If \( p \mid N \), replace by a larger prime such that \( p \nmid N \).
2. \( T_p := \text{HeckeOperator}(M^\vee, p) \), \( D := \{\} \).
3. \( f := \text{CharPoly}(T_p) \). Write \( f = \prod_{i=1}^l f_i^{a_i} \).
4. for \( i \in \{1, 2, \ldots, l\} \) do
   (a) \( V := \ker(f_i(T_p)^{a_i}) \subseteq M^\vee \), \( W := V^\vee \subseteq M \).
   (b) If \( \text{IsIrreducible}(W) \) then \( \text{Append}(D, W) \) else
      (i) if \( W = M \) then \( q := \text{NextPrime}(p) \) else \( q := 2 \).
      (ii) \( D := D \cup \text{Decompose}(W, q) \).
5. Return \( D \).

Here Decompose is a recursive call to the function, and \( \text{IsIrreducible} \) is creating a random linear combination of Hecke operators, and checks that its characteristic polynomial (on the plus subspace) is irreducible.

Calling Algorithm [6.2.3] with \( M = S_k(\Gamma) \), \( p = 2 \) we obtain the decomposition we wanted.

Note that the algorithm terminates if there are no eigenvectors with multiplicities, that is there are no oldforms arising as images of forms of lower levels. In that case, the number of steps in the algorithm can be bounded using [6.2.1].

In order to deal with the images of oldforms, we may begin by identifying the new subspace, and continuing recursively. The new subspace is computed by noting that it is dual to \( S_k(\Gamma_G)^\text{new} := \bigcap_{G' \in \text{Over}(G), t \in T_{G'}} \ker(\alpha^\vee) \).

Also, in order to get the old subspace one need not compute the Petersson inner product, but instead note that it is generated by the images of the \( \beta_t^\vee \), \( S_k(\Gamma_G)^\text{old} := \langle \text{Im}(\beta_t^\vee)_{G' \in \text{Over}(G), t \in T_{G'}} \rangle \).
6.3. **Computing the q-Expansions.** Finally, for a subspace $V \subseteq S_k(\Gamma_G)$ such that $V^\vee$ is an irreducible Hecke module, we can compute the $q$-expansion of an eigenform (and hence of all eigenforms in $V^\vee$).

**Algorithm 6.3.1.** $q$Eigenform$(V, L)$. Compute the $q$-expansion of an eigenform.

**Input :**
- $V \subseteq S_k(\Gamma_G)$ such that $V^\vee$ is an irreducible Hecke module.
- $L$ - a positive integer.

**Output :** $F \in \mathbb{Q}[[q]]$ such that $f = F(e^{2\pi i \tau / N}) = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma)$ and $T_n f = a_n f$ for all $n$ (set $a_1 = 1$), specified to precision $q^L$.

1. Find (randomly) a linear combination of Hecke operators which is irreducible on $(V^\vee)^+$, denote it by $T$.
2. Let $v$ be an eigenvector of $T$ over the field $\mathbb{Q}[x]/f_T(x)$, where $f_T = \text{CharPoly}(T)$.
3. Write $v = \sum_{j=i}^{d} c_j e^\vee_j$, where $e_j$ are the basis vectors of $M_k(\Gamma)$ given by Manin symbols, and $c_i \neq 0$.
4. For primes $p < L$, set $a_p := \frac{1}{c_i} \langle T_p(e_i), v \rangle$. Complete the other $a_n$'s using multiplicativity.
5. Return $\sum_{n=1}^{L-1} a_n \cdot q^n + O(q^L)$.

We start by showing that the algorithm returns the correct output, and measure its complexity.

**Lemma 6.3.2.** Let $G \subseteq GL_2(\mathbb{Z}/N\mathbb{Z})$ be a group of real type of index $I_G$ with surjective determinant such that $T_p$ is effectively computable for all $p \mid N$. There exists an algorithm that given a subspace $V$ of $S_k(\Gamma_G)$ of dimension $d$, such that $V^\vee$ is irreducible as a module for the Hecke algebra, and an integer $L$, returns the $q$-expansion of an eigenform up to precision $q^L$ in $O(C \log N (L \log L + N) + N I_G^2 + d^3)$, where $C$ is the cost of a basic CosetIndex operation.

**Proof.** We apply Algorithm 6.3.1. Since $G$ is of real type, by Corollary 4.1.9, the Hecke operator commutes with the star involution, so that step (1) makes sense. For any $p < L$ we have

$$\sum_{j=i}^{d} a_p c_j e^\vee_j = a_p v = T_p^\vee v$$

In particular,

$$\langle T_p(e_i), v \rangle = \langle e_i, T_p^\vee v \rangle =$$

$$= \langle e_i, \sum_{j=i}^{d} a_p c_j e^\vee_j \rangle = a_p \cdot c_i$$

For computation of the Hecke operators away from the level, we apply Corollary 4.5.11, while for the Hecke operators at primes $p \mid N$, we apply Theorem 4.2.10. Note that as the $T_p$ are effectively computable for all $p \mid N$, we may do so with no additional cost to our complexity.
Therefore, the complexity of computing the Hecke operators is given by
\[
C \cdot \sum_{p \leq L, p \nmid N} p \log p + C \cdot \sum_{p \leq L, p \mid N} p \log(N^{2} \cdot p) + \sum_{p \leq L, p \mid N} I_{G}^{2} \cdot \text{In} = \\
= C \cdot \sum_{p \leq L} p \log p + 2C \cdot \log N \sum_{p \leq L, p \mid N} p + N I_{G}^{2} \cdot \text{In} = \\
= O(C \cdot (L \log L + N) \log N + N I_{G}^{2} \cdot \text{In}).
\]
Finally, the \(d^{3}\) contribution comes from the linear algebra operations. \(\square\)

\[\text{We can now pack everything together to produce the following corollary.}\]

**Corollary 6.3.3.** There exists an algorithm that given a group of real type \(G \subseteq GL_{2}(\mathbb{Z}/N\mathbb{Z})\) with surjective determinant such that for all \(p \mid N\) the Hecke operator is effectively computable, an integer \(k \geq 2\), and a positive integer \(L\), returns the \(q\)-expansions of a basis of eigenforms for \(S_{k}(\Gamma_{G})\) using

\[O(d(C \log N (L \log L + N) + N I_{G}^{2} \cdot \text{In} + k I_{G} \log(k I_{G})) + d^{3})\]

field operations, where \(d := \dim S_{k}(\Gamma_{G}), I_{G} := [SL_{2}(\mathbb{Z}) : \Gamma_{G}], C\) is the cost of a CosetIndex operation, and \(\text{In}\) is the cost of membership testing in \(G\).

**Proof.** We first compute a basis for \(S_{k}(\Gamma)\) using Theorem 3.5.1. Then we can apply Algorithm 6.2.3 to decompose it to irreducible subspaces. Here, we have to compute the matrices of the Hecke operators up to the Sturm bound (6.2.2), which is linear in \(k I_{G}\).

Since the number of primes dividing the level \(N\) is finite, in order to compute the complexity it is enough to use Corollary 4.5.11. As a single computation of \(T_{p}\) costs \(O(p \log p)\), computing its matrix will cost \(O(d \cdot p \log p)\).

Summing these over all primes up to \(I_{G}\), we get \(O(d \cdot k I_{G} \log(k I_{G}))\).

Finally, for each of the irreducible subspaces, we should apply Algorithm 6.3.1 which by 6.3.2 costs \(O(L \log L)\) basic CosetIndex operations. The \(d^{3}\) additional contribution comes from the linear algebra operations on the matrices of operators on \(S_{k}(\Gamma)\). \(\square\)

7. Applications

In this section, we present a few of the applications that the above result contributed to, and also some future applications. All time measurements were taken on a MacBook with 2.3 Ghz 8-Core Intel Core i9 processor, and 16 GB 2400 MHz DDR4 memory.

7.1. Classification of 2-adic images of Galois representations associated to elliptic curves over \(\mathbb{Q}\). In [23], the authors compute the maximal tower of 2-power level modular curves containing a non-cuspidal non-CM rational point, and develop special techniques to compute their equations. Using our implementation, one can obtain the \(q\)-expansions of a basis for \(S_{2}(\Gamma)\), and hence a projective embedding, for each of those groups.

For example, in [23 Example 6.1] the authors present a curious example associated with the group \(H_{155}\), which is of index 24 and level 16, whose image in \(GL_{2}(\mathbb{Z}/16\mathbb{Z})\) is generated
Then a quick calculation of 1.78 seconds with our implementation yields the following.

```plaintext
> tt := Cputime();
> gens := [[1,3,12,3],[1,1,12,7],[1,3,0,3],[1,0,2,3]];
> N := 16;
> H_N := sub<GL(2,Integers(N)) | gens>;
> H := PSL2Subgroup(H_N);
> M := ModularSymbols(H, 2, Rationals(), 0);
> S := CuspidalSubspace(M);
> D := Decomposition(S, HeckeBound(S));
> qEigenform(D[1],100);
q - 4*q^5 - 3*q^9 - 4*q^13 - 2*q^17 + 11*q^25 - 4*q^29 +
12*q^37 - 10*q^41 + 12*q^45 - 7*q^49 - 4*q^53 + 12*q^61 +
16*q^65 - 6*q^73 + 9*q^81 + 8*q^85 + 10*q^89 - 18*q^97 + O(q^100)
> Cputime(tt);
1.780
```

Proceeding to compute the invariants of the elliptic curve associated to this eigenform, we see that the modular curve $X_H$ is the elliptic curve labelled [256b1], defined by the equation $y^2 = x^3 - 2x$ over $\mathbb{Q}$, for which $X_H(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ is generated by $(0, 0)$ and $(-1, -1)$.

**Remark 7.1.1.** For this application, and some of the applications that follow, it is important to compute the $j$-map when the resulting curve is an elliptic curve. This can be done by computing the $q$-expansion of Eisenstein modular forms, and at the time of writing this paper is yet to be implemented.

### 7.2. Modular curves of prime-power level with infinitely many rational points.

In [31, Corollary 1.6], the authors introduce a finite set of subgroups $G$ of $GL_2(\mathbb{Z}_l)$ that arise as the image of a Galois representation for infinitely many elliptic curves over $\mathbb{Q}$ with distinct $j$-invariants.

In section 6, they consider the 250 cases where $G$ has genus 1 and show that the Jacobian of $X_G$, $J_G$, for each such group is isogenous to one of a certain finite set of elliptic curves. Then they compute point counts of the reduction modulo several primes to pinpoint the correct isogeny class over $\mathbb{Q}$, and show that one needs only consider a certain set of 28 groups.

These are included in the examples computed in [23], and again, using our implementation, it is possible to construct for each of these subgroups $G$ a basis of $q$-eigenforms for the modular curve $X_G$, and in particular find a projective embedding for it.

The following example took 2.120 seconds.

```plaintext
> tt := Cputime();
> gens := [[2,1,3,2],[0,3,5,8],[1,0,0,5],[1,8,0,3]];
> N := 16;
> H_N := sub<GL(2,Integers(N)) | gens>;
> H := PSL2Subgroup(H_N);
```
> M := ModularSymbols(H, 2, Rationals(), 0);
> S := CuspidalSubspace(M);
> D := Decomposition(S, HeckeBound(S));
> qEigenform(D[1],100);
q - 2*q^3 + q^9 - 6*q^11 - 6*q^17 - 2*q^19 - 5*q^25 + 4*q^27 + 12*q^33 + 6*q^41
  + 10*q^43 - 7*q^49 + 12*q^51 + 4*q^57 - 6*q^59 + 14*q^67 - 2*q^73 + 10*q^75
  - 11*q^81 - 18*q^83 - 18*q^89 + 10*q^97 - 6*q^99 + O(q^100)
> Cputime(tt);
2.120

This is the newform \([256.2.2.a]\). Computing the elliptic invariants, we see that this is the
curve \([256a2]\) defined by \(y^2 = x^3 + x^2 - 13x - 21\).

7.3. **Efficient computation of \(q\)-eigenforms for \(X_{ns}(N)\) and \(X^+_{ns}(N)\).** Using the Merel
pair introduced in Example 4.5.6 and a similar one for \(\Gamma^+_{ns}(p)\), its normalizer, one obtains
a highly efficient implementation of the Hecke operators \(T_n\) for all \(n\). This allows one to
compute the \(q\)-eigenform and verify the results of [12] for \(X_{ns}(11)\), [5] for \(X^+_{ns}(13)\), and [18]
for \(X^+_{ns}(17), X^+_{ns}(19)\) and \(X^+_{ns}(23)\), as follows.

> tt := Cputime();
> G := GammaNSplus(13);
> M := ModularSymbols(G, 2, Rationals(), 0);
> S := CuspidalSubspace(M);
> D := Decomposition(S, HeckeBound(S));
> qEigenform(D[1], 20);
q + a*q^2 + (-a^2 - 2*a)*q^3 + (a^2 - 2)*q^4 +
(a^2 + 2*a - 2)*q^5 + (-a - 1)*q^6 + (a^2 - 3)*q^7 +
(-2*a - 2*a + 1)*q^8 + (a^2 + 3*a - 1)*q^9 + (-a + 1)*q^10 + (-a^2 - 2*a - 2)*q^11 + (a^2 + 3*a)*q^12 +
(-2*a^2 - 2*a + 1)*q^14 + (a^2 + 2*a - 2)*q^15 +
(-a^2 - a + 2)*q^16 + (-a^2 + a + 2)*q^17 +
(a^2 + 1)*q^18 + (-2*a^2 - 2*a - 2)*q^19 + O(q^20)
> BaseRing(Parent($1));
Number Field with defining polynomial \(x^3 + 2*x^2 - x - 1\)
over the Rational Field
> Cputime(tt);
0.160

This coincides with the result in section 4 in [5], and took merely 0.16 seconds.

Similarly, in only 0.33 seconds we get the result for \(X^+_{ns}(17)\).

> tt := Cputime();
> G := GammaNSplus(17);
> M := ModularSymbols(G, 2, Rationals(), 0);
> S := CuspidalSubspace(M);
> D := Decomposition(S, HeckeBound(S));
> [*qEigenform(d, 20) : d in D*];
[*
q - q^2 - q^4 + 2*q^5 - 4*q^7 + 3*q^8 - 3*q^9 - 2*q^10 - 2*q^13 + 4*q^14 - q^16 + 3*q^18 - 4*q^19 + O(q^20),
q + a*q^2 + (-a - 1)*q^3 + (-a + 1)*q^4 + a*q^5 - 3*q^6 + (-a - 2)*q^7 + 3*q^8 + (a + 1)*q^9 + (-a + 3)*q^10 - 3*q^11 + (-a + 2)*q^12 + (a - 3)*q^13 + (-a - 3)*q^14 - 3*q^15 + (-a - 2)*q^16 + 3*q^18 + (-3*a - 1)*q^19 + O(q^20),
q + a*q^2 + (-a^2 + 1)*q^3 + (a^2 - 2)*q^4 + (-a - 2)*q^5 + (-2*a + 1)*q^6 + (a^2 - 2)*q^7 + (-a - 2)*q^8 + (a - 2)*q^9 + (a - 2 - 2*a)*q^10 + (2*a^2 - 2*a - 6)*q^11 + (-a^2 + a - 1)*q^12 + (-2*a^2 - 3*a + 6)*q^13 + (a - 1)*q^14 + (2*a^2 + 2*a - 3)*q^15 + (-3*a^2 - 2*a - a + 4)*q^16 + (-a - 2 + a - 1)*q^18 + a*q^19 + O(q^20)
]*
> BaseRing(Parent(qEigenform(D[2])));
Number Field with defining polynomial x^2 + x - 3
over the Rational Field
> BaseRing(Parent(qEigenform(D[3])));
Number Field with defining polynomial x^3 - 3*x + 1
over the Rational Field
> Cputime(tt);
0.330

which coincides with the results of [18].

In [4], section 6, there is a description of coset representatives for \(\Gamma_{ns}(N)\) and \(\Gamma_{ns}^+(N)\) for general \(N\), which again makes explicit the implementation via modular symbols.

### 7.4. Decomposition of the Jacobian of \(X_{ns}^+(p)\).

Applying Algorithm [6.2.3] to the space \(S_2(\Gamma_{ns}^+(p))\), where \(\Gamma_{ns}^+(p)\) yields a decomposition of the Jacobian of \(X_{ns}^+(p)\). This has been computed using our code for all \(p \leq 97\). (see [2])

### 7.5. Computation of \(q\)-Eigenforms for \(X_{G}(p)\) when \(G\) is exceptional.

Similarly, the code can be used to compute \(q\)-eigenforms for the modular curves \(X_{G}(N)\), when \(G\) is an exceptional subgroup of \(GL_2(\mathbb{F}_p)\), e.g. running it for \(X_{S_4}(13)\) we recover [3, Theorem 1.8], which in the original paper is the content of entire section 4 consisting of 7 steps, in just 0.43 seconds. (We use the embedding of \(S_4 \hookrightarrow GL_2(\mathbb{F}_p)\) using the quaternions as describes in [33, Section 11.5]).

> tt := Cputime();
> p := 13;
> O := QuaternionOrder([1,i,j,k]);
> _, mp := pMatrixRing(O,p);
> S4tp := sub,GL(2,p) | [mp(1+s) : s in [i,j,k]]
> cat [mp(1-s) : s in [i,j,k]] cat [-1]>;
> H_S4 := sub,GL(2,Integers(p)) | Generators(S4tp)>
> G_S4 := PSL2Subgroup(H_S4);
> M := ModularSymbols(G_S4, 2, Rationals(), 0);
> S := CuspidalSubspace(M);
> D := Decomposition(S, HeckeBound(S));
> [*qEigenform(d,20) : d in D*];
> [
> q + a*q^2 + (-a^2 - 2*a)*q^3 + (a^2 - 2)*q^4 +
> (a^2 + 2*a - 2)*q^5 + (-a - 1)*q^6 + (a^2 - 3)*q^7 +
> (-2*a^2 - 3*a + 1)*q^8 + (a^2 + 3*a - 1)*q^9 +
> (-a + 1)*q^10 + (-a^2 - 2*a - 2)*q^11 + (a^2 + 3*a)*q^12 +
> (-2*a^2 - 2*a + 1)*q^14 + (a^2 + a - 2)*q^15 +
> (-a^2 - a + 2)*q^16 + (-a^2 - 2*a - 2)*q^17 +
> (a^2 + 1)*q^18 + (-2*a^2 - a + 2)*q^19 + O(q^20)
> *]
> BaseRing(Parent(qEigenform(D[1])));
Number Field with defining polynomial x^3 + 2*x^2 - x - 1
over the Rational Field
> Cputime(tt);
0.430

7.6. Smooth plane models for modular curves. Modular curves of genus 0, 1 obviously admit a smooth plane model always. In general, if a curve admit such a model, of degree $d$, then its genus would be $g = \frac{(d-1)(d-2)}{2}$, so the next numbers to check are $g = 3, 6$. When $g = 3$, by [13], the generic case is a smooth plane quartic, so the answer is "almost always".

However, for $g > 3$, a generic curve does not admit a smooth plane model.

The smallest case for which we do not know the answer is $g = 6$. Thus, computing a basis of eigenforms of $S_2(\Gamma)$ for each of the congruence subgroups $\Gamma$ of genus 6 (see [10]), will help us to get a canonical model and from it (maybe) decide whether there exists a plane model.

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