Abstract
We study the trinified model, SU(3)_C × SU(3)_L × SU(3)_R × Z_3, with the minimal Higgs sector required for symmetry breaking. There are five Higgs doublets, and gauge-coupling unification results if all five are at the weak scale, without supersymmetry. The radiative see-saw mechanism yields sub-eV neutrino masses, without the need for intermediate scales, additional Higgs fields, or higher-dimensional operators. The proton lifetime is above the experimental limits, with the decay modes p → νK^+ and p → µ^+K^0 potentially observable. We also consider supersymmetric versions of the model, with one or two Higgs doublets at the weak scale. The radiative see-saw mechanism fails with weak-scale supersymmetry due to the nonrenormalization of the superpotential, but operates in the split-SUSY scenario.
1 Introduction

Grand unification of the strong, weak, and electromagnetic interactions into a simple gauge group is a very appealing idea that has been vigorously pursued for many years \[1\]. In recent years most effort has gone into \(SU(5)\) and \(SO(10)\) grand-unified models, as well as \(E_6\). Among its most notable successes, grand unification predicted neutrino masses in the \(10^{-5} - 10^2\) eV range \[2\], compatible with the masses deduced from neutrino oscillation experiments \[3\].

An alternative to unification into a simple gauge group is unification into a product group, with identical factor groups. The simplest and most promising theory is the trinified model, \([SU(3)]^3 = SU(3)_C \times SU(3)_L \times SU(3)_R\) \[4, 5\]. The equality of the three gauge couplings is enforced by a discrete symmetry, such as \(\mathbb{Z}_3\). Alternatively, the equality of the gauge couplings may be a result of some additional structure at the unification scale, such as string theory \[6\].

\([SU(3)]^3\) is a subgroup of \(E_6\), as are \(SU(5)\) and \(SO(10)\). All of these groups share the common feature that \(\sin^2 \theta_W = \frac{3}{8}\) at the grand-unified scale. The \([SU(3)]^3\) model, however, is less unified than the models based on simple groups. This is both its weakness and its strength: it makes fewer predictions, but it also does not run into phenomenological difficulties as readily.

The implications of (non-supersymmetric) trinification were first studied in some detail in Ref. \[7\], but with restricted Yukawa couplings to leptons.\(^1\) Those couplings not only determine the lepton masses but also the couplings of colored Higgs bosons to quarks, which yield proton decay. In this paper we revisit this model, and completely elucidate its structure without making any restrictions on the Yukawa couplings. Furthermore, Ref. \[7\] used a crude technique to calculate the proton decay branching ratios. We will show that with unrestricted lepton Yukawa couplings and a better technique for the calculation of proton decay, the branching ratios are significantly different than those previously obtained.

Trinification has a number of nice features beyond those of models based on simple gauge groups:

- Baryon number is conserved by the gauge interactions. This allows the possibility of lowering the grand-unified scale. The (non-supersymmetric) standard model with six Higgs doublets yields gauge-coupling unification (with \(\sin^2 \theta_W = \frac{3}{8}\)) at \(M_U \simeq 10^{14}\) GeV \[10\]. As we shall see, the minimal\(^2\) trinified model can have as many as five light Higgs doublets, which is also sufficient for gauge-coupling unification. As usual, the supersymmetric version of the theory needs just two light Higgs doublets (and their superpartners), or even just one in the split-SUSY scenario \[11\].

- In the minimal model, the only Higgs representations needed to break the gauge group to the standard model and to generate realistic quark and lepton masses are built from the defining representations of \(SU(3)\). In particular, no adjoint Higgs field is needed. Furthermore, the light Higgs doublets that break the electroweak symmetry lie in the same representations as the Higgs field that break the grand-unified symmetry.

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\(^1\)The supersymmetric model has been studied in Refs. \[8\], recently also in the context of orbifold GUTs \[9\].

\(^2\)By minimal we mean a Higgs sector that contains just the fields needed to break \([SU(3)]^3\) to the standard model.
The minimal renormalizable model is sufficiently flexible to accommodate any quark and lepton masses and mixing angles. This may be viewed as a strength or a weakness, as mentioned above. Neutrinos acquire eV-scale masses via a “radiative see-saw” mechanism \[12\], both in the non-supersymmetric and the split-SUSY versions of the model. If the mass differences of the SUSY partners is \(O(1 \text{ TeV})\), this mechanism fails.

As mentioned above, \([SU(3)]^3\) is a subgroup of \(E_6\), and thus also a subgroup of \(E_8 \times E_8\), which makes it a candidate for embedding in the heterotic string. However, \([SU(3)]^3\) is a candidate for heterotic string constructions itself, since adjoint fields are not needed to break this gauge group to the standard model \[5\]. In addition, the smallest \(E_6\) Higgs representation capable of breaking \(E_6\) to \([SU(3)]^3\) is the 650. In this paper we study the \([SU(3)]^3\) model on its own.

This paper is organized as follows. In Section 2 we review the trinified model and derive the fermion masses and eigenstates for one generation. In Section 3 we consider the case of three generations, in particular quark mixing and the hierarchy of neutrino masses. Section 4 is devoted to proton decay, where we estimate the dominant decay modes. Conclusions are given in Section 5. Additional details are given in several Appendices.

## 2 Trinified Model

We begin by briefly reviewing the \(SU(3)_C \times SU(3)_L \times SU(3)_R \times Z_3\) model \[4-7\]. The \(Z_3\) symmetry guarantees that the three gauge couplings are equal at and above the grand-unified scale. The gauge bosons are assigned to the adjoint representation, the fermions to

\[
\psi_l \rightarrow (1, 2, \frac{1}{2}) \oplus 2 (1, 2, -\frac{1}{2}) \oplus (1, 1, 1) \oplus 2 (1, 1, 0),
\]

\[
\psi_{l^c} \rightarrow (3^*, 1, -\frac{2}{3}) \oplus 2 (3^*, 1, \frac{1}{3}),
\]

\[
\psi_q \rightarrow (3, 2^*, \frac{1}{6}) \oplus (3, 1, -\frac{1}{3}).
\]

Thus we find the fifteen left-chiral fermions of the standard model plus twelve additional fermions. More explicitly,

\[
\psi_l = \begin{pmatrix}
(e^{c}) \\
N_1 \\
E^{c}
\end{pmatrix}, \quad \psi_{l^c} = \begin{pmatrix}
D^{c} \\
e^{c}
\end{pmatrix}, \quad \psi_q = \begin{pmatrix}
(-d \quad u) \\
B^c
\end{pmatrix}.
\]

The field \((-d, u)\) is the (conjugate of the) usual quark doublet \(Q = (u)\), while \(B\) is an additional color-triplet, weak-singlet quark. The field \(e^c\) is the usual up-conjugate quark field, while \(D^c\) and \(B^c\) have the quantum numbers of the down-conjugate quark field. The actual down-conjugate field

\[3\]Instead of \(Z_3\), Refs. \[13\] use \(S_3\).

\[4\]Our notation differs from that in Refs. \[4-7\], since \(\psi_l\) is an SU(3)_L-triplet, which contains SU(2)_L-doublets, whereas \(\psi_q\) is an SU(3)_L-anti-triplet. In the other notation, \(\psi_l\) contains SU(2)_L-anti-doublets while \(\psi_q\) contains SU(2)_L-doublets. The resulting fermion masses and mixings agree, however.
$d^c$ is a linear combination of the two, as we shall see. The field $e^c$ is the usual positron field, and the lepton doublet is a linear combination of $\mathcal{L}$ and $\mathcal{E}$. The field $E^c$ denotes a lepton doublet with the opposite hypercharge, and $\mathcal{N}_1$ and $\mathcal{N}_2$ are sterile (with respect to the standard model) fermions.

The gauge interactions have an accidental $U(1)_Q \times U(1)_{Q^c} \times U(1)_L$ global symmetry corresponding to phase rotations of the fermion multiplets $\psi_u, \psi_{Q^c}, \psi_L$. The linear combination $U(1)_{Q-Q^c}$ is proportional to baryon number, so proton decay is not mediated by gauge interactions. As we shall see, proton decay is mediated by Yukawa interactions. The global symmetry $U(1)_L$ is not lepton number, as the lepton multiplet $\psi_L$ contains both leptons and antileptons.

$[SU(3)]^3 \times \mathbb{Z}_3$ is broken by a pair of $(1,3,3^*)$ Higgs fields, which we denote by $\Phi^1_L, \Phi^2_L$,

$$\Phi^a_L = \begin{pmatrix} \phi^a_1 \\ S^a_1 \\ S^a_2 \\ S^a_3 \end{pmatrix}, \quad \langle \Phi^1_L \rangle = \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & v_1 \end{pmatrix}, \quad \langle \Phi^2_L \rangle = \begin{pmatrix} n_1 & 0 & n_3 \\ 0 & n_2 & 0 \\ v_2 & 0 & v_3 \end{pmatrix}, \quad (3)$$

with $v_i = \mathcal{O}(M_U)$ and $u_i, n_i = \mathcal{O}(M_{EW})$, where $M_U$ and $M_{EW}$ are the unification and electroweak scales, respectively. Both $v_1$ $(v_3)$ and $v_2$ break $SU(3)_L \times SU(3)_R$ to $SU(2)_L \times SU(2)_R \times U(1)$, but the $SU(2)_R \times U(1)$ are different. Together they break $[SU(3)]^3 \times \mathbb{Z}_3$ to $SU(3)_C \times SU(2)_L \times U(1)_Y$.

Of the six Higgs doublets $(\phi_i, \psi_i$, each in $\Phi^1_L, \Phi^2_L)$, one linear combination$^6$ is eaten by the gauge bosons that acquire unification-scale masses. If the remaining five doublets have electroweak-scale masses, then gauge-coupling unification results at $M_U \simeq 10^{14}$ GeV without supersymmetry$^10$. In general it would take several fine-tunings to arrange this, so it is an even more acute form of the usual hierarchy problem. We do not address this problem further in this paper.

Another potential drawback of five light Higgs doublets is Higgs-mediated flavor-changing neutral currents, which are present whenever a fermion of a given electric charge couples to more than one Higgs field$^14$. Whether such interactions are present at an acceptable level depends on the details of fermion mass generation, which we do not pursue in this paper. Higgs-mediated flavor-changing neutral currents may be suppressed by small Yukawa couplings$^15$.

Alternatively, one may consider the supersymmetric version of the theory, in which case just two light Higgs doublets (and their superpartners) are required to yield successful gauge-coupling unification$^16$. However, the supersymmetric version of trinification, in its minimal incarnation, does not provide a see-saw mechanism for neutrino masses. This problem is ameliorated in the split-SUSY version of the model$^17$, as we will discuss.

The electroweak symmetry is broken to $U(1)_{EM}$ when any of the five Higgs doublets acquires an electroweak-scale vev. These are indicated by the vevs $u_i, n_i$ in Eq. (3).

For simplicity, we set $v_3 = 0$ henceforth; this does not have any effect on the qualitative aspects of the model. In order to generate masses for up-type quarks, we need $u_2$ and/or $n_2$ nonzero; for down-type quarks and charged leptons, $u_1, n_1, n_3$ are the relevant vevs. Henceforth we choose $u_1$ and $u_2$ nonzero, and set $n_i = 0$. Again, this does not affect the qualitative aspects of the model. The general expressions for the fermion masses, with all vevs nonzero, are given in Appendix A.

$^5$With three generations, the accidental global symmetry is $U(3)_Q \times U(3)_{Q^c} \times U(3)_L$. However, only the $U(1)$ subgroups will be important for later discussion.

$^6G \propto v_1 \phi^3_1 + v_2 \phi^3_2 + v_3 \phi^3_3$. 

4
2.1 Yukawa Interactions

With the Higgs fields $\Phi_L^a (1, 3, 3^*)$ (plus cyclic permutations), two types of Yukawa couplings are allowed, namely $\psi_{qL} \psi_q \Phi_L^a \equiv (\psi_{qL})^i_j (\Phi_L^a)_i^j$ for the quarks and $\psi_{lL} \psi_l \Phi_L^a \equiv \epsilon^{ijk} \epsilon_{ret} (\psi_{lL})^i_j (\Phi_L^a)_k^j$ for the leptons. The former read

$$\mathcal{L}_q = \psi_{qL} \psi_q \left( g_1 \Phi_L^1 + g_2 \Phi_L^2 \right) + \text{cyclic + h.c.}, \quad (4a)$$

where

$$\psi_{qL} \psi_q \Phi_L = \mathcal{D} Q \phi_1 + u^e Q \phi_2 + \mathcal{B} B \phi_3 + \mathcal{D} S_1 + u^e B S_2 + \mathcal{B} B S_3 . \quad (4b)$$

In Eq. (4b) we suppress both the superscript on the Higgs field and $\epsilon = i\sigma_2$, which is implicit between two SU(2)$_L$ doublets here and throughout the paper. When $S_3^1$ and $S_3^2$ acquire the vevs $v_1$ and $v_2$, respectively (see Eq. (3)), $B$ pairs up with a linear combination of $\mathcal{D}$ and $\mathcal{B}$ to form a Dirac fermion with a mass at the unification scale,

$$m_B = \sqrt{g_1^2 v_1^2 + g_2^2 v_2^2} . \quad (5)$$

The mass eigenstates are

$$d^c = -s_\alpha \mathcal{D} c + c_\alpha \mathcal{B} c, \quad B^c = c_\alpha \mathcal{D} c + s_\alpha \mathcal{B} c, \quad \tan \alpha = \frac{g_1 v_1}{g_2 v_2} , \quad (6)$$

where $s \equiv \sin$, $c \equiv \cos$. We can express the Yukawa couplings of these fields to the SU(2)$_L$-doublet Higgs fields as

$$\mathcal{L}_q = m_B B^c B + \sum_{a=1}^2 g_a \left[ (-s_\alpha d^c + c_\alpha B^c) Q \phi_1^a + u^e Q \phi_2^a + (c_\alpha d^c + s_\alpha B^c) Q \phi_3^a \right] + \text{h.c.} \quad (7)$$

When $\phi_{1,2}^1$ acquire the vevs $u_{1,2}$, the light quarks acquire masses

$$m_u = g_1 u_2 , \quad m_d = g_1 u_1 s_\alpha . \quad (8)$$

The heavy $B$ quark mass, as well as the quark mass eigenstates of Eq. (5), obtain tiny corrections $\mathcal{O}(\frac{1}{v^2})$, which we neglect here and throughout the paper.

The Yukawa couplings for the leptons are

$$\mathcal{L}_l = \frac{1}{2} \psi_{lL} \psi_l \left( h_1 \Phi_L^1 + h_2 \Phi_L^2 \right) + \text{cyclic + h.c.}, \quad (9)$$

where

$$\frac{1}{2} \psi_{lL} \psi_l \Phi_L = - (E^c N_2 - \mathcal{L} \, e^c) \phi_1 + (\mathcal{E} N_2 - \mathcal{L} \, N_1) \phi_2 + (E^c N_1 - \mathcal{E} e^c) \phi_3$$

$$+ E^c \mathcal{L} S_1 - \mathcal{E} \mathcal{L} S_2 - E^c \mathcal{E} S_3 . \quad (10)$$

Only the first term in Eq. (10) is used in Ref. [4], which is the restriction on the leptonic Yukawa couplings mentioned in the Introduction. The doublets $\mathcal{E}$ and $\mathcal{L}$ as well as the singlets $N_1$ and $N_2$ mix to

$$E = -s_\beta \, \mathcal{E} + c_\beta \, \mathcal{L} , \quad L = c_\beta \, \mathcal{E} + s_\beta \, \mathcal{L} , \quad \tan \beta = \frac{h_1 v_1}{h_2 v_2} . \quad (11a)$$
\[ N_1 = s_\beta N_1 - c_\beta N_2, \quad N_2 = -c_\beta N_1 - s_\beta N_2, \]  

so that

\[
\mathcal{L}_l = -m_E E^c E + \sum_{a=1}^{2} h_i \left\{ - [E^c (-c_\beta N_1 - s_\beta N_2) - (c_\beta E + s_\beta L) e^c] \phi_1^a + (E N_2 - L N_1) \phi_2^a \right. \\
+ \left[ E^c (s_\beta N_1 - c_\beta N_2) - (-s_\beta E + c_\beta L) e^c] \phi_3^a \right\} + \text{h.c.} \quad (12)
\]

The masses of the leptons are given by

\[
m_E = \sqrt{h_1^2 v_1^2 + h_2^2 v_2^2}, \quad m_e = h_1 u_1 s_\beta, \quad m_{\nu, N_1} = h_1 u_2, \quad m_{N_2} \simeq \frac{h_1^2 u_1 u_2 s_\beta}{m_E}. \quad (13)
\]

The general formulae for fermion masses, with all vevs nonzero, are given in Appendix A.

These results for the fermion masses show that, even in the minimal model, there is no relation between the masses of the quarks and leptons, since they depend on five independent parameters \(g_1, h_1, \frac{u_1}{u_2}, s_\alpha, s_\beta\). This is in contrast to minimal \(SU(5)\), which yields \(m_d = m_e\), or minimal \(SO(10)\), which yields \(m_d = m_e\) and \(m_u = m_\nu\). Thus the minimal trinification model is sufficient to describe the masses of the quarks and charged leptons. The additional matter charged under the standard model group is vectorlike and superheavy; at tree level, however, the model yields an active Dirac neutrino at the electroweak scale and a sterile Majorana neutrino \(N_2\) at the eV scale, both in (potential) conflict with observation. As we shall see, this may be corrected at one loop, via the “radiative see-saw” mechanism.

It is useful to understand the electroweak-scale Dirac neutrino \(\nu, N_1\), and the eV-scale sterile neutrino \(N_2\), in terms of accidental global symmetries. The accidental \(U(1)_Q \times U(1)_{Qc} \times U(1)_L\) global symmetry of the gauge interactions is violated by the Yukawa interactions, except for a \(U(1)_X\) subgroup in which the fermion multiplets carry charge \(\frac{1}{2}\) and the Higgs fields carry charge \(-1\). This symmetry is broken when the Higgs fields acquire unification-scale vevs \(v_1\), but there remains an unbroken global \(U(1)\) which is a linear combination of \(U(1)_X\) and a broken gauge symmetry. This unbroken global symmetry is then broken by the electroweak-scale vev \(u_2\), but again an unbroken global \(U(1)\) survives. The fields \(\nu, N_1\) pair up to form a Dirac neutrino with a mass proportional to \(u_2\) because they have equal and opposite charges under this global \(U(1)\), while the field \(N_2\), which is also charged, remains massless. If both \(u_1\) and \(u_2\) are nonzero, then no global symmetry survives. This explains why the sterile neutrino \(N_2\) acquires a Majorana mass proportional to \(u_1 u_2\).

A detailed discussion of the accidental global symmetries is contained in Appendix B.

### 2.2 One-loop Corrections to Neutrino Masses

As discussed at the end of the previous section, we can understand why \(\nu, N_1\) pair up to form an electroweak-scale Dirac neutrino, and \(N_2\) acquires an eV-scale Majorana mass, in terms of accidental global symmetry. If this symmetry is not respected by the Higgs potential, however, there are large
radiative contributions to these masses. The gauge symmetries allow the cubic Higgs couplings 
\( \Phi_q \Phi_q \Phi_l \) and \( \Phi_l \Phi_l \Phi_l + \text{cyclic} \), and these violate the accidental global \( U(1)_X \) symmetry of the gauge and Yukawa interactions, under which the Higgs fields carry charge \(-1\). Thus there are large (unification-scale) radiative contributions to neutrino masses due to Higgs exchange.

In order to see the one-loop diagrams that are responsible for the large radiative contributions to the neutrino masses, we must consider the Yukawa interactions that are obtained from the cyclic permutation of Eq. (4). Concentrating on just one of the Higgs fields, we have

\[
\mathcal{L}_q = g (\psi_Q \psi_Q \Phi_L + \psi_L \psi_Q \Phi_Q + \psi_Q \psi_L \Phi_Q) + \text{h.c.} \tag{14}
\]

These three interactions may be used to construct the one-loop diagram in Fig. 1(a). This diagram also makes use of the cubic Higgs coupling \( \Phi_q \Phi_q \Phi_l \) (with the Higgs field \( \Phi_l \) acquiring a unification-scale vev), as dictated by the symmetry argument above. A similar diagram, which instead makes use of the Yukawa coupling \( \psi_l \psi_l \Phi_l \) and the cubic Higgs coupling \( \Phi_l \Phi_l \Phi_l \), is shown in Fig. 1(b).

The diagram in Fig. 1(a) is in the interaction basis. To calculate the contribution of this diagram to the neutrino masses, we must work in the mass-eigenstate basis. In order to perform such a calculation, one must specify the Higgs potential, which determines the Higgs-field mass eigenstates. The potential has many terms, which have been identified in Ref. [7]. Rather than pursue such a calculation in gory detail, we idealize the situation to make the calculation tractable, yet maintain all the qualitative features of a full calculation.

We make the following simplifications. First, we consider just one of the two Higgs fields, \( \Phi^1 \equiv \Phi \). Second, we consider only the dimension-two and -three terms in the Higgs potential, and ignore the quartic interactions. These terms are

\[
\mathcal{L}_h = m^2 (\Phi_Q^* \Phi_Q + \Phi_Q^* \Phi_Q + \Phi_L^* \Phi_L) + [\gamma_1 \Phi_Q^* \Phi_Q \Phi_L + \gamma_2 (\Phi_L \Phi_L \Phi_L + \text{cyclic}) + \text{h.c.}], \tag{15}
\]

with \( m, \gamma_i = \mathcal{O} (M_U) \).\(^8\)

\(^8\)In Ref. [7] it is claimed that the cubic coupling \( \gamma_i \) must be small compared to \( v \) in order to justify a one-loop perturbative calculation. No such restriction appears to be necessary.
We use the following notation for the colored Higgs bosons in terms of component fields:

\[ \Phi_Q = (-\mathcal{D}_h \mathcal{W}_h \mathcal{B}_h), \quad \Phi_{Q^c} = \begin{pmatrix} \mathcal{D}^c_h \\ \mathcal{W}^c_h \\ \mathcal{B}^c_h \end{pmatrix}. \] (16)

The two cubic couplings, \( \gamma_1 \Phi_Q \Phi_Q \Phi_L \) and \( \gamma_2 \Phi_L \Phi_L \Phi_L \), are given in terms of component fields analogously to Eqs. (14) and (10).

We first consider the contribution to the neutrino masses from the cubic coupling \( \gamma_1 \Phi_Q \Phi_Q \Phi_L \). We see from Fig. 1(a) that the diagram is dominated by the quark that acquires a unification-scale mass, namely the heavy \( B \) quark. Therefore the relevant scalar fields in the loop are the \( SU(2)_L \)-singlet, down-type Higgs fields \( \mathcal{D}_{c,h}, \mathcal{B}_{c,h} \). In the idealized potential, Eq. (15), only \( \mathcal{B}_{c,h} \) and \( \mathcal{B}^c_{c,h} \) mix, but not \( \mathcal{D}^c_{c,h} \). Thus the mass eigenstates are

\[ B_{1,2h} = \frac{1}{\sqrt{2}} (\pm \mathcal{B}_h + \mathcal{B}^c_{c,h}), \quad B_{3h} = \mathcal{B}^c_{c,h}, \] (17a)

with masses

\[ m^2_{B_{1,2h}} = m^2 \pm \gamma_1 v_1, \quad m^2_{B_{3h}} = m^2, \] (17b)

where we neglect the tiny contribution from electroweak-scale vevs.

We now derive the vertices of the dominant contributions in terms of these mass eigenstates (see Fig. 2(a)). The second term of Eq. (14) gives

\[ \psi_L \psi_Q \Phi_Q = \left[ - \left( E^c u^c + \mathcal{E} \mathcal{D}^c + \mathcal{L} \mathcal{B}^c \right) Q_h + (e^c u^c + \mathcal{N}_1 \mathcal{D}^c + \mathcal{N}_2 \mathcal{B}^c) \mathcal{B}_h \right], \] (18)

with \( Q_h \equiv (\mathcal{W}_h, \mathcal{D}_h) \). The final two terms yield the relevant neutrino interactions, written in terms of the mass eigenstates as

\[ \psi_L \psi_Q \Phi_Q \ni \left[ (s_\beta N_1 - c_\beta N_2) (-s_\alpha d^c + c_\alpha B^c) - (s_\beta N_1 + s_\beta N_2) (c_\alpha d^c + s_\alpha B^c) \right] \frac{1}{\sqrt{2}} (B^*_{1h} - B^*_{2h}). \] (19)

The singlet fields \( N_{1,2} \) couple to \( B_{1,2h} \), but \( \nu \) does not, and none of the fermion fields couple to \( B_{3h} \). The third term of Eq. (14) gives

\[ \psi_Q \psi_L \Phi_{Q^c} = \left[ (Q E^c + B e^c) U^c_h + (Q \mathcal{E} + B \mathcal{N}_1) \mathcal{D}^c_h + (Q \mathcal{L} + B \mathcal{N}_2) \mathcal{B}^c_h \right], \] (20)

the last two terms of which yield the neutrino interactions

\[ \psi_Q \psi_L \Phi_{Q^c} \ni \left[ -d_s B \nu - B (c_\beta N_1 + s_\beta N_2) \right] \frac{1}{\sqrt{2}} (B^*_{1h} + B^*_{2h}). \] (21)

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9With a more general potential, the mass eigenstates are linear combinations of \( \mathcal{B}_h, \mathcal{B}^c_h, \) and \( \mathcal{D}^c_h \).
Figure 2: Dominant one-loop diagrams, in the mass-eigenstate basis, that contribute to the sterile neutrino masses via (a) colored Higgs and fermion fields and (b) color-singlet Higgs and fermion fields. In both cases there are actually two diagrams, which differ in the direction that fermion number flows in the internal fermion propagator.

With these vertices, we construct the one-loop contribution to the $N_{1,2}$ two-point functions shown in Fig. 2(a). There are actually two diagrams, which differ in the direction that fermion number flows in the internal fermion propagator. The loop integral gives

$$f(m_s, m_f) = \int \frac{d^n k}{(2\pi)^4} \frac{i}{k^2 - m_s^2} \left[ \frac{1 - \gamma_5}{2} \frac{\not{p} + \not{k} + m_f}{(p + k)^2 - m_f^2} \frac{1 - \gamma_5}{2} + \frac{1 + \gamma_5}{2} \frac{-\not{p} - \not{k} + m_f}{(p + k)^2 - m_f^2} \frac{1 + \gamma_5}{2} \right]$$

$$= -\frac{m_f}{(4\pi)^2} \left\{ \frac{2}{4 - n} - \gamma_E + \log(4\pi) - \frac{m_f^2}{m_f^2 - m_s^2} \log m_f^2 + \frac{m_s^2}{m_f^2 - m_s^2} \log m_s^2 + 1 \right\}. \quad (22)$$

The integral is proportional to the fermion mass, as anticipated from Fig. 1(a). The two-point function has contributions from both $B_{1h}$ and $B_{2h}$, and they enter with the opposite sign (see Eq. (19)). Thus the total contribution to the two-point function is proportional to

$$F_B = \frac{1}{2} \left[ f(m_{B_{2h}}, m_B) - f(m_{B_{1h}}, m_B) \right]$$

$$= \frac{m_B}{(4\pi)^2} \frac{1}{2} \left( \frac{m_{B_{1h}}^2}{m_B^2 - m_{B_{1h}}^2} \log m_{B_{1h}}^2 - \frac{m_{B_{2h}}^2}{m_B^2 - m_{B_{2h}}^2} \log m_{B_{2h}}^2 \right). \quad (23)$$

The ultraviolet divergences cancel, leaving a contribution to the sterile neutrino masses proportional to the fermion mass and the Higgs mass in the loop.

The analogous contribution to the neutrino two-point function induced by the cubic Higgs coupling $\Phi_L \Phi_L \Phi_L$ is shown in Fig. 2(b). The quark $B$ is replaced by the lepton doublet $E$, and the Higgs fields $B_{1,2h}$ are replaced by SU(2)$_L$-doublet, color-singlet Higgs fields. Recall that in order to have successful gauge unification (in the non-supersymmetric model), these Higgs doublet fields must lie at the weak scale. Since the total contribution to the neutrino two-point function, Eq. (23), is proportional to the mass of the Higgs fields in the loop, this contribution is negligible. In the split-SUSY scenario, this contribution would be the same order as the contribution of Fig. 2(a).

Including the large one-loop contributions, the neutrino mass matrix, in the $(\nu, N_1, N_2)$ basis, is

$$M_N^{1\text{-loop}} \simeq \begin{pmatrix} 0 & -h_1 u_2 & 0 \\ -h_1 u_2 & s_{\alpha - \beta} c_{\beta} g^2 F_B & (s_{2\beta} s_{\alpha} - c_{\alpha}) g^2 F_B \\ 0 & (s_{2\beta} s_{\alpha} - c_{\alpha}) g^2 F_B & c_{\alpha - \beta} s_{\beta} g^2 F_B \end{pmatrix}. \quad (24)$$
where we neglect the tiny tree-level Majorana mass of $N_2$ (Eq. (13)). This matrix (after factoring out $g^2 F_B$) is of the form \[ \begin{pmatrix} 0 & \epsilon & 0 \\ \epsilon & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \] where $\epsilon \sim \frac{h_1 u_2}{g^2 F_B}$. It has two eigenvalues $O(1)$ and one $O(\epsilon^2)$. Thus the two sterile neutrinos acquire unification-scale masses at one loop, while the active neutrino acquires a “radiative see-saw” Majorana mass

\[ m_{N_{1,2}} \sim g^2 F_B, \quad m_\nu \sim \frac{h_1^2 u_2^2}{g^2 F_B}. \] (26)

In order to obtain the correct values for the tau and top masses, we expect $h_1 \simeq 0.1$, $g \simeq 1$ and $u_2 = O(10^2 \text{GeV})$. Since $F_B \simeq \frac{1}{(4\pi)^2} M_U$, the mass of the light neutrino is then $O(0.1 \text{eV})$, consistent with the experimental constraints \[ \text{[3]}. \]

There is also a one-loop diagram that couples $\nu$ to $N_{1,2}$, of the form of Fig. 1(a) but with the heavy $B$ quark replaced by $d$. This diagram is of order $\frac{g^2}{(4\pi)^2} m_d$, which could be comparable to the tree-level Dirac mass $h_1 u_2$. In any case, it does not qualitatively change the radiative see-saw mechanism.\[ \text{[10]} \]

The radiative see-saw mechanism is absent in models with weak-scale supersymmetry, since the one-loop contributions are reduced to $O(1 \text{TeV})$ (due to the non-renormalization of the superpotential in the limit of exact supersymmetry) so that the nonvanishing entries in the neutrino mass matrix are all of the same order. This is analogous to the absence of the radiative seesaw mechanism in supersymmetric SO(10) \[ \text{[18]}. \] In the supersymmetric model, one must add higher-dimensional operators or additional Higgs representations to obtain a light, active neutrino \[ \text{[8]}. \] On the other hand, the radiative seesaw mechanism is present if the mass difference between scalars and fermions is comparable to the grand-unified scale, as is the case in split supersymmetry.\[ \text{[11]} \] The lifetime of the gluino restricts the sfermion masses, $m_s \lesssim 10^{14} \text{GeV} \simeq M_U \text{[20]}$; thus sfermions near the upper bound yield neutrino masses in the desired range, as in the non-supersymmetric case.

We derived the neutrino mass matrix, Eq. (24), in a simplified model, both in the sense of the weak-scale vevs ($n_i = 0$, see Eq. (3)) and the Higgs potential (see Eq. (15)). However, the radiative see-saw mechanism is independent of these details, since it is governed by the violation of the accidental global symmetry by the cubic terms in the Higgs potential.

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\[ \text{[10]} \] It is claimed in Ref. [7] that one should also consider the two-loop contribution to the $\nu-\nu$ two-point function obtained by sewing together two one-loop $\nu-N_{1,2}$ diagrams; however, such a diagram is one-particle reducible, and does not contribute to the mass matrix.

\[ \text{[11]} \] This was recently discussed for SO(10) in Ref. [19].
3 Three Generations

As we showed in Section 2.1, with one generation there is no relation between the masses of the quarks and the charged lepton. The same continues to be true when we extend the model to three generations. The Yukawa couplings $g_{1,2}$ and $h_{1,2}$ become three-by-three matrices, and there are many more parameters than there are constraints. The one qualitative prediction of the model, the radiative seesaw mechanism for the light neutrino mass, extends to three generations, as we discuss in Section 3.2. In the following section, we consider what the model predicts for the CKM matrix.

3.1 Quark Mixing

The breaking of the electroweak symmetry leads to mixing of $d$ and $B$, so we do not expect the CKM matrix to be exactly unitary. Here we show that it is unitary to a very good approximation, up to corrections $O\left(\frac{M_{\text{ew}}}{M_{\mu}}\right)$. However, like the quarks masses, the model does not make any prediction for the CKM matrix.

From Eqs. (30), we read off the $6 \times 6$ mass matrix of the down-type quarks for three generations,

$$M^D = \begin{pmatrix} -g_1 u_1 & g_2 v_2 \\ 0 & g_1 v_1 \end{pmatrix},$$

where $g_{1,2}$ are now $3 \times 3$ matrices. $M^D$ can be diagonalized by unitary rotations $U_D$ and $V_D$, where\textsuperscript{12}

$$U_D^\dagger M^D U_D = \begin{pmatrix} (M^d_{\text{diag}})^2 & 0 \\ 0 & (M^B_{\text{diag}})^2 \end{pmatrix}, \quad U_D = \begin{pmatrix} X & Y_1^\dagger \\ Y_2 & Z \end{pmatrix}. \quad (28)$$

Due to the unitarity of $U_D$, the four $3 \times 3$ submatrices fulfill

$$XX^\dagger + Y_1 Y_1^\dagger = 1, \quad XY_2^\dagger + Y_1 Z^\dagger = 0, \quad X^\dagger Y_1 + Y_2 Z = 0, \quad Y_2 Y_2^\dagger + ZZ^\dagger = 1. \quad (29)$$

Combining Eqs. (27) and (28) gives the relations

$$X^\dagger \left( u_{11}^2 g_1^1 g_1 X - u_{12} v_2 g_1^1 g_2 Y \right) + Y_2^\dagger \left( -u_{11} v_2 g_2^1 g_1 Y_2 + v_2^2 g_1^1 g_1 Y_2 + v_2^2 g_2 Y_2 \right) = (M^d_{\text{diag}})^2, \quad (30a)$$

$$X^\dagger \left( u_{11}^2 g_1^1 g_1 Y_1 - u_{12} v_2 g_1^1 g_2 Z \right) + Y_2^\dagger \left( -u_{11} v_2 g_2^1 g_1 Y_1 + v_2^2 g_1^1 g_1 Z + v_2^2 g_2 Y_2 \right) = 0, \quad (30b)$$

$$Y_1^\dagger \left( u_{11}^2 g_1^1 g_1 X - u_{12} v_2 g_1^1 g_2 Z \right) + Z^\dagger \left( -u_{11} v_2 g_2^1 g_1 Y_1 + v_2^2 g_1^1 g_1 Z + v_2^2 g_2 Y_2 \right) = 0, \quad (30c)$$

$$Y_1^\dagger \left( u_{11}^2 g_1^1 g_1 Y_1 - u_{12} v_2 g_1^1 g_2 Z \right) + Z^\dagger \left( -u_{11} v_2 g_2^1 g_1 Y_1 + v_2^2 g_1^1 g_1 Z + v_2^2 g_2 Y_2 \right) = (M^B_{\text{diag}})^2. \quad (30d)$$

One linear combination of Eqs. (30) leads to the simple relation\textsuperscript{13}

$$X \left( M^d_{\text{diag}} \right)^2 X^\dagger + Y_1 \left( M^B_{\text{diag}} \right)^2 Y_1^\dagger = u_1^2 g_1^1 g_1, \quad (31)$$

\textsuperscript{12}For the mixing among the right-handed states, see Appendix C. $V_D$ is displayed in Eq. (53).

\textsuperscript{13}The combination reads $X$ [Eq. (30a)] $X^\dagger + X$ [Eq. (30b)] $Y_2^\dagger + Y_1$ [Eq. (30c)] $X^\dagger + Y_1$ [Eq. (30d)] $Y_1^\dagger$. 

11
where we have used Eqs. (29). Since two of the three terms in this equation are manifestly $O(M^2_{EW})$, we see that $Y_1$ must be $O\left(\frac{M_{EW}}{M_U}\right)$ in order to compensate the $O(M_U)$ entries from $M^U_{\text{diag}}$. Thus the mixing between $d$ and $B$ is $O\left(\frac{M_{EW}}{M_U}\right)$.

Since $Y_1$ is very small, the first relation of Eq. (29) implies that the matrix $X$ is approximately unitary; this matrix represents the generational mixing amongst the $d$ states. The CKM matrix is given by

$$U^\dagger_u X \equiv V_{\text{CKM}}.$$ 

(32)

where $U_u$ is the unitary matrix that diagonalizes the up-type quark mass matrix,

$$U^\dagger_u \left(u^2 g_1 g_1\right) U_u = (M^u_{\text{diag}})^2.$$ 

(33)

Since $X$ is approximately unitary, the CKM matrix is unitary up to terms $O\left(\frac{M_{EW}}{M_U}\right)$.

3.2 Neutrino masses

In Ref. [5], it is claimed that the light neutrino masses would naturally have an inverted hierarchy. This can be understood as follows. As seen in Section 2.2, the eigenvalues are proportional to $h^2/g$. Neglecting the small mixing between the quarks, and assuming the hierarchy of $h$ is not stronger than that of $g$, $h^2/g$ decreases from the first to third generation yielding an inverted hierarchy. In contrast, in Ref. [7], the two-loop contribution (see Footnote 10) is claimed to result in a normal hierarchy. In this section we will show that our present understanding of the Yukawa couplings naturally gives either quasi-degenerate masses or a normal hierarchy.

For our discussion, we consider models with family symmetries, which have been used extensively to study neutrino masses and mixings [17]. A model based on $[SU(3)]^3 \times U(1)_F$ was introduced in Ref. [21], with the up-quark mass matrix for three generations

$\begin{pmatrix}
\epsilon^4 & \epsilon^3 & 0 \\
\epsilon^3 & \epsilon^2 & 0 \\
\epsilon & 1 & 1
\end{pmatrix}$

This choice of $M^u$, together with a similar mass matrix for the down quarks, leads to a viable CKM matrix. In the following, let us assume that $g$ (the matrix in Eq. (34), see Eq. (8)) is of this form. To obtain a hierarchical structure for the up and down quarks, both $g_1$ and $g_2$ will generally be hierarchical. Then, from Eq. (6), we expect the $B$ quarks to have a similar hierarchy.

Since the one-loop contributions to the neutrino masses are proportional to the fermion mass in the loop (see Eq. (29)), those with the heaviest quark, $B_3$, are dominant. With $g$ as given in Eq. (34), the three-generational mass matrix for the sterile neutrinos (both $N_1$ and $N_2$) reads (see Fig. 2(a))

$\begin{pmatrix}
\epsilon^4 & \epsilon^3 & 0 \\
\epsilon^3 & \epsilon^2 & 0 \\
\epsilon & 1 & 1
\end{pmatrix}$

$F_{B_3} \sim \begin{pmatrix}
\epsilon^4 & \epsilon^3 & \epsilon \\
\epsilon^3 & \epsilon^2 & 1 \\
\epsilon & 1 & 1
\end{pmatrix}$

(35)
with the eigenvalues
\[ m_3^N \sim m_2^N \sim F_{B_3}, \quad m_1^N \sim \epsilon^4 F_{B_3}, \] (36)

where \( m_i^N \) denote the quasi-degenerate masses of both \( N_1 \) and \( N_2 \) (of the \( i \)-th generation). Since \( F_{B_3} \approx \frac{1}{(4\pi)^2} M_U \), the masses of the sterile neutrinos are \( m_3^N \sim m_2^N \sim 10^{12} \text{ GeV} \) and \( m_1^N \sim 10^8 \text{ GeV}. \footnote{The mass \( m_1^N \) is comparable to the mass of the lightest sterile neutrino in thermal leptogenesis \cite{22}.}

Let us turn to the light neutrinos. To get a qualitative picture, we first consider only the second and third generation. Then we need to find the two weak-scale eigenvalues of the effective 6 \times 6 neutrino mass matrix, given by the generalization of Eqs. (24) and (25) to two generations. After a straightforward calculation, we find
\[ m_{2,3}^\nu \sim \frac{u_1^2}{F_{B_3}} \left\{ \left[ (h^{22})^2 + h^{22}h^{23} + (h^{23})^2 + h^{23}h^{33} + (\epsilon h^{33})^2 \right] \right. \\
\left. \pm \sqrt{\left[ (h^{22})^2 + h^{22}h^{23} + (h^{23})^2 + h^{23}h^{33} + (\epsilon h^{33})^2 \right]^2 + [h^{22}h^{33} + (h^{23})^2]^2} \right\}. \] (37)

Hence, if \( h^{23} \sim h^{33} \), the masses are almost degenerate; if \( h^{23} \) is smaller than \( h^{33} \) and \( h^{22} \ll \epsilon h^{23} \), they are hierarchical.

This result holds for the three-generational case as well. The eigenvalues are only proportional to \( \frac{h^2}{g^2} \) due to the common loop-integral, where \( g^2 \) is given by the third column of the symmetric matrix in Eq. (35). Since this hierarchy is weak, the neutrino hierarchy is determined by the hierarchy of \( h \) and we find either quasi-degenerate masses or a normal hierarchy.

4 Proton Decay

The gauge interactions conserve baryon number, and therefore do not mediate proton decay. Let us consider the Yukawa interactions. The Yukawa coupling that generates quark masses, Eq. (4), plus its cyclic permutations are displayed in Eq. (14),
\[ \mathcal{L}_q = g (\psi_L \psi_Q \Phi_L + \psi_L \psi_Q \Phi_Q + \psi_Q \psi_L \Phi_Q^c) + \text{h.c.} \] (38)
If we assign baryon number 0 to \( \Phi_L \), \( \frac{1}{3} \) to \( \Phi_Q \), and \( -\frac{1}{3} \) to \( \Phi_Q^c \), then this Yukawa interaction conserves baryon number. However, consider the Yukawa interaction that generates lepton masses, Eq. (9), plus its cyclic permutations:
\[ \mathcal{L}_l = h (\psi_L \psi_Q \Phi_L + \psi_Q \psi_Q \Phi_Q + \psi_Q \psi_Q \Phi_Q) + \text{h.c.} \] (39)
In this case baryon number is conserved if we assign baryon number 0 to \( \Phi_L \), \( -\frac{2}{3} \) to \( \Phi_Q \), and \( \frac{2}{3} \) to \( \Phi_Q^c \). Thus, with both Yukawa interactions present, baryon number is not conserved.\footnote{With both Yukawa interactions present, only a \( \mathbb{Z}_3 \) subgroup of baryon number is respected, where the fermion fields \( (\psi_L, \psi_Q, \psi_Q^c) \) and Higgs fields \( (\Phi_L, \Phi_Q, \Phi_Q^c) \) carry charges \( (0,1,2) \).}

The amplitude for proton decay is therefore proportional to \( gh \). Proton-decay diagrams are generated...
either by putting together two interactions containing the same Higgs field, such as $\psi_L \psi^c \Phi_q$ and $\psi_q \psi^c \Phi_q$ (Fig. 3(a)), or two interactions containing different Higgs fields, connected via the cubic term $\Phi_q^c \Phi_q \Phi_l$ in the potential (see Eq. (15)), with $\Phi_l$ acquiring a unification-scale vev (Fig. 3(b)). The Higgs field $\Phi_l$ does not mediate proton decay because it carries the same baryon number in both the quark and leptonic Yukawa interactions. Thus only the colored Higgs fields, $\Phi_q$ and $\Phi_q^c$, mediate proton decay.

As discussed in the Introduction, the analysis in Ref. [7] has two shortcomings. First, the authors use the technique for a three-body decay. Second, they use only one of the couplings in Eq. (9), so the leptonic Yukawa couplings are diagonal. As a consequence the decay channel $p \to \mu^+ K^0$ is absent, because a diagonal matrix $h$ forbids the vertex $us$ present in the exchange diagram. With both leptonic Yukawa couplings present in Eq. (9) the couplings are not diagonal, so the exchange diagram is allowed. We will show that $p \to \mu^+ K^0$ is one of the dominant decay modes.

Proton decay is described by four effective operators of dimension six. To derive them, consider the Yukawa couplings with the heavy colored Higgs bosons (Eqs. (18), (20)) and recall that only $\mathcal{B}_h^c$ and $\mathcal{B}_h$ mix with each other (Eq. (17)) when using the idealized potential of Eq. (15). Thus the relevant couplings for proton decay in Eq. (38) are

$$\mathcal{L}_q \supset g \left[ e^c u^c \frac{1}{\sqrt{2}} (B_{1h} - B_{2h}) + Q \hat{s}_\beta L \frac{1}{\sqrt{2}} (B_{1h}^* + B_{2h}^*) \right] + \text{h.c.}, \quad (40)$$

where we use the mass eigenstates and $\hat{s}_\beta$ denotes the three-generational analogue of the mixing between $\mathcal{E}$ and $\mathcal{L}$ (see Eqs. (62) in Appendix C). Similarly in Eq. (39), where

$$\psi_q \psi_q \Phi_q = QQ \mathcal{B}_h + BQ \mathcal{Q}_h , \quad \psi_q^c \psi_q^c \Phi_q^c = \mathcal{D}^c u^c \mathcal{B}_h^c + u^c \mathcal{B} \mathcal{R}_h^c + \mathcal{B}^c \mathcal{R}_h^c \mathcal{W}_h^c , \quad (41)$$

the relevant couplings are

$$\mathcal{L}_l \supset h \left[ QQ \frac{1}{\sqrt{2}} (B_{1h} - B_{2h}) + d^c \left( -\hat{s}_\alpha^T \right) u^c \frac{1}{\sqrt{2}} (B_{1h}^* + B_{2h}^*) \right] + \text{h.c.}, \quad (42)$$

with $\hat{s}_\alpha$, the three-generational analogue of the mixing between $\mathcal{D}^c$ and $\mathcal{B}^c$, as given in Eqs. (59) in

\footnote{The annihilation diagram is absent because there are no Higgs particles with charges $\pm \frac{4}{3}$.}
\[
\Gamma(p \to e_i^+ \pi^0) = \frac{m_p}{32\pi f_\pi^2} \left(1 + D + F\right)^2 |C_{udue}|^2
\]

\[
\Gamma(p \to \bar{\nu}_i \pi^+) = \frac{m_p}{32\pi f_\pi^2} \left(1 + D + F\right)^2 |C_{uddv}|^2
\]

\[
\Gamma(p \to e_i^+ K^0) = \frac{m_p}{32\pi f_\pi^2} \left(1 - \frac{m_{K^0}^2}{m_p^2}\right)^2 \left(1 + (D - F) \frac{m_p}{m_B}\right)^2 |C_{usue}|^2
\]

\[
\Gamma(p \to \bar{\nu}_i K^+) = \frac{m_p}{32\pi f_\pi^2} \left(1 - \frac{m_{K^+}^2}{m_p^2}\right)^2 \left(\frac{2}{3} D \frac{m_p}{m_B}\right) C_{usdv} + \left(1 + \frac{D + 3F}{3} \frac{m_p}{m_B}\right) C_{usdv} |^2
\]

Table 1: Partial widths of the proton decay channels \[27\].

Appendix \[C\] We now integrate out the two heavy scalars and obtain the effective operators

\[
\mathcal{L}_{\text{eff}} = \frac{1}{\gamma_1 v_1^2 - m^4} \left[ \frac{\gamma_1 v_1}{2} \left( (g \hat{s}_\beta)^{ij} h^{mn} Q_m Q_n Q_i L_j + g^{ij} \left(-\hat{s}_\alpha^T h\right)^{mn} d^c_m u^c_i e^c_j u^c_j \right) \right. \\
- m^2 \left( g^{ij} h^{mn} Q_m Q_n e^c_i u^c_j + (g \hat{s}_\beta)^{ij} \left(-\hat{s}_\alpha^T h\right)^{mn} d^c_m u^c_i e^c_j L_k \right) \left] + \text{h.c.} \right. \quad (43)
\]

where we have used

\[
\frac{1}{m_{B_{1u}}^2} + \frac{1}{m_{B_{2u}}^2} = - \frac{2m^2}{\gamma_1 v_1^2 - m^4}, \quad \frac{1}{m_{B_{1u}}^2} - \frac{1}{m_{B_{2u}}^2} = \frac{2\gamma_1 v_1}{\gamma_1 v_1^2 - m^4}, \quad (44)
\]

and we now explicitly display the generation indices. The first two operators, involving only left or right-handed fields, are called $LLLL$ and $RRRR$ operators, respectively. They are proportional to $\gamma_1$ because they arise from the coupling of two different Higgs fields via the cubic interaction in the Higgs potential, Eq. \(15\) (see Fig. \(3b\)). The other operators include both left-handed and right-handed fields, and are labeled as $LLRR$ and $RRLL$. They are proportional to $m^2$ because they arise from the coupling of a Higgs field to itself via the mass term in the Higgs potential (see Fig. \(3a\)). The quark-lepton vertex is determined by the coupling $g$ while the quark-quark vertex is given by $h$.

To calculate the decay rates, we evolve the operators from $M_U$ down to the hadronic scale, $M_{\text{had}}$. The ratio of the Wilson coefficients at $M_U$ and $M_{\text{had}}$ is described by a factor $A = \mathcal{O}(1)$ \[23\ \[24\]. At $M_{\text{had}}$, we switch to the hadronic level with the aid of chiral perturbation theory \[25\]. The hadron matrix elements $\langle PS|\mathcal{O}|p\rangle$, which describe the transition of the proton to a pseudoscalar meson via the three-quark operator $\mathcal{O}$, depend on two coefficients in the chiral Lagrangian, namely $\alpha$ for the $LLRR$ and $RRLL$ operators, and $\beta$ for the $LLLL$ and $RRRR$ operators. These coefficients describe the transition of the proton to the vacuum via the operator $\mathcal{O}$, and are calculated by means of lattice QCD, yielding $|\alpha| = |\beta| \simeq 0.01$ GeV\(^3\) \[26\].

The decay rates are given in Table \[1\]. Here $m_p$ and $m_K$ denote the masses of proton and kaon, respectively, and $f_\pi = 131$ MeV is the pion decay constant; $m_B = 1.15$ GeV is an average baryon mass according to contributions from diagrams with virtual $\Sigma$ and $\Lambda$; $D = 0.80$ and $F = 0.46$ are
The coefficients $C$ in Table 2 are given by

$$C = \mathcal{C} \frac{1}{\gamma_i^2 v_i^2 - m^2} A \begin{pmatrix} \gamma_1 v_1 \beta \\
-m^2 \alpha \end{pmatrix},$$

where $\mathcal{C}$ as given in Table 2. Generally, $\gamma_1 v_1$ and $m^2$ are $\mathcal{O}(M_{\nu})$. In the following discussion we approximate $C \simeq \frac{h}{M_{\nu}}$.

Let us now estimate the lifetime in the different decay modes with $g$ as given by the matrix in Eq. (34), which is displayed again in Table 4(b). The experimental limits are listed in Table 4(a). The most stringent experimental limit is on $p \rightarrow e^+\pi^0$. Using the numerical values for the various particle masses and constants, discussed above, we can derive an upper limit on the product of the Yukawa coupling matrices,

$$\tau \simeq \left( \frac{1}{g h} \right)^2 \times 10^{28} \text{ years} \quad \Rightarrow \quad g h \lesssim 10^{-3}.$$  

The decay involves particles of the first generation only, so $g = g^{11} \sim \epsilon^4 \sim 10^{-4}$. Hence the decay rate is well below the experimental limit, regardless of the (presumably small) value of $h^{11}$.

Table 3: (a) Dominant coefficients for the various decay channels and their current experimental limit [29]; (b) matrix $g$ of our discussion [21].

| mode      | dominant coeff. | $g$   | limit [years] |
|-----------|-----------------|-------|---------------|
| $e^+\pi^0$ | $g^{11}h^{11}$  | $\epsilon^4$ | $5.4 \times 10^{33}$ |
| $\mu^+\pi^0$ | $g^{12}h^{11}$  | $\epsilon^3$ | $4.3 \times 10^{33}$ |
| $\bar{\nu}\pi^+$ | $g^{13}h^{11}$  | $\epsilon^3$ | $2.5 \times 10^{31}$ |
| $e^+K^0$ | $g^{11}h^{12}$  | $\epsilon^4$ | $1.1 \times 10^{33}$ |
| $\mu^+K^0$ | $g^{12}h^{12}$  | $\epsilon^3$ | $1.4 \times 10^{33}$ |
| $\bar{\nu}K^+$ | $g^{23}h^{11}$  | $\epsilon^2$ | $2.3 \times 10^{33}$ |
|           | $g^{13}h^{12}$  | $\epsilon^3$ |               |

$g \sim \begin{pmatrix} \epsilon^4 & \epsilon^3 & \epsilon^3 \\ \epsilon^3 & \epsilon^2 & \epsilon^2 \\ \epsilon & 1 & 1 \end{pmatrix}$ (b)
Since the Yukawa couplings of the second generation are larger and the Yukawa matrices are not diagonal, we expect flavor non-diagonal decays to be dominant, in particular \( p \to \bar{\nu}K^+ \) and \( p \to \mu^+K^0 \). The constraints on the product \( gh \) are slightly less than that of Eq. (46) for two reasons: the experimental limits are lower (Table 4(a)), and the kaon mass suppresses the decay rate with respect to that of a pionic decay by a factor of about 2 (Table 1). We show below that the dominant decay mode is \( p \to \bar{\nu}K^+ \), with \( p \to \mu^+K^0 \) less than or comparable to it.

Let’s start with \( p \to \mu^+K^0 \). The Yukawa couplings, corresponding to an exchange diagram, are given by \( g^{12}h^{12} \) (LLLL/RRLL) and \( g^{21}h^{12} \) (LLRR/RRRR) (there is no annihilation diagram, as already mentioned). Since \( g^{12} \sim g^{21} \sim \epsilon^3 \), the decay width is at least two orders of magnitude larger than that of \( p \to e^+\pi^0 \), assuming \( h^{12} \gtrsim h^{11} \).

Next consider \( p \to \bar{\nu}K^+ \). The experimental constraint applies to the sum of all neutrino species. We see from Table 4(b) that the couplings to the muon and tau neutrinos are of the same size, and are much greater than the coupling to the electron neutrino, independent of the quark generation. The decay into neutrinos is mediated by the LLLL and RRLL operators only.

The channel \( p \to \bar{\nu}K^+ \) is the only channel with two different coefficients, viz. \( uds\nu_i \) and \( usd\nu_i \) (see Table 1). For \( uds\nu_i \) we obtain \( g^{21}h^{11} \), whereas \( usd\nu_i \) gives \( g^{11}h^{12} \). The latter is the same magnitude as the coupling for \( p \to \mu^+K^0 \), so the width is comparable to or larger than that of \( p \to \mu^+K^0 \). Thus \( p \to \bar{\nu}K^+ \) is the dominant decay mode. The lifetime is above the experimental limit both because the \( usd\nu_i \) coefficient is suppressed by \( g^{11} \lesssim \epsilon^3 \sim 1 \times 10^{-3} \), and because \( g^{21} \lesssim \epsilon^2 \) and \( \frac{h^{11}}{m_{\pi}} \sim \frac{m_{\pi}}{m_{\tau}} \) sufficiently suppress the \( uds\nu_i \) coefficient.

The remaining channels are suppressed relative to the dominant ones. For \( p \to e^+K^0 \), we read off \( g^{11}h^{12} \), one factor of \( \epsilon \) less than the coefficient for \( p \to \mu^+K^0 \); the width is therefore about two orders of magnitude smaller. The couplings involved in \( p \to \bar{\nu}\pi^+ \) and \( p \to \mu^+\pi^0 \) are given by \( g^{11}h^{11} \) and \( g^{12}h^{11} \), respectively, both \( \epsilon^3h^{11} \) since \( g^{12} \sim g^{13} \sim \epsilon^3 \). These coefficients are one factor of \( \epsilon \) smaller than the coefficient \( g^{21}h^{11} \) of \( p \to \bar{\nu}K^+ \). Note that \( p \to \mu^+\pi^0 \) was claimed to be one of the dominant decay modes in Ref. [7].

The decay channels are summarized in Table 4(a). If we assume that \( h \) is hierarchical in order to have hierarchical charged leptons, then we see – even without a specific form of \( h \) – that the decay into kaons is favored.

Let us finally discuss proton decay in the presence of supersymmetry. For the mixed operators, LLRR and RRLL, the discussion remains unchanged, since they arise from D terms. The unification scale is increased by two orders of magnitude, and so the lifetime by eight. Therefore proton decay via these operators will not even be observable in future experiments which aim to reach a lifetime of \( 10^{35–36} \) years [30].

On the other hand, the LLLL and RRRR operators stem from F terms and have mass-dimension five. When the sfermions are integrated out, they give rise to effective four-fermion operators of dimension six. Thus the operators are suppressed by \( (m_sM_U)^2 \) instead of \( M_U^4 \). The decay rate is naturally consistent with the experimental limit if the sfermion masses, \( m_s \), are above a few hundred TeV [31]. Hence, the model with weak-scale SUSY needs fine-tuning among the Yukawa couplings (which is similar to models such as SU(5) [23]), whereas proton decay is unobservable in the split-SUSY case.
5 Conclusions

In this paper we studied the trinified model, $SU(3)_C \times SU(3)_L \times SU(3)_R \times Z_3$, with the minimal Higgs sector required for symmetry breaking, namely two copies of $(1, 3, 3^*)$ and its cyclic permutations. After breaking to the standard model there are five Higgs doublets, and gauge-coupling unification results at $M_U \simeq 10^{14}$ GeV if all five are at the weak scale, without supersymmetry. Baryon number is conserved by the gauge interactions, so such a low unification scale is not forbidden by proton decay. Unlike other grand-unified theories, such as $SU(5)$ or $SO(10)$, the minimal model is able to correctly describe the fermion masses and mixing angles without the need to introduce intermediate scales, additional Higgs fields, or higher-dimensional operators. Indeed, with a relatively low unification scale, it is plausible that the effects of higher-dimensional operators induced by Planck-scale physics are negligible.

Light, active neutrinos are naturally generated at one loop via the radiative seesaw-mechanism. The additional matter, which is either vectorlike or sterile, is superheavy with masses above $10^8$ GeV. Thus no additional particles are present at the weak scale.

Proton decay is mediated by colored Higgs bosons. We found that the proton lifetime is above the experimental bounds in all the possible decay modes due to the small Yukawa couplings. The dominant decay modes, $p \rightarrow \bar{\nu}K^+$ and $p \rightarrow \mu^+K^0$, are potentially observable in future experiments.

Minimal trinification is perhaps the simplest viable nonsupersymmetric unified theory. There are also viable models based on $SO(10)$, but they require intermediate scales as well as non-minimal Higgs sectors and/or higher-dimensional operators [32].

We also considered the minimal model with supersymmetry. The scenario with weak-scale SUSY fails both because it requires additional Higgs fields or higher-dimensional operators to generate viable neutrino masses, and because fine-tuning is needed in order to avoid too rapid proton decay. In contrast, the radiative see-saw mechanism for neutrino masses operates if the mass scale of the sfermions is around $10^{14}$ GeV, as in split-SUSY, which is near their upper bound based on the gluino lifetime. These large sfermion masses suppress proton decay such that it is unobservable.

The dominant decay modes in minimal trinification are similar to those in grand-unified models with weak-scale supersymmetry, where the decay is dominated by dimension-five operators. In $SU(5)$ or $SO(10)$, dimension-six operators mediated by gauge bosons are suppressed but potentially observable since they yield decay into pions; the estimated lifetime for $p \rightarrow e^+\pi^0$ is $10^{35\pm1}$ years [33]. Therefore the different types of models would be distinguished by the presence or absence of supersymmetric particles and the number of Higgs doublets at the weak scale, together with the observation of specific proton decay modes [34].

The smoking gun for minimal nonsupersymmetric trinification would be the discovery of five Higgs doublets at the weak scale, together with the observation of proton decay into final states containing kaons. The split-SUSY version of the theory would be difficult to prove, as its main difference with split-SUSY $SU(5)$ or $SO(10)$ would be the absence of observable proton decay. We look forward to probing electroweak symmetry breaking at the Fermilab Tevatron and the CERN Large Hadron Collider in the near future.
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A Fermion masses in the general case

For good measure, we present the general formulae for the fermion masses in one generation where the five Higgs doublets all acquire non-vanishing vevs \( u_i \) and \( n_i \).

Then the quark masses read up to corrections \( \mathcal{O}\left(\frac{M_{EW}}{M_U}\right) \)

\[
m_B = \sqrt{(g_1 v_1 + g_2 v_3)^2 + (g_2 v_2)^2},
\]

\[
m_u = g_1 u_2 + g_2 n_2,
\]

\[
m_d = \frac{(g_1 u_1 + g_2 n_1)(g_1 v_1 + g_2 v_3) - g_2^2 n_3 v_2}{m_B},
\]

and those of the leptons are

\[
m_E = \sqrt{(h_1 v_1 + h_2 v_3)^2 + (h_2 v_2)^2},
\]

\[
m_e = \frac{(h_1 u_1 + h_2 n_1)(h_1 v_1 + h_2 v_3) - h_2^2 n_3 v_2}{m_E},
\]

\[
m_{\nu,N_1} = h_1 u_2 + h_2 n_2,
\]

\[
m_{N_2} = \frac{(h_1 u_2 + h_2 n_2)[(h_1 u_1 + h_2 n_1)(h_1 v_1 + h_2 v_3) - h_2^2 v_2 n_3]}{m_E^2}.
\]

If \( v_3 \) is smaller than \( \mathcal{O}(M_U) \), it can be neglected.

B Global U(1) Symmetries

The gauge sector of \( SU(3)_C \times SU(3)_L \times SU(3)_R \times Z_3 \) has an accidental global \( U(1)_Q \times U(1)_{Q^c} \times U(1)_L \) symmetry, as discussed in Section 2. This global symmetry is violated by the Yukawa couplings of Eqs. (4), (9), leaving a single \( U(1)_X \) global symmetry

\[
X(\Phi) = -1, \quad X(\psi) = \frac{1}{2}.
\]

Cubic Higgs couplings do not respect this global symmetry. When \( \Phi_{l1,2}^4 \) acquire their vevs, this global symmetry is broken; however, combinations of this symmetry with one or more of the broken diagonal generators of the gauge symmetry will survive as new global symmetries. We explicitly orthogonalize the global symmetry from any local ones.

If we impose only \( v_1 \), then the generators \( X, \lambda_{SL}, \) and \( \lambda_{SR} \) are individually broken; however, the combination \( (\lambda_{SL} + \lambda_{SR})/\sqrt{2} \) remains as an unbroken local symmetry, \( U(1)_{L+R} \). We then form an orthogonal global symmetry by combining the local generator orthogonal to \( U(1)_{L+R} \) with the global
the following charges:

\[
\Phi^1_{L^2} : \frac{3}{4} \begin{pmatrix} -2 & -2 & -1 \\ -2 & -2 & -1 \\ -1 & -1 & 0 \end{pmatrix}, \quad \psi_L : \frac{3}{4} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad \psi_{Q^c} : \frac{3}{4} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \psi_Q : \frac{3}{4} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}. \quad (49)
\]

If we now add \(v_2\), then this global symmetry is broken along with \(\lambda_{3R}\) and \(U(1)_{L+R}\). The combination \((\lambda_{SR} + \lambda_{SL} - \sqrt{3}\lambda_{3R}) / \sqrt{2}\) remains as an unbroken local symmetry, which is proportional to hypercharge. The orthogonal unbroken global charge is then given by \(-\frac{3}{10} \left( \frac{\sqrt{2}}{2} (\lambda_{SR} + \lambda_{SL}) + \lambda_{3R} \right) + X_1\) which we can rewrite as \(\frac{\sqrt{2}}{10} \lambda_{SR} - \frac{2\sqrt{2}}{5} \lambda_{SL} - \frac{3}{10} \lambda_{3R} + X \equiv X_2\), where the fields carry the charges

\[
\Phi^1_{L^2} : \frac{3}{5} \begin{pmatrix} -2 & -3 & -2 \\ -2 & -3 & -2 \\ 0 & -1 & 0 \end{pmatrix}, \quad \psi_L : \frac{3}{10} \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 5 & 3 & 5 \end{pmatrix}, \quad \psi_{Q^c} : \frac{3}{10} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \quad \psi_Q : \frac{3}{10} \begin{pmatrix} 3 & 3 & -1 \end{pmatrix}. \quad (50)
\]

We can now add either \(u_1\) or \(u_2\). Both will break \(\lambda_{3L}\), hypercharge and the global symmetry \(X_2\), and we still have a preserved local symmetry, which is of course electric charge, proportional to \(\lambda_{SR} + \lambda_{SL} - \sqrt{3}(\lambda_{3R} + \lambda_{3L})\). In the \(u_2\) case, we construct the residual global symmetry \(-\frac{2}{5} \left( \frac{\sqrt{2}}{2} (\lambda_{SR} + \lambda_{SL}) - \frac{3}{5} \lambda_{3R} + \lambda_{3L} \right) + X_2 = \frac{1}{5} \left( \sqrt{3}(-\lambda_{SR} - 5\lambda_{SL}) + 3\lambda_{3R} - 9\lambda_{3L} \right) + X\). This yields the charges:

\[
\Phi^1_{L^2} : \frac{3}{4} \begin{pmatrix} -4 & -3 & -4 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \psi_L : \frac{3}{4} \begin{pmatrix} -2 & -1 & -2 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}, \quad \psi_{Q^c} : \frac{3}{4} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \psi_Q : \frac{3}{4} \begin{pmatrix} 3 & 0 & -1 \end{pmatrix}. \quad (51)
\]

Alternatively, in the \(u_1\) case, \(\frac{2}{5} \left( \frac{\sqrt{2}}{2} (\lambda_{SR} + \lambda_{SL}) - \frac{3}{5} \lambda_{3R} + \lambda_{3L} \right) + X_2\) is the unbroken global symmetry. Collecting terms gives \(\frac{\sqrt{2}}{4} \lambda_{SR} - \frac{\sqrt{2}}{4} \lambda_{SL} - \frac{3}{4} \lambda_{3R} + \frac{3}{4} \lambda_{3L} + X\). This gives an alternate set of global charges,

\[
\Phi^1_{L^2} : \frac{3}{2} \begin{pmatrix} 0 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \psi_L : \frac{3}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \psi_{Q^c} : \frac{3}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \psi_Q : \frac{3}{2} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}. \quad (52)
\]

Both \(u_1\) and \(u_2\) break the electroweak symmetry to \(U(1)_{EM}\) but yield different charges for the global symmetry. Hence, with both \(u_1\) and \(u_2\), the global symmetry is broken. This result is independent of which \(\Phi_L\) contains a specific vev, since both \(\Phi_L^{1,2}\) transform identically under the gauge and global symmetries. Furthermore, the vevs \(n_i\), which we have set to zero, do not alter the discussion.

C  Mixing among Heavy and Light States

In Section 3.1, we discussed the mixing among the left-handed down-type quarks for three generations; we now consider the right-handed fields as well as the mixing among the charged leptons.
We can write the mixing among the right-handed fields similarly to that of the left-handed fields (cf. Eq. (28)),
\[
\begin{pmatrix}
\mathcal{S}^c \\
\mathcal{B}^c
\end{pmatrix} \rightarrow V_D \begin{pmatrix}
d^c \\
B^c
\end{pmatrix}, \quad V_D = \begin{pmatrix}
X_\alpha & Y_{a1} \\
Y_{a2} & Z_\alpha
\end{pmatrix}.
\] (53)

where \(X_\alpha\), the rotation of \(\mathcal{S}^c\) into \(d^c\), is equivalent to \(-\hat{s}_\alpha\) in Eq. (42). For a single generation, the comparison with Eq. (6) yields \(X_\alpha = -Z_\alpha = -\sin \alpha\) and \(Y_{a1} = Y_{a2} = \cos \alpha\).

\(M^D\) is diagonalized by \(V_D^* M^D U_D\) (cf. Eq. (28)),
\[
\begin{pmatrix}
X_\alpha^T & Y_{a1}^T \\
Y_{a2}^T & Z_\alpha^T
\end{pmatrix} \begin{pmatrix}
g_1 u_1 & g_2 v_2 \\
0 & g_1 v_1
\end{pmatrix} \begin{pmatrix}
X_\alpha \\
Y_{a1} \\
Y_{a2} \\
Z_\alpha
\end{pmatrix} = \begin{pmatrix}
M_{\text{diag}}^d & 0 \\
0 & M_{\text{diag}}^B
\end{pmatrix}.
\] (54)

however, as discussed in Section 2.1, the mixing among \(\mathcal{S}^c\) and \(\mathcal{B}^c\) is dominated by the unification-scale vevs and \(u_1\) can be neglected. We are then free to rotate \(d\) and \(B\) independently, \(d \rightarrow U_d d\) and \(B \rightarrow U_B B\), with \(U_d = X\) and \(U_B = Z\). In this approximation, the transformations above reduce the mass terms to three supermassive and three independent massless fields.

Neglecting \(u_1\), we obtain from Eq. (54)
\[
(X_\alpha^T g_2 v_2 + Y_{a2}^T g_1 v_1) U_B = 0, \quad (Y_{a1}^T g_2 v_2 + Z_\alpha^T g_1 v_1) U_B = M_{\text{diag}}^B.
\] (55)

We must also satisfy the unitarity relations for \(V_D\),
\[
X_\alpha X_\alpha^T + Y_{a1} Y_{a1}^T = 1, \quad X_\alpha Y_{a2} + Y_{a1} Z_\alpha^T = 0, \quad X_\alpha^T Y_{a1} + Y_{a2} Z_\alpha = 0, \quad Y_{a2} Y_{a2}^T + Z_\alpha Z_\alpha^T = 1.
\] (56)

Eqs. (55) and (56) together yield
\[
g_2 v_2 U_B = Y_{a1} M_{\text{diag}}^B, \quad g_1 v_1 U_B = Z_\alpha M_{\text{diag}}^B, \quad U_B^* \left(v_1^2 g_1^2 g_1 + v_2^2 g_2^2 g_2\right) U_B = (M_{\text{diag}}^B)^2,
\] (57)

thus \(U_B\) diagonalizes \(v_1^2 g_1^2 g_1 + v_2^2 g_2^2 g_2\).

It is also useful to define unitary matrices \(L_\alpha\) and \(R_\alpha\) that diagonalize \(g_2 g_1^{-1}\),
\[
L_\alpha g_2 g_1^{-1} R_\alpha = \frac{v_1}{v_2} D_\alpha,
\] (58)

where \(D_\alpha\) is a diagonal matrix. With these definitions it can be shown that a solution to the eigenvalue problem above with appropriate unitary relations is:
\[
X_\alpha = -v_1 L_\alpha^* R_\alpha g_1 U_B (M_{\text{diag}}^B)^{-1} V_d \equiv -\hat{s}_\alpha, \quad Y_{a1} = v_2 g_2 U_B^* (M_{\text{diag}}^B)^{-1},
\]
\[
Y_{a2} = v_2 R_\alpha^* L_\alpha^* g_2 U_B (M_{\text{diag}}^B)^{-1} V_d, \quad Z_\alpha = v_1 g_1 U_B^* (M_{\text{diag}}^B)^{-1}.
\] (59)

Here \(V_d\) is an arbitrary unitary matrix, which accounts for the freedom we would expect to rotate the three degenerate massless fields. It will be specified when the electroweak vevs are turned on.

We can perform a similar analysis of the \(E\) and \(L\) mixing:
\[
\left(\mathcal{L}^c, \mathcal{E}\right) \rightarrow U_E \left(\begin{array}{c}
L \\
E
\end{array}\right), \quad U_E = \begin{pmatrix}
X_\beta & Y_{\beta 1} \\
Y_{\beta 2} & Z_\beta
\end{pmatrix}, \quad E^c \rightarrow V_E E^c,
\] (60)
where $X_\beta \equiv \hat{s}_\beta$ (cf. Eq. (30)). With

$$
V_E^\dagger \left( v_1^2 h_1^1 h_1^1 + v_2^2 h_2 h_2^\dagger \right) V_E = (M^E_{\text{diag}})^2, \quad R^\dagger_\beta h_1^1 h_2 L_\beta = \frac{v_1}{v_2} D_\beta , \quad (61)
$$

we can write

$$
X_\beta = v_1 L_\beta R^\dagger_\beta h_1^1 V_E (M^E_{\text{diag}})^{-1} U_\ell \equiv \hat{s}_\beta , \quad Y_{\beta 1} = v_2 h_2^\dagger V_\ell^* (M^E_{\text{diag}})^{-1} ,
$$

$$
Y_{\beta 2} = v_2 R^\dagger_\beta L_\beta h_2 V_E (M^E_{\text{diag}})^{-1} U_\ell , \quad Z_\beta = -v_1 h_1^\dagger V_\ell^* (M^E_{\text{diag}})^{-1} . \quad (62)
$$

Again, the matrix $U_\ell$ is unspecified as long as the electroweak vevs are zero.

When the electroweak symmetry is broken, all the above equations obtain tiny corrections $O\left( \frac{M_{\text{ew}}}{M_U} \right)$, and $V_d$ and $U_\ell$ are determined.

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