Analytical construction of non local operator for n-qubit Dicke state

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Entanglement in multipartite quantum systems is much more elusive than its bipartite counterpart. In recent past the usefulness of multipartite entangled states in several information theoretic tasks have been demonstrated. Being a resource, the detection of multipartite entanglement is an imperative necessity. Among the different classes of multipartite entangled states the Dicke state has found importance in several tasks due to its permutation symmetric nature. In this work we propose a simple and elegant way of detecting n-qubit Dicke states using permutation symmetric Bell operators. We conjecture that maximal expectation value of the operator corresponds to the detection of Dicke states.

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I. INTRODUCTION

It would not be an exaggeration to say that quantum entanglement has passed the test of time as a fundamental concept in both quantum foundation as well as information. Although it can be considered as ‘quantum’ in nature, its widespread use in information theoretic and communication tasks call for a device-independent characterization. Till date a significant amount of labour has been devoted in this direction. This work humbly intends to contribute to this field, more specifically on the topic of the device independent characterization of a class of permutationally symmetric entangled states.

In the usual tomography task it is possible to estimate the unknown quantum state of system when many identical copies of the state are available so that different measurements can be performed on each copy. Not only for single system, it is also valid for correlated systems. But in this task the experimenter has to trust the quantumness of the devices which may not be true in reality. To avoid this difficulty one should be able to device the experiment in such a way that conclusions can be drawn even if the experimenter does not have control over the devices.

Recently preparation and characterization of multipartite states have acquired research interest keeping in view the technological implementation of information theoretic task in a distributive scenario. 14 entangled qubits were prepared via ion-trap experiments very recently. To do full tomography of this multipartite state one needs more information about this state and this procedure has been used for states of low rank, matrix product state and permutationally invariant state. Now this extra knowledge involves the trust the measurement devices. So a different method i.e Device - independent way has been approached in that scenario one does not have detailed knowledge about the experimental apparatus. In this scenario the experimental set up can be realised ad Black box scenario with some input knobs and output data counter. This device independent method is useful for randomness certification, quantum key distribution, dimension witness, certification of entangled measurements , Bit commitment and random number generation. Now this tomography via device independent initially coined as self testing where one multipartite state characterize via only statistical data. Later on large improvisation take place in self testing such as robustness of self testing(both in ideal and in presence of noise).

Among the different types of multiparticle quantum states, Dicke states have attracted a lot of attention. These states were first investigated in 1954 by R. Dicke for describing light emission from a cloud of atoms [4], and recently several other features have been studied: Dicke states are relatively robust to decoherence[5], their permutational symmetry allows to simplify the task of state tomography[6, 7] and entanglement characterization[8–11]. In addition, they are the symmetric states which are in some sense far away from the separable states[12]. Finally, they are relatively easy to generate in photon experiments, and Dicke states with up to six photons have been observed experimentally[13–15].

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They propose a method for generating all symmetric Dicke states, either in the long-lived internal levels of N massive particles or in the polarization degrees of freedom of photonic qubits, using linear optical tools only. They also discuss how our method can also be used to prepare symmetric Dicke states in the polarization degree of freedom of photon qubits. Actually this kind of study motivates to investigate the properties of Dicke states.

In [16, 17] authors propose the experimental realization of self testing for quantum states using swap method. As an example authors certify singlet fidelity of more than 70% for bipartite case having a CHSH [19] violation of 2.57. In [18], making use of the SWAP method, they extend the concept from the bipartite scenario to the multipartite for the W state [20], the 3 and 4-qubit GHZ states [21] and the 4-qubit cluster state [22]. In case of self testing there is no need of any knowledge regarding the specific workings of the experimental devices. In [18] they left the question of self testing Dicke states open. We have tried to addressed this question by constructing a permutationally symmetric Bell operator which n-qubit Dicke states.

The rest of the article is structured in the following way- in Sec. (II) we review the concept of Parity invariant Bell inequalities to detect multipartite entanglement in a Device independent(DI) way. Here we propose and demonstrate a Bell operator that can detect three qubit W state followed by constructing a witness of n qubit Dicke state by providing a permutation invariant Bell operator.

II. PRELIMINARIES

A. PI Bell inequalities

Bell-type inequalities are the central tool of our investigations [13]. We shall focus on multipartite Bell polynomials which are permutationally invariant, that is, they are symmetric under any permutation of the parties. Each observer can choose between two possible measurements featuring binary outputs. We use the following simplified notation to represent such a PI Bell inequality:

\[
[\alpha_1 \alpha_2, \alpha_{11} \alpha_{12} \alpha_{22}] = \alpha_1(A_1 + B_1) + \alpha_2(A_2 + B_2) + \alpha_{11}A_1B_1 + \alpha_{12}(A_1B_2 + A_2B_1) + \alpha_{22}A_2B_2,
\]

where \(A_i = \pm 1\) denotes the outcome of Alice’s measurement settings \(i = 1, 2\). Likewise for Bob’s settings. The extension to more parties is straightforward. For instance, for \(N = 3\) parties, the Mermin inequality [23], usually written as

\[
M_3 = A_1B_1C_1 - A_1B_2C_2 - A_2B_1C_2 - A_2B_2C_1 \leq 2
\]

now reads

\[
M_3 = [0 0 ; 0 0 0 ; 1 0 -1 0] \leq 2.
\]

Here the maximum algebraic sum of \(M_3 = 4\), corresponding to the set of correlations attained with a three-qubit GHZ state [21]:

\[
GHZ_3 = (|000\rangle + |111\rangle)/\sqrt{2}.
\]

and Pauli \(\hat{X}\) and \(\hat{Y}\) measurements.

Let us turn to the case of 4 parties. The generalized Mermin-Ardehali-Belinskii-Klyshko [24] (MABK) Bell inequality for \(N = 4\) is given by

\[
M_4 = [0 0 ; 0 0 0 ; 0 0 0 ; 1 1 -1 -1 1] \leq 4.
\]

Here the quantum maximum reads \(8\sqrt{2}\), which can be obtained by using \(\hat{X}\) and \(\hat{Y}\) Pauli measurements and a four-qubit GHZ state [21]:

\[
GHZ_4 = (|0000\rangle + |1111\rangle)/\sqrt{2}.
\]
B. n-qubit Dicke state

Here we introduce a class of permutationally symmetric states important from the perspective of quantum information. The $n$-qubit symmetric Dicke state of weight $m$ ($1 \leq m < n$) is defined by,

$$| m,n \rangle = \frac{1}{\sqrt{(\binom{n}{m})}} \left[ \sum_{j=1}^{\Pi} \Pi_j(|111\ldots1000\ldots0\rangle_{n-m}) \right],$$  

(7)

where $\{\Pi_j(|111\ldots1000\ldots0\rangle_{n-m})\}$ is the set of all possible distinct permutations of $m$ 1’s and $n-m$ 0’s [2]. For $m = 1$, the state given in (7) is generally called an $n$-qubit W state.

III. DI WITNESS OF MULTIPARTITE ENTANGLED STATES

A. Witnessing W state

In the case of PI Bell inequalities with two binary settings per party, there are nine independent Bell coefficients and we can write the Bell inequality in the notation of section II A as:

$$B = [b_1 \ b_2 ; b_3 \ b_4 \ b_5 ; b_6 \ b_7 \ b_8 \ b_9] \leq L,$$

(8)

where $L$ is the local maximum.

Our aim is to construct a Bell operator which gives maximum expectation value for the 3-qubit W state [20]:

$$|W\rangle \equiv \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle).$$

(9)

The operators of the measurements we have taken are the same for each party, that is $\hat{A}_1 = \hat{B}_1 = \hat{C}_1 = \hat{M}_1$ and $\hat{A}_2 = \hat{B}_2 = \hat{C}_2 = \hat{M}_2$. With this choice, observing the permutation symmetry of the W state the Bell operator may be written as

$$\hat{B} = \sum_{i=1}^{6} \hat{G}_i,$$

(10)

where

$$\hat{G}_1 \equiv \hat{\sigma}_z \hat{\sigma}_x \hat{\sigma}_x$$

$$\hat{G}_2 \equiv \hat{\sigma}_x \hat{\sigma}_z \hat{\sigma}_x$$

$$\hat{G}_3 \equiv \hat{\sigma}_x \hat{\sigma}_x \hat{\sigma}_z$$

$$\hat{G}_4 \equiv \hat{\sigma}_z \hat{\sigma}_y \hat{\sigma}_y$$

$$\hat{G}_5 \equiv \hat{\sigma}_y \hat{\sigma}_z \hat{\sigma}_y$$

$$\hat{G}_6 \equiv \hat{\sigma}_y \hat{\sigma}_y \hat{\sigma}_z$$

(11)

Note above we used the shorthand $\hat{\sigma}_i \hat{\sigma}_j \hat{\sigma}_k$ for denoting the tensor product $\hat{\sigma}_i \otimes \hat{\sigma}_j \otimes \hat{\sigma}_k$. The state giving the maximum quantum violation is the eigenstate belonging to the largest eigenvalue of the Bell-operator with the measurements chosen optimally. Therefore, we must make sure that the W state is an eigenstate of the Bell operator, that is $\langle W | \hat{B} | W \rangle = 0$ for all states $| \psi \rangle$ orthogonal to $| W \rangle$. From $\hat{\sigma}_z |0\rangle = |0\rangle$, $\hat{\sigma}_z |1\rangle = -|1\rangle$, $\hat{\sigma}_x |0\rangle = |1\rangle$, $\hat{\sigma}_x |1\rangle = |0\rangle$, $\hat{\sigma}_y |0\rangle = \hat{\sigma}_y |1\rangle = 0$, and $\hat{\sigma}_y |1\rangle = \hat{\sigma}_y |0\rangle = 0$, we conclude that the W state is an eigenstate of $\hat{B}$.
\( \hat{\sigma}_y |0 \rangle = i |1 \rangle \) and \( \hat{\sigma}_y |1 \rangle = -i |0 \rangle \) it is not difficult to derive:

\[
\hat{G}_1 |W \rangle = |W \rangle - \frac{|100 \rangle + |111 \rangle}{\sqrt{3}} \\
\hat{G}_2 |W \rangle = |W \rangle - \frac{|010 \rangle + |111 \rangle}{\sqrt{3}} \\
\hat{G}_3 |W \rangle = |W \rangle - \frac{|001 \rangle + |111 \rangle}{\sqrt{3}} \\
\hat{G}_4 |W \rangle = |W \rangle - \frac{|100 \rangle - |111 \rangle}{\sqrt{3}} \\
\hat{G}_5 |W \rangle = |W \rangle - \frac{|010 \rangle - |111 \rangle}{\sqrt{3}} \\
\hat{G}_6 |W \rangle = |W \rangle - \frac{|001 \rangle - |111 \rangle}{\sqrt{3}},
\]

(12)

From Eqs. (10) and (12) it follows that \( |W \rangle \) is an eigenstate of \( \hat{B} \). The expectation value of \( \hat{B} \) is:

\[
q \equiv \langle W | \hat{B} | W \rangle = 4.
\]

(13)

Let us stress that the constraints we have derived are only necessary conditions for the W state to be the one which violates the Bell inequality maximally. For the right solution the W state must be the eigenstate belonging to the maximum eigenvalue, and there must not exist another state with some different measurement operators giving the same or larger violation. This extra condition, for instance, is not guaranteed by our procedure.

At this point a pertinent question would be whether one can devise a permutation invariant Bell quantity which is maximally violated by a larger class of permutationally symmetric states. We provide such a construction in the next section.

B. Witness of n qubit Dicke states using a PI Bell operator

Now let us consider a more general scenario where there are n-parties spatially separated from each other. All of them are allowed to perform spin measurements in either X-direction or Y-direction or Z-direction on their respective systems. In such a situation we propose the following construction of a generalized parity invariant Bell operator with an ubiquitous property of witnessing Dicke states:

**Theorem 1.** Let us define a Bell operator for an n-qubit system as

\[
B_n = \sum_{j=1}^{\binom{n}{2}} \Pi_j (Z \ldots Z X \ldots Z X \ldots Z) + \sum_{j=1}^{\binom{n}{2}} \Pi_j (Z \ldots Z Y \ldots Z Y \ldots Z),
\]

(14)

where \( \{ \Pi_j (Z \ldots Z X \ldots Z X \ldots Z) \} \) is the set of all possible distinct permutations of two X’s and \((n-2)\) Z’s and \( U \equiv \sigma_U \) for \( U \in \{X,Y,Z\} \), then one must have

\[
\langle m, n | B_n | m, n \rangle = \lambda_n
\]

(15)

**Proof.** Here we prove that \(|m, n\rangle\) is an eigenstate of the operator \(B_n\).

Let us define n-qubit operators

\[
X_{ij} = Z \ldots Z X \ldots Z \quad \text{for} \quad i-th \quad \text{and} \quad j-th,
\]

\[
Y_{ij} = Z \ldots Z Y \ldots Z \quad \text{for} \quad i-th \quad \text{and} \quad j-th.
\]
Hence 

$$\begin{align*}
X_{i,j}|m,n\rangle_{0,0_j} &= (-1)^m|m+2,n\rangle_{1,1_j} \\
Y_{i,j}|m,n\rangle_{0,0_j} &= (-1)^{m+1}|m+2,n\rangle_{1,1_j} \\
X_{i,j}|m,n\rangle_{1,1_j} &= (-1)^{m-2}|m-2,n\rangle_{0,0_j} \\
Y_{i,j}|m,n\rangle_{1,1_j} &= (-1)^{m-1}|m-2,n\rangle_{0,0_j} \\
X_{i,j}|m,n\rangle_{0,1_j} &= (-1)^{m-1}|m,n\rangle_{1,0_j} \\
Y_{i,j}|m,n\rangle_{0,1_j} &= (-1)^{m-1}|m,n\rangle_{1,0_j}
\end{align*}$$

(16)

Thus

$$\begin{align*}
X_{i,j}|m,n\rangle &= (-1)^{m-1}|m,n\rangle \\
&\quad +(-1)^{m-2} \sum \Pi_{i\neq j} (|m,n\rangle_{1,1_j}) \\
&\quad +(-1)^{m-2} \sum \Pi_{i\neq j} (|m,n\rangle_{0,0_j}) \\
&\quad +(-1)^{m-2} \sum \Pi_{i\neq j} (|m+2,n\rangle_{1,1_j}) \\
&\quad +(-1)^{m-2} \sum \Pi_{i\neq j} (|m-2,n\rangle_{0,0_j}) \\
Y_{i,j}|m,n\rangle &= (-1)^{m-1}|m,n\rangle \\
&\quad +(-1)^{m-2} \sum \Pi_{i\neq j} (|m,n\rangle_{1,1_j}) \\
&\quad +(-1)^{m-2} \sum \Pi_{i\neq j} (|m,n\rangle_{0,0_j}) \\
&\quad +(-1)^{m-1} \sum \Pi_{i\neq j} (|m+2,n\rangle_{1,1_j}) \\
&\quad +(-1)^{m-1} \sum \Pi_{i\neq j} (|m-2,n\rangle_{0,0_j})
\end{align*}$$

Thus

$$\begin{align*}
(X_{i,j} + Y_{i,j})|m,n\rangle &= (-1)^{m-1}2|m,n\rangle \\
&\quad +(-1)^{m-2}2 \sum \Pi_{i\neq j} (|m,n\rangle_{1,1_j}) \\
&\quad +(-1)^{m-2}2 \sum \Pi_{i\neq j} (|m,n\rangle_{0,0_j})
\end{align*}$$

Hence

$$\begin{align*}
B_{n}|m,n\rangle &= (-1)^{m-12} \sum_{i\neq j} |m,n\rangle \\
&\quad +(-1)^{m-2}2 \sum_{i\neq j} \Pi_{i\neq j} (|m,n\rangle_{1,1_j}) \\
&\quad +(-1)^{m-2}2 \sum_{i\neq j} \Pi_{i\neq j} (|m,n\rangle_{0,0_j}) \\
&= (-1)^{m-12}m(n-m)|m,n\rangle
\end{align*}$$

Analytically we have calculated the eigenvalues of the proposed Bell quantity $B_n$ corresponding to the n-qubit Dicke states as eigenvectors up to $n = 10$ (see Appendix A). Interestingly, the eigenvalues are extremal for the cases $m = \left[n \over 2\right], \left[\frac{n}{2} \right]$. One can easily check that the eigenvalues corresponding to $m = \left[n \over 2\right], \left[\frac{n}{2} \right]$ are the maximum among all of the possible eigenstates. Based on the observations presented in Appendix A we propose the following conjecture-
For $m = \lceil \frac{n}{2} \rceil$ and $m = \lfloor \frac{n}{2} \rfloor$, $|\lambda_n|$ is the largest eigenvalue of $B_n$ and given by

$$|\lambda_n| = 2 \sum_{j=0}^{n-2} n^{-j} C_2 (-1)^j$$

with corresponding eigenvector $|m,n\rangle$.

IV. DISCUSSIONS

We have derived an analytical witness operator for n-qubit Dicke state in some restricted scenario. Construction of such type of non-local witness operators in general scenario can be further investigated. The work can be extended to other multipartite state. Generalization of this notion to sub-systems having higher dimension can be an interesting line of future research. Our method shall open a way to detect that multipartite dicke state in device independent manner.

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[1] Karoly F. Pal, Tamas Vertesi, and Miguel Navascues, 'Device-independent tomography of multipartite quantum states', arXiv:1407.5911v2 (2014).
[2] R. Rahaman and M. G. Parker, 'Quantum scheme for secret sharing based on local distinguishability', Phys. Rev. A 91 022330 (2015).
[3] R. H. Dicke, Phys. Rev. 93, 99 (1954).
[4] R. H. Dicke, Phys. Rev. 93, 99 (1954).
[5] O. Gühne, F. Bodoky, and M. Blaauboer, Phys. Rev. A 78, 060301(R) (2008).
[6] G. Tóth et al., Phys. Rev. Lett. 105, 250403 (2010).
[7] T. Moroder et al., New J. Phys. 14, 105001 (2012).
[8] G. Tóth, J. Opt. Soc. Am. B 24, 275 (2007).
[9] G. Tóth et al., New J. Phys. 11, 083002 (2009).
[10] M. Huber et al., Phys. Rev. A 83, 040301(R) (2011); Erratum: Phys. Rev. A 84, 039906(E) (2011).
[11] L. Novo, T. Moroder, and O. Gühne, Phys. Rev. A 88, 012305 (2013).
[12] M. Christandl, R. König, G. Mitchison, and R. Renner, Comm. Math. Phys. 273, 473 (2007).
[13] H. Häffner et al., Nature 438, 643 (2005).
[14] N. Kiesel et al., Phys. Rev. Lett. 98, 063604 (2007).
[15] W. Wieczorek et al., Phys. Rev. Lett. 103, 020504 (2009).
[16] Tzyh Haur Yang and Miguel Navascus, Phys. Rev. A 87, 050102(R)
[17] Tzyh Haur Yang, Tams Vrtesi, Jean-Daniel Bancal, Valerio Scarani, and Miguel Navascus, Phys. Rev. Lett. 113, 040401
[18] Karoly F. Pal, Tamas Vertesi, Miguel Navascues, Phys. Rev. A 90, 042340 (2014)
[19] John F. Clauser, Michael A. Horne, Abner Shimony, and Richard A. Holt, Phys. Rev. Lett. 23, 880
[20] W. Dr. G. Vidal, and J.I. Cirac, Phys. Rev. A 62, 062314 (2000).
[21] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. 58, 1131 (1990).
[22] H. J. Briegel and R. Raussendorf, "Persistent Entanglement in Arrays of Interacting Particles", Phys. Rev. Lett. 86, 910(2001).
[23] N.D. Mermin, Phys. Rev. Lett. 65, 1838-1840(1990).
[24] M. Ardehali, Phys. Rev. A 46, 5375 (1992); A.V. Belinskii and D.N. Klyshko, Phys. Usp. 36, 653 (1993).
Appendix A: Analytic results for the extremal eigenvalues and corresponding eigenstates of the Bell operator (Eq. 14) for $n = 3$ to 10

| $n$ | $\lambda_n$ | Eigenstate associated with $\lambda_n$ |
|-----|--------------|-----------------------------------------|
| 3   | $-4$         | $\frac{1}{\sqrt{3}} \left[ |001\rangle + |010\rangle + |100\rangle \right]$ |
|     | $+4$         | $\frac{1}{\sqrt{3}} \left[ |011\rangle + |101\rangle + |110\rangle \right]$ |
| 4   | $-8$         | $\frac{1}{\sqrt{6}} \left[ |0011\rangle + |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle \right]$ |
| 5   | $-12$        | $\frac{1}{\sqrt{2}} \left[ \Pi( |00011\rangle) \right]$ |
|     | $+12$        | $\frac{1}{\sqrt{2}} \left[ \Pi( |00111\rangle) \right]$ |
| 6   | $-18$        | $\frac{1}{\sqrt{3}} \left[ \Pi( |000111\rangle) \right]$ |
| 7   | $-24$        | $\frac{1}{\sqrt{4}} \left[ \Pi( |0000111\rangle) \right]$ |
|     | $+24$        | $\frac{1}{\sqrt{4}} \left[ \Pi( |0001111\rangle) \right]$ |
| 8   | $-32$        | $\frac{1}{\sqrt{4}} \left[ \Pi( |00001111\rangle) \right]$ |
| 9   | $-40$        | $\frac{1}{\sqrt{4}} \left[ \Pi( |000001111\rangle) \right]$ |
|     | $+40$        | $\frac{1}{\sqrt{4}} \left[ \Pi( |000011111\rangle) \right]$ |
| 10  | $-50$        | $\frac{1}{\sqrt{5}} \left[ \Pi( |0000011111\rangle) \right]$ |