q-DIFFERENCE EQUATIONS FOR HOMOGENEOUS q-DIFFERENCE OPERATORS AND THEIR APPLICATIONS

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Abstract.

In this short paper, we show how to deduce several types of generating functions from Srivastava et al [Appl. Set-Valued Anal. Optim. 1 (2019), pp. 187-201.] by the method of q-difference equations. Moreover, we build relations between transformation formulas and homogeneous q-difference equations.

Keywords. Basic (q-) hypergeometric series; q-difference equation; Homogeneous q-difference operator; Cauchy polynomials; Hahn polynomials; Generating functions.

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1. INTRODUCTION AND BASICS PROPERTIES

In this paper, we adopt the common conventions and notations on q-series. For the convenience of the reader, we provide a summary of the mathematical notations, basics properties and definitions to be used in the sequel. We refer to the general references (see [16]) for the definitions and notations. Throughout this paper, we assume that |q| < 1.

For complex numbers a, the q-shifted factorials are defined by:

\[(a; q)_n = \begin{cases} 
1 & \text{if } n = 0 \\
(1 - a)(1 - aq) \cdots (1 - aq^{n-1}) & \text{if } n = 1, 2, 3, \ldots
\end{cases} \tag{1.1} \]

and for tends to infinity, we have

\[(a; q)_{\infty} := \prod_{k=0}^{\infty} (1 - aq^k). \tag{1.2} \]

The following easily verified identities will be frequently used in this paper:

\[(a; q)_n = \frac{(a; q)_n}{(aq^n; q)_n} (a; q)_n = (a; q)_n (a; q; q)_n \]

and \[(a_1, a_2, \ldots, a_r; q)_m = (a_1; q)_m (a_2; q)_m \cdots (a_r; q)_m, m \in \{0, 1, 2, \ldots \}. \]

The q-binomial coefficients are given by

\[\binom{n}{k}_q := \begin{cases} 
\frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} & \text{if } 0 \leq k \leq n \\
0 & \text{otherwise}. \end{cases} \]

The basic (or q-) hypergeometric function of the variable z and with r numerator and s denominator parameters (see, for details, the monographs by Slater [25 Chapter 3] and by Srivastava and Karlsson E-mail addresses: rjksama2008@gmail.com (Sama Arjika).
where \( q \neq 0 \) when \( r > s + 1 \). Note that:

\[
\Phi_t \left[ \begin{array}{c}
a_1, a_2, \ldots, a_r; \\
b_1, b_2, \ldots, b_s;
\end{array} \right] q; z = \sum_{n=0}^{\infty} \left( \frac{(-1)^n q(\ell)}{b_1, b_2, \ldots, b_s; z^n} \right) \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n} (q; q)_n
\]

Here, in our present investigation, we are mainly concerned with the Cauchy polynomials \( p_n(x, y) \) as given below (see [6, 9]):

\[
p_n(x, y) := (x - y)(x - qy) \cdots (x - q^{n-1}y) = (y/x; q)_n x^n
\]

with the generating function [6]

\[
\sum_{n=0}^{\infty} p_n(x, y) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_\infty}{(xt; q)_\infty},
\]

where [6]

\[
p_n(x, y) = (-1)^n q(\ell) p_n(y, q^{1-n}x),
\]

and

\[
p_{n-k}(x, q^{1-n}y) = (-1)^{n-k} q^{\ell(\ell-1)/2} p_{n-k}(y, q^k x)
\]

which naturally arise in the \( q \)-umbral calculus [2], Goldman and Rota [10], Ihrig and Ismail [14], Johnson [15] and Roman [22]. The generating function (1.4) is also the homogeneous version of the Cauchy identity or the \( q \)-binomial theorem [9]

\[
\sum_{k=0}^{\infty} \frac{(a; q)_k z^k}{(q; q)_k} = \Phi_0 \left[ \begin{array}{c}
a \\
q; z
\end{array} \right] = \frac{(aq; q)_\infty}{(z; q)_\infty} |z| < 1.
\]

Putting \( a = 0 \), the relation (1.5) becomes Euler’s identity [9]

\[
\sum_{k=0}^{\infty} \frac{z^k}{(q; q)_k} = \frac{1}{(z; q)_\infty} |z| < 1
\]

and its inverse relation [9]

\[
\sum_{k=0}^{\infty} \frac{(-1)^k q^{\ell(\ell-1)/2} z^k}{(q; q)_k} = (z; q)_\infty.
\]

The following two \( q \)-difference operators are defined by [7, 27, 23]

\[
D_q \{ f(x) \} = \frac{f(x) - f(qx)}{x}, \quad \theta_x = \theta_{xy} |_{y=0}, \quad \theta_{xy} \{ f(x, y) \} := \frac{f(q^{-1}x, y) - f(x, qy)}{q^{-1}x - y}.
\]

The Leibniz rule for the \( D_q \) is the following identity [22]

\[
D_q^n \{ f(x)g(x) \} = \sum_{k=0}^{n} \binom{n}{k} q^{k(n-k)} D_q^k \{ f(x) \} D_q^{n-k} \{ g(q^k x) \}
\]

where \( D_q^0 \) is understood as the identity. For \( f(x) = x^k \) and \( g(x) = 1/(xt; q)_\infty \), we have

\[
D_q^k \left\{ \frac{x^k}{(xt; q)_\infty} \right\} = \frac{(q; q)_k}{(xt; q)_\infty} \sum_{j=0}^{n} \binom{n}{j} \frac{(xt; q)_j}{(q; q)_{k-j}} q^{n-j} x^{k-j}.
\]
Saad and Sukhi [23, 24] and Chen and Liu [7, 8] employed the technique of parameter augmentation by constructing the following $q$-exponential operators
\[
R(bD_q) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{k(k+1)}{2}}}{(q; q)_k} (bD_q)^k, \quad \Xi(b \theta_0) = \sum_{k=0}^{\infty} \frac{q^{\frac{k(k+1)}{2}}}{(q; q)_k} (b \theta_0)^k.
\] (1.11)

**Theorem 1.1.** ([17, Theorem 2]) Let $f(a, b)$ be a two-variable analytic function in a neighborhood of $(a, b) = (0, 0) \in \mathbb{C}^2$. If $f(a, b)$ satisfies the $q$-difference equation
\[
a f(aq, b) - b f(a, bq) = (a - b) f(aq, bq)
\] (1.12)
then we have:
\[
f(a, b) = \Xi(b \theta_0) \left\{ f(a, 0) \right\}.
\] (1.13)

Liu [17, 18] initiated the method of $q$-difference equations and deduced several results involving Bailey’s $\psi$, $q$-Mehler formulas for Rogers-Szegő polynomials and $q$-integral of Sears transformation.

Recently, Srivastava, Arjika and Kelil [29] introduced two homogeneous $q$-difference operators $\tilde{E}(a, b; D_q)$ and $\tilde{L}(a, b; \theta_3y)$
\[
\tilde{E}(a, b; D_q) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{k(k+1)}{2}} (a; q)_k}{(q; q)_k} (bD_q)^k, \quad \tilde{L}(a, b; \theta_3y) = \sum_{k=0}^{\infty} \frac{q^{\frac{k(k+1)}{2}} (a; q)_k}{(q; q)_k} (b \theta_3y)^k.
\] (1.14)
which turn out to be suitable for dealing with a generalized Cauchy polynomials $p_n(x, y, a)$ [29]
\[
p_n(x, y, a) = \tilde{E}(a, y; D_q) \{ x^n \}.
\] (1.15)

The method of $q$-exponential operator is a rich and powerful tool for $q$-series, especially it makes many famous results easily fall into this framework. In this paper, we use this method to derive some results such as: generating functions, Srivastava-Agarwal type generating functions and transformational identity involving the generalized Cauchy polynomials.

The paper is organized as follows: In Section 2, we state and prove two theorems on $q$-difference equations. We give generating functions for generalized Cauchy polynomials $p_n(x, y, a)$ by using the perspective of $q$-difference equations, in Section 3. In Section 4, we derive Srivastava-Agarwal type generating functions involving the generalized Cauchy polynomials. Finally, we obtain a transformational identity involving generating functions for generalized Cauchy polynomials by the method of homogeneous $q$-difference equations in Section 5.

### 2. $q$-DIFFERENCE EQUATIONS

In this section, we give and prove two theorems to be used in the sequel.

**Theorem 2.1.** Let $f(a, x, y)$ be a three-variable analytic function in a neighborhood of $(a, x, y) = (0, 0, 0) \in \mathbb{C}^3$. If $f(a, x, y)$ can be expanded in terms of $p_n(x, y, a)$ if and only if
\[
x \left[ f(a, x, y) - f(a, x, qy) \right] = y \left[ f(a, qx, y) - f(a, x, qy) \right] - ay \left[ f(a, qx, q^2y) - f(a, x, q^2y) \right].
\] (2.1)

To determine if a given function is an analytic function in several complex variables, we often use the following Hartogs’s Theorem. For more information, please refer to Taylor [30] p. 28 and Liu [19] Theorem 1.8].
Lemma 2.1. [Hartogs’s Theorem [11] p.15] If a complex-valued function is holomorphic (analytic) in each variable separately in an open domain $D \subset \mathbb{C}^n$, then it is holomorphic (analytic) in $D$.

Lemma 2.2. [20] p. 5 Proposition 1] If $f(x_1, x_2, \ldots, x_k)$ is analytic at the origin $(0, 0, \ldots, 0) \in \mathbb{C}^k$, then $f$ can be expanded in an absolutely convergent power series

$$f(x_1, x_2, \ldots, x_k) = \sum_{n_1, n_2, \ldots, n_k=0}^{\infty} \alpha_{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}. \quad (2.2)$$

Proof of Theorem 2.1. From the Hartogs’s Theorem and the theory of several complex variables (see Lemmas 2.1 and 2.2), we assume that

$$f(a, x, y) = \sum_{k=0}^{\infty} A_k(a, x) y^k. \quad (2.3)$$

Substituting (2.3) into (2.1) yields

$$x \sum_{k=0}^{\infty} (1 - q^k) A_k(a, x) y^k = -\sum_{k=0}^{\infty} (1 - aq^k) q^k \left[ A_k(a, x) - A_k(a, qx) \right] y^{k+1}. \quad (2.4)$$

Comparing coefficients of $y^k, k \geq 1$, we readily find that

$$x(1 - q^k) A_k(a, x) = -(1 - aq^{k-1}) q^{k-1} \left[ A_{k-1}(a, x) - A_{k-1}(a, qx) \right] \quad (2.5)$$

which equals to

$$A_k(a, x) = -q^{k-1} \frac{1 - aq^{k-1}}{1 - q^k} D_q \left\{ A_{k-1}(a, x) \right\}. \quad (2.6)$$

By iteration, we gain

$$A_k(a, x) = (-1)^k q^{\frac{k(k+1)}{2}} \left( \frac{a; q}{q; q} \right)_k D_q \left\{ A_0(a, x) \right\}. \quad (2.7)$$

Letting $f(a, x, 0) = A_0(a, x) = \sum_{n=0}^{\infty} \mu_n x^n$, we have

$$A_k(a, x) = (-1)^k q^{\frac{k(k+1)}{2}} \left( \frac{a; q}{q; q} \right)_k \sum_{n=0}^{\infty} \mu_n \frac{(q; q)_n}{(q; q)_{n-k}} x^{n-k}. \quad (2.8)$$

Replacing (2.8) in (2.5), we have:

$$f(a, x, y) = \sum_{k=0}^{\infty} (-1)^k q^{\frac{k(k+1)}{2}} \left( \frac{a; q}{q; q} \right)_k \sum_{n=0}^{\infty} \mu_n \frac{(q; q)_n}{(q; q)_{n-k}} x^{n-k} y^k$$

$$= \sum_{n=0}^{\infty} \mu_n \sum_{k=0}^{n} \left[ \binom{n}{k} (-1)^k \left( \frac{a; q}{q; q} \right)_k x^{n-k} y^k. \quad (2.9)$$

On the other hand, if $f(a, x, y, z)$ is a four-variable analytic function in a neighborhood of $(a, x, y, z) = (0, 0, 0, 0) \in \mathbb{C}^4$. 

Theorem 2.2. Let $f(a, x, y, z)$ be a four-variable analytic function in a neighborhood of $(a, x, y, z) = (0, 0, 0, 0) \in \mathbb{C}^4$. □
(1) If \( f(a,x,y) \) satisfies the \( q \)-difference equation
\[
x f(a,x,y) - f(a,x,qy) = y f(a,qx,qy) - f(a,x,qy) - ay f(a,qx,q^2 y) - f(a,x,q^2 y)
\] (2.10)
then we have:
\[
f(a,x,y) = \bar{E}(a,y;D_q) \left\{ f(a,x,0) \right\}.
\] (2.11)

(2) If \( f(a,x,y,z) \) satisfies the \( q \)-difference equation
\[
(q^{-1}x-y) f(a,x,y,z) - f(a,x,y,qz) = z f(a,q^{-1}x,y,qz) - f(a,x,qy,qz) + az f(a,x,qy,q^2 z) - f(a,q^{-1}x,y,q^2 z)
\] (2.12)
then we have:
\[
f(a,x,y,z) = \bar{L}(a,z;\theta_y) \left\{ f(a,x,y,0) \right\}.
\] (2.13)

**Corollary 2.1.** Let \( f(a,b) \) be a two-variable analytic function in a neighborhood of \( (a,b) = (0,0) \in \mathbb{C}^2 \).
If \( f(a,b) \) satisfies the \( q \)-difference equation
\[
a f(a,b) - b f(qa,qb) = (a-b) f(a,qb)
\] (2.14)
then we have:
\[
f(a,b) = \tilde{R}(bD_q) \left\{ f(a,0) \right\}.
\] (2.15)

**Remark 2.1.** For \( x = a, y = b \) and \( z = 0 \), the relation (2.10) reduces to (2.14).
For \( a = 0, x = a, y = 0 \) and \( z = b \), the \( q \)-difference equation (2.12) reduces to (1.12).

**Proof of Theorem 2.2.** From the theory of several complex variables [21], we begin to solve the \( q \)-difference equation (2.10). First we may assume that
\[
f(a,x,y) = \sum_{k=0}^{\infty} A_k(a,x) y^k,
\] (2.16)
Substituting this equation into (2.10) and comparing coefficients of \( y^k, k \geq 1 \), we readily find that
\[
x(1-q^k)A_k(a,x) = -(1-aq^{k-1})q^{k-1} \left[ A_{k-1}(a,x) - A_{k-1}(a,qx) \right]
\] (2.17)
which equals to
\[
A_k(a,x) = -q^{k-1} \frac{1-aq^{k-1}}{1-q^k} D_q \left\{ A_{k-1}(a,x) \right\}.
\] (2.18)
By iteration, we gain
\[
A_k(a,x) = (-1)^k q^{k\left(k-1\right)} (a;q)_k (q;q)_k D_q^k \left\{ A_0(a,x) \right\}.
\] (2.19)
Now we return to calculate \( A_0(a,x) \). Just taking \( y = 0 \) in (2.16), we immediately obtain \( A_0(a,x) = f(a,x,0) \). The proof of the assertion (2.11) of Theorem 2.2 is now completed by substituting (2.19) back into (2.16).
Similarly, we begin to solve the \( q \)-difference equation (2.12). First we may assume that
\[
f(a,x,y,z) = \sum_{n=0}^{\infty} B_n(a,x,y) z^n.
\] (2.20)
Then substituting the above equation into (2.12), we have:

\[
(q^{-1} x - y) \sum_{n=0}^{\infty} (1 - q^n) B_n(a, x, y) z^n = \sum_{n=0}^{\infty} q^n (1 - aq^n)[B_n(a, q^{-1} x, y) - B_n(a, x, qy)] z^{n+1}
\]  

(2.21)

Comparing coefficients of \(z^n, n \geq 1\), we readily find that

\[
(q^{-1} x - y)(1 - q^n) B_n(a, x, y) = q^{n-1}(1 - aq^{n-1})[B_{n-1}(a, q^{-1} x, y) - B_{n-1}(a, x, qy)].
\]

(2.22)

After simplification, we get

\[
B_n(a, x, y) = q^{n-1} \frac{1 - aq^{n-1}}{1 - q^n} \theta_{xy} \left\{ B_{n-1}(a, x, y) \right\}.
\]

(2.23)

By iteration, we gain

\[
B_n(a, x, y) = \frac{q^{(n)}(a; q)_n}{(q; q)_n} q^n \theta_{xy} \left\{ B_0(a, x, y) \right\}.
\]

(2.24)

Now we return to calculate \(A_0(a, x, y)\). Just taking \(z = 0\) in (2.20), we immediately obtain \(A_0(a, x, y) = f(a, x, y, 0)\). The proof of the assertion (2.13) of Theorem 2.2 is now completed by substituting (2.24) back into (2.20).

\[\square\]

3. Generating functions for generalized Cauchy polynomials

The generalized Cauchy polynomials \(p_n(x, y, a)\) are defined as

\[
p_n(x, y, a) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{(2)}(a; q)_k x^{n-k} y^k
\]

(3.1)

and their generating function

**Lemma 3.1.** [29] Eq. (2.21)] Suppose that \(|x| < 1\), we have:

\[
\sum_{n=0}^{\infty} p_n(x, y, a) \frac{t^n}{(q; q)_n} = \frac{1}{(xt; q)_\infty} \Phi_1 \left[ \begin{array}{c} a; \\ q; yt \\ 0; \end{array} \right].
\]

(3.2)

For \(a = 0\), in Lemma 3.1, we get the following

**Lemma 3.2.** [6] Suppose that \(|x| < 1\), we have:

\[
\sum_{n=0}^{\infty} p_n(x, y) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_\infty}{(xt; q)_\infty}.
\]

(3.3)

In this section, we use the representation (3.1) to derive another generating function for generalized Cauchy polynomials by the method of homogeneous \(q\)-difference equations.

**Theorem 3.1.** Suppose that \(|rx| < 1\), we have:

\[
\sum_{n=0}^{\infty} p_n(x, y, a) \frac{(s/r; q)_n r^n}{(q; q)_n} = \frac{(sx; q)_\infty}{(rx; q)_\infty} 2\Phi_2 \left[ \begin{array}{c} a, s/r; \\ q; ry \\ sx, 0; \end{array} \right].
\]

(3.4)

**Corollary 3.1.**

\[
\sum_{n=0}^{\infty} p_n(x, y, a)(-1)^n q^{(2)} \frac{s^n}{(q; q)_n} = (sx; q)_\infty \Phi_2 \left[ \begin{array}{c} a; \\ q; sy \\ sx, 0; \end{array} \right].
\]

(3.5)
Remark 3.1. For $s = 0$ and $r = t$ in Theorem 3.1 (3.4) reduces to (3.2). For $s = 0$, $r = t$ and $a = 0$ in Theorem 3.1 (3.4) reduces to (3.3). For $r = 0$ in Theorem 3.1 (3.4) reduces to (3.5).

Proof of Theorem 3.1. By denoting the right-hand side of (3.4) by $f(a,x,y)$, we can verify that $f(a,x,y)$ satisfies (2.1). So, we have

$$f(a,x,y) = \sum_{n=0}^{\infty} \mu_n p_n(x,y,a)$$

(3.6)

and

$$f(a,x,0) = \sum_{n=0}^{\infty} \mu_n x^n = \sum_{n=0}^{\infty} \frac{(s/r;q)_n (rx)^n}{(q;q)_n}.$$

(3.7)

So, $f(a,x,y)$ is equal to the right-hand side of (3.4).

Theorem 3.2. For $k \in \mathbb{N}$ and $|xt| < 1$, we have:

$$\sum_{n=0}^{\infty} p_{n+k}(x,y,a) = \frac{t^n}{(q;q)_n} = \frac{x^k}{(xt;q)_n} \sum_{n=0}^{\infty} \frac{(q^{-k},xt,a;q)_n (yx^{-1} q^k)_n}{(q;q)_n}, \Phi_1\left[\begin{array}{c} aq^n; \\ q;ytq^n \end{array}\right].$$

(3.8)

Remark 3.2. For $k = 0$, in Theorem 3.2 (3.8) reduces to (3.2).

Proof of Theorem 3.2. Denoting the right-hand side of equation (3.8) equivalently by

$$f(a,x,y) = x^k \sum_{n=0}^{\infty} \frac{(q^{-k},xt,a;q)_n (yx^{-1} q^k)_n}{(q;q)_n} \frac{1}{(qt^{q^n};q)_n} \Phi_1\left[\begin{array}{c} aq^n; \\ q;ytq^n \end{array}\right].$$

(3.9)

and it is easy to check that (3.9) satisfies (2.10), so we have:

$$f(a,x,y) = \sum_{n=0}^{\infty} \mu_n p_n(x,y,a).$$

(3.10)

Setting $y = 0$ in (3.9), it becomes

$$f(a,x,0) = \sum_{n=0}^{\infty} \mu_n x^n = \sum_{n=0}^{\infty} x^{n+k} = \sum_{n=0}^{\infty} x^n \frac{t^{n-k}}{(q;q)_{n-k}}.$$

(3.11)

Hence

$$f(a,x,y) = \Phi(a, \mu; D_q) \sum_{n=0}^{\infty} x^n \frac{t^{n-k}}{(q;q)_{n-k}} = \sum_{n=0}^{\infty} p_n(x,y,a) \frac{t^{n-k}}{(q;q)_{n-k}} = \sum_{n=0}^{\infty} p_{n+k}(x,y,a) \frac{t^n}{(q;q)_n},$$

(3.12)

which is the left-hand side of (3.8).

4. SRIVASTAVA-AGARWAL TYPE GENERATING FUNCTIONS INVOLVING GENERALIZED CAUCHY POLYNOMIALS

The Hahn polynomials \cite{12, 13} (or Al-Salam and Carlitz polynomials \cite{11}) are given by

$$\phi_n^{(a)}(x|q) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array}\right]_q (a;q)_k x^k.$$

(4.1)

Srivastava and Agarwal deduced the following generating function (also called Srivastava-Agarwal type generating functions).
Theorem 4.1. \cite{28} eq. (3.20)]
\[
\sum_{n=0}^{\infty} \phi_n^{(\alpha)}(x|q)(\lambda;q)_n t^n (q;q)_n = \frac{(\lambda t;q)_\infty}{(t;q)_\infty} \sum_{\lambda, \alpha; q; xt} \Phi_1 (\lambda, \alpha; q; xt)_{-\lambda t; 1} \left( \frac{a q^k}{q;\mu t q^k} \right), \quad \max\{|t|,|xt|\} < 1. \quad (4.2)
\]

For \(\lambda = 0\), we have:

Lemma 4.1. \cite{5} eq.(1.14)]
\[
\sum_{k=0}^{\infty} \phi_k^{(\alpha)}(x|q) t^k (q;q)_k = \frac{(\alpha x t;q)_\infty}{(x t; q)_\infty} \sum_{\alpha; q; xt} \Phi_1 (\alpha, q; xt)_{0; q; xt} \left( \frac{a q^k}{q;\mu t q^k} \right), \quad \max\{|xt|,|x|\} < 1. \quad (4.3)
\]

For more information about Srivastava-Agarwal type generating functions for Al-Salam-Carlitz polynomials, please refer to \cite{28} \cite{3}.

In this section, we use the representation (3.1) to derive Srivastava-Agarwal type generating function for generalized Cauchy polynomials by the method of homogeneous \(q\)-difference equations.

Theorem 4.1. For \(M \in \mathbb{N}, \) if \(\alpha = q^{-M}\) and \(\max\{|\lambda t|,|\lambda xt|\} < 1\), we have:
\[
\sum_{n=0}^{\infty} \phi_n^{(\alpha)}(x|q)H_n(\lambda, \mu, \alpha) t^n (q;q)_n = \frac{(\alpha x t;q)_\infty}{(x t; q)_\infty} \sum_{k=0}^{\infty} \left( -1 \right)^k \sum_{\alpha; q; xt} \Phi_1 (\alpha, q; xt) \left( \frac{a q^k}{q;\mu t q^k} \right) \left( \frac{(\alpha x t q^k;q)_\infty}{(x t q^k; q)_\infty} \right). \quad (4.4)
\]

Remark 4.1. Setting \(a = 0, \lambda = 1\) and \(\mu = 0\), formula (4.4) reduces to (4.3). For \(a = 0, \lambda = 1\) and \(\mu = \lambda\), formula (4.4) reduces to (4.2).

Proof of Theorem 4.1] Denoting the right-hand side of equation (4.4) by \(H(a, \lambda, \mu, \alpha, x)\), then we have:
\[
H(a, \lambda, \mu, \alpha, x) = \frac{1}{(\lambda x t;q)_\infty} \sum_{n=0}^{\infty} \left( -1 \right)^n \sum_{\alpha; q; xt} \Phi_1 (\alpha, q; xt) \frac{(\alpha x t q^n;q)_\infty}{(x t q^n; q)_\infty} \left( \frac{(\alpha x t q^n;q)_\infty}{(x t q^n; q)_\infty} \right). \quad (4.5)
\]

We suppose that the operator \(D_q\) acts upon the variable \(\lambda\). Because equation (4.5) satisfies (2.10), we have:
\[
H(a, \lambda, \mu, \alpha, x) = \tilde{E}(a, \mu; D_q) \left\{ H(a, \lambda, 0, \alpha, x) \right\} = \tilde{E}(a, \mu; D_q) \left\{ \frac{(\alpha x t;q)_\infty}{(x t; q)_\infty} \right\} = \tilde{E}(a, \mu; D_q) \left\{ \sum_{k=0}^{\infty} \Phi_k^{(\alpha)}(x|q) \frac{(\lambda t)^k}{(q;q)_k} \right\} = \sum_{k=0}^{\infty} \Phi_k^{(\alpha)}(x|q) \frac{t^k}{(q;q)_k} \tilde{E}(a, \mu; D_q) \{\lambda^k\}
\]

which is the left-hand side of (4.5). The proof is complete. \(\Box\)
5. A TRANSFORMATIONAL IDENTITY INVOLVING GENERATING FUNCTIONS FOR GENERALIZED CAUCHY POLYNOMIALS

In this section we deduce the following transformational identity involving generating functions for generalized Cauchy polynomials by the method of homogeneous $q$-difference equation.

**Theorem 5.1.** Let $A(k)$ and $B(k)$ satisfy

$$\sum_{k=0}^{\infty} A(k)x^k = \sum_{k=0}^{\infty} B(k)\frac{1}{(xtq^k; q)_\infty}$$  \(5.1\)

and we have

$$\sum_{k=0}^{\infty} A(k)p_k(x, y, a) = \sum_{k=0}^{\infty} B(k)\frac{1}{(xtq^k; q)_\infty} \Phi_1 \left[ \begin{array}{c} a; \\ q; ytq^k \\ 0; \end{array} \right]$$  \(5.2\)

supposing that (5.1) and (5.2) are convergent.

**Proof.** We denote the right-hand side of (5.2) by $f(a, x, y)$ and we can check that $f(a, x, y)$ satisfies (2.10). We then obtain

$$f(a, x, y) = \sum_{k=0}^{\infty} \mu_k p_k(x, y, a)$$  \(5.3\)

and

$$f(a, x, 0) = \sum_{k=0}^{\infty} \mu_k x^k = \sum_{k=0}^{\infty} B(k)\frac{1}{(xtq^k; q)_\infty} (by \ 5.1)$$

$$= \sum_{k=0}^{\infty} A(k)x^k. \quad (5.4)$$

Hence

$$f(a, x, y) = \sum_{k=0}^{\infty} A(k)p_k(x, y, a),$$  \(5.5\)

which is the left-hand side of (5.2). The proof of Theorem 5.1 is thus completed. \(\square\)

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