An approach to traces of random walks on the boundary of a hyperbolic group via reflected Dirichlet spaces

Pierre Mathieu∗ Yuki Tokushige†

Abstract

We show the existence of a trace process at infinity for random walks on hyperbolic groups of conformal dimension $< 2$. To do so, we employ the theory of Dirichlet forms which connects the theory of symmetric Markov processes to functional analytic perspectives. We introduce a family of Besov spaces associated to random walks and prove that they are isomorphic to some of the Besov spaces constructed from the co-homology of the group studied in Bourdon-Pajot (2003). We also study the regularity of harmonic measures of random walks on hyperbolic groups using the potential theory associated to Dirichlet forms.

1 Introduction

In this introduction, we will explain jump processes on a boundary induced by a stochastic process (e.g. Brownian motions or random walks) on an inner domain (or graph). The induced jump process on a boundary and the corresponding Dirichlet form illustrate probabilistic and analytic aspects of the problem we consider in this paper. We start with two motivational examples.

Jump process on the circle. Consider a reflecting Brownian motion $(BM_t)$ on a 2-dimensional closed disc $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ started at the origin. The path of $(BM_t)$ after its first hitting to the circle $S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ can be decomposed into countably many boundary excursions.

Intuitively speaking, by forgetting how $(BM_t)$ moves in the interior of $D$, we obtain the jump process on $S^1$ that jumps from starting points of boundary excursions to their endpoints.

∗Aix-Marseille Université, CNRS, Centrale Marseille, I2M UMR 7373, 13453 Marseille, FRANCE.

†Kyushu University, Institute of Mathematics for Industry, Fukuoka Prefecture, JAPAN.

1An excursion is a continuous path that starts from some point on $S^1$, also ends on $S^1$ and lies in the interior of $D$ in the meantime. The original path of $(BM_t)$ can be reconstructed by gluing together the sequence of its excursions off $S^1$ on the right time-scale. We refer to [3] for details.
In order to turn this intuition into a rigorous construction, we will use the theory of Dirichlet forms. See [FOT] and [CF] for background. Specifically, Example 1.2.3 in [FOT] is particularly relevant to what we will explain below.

The Dirichlet form corresponding to \((BM_t)\) on \(\mathcal{D}\) is given by the Dirichlet integral

\[
E_D(f,g) := \frac{1}{2} \int_{\mathcal{D}} \left( \frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial g}{\partial y} \right) dxdy,
\]

with domain

\[
W^{1,2}(\mathcal{D}) := \left\{ f \in L^2(\mathcal{D}) : \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \in L^2(\mathcal{D}) \right\}.
\]

In order to introduce the Dirichlet form corresponding to the jump process on \(S^1\) induced by \((BM_t)\), we recall the Poisson integral, which gives the harmonic extension of some prescribed boundary value. For a function \(\phi : S^1 \to \mathbb{R}\), define its Poisson integral, denoted by \(H^D\phi\), as follows:

\[
H^D\phi(z) := \int_0^{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - \theta') + r^2} \phi(\theta') \frac{d\theta'}{2\pi},
\]

where \(z = re^{i\theta}\) and \(\frac{d\theta'}{2\pi}\) is the uniform probability measure on \(S^1\). It is well-known that the following Dirichlet form corresponds to the jump process on \(S^1\) described above:

\[
\mathcal{F}^{S^1} := \{ \phi \in L^2(S^1, d\theta/2\pi) : H\phi \in W^{1,2}(\mathcal{D}) \},
\]

\[
\mathcal{E}^{S^1}(\phi, \psi) := E_D(H^D\phi, H^D\psi) \quad \text{for } \phi, \psi \in \mathcal{F}^{S^1}.
\]

Moreover, \(\mathcal{E}^{S^1}\) has a more explicit expression called the Douglas integral:

\[
\mathcal{E}^{S^1}(\phi, \psi) := \frac{\pi}{4} \int_0^{2\pi} \int_0^{2\pi} \sin^{-2} \left( \frac{\theta - \theta'}{2} \right) \left( \phi(\theta) - \phi(\theta') \right) \left( \psi(\theta) - \psi(\theta') \right) \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi}.
\]

The probability measure \(\frac{d\theta}{2\pi}\) on \(S^1\) is the first hitting distribution of \((BM_t)\) started at the origin. We remark that the integral kernel \(\sin^{-2} \left( \frac{\theta - \theta'}{2} \right)\) in (3) is comparable to \(|\theta - \theta'|^{-2}\) when \(|\theta - \theta'| \ll 1\). This shows a glimpse of the fact that certain Besov spaces naturally appear in the study of induced jump processes on a boundary.

The Markov process corresponding to the Dirichlet form \((\mathcal{E}^{S^1}, \mathcal{F}^{S^1})\) is the jump process on \(S^1\) induced by a reflecting Brownian motion on \(\mathcal{D}\). It is called the boundary trace of reflecting Brownian motion in the literature.

Another rigorous description of this induced jump process, which is more directly related to what we intuitively explained at the beginning, is to use the theory of time-changes of Markov processes. As a matter of fact, it is known that the induced jump process on \(S^1\) is a reflecting Brownian motion time-changed by the inverse of its boundary local time on \(S^1\). See Section 5.3, Example (3°) in [CF] for detailed discussions.

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2 The boundary local time, \(L_t\), is a non-decreasing continuous additive functional of \((BM_t)\) that only increases at times when the reflected Brownian motion belongs to \(S^1\). Let \((L_t^{-1})\) be the right-continuous inverse of the function \(t \to L_t\). Then the time-changed process \(t \to BM \circ L_t^{-1}\) is a jump process on \(S^1\) and its Dirichlet form is the one defined in [2].
The example of a reflecting Brownian motion on \( \mathcal{D} \) already has a hyperbolic flavor. Note that an Euclidean Brownian motion on \( \mathcal{D} \) is a time change of a hyperbolic Brownian motion on the Poincaré disc \( \mathbb{D} \). Since the operation of taking a boundary trace is independent the time-scale of the process, the boundary jump process of reflecting Brownian motion we already discussed can also be constructed as the boundary process of hyperbolic Brownian motion.

Let us finally mention a link with Calderón’s inverse problem related from Electrical Impedance Tomography. Calderón’s problem is to recover the unknown conductivity \( \kappa \) in the elliptic equation

\[
\nabla \cdot (\kappa \nabla f) = 0 \quad \text{in } \Omega
\]

from the measurements on \( \partial(\Omega) \) expressed by the Dirichlet-to-Neumann map. In the paper \cite{PS} the authors give a probabilistic interpretation of Calderón’s inverse problem using the boundary trace process of a reflecting diffusion process on an Euclidean domain \( \Omega \subset \mathbb{R}^d \) (\( d \geq 2 \)).

Jump process on the boundary at infinity of an infinite tree. Motivated by the example explained above, Kigami \cite{Ki} studied an analogous problem for transient random walks on an infinite tree. Suppose that we have a rooted infinite tree \( T \). For simplicity, we consider a simple random walk \( (\mathbf{R}_n^\infty) \), \( \omega \in T \), on \( T \) in what follows. Furthermore, assume that \( (\mathbf{R}_n^\infty) \) is transient. Because of transience, roughly speaking, \( (\mathbf{R}_n^\infty) \) escapes to ”infinities” as \( n \rightarrow \infty \). One way to make this loose description rigorous is to use the geometric boundary \( \Sigma \) of \( T \), which is the collection of infinite geodesics emanating from the root. The simple random walk \( (\mathbf{R}_n^\infty) \) almost surely converges as \( n \rightarrow \infty \) to a random point of \( \Sigma(T) \), say \( \mathbf{R}^\infty \), which is given by the loop-erasure of its trajectory. Denote by \( \nu_T \) the distribution of \( \mathbf{R}^\infty \); this probability measure

3 The Poincaré disc is the interior of \( \mathcal{D} \) equipped with the Riemannian metric
\[
\frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2}
\]
and the volume form \( \frac{4dx\,dy}{(1-x^2-y^2)^2} \). The hyperbolic Brownian motion in the Poincaré disc, say \( \mathbf{w}_t \), can be constructed as the solution of the stochastic differential equation

\[
d\mathbf{w}_t = \frac{1}{2} (1 - \|\mathbf{w}_t\|^2) dB_t,
\]

where \( \|\cdot\| \) is the Euclidean norm. Define

\[
A(t) = \int_0^t \frac{(1 - \|\mathbf{w}_s\|^2)^2}{4} ds.
\]

Then \( A \) is an increasing bijection from \( \mathbb{R}_+ \) to the interval \([0, A(\infty))\). Let \( A^{-1} \) be its inverse. Then the process \( t \to \mathbf{w}_{A^{-1}(t)} \) turns out to be a Euclidean Brownian motion considered up to its hitting time of \( S^1 \) i.e. it has the same law as the process \( (B_t) \) up to time \( T_1 = \inf \{ t > 0 : \|B_t\| = 1 \} \).

4 Recall from Footnote 3 that the hyperbolic Brownian motion is in fact a time-change of planar Brownian motion only up to its first hitting of \( S^1 \). Therefore the construction of the boundary process from hyperbolic Brownian motion involves as an extra step the reconstruction of reflected Brownian motion from Brownian motion killed on \( S^1 \). A similar issue arises in the case of jump processes on boundaries of trees discussed in the next paragraph. We address it in our context in Part 8 of the paper.
on $\Sigma(T)$ is called the \textit{harmonic measure} of $(\mathbb{R}^n)$. We next define the energy form $(\mathcal{E}^T, \mathcal{F}^T)$ by

$$\mathcal{F}^T := \{ f : T \to \mathbb{R} : \mathcal{E}^T(f, f) < \infty \},$$

where

$$\mathcal{E}^T(f, g) := \frac{1}{2} \sum_{x, y \in T : x \sim y} (f(x) - f(y))(g(x) - g(y)) \text{ for } f, g \in \mathcal{F}^T.$$  

Analogously to the Poisson integral \( [1] \), we define a linear operator $H^T$ which transforms functions on $\Sigma(T)$ into harmonic functions on $T$ in the following way: for a function $u : \Sigma(T) \to \mathbb{R}$ and $x \in T$, let

$$H^T u(x) := \mathbb{E}[u(\mathbb{R}^n_\infty)|\mathbb{R}^n_0 = x].$$

(4)

We finally introduce the Dirichlet form $(\mathcal{E}^{\Sigma(T)}, \mathcal{F}^{\Sigma(T)})$ on $\Sigma(T)$ in an analogous way to \( [2] \) as follows:

$$\mathcal{F}^{\Sigma(T)} := \{ u \in L^2(\Sigma(T), \nu^T) : H^T u \in \mathcal{F}^T \},$$

$$\mathcal{E}^{\Sigma(T)}(u, v) := \mathcal{E}^T(H^T u, H^T v) \text{ for } u, v \in \mathcal{F}^{\Sigma(T)}.$$  

(5)

It is shown in \( [K1] \), among other results, that $(\mathcal{E}^{\Sigma(T)}, \mathcal{F}^{\Sigma(T)})$ is \textit{regular} on $L^2(\Sigma(T), \nu^T)$, which is a condition that guarantees the existence of the Markov process on $\Sigma(T)$ corresponding to $(\mathcal{E}^{\Sigma(T)}, \mathcal{F}^{\Sigma(T)})$. Moreover, an explicit formula for $\mathcal{E}^{\Sigma(T)}(u, v)$, which is reminiscent of the Douglas integral \( [3] \), is also obtained. See Theorem 5.6 in \( [K1] \). Since $\Sigma(T)$ is homeomorphic to the Cantor set under suitable assumptions on the geometry of $T$, the Markov process corresponding to $(\mathcal{E}^{\Sigma(T)}, \mathcal{F}^{\Sigma(T)})$ can be viewed as a jump process on the Cantor set. We finally notice that many arguments in \( [K1] \) heavily utilize explicit computations peculiar to tree structure. Therefore it seems very difficult to extend his arguments to graphs that have more robust geometric structure.

\textbf{Jump process on the Gromov boundary at infinity of a hyperbolic group.} Motivated by two examples summarized above, we will consider in this paper an analogous problem for hyperbolic groups. Let $\Gamma$ be a non-elementary hyperbolic group, then $\Gamma$ is infinite, countable, non-amenable and discrete. Let $\mu$ be a symmetric probability measure on $\Gamma$ the support of which generates $\Gamma$. We need to assume a certain moment condition of $\mu$ for a reason we will explain later. Define a quadratic form $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ by

$$\mathcal{F}^\mu := \{ f : \Gamma \to \mathbb{R} : \mathcal{E}^\mu(f, f) < \infty \},$$

where

$$\mathcal{E}^\mu(f, g) := \frac{1}{2} \sum_{x, y \in \Gamma} \mu(x^{-1} y)(f(x) - f(y))(g(x) - g(y)) \text{ for } f, g \in \mathcal{F}^\mu.$$  

Let $(\mathbb{R}^n_\infty)$ be the RW driven by $\mu$ started at the identity $id$ of $\Gamma$. It is shown in \( [Ka] \) that almost all trajectories of the random walk $(\mathbb{R}^n_\infty)$ converge to some limit point $Z_\infty$ on the \textit{Gromov boundary} $\partial \Gamma$ of $\Gamma$. The law of the random variable $Z_\infty$ is called the harmonic measure of $(\mathbb{R}^n_\infty)$ and denoted by $\nu$.  

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By this result, analogously to (1) and (4), we have a linear operator $H$ that maps functions on $\partial \Gamma$ to harmonic functions on $\Gamma$:

$$Hu(x) = \mathbb{E}[u(x \cdot Z_\infty)] = \int u(x \cdot \xi) \, d\nu(\xi),$$

where $\mathbb{E}$ denotes the expectation with respect to the law of the random walk and $x \cdot \xi$ is the natural action of $\Gamma$ on its boundary. Following the analogy with (2) and (3), define the Dirichlet form on $L^2(\partial \Gamma, \nu)$ as follows:

$$\mathcal{F}^{\partial \Gamma, \mu} := \{ u \in L^2(\partial \Gamma, \nu) : Hu \in \mathcal{F}^\mu \},$$

$$\mathcal{E}^{\partial \Gamma, \mu}(u, v) := \mathcal{E}^\mu(Hu, Hv) \text{ for } u, v \in \mathcal{F}^{\partial \Gamma, \mu}.$$

It follows from results in [Na] and [Sil] that $\mathcal{E}^{\partial \Gamma, \mu}(u, v)$ has an expression similar to the Douglas integral (3). It reads

$$\mathcal{E}^{\partial \Gamma, \mu}(u, v) = \int \int_{\partial \Gamma \times \partial \Gamma} (u(\xi) - u(\eta))(v(\xi) - v(\eta))\Theta^\mu(\xi, \eta)\,d\nu(\xi)\,d\nu(\eta) \text{ for } u, v \in \mathcal{F}^{\partial \Gamma, \mu},$$

where

$$\mathcal{F}^{\partial \Gamma, \mu} = B_2(\mu) := \left\{ u \in L^2(\partial \Gamma, \nu) : \int \int_{\partial \Gamma \times \partial \Gamma} (u(\xi) - u(\eta))^2\Theta^\mu(\xi, \eta)\,d\nu(\xi)\,d\nu(\eta) < +\infty \right\}.$$

See Proposition 5.7 and Definition 5.8. The integral kernel $\Theta^\mu(\cdot, \cdot)$ appearing in (6) is the Naïm kernel introduced in [Na], see Definition 5.4.

The expression (6) by itself is not enough to show the regularity of $(\mathcal{E}^{\partial \Gamma, \mu}, B_2(\mu))$ on $L^2(\partial \Gamma, \nu)$. To overcome this problem, we consider two classes of Besov spaces consisting of (a) Besov spaces associated to symmetric probability measures with finite second moments, and (b) Besov spaces associated to metrics in the Ahlfors-regular conformal gauge introduced in [BP]. Under the assumption that the Ahlfors-regular conformal dimension of $\partial \Gamma$ is less than 2, through a careful comparison between Besov spaces in these classes using martingale arguments, see Theorem 6.1, we will obtain the regularity at once. This is where we need to assume that $\mu$ has a finite second moment, since this enables us to compare energies of functions on $\Gamma$ up to multiplicative constants. See Proposition 5.10.

Having established the regularity property, we prove that the jump process on $\partial \Gamma$ corresponding to $(\mathcal{E}^{\partial \Gamma, \mu}, B_2(\mu))$ is a certain time-change of a reflecting random walk in a sense to be made precise in Section 8. This last result completes the analogy with the case of a reflecting Brownian motion on $D$. Finally, we prove that harmonic measures of random walks with a finite first moment are smooth in a potential theoretic sense.

At the end of the paper, we will construct a boundary jump process that, as in the two motivating examples, is a time-change of a reflected random walk under the assumption of the Ahlfors-regular conformal dimension being less than 2. The boundary of a regular tree is a Cantor set of Ahlfors-regular conformal dimension 0. The boundary of the two-dimensional disc considered in the first paragraph of this introduction is the circle $S^1$ whose Ahlfors-regular conformal dimension is 1. Thus our condition on the Ahlfors-regular conformal dimension being less than 2 extends to new examples the construction of boundary jump processes.
2 Detailed summary of the paper

The paper is based on the interplay of the following three subjects: hyperbolic geometry, analysis on metric spaces, and the theory of Dirichlet forms and symmetric Markov processes associated to them. We begin this summary with briefly explaining the paper [BP], which is the starting point of our study and explains the interplay of the first two of the three subjects mentioned above.

**Besov spaces.** Let \((Z, \rho)\) be a uniformly perfect compact metric space which carries a doubling measure. In [BP], the authors introduced a class of Besov spaces which is canonically associated to a certain conformal structure of \((Z, \rho)\). For any metric \(d\) in the Ahlfors-regular conformal gauge \(J_{AR}(Z, \rho)\) of \((Z, \rho)\), we define a Besov space \((E^Z,d, B_2(d))\) on \(Z\) by

\[
E^Z,d(u, v) := \int \int_{Z \times Z} \frac{(u(\xi) - u(\eta))(v(\xi) - v(\eta))}{d(\xi, \eta)^{2q}} \, d\mathcal{H}_d(\xi) \, d\mathcal{H}_d(\eta),
\]

\[
B_2(d) := \{u \in L^2(Z, \mathcal{H}_d) ; E^Z,d(u, u) < \infty\},
\]

where \(q\) is the Hausdorff dimension of \((Z, d)\), and \(\mathcal{H}_d\) is a Hausdorff measure of \(d\). In [BP], the authors constructed a hyperbolic graph \(\Gamma_d\) whose Gromov boundary is equivalent to \((Z, d)\). They showed that the Besov space associated to a metric \(d\) in \(J_{AR}(Z, \rho)\) is Banach isomorphic to the set of boundary values of the elements in the \(\ell_2\)-cohomology group of \(\Gamma_d\). They also showed that for any metrics \(d, d'\) in \(J_{AR}(Z, \rho)\), the two graphs \(\Gamma_d\) and \(\Gamma_{d'}\) are quasi-isometric, and this quasi-isometry induces an isomorphism between the \(\ell_2\)-cohomologies of \(\Gamma_d\) and \(\Gamma_{d'}\). These results imply that all the Besov spaces associated to metrics \(d\) in \(J_{AR}(Z, \rho)\) are Banach isomorphic, and thus are canonically associated to the conformal structure on \(Z\).

In this paper, we shall consider the Besov spaces \(B_2(d)\) when the compact metric space \((Z, \rho)\) is the Gromov boundary of a non-elementary hyperbolic group \(\Gamma\) equipped with a visual metric, see Definition 4.3.

In the first part of this paper, we also define Besov spaces associated to random walks on \(\Gamma\). We will always assume that the driving measure \(\mu\) of a random walk on \(\Gamma\) is symmetric and admissible, which means that the support of \(\mu\) generates \(\Gamma\). For \(k \geq 1\), define \(M_k\) to be the set of all symmetric admissible probability measures on \(\Gamma\) with finite \(k\)-th moment. For \(\mu \in M_1\), we define a Besov space \((E^{\partial \Gamma, \mu}, B_2(\mu))\) associated to \(\mu\) by

\[
E^{\partial \Gamma, \mu}(u, v) := \int \int_{\partial \Gamma \times \partial \Gamma} (u(\xi) - u(\eta))(v(\xi) - v(\eta)) \Theta^\mu(\xi, \eta) \, d\nu(\xi) \, d\nu(\eta),
\]

\[
B_2(\mu) := \{u \in L^2(\partial \Gamma, \nu) ; E^{\partial \Gamma, \mu}(u, u) < \infty\},
\]

where \(\nu\) is the harmonic measure of the random walk driven by \(\mu\), and \(\Theta^\mu(\cdot, \cdot)\) is the Naim kernel associated to \(\mu\). We prove that the Besov spaces associated to different random walks are all isomorphic with each others and isomorphic to the Besov spaces in [BP]. (See Proposition 5.14 and Theorem 5.15) In the construction of the Besov spaces associated to random walks on \(\Gamma\), the role of the \(\ell_2\)-cohomology of \(\Gamma_d\) in [BP] is played by the set of harmonic functions with a finite energy. Moreover, the role of the quasi-isometry between \(\Gamma_d\) and \(\Gamma_{d'}\) will be played by a stability result for bilinear forms of random walks on a group established in [PSC].
Dirichlet forms. The second purpose of this paper is to further investigate the probabilistic aspects of the Besov spaces introduced above by using the theory of Dirichlet forms. A Dirichlet form is a closed symmetric bilinear form on an $L^2$ space which satisfies a certain contraction property, called the Markovian property. In particular, for Dirichlet forms satisfying the regularity property (which roughly means that the domain of the form contains sufficiently many continuous functions, see Definition 2.6.), there is a well-known correspondence between regular Dirichlet forms and symmetric Markov processes. We will explain basic facts on Dirichlet forms in Section 2.

In Section 6.1, we will prove that, under the assumption that the Ahlfors-regular conformal dimension of $(\partial \Gamma, \rho_{\Gamma})$ is strictly less than 2, the Besov spaces $(\mathcal{E}^{\partial \Gamma, d}, B_2(d))$ and $(\mathcal{E}^{\partial \Gamma, \mu}, B_2(\mu))$ are regular Dirichlet forms for any $d \in J_{AR}(\partial \Gamma)$ and $\mu \in M_2$. This result will allow us to construct symmetric Markov processes associated to them.

**Theorem 6.1.** Assume the Ahlfors-regular conformal dimension of $\partial \Gamma$ is strictly less than 2. Then for any $d \in J_{AR}(\partial \Gamma)$ and any $\mu \in M_2$, $(\mathcal{E}^{\partial \Gamma, d}, B_2(d))$ and $(\mathcal{E}^{\partial \Gamma, \mu}, B_2(\mu))$ are regular Dirichlet forms on $L^2(\partial \Gamma, H_d)$ and on $L^2(\partial \Gamma, \nu)$ respectively.

Here is a brief sketch of the argument leading to Theorem 6.1. Under our assumption on the Ahlfors-regular conformal dimension, there exists a metric $d_0$ belonging to $J_{AR}(\partial \Gamma)$ such that $\text{dim}(\partial \Gamma, d_0) < 2$, where $\text{dim}$ is the Hausdorff dimension. It is easy to see that $(\mathcal{E}^{\partial \Gamma, d}, B_2(d_0))$ is regular.

Then, after a careful look at the isomorphism between different Besov spaces, we can deduce that all Lipschitz functions with respect to $d_0$ belong to all the Besov spaces of the form $(\mathcal{E}^{\partial \Gamma, d}, B_2(d))$ and $(\mathcal{E}^{\partial \Gamma, \mu}, B_2(\mu))$ as in the theorem. We thus obtain the regularity of all these Besov spaces.

Examples of hyperbolic groups with Ahlfors-regular conformal dimension less than 2 are free groups, cocompact Fuchsian groups and carpet groups. In particular, it is proved in [Ha1] that for any non-elementary hyperbolic group $G$ with planar boundary non-homeomorphic to the full sphere, the Ahlfors-regular conformal dimension of the boundary is strictly less than 2 if and only if $G$ is virtually isomorphic to a convex-cocompact Kleinian group. We refer to [Ha1] and its references for other results on hyperbolic groups with planar boundaries.

As a consequence of Theorem 5.1, by the general correspondence between regular Dirichlet forms and Markov processes, we conclude that each of the Besov spaces gives rise to a Markov process on $\partial \Gamma$. The Besov space $(\mathcal{E}^{\partial \Gamma, d}, B_2(d))$ corresponds to a strong Markov process (Hunt process) whose reference measure is the Hausdorff measure $\mathcal{H}_d$ and whose jumping kernel is $d(\cdot, \cdot)^{-2q}$, and the Besov space $(\mathcal{E}^{\partial \Gamma, \mu}, B_2(\mu))$ corresponds to a strong Markov process whose reference measure is the harmonic measure and whose jumping kernel is the Naïm kernel.

At the end of this paper, in Part 8, we will give a further probabilistic interpretation, now at the level of processes themselves. More precisely, we show that there exists a Markov process $(W_t)$ with state space $\Gamma \cup \partial \Gamma$ that satisfies the following properties:
(i) almost all trajectories of \((W_t)\) hit the boundary \(\partial \Gamma\) in finite time;
(ii) until the hitting time of \(\partial \Gamma\), the trajectories of \((W_t)\) behave in distribution like a time change of the random walk with driving measure \(\mu\) (See Proposition 8.2.);
(iii) the process \((W_t)\) has a trace on the boundary \(\partial \Gamma\) whose Dirichlet form is given by \((\mathcal{E}^\partial \Gamma, \mu, B_2(\mu))\) (See Theorem 8.3.).

The trace process is defined as the time-change of \((W_t)\) using the positive continuous additive functional whose Revuz measure is the harmonic measure \(\nu\).

The proofs of these claims use the notions of smooth measures (to be discussed in the next paragraph) and reflected Dirichlet spaces. A reflected Dirichlet space can be associated to any nice transient Markov process, [Sil], [Ch], [CF]. Roughly speaking, in our context, it consists in speeding up the original random walk so that it now hits the boundary in finite time and then prolongating its life in such a way that the resulting process still behaves as the initial random walk when it is in \(\Gamma\).

Under the assumption of the Ahlfors-regular conformal dimension being less than 2, we show that the construction of such a reflected random walk is possible with state space \(\Gamma \cup \partial \Gamma\). Properties (i), (ii) and (iii) follow from general results from the theory of Dirichlet forms. Observe however that the regularity stated in Theorem 6.1 and its close companion Theorem 8.1 are a crucial step to use Dirichlet form theory.

Potential theoretic properties of harmonic measures. Harmonic measures of random walks on a non-elementary hyperbolic groups have been extensively studied and it is known that their behavior strongly depends on moment assumptions of the driving measures of the random walk. For instance, when the driving measure is finitely supported, it is shown in [BHM] that the associated Green metric on \(\Gamma\) is hyperbolic and the corresponding harmonic measure belongs to the Patterson-Sullivan class (See [Coo] and [Ha2].) determined by the Green metric; the hyperbolicity of the Green metric is equivalent to Ancona’s inequality, which roughly means that the Green function is submultiplicative along geodesics.

Properties of harmonic measures are not so well understood when we only assume a weaker moment condition. For instance it is shown in [Gou] that for any non-elementary hyperbolic group \(\Gamma\), there exists a symmetric probability measure on \(\Gamma\) with some finite exponential moment for which Ancona’s inequality fails. Therefore, we cannot conclude in general that a harmonic measure belongs to the Patterson-Sullivan class determined by the Green metric. We mention here that results in [T] imply that one can still compute the Hausdorff dimension of a harmonic measure on \(\partial \Gamma\) as long as the driving measure has a finite first moment.

When we are given a regular Dirichlet form and the corresponding Markov processes, we have potential theoretic objects associated to it such as capacities. Measures which do not charge sets of zero capacity are said to be smooth, and there is another potential theoretic notion for measures, called measures of finite energy integral, which is a stronger property than smoothness. Both notions are related to time changes of symmetric Markov processes. (See Subsection 5.2 for details.)

After proving the regularity of the Besov spaces, we study the smoothness property of harmonic measures of random walks on \(\Gamma\). For \(\mu \in M_2\), we will introduce the set of all smooth measures (resp. the set of all measures of finite energy integral) with respect to the regular
Dirichlet form \((\mathcal{E}^{\partial \Gamma, \mu}, B_2(\mu))\); we denote it with \(\mathcal{S}(\partial \Gamma, \mu)\). (resp. \(\mathcal{S}_0(\partial \Gamma, \mu)\)) Similarly, for a metric \(d\) in \(J_{AR}(\partial \Gamma)\), we will define \(\mathcal{S}(\partial \Gamma, d)\) \((\mathcal{S}_0(\partial \Gamma, d)\), resp.) to be the set of all smooth measures (resp. the set of all measures of finite energy integral) with respect to \((\mathcal{E}^{\partial \Gamma, d}, B_2(d))\). (See Definition 6.9.) In Theorem 6.11 we will prove that for any \(d \in J_{AR}(\partial \Gamma)\) and \(\mu \in M_2\) we have

\[
\mathcal{S}(\partial \Gamma, d) = \mathcal{S}(\partial \Gamma, \mu), \quad \text{and} \quad \mathcal{S}_0(\partial \Gamma, d) = \mathcal{S}_0(\partial \Gamma, \mu).
\]

(7) We will denote the common set by \(\mathcal{S}(\partial \Gamma)\) and by \(\mathcal{S}_0(\partial \Gamma)\) respectively. We finally use the above coincidence (7) to study harmonic measures under very weak moment condition.

**Theorem 9.4.** Assume the Ahlfors-regular conformal dimension of \(\partial \Gamma\) is strictly less than 2. Then, both of \(\mathcal{S}(\partial \Gamma)\) and \(\mathcal{S}_0(\partial \Gamma)\) contain the harmonic measure \(\nu\) of any random walk driven by \(\mu \in M_1\).

When \(\mu \in M_2\), the above claim can be relatively easily deduced from Poincaré-type inequalities on \(\partial \Gamma\) as in Proposition 6.10 (See Theorem 6.11.) To relax the moment condition to a finite first moment, in Section 9 we combine heat kernel estimates for jump processes from [GHH] (See also [CK, CKW].) and deviation inequalities from [MS].

The paper is organized as follows. In Section 3 we explain definitions and basic facts about Dirichlet forms. We also introduce several examples of Dirichlet forms and the corresponding Markov processes. Section 4 is devoted to the explanation of the paper [BP] including their construction of Besov spaces. In Section 5 we first introduce Besov spaces associated to random walks on a non-elementary hyperbolic group \(\Gamma\). We then prove that Besov spaces introduced here and in [BP] are all isomorphic. Section 6 starts with the proof of Theorem 6.1. Then we further develop the potential theory of Dirichlet forms and prove the smoothness property of harmonic measures. In Section 7 we first introduce several general facts about time-change techniques in the theory of Dirichlet forms. We next give a brief explanation of reflected Dirichlet spaces. In Section 8 we provide an interpretation of the Markov processes which correspond to the Besov spaces associated to random walks using reflected random walks. In Section 9, we pursue further the potential theoretic aspect of harmonic measures using heat kernel estimates and deviation inequalities and we prove Theorem 9.4.

### 3 Preliminary facts on Dirichlet forms

In this section, we briefly explain several basic facts about Dirichlet forms including their connection to probability theory. See [FOT, CF] for details of the theory of Dirichlet forms and their probabilistic aspects, especially the theory of symmetric Markov processes. A Dirichlet form is a closed symmetric bilinear form on an \(L^2\)-space which satisfies a kind of contraction property, called the Markovian property. It is a general fact in functional analysis that there is a one to one correspondence between the collection of closed symmetric bilinear form defined on a Hilbert space \(H^1\) and the collection of non-positive definite self-adjoint operators on \(H^1\). Thus, we can associate a strongly continuous semigroup to a given closed symmetric form. In particular, when a given closed symmetric form satisfies the Markovian property, then the
corresponding semigroup also has a kind of positivity preserving property, which is also called the *Markovian* property. We will explain below what we quickly sketched out in detail.

Let $E$ be a locally compact Hausdorff space, and $m$ be a positive Radon measure on $E$ with full support.

**Definition 3.1.** (1) We say that a bilinear form $(\mathcal{E}, \mathcal{F})$ on a real Hilbert space $\text{Hil}$ is a symmetric closed form if the following conditions are satisfied:

- $\mathcal{F}$ is a dense linear subspace of $\text{Hil}$, and $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ is non-negative definite, symmetric and bilinear,
- for any $\alpha > 0$, $(\mathcal{F}, (\mathcal{E}_\alpha)^{1/2})$ is a Hilbert space, where
  \[ \mathcal{E}_\alpha(u, v) := \mathcal{E}(u, v) + \alpha(u, v)_H, \quad u, v \in \mathcal{F}. \]

(2) A bilinear form $(\mathcal{E}, \mathcal{F})$ is called a Dirichlet form on $L^2(E, m)$ if $(\mathcal{E}, \mathcal{F})$ is a closed symmetric form on $L^2(E, m)$, and for any $u \in \mathcal{F}$, we have that $v := (0 \lor u) \land 1 \in \mathcal{F}$ and $\mathcal{E}(v, v) \leq \mathcal{E}(u, u)$. The latter condition is called the Markovian property.

(3) A linear operator $U : L^2(E, m) \to L^2(E, m)$ is said to be Markovian if for any $v \in L^2(E, m)$ with $0 \leq v \leq 1$ $m$-a.e., we have that $0 \leq Uv \leq 1$ $m$-a.e.

**Theorem 3.2.** [FOT, Theorem 1.3.1, Lemma 1.3.2, Theorem 1.4.1]

(1) There is a one to one correspondence between the collection of closed symmetric forms $(\mathcal{E}, \mathcal{F})$ on a real Hilbert space $\text{Hil}$ and the collection of non-positive definite self-adjoint operators $A$ on $\text{Hil}$. This correspondence is characterized by

\[
\begin{align*}
\{ & \text{Dom}(A) \subset \mathcal{F}, \\
& \mathcal{E}(u, v) = (-Au, v)_H, \quad u \in \text{Dom}(A), v \in \mathcal{F}. \}
\end{align*}
\]

(2) Let $A$ be a non-positive definite self-adjoint operator on $\text{Hil}$. Then, $(T_t)_{t>0} := (\exp(tA))_{t>0}$ is a strongly continuous semigroup on $\text{Hil}$, and the generator of $(T_t)_{t>0}$ coincides with $A$. Moreover, there is a unique strongly continuous semigroup whose generator is $A$.

(3) Let $(\mathcal{E}, \mathcal{F})$ be a closed symmetric form on $\text{Hil}$ and $(T_t)_{t>0}$ be the corresponding strongly continuous semigroup on $\text{Hil}$. Then, $(\mathcal{E}, \mathcal{F})$ is Markovian if and only if $T_t$ is Markovian for any $t > 0$.

We next define an extended Dirichlet space, which will be used in what follows.

**Definition 3.3.** Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(E, m)$. We denote by $\mathcal{F}_e$ the set of all $m$-measurable functions $u$ with the following properties:

- $|u| < \infty$ $m$-a.e. and
Hunt processes on $E$.

Remark 3.5. We recommend interested readers to consult textbooks such as [FOT, CF] for details including the precise definition of Hunt processes.

Definition 3.6. A Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E, m)$ is called regular if $C_0(E) \cap \mathcal{F}$ is dense both in $(C_0(E), \| \cdot \|_\infty)$ and $(\mathcal{F}, (\mathcal{E}_1)^{1/2})$.

FACT. There exists a correspondence, which is one to one in a certain sense, between regular Dirichlet forms on $L^2(E, m)$ and $m$-symmetric Hunt processes on $E$. A Hunt process is a strong Markov process which has cadlag sample paths and certain additional properties. See [FOT, Appendix A.2] for the precise definition.

For a given regular Dirichlet form $(\mathcal{E}, \mathcal{F})$, let $A$ be the self-adjoint operator determined by $\mathcal{E}$, and $(T_t)$ be the semigroup whose generator is $A$. Then, the corresponding Hunt process $(X_t)$ satisfies (9). See [FOT, Chapter 7] and [CF, Theorem 1.5.1] for the precise statement.

In the rest of this section, we will give two examples of regular Dirichlet forms and explain their probabilistic interpretation. The latter example will play a very important role in what follows.
Example 3.7. Consider the standard Dirichlet energy \( \frac{1}{2} \int_{\mathbb{R}^n} (\nabla f \cdot \nabla g) \, dx \) on the Euclidean space \( \mathbb{R}^n \), where \( u, v \in W^{1,2}(\mathbb{R}^n) := \{ f \in L^2(\mathbb{R}^n, dx) ; \frac{\partial f}{\partial x_i} \in L^2(\mathbb{R}^n, dx), \text{ for } i = 1, \ldots, n \} \). Then it is well-known that \( \left( \frac{1}{2} \int_{\mathbb{R}^n} (\nabla f \cdot \nabla g) \, dx, W^{1,2}(\mathbb{R}^n) \right) \) is a regular Dirichlet form on \( L^2(\mathbb{R}^n, dx) \). Moreover, the relation (8) implies that the corresponding non-positive self-adjoint operator is given by \( \frac{1}{2} \Delta = \frac{1}{2} \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \). Thus, the corresponding symmetric Hunt process is a standard \( n \)-dim Brownian motion on \( \mathbb{R}^n \).

Example 3.8. Let \( \mathcal{V} \) be either a finite set or a countable set. Let \( c : \mathcal{V} \times \mathcal{V} \to \mathbb{R}_{\geq 0} \) be a weight function which is symmetric (i.e., \( c(x,y) = c(y,x) \)). We define a measure \( m \) on \( \mathcal{V} \) by \( m(x) := \sum_{y \in \mathcal{V}} c(x,y) \), and assume that \( \text{supp}(m) = \mathcal{V} \) and \( \sup_{x \in \mathcal{V}} m(x) < \infty \). (10)

Now we define a bilinear form \( (\mathcal{E}, \mathcal{F}) \) on \( L^2(\mathcal{V}, m) \) by

\[
\mathcal{E}(u, v) := \frac{1}{2} \sum_{x,y \in \mathcal{V}} c(x,y)(u(x) - u(y))(v(x) - v(y)),
\]

\[
\mathcal{F} = L^2(\mathcal{V}, m),
\]

Then, it is shown in [CF, Theorem 2.2.2] that \( (\mathcal{E}, \mathcal{F}) \) is a regular Dirichlet form on \( L^2(\mathcal{V}, m) \), and the corresponding \( m \)-symmetric Hunt process is the continuous time random walk \( (X_t)_{t \geq 0} \) which is defined as follows: define \( p(x,y) := c(x,y)/m(x) \), and let \( (R_n)_{n \in \mathbb{N}} \) be a discrete time Markov chain with transition probabilities \( (p(x,y))_{x,y \in \mathcal{V}} \). Let \( (N_t)_{t \geq 0} \) be a Poisson process with intensity 1 that is independent of \( (R_n) \). Then, the \( m \)-symmetric Markov process \( (X_t)_{t \geq 0} \) is given by \( X_t := Y_{N_t} \). This construction of \( (X_t) \) is equivalent to the fact that \( (X_t) \) has random holding times given by i.i.d. exponential distributions with mean 1 at all vertices. For this reason, the process \( (X_t)_{t \geq 0} \) is often called the “constant speed random walk”. See [CF, Section 2.2.1] for detail.

4 Besov spaces constructed by Bourdon and Pajot

In this section, we will give a summary of some results in [BP], in particular the construction of Besov spaces on a compact metric space.

4.1 \( \ell_p \)-cohomology of simplicial complexes and its invariance by quasi-isometries

We consider a simplicial complex \( K \) equipped with a length metric, denoted by \( | \cdot - \cdot | \), such that

- there exists a constant \( C > 0 \) such that the diameter of all simplexes of \( K \) are bounded by \( C \), and
Simplicial complexes satisfying the above properties are called geometric. Now we define the \( \ell_p \)-cohomology of \( K \). We will say that \( K \) is uniformly contractible if it is contractible and there exists a function \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) such that all balls \( B_K(x, r) \) are contractible in \( B_K(x, \phi(r)) \). Let \( K_i \) be the set of \( i \)-simplexes of \( K \) and \( \ell_p C^i(K) \) (\( p \in [1, \infty] \)) be the Banach space consisting of \( \ell_p \)-functions on \( K_i \). Define the coboundary operator \( \delta_i : \ell_p C^i(K) \to \ell_p C^{i+1}(K) \) by \( (\delta_i \tau)(\sigma) := \tau(\partial \sigma) \), where \( \tau \in \ell_p C^i(K) \) and \( \sigma \in K_{i+1} \). Note that if \( K \) is a geometric simplicial complex, then \( \delta_i \) is a bounded operator. The \( i \)-th \( \ell_p \)-cohomology group of \( K \) is defined by

\[
\ell_p H^i(K) := \ker \delta_i / \text{Im} \delta_{i-1}.
\]

The following theorem asserts the invariance of \( \ell_p H^i(K) \) by quasi-isometries.

**Theorem 4.1.** [FP Theorem 1.1] Let \( K \) and \( K' \) be geometric uniformly contractible simplicial complexes. If \( F : K \to K' \) is a quasi-isometry, then it induces an isomorphism of topological vector spaces \( N^* : \ell_p H^*(K) \to \ell_p H^*(K') \).

We give a brief sketch of the construction of \( N^1 \), which will be used later. Let \( C_i(K) \) be the vector space spanned by elements of \( K_i \). First, define a map

\[
c_0 : K_0 \to C_0(K')
\]

by choosing an element of \( K_0' \) uniformly close to \( F(x) \) for each \( x \in K_0 \). Next, define a map \( c_1 : K_1 \to C_1(K') \), satisfying \( \partial c_1(\sigma) = c_0(\partial \sigma) \) for any \( \sigma \in K_1 \) in the following way: for an edge \( a \in K_1 \), denote its end points by \( a_+, a_- \in K_0 \). Then we can find an element \( c_1(a) \) of \( C_1(K') \) with \( \partial c_1(a) = c_0(a_+) - c_0(a_-) \). For \( \tau \in \ell_p C^1(K') \), define a map \( N^*(\tau) : K_1 \to \mathbb{R} \) by

\[
N^*(\tau)(\sigma) := \tau(c_1(\sigma)), \quad \sigma \in K_1.
\]

Then the isomorphism \( N^1 : \ell_p H^1(K') \to \ell_p H^1(K) \) is induced by the linear map \( N^* : \ell_p C^1(K') \to \ell_p C^1(K) \).

### 4.2 The hyperbolic fillings by Bourdon and Pajot

In what follows, we always assume that a compact metric space \((Z, \rho)\) satisfies the following properties:

(i) \((Z, \rho)\) is uniformly perfect, namely there exists a constant \( C > 1 \) such that for any \( x \in Z \) and any \( 0 < r \leq \text{diam}(Z, \rho) \), we have that

\[
B_\rho(x, r) \setminus B_\rho(x, r/C) \neq \emptyset,
\]

where \( B_\rho(x, r) := \{z \in Z : \rho(x, z) < r\} \).
(ii) \((Z, \rho)\) carries a doubling measure, namely there exists a Borel measure \(\theta\) on \(Z\) such that there exists a constant \(C' > 1\) such that for any \(x \in Z\) and any \(r > 0\), we have that

\[
\theta(B_\rho(x, 2r)) \leq C' \theta(B_\rho(x, r)).
\]

We now introduce several definitions which will be important later.

**Definition 4.2.**
(1) A metric \(d\) on \(Z\) is called Ahlfors-regular if there exists constants \(C, C' > 0\) such that for any \(\xi \in Z\) and any \(0 < r < \text{diam}(Z, d)\), we have that

\[
Cr^q \leq H_d(B_d(x, r)) \leq C'r^q,
\]

where \(H_d\) is a Hausdorff measure of \(d\), and \(q = \dim(Z, d)\).

(2) Two metrics \(d, d'\) on \(Z\) are called quasi-symmetric if there exists an increasing homeomorphism \(\alpha : [0, \infty) \to [0, \infty)\) such that for any distinct triple \(\xi, \eta, \omega\) of \(Z\) we have that

\[
\frac{d(\xi, \eta)}{d(\xi, \omega)} \leq \alpha \left( \frac{d'(\xi, \eta)}{d'(\xi, \omega)} \right).
\]

(3) We denote by \(J(Z, \rho)\) (called the conformal gauge of \((Z, \rho)\)) the set of all metrics on \(Z\) which are quasi-symmetric to \(\rho\), and by \(J_{AR}(Z, \rho)\) (called the Ahlfors-regular conformal gauge of \((Z, \rho)\)) the set of all Ahlfors regular metrics in \(J(Z, \rho)\).

**Definition 4.3.** Let \(\tilde{\Gamma}\) be a proper geodesic hyperbolic space in the sense of Gromov. A metric \(d\) on the Gromov boundary \(\partial \tilde{\Gamma}\) of \(\tilde{\Gamma}\) is called a visual metric if the following condition holds: there exists a constant \(C > 0\) and \(a > 0\) such that for any \(\xi, \eta \in \partial \tilde{\Gamma}\) we have that

\[
C^{-1} e^{-a(\xi|\eta)O} \leq d(\xi, \eta) \leq Ce^{-a(\xi|\eta)O},
\]

where \((\xi|\eta)O\) is the Gromov product extended to \(\partial \tilde{\Gamma}\) with a fixed base point \(O\). Notice that the notion of visual metrics defines a class of metrics on \(\partial \tilde{\Gamma}\) whose members are all quasi-symmetric. See [GH] for notions that appear here.

For a uniformly perfect compact metric space \((Z, \rho)\) which carries a doubling measure, in [BP], it is shown that for any \(d \in J(Z, \rho)\) we can construct a geometric uniformly contractible simplicial complex \(K_d\) and a graph \(\Gamma_d\) which is the 1-skeleton of \(K_d\) with the following properties. Precise definitions of \(\Gamma_d\) and \(K_d\) will be given later in this subsection.

**Theorem 4.4.** [BP, Proposition 2.1, Corollary 2.4.]
Let \(d \in J(Z, \rho)\).

- The graph \(\Gamma_d\) is of bounded degree.
- The graph \(\Gamma_d\) is hyperbolic in the sense of Gromov. Moreover, the Gromov boundary of \(\Gamma_d\) equipped with a visual metric is quasi-symmetric to \((Z, d)\).
• For two metrics \(d, d' \in J(Z, \rho)\), there exists a quasi-isometry \(F : \Gamma_d \to \Gamma_{d'}\) which can be continuously extended to the identity map on \(Z\).

• Let \(\tilde{\Gamma}\) be a proper and geodesic hyperbolic space in the sense of Gromov. Suppose that there exists a point \(O \in \tilde{\Gamma}\) and a constant \(C \geq 0\), such that all points in \(\tilde{\Gamma}\) are within distance \(C\) from some geodesic ray starting at \(O\). Assume that the Gromov boundary of \(\tilde{\Gamma}\) is \(Z\) and that \(\rho\) is quasi-symmetric to a visual metric on \(Z\) induced by the hyperbolic structure of \(\tilde{\Gamma}\). Then for any \(d \in J(Z, \rho)\), there exists a quasi-isometry \(F : \Gamma_d \to \tilde{\Gamma}\) which can be continuously extended to the identity map on \(Z\).

It follows from Theorem 4.1 and the third claim of Theorem 4.4 that \(\ell_p H^\bullet(K_d)\) and \(\ell_p H^\bullet(K_{d'})\) are isomorphic topological vector spaces. Hence these topological vector spaces can be considered to be an invariant with respect to quasi-symmetry, and we will denote it by \(\ell_p H^\bullet(J(Z, \rho))\).

Here we explain the constructions of \(K_d\) and \(\Gamma_d\). Normalize the metric \(d\) in such a way that \(\text{diam}(Z, d) = 1/2\). For each \(l \geq 0\), choose points \(z_1^l, ..., z_{k(l)}^l\) in \(Z\) in such a way that for any \(i, j \in \{1, ..., k(l)\}\) with \(i \neq j\), we have \(d(z_i^l, z_j^l) \geq e^{-l}\), and for each \(l \geq 0\), the balls \(B_i^l := B_d(z_i^l, e^{-l})\), \(1 \leq i \leq k(l)\), cover \((Z, \rho)\). Denote by \(S_l\) the cover \(\{B_i^l ; i \in \{1, ..., k(l)\}\}\).

Remark that \(S_0\) must be the singleton \(\{B_1^0\}\) because of the normalization of the diameter. Now define \(\Gamma_d\) as follows. The vertex set \(V(\Gamma_d)\) is the collection of balls \(\{B_i^l ; n \geq 0, i \in \{1, ..., k(l)\}\}\), and two distinct vertex \(B, B'\) are connected by an edge if

1. both \(B\) and \(B'\) belong to \(S_l\) and \(B \cap B' \neq \emptyset\), or if
2. one of them belongs to \(S_l\), the other belongs to \(S_{l+1}\) and \(B \cap B' \neq \emptyset\).

We equip \(\Gamma_d\) with a length metric, denoted by \(|·−·|\), by identifying each edge with the Euclidean segment of length 1. Denote the vertex \(B_0^1\) by \(O\) and for each \(x \in V(\Gamma_d)\), let \(B(x)\) be the set of all infinite geodesic rays starting at \(O\) and passing through \(x\). Now a simplicial complex \(K_d\) is defined as follows: for \(n \in \mathbb{N}\), the \(n\)-th Rips complex of \(\Gamma_d\) is a simplicial complex whose \(k\)-simplexes are sets of vertices \(\{x_1, ..., x_{k+1}\}\) \((x_i \in V(\Gamma_d), 1 \leq i \leq k+1\) with \(|x_i − x_j| \leq n\) for any \(i, j \in \{1, ..., k+1\}\). It is known that for \(n\) large enough, the \(n\)-th Rips complex of \(\Gamma_d\) is geometric and uniformly contractible, thus we let \(K_d\) be the \(n\)-th Rips complex of \(\Gamma_d\) for \(n\) large enough. See [BH] Section 3 and [BP] for the proof of these facts.

Note that it is shown in [BP] that for \(d \in J(Z, \rho)\),

\[
\ell_p H^1(J(Z, \rho)) \simeq \{f : V(\Gamma_d) \to \mathbb{R} ; df \in \ell_p(E(\Gamma_d))/\ell_p(V(\Gamma_d))\} + \mathbb{R}. \tag{14}
\]

Relying on Theorem 4.4 in [BP], the authors introduced the following Besov space associated to each metric in \(J_{AR}(Z, \rho)\), and it is shown that the set of boundary values of elements in \(\ell_p H^1(J(Z, \rho))\) coincides with the Besov space.

**Definition 4.5.** (1) Let \(p \in [1, \infty)\). For a function \(u : Z \to \mathbb{R}\) and a metric \(d \in J_{AR}(Z, \rho)\) of dimension \(q\), define

\[
\|u\|_{p, d} := \left( \int_{Z×Z} \frac{|u(\xi) − u(\eta)|^p}{d(\xi, \eta)^{2q}} d\mathcal{H}_d(\xi)d\mathcal{H}_d(\eta) \right)^{1/p},
\]
where $\mathcal{H}_d$ is the Hausdorff measure of $d$. Define $B_p(Z, d) := \{u : Z \to \mathbb{R}; \|u\|_{p,d} < \infty\}$. We will call $(\| \cdot \|_{p,d}, B_p(Z, d))$ a $p$-Besov space on $Z$ associated to $d$. Then $(B_p(Z, d)/ \sim, \| \cdot \|_{p,d})$ is a Banach space, where $u \sim v$ means $u(\xi) - v(\xi)$ is a constant for $\mathcal{H}_d$-a.e $\xi$.

In the rest of the paper, we will write $B_p(Z, d) = B_p(d)$ when the choice of the space $Z$ is clear from the context.

(2) In what follows, we will particularly focus on the $L^2$ case ($p = 2$), and thus employ the following special notation: let $d \in J_{AR}(Z, \rho)$ be a metric of dimension $q$, and $u, v \in B_2(d)$.

Define

$$\mathcal{E}^{Z,d}(u, v) := \int \int_{Z \times Z} \frac{(u(\xi) - u(\eta))(v(\xi) - v(\eta))}{d(\xi, \eta)^{2q}} d\mathcal{H}_d(\xi) d\mathcal{H}_d(\eta).$$

(3) For $d \in J(Z, \rho)$ and $f : V(\Gamma_d) \to \mathbb{R}$ with $df \in \ell_p(E(\Gamma_d))$, define $f_\infty : Z \to \mathbb{R}$ (if it exists)

$$f_\infty(\xi) := \lim_{n \to \infty} f(r(n)), \quad \xi \in Z,$$

where $r$ is a geodesic ray of $\Gamma_d$ starting at $O$ and converging to $\xi$. Note that if the limit exists, it does not depend on the choice of $r$.

In this paper, a linear map $T : D_1 \to D_2$ between two Banach spaces $(D_1, \| \cdot \|_1)$ and $(D_2, \| \cdot \|_2)$ is said to be a Banach isomorphism if $T$ is a bijection and a linear operator such that for any $u \in D_1$, we have

$$C^{-1}\|u\|_1 \leq \|Tu\|_2 \leq C\|u\|_1$$

for some constant $C > 1$ which is independent of $u$.

**Theorem 4.6.** [BP, Theorem 0.1, Theorem 3.1, Theorem 3.4.]

(1) Let $d \in J(Z, \rho)$ and $f : V(\Gamma_d) \to \mathbb{R}$ be a function with $df \in \ell_p(E(\Gamma_d))$. Then, for $\mathcal{H}_d$-a.e. $\xi \in Z$, the limit $f_\infty(\xi)$ exists and $f_\infty \in L^p(Z, \mathcal{H}_d)$.

(2) For $d \in J(Z, \rho)$, the linear maps

$$I^d : \{f : V(\Gamma_d) \to \mathbb{R}; df \in \ell_p(E(\Gamma_d))\} \to L^p(Z, \mathcal{H}_d)$$

$$f \mapsto f_\infty$$

$$\bar{I}^d : \{f : V(\Gamma_d) \to \mathbb{R}; df \in \ell_p(E(\Gamma_d))\}/\mathbb{R} \to L^p(Z, \mathcal{H}_d)/\mathbb{R}$$

$$[f] \mapsto f_\infty \mod \mathbb{R}$$

are continuous. Moreover, $\text{Ker}(I^d) = \ell_p(V(\Gamma_d))$ and $\text{Ker}(\bar{I}^d) = \ell_p(V(\Gamma_d)) + \mathbb{R}$.

(3) When $d \in J_{AR}(Z, \rho)$, the map $\bar{I}^d$ induces a Banach isomorphism $\bar{I}^d$ between $\ell_pH^1(J(Z, \rho))$ and $B_p(Z, d)/ \sim$. 

16
5 Besov spaces associated to random walks on hyperbolic groups

In the previous section, we explained how to construct Besov spaces associated to metrics in the Ahlfors-regular conformal gauge of a given compact metric space. In this section, we will choose, as a compact metric space, the Gromov boundary $\partial \Gamma$ of a non-elementary word hyperbolic group $\Gamma$ equipped with a visual metric, and we will introduce Besov spaces on $\partial \Gamma$ associated to random walks driven by probability measures with a finite second moment. Those Besov spaces will be introduced as sets of boundary values of harmonic functions on $\Gamma$. Moreover, we will show that sets of continuous functions in those Besov spaces, which are associated either to metrics in the Ahlfors-regular conformal gauge or to random walks driven by probability measures with finite second moment, are canonical. Namely, sets of continuous functions in Besov spaces do not depend on the choice of metrics in the Ahlfors-regular conformal gauge nor on probability measures with finite second moment. In addition, we will prove that for any choice of two Besov spaces among them, there exists a Banach isomorphism which coincides with the identity on the set of continuous functions.

Notation

Let $\Gamma$ be a non-elementary word hyperbolic group. We denote the neutral element by $id$. We will denote by $| \cdot - \cdot |_{\Gamma}$ a left-invariant word metric with respect to a fixed finite symmetric generating set, and let $\rho_{\Gamma}$ be a visual metric on the Gromov boundary $\partial \Gamma$ constructed from $| \cdot - \cdot |_{\Gamma}$. In what follows, we will write $J_{AR}(\partial \Gamma) := J_{AR}(\partial \Gamma, \rho_{\Gamma})$. For $d \in J_{AR}(\partial \Gamma)$, let $H_d$ be the Hausdorff measure and $q$ be its Hausdorff dimension. Recall that $H_d$ is $q$-Ahlfors regular, namely, there exists a constant $C > 1$ such that

$$C^{-1}r^q \leq H_d(B_d(\xi, r)) \leq Cr^q$$

for any $\xi \in \partial \Gamma$ and any $0 < r < \text{diam}(\partial \Gamma, \rho_{\Gamma})$. Let $B_2(d)$ be the Besov space on the Gromov boundary $\partial \Gamma$ constructed by using $d$ as in [BP].

Let $\mu$ be a probability measure on $\Gamma$. Recall that we always assume that $\mu$ is symmetric (i.e., $\mu(x) = \mu(x^{-1})$ for $x \in \Gamma$), and admissible, which means that the support of $\mu$ generates $\Gamma$. For $k \in \mathbb{N}$, let $M_k$ be the set of symmetric admissible probability measures on $\Gamma$ with finite $k$-th moment (i.e., $\sum_{x \in \Gamma} |id - x|^k \mu(x) < \infty$).

We consider the random walk with driving measure $\mu$: let $R_n$ be the position of the walk at time $n$. We denote with $(\mathcal{G}_n)$ the filtration generated by the sequence $(R_n)$. Let $\mathbb{P}_x^\mu$ be the law of the random walk $(R_n)$ starting at $x \in \Gamma$. For $x, y \in \Gamma$, we have $\mathbb{P}_x^\mu(R_n = y) = \mu^n(x^{-1}y)$, where $\mu^n$ is the $n$-th fold convolution power of $\mu$. Let $\mathbb{E}_x^\mu$ be the corresponding expectation. Define $\mathbb{P}^\mu = \mathbb{P}^\mu_{id}$ and $\mathbb{E}^\mu = \mathbb{E}^\mu_{id}$. We use the notation $G^\mu(x) := \sum_{k \geq 0} \mu^k(x)$ for the Green function associated to $\mu$.

It is shown in [Ka] that when $\mu \in M_1$ and $x \in \Gamma$, the random walk $(R_n)$ $\mathbb{P}_x^\mu$-almost surely converges to a random point $R_\infty \in \partial \Gamma$ in the topology of $\Gamma \cup \partial \Gamma$. Denote by $\nu_x$ the distribution of $R_\infty$ under the law $\mathbb{P}_x^\mu$. Then $\nu_x$ is a probability measure on $\partial \Gamma$; it is called the harmonic
measure of \((R_n)\) starting at \(x\). Define \(\nu := \nu_{id}\). Note that
\[
\int_{\partial\Gamma} u(\xi) \, d\nu_x(\xi) = E^\mu_x[u(R_\infty)] = E^\mu_u(x \cdot R_\infty)
\]
for any positive measurable function \(u\) on \(\partial\Gamma\). It is known that, for all \(x, y \in \Gamma\), the harmonic measures \(\nu_x\) and \(\nu_y\) are equivalent with a bounded density. The density of \(\nu_x\) with respect to \(\nu\) is given by the Martin kernel
\[
\mathcal{K}_\mu(x, \xi) = \frac{d\nu_x}{d\nu}(\xi)
\]
see Definition 5.4 and Lemma 5.5.

Consider the bilinear form
\[
\mathcal{E}^\mu(f, g) = \frac{1}{2} \sum_{x, y \in \Gamma} \mu(x^{-1}y)(f(x) - f(y))(g(x) - g(y)),
\]
and its full domain
\[
\mathcal{F}^\mu = \{ f : \Gamma \to \mathbb{R} ; \mathcal{E}^\mu(f, f) < \infty \}.
\]
Define
\[
\ell_2(\Gamma) := \left\{ f : \Gamma \to \mathbb{R} ; \|f\|_{\ell_2(\Gamma)}^2 := \sum_{x \in \Gamma} f(x)^2 < \infty \right\},
\]
then for any \(f \in \mathcal{F}^\mu\), we have that
\[
\mathcal{E}^\mu(f, f) = \frac{1}{2} \sum_{x, y \in \Gamma} \mu(x^{-1}y)(f(x) - f(y))^2 \leq \sum_{x, y \in \Gamma} \mu(x^{-1}y)(f(x)^2 + f(y)^2) = 2 \|f\|_{\ell_2(\Gamma)}^2.
\]
(15)
Therefore \(\ell_2(\Gamma) \subset \mathcal{F}^\mu\).

By Example 3.8 and Proposition 5.1 we know that \((\mathcal{E}^\mu, \ell_2(\Gamma))\) is a regular Dirichlet form on \(\ell_2(\Gamma)\) (note that \(m(x) = \sum_y \mu(x^{-1}y) = 1\) for any \(x \in \Gamma\), and the corresponding Hunt process is the constant speed random walk \((X_t)\) on \(\Gamma\), which is given by \((X_t) = (R_{N_t})\), where \((N_t)\) is an independent Poisson process with intensity 1. Notice that the trajectories of \((X_t)\) and \((R_{N_t})\) are the same. As a consequence, for \(\mu \in M_1\) and \(x \in \Gamma\), when \(t\) tends to \(\infty\), \(X_t\) also \(\mathbb{P}^\mu\)-almost surely converges to a random point in \(\partial\Gamma\) whose distribution is the harmonic measure \(\nu_x\).

We also define the discrete Laplacian \(\Delta_\mu : \{ f : \Gamma \to \mathbb{R} \} \to \{ f : \Gamma \to \mathbb{R} \}\) by
\[
\Delta_\mu f(x) := \sum_{y \in \Gamma} \mu(x^{-1}y)f(y) - f(x).
\]
We will say that \(f : \Gamma \to \mathbb{R}\) is \(\mu\)-harmonic on \(A \subset \Gamma\) if \(\Delta_\mu f(x) = 0\) for any \(x \in A\). We have We introduce the space of harmonic-Dirichlet functions:
\[
\mathcal{HD}^\mu(\mu) := \{ f \in \mathcal{F}^\mu ; \Delta_\mu f = 0 \text{ on } \Gamma \}.
\]
We have the following decomposition for \(\mathcal{F}^\mu\), see also [Soa, Theorem 3.69].
Proposition 5.1. We have that $\ell_2(\Gamma) \subset \mathcal{F}^\mu$, and for every $f \in \mathcal{F}^\mu$, there exists a unique pair of functions $(f_0, f_{\text{HD}}) \in \ell_2(\Gamma) \times \mathbb{HID}(\mu)$ such that $f = f_0 + f_{\text{HD}}$. For such a pair of functions $(f_0, f_{\text{HD}})$, we have that $\mathcal{E}^\mu(f, f) = \mathcal{E}^\mu(f_0, f_0) + \mathcal{E}^\mu(f_{\text{HD}}, f_{\text{HD}})$. In other words, the following orthogonal decomposition holds:

$$\mathcal{F}^\mu = \ell_2(\Gamma) \bigoplus \mathbb{HID}(\mu).$$

Moreover, we have that $\ell_2(\Gamma) = C_0(\Gamma)$, where $C_0(\Gamma) := \{ f' : \Gamma \to \mathbb{R} ; \supp(f') < \infty \}$.

Proof.

It is easy to see that the space $\mathcal{F}^\mu/\mathbb{R}$ equipped with the norm $\sqrt{\mathcal{E}^\mu(\cdot, \cdot)}$ is a Hilbert space. Let $\mathcal{F}_0^\mu$ be the closure of $C_0(\Gamma)$ with respect to the metric $\sqrt{\mathcal{E}^\mu(\cdot, \cdot)}$.

Note that for any $f \in C_0(\Gamma)$ and any $g \in \mathcal{F}^\mu$, we have that

$$\mathcal{E}^\mu(f, g) = -\sum_{x \in \Gamma} f(x)(\Delta_{\mu}g)(x).$$

Therefore the spaces $\mathcal{F}_0^\mu$ and $\mathbb{HID}(\mu)$ are orthogonal for the scalar product $\mathcal{E}^\mu$ and

$$\mathcal{F}^\mu = \mathcal{F}_0^\mu \bigoplus \mathbb{HID}(\mu).$$

We still have to prove that $\mathcal{F}_0^\mu = \ell_2(\Gamma)$. We already noticed that $\ell_2(\Gamma) \subset \mathcal{F}_0^\mu$. On the other hand, every non-elementary hyperbolic group satisfies a linear isoperimetric inequality: there exists a constant $C > 0$ such that

$$\|f\|^2_{\ell_2(\Gamma)} \leq C \mathcal{E}^\mu(f, f)$$

for any $f \in C_0(\Gamma)$. Thus $\mathcal{F}_0^\mu$ is also the closure of $C_0(\Gamma)$ in $\ell_2(\Gamma)$ and therefore coincides with $\ell_2(\Gamma)$.

We next study boundary values of functions in $\mathcal{F}^\mu$. A natural way to define a boundary value for $f \in \mathcal{F}^\mu$ is to take a limit of $f(R_n)$. Here we will use the discrete-time process $(R_n)$ for simplicity of notation, but the same results hold for the continuous-time process $(X_t)$. By Proposition 5.1, for any $f \in \mathcal{F}^\mu$, there exists a unique pair of functions $(f_0, f_{\text{HD}}) \in \ell_2(\Gamma) \times \mathbb{HID}(\mu)$ such that

$$f = f_0 + f_{\text{HD}}.$$ 

By the definition of $\ell_2(\Gamma)$, for any $\varepsilon > 0$ there exists a finite set $B \subset \Gamma$ such that

$$\sum_{x \in \Gamma \setminus B} f_0(x)^2 < \varepsilon.$$ 

Since $(R_n)$ is transient, this implies that

$$\lim_{n \to \infty} f_0(R_n)(= \lim_{t \to \infty} f_0(X_t)) = 0$$

19
\(\mathbb{P}_x\)–almost surely for any \(x \in \Gamma\). Thus, we only need to consider the limit of \(f_{\overline{\text{HD}}}(R_n)\). Since \(f_{\overline{\text{HD}}}\) is \(\mu\)-harmonic, we have that for any \(x \in \Gamma\), \((f_{\overline{\text{HD}}}(R_n))\) is a martingale under \(\mathbb{P}_x^\mu\). It is shown in Theorem 9.11 in [LP] that for any \(x \in \Gamma\) we have that
\[
\sup_{n \in \mathbb{N}} \mathbb{E}_x^\mu[f_{\overline{\text{HD}}}(R_n)] \leq f_{\overline{\text{HD}}}(x) + 2G^\mu(id)\mathcal{E}^\mu(f_{\overline{\text{HD}}}, f_{\overline{\text{HD}}}),
\]
where \(G^\mu(x) := \sum_{k \geq 0} \mu^{*k}(x)\) is the Green function associated to \(\mu\). Therefore, under \(\mathbb{P}_x^\mu\), \((f_{\overline{\text{HD}}}(R_n))\) is a martingale which is bounded in \(L^2\). By Doob’s theorem, it converges almost surely and in \(L^2\). Since \((\partial \Gamma, \nu)\) is the Poisson boundary of the random walk, see [Ka], there exists the unique function \(u \in L^2(\partial \Gamma, \nu)\) such that
\[
\lim_{n \to \infty} f_{\overline{\text{HD}}}(R_n) = u(R_\infty) \text{ a.s.,} \tag{19}
\]
and
\[
f_{\overline{\text{HD}}}(x) = \int_{\partial \Gamma} u(\xi)d\nu_\Gamma(\xi) =: Hu(x). \tag{20}
\]

We summarize the discussions from the previous page in the following corollary.

**Corollary 5.2.** For \(f \in \mathcal{F}^\mu\), let \(f = f_0 + f_{\overline{\text{HD}}} \quad \big(f_0 \in \ell_2(\Gamma), f_{\overline{\text{HD}}} \in \overline{\text{HD}}(\mu)\big)\) be its Royden decomposition as in (16). Then the limit \(\lim_{n \to \infty} f(R_n) = \lim_{n \to \infty} f_{\overline{\text{HD}}}(R_n)\) almost surely exists and the limiting function \(u\) defined in (19) belongs to \(L^2(\partial \Gamma, \nu)\).

Let \(u \in L^2(\partial \Gamma, \nu)\) and define \(Hu\) as in (20). Note that this definition makes sense since \(\nu_\Gamma\) is absolutely continuous with respect to \(\nu\) with a bounded density, see Lemma 5.3 and therefore \(u \in L^2(\partial \Gamma, \nu_\Gamma)\) for all \(x \in \Gamma\).

**Lemma 5.3.** For \(u \in L^2(\partial \Gamma, \nu)\), the sequence \((Hu(R_n))\) forms a martingale that is bounded in \(L^2\) and converges almost surely and in \(L^2\) towards \(u(R_\infty)\).

**Proof.**
Observe that
\[
Hu(R_n) = \mathbb{E}_{R_n}^\mu[u(R_\infty)] = \mathbb{E}^\mu[u(R_\infty)|G_n]. \tag{21}
\]
The first equality is the definition of \(Hu\). The second equality is the Markov property applied to the random variable \(u(R_\infty)\).

Since \(u \in L^2(\partial \Gamma, \nu)\), the martingale \(\mathbb{E}^\mu[u(R_\infty)|G_n]\) is bounded in \(L^2\). By Doob’s theorem, it converges almost surely and in \(L^2\) towards \(\mathbb{E}^\mu[u(R_\infty)|G_\infty] = u(R_\infty)\). \(\square\)

The following result gives a motivation to introduce a class of Besov spaces associated to \(\mu \in M_1\). We will see below that the Besov space associated to \(\mu\) gives an alternative description of the collection of \(u\)’s in \(L^2(\partial \Gamma, \nu)\) such that \(Hu \in \overline{\text{HD}}(\mu)\).

**Definition 5.4.** For \(x, y \in \Gamma\), define the Martin kernel \(K^\mu(x, \cdot)\) by
\[
K^\mu(x, y) := \frac{G^\mu(x^{-1}y)}{G^\mu(y)}.
\]
It is shown in [Dyn] that $K^\mu$ can be extended to $\Gamma \times M^\mu$, where $M^\mu$ is the Martin boundary of $(\Gamma, \mu)$.

For $x, y \in \Gamma$, define the Na"im kernel $\Theta^\mu(\cdot, \cdot)$ by
\[
\Theta^\mu(x, y) := \frac{G^\mu(x^{-1}y)}{G^\mu(x)G^\mu(y)}.
\]

It is shown in [Sil] (See also [Na].) that $\Theta^\mu$ can be extended to $M^\mu \times M^\mu \setminus \{(\omega, \omega') \in M^\mu \times M^\mu : \omega = \omega'\}$.

It is also shown in [Dyn] that the restriction of $K^\mu(x, \cdot)$ to $M^\mu$ is a version of the Radon-Nikodym derivative of $\nu_x$ with respect to $\nu$.

Lemma 5.5. The Martin kernel $K^\mu(x, y) (x, y \in \Gamma)$ has the following lower bound:
\[
\frac{G^\mu(id)}{G^\mu(x)} \leq K^\mu(x, y) \geq \frac{G^\mu(x)}{G^\mu(id)} \quad \text{for any } x, y \in \Gamma.
\]

The Na"im kernel $\Theta^\mu(x, y) (x, y \in \Gamma)$ has the following lower bound:
\[
\Theta^\mu(x, y) \geq 1/G^\mu(id) \quad \text{for any } x, y \in \Gamma.
\]

We will give the proof for the sake of completeness. We will need this lemma to ensure that the Besov space associated to $\mu$ is included in $L^2(\partial \Gamma, \nu)$. See Definition 5.8 below.

Proof.
By the Markov property, we have
\[
G^\mu(z) = G^\mu(id)P^\mu(Z_k = z \text{ for some } k \geq 1).
\]
Therefore
\[
K^\mu(x, y) = \frac{P^\mu(Z_k = y \text{ for some } k \geq 1)}{P^\mu_{id}(Z_k = y \text{ for some } k \geq 1)}.
\]
But
\[
P^\mu_{id}(Z_k = y \text{ for some } k \geq 1)
\geq P^\mu_{id}(Z_k = x \text{ for some } k \geq 1)P^\mu_x(Z_k = y \text{ for some } k \geq 1).
\]
Therefore
\[
K^\mu(x, y) \leq \frac{1}{P^\mu_{id}(Z_k = x \text{ for some } k \geq 1)} = \frac{G^\mu(id)}{G^\mu(x)}.
\]
The lower bound on $K^\mu(x, y)$ is proved the same way. The bound on $\Theta^\mu(x, y)$ follows at once. □

Remark 5.6. The Na"im kernel is pointwisely defined on the Martin boundary. In Definition 5.8 and thereafter, we use a version of the Na"im kernel that is $\nu \times \nu$-almost surely defined on the Gromov boundary. It is shown in [Ka] that the Gromov boundary equipped with the harmonic measure is isomorphic to the Poisson boundary of the walk, therefore it is measurably isomorphic to the Martin boundary equipped with the harmonic measure. See [Woe, Chapter VI] for details.
By (20) and [Sil, Theorem 3.5], we have the following result.

Proposition 5.7. [Sil, Theorem 3.5] Suppose that we have a function \( u \in L^2(\partial \Gamma, \nu) \) Then we have

\[
\mathcal{E}^\mu(Hu, Hu) = \int_{\partial \Gamma \times \partial \Gamma} (u(\xi) - u(\eta))^2 \Theta^\mu(\xi, \eta)d\nu(\xi)d\nu(\eta), \tag{22}
\]

where \( \Theta^\mu \) is the Naïm kernel and with the understanding that the right-hand side of (22) is finite if and only if \( Hu \in \mathcal{H}^D(\mu) \).

We now introduce the following bilinear form associated to \( \mu \).

Definition 5.8. Define

\[
\mathcal{E}^{\partial \Gamma, \mu}(u, u) = \int_{\partial \Gamma \times \partial \Gamma} (u(\xi) - u(\eta))^2 \Theta^\mu(\xi, \eta)d\nu(\xi)d\nu(\eta),
\]

with domain \( B_2(\mu) := \{ u : \partial \Gamma \to \mathbb{R} ; \mathcal{E}^{\partial \Gamma, \mu}(u, u) < \infty \} \). We will call \( (\mathcal{E}^{\partial \Gamma, \mu}, B_2(\mu)) \) the Besov space associated to \( \mu \in M_2 \).

Note that \( B_2(\mu) \subset L^2(\partial \Gamma, \nu) \) by Lemma 5.5. By Proposition 5.7, we have

\[
B_2(\mu) = \{ u \in L^2(\partial \Gamma, \nu) ; Hu \in \mathcal{H}^D(\mu) \}. \tag{23}
\]

Thus if \( u \in B_2(\mu) \) then \( Hu \in \mathcal{F}^\mu \) and Lemma 5.3 implies that \( \lim_{n \to \infty} f(R_n) = u(R_\infty) \). Reciprocally, if \( f \in \mathcal{F}^\mu \) then \( f_{\mathcal{H}^D} = Hu \) for some \( u \) in \( B_2(\mu) \) and Corollary 5.2 and Lemma 5.3 imply that \( \lim_{n \to \infty} f(R_n) = u(R_\infty) \). We summarize these findings in the next corollary.

Corollary 5.9. For \( f \in \mathcal{F}^\mu \), let \( \tilde{U}(\mu)f \) be the function on \( \partial \Gamma \) defined by

\[
\lim_{n \to \infty} f(R_n) = \tilde{U}(\mu)f(R_\infty) \text{ a.s.}
\]

Then \( \tilde{U}(\mu) \) defines a surjective linear map from \( \mathcal{F}^\mu \) to \( B_2(\mu) \) and \( \text{Ker}(\tilde{U}(\mu)) = \ell_2(\Gamma) \).

We will study the relations between \( B_2(\mu) \) and \( B_2(d) \). Hereafter, we assume \( \mu \in M_2 \). The next proposition plays in our probabilistic construction a similar role to quasi-isometries in the geometric context. In what follows, if \( f \) and \( g \) are two functions defined on a set \( A \), \( f \asymp g \) means that there exists a constant \( C > 1 \) such that \( C^{-1}g(a) \leq f(a) \leq Cg(a) \) for any \( a \in A \).

Proposition 5.10. [PSC, Lemma 2.1] For any \( \mu, \mu' \in M_2 \), we have that

\[
\mathcal{F}^\mu = \mathcal{F}^{\mu'}, \text{ and } \mathcal{E}^\mu(f, f) \asymp \mathcal{E}^{\mu'}(f, f) \text{ for any } f \in \mathcal{F}^\mu = \mathcal{F}^{\mu'}.
\]

Before giving the main result of this section, we will prove the following results.

Lemma 5.11. For any \( d \in J_{AR}(\partial \Gamma) \) and any \( \mu \in M_2 \), we have that \( C(\partial \Gamma) \cap B_2(\mu) = C(\partial \Gamma) \cap B_2(d) \). We will denote the common set by \( \mathcal{C} \).
**Proof.** Let $\mathcal{E}^{\text{SRW}}$ be the bilinear form associated to the simple random walk on $\Gamma$ with respect to a fixed finite symmetric generating set. Choose $d \in J_{AR}(\partial \Gamma)$ and $\mu \in M_2$ arbitrarily. Take $u \in C(\partial \Gamma) \cap B_2(d)$. Define $g : V(\Gamma_d) \to \mathbb{R}$ by

$$g(x) := \frac{1}{\mathcal{H}_d(B(x))} \int_{B(x)} u(\xi) d\mathcal{H}_d(\xi).$$

(24)

It is shown in the proof of Theorem 3.4 in [BP] that

$$\mathcal{E}^{\partial \Gamma,d}(u, u) \propto \|dg\|_{\ell^2(\mathbb{E}(\Gamma_d))}^2,$$

and $I^d(g) = u$, $\mathcal{H}_d$-a.e.

Let $F_d : \Gamma \to V(\Gamma_d)$ be a quasi-isometry which continuously extends to the identity on $\partial \Gamma$, as was shown to exist in Theorem 4.4. We have that

$$\mathcal{E}^{\partial \Gamma,d}(g \circ F_d, g \circ F_d) \propto \|dg\|_{\ell^2(\mathbb{E}(\Gamma_d))},$$

by the stability of Dirichlet forms under quasi-isometries ([Woe, Theorem 3.10]). Note that $\Gamma_d$ satisfies the assumption in [Woe, Theorem 3.10] since $\Gamma$ is of bounded degree. See Theorem 4.4. By Proposition 5.10 for $\mu \in M_2$ we have that

$$\mathcal{E}^{\text{SRW}}(g \circ F_d, g \circ F_d) \propto \mathcal{E}^{\mu}(g \circ F_d, g \circ F_d).$$

Thus $g \circ F_d \in \mathcal{F}^\mu$. By Proposition 5.7, $g \circ F_d$ has a limit along a path of the random walk driven by $\mu$ with probability 1, and the limiting function $v : \partial \Gamma \to \mathbb{R}$ belongs to $B_2(\mu)$. Since it is shown in [Ka] that $(R_n)$ converges to a random element $R_\infty \in \partial \Gamma$ in the topology of the compactified space $\Gamma \cup \partial \Gamma$, the sequence $(F_d(R_n))$ also converges to $R_\infty$. On the other hand, by the continuity of $u$ and the definition of $g$, for any sequence $(h_n) \subset \Gamma_d$ converging to a point $\eta \in \partial \Gamma$ we get that $\lim_{n \to \infty} g(h_n) = u(\eta)$. This observation together with the above argument implies that $v(R_\infty) = \lim_{n \to \infty} g \circ F_d(R_n) = u(R_\infty)$ $\mathbb{P}^\mu$-a.s. Thus we get that $v = u \nu$-a.s., and this implies that $u \in B_2(\mu)$.

For $v' \in C(\partial \Gamma) \cap B_2(\mu)$, define its harmonic extension $Hv' : \Gamma \to \mathbb{R}$ with respect to $\mu$ as in (20). Then by Proposition 5.7, we have that $Hv' \in \mathbb{H}^d(\mu)$ and $\mathcal{E}^{\mu}(Hv', Hv') = \mathcal{E}^{\partial \Gamma,d}(v', v')$. Let $\tilde{F}_d : V(\Gamma_d) \to \Gamma$ be a quasi-isometry such that $F_d \circ \tilde{F}_d$ and $\tilde{F}_d \circ F_d$ are within bounded distance from $id_{\Gamma_d}$ and $id_\Gamma$ respectively. Since $F_d$ continuously extends to the identity on $\partial \Gamma$, $\tilde{F}_d$ does so as well. Then we have that

$$\mathcal{E}^{\mu}(Hv', Hv') \propto \mathcal{E}^{\text{SRW}}(Hv', Hv') \propto \|d(Hv' \circ \tilde{F}_d)\|_{\ell^2(\mathbb{E}(\Gamma_d))}^2 < +\infty.$$

By Theorem 4.6, the function $Hv' \circ \tilde{F}_d$ has a limit along $\mathcal{H}_d$-almost every geodesics and the limiting function $u' : \partial \Gamma \to \mathbb{R}$ belongs to $B_2(d)$. On the other hand, by [Ka, Lemma 2.2], for any sequence $(g_n) \subset \Gamma$ converging to a point $\eta \in \partial \Gamma$, we have that $\lim_{n \to \infty} H\xi'(g_n) = v'(\eta)$. Now $u'$ is the limit of $Hv' \circ \tilde{F}_d$ along $\mathcal{H}_d$-almost every geodesics, and $(\tilde{F}_d(g_n))$ converges to $\eta$ whenever $(g_n)$ converges to $\eta$. Hence we get that $u' = v' \mathcal{H}_d$-a.e., and this implies the conclusion. \( \square \)

Now we wish to relate the two Besov spaces $B_2(d)$ and $B_2(d')$ for $d, d' \in J_{AR}(\partial \Gamma)$. To do so, we use a bijective linear map between them given in [BP], and show that it coincides with the identity map on $C$. We now recall the construction of a Banach isomorphism between
\((B_2(d) \sim, \mathcal{E}^{\partial^r,d})\) and \((B_2(d') \sim, \mathcal{E}^{\partial^r,d'})\) in \([BP]\). Let \(u \in B_2(d)\), and define \(g : V(\Gamma_d) \to \mathbb{R}\) as in \([21]\). Now we define \(\bar{T}(d \to d') : B_2(d) \to B_2(d')\) as follows:
\[
\bar{T}(d \to d')u := I^{d'}(g \circ c_0) \in B_2(d').
\]
See \([12]\) for the definition of \(c_0\).

**Lemma 5.12.** For any \(d, d' \in J_{AR}(\partial \Gamma)\), the linear map \(\bar{T}(d \to d') : B_2(d) \to B_2(d')\) introduced above satisfies \(\bar{T}(d \to d')|_C = \text{Id}_C\). Moreover, it induces a Banach isomorphism \(T(d \to d')\) between \((B_2(d) \sim, \mathcal{E}^{\partial^r,d})\) and \((B_2(d') \sim, \mathcal{E}^{\partial^r,d'})\).

**Proof.** It is shown in \([BP]\) that the linear map \(\bar{T}(d \to d')\) induces a Banach isomorphism between \((B_2(d) \sim, \mathcal{E}^{\partial^r,d})\) and \((B_2(d') \sim, \mathcal{E}^{\partial^r,d'})\). Therefore, it suffices to prove that \(\bar{T}(d \to d')|_C = \text{Id}_C\).

Under the identification \([14]\), we have that \(N^*(g) = g \circ c_0\), where \(c_0 : V(\Gamma_{d'}) \to V(\Gamma_d)\) is a quasi-isometry which continuously extends to the identity on \(\partial \Gamma\). See the third statement in Theorem \([4.3]\). When \(u \in C\), it is obvious that for any sequence \((h_n) \subset \Gamma_d\) converging to \(\eta \in \partial \Gamma\), we have that \(\lim_{n \to \infty} g(h_n) = u(\eta)\). This implies that \(\bar{T}(d \to d')|_C = \text{Id}_C\). It is shown in \([BP]\) that the linear map \(T(d \to d')\) induces a Banach isomorphism between \((B_2(d) \sim, \mathcal{E}^{\partial^r,d})\) and \((B_2(d') \sim, \mathcal{E}^{\partial^r,d'})\). \(\square\)

We next prove that for any \(\mu, \mu' \in M_2\), there exists an isomorphism between the two Besov spaces \((\mathcal{E}^{\partial^r,\mu}, B_2(\mu))\) and \((\mathcal{E}^{\partial^r,\mu'}, B_2(\mu'))\) which coincides with the identity map on \(C\). Before proving this claim, we need some preparation.

**Lemma 5.13.** (1) The functional space \((\mathcal{E}^\mu, \mathcal{H}D(\mu) / \sim)\) is a Hilbert space.

(2) The linear map \(\bar{U}(\mu)|_{\mathcal{H}D(\mu)} : \mathcal{H}D(\mu) \to B_2(\mu)\) induces a Banach isomorphism \(U(\mu)\) between the two Hilbert spaces \((\mathcal{H}D(\mu) / \sim, \mathcal{E}^\mu)\) and \((B_2(\mu) / \sim, \mathcal{E}^{\partial^r,\mu})\). See Corollary \([5.9]\) for the definition of \(U(\mu) : \mathcal{F}^\mu \to B_2(\mu)\).

**Proof.** We first prove the first claim. Recall that by Proposition \([5.1]\) \(\ell_2(\Gamma)\) is a closed subspace of the Hilbert space \((\mathcal{F}^\mu / \sim, \mathcal{E}^\mu)\). This fact together with the decomposition \([16]\) implies the result.

We next prove the second claim. By Corollary \([5.9]\) and the decomposition \([16]\), \(\bar{U}(\mu)|_{\mathcal{H}D(\mu)}\) is bijective. Moreover, by Proposition \([5.7]\) for \(f \in \mathcal{H}D(\mu)\) we have that
\[
\mathcal{E}^\mu(f, f) = \mathcal{E}^{\partial^r,\mu}(\bar{U}(\mu)f, \bar{U}(\mu)f).
\]
Thus, \(\bar{U}(\mu)|_{\mathcal{H}D(\mu)}\) induces a Banach isomorphism between \((B_2(\mu) / \sim, \mathcal{E}^{\partial^r,\mu})\) and \((\mathcal{H}D(\mu) / \sim, \mathcal{E}^\mu)\). \(\square\)

We now construct a bijective linear map between \(B_2(\mu)\) and \(B_2(\mu')\). By Lemma \([5.13]\) we already have two bijective linear maps \(\bar{U}(\mu)|_{\mathcal{H}D(\mu)} : \mathcal{H}D(\mu) \to B_2(\mu)\) and \(\bar{U}(\mu')|_{\mathcal{H}D(\mu')} : \mathcal{H}D(\mu') \to B_2(\mu')\). Therefore, we wish to relate \(\mathcal{H}D(\mu)\) and \(\mathcal{H}D(\mu')\). We define a linear map \(HD(\mu \to \mu') : \mathcal{H}D(\mu) \to \mathcal{H}D(\mu')\) as follows: for \(f \in \mathcal{H}D(\mu)\), let \(f = f_1 + f_2\) be its Royden decomposition with respect to \(\mu'\), where \(f_1 \in \ell_2(\Gamma)\) and \(f_2 \in \mathcal{H}D(\mu')\). Now we define
\[
HD(\mu \to \mu')f := f_2.
\]
and \( \bar{T}(\mu \to \mu') : B_2(\mu) \to B_2(\mu') \) by

\[
\bar{T}(\mu \to \mu') := U(\mu') \circ HD(\mu \to \mu') \circ (\bar{U}(\mu)|_{\text{HD}(\mu)})^{-1}
\]

(25)

**Proposition 5.14.** For any \( \mu, \mu' \in M_2 \), the linear map \( \bar{T}(\mu \to \mu') : B_2(\mu) \to B_2(\mu') \) constructed above satisfies \( \bar{T}(\mu \to \mu')|_{\mathcal{C}} = \text{Id}_\mathcal{C} \). Moreover, it induces a Banach isomorphism \( T(\mu \to \mu') : (B_2(\mu)/ \sim, \mathcal{E}^{\partial \Gamma, \mu}) \to (B_2(\mu')/ \sim, \mathcal{E}^{\partial \Gamma, \mu'}) \).

**Proof.** We first show that \( HD(\mu \to \mu') \) induces an isomorphism between \( (\text{HD}(\mu)/ \sim, \mathcal{E}^{\mu}) \to (\text{HD}(\mu')/ \sim, \mathcal{E}^{\mu'}) \), which implies the second claim. Recall that by Proposition 5.10, we have \( \mathcal{F}^\mu = \mathcal{F}^{\mu'} \) and \( \mathcal{E}^{\mu}(f, f) = \mathcal{E}^{\mu'}(f, f) \) for \( f \in \mathcal{F}^\mu = \mathcal{F}^{\mu'} \). When \( HD(\mu \to \mu') f = 0 \), we have \( f \in \ell_2(\Gamma) \cap \text{HD}(\mu) \), hence \( f = 0 \). Thus \( HD(\mu \to \mu') \) is injective. On the other hand, take \( g \in \text{HD}(\mu') \) arbitrarily. Let \( g = g_1 + g_2 \) be its Royden decomposition with respect to \( \mu \), where \( g_1 \in \ell_2(\Gamma) \) and \( g_2 \in \text{HD}(\mu) \). Then \( g_2 = -g_1 + g \), hence we have \( HD(\mu \to \mu')g_2 = g \). Therefore \( HD(\mu \to \mu') \) is surjective. Moreover, for \( f \in \text{HD}(\mu) \) we have

\[
\mathcal{E}^{\mu}(f, f) = \min_{h \in \ell_2(\Gamma)} \mathcal{E}^{\mu}(f + h, f + h) \leq \min_{h \in \ell_2(\Gamma)} \mathcal{E}^{\mu'}(f + h, f + h) = \mathcal{E}^{\mu'}(HD(\mu \to \mu')f, HD(\mu \to \mu')f).
\]

Thus, \( HD(\mu \to \mu') \) induces an isomorphism between \( (\text{HD}(\mu)/ \sim, \mathcal{E}^{\mu}) \to (\text{HD}(\mu')/ \sim, \mathcal{E}^{\mu'}) \).

We next show that \( \bar{T}(\mu \to \mu')|_{\mathcal{C}} = \text{Id}_\mathcal{C} \). Let \( u \in \mathcal{C} \). By Lemma 2.2 in [Ka], we have that for any sequence \( (g_n)_n \subset \Gamma \) converging to \( \eta \in \partial \Gamma \), \( \lim_{n \to \infty} Hu(g_n) = u(\eta) \). This implies that \( u = \bar{U}(\mu)(Hu) \) and \( u = \bar{U}(\mu')(Hu) \). Let \( Hu = h_1 + h_2 \) be the Royden decomposition with respect to \( \mu' \), where \( h_1 \in \ell_2(\Gamma) \) and \( h_2 \in \text{HD}(\mu') \). Then by Proposition 5.7, we have that

\[
u = \bar{U}(\mu')(Hu) = \bar{U}(\mu')h_2 = \bar{U}(\mu') \circ HD(\mu \to \mu')(Hu)
\]

\[
= \bar{U}(\mu') \circ HD(\mu \to \mu') \circ (\bar{U}(\mu)|_{\text{HD}(\mu)})^{-1} u.
\]

Therefore, we get the conclusion. \( \square \)

We will give the main results of this section below.

**Theorem 5.15.** For any \( d \in J_{AR}(\partial \Gamma) \) and \( \mu \in M_2 \), there exist linear maps \( \bar{T}(d \to \mu) : B_2(d) \to B_2(\mu) \) and \( \bar{T}(\mu \to d) : B_2(\mu) \to B_2(d) \) with \( \bar{T}(\mu \to d)|_{\mathcal{C}} = \bar{T}(d \to \mu)|_{\mathcal{C}} = \text{Id}_\mathcal{C} \) which induce Banach isomorphisms \( \bar{T}(d \to \mu) : (B_2(d)/ \sim, \mathcal{E}^{\partial \Gamma, d}) \to (B_2(\mu)/ \sim, \mathcal{E}^{\partial \Gamma, \mu}) \) and \( \bar{T}(\mu \to d) : (B_2(\mu)/ \sim, \mathcal{E}^{\partial \Gamma, \mu}) \to (B_2(d)/ \sim, \mathcal{E}^{\partial \Gamma, d}) \).

**Proof.** Take a probability measure \( \mu' \) on \( \Gamma \) with a finite support. It is shown in [BHM] that there exists a visual metric on \( \partial \Gamma \) called the Green visual metric and denoted with \( \rho(G^{\mu'}) \) which belongs to the Ahlfors-regular conformal gauge \( J_{AR}(\partial \Gamma) \) and is such that \( \mathcal{E}^{\partial \Gamma, \mu'} = (\mathcal{E}^{\partial \Gamma, \mu}(G^{\mu'}), B_2(\rho(G^{\mu'}))) \). See Corollary 1.2 and Section 3.2 in [BHM]. Now we choose \( d \in J_{AR}(\partial \Gamma) \) and \( \mu \in M_2 \) arbitrarily. We define \( \bar{T}(d \to \mu) : B_2(d) \to B_2(\mu) \) and \( \bar{T}(\mu \to d) : B_2(\mu) \to B_2(d) \) by

\[
\bar{T}(d \to \mu) := \bar{T}(\mu \to \mu') \circ \bar{T}(d \to \rho(G^{\mu'})),
\]

\[
\bar{T}(\mu \to d) := \bar{T}(\rho(G^{\mu'}) \to d) \circ \bar{T}(\mu \to \mu'),
\]

respectively. By Lemma 5.12 and Proposition 5.14, it is obvious that the above two linear maps coincide with the identity on \( \mathcal{C} \) and induce Banach isomorphisms. \( \square \)
6 Besov spaces associated to random walks and the theory of Dirichlet forms

In this section, we first prove that when the Ahlfors-regular conformal dimension of the Gromov boundary $\partial \Gamma$ is strictly less than 2, Besov spaces on $\partial \Gamma$ associated either to metrics $d \in J_{AR}(\partial \Gamma)$ and to random walks driven by $\mu \in M_2$ give rise to regular Dirichlet forms on the boundary. Secondly, we will study a potential theoretic property of Hausdorff measures of metrics in $J_{AR}(\partial \Gamma)$ and harmonic measures of random walks on $\Gamma$. Specifically, we will prove that those Hausdorff measures and harmonic measures are smooth in a potential theoretic sense with respect to any regular Dirichlet form on the boundary given by the Besov spaces.

6.1 Regularity of Besov spaces and smoothness of harmonic measures

From now on, we will assume that there exists a metric $d_0 \in J_{AR}(\partial \Gamma)$ such that $q_0 := \dim(\partial \Gamma, d_0) < 2$. In other words, we will assume that the Ahlfors-regular conformal dimension of $(\partial \Gamma, \rho_\Gamma)$ is strictly less than 2. Let $\text{Lip}_0$ be the set of Lipschitz functions with respect to $d_0$. By a straightforward computation, we can check that $\text{Lip}_0 \subset B_2(d_0)$, thus $\text{Lip}_0 \subset C$. In Proposition 5.14, we also checked that isomorphisms between Besov spaces can be arranged in such a way that functions in $\text{Lip}_0$ are invariant. We now claim the regularity of Dirichlet forms associated to $d \in J_{AR}(\partial \Gamma)$ and $\mu \in M_2$.

**Theorem 6.1.** Assume the Ahlfors-regular conformal dimension of $\partial \Gamma$ is strictly less than 2. Then for any $d \in J_{AR}(\partial \Gamma)$ and any $\mu \in M_2$, $(E_{\partial \Gamma}, d, B_2(d))$ and $(E_{\partial \Gamma}, \mu, B_2(\mu))$ are regular Dirichlet forms on $L^2(\partial \Gamma, \mathcal{H}_d)$ and on $L^2(\partial \Gamma, \nu)$ respectively.

Before giving the proof, we introduce the following estimates which are reminiscent of the Poincaré inequality.

**Lemma 6.2.** Let $d \in J_{AR}(\partial \Gamma)$. For $u \in B_2(d)$, let $g : V(\Gamma_d) \to \mathbb{R}$ be the function defined in (24). Then, there exists a constant $C > 0$ such that for any $u \in B_2(d)$ we have that

$$\|u - g(O)\|^2_{L^2(\mathcal{H}_d)} \leq C E_{\partial \Gamma, d}(u, u).$$

**Proof.** The claim immediately follows from Theorem 3.1 and Lemma 3.2 in [BP]. See the argument below Lemma 3.2 in [BP].

**Lemma 6.3.** Let $(R_k)_{k \geq 0}$ be a random walk on a non-amenable finitely generated group driven by a symmetric admissible probability measure $\mu$. Then, there exist constants $a > 1$ and $C > 0$ such that for any $g \in \mathcal{F}^\mu$ we have that

$$\sum_{k=0}^{\infty} \mathbb{E}[(g(R_{k+1}) - g(R_k))^2] a^k \leq C \mathcal{E}^\mu(g, g).$$

(26)
Choosing $c$, moreover, using the elementary inequality $c > x$, Theorem 10.3 in [Woe], there exists a constant $a > 0$ such that $\mu^k(id) \leq e^{-ck}$ for any $k \in \mathbb{N}$. Moreover, since $\mu^k(x) \leq \sqrt{\mu^{2k}(id)}$, we have that $\mu^k(x) \leq e^{-ck}$ for any $x \in \Gamma$ and any $k \in \mathbb{N}$. On the other hand, we have that
\[ \sum_k \mathbb{E}[(g(R_{k+1}) - g(R_k))^2]a^k = \sum_k a^k \sum_x \mu^k(x) \sum_y \mu(x^{-1}y)(g(y) - g(x))^2. \]

Combining this formula with the exponential decay of $\{\mu^k(x)\}_{k \geq 1}$, we get the desired estimate for sufficiently small $a > 1$ and some constant $C > 0$. □

**Proof of Theorem 6.1.** Since $\text{Lip}_0 \subset C$, the set $C$ separates points. Thus, $C$ is $\| \cdot \|_{\infty}$-dense in $C(\partial \Gamma)$ by the Stone-Weierstrass theorem. Therefore, we only need to show that $C$ is dense both in $(B_2(d), (C_1^{0\Gamma,d})^{1/2})$ and $(B_2(\mu), (C_1^{0\Gamma,\mu})^{1/2})$. We first prove the claim for $d \in J_{AR}(\partial \Gamma)$. By [Cos, Proposition 3.13], for any $u \in B_2(d)$ there exists a sequence $(w_n) \subset C$ such that $C^{0\Gamma,d}(u - w_n, u - w_n) \to 0$. Since $C^{0\Gamma,d}(v - c', v - c') = C^{0\Gamma,d}(v, v)$ for any $v \in B_2(d)$ and $c' \in \mathbb{R}$, it suffices to show that there exists a constant $c_n \in \mathbb{R}$ such that
\[ \lim_{n \to \infty} \|u - w_n - c_n\|_{L^2(\mathcal{H}_d)} = 0. \] (27)

By Lemma 6.2 there exists a constant $C > 0$ such that for any $v \in B_2(d)$
\[ \|v - c(v)\|^2_{L^2(\mathcal{H}_d)} \leq C C^{0\Gamma,d}(v, v), \text{ where } c(v) = \frac{1}{\mathcal{H}_d(\partial \Gamma)} \int_{\partial \Gamma} v(\xi)d\mathcal{H}_d(\xi). \]

Choosing $c_n := c(u - w_n)$, the above inequality together with (27) implies that $C$ is dense in $(B_2(d), (C_1^{0\Gamma,d})^{1/2})$ for $d \in J_{AR}(\partial \Gamma)$.

We next prove the claim for $\mu \in M_2$. Take $v \in B_2(\mu)$ and let $Hv$ be its harmonic extension with respect to $\mu$ as in (20). By applying Lemma 6.3 to $Hv$, we get that there exist constants $a > 1$ and $C > 0$ such that
\[ \sum_k \mathbb{E}^\mu[(Hv(R_{k+1}) - Hv(R_k))^2]a^k \leq C C^{0\Gamma,\mu}(Hv, Hv) = C C^{0\Gamma,\mu}(v, v). \]

Since the limit of $Hv$ along a path of $(R_n)$ coincides with $v$, by the Cauchy-Schwarz inequality we get we obtain that
\[ \|v - Hv(id)\|^2_{L^2(\nu)} = \mathbb{E}^\mu[(v(R_\infty) - Hv(id))^2] \]
\[ \leq \mathbb{E}^\mu \left[ \left( \sum_{k=0}^\infty (Hv(R_{k+1}) - Hv(R_k)) \right)^2 \right] \]
\[ \leq \mathbb{E}^\mu \left[ \left( \sum_{k=0}^\infty (Hv(R_{k+1}) - Hv(R_k))a^{k/2} \cdot a^{-k/2} \right)^2 \right] \]
\[ \leq \left( \sum_{k=0}^\infty a^{-k} \right) \cdot \sum_{k=0}^\infty \mathbb{E}^\mu [(Hv(R_{k+1}) - Hv(R_k))^2]a^k \]
\[ \leq C \sum_{k=0}^\infty \mathbb{E}^\mu [(Hv(R_{k+1}) - Hv(R_k))^2]a^k. \] (28)
Therefore, we get that
\[ \|v - Hv(id)\|_{L^2(\nu)}^2 \leq CE^{\partial \Gamma, \mu}(v, v). \] (29)

Thus, we can get the density of \( C \) in \( (B_2(\mu), (E^{\partial \Gamma, \mu})^{1/2}) \) by a similar argument as for the density in \( (B_2(d), (E^{\partial \Gamma, d})^{1/2}) \).

### 6.2 Smooth measures and measures of finite energy integral

In what follows, we will study some potential theoretic property of Hausdorff measures associated to the Ahlfors-regular conformal gauge \( J_{AR}(\partial \Gamma) \) and harmonic measures associated to \( M_2 \). In this subsection, we give several general definitions about measures and potential theory of Dirichlet forms. We will explain later their probabilistic interpretation, especially how those measures arise in the study of symmetric Markov processes and their time changes.

Let \( E \) be a locally compact Hausdorff space, and \( m \) be a positive Radon measure on \( E \). Assume that we have a regular Dirichlet form \( (\mathcal{E}, \mathcal{F}) \) on \( L^2(E, m) \).

**Definition 6.4.** For an open subset \( U \subseteq E \), we define
\[ L_U := \{ u \in \mathcal{F}; \ u \geq 1 \text{ m-a.e. on } U \}, \]
and
\[ \text{Cap}(U) := \begin{cases} \inf_{u \in L_U} \mathcal{E}_1(u, u), & \text{if } L_U \neq \emptyset \\ \infty, & \text{if } L_U = \emptyset. \end{cases} \]

For any subset \( A \subseteq E \), we define
\[ \text{Cap}(A) = \inf_{U: \text{open}, A \subseteq U} \text{Cap}(U). \]

The value of \( \text{Cap}(A) \) is called (1-)capacity of \( A \).

**Definition 6.5.** Let \( \kappa \) be a positive Borel measure on \( E \). We say that \( \kappa \) is smooth with respect to a regular Dirichlet form \( (\mathcal{E}, \mathcal{F}) \) when the following conditions are satisfied.

1. \( \kappa(B) = 0 \) whenever \( \text{Cap}(B) = 0 \) and
2. there exists an increasing sequence \( (C_n) \) of closed subsets of \( E \) such that
   \[ \kappa(C_n) < \infty \text{ for any } n \in \mathbb{N} \text{ and } \lim_{n \to \infty} \text{Cap}(K \setminus C_n) = 0 \text{ for any compact subset } K \subseteq E. \]

**Remark 6.6.** In what follows, we will choose the Gromov boundary \( \partial \Gamma \) as the state space \( E \), thus the second condition of smoothness is not important in this paper.
Definition 6.7. Let $\kappa$ be a positive Radon measure on $E$. We say that $\kappa$ is of finite energy integral with respect to a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ if there exists a constant $C > 0$ such that for any $v \in \mathcal{F} \cap C_0(E)$ we have that
\[
\int_E |v(x)| \, d\kappa(x) \leq C \sqrt{\mathcal{E}_1(v, v)}.
\]
Note that by Riesz’s representation theorem, a positive Radon measure $\kappa$ on $E$ is of finite energy integral with respect to a regular Dirichlet form $(E, F)$ if and only if for each $\alpha > 0$, there exists a unique function $U_{\alpha} \kappa \in F$ such that
\[
\int_E v(x) \, d\kappa(x) = \mathcal{E}_\alpha(U_{\alpha} \kappa, v)
\]
for any $v \in F \cap C_0(E)$.

Measures of finite energy integral are known to form a subclass of smooth measures.

Proposition 6.8. \cite[Section 2.2]{FOT} Any positive Radon measure $\kappa$ which is of finite energy integral with respect to a regular Dirichlet form $(E, F)$ is smooth with respect to $(E, F)$.

Now we introduce the following notions concerning capacity and smoothness of measures with respect to regular Dirichlet forms associated to $d \in J_{AR}(\partial \Gamma)$ and $\mu \in M_2$.

Definition 6.9. (1) Thanks to Theorem 6.1, for any $d \in J_{AR}(\partial \Gamma)$ and any $\mu \in M_2$, we can define $S(\partial \Gamma, d)$ ($S(\partial \Gamma, \mu)$, resp.) as the collection of all smooth measures with respect to the regular Dirichlet form $(\mathcal{E}^{\partial \Gamma, d}, B_2(d))$ on $L^2(\partial \Gamma, \mathcal{H}_d)$ ($(\mathcal{E}^{\partial \Gamma, \mu}, B_2(\mu))$ on $L^2(\partial \Gamma, \nu)$, resp.).

(2) Similarly, we define $S_0(\partial \Gamma, d)$ ($S_0(\partial \Gamma, \mu)$, resp.) as the collection of all measures of finite energy integral with respect to the regular Dirichlet form $(\mathcal{E}^{\partial \Gamma, d}, B_2(d))$ on $L^2(\partial \Gamma, \mathcal{H}_d)$ ($(\mathcal{E}^{\partial \Gamma, \mu}, B_2(\mu))$ on $L^2(\partial \Gamma, \nu)$, resp.).

(3) We also define $0(\partial \Gamma, d)$ ($0(\partial \Gamma, \mu)$, resp.) as the collection of all subsets of $\partial \Gamma$ with zero capacity with respect to the regular Dirichlet forms $(\mathcal{E}^{\partial \Gamma, d}, B_2(d))$ on $L^2(\partial \Gamma, \mathcal{H}_d)$ and $(\mathcal{E}^{\partial \Gamma, \mu}, B_2(\mu))$ on $L^2(\partial \Gamma, \nu)$.

6.3 Poincaré-type inequalities and sets of $0$ capacity

Below we consider two random walks with respective driving measures $\mu$ and $\mu'$ (always in $M_2$) and respective harmonic measures $\nu$ and $\nu'$. Now recall that by Theorem 5.15, for any $\mu \in M_2$, $d \in J_{AR}(\partial \Gamma)$ and any $u \in \mathcal{C}$, $\mathcal{E}^{\partial \Gamma, \mu}(u, u)$ and $\mathcal{E}^{\partial \Gamma, d}(u, u)$ are comparable up to multiplicative constants.

Proposition 6.10. Let $\mu, \mu' \in M_2$ and $d \in J_{AR}(\partial \Gamma)$. Denote by $\nu$ and $\nu'$ the harmonic measures of the random walks driven by $\mu$ and $\mu'$ respectively. Then, there exists a constant
\( C > 0 \) such that for any \( u \in \mathcal{C} \), we have that
\[
\left( \int_{\partial \Gamma} u \, d\nu - \int_{\partial \Gamma} u \, d\nu' \right)^2 \leq C \mathcal{E}^{\partial \Gamma, \mu}(u, u), \quad \text{and}
\]
\[
\left( \int_{\partial \Gamma} u \, d\nu - \mathcal{H}_d(\partial \Gamma)^{-1} \int_{\partial \Gamma} u \, d\mathcal{H}_d \right)^2 \leq C \mathcal{E}^{\partial \Gamma, \mu}(u, u).
\]

**Proof.** Let \( u \in \mathcal{C} \) and \( F_d : \Gamma \rightarrow V(\Gamma_d) \) be a quasi-isometry which continuously extends to the identity on \( \partial \Gamma \). Moreover, we will assume that \( F_d(id) = O \) without loss of generality. Then if we define a function \( g : V(\Gamma_d) \rightarrow \mathbb{R} \) as in (24), we have \( g \circ F_d \in \mathcal{F}^\mu \). Since the limit of \( g \circ F_d \) along a path of \( (R_n) \) coincides with \( u \), by the same argument as in (28), we get that
\[
\| u - g(O) \|_{L^2(\nu)}^2 = \| u - g \circ F_d(id) \|_{L^2(\nu)}^2 = \mathbb{E}^\mu[(u(R_\infty) - g \circ F_d(id))^2] \\
\leq C \sum_{k=0}^{\infty} \mathbb{E}^\mu[(g \circ F_d(R_{k+1}) - g \circ F_d(R_k))^2] a^k.
\]

By Lemma 6.3, we get that
\[
\| u - g(O) \|_{L^2(\nu)}^2 \leq C \mathcal{E}^\mu(g \circ F_d, g \circ F_d).
\]

Since
\[
\mathcal{E}^\mu(g \circ F_d, g \circ F_d) \asymp \| dg \|_{L^2(\mathcal{E}(\Gamma_d))} \asymp \mathcal{E}^{\partial \Gamma, d}(u, u),
\]
we get that
\[
\| u - g(O) \|_{L^2(\nu)}^2 \leq C \mathcal{E}^{\partial \Gamma, d}(u, u).
\] (30)

Recalling \( g(O) := \mathcal{H}_d(\partial \Gamma)^{-1} \int_{\partial \Gamma} u \, d\mathcal{H}_d \), we get that
\[
\left( \int_{\partial \Gamma} u \, d\nu - \mathcal{H}_d(\partial \Gamma)^{-1} \int_{\partial \Gamma} u \, d\mathcal{H}_d \right)^2 \leq C \mathcal{E}^{\partial \Gamma, d}(u, u).
\] (31)

We obtain the first estimate by applying the inequality (31) to two harmonic measures \( \nu, \nu' \) and combining them with the triangle inequality. \( \square \)

**Theorem 6.11.** For any \( d \in J_{AR}(\partial \Gamma) \) and \( \mu \in M_2 \), we have that \( \mathcal{S}_0(\partial \Gamma, d) = \mathcal{S}_0(\partial \Gamma, \mu) \). The similar statement holds for \( \mathcal{S}(\partial \Gamma, d) \) and \( \mathcal{S}(\partial \Gamma, \mu) \), also for \( 0(\partial \Gamma, d) \) and \( 0(\partial \Gamma, \mu) \). We will denote those common sets by \( \mathcal{S}_0(\partial \Gamma), \mathcal{S}(\partial \Gamma) \) and \( 0(\partial \Gamma) \) respectively.

**Proof.** Choose \( d \in J_{AR}(\partial \Gamma) \) and \( \mu \in M_2 \) arbitrarily. The first claim together with Theorem 2.2.3 in [FO1] implies the third one. The second claim follows from the third one and the definition of smooth measures. See Definition 6.5 and Remark 6.6. Therefore, it suffices to prove the first claim. Let \( \kappa \) be a positive Radon measure and assume that \( \kappa \in \mathcal{S}_0(\partial \Gamma, d) \), namely, there exists a constant \( C > 0 \) such that for any \( v \in \mathcal{C} \)
\[
\int_{\partial \Gamma} |v(\xi)| \, d\kappa(\xi) \leq C \left( \mathcal{E}^{\partial \Gamma, d}_1(v, v) \right)^{1/2}.
\] (32)
We will show that $\kappa \in S_0(\partial \Gamma, \mu)$. Recall that for any $v \in C$, we have that $\mathcal{E}^{\partial \Gamma, d}(v, v) = \mathcal{E}^{\partial \Gamma, \mu}(v, v)$. By Lemma 6.2, we have that

$$\inf_c \|v - c\|_{L^2(H_d)}^2 \leq C \mathcal{E}^{\partial \Gamma, d}(v, v) \leq C' \mathcal{E}^{\partial \Gamma, \mu}(v, v),$$

for a constant $C' > 0$ independent of $v$. On the other hand, we have that

$$\inf_c \|v - c\|_{L^2(H_d)}^2 = \inf_c \left( \mathcal{H}_d(\partial \Gamma) \cdot c^2 - 2c \left( \int_{\partial \Gamma} vdH_d \right) + \|v\|_{L^2(H_d)}^2 \right)$$

$$= \|v\|_{L^2(H_d)}^2 - \mathcal{H}_d(\partial \Gamma)^{-1} \left( \int_{\partial \Gamma} vdH_d \right)^2.$$

Thus, by Proposition 6.10 we get that

$$\|v\|_{L^2(H_d)}^2 \leq C' \mathcal{E}^{\partial \Gamma, \mu}(v, v) + \mathcal{H}_d(\partial \Gamma)^{-1} \left( \int_{\partial \Gamma} vdH_d \right)^2$$

$$\leq C' \mathcal{E}^{\partial \Gamma, \mu}(v, v) + \mathcal{H}_d(\partial \Gamma) \cdot \left( \|v\|_{L^1(\nu)} + \sqrt{\mathcal{E}^{\partial \Gamma, \mu}(v, v)} \right)^2$$

$$\leq (C' + 2C \mathcal{H}_d(\partial \Gamma)) \cdot \mathcal{E}^{\partial \Gamma, \mu}(v, v) + 2\mathcal{H}_d(\partial \Gamma)\|v\|_{L^1(\nu)}^2$$

$$\leq (C' + 2C \mathcal{H}_d(\partial \Gamma)) \cdot \mathcal{E}^{\partial \Gamma, \mu}(v, v) + 2\mathcal{H}_d(\partial \Gamma)\|v\|_{L^2(\nu)}^2,$$

where we used Jensen’s inequality in the last step. By substituting the above estimate for the inequality (32), we get that $\kappa \in S_0(\partial \Gamma, \mu)$. By using the estimate (29) and Proposition 6.10, the converse claim can be proved similarly. □

**Remark 6.12.** Note that both of $S(\partial \Gamma)$ and $S_0(\partial \Gamma)$ contain any Hausdorff measure $\mathcal{H}_d$ of a metric $d \in \mathcal{J}_{AR}(\partial \Gamma)$ and any harmonic measure $\nu$ of a random walk driven by a probability measure $\mu \in M_2$ since $\mathcal{H}_d$ is smooth with respect to $(\mathcal{E}^{\partial \Gamma, d}, B_2(d))$ and $\nu$ is smooth with respect to $(\mathcal{E}^{\partial \Gamma, \mu}, B_2(\mu))$.

### 7 Time changes of processes associated with Dirichlet forms

In Section 4, we introduced notions such as smooth measures and measures of finite energy integral, which concern the relation between measures and potential theory of Dirichlet forms. In this section, we will introduce several general facts on Dirichlet forms to explain the probabilistic interpretation of smooth measures and measures of finite energy integral, which are heavily related to time changes of symmetric Markov processes.

#### 7.1 Positive continuous additive functionals

Let $E$ be a locally compact separable metric space and $m$ be a positive Radon measure on $E$ with full support. If we are given a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E, m)$, we can associate
an \( m \)-symmetric Hunt process \((X_t, P_x)\) on \( E \). In the theory of Dirichlet forms, it is well-known that there is a relationship, called the Revuz correspondence, between smooth measures on \( E \) and positive continuous additive functionals (PCAFs in short). See [FOT, CF] for the precise definition of PCAFs. We denote the set of smooth measures on \( E \) by \( S \), and the set of PCAFs of the Hunt process \( X \) by \( A^+_c \). In what follows, we denote the extended Dirichlet space of \((E, \mathcal{F})\) by \((\mathcal{F}_e, \mathcal{E})\).

For a given \( A \in A^+_c \), define the measure \( \kappa_A \), called the Revuz measure of \( A \), by the following formula: for any \( f \in \mathcal{B}_+(E) := \{ f : E \to \mathbb{R}_{\geq 0} ; f \text{ is Borel measurable} \} \),

\[
\langle \kappa_A, f \rangle = \lim_{t \downarrow 0} \frac{1}{t} E_{\cdot} \left[ \int_0^t f(X_s) dA_s \right],
\]

where \( E_{\cdot} \) is the measure of \( \int E_{\cdot}[\cdot] m(dx) \). Two positive additive functionals \( A, B \in A^+_c \) are called \( m \)-equivalent if \( P_m(A_t = B_t) = 1 \) for every \( t > 0 \).

**Theorem 7.1.** [CF, Theorem 4.1.1]  

1. For any \( A \in A^+_c \), \( \kappa_A \in S \).
2. For any \( \kappa \in S \), there exists \( A \in A^+_c \) which satisfies \( \kappa_A = \kappa \) uniquely up to \( m \)-equivalence.
3. For \( A \in A^+_c \) and \( \kappa \in S \), the following conditions are equivalent.
   
   (a) \( \kappa_A = \kappa \)

   (b) For any \( f, h \in \mathcal{B}_+(E) \) and any \( t > 0 \),

   \[
   E_{h \cdot m} \left[ \int_0^t f(X_s) dA_s \right] = \int_0^t \langle T_s h, f \cdot \kappa \rangle ds,
   \]

   where \( (T_s) \) is the semigroup associated to the process \( X \).

**Example 7.2.** Let \((E, \mathcal{F})\) be a regular Dirichlet form on \( L^2(E, m) \) and \((X_t)\) be the corresponding \( m \)-symmetric Hunt process. Then for a nonnegative function \( g \in L^1(E, m) \), the absolutely continuous measure

\[
g \cdot m(dx) := g(x)m(dx)
\]

is smooth with respect to \((E, \mathcal{F})\). This immediately follows from the absolute continuity and the fact that \( m \) itself is a smooth measure with respect to \((E, \mathcal{F})\).

Let us prove that the Revuz correspondence relates \( g \cdot m \) to the PCAF

\[
A^g_t := \int_0^t g(X_s) ds.
\]

We will verify (33). Let \( f, h \in \mathcal{B}_+(E) \) and \( t > 0 \). By Fubini’s theorem, we have that

\[
E_{h \cdot m} \left[ \int_0^t f(X_s) dA^g_s \right] = E_{h \cdot m} \left[ \int_0^t f(X_s) g(X_s) ds \right] = \int_0^t ds E_{h \cdot m}[(fg)(X_s)].
\]

32
Using the $m$-symmetry of $X$, we finally get that

$$
\int_0^t ds \, E_{h,m}[(fg)(X_s)] = \int_0^t ds \, \langle T_s(fg), h \cdot m \rangle = \int_0^t ds \, \langle T_s h, f g \cdot m \rangle,
$$

which is (33).

The following theorem gives the probabilistic interpretation of sets of zero capacity.

**Theorem 7.3.** Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(E, m)$ and $(X_t)$ be the $m$-symmetric Hunt process which corresponds to $(\mathcal{E}, \mathcal{F})$. Then for a set $C \subseteq E$, we have that $\text{Cap}(C) = 0$ if and only if

$$
P_x(X_t \notin C \text{ for any } t \in [0, \infty)) = 1
$$

for $m$-almost every $x$.

For a PCAF $A^\kappa \in A_c^+$ whose Revuz measure is $\kappa \in S$, define its right continuous inverse $(A^\kappa)^{-1}_t$ by

$$(A^\kappa)^{-1}_t := \inf\{s > 0; A^\kappa_s > t\}.$$ 

Then the time-changed process $Y_t := X \circ (A^\kappa)^{-1}_t$ is a $\kappa$-symmetric Markov process. The Dirichlet form of $Y_t$ is characterized by the following theorem.

**Theorem 7.4.** For $B \in \mathcal{B}(E)$, define the hitting time $\sigma_B$ of $B$ by

$$
\sigma_B := \inf\{t > 0; X_t \in B\}.
$$

Let $A^\kappa$ be a PCAF whose Revuz measure is $\kappa \in S$, and $(A^\kappa)^{-1}_t$ be the right continuous inverse of $A^\kappa$. Denote the support of $A^\kappa$ by $F$. (See [FOT, page 234] for the definition.) Define $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ by

$$
\tilde{\mathcal{F}} := \{u \in L^2(F, \kappa) ; u = \tilde{\varphi} \text{ } \kappa\text{-a.e. on } F \text{ for some } \varphi \in \mathcal{F}_c\},
$$

$$
\tilde{\mathcal{E}}(u, v) := \mathcal{E}(H_F u, H_F v) \text{ for } u, v \in \tilde{\mathcal{F}},
$$

where $\tilde{\varphi}$ is a quasi-continuous modification of $\varphi$ (See Theorem 2.1.3 in [FOT]).) and $H_F f(x) := E_x[f(X_{\sigma_F}); \sigma_F < \infty], x \in E, f \in \mathcal{B}_+(E)$. Then $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is the regular Dirichlet form on $L^2(F, \kappa)$ which corresponds to $Y_t := X \circ (A^\kappa)^{-1}_t$. We will call $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ the trace of $(\mathcal{E}, \mathcal{F})$ on $F$ with respect to $\kappa$.

### 7.2 The reflected Dirichlet spaces

In [CH], it is shown how to associate to any regular transient Dirichlet form, its reflected Dirichlet space. We will use the notion of reflected Dirichlet space in Part 8 to define a reflected random walk on $\Gamma \cup \partial \Gamma$ and identify the Dirichlet form of its trace on $\partial \Gamma$.

In this paper, we will only deal with reflected Dirichlet spaces of regular Dirichlet forms on discrete graphs corresponding to random walks on them. The reflected Dirichlet spaces which arise in discrete settings are characterized as in the following example. See [CF, Section 6.5] for details.
Example 7.5. (See Example 3.8 for notation used here.) Let $\mathbb{V}$ be either a finite set or a countable set. Then, under the assumption (10), the constant speed random walk $(X_t)$ corresponds to the following regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{V}, \mu)$:

$$\mathcal{E}(u, v) := \frac{1}{2} \sum_{x, y \in \mathbb{V}} c(x, y)(u(x) - u(y))(v(x) - v(y)),$$

$$\mathcal{F} = L^2(\mathbb{V}, \mu).$$

It is shown in [CF, Section 6.5] that when the constant speed random walk $(X_t)$ on $\mathbb{V}$ is transient, the reflected Dirichlet space $(\mathcal{E}_{\text{ref}}, \mathcal{F}_{\text{ref}})$ of $(\mathcal{E}, \mathcal{F})$ is given by

$$\mathcal{E}_{\text{ref}} := \mathcal{E},$$

$$\mathcal{F}_{\text{ref}} := \{u : \mathbb{V} \to \mathbb{R} ; \mathcal{E}(u, u) < \infty\}.$$

Now we introduce the following theorem, which is fundamental in the theory of reflected Dirichlet forms.

Theorem 7.6. [Ch, Theorem 3.10][CF, Theorem 6.2.14] Let $(\mathcal{E}, \mathcal{F})$ be a regular transient Dirichlet form on $L^2(\mathbb{E}, \mu)$ and denote its reflected Dirichlet space by $(\mathcal{E}_{\text{ref}}, \mathcal{F}_{\text{ref}})$. Define $(\mathcal{F}_{\text{ref}})_a := \mathcal{F}_{\text{ref}} \cap L^2(\mathbb{E}, \mu)$. Then $(\mathcal{E}_{\text{ref}}, (\mathcal{F}_{\text{ref}})_a)$ is a Dirichlet form on $L^2(\mathbb{E}, \mu)$.

In what follows, we will utilize the following results about reflected Dirichlet spaces.

Theorem 7.7. [CF, Proposition 6.4.6, Theorem 6.6.10] Let $(\mathcal{E}, \mathcal{F})$ be a transient regular Dirichlet form on $L^2(\mathbb{E}, \mu)$. Then the following results hold.

(1) If $\mu$ is a finite measure on $\mathbb{E}$, then the extended Dirichlet space of $(\mathcal{E}_{\text{ref}}, (\mathcal{F}_{\text{ref}})_a)$ coincides with $(\mathcal{F}_{\text{ref}}, \mathcal{E})$.

(2) For any $\kappa \in S$ with full support, let $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ be the trace of $(\mathcal{E}, \mathcal{F})$ with respect to $\kappa$. If we denote the reflected Dirichlet space of $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ by $(\tilde{\mathcal{E}}_{\text{ref}}, \tilde{\mathcal{F}}_{\text{ref}})$, then we have $(\tilde{\mathcal{E}}_{\text{ref}}, \tilde{\mathcal{F}}_{\text{ref}}) = (\mathcal{E}_{\text{ref}}, \mathcal{F}_{\text{ref}})$.

Remark 7.8. In most examples, the Dirichlet space $(\mathcal{F}_{\text{ref}})_a$ is not regular. For an infinite connected locally finite graph as in Example 7.5, all functions defined on $\mathbb{V}$ are continuous and, except in very special situations, the Dirichlet space $(\mathcal{F}_{\text{ref}})_a$ will not be rich enough to contain a subset that is dense in the set of all continuous functions. It is then not possible to construct a Hunt process on $\mathbb{V}$ whose Dirichlet form is the reflected one.

It follows from general theorems, that there always exists a regular Dirichlet form on an extended state space that is equivalent to the reflected Dirichlet form $(\mathcal{F}_{\text{ref}})_a$, see [FOT, Appendix A.4].

In the next part of this paper, rather than using the general theory, we will show that one can define the reflected random walk on the group $\Gamma$ enlarged with its Gromov boundary $\partial \Gamma$. In other words we obtain a regular Dirichlet form that is equivalent to the reflected one on the state space $\Gamma \cup \partial \Gamma$. 

34
8 Reflected random walks on $\Gamma \cup \partial \Gamma$ and their trace processes on $\partial \Gamma$

In this section, we will give a probabilistic interpretation of the strong Markov process of jump type associated to the regular Dirichlet form $(B_2(\mu), \mathcal{E}^{\partial \Gamma, \mu})$. We consider the random walk $(R_n)$ on $\Gamma$ driven by $\mu$ with $\mu \in M_2$. We first construct a reflected random walk on $\Gamma \cup \partial \Gamma$ using the notion of reflected Dirichlet spaces introduced in the previous subsection. Then we prove that the trace process of the reflected random walk on $\partial \Gamma$ coincides with the jump process associated to $(B_2(\mu), \mathcal{E}^{\partial \Gamma, \mu})$.

Let us first explain the construction of the reflected random walk on $\Gamma \cup \partial \Gamma$. Recall the definition of the constant speed random walk from Example 3.8. Let $\mu \in M_2$, and $(X_t)$ be the constant speed random walk associated to the regular Dirichlet form $(\mathcal{E}^\mu, \lambda_2(\Gamma))$ on $\lambda_2(\Gamma)$. The process $(X_t)$ does not reach $\partial \Gamma$ in finite time since its holding time at any vertex in $\Gamma$ is distributed as the exponential distribution with mean 1.

We first take a time change of $(X_t)$ in such a way that the time-changed process reaches the boundary $\partial \Gamma$ within finite time. This can be done as follow: let $\omega$ be a finite measure on $\Gamma$ with full support. Define $(Y_t)$ by taking a time change of $(X_t)$ such that at a vertex $x \in \Gamma$, $(Y_t)$ has a holding time distributed as the exponential distribution with mean $\omega(x)$. Then, $(Y_t)$ corresponds to the regular Dirichlet form $(\mathcal{E}^\mu, \lambda_2(\Gamma))$ on $L^2(\Gamma, \omega)$, and $(Y_t)$ is the time change of $(X_t)$ by the PCAF $\int_0^t \omega(X_s)ds$ which is related to $\omega$ by the Revuz correspondence as in Part 7.1. (Since $\omega$ has full support, it is obvious that $\omega$ is smooth with respect to the regular Dirichlet form $(\mathcal{E}^\mu, \lambda_2(\Gamma))$ on $\lambda_2(\Gamma)$.)

According to the second claim of Theorem 7.4, the reflected extension of the Dirichlet form of $(Y_t)$ is given by

$$
\mathcal{E}^\mu_{\text{ref}} := \mathcal{E}^\mu,
$$

$$
\mathcal{F}^\mu_{\text{ref}} := \mathcal{F}^\mu(= \{f : \Gamma \to \mathbb{R} ; \mathcal{E}^\mu(f, f) < \infty\}),
$$

$$(\mathcal{F}^\mu_{\text{ref}})_a := \mathcal{F}^\mu \cap L^2(\Gamma, \omega).$$

and $\mathcal{F}^\mu_{\text{ref}} = \mathcal{F}^\mu$ coincides with the extended Dirichlet space of

$$(\mathcal{E}^\mu_{\text{ref}}, (\mathcal{F}^\mu_{\text{ref}})_a) = (\mathcal{E}^\mu, \mathcal{F}^\mu \cap L^2(\Gamma, \omega)).$$

By Theorem 7.6 $(\mathcal{E}^\mu, \mathcal{F}^\mu \cap L^2(\Gamma, \omega))$ is a Dirichlet form on $L^2(\Gamma, \omega)$.

Observe that $(\mathcal{E}^\mu, \mathcal{F}^\mu \cap L^2(\Gamma, \omega))$ is not regular on $L^2(\Gamma, \omega)$. Indeed here $\Gamma$ is equipped with the discrete topology; $C_0(\Gamma)$ is the set of functions on $\Gamma$ with a finite support and $C_0(\Gamma) \cap \mathcal{F}^\mu \cap L^2(\Gamma, \omega)$ is not large enough to be dense in $\left(\mathcal{F}^\mu \cap L^2(\Gamma, \omega), (\mathcal{E}^\mu)^{1/2}\right)$ (See Remark 7.8).

We shall resort to the Gromov compactification to obtain the regularity on the larger state space $\Gamma \cup \partial \Gamma$ in the following manner: consider $\omega$ as a measure on $\Gamma \cup \partial \Gamma$ that gives zero weight to the boundary. Note that $\omega$ still has full support. Then we look at functions defined on $\Gamma \cup \partial \Gamma$ whose restrictions to $\Gamma$ are in $L^2(\Gamma, \omega)$ and in $\mathcal{F}^\mu$. Recall that for any function in $\mathcal{F}^\mu$, we can define its boundary value on $\partial \Gamma$ thanks to the discussion around (1.9). We now introduce
some new notation. We denote by \( \tilde{\mathcal{F}}^\mu \) the set of all functions on \( \Gamma \cup \partial \Gamma \) that are extensions of functions in \( \mathcal{F}^\mu \). We define

\[
\tilde{\mathcal{F}}^\mu_a(\omega) := \{ f \in \tilde{\mathcal{F}}^\mu; \ f|_{\Gamma} \in L^2(\Gamma, \omega) \},
\]

\[
\tilde{\mathcal{E}}^\mu(f, f) := \mathcal{E}^\mu(f|_{\Gamma}, f|_{\Gamma}) \text{ for } f \in \tilde{\mathcal{F}}^\mu.
\]

The above extension does not change the \( L^2 \) norm, nor the Dirichlet norm or the \( L^\infty \) norm. In other words, the two Dirichlet forms \( (\mathcal{E}^\mu, \mathcal{F}^\mu \cap L^2(\Gamma, \omega)) \) and \( (\tilde{\mathcal{E}}^\mu, \tilde{\mathcal{F}}^\mu_a(\omega)) \) are equivalent in the sense in [FOT] Appendix A.4, p 422. Moreover, we deduce that \( (\tilde{\mathcal{E}}^\mu, \tilde{\mathcal{F}}^\mu_a(\omega)) \) is a Dirichlet form on \( L^2(\Gamma \cup \partial \Gamma, \omega) \) and the reflected Dirichlet space of \( (\tilde{\mathcal{E}}^\mu, \tilde{\mathcal{F}}^\mu_a(\omega)) \) is \( \tilde{\mathcal{F}}^\mu \) since similar claims hold for \( (\mathcal{E}^\mu, \mathcal{F}^\mu \cap L^2(\Gamma, \omega)) \). We now prove that \( (\tilde{\mathcal{E}}^\mu, \tilde{\mathcal{F}}^\mu_a(\omega)) \) is a regular Dirichlet form on \( L^2(\Gamma \cup \partial \Gamma, \omega) \).

**Theorem 8.1.** Assume that the Ahlfors-regular conformal dimension of \( \partial \Gamma \) is strictly less than 2. Then for \( \mu \in M_2 \), \( (\tilde{\mathcal{E}}^\mu, \tilde{\mathcal{F}}^\mu_a(\omega)) \) is a regular Dirichlet form on \( L^2(\Gamma \cup \partial \Gamma, \omega) \).

**Proof.** In the following proof, we will write \( \tilde{\mathcal{F}}^\mu_a \) for \( \tilde{\mathcal{F}}^\mu_a(\omega) \) for simplicity of notation. In order to prove the theorem, it suffices to show that \( C(\Gamma \cup \partial \Gamma) \cap \tilde{\mathcal{F}}^\mu_a \) is dense both in \( (C(\Gamma \cup \partial \Gamma), \| \cdot \|_\infty) \) and in \( (\tilde{\mathcal{F}}^\mu_a, (\tilde{\mathcal{E}}^\mu_1)^{1/2}) \). Note that

\[
C(\Gamma \cup \partial \Gamma) \cap \tilde{\mathcal{F}}^\mu_a = C(\Gamma \cup \partial \Gamma) \cap \tilde{\mathcal{F}}^\mu,
\]

since \( \omega \) is a finite measure on \( \Gamma \cup \partial \Gamma \).

We first prove that \( C(\Gamma \cup \partial \Gamma) \cap \tilde{\mathcal{F}}^\mu \) is dense in \( (C(\Gamma \cup \partial \Gamma), \| \cdot \|_\infty) \). By Stone-Weierstrass theorem, we just need to prove that \( C(\Gamma \cup \partial \Gamma) \cap \tilde{\mathcal{F}}^\mu \) separates points in \( \Gamma \cup \partial \Gamma \). Obviously, it is enough to prove the claim for points in \( \partial \Gamma \).

Recall the definition of the space \( \text{Lip}_0 \) from Part 6.1. For any \( \eta, \xi \in \partial \Gamma \) with \( \eta \neq \xi \), there exists a function \( u \in \text{Lip}_0 \) such that \( u(\eta) = 1 \) and \( u(\xi) = 0 \). Note that \( u \in C \) since \( \text{Lip}_0 \subset C \). Let \( Hu : \Gamma \to \mathbb{R} \) be the harmonic extension of \( u \) as in (20). Then by Proposition 5.1 and Theorem 8.1, we get \( Hu \in \mathcal{F}^\mu \).

Take any sequence \( (g_n) \subset C \) converging to \( \tau \in \partial \Gamma \). Then, by [Ka] Lemma 2.2, the sequence of harmonic measures \( (\nu_{g_n}) \) weakly converges to the dirac measure of \( \tau \). Since \( u \) is bounded and continuous, this implies \( \lim_{n \to \infty} Hu(g_n) = u(\tau) \). Therefore the function \( Hu \cdot 1_{\Gamma} + u \cdot 1_{\partial \Gamma} \) belongs to \( C(\Gamma \cup \partial \Gamma) \). This is enough to prove that \( C(\Gamma \cup \partial \Gamma) \cap \tilde{\mathcal{F}}^\mu \) is dense in \( (C(\Gamma \cup \partial \Gamma), \| \cdot \|_\infty) \).

We next prove that \( C(\Gamma \cup \partial \Gamma) \cap \tilde{\mathcal{F}}^\mu \) is dense in \( (\tilde{\mathcal{F}}^\mu_a, (\tilde{\mathcal{E}}^\mu_1)^{1/2}) \). Take \( h \in \tilde{\mathcal{F}}^\mu_a \). Note that by Theorem 1.4.2 (ii),(iii) in [FOT], we can assume that \( h \) is a bounded function without loss of generality. By applying the Royden decomposition to \( h \), we get that

\[
h = h_0 + h_1, \text{ where } h_0 \in \ell_2(\Gamma) \text{ and } h_1 \in \mathbb{H}D(\mu).
\]

By the definition of \( \ell_2(\Gamma) \), for any \( \varepsilon > 0 \) there exists \( k \in \mathbb{N} \) such that

\[
\sum_{x \in \Gamma: |x| > k} h_0(x)^2 < \varepsilon.
\]

This implies that \( h_0 \) can be extended to a continuous function on \( \Gamma \cup \partial \Gamma \) by setting \( h_0|_{\partial \Gamma} = 0 \). Notice that \( \ell_2(\Gamma) \subset \tilde{\mathcal{F}}^\mu_a \) since functions in \( \ell_2(\Gamma) \) are bounded and belong to \( \mathcal{F}^\mu \) by Proposition 8.1.
Thus we have that $h_0 \in \tilde{\mathcal{F}}^\mu_\alpha$. Therefore, it suffices to show the claim for $h_1$. We know that $h_1$ is bounded and belongs to $\mathcal{HD}(\mu) \cap \tilde{\mathcal{F}}^\mu_\alpha$.

Define

$$C_1 := \sup_{x \in \Gamma} h_1(x).$$

We let $v : \partial \Gamma \to \mathbb{R}$ be such that $\lim_{t \to \infty} h_1(X_t) = v(R_\infty)$ as in (19). Then $v \in B_2(\mu)$. Moreover, by Corollary 5.2 we have that $h_1 = Hv$. Since $\mathcal{C}$ is dense in $(B_2(\mu), (\mathcal{E}^{\partial \Gamma, \mu})^{1/2})$, we can take a sequence $(w_n) \subset \mathcal{C}$ which converges to $v$ in $(B_2(\mu), (\mathcal{E}^{\partial \Gamma, \mu})^{1/2})$. Moreover, by Theorem 1.4.2 (v) in [FOT], we have the same convergence for the sequence $w'_n := w_n \wedge C_1$, namely

$$\mathcal{E}^{\partial \Gamma, \mu}(v - w'_n, v - w'_n) + \|v - w'_n\|_{L^2(\nu)}^2 \to 0. \quad (34)$$

Since we have that $\mathcal{E}^{\partial \Gamma, \mu}(u, u) = \mathcal{E}^\mu(Hu, Hu)$ for any $u \in B_2(\mu)$ and $h_1 = Hv$, this implies that

$$\mathcal{E}^\mu(h_1 - Hw'_n, h_1 - Hw'_n) \to 0. \quad (35)$$

It is easy to see that (35) implies that

$$\lim_{n \to \infty} (h_1(x) - Hw'_n(x) - c_n) = 0 \quad \text{for any } x \in \Gamma, \quad (36)$$

where

$$c_n := h_1(id) - Hw'_n(id).$$

First observe that

$$\mathcal{E}^\mu(h_1 - Hw'_n - c_n, h_1 - Hw'_n - c_n) = \mathcal{E}^\mu(h_1 - Hw'_n, h_1 - Hw'_n)$$

still converges to 0. It remains to check that $\|h_1 - Hw'_n - c_n\|_{L^2(\partial \Gamma, \omega)} \to 0$.

By (20) and (34) we have that

$$|c_n| = |h_1(id) - Hw'_n(id)| = |H(v - w'_n)(id)| \leq \int_{\partial \Gamma} |v(\xi) - w'_n(\xi)| d\nu(\xi) \leq \|v - w'_n\|_{L^1(\nu)} \leq \|v - w'_n\|_{L^2(\nu)}.$$

Therefore the sequence $(c_n)$ converges to 0 as $n \to \infty$ and $\sup_{m \geq 1} |c_m| < \infty$.

We have that for any $x \in \Gamma$

$$|h_1(x) - Hw'_n(x) - c_n| \leq 2C_1 + \sup_{m \geq 0} |c_m| < \infty.$$

Since $\omega$ is a finite measure on $\Gamma$, the dominated convergence theorem implies the conclusion. □

By Theorem 5.1 there exists a $\Gamma \cup \partial \Gamma$-valued, $\omega$-symmetric process $(W_t)$ associated to the regular Dirichlet form $(\mathcal{E}^\mu, \mathcal{F}^\mu_\alpha)$. Note that $1 \in \tilde{\mathcal{F}}^\mu_\alpha$ and therefore the process $(W_t)$ is recurrent. (See Theorem 1.6.3 in [FOT].)

For a process $(S_t)$ on $\Gamma \cup \partial \Gamma$ and a subset $A \subset \Gamma \cup \partial \Gamma$, we define

$$\sigma_S(A) := \inf\{t > 0 : S_t \notin A\}.$$

The next result shows that $(W_t)$ is an extension of $(Y_t)$. 37
Proposition 8.2. \((Y_t \mid 0 \leq t \leq \sigma_Y(\Gamma)) \overset{(d)}{=} (W_t \mid 0 \leq t \leq \sigma_W(\Gamma))\).

Proof. By Theorem 3.3, it suffices to prove that the extended Dirichlet spaces associated to the above two processes coincide. By the inequality (18), there exists a constant \(C > 0\) such that

\[
\|f\|_{\ell^2(\Gamma)} \leq C\mathcal{E}^\mu(f, f)
\]

for any \(f \in \ell^2(\Gamma)\). Hence the extended Dirichlet space associated to \((Y_t \mid 0 \leq t \leq \sigma_Y(\Gamma))\) is \(\ell^2(\Gamma)\).

On the other hand, by Theorem 3.4.9 in [CF], the extended Dirichlet space associated to \((W_t \mid 0 \leq t \leq \sigma_W(\Gamma))\) is given by \(\{f \in \tilde{F}^\mu : \tilde{f} = 0\text{ q.e. on } \partial\Gamma\}\), where \(\tilde{f}\) is a quasi-continuous modification of \(f\). Now we have the decomposition \(\mathcal{F}^\mu = \ell^2(\Gamma) \oplus \mathbb{H}D(\mu)\) and any function in \(\mathbb{H}D(\mu)\) with zero boundary value should be identically zero. Thus we get the conclusion. \(\square\)

Finally, the next theorem gives a probabilistic interpretation of the regular Dirichlet form \((\mathcal{E}^{\partial\Gamma, \mu}, B_2(\mu))\) on \(L^2(\partial\Gamma, \nu)\). Recall that by Remark 6.12, \(\nu\) is smooth with respect to \((\mathcal{E}^{\partial\Gamma, \mu}, B_2(\mu))\), therefore there exists a PCAF \(A^\nu\) related to \(\nu\) by the Revuz correspondence.

Theorem 8.3. Assume that the Ahlfors-regular conformal dimension of \(\partial\Gamma\) is less than 2. Then for \(\mu \in M_2\), the regular Dirichlet form \((\mathcal{E}^{\partial\Gamma, \mu}, B_2(\mu))\) on \(L^2(\partial\Gamma, \nu)\) coincides with the trace of \((\tilde{\mathcal{E}}^\mu, \tilde{\mathcal{F}}^\mu_a)\) on \(\partial\Gamma\) with respect to \(\nu\). In other words, the regular Dirichlet form \((\mathcal{E}^{\partial\Gamma, \mu}, B_2(\mu))\) on \(L^2(\partial\Gamma, \nu)\) corresponds to the \(\nu\)-symmetric Hunt process \(W \circ (A^\nu)_t^{-1}\) on \(\partial\Gamma\).

Remark 8.4. Notice that the trace of \((\tilde{\mathcal{E}}^\mu, \tilde{\mathcal{F}}^\mu_a)\) on \(\partial\Gamma\) with respect to \(\nu\) does not depend on the choice of \(\omega\).

Proof. Let \((\tilde{E}, \tilde{F})\) be the trace of \((\tilde{\mathcal{E}}^\mu, \tilde{\mathcal{F}}^\mu_a)\) on \(\partial\Gamma\) with respect to \(\nu\). Then by Theorem 7.3, \((\tilde{E}, \tilde{F})\) is given by

\[
\tilde{E}(u, u) := \mathcal{E}^\mu(H_{\partial\Gamma}u, H_{\partial\Gamma}u),
\tilde{F} := \{u \in L^2(\partial\Gamma, \nu) : u = \tilde{g} \text{ } \nu\text{-a.e. on } \Gamma \text{ for some } g \in \tilde{\mathcal{F}}^\mu\},
\]

where \(\tilde{g}\) is a quasi-continuous modification of \(g\) and \(H_{\partial\Gamma}u(g) := \mathbb{E}_g[u(W_{\sigma_W(\Gamma)})]\). By Proposition 8.2 we get that

\[
\mathbb{E}_g[u(W_{\sigma_W(\Gamma)})] = \int_{\partial\Gamma} u(\eta) d\nu_\eta(\eta) = Hu(g).
\]

Thus, it suffices to prove that \(\tilde{F} = B_2(\mu)\). We first show that \(\tilde{F} \supseteq B_2(\mu)\). Take \(v \in B_2(\mu)\) arbitrarily. By Proposition 5.7, there exists \(f \in \tilde{\mathcal{F}}^\mu\) such that \(\lim_{t \to \infty} f(X_t) = v(X_\infty)\) \(\mathbb{P}^\mu\text{-a.s.}\). This implies that \(\tilde{f} = v\) \(\nu\text{-a.e. on } \partial\Gamma\).

We next show that \(\tilde{F} \subseteq B_2(\mu)\). Take \(u \in \tilde{F}\) arbitrarily. Then, we have that \(u = \tilde{g}\) \(\nu\text{-a.e. on } \partial\Gamma\) for some \(g \in \tilde{\mathcal{F}}^\mu\). On the other hand, by Proposition 5.7 again, there exists \(w \in B_2(\mu)\) such that \(\lim_{t \to \infty} g(X_t) = w(X_\infty)\) \(\mathbb{P}^\mu\text{-a.s.}\). Hence, we get that \(u = w\) \(\nu\text{-a.e.}\), and this implies that \(u \in B_2(\mu)\). \(\square\)
9 More on the potential theory of harmonic measures

9.1 Measures of finite energy integral and heat kernel estimates

In this subsection, we will prove an integral condition for measures to be of finite energy integral, see (38). We will use the heat kernel estimates for non-local Dirichlet forms from \[\text{GHH, CKW, CK}\], which we state below. Recall that the metric \(d_0\) belongs to the Ahlfors-regular conformal gauge \(J_{\text{AR}}(\partial \Gamma)\), and \(q_0 := \dim(\partial \Gamma, d_0) < 2\). This last condition ensures that results obtained in \[\text{GHH}\] can be applied to the regular Dirichlet form \((\mathcal{E}_{\partial \Gamma, d_0}, B_2(d_0))\) on \(L^2(\partial \Gamma, \mathcal{H}_{d_0})\). We refer to \[\text{GHH}\], in particular statement (1.16), for the following

**Theorem 9.1.** There exists a jointly measurable transition density function \(p^0_t(\cdot, \cdot)\) on \((0, \infty) \times \partial \Gamma \times \partial \Gamma\) associated to the regular Dirichlet form \((\mathcal{E}_{\partial \Gamma, d_0}, B_2(d_0))\) on \(L^2(\partial \Gamma, \mathcal{H}_{d_0})\), and \(p^0_t(\cdot, \cdot)\) satisfies the following estimates: define \(P_t(\cdot, \cdot) : (0, \infty) \times \partial \Gamma \times \partial \Gamma \to \mathbb{R}_{\geq 0}\) by

\[
P_t(\xi, \eta) = \begin{cases} 
  t^{-1}, & \text{when } t \geq d_0(\xi, \eta)^{q_0}, \\
  \frac{t}{d_0(\xi, \eta)^{2q_0}}, & \text{when } 0 < t \leq d_0(\xi, \eta)^{q_0},
\end{cases}
\]

then there exist constants \(C_1, C_2 > 0\) such that

\[
(C_1)^{-1}P_t(\xi, \eta) \leq p^0_t(\xi, \eta) \leq C_1P_t(\xi, \eta)
\]

for any \(t \in (0, 1)\) and any \(\xi, \eta \in \partial \Gamma\), and such that

\[
(C_2)^{-1} \leq p^0_t(\xi, \eta) \leq C_2
\]

for any \(t \geq 1\) and any \(\xi, \eta \in \partial \Gamma\).

We now introduce the following criterion using \(p^0_t(\cdot, \cdot)\) for measures of finite energy integral.

**Lemma 9.2.** Let \(\kappa\) be a positive Radon measure on \(\partial \Gamma\). Then \(\kappa \in \mathcal{S}_0(\partial \Gamma)\) if and only if

\[
\int_{\partial \Gamma} \int_{\partial \Gamma} \left( \int_0^\infty e^{-t}p^0_t(\xi, \eta)dt \right) d\kappa(\xi)d\kappa(\eta) < \infty.
\]

(37)

**Proof.** The above equivalence immediately follows from Theorem 6.11 and Problem 4.2.1 in \[\text{FOT}\]. \(\Box\)

Using the estimates of \(p^0_t(\cdot, \cdot)\) given in Theorem 9.1 and computing the left hand side of (37), we obtain the following criterion for measures of finite energy integral.

**Proposition 9.3.** Let \(\kappa\) be a positive Radon measure on \(\partial \Gamma\). Then \(\kappa \in \mathcal{S}_0(\partial \Gamma)\) if and only if

\[
\int_{\partial \Gamma} \int_{\partial \Gamma} |\log d(\xi, \eta)| d\kappa(\xi)d\kappa(\eta) < \infty,
\]

(38)

for some \((\Leftrightarrow \text{any})\) metric \(d \in J_{\text{AR}}(\partial \Gamma)\).
Proof. By Corollary 11.5 in [Hei], we know that the finiteness of the integral (38) does not depend on the choice of $d \in J_{AR}(\partial \Gamma)$. Thus, in the light of Lemma 9.2 we only need to compare the two integrals (37) and (38) when we choose $d = d_0$ in (38). Since $\text{diam}(\partial \Gamma, d_0) < \infty$ and $\kappa$ is a positive Radon measure, we have that

$$\int_{\partial \Gamma} \int_{\partial \Gamma} |\log d_0(\xi, \eta)| 1_{\{d_0(\xi, \eta) > 1\}} d\kappa(\xi) d\kappa(\eta) < \infty.$$ 

By using the estimates in Theorem 9.1, it is immediate to check that

$$\int_{\partial \Gamma} \int_{\partial \Gamma} \left(\int_0^\infty e^{-t} p_t^0(\xi, \eta) dt\right) 1_{\{d_0(\xi, \eta) > 1\}} d\kappa(\xi) d\kappa(\eta) < \infty.$$ 

Note that by Theorem 9.1 $p_t^0(\cdot, \cdot)$ is of constant order for $t \geq 1$. Thus in order to show the converse claim, it suffices to prove that

$$\int_{\partial \Gamma} \int_{\partial \Gamma} |\log d_0(\xi, \eta)| 1_{\{d_0(\xi, \eta) \leq 1\}} d\kappa(\xi) d\kappa(\eta) < \infty,$$

if and only if

$$\int_{\partial \Gamma} \int_{\partial \Gamma} \left(\int_0^1 e^{-t} p_t^0(\xi, \eta) dt\right) 1_{\{d_0(\xi, \eta) \leq 1\}} d\kappa(\xi) d\kappa(\eta) < \infty$$

or equivalently if and only if

$$\int_{\partial \Gamma} \int_{\partial \Gamma} \left(\int_0^1 p_t^0(\xi, \eta) dt\right) 1_{\{d_0(\xi, \eta) \leq 1\}} d\kappa(\xi) d\kappa(\eta) < \infty.$$ 

Let $\xi$ and $\eta$ be such that $d_0(\xi, \eta) \leq 1$. It immediately follows from the bounds in Theorem 9.1 that

$$(C_1)^{-1} \left(\frac{1}{2} - q_0 \log d_0(\xi, \eta)\right) \leq \int_0^1 p_t^0(\xi, \eta) dt \leq C_1 \left(\frac{1}{2} - q_0 \log d_0(\xi, \eta)\right).$$

Thus, we get the conclusion. \(\square\)

We next give an alternative proof of one implication of Proposition 9.3 which does not involve either the heat kernel estimates or the assumption $q_0 < 2$. We will prove that if (38) is finite then $\kappa \in S_0(\partial \Gamma)$.

Proof. Let $v \in \mathcal{C}$ and define $g : V(\Gamma_d) \to \mathbb{R}$ as in (24). Notice that there exists a surjective continuous map from the set of geodesic rays $\mathcal{R}$ in $\Gamma_d$ emanating from $O$ to $\partial \Gamma$, which is defined by $r \mapsto \lim_{t \to \infty} r(t)$ ($r \in \mathcal{R}$). Now we take a measurable section $\xi \in \partial \Gamma \to \xi \in \mathcal{R}$. See [Par, Chapter I] for the existence of such a measurable section. Remark that by the continuity of $v$, for any sequence $(x_n) \subset V(\Gamma_d)$ and $\xi \in \partial \Gamma$ with $\lim_{n \to \infty} x_n = \xi$ we have that $\lim_{n \to \infty} g(x_n) = v(\xi)$. Thus for any positive Radon measure $\kappa$ on $\partial \Gamma$, we have that

$$\int_{\partial \Gamma} v d\kappa - \int_{\partial \Gamma} v d\mathcal{H}_d = \int_{\partial \Gamma} d\kappa(\xi) \left(\sum_{e \in \beta(\xi)} dg(e)\right),$$

40
Theorem 9.4. controls how a path of a random walk on $\Gamma$ deviates from geodesics.

By the Cauchy-Schwartz inequality, we get that

$$\int_{\partial \Gamma} d\kappa(\xi) \left( \sum_{e \in \beta(\xi)} dg(e) \right) = \sum_{e \in E(\Gamma_d)} \left( \int_{\partial \Gamma} d\kappa(\xi) 1_{\{e \in \beta(\xi)\}} \right) dg(e) \leq \|dg\|_{\ell_2(E(\Gamma_d))} \cdot \sqrt{\sum_{e \in E(\Gamma_d)} \left( \int_{\partial \Gamma} d\kappa(\xi) 1_{\{e \in \beta(\xi)\}} \right)^2}.$$

For any fixed $e \in E(\Gamma_d)$, we have that

$$\left( \int_{\partial \Gamma} d\kappa(\xi) 1_{\{e \in \beta(\xi)\}} \right)^2 = \int_{\partial \Gamma} d\kappa(\xi) \int_{\partial \Gamma} d\kappa(\eta) 1_{\{e \in \beta(\xi) \cap \beta(\eta)\}}.$$

Thus,

$$\sum_{e \in E(\Gamma_d)} \left( \int_{\partial \Gamma} d\kappa(\xi) 1_{\{e \in \beta(\xi)\}} \right)^2 = \int_{\partial \Gamma} d\kappa(\xi) \int_{\partial \Gamma} d\kappa(\eta)|\beta(\xi) \cap \beta(\eta)|,$$

where $|A|$ denotes the cardinality of the set $A$. By using the tree approximation for $d$-hyperbolic metric spaces (for instance Theorem 1.1 in [Coo], which refers to [Gro]), we know that there exists a constant $C > 0$ such that $|\beta(\xi) \cap \beta(\eta)|$ is bounded above by $(|\xi| \eta)^d_\partial + C$, where $(|\xi| \eta)^d_\partial$ is the Gromov product of $\xi$ and $\eta$ on $\Gamma_d$ with respect to the base point $O$. Noticing that by Proposition 2.1 in [BP], $\exp(-(|\xi| \eta)^d_\partial)$ is comparable to $d(\xi, \eta)$, hence $|\beta(\xi) \cap \beta(\eta)|$ is bounded above by $|\log d(\xi, \eta)| + C$. Thus we get the desired result. \(\square\)

### 9.2 Driving measures with a finite first moment

We only proved so far that harmonic measures with driving measures in $M_2$ are of finite energy integral. Now we extend Theorem 9.11 to a harmonic measure of a random walk driven by $\mu \in M_1$. The proof uses Proposition 9.3 and the deviation inequality shown in [MS] which controls how a path of a random walk on $\Gamma$ deviates from geodesics.

**Theorem 9.4.** Assume the Ahlfors-regular conformal dimension of $\partial \Gamma$ is strictly less than 2. Then, both of $\mathcal{S}(\partial \Gamma)$ and $\mathcal{S}_0(\partial \Gamma)$ contain any harmonic measure $\nu$ of a random walk driven by a probability measure $\mu \in M_1$.

**Proof.** Denote by $(R_n)$ the random walk driven by $\mu \in M_1$. If we choose as $d$ a visual metric $\rho_\Gamma$ on $\partial \Gamma$ for [33], the integral [33] is finite if and only if

$$\mathbb{E}^\mu[(R_\infty, R_\infty')_{id}] < \infty,$$

where $(\cdot, \cdot)_{id}$ is the Gromov product with respect to the base point $id$ which is computed in the word metric, and $(R_n')$ is a random walk driven by $\mu$ starting at $id$ which is independent of $(R_n)$. By the symmetry of $(R_n)$ and $(R_n')$, we observe that for any $n, m \in \mathbb{N}$

$$\mathbb{E}^\mu[(R_n, R_m')_{id}] = \mathbb{E}^\mu[(id, R_{n+m})_{R_n}].$$
By the second statement in [MS, Theorem 11.1], we have that

\[ \sup_{n,m \in \mathbb{N}} \mathbb{E}^\mu[(id, R_{n+m})_{R_n}] < \infty, \]

which implies the conclusion. \( \square \)

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