PAC Battling Bandits in the Plackett-Luce Model

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Abstract
We introduce the probably approximately correct (PAC) Battling-Bandit problem with the Plackett-Luce (PL) subset choice model—an online learning framework where at each trial the learner chooses a subset of \(k\) arms from a fixed set of \(n\) arms, and subsequently observes a stochastic feedback indicating preference information of the items in the chosen subset, e.g., the most preferred item or ranking of the top \(m\) most preferred items etc. The objective is to identify a near-best item in the underlying PL model with high confidence. This generalizes the well-studied PAC Dueling-Bandit problem over \(n\) arms, which aims to recover the best-arm from pairwise preference information, and is known to require \(O(n \epsilon^2 \ln \frac{1}{\delta})\) sample complexity (Szörenyi et al., 2015; Busa-Fekete et al., 2013). We study the sample complexity of this problem under various feedback models: (1) Winner of the subset (WI), and (2) Ranking of top-\(m\) items (TR) for \(2 \leq m \leq k\). We show, surprisingly, that with winner information (WI) feedback over subsets of size \(2 \leq k \leq n\), the best achievable sample complexity is still \(O(n \epsilon^2 \ln \frac{1}{\delta})\), independent of \(k\), and the same as that in the Dueling Bandit setting (\(k = 2\)). For the more general top-\(m\) ranking (TR) feedback model, we show a significantly smaller lower bound on sample complexity of \(\Omega(n^m \epsilon^2 \ln \frac{1}{\delta})\), which suggests a multiplicative reduction by a factor \(m\) owing to the additional information revealed from preferences among \(m\) items instead of just 1. We also propose two algorithms for the PAC problem with the TR feedback model with optimal (upto logarithmic factors) sample complexity guarantees, establishing the increase in statistical efficiency from exploiting rank-ordered feedback.

Keywords: Online Learning, PAC, Subsetwise Preference, Sample Complexity, Lower Bound, Plackett-Luce, Random Utility Choice Models, Dueling Bandit, Multi-armed Bandit.

1. Introduction
The dueling bandit problem has recently gained attention in the machine learning community (Yue et al., 2012; Ailon et al., 2014; Zoghi et al., 2014; Szörenyi et al., 2015). This is a variant of the multi-armed bandit problem (Auer et al., 2002) in which the learner needs to learn an ‘best arm’ from pairwise comparisons between arms. In this work, we consider a natural generalization of the dueling bandit problem where the learner can adaptively
select a subset of \( k \) arms \((k \geq 2)\) in each round, and observe relative preferences in the subset following a Plackett-Luce (PL) feedback model (Marden, 1996), with the objective of learning the ‘best arm’. We call this the \textit{battling bandit} problem with the Plackett-Luce model.

The \textit{battling bandit} decision framework (Saha and Gopalan, 2018; Chen et al., 2018) models several application domains where it is possible to elicit feedback about preferred options from among a general set of offered options, instead of being able to compare only two options at a time as in the dueling setup. Furthermore, the phenomenon of competition – that an option’s utility or attractiveness is often assessed relative to that of other items in the offering – is captured effectively by a subset-dependent stochastic choice model such as Plackett-Luce. Common examples of learning settings with such feedback include recommendation systems and search engines, medical interviews, tutoring systems—any applications where relative preferences from a chosen pool of options are revealed.

We consider a natural probably approximately correct (PAC) learning problem in the \textit{battling bandit} setting: Output an \( \epsilon \)-approximate best item (with respect to its Plackett-Luce parameter) with probability at least \((1 - \delta)\), while keeping the total number of adaptive exploration rounds small. We term this the \((\epsilon, \delta)\)-PAC objective of searching for an approximate winner or top-1 item.

Our primary interest lies in understanding how the subset size \( k \) influences the sample complexity of achieving \((\epsilon, \delta)\)-PAC objective in subset choice models for various feedback information structures, e.g., winner information (WI), which returns only a single winner of the chosen subset, or the more general top ranking (TR) information structure, where an ordered tuple of \( m \) ‘most-preferred’ items is observed. More precisely, we ask: Does being able to play size-\( k \) subsets help learn optimal items faster than in the dueling setting \((k = 2)\)? How does this depend on the subset size \( k \), and on the feedback information structure? How much, if any, does rank-ordered feedback accelerate the rate of learning, compared to only observing winner feedback? This paper takes a step towards resolving such questions within the context of the Plackett-Luce choice model. Among the contributions of this paper are:

1. We frame a \textit{PAC version of Battling Bandits} with \( n \) arms – a natural generalization of the PAC-Dueling-Bandits problem (Szörenyi et al., 2015) – with the objective of finding an \( \epsilon \)-approximate best item with probability at least \( 1 - \delta \) with minimum possible sample complexity, termed as the \((\epsilon, \delta)\)-PAC objective (Section 3.2).

2. We consider learning with winner information (WI) feedback, where the learner can play a subsets \( S_t \subseteq [n] \) of exactly \( |S_t| = k \) distinct elements at each round \( t \), following which a winner of \( S_t \) is observed according to an underlying, unknown, Plackett-Luce model. We show an information-theoretic lower bound on sample complexity for \((\epsilon, \delta)\)-PAC of \( \Omega \left( \frac{n^2}{\epsilon^2} \ln \frac{1}{\delta} \right) \) rounds (Section 4.1), which is of the same order as that for the dueling bandit \((k = 2)\) (Yue and Joachims, 2011). This implies that, despite the increased flexibility of playing sets of potentially large size \( k \), with just winner information feedback, one cannot hope for a faster rate of learning than in the case of pairwise selections. Intuitively, competition among a large number \((k)\) of elements vying for the top spot at each time exactly offsets the potential gain that being able to test more alternatives together brings. On the achievable side, we design two
algorithms (Section 4.2) for the $(\epsilon, \delta)$-PAC objective, and derive sample complexity guarantees which are optimal within a logarithmic factor of the lower bound derived earlier. When the learner is allowed to play subsets of sizes $1, 2, \ldots$ up to $k$, which is a slightly more flexible setting than above, we design a median elimination-based algorithm with order-optimal $O\left(\frac{n}{m^2 \ln \frac{1}{\delta}}\right)$ sample complexity which, when specialize  

to $k = 2$, improves upon existing sample complexity bounds for PAC-dueling bandit algorithms, e.g. Yue and Joachims (2011); Szörényi et al. (2015) under the PL model (Section 4.3).

3. We next study the $(\epsilon, \delta)$-PAC problem in a more general top-ranking (TR) feedback model where the learner gets to observe the ranking of top $m$ items drawn from the Plackett-Luce distribution, $2 \leq m \leq k$ (Section 3.1), departing from prior work. For $m = 1$, the setting simply boils down to WI feedback model. In this case, we are able to prove a sample complexity lower bound of $\Omega\left(\frac{n}{m^2 \ln \frac{1}{\delta}}\right)$ (Theorem 10), which suggests that with top-$m$ ranking (TR) feedback, it may be possible to aggregate information $m$ times faster than with just winner information feedback. We further present two algorithms (Section 5.2) for this problem which, are shown to enjoy optimal (upto logarithmic factors) sample complexity guarantees. This formally shows that the $m$-fold increase in statistical efficiency by exploiting richer information contained in top-$m$ ranking feedback is, in fact, algorithmically achievable.

4. From an algorithmic point of view, we elucidate how the structure of the Plackett-Luce choice model, such as its independent of irrelevant attributes (IIA) property, play a crucial role in allowing the development of parameter estimates, together with tight confidence sets, which form the basis for our learning algorithms. It is indeed by leveraging this property (Lemma 1) that we afford to maintain consistent pairwise preferences of the items by applying the concept of Rank Breaking to subsetwise preference data. This significantly alleviates the combinatorial explosion that could otherwise result if one were to keep more general subset-wise estimates.

**Related Work:** Statistical parameter estimation in Plackett-Luce models has been studied in detail in the offline batch (non-adaptive) setting (Chen and Suh, 2015; Khetan and Oh, 2016; Jang et al., 2017).

In the online setting, there is a fairly mature body of work concerned with PAC best-arm (or top-$\ell$ arm) identification in the classical multi-armed bandit (Even-Dar et al., 2006; Audibert and Bubeck, 2010; Kalyanakrishnan et al., 2012; Karnin et al., 2013; Jamieson et al., 2014), where absolute utility information is assumed to be revealed upon playing a single arm or item. Though most work on dueling bandits has focused on the regret minimization goal (Zoghi et al., 2014; Ramamohan et al., 2016), there have been recent developments on the PAC objective for different pairwise preference models, such as those satisfying stochastic triangle inequalities and strong stochastic transitivity (Yue and Joachims, 2011), general utility-based preference models (Urov et al., 2013), the Plackett-Luce model (Szörényi et al., 2015), the Mallows model (Busa-Fekete et al., 2014a), etc. Recent work in the PAC setting focuses on learning objectives other than identifying the single (near) best arm, e.g. recovering a few of the top arms (Busa-Fekete et al., 2013; Mohajer et al., 2017;
Chen et al., 2017), or the true ranking of the items (Busa-Fekete et al., 2014b; Falahatgar et al., 2017).

The work which is perhaps closest in spirit to ours is that of Chen et al. (2018), which addresses the problem of learning the top-$\ell$ items in Plackett-Luce battling bandits. Even when specialized to $\ell = 1$ (as we consider here), however, this work differs in several important aspects from what we attempt. Chen et al. (2018) develop algorithms for the probably exactly correct objective (recovering a near-optimal arm is not favored), and, consequently, show instance-dependent sample complexity bounds, whereas we allow a tolerance of $\epsilon$ in defining best arms, which is often natural in practice Sz"{o}r"{e}nyi et al. (2015); Yue and Joachims (2011). As a result, we bring out the dependence of the sample complexity on the specified tolerance level $\epsilon$, rather than on purely instance-dependent measures of hardness. Also, their work considers only winner information (WI) feedback from the subsets chosen, whereas we consider, for the first time, general $m$ top-ranking information feedback.

A related battling-type bandit setting has been studied as the MNL-bandits assortment-optimization problem by Agrawal et al. (2016), although it takes prices of items into account when defining their utilities, due to which the notion of a ‘best item’ is different from ours (Condorcet winner), and the two settings are in general incomparable.

2. Preliminaries

Notation. We denote by $[n]$ the set $\{1, 2, ..., n\}$. For any subset $S \subseteq [n]$, let $|S|$ denote the cardinality of $S$. When there is no confusion about the context, we often represent (an unordered) subset $S$ as a vector, or ordered subset, $S$ of size $|S|$ (according to, say, a fixed global ordering of all the items $[n]$). In this case, $S(i)$ denotes the item (member) at the $i$th position in subset $S$. $\Sigma_S = \{\sigma \mid \sigma$ is a permutation over items of $S\}$, where for any permutation $\sigma \in \Sigma_S$, $\sigma(i)$ denotes the element at the $i$-th position in $\sigma$, $i \in [|S|]$. $1(\varphi)$ is generically used to denote an indicator variable that takes the value 1 if the predicate $\varphi$ is true, and 0 otherwise. $x \vee y$ denotes the maximum of $x$ and $y$, and $Pr(A)$ is used to denote the probability of event $A$, in a probability space that is clear from the context.

2.1 Discrete Choice Models and Plackett-Luce (PL)

A discrete choice model specifies the relative preferences of two or more discrete alternatives in a given set. A widely studied class of discrete choice models is the class of Random Utility Models (RUMs), which assume a ground-truth utility score $\theta_i \in \mathbb{R}$ for each alternative $i \in [n]$, and assign a conditional distribution $\mathcal{D}_i(\cdot|\theta_i)$ for scoring item $i$. To model a winning alternative given any set $S \subseteq [n]$, one first draws a random utility score $X_i \sim \mathcal{D}_i(\cdot|\theta_i)$ for each alternative in $S$, and selects an item with the highest random score.

One widely used RUM is the Multinomial-Logit (MNL) or Plackett-Luce model (PL), where the $\mathcal{D}_i$s are taken to be independent Gumbel distributions with parameters $\theta'_i$ (Azari et al., 2012), i.e., with probability densities $\mathcal{D}_i(x_i|\theta'_i) = e^{-(x_i-\theta'_i)}e^{-e^{-(x_i-\theta'_i)}}$, $\theta'_i = \ln \theta_i$ and $\theta_i > 0, \forall i \in [n]$. In this case, the probability that alternative $i$ emerges the winner in the set $S \ni i$ is simply proportional to its (exponentiated) parameter value:

$$Pr(i|S) = \frac{\theta_i}{\sum_{j \in S} \theta_j}.$$ (1)
Clearly, any PL-based choice model induces a total ordering on the arm set \([n]\): If \(p_{ij} = P(i \succ j) = Pr(i|\{i, j\}) = \frac{\theta_i}{\theta_i + \theta_j}\) denotes the pairwise probability of item \(i\) being preferred over item \(j\), then \(p_{ij} \geq \frac{1}{2}\) if and only if \(\theta_i \geq \theta_j\), or in other words if \(p_{ij} \geq \frac{1}{2}\) and \(p_{jk} \geq \frac{1}{2}\) then \(p_{ik} \geq \frac{1}{2}\), \(\forall i, j, k \in [n]\) (Ramamohan et al., 2016).

Other families of discrete choice models can be obtained by imposing different probability distributions over the utility scores \(X_i\), e.g. if \((X_1, \ldots, X_n) \sim \mathcal{N}(\theta, \Lambda)\) are jointly normal with mean \(\theta = (\theta_1, \ldots, \theta_n)\) and covariance \(\Lambda \in \mathbb{R}^{n \times n}\), then the corresponding RUM-based choice model reduces to the Multinomial Probit (MNP). Unlike MNL, though, the choice probabilities \(Pr(i|S)\) for the MNP model do not admit a closed-form expression (Vojacek et al., 2010).

### 2.2 Independence of irrelevant alternatives

A choice model \(Pr\) is said to possess the Independence of Irrelevant Attributes (IIA) property if the ratio of probabilities of choosing any two items, say \(i_1\) and \(i_2\) from within any choice set \(S \supseteq i_1, i_2\) is independent of a third alternative \(j\) present in \(S\) (Benson et al., 2016). More specifically, \(\frac{Pr(i_1|S)}{Pr(i_2|S)} = \frac{Pr(i_1|S_1)}{Pr(i_2|S_2)}\) for any two distinct subsets \(S_1, S_2 \subseteq [n]\) that contain \(i_1\) and \(i_2\). One example of such a choice model is Plackett-Luce.

**Remark 1** IIA turns out to be very valuable in estimating the parameters of a PL model, with high confidence, via Rank-Breaking – the idea of extracting pairwise comparisons from (partial) rankings and applying estimators on the obtained pairs, treating each comparison independently. Although this technique has previously been used in batch (offline) PL estimation (Khetan and Oh, 2016), we show that it can be used in online problems for the first time. We crucially exploit this property of the PL model in the algorithms we design (Algorithms 1–3), and in establishing their correctness and sample complexity guarantees.

**Lemma 1 (Deviations of pairwise win-probability estimates for the PL model)**

Consider a Plackett-Luce choice model with parameters \(\theta = (\theta_1, \ldots, \theta_n)\), and fix two distinct items \(i, j \in [n]\). Let \(S_1, \ldots, S_T\) be a sequence of (possibly random) subsets of \([n]\) of size at least 2, where \(T\) is a positive integer, and \(i_1, \ldots, i_T\) a sequence of random items with each \(i_t \in S_t\), \(1 \leq t \leq T\), such that for each \(1 \leq t \leq T\), (a) \(S_t\) depends only on \(S_{t-1}\), (b) \(i_t\) is distributed as the Plackett-Luce winner of the subset \(S_t\), given \(S_1, i_1, \ldots, S_{t-1}, i_{t-1}\) and \(S_t\), and (c) \(\forall t : \{i, j\} \subseteq S_t\) with probability 1. Let \(n_i(T) = \sum_{t=1}^{T} 1(i_t = i)\) and \(n_{ij}(T) = \sum_{t=1}^{T} 1(\{i_t \in \{i, j\}\})\). Then, for any positive integer \(v\), and \(\eta \in (0, 1)\),

\[
Pr \left( \frac{n_i(T)}{n_{ij}(T)} - \frac{\theta_i}{\theta_i + \theta_j} \geq \eta, n_{ij}(T) \geq v \right) \vee Pr \left( \frac{n_i(T)}{n_{ij}(T)} - \frac{\theta_i}{\theta_i + \theta_j} \leq -\eta, n_{ij}(T) \geq v \right) \leq e^{-2v\eta^2}.
\]

Note: The result gives an exponential deviation inequality for the estimate \(\frac{n_i(T)}{n_{ij}(T)}\) of \(\frac{\theta_i}{\theta_i + \theta_j}\). Although it is tempting to conclude that \(\frac{n_i(T)}{n_{ij}(T)}\) is an unbiased estimate of \(\frac{\theta_i}{\theta_i + \theta_j}\), it is unclear if this holds for any finite time horizon \(T\) due to the estimate being a ratio of two random variables. In an asymptotic sense, as \(n_{ij}(T) \to \infty\), the bias can indeed be seen to vanish by a renewal theory argument.
Proof (sketch). The proof uses a novel coupling argument to work in an equivalent probability space for the PL model with respect to the item pair $i, j$, as follows. Let $Z_1, Z_2, \ldots$ be a sequence of iid Bernoulli random variables with success parameter $\theta_i/(\theta_i + \theta_j)$. A counter $C$ is first initialized to 0. At each time $t$, given $S_1, i_1, \ldots, S_{t-1}, i_{t-1}$ and $S_t$, an independent coin is tossed with probability of heads $(\theta_i + \theta_j)/\sum_{k \in S_t} \theta_k$. If the coin lands tails, then $i_t$ is drawn as an independent sample from the Plackett-Luce distribution over $S_t \setminus \{i, j\}$, else, the counter is incremented by 1, and $i_t$ is returned as $i$ if $Z(C) = 1$ or $j$ if $Z(C) = 0$. This construction yields the correct joint distribution for the sequence $i_1, S_1, \ldots, i_T, S_T$, because of the IIA property of the PL model:

$$Pr(i_t = i|i \in \{i, j\}, S_t) = \frac{Pr(i_t = i|S_t)}{Pr(i_t \in \{i, j\}|S_t)} = \frac{\theta_i / \sum_{k \in S_t} \theta_k}{(\theta_i + \theta_j) / \sum_{k \in S_t} \theta_k} = \frac{\theta_i}{\theta_i + \theta_j}.$$ 

The proof now follows by applying Hoeffding’s inequality on prefixes of the sequence $Z_1, Z_2, \ldots$. □

3. Problem Setup

We consider the PAC version of the sequential decision-making problem of finding the best item in a set of $n$ items by making subset-wise comparisons. Formally, the learner is given a finite set $[n]$ of $n > 2$ arms. At each decision round $t = 1, 2, \ldots$, the learner selects a subset $S_t \subseteq [n]$ of $k$ distinct items, and receives (stochastic) feedback depending on (a) the chosen subset $S_t$, and (b) a Plackett-Luce (PL) choice model with parameters $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ a priori unknown to the learner. The nature of the feedback can be of several types as described in Section 3.1. Without loss of generality, we will henceforth assume $\theta_i \in [0, 1], \forall i \in [n]$, since the PL choice probabilities are positive scale-invariant by (1). We also let $\theta_1 > \theta_i, \forall i \in [n] \setminus \{1\}$ for ease of exposition\(^1\). We call this decision-making model, parameterized by a PL instance $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ and a playable subset size $k$, as Battling Bandits (BB) with the Plackett-Luce (PL), or BB-PL in short. We define a best item to be one with the highest score parameter: $i^* \in \arg\max_{i \in [n]} \theta_i$. Under the assumptions above, $i^* = 1$ uniquely. Note that here we have $p_{1i} = P(1 > i) > \frac{1}{2}, \forall i \in [n] \setminus \{1\}$, so item 1 is the Condorcet Winner (Ramamohan et al., 2016) of the PL model.

3.1 Feedback models

By feedback model, we mean the information received (from the ‘environment’) once the learner plays a subset $S \subseteq [n]$ of $k$ items. We define three different types of feedback in the PL battling model:

- **Winner of the selected subset** (WI): The environment returns a single item $I \in S$, drawn independently from the probability distribution $Pr(I = i|S) = \frac{\theta_i}{\sum_{j \in S} \theta_j} \quad \forall i \in S$.

- **Full ranking selected subset of items** (FR): The environment returns a full ranking $\sigma \in \Sigma_S$, drawn from the probability distribution $Pr(\sigma = \sigma|S) = \prod_{i=1}^{\left|S\right|} \frac{\theta_{\sigma(i)}}{\sum_{j=1}^{\left|S\right|} \theta_{\sigma(j)}}, \sigma \in \Sigma_S$.

\(^1\) We naturally assume that this knowledge is not known to the learning algorithm, and note that extension to the case where several items have the same highest parameter value is easily accomplished.
Suppose $\theta \equiv (\theta_1, \ldots, \theta_n)$ and $k \leq n$ define a BB-PL instance with best arm $i^* = 1$, and $0 < \epsilon \leq 1/2$, $0 < \delta \leq 1$ are given constants. An arm $i \in [n]$ is said to be $\epsilon$-optimal\(^2\) if the probability that $i$ beats 1 is over $\frac{1}{2} - \epsilon$, i.e., if $p_{i1} := Pr(i\{1, i\}) > \frac{1}{2} - \epsilon$. A sequential algorithm that operates in this BB-PL instance, using feedback from an appropriate subset-wise feedback model (e.g., WI, FR or TR), is said to be $(\epsilon, \delta)$-PAC if (a) it stops and outputs an arm $I \in [n]$ after a finite number of decision rounds (subset plays) with probability 1, and (b) the probability that its output $I$ is an $\epsilon$-optimal arm is at least $1 - \delta$, i.e, $Pr(I \text{ is } \epsilon\text{-optimal}) \geq 1 - \delta$. Furthermore, by sample complexity of the algorithm, we mean the expected time (number of decision rounds) taken by the algorithm to stop.

Note that $p_{ij} > \frac{1}{2} + \epsilon \iff \theta_i > \frac{\theta_j + \epsilon}{\frac{1}{2} + \epsilon}$, $\forall i, j \in [n]$, so the score parameter $\theta_i$ of a near-best item must be at least $\frac{1/2 - \epsilon}{\frac{1}{2} + \epsilon}$ times $\theta_1$.

### 4. Analysis with Winner Information (WI) feedback

In this section we consider the PAC-WI goal with the WI feedback information model in BB-PL instances of size $n$ with playable subset size $k$. We start by showing that a sample complexity-lower bound for any $(\epsilon, \delta)$-PAC algorithm with WI feedback is $\Omega\left(\frac{n}{\epsilon^2} \ln \frac{1}{\delta}\right)$ (Theorem 2). This bound is independent of $k$, implying that playing a dueling game ($k = 2$) is as good as the battling game as the extra flexibility of $k$-subsetwise feedback does not result in a faster learning rate. We next propose two algorithms for $(\epsilon, \delta)$-PAC, with WI feedback, with optimal (upto a logarithmic factor) sample complexity of $O\left(\frac{n}{\epsilon^2} \ln \frac{k}{\delta}\right)$ (Sec. 4.2). We also analyze a slightly different setting allowing the learner to play subsets $S_i$ of any size $1, 2, \ldots, k$, rather than a fixed size $k$ – this gives somewhat more flexibility to the learner, resulting in algorithms with improved sample complexity guarantees of $O\left(\frac{n}{\epsilon^2} \ln \frac{1}{\delta}\right)$, without the $\ln k$ dependency as before (Sec. 4.3).

\(^2\) informally, a ‘near-best’ arm
4.1 Lower Bound for Winner Information (WI) feedback

Theorem 2 (Lower bound on Sample Complexity with WI feedback) Given $\epsilon \in (0, \frac{1}{\sqrt{8}})$ and $\delta \in (0, 1)$, and an $(\epsilon, \delta)$-PAC algorithm $A$ for BB-PL with feedback model WI, there exists a PL instance $\nu$ such that the sample complexity of $A$ on $\nu$ is at least $\Omega\left(\frac{n}{\epsilon^2}\ln(\frac{1}{\delta})\right)$.

Proof (sketch). The argument is based on a change-of-measure argument (Lemma 1) of Kaufmann et al. (2016), restated below for convenience:

Consider a multi-armed bandit (MAB) problem with $n$ arms or actions $A = [n]$. At round $t$, let $A_t$ and $Z_t$ denote the arm played and the observation (reward) received, respectively. Let $\mathcal{F}_t = \sigma(A_1, Z_1, \ldots, A_t, Z_t)$ be the sigma algebra generated by the trajectory of a sequential bandit algorithm upto round $t$.

Lemma 3 (Lemma 1, Kaufmann et al. (2016)) Let $\nu$ and $\nu'$ be two bandit models (assignments of reward distributions to arms), such that $\nu_i$ (resp. $\nu'_i$) is the reward distribution of any arm $i \in A$ under bandit model $\nu$ (resp. $\nu'$), and such that for all such arms $i$, $\nu_i$ and $\nu'_i$ are mutually absolutely continuous. Then for any almost-surely finite stopping time $\tau$ with respect to $(\mathcal{F}_t)_{t \geq 1}$

$$\sum_{i=1}^{n} \mathbb{E}_{\nu}[N_i(\tau)]KL(\nu_i, \nu'_i) \geq \sup_{\mathcal{E} \in \mathcal{F}_\tau} k(l(Pr_{\nu}(\mathcal{E}), Pr_{\nu'}(\mathcal{E}))),$$

where $k(l(x, y) := x \log(x) + (1-x) \log(\frac{1}{1-x})$ is the binary relative entropy, $N_i(\tau)$ denotes the number of times arm $i$ is played in $\tau$ rounds, and $Pr_{\nu}(\mathcal{E})$ and $Pr_{\nu'}(\mathcal{E})$ denote the probability of any event $\mathcal{E} \in \mathcal{F}_\tau$ under bandit models $\nu$ and $\nu'$, respectively.

To employ this result, note that in our case, each bandit instance corresponds to an instance of the BB-PL problem with the arm set containing all subsets of $[n]$ of size $k$: $A = \{S = (S(1), \ldots, S(k)) \subseteq [n] \mid |S(i)| < |S(j)|, \forall i < j\}$. The key part of our proof relies on carefully crafting a true instance, with optimal arm 1, and a family of slightly perturbed alternative instances $\{\nu^a : a \neq 1\}$, each with optimal arm $a \neq 1$.

We choose the true problem instance $\nu^1$ as the Plackett-Luce model with parameters

$$\theta_j = \theta \left(\frac{1}{2} - \epsilon\right), \forall j \in [n] \setminus \{1\}, \text{ and } \theta_1 = \theta \left(\frac{1}{2} + \epsilon\right), \text{ (true instance)}$$

for some $\theta \in \mathbb{R}_+, \epsilon > 0$. Corresponding to each suboptimal item $a \in [n] \setminus \{1\}$, we now define an alternative problem instance $\nu^a$ as the Plackett-Luce model with parameters

$$\theta'_j = \theta \left(\frac{1}{2} - \epsilon\right)^2, \forall j \in [n] \setminus \{a, 1\}, \theta'_1 = \theta \left(\frac{1}{4} - \epsilon^2\right), \theta'_a = \theta \left(\frac{1}{2} + \epsilon\right)^2. \text{ (alternative instance)}$$

The result of Theorem 2 is now obtained by applying Lemma 3 on pairs of problem instances $(\nu, \nu^{(a)})$, with suitable upper bounds on the KL-divergence terms, and the observation that $k(l(\delta, 1 - \delta) \geq \ln(\frac{1}{\delta})$. The complete proof is given in Appendix B.1.

Remark 2 Theorem 2 shows, rather surprisingly, that the PAC sample complexity of identifying a near-optimal item with only winner feedback information from $k$-size subsets, does not reduce with $k$, implying that there is no reduction in hardness of learning from the
pairwise comparisons case \( (k = 2) \). On one hand, one may expect to see improved sample complexity as the number of items being simultaneously tested in each round is large \((k)\). On the other hand, the sample complexity could also worsen, since it is intuitively ‘harder’ for a good (near-optimal) item to win and show itself, in just a single winner draw, against a large population of \( k - 1 \) other competitors. The result, in a sense, formally establishes that the former advantage is nullified by the latter drawback. A somewhat more formal, but heuristic, explanation for this phenomenon is that the number of bits of information that a single winner draw from a size-\( k \) subset provides is \( O(\ln k) \), which is not significantly larger than when \( k > 2 \), thus an algorithm cannot accumulate significantly more information per round compared to the pairwise case.

4.2 Algorithms for Winner Information (WI) feedback model

This section describes our proposed algorithms for the \((\epsilon, \delta)\)-PAC objective with winning item (WI) feedback.

**Principles of algorithm design.** The key idea on which all our learning algorithms are based is that of maintaining estimates of the pairwise win-loss probabilities \( p_{ij} = \Pr(i|j) \) in the Plackett-Luce model. This helps circumvent an \( O(n^k) \) combinatorial explosion that would otherwise result if we directly attempted to estimate probability distributions for each possible \( k \)-size subset. However, it is not obvious if consistent and tight pairwise estimates can be constructed in a general subset-wise choice model, but the special form of the Plackett-Luce model again comes to our rescue. The IIA property that the PL model enjoys, allows for accurate pairwise estimates via interpretation of partial preference feedback as a set of pairwise preferences, e.g., a winner \( a \) sampled from among \( a, b, c \) is interpreted as the pairwise preferences \( a \succ b, a \succ c \). Lemma 1 formalizes this property and allows us to use pairwise win/loss probability estimators with explicit confidence intervals for them.

**Algorithm 1:** (Trace-the-Best). Our first algorithm Trace-the-Best is based on the simple idea of tracing the empirical best item—specifically, it maintains a running winner \( r_\ell \) at every iteration \( \ell \), making it battle with a set of \( k - 1 \) arbitrarily chosen items. After battling long enough (precisely, for \( \frac{\delta}{2\epsilon^2} \ln \frac{2n}{\delta} \) many rounds), if the empirical winner \( c_\ell \) turns out to be more than \( \frac{1}{2} + \epsilon \)-favorable than the running winner \( r_\ell \), in term of its pairwise preference score: \( \hat{p}_{r_\ell r_\ell} > \frac{1}{2} + \epsilon \), then \( c_\ell \) replaces \( r_\ell \), or else \( r_\ell \) retains its place and status quo ensues.

**Theorem 4** (Trace-the-Best: Correctness and Sample Complexity with WI) Trace-the-Best (Algorithm 1) is \((\epsilon, \delta)\)-PAC with sample complexity \( O(\frac{n}{\epsilon^2} \log \frac{n}{\delta}) \).

**Proof (sketch).** The main idea is to retain an estimated best item as a ‘running winner’ \( r_\ell \), and compare it with the ‘empirical best item’ \( c_\ell \) of \( A \) at every iteration \( \ell \). The crucial observation lies in noting that at any iteration \( \ell, r_\ell \) gets updated as follows:

**Lemma 5** At any iteration \( \ell = 1, 2 \ldots \left\lfloor \frac{n}{k-1} \right\rfloor \), with probability at least \((1 - \frac{\delta}{2\epsilon^2})\), Algorithm 1 retains \( r_{\ell+1} \leftarrow r_\ell \) if \( p_{c_\ell r_\ell} \leq \frac{1}{2} \), and sets \( r_{\ell+1} \leftarrow c_\ell \) if \( p_{c_\ell r_\ell} \geq \frac{1}{2} + \epsilon \).

This leads to the claim that between any two successive iterations \( \ell \) and \( \ell + 1 \), we must have, with high probability, that \( p_{r_\ell r_\ell} \geq \frac{1}{2} \) and, \( p_{r_{\ell+1} c_\ell} \geq \frac{1}{2} - \epsilon \), showing that the estimated ‘best’ item \( r_\ell \) can only get improved per iteration as \( p_{r_{\ell+1} r_\ell} \geq \frac{1}{2} \) (with high probability at
least $1 - \frac{(k-1)\delta}{2n}$. Repeating this above argument for each iteration $\ell \in \left\lfloor \frac{n}{k-1} \right\rfloor$ results in the desired correctness guarantee of $p_{r_{\ell+1}} \geq \frac{1}{2} - \epsilon$. The sample complexity bound follows easily by noting the total number of possible iterations can be at most $\left\lceil \frac{n}{k-1} \right\rceil$, with the per-iteration sample complexity being $t = \frac{2k}{\epsilon^2} \ln \frac{2n}{\delta}$.

**Algorithm 1** Trace-the-Best

1: Input: 
2: Set of items: $[n]$, Subset size: $n \geq k > 1$
3: Error bias: $\epsilon > 0$, Confidence parameter: $\delta > 0$
4: Initialize:
5: $r_1 \leftarrow$ Any (random) item from $[n]$, $A \leftarrow$ Randomly select $(k-1)$ items from $[n] \setminus \{r_1\}$
6: Set $A \leftarrow A \cup \{r_1\}$, and $S \leftarrow [n] \setminus A$
7: while $\ell = 1, 2, \ldots$ do
8: Play the set $A$ for $t := \frac{2k}{\epsilon^2} \ln \frac{2n}{\delta}$ rounds
9: $w_i \leftarrow$ Number of times $i$ won in $t$ plays of $A$, $\forall i \in A$
10: Set $c_\ell \leftarrow \arg\max_{i \in A} w_i$, and $\hat{p}_{ij} \leftarrow \frac{w_i}{w_i + w_j}$, $\forall i, j \in A, i \neq j$
11: if $\hat{p}_{c_\ell, r_\ell} > \frac{1}{2} + \frac{\epsilon}{2}$, then set $r_{\ell+1} \leftarrow c_\ell$; else $r_{\ell+1} \leftarrow r_\ell$
12: if $(S = \emptyset)$ then
13: Break (exit the while loop)
14: else if $|S| < k-1$ then
15: $A \leftarrow$ Select $(k-1 - |S|)$ items from $A \setminus \{r_\ell\}$ uniformly at random
16: $A \leftarrow A \cup \{r_\ell\} \cup S$; and $S \leftarrow \emptyset$
17: else
18: $A \leftarrow$ Select $(k-1)$ items from $S$ uniformly at random
19: $A \leftarrow A \cup \{r_\ell\}$ and $S \leftarrow S \setminus A$
20: end if
21: end while
22: Output: $r_* = r_\ell$ as the $\epsilon$-optimal item

**Remark 3** The sample complexity of Trace-the-Best, is order wise optimal when $\delta < \frac{1}{n}$, as follows from our derived lower bound guarantee (Thm. 2).

When $\delta \geq \frac{1}{n}$, the sample complexity guarantee of Trace-the-Best is off by a factor of $\ln n$. We now propose another algorithm, Divide-and-Battle (Algorithm 2) that enjoys an $(\epsilon, \delta)$-PAC sample complexity of $O\left(\frac{n}{\epsilon^2} \log \frac{k}{\delta}\right)$.

**Algorithm 2**: (Divide-and-Battle). Divide-and-Battle first divides the set of $n$ items into groups of size $k$, and plays each group long enough so that a good item in the group stands out as the empirical winner with high probability (Line 11). It then retains the empirical winner per group (Line 13) and recurses on the retained set of the winners, until it is left with only a single item, which is finally declared as the $\epsilon$-optimal item. The pseudo code of Divide-and-Battle is given in Appendix B.3.

**Theorem 6** (Divide-and-Battle: Correctness and Sample Complexity with WI) Divide-and-Battle (Algorithm 2) is $(\epsilon, \delta)$-PAC with sample complexity $O\left(\frac{n}{\epsilon^2} \log \frac{k}{\delta}\right)$.

**Proof (sketch)**. The crucial observation here is that at any iteration $\ell$, for any set $G_g$ ($g = 1, 2, \ldots, G$), the item $c_g$ retained by the algorithm is likely to be not more than $\epsilon$-worse than the best item of the set $G_g$, with probability at least $(1 - \delta_\ell)$. Precisely, we show that:
Lemma 7 At any iteration \( \ell \), for any \( G_g \), if \( i_g := \arg \max_{i \in G_g} \theta_i \), then with probability at least 
\[ (1 - \delta_{\ell}), p_{c_{i_g}} > \frac{1}{2} - \epsilon_{\ell}. \]

This guarantees that, between any two successive rounds \( \ell \) and \( \ell + 1 \), we do not lose out by more than an additive factor of \( \epsilon_{\ell} \) in terms of highest score parameter of the remaining set of items. Aggregating this claim over all iterations can be made to show that \( p_{r+1} > \frac{1}{2} - \epsilon \), as desired. The sample complexity bound follows by carefully summing the total number of times \( (t = c_{\ell} \ln \frac{k}{\delta_{\ell}}) \) a set \( G_g \) is played per iteration \( \ell \), with the maximum number of possible iterations being \( \lceil \ln_k n \rceil \).

Remark 4 The sample complexity of Divide-and-Battle is order-wise optimal in the ‘small-\( \delta \)’ regime \( \delta \ll \frac{1}{k} \) by the lower bound result (Thm. 2). However, for the ‘moderate-\( \delta \)’ regime \( \delta \gtrsim \frac{1}{k} \), we conjecture that the lower bound is loose by an additive factor of \( \frac{n \ln k}{\epsilon^2} \), i.e., that a improved lower bound of \( \Omega(\frac{n \ln k}{\epsilon^2}) \) holds. This is primarily because we believe that the error probability \( \delta \) of any typical, label-invariant PAC algorithm ought to be distributed roughly uniformly across misidentification of all the items, allowing us to use \( \delta/k \) instead of \( \delta \) on the right hand side of the change-of-measure inequalities of Lemma 3, resulting in the improved quantity \( \ln(k/2.4\delta) \). This is perhaps in line with recent work in multi-armed bandits (Simchowitz et al., 2017) that points to an increased difficulty of PAC identification in the moderate-confidence regime.

We now consider a variant of the BB-PL decision model which allows the learner to play sets of any size 1, 2, …, \( k \), instead of a fixed size \( k \). In this setting, we are indeed able to design an \((\epsilon, \delta)\)-PAC algorithm that enjoys an order-optimal \( O(\frac{n}{\epsilon^2} \ln \frac{n}{\delta}) \) sample-complexity.

4.3 BB-PL2: A slightly different battling bandit decision model

The new winner information feedback model BB-PL-2 is formally defined as follows: At each round \( t \), here the learner is allowed to select a set \( S_t \subseteq [n] \) of size 2, 3, …, upto \( k \). Upon receiving any set \( S_t \), the environment returns the index of the winning item as \( I \in |S_t| \) such that, \( P(I = i|S_t) = \frac{\theta_{S(i)}}{\sum_{j=1}^{|S_t|} \theta_{S(j)}} \forall i \in |S| \).

On applying existing PAC-Dueling-Bandit strategies. Note that given the flexibility of playing sets of any size, one might as well hope to apply the PAC-Dueling Bandit algorithm \( PLPAC(\epsilon, \delta) \) of Szőrényi et al. (2015) which plays only pairs of items per round. However, their algorithm is shown to have a sample complexity guarantee of \( O\left( \frac{n}{\epsilon^2} \ln \frac{n}{\delta^2} \right) \), which is suboptimal by an additive \( O\left( \frac{n}{\epsilon^2} \ln \frac{n}{\delta^2} \right) \) as our results will show. A similar observation holds for the Beat-the-Mean (BTM) algorithm of Yue and Joachims (2011), which in fact has a even worse sample complexity guarantee of \( O\left( \frac{n}{\epsilon^2} \ln \left( \frac{n}{\epsilon^2} \ln \frac{n}{\delta^2} \right) \right) \).

Algorithm 3: Halving-Battle. We here propose a Median-Elimination-based approach (Even-Dar et al., 2006) which is shown to run with optimal sample complexity \( O\left( \frac{n}{\epsilon^2} \ln \frac{n}{\delta^2} \right) \) rounds (Theorem 8). (Note that an \( O\left( \frac{n}{\epsilon^2} \ln \frac{n}{\delta^2} \right) \) fundamental limit on PAC sample complexity for BB-PL2-WI can easily be derived using an argument along the lines of Theorem 2; we omit the explicit derivation.) The name Halving-Battle for the algorithm is because it
is based on the idea of dividing the set of items into two partitions with respect to the empirical median item and retaining the ‘better half’. Specifically, it first divides the entire item set into groups of size \(k\), and plays each group for a fixed number of times. After this step, only the items that won more than the empirical median \(h_g\) are retained and rest are discarded. The algorithm recurses until it is left with a single item. The intuition here is that some \(\epsilon\)-best item is always likely to beat the group median and can never get wiped off.

**Algorithm 3 Halving-Battle**

1: **Input:**
2:  
3: **Initialize:**
4:  
5: **while** \(\ell = 1, 2, \ldots\) **do**
6:  
7: **for** \(g = 1, 2, \ldots, G\) **do**
8:  
9: **end for**
10:  
11: **end while**
12: **Output:** \(r_*\) as the \(\epsilon\)-optimal item, where \(S = \{r_*\}\)

**Theorem 8 (Halving-Battle: Correctness and Sample Complexity with WI)** Halving-Battle (Algorithm 3) is \((\epsilon, \delta)\)-PAC with sample complexity \(O\left(\frac{n}{\epsilon^2} \log \frac{1}{\delta}\right)\).

**Proof (sketch).** The sample complexity bound follows by carefully summing the total number of times \((t = \frac{k}{2\epsilon^4} \ln \frac{1}{\delta})\) a set \(G_g\) is played per iteration \(\ell\), with the maximum number of possible iterations being \(\lceil \ln n \rceil\) (this is because the size of the set \(S\) of remaining items gets halved at each iteration as it is pruned with respect to its median).

The key intuition in proving the correctness property of Halving-Battle lies in showing that at any iteration \(\ell\), Halving-Battle always carries forward at least one ‘near-best’ item to the next iteration \(\ell + 1\).

**Lemma 9** At any iteration \(\ell\), for any set \(G_g\), let \(i_g \leftarrow \arg \max_{i \in G_g} \theta_i\), and consider any sub-optimal item \(b \in G_g\) such that \(p_{ib} < \frac{1}{2} - \epsilon_\ell\). Then with probability at least \(1 - \frac{\delta_\ell}{4}\), the empirical win count of \(i_g\) lies above that of \(b\), i.e. \(w_{i_g} \geq w_b\) (equivalently \(\hat{p}_{i_gb} = \frac{w_{i_g}}{w_{i_g} + w_b} \geq \frac{1}{2}\)).
Using the property of the median element $h_g$ along with Lem. 9 and Markov’s inequality, we show that we do not lose out more than an additive factor of $\epsilon \ell$ in terms of highest score $\theta_i$ of the remaining set of items between any two successive iterations $\ell$ and $\ell + 1$. This finally leads to the desired $(\epsilon, \delta)$-PAC correctness of Halving-Battle.

5. Analysis with Top Ranking (TR) feedback

We now proceed to analyze the BB-PL problem with Top-m Ranking (TR) feedback (Sec. 3.1). We first show that unlike WI feedback, the sample complexity lower bound here scales as $\Omega \left( \frac{n}{m^2} \ln \frac{k}{\delta} \right)$ (Thm. 10), which is a factor $m$ smaller than that in Thm. 2 for the WI feedback model. At a high level, this is because TR reveals the preference information of $m$ items per feedback step (round of battle), as opposed to just a single (noisy) information sample of the winning item (WI). Following this, we also present two algorithms for this setting which are shown to enjoy an optimal (upto logarithmic factors) sample complexity guarantee of $O \left( \frac{n}{m^2} \ln \frac{k}{\delta} \right)$ (Sec. 5.2).

5.1 Lower Bound for Top-m Ranking (TR) feedback

**Theorem 10 (Sample Complexity Lower Bound for TR)** Given $\epsilon \in (0, \frac{1}{32}]$ and $\delta \in (0, 1]$, and an $(\epsilon, \delta)$-PAC algorithm $A$ with top-m ranking (TR) feedback ($2 \leq m \leq k$), there exists a PL instance $\nu$ such that the expected sample complexity of $A$ on $\nu$ is at least $\Omega \left( \frac{n}{m^2} \ln \frac{k}{\delta} \right)$.

**Remark 5** The sample complexity lower for PAC-WI objective for BB-PL with top-m ranking (TR) feedback model is $\frac{1}{m}$-times that of the WI model (Thm. 2). Intuitively, revealing a ranking on $m$ items in a $k$-set provides about $\ln \left( \frac{k}{m} \right) = O(m \ln k)$ bits of information per round, which is about $m$ times as large as that of revealing a single winner, yielding an acceleration of $m$.

**Corollary 11** Given $\epsilon \in (0, \frac{1}{32}]$ and $\delta \in (0, 1]$, and an $(\epsilon, \delta)$-PAC algorithm $A$ with full ranking (FR) feedback ($m = k$), there exists a PL instance $\nu$ such that the expected sample complexity of $A$ on $\nu$ is at least $\Omega \left( \frac{n}{k^2} \ln \frac{1}{2.3\delta} \right)$.

5.2 Algorithms for Top-m Ranking (TR) feedback model

This section presents two algorithms for $(\epsilon, \delta)$-PAC objective for BB-PL with top-m ranking feedback. We achieve this by generalizing our earlier two proposed algorithms (see Algorithm 1 and 2, Sec. 4.2 for WI feedback) to the top-m ranking (TR) feedback mechanism.

**Rank-Breaking.** The main trick we use in modifying the above algorithms for TR feedback is Rank Breaking (Soufiani et al., 2014), which essentially extracts pairwise comparisons.

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3. Our third algorithm *Halving-Battle* is not applicable to TR feedback as it allows the learner to play sets of sizes $1, 2, 3, \ldots$ up to $k$, whereas the TR feedback is defined only when the size of the subset played is at least $m$. The lower bound analysis of Thm. 10 also does not apply if sets of size less than $m$ is allowed.
from multiwise feedback information. Formally, given any set $S$ of size $k$, if $\sigma \in \Sigma_{S_m}$, $(S_m \subseteq S, |S_m| = m)$ denotes a possible top-$m$ ranking of $S$, the Rank Breaking subroutine considers each item in $S$ to be beaten by its preceding items in $\sigma$ in a pairwise sense. For instance, given a full ranking of a set of 4 elements $S = \{a, b, c, d\}$, say $b \succ a \succ c \succ d$, Rank-Breaking generates the set of 6 pairwise comparisons: $\{(b \succ a), (b \succ c), (b \succ d), (a \succ c), (a \succ d), (c \succ d)\}$. Similarly, given the ranking of only 2 most preferred items say $b \succ a$, it yields the 5 pairwise comparisons $(b, a \succ c), (b, a \succ d)$ and $(b \succ a)$ etc. See Algorithm 4 for detailed description of the Rank-Breaking procedure.

**Lemma 12 (Rank-Breaking Update)** Consider any subset $S \subseteq [n]$ with $|S| = k$. Let $S$ be played for $t$ rounds of battle, and let $\sigma_\tau \in \Sigma_{S_{m_\tau}}$, $(S_{m_\tau} \subseteq S, |S_{m_\tau}| = m)$, denote the TR feedback at each round $\tau \in [t]$. For each item $i \in S$, let $q_i := \sum_{\tau=1}^{t} 1(i \in S'_{m_\tau})$ be the number of times $i$ appears in the top-$m$ ranked output in $t$ rounds. Then, the most frequent item(s) in the top-$m$ positions must appear at least $\frac{m t}{k}$ times, i.e. $\max_{i \in S} q_i \geq \frac{m t}{k}$.

**Algorithm 4** Rank-Breaking (for updating the pairwise win counts $w_{ij}$ for TR feedback)

1: **Input:** STATE Subset $S \subseteq [n], |S| = k$ ($n \geq k$)
2: A top-$m$ ranking $\sigma \in \Sigma_{S_m}$, $S_m \subseteq [n]$, $|S_m| = m$
3: Pairwise (empirical) win-count $w_{ij}$ for each item pair $i, j \in S$
4: while $\ell = 1, 2, \ldots m$ do
5: Update $w_{\sigma(\ell)j} \leftarrow w_{\sigma(\ell)j} + 1$, for all $i \in S \setminus \{\sigma(1), \ldots, \sigma(\ell)\}$
6: end while

**Proposed Algorithms for TR feedback.** The formal descriptions of our two algorithms, Trace-the-Best and Divide-and-Battle, generalized to the setting of TR feedback, are given as Algorithm 5 and Algorithm 6 respectively. They essentially maintain the empirical pairwise preferences $\hat{p}_{ij}$ for each pair of items $i, j$ by applying Rank Breaking on the TR feedback $\sigma$ after each round of battle. Of course in general, Rank Breaking may lead to arbitrarily inconsistent estimates of the underlying model parameters (Azari et al., 2012). However, owing to the IIA property of the Plackett-Luce model, we get clean concentration guarantees on $p_{ij}$ using Lem. 1. This is precisely the idea used for obtaining the $\frac{1}{m}$ factor improvement in the sample complexity guarantees of our proposed algorithms along with Lem. 12 (see proofs of Thm. 13 and 14).

**Theorem 13** (Trace-the-Best: Correctness and Sample Complexity with TR) With top-$m$ ranking (TR) feedback model, Trace-the-Best (Algorithm 5) is $(\epsilon, \delta)$-PAC with sample complexity $O(\frac{m}{\epsilon \delta} \log \frac{n}{\delta})$.

**Theorem 14** (Divide-and-Battle: Correctness and Sample Complexity with TR) With top-$m$ ranking (TR) feedback, Divide-and-Battle (Algorithm 6) is $(\epsilon, \delta)$-PAC with sample complexity $O(\frac{m}{\epsilon \delta} \log \frac{k}{\delta})$.

**Remark 6** The sample complexity bounds of the above two algorithms are $\frac{1}{m}$ fraction lesser than their corresponding counterparts for WI feedback, as follows comparing Thm. 4 vs. 13, or Thm. 6 vs. 14, which admit a faster learning rate with TR feedback. Similar to the case with WI feedback, sample complexity of Divide-and-Battle is still orderwise optimal for any $\delta \leq \frac{1}{k}$, as follows from the lower bound guarantee (Thm. 10). However, we believe that the above lower bound can be tightened by a factor of $\ln k$ for ‘moderate’ $\delta \geq \frac{1}{k}$, for reasons similar to those stated in Remark 4.
Algorithm 5 *Trace-the-Best* (for TR feedback)

1: **Input:**
2: Set of items: \([n]\), and subset size: \(k > 2\) (\(n \geq k \geq m\))
3: Error bias: \(\epsilon > 0\), and confidence parameter: \(\delta > 0\)
4: **Initialize:**
5: \(r_1 \leftarrow \text{Any (random) item from } [n]\), \(A \leftarrow \text{Randomly select } (k-1) \text{ items from } [n] \setminus \{r_1\}\)
6: **while** \(\ell = 1, 2, \ldots\) **do**
7: Initialize pairwise (empirical) win-count \(w_{ij} \leftarrow 0\), for each item pair \(i, j \in A\)
8: **for** \(\tau = 1, 2, \ldots t := \frac{2k}{m\epsilon} \ln \frac{2n}{\delta}\) **do**
9: Play the set \(A\) (one round of battle)
10: Receive TR feedback: \(\sigma \in \Sigma_{A^*_m}\), where \(A^*_m \subseteq A\) such that \(|A^*_m| = m\)
11: Update pairwise win-counts \(w_{ij}\) of each item pair \(i, j \in A\) using Rank-Breaking\((A, \sigma)\)
12: **end for**
13: \(B_\ell \leftarrow \text{argmax}\{i \in A | \sum_{j \in A \setminus \{i'} 1(w_{ij} \geq w_{j'i'})\}\},\)
14: \(\hat{p}_{ij} \leftarrow \frac{w_{ij}}{w_{ij} + w_{j'i}}, \forall i, j \in A, i \neq j\)
15: if \(\exists \ell \in B_\ell\) such that \(\hat{p}_{c_\ell r_\ell} > \frac{1}{2} + \frac{\epsilon}{2}\), then set \(r_{\ell+1} \leftarrow c_\ell\); else set \(r_{\ell+1} \leftarrow r_\ell\)
16: **if** \((S == \emptyset)\) **then**
17: Break (go out of the while loop)
18: **else if** \(|S| < k - 1\) **then**
19: \(A \leftarrow \text{Randomly select } (k - 1 - |S|) \text{ items from } A \setminus \{r_\ell\}\)
20: \(A \leftarrow A \cup \{r_\ell\} \cup S; \text{ and } S \leftarrow \emptyset\)
21: **else**
22: \(A \leftarrow \text{Randomly select } (k - 1) \text{ items from } S\)
23: \(A \leftarrow A \cup \{r_\ell\} \text{ and } S \leftarrow S \setminus A\)
24: **end if**
25: **end while**
26: **Output:** \(r_* = r_\ell\) as the \(\epsilon\)-optimal item

6. Conclusion and Future Directions

We have developed foundations for probably approximately correct (PAC) online learning with subset choices: introducing the problem of Battling-Bandits (BB) with subset choice models – a novel generalization of the well-studied Dueling-Bandit problem, where the objective is to find the ‘best item’ by successively choosing subsets of \(k\) alternatives from \(n\) items, and subsequently receiving a set-wise feedback information in an online fashion. We have specifically studied the Plackett-Luce (PL) choice model along with winner information (WI) and top ranking (TR) feedback, with the goal of finding an \((\epsilon, \delta)\)-PAC item: an \(\epsilon\)-approximation of the best item with probability at least \((1 - \delta)\). Our results show that with just the WI feedback, playing a battling game is just as good as that of a dueling game \((k = 2)\), as in this case the required sample complexity of the PAC learning problem is independent of the subset set \(k\). However with TR feedback, the battling framework provides a \(\frac{1}{m}\)-times faster learning rate, leading to an improved performance guarantee.
owing to the information gain with top-\(m\) ranking feedback, as intuitively well justified as well.

**Algorithm 6** Divide-and-Battle (for TR feedback)

1: **Input:**
2:     Set of items: \([n]\), and subset size: \(k > 2\) (\(n \geq k \geq m\))
3:     Error bias: \(\epsilon > 0\), and confidence parameter: \(\delta > 0\)
4: **Initialize:**
5:     \(S \leftarrow \lfloor \frac{n}{k} \rfloor\), \(\epsilon_0 \leftarrow \frac{\epsilon}{2}\), and \(\delta_0 \leftarrow \frac{\delta}{2}\)
6:     Divide \(S\) into \(G := \lceil \frac{n}{k} \rceil\) sets \(G_1, G_2, \ldots G_G\) such that \(\bigcup_{j=1}^{G_j} G_j = S\) and \(G_j \cap G_{j'} = \emptyset\), \(\forall j, j' \in [G]\), \(|G_j| = k\), \(\forall j \in [G - 1]\). If \(|G| < k\), then set \(R_1 \leftarrow G_G\) and \(G = G - 1\).
7: while \(\ell = 1, 2, \ldots\) do
8:     Set \(S \leftarrow \emptyset\), \(\delta_\ell \leftarrow \frac{\delta_\ell}{2}\), \(\epsilon_\ell \leftarrow \frac{3}{4}\epsilon_{\ell-1}\)
9:     for \(g = 1, 2, \ldots G\) do
10:        Initialize pairwise (empirical) win-count \(w_{ij} \leftarrow 0\), for each item pair \(i, j \in G_g\)
11:    for \(\tau = 1, 2, \ldots t := \lfloor \frac{4k}{m_\tau} \ln \frac{2k}{\alpha'} \rfloor\) do
12:        Play the set \(G_g\) (one round of battle)
13:        Receive feedback: The top-\(m\) ranking \(\sigma \in \Sigma_{G_{gm}}\), where \(G_{gm} \subseteq G_g\), \(|G_{gm}| = m\)
14:        Update win-count \(w_{ij}\) of each item pair \(i, j \in G_g\) using Rank-Breaking\((G_g, \sigma)\)
15:    end for
16:    Define \(\hat{p}_{i,j} = \frac{w_{ij}}{w_{ij} + w_{j,i}}\), \(\forall i, j \in G_g\)
17:    If \(\exists i \in G_g\) such that \(\hat{p}_{ij} + \frac{\epsilon_\ell}{2} \geq \frac{1}{2}, \forall j \in G_g\), then set \(c_g \leftarrow i\), else select \(c_g \leftarrow \) uniformly at random from \(G_g\), and set \(S \leftarrow S \cup \{c_g\}\)
18: end for
19: \(S \leftarrow S \cup R_\ell\)
20: if \(|S| = 1\) then
21:     Break (go out of the while loop)
22: else if \(|S| \leq k\) then
23:     \(S' \leftarrow \) Randomly sample \(k - |S|\) items from \([n] \setminus S\), and \(S \leftarrow S \cup S', \epsilon_\ell \leftarrow \frac{2\epsilon}{3}, \delta_\ell \leftarrow \delta\)
24: else
25:     Divide \(S\) into \(G := \lceil \frac{|S|}{k} \rceil\) sets \(G_1, \ldots G_G\) such that \(\bigcup_{j=1}^{G_j} G_j = S\), \(G_j \cap G_{j'} = \emptyset\), \(\forall j, j' \in [G]\), \(|G_j| = k\), \(\forall j \in [G - 1]\). If \(|G| < k\), then set \(R_{\ell+1} \leftarrow G_G\) and \(G = G - 1\).
26: end if
27: end while
28: **Output:** \(r_\ast\) as the \(\epsilon\)-optimal item, where \(S = \{r^\ast\}\)

**Future Directions.** Our proposed framework of Battling Bandits opens up a set of new directions to pursue - with different feedback mechanisms, choice models (e.g. Multinomial Probit, Mallows, nested logit, generalized extreme-value models etc.), other learning objectives, etc. It is an interesting open problem to analyse the trade-off between the subset size \(k\) and the learning rate for other choice models with different feedback mechanisms. Another relevant direction to pursue within battling bandits could be to extend it to more general settings such as revenue maximization (Agrawal et al., 2016), learning with cost budgets (Xia et al., 2016; Zhou and Tomlin, 2017), feature-based preference information and adversarial choice feedback (Gajane et al., 2015).
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Appendix A. Appendix for Section 2.2

A.1 Proof of Lemma 1

Lemma 1 (Deviations of pairwise win-probability estimates for the PL model)
Consider a Plackett-Luce choice model with parameters \( \theta = (\theta_1, \theta_2, \ldots, \theta_n) \), and fix two distinct items \( i, j \in [n] \). Let \( S_1, \ldots, S_T \) be a sequence of (possibly random) subsets of \([n]\) of size at least 2, where \( T \) is a positive integer, and \( i_1, \ldots, i_T \) a sequence of random items with each \( i_t \in S_t, 1 \leq t \leq T \), such that for each \( 1 \leq t \leq T \), (a) \( S_t \) depends only on \( S_1, \ldots, S_{t-1} \), and (b) \( i_t \) is distributed as the Plackett-Luce winner of the subset \( S_t \), given \( S_1, i_1, \ldots, S_{t-1}, i_{t-1} \) and \( S_t \), and (c) \( \forall t : \{i, j\} \subseteq S_t \) with probability 1. Let \( n_i(T) = \sum_{t=1}^{T} 1(i_t = i) \) and \( n_{ij}(T) = \sum_{t=1}^{T} 1(i_t \in \{i, j\}) \). Then, for any positive integer \( v \), and \( \eta \in (0, 1) \),

\[
\Pr \left( \frac{n_i(T)}{n_{ij}(T)} - \frac{\theta_i}{\theta_i + \theta_j} \geq \eta, \ n_{ij}(T) \geq v \right) \vee \Pr \left( \frac{n_i(T)}{n_{ij}(T)} - \frac{\theta_i}{\theta_i + \theta_j} \leq -\eta, \ n_{ij}(T) \geq v \right) \leq e^{-2\eta v^2}.
\]

Proof: We prove the lemma by using a coupling argument. Consider the following ‘simulator’ or probability space for the Plackett-Luce choice model that specifically depends on the item pair \( i, j \), constructed as follows. Let \( Z_1, Z_2, \ldots \) be a sequence of iid Bernoulli random variables with success parameter \( \theta_i/\theta_i + \theta_j \). A counter is first initialized to 0. At each time \( t \), given \( S_1, i_1, \ldots, S_{t-1}, i_{t-1} \) and \( S_t \), an independent coin is tossed with probability of heads \( \theta_i/\theta_i + \theta_j \). If the coin lands tails, then \( i_t \) is drawn as an independent sample from the Plackett-Luce distribution over \( S_t \setminus \{i, j\} \), else, the counter is incremented by 1, and \( i_t \) is returned as \( i \) if \( Z(C) = 1 \) or \( j \) if \( Z(C) = 0 \) where \( C \) is the present value of the counter.

It may be checked that the construction above indeed yields the correct joint distribution for the sequence \( i_1, S_1, \ldots, i_T, S_T \) as desired, due to the independence of irrelevant alternatives (IIA) property of the Plackett-Luce choice model:

\[
\Pr(i_t = i | i_t \in \{i, j\}, S_t) = \frac{\Pr(i_t = i | S_t)}{\Pr(i_t \in \{i, j\} | S_t)} = \frac{\theta_i / \sum_{k \in S_t} \theta_k}{(\theta_i + \theta_j) / \sum_{k \in S_t} \theta_k} = \frac{\theta_i}{\theta_i + \theta_j}.
\]

Furthermore, \( i_t \in \{i, j\} \) if and only if \( C \) is incremented at round \( t \), and \( i_t = i \) if and only if \( C \) is incremented at round \( t \) and \( Z(C) = 1 \). We thus have

\[
\Pr \left( \frac{n_i(T)}{n_{ij}(T)} - \frac{\theta_i}{\theta_i + \theta_j} \geq \eta, \ n_{ij}(T) \geq v \right) = \Pr \left( \sum_{\ell=1}^{n_{ij}(T)} \frac{Z_{\ell}}{n_{ij}(T)} - \frac{\theta_i}{\theta_i + \theta_j} \geq \eta, \ n_{ij}(T) \geq v \right)
\]

\[
= \sum_{m=v}^{T} \Pr \left( \sum_{\ell=1}^{m} \frac{Z_{\ell}}{m} - \frac{\theta_i}{\theta_i + \theta_j} \geq \eta, \ n_{ij}(T) = m \right)
\]

\[
= \sum_{m=v}^{T} \Pr \left( \sum_{\ell=1}^{m} Z_{\ell} \geq \eta, \ n_{ij}(T) = m \right)
\]

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\[
\begin{align*}
\text{(a)} & \sum_{m=v}^{T} Pr \left( \frac{\sum_{\ell=1}^{m} Z_{\ell}}{m} - \frac{\theta_i}{\theta_i + \theta_j} \geq \eta \right) Pr(n_{ij}(T) = m) \\
\text{(b)} & \leq \sum_{m=v}^{T} Pr(n_{ij}(T) = m) e^{-2m\eta^2} \leq e^{-2\eta^2},
\end{align*}
\]

where (a) uses the fact that \(S_1, \ldots, S_T, X_1, \ldots, X_T\) are independent of \(Z_1, Z_2, \ldots,\) and so \(n_{ij}(T) \in \sigma(S_1, \ldots, S_T, X_1, \ldots, X_T)\) is independent of \(Z_1, \ldots, Z_m\) for any fixed \(m,\) and (b) uses Hoeffding’s concentration inequality for the iid sequence \(Z_i.\)

Similarly, one can also derive
\[
Pr\left(\frac{n_i(T)}{n_{ij}(T)} - \frac{\theta_i}{\theta_i + \theta_j} \leq -\eta, n_{ij}(T) \geq \nu\right) \leq e^{-2\eta^2},
\]

which concludes the proof. \(\blacksquare\)

Appendix B. Appendix for Section 4

B.1 Proof of Theorem 2

Theorem 2 (Lower bound on Sample Complexity with WI feedback) Given \(\epsilon \in (0, \frac{1}{\sqrt{8}},\) and \(\delta \in (0, 1],\) and an \((\epsilon, \delta)-PAC\) algorithm \(A\) for BB-PL with feedback model WI, there exists a PL instance \(\nu\) such that the sample complexity of \(A\) on \(\nu\) is at least
\[
\Omega\left(\frac{n^2}{\epsilon^2 \ln \frac{1}{2\delta}}\right).
\]

Proof We will apply Lemma 3 to derive the desired lower bounds of Theorem 2 for BB-PL with WI feedback model.

Let us consider a bandit instance with the arm set containing all subsets of size \(k:\n\]
\[
\mathcal{A} = \{S = (S(1), \ldots S(k)) \subseteq [n] \mid S(i) < S(j), \forall i < j\}.\]

Let \(\nu^1\) be the true distribution associated with the bandit arms, given by the Plackett-Luce parameters:

**True Instance** \((\nu^1) : \theta_j^1 = \theta\left(\frac{1}{2} - \epsilon\right), \forall j \in [n] \setminus \{1\}, \) and \(\theta_1^1 = \theta\left(\frac{1}{2} + \epsilon\right),\)

for some \(\theta \in \mathbb{R}_+, \epsilon > 0.\) Now for every suboptimal item \(a \in [n] \setminus \{1\},\) consider the modified instances \(\nu^a\) such that:

**Instance–a** \((\nu^a) : \theta_j^a = \theta\left(\frac{1}{2} - \epsilon\right)^2, \forall j \in [n] \setminus \{a, 1\}, \) \(\theta_1^a = \theta\left(\frac{1}{4} - \epsilon^2\right),\) and \(\theta_a^a = \theta\left(\frac{1}{2} + \epsilon\right)^2.\)

For problem instance \(\nu^a, a \in [n] \setminus \{1\},\) the probability distribution associated with arm \(S \in \mathcal{A}\) is given by

\[\nu_S^a \sim \text{Categorical}(p_1, p_2, \ldots, p_k),\] where \(p_i = Pr(i|S), \forall i \in [k], \forall S \in \mathcal{A},\)

where \(Pr(i|S)\) is as defined in Section 3.1. Note that the only \(\epsilon\)-optimal arm for **Instance-a** is arm \(a.\) Now applying Lemma 3, for some event \(E \in \mathcal{F}_r\) we get,
\[
\sum_{\{S \in A:a \in S\}} E_{\nu^1}[N_S(\tau_A)]KL(\nu^1_S, \nu^2_S) \geq kl(P_{\nu}(\mathcal{E}), P_{\nu'}(\mathcal{E})).
(2)
\]

The above result holds from the straightforward observation that for any arm \(S \in A\) with \(a \notin S\), \(\nu^1_S\) is same as \(\nu^a_S\), hence \(KL(\nu^1_S, \nu^2_S) = 0\), \(\forall S \in A, a \notin S\). For notational convenience, we will henceforth denote \(S^a = \{S \in A:a \in S\}\).

Now let us analyse the left hand side of (2) for \(r = 1(1 \in S)\), \(q = (k - 1 - r)\), and \(R = \frac{1 + \epsilon}{2^r}\). Note that

\[
\nu^1_S(i) = \begin{cases} 
\frac{R}{rR + (k - r)} & \text{if } S(i) = 1, \\
\frac{R^2}{rR + q} & \text{if } S(i) = a, \\
\frac{R}{rR + q} & \text{otherwise.}
\end{cases}
\]

On the other hand, for problem \textbf{Instance-a}, we have that:

\[
\nu^2_S(i) = \begin{cases} 
\frac{R}{rR + (k - r) + q} & \text{if } S(i) = 1, \\
\frac{R^2}{rR + q} & \text{if } S(i) = a, \\
\frac{R}{rR + q} & \text{otherwise.}
\end{cases}
\]

Now using the following upper bound on \(KL(p_1, p_2) \leq \sum_{x \in \mathcal{X}} p_2(x) - 1\), \(p_1\) and \(p_2\) be two probability mass functions on the discrete random variable \(\mathcal{X}\) (Popescu et al., 2016) we get:

\[
KL(\nu^1_S, \nu^2_S) \leq \frac{rR + R^2 + q}{(rR + k - r)^2} (rR + \frac{1}{R} + q) - 1.
\]

Replacing \(q\) by \((k - 1 - r)\) and re-arranging terms, we get

\[
KL(\nu^1_S, \nu^2_S) \leq \frac{(rR + k - r) + (R^2 - 1)(rR + k - r) + (R^{-2} - 1)}{(rR + k - r)^2} - 1
= \frac{(rR + k - r - 1)}{(rR + k - r)^2} \left(R - \frac{1}{R}\right)^2 \leq \frac{1}{k} \left(R - \frac{1}{R}\right)^2 [\text{since } s \geq 0, \text{ and } R > 1].
(3)
\]

Note that the only \(\epsilon\)-optimal arm for any \textbf{Instance-a} is arm \(a\), for all \(a \in [n]\). Now, consider \(\mathcal{E}_0 \in \mathcal{F}_r\) be an event such that the algorithm \(A\) returns the element \(i = 1\), and let us analyse the left hand side of (2) for \(\mathcal{E} = \mathcal{E}_0\). Clearly, \(A\) being an \((\epsilon, \delta)\)-PAC algorithm, we have \(P_{r\nu}(\mathcal{E}_0) > 1 - \delta\), and \(P_{r\nu'}(\mathcal{E}_0) < \delta\), for any suboptimal arm \(a \in [n] \setminus \{1\}\). Then we have

\[
kl(P_{r\nu}(\mathcal{E}_0), P_{r\nu'}(\mathcal{E}_0)) \geq kl(1 - \delta, \delta) \geq \frac{1}{2.4\delta}
(4)
\]
where the last inequality follows from Kaufmann et al. (2016, Equation (3)).

Now applying (2) for each modified bandit \( \nu^a \), and summing over all suboptimal items \( a \in [n] \setminus \{1\} \) we get,

\[
\sum_{a=2}^{n} \sum_{\{S \in A| a \in S\}} E_{\nu^1}[N_S(\tau_A)]KL(\nu^1_S, \nu^2_S) \geq (n - 1) \frac{1}{2.4\delta}.
\]  \( (5) \)

Moreover, using (3), the term of the right hand side of (5) can be further upper bounded as

\[
\sum_{a=2}^{n} \sum_{\{S \in A| a \in S\}} E_{\nu^1}[N_S(\tau_A)]KL(\nu^1_S, \nu^2_S) \leq \sum_{S \in A} E_{\nu^1}[N_S(\tau_A)] \sum_{\{a \in S| a \neq 1\}} \frac{1}{k} \left( R - \frac{1}{R} \right)^2
\]

\[
= \sum_{S \in A} E_{\nu^1}[N_S(\tau_A)] \frac{k - (1(1 \in S))}{k} \left( R - \frac{1}{R} \right)^2
\]

\[
\leq \sum_{S \in A} E_{\nu^1}[N_S(\tau_A)](256\epsilon^2) \quad \text{[since \( R - \frac{1}{R} = \frac{8\epsilon}{(1 - 4\epsilon^2)} \leq 16\epsilon, \forall \epsilon \in [0, \frac{1}{\sqrt{8}}] \).} \]  \( (6) \)

Finally noting that \( \tau_A = \sum_{S \in A}[N_S(\tau_A)] \), combining (6) and (5), we get

\[
(256\epsilon^2)E_{\nu^1}[\tau_A] = \sum_{S \in A} E_{\nu^1}[N_S(\tau_A)](256\epsilon^2) \geq (n - 1) \frac{1}{2.4\delta}.
\]

Thus above construction shows the existence of a problem instance \( \nu = \nu^1 \), such that \( E_{\nu^1}[\tau_A] = \Omega(\frac{n}{\epsilon^2} \frac{1}{2.4\delta}) \), which concludes the proof.

\[ \square \]

**B.2 Proof of Theorem 4**

**Theorem 4 (Trace-the-Best: Correctness and Sample Complexity with WI)** Trace-the-Best (Algorithm 1) is \((\epsilon, \delta)-\text{PAC} \) with sample complexity \( O(\frac{n}{\epsilon^2} \log \frac{1}{\delta}) \).

**Proof** We start by analyzing the required sample complexity first. Note that the ‘while loop’ of Algorithm 1 always discards away \( k - 1 \) items per iteration. Thus, \( n \) being the total number of items the loop can be executed is at most for \( \left\lceil \frac{n}{k-1} \right\rceil \) many number of iterations. Clearly, the sample complexity of each iteration being \( t = \frac{2k}{\epsilon^2} \ln \frac{2\delta}{n} \) (as follows from Line 8), the total sample complexity of the algorithm becomes \( \left( \left\lceil \frac{n}{k-1} \right\rceil \right) \frac{2k}{\epsilon^2} \ln \frac{2\delta}{n} \leq \left( \frac{n}{k-1} + 1 \right) \frac{2k}{\epsilon^2} \ln \frac{2\delta}{n} = (n + \frac{n}{k-1}) \frac{2k}{\epsilon^2} \ln \frac{2\delta}{n} = O(\frac{n}{\epsilon^2} \log \frac{1}{\delta}) \).

We now prove the \((\epsilon, \delta)-\text{PAC} \) correctness of the algorithm. As argued before, the ‘while loop’ of Algorithm 1 can run for maximum \( \left\lceil \frac{n}{k-1} \right\rceil \) many number of iterations. We denote the iterations by \( \ell = 1, 2, \ldots \left\lceil \frac{n}{k-1} \right\rceil \), and the corresponding set \( A \) of iteration \( \ell \) by \( A_\ell \).
Note that our idea is to retain the estimated best item in ‘running winner’ \( r_\ell \) and compare it with the ‘empirical best item’ \( c_\ell \) of \( A_\ell \) at every iteration \( \ell \). The crucial observation lies in noting that at any iteration \( \ell \), \( r_\ell \) gets updated as follows:

**Lemma 5** At any iteration \( \ell = 1, 2 \ldots \lfloor \frac{n}{k-1} \rfloor \), with probability at least \((1 - \frac{\delta}{2n})\), Algorithm 1 retains \( r_{\ell+1} \leftarrow r_\ell \) if \( p_{c_\ell r_\ell} \leq \frac{1}{2} \), and sets \( r_{\ell+1} \leftarrow c_\ell \) if \( p_{c_\ell r_\ell} \geq \frac{1}{2} + \epsilon \).

**Proof** Consider any set \( A_\ell \), by which we mean the state of \( A \) in the algorithm at iteration \( \ell \). The crucial observation to make is that since \( c_\ell \) is the empirical winner of \( t \) rounds of battle, then \( w_{c_\ell} \geq \frac{t}{k} \). Thus \( w_{c_\ell} + w_{r_\ell} \geq \frac{t}{k} \). Let \( n_{ij} := w_i + w_j \) denotes the total number of pairwise comparisons between item \( i \) and \( j \) in \( t \) rounds, for any \( i, j \in A_\ell \). Then clearly, \( 0 \leq n_{ij} \leq t \) and \( n_{ij} = n_{ji} \). Specifically we have \( \hat{p}_{rrc} = \frac{w_{r_\ell}}{w_{r_\ell} + w_{c_\ell}} = \frac{w_{r_\ell}}{n_{r_\ell c_\ell}} \). We prove the claim by analyzing the following cases:

**Case 1.** (If \( p_{c_\ell r_\ell} \leq \frac{1}{2} \), Trace-the-Best retains \( r_{\ell+1} \leftarrow r_\ell \)): Note that Trace-the-Best replaces \( r_{\ell+1} \) by \( c_\ell \) only if \( \hat{p}_{c_\ell r_\ell} > \frac{1}{2} + \frac{\epsilon}{2} \), but this happens with probability:

\[
Pr \left( \left\{ \hat{p}_{c_\ell r_\ell} > \frac{1}{2} + \frac{\epsilon}{2} \right\} \right) = Pr \left( \left\{ \hat{p}_{c_\ell r_\ell} > \frac{1}{2} + \frac{\epsilon}{2} \right\} \cap \left\{ n_{c_\ell r_\ell} \geq \frac{t}{k} \right\} \right) + Pr \left( \left\{ \hat{p}_{c_\ell r_\ell} < \frac{t}{k} \right\} \right) Pr \left( \left\{ \hat{p}_{c_\ell r_\ell} > \frac{1}{2} + \frac{\epsilon}{2} \right\} \right) \cap \left\{ n_{c_\ell r_\ell} < \frac{t}{k} \right\} \right) \\
\leq Pr \left( \left\{ \hat{p}_{c_\ell r_\ell} - p_{c_\ell r_\ell} > \frac{\epsilon}{2} \right\} \cap \left\{ n_{c_\ell r_\ell} \geq \frac{t}{k} \right\} \right) \leq \exp \left( -2 \frac{t}{k} \left( \frac{\epsilon}{2} \right)^2 \right) = \frac{\delta}{2n},
\]

where the first inequality follows as \( p_{c_\ell r_\ell} \leq \frac{1}{2} \), and the second inequality is by applying Lem. 1 with \( \eta = \frac{\epsilon}{2} \) and \( v = \frac{t}{k} \). We now proceed to the second case:

**Case 2.** (If \( p_{c_\ell r_\ell} \geq \frac{1}{2} + \epsilon \), Trace-the-Best sets \( r_{\ell+1} \leftarrow c_\ell \)): Recall again that Trace-the-Best retains \( r_{\ell+1} \leftarrow r_\ell \) only if \( \hat{p}_{c_\ell r_\ell} \leq \frac{1}{2} + \frac{\epsilon}{2} \). This happens with probability:

\[
Pr \left( \left\{ \hat{p}_{c_\ell r_\ell} \leq \frac{1}{2} + \frac{\epsilon}{2} \right\} \right) = Pr \left( \left\{ \hat{p}_{c_\ell r_\ell} \leq \frac{1}{2} + \frac{\epsilon}{2} \right\} \cap \left\{ n_{c_\ell r_\ell} \geq \frac{t}{k} \right\} \right) + Pr \left( \left\{ \hat{p}_{c_\ell r_\ell} < \frac{t}{k} \right\} \right) Pr \left( \left\{ \hat{p}_{c_\ell r_\ell} \leq \frac{1}{2} + \frac{\epsilon}{2} \right\} \right) \cap \left\{ n_{c_\ell r_\ell} < \frac{t}{k} \right\} \right) \\
= Pr \left( \left\{ \hat{p}_{c_\ell r_\ell} \leq \frac{1}{2} + \frac{\epsilon}{2} \right\} \cap \left\{ n_{c_\ell r_\ell} \geq \frac{t}{k} \right\} \right) \\
\leq Pr \left( \left\{ \hat{p}_{c_\ell r_\ell} - p_{c_\ell r_\ell} \leq \frac{\epsilon}{2} \right\} \cap \left\{ n_{c_\ell r_\ell} \geq \frac{t}{k} \right\} \right) \leq \exp \left( -2 \frac{t}{k} \left( \frac{\epsilon}{2} \right)^2 \right) = \frac{\delta}{2n},
\]

where the first inequality holds as \( p_{c_\ell r_\ell} \geq \frac{1}{2} + \epsilon \), and the second one by applying Lem. 1 with \( \eta = \frac{\epsilon}{2} \) and \( v = \frac{t}{k} \). Combining the above two cases concludes the proof. ■
Given Algorithm 1 satisfies Lemma 5, and taking union bound over \((k-1)\) elements in 
\(A_\ell \setminus \{r_\ell\}\), we get that with probability at least \(1 - \frac{(k-1)\delta}{2n}\),

\[
p_{r_{\ell+1} r_\ell} \geq \frac{1}{2} \quad \text{and} \quad p_{r_{\ell+1} c_\ell} \geq \frac{1}{2} - \epsilon.
\] (7)

Above suggests that for each iteration \(\ell\), the estimated ‘best’ item \(r_\ell\) only gets improved as \(p_{r_{\ell+1} r_\ell} \geq \frac{1}{2}\). Let, \(\ell_*\) denotes the specific iteration such that \(1 \in A_\ell\) for the first time, i.e. 
\(\ell_* = \min\{\ell \mid 1 \in A_\ell\}\). Clearly \(\ell_* \leq \lceil \frac{n}{k-1} \rceil\). Now (7) suggests that with probability at least \((1 - \frac{(k-1)\delta}{2n})\), \(p_{r_{\ell_*+1} r_\ell} \geq \frac{1}{2} - \epsilon\). Moreover (7) also suggests that for all \(\ell > \ell_*\), with probability at least \((1 - \frac{(k-1)\delta}{2n})\), \(p_{r_{\ell_*+1} r_\ell} \geq \frac{1}{2}\), which implies for all \(\ell > \ell_*\), \(p_{r_{\ell_*+1} r_\ell} \geq \frac{1}{2} - \epsilon\) as well – This holds due to the following transitivity property of the Plackett-Luce model: For any three items \(i_1, i_2, i_3 \in [n]\), if \(p_{i_1 i_2} \geq \frac{1}{2}\) and \(p_{i_2 i_3} \geq \frac{1}{2}\), then we have \(p_{i_1 i_3} \geq \frac{1}{2}\) as well.

This argument finally leads to \(p_{r_{\ell_*} r_\ell} \geq \frac{1}{2} - \epsilon\). Since failure probability at each iteration \(\ell\) is at most \((\frac{(k-1)\delta}{2n})\), and Algorithm 1 runs for maximum \(\lceil \frac{n}{k-1} \rceil\) many iterations, using union bound over \(\ell\), the total failure probability of the algorithm is at most
\(\lceil \frac{n}{k-1} \rceil \frac{(k-1)\delta}{2n} \leq (\frac{n}{k-1} + 1) \frac{(k-1)\delta}{2n} = \delta \left( \frac{n + (k-1)}{2n} \right) \leq \delta\) (since \(k \leq n\)). This concludes the correctness of the algorithm showing that it indeed satisfies the \((\epsilon, \delta)\)-PAC objective.

**B.3 Proof of Theorem 6**

**Theorem 6** (*Divide-and-Battle: Correctness and Sample Complexity with WI*) *Divide-and-Battle* (Algorithm 2) is \((\epsilon, \delta)\)-PAC with sample complexity \(O\left(\frac{n}{\epsilon^2} \log \frac{1}{\delta}\right)\).

**Proof** For the notational convenience we will use \(\tilde{p}_{ij} = p_{ij} - \frac{1}{2}\), \(\forall i, j \in [n]\). We start by proving the following lemma which would be used crucially in the analysis:

**Lemma 15** For any three items \(a, b, c \in [n]\) such that \(\theta_a > \theta_b > \theta_c\). If \(\tilde{p}_{ba} > -\epsilon_1\), and \(\tilde{p}_{cb} > -\epsilon_2\), where \(\epsilon_1, \epsilon_2 > 0\), and \((\epsilon_1 + \epsilon_2) < \frac{1}{2}\), then \(\tilde{p}_{ca} > -(\epsilon_1 + \epsilon_2)\).

**Proof** Note that \(\tilde{p}_{ba} > -\epsilon_1 \implies \frac{\theta_a - \theta_b}{2(\theta_b + \theta_a)} > -\epsilon_1 \implies \frac{\theta_a}{\theta_b} > \frac{(1-2\epsilon_1)}{(1+2\epsilon_1)}\).

Similarly we have \(\tilde{p}_{cb} > -\epsilon_2 \implies \frac{\theta_c}{\theta_b} > \frac{(1-2\epsilon_2)}{(1+2\epsilon_2)}\). Combining above we get

\[
\frac{\theta_c}{\theta_a} > \frac{(1-2\epsilon_1)(1-2\epsilon_2)}{(1+2\epsilon_1)(1+2\epsilon_2)} = \frac{1 - 2(\epsilon_1 + \epsilon_2) + \epsilon_1 \epsilon_2}{1 + 2(\epsilon_1 + \epsilon_2) + \epsilon_1 \epsilon_2} > \frac{1 - 2(\epsilon_1 + \epsilon_2)}{1 + 2(\epsilon_1 + \epsilon_2)} \quad \left(\text{since}, (\epsilon_1 + \epsilon_2) < \frac{1}{2}\right)
\]

\[
\implies \tilde{p}_{ca} = \frac{\theta_c - \theta_a}{2(\theta_c + \theta_a)} > -(\epsilon_1 + \epsilon_2),
\]

which concludes the proof.

We now analyze the required sample complexity of *Divide-and-Battle*. For clarity of notation, we will denote the set \(S\) at iteration \(\ell\) by \(S_\ell\). Note that at any iteration \(\ell\), any set
Algorithm 2 *Divide-and-Battle*

1: **Input:**
2: Set of items: \([n]\), Subset size: \(n \geq k > 1\)
3: Error bias: \(\epsilon > 0\), Confidence parameter: \(\delta > 0\)
4: **Initialize:**
5: \(S \leftarrow [n]\), \(e_0 \leftarrow \frac{\delta}{8}\), and \(\delta_0 \leftarrow \frac{\delta}{2}\)
6: Divide \(S\) into \(G := \left\lfloor \frac{n}{k} \right\rfloor\) sets \(G_1, G_2, \ldots, G_r\) such that \(\bigcup_{j=1}^{G} G_j = S\) and \(G_j \cap G_{j'} = \emptyset\), \(\forall j, j' \in [G]\), where \(|G_j| = k\), \(\forall j \in [G-1]\)
7: **If** \(|G_G| < k\), **then** set \(R_1 \leftarrow G_G\) and \(G = G - 1\)
8: **while** \(\ell = 1, 2, \ldots\) **do**
9: Set \(S \leftarrow \emptyset\), \(\delta_\ell \leftarrow \frac{\delta}{2^{\ell-1}}, \epsilon_\ell \leftarrow \frac{\epsilon}{2^\ell}\)
10: **for** \(g = 1, 2, \ldots, G\) **do**
11: Play the set \(G_g\) for \(t := \frac{k}{2\epsilon_\ell} \ln \frac{k}{\delta_\ell}\) rounds
12: \(w_i \leftarrow \text{Number of times } i \text{ won in } t \text{ plays of } G_g, \forall i \in G_g\)
13: Set \(c_g \leftarrow \arg \max_{i \in A} w_i\) and \(S \leftarrow S \cup \{c_g\}\)
14: **end for**
15: \(S \leftarrow S \cup R_\ell\)
16: **if** \(|S| = 1\) **then**
17: Break (go out of the while loop)
18: **else if** \(|S| \leq k\) **then**
19: \(S' \leftarrow \text{Randomly sample } k - |S| \text{ items from } [n] \setminus S\), and \(S \leftarrow S \cup S', \epsilon_\ell \leftarrow \frac{2\epsilon}{3}, \delta_\ell \leftarrow \delta\)
20: **else**
21: Divide \(S\) into \(G := \left\lfloor \frac{|S|}{k} \right\rfloor\) sets \(G_1, G_2, \ldots, G_r\), such that \(\bigcup_{j=1}^{G} G_j = S\), and \(G_j \cap G_{j'} = \emptyset\), \(\forall j, j' \in [G]\), where \(|G_j| = k\), \(\forall j \in [G-1]\)
22: **If** \(|G_G| < k\), **then** set \(R_{\ell+1} \leftarrow G_G\) and \(G = G - 1\)
23: **end if**
24: **end while**
25: **Output:** \(r^*_s\) as the \(\epsilon\)-optimal item, where \(S = \{r^*_s\}\)

\(G_g\) is played for exactly \(t = \frac{k}{2\epsilon_\ell} \ln \frac{k}{\delta_\ell}\) many number of rounds. Also since the algorithm discards away exactly \(k - 1\) items from each set \(G_g\), hence the maximum number of iterations possible is \([\ln_k n]\). Now at any iteration \(\ell\), since \(G = \left\lceil \frac{|S|}{k} \right\rceil < \frac{|S|}{k}\), the total sample complexity for iteration \(\ell\) is at most \(\frac{|S|}{k} t \leq \frac{k}{2\epsilon_\ell} \ln \frac{k}{\delta_\ell}\), as \(|S_\ell| \leq \frac{n}{k^2}\) for all \(\ell \in [\ln_k n]\).

Also note that for all but last iteration \(\ell \in [\ln_k n]\), \(\epsilon_\ell = \frac{\epsilon}{2^\ell}\), and \(\delta_\ell = \frac{\delta}{2^{\ell-1}}\). Moreover for the last iteration \(\ell = [\ln_k n]\), the sample complexity is clearly \(t = \frac{2k}{\epsilon^2} \ln \frac{2k}{\delta}\), as in this case \(\epsilon_\ell = \frac{\epsilon}{2}, \delta_\ell = \frac{\delta}{2}, \) and \(|S| = k\). Thus the total sample complexity of Algorithm 2 is given by

\[
\sum_{\ell=1}^{\left\lceil \ln_k n \right\rceil} \frac{|S_{\ell}|}{2e_\ell^2} \ln \frac{k}{\delta_\ell} \leq \sum_{\ell=1}^{\left\lceil \ln_k n \right\rceil} \frac{n}{2k^{\ell+1}} \frac{\epsilon^2 \ln \frac{2k}{\delta}}{\delta} + \frac{2k}{\epsilon^2} \ln \frac{2k}{\delta}
\]
Lemma 7. At any iteration $\ell$, for any $G_g$, if $i_g := \arg\max_{i \in G_g} \theta_i$, then with probability at least $(1 - \delta_\ell)$, \( p_{c_g i_g} > \frac{1}{2} - \epsilon_\ell \).

Proof. Let us define \( \hat{p}_{ij} = \frac{w_i}{w_i + w_j} \), \( \forall i, j \in G_g, i \neq j \). Then clearly \( \hat{p}_{c_g i_g} \geq \frac{1}{2} \), as \( c_g \) is the empirical winner in \( t \) rounds, i.e. \( c_g \leftarrow \arg\max_i w_i \) (Line 13). Moreover \( c_g \) being the empirical winner of \( G_g \) we also have \( w_{c_g} \geq \frac{t}{k} \), and thus \( w_{c_g} + w_{i_g} \geq \frac{2t}{k} \) as well. Let \( n_{ij} := w_i + w_j \) denotes the number of pairwise comparisons of item \( i \) and \( j \) in \( t \) rounds, \( i, j \in G_g \). Clearly \( 0 \leq n_{ij} \leq t \). Then let us analyze the probability of a ‘bad event’ where \( c_g \) is indeed such that \( p_{c_g i_g} < \frac{1}{2} - \epsilon_\ell \) but we have \( c_g \) beating \( i_g \) empirically:

\[
Pr\left( \left\{ \hat{p}_{c_g i_g} \geq \frac{1}{2} \right\} \right) \\
= Pr\left( \left\{ \hat{p}_{c_g i_g} \geq \frac{1}{2} \right\} \cap \left\{ n_{c_g i_g} \geq \frac{t}{k} \right\} \right) + Pr\left\{ n_{c_g i_g} < \frac{t}{k} \right\} Pr\left( \left\{ \hat{p}_{c_g i_g} \geq \frac{1}{2} \right\} \left| \left\{ n_{c_g i_g} < \frac{t}{k} \right\} \right. \right) \\
= Pr\left( \left\{ \hat{p}_{c_g i_g} - \epsilon_\ell \geq \frac{1}{2} - \epsilon_\ell \right\} \cap \left\{ n_{c_g i_g} \geq \frac{t}{k} \right\} \right) \\
\leq Pr\left( \left\{ \hat{p}_{c_g i_g} - p_{c_g i_g} \geq \epsilon_\ell \right\} \cap \left\{ n_{c_g i_g} \geq \frac{t}{k} \right\} \right) \\
\leq \exp\left( -2\frac{t}{k}(\epsilon_\ell)^2 \right) = \frac{\delta_\ell}{k},
\]

where the first inequality holds as \( p_{c_g i_g} < \frac{1}{2} - \epsilon_\ell \), and the second inequality is by applying Lem. 1 with \( \eta = \epsilon_\ell \) and \( v = \frac{t}{k} \). Now taking union bound over all \( \epsilon_\ell \)-suboptimal elements \( i' \) of \( G_g \) (i.e. \( p_{i' i_g} < \frac{1}{2} - \epsilon_\ell \)), we get

\[
Pr\left( \left\{ \exists i' \in G_g | p_{i' i_g} < \frac{1}{2} - \epsilon_\ell, \text{and} c_g = i' \right\} \right) \leq \frac{\delta_\ell}{k} \left\{ \exists i' \in G_g | p_{i' i_g} < \frac{1}{2} - \epsilon_\ell, \text{and} c_g = i' \right\} \leq \delta_\ell,
\]

as \( |G_g| = k \), and the claim follows henceforth.
Remark 7 For the last iteration $\ell = \lceil \ln k \rceil$, since $\epsilon_\ell = \frac{\delta}{\ell}$, and $\delta_\ell = \frac{\delta}{2}$, applying Lemma 7 on $S$, we get that $\Pr \left( \ell_{r, i_0} < \frac{1}{2} - \frac{\delta}{2} \right) \leq \frac{\delta}{2}$.

Now for each iteration $\ell$, let us define $g_\ell \in [G]$ to be the set that contains best item of the entire set $S$, i.e. $\arg \max_{i \in S} \theta_i \in G_{g_\ell}$. Then applying Lemma 7, with probability at least $(1 - \delta_\ell)$, $\tilde{p}_{c_{g_\ell} \ell} > -\epsilon_\ell$. Then, for each iteration $\ell$, applying Lemma 15 and Lemma 7 to $G_{g_\ell}$, we finally get $\tilde{p}_{r, 1} > -\left( \frac{\delta}{n} + \frac{\delta}{8} \left( \frac{3}{4} \right)^1 + \cdots + \frac{\delta}{2} \left( \frac{3}{4} \right)^{\lfloor \ln k \rfloor} \right)$, since $\ell_0 = \min\{l \mid 1 \notin R_l, l \geq 1\}$ because prior considering item 1, it does not matter even if the algorithm mistakes in any of the iteration $\ell < \ell_0$. Thus assuming the algorithm does not fail in any of the iteration $\ell$, we finally have that $\tilde{p}_{r, 1} > \frac{1}{2} - \epsilon$.

Finally since at each iteration $\ell$, the algorithm fails with probability at most $\delta_\ell$, the total failure probability of the algorithm is at most $\left( \frac{\delta}{8} + \frac{\delta}{8} + \cdots + \frac{\delta}{2^{\lfloor \ln k \rfloor}} \right) + \frac{\delta}{2} \leq \delta$. This concludes the correctness of the algorithm showing that it indeed satisfies the $(\epsilon, \delta)$-PAC objective.

B.4 Proof of Theorem 8

**Theorem 8** (Halving-Battle: Correctness and Sample Complexity with WI) Halving-Battle (Algorithm 3) is $(\epsilon, \delta)$-PAC with sample complexity $O\left( \frac{n}{\epsilon^2} \log \frac{1}{\delta} \right)$.

**Proof** For clarity of notation, we will denote the set of remaining items $S$ at iteration $\ell$ by $S_\ell$. We start by observing that at each iteration $\ell = 1, 2, \ldots$, the size of the set of remaining items $S_{\ell+1}$ gets halved compared to that of the previous iteration $S_\ell$, since the algorithm discards away all the elements below the median item $h_g$ (Line 12), as follows from the definition of median. This implies that the maximum number of iterations possible is $\ell = \lceil \ln n \rceil$, after which $|S| = 1$ and the algorithm returns $r_\ast$ (Line 15, 20).

We first analyze the sample complexity of the algorithm. Clearly each iteration $\ell$ uses a sample complexity of $t = \frac{k}{2\ell} \ln \frac{1}{\delta_\ell}$ (Line 10), and as argued before $\ell$ can be at most $\lceil \ln n \rceil$ which makes the total sample complexity of the algorithm:

\[
\sum_{\ell=1}^{\lceil \ln n \rceil} \frac{|S_\ell|}{2\ell} \ln \frac{1}{\delta_\ell} \leq \sum_{\ell=1}^{\lceil \ln n \rceil} \frac{n}{2k\ell} \left( \frac{4}{\ell} \right)^{\ell-1} \ln \frac{4}{\ell} \leq \frac{16n}{2\ell^2} \sum_{\ell=1}^{\lceil \ln n \rceil} \frac{16^{\ell-1}}{(9k)^{\ell-1}} \left( \ln \frac{4}{\ell} + \ell \right) \\
\leq \frac{8n}{\epsilon^2} \ln \frac{4}{\delta} \sum_{\ell=1}^{\lceil \ln n \rceil} \frac{16^{\ell-1}}{9k^{\ell-1}} (2\ell) = O\left( \frac{n}{\epsilon^2} \ln \frac{1}{\delta} \right) \text{ [for any } k > 1\text{].}
\]

This ensures the sample complexity of Thm. 8 holds good.

We are now only left with verifying the $(\epsilon, \delta)$-PAC property of the algorithm where lies the main difference of the analysis of Halving-Battle from Divide-and-Battle. Consider any iteration $\ell \in \lceil \lceil \ln n \rceil \rceil$. The crucial observation is that, with high probability of at least
\[(1 - \delta_\ell)\] for any such \(\ell\), and any set \(G_g (g = 1, 2, \ldots G)\), some \(\epsilon_\ell\)-approximation of the 'best-item' (the one with the highest score parameter \(\theta_i\)) of \(G_g\) must lie above the median in terms of the empirical win count \(w_i\), and hence must be retained by the algorithm till the next iteration \(\ell + 1\). We prove this formally below.

Our first claim starts by showing that for any set \(G_g\), the empirical win count estimate \(w_i\) of the best item \(i_g := \arg \max_{i \in G_g} \theta_i\) (i.e. the one with highest score parameter \(\theta_i\)) can not be too small, as shown in Lemma 16:

**Lemma 16** Consider any particular set \(G_g\) at any iteration \(\ell \in [\ln n]\). If \(i_g := \arg \max_{i \in G_g} \theta_i\), then with probability atleast \((1 - \frac{\delta_\ell}{4})\), the empirical win count \(w_{i_g} > (1 - \eta) \frac{1}{\ell}\), for any \(\eta \in (\frac{3}{16}, 1]\).

**Proof** The proof follows from an straightforward application of Chernoff-Hoeffding’s inequality Boucheron et al. (2013). Note that the algorithm plays each set \(G_g\) for \(t = \frac{1}{2\epsilon_\ell} \ln \frac{1}{\delta_\ell}\) number of times. Fix any iteration \(\ell\) and a set \(G_g\), \(g \in 1, 2, \ldots G\). Suppose \(i_\tau\) denotes the winner of \(\tau\)-th play of \(G_g\), \(\tau \in [t]\). Then clearly, for any item \(i \in G_g\), \(w_i = \sum_{\tau=1}^{t} 1(i_\tau == i)\), where \(1(i_\tau == i)\) is a Bernoulli random variable with parameter \(\frac{\theta_i}{\sum_{j \in G_g} \theta_j}\), by definition of WI feedback model. Also for \(i = i_g\), we have \(Pr(\{i_\tau == i_\gamma\}) = \frac{\theta_i}{\sum_{j \in G_g} \theta_j} \geq \frac{1}{k}, \forall \tau \in [t]\), as follows from the definition \(i_g := \arg \max_{i \in G_g} \theta_i\). Hence \(E[w_{i_g}] = \sum_{\tau=1}^{t} E[1(i_\tau == i)] \geq \frac{1}{k}\). Now applying multiplicative Chernoff-Hoeffdings bound for \(w_{i_g}\), we get that for any \(\eta \in (\frac{3}{16}, 1]\),

\[
Pr\left( w_{i_g} \leq (1 - \eta) E[w_{i_g}] \right) \leq \exp \left( - \frac{E[w_{i_g}]\eta^2}{2} \right) \leq \exp \left( - \frac{t\eta^2}{2k} \right) \leq \exp \left( - \frac{\eta^2}{\epsilon_\ell^2} \ln \left( \frac{4}{\delta_\ell} \right) \right) \leq \exp \left( - \ln \left( \frac{4}{\delta_\ell} \right) \right) = \frac{\delta_\ell}{4},
\]

where the second last inequality holds as \(\eta > \frac{3}{16}\) and \(\epsilon_\ell \leq \frac{3}{16}\), for any iteration \(\ell \in [\ln n]\); in other words for any \(\eta \geq \frac{1}{4}\), we have \(\frac{\eta}{\epsilon_\ell} > 1\) which leads to the second last inequality, and the proof follows henceforth. \(\square\)

In particular, fixing \(\eta = \frac{1}{4}\) in Lemma 16, we get that with probability atleast \((1 - \frac{\delta_\ell}{4})\), \(w_{i_g} > (1 - \frac{1}{4}) E[w_{i_g}] > \frac{t}{2k}\). We now prove that for any set \(G_g\), given its best item \(i_g\) is selected as the winner for atleast \(\frac{t}{2k}\) times out of \(t\) plays of \(G_g\), the empirical estimate of \(p_{i_gb}\), defined as \(\hat{p}_{i_g} = \frac{w_{i_g}}{w_{i_g} + w_b}\), for any suboptimal element \(b \in G_g\) (such that \(p_{i_gb} > \frac{1}{2} + \epsilon\)) can not be too misleading where empirical win count of \(b\) exceeds that of \(i_g\), i.e. \(w_b > w_{i_g}\). The formal claim is as follows:

**Lemma 9** At any iteration \(\ell\), for any set \(G_g\), let \(i_g \leftarrow \arg \max_{i \in G_g} \theta_i\), and consider any suboptimal item \(b \in G_g\) such that \(p_{i_gb} < \frac{1}{2} - \epsilon_\ell\). Then with probability at least \((1 - \frac{\delta_\ell}{4})\), the empirical win count of \(i_g\) lies above that of \(b\), i.e. \(w_{i_g} \geq w_b\) (equivalently \(\hat{p}_{i_g} = \frac{w_{i_g}}{w_{i_g} + w_b} \geq \frac{1}{2}\)).
Proof First note since \( w_{ib} \geq \frac{t}{2k} \), this implies \( w_i + w_j \geq \frac{t}{2k} \) as well. Let us define \( n_{ij} = w_i + w_j \) to be the number of pairwise comparisons of item \( i \) and \( j \) in \( t \) rounds, for any \( i, j \in \mathcal{G}_g \), and \( \hat{p}_{ij} = \frac{w_i}{w_i + w_j} \) to be the empirical estimate of pairwise probability of item \( i \) and \( j \). Then,

\[
Pr\left( \left\{ w_b \geq w_{ig} \right\} \cap \left\{ n_{ig} \geq \frac{t}{2k} \right\} \right) = Pr\left( \left\{ \hat{p}_{bi} \geq \frac{1}{2} \right\} \cap \left\{ n_{ig} \geq \frac{t}{2k} \right\} \right) \\
= Pr\left( \left\{ \hat{p}_{bi} - \epsilon_{\ell} \geq \frac{1}{2} - \epsilon_{\ell} \right\} \cap \left\{ n_{ig} \geq \frac{t}{2k} \right\} \right) \\
\leq Pr\left( \left\{ \hat{p}_{bi} - p_{bi} \geq \epsilon_{\ell} \right\} \cap \left\{ n_{ig} \geq \frac{t}{2k} \right\} \right) \\
\leq \exp\left( -2 \frac{t}{2k} (\epsilon_{\ell})^2 \right) \leq \frac{\delta_{\ell}}{4},
\]

where the second last inequality holds since \( p_{bi} < \frac{1}{2} - \epsilon_{\ell} \). The last inequality follows by applying Lem. 1 with \( \eta = \epsilon_{\ell} \) and \( v = \frac{t}{2k} \).

Using the results from Lemma 16 we further get that for any such suboptimal element \( b \in C_g \) with \( p_{bi} < \frac{1}{2} - \epsilon_{\ell} \),

\[
Pr\left( \left\{ w_b \geq w_{ig} \right\} \right) \\
= Pr\left( \left\{ w_b \geq w_{ig} \right\} \cap \left\{ w_{ig} < \frac{t}{2k} \right\} \right) + Pr\left( \left\{ w_b \geq w_{ig} \right\} \cap \left\{ w_{ig} \geq \frac{t}{2k} \right\} \right) \\
\leq Pr\left( \left\{ w_{ig} < \frac{t}{2k} \right\} \right) + Pr\left( \left\{ w_b \geq w_{ig} \right\} \cap \left\{ n_{ig} \geq \frac{t}{2k} \right\} \right) \\
\leq \frac{\delta_{\ell}}{4} + \frac{\delta_{\ell}}{4} \left[ \text{Applying Lem. 16 with } \eta = \frac{1}{2} \right] \leq \frac{\delta_{\ell}}{2}
\]

Now for any particular \( \mathcal{G}_g \), and for all suboptimal element \( b \in \mathcal{G}_g \), let us define an indicator random variable \( F_b := 1( w_b > w_{ig} ) \). Note that by above claim we have \( E[F_b] = Pr(F_b) = Pr(w_b > w_{ig}) \leq \frac{\delta_{\ell}}{2} \). Moreover if \( B = \{ b \in \mathcal{G}_g \mid p_{bi} < \frac{1}{2} - \epsilon \} \) denotes the set of all \( \epsilon_{\ell} \)-suboptimal elements of \( \mathcal{G}_g \) (with respect to the best item \( i_g \) of \( \mathcal{G}_g \)), then clearly \( |B| < |\mathcal{G}_g| \), and thus we have \( E[\sum_{b \in B} F_b] \leq |\mathcal{G}_g| \frac{\delta_{\ell}}{2} \). Now using Markov’s inequality Boucheron et al. (2013) we get:

\[
Pr\left[ \sum_{b \in B} F_b \geq \frac{|\mathcal{G}_g|}{2} \right] \leq \frac{E[\sum_{b \in B} F_b]}{|\mathcal{G}_g|/2} \leq \frac{|\mathcal{G}_g|}{|\mathcal{G}_g|/2} \frac{\delta_{\ell}}{2} = \delta_{\ell}.
\]

1 Above immediately implies that at any iteration \( \ell \), and for any set \( \mathcal{G}_g \) in \( \ell \), more than \( \frac{|\mathcal{G}_g|}{2} \) of the suboptimal elements of \( \mathcal{G}_g \) can not beat the best item \( i_g \) in terms of empirical
Appendix C. Appendix for Section 5

C.1 Proof of Theorem 10

Theorem 10 (Sample Complexity Lower Bound for TR) Given \( \epsilon \in (0, \frac{1}{32}] \) and \( \delta \in (0, 1] \), and an \((\epsilon, \delta)\)-PAC algorithm \( A \) with top-\( m \) ranking (TR) feedback \((2 \leq m \leq k)\), there exists a PL instance \( \nu \) such that the expected sample complexity of \( A \) on \( \nu \) is at least

\[
\Omega \left( \frac{n}{m \epsilon^2} \ln \frac{1}{2\delta} \right).
\]

Proof In this case too, we will use Lemma 3 to derive the desired lower bounds of Theorem 2 for BB-PL with TR feedback model.

Let us consider a bandit instance with the arm set containing all subsets of size \( k \):

\[ A = \{ S = \{S(1), \ldots, S(k)\} \mid S(i) < S(j), \forall i < j \}. \]

Let \( \nu^1 \) be the true distribution associated with the bandit arms, given by the Plackett-Luce parameters:

\[
\text{True Instance } (\nu^1): \theta_j^1 = \theta \left( \frac{1}{2} - \epsilon \right), \forall j \in [n] \setminus \{1\}, \text{ and } \theta_1^1 = \theta \left( \frac{1}{2} + \epsilon \right),
\]

for some \( \theta \in \mathbb{R}_+ \), \( \epsilon > 0 \). Now for every suboptimal item \( a \in [n] \setminus \{1\} \), consider the modified instances \( \nu^a \) such that:

\[
\text{Instance–a } (\nu^a): \theta_j^a = \theta \left( \frac{1}{2} - \epsilon \right), \forall j \in [n] \setminus \{a, 1\}, \theta_1^a = \theta \left( \frac{1}{4} - \epsilon^2 \right), \text{ and } \theta_a^a = \theta \left( \frac{1}{2} + \epsilon \right)^2.
\]

Note that for any problem instance say \( \nu^a \), \( a \in [n] \), each arm \( S \subseteq [n] \) is associated to \( \binom{k}{m} (m!) \) number of possible outcomes, each representing one possible ranking of set of \( m \) items of \( S \), say \( S_m \). Also the probability of any permutation \( \sigma \in \Sigma S_m \) is given by

\[
\nu^S_\sigma(\sigma) = Pr_{\nu^a}(\sigma|S),
\]

where \( Pr_{\nu^a}(\sigma|S) \) is as defined in Section 3.1. For ease of analysis let us first assume \( 1 \notin S \). Thus we get

...
\[ \nu^1_S(\sigma) = \prod_{i=1}^{m} \frac{\theta_{\sigma(i)}}{\sum_{j=1}^{m} \theta_{\sigma(j)} + \sum_{j \in S, \sigma(j)} \theta_{\sigma(j)}} = \frac{1}{k(k-1)(k-2) \cdots (k-m+1)}. \]

On the other hand, for problem **Instance-a**, we have that:

\[ \nu^2_S(\sigma) = \frac{R^2}{(R^2 + k - 1)(R^2 + k - 2) \cdots (R^2 + k - i)(k - i - 1) \cdots (k - m + 1)}, \]

for any \( \sigma \in \Sigma_S \) such that \( \sigma(i) = a, \forall i \in [m] \) Similarly applying Lemma 3, for some event \( E \in F_r \) we get,

\[ \sum_{\{S \in A : a \in S\}} E_{r1} [N_S(\tau_A)] KL(\nu^1_S, \nu^2_S) \geq kl(Pr_{r}(E), Pr_{r'}(E)). \tag{8} \]

The above result holds from the straightforward observation that for any arm \( S \in A \) with \( a \notin S \), \( \nu^1_S \) is same as \( \nu^2_S \), hence \( KL(\nu^1_S, \nu^2_S) = 0, \forall S \in A, a \notin S \). For the notational convenience we will henceforth denote \( S^a = \{S \in A : a \in S\} \).

We further denote \( R = \frac{1}{2} + \frac{\epsilon}{2} \). Using above, we now analyse the right hand side of (8), for any set \( S \in S^a \). Now using the following upper bound on \( KL(p, q) \leq \sum_{x \in V} p(x)^2 \leq 1 \), \( p \) and \( q \) be two probability mass functions on the discrete random variable \( X \) Popescu et al. (2016) we get:

\[
KL(\nu^1_S, \nu^2_S) \leq \sum_{i=1}^{m} \sum_{\{\sigma \in \Sigma_S | \sigma(i) = a\}} \frac{(\nu^1_S(\sigma))^2}{\nu^2_S(\sigma)} - 1
\]

\[
= \sum_{i=1}^{m} \sum_{\{\sigma \in \Sigma_S | \sigma(i) = a\}} \frac{1}{(k(k-1) \cdots (k-m+1))^2} \frac{(R^2 + k - 1)(R^2 + k - 2) \cdots (R^2 + k - i)(k - i - 1) \cdots (k - m + 1)}{R^2} - 1
\]

\[
= \frac{1}{k} \sum_{i=1}^{m} \frac{(R^2 + k - 1)(R^2 + k - 2) \cdots (R^2 + k - i)(k - i - 1) \cdots (k - m + 1)}{k(k-1) \cdots (k-m+1)} \frac{1}{(k-1) \cdots (k-i+1)} - 1
\]

\[
= \frac{1}{kR^2} \sum_{i=1}^{m} \frac{(R^2 + k - 1)(R^2 + k - 2) \cdots (R^2 + k - i)}{k(k-1) \cdots (k-i+1)} - 1
\]

\[
= \frac{1}{kR^2} \sum_{i=1}^{m} \frac{(R^2 + k - 1)(R^2 + k - 2) \cdots (R^2 + k - i)}{k(k-1) \cdots (k-i+1)} - 1
\]
\[
= \frac{1}{kR^2} \sum_{i=1}^{m} \left( 1 + \frac{R^2 - 1}{k} \right) \left( 1 + \frac{R^2 - 1}{k-1} \right) \cdots \left( 1 + \frac{R^2 - 1}{k-i+1} \right) - 1
\]

\[
= \frac{1}{kR^2} \sum_{i=1}^{m} \left( 1 + (R^2 - 1) \left( \frac{1}{k} + \frac{m-1}{k-1} + \cdots + \frac{1}{k-i+1} \right) + (R^2 - 1)^2 \left( \frac{1}{k(k-1)} + \cdots + \frac{1}{(k-i+2)(k-i+1)} \right) + \cdots + (R^2 - 1)^i \left( \frac{1}{k(k-1)(k-2)} + \cdots + \frac{1}{(k-i+3)(k-i+2)(k-i+1)} \right) + \cdots + \left( \frac{(R^2 - 1)^i}{k \cdots (k-i+1)} \right) \right) - 1
\]

\[
= \frac{1}{kR^2} \left[ m + (R^2 - 1) \left( \frac{m}{k} + \frac{m-1}{k-1} + \cdots + \frac{1}{k-m+1} \right) + (R^2 - 1)^2 \left( \frac{m-1}{k(k-1)} + \cdots + \frac{1}{(k-m+2)(k-m+1)} \right) + \cdots + (R^2 - 1)^i \left( \frac{m-2}{k(k-1)(k-2)} + \cdots + \frac{1}{(k-m+3)(k-m+2)(k-m+1)} \right) + \cdots + \left( \frac{(R^2 - 1)^m}{k \cdots (k-m+1)} \right) \right] - 1
\]

\[
\leq \frac{1}{kR^2} \left[ m + m((R^2 - 1) + (R^2 - 1)^2 + \cdots (R^2 - 1)^m) \right] - 1
\]

\[
\leq \frac{m}{k} \left( R - \frac{1}{R} \right)^2 + \frac{m}{kR^2} \left[ (R^2 - 1)^3 + (R^2 - 1)^4 + \cdots + (R^2 - 1)^m \right]
\]

\[
\leq \frac{m}{k} \left( R - \frac{1}{R} \right)^2 + \frac{m}{k} \left[ \frac{(R^2 - 1)^3 + (R^2 - 1)^5 + \cdots}{(R^2 - 1)^2} \right] \quad \text{[since, } R^2 \geq 1] \]

\[
\leq \frac{m}{k} \left( R - \frac{1}{R} \right)^2 + \frac{m}{k} \left[ \frac{(16\epsilon)^3 + (16\epsilon)^5 + \cdots}{(1 - (16\epsilon)^2)} \right] \quad \text{[since, } (R^2 - 1) = \frac{8\epsilon}{(1 - 2\epsilon)^2} \leq 16\epsilon, \forall \epsilon \in [0, \frac{1}{32}] \]

\[
\leq \frac{m}{k} \left( (16\epsilon)^2 + (16\epsilon)^3 \frac{1}{(1 - (16\epsilon)^2)} \right) \leq \frac{m}{k} 2(16\epsilon)^2 = \frac{m}{k} (512\epsilon^2),
\]

where the second last inequality follows from the fact that \( \left( R - \frac{1}{R} \right)^2 = \frac{8\epsilon}{(1 - 4\epsilon^2)} \leq 16\epsilon, \)

and \( (R^2 - 1) = \frac{8\epsilon}{(1 - 2\epsilon)^2} \leq 16\epsilon < 1, \) for any \( \epsilon \in [0, \frac{1}{32}] \). The last inequality follows from \( \frac{16\epsilon}{(1 - (16\epsilon)^2)} \leq 1, \) whenever \( \epsilon \leq \frac{1}{32} \). An exact similar analysis also gives the same result for any set \( S \in S^a \) such that \( 1 \in S \). Hence we show that, for any \( \epsilon \in [0, \frac{1}{32}] \), and any \( S \in S^a \),

\[
KL(\nu^1_S, \nu^0_S) \leq \frac{m}{k} 512\epsilon^2.
\]

Note that the only \( \epsilon \)-optimal arm for any **Instance-a** is arm \( a \), for all \( a \in [n] \). Now, consider \( E_0 \in \mathcal{F}_r \) be an event such that the algorithm \( A \) returns the element \( i = 1 \), and let us analyse the left hand side of (8) for \( E = E_0 \). Clearly, \( A \) being an \( (\epsilon, \delta) \)-PAC algorithm, we have \( Pr_{\nu^1}(E_0) > 1 - \delta \), and \( Pr_{\nu^0}(E_0) < \delta \), for any suboptimal arm \( a \in [n] \setminus \{1\} \). Then we have

\[
kl(Pr_{\nu^1}(E_0), Pr_{\nu^0}(E_0)) \geq kl(1 - \delta, \delta) \geq \frac{1}{2.4\delta}
\]

where the last inequality follows due to Equation (3) of Kaufmann et al. (2016).
Now applying (8) for each modified bandit Instance-$\nu^a$, and summing over all suboptimal items $a \in [n] \setminus \{1\}$ we get,

$$\sum_{a=2}^{n} \sum_{S \in A | a \in S} \mathbb{E}_{\nu^a} [N_S(\tau_A)] KL(\nu^1_S, \nu^a_S) \geq (n-1) \frac{1}{2.4\delta}. \quad (11)$$

Moreover, using (9), the term of the right hand side of (11) can be further upper bounded as:

$$\sum_{a=2}^{n} \sum_{S \in A | a \in S} \mathbb{E}_{\nu^a} [N_S(\tau_A)] KL(\nu^1_S, \nu^a_S) \leq \sum_{S \in A} \mathbb{E}_{\nu^1} [N_S(\tau_A)] \sum_{\{a \in S | a \neq 1\}} m \frac{512\epsilon^2}{k}$$

$$= \sum_{S \in A} \mathbb{E}_{\nu^1} [N_S(\tau_A)] (k-1(1 \in S)) \left( \frac{m}{k} \frac{512\epsilon^2}{k} \right) \leq \sum_{S \in A} \mathbb{E}_{\nu^1} [N_S(\tau_A)] m \left( \frac{512\epsilon^2}{k} \right) \quad (12)$$

Finally noting that $\tau_A = \sum_{S \in A} [N_S(\tau_A)]$, combining (12) and (11), we get

$$m \left( \frac{512\epsilon^2}{k} \right) \mathbb{E}_{\nu^1} [\tau_A] = \sum_{S \in A} \mathbb{E}_{\nu^1} [N_S(\tau_A)] m \left( \frac{512\epsilon^2}{k} \right) \geq (n-1) \frac{1}{2.4\delta}.$$

Thus above construction shows the existence of a problem instance $\nu = \nu^1$, such that $\mathbb{E}_{\nu^1} [\tau_A] = \Omega(\frac{n}{m\epsilon^2} \frac{1}{2.4\delta})$, which concludes the proof.

C.2 Proof of Lemma 12

Lemma 12 (Rank-Breaking Update) Consider any subset $S \subseteq [n]$ with $|S| = k$. Let $S$ be played for $t$ rounds of battle, and let $\sigma_\tau \in \Sigma_{S_m^\tau}$, $(S_m^\tau \subseteq S, |S_m^\tau| = m)$, denote the TR feedback at each round $\tau \in [t]$. For each item $i \in S$, let $q_i := \sum_{\tau=1}^{t} \mathbb{1}(i \in S_m^\tau)$ be the number of times $i$ appears in the top-$m$ ranked output in $t$ rounds. Then, the most frequent item(s) in the top-$m$ positions must appear at least $\frac{mt}{k}$ times, i.e. $\max_{i \in S} q_i \geq \frac{mt}{k}$.

Proof Let us denote $\hat{i} := \arg \max_{i \in S} q_i$ to be the item (note that it need not be unique) that appears in the top-$m$ set for maximum number of times in $t$ rounds of battle. Note that, after the battle of any round $\tau \in [t]$, $\sigma_\tau$ chooses exactly $m$ distinct items in the top-$m$ set $S_m^\tau \subseteq S$. Thus $t$ rounds of feedback places exactly $mt$ items in the top-$m$ slots, i.e. $\sum_{i \in S} q_i = mt$. Now at any round $\tau$, since an item $i \in S$ can appear in $S_m^\tau$ at most once, and $\sum_{i \in S} q_i = mt$, item $\hat{i}$ must be selected for at least $\frac{mt}{k}$ many rounds in the top-$m$ set implying that $q_{\hat{i}} \geq \frac{mt}{k}$ (as we have $|S| = k$).
C.3 Proof of Theorem 13

Theorem 13 (Trace-the-Best: Correctness and Sample Complexity with TR) With top-\(m\) ranking (TR) feedback model, Trace-the-Best (Algorithm 5) is \((\epsilon, \delta)\)-PAC with sample complexity \(O(\frac{m}{m^*} \log \frac{2}{\delta})\).

Proof We start by analyzing the required sample complexity first. Note that the ‘while loop’ of Algorithm 5 always discards away \(k - 1\) items per iteration. Thus, \(n\) being the total number of items, the ‘while loop’ can be executed for at most \(\left\lceil \frac{n}{k-1} \right\rceil\) many number of iterations. Clearly, the sample complexity of each iteration being \(t = \frac{2k}{m^*} \ln \frac{2n}{\delta}\) (Line 11), the total sample complexity of the algorithm becomes \(\left(\left\lceil \frac{n}{k-1} \right\rceil\right) \frac{2k}{m^*} \ln \frac{2n}{\delta} \leq \left(\frac{n}{k-1} + 1\right) \frac{2k}{m^*} \ln \frac{2n}{m^*} = (n + \frac{n}{k-1} + k) \frac{2k}{m^*} \ln \frac{2n}{\delta} = O(\frac{n}{m^*} \ln \frac{\delta}{n})\).

We now proceed to prove the \((\epsilon, \delta)\)-PAC correctness of the algorithm. As argued before, the ‘while loop’ of Algorithm 5 can run for maximum \(\left\lceil \frac{n}{k-1} \right\rceil\) many number of iterations, say \(\ell = 1, 2, \ldots \left\lceil \frac{n}{k-1} \right\rceil\), and let us denote the corresponding set \(A_{\ell}\) of iteration \(\ell\) as \(A_{\ell}\). Same as before, our idea is to retain the estimated best item as the ‘running winner’ in \(r_{\ell}\) and compare it with the ‘empirical best item’ of \(A_{\ell}\) at every \(\ell\). We start by noting the following important property of item \(c_{\ell}\) for any iteration \(\ell\):

Lemma 17 Suppose \(q_i := \sum_{t=1}^{\ell} 1(i \in A_{\ell}^t)\) denotes the number of times item \(i\) appeared in the top-\(m\) ranking in \(t\) iterations, and let \(B_{\ell} \subseteq A_{\ell}\) is defined as \(B_{\ell} := \{i \in A_{\ell} \mid q_i = \max_{j \in A_{\ell}} q_j\}\), that denotes the subset of items in \(A_{\ell}\) which are selected in the top-\(m\) ranking for maximum number of times in \(t\) rounds of battle on set \(A_{\ell}\). Then \(c_{\ell} \in B_{\ell}\).

Proof We prove the claim by contradiction. Suppose, \(c_{\ell} \notin B_{\ell}\) and consider any item \(i \in B_{\ell}\). Then by definition, \(q_i > q_{c_{\ell}}\). But in that case following our rank breaking update (see Algorithm 4) implies that \(w_{i \cap_{\ell}} > w_{c_{\ell} \cap_{\ell}}\), since item \(i\) is ranked higher than item \(c_{\ell}\) for at least \((q_i - q_{c_{\ell}}) > 0\) many rounds of battle. Now consider any other item \(j \in A_{\ell}\). Note that \(j\) can belong to either of these two cases:

**Case 1.** \((j \notin B_{\ell})\) Following the same argument as above (i.e. for \(i\) vs \(c_{\ell}\)) we again have \(w_{ij} > w_{ij}\), whereas for \(c_{\ell}\) vs \(j\), either \(w_{c_{\ell} j} > w_{j c_{\ell}}\), or \(w_{c_{\ell} j} < w_{j c_{\ell}}\), both cases are plausible. Thus we get: \(1(w_{ij} > w_{ij}) = 1 \geq 1(w_{c_{\ell} j} > w_{j c_{\ell}})\).

**Case 2.** \((j \in B_{\ell})\) In this case since \(j \in B_{\ell}\) again following the same argument as for \(i\) vs \(c_{\ell}\), we here have \(w_{ij} > w_{c_{\ell} j}\); whereas for \(i\) vs \(j\), either \(w_{ij} > w_{i j}\), or \(w_{ij} < w_{i j}\), both cases are plausible. Thus we get: \(1(w_{ij} > w_{ij}) \geq 0 = 1(w_{c_{\ell} j} > w_{j c_{\ell}})\).

Combining the results of Case 1 and 2 along with the fact that \(w_{c_{\ell} c_{\ell}} > w_{c_{\ell} c_{\ell}}\), we get \(\sum_{j \in A_{\ell} \setminus \{i\}} 1(w_{ij} \geq w_{ij}) > \sum_{j \in A_{\ell} \setminus \{c_{\ell}\}} 1(w_{c_{\ell} j} \geq w_{j c_{\ell}})\). But this violates the fact that \(c_{\ell}\) is defined as \(c_{\ell} := \arg \max_{i \in A_{\ell}} \sum_{j \in A_{\ell} \setminus \{i\}} 1(w_{ij} \geq w_{ij})\) (Line 15) which leads to a contradiction.

Then our initial assumption has to be wrong and \(c_{\ell} \in B_{\ell}\), which concludes the proof.

The next crucial observation lies in noting that, the estimated best item \(r\) (‘running winner’) gets updated as per the following lemma:

Lemma 18 At any iteration \(\ell = 1, 2, \ldots \left\lceil \frac{n}{k-1} \right\rceil\), for any set \(A_{\ell}\), with probability at least \((1 - \frac{\delta}{2n})\), Algorithm 1 retains \(r_{\ell+1} \leftarrow r_{\ell}\) if \(p_{c_{\ell} c_{\ell}} \leq \frac{1}{2}\), and set \(r_{\ell+1} \leftarrow c_{\ell}\) if \(p_{c_{\ell} r_{\ell}} \geq \frac{1}{2} + \epsilon\).
Proof The main observation lies in proving that at any iteration $\ell$, $w_{cr_\ell} + w_{ri_\ell} \geq \frac{mt}{k}$. We argue this as follows: Firstly note that by Lemma 12 and 17, $c_\ell \in B_\ell$ (Lem. 17) and hence it must have appeared in top-$m$ positions for at least $\frac{mt}{k}$ times (Lem 12). But the rank breaking update (Line 13) ensures that every element in top-$m$ position gets updated for exactly $k$ times (it loses to all elements preceding it in the top-$m$ ranking and wins over the rest). Define $n_{ij} = w_{ij} + w_{ji}$ to be the number of times item $i$ and $j$ are compared after rank-breaking, $i,j \in A_\ell$. Clearly $0 \leq n_{ij} \leq tk$ and $n_{ij} = n_{ji}$. Now using above argument we have that $n_{cr_\ell} = w_{cr_\ell} + w_{ri_\ell} \geq \frac{mt}{k}$. We are now proof the claim using the following two case analyses:

Case 1. (If $p_{cr_\ell} \leq \frac{1}{2}$, Trace-the-Best retains $r_{\ell+1} \leftarrow r_\ell$): Note that Trace-the-Best replaces $r_{\ell+1}$ by $c_\ell$ only if $p_{cr_\ell} > \frac{1}{2} + \frac{\epsilon}{2}$, but this happens with probability:

$$Pr \left( \left\{ p_{cr_\ell} > \frac{1}{2} + \frac{\epsilon}{2} \right\} \right)$$

$$= Pr \left( \left\{ p_{cr_\ell} > \frac{1}{2} + \frac{\epsilon}{2} \right\} \cap \left\{ n_{cr_\ell} \geq \frac{mt}{k} \right\} \right) + Pr \left( n_{cr_\ell} < \frac{mt}{k} \right) Pr \left( \left\{ p_{cr_\ell} > \frac{1}{2} + \frac{\epsilon}{2} \right\} \left| n_{cr_\ell} < \frac{mt}{k} \right. \right)$$

$$\leq Pr \left( \left\{ p_{cr_\ell} - p_{cr_\ell} > \frac{\epsilon}{2} \right\} \cap \left\{ n_{cr_\ell} \geq \frac{mt}{k} \right\} \right) \leq \exp \left( -2 \frac{mt}{k} \left( \frac{\epsilon}{2} \right)^2 \right) = \frac{\delta}{2n},$$

where the first inequality follows as $p_{cr_\ell} \leq \frac{1}{2}$, and the second inequality is simply by applying Lem. 1 with $\eta = \frac{\epsilon}{2}$ and $v = \frac{mt}{k}$. We now proceed to the second case:

Case 2. (If $p_{cr_\ell} \geq \frac{1}{2} + \epsilon$, Trace-the-Best sets $r_{\ell+1} \leftarrow c_\ell$): Again recall that Trace-the-Best retains $r_{\ell+1} \leftarrow r_\ell$ only if $p_{cr_\ell} \leq \frac{1}{2} + \frac{\epsilon}{2}$. In this case, that happens with probability:

$$Pr \left( \left\{ p_{cr_\ell} \leq \frac{1}{2} + \frac{\epsilon}{2} \right\} \right)$$

$$= Pr \left( \left\{ p_{cr_\ell} \leq \frac{1}{2} + \frac{\epsilon}{2} \right\} \cap \left\{ n_{cr_\ell} \geq \frac{mt}{k} \right\} \right) + Pr \left( n_{cr_\ell} < \frac{mt}{k} \right) Pr \left( \left\{ p_{cr_\ell} \leq \frac{1}{2} + \frac{\epsilon}{2} \right\} \left| n_{cr_\ell} < \frac{mt}{k} \right. \right)$$

$$= Pr \left( \left\{ p_{cr_\ell} \leq \frac{1}{2} + \epsilon - \frac{\epsilon}{2} \right\} \cap \left\{ n_{cr_\ell} \geq \frac{mt}{k} \right\} \right)$$

$$\leq Pr \left( \left\{ p_{cr_\ell} - p_{cr_\ell} \leq -\frac{\epsilon}{2} \right\} \cap \left\{ n_{cr_\ell} \geq \frac{mt}{k} \right\} \right) \leq \exp \left( -2 \frac{mt}{k} \left( \frac{\epsilon}{2} \right)^2 \right) = \frac{\delta}{2n},$$

where the first inequality holds as $p_{cr_\ell} \geq \frac{1}{2} + \epsilon$, and the second one is simply by applying Lem. 1 with $\eta = \frac{\epsilon}{2}$ and $v = \frac{mt}{k}$. Combining the above two cases concludes the proof.

The rest of the proof follows exactly same as that of Thm. 4. We include the details for completeness. Given Algorithm 5 satisfies Lemma 18, and taking union bound over $(k - 1)$ elements in $A_\ell \setminus \{r_\ell\}$, we get that with probability at least $\left( 1 - \frac{(k-1)k}{2n} \right)$,
Above clearly suggests that for each iteration \( \ell \), the estimated ‘best’ item \( r_\ell \) only gets improved as \( p_{r_{\ell+1} r_\ell} \geq 0 \). Let, \( \ell_* \) denotes the specific iteration such that \( 1 \in A_\ell \) for the first time, i.e. \( \ell_* = \min \{ \ell \mid 1 \in A_\ell \} \). Clearly \( \ell_* \leq \lceil \frac{n}{k-1} \rceil \).

Now (13) suggests that with probability at least \( 1 - (\frac{(k-1)\delta}{2n}) \), \( p_{r_{\ell_*+1} r_{\ell_*}} \geq \frac{1}{2} - \epsilon \). Moreover (13) also suggests that for all \( \ell > \ell_* \), with probability at least \( 1 - (\frac{(k-1)\delta}{2n}) \), \( p_{r_{\ell+1} r_\ell} \geq \frac{1}{2} \), which implies for all \( \ell > \ell_* \), \( p_{r_{\ell+1} r_{\ell+1}} \geq \frac{1}{2} - \epsilon \) as well.

Note that above holds due to the following transitivity property of the Plackett-Luce model: For any three items \( i_1, i_2, i_3 \in [n] \), if \( p_{i_1 i_2} \geq \frac{1}{2} \) and \( p_{i_2 i_3} \geq \frac{1}{2} \), then we have \( p_{i_1 i_3} \geq \frac{1}{2} \) as well. This argument finally leads to \( p_{r_{\ell,1}} \geq \frac{1}{2} - \epsilon \). Since failure probability at each iteration \( \ell \) is at most \( (\frac{k-1}\delta)^{k-1} \), and Algorithm 5 runs for maximum \( \lceil \frac{n}{k-1} \rceil \) many number of iterations, using union bound over \( \ell \), the total failure probability of the algorithm is at most \( \sum_{\ell=1}^{\lceil \frac{n}{k-1} \rceil} (\frac{k-1}\delta)^{k-1} \leq \delta \left( \frac{n^k}{2n} \right) \leq \delta \) (since \( k \leq n \)). This concludes the correctness of the algorithm showing that it indeed returns an \( \epsilon \)-best element \( r_* \) such that \( p_{r_{\ell,1}} \geq \frac{1}{2} - \epsilon \) with probability at least \( 1 - \delta \).

C.4 Proof of Theorem 14

**Theorem 14** (**Divide-and-Battle: Correctness and Sample Complexity with TR**) With top-\( m \) ranking (TR) feedback, Divide-and-Battle (Algorithm 6) is \((\epsilon, \delta)\)-PAC with sample complexity \( O\left( \frac{n}{m^2} \log k \right) \).

**Proof** For the notational convenience we will use \( \hat{p}_{i,j} = p_{i,j} - \frac{1}{2} \), \( \forall i, j \in [n] \).

We first analyze the required sample complexity of the algorithm. For clarity of notation, we will denote the set \( S \) at iteration \( \ell \) by \( S_\ell \). Note that at any iteration \( \ell \), any set \( G_\ell \) is played for exactly \( t = \frac{4k}{m\epsilon^2} \ln \frac{2k}{\delta \ell} \) many number of times. Also since the algorithm discards away exactly \( k-1 \) items from each set \( G_\ell \), hence the maximum number of iterations possible is \( \lceil \ln_k n \rceil \). Now at any iteration \( \ell \), since \( G = \left\lfloor \frac{|S_\ell|}{k} \right\rfloor \leq \left\lfloor \frac{|S_{\ell-1}|}{k} \right\rfloor \), the total sample complexity for iteration \( \ell \) is at most \( \sum_{\ell=1}^{\lceil \ln_k n \rceil} \left( \frac{|S_\ell|}{m(\epsilon^2/2)^2} \right) \ln \frac{2k}{\delta \ell} \), as \( |S_\ell| \leq \frac{n}{k^\ell} \) for all \( \ell \in [\lceil \ln_k n \rceil] \). Also note that for all but last iteration \( \ell \in [\lceil \ln_k n \rceil] \), \( \epsilon_\ell = \frac{\epsilon}{3^{\ell-1}} \), and \( \delta_\ell = \frac{\delta}{2^{\ell-1}} \). Moreover for the last iteration \( \ell = \lceil \ln_k n \rceil \), the sample complexity is clearly \( t = \frac{4k}{m(\epsilon^2/2)^2} \ln \frac{4k}{\delta \ell} \), as in this case \( \epsilon_\ell = \frac{\epsilon}{2} \), and \( \delta_\ell = \frac{\delta}{2} \), and \( |S| = k \). Thus the total sample complexity of Algorithm 6 is given by

\[
\sum_{\ell=1}^{\lceil \ln_k n \rceil} \left( \frac{|S_\ell|}{m(\epsilon_\ell/2)^2} \right) \ln \frac{2k}{\delta_\ell} \leq \sum_{\ell=1}^{\infty} \frac{4n}{mk^\ell} \left( \frac{3}{\epsilon} \right)^{\ell-1} k \ln \frac{k^{\ell+1}}{\delta} + \frac{16k}{m} \ln \frac{4k}{\delta}.
\]
when items in the set with probability at least \( \eta \) where the second last inequality holds as \( m \) equality. Thus we finally derive that with probability at least \( q \) the set gets compared with rest of the \( G \).

Now prove that for any set \( G \),

\[
\Pr\left( \sum_{\ell=1}^{\infty} \frac{16^\ell}{(9k)^{\ell-1}} \left( \ln \frac{k}{\delta} + (\ell + 1) \right) + \frac{16k}{m^2} \ln \frac{4k}{\delta} \leq \frac{256n}{m^2} \ln \frac{k}{\delta} + \frac{16k}{m^2} \ln \frac{4k}{\delta} \right) = O\left( \frac{n}{m^2} \ln \frac{k}{\delta} \right) \text{ [for any } k > 1].
\]

Above proves the sample complexity bound of Theorem 14. We now proceed to prove the \((\epsilon, \delta)\)-PAC correctness of the algorithm. We start by making the following observations:

**Lemma 19** Consider any particular set \( G_g \) at any iteration \( \ell \in [\frac{n}{k}] \) and define \( q_i := \sum_{\tau=1}^{\ell} 1(i \in G_{gm}) \) as the number of times any item \( i \in G_g \) appears in the top-\( m \) rankings when items in the set \( G_g \) is made to battle for \( t \) rounds. Then if \( i_g := \arg \max_{i \in G_g} \theta_i \), then with probability at least \( \left( 1 - \frac{\epsilon}{2^k} \right) \), one can show that \( q_{i_g} > (1 - \eta) \frac{mt}{k} \), for any \( \eta \in \left( \frac{3}{32\sqrt{2}}, 1 \right] \).

**Proof** Fix any iteration \( \ell \) and a set \( G_g, g \in 1, 2, \ldots, G \). Define \( i^* := 1(i \in G_{gm}) \) as the indicator variable if \( i^* \)th element appeared in the top-\( m \) ranking at iteration \( \tau \in [t] \). Recall the definition of TR feedback model (Sec. 3.1). Using this we get \( \mathbb{E}[i^*_g] = \Pr\{i_g \in G_{gm} \} = \Pr(\exists j \in [m] \mid \sigma(j) = i_g) = \sum_{j=1}^{m} \mathbb{E}[\sigma(j) = i_g] = \sum_{j=0}^{m-1} \frac{1}{k^j} \geq \frac{m}{k} \), since \( \Pr \{i_g \mid S\} = \frac{\theta_{i_g}}{\sum_{j \in S} \theta_j} \geq \frac{1}{|S|} \) for any \( S \subseteq [G_g] \), as \( i_g := \arg \max_{i \in G_g} \theta_i \) is the best item of set \( G_g \). Hence \( \mathbb{E}[q_{i_g}] = \sum_{\tau=1}^{\ell} \mathbb{E}[i^*_g] \geq \frac{mt}{k} \).

Now applying Chernoff-Hoeffdings bound for \( w_{i_g} \), we get that for any \( \eta \in \left( \frac{3}{32\sqrt{2}}, 1 \right] \),

\[
\Pr\left( q_{i_g} \leq (1 - \eta) \mathbb{E}[q_{i_g}] \right) \leq \exp\left( -\frac{\mathbb{E}[q_{i_g}] \eta^2}{2} \right) \leq \exp\left( -\frac{mt \eta^2}{2k} \right)
\]

\[
= \exp\left( -\frac{2n^2}{\epsilon^2} \ln \left( \frac{2k}{\delta} \right) \right) = \exp\left( -\frac{(\sqrt{2} \eta)^2}{\epsilon^2} \ln \left( \frac{2k}{\delta} \right) \right)
\]

\[
\leq \exp\left( -\ln \left( \frac{2k}{\delta} \right) \right) \leq \frac{\delta^k}{2k},
\]

where the second last inequality holds as \( \eta \geq \frac{3}{32\sqrt{2}} \) and \( \epsilon \leq \frac{3}{32} \), for any iteration \( \ell \in [\ln n] \); in other words for any \( \eta \geq \frac{3}{32\sqrt{2}} \), we have \( \frac{\sqrt{2} \eta}{\epsilon} \geq 1 \) which leads to the second last inequality. Thus we finally derive that with probability at least \( \left( 1 - \frac{\epsilon}{2^k} \right) \), one can show that \( q_{i_g} > (1 - \eta) \mathbb{E}[q_{i_g}] \geq (1 - \eta) \frac{mt}{k} \), and the proof follows henceforth.

In particular, fixing \( \eta = \frac{1}{k} \) in Lemma 16, we get that with probability at least \( \left( 1 - \frac{\epsilon}{2^k} \right) 
\]

\[
q_{i_g} > (1 - \frac{1}{k}) \mathbb{E}[w_{i_g}] > \frac{mt}{2k} \text{. Note that, for any round } \tau \in [t], \text{ whenever an item } i \in G_g \text{ appears in the top-}m \text{ set } G_{gm}, \text{ then the rank breaking update ensures that every element in the top-}
\]

\( m \) set gets compared with rest of the \( k - 1 \) elements of \( G_g \). Based on this observation, we now prove that for any set \( G_g \), its best item \( i_g \) is retained as the winner \( c_g \) with probability at least \( \left( 1 - \frac{4}{k} \right) \). More formally, first thing to observe is:
Lemma 20 Consider any particular set \( G_g \) at any iteration \( \ell \in [\frac{n}{k}] \). If \( i_g \leftarrow \arg \max_{i \in G_g} \theta_i \), then with probability at least \((1 - \delta_{\ell})\), \( \hat{p}_{i_gj} + \frac{\epsilon_{\ell}}{2} \geq \frac{1}{2} \) for all \( \epsilon_{\ell} \)-optimal item \( \forall j \in G_g \) such that \( p_{i_gj} \in \left( \frac{1}{2}, \frac{1}{2} + \epsilon_{\ell} \right) \), and \( \hat{p}_{i_gj} - \frac{\epsilon_{\ell}}{2} \geq \frac{1}{2} \) for all non \( \epsilon_{\ell} \)-optimal item \( \forall j \in G_g \) such that \( p_{i_gj} > \frac{1}{2} + \epsilon_{\ell} \).

Proof With top-\( m \) ranking feedback, the crucial observation lies in the fact that at any round \( \tau \in [t] \), whenever an item \( i \in G_g \) appears in the top-\( m \) set \( G_{g_{\tau m}} \), then the rank-breaking update ensures that every element in the top-\( m \) set gets compared with each of the rest of the \( k - 1 \) elements of \( G_g \) - it defeats to every element preceding item in \( \sigma \in \Sigma_{g_{\tau m}} \), and wins over the rest. Therefore defining \( n_{ij} = w_{ij} + w_{ji} \) to be the number of times item \( i \) and \( j \) are compared after rank-breaking, \( i, j \in G_g \). Clearly \( n_{ij} = n_{ji} \), and \( 0 \leq n_{ij} \leq tk \). Moreover, from Lemma 19 with \( \eta = \frac{1}{2} \), we have that \( n_{ij} \geq \frac{mt}{2k} \). Given the above arguments in place let us analyze the probability of a ‘bad event’ that indeed:

Case 1. \( j \) is \( \epsilon_{\ell} \)-optimal with respect to \( i_g \), i.e. \( p_{i_gj} \in \left( \frac{1}{2}, \frac{1}{2} + \epsilon_{\ell} \right) \). Then we have

\[
Pr \left( \left\{ \hat{p}_{i_gj} + \frac{\epsilon_{\ell}}{2} < \frac{1}{2} \right\} \cap \left\{ n_{i_gj} \geq \frac{mt}{2k} \right\} \right) = Pr \left( \left\{ \hat{p}_{i_gj} < \frac{1}{2} - \frac{\epsilon_{\ell}}{2} \right\} \cap \left\{ n_{i_gj} = \frac{mt}{2k} \right\} \right) \\
\leq Pr \left( \left\{ \hat{p}_{i_gj} - p_{i_gj} < -\frac{\epsilon_{\ell}}{2} \right\} \cap \left\{ n_{i_gj} = \frac{mt}{2k} \right\} \right) \\
\leq \exp \left( -\frac{2mt (\epsilon_{\ell}/2)^2}{2k} \right) = \frac{\delta_{\ell}}{2k},
\]

where the first inequality follows as \( p_{i_gj} > \frac{1}{2} \), and the second inequality follows from Lem. 1 with \( \eta = \frac{1}{2} \) and \( v = \frac{mt}{2k} \).

Case 2. \( j \) is non \( \epsilon_{\ell} \)-optimal with respect to \( i_g \), i.e. \( p_{i_gj} > \frac{1}{2} + \epsilon_{\ell} \). Similar to before, we have

\[
Pr \left( \left\{ \hat{p}_{i_gj} - \frac{\epsilon_{\ell}}{2} < \frac{1}{2} \right\} \cap \left\{ n_{i_gj} \geq \frac{mt}{2k} \right\} \right) = Pr \left( \left\{ \hat{p}_{i_gj} < \frac{1}{2} + \frac{\epsilon_{\ell}}{2} \right\} \cap \left\{ n_{i_gj} = \frac{mt}{2k} \right\} \right) \\
\leq Pr \left( \left\{ \hat{p}_{i_gj} - p_{i_gj} < -\frac{\epsilon_{\ell}}{2} \right\} \cap \left\{ n_{i_gj} = \frac{mt}{2k} \right\} \right) \\
\leq \exp \left( -\frac{2mt (\epsilon_{\ell}/2)^2}{2k} \right) = \frac{\delta_{\ell}}{2k},
\]

where the third last inequality follows since in this case \( p_{i_gj} > \frac{1}{2} + \epsilon_{\ell} \), and the last inequality follows from Lem. 1 with \( \eta = \frac{1}{2} \) and \( v = \frac{mt}{2k} \).

Let us define the event \( E := \{ \exists j \in G_g \text{ such that } \hat{p}_{i_gj} + \frac{\epsilon_{\ell}}{2} < \frac{1}{2}, p_{i_gj} \in \left( \frac{1}{2}, \frac{1}{2} + \epsilon_{\ell} \right) \} \) or \( \hat{p}_{i_gj} - \frac{\epsilon_{\ell}}{2} < \frac{1}{2}, p_{i_gj} > \frac{1}{2} + \epsilon_{\ell} \). Then by combining Case 1 and 2, we get

\[
Pr \left( E \right) = Pr \left( E \cap \left\{ n_{i_gj} \geq \frac{mt}{2k} \right\} \right) + Pr \left( E \cap \left\{ n_{i_gj} < \frac{mt}{2k} \right\} \right)
\]
\[ \leq \sum_{j \in G_g \text{ s.t. } p_{i,j} \in \left( \frac{1}{2} + \epsilon \right)} \Pr\left( \left\{ \hat{p}_{i,j} + \frac{\epsilon}{2} < \frac{1}{2} \right\} \right) \]

\[ + \sum_{j \in G_g \text{ s.t. } p_{i,j} > \frac{1}{2} + \epsilon} \Pr\left( \left\{ \hat{p}_{i,j} - \frac{1}{2} < \frac{1}{2} \right\} \right) \leq \frac{(k - 1)\delta}{2k} + \frac{\delta}{2k} \leq \delta \]

where the last inequality follows from the above two case analyses and Lem. 19.

Given Lem. 20 in place, let us now analyze with what probability the algorithm can select a non-\( \epsilon \)-optimal item \( j \in G_g \) as \( c_g \) at any iteration \( \ell \in \left[ \frac{n}{k} \right] \). For any set \( G_g \) (or set \( S \) for the last iteration \( \ell = \left[ \frac{n}{k} \right] \)), we define the set of non-\( \epsilon \)-optimal element \( O_g = \{ j \in G_g \mid p_{i,j} > \frac{1}{2} + \epsilon \} \), and recall the event \( E := \{ \exists j \in G_g \text{ such that } \hat{p}_{i,j} + \frac{\epsilon}{2} < \frac{1}{2}, p_{i,j} \in \left( \frac{1}{2} + \epsilon \right) \text{ or } \hat{p}_{i,j} - \frac{\epsilon}{2} < \frac{1}{2}, p_{i,j} > \frac{1}{2} + \epsilon \} \). Then

\[ \Pr(c_g \in O_g) \leq \Pr\left( \{ \exists j \in G_g, \hat{p}_{i,j} + \frac{\epsilon}{2} < \frac{1}{2} \} \cup \{ \exists j \in O_g, \hat{p}_{j,i} + \frac{\epsilon}{2} \geq \frac{1}{2} \} \right) \]

\[ \leq \Pr\left( E \cup \{ \exists j \in O_g, \hat{p}_{j,i} + \frac{\epsilon}{2} \geq \frac{1}{2} \} \right) \]

\[ = \Pr\left( E \right) + \Pr\left( \{ \exists j \in O_g, \hat{p}_{j,i} + \frac{\epsilon}{2} \geq \frac{1}{2} \} \cap E^c \right) \]

\[ = \Pr\left( E \right) + \Pr\left( \{ \exists j \in O_g, \hat{p}_{j,i} + \frac{\epsilon}{2} \geq \frac{1}{2} \} \cap E^c \right) \leq \delta + 0 = \delta, \quad (14) \]

where the last inequality follows from Lemma 20, and the fact that \( \hat{p}_{i,j} - \frac{\epsilon}{2} \geq \frac{1}{2} \iff \hat{p}_{j,i} + \frac{\epsilon}{2} < \frac{1}{2} \). The proof now follows combining all the above parts together.

More formally, for each iteration \( \ell \), let us define \( g_{\ell} \in [G] \) to be the set that contains best item of the entire set \( S \), i.e. \( \arg \max_{i \in S} \theta_i \in G_{g_{\ell}} \). Then from (14), with probability at least \((1 - \delta)\), \( \tilde{p}_{c_{g_{\ell}}} > -\epsilon \). Now for each iteration \( \ell \), recursively applying (14) and Lemma 15 to \( G_{g_{\ell}} \), we get that \( \tilde{p}_{r_{\ell + 1}} > -\left( \frac{\epsilon}{\delta} + \left( \frac{3}{8} \right) + \cdots + \left( \frac{3}{8} \right)^{\ell} \right) \) + \( \frac{3}{8} \geq -\frac{1}{8} \left( \sum_{i=0}^{\infty} \left( \frac{3}{8} \right)^i \right) + \frac{3}{8} = \epsilon \).

(Note that, for above analysis to go through, it is in fact sufficient to consider only the set of iterations \( \ell \geq \ell_0 \mid \ell_0 = \min\{ l \mid 1 \notin R_l, l \geq 1 \} \) because prior considering item 1, it does not matter even if the algorithm mistakes in any of the iteration \( \ell < \ell_0 \).) Thus assuming the algorithm does not fail in any of the iteration \( \ell \), we have that \( p_{r_{\ell + 1}} > 1 - \epsilon \).

Finally, since at each iteration \( \ell \), the algorithm fails with probability at most \( \delta \), the total failure probability of the algorithm is at most \( \left( \frac{\delta}{4} + \frac{\delta}{8} + \cdots + \frac{\delta}{2^{\ell + 1}} \right) + \frac{\delta}{2} \leq \delta \). This concludes the correctness of the algorithm showing that it indeed returns an \( \epsilon \)-best element.
$r_*$ such that $p_{r_*,1} \geq \frac{1}{2} - \epsilon$ with probability at least $1 - \delta$. 

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