Correction-to-Scaling Exponents for Two-Dimensional Self-Avoiding Walks

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Abstract

We study the correction-to-scaling exponents for the two-dimensional self-avoiding walk, using a combination of series-extrapolation and Monte Carlo methods. We enumerate all self-avoiding walks up to 59 steps on the square lattice, and up to 40 steps on the triangular lattice, measuring the mean-square end-to-end distance, the mean-square radius of gyration and the mean-square distance of a monomer from the endpoints. The complete endpoint distribution is also calculated for self-avoiding walks up to 32 steps (square) and up to 22 steps (triangular). We also generate self-avoiding walks on the square lattice by Monte Carlo, using the pivot algorithm, obtaining the mean-square radii to $\approx 0.01\%$ accuracy up to $N = 4000$. We give compelling evidence that the first non-analytic correction term for two-dimensional self-avoiding walks is $\Delta_1 = 3/2$. We compute several moments of the endpoint distribution function, finding good agreement with the field-theoretic predictions. Finally, we study a particular invariant ratio that can be shown, by conformal-field-theory arguments, to vanish asymptotically, and we find the cancellation of the leading analytic correction.

KEY WORDS: Self-avoiding walk; polymer; exact enumeration; series expansion; Monte Carlo; pivot algorithm; corrections to scaling; critical exponents; conformal invariance.
1 Introduction

The study of the universal properties of self-avoiding walks (SAWs) in the long-chain limit has been a central problem in both statistical mechanics and polymer physics for more than three decades [1–3]. In an \( N \)-step chain, the mean value of any global observable \( \mathcal{O} \) typically has an asymptotic expansion as \( N \to \infty \) of the form

\[
\langle \mathcal{O} \rangle_N = AN^{p_{\mathcal{O}}} \left[ 1 + \frac{a_1}{N} + \frac{a_2}{N^2} + \cdots + \frac{b_0}{N^{\Delta_1}} + \frac{b_1}{N^{\Delta_1+1}} + \frac{b_2}{N^{\Delta_1+2}} + \cdots \\
+ \frac{c_0}{N^{\Delta_2}} + \frac{c_1}{N^{\Delta_2+1}} + \frac{c_2}{N^{\Delta_2+2}} + \cdots \right],
\]

where the leading exponent \( p_{\mathcal{O}} \) and the correction-to-scaling exponents \( \Delta_1 < \Delta_2 < \ldots \) are universal, i.e. they depend on the spatial dimension \( d \) but not on the microscopic details of the interactions (provided that the interactions are short-ranged and primarily repulsive). This universality justifies the intense efforts that have been devoted to determining these universal exponents, using a variety of analytical and numerical approaches.

In this paper we will address the problem of determining the leading non-analytic correction-to-scaling exponent \( \Delta_1 \) for the two-dimensional self-avoiding walk and for the closely related problem of self-avoiding polygons (SAPs). At least two different theoretical predictions have been made for the purportedly exact value of this exponent: \( \Delta_1 = 3/2 \) based on Coulomb-gas arguments [4, 5], and \( \Delta_1 = 11/16 \) based on conformal-invariance methods [6]. In addition, several numerical methods have been employed to estimate \( \Delta_1 \), notably exact enumeration and extrapolation (series analysis) [7–19] and Monte Carlo [18, 20–23]. The estimates of \( \Delta_1 \) resulting from these numerical works are, for the most part, wildly contradictory: even when one compares estimates produced by a single method, such as series analysis, they range from \( \approx 0.5 \) [18] to \( \approx 0.65 \) [8,10,13,14] to \( \approx 0.85 \) [15] to \( \approx 1 \) [7,14] to 1.5 [16,17,19]. Similar variation can be found in estimates of \( \Delta_1 \) obtained from Monte Carlo studies, ranging from \( \approx 0.5 \) [18] to \( \approx 0.6 \) [23] to \( \approx 0.84 \) [21] to \( \approx 1.1 \) [23] to \( \approx 1.2 \) [20].

Other models in the same universality class have also been considered, yielding results in contrast with those for the SAW. For instance, for lattice trails (connected paths where the self-avoidance constraint is applied only to bonds, not vertices) it was shown by a transfer-matrix study [24] that the correction-to-scaling exponent is indeed \( \approx 11/16 \), confirming an earlier result based on series analysis [25]. This same transfer-matrix study also found \( \Delta_1 \approx 3/2 \) for SAWs. What remains to be understood is why the contribution with \( \Delta_1 = 11/16 \) seems to be absent for square- and triangular-lattice SAWs, yet present for trails.

To further confuse the subject, we should mention the recent results of Jensen [26] for osculating SAPs on the square lattice: these are a superset of SAPs in which bonds may touch at a vertex but not cross. In a sense they interpolate between SAPs, in which intersections are strictly forbidden, and closed trails, in which crossing at a vertex is allowed. A careful analysis of the corresponding series [26] shows very convincingly that \( \Delta_1 = 3/2 \) and finds no correction corresponding to \( \Delta_1 = 11/16 \).
To add to our list of unexplained phenomena, we remark on a recent Monte Carlo study of SAWs on the Manhattan lattice [27], where it was found that the critical exponent $\gamma$ has the same value $43/32$ as for SAWs on regular lattices provided that the value $\Delta_1 = 11/16$ is used in the analysis. It is quite unclear why a different value of $\Delta_1$ should arise for the Manhattan lattice than is found for the square lattice.

Returning now to the simplest case of SAWs and SAPs on regular lattices, the gross disparities among the extant estimates of the correction-to-scaling exponent might lead one to suspect that different methods are computing different quantities. For example, it might be that some methods are measuring (or predicting) the leading correction exponent $\Delta_1$, while others are measuring (or predicting) a subleading correction exponent $\Delta_2$ or $\Delta_3$, and still others are measuring some sort of “effective” exponent $\Delta_{\text{eff}}$ that represents phenomenologically the observed corrections to scaling in some specified interval of walk length $N$ (and arising in reality from the sum of two or more correction-to-scaling terms).

Moreover, it may even be true that different observables produce different patterns of nonvanishing corrections to scaling. For instance, the $\Delta_1 = 3/2$ correction term appears to be present for SAWs but absent for SAPs. While this may at first sight be considered a violation of universality, we show below that it is not.

Two recent analyses [17, 19] based on very long series for square-lattice SAWs and SAPs have, however, yielded a consistent and convincing picture of the corrections to scaling: the first non-analytic correction-to-scaling exponent is indeed just $\Delta_1 = 3/2$, as predicted by Nienhuis [4, 5]; but there are also analytic corrections to scaling proportional to integer powers of $1/N$, the first of which dominates asymptotically. More precisely, a careful numerical study based on a 51-term SAW series [17] found that the number of $N$-step SAWs on the square lattice is given asymptotically by

$$c_N \sim \mu^N N^{11/32}[1.177043 + 0.5500/N - 0.140/N^{3/2} - 0.12/N^2 + \cdots]$$

$$+ (-\mu)^N N^{-3/2}[-0.1899 + 0.175/N - 1.51/N^2 + \cdots]. \quad (1.2)$$

It is likely that previous studies identified some sort of effective exponent that reflects a combination of the effects of the $1/N$ and $1/N^{3/2}$ correction-to-scaling terms (see Section 2.2 for further discussion of this point). Similarly, a careful numerical study based on a 90-term SAP series [19] found that the number of $2N$-step SAPs on the square lattice is given asymptotically by

$$p_{2N} \sim \mu^{2N} N^{-5/2}[0.0994018 - 0.02751/N + 0.0255/N^2 + 0.12/N^3 + \cdots]; \quad (1.3)$$

note that here there is no $N^{-3/2}$ (or $N^{-5/2}$) correction term. Finally, a recent transfer-matrix analysis [24] of SAWs on the square lattice also found compelling numerical evidence in favour of the value $\Delta_1 = 3/2$, and against all values $\Delta_1$ significantly less than $3/2$.

One possible cause of some confusion is that, because the value of the leading critical exponent of the SAP generating function is $2 - \alpha = 3/2$, any correction-to-scaling term with $\Delta =$ half-integer “folds into” the analytic background term and is therefore undetectable! In other words, no corrections proportional to $N^{-\Delta}$ appear in the coefficients $p_N$. In order to understand this point, let us recall that the critical exponent $\alpha$ is defined
by the leading asymptotic behavior $p_N \propto \mu^N N^{\alpha-3}$ of the polygon counts, corresponding to a leading behavior $P(x) = \sum_{N \geq 0} p_N x^N \sim \text{const} \times (1 - x/x_c)^{2-\alpha} \text{as } x \uparrow x_c = 1/\mu$ for the polygon generating function. If we now include both analytic and non-analytic corrections to scaling, the polygon generating function can be written generically as

$$ P(x) = \sum_{N \geq 0} p_N x^N \sim A(x) + B(x)(1 - x/x_c)^{2-\alpha}[1 + c(1 - x/x_c)^{\Delta_1} + \cdots] \quad (1.4) $$

with $A(x)$ and $B(x)$ analytic in the neighbourhood of the critical point $x_c = 1/\mu$. Since $\alpha = 1/2$ for two-dimensional SAPs [4,5,19], if $\Delta_1$ = half-integer the above equation may be rewritten as

$$ P(x) = \sum_{N \geq 0} p_{2N} x^N \sim \hat{A}(x) + \hat{B}(x)(1 - x/x_c)^{3/2}[1 + \cdots] \quad (1.5) $$

with the $(1 - x/x_c)^{\Delta_1}$ correction term absorbed into the analytic background term $\hat{A}(x)$. Therefore, if (1.4) holds, no correction of the form $N^{-\Delta_1}$ is present. On the other hand, if the polygon counts were to exhibit a behaviour of the form $p_N \propto \mu^N N^{\alpha-3}[1 + \ldots + a/N^{\Delta_1} + \ldots]$ with $\alpha = 1/2$ and $\Delta_1 = 3/2$ — and hence include a term $\propto \mu^N N^{-4}$ — then the generating function $P(x)$ would exhibit, on top of the $(1 - x/x_c)^{3/2}$ leading behaviour, a non-analytic confluent term of the form $(1 - x/x_c)^3 \log(1 - x/x_c)$ in addition to the analytic term $(1 - x/x_c)^3$. However, as discussed in some detail in [19], there is no evidence for a term of the form $a/N^{3/2}$ in the analysis of the SAP count series, and indeed there is considerable evidence for the absence of such a term. There is, however, abundant evidence of such a term in the radius-of-gyration series of SAPs [28].

In the present paper we make some further progress in supporting the assertion that $\Delta_1 = 3/2$ for SAWs on regular two-dimensional lattices (here square and triangular). First, we make a conventional analysis of corrections to scaling in the standard observables $\langle R^2_g \rangle$, $\langle R^2 \rangle$ and $\langle R^2_m \rangle$; our contribution here is to present and use extended series expansions and a more efficient Monte Carlo algorithm. The results of this analysis are consistent with other recent work in supporting the conclusion that $\Delta_1 = 3/2$. In the course of this analysis we point out that, for certain observables, pairs of correction terms of opposite sign can (and do) conspire to give an effective exponent that is smaller than both of the individual exponents; this explains the apparent exponents $\Delta < 1$ observed in some earlier work. Second—and this is perhaps our main contribution—we point out several observables in which a correction-to-scaling term becomes the leading term. These include: (a) the combination $\frac{246}{31} \langle R^2_g \rangle - 2 \langle R^2_m \rangle + \frac{1}{2} \langle R^2_c \rangle$, which arises in the conformal-invariance theory [29,30]; and (b) quantities related to the breaking of Euclidean invariance down to the lattice symmetry group, the simplest of which is (on the square lattice) the fourth-order moment $\langle r^4 \cos 4\theta \rangle = \langle x^4 - 6x^2y^2 + y^4 \rangle$. Analysis of these quantities by Monte Carlo methods yields only a modest improvement over the analysis of conventional quantities—the trouble is that the new quantities exhibit a low “signal-to-noise ratio”—but the series analysis is quite precise.

The plan of this paper is as follows. In Section 2 we define the quantities to be studied and collect some theoretical results that will be used or tested in the following sections. Section 3 reports the results of our series analysis: first, we analyze the SAW
counts (Section 3.3); then, we analyze the radius of gyration, the end-to-end distance and the average distance of a monomer from the endpoints, along with their invariant ratios (Section 3.4); finally, we analyze the higher-order rotationally-invariant moments of the endpoint distribution function (Section 3.5) and the corresponding non-rotationally-invariant moments (Section 3.6). For each of them, we determine the asymptotic behaviour as \( N \to \infty \), focusing in particular on the correction-to-scaling exponent \( \Delta_1 \) and on the behaviour at the antiferromagnetic singularity (in the case of the square lattice). In Section 4 we report the analyses of our Monte Carlo data, confirming the absence of a correction-to-scaling exponent \( \Delta_1 = \frac{11}{16} \). Finally, in Section 5 we draw our conclusions.

2 Definitions and theoretical background

In this section we review briefly the basic facts and conjectures about the SAW that will be used (or tested) in the remainder of the paper.

2.1 Definitions and notation

Let \( \mathcal{L} \) be some regular \( d \)-dimensional lattice. Then an \( N \)-step self-avoiding walk (SAW) \( \omega \) on \( \mathcal{L} \) is a sequence of distinct points \( \omega_0, \omega_1, \ldots, \omega_N \) in \( \mathcal{L} \) such that each point is a nearest neighbour of its predecessor. We assume all walks to begin at the origin (\( \omega_0 = 0 \)) unless stated otherwise.

First we define the quantities relating to the number (or “entropy”) of SAWs. Let \( c_N \) [resp. \( c_N(x) \)] be the number of \( N \)-step SAWs on \( \mathcal{L} \) starting at the origin and ending anywhere [resp. ending at \( x \)]. Then \( c_N \) and \( c_N(x) \) are believed to have the asymptotic behaviour

\[
c_N \sim \text{const} \times \mu^N N^{\gamma - 1} \quad \text{as} \quad N \to \infty
\]

\[
c_N(x) \sim \text{const} \times \mu^N N^{\alpha - 2} \quad (x \text{ fixed } \neq 0)
\]

as \( N \to \infty \); here \( \mu \) is called the connective constant of the lattice, and \( \gamma \) and \( \alpha \) are critical exponents. The critical exponents are believed to be universal among lattices of a given dimension \( d \). For rigorous results concerning the asymptotic behaviour of \( c_N \) and \( c_N(x) \), see [31–34].

Next we define several measures of the size of an \( N \)-step SAW:

- The squared end-to-end distance

\[
R_e^2 = \omega_N^2 .
\]

- The squared radius of gyration

\[
R_g^2 = \frac{1}{2(N+1)^2} \sum_{i,j=0}^{N} (\omega_i - \omega_j)^2 .
\]

- The mean-square distance of a monomer from the endpoints

\[
\]
\[ R_m^2 = \frac{1}{2(N+1)} \sum_{i=0}^{N} \left[ \omega_i^2 + (\omega_i - \omega_N)^2 \right]. \] (2.5)

We then consider the mean values \( \langle R_e^2 \rangle_N, \langle R_g^2 \rangle_N \) and \( \langle R_m^2 \rangle_N \) in the probability distribution that gives equal weight to each \( N \)-step SAW. Very little has been proven rigorously about these mean values, but they are believed to have the leading asymptotic behaviour

\[ \langle R_e^2 \rangle_N, \langle R_g^2 \rangle_N, \langle R_m^2 \rangle_N \sim \text{const} \times N^{2\nu} \] (2.6)
as \( N \to \infty \), where \( \nu \) is another (universal) critical exponent. Hyperscaling [35] predicts that

\[ d\nu = 2 - \alpha. \] (2.7)

For SAWs in two dimensions, Coulomb-gas arguments [4, 5] as well as arguments based on stochastic Loewner evolution (SLE) [36] predict that \( \nu = 3/4, \alpha = 1/2 \) and \( \gamma = 43/32 \). Prior numerical studies have confirmed these values to high precision [17, 19, 37]; in this paper we take them for granted.

The amplitude ratios

\[ A_N = \frac{\langle R_g^2 \rangle_N}{\langle R_e^2 \rangle_N} \] (2.8)
\[ B_N = \frac{\langle R_m^2 \rangle_N}{\langle R_e^2 \rangle_N} \] (2.9)

are expected to approach universal values in the limit \( N \to \infty \), which we call \( A \) and \( B \); one of our goals is to estimate these limiting amplitude ratios. Many other universal amplitude combinations (notably involving SAPs) are discussed in [38, 39].

Of particular interest is the linear combination [29, 30]

\[ F_N \equiv \left( 2 + \frac{y_t}{y_h} \right) A_N - 2B_N + \frac{1}{2} \] (2.10)

and the corresponding unnormalized quantity

\[ f_N \equiv F_N \langle R_e^2 \rangle_N \equiv \left( 2 + \frac{y_t}{y_h} \right) \langle R_g^2 \rangle_N - 2\langle R_m^2 \rangle_N + \frac{1}{2} \langle R_e^2 \rangle_N, \] (2.11)

where \( y_t = 1/\nu \) and \( y_h = 1+\gamma/(2\nu) \) are the thermal and magnetic renormalization-group eigenvalues, respectively, of the \( n \)-vector model at \( n = 0 \). In two dimensions — where \( y_t = 4/3 \) and \( y_h = 91/48 \), hence \( 2+y_t/y_h = 246/91 \)— Cardy and Saleur [29] (as corrected by Caracciolo, Pelissetto and Sokal [30]) have predicted, using conformal field theory, that \( \lim_{N \to \infty} F_N = 0 \). We shall henceforth refer to this relation as the CSCPS relation. This conclusion has been confirmed by previous high-precision Monte Carlo work [30] as well as by series extrapolations [40]. It is therefore of interest to examine the rate at which \( F_N \) tends to zero, as this gives information on the correction-to-scaling terms. We will discuss this from a theoretical point of view near the end of Section 2.2, and from a numerical point of view in Sections 3.4.2 and 4.2.
It turns out that $\lim_{N \to \infty} F_N = 0$ holds not only for the ordinary square-lattice SAW, but also for SAWs with nearest-neighbour interactions, right up to (but not at) the theta point [41]. Moreover, the relation appears to hold at the theta point if $2 + y_t/y_h$ is given its theta-point value $23/8$ instead of $246/91$. This observation is used in [41] to locate the theta point more precisely.

We shall also consider higher moments of the end-to-end distance. Limiting ourselves to two-dimensional lattices, let us write

\[ \omega_N \equiv (x, y) \equiv (r \cos \theta, r \sin \theta). \]  

The Euclidean-invariant moments $\langle r^k \rangle_N$ are of course expected to behave as

\[ \langle r^k \rangle_N \sim \text{const} \times N^{k \nu} \]  

as $N \to \infty$. One can consider the dimensionless ratios

\[ M_{2k,N} = \frac{\langle r^{2k} \rangle_N}{\langle r^2 \rangle_N^k}, \]  

which approach finite limits for $N \to \infty$; these limiting ratios $M_{2k,\infty}$ are universal quantities that characterize the end-to-end distribution function. Estimates of $M_{2k,\infty}$ have been obtained in [42] using field theory and the Laplace–deGennes transform method.

\[ \tilde{D}(q)^{-1} = 1 + q^2 + \sum_{n=2}^{\infty} b_n q^{2n}, \]  

one has $1 \gg |b_2| \gg |b_3| \gg |b_4| \gg \ldots$. One obtains [42]

\[ M_{2k,\infty} = \frac{\Gamma(\gamma + 2\nu)^k}{\Gamma(\gamma + 2k\nu) \Gamma(\gamma)^{k-1}} [1 - b_2(k - 1) + R_k] \frac{k^{-1}}{k!} \prod_{j=0}^{k-1} \left( 1 + \frac{2j}{d} \right), \]  

where $R_k$ is a very small correction (unless $k$ is very large) that involves the constants $b_3, b_4, \ldots$ as well as higher powers $b_t b_j, b_t b_j b_k, \ldots$. Explicitly,

\[ R_k = \sum_{n=3}^{k} (-1)^{n+1} (k + 1 - n)b_n + \frac{1}{2}(k - 2)(k - 3)b_2^2 + \cdots \]  

Note that $R_2 = 0$ exactly. The universal nonperturbative constants $b_2, b_3, \ldots$ have been obtained from the analysis of exact-enumeration series on the square, triangular and hexagonal lattices [44]. Numerically, it is found [44] that $b_2$ is extremely small, 

\[^1\tilde{D}(q)\) is the Fourier transform of the two-point correlation function, rescaled so that the first two terms at small $q$ are $1 - q^2 + O(q^4)$.
\( b_2 = 0.00015(20) \), and that \( b_3 \) is even smaller, \( |b_3| \lesssim 3 \times 10^{-5} \). Using the estimate of \( b_2 \) in (2.16) and neglecting \( R_k \), we obtain for the lowest values of \( k \):

\[
\begin{align*}
M_{4,\infty} & = 1.44574(28) \quad (2.18) \\
M_{6,\infty} & = 2.5876(10) \quad (2.19) \\
M_{8,\infty} & = 5.3805(32) \quad (2.20) \\
M_{10,\infty} & = 12.557(10) \quad . (2.21)
\end{align*}
\]

A second class of interesting observables are moments that are invariant under the symmetry group of the lattice but not under the full Euclidean group: examples are the moments \( \langle r^k \cos n\theta \rangle \) with \( n \neq 0 \), where for the square (resp. triangular) lattice \( n \) must be a multiple of 4 (resp. 6). We expect these non-Euclidean-invariant moments to behave as

\[
\langle r^k \cos n\theta \rangle \sim \text{const} \times N^{k\nu - \Delta_{nr}},
\]

where \( \Delta_{nr} > 0 \) is a new correction-to-scaling exponent [43] associated with the breaking of full rotation invariance down to the lattice rotation group: it thus depends on the lattice in question (e.g., square or triangular) and is in general different from the leading correction-to-scaling exponent \( \Delta_1 \) (which corresponds to a Euclidean-invariant irrelevant operator).

For Gaussian models—and thus also for \( n \)-vector models (including the SAW case \( n = 0 \)) in dimension \( d \geq 4 \)—we have \( \Delta_{nr} = 2\nu = 1 \) on any hypercubic lattice. For \( n \)-vector models in dimension \( d = 4 - \epsilon \), this relation is modified at order \( \epsilon^2 \) [43]:

\[
\Delta_{nr} = \nu \left[ 2 + \frac{7}{20} \frac{n + 2}{(n + 8)^2} \epsilon^2 + O(\epsilon^3) \right].
\]

In dimension \( d = 3 \), several alternative methods—field theory and exact-enumeration analysis—show that \( \Delta_{nr} \) is very close to \( 2\nu \), though not exactly equal [43]. In two dimensions on the square lattice, \( \Delta_{nr} = 2\nu \) exactly for the Ising model and for the \( n \)-vector model with \( n \geq 3 \) (in the latter case with logarithmic corrections) [43]. For the triangular lattice, similar arguments [43] predict \( \Delta_{nr} = 4\nu \). For the Ising model, these predictions can be obtained using conformal field theory (see [45,46] for the classification of the subleading operators appearing in the Ising model); they can be checked explicitly [43] for at least one specific observable, using the analytic expression for the mass gap [44,47,48].

\[\text{\underline{2}}\] For the hexagonal lattice, \( \Delta_{nr} = 4\nu \) for observables that break rotational invariance but are invariant under interchange of the two sublattices, while \( \Delta_{nr} = 3\nu \) for observables that distinguish the two sublattices [43].

\[\text{\underline{3}}\] For instance, consider on a square lattice the mass gap \( m(\hat{n}) \) in the direction \( \hat{n} \) defined as

\[
m(\hat{n}) = -\lim_{r \to \infty} \frac{1}{|r|} \log \left( \sum_{\vec{x}, \vec{\alpha} = r} G(\vec{x}) \right),
\]

were \( \hat{n} = (\cos \theta, \sin \theta) \) is a unit vector and the summation runs over all \( \vec{x} \) such that \( \vec{x} \cdot \hat{n} = r \). From the
and \( \Delta_{nr} = 3 \) on the triangular lattice. In Section 3.6 we will test (and confirm) this conjecture, by series analysis, for both square-lattice and triangular-lattice SAWs.

### 2.2 Corrections to scaling

Let us now make some general remarks concerning corrections to scaling. Clearly, (2.1)/(2.2)/(2.6) are only the leading term in a large-\( N \) asymptotic expansion. According to renormalization-group theory [49], the mean value of any global observable \( \mathcal{O} \) behaves as \( N \to \infty \) as

\[
\langle \mathcal{O} \rangle_N = A N^{p_\mathcal{O}} \left[ 1 + \frac{a_1}{N} + \frac{a_2}{N^2} + \cdots + \frac{b_0}{N^{\Delta_1}} + \frac{b_1}{N^{\Delta_1+1}} + \frac{b_2}{N^{\Delta_1+2}} + \cdots \right. \\
\left. + \frac{c_0}{N^{\Delta_2}} + \frac{c_1}{N^{\Delta_2+1}} + \frac{c_2}{N^{\Delta_2+2}} + \cdots \right].
\]

(2.24)

Thus, in addition to “analytic” corrections to scaling of the form \( a_k/N^k \), there are “non-analytic” corrections to scaling of the form \( b_k/N^{\Delta_1+k}, c_k/N^{\Delta_2+k} \) and so forth, as well as more complicated terms [not shown in (2.24)] which have the general form \( \text{const}/N^{k_1\Delta_1+k_2\Delta_2+\cdots+l} \) where \( k_1, k_2, \ldots \) and \( l \) are non-negative integers. The leading exponent \( p_\mathcal{O} \) and the correction-to-scaling exponents \( \Delta_1 < \Delta_2 < \cdots \) are universal; \( p_\mathcal{O} \) of course depends on the observable \( \mathcal{O} \) in question, but the \( \Delta_i \) do not. The various amplitudes (both leading and subleading) are all nonuniversal (and of course also depend on the observable\(^4\)). However, ratios of the corresponding amplitudes \( A \), \( b_0 \) and \( c_0 \) (but not \( a_k \) or the higher \( b_k, c_k \)) for different observables are universal [50, 51].

In fact, (2.24) is incomplete, as there are “mixing” terms arising from the fact that the temperature deviation from criticality is a smooth but nonlinear function of the nonlinear scaling fields \( q_t \) and \( q_h \). This has the consequence [35, 52–55] that the susceptibility (or SAW generating function), which has a leading singularity \((x_c - x)^{-\gamma} \), also contains an additive term proportional to the energy, of order \((x_c - x)^{1-\alpha} \). In the case of the two-dimensional Ising model, we have \( \alpha = 0 \), and this term is responsible for the logarithmic terms in the susceptibility, as was recently exhaustively studied in [56]. For the two-dimensional SAW, we have \( \alpha = 1/2 \), and so one would expect a term \( A(x)(x_c - x)^{1/2} \) in the SAW generating function. To incorporate this term requires that the naively expected asymptotic form

\[
c_N \sim \mu^N N^{11/32} [a_0 + a_1/N + a_2/N^{3/2} + a_3/N^2 + a_4/N^{5/2} + \cdots]
\]

(2.25)

be modified to read

\[
c_N \sim \mu^N N^{11/32} [a_0 + a_1/N + a_2/N^{3/2} + a_3/N^2 + a_4/N^{5/2} + \cdots] \\
+ \mu^N N^{-3/2} [\tilde{a}_0 + \tilde{a}_1/N + \cdots].
\]

(2.26)

exact solution [47], one can easily see that, for \( \beta \to \beta_c \),

\[
m(\hat{u}) = m_0(\beta_c - \beta)^{-1} \left[ 1 + (\beta_c - \beta)^2 (a_0 + b_0 \cos 4\theta) + \cdots \right]
\]

with \( b_0 \neq 0 \). This result shows explicitly that \( \Delta_{nr} = 2 = 2\nu \).

\(^4\)Sometimes a particular correction-to-scaling amplitude will vanish for some observables but not for others (e.g. for symmetry reasons).
For loose-packed (i.e., bipartite) lattices, such as the square and simple-cubic lattices, there is an additional set of terms arising from the antiferromagnetic singularity, of the form

$$(-1)^N N^q \left[ d_0 + \frac{d_1}{N} + \frac{d_2}{N^2} + \ldots + \frac{e_0}{N^{\Delta_1 AF}} + \frac{e_1}{N^{\Delta_1 AF + 1}} + \frac{e_2}{N^{\Delta_1 AF + 2}} + \ldots \right],$$

(2.27)

where the exponent $q$ of course depends on the observable. We know of no theoretical argument that predicts the value of the exponent $\Delta_{AF}$. For the exponent $q$, in the closely related problem of the Ising-model susceptibility in two and three dimensions, Sykes [57] has given a configurational “counting theorem” that enables one to guess that the antiferromagnetic susceptibility behaves as the internal energy. This reasoning is discussed in greater detail in [58, 59].

It follows that there should be a term in the susceptibility of the form

$$D(x)(1 + x/x_c)^{1-\alpha},$$

where $D(x)$ is analytic in a neighbourhood of the antiferromagnetic critical point $x = -x_c$ and thus a term $(-x_c)^{-N N^{\alpha - 2}}$ in the high-temperature-series coefficients. This result can be put on more solid ground [60] by noting that at the antiferromagnetic critical point the (unstaggered) magnetic field is an irrelevant variable, so that the leading contribution to the free energy is

$$F(x, h) = a_{gt}(x, h)^{2-\alpha} + F_{reg}(x, h),$$

(2.28)

where $x$ is the inverse temperature and $g_c(x, h)$ is the nonlinear scaling field associated with the temperature at the antiferromagnetic critical point. Since

$$g_t(x, h) = (1 + x/x_c) + a_t h^2 + \ldots,$$

(2.29)

by performing the appropriate derivatives we obtain the result reported above (provided of course that $a_t \neq 0$). This argument is very general and applies to any $n$-vector model; in particular, it applies for $n = 0$, i.e. to the SAW. Thus, for the SAW counts $c_N$ we expect a term $(-\mu)^N N^{\alpha - 2}$, so that $q = \alpha - 2$ for this observable [61].

The argument of Sykes can be generalized to higher moments of the two-point function, i.e., $\sum_r |r|^{2k} G(r)$. Also in this case one can identify two terms: one is proportional to the energy, while the other is conjectured to give an algebraically small (i.e., noncritical) correction at the antiferromagnetic critical point. Such a conjecture was numerically verified in [62] for the three-dimensional Ising model. As for the susceptibility, this implies that asymptotically the moments have the form $D(x)(1 + x/x_c)^{1-\alpha}$, with $D(x)$ analytic, for any $k$. Therefore, a term $(-\mu)^N N^{\alpha - 2}$ should be present in their high-temperature-series coefficients. Extending this conjecture to the $n$-vector model and in particular to the SAW ($n = 0$), we predict

$$c_N \sim \mu^N N^{\gamma - 1}[a_0 + \ldots] + (-\mu)^N N^{\alpha - 2}[d_0 + \ldots],$$

(2.30)

$$c_N \langle r^{2k} \rangle_N \sim \mu^N N^{2k \nu + \gamma - 1}[a_0 + \ldots] + (-\mu)^N N^{\alpha - 2}[d_0 + \ldots].$$

(2.31)

5The basic idea is that the susceptibility can be rewritten as the sum of two terms: one proportional to the energy, and a second one which can be argued (by series analysis) to give an algebraically small contribution near the antiferromagnetic critical point.
For $k = 1$, (2.31) gives the behaviour of $c_N \langle R^2 \rangle_N$. It may seem natural to generalize the expression (2.31) to the other metric quantities, namely $R^2_m$ and $R^2_g$. Surprisingly (to us), our subsequent analysis (see Section 3.4.2) shows that, while the unnormalized second-moment series of the end-to-end distance series behaves precisely as expected in (2.31), the unnormalized series corresponding to both the radius of gyration and the mean monomer-endpoint distance behave a little differently. We find

$$c_N \langle R^2_{g,m} \rangle_N \sim \mu^N N^{2\nu+\gamma-1} [a'_0 + \cdots] + (-\mu)^N [d'_0 + \cdots]. \quad (2.32)$$

That is to say, the antiferromagnetic exponent is different in the latter cases, namely 0 instead of $\alpha - 2 = -3/2$. Nevertheless, by taking the quotient of either (2.31) or (2.32) by (2.30), we obtain in all cases

$$\langle R^2 \rangle_N \sim N^{2\nu} [a''_0 + \cdots] + (-1)^N N^q [d''_0 + \cdots] \quad (2.33)$$

with $q = 2\nu + \alpha - 1 - \gamma$. [For the end-to-end distance, the dominant antiferromagnetic correction always comes only from (2.30); for the other two metric quantities, it comes from both (2.30) and (2.32).] For the end-to-end distance only, we have the additional relation

$$\frac{a''_0}{a'_0} = -\frac{a_0}{d'_0}. \quad (2.34)$$

Similarly, the rotationally-invariant higher moments $\langle r^{2k} \rangle_N$ are expected to behave as

$$\langle r^{2k} \rangle_N \sim N^{2k\nu} [a'''_0 + \cdots] + (-1)^N N^{q_k} [d'''_0 + \cdots] \quad (2.35)$$

with $q_k = 2k\nu + \alpha - 1 - \gamma$. The coefficients $a'''_0$ and $d'''_0$ also satisfy a relation analogous to (2.34). Our numerical analysis, described below, provides supporting evidence that the corresponding exponents are indeed $q_k = 3k/2 - 59/32$ in two dimensions (see Sections 3.4.2 and 3.5).

Finally, the non-analytic correction-to-scaling exponent $\Delta^\text{AF}_1$ was found numerically, in the case of the square-lattice SAW counts, to be 1 [17]. It would seem likely that this value should also hold for other properties, such as the metric quantities $\langle R^2 \rangle_N$. Our numerical studies, discussed below, are consistent with this conjecture—or, put another way, they are insufficiently sensitive to refute this obvious first guess.

Let us now return to the question of the corrections to the CSCPS relation $\lim_{N \to \infty} F_N = 0$ [cf. (2.10)]. Series analysis and Monte Carlo simulations (see Sections 3.4.2 and 4.2 below) indicate that $F_N \propto N^{-3/2}$, i.e. that the leading analytic correction cancels. Such a cancellation may seem surprising, but it can be understood by means of a standard renormalization-group argument. Consider the continuum $O(n)$ model with Hamiltonian

$$\mathcal{H} = \mathcal{H}^* + \int d^2 r \left[ tE(r) + hs^1(r) \right], \quad (2.36)$$

where $\mathcal{H}^*$ is the fixed-point Hamiltonian, and $E(r)$ and $s^i(r)$ are the energy and spin operators, respectively. The CSCPS relation [29,30] is a consequence of the sum rule

$$\int d^2 r \langle \Theta(0) \rangle \langle \Theta(r) \rangle = 0, \quad (2.37)$$

11
where $\Theta(r)^\text{cont}$ is the trace of the continuum stress-energy tensor, and of course we must set $n = 0$ to obtain SAWs. In order to translate this continuum relation into a lattice one, we must relate the continuum operator to its lattice counterpart. It is natural to assume that the trace of the lattice stress-energy tensor, $\Theta(r)^\text{latt}$, whose explicit form is given in [29], behaves as

$$\Theta(r)^\text{latt} = Z(t, h)\Theta(r)^\text{cont} + \ldots,$$

where $Z(t, h)$ is a smooth function of $t$ and $h$, and the dots represent the contributions of the subleading operators. As a consequence of (2.38) we have

$$\int d^2r \langle \Theta(0)^\text{latt}\Theta(r)^\text{latt} \rangle = O(t^{\Delta_1}, h^{(y_T/y_H)\Delta_1}).$$

No corrections of order $t$ appear in the previous relation. Eq. (2.39) therefore implies the absence of the analytic corrections in the CSCPS relation $F_N \to 0$.

The observation that $F_N \propto N^{-3/2}$ implies a constraint on the subdominant amplitudes. More precisely, if we write

$$\langle R^2_e \rangle_N \sim a_e N^{3/2} + b_e N^{1/2} + c_e + O(1/\sqrt{N})$$

$$\langle R^2_g \rangle_N \sim a_g N^{3/2} + b_g N^{1/2} + c_g + O(1/\sqrt{N})$$

$$\langle R^2_m \rangle_N \sim a_m N^{3/2} + b_m N^{1/2} + c_m + O(1/\sqrt{N}),$$

then the original CSCPS relation $F_N \to 0$ implies

$$91a_e = 364a_m - 492a_g,$$

while the absence of a $1/N$ term in $F_N$ means that the leading subdominant terms also satisfy an amplitude relationship analogous to (2.43), namely

$$91b_e = 364b_m - 492b_g.$$

Note too from (2.11) that

$$f \equiv \lim_{N \to \infty} f_N = (2 + y_T/y_H)c_e - 2c_m + c_g/2.$$

Let us conclude by discussing briefly the behaviour of “effective exponents”. Given a function $f(N)$, let us define $\Delta_{\text{eff}}(N)$ by fitting locally to the Ansatz $f(N) = a + b/N^\Delta$: this gives

$$\Delta_{\text{eff}}(N) \equiv -\frac{d \log f'(N)}{d \log N} - 1.$$

Applying this to $f(N) = a_0 + a_1/N^{\Delta_1} + a_2/N^{\Delta_2}$, we obtain

$$\Delta_{\text{eff}}(N) = \frac{a_1\Delta_1^2 N^{-\Delta_1} + a_2\Delta_2^2 N^{-\Delta_2}}{a_1\Delta_1 N^{-\Delta_1} + a_2\Delta_2 N^{-\Delta_2}}.$$

Thus, if $a_1$ and $a_2$ have the same sign, the “effective” exponent $\Delta_{\text{eff}}(N)$ lies between $\Delta_1$ and $\Delta_2$ for all $N$, and decreases monotonically to $\Delta_1$ as $N \to \infty$. (This is the behaviour
one would expect intuitively for an “effective exponent”.) By contrast, if $a_1$ and $a_2$ have opposite signs, then $\Delta_{\text{eff}}(N)$ starts above $\Delta_2$ for small $N$, increases monotonically and reaches $+\infty$ at some finite value of $N$; it then jumps to $-\infty$, after which it continues to increase monotonically, tending asymptotically to $\Delta_1$ from below as $N \to \infty$. Thus, the qualitative behaviour of the effective exponents depends crucially on the sign (and magnitude) of $a_1/a_2$, which can vary from one observable to another. We shall see this phenomenon quite clearly in the two-dimensional SAW.

3 Series analysis

3.1 Summary of our data

We have previously reported enumerations of square-lattice SAWs up to 29 steps for $c_N$, $\langle R_2^2 \rangle_N$, $\langle R_4^2 \rangle_N$, and $\langle R_6^2 \rangle_N$ [63, 64], and of triangular-lattice SAWs up to 22 steps for $c_N$ and $\langle R_2^2 \rangle_N$ [63] and up to 19 steps for $\langle R_4^2 \rangle_N$ and $\langle R_6^2 \rangle_N$ [64]. Analysis of these series can be found in [63,64]. We have also previously presented the square-lattice SAW counts $c_N$ up to 51 steps [17] and the square-lattice polygon counts $p_{2N}$ up to 90 steps [19]. Analysis of these SAW series [17,19] provided good evidence that the non-analytic correction-to-scaling exponent is exactly $\Delta_1 = 3/2$ as predicted by Nienhuis [4, 5], and that there is also the expected analytic term of leading order $1/N$ (as well as $1/N^2, 1/N^3, \ldots$). For SAPs we found compelling evidence for purely analytic correction-to-scaling terms. We have thus far found no numerical evidence of a second non-analytic correction-to-scaling exponent $\Delta_2$, although it is reasonable to expect that one exists.

In the present paper, we have extended the previous work by enumerating all SAWs on the square lattice up to 59 steps, and on the triangular lattice up to 40 steps, using refinements of the finite-lattice method (FLM) due to Rogers (unpublished) and Jensen [65]. The results for $c_N$, $\langle R_2^2 \rangle_N$, $\langle R_4^2 \rangle_N$, and $\langle R_6^2 \rangle_N$ are collected in Tables 1 and 2.

For square-lattice SAPs, the counts are now known up to 110 steps [66], and the radii of gyration up to 100 steps [66]. For triangular-lattice SAPs, the counts $p_N$ were previously known up to $N = 35$ [67]; in as-yet-unpublished work, one of us (ANR) has extended them up to $N = 40$ [68], while even more recently another of us (IJ) has extended the series to $N = 60$ steps [69].

As the FLM does not enable us to record the full end-point distribution, nor higher moments (at least not with the amount of memory available to us), we also programmed a conventional backtracking algorithm and recorded the full end-point distribution $c_N(x)$ for $N \leq 32$ (square lattice) and for $N \leq 22$ (triangular lattice). As a result, we are able to study arbitrary moments. We refrain here from inundating the reader with the complete tables of $c_N(x)$; they are available on www.ms.unimelb.edu.au/~iwan. However, we do list here most of those series that we subsequently analyse. For the square lattice, we give in Table 3 the rotationally invariant moments $\langle r^4 \rangle_N$, $\langle r^6 \rangle_N$ and $\langle r^8 \rangle_N$, and in Table 4 the corresponding non-rotationally-invariant moments $\langle r^4 \cos 4\theta \rangle_N$, $\langle r^6 \cos 4\theta \rangle_N$, and $\langle r^8 \cos 4\theta \rangle_N$. For the triangular lattice, we give in Table 5 the rotationally invariant moments $\langle r^4 \rangle_N$, $\langle r^6 \rangle_N$, and $\langle r^8 \rangle_N$, and in Table 6 the corresponding non-rotationally-
invariant moments \( \langle r^6 \cos 6\theta \rangle_N \) and \( \langle r^8 \cos 6\theta \rangle_N \).

### 3.2 Method of analysis

In this subsection we explain in detail the method we used to analyse the data, the results of which are reported in subsequent subsections. For the triangular lattice, we expect the series coefficients of any generic quantity, such as the SAW generating function, to have an asymptotic expansion of the form

\[
\mu^N N^{\gamma-1} \left( a_0 + \sum_{i=1}^{k} \frac{a_i}{N^i} \right),
\]

where \( \mu \) is the connective constant and \( \gamma \) is the critical exponent. Likewise, we expect the square-lattice series coefficients to have an asymptotic expansion of the form

\[
\mu^N N^{\gamma-1} \left( a_0 + \sum_{i=1}^{k} \frac{a_i}{N^i} \right) + (-\mu)^N N^{\alpha-2} \left( b_0 + \sum_{i=1}^{m} \frac{b_i}{N^i} \right),
\]

where \( \alpha \) is the critical exponent occurring in the polygon generating function. Similar expansions hold for metric quantities, and involve also the critical exponent \( \nu \). Since the exact values \( \gamma = 43/32 \), \( \alpha = 1/2 \) and \( \nu = 3/4 \) are well established, we shall use them throughout this paper.

Given the calculated terms of the series up to some order \( N_{\text{max}} \), we proceed as follows: First we decide how many correction terms \( \{a_i\} \) and \( \{b_i\} \) we wish to include (i.e., we fix the numbers \( k \) and \( m \)); then we make some assumption for the values of \( \mu, \Delta_i \) and \( \Delta_{i,\text{AF}} \); finally, we fit the data to (3.1) or (3.2) by taking \( (k + m + 2) \)-tuples of successive values of \( N \) and solving for \( \{a_i\} \) and \( \{b_i\} \). This can be done by solving a system of linear equations.

By using \( (k + m + 2) \)-tuples at steadily larger values of \( N \), many estimates for the \( \{a_i\} \) and \( \{b_i\} \) are found. If the different estimates seem stable as \( N \) grows, we presume that they provide an acceptably accurate estimate of the actual asymptotic coefficients.

A noteworthy feature of the method is that, if a blatantly-too-low correction-to-scaling exponent is given as input (for example, specifying \( \Delta_1 = 1/2 \) for the two-dimensional SAW), the sequence of amplitude estimates for the term corresponding to that exponent will converge rapidly to zero, giving a very strong signal that the exponent in question is absent. (Of course, if such a term were to occur with an amplitude several orders of magnitude smaller than the amplitudes of the other terms, one could be fooled into thinking such a term is absent. Our analysis assumes the absence of such pathologies.)

Another point to bear in mind is that even if one knows the precise asymptotic form, with a limited number of series coefficients one can fit only to a small number of asymptotic terms (i.e., \( k \) and \( m \) cannot be taken too large). Beyond a certain number of terms in the asymptotic form, the quality of the fit visibly deteriorates. The more series coefficients are available, the more terms can be included in the Ansatz (provided that sufficient numerical precision is retained during the analysis).
A universally observed feature of the method is that the apparent accuracy of the amplitude estimates decreases rapidly as we move to higher-order terms in the asymptotic expansions. That is to say, the apparent accuracy of the estimate of amplitude $a_{i+1}$ is significantly less than that of $a_i$. Moreover, adding further terms in the assumed asymptotic form (i.e., increasing $k$ and $m$) improves convergence of the low-order amplitudes $a_i$ provided that $k + m$ does not get too large, but after a certain point actually slows the convergence. In the case at hand, allowing more than 2–5 terms (these being the values of $k + 1$ and $m + 1$ separately) in the assumed asymptotic form led to a deteriorating (i.e., less stable) fit.

As the series data at very small $N$ are probably not reflective of asymptotic behaviour, and we have here the luxury of access to many terms (i.e., quite large $N_{\text{max}}$), the first $19 - k - m$ terms of the series will not be used in any of our analyses here.

Our analysis thus comprises two phases. In the first phase, we determine the correct connective constant $\mu$ and the correct exponents $\Delta_i$ and $\Delta_{i}^{AF}$ for the asymptotic expansion, as just described. In the second phase, we determine how many terms in the asymptotic expansion we can reliably use. We now describe our procedure for the second phase of the analysis.

We begin by fitting for only one correction coefficient, $a_1$. Then we add further asymptotic terms until the estimates obtained do not appear to be converging as $N \to \infty$ to a value that is consistent with the previous estimates given by fits with one fewer asymptotic term. We define “consistent” by the requirement that estimates of all included asymptotic coefficients be well-converged and of the same sign and within a factor $F = 2.4$ of the previous estimates. More specifically, we invoke this requirement as follows: setting $k = K$ gives estimates of $a_0, \ldots, a_K$; repeating the analysis with $k = K + 1$ yields estimates of $a_0, \ldots, a_{K+1}$. We require that the coefficients $a_0, \ldots, a_K$ from the two fits agree in sign and in magnitude within a factor of $F$; otherwise, we reject the fit with $k = K + 1$ and stop at $k = K$. Note that the choice of the value of $F$ is somewhat arbitrary. Realistically, one can reasonably make any choice in the range $1.5 \lesssim F \lesssim 3$, the lower value being more conservative. We chose a value in this range that included most data sets with small values of $k$ and $m$, and excluded those with higher values.

Please note that the convergence (as $N$ grows) of each fit is here judged by traditional intuitive (and thus somewhat subjective) methods. It would be an interesting project to find a precise definition of “well-converged” (or its synonym, “stable”) that accords satisfactorily with our intuitive judgments and gives good results on test series; this would allow the series analysis to be converted into a precise algorithm. But we do not purport to carry out such a project here.

For the triangular lattice, this procedure is thus relatively simple to implement. We compute fits initially with $k = 0$, incrementing $k$ by 1 until a non-stable or inconsistent estimate (as defined in the preceding paragraph) is found; we then revert to the previous group of stable and consistent estimates. The final entries (i.e., those corresponding to the maximum $N$) in the largest stable group are taken as our final estimates.

For the square lattice, the procedure is more complicated, as it is not clear a priori whether terms involving $\Delta_i$ or $\Delta_{i}^{AF}$ should be added to a given group. Empirically we have found that groups containing approximately equal numbers of $\Delta_i$ and $\Delta_{i}^{AF}$ terms,
or slightly more $\Delta_i$ terms, are more stable than estimates with significantly different values of $k$ and $m$. Hence we begin by exploring groups with equal numbers of $\Delta_i$ and $\Delta_i^{AF}$ terms, that is with $k = m$, adding one coefficient to each group at every stage. Next we try groups with one more $\Delta_i$ term than $\Delta_i^{AF}$ terms, so that $k = m + 1$; and finally we try groups with two more $\Delta_i$ terms, so that $k = m + 2$. Again, the largest group that provides stable and consistent estimates is selected. As always, the given estimates are taken from the fits to the largest available value of $N$, which is $N_{max}$, since these should best reflect the asymptotic regime.

The estimated error is calculated as the change between the estimate given by the longest series and the series ten terms shorter, multiplied by a fact or reflective of the expected rate of convergence of the estimates. This latter factor is determined by assuming that the error in the estimates is principally given by the first omitted $\Delta_i$ or $\Delta_i^{AF}$ term. The difference in the exponents between the term in question and the first omitted term is then used to predict the value of the estimate on a fit to an infinite series.

To illustrate these procedures, we show below the output from fitting the triangular-lattice series for $c_N\langle R_g^2 \rangle_N$ with increasing numbers of correction-to-scaling terms. We make the Ansatz

$$(N + 1)^2 c_N \langle R_g^2 \rangle_N / 6 \sim \mu^N N^{123/32} [a_0 + a_1/N + a_2/N^{3/2} + a_3/N^2 + a_4/N^{5/2} + \cdots] \quad (3.3)$$

and obtain fits as follows:

| $N$ | $a_0$       | $a_1$     |
|-----|-------------|-----------|
| 21  | 0.01929438  | 0.08276963|
| 22  | 0.01932692  | 0.08208623|
| 23  | 0.01935556  | 0.08145624|
| 24  | 0.01938091  | 0.08087327|
| 25  | 0.01940346  | 0.08033194|
| 26  | 0.01942363  | 0.07982768|
| 27  | 0.01944175  | 0.07935660|
| 28  | 0.01945809  | 0.07891535|
| 29  | 0.01947289  | 0.07850104|
| 30  | 0.01948633  | 0.07811114|
| 31  | 0.01949859  | 0.07774345|
| 32  | 0.01950980  | 0.07739603|
| 33  | 0.01952007  | 0.07706717|
| 34  | 0.01952952  | 0.07675536|
| 35  | 0.01953823  | 0.07645924|
| 36  | 0.01954628  | 0.07617761|
| 37  | 0.01955373  | 0.07590938|
| 38  | 0.01956064  | 0.07565359|
| 39  | 0.01956707  | 0.07540935|
| 40  | 0.01957306  | 0.07517587|
| N  | $a_0$     | $a_1$     | $a_2$     | $a_3$     |
|----|-----------|-----------|-----------|-----------|
| 22 | 0.01976614| 0.05375917| 0.08754307|           |
| 23 | 0.01976114| 0.05408172| 0.08654625|           |
| 24 | 0.01975680| 0.05437453| 0.08562049|           |
| 25 | 0.01975301| 0.05464205| 0.08475611|           |
| 26 | 0.01974966| 0.05488785| 0.08394517|           |
| 27 | 0.01974669| 0.05511490| 0.08318096|           |
| 28 | 0.01974404| 0.05532562| 0.08245789|           |
| 29 | 0.01974166| 0.05552207| 0.08177122|           |
| 30 | 0.01973951| 0.05570593| 0.08111693|           |
| 31 | 0.01973756| 0.05587867| 0.08049156|           |
| 32 | 0.01973578| 0.05604149| 0.07989215|           |
| 33 | 0.01973415| 0.05619545| 0.07931614|           |
| 34 | 0.01973265| 0.05634145| 0.07876131|           |
| 35 | 0.01973127| 0.05648027| 0.07822571|           |
| 36 | 0.01972999| 0.05661259| 0.07770765|           |
| 37 | 0.01972881| 0.05673898| 0.07720563|           |
| 38 | 0.01972770| 0.05685998| 0.07671833|           |
| 39 | 0.01972667| 0.05697603| 0.07624459|           |
| 40 | 0.01972571| 0.05708754| 0.07578337|           |

| N  | $a_0$     | $a_1$     | $a_2$     | $a_3$     |
|----|-----------|-----------|-----------|-----------|
| 23 | 0.01970992| 0.06084150| 0.04428770| 0.07428718|
| 24 | 0.01971020| 0.06080458| 0.04451851| 0.07388144|
| 25 | 0.01971034| 0.06078451| 0.04464682| 0.07365079|
| 26 | 0.01971039| 0.06077784| 0.04469036| 0.07357085|
| 27 | 0.01971036| 0.06078284| 0.04465705| 0.07363327|
| 28 | 0.01971026| 0.06079722| 0.04455931| 0.07382009|
| 29 | 0.01971013| 0.06081948| 0.04440510| 0.07412046|
| 30 | 0.01970995| 0.06084380| 0.04420187| 0.07452360|
| 31 | 0.01970976| 0.06082254| 0.04395605| 0.07501986|
| 32 | 0.01970954| 0.06092130| 0.04367303| 0.07560099|
| 33 | 0.01970931| 0.06096380| 0.04335757| 0.07625946|
| 34 | 0.01970908| 0.06100940| 0.04301375| 0.07698862|
| 35 | 0.01970883| 0.06105753| 0.04264515| 0.07778247|
| 36 | 0.01970859| 0.06110773| 0.04225488| 0.07863562|
| 37 | 0.01970834| 0.06115962| 0.04184568| 0.07954324|
| 38 | 0.01970809| 0.06121284| 0.04141995| 0.08050094|
| 39 | 0.01970785| 0.06126712| 0.04097980| 0.08150474|
| 40 | 0.01970761| 0.06132220| 0.04052709| 0.08255105|
We observe in these fits the following behaviour: As $N$ increases, the estimates of the amplitudes $a_i$ appear to be converging in each set, until we reach the set with five asymptotic coefficients ($a_0, \ldots, a_4$). In this latter fit, we see that the estimate of $a_4$ appears to be diverging. Further, the estimates of $a_1$, $a_2$ and $a_3$ have deteriorating apparent convergence as we go from a four-term to a five-term fit. By contrast, going from a two-term to a three-term fit, and from a three-term to a four-term fit, improved the apparent convergence of the amplitude sequences. Thus we reject the five-term fit, and base our estimates on the four-term fit.

Finally, the estimated error from a series of length $N$ is taken to be the appropriately scaled difference between the values obtained from this series and those obtained from the series of length $N - 10$. This difference is scaled by a factor dependent on the difference between the exponent in question and the first omitted exponent. The scaling factor follows from our assumption that the error is given principally by the first neglected term, $c/N^{\Delta k+1}$ (or similarly with $\Delta_{m+1}$). Hence, if the actual value of the coefficient in question is $a_i$ and the two estimates are $a_i^{(N)}$ and $a_i^{(N-10)}$, we expect that

\[
\frac{a_i^{(N)}}{N^{\Delta_i}} = \frac{a_i}{N^{\Delta_i}} + \frac{c}{N^{\Delta_{k+1}}}
\]

\[
\frac{a_i^{(N-10)}}{(N-10)^{\Delta_i}} = \frac{a_i}{(N-10)^{\Delta_i}} + \frac{c}{(N-10)^{\Delta_{k+1}}}
\]

Simple algebra then yields

\[
a_i^{(N)} - a_i = -\frac{a_i^{(N)} - a_i^{(N-10)}}{(N/N^{10})^{\Delta_{k+1} - \Delta_i} - 1}.
\]

Therefore, $a_i$ is estimated by $a_i^{(N)}$ with error quoted as

\[
2|a_i^{(N)} - a_i^{(N-10)}|/[(N/N^{10})^{\Delta_{k+1} - \Delta_i} - 1].
\]
The factor of 2 is included to make our errors more conservative. More adventurous readers may choose to reduce this factor.

In the above example, the first omitted term is \( O(N^{-5/2}) \). The difference in the estimate of \( a_0 \) from the \( N = 40 \) series with four asymptotic coefficients (0.019707611) and that from a \( N = 30 \) series (0.019709959) is 2.348 \( \times 10^{-6} \). Thus the error is quoted as \( 2 \times 2.348 \times 10^{-6}/((40/30)^{2.5} - 1) = 4.5 \times 10^{-6} \). Our amplitude estimate is then \( a_0 = 0.019708 \pm 0.000005 \). Similarly, \( a_1 = 0.0613 \pm 0.0018 \), where the error is given by \( 2 \times 0.0004739/((40/30)^{1.5} - 1) = 4.5 \times 10^{-6} \).

\[ a_0 = 0.019708 \pm 0.000005, \quad a_1 = 0.0613 \pm 0.0018, \quad a_2 = 0.04 \pm 0.02, \quad a_3 = 0.08 \pm 0.1. \]

### 3.3 SAW counts

In this subsection we discuss the analysis of the newly extended series for SAW counts on the square and triangular lattices. Here we give only a brief analysis, as fuller details will be published elsewhere [65], along with a discussion of the series derivation.

Let us begin with the triangular lattice. We first analysed the extended SAP series using biased differential approximants using the known exponent \( \alpha = 1/2 \) (see [19] for the method). We obtained

\[ x_c = 1/\mu = 0.240 \, 917 \, 574 \pm 0.000 \, 000 \, 004, \quad (3.7) \]

which we will use in subsequent analyses.

We also performed a similar analysis using the extended SAW series, biasing the estimate with the known exponent \( \gamma = 43/32 \). We obtained the estimate \( x_c = 1/\mu = 0.240 \, 917 \, 579 \pm 0.000 \, 000 \, 008 \), in agreement with the SAP result but less precise.

Using the estimate (3.7) of \( x_c \), we proceeded as described in Section 3.2 to fit the series coefficients to various asymptotic forms. For triangular-lattice SAWs, we expect, based on earlier investigations of the corresponding square-lattice series [17], that

\[ c_N \sim \mu^N N^{11/32} \left[ a_0 + a_1/N + a_2/N^{3/2} + a_3/N^2 + a_4/N^{5/2} + a_5/N^3 + \cdots \right]. \quad (3.8) \]

However, as discussed in Section 2.2, renormalization-group theory predicts an additional "energy-like" term arising from the mixing between nonlinear scaling fields. For the two-dimensional SAW (\( \alpha = 1/2 \)), incorporating this term requires that (3.8) be modified to read

\[ c_N \sim \mu^N N^{11/32} \left[ a_0 + a_1/N + a_2/N^{3/2} + \tilde{a}_0/N^{59/32} + a_3/N^2 + a_4/N^{5/2} + \cdots \right]. \quad (3.9) \]

[cf. (2.26)].

In our analysis, we tried both the asymptotic forms (3.8) and (3.9). Since the exponents associated with amplitudes \( \tilde{a}_0 \) and \( a_3 \) are numerically close (1.84375 and 2, respectively), we expect that it will be very difficult to distinguish numerically between the Ansätze (3.8) and (3.9). This is indeed the case. Under the assumption (3.8), we found that the sequences corresponding to the amplitudes are well converged up to \( k = 4 \) for triangular-lattice SAW and we estimate \( a_0 = 1.183966(1), \ a_1 = 0.5960(4), \ a_2 = -0.274(6), \ a_3 = -0.14(4), \) and \( a_4 = 0.09(10) \). Our errors are calculated as described in Section 3.2 and are given, in parentheses, as the uncertainty in the last quoted
digit(s). Under the alternative Ansatz (3.9), we find that the fit is neither better nor worse. We observed that the sequences of estimates of the corresponding amplitudes $\tilde{a}_0$ and $a_3$ appear to be correlated: they are monotonically increasing in magnitude but are of opposite sign, the sum $\tilde{a}_0 + a_3$ being almost constant.

To investigate this point further, we constructed a test series, with known asymptotic behaviour, similar to that in (3.9), namely

$$d_N = N^{11/32}[1 + 1/N + 0.7/N^{3/2} + 1.25/N^{59/32} + 3/N^2 - 4/N^{5/2} + 5/N^3 - 6/N^{7/2}] + (0.5)^N.$$  

(3.10)

The last term is included to incorporate the fact that there are other singularities in the complex plane, beyond $x_c$, which will make an exponentially decaying contribution to the asymptotics. We generated the first 1000 terms of this sequence and analysed them as above, including either a term $N^{-59/32}$ or a term $N^{-2}$ or both. The analyses using either one of these two terms behaved similarly. The analysis using both terms gave *inferior* estimates of the first three amplitudes, and the *wrong sign* for the amplitude of the $N^{-2}$ term, when $N \lesssim 240$. Only beyond this point does the analysis using both terms give superior estimates of the first three amplitudes, along with the right sign for the $N^{-2}$ term (the two issues clearly go together). We conclude that using series of the length available to us ($N \lesssim 40$), it is unfeasible to determine whether a term $N^{-59/32}$ is present or absent.

In conclusion, our analysis is unable to resolve the question of whether the “energy-like” term $N^{-59/32}$ is present or not. Therefore, for the subsequent analysis of the metric quantities, reported in the next subsection, we have assumed for simplicity the absence of this term, and just assumed the asymptotic form (2.24) with one correction-to-scaling exponent $\Delta_1 = 3/2$.

On the square lattice, the situation is complicated by the presence of an “antiferromagnetic” singularity at $x = -1/\mu$. From (2.27) ff. we recall that the asymptotic form of the coefficients given in (3.8) and (3.9) is modified by the additional term

$$(-1)^N \mu^N N^{-3/2}[d_0 + d_1/N + d_2/N^2 + d_3/N^3 + \cdots].$$  

(3.11)

Analysing the square-lattice SAW data using as our estimate of $x_c$ the positive real root of

$$581x^4 + 7x^2 - 13 = 0,$$  

(3.12)

which is a useful mnemonic for the current best estimate $x_e^2 = 0.14368062927(1)$ [19], we obtain a very convincing fit with $k = 3$ and $m = 2$, enabling the following amplitude estimates to be made: $a_0 = 1.1770425(7)$, $a_1 = 0.5501(2)$, $a_2 = -0.1402(3)$ and $a_3 = -0.12(2)$ for the “ferromagnetic” amplitudes, and $d_0 = -0.189848(3)$, $d_1 = 0.17473(9)$ and $d_2 = -1.51(1)$ for the “antiferromagnetic” amplitudes. This is in close agreement with the earlier estimates (1.2) based on slightly shorter series; here we have obtained a slight improvement in the precision of the estimates for the leading amplitudes $a_0$, $d_0$ and $d_1$.

If instead we assume the asymptotic form (3.9) for the “ferromagnetic” term, we find that estimates of $a_3$ are small (less than 0.03 in magnitude) and tending toward zero. Estimates of $a_4$ are tending toward the estimate for this term obtained in the absence of
the additional term with amplitude $a_3$. Once again, we are unable to distinguish (3.8) from (3.9).

### 3.4 Metric quantities $\langle R_e^2 \rangle$, $\langle R_g^2 \rangle$ and $\langle R_m^2 \rangle$

In this subsection we shall analyse the metric quantities $\langle R_e^2 \rangle_N$, $\langle R_g^2 \rangle_N$ and $\langle R_m^2 \rangle_N$, any one of which we shall generically denote by $\langle R^2 \rangle_N$. As discussed in Section 2.2 [cf. (2.24)/(2.27)], their asymptotic behaviour is expected in the first instance to be

$$\langle R^2 \rangle_N \sim N^{2\nu}[a_0 + a_1/N + a_2/N^{\Delta_1} + a_3/N^2 + \cdots]$$

(3.13)

for the triangular lattice, and

$$\langle R^2 \rangle_N \sim N^{2\nu}[a_0 + a_1/N + a_2/N^{\Delta_1} + a_3/N^2 + \cdots] + (-1)^N N^q[b_0 + b_1/N^{\Delta_1AF} + b_2/N + \cdots]$$

(3.14)

for the square lattice. As mentioned earlier, there is overwhelming numerical evidence [17,37] that the leading exponent $2\nu$ equals $3/2$ exactly, as predicted by Coulomb-gas arguments [4,5]; we shall henceforth take this fact for granted. We also expect $\Delta_1 = 3/2$, as predicted by Nienhuis [4] and confirmed numerically for the SAW counts. Furthermore, for the square lattice we predict $q = -11/32$ [cf. (2.33)]. Finally, in Section 2.2 we pointed out that renormalization-group theory predicts an additional “energy-like” term in the susceptibility [cf. (3.9)], though alas we were unable to distinguish it numerically (see Section 3.3); it is reasonable to guess that there may be a corresponding term also in the series for the unnormalized second moments $c_N\langle R^2 \rangle_N$. Whether or not the latter term is present, the existence of an “energy-like” term in $c_N$ will induce in $\langle R^2 \rangle_N$ additional correction-to-scaling terms $N^{-59/32}, N^{-91/32}, \ldots$ beyond those included in (3.13)/(3.14).

In addition to the normalized metric quantities $\langle R^2 \rangle_N$, we also studied the corresponding unnormalized quantities $c_N\langle R^2 \rangle_N$, whose expected asymptotic form is

$$c_N\langle R^2 \rangle_N \sim \mu^N N^{2\nu+\gamma-1}[b_0 + b_1/N + b_2/N^{\Delta_1} + b_3/N^2 + \cdots],$$

(3.15)

with appropriate additional antiferromagnetic terms (2.31) when analysing the square-lattice data. The latter quantities have the disadvantage that the analysis depends sensitively on an input estimate of $\mu$; but, for loose-packed lattices and for $\langle R_e^2 \rangle_N$ only, they have the advantage that the effect of the antiferromagnetic singularity is weaker. To see this, compare (2.30)-(2.33): the antiferromagnetic contribution in $c_N\langle R_e^2 \rangle_N$ is relatively weaker than that in $c_N$; but the antiferromagnetic contribution in $\langle R_e^2 \rangle_N$ is dominated by that in $c_N$. Therefore, the antiferromagnetic contribution is relatively weaker in $c_N\langle R_e^2 \rangle_N$ than in $\langle R_e^2 \rangle_N$.

Our method of analysis is based on directly fitting $\langle R^2 \rangle_N$ and $c_N\langle R^2 \rangle_N$ to the assumed asymptotic form (3.13)/(3.14)/(3.15), as described in Section 3.2. The values of the exponents $\nu, q, \Delta_1$ and $\Delta_1^{AF}$ are assumed, and the appropriate system of linear equations is solved to give estimates of the amplitudes $\{a_i\}$ and $\{b_i\}$. In applying the method to metric quantities (see, for example, the table in Section 3.2), the fit to the
leading amplitude is rather stable, that to the first analytic correction term is moderately stable, while the fit to the amplitude of the assumed correction-to-scaling term $N^{-3/2}$ converges less impressively for both the normalized and unnormalized metric quantities. As already noted, adding further terms in the assumed asymptotic form beyond the first initially improved convergence, but this improvement is not sustained. That is to say, allowing more than between two and five terms in the assumed asymptotic form led to an apparently deteriorating fit.

### 3.4.1 Triangular lattice

With our 40-term triangular-lattice series we found that we could fit to $a_0$, $a_1$, $a_2$ and sometimes $a_3$. For the normalized and unnormalized metric quantities, the estimate of $a_0$ could usually be made to four-digit precision, while the estimate of $a_1$ could be made only to one or two significant digits, and the estimate of $a_2$ is accurate only to at best one significant digit. For $a_3$ the error is comparable to or greater than the estimate.

We have applied this analysis method to the triangular-lattice data, using a 40-term series for all metric quantities. Because the triangular lattice is close-packed, there is only one singularity on the circle of convergence, which makes the analysis simpler than for the square lattice [compare (3.13) to (3.14)].

The tables of estimates for the metric quantities obtained according to the procedure described in Section 3.2 are shown in Table 7. The results are:

\[
\langle R^2_e \rangle_N \sim N^{3/2}[0.71174(32) + 0.95(12)/N - 2.6(1.6)/N^{3/2} + 3(7)/N^2 + O(1/N^{5/2})]
\] (3.16)

\[
\langle R^2_g \rangle_N \sim N^{3/2}[0.09989(4) + 0.056(16)/N + 0.3(2)/N^{3/2} - 0.2(1.0)/N^2 + O(1/N^{5/2})]
\] (3.17)

\[
\langle R^2_m \rangle_N \sim N^{3/2}[0.3133(4) + 0.24(12)/N - 2(1.0)/N^{3/2} + O(1/N^2)].
\] (3.18)

These were obtained with $k = 3, k = 3$, and $k = 2$ respectively. Unfortunately the uncertainties in the coefficients of the $O(1/N^{3/2})$ are so great as to be comparable to (or, in the case of $\langle R^2_m \rangle$, larger than) the coefficient itself. Further, the analysis of the $\langle R^2_g \rangle_N$ series violates the convergence criterion we have set, in that the coefficient of $a_1$ differs by nearly a factor of 3 in going from a two-term fit ($k = 1$) to a three-term fit ($k = 2$). We have nevertheless presented results for $k = 3$. Our justification for this is twofold. Firstly, the estimate of $a_1$ stabilises if we then go to a four-term fit. Secondly, as we have already seen, the data for the essentially equivalent series $(N + 1)^2c_N \langle R^2_g \rangle_N/6$ supports a four-term fit. For the reader unconvinced by these arguments, the corresponding analysis with $k = 0$ (a one-term fit, as would be justified by strict adherence to the convergence criteria we have set), gives $a_0 = 0.106 \pm 0.012$.

We can do somewhat better from a similar analysis of the unnormalized metric quantities, using the estimate $\mu = 4.15079723$ from (3.7), which gave

\[
c_N \langle R^2_e \rangle_N/6 \sim \mu^N N^{59/32}[0.14045(6) + 0.256(26)/N - 0.53(32)/N^{3/2} + 0.6(1.6)/N^2 + O(1/N^{5/2})]
\] (3.19)
\[(N+1)^2c_N\langle R_g^2 \rangle_N/6 \sim \mu^N N^{123/32}[0.019708(5) + 0.0613(18)/N + 0.04(2)/N^{3/2} + 0.08(10)/N^2 + O(1/N^{5/2})] \]
\[(N+1)c_N\langle R_m^2 \rangle_N/6 \sim \mu^N N^{31/32}[0.06183(12) + 0.136(28)/N - 0.02(24)/N^{3/2} + O(1/N^2)] \]

These estimates were also obtained with \(k = 3\), \(k = 3\) and \(k = 2\) respectively. From these we can estimate the amplitudes of the metric quantities by dividing through by the asymptotic form (3.8) for \(c_N\), and accounting for the factor of 6 and appropriate factors of \((N+1)\). In this way we obtain a second set of amplitude estimates,

\[\langle R_e^2 \rangle_N \sim N^{3/2}[0.71176(15) + 0.94(7)/N - 2.5(8)/N^{3/2} + 3(3)/N^2 + O(1/N^{5/2})] \]
\[\langle R_g^2 \rangle_N \sim N^{3/2}[0.09987(2) + 0.061(5)/N + 0.23(5)/N^{3/2} - 0.5(5)/N^2 + O(1/N^{5/2})] \]
\[\langle R_m^2 \rangle_N \sim N^{3/2}[0.3133(3) + 0.22(7)/N - 0.03(60)/N^{3/2} + O(1/N^2)]. \]

These differ from the directly measured amplitudes only within the quoted errors for each amplitude, consistent with our claimed errors.

One immediate observation is that for \(\langle R_e^2 \rangle_N\) and \(\langle R_m^2 \rangle_N\) the correction-to-scaling amplitudes corresponding to the \(1/N\) and \(1/N^{3/2}\) terms are of opposite sign, while for \(\langle R_g^2 \rangle_N\) they are of the same sign (though the errors associated with the estimates of the amplitude of the \(1/N^{3/2}\) term are rather large). Note too that for both \(\langle R_e^2 \rangle_N\) and \(\langle R_g^2 \rangle_N\) the amplitude of the \(1/N^{3/2}\) term is larger (in magnitude) than the amplitudes of both the leading term and first analytic correction; for \(\langle R_m^2 \rangle\), by contrast, the error in the \(1/N^{3/2}\) term is too great to comment on the relative size of this term.

As a consequence, the “effective” exponent \(\Delta_{\text{eff}}\); based on fitting to a given range of \(N\) behaves differently as a function of \(N\) for the different observables. For \(\langle R_g^2 \rangle_N\), \(\Delta_{\text{eff}}\) lies between 1 and 3/2 for all \(N\), and decreases monotonically to 1 as \(N \to \infty\). For \(\langle R_e^2 \rangle_N\) and \(\langle R_m^2 \rangle_N\), by contrast, \(\Delta_{\text{eff}}\) starts above 3/2 for very small \(N\), then increases monotonically, reaching +\(\infty\) at some finite value of \(N\) (here \(\approx 15\)); then it jumps to \(-\infty\), after which it continues to increase monotonically, tending asymptotically to 1 as \(N \to \infty\). These observations are in accordance with previous studies. Most studies of \(\langle R_e^2 \rangle_N\) resulted in estimates for \(\Delta\) of \(\approx 0.65\) [8, 10, 13, 14, 23], while most studies of \(\langle R_m^2 \rangle_N\) resulted in estimates for \(\Delta\) in the range 1.05–1.2 [13, 14, 20, 23]. (There have been few previous studies of \(\langle R_m^2 \rangle_N\).) This is clearly a source—indeed, probably the major source—of the long-standing difficulty in the analysis of these quantities for the correction-to-scaling term.

The amplitude ratios \(A\) and \(B\) [defined as the \(N \to \infty\) limit of (2.8)] follow immediately from (3.22)–(3.24) as \(A = 0.14031\) and \(B = 0.4398\), and from (3.16)–(3.18) as \(A = 0.14033\) and \(B = 0.4402\).

We also estimated the ratios \(A\) and \(B\) by direct extrapolation of the appropriate coefficient quotients, using the following method [41]: Given a sequence \(\{a_n\}\) defined for \(n \geq 1\), which is known or assumed to converge to a limit \(a_{\infty}\) with corrections of the form
\[ a_n \sim a_\infty (1 + b/n + \ldots) \], we first construct a new sequence \( \{ h_n \} \) defined by \( h_n = \prod_{m=1}^{n} a_m \). Then the generating function \( \sum h_n x^n \sim (1 - a_\infty x)^{-(1+b)} \). We obtain estimates of the required limit \( a_\infty \) and parameter \( b \) by analysing this generating function by the standard method of differential approximants. (The value of the parameter \( b \) can also be obtained numerically from the amplitude estimates given in (3.22)–(3.24) above.) In this way, we obtain the estimates
\[
A = 0.140296(6) \quad \text{(3.25)} \\
B = 0.439649(9) \quad \text{(3.26)}
\]

### 3.4.2 Square lattice

Let us now consider the square-lattice data. We first analysed the three metric quantities \( \langle R^2_e \rangle_N \), \( \langle R^2_g \rangle_N \) and \( \langle R^2_m \rangle_N \) by a method similar to that leading to (3.16)–(3.18), but including the contribution of the antiferromagnetic singularity. We imposed the exponent values \( \nu = 3/4 \), \( \Delta_1 = 3/2 \), \( q = -11/32 \) and \( \Delta_1^{AF} = 1 \); the justification for these choices has already been given above. The sequences of amplitude estimates are shown in Tables 8, 9 and 10. In this way, we obtain the following results: As discussed in Section (3.2), some experimentation was needed to determine the maximum values of the parameters \( m \) and \( k \) in (3.2). The results given below are given by \( m = 4 \), \( k = 2 \) for \( \langle R^2_e \rangle_N \), by \( m = 2 \), \( k = 1 \) for \( \langle R^2_g \rangle_N \), and by \( m = 3 \), \( k = 1 \) for \( \langle R^2_m \rangle_N \). We find
\[
\langle R^2_e \rangle_N \sim N^{3/2} [0.77124(5) + 1.159(38)/N - 3.13(74)/N^{3/2} + 6(6)/N^2 - 6(24)/N^{5/2} \\
+ 0.4(4.0)/N^3 + O(1/N^{7/2})] + (-1)^N N^{-1/32} [0.12451(17) - 0.027(24)/N + O(1/N^2)] \quad \text{(3.27)}
\]
\[
\langle R^2_g \rangle_N \sim N^{3/2} [0.108230(1) + 0.1019(1)/N + 0.1082(4)/N^{3/2} - O(1/N^2)] \\
+ (-1)^N N^{-1/32} [0.008364(19) + 0.0031(21)/N + O(1/N^2)] \quad \text{(3.28)}
\]
\[
\langle R^2_m \rangle_N \sim N^{3/2} [0.33913(8) + 0.426(17)/N - 1.1(1.1)/N^{3/2} + 2(4)/N^2 + O(1/N^{5/2})] \\
+ (-1)^N N^{-1/32} [0.03652(11) + 0.015(12)/N + O(1/N^2)] . \quad \text{(3.29)}
\]

A similar analysis of the unnormalized quantities, using the estimate \( \mu = 2.63815853034174086843 \) from (3.12), was made. The results below are given by \( m = 4 \), \( k = 2 \) for the first two quantities, and \( m = 3 \), \( k = 1 \) for the third. For the first two quantities we have not given our estimate of \( a_4 \) as the associated error is significantly bigger than the estimate. We find
\[
c_N \langle R^2_e \rangle_4/4 \sim N^{59/32} [0.226945(14) + 0.4471(11)/N - 0.95(22)/N^{3/2} + 2(2)/N^2 + O(1/N^{5/2})] \\
+ (-1)^N N^{-3/2} [0.019098(1) + 0.0415(41)/N - 0.08(45)/N^2 + O(1/N^3)] \quad \text{(3.30)}
\]
\[
c_N \langle R^2_g \rangle_N \sim N^{59/32} [0.127388(31) + 0.181(17)/N + 0.10(26)/N^{3/2} + 0.1(1.0)/N^2 + O(1/N^{5/2})] \\
+ (-1)^N [-0.010688(15) + 0.0047(17)/N - 0.20(5)/N^2 + O(1/N^3)] \quad \text{(3.31)}
\]
\[
c_N \langle R^2_m \rangle_N \sim N^{59/32} [0.39917(11) + 0.686(44)/N - 1.3(6)/N^{3/2} + 2(2)/N^2 + O(1/N^{5/2})] \\
+ (-1)^N [-0.021383(45) + 0.028(5)/N + O(1/N^2)] . \quad \text{(3.32)}
\]
From these we can estimate the amplitudes of the normalized metric quantities by dividing through by the asymptotic form (2.30) for \( c_N \). In this way we obtain a second set of amplitude estimates,

\[
\langle R^2_e \rangle_N \sim N^{3/2}[0.77124(5) + 1.1593(38)/N - 3.13(75)/N^{3/2} + 6(8)/N^2 + O(1/N^{5/2})] \\
+ (-1)^N N^{-11/32}[0.12439(4) - 0.0144(9)/N + O(1/N^2)]
\]

(3.33)

\[
\langle R^2_g \rangle_N \sim N^{3/2}[0.108227(50) + 0.103(17)/N + 0.098(160)/N^{3/2} - O(1/N^2)] \\
+ (-1)^N N^{-11/32}[0.008376(20) + 0.0006(30)/N + O(1/N^2)]
\]

(3.34)

\[
\langle R^2_m \rangle_N \sim N^{3/2}[0.33913(8) + 0.424(19)/N - 1(1)/N^{3/2} + O(1/N^2)] \\
+ (-1)^N N^{-11/32}[0.03654(7) + 0.027(8)/N + O(1/N^2)]
\]

(3.35)

These differ from the directly analysed amplitudes only in the last quoted digits for all but the least significant amplitudes, and are consistent with our quoted errors in all cases. In the notation of (2.34), from (3.33) we have \( a'_0/d'_0 = 6.200(3) \), while from the amplitudes quoted below (3.12) we have \( a_0/d_0 = 6.1999(1) \), in complete agreement.

The amplitude ratios \( A \) and \( B \) follow immediately from (3.27)–(3.29) as \( A = 0.14033 \) and \( B = 0.43971 \). From (3.33)–(3.35) we obtain the almost identical values, \( A = 0.14033 \) and \( B = 0.43972 \).

We also analysed these amplitude ratios directly, using the same method as discussed above for the analysis of the triangular-lattice data. We obtained

\[
A = 0.140299(6)
\]

(3.36)

\[
B = 0.439647(6).
\]

(3.37)

Comparison with the corresponding estimates (3.25)–(3.26) for the triangular lattice is entirely consistent with the belief that these ratios are lattice-independent [25].

These amplitude ratios are also consistent with the CSCPS relation (2.10): using our best estimates (3.36)–(3.37), we find \( F \equiv \lim_{N \to \infty} F_N = -0.000024 \pm 0.000025 \) for the square lattice; and using (3.25)–(3.26), we find \( F \equiv \lim_{N \to \infty} F_N = -0.000036 \pm 0.000036 \) for the triangular lattice. A direct analysis of the sequence \( \{F_N\} \) was also undertaken, but that sequence was found difficult to extrapolate; and our estimate of the limit, while entirely consistent with zero, was a factor of 10 less precise than the one just quoted.

If it is in fact true (as certainly seems to be the case) that \( F_N \to 0 \) as \( N \to \infty \), then it is of interest to investigate the rate at which \( F_N \to 0 \). If \( F_N \propto N^{-\delta} \), then \( f_N = F_N(R^2_e)_N \propto N^{3/2-\delta} \). For both square and triangular lattices, we find that \( \delta = 3/2 \), i.e. that \( f_N \) approaches a nonzero constant \( f \equiv \lim_{N \to \infty} f_N \) as \( N \to \infty \). This behaviour was initially surprising to us, because it implies that there is no analytic correction-to-scaling term \( 1/N \) in \( f_N \), even though such terms are manifestly present in each of the three individual metric quantities \( \langle R^2 \rangle \). Moreover, this remarkable result appears to hold for both lattices.\(^6\) However, we were subsequently able to provide a renormalization-

\(^6\)Indeed, as shown in [41] it appears to hold even in the case of interacting SAWs within the good-solvent regime (i.e., above the theta temperature). Of course, the limiting constant \( f \) depends on the interaction. For repulsive nearest-neighbour interactions, \( f \) increases from 0.78 to an asymptotic value of about 1.6 as the repulsion gets very strong.
group argument for this cancellation (see Section 2.2 above). Our estimates of the amplitude are \( f = 0.78 \pm 0.03 \) [64] for the square lattice and \( f = 0.96 \pm 0.04 \) for the triangular lattice. These estimates are based on extrapolation of the sequences \( f_N \) using a variety of extrapolation algorithms, including Levin’s \( u \) transform, Brezinski’s \( \theta \) algorithm, Neville-Aitken extrapolation and Wynn’s \( \epsilon \) algorithm. Details of these and other algorithms, as well as programs for their implementation, can be found in [70].

As noted in Section 2.2, the observation that \( \delta = 3/2 \) implies the constraint (2.44) on the subdominant amplitudes arising in (2.40)–(2.42). Our series estimates (3.22)–(3.24) and (3.27)–(3.29) are consistent with this prediction, as are our Monte Carlo estimates (4.3)–(4.11). Furthermore, our series and Monte Carlo estimates of \( f \) are consistent with the relation (2.45); but the associated error bars are very large, so this is not a stringent test.

Finally, we note the fact that \( \delta = 3/2 \) is another indicator that the correction-to-scaling exponent is indeed \( 3/2 \). If it were less than this, then the leading non-analytic correction-to-scaling term would have to cancel miraculously (as the \( 1/N \) term does) in the combination (2.11) for \( f_N \). This seems \textit{a priori} unlikely.

3.5 Euclidean-invariant moments of the distribution function

We have also analysed the series for rotationally-invariant and non-rotationally-invariant moments of the endpoint distribution function, given in Tables 3–6, using methods similar to those just described for the analysis of \( \langle R^2 \rangle \). Let us start with the rotationally-invariant moments \( \langle r^{2k} \rangle_N \), for which we expect an asymptotic behaviour of the form

\[
\langle r^{2k} \rangle_N \sim N^{2k\nu}[c_{0,k} + c_{1,k}/N + c_{2,k}/N^{\Delta_1} + c_{3,k}/N^2 + \ldots]
\]  

(3.38)

for the triangular lattice, and

\[
\langle r^{2k} \rangle_N \sim N^{2k\nu}[c_{0,k} + c_{1,k}/N + c_{2,k}/N^{\Delta_1} + c_{3,k}/N^2 + \ldots]
\]

\[+ (-1)^N N^{q_k}[d_{0,k} + d_{1,k}/N^{\Delta_{AF}^1} + d_{2,k}/N + \ldots]\]  

(3.39)

for the square lattice. In Section 2.2 we gave arguments predicting that \( q_k = 2k\nu + \alpha - 1 - \gamma = 3k/2 - 59/32 \). Furthermore, it is reasonable to expect \( \Delta_{AF}^1 = 1 \) as was already observed for the SAW counts [17] and for the metric quantities \( \langle R^2 \rangle \). Our numerical results are consistent with these predictions.

We began by analysing the moments \( \langle r^{2k} \rangle_N \) using the method of differential approximants [70, 71], with the aim of confirming the predicted leading exponents \( 2k\nu = 3k/2 \) and \( q_k = 3k/2 - 59/32 \). It is a previously observed feature of the method of differential approximants (DA) that its application to the analysis of SAW moment series is less accurate than might be expected [71]. For example, DA analysis of a 27-term square-lattice \( \langle R^2 \rangle_N \) series, biased at a critical point of 1, produces estimates of \( 2\nu \) in the range 1.495–1.497 [71], whereas an \textit{unbiased} analysis of a 27-term SAW series on the same lattice yields exponent estimates of \( \gamma = 1.34364 \pm 0.00088 \), which is rather more accurate, as well as more precise. This behaviour is most likely connected to the fact that the method of differential approximants tacitly assumes that the function is well approximated by a differentially finite (D-finite) function, i.e. the solution of a linear ordinary
differential equation with polynomial coefficients [72]. While there is strong evidence that neither SAWs nor SAPs are D-finite [72], it nevertheless appears that the SAW and SAP counts are well approximated by a D-finite function, while the generating functions for SAW and SAP metric properties (such as $\langle R^2 \rangle_N$) appear not to be. Evidence for this remark includes the telling fact that most of the differential approximants for $\langle R^2 \rangle$ (for both SAWs and SAPs) are defective [71], which is usually a signal that the function being approximated is not of the type tacitly assumed by the analysis. For this reason, our DA analysis gives only moderately accurate estimates of the leading exponents, both ferromagnetic and antiferromagnetic, but no reliable information as to the value of the subleading exponents.

Our DA analysis confirmed the expected leading behaviour $\langle r^{2k} \rangle_N \propto N^{3k/2}$, the exponents being identified as 1.4997(5), 2.998(6), 4.496(9), 5.996(12), and 7.496(12) for $k = 1, 2, 3, 4, 5$ on the square lattice, and 1.4975(5), 2.9967(9), 4.4959(9), 5.9971(2), and 7.5001(10) for $k = 1, 2, 3, 4, 5$ on the triangular lattice. DA analysis also gave reasonable estimates of the leading antiferromagnetic exponent on the square lattice: we found $q_k = 3k/2 - 59/32$ to an accuracy of approximately 0.01, 0.04, 0.05, 0.05, 0.06 for $k = 1, 2, 3, 4, 5$, respectively. It is possible that higher moments ($k > 1$) may have non-analytic correction-to-scaling terms with exponent $\Delta_1 < 1.5$ which would then be more prominent than the leading correction-to-scaling term of the second-moment quantities $\langle R^2 \rangle$. A more prominent such singularity would also explain the relatively poor accuracy of the DA analysis. We allow for this possibility in our analysis, described immediately below, but find no evidence for such a term.

We next proceeded to fit $\langle r^{2k} \rangle_N$ to the asymptotic forms (3.38) and (3.39), setting $\nu = 3/2$ and $q_k = 3k/2 - 59/32$ and investigating the quality of the fit for a variety of possible values of $\Delta_1$ and $\Delta_1^{AF}$. Among other things, we considered the possibility that $\Delta_1 < 1$, even though no such term is observed in the metric quantities $\langle R^2 \rangle$.

We first fitted the available series to the above forms with $\Delta_1 = 1/2$. Estimates of the associated amplitude were, in all cases, monotonically decreasing toward zero. Furthermore, as we increased the number of subdominant terms included in the fit, this amplitude approached zero more and more closely. The data are insufficient to judge whether the rate of approach to zero increased, but the entries were numerically smaller. Also, estimates of the leading amplitude $c_{0,k}$ did not display the sort of convergence we found in the analysis of $\langle R^2 \rangle_N$; rather, the convergence deteriorated as we increased the number of subdominant terms included in the fit. Both of these observations suggest that there is no term $c_{2,k}/N^{\Delta_1}$ with $\Delta_1 \approx 1/2$. Similar behaviour was observed with $\Delta_1 = 11/16$, though we cannot say whether the effect was stronger or weaker. That is to say, the analysis is insensitive to this level of exponent change for these series. This is consistent with the situation found in the analysis of $\langle R^2 \rangle$. We conclude that there is no evidence of a correction term $c_{2,k}/N^{\Delta_1}$ with $\Delta_1 < 1$.

We then reanalysed the data assuming that the only correction-to-scaling term, other than integer powers of $1/N$, was that with exponent $\Delta_1 = 3/2$, exactly as found for the second-moment series. As for the second-moment series, we retained only analytic correction terms to the antiferromagnetic singularity. As we increased the order of the fit, the leading amplitudes $c_{0,k}$ and $d_{0,k}$ displayed improved convergence. This is usually an indicator that the guessed asymptotic form is correct. The higher-order amplitudes
displayed less convincing convergence, but we ascribe this to a lack of adequate data. For the second moment, we have a 59-term series, which converged rather well, as can be seen from Table 8. But the convergence is much less impressive after only 32 terms—which is all we have available for the higher moments.

Taken together, our results favour the most obvious conjecture, which is that the subdominant behaviour is characterised by the same exponent set as is observed for $\langle R^2 \rangle_N$.

In order to test the conclusion that the leading correction term in all these series is the $1/N$ term, we used the method of differential approximants on a modified series obtained by subtracting the estimated leading-order term from the original series: that is, we analysed $\langle r^{2k} \rangle_N - c_{0,k}N^{2k\nu}$. We found that the series coefficients behave like $N^{2k\nu-1}$, consistent with the conclusion that the leading correction term is $1/N$ and that the non-analytic correction-to-scaling term(s), have exponent $\Delta_1 > 1$, consistent with our view that $\Delta_1 = 3/2$.

With the foregoing observations in mind, we obtained the following estimates for the corresponding amplitudes for the square-lattice moments $\langle r^{2k} \rangle_N$, where we have assumed a single correction-to-scaling exponent $\Delta_1 = 3/2$ associated with the ferromagnetic singularity, and otherwise only analytic corrections to both the ferromagnetic and antiferromagnetic singularities:

\begin{align}
    k = 2: & \quad c_{0,2} = 0.860(2), \quad c_{1,2} = 1.9(2), \\
    & \quad d_{0,2} = 0.139(5), \quad d_{1,2} = -0.03(2), \quad (3.40) \\
    k = 3: & \quad c_{0,3} = 1.184(5), \quad c_{1,3} = 3(1), \\
    & \quad d_{0,3} = 0.193(5), \quad d_{1,3} = -2(1), \quad (3.41) \\
    k = 4: & \quad c_{0,4} = 1.907(10), \quad c_{1,4} = 2.5(5), \\
    & \quad d_{0,4} = 0.310(3), \quad d_{1,4} = -0.46(5), \quad (3.42) \\
    k = 5: & \quad c_{0,5} = 3.434(10), \quad c_{1,5} = -3(1), \\
    & \quad d_{0,5} = 0.551(3), \quad d_{1,5} = -1.6(2). \quad (3.43)
\end{align}

As a check we verify relation (2.34). Our results for $c_N$ predict $d_{0,k}/c_{0,k} = 0.161292(2)$, a relation that is well satisfied by our results for $c_{0,k}$ and $d_{0,k}$.

We can now provide a direct estimate of the invariant ratios $M_{2k,N} = \langle r^{2k} \rangle_N/\langle r^2 \rangle_N^{k}$ in the limit $N \to \infty$. From the above amplitude estimates, we have, for the square lattice,

\begin{align}
    M_{4,\infty} & = 1.446(3) \quad (3.44) \\
    M_{6,\infty} & = 2.581(11) \quad (3.45) \\
    M_{8,\infty} & = 5.391(28) \quad (3.46) \\
    M_{10,\infty} & = 12.59(4). \quad (3.47)
\end{align}

These estimates agree well with the estimates (2.18)–(2.21) obtained previously [42], but are a factor 5–10 less precise.
For the triangular lattice, there is of course no “antiferromagnetic” singularity, so that the terms corresponding to the amplitudes $d_{j,k}$ are absent. We find from the triangular lattice data:

\[
\begin{align*}
  k = 2: & \quad c_{0,2} = 0.7330(9), \quad c_{1,2} = 1.2(2), \quad c_{2,2} = -5(1), \\
  k = 3: & \quad c_{0,3} = 0.934(2), \quad c_{1,3} = 1.5(5), \\
  k = 4: & \quad c_{0,4} = 1.383(3), \quad c_{1,4} = 2(1), \\
  k = 5: & \quad c_{0,5} = 2.31(3), \quad c_{1,5} = -8.4(5). \tag{3.48}
\end{align*}
\]

From the above amplitude estimates, we have, for the triangular lattice,

\[
\begin{align*}
  M_{4,\infty} &= 1.446(2) \tag{3.52} \\
  M_{6,\infty} &= 2.588(5) \tag{3.53} \\
  M_{8,\infty} &= 5.381(12) \tag{3.54} \\
  M_{10,\infty} &= 12.64(15). \tag{3.55}
\end{align*}
\]

These estimates agree well with those found for the square lattice, confirming the expected universality. Therefore they are also in agreement with the field-theory estimates (2.18)–(2.21), though less precise.

### 3.6 Non-Euclidean-invariant moments of the distribution function

In this section we discuss the behaviour of the non-rotationally-invariant moments: $\langle r^{2k} \cos 4\theta \rangle_N$ with $k = 2, 3, 4$ and $\langle r^8 \cos 8\theta \rangle_N$ for the square lattice, and $\langle r^{2k} \cos 6\theta \rangle_N$ with $k = 3, 4$ for the triangular lattice.

Let us first consider the triangular lattice. We began by analysing the series using the method of differential approximants, with the aim of determining the leading exponent. For the triangular lattice we write

\[
\langle r^{2k} \cos 6\theta \rangle_N \sim N^{2k\nu - \Delta_{nr}} [a_{0,k} + a_{1,k}/N^{\Delta} + a_{2,k}/N + \cdots]. \tag{3.56}
\]

Using first- and second-order differential approximants, we found $\Delta_{nr} = 3.00 \pm 0.10$ for $k = 3$ and $\Delta_{nr} = 2.95 \pm 0.10$ for $k = 4$, from which we conjecture that $\Delta_{nr} = 4\nu = 3$ exactly, as predicted in Section 2.1.

Fitting the triangular-lattice data to the asymptotic form (3.56), we found good convergence only if we set the correction-to-scaling exponent to a value $\Delta \approx 1/2$—in sharp contrast to situation for the corresponding rotationally invariant moments, where we found $\Delta = 3/2$. To test the conjecture that the leading correction is $N^{-1/2}$, we subtracted the estimated leading term $a_{0,k}N^{2k\nu - \Delta_{nr}}$ from $\langle r^{2k} \cos 6\theta \rangle_N$ and analysed the resulting series. It was found to behave as $a_{1,k}N^{2k\nu - \Delta_{nr} - 0.50 \pm 0.10}$, implying that $\Delta = 0.5 \pm 0.10$. At this stage, we have no theoretical explanation for this numerical
obtained the following estimates for the square-lattice amplitudes:

\[ k = 3: \quad a_{0,3} = 1.120(3), \ a_{1,3} = -1.95(5), \ a_{2,3} = 1.7(4) \]
\[ \quad k = 4: \quad a_{0,4} = 4.05(5), \ a_{1,4} = -9(1), \ a_{2,4} = 20(4). \]

For the square lattice, equation (3.56) needs to be modified by the addition of a term representing the antiferromagnetic singularity, so we write

\[
\langle r^{2k} \cos 4\theta \rangle_N \sim N^{2k\nu-\Delta_{nr}}[a_{0,k} + a_{1,k}/N^\Delta + a_{2,k}/N + \cdots] + (-1)^NN^\psi[b_{0,k} + b_{1,k}/N^1 + b_{2}/N^2 + \cdots].
\]

From first- and second-order differential approximants applied to the square-lattice data, we found \( \Delta_{nr} = 1.46 \pm 0.03, \Delta_{nr} = 1.45 \pm 0.06 \) and \( \Delta_{nr} = 1.44 \pm 0.09 \) for \( k = 2, 3, 4 \), respectively. From these results we conjecture that \( \Delta_{nr} = 2\nu = 3/2 \) exactly, as predicted in Section 2.1. Differential-approximant analysis also gave reasonable estimates of the the leading antiferromagnetic exponent: we found \( \psi = 2k\nu - \Delta_{nr} - 3 + \gamma = 2k\nu - 101/32 \), accurate to \( \pm 0.013 \) for \( k = 2 \), \( \pm 0.05 \) for \( k = 3 \), and \( \pm 0.15 \) for \( k = 4 \). This expression for \( \psi \) is different from the one which is the natural generalization of the result for the Euclidean-invariant moments, \( 2k\nu - \Delta_{nr} + \alpha - 1 - \gamma = 2k\nu - 107/32 \), which is excluded from the analysis: the difference is \( 6/32 = 0.1875 \), much larger than the errors.

Fitting the data to the above asymptotic form (3.59) with \( \Delta_{nr} = 3/2 \) and \( \psi = 2k\nu - 101/32 \), and assuming only analytic corrections to scaling at the antiferromagnetic critical point, as found for all the other series, we again found good convergence only if we set the ferromagnetic correction-to-scaling exponent \( \Delta \) to approximately \( 1/2 \), just as was found in the analysis of the triangular-lattice data. As in the triangular-lattice analysis, we verified this conjecture by subtracting the estimated leading term \( a_{0,k}N^{2k\nu-\Delta_{nr}} \) from \( \langle r^{2k} \cos 4\theta \rangle_N \) and analysing the resulting series, which was found to behave as \( a_{1,k}N^{2k\nu-\Delta_{nr}-0.497 \pm 0.005} \). This is strong support for an \( N^{-1/2} \) correction.

Setting \( \Delta_{nr} = 3/2, \psi = 2k\nu - 101/32 \) and \( \Delta = 1/2 \) and assuming subsequent half-integer terms in the asymptotic expansion of the ferromagnetic singularity (3.59), we obtained the following estimates for the square-lattice amplitudes:

\[ k = 2: \quad a_{0,2} = 1.148(6), \ a_{1,2} = -1.70(5), \ a_{2,2} = 2.7(3), \]
\[ \quad b_{0,2} = 0.060(5), \ b_{1,2} = 0.6(2), \]
\[ k = 3: \quad a_{0,3} = 3.200(10), \ a_{1,3} = -6.25(10), \ a_{2,3} = 12.0(5), \]
\[ \quad b_{0,3} = 0.175(10), \ b_{1,3} = 1.2(4), \]
\[ k = 4: \quad a_{0,4} = 8.90(10), \ a_{1,4} = -20(2), \ a_{2,4} = 50(8), \]
\[ \quad b_{0,4} = 0.47(5), \ b_{1,4} = 3(1). \]

Finally, we analyzed the series \( \langle r^8 \cos 8\theta \rangle_N \) on the square lattice. A differential-approximant analysis gave \( \langle r^8 \cos 8\theta \rangle_N \propto N^{3.07 \pm 0.1} \). We conjecture that the exponent is
exactly 3, consistent with the behaviour $\langle r^8 \cos 8\theta \rangle_N \propto N^{8 \nu - \Delta_{nr,8}}$ with $\Delta_{nr,8} = 3$. This is exact at the mean-field level and also for the two-dimensional Ising model. In the same spirit that we previously conjectured that $\Delta_{nr} = 2 \nu$, we now conjecture that $\Delta_{nr,8} = 4 \nu$.

The antiferromagnetic exponent in $\langle r^8 \cos 8\theta \rangle_N$ was estimated to be $\psi = 1.35 \pm 0.05$, which, by analogy with the antiferromagnetic exponents for $\langle r^{2k} \cos k\theta \rangle_N$, we conjecture is exactly $8 \nu - \Delta_{nr,8} - 3 + \gamma = 43/32$. We found the subsequent analysis consistent with only analytic corrections at the antiferromagnetic critical point. At the ferromagnetic critical point, the data were again consistent with a leading $N^{-1/2}$ correction-to-scaling term. In an identical notation to that used above, we find the amplitudes to be:

$$a_0 = 135(2), a_1 = -540(10), a_2 = 1440(50),$$
$$b_0 = 6.5(1), b_1 = -11(3).$$

(3.63)

4 Monte Carlo analysis

4.1 Summary of our data

We have also generated extensive Monte Carlo data, using the pivot algorithm [37,73–75], for SAWs on the square lattice at selected values of $N$ in the range $40 \leq N \leq 4000$, measuring the observables $\langle R_e^2 \rangle$, $\langle R_g^2 \rangle$ and $\langle R_m^2 \rangle$. The results are collected in Tables 11 and 12. Unfortunately, in some runs we did not measure all observables: in particular, for larger values of $N$, the statistics available for $\langle R_m^2 \rangle$ are much smaller than for the other radii. The statistics range from $10^9$ to $10^{10}$ pivot iterations per point, or approximately $10^7$ to $10^9$ times the integrated autocorrelation time $\tau_{int,A}$ for these observables.\footnote{High-precision Monte Carlo data for $\langle R_e^2 \rangle_N$ have been kindly provided by Peter Grassberger. His results have been merged with ours and appear in Tables 11 and 12.}

We begin by analyzing the Monte Carlo data in an unbiased way, in order to extract the leading amplitudes and the correction-to-scaling exponent(s) and amplitude(s) (Section 4.2). Then we compare the Monte Carlo data, which lie at relatively high $N$ but are afflicted by statistical errors, with the formulae (3.27)–(3.29) obtained by extrapolating the series data, which lie at much smaller $N$ but are exact (Section 4.3).

4.2 Data analysis

In order to determine the leading amplitudes and the correction-to-scaling exponent(s) and amplitude(s), we have analysed the three quantities $\langle R_e^2 \rangle$, $\langle R_g^2 \rangle$ and $\langle R_m^2 \rangle$. We first tried nonlinear least-squares fits of the form\footnote{Note that we have rescaled the length $N$ by a fixed parameter $N_0$ that has always been taken equal to $N_0 = 750$. The purpose of this rescaling is to minimize the covariance between the estimates of $b$ and $\Delta$. (The optimal choice is to take $N_0$ equal to the weighted geometric mean of the $N$ values occurring as data points.) As a consequence, the relative error on $b$ is a factor of 3–4 smaller than in fits with $N_0 = 1$. The error on $\Delta$ does not depend on the choice of $N_0$.}

$$\langle R_e^2 \rangle_N/N^{2\nu} = a + b(N/N_0)^{-\Delta},$$

(4.1)
in which $\nu$ has been fixed equal to $3/4$ and the parameters $a, b, \Delta$ are free. In these fits we use only the data with $N \geq$ some cutoff value $N_{\text{min}}$; we then vary $N_{\text{min}}$ systematically and investigate the quality of the fit (see Tables 13, 14 and 15). For $N_{\text{min}} \gtrsim 60$ the $\chi^2$ values are reasonable. The fits are stable and the error bars reasonable up to $N_{\text{min}} \approx 700$; after this, the error bars increase drastically and the estimates should be considered unreliable.

Let us consider first $\langle R_e^2 \rangle$ and $\langle R_g^2 \rangle$, the radii for which we have the best statistics. The fit for the radius of gyration is extremely stable\(^9\) and clearly suggests $\Delta = 1$. No subleading exponent with $\Delta < 1$ appears to be present; in particular, this excludes $\Delta = 11/16 = 0.6875$ unless the corresponding amplitude is extremely small. By contrast, the fit of $\langle R_e^2 \rangle$ gives estimates that vary with $N_{\text{min}}$: the estimate of $\Delta$ at first increases with $N_{\text{min}}$, then flattens off at $\Delta \approx 0.9$. The theoretical prediction $\Delta = 11/16$ is again excluded, but in this case a non-analytic correction $\Delta_1 < 1$ is still possible a priori. However, we believe that a subleading exponent $\Delta_1 \approx 0.9$ is unlikely: after all, it does not agree with any theoretical prediction; and the observed behaviour can be explained equally well, as noted earlier, by the competition between two correction terms of opposite sign, provided that both terms are still sizable in the range of $N$ that we are considering. The fact that a range $500 \leq N \leq 4000$ is insufficient to see clearly that $\Delta = 1$ shows how difficult is the determination of $\Delta$ and explains the wide range of contradictory results found in previous work.

We can also consider $\langle R_m^2 \rangle$. The behaviour is similar to that observed for $\langle R_e^2 \rangle$, although the errors are larger. Again, the data are compatible with $\Delta \approx 0.9$, but, as before, we believe that what we are observing is simply an effective exponent arising from the competition between two correction terms of opposite sign. Moreover, we are here probing shorter walks than in the case of $\langle R_e^2 \rangle$, because of the lack of statistics for larger $N$: the fit is effectively dominated by the data in the range $300 \lesssim N \lesssim 1000$.

Since we have little evidence that $\Delta_1 = 11/16$, the most likely possibility is that $\Delta_1 = 3/2$. We have therefore checked whether our data can be fitted by the simple Ansatz

$$\langle R^2 \rangle_N = N^{2\nu}(a + bN^{-1} + cN^{-3/2}),$$

with $a, b, c$ free parameters. As before, we perform several fits using only data with $N \geq N_{\text{min}}$, varying $N_{\text{min}}$ systematically. The results are reported in Tables 16, 17 and 18. The fit quality is good even for the lowest value of $N_{\text{min}}$: additional corrections do not play much role for $N \gtrsim 60$. Notice that for $\langle R_g^2 \rangle$ the constant $c$ is very small, explaining why the previous fit gave $\Delta = 1$ essentially without corrections. For $\langle R_e^2 \rangle$ and $\langle R_m^2 \rangle$, $c$ is instead sizable and of opposite sign with respect to $b$, giving in fit (4.1) an effective exponent $\Delta < 1$.

The leading amplitudes are extremely stable and, using the data with $N_{\text{min}} = 200$,
we can estimate

\begin{align*}
    a_e &= 0.77121 \pm 0.00004 \\
    b_e &= 1.17 \pm 0.05 \\
    c_e &= -2.8 \pm 0.7, \\
    a_g &= 0.108207 \pm 0.000007 \\
    b_g &= 0.108 \pm 0.010 \\
    c_g &= 0.0 \pm 0.2, \\
    a_m &= 0.33903 \pm 0.00004 \\
    b_m &= 0.46 \pm 0.05 \\
    c_m &= -1.1 \pm 0.7. 
\end{align*}

where the error bars are 68% confidence limits. We can compare these results with the series estimates (3.27)–(3.29), and note that they agree well within quoted errors with one exception: $a_g$ differs by three error bars from one set of the corresponding series estimate. We consider the stated errors of this series estimate to be anomalously low, compared to all other error estimates, and thus not to be taken literally.

We have also considered the universal amplitude ratios $A_N$ and $B_N$. The most notable feature of the raw data for $A_N$ (Table 12) is its nonmonotonicity: at first $A_N$ decreases, reaching a minimum at $N \approx 130$; then it increases. This immediately suggests the presence of two correction-to-scaling terms of opposite sign, in agreement with the analysis presented above. We have therefore analysed the ratios $A_N$ and $B_N$ by performing a fit of the form

\[ O_N = a + bN^{-1} + cN^{-3/2}. \]  

The results are reported in Tables 19 and 20. Again the quality of the fits is quite good, and we obtain the final estimates (again we use conservatively the data for $N_{\text{min}} = 200$)

\begin{align*}
    A &= 0.140310 \pm 0.000011 \\
    B &= 0.439614 \pm 0.000050,
\end{align*}

where the error bars are 68% confidence limits.\(^\text{10}\)

\(^{10}\)Cardy and Mussardo [39] have used the form-factor method, applied to the exact $S$-matrix of the massive $O(n)$ model, to derive the estimates $A \approx 0.126$ and $B \approx 0.420$. This is impressive accuracy for a first-principles theoretical calculation; the approximately 5% error is about what one expects from the one-particle approximation used in this computation.
constant as $N \to \infty$. This means, as noted earlier, that the $1/N$ correction is absent within our errors. A fit to a constant gives the results reported in Table 22. The results show an initial downward trend with $N_{\text{min}}$ and then increase again. A stable region may be identified for $N_{\text{min}} \geq 250$. For $N_{\text{min}} = 250$ we have

$$f = 0.68 \pm 0.14.$$ (4.15)

This is in agreement with, but less precise than, the result $f = 0.79 \pm 0.03$ obtained from series analysis.

Finally, we have tried to see whether our estimated values of $A$ and $B$ are consistent with simple rational values that satisfy the CSCPS formula. If we require the denominators to be $\leq 1000$, then the only possible values anywhere near our estimates are

$$A = \frac{23}{164} \approx 0.1402439, \quad B = \frac{40}{91} \approx 0.4395604$$ (4.16)

and

$$A = \frac{91}{648} \approx 0.1404321, \quad B = \frac{95}{216} \approx 0.4398148.$$ (4.17)

Our data are, however, precise enough to clearly exclude both guesses. We therefore conjecture that $A$ and $B$ do not take simple rational values, even though one particular linear combination of them does.

### 4.3 Comparison to series-extrapolation predictions

We can also directly compare our raw Monte Carlo data to the extrapolation formulas (3.27)–(3.29). For this purpose we compute

$$\chi^2 = \sum_{\text{MC data}} \frac{(R_{\text{MC}}^2 - R_{\text{series}}^2)^2}{\sigma_{\text{MC}}^2},$$ (4.18)

where $R_{\text{MC}}^2$ is the Monte Carlo estimate, $\sigma_{\text{MC}}$ the corresponding error, and $R_{\text{series}}^2$ the prediction of the extrapolations (3.27)–(3.29). For $\langle R_g^2 \rangle$ we find that (3.27) describes the numerical data rather well. Indeed, $\chi^2 = 29$ for 19 data points. The small remaining discrepancy is mainly due to the error on the coefficients. Indeed, if we use for the ferromagnetic part the results of the Monte Carlo fit with $N_{\text{min}} = 40$ reported in Table 16—these estimates are compatible with the exact-enumeration ones reported in (3.27)—the $\chi^2$ drops to 15. Analogous discussion applies to $\langle R_m^2 \rangle$. If we use (3.29) we obtain $\chi^2 = 55$ with 15 data points. But again it is enough to replace the ferromagnetic coefficients obtained using exact enumeration with those obtained in the Monte Carlo fit, see Table 18—they are fully compatible—to have $\chi^2 = 19$ with 15 points. The situation is worse for $\langle R_g^2 \rangle$. Using all data we obtain $\chi^2 = 80$ with 18 data points. Such a result does not improve significantly if we change the coefficients in (3.28) within error bars. This is related to the fact we have already noticed that the leading amplitude for $\langle R_g^2 \rangle_N$ reported in (3.28) significantly differs from the Monte Carlo estimate obtained for any value of $N_{\text{min}}$. 

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5 Conclusions and open questions

In this study of the SAW correction-to-scaling exponents, we have seen a consistent picture emerging, given independent support by both Monte Carlo and series analysis. We have presented compelling evidence that the first non-analytic correction term in the generating function for both SAWs and SAPs, as well as in several Euclidean-invariant metric properties, is $\Delta_1 = 3/2$, as predicted by Nienhuis some 20 years ago [4, 5]. We find no evidence for the presence of an exponent $\Delta_1 = 11/16$ in SAWs and SAPs on the square and triangular lattices. Our analysis of the interplay between dominant and subdominant correction-to-scaling terms also enables us to explain quantitatively why many earlier analyses gave incorrect conclusions, predicting exponents $\Delta_1 < 1$. For certain observables, we find pairs of correction terms of opposite sign that conspire to give effective exponents that are smaller than both of the individual exponents. Thus, corrections behaving as $a/N + b/N^{3/2}$ with $ab < 0$ were incorrectly identified with a single correction term $c/N^{\Delta_1}$ with $\Delta_1 < 1$.

Monte Carlo and series analysis turn out to complement each other well. Series provide a basis for calculating the amplitudes of several subdominant asymptotic terms, while the Monte Carlo data frequently provide greater accuracy for the estimate of the leading amplitudes.

We have also studied the asymptotic behaviour of several non-Euclidean-invariant quantities. Their leading behaviour is characterized by a new exponent $\Delta_{nr}$. We find compelling evidence that $\Delta_{nr} = 2\nu$ on the square lattice and $\Delta_{nr} = 4\nu$ on the triangular lattice, confirming the conjecture of [43, 44]. We also computed the leading correction-to-scaling exponent in these observables, finding $\Delta_1 \approx 0.5$. We are unaware of any theoretical prediction for this quantity.

We have also determined the dominant and subdominant exponents characterizing the “antiferromagnetic singularity” of the square lattice. These exponent predictions are for the most part new.

We also tested the CSCPS relation $\lim_{N \to \infty} F_N = 0$ [cf. (2.10)], which arises from conformal field theory [29, 30]. Both our Monte Carlo and series work are completely consistent with the CSCPS relation. Further, we find that the $1/N$ correction term in $F_N$ is absent, so that $F_N \sim \text{const} \times N^{-\Delta_1}$ with $\Delta_1 = 3/2$. The absence of this analytic correction-to-scaling term implies a new amplitude relation (2.44).

Finally, we remark that our numerical estimates for the universal amplitude ratios $A$ and $B$ are now so precise as to allow us to rule out the possibility that these are rational numbers with small integer denominators. Some other universal amplitude ratios include powers of $\pi$ [38], so it is possible that $A$ and $B$ are combinations of $\pi$ and rational numbers; but there is no $a$ priori reason why powers of $\pi$ should enter into the amplitude ratios $A$ and $B$.

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Table 1: Exact enumeration data for SAWs on the square lattice.
Table 2: Exact enumeration data for SAWs on the triangular lattice.

| \(N\) | \(c_N\) | \(\frac{1}{6}c_N(R_e^2)_N\) | \(\frac{1}{6}(N + 1)^2c_N(R_g^2)_N\) | \(\frac{1}{6}(N + 1)c_N(R_m^2)_N\) |
|---|---|---|---|---|
| 1 | 6 | 1 | 1 | 1 |
| 2 | 30 | 12 | 22 | 17 |
| 3 | 138 | 97 | 282 | 7039 |
| 4 | 618 | 654 | 2778 | 7039 |
| 5 | 2730 | 3977 | 3305 | 10067 |
| 6 | 11946 | 22624 | 175194 | 7039 |
| 7 | 51882 | 44088 | 2959366 | 7039 |
| 8 | 224130 | 644082 | 7939156 | 7039 |
| 9 | 964134 | 3288739 | 213650877 | 7039 |
| 10 | 4133166 | 1649468 | 89768737 | 7039 |
| 11 | 17668938 | 80783857 | 171756604 | 7039 |
| 12 | 75355206 | 39110240 | 969740818 | 7039 |
| 13 | 320743686 | 187263387 | 535313021 | 7039 |
| 14 | 1362791250 | 887096422 | 289769688 | 7039 |
| 15 | 5781775582 | 4147686501 | 1541876281342 | 7039 |
| 16 | 24497103022 | 19401270964 | 808186877224 | 7039 |
| 17 | 10673947682 | 8976397621 | 418012603145 | 7039 |
| 18 | 438296739594 | 412790427930 | 2136508771740 | 7039 |
| 19 | 1851231367347 | 18879836654237 | 108804709625886 | 7039 |
| 20 | 781243962678 | 2543926453928 | 54115151508565 | 7039 |
| 21 | 332492955934 | 389382874604291 | 266589044415781 | 7039 |
| 22 | 138825972053046 | 175738304567340 | 132388831054338256 | 7039 |
| 23 | 584633909268402 | 790253525660965 | 64706441113059344 | 7039 |
| 24 | 2460608873866142 | 3517121590633144 | 3249528461015512 | 7039 |
| 25 | 10350605422753064 | 42078042797380 | 15172247917363793 | 7039 |
| 26 | 43518144146729466 | 705908848577456242 | 7282367061554891905 | 7039 |
| 27 | 18285610165537558 | 3121749279518281231 | 3477214816277646768 | 7039 |
| 28 | 76823894470193747 | 389382874604291 | 165212611750977644768 | 7039 |
| 29 | 3225816257263972170 | 6141582710719862263 | 781383924149601017943 | 7039 |
| 30 | 1354001155814409774 | 271046328208157919578 | 367982305986878952874 | 7039 |
| 31 | 568128783484781895282 | 119383906036544838615 | 17260375240686493096 | 7039 |
| 32 | 238303459915216614558 | 5248569464005058190720 | 80650931507788360195202 | 7039 |
| 33 | 999260857726920753905 | 20344474284176644819305 | 375562794128347463256 | 7039 |
| 34 | 418899172155056797837847 | 10093547960002949033216 | 174297795283129034728776 | 7039 |
| 35 | 17555621735769416377909 | 44152425284336423359991 | 806362316808439340571104 | 7039 |
| 36 | 735310757481339206205896 | 1928731794198955230144424 | 371943975622772326856339418 | 7039 |
| 37 | 30880400620723417094182 | 84113742430456823044291 | 17108134061503861888460075 | 7039 |
| 38 | 1290171266649477408776990 | 36655277827728849374240 | 78481814499900179700643892 | 7039 |
| 39 | 540167866643658042327390 | 1594964189022085760831541 | 3591184984367024911943170002 | 7039 |
| 40 | 22610911672575420510653226 | 6931745697625513186301776 | 1639290849786142570127812 | 7039 |
Table 3: Exact enumeration data for SAWs on the square lattice. Rotationally invariant moments.

|   | $\frac{1}{T}c_N\langle r^4 \rangle_N$ | $\frac{1}{T}c_N\langle r^6 \rangle_N$ | $\frac{1}{T}c_N\langle r^8 \rangle_N$ |
|---|----------------------------------|----------------------------------|----------------------------------|
| 1 | 1                                | 1                                | 1                                |
| 2 | 24                               | 80                               | 288                              |
| 3 | 233                              | 1481                             | 10313                            |
| 4 | 1552                             | 15584                            | 171712                           |
| 5 | 8261                             | 118741                           | 1876421                          |
| 6 | 40128                            | 761824                           | 15997248                         |
| 7 | 174687                           | 4216895                          | 113009823                        |
| 8 | 711744                           | 21139264                         | 699292800                        |
| 9 | 2756691                          | 98246971                         | 3911019843                       |
| 10| 10258032                         | 430155712                        | 20197992960                      |
| 11| 36953225                         | 1794576465                       | 97801373081                      |
| 12| 12959552                         | 7194227712                       | 449049597184                     |
| 13| 444358551                        | 27891276903                      | 197183584795                     |
| 14| 1494601312                       | 105092615072                     | 8336039677888                    |
| 15| 4944384005                       | 386372087101                     | 34107295967573                   |
| 16| 16121969312                      | 1390424839040                    | 135635987698688                  |
| 17| 51903980173                      | 4910490964373                    | 526075527334141                  |
| 18| 165229382704                     | 17055786755328                   | 1995781800318592                |
| 19| 520720306083                     | 58367380590987                   | 7423576318235379                |
| 20| 1626289219696                     | 1970978715526208                | 27128806075092160               |
| 21| 5037880731363                     | 657614956490835                  | 97570424122840995               |
| 22| 15491105783776                    | 2170327643009376                 | 34587799231828288               |
| 23| 47313566966717                    | 7091919679833573                 | 1210056084152236397             |
| 24| 143616941038800                   | 22964364302956192                | 4182680502669028416             |
| 25| 433471181567175                   | 73742760159367607                | 1249878834735619530             |
| 26| 1301492251611088                  | 234986541658461504               | 4838572587437034944             |
| 27| 3888842767461272                 | 743493757302422163               | 162197418768856363467           |
| 28| 11567743361677920                | 2336936884325400320              | 538986352207098913536           |
| 29| 34265929488742837                | 730504880342236965               | 177658392652252100533           |
| 30| 101107717070386128               | 22676475755899170368             | 5811758654106692557056          |
| 31| 2972517196011411                | 70061223151862034731              | 188781408088883145579           |
| 32| 870928677714199072               | 215380608263460514688             | 60916738136365328424448         |
| $N$ | $\frac{1}{4}c_N\langle r^4 \cos 4\theta \rangle_N$ | $\frac{1}{4}c_N\langle r^6 \cos 4\theta \rangle_N$ | $\frac{1}{4}c_N\langle r^8 \cos 4\theta \rangle_N$ |
|----|------------------------------------------------|------------------------------------------------|------------------------------------------------|
| 1  | 1                                              | 1                                              | 1                                              |
| 2  | 8                                              | 48                                             | 224                                            |
| 3  | 41                                             | 521                                            | 5513                                           |
| 4  | 176                                            | 3616                                           | 64832                                          |
| 5  | 517                                            | 18581                                          | 515461                                         |
| 6  | 2464                                           | 96672                                          | 3460288                                        |
| 7  | 8543                                           | 424767                                         | 19559327                                       |
| 8  | 28672                                          | 1744320                                       | 99520384                                       |
| 9  | 93715                                          | 6804987                                       | 467838211                                      |
| 10 | 300016                                         | 25497024                                      | 2067530752                                     |
| 11 | 943881                                         | 92462801                                      | 8694947865                                     |
| 12 | 2927136                                        | 326371072                                      | 35106653952                                    |
| 13 | 8966103                                        | 1126120359                                     | 136996603671                                   |
| 14 | 27176192                                       | 3810903520                                    | 519344946752                                   |
| 15 | 81614149                                       | 12681966461                                    | 1920357610645                                  |
| 16 | 243136160                                       | 41589494144                                    | 6948572721152                                  |
| 17 | 719161805                                       | 134640733141                                   | 24668510505533                                 |
| 18 | 2113740144                                     | 430916393344                                   | 86113474737024                                 |
| 19 | 6177297699                                     | 1365089628939                                  | 296122738362483                                |
| 20 | 17960659728                                    | 4284776312224                                  | 1004647110417216                               |
| 21 | 51978553251                                     | 13337579454483                                 | 3367181961982563                               |
| 22 | 149793700032                                    | 41203857605920                                 | 11161461862577856                              |
| 23 | 430013901309                                    | 126414733312805                                | 36627023206822344                              |
| 24 | 1230085625008                                   | 385396000445280                                | 119090886021960640                             |
| 25 | 3507275950151                                   | 116811892213531                                | 383951817561778304                             |
| 26 | 99700803693360                                   | 3521536905190720                               | 1228239234255697152                            |
| 27 | 28262765992155                                   | 10563701297658838                              | 3900781003385081163                            |
| 28 | 79911109071584                                   | 31542297194620416                              | 12305781139629052160                           |
| 29 | 22539846017269                                   | 93778343662150501                              | 38579685692175173877                           |
| 30 | 634334283147728                                  | 277695329500224576                             | 12024924984158963968                           |
| 31 | 1781434024153067                                 | 819226661598869419                             | 372773226138821611691                          |
| 32 | 4993035148467488                                 | 2408303159048790400                            | 1149726544611189212672                        

Table 4: Exact enumeration data for SAWs on the square lattice. Non-rotationally-invariant moments.
Table 5: Exact enumeration data for SAWs on the triangular lattice. Rotationally invariant moments.

| \(N\) | \(\frac{1}{6}c_N\langle r^4 \rangle_N\) | \(\frac{1}{6}c_N\langle r^6 \rangle_N\) | \(\frac{1}{6}c_N\langle r^8 \rangle_N\) |
|-----|----------------------------------|----------------------------------|----------------------------------|
| 1   | 1 1 1 1                          | 1 1 1 1                          | 1 1 1 1                          |
| 2   | 36 120 420                       | 2293                            |
| 3   | 529 3337 2293                    |                                 |
| 4   | 5454 53358 579918                |                                 |
| 5   | 46169 633185 9725849             |                                 |
| 6   | 344428 6221884 126456796         |                                 |
| 7   | 2352769 53647549 1380311377      |                                 |
| 8   | 15060090 420194610 13256099610   |                                 |
| 9   | 91701871 3057404227 115436446639 |                                 |
| 10  | 536695548 20985811596 930424151244 |                                 |
| 11  | 3041620465 137386509145 7043622904369 |                                 |
| 12  | 16784388968 864860517248 5062426215432 |                                 |
| 13  | 90564392107 5267777416675 348381660817291 |                                 |
| 14  | 479388030946 31195330864090 2309752803978322 |                                 |
| 15  | 2495911050345 180302936975925 14829889785198921 |                                 |
| 16  | 12808562012852 1020273312831596 92588968138883348 |                                 |
| 17  | 6490124792059 5666785971562159 56401808195524619 |                                 |
| 18  | 325170745810666 30958613694252346 3361656410695492858 |                                 |
| 19  | 1612871078099977 166655904211475269 196498101539028217 |                                 |
| 20  | 7927964773508104 885333404431705216 11286793534532712424 |                                 |
| 21  | 38652473796950531 4647266177110051355 638156051853903239891 |                                 |
| 22  | 187056802703356296 24130876287242419704 3556820174217345377400 |                                 |
| \( N \) | \( \frac{1}{6}c_N \langle r^6 \cos 6 \theta \rangle_N \) | \( \frac{1}{6}c_N \langle r^8 \cos 6 \theta \rangle_N \) |
| --- | --- | --- |
| 1 | 1 | 1 |
| 2 | 12 | 96 |
| 3 | 97 | 1609 |
| 4 | 654 | 17454 |
| 5 | 3977 | 151649 |
| 6 | 22684 | 1149148 |
| 7 | 123721 | 7935829 |
| 8 | 652842 | 51236610 |
| 9 | 3357439 | 314319571 |
| 10 | 16914348 | 1852261068 |
| 11 | 83777857 | 10566164665 |
| 12 | 409089560 | 58677117008 |
| 13 | 1973505067 | 318573263587 |
| 14 | 9421326322 | 169658322746 |
| 15 | 44567944521 | 8885936605365 |
| 16 | 209144745044 | 45868003547852 |
| 17 | 974497840243 | 233746526628199 |
| 18 | 4511869867210 | 1177691322037546 |
| 19 | 20770914530257 | 587341983148317 |
| 20 | 95130303643048 | 29024633960838784 |
| 21 | 433664585252891 | 142245181072370291 |
| 22 | 1968525488778840 | 691879404495232056 |

Table 6: Exact enumeration data for SAWs on the triangular lattice. Non-rotationally-invariant moments.
Table 7: Fit to $\langle R^2 \rangle = N^{3/2}(a_0 + a_1 N^{-1} + a_2 N^{-3/2} + a_3 N^{-2})$ for SAWs on the triangular lattice. Data is for $\langle R^2 \rangle_N$ at top, then $\langle R^2 \rangle_{2N}$ at middle, then $\langle R^2 \rangle_{3N}$ at bottom. The fit for $\langle R^2 \rangle_{mN}$ includes only terms up to order $(N^{-3/2})$.  

| N   | $a_0$ | $a_1$ | $a_2$ | $a_3$ |
|-----|-------|-------|-------|-------|
| 19  | 0.099915 | 0.050034 | 0.335467 | 0.712401 | 0.847374 |
| 20  | 0.099919 | 0.049529 | 0.338206 | 0.712338 | 0.854132 |
| 21  | 0.099917 | 0.049529 | 0.338252 | 0.712262 | 0.862841 |
| 22  | 0.099915 | 0.049529 | 0.338252 | 0.712207 | 0.869360 |
| 23  | 0.099916 | 0.049529 | 0.338252 | 0.712154 | 0.876015 |
| 24  | 0.099917 | 0.049529 | 0.338252 | 0.712108 | 0.882131 |
| 25  | 0.099919 | 0.049529 | 0.338252 | 0.712067 | 0.887809 |
| 26  | 0.099921 | 0.049529 | 0.338252 | 0.712030 | 0.893200 |
| 27  | 0.099923 | 0.049529 | 0.338252 | 0.711996 | 0.898252 |
| 28  | 0.099925 | 0.049529 | 0.338252 | 0.711965 | 0.903032 |
| 29  | 0.099928 | 0.049529 | 0.338252 | 0.711937 | 0.907558 |
| 30  | 0.099931 | 0.049529 | 0.338252 | 0.711912 | 0.911851 |
| 31  | 0.099934 | 0.049529 | 0.338252 | 0.711888 | 0.915933 |
| 32  | 0.099937 | 0.049529 | 0.338252 | 0.711867 | 0.919820 |
| 33  | 0.099940 | 0.049529 | 0.338252 | 0.711847 | 0.923529 |
| 34  | 0.099943 | 0.049529 | 0.338252 | 0.711828 | 0.927071 |
| 35  | 0.099945 | 0.049529 | 0.338252 | 0.711811 | 0.930464 |
| 36  | 0.099947 | 0.049529 | 0.338252 | 0.711795 | 0.933713 |
| 37  | 0.099949 | 0.049529 | 0.338252 | 0.711780 | 0.936831 |
| 38  | 0.099951 | 0.049529 | 0.338252 | 0.711766 | 0.939827 |
| 39  | 0.099953 | 0.049529 | 0.338252 | 0.711753 | 0.942708 |
| 40  | 0.099955 | 0.049529 | 0.338252 | 0.711741 | 0.945482 |

| N   | $a_0$ | $a_1$ | $a_2$ | $a_3$ |
|-----|-------|-------|-------|-------|
| 41  | 0.099956 | 0.049529 | 0.338252 | 0.711731 | 0.948389 |
| 42  | 0.099958 | 0.049529 | 0.338252 | 0.711721 | 0.951317 |
| 43  | 0.099960 | 0.049529 | 0.338252 | 0.711712 | 0.954296 |
| 44  | 0.099962 | 0.049529 | 0.338252 | 0.711703 | 0.957247 |
| 45  | 0.099964 | 0.049529 | 0.338252 | 0.711696 | 0.960204 |
| 46  | 0.099966 | 0.049529 | 0.338252 | 0.711688 | 0.963171 |
| 47  | 0.099968 | 0.049529 | 0.338252 | 0.711681 | 0.966148 |
| 48  | 0.099970 | 0.049529 | 0.338252 | 0.711674 | 0.969137 |
| 49  | 0.099972 | 0.049529 | 0.338252 | 0.711667 | 0.972142 |
| 50  | 0.099974 | 0.049529 | 0.338252 | 0.711660 | 0.975169 |

| N   | $a_0$ | $a_1$ | $a_2$ | $a_3$ |
|-----|-------|-------|-------|-------|
| 51  | 0.099976 | 0.049529 | 0.338252 | 0.711653 | 0.978217 |
| 52  | 0.099978 | 0.049529 | 0.338252 | 0.711646 | 0.981286 |
| 53  | 0.099980 | 0.049529 | 0.338252 | 0.711640 | 0.984377 |
| 54  | 0.099982 | 0.049529 | 0.338252 | 0.711634 | 0.987488 |
| 55  | 0.099984 | 0.049529 | 0.338252 | 0.711629 | 0.990619 |
| 56  | 0.099986 | 0.049529 | 0.338252 | 0.711624 | 0.993763 |
| 57  | 0.099988 | 0.049529 | 0.338252 | 0.711619 | 0.996928 |
| 58  | 0.099990 | 0.049529 | 0.338252 | 0.711615 | 0.999114 |
| 59  | 0.099992 | 0.049529 | 0.338252 | 0.711611 | 0.999114 |
| 60  | 0.099994 | 0.049529 | 0.338252 | 0.711607 | 0.999114 |
Table 8: Fit $\langle R^2_N \rangle = N^{3/2}(a_{0,e} + a_{1,e}N^{-1} + a_{2,e}N^{-3/2} + a_{3,e}N^{-2} + a_{4,e}N^{-5/2}) + (-1)^NN^{-11/32}(b_{0,e} + b_{1,e}/N + b_{2,e}/N^2)$ for SAWs on the square lattice.
| $N$  | $a_{0,g}$ | $a_{1,g}$ | $a_{2,g}$ | $b_{0,g}$ | $b_{1,g}$ |
|------|-----------|-----------|-----------|-----------|-----------|
| 33   | 0.108213  | 0.103702  | 0.100792  | 0.008333  | 0.004174  |
| 34   | 0.108216  | 0.103450  | 0.101736  | 0.008370  | 0.003016  |
| 35   | 0.108218  | 0.103242  | 0.102526  | 0.008394  | 0.003994  |
| 36   | 0.108220  | 0.103040  | 0.103306  | 0.008369  | 0.003017  |
| 37   | 0.108221  | 0.102873  | 0.103959  | 0.008345  | 0.003842  |
| 38   | 0.108223  | 0.102712  | 0.104599  | 0.008368  | 0.003025  |
| 39   | 0.108224  | 0.102579  | 0.105134  | 0.008349  | 0.003715  |
| 40   | 0.108225  | 0.102452  | 0.105653  | 0.008367  | 0.003039  |
| 41   | 0.108226  | 0.102347  | 0.106085  | 0.008352  | 0.003607  |
| 42   | 0.108227  | 0.102249  | 0.106500  | 0.008366  | 0.003058  |
| 43   | 0.108228  | 0.102168  | 0.106842  | 0.008355  | 0.003515  |
| 44   | 0.108228  | 0.102092  | 0.107166  | 0.008365  | 0.003078  |
| 45   | 0.108229  | 0.102032  | 0.107429  | 0.008357  | 0.003436  |
| 46   | 0.108229  | 0.101976  | 0.107674  | 0.008365  | 0.003100  |
| 47   | 0.108230  | 0.101933  | 0.107867  | 0.008359  | 0.003367  |
| 48   | 0.108230  | 0.101894  | 0.108043  | 0.008364  | 0.003122  |
| 49   | 0.108230  | 0.101865  | 0.108175  | 0.008360  | 0.003307  |
| 50   | 0.108230  | 0.101840  | 0.108290  | 0.008363  | 0.003145  |
| 51   | 0.108230  | 0.101823  | 0.108367  | 0.008361  | 0.003254  |
| 52   | 0.108230  | 0.101810  | 0.108428  | 0.008363  | 0.003166  |
| 53   | 0.108230  | 0.101804  | 0.108456  | 0.008362  | 0.003207  |
| 54   | 0.108230  | 0.101801  | 0.108470  | 0.008363  | 0.003187  |
| 55   | 0.108230  | 0.101804  | 0.108455  | 0.008363  | 0.003165  |
| 56   | 0.108230  | 0.101810  | 0.108427  | 0.008362  | 0.003207  |
| 57   | 0.108230  | 0.101821  | 0.108373  | 0.008364  | 0.003128  |
| 58   | 0.108230  | 0.101834  | 0.108307  | 0.008362  | 0.003226  |
| 59   | 0.108230  | 0.101852  | 0.108219  | 0.008364  | 0.003094  |

Table 9: Fit $\langle R_g^2 \rangle_N = N^{3/2}(a_{0,g} + a_{1,g}N^{-1} + a_{2,g}N^{-3/2}) + (-1)^NN^{-11/32}(b_{0,g} + b_{1,g}/N)$ for SAWs on the square lattice.
| $N$ | $a_{0,m}$ | $a_{1,m}$ | $a_{2,m}$ | $a_{3,m}$ | $b_{0,m}$ | $b_{1,m}$ |
|-----|----------|----------|----------|----------|----------|----------|
| 33  | 0.339223 | 0.401114 | -0.849196| 1.011789 | 0.036366 | 0.020608 |
| 34  | 0.339252 | 0.395837 | -0.810037| 0.930082 | 0.036379 | 0.020222 |
| 35  | 0.339210 | 0.403819 | -0.870217| 1.057665 | 0.036397 | 0.019642 |
| 36  | 0.339233 | 0.399343 | -0.835945| 0.983880 | 0.036407 | 0.019318 |
| 37  | 0.339198 | 0.406364 | -0.890517| 1.103143 | 0.036422 | 0.018812 |
| 38  | 0.339216 | 0.402541 | -0.860364| 1.036281 | 0.036430 | 0.018538 |
| 39  | 0.339188 | 0.408723 | -0.909807| 1.147474 | 0.036442 | 0.018096 |
| 40  | 0.339202 | 0.405479 | -0.883503| 1.087501 | 0.036448 | 0.017866 |
| 41  | 0.339178 | 0.401928 | -0.928281| 1.190967 | 0.036452 | 0.017480 |
| 42  | 0.339190 | 0.408177 | -0.905379| 1.087501 | 0.036448 | 0.017866 |
| 43  | 0.339170 | 0.413005 | -0.946085| 1.233861 | 0.036472 | 0.016947 |
| 44  | 0.339180 | 0.410665 | -0.926109| 1.185914 | 0.036476 | 0.016783 |
| 45  | 0.339163 | 0.414963 | -0.963242| 1.276125 | 0.036483 | 0.016483 |
| 46  | 0.339170 | 0.412973 | -0.945843| 1.233555 | 0.036486 | 0.016345 |
| 47  | 0.339156 | 0.416811 | -0.979781| 1.317746 | 0.036492 | 0.016080 |
| 48  | 0.339162 | 0.415122 | -0.964678| 1.279766 | 0.036495 | 0.015963 |
| 49  | 0.339150 | 0.418559 | -0.995753| 1.358778 | 0.036500 | 0.015727 |
| 50  | 0.339155 | 0.417130 | -0.982689| 1.325201 | 0.036502 | 0.015629 |
| 51  | 0.339144 | 0.420217 | -1.011200| 1.399256 | 0.036506 | 0.015418 |
| 52  | 0.339148 | 0.419011 | -0.999949| 1.369730 | 0.036508 | 0.015336 |
| 53  | 0.339139 | 0.421791 | -1.026155| 1.439200 | 0.036512 | 0.015148 |
| 54  | 0.339142 | 0.420779 | -1.016525| 1.413417 | 0.036513 | 0.015079 |
| 55  | 0.339134 | 0.423289 | -1.040647| 1.478633 | 0.036516 | 0.014910 |
| 56  | 0.339137 | 0.422446 | -1.032471| 1.456316 | 0.036517 | 0.014854 |
| 57  | 0.339130 | 0.424716 | -1.054707| 1.517578 | 0.036520 | 0.014702 |
| 58  | 0.339132 | 0.424021 | -1.047836| 1.498474 | 0.036521 | 0.014655 |
| 59  | 0.339126 | 0.426078 | -1.068359| 1.556056 | 0.036523 | 0.014518 |

Table 10: Fit $\langle R_m^2 \rangle_N = N^{3/2}(a_{0,m} + a_{1,m} N^{-1} + a_{2,m} N^{-3/2} + a_{3,m} N^{-2}) + (-1)^N N^{-11/32}(b_{0,m} + b_{1,m}/N)$ for SAWs on the square lattice.
| $N$ | $\langle R^2 \rangle_N$ | $\langle R^4 \rangle_N$ | $\langle R^2_m \rangle_N$ | $F_N \langle R^2 \rangle_N$ |
|-----|------------------|------------------|------------------|------------------|
| 40  | 200.106 ± 0.011  |                  |                  |                  |
| 60  | 364.977 ± 0.012  | 51.2026 ± 0.0031 | 160.10 ± 0.04   | 0.71 ± 0.08     |
| 80  | 559.656 ± 0.033  | 78.4660 ± 0.0051 | 245.51 ± 0.06   | 0.91 ± 0.13     |
| 100 | 780.245 ± 0.028  | 109.351 ± 0.007  | 342.34 ± 0.09   | 1.06 ± 0.18     |
| 120 | 1023.812 ± 0.064 | 143.488 ± 0.010  | 449.39 ± 0.12   | 1.02 ± 0.24     |
| 140 | 1288.631 ± 0.064 | 180.599 ± 0.013  | 565.93 ± 0.16   | 0.67 ± 0.32     |
| 150 | 1428.367 ± 0.035 | 200.185 ± 0.015  | 627.51 ± 0.18   | 0.32 ± 0.36     |
| 180 | 1875.399 ± 0.081 | 262.858 ± 0.020  | 823.95 ± 0.13   | 0.38 ± 0.26     |
| 200 | 2195.00 ± 0.11   | 307.676 ± 0.024  | 964.36 ± 0.13   | 0.52 ± 0.27     |
| 250 | 3064.11 ± 0.11   | 429.499 ± 0.024  | 1346.26 ± 0.19  | 0.59 ± 0.39     |
| 300 | 4024.66 ± 0.17   | 564.226 ± 0.021  | 1768.44 ± 0.07  | 0.71 ± 0.18     |
| 400 | 6190.01 ± 0.26   | 867.890 ± 0.033  | 2720.27 ± 0.12  | 0.62 ± 0.28     |
| 500 | 8645.61 ± 0.31   | 1212.26 ± 0.07   | 3799.66 ± 0.25  | 0.57 ± 0.57     |
| 700 | 14311.70 ± 0.57  | 2007.10 ± 0.11   |                  |                  |
| 1000| 24421.12 ± 0.85  | 3425.28 ± 0.11   | 10734.5 ± 0.53  | 1.15 ± 1.20     |
| 1400| 40439.29 ± 2.11  | 5671.95 ± 0.42   |                  |                  |
| 2000| 69028.33 ± 3.82  | 9684.17 ± 0.79   | 30347.0 ± 3.5   | −0.6 ± 7.5      |
| 3000| 126789.4 ± 7.9   | 17790.3 ± 2.1    |                  |                  |
| 4000| 195162.3 ± 12.2  | 27376.1 ± 2.7    | 85773.6 ± 13.0  | 40 ± 27         |

Table 11: Our Monte Carlo data for radii $\langle R^2 \rangle_N$ as a function of walk length $N$. Errors (one standard deviation) are shown in parentheses.
| \( N \) | \( a_N \) (10) | \( b_N \) (100) | \( c_N \) (76) | \( f_N \) (200) |
|---|---|---|---|---|
| 60 | 0.140290 (10) | 0.438644 (100) | 0.319826 (76) | 0.001957 (200) |
| 80 | 0.140204 (12) | 0.438689 (120) | 0.319598 (88) | 0.001635 (230) |
| 100 | 0.140149 (11) | 0.438756 (120) | 0.319425 (88) | 0.001354 (240) |
| 120 | 0.140150 (13) | 0.438938 (120) | 0.319294 (88) | 0.000992 (240) |
| 140 | 0.140148 (12) | 0.439171 (130) | 0.319118 (93) | 0.000518 (250) |
| 150 | 0.140149 (11) | 0.439320 (130) | 0.319014 (94) | 0.000225 (250) |
| 180 | 0.140161 (12) | 0.43947 (71) | 0.319022 (55) | 0.000204 (140) |
| 200 | 0.140171 (13) | 0.439344 (63) | 0.319047 (49) | 0.000236 (120) |
| 250 | 0.140171 (9) | 0.439365 (64) | 0.319030 (48) | 0.000193 (130) |
| 300 | 0.140192 (8) | 0.439402 (26) | 0.319052 (18) | 0.000177 (45) |
| 400 | 0.140258 (13) | 0.439499 (71) | 0.319167 (57) | 0.000204 (140) |
| 500 | 0.140259 (7) | 0.439557 (27) | 0.319091 (19) | 0.000047 (48) |
| 1000 | 0.140258 (13) | 0.439630 (56) | 0.319115 (45) | 0.000008 (110) |
| 1400 | 0.140293 (14) | 0.439630 (56) | 0.319115 (45) | 0.000008 (110) |
| 2000 | 0.140274 (16) | 0.439499 (71) | 0.319167 (57) | 0.000204 (140) |
| 3000 | 0.140314 (19) | 0.439499 (71) | 0.319167 (57) | 0.000204 (140) |

Table 12: Our Monte Carlo data for amplitude ratios as a function of walk length. Errors (one standard deviation) are shown in parentheses.

\[
\langle R^2 \rangle = N^{2\nu}[a_e + b_e(N/750)^{\Delta}].
\]

| \( N_{\text{min}} \) | \( a_e \) | \( b_e \) | \( \Delta \) | \( \chi^2 \) | DF | CL |
|---|---|---|---|---|---|---|
| 40 | 0.770998 (23) | 0.001644 (16) | 0.855 (3) | 40.9 | 16 | 0.1% |
| 60 | 0.771054 (25) | 0.001589 (19) | 0.869 (4) | 14.9 | 15 | 46.2% |
| 80 | 0.771081 (30) | 0.001559 (26) | 0.878 (7) | 12.2 | 14 | 59.0% |
| 100 | 0.771106 (32) | 0.001532 (29) | 0.887 (8) | 8.4 | 13 | 82.0% |
| 120 | 0.771122 (37) | 0.001512 (36) | 0.894 (12) | 7.6 | 12 | 81.8% |
| 140 | 0.771140 (38) | 0.001491 (38) | 0.902 (13) | 4.8 | 11 | 93.9% |
| 150 | 0.771140 (39) | 0.001492 (39) | 0.902 (13) | 4.8 | 10 | 90.2% |
| 180 | 0.771160 (45) | 0.001467 (47) | 0.914 (19) | 4.1 | 9 | 90.7% |
| 200 | 0.771148 (49) | 0.001481 (54) | 0.906 (23) | 3.7 | 8 | 88.1% |
| 250 | 0.771155 (55) | 0.001472 (62) | 0.911 (29) | 3.6 | 7 | 82.0% |
| 300 | 0.771148 (64) | 0.001481 (73) | 0.905 (39) | 3.6 | 6 | 73.2% |
| 400 | 0.771136 (82) | 0.001495 (94) | 0.895 (58) | 3.5 | 5 | 61.8% |
| 500 | 0.771161 (93) | 0.001469 (104) | 0.921 (80) | 3.3 | 4 | 51.0% |
| 700 | 0.771249 (110) | 0.001396 (109) | 1.051 (156) | 2.3 | 3 | 51.1% |
| 1000 | 0.770975 (410) | 0.001571 (344) | 0.692 (332) | 0.9 | 2 | 62.4% |

Table 13: Fit \( \langle R^2 \rangle = N^{2\nu}[a_e + b_e(N/750)^{\Delta}] \). DF is the number of degrees of freedom and CL is the confidence level of the fit.
| $N_{\text{min}}$ | $a_g$    | $b_g$    | $\Delta$ | $\chi^2$ | DF | CL  |
|-----------------|----------|----------|-----------|----------|----|-----|
| 60              | 0.108213(4) | 0.000142(3) | 1.037(8)  | 19.9     | 15 | 17.5% |
| 80              | 0.108209(5) | 0.000146(4) | 1.023(11) | 16.8     | 14 | 26.7% |
| 100             | 0.108205(5) | 0.000151(5) | 1.007(15) | 14.2     | 13 | 35.9% |
| 120             | 0.108205(6) | 0.000150(5) | 1.008(18) | 14.2     | 12 | 28.8% |
| 140             | 0.108207(6) | 0.000148(6) | 1.018(22) | 13.5     | 11 | 25.9% |
| 150             | 0.108207(7) | 0.000148(7) | 1.019(26) | 13.5     | 10 | 19.5% |
| 180             | 0.108210(8) | 0.000145(8) | 1.033(34) | 13.1     | 9  | 15.7% |
| 200             | 0.108208(9) | 0.000147(9) | 1.023(41) | 12.9     | 8  | 11.4% |
| 250             | 0.108205(10)| 0.000151(11)| 1.000(53) | 12.5     | 7  | 8.5%  |
| 300             | 0.108208(11)| 0.000147(13)| 1.025(66) | 12.1     | 6  | 6.0%  |
| 400             | 0.108201(15)| 0.000155(17)| 0.966(100)| 11.5     | 5  | 4.2%  |
| 500             | 0.108198(19)| 0.000158(21)| 0.937(146)| 11.5     | 4  | 2.2%  |
| 700             | 0.108213(20)| 0.000146(19)| 1.161(286)| 10.6     | 3  | 1.4%  |
| 1000            | 0.107954(1213)| 0.000384(1200)| 0.206(779)| 8.8     | 2  | 1.3%  |

Table 14: Fit $\langle R_g^2 \rangle = N^{2\nu}[a_g + b_g(N/750)^\Delta]$. DF is the number of degrees of freedom and CL is the confidence level of the fit.

| $N_{\text{min}}$ | $a_m$    | $b_m$    | $\Delta$ | $\chi^2$ | DF | CL  |
|-----------------|----------|----------|-----------|----------|----|-----|
| 60              | 0.338957(32) | 0.0000635(27) | 0.850(17) | 15.4     | 12 | 21.9% |
| 80              | 0.338932(38) | 0.0000661(33) | 0.828(23) | 13.3     | 11 | 27.1% |
| 100             | 0.338940(41) | 0.0000652(38) | 0.836(29) | 13.1     | 10 | 21.7% |
| 120             | 0.338971(43) | 0.0000618(41) | 0.871(36) | 10.2     | 9  | 33.6% |
| 140             | 0.339008(44) | 0.0000577(42) | 0.917(42) | 5.7      | 8  | 68.1% |
| 150             | 0.339018(45) | 0.0000566(44) | 0.930(47) | 5.3      | 7  | 62.7% |
| 180             | 0.339013(48) | 0.0000571(48) | 0.923(52) | 5.2      | 6  | 52.1% |
| 200             | 0.338990(56) | 0.0000598(58) | 0.890(62) | 4.2      | 5  | 51.7% |
| 250             | 0.338962(71) | 0.0000630(75) | 0.852(79) | 3.7      | 4  | 45.3% |
| 300             | 0.338955(75) | 0.0000638(81) | 0.842(84) | 3.6      | 3  | 31.3% |
| 400             | 0.338915(115)| 0.0000683(126)| 0.784(136)| 3.3      | 2  | 19.2% |
| 500             | 0.338896(146)| 0.0000702(155)| 0.752(186)| 3.2      | 1  | 7.2%  |

Table 15: Fit $\langle R_m^2 \rangle = N^{2\nu}[a_m + b_m(N/750)^\Delta]$. DF is the number of degrees of freedom and CL is the confidence level of the fit.
Table 16: Fit $\langle R^2_e \rangle = N^{2r}(a_e + b_e N^{-1} + c_e N^{-3/2})$. DF is the number of degrees of freedom and CL is the confidence level of the fit.

| $N_{\text{min}}$ | $a_e$       | $b_e$       | $c_e$       | $\chi^2$ | DF | CL   |
|------------------|-------------|-------------|-------------|-----------|----|------|
| 40               | 0.771261(16)| 1.093(7)    | -1.93(4)    | 15.4      | 16 | 49.4%|
| 60               | 0.771247(18)| 1.104(9)    | -2.03(7)    | 12.1      | 15 | 67.1%|
| 80               | 0.771224(20)| 1.131(15)   | -2.30(14)   | 6.7       | 14 | 94.6%|
| 100              | 0.771224(22)| 1.131(18)   | -2.30(18)   | 6.7       | 13 | 91.7%|
| 120              | 0.771211(25)| 1.150(26)   | -2.52(29)   | 5.7       | 12 | 93.0%|
| 140              | 0.771218(27)| 1.139(29)   | -2.39(32)   | 5.0       | 11 | 93.1%|
| 150              | 0.771216(27)| 1.143(30)   | -2.44(34)   | 4.8       | 10 | 90.1%|
| 180              | 0.771218(31)| 1.137(42)   | -2.36(54)   | 4.8       | 9  | 85.0%|
| 200              | 0.771206(33)| 1.165(52)   | -2.79(70)   | 3.9       | 8  | 86.6%|
| 250              | 0.771204(37)| 1.170(65)   | -2.86(95)   | 3.9       | 7  | 79.3%|
| 300              | 0.771194(42)| 1.199(87)   | -3.38(1.41) | 3.6       | 6  | 72.6%|
| 400              | 0.771177(53)| 1.257(136)  | -4.50(2.46) | 3.3       | 5  | 64.9%|
| 500              | 0.771187(62)| 1.218(186)  | -3.68(3.65) | 3.2       | 4  | 51.9%|
| 700              | 0.771244(85)| 0.936(340)  | 3.16(7.80)  | 2.3       | 3  | 52.2%|
| 1000             | 0.771118(142)| 1.680(755) | -16.96(19.82) | 1.0       | 2  | 59.7%|

Table 17: Fit $\langle R^2_g \rangle = N^{2r}(a_g + b_g N^{-1} + c_g N^{-3/2})$. DF is the number of degrees of freedom and CL is the confidence level of the fit.

| $N_{\text{min}}$ | $a_g$      | $b_g$      | $c_g$      | $\chi^2$ | DF | CL   |
|------------------|-------------|-------------|-------------|-----------|----|------|
| 60               | 0.108209(3) | 0.107(2)    | 0.08(2)     | 17.8      | 15 | 27.5%|
| 80               | 0.108207(4) | 0.109(3)    | 0.06(3)     | 16.3      | 14 | 29.3%|
| 100              | 0.108204(4) | 0.113(4)    | 0.02(4)     | 14.3      | 13 | 35.6%|
| 120              | 0.108204(5) | 0.113(5)    | 0.02(5)     | 14.3      | 12 | 28.5%|
| 140              | 0.108206(5) | 0.110(6)    | 0.05(6)     | 13.5      | 11 | 25.9%|
| 150              | 0.108206(5) | 0.110(6)    | 0.06(8)     | 13.5      | 10 | 19.6%|
| 180              | 0.108208(6) | 0.106(8)    | 0.11(11)    | 13.0      | 9  | 16.4%|
| 200              | 0.108207(7) | 0.108(10)   | 0.08(14)    | 12.9      | 8  | 11.7%|
| 250              | 0.108205(8) | 0.113(13)   | 0.00(19)    | 12.5      | 7  | 8.5% |
| 300              | 0.108208(9) | 0.106(17)   | 0.12(25)    | 12.0      | 6  | 6.1% |
| 400              | 0.108203(11)| 0.120(26)   | -0.14(45)   | 11.5      | 5  | 4.2% |
| 500              | 0.108201(13)| 0.128(37)   | -0.30(72)   | 11.5      | 4  | 2.2% |
| 700              | 0.108212(17)| 0.076(64)   | 0.95(1.46)  | 10.5      | 3  | 1.5% |
| 1000             | 0.108185(30)| 0.227(156)  | -3.03(4.03) | 9.4       | 2  | 0.9% |
Table 18: Fit $\langle R_m^2 \rangle = N^{2\nu} (a_m + b_m N^{-1} + c_m N^{-3/2})$. DF is the number of degrees of freedom and CL is the confidence level of the fit.

| $N_{\text{min}}$ | $a_m$     | $b_m$     | $c_m$     | $\chi^2$ | DF | CL  |
|------------------|-----------|-----------|-----------|----------|----|-----|
| 60               | 0.339040(21) | 0.447(14) | -0.99(12) | 18.6     | 12 | 9.9% |
| 80               | 0.339013(23) | 0.479(18) | -1.37(18) | 11.3     | 11 | 41.7% |
| 100              | 0.339008(25) | 0.487(23) | -1.47(26) | 11.0     | 10 | 35.7% |
| 120              | 0.339019(27) | 0.469(28) | -1.23(34) | 9.8      | 9  | 36.4% |
| 140              | 0.339038(29) | 0.436(33) | -0.76(42) | 6.3      | 8  | 61.7% |
| 150              | 0.339044(30) | 0.426(37) | -0.61(48) | 5.9      | 7  | 55.6% |
| 180              | 0.339041(32) | 0.431(41) | -0.69(55) | 5.8      | 6  | 44.8% |
| 200              | 0.339028(34) | 0.457(49) | -1.09(69) | 4.9      | 5  | 43.1% |
| 250              | 0.339010(41) | 0.496(68) | -1.69(1.01) | 4.2    | 4  | 38.0% |
| 300              | 0.339005(43) | 0.508(75) | -1.88(1.13) | 4.1    | 3  | 25.5% |
| 400              | 0.338976(62) | 0.583(140) | -3.21(2.37) | 3.6    | 2  | 16.1% |
| 500              | 0.338964(74) | 0.623(194) | -4.04(3.62) | 3.6    | 1  | 5.9% |

Table 19: Fit $A_N = a_A + b_A N^{-1} + c_A N^{-3/2}$. DF is the number of degrees of freedom and CL is the confidence level of the fit.

| $N_{\text{min}}$ | $a_A$     | $b_A$     | $c_A$     | $\chi^2$ | DF | CL  |
|------------------|-----------|-----------|-----------|----------|----|-----|
| 60               | 0.140305(6) | -0.062(3) | 0.47(2)   | 8.4      | 15 | 90.7% |
| 80               | 0.140306(6) | -0.064(5) | 0.49(4)   | 8.1      | 14 | 88.2% |
| 100              | 0.140303(7) | -0.060(6) | 0.45(6)   | 7.1      | 13 | 89.6% |
| 120              | 0.140306(8) | -0.064(8) | 0.49(9)   | 6.6      | 12 | 88.2% |
| 140              | 0.140306(8) | -0.064(9) | 0.50(10)  | 6.6      | 11 | 83.0% |
| 150              | 0.140307(9) | -0.065(10) | 0.52(12)  | 6.5      | 10 | 77.0% |
| 180              | 0.140309(10) | -0.071(13) | 0.59(17)  | 6.2      | 9  | 72.5% |
| 200              | 0.140310(11) | -0.072(16) | 0.62(22)  | 6.1      | 8  | 63.4% |
| 250              | 0.140307(12) | -0.065(21) | 0.51(31)  | 5.9      | 7  | 55.7% |
| 300              | 0.140312(14) | -0.078(27) | 0.72(42)  | 5.3      | 6  | 50.3% |
| 400              | 0.140310(18) | -0.072(42) | 0.61(74)  | 5.3      | 5  | 38.2% |
| 500              | 0.140304(21) | -0.049(59) | 0.12(1.15) | 5.0     | 4  | 28.9% |
| 700              | 0.140306(27) | -0.060(104) | 0.39(2.38)| 5.0    | 3  | 17.4% |
| 1000             | 0.140297(47) | -0.007(244) | -1.03(6.34) | 4.9    | 2  | 8.6% |
| $N_{\text{min}}$ | $a_B$         | $b_B$         | $c_B$         | $\chi^2$ | DF  | CL   |
|----------------|--------------|--------------|--------------|----------|-----|------|
| 60             | 0.439619(31) | -0.062(21)  | -0.02(17)    | 12.8     | 12  | 38.1%|
| 80             | 0.439587(34) | -0.024(27)  | -0.45(27)    | 8.3      | 11  | 68.3%|
| 100            | 0.439578(38) | -0.012(34)  | -0.61(37)    | 8.0      | 10  | 63.2%|
| 120            | 0.439596(41) | -0.040(42)  | -0.24(49)    | 6.6      | 9   | 67.5%|
| 140            | 0.439620(43) | -0.082(49)  | 0.34(61)     | 4.0      | 8   | 85.6%|
| 150            | 0.439629(45) | -0.098(55)  | 0.59(70)     | 3.5      | 7   | 83.4%|
| 180            | 0.439626(48) | -0.091(61)  | 0.48(80)     | 3.4      | 6   | 75.2%|
| 200            | 0.439614(51) | -0.068(73)  | 0.12(1.01)   | 3.1      | 5   | 68.5%|
| 250            | 0.439590(61) | -0.014(102) | -0.72(1.52)  | 2.5      | 4   | 63.6%|
| 300            | 0.439583(65) | 0.004(118)  | -1.02(1.79)  | 2.4      | 3   | 48.5%|
| 400            | 0.439558(89) | 0.073(207)  | -2.27(3.55)  | 2.3      | 2   | 31.9%|
| 500            | 0.439528(106)| 0.174(284)  | -4.35(5.35)  | 2.0      | 1   | 15.6%|

Table 20: Fit $B_N = a_B + b_B N^{-1} + c_B N^{-3/2}$. DF is the number of degrees of freedom and CL is the confidence level of the fit.

| $N_{\text{min}}$ | $a_F$         | $b_F$         | $c_F$         | $\chi^2$ | DF  | CL   |
|----------------|--------------|--------------|--------------|----------|-----|------|
| 60             | 0.000043(59) | -0.039(39)  | 1.30(33)     | 11.5     | 12  | 49.0%|
| 80             | 0.000105(65) | -0.112(52)  | 2.14(51)     | 6.7      | 11  | 81.9%|
| 100            | 0.000117(71) | -0.129(65)  | 2.36(71)     | 6.5      | 10  | 76.8%|
| 120            | 0.000087(77) | -0.082(79)  | 1.72(94)     | 5.4      | 9   | 79.4%|
| 140            | 0.000041(82) | -0.004(93)  | 0.64(1.17)   | 3.0      | 8   | 93.5%|
| 150            | 0.000026(86) | 0.023(104)  | 0.24(1.34)   | 2.6      | 7   | 91.7%|
| 180            | 0.000037(90) | 0.002(116)  | 0.54(1.54)   | 2.5      | 6   | 87.2%|
| 200            | 0.000064(98) | -0.052(139) | 1.38(1.94)   | 2.0      | 5   | 85.3%|
| 250            | 0.000110(116)| -0.153(195) | 2.95(2.89)   | 1.4      | 4   | 83.9%|
| 300            | 0.000133(124)| -0.205(220) | 3.80(3.32)   | 1.2      | 3   | 76.2%|
| 400            | 0.000167(174)| -0.298(398) | 5.45(6.75)   | 1.1      | 2   | 58.1%|
| 500            | 0.000189(206)| -0.372(545) | 6.98(10.23)  | 1.0      | 1   | 30.7%|

Table 21: Fit $F_N = a_F + b_F N^{-1} + c_F N^{-3/2}$. DF is the number of degrees of freedom and CL is the confidence level of the fit.
Table 22: Fit $F_{N}(R^2_{\epsilon}) = f$. DF is the number of degrees of freedom and CL is the confidence level of the fit.