Sparse control for continuous-time systems

Masaaki Nagahara

Institute of Environmental Science and Technology, The University of Kitakyushu, Fukuoka, Japan

Correspondence
Masaaki Nagahara, The University of Kitakyushu, Fukuoka 808-0135, Japan.
Email: nagahara@ieee.org

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Abstract
In this survey article, we give a comprehensive review of sparse control for continuous-time systems, called maximum hands-off control. The maximum hands-off control is the $L^0$ optimal control, for which we introduce fundamental properties such as necessary conditions, existence, and equivalence to the $L^1$ optimal control. We also show an efficient numerical computation algorithm for the maximum hands-off control based on the time discretization and ADMM (alternating direction method of multipliers). A numerical example is shown with an available MATLAB program.

KEYWORDS
compressed sensing, convex optimization, maximum hands-off control, optimal control, sparse control

1 | INTRODUCTION

The method of sparsity becomes more and more important in science and technology, such as signal processing, machine learning, and statistics. The method is also known as compressed sensing, compressive sampling, sparse representation, or sparse modeling. A number of text books on this topic have been published, for example, by Mallat, Elad, Eldar and Kutyniok, Starck et al., Temlyakov, Unser and Tafti, and Vidyasagar.

The idea of the method of sparsity is to reduce the number of parameters in estimation or approximation by minimizing the $\ell^0$ norm (i.e., the number of nonzero elements) of the parameter vector. However, the $\ell^0$ optimization is very hard to solve for large scale problems, since it is NP hard. Instead, the $\ell^1$ norm has been widely used to obtain a sparse solution. This method is called LASSO (least absolute shrinkage and selection operator) in statistics, and basis pursuit in signal processing. The method became very popular in particular after the pioneering works of compressed sensing by Donoho and Candes and Tao, which mathematically proved the validity of the use of the $\ell^1$ norm as an approximation of the $\ell^0$ norm in the optimization.

The method of sparsity has also been applied to networked control systems. In a networked control system, sensor data from the controlled object is sent to the controller through a rate-limited, unreliable, and noisy communication network. For networked control systems, sparsity methods play an important role to realize resource-aware control that can significantly reduce the communication and computational burden. In recent papers, sparse control is proposed by using the $\ell^1$ norm minimization to reduce the size of control packets that are sent through rate-limited communication networks. Minimum actuator placement is also an important method of sparsity for resource-aware control by minimizing the number of actuators (or control inputs) that achieve a control objective (e.g., controllability). For state feedback control, the control gain matrix is also sparsified. The obtained feedback controller is sparsely structured and the design should achieve an optimal tradeoff between closed-loop performance and sparsity. See a review paper by Jovanović and Dhingra for detailed discussion on this topic.
Yet another important application of the sparsity method to control theory is sparse control for continuous-time systems. In the pioneering papers, the $L^0$ norm optimization for continuous-time systems has been proposed to obtain maximum hands-off control, the sparsest control that has the minimum length of time on which the control value is nonzero to achieve a control objective with constraints. In other words, hands-off control has a significant time duration over which the control is exactly zero. Such control is also called gliding or coasting, which is actually used in practical control systems. For example, a stop-start system in automobiles is hands-off control, which automatically shuts down the engine to avoid it idling for a long duration of time. In a hybrid vehicle, the internal combustion engine is stopped when the vehicle is at a stop or a low speed, and the electric motor is alternatively used. In these systems, CO or CO2 emissions as well as fuel consumption are effectively reduced. Other examples of hands-off control are found in railway vehicles and free-flying robots. Hands-off control is also desirable for networking and embedded systems since the communication can be stopped during a period of zero-valued control. This property is advantageous in particular for wireless communications. By these properties hands-off control is also known as green control.

An important mathematical property of the maximum hands-off control (or the $L^0$ optimal control) is that it is equivalent to the $L^1$ optimal control, also known as minimum fuel control, under some assumptions. The $L^1$ optimal control is much easier to solve than the $L^0$ optimal control, and hence such equivalence is useful for designing maximum hands-off control. Other mathematical properties of the maximum hands-off control have been investigated for the value function, necessary conditions, and existence. The maximum hands-off control has also been extended to time-optimal control, control in the presence of denial-of-service attacks, distributed control, infinite-dimensional systems, optimal multiplexing, stochastic control and time-space sparse control. Efficient algorithms to numerically obtain the maximum hands-off control have been proposed in recent papers. Discrete-time hands-off control is also important in digital control systems, on which a recent line of research has focused. Finally, practical applications have been proposed for electrically tunable lens, spacecraft maneuvering, and thermally activated building systems, to name a few.

This article summarizes the recent researches of maximum hands-off control for continuous-time systems. In particular, we give mathematical formulation of $L^0$ optimal control and its mathematical properties. An efficient numerical computation method is also shown to obtain maximum hands-off control for linear systems, based on time discretization and ADMM (alternating direction method of multipliers) algorithm. This algorithm is implemented using MATLAB, and the code is available (see Section A).

The remainder of the article is structured as follows. Section 2 reviews the control problem of state transfer, which we discuss in this article for maximum hands-off control, and discuss the feasibility of the problem. Section 3 gives mathematical definition of the $\ell^0$ norm for finite-dimensional vectors, and the $L^0$ norm for continuous-time signals (or functions), and show their basic properties. Section 4 is the main section of this article, where we formulate the maximum hands-off control as the $L^0$ optimal control, and give its fundamental properties of necessary conditions, existence, and equivalence to the $L^1$ optimal control. Section 5 shows an efficient algorithm to obtain the maximum hands-off control, with which we show a numerical example of maximum hands-off control in Section 6. In Section 7, we will make conclusions.

## 2 STATE TRANSFER PROBLEM AND CONTROLLABILITY

Before discussing sparse control, we here consider the problem of state transfer for linear time-invariant systems. The plant model is assumed to be given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \quad x(0) = \xi,$$

where $x(t) \in \mathbb{R}^d$ is the state, $u(t) \in \mathbb{R}$ is the control, and $A \in \mathbb{R}^{d \times d}$ and $B \in \mathbb{R}^{d \times 1}$ are state-space matrices. For this linear system, we consider the problem of state transfer described as follows.

**Problem 1** (State transfer). Fix terminal time $T > 0$. Find a control $\{u(t) : t \in [0, T]\}$ that satisfies the control magnitude constraint

$$\|u\|_\infty \triangleq \text{ess sup}_{t \in [0,T]} |u(t)| \leq 1. \quad (2)$$

*The essential supremum is defined with the Lebesgue measure $\mu$ on $\mathbb{R}$.}
and steers the state $x(t)$ in (1) from $x(0) = \xi$ to

$$x(T) = 0.$$ \hfill (3)

The magnitude constraint (2) is natural for real systems where the actuator power is limited. Note that the constraint (2) is normalized. If one wants to consider a constraint as $\|u\|_{\infty} \leq u_{\text{max}}$, then taking $B$ and $u(t)$ respectively as $Bu_{\text{max}}$ and $u(t)/u_{\text{max}}$ results in the normalized constraint (2). We call $u(t)$ that satisfies (2) an admissible control. The terminal state condition (3) is also generalized to $x(T) = \eta \in \mathbb{R}^d$, for which we redefine the state as $x(t) - \eta$ to shift the terminal state to the origin.

We first consider the feasibility of the state transfer problem. For this, we define the $T$-controllable set.\textsuperscript{74,75}

**Definition 1** (T-controllable set). Let $T > 0$. The set of initial states that can be transferred to the origin by some admissible control $\{u(t) : t \in [0, T], \|u\|_{\infty} \leq 1\}$ is called the $T$-controllable set, and denoted by $R(T)$.

For the plant (1), the $T$-controllable set $R(T)$ is given by

$$R(T) = \left\{ -\int_0^T e^{-AT} Bu(t) dt : \|u\|_{\infty} \leq 1 \right\}. \hfill (4)$$

This is easily shown using the solution formula of the linear differential Equation 1, that is,

$$x(t_1) = e^{A(t_1-t_0)}x(t_0) + \int_{t_0}^{t_1} e^{A(t_1-t)}Bu(t) dt, \quad 0 \leq t_0 \leq t_1,$$ \hfill (5)

with $t_1 = T$ and $t_0 = 0$. The $T$-controllable set has nice properties as described in the following theorem[76, Theorem 8.1]:

**Theorem 1.** Let $T > 0$. Then the $T$-controllable set $R(T)$ is a bounded, closed, and convex set. Also, if $T_1 < T_2$ then $R(T_1) \subset R(T_2)$.

Fix $T > 0$. From the definition of $R(T)$, if initial state $\xi$ is in the $T$-controllable set $R(T)$, then the state can be transferred to the origin by an admissible control. We call this control a feasible control for the state transfer problem. We denote by $U(T, \xi)$ the set of all feasible controls that are admissible (i.e., $\|u\|_{\infty} \leq 1$) and transfer the initial state $\xi \in \mathbb{R}^d$ to the origin within time $T$. The feasible set is denoted by

$$U(T, \xi) = \left\{ u : \|u\|_{\infty} \leq 1, \quad \xi = -\int_0^T e^{-AT} Bu(t) dt \right\}. \hfill (6)$$

This is also easily shown by using (5). The relation between $R(T)$ and $U(T, \xi)$ is given in the following theorem.

**Theorem 2.** Suppose that terminal time $T > 0$ and initial state $\xi \in \mathbb{R}^d$ are given. Then, $\xi \in R(T)$ if and only if there exists $u \in U(T, \xi)$.

The proof is easily done using the representations (4) and (6) respectively of $R(T)$ and $U(T, \xi)$.

From Theorem 2, if $\xi \not\in R(T)$, then there is no admissible control, and hence $U(T, \xi)$ is empty. In this case, if we take sufficiently large $\tilde{T} > T$, then $U(\tilde{T}, \xi)$ may be nonempty. An interesting question is to find the minimum value of $T$ such that there exists $u \in U(T, \xi)$ for given initial state $\xi \in \mathbb{R}^d$. This is called the minimum time, which is mathematically defined as

$$T^*(\xi) \triangleq \inf \{ T \geq 0 : \exists u, u \in U(T, \xi) \}. \hfill (7)$$

The following theorem is important to guarantee the existence of feasible controls.

**Theorem 3.** Suppose that $T^*(\xi) < \infty$. Then, there exists a minimum-time control $u^* \in U(T^*(\xi), \xi)$. Moreover, for any $T > T^*(\xi)$, the set $U(T, \xi)$ is nonempty.

The proof can be found in Section 6–8 of the book by Athans and Falb,\textsuperscript{47} or Section III.19 of Pontryagin’s book.\textsuperscript{77} The condition $T^*(\xi) < \infty$ is also characterized by the controllable set $R$ defined by
\[ R \triangleq \bigcup_{T > 0} R(T). \]  

**Theorem 4.** Let \( \xi \in \mathbb{R}^d \). Then, \( T^*(\xi) < \infty \) if and only if \( \xi \in R \).

This theorem can be directly proved by Theorem 3 and definition (8). We should note that the controllable set \( R \) may not be equivalent to \( \mathbb{R}^d \). That is, \( R \) may be a strict subset of \( \mathbb{R}^d \) and there may exist \( \xi_0 \in \mathbb{R}^d \) such that \( \xi_0 \notin R \). In this case, there is no admissible control that achieves \( x(T) = 0 \) from \( \xi_0 \) in finite time \( T \), for which we write \( T^*(\xi_0) = \infty \). A sufficient condition for \( R \) to be \( \mathbb{R}^d \) is given in the following theorem [76, Theorem 17.6]:

**Theorem 5.** Suppose that \((A, B)\) is controllable and the eigenvalues of \( A \) are all in the left half plane \( \{z \in \mathbb{C} : \text{Re } z \leq 0\} \) (i.e., \( A \) is stable). Then the controllable set \( R \) is \( \mathbb{R}^d \), and the minimum time \( T^*(\xi) \) is finite for any \( \xi \in \mathbb{R}^d \).

From now on, we assume that the initial state \( \xi \) is in \( R \) (or equivalently \( T^*(\xi) < \infty \)), and \( T > T^*(\xi) \), with which the feasible set \( U(T, \xi) \) is nonempty.

**Assumption 1.** The initial state \( \xi \) satisfies that \( \xi \in R \) (or equivalently \( T^*(\xi) < \infty \)), and the terminal time \( T > T^*(\xi) \).

### 3 \ L^0 Norm for Continuous-Time Signals

Before defining the \( L^0 \) norm for continuous-time signals, we recall the definition of the \( \ell^0 \) norm for finite-dimensional vectors. For a vector \( v \in \mathbb{R}^n \), the \( \ell^0 \) norm is defined as

\[ ||v||_{\ell^0} \triangleq \#(\text{supp}(v)), \]  

where \( \text{supp}(v) \) is the support set of vector \( v \), which is defined by

\[ \text{supp}(v) \triangleq \{ i \in \{1, \ldots , n\} : v_i \neq 0 \}, \]  

and \( \#(\cdot) \) is the number of elements in the argument set. It is easily understood from (9) that the \( \ell^0 \) norm \( ||v||_{\ell^0} \) is the number of nonzero elements in \( v \). If \( ||v||_{\ell^0} \) is sufficiently smaller than \( n \), then the vector is said to be sparse. Note that the \( \ell^0 \) norm is not a proper norm since it does not satisfy the property of absolute homogeneity. To see this, let us consider nonzero vectors \( v \in \mathbb{R}^n \) and \( 2v \). Then, we have

\[ ||2v||_{\ell^0} = ||v||_{\ell^0} \neq 2||v||_{\ell^0}. \]

We also note that the triangle inequality

\[ ||v + w||_{\ell^0} \leq ||v||_{\ell^0} + ||w||_{\ell^0} \]  

holds for any \( v, w \in \mathbb{R}^n \). Also, \( ||v||_{\ell^0} = 0 \) if and only if \( v = 0 \). Since the \( \ell^0 \) norm is discontinuous and nonconvex, the \( \ell^1 \) norm is often used as a convex surrogate of the \( \ell^0 \) norm in optimization:

\[ ||v||_{\ell^1} \triangleq \sum_{i=1}^{n} |v_i|. \]

Now, we define the \( L^0 \) norm for continuous-time signals, more precisely Lebesgue measurable functions over interval \([0, T] \subset \mathbb{R}\) with fixed \( T > 0 \). The \( L^0 \) norm of a measurable function \( u \) is defined by

\[ ||u||_0 \triangleq \mu (\text{supp}(u)), \]  

where \( \mu \) is the Lebesgue measure, and \( \text{supp}(u) \) is the support set of function \( u \), that is,

\[ \text{supp}(u) \triangleq \{ t \in [0, T] : u(t) \neq 0 \}. \]
The $L^0$ norm of continuous-time signal $u(t)$ is the time length over which the signal takes nonzero values. If $\|u\|_0$ is much smaller than $T$, then this signal is said to be sparse. The $L^0$ norm is also represented by

$$\|u\|_0 = \int_0^T |u(t)|^0 dt,$$  

(16)

where the zero exponent is defined as

$$z^0 \triangleq \begin{cases} 1, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases}$$  

(17)

The $L^0$ norm is discontinuous and nonconvex, as the $\ell^0$ norm, and the following $L^1$ norm is used as a convex surrogate:

$$\|u\|_1 \triangleq \int_0^T |u(t)| dt,$$  

(18)

for $u \in L^1[0, T]$, the linear space of functions over $[0, T]$ whose $L^1$ norms are finite. The relation between the $L^0$ norm and the $L^1$ norm is understood comparing Equations (16) and (18). In fact, we have the following theorem[78, Section 1.3]:

**Theorem 6.** Suppose $u \in L^1[0, T]$ with fixed $T > 0$. Then, for any $p \in (0, 1)$, the $L^p$ norm

$$\|u\|_p \triangleq \left\{ \int_0^T |u(t)|^p dt \right\}^{1/p}$$  

(19)

is finite, and

$$\lim_{p \to 0^+} \|u\|_p^p = \lim_{p \to 0^+} \int_0^T |u(t)|^p dt = \|u\|_0$$  

(20)

holds.

## 4 | Maximum Hands-Off Control

Under Assumption 1, the feasible set $U(T, \xi)$ is nonempty, and we have at least one feasible control. Moreover, if $T$ is sufficiently larger than $T^* (\xi)$, then there may be infinitely many feasible controls. The maximum hands-off control is the minimum $L^0$ norm control among the feasible solutions. More precisely, the maximum hands-off control is formulated as

$$\begin{align*}
\text{minimize} \quad & \|u\|_0 \\
\text{subject to} \quad & \dot{x}(t) = Ax(t) + Bu(t), \forall t \in [0, T], \\
& x(0) = \xi, x(T) = 0, \\
& \|u\|_\infty \leq 1,
\end{align*}$$  

(21)

or equivalently,

$$\begin{align*}
\text{minimize} \quad & \|u\|_0 \quad \text{subject to} \quad u \in U(T, \xi).
\end{align*}$$  

(22)

In this section, we review fundamental properties of maximum hands-off control.

### 4.1 Necessary conditions

Here we derive necessary conditions of the optimal control in (21) or (22) via the nonsmooth maximum principle.\textsuperscript{49,79}
Theorem 7. Associated to every solution \([0, T] \ni t \mapsto (x^*(t), u^*(t))\) to the maximum hands-off control problem \((21)\), there exist a number \(\eta \in \{0, 1\}\) and an absolutely continuous curve \(\{p^*(t) : t \in [0, T]\} \subset \mathbb{R}^d\) such that for almost all \(t \in [0, T]\):

\[
\dot{x}^*(t) = Ax^*(t) + Bu^*(t),
\]

\[
p^*(t) = -A^T p^*(t),
\]

and if \(\eta = 1\), the optimal control \(u^*\) satisfies

\[
u^*(t) = \begin{cases} 
1, & \text{if } p^*(0)^T e^{A(T-t)} B > 1, \\
0, & \text{if } -1 < p^*(0)^T e^{A(T-t)} B < 1, \\
-1, & \text{if } p^*(0)^T e^{A(T-t)} B < -1, 
\end{cases}
\]

\[
u^*(t) \in \{1, 0\}, \quad \text{if } p^*(0)^T e^{A(T-t)} B = 1,
\]

\[
u^*(t) \in \{0, -1\}, \quad \text{if } p^*(0)^T e^{A(T-t)} B = -1,
\]

otherwise (i.e., \(\eta = 0\)), then \(\|p^*\|_\infty > 0\) and the optimal control satisfies

\[
u^*(t) = \begin{cases} 
1, & \text{if } p^*(0)^T e^{A(T-t)} B > 0, \\
-1, & \text{if } p^*(0)^T e^{A(T-t)} B < 0, 
\end{cases}
\]

\[
u^*(t) \in \{1, -1\}, \quad \text{if } p^*(0)^T e^{A(T-t)} B = 0.
\]

The optimal control when \(\eta = 0\) is called abnormal,\(^79\) and it is equivalent to the minimum-time control from \(\xi\) to the origin. In this case, the solution is no sparse, and actually the optimal control does not depend on the cost function. We do not consider this case, since this does not lead to any sparse control. A sufficient condition for the optimal control to be normal (i.e., \(\eta = 1\)) is given in the following theorem.

Theorem 8. Suppose that Assumption 1 holds, and the pair \((A, B)\) is controllable. Then the solution of the maximum hands-off control \((21)\) cannot be abnormal.

Proof. Assume \(\eta = 0\). From Theorem 7, we have \(\|p^*\|_\infty > 0\), or \(p^*\) is not identically zero. Also from \((23)\), \(p^*\) is given by

\[
p^*(t) = e^{-A^T} p^*(0),
\]

and hence \(p^*(0) \neq 0\). Since \((A, B)\) is controllable, we have

\[
\mu \{ t \in [0, T] : p^*(0)^T e^{A(T-t)} B = 0 \} = 0.
\]

It follows that the maximum hands-off control \(u^*(t)\) takes \(\pm 1\) for almost all \(t \in [0, T]\). Therefore, we have

\[
\|u^*\|_0 = \int_0^T |u(t)| dt = T.
\]

On the other hand, if \((A, B)\) is controllable, the minimum-time control from \(\xi\) to the origin is (if it exists) unique and takes \(\pm 1\) for almost all \(t \in [0, T^*(\xi)]\) (i.e., bang-bang control)\(^76\), Section II.16. Indeed, from Assumption 1, there exists the minimum-time control \(u^*\). Then we define the following control:

\[
\bar{u}(t) = \begin{cases} 
\nu^*(t), & \text{if } 0 \leq t \leq T^*(\xi), \\
0, & \text{if } T^*(\xi) < t \leq T.
\end{cases}
\]

It is easily shown that \(\bar{u}\) is a feasible control, that is \(\bar{u} \in U(T, \xi)\). Also, with this \(\bar{u}\), we have

\[
\|\bar{u}\|_0 = \int_0^T |\bar{u}(t)| dt = \int_0^{T^*(\xi)} |u^*(t)| dt = T^*(\xi) < T = \|u^*\|_0.
\]
It follows that the control $u^*(t)$ can never be $L^0$ optimal, and hence $\eta \neq 0$ and the optimal control cannot be abnormal. □

**Remark 1.** In the classical book by Athans and Falb⁴⁷ and the original paper of maximum hands-off control, ³⁶ the normality is differently defined for the system (1) such that the pair $(A, B)$ is controllable and $A$ is nonsingular. To avoid confusion, we call such a system *nonsingular*, as proposed in a recent book.⁷⁵

From Theorems 7 and 8, the maximum hands-off control (or the $L^0$ optimal control) always takes only three values of $±1$ or $0$ if $(A, B)$ is controllable. Such control is called *bang-off-bang control*. Figure 1 shows an example of bang-off-bang control. This property is important in proving the equivalence between the $L^0$ and $L^1$ optimal controls (see Section 4.3).

### 4.2 | Existence

To consider the existence of maximum hands-off control, a solution of the $L^0$ optimal control problem (21), we introduce the $L^p$ optimal control with $p \in (0, 1)$:

$$\min_u \|u\|_p \quad \text{subject to } u \in U'(T, \xi).$$  \hfill (31)

It is known that the $L^p$-optimal solution with $p \in (0, 1)$ is bang-off-bang.⁸⁰ Also, it is easily shown that the $L^p$-optimal control is always equivalent to the $L^0$-optimal control (21) as shown in the following theorem:⁸⁰

**Theorem 9.** Suppose that an $L^p$-optimal control with $p \in (0, 1)$ exists. Then, the set of $L^0$ solutions of (21) is equivalent to the set of $L^p$ solutions of (31).

We now prove the existence of $L^p$-optimal control with $p > 0$.

**Theorem 10.** Suppose Assumption 1 holds. Then there exists an $L^p$-optimal control with $p > 0$.

**Proof.** From Assumption 1, there exists $u \in U'(T, \xi)$. Then we define

$$J^*_p \triangleq \inf \left\{ \|u\|_p^p : u \in U'(T, \xi) \right\}. \hfill (32)$$

Since $u \in U'(T, \xi)$ satisfies $\|u\|_\infty \leq 1$, we have $J^*_p < \infty$. Then, from the definition of $J^*_p$, there exists a sequence $(u_l)_{l \in \mathbb{N}} \subset U'(T, \xi)$ such that $\|u_l\|_p \to J^*_p$ as $l \to \infty$. Now, since $u_l \in U'(T, \xi)$, we have $\|u_l\|_\infty \leq 1$, and hence $(u_l)_{l \in \mathbb{N}} \subset B_{\infty} \triangleq \{ u \in L^\infty[0, T] : \|u\|_\infty \leq 1 \}$. It is known that the unit ball $B_{\infty}$ is sequentially compact in the weak* topology of $L^\infty[0, T]$ [⁸¹, Theorem A.9]. That is, there exists a subsequence $(u_{l_k})_{k \in S}, S \subset \mathbb{N}$, such that there exists $u_\infty \in B_{\infty}$ and

$$\lim_{l \to \infty} \frac{1}{T} \int_0^T f(t)(u(t) - u_\infty(t)) \, dt = 0, \hfill (33)$$

---

**FIGURE 1** Bang-off-bang control that takes $±1$ and $0$.
for any \( f \in L^1[0, T] \). Now, since \( u_f \in U(T, \xi) \), we have

\[
\xi = - \int_0^T e^{-At}Bu_f(t)dt
\]  

(34)

for all \( l' \in S \). On the other hand, from (33) with \( f(t) = e^{-At}B \), we have

\[
\lim_{T \to \infty} \int_0^T e^{-At}Bu_f(t)dt = \int_0^T e^{-At}Bu_\infty(t)dt.
\]  

(35)

From (34), it follows that

\[
\xi = - \int_0^T e^{-At}Bu_\infty(t)dt.
\]  

(36)

Also since \( u_\infty \in B_\infty \), we have \( \|u_\infty\|_\infty \leq 1 \). Therefore, \( u_\infty \in U(T, \xi) \).

Next, from (33) with \( f(t) = \text{sgn}(u_f(t) - u_\infty(t)) \), where \( \text{sgn} \) is the sign function defined by

\[
\text{sgn}(v) = \begin{cases} 
-1, & \text{if } v < 0, \\
0, & \text{if } v = 0, \\
1, & \text{if } v > 0,
\end{cases}
\]  

(37)

we have

\[
\lim_{l' \to \infty} \int_0^T |u_f(t) - u_\infty(t)|dt = 0.
\]  

(38)

Then, the following inequalities hold

\[
\|u_f\|_p^p - \|u_\infty\|_p^p \leq \int_0^T |u_f(t)|^p - |u_\infty(t)|^p dt \leq \int_0^T |u_f(t) - u_\infty(t)|^p dt \leq \left( \int_0^T |u_f(t) - u_\infty(t)|dt \right)^p T^{1-p},
\]  

(39)

where the last inequality is from Hölder’s inequality. It follows from (38) and (39) that

\[
\lim_{l' \to \infty} \|u_f\|_p^p = \|u_\infty\|_p^p.
\]  

(40)

The left-hand side of the above equation is equivalent to \( J_p^* \) by definition, and hence \( \|u_\infty\|_p^p = J_p^* \). That is, \( u_\infty \in U(T, \xi) \) is an \( L^p \)-optimal control.

From Theorems 9 and 10, we have the existence theorem for the \( L^0 \)-optimal control of (21):

**Theorem 11.** Suppose Assumption 1 holds. Then, there exists an \( L^0 \) solution of (21).

### 4.3 \( L^1 \) approximation and equivalence

Since the maximum hands-off control problem is highly nonconvex due to the \( L^0 \) norm, it is not easily solved. In this section, we approximate the maximum hands-off control problem by using the \( L^1 \) norm (18). Namely, we consider the following problem called the \( L^1 \) optimal control, also known as minimum fuel control:

\[
\begin{align*}
\text{minimize} & \quad \|u\|_1 \\
\text{subject to} & \quad \dot{x}(t) = Ax(t) + Bu(t), \forall t \in [0, T], \\
& \quad x(0) = \xi, x(T) = 0, \\
& \quad \|u\|_\infty \leq 1.
\end{align*}
\]  

(41)
Note that if Assumption 1 holds, then there exists an \( L^1 \) optimal control from Theorem 10. The most important merit of the \( L^1 \) approximation is that the \( L^1 \) problem (41) is convex and easily solved, by using numerical computation shown in Section 5.

An important question is when the \( L^1 \) optimal control is equivalent to the maximum hands-off control (i.e., the \( L^0 \) optimal control). In this section, we show their equivalence.

First, the following theorem is a fundamental theorem for the equivalence.

**Theorem 12.** Suppose that there exists an \( L^1 \)-optimal control that takes \( \pm 1 \) or 0 for almost all \( t \in [0, T] \) (i.e., bang-off-bang control). Then it is also \( L^0 \) optimal.

**Proof.** Define \( J_0(u) \triangleq \| u \|_0 \) and \( J_1(u) \triangleq \| u \|_1 \). From the assumption, there exists an \( L^1 \)-optimal control \( u^*_1 \) that is bang-off-bang. Since \( u^*_1 \) is a feasible control for (41), the set of feasible controls \( \mathcal{U}(T, \xi) \) is nonempty. Then, for any \( u \in \mathcal{U}(T, \xi) \) we have

\[
J_1(u) = \int_0^T |u(t)| \, dt = \int_{\text{supp}(u)} |u(t)| \, dt \leq \int_{\text{supp}(u)} 1 \, dt = J_0(u). \tag{42}
\]

Since \( u^*_1 \) is bang-off-bang, we have

\[
J_1(u^*_1) = \int_0^T |u^*_1(t)| \, dt = \int_{\text{supp}(u^*_1)} 1 \, dt = J_0(u^*_1). \tag{43}
\]

From (42) and (43), we have \( J_0(u^*_1) = J_1(u^*_1) \leq J_1(u) \leq J_0(u) \) for any \( u \in \mathcal{U}(T, \xi) \), and hence \( u^*_1 \) minimizes \( J_0(u) \). That is, \( u^*_1 \) is also \( L^0 \) optimal.

The following theorem gives a sufficient condition for the \( L^1 \) optimal control to be bang-off-bang [47, Theorems 6–13].

**Theorem 13.** Suppose that \( (A, B) \) is controllable, and \( A \) is nonsingular. Then the \( L^1 \) optimal control is (if it exists) bang-off-bang.

From this theorem, if \( (A, B) \) is controllable and \( A \) is nonsingular, then the \( L^1 \)-optimal control is bang-off-bang, that is, the optimal control \( u^*(t) \) takes values 0 or \( \pm 1 \) for almost all \( t \in [0, T] \). From this property, we obtain the following theorem.

**Theorem 14.** Suppose that Assumption 1 holds. Suppose also that \( (A, B) \) is controllable and \( A \) is nonsingular. Then the set of \( L^0 \)-optimal controls is equivalent to the set of \( L^1 \)-optimal controls.

**Proof.** Let \( U^*_0 \) and \( U^*_1 \) be the sets of \( L^0 \) and \( L^1 \) optimal controls, respectively. From Assumption 1 and Theorem 10, \( U^*_1 \) is nonempty. Take \( u^*_1 \in U^*_1 \) arbitrarily. Then, from Theorem 13, \( u^*_1 \) is bang-off-bang. It follows from Theorem 12 that \( u^*_1 \in U^*_0 \), and hence \( U^*_1 \subseteq U^*_0 \).

Then we prove \( U^*_0 \subseteq U^*_1 \). Take \( u^*_0 \in U^*_0 \subseteq \mathcal{U}(T, \xi) \) arbitrarily. Take also \( u^*_1 \in U^*_1 \subseteq \mathcal{U}(T, \xi) \) independently. From (43) and the \( L^1 \) optimality of \( u^*_1 \), we have

\[
J_0(u^*_1) = J_1(u^*_1) \leq J_1(u^*_0). \tag{44}
\]

On the other hand, from (42) and the \( L^0 \) optimality of \( u^*_0 \), we have

\[
J_1(u^*_0) \leq J_0(u^*_0) \leq J_0(u^*_1). \tag{45}
\]

From (44) and (45), we have

\[
J_0(u^*_1) = J_1(u^*_1) \leq J_1(u^*_0) \leq J_0(u^*_0) \leq J_0(u^*_1). \tag{46}
\]

It follows that \( J_1(u^*_1) = J_1(u^*_0) \), and \( u^*_0 \) minimizes \( J_1(u) \). That is, we have \( u^*_0 \in U^*_1 \) and hence \( U^*_0 \subseteq U^*_1 \).
5 | NUMERICAL COMPUTATION

In this section, we show an efficient numerical computation algorithm that solves the $L^1$ optimal control problem (41).

5.1 | Time discretization and discrete-time problem

Fix $T > 0$. We first discretize the time interval $[0, T]$ into $n$ subintervals as

$$[0, T] = [0, h) \cup [h, 2h) \cup \cdots \cup [nh - h, nh],$$

(47)

where $h > 0$ is the sampling time, and $n \in \mathbb{N}$ is the number of subintervals such that $T = nh$.

We assume that the control is digital such that the signal $u(t)$ is constant on each subinterval. More precisely, we assume the control is the output of the zero-order hold as

$$u(t) = u(kh) = u_d[k], \quad t \in [kh, (k+1)h), \quad k = 0, 1, 2, \ldots, n - 1,$$

(48)

for a discrete-time signal $\{u_d[0], u_d[1], \ldots, u_d[n-1]\}$ (see Figure 2). The zero-order hold assumption is actually reasonable for networked control systems where control values $u_d[k], k = 0, 1, 2, \ldots, n - 1$, are computed in a digital computer, transmitted through a wireless communication network, and applied to an actuator through a D/A converter (zero-order hold).

Let us compute the state transition in (1) under the zero-order assumption on the control. From the solution formula in (5) with

$$t_0 = kh, \quad t_1 = kh + h, \quad k = 0, 1, 2, \ldots, n - 1,$$

(49)

we have

$$x(kh + h) = e^{Ah}x(kh) + \int_{kh}^{kh+h} e^{A(kh+h-t)}Bu(t)dt = e^{Ah}x(kh) + \int_{0}^{h} e^{A(h-t)}Bu(t+kh)dt.$$

(50)

Then we define

$$x_d[k] \triangleq x(kh), \quad u_d[k] \triangleq u(kh), \quad k = 0, 1, \ldots, n - 1, \quad x_d[n] \triangleq x(T).$$

(51)

From the zero-order-hold assumption (48), the control $u(t)$ takes a constant value $u_d[k] = u(kh)$ on the subinterval $[kh, kh + h), k = 0, 1, 2, \ldots, n - 1$, as shown in Figure 2. Then from (50) we have

$$x_d[k + 1] = e^{Ah}x_d[k] + \left( \int_{0}^{h} e^{A(h-t)}Bu(t+kh)dt \right) u_d[k].$$

(52)

It follows that the state-space Equation 1 is transformed into a discrete-time system described as

$$x_d[k + 1] = A_dx_d[k] + B_du_d[k], \quad k = 0, 1, \ldots, n - 1,$$

(53)

where

$$A_d \triangleq e^{Ah}, \quad B_d \triangleq \int_{0}^{h} e^{A(t)}B dt.$$

(54)

We note that the discrete-time system in (53) is linear and time-invariant.
Next, we define the control vector as

$$u_d = \begin{bmatrix} u_d[0] \\ u_d[1] \\ \vdots \\ u_d[n-1] \end{bmatrix} \in \mathbb{R}^n. \quad (55)$$

The terminal state $x(T)$ is then represented as

$$x(T) = x_d[n] = -\zeta + \Phi u_d, \quad (56)$$

where

$$\Phi = \begin{bmatrix} A_d^{n-1}B_d & A_d^{n-2}B_d & \cdots & B_d \end{bmatrix} \in \mathbb{R}^{d \times n}. \quad (57)$$

The discrete-time system (53) is called the zero-order-hold discretization, also known as step-invariant discretization\(^ {82} \) of the continuous-time system (1). See Figure 3 for the block diagram of the zero-order-hold discretization. In this block diagram, $H_h$ is the zero-order hold with sampling time $h > 0$, which transforms a discrete-time signal $u_d[k], k = 0, 1, 2, \ldots$ into a piecewise constant signal $u(t)$ defined in (48). Also, $S_h$ is the ideal sampler that transforms the continuous-time signal $x(t)$ into sampled data $x_d[k] = x(kh), k = 0, 1, 2, \ldots$.

5.2 Reduction to Finite-dimensional Optimization

By the time discretization, we can reduce the $L^1$ optimal control problem (41) into a finite-dimensional $\ell^1$ optimization problem.

First, the constraint (2) on the control $u$ is equivalently represented by

$$\|u_d\|_{\ell^1} = \max_{k=1,2,\ldots,n} |u_d[k]| \leq 1, \quad (58)$$
with the zero-order-hold control assumption (48).

Next, under the zero-order-hold assumption, the $L^1$ cost function becomes

$$
||u||_1 = \sum_{k=0}^{n-1} \int_{kh}^{(k+1)h} |u(t)| dt = \sum_{k=0}^{n-1} |u_d[k]| h = \sum_{k=0}^{n-1} |u_d[k]| h ||u_d||_{\ell^1},
$$

(59)

where the $\ell^1$ norm is defined in (13).

Now the $L^1$ optimal control problem (41) is reduced to the following finite-dimensional $\ell^1$ optimization problem:

$$
\min_{u_d \in \mathbb{R}^n} ||u_d||_{\ell^1} \text{ subject to } \Phi u_d = \zeta, \ ||u_d||_{\ell^\infty} \leq 1.
$$

(60)

This optimization problem is a convex optimization since the cost function ($\ell^1$ norm) is a convex function, and the constraint set

$$
C \triangleq \{ u_d \in \mathbb{R}^n : \Phi u_d = \zeta, \ ||u_d||_{\ell^\infty} \leq 1 \}
$$

(61)

is a convex set in $\mathbb{R}^n$.

### 5.3 Fast $\ell^1$ optimization algorithm

The discretized problem (60) is a convex optimization problem, and hence we can use efficient algorithms such as interior point methods [83, Chapter 11], with easy numerical computation softwares such as CVX on MATLAB. However, in general, interior point methods require significant computational time if the problem size is large. This becomes a drawback in feedback control, where we should solve the problem in real time. Also, in real systems, the control algorithm should be implemented in a microcomputer, which often has just a cheap computational ability. In such a case, we need to implement a fast and simple algorithm for the specific $\ell^1$ optimization problem (60). For this purpose, we can use an efficient algorithm called ADMM (Alternating Direction Method of Multipliers) [72] to solve (60). The ADMM algorithm for (60) is given as follows:

$$
u[l + 1] = M(\nu[l] - \nu[l]),
$$

$$
\nu[l + 1] = S_\gamma(u[l + 1] + \nu_0[l]),
$$

$$
\nu[l + 1] = \nu[l] + \Psi u[l + 1] - \nu[l + 1], \quad l = 0, 1, 2, \ldots,
$$

(62)

where $M$ is a matrix given by

$$
M \triangleq (\Psi^T \Psi)^{-1} \Psi^T, \quad \Psi \triangleq \begin{bmatrix} I_n \Phi \\ I_n \end{bmatrix} \in \mathbb{R}^{(2n+d) \times n},
$$

(63)

and $v[l] = [v_0[l]^T, v_1[l]^T, v_2[l]^T]^T$ with the same partition as $\Psi$ in (63). $S_\gamma$ is the soft-thresholding function [75] defined by

$$
[S_\gamma(u)]_k \triangleq \begin{cases} 
    u_k - \gamma, & \text{if } u_k \geq \gamma, \\
    0, & \text{if } |u_k| < \gamma, \\
    u_k + \gamma, & \text{if } u_k \leq -\gamma.
\end{cases}
$$

(64)

$I_n$ is the identity matrix of size $n \times n$. 
where $[\cdot]_k$ denotes the $k$th element of the argument vector, and $u_k$ is the $k$th element of vector $u$. The function $\text{sat}$ is the saturation function defined as

$$[\text{sat}(u)]_k \triangleq \text{sgn}(u_k) \min\{|u_k|, 1\}. \quad (65)$$

Figure 4 shows the graphs of the soft-thresholding function and the saturation function.

Note that $\Psi^T \Psi = 2I_n + \Phi^T \Phi$ is nonsingular and the matrix $M$ can be computed off-line (i.e., outside the iteration). The size of $\Psi^T \Psi$ is $n \times n$, and if the number $n$ of time discretization is very large, then the computation of the inversion may take large computational time. In this case, we can adopt the matrix inversion lemma

$$(X + UYV)^{-1} = X^{-1} - X^{-1}U(Y^{-1} + VX^{-1}U)^{-1}VX^{-1}. \quad (66)$$

By this, the inverse matrix $(\Psi^T \Psi)^{-1}$ can be rewritten as

$$(\Psi^T \Psi)^{-1} = (2I_n + \Phi^T \Phi)^{-1} = \frac{1}{2}I_n - \frac{1}{2} \Phi^T (2I_d + \Phi \Phi^T)^{-1} \Phi. \quad (67)$$

This requires inversion of matrix $2I_d + \Phi \Phi^T$ of size $d \times d$, and if $d \ll n$ then the computational time can be significantly reduced.

ADMM algorithm is very fast and needs just some dozens of iterations to obtain a solution with a sufficient precision. This property is very important if you adapt the finite-horizon $L^1$ optimal control to the model predictive control, where real-time computation is essential.

### 6 NUMERICAL EXAMPLE

Here we show a numerical example of maximum hands-off control. We consider the following linear system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t), \quad x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \quad (68)$$

We assume the horizon length $T = 10$, with which we compute the maximum hands-off control to achieve $x(10) = 0$ with the magnitude constraint (2) on the control $u$. We use the ADMM algorithm (62) to numerically obtain the $L^1$ optimal control. Note that the $A$-matrix in (68) is singular, and we cannot say the $L^1$ optimal control is equivalent to the $L^0$ optimal control at this time. Figure 5 shows the obtained maximum hands-off control. We can see that the control is bang-off-bang, and from Theorem 12, this is also $L^0$ optimal. The MATLAB code to compute this numerical example is available (See Appendix A).
FIGURE 5 Maximum hands-off control for (68)

7 CONCLUSION

In this survey article, we have shown the mathematical formulation of maximum hands-off control and its fundamental properties. Also, we have shown an efficient numerical computation algorithm for the $L^1$ optimal control based on time discretization and ADMM. The research on sparse control for continuous-time systems has just started, and remains many open problems, such as efficient algorithms for the $L^0$ optimal control when it is not equivalent to the $L^1$ optimal control (e.g., the $L^1$ control is not bang-off-bang), equivalence between $L^0$ and $L^1$ optimal control problems for nonlinear system, and infinite-horizon control (or feedback control).

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CONFLICT OF INTEREST

The author declares no conflict of interest.

DATA AVAILABILITY STATEMENT

The MATLAB program that can produce the data shown in Section 6 is openly available at https://nagahara-masaaki.github.io/IJRN2021.

ORCID

Masaaki Nagahara https://orcid.org/0000-0002-8992-2495

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APPENDIX A: MATLAB PROGRAM

The MATLAB program that can produce the data shown in Section 6 is openly available at https://nagahara-masaaki.github.io/IJRN2021

Other MATLAB programs for sparse control, as well as compressed sensing are found in the author’s recent book. 75