ON LERCH’S FORMULA AND ZEROS OF THE QUADRILATERAL ZETA FUNCTION

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Abstract. Let $0 < a \leq 1/2$ and define the quadrilateral zeta function by $2Q(s, a) := \zeta(s, a) + \zeta(s, 1-a) + \text{Li}_s(e^{2\pi ia}) + \text{Li}_s(e^{2\pi(1-a)})$, where $\zeta(s, a)$ is the Hurwitz zeta function and $\text{Li}_s(e^{2\pi a})$ is the periodic zeta function.

In the present paper, we show that there exists a unique real number $a_0 \in (0, 1/2)$ such that $Q(\sigma, a_0)$ has a unique double real zero at $\sigma = 1/2$ when $\sigma \in (0, 1)$, for any $a \in (a_0, 1/2]$, the function $Q(\sigma, a)$ has no zero in the open interval $\sigma \in (0, 1)$ and for any $a \in (0, a_0)$, the function $Q(\sigma, a)$ has at least two real zeros in $\sigma \in (0, 1)$.

Moreover, we prove that $Q(s, a)$ has infinitely many complex zeros in the region of absolute convergence and the critical strip when $a \in \mathbb{Q} \cap (0, 1/2) \setminus \{1/6, 1/4, 1/3\}$. The Lerch formula, Hadamard product formula, Riemann-von Mangoldt formula for $Q(s, a)$ are also shown.

1. Introduction, Main results and Some remarks

1.1. Introduction. For $0 < a \leq 1$, we define the Hurwitz zeta function $\zeta(s, a)$ and the periodic zeta function $\text{Li}_s(e^{2\pi a})$ by

$$\zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad \text{Li}_s(e^{2\pi a}) := \sum_{n=1}^{\infty} \frac{e^{2\pi ina}}{n^s}, \quad \sigma > 1,$$

respectively. The both Dirichlet series of $\zeta(s, a)$ and $\text{Li}_s(e^{2\pi a})$ converge absolutely in the half-plane $\sigma > 1$ and uniformly in each compact subset of this region. The Hurwitz zeta function $\zeta(s, a)$ can be extended for all $s \in \mathbb{C}$ except $s = 1$, where there is a simple pole with residue 1 (see for instance [2, Section 12]). On the other hand, the Dirichlet series of the function $\text{Li}_s(e^{2\pi a})$ with $0 < a < 1$ converges uniformly in each compact subset of the half-plane $\sigma > 0$ (see for example [11, p. 20]). The function $\text{Li}_s(e^{2\pi a})$ with $0 < a < 1$ is analytically continuable to the whole complex plane (see for instance [11, Section 2.2]).

Obviously, one has $\zeta(s, 1) = \text{Li}_s(1) = \zeta(s)$, where $\zeta(s)$ is the Riemann zeta function. We can easily see that $\zeta(\sigma) > 0$ when $\sigma > 1$ form the series expression of $\zeta(s)$. Since (see for instance [22, (2.12.4)])

$$(1 - 2^{1-\sigma})\zeta(\sigma) = 1 - \frac{1}{2^\sigma} + \frac{1}{3^\sigma} - \frac{1}{4^\sigma} + \cdots > 0, \quad 0 < \sigma < 1,$$

one has $\zeta(\sigma) < 0$ if $0 < \sigma < 1$. Moreover, we have $\zeta(0) = -1/2 < 0$ (see for example [22 (2.4.3)]). Therefore, from the functional equation of $\zeta(s)$, we have the following (see for example [9, Theorem 1.6.1] or [22, Section 2.12]).

Theorem A. All real zeros of $\zeta(s)$ are simple and at only the negative even integers.
The following Lerch’s formula is well-known (see for example [11, Theorem 1.3.4])

\[ \zeta'(0, a) := \left[ \frac{d}{ds} \zeta(s, a) \right]_{s=0} = \log \frac{\Gamma(a)}{\sqrt{2\pi}} \]

(1.1)

Especially, the value \( \zeta'(0, 1) = -(\log 2\pi)/2 \) plays a crucial role in the theory of the Riemann zeta function. For example, the approximation of \( \zeta(s) \) near \( s = 1 \) is given by

\[ \left[ \zeta(s) - \frac{1}{s-1} \right]_{s=1} - 2\zeta'(0) = \gamma_E + \log 2\pi, \]

(1.2)

where \( \gamma_E \) is the Euler constant (see [22, (2.1.16)]). Furthermore, the value \( \zeta'(0) \) appears in the Hadamard product formula for the completed Riemann zeta function \( \xi(s) \). Put \( 2\xi(s) := s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) \). Then we have

\[ \xi(s, a) = e^{A+Bs} \prod_{\rho} \left( 1 - \frac{s}{\rho} \right) e^{s/\rho}, \]

where the product is over the zeros \( \rho \) of \( \zeta(s) \), the constants \( A \) and \( B \) are given as

\[ e^A = \frac{1}{2}, \quad \text{and} \quad B = \frac{\zeta'(0)}{\zeta(0)} - 1 - \frac{\gamma_E + \log \pi}{2} \]

(1.3)

(see [22, Chapter 2.12]). Moreover, we have the following Riemann-von Mangoldt formula for \( \zeta(s) \) (see [9, Theorem 1.8.1] and [22, Theorem 9.4]). Let \( N(T, F) \) count the number of non-real zeros of a function \( F(s) \) having \( |\Im(s)| < T \). Then we have

\[ N(T, \zeta(s)) = \frac{T}{\pi} \log \frac{T}{2\pi} - \frac{T}{\pi} + O(\log T). \]

1.2. Main results. For \( 0 < a \leq 1/2 \), define the quadrilateral zeta function \( Q(s, a) \) by

\[ 2Q(s, a) := \zeta(s, a) + \zeta(s, 1-a) + \operatorname{Li}_a(e^{2\pi i a}) + \operatorname{Li}_a(e^{2\pi i (1-a)}). \]

The function \( Q(s, a) \) can be continued analytically to the whole complex plane except \( s = 1 \). In [14, Theorem 1.1], the author proved the functional equation

\[ Q(1-s, a) = \Gamma_{\cos}(s)Q(s, a), \quad \Gamma_{\cos}(s) := \frac{2\Gamma(s)}{(2\pi)^s} \cos\left( \frac{\pi s}{2} \right). \]

(1.4)

We remark that the gamma factor in [14] completely coincides with that of the functional equation for \( \zeta(s) \) (see [14, Section 1.3]). Let \( N_{Q}^{\mathrm{CL}}(T) \) the number of the zeros of \( Q(s, a) \) on the line segment from \( 1/2 \) to \( 1/2 + iT \). In [14, Theorem 1.2], he showed that for any \( 0 < a \leq 1/2 \), there exist positive constants \( A(a) \) and \( T_0(a) \) such that

\[ N_{Q}^{\mathrm{CL}}(T) \geq A(a)T \quad \text{whenever} \quad T \geq T_0(a). \]

In the present paper, we show the following statements on Lerch’s formula, the Hadamard product formula, Riemann-von Mangoldt formula and real and complex zeros of \( Q(s, a) \). First, we state results on real zeros of \( Q(s, a) \). By Mathematica 11.3, the real number \( a_0 \in (0, 1/2) \) appeared in the next theorem on real zeros of \( Q(s, a) \) is

\[ a_0 = 0.1183751396152722935827190345521191297147176999905314554591427859384268411483278906208314018589873082... \]

(1.5)

**Theorem 1.1.** There exists the unique real number \( a_0 \in (0, 1/2) \) such that

1. the function \( Q(\sigma, a_0) \) has a unique real zero at \( \sigma = 1/2 \) when \( \sigma \in (0, 1) \),
2. for any \( a \in (a_0, 1/2] \), the function \( Q(\sigma, a) \) has no real zero in \( \sigma \in (0, 1) \),
3. for any \( a \in (0, a_0) \), the function \( Q(\sigma, a) \) has at least two real zeros in \( \sigma \in (0, 1) \).
By this theorem, we have the following as an analogue of Theorem A.

**Corollary 1.2.** All real zeros of the quadrilateral zeta function $Q(s, a)$ are simple and are located only at the negative even integers just like $\zeta(s)$ if and only if $a_0 < a \leq 1/2$.

For non-real zeros of $Q(s, a)$, we have the following propositions. We first consider the cases $a = 1/6, 1/4, 1/3$ or $1/2$.

**Proposition 1.3.** Suppose that $a = 1/6, 1/4, 1/3$ or $1/2$. Then the Riemann hypothesis is true if and only if all non-real zeros of $Q(s, a)$ are on the critical line $\sigma = 1/2$.

On the other hand, we can see that $Q(s, a)$ with $a \in \mathbb{Q} \cap (0, 1/2) \setminus \{1/6, 1/4, 1/3\}$ does not satisfy an analogue of the Riemann hypothesis.

**Proposition 1.4.** Let $a \in \mathbb{Q} \cap (0, 1/2) \setminus \{1/6, 1/4, 1/3\}$. Then for any $\delta > 0$, there exist positive constants $C^0_a(\delta)$ and $C^2_a(\delta)$ such that the function $Q(s, a)$ has more than $C^0_a(\delta)T$ and less than $C^2_a(\delta)T$ complex zeros in the rectangles $1 < \sigma < 1 + \delta$ and $0 < t < T$, and $-\delta < \sigma < 0$ and $0 < t < T$ if $T$ is sufficiently large. Furthermore, for any $1/2 < \sigma_1 < \sigma_2 < 1$, there are positive numbers $C^0_a(\sigma_1, \sigma_2)$ and $C^2_a(\sigma_1, \sigma_2)$ such that $Q(s, a)$ has more than $C^2_a(\sigma_1, \sigma_2)T$ non-trivial zeros in the rectangles $\sigma_1 < \sigma < \sigma_2$ and $0 < t < T$, $1 - \sigma_2 < \sigma < 1 - \sigma_1$ and $0 < t < T$ when $T$ is sufficiently large.

Next we consider Lerch’s formula for $Q(s, a)$ as a analogue of (1.1). We remark that we have $Q(0, a) = -1/2 = \zeta(0)$ by (2.1).

**Theorem 1.5.** Let $\psi(a)$ be the digamma function. Then one has

$$Q'(0, a) = \frac{1}{4} \left( -2 \log(\sin \pi a) - 2 \log 4\pi - 2\gamma_E - \psi(a) - \psi(1 - a) \right). \quad (1.6)$$

We have the following corollary. The equation (1.9) below gives the approximation of $Q(s, a)$ near $s = 1$. It should be noted that the right hand side of (1.2) and (1.7) below completely coincide. The second statement is an asymptotic expansion of $Q'(0, a)$ as $a \to +0$. The equation (1.9) implies that $Q'(0, a)$ is written by only the logarithm and trigonometric functions when $0 < a \leq 1/2$ is rational.

**Corollary 1.6.** We have

$$Q(s, a) - \frac{1}{s - 1} \bigg|_{s=1} - 2Q'(0, a) = \gamma_E + \log 2\pi. \quad (1.7)$$

One has $Q'(0, a) \to \infty$ when $a \to +0$. More precisely,

$$Q'(0, a) = \frac{1}{4} \left( \frac{1}{a} - 2 \log a \right) + O(1), \quad a \to +0. \quad (1.8)$$

Let $r$ and $q$ be relatively prime natural numbers and $0 < r/q \leq 1/2$. Then we have

$$Q'(0, r/q) = \frac{1}{2} \left( \log \frac{q}{4\pi} - \log 2 \sin \frac{\pi r}{q} - \sum_{n=1}^{q-1} \cos \frac{2\pi r n}{q} \log 2 \sin \frac{\pi n}{q} \right). \quad (1.9)$$

For $0 < a \leq 1/2$, define the function $\xi_Q(s, a)$ by

$$2\xi_Q(s, a) := s(s - 1)\pi^{-s/2}\Gamma(s/2)Q(s, a).$$

The value $Q'(0, a)$ plays important role in the following Hadamard product formula for $\xi_Q(s, a)$. Note that the constant $B$ in (1.3) and $B(a)$ given by (1.11) completely coincide if we replace $\zeta(s)$ by $Q(s, a)$. Moreover, we can show an approximate formula for
$Q'(s,a)/Q(s,a)$ in terms of zeros of $Q(s,a)$ near to $s$ in Proposition 2.5 as a consequence of the Hadamard product formula for $\xi_Q(s,a)$.

**Proposition 1.7.** Let $0 < a \leq 1/2$ and $\rho_a$ be the zeros of $\xi_Q(s,a)$. Then we have

$$\xi_Q(s,a) = e^{A + B(a)s} \prod_{\rho_a} \left(1 - \frac{s}{\rho_a}\right)e^{s/\rho_a},$$

where $A$ and $B(a)$ are defined as

$$e^A = \frac{1}{2}, \quad \text{and} \quad B(a) = \frac{Q'(0,a)}{Q(0,a)} - 1 - \frac{\gamma_E + \log \pi}{2}.$$  \hspace{1cm} (1.10)

Furthermore, we have the following Riemann-von Mangoldt formula for $Q(s,a)$.

**Proposition 1.8.** For $0 < a \leq 1/2$ and $T > 2$, we have

$$N(T, Q(s,a)) = \frac{T}{\pi} \log \frac{T}{2\pi} - \frac{T}{\pi} - \frac{2T}{\pi} \log a + O_a(\log T).$$

In the next subsection, we give some remarks for Theorem 1.1, Corollary 1.2 and 1.6, Propositions 1.4 and 1.8, and the Epstein zeta function $\zeta_B(s)$ in order to explain that the properties of $Q(s,a)$ are “better” than those of $\zeta_B(s)$.

1.3. The Epstein zeta function. As we mention below, the quadrilateral zeta function $Q(s,a)$ has many analytical properties in common with the Epstein zeta function $\zeta_B(s)$ defined below, for example, the functional equation (1.11), the Riemann-von Mangoldt formula (1.12) and zeros on the critical line (14, Theorem 1.2).

Let $B(x,y) = ax^2 + bxy + cy^2$ be a positive definite integral binary quadratic form, and denote by $r_B(n)$ the number of solutions of the equation $B(x,y) = n$ in integers $x$ and $y$. Then the Epstein zeta function for the binary quadratic form $B$ is defined by the series

$$\zeta_B(s) := \sum_{(x,y) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{B(x,y)^s} = \sum_{n=1}^{\infty} \frac{r_B(n)}{n^s}$$

for $\sigma > 1$. It is widely known that the function $\zeta_B(s)$ admits analytic continuation into the entire complex plane except for a simple pole at $s = 1$ with residue $2\pi(-D)^{-1/2}$, where $D := b^2 - 4ac < 0$. Moreover, the function $\zeta_B(s)$ fulfills the functional equation

$$\left(\frac{\sqrt{-D}}{2\pi}\right)^s \Gamma(s)\zeta_B(s) = \left(\frac{\sqrt{-D}}{2\pi}\right)^{1-s} \Gamma(1-s)\zeta_B(1-s).$$  \hspace{1cm} (1.11)

Denoted by $N_{\text{Ep}}^{\text{CL}}(T)$ the number of the zeros of the Epstein zeta function $\zeta_B(s)$ on the critical line and whose imaginary part is smaller than $T > 0$. Potter and Titchmarsh [16] showed $N_{\text{Ep}}^{\text{CL}}(T) \gg T^{1/2-\varepsilon}$. The current (June, 2022) best result is $N_{\text{Ep}}^{\text{CL}}(T) \gg T^{4/7-\varepsilon}$, shown by Baier, Srinivas and Sangale [3].

The distribution of zeros of $\zeta_B(s)$ off the critical line depends on the value of the class number $h(D)$ of the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$. If $h(D) = 1$, it is expected that $\zeta_B(s)$ satisfies an analogue of the Riemann hypothesis since $\zeta_B(s)$ has an Euler product. When $h(D) \geq 2$, Davenport and Heilbronn [7] proved that $\zeta_B(s)$ has infinitely many zeros in the region of absolute convergence $\Re(s) > 1$. Voronin [21] proved that if $h(D) \geq 2$, then there exists a positive constant $C_B^0(\sigma_1, \sigma_2)$ such that the function $\zeta_B(s)$ has more than $C_B^0(\sigma_1, \sigma_2)T$ non-trivial zeros in the rectangle $\sigma_1 < \sigma < \sigma_2$ and $0 < t < T$, when $T$ is sufficiently large (see also [9, Theorem 7.4.3]). Recently, for example, Lee [12] and Lamzouri [10] improve Voronin’s theorem.
Hereafter, let $B(x, y) = ax^2 + bxy + cy^2$ be a positive definite binary quadratic form, namely, $a, b, c \in \mathbb{R}$, $a > 0$ and $d := b^2 - 4ac < 0$. And define $\kappa > 0$ by putting

$$\kappa^2 := \frac{|d|}{4a^2} = \frac{4ac - b^2}{4a^2} = \frac{c}{a} - \left(\frac{b}{2a}\right)^2.$$ 

Bateman and Grosswald [4] showed the following theorem. It should be noted that this result was announced by Chowla and Selberg [6] without a proof.

**Theorem B.** Let $\kappa \geq \sqrt{3}/2$. Then one has

$$\zeta_B(1/2) > 0 \quad \text{if} \quad \kappa \geq 7.00556$$

(or if $4\kappa^2 = |d|/a^2 \geq 199.2$) but,

$$\zeta_B(1/2) < 0 \quad \text{if} \quad \sqrt{3}/2 \leq \kappa \leq 7.00554$$

(or if $3 \leq 4\kappa^2 = |d|/a^2 \leq 199.1$).

There are no result for the sign of ‘the central value’ $\zeta_B(1/2)$ when $7.00554 < \kappa < 7.00556$ at present. It should be mentioned that $\zeta_B(s)$ vanishes in the interval $(1/2, 1)$ if $\kappa \geq 7.00556$ by the theorem above, the intermediate value theorem and $\lim_{\sigma \to 1^-} \zeta_B(\sigma) = -\infty$. Moreover, it is probable that $\zeta_B(\sigma) < 0$ for all $\sigma \in (0, 1)$ if $\sqrt{3}/2 \leq \kappa \leq 7.00554$.

Furthermore, we have the following Riemann-von Mangoldt formula for $\zeta_B(s)$ (see for example [18, p. 692]). Denote by $m(B)$ the minimum of the values of the quadratic form $B(x, y)$ for $(x, y) \neq (0, 0)$. Then we have

$$N(T, \zeta_B(s)) = \frac{2T}{\pi} \log \frac{|d|^{1/2}T}{2\pi e m(B)} + O(\log T). \quad (1.12)$$

**Remark.** It is known to be difficult to prove the non-vanishing of central values of zeta or $L$-functions. We can regard Theorem 1.1 as an analogue or “improvement” of Theorem B since Theorem 1.1 implies that

$$Q(1/2, a) > 0 \quad \text{if and only if} \quad 0 < a < a_0.$$ 

In other words, there is the unique absolute threshold $a_0$ that determines ‘the central value’ $Q(1/2, a)$ is positive, negative or zero. However, for Epstein zeta functions, there does not exist such an absolute threshold since we have no result for the sign of $\zeta_B(1/2)$ when $7.00554 < \kappa < 7.00556$.

**Remark.** The first statement of Corollary 1.6 is the analogue of the approximation of $\zeta_B(s)$ near $s = 1$ (see [10, Theorem 1]). Proposition 1.4 can be regarded as an analogue of theorems proved by Davenport & Heilbronn [7] and Voronin [21]. Proposition 1.8 is an analogue of (1.12).

**Remark.** It should be emphasised that one has $N_{Ep}^{\text{CL}}(T) \gg T^{4/7-\varepsilon}$ but $N_Q^{\text{CL}}(T) \gg T$. The values of $\zeta_B(s)$ at positive integers are given in [20, Theorem 1]. On the other hand, it is proved in [15, Theorem 1.2] that $\pi^{-2n}Q(2n, a)$, where $n \in \mathbb{N}$, can be expressed as a rational function with rational coefficients of $e^{2\pi ia}$.

The rest of this paper is organized as follows. Section 2.1 is devoted to the proofs of Theorem 1.1 and Corollary 1.2. The proofs of Propositions 1.3 and 1.4 are given in Section 2.2. We prove Theorem 1.5 and Corollary 1.6 in Section 2.3. We give proofs of Propositions 1.7 and 1.8 in Section 2.4 and 2.5, respectively.
2. Proofs

2.1. Proofs of Theorem 1.1 and Corollary 1.2

For \(0 < a \leq 1/2\), put
\[ Z(s, a) := \zeta(s, a) + \zeta(s, 1 - a), \quad P(s, a) := \text{Li}_s(e^{2\pi i a}) + \text{Li}_s(e^{2\pi i (1 - a)}) \]

Note that one has \(2Q(s, a) = Z(s, a) + P(s, a)\). By [14, Lemma 4.1], we have
\[ Z(1 - s, a) = \Gamma_{\cos}(s)P(s, a), \quad P(1 - s, a) = \Gamma_{\cos}(s)Z(s, a), \tag{2.1} \]

where \(\Gamma_{\cos}(s)\) is defined in (1.4). In [14, Lemma 4.4], it is shown that
\[ \frac{\partial}{\partial a}Z(\sigma, a) < 0, \quad \frac{\partial}{\partial a}P(\sigma, a) < 0, \quad 0 < \sigma < 1. \]

Hence we have the following inequality.

**Lemma 2.1.** Let \(0 < a < 1/2\). Then it holds that
\[ \frac{\partial}{\partial a}Q(\sigma, a) < 0, \quad 0 < \sigma < 1. \tag{2.2} \]

From [13, Theorem 1.3 and Proposition 4.5], we have the following.

**Lemma 2.2.** One has

(1) Let \(1/6 \leq a \leq 1/2\). Then one has \(Z(\sigma, a) < 0\) for \(0 < \sigma < 1\).

(2) When \(0 < a < 1/6\), the function \(Z(\sigma, a)\) has precisely one simple zero in \((0, 1)\). Let \(\beta_Z(a)\) denote the unique zero of \(Z(\sigma, a)\) in \((0, 1)\). Then the function \(\beta_Z(a) : (0, 1/6) \to (0, 1)\) is a strictly decreasing \(C^\infty\)-diffeomorphism.

**Proof of Theorem 1.1.** From Lemma 2.2, there is the unique \(0 < a_0 < 1/6\) such that \(Z(1/2, a_0) = 0\). By the functional equations (2.1), we have \(P(1/2, a_0) = 0\). Hence, from (2.2), the exists the unique \(0 < a_0 < 1/6\) such that
\[ 2Q(1/2, a_0) = Z(1/2, a_0) + P(1/2, a_0) = 0. \]

It should be noted that the \(0 < a_0 < 1/6\) above is given by (1.5) numerically. By the functional equation (1.4), we have
\[ Q(1/2 - \varepsilon, a_0)Q(1/2 + \varepsilon, a_0) \geq 0 \]
if \(\varepsilon > 0\) is sufficiently small. The inequality above implies that
\[ Q'(1/2, a_0) = 0, \quad Q'(\sigma, a_0) := \frac{d}{d\sigma}Q(\sigma, a_0). \tag{2.3} \]
Hereafter, we only consider the open interval \((0, 1/2)\) in virtue of (1.4). We have
\[ 2Q(0, a) = Z(0, a) + P(0, a) = 0 - 1 = -1, \quad 0 < a \leq 1/2 \tag{2.4} \]
from [14] (4.11) and (4.12). Put
\[ a_5 := 0.11837513961 < a_0 < 0.11837513962 =: a_2. \]

Then we have the following ten figures by Mathematica 11.3.

We can see that \(Q''(\sigma, a_0) < -2 < 0\) for all \(1/3 \leq \sigma \leq 1/2\) by the seventh, eighth and tenth figures. Hence we have
\[ Q(\sigma, a_0) < 0, \quad \sigma \in (1/3, 1/2). \]

In addition, we can see that
\[ Q(\sigma, a_0) < 0, \quad \sigma \in [0, 1/3] \]
from (2.2), $Q(0, a) = -1/2$ and the third and forth figures. Therefore, by (2.3) and the functional equation (1.4), we have

$$Q(1/2, a_0) = Q'(1/2, a_0) = 0,$$  \quad $$Q(\sigma, a_0) < 0, \quad \sigma \in (0, 1/2) \cup (1/2, 1)$$
which imply the first statement of Theorem 1.1. Moreover, from (1.4), (2.2) and the first statement of this theorem, one has

\[ Q(\sigma, a) < 0, \quad \sigma \in (0, 1), \quad a_0 < a \leq 1/2. \]  

(2.5)

which implies the second statement of Theorem 1.1.

Next suppose \( 0 < a < a_0 \). Then we have \( Q(1/2, a) > 0 \) from (2.2). On the other hand, one has \( Q(0, a) = -1/2 \) for all \( 0 < a \leq 1/2 \). Thus we have the third statement of Theorem 1.1 according to the intermediate value theorem.

\( \square \)

Proof of Corollary 1.2. When \( \sigma > 1 \) and \( 0 < a \leq 1/2 \), we have

\[ 2Q(\sigma, a) = Z(\sigma, a) + P(\sigma, a) > \sum_{n=0}^{\infty} \left( \frac{1}{(n+a)^\sigma} + \frac{1}{(n+1-a)^\sigma} - \frac{2}{(n+1)^\sigma} \right) > 0. \]  

(2.6)

Moreover, we have \( Q(\sigma, a) < 0 \) if \( \sigma \in (0, 1) \) and \( a_0 < a \leq 1/2 \) from (2.2). Recall that one has \( Q(0, a) = -1/2 \) by (2.3) or [14] (4.11) and (4.12). Hence, all real zeros of \( Q(1-s, a) \) with \( a_0 < a \leq 1/2 \) and \( \sigma > 1 \) come from \( \cos(\pi s/2) = 0 \) with \( \sigma > 1 \) from (1.4) and the fact that \( \Gamma(\sigma) > 0 \) when \( 0 < \sigma < 1 \). Hence every real zero of \( Q(s, a) \) with \( a_0 < a \leq 1/2 \) and \( \sigma < 1 \) is caused by \( \cos(\pi(1-s)/2) = 0 \) with \( \sigma < 0 \), which is equivalent to that \( s \) is a negative even integer.

\( \square \)

2.2. Proofs of Propositions 1.3 and 1.4. From (1.4), (4.1), (4.2), (4.3), (4.4), (4.5), (4.6), (4.7) and (4.8), it holds that

\[ Z(s, 1/2) = 2(2^s - 1)\zeta(s), \quad P(s, 1/2) = 2(2^{1-s} - 1)\zeta(s), \]  

(2.7)

\[ Z(s, 1/3) = (3^s - 1)\zeta(s), \quad P(s, 1/3) = (3^{1-s} - 1)\zeta(s), \]  

(2.8)

\[ Z(s, 1/4) = 2^s(2^s - 1)\zeta(s), \quad P(s, 1/4) = 2^{1-s}(2^{1-s} - 1)\zeta(s), \]  

(2.9)

\[ Z(s, 1/6) = (2^s - 1)(3^s - 1)\zeta(s), \quad P(s, 1/6) = (2^{1-s} - 1)(3^{1-s} - 1)\zeta(s). \]  

(2.10)

Proof of Proposition 1.3. When \( a = 1/2 \), we have

\[ 2Q(s, 1/2) = 2(2^s - 1)\zeta(s) + 2(2^{1-s} - 1)\zeta(s) = 2(X - 2 + 2X^{-1})\zeta(s), \]  

where \( 0 \neq X := 2^s \), form (2.7). The solutions of \( X^2 - 2X + 2 = 0 \) are \( X = 1 \pm i \). Obviously, the all solutions of \( 2^s = 1 \pm i \) are on the line \( \sigma = 1/2 \).

Assume that \( a = 1/3 \). By using (2.8), we have

\[ 2Q(s, 1/3) = (3^s - 1)\zeta(s) + (3^{1-s} - 1)\zeta(s) = (Y - 2 + 3Y^{-1})\zeta(s), \]  

where \( 0 \neq Y := 3^s \). We can easily see that \( Y = 1 \pm i \sqrt{2} \) are the roots of \( Y^2 - 2Y + 3 = 0 \) and the all solutions of \( 3^s = 1 \pm i \sqrt{2} \) are on the line \( \sigma = 1/2 \).
Let $a = 1/4$. By (2.9), it holds that
\[
2Q(s, 1/4) = 2^s(2^s - 1)\zeta(s) + 2^{1-s}(2^{1-s} - 1)\zeta(s) = (X^2 - X + 4X^{-2} - 2X^{-1})\zeta(s).
\]
The solutions of $X^2 - X + 4X^{-2} - 2X^{-1} = 0$ are
\[
X = \frac{1}{4}\left(1 + \sqrt{17} \pm i\sqrt{2(7 - \sqrt{17})}\right), \quad \frac{1}{4}\left(1 - \sqrt{17} \pm i\sqrt{2(7 + \sqrt{17})}\right).
\]
The absolute value of the numbers above are $\sqrt{2}$. Therefore, the all roots of $X^2 - X + 4X^{-2} - 2X^{-1} = 0$, where $X := 2^s$, are on the line $\sigma = 1/2$.

Finally, suppose that $a = 1/6$. From (2.10), one has
\[
2Q(s, 1/6) = ((2^s - 1)(3^s - 1) + (2^{1-s} - 1)(3^{1-s} - 1))\zeta(s)
= (3^s - 1)(2^{1-s} - 1)(g_2^+(s) + g_3^+(1-s))\zeta(s),
\]
where the function $g_p^\pm(s)$ is defined as
\[
g_p^\pm(s) := \frac{p^s \pm 1}{p^{1-s} \pm 1}, \quad p = 2, 3.
\]
Obviously, we have $g_p^\pm(1-s) = 1/g_p^\pm(s)$ from the definition. At the end of [14, Section 3.1], it is proved that $|g_2^+(s)| = 1$ for $\sigma = 1/2$, $|g_2^-(s)| < 1$ if $\sigma > 1/2$ and $|g_2^+(s)| > 1$ if $\sigma < 1/2$. By modifying the proof of this fact, we can prove that
\[
|g_p^+(s)| = 1, \quad |g_p^-(s)| > 1/2, \quad \sigma > 1/2, \quad |g_p^-(s)| < 1/2, \quad \sigma < 1/2.
\]
Hence $(2^s - 1)(3^s - 1) + (2^{1-s} - 1)(3^{1-s} - 1)$ does not vanish when $\sigma \neq 1/2$. \hfill $\Box$

Let $\varphi$ be the Euler totient function, $\chi$ be a primitive Dirichlet character of conductor of $q \in \mathbb{N}$ and $L(s, \chi)$ be the Dirichlet $L$-function. Let $G(r, \chi)$ denote the (generalized) Gauss sum $G(r, \chi) := \sum_{n=1}^{q} \chi(n)e^{2\pi i n/r}$ associated to a Dirichlet character $\chi$. When $0 < r/q \leq 1/2$, where $q$ and $r$ are relatively prime integers, we have
\[
Q(s, r/q) = \frac{1}{2\varphi(q)} \sum_{\chi \text{ mod } q} (1 + \chi(-1)) (\chi(r) q^s + G(r, \chi)) L(s, \chi)
\] (2.11)
from [14, (2.3)]. It is well-know that $\varphi(q) \leq 2$ if and only if $q = 1, 2, 3, 4, 6$.

Proof of Proposition [14] The upper bound for the number of complex zeros of $Q(s, a)$ is proved by the Bohr–Landau method (see for instance [22, Theorem 9.15 (A)]), the mean square of $\zeta(s, a)$ and $\operatorname{Li}_s(e^{2\pi i a})$ (see [11, Theorem 4.2.1]) and the inequality
\[
\int_2^T |Q(\sigma + it, a)|^2 dt \leq \int_2^T |\zeta(\sigma + it, a)|^2 dt + \int_2^T |\zeta(\sigma + it, 1-a)|^2 dt
+ \int_2^T |\operatorname{Li}_{\sigma+it}(e^{2\pi i a})|^2 dt + \int_2^T |\operatorname{Li}_{\sigma+it}(e^{2\pi i (1-a)})|^2 dt, \quad \sigma > 1/2.
\]
The lower bounds for the number of zeros of $Q(s, a)$ in the half-planes $1 < \sigma < 1 + \delta$ and $-\delta < \sigma < 0$ are prove by [14], [21] and [17, Corollary]. Furthermore, the lower bounds for the number of zeros of $Q(s, a)$ in the half-planes $\sigma_1 < \sigma < \sigma_2$ and $1 - \sigma_2 < \sigma < 1 - \sigma_1$ are shown by [14], (2.11) and [8, Theorem 2] and the definition of $Q(s, a)$. \hfill $\Box$
2.3. **Proofs of Theorem 1.5 and Corollary 1.6**

Clearly, the functional equations in (2.1) imply

\[ \Gamma_{\cos}(s) \cdot \Gamma_{\cos}(1 - s) = 1. \] (2.12)

Note that this equality is also proved by Euler’s reflection formula

\[ \Gamma(s) \Gamma(1 - s) = \frac{\pi}{\sin \pi s}, \quad s \notin \mathbb{Z}. \]

**Proof of Theorem 1.5.** From Lerch’s formula (1.1) and Euler’s reflection formula, we have

\[ \left[ \frac{d}{ds} Z(s, a) \right]_{s=0} = \log \left( \frac{\Gamma(a)\Gamma(1 - a)\pi}{2\pi} \right) = -\log(\sin \pi a) - \log 2. \]

By (2.1) and (2.12), we can see that

\[ \left[ \frac{d}{ds} P(s, a) \right]_{s=0} = \left[ \frac{d}{ds} \left( \Gamma_{\cos}(1 - s) Z(1 - s, a) \right) \right]_{s=0}. \]

In [5, Theorem 1], Berndt show that

\[ \zeta(s, a) = \frac{1}{s - 1} + \sum_{n=0}^{\infty} \gamma_n(a)(s - 1)^n = \sum_{n=1}^{\infty} \gamma_n(a)(s - 1)^n, \] (2.13)

where \( \gamma_{-1}(a) := 1 \) and \( \gamma_n(a) \) are given by

\[ \gamma_n(a) := \frac{(-1)^n}{n!} \lim_{l \to \infty} \left( \sum_{k=0}^{l} \frac{\log^n(k + a)}{k + a} - \frac{\log^{n+1}(l + a)}{n + 1} \right), \quad n \geq 0. \]

Note that \( \gamma_0(a) = -\psi(a) \), where \( \psi(a) \) is the digamma function defined as the logarithmic derivative of the gamma function. We can easily show that

\[ \frac{d}{ds} \left( \Gamma_{\cos}(1 - s) \right) = 2 \frac{d}{ds} \left( \frac{\Gamma(1 - s)}{(2\pi)^{1 - s} \sin \left( \frac{\pi s}{2} \right)} \right) = D_1(s) + D_2(s) + D_3(s), \]

where the functions \( D_1(s), D_2(s) \) and \( D_3(s) \) are defined as

\[
D_1(s) := 2(2\pi)^{s - 1} \Gamma(1 - s) \sin \left( \frac{\pi s}{2} \right) \log 2\pi, \\
D_2(s) := -2(2\pi)^{s - 1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1 - s) \psi(1 - s), \\
D_3(s) := \pi(2\pi)^{s - 1} \Gamma(1 - s) \cos \left( \frac{\pi s}{2} \right).
\]

From the expansion (2.13), we have

\[ Z(1 - s, a) = \sum_{n=-1}^{\infty} \delta_n(a)(-s)^n, \quad \frac{d}{ds} Z(1 - s, a) = \sum_{n=-1}^{\infty} n\delta_n(a)(-s)^{n-1}, \]

where \( \delta_n(a) := \gamma_n(a) + \gamma_n(1 - a) \). Hence one has

\[
\left[ D_1(s) Z(1 - s, a) \right]_{s=0} = \frac{\log 2\pi}{\pi} \left[ \sin \left( \frac{\pi s}{2} \right) Z(1 - s, a) \right]_{s=0} = \frac{\log 2\pi}{\pi} \left[ \sin \left( \frac{\pi s}{2} \right) \cdot \frac{2}{-s} \right]_{s=0} = -\log 2\pi, \\
\left[ D_2(s) Z(1 - s, a) \right]_{s=0} = -\frac{\psi(1)}{\pi} \left[ \sin \left( \frac{\pi s}{2} \right) Z(1 - s, a) \right]_{s=0} = \psi(1) = -\gamma_E,
\]
Thus we obtain (1.6) by $\gamma_0(a) = -\psi(a)$ and $\delta_0(a) := \gamma_n(a) + \gamma_n(1-a)$.

**Proof of Proposition 1.7.** According to (2.13), we can easily see that
\[\frac{2}{s-1} \sum_{n=1}^{\infty} \delta_n(a)(-s)^n - \frac{1}{\pi} \sin\left(\frac{\pi s}{2}\right) \sum_{n=1}^{\infty} n\delta_n(a)(-s)^{n-1}\]$_{s=0} = \frac{\delta_0(a)}{2}.

Thus we obtain (1.6) by $\gamma_0(a) = -\psi(a)$ and $\delta_0(a) := \gamma_n(a) + \gamma_n(1-a)$.

**Proof of Corollary 1.6.** According to (2.13), we can easily see that
\[\frac{2}{s-1} \sum_{n=1}^{\infty} \delta_n(a)(-s)^n - \frac{1}{\pi} \sin\left(\frac{\pi s}{2}\right) \sum_{n=1}^{\infty} n\delta_n(a)(-s)^{n-1}\]$_{s=0} = \delta_0(a) = -\psi(a) - \psi(1-a).

On the other hand, it is proved
\[P(1, a) = -2\log(2\sin \pi a) = -2\log(\sin \pi a) - 2\log 2\]
in [13, (4.12)]. Therefore, we obtain
\[Q(s, a) - \frac{1}{s-1} = 2Q'(0, a) + \gamma_E + \log 2\pi.\] \tag{2.14}

We can find that
\[\zeta(s) - \frac{1}{s-1} = \gamma_E \quad \text{and} \quad \zeta'(0) = -\frac{\log 2\pi}{2}\]
in [22, (2.1.16)] and [22, (2.4.5)], respectively. Thus we have (1.7). In addition, one has
\[\psi(a) = \frac{\Gamma'(1)}{\Gamma(1)} + \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+a}\right), \quad \psi(a) = -\frac{1}{a} + \psi(1+a)
\]
(see [1, (1.2.13) and (1.2.15)]). Thus we have
\[\psi(1+a), \psi(1-a) = O(1), \quad 0 \leq a \leq 1/2.
\]
Hence we obtain (1.8). It is shown that
\[\psi(r/q) + \psi(1-r/q) = -2\gamma_E - 2\log q + 2\sum_{n=1}^{q-1} \cos \frac{2\pi rn}{q} \log \left(2\sin \frac{\pi n}{q}\right)\]
in [1, (1.2.17)]. The equation above and Theorem 1.5 imply (1.9).

2.4. **Proof of Proposition 1.7.** First we show the following zero-free region of the quadrilateral zeta function.

**Lemma 2.3.** For any $0 < a \leq 1/2$ and $\eta > 0$, there exists $\sigma'_a(\eta) \geq 3/2$ such that $\abs{Q(s, a)} > \eta$ for all $\Re(s) \geq \sigma'_a(\eta)$.

**Proof.** Fix $0 < a < 1/2$. When $\sigma \geq 3/2$ we have
\[2\abs{Q(s, a)} \geq a^{-\sigma} - \sum_{n=1}^{\infty} \frac{1}{(n+a)^\sigma} - \sum_{n=0}^{\infty} \frac{1}{(n+1-a)^\sigma} - 2\sum_{n=1}^{\infty} \frac{\abs{\cos 2\pi na}}{n^\sigma}\]
\[\geq a^{-\sigma} - (1-a)^{-\sigma} - \zeta(\sigma, 1+a) - \zeta(\sigma, 2-a) - 2\zeta(\sigma)\]
\[\geq a^{-\sigma} - (1-a)^{-\sigma} - \zeta(\sigma) - \zeta(\sigma) - 2\zeta(\sigma)\]
\[\geq a^{-\sigma} - (1-a)^{-\sigma} - 4\zeta(3/2)\]
by the series expression of $Q(s, a)$. Note that the function $a^{-\sigma} - (1-a)^{-\sigma}$ is monotonically increasing with respect to $\sigma > 1$. Hence the inequality above implies Lemma 2.3 with $0 < a < 1/2$. When $a = 1/2$ and $\sigma \geq 3/2$, one has
\[
2|Q(s, 1/2)| \geq 2(1/2)^{-\sigma} - 2 \sum_{n=1}^{\infty} \frac{1}{(n + 1/2)^\sigma} - 2 \sum_{n=1}^{\infty} \frac{|\cos \pi n|}{n^\sigma} 
\geq 2^{\sigma+1} - 2\zeta(\sigma, 3/2) - 2\zeta(\sigma) \geq 2^{\sigma+1} - 4\zeta(3/2). 
\]
Hence we have this lemma for $0 < a \leq 1/2$. □

**Lemma 2.4.** Let $0 < a \leq 1/2$. Then $\xi_Q(s, a)$ is an entire function of order 1.

**Proof.** By the functional equation (1.4), it suffices to estimate $|\xi_Q(s, a)|$ on the half-plane $\Re(s) \geq 1/2$. From the approximate functional equations of $\zeta(s, a)$ and $\text{Li}_s(e^{2\pi i a})$ (see [11, Theorems 3.1.2 and 3.1.3]), it holds that
\[
|s(s-1)Q(s, a)| = O_a(|s|^3). 
\]
According to the inequalities
\[
|\Gamma(s)| \leq \exp(c_1|s| \log |s|), \quad |\pi^{-s/2}| \leq \exp(c_2|s|),
\]
where $c_1$ and $c_2$ are some positive constants (see for instance [9, p. 20]), $\xi_Q(s, a)$ is a function of at most 1. The order is exactly 1 since one has
\[
4Q(\sigma, a) \geq a^{-\sigma}, \quad \log \Gamma(\sigma) \geq (\sigma - 1/2) \log \sigma - 2\sigma,
\]
where $\sigma > 0$ is sufficiently large, by the general Dirichlet series expression of $Q(s, a)$ and Stirling’s formula. □

**Proof of Proposition 1.7** By Lemma 2.4 and Hadamard’s factorization theorem, we only determine the constants $A$ and $B(a)$. From (2.4), one has
\[
\xi_Q(0, a) = -Q(0, a) = 1/2 = e^A.
\]
Hence it holds that
\[
Q(s, a) = \frac{e^{B(a)s} \pi^{s/2}}{s(s-1)\Gamma(s/2)} \prod_{\rho_a} \left(1 - \frac{s}{\rho_a}\right) e^{s/\rho_a} = \frac{e^{(B(a)+(\log \pi)/2)s}}{2(s-1)\Gamma(s/2+1)} \prod_{\rho_a} \left(1 - \frac{s}{\rho_a}\right) e^{s/\rho_a}. 
\]
By taking the logarithmic derivative of the formula above, we obtain
\[
\frac{Q'(s, a)}{Q(s, a)} = B(a) + \frac{\log \pi}{2} - \frac{1}{s-1} - \frac{1}{2} \frac{\Gamma'(s/2 + 1)}{\Gamma(s/2 + 1)} + \sum_{\rho_a} \left(\frac{1}{s - \rho_a} + \frac{1}{\rho_a}\right) 
\]
(2.15)
By making $s \to 0$, we have
\[
\frac{Q'(0, a)}{Q(0, a)} = B(a) + 1 + \frac{\gamma_E + \log \pi}{2}
\]
which implies Proposition 1.7. □
2.5. **Proof of Proposition 1.8.** By using the Hadamard product formula for \(Q(s, a)\), we show the following approximate formula for \(Q'(s, a)/Q(s, a)\) in terms of zeros near to \(s\) which is an analogue of [9, Corollary 1.6.3] or [22, Theorem 9.6].

**Proposition 2.5.** Let \(0 < a \leq 1/2, \sigma_a := \sigma'_a + 1\), where \(\sigma'_a := \sigma'_a(\eta) \geq 3/2\) is given in Lemma 2.3, and \(\gamma_a\) be the imaginary part of the zeros of \(\xi_Q(s, a)\). Then for \(1 - \sigma_a \leq \sigma \leq \sigma_a\) and \(s = \sigma + it\), it holds that

\[
\frac{Q'(s, a)}{Q(s, a)} = -\frac{1}{s-1} + \sum_{|t - \gamma_a| \leq 1} \frac{1}{s - \rho_a} + O_a(\log(|t| + 2)).
\]

**Proof.** From (2.15) and the Stirling formula

\[
\frac{\Gamma'(s/2)}{\Gamma(s/2)} = \log(s/2) + O_a(|s|^{-1}),
\]

it holds that

\[
\frac{Q'(s, a)}{Q(s, a)} = -\frac{1}{s-1} + \sum_{\rho_a} \left( \frac{1}{s - \rho_a} + \frac{1}{\rho_a} \right) + O_a(\log(|t| + 2)).
\]

By putting \(s = \sigma_a + it\), we obtain

\[
\sum_{\rho_a} \left( \frac{1}{\sigma_a + it - \rho_a} + \frac{1}{\rho_a} \right) = O_a(\log(|t| + 2))
\]

since the general Dirichlet series of \(Q(s, a)\) converges absolutely and does not vanish when \(\sigma \geq \sigma_a = \sigma'_a + 1\) from Lemma 2.3. Let \(\rho_a := \beta_a + i\gamma_a\). Then, the real part of each term in the infinite summation above can be expressed as

\[
\frac{\sigma_a - \beta_a}{(\sigma_a - \beta_a)^2 + (t - \gamma_a)^2} + \frac{\beta_a}{\beta_a^2 + \gamma_a^2} \geq \frac{\sigma_a - \beta_a}{(\sigma_a - \beta_a)^2 + (t - \gamma_a)^2} \gg \frac{1}{1 + (t - \gamma_a)^2}.
\]

Therefore, we have

\[
\sum_{\rho_a} \frac{1}{1 + (t - \gamma_a)^2} \ll_a \log(|t| + 2). \tag{2.19}
\]

We have the left hand side of (2.19) \(\gg N(T + 1, Q(s, a)) - N(T, Q(s, a))\) since one has \((1 + (t - \gamma_a)^2)^{-1} \gg 1\) when \(0 < t \leq \gamma_a \leq t + 1\). Thus we obtain

\[
N(T + 1, Q(s, a)) - N(T, Q(s, a)) \ll_a \log(|t| + 2). \tag{2.20}
\]

From (2.18) and we subtract the same equality with \(s = \sigma_a + it\),

\[
\frac{Q'(s, a)}{Q(s, a)} = \sum_{\rho_a} \left( \frac{1}{s - \rho_a} - \frac{1}{\sigma_a + it - \rho_a} \right) + O_a(\log(|t| + 2)) \tag{2.21}
\]

According to the assumption \(1 - \sigma_a \leq \sigma \leq \sigma_a\), one has

\[
\left| \frac{1}{s - \rho_a} - \frac{1}{\sigma_a + it - \rho_a} \right| = \frac{\sigma_a - \sigma}{(|s - \rho_a)(\sigma_a + it - \rho_a)|} \leq \frac{2\sigma_a}{|t - \gamma_a|^2}.
\]

Hence the the infinite summation in (2.21) with \(|t - \gamma_a| > 1\) is \(O_a(\log(|t| + 2))\) by (2.19). Moreover, it holds that

\[
\sum_{|t - \gamma_a| \leq 1} \frac{1}{\sigma_a + it - \rho_a} = O_a(\log(|t| + 2))
\]
from (2.20) and the inequality $|\sigma_a + it - \rho_a| \geq 1$ which is proved by Lemma 2.3 and the definitions of $\sigma'_a$ and $\sigma_a$. Therefore, we have (2.20) by (2.21).

**Proof of Proposition 2.8** The proof method is using the argument principle (see for instance [9] Section 1.8 and [22] Section 9.3). Fix $0 < a \leq 1/2$ and assume that $Q(s, a)$ does not vanish on the line $\Im(s) = T$. Let $C$ be the rectangular contour with vertices $s = \sigma_a \pm T, s = 1 - \sigma_a \pm T$, where $\sigma_a > 0$ is given in Proposition 2.5. By Theorem 1.1 and Littlewood’s Lemma, it holds that

$$N(T, Q(s, a)) = \frac{1}{2\pi i} \int_C \xi'_Q(s, a) \xi_Q(s, a) ds + O_a(1). \quad (2.22)$$

From the definition of $\xi_Q(s, a)$, one has

$$\frac{\xi'_Q(s, a)}{\xi_Q(s, a)} = \frac{1}{s} + \frac{1}{s - 1} - \frac{\log \pi}{2} + \frac{1}{2} \frac{\Gamma'(s/2)}{\Gamma(s/2)} + \frac{Q'(s, a)}{Q(s, a)}. \quad (2.23)$$

The equation (2.22) can be expressed as

$$N(T, Q(s, a)) + O_a(1) = \frac{1}{2\pi} \int_{-T}^T \left( \frac{\xi'_Q(s, a + it, a)}{\xi_Q(s, a + it, a)} - \frac{\xi'_Q(1 - \sigma_a + it, a)}{\xi_Q(1 - \sigma_a + it, a)} \right) dt$$

$$+ \frac{1}{2\pi i} \int_{1 - \sigma_a}^{\sigma_a} \left( \frac{\xi'_Q(s - iT, a)}{\xi_Q(s - iT, a)} - \frac{\xi'_Q(s + iT, a)}{\xi_Q(s + iT, a)} \right) ds \quad (2.24)$$

where $I_1$ and $I_2$ are the first and second integrals in the last formula.

First we find an upper bound of for $|I_2|$. By (2.17) and the definition of $I_2$, we have

$$I_2 = \frac{1}{2\pi i} \int_{1 - \sigma_a}^{\sigma_a} \left( \frac{Q'(s - iT, a)}{Q(s - iT, a)} - \frac{Q'(s + iT, a)}{Q(s + iT, a)} \right) ds + O_a(\log T).$$

From Proposition 2.5 one has

$$\frac{Q'(s, a)}{Q(s, a)} = \sum_{|T - \gamma_a| \leq 1} \frac{1}{\sigma + iT - \rho_a} + O_a(\log T), \quad (2.25)$$

where $\rho_a$ are the zeros of $\xi_Q(s, a)$. Now let $C'$ be the rectangular contour with vertices $s = 1 - \sigma_a + iT, s = \sigma_a + iT, s = \sigma_a + i(T - 2), s = 1 - \sigma_a + i(T - 2)$. Note that the number of $\rho_a$ satisfying $T - 2 \leq \Im(\rho_a) \leq T$ is $O_a(\log T)$ from (2.20). Hence, we obtain

$$\int_{C'} \left( \sum_{|T - \gamma_a| \leq 1} \frac{1}{s - \rho_a} \right) ds = O_a(\log T).$$

This integral over $C'$ can also be written as

$$\int_{C'} \left( \sum_{|T - \gamma_a| \leq 1} \frac{1}{s - \rho_a} \right) ds = \int_{T - 1\leq \gamma_a \leq T + 1} \int_{C'} \frac{ds}{s - \rho_a} =$$

$$+ \sum_{T - 1\leq \gamma_a \leq T + 1} \int_{1 - \sigma_a}^{\sigma_a} \frac{d\sigma}{\sigma + iT - \rho_a} + i \sum_{T - 1\leq \gamma_a \leq T + 1} \int_{T - 2}^{T} \frac{dt}{\sigma + iT - \rho_a}$$

$$- i \sum_{T - 1\leq \gamma_a \leq T + 1} \int_{T - 2}^{T} \frac{dt}{1 - \sigma_a + iT - \rho_a} + \sum_{T - 1\leq \gamma_a \leq T + 1} \int_{1 - \sigma_a}^{\sigma_a} \frac{d\sigma}{\sigma + i(T - 2) - \rho_a}$$
We can easily see that the last three sums are \(O_a(\log T)\). Hence the first sum is also \(O_a(\log T)\). From this fact and (2.23), we can conclude that

\[ I_2 = O_a(\log T). \]

Next we estimate \(I_1\), making use of (2.23) and the relation

\[ \frac{\xi_Q'(s,a)}{\xi_Q(s,a)} = -\frac{\xi_Q'(1-s,a)}{\xi_Q(1-s,a)}. \]

From (2.17), (2.23) and the formula above, we have

\[
\frac{\xi_Q'(\sigma_a + it,a)}{\xi_Q(\sigma_a + it,a)} - \frac{\xi_Q'(1-\sigma_a + it,a)}{\xi_Q(1-\sigma_a + it,a)} = \frac{\xi_Q'(\sigma_a + it,a)}{\xi_Q(\sigma_a + it,a)} + \frac{\xi_Q'(\sigma_a - it,a)}{\xi_Q(\sigma_a - it,a)}
\]

\[
= \frac{\sigma_a^2 + t^2}{\frac{1}{2} \log \pi + 1} - 1 - \log \pi + 1 \log \sigma_a^2 + t^2 + O_a((\sigma_a^2 + t^2)^{-1/2})
\]

\[
+ \frac{Q'(\sigma_a + it,a)}{Q(\sigma_a + it,a)} + \frac{Q'(\sigma_a - it,a)}{Q(\sigma_a - it,a)}.
\]

Obviously, one has

\[
\int_{-T}^{T} \frac{Q'(\sigma_a + it,a)}{Q(\sigma_a + it,a)}dt = \int_{-T}^{T} \frac{(a^{-\sigma_a-it})'}{a^{-\sigma_a-it}}dt + \int_{-T}^{T} \frac{(a^{\sigma_a+it}Q(\sigma_a + it,a))'}{a^{\sigma_a+it}Q(\sigma_a + it,a)}dt.
\]

For the first integral, we have

\[
\int_{-T}^{T} \frac{(a^{-\sigma_a-it})'}{a^{-\sigma_a-it}}dt = -i \left[ \log a^{-\sigma_a-it} \right]_{-T}^{T} = -2T \log a.
\]

For some \(\theta > 0\), without loss of generality we can assume that \(\sigma_a\) satisfies

\[ 2\Re(a^{\sigma_a+it}Q(\sigma_a + it,a)) \geq 1 - \left( \frac{a}{1-a} \right) a^\theta - 4a \zeta(\sigma_a) > \theta \]

when \(0 < a < 1/2\) by modifying the proof of Lemma 223. One can assume similarly when \(a = 1/2\) (see the proof of Lemma 223). Hence we obtain

\[
\int_{-T}^{T} \frac{(a^{\sigma_a+it}Q(\sigma_a + it,a))'}{a^{\sigma_a+it}Q(\sigma_a + it,a)}dt = -i \left[ \log a^{\sigma_a+it}Q(\sigma_a + it,a) \right]_{-T}^{T} = O_a(1).
\]

Therefore, it holds that

\[
I_1 = \frac{1}{2\pi} \int_{-T}^{T} \left( \frac{\xi_Q'(\sigma_a + it,a)}{\xi_Q(\sigma_a + it,a)} + \frac{\xi_Q'(\sigma_a - it,a)}{\xi_Q(\sigma_a - it,a)} \right) dt
\]

\[
= -\frac{T}{\pi} \log \pi - \frac{T}{\pi} \log 2 + \frac{T}{\pi} \log T - \frac{T}{\pi} - \frac{2T}{\pi} \log a + O_a(\log T)
\]

\[
= \frac{T}{\pi} \log T - \frac{T}{\pi} \log(2e\pi a^2) + O_a(\log T).
\]

The theorem in this case follows from the formula above and the bound for \(I_2\). If we suppose that \(Q(s,a)\) has zeros on the line \(\Re(s) = T\), then the theorem follows from the case above of the theorem along with (2.20). \(\square\)

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