Abstract—The Pliable Index CODing (PICOD) problem is a variant of the index coding problem, where the desired messages by the users, who are equipped with message side information, is part of the optimization. This paper studies a special class of private PICOD where 1) the side information structure is circular, and 2) each user can decode only one message. The first condition is a special case of the “circular-arc network topology hypergraph” class of PICOD studied in [6], for which an optimal solution was given without the second constraint. The second condition was first studied in [8] and was motivated by the need to keep content privacy in some distribution networks. An achievable scheme is proposed in this paper. This scheme not only strictly outperforms the one in [8] for some values of the system parameters, but it is also information theoretically optimal in some settings. For the remaining cases, the proposed codes are shown to require at most one more transmission than the optimal linear codes.

I. INTRODUCTION

a) PICOD: Pliable Index CODing (PICOD) is a variant of the Index Coding (IC) problem first introduced in [2]. In PICOD, the messages to be decoded by the users, who have message side information, are not part of the problem definition. Instead, in PICOD, the sender assigns to the users the messages they need to decode so that (i) the assigned messages were not already present in the local side information, and (ii) the length of the code that allows every user to recover the assigned message has the shortest possible length. The PICOD problem formulation captures the nature of some content delivery applications, where there is flexibility in the choice of the desired messages to be delivered to the users. This flexibility allows to reduce the number of transmissions compared to an IC with the same side information structure.

The IC problem in its general form is known to be hard [11]. The general PICOD problem is not simpler to solve than the IC problem. For instance, the linear PICOD (here the sender is restricted to use linear codes) is still NP-hard [9]. Some efficient algorithms to solve the general PICOD were proposed in [10]. For the case where the side information structure of the PICOD has “symmetry,” we found the optimal code length (under no restriction on what type of encoding the sender can use) in [6]. However, the general PICOD problem is open.

b) Private PICOD: The problem of security and privacy in IC has been studied from different perspectives. In [3], the Authors proposed an IC model where an eavesdropper has a limited access to the side information sets and to the transmitted codeword; the goal here is to prevent the eavesdropper from obtaining any new information. In [5], the Authors considered an IC model where the transmitter needs to design a code that allows users to decode their desired message, but at the same time prevent them from obtaining any information about the side information or the desired messages of the other users. This latter model has the flavor of the private information retrieval problem [11], where a user wants to hide its desired message from the other users and the server. Along this line, the Authors of [7] formulated the private IC problem, where a user should only be able to decode its own desired messages upon receiving the codewords but no other message.

Recently, in [8], the Authors extended the private IC problem to the PICOD framework. Only the case where the side information structure is “circular,” and where each user can decode one and only one message was considered. Several achievable schemes were proposed, and it was shown that they provide the desired level of privacy.

c) Contributions and Paper Organization: In this paper we study a generalization (in terms of the form of the side information sets) of the private PICOD model from [8], as formally described in Section III. We provide both achievable schemes and the converse bounds, where past work only focused on linear achievable schemes; the main
result of this paper is discussed in Section III. In Section IV we derive both information theoretical and linear restricted converse bounds. We also provide some linear achievable schemes and show they are either information theoretical optimal, or differ from the linear optimal by at most one transmission. Section V concludes the paper.

II. SYSTEM MODEL

A private \((n, m, A)\) PICOD\((t)\) is defined as follows. There are \(n \in \mathbb{N}\) users and one central transmitter. The user set is denoted as \(U := \{u_1, u_2, \ldots, u_n\}\). There are \(m \in \mathbb{N}\) independent and uniformly distributed binary messages of \(k \in \mathbb{N}\) bits each. The message set is denoted as \(W := \{w_1, w_2, \ldots, w_m\}\).

The central transmitter has knowledge of all messages \(W\). User \(u_i\) has available to it locally the messages indexed by its side information set \(A_i \subset [m], i \in [n]\). The messages index by \(A_i\) are denoted as \(W_{A_i}\). The collection of all side information sets is denoted as \(A := \{A_1, A_2, \ldots, A_n\}\), which is assumed globally known at all users and the transmitter.

The sender and the users are connected by an error-free broadcast link. The sender transmits the codeword \(x^{\kappa \ell} := \text{ENC}(W, A)\), where \(\text{ENC}\) is the encoding function.

The decoding function for user \(u_j\) is

\[
\{\hat{\omega}_1^{(j)}, \ldots, \hat{\omega}_t^{(j)}\} := \text{DEC}_j(W_{A_j}, x^{\kappa \ell}), \quad \forall j \in [n],
\]

where \(t\) is the number of messages desired by a user and not already included in \(A_j\). In other words, the decoding function at \(u_j\) is \(\text{DEC}_j, j \in [n]\), such that

\[
\Pr[\exists \{d_{j,1}, \ldots, d_{j,t}\} \cap A_j = \emptyset : \{\hat{\omega}_1^{(j)}, \ldots, \hat{\omega}_t^{(j)}\} \neq \{w_{d_{j,1}}, \ldots, w_{d_{j,t}}\}] \leq \epsilon, \tag{1}
\]

for some \(\epsilon \in (0, 1)\) and some \(D_j := \{d_{j,1}, \ldots, d_{j,t}\} \subseteq [m]\setminus A_i\). We set \(D_j\) contains the indices of the desired messages by \(u_j\).

Up to this point, the system definition is that of a classical PICOD problem. We introduce now the privacy constraint. Privacy is modeled here as follows: user \(u_j\) can not decode any messages other than the \(t\) messages indexed by \(D_j\). Specifically, we impose that

\[
H(W_{[m] \setminus (D_j \cup A_j)}|x^{\kappa \ell}, W_{A_j}) \\
> \min H(W_{[m] \setminus (D_j \cup A_j)}) - (m - t - |A_j|)\epsilon. \tag{2}
\]

A code is called valid for the private \((n, m, A)\) PICOD\((t)\) if and only if it satisfies the conditions in (1) and (2). The goal is to find a valid code and a desired message assignment that results in the smallest possible codelength, i.e,

\[
\ell^* := \min \ell : \exists \text{ a valid } x^{\kappa \ell} \text{ for some } \kappa.
\]

Finally, if the encoding function at the sender is restricted to be a linear map from the message set, the length of shortest possible such valid codewords is denoted as \(\ell^*_{\text{lin}}\).

A. Network Topology Hypergraph (NTH) and size-s circular-h shift Side Information

In the rest of the paper we shall consider a class of \((n, m, A)\) PICOD\((t)\) problems with structure on \(A\). Such class is a generalization of the one studied in past work [8], which is a special case of the circular-arc NTH that we studied in [6] and fully solved for the case \(t = 1\) (without the privacy constraint). The rest of the section contains graph definition that will be used later on.

Let \(H = (V, \mathcal{E})\) denote a hypergraph with vertex set \(V\) and edge set \(\mathcal{E}\), where an edge \(E \in \mathcal{E}\) is a subset of \(V\), i.e., \(E \subseteq V\). The NTH, first introduced in [6], is a generalization of network topology graph for the IC problem [4]. In a NTH, messages are the hyperedges, while the users are the vertices. A user does NOT have a message in its side information set if and only if its corresponding vertex is incident to the hyperedge that represents the message. A 1-factor of \(H\) is a spanning edge induced subgraph of \(H\) that is 1-regular. A hypergraph \(H\) is called an circular-arc hypergraph if there exists an ordering of the vertices \(v_1, v_2, \ldots, v_n\) such that if \(v_i, v_j, i < j\), then the \(v_q\) for either all \(i \leq q \leq j\), or all \(q \leq i\) and \(q \geq j\), are incident to an edge \(E\).

In this paper we study the private \((n, m, A)\) PICOD\((1)\) with a special side information set structure: the sets in \(A\) are size-\(s\) circular-h shift of the message set. More precisely, The side information set of user \(u_i\) is of the form

\[
A_i = \{(i-1)h + 1, \ldots, (i-1)h + s\}, \tag{3}
\]

for \(i \in [n]\) where all indices are intended modulo the size of the message set, i.e., denoted as \((\text{mod } m)\).

Note that \(0 \leq s \leq m - 1\) in order to make the users have messages that are not in their side information
sets. \( h \geq 1 \) does not have any constraint for the circular structure of the side information sets.

Let \( g := \gcd(m, h) \). In this PICOD(1) there are \( n = m/g \) users, since all users have distinct side information sets. Note that the size-\( s \) circular-\( h \) shift side information setup is a special case of the side information structure with circular-arc we introduced in [6]. Also, the model studied in [8] is the special case when \( g = 1 \) (and thus \( n = m \)).

III. MAIN RESULT

For the size-\( s \) circular-\( h \) shift side information private PICOD(1) problem, we have the following main result.

**Theorem 1.** Consider the private PICOD(1) where the side information sets are size-\( s \) circular-\( h \) shifts of the \( m \) messages (as in [3]). When \( m \) is odd, \( g = 1 \), and either \( s = m - 2 \) or \( s = 1 \), a valid code does not exists (i.e., it is not possible to satisfy the privacy constraint). For the remaining cases:

- When \( s \geq m/2 \) and either \( 1 \leq s < m/2 \), \( g \geq 3 \) or \( 1 \leq s < m/2 \), \( s \neq 2 \), \( g = 2 \), we have
  \[
  \ell^* = \begin{cases} 
  1, & \text{if the NTH has 1-factor,} \\
  2, & \text{otherwise.}
  \end{cases}
  \] (4)

- When either \( 1 \leq s < m/2 \) and either \( g = 1 \) or \( s = g = 2 \), we have
  \[
  \left\lfloor \frac{m}{s} \right\rfloor / 2 \leq \ell_{s, \text{lin}} \leq \begin{cases} 
  \left\lfloor \frac{m}{s} \right\rfloor / 2, & \frac{m}{s} \in \mathbb{Z}, \\
  \left\lfloor \frac{m}{s} \right\rfloor / 2 + 1, & \frac{m}{s} \notin \mathbb{Z}.
  \end{cases}
  \] (5)

When \( s \geq m/2 \), the achievable scheme provided in [8] is indeed information theoretical optimal. This can be shown by comparing it with our converse bound in [6, Theorem 3]. Therefore, our main contribution in Theorem[1] is three-fold: 1) for \( s \geq m/2 \) we provide information theoretic optimality of the scheme in [8]; 2) for \( s < m/2 \) we provide a new achievable scheme, and show it is almost linear optimal; 3) we generalize the side information structure to any \( g > 1 \).

**Remark 1.** In [5], if we fix \( s \) and \( g \), \( \left\lfloor \frac{m}{s} \right\rfloor \) is monotonic in the message set size \( m \). One interesting observation is that, although the lower bound on \( \ell_{s, \text{lin}} \) is monotonic with \( m \), the upper bound is not. For instance, consider the case \( s = 2 \), \( g = 1 \); when \( m = 10 \) or \( m = 12 \), we have \( \ell_{s, \text{lin}} \leq 3 \), while when \( m = 11 \) we have \( \ell_{s, \text{lin}} \leq 4 \). In other words, from the point of \( m = 11 \), both increasing and decreasing the message set size may result in an increase of the required number of transmissions. Note that this is the point where the upper and the lower bounds differ. It is not clear at this point is whether this means the achievable scheme is not optimal, or the optimal private linear PICOD solution is not monotonic in \( m \).

IV. PROOF OF THEOREM 1

We divide the whole class into several sub-cases to show the proof of Theorem 1. Our focus of this paper is the case \( s < m/2 \), \( g = 1 \). Due to the page limitation, we omit some technical proofs of the propositions and the detailed proofs for the other cases. The reader might refer Appendix for the complete proof.

A. Case: \( s < m/2 \), \( g = 1 \)

In this case we show
\[
\left\lfloor \frac{m}{s} \right\rfloor / 2 \leq \ell_{s, \text{lin}} \leq \begin{cases} 
  \left\lfloor \frac{m}{s} \right\rfloor / 2, & \frac{m}{s} \in \mathbb{Z}, \\
  \left\lfloor \frac{m}{s} \right\rfloor / 2 + 1, & \frac{m}{s} \notin \mathbb{Z}.
  \end{cases}
\] (5)

1) Achievability: Let \( m = 2sq + r \) for some \( q, r \in \mathbb{Z} \) such that \( 0 \leq r < 2s \), i.e., \( r \) is the remainder of \( m \) modulo \( 2s \), and \( g \) is the maximum number of users who can have disjoint side information sets. We can have \( 2q + \lfloor \frac{r}{2} \rfloor \) groups of \( s \) users such that the users in each group have at least one message in common in their side information sets. Also, \( r - s \lfloor \frac{r}{s} \rfloor \) is the number of users that are not contained in any of these groups.

The intuition of our achievable scheme is as follows. Under the privacy constraint, we can satisfy the users in two groups with one transmission, therefore \( 2sq \) users can be satisfied by \( q \) transmissions. If \( r = 0 \), \( q \) transmissions suffice; if \( 0 < r < s \), we can satisfy the remaining \( r \) users by one transmission; and if \( s < r < 2s \), we can satisfy the remaining \( r \) users by two transmissions. Therefore the total number of transmissions is \( q + \lfloor \frac{r}{s} \rfloor \). Based on this intuition, we distinguish three sub-cases: a) \( r = 0 \); b) \( 0 < r < s \); and c) \( s < r < 2s \).

Case \( r = 0 \): This is the case where \( m \) is divisible by \( 2s \), therefore is divisible by \( s \). We group the users into groups \( G_1, G_2, \ldots, G_{2q} \) such that all users in \( G_i \) have \( w_{is} \) in their side information. Set the desired message of the users in \( G_{2i}, i \in [q] \) to
be \( w_{(2i-1)s}, \) and the desired message of the users in \( G_{2i-1}, i \in [q] \) to be \( w_{2is} \). There are \( q \) transmissions, each of them is \( w_{2is} + w_{(2i-1)s}, i \in [q] \), that satisfies the users in \( G_i \) and \( G_{i+1} \) while it does not provide any useful information for the users in other groups. Therefore, \( q = \frac{m}{s} \) transmissions satisfy all \( m \) users.

Case 0 < \( r \leq s \): We divide the users into \( 2q + 1 \) groups. Similar to the case above, the first \( 2q \) groups contain \( s \) users. The users in \( G_i \) all have \( w_{is} \) in their side information. Group \( G_{2q+1} \) has \( r \) users. The first \( q \) transmissions are \( w_{2is} + w_{(2i-1)s}, i \in [q] \), and satisfy the \( 2sq \) users in groups \( G_i, i \in [q] \).

If \( r = 1 \), we have \( G_{2q+1} = \{ w_m \} \). Let \( d_m = s + 1 \) and the \((q + 1)\)-th transmission be \( w_{s+1} + \sum_{j \in A_m} w_j \). Note that in this case \( s \geq 2 \), therefore user \( u_m \) can decode \( w_{s+1} \) while the other users cannot decode any new messages one they receive the last transmission.

If \( r \geq 2 \), the users in \( G_{2q+1} \) all have \( W_{[1s-r] \cup [m]} \) in their side information. Let \( d_{2sq+1} = s + r + 1 \) and \( d_j = 2sq + 1, j \in [2sq + 2 : m] \). The \((q + 1)\)-th transmission is \( w_{2sq+1} + w_m + \sum_{j=1}^{s-r+1} w_j \). Since \( w_{2sq+1} \) can compute \( w_{2sq+1} + w_m + \sum_{j=1}^{s-r} w_j \) and \( w_j, j \in [2sq + 2 : m] \), can compute \( w_m + \sum_{j=1}^{s-r+1} w_j \), these users have the message that is not in their side information set as their desired message. All the other users who are not in \( G_{2q+1} \) have at least two messages unknown in the transmission and this cannot decode it. Therefore, each user can decode only one message by the achievable scheme with \( q + 1 \) transmissions. If \( m \) is divisible by \( s \), then \( r = s \) and \( q + 1 \) is an integer. If \( m \) is not divisible by \( s \), \( q + 1 = \lceil \frac{m}{s} \rceil / 2 + 1 \).

Case 0 < \( s < r < 2s \): We group the users into \( 2q + 2 \) groups, \( G_1, \ldots, G_{2q+2} \). The users in \( G_i, i \in [2q + 1] \), all have message \( w_{is} \), while the users in \( G_{2q+2} \) all have \( W_{[1,2s-r] \cup [m]} \). We satisfy the first \( 2q \) groups by sending \( w_{2is} + w_{(2i-1)s}, i \in [q] \). We satisfy all users in \( G_{2q+1} \) by sending \( w_{2sq+1} + w_{2sq+s} + \sum_{j \in A_m} w_j \). If \( r = s + 1 \), \( G_{2q+2} = \{ w_m \} \). We let \( d_m = s + 1 \) and send as last transmission \( w_{s+1} + \sum_{j \in A_m} w_j \). Otherwise, let \( d_{2sq+s+1} = 2s - r + 1 \) and \( d_j = 2sq + s + 1, j \in [2sq + s + 1 : m] \) and send \( w_{2sq+s+1} + \sum_{j=1}^{s-r+1} w_j \). One can verify that all users can decode only one message. We use \( q + 2 = \lceil \frac{m}{s} \rceil / 2 + 1 \) transmissions.

2) Converse: Messages are bit vectors of length \( \kappa \), for some \( \kappa \); we thus see each message as an element in \( \mathbb{F}_{2^\kappa} \). When the sender uses a linear code (on \( \mathbb{F}_{2^\kappa} \), we can write the transmitted codeword as \( x^E = Ew^m \), where \( w^m = (w_1, w_2, \ldots, w_m) \) is the vector containing all the messages, and where \( E \in \mathbb{F}_{2^\kappa \times m} \) is the generator matrix of the code. We denote the linear span of the row vectors of \( E \) as \( \text{Span}(E) \). Recall that in this setting, user \( u_i, i \in [n] \), must be able to decode one and only one message outside its side information set \( A_i \); the index of the decoded message is \( d_i \). Let \( v_{i,j} \) be a vector whose \( j \)-th element is non-zero and all other elements with index not in \( A_i \) are zeros. A valid generator matrix \( E \) must satisfy the following conditions:

- **Decodability:** \( v_{i,d_i} \in \text{Span}(E) \), for all \( i \in [m] \);
- **Privacy:** \( v_{i,j} \notin \text{Span}(E) \) for all \( i \in [m], j \in [m] \setminus (A_i \cup \{d_i\}) \).

The decodability condition guarantees successful decoding of the desired message \( w_{d_i} \) by user \( u_i \). The privacy condition must hold because the existence of a vector \( v_{i,j} \in \text{Span}(E) \) for some \( j \in [m] \setminus (A_i \cup \{d_i\}) \) implies that user \( u_i \) is able to decode message \( w_j \) in addition to message \( w_{d_i} \).

The optimal linear code length \( \ell_{\text{lin}} \) is the smallest rank of the generator matrix \( E \), which by definition is the maximum number of pairwise linearly independent vectors in \( \text{Span}(E) \). We prove the converse bound by showing a lowered bound on the maximum number of pairwise linearly independent vectors in \( \text{Span}(E) \). To do so, we provide the following two propositions.

**Proposition 1.** In a working system (where every user can decode without violating the privacy condition) with \( g = 1 \) we must have \( c_i \notin \text{Span}(E) \) for all \( i \in [m] \), where \( c_i \) are standard basis of \( m \)-dim space.

**Proposition 2.** For a working system with \( g = 1 \), among all \( n = m \) users, consider \( k \) users whose side information sets are pairwise disjoint. The number of transmissions of linear code to satisfy these \( k \) users must be \( \ell_{\text{lin}} \geq k/2 \).

Because of the lack of space we omit the detailed proof here. The full proofs can be found in Appendix-A.

Proposition 1 states that a trivial ‘uncoded scheme’ (that consists of sending \( \ell_{\text{lin}} \) messages one by one) violates the privacy constraint. Proposition 2 provides a lower bound on the linear code words length for a subset of the users in the system, thus for all users. Therefore, among all \( m \) users
in the system, there are \( \lfloor \frac{m}{2} \rfloor \) users with pairwise disjoint side information sets. By Proposition 2, we need at least \( \lfloor \frac{m}{2} \rfloor / 2 \) transmissions to satisfy these users. Therefore, in order to satisfy all the users in the system, we must have \( \ell^{\text{lin}} \geq \ell^{\text{lin}} \geq \lfloor \frac{m}{2} / 2 \rfloor \). This provides the claimed lower bound.

B. Brief Proofs for the Remaining Cases

Due to the page limitation, here we provide brief proofs for the rest of the cases. The detailed proofs can be found in Appendix B.

1) Case: \( s < m/2, g = 2, \) and \( s = 2 \): The achievable scheme is the same as the case of \( s \leq m/2, s = 2, g = 1 \). For the converse proof, we extend Proposition 1 and show that the same statement holds in the case of \( s < m/2, g = 2 \), and \( s = 2 \) as well. Then converse follows the similar reasoning as in Section V-A.

2) Cases: \( s < m/2 \), either \( g = 2 \), \( s \neq 2 \) or \( g \geq 3 \): We provide achievable scheme to show that two transmissions are always possible to satisfying both the decoding and the privacy constraint. The converse follows the converse bound in [6, Theorem 3], since the extra privacy constraint clearly does not relax the converse bound.

3) Case: \( s \geq m/2 \): The achievable scheme in this case is the one proposed in [8], where \( g = 1 \) is considered. For the cases where \( g > 1 \), the users in the system is a proper subset of the users of \( g = 1 \). Therefore the scheme is still valid in both decoding and privacy constraint. The converse bound follows [6, Theorem 3].

4) Case: \( m \) is odd, \( g = 1 \), and either \( s = m-2 \) or \( s = 1 \): For these two cases we show that the privacy constraint cannot be satisfied. In other words, for all possible choices of desired messages at the users, there always exists one user who can decode more than one messages. The idea of the proof is to check the “decoding chain”, which consists of two users. The first user, by decoding its desired message, can have all side information of the second user, thus be able to decode the desired message of the second user. If the desired message of the second user does not belong to the side information set of the first user, the first user is able to decode two messages. We can show that such decoding chain exists regardless of the choices of the desired messages in these two cases. Therefore, the privacy constraint can not be satisfied. This technique was used in [6] for the converse proof of consecutive complete--PICOD(t).

V. Conclusion

In this paper we gave both achievable and converse bounds for the private PICOD(1) problem with circular side information sets. We showed that our linear achievable scheme is information theoretical optimal for some parameters, or it requires at most one more transmission compared to a converse under the constraint that the sender can only use linear codes. Proving, or disproving, that our linear codes are actually information theoretic optimal is subject of current investigation.

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A. Proof of Propositions 1 and 2

Proof of Proposition 1: Recall that, for \( g = 1 \), the side information sets are \( A_i = (i, \ldots, i + s - 1) \) for all \( i \in [m] \), as here \( n = m \). The proof is by contradiction. Assume without loss of generality (wlog) that we have a working systems with \( e_1 \in \text{Span}(E) \), that is, every user can decode message \( w_1 \) without even using its side information. Then, all users \( u_i, i \in [2 : m - s + 1] \) (who do not have \( w_1 \) in their side information sets) must have desired message \( w_1 \), in order to make sure that privacy constraint is not violated. This implies Fact 1: user \( u_1 \) can only have \( w_{d_1} = w_{s+1} \) as desired message.

Fact 1 is true because \( u_2 \) desires \( w_1 \), therefore \( A_2 \cup \{d_2\} \supset A_1 \). After decoding \( w_1 \), user \( u_2 \) can mimic user \( u_1 \) and thus decode message \( d_2 \). Since user \( u_2 \) can decode only one message, then \( d_1 \in A_2 \setminus A_1 = \{s + 1\} \). Therefore \( d_1 = s + 1 \). By taking \( d_1 = s + 1 \), we conclude that there must exist vector \( v_{1,d_1} = v_{1,s+1} = c + \alpha e_{s+1} \), where \( \alpha \in \mathbb{F}_2^s \), \( \alpha \neq 0 \) and \( c \in \text{Span}(A_1) \), where with an abuse of notation we let \( \text{Span}(A_i) \) denote \( \text{Span}\{e_j : j \in A_i\} \).

Given that we established Fact 1, let \( j \) be the position of the first non-zero element in the so found \( v_{1,s+1} \). Clearly, \( j \leq s + 1 \) since the \((s+1)\)-th element of \( v_{1,s+1} \) is \( \alpha e_{s+1} \neq 0 \). We have the following cases:

1) If \( j = s + 1 \), all the users who do not have \( w_{s+1} \) in their side information sets, can decode \( w_{s+1} \). This is because in this case \( v_{1,s+1} = \alpha e_{s+1} \), since all \( i \)-th elements of \( v_{1,s+1} \) are zeros with \( i \geq s + 2 \). Thus user \( u_{s+2} \), who has neither \( w_1 \) nor \( w_{s+1} \) in its side information set, can thus decode both \( w_1 \) and \( w_{s+1} \).

2) If \( 1 < j < s + 1 \), then user \( u_{j+1} \) can decode \( w_j \), since \( s + 1 \in A_j \). But user \( u_{j+1} \) decodes \( w_1 \) by assumption. Therefore, user \( u_j \) can decode both \( w_1 \) and \( w_j \).

3) If \( j = 1 \), user \( u_{s+2} \) can decode both \( w_{s+1} \) and \( w_1 \). Therefore, \( u_{s+2} \) can decode two messages.

In all the three above cases, there exists at least one user who can decode at least two messages, thus violating the privacy constraint. Therefore, the original assumption \( e_1 \in \text{Span}(E) \) must be impossible in a working system. The same reasoning applies to any \( e_j, j \in [m] \). This proves the claim.

Proof of Proposition 2: Let the messages desired by at least one user be \( w_1, w_2, \ldots, w_q \), for some \( q \leq k \). Partition the \( k \) users into \( q \) groups, such that the users in group \( G_i := \{u_{i,1}, u_{i,2}, \ldots, u_{i,q_i}\} \) desire message \( w_i \), with \( G_i \cap G_j = \emptyset \) for all \( i \neq j \) and such that \( \bigcup_{i=1}^{q} G_i = \{k\} \).

By Proposition 1 for all \( i \in [k] \) there exists \( v_{i,d_i} = \alpha_i e_{d_i} + c_i \in \text{Span}(E) \), where \( c_i \in \text{Span}(A_i) \) and \( \alpha_i \neq 0 \). Since the side information sets \( A_i \) are assumed to be disjoint, the vectors \( c_i \) are linearly independent. \( v_{i,d_i} \) are linearly dependent only if \( d_i \in A_j \) and \( d_j \in A_i \) for some \( i \neq j \). In other words, there exists a “loop” between \( u_i \) and \( u_j \). Note that since the side information sets are disjoint, one user can be in at most one “loop”, and the number of “loops” is at most \([k/2]\). Therefore the number of \( v_{i,d_i} \) that are linearly dependent is at most \([k/2]\), and thus the number of linearly independent \( v_{i,d_i} \) is at least \( k - [k/2] = [k/2] \). Therefore, the number of transmissions that is needed to satisfy \( k \) users with disjoint side information sets must satisfy \( \ell = \text{rk}(E) \geq [k/2] \).

B. Proof for sub-cases

1) Case: \( s < m/2, g = 2 \), and \( s = 2 \): In this case we show \( \ell^{\text{lin}} [m/4] \).

a) Achievability: We use the achievable scheme for case \( s = 2 < m/2 \) and \( g = 1 \). We need \([m/4] \) to satisfy all \( n = m \) users for the case \( g = 1 \). Since in the case \( g = 2 \) we have \( n = m/2 \), the users are in a proper subset of the users in the case \( g = 1 \). The achievable scheme for \( g = 1 \) still satisfies all users and meets the privacy constraint. We have \( \ell \leq [m/4] \) in this case.

b) Converse: We show \( \ell^{\text{lin}} \geq [m/4] \) as a linear converse bound. Therefore, the achievable scheme proposed in Section IV-A2 is tight under the linear encoding constraint.

The converse proof in Section IV-A2 does not directly apply in this case, mainly because Proposition 1 requires \( g = 1 \). Here we show that when \( g = s = 2 \), the statement of Proposition 1 holds as well.

Proposition 3. In a working system (where every user can decode without violating the privacy condition) with \( g = s = 2 \) we must have \( e_i \notin \text{Span}(E) \) for all \( i \in [m] \), where \( e_i \) are standard basis of \( m \)-dim space.

Proof of Proposition 3: Similar to the proof of Proposition 1 the proof is done by contradiction. W.l.o.g., assume \( e_1 \in \text{Span}(E) \). All users \( u_i, i \in \)
[2 : m − s + 1] in this case need to desire message \(w_1\). Let \(d_1 \in A_j\), for some \(j \neq 1\) For the decoding at \(u_1\), there exists a vector \(v_{1,d_1} \in \text{Span}(E)\) such that: 1) the \(d_1\)-th element is non-zero; 2) all elements with indices that are not 1, 2 or \(d_1\) are zeros.

We check the first and second element of \(v_{1,d_1}\) and have the following cases:

1) Both the first and second elements of \(v_{1,d_1}\) are zeros, \(v_{1,d_1} = e_{d_1}\). Therefore all users without \(w_{d_1}\) in their side information sets can decode \(w_{d_1}\).

2) The first element is zero while the second element is non-zero. By \(v_{1,d_1}\) the user \(u_j\) is able decode \(w_2\) since \(u_j\) already decodes \(w_1\) and has \(w_{d_1}\) in its side information sets. \(u_j\) can decode two messages.

3) The first element is non-zero while the second element is zero. Since all users that do not have \(w_1\) can decode \(w_1\), all users can generate \(e_{d_1}\) by \(w_1\) and decode \(w_{d_1}\) if they do not have it in their side information sets.

4) Both the first and second elements of \(v_{1,d_1}\) are non-zeros. \(u_j\) decodes \(w_1\) by assumption. It also has \(w_{d_1}\) in its side information set. Therefore \(u_j\) can decode \(w_2\).

All possible cases show that there exists at least one user that can decode at least two messages. The assumption that \(e_1\) is in \(\text{Span}(E)\) is impossible. The reasoning applies to all \(e_j, j \in [m]\). Therefore we conclude that \(e_i \notin \text{Span}(E)\) for all \(i \in [m]\).

By Proposition 3 the proof follows the same argument in Section IV-A2 by replacing Proposition 1 with Proposition 3. We show that for \(k\) user with pairwise disjoint side information sets, \([k/2]\) transmissions are needed for this case under the linear encoding restriction. Note that in this case all \(n = m/2\) users are with pairwise disjoint side information sets. Therefore, the total number of transmissions that satisfy all users is at least \(m/4\).

2) Case: \(s < m/2\), \(g = 2\), and \(s \neq 2\): In this case we prove

\[ \ell^* = \begin{cases} 1, & \text{if the NTH has 1-factor,} \\ 2, & \text{otherwise.} \end{cases} \]

a) Achievability: If \(s = 1\), the NTH has 1-factor. Thus \(\ell^* = 1\), in which case we send the sum of all messages. If \(2 < s < m/2\), we send \(w_{s+1}\) as the first transmission. This transmission satisfies all users but \(u_i, i = 2, \ldots, [s/2] + 1\), since they all have \(w_{s+1}\) in their side information set. When \(s\) is even, they have common side information set \([s + 1, s + 2]\). We send the second transmission as \(w_3 + w_{s+1} + w_{s+2} + w_{s+3}\). \(u_2\) can decode \(w_{s+3}\), \(u_i, i = 3, \ldots, [s/2] + 1\) can decode \(w_3\). All the other users, after decoding \(w_{s+1}\), still have at least two messages known in the summation, therefore can not decode any more messages. When \(s\) is odd, we send the second transmission as \(w_3 + w_{s} + w_{s+1} + w_{s+2} + w_{s+3}\). By similar argument we can show that \(u_i, i = 2, \ldots, [s/2] + 1\) can decode one messages from the second transmission while the other users can not. By the converse for the circular-arc NTH in [6], the proposed achievable scheme is information theoretical optimal.

b) Converse: By the converse bound in [6, Theorem 3] for circular-arc PICOD, without the privacy constraint, \(\ell^* \geq 1\) when the NTH has 1-factor, and \(\ell^* \geq 2\) when the NTH has no 1-factor. Since extra privacy constraint can not relax the converse bound, we see that the achievable scheme is information theoretical optimal.

3) Case: \(s < m/2, g \geq 3\): In this case we prove

\[ \ell^* = \begin{cases} 1, & \text{if the NTH has 1-factor,} \\ 2, & \text{otherwise.} \end{cases} \]

a) Achievability: It is trivial that if the NTH has 1-factor we have \(\ell^* = 1\), in which case we send the sum of all messages. Therefore, we show that if the NTH does not have 1-factor we can satisfy all users with two transmissions while satisfying the privacy constraint.

Send \(w_{s+1}\) as the first transmission. All users who do not have \(w_{s+1}\) in the side information sets are satisfied. The users that have \(w_{s+1}\) in the side information sets are \(u_i, i = 2, \ldots, [s/g], [s/g] + 1\). They have common side information set \([[s/g]g + 1 : s + g], [s+2 : s+g]] \geq 2\) since \(g \geq 3\). For the second transmission we send \(w_{m + \sum_{i=s+2}^{s+g} w_i}\). By the condition \(s < m/2\), all users \(u_i, i = 2, \ldots, [s/g], [s/g] + 1\) do not have \(w_m\) in the side information sets. Therefore these users can decode \(w_m\) as the desired message. For the second transmission, all the other users have at least two messages known in the summation, therefore can not decode any information from the second transmission. The privacy constraint is satisfied. This scheme is information theoretically optimal.

b) Converse: The reasoning is the same as it is in Appendix B2C.
4) **Case**: \( s \geq m/2 \): In this case we prove 

\[
\ell^* = \begin{cases} 
1, & \text{if the NTH has } 1\text{-factor,} \\
2, & \text{otherwise.} 
\end{cases}
\]

**a) Achievability:** We use the proposed achievable scheme in [8]. When \( g > 1 \), the users are in a proper subset of the users of \( g = 1 \), which is the case considered in [8]. Therefore the users can still be satisfied by the scheme that can satisfy strictly more users. The achievable scheme can achieve \( \ell = 1 \) when NTH has 1-factor, and \( \ell = 2 \) when NTH does not have 1-factor.

**b) Converse:** The reasoning is the same as it is in Appendix-B2b.

For the next two cases we show that the privacy constraint cannot be satisfied. The proof under the linear encoding constraint was provided in [8]. Here we provide a simple information theoretic proof of the same.

5) **Case** \( m \) is odd, \( s = m - 2 \), and \( g = 1 \): User \( u_i \) has two possible choices of desired message, \( d_i = (i + s) \pmod{m} \) or \( d_i = (i - 1) \pmod{m} \). If \( d_i = (i + s) \pmod{m} \), by decoding \( w_{d_i} \), user \( u_i \) can mimic \( u_{(i-1)} \pmod{m} \) since \( A_{(i-1)} \pmod{m} \subseteq \{(i + s) \pmod{m}\} \cup A_i \). Therefore, user \( u_i \) can decode \( w_{d_{(i-1)}} \pmod{m} \). To make sure user \( u_i \) can decode only one message, we need \( d_{(i-1)} \pmod{m} \in A_i \) so that user \( u_i \) does not decode another message that is not in its side information set. We thus have \( d_i \in A_{(i-1)} \pmod{m} \) and \( d_{(i-1)} \pmod{m} \in A_i \), i.e., \( u_i \) and \( u_{(i-1)} \pmod{m} \) form a “loop” where they can mimic one another. The same argument holds for the other choice of \( d_i \) as well. To make sure all users can decode only one message only, every user must be in a “loop”. However, note that one user can be in only one loop; thus, there must be one user that is not contained in any loop because \( m \) is odd. Therefore, there exists one user that can mimic another user and decode in total two messages, which violates the privacy constraint.

6) **Case** \( m \) is odd, \( s = 1 \), and \( g = 1 \): User \( u_i \), by decoding its desired message \( d_i = j, j \neq i \), can mimic user \( u_j \) and thus also decode \( d_j \). To make sure user \( u_i \) can decode only one message, we must have \( d_j = i \). Therefore user \( u_i \) and \( u_j \) form a “loop”. Similarly, every user can be in one loop. We need all users to be in a loop to make sure that every user can decode at most one message. Since \( m \) is odd, this is impossible and there must exists one user that can decode at least two messages, which violates the privacy constraint.