Supersolvability and Freeness for $\psi$-Graphical Arrangements

Lili Mu$^{1,2}$ · Richard P. Stanley$^3$

Abstract Let $G$ be a simple graph on the vertex set $\{v_1, \ldots, v_n\}$ with edge set $E$. Let $K$ be a field. The graphical arrangement $A_G$ in $K^n$ is the arrangement $x_i - x_j = 0$, $v_i v_j \in E$. An arrangement $\mathcal{A}$ is supersolvable if the intersection lattice $L(c(\mathcal{A}))$ of the cone $c(\mathcal{A})$ contains a maximal chain of modular elements. The second author has shown that a graphical arrangement $A_G$ is supersolvable if and only if $G$ is a chordal graph. He later considered a generalization of graphical arrangements which are called $\psi$-graphical arrangements. He conjectured a characterization of the supersolvability and freeness (in the sense of Terao) of a $\psi$-graphical arrangement. We provide a proof of the first conjecture and state some conditions on free $\psi$-graphical arrangements.

Keywords Graphical arrangement · Supersolvable arrangement · Free arrangement · Chordal graph

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1 Introduction

A finite hyperplane arrangement $\mathcal{A}$ is a finite set of affine hyperplanes in some vector space $V \cong K^n$, where $K$ is a field. The intersection poset $L(\mathcal{A})$ of $\mathcal{A}$ is the set of all nonempty intersections of hyperplanes in $\mathcal{A}$, including $V$ itself as the intersection over the empty set, ordered by reverse inclusion. Define the order relationship $x \leq y$ in $L(\mathcal{A})$ if $x \supseteq y$ in $V$.

Let $G$ be a graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set $E$. The graphical arrangement $\mathcal{A}_G$ in $K^n$ is the arrangement with hyperplane $x_i - x_j = 0$, $v_iv_j \in E$. We will use poset notation and terminology from [6, Ch. 3]. In particular, the intersection poset of the graphical arrangement $\mathcal{A}_G$ (or of any central arrangement) is geometric. (An arrangement $\mathcal{A}$ is central if $\bigcap_{H \in \mathcal{A}} H \neq \emptyset$.) Let $2^P$ denote the set of all subsets of $\mathbb{P}$, and let $\psi : V \to 2^P$ satisfy $|\psi(v)| < \infty$ for all $v \in V$. Define the $\psi$-graphical arrangement $\mathcal{A}_{G,\psi}$ to be the arrangement in $\mathbb{R}^n$ with hyperplanes $x_i = x_j$ whenever $v_iv_j \in E$, together with $x_i = \alpha_j y$ if $\alpha_j \in \psi(v_i)$.

In general, $\mathcal{A}_{G,\psi}$ is not a central arrangement and the intersection poset $L(\mathcal{A}_{G,\psi})$ of $\mathcal{A}_{G,\psi}$ is not a geometric lattice. Instead of $\mathcal{A}_{G,\psi}$, we consider the cone $c(\mathcal{A}_{G,\psi})$ with coordinates $x_1, \ldots, x_n, y$. The cone $\psi$-graphical arrangement $c(\mathcal{A}_{G,\psi})$ is the arrangement with hyperplanes $x_i = x_j$ whenever $v_iv_j \in E$, together with $y = 0$ and $x_i = \alpha_j y$ if $\alpha_j \in \psi(v_i)$.

An element $x$ of a geometric lattice $L$ is modular if $rk(x) + rk(y) = rk(x \land y) + rk(x \lor y)$ for all $y \in L$, where $rk$ denotes the rank function of $L$. A geometric lattice $L$ is supersolvable if there exists a modular maximal chain, i.e., a maximal chain $\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$ such that each $x_i$ is modular. A central arrangement $\mathcal{A}$ is supersolvable if its intersection lattice $L(\mathcal{A})$ is supersolvable.

A graph is chordal if each of its cycles of four or more vertices has a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle. Equivalently, every induced cycle in the graph should have exactly three vertices. A graphical arrangement $\mathcal{A}_G$ is supersolvable if and only if $G$ is a chordal graph [5, Cor. 4.10].

It is well known that the elements $X\pi$ of $L(\mathcal{A}_G)$ correspond to the connected partitions $\pi$ of $V(G)$, i.e., the partitions $\pi = \{B_1, \ldots, B_k\}$ of $V(G)$ such that the restriction of $G$ to each block $B_i$ is connected.

We have $X\pi \leq X\sigma$ in $L(\mathcal{A}_G)$ if and only if every block of $\pi$ is contained in a block of $\sigma$. Hence $L(\mathcal{A}_G)$ is isomorphic to an induced subposet $L_G$ of $\Pi_n$, the lattice of partitions of the set $\{1, 2, \ldots, n\}$. From the definition of $L(c(\mathcal{A}_{G,\psi}))$, it is easy to see that $L(\mathcal{A}_G)$ is an interval of $L(c(\mathcal{A}_{G,\psi}))$, namely, the interval from the bottom element $\hat{0}$ (the ambient space $K^n$) to the intersection of all the hyperplanes $x_i = x_j$ of $c(\mathcal{A}_{G,\psi})$. For brevity, an element $X\pi = (x_1, \ldots, x_{i-1}, \alpha_i y, x_{i+1}, \ldots, x_{j-1}, \alpha_j y, x_{j+1}, \ldots, x_n, y)$ ($\alpha_i \in \psi(v_i)$ or $\alpha_j \in \psi(v_j)$) of $L(c(\mathcal{A}_{G,\psi}))$ is written as $\sigma : v_i = v_j = \alpha_i y$, or more briefly as $\sigma = \{v_iv_j\alpha_i y\}$, and an element $X_\delta = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n, 0)$.
is written as $\delta : v_i = v_j = y = 0$, or more briefly as $\delta = \{v_i v_j y 0\}$. The following sufficient condition for the supersolvability of a $\psi$-graphical arrangement is stated in [7] without proof.

**Theorem 1** Let $(G, \psi)$ be as above. Suppose that we can order the vertices of $G$ as $v_1, v_2, \ldots, v_n$ such that

1. $v_{i+1}$ connects to previous vertices along a clique (so by Lemma 1, $G$ is chordal).
2. If $i < j$ and $v_i$ is adjacent to $v_j$, then $\psi(v_j) \subseteq \psi(v_i)$.

Then $A_G, \psi$ is supersolvable.

**Proof** To prove that $A_G, \psi$ is supersolvable, we need to find a modular maximal chain in $L(c(A_G, \psi))$. We will show that a modular maximal chain is given by $\hat{0} < \pi_1 < \cdots < \pi_n < \hat{1}$, where $\pi_i = \{v_1 v_2 \cdots v_i v_{i+1} y 0\}$. First we prove that $\pi_n = \{v_1 v_2 \cdots v_{n-1} v_n y 0\}$ is a modular element. For any $\sigma = \{B_1, B_2, \ldots, B_t\} \in L(c(A_G, \psi))$, we only need to consider the block $B_t$ which contains $v_n$. If $B_t = \{v_n\}$, then $\sigma < \pi_n$. Hence $\rk(\pi_n) = \rk(\sigma) < \rk(\pi_n \cap \sigma) + \rk(\pi_n \cap \sigma)$.

If $B_t = \{v_1 \cdots v_m v_n y 0\}$, then $\pi_n \cap \sigma = \hat{1}$. Since $v_n$ connects to previous vertices along a clique, the block $B'_t = \{v_1 \cdots v_m v_n\}$ exists. Then $\pi_n \cap \sigma = \{B_1, \ldots, B_{t-1}, B_{t+1}, \ldots, B_t, B'_t, v_n\}$. Hence $\rk(\pi_n \cap \sigma) = \rk(\sigma) - 1$ and $\rk(\pi_n) + \rk(\sigma) = \rk(\pi_n \cap \sigma) + \rk(\pi_n \cap \sigma)$.

If $B_t = \{v_1 \cdots v_m v_n y 0\}$, then $\pi_n \cap \sigma = \hat{1}$ and

$$\pi_n \cap \sigma = \{B_1, \ldots, B_{t-1}, B_{t+1}, \ldots, B_t, v_1 \cdots v_m y 0, v_n\}.$$  

Hence $\rk(\pi_n \cap \sigma) = \rk(\sigma) - 1$ and $\rk(\pi_n) + \rk(\sigma) = \rk(\pi_n \cap \sigma) + \rk(\pi_n \cap \sigma)$.

If $B_t = \{v_1 \cdots v_{m+1} \alpha_j \psi(v_n) \cdots v_n \alpha_j\}$, then $\pi_n \cap \sigma = \hat{1}$. Since $\psi(v_n) \subseteq \psi(v_j)$ if $v_j v_n \in E$, i.e., if $\alpha_j \in \psi(v_n)$, we have $\alpha_j \in \psi(v_j)$.

Hence the block $B'_t = \{v_1 \cdots v_m \alpha_j \psi(v_n) \cdots v_n \alpha_j\}$ exists and $\pi_n \cap \sigma = \{B_1, \ldots, B_{t-1}, B_{t+1}, \ldots, B_t, B'_t, v_n\}$. Then $\rk(\pi_n \cap \sigma) = \rk(\sigma) - 1$ and $\rk(\pi_n) + \rk(\sigma) = \rk(\pi_n \cap \sigma) + \rk(\pi_n \cap \sigma)$.

Hence we get that $\pi_n = \{v_1 v_2 \cdots v_{n-1} y 0\}$ is a modular element. Now if $\pi_{n-1} = \{v_1 v_2 \cdots v_{n-1} y 0\}$ is modular in the interval $[\hat{0}, \pi_n]$, then it is modular in $L(c(A_G, \psi))$ [5, Prop. 4.10(b)]. Therefore, we just need to show that $\pi_{n-1}$ is modular in the interval $[\hat{0}, \pi_n]$.

Since all elements $\sigma$ in $[\hat{0}, \pi_n]$ must satisfy that $\sigma$ has a block $B_t = \{v_n\}$, we can ignore the block $B_t = \{v_n\}$. In the same way, we can get that $\pi_{n-1} = \{v_1 v_2 \cdots v_{n-2} y 0\}$ is a modular element in the interval $[\hat{0}, \pi_n]$. Continuing the procedure, we get the modular maximal chain $\hat{0} < \pi_1 < \cdots < \pi_n < \hat{1}$. □

Our main result is the converse to Theorem 1.

**Theorem 2** The sufficient condition in Theorem 1 for the supersolvability of $A_G, \psi$ is also necessary.

Before we prove Theorem 2, the following two results of Dirac [1] are required. A vertex is simplicial in a graph if its neighbors form a complete subgraph. A graph
is recursively simplicial if it consists of a single vertex, or if it contains a simplicial vertex \( v \) and when \( v \) is removed the subgraph that remains is recursively simplicial. It is well known and easy to see that if \( G \) is recursively simplicial and \( v \) is any vertex, then \( G - v \) is recursively simplicial.

**Lemma 1** \( G \) is chordal if and only if \( G \) is recursively simplicial.

**Lemma 2** Every chordal graph \( G \) that is not a complete graph has at least two nonadjacent simplicial vertices.

**Proof** (of Theorem 2) Condition (1) is easy to check, because \( L(\mathcal{A}_G) \) is an interval of \( L(c(\mathcal{A}_G, \psi)) \). Since intervals of supersolvable lattices are supersolvable ([4, Prop. 3.2]), we have that \( L(c(\mathcal{A}_G)) \) is supersolvable. Hence by [4, Prop. 2.8] \( G \) is chordal.

By Lemma 2, we know that there are at least two nonadjacent simplicial vertices in the chordal graph \( G \). Suppose that there is a simplicial vertex, say \( v_{in} \), which satisfies the following condition:

\[
\psi(v_{in}) \subseteq \psi(v_{ij}) \quad \text{for all} \quad v_{ij} v_{in} \in E. \tag{1.1}
\]

Then we label \( v_{in} \) as \( v_n \) and remove this vertex. By Lemma 1, we know that the remaining graph is still recursively simplicial. Continuing in this way, suppose that there is a simplicial vertex, which we label as \( v_{n-1} \) and then remove it. Continue this procedure. If condition (2) is not necessary, then that means there exists one step \( m \) in the above procedure such that all the remaining simplicial vertices do not satisfy condition (1.1). Then we will show that there is no modular maximal chain in \( L(c(\mathcal{A}_G, \psi)) \).

Next, we show that among all the coatoms, only \( \sigma_i = \{v_1 v_2 \cdots v_{i-1} y 0\} \) and \( \delta_i = \{v_1 v_2 \cdots v_n \alpha_i y\} \), \( \alpha_i \in \psi(v_{ij}), 1 \leq i \leq n \), could be the modular elements of \( L(c(\mathcal{A}_G, \psi)) \). We claim that a coatom is not modular if it has more than two blocks or it has two blocks but the cardinalities of both of the blocks are greater than 1.

First, it is easy to check that any coatom \( \sigma \) is not modular if it has more than two blocks. Suppose \( \sigma = \{A, B, C\} \) is a coatom. Since \( \text{rk}(\sigma) = n - 1 \), i.e., \( \dim(\sigma) = 1 \), \( A, B, \) and \( C \) can only be \( \{v_1 v_2 \cdots v_{i-1} \alpha_i y\} \), where \( i = 1, 2, 3 \) and \( \alpha_i \in \psi(v_{in}), m = 1, 2, \ldots, j_i \). Then \( \gamma = \{v_1 v_2 \cdots v_n, y 0\} \), \( \text{rk}(\sigma) = \text{rk}(\gamma) = n - 1 \), \( \text{rk}(\sigma \lor \gamma) = n \) but \( \text{rk}(\sigma \land \gamma) < n - 2 \). Hence \( \sigma \) is not modular.

Moreover if \( \sigma = \{A, B\} \) is a coatom such that \( |A| > 1 \) and \( |B| > 1 \), then \( \sigma \) is also not modular. Without loss of generality, assume that there exist \( u, v \in A \) and \( u', v' \in B \) such that \( u \neq v' \) and \( u' \neq v \), \( uu' \in E(G) \), and \( vv' \in E(G) \). Let \( \gamma = \{(A \cup u') \lor v, (B \cup v) \land u'\} \). Then \( \text{rk}(\sigma) = \text{rk}(\gamma) = n - 1 \), \( \text{rk}(\sigma \lor \gamma) = n \) but \( \text{rk}(\sigma \land \gamma) < n - 2 \). Hence \( \sigma \) is not modular.

Therefore, among all coatoms, only \( \sigma_i = \{v_1 v_2 \cdots v_{i-1} y 0\} \) and \( \delta_i = \{v_1 v_2 \cdots v_n \alpha_i y, \alpha_i \in \psi(v_{ij}), 1 \leq i \leq n\} \) could be the modular elements. Similarly, among all the elements which \( \sigma_i \) covers, only \( \{(v_1 v_2 \cdots v_{n-1} \setminus v_{ij}) y 0\} \) could be modular.
If \(v_i\) is not a simplicial vertex, then we show that \(\sigma_i\) is not modular. Without loss of generality, assume \(v_i, v_{i+1} \in E\) and \(v_i, v_{i+1} \notin E\). Let \(\gamma = \{(v_i, v_{i+1}) \setminus \{v_i, v_{i+1}\}\} \cup \{v_i, v_{i+1}\}\). Then \(\text{rk}(\sigma) = \text{rk}(\gamma) = n - 1\), \(\text{rk}(\sigma \wedge \gamma) = n\) but \(\text{rk}(\sigma \vee \gamma) < n - 2\). Hence \(\sigma_i\) is not modular if \(v_i\) is not a simplicial vertex.

We now show that if \(v_i\) is a simplicial vertex but does not satisfy condition (1.1), then \(\sigma_i\) is not modular. Without loss of generality, assume that \(\alpha_i \in \psi(v_i)\) but \(\alpha_i \notin \psi(v_j)\) for \(v_i, v_{i+1} \in E\). Then \(\gamma = \{v_i, v_{i+1}, \alpha_i\}\). Let \(\text{rk}(\sigma_i) = \text{rk}(\gamma) = n - 1\), \(\text{rk}(\sigma_i \wedge \gamma) = n\) but \(\text{rk}(\sigma_i \vee \gamma) = 0\). From the above discussion, if condition (2) is not necessary, then there exists one step \(m\) such that all the remaining simplicial vertices do not satisfy condition (1.1). It means that all \(\{v_1, v_2, \ldots, v_{n-m}\}\) are not modular elements. Hence there is no modular maximal chain from 0 to \(\sigma_i\).

We now show that if \(\delta_i\) is modular, then \(\alpha_i \in \psi(v_i)\) for all \(i \in [n]\). Equivalently, we show that if there exists some \(v_m\) such that \(\alpha_i \notin \psi(v_m)\), then \(\delta_i\) is not modular. Let \(\gamma = \{v_1, v_2, \ldots, v_n \setminus v_m, v_m, y\}\). Hence \(\text{rk}(\delta_i) = \text{rk}(\gamma) = n - 1\), \(\text{rk}(\delta_i \wedge \gamma) = n\) but \(\text{rk}(\delta_i \vee \gamma) < n - 2\). From the above discussion, if condition (2) is not necessary, then there are at least two nonadjacent simplicial vertices, say \(v_s\) and \(v_t\), which do not satisfy condition (1.1). It means that there exist \(\alpha_s \in \psi(v_s)\), \(\alpha_t \in \psi(v_t)\), and \(\alpha_s, \alpha_t \neq \alpha_i\). If \(\alpha_s = \alpha_t\), then let \(\gamma = \{v_1, v_2, \ldots, v_n \setminus v_s, v_t, v_s, v_t\}\). Hence \(\text{rk}(\delta_i) = \text{rk}(\gamma) = n - 1\), \(\text{rk}(\delta_i \wedge \gamma) = n\) but \(\text{rk}(\delta_i \vee \gamma) < n - 2\), so \(\delta_i\) is not modular.

We call \(v_1, \ldots, v_n\) a vertex elimination order for \(G\) if \(v_{i+1}\) connects to previous vertices along a clique. For any supersolvable arrangement \(A\) of rank \(n\), the characteristic polynomial of \(A\) (defined, e.g., in [5, §1.3] or [6, §3.11.2]) factors as \(\chi_G(q) = \prod_{i=1}^{n}(q - a_i)\), where \(a_1, \ldots, a_n\) are nonnegative integers, called the exponents of \(A\). There is a simple combinatorial interpretation of the exponents of \(A\) when \(G\) is chordal.

Proposition 1 [2, Lemma 3.4] Let \(G\) be a chordal graph with vertex elimination order \(\{v_1, \ldots, v_n\}\). For \(1 \leq i \leq n\), let \(b_i\) be the degree of \(v_i\) in the graph \(G - \{v_n, \ldots, v_{i+1}\}\). Then \(b_1, \ldots, b_n\) are the exponents of the supersolvable arrangement \(A(G)\).

It is not hard to get a similar property for the supersolvable arrangement \(A_G, \psi\). We omit the proof of this proposition.

Proposition 2 Let \((G, \psi)\) be a chordal graph with vertex elimination order \(\{v_1, \ldots, v_n\}\). Assume that for any \(v_i, v_j \in E(G)\) such that \(i < j\), we have \(\psi(v_j) \subseteq \psi(v_i)\). For \(1 \leq i \leq n\), let \(b_i\) be the sum of \(|\psi(v_j)|\) and the degree of \(v_i\) in the graph \(G - \{v_n, \ldots, v_{i+1}\}\). Then \(b_1, \ldots, b_n\) are the exponents of the supersolvable arrangement \(A_G, \psi\).

There is another conjecture in [7]. It is well known that every supersolvable arrangement is free (in the sense of Terao [3, §6.3]) and every free graphical arrangement is supersolvable. Thus the second author proposed the following conjecture.
Conjecture 1 If $A_{G,\psi}$ is a free $\psi$-graphical arrangement, then $A_{G,\psi}$ is supersolvable.

We are unable to prove this conjecture, but we do have the following weaker result, which we simply state without proof. The proof involves the inheritance of freeness under localization of arrangements and a result of Yoshinaga [8] on the freeness of 3-arrangements.

Theorem 3 The $\psi$-graphical arrangement $A_{G,\psi}$ is not free if there is an edge $v_iv_j \in E(G)$ such that $\psi(v_i) \not\subseteq \psi(v_j)$ and $\psi(v_j) \not\subseteq \psi(v_i)$.

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