Superpotentials and Membrane Instantons

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Abstract

We investigate nonperturbative effects in $M$-theory compactifications arising from wrapped membranes. In particular, we show that in $d = 4, \mathcal{N} = 1$ compactifications along manifolds of $G_2$ holonomy, membranes wrapped on rigid supersymmetric 3-cycles induce nonzero corrections to the superpotential. Thus, membrane instantons destabilize many $M$-theory compactifications. Our computation shows that the low energy description of membrane physics is usefully described in terms of three-dimensional topological field theories, and the superpotential is expressed in terms of topological invariants of the 3-cycle. We discuss briefly some applications of these results. For example, using mirror symmetry we derive a counting formula for supersymmetric three-cycles in certain Calabi-Yau manifolds.

July 4, 1999
1. Introduction

The importance of instanton computations in string theory and in M-theory can hardly be overstated. To cite but one reason, the understanding of instanton effects is a necessary ingredient in attempts to make realistic M/string theory models which address the problems of vacuum selection and supersymmetry breaking.

Compared to instantons in gauge theory and string theory, M-brane instantons have not been so thoroughly discussed. These effects were first discussed in the fundamental paper of Becker, Becker, and Strominger [1]. There has been some work on non-perturbative corrections in compactifications of M-theory with 4 unbroken supersymmetries beginning with Witten’s investigation of five-brane instantons [2]. Since M2 branes can sometimes be interpreted in terms of world-sheet instantons in IIA string theory (see, e.g. [3][4]) a fair amount is known in this case. Nevertheless, we believe that much remains to be understood concerning the general computation of M-brane instantons, and this paper is a modest step in that direction.

Specifically, in the present paper we discuss instanton effects associated with wrapped M2 branes in M theory compactifications to three and four dimensions. Throughout most of the paper we focus on the example of M-theory compactification on smooth seven-manifolds of G2 holonomy (henceforth abbreviated as G2-manifolds). We will show that in such compactifications membrane instantons can induce a nonzero superpotential. As an example, rigid membranes lead to a contribution to the superpotential given in equations (2.13) and (6.10) below.

Here is a brief overview of the paper. In section 2 we review a few aspects of Kaluza-Klein reduction of 11-dimensional supergravity on G2 manifolds. In section 3 we complain about the absence of a clear set of rules for computing M-brane instanton effects (due to the want of a fundamental formulation of M-theory), and outline the practical procedure we will adopt.

In sections 4 and 5 we consider the low energy action of a membrane in curved superspace. The upshot of our discussion is that the low energy fluctuations of a membrane wrapped on a supersymmetric 3-cycle are usefully thought of in terms of certain three-dimensional topological field theories. It is well-known that in string theory two-dimensional topological sigma models (the A- and B- models, and their couplings to topological gravity) are quite relevant to the computation of superpotentials [5][6]. Not surprisingly, we are finding that in M-theory an analogous role is played by three-dimensional topological field theory.
Section 6 contains the key results of the paper. We derive (following the provisional rules of section 3) the contribution to the superpotential from a rigid membrane instanton, and derive one-loop determinants and zeromodes needed in the more general case of arbitrary membrane instantons. We discuss briefly the relation of the phase of the instanton contribution to the one-loop determinants, following a similar discussion by Witten in the example of D1-brane instantons [7].

In the remainder of the paper we turn to examples and applications of our results. In section 7 we discuss a simple class of $G_2$ manifolds and show that there do exist examples of the kinds of effects we have discussed. (We also give a relevant example in appendix C.) Section 8 contains some brief comments on the extension of this work to compactifications of IIA string theory on $G_2$ manifolds. Section 9 concerns generating functions counting supersymmetric 3-cycles in Calabi-Yau 3-folds admitting a real structure. Section 10 examines consequences for nonperturbative effects in the heterotic string. Section 11 addresses some effects of open membrane instantons. Many conventions and notations can be found in appendices A and B.

Some closely related issues to those discussed in this paper have been addressed by B.S. Acharya in [8]. Acharya focuses on certain interesting singular $G_2$ manifolds. In the present paper we usually work with the case of smooth $G_2$ manifolds.

2. Compactification on $G_2$ manifolds

The low energy limit of $M$-theory is described by 11-dimensional supergravity. This theory consists of a metric $g$, a 3-form $C$, and a gravitino $\Psi$, defined on an eleven-dimensional Lorentzian spin manifold $M_{11}$. The bosonic action enters the path integral through

$$\exp\left[\frac{i}{(2\pi)^2 \ell^9} \int \text{vol}(g) R(g) + \frac{i}{(2\pi)^2 \ell^3} \int \frac{1}{2} dC \wedge *dC - \frac{i}{6 \cdot (2\pi)^2} \int_{M_{11}} G \wedge G \wedge C + \cdots \right]$$

(2.1)

where $\ell$ is the eleven-dimensional Planck length and $G = dC$. The ellipsis indicates the presence of higher order terms in the low energy expansion. These will be ignored in what follows. The field strength $G$ is normalized by requiring that

$$\int_{S^4} \frac{G}{2\pi} = 1$$

(2.2)

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1 An explanation of our conventions for scales and dimensions is in appendix B.
where the integral is taken around a 4-sphere linking the basic five-brane in $\mathbb{R}^{1,10}$.

In this paper $M_{11}$ will be taken to be smooth, and all relevant lengthscales are large compared to $\ell$. Moreover, we focus on vacua with smooth Ricci flat direct product metrics:

$$M_{11} = M_4 \times X \quad (2.3)$$

$$(0 \leq \mu, \nu \leq 3, 1 \leq m, n \leq 7.)$$

We take $X$ to be a manifold of $G_2$ holonomy. In particular we assume $X$ has a covariantly constant spinor $\nabla_m \theta = 0$, unique up to scale. Thus, compactification on $X$ leads to a theory with $d = 4, \mathcal{N} = 1$ supersymmetry. When we work in Euclidean signature we will take $M_4$ to be a hyperkähler manifold.

We complete the specification of the background by taking the background value of $C$ to be a real harmonic 3-form on $X$, and thus $G = dC = 0$.\footnote{We are ignoring an important subtlety here. The following remarks were explained to us by E. Witten. In compactifications of $M$ theory on $X$ the quantization condition is shifted\footnote{[1]} and becomes the statement that $[G/2\pi] - \frac{\lambda}{2} \in H^4(X, \mathbb{Z}) \quad (2.4)$ where $[G/2\pi]$ is the cohomology class of $G/2\pi$ and $\lambda(X) = p_1(X)/2$. For a $G_2$-manifold $\lambda$ is always even, so that it is consistent to set $G = 0$. The argument is the following: Note that $p_1(X) = p_1(X \times S^1)$. Now $X \times S^1$ is an eight-dimensional spin manifold and $\lambda$ is even or odd on an eight-dimensional spin manifold according to whether the intersection form is even or odd \footnote{[9]}. Now observe that the intersection form of $X \times S^1$ is obviously even.} We are ignoring an important subtlety here. The following remarks were explained to us by E. Witten. In compactifications of $M$ theory on $X$ the quantization condition is shifted\footnote{[1]} and becomes the statement that $[G/2\pi] - \frac{\lambda}{2} \in H^4(X, \mathbb{Z}) \quad (2.4)$ where $[G/2\pi]$ is the cohomology class of $G/2\pi$ and $\lambda(X) = p_1(X)/2$. For a $G_2$-manifold $\lambda$ is always even, so that it is consistent to set $G = 0$. The argument is the following: Note that $p_1(X) = p_1(X \times S^1)$. Now $X \times S^1$ is an eight-dimensional spin manifold and $\lambda$ is even or odd on an eight-dimensional spin manifold according to whether the intersection form is even or odd \footnote{[9]}. Now observe that the intersection form of $X \times S^1$ is obviously even.}
Let us now consider the chiral multiplets. Using the covariantly constant spinor one can construct a “$G_2$ calibration,” i.e., a covariantly constant 3-form: $\Phi_{mnp} = \bar{\gamma}_{mnp} \vartheta$. In an appropriate local orthonormal frame the components of $\Phi$ are the structure constants of the octonions $\mathbb{O}$. Conversely, such a covariantly constant 3-form can be used to characterize the $G_2$ structure [11].

We may thus define real scalar fields $S^i(x)$ by choosing a base point $G_2$ structure $\Phi_0$ and associating the $G_2$ metric to a torsion free calibration

$$\Phi = \Phi_0 + \ell^{-3} \sum_{i=1}^{b_3(X)} S^i(x) \omega_i^{(3)}(y)$$ (2.6)

By straightforward reduction of the supergravity action (described below) one finds that the Kähler target space has holomorphic tangent space $T^{1,0} \mathcal{M} = H^3(X; \mathbb{R}) \oplus H^3(X; \mathbb{R})$, with the obvious complex structure. The holomorphic coordinates on this space are given by

$$C + i\ell^{-3} \Phi = i\Phi_0 + \ell^{-3} \sum_{i=1}^{b_3(X)} z^i(x) \omega_i^{(3)}(y)$$ (2.7)

with $z^i = P^i + iS^i$. This is the analog for $G_2$ manifolds of the complexified Kähler class familiar from string compactification on Calabi-Yau 3-folds. Holomorphy in $C + i\ell^{-3} \Phi$ will play an important role in the discussion below.

The Kaluza-Klein expansion of the gravitino is

$$dx^M \Psi_M = dx^\mu \Psi_\mu + dx^m \Psi_m$$

$$\Psi_\mu(x, y) = \psi_\mu(x) \otimes \vartheta(y)$$

$$\Psi_m(x, y) = \ell^3 \sum_{i=1}^{b_3} \omega_i^{(3)}(y) \Gamma_{pq}^i \chi^i(x) \otimes \vartheta(y)$$

$$+ \ell^2 \sum_{I=1}^{b_2} \omega_I^{(2)}(y) \Gamma_{p}^I (x) \otimes \vartheta(y) + \cdots$$ (2.8)

Now let us consider the low energy effective action. The effective action at the two-derivative order is determined by a choice of a Kähler target space $\mathcal{M}$ for the chiral scalars $z^i$, a holomorphic gauge kinetic function $\tau_{IJ}(z)$, and a holomorphic superpotential $W(z)$.

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3 See particularly [12], Theorem C. Much useful background material can be found in [13] [14].

4 The term $\int C \wedge I_8$ does contribute, but at 4th order in derivatives.
The gauge kinetic term, entering the 4D action through (among other terms)
\[
\frac{1}{32\pi} \int_{M_4} \text{Im} \left[ \tau_{IJ} (F^I - i\tilde{F}^I)(F^J - i\tilde{F}^J) \right],
\]
(2.9)
comes from the kinetic terms and Chern-Simons term for $C$. It is given by
\[
\tau_{IJ} = \frac{2}{\pi} \int_X (C + i\ell^{-3}\Phi) \wedge \omega_I^{(2)} \wedge \omega_J^{(2)}.
\]
(2.10)
Harmonic forms on $X$ transform in the 14 of the holonomy group and therefore satisfy
\[*\omega_I^{(2)} = -\Phi \wedge \omega_I^{(2)}\]. It follows that $\text{Im}\tau_{IJ}$ is definite when $X$ is smooth.

The Kähler potential can be deduced by looking at the kinetic terms of the scalars $P^i$. Using identities similar to the Calabi-Yau case one obtains:
\[
K = -\log \left[ \frac{\int_X \sqrt{\det g}}{\int_X \sqrt{\det g_0}} \right]
\]
(2.11)
where $g_0$ corresponds to the basepoint $\Phi_0$. We have chosen the additive constant in $K$ so that the 4D Newton constant is $\pi\ell^9/(4\int_X \sqrt{\det g_0})$.

The superpotential is zero in the low energy supergravity approximation. This is proved using an argument analogous to that in [5]. As shown in [13,16] the superpotential $W$ is a global holomorphic section of a negative Hermitian line bundle $L \to M$ with
\[
c_1(L) = -\frac{i}{2\pi} \partial \bar{\partial} \log \|W\|^2 := -\frac{i}{2\pi} \partial \bar{\partial} \log e^K|W|^2.
\]
(2.12)
In the absence of membrane instantons there is a continuous Peccei-Quinn symmetry $C \to C + \omega$, $\omega \in \mathcal{H}^3(M_{11};\mathbb{R})$. On the other hand, the overall volume $\text{vol}(X)$ is equal to $\frac{1}{7} \int \Phi \wedge *\Phi$, so the energy expansion is an expansion in inverse powers of $\text{Im}(C + i\ell^{-3}\Phi$).

Now we use holomorphy of $W$ to conclude that there are no corrections.

On the other hand, membrane instantons from Euclidean M2 branes wrapping 3-cycles $\Sigma \subset X$ violate the PQ symmetry, breaking it to some discrete subgroup of $\mathcal{H}^3(M_{11};\mathbb{Z})$. Thus membrane instantons can induce a superpotential. We will show that indeed the contribution to the superpotential of a rigid supersymmetric rational homology sphere in the $G_2$ manifold is
\[
\Delta W \propto |H_1(X;\mathbb{Z})| \exp[i\int_{\Sigma} (C + i\Phi)]
\]
(2.13)
\footnote{Again, the subtlety mentioned in footnote 2 above is important at this point. The axion terms associated with $C$ are only really well-defined when combined with the one-loop Rarita-Schwinger determinant, as in [7].}
The proportionality constant (independent of $C, \Phi$) is positive and associated with bulk supergravity determinants. A more precise result for this constant awaits the solution of some conceptual problems described in the next section.

We should also worry about 5-brane instanton effects. In [12] (II. Proposition 1.1.1) Joyce shows that $\pi_1(X)$ (and hence $H_6(X; \mathbb{Z})$) is finite if the holonomy is all of $G_2$. If the holonomy group is smaller the group $H_6(X; \mathbb{Z})$ might or might not be finite. Even when it is finite there might be interesting effects. However, generically, there will be no 5-brane instantons.

3. On the rules for computing membrane instanton effects

3.1. Divertimento

Salviati: Sagredo! When are we going to write that paper on membrane instantons!? Basta! Basta!

Sagredo: Salviati! How can we write a paper on M-theory instantons when we don’t understand the fundamental formulation of M-theory?!

Salviati: Piano, piano. Look. We have now learned the lesson of $p$-brane democracy. $M$-theory is a theory of fundamental M2-branes, or, by duality, a theory of fundamental M5-branes. Eleven-dimensional supergravity is just a collective excitation, as in string theory. Indeed, if we study the membrane solution in supergravity we find a timelike singularity in the metric! Thus, as for the string solution of ten-dimensional supergravity, the membrane is fundamental. Therefore, we simply need to follow the obvious generalization of the rules of string theory and reduce the computation of fermion bilinears in spacetime to a computation of the correlation function of vertex operators in the M2-brane theory. The gravitino vertex operator has even been computed for us in [1].

Sagredo: No. I cannot agree. There is no “obvious generalization of the rules of string theory.” No one has found the graviton by quantizing membranes. On the other hand, Matrix Theory is a proposal for a fundamental formulation. Furthermore, in matrix theory the M2-brane is a collective excitation composed of D0-branes. So let’s do the computation directly in Matrix Theory!

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6 With apologies to S. Coleman, and G. Galilei.
Salviati: For membranes wrapped on supersymmetric cycles in $G_2$ manifolds...? Prego: “to appear” or “in preparation”? 

Sagredo: O.K. Perhaps this is not necessary. Still, membranes are not fundamental, but simply collective excitations of some fundamental degrees of freedom in some fundamental formulation of M theory. The best we can do is take the supergravity soliton description seriously and think of the membrane modes as collective coordinates in the solitonic description. The computation of membrane instantons indeed involves a path integral of a 3D field theory, but this should simply be regarded as an integral over collective coordinates. After all, the saddle point technique is just a way of reducing the number of integrals you have to do, and we are doing nothing more than a saddle point approximation to the “path integral for $M$-theory” (if that is even the right formulation). The determinants of the saddle-point approximation are just the 11D supergravity determinants in the membrane solution background.

Salviati: Alas, Sagredo, I fear you have erred in two ways. First, recall that in ’t Hooft’s calculation of instanton effects in Yang-Mills theory the space of collective coordinates is finite dimensional, and the action is constant on this space of collective coordinates. Do you really want to identify all of the superembeddings $(X(s), \Theta(s))$ of the membrane as collective coordinates? This cannot be, for the action is not constant on this space. Second, the singularity in the membrane metric makes the supergravity determinants you propose to calculate ill-defined. One needs to specify boundary conditions in a second asymptotic regime: How do you propose to choose them!? 

Sagredo: What you say sounds correct. Of course $(X(s), \Theta(s))$ are not collective coordinates. Nevertheless, for small derivatives $\partial_s X$ they do describe low-energy excitations of the membrane. There are, after all, several scales in our problem: that of the 4D effective theory, $\ell_{4D}$, that of the Kaluza-Klein compactification $\ell_{KK}$, in addition to the 11D Planck scale $\ell$. In the domain $\ell \ll \ell_{KK} \ll \ell_{4D}$ the data of the UV degrees of freedom needed to make sense of the supergravity solution are summarized at long distances by the embedding coordinates $(X(s), \Theta(s))$. If we only consider the modes of the 11D gravitino $\Psi$ at scales $\ell_{4D}$ then there is a clear distinction between $\Psi|_{\Sigma}$ and $\Theta(s)$. This coupling of degrees of freedom from different scales is summarized by the BST supermembrane action.

Salviati: Allora, non è niente altro da fare. What other formalism could we use besides that advocated by BBS in [1]? So, you suggest we follow the Copenhagen interpretation?

Sagredo: Which is?

Salviati: Shut up and calculate!

Sagredo: Si! Andiamo!
3.2. The procedure in this paper

Unfortunately, the authors of this paper are just as confused as the Tuscan twosome. The rules for computing brane-instanton effects have not been clearly formulated. Indeed, upon reflection one finds many unanswered and vexing conceptual issues. These issues become important when one attempts to compute sub-leading effects in an instanton sector.

In this note we take a practical approach to the problem and follow the procedure implicitly followed in the original paper of Becker, Becker, and Strominger [1]. As in, for example, the calculations of [17,18], we will extract the effective superpotential by computing instanton-induced fermion bilinears and then comparing to the fermion bilinears in the low energy effective supergravity action. The latter are given by [19,16]

\[
e^{K/2} W \bar{\psi}_\mu \gamma^{\mu\nu} \bar{\psi}_\nu + \frac{i}{4} e^{K/2} (D_i W) g^{ij} (\partial_j \tau_{IJ})^* \bar{\lambda}^I \bar{\lambda}^J
\]

\[
- e^{K/2} (D_i D_j W) \chi^i \chi^j + e^{K/2} (D_i W) \bar{\psi}_\mu \gamma^\mu \chi^i + \text{cplxconj}
\]

(3.1)

where \( D_i \) is the metric covariant derivative for both \( M \) and \( L \).

In an instanton sector an \( M2 \) brane wraps a 3-cycle \( \Sigma \subset M_{11} \). The low energy effective theory of the membrane is a three-dimensional field theory with degrees of freedom described by a super-embedding: \( Z : \Sigma^{3|0} \rightarrow M^{11|32} \). Here \( \Sigma^{3|0} \) is the oriented brane worldvolume, and \( M^{11|32} \) is eleven-dimensional space-time, thought of as super-manifolds. The target space is equipped with an on-shell supergravity background.

Following [1] we formulate correlators in an instanton sector as follows. If we wish to compute the 2-point function of generic four-dimensional spacetime fermions \( F_1, F_2 \) at positions \( x_1, x_2 \) then we must compute a path integral within a path integral:

\[
\langle F_1(x_1)F_2(x_2) \rangle_{\Sigma} := \int [Dg_{\mu\nu}(x)D\psi_{\mu}(x)Dz^iD\lambda(x)DA(x)]e^{-S_{4D\text{augra}}}.
\]

\[
\cdot F_1(x_1)F_2(x_2) \cdot \int [DZ(s)]e^{-S_{M2}[Z(s);g,C,\Psi]}.
\]

(3.2)

The first path integral is that of the effective four-dimensional supergravity described above. The second path integral is that of a three-dimensional field theory on the \( M2 \)-brane, described below. The \( M2 \) brane couples to the background fields \( g,C,\Psi \) of 11D supergravity, and we substitute the Kaluza-Klein ansatz. While the above procedure surely receives many corrections when computing generic correlation functions, it might well be exact for the special correlators that determine the superpotential. It would be
highly desirable to have a firmer foundation for brane-instanton computations in M-theory. Nevertheless, we maintain our practical attitude and continue.

As in [1] the amplitude (3.2) is evaluated by contracting $F_1, F_2$ with the coupling of $\Psi$ to the brane through the “gravitino vertex operator” $\sim \int_{\Sigma} \bar{\Psi} V$. The contractions of $F_1, F_2$ with $\Psi$ are carried out by first substituting the Kaluza-Klein reduction (2.8). The fermion propagators are then amputated, as in [18]. We are then left with the computation of the two-point function $\langle V(s_1)V(s_2) \rangle$ of the “gravitino vertex operator” in the three-dimensional membrane theory.

In order to contribute to the superpotential the membrane instanton must leave at least two fermion zero-modes unbroken. In the following we will evaluate the contribution in the case when there are exactly two fermion zero-modes. The possible contribution of configurations with more fermion zero-modes is left for future work. In order to examine the fermion zero-modes, and compute determinants we need to understand in detail the expansion of the membrane Lagrangian in curved spaces. We describe this in the next section.

4. Effective actions for $M_2$ branes in curved spaces

As above, we regard an $M2$-brane as a dynamical object described by a super-embedding $Z: \Sigma^{3|0} \rightarrow M^{11|32}$. Separating bosonic and fermionic coordinates we have

$$Z^M(s) = \left( X^M(s), \Theta^\mu(s) \right)$$

where $\Theta(s) \in \Gamma[\mathcal{S}(TM_{11})|\Sigma]$ is a section of a pulled-back spinor bundle. These degrees of freedom are governed by the super-membrane action of Bergshoeff, Sezgin, and Townsend [20]. In Euclidean signature the action enters the path integral through

$$\exp \left\{ - \int_{\Sigma} d^3 s \left[ \ell^{-3} \sqrt{\det g_{ij}} - \frac{i}{3!} \epsilon^{ijk} \partial_i Z^M \partial_j Z^N \partial_k Z^P C_{PNM}(X(s), \Theta(s)) \right] \right\}$$

where

$$\Pi_i^A = \partial_i Z^M \epsilon^M_E^A$$

$$g_{ij} = \Pi_i^A \Pi_j^B \eta_{AB}$$

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7 Actually, there are also $\Psi^2$ interactions in the expansion of $S_{M2}$ in powers of $\Psi$. These in fact do contribute to fermion bilinears and are important in sorting out contact terms. These (important!) subtleties will not be discussed in the present paper.
The sign of the WZ term depends on the orientation of $\Sigma$.

The action (4.2) has bosonic and fermionic gauge invariances, $\text{diff}(\Sigma) \oplus \text{diff}_\kappa$, where $\text{diff}_\kappa$ refers to "$\kappa$-supersymmetry" [20]. The induced metric for any embedding of $\Sigma$ gives an orthogonal decomposition $TM_{11}|_\Sigma = T\Sigma \perp N$ in terms of tangent and normal bundles. This leads to a reduction of the structure group $\text{Spin}(1,10) \supset \text{Spin}(1,2) \parallel \times \text{Spin}(8) \perp$ under which $32_{\mathbb{R}} = (2; 8^-) \oplus (2; 8^+)$. Therefore the spinor bundles can be reduced as

$$S(TM_{11})|_\Sigma = S(T\Sigma) \otimes S^-(N) \oplus S(T\Sigma) \otimes S^+(N).$$

(4.4)

In this language, the meaning of $\kappa$-supersymmetry is that the physical degrees of freedom are in $S(T\Sigma) \otimes S^-(N)$. Hence, after fixing static gauge the physical degrees of freedom are given by

$$y \in \Gamma[N \to \Sigma]$$

$$\Theta \in \Gamma[S(T\Sigma) \otimes S^-(N)]$$

(4.5)

where $N$ is the normal bundle to $\Sigma$ in the full eleven-dimensional space-time, and $S^-(N)$ is the negative chirality spinor bundle associated to $N$.

In order to describe the action for the physical degrees of freedom in a way useful for our purposes we need to expand the action in powers of $y(s)\Theta(s)$. Let us first focus on the purely bosonic action. Choose a local coordinate system $X^M = (x^m, y^m)$ so that $\Sigma$ is described by the equation $y^m = 0$. In general we will denote tangent indices with a prime and normal indices with a double-prime. We choose coordinates so that the metric tensor on $(TM_{11})|_\Sigma$ is of the form

$$g|_\Sigma = \begin{pmatrix} h_{m'n'}(x) & 0 \\ 0 & h_{m''n''}(x) \end{pmatrix}.$$ 

(4.6)

Expanding the induced area we obtain the standard result that the area is stationary if $\Sigma$ is a minimal sub-manifold, i.e., if the trace of the second fundamental form is zero. Choosing static gauge $x^m(s) = s^i \delta_i^m$ we find that the expansion around a minimal sub-manifold is:

$$\int_{\Sigma} d^8 s \sqrt{\det[\partial_{m'}X^M \partial_{n'}X^N g_{MN}(s, y(s))]} = \int_{\Sigma} ds \sqrt{\det h_{m'n'}(s)}$$

$$+ \int_{\Sigma} ds \sqrt{\det h_{m'n'}(s)} \left( \frac{1}{2}(Dy)^2 - \left[ \frac{1}{2}R^m_{k'n'm'l'} + \frac{1}{8}Q^m_{k'n'}Q_{m'n'l'} \right] y^k y^l + O(y^3) \right)$$

(4.7)
where $D$ is the induced connection on the normal bundle, $Q_{m'n'n'}$ is the second fundamental form, and $R_{m'k'n'm'n'n'}$ is the ambient curvature tensor restricted to the brane.

To obtain the action for the fermions we expand the on-shell supergravity background in powers of the super-coordinate $\Theta$. The torsion constraint for 11D on shell super-space were found in [21]. The frame to order $\Theta$ including the gravitino but putting $G_{ABCD}| = G_{ABCD} = 0$ is:

$$e_M^A = \left( e_M^A - i \bar{\Psi}_M \Gamma^A \Theta - i \Gamma^A_{\mu} \Theta^\mu \right) + O(\Theta^3)$$

(4.8)

where $\omega^C_M$ is the Riemannian spin connection. In addition, we will need

$$E_M^A = e_M^A - \frac{i}{4} \Theta \Gamma^{ACD} \Theta \omega_{M,CD} + O(\Theta^3, \Psi)$$

(4.9)

$$\mathbb{C}_{MNP} = C_{MNP}(x) + i \left( \bar{\Theta} \Gamma_{MN} \Psi_P + \bar{\Theta} \Gamma_{PM} \Psi_N + \bar{\Theta} \Gamma_{NP} \Psi_M \right)$$

$$+ \frac{3i}{4} \bar{\Theta} \Gamma_{[MN} \Gamma^{CD} \Theta \omega_{P]},CD + O(\Theta^3)$$

(4.10)

$$\mathbb{C}_{M\nu}\rho = O(\Theta^2)$$

$$\mathbb{C}_{\mu\nu}\rho = O(\Theta^3)$$

Using the above results on the frame we find:

$$\Pi_i^A = e_M^A \partial_i X^M - i \bar{\Psi}_M \Gamma^A \left( D_i \Theta + \partial_i X^M \Psi_M \right) + O(\Theta^4, \Theta^3 \Psi)$$

(4.11)

$$\Pi_i^\alpha = (D_i \Theta^\mu) \delta^\alpha_\mu + \frac{1}{2} \partial_i X^M \Psi_M^\alpha + O(\Theta^3, \Theta^2 \Psi)$$

where $D_i \Theta$ is the pullback of the spin connection from the ambient space:

$$(D_i \Theta)^\mu := \partial_i \Theta^\mu + \partial_i X^M \omega^{CD}_M \frac{1}{4} \left( \Gamma_{CD} \right)^\mu_v \Theta_v.$$  

(4.12)

We now consider the expansion of (4.2) in fluctuations around $y = \Theta = 0$. Treating the brane as an elementary object the leading order term is simply

$$\exp\left[ i \int_\Sigma (C + i \ell^{-3} \text{vol}(h)) \right].$$

(4.13)

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\footnote{After working out these superspace expansions we learned of the paper [22] which also works out the frame to order $\Theta^2$. The conventions are different but up to numerical coefficients our expressions agree, at least up to order $O(\Theta^2 \Psi)$. See also [23].}
Expanding around a minimal sub-manifold and including the fermions we find the quadratic action

\[
\ell^{-3} \int_{\Sigma} ds \left[ \sqrt{\det h} \left( h^{ij}(D_i y^{m''})(D_j y^{n''}) h_{m''n''} - y^{m''} U_{m''n''} y^{n''} \right) \right. \\
- i\sqrt{\det h} h^{ij} e_i^a \left( \bar{\Theta} \Gamma^a D_j \Theta \right) + \frac{1}{2} \epsilon^{ijk} \bar{\Theta} \Gamma_{ij} D_k \Theta \right] 
\]  

(4.14)

where \( \Gamma_i := \partial_i X^M \Gamma_M \), and the covariant derivatives are those determined by the above bundles. (There are terms in the pulled-back spinor connection involving the second fundamental form. These can be shown to vanish using the fact that \( \Sigma \) is a minimal submanifold.) The “mass term” \( U \) depends on the curvatures and second fundamental form, as in (4.7).

In Euclidean space the two fermion kinetic terms are equal, to lowest order in interactions. If we keep the gravitino in the expansion (4.8)-(4.9) we find the coupling to the background gravitino is

\[
i\ell^{-3} \int_{\Sigma} ds \sqrt{\det h} \bar{\Psi}_M V^M + \mathcal{O}(\Psi^2), \]

where \( V^M \) is the gravitino vertex operator

\[
V^M = h^{ij} \partial_i X^M \partial_j X^N \Gamma_N \Theta + \frac{1}{2} \epsilon^{ijk} \partial_i X^M \partial_j X^N \partial_k X^P \Gamma_{PN} \Theta. 
\]  

(4.15)

Let us now consider the supersymmetries of the action. These will be quite useful in sorting out topological field theories in the next section. The unbroken super-isometries of the on-shell 11D background become global supersymmetries of the M2 theory. The super-isometries are defined by a super-vector \( K^M = (k^M, \kappa^\mu) \) satisfying the equations:

\[
\delta Z^M = -K^M \\
\delta E^A_M = K^N \partial_N E^A_M + \partial_M K^N E^A_N = 0 
\]  

(4.16)

Setting the background \( \Psi = 0 \) the isometry can be given as an expansion in powers of \( \Theta \):

\[
k^\mu = \epsilon^\mu + (\Delta \epsilon)^\mu \\
= \epsilon^\mu - i(\Gamma_{CD} \Theta)^\mu (\bar{\Theta} \Gamma^M \epsilon) \omega_M^{CD} + \mathcal{O}(\Theta^4 \epsilon) \\
k^N = i\bar{\Theta} \Gamma^N \epsilon + (\Delta k)^N \\
= i\bar{\Theta} \Gamma^N \epsilon + \mathcal{O}(\Theta^3 \epsilon) 
\]  

(4.17)

where \( \epsilon \) is a covariantly constant spinor for the bosonic (\( \Theta = 0 \)) background. Thus, including the gauge transformations, the action is invariant under the following transformations:

\[
\delta \Theta(s) = \epsilon + (\Delta \epsilon) + \delta_\kappa \Theta + v^i(s) \partial_i \Theta + \mathcal{O}(\Theta^4 \kappa) \\
\delta X^M(s) = i\bar{\Theta} \Gamma^M \epsilon + (\Delta k^M) - i\bar{\Theta} \Gamma^M \delta_\kappa \Theta + v^i(s) \partial_i X^M + \mathcal{O}(\Theta^3 \epsilon) \\
\delta_\kappa \Theta := (1 + \Gamma || [X, \Theta]) \kappa(s) 
\]  

(4.18)
where \( v^i(s) \) is a diffeomorphism of \( \Sigma \), \( \kappa(s) \) is an arbitrary spinor and
\[
\Gamma_{\parallel}[X, \Theta] := \frac{1}{3!} \frac{1}{\sqrt{\det g}} \epsilon^{ijk} \Pi_i^A \Pi_j^B \Pi_k^C \Gamma_{ABC}
\]
is the induced Clifford volume element. One easily shows that \( \Gamma_{\parallel}[X, \Theta]^2 = -\frac{\det g}{\det g} \), so, working in Minkowskian signature \( \frac{1}{2}(1 + \Gamma_{\parallel}) \) is a projector. To see how these transformations act on physical degrees of freedom we fix \( \text{diff}(\Sigma) \) symmetry by choosing static gauge. Then, to fix \( \kappa \) supersymmetry we choose a decomposition of spinors under \( \text{Spin}(1,10) \supset \text{Spin}(1,2) \times \text{Spin}(8) \) such that \( 32 = (2,8^-) \oplus (2,8^+) \). We choose a representation of the Clifford algebra, and spinor conventions as in appendix A. In this representation the spinor degrees of freedom may be written as:
\[
\Theta = \begin{pmatrix} \Theta^{Aa}_1 \\ \Theta^{A\dot{a}}_2 \end{pmatrix}.
\]
For the configuration \( y = \Theta = 0 \) the induced Clifford volume is
\[
\Gamma_{\parallel}[y = 0, \Theta = 0] = 1_2 \otimes \begin{pmatrix} -1_8 & 0 \\ 0 & 1_8 \end{pmatrix}
\]
so we fix the gauge by taking:
\[
\Theta = \begin{pmatrix} \Theta^{Aa}_1(s) \\ 0 \end{pmatrix}.
\]

In order to get something useful from (4.18) we introduce the low energy expansion which is an expansion in degrees defined by: \([s^i] = -1, [y^{m\prime\prime}] = 0, [\partial_i y^{m\prime\prime}] = 1, [\Theta_1] = 1/2\). (Note that this differs from the energy expansion in bulk supergravity, where \([\Theta] \) has weight \(-1/2\).) The way in which we assign degrees to the geometrical objects \([e^A_M], [\omega^AB_M], [R_{MNPQ}] \) requires discussion. In general we must assign degree one to \( \frac{\partial \Theta}{\partial s^i} \). If some set of normal directions has a direct product structure then we can consistently assign degree zero to \([e], [\omega], [R] \) in those normal directions. However, if there are off-diagonal terms in the metric, e.g. second fundamental forms or connections on the normal bundle, then we must assign degree one to \([\omega^AB_M], [R_{MNPQ}] \).

We now decompose the covariantly constant spinor in terms of \( \text{Spin}(8) \) chirality as in (4.20):
\[
\epsilon = \begin{pmatrix} \epsilon^{Aa}_1 \\ \epsilon^{\dot{A}\dot{a}}_2 \end{pmatrix}
\]
The spinors \( \epsilon_1 \) and \( \epsilon_2 \) generate a broken and an unbroken supersymmetry in the membrane theory, respectively. We consider the supersymmetry parameter to have degree \(+1/2\).
Then, the degree expansion of the broken $\delta_1$, and unbroken $\delta_2$ supersymmetries takes the form
\[
\delta_1 = [\delta_1]^0 + [\delta_1]^2 + \cdots \\
\delta_2 = [\delta_2]^1 + [\delta_2]^2 + \cdots
\] (4.23)
where the superscript denotes the change in degree. Explicitly, the broken supersymmetry is given by
\[
\delta_1 y^{m''} = 0 \\
\delta_1 \Theta_1 = \epsilon_1 + (\Delta \epsilon_1)_1 \\
(\Delta \epsilon_1)_1 = -\frac{i}{4}(\bar{\Theta}_1 \gamma^a'' \epsilon_2)(\omega^c_d'' \gamma_{(\gamma} c'' \gamma_{d') \Theta_1} + \omega^c_d'' \gamma_{c') \gamma_{d') \Theta_1}) \\
\] (4.24)
while the unbroken supersymmetry is
\[
\delta_2 y^{m''} = -i(\bar{\Theta}_1 \gamma^a'' \epsilon_2)e_{a''}^{m''}(s, y(s)) \\
\delta_2 \Theta_1 = -[B]^1 \epsilon_2 + (\Delta \epsilon_2)_1 \\
(\Delta \epsilon_2)_1 = \frac{i}{4}(\Theta_1 \gamma^a'' \epsilon_2)(\omega^c_d'' \gamma_{(\gamma} c'' \gamma_{d') \Theta_1} + \omega^c_d'' \gamma_{c') \gamma_{d') \Theta_1}) \\
[B]^1 = -(\partial_i y^{m''} e_{m''}^{a''} + \frac{i}{2}(\bar{\Theta}_1 \gamma^c' \gamma^d'' \bar{\gamma} \Theta_1)(\omega_i)^{c'd''}) \tau^i \gamma_{a''}
\] (4.25)

The unbroken transformations are closely related to the transformations of a supersymmetric 3d sigma model. In particular note the terms:
\[
\delta_2 y^{m''} = -i(\bar{\Theta}_1 \gamma^a'' \epsilon_2)e_{a''}^{m''}(s, y(s)) \\
\delta_2 \Theta_1 = (\partial_i y^{m''} e_{m''}^{a''} \tau^i \gamma_{a''} \epsilon_2 - \frac{1}{4} \omega^c_d'' \gamma_{c') \gamma_{d') \Theta_1}(\delta_2 y^{m''})
\] (4.26)
The extra terms in (4.25) are related to second fundamental forms and connections on the normal bundle. They do not occur in the usual treatment of supersymmetric sigma models because the metric on the world-volume and the metric in target space is a product metric in the standard sigma model.

5. Membranes in a $G_2$ manifold

We now consider manifolds of the form $M_4 \times X$ where $M_4$ is hyperkähler and $X$ has $G_2$ holonomy. In order to induce a superpotential the brane instanton should leave at least 2 unbroken supersymmetries. A cycle $\Sigma \subset X$ such that $X$ has a covariantly constant spinor $\epsilon$ with $(1 + \Gamma_{\parallel}[X, \Theta = 0]) \epsilon = 0$ is a “supersymmetric cycle” [1]. A key result of [1][24] is that such cycles are the “calibrated sub-manifolds” of Harvey and Lawson [25]. Therefore,
we specialize to the case of an associative 3-fold $\Sigma$ in $X$. Under these circumstances the formulae of the previous section simplify considerably.

We begin this simplification by noting that in the above circumstances there is a reduction of the normal bundle structure group

$$\text{Spin}(8) \ni \text{Spin}(4)_{1,2,3,4} \times \text{Spin}(4)_{1,2,3,4} \quad (5.1)$$

Under this reduction the general Spin(8) spinor can be written as

$$\psi = ((\psi_{-\alpha}^Y, (\psi_{+\alpha})_\dot{Y}; (\psi_{+\dot{\alpha}}^Y, (\psi_{-\dot{\alpha}})^\alpha_\dot{Y}) \quad (5.2)$$

where the $+,-$ refers to the chiralities in the first and second Spin(4) factors. In this notation, after we fix $\kappa$-symmetry, and restore the $\text{Spin}(1,2)$ spinor index $\alpha$ for the structure group of the spinor bundle $\mathcal{S}(T\Sigma)$, we get the gauge fixed spinor degree of freedom:

$$\Theta = ((\Theta_{-\alpha})_\alpha^A, (\Theta_{+\dot{\alpha}}^Y, (\Theta_{-\dot{\alpha}})^\alpha_{\dot{A}}; 0, 0) \quad (5.3)$$

Now we must review a few consequences of $G_2$ holonomy, in down-to-earth terms. On a $G_2$ manifold the $\text{Spin}(7)$ structure group of the tangent bundle is reduced to $G_2$. We choose an identification

$$(\Phi, TX) \cong (\varphi_0, \mathbb{R}^7 = \mathbb{O}) \quad (5.4)$$

where in a local orthonormal frame $e^{1,2,\ldots,7}$ for $T^*X$ we have

$$\varphi_0 = e^{123} + e^{145} + e^{176} + e^{246} + e^{257} + e^{347} + e^{653}. \quad (5.5)$$

We may identify the spin representation of spin(7) with the octonions $\mathbb{O}$. These, in turn, are identified with pairs of quaternions $\mathbb{H}$. Along an associative 3-fold we have a covariantly constant identification of $T_s\Sigma \subset T_sX$, $s \in \Sigma$, with $(\text{Im}\mathbb{H}, 0) \subset \text{Im}\mathbb{O}$. The structure group of $T\Sigma$ has spin indices $A, B, \ldots, = 1, 2$ while that of the normal bundle has $SU(2) \times SU(2)$ spin indices $Y, \dot{Y}$, respectively. The corresponding spin bundles are denoted $E_\pm$. The spin connection on $X$, restricted to $\Sigma$, may be written as

$$\omega = \begin{pmatrix} (\omega_{\parallel})^{a'b'} & \omega^{a'b''} \\ -\omega^{b'a''} & (\omega_{\perp})^{a''b''} \end{pmatrix} \quad (5.6)$$

where $a', b' = 1,2,3$ are tangent frame indices and $a'', b'' = 4,5,6,7$ are normal bundle orthonormal frame indices. This corresponds to the decomposition of the adjoint of $so(7)$
under the $so(3)_{∥} \oplus so(4)_{⊥}$ subgroups. The adjoint of $so(4)$ can be further decomposed in terms of its self-dual and anti-self-dual parts with spinor indices $(\omega_+)_{Y,Y'}$ and $(\omega_-)_{Y,Y'}$. Similarly, the off-diagonal connection has spinor indices $(\omega)_{Y,Y'AB}$ symmetric in $AB$. Writing a spinor of $spin(7)$ as

$$\psi = ((\psi^-)_A^Y, (\psi^+_A)^Y)$$

(5.7)

( regarded as a pair of quaternions) the covariantly constant spinor is $\vartheta = (\delta^A_Y, 0)$. Writing out the covariant constancy condition $\nabla_m \vartheta = 0$ we find the spin connection satisfies

$$(\omega_{∥})_A^Y - (\omega_-)_A^Y = 0$$

(5.8)

and

$$\omega_{YY}^A = 0,$$

(5.9)

that is, $(\omega)_{Y,Y'AB}$ is totally symmetric in the undotted indices. The connection $\omega_+$ is unconstrained. These equations describe the decomposition of the adjoint $14$ of $g_2 \subset spin(7)$ under the $so(3) \oplus so(4)$ subgroup of $spin(7)$.

The condition (5.8) is very important and is one concrete way to understand the connection of brane actions to topological field theory, (This was predicted in [27] based on R-symmetry considerations.) The identification of connections for the tangent bundle $\omega_{∥}$ and the normal bundle, $\omega_-$ (with “R-symmetry” structure group) is a standard approach to formulating the procedure of “topological twisting” [28]. (See [29,30,31] for reviews.)

It follows from (5.8) that the $G_2$ structure allows us to identify the spin groups transforming the indices $A$ and $Y$. Thus we may may trade in the fermions

$$(\Theta_-)_{\alpha}^{A Y} = \eta_{\alpha}\epsilon^{A Y} + \chi_{j\alpha}(i\tau^j)^{A Y}$$

(5.10)

for zero-forms $\eta$ and 1-forms $\chi$ on $\Sigma$, as is familiar in topological field theory. (These still carry an index $\alpha$ since they are spinors on $M_4$.) Moreover, again because of (5.8), the normal bundle directions for motion of $\Sigma$ within $X$, $y^\tilde{m}$, can be written as a bispinor $y^{A\tilde{Y}}$, that is, as sections of $S(T\Sigma) \otimes E_+$. Thus, the physical degrees of freedom of the $M2$ brane naturally split into two multiplets reflecting the reduction of the structure group (5.1).

The first multiplet, whose bosonic degrees of freedom describe normal motion in $M_4$ is

$$(y^{\alpha\tilde{\alpha}}, \eta_{\alpha}, \chi_{j\alpha})$$

(5.11)
We call this the “Rozansky-Witten multiplet” (or just the RW multiplet). Denoting $\nu_{\dot{\alpha}} \dot{Y} = (\Theta_{++})_{\dot{\alpha}A}^A$ the second multiplet is $(y^A, \nu_{\dot{\alpha}} \dot{Y})$. We call this the “McLean multiplet.”

We will now explain why these names are appropriate. Because of the $G_2$ holonomy, the covariantly constant spinor on $M_4 \times X$ must have the form:

$$\epsilon = (\epsilon_1; \epsilon_2) = (\epsilon^- \varepsilon^{AY}, 0; \epsilon^+ \varepsilon^{AY}, 0)$$

(5.12)

where $\epsilon^-, \epsilon^+$ are negative and positive chirality covariantly constant spinors on the hyperkähler manifold $M_4$, respectively. As in the previous section, these give linear and nonlinear supersymmetries.

Under the nonlinear supersymmetry all fields transform to zero except $\delta \eta_{\alpha} = \epsilon^-_{\alpha}$. Under the linear supersymmetry the RW multiplet transforms as:

$$\delta_{\epsilon} y^{\alpha_\beta} = -2i \eta^\alpha \epsilon^\beta_-$$

$$\delta_{\epsilon} (\Theta_{--})_{\alpha}^{AY} = -(\partial_{\alpha} y)^{(\beta)} (\omega_{\beta})_{\alpha}^{AY}$$

(5.13)

while from (4.25) we find the McLean multiplet transforms as:

$$\delta y^{AY} = i \nu_{\dot{A}} \varepsilon^{AY} \epsilon_{+\dot{\alpha}}$$

$$\delta \nu_{\dot{\alpha}} \dot{A}^{AY} = - (D y)^{AY} \epsilon_{+\dot{\alpha}}$$

(5.14)

The transformations (5.13) are just the transformations of Rozansky and Witten [32]. Moreover, from the second line of (5.14) we immediately recover the fact that the tangent space to the moduli of associative threefolds at $\Sigma$ is identified with the space of zero-modes of the twisted Dirac operator. This is Theorem 5.2 of [33].

The quadratic action likewise splits up as a sum for the Rozansky-Witten and McLean multiplets. The action for the RW multiplet is [32]:

$$\int \Sigma dy^{\alpha\dot{\alpha}} \wedge *dy_{\alpha\dot{\alpha}} + \epsilon_{\alpha\beta} (\chi^\alpha \wedge *d\theta^\beta + \chi^\alpha \wedge d\chi^\beta) + R_{\alpha\beta\gamma\delta} \eta^\alpha \chi^\beta \wedge \chi^\gamma \wedge \chi^\delta + \ldots$$

(5.15)

where $R_{\alpha\beta\gamma\delta}$ is the self-dual part of the curvature, and we have not been careful about numerical coefficients. We have only shown terms up to degree 2. Although we only worked out the quadratic action in (4.14), invariance under (5.13) requires the curvature term.

Similarly, the quadratic fluctuations for the McLean multiplet are given by:

$$\int [(D_{E_+} y)^2 + \bar{\nu} D_{E_+} \nu + \ldots]$$

(5.16)

9 Equation 2.18 in [32] has a misprint.
here $\mathcal{D}_{E^+}$ is the Dirac operator twisted by $E_+$ (and again, we have not been careful about numerical coefficients). The bosonic terms in the action correspond to Theorem 5.3 of [33]. It follows from (4.23) that the higher order terms in (5.16) are of degree 3.

It is interesting to note what happens when we specialize the holonomy of $X$ further and take $X = Z \times S^1$ where $Z$ is a Calabi-Yau 3-fold and $\Sigma \subset Z$ is a special Lagrangian submanifold. Then the Spin(4) structure group of $\mathcal{N}(\Sigma \hookrightarrow X)$ is reduced to the Spin(3) structure group of $\mathcal{N}(\Sigma \hookrightarrow Z)$. The group $Spin(3) = SU(2)$ is embedded diagonally in $Spin(4)$ so we have another identification of connections $\omega_+ = \omega_-$. We can therefore introduce a topological twisting of the McLean multiplet

$$\Theta^{\dot{A}Y} = \bar{\eta}^{\dot{\alpha}} \varepsilon^{AY} + \bar{\chi}^{\dot{\alpha}}(i\tau^j)^{AY}. \quad (5.17)$$

The first order deformations of $\Sigma$ in $Z$ are now given by zeromodes $\bar{\chi} \in \mathcal{H}^1(\Sigma; \mathbb{R})$. In this way we can reproduce McLean’s theorem 3-6 on deformations of special Lagrangian submanifolds of $Z$ [33].

6. Computation of the superpotential

6.1. Zeromodes and Determinants

In this section we compute the contributions from M2 instantons to the superpotential for $G_2$ compactification for the choice $M_4 = \mathbb{R}^4$. We have seen that at degree 2 the M2 theory is a product of a theory for the RW multiplet and for the McLean multiplet. We now describe the zeromodes and one-loop determinants for these multiplets.

The zero-modes in the path integral for the RW multiplet are: (1.) The constant scalars $y^{\dot{\alpha}0}(s) = y^{\dot{\alpha}}_0$. These simply correspond to the position of the susy 3-cycle in $\mathbb{R}^4$. (2.) The 2 constant fermion zero-mode partners $\eta^0(s) = \vartheta^0_0$. (3.) Harmonic 1-form zero-modes for $\chi$. There are $b_1(\Sigma)$ such linearly independent zero-modes.

As discussed in [32] the path integral is a measure on the space of zero-modes $\Lambda^{max}(2H_0(\Sigma; \mathbb{R})) \otimes \Lambda^{max}(H_1(\Sigma; \mathbb{R}))$. The measure is the product of the natural measure given by the metric with the $\zeta$-regularized determinants. As discussed in [32] the path integral takes the form

$$\frac{\det' L_{-}}{(\det' \Delta^0)^2}. \quad (6.1)$$
where $\Delta^{(0)}$ is the Laplacian on scalars and $L_-$ is an operator appearing in Chern-Simons perturbation theory $^{[34]}$. The absolute value of the determinants is the Ray-Singer torsion. The phase is more subtle and is discussed below.

We now turn to the McLean multiplet. The path integral gives a measure on the space of zero-modes $[dy_0^{A\bar{Y}} dv_0]$ and therefore gives a measure on the moduli space of supersymmetric 3-cycles deformable to $\Sigma$. In this case the quadratic fluctuation determinant is just the phase of the Dirac determinant

$$ \det'(D_{E+}) \over \sqrt{\det' D_{E+} D_{E+}}. \quad (6.2) $$

Again this phase is subtle, and discussed below.

Let us now specialize to $\Sigma$‘s which are rigid and topologically rational homology spheres. Then the measure becomes

$$ \int dy^\alpha_0 d\bar{y}^\alpha_0 |H_1(\Sigma; \mathbb{Z})| e^{i\alpha(\Sigma; N)} \exp[i \int_{\Sigma} (C + i\ell^{-3}\text{vol}(h))]. \quad (6.3) $$

Here we have used the standard result that the Ray-Singer torsion of a rational homology sphere is just the order of the finite group $H_1(\Sigma; \mathbb{Z})$ $^{[32]}$. Here $e^{i\alpha(\Sigma; N)}$ is the phase from the fermion determinants.

6.2. The phase of the one-loop measure

The phase of the Dirac determinant coupled to a vector bundle in an odd-dimensional space-time is discussed in $^{[35,36,37,38,39]}$. One of the standard approaches to the subject is to use a Pauli-Villars regulator. This leads to the result

$$ \frac{\det' D_{E}}{|\det' D_{E}|} = \exp\left[\mp \frac{i\pi}{2} \eta(D_{E})\right] \quad (6.4) $$

where $\eta$ is the Atiyah-Patodi-Singer (APS) invariant. The sign in the exponent depends on the choice of sign of the mass of a Pauli-Villars regulator.

If we use this result in the present problem then we encounter a problem. The resulting expression for the superpotential violates holomorphy of $W$ as a function of $C + i\Phi$, because the $\eta$-invariants such as (6.4) are nontrivial functions of the $G_2$ structure, but are not functions of $C$. One can approach this problem by adding local Chern-Simons terms in $\omega_\pm$ to cancel the holomorphy anomaly, but then one runs into thorny issues related to correctly defining the Chern-Simons terms (which have half-integer coefficients). A better
way to define the phase of the one loop determinants has been described by Witten in \[7\] in the case of D1 instantons. His method is easily adapted to our problem.

Quite generally, when considering membrane instantons we have a closed oriented three-manifold $\Sigma$ in an oriented, spin, 11-manifold $M_{11}$ with Ricci flat metric $g$ and real rank 32 spin bundle $S(TM_{11})$. Restricting $S(TM_{11})$ to $\Sigma$ we have the splitting \[4.4\]. Consider the Dirac operator $\slashed{D}$ on $\Sigma$ coupled to the induced connection on $S(T\Sigma) \otimes S^-(N)$ from the ambient metric $g$. The phase of the membrane determinant involves

$$\text{Pfaff}(\slashed{D} S^-(N)) \exp[i \int_{\Sigma} (C + i\Phi)]$$

As stressed in \[9\][7], because of the subtle geometrical nature of the field $C$, neither factor in \[6.5\] is separately well-defined, in general. The fermion determinant is not well-defined because of global anomalies. Nevertheless, the product is well-defined. (In the example of a $G_2$ manifold each factor is well-defined, but not canonically so.)

Let us choose a basis of cycles $\Sigma_i$ for $H_3(X; \mathbb{Z})$. Then the phases $e^{i\theta_i}$ for the expression \[6.5\] evaluated for the cycles $\Sigma_i$ constitutes part of the data determining the $M$-theory background. Once this data has been specified the phases of all other membrane instantons can be expressed in terms of $e^{i\theta_i}$. To see this consider an arbitrary sum of cycles $\Sigma_i$ trivial in homology, that is, such that there is an open 4-manifold such that $\sum [\Sigma_i] = \partial M_4$. By a natural extension of the result of \[10\] there should be a canonical trivialization of the Pfaffian line bundles $T : \otimes_i \text{PFAFF}(\slashed{D} S^-(N), \Sigma_i) \to \mathbb{R}$. Then the phase of the product of the determinants for $\Sigma_i$ is that of

$$T \left[ \otimes_i \text{Pfaff}(\slashed{D} S^-(N), \Sigma_i) \right] \exp[i \int_{M_4} G]$$

and the expression \[6.6\] may be shown to be independent of the choice of cobordism $M_4$.

The upshot of this discussion is that, if we are willing to ignore the correct geometrical status of $C$ then we can absorb the phase $e^{i\alpha(\Sigma,N)}$ in \[6.3\] into the definition of the object $\exp[i \int_{\Sigma} C]$.

6.3. Computing $W$

Let us now turn to the computation of the fermion two-point functions in the four-dimensional effective action. The computation of gaugino and gravitino mass terms is tricky because of contact terms. The cleanest term to evaluate is the part of the chiral multiplet mass term proportional to $\partial_i \partial_j W$. Computing the two-point function $\langle \chi^i(x_1) \chi^j(x_2) \rangle$
for $|x_1 - x_2| \gg \ell$ using the instanton approximation in (3.2) we find (after truncating the fermion propagators) the mass term:

$$-D_{\Sigma}^{\text{sugra}} \chi^i \chi^j v_i v_j \left| \frac{\det' L_{-}}{(\det' \Delta^{(0)})^2} \right| \exp[i \int_{\Sigma} (C + i\ell^{-3} \text{vol}(h))] \quad (6.7)$$

Here $D_{\Sigma}^{\text{sugra}}$ stand for the 11D bulk supergravity determinants of $g, C, \Psi$ are are discussed further below. We also have:

$$v_i = 4 \int_{\Sigma} \omega_i^{(3)} \quad (6.8)$$

and we have taken the harmonic three-forms $\omega_i^{(3)}$ to be in the 27 representation of $G_2,$ again to avoid subtleties with contact terms.

This computation shows that a rigid rational homology 3-sphere gives a contribution to the superpotential of:

$$\Delta W \propto D_{\Sigma}^{\text{sugra}} |H_1(\Sigma; \mathbb{Z})| \exp[i \int_{\Sigma} (C + i\ell^{-3} \Phi)] \quad (6.9)$$

where we have used the fact that $\Phi|_{\Sigma} = \text{vol}(h)$ for an associative 3-cycle. We only write $\propto$ because we have not been careful about the overall normalization of the Kähler potential. (This will involve a purely numerical factor with powers of 2 and $\pi.$) Note that, since we are working in a Green-Schwarz-like formalism we do not sum over spin structures on $\Sigma.$ Rather, the spin structure on $\Sigma$ is induced by that of the ambient supergravity background.

When we consider the supergravity determinants $D_{\Sigma}^{\text{sugra}}$ we run into some of the conceptual problems discussed in section three. If we treat the membrane as an elementary brane in the $G_2$ manifold $\mathbb{R}^4 \times X_7$ with no account of backreaction then we can simply take $D_{\Sigma}^{\text{sugra}}$ to be a constant, independent of $\Sigma.$ Therefore, if we treat the membrane as an elementary membrane in a smooth supergravity background, then we can choose an additive constant in the Kahler potential so that rigid membranes contribute to the superpotential

$$\Delta W = |H_1(\Sigma; \mathbb{Z})| \exp[i \int_{\Sigma} (C + i\ell^{-3} \Phi)]. \quad (6.10)$$

6.4. Further development

An important check on (6.9) will be the computation of other terms in (3.1). These other terms will be more difficult to compute because of contact terms. (For example, the gravitino mass term is entirely due to contact terms.)
The above discussion should also be extended to the case of supersymmetric 3-cycles with \( b_1(\Sigma) > 0 \) and to non-rigid supersymmetric 3-cycles. Since wrapped membrane instantons can sometimes be related to worldsheet instantons it is clear that in general nonisolated membranes can contribute to nonperturbative effects. It is quite likely that higher order interactions from the expansion of the DBI action can come down from the exponential and soak up the extra zero-modes. We expect that this will occur already at degree three in the low energy expansion.

Finally, one needs to address the issue of a multiple-cover formula. We give an example below which shows that in general there must be contributions from such multiple covers. Based on the known multiple-cover formula for the contribution of worldsheet instantons one may guess that a \( k \)-fold covering membrane enters as \( m(k, b_1(\Sigma)) e^{-k \text{vol}(\Sigma)} \), where \( m \) is a universal function, but we do not have much evidence for this.

7. Examples with rigid supersymmetric 3-cycles

In this section we give an example of a class of compactifications in which one can show that the effects we have discussed above really exist.

7.1. Barely \( G_2 \) manifolds and their moduli

We now focus on a very special class of \( G_2 \) manifolds which we call barely \( G_2 \) manifolds. These are, by definition, \( G_2 \) manifolds of the form:

\[
X = (Z \times S^1)/\mathbb{Z}_2
\]  

Here \( Z \) is a Calabi-Yau 3-fold. \( \mathbb{Z}_2 \) acts by \((\sigma_Z, -1)\). \( \sigma_Z \) is a real structure, by which we mean an anti-holomorphic involutive isometry of the Calabi-Yau metric taking

\[
\sigma_Z^*(J) = -J, \\
\sigma_Z^*(\Omega) = +\bar{\Omega}.
\]  

We will take \( \sigma_Z \) to act without fixed points so \( X \) is smooth. For generic holonomy of \( Z \) the holonomy is merely \( SU(3) \ltimes \mathbb{Z}_2 \). Nevertheless, there is only one covariantly constant spinor, so for many physical purposes \( X \) is a genuine \( G_2 \) manifold. Examples of smooth barely \( G_2 \) manifolds may be easily constructed from complete intersection Calabi-Yau

\[\text{[^{10}] These manifolds are discussed briefly in [12].}\]
manifolds in weighted projective spaces defined by polynomials with real coefficients. The case where $\sigma_Z$ does have fixed points can presumably be smoothed, as discussed in [12]. In such manifolds, some of the following considerations will still apply.

The cohomology of barely $G_2$ manifolds is easily computed in terms of that of $\mathbb{Z}$:

$$H^3(X; \mathbb{R}) = H^2(Z; \mathbb{R})^- \oplus H^3(Z; \mathbb{R})^+$$  

(7.3)

where the superscripts denote the $\pm$ eigenspaces under the action of $\sigma^*_Z$. Now suppose that $Z$ is a Calabi-Yau obtained from a hypersurface of a toric variety. Then $H^2(Z; \mathbb{R})^-$ can be interpreted as the space of Kähler deformations inherited from the ambient quasi-projective spaces, while $H^3(Z; \mathbb{R})^+$ may be identified with the complex structure deformations preserving the real structure. Thus, we expect that the Calabi-Yau’s with anti-holomorphic involution $\sigma^*_Z$ form a real sub-manifold of $M_{2,1,1} \times M_{1,1}$, of real dimension $h^{2,1}(Z) + h^{1,1}(Z)$,

$$M_{2,1}^\mathbb{R} \times M_{1,1}^\mathbb{R} \subset M_{2,1,1} \times M_{1,1}.$$  

(7.4)

Indeed we expect (7.4) to be a natural Lagrangian sub-manifold of the full CY moduli $M_{2,1,1} \times M_{1,1}$ (very much in the spirit of the discussion in [11]).

We can describe the moduli space of barely $G_2$ manifolds. For $G_2$ manifolds the tangent to moduli space is $H^3(X; \mathbb{R})$. We now see that the deformation space has a natural interpretation in terms of the Kähler and complex structure deformations preserving the isometry $\sigma^*_Z$. Note that $H^3_{-}$ has real dimension $h^{2,1}(Z) + 1$. The extra $(+1)$ dimension (relative to dim $M_{2,1}$) corresponds the normalization of the $(3,0)$ form $\Omega$. The latter is relevant since, in a local orthonormal frame the calibration must look like

$$\varphi_0 = e^{123} + e^{145} e^{176} + e^{246} + e^{257} + e^{347} + e^{653} = J \wedge e^1 + \text{Re}[\Omega]$$  

(7.5)

and the normalization of $\Omega$ is fixed by $\frac{1}{2} \Omega \wedge \Omega = \frac{1}{6} J \wedge J \wedge J$. Here we are using a real orthonormal frame $e^{1,2,\ldots,7}$ for $T^*X$ from which we construct an orthonormal frame for $T^{1,0}Z$ given by $w^1 = e^2 + ie^3, w^2 = e^4 + ie^5, w^3 = e^6 - ie^7$ and $\Omega = w^1 \wedge w^2 \wedge w^3$. The extra modulus can also be identified with the extra degree of freedom in the radius $R$ of the circle cotangent to $e^1$. In any case, the moduli space of barely $G_2$ manifolds is a real line bundle

$$\mathbb{R} \rightarrow M_{G_2} \rightarrow M_{2,1}^\mathbb{R} \times M_{1,1}^\mathbb{R}.$$  

(7.6)
A word of caution is required here. In applications to physics we would regard \( M_{G_2} \) as a real submanifold of the Kähler target \( \mathcal{M} \) for the chiral scalars. Such a description is valid near a large Kähler and complex structure boundary, but will receive corrections away from such boundaries because quantum corrections will change \( K \) and could even change the topology of the moduli space \( \mathcal{M} \).

7.2. Supersymmetric cycles

The associative 3-cycles in barely \( G_2 \) manifolds are also easily described. These fall into two classes analogous to the divisors of types b,a (respectively) in sec. 3 of [2]. The first class, which we refer to as “holomorphic 3-cycles” are of the form

\[
[\Sigma^{\text{hol}}] \equiv ([\Sigma_2^-] \times S^1)/\mathbb{Z}_2
\]

(7.7)

where \([\Sigma_2^-]\) is a holomorphic cycle in \( Z \) mapped to \(-[\Sigma_2^-]\) by \( \sigma_Z \). That is, \( \sigma_Z \) preserves \( \Sigma^{\text{hol}} \) setwise and is orientation reversing. These cycles will be rigid if \([\Sigma_2^-]\) is a rigid rational curve in \( Z \). If \( \Sigma_2^- \) is a rational curve then \( \Sigma^{\text{hol}} \) has the topology of the nontrivial circle bundle over \( \mathbb{R}P^2 \) and is hence a rational homology sphere. The second class, which we refer to as “Lagrangian 3-cycles,” are of the form:

\[
\Sigma^{\text{lag}} = \Sigma^+/\mathbb{Z}_2
\]

(7.8)

where \([\Sigma^+]\) is a cycle mapped to \(+[\Sigma^+]\) by \( \sigma_Z \). Examples of such cycles are provided by special Lagrangian spheres in \( Z \) on which \( \sigma_Z \) acts nontrivially. In appendix C we give an example of such a rigid Lagrangian 3-cycle in a barely \( G_2 \) manifold.

7.3. \( M \)-theory on barely \( G_2 \) manifolds

We now consider the superpotential for \( M \)-theory compactification on a barely \( G_2 \) manifold \( X \). Note that the real structure \( \sigma_Z \) is orientation reversing on the 6-cycle \( Z \), and therefore \( H_6(X;\mathbb{Q}) = 0 \). Thus, there are no 5-brane instantons in these compactifications. Therefore, the only other nonperturbative contributions (coming from wrapped branes) are from membranes, and this gives a sum over the two sets of cycles described above:

\[
W = W^{\text{hol}}(Z) + W^{\text{lag}}(Z)
\]

(7.9)

where \( W^{\text{hol}}(Z) \) is the sum over holomorphic curves in \( Z \) and \( W^{\text{lag}}(Z) \) is the sum over susy 3-cycles in \( Z \) with the above orientation properties. Note these can be expressed purely in terms of the data of \( Z \) and its holomorphic \((3,0)\) form.
8. Superpotentials in Type IIA compactifications

The compactifications we have discussed are closely related to compactifications of IIA string theory on $G_2$ manifolds to three dimensions. Once again, there will be instanton-induced superpotentials, arising from many new effects. In the first place, there can now be worldsheet instanton effects. Moreover, there will be D-brane instanton effects associated to $p = 0, 2, 4, 6$ branes wrapping nontrivial classes in $H_1, H_3, H_5, H_7$, respectively. If we make the mild assumption that $H_1(X) = H_2(X) = H_5(X) = 0$ then there are no worldsheet instanton effects, and no $D0$ or $D4$ instanton effects. The computation of $D2$ instantons must be equivalent to the above results since the M2 and D2 brane Lagrangians can be mapped into one another by a change of variables \[42\]. Thus, the main new unavoidable ingredient is the effect of $D6$ instantons.

Let us consider briefly the effects of singly-wrapped $D6$ instantons, temporarily relaxing the restriction $H^2(X) = 0$. The action for an elementary $D6$ brane wrapped on $X$ is

$$e^{i\left(\gamma + i\ell^{-7}\text{vol}(X)\right)}$$

(8.1)

where $\gamma = \int C^{(7)}$ and the volume is measured in $M$-theory units. This follows from the Born-Infeld action in IIA theory. It can also be obtained directly from $M$-theory using the identification between the D6 brane of type IIA supergravity and the Kaluza-Klein monopole of 11D supergravity, $TN \times X$, where $TN$ is the singly charged Taub-NUT space \[43\]. The metric at infinity asymptotes to the metric of $\mathbb{R}^3 \times S^1 \times X$. Putting a boundary $S^2 \times S^1 \times X$ near infinity where $S^2$ has radius $r$ we compute the $M$-theory Einstein-Hilbert action:

$$\lim_{r \to \infty} \left[ \frac{1}{(2\pi)^2\ell^9} \left( \int_{\mathbb{R}^3 \times S^1 \times X} \sqrt{g} R(g) + \int_{S^2 \times S^1 \times X} Q \right) - \frac{1}{(2\pi)^2\ell^9} \left( \int_{TN \times X} \sqrt{g} R(g) + \int_{\partial TN \times X} Q \right) \right] = \text{vol}(X)/\ell^7$$

(8.2)

where in the first term we have the flat metric on $\mathbb{R}^3 \times S^1$, and $Q$ is the second fundamental form (normalized as in (A.1)).

\[11\] For example, for a barely $G_2$ manifold it suffices to take $b_1(Z) = 0$ and $H^{2,+}(Z) = 0$. The latter condition follows if all the Kähler classes are induced from an ambient projective space.
In addition a single D6 brane carries an abelian super Maxwell theory. When the fieldstrength \( \frac{1}{(2\pi)^2} \mathcal{F} \) is in \( H^2(X; \mathbb{Z}) \) the Born-Infeld action gives a correction to the instanton action. Assuming the background IIA field \( C^{(5)} = 0 \), the correction is

\[
\frac{i}{8\pi^2} \int (C + i\ell^{-3}\Phi) \mathcal{F} \wedge \mathcal{F} - \frac{i}{48} \int (C + i\ell^{-3}\Phi) \wedge p_1(TX)
\]

where we have transcribed the result to the \( M \)-theory variables \( C, \Phi \). The imaginary part comes from the standard D-brane Chern-Simons coupling. In the second term we have added the real part dictated by holomorphy. Note that from [12], Lemma 1.1.2, \( \int \Phi \wedge p_1(TX) < 0 \), for \( G_2 \)-manifolds.

Now let us consider determinants and zeromodes. When \( \frac{\mathcal{F}}{(2\pi)} \) is harmonic there are exactly two fermion zeromodes (if \( X \) has only one covariantly constant spinor) and the instanton can contribute to the superpotential. The determinants for the fields in the super Maxwell multiplet are just

\[
\text{det}' \mathcal{D} \cdot \frac{\text{det}' \Delta^{(0)}}{(\text{det} \Delta^{(1)})^{1/2}} \cdot \frac{1}{(\text{det}' \Delta^{(0)})^{3/2}} = \frac{\text{det}' \mathcal{D}}{|\text{det}' \mathcal{D}|}
\]  

(8.4)

Here \( \text{det} \Delta^{(1)} \) is the Laplacian on one-forms and we assume there are no harmonic one-forms on \( X \). Equation (8.4) is established by noting that because \( R_{ij} = 0 \) on \( X \) we have \( \mathcal{D}^2 = \nabla^2 \) where \( \nabla \) is the spinor covariant derivative. Since \( S(TX) \cong \Omega^0(X) \oplus \Omega^1(X) \) on a \( G_2 \) manifold the spinor covariant derivative may be identified with the ordinary one and the above partition function reduces to a pure phase. As in our discussion of the phase of membrane determinants one could use Pauli-Villars regularization to get a phase in terms of the \( \eta \) invariant, \( \exp[\pm \frac{i\pi}{2} \eta(\mathcal{D})] \), where \( \mathcal{D} \) is the Dirac operator on \( X \). However, this then requires the addition of counterterms to cancel holomorphy anomalies. Again we can instead define the determinant to be real, up to a sign, and the sign ambiguity cancels against that of (8.1), as in [7].

Putting all these remarks together we find the contribution of the wrapped D6-brane is thus essentially a \( \Theta \)-function:

\[
Z_1(X) = e^{i\rho} \sum_{\lambda \in H^2(X; \mathbb{Z})} e^{i\frac{\pi}{4} \tau_{ij} \lambda^i \lambda^j}
\]

(8.5)

where

\[
e^{i\rho} = \exp \left[ i(\gamma + i\ell^{-7}\text{vol}(X)) - \frac{i}{48} \int (C + i\ell^{-3}\Phi) \wedge p_1(TX) \right]
\]

(8.6)
In addition to singly wrapped $D_6$ instantons there can be multiply-wrapped instantons. In $M$-theory these will be $A_n$ Taub-NUT singularities with instanton action

$$e^{in\left(\gamma + i\ell^{-7}\text{vol}(X)\right)}$$

(8.7)

The one-loop contributions $Z_n(X)$ for $n > 1$ could be extremely interesting and involve the analog of Donaldson theory partition functions for the octonionic instanton equations in seven dimensions [44,45,46,47,48]. It is worth noting that if $Z$ has a real structure then Hermitian Yang-Mills connections invariant under $\sigma_Z$ provide nontrivial examples of solutions to the nonabelian octonionic instanton equations on the associated barely $G_2$ manifold $X$. This gives the first example (of which we are aware) where these instanton moduli spaces are nonempty.

In addition there will be mixed $D_2 + D_6$ instanton effects. These correspond to "pointlike instantons" in the abelian gauge theory, but can be more directly analyzed by considering M2 branes wrapping $\Sigma \subset X$ in $M$-theory on $TN(n) \times X$ where $TN(n)$ is a (multi-)Taub-NUT space. Our analysis of section 5 continues to apply because $TN(n)$ is hyperKähler. By holomorphy, only one orientation of M2 brane can contribute to $W$. Then, in order to have two fermion zero-modes, the Taub-NUT space must be oriented so that its covariantly constant spinor is of chirality $\epsilon^\alpha$. Naively the path integral factorizes as the product of D2 and D6 instanton effects described below, but we expect D2 D6 interactions to spoil this.

The superpotential now takes the form:

$$W = \sum_{\Sigma \subset X} \tau(\Sigma) e^{i \int_{\Sigma} (C + i\Phi)} + \sum_{TN(n)} Z_n(X) + D2D6 \text{ instantons}$$

(8.8)

where in the first sum $\tau(\Sigma)$ is the path integral from RW and McLean multiplets, as described above.

9. A mathematical application: Counting supersymmetric 3-cycles

Let us now combine the above results with mirror symmetry. The family of barely $G_2$ manifolds is preserved by the mirror map at large complex and Kähler structure. That is, if $Z$ is a Calabi-Yau with a real structure $\sigma_Z$, then for zero axion (i.e. pure imaginary complexified Kähler form) the mirror $\tilde{Z}$ has the same properties. For toric Calabi-Yau manifolds this can be proven rather directly by using the explicit formulae for periods
and special coordinates given in [49]. It also follows from the gauged linear sigma model approach to mirror symmetry described in [50]. Physically the assertion is not at all surprising since the existence of a real structure on $Z$ implies CP invariance of heterotic string compactified on $Z$ [51]. We expect that if $\sigma_Z$ has no fixed points then neither does $\tilde{\sigma}_{\tilde{Z}}$ for a reason given below.

Let us now consider two mirror Calabi-Yau 3-folds $Z$ and $\tilde{Z}$. A choice of $Z$ is determined by a choice of complexified Kähler class $B + iJ$ and complex structure, the latter encoded in the holomorphic $(3,0)$ form $\Omega$. If $\tilde{\sigma}_{\tilde{Z}}$ acts freely then by strong mirror symmetry we have:

$$IIA\left[\frac{Z(B+iJ,\Omega)}{(\sigma_Z,-1)}; g_s; \ell_s\right] = IIA\left[\frac{\tilde{Z}(\tilde{B}+i\tilde{J},\tilde{\Omega})}{(\tilde{\sigma}_{\tilde{Z}},-1)}; \tilde{g}_s; \ell_{\tilde{s}}\right]$$

(9.1)

where the freely acting anti-holomorphic involutions $\sigma_Z, \tilde{\sigma}_{\tilde{Z}}$ are mapped into each other by mirror symmetry. Note that if $\sigma_Z$ acts freely then so should $\tilde{\sigma}_{\tilde{Z}}$. For, if $\tilde{\sigma}_{\tilde{Z}}$ did not act freely, then the fixed point locus could be resolved by blowing up $A_1$ singularities as in [12]. In that case, there would be an enhanced nonabelian $SU(2)$ gauge symmetry in the mirror theory. (We cannot use mirror symmetry to prove that $\tilde{\sigma}_{\tilde{Z}}$ acts freely because the reasoning would be circular.)

The superpotentials in the three-dimensional effective supergravities based on $Z, \tilde{Z}$ must be equal. Since we can take the volumes $V, \tilde{V}$ to be both large (by going to both large Kähler and large complex structure moduli of $Z$) we must have equality for the sum over 3-cycles. Since Kähler and complex structure moduli are exchanged by mirror symmetry we conclude that

$$W_{\text{lag}}(Z) = W_{\text{hol}}(\tilde{Z}).$$

(9.2)

It is widely expected that under mirror symmetry there should be a connection between holomorphic curves and Lagrangian 3-cycles. This has been discussed in [52] and is closely connected to the SYZ construction [53]. For further recent discussion see [54, 55, 56]. As far as we know, the “explicit” counting formula (9.2) is new. (Similar proposals were of course made in [1]. These concern the curvature of hypermultiplet moduli spaces. In order to make the proposal of [1] more explicit we would need to investigate the one-loop measures for several different kinds of brane-instantons.)
10. Heterotic Duals

It is interesting to consider the interpretation of membrane-induced superpotentials in the context of dual heterotic models. Some conditions for the existence of such dual pairs were investigated in [10, 57, 58, 59, 60]. Here we focus on barely \( G_2 \) manifolds \( X = (Z \times S^1)/\mathbb{Z}_2 \). If \( Z \) admits both a K3 fibration \( p : Z \to \mathbb{P}^1 \) and an elliptic fibration with section then

\[ M[Z \times S^1] = IIA[Z] = HET[S_H \times T^2, V] \]  

where \( S_H \) is a K3 surface and \( V \) is an \( E_8 \times E_8 \) gauge bundle. (See [61] for a review.) Taking a further quotient by \( \mathbb{Z}_2 \) we may expect \( M[Z/\mathbb{Z}_2] = HET[Z_H, V_H] \) for a Calabi-Yau \( Z_H \) and gauge bundle \( V_H \). We can give a heterotic string interpretation of the membrane instantons \([\Sigma^{\text{hol}}]\) and \([\Sigma^{\text{lag}}]\) of section seven by examining the coupling constant dependence of the instanton action, as in sec. 4 of [2].

Let us consider first \([\Sigma^{\text{hol}}]\). From (7.5) we find

\[ \int_{\Sigma^{\text{hol}}} \Phi = \frac{1}{2} \left( R/\ell \right) \left( \int_{\Sigma_{-2}} J \right) \]  

where \( R \) is the radius of the \( M \)-theory circle, and \( J \) is the Kähler class of \( Z \). Making the standard Weyl rescaling to the IIA string metric we get the action \( \frac{1}{2} \left( \int_{\Sigma_{-2}} J \right) /\ell_{IIA}^2 \) where the Kähler class is now with respect to the IIA string metric and \( \ell_{IIA} \) is the string scale. Now we choose a basis of Kähler forms for \( Z \) so that the complexified Kähler moduli \( \tau_H, y^i \) correspond under \((10.1)\) to the heterotic axiodil and Wilson lines. Under string duality, \( \tau_H \) is the Kähler class of a section \( \sigma : \mathbb{P}^1 \to Z \) [61]. Let \([\Sigma^i]\) be dual to \( y^i \). If \( [\Sigma_{-2}] = n_0 [\sigma(\mathbb{P}^1)] + n_i [\Sigma^i] \) then the instanton contributes an effect \( \Delta W \) which depends on vectormultiplet moduli of \( IIA[Z] \) according to:

\[ \exp \left[ i(n_0 \tau_H + n_i y^i) \right]. \]  

(10.3)

If \( n_0 = 0, n_i \neq 0 \) we may interpret these effects in the heterotic theory as worldsheet instantons localized in \( S_H \). If \( n_0 \neq 0, n_i = 0 \) we may interpret these effects as spacetime instantons, corresponding to small heterotic 5-branes. If both \( n_0, n_i \neq 0 \) we have mixed instantons.

Let us now turn to Lagrangian-type cycles \([\Sigma^{\text{lag}}]\). The action only depends on hypermultiplet moduli of \( IIA[Z] \) and in IIA units at large volume is given by [61]:

\[ \exp \left[ -\frac{1}{2g_s} \frac{\left| \int_{\Sigma} \Omega \right|}{\sqrt{i \int_{\Sigma} \Omega \wedge \overline{\Omega}}} + i \int C^{(3)} \right]. \]  

(10.4)
Here $g_s$ is the 4-dimensional string coupling for $IIA[Z]$, $\Omega$ is a holomorphic $(3,0)$ form on $Z$ and $C^{(3)}$ is now a RR potential. Further orbifolding by $\mathbb{Z}_2$ should not introduce dependence on the heterotic axiodil (part of a vectormultiplet) and hence we conclude that these instanton effects correspond to heterotic worldsheet instantons.

This discussion raises several questions. First, it would be nice to understand how the expression (10.4) becomes a holomorphic function of the complex structure and bundle moduli of $[Z_H, V_H]$. (This is partially answered in [62].) Second, it is interesting to note that in the gauged linear sigma model approach to $(0,2)$ models [63] the complexified Kähler moduli have a very different origin (FI terms) from the complex structure and bundle moduli (superpotential terms). It would be interesting to see if this distinction is related to the above distinction of types of membrane instantons. Finally, we note that, as in [3], it follows that world-sheet instantons can indeed destabilize $(0,2)$ theories. It should be interesting to reconcile (2.13) with the phenomena discussed in [75].

11. Open membrane instantons

Another interesting application of our results is to open membrane instanton effects. These effects are not well-understood, but should be important in learning about mirror symmetry through the SYZ construction [53], the physics of the D1D5 system (via the papers [66] [67]), and in obtaining a deeper understanding of $(0,4)$ models of supersymmetric black holes [68] [69]. They should also play an important role in the models of low energy physics discussed in [70] [71] [72] [73] [74] [75]. There are many nontrivial issues one must discuss in order to do detailed computations. We hope to return to these elsewhere and limit ourselves here to some brief preliminary discussion.

We consider the strongly coupled $E_8 \times E_8$ heterotic string, realized as $M$-theory on $\mathbb{R}^4 \times S^1/\mathbb{Z}_2 \times Z$. There can also be 5-branes wrapping $\mathbb{R}^4 \times \{x\} \times S$, where $S \subset Z$ is a holomorphic curve. There are now several kinds of membrane instantons. In addition to the closed 3-manifolds discussed thus far in this paper there are membranes ending on 5-branes and/or 9-branes. The general form of the superpotential will be

$$W = W_c + W_{55} + W_{59} + W_{99} + W_{gc}$$

(11.1)

Here $W_c$ is the superpotential arising from closed membrane instantons. $W_{gc}$ is the superpotential generated by strong infrared dynamics of the boundary gauge theories (discussed in [72] [73] [74]). $W_{99}$ correspond to the worldsheet instantons [70].
Let us now consider briefly open membranes ending on 5branes. The physical "gerbe connection" on the 5brane is \( C - d\beta \) where \( \beta \) is the chiral 2-form on the 5-brane worldvolume. It follows that the instanton action for a membrane ending on a 5brane is given by

\[
q_{\text{membrane}} := \exp \left[ i \left( \int_{\partial \Sigma} \beta + i \int_{\Sigma} \ell^{-3} \text{vol} \right) \right] \tag{11.2}
\]

The parameter (11.2) will play the role for membranes analogous to the complexified Kähler expansion parameter for open worldsheet instantons. In particular, consider the term \( W_{55} \) corresponding to membranes stretching between different 5-branes. Reduction of the chiral 2-form on each 5brane results in a complex scalar \( a_i + ix_i \) in a chiral multiplet, where \( x_i \) is the position of the 5-branes on the interval \( S^1/\mathbb{Z}_2 \). The superpotential \( W_{55} \) will come from membrane instantons stretching between holomorphic curves in the 5brane worldvolumes. These will have the topology \( \Sigma = S \times [0, 1] \) and therefore contribute

\[
W_{55} = \sum_{x_i > x_j} e^{[a_i - a_j + i(x_i - x_j)](z \cdot S)} t_{55}^{(ij)} \tag{11.3}
\]

Here \( z \cdot S \) is the chiral modulus associated with the size of the holomorphic cycle \( S \subset Z \) on which the membrane ends, \( z \cdot S := \int_S (B + iJ) \), where \( B = \iota(\frac{\partial}{\partial x})C \) is the superpartner of \( J \). We also expect that the coefficients \( t_{55}^{(ij)} \) will be essentially given by transition amplitudes in the RW and McLean topological field theories discussed above.

The expression (11.3) only holds for \( x_i - x_j \gg \ell \). This raises the interesting question of holomorphy since for the other order the instanton action must be suppressed by \( \sim \exp[-|x_i - x_j|] \). Thus there appears to be nonanalytic behavior on a real codimension one locus. The resolution of this problem is probably that there is a nontrivial multiple cover formula for the 55 instantons. Thus, we expect a series in \( q \) defined by (11.2) and different power series expansions dominate in the regions \( x_i - x_j \gg \ell \) and \( x_j - x_i \gg \ell \). Note that this implies there should be a pole in \( W_{55} \) so it certainly cannot always be neglected relative to other terms (like \( W_{gc} \)).

Similar considerations hold for \( W_{59} \). Toy models (i.e., choosing a simple ansatz for the Kähler potential) indicate that one can generate interesting potentials for the 5-brane position moduli.

\[\text{References to the substantial literature on this subject will be given in the hypothetical future paper mentioned above.}\]
12. Conclusions

12.1. Applications

One immediate application of the above results is that certain $G_2$ compactifications of $M$-theory are unstable quantum mechanically in the regime where our calculation is valid, that is for 3-cycles whose volume is large compared to the Planck scale. This class includes smooth $G_2$-manifolds which admit susy 3-cycles which are rational homology spheres. Note that we do not expect any further corrections to the superpotential from non-trivial low-energy dynamics since the low-energy gauge theory at generic points in the moduli space consists of $U(1)$ gauge theory with non-chiral matter. Thus, (as is hardly surprising), there is an M-theoretic Dine-Seiberg problem [76].

12.2. Potential applications

We think the mathematical applications for enumerating supersymmetric 3-cycles in Calabi-Yau 3-folds are very promising. We anticipate some fairly amazing identities for sums over supersymmetric 3-cycles weighted by topological invariants such as Ray-Singer torsion and the Casson invariant.

We hope that some of the technical results found above will be useful in further investigations of brane-induced instanton effects in other compactifications. As one example, it should be possible to be more explicit about the nonperturbative corrections to hypermultiplet geometry discussed in [1]. It follows, for example, that for a rigid special Lagrangian 3-cycle $\Sigma$ the factor $N$ in [1] eq. 2.49 is just the order of the finite group: $|H_1(\Sigma; \mathbb{Z})|$. Our methods suggest some interesting formulae for instanton/anti-instanton effects such as $D2\overline{D6}$ effects. Indeed, it follows from [32] that such effects will involve the Casson invariant of $\Sigma$. These effects contribute to the induced potential. On the other hand, we know from supergravity that

$$V = e^K \left( |DW|^2 - 3|W|^2 \right) \quad (12.1)$$

It is possible that by comparing expressions we will learn something about quantum corrections to the Kähler potential in $d = 4, \mathcal{N} = 1$ M-theory compactifications. Of course, this requires first an understanding of perturbative contributions to $K$. In any case, an understanding of the Kähler potential is essential to addressing questions of supersymmetry breaking, and is sadly lacking.
12.3. Some Unfinished Business

The present paper has, regrettably, a somewhat programmatic flavor. Several points related to the above discussion should be looked into much more thoroughly than we have done here. First, and foremost, the rules for systematic computation of membrane instanton effects should be clarified.

Second, we determined $W$ through the chiral multiplet mass matrix. When one considers the gravitino and gaugino mass matrices one encounters important contact interactions. These need to be sorted out. It is quite likely that doing so will shed light on nonperturbative corrections to the gauge kinetic function $\tau_{IJ}(z)$.

Third, we have only stated a complete claim for rigid membrane instantons. In general there might be contributions from nonisolated instantons. The case of worldsheet instantons can give us some guidance, but the full story remains to be understood. Similarly, the effects of open membrane instantons are potentially very important, but much work remains to be done here.

Appendix A. Conventions and notations

A.1. Index conventions

We use AMS fonts for superspace indices and for superfields (e.g., $Z^M$). $M = 0, \ldots, 10$ is a bosonic world index and $\mu = 1, \ldots, 32$ is a fermionic world index. (We also use $\mu = 0, 1, 2, 3$ for bosonic world indices in the noncompact spacetime $M_4$, but the distinction is clear from context.) $A, \alpha$ are the tangent frame counterparts of $M, \mu$. $e_M^A$ is a frame for the metric at $\Theta = 0$. $\Gamma_{MN...}$ are antisymmetrized gamma matrices of weight one. Worldvolume indices on $\Sigma$ are denoted by $1 \leq i, j, k \leq 3$, and a generic worldvolume coordinate system is denoted $s^i$. Our Lorentzian signature is mostly $+$.

A.2. Differential geometry

Our conventions for differential forms are $C = \frac{1}{3!} C_{MNP} dx^M dx^N dx^P$. Our normalization of the second fundamental form is:

$$\Gamma_{k'l'}^{m''} = -\frac{1}{2} h^{m'n''} Q_{k'l'n''}$$  \hspace{1cm} (A.1)

where $\Gamma_{k'l'}^{m''}$ is the Christoffel connection in the ambient space, restricted to $\Sigma$. Real harmonic $p$-forms on a manifold $M$ are denoted $\mathcal{H}^p(M; \mathbb{R})$. The forms corresponding to integral classes under the Hodge-DeRham isomorphism are $\mathcal{H}^p(M; \mathbb{Z})$. 33
If $V$ is a vector bundle with metric we let $\mathcal{S}(V)$ (or $\mathcal{S}^{\pm}(V)$) denote the minimal dimension associated spin bundle (there is an implicit choice of spin structure).

We often abbreviate Calabi-Yau to CY.

### A.3. Some spinor conventions

In section four we use the following Clifford algebra conventions. The three-dimensional Clifford algebra on the M2 world-volume is

$$\tau^0 = i\sigma^2 \quad \tau^1 = \sigma^1 \quad \tau^2 = \sigma^3$$  \hspace{1cm} (A.2)

The orientation is $\epsilon_{012} = -\epsilon^{012} = +1$. The Euclidean continuation is $\tau^3 = \sigma^2$.

The 7-dimensional Euclidean Clifford algebra $C\ell_7$ is nicely realized using the octonions $\mathbb{O}$. In the Cayley-Dickson description the octonions can be thought of as pairs of quaternions with multiplication

$$(a, b) \cdot (c, d) \equiv (ac - \bar{d}b, da + b\bar{c}).$$  \hspace{1cm} (A.3)

Choosing an isomorphism of $\text{Im} \mathbb{O} \cong \mathbb{R}^7$ we define Clifford multiplication by an orthonormal basis to be octonionic multiplication by the imaginary units. This gives a representation of $\gamma_i$ by $8 \times 8$ real antisymmetric matrices.

The Clifford algebra $C\ell_8$, used for the normal directions to the membrane, is represented by

$$\Sigma^{1,\ldots,8} = \left( \begin{array}{cc} 0 & \gamma^{1,\ldots,8} \\ \bar{\gamma}^{1,\ldots,8} & 0 \end{array} \right)$$  \hspace{1cm} (A.4)

where $\gamma^i$ is the representation of $C\ell_7$ described above, and $\bar{\gamma}^{1,\ldots,7}$ is the other inequivalent representation of $C\ell_7$. We take $\gamma^8_{a\dot{a}} = \delta_{a\dot{a}}$. The chirality operator is

$$\bar{\Sigma} = \left( \begin{array}{cc} -1_8 & 0 \\ 0 & 1_8 \end{array} \right)$$  \hspace{1cm} (A.5)

Note that $(\Sigma^i)^T = \Sigma^i$, and $(\Sigma^i)^* = \Sigma^i$, $i = 1,\ldots,8$.

We take the representation of the Clifford algebra $C\ell(1,10)$ to be

$$\Gamma^{m'} = \tau^{m'} \otimes \Sigma$$  \hspace{1cm} (A.6)
It is often convenient to introduce spinor index notation: $A = 1, 2, a, \dot{a} = 1, \ldots, 8$. The above matrices are written

$$\begin{pmatrix} \tau^m' \end{pmatrix}_B \quad \tilde{\gamma}^{m''}_{a\dot{a}} \quad \tilde{\gamma}^{m''}_{a\dot{a}}$$

Note that $\tau_{AB} = (\tau^{0} \tau^m')_{AB} = \tau^m_{BA}$ is symmetric. We raise/lower spinor indices with $\varepsilon^{AB} = (i\sigma^2)^{AB}$. The Euclidean continuation is obtained by $\tau^3 = \sigma^2$. Then the charge conjugation matrix $(\Gamma^M)^T = -C \Gamma^M C^{-1}$ is given by $C = i\sigma^2 \otimes \bar{\Sigma}$.

**Appendix B. Scales and dimensions**

Our conventions on units and scales are the following: $\ell$ is the 11-dimensional Planck length. $g_{MN} = \eta_{MN} + h_{MN}$ is dimensionless. Local coordinate differentials $dx^M$ have dimensions of length. Thus $[\text{vol}(g)] = L^d$ on a $d$-dimensional manifold. A field (like $C$) which is a differential form is dimensionless. Thus

$$C = \frac{1}{3!} C_{MNP} dx^M dx^N dx^P$$

means $C_{MNP}$ has dimensions $L^{-3}$. The gravitino $\Psi_M$ has dimension $[\Psi_M] = L^{-1/2}$. With these conventions, the Hodge star $*$ acting: $\Omega^k \rightarrow \Omega^{n-k}$ has dimensions of $L^{n-2k}$. Since the components of $\Phi_{mnp}$ are, in an orthonormal frame, the dimensionless structure constants of the octonions we have $[\Phi] = L^3$.

Our normalization of the 11D supergravity action is nonstandard. To translate to the normalizations used, for example, in [70] one sets:

$$C_{MNP}^{\text{here}} = \frac{3\sqrt{2}}{\ell^3} C_{MNP}^{\text{HW}}$$

$$\left(4\pi\right)^2 \ell^6 = \kappa^2$$

Also, note that we use standard normalizations for differential forms so $(dC)_{MNPQ} = \partial_M C_{NPQ} + 3 \text{terms}$. Finally we made a constant rescaling by $g \rightarrow 2^{2/3} g$.  

35
Appendix C. Some explicit susy 3-cycles

One well-known way to find supersymmetric 3-cycles is as the fixed point set of a real structure in a Calabi-Yau manifold \( Z \). At first sight it appears that one cannot use this idea to construct susy 3-cycles in barely \( G_2 \) manifolds, because \( X_7 \) is only smooth if \( \sigma_Z \) acts without fixed points - but then we lose the susy cycle! Of course, a CY can have several different real structures. Any two differ by a holomorphic isometry. Therefore, using CY’s with both real structures and symmetries we can use one real structure \( \sigma_Z \) (fixed point free) to make a barely \( G_2 \) manifold and study the fixed points of another real structure \( \tilde{\sigma}_Z \) to find susy 3-cycles.

We now give an explicit example of this technique. We consider the family in \( \mathbb{P}^5[2, 4] \):

\[
\begin{align*}
\sum_{i=1}^{6} X_i^2 &= 0 \\
\sum_{i=1}^{6} a_i X_i^4 &= 0
\end{align*}
\]

We assume \( a_i \neq 0 \). The discriminant locus is then defined by the equations:

\[
\sum_{i \in I_\alpha} \frac{1}{a_i} = 0
\]

where \( I_\alpha \) are subsets of \{1, 2, \ldots, 6\} with 2 or more elements. We will take \( a_i \) so that the CY is smooth. without loss of generality we may take \( a_2 + a_3 = 1 \).

If \( a_i \) are real we have the standard real structure \( \sigma_Z : X_i \rightarrow \bar{X}_i \). The fixed points are all real, and there are clearly no real solutions to (C.1). Thus we can build a smooth barely \( G_2 \) manifold. We will study the alternative real structure with \( \tilde{\sigma}_Z(X_i) = \bar{X}_i, \ i = 1, 2 \) and \( \tilde{\sigma}_Z(X_i) = -\bar{X}_i, \ i = 3, \ldots, 6 \). The fixed point locus is:

\[
\begin{align*}
X_{1,2} &= Y_{1,2} \\
X_{3,4,5,6} &= i Y_{3,4,5,6}
\end{align*}
\]

where the coordinates \( Y_i \) are real. Thus, the supersymmetric cycle is the intersection of the fixed point locus (C.3) and (C.1) and is given by the solutions with \( Y_i \) real to

\[
\begin{align*}
Y_1^2 + Y_2^2 &= Y_3^2 + r^2 \\
a_1 Y_1^4 + a_2 Y_2^4 + a_3 Y_3^4 + f_4 &= 0
\end{align*}
\]
Here \( r^2 = Y_4^2 + Y_5^2 + Y_6^2 \) and \( f_4 \) is the quartic defined above. It is useful to define: \( f_4 := r^4 \tilde{f}_4 \), where \( \tilde{f}_4 \) is just a function of the polar angles. We assume \( a_2 + \tilde{f}_4 > 0 \) for all angles.

Now, it is not difficult to see that the region:

\[
(a_1 + a_2(r^2 - 1)^2 + f_4) < 0
\]  

(C.5)

is a connected bounded region \( B_* \) near the origin of \( \mathbb{R}^3 \) and diffeomorphic to \( B^3 \), the 3-ball. The boundary of this region is the surface \( \Sigma_* \) diffeomorphic to \( S^2 \). In the region \( B_* \) there are only two real roots for \( Y_3 \). These two real roots collide and vanish along the surface \( \Sigma_* \). Moreover, if \( 2a_2 > 1 \) then the two roots for \( Y_2 \) are both real and never vanish in the region \( B_* \).

We are now in a position to describe our real solution set. There are 4 copies of the ball \( B_* \) labelled by the 4 real roots \( (\pm Y_2, \pm Y_3) \). Along the surface \( \Sigma_* \) the two roots \( Y_3 \) vanish, but the roots \( \pm Y_2 \) are bounded away from zero. Therefore, for each root of \( Y_2 \), we glue two copies of \( B_* \) together along \( \Sigma_* \). We get, topologically, two disjoint copies of \( S^3 \). Now, when we take the quotient by \( \sigma_Z \) this acts by taking \( (Y_1, Y_2) \rightarrow (Y_1, Y_2) \), but \( (Y_3, \ldots, Y_6) \rightarrow -(Y_3, \ldots, Y_6) \). This is clearly homotopic to the antipodal map on \( S^3 \) acting separately for each disjoint copy. We thus get two disjoint copies of \( \mathbb{R}P^3 \) in the barely \( G_2 \) manifold. These are rigid Lagrangian 3-cycles.

**Acknowledgements**

Some of these results were presented in the Penn Math-Physics seminar Nov. 4, 1998 and again at the conference “New ideas in particle physics and cosmology,” at the University of Pennsylvania, May 21, 1999. GM would like to thank the organizing committee for the invitation to speak. We would like to thank M. Mariño, R. Minasian, D. Morrison, T. Pantev, N. Seiberg, E. Silverstein, I. Singer, A. Strominger, and E. Witten for discussions. We are particularly grateful to E. Witten for important remarks on and corrections to a preliminary version of this paper. GM and JH would like to acknowledge the hospitality of the Aspen Center for Physics and GM would like to thank the Institute for Advanced Study for hospitality and the Monell foundation for support. The work of JH is supported by NSF Grant No. PHY 9600697, GM is supported by DOE grant DE-FG02-92ER40704.
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