Topology and Staggered Fermion Action Improvement

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It is conventional wisdom that staggered fermions do not feel gauge field topology. However, the response of staggered fermion eigenmodes to the topology of the gauge field can depend quite sensitively on the way in which the staggered fermion action is improved. We study this issue using a variety of improved staggered quark actions. We observe that the separation between the “would be” zero modes and the non-chiral modes increases with the level of improvement. This enables the “zero modes” to be identified unambiguously. The distribution of the remaining non-chiral modes is compared with the predictions of Random Matrix Theory. Satisfactory agreement is obtained.

It is a long standing problem that staggered fermions do not feel gauge field topology on coarse lattices. In particular, studies of the microscopic eigenvalue spectrum of the staggered Dirac operator, upon comparison with the analytic results predicted by Random Matrix Theory (RMT), show that only the trivial topological charge sector is probed by the operator \cite{1}. On the other hand, it has been suggested that lattice artifacts, in particular flavour changing interactions associated with the staggered quark action, are the main cause for this failure and other related topological problems \cite{2}. The staggered quark action describes four quark flavours in the continuum limit and the eigenvalue spectrum has a 4-fold degeneracy in this limit. At finite lattice spacing, the flavour changing interactions break the flavour symmetry and the degeneracy is lifted. Consequently, staggered fermions do not have exact zero modes at finite lattice spacing because the continuum chiral modes (if there are any) are scattered on the lattice. One thus expects staggered fermions to show better topological properties if flavour changing effects can be suppressed.

In this project, this issue is examined using a variety of improved staggered quark operators, which are designed to suppress flavour changing effects by smoothing out the quark-gluon interaction vertex \cite{3}. We observe that the “would be” zero modes are visibly decoupled from the non-chiral modes as the level of improvement increases. The effect of changing the lattice spacing on the eigenvalue spectrum is also studied. We again observe that separation between the “zero modes” and the non-chiral modes increases as one approaches the continuum limit. This enables the “zero modes” and subsequently the gauge field topological indices to be identified. Although the charge indices obtained by using different operators do not agree on a configuration by configuration basis, the charge distributions are found to have no significant difference. Finally, distribution of the remaining non-chiral modes is compared with the predictions of RMT.

The unimproved staggered Dirac operator is

\[ D_{x,y} = \frac{1}{2} \sum_\mu \eta_\mu(x) \left[ U_\mu(x) \delta_{x+\mu,y} - U_\mu^\dagger(y) \delta_{x-\mu,y} \right], \] (1)

where \( \eta_\mu(x) = (-1)^{x_1+\ldots+x_\mu-1} \) is the standard fermion phase. A variety of improvement schemes are considered here. The basic improved version is the \( \mathcal{O}(a^2) \) improved Asq operator \cite{4}, which includes an additional 3-link Naik term and replaces the gauge field in \cite{4} with Fat7 effective links (sum of the original link and the nearby paths, up to 7-link staples). Further improvement iterates this fattening procedure with an additional SU(3) unitarization step between successive iterations, giving the improved Asq op-
operators (UFat7)$n \times$ Asq. The HYP-improved operators (HYP)$n$ are constructed in the similar fashion but only those links within the hypercube containing the original link are included in the fattening process. We refer the readers to the respective papers for further details.

Simulations are done with the standard Wilson gauge field action at three values of coupling, $\beta = 5.85, 6.0$ and $6.2$. About 1000 configurations are generated for each $\beta$. The lattice sizes are $10^4$, $12^4$ and $16^4$ respectively so that the physical volumes are $\sim (1.2 fm)^4$ in all cases. We compute the eigenvalues ($\lambda^2$) and chirality ($\chi$) of the lowest 10 eigenstates of $-D^2$ for all operators listed above. The corresponding eigenvalues of $D$ are $\pm i \lambda$. Since $D^2$ connects either even-even or odd-odd sites on the lattice, only half of the spectrum is computed. For comparison, the lowest 5 eigenvalues (in each chiral sector) are also computed for the overlap operator on the $10^4$ lattice.

The effect of improvement on the eigenvalue spectrum is shown in figure 1 where $|\chi|$ is plotted against $\lambda$ for the different operators at a fixed coupling $\beta = 5.85$. It is observed that the “would be” zero modes separate from the non-chiral modes with higher level of improvement. In addition, as the level of improvement increases, the continuum 4-fold degeneracy emerges where the scattered eigenmodes begin to form quartets (remember that only half of the spectrum is computed so the data points form doublets). Note also that at this coupling the Asq operator still retains large lattice artifacts so the “would be” zero modes cannot be identified. Further improvement is required.

We next study the dependence of the eigenvalue spectrum on $\beta$. The spectra of the UFat7 $\times$ Asq operator at $\beta = 6.0$ and $6.2$ are given in figure 2. Again, the “would be” zero modes are well separated from the non-chiral modes and the continuum 4-fold degeneracy is better realized as one approaches the continuum limit. To quantify the separation between the “zero modes” and the non-chiral modes, the ratio of eigenvalues between the smallest non-chiral mode and the largest “zero mode” is plotted in figure 3. The ratios increase from one order of magnitude at $\beta = 5.85$ to three orders of magnitude at $\beta = 6.2$ for the improved staggered operators. Theoretically, the ratio is infinite for the overlap operator because exact zero modes exist on the lattice for overlap fermions. It is finite here solely because of computational precision. Results here also show that improvement using fat-links or hyper-cubic blocking are equally efficient.

Because of lattice artifacts, the topological charge indices obtained by using different operators do not agree on a configuration by configuration basis. In the present case we find that the charge indices determined by using different operators agree at about 60-70%, compared to 28% if the values were completely random. It is then important to check whether the charge distributions are also different because physical ob-
Figure 3. Ratio between the eigenvalues of the smallest non-chiral modes and the largest “zero mode”.

Figure 4. Normalized topological charge distributions (a) UFat7 × Asq (b) HYP (c) overlap.

Figure 5. The cumulative distributions of the smallest non-chiral mode in different topological sectors. Solid lines are predictions of RMT. The eigenvalues are rescaled by the infinite-volume chiral condensate $\Sigma$, $z = \lambda V \Sigma$, where $\Sigma$ is a fitting parameter here.

Two related works with similar conclusions were also reported at this conference [9].

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