Chiral Sum Rules and Duality in QCD

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Abstract

The ALEPH data on the vector and axial-vector spectral functions, extracted from tau-lepton decays, is used in order to test local and global duality, as well as a set of four QCD chiral sum rules. These are the Das-Mathur-Okubo sum rule, the first and second Weinberg sum rules, and a relation for the electromagnetic pion mass difference. We find these sum rules to be poorly saturated, even when the upper limit in the dispersion integrals is as high as 3 GeV\(^2\). Since perturbative QCD, plus condensates, is expected to be valid for \(|q^2| \geq \mathcal{O}(1 \text{ GeV}^2)\) in the whole complex energy plane, except in the vicinity of the right hand cut, we propose a modified set of sum rules with weight factors that vanish at the end of the integration range on the real axis. These sum rules are found to be precociously saturated by the data to a remarkable extent. As a byproduct, we extract for the low energy renormalization constant \(\bar{L}_{10}\) the value \(-4\bar{L}_{10} = 2.43 \times 10^{-2}\), to be compared with the standard value \(-4\bar{L}_{10} = (2.73 \pm 0.12) \times 10^{-2}\). This in turn leads to a pion polarizability \(\alpha_E = 3.7 \times 10^{-4} \text{ fm}^3\).

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There is a set of sum rules involving the difference between vector and axial-vector spectral functions, first discovered in the framework of current algebra \( [1] \), which are now understood as consequences of the underlying chiral symmetry of QCD. These spectral functions are related to the discontinuities in the complex energy plane of the two-point functions involving the vector and axial-vector currents

\[
\Pi_{\mu\nu}^{VV}(q^2) = i \int d^4x \, e^{iqx} \langle 0 | T(V_{\mu}(x) \, V^\dagger(0)) | 0 \rangle \\
= (-g_{\mu\nu} q^2 + q_\mu q_\nu) \, \Pi_V(q^2),
\]

\( (1) \)

\[
\Pi_{\mu\nu}^{AA}(q^2) = i \int d^4x \, e^{iqx} \langle 0 | T(A_{\mu}(x) \, A^\dagger(0)) | 0 \rangle \\
= (-g_{\mu\nu} q^2 + q_\mu q_\nu) \, \Pi_A(q^2) - q_\mu q_\nu \, \Pi_0(q^2),
\]

\( (2) \)

where \( V_{\mu}(x) = : \bar{q}(x) \gamma_\mu q(x) : \), \( A_{\mu}(x) = : \bar{q}(x) \gamma_\mu \gamma_5 q(x) : \), and \( q = (u, d) \). In this note we will study specifically the chiral correlator \( \Pi_{V-A} \equiv \Pi_V - \Pi_A \). This correlator vanishes identically in the chiral limit \( (m_q = 0) \), to all orders in QCD perturbation theory. Renormalon ambiguities are thus avoided. Non-perturbative contributions to this two-point function start at dimension \( d = 6 \), and involve the four-quark condensate. In the factorization approximation, and at high momentum transfer, this contribution is given by

\[
\Pi_{V-A}(q^2) = \frac{32\pi}{9} \, \frac{\alpha_s < \bar{q}q >^2}{q^6} \left\{ 1 + \frac{\alpha_s(q^2)}{4\pi} \left[ \frac{247}{12} + \ln \left( \frac{\mu^2}{-q^2} \right) \right] \right\} + \mathcal{O}(1/q^8) .
\]

\( (3) \)

The low energy behaviour of \( \Pi_{V-A}(q^2) \) is governed by chiral perturbation theory; to one loop order this is given by

\[
\Pi_{V-A}(q^2) = \frac{f_\pi^2}{q^2 - \mu_\pi^2} + \frac{1}{48\pi^2} \left[ \sigma^2 \left( \sigma \ln \frac{\sigma - 1}{\sigma + 1} + 2 \right) + \frac{2}{3} \right] + 4\bar{L}_{10} ,
\]

\( (4) \)

where \( \mu_\pi \) is the pion mass, \( f_\pi \) the pion decay constant, \( f_\pi = 92.4 \pm 0.26 \) MeV \( [4] \), \( \sigma \equiv (1 - 4\mu_\pi^2/q^2)^{1/2} \), and \( \bar{L}_{10} \) is the scale independent part of the coupling constant of the relevant operator in the \( \mathcal{O}(E^4) \) Lagrangian of chiral perturbation theory \( [3] \). The latter is related to the following combination of hadronic parameters

\[
\bar{L}_{10} = -\frac{1}{4} \left[ \frac{1}{3} f_\pi^2 < r_\pi^2 > - F_A \right] ,
\]

\( (5) \)
where $< r_\pi^2 >$ is the electromagnetic mean squared radius of the pion, $< r_\pi^2 > = 0.439 \pm 0.008 \text{fm}^2$ [4], and $F_A$ is the axial-vector coupling measured in radiative pion decay, $F_A = 0.0058 \pm 0.0008$ [2]. In the sixties, a number of sum rules were derived for the discontinuity of $\Pi_{V-A}(q^2)$, in the complex energy plane, from current algebra and an assumed asymptotic behaviour. The first sum rule is that of Das-Mathur-Okubo (DMO) [5]

$$W_0 \equiv \int_0^\infty \frac{ds}{s} [\rho_V(s) - \rho_A(s)] = \frac{1}{3} f_\pi^2 < r_\pi^2 > - F_A = -4 \bar{L}_{10} = (2.73 \pm 0.12) \times 10^{-2},$$

(6)

where $\rho_{V,A}(s)$ are related to the imaginary parts of $\Pi_{V,A}$, and normalized according to

$$\rho_{V,A}(s) = \frac{1}{8\pi^2} \left[ 1 + \mathcal{O}(\alpha_s) \right].$$

(7)

The second and third relations we consider are the first and second Weinberg sum rules [6]

$$W_1 \equiv \int_0^\infty ds \ [\rho_V(s) - \rho_A(s)] = f_\pi^2,$$

(8)

$$W_2 \equiv \int_0^\infty ds \ s \ [\rho_V(s) - \rho_A(s)] = 0.$$

(9)

The final sum rule is a relation for the electromagnetic pion mass difference [7]

$$W_3 \equiv \int_0^\infty ds \ s \ln \left( \frac{s}{\mu^2} \right) [\rho_V(s) - \rho_A(s)] = -\frac{16}{3} \frac{\pi^2 f_\pi^2}{e^2} \left( \mu_{\pi^\pm}^2 - \mu_{\pi^0}^2 \right).$$

(10)

Notice that this sum rule is actually independent of the arbitrary scale $\mu$, by virtue of the second Weinberg sum rule, Eq.(9). The above set of sum rules are exact relations in QCD, in the chiral $SU(2)_L \times SU(2)_R$ limit, i.e. for vanishing up- and down-quark masses [8].

With the advent of the ARGUS data on semi-leptonic tau-lepton decays [9], a reconstruction of the vector and axial-vector spectral functions became possible, albeit in the restricted kinematical range limited by the tau-lepton mass, thus allowing for the above sum rules to be confronted with experiment. This was first attempted in [10]. Shortly after, an improved parametrization of the tau-lepton decay data was discussed in [11], and used in [12] to check the first three sum rules above. Since kinematically $s < M_\tau^2 < \infty$, the question is how well does one actually verify these sum rules. It turns out that by
invoking quark-hadron duality, all three sum rules do effectively become Finite Energy
Sum Rules (FESR). In other words, the upper limit of integration may be taken at a finite
energy squared \( s = s_0 \), where the asymptotic freedom threshold \( s_0 \) signals the end of the
resonance region and the start of the hadronic continuum, which is well approximated by
perturbative QCD. The conclusions of \cite{10} and \cite{12} were that the ARGUS data appeared
to saturate the sum rules reasonably well. In this paper we make use of the recent, and
much more accurate, ALEPH data \cite{13} on the vector and axial-vector spectral functions,
extracted from tau-lepton decays, to test the validity of (a) local and global duality, and
(b) the four chiral sum rules Eqs.(6),(8)-(10)(for earlier related work see e.g. \cite{14}).

The concept of duality is well defined in QCD. The starting point is Wilson’s Operator
Product Expansion (OPE), extended beyond perturbative QCD by the addition of power
corrections involving quark and gluon vacuum condensates \cite{15}. This OPE is expected to
be valid in the whole complex energy plane, except in the resonance region. In the space-
like region \( q^2 < 0 \) the relevant scale of the OPE is determined by \( \Lambda_{QCD} \simeq 300 - 400 \)
MeV, on the perturbative side, and by the scale of the condensates, also roughly 300 MeV,
on the non-perturbative side. It is therefore reasonable to use the OPE in the region
\( -q^2 \geq 1 \text{ GeV}^2 \). However, in the time-like region the relevant scale is much higher, as it is
set by the masses of resonances in the range 1 - 2 GeV. In this region, the use of the OPE
may be further impaired by instanton and threshold effects. Hence, the OPE may be used
in the time-like region only at much higher scales than in the space-like region. To account
for this fact, one distinguishes different types of duality, to wit. (i) Local duality

\[
\text{Im } \Pi (s) \simeq \text{Im } \Pi (s)|_{QCD},
\]  

valid in QCD at very high \( -q^2 \). (ii) Global Duality, which is an integral statement based
on Cauchy’s theorem

\[
\int_{s_\text{th}}^{s_0} \frac{1}{\pi} f(s) \text{Im } \Pi(s) \, ds = -\frac{1}{2\pi i} \oint f(s) \text{Pi}(s) \, ds
\]

\[
\approx -\frac{1}{2\pi i} \oint_{|s|=s_0} f(s) \Pi(s)|_{QCD} \, ds.
\]
Alternatively, since Cauchy’s theorem is valid in perturbative QCD,
\[ \int_{s_{th}}^{s_0} \frac{1}{\pi} f(s) \text{Im} \Pi(s) \, ds = \int_{s_{QCD}}^{s_0} \frac{1}{\pi} f(s) \text{Im} \Pi(s)|_{QCD} \, ds . \] (13)

In the above equations, \( s_{th} \) is the physical threshold, \( s_{QCD} \) the QCD threshold, and \( s_0 > s_{th} \) is the so-called asymptotic freedom threshold. The latter must be chosen large enough so that \( \Pi(s) \simeq \Pi(s)_{QCD} \) on the circle \( |s| = s_0 \). The two equations above hold only if the weight function \( f(s) \) is analytic in the domain \( |s| \leq s_0 \); for instance, if \( f(s) \) is a polynomial.

Global duality, therefore, relates weighted integrals of spectral functions (connected to cross sections through the optical theorem) to integrals of the QCD correlator over a large circle. Global duality in the interval \( s_{th} - s_0 \) implies that local duality should be valid for \( s > s_0 \). This may be seen by invoking Cauchy’s theorem for a contour made of two circles of radii \( s_0 \) and \( s_1 > s_0 \), together with the connecting lines along the cut. In practice, global duality may often hold to a much better degree than local duality, because oscillations can lead to cancellations.

(iii) Finally, we wish to introduce a form of global duality, let us call it \textit{restricted global duality}, expected to set in much sooner than ordinary global duality. This can be accomplished if the weight functions in Eq.(12) are constrained to vanish at \( s = s_0 \). A well known example is the total hadronic decay rate of the \( \tau \)-lepton. Below, we shall make use of this form of duality to write four modified chiral sum rules, as alternatives to Eqs. (6), (8)-(10).

A few comments on the above chiral sum rules are in order. The left hand side of Equations (8) and (9) can be related in a straightforward manner to contour integrals of the QCD correlator. The same is true of Eq.(6), provided one takes into account that \( \Pi(0) \) is known, i.e.
\[ \Pi_{V-A}(0) = -\frac{f_\pi^2}{\mu_\pi^2} + 4 \tilde{L}_{10} . \] (14)

In principle, duality arguments cannot be applied to Eq.(10) because of the presence of the logarithm. As this is only mildly singular, and it can be approximated rather well in the integration interval by an analytic function, one could still invoke duality. However, one should keep in mind that in this case duality may only be approximately valid. From a theoretical point of view, the four chiral sum rules are expected to be saturated for rela-
tively low values of $s_0$, on account of the very rapid fall-off of the correlator, cf. Eq.(3).

In Fig. 1 we show the ALEPH experimental data points \[13\] on the vector and axial-vector spectral functions, together with our fit to these data (solid lines) and the perturbative QCD prediction (dash line). The normalization of these spectral functions has been changed (Eq.(7) has been multiplied by $4\pi^2$) in order to simplify comparisons with Ref. \[13\]. Experimental errors are small, of the size of the dots in the figure, except near the end point. Even after taking these errors into account, the agreement between data and theory, which measures the validity of local duality, is not quite satisfactory. Furthermore, because of cancellations, the errors on the difference $\rho_V - \rho_A$ should be smaller than the vector and axial-vector spectral function errors taken in quadrature. In view of this, one would expect a similar unsatisfactory saturation of the four chiral sum rules, Eqs.(6), (8)-(10), and in fact, this is what we find. Before illustrating our results, let us write down a set of four chiral sum rules modified according to the principle of restricted global duality introduced above

\begin{align}
\bar{W}_0 &\equiv \int_{s_0}^{\infty} ds \left( 1 - \frac{s}{s_0} \right) \left( \rho_V(s) - \rho_A(s) \right) = -4 L_{10} - \frac{f_\pi^2}{s_0}, \\
\bar{W}_1 &\equiv \int_{0}^{\infty} ds \left( 1 - \frac{s}{s_0} \right) \left( \rho_V(s) - \rho_A(s) \right) = f_\pi^2, \\
\bar{W}_2 &\equiv \int_{0}^{\infty} ds \left( 1 - \frac{s}{s_0} \right)^2 \left( \rho_V(s) - \rho_A(s) \right) = -4 L_{10} - \frac{2 f_\pi^2}{f_\pi}, \\
\bar{W}_3 &\equiv \int_{0}^{\infty} ds \ln \left( \frac{s}{s_0} \right) \left( \rho_V(s) - \rho_A(s) \right) = -\frac{16}{3} \frac{\pi^2 f_\pi^2}{\rho^2} \left( \mu_{\pi} - \mu_{\rho} \right),
\end{align}

where Eq.(15) is a combination of the DMO sum rule and the first Weinberg sum rule, Eq.(17) is a combination of the DMO sum rule and the first and second Weinberg sum rules, and the arbitrary scale in Eq.(18) has been fixed to $s_0$, which becomes the upper limit of integration in the four sum rules. We now show that these modified sum rules are saturated far better than the original sum rules Eqs.(6), (8)-(10). In Fig.2 we plot the left hand side (l.h.s.) of Eq.(6) computed using the fit to the data (curve(a)), and the right hand side (r.h.s.) (curve(b)). Agreement with the data can be considerably improved by
rescaling the r.h.s. of Eq.(6) from the value $2.73 \times 10^{-2}$ to $2.43 \times 10^{-2}$. Figure 3 shows the l.h.s. of the modified DMO sum rule Eq.(15) (curve(a)) compared to the r.h.s. (curve (b)) after performing the above rescaling. This value of $W_0$ has implications for the pion polarizability, i.e.

$$\alpha_E = \frac{\alpha_{EM}}{\mu_\pi} \left( \frac{< r^2_\pi >}{3} - \frac{W_0}{f^2_\pi} \right) = 3.7 \times 10^{-4} \text{ fm}^3,$$

(19)

where the numerical value above is the result of using the rescaled value $W_0 = 2.43 \times 10^{-2}$.

In Fig.4 we plot the l.h.s. of the first Weinberg sum rule Eq.(8) (curve(a)), its modified version, Eq.(16) (curve(b)), and their r.h.s. (curve(c)). Figure 5 shows the l.h.s. of the second Weinberg sum rule Eq.(9) (curve(a)), compared to its r.h.s. (curve(b)), and Fig. 6 the corresponding curves for the modified sum rule Eq.(17). Finally, in Fig. 7 we plot the l.h.s. of the sum rule Eq.(10) (curve(a)), the l.h.s. of the modified sum rule, Eq.(18) (curve (b)), and their r.h.s. (curve(c)). An inspection of these results clearly indicates that, in line with the unsatisfactory check of local duality, the original chiral sum rules do not appear well saturated by the data. While an overall constant rescaling of the experimental data can result in a better saturation of the DMO sum rule, this would not help with the other three sum rules. Hence, the problem cannot be blamed on a systematic overall normalization uncertainty in the data. On the other hand, by using restricted global duality, the four modified chiral sum rules Eqs.(15)-(18) are extremely well saturated by the data.

In this note we have addressed the issue of the range of applicability of perturbative QCD, augmented by condensate terms (or higher twists) using the chiral correlator $\Pi_{V-A}$ as an example. We have argued that the notions of duality or precocious scaling apply, in the space-like region, at remarkably low momentum transfers, but they must be used with great care when time-like momenta are involved (for discussions on the possible breakdown of local duality in QCD see e.g. [16]-[17]). In fact, even at a $q^2$ as large as $q^2 \simeq 3 \text{ GeV}^2$ neither the experimental spectral function itself, nor its moments are satisfactorily given by their QCD counterpart. However, by taking linear combinations of moments, such that
they vanish at the upper limit of integration (i.e. at $s = s_0$, the continuum threshold), a remarkable agreement with the QCD prediction is achieved. This we referred to as restricted duality. Clearly, if the (finite range) integral over the spectral function is equated to an integral over a large circle in the complex energy plane, the choice of polynomial integral kernels that vanish at the upper limit of integration causes the contribution in the vicinity of the cut to vanish. We therefore conclude that perturbative QCD may be expected to hold in the complex energy plane for $|q^2| \geq s_0$, with $s_0$ as low as 1 GeV$^2$, except near the cut, where the corresponding radius appears to be definitely much larger than 3.5 GeV$^2$.

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Figure Captions

Figure 1. Our fits to the vector (curve(a)) and axial-vector (curve(b)) spectral functions together with the ALEPH experimental data [13]. Normalization is according to: \( \rho_{V,A}(s) = \frac{1}{2} \left[ 1 + \mathcal{O}(\alpha_s) \right] \).

Figure 2. The left hand side of Eq.(6) computed using the fits to the data (curve(a)), and its right hand side (curve(b)).

Figure 3. The left hand side of Eq.(15) computed using the fits to the data (curve(a)), and its right hand side (curve(b)) with \(-4\bar{L}_{10}\) rescaled to \(2.43 \times 10^{-2}\).

Figure 4. The left hand sides of Eq.(8) and Eq.(16) computed using the fits to the data (curves(a) and (b), respectively), and their right hand side (curve(c)).

Figure 5. The left hand side of Eq.(9) computed using the fits to the data (curve(a)), and its right hand side (curve(b)).
Figure 6. The left hand side of Eq.(17) computed using the fits to the data (curve(a)), and its right hand side (curve(b))

Figure 7. The left hand sides of Eq.(10) and Eq.(18) computed using the fits to the data (curves(a) and (b), respectively), and their right hand side (curve(c)).
