CONVEXITY WITH RESPECT TO FAMILIES OF MEANS

GYULA MAKSA AND ZSOLT PÁLES

Dedicated to the 90th birthday of Professor János Aczél

Abstract. In this paper we investigate continuity properties of functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ that satisfy the $(p, q)$-Jensen convexity inequality

$$f(H_p(x, y)) \leq H_q(f(x), f(y)) \quad (x, y > 0),$$

where $H_p$ stands for the $p$th power (or Hölder) mean. One of the main results shows that there exist discontinuous multiplicative functions that are $(p, p)$-Jensen convex for all positive rational number $p$. A counterpart of this result states that if $f$ is $(p, p)$-Jensen convex for all $p \in P \subseteq \mathbb{R}_+$, where $P$ is a set of positive Lebesgue measure, then $f$ must be continuous.

1. Introduction

Given two intervals $I$ and $J$ in $\mathbb{R}$ and two two-variable means $M : I^2 \to I$ and $N : J^2 \to J$, a real function $f : I \to J$ is called $(M, N)$-convex if

$$f(M(x, y)) \leq N(f(x), f(y)) \quad (x, y \in I).$$

In the particular case when $M$ and $N$ are the arithmetic means, $(M, N)$-convexity is termed Jensen convexity. Jensen convex functions, in general, are not necessarily continuous. In view of the existence of a Hamel base of $\mathbb{R}$ considered as a vector space over $\mathbb{Q}$, one can construct a discontinuous additive function ([4], [9]), such a function is automatically a discontinuous Jensen convex function. On the other hand, certain weak regularity properties of Jensen convex functions ensure that they are also convex and henceforth continuous. Jensen [7], [8] verified that continuous Jensen convex functions are convex. Bernstein and Doetsch [3] showed that upper boundedness on a nonempty open set and Jensen convexity imply convexity. Sierpiński [13] proved that upper boundedness on a set of positive Lebesgue measure and Jensen convexity are also sufficient conditions for convexity. In the context of $(M, N)$-convexity, Zgraja [14] proved that with sufficiently regular means $M$ and $N$, weak regularity properties of $(M, N)$ convex functions imply their continuity.

Date: December 24, 2015.

2010 Mathematics Subject Classification. Primary 39B62; Secondary 26D07, 26D15.

The research of the second author was realized in the frames of TÁMOP 4.2.4. A/2-11-1-2012-0001 "National Excellence Program – Elaborating and operating an inland student and researcher personal support system". This project was subsidized by the European Union and co-financed by the European Social Fund. The research of both authors was also supported by the Hungarian Scientific Research Fund (OTKA) Grant NK 81402.
In this paper we will consider the particular case of $(M, N)$-convexity when $I = J = \mathbb{R}_+ := ]0, \infty[$ and the means $M$ and $N$ belong to the class of power (or Hölder) means. We recall that, for $p \in \mathbb{R}$ the $p$th power mean $H_p$ on $\mathbb{R}_+$ is defined by

$$H_p(x, y) := \begin{cases} \left( \frac{x^p + y^p}{2} \right)^{\frac{1}{p}} & \text{if } p \neq 0 \\ \sqrt{xy} & \text{if } p = 0 \end{cases} \quad (x, y \in \mathbb{R}_+)$$

(cf. [5], [6]). In the sequel $(H_p, H_q)$-convex functions will simply be called $(p, q)$-Jensen convex. Obviously, $(1, 1)$-Jensen convexity is equivalent to Jensen convexity.

The results of this paper are motivated by the following question of Matkowski [11] formulated at the 30th ISFE in 1992: Is it true that $(0, 0)$-Jensen convexity and $(1, 1)$-Jensen convexity imply continuity? During the symposium, a negative answer was given by Maksa [10] who constructed a discontinuous $(0, 0)$-Jensen convex and $(1, 1)$-Jensen convex function.

The aim of this paper is to investigate continuity properties of functions that are $(p, q)$-Jensen convex for $(p, q) \in \Pi$, where $\Pi \subseteq \mathbb{R}^2$ is a nonempty set. Such functions will also be termed $\Pi$-Jensen convex.

2. Preliminary observations

We start with a characterization of $(p, q)$-Jensen convexity. To formulate this result, for $(p, q) \in \mathbb{R}^2$ and for $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$, define the function $f_{p,q} : I_p \to \mathbb{R}$ by

$$f_{p,q}(x) := \begin{cases} \text{sign}(q) \left( f \left( \frac{1}{p} x \right) \right)^q & \text{if } p \neq 0, \ q \neq 0 \\ \text{sign}(q) \left( f \circ \exp(x) \right)^q & \text{if } p = 0, \ q \neq 0 \\ \log f \left( \frac{1}{p} x \right) & \text{if } p \neq 0, \ q = 0 \\ \log f \circ \exp(x) & \text{if } p = 0, \ q = 0 \end{cases} \quad (x \in I_p),$$

where

$$I_p := \begin{cases} I^p := \{ t^p \mid t \in I \} & \text{if } p \neq 0, \\ \log(I) := \{ \log t \mid t \in I \} & \text{if } p = 0. \end{cases}$$

The following result is very well known in the mathematical folklore. It could easily be deduced from the results of Aczél [1].

**Theorem 1.** Let $I \subseteq \mathbb{R}_+$ be an interval. Then, for any $(p, q) \in \mathbb{R}^2$, a function $f : I \to \mathbb{R}_+$ is $(p, q)$-Jensen convex if and only if $f_{p,q}$ is Jensen convex on $I_p$.

**Proof.** We prove the statement only in the case $pq \neq 0$. In this case, the $(p, q)$-Jensen convexity of $f$ means that

$$f \left( \left( \frac{x^p + y^p}{2} \right)^{\frac{1}{p}} \right) \leq \left( \frac{(f(x))^q + (f(y))^q}{2} \right)^{\frac{1}{q}} \quad (x, y \in I).$$
Taking the $q$th power of both sides and substituting $x = u^{\frac{1}{p}}$ and $y = v^{\frac{1}{p}}$ (where $u, v \in I_p$), this inequality can be rewritten as

$$\text{sign}(q) \left( f \left( \left( \frac{u + v}{2} \right)^{\frac{1}{p}} \right) \right)^q \leq \text{sign}(q) \frac{(f(u^{\frac{1}{p}}))^q + (f(v^{\frac{1}{p}}))^q}{2}$$

$(u, v \in I_p)$. This inequality, however, is equivalent to the Jensen convexity of the function $f_{p,q}$.

The proof in the case $pq = 0$ is completely analogous. If $p = 0$ then the substitution $x = \exp(u)$ and $y = \exp(v)$ should be applied. If $q = 0$, then the logarithm of the two sides of the $(p,q)$-Jensen convexity inequality should be taken.

**Corollary 2.** Let $I \subseteq \mathbb{R}_+$ be an open interval. If $f : I \to \mathbb{R}_+$ is $(p,q)$-Jensen convex for some $q < 0$ and $p \in \mathbb{R}$, then $f_{p,q}$ is convex on $I_p$ and, consequently $f$ is continuous, moreover locally Lipschitz on $I$.

**Proof.** If $f$ is $(p,q)$-Jensen convex, then, by the previous theorem, $f_{p,q}$ is Jensen convex on $I_p$. On the other hand, for $q < 0$, we have that $f_{p,q} \leq 0$ on $I_p$. Therefore, by the Bernstein–Doetsch theorem [3], $f_{p,q}$ is convex. The interval $I_p$ is open, hence $f_{p,q}$ is locally Lipschitz on $I_p$. This implies that $f$ is locally Lipschitz on $I$. \qed

In view of the above corollary, we restrict ourselves to investigate the continuity properties of $\Pi$-Jensen convex functions, where $\Pi \subseteq \mathbb{R} \times [0, \infty]$ is a nonempty set.

### 3. Main Results

We recall that the real functions $a : \mathbb{R} \to \mathbb{R}$, $e : \mathbb{R} \to \mathbb{R}_+$, $\ell : \mathbb{R}_+ \to \mathbb{R}$, and $m : \mathbb{R}_+ \to \mathbb{R}_+$ are called **additive**, **exponential**, **logarithmic**, and **multiplicative** if the following Cauchy type functional equations

$$a(x + y) = a(x) + a(y), \quad e(x + y) = e(x)e(y) \quad (x, y \in \mathbb{R}),$$

$$\ell(xy) = \ell(x) + \ell(y), \quad m(xy) = m(x)m(y) \quad (x, y \in \mathbb{R}_+)$$

are satisfied, respectively. The basic properties and of these classes of functions are described in details in the monographs [2] and [9].

A function $a : \mathbb{R} \to \mathbb{R}$ is said to be a **derivation** if it is additive and satisfies the **Leibniz Rule**, that is, if $d$ fulfills the following functional equation

$$d(xy) = xd(y) + yd(x) \quad (x, y \in \mathbb{R}).$$

By the Leibniz Rule $d(1) = 0$ easily follows. Hence derivations vanish on $\mathbb{Q}$ (the rationals). Thus, a continuous derivation is identically zero. It is a nontrivial fact that there exist nonzero derivations. Moreover, by taking an algebraic base of $\mathbb{R}$, any real-valued function can uniquely be extended to a derivation (cf. [9]).

The next theorem is one of the main results of this paper.

**Theorem 3.** For any derivation $d : \mathbb{R} \to \mathbb{R}$ and $\alpha > 0$, the function $F_{d,\alpha} : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$F_{d,\alpha}(x) := x^\alpha \exp \left( \frac{d(x)}{x} \right) \quad (x \in \mathbb{R}_+)$$

is locally Lipschitz on $\mathbb{R}_+$.
is multiplicative and $(p, \alpha^{-1}p)$-Jensen convex for all $p \in \mathbb{Q}_+$ (the positive rationals). Hence, if $d$ is a nonzero derivation, then $F_{d, \alpha}$ is a discontinuous function on $\mathbb{R}_+$ which is $(p, \alpha^{-1}p)$-Jensen convex for all $p \in \mathbb{Q}_+$.

Proof. Observe that if $d$ satisfies the Leibniz Rule then the function $\ell(x) := d(x)/x$ is logarithmic and therefore $x \mapsto \exp(d(x)/x)$ is multiplicative. The product of two multiplicative functions is again multiplicative, hence $F_{d, \alpha}$ is multiplicative.

The multiplicativity of $F_{d, \alpha}$ implies that $a := \log \circ F_{d, \alpha} \circ \exp$ is additive. Thus, by well-known properties of additive functions, $a$ is $\mathbb{Q}$ homogeneous, i.e., $a(px) = pa(x)$ holds for all $x \in \mathbb{R}$ and for all $p \in \mathbb{Q}$. This directly yields that

$$F_{d, \alpha}(x^p) = (F_{d, \alpha}(x))^p \quad (x \in \mathbb{R}_+, p \in \mathbb{Q}). \tag{1}$$

First we show $F = F_{d, \alpha}$ is $(1, \alpha^{-1})$-Jensen convex. Indeed, using the inequality between the weighted geometric mean and the weighted Hölder mean of order $\alpha^{-1}$, we get

$$F(H_1(x, y)) = F\left(\frac{x + y}{2}\right) = \left(\frac{x + y}{2}\right)^\alpha \exp\left(\frac{d(x) + d(y)}{x + y}\right)$$

$$= \left(\frac{x + y}{2}\right)^\alpha \exp\left(\frac{x}{x + y} d(x) + \frac{y}{x + y} d(y)\right)$$

$$= \left(\frac{x + y}{2}\right)^\alpha \left(\frac{F(x)}{x^{\alpha}}\right)^{\frac{x}{x+y}} \cdot \left(\frac{F(y)}{y^{\alpha}}\right)^{\frac{y}{x+y}}$$

$$\leq \left(\frac{x + y}{2}\right)^\alpha \left(\frac{x}{x + y} \left(\frac{F(x)}{x^{\alpha}}\right)^{\frac{1}{\alpha}} + \frac{y}{x + y} \left(\frac{F(y)}{y^{\alpha}}\right)^{\frac{1}{\alpha}}\right)^\alpha$$

$$= \left(\frac{(F(x))^{\frac{1}{\alpha}} + (F(y))^{\frac{1}{\alpha}}}{2}\right)^\alpha = H_{\alpha^{-1}}((F(x), F(y))) \quad (x, y \in \mathbb{R}_+).$$

Now, applying (1) and the $(1, \alpha^{-1})$-Jensen convexity, for $p \in \mathbb{Q}_+$, we obtain

$$F(H_p(x, y)) = F\left(\frac{x^p + y^p}{2}\right)^{\frac{1}{p}} = \left(\frac{x^p + y^p}{2}\right)^{\frac{1}{p}} = \left(\frac{x^p + y^p}{2}\right)^{\frac{1}{p}} \leq \left(\frac{(F(x^p), F(y^p))^{\frac{1}{p}}}{2}\right)^{\frac{1}{p}}$$

$$\leq \left(\frac{H_{\alpha^{-1}}((F(x), F(y))^p)}{2}\right)^{\frac{1}{p}} = H_{\alpha^{-1}}((F(x)^p, F(y)^p))^{\frac{1}{p}} = H_{\alpha^{-1}}(F(x), F(y)).$$

This shows that $F_{d, \alpha}$ is $(p, \alpha^{-1}p)$-Jensen convex for all $p \in \mathbb{Q}_+$. \hspace{1em} \square

**Corollary 4.** For any derivation $d : \mathbb{R} \to \mathbb{R}$ and $\alpha \geq 1$, the function $F_{d, \alpha} : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$F_{d, \alpha}(x) := x^\alpha \exp\left(\frac{d(x)}{x}\right) \quad (x \in \mathbb{R}_+)$$

is a multiplicative and Jensen convex. Hence, if $d$ is a nonzero derivation, then $F_{d, \alpha}$ is a discontinuous function which is Jensen convex and multiplicative.

In the next result we characterize those multiplicative functions that are also Jensen convex.
Theorem 5. A multiplicative function $m : \mathbb{R}_+ \to \mathbb{R}_+$ is Jensen convex if and only if there exists an additive function $a : \mathbb{R} \to \mathbb{R}$ such that

$$m(t) \geq 1 + a(t - 1) \quad (t \in \mathbb{R}_+). \quad (2)$$

Proof. If $m$ is a multiplicative function, then we have that $m(1) = 1$.

If $m$ is Jensen convex then, as a consequence of Rodé’s Theorem [12], for every $p > 0$ there exists an additive function $a_p : \mathbb{R} \to \mathbb{R}$ such that

$$m(t) \geq m(p) + a_p(t - p) \quad (t > 0).$$

Therefore, (2) holds with $a = a_1$.

No assume that there exists an additive function $a : \mathbb{R} \to \mathbb{R}$ satisfying (2). Let $x, y > 0$ and apply (2) for $t := \frac{2x}{x+y}$ and $t := \frac{2y}{x+y}$. Adding up the inequalities so obtained side by side, we get

$$m\left(\frac{2x}{x+y}\right) + m\left(\frac{2y}{x+y}\right) \geq 2 + a\left(\frac{2x}{x+y} - 1\right) + a\left(\frac{2y}{x+y} - 1\right)$$

$$= 2 + a\left(\frac{2x}{x+y} + \frac{2y}{x+y} - 2\right) = 2.$$

Now multiplying both sides of this inequality by $m\left(\frac{x+y}{2}\right)$ and using the multiplicativity of $m$, it follows that

$$m(x) + m(y) \geq 2m\left(\frac{x+y}{2}\right),$$

which proves that $m$ is Jensen convex. \hfill \square

The following problem is open: Find or characterize those multiplicative functions $m$ and additive functions $a$ such that (2) be valid.

Theorem 6. Let $P \subseteq \mathbb{R}_+$ be a set whose internal Lebesgue measure is positive. Let $q : P \to \mathbb{R}$ be an arbitrary function. If $f : \mathbb{R}_+ \to \mathbb{R}_+$ is $(p, q(p))$-Jensen convex for all $p \in P$, then $f$ is continuous (and hence, $f_{p,q(p)}$ is convex for all $p \in P$).

Proof. We may assume that $P$ is measurable and we may also assume that $q$ is nonnegative because if, for some $p$, the value of $q(p)$ were negative, then Theorem [11] could be applied. Let $0 < x < y$ be fixed. Then the image $S$ of the set $P$ by the strictly increasing and continuously differentiable mapping

$$p \mapsto H_p(x, y)$$

is of positive measure. On the other hand, in view of the $(p, q(p))$-Jensen convexity of $f$,

$$f(H_p(x, y)) \leq H_{q(p)}(f(x), f(y)) \leq \max(f(x), f(y)).$$

Therefore, $f$ is bounded from above on $S$ by $\max(f(x), f(y))$. Thus, for every $p \in P \setminus \{0\}$, $f_{p,q(p)}$ is bounded from above on $S^p := \{s^p | s \in S\}$ by $\max(f^{q(p)}(x), f^{q(p)}(y))$ if $q(p) > 0$ and by $\max(\log(f(x)), \log(f(y)))$ if $q(p) = 0$.

Consequently, by Sierpiński’s generalization [13] of the Bernstein–Doetsch Theorem [3], for every $p \in P$, $f_{p,q(p)}$ is continuous and hence it is also convex. \hfill \square
References

[1] J. Aczél. A generalization of the notion of convex functions. *Norske Vid. Selsk. Forh., Trondhjem*, 19(24):87–90, 1947.

[2] J. Aczél. *Lectures on Functional Equations and Their Applications*, volume 19 of Mathematics in Science and Engineering. Academic Press, New York–London, 1966.

[3] F. Bernstein and G. Doetsch. Zur Theorie der konvexen Funktionen. *Math. Ann.*, 76(4):514–526, 1915.

[4] G. Hamel. Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung $f(x + y) = f(x) + f(y)$. *Math. Ann.*, 60:459–462, 1905.

[5] G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge University Press, Cambridge, 1934. (first edition), 1952 (second edition).

[6] O. Hölder. Über einen Mittelwerthsatz. *Nachr. Ges. Wiss. Göttingen*, page 38–47, 1889.

[7] J. L. W. V. Jensen. Om konvekse funktioner og uligheder imellem middelværdier. *Nyt. Tidskrift for Matematik*, 16 B:49–69, 1905.

[8] J. L. W. V. Jensen. Sur les fonctions convexes et les inégalités entre les valeurs moyennes. *Acta Math.*, 30:175–193, 1906.

[9] M. Kuczma. *An Introduction to the Theory of Functional Equations and Inequalities*, volume 489 of Prace Naukowe Uniwersytetu Śląskiego w Katowicach. Państwowe Wydawnictwo Naukowe — Uniwersytet Śląski, Warszawa–Kraków–Katowice, 1985. 2nd edn. (ed. by A. Gilányi), Birkhäuser, Basel, 2009.

[10] Gy. Maksa. 16. Remark (Solution of J. Matkowski’s problem 14), Report of Meeting: The Thirtieth International Symposium on Functional Equations, September 20–26, 1992, Oberwolfach, Germany, *Aequationes Math.* 46 1993, p. 292.

[11] J. Matkowski. 14. Problem Report of Meeting: The Thirtieth International Symposium on Functional Equations, September 20–26, 1992, Oberwolfach, Germany, *Aequationes Math.* 46 1993, p. 291.

[12] G. Rodé. Eine abstrakte Version des Satzes von Hahn–Banach. *Arch. Math. (Basel)*, 31:474–481, 1978.

[13] W. Sierpiński. Sur les fonctions convexes mesurables. *Fund. Math.*, 1:125–128, 1920.

[14] T. Zgraja. Continuity of functions which are convex with respect to means. *Publ. Math. Debrecen*, 63(3):401–411, 2003.

Institute of Mathematics, University of Debrecen, H-4010 Debrecen, Pf. 12, Hungary
E-mail address: {maksa,pales}@science.unideb.hu