Reconstruction of black hole metric perturbations from Weyl curvature: II. The Regge–Wheeler gauge

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Abstract
Perturbation theory of rotating black holes is described in terms of the Weyl scalars $\psi_4$ and $\psi_0$, each satisfying Teukolsky’s complex master wave equation with spin $s = \mp 2$, and respectively representing outgoing and ingoing radiation. We explicitly construct the metric perturbations out of these Weyl scalars in the Regge–Wheeler gauge in the non-rotating limit. We propose a generalization of the Regge–Wheeler gauge for a Kerr background in Newman–Penrose language and discuss the approach for building up the perturbed spacetime of a rotating black hole. We also provide two-way relationships between waveforms defined in the metric and curvature approaches in the time domain, also known as (inverse) Chandrasekhar transformations, generalized to include matter.

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1. Introduction
There is a formulation of the perturbation problem derived from the Newman–Penrose formalism [20] that is valid for perturbations of rotating black holes [25]. This formulation fully exploits the null structure of black holes to decouple the curvature perturbation equations into a single master wave equation that, in Boyer–Lindquist coordinates $(t, r, \theta, \phi)$, can be written as

\[
\left\{ \begin{aligned}
\left[ a^2 \sin^2 \theta - \frac{(r^2 + a^2)^2}{\Delta} \right] \partial_t - \frac{4Mar}{\Delta} \partial_\phi - 2s \left[ (r + ia \cos \theta) - \frac{M(r^2 - a^2)}{\Delta} \right] \partial_t \\
+ \Delta^{-1} \partial_r \left( \Delta^{s+1} \partial_t \right) + \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \left( \frac{1}{\sin^2 \theta} - \frac{a^2}{\Delta} \right) \partial_\phi \\
+ 2s \left[ \frac{a(r - M)}{\Delta} + \frac{i \cos \theta}{\sin^2 \theta} \right] \partial_\phi - (s^2 \cot^2 \theta - s) \right) \right) \Psi = 4\pi \Sigma T,
\end{aligned} \right.
\]
$M$ is the mass of the black hole, $a$ is its angular momentum per unit mass, $s$ is the spin of the perturbation, $\Sigma \equiv r^2 + a^2 \cos^2 \theta$ and $\Delta \equiv r^2 - 2Mr + a^2$. The source term $T$ is built up from the energy–momentum tensor [25]. Gravitational perturbations, corresponding to $s = \pm 2$, are compactly described in terms of contractions of the Weyl tensor with a null tetrad. The components of the tetrad (also given in [25]) are appropriately chosen along the repeated principal null directions of the background spacetime (see equation (3) below). The resulting (infinitesimal) coordinate and tetrad invariant components of the Weyl curvature are given by

$$\Psi_1 = \begin{cases} \rho K^{-3} \psi_4 \equiv -\rho K^{-4} C_{nmnm}, & \text{for } s = -2, \\ \psi_0 \equiv -C_{rmlm}, & \text{for } s = +2, \end{cases}$$

(2)

where an overbar means complex conjugation and $\rho K$ is given in equation (5) below. Asymptotically, the leading behaviour of the field $\Psi_1$ represents either the outgoing radiative part of the perturbed Weyl tensor ($s = -2$) or the ingoing radiative part ($s = +2$).

The components of the Boyer–Lindquist null tetrad for the Kerr background are given by

$$\begin{align*}
(l_K^a) &= \left( \frac{r^2 + a^2}{\Delta}, 1, 0, \frac{a}{\Delta} \right), \\
(n_K^a) &= \frac{1}{2 (r^2 + a^2 \cos^2 \theta)} \left( r^2 + a^2, -\Delta, 0, a \right), \\
(m_K^a) &= \frac{1}{\sqrt{2} (r + ia \cos \theta)} \left( ia \sin \theta, 0, 1, i/sin \theta \right).
\end{align*}$$

(3)

One can also define directional derivatives as

$$\delta = m_K^\mu \partial_\mu, \quad \hat{\Delta} = n_K^\mu \partial_\mu, \quad \hat{D} = l_K^\mu \partial_\mu.$$  

(4)

With the above choice of the tetrad, the nonvanishing spin coefficients are (where an overbar stands for complex conjugation)

$$\begin{align*}
\rho_K &= -\frac{1}{(r - ia \cos \theta)}, & \beta_K &= -\frac{\rho K \cot \theta}{2 \sqrt{2}}, \\
\pi_K &= ia \rho K^2 \frac{\sin \theta}{\sqrt{2}}, & \tau_K &= -ia \rho K \beta K \frac{\sin \theta}{\sqrt{2}}, \\
\mu_K &= \rho K^2 \frac{\Delta}{2}, & \alpha_K &= \pi K - \beta K, \\
\gamma &= \mu + \rho K \beta K \frac{(r - M)}{2}, \quad \text{and the only nonvanishing Weyl scalar in the background is} \\
\psi_2 &= M \rho K^3.
\end{align*}$$

(5)

As we mentioned above, the Weyl scalars $\psi_4$ and $\psi_0$ allow a direct computation of the radiation escaping to infinity [8] and going down the horizon. The time-domain formulation is particularly well suited for interfacing with full numerical relativity techniques [1–5]. There are, besides other physical phenomena of interest such as the self-force on a particle orbiting the hole [16], studies of the horizon structure and second-order perturbations [8], which require the computation of the metric perturbations.

Starting from the pioneering work of Chrzanowski [10], there is a series of papers [15, 27, 24, 21] that deal with the problem of metric reconstruction by the introduction of a potential that satisfies the Teukolsky equation, neither $\psi_4$ nor $\psi_0$ describing the physical situation under study. The problem of relating the potentials introduced to describe the radiation gauges and
the physical $\psi_4$ and $\psi_0$ has been recently studied in [18]. There, the results are explicitly given for vacuum metric perturbations on the Schwarzschild, i.e. a non-rotating black hole background.

Given the difficulties in obtaining by these methods the explicit metric expression for perturbations around a Kerr, i.e. a rotating black hole background, we present here an alternative approach. In this paper we give explicit formulae for the metric reconstruction in the Regge–Wheeler gauge, still for a non-rotating background but allowing for source terms, and bearing in mind, for instance, the applications to the radiation reaction problem. In the next section we will give explicitly the form $\psi_4$ and $\psi_0$ take in terms of metric perturbation, in the Regge–Wheeler gauge, making explicit use of the multipole decomposition of the metric. In order to invert these expressions, in section 3 we introduce the symmetric and antisymmetric components of the Weyl scalar under the discrete parity transformation. With the help of the field equations of general relativity in the Regge–Wheeler gauge (reviewed in appendix A), we succeed in expressing the metric perturbations in terms of $\psi_4$ and $\psi_0$, including matter terms. In the final section of the paper we describe how to generalize the first few of these steps to the Kerr background case, and speculate about the completion of this programme.

2. Weyl scalars

The first step in explicitly constructing the metric perturbations is actually computing the inverse relation that makes use of the definition of the Weyl scalars (2) in terms of the Weyl tensor. Chrzanowski [10] made this computation explicitly relating the perturbed Weyl scalars to the metric perturbations $\psi_4 = \frac{1}{16} \left\{ (\delta + 3\alpha + \beta - \tau)(\delta + 2\alpha + 2\beta - \tau)h_{nn} \right. \\
+ (\hat{\Delta} + \Pi + 3\gamma - \varpi)(\hat{\Delta} + \Pi + 2\gamma - 2\varpi)h_{mm} \\
- [(\hat{\Delta} + \Pi + 2\gamma - 2\varpi + 2\alpha) \\
+ (\delta + 3\alpha + \beta - \tau)(\hat{\Delta} + 2\Pi + 2\gamma)]h_{nm} \} \\
and \\
\psi_0 = \frac{1}{16} \{ (\delta - \alpha - 3\beta + \Pi)(\delta - 2\alpha - 2\beta + \Pi)h_{ll} \\
+ (\hat{\Delta} - \Pi - 3\epsilon + \tau)(\hat{\Delta} - \Pi - 2\epsilon + 2\tau)h_{mm} \\
- [(\hat{\Delta} - \Pi - 2\epsilon + 2\tau)(\delta + 2\Pi - 2\beta) \\
+ (\delta - \alpha - 3\beta + \Pi)(\hat{\Delta} - 2\Pi - 2\epsilon)]h_{lm} \},

(7)

(8)

where $h_{nn} = n^\mu n^\nu h_{\mu\nu}$, $h_{lm} = l^\mu m^\nu h_{\mu\nu}$, etc.

Many simplifications are possible in the analysis when the background has spherical symmetry. In the Schwarzschild black hole case, expressions (7) and (8) reduce to

$\psi_4 = \frac{1}{16} \left\{ \frac{1}{r^2} \left( \partial_\theta - \cot \theta - \frac{i}{\sin \theta} \partial_\varphi \right) \left( \partial_\theta - \frac{i}{\sin \theta} \partial_\varphi \right) \left[ h_{tt} - 2h_{t\varphi} + h_{\varphi \varphi} f^2 \right] \\
+ \left( \partial_t - f \partial_r + f' \right) \left( \partial_t - f \partial_r \right) \frac{1}{r^2} \left[ h_{\theta \theta} - \frac{h_{\varphi \varphi}}{\sin^2 \theta} - 2f h_{\varphi \theta} \right] \right\} \\
- \frac{2}{r^2} \left( \partial_\theta - f \partial_r + f' \right) \left( \partial_\theta - \cot \theta - \frac{i}{\sin \theta} \partial_\varphi \right) \left[ h_{t\theta} - \frac{h_{\varphi \varphi}}{\sin \theta} - f \left( h_{\theta \varphi} - i \frac{h_{\varphi \theta}}{\sin \theta} \right) \right] \}

(9)
\[ \psi_0 = \frac{1}{4} \left\{ \frac{1}{r^2} \left( \partial_\theta - \cot\theta + \frac{i}{\sin\theta} \partial_\phi \right) \left( \partial_\theta + \frac{i}{\sin \theta} \partial_\phi \right) \left[ h_{rr}, f^{-2} + 2 h_{rr} f^{-1} + h_{rr} \right] \right. \\
+ \left( f^{-1} \partial_t + \partial_r + \frac{2}{r} \right) \left( f^{-1} \partial_t + \partial_r + \frac{i}{\sin \theta} \partial_\phi \right) \left[ h_{\phi\phi} - \frac{h_{\psi\psi}}{\sin^2 \theta} + 2i \frac{h_{\theta\phi}}{\sin \theta} \right] \left[ h_{t\theta} + i h_{t\phi} \sin \theta \right] \left[ h_{r\phi} + i h_{r\theta} \sin \theta \right] \right\} \]
\[ \times \left[ f^{-1} \left( h_{t\theta} + i h_{t\phi} \sin \theta \right) + h_{r\phi} + i h_{r\theta} \sin \theta \right], \] (10)

where \( f = 1 - 2M/r \) and \( f' = 2M/r^2 \).

The imposition of spherical symmetry also carries the following computational advantage: the multipole decomposition of the metric perturbations in terms of spin-weighted harmonics \(-2Y_{\ell m}(\theta)\) can be performed \([22, 19]\), and even and odd parity perturbations decouple so they can be considered independently. Below, we shall decompose all metric perturbations in multipoles with index \( \ell m \) (not to be confused with the tetrad vectors).

From equations (9) and (10) in the Regge–Wheeler gauge \( (h_{\ell m}^{(e)} = h_{\ell m}^{(o)} = G_{\ell m} = 0 = \langle \text{odd} \rangle H_{\ell m}^{(e)} \) ), we get

\[ \psi_4 \dot{=} \sum_{\ell m} \psi_4^{(e)}_{\ell m} - 2Y_{\ell m} = - \sum_{\ell m} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \left\{ \frac{-f}{16r^2} \left( H_{0}^{(e)} - 2 H_{1}^{(e)} + H_{2}^{(e)} \right) \right. \]
\[ \left. - \frac{i}{8r^2} \left[ \partial_t - f \partial_r + f' \left( \langle \text{odd} \rangle h_{0}^{(e)} - f \langle \text{odd} \rangle h_{1}^{(e)} \right) \right] \left\} - 2Y_{\ell m} \right\} \] (11)

and

\[ \psi_0 \dot{=} \sum_{\ell m} \psi_0^{(e)}_{\ell m} + 2Y_{\ell m} = - \sum_{\ell m} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \left\{ \frac{-1}{4fr^2} \left( H_{0}^{(e)} + 2 H_{1}^{(e)} + H_{2}^{(e)} \right) \right. \]
\[ \left. - \frac{i}{2fr^2} \left[ \partial_t + f \partial_r - f' \left( \langle \text{odd} \rangle h_{0}^{(e)} + f \langle \text{odd} \rangle h_{1}^{(e)} \right) \right] \left\} + 2Y_{\ell m} \right\}. \] (12)

These represent our basic equations (real and imaginary parts) that we will use in the next section to express the metric perturbations in terms of \( \psi_4 \) and \( \psi_0 \).

3. Explicit solution in the Regge–Wheeler gauge

Two key elements are introduced here in order to complete the inversion of the metric coefficients from equations (11) and (12). The first is the decomposition of the Weyl field into its symmetric and antisymmetric parts with respect to the parity transformation. This allows us to separate the even and odd parity perturbations from the multipole decomposed \( \psi_4, \psi_0 \) and metric perturbations. For the even parity case, this allows us to obtain directly two of the four metric coefficients. In order to obtain the other two, we have to resort to the general relativity field equations, which represent the other key element in the inversion process. For the odd parity case, one ends up with first-order differential relations that can be brought to explicit integrals (previous simplification by making use of the odd parity field equations).
3.1. Even parity

Let us define the symmetric and antisymmetric Weyl scalar fields as [18]

$$\psi^\pm = \frac{1}{2} [\psi^{\ell, m} \pm (-)^m \bar{\psi}^{\ell, -m}],$$

where for notational simplicity we drop the $\ell, m$ indices. Thus, given the symmetric nature of the even parity metric perturbations, equations (11) and (12) take the form

$$\psi^+_4 = \frac{f}{16r^2} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} (H^{0}_{\ell m} - 2H_{1m}^{\ell m} + H_{2m}^{\ell m})$$

and

$$\psi^+_0 = \frac{1}{4f r^2} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} (H^{0}_{\ell m} + 2H_{1m}^{\ell m} + H_{2m}^{\ell m}).$$

From equations (14) and (15), we can obtain the $\ell m$ components of the metric perturbations as follows:

$$H^{\ell m}_1(r, t) = -\frac{4r^2}{f} \sqrt{\frac{(\ell + 2)!}{(\ell - 2)!}} \left[ \psi^+_4 - \frac{f^2}{4} \psi^+_0 \right]$$

and

$$H^{\ell m}_0(r, t) + H^{\ell m}_2(r, t) = \frac{8r^2}{f} \sqrt{\frac{(\ell + 2)!}{(\ell - 2)!}} \left[ \psi^+_4 + \frac{f^2}{4} \psi^+_0 \right].$$

We now bring into the play the Hilbert–Einstein equations in the Regge–Wheeler gauge: equation (A.14) gives

$$H^{\ell m}_0(r, t) - H^{\ell m}_2(r, t) = \frac{16\pi r^2}{\sqrt{2\lambda(\lambda + 1)}} F^{\ell m},$$

where $F^{\ell m}$ is a source term given in table 3 of [29] (see also appendix B). This allows us to find the metric perturbations for $H^{\ell m}_0, H^{\ell m}_1$ and $H^{\ell m}_2$. The last metric coefficient has to be found by use of the Hilbert–Einstein equations ($K^{\ell m}$ and $H^{\ell m}_0 - H^{\ell m}_2$ give a measure of the trace of the even parity sector in the Regge–Wheeler gauge, so it does not appear in the Weyl scalars since the Weyl tensor is traceless).

Using equation (A.12), we can solve for $\partial_r K^{\ell m}$ and then replace it in equation (A.8) to find $K^{\ell m}$ in terms of the other even parity metric coefficients and source terms

$$K(r, t)^{\ell m} = 2 \frac{(r - M) \frac{\partial}{\partial r} H^{\ell m}_0(r, t) + (r - 2M) \frac{\partial^2}{\partial r^2} H^{\ell m}_0(r, t)}{\lambda}$$

$$+ \frac{r^2 \frac{\partial}{\partial r} H^{\ell m}_1(r, t)}{\lambda} + \frac{M \frac{\partial}{\partial r} H^{\ell m}_2(r, t)}{\lambda} - \frac{(2r^2 - 8rM + 9M^2) H^{\ell m}_0(r, t)}{r\lambda(r - 2M)}$$

$$+ \frac{(-r^2 \lambda + 2rM\lambda + 3M^2 - 2rM) H^{\ell m}_1(r, t)}{r\lambda(r - 2M)} + \frac{r(-3r + 7M) \frac{\partial}{\partial r} H^{\ell m}_1(r, t)}{\lambda(r - 2M)}$$

$$+ \frac{8(r - 2M)\pi r^2 \frac{\partial}{\partial r} B^{\ell m}(r, t)}{\sqrt{\lambda + 1\lambda}} - \frac{8(7M - 4r\pi r B^{\ell m}(r, t)}{\sqrt{\lambda + 1\lambda}} = \frac{A^{\ell m}_0(r, t) r^3}{\lambda(r - 2M)}.\)
equation (A.12)

\begin{equation}
K_{\ell m} = H_{0m} + \int_{2M}^{r} \frac{dr}{1 - \frac{2M}{r}} \left[ - \frac{\partial H_{1m}^{\ell}}{\partial r} + \frac{2M}{r^2} H_{0m}^{\ell} - 16\pi (r - 2M) \frac{F_{\ell m}^{\ell}}{\sqrt{2\lambda(\lambda + 1)}} - \frac{8\pi (r - 2M)}{\sqrt{\lambda + 1}} B_{\ell m}^{\ell} \right].
\end{equation}

(20)

It is worth mentioning here that if the source is modelled as a particle (represented by a Dirac delta), the above metric coefficients are continuous (C\(^0\)) for head-on collisions [16] at the location of the particle. However, for more general orbits, they do not all appear to be continuous but some of them behave as a Dirac delta. For instance, one can see that from expression (18). \(F_{\ell m}^{\ell}\) is proportional to a Dirac delta, as given in table I; hence, at least \(H_{2m}^{\ell}\) or \(H_{2m}^{\ell}\) have to behave as \(\delta[r - R(t)]\).

3.2. Odd parity

From equations (11) and (12), given the antisymmetric behaviour of the odd parity metric coefficients, we get

\begin{equation}
\psi_{-4} = \frac{i}{8\sqrt{2}} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \left[ \partial_t f \frac{\partial r}{\partial t} + f' \right] (\text{odd}) h_{0m}^{\ell} f (\text{odd}) h_{1m}^{\ell}.
\end{equation}

(21)

and

\begin{equation}
\psi_{-0} = \frac{i}{2\sqrt{2}} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \left[ \partial_t f \frac{\partial r}{\partial t} + f' \right] (\text{odd}) h_{0m}^{\ell} f (\text{odd}) h_{1m}^{\ell}.
\end{equation}

(22)

A linear combination of these previous equations produces

\begin{equation}
\psi_{-4} + \frac{f^2}{4} \psi_{-0} = \frac{i}{4\sqrt{2}} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \left[ \partial_t f \frac{\partial r}{\partial t} + f' \right] (\text{odd}) h_{0m}^{\ell} f (\text{odd}) h_{1m}^{\ell}.
\end{equation}

(23)

and

\begin{equation}
\psi_{-4} - \frac{f^2}{4} \psi_{-0} = -\frac{i}{4\sqrt{2}} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \left[ \partial_t f \frac{\partial r}{\partial t} + f' \right] (\text{odd}) h_{0m}^{\ell} f (\text{odd}) h_{1m}^{\ell}.
\end{equation}

(24)

From equation (A.17) we can substitute \(\partial_t h_{0m}^{\ell}\) into equation (23) leading to the equation

\begin{equation}
\partial_r h_{1m}^{\ell} + \left( \frac{f'}{2f} \right) h_{1m}^{\ell} = S_{1m}^{\ell}(r, t) \pm \frac{2\pi r^2}{\sqrt{(\ell + 2)!}} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \left( \psi_{-4} + \frac{f^2}{4} \psi_{-0} \right) - \frac{2\pi r^2 D_{\ell m}}{\sqrt{\lambda(\lambda + 1)}}.
\end{equation}

(25)

which integrated produces

\begin{equation}
h_{1m}^{\ell} = \frac{1}{\sqrt{1 - \frac{2M}{r}}} \left[ \int_{r'}^{r} S_{1m}^{\ell}(r', t) \sqrt{1 - \frac{2M}{r'}} dr' + C_{1m}^{\ell}(t) \right],
\end{equation}

(26)

where \(C_{1m}^{\ell}\) is an integration constant that in vacuum and with vanishing \(\psi_{0}\) and \(\psi_{4}\) can be taken to vanish [26].

Knowing now the form of \(h_{1m}^{\ell}\), we can use equation (24) to find a differential equation for \(h_{0m}^{\ell}\):

\begin{equation}
\partial_r h_{0m}^{\ell} - \left( \frac{f'}{f} \right) h_{0m}^{\ell} = S_{0m}^{\ell}(r, t) \pm \frac{4\pi r^2}{\sqrt{(\ell + 2)!}} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \left( \psi_{-4} - \frac{f^2}{4} \psi_{-0} \right) - \partial_t h_{1m}^{\ell}.
\end{equation}

(27)
with solution
\[ h_{0}^{\ell m} = \left( 1 - \frac{2M}{r} \right) \left\{ \int_{2M}^{r} \frac{S_{0}^{\ell m}(r', t)}{1 - \frac{2M}{r'}} \, dr' + C_{0}^{\ell m}(t) \right\}. \] (28)

Again, \( C_{0}^{\ell m} \) is an integration constant that in vacuum and with vanishing \( \psi_{0} \) and \( \psi_{4} \) can be taken to vanish.

This essentially completes the work of expressing the metric perturbations in terms of the computed \( \psi_{0} \) and \( \psi_{4} \) expressed in the time domain. For the even parity case, it contains second derivatives of the Weyl scalars (unlike the corresponding expressions for the radiation gauge [18]). For the odd parity case, solutions (26) and (28) are written in an integral form.

A last observation applies here, since the spin weight of the Weyl scalars are \( s = \pm 2 \) they do not contain multipole modes \( \ell = 0 \) and \( \ell = 1 \), hence we need to give them by directly solving the field equations for the metric coefficients. The Regge–Wheeler gauge do not completely allow us to determine them because there is one extra degree of freedom for \( \ell = 1 \) and two degrees of freedom for \( \ell = 0 \). Zerilli [29] made choices to fix this extra freedom that allowed him to solve analytically for the metric coefficients. In [6], a different choice was made to make those coefficients continuous in the head-on collision of extreme mass black holes. Finally in [12], the metric coefficients for \( \ell = 0, 1 \) have been found in the harmonic gauge, for particles in circular orbits.

4. Discussion of Kerr perturbations

A possible generalization of the Regge–Wheeler gauge conditions for spherically symmetric backgrounds, but where perturbations are not decomposed into multipoles is [7]

\[ (\sin \theta)^{2} h_{\theta \theta} - h_{\phi \phi} = 0, \] (29)
\[ h_{\theta \phi} = 0, \] (30)
\[ \sin \theta \partial_{\theta} (\sin \theta h_{\theta \phi}) + \partial_{\phi} h_{\theta \phi} = 0, \] (31)
\[ \sin \theta \partial_{\theta} (\sin \theta h_{r \phi}) + \partial_{\phi} h_{r \phi} = 0. \] (32)

The first equation above leads to the condition \( G_{\ell m} = 0 \). The second then gives \( (\text{odd}) h_{0}^{\ell m} = 0 \). The other two differential conditions are chosen such that they lead to \( (\text{even}) h_{0}^{\ell m} = 0 = (\text{even}) h_{1}^{\ell m} \), but allow \( (\text{odd}) h_{0}^{\ell m} \neq 0 \) and \( (\text{odd}) h_{1}^{\ell m} \neq 0 \) to be unconstrained.

Now we will consider the generalization of the Regge–Wheeler gauge in the Newman–Penrose formalism. In this formalism, the first two Regge–Wheeler conditions, equations (29) and (30), have a simple generalization

\[ h_{\mu \nu} = m^{\mu} m^{\nu} h_{\mu \nu} = 0. \] (33)

Note that requiring that the real and imaginary parts vanish contains both conditions. Obviously,

\[ h_{\mu \nu} = \bar{m}^{\mu} m^{\nu} h_{\mu \nu} = 0 \] (34)

also holds. Note that conditions (33) and (34) are invariant under type III (spin–boosts) transformations of the background tetrad

\[ l \rightarrow A \hat{l}, \quad n \rightarrow A^{-1} n, \quad m \rightarrow e^{2i\epsilon} m, \quad \bar{n} \rightarrow e^{-2i\epsilon} \bar{n}. \] (35)

This is an important feature since the Kinnersley choice of the tetrad, with the spin coefficient \( \epsilon = 0 \), is just a simple but arbitrary way of fixing the spin–boost freedom.
In contrast, the convenient choice of the \( l \) and \( n \) tetrad vectors along the repeated principal null directions of the Kerr background allows us to single out wave equations for the perturbations of \( \psi_4 \) and \( \psi_0 \).

To generalize the differential conditions (31) and (32), one can resort to the type III transformation properties of the \( \delta \) and \( \bar{\delta} \), as well as spin coefficient operators in the Kerr background acting on the metric coefficients \( h_{lm} \) and \( h_{lmn} \). The objects
\[
(\delta - 2\bar{\alpha})h_{lmn} \rightarrow \Lambda^2(\delta - 2\bar{\alpha})h_{lmn}
\]
and
\[
(\bar{\delta} + 2\bar{\beta})h_{mn} \rightarrow \Lambda^{-2}(\bar{\delta} + 2\bar{\beta})h_{mn}
\]
transform as objects of spin 0 and boost weight +1 and −1, respectively, under type III transformations of the background tetrad (35).

In order to reproduce the differential conditions (31) and (32) in the Schwarzschild limit, one then requires
\[
\text{Re}\left[(\delta - 2\bar{\alpha} + a\tau - b\bar{\pi})h_{lmn}\right] = 0
\]
and
\[
\text{Re}\left[(\bar{\delta} + 2\bar{\beta} + c\bar{\tau} - d\pi)h_{mn}\right] = 0,
\]
where \( \text{Re} \) is the real part and where, following [28], we have introduced additional terms containing spin coefficients with spin \( \pm 1 \), respectively, and boost 0 multiplied by constants \( a, b, c, d \) to allow for a more general choice of the gauge. These constants can be readily chosen to facilitate the metric reconstruction or, in other contexts, to impose further symmetries or facilitate the numerical integration of general relativity field equations, etc. It also stresses the ambiguities in generalizing the Regge–Wheeler gauge on the Kerr background.

Note also that conditions (38) and (39) are invariant under type III transformations as well. A crucial role in achieving this is played by the spin-0 transformation properties of the constructed object, allowing invariance of its real part.

Independently, we can try to proceed along the lines of the previous sections with now a simple mode decomposition of the metric coefficients. For instance,
\[
h_{lmn}(t, r, \theta, \phi) = \sum_m e^{im\phi}h_{lm}^m(t, r, \theta).
\]

We can now replace this decomposition directly into equations (7) and (8) for \( \psi_4 \) and \( \psi_0 \) or the following more convenient form making use of equation (2.11) of [25], where we have the following identity:
\[
[\hat{D} - (p + 1)\epsilon + \bar{\epsilon} + q\rho - \bar{\rho}](\delta - p\beta + q\tau) = [\delta - (p + 1)\beta - \alpha + q\tau + \pi](\hat{D} - p\epsilon + q\rho),
\]
and the identity derived from it exchanging tetrads \( l \rightarrow n \) and \( m \rightarrow \bar{m} \):
\[
[\hat{\Delta} + (p + 1)\gamma - \bar{\gamma} - q\mu + \bar{\mu}](\delta + p\alpha - q\pi) = [\delta + (p + 1)\alpha + \bar{\beta} - q\pi - \bar{\tau}](\hat{\Delta} + p\gamma - q\mu).
\]

Using \( p = 2 \) and \( q = 0 \) in the identities above allows us to rewrite equations (7) and (8) as
\[
\psi_4 = \frac{1}{2}\left\{[\delta + 3\alpha + \bar{\beta} + \bar{\gamma}](\delta + 2\alpha + 2\bar{\beta} + 2\bar{\gamma})h_{mn} + (\hat{\Delta} + \bar{\pi} + 3\gamma - \bar{\gamma})(\hat{\Delta} + \bar{\pi} + 2\gamma - 2\bar{\gamma})h_{mn}ight. \\
- [2(\hat{\Delta} + \bar{\pi} + 3\gamma - \bar{\gamma})(\delta + 3\alpha + \bar{\beta} + 2\bar{\pi}) + 2(\delta + 3\alpha + \bar{\beta} + 2\bar{\pi})h_{mn}\}
\]
(44)
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\[ \psi_0 = \frac{1}{2} \left\{ (\delta - \alpha - 3\beta + \pi)(\delta - 2\alpha - 2\beta + \pi)h_{ll} + (\dot{\alpha} - \pi - \alpha - 3\beta + \pi)(\dot{\alpha} - \pi - 2\beta + 2\pi)h_{mm} \right. \\
- \left. 2(\dot{\alpha} - \pi - 3\epsilon + \pi)(\delta + \pi - 2\beta) - 2(\delta - \alpha - 3\beta + \pi)(\pi)h_{lm} \right\}. \quad (45) \]

A choice of the symmetric tetrad (appendix D) will further simplify the appearance of the equations. At this point, we impose our gauge condition on \( h_{lm} + (lm) \) and \( h_{nm} + (nm) \). Then, paralleling the work done in the non-rotating limit, we could make further progress by writing explicitly the Newman–Penrose equations in terms of metric perturbations, for finally using these equations to obtain decoupled expressions for some metric coefficients. The completion of this programme remains an open issue and goes beyond the scope of this paper. We leave this for future research.

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**Appendix A. Hilbert–Einstein equations in the Regge–Wheeler gauge**

Because of some misprints in the original Zerilli paper [29], we reproduce here the relevant equations for our discussions (see also [13, 23]).

The metric perturbations on a Schwarzschild background

\[ ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (A.1) \]

can be decomposed into spherical harmonics [29]

\[ h^{\text{even}}_{\mu\nu} = \begin{pmatrix} H_{0m}^m(t, r) & h_{1m}^m(t, r) & h_{2m}^m(t, r) & h_{0m}^m(t, r) \\
H_{1m}^m(t, r) & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\
H_{2m}^m(t, r) & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\
\end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix} \begin{pmatrix} Y_{lm}(\theta, \phi) \\
Y_{lm}(\theta, \phi) \\
Y_{lm}(\theta, \phi) \\
\end{pmatrix} \quad (A.2) \]

for the even parity modes and

\[ h^{\text{odd}}_{\mu\nu} = \begin{pmatrix} h_{0m}^m(t, r) & h_{1m}^m(t, r) & h_{2m}^m(t, r) & h_{0m}^m(t, r) \\
-\frac{\partial}{\sin \theta \partial \phi} & -\frac{\partial}{\sin \theta \partial \phi} & -\frac{\partial}{\sin \theta \partial \phi} & -\frac{\partial}{\sin \theta \partial \phi} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix} \begin{pmatrix} Y_{lm}(\theta, \phi) \\
Y_{lm}(\theta, \phi) \\
Y_{lm}(\theta, \phi) \\
Y_{lm}(\theta, \phi) \\
\end{pmatrix} \quad (A.3) \]

for the odd parity modes, where \( g_{\mu\nu} = g_{\text{Schw}} + h_{\mu\nu} \).
Above, we used Zerilli’s notation
\[ \dot{X}_{\ell m} = 2 \frac{\partial}{\partial \varphi} \left( \frac{\partial}{\partial \theta} - \cot \theta \right), \]
(A.4)
\[ \dot{W}_{\ell m} = \left( \frac{\partial^2}{\partial \theta^2} - \cot \theta \frac{\partial}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right), \]
(A.5)
\[ \dot{Z}_{\ell m} = \left( \frac{\partial^2}{\partial \varphi^2} + \sin \theta \cos \theta \frac{\partial}{\partial \theta} \right). \]
(A.6)

We will also introduce Zerilli’s \( \lambda \)
\[ \lambda = (\ell - 1)(\ell + 2)/2. \]
(A.7)

A.1. Even parity

Zerilli’s equations (C.7a)–(C.7g) with corrections are
\[ \left( 1 - \frac{2M}{r} \right)^2 \frac{\partial^2 K_{\ell m}^0}{\partial r^2} + \frac{1}{r} \left( 1 - \frac{2M}{r} \right) \left( 3 - \frac{5M}{r} \right) \frac{\partial K_{\ell m}^n}{\partial r} \]
\[ = -8\pi A_{\ell m}^{(0)}, \]
(A.8)
\[ \frac{\partial}{\partial t} \left[ \frac{\partial K_{\ell m}^n}{\partial r} + \frac{1}{r} \left( K_{\ell m}^n - H_{\ell m}^n \right) - \frac{M}{r(r - 2M)} K_{\ell m}^n \right] - \frac{(\lambda + 1)}{r^2} H_{\ell m}^n = -4\sqrt{2} \pi i A_{\ell m}^{(1)}, \]
(A.9)
\[ \left( \frac{r}{r - 2M} \right)^2 \frac{\partial^2 K_{\ell m}^n}{\partial t^2} - \frac{r - M}{r(r - 2M)} \frac{\partial K_{\ell m}^n}{\partial r} - \frac{2}{r - 2M} \frac{\partial H_{\ell m}^n}{\partial t} \]
\[ + \frac{1}{r} \frac{\partial H_{\ell m}^n}{\partial r} + \frac{1}{r(r - 2M)} \left( H_{\ell m}^n - K_{\ell m}^n \right) \]
\[ = -8\pi A_{\ell m}, \]
(A.10)
\[ \frac{\partial}{\partial r} \left[ \left( 1 - \frac{2M}{r} \right) H_{\ell m}^1 \right] - \frac{\partial}{\partial t} \left( H_{\ell m}^2 + K_{\ell m}^n \right) = \frac{8\pi i r}{\sqrt{\lambda + 1}} B_{\ell m}^{(0)}, \]
(A.11)
\[ - \frac{\partial H_{\ell m}^n}{\partial t} + \left( 1 - \frac{2M}{r} \right) \frac{\partial}{\partial r} \left( H_{\ell m}^n - K_{\ell m}^n \right) + \frac{2M}{r^2} H_{\ell m}^n \]
\[ + \frac{1}{r} \left( 1 - \frac{M}{r} \right) \left( H_{\ell m}^2 - H_{\ell m}^0 \right) = \frac{8\pi (r - 2M)}{\sqrt{\lambda + 1}} B_{\ell m}, \]
(A.12)
\[ \frac{\partial}{\partial r} \left[ \left( 1 - \frac{2M}{r} \right) H_{\ell m}^0 \right] + \frac{\partial}{\partial t} \left( H_{\ell m}^2 + K_{\ell m}^n \right) \]
\[ = \frac{8\pi i r}{\sqrt{\lambda + 1}} B_{\ell m}^{(0)}, \]
(A.13)
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Table 1. Energy–momentum–stress tensor in terms of tensor harmonics.

| Term | Formula |
|------|---------|
| $A_{lm}(r,t) = m_0 U^0(t) \frac{d}{dr}(r - 2M)^{-2} \delta [r - R(t)] T_{lm}(\Omega_\nu(t))$ |
| $A_{lm}^{(0)} = m_0 U^0(t) \frac{1}{r^2} \delta [r - R(t)] T_{lm}(\Omega_\nu(t))$ |
| $A_{lm}^{(1)} = \sqrt{2} m_0 U^0(t) \frac{dR}{dt} r^{-2} \delta [r - R(t)] T_{lm}(\Omega_\nu(t))$ |
| $B_{lm}^{(0)} = [\lambda + 1]^{-1/2} m_0 U^0(t) \frac{d}{dr} \delta [r - R(t)] T_{lm}(\Omega_\nu(t))$ |
| $B_{lm}^{(1)} = [\lambda + 1]^{-1/2} m_0 U^0(t) \frac{d}{dr} \delta [r - R(t)] T_{lm}(\Omega_\nu(t))$ |
| $Q_{lm}^{(0)} = [\lambda + 1]^{-1/2} m_0 U^0(t) \frac{d}{dr} \delta [r - R(t)]$ |

The Einstein equation in the Regge–Wheeler gauge for the odd parity sector are (see Zerilli’s equations (A.2a)–(A.2c) [30]. Note the corrections to the source terms.)

\[\frac{\partial^2 h^{0lm}_m}{\partial r^2} - \frac{\partial h^{0lm}_m}{\partial r} - \frac{2}{r} \frac{\partial h^{0lm}_m}{\partial t} + \frac{4M}{r^2} - \frac{2(\lambda + 1)}{r} = \frac{8\pi r Q_{lm}^{(0)}}{(r - 2M)\sqrt{\lambda + 1}}, \tag{A.15}\]

\[\frac{\partial^2 h^{1lm}_m}{\partial t^2} - \frac{\partial h^{1lm}_m}{\partial t} + \frac{2}{r} \frac{\partial h^{1lm}_m}{\partial r} + 2\lambda (r - 2M) \frac{h^{1lm}_m}{r^3} = \frac{8\pi i (r - 2M) Q_{lm}}{\sqrt{\lambda + 1}}, \tag{A.16}\]

\[\left(1 - \frac{2M}{r}\right) \frac{\partial h^{1lm}_m}{\partial r} - \frac{1}{(1 - \frac{2M}{r})} \frac{\partial h^{1lm}_m}{\partial t} + \frac{2M}{r^2} h^{1lm}_m = \frac{4\pi ir^2 D_{lm}}{\sqrt{\lambda + 1}} \tag{A.17}\]

A2. Odd parity

The Einstein equation in the Regge–Wheeler gauge for the odd parity sector are (see Zerilli’s equations (C.6a)–(C.6c) [30]. Note the corrections to the source terms.)

A3. Source terms

Table 1 gives the source terms produced by an orbiting particle in the Schwarzschild background after decomposition of the stress–energy tensor into tensor harmonics. There,
Appendix B. Reconstruction in terms of metric perturbation waveforms

Here we recall the metric reconstruction in the original Schwarzschild perturbations approach based on waveforms for the even and odd parity perturbations (Zerilli’s and Regge–Wheeler, respectively). We first introduce the gauge invariant expressions for these waveforms. We then make use of the general relativistic field equations, in the Regge–Wheeler gauge, to solve for the metric perturbations, including nonvanishing matter terms.

B.1. Even parity

We consider the following waveform [17] in terms of generic metric perturbations in the Regge–Wheeler notation:

\[
\psi_{\ell m}^{\text{even}}(r, t) = \frac{r}{(\lambda + 1)} \left[ K_{\ell m} + \frac{r - 2M}{\lambda r + 3M} \left( H_{2}^{\ell m} - r \partial_r K_{\ell m} \right) \right] + \frac{r - 2M}{\lambda r + 3M} \left[ r^2 \partial_r G_{\ell m} - 2h_1^{\ell m} \right].
\]

(B.1)

This is related to Zerilli’s [30] even parity waveforms, \( \psi_{\ell m}^{\text{even}} \), by

\[
\partial_t \psi_{\ell m}^{\text{even}} = \psi_{\text{Zer}}^{\ell m} + \frac{4\pi i \sqrt{2} r^2 (r - 2M) A_{\ell m}^{(1)}}{(\lambda + 1)(\lambda r + 3M)},
\]

(B.2)

where for an orbiting particle [30]

\[
A_{\ell m}^{(1)} = i m_0 \sqrt{2} \left( \frac{U^0(t)}{r^2} \right) \left( \frac{dR}{dt} \right) \bar{Y}_{\ell m} \delta(r - R(t)),
\]

(B.3)

and it relates to Moncrief’s [19] waveform, \( \psi_{\text{Mon}}^{\ell m} \), by

\[
\psi_{\ell m}^{\text{even}} = \psi_{\text{Mon}}^{\ell m} \frac{\lambda}{\lambda + 1}.
\]

(B.4)

The \( tt \) component of the Hilbert–Einstein equations gives us the Hamiltonian constraint. In the Regge–Wheeler gauge \( h_1^{\ell m} = h_0^{\ell m} = G_{\ell m} = 0 \), it is given by equation (A.8). Only two metric coefficients \( K_{\ell m} \) and \( H_{2}^{\ell m} \) appear in this equation and none of its time derivatives. Considering the Regge–Wheeler gauge, the definition of \( \psi_{\ell m}^{\text{even}} \) (see equation (B.1)) and the Hamiltonian constraint, equation (A.8), we can express these two metric coefficients in terms of \( \psi_{\ell m}^{\text{even}} \) (and source terms) only

\[
K_{\ell m} = \frac{6M^2 + 3M \lambda r + \lambda (\lambda + 1) r^2}{r^2(\lambda r + 3M)^2} \psi_{\ell m}^{\text{even}} + \left( 1 - \frac{2M}{r} \right) \partial_r \psi_{\ell m}^{\text{even}} - \frac{8\pi r^3 A_{\ell m}^{(0)}}{(\lambda + 1)(\lambda r + 3M)}
\]

(B.5)

and

\[
H_{2}^{\ell m} = -\frac{9M^2 + 9\lambda M r + 3\lambda^2 M r^2 + \lambda^2 (\lambda + 1) r^3}{r^2(\lambda r + 3M)^2} \psi_{\ell m}^{\text{even}}
+ \frac{3M^2 - \lambda M r + \lambda^2}{r(\lambda r + 3M)} \partial_r \psi_{\ell m}^{\text{even}} + (r - 2M) \partial^2 \psi_{\ell m}^{\text{even}} - \frac{8\pi r^4 A_{\ell m}^{(0)}}{(\lambda + 1)(\lambda r + 3M)} \partial_r A_{\ell m}^{(0)}
+ \frac{8\pi r^4 (\lambda^2 r^2 - 2\lambda^2 + 10\lambda M r - 9r M + 27M^2)}{(\lambda + 1)(r - 2M)(\lambda r + 3M)^2} A_{\ell m}^{(0)}.
\]

(B.6)
From equation (A.9) and the expressions for \( \partial_t K_{\ell m} \) and \( \partial_t H_{\ell m}^2 \) in terms of \( \partial_t \psi_{\text{even}} \), we find the \( H_{\ell m}^1 \) metric coefficient in the Regge–Wheeler gauge

\[
H_{\ell m}^1 = r \partial_r \left( \partial_t \psi_{\text{even}} \right) + \frac{\lambda r^2 - 3 M \lambda r - 3 M^2}{(r - 2 M)(\lambda r + 3 M)} \partial_r \psi_{\text{even}} \\
- \frac{8 \pi r^5}{(\lambda + 1)(r - 2 M)(\lambda r + 3 M)} \partial_r A_{\ell m}^{(0)} + \frac{4 \sqrt{2 \pi} r^2}{(\lambda + 1)} A_{\ell m}^{(1)}.
\]

(B.7)

These equations together with

\[
H_{\ell m}^0 = H_{\ell m}^2 + \frac{16 \pi r^2 F_{\ell m}}{\sqrt{2 \lambda (\lambda + 1)}},
\]

(B.8)
give us all metric perturbations on the chosen hypersurface in terms only of \( \psi_{\text{even}} \) and \( \partial_t \psi_{\text{even}} \) (and the source) (see also [14] for general source expressions).

For the specific case of interest of a point-like particle, we have

\[
K_{\ell m} = \frac{6 M^2 + 3 M \lambda r + \lambda (\lambda + 1) r^2}{r^2(\lambda r + 3 M)} \psi_{\text{even}} + \left( 1 - \frac{2 M}{r} \right) \partial_r \psi_{\text{even}} \\
- \frac{8 \pi m_0 Y_{\ell m}(t) U_0(t)(r - 2 M)^2}{(\lambda + 1)(\lambda r + 3 M)r} \delta[r - R(t)],
\]

(B.9)

\[
H_{\ell m}^2 = -\frac{9 M^3 + 9 \lambda M^2 r + 9 \lambda^2 M r^2 + \lambda^2 (\lambda + 1) r^3}{r^2(\lambda r + 3 M)} \psi_{\text{even}} \\
+ \frac{3 M^2 - \lambda \lambda M r + \lambda r^2}{r(\lambda r + 3 M)} \partial_r \psi_{\text{even}} \partial_t \psi_{\text{even}} + (r - 2 M) \partial_r \psi_{\text{even}} \\
+ \frac{8 \pi m_0 Y_{\ell m}(t) U_0(t)(1 - \frac{2 M}{r})[\lambda^2 r^2 + 2 \lambda \lambda M r - 3 \lambda M r + 3 M^2]}{\delta[r - R(t)]}
\]

(B.10)

\[
H_{\ell m}^1 = r \partial_r \left( \partial_t \psi_{\text{even}} \right) + \frac{\lambda r^2 - 3 M \lambda r - 3 M^2}{(r - 2 M)(\lambda r + 3 M)} \partial_r \psi_{\text{even}} \\
- \frac{8 \pi m_0 Y_{\ell m}(t) U_0(t)\bar{r}_p(\lambda r + M)}{(\lambda + 1)(\lambda r + 3 M)} \delta[r - R(t)] \\
+ \frac{8 \pi m_0 Y_{\ell m}(t) U_0(t)\bar{r}_p(r - 2 M)}{(\lambda + 1)(\lambda r + 3 M)} \delta[r - R(t)]
\]

(B.11)

and

\[
H_{\ell m}^0 = H_{\ell m}^2 + 16 \pi r^2 m_0 U_0(t) \text{ang1}(t) \delta[r - R(t)],
\]

(B.12)

where

\[
\text{ang1}(t) = \frac{1}{2} \left[ \left( \frac{d\Theta}{dr} \right)^2 - \sin^2 \Theta \left( \frac{d\Phi}{dr} \right)^2 \right] \bar{W}_{\ell m} + \frac{d\Phi}{dr} \frac{d\Theta}{dr} \bar{X}_{\ell m}. \]

(B.13)

\[
\bar{X}_{\ell m} = 2 \partial_\phi (\partial_\phi - \cot \theta) \bar{Y}_{\ell m},
\]

(B.14)

\[
\bar{W}_{\ell m} = \left( \partial_\phi^2 - \cot \theta \partial_\phi \right) \bar{Y}_{\ell m}.
\]

(B.15)
B.2. Odd parity

We consider the following waveform in terms of generic metric perturbations in the Regge–Wheeler notation:

\[
\psi_{\ell m}^{\text{odd}}(r, t) = \frac{r}{\lambda} \left[ r^2 \partial_r \left( \frac{h_{0m}^{\text{em}}(r, t)}{r^2} \right) - \partial_t h_{1m}^{\text{em}}(r, t) \right] = \frac{2r}{\lambda} \sqrt{1 - \frac{2M}{r}} K_{\theta}.
\]  

(B.16)

This waveform is related to Zerilli’s [30] and Moncrief’s [19] odd parity waveforms

\[
\psi_{\text{Zer}}^{\text{odd}} = \psi_{\text{Mon}}^{\text{odd}}
\]

by (see equation (A.16))

\[
\partial_t \psi_{\text{odd}}^{\text{em}} = 2 \psi_{\text{Zer}}^{\text{odd}} - \frac{8\pi i r(r - 2M)Q_{\ell m}}{\lambda \sqrt{\lambda + 1}},
\]

(B.18)

to the Cunningham et al [11] waveform, \(\psi_{G}^{\ell m}\), by

\[
\psi_{\text{odd}}^{\text{em}} = -2 \frac{(\ell - 2)!}{(\ell + 2)!} \psi_{G}^{\ell m} = -\frac{1}{2} \psi_{G}^{\ell m} - \frac{1}{2} \psi_{G}^{\ell m},
\]

(B.19)

and to the Weyl scalar, \(\Psi_2\),

\[
\Psi_2 = \frac{(\ell + 2)!}{8(\ell - 2)!} \psi_{\text{odd}}^{\ell m}.
\]

(B.20)

(Here we used the Kinnersley tetrad in the Schwarzschild background and decomposed \(\Psi_2\) into spherical harmonics.)

One can use the field equations to write the metric perturbation in the Regge–Wheeler gauge

\[
h_{0m}^{\ell m}(r, t) = \frac{1}{2} \left( 1 - \frac{2M}{r} \right) \partial_t (\psi_{\text{odd}}^{\ell m}) + \frac{4\pi r^3 Q_{\ell m}^{(0)}}{\lambda \sqrt{\lambda + 1}},
\]

(B.21)

\[
h_{1m}^{\ell m}(r, t) = \frac{1}{2} \left( 1 - \frac{2M}{r} \right) \partial_t \psi_{\text{odd}}^{\ell m} + \frac{4\pi r^3 Q_{\ell m}}{\lambda \sqrt{\lambda + 1}}.
\]

For a source term represented by a particle, the corresponding metric perturbations in the Regge–Wheeler gauge are

\[
h_{0m}^{\ell m}(r, t) = \frac{1}{2} \left( 1 - \frac{2M}{r} \right) \partial_t (\psi_{\text{odd}}^{\ell m}) + \frac{4\pi m r (r - 2M) U^0(t) \text{ang}(t) \delta[r - R(t)]}{\lambda \sqrt{\lambda + 1}},
\]

(B.22)

\[
h_{1m}^{\ell m}(r, t) = \frac{1}{2} \left( 1 - \frac{2M}{r} \right) \partial_t \psi_{\text{odd}}^{\ell m} - \frac{4\pi m r^3 U^0(t) \left( \frac{d}{dt} R \right) \text{ang}(t) \delta[r - R(t)]}{(r - 2M) \lambda \sqrt{\lambda + 1}},
\]

(B.23)

where

\[
\text{ang}(t) = \frac{1}{\sin \Theta} \left( \frac{d\Theta}{dt} \right) \partial_\Theta \bar{Y}^{\ell m}(\Theta, \Phi) - \sin \Theta \left( \frac{d\Phi}{dt} \right) \partial_\Phi \bar{Y}^{\ell m}(\Theta, \Phi),
\]

(B.24)

and \(R(t), \Theta, \Phi\) define the trajectory of the orbiting particle in spherical coordinates.
Appendix C. (Inverse) Chandrasekhar transformations in the time domain

Chandrasekhar transformations deal with the expressions that relate the waveforms in the metric perturbation picture (Regge–Wheeler and Zerilli’s) and the Weyl scalars that describe the curvature perturbations (Teukolsky’s). Chandrasekhar [9] found the transformations in the frequency domain. Those expressions can be generalized to the time domain to describe local transformations, and take into account matter terms such as a particle. Below, we give the explicit expressions for the transformations (we drop the \((\ell m)\) superscript in the waveforms for the sake of notational simplicity).

C.1. From waveforms to Weyl scalars

To obtain the Weyl scalars from the waveforms \((B^{+}_1)\) and \((B^{2}_1)\) for even and odd parity respectively, we simply substitute into equations \((14), (15), (21)\) and \((22)\) the expressions \((B^{0}_5)–(B^{0}_8)\) and \((B^{0}_{21})–(B^{0}_{22})\) for the metric coefficients in the Regge–Wheeler gauge. The result is

\[
\psi^+_{4} = \frac{1}{16r} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \left\{ \begin{array}{l}
2\psi^{\text{even}}_{tr} - 2\psi^{\text{even}}_{tr} + W^+ (\psi^{\text{even}}_{tr} - \psi^{\text{even}}_{t}) - V^+ \psi^{\text{even}} \\
+ \frac{16\pi r^3}{(\lambda r + 3M)(\lambda + 1)} \left( \partial_r A^{(0)}_{\ell m} - \partial_r A^{(0)}_{\ell m} \right) - \frac{8i(r - 2M)\sqrt{2}\pi A^{(1)}_{\ell m}(r, t)}{\lambda + 1} \\
+ 16\frac{\pi r (\lambda r^2 - 2\lambda r^2 + 10\lambda r M - 9r M + 27M^2)A^{(0)}_{\ell m}(r, t)}{(\lambda + 1)(\lambda r + 3M)^2} \\
- 8 \frac{F_{\ell m}(r, t)\sqrt{2}\pi (r - 2M)}{\sqrt{\lambda (\lambda + 1)}} \right\},
\]

\(\psi^-_{4} = \frac{-i}{16r} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \left\{ \begin{array}{l}
2\psi^{\text{odd}}_{tr} - 2\psi^{\text{odd}}_{tr} + W^- (\psi^{\text{odd}}_{tr} - \psi^{\text{odd}}_{t}) - V^- \psi^{\text{odd}} \\
- \frac{16\pi r^3}{\lambda (\lambda + 1)} \left( \partial_r Q^{(0)}_{\ell m} - \partial_r Q^{(0)}_{\ell m} \right) + \frac{16i\pi r (r - 2M)}{\lambda (\lambda + 1)} \left( \partial_r Q^{(0)}_{\ell m} - \partial_r Q^{(0)}_{\ell m} \right) \\
- \frac{48\pi (r - 2M)^2 Q^{(0)}_{\ell m}}{\lambda \sqrt{\lambda + 1}} + \frac{16\pi (3r - 8M)Q^{(0)}_{\ell m}}{\lambda \sqrt{\lambda + 1}} - S^- \right\},
\]

\[
\psi^0_{0} = \frac{1}{4f^2r} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \left\{ \begin{array}{l}
2\psi^{\text{even}}_{tr} + 2\psi^{\text{even}}_{tr} + W^+ (\psi^{\text{even}}_{tr} + \psi^{\text{even}}_{t}) - V^+ \psi^{\text{even}} \\
- \frac{16\pi r^3}{(\lambda r + 3M)(\lambda + 1)} \left( \partial_r A^{(0)}_{\ell m} + \partial_r A^{(0)}_{\ell m} \right) + \frac{8i(r - 2M)\sqrt{2}\pi A^{(1)}_{\ell m}(r, t)}{\lambda + 1} \\
+ 16\frac{\pi r (\lambda r^2 - 2\lambda r^2 + 10\lambda r M - 9r M + 27M^2)A^{(0)}_{\ell m}(r, t)}{(\lambda + 1)(\lambda r + 3M)^2} \\
- 8 \frac{F_{\ell m}(r, t)\sqrt{2}\pi (r - 2M)}{\sqrt{\lambda (\lambda + 1)}} \right\},
\]

\(\psi^{-} = \frac{-i}{16r} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \left\{ \begin{array}{l}
2\psi^{\text{odd}}_{tr} - 2\psi^{\text{odd}}_{tr} + W^- (\psi^{\text{odd}}_{tr} - \psi^{\text{odd}}_{t}) - V^- \psi^{\text{odd}} \\
- \frac{16\pi r^3}{\lambda (\lambda + 1)} \left( \partial_r Q^{(0)}_{\ell m} - \partial_r Q^{(0)}_{\ell m} \right) - \frac{16i\pi r (r - 2M)}{\lambda (\lambda + 1)} \left( \partial_r Q^{(0)}_{\ell m} - \partial_r Q^{(0)}_{\ell m} \right) \\
+ \frac{48\pi (r - 2M)^2 Q^{(0)}_{\ell m}}{\lambda \sqrt{\lambda + 1}} - \frac{16\pi (3r - 8M)Q^{(0)}_{\ell m}}{\lambda \sqrt{\lambda + 1}} - S^- \right\},
\]
\[
\psi^-_0 = \frac{i}{4f^2r} \left\{ \psi^{\text{odd}}_{r, r^*} + 2\psi^{\text{odd}}_{r, r^*} + W^- \left( \psi^{\text{odd}}_{r, r^*} + \psi^{\text{odd}}_{r, \ell} \right) - V^- \psi^{\text{odd}}_{r, r^*} \right\}
\]

\[
\times \frac{16\pi r^2}{\lambda(\lambda + 1)} \left( \partial_r Q^{(0)}_{\ell m} + \partial_{r^*} Q^{(0)}_{\ell m} \right) + \frac{16\pi r(r - 2M)}{\lambda(\lambda + 1)} \left( \partial_r Q_{\ell m} + \partial_{r^*} Q_{\ell m} \right)
\]

\[
+ \frac{48\pi (r - 2M)^2 Q_{\ell m}}{\lambda \sqrt{\lambda + 1} r} + \frac{16\pi (3r - 8M) Q^{(0)}_{\ell m}}{\lambda \sqrt{\lambda + 1}} = S^- \right\}, \tag{C.4}
\]

where we have introduced the Chandrasekhar notation for

\[
V^+ = 2 \left( 1 - \frac{2M}{r} \right) \left[ \frac{1}{r^3} \left( \frac{\lambda^2}{r^2} + 3 r M^2 r + 9 M^2 r + 9 M^3 \right) \right], \tag{C.5}
\]

\[
V^- = 2 \left( 1 - \frac{2M}{r} \right) \left( \frac{\lambda^2}{r^2} - 3 M r \right), \tag{C.6}
\]

\[
W^+ = 2 \left( \frac{\lambda^2 r^2 - 3 M^2 M}{r^2 \lambda} \right), \tag{C.7}
\]

\[
W^- = 2 \left( \frac{r - 3 M}{r^2} \right) \tag{C.8}
\]

and

\[
S^- = \frac{8\pi (r - 2M)}{\lambda} \left[ \frac{\partial}{\partial r} (r Q^{(0)}(r, t)) - ir \frac{\partial}{\partial t} Q(r, t) \right] \tag{C.9}
\]

is the source term for the Regge–Wheeler wave equation.

These relations are local and in the time domain. Compare these to the expressions in the frequency domain of \cite{9}, equations (345) and (353) in chapter 4.

\subsection*{C.2. From Weyl scalars to waveforms}

To obtain the inverse Chandrasekhar relations, we make use of equations (16)--(20) and (26)--(28) for the metric coefficients entering the definitions of the even and odd parity waveforms, equations (B.1) and (B.16) respectively.

\[
\psi^{\text{even}} = \frac{r^2(r - 2M) \partial^2}{\partial r^2} H_0(r, t) - \frac{r^3 \partial^2}{\partial r^2} H_1(r, t)
\]

\[
+ \frac{r(r M - 3 M^2 + r^2 \lambda + 6 r M)}{(\lambda + 1)(\lambda + 3 M)} \frac{\partial}{\partial r} H_0(r, t)
\]

\[
+ \frac{(2 r^2 \lambda - 5 r M \lambda - 21 M^2 + 9 r M^2) \partial^2}{\partial r^2} H_1(r, t)
\]

\[
- \frac{2(\lambda^2 M^2 + r^2 M - 2 r^2 \lambda + 2 r^2 M^2 - r^3 \lambda^2 - 63 M^3) H_0(r, t)}{(\lambda + 1)\lambda(\lambda + 3 M)(r^2 M - r^2 \lambda + 6 M^3)}
\]

\[
- 4 \frac{r^2 \sqrt{2 \lambda + 2}(-5 r^2 \lambda - 12 r M \lambda + 21 M^2)^{1/2} B(r, t)}{(\lambda + 1)(\lambda + 3 M)(r + 2 M) \lambda}
\]

\[
+ 4 \frac{\sqrt{2} \sqrt{\lambda(\lambda + 1)^2} r^2 \pi (2 r M - 11 M^2 - r^2 \lambda + 2 r M \lambda) F(r, t)}{(r + 2 M)(\lambda + 1)^2 \lambda^2}
\]

\[
- 8 \frac{8 (r + 2 M)^2 \pi \gamma \gamma B(r, t)}{(\lambda + 1)^{2/3}} - 8 \frac{r^3 M \pi \sqrt{2} \sqrt{\lambda} F(r, t)}{(\lambda + 1)\lambda(\lambda + 1) \lambda} \tag{C.10}
\]
and
\[
\psi_{\text{odd}} = \frac{r}{\lambda} \left\{ \frac{2}{r} \left( \frac{1 - M}{r} \right) b_0 + S_0 - \partial_t h_1 \right\}.
\] (C.11)

So, finally
\[
\psi_{\text{odd}} = \frac{r}{\lambda} \left\{ -\frac{2}{r} \left( 1 - \frac{M}{r} \right) \int_{2M}^{r} S_0(r', t) \frac{4i r^2}{f} \sqrt{(\ell + 2)!} (\ell - 2)! \left( \psi_4 f^2 - \psi_0 \right) \right. \\
- \left. \frac{2}{\sqrt{1 - 2M/r}} \left[ \int_{2M}^{r} \partial_t S_1(r', t) \sqrt{1 - \frac{2M}{r'}} \, dr' \right] \right\}.
\] (C.12)

**Appendix D. Symmetric tetrad**

A further algebraic simplification of the expressions can be achieved by choosing the background tetrad such that we treat \( \psi_4 \) and \( \psi_0 \) on the same footing, thus allowing simple linear combinations of the sort \( \psi_S^4 \pm \psi_S^0 \) in the expression in section 3. The components of the symmetric null tetrad for the Kerr background are given by

\[
(l_\alpha^a) = \left( \frac{r^2 + a^2}{\sqrt{2\Delta \Sigma}}, \frac{\Delta}{2\Sigma}, 0, -\frac{a}{\sqrt{2\Delta \Sigma}} \right),
\] (D.1a)

\[
(n_\alpha^a) = \left( \frac{r^2 + a^2}{\sqrt{2\Delta \Sigma}}, -\frac{\Delta}{2\Sigma}, 0, -\frac{a}{\sqrt{2\Delta \Sigma}} \right),
\] (D.1b)

\[
(m_\alpha^a) = \frac{1}{\sqrt{2(r + ia \cos \theta)}} \left( ia \sin \theta, 0, 1, \frac{i}{\sin \theta} \right).
\] (D.1c)

With the above choice of the tetrad, the spin coefficients are

\[
\nu_S = 0, \quad \sigma_S = 0, \quad \lambda_S = 0, \quad \kappa_S = 0,
\]

\[
\pi_S = \pi_K = ia \rho_K^2 \frac{\sin \theta}{\sqrt{2}}, \quad \tau_S = \tau_K = -ia \rho_K \bar{\rho}_K \frac{\sin \theta}{\sqrt{2}},
\]

\[
\rho_S = \mu_S = \sqrt{\frac{\Delta}{2\Sigma}} \rho_K \quad \epsilon_S = \frac{[M(r^2 - a^2 \cos^2 \theta) - a^2 r \sin^2 \theta]}{2\sqrt{2}\Delta \Sigma^3}
\]

\[
\gamma_S = \epsilon_S - ia \cos \theta \sqrt{\frac{\Delta}{2\Sigma^3}}
\]

\[
\alpha_S = \frac{[(r^2 + a^2) \cos \theta - 2iar \sin^2 \theta]}{2\sqrt{2} \sin \theta} \left( \rho_K^2 \bar{\rho}_K \right)
\]

\[
\beta_S = -\frac{[(r^2 + a^2) \cot \theta]}{2\sqrt{2}} \left( \bar{\rho}_K^2 \rho_K \right), \quad \alpha_S - \beta_S = \alpha_K - \beta_K,
\]

where an overbar stands for complex conjugation.
The Weyl scalars computed with the symmetric tetrad relate to those computed with the Kinnersley tetrad as follows:

\[
\psi^S_4 = \left(\frac{2\Sigma}{\Delta}\right) \psi^K_4, \quad \psi^S_3 = \sqrt{\frac{2\Sigma}{\Delta}} \psi^K_3, \quad \psi^S_2 = \psi^K_2, \\
\psi^S_1 = \sqrt{\frac{\Delta}{2\Sigma}} \psi^K_1, \quad \psi^S_0 = \left(\frac{\Delta}{2\Sigma}\right) \psi^K_0.
\]

(D.3)

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