Airy complex variable function Gaussian beams

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Abstract. We have presented a novel family of paraxial laser beams, which are called Airy complex variable function Gaussian (ACVF–Gaussian) beams. The ACVF–Gaussian beams are formed by the product of an Airy complex variable function and a Gaussian function. Unlike Airy–Gaussian beams, which are nondiffracting, the ACVF–Gaussian beams, which have a rotating phase term are diffracting and rotating. The rotating velocity decreases when the propagation distance increases. We have analyzed the evolution of the Poynting vector and angular momentum density of the ACVF–Gaussian beams as they propagate in free space.

Paraxial approximation plays an important role in studying the optical beam propagation [1]. Various closed-form solutions of the paraxial wave equation have been obtained, including the standard and elegant Hermite–Gaussian beams [1, 2], Laguerre–Gaussian beams [3, 4], Ince–Gaussian beams [5, 6], Hermite–Laguerre–Gaussian beams [7, 8], Helmholtz–Gauss and Laplace–Gauss beams [9], Airy beams [10]–[13], hypergeometric Gaussian beams [14], Cartesian beams and circular beams [15, 16]. Optical vortices [17, 18] have drawn increased attention due to their various applications, such as diffractive optical implementation of point transforms [19]–[21], and trapping and manipulation of small particles [22]. The physical nature and mechanisms of transverse energy circulation in light vortex beams [23]–[26] have been studied extensively, which has given rise to remarkable interest in so-called rotating beams where this energy circulation seems to be especially obvious and spectacular [26]–[28]. Within the variety of such rotating beams, an important class is formed by superpositions of coaxial Laguerre–Gaussian modes [18, 23, 27, 28]. However, to the best of our knowledge, the exact closed-form Airy complex variable function Gaussian (ACVF–Gaussian) solution of the paraxial wave equation in free space has not been reported so far.

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In this paper, a novel family of paraxial laser beams, called ACVF–Gaussian beams, is introduced. Unlike Airy–Gaussian beams [29], which are nondiffracting, the ACVF–Gaussian beams, which have a rotating phase term, are diffracting and rotating. The evolution of the Poynting vector and angular momentum (AM) density of the ACVF–Gaussian beams as they propagate in free space has been analyzed.

In free space, consider an \( \hat{x} \)-polarized vector potential \( \vec{A} = \hat{e}_x F(x, y, z) \exp(ikz - i\omega t) \), where \( \hat{e}_x \) is the unit vector along the \( x \)-direction, and the scalar light field \( \Phi(x, y, z) \) satisfies the following paraxial wave equation:

\[
\nabla_z^2 \Phi + 2ik \partial \Phi / \partial z = 0,
\]

where \( \nabla_z^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 \) is the transverse Laplace operator. Assuming the solutions of equation (1) can be written in the form

\[
\Phi(r, z) = \Phi_e(r, z) \Phi_G(r, z),
\]

where \( \Phi_G(r, z) = q_0 / q(z) \exp[ikr^2 / (2q(z))] \), \( q(z) = z - iz_R, q_0 = q(0) = -iz_R, z_R = kw_0^2 \) is the Rayleigh range of the Gaussian beam. Substituting equation (2) into equation (1), one obtains

\[
\nabla_z^2 \Phi_e + \frac{2ik}{q(z)} \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \Phi_e + 2ik \frac{\partial \Phi_e}{\partial z} = 0.
\]

In the transverse plane perpendicular to \( z \), one can introduce the rotating coordinate system where the rotating coordinate is assumed to be

\[
\begin{align*}
    x' &= w_0 [x \cos(\vartheta(z)) + y \sin(\vartheta(z))] / w(z), \\
    y' &= w_0 [y \cos(\vartheta(z)) - x \sin(\vartheta(z))] / w(z),
\end{align*}
\]

where \( w(z) = w_0 \sqrt{1 + z^2 / z_R^2} \) and \( z = z, \vartheta(z) \) is the rotating angle of the rotating coordinate and \( d\vartheta(z) / dz \) denotes the angular velocity (AV) of the rotating coordinate. When \( \vartheta(z) = 0 \), the solutions of equation (3) become ordinary (nonrotating) beams. In the rotating coordinate system, equation (3) can be changed into

\[
\frac{1}{w^2(z)} \Delta_{z} \Phi_e + 2k \frac{d\vartheta(z)}{dz} \dot{L}_z \Phi_e - \frac{2}{w^2(z)} \left( x \frac{\partial \Phi_e}{\partial x'} + y' \frac{\partial \Phi_e}{\partial y'} \right) = 0,
\]

where \( \Delta_z = \partial^2 / \partial x'^2 + \partial^2 / \partial y'^2 \), \( \dot{L}_z = -i(x' \partial / \partial y' - y' \partial / \partial x') \) denotes the \( z \)-component of the AM operator, familiar in quantum mechanics [30].

The solutions of equation (5) can be written in the form

\[
\Phi_e(r') = f \left( \frac{x' \pm iy'}{bw_0} \right), \vartheta(z) = \pm \arctan \left( \frac{z}{z_R} \right),
\]

where \( f(\cdot) \) is an arbitrary function. Here we are interested in taking \( f(\cdot) \) as an Airy Ai(\cdot) or Bi(\cdot) [31] function, \( b \neq 0 \) is a distribution factor, and ‘+(-)’ offers the rotation direction, which agrees with the right-hand (left-hand) rule relative to the propagation direction. The rotation of the beam in space is determined by the Gouy phase evolution [32], so \( \vartheta(z) = \pm \arctan(z_R) \). Substituting equation (6) into equation (2), the solutions of equation (1) can be obtained in the laboratory frame:

\[
\Phi(r, z) = \frac{Cq_0 f(\xi \pm)}{q(z)} \exp \left[ \frac{ikr^2}{2q(z)} \right],
\]

where \( f(\cdot) \) is an Airy Ai(\cdot) or Bi(\cdot) function.
where $C$ is a constant, $\zeta_\pm = [(x \pm iy)/(bw(z))] \exp[-i\vartheta(z)]$. Because $f(\xi)$ is an Airy $\text{Ai}(\xi)$ or $\text{Bi}(\xi)$ complex variable function, we call the solutions ACVF–Gaussian beams. The structure of the beams is constructed by the product of the Airy function $f(\xi)$ and a Gaussian function. The AV of the optical beams is $d\vartheta(z)/dz = \pm 1/(kw^2(z))$, which is dependent on the wavelength and the beam width and shows that the beam rotation does not depend on the background amplitude. This means that the singular beam rotates, as a whole, around the $z$-axis and the mechanical AM can be attributed to this rotation [26]. Similar to mechanical attributes, the optical beam possesses the AV, the moment of inertia that is proportional to the total beam equivalent mass, which equals the beam power, and to the squared mean radius, and the AM that is the product of the AV and the moment of inertia. Just like the movement of a rigid body in mechanics, this shows that if the beam is narrowed or broadened while keeping the total ‘mass’, its AV increases or decreases. In this way, the result of AV can be clarified physically and is seen in figure 2. Passing from $z = 0$ to $z_R$, the beam rotates through almost the same angle as passing from $z = z_R$ to $50z_R$; the second distance is much longer, but the beam expands due to diffraction and rotates much more slowly. Unlike those Airy waves that tend to accelerate or self-bend during propagation as found in [11], the ACVF–Gaussian beams tend to rotate and the rotating velocity decelerates during propagation.

The Airy function $\text{Ai}(\zeta_\pm)$ or $\text{Bi}(\zeta_\pm)$ can be expressed in terms of series [33],

$$\text{Ai}(\zeta_\pm) = \frac{1}{3^{2/3}\pi} \sum_{n=0}^{\infty} \frac{\Gamma[(n+1)/3]}{n!} (3^{1/3}\zeta_\pm)^n \sin \left[ \frac{2(n+1)\pi}{3} \right]$$

and

$$\text{Bi}(\zeta_\pm) = \frac{1}{3^{1/6}\pi} \sum_{n=0}^{\infty} \frac{\Gamma[(n+1)/3]}{n!} (3^{1/3}\zeta_\pm)^n \bigg| \sin \left[ \frac{2(n+1)\pi}{3} \right] \bigg|.$$
The ACVF–Gaussian beams can be regarded as the superposition of a series of singly ringed Laguerre–Gaussian (LG) modes with topological charge \( n \). LG beams have been generated in experiment by many authors [34, 35]. Hence, the ACVF–Gaussian beams may be created experimentally if one superposes different mode singly ringed LG beams.

Figure 1 gives the transverse normalized intensity distribution of the ACVF–Gaussian beam when \( f(\zeta) = \text{Ai}(\zeta) \) for different \( b \). Such an ACVF–Gaussian beam is clustered more closely around the beam center with increasing \( b \). The free-space evolution of the transverse intensity distribution of the ACVF–Gaussian beam with \( f(\zeta) = \text{Ai}(\zeta) \) for \( b = 0.6 \) is presented in the background of each frame of figure 2, which shows that, for a given \( f(\zeta) \) and \( b \), the beam transverse patterns are stable, except that the overall scale increases, as does the pattern rotation as a whole with increasing \( z \). The total beam rotation angle amounts to \( \pi/2 \) during the beam propagation from \( z = 0 \) to \( \infty \). In fact, the total beam rotation angle becomes almost \( \pi/2 \) when \( z = 50z_R \), as shown in figure 2. The beam pattern of the ACVF–Gaussian beam is composed of a rotating fabaceous beam when \( b = 0.6 \), as is shown in the background of each frame of figures 2(a)–(d).

The propagation properties of electromagnetic fields are closely related to their local energy flow, which is usually expressed in terms of the Poynting vector. The Poynting vector has a magnitude of energy per unit area (or per unit time), and a direction which represents the energy flow at any point in the field. The Poynting vector is defined as [36] \( \vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} \), where \( c \) is the speed of light in vacuum, and \( \vec{E} \) and \( \vec{B} \) are the electric and magnetic fields, respectively. In the

**Figure 2.** Transverse normalized intensity (background) and transverse energy flow (red arrows) for ACVF–Gaussian beams. The parameters are the same as those in figure 1(c) except (a) \( z = 0 \); (b) \( z = z_R \); (c) \( z = 2z_R \); (d) \( z = 50z_R \).
Lorenz gauge, the time-averaged Poynting vector can be expressed as \[ S = \frac{c}{4\pi} \langle \vec{E} \times \vec{B} \rangle \]
\[ = \frac{c}{8\pi} \left[ i\omega (\Phi \nabla_\perp \Phi^* - \Phi^* \nabla_\perp \Phi) + 2\omega k |\Phi|^2 \hat{e}_z \right], \tag{8} \]

where \( \nabla_\perp = \partial/\partial x \hat{e}_x + \partial/\partial y \hat{e}_y \), \( \hat{e}_y \) and \( \hat{e}_z \) are the unit vectors along the \( y \)- and \( z \)-directions, respectively, and * denotes the complex conjugate. It is easy to see from equation (8) that the energy flow of the \( z \)-direction is just proportional to the light intensity. In figure 2, the direction and magnitude of the arrows agree with the direction and magnitude of the energy flow in the transverse plane, and the energy flow of the main peak changes during propagation. Figure 2(a) shows that the Poynting vector is initially aiming at the \( x \)-direction on the tail of the ACVF–Gaussian beam; in figure 2(b), the direction of the Poynting vector begins to move partially towards the \( x \)-direction; and in figures 2(c) and (d), as the beam propagates further, the direction turns around even more towards the \( x \)-direction. In figure 2(d), the direction of the Poynting vector points outwards from the beam center. The change in the Poynting vector can be used to explain the physics behind the rotation phenomenon of the beams as the inhomogeneity of the energy flow within the beam cross section, which results in a transverse energy redistribution over the beam cross section [27].
As in mechanics, the time-averaged AM density for the electromagnetic field is the AM per unit area (per unit time), obtained by forming the cross product of the position vector with the time-averaged momentum density \( p \propto \vec{E} \times \vec{B} \) [13],

\[
\langle \vec{j} \rangle = \vec{r} \times \langle \vec{E} \times \vec{B} \rangle = \frac{\omega}{2} \left[ \hat{e}_x (2yk|\Phi|^2 - zS_y) + \hat{e}_y (ziS_x - 2xk|\Phi|^2) + \hat{e}_z i(xS_y - yS_x) \right],
\]

where \( S_y = \Phi \partial \Phi^*/\partial y - \Phi^* \partial \Phi/\partial y \), \( S_x = \Phi \partial \Phi^*/\partial x - \Phi^* \partial \Phi/\partial x \). Figure 3 shows the longitudinal normalized AM density (background) and transverse AM density flow (arrows) for ACVF–Gaussian beams. It is not difficult to see from figure 3 that the distribution of the longitudinal normalized AM density and the transverse AM density flow rotates during propagation. The rotation of the longitudinal normalized AM density flow agrees with that of the transverse normalized intensity distribution of the ACVF–Gaussian beam. The transverse AM density flow is initially clockwise and turns counterclockwise in the far field.

In conclusion, we have presented a family of paraxial laser beams called ACVF–Gaussian beams. The product of an Airy complex variable function and a Gaussian function constitutes the ACVF–Gaussian beams. The distribution of the ACVF–Gaussian beams is farther from the beam center for smaller \( b \) and is clustered more closely around the beam center for larger \( b \). The direction of the Poynting vector of ACVF–Gaussian beams varies during propagation. The distribution of the longitudinal normalized AM density and the transverse AM density flow rotates during propagation.

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