Supplemental Material for “Monopole Superconductivity in Magnetically Doped Cd$_3$As$_2$”

CONTENTS

I. $k \cdot p$ models of Cd$_3$As$_2$  
   A. Symmetry analysis for Dirac semimetal Cd$_3$As$_2$  
   B. Model parameters for magnetically doped Cd$_3$As$_2$  

II. Monopole charge and Chern number  
   A. Monopole charge and Chern number for Bloch states  
   B. Projected gap function and pair monopole charge  

III. Analytical derivation of Chern numbers of Fermi surfaces in a model of magnetically doped Cd$_3$As$_2$ with $\beta_s = \beta_p$  

IV. Monopole superconducting order for different irreps in the model of magnetically doped Cd$_3$As$_2$ with $\beta_s = \beta_p$  

References

I. $k \cdot p$ MODELS OF Cd$_3$As$_2$

A. Symmetry analysis for Dirac semimetal Cd$_3$As$_2$

We review the form of the $k \cdot p$ Hamiltonian for the Dirac semimetal Cd$_3$As$_2$ [1]. The $D_{4h}$ point group together with time reversal symmetry of the material in the absence of magnetic dopants restricts the structure of the $k \cdot p$ Hamiltonian [2]. Following the theory of invariants [3], the symmetry condition $D(g)\mathcal{H}(g^{-1}k)D(g)^{-1} = \mathcal{H}(k)$ implies that for a particular basis of matrices, the Hamiltonian matrix is a sum of matrices paired with functions of $k$ that both belong to the same irreducible representation of $D_{4h}$.

As discussed in Ref. [1], the bands relevant to the Dirac crossing constitute the spin-orbit coupled basis $\{|S_{J = \frac{1}{2}}, J_z = \frac{1}{2}\rangle, |P_{\frac{1}{2}}, \frac{3}{2}\rangle, |S_{\frac{1}{2}}, -\frac{1}{2}\rangle, |P_{\frac{1}{2}}, -\frac{3}{2}\rangle\}^T$. The symmetry group is generated by time reversal $\Theta$, inversion $\Pi$, rotation $C_4$ about the $z$ axis, and rotation $C_2'$ about the $x$ axis, which in the spin-orbit coupled basis can be written as

$$
D(\Theta) = i\sigma_y \otimes I \circ K,
D(\Pi) = I \otimes \tau_z,
D(C_4) = e^{-i\frac{\pi}{4}\Sigma} \equiv e^{-i\frac{\pi}{4}\sigma_x \otimes (2I - \tau_z)},
D(C_2') = -i\sigma_x \otimes \tau_z,
$$

with the $\tau_i$ Pauli matrices acting on the orbital part of the basis $(s/p)$ and $\sigma_i$ acting on the spin part, with spin $\uparrow/\downarrow$ corresponding to $\text{sgn}(J_z) = \pm$. Here $K$ is the complex conjugation operator. The full double group need not be considered since $2\pi$ spin rotation acts as $-I \otimes I$ and thus commutes with $\mathcal{H}(k)$. The presence of both time reversal and parity requires that the gamma matrices appearing in $\mathcal{H}(k)$ be even under the product operation $\Theta \Pi$, since $k$ is even under this combined symmetry. This restricts the basis of allowed matrices to $\Gamma_i$ for $i = 0, \ldots, 5$ with $\Gamma_0$ the identity, $\Gamma_{1,2,3} = \sigma_{x,y,z} \otimes \tau_x$, and $\Gamma_{4,5} = I \otimes \tau_{y,z}$.

The characters of $\Gamma_i$ under the symmetry transformations $D(\Theta), D(\Pi)$, or $D(C_2')$ are listed in Table I. $\Gamma_1, \Gamma_2$, and $\Gamma_5$ form one-dimensional representations while $(\Gamma_3, \Gamma_4)$ is an irreducible two-dimensional representation, since $C_2'$ and $C_4$ respectively act as $\sigma_z$ and $-i\sigma_y$ on $(\Gamma_3, \Gamma_4)$ and thus cannot be simultaneously diagonalized. As a result, comparison to the character table [4] for $D_{4h}$ uniquely identifies the corresponding irreducible representations, listed in Table I along with the corresponding basis functions to lowest nonzero order in $k$. 

References

1. A. Symmetry analysis for Dirac semimetal Cd$_3$As$_2$  
2. B. Model parameters for magnetically doped Cd$_3$As$_2$  
3. C. Projected gap function and pair monopole charge  
4. D. The symmetry group is generated by time reversal $\Theta$, inversion $\Pi$, rotation $C_4$ about the $z$ axis, which in the spin-orbit coupled basis can be written as

$$
D(\Theta) = i\sigma_y \otimes I \circ K,
D(\Pi) = I \otimes \tau_z,
D(C_4) = e^{-i\frac{\pi}{4}\Sigma} \equiv e^{-i\frac{\pi}{4}\sigma_x \otimes (2I - \tau_z)},
D(C_2') = -i\sigma_x \otimes \tau_z,
$$

with the $\tau_i$ Pauli matrices acting on the orbital part of the basis $(s/p)$ and $\sigma_i$ acting on the spin part, with spin $\uparrow/\downarrow$ corresponding to $\text{sgn}(J_z) = \pm$. Here $K$ is the complex conjugation operator. The full double group need not be considered since $2\pi$ spin rotation acts as $-I \otimes I$ and thus commutes with $\mathcal{H}(k)$. The presence of both time reversal and parity requires that the gamma matrices appearing in $\mathcal{H}(k)$ be even under the product operation $\Theta \Pi$, since $k$ is even under this combined symmetry. This restricts the basis of allowed matrices to $\Gamma_i$ for $i = 0, \ldots, 5$ with $\Gamma_0$ the identity, $\Gamma_{1,2,3} = \sigma_{x,y,z} \otimes \tau_x$, and $\Gamma_{4,5} = I \otimes \tau_{y,z}$.

The characters of $\Gamma_i$ under the symmetry transformations $D(\Theta), D(\Pi)$, or $D(C_2')$ are listed in Table I. $\Gamma_1, \Gamma_2$, and $\Gamma_5$ form one-dimensional representations while $(\Gamma_3, \Gamma_4)$ is an irreducible two-dimensional representation, since $C_2'$ and $C_4$ respectively act as $\sigma_z$ and $-i\sigma_y$ on $(\Gamma_3, \Gamma_4)$ and thus cannot be simultaneously diagonalized. As a result, comparison to the character table [4] for $D_{4h}$ uniquely identifies the corresponding irreducible representations, listed in Table I along with the corresponding basis functions to lowest nonzero order in $k$. 

References
The symmetry-allowed $\mathcal{H}(\mathbf{k})$ is then
\[
\mathcal{H}(\mathbf{k}) = \left( e_0 + e_1 k_z^2 + e_2 (k_x^2 + k_y^2) \right) \Gamma_0
+ \left( M_0 + M_1 k_z^2 + M_2 (k_x^2 + k_y^2) \right) \Gamma_5
+ A(k_x \Gamma_3 - k_y \Gamma_4)
+ B_1 k_z^2 \Gamma_1 + B_2 2k_x k_y \Gamma_2,
\]
with all coefficients real due to time reversal symmetry and the factor of 2 on the $B_2$ term added for convenience. The appropriate basis vector in the two-dimensional $E_u$ subspace is selected by requiring $D(g) H(g^{-1} \mathbf{k}) D(g)^{-1} = H(\mathbf{k})$.

The off-diagonal terms in Eq. (2) are
\[
\mathcal{H}_{A,B}(\mathbf{k}) = \begin{pmatrix}
0 & A k_+ & 0 & f_B(\mathbf{k}) \\
A k_- & 0 & f_B(\mathbf{k}) & 0 \\
f_B^*(\mathbf{k}) & 0 & -A k_- & 0
\end{pmatrix}
\]
with $f_B(\mathbf{k}) \equiv G k_z^2 + G' k_x^2 k_y^2$ and $G, G' = \frac{B_1 \pm B_2}{2}$ for $k_z = k_x \pm ik_y$. In particular, $f_B(\mathbf{k})$ includes only the $k_z^2$ term when $B_1 = B_2$, which is not generally required by the $D_{4h}$ symmetry.

As noted in Ref. [5], the $G'$ term arises due to weak crystal field effects and can be taken to be zero. In fact, the inclusion of a small $G' < G$ does not affect topological properties of the magnetically doped model, such as the Chern number and hence the number of BdG nodes required by the topological pairing.

**B. Model parameters for magnetically doped Cd$_3$As$_2$**

In the presence of magnetic dopants, modeled by a Zeeman field along the $z$ direction, the Weyl points near $k_z = k_D$, the location of the Dirac node in the absence of magnetic dopants, are described by
\[
\mathcal{H}(\mathbf{k}) = (\epsilon_D - \mu) I + \begin{pmatrix}
v_s \hat{k}_y + \beta_s \hbar & A k_+ & 0 & G k_z^2 \\
A k_- & -v_p \hat{k}_y + \beta_p \hbar & G k_z^2 & 0 \\
0 & G k_z^2 & v_s \hat{k}_z - \beta_s \hbar & -A k_- \\
G k_z^2 & 0 & -A k_+ & -v_p \hat{k}_z - \beta_p \hbar
\end{pmatrix},
\]
as studied in Ref. [5], with fitting parameters enumerated in Table II for convenience.

| $v_s$ (eV Å) | $v_p$ (eV Å) | $A$ (eV Å) | $G$ (eV Å$^2$) |
|-------------|-------------|-----------|----------------|
| 2.68        | 0.36        | 0.99      | 10             |

| $\beta_s$ (eV/Å$^{-1}$) | $\beta_p$ (eV/Å$^{-1}$) | $\hbar$ (T) | $k_D$ (Å$^{-1}$) |
|------------------------|------------------------|-------------|----------------|
| $5.4 \times 10^{-4}$   | $1.15 \times 10^{-4}$  | 100         | 0.037          |

TABLE II. Parameters in the $\mathbf{k} \cdot \mathbf{p}$ Hamiltonian Eq. (4), as calculated in Ref. [5].

**II. MONOPOLE CHARGE AND CHERN NUMBER**

In this section, we review the relationship between monopole charge and Chern number associated with topological Bloch states and the corresponding monopole pairing order. In particular, we review how an obstruction to smoothly
defining the gauge for Bloch states over a Fermi surface indicates a nonzero Chern number as well as a nonzero monopole charge associated with the corresponding pairing order.

A. Monopole charge and Chern number for Bloch states

The topological structure of a Fermi surface with a nonzero Chern number closely resembles that of a Dirac magnetic monopole. Consider a Bloch state $|\psi_k\rangle$ on a spherical Fermi surface. For a Fermi surface with Chern number $C$, the Bloch state at the Fermi surface experiences a fictitious magnetic field $B_k = q \hat{\phi}^k$ from a momentum-space monopole charge $q \equiv \frac{1}{4\pi} \int_{S} \nabla \times A_k \equiv C/2$. The corresponding vector potential, the Berry connection $A_k = i \langle \psi_k | \nabla_k | \psi_k \rangle$, is piecewise well defined over the northern and the southern hemispheres. The Bloch states $\psi_k$ cannot be smoothly defined globally over the entire Fermi surface when $q \neq 0$ but can be smoothly defined locally over the north and the south hemispheres under appropriate gauges related at the equator by

$$|\psi^N\rangle = e^{-2i\phi^k} |\psi^S\rangle,$$

Here $N$ and $S$ refer to the gauges for which the states are smooth over the patches $\theta^k < \pi/2 + \delta$ covering the northern hemisphere and $\theta^k > \pi/2 - \delta$ covering the southern hemisphere for a small angle $\delta > 0$. Eq. (5) describing the transition function then holds on the overlap, $\pi/2 + \delta > \theta^k > \pi/2 - \delta$. The components of $|\psi\rangle$ can be understood as sections of a complex line bundle and written in terms of monopole harmonics [6].

The Chern number can be calculated from the Berry connection $A^j = \langle \psi^j | i \nabla_k | \psi^j \rangle$ in each patch $j = N, S$. From the transition function, $A^N = A^S + \frac{2q}{\sin \theta^k} \hat{\phi}$. The Chern number is then

$$C \equiv \frac{1}{2\pi} \left[ \int_{S^N} \nabla \times A^N + \int_{S^S} \nabla \times A^S \right] = \frac{1}{2\pi} \oint_{C_{E}} A^N - A^S = 2q,$$

where $S^N = \{ (\theta^k, \phi^k) | \theta^k \leq \pi/2, \phi^k \in [0, 2\pi) \}$ and $S^S = \{ (\theta^k, \phi^k) | \theta^k \geq \pi/2, \phi^k \in [0, 2\pi) \}$ are the northern and southern hemispheres and $C_{E} = \{ (\theta^k, \phi^k) | \theta^k = \pi/2, \phi^k \in [0, 2\pi) \}$ is the equator oriented in the $+\hat{\phi}$ direction so that the boundary of $S^N$ is $C_{E}$ and the boundary of $S^S$ is $-C_{E}$. Thus, the Chern number can be read immediately from the transition function, and a nontrivial transition function implies a nonzero Chern number. It is important to note that the above Chern number calculation is for an electron pocket, where the normal direction points out of the sphere. For a hole pocket, with normal direction pointing into the sphere, the Chern number for the same transition function is $-2q$.

B. Projected gap function and pair monopole charge

The projected gap function follows from the paired band eigenstates at the Fermi surface and the form of the pairing matrix. For a pairing matrix $\Delta$, the BdG matrix for zero center-of-mass momentum pairing in a parity-symmetric system takes the form

$$\mathcal{H}_{BdG} = \begin{pmatrix} \mathcal{H}(k) & \hat{\Delta} \\ \hat{\Delta}^\dagger & -\mathcal{H}^*(-k) \end{pmatrix},$$

with the $n$-band system described in a spin-orbital or pseudospin basis by an $n \times n$ matrix $\mathcal{H}$. The particle, $\mathcal{H}(k)$, and hole, $-\mathcal{H}^*(-k)$, blocks can be diagonalized by unitary matrices $U_k$ and $U_{-k}^\dagger$, where the $j$th column of $U_k$ is the $j$th eigenstate of $\mathcal{H}(k)$. Importantly, since eigenstates describing Fermi surfaces with nonzero Chern number require two gauges on two patches covering the Fermi surface to be smoothly defined, the unitary matrix will generally require two gauges as well, related by $U^N_k = U^S_k D^{SN}$ on the patch overlap. The $j$th diagonal element of the diagonal matrix $D^{SN}$ is the transition function $e^{-2i\phi^k}$ for eigenstate $j$. Using $U^{N/S}_{BdG,k} = \text{diag}[U^{N/S}_k, (U^{N/S}_{-k})^\dagger]$ to diagonalize the band and hole blocks,

$$U^{N/S}_{BdG,k} \mathcal{H}_{BdG} U^{N/S}_{BdG,k}^\dagger = \begin{pmatrix} \mathcal{E}(k) & U^{N/S}_{-k} \hat{\Delta} U^{N/S}_{-k}^\dagger \\ (U^{N/S}_k \hat{\Delta} U^{N/S}_{-k})^\dagger & -\mathcal{E}(-k) \end{pmatrix},$$

Equation (8)
where $\mathcal{E}(k)$ is an $n \times n$ diagonal matrix with $j$th diagonal element the energy $E_j(k)$ of the $j$th eigenstate of $\mathcal{H}(k)$. In this band-diagonal basis, the Fermi surface projection identifies the pairing between states at the Fermi surface as the projected gap function. For a Fermi surface described by eigenstate $j$, $E_j(k) = 0$, and the projected gap function to lowest order is the component $\left(U_{k}^{N/S} \Delta U_{k}^{-N/S}\right)_{jj}$. Noting that the $j$th column of $U_{k}^{N/S}$ is the eigenvector $|\psi_{j}^{N/S}(k)\rangle$ of $\mathcal{H}(k)$ and the $j$th column of $U_{k}^{-N/S}$ is the eigenvector $|\psi_{h,j}^{N/S}(k)\rangle$ of the hole block $-\mathcal{H}^*(-k)$, the projected gap function is

$$\Delta_{\text{proj}}^{N/S} = \langle \psi_{j}^{N/S}(k) | \Delta | \psi_{h,j}^{N/S}(k) \rangle$$

$$= \langle \psi_{j}^{N/S}(k) | \Delta \Pi \mathcal{K} | \psi_{j}^{N/S}(k) \rangle.$$  

(9)

Here $\mathcal{K}$ is the complex conjugation operator and $\Pi$ is the appropriate representation of the parity operator in the spin-orbital or pseudospin basis. Thus, assuming parity symmetry, we write $|\psi_{j}^{N/S}(k)\rangle = \mathcal{K} |\psi_{j}^{N/S}(-k)\rangle = \Pi \mathcal{K} |\psi_{j}^{N/S}(k)\rangle$. The projected gap function then inherits a transition function from the single-particle eigenstates,

$$\Delta_{\text{proj}}^{N} = e^{4i\theta} \Delta_{\text{proj}}^{S}.$$  

(10)

Thus, the projected gap function is described by monopole harmonics $Y_{qlm}$ [6] with $q = 2q_{j} = C_{j}$. Similarly to how the Chern number can be identified from the transition function of a single-particle state, the pair monopole charge can be identified from the transition function of the projected gap function. The pair monopole charge $q_{p}$ follows from integrating the pair Berry curvature over the Fermi surface and is related to the total vorticity $\nu_{\text{tot}} = 2q_{p}$ [7, 8]. As the total vorticity is defined in terms of the integral of the circulation field $v_{k} = \nabla_{k} \varphi_{k} - A_{p}(k)$, the gauge invariance of the circulation field, and hence the vorticity, follows from the pair Berry connection $A_{p}(k)$ and pairing phase $\varphi_{k}$ transforming the same way. Explicitly, in terms of the projected gap function, the pairing phase is the complex argument $\varphi_{k} = \text{arg}\{\Delta_{\text{proj}}(k)\}$, and $\nabla_{k} \varphi_{k}^{N/S} = \nabla_{k} \varphi_{k}^{S} + \frac{4q_{j}}{k \sin \theta_{j}} \hat{\phi}_{k}$. The pair state transforms with a transition function given by the product of the single-particle transition functions, and thus $A_{p}^{N}(k) = A_{p}^{S}(k) + \frac{4q_{j}}{k \sin \theta_{j}} \hat{\phi}_{k}$. Integrating the Berry connection $\nabla \times A_{p}$ over the Fermi surface, a calculation similar to Eq. (6) shows $q_{p} = 2q_{j}$, and the pair monopole charge can thus be read from the transition function of $\Delta_{\text{proj}}^{N}(k)$. 

### III. ANALYTICAL DERIVATION OF CHERN NUMBERS OF FERMI SURFACES IN A MODEL OF MAGNETICALLY DOPED Cd$_{x}$As$_{2}$ WITH $\beta_{s} = \beta_{p}$

We consider a deformation of the model in Eq. (4) with $\beta_{s} = \beta_{p} = 5.4 \times 10^{-5} \text{eV/T}$. This simplification allows the eigenstates to be written in a simpler form where the appropriate gauges for smooth eigenstates near the poles of each Fermi pocket be can be easily identified. This model is a continuous deformation of Eq. (4) that preserves the topological structure of the Fermi pockets.

The Hamiltonian matrix kernel for $k_{z} > 0$ can be written as

$$\mathcal{H}_{+}(k) = |v_{-}(k_{z} - k_{D}) - \mu| I + \beta_{-} h_{x} \otimes \tau_{z} + \beta_{+} h_{z} \otimes I + v_{+}(k_{z} - k_{D}) |I \otimes \tau_{z} - A_{k} \tau_{y} + A_{k} h_{z} \otimes \tau_{x} + G(k_{x}^{2} - k_{y}^{2}) \sigma_{x} \otimes \tau_{x} + G k_{x} k_{y} \sigma_{y} \otimes \tau_{x} + \sum_{i=1}^{5} d_{i} \Gamma_{i}$$

(11)

where $v_{\pm} = \frac{v_{x, y} \pm v_{z}}{2}$, $\beta_{\pm} = \frac{\beta_{s} \pm \beta_{p}}{2}$, and $B = \beta_{+} h$. With $\beta_{s} \neq \beta_{p}$, this model is exactly the $k \cdot p$ model in Eq. (4), here labeled $\mathcal{H}_{+}$ to emphasize that this linearized model is valid for $k_{z} > 0$. The five anticommuting gamma matrices are $\Gamma_{1,2,3} = \sigma_{x,y,z} \otimes \tau_{z}$ and $\Gamma_{4,5} = I \otimes \tau_{y,z}$, as in Sec. 1A, and we denote products of gamma matrices by $\Gamma_{nm} \equiv -i \Gamma_{n} \Gamma_{m}$.

Thus, $f_{0}(k_{z}) = v_{-}(k_{z} - k_{D}) - \mu$, $d_{1} = G(k_{x}^{2} - k_{y}^{2})$, $d_{2} = G k_{x} k_{y}$, $d_{3} = A_{k} h_{x}$, $d_{4} = - A_{k} h_{y}$, and $d_{5} = v_{+}(k_{z} - k_{D})$. For the simplified model with $\beta_{s} = \beta_{p}$, it involves only six gamma matrices, as $\beta_{-} = 0$. With $\beta_{-} = 0$, the energies are

$$E_{s_{1}, s_{2}} = f_{0}(k_{z}) + s_{1} \sqrt{B^{2} + d^{2}} + s_{2} 2Bd^{(3)},$$

(12)

labeled by the signs $s_{1}, s_{2} = \pm 1$, and with $d^{(3)} \equiv \sqrt{d_{3}^{2} + d_{4}^{2} + d_{5}^{2}}$ and $d \equiv \sqrt{d_{1}^{2} + d_{2}^{2} + (d^{(3)})^{2}}$.

For $k_{z} < 0$, the parity-related Hamiltonian is $\mathcal{H}_{-}(k) = \Pi \mathcal{H}_{+}(-k)\Pi$, with $\Pi = \Gamma_{5} = I \otimes \tau_{x}$. In other words, we consider the model in Eq. (4) with $\beta_{s} = \beta_{p}$ and extended to all $k$ by requiring parity symmetry,
The BdG Hamiltonian, for $k_z > 0$, is thus

$$\mathcal{H}_B(k) = \begin{cases} \mathcal{H}_+(k) & k_z > 0 \\ \mathcal{H}_-(k) = \Pi \mathcal{H}_-(-k) \Pi & k_z < 0. \end{cases}$$

The BdG Hamiltonian, for $k_z > 0$, is thus

$$\mathcal{H}_B(k) = \begin{pmatrix} \mathcal{H}(k) & \hat{\Delta}(k) \\ \hat{\Delta}(k)^\dagger & -\mathcal{H}^*(k) \end{pmatrix} = \begin{pmatrix} \mathcal{H}_+(k) & \hat{\Delta}(k) \\ \hat{\Delta}(k)^\dagger & -\Pi \mathcal{H}_+^+(k) \Pi \end{pmatrix},$$

where, in terms of $d_i$, the hole block is

$$\mathcal{H}_h(k) = -\Pi \mathcal{H}_+^+(k) \Pi = -f_0(k_z) - B \Gamma_{12} + (d_1, -d_2, d_3, -d_4, -d_5) \cdot \Gamma.$$

The eigenvectors of Eq. (11) are, up to an overall phase choice,

$$\psi_{s_1, s_2} = \frac{1}{N} \left( \frac{s_2 d_1 - i d_2}{d_1 \Sigma_{34} g_{s_1, s_2}} \left( \begin{array}{c} d_{34} g_{s_1, s_2}^3 \\ 0 \end{array} \right), \frac{s_2 (d_1 - i d_2)}{d_1 \Sigma_{34} g_{s_1, s_2}} \left( \begin{array}{c} d_{34} g_{s_1, s_2}^3 \\ 0 \end{array} \right) \right)$$

$$N = \sqrt{\left( \frac{d_{34} g_{s_1, s_2}^3}{d_{34} g_{s_1, s_2}} \right)^2 + \left( \frac{d_{34} + (d_3 - s_2 d_5) g_{s_5, s_2}}{d_{34} g_{s_1, s_2}} \right)^2 + \left( \frac{d_{12} (d_3 - s_2 d_5)}{d_{34} g_{s_1, s_2}} \right)^2 + d_{34}^2 d_{12}^2}$$

with $s_1, s_2 = \pm 1$, $d_{ij} = \sqrt{d_i^2 + d_j^2}$, $g_{s_1, s_2} = s_2 B + d_3 + s_2 s_1 \sqrt{B^2 + d_2^2 + s_2^2 B d_3^2}$, and $g_{s_5, s_2} = g_{s_1, s_2} - d_3 - s_2 d_5$.

The spectrum along the $k_z$ axis is shown near $k_D$ in Fig. 1. There are four Weyl nodes at $k^{s_1}_{w_1} = k_D$, $k^{s_2}_{w_2} = k_D - B/v_+$, $k^{s_2}_{w_3} = k_D + B/v_+$, $k^{s_2}_{w_4} = k_D$ with energies $E_{w_1} = -B$, $E_{w_2} = -B v_- / v_+$, $E_{w_3} = B v_- / v_+$, $E_{w_4} = B$.

For a Fermi pocket defined by $E_{s_1, s_2} = 0$, the pocket is described by the eigenstate $|\psi_{s_1, s_2}\rangle$ of $\mathcal{H}(k)$. The Fermi surface projection selects the zero-energy eigenstates of $\mathcal{H}(k)$ and $\mathcal{H}_h(k)$, the latter of which has eigenstates

$$|\psi_{s_1, s_2}^h\rangle = -\Pi K |\psi_{s_1, s_2}\rangle,$$

where $K$ is the complex conjugation operator, at energy $E_{s_1, s_2}^h = -E_{s_1, s_2} = 0$. The overall phase in Eq. (17) is chosen for convenience to keep the final component of the state positive. The projected gap function over the $E_{s_1, s_2} = 0$ Fermi pocket to lowest order is then

$$\Delta_{proj}(k) = \langle \psi_{s_1, s_2}|\hat{\Delta}(k)|\psi_{s_1, s_2}^h\rangle = -\langle \psi_{s_1, s_2}|\hat{\Delta}(k)\Pi K |\psi_{s_1, s_2}\rangle.$$
and we refer to the point on the $k_z$ axis furthest from the $\Gamma$ point ($k = 0$) as the north pole and the point on the $k_z$ axis closest to the $\Gamma$ point as the south pole. With two such pockets, we call the pocket with the south pole closest to $\Gamma$ $FS_1$ and the other pocket $FS_2$, and the locations of the north and south poles are listed in Table III. The Fermi pockets at $k_z < 0$ are related by parity. When $\mu > \mu_{l,1} \equiv k_D v_p - B$ or $\mu < \mu_{l,1} \equiv -(k_D v_s - B)$, $FS_1$ has merged with the parity-related pocket and is trivial. For $\mu > \mu_{l,2} \equiv k_D v_p + B$ or $\mu < \mu_{l,2} \equiv -(k_D v_s + B)$, there are only two Fermi pockets in the entire Brillouin zone, and both are trivial. The eigenvectors in Eq. (16) can only have phase winding singularities along the $k_z$ axis and can be smoothly defined on any Fermi pocket using at most two patches. The eigenvectors at each Fermi pocket are shown in Table IV in the appropriate smooth gauge at each pole. As in Sec. II A, the transition function, and hence the Chern number, can be read from the gauge at each pole together with knowledge of whether the Fermi pocket is electron- or hole-type. For example, at $\mu_{U,1} > \mu > B$, the smaller Fermi pocket, $FS_2$, is electron-type and has $|\psi^{N}_{++}\rangle = e^{-i\phi_k}|\psi^{S}_{++}\rangle$, indicating a transition function with $q = 1/2$ in Eq. (5) and hence the Chern number $C = +1$. For the hole pocket $FS_2$ at $B_{-}^{+} > \mu > -B_{-}^{+}$, the appropriate gauges satisfy $|\psi^{N}_{-}\rangle = e^{2i\phi_k}|\psi^{S}_{-}\rangle$, which, due to the hole pocket reversing the normal direction, corresponds to $C = +2$.

| $\mu$ | $k_{z,1}^{S}$ | $k_{z,1}^{N}$ | $k_{z,2}^{S}$ | $k_{z,2}^{N}$ |
|------|-------------|-------------|-------------|-------------|
| $-B > \mu > \mu_{L,1}$ | $k_D - \frac{\mu - \mu_{L,1}}{v_p}$ | $k_D + \frac{\mu - \mu_{L,1}}{v_p}$ | $k_D + \frac{\mu - \mu_{L,1}}{v_p}$ | $k_D - \frac{\mu - \mu_{L,1}}{v_p}$ |
| $-B_{-}^{+} > \mu > -B_{-}^{+}$ | $k_D - \frac{\mu - \mu_{L,1}}{v_p}$ | $k_D + \frac{\mu - \mu_{L,1}}{v_p}$ | $k_D + \frac{\mu - \mu_{L,1}}{v_p}$ | $k_D - \frac{\mu - \mu_{L,1}}{v_p}$ |
| $B_{+}^{+} > \mu > -B_{+}^{+}$ | $k_D - \frac{\mu - \mu_{L,1}}{v_p}$ | $k_D + \frac{\mu - \mu_{L,1}}{v_p}$ | $k_D + \frac{\mu - \mu_{L,1}}{v_p}$ | $k_D - \frac{\mu - \mu_{L,1}}{v_p}$ |
| $\mu_{U,1} > \mu > B$ | $k_D - \frac{\mu - \mu_{L,1}}{v_p}$ | $k_D + \frac{\mu - \mu_{L,1}}{v_p}$ | $k_D + \frac{\mu - \mu_{L,1}}{v_p}$ | $k_D - \frac{\mu - \mu_{L,1}}{v_p}$ |

TABLE III. $k_z$ coordinates for the nodes of the two $k_z > 0$ Fermi pockets for $\mu_{U,1} > \mu > \mu_{L,1}$. For larger or smaller $\mu$, $FS_1$ has the north pole $k_{z,1}^{S}$ given respectively in the first or last row with the remaining pole at $-k_{z,1}^{N}$. For $\mu > \mu_{U,2}$ or $\mu < \mu_{L,2}$, the poles of $FS_2$ are similarly at $\pm k_{z,2}^{N}$.

| $\mu$ | $FS_1$ $S$ pole state | $FS_1$ $N$ pole state | $FS_2$ $S$ pole state | $FS_2$ $N$ pole state |
|------|----------------|----------------|----------------|----------------|
| $-B > \mu > \mu_{L,1}$ | $e^{3i\phi_k}|\psi_{--}\rangle$ | $e^{i\phi_k}|\psi_{--}\rangle$ | $e^{i\phi_k}|\psi_{+}\rangle$ | $|\psi_{--}\rangle$ |
| $-B_{+}^{+} > \mu > -B_{+}^{+}$ | $e^{3i\phi_k}|\psi_{--}\rangle$ | $e^{i\phi_k}|\psi_{--}\rangle$ | $|\psi_{--}\rangle$ | $e^{3i\phi_k}|\psi_{--}\rangle$ |
| $B_{+}^{+} > \mu > -B_{+}^{+}$ | $|\psi_{+}\rangle$ | $e^{3i\phi_k}|\psi_{+}\rangle$ | $e^{i\phi_k}|\psi_{--}\rangle$ | $e^{3i\phi_k}|\psi_{--}\rangle$ |
| $\mu_{U,1} > \mu > B$ | $|\psi_{+}\rangle$ | $e^{3i\phi_k}|\psi_{+}\rangle$ | $e^{i\phi_k}|\psi_{+}\rangle$ | $e^{3i\phi_k}|\psi_{+}\rangle$ |

TABLE IV. States describing the two $k_z > 0$ Fermi pockets for $\mu_{U,1} > \mu > \mu_{L,1}$ in the appropriate smooth gauge at each pole.
IV. MONOPOLE SUPERCONDUCTING ORDER FOR DIFFERENT IREPS IN THE MODEL OF MAGNETICALLY DOPED Cd₃As₂ WITH $\beta_s = \beta_p$

The possible pairings in irreducible representations of the $C_{4h}$ point group symmetry of the magnetically doped model in Eq. (4) can be constructed from products of momentum and matrix irreps in the spin-orbital basis. The representation of the symmetry group acting on the $4 \times 4$ spin-orbital basis is generated by $D(i) = I_{2 \times 2} \otimes \tau_z$ and $D(C_1) = e^{-i \frac{\pi}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$, $D(C_2) = e^{-i \frac{\pi}{4} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}}$, $D(C_3) = e^{-i \frac{\pi}{4} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}}$, and $D(C_4) = e^{-i \frac{\pi}{4} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}}$. Since $C_{4h}$ is abelian, the irreps are one-dimensional, and each matrix $M$ that is in an irrep satisfies $D(g)Mg(g)^T = \chi(g)M$ for each group element $g$. Note that the transpose $D(g)^T$ is used rather than the conjugate transpose since the gap function mixes particle and hole blocks of the BdG matrix. In terms of $M_{ij} = \sigma_i \otimes \tau_j$, with 0, 1, 2, 3 labeling the identity and $x$, $y$, and $z$ Pauli matrices, the sixteen matrices in Table Vb span the space of $4 \times 4$ matrices and are irreps of $C_{4h}$, with the character table reproduced in Table Vb for convenience [9].

| Irrep | Matrices       | Irreps | $E$ | $C_4$ | Basis Functions   |
|-------|----------------|--------|-----|-------|------------------|
| $A_g$ | $M_{10}$, $M_{13}$, $M_{20}$, $M_{23}$ | $A_g$ | 1   | 1     | $k_x^2 + k_y^2$  |
| $E_g$ | $M_{00}$ + $M_{33}$, $M_{63}$ + $M_{20}$ | $B_g$ | 1   | 1     | $ky$, $k_y - k_x^2$ |
| $E_g$ | $M_{30}$ + $M_{33}$, $M_{63}$ + $M_{30}$ | $e^{iB_g}$ | 1   | 1     | $k_x(k_y - ik_y)$ |
| $B_u$ | $M_{01}$, $M_{02}$, $M_{31}$, $M_{32}$ | $e^{iE_g}$ | 1   | 1     | $k_x(k_y + ik_y)$ |
| $E_u$ | $M_{11}$ + $M_{22}$, $M_{12} - M_{21}$ | $A_u$ | $-1$ | 1     | $k_x$               |
| $E_u$ | $M_{11}$ + $M_{22}$, $M_{12} - M_{21}$ | $B_u$ | $-1$ | 1     | $k_y$, $k_x(k_x^2 - k_y^2)$ |
| $E_u$ | $M_{11}$ + $M_{22}$, $M_{12} - M_{21}$ | $e^{iE_u}$ | $-1$ | 1     | $k_y - ik_y$       |

Table V. Irrep tables listing (a) the gap function matrices in each irreducible representation of $C_{4h}$ under the transformation $D(g)Mg(g)^T$ and (b) the character table for $C_{4h}$. In (a), the matrices are expressed as tensor products of Pauli matrices, $M_{ij} = \sigma_i \otimes \tau_j$. The character table in (b) includes the characters for the identity $E$ and group generators, parity $\Pi$ and $\pi/2$ clockwise rotation about the $z$ axis $C_4$.

For any matrix part, the momentum irrep must be chosen so that the condition required by Fermi statistics, $\Delta_{\beta}(k) = -\Delta_{\beta}(\mathbf{k} - \mathbf{k})$, is satisfied. Further, the projected gap function vanishes for pairings with matrix part $M_{01}$, $M_{10}$, $M_{11} \pm M_{22}$, $M_{20}$, $M_{23}$, or $M_{31}$. Note that the projected gap function to lowest order is calculated from $\Delta_{\text{proj}}(\mathbf{k}) = \langle \psi_{s_1, s_2} | \Delta(\mathbf{k}) | \psi_{s_1, s_2}\rangle = \langle \psi_{s_1, s_2} | \Delta(\mathbf{k})(-\Pi)K|\psi_{s_1, s_2}\rangle$, with $\Pi = M_{03}$ the parity operator and $K$ the complex conjugation operator. The projected gap function to lowest order is thus of the form $v^T M v$ and vanishes when the matrix $M = -\Delta(\mathbf{k})\Pi$ is antisymmetric, $M = -M^T$, as is the case for $\Delta(\mathbf{k}) \propto M_{01}$, $M_{11} \pm M_{22}$, $M_{20}$, $M_{23}$ and $M_{31}$. These gap functions thus vanish as a result of parity symmetry. The final vanishing gap function, $\Delta(\mathbf{k}) \propto M_{10}$, is a special case for the $\beta_s = \beta_p$ model that vanishes since $\langle \psi_{s_1, s_2} | (\psi_{s_1, s_2}^\dagger)^{(1)} \psi_{s_1, s_2}^\dagger | \psi_{s_1, s_2} \rangle = 0$, where $\psi_{s_1, s_2}^\dagger$ is the $j$th component of $|\psi_{s_1, s_2}\rangle$ in the spin-orbital basis. In the full model, Eq. (4) with $\beta_s \neq \beta_p$, we expect a pairing with matrix $M_{10}$ to have qualitatively similar projected gap function structure to one with $M_{13}$, both of which are in the $A_g$ representation.

Accounting for Fermi statistics, the pairings $\Delta(\mathbf{k})$ that lead to nonvanishing projected gap functions can be in one of four irreps, $A_u$, $B_u$, $E_u$, and $E_u$. In the following tables, the phase winding patterns are shown for the nine $M_{ij}$ matrices leading to nonvanishing projected gap functions in the $\beta_s = \beta_p$ model. For simplicity, only momentum-independent pairings and pairings with $k_z/k_D$ momentum dependence are considered. Considering different momentum dependence amounts to simply multiplying by a different function of momentum. The tables include descriptions of the nodes and phase winding from which the vorticity can be read. For rows with multiple pairing matrices listed, the pattern is plotted for the first pairing matrix listed, and the patterns for the rest are qualitatively similar with any differences noted in the description. The projected gap function dispersion near the $k_z$-axis nodes is written in terms of $k_{||} = \sqrt{k_x^2 + k_y^2}$.

Table VI lists phase winding patterns for the smaller, topologically trivial electron pocket at $\mu = 35$meV, which is described by the $|\psi_{++}\rangle$ state in Eq. (16). As this pocket is topologically trivial, $C = 0$, the total vorticity vanishes for all pairings. Table VII lists phase winding patterns for the $C = 1$ electron pocket at $k_z > 0$ for $\mu > 20$meV, described by the $|\psi_{++}\rangle$ state. Table VIII lists phase winding patterns for the $C = 2$ hole pocket at $k_z > 0$ for $\mu = 1$meV, described by the $|\psi_{--}\rangle$ state. The total vorticity in all cases can be seen to be $2C$. Importantly, the vorticity should be read from the counterclockwise phase winding number with respect to the local normal direction, which points out of the surface shown for electron pockets and into the surface for hole pockets.
| Pairing | Irrep | $\Delta_{proj}(k)$ Phase Winding | $\Delta_{proj}(k)$ Node Locations |
|---------|-------|---------------------------------|---------------------------------|
| $\Delta_0 \frac{k_z}{k_D} M_{13}$ | $A_u$ | $k^3 e^{-i\phi}$ nodes on $k_z$ axis. |
| $\Delta_0 M_{02}$ $\Delta_0 M_{32}$ | $B_u$ | $k|| e^{i\phi}$ nodes on $k_z$ axis. |
| $\Delta_0 (M_{12} + M_{21})$ $\Delta_0 \frac{k_z}{k_D} (M_{00} - M_{33})$ $\Delta_0 \frac{k_z}{k_D} (M_{03} - M_{30})$ | $^2 E_u$ | $k^2 e^{2i\phi}$ nodes on $k_z$ axis. |
| $\Delta_0 (M_{12} - M_{21})$ | $^1 E_u$ | $k^4$ nodes on $k_z$ axis. |
| $\Delta_0 \frac{k_z}{k_D} (M_{00} + M_{33})$ $\Delta_0 \frac{k_z}{k_D} (M_{03} + M_{30})$ | $^1 E_u$ | Gapped on $k_z$ axis. Gap function only vanishes at $k_z = 0$. |
TABLE VII: Pairing phase winding and node behavior for the $C = 1$
Fermi pocket at $\mu = 20\text{meV}$.

| Pairing                  | Irrep | $\Delta_{\text{proj}}(k)$ Phase Winding (North View) | $\Delta_{\text{proj}}(k)$ Phase Winding (South View) | $\Delta_{\text{proj}}(k)$ Node Locations |
|--------------------------|-------|------------------------------------------------------|------------------------------------------------------|------------------------------------------|
| $\Delta_0 \frac{k_z}{k_D} M_{13}$ | $A_u$ | ![Image](image1.png)                                  | ![Image](image2.png)                                  | $k_3 e^{-i\phi}$ node at $N$, $k_3 e^{-3i\phi}$ node at $S$. |
| $\Delta_0 M_{02}$       | $B_u$ | ![Image](image3.png)                                  | ![Image](image4.png)                                  | $k_3 e^{i\phi}$ node at $N$, $k_3 e^{-i\phi}$ node at $S$. |
| $\Delta_0 M_{32}$       |       |                                                      |                                                      |                                          |
| $\Delta_0 (M_{12} + M_{21})$ | $^2E_u$ | ![Image](image5.png)                                  | ![Image](image6.png)                                  | $k_3 e^{-2i\phi}$ node at $N$, $k_3 e^{-4i\phi}$ node at $S$. |
| $\Delta_0 \frac{k_z}{k_D} (M_{00} - M_{33})$ | $^2E_u$ | ![Image](image7.png)                                  | ![Image](image8.png)                                  | $k_3 e^{2i\phi}$ node at $N$, gapped at $S$. |
| $\Delta_0 \frac{k_z}{k_D} (M_{03} - M_{30})$ |       |                                                      |                                                      |                                          |
| $\Delta_0 (M_{12} - M_{21})$ | $^1E_u$ | ![Image](image9.png)                                  | ![Image](image10.png)                                 | $k_3 e^{-2i\phi}$ node at $N$, $k_3 e^{-4i\phi}$ node at $S$. |
\[ \Delta_0 \frac{k_x}{k_D} (M_{00} + M_{33}) \]  
\[ \Delta_0 \frac{k_x}{k_D} (M_{03} + M_{30}) \]  
$1E_u$

Gapped at N, $k_x^2 e^{-2i\phi}$ node at S.

### TABLE VIII: Pairing phase winding and node behavior for the $C = 2$ hole pocket at $\mu = 0$ meV.

| Pairing | Irrep | $\Delta_{\text{proj}}^N(k)$ Phase Winding (North View) | $\Delta_{\text{proj}}^S(k)$ Phase Winding (South View) | $\Delta_{\text{proj}}(k)$ Node Locations |
|---------|-------|------------------------------------------------------|------------------------------------------------------|----------------------------------------|
| $\Delta_0 \frac{k_x}{k_D} M_{13}$ | $A_u$ | ![North View](image1) | ![South View](image2) | $k_x^3 e^{-3i\phi}$ node at N, $k_x^3 e^{i\phi}$ node at S. |
| $\Delta_0 M_{02}$  
$\Delta_0 M_{32}$ | $B_u$ | ![North View](image3) | ![South View](image4) | $k_x e^{-i\phi}$ nodes at N and S poles.  
Four nodes with $e^{i\phi'}$ local winding at $\phi = \pm \pi/4, \pm 3\pi/4$ around concave region near S.  
For $M_{32}$, these nodes are at $\phi = 0, \pm \pi/2, \pi$ instead. |
| $\Delta_0(M_{12} + M_{21})$ | $2E_u$ | ![North View](image5) | ![South View](image6) | $k_x^3 e^{-4i\phi}$ node at N, $k_x^4$ node at S. |
| $\Delta_0 \frac{k_x}{k_D} (M_{00} - M_{33})$  
$\Delta_0 \frac{k_x}{k_D} (M_{03} - M_{30})$ | $2E_u$ | ![North View](image7) | ![South View](image8) | Gapped at N and S. Four nodes with $e^{i\phi'}$ local winding at $\phi = 0, \pm \pi/2, \pi$ around the concave region near S.  
For $M_{03} - M_{30}$, these nodes are at $\phi = \pm \pi/4, \pm 3\pi/4$ instead. |

\[ \Delta_0 \frac{k_x}{k_D} (M_{00} + M_{33}) \]  
\[ \Delta_0 \frac{k_x}{k_D} (M_{03} + M_{30}) \]  
$1E_u$
\[ \Delta_0(M_{12} - M_{21}) \quad \text{and} \quad \Delta_0 \frac{E_u}{k_0} (M_{00} + M_{33}) \quad \Delta_0 \frac{E_u}{k_0} (M_{03} + M_{30}) \]

\[ k_0^2 e^{-2i\phi} \text{ node at } N, \quad k_0^2 e^{2i\phi} \text{ node at } S. \]

\[ k_0^2 e^{-2i\phi} \text{ nodes at } N \text{ and } S \]

poles. Four nodes with \( e^{i\phi'} \) local winding at \( \phi = 0, \pm \pi/2, \pi \)

around concave region near S. For \( M_{03} - M_{30} \), these nodes are at \( \phi = \pm \pi/4, \pm 3\pi/4 \) instead.

[1] Z. Wang, H. Weng, Q. Wu, X. Dai, and Z. Fang, Phys. Rev. B 88, 125427 (2013).
[2] C.-X. Liu, X.-L. Qi, H. Zhang, X. Dai, Z. Fang, and S.-C. Zhang, Phys. Rev. B 82, 045122 (2010).
[3] R. Winkler, Spin-Orbit Coupling Effects in Two-Dimensional Electron Systems (Springer) pp. 18–20.
[4] S. Altmann and P. Herzig, Point-group Theory Tables, Oxford science publications (Clarendon Press, 1994).
[5] S. Baidya and D. Vanderbilt, Phys. Rev. B 102, 165115 (2020).
[6] T. T. Wu and C. N. Yang, Nucl. Phys. B 107, 365 (1976).
[7] S. Murakami and N. Nagaosa, Phys. Rev. Lett. 90, 057002 (2003).
[8] Y. Li and F. D. M. Haldane, Phys. Rev. Lett. 120, 067003 (2018).
[9] M. I. Aroyo, J. M. Perez-Mato, C. Capillas, E. Kroumova, S. Ivantchev, G. Madariaga, A. Kirov, and H. Wondratschek, Zeitschrift für Kristallographie - Crystalline Materials 221, 15 (2006).