COMPUTING UPPER BOUNDS FOR OPTIMAL DENSITY OF \((t, r)\) BROADCASTS ON THE INFINITE GRID

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Abstract. The domination number of a finite graph \(G\) with vertex set \(V\) is the cardinality of the smallest set \(S \subseteq V\) such that for every vertex \(v \in V\) either \(v \in S\) or \(v\) is adjacent to a vertex in \(S\). A set \(S\) satisfying these conditions is called a dominating set. In 2015 Blessing, Insko, Johnson, and Mauretour introduced \((t, r)\) broadcast domination, a generalization of graph domination parameterized by the nonnegative integers \(t\) and \(r\). In this setting, we say that the signal a vertex \(v \in V\) receives from a tower of strength \(t\) located at vertex \(T\) is defined by \(\text{sig}(v, T) = \max(t - \text{dist}(v, T), 0)\). Then a \((t, r)\) broadcast dominating set on \(G\) is a set \(S \subseteq V\) such that the sum of all signal received at each vertex \(v \in V\) is at least \(r\). In this paper, we consider \((t, r)\) broadcasts of the infinite grid and present a Python program to compute upper bounds on the minimal density of a \((t, r)\) broadcast on the infinite grid. These upper bounds allow us to construct counterexamples to a conjecture by Blessing et al. that the \((t, r)\) and \((t + 1, r + 2)\) broadcasts are equal whenever \(t, r \geq 1\).

1. Introduction

Let \(G\) be a finite graph with vertex set \(V\) and edge set \(E\) on which the distance between two vertices \(u\) and \(v\) in \(V\), denoted \(\text{dist}(u, v)\), is defined as the length of the shortest path in \(G\) between \(u\) and \(v\). A set \(S \subseteq V\) is called a dominating set of \(G\) if for any vertex \(v \in V\) either \(v \in S\) or \(d(u, v) = 1\) for some \(u \in S\). The cardinality of a smallest dominating set is called the domination number of \(G\) and is denoted \(\delta(G)\).

We let \(G_{m,n}\) denote the finite grid graph of dimension \(m \times n\). More precisely,

\[ G_{m,n} = (V, E) \] with

\[ V = \{v_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\} \]

\[ E = \{(v_{i,j}, v_{i+1,j}), (v_{i,j}, v_{i,j+1}) : 1 \leq i < m, 1 \leq j < n\}. \]

Determining the domination number of finite graphs, in particular grid graphs, has received much attention in the graph theory literature; for an overview of the subject, refer to [7]. The 2011 work of Gonçalves, Pinlou, Rao, and Thomassé [5] finally confirmed in the affirmative Chang's 1992 conjecture [2] that for every \(16 \leq n \leq m\), \(\delta(G_{m,n}) = \left\lceil \frac{(n+2)(m+2)}{8} \right\rceil\), thereby establishing the domination number for all finite grid graphs.

Other work in this area expands domination theory by considering variations on domination and includes work on \(r\)-domination and broadcast domination [1] [6]. In 2015, Blessing, Insko, Johnson, and Mauretour introduced \((t, r)\) broadcast domination, another generalization of graph domination defined by the nonnegative integral parameters \(r\) and \(t\) [1]. In this

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setting, we say that the signal a vertex \( v \in V \) receives from a tower of strength \( t \) located at vertex \( T \) is defined by \( \text{sig}(v, T) = \max(t - \text{dist}(v, T), 0) \). A \((t, r)\) broadcast dominating set on \( G \) is a set \( S \subseteq V \) such that the sum of all signal received at each vertex \( v \in V \) is at least \( r \). The cardinality of the smallest \((t, r)\) broadcast on a finite graph \( G \) is called the \((t, r)\) broadcast domination number of \( G \). Note that the \((2, 1)\) broadcast domination number of a graph \( G \) is exactly the classical domination number of \( G \).

In this paper, we consider \((t, r)\) broadcasts on the integer lattice graph \( G_\infty = \mathbb{Z} \times \mathbb{Z} \), which we refer to as the infinite grid. Since the cardinality of any \((t, r)\) broadcast \( T \) on \( G_\infty \) will be infinite, we instead compute the density of a \((t, r)\) broadcast, which is intuitively defined as the proportion of the vertices of \( G_\infty \) contained in a \((t, r)\) broadcast \( T \).

**Definition 1.** Given a \((t, r)\) broadcast \( T \) on \( G_\infty \), consider the vertex set \( V \) of the subgraph \( G_{2n+1,2n+1} \) with its central vertex located at \((0,0)\). Then the broadcast density of a \((t, r)\) broadcast on \( G_\infty \) is defined as

\[
\lim_{n \to \infty} \frac{|T \cap V|}{|V|}.
\]

The optimal density of a \((t, r)\) broadcast on \( G_\infty \), denoted \( \delta_{t,r}(G_\infty) \), is the minimum broadcast density over all \((t, r)\) broadcasts and we say that a \((t, r)\) broadcast is optimal if its density is optimal. In previous work, the authors have established the optimal densities of \((t, r)\) broadcasts for all \( t \geq 1 \) and \( r = 1, 2 \).

**Theorem 1** (Theorems 1, 2, 3 in [3]). If \( t, r \in \mathbb{Z}^+ \), then

- \( \delta_{t,1}(G_\infty) = \frac{1}{2^{2^t-2t+1}} \) if \( t = 2 \)
- \( \delta_{t,2}(G_\infty) = \begin{cases} \frac{1}{3} & \text{if } t = 2 \\ \frac{1}{2(t-1)^2} & \text{if } t > 2 \end{cases} \)
- \( \delta_{t,3}(G_\infty) \leq \delta_{t-1,1}(G_\infty) \).

Unfortunately, the methods employed in [3] to establish statements 1 and 2 of Theorem 1 do not easily extend to compute optimal \((t, r)\) broadcast densities for \( r \geq 3 \). Hence, a more computational approach is necessary and motivates this work. In this paper, we present a Python program\(^1\) to compute upper bounds for the optimal density of \((t, r)\) broadcast domination of \( G_\infty \) for any \( t > r \geq 1 \).

To compute an upper bound on the \((t, r)\) broadcast density for given \( t \) and \( r \), our program systematically checks sets of vertices of known broadcast densities and returns the \((t, r)\) broadcast with the lowest density among them. The sets of vertices used by our program are called the standard patterns, which we define as follows.

**Definition 2.** The standard pattern is defined by the positive integers \( d \) and \( e \) as

\[
p(d, e) = \{(dx + ey, y) : x, y \in \mathbb{Z}\}.
\]

The standard patterns \( p(4, 2) \) and \( p(13, 5) \) are depicted in Figure 1. Since, for any positive integer \( d \), the standard pattern \( p(d, e) \) uses one out of every \( d \) vertices on each horizontal line of the grid, it follows that the density of the standard pattern \( p(d, e) \) is \( \frac{1}{d} \). We use the notation \( \mathbb{T}(d, e) \) to refer to the set of vertices on \( G_\infty \) corresponding to the standard pattern

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\(^1\)The program is available for download from GitHub [3].
The standard pattern \( p(4,2) \).

The standard pattern \( p(13,5) \).

Figure 1. Examples of standard patterns on the infinite grid.

\( p(d,e) \). Note that \( T(d,e) \) is not necessarily a \((t,r)\) broadcast for any values of \( t \) and \( r \). When \( T(d,e) \) is in fact a broadcast we call it a \( \text{standard broadcast} \).

The standard patterns are convenient for computing upper bounds for the optimal density of \((t,r)\) broadcasts because of their regular structure. However, not all standard patterns are \((t,r)\) broadcasts for a given \( t \) and \( r \). To determine if \( T(d,e) \) is in fact a broadcast, we need only to ensure that the sum of signal received is at least \( r \) at \( d \) specific vertices. To make this observation precise, we prove the following proposition.

**Proposition 1.** Let \( t \) and \( r \) be positive integers. Then \( T(d,e) \) is a \((t,r)\) broadcast if and only if the sum of all signal is at least \( r \) for every vertex in the set \( \{v_{i,0} : 0 \leq i < d\} \).

**Proof.** The sufficiency condition follows directly from the definition of a \((t,r)\) broadcast.

To establish the necessity condition suppose that under the set of broadcast towers \( T(d,e) \) the sum of all signal is at least \( r \) for every vertex in the set \( \{v_{i,0} : 0 \leq i < d\} \), and let \( v_{x',y} \) be a vertex of \( G_\infty \). By the definition of a standard pattern, there exist integers \( x \) and \( k \), \( 0 \leq k < d \), such that

\[
x' = dx + ey + k.
\]

Thus \( v_{x',y} = v_{dx+ey+k,y} \) for some integers \( x \) and \( k \). For every tower \( T_0 \) located at the point \( (p,q) \), by the definition of a standard broadcast, there exists a tower \( T_1 \) located at the point \( (dx+ey+p,y+q) \) with \( sig(v_{dx+ey+k,y}, T_1) = sig(v_{k,0}, T_0) \). As the sum of all signal at \( v_{k,0} \) is at least \( r \), the sum of all signal at \( v_{x',y} \) is at least \( r \).

Our program computes an upper bound on the optimal \((t,r)\) broadcast equal to the minimum density over all standard \((t,r)\) broadcasts. While this result is not necessarily optimal, most known optimal \((t,r)\) broadcasts do correspond to standard patterns, including optimal broadcasts for \((t,1)\) with \( t \geq 1 \), \((t,2)\) with \( t \geq 2 \), and \((3,3)\) \[1\, 3\]. The \( t = 2, r = 3 \) case is the only known \((t,r)\) pair for which there exists an optimal non-standard \((t,r)\) broadcast and no optimal standard \((t,r)\)-broadcast \[3\].

We end this section by remarking that the third statement of Theorem \[1\] provides further evidence for the conjecture of Blessing et al. that the optimal \((t,1)\) and \((t+1,3)\) broadcasts are identical for all \( t \geq 3 \). However, their conjecture stated more broadly that the optimal \((t,r)\) and \((t+1,r+2)\) broadcasts were identical for all \( t \geq 3 \), which is false in general. As an application of our program, we present counterexamples to this conjecture in Section \[2.2\].
2. Computer implementation

2.1. Implementation Details. The program takes as input positive integers \( t \) and \( r \) and computes the density of the optimal standard \((t, r)\) broadcast as detailed in Algorithm 1.

**Algorithm 1** Optimal Standard \((t, r)\) Broadcast

1: procedure \text{MinDensity}(t, r) 
2: \hspace{1em} \( d_{\text{max}} = \text{DMax}(t, r) \) 
3: \hspace{1em} \( d_{\text{best}} \leftarrow 0 \) 
4: \hspace{1em} \( d \leftarrow 1 \) 
5: \hspace{1em} while \( d \leq d_{\text{max}} \) do 
6: \hspace{2em} \( e \leftarrow 0 \) 
7: \hspace{2em} while \( e < d \) do 
8: \hspace{3em} if \text{IsBroadcast}(t, r, d, e) then 
9: \hspace{4em} \( d_{\text{best}} \leftarrow d \) 
10: \hspace{3em} end if 
11: \hspace{2em} \( e \leftarrow e + 1 \) 
12: \hspace{1em} end while 
13: \hspace{1em} \( d \leftarrow d + 1 \) 
14: \hspace{1em} end while 
15: \hspace{1em} return \( \frac{1}{d_{\text{best}}} \) 
16: end procedure

To compute \( d_{\text{max}} \), we first compute the *usable signal* emitted by a broadcast tower of strength \( t \) on the infinite grid \( G_{\infty} = (V_{\infty}, E_{\infty}) \), which is given by the equation

\[
\sum_{v \in V_{\infty}} \min(\text{sig}(v, T), r).
\]

This value represents the total signal generated by the tower for all nearby vertices, not including the unnecessary signal provided to each vertex \( v \) with \( \text{sig}(v, T) > r \). We divide the usable signal by \( r \) to get a conservative lower bound \( \delta_{\text{min}}(t, r) \) on the optimal \((t, r)\) broadcast density. Hence, the maximum distance between tower vertices on the same horizontal line in a standard \((t, r)\) broadcast, is equal to

\[
d_{\text{max}} = \left\lfloor \frac{1}{\delta_{\text{min}}(t, r)} \right\rfloor.
\]

For each \( d \in \{1, 2, ... d_{\text{max}}\} \), the program iterates through each \( e \in \{0, 1, ..., d - 1\} \) and checks to see if \( T(d, e) \) is a standard \((t, r)\) broadcast. The program then returns the optimal standard broadcast and its density.

To check if a set of vertices \( T(d, e) \) is a \((t, r)\) broadcast, the program creates a grid window \( W \) consisting of every vertex within distance \( t \) of any vertex \( v \in \{v_{i,0} : 0 \leq i < d\} \). Thus every tower in \( T(d, e) \) that contributes signal to a vertex in \( \{v_{i,0} : 0 \leq i < d\} \) is included in the window. The program then places towers at each vertex \( T \in W \cap T(d, e) \) and computes the total signal of each vertex \( v \in \{v_{i,0} : 0 \leq i < d\} \). By Proposition 1, if each vertex in this set has total signal at least \( r \), then \( T(d, e) \) is a standard \((t, r)\) broadcast.
2.2. **Computations.** Blessing et al. prove that the optimal (3, 3) and (2, 1) broadcast densities are equal on large grids [1], and further conjecture that the optimal \((t, r)\) and \((t + 1, r + 2)\) broadcast densities are equal for all \((t, r)\). However, with the exception of the \(r = 1\) case, the conjecture is broadly false.

![Image](https://www.example.com/image.png)

**Table 1.** Best standard \((t, r)\) broadcasts for \(1 \leq t \leq 15\) and \(1 \leq r \leq 8\).

Table 1 lists the best standard \((t, r)\) broadcasts for all \(1 \leq r \leq 8\) and \(1 \leq t \leq 15\), as computed by our program. If multiple standard broadcasts achieve the best broadcast density, the broadcast \(T(d, e)\) with the lowest offset value \(e\) is listed. For instance, although \(T(5, 1)\), \(T(5, 2)\), \(T(5, 3)\), and \(T(5, 4)\) are optimal \((3, 3)\) broadcasts, only \(T(5, 1)\) is listed. Recall that the density of the standard \((t, r)\) broadcast \(T(d, e)\) is \(\frac{1}{7}\).

Note that the first two statements of Theorem 1 confirm that the standard \((t, 1)\) and \((t, 2)\) broadcasts listed in the table are optimal, as proved in 3. The data provide further evidence that the conjecture of Blessing et al. is true in some cases. First, the optimal \((t, 1)\) broadcasts are best standard \((t + 1, 3)\) broadcasts, providing support for the limited conjecture that the optimal \((t, r)\) and \((t + 1, r + 2)\) broadcast densities are equal in the \(r = 1\) case 3. Furthermore, the optimal \((t, 2)\) broadcasts appear to be equal in density to the best standard \((t + 1, 4)\) broadcasts when \(t\) is even and greater than 3, indicating a second case in which the conjecture of Blessing et al. might hold.

However, Table 1 provides a variety of counterexamples that indicate the conjecture is false. Although the conjecture would entail that the optimal \((t, 1)\) and \((t + 2, 5)\) broadcast densities are equal, the densities of the optimal \((t, 1)\) broadcasts are in fact consistently greater than the corresponding densities of the best standard \((t + 2, 5)\) broadcasts. Figure
2 displays an optimal (2,1) broadcast and a (4,5) broadcast with lower broadcast density. Additionally, the densities of the optimal (t, 2) broadcasts are often greater than those of the best standard (t + 1, 4) broadcasts and consistently greater than those of the best standard (t + 2, 6) broadcasts. Thus, establishing that the conjecture of Blessing et al. is false in the general case, the best standard broadcasts provide tighter upper bounds on the optimal (t, r) broadcasts.

![Diagram of broadcasts](image)

**Figure 2.** The standard (4, 5) broadcast $T(8, 2)$ has a lower broadcast density than the optimal (2, 1) broadcast $T(5, 3)$.

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