Multi-scale regularity of axisymmetric Navier-Stokes equations

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Abstract

By applying the delicate \textit{a priori} estimates for the equations of \((\Phi, \Gamma)\), which is introduced in the previous work, we obtain some multi-scale regularity criteria of the swirl component \(u^{\theta}\) for the 3D axisymmetric Navier-Stokes equations. In particularly, the solution \(u\) can be continued beyond the time \(T\), provided that \(u^{\theta}\) satisfies

\[
\begin{align*}
u^{\theta} & \in L_{T}^{p}L_{w}^{q_{v}}L_{h}^{q_{h}}, \quad \frac{2}{p} + \frac{1}{q_{v}} + \frac{2}{q_{h}} \leq 1, \quad 2 < q_{h} \leq \infty, \quad \frac{1}{q_{v}} + \frac{2}{q_{h}} < 1.
\end{align*}
\]

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1. Introduction

This article aims at presenting some new regularity criteria of the swirl component \(u^{\theta}\), in the framework of anisotropic Lebesgue space, which improve in that of [7].

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Consider the Cauchy problem of the 3D Navier-Stokes equations:

\begin{equation}
\begin{aligned}
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \Delta \mathbf{u} + \nabla p &= 0, \\
\nabla \cdot \mathbf{u} &= 0, \\
\mathbf{u}|_{t=0} &= \mathbf{u}_0.
\end{aligned}
\tag{1.1}
\end{equation}

where \( \mathbf{u}(t, x) = (u^1, u^2, u^3) \), \( p(t, x) \) and \( \mathbf{u}_0 \) denote the fluid velocity field, the pressure, and the given initial velocity field, respectively.

For given \( \mathbf{u}_0 \in L^2(\mathbb{R}^3) \) with \( \text{div} \; \mathbf{u}_0 = 0 \) in the sense of distribution, a global weak solution \( \mathbf{u} \) to the Navier-Stokes equations was constructed by Leray [31] and Hopf [20], which is called Leray-Hopf weak solution. The regularity of such Leray-Hopf weak solution in three dimension plays an important role in the mathematical fluid mechanics. One essential work is usually referred as Prodi-Serrin (P-S) conditions (see [10, 11, 18, 38, 40, 41, 42]), i.e. if in addition, the weak solution \( \mathbf{u} \) belongs to \( L^p((0, T); L^q(\mathbb{R}^3)) \), where \( \frac{2}{p} + \frac{3}{q} \leq 1 \), \( 3 \leq q \leq \infty \), then the weak solution becomes regular.

In this paper, we assume that the solution \( \mathbf{u} \) of the system (1.1) has the axisymmetric form

\begin{equation}
\mathbf{u}(t, x) = u^r(t, r, x_3)\mathbf{e}_r + u^\theta(t, r, x_3)\mathbf{e}_\theta + u^3(r, x_3)\mathbf{e}_3,
\tag{1.2}
\end{equation}

where

\[
\mathbf{e}_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \quad \mathbf{e}_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \quad \mathbf{e}_3 = (0, 0, 1), \quad r = \sqrt{x_1^2 + x_2^2}.
\]

In above, \( u^\theta \) is usually called the swirl component. And if \( u^\theta = 0 \), the solution \( \mathbf{u} \) is without swirl.

For the axisymmetric solutions of Navier-Stokes system, we can equiva-
lently reformulate (1.1) as

\[
\begin{aligned}
\partial_t u^r + (u^r \partial_r + u^3 \partial_3) u^r - (\partial_r^2 + \partial_3^2 + \frac{1}{r} \partial_r - \frac{1}{r^2}) u^r - \frac{(u^\theta)^2}{r} + \partial_r p &= 0, \\
\partial_t u^\theta + (u^r \partial_r + u^3 \partial_3) u^\theta - (\partial_r^2 + \partial_3^2 + \frac{1}{r} \partial_r - \frac{1}{r^2}) u^\theta + \frac{u^\theta u^r}{r} &= 0, \\
\partial_t u^3 + (u^r \partial_r + u^3 \partial_3) u^3 - (\partial_r^2 + \partial_3^2 + \frac{1}{r} \partial_r) u^3 + \partial_3 p &= 0, \\
\partial_r u^r + \frac{1}{r} u^r + \partial_3 u^3 &= 0, \\
(u^r, u^\theta, u^3)|_{t=0} &= (u^r_0, u^\theta_0, u^3_0).
\end{aligned}
\]

(1.3)

For the axisymmetric velocity field $u$, we can also compute the vorticity $\omega = \text{curl } u$ as follows,

\[
\omega = \omega^r e_r + \omega^\theta e_\theta + \omega^3 e_3,
\]

(1.4)

with $\omega^r = -\partial_3 u^\theta$, $\omega^\theta = \partial_3 u^r - \partial_r u_3$, $\omega^3 = \partial_r u^\theta + \frac{u^\theta}{r}$. Furthermore, $(\omega^r, \omega^\theta, \omega^3)$ satisfy

\[
\begin{aligned}
\partial_t \omega^r + (u^r \partial_r + u^3 \partial_3) \omega^r - (\partial_r^2 + \partial_3^2 + \frac{1}{r} \partial_r - \frac{1}{r^2}) \omega^r - (\omega^r \partial_r + \omega^3 \partial_3) u^r &= 0, \\
\partial_t \omega^\theta + (u^r \partial_r + u^3 \partial_3) \omega^\theta - (\partial_r^2 + \partial_3^2 + \frac{1}{r} \partial_r - \frac{1}{r^2}) \omega^\theta - \frac{2u^\theta \partial_3 u^\theta}{r} - \frac{u^r \omega^\theta}{r} &= 0, \\
\partial_t \omega^3 + (u^r \partial_r + u^3 \partial_3) \omega^3 - (\partial_r^2 + \partial_3^2 + \frac{1}{r} \partial_r) \omega^3 - (\omega^r \partial_r + \omega^3 \partial_3) u^3 &= 0, \\
(\omega^r, \omega^\theta, \omega^3)|_{t=0} &= (\omega^r_0, \omega^\theta_0, \omega^3_0).
\end{aligned}
\]

(1.5)

The bounded property of $ru^\theta$ preserves as the time grows, i.e. $ru^\theta \in L^\infty([0, +\infty); L^\infty(\mathbb{R}^3))$, if $ru^\theta_0 \in L^\infty(\mathbb{R}^3)$, see [37, 35] etc. It is an essential ingredient for the axisymmetric Navier-Stokes equations. And it makes us to consider the regularity criteria of $u^\theta$ in the critical case for the axisymmetric Navier-Stokes equations.

We recall that global well-posedness result was firstly proved under no swirl assumption, i.e. $u^\theta = 0$, independently by Ukhovskii and Yudovich [43], and Ladyzhenskaya [26], also [30] for a refined proof. When the angular velocity $u^\theta$ is not trivial, the global well-posedness problem is still open. Much attentions has been draw for decades and tremendous efforts
and interesting progress have been made on the regularity problem of the axisymmetric Navier-Stokes equations\cite{3, 4, 5, 6, 7, 23, 25, 28, 45} etc. . In \cite{4, 5}, Chen, Strain, Tsai and Yau proved that the suitable weak solutions are smooth if the velocity field $u$ satisfies $r|u| \leq C < \infty$. Applying the Liouville type theorem for the ancient solutions of Navier-Stokes equations, Z. Lei and Qi S. Zhang \cite{28} obtained the similar result in the case $b = u^r(t, r, x_3)e_r + u^3(t, r, x_3)e_3 \in L^\infty((0, T); BMO^{-1})$. And we promote in \cite{7} that the solution $u$ is smooth in $(0, T) \times \mathbb{R}^3$, if $r^d u^\theta \in L^p((0, T); L^q(\mathbb{R}^3))$, where

$$\frac{2}{p} + \frac{3}{q} \leq 1 - d, \quad 0 \leq d < 1, \quad \frac{3}{1-d} < q \leq \infty, \quad \frac{2}{1-d} \leq p \leq \infty.$$

The above regularity criteria of $u^\theta$, which is scaling invariant, greatly develop the corresponding regularity criteria in \cite{24, 25, 37, 45}. Unfortunately, it fails in the critical case $d = 1$, which is the ideal goal, since the conservation law of $r u^\theta$. Since then, there are some significant improvements and applications \cite{29, 44, 6, 12}.

In this paper, we introduce an anisotropic Lebesgue space $L^p_T L^q_v L^{q_h,w}_h$, since the solutions behavior anisotropic on the variable $r$ and $x_3$. By applying the delicate a priori estimations for the equations of $(\Phi, \Gamma)$, we can obtain the regularity criteria

$$u^\theta \in L^p_T L^q_v L^{q_h,w}_h, \quad \frac{2}{p} + \frac{1}{q_v} + \frac{2}{q_h} \leq 1, \quad 2 < q_h \leq \infty, \quad \frac{1}{q_v} + \frac{2}{q_h} < 1.$$

It improves the regularity criteria in \cite{7}. Moreover it provides us a new perspective to the open problem, instead of the weighted Lebesgue space in \cite{7}. For instance, we assume $r u^\theta$ is Hölder for the variable $r$, i.e. $|u^\theta| \leq C r^{\alpha-1}$, $0 < \alpha \leq 1$. Therefore, the solution $u$ is regular, since $u^\theta \in L^\infty_T L^\infty_v L^{2,\infty}_h$. The authors in \cite{4, 5, 7, 28} drew a similar argument. And there are some detail discussions in Remark 1 for the extreme points.

**Notations.** Throughout this paper, $L^{q,r}(\mathbb{R}^n)$ stands for Lorentz space, while $L^{q,w} = L^{q,\infty}$.

Moreover, we introduce the Banach space $L^p_T L^q_v L^{q_h,w}_h$, equipped with norm

$$\|f\|_{L^p_T L^q_v L^{q_h,w}_h} = \||f(t, x_1, x_2, x_3)\|_{L^{q_h,w}((\mathbb{R}^3, dx_1dx_2))} \|L^{q_v}(\mathbb{R}, dx_3)\|_{L^p((0,T), dt)}.$$
And we denote $\dot{H}^{s,p}$ and $\dot{B}^{s,q}_{p,q}$ for the homogeneous Sobolev space and homogeneous Besov space, respectively. For simplicity, we denote $\dot{H}^{s} = \dot{H}^{s}(\mathbb{R}^2, dx_1dx_2)$, $L^p_x = L^p(\mathbb{R}^3, dx)$. And the other ones are similar.

We note $b = u^r(t, r, x_3)e_r + u^3(t, r, x_3)e_3$, and $(\Phi, \Gamma) = (\frac{\omega^r}{r}, \frac{\omega^\theta}{r})$, while $\omega^r, \omega^\theta$ is defined in (1.4).

Finally, we note $C$ the arbitrary constant.

2. Main Result

**Theorem 2.1.** Let $u \in C([0, T); H^2(\mathbb{R}^3)) \cap L^2_{loc}([0, T); H^3(\mathbb{R}^3))$ be the unique axisymmetric solution of the Navier-Stokes equations with the axisymmetric initial data $u_0 \in H^2(\mathbb{R}^3)$ and $\text{div } u_0 = 0$. If $ru^\theta_0 \in L^\infty$ and the time $T < \infty$, the solution $u$ can be continued beyond the time $T$, provided that the swirl $u^\theta$ satisfies

$$r^d u^\theta \in L_T^p L_v^{q_v} L_h^{q_h,w}, \quad \frac{2}{p} + \frac{1}{q_v} + \frac{2}{q_h} \leq 1 - d, \quad -1 \leq d < 1, \quad \frac{2}{1-d} < q_h \leq \infty, \quad \frac{1}{q_v} + \frac{2}{q_h} < 1 - d.$$ 

Set $d = 0$ in Theorem 2.1, and the following corollary is derived straightforward.

**Corollary 2.2.** Let $u \in C([0, T); H^2(\mathbb{R}^3)) \cap L^2_{loc}([0, T); H^3(\mathbb{R}^3))$ be the unique axisymmetric solution of the Navier-Stokes equations with the axisymmetric initial data $u_0 \in H^2(\mathbb{R}^3)$ and $\text{div } u_0 = 0$. If $ru^\theta_0 \in L^\infty$ and the time $T < \infty$, the solution $u$ can be continued beyond the time $T$, provided that the swirl $u^\theta$ satisfies

$$u^\theta \in L_T^p L_v^{q_v} L_h^{q_h,w}, \quad \frac{2}{p} + \frac{1}{q_v} + \frac{2}{q_h} \leq 1, \quad 2 < q_h \leq \infty, \quad \frac{1}{q_v} + \frac{2}{q_h} < 1. \quad (2.1)$$

**Remark 1.** At the extreme points $\{p = \infty, \frac{1}{q_v} + \frac{2}{q_h} = 1, \quad 2 < q_h \leq \infty\}$ in Corollary 2.2, we can still derive regularity criteria with an additional smallness assumption

$$\|u^\theta\|_{L_T^\infty L_v^{q_v} L_h^{q_h,w}} \leq \epsilon,$$

where $\epsilon$ is a sufficiently small constant. The precise proof can be dealt with in an analogous process in Section 4.

As a matter of fact, $u^\theta \in L_T^\infty L_v^{\infty} L_h^{2,w}$, since $|ru^\theta| \leq C\|ru^\theta_0\|_{L^\infty(\mathbb{R}^3)}$. And the global regularity of the solutions of axisymmetric Navier-Stokes equations can be solved, if the extreme point $(p, q_v, q_h) = (\infty, \infty, 2)$ in Corollary 5.
2.2 is settled. However, the problems remain open. Recently, D. Wei [44] established regularity criterion of the form \(|u^\theta| \leq \frac{C}{r|\ln r|^2}, r < \frac{1}{2}\). And the function \(\frac{1}{r|\ln r|^2}|_{r<\frac{1}{2}} \in L_T^\infty L_v^\infty L_h^{2,\beta}, \beta > \frac{2}{3}\). Therefore, it remains gaps between \(L_h^{2,\frac{3}{2}}\) and \(L_h^{2,w}\) in a certain sense.

**Remark 2.** The Corollary 2.2 still holds if we replace the regularity criteria by

\[
u^\theta|_{r<\delta} \in L_T^p L_v^{q_v} L_h^{q_h,w},
\]

where \(\frac{2}{p} + \frac{1}{q_v} + \frac{2}{q_h} \leq 1, \ 2 < q_v \leq \infty, \ \frac{1}{q_v} + \frac{2}{q_h} < 1\) and \(\delta > 0\) is an arbitrary constant.

Inspired by [4, 5, 23, 28], we have the following theorem in the critical space \(L_T^\infty L_v^\infty L_h^{2,w}\).

**Theorem 2.3.** Let \(u\) be an axisymmetric suitable weak solution of the Navier-Stokes equations (1.1) with the axisymmetric initial data \(u_0 \in L^2(\mathbb{R}^3), \ \text{div} \ u_0 = 0\), and \(ru^\theta_0 \in L^\infty(\mathbb{R}^3)\). Suppose \(b \in L_T^p L_v^{q_v} L_h^{q_h,w}\), then \(u\) is smooth in \((0,T] \times \mathbb{R}^3\).

**Remark 3.** The proof is analogously to [28], and we omit the details here.

### 3. Preliminaries

We will give some useful *a priori* estimates in the axisymmetric Navier-Stokes equations, and refer to [7, 35, 37, 44] for details.

**Lemma 3.1.** Assume \(u\) is the smooth axisymmetric solution of (1.1) on \([0,T]\). If in addition, \(ru^\theta_0 \in L^\infty(\mathbb{R}^3)\), then \(|ru^\theta| \leq C\|ru^\theta_0\|_{L^\infty(\mathbb{R}^3)}\).

**Lemma 3.2 ([44]).**

\[
\|\nabla u^\theta_r\|_{L^2(\mathbb{R}^3)} \leq \|\Gamma\|_{L^2(\mathbb{R}^3)}, \quad \|\nabla^2 u^\theta_r\|_{L^2(\mathbb{R}^3)} \leq \|\partial_3 \Gamma\|_{L^2(\mathbb{R}^3)}.
\]

**Lemma 3.3.** [7]

Let \(u \in C([0,T); H^2(\mathbb{R}^3)) \cap L^2_{loc}([0,T); H^3(\mathbb{R}^3))\) be the unique axisymmetric solution of the Navier-Stokes equations with the axisymmetric initial data \(u_0 \in H^2(\mathbb{R}^3)\) and \(\text{div} \ u_0 = 0\). If in addition, \(T < \infty\) and \(\|\Gamma\|_{L^\infty([0,T);L^2(\mathbb{R}^3))} < \infty\), then \(u\) can be continued beyond \(T\).
For convenience of readers, we will list some basic properties of Lorentz space.

**Lemma 3.4.** We denote $L^{p,q}, 0 < p, q \leq \infty$ the Lorentz space.

(i) If $0 < p, r < \infty, 0 < q \leq \infty$,

$$\| |g|^r \|_{L^{p,q}} = \| g \|_{L^{pr,qr}}^r. \quad (3.2)$$

(ii) [pointwise product] Let $1 < p < \infty, 1 \leq q \leq \infty$, $1 + \frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1$. Then pointwise multiplication is a bounded bilinear operator:

a) from $L^{p,q} \times L^{\infty}$ to $L^{p,q}$;

b) from $L^{p,q} \times L^{p',q'}$ to $L^1$;

c) from $L^{p,q} \times L^{p_1,q_1}$ to $L^{p_2,q_2}$, for $1 < p_1, p_2 < \infty, \frac{1}{p_2} = \frac{1}{p} + \frac{1}{p_1}, \frac{1}{q_2} = \frac{1}{q} + \frac{1}{q_1}$;

d) if $q < \infty$, the dual space of $L^{p,q}$ is $L^{p',q'}$.

(iii) [convolution] Let $1 < p < \infty, 1 \leq q \leq \infty$, $1 + \frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1$. Then convolution is a bounded bilinear operator:

a) from $L^{p,q} \times L^1$ to $L^{p,q}$;

b) from $L^{p,q} \times L^{p',q'}$ to $L^\infty$;

c) from $L^{p,q} \times L^{p_1,q_1}$ to $L^{p_2,q_2}$, for $1 < p_1, p_2 < \infty, 1 + \frac{1}{p_2} = \frac{1}{p} + \frac{1}{p_1}, \frac{1}{q_2} = \frac{1}{q} + \frac{1}{q_1}$.

We give a general Sobolev-Hardy-Littlewood inequality.

**Lemma 3.5.** We assume $2 \leq p < \infty, 0 \leq s < \frac{n}{p}, 1 \leq r \leq \infty$. For all $f \in B^{s+n\left(\frac{1}{2} - \frac{1}{p}\right)}_{2,r}(\mathbb{R}^n)$, we have

$$\| f \|_{L^r(\mathbb{R}^n)} \leq C \| f \|_{B^{s+n\left(\frac{1}{2} - \frac{1}{p}\right)}_{2,r}(\mathbb{R}^n)}. \quad (3.3)$$

Proof. Set \( \frac{1}{p} = \frac{\alpha}{n} + \frac{1}{q} \), \( p < q < \infty \). Apply Lemma 3.4 and interpolation, successively, we have

\[
\left\| \frac{f}{|x|^s} \right\|_{L^{p,r}(\mathbb{R}^n)} \leq C \left\| \frac{1}{|y|^s} \right\|_{L^{\frac{n}{2},r}(\mathbb{R}^n)} \left\| f \right\|_{L^{q,r}(\mathbb{R}^n)} \\
\leq C \left\| f \right\|_{L^{q,r}(\mathbb{R}^n)} \\
\leq C \left\| f \right\|_{B^{\frac{n}{2}}_{q,r}(\mathbb{R}^n)} \\
\leq C \left\| f \right\|_{B^{\frac{n}{2} + \alpha \left(\frac{1}{2} - \frac{1}{p}\right)}_{q,r}(\mathbb{R}^n)}. \]
\]

Lemma 3.6 (Trace Operator). For all \( f \in \dot{H}^1(\mathbb{R}^3) \), we have

\[
\operatorname{esssup}_{x_3 \in \mathbb{R}} \left\| f(\cdot, x_3) \right\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)} \leq C \left\| f(\cdot) \right\|_{\dot{H}^1(\mathbb{R}^3)}, \tag{3.4}
\]

Proof. Since the translation invariance and dense embedding, it is sufficiently to show that

\[
\left\| f(\cdot, 0) \right\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)} \leq C \left\| f(\cdot) \right\|_{\dot{H}^1(\mathbb{R}^3)}, \quad \forall f \in S(\mathbb{R}^3).
\]

Set \( x' = (x_1, x_2) \), \( \xi' = (\xi_1, \xi_2) \) and \( \gamma(f)(x_1, x_2) = f(x_1, x_2, 0) = f(x', 0) \),

\[
\gamma(f)(x_1, x_2) = (2\pi)^{-3} \int \int \int e^{ix' \cdot \xi'} \hat{f}(\xi) \ d\xi, \\
\gamma(\hat{f})(\xi_1, \xi_2) = (2\pi)^{-1} \int \hat{f}(\xi) \ d\xi_3 \\
\leq C \int \hat{f} |\xi| |\xi|^{-1} \ d\xi_3 \\
\leq C \left( \int |\hat{f}|^2 |\xi|^2 \ d\xi_3 \right)^{\frac{1}{2}} \left( \int |\xi|^{-2} \ d\xi_3 \right)^{\frac{1}{2}} \\
\leq C \left( \int |\hat{f}|^2 |\xi|^2 \ d\xi_3 \right)^{\frac{1}{2}} |\xi'|^{-\frac{1}{2}},
\]

Thus

\[
|\xi'||\hat{\gamma f}|^2 \leq C \int |\hat{f}|^2 |\xi|^2 \ d\xi_3.
\]

Finally, integrating both side with \( \int \int \ d\xi' \), we will derive the result. \( \square \)
We present an essential \textit{a priori} estimate below for Theorem 2.1.

\textbf{Lemma 3.7.} Assume $\frac{2}{p} + \frac{1}{q_v} + \frac{2}{q_h} = 1 - d$, $-1 \leq d < 1$, $\frac{2}{1-d} < q_h \leq \infty$, $\frac{1}{q_v} + \frac{2}{q_h} < 1 - d$. For a sufficiently small constant $\epsilon > 0$, we have

$$
\int_{\mathbb{R}^3} \frac{|u^\theta|}{r}|f|^2 \, dx \leq C_{\epsilon} \|r^d u^\theta\|_{L^{q_v}_w L^{q_h}_w}(\mathbb{R}^3)^2 \|f\|^2_{L^2(\mathbb{R}^3)} + \epsilon \|\nabla f\|^2_{L^2(\mathbb{R}^3)}. \tag{3.5}
$$

\textbf{Proof.} Set

$$
a = \frac{2\tau}{p}, \quad b = \frac{2\tau}{q_v}, \quad c = 2 - a - b, \quad \frac{2}{\gamma} = 1 - \frac{\tau}{q_h}, \quad \tau = \begin{cases} 1, & 0 \leq d < 1 \\ \frac{1}{1-d}, & -1 \leq d < 0 \end{cases}.
$$

Thus

$$0 \leq a, b, c \leq 2, \quad a \neq 0, \quad 2 \leq \gamma < 4.$$

It is appeared to see that $\gamma > 2$ if $q_h < \infty$, and $\gamma = 2$ if $q_h = \infty$. By applying Lemma 3.4, Lemma 3.1, Lemma 3.5, Lemma 3.6 and interpolation, we can deduce the following estimates, respectively.

$$
\int_{\mathbb{R}^3} \frac{|u^\theta|}{r}|f|^2 \, dx = \int_{\mathbb{R}^3} \left|(ru^\theta)^{1-\tau}(r^d u^\theta)^\gamma \right| \frac{f^2}{r^{2+(d-1)\tau}} \, dx
\leq C \int_{\mathbb{R}} \|r^d u^\theta\|_{L^{q_h}_w} \|f\|_{L^{\frac{2}{1+(d-1)\tau}}_{L^2}}^2 \, dx_3
\leq C \int_{\mathbb{R}} \|r^d u^\theta\|_{L^{q_h}_w} \|f\|_{L^{\frac{1}{1+(d-1)\tau}}_{L^2}}^2 \, dx_3
\leq C \int_{\mathbb{R}} \|r^d u^\theta\|_{L^{q_v}_w} \|f(\cdot, x_3)\|_{B^\frac{(d-1)\tau}{2}_{\infty, \frac{d}{2}(\mathbb{R}^2)}}^2 \, dx_3
\leq C \int_{\mathbb{R}} \|r^d u^\theta\|_{L^{q_v}_w} \|f(\cdot, x_3)\|_{L^\frac{1}{1+(d-1)\tau}_{L^2}}^b \|f(\cdot, x_3)\|_{L^\frac{1}{1+(d-1)\tau}_{L^2}}^b \|f(\cdot, x_3)\|_{L^\frac{1}{1+(d-1)\tau}_{L^2}}^b \, dx_3
\leq C \|r^d u^\theta\|_{L^{q_v}_w L^{q_h}_w} \|f\|_{L^\frac{1}{1+(d-1)\tau}_{L^2}}^b \|f\|_{L^\frac{1}{1+(d-1)\tau}_{L^2}}^b \|f\|_{L^\frac{1}{1+(d-1)\tau}_{L^2}}^b \|\nabla f\|^2_{L^2}\tag{3.5}
\leq C_{\epsilon} \|r^d u^\theta\|_{L^{q_v}_w L^{q_h}_w} \|f\|_{L^\frac{1}{1+(d-1)\tau}_{L^2}}^2 + \epsilon \|\nabla f\|^2_{L^2}.
$$

$\square$
4. Proof of Theorem 2.1

• For simplicity, we only prove the Theorem 2.1 in the critical case, i.e.
\[ r^d u^\theta \in L_T^p L_v^q L_h^{q_h,w}, \quad \frac{2}{p} + \frac{1}{q_v} + \frac{2}{q_h} = 1 - d, \quad -1 \leq d < 1, \quad \frac{2}{1 - d} < q_h \leq \infty, \quad \frac{1}{q_v} + \frac{2}{q_h} < 1 - d. \]  

(4.1)

Otherwise, we can find \( p_* < p \), such that
\[
\| r^d u^\theta \|_{L_T^{p_*} L_v^q L_h^{q_h,w}} \leq T \frac{1}{p} \| r^d u^\theta \|_{L_T^{p} L_v^q L_h^{q_h,w}} < \infty, \\
\frac{2}{p_*} + \frac{1}{q_v} + \frac{2}{q_h} = 1 - d, \quad -1 \leq d < 1, \quad \frac{2}{1 - d} < q_h \leq \infty, \quad \frac{1}{q_v} + \frac{2}{q_h} < 1 - d.
\]

(4.2)

Therefore, we can calculate analogously below with the condition (4.2).

• As in [7], we introduce the ingredient \((\Phi, \Gamma) = (\omega^r, \omega^\theta)r\), which satisfy the following equations
\[
\begin{aligned}
\partial_t \Phi + (b \cdot \nabla) \Phi - (\Delta + \frac{2}{r} \partial_r) \Phi - (\omega^r \partial_r + \omega^\theta \partial_\theta) \frac{u^r}{r} &= 0, \\
\partial_t \Gamma + (b \cdot \nabla) \Gamma - (\Delta + \frac{2}{r} \partial_r) \Gamma + 2 \frac{u^\theta}{r} \Phi &= 0.
\end{aligned}
\]

(4.3)

We show that
\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \| \Phi \|_{L_x^2}^2 + \| \nabla \Phi \|_{L_x^2}^2 &= \int_{\mathbb{R}^3} u^\theta (\partial_r \frac{u^r}{r} \partial_\theta \Phi - \partial_\theta \frac{u^r}{r} \partial_r \Phi) \, dx \\
&\leq \frac{1}{2} \int_{\mathbb{R}^3} |u^\theta|^2 |\nabla \frac{u^r}{r}|^2 \, dx + \frac{1}{2} \| \nabla \Phi \|_{L_x^2}^2 \\
&\leq C \int_{\mathbb{R}^3} \frac{|u^\theta|}{r} |\nabla \frac{u^r}{r}|^2 \, dx + \frac{1}{2} \| \nabla \Phi \|_{L_x^2}^2.
\end{aligned}
\]

(4.4)

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \| \Gamma \|_{L_x^2}^2 + \| \nabla \Gamma \|_{L_x^2}^2 &= -2 \int_{\mathbb{R}^3} \frac{u^\theta}{r} \Gamma \Phi \, dx \\
&\leq \int_{\mathbb{R}^3} \frac{|u^\theta|}{r} |\Gamma|^2 \, dx + \int_{\mathbb{R}^3} \frac{|u^\theta|}{r} |\Phi|^2 \, dx.
\end{aligned}
\]

(4.5)
Applying Lemma 3.2 and Lemma 3.7 in (4.4), we have
\[ \frac{d}{dt} \| \Phi \|_{L^2_x}^2 + \| \nabla \Phi \|_{L^2_x}^2 \leq C \int_{\mathbb{R}^3} \frac{|u^\theta|}{r} |\nabla \frac{u^r}{r}|^2 \, dx \]
\[ \leq C \| r^d u^\theta \|_{L^p_{t,x} L^q_{h,x}}^p \| \nabla \frac{u^r}{r} \|_{L^2_x}^2 + \frac{1}{4} \| \nabla^2 \frac{u^r}{r} \|_{L^2_x}^2 \]
\[ \leq C \| r^d u^\theta \|_{L^p_{t,x} L^q_{h,x}}^p \| \Gamma \|_{L^2_x}^2 + \frac{1}{4} \| \nabla \Gamma \|_{L^2_x}^2. \tag{4.6} \]

Analogously, applying Lemma 3.7 in (4.5), it is easy to obtain that,
\[ \frac{1}{2} \frac{d}{dt} \| \Gamma \|_{L^2_x}^2 + \| \nabla \Gamma \|_{L^2_x}^2 \leq C \| r^d u^\theta \|_{L^p_{t,x} L^q_{h,x}}^p \left( \| \Phi \|_{L^2_x}^2 + \| \Gamma \|_{L^2_x}^2 \right) + \frac{1}{4} \| \nabla \Phi \|_{L^2_x}^2 + \frac{1}{4} \| \nabla \Gamma \|_{L^2_x}^2. \tag{4.7} \]

Summing up (4.6) and (4.7), we have
\[ \frac{d}{dt} \left( \| \Phi \|_{L^2_x}^2 + \| \Gamma \|_{L^2_x}^2 \right) + \| \nabla \Phi \|_{L^2_x}^2 + \| \nabla \Gamma \|_{L^2_x}^2 \leq C \| r^d u^\theta \|_{L^p_{t,x} L^q_{h,x}}^p \left( \| \Phi \|_{L^2_x}^2 + \| \Gamma \|_{L^2_x}^2 \right). \]

Using Gronwall’s inequality, we have
\[ \sup_{t \in [0, T^*]} \| \Gamma \|_{L^2_{t,x}(0,T^*;L^2_x)}^2 \leq \left( \| \Phi_0 \|_{L^2_x}^2 + \| \Gamma_0 \|_{L^2_x}^2 \right) \exp \left( C \| u^\theta \|_{L^p_{t,x} L^q_{h,x}}^p \right) < \infty. \tag{4.8} \]

Applying Lemma 3.3, we obtain that \( u \) can be continued beyond \( T \).

\[ \square \]

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