Quantized Anomalous Hall Effects in Skyrmion Crystal

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We theoretically study the quantized anomalous Hall effect (QAHE) in skyrmion crystal (SkX) without external magnetic field. The emergent magnetic field in SkX could be gigantic as much as \( \sim 4000T \) when its lattice constant is \( \sim 1nm \). The band structure of SkX is not flat but has a finite gap in the low electron-density regime. We also study the conditions to realize the QAHE for the skyrmion size, carrier density, disorder strength and temperature. Comparing the SkX and the system under the corresponding uniform magnetic field, the former is more fragile against the temperature compared with the latter since the gap is reduced by a factor of \( \sim 1/5 \), while they are almost equally robust against the disorder. Therefore, it is expected that the QAHE of SkX system is realized even with strong disorder at room temperature when the electron density of the order of one per a skyrmion.

Magnetic skyrmion is a topological spin texture in ferromagnet. After the early theoretical proposals in magnet, the study of magnetic skyrmion is growing rapidly since it was discovered experimentally. The periodic array of skyrmions, i.e., a skyrmion crystal (SkX), is realized at interfaces or in bulk chiral magnets such as B20 compounds. An emergent magnetic field, generated in the background skyrmion spin texture. Namely, a skyrmion has one magnetic flux \( \phi_0 = h/e \) acting on the conduction electrons coupled to it. When the skyrmions form a periodic lattice, i.e., a SkX, the emergent magnetic field reaches \( \sim 4000T \) assuming the uniform averaged flux for the skyrmion size and the lattice constant of SkX of the order of \( \sim 1nm \). The effective magnetic field is proportional to \( \lambda^{-2} \), where \( \lambda \) is the skyrmion radius. Since the size of the skyrmion is 3nm for MnGe, 18nm for MnSi, and 70nm for FeGe, the corresponding emergent magnetic field is \( \sim 1100T, 28T, \) and \( 1T \), respectively.

This emergent magnetic field leads to the Hall effect. Most of the studies focus on the Hall effect in metallic systems with large electron density. This so-called topological Hall conductivity \( \sigma_{xy} \) is usually small compared with the longitudinal conductivity \( \sigma_{xx} \), i.e., the Hall angle \( \sigma_{xy}/\sigma_{xx} \) is typically of the order of \( 10^{-2} \) at most.

Up to now, we regard skyrmions as the source of the real space emergent magnetic field. When the size of the skyrmion becomes comparable to the mean free path, it is expected that the crossover from the real to momentum space Berry curvature occurs. One can regard the case of pyrochlore ferromagnet as the limit of large emergent magnetic field, where the spin chirality is defined for each unit cell of tetrahedron. In this case, there is no Landau Level (LL) formation, but the band structure is formed by taking into account the solid angle of the spin, and the intrinsic anomalous Hall effect appears whose conductance is given by the integral of the Berry curvature in momentum space. A more drastic example is the quantized anomalous Hall effect (QAHE) in magnetic topological insulator (TI), where the surface state with gap opening due to the exchange coupling to the magnetic ions produces the quantized Hall conductance of \( e^2/h \) without the external magnetic field.

It is expected that the SkX offers an ideal laboratory to study QAHE from a unified viewpoint since one can change the size of the skyrmion, the mean free path, and even the carrier concentration by gating at the interface, to reveal the crossover between real and momentum space Berry curvature and stability of the quantized Hall conductance as these conditions are changed.

In this paper we theoretically explore the emergence of the QAHE in the SkX without external magnetic field. Landau Levels are not formed in the present system since the emergent magnetic field is not uniform. Nevertheless, the band structure contains several well separated bands in the low electron-density regime, where each band has a Chern number \( C \approx -1 \). Consequently the emergence of the QAHE is predicted. We point out that the lowest and next-lowest bands are well described by the Dirac theory. On the other hand, Hall plateaux disappear in the large electron-density regime because of the overlap of bands. We also clarify the conditions of QAHE for the skyrmion size, carrier density, disorder strength and temperature.

The model: We start with a free electron system coupled with the background spin texture \( n_s \) by Hund’s coupling. The Hamiltonian is given by the double-exchange model,

\[
H = \sum_{ij} t_{ij} c_{i\uparrow} c_{j\downarrow} - J \sum n_i c_{i\uparrow} c_{i\downarrow} \sigma_{i\uparrow},
\]

where \( c_{i\uparrow} (c_{i\downarrow}) \) is the two-component (spin up and spin down) creation (annihilation) operator at the \( i \) site, \( t_{ij} \) is the transfer integral between nearest-neighbor sites, \( J \) is the Hund’s coupling strength between the electron spin and background spin texture and \( \sigma \) denotes the Pauli matrix. When Hund’s coupling is strong enough \( J \gg t_{ij} \), the spin of the hopping electron is forced to align parallel to the spin texture. The wave-function \( |\chi(r)\rangle \) of the conduction electron at \( r \) corresponding to the localized spin \( n(r) \) is given by

\[
|\chi(r)\rangle = \left(\cos \frac{\theta(r)}{2}, e^{i\phi(r)} \sin \frac{\theta(r)}{2}\right),
\]

where we have introduced the polar coordinate of the spin configuration \( n = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \). Then the effective transfer integral is obtained,

\[
t_{ij}^{\text{eff}} = t_{ij} n_i n_j \cos \frac{\theta_{ij}}{2},
\]
FIG. 1: The band structure in the presence of SkX. We have set \( \lambda = 2 \). The horizontal axes are momentum \( k_x \) and \( k_y \), while the vertical axis is the energy. There are \( 4\lambda^2 = 16 \) bands. We show (a) a bird’s eye view and (b) a cross section of the lowest 8 bands. (c) The band structure of a tight-binding model with the uniform magnetic flux, which forms the almost flat Landau levels except near the center of the energy 0.

where \( a_{ij} = (\phi_i - \phi_j)(1 - \cos \theta_i \theta_j)/2 \) is the vector potential generated by the spin between the \( i \) and \( j \) sites, and \( \theta_{ij} \) is the angle between the two spins. The effective tight-binding Hamiltonian is obtained as

\[
H = \sum_{ij} t_{ij}^d c_i^d d_j^d, \tag{4}
\]

where \( d_i^d \) (\( d_i \)) is the spinless creation (annihilation) operator at the \( i \) site. One can easily see that the spin chirality is absent for collinear and coplanar spin alignment.

**Skyrmion crystal:** We consider a background spin texture \( n(r) \) made of a square SkX. Each skyrmion has a nontrivial topological number. The skyrmion profile is well assumed as \( \theta(r) = \pi (1 - r/\lambda) \) for \( r < \lambda \) and \( \theta(r) = 0 \) for \( r > \lambda \). The emergent magnetic field is produced by the spin texture since it has a finite solid angle. \[ b_z(r) = \frac{\hbar}{2e} n \cdot (\partial_r n \times \partial_z n) = \frac{\hbar}{2e} \frac{\pi}{r \lambda} \sin(\pi (1 - r/\lambda)) \tag{5} \]

for \( r < \lambda \) and \( b_z = 0 \) for \( r > \lambda \). It does not depend on the azimuthal angle \( \phi \). The total magnetic flux is

\[
\int_0^\lambda d^2 rb_z(r) = \Phi_0, \tag{6}
\]

with \( \Phi_0 = \hbar/e \), and is independent of the skyrmion radius \( \lambda \).

**Band structure:** The band structure can be obtained by numerically diagonalizing the Hamiltonian (4) in the unit cell with the size \( 2\lambda \times 2\lambda \) in the presence of the SkX. The Brillouin zone is given by \( -\pi/(2\lambda < k_x < \pi/2\lambda \) and \( -\pi/(2\lambda < k_y < \pi/2\lambda \). Here we take the unit with the lattice constant \( a = 1 \). Then the unit cell has dimensionless area \( (2\lambda)^2 \). We show the band structure in the presence of SkX in Figs.1(a) and (b), which are warped and the number of bands is \( 4\lambda^2 \). There are finite gaps between two successive bands for lower bands. However, the band overlap starts at a higher band. In

FIG. 2: (a) Band gaps and overlaps for various sizes \( \lambda \) of the skyrmions. (b) The \( (2\lambda)^{-2} \) dependence of the lowest band gap \( \Delta \). The gap is proportional to \( (2\lambda)^{-2} \). The band gap of the system with the corresponding uniform mean magnetic flux is shown in the blue line.

Fig.1(c) shows the band structure of the corresponding uniform mean magnetic field. Here, since one skyrmion exists per area \( 4\lambda^2 \), the mean magnetic field \( \Phi \) is given by \( \Phi_0/4\lambda^2 \). It is seen that there is almost no energy dispersion, i.e., the Landau level formation, with the energy separation \( \Delta_0 \) which slightly depends on the Landau level index.

We show the gaps and overlaps of bands in Fig.2(a) for various skyrmion radius \( \lambda \), where the bands are marked in color bar and the band gaps are denoted by the white blanks. We find that the band gap shows a scaling behavior for the skyrmion radius. The band gap between the lowest and second-lowest bands always opens.

We show the \( \lambda \) dependence of the lowest band gap \( \Delta \) in Fig.2(b). It is proportional to \( 1/\lambda^2 \). There is some deviation from the linear fit in small \( \lambda \), which is probably due to the finite size effect. We can compare this band gap with the Landau level separation \( \Delta_0 \) shown in Fig.1(c). The lowest Landau level gap \( \Delta_0 \) is given by 0.71t for \( \lambda = 2 \), while the lowest gap of the QAHE in SkX is given by \( \Delta = 0.11t \). The linear relation in the small \( 1/4\lambda^2 \) region of Fig.2(b) indicates the relation \( \Delta \propto \Delta_0/5 \).

**Berry curvature:** We focus on the lowest and second-lowest bands [Fig.3(a)]. We show the spin direction determined by \( \langle \psi(k) | \sigma | \psi(k) \rangle \) for conduction and valence bands in Fig.3(c) and (d), respectively, where the anti-vortex structure is evident at the \( M \) point. It implies that the spin texture has a nontrivial Berry curvature.

We may define a “gauge potential” in the momentum space, \( a_k(k) = -i \langle \psi(k) | \partial_k | \psi(k) \rangle \) for Bloch state \( | \psi(k) \rangle \), which is properly called the Berry connection. Then we may define the “magnetic field” or the Berry curvature by \( b_z(k) = \partial_k a_y(k) - \partial_k a_x(k) \). The Chern number is the integral of the Berry curvature over the first Brillouin zone, \( C = \frac{1}{2\pi} \int d^2kb_z(k) \).

We show the momentum dependence of the Berry curvature in Fig.3. It takes a large value in the vicinity of the \( M \) point, which is \( (k_x, k_y) = (\pi/2\lambda, \pi/2\lambda) \). The sign of the Berry curvature is negative for the lowest band, while it is positive for the second-lowest band. Accordingly, the sign of the Berry curvatures are opposite between the lowest and second-lowest bands.

Each band has one unit Chern number \( C = -1 \). This fact
FIG. 3: (a) Band structure of the lowest and second-lowest bands in the vicinity of the $M$ point. It has the Dirac-cone shape. (b) The momentum distribution of the Berry curvature in the lowest and second-lowest bands. The Berry curvature takes the largest value at the $M$ point, where the Chern numbers $C = -1/2$ and $1/2$ are generated for these bands, respectively. The residual Chern numbers $C = -1/2$ and $-3/2$ arise away from the $M$ point. The spin directions of the band structure of (c) the conduction and (d) valence bands. They clearly exhibit the anti-vortex structure at the $M$ point.

can be understood in terms of adiabatic connection from the Landau level. In our system, the emergent magnetic field has a space dependence. We consider an adiabatic pass from the uniform magnetic field to the space-dependent magnetic field induced by SkX. This deformation has no singularity. Thus the band structure in Fig.1 can be obtained from the deformation of the Landau level. Each Landau level has the same Chern number. Accordingly, each band of our system also has the same Chern number, which is $C = -1$.

The structure of the lowest and second-lowest bands has precisely the shape of the Dirac cone in the vicinity of the $M$ point $(k_x, k_y) = (\pi/2\lambda, \pi/2\lambda)$ as in Fig.3(a). This suggests that electrons are described by the Dirac theory $H = \hbar v(\mathbf{k} \cdot \mathbf{\sigma}) + m\mathbf{\sigma} \cdot \mathbf{\sigma}$, where the mass $m$ is related to the gap $\Delta$ between the lowest and second-lowest bands by $\Delta = 2|m|$.  

**Hall conductance:** The conductance is calculated by the Kubo formula:

$$\sigma_{xy} = -\frac{ie^2}{\hbar} \sum_{n,k} f(E_{nk})$$

$$\times \sum_{m \neq n} \frac{\langle nk | \frac{\partial H}{\partial k_x} | mk \rangle \langle mk | \frac{\partial H}{\partial k_y} | nk \rangle - (n \leftrightarrow m)}{(E_{nk} - E_{mk})^2}$$

(7)

where $n$ and $m$ are the band indices and $f(x)$ is the Fermi distribution function. We show the Hall conductance in Fig 4 calculated using the Kubo formula (7). At the zero temperature, it is quantized to be the Chern number when the Fermi energy is inside the gap, $\sigma_{xy} = \frac{e^2}{\hbar} \sum_{\text{filled}} C_n$, where $C_n$ is the Chern number of the $n$th band. Below the band gap, the Hall conductance decreases, while it increases above the band gap. This is due to the fact that the sign of the Berry curvature is opposite between the two adjacent bands. The total Chern number can be very large for large $\lambda$, which is distinct from the QAHE in magnetic topological insulators.[16]

This behavior of the Hall conductance across the gap can be interpreted by the Dirac theory, which gives

$$\sigma_{xy} = \begin{cases} 
-1/2 & \text{for } |\mu| < |m| \\
-m/(2|\mu|) & \text{for } |\mu| > |m| 
\end{cases}$$

(8)

It describes the peculiar behavior of the Hall conductance quite well as in Fig 4(b), where the conductance is quantized inside the gap and continuously changes outside the gap showing a dip structure.

**Finite temperature:** We show the Hall conductance at finite temperature in Fig 5. It is evident that the temperature scale is given by the gap $\Delta$ in Fig. 1(b), which is around 0.1t for $\lambda = 2$. Although the quantization of plateau is broken at $k_B T = 0.01t \simeq 0.01eV \simeq 100K$, the peculiar dip structure toward the formation of plateau is clearly visible even at $k_B T \simeq 400K$. Consequently, our prediction of QHE without external magnetic field can be experimentally observable at room temperature.

**Disorder effect:** Up to now, we focus on the pure system, and the Berry curvature in momentum space. Now we introduce the disorder potential as given by

$$H_{\text{imp}} = \sum_i U_i d_i^\dagger d_i$$

(9)
where $U_i$ takes a uniform random distribution for $-V < U_i < V$. We have calculated the Hall conductivity $\sigma_{xy}$ for the system of size $8 \times 8$, including 4 unit cells. Figure 6 shows the numerical results, which clearly shows the plateau transition occurs at $V \approx t$ for the lowest band, i.e., $\sigma_{xy} = -e^2/h$. Note that when $|\sigma| > \frac{1}{2} \cdot \frac{e^2}{h}$, the two-parameter scaling trajectory converges to the quantized Hall state. Taking this criterion, the lowest occupied band, which is most promising candidate to observe the QHE in SkX, can support the plateau up to $V \approx t$. This magnitude of the random potential is similar to the separation between LLs in Fig. 1(c) and is much larger than the thermal excitation is reduced.

$\Delta$ is estimated as $\Delta \approx \frac{V}{t}$ as shown in Fig.6(a). We also calculated quantity $P_n$ for the eigenfunction $\langle \psi_n | \psi_n \rangle$ with energy $\varepsilon_n$ defined by

$$ P_n = \sum_i |\langle \psi_n | \psi_i \rangle|^4, \quad (10) $$

which measures the extent of the wavefunction. Namely, when $\langle \psi_n | \psi_n \rangle$ extends over the M sites, $P_n \sim M^{-1}$. Therefore, the localization length is estimated by $\lambda = \frac{1}{P_n^{1/2}}$ in 2D. We show in the inset of Fig. 6 the averaged $P_{\lambda_n}^{-1/2}$ for the lowest band (purple curve) and all the bands (black curve) as a function of $V$. It seems that the disappearance of the QHE roughly corresponds to the localization of the wavefunctions less than 4, where $\lambda$ is the size of the unit cell (skyrmion). Therefore, since the mean free path $\lambda$ is always longer than the localization length, the QAHE is observed only in the case where $\lambda > 2\lambda$, i.e., the Berry curvature in momentum space is more appropriate picture rather than that in the real space.

**Discussions:** We have shown that the QAHE occurs in metallic SkX due to the emergent magnetic field induced by the spin texture. It is shown that there is a gap between the lowest and second-lowest bands. The electron density should be of the order of one electron per one skyrmion, whose electron density is given by $1/4\lambda^2$. In order to realize experimental situations, it is necessary to reduce the number of electrons. In this sense, an interface of dilute magnetic semiconductors will be promising. The emergent magnetic field and the associated energy gap is proportional to $1/\lambda^2$. Namely, skyrmions with smaller radius are better for realizing QAHE.

The QAHE is robust against disorder as in the case of system with corresponding uniform magnetic field with Landau levels. Considering the very large mean magnetic field of $\sim 4000T$ for the skyrmion size $\lambda \sim 1nm$, the disorder does not destroy the QAHE so seriously. This is because the change in the Chern numbers occurs only via the pair annihilation, and the appreciable energy dispersion of the band in SkX protects the Chern number. On the other hand, the band gap $\Delta$ of SkX is smaller than $\Delta_0$, i.e., that of the corresponding system with uniform magnetic field, and hence the stability against the thermal excitation is reduced. $\Delta$ is estimated as $\Delta_0/5$, but as mentioned above, the emergent magnetic flux induced by SkX is gigantic and even the QAHE even at room temperature is expected.

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