THE STRONG CHOWLA-MILNOR SPACES AND A CONJECTURE OF GUN, MURTY AND RATH

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Abstract. In a recent work, Gun, Murty and Rath formulated the Strong Chowla-Milnor conjecture and defined the Strong Chowla-Milnor space. In this paper, we prove a non-trivial lower bound for the dimension of these spaces. We also obtain a conditional improvement of this lower bound and noted that an unconditional improvement of this lower bound will lead to irrationality of both $\zeta(k)$ and $\zeta(k)/\pi^k$ for all odd positive integers $k > 1$. Following Gun, Murty and Rath, we define generalized Zagier spaces $V_p(K)$ for multiple zeta values over a number field $K$. We prove that the dimension of $V_{4d+2}(K)$ for $d \geq 1$, is at least 2, assuming a conjecture of Gun, Murty and Rath.

1. Introduction

Throughout the paper, we consider $s \in \mathbb{C}$ with $\Re(s) > 1$, unless otherwise stated. The Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \text{ for } \Re(s) > 1.$$ 

The Riemann zeta function can be extended holomorphically to the whole complex plane except at $s = 1$, where it has a simple pole with residue 1. Hurwitz studied the function, now called the Hurwitz zeta function, which is defined as

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}$$

where $0 < x \leq 1$ and $s \in \mathbb{C}$ with $\Re(s) > 1$. He proved that $\zeta(s, x)$ can be extended holomorphically to the entire complex plane except at $s = 1$, where it has a simple pole with residue 1. Note that $\zeta(s, 1) = \zeta(s)$, the classical Riemann zeta function.

For a periodic arithmetic function $f$ with period $q > 1$ and $s \in \mathbb{C}$, the $L$-function associated to $f$ is defined as

$$L(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$
Since \( f \) is periodic with period \( q \), we have

\[
L(s, f) = q^{-s} \sum_{a=1}^{q} f(a) \zeta(s, a/q),
\]

Hence \( L(s, f) \) extends holomorphically to the whole complex plane with a possible simple pole at \( s = 1 \) with residue \( q^{-1} \sum_{a=1}^{q} f(a) \).

For an algebraic valued periodic function \( f \) with period \( q \), the transcendental nature of \( L(1, f) \), whenever it exists, is discussed in the paper [5].

We got our motivation for this work from a conjecture formulated by Gun, Murty and Rath in [1] which is a generalization of the Chowla-Milnor conjecture stated below.

**Conjecture 1 (Chowla-Milnor).** Let \( k > 1 \) and \( q > 1 \), be integers. Then the following \( \varphi(q) \) real numbers are linearly independent over \( \mathbb{Q} \):

\[
\zeta(k, a/q) \quad \text{with} \quad 1 \leq a < q, \quad (a, q) = 1.
\]

The authors in [1] generalized the above conjecture in the following manner:

**Conjecture 2 (Strong Chowla-Milnor).** For any integers \( k > 1 \) and \( q > 1 \), the following \( \varphi(q) + 1 \) real numbers

\[
1, \zeta(k, a/q) \quad \text{with} \quad 1 \leq a < q, \quad (a, q) = 1
\]

are \( \mathbb{Q} \)-linearly independent.

For a rational valued periodic function \( f \) with period \( q \) and satisfying \( f(a) = 0 \) for \( 1 < (a, q) < q \), the Strong Chowla-Milnor conjecture is equivalent to the irrationality of \( L(k, f) \), unless

\[
f(a) = -\frac{f(q)q^{-k}}{\prod_{\substack{p \in P, \quad p|q \quad (p|q)}} (1 - p^{-k})},
\]

for \( 1 \leq a < q, (a, q) = 1 \). Here \( P \) denote the set of primes.

In order to see that the above statement is equivalent to the Strong Chowla-Milnor conjecture, we consider the identity

\[
\zeta(k) \prod_{\substack{p \in P, \quad p|q \quad (p|q)}} (1 - p^{-k}) = q^{-k} \sum_{\substack{a=1 \quad (a,q)=1}}^{q-1} \zeta(k, a/q).
\]
Substituting this in the above expression for $L(s, f)$ and using $f(a) = 0$ for $1 < (a, q) < q$, we get

$$L(k, f) = q^{-k} \sum_{a=1}^{q-1} \left[ f(a) + f(q)q^{-k} \prod_{p|q} (1 - p^{-k}) \right] \zeta(k, a/q).$$

(1)

If $L(k, f)$ is rational, then the above equation shows that $1, \zeta(k, a/q)$ for $1 \leq a < q, (a, q) = 1$ are linearly dependent over the rationals since $f$ is a rational-valued function. Conversely, if $1, \zeta(k, a/q)$ with $1 \leq a < q, (a, q) = 1$ are linearly dependent over the rationals, then there are rational numbers $c_0, c_a, 1 \leq a < q, (a, q) = 1$, not all zero, such that

$$c_0 + \sum_{a=1}^{q-1} c_a \zeta(k, a/q) = 0.$$

Now define the following rational-valued periodic function $f$ with period $q$. Set $f(a) = 0$ for $1 < a \leq q, (a, q) > 1$ and $f(a) = c_a$ for $1 \leq a < q, (a, q) = 1$. Then, our identity shows that $q^k L(k, f) = -c_0$ so that $L(k, f)$ is rational.

This shows that the above statement is equivalent to the Strong Chowla-Milnor conjecture.

**Definition 3.** For an integer $m \geq 1$ and complex numbers $z$ with $|z| \leq 1$, the polylogarithm function $Li_m(z)$ is defined by

$$Li_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m}.$$

Note that for $m = 1$, the above series is equal to $-\log(1 - z)$ for $|z| < 1.$

In a recent work [1] Gun, Murty and Rath formulated the Polylog conjecture. Here we have following strong conjecture about polylogarithms, which generalize Baker’s theorem.

**Conjecture 4 (Strong Polylog Conjecture).** Suppose $\alpha_1, ..., \alpha_n$ are algebraic numbers with $|\alpha_i| \leq 1$ for $1 \leq i \leq n$, such that $Li_m(\alpha_1), ..., Li_m(\alpha_n)$ are linearly independent over $\mathbb{Q}$. Then $1, Li_m(\alpha_1), ..., Li_m(\alpha_n)$ are linearly independent over $\overline{\mathbb{Q}}$.

Clearly for $m = 1$ the above conjecture reduces to a consequence of Baker’s theorem about linear forms in logarithms. In section 2, we establish a link between the Strong Polylog conjecture and the Strong Chowla-Milnor conjecture in a form of a theorem stated as follows.

**Theorem 5.** The Strong Polylog conjecture implies the Strong Chowla-Milnor conjecture for all $q > 1$ and $k > 1.
Definition 6. For any integer $k > 1$ and $q \geq 2$, define the Strong Chowla-Milnor space $\hat{V}_k(q)$ by
\[
\hat{V}_k(q) := \mathbb{Q} - \text{span of } \{1, \zeta(k, a/q) : 1 \leq a < q, \ (a, q) = 1\}.
\]

In a recent work [1], authors have shown that for an odd integer $k > 1$ and two co-prime integers $q, r > 2$, either
\[
\dim_{\mathbb{Q}} V_k(q) \geq \frac{\varphi(q)}{2} + 1
\]
or
\[
\dim_{\mathbb{Q}} V_k(r) \geq \frac{\varphi(r)}{2} + 1.
\]
where the Chowla-Milnor space $V_k(q)$ is defined by
\[
V_k(q) := \mathbb{Q} - \text{span of } \{\zeta(k, a/q) : 1 \leq a < q, \ (a, q) = 1\}.
\]

One can ask a similar type of question for the Strong Chowla-Milnor space i.e., for an odd integer $k > 1$ and two co-prime integers $q, r > 2$, whether or not either
\[
\dim_{\mathbb{Q}} \hat{V}_k(q) \geq \frac{\varphi(q)}{2} + 2
\]
or
\[
\dim_{\mathbb{Q}} \hat{V}_k(r) \geq \frac{\varphi(r)}{2} + 2.
\]

But if the above statement is true, then clearly the Strong Chowla-Milnor conjecture is true for either $q = 3$ or $q = 4$ i.e. either $\dim_{\mathbb{Q}} \hat{V}_k(3) = 3$ or $\dim_{\mathbb{Q}} \hat{V}_k(4) = 3$. Then from the proposition 7 of [1] we get that $\zeta(k)$ is irrational for all odd $k > 1$. In general we do not know for all odd integers $k$ whether $\zeta(k)$ is irrational or not. It is known, thanks to Apery, that $\zeta(3)$ is irrational. On the other hand by a theorem of K. Ball and T. Rivoal (see [6] and [7]), it is known that $\zeta(k)$ is irrational for infinitely many odd $k > 1$.

In section 3, we will prove the following theorem.

Theorem 7. Let $k > 1$ be an odd integer with $\zeta(k)$ irrational and $q, r > 2$ be two co-prime integers. Then either
\[
\dim_{\mathbb{Q}} \hat{V}_k(q) \geq \frac{\varphi(q)}{2} + 2
\]
or
\[
\dim_{\mathbb{Q}} \hat{V}_k(r) \geq \frac{\varphi(r)}{2} + 2.
\]

In a recent work [2], Gun, Murty and Rath formulated a variant of the Chowla-Milnor conjecture which is the following.
**Conjecture** 8 (Gun, Murty and Rath). Let $K$ be a number field and $k > 1, q \geq 2$ be integers such that $K \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Then the $\varphi(q)$ real numbers $\zeta(k, a/q), 1 \leq a < q, (a, q) = 1$ are linearly independent over $K$.

In section 4, we investigate the values of $L(k, \chi)$, as $\chi$ ranges over non-trivial primitive Dirichlet characters mod $q$, over a certain family of algebraic number fields assuming the above conjecture.

**Theorem** 9. Let $K$ be an algebraic number field and $K_1 = K(e^{2\pi i/\varphi(q)})$. Suppose that $K_1 \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Then the values $L(k, \chi)$, as $\chi$ ranges over non-trivial primitive Dirichlet characters mod $q$, are linearly independent over $K$ for all $k \geq 1$, if the Gun, Murty and Rath conjecture is true.

In fact for $k > 1$ the above theorem is true for any primitive Dirichlet characters mod $q$, i.e. one can include the principal character mod $q$.

In section 5, we establish a link between the Strong Chowla-Milnor conjecture and the multiple zeta values (MZVs).

**Definition** 10. Let $k, s_1, \ldots, s_k$ be positive integers with $s_1 > 1$. Then the *multiple zeta values* (MZVs) are defined as

$$\zeta(s_1, \ldots, s_k) = \sum_{n_1 > \ldots > n_k \geq 1} \frac{1}{n_1^{s_1} \ldots n_k^{s_k}}.$$  

Clearly $k = 1$ gives the classical Riemann zeta function. The sum $s_1 + \ldots + s_k$ is called the weight of the multiple zeta value $\zeta(s_1, \ldots, s_k)$ while $k$ is called the length of $\zeta(s_1, \ldots, s_k)$.

**Definition** 11. Let $K$ be a number field such that $K \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. For any integer $p > 1$, we define the generalized Zagier space as the $K$-linear space $V_p(K)$ defined by

$$V_p(K) = K - \text{span of } \{\zeta(s_1, \ldots, s_k)|s_1 + \ldots + s_k = p\}.$$  

In [1], authors have shown that the dimension of Zagier spaces $W_{4d+2}$ is at least 2 for all $d \geq 1$ assuming the Chowla-Milnor conjecture, where the Zagier space is defined by

$$W_p = \mathbb{Q} - \text{span of } \{\zeta(s_1, \ldots, s_k)|s_1 + \ldots + s_k = p\}.$$  

In this section we will prove the following theorem analogous to the theorem 3 in [1].

**Theorem** 12. Let $d$ be a positive integer. Then the Gun, Murty and Rath conjecture implies

$$\dim_K V_{4d+2}(K) \geq 2.$$
2. Proof of Theorem 5

For the proof of Theorem 5, we shall need the following lemma (see S. Lang [10], p.548).

**Lemma 13.** Let $G$ be any finite abelian group of order $n$ and $F : G \to \mathbb{C}$ be any complex-valued function on $G$. The determinant of the $n \times n$ matrix given by $(F(xy^{-1}))$ as $x, y$ range over the group elements is called the *Dedekind determinant* and is equal to

$$\prod_{\chi} \left( \sum_{x \in G} \chi(x)F(x) \right),$$

where the product is over all characters $\chi$ of $G$.

**Proof of Theorem 5.** Let $k > 1$ and $q > 1$. Let $f$ be a rational valued period function with period $q$ satisfying $f(a) = 0$ for $1 < (a, q) < q$. Suppose that $L(k, f) = r$ is a rational number. Then we have

$$L(k, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^k} = r$$

As $f$ is periodic function, we have the Fourier transformation of $f$ given by

$$\hat{f}(n) = \frac{1}{q} \sum_{a=1}^{q} f(a) \zeta_q^{-an}$$

where $\zeta_q = e^{\frac{2\pi i}{q}}$ and hence we have the Fourier inversion formula

$$f(n) = \sum_{a=1}^{q} \hat{f}(a) \zeta_q^{an}.$$ 

Then we have

$$L(k, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^k} = \sum_{n=1}^{\infty} \frac{1}{n^k} \sum_{a=1}^{q} \hat{f}(a) \zeta_q^{an} = r$$

and hence

$$\sum_{a=1}^{q} \hat{f}(a) Li_k(\zeta_q^a) - r = 0.$$

Let $Li_k(\alpha_1), ..., Li_k(\alpha_t)$ be a maximal linearly independent subset of

$$\{Li_k(\zeta_q^a) | 1 \leq a < q\}$$

over $\mathbb{Q}$. 

Then

\[ L_i_k(\zeta_q^n) = \sum_{b=1}^{t} C_{ab}L_i_k(\alpha_b) \]

for some \( C_{ab} \in \mathbb{Q} \). So we have

\[ \sum_{b=1}^{t} \beta_bL_i_k(\alpha_b) - r = 0 \]

where

\[ \beta_b = \sum_{a=1}^{q} \hat{f}(a)C_{ab}. \]

Since \( f \) is rational valued, \( \hat{f} \) is algebraic valued. So by the Strong Polylog conjecture, we have

\[ r = 0 \text{ and } \beta_b = \sum_{a=1}^{q} \hat{f}(a)C_{ab} = 0, \ 1 \leq b \leq t. \]

Now for any automorphism \( \sigma \) of the field \( \overline{\mathbb{Q}} \) over \( \mathbb{Q} \), we have

\[ \sum_{a=1}^{q} \sigma(\hat{f}(a))C_{ab} = 0, \ 1 \leq b \leq t, \]

and hence

\[ \sum_{a=1}^{q} \sigma(\hat{f}(a))L_i_k(\zeta_q^n) = 0. \]

In particular, if for \( 1 \leq h < q, (h, q) = 1 \), \( \sigma_h \) is the element of the Galois group of \( \mathbb{Q}(\zeta_q) \) over \( \mathbb{Q} \) such that

\[ \sigma_h(\zeta_q) = \zeta_q^h, \]

then we have,

\[ \sigma_h(\hat{f}(n)) = \hat{f}_h(n) \]

where

\[ f_h(n) = f(nh^{-1}). \]

Thus, we have

\[ L(k, f_h) = \sum_{n=1}^{\infty} \frac{f_h(n)}{n^k} \]

\[ = \sum_{a=1}^{q} \hat{f}_h(a)L_i_k(\zeta_q^n) \]

\[ = \sum_{a=1}^{q} \sigma_h(\hat{f}(a))L_i_k(\zeta_q^n) = 0 \]
for all $1 \leq h < q, (h, q) = 1$. Thus by equation (1), we get

$$L(k, f_h) = q^{-k} \sum_{a=1}^{q-1} \left[ f_h(a) + \frac{f_h(q)q^{-k}}{\prod_{p|q}(1 - p^{-k})} \right] \zeta(k, a/q) = 0$$

for all $1 \leq h < q, (h, q) = 1$.

Now, putting $ah^{-1} = b$ and noting that $f_h(q) = f(q)$, we have

$$L(k, f_h) = q^{-k} \sum_{b=1}^{q-1} \left[ f(b) + \frac{f(q)q^{-k}}{\prod_{p|q}(1 - p^{-k})} \right] \zeta(k, bh/q) = 0 \quad (2)$$

for all $1 \leq h < q, (h, q) = 1$.

Thus we get a matrix equation with $M$ being the $\varphi(q) \times \varphi(q)$ matrix whose $(b, h)$-th entry is given by $\zeta(k, bh/q)$. Then by the evaluation of the Dedekind determinant as in Lemma 13, we get

$$\text{Det}(M) = \pm \prod_{\chi} q^k L(k, \chi) \neq 0.$$ 

Thus the matrix $M$ is invertible and hence by the equation (2), we have

$$f(a) + \frac{f(q)q^{-k}}{\prod_{p|q}(1 - p^{-k})} = 0, \quad 1 \leq a < q, (a, q) = 1$$

and hence

$$f(a) = -\frac{f(q)q^{-k}}{\prod_{p|q}(1 - p^{-k})}$$

for all $1 \leq a < q, (a, q) = 1$. This completes the proof of theorem 5.

### 3. Dimension of Strong Chowla-Milnor Spaces

The following lemma 14, due to Okada [3] about the linear independence of co-tangent values at rational arguments, plays a significant role in proving the theorem 7.

**Lemma 14.** Let $k$ and $q$ be positive integers with $k \geq 1$ and $q > 2$. Let $T$ be a set of $\varphi(q)/2$ representations mod $q$ such that the union $T \cup (-T)$ constitutes a complete set of co-prime residue classes mod $q$. Then the set of real numbers

$$\frac{d^{k-1}}{dz^{k-1}} \cot(\pi z)|_{z=a/q}, \quad a \in T$$

is linearly independent over $\mathbb{Q}$.

We first start with a proposition.
Proposition 15. \( \frac{\varphi(q)}{2} + 1 \leq \dim_{\mathbb{Q}} \hat{V}_k(q) \leq \varphi(q) + 1 \).

Proof. Clearly from the definition of the Strong Chowla-Milnor space \( \dim_{\mathbb{Q}} \hat{V}_k(q) \leq \varphi(q) + 1 \). Note that the space \( \hat{V}_k(q) \) is also spanned by the following sets of real numbers:

1. \( \{ \zeta(k, a/q) + \zeta(k, 1 - a/q) \} | (a, q) = 1, \ 1 \leq a < q/2 \}, \)
2. \( \{ \zeta(k, a/q) - \zeta(k, 1 - a/q) \} | (a, q) = 1, \ 1 \leq a < q/2 \}. \)

Then we have the following (see [4] and [1], for instance)

\[ \zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q) = \frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} (\pi \cot \pi z)|_{z = a/q} \text{ for } k \geq 2 \]

and by Okada([3]), we get

\[ \zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q) \]

for \( 1 \leq a < q/2, (a, q) = 1 \), are linearly independent over \( \mathbb{Q} \). Again by induction, we have

\[ \frac{d^{k-1}}{dz^{k-1}} (\pi \cot \pi z) = \pi^k \times \mathbb{Z} \text{ linear combination of } (\csc \pi z)^{2l} (\cot \pi z)^{k-2l}, \]

for some non-negative integer \( l \) and for an integer \( k \geq 1 \). Since \( \csc \pi z \) and \( \cot \pi z \) are algebraic at rationals, we have all the numbers \( \zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q) \) are transcendental for any \( k \). Hence \( \dim_{\mathbb{Q}} \hat{V}_k(q) \geq \frac{\varphi(q)}{2} + 1 \).

Proof of Theorem 7. Suppose not, then we have

\[ \dim_{\mathbb{Q}} \hat{V}_k(q) = \frac{\varphi(q)}{2} + 1. \]

This gives that the numbers

\[ 1, \zeta(k, a/q) - \zeta(k, 1 - a/q), \text{ where } (a, q) = 1, \ 1 \leq a < q/2 \]

generate \( \hat{V}_k(q) \).

Since \( k \) is odd, then by Hecke [9] (See also paper 41 of E. Hecke, Mathematische Werke, Dritte Auflage, Vandenhoeck and Ruprecht, Gottingen, 1983), we have

\[ \frac{\zeta(k, a/q) - \zeta(k, 1 - a/q)}{(2\pi i)^k} \in \mathbb{Q}(\zeta_q). \]  \hspace{1cm} (3)

Again we know that

\[ \zeta(k) \prod_{p \in P, \ p \mid q} (1 - p^{-k}) = q^{-k} \sum_{a=1, (a,q)=1}^{q-1} \zeta(k, a/q) \in \hat{V}_k(q), \]
where $P$ be the set of primes.
Thus $\zeta(k) \in \hat{V}_k(q)$ and hence we have

$$\zeta(k) = q_1 + \sum_{\substack{(a,q) = 1 \\ 1 \leq a < q/2}} \lambda_a [\zeta(k, a/q) - \zeta(k, 1 - a/q)]$$

for some $q_1, \lambda_a \in \mathbb{Q}$
so that

$$\frac{\zeta(k) - q_1}{(2\pi i)^k} = \sum_{\substack{(a,q) = 1 \\ 1 \leq a < q/2}} \lambda_a [\zeta(k, a/q) - \zeta(k, 1 - a/q)] \frac{1}{(2\pi i)^k}.$$  

Thus by (3)

$$\frac{\zeta(k) - q_1}{i\pi^k} = a_1 \text{(say)} \in \mathbb{Q}(\zeta_q).$$

Similarly, if

$$\dim_{\mathbb{Q}} \hat{V}_k(r) = \frac{\varphi(r)}{2} + 1,$$

then

$$\frac{\zeta(k) - q_2}{i\pi^k} = a_2 \text{(say)} \in \mathbb{Q}(\zeta_r), \text{ with } q_2 \in \mathbb{Q}.$$

So we have

$$a_1 i\pi^k + q_1 = a_2 i\pi^k + q_2$$

which implies

$$(a_1 - a_2)i\pi^k = q_2 - q_1.$$  

The L.H.S of the above equation is algebraic number times transcendental number hence transcendental and the R.H.S is a rational number. Hence we get that $q_1 = q_2$ and $a_1 = a_2$.
Thus we have

$$\frac{\zeta(k) - q_1}{i\pi^k} \in \mathbb{Q}(\zeta_q) \cap \mathbb{Q}(\zeta_r) = \mathbb{Q}.$$  

Let

$$\frac{\zeta(k) - q_1}{i\pi^k} = a \in \mathbb{Q}.$$  

Since L.H.S of the above equation is purely imaginary and R.H.S is rational, we have $a = 0$ and $\zeta(k) = q_1$, a rational number. This is a contradiction to the irrationality of $\zeta(k)$. Thus either

$$\dim_{\mathbb{Q}} \hat{V}_k(q) \geq \frac{\varphi(q)}{2} + 2$$

or

$$\dim_{\mathbb{Q}} \hat{V}_k(r) \geq \frac{\varphi(r)}{2} + 2.$$  

This completes the proof of the theorem.

**Proposition** 16. $2 \leq \dim_{\mathbb{Q}} \hat{V}_k(q) \leq \frac{\varphi(q)}{2} + 2.$
Proof. We know that \( \zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q) \in \pi_k \overline{\mathbb{Q}} \) for \( 1 \leq a < q/2, \ (a, q) = 1 \). As 1 and \( \pi \) are linearly independent over \( \overline{\mathbb{Q}} \), we have \( \dim_{\overline{\mathbb{Q}}} \hat{V}_k(q) \geq 2 \).

As \( \zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q) \) for \( 1 \leq a < q/2, \ (a, q) = 1 \), are linearly dependent over \( \overline{\mathbb{Q}} \), they contribute at most 1 in the dimension. Hence we have \( \dim_{\overline{\mathbb{Q}}} \hat{V}_k(q) \leq \frac{\varphi(q)}{2} + 2 \).

4. Proof of Theorem 9

Proof. The case \( k = 1 \) is a theorem of M. Ram Murty and N. Saradha [4]. Let \( k > 1 \) and assume that

\[
\sum_{\chi} c_{\chi} L(k, \chi) = 0, \ c_{\chi} \in K_1
\]

where the summation is over all primitive Dirichlet characters mod \( q \).

Again we know that

\[
L(k, \chi) = q^{-k} \sum_{1 \leq a < q \atop (a, q) = 1} \chi(a) \zeta(k, a/q).
\]

So from the above equation we get

\[
q^{-k} \sum_{\chi} c_{\chi} \sum_{1 \leq a < q \atop (a, q) = 1} \chi(a) \zeta(k, a/q) = 0,
\]

and hence we have

\[
\sum_{1 \leq a < q \atop (a, q) = 1} \zeta(k, a/q) \sum_{\chi} c_{\chi} \chi(a) = 0.
\]

The values of \( \chi \) lie in the field \( K_1 \) so that the sum \( \sum_{\chi} c_{\chi} \chi(a) \in K_1 \) which is disjoint from \( \mathbb{Q}(\zeta_q) \). Hence using Gun, Murty and Rath conjecture, we get

\[
\sum_{\chi} c_{\chi} \chi(a) = 0
\]

for all \( a \in (\mathbb{Z}/q\mathbb{Z})^* \). Then by the orthogonality of characters, we have \( c_{\chi} = 0 \) for all \( \chi \). This completes the proof.

5. Dimension of generalized Zagier spaces

Before we proceed to the proof of the above theorem 12, we review some basic facts about Kronecker symbols and associated Gauss sums. If \( \Delta \) be a fundamental discriminant then we have \( \Delta \equiv 0, 1 \pmod{4} \). Then the Kronecker symbol \( (\Delta \over n) \) is defined by the following relations:
(i) \( \left( \frac{\Delta}{p} \right) = 0 \) when \( p | \Delta \) and \( p \) prime,

(ii) \( \left( \frac{\Delta}{2} \right) = \begin{cases} 1 & \text{when } \Delta \equiv 1 \pmod{8}, \\ -1 & \text{when } \Delta \equiv 5 \pmod{8}. \end{cases} \)

(iii) \( \left( \frac{\Delta}{p} \right) = \left( \frac{\Delta}{p} \right)_L \), the Legendre symbol, when \( p > 2 \),

(iv) \( \left( \frac{\Delta}{-1} \right) = \begin{cases} 1 & \text{when } \Delta > 0, \\ -1 & \text{when } \Delta < 0, \end{cases} \)

(v) \( \left( \frac{\Delta}{n} \right) \) is completely multiplicative function of \( n \).

In the proof of theorem 12, we use the following theorem (see [8], p.297).

**Theorem 17.** Let \( \Delta \) be a fundamental discriminant. Then \( \chi_{\Delta}(n) = \left( \frac{\Delta}{n} \right) \) is a primitive quadratic character modulo \( |\Delta| \).

Clearly from the definition \( \chi_{\Delta}(n) \) is an odd character if \( \Delta < 0 \).

We first start with a lemma.

**Lemma 18.** Let \( K \) be an algebraic number field. Then \( \left[ \zeta(2d+1) \pi^{2d+1} \right]^2 \notin K \) implies \( \dim_K V_{4d+2}(K) \geq 2 \).

**Proof.** Using the definition of multiple zeta values we get

\( \zeta(s_1) \zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2) \).

Thus we have

\( \zeta(2d+1)^2 = 2\zeta(2d+1, 2d+1) + \zeta(4d+2) \)

and

\( \left[ \zeta(2d+1) \pi^{2d+1} \right]^2 = 2 \zeta(2d+1, 2d+1) \pi^{4d+2} + \zeta(4d+2) \pi^{4d+2} \).

Since

\( \left[ \zeta(2d+1) \pi^{2d+1} \right]^2 \notin K \) and \( \zeta(4d+2) \pi^{4d+2} \in K \)

it follows that \( \zeta(2d+1, 2d+1) \) is not in the \( K \)-span of \( \zeta(4d+2) \) and hence the \( K \)-dimension of the space \( V_{4d+2}(K) \) \( \geq 2 \).

**Lemma 19.** Suppose the Gun, Murty and Rath conjecture is true. Then \( \left[ \zeta(2d+1) \pi^{2d+1} \right]^2 \notin K \), for all \( d \geq 1 \).

**Proof.** Let \( \Delta < 0 \) be a fundamental discriminant. Then the Kronecker symbol \( \chi_{\Delta}(n) = \left( \frac{\Delta}{n} \right) \) is an odd, primitive, quadratic character modulo \( |\Delta| \).

Let \( q = |\Delta| \).
Now from the theory of Gauss sums we know that the Gauss sums $\tau(\chi_\Delta)$ associated with the Kronecker symbol $\chi_\Delta$ (see [8], p.300) is given by

$$\tau(\chi_\Delta) = \sum_{a=1}^{q} \chi_\Delta(a)\zeta_q^a = i\sqrt{q}.$$  

Again using the primitivity of $\chi_\Delta$, we have

$$\tau(\chi_\Delta, b) = \sum_{a=1}^{q} \chi_\Delta(a)\zeta_q^{ab} = \overline{\chi_\Delta}(b)i\sqrt{q}.$$  

Since $\chi_\Delta$ is an odd character, we have

$$\frac{q}{2} \sum_{a=1}^{q/2} \chi_\Delta(a)(\zeta_q^{ab} - \zeta_q^{-ab}) = \overline{\chi_\Delta}(b)i\sqrt{q}.$$  

Let $B_l(x)$ be the $l$th Bernoulli polynomial. Multiplying both sides of the above equation by $B_{2d+1}(b/q)$ and taking sum over $b = 1$ to $q$ we get,

$$\frac{q}{2} \sum_{a=1}^{q/2} \chi_\Delta(a) \sum_{b=1}^{q} (\zeta_q^{ab} - \zeta_q^{-ab})B_{2d+1}(b/q) = i\sqrt{q} \sum_{b=1}^{q} \overline{\chi_\Delta}(b)B_{2d+1}(b/q).$$  

Let $k = 2d + 1$. Then from proposition 1 of [1], we have

$$\zeta(k, a/q) - \zeta(k, 1 - a/q) = \frac{q^{k-1}}{(2\pi i)^k} \sum_{b=1}^{q} (\zeta_q^{ab} - \zeta_q^{-ab})B_k(b/q)$$

for any $(a, q) = 1$ and $1 \leq a < q/2$. As $\chi_\Delta$ is a quadratic character we get that the number $i\sqrt{q}$ lies in the $K$-linear space generated by the real numbers

$$\zeta(k, a/q) - \zeta(k, 1 - a/q)$$

with $(a, q) = 1$ and $1 \leq a < q/2$.

Again we know that

$$\zeta(k) \prod_{p \notin P, \ p|q} (1 - p^{-k}) = q^{-k} \sum_{a=1}^{q-1} \zeta(k, a/q)$$

where $P$ be the set of primes. So that

$$\zeta(k) \prod_{p \notin P, \ p|q} (1 - p^{-k}) = q^{-k/2} \sum_{a=1}^{q/2} [\zeta(k, a/q) + \zeta(k, 1 - a/q)].$$
Hence $\zeta(k)/(2\pi i)^k$ lies in the $K$-linear space generated by the real numbers
\[
\frac{\zeta(k, a/q) + \zeta(k, 1 - a/q)}{(2\pi i)^k}
\]
with $(a, q) = 1$ and $1 \leq q < q/2$.

Thus the Gun, Murty and Rath conjecture for the modulus $q$ implies that $i\sqrt{q}$ and $\zeta(k)/(2\pi i)^k$ lie in two disjoint $K$-spaces. Hence for any such $q$, we have
\[
\zeta(2d + 1)\sqrt{q}/\pi^{2d+1} \notin K.
\]
Thus, if the Gun, Murty and Rath conjecture true for all modulus, then
\[
\left[\frac{\zeta(2d + 1)}{\pi^{2d+1}}\right]^2 \notin K
\]
for all $d \geq 1$.

**Proof of theorem 12.** Suppose the Gun, Murty and Rath conjecture is true. Then lemma 19 implies
\[
\left[\frac{\zeta(2d + 1)}{\pi^{2d+1}}\right]^2 \notin K.
\]
Hence from lemma 18, we get
\[
\dim_K V_{4d+2}(K) \geq 2.
\]
This completes the proof of theorem 12.

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