1 Introduction-What is complexity?

The complex system has been considered by Casity in Santafe research center as follows:
(1) A system is composed of several elements called agents. The size of the system (the number of the elements) is medium.
(2) The agent has intelligence.
(3) Each agent has interaction due to local information. The decision of each agent is determined by not all information but the limited information of the system.

Under a small modification, I define the complex system as follows:
(1) A system is composed of several elements. The scale of the system is often large but not always, in some cases one.
(2) Some elements of the system have special (self) interactions (relations), which produce a dynamics of the system.
(3) The system shows a particular character (not sum of the characters of all elements) due to (2).

Definition 1 A system having the above three properties is called “complex system”. The “complexity” of such a complex system is a quantity measuring that complexity, and its change describes the appearance of the particular character of the system.
There exist such measures describing the complexity for a system, for instance, variance, correlation, level - statistics, fluctuation, randomness, multiplicity, entropy, fuzzy, fractal dimension, ergodicity (mixing, flow), bifurcation, localization, computational complexity (Kolmogorov’s or Chaitin’s), catastrophe, dynamical entropy, Lyapunov exponent, etc. These quantities are used case by case and they are often difficult to compute. Moreover, the relations among these are lacking (not clear enough). Therefore it is important to find common property or expression of these quantities. In this paper, we introduce such a common degree to describe the chaotic aspect of quantum dynamical systems. Further we describe the function of brain in the framework of information dynamics [6, 14] (ID for short) and we discuss the value of information attached to the brain in terms of the complexity in ID and the chaos degree [9, 11, 12].

2 Quantum Information Dynamics

There are two aspects for the complexity, that is, the complexity of a state describing the system itself and that of a dynamics causing the change of the system (state). The former complexity is simply called the ”complexity” of the state, and the later is called the ”chaos degree” of the dynamics in this paper. Therefore the examples of the complexity are entropy, fractal dominion, and those of the chaos degree are Lyapunov exponent, dynamical entropy, computational complexity. Let us discuss a common quantity measuring the complexity of a system so that we can easily handle. The complexity of a general quantum state was introduced in the frame of ID [6, 11, 14] and the quantum chaos degree was defined in [2], which we will review in this section.

Information Dynamics is a synthesis of dynamics of state change and complexity of state. More precisely, let \((\mathcal{A}, \mathcal{G}, \alpha(G))\) be an input (or initial) system and \((\overline{\mathcal{A}}, \overline{\mathcal{G}}, \overline{\alpha(G)})\) be an output (or final) system. Here \(\mathcal{A}\) is the set of all objects to be observed and \(\mathcal{G}\) is the set of all means for measurement of \(\mathcal{A}\), \(\alpha(G)\) is a certain evolution of system. Once an input and an output systems are set, the situation of the input system is described by a state, an element of \(\mathcal{G}\), and the change of the state is expressed by a mapping from \(\mathcal{G}\) to \(\overline{\mathcal{G}}\), called a channel, \(\Lambda^* : \mathcal{G} \rightarrow \overline{\mathcal{G}}\). Often we have \(\mathcal{A} = \overline{\mathcal{A}}, \mathcal{G} = \overline{\mathcal{G}}, \alpha = \overline{\alpha}\), which is assumed in the sequel. Thus we claim

[Giving a mathematical structure to input and output triples]
≡ Having a theory]

For instance, when $\mathcal{A}$ is the set $M(\Omega)$ of all measurable functions on a measurable space $(\Omega, \mathcal{F})$ and $\mathcal{G}(\mathcal{A})$ is the set $P(\Omega)$ of all probability measures on $\Omega$, we have usual probability theory, by which the classical dynamical system is described. When $\mathcal{A} = B(\mathcal{H})$, the set of all bounded linear operators on a Hilbert space $\mathcal{H}$, and $\mathcal{G}(\mathcal{A}) = \mathcal{G}(\mathcal{H})$, we have a usual quantum dynamical system. In this paper, we assume that both the input and output triple $(\mathcal{A}, \mathcal{G}, \alpha(G))$ is a C*-dynamical system or the usual quantum system as above, and a channel, $\Lambda^* : \mathcal{G} \rightarrow \mathcal{G}$ is a completely positive map.

There exist two complexities in ID, which are axiomatically given as follows:

Let $(\mathcal{A}_t, \mathcal{G}_t, \alpha_t(G'))$ be the total system of both input and output systems; $\mathcal{A}_t \equiv \mathcal{A} \otimes \mathcal{A}, \mathcal{G}_t \equiv \mathcal{G} \otimes \mathcal{G}, \alpha_t \equiv \alpha \otimes \alpha$ with suitable tensor products $\otimes$. Further, let $C(\varphi)$ be the complexity of a state $\varphi \in \mathcal{G}$ and $T(\varphi; \Lambda^*)$ be the transmitted complexity associated with the state change $\varphi \rightarrow \Lambda^* \varphi$. These complexities $C$ and $T$ are the quantities satisfying the following conditions:

(i) For any $\varphi \in \mathcal{G}$,
$$C(\varphi) \geq 0, \ T(\varphi; \Lambda^*) \geq 0.$$

(ii) For any orthogonal bijection $j : \text{ex}\mathcal{G} \rightarrow \text{ex}\mathcal{G}$ (the set of all extreme points in $\mathcal{G}$),
$$C(j(\varphi)) = C(\varphi),$$
$$T(j(\varphi); \Lambda^*) = T(\varphi; \Lambda^*).$$

(iii) For $\Phi \equiv \varphi \otimes \psi \in \mathcal{G}_t$,
$$C(\Phi) = C(\varphi) + C(\psi).$$

(iv) For any state $\varphi$ and a channel $\Lambda^*$,
$$T(\varphi; \Lambda^*) \leq C(\varphi).$$

(v) For the identity map “id” from $\mathcal{G}$ to $\mathcal{G}$,
$$T(\varphi; \text{id}) = C(\varphi).$$
Definition 2: Quantum Information Dynamics (QID) is defined by
\((\mathcal{A}, \mathcal{G}, \alpha(G); \Lambda^*; C(\varphi), T(\varphi; \Lambda^*))\)
and some relations \(R\) among them.

There are several examples of the above complexities \(C\) and \(T\) such as quantum entropy and quantum mutual entropy \([4, 8]\). Information Dynamics can be applied to the study of chaos in the following sense:

Definition 3 \([9, 11, 1]\): \(\psi\) is more chaotic than \(\varphi\) as seen from the reference system \(S\) if \(C(\psi) \geq C(\varphi)\). When \(\varphi\) changes to \(\Lambda^* \varphi\), the degree of chaos associated to this state change (dynamics) \(\Lambda^*\) is given by

\[
D(\varphi; \Lambda^*) = \inf \left\{ \int_{\mathcal{S}} C(\Lambda^* \omega) d\mu; \mu \in M(\varphi) \right\},
\]

where \(\varphi = \int_{\mathcal{S}} \omega d\mu\) is a maximal extremal decomposition of \(\varphi\) equipped with a certain topology in the state space \(\mathcal{S}\) and \(M(\varphi)\) is the set of such measures. In some cases such that \(\Lambda^*\) is linear, this chaos degree \(D(\varphi; \Lambda^*)\) can be written as \(C(\Lambda^* \varphi) - T(\varphi; \Lambda^*)\).

Since ID has hierarchy (hierarchical structure), it can be applied several open systems. Here we apply ID to Brain Dynamics.

3 Entropic chaos degree

In the context of information dynamics, a chaos degree associated with a dynamics in classical systems was introduced in \([9]\). It has been applied to several dynamical maps such logistic map, Baker’s transformation and Tinkerbel map with successful explanations of their chaotic characters \([9]\). This chaos degree has several merits compared with usual measures such as Lyapunov exponent.

Here we discuss the quantum version of the classical chaos degree, which is defined by quantum entropies in Section 2, and we call the quantum chaos degree the entropic quantum chaos degree. In order to contain both classical and quantum cases, we define the entropic chaos degree in \(C^*\)-algebraic terminology. This setting will not be used in the sequel application, but for mathematical completeness we first discuss the \(C^*\)-algebraic setting.

Let \((\mathcal{A}, \mathcal{G})\) be an input \(C^*\) system and \((\overline{\mathcal{A}}, \overline{\mathcal{G}})\) be an output \(C^*\) system; namely, \(\mathcal{A}\) is a \(C^*\) algebra with unit \(I\) and \(\mathcal{G}\) is the set of all states on \(\mathcal{A}\). We
assume $\mathcal{A} = \mathcal{A}$ for simplicity. For a weak* compact convex subset $S$ (called the reference space) of $\mathcal{S}$, take a state $\varphi$ from the set $S$ and let

$$\varphi = \int_S \omega d\mu_\varphi$$

be an extremal orthogonal decomposition of $\varphi$ in $S$, which describes the degree of mixture of $\varphi$ in the reference space $S$ [5, 16]. The measure $\mu_\varphi$ is not uniquely determined unless $S$ is the Schouk simplex, so that the set of all such measures is denoted by $M_\varphi (S)$. The entropic chaos degree with respect to $\varphi \in S$ and a channel $\Lambda^*$ is defined by

$$D^S (\varphi; \Lambda^*) \equiv \inf \left\{ \int_S S^S (\Lambda^* \varphi) d\mu_\varphi; \mu_\varphi \in M_\varphi (S) \right\} \quad (3.1)$$

where $S^S (\Lambda^* \varphi)$ is the mixing entropy of a state $\varphi$ in the reference space $S$ [7, 11]. When $S = \mathcal{S}$, $D^S (\varphi; \Lambda^*)$ is simply written as $D (\varphi; \Lambda^*)$. This $D^S (\varphi; \Lambda^*)$ contains both the classical chaos degree and the quantum one.

The classical entropic chaos degree is the case that $\mathcal{A}$ is abelian and $\varphi$ is the probability distribution of a orbit generated by a dynamics (channel) $\Lambda^*$;

$$\varphi = \sum_k p_k \delta_k,$$

where $\delta_k$ is the delta measure such as

$$\delta_k (j) \equiv \begin{cases} 1 & (k = j) \\ 0 & (k \neq j) \end{cases}.$$

Then the classical entropic chaos degree is

$$D_c (\varphi; \Lambda^*) = \sum_k p_k S (\Lambda^* \delta_k),$$

where $S$ equals to von Neumann’s entropy, equivalently Shannon’s one [16].

We explain the entropic chaos degree of a quantum system described by a density operator. Let $F^*$ be a channel sending a state to a state and $\rho$ be an intial state. After time $n$, the state is $F^{*n} \rho$, whose Schatten decomposition is denoted by $\sum_k \lambda_k^{(n)} E_k^{(n)}$. Then define a channel $\Lambda^*_m$ on $\otimes^n_1 \mathcal{H}$ by

$$\Lambda^*_m \sigma = F^* \sigma \otimes \cdots \otimes F^{*m} \sigma, \quad \sigma \in \mathcal{S} (\mathcal{H}),$$

from which the entropic chaos degree (3.1) for the channels $F^*$ and $\Lambda^*_n$ are written as
\[ D_q (\rho; F^*) = \inf \left\{ \sum_k \lambda_k^{(n)} \mathcal{S} \left( F^* E_k^{(n)} \right); \{ E_k^{(n)} \} \right\}, \]
\[ D_q (\rho; \Lambda^* m) = \inf \left\{ \frac{1}{m} \sum_k \lambda_k^{(n)} \mathcal{S} \left( \Lambda^* m E_k^{(n)} \right); \{ E_k^{(n)} \} \right\}, \]

where the supremum is taken over all Schatten decompositions of \( F^{*n} \rho \).

We can judge whether the dynamics \( F^* \) causes a chaos or not by the value of \( D \) as

\[ D > 0 \text{ and not constant} \iff \text{chaotic}, \]
\[ D = \text{constant} \iff \text{weak stable}, \]
\[ D = 0 \iff \text{stable}. \]

The classical version of this degree was applied to study the chaotic behaviors of several nonlinear dynamics \[3, 9\]. The quantum entropic chaos degree is applied to the analysis of quantum spin system[2] and quantum Baker’s type transformation, and we could measure the chaos of these systems. The information theoretical meaning of this degree was explained in \[11, 14\].

## 4 Quantum Chaos Degree

In this section, we apply it to study the appearance of chaos in quantum spin systems[2].

The quantum chaos degree has the following properties.

**Theorem 1** For any \( \rho \in \mathcal{S}(\mathcal{H}), F^* : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H}) \) and \( \Lambda^* m : \mathcal{S}(\otimes^m \mathcal{H}) \rightarrow \mathcal{S}(\otimes^m \mathcal{H}) \) defined above, we have:

1. Let \( U_t \) be an unitary operator satisfying \( U_t = \exp (itH) \) for any \( t \in \mathbb{R} \).
   
   If \( F^* \rho = \text{Ad} U_t (\rho) \equiv U_t \rho U_t^* \), \( D_q (\rho; F^*) = D_q (\rho; \Lambda^* m) = 0 \).
   
2. Let \( \rho_0 \) be a fixed state on \( \mathcal{H} \). If \( F^* \rho = \rho_0 \), \( D_q (\rho; F^*) = D_q (\rho; \Lambda^* m) = S (\rho_0) \).
(3) Let \( \lambda \) be a fixed positive real number. If \( F^* \rho = e^{-\lambda} \rho + (1 - e^{-\lambda}) \rho_0 \), \( D_q(\rho; F^*) = D_q(\rho; \Lambda^*_m) = S(\rho_0) \).

(4) Let \( \{P_k\} \) be the positive operated measure and \( F^* \rho = \sum_k P_k \rho P_k \). Then \( D_q(\rho; F^*) = D_q(\rho; \Lambda^*_m) \) = constant, that is, \( F^* \) is weak stable. Moreover, if \( [P_k, \rho] = 0 \), then \( D_q(\rho; F^*) = D_q(\rho; \Lambda^*_m) = 0 \), that is, \( F^* \) is stable.

The proof of this theorem is given in \([2]\). We will apply the quantum entropic chaos degree to spin 1/2 system. See \([2]\) again for the details.

Let \( \vec{X} = (x_1, x_2, x_3) \) be a vector in \( \mathbb{R}^3 \) satisfying \( \|\vec{X}\| = \sqrt{\sum_{i=1}^{3} x_i^2} \leq 1 \) and \( I \) be the identity \( 2 \times 2 \) matrix. Any state \( \rho \) in a spin 1/2 system is expressed as

\[
\rho = \frac{1}{2} \left( I + \vec{\sigma} \cdot \vec{X} \right),
\]

where \( \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \) is the Pauli spin matrix vector;

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Let \( f \) be a non-linear map from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \) satisfying \( \|f(\vec{X})\| \leq 1 \) for any \( \vec{X} \in \mathbb{R}^3 \) with \( \|\vec{X}\| \leq 1 \). A channel \( F^* \) is defined by

\[
F^* \rho = \frac{1}{2} \left( I + \vec{\sigma} \cdot f(\vec{X}) \right)
\]

for any state \( \rho \).

We now define Baker’s type map and see whether this map produces the chaos. For any vector \( \vec{X} = (x_1, x_2, x_3) \) on \( \mathbb{R}^3 \), we consider the following map \( f : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \);

\[
f(x_1, x_2, x_3) = \begin{cases} 
  f_1(x_1, x_2, x_3) & \left( -\frac{1}{\sqrt{2}} \leq x_1 < 0 \right) \\
  f_2(x_1, x_2, x_3) & \left( 0 \leq x_1 \leq \frac{1}{\sqrt{2}} \right)
\end{cases}
\]

where
Whenever \( \frac{1}{\sqrt{2}} \leq |x_1| \leq 1 \) (resp. \( \frac{1}{\sqrt{2}} \leq |x_2| \leq 1 \)), we replace \( x_1 \) with 0 (resp. \( x_2 = 0 \)).

The entropic chaos degree \( D(\rho^{(n)}; F^*) \) and \( D(\rho^{(n)}; \Lambda^*_m) \) can be computed \[2], and the result of \( D(\rho^{(n)}; \Lambda^*_m) \) is shown in Fig. 4.1 for an initial value \( \vec{X} = (0.3, 0.3, 0.3) \). We took 740 different \( a \)'s between 0 and 1 with \( m = 1000, n = 2000 \).

Figure 4.1: The change of \( D(\rho^{(n)}; \Lambda^*_m) \) w.r.t. \( a \)

The result shows that the quantum dynamica constructed by Baker’s type transformation is stable in \( 0 \leq a \leq 0.5 \) and chaotic in \( 0.5 < a \leq 1.0 \). Though there are several approaches to study chaotic behaviors of quantum systems, we used a new quantity to measure the degree of chaos for a quantum system. Our chaos degree has the following merits: (1) once the channel \( \Lambda^* \), describing the dynamics of a quantum system, is given, it is easy to compute this degree numerically; (2) the algorithm computing the degree is easily set for any quantum state.

5 Quantum Information Dynamic Description of Brain

The complexity and the chaos degree can be used to examine the chaotic aspects of not only several nonlinear classical and quantum physical physics but also life sciences. We will construct a model describing the function of brain in the context of QID.

The brain system \( BS = \mathcal{X} \) is supposed to be described by a triple \( (B(\mathcal{H}), \mathcal{S}(\mathcal{H}), \Lambda(G)) \) on a certain Hilbert space \( \mathcal{H} \).

Further we assume the followings:

(1) \( BS \) is described by a quantum state and the brain itself is divided into several parts, each of which corresponds to a Hilbert space, so that \( \mathcal{H} = \oplus_k \mathcal{H}_k \) and \( \varphi = \oplus_k \varphi_k \), \( \varphi_k \in \mathcal{S}(\mathcal{H}_k) \).
(2) Each $\varphi_k$ is an entangled state.
(3) The function (action) of is described by a channel $\Lambda^*=\oplus_k \Lambda_k^*$.
(4) $BS$ is composed of two parts; information processing part "$P"$ and others "$O"$ (consciousness, memory) so that $\mathcal{X} = \mathcal{X}_P \otimes \mathcal{X}_O$, $\mathcal{H} = \mathcal{H}_P \otimes \mathcal{H}_O$.

Thus in our model, the brain may be considered as a parallel quantum computer. We will briefly explain mathematical structure of our model.

Let be $s = \{s_1, s_2, \ldots, s^n\}$ a given (input) signal (perception) and be $\bar{s} = \{\bar{s}_1, \bar{s}_2, \ldots, \bar{s}^n\}$ the output signal. After the signal $s$ enters in the brain, each element $s^j$ of $s$ is coded into a proper quantum state $\varphi^j \in \mathcal{H}_P$, so that the state corresponding the signal $s$ is $\varphi = \sum_j \lambda_j \varphi^j$ with some suitable weight $\{\lambda_j\}$. This state may be regarded as a state processed by the brain and it is coupled to a state stored $\varphi_O$ as a memory (pre-consciousness) in the brain. The processing in the brain is expressed by a quantum channel $\Lambda^*$ (or $\Lambda^*_P \otimes \Lambda^*_O$) properly chosen by the form a state $\varphi$ entering the brain, through which the outcome becomes $\Lambda^* \varphi \otimes \varphi_O \equiv \varphi$ (it may have different symmetry to $\varphi$). The channel is determined by the form of the network of neurons and some other biochemical actions, and its function is like a (quantum) gate in computer. The outcome state $\varphi$ contacts with an operator $Q$ describing the work as noema of consciousness, after the contact a certain reduction of state is occurred, which corresponds to the noesis of consciousness. A part of the reduced state is stored in the brain as a memory. The scheme of our model is represented in the following Figure 5.1.
The complex system responds to the information and has a particular role to choose the information (value of information). That value should be estimated by a function of the state $\varphi \otimes \varphi_O$, the channel $\Lambda^*$ and the operator $Q$, so that the function $V(\varphi \otimes \varphi_O, \Lambda^*, Q)$ estimating the effect of a signal and a function of brain is defined as follows:

**Definition 4 Value of Information and Function:**

1. $s = \{s^1, s^2, \ldots, s^n\}$ is more valuable than $s' = \{s'^1, s'^2, \ldots, s'^n\}$ for $\Lambda^*$ and $Q$ iff
   \[ V(\varphi \otimes \varphi_O, \Lambda^*, Q) \geq V(\varphi' \otimes \varphi'_O, \Lambda^*, Q). \]
(2) $\Lambda^*$ is more valuable than $\Lambda^*$ for given $s = \{s^1, s^2, \ldots, s^n\}$ and $Q$ iff

$$V(\varphi \otimes \varphi, \Lambda^*, Q) \geq V(\varphi \otimes \varphi, \Lambda^*, Q).$$

The details of this estimator is discussed in [12], where there exist some relations between the information of value and the complexity or the chaos degree.

References

[1] R.S. Ingarden, A. Kossakowski and M. Ohya, Information Dynamics and Open Systems, Kluwer Academic Publishers, 1997.

[2] K. Inoue, A. Kossakowski and M. Ohya, On quantum chaos in a spin system, preprint.

[3] K. Inoue, M. Ohya and K. Sato, Application of chaos degree to some dynamical systems, to appear in Chaos, Solitons & Fractals.

[4] M. Ohya, On compound state and mutual information in quantum information theory, IEEE Transaction of Information Theory, 29, pp.770-777, 1983.

[5] M. Ohya, Some aspects of quantum information theory and their applications to irreversible processes, Reports on Mathematical Physics, 27, pp.19-47, 1989.

[6] M. Ohya, Information dynamics and its application to optical communication processes, Springer Lecture Notes in Physics, 378, pp.81-92, 1991.

[7] M. Ohya, Entropy transmission in $C^*$-dynamical systems, J. Math. Anal. Appl., 100, pp.222-235, 1984.

[8] M. Ohya, Complexity and fractal dimensions for quantum states, Open Systems and Information Dynamics, 4, pp.141-157, 1997.

[9] M. Ohya, Complexities and their applications to characterization of chaos, International Journal of Theoretical Physics, 37, No.1, pp.495-505 (1998).

[10] M. Ohya, Mathematical Foundation of Quantum Computer, Maruzen Publ. Company, 1999.
[11] M. Ohya, Complexity in dynamics and computation, to appear in The Proceedings of Classical and Quantum White Noise, Kluwer Academic Publishers.

[12] M. Ohya, A mathematical model of the function of brain and the value of information, SUT preprint.

[13] M. Ohya, Mathematical Foundation of Quantum Computer, Maruzen Publ., 1999.

[14] M.Ohya, Quantum Information Dynamics and Quantum Chaos, Springer-Tokyo, in preparation.

[15] M.Ohya and A.Masuda, NP problem in quantum algorithm, quant-ph/9809075 Sep.,1998.

[16] M.Ohya and D.Petz, Quantum Entropy and Its Use, Springer-Verlag, 1993.