Optimal control/Game theory

Bang–bang-type Nash equilibrium point for Markovian nonzero-sum stochastic differential game

Sur un jeu différentiel stochastique de somme non nulle avec contrôles de type bang–bang

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A B S T R A C T

In this Note, we solve a nonzero-sum stochastic differential game (NZSDG) with bang-bang-type equilibrium controls by using backward stochastic differential equations (BSDEs). The generator is multi-dimensional and discontinuous with respect to z.

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Version française abrégée

Jeux différentiels stochastiques de somme non nulle de type bang–bang et EDSRs multidimensionnelles à générateurs discontinus

Cette Note traite des jeux différentiels stochastiques de somme non nulle de type bang–bang, que nous allons décrire et étudier à travers un cas spécifique, mais dont les idées sont assez générales et transposables à d’autres situations. Soit \((\Omega, F, P)\) un espace de probabilité sur lequel est défini un mouvement brownien standard \(B := (B_t)_{t \leq T}\) 1-dimensionnel et qui est de filtration naturelle complétée \((\mathcal{F}_t)_{t \leq T}\). Nous considérons le cas de deux joueurs \(\pi_1\) et \(\pi_2\), la généralisation à plusieurs joueurs étant une simple formalité. Pour \((t, x) \in [0, T] \times \mathbb{R}\), soit \((X_{s+}^t, x)_{s \leq T}\) le processus stochastique défini comme suit :

\[
\forall s \leq T, \quad X_{s+}^t = x + (B_{s+t} - B_t).
\]
Le processus $X^{0,x}$ est la dynamique stochastic d’un système lorsqu’il n’est pas contrôlé. Maintenant, soit $U = [0, 1]$, $V = [-1, 1]$ et soit $M_1$ (resp. $M_2$) l’ensemble des processus $F_t$-progressivement mesurables $u = (u_t)_{t \leq T}$ (resp. $v = (v_t)_{t \leq T}$) sur $[0, T] \times \Omega$ et à valeurs dans $U$ (resp. $V$). L’ensemble $M := M_1 \times M_2$ est dit des contrôles admissibles pour les deux joueurs $\pi_1$ et $\pi_2$.

Soit $f : (t, x) \in [0, T] \times R \rightarrow f(t, x) \in R$ une fonction mesurable et à croissance linéaire, i.e. telle qu’il existe une constante $C$ telle que $|f(t, x)| \leq C(1 + |x|)$, $\forall (t, x) \in [0, T] \times R$, et soit $\Gamma$ la fonction qui à $(t, x, u, v) \in [0, T] \times R \times U \times V$ associe $f(t, x) = f(t, x) + u + v$.

À $(u, v) \in M$ on associe la mesure positive $P^{u, v}$ sur $(\Omega, F)$ définie comme suit :

$$dP^{u, v} = P^{u, v} dP$$
avec $\xi_t = 1 + \int_0^t \Gamma(s, X_s^{0,x}, u_s, v_s) \xi_s dB_s, \quad t \leq T.$

Noter que $P^{u, v}$ est une probabilité, puisque $\Gamma$ est à croissance linéaire en $x$ uniformément en $(u, v) \in U \times V$. Par ailleurs, il existe $p > 1$ tel que $E[(\xi_T^{u,v})^p] < \infty$ ([16], p. 14). Enfin, sous $P^{u, v}$ le processus $B^{u, v} = (B_t - J_0^t \Gamma(r, X_r^{0,x}, u_r, v_r) dr)_{t \leq T}$ est un $(F_t, P^{u, v})$-mouvement brownien et le processus $X_0^{0,x}$ vérifie :

$$\forall s \in [0, T], \quad dX_s^{0,x} = \Gamma(s, X_s^{0,x}, u_s, v_s) ds + dB_s^{u, v} \quad \text{et} \quad X_0^{0,x} = x,$$

i.e., $X_0^{0,x}$ est solution faible de l’équation différentielle stochastique (2). Lorsque le système est contrôlé avec $u = (u_t)_{t \leq T}$ (resp. $v = (v_t)_{t \leq T}$) par $\pi_1$ (resp. $\pi_2$), la loi de sa dynamique est la même que celle de $X_0^{0,x}$ sous $P^{u, v}$.

Soient $g_i, i = 1, 2$, deux fonctions de $R$ dans $R$ bornées et à croissance polynomiale, i.e. vérifiant $|g_i(x)| + |g_2(x)| \leq C(1 + |x|^\gamma)$, $\forall x \in R$ avec $C$ et $\gamma > 0$ constantes fixes. Lorsque le système est contrôlé par les deux joueurs avec $(u, v) \in M$, on associe à $\pi_1$ (resp. $\pi_2$) un payoff égal à $J_1(u, v) := E^{u, v}[g_1(X_T^{0,x})]$ (resp. $J_2(u, v) := E^{u, v}[g_2(X_T^{0,x})]$), où $E^{u, v}$ est l’espérance sous la probabilité $P^{u, v}$.

Dans cette Note, nous nous intéressons à l’existence d’un équilibre de Nash pour ce jeu différentiel de somme non nulle, i.e. un couple de contrôle $(u^*, v^*) \in M$ vérifiant $J_1(u^*, v^*) \geq J_1(u, v^*)$ et $J_2(u^*, v^*) \geq J_2(u^*, v)$, pour tout $(u, v) \in M$.

Si $\Gamma$ ne dépend pas de $v$, alors le jeu devient simplement un problème de contrôle stochastique admet un contrôle optimal de type bang-bang, car ce dernier ne prend ses valeurs qu’à la frontière du domaine en fonction du signe du gradient de la fonction valeur. Aussi, par analogie avec le problème de contrôle stochastique standard, dès lors que $J_1$ et $J_2$ sont sans payoffs instantanés, l’équilibre de Nash pour ce jeu, s’il existe, sera en général de type bang-bang.

Soient $H_1$ et $H_2$ les fonctions hamiltoniennes de ce jeu, i.e. les fonctions ne dépendant pas de $\omega$, qui à $(t, x, p, q, u, v) \in [0, T] \times R \times R \times R \times U \times V$ associe :

$$H_1(t, x, p, u, v) := p \Gamma(t, x, u, v) = p(f(t, x) + u + v) \quad \text{et} \quad H_2(t, x, q, u, v) := H_1(t, x, q, u, v).$$

Soient maintenant $\tilde{u}$ et $\tilde{v}$ les fonctions définies comme suit, respectivement sur $R \times U$ et $R \times V$ et à valeurs dans $U$ et $V$ :

$$\forall p, q \in R, \quad \tilde{u}(p, \epsilon_1) = \begin{cases} 1 & \text{si } p > 0, \\ 0 & \text{si } p < 0, \end{cases} \quad \text{et} \quad \tilde{v}(q, \epsilon_2) = \begin{cases} 1 & \text{si } q > 0, \\ -1 & \text{si } q < 0, \\ \epsilon_2 & \text{si } q = 0. \end{cases}$$

Alors $\tilde{u}$ et $\tilde{v}$ vérifient la condition suivante, dite de Isaacs généralisée : pour tout $(t, x, p, q, u, v) \in [0, T] \times R \times R \times R \times U \times V$ et $(\epsilon_1, \epsilon_2) \in U \times V$,

$$H_1^*(t, x, p, q, \epsilon_2) := H_1(t, x, p, \tilde{u}(p, \epsilon_1), \tilde{v}(q, \epsilon_2)) \geq H_1(t, x, p, u, \tilde{v}(q, \epsilon_2)) \quad \text{et} \quad H_2^*(t, x, q, \epsilon_1) := H_2(t, x, q, \tilde{u}(p, \epsilon_1), \tilde{v}(q, \epsilon_2)) \geq H_2(t, x, q, \tilde{u}(p, \epsilon_1), v).$$

On notera que la fonction $H_1^*$ (resp. $H_2^*$) ne dépend pas de $\epsilon_1$ (resp. $\epsilon_2$), puisque $p\tilde{u}(p, \epsilon_1)$ (resp. $q\tilde{v}(q, \epsilon_2)$) ne dépend pas de $\epsilon_1$ (resp. $\epsilon_2$).

Pour étudier ce problème de jeu différentiel stochastique de somme non nulle, nous allons utiliser les équations différentielles stochastiques rétrogrades comme dans plusieurs travaux antécédants sur ce sujet [4,5]. Le lien est donné par les deux résultats suivants :

**Proposition 1.** Pour tout $(u, v) \in M$, pour $i = 1, 2$, il existe un couple de processus $(Y_i^{u,v}, Z_i^{u,v})$, $F_t$-progressivement mesurables et à valeurs dans $R \times R$, vérifiant :

(i) pour tout $q \geq 1$,

$$E^{u, v} \left[ \sup_{0 \leq s \leq T} \left| Y_s^{u,v} \right|^q + \left( \int_0^T \left| Z_s^{u,v} \right|^2 ds \right)^{q/2} \right] < \infty.$$
Proof. Step 1. First let us point out that the functions \( p \in \mathbb{R} \mapsto p\tilde{u}(p, \epsilon_1) \) and \( q \in \mathbb{R} \mapsto q\tilde{v}(q, \epsilon_2) \) are Lipschitz uniformly with respect to \( \epsilon_1 \) and \( \epsilon_2 \), since \( p\tilde{u}(p, \epsilon_1) = p\tilde{u}(p, 0) = \sup_{u \in U} pu \) and \( q\tilde{v}(q, \epsilon_2) = q\tilde{v}(q, 0) = \sup_{v \in V} qv \). Hereafter \( \tilde{u}(p, 0) \) (resp. \( \tilde{v}(q, 0) \)) will be simply denoted by \( \bar{u}(p) \) (resp. \( \bar{v}(p) \)). Next for \( n \geq 1 \), let \( \bar{u}^n \) and \( \bar{v}^n \) be the functions defined as follows:

\[
\bar{u}^n(p) = \begin{cases} 
0 & \text{if } p \leq -1/n, \\
1 & \text{if } p \geq 0, \\
np + 1 & \text{if } p \in (-1/n, 0),
\end{cases} \quad \text{and} \quad \bar{v}^n(q) = \begin{cases} 
-1 & \text{if } q \leq -1/n, \\
1 & \text{if } q \geq 1/n, \\
nq & \text{if } q \in (-1/n, 1/n).
\end{cases}
\]

Finally let \( \Phi_n \) be the function \( x \in \mathbb{R} \mapsto \Phi_n(x) = (x \wedge n) \vee (-n) \in \mathbb{R} \). Note that \( \Phi_n \) is bounded by \( n \) while \( \bar{u}^n \) and \( \bar{v}^n \) are Lipschitz in \( p \) and \( q \) respectively and roughly speaking they are approximations of \( \bar{u} \) and \( \bar{v} \).
Now for $n \geq 1 \text{ let } (Y^{i,n,t,x}, Z^{i,n,t,x}, i = 1, 2, \text{ be the solution in } S_2^T(\mathbb{R}) \times \mathcal{H}_2^T(\mathbb{R}) \text{ of the following BSDE of dimension two. The solution exists thanks to the result by Pardoux and Peng } [9] \text{ since the generator of the BSDE is Lipschitz in } (p, q). \text{ For any } s \in [t, T],
\begin{align*}
Y^{1,n,t,x}_s &= g_1(X^{1,x}_T) + \int_t^T \Phi_n(Z^{1,n,t,x}_r) f(r, X^{1,x}_r) + \Phi_n(Z^{1,n,t,x}_r) \bar{u}(Z^{1,n,t,x}_r) \, dr - \int_t^T Z^{1,n,t,x}_r \, dB_r, \\
Y^{2,n,t,x}_s &= g_2(X^{1,x}_T) + \int_t^T \Phi_n(Z^{2,n,t,x}_r) f(r, X^{1,x}_r) + \Phi_n(Z^{2,n,t,x}_r) \bar{v}(Z^{2,n,t,x}_r) \, dr - \int_t^T Z^{2,n,t,x}_r \, dB_r.
\end{align*}
(9)

Using the result by El-Karoui et al. ([3], p. 46, Theorem 4.1), there exist measurable deterministic functions $u^{i,n}$ and $d^{i,n}$ of $(r, x) \in [0, T] \times \mathbb{R}, i = 1, 2 \text{ and } n \geq 1, \text{ such that:}
\begin{align*}
\forall s \in [t, T], \quad Y^{i,n,t,x}_s &= u^{i,n}(s, X^{1,x}_s) \quad \text{and} \quad Z^{i,n,t,x}_s = d^{i,n}(s, X^{1,x}_s).
\end{align*}
(10)

Moreover, for $n \geq 1 \text{ and } i = 1, 2, \text{ the functions } u^{i,n} \text{ verify: } \forall (t, x) \in [0, T] \times \mathbb{R},
\begin{align*}
u^{i,n}(t, x) &= \mathbb{E} \left[ g_i(X^{1,x}_T) + \int_t^T H^{n}_{ir}(r, X^{1,x}_r) \, dr \right]
\end{align*}
(11)

with, for any $(s, x) \in [0, T] \times \mathbb{R},
\begin{align*}
H^{1}_{ir}(s, x) &= \Phi_n(d^{1,n}(s, x)) \Phi_n(f(s, x)) + \Phi_n(d^{1,n}(s, x)) \bar{u}(d^{1,n}(s, x)) + \Phi_n(d^{1,n}(s, x)) \bar{v}(d^{1,n}(s, x)), \\
H^{2}_{ir}(s, x) &= \Phi_n(d^{2,n}(s, x)) \Phi_n(f(s, x)) + \Phi_n(d^{2,n}(s, x)) \bar{u}(d^{2,n}(s, x)) + \Phi_n(d^{2,n}(s, x)) \bar{v}(d^{1,n}(s, x)).
\end{align*}
(12)

Step 2. We provide, in this step, the uniform bound of processes $(Y^{i,n,t,x}, Z^{i,n,t,x})$ with respect to $n$ for $i = 1, 2, (t, x) \in [0, T] \times \mathbb{R}. \text{ First, note that for } i = 1, 2, H^{i}_{ir}(s, X^{1,x}_s) \text{ are the generators of BSDE } (9) \text{ and } |H^{i}_{ir}(s, X^{1,x}_s)| \leq \Phi_n(C(1 + |X^{1,x}_s|)^2)Z^{i,n,t,x}_s + C|Z^{i,n,t,x}_s| \text{ for } n \geq 1, (t, x) \in [0, T] \times \mathbb{R}. \text{ Let us now consider the following BSDEs, for } i = 1, 2 \text{ and } s \in [t, T],
\begin{align*}
\tilde{Y}^{i,n}_s &= g_i(X^{1,x}_T) + \int_t^T \Phi_n(C(1 + |X^{1,x}_r|))^2 |\tilde{Z}^{i,n}_r| + C|\tilde{Z}^{i,n}_r| \, dr - \int_t^T \tilde{Z}^{i,n}_r \, dB_r.
\end{align*}
(13)

Observing that the application $z \in \mathbb{R} \mapsto \Phi_n(C(1 + |X^{1,x}_s|))|z| + C|z|$ is Lipschitz continuous, therefore the solution $(\tilde{Y}^{i,n}, \tilde{Z}^{i,n})$ of the above BSDE exists on space $S_2^T(\mathbb{R}) \times \mathcal{H}_2^T(\mathbb{R})$ and is unique. If we show the uniform estimate for $\tilde{Y}^{i,n}$ w.r.t. $n$, then the estimate for $Y^{i,n,t,x}$ will be straightforward consequence of the comparison theorem of BSDEs. Below, we will focus on the property of $\tilde{Y}^{i,n}$. Using again the result by El-Karoui et al. ([3], p. 46, Theorem 4.1) yields that there exist deterministic measurable functions $\tilde{u}^{i,n} : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ such that, for any $s \in [t, T],
\begin{align*}
\tilde{Y}^{i,n}_s &= \tilde{u}^{i,n}(s, X^{1,x}_s), \quad i = 1, 2.
\end{align*}
(14)

Next let us consider the process $B^{i,n}_t = B_t - \int_t^T \Phi_n(C(1 + |X^{1,x}_s|)) + C \text{ sign}(Z^{i,n}_s) \, ds, t \leq s \leq T, i = 1, 2, \text{ which is, thanks to Girsanov’s Theorem, a Brownian motion under the probability } P^{i,n} \text{ on } (\Omega, \mathcal{F}) \text{ whose density with respect to } P \text{ is } \zeta_t \left( |\Phi_n(C(1 + |X^{1,x}_s|)) + C |Z^{i,n}_s| \right) \text{ where for any } z \in \mathbb{R}, \text{ sign}(z) = 1_{\{z \neq 0\}} \frac{z}{|z|} \text{ and } \zeta_t(.) \text{ verifies } \zeta_t(\Theta) = 1 + \int_0^t \Theta_s \zeta_s \, dB_s, t \leq T. \text{ Then the BSDE (13) will be simplified into}
\begin{align*}
\tilde{Y}^{i,n}_s &= g_i(X^{1,x}_T) - \int_s^T \tilde{Z}^{i,n}_r \, dB^{i,n}_r, \quad s \in [t, T], \quad i = 1, 2.
\end{align*}
In view of (14), we obtain $\tilde{u}^{i,n}(t,x) = E^{i,n}[g_i(X^{i,x}_T)|F_t], i = 1, 2$, where $E^{i,n}$ is the expectation under probability $P^{i,n}$. By taking the expectation on both sides under the probability $P^{i,n}$ and considering $\tilde{u}^{i,n}(t,x)$ is deterministic, we arrive at

$$\tilde{u}^{i,n}(t,x) = E^{i,n}[E^{i,n}[g_i(X^{i,x}_T)|F_t]] = E^{i,n}[g_i(X^{i,x}_T)], \quad i = 1, 2.$$ 

The functions $g_i, i = 1, 2$, are of polynomial growth, combining with the result of U.G. Haussmann ([6], p. 14) gives that there exists a constant $p_0 \in (1, 2)$ (which does not depend on $(t,x)$) such that:

$$|\tilde{u}^{i,n}(t,x)| \leq CE^{i,n}\sup_{x \in [T,T]}(1 + |X^{i,x}_t|^\lambda) = \frac{C}{\sqrt{\lambda}}E\left[\sup_{x \in [0,T]}(1 + |X^{i,x}_t|^\lambda)\right].$$

with the constant $\lambda \geq 1$. It follows from comparison of solutions of BSDEs, for any $s \in [T,T]$ and $i = 1, 2, \tilde{v}^{i,n}_s = \tilde{u}^{i,n}(s, X^{i,x}_s) \geq Y^{i,n}_s = u^{i,n}(s, X^{i,x}_s)$. Choosing $s = t$ leads to $u^{i,n}(t,x) \leq C(1 + |x|^k), (t,x) \in [0,T] \times \mathbb{R}$. In a similar way, we can show that $u^{i,n}(t,x) \geq \sup_{s \in [T,T]}X^{i,x}_s$. Hence, $u^{i,n}, i = 1, 2$ are of polynomial growth with respect to $(t,x)$ uniformly in $n$, i.e., $|u^{i,n}(t,x)| \leq C(1 + |x|^k), i = 1, 2, (t,x) \in [0,T] \times \mathbb{R}$. Then, it follows from (10) and the polynomial growth of function $u^{i,n}, i = 1, 2$, that there exists a constant $C$ that does not depend on $n$, such that, for any $\alpha \geq 1$,

$$E\left[\sup_{t \leq s \leq T} |Y^{i,n,t,x}_s|^{\alpha}\right] \leq C.$$  

Next using Itô’s formula with $(Y^{i,n,t,x}_s)^2$ yields the existence of a constant $C$ which is not depending on $n$ such that,

$$E\left[\int_0^T |Z^{i,n,t,x}_s|^2 ds\right] \leq C.$$  

**Step 3.** We show the convergence result in this step. Observing (12) and (16), we infer that the processes $H^i_0, X^{0,x}_s, i = 1, 2$ are uniformly bounded in $L^q(dt \otimes dP)$ for any constant $q \in (1, 2)$ and fixed $x \in \mathbb{R}$, i.e.,

$$E\left[\int_0^T \left|H^i_0(s, X^{0,x}_s)\right|^q ds\right] = \int_{[0,T] \times \mathbb{R}} \left|H^i_0(s, y)\right|^q \mu(0,x; s, dy) ds \leq C,$$

where $\mu(0,x; s, dx)$ is the law of $X^{0,x}_s$. As a result, there exists a sub-sequence $\{n_k\}$ (we still denote it by $n$ for simplicity) and a $B((0,T) \otimes \mathcal{B}(\mathbb{R}))$-measurable deterministic function $H_i(s,x)$ such that:

$$H^i_0 \rightarrow H_i \text{ weakly in } L^q([0,T] \times \mathbb{R}; \mu(0,x; s, dx) ds), \quad i = 1, 2, q \in (1, 2).$$

Let us now illustrate how we pass from the weak convergence to strong convergence. Take $i = 1$ for example. For each $(t,x) \in [0,T] \times \mathbb{R}, \delta > 0$ and integer $k, m, n$, using (11) yields,

$$|u^{1,n}(t,x) - u^{1,m}(t,x)| \leq E\int_t^{t+\delta} H^1_0(s, X^{1,x}_s) - H^1_0(s, X^{1,x}_s) ds$$

$$+ E\int_{t+\delta}^T (H^1_0(s, X^{1,x}_s) - H^1_0(s, X^{1,x}_s))1_{(|X^{1,x}_s| \leq k)} ds$$

$$+ E\int_{t+\delta}^T (H^1_0(s, X^{1,x}_s) - H^1_0(s, X^{1,x}_s))1_{(|X^{1,x}_s| > k)} ds.$$ 

The first term on the right-hand side is bounded by $C\delta^{\frac{k+1}{2}}$, which is obtained by Young’s inequality and (17). For the third component, it follows from Young’s inequality, Markov inequality and the result (17) that it is bounded by $Ck^{\frac{k+1}{2}}$, while the second part is exactly the following one,
\[
\left| \int_{\mathbb{R}}^{T} \left( H_1^n(s, \eta) - H_1^m(s, \eta) \right) 1_{|\eta| \leq k} \frac{1}{\sqrt{2\pi (s-t)}} e^{-\frac{(s-t)^2}{2\sigma^2}} \, ds \, d\eta \right|
= \left| \int_{\mathbb{R}}^{T} \left( H_1^n(s, \eta) - H_1^m(s, \eta) \right) 1_{|\eta| \leq k} \frac{\sqrt{\sigma}}{\sqrt{2\pi s}} e^{-\frac{\sigma^2}{2s}} \, ds \, d\eta \right|.
\]

But for any \((t, x)\) the function \((s, \eta) \in [t + \delta, T] \times \mathbb{R} \mapsto \frac{x^2}{2s} e^{-\frac{(s-t)^2}{2\sigma^2}} \) is bounded. Thus by the weak convergence result (18), we have that the term above converges to 0 as \(n, m \to \infty\). Therefore, \((u^{1,n}(t, x))_{n \geq 1}\) is a Cauchy sequence for each \((t, x) \in [0, T] \times \mathbb{R}\). Then there exists a measurable function \(u^1\) on \([0, T] \times \mathbb{R}\) such that \(\lim_{n \to \infty} u^{1,n}(t, x) = u^1(t, x)\) for each \((t, x) \in [0, T] \times \mathbb{R}\).

Taking into account (10) and the polynomial growth property of function \(u^{1,n}\), we obtain by Lebesgue dominated convergence theorem that \((u^{1,n,0,x}_s = u^{1,n}(s, X^{0,x}_s))_{s \leq T, n \geq 1}\) converges to \((Y^1_s = u^1(s, X^0_s))_{s \leq T}\) in \(H^2_T(\mathbb{R})\). Moreover, using Itô’s formula with \((Y^{1,n,0,x}_s = Y^{1,n,0,x}_s)_{s \leq T}\), we get in a standard way that \((Z^{1,n,0,x}_s = d^{1,n}(s, X^{0,x}_s))_{s \leq T, n \geq 1}\) is convergent to a process \((Z^1_{s \leq T})\) in \(H^2_T(\mathbb{R})\). Then, we extract a subsequence still denoted by \((Z^{1,n}_s)_{n \geq 1}\) that converges to \(Z^1\), \(dt \otimes d\mathbb{P}\)-a.e. and such that \(\sup_{n \geq 1} |Z^{1,n}_s(\omega)| \in H^2_T(\mathbb{R})\). Repeating the previous procedure for processes \((Y^{2,n,0,x}_s = Y^{2,n,0,x}_s)_{s \leq T}\) of player \(\pi_2\) gives the existence of processes \((Y^2_{s \leq T}, Z^2_{s \leq T})\). Finally, we have also the convergence in \(S^2_T(\mathbb{R})\) of the sequence \((Y^{1,n,0,x}_s)_{n \geq 0}\) (resp. \((Y^{2,n,0,x}_s)_{n \geq 0}\)) toward \(Y^1\) (resp. \(Y^2\)) and thus \(Y^1\) and \(Y^2\) are continuous processes.

**Step 4.** Finally, let us show the existence of processes \(\theta^2\) (resp. \(\theta^1\)) valued on \(V\) (resp. \(U\)) such that \(Y^1, Z^1, u^1, \theta^1, i = 1, 2, 1\), verify the requirements of Proposition 2. In this step, we delete the superscript \(0, x\) for convenience.

First recall \(H^2_T\) of (12) and note that \(\Phi_n(Z^{1,n}_s) \Phi_n(f(s, X_s)) + \Phi_n(Z^{2,n}_s) \Phi_n(f(s, X_s)) - \Phi_n(Z^{1}_s) f(s, X_s) + Z^{2}_s f(s, X_s)\) is \(dt \otimes d\mathbb{P}\)-a.e. since \(Z^{1,n}_{s \to \infty} \to Z^1, d\mathbb{P} \otimes \mathbb{P}\), for any \(x \in \mathbb{R}\). \(\Phi_n(x) \to \rightarrow \infty\) and finally by the continuity of \(p \in \mathbb{R} \mapsto p\). The rest part in the expression of \(H^2_T\) is:

\[
\Phi_n(Z^{1,n}_s) \overline{v}(Z^{2,n}_s) = \Phi_n(Z^{1,n}_s) \overline{v}(Z^{2,n}_s) 1_{|Z^{2,\omega}_s| = 0} + \Phi_n(Z^{1,n}_s) \overline{v}(Z^{2,n}_s) 1_{|Z^{2,\omega}_s| = 0}.
\]

The term \(\Phi_n(Z^{1,n}_s) \overline{v}(Z^{2,n}_s) 1_{|Z^{2,\omega}_s| = 0}\) converges to \(Z^1 \overline{v}(Z^2_{s \infty}) 1_{|Z^{2,\omega}_s| = 0}\) \(d\mathbb{P} \otimes \mathbb{P}\)-a.e., since \(\overline{v}\) is continuous at any point different from 0 and \(Z^{2,n}_s \to Z^2_{s \infty}\) \(d\mathbb{P} \otimes \mathbb{P}\)-a.e. Let us next define an \(F_t\)-progressively measurable process \((\theta^1_{s \leq T})\) valued on \(V\) as the weak limit in \(H^2_T(\mathbb{R})\) of some subsequence \((\overline{v}(Z^{2,\omega}_s) 1_{|Z^{2,\omega}_s| = 0})_{s \leq T, k \geq 0}\). The weak limit exists since \(\overline{v}(Z^{2,\omega}_s)\) is bounded. Then the sequence \((\Phi_n(Z^{1,n}_s) \overline{v}(Z^{2,n}_s) 1_{|Z^{2,\omega}_s| = 0})_{s \leq T, k \geq 0}\) converges weakly in \(H^2_T(\mathbb{R})\) to \((Z^2_{s \infty})_{s \leq T, k \geq 0}\). Therefore \((\overline{v}(Z^{2,\omega}_s) 1_{|Z^{2,\omega}_s| = 0})_{s \leq T, k \geq 0}\) converges weakly to \(H^2_T(\mathbb{R})\) in \(H^2_T(\mathbb{R})\).

Let us now show for any stopping time \(\tau\), we have:

\[
\left| \int_{0}^{\tau} H^1_{s \leq T}(s, X_s) \, ds \right| \to_{k \to \infty} \left| \int_{0}^{\tau} H^1_{s \leq T}(s, X_s) \, Z^1_{s \infty}, \theta^2 \left| ds \text{ weakly in } L^2(\Omega, d\mathbb{P}) \right. \right|
\]

As explained at the beginning of this step, we only need to show the weak convergence of \(\int_{0}^{\tau} \Phi_n(Z^{1,n}_s) \overline{v}(Z^{2,n}_s) 1_{|Z^{2,\omega}_s| = 0}\) \(d\mathbb{P} \otimes \mathbb{P}\) as \(k \to \infty\). Obviously, we have:

\[
\int_{0}^{\tau} \Phi_n(Z^{1,n}_s) \overline{v}(Z^{2,n}_s) 1_{|Z^{2,\omega}_s| = 0}\, ds = \int_{0}^{\tau} \left( \Phi_n(Z^{1,n}_s) - Z^{1}_s \right) \overline{v}(Z^{2,n}_s) 1_{|Z^{2,\omega}_s| = 0}\, ds + \int_{0}^{\tau} Z^{2}_s \overline{v}(Z^{2,n}_s) 1_{|Z^{2,\omega}_s| = 0}\, ds.
\]

On the right-hand side, the first term converges to 0 by Lebesgue’s dominated convergence theorem. Below, we will give the weak convergence in \(L^2(\Omega, d\mathbb{P})\) of the second term. For any random variable \(\xi \in L^2(\Omega, F_T, d\mathbb{P})\), we need to show:

\[
E \left[ \int_{0}^{\tau} Z^1_{s \leq T} \overline{v}(Z^{2,n}_s) 1_{|Z^{2,\omega}_s| = 0}\, ds \right] \to E \left[ \int_{0}^{\tau} Z^1_{s \leq T} \overline{v}(Z^{2,n}_s) 1_{|Z^{2,\omega}_s| = 0}\, ds \right] \text{ as } k \to \infty.
\]

Thanks to martingale representation theorem, there exists an \(F_T\)-progressively measurable process \((\eta_k)_{s \leq T} \in H^2_T(\mathbb{R})\) such that, \(\xi = E[\xi] + \int_{0}^{T} \eta_k dB_s\). Therefore,
\[
E\left[ \xi \int_0^T Z_s^1 \tilde{\nu}_n \left( Z_s^2 \right) 1_{\left\{ Z_s^2 = 0 \right\}} ds \right] = E\left[ E[\xi] \int_0^T Z_s^1 \tilde{\nu}_n \left( Z_s^2 \right) 1_{\left\{ Z_s^2 = 0 \right\}} ds \right]
\]
\[
+ E\left[ \int_0^T \eta_s dB_s \int_0^T Z_s^1 \tilde{\nu}_n \left( Z_s^2 \right) 1_{\left\{ Z_s^2 = 0 \right\}} ds \right].
\]

Notice that \( E[\xi] E\left[ \int_0^T Z_s^1 \tilde{\nu}_n \left( Z_s^2 \right) 1_{\left\{ Z_s^2 = 0 \right\}} ds \right] \) converges to \( E[\xi] E\left[ \int_0^T Z_s^1 \tilde{\nu}_n \left( Z_s^2 \right) 1_{\left\{ Z_s^2 = 0 \right\}} ds \right] \) as \( k \to \infty \), since \( \left( Z_s^1 \right)_{s \leq T} \in \mathcal{H}^2 \) and \( \tilde{\nu}_n \left( Z_s^2 \right) 1_{\left\{ Z_s^2 = 0 \right\}} \to \theta_s^2 \) weakly in \( \mathcal{H}^1 \). Next, by Itô's formula,
\[
E\left[ \int_0^T \eta_s dB_s \int_0^T Z_s^1 \tilde{\nu}_n \left( Z_s^2 \right) 1_{\left\{ Z_s^2 = 0 \right\}} ds \right] = E\left[ \int_0^T \left( \int_0^T \eta_s dB_u \right) Z_s^1 \tilde{\nu}_n \left( Z_s^2 \right) 1_{\left\{ Z_s^2 = 0 \right\}} ds \right]
\]
\[
+ E\left[ \int_0^T \left( \int_0^T Z_u^1 \tilde{\nu}_n \left( Z_u^2 \right) 1_{\left\{ Z_u^2 = 0 \right\}} du \right) \eta_s dB_s \right].
\]

The later one on the right-hand side is 0, since \( \int_0^T \eta_s dB_u \) by \( \psi_s \) for any \( s \in [0, \tau] \). Then for any integer \( \kappa > 0 \), we have:
\[
E\left[ \int_0^T \psi_s Z_s^1 \left( \tilde{\nu}_n \left( Z_s^2 \right) - \theta_s^2 \right) 1_{\left\{ Z_s^2 = 0 \right\}} ds \right] \leq E\left[ \int_0^T \psi_s Z_s^1 \left( \tilde{\nu}_n \left( Z_s^2 \right) - \theta_s^2 \right) 1_{\left\{ \left| \psi_s \right| \leq \kappa \right\}} 1_{\left\{ Z_s^2 = 0 \right\}} ds \right]
\]
\[
+ E\left[ \int_0^T \psi_s Z_s^1 \left( \tilde{\nu}_n \left( Z_s^2 \right) - \theta_s^2 \right) 1_{\left\{ \left| \psi_s \right| \geq \kappa \right\}} 1_{\left\{ Z_s^2 = 0 \right\}} ds \right].
\]

On the right-hand side of the above equation, the first component converges to 0, which is the consequence of \( \tilde{\nu}_n \left( Z_s^2 \right) 1_{\left\{ Z_s^2 = 0 \right\}} \to \kappa \to \infty \theta_s^2 \) weakly in \( \mathcal{H}^1 \). For the second component, considering both \( \left( \tilde{\nu}_n \left( Z_s^2 \right) \right)_{s \leq \tau} \) and \( \theta_s^2 \) are bounded, it is smaller than \( C E\left[ \int_0^T \left| \psi_s Z_s^1 \right| 1_{\left\{ \left| \psi_s \right| \geq \kappa \right\}} ds \right] \), which obviously converges to 0 as \( \kappa \to \infty \). Thus (19) holds.

Besides, we also have \( \int_0^T Z_s^1 dB_s \to k \to \infty \int_0^T Z_s^1 dB_s \) in \( L^2 (\Omega, d\mathbb{P}) \), which is obtained from the convergence of \( \left( Z_s^1 \right)_{k \geq 0} \) to \( Z \) in \( \mathcal{H}^2 \) and the isometric property. Then by observing the approximation BSDE (9) in a forward way, i.e. for any stopping time \( \tau \),
\[
Y_\tau^1 = Y_0^1 - \int_0^\tau H_t^0 (s, X_s, Z_s^1, \theta_s^2) ds + \int_0^\tau Z_s^1 dB_s,
\]
combing with the convergence of \( \left( Y_s^1 \right)_{s \leq \tau} \) to \( (Y_s^1)_{s \leq \tau} \) in \( S^2_t (\mathcal{F}) \), we infer that
\[
P \text{-a.s. } Y_\tau^1 = Y_0^1 - \int_0^\tau H_t^0 (s, X_s, Z_s^1, Z_s^2, \theta_s^2) ds + \int_0^\tau Z_s^1 dB_s \quad \text{for every stopping time } \tau.
\]

As \( \tau \) is arbitrary, then the processes appearing in the two sides are indistinguishable, i.e., \( P \)-a.s.
\[
\forall t \leq T, \quad Y_t^1 = Y_0^1 - \int_0^t H_s^0 (s, X_s, Z_s^1, Z_s^2, \theta_s^2) ds + \int_0^t Z_s^1 dB_s.
\]
On the other hand, \( Y_t^1 = g_1 (X_T) \), then
\[
P \text{-a.s. } \forall t \leq T, \quad Y_t^1 = g_1 (X_T) + \int_t^T H_t^0 (s, X_s, Z_s^1, Z_s^2, \theta_s^2) ds - \int_t^T Z_s^1 dB_s.
\]
Similarly, for player \( \pi_2 \), there exists an \( \mathcal{F}_t \)-progressively measurable process \( \left( \theta_t^1 \right)_{s \leq \tau} \) valued on \( U \), such that
\[ P\text{-a.s. } \forall t \leq T, \quad Y^2_t = g_2(X_T) + \int_t^T H^*_2(s, X_s, Z^1_s, Z^2_s, \theta^1_s) \, ds - \int_t^T Z^2_s \, dB_s. \]

The proof is completed. \(\square\)

As a result of Theorem 1 and Proposition 2, we obtain the main result of this Note.

**Theorem 2.** The bang–bang nonzero-sum stochastic differential game has a Nash equilibrium point.

**Remark 1.** Theorem 2 can be generalized to the following frameworks:

(i) in the drift term \(\Gamma\), one can replace \(u\) (resp. \(v\)) by \(h(u)\) (resp. \(l(v)\)) where \(h\) and \(l\) are continuous functions;

(ii) the dynamics of the process \(X^0, x\) of (1) may contain a diffusion term \(\sigma\) with appropriate properties;

(iii) in the same way one can deal with multidimensional diffusion processes \(X^0, x\).

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