Non-Archimedean Whitney stratifications

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Abstract

We define ‘t-stratifications’, a strong notion of stratifications for Henselian-valued fields $K$ of equi-characteristic 0, of equi-characteristic 0, and prove that they exist. In contrast to classical stratifications in Archimedean fields, t-stratifications also contain non-local information about the stratified sets. For example, they do not only see the singularities in the valued field, but also see those in the residue field.

Like Whitney stratifications, t-stratifications exist for different classes of subsets of $K^n$, for example, algebraic subvarieties or certain classes of analytic subsets. The general framework are definable sets (in the sense of model theory) in a language that satisfies certain hypotheses.

We give two applications. First, we show that t-stratifications in suitable valued fields $K$ induce classical Whitney stratifications in $\mathbb{R}$ or $\mathbb{C}$; in particular, the existence of t-stratifications implies the existence of Whitney stratifications.

Second, we show how, using t-stratifications, one can determine the ultra-metric isometry type of definable subsets of $\mathbb{Z}_p^n$ for $p$ sufficiently big. For those $p$, this proves a conjecture stated in a previous article. In particular, this yields a geometric explanation of why Poincaré series are rational.

1. Introduction

Over the fields $\mathbb{R}$ and $\mathbb{C}$, a very useful tool to describe singularities of algebraic or analytic sets are Whitney stratifications; see, for example, [1, 14]. The definition of a Whitney stratification can be translated in a straightforward way to non-Archimedean local fields and in [2], it has been proved that in this sense, Whitney stratifications also exist in the $p$-adics $\mathbb{Q}_p$. In this article, we introduce t-stratifications: another kind of stratifications which exist in Henselian-valued fields and whose regularity conditions are, in a certain sense, much stronger and in particular strictly non-local. To get a first impression, suppose that we have a set $X \subseteq K^n$ for some valued field $K$. Any ball $B \subseteq K$ around 0 is an additive subgroup of $K$ and we can consider the image $X_B$ of $X$ in the quotient $(K/B)^n$. A t-stratification for $X$ simultaneously describes the singularities of all those images $X_B$ (in some sense which we will make precise).

If we take $B$ to be the maximal ideal of the valuation ring, then the residue field $k$ of $K$ is a subset of $K/B$, so in particular, if $k \subseteq K$ and $X = V(K) \subseteq K^n$ is an affine variety defined over $k$, then a t-stratification of $X$ induces a stratification of $V(k) = X_B \cap k^n$. If $k$ is $\mathbb{R}$ or $\mathbb{C}$, and under some additional assumptions about $K$, one can show that this stratification of $V(k)$ is a Whitney stratification in the classical sense. In this way, although t-stratifications and classical Whitney stratifications live in different worlds, we will prove that the existence of t-stratifications genuinely implies the existence of Whitney stratifications.

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A different point of view is that the existence of a t-stratification of a set $X \subseteq K^n$ is a strong statement about the isometry type of $X$ (with respect to the ultrametric induced by the valuation on $K$). This leads to the original motivation for this article. The main conjecture of [9] is a complete classification of algebraic (or, more precisely, definable) sets $X \subseteq \mathbb{Z}_p^n$ up to isometry. The present results yield a proof of this conjecture for big $p$ (depending on $X$). To such a set $X \subseteq \mathbb{Z}_p^n$, one can associate a Poincaré series, a formal powers series, which has been proved to be a rational function [6]. This Poincaré series only depends on the isometry type of $X$ and the classification of isometry types yields a new, geometric proof of its rationality for big $p$.

The main result of this article is the existence of t-stratifications in various contexts. To formulate this precisely, we introduce some notation. Let $K$ be a Henselian-valued field of equi-characteristic 0 (that is, both, $K$ and its residue field have characteristic 0). We will later fix a suitable class $\mathcal{C}$ of subsets of $K^n$. One example of a suitable class is the class of all subvarieties of $K^n$ (not necessarily closed or irreducible; so ‘subvariety’ means: locally closed in the Zariski topology); other possible classes $\mathcal{C}$ are the class of definable subsets in a suitable first-order language. In particular, there are classes $\mathcal{C}$ including analytic subsets of $K^n$.

The goal is to understand the singular locus of sets $X \in C$. Classically, the main theorem states that given such a set $X \subseteq K^n$, $K^n$ can be partitioned into subsets $S_0, \ldots, S_n \in C$ with $\dim S_d = d$ such that near any point $x \in S_d$, $X$ is ‘non-singular in $d$ directions’. Classically, this statement is formalized using local trivializations. Our result is of a similar nature; we obtain that on a suitable ball $B$ around $x \in S_d$, the family of sets $(S_d, S_{d+1}, \ldots, S_n, X)$ is ‘$d$-translatable’, which is a pretty strong notion of trivialization. However, in contrast to local trivializations in the classical sense, we require that $B$ be rather cumbersome; so, let me just say that definable means something like ‘piecewise algebraic’; in particular, if $\phi$ is a definable map with $n$-dimensional domain, then the Zariski closure of its graph is also $n$-dimensional.

We call the family of sets $(S_d, S_{d+1}, \ldots, S_n, X)$ ‘$d$-translatable’ on a ball $B \subseteq K^n$ if there exists a definable isometry $\phi: B \to B$ and a $d$-dimensional vector space $V \subseteq K^n$ such that each set $\phi(S_d \cap B), \ldots, \phi(S_n \cap B), \phi(X \cap B)$ is, as a subset of $B$, translation invariant in direction $V$, that is, it is the intersection of $B$ with a union of cosets of $V$. In less formal terms, $d$-translatability means that there exists a local trivialization via a risometry.

Now, we can formulate a first precise version of the main theorem.
Theorem 1.1. For every set $X \subseteq K^n$ in the class $\mathcal{C}$, there exists a t-stratification of $K^n$ reflecting $X$, that is, a partition $(S_i)_{0 \leq i \leq n}$ of $K^n$ with $S_i \in \mathcal{C}$ such that for each $d \leq n$, we have the following:

(i) $\dim S_d = d$ or $S_d = \emptyset$;
(ii) for any ball $B \subseteq S_d \cup \cdots \cup S_n$, the family $(S_d, \ldots, S_n, X)$ is $d$-translatable on $B$.

The ‘full version’ of this theorem (formulated in the language of model theory) is Theorem 4.12. Corollary 4.13 is a reformulation which is uniform in the field $K$ and which also works in sufficiently large positive characteristic. For readers not familiar with model theory, Theorem 6.6 is a reformulation of Corollary 4.13 in a purely algebraic context. Note that these full versions (Theorems 4.12 and 6.6 and Corollary 4.13) also yield the existence of t-stratifications uniformly in families.

One of the strengths of Theorem 1.1 is that we obtain $d$-translatability not just on a neighborhood of each point of $S_d$ but on any ball disjoint from $S_0 \cup \cdots \cup S_{d-1}$. However, a drawback of this is that the choice of the $S_d$ can be pretty uncanonical. (I have been told that in Archimedean fields, Mostowski’s Lipschitz stratifications have a similar kind of uncanonicity. Indeed, there seems to be a link between t-stratifications and Lipschitz stratifications.) Here are some examples.

Example 1.2. For $X = \{(x, y) \in K^2 \mid xy = 0\}$, the obvious stratification works: $S_0 = \{(0, 0)\}$, $S_1 = X \setminus S_0$, and $S_2 = K^2 \setminus X$. Now consider the curve $X' = \{(x, y) \in K^2 \mid xy = a\}$ for some $a \in K \setminus \{0\}$. Then $X'$ is smooth (in some naive sense), and this fits to the fact that each $x \in X'$ has a 1-translatable neighborhood. However, one can check that for any ball $B$ around $(0, 0)$ with $B \cap X' \neq \emptyset$, $X'$ is not 1-translatable on $B$. Thus to obtain a t-stratification, we are forced to choose a point $s_0 \in B$ and to set $S_0 := \{s_0\}$. This choice is very uncanonical: if, say, $v(a) = 0$, then any $s_0 \in C_K$ works. (After that, $S_1$ and $S_2$ can be defined as expected: $S_1 := X'$, $S_2 :=$ the remainder.) The intuition behind this phenomenon is that ‘from far away, $X'$ looks very similar to $X$', which has a singularity at $(0, 0)$. Thus, $X'$ has an ‘almost singularity’ near $(0, 0)$, and such almost singularities are detected by t-stratifications. A more precise formulation of the similarity between $X$ and $X'$ is that for $B$ as above, the images $X_B, X'_B \subseteq (K/B)^2$ from the beginning of the introduction are equal.

In a model theoretic setting, where $\mathcal{C}$ the class of $\emptyset$-definable sets in a suitable language, even for $X \subseteq K$, the existence of t-stratifications is not entirely trivial, as the next example shows.

Example 1.3. Suppose that $X$ is a $\emptyset$-definable ball in $K$. Then $X$ is not 1-translatable on any ball $B$ which strictly contains $X$, hence for $(S_0, S_1)$ to be a t-stratification reflecting $X$, any such $B$ must contain an element of $S_0$. This could be achieved by putting some element of $X$ into $S_0$, but it is well possible that $X \cap \text{ncl}(\emptyset) = \emptyset$, that is, $X$ is disjoint from every finite $\emptyset$-definable set. However, one can show (for example, using some version of cell decomposition) that this can happen only if $X$ is an open ball, say, $X = \{x \in K \mid v(x - a) > \lambda\}$; moreover, one can then find a finite $\emptyset$-definable set $S_0$ containing an element $x$ with $v(x - a) \geq \lambda$. In particular, $S_0 \cap B \neq \emptyset$ for every ball $B \supseteq X$, so this does the job.

Now let us compare Theorem 1.1 to classical Whitney stratifications (their definition is recalled in Subsection 7.2). Fix a point $x$ in some stratum $S_d$ of a t-stratification and suppose that $B$ is a ball containing $x$ on which we have $d$-translatability, that is, we have a $d$-dimensional vector space $V \subseteq K^n$ as explained above Theorem 1.1. In Whitney stratifications, each stratum is smooth, so tangent spaces exist. With t-stratifications, we do not
know whether actual tangent spaces exist, but $V$ can be seen as an ‘approximate tangent space’ of $S_j$ at $x$ (approximate tangent spaces are made more precise in Subsection 3.1). Moreover, for any $j \geq d$ and any $y \in B \cap S_j$, $d$-translatability on $B$ also implies that an approximate tangent space of $S_j$ at $y$ approximately contains $V$. Thus, formulated sloppily, we have: for any $x \in S_d$, any $j \geq d$, and any $y \in S_j$ close enough to $x$, $T_yS_j$ approximately contains $T_xS_d$. If one replaces ‘approximately contains’ by ‘contains in the limit for $y \to x$’, then this statement is essentially the classical Condition (a) of Whitney.

‘Containing approximately’ sounds like a weaker condition and indeed, t-stratifications do not necessarily satisfy the straightforward translation of the Whitney conditions to non-Archimedean fields introduced in [2]. However, to be able to apply methods from non-standard analysis, this is exactly the right statement. As a consequence, if we let $K$ be a non-standard model of $\mathbb{R}$ or $\mathbb{C}$ (that is, a particular valued field whose residue field $k$ is equal to $\mathbb{R}$ or $\mathbb{C}$, respectively), then any t-stratification of $k^n$ induces a stratification of $k^n$ that satisfies Condition (a).

Using this method, we will prove (Theorem 7.11) that t-stratifications induce classical Whitney stratifications. For this, we also need our t-stratifications to satisfy a non-standard version of Whitney’s Condition (b). A priori, this is not true: using a kind of non-Archimedean logarithmic spiral, one can construct a t-stratification violating Condition (b). However, we are only considering t-stratifications consisting of sets in the class $C$, and inside this class, such counter-examples are excluded by the following result.

**Theorem 1.4.** For every set $X \subseteq K^n$ in our class $C$ and every $x \in K^n$, there exists a finite subset $M_x \subseteq \Gamma$ of the value group such that for any $y \in K^n$ with $\hat{v}(y-x) \notin M_x$ and any ball $B$ containing $y$ but not $x$, $X$ is ‘translatable on $B$ in direction $K \cdot (y-x)$’, that is, there exists a definable risometry $\phi: B \to B$ such that $\phi(X \cap B)$ is translation invariant in direction $K \cdot (y-x)$.

The ‘full version’ of this theorem is Theorem 7.4 and Corollary 7.6 is a non-standard version of Whitney’s Condition (b). Whereas Theorem 1.1 yields only the existence of translatability, Theorem 1.4 is a strong result about translatability in specific directions. Indeed, formulated sloppily, it says the following. Fix any $x \in K^n$. Then for almost any $y \in X$ (more precisely: for $y \in X$ at almost any distance from $x$), the approximate tangent space $T_yX$ approximately contains the line $K \cdot (y-x)$. It is important that the set of permitted distances can depend on $x$; otherwise, we would obtain that almost every tangent space approximately contains almost every line, which, of course, is absurd.

In the Archimedean setting, given a finite family of subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$, one can find a single Whitney stratification that simultaneously fits to all those sets. In valued fields, we can even treat ‘small’ infinite families with a single t-stratification. Here, ‘small’ is not meant in the sense of cardinality; instead, a family of sets is small roughly if it is parameterized by a subset of $k^t \times \Gamma^{t'}$ for some $t, t' \in \mathbb{N}$. (In contrast, a family parameterized by the valued field $K$ would be large.) To make sense of this, one needs the language of model theory, that is, the class $C$ should be a suitable class of definable sets. Being able to treat such infinite families will be crucial in the proof of Theorem 1.1 to make the induction work.

Now let us come back to the original motivation for this article, namely understanding sets up to isometry. The existence of a t-stratification $(S_i)_i$, reflecting a set $X$ implies that the isometry type of $X$ is rather simple (and even its risometry type), since all risometries appearing in Theorem 1.1 can be pieced together to one single risometry $\phi: K^n \to K^n$ such that $\phi(X)$ translation invariant in $d$ directions on any ball $B \subseteq S_d \cup \cdots \cup S_n$. (Note, however, that this $\phi(X)$ will almost never lie in $C$.) In particular, only few different risometry types
occur at all in $\mathcal{C}$, and $t$-stratifications help understanding them. This can be made more precise as follows.

Suppose we have a uniform family of sets $X_q \subseteq K^n$ in $\mathcal{C}$, parameterized by $q \in Q$ ($Q \subseteq \mathcal{C}$) and suppose we want to decide for which $q, q' \in Q$ there exists a risometry $K^n \rightarrow K^n$ sending $X_q$ to $X_{q'}$. A priori, this is a difficult task; in model theoretic terms, the induced equivalence relation on $Q$ is not definable in general. However, if we assume that each $X_q$ is equipped with a $t$-stratification $(S_{i,q})$, and we ask that these $t$-stratifications are also respected by the risometries, then the corresponding refined equivalence relation on $Q$ is definable (Proposition 3.19), and the risometry type of $(X_q, (S_{i,q}))$ can be described by a ‘finite amount of data living only in $k$ and $\Gamma$’. A slightly weaker but purely algebraic version of this statement is given in Corollary 6.7; roughly, there exist finitely many regular functions on $Q$ such that the risometry type only depends on the leading term of these functions.

The exact data needed to describe the risometry type of $(X_q, (S_{i,q}))$ can be extracted from the proof of Proposition 3.19. In this way, one could obtain a complete (but long and technical) classification of all possible risometry types. In [9], such a classification statement has been formulated for isometries instead of risometries in the case $K = \mathbb{Q}_p$. As already mentioned, we will prove that conjecture for $p$ sufficiently big (Theorem 8.3). In fact, our proof works for more general $K$; Definition 8.5 is the classification statement in such a generalized setting. To make ‘for $p$ sufficiently big’ precise, we assume that $X(K) \subseteq K^n$ is given uniformly for all $K$; then we obtain the result for all $K$ with big enough residue characteristic, where the bound may depend on $X$.

Now let me describe for which classes $\mathcal{C}$ Theorems 1.1 and 1.4 hold. In most of the article (in particular for the main Theorems 4.12 and 7.4 and their proofs), $\mathcal{C}$ will be the class of definable sets in a suitable language $\mathcal{L}$ expanding the language $\mathcal{L}_{\text{Hen}}$ of valued fields (see Definition 2.16). Hypotheses 2.21 and 7.1 list the precise conditions on $\mathcal{L}$ we will need (Hypothesis 7.1 is only needed for Theorem 1.4). We will prove that these Hypotheses hold in a quite general setting, namely in any expansion $\mathcal{L}$ of $\mathcal{L}_{\text{Hen}}$ by an analytic structure in the sense of [3] (see Propositions 5.6, 5.12 and 7.2). A concrete example of such an analytic structure on a complete valued field of rank 1 is given in Example 5.2; it includes all functions given by restricted power series with integer coefficients. Many more examples of analytic structures can be found in [3].

Most of the assumptions in Hypotheses 2.21 and 7.1 are immediate consequences of [3]; in the case $\mathcal{L}_{\text{Hen}}$, they also follow easily from classical results like quantifier elimination and cell decomposition. However, the ‘higher-dimensional Jacobian property’ is more subtle and to prove it, we will inductively use the existence of $t$-stratifications in lower dimensions. To make this possible, the formulation of Theorem 4.12 states precisely in which dimension the Jacobian property is needed as a prerequisite. Even in the base case $\mathcal{L}_{\text{Hen}}$, I do not know a better proof than the above one, where we work with the ‘trivial’ analytic structure (whose existence, by the way, is not so trivial; see [3, Section 4.6]).

I promised that for $\mathcal{C}$, we can also take the class of varieties. Varieties are definable in any of the above languages, but the question is whether we can obtain a $t$-stratification consisting of varieties and not just of definable sets. We will show that for $\mathcal{L} = \mathcal{L}_{\text{Hen}}$, this is indeed true, even if the set $X$ we started with is only definable; Corollary 6.5 is a model-theoretic formulation of that result, and Theorem 6.6 is essentially a precise algebraic formulation of Theorem 1.1 in the case of varieties.

Here is an overview over the article.

In Section 2, we introduce the basic notation and tools. The first three subsections are independent of model theory. In particular, in Subsection 2.2, we define and describe the ‘higher dimensional leading term structures’ $\mathcal{R}^n$, which are ubiquitous in this article. In Subsections 2.4 and 2.5, we fix the model theoretic setup and assumptions.

The purpose of Section 3 is to define translatability and $t$-stratifications and to prove some first properties. The most important ones are that the restriction of a $t$-stratifications to a
suitable affine subspace of a ball is again a t-stratification (Lemma 3.16), that being a t-stratification is a first-order property (Proposition 3.19(1)) and that t-stratifications can be used to understand risometry types (Proposition 3.19(2)).

The next section contains the main part of the proof of Theorem 4.12, namely that under Hypothesis 2.21, t-stratifications do exist. There is a sketch of the proof at the beginning of the section. Subsection 4.4 contains some direct corollaries and the last subsection gives other characterizations of what it means for a t-stratification to reflect a set; these will be useful for applications and for the inductive proof of the higher-dimensional Jacobian property.

Up to there, we do not yet know whether Hypothesis 2.21 can be satisfied at all; in Section 5, we will show that it holds in any field with analytic structure in the sense of [3]. The easy part is Proposition 5.6, which treats everything except the Jacobian property; the latter is proved in Proposition 5.12.

The remaining sections give some variants and applications of the main result, mostly under some additional assumptions. In Section 6, we show how to obtain t-stratifications such that for each $d$, $S_0 \cup \cdots \cup S_d$ is closed in a suitable topology. In the pure valued field language $\mathcal{L}_{\text{Hen}}$, this can be applied to the Zariski topology, which yields the algebraic version of the main result.

In the next section, we show how our result implies the existence of classical Whitney stratifications (Theorem 7.11). To this end, we first prove the valued field version of Whitney’s Condition (b) (Theorem 7.4, Corollary 7.6); this needs an additional hypothesis on the language $\mathcal{L}$ (Hypothesis 7.1), which also holds in any field with analytic structure (Proposition 7.2).

Finally, we show how the existence of t-stratifications implies the main conjecture of [9] about sets up to isometry in $\mathbb{Q}_p$ for $p \gg 1$ (Section 8) and we list some open questions concerning enhancements of the main result (Section 9).

2. The setting

2.1. Basic notation

In most of the article, we will work in a fixed valued field. We use the following notation.

**Notation 2.1.** Throughout the article, $K$ is a valued field, $\mathcal{O}_K$ is the valuation ring, $\mathcal{M}_K \subseteq \mathcal{O}_K$ is its maximal ideal, $k$ is the residue field, $\Gamma$ is the value group, $v: K \to \Gamma \cup \{\infty\}$ is the valuation and $\text{res}: \mathcal{O}_K \to k$ is the residue map.

Moreover, we define $\hat{v}: K^n \to \Gamma \cup \{\infty\}, \hat{v}(x_1, \ldots, x_n) := \min_i v(x_i)$ (as in the introduction). We also write $\text{res}$ for the canonical map $\mathcal{O}_K^n \to k^n$, and, more generally, for $X(\mathcal{O}_K) \to X(k)$, where $X$ is any variety defined over $\mathcal{O}_K$.

We apply the map $\text{res}$ to subvector spaces of $K^n$ as follows.

**Definition 2.2.** A vector space $V \subseteq K^n$ can be considered as an element of the Grassmannian $\mathbb{G}_{n,d}(K) = \mathbb{G}_{n,d}(\mathcal{O}_K)$ (where $d = \dim V$). We write $\text{res}(V)$ for its image in $\mathbb{G}_{n,d}(k)$, considered as a subvector space of $k^n$. Equivalently, we have $\text{res}(V) = \{\text{res}(x) \mid x \in V \cap \mathcal{O}_K^n\}$.

Vice versa, if $V \subseteq k^n$ is a vector space, then any vector space $\tilde{V} \subseteq K^n$ with $\text{res}(\tilde{V}) = V$ will be called a lift of $V$.

The map $\hat{v}$ on $K^n$ satisfies the ultrametric triangle inequality. Balls in $K^n$ are defined using this 'metric', that is, a ball is the same as a 'cube': a product of $n$ balls in $K$ of the same radius. Here is the precise notion of balls we will use.
DEFINITION 2.3. (1) An open ball in $K^n$ is a set of the form $B(a, \delta) := \{x \in K^n \mid \hat{v}(x-a) > \delta\}$ for $a \in K^n$ and $\delta \in \Gamma \cup \{-\infty\}$.

(2) A closed ball is a set of the form $B(a, \delta) := \{x \in K^n \mid \hat{v}(x-a) \geq \delta\}$ for $a \in K^n$ and $\delta \in \Gamma$.

(3) A ball is either an open or a closed ball.

(4) The radius of a ball $B$ is the above $\delta$; we denote it by $\text{rad}_B(B)$ if $B$ is an open ball and by $\text{rad}_c(B)$ if $B$ is a closed ball.

Thus, do consider $K^n$ as a ball (an open one), but we do not consider points as balls, and neither do we allow arbitrary cuts in $\Gamma$ as radii of balls. The reason to have different notation $\text{rad}_B$ and $\text{rad}_c$ is that if $\Gamma$ is discrete, then any ball $B \neq K^n$ can be considered both as an open or as a closed ball and $\text{rad}_B(B) < \text{rad}_c(B)$.

From time to time, given a ball $B$ we will need to consider the ball $B'$ of the same radius containing the origin. We do not introduce a special notation for this; instead, note that $B' = B - B = \{ b-b' \mid b, b' \in B \}$.

We will work a lot with projections $\pi: K^n \to K^d$ to some subset of the coordinates. The corresponding projection $k^n \to k^d$ at the level of the residue field will be denoted by $\bar{\pi}$. By $\pi^\vee: K^n \to K^{n-d}$, we will denote the projection to the complementary set of coordinates. Often, we will consider restricted coordinate projections $\pi: B \to K^d$ for some subset $B \subseteq K^n$ (most of the time, a ball); in that case, $\bar{\pi}$ still denotes the entire projection $k^n \to k^d$.

Given a coordinate projection $\pi: B \to K^n$, any fiber $\pi^{-1}(x)$ (for $x \in \pi(B)$) can be identified with a subset of $K^{n-d}$ via $\pi^\vee$. Using this, any definition made for $K^{n-d}$ can also be applied to fibers of coordinate projections. As an example, this yields a notion of a ball inside a fiber $\pi^{-1}(x)$.

2.2. Higher dimensional leading term structures and their linear algebra

The ‘leading term structure’ is usually defined as $RV := \{0\} \cup K^\times/(1 + M_K)$. We will need the following higher dimensional version of it.

DEFINITION 2.4. Define $RV(n) := K^n/\sim$, where $x \sim y \Leftrightarrow (\hat{v}(x-y) > \hat{v}(x) \vee x = y = 0)$; we write $\hat{v}_n: K^n \to RV(n)$ for the canonical map and $\hat{v}_{RV}$ for the map from $RV(n)$ to $\Gamma \cup \{\infty\}$ satisfying $\hat{v}_{RV} \circ \hat{v}_n = \hat{v}$. Instead of $RV(1)$, we also write $RV$.

The following is a (more general) coordinate free version of this definition.

DEFINITION 2.5. Let $L$ be a free $O_K$-module and set $V_L := K \otimes_{O_K} L$. First, define the valuation $\hat{v}: V_L \to \Gamma \cup \{\infty\}$ by setting $\hat{v}(rx) := \hat{v}(r)$ for any $r \in K, x \in L \setminus M_K L$. (This is well-defined and satisfies the ultrametric triangle inequality.) Then set $RV_L := V_L/\sim$ where $x \sim y \Leftrightarrow (\hat{v}(x-y) > \hat{v}(x) \vee x = y = 0)$, write $\hat{v}_L: V_L \to RV_L$ for the canonical projection, and write $\hat{v}_{RV}$ for the map from $RV_L$ to $\Gamma \cup \{\infty\}$ satisfying $\hat{v}_{RV} \circ \hat{v}_L = \hat{v}$.

It is easy to check that we have $RV(n) = RV_{O^{\text{ab}}}$ and that the two definitions of $\hat{v}$ and $\hat{v}_n$ on $K^n$ coincide. In most of the article, we will work with coordinates anyway, so we will not bother giving coordinate free definitions of everything. Note, however, that using Definition 2.5, one obtains the following for free.

LEMMA 2.6. If $L$ is a free $O_K$-module, then any $\phi \in \text{Aut}(L)$ induces a map $\phi: RV_L \to RV_L$ satisfying $\phi(\hat{v}_L(x)) = \hat{v}_L(\phi(x))$ for $x \in V_L$ (where we also write $\phi$ for the induced map $V_L \to V_L$). In particular, any $M \in \text{GL}_n(O_K)$ induces a map $M: RV(n) \to RV(n)$. 
We will need one more notation.

**Definition 2.7.** For \( x \in K^n \setminus \{0\} \), let the direction of \( x \) be the one-dimensional subspace \( \text{dir}(x) := \text{res}(K \cdot x) \) of \( k^n \), considered as an element of the projective space \( \mathbb{P}^{n-1}k \). Notationally, we almost always treat \( \text{dir}(x) \) as a representative \( y \in \text{res}(K \cdot x) \) of the actual direction. Whenever we use this notation, we make sure that the particular choice of \( y \) does not matter.

One easily verifies that the direction map factors over \( \Gamma V(n) \); we write \( \text{dir}_{\Gamma V} \) for the corresponding map \( \Gamma V(n) \to \mathbb{P}^{n-1}k \) (that is, \( \text{dir}_{\Gamma V} \circ \hat{v} = \text{dir} \)).

A lot of commutative diagrams can be drawn, showing how all these maps fit together. The following two lemmas shows only some of them. Lemma 2.8 in particular shows that \( \Gamma V(n) \) can also be defined as the quotient of \( K^n \) by a suitable group action, generalizing the one-dimensional case \( \Gamma V = K/(1 + M_K) \).

**Lemma 2.8.** Let \( U_n \) be the kernel of the map \( \text{res}: \text{GL}_n(\mathcal{O}_K) \to \text{GL}_n(k) \). Then we have the following (commutative) diagrams, where \( G \ract X \) means that \( G \) acts on \( X \), and each straight line \( G \ract X \to Y \) is exact in the sense that \( Y \) is the quotient of \( X \) by the action of \( G \).

\[
\begin{array}{ccc}
U_n & \xrightarrow{\text{res}} & \text{GL}_n(k) \\
\downarrow \text{id} & & \downarrow \text{res} \\
K^n & \ract_{\hat{v}} & \Gamma \cup \{\infty\} \\
\downarrow \hat{v} & & \downarrow \hat{v}_{\Gamma V} \\
\text{RV}(n) & \xrightarrow{\text{res}} & \mathcal{G}_{n,d}(k) \\
\end{array}
\]

**Proof.** None of this is difficult to show. As an example, let us verify that if \( V, V' \subseteq K^n \) are \( d \)-dimensional vector spaces with \( \text{res}(V) = \text{res}(V') \), then there exists \( M \in U_n \) with \( MV = V' \).

Choose any basis \((b_i)_{i \leq d}\) of \( \text{res}(V) \) and extend it to a basis \((b_i)_{i \leq n}\) of \( k^n \). Choose preimages \((v_i)_{i \leq n}\) and \((v'_i)_{i \leq n}\) of \((b_i)_{i \leq n}\) in \( \mathcal{O}_K^n \) such that for \( i \leq d \), we have \( v_i \in V \) and \( v'_i \in V' \). Then the linear map sending \((v_i)_{i \leq n}\) to \((v'_i)_{i \leq n}\) sends \( V \) to \( V' \), and it lies in \( U_n \) since it induces the identity on \( k^n \).

**Lemma 2.9.** We also have the following commutative diagram, where each straight line is exact. Here, \( \Gamma V(0) \ract \Gamma V(n) \setminus \{0\} \) is the action induced by the scalar multiplication \( K^\times \ract K^n \setminus \{0\} \) and \( \Gamma \ract \Gamma \) is the action by translation. The middle horizontal line is exact in the sense that \( k^n \setminus \{0\} = \hat{v}_{\Gamma V}^{-1}(0) \).

\[
\begin{array}{ccc}
K^\times & \ract & \Gamma V(0) \to \Gamma \\
\downarrow \text{id} & & \downarrow \text{id} \\
k^n \setminus \{0\} & \ract & \Gamma V(n) \setminus \{0\} \\
\downarrow \text{dir}_{\Gamma V} & & \downarrow \text{dir}_{\Gamma V} \\
\mathbb{P}^{n-1}k & = & \mathbb{P}^{n-1}k
\end{array}
\]

**Proof.** Easy.
Note that the top-right square of the diagram implies that \( \hat{v}_{RV} : RV^{(n)} \setminus \{0\} \to \Gamma \) is a fibration with fibers ‘isomorphic’ to \( k^n \setminus \{0\} \).

Here are some more basic properties of \( RV^{(n)} \) and the maps defined above.

**Lemma 2.10.**

1. If \( a_1, a_2 \in K^n \) satisfy \( \hat{v}(a_1 + a_2) = \min\{\hat{v}(a_1), \hat{v}(a_2)\} \), then \( \hat{v}(a_1) \) and \( \hat{v}(a_2) \) together determine \( \hat{v}(a_1 + a_2) \), that is, for any other \( a_1', a_2' \in K^n \) with \( \hat{v}(a_1') = \hat{v}(a_1) \), we have \( \hat{v}(a_1' + a_2') = \hat{v}(a_1 + a_2) \).

2. If \( a_1, a_2 \in K^n \) satisfy \( \text{dir}(a_1) \neq \text{dir}(a_2) \), then \( \text{dir}(a_1 + a_2) \) lies in the \( k \)-vector space spanned by \( \text{dir}(a_1) \) and \( \text{dir}(a_2) \).

3. Suppose that \( \pi : K^n \to K^d \) is a coordinate projection, \( \bar{\pi} : k^n \to k^d \) is the corresponding projection at the level of the residue field, and \( a \in K^n \setminus \{0\} \). Then we have \( \hat{v}(\pi(a)) = \hat{v}(a) \) if and only if \( \bar{\pi}(\text{dir}(a)) \neq 0 \). Moreover, in that case \( \bar{\pi}(\text{dir}(a)) = \text{dir}(\pi(a)) \), and if \( a' \in K^n \) is another element with \( \pi(a') = \pi(a) \) and \( \text{dir}(a') = \text{dir}(a) \), then we have \( \hat{v}(a') = \hat{v}(a) \).

4. For any subvector space \( V \subseteq K^n \), we have \( \hat{v}(V) = \hat{v}(\text{res}(V)) \).

5. Let \( \langle \cdot, \cdot \rangle \) denote the standard scalar product, both on \( K^n \) and on \( k^n \). Then for any \( a, b \in K^n \) we have \( \hat{v}(\langle a, b \rangle) \geq \hat{v}(a) + \hat{v}(b) \) and we have the equivalence \( \hat{v}(\langle a, b \rangle) > \hat{v}(a) + \hat{v}(b) \iff \langle \text{dir}(a), \text{dir}(b) \rangle = 0 \).

**Proof.** Easy. \( \square \)

2.3. Risometries

Let us now have a look at the notion of risometry, which already appeared in the introduction. Its definition can be written down nicely using the multidimensional \( \hat{v} \)-map introduced in Definition 2.4.

**Definition 2.11.** For \( X_1, X_2 \subseteq K^n \), a risometry from \( X_1 \) to \( X_2 \) is a bijection \( \phi : X_1 \to X_2 \) satisfying \( \hat{v}(\phi(x) - \phi(x')) = \hat{v}(x - x') \) for any \( x, x' \in X_1 \). If \( \phi \) is such a risometry, we use the following terminology.

(i) For maps \( \chi_i \), with domain \( X_i \) (for \( i = 1, 2 \)), we say that \( \phi \) is a risometry from \( \chi_1 \) to \( \chi_2 \) (or: \( \phi \) sends \( \chi_1 \) to \( \chi_2 \)) if \( \chi_1 = \phi \circ \chi_2 \). More generally, if \( (\theta_{1, \nu})_\nu \) and \( (\theta_{2, \nu})_\nu \) are tuples where for each \( \nu, \theta_{1, \nu} \) and \( \theta_{2, \nu} \) are either maps with domain \( X_1 \) and \( X_2 \) or subsets of \( X_1 \) and \( X_2 \), then we say that \( \phi \) sends \( (\theta_{1, \nu})_\nu \) to \( (\theta_{2, \nu})_\nu \) (and we sometimes write \( \phi : (\theta_{1, \nu})_\nu \to (\theta_{2, \nu})_\nu \) if \( \phi \) sends \( \theta_{1, \nu} \) to \( \theta_{2, \nu} \) for each \( \nu \).

(ii) If \( \chi \) is a map whose domain contains \( X_1 \cup X_2 \), we say that \( \phi \) respects \( \chi \) if it sends \( \chi|_{X_1} \) to \( \chi|_{X_2} \), and \( \chi \) respects a set \( Y \subseteq K^n \) if it sends \( Y \cap X_1 \) to \( Y \cap X_2 \).

As in ‘\( rv \)’, the ‘\( r \)’ in ‘risometry’ stands for ‘residue field’. The condition about \( \hat{v} \) in Definition 2.11 already implies injectivity, so any map satisfying that condition is a risometry from its domain to its image. Note also that the composition of risometries is again a risometry; in particular, the risometries from a set to itself form a group.

**Remark 2.12.** Lemma 2.6 implies that if \( \phi : X \to Y \) is a risometry and \( M \in \text{GL}_n(\mathcal{O}_K) \), then we also have a risometry \( M \circ \phi \circ M^{-1} : M(X) \to M(Y) \). This will be used from time to time to ‘without loss change coordinates’. In particular, any matrix \( M \in \text{GL}_n(k) \) can be lifted to a matrix \( \tilde{M} \in \text{GL}_n(\mathcal{O}_K) \), so we can apply any coordinate transformation at the level of the residue field.
Remark 2.13. The group $U_n \subseteq \text{GL}_n(\mathcal{O}_K)$ introduced in Lemma 2.8 consists exactly of those linear maps $K^n \to K^n$ which are isometries. In particular, if $V_1, V_2 \subseteq K^n$ are vector spaces with $\text{res}(V_1) = \text{res}(V_2)$, then there exists a isometry $K^n \to K^n$ sending $V_1$ to $V_2$.

Cartesian products of isometries are again isometries; the following lemma strengthens this a bit. Each of its statements is almost trivial (so we omit the proof), but together, they will be useful to construct isometries.

Lemma 2.14. Let $V_1, V_2 \subseteq K^n$ be subvector spaces such that we have a direct sum decomposition $\text{res}(V_1) \oplus \text{res}(V_2) = K^n$; write $\pi_i: K^n \to K^n$ for the projection with image $V_i$ and kernel $V_{3\ldots i}$ (for $i = 1, 2$).

1) Suppose that $X \subseteq K^n$ and that $\phi_1, \phi_2: X \to K^n$ are maps satisfying

$$\hat{v}(\phi_1(x) - \phi_1(x') - \pi_i(x - x')) > \hat{v}(x - x') \quad \text{for every } x, x' \in X, \ x \neq x'. \quad (\ast_i)$$

Then the map $x \mapsto \phi_1(x) + \phi_2(x)$ is a isometry from $X$ to its image.

2) If $\phi$ is a map satisfying $(\ast_i)$ and $\psi$ is a isometry (with suitable domain and image), then $\phi \circ \psi$ and $\psi \circ \phi$ also satisfy $(\ast_i)$; in particular, $\pi_i \circ \psi$ and $\psi \circ \pi_i$ satisfy $(\ast_i)$.

Next, we describe isometries between finite sets and how such isometries can be extended to larger sets. In the following, for $x \in K^n$ and $T \subseteq K^n$, the notation $\hat{r}v(x - T)$ means $\{\hat{r}v(x - t) \mid t \in T\}$.

Lemma 2.15. Let $T \subseteq K^n$ be a finite set.

1) The only isometry $T \to T$ is the identity. (In particular, between two different finite sets, there is at most one isometry.)

2) For $x_1, x_2 \in K^n$ with $x_1 \neq x_2$, the following are equivalent:

(a) there exists a isometry $\phi: K^n \to K^n$ with $\phi(T) = T$ and $\phi(x_1) = x_2$;

(b) $B(x_1, \geq \hat{v}(x_1 - x_2)) \cap T = \emptyset$;

(c) $\hat{r}v(x_1 - T) = \hat{r}v(x_2 - T)$.

In particular, the fibers of the map sending $x \in K$ to the set $\hat{r}v(x - T)$ are exactly the singletons $\{t\}$ for $t \in T$ and the maximal balls $B \subseteq K$ that are disjoint from $T$.

3) A map $\phi: K^n \to K^n$ which is the identity on $T$ is a isometry if and only if for each maximal ball $B \subseteq K^n \setminus T$, the restriction $\phi|_B$ is a isometry from $B$ to itself.

Proof. (2) ‘(a) $\Rightarrow$ (c)’ and ‘(b) $\Rightarrow$ (a)’ are trivial. (For the latter, define $\phi$ to be the translation by $x_2 - x_1$ on $B(x_1, \geq \hat{v}(x_1 - x_2))$ and the identity everywhere else.)

‘(c) $\Rightarrow$ (b)’: Without loss, $\hat{v}(x_1 - x_2) = 0$ and $x_1, x_2 \in \mathcal{O}_K^n$. Suppose for contradiction that $T_0 := T \cap \mathcal{O}_K^n$ is non-empty. The assumption (c) implies $\hat{r}v(x_1 - T_0) = \hat{r}v(x_2 - T_0)$ and hence $\text{res}(x_1 - T_0) = \text{res}(x_2 - T_0)$. This implies

$$\sum_{\tilde{t} \in \text{res}(T_0)} (\text{res}(x_1) - \tilde{t}) = \sum_{\tilde{t} \in \text{res}(T_0)} (\text{res}(x_2) - \tilde{t}).$$

Adding $\sum_{\tilde{t} \in \text{res}(T_0)} \tilde{t}$ and then dividing by $|\text{res}(T_0)|$ on both sides yields $\text{res}(x_1) = \text{res}(x_2)$, which contradicts $\hat{v}(x_1 - x_2) = 0$.

The ‘in particular’ part of (2) follows from (b) $\Rightarrow$ (c).

1) If $\phi: T \to T$ is a isometry, then for any $t \in T$ we have $\hat{r}v(t - T) = \hat{r}v(\phi(t) - (T)) = \hat{r}v(\phi(t) - T)$. Suppose that $\phi(t) \neq t$. Then (2) ‘(c) $\Rightarrow$ (b)’ yields $B(t, \geq \hat{v}(t - \phi(t))) \cap T = \emptyset$, which contradicts $t \in T$. 
Unless specified otherwise, ‘definable’ will always mean definable with parameters. There will be some results concerning $\emptyset$-definable sets (of the form: for some $\emptyset$-definable $X$, there exists a $\emptyset$-definable $Y$ . . .). Our general assumptions about the language and the theory will always allow to add parameters to the language (see Remark 2.22), so the reason to write ‘$\emptyset$-definable’ is only to emphasize that $Y$ is definable over the same parameters as $X$.

We start by fixing a basic language $L_{\text{Hen}}$ for valued fields and a corresponding theory $T_{\text{Hen}}$. In almost all of the article, we care only about the language up to interdefinability; however, we will have to be precise about the sorts of the language. We use the notation introduced in Subsections 2.1 and 2.2.

**Definition 2.16.** (1) Let $L_{\text{Hen}}$ be the language consisting of one sort $K$ for the valued field with the ring language, all sorts $RV^{eq}$ with the corresponding canonical maps between them, and the map $rv: K \to RV$. More precisely, the sorts of $RV^{eq}$ are all sets of the form $X/\sim$, where $X \subseteq RV^{k}$ is $\emptyset$-definable and $\sim$ is a $\emptyset$-definable equivalence relation on $X$, and the corresponding canonical map is $X \to X/\sim$.

(2) We call $RV^{eq}$ the auxiliary sorts. By an auxiliary set resp. element, we mean a subset resp. element of an auxiliary sort.

(3) Let $T_{\text{Hen}}$ be the theory of Henselian-valued fields of equi-characteristic 0 in the language $L_{\text{Hen}}$.

Notationally, we will often treat $RV^{eq}$ as the union of all auxiliary sorts. In particular, by a ‘definable map $\chi: K^{n} \to RV^{eq}$’, we mean a definable map whose target is an arbitrary auxiliary sort (and similarly for definable sets $Q \subseteq RV^{eq}$).

**Remark 2.17.** One easily checks that $k$, $\Gamma$, and $RV^{(n)}$ are auxiliary sorts. (For the latter, note that the map $K^{n} \to RV^{(n)}$ factors over $RV^{n}$.)

**Notation 2.18.** If $(X_{q})_{q \in Q}$ is a definable family of sets (or maps), then $\lbrack X_{q} \rbrack$ denotes a ‘code’ for $X_{q}$. More precisely, if $X_{q}$ is defined by a formula $\phi(x, q)$, then there exists a definable map $f: Q \to Q'$ for some definable set $Q'$ (possibly imaginary) and a formula $\psi(x, y)$ such that $\psi(x, f(q))$ also defines $X_{q}$ and such that $f(q)$ is a canonical parameter for $X_{q}$. We set $\lbrack X_{q} \rbrack := f(q)$. (Of course, this involves some choices.)

Most of the time when we use Notation 2.18, we make sure that $Q'$ can be chosen in a non-imaginary sort of $L_{\text{Hen}}$.

**2.5. Requirements on the theory**

In most of the article, we will not work with $T_{\text{Hen}}$ and $L_{\text{Hen}}$ themselves, but with an expansion $T$ of $T_{\text{Hen}}$ in a language $L \supseteq L_{\text{Hen}}$ (which has the same sorts as $L_{\text{Hen}}$). In particular, the main theorem will be proved in any expansion $T$ of $T_{\text{Hen}}$ having certain properties, which will be listed in Hypothesis 2.21. Variants of these properties have already been introduced and
described in [3, 4]: ‘b-minimality’ is a list of axioms designed to yield cell decomposition and a notion of dimension, and the ‘Jacobian property’ imposes conditions on definable functions in one variable. Our version of the Jacobian property includes definable functions in several variables. In addition to these two properties, we will require that zero-dimensional sets are finite and that RV is stably embedded.

We start by defining our version of the Jacobian property. Note that even in the one-variable case, it does not entirely agree with [3, Definitions 6.3.5 and 6.3.6].

**Definition 2.19.** (1) Let $X \subseteq K^n$ be a set. We say that a map $f : X \to K$ has the Jacobian property (on $X$), if either it is constant, or there exists a $z \in K^n \setminus \{0\}$ such that for every $x, x' \in X$ with $x \neq x'$, we have

$$v(f(x) - f(x') - \langle z, x - x' \rangle) > \hat{v}(z) + \hat{v}(x - x').$$

(2) We say that an expansion $T$ of $\text{Hen}$ has the Jacobian property if for every model $K \models T$, for every set $A \subseteq K \cup \text{RV}^{eq}$, for every $n \in \mathbb{N}$, and for every $A$-definable map $f : K^n \to K$, there exists an $A$-definable map $\chi : K^n \to Q \subseteq \text{RV}^{eq}$ such that for each $q \in Q$, if $\chi^{-1}(q)$ contains a ball, then $f|_{\chi^{-1}(q)}$ has the Jacobian property.

(3) If (2) only holds for $n \leq n_0$ (where $n_0 \in \mathbb{N}$), then we say that $T_{\text{Hen}}$ has the Jacobian property up to dimension $n_0$.

We will associate dimensions to definable sets in Definition 2.27; the condition in (2) that $\chi^{-1}(q)$ contains a ball will be equivalent to $\dim(\chi^{-1}(q)) = n$.

**Remark 2.20.** In Definition 2.19(1), replacing $z$ by any other $z' \in K^n$ satisfying $\hat{rv}(z) = \hat{rv}(z')$ does not change the validity of the inequation, since $v((z, x - x') - (z', x - x')) > \hat{v}(z) + \hat{v}(x - x')$ by Lemma 2.10(5).

Now we can summarize the prerequisites needed for Theorem 4.12. We also introduce a notation for a weakening of the hypothesis, which we will need in an inductive argument.

**Hypothesis 2.21.** We assume that $T$ is an expansion of $T_{\text{Hen}}$ in a language $\mathcal{L} \supseteq \mathcal{L}_{\text{Hen}}$ that has the same sorts as $\mathcal{L}_{\text{Hen}}$, with the following properties.

1. The sort $\text{RV}$ is stably embedded, that is, in every model of $T$, every definable subset of $\text{RV}^n$ is definable using only parameters from $\text{RV}$.
2. In every model $K \models T$, every definable map from $\text{RV}$ to $K$ has finite image.
3. For every model $K \models T$, for every set $A \subseteq K \cup \text{RV}^{eq}$ of parameters, and for every $A$-definable set $X \subseteq K$, there exists a finite, $A$-definable set $S_0 \subseteq K$ such that every ball $B \subseteq K \setminus S_0$ is either contained in $X$ or disjoint from $X$.
4. The theory $T$ has the Jacobian property (Definition 2.19).

For $n \in \mathbb{N}$, we write ‘Hypothesis 2.21$_n$’ for the following weakening of this hypothesis.

1. (4) $T$ has the Jacobian property up to dimension $n$.

2. (4') For every model $K \models T$, for every set $A \subseteq K \cup \text{RV}^{eq}$, and for every $A$-definable map $f : K \to K$, there exists an $A$-definable map $\chi : K^n \to Q \subseteq \text{RV}^{eq}$ such that for each $q \in Q$, $f|_{\chi^{-1}(q)}$ is either injective or constant.
Note that Condition (4″) is relevant only in the case \( n = 0 \), since it follows from the Jacobian property in dimension 1.

**Remark 2.22.** All conditions in Hypothesis 2.21 remain true if we add constant symbols to the language. In particular, any result proved for \( \emptyset \)-definable sets automatically also holds over any parameter set \( A \). This will be used throughout the paper without further mentioning.

**Remark 2.23.** By Hypothesis 2.21(1), for any definable set \( Q \subseteq RV^{eq} \), we may assume \( \Gamma Q \subseteq RV^{eq} \), and similarly \( \Gamma f \subseteq RV^{eq} \) for a definable map \( f: Q \to RV^{eq} \).

**Remark 2.24.** Hypothesis 2.21(3) exactly says that t-stratifications exist for subsets of \( K \).

Note also that by Lemma 2.15, the condition relating \( S_0 \) and \( X \) is equivalent to: \( X \) is a union of fibers of the map \( K \to RV^{eq}, x \mapsto \Gamma rv(x - S_0) \).

Hypothesis 2.21 does not mention the notion of b-minimality from [4] explicitly. Since we will use results from [4] about the existence of a good notion of dimension, we conclude this subsection by proving that Hypothesis 2.21 implies b-minimality. The following definition is [4, Definition 2.2.1], applied to the context of valued fields with auxiliary sorts \( RV^{eq} \). Note that it is not exactly the same as [3, Definitions 6.3.1], since there, only \( RV \) is used as an auxiliary sort. We will come back to this difference when it becomes an issue, namely in Section 5, when we prove that analytic structures in the sense of [3] satisfy Hypothesis 2.21.

**Definition 2.25.** An expansion \( T \) of \( T_{Hen} \) is b-minimal over \( RV^{eq} \) if for every model \( K \models T \) and every set \( A \subseteq K \cup RV^{eq} \) of parameters, the following holds.

1. For every \( A \)-definable set \( X \subseteq K \), there exists an \( A \)-definable map \( \chi: X \to Q \subseteq RV^{eq} \) such that every fiber \( \chi^{-1}(q) \) (for \( q \in Q \)) is either a point or an open ball.
2. There exists no surjective definable map from an auxiliary set to an open ball \( B \subseteq K \).
3. For every \( A \)-definable \( X \subseteq K \) and \( f: X \to K \), there exists an \( A \)-definable map \( \chi: X \to Q \subseteq RV^{eq} \) such that for each \( q \in Q \), \( f|_{\chi^{-1}(q)} \) is either injective or constant.

**Lemma 2.26.** Hypothesis 2.21 implies b-minimality over \( RV^{eq} \).

**Proof.** (2) follows from Hypothesis 2.21(2), (3) is exactly (4″) from Hypothesis 2.21, and (1) can be deduced as follows. Let \( X \subseteq K \) be given and let \( S_0 \subseteq K \) be a finite set as in Hypothesis 2.21(3). Then by Lemma 2.15, \( X \) is a union of fibers of the map \( \chi: K \to RV^{eq}, x \mapsto \Gamma rv(x - S_0) \) and each such fiber is either a point or a ball; thus \( \chi|_X \) does the job.

As mentioned in the introduction, we will prove (Proposition 5.12) that Henselian-valued fields with analytic structure in the sense of [3] satisfy Hypothesis 2.21. The proof of the Jacobian property will inductively use the existence of t-stratifications in lower dimensions. To make this precise, everywhere in the proof of the existence of t-stratifications, we will specify for which \( n \) Hypothesis 2.21\(_n\) is needed. (Most of the time, Hypothesis 2.21\(_0\) will already be enough.)

### 2.6. First consequences: dimension and spherically completeness

We assume Hypothesis 2.21. By Lemma 2.26, this implies b-minimality (over \( RV^{eq} \)), and by [4], this implies the existence of a good notion of dimension of definable sets, which in particular satisfies the axioms given in [13].
**Definition 2.27.** Let $X \subseteq K^n$ be a definable set. The *dimension* $\dim X$ is the maximal $d$ such that there exists a coordinate projection $\pi : K^n \to K^d$ such that $\pi(X)$ contains a ball. We set $\dim \emptyset := -\infty$. For $x \in K^n$, the *local dimension* of $X$ at $x$ is $\dim_x X := \min\{\dim(X \cap B(x, \gamma)) \mid \gamma \in \Gamma\}$.

**Remark 2.28.** By Hypothesis 2.21(3), any subset of $K$ is either finite or it contains a ball, hence zero-dimensional subsets of $K$ are finite. This is also true for subsets of $K^n$, as one sees by applying coordinate projections $K^n \to K$.

It is clear that dimension is definable, that is, if $X_q \subseteq K^n$ is a $\emptyset$-definable family of sets (for $q \in Q$), then $\{q \in Q \mid \dim X_q = d\}$ is $\emptyset$-definable for every $d$. Moreover, we have the following.

**Lemma 2.29** [4, 13]. *Dimension has the following properties.*

1. If $\{X_q\}_{q \in Q}$ is a definable family of subsets of $K^n$ and $Q \subseteq RV^{\text{ext}}$ is auxiliary, then $\dim \bigcup_{q \in Q} X_q = \max_q \dim X_q$.
2. If $X \subseteq K^n$, $Y \subseteq K^n$ and $f : X \to Y$ are definable and each fiber $f^{-1}(y)$ has dimension $d$ (in particular, $f$ is surjective) then $\dim X = \dim Y + d$.

The following property of local dimension is [8, Theorem 3.1]. (That theorem only requires dimension to satisfy some very general axioms which follow directly from our definition and Lemma 2.29(2).)

**Lemma 2.30.** Let $X \subseteq K^n$ be a definable set and set $Y := \{x \in X \mid \dim_x X < \dim X\}$. Then $\dim Y < \dim X$.

Hypothesis 2.21 also implies that $K$ is ‘definable spherically complete’ in the following sense.

**Lemma 2.31.** For every definable family $\{B_q\}_{q \in Q}$ of balls $B_q \subseteq K$ which form a chain with respect to inclusion, the intersection $\bigcap_{q \in Q} B_q$ is non-empty.

**Proof.** Let such a family $\{B_q\}_{q \in Q}$ be given. We can assume that there is no smallest ball.

Let $S_q$ be the finite set obtained by applying Hypothesis 2.21(3) to $B_q$ and set $S'_q := \{x \in S_q \mid \exists \xi \in RV : x + rv^{-1}(\xi) \subseteq B_q\}$. Using compactness, we assume that $S_q$ (and hence $S'_q$) is definable uniformly in $q$; thus, the intersection $S' := \bigcup_{q \in Q} S'_q$ is zero-dimensional by Lemma 2.29(1) and hence finite by Remark 2.28.

Using that $B_q$ is a union of fibers of the map $x \mapsto rv(x - S_q)$ (by Lemma 2.15), one obtains $S'_q \neq \emptyset$ for every $q \in Q$. Choose an element $x_0 \in S'$ such that $Q' := \{q \in Q \mid x_0 \in S'_q\}$ is co-final in $Q$ (with respect to inclusion of the corresponding balls). Now consider any $q' \in Q'$ and choose $\xi \in RV$ with $C := x_0 + rv^{-1}(\xi) \subseteq B_{q'}$. Then every ball strictly containing $C$ contains $x_0$, hence in particular $x_0 \in B_q$ for every $q \in Q$ with $B_q \supseteq B_{q'}$. Since $Q'$ is co-final in $Q$, we obtain $x_0 \in \bigcap_{q \in Q} B_q$.

It will be important to us that there is no risometry from a ball $B$ to a proper subset of $B$. In spherically complete valued fields, this is true in general. We will need it only for definable risometries; to obtain that, definable spherically completeness is be enough. The proof goes via the following ‘definable Banach fixed point theorem’.
Lemma 2.32. Let $B \subseteq K^n$ be a ball and suppose that $f : B \rightarrow B$ is definable and contracting in the sense that for any $x_1, x_2 \in B$ with $x_1 \neq x_2$, $\hat{v}(f(x_1) - f(x_2)) > \hat{v}(x_1 - x_2)$. Then $f$ has (exactly) one fixed point.

Proof. Suppose that $f(x) \neq x$ for all $x \in B$. For $x \in B$, set

$$B_x := B(x, \hat{v}(x - f(x))).$$

For two different points $x, x' \in B$, the assumption $\hat{v}(f(x) - f(x')) > \hat{v}(x - x')$ implies $\hat{v}(x - x') \geq \min\{\hat{v}(x - f(x)), \hat{v}(x' - f(x'))\}$ and hence either $B_x$ contains $x'$ or vice versa. In particular, $B_x \cap B_{x'} \neq \emptyset$, so all balls $B_x$ form a chain under inclusion. By Lemma 2.31 (applied to each coordinate projection), their intersection $\bigcap_{x \in B} B_x$ contains an element $x_0$. Then $f(x_0) = x_0$, since otherwise, the assumption $\hat{v}(f(x_0) - f(f(x_0))) > \hat{v}(x_0 - f(x_0))$ implies $x_0 \notin B_f(x_0)$.

Lemma 2.33. Let $B \subseteq K^n$ be a ball and let $f : B \rightarrow X$ be a definable risometry with $X \subseteq B$. Then $X = B$.

Proof. Let $x_0 \in B$ be given; the idea is to find a preimage of $x_0$ by Newton-approximation (although $f$ might not be differentiable, it behaves as if the derivative would be approximately 1). For $x \in B$, set $g(x) := x + x_0 - f(x)$. Obviously, a fixed point of $g$ is a preimage of $x_0$, so we just need to verify that $g$ is contracting. Indeed,

$$g(x_1) - g(x_2) = (x_1 - x_2) - (f(x_1) - f(x_2)),$$

and since $f$ is a risometry, we have $\hat{r}v(x_1 - x_2) = \hat{r}v(f(x_1) - f(x_2))$ and thus $\hat{v}(g(x_1) - g(x_2)) > \hat{v}(x_1 - x_2)$.

3. t-Stratifications

In this section, we will make the definition of $t$-stratification precise and we will prove a bunch of basic properties. Throughout the section, we assume that $T$ is a theory satisfying Hypothesis 2.210 and that $K$ is a model of $T$. We start by looking more closely at the notion of translatability.

3.1. Translatability

Recall that a lift of a subspace $V \subseteq k^n$ is any subspace $\hat{V} \subseteq K^n$ with $\text{res}(\hat{V}) = V$.

Definition 3.1. Suppose that $B_0 \subseteq K^n$ is any definable set, $\chi : B_0 \rightarrow \text{RV}^{\text{res}}$ is a definable map, and $B \subseteq B_0$ is a ball (open or closed).

(1) For a subspace $\hat{V} \subseteq K^n$, we say that $\chi$ is $\hat{V}$-translation invariant on $B$ if for any $x, x' \in B$ with $x \neq x' \in \hat{V}$, we have $\chi(x) = \chi(x')$.

(2) For a subspace $V \subseteq k^n$, we say that $\chi$ is $V$-translatable on $B$ if there exists a lift $\hat{V} \subseteq K^n$ of $V$ and a definable risometry $\phi : B \rightarrow B$ such that $\chi \circ \phi$ is $\hat{V}$-translation invariant on $B$; $\phi$ will be called a straightener (of $\chi$ on $B$).

(3) For an integer $d \in \{0, \ldots, n\}$, we say that $\chi$ is $d$-translatable on $B$ if there exists a $d$-dimensional $V \subseteq k^n$ such that $\chi$ is $V$-translatable on $B$.

By Remark 2.13, in (2) the choice of $\hat{V}$ doesn’t matter, that is, if $\chi$ is $V$-translatable, then for any lift $\hat{V}$ of $V$, we can find a straightener $\phi$ such that $\chi \circ \phi$ is $\hat{V}$-translation invariant.
One can easily modify Definition 3.1 to obtain a notion of translatability (on a ball $B$) for definable sets $X \subseteq K^n$. More generally, we will use the following convention, which fits to the terminology introduced in Definition 2.11.

**Convention 3.2.** Several definitions of properties $P$ of definable maps $\chi: B_0 \to RV^\text{eq}$ for $B_0 \subseteq K^n$ (like Definition 3.1) will also be applied to subsets of $B_0$ and to tuples of maps and sets, by first turning such an object into a map, as follows.

1. A definable set $X \subseteq B_0$ has property $P$ if and only if the map $\chi: B_0 \to RV^\text{eq}$ sending $X$ to $0 \in k$ and $B_0 \setminus X$ to $1 \in k$ has property $P$.

2. If $\theta = (\theta_1, \ldots, \theta_\ell)$ is a tuple of maps and sets, then we first replace each set $\theta_i$ by the corresponding map as in (1) and then consider the map $\chi: x \mapsto (\theta_1(x), \ldots, \theta_\ell(x)); \theta$ has property $P$ if and only if $\chi$ has it.

Note that with this convention, the notion of translatability of a tuple of sets agrees with the one introduced before Theorem 1.1.

It is clear that if a map $\chi$ is $V$-translatable on a ball $B$ for some $V \subseteq K^n$, then it is also $V'$-translatable on $B$ for any subspace $V' \subseteq V$. Also, since risometries preserve balls, $V$-translatability on $B$ implies $V$-translatability on $B'$ for any sub-ball $B' \subseteq B$. The following fact is less obvious than it looks. (Its proof needs definability of the involved maps.)

**Lemma 3.3.** Suppose that a definable map $\chi: B \to RV^\text{eq}$ is both, $V_1$ and $V_2$-translatable for some $V_1, V_2 \subseteq k^n$. Then $\chi$ is $(V_1 + V_2)$-translatable.

**Proof.** Without loss, $V_1 \cap V_2 = 0$. Choose a complement $V_3$ of $V_1$ in $k^n$ with $V_2 \subseteq V_3$, and let $\tilde{V}_i$ be lifts of $V_i$ for $i = 1, 2, 3$ with $\tilde{V}_2 \subseteq \tilde{V}_3$. For $i = 1, 3$, let $\pi_i: K^n \to K^n$ be the projection to $\tilde{V}_i$ with kernel $\tilde{V}_{4-i}$.

Without loss, $\chi$ is $\tilde{V}_1$-translation invariant and $0 \in B$. Let $\phi: B \to B$ be a definable risometry such that $\chi \circ \phi$ is $\tilde{V}_2$-translation invariant and define $\psi: B \to B$, $x \mapsto \phi(\pi_3(x)) + \pi_1(x)$. By Lemma 2.14, $\psi$ is a risometry. Its image is contained in $B$, so by Lemma 2.33, it is equal to $B$. We claim that $\chi \circ \psi$ is $(\tilde{V}_1 + \tilde{V}_2)$-translation invariant. Indeed, suppose that $x, x' \in B$ satisfy $x - x' \in \tilde{V}_1 + \tilde{V}_2$ (or, equivalently, $\pi_3(x - x') \in \tilde{V}_2$). Then $\chi(\psi(x)) = \chi(\phi(\pi_3(x)) + \pi_1(x)) = \chi(\phi(\pi_3(x'))) = \chi(\phi(\pi_3(x') + \pi_1(x')) = \chi(\psi(x'))$. \hfill \Box

By this lemma, for every definable map $\chi: B_0 \to RV^\text{eq}$ and every ball $B \subseteq B_0$, there exists a (unique) maximal space in which $\chi$ is translatable on $B_0$; we fix a notation for it.

**Definition 3.4.** Let $\chi: B_0 \to RV^\text{eq}$ be a definable map for some $B_0 \subseteq K^n$ and let $B \subseteq B_0$ be a ball. We write $\text{tsp}_B(\chi)$ for the maximal subvector space of $k^n$ such that $\chi$ is $\text{tsp}_B(\chi)$-translatable on $B$ (‘tsp’ stands for ‘translatability space’). Using Convention 3.2, we also define $\text{tsp}_B(\theta)$ when $\theta$ is a set or a tuple of sets and maps.

Using this definition, we have: $\chi$ is $V$-translatable on $B$ if and only if $V \subseteq \text{tsp}_B(\chi)$, and $\chi$ is $d$-translatable if and only if $\dim \text{tsp}_B(\chi) \geq d$.

To understand a $V$-translatable map $\chi$, we will often choose a projection $\pi: K^n \to K^d$ and work fiberwise. This only works well if the fibers of $\pi$ are ‘sufficiently transversal’ to lifts of $V$. It will be handy to fix, once and for all, a finite set of $\theta$-definable projections which work for all $V$. This is the purpose of the following definition.
Definition 3.5. Let $V \subseteq k^n$ be a subvector space. An exhibition of $V$ is a coordinate projection $\pi : K^n \to K^d$ inducing an isomorphism $\tilde{\pi} : V \xrightarrow{i} k^d$ (in particular, $d = \dim V$). We also say that $\pi$ exhibits $V$. If $B \subseteq K^n$ is a subset (usually a ball), then the restriction $\pi|_B$ will also be called an exhibition of $V$.

The following lemma summarizes basic facts needed to work fiberwise.

Lemma 3.6. Suppose that $B \subseteq K^n$ is a ball, $V \subseteq k^n$ is a subvector space of dimension $d$, and $\chi : B \to RV^{eq}$ is a $V$-translatable definable map. Fix an exhibition $\pi : B \to K^d$ of $V$ and a lift $\tilde{V} \subseteq K^n$ of $V$. Then we have the following.

1. There exists a definable risometry $\phi : B \to B$ satisfying $\pi \circ \phi = \pi$ such that $\chi \circ \phi$ is $\tilde{V}$-translating invariant. (In other words, $\phi$ is a straightener respecting the fibers of $\pi$.)

2. For any definable risometry $\psi : B \to B$ and any $\pi$-fiber $F = \pi^{-1}(x) \subseteq B$ (where $x \in \pi(B)$), there exists a definable risometry $\psi' : F \to F$ such that $(\chi \circ \psi)|_F = (\chi|_F) \circ \psi'$.

In (2), one can think of $\chi$ and $\chi \circ \psi$ as two different but risometric maps; from that point of view, the statement is that the restrictions of two risometric maps to a $\pi$-fiber are also risometric.

Proof of Lemma 3.6. Let $\pi_1, \pi_2 : K^n \to K^n$ be the projections with $\im \pi_1 = \ker \pi_2 = \tilde{V}$ and $\ker \pi_1 = \im \pi_2 = \ker \pi$.

1. Let $\psi$ be a straightener of $\chi$ and consider the map $x \mapsto \pi_1(x) + \pi_2(\psi^{-1}(x))$. By Lemma 2.14, it is a risometry and by Lemma 2.33, it goes from $B$ onto $B$. Its inverse is the desired straightener $\phi$.

2. Let $\phi$ be a straightener of $\chi$ satisfying $\pi \circ \phi = \pi$ and define $\phi'(x) := \pi_1(x) + \pi_2(\phi^{-1}(\psi(x)))$. By Lemmas 2.14 and 2.33, $\phi'$ is a risometry from $B$ to $B$ and by definition, we have $\pi \circ \phi' = \pi$ and (using that $\chi \circ \phi$ factors over $\pi_2$) $(\chi \circ \phi) \circ \phi^{-1} \circ \psi = (\chi \circ \phi) \circ \phi'$, so we can define $\psi'$ to be the restriction of $\phi \circ \phi'$ to $F$.

Using this, we can give an alternative characterization of translatability. Recall that for a ball $B$, $B - B$ is the ball of the same radius containing the origin and that $\dir$ was introduced in Definition 2.7.

Lemma 3.7. Let $\chi : B \to RV^{eq}$ be a definable map, $V \subseteq k^n$ a subspace and $\pi : K^n \to K^d$ an exhibition of $V$. Then $\chi$ is $V$-translatable if and only if there exists a definable family of risometries $\alpha_x : B \to B$, where $x$ runs over $\pi(B - B)$, with the following properties (for all $x, x' \in \pi(B - B)$ and all $z \in B$):

1. $\chi \circ \alpha_x = \chi$;
2. $\alpha_x \circ \alpha_{x'} = \alpha_{x+x'}$;
3. $\pi(\alpha_x(z) - z) = x$;
4. $\dir(\alpha_x(z) - z) \in V$ if $x \neq 0$.

Proof. \(\Rightarrow\): Choose a straightener $\phi$ respecting the fibers of $\pi$ (using Lemma 3.6(1)) and let $\tilde{V}$ be the corresponding lift of $V$. For any $x \in \pi(B - B)$, denote by $\alpha'_x : B \to B$ the translation by the unique element of $\pi^{-1}(x) \cap \tilde{V}$. Then $\alpha'_x$ satisfies $\chi \circ \phi \circ \alpha'_x = \chi \circ \phi$ and (2)–(4), and from this, one deduces that $\alpha_x := \phi \circ \alpha'_x \circ \phi^{-1}$ satisfies (1)–(4).

\(\Leftarrow\): Without loss, $0 \in B$, $V = k^d \times \{0\}^{n-d}$, and $\pi$ is the projection to the first $d$ coordinates. Write elements of $B$ as $(x, y) \in K^d \times K^{n-d}$. We claim that $\phi(x, y) := \alpha_x(0, y)$ is a straightener.
By (1), $\chi \circ \phi$ is $(K^d \times \{0\}^{n-d})$-translation invariant. To check that $\phi$ is a risometry, consider $(x_1,y_1), (x_2,y_2) \in B$ and set $x := x_2 - x_1$. We have $\phi(x_1,y_1) = \alpha_{x_1}(0,y_1)$ and (2) implies $\phi(x_2,y_2) = \alpha_{x_1}(\alpha_{x_1}(0,y_2))$, so since $\alpha_{x_1}$ is a risometry, it suffices to check that $\hat{v}((x_1,y_1) - (x_2,y_2)) = \hat{v}((0,y_1) - \alpha_{x_1}(0,y_2))$, but this follows from $\hat{v}(y_2 - \pi(x_1,0)) > \hat{v}(x)$, which in turn follows from (3) and (4). (Recall that $\pi: K^n \to K^{n-d}$ is the ‘complement’ of $\pi$.) \hfill \Box

**Definition 3.8.** Let $B \subseteq K^n$ be a ball, $\pi: B \to K^d$ a coordinate projection, and $\chi: B \to \text{RV}^\text{eq}$ a definable map. A definable family of risometries $(\alpha_x)_{x \in \pi(B-B)}$ from $B$ to itself satisfying (1)–(4) of Lemma 3.7 will be called a translater of $\chi$ (on $B$, with respect to $\pi$). We also apply Convention 3.2.

Characterizing translatability via translaters has the disadvantage of being more technical, but one advantage is that it avoids the (uncanonical) lift $\hat{V}$ appearing in Definition 3.1.

The following lemma says how (and when) translatability is preserved under the restriction to an affine subspace.

**Lemma 3.9.** Suppose that $\chi: B \to \text{RV}^\text{eq}$ is a $V$-translatable definable map (where $B \subseteq K^n$ is a ball and $V \subseteq K^n$) and $\rho: B \to K^d$ is a coordinate projection with $\rho(V) = k^d$ (in particular $\dim V \geq d$). Then the restriction of $\chi$ to any fiber $\rho^{-1}(y)$ (for $y \in \rho(B)$) is $(V \cap \ker \rho)$-translatable.

*Proof.* Choose an exhibition $\pi: B \to K^d$ of $V$ satisfying $\ker \pi \subseteq \ker \rho$ and let $\phi: B \to B$ be a straightener of $\chi$ respecting the fibers of $\pi$. Then $\phi$ sends any $\rho$-fiber $\rho^{-1}(y)$ to itself and thus $\phi|_{\rho^{-1}(y)}$ is a straightener of $\chi|_{\rho^{-1}(y)}$ proving $(V \cap \ker \rho)$-translatability. \hfill \Box

The next two lemmas state that translatability behaves as one would expect with respect to dimension and topological closure (using the valued field topology); we write $X^{\text{top}}$ for the topological closure of a set $X$.

**Lemma 3.10.** Suppose that $B \subseteq K^n$ is a ball, that $X \subseteq B$ is a definable set which is $V$-translatable on $B$ for some $V \subseteq K^n$, and that $\pi: B \to K^d$ exhibits $V$. Then for any $x \in \pi(B)$, we have

$$\dim X = \dim(X \cap \pi^{-1}(x)) + d.$$  

In particular, $\dim X \geq \dim V$.

*Proof.* The translaters of Lemma 3.7 can be restricted to definable bijections between the fibers $X \cap \pi^{-1}(x)$, so all of them have the same dimension. Now use Lemma 2.29(2). \hfill \Box

**Lemma 3.11.** If $X \subseteq K^n$ is $V$-translatable on a ball $B \subseteq K^n$, then so is $(X, X^{\text{top}})$.

*Proof.* Since risometries are homeomorphisms, a straightener for $X$ also straightens $X^{\text{top}}$. \hfill \Box

### 3.2. Definition of t-stratifications

We now give the general definition of a t-stratification and prove basic properties. (The ‘t’ in ‘t-stratification’ stands for ‘translatable’.) Recall that translatability has been defined precisely in Definition 3.1 and Convention 3.2.
DEFINITION 3.12. Let $B_0 \subseteq K^n$ be a ball. A t-stratification of $B_0$ is a partition of $B_0$ into definable sets $S_0, \ldots, S_n$ with the properties listed below. We write $S_{\leq d}$ for $S_0 \cup \cdots \cup S_d$ and $S_{\geq d}$ for $S_d \cup \cdots \cup S_n$:

1. $\dim S_d \leq d$;
2. for each $d$ and each ball $B \subseteq S_{\geq d}$ (open or closed), $(S_i)_{i \leq n}$ is $d$-translatable on $B$.

We say that a t-stratification $(S_i)_{i \leq n}$ reflects a definable map $\chi: B_0 \rightarrow \text{RV}^\text{eq}$ if the following stronger version of (2) holds.

(2') For each $d$ and each ball $B \subseteq S_{\geq d}$ (open or closed), $((S_i)_{i \leq n}, \chi)$ is $d$-translatable on $B$.

We define when a t-stratification reflects a set or a tuple of sets and maps using Convention 3.2.

In other words, for any ball $B \subseteq B_0$, we let $d$ be minimal with $B \cap S_d \neq \emptyset$ and require $d$-translatability on $B$. Note that this is as much as one can get: since $\dim S_d \leq d$, Lemma 3.10 implies that $(S_i)_i$ is not $(d+1)$-translatable on $B$. In particular, we have $\text{tsp}_B(S_d) = \text{tsp}_B((S_i)_i)$.

In general, if, for some $V \subseteq k^n$, two maps $\chi$ and $\chi'$ are both $V$-translatable on the same ball $B$, this does not imply that $(\chi, \chi')$ is $V$-translatable on $B$. However, we will see in Remark 4.18 that if a t-stratification $(S_i)_i$ reflects both $\chi$ and $\chi'$, then it also reflects $(\chi, \chi')$.

REMARK 3.13. If $(S_i)_i$ is a t-stratification of $B_0$ (reflecting $\chi$), then the restriction to any sub-ball of $B_0$ is also a t-stratification (reflecting the restriction of $\chi$). In the other direction, a t-stratification of $B_0 \subseteq K^n$ can be extended to a t-stratification of $K^n$ by replacing $S_n$ with $S_n \cup (K^n \setminus B_0)$, but only under the assumption that $S_0 \neq \emptyset$. This assumption is needed because in general, $(S_i)_i$ will not be 1-translatable on any ball strictly bigger than $B_0$.

In general, even if $\chi: K^n \rightarrow \text{RV}^\text{eq}$ is definable, the map $B \mapsto \text{tsp}_B(\chi)$ does not need to be definable (see Example 3.15). However, for t-stratifications, the corresponding map is definable as the following lemma states.

LEMMA 3.14. For a fixed t-stratification $(S_i)_i$ of $K^n$, the map $B \mapsto \text{tsp}_B((S_i)_i)$ is definable, uniformly for all models $K \models T$ (that is, if each $S_i$ is given by a formula, then then there exists a formula defining $B \mapsto \text{tsp}_B((S_i)_i)$ in all models).

Proof. The dimension of $\text{tsp}_B((S_i)_i)$ can be defined as the minimal $d$ with $B \cap S_d \neq \emptyset$. We claim that then, for any $x \in S_d \cap B$ and any sufficiently small ball $B'$ containing $x$, we have $\text{tsp}_B((S_i)_i) = \{ \text{dir}(x_1 - x_2) \mid x_1, x_2 \in S_d \cap B' \}$.

To prove this claim, fix an exhibition $\pi: B \rightarrow K^d$ of $W := \text{tsp}_B((S_i)_i)$. For each $\pi$-fiber $F = \pi^{-1}(y)$ (with $y \in \pi(B)$), $S_d \cap F$ is zero-dimensional by Lemma 3.10 and hence finite by Remark 2.28, so for any $x \in S_d \cap F$, we can find a ball $B' \subseteq B$ such that $B' \cap S_d \cap F = \{x\}$. By $W$-translatability on $B'$, $B' \cap S_d \cap \pi^{-1}(y')$ is a singleton for any $y' \in \pi(B')$ and we obtain the claim.

The following is an example of a set $X \subseteq K^3$ such that $\text{tsp}_B(X)$ does not depend definably on $B$. The key ingredient is that if the residue field $k$ is ‘sufficiently evil’, then whether a definable map $k \rightarrow k$ can be lifted to a definable map $O_K \rightarrow O_K$ is not a definable condition.
EXAMPLE 3.15. Recall that in the field \( \mathbb{Q} \), the subset \( \mathbb{N} \) and the exponential map \( \mathbb{Q} \times \mathbb{N} \to \mathbb{Q}, (x, y) \mapsto x^y \) are definable. Fix an elementary extension \( \mathbb{Q} \models \mathbb{Q} \) and write \( \mathbb{N} \) for the interpretation of \( \mathbb{N} \) in \( \mathbb{Q} \). Set \( K := \mathbb{Q}(t) \) and define

\[
X := \{(t^{-1}x, y, tz) \in t^{-1}\mathcal{O}_K \times \mathcal{O}_K \times t\mathcal{O}_K \mid \text{res}(z) = y^{\text{res}(x)} \}.
\]

We claim that for \( a \in \mathbb{N} \), \( X \) is \( \mathbb{Q}(0) \times \{0\} \)-translatable on the ball \( B_a := t^{-1}\text{res}^{-1}(a) \times \mathcal{O}_K^2 \) if and only if \( a \in \mathbb{N} \), which is not a definable set.

Fix \( a \in \mathbb{N} \). Since on \( B_a \), \( X \) is \( (K \times \{0\}) \)-translation invariant, the question is only whether the fiber \( X_a = \{(y, tz) \in \mathcal{O}_K \times t\mathcal{O}_K \mid \text{res}(z) = y^a \} \) is \( \mathbb{Q}(0) \times \{0\} \)-translatable on \( \mathcal{O}_K^2 \). If \( a \in \mathbb{N} \), then an easy computation shows that the map \( \mathcal{O}_K^2 \to \mathcal{O}_K^2, (y, z) \mapsto (y, z + ty^a) \) is a straightener of \( X_a \).

Now suppose that \( X_a \) is \( \mathbb{Q} \times \{0\} \)-translatable on \( \mathcal{O}_K^2 \) for some \( a \in \mathbb{N} \). Then we can find a 1-dimensional definable subset \( Y \subseteq X_a \) whose projection to the first coordinate is equal to \( \mathcal{O}_K \).

(Choose a straightener \( \phi : \mathcal{O}_K \to \mathcal{O}_K^2 \) and set \( Y := \phi(\mathcal{O}_K \times \{z_0\}) \) for a suitable \( z_0 \in \mathcal{O}_K \).) Set \( Y' := \{(y, z) \in \mathcal{O}_K^2 \mid (y, tz) \in Y \} \). Since \( \dim Y' = 1 \), there exists a polynomial \( f \in \mathcal{O}_K[y, z] \) vanishing on \( Y' \) (for example, by Lemma 6.3), and thus \( f := \text{res}(f) \) is a non-zero polynomial vanishing on \( \text{res}(Y') = \{(y, y^a) \mid y \in \mathbb{Q} \} \). However, for any \( d \in \mathbb{N} \), we have

\[
\mathbb{Q} \models \forall k \in \mathbb{N}, k > d : \text{no polynomial of degree at most } d \text{ vanishes on } \{(y, y^k) \mid y \in \mathbb{Q} \}.
\]

Since this also holds in \( \mathbb{Q} \) and since \( a > d \) for all \( d \in \mathbb{N} \), we obtain a contradiction.

A property of t-stratifications which is important for inductive arguments is that on an affine subspace of a ball which is transversal to the translatability space on that ball, they again induce t-stratifications. This is the statement of the following lemma. (It is formulated for a t-stratification reflecting a map, but of course, we can apply it to the trivial map if we are interested in a ‘pure’ t-stratification.)

**Lemma 3.16.** Let \( (S_i) \) be a t-stratification of \( B_0 \subseteq K^n \) reflecting a definable map \( \chi : B_0 \to \text{RV}^{\text{res}} \), and let \( B \subseteq B_0 \) be a ball. Let \( \pi : B \to K^d \) be an exhibition of \( \text{tsp}_B((S_i)) \) and suppose that \( F = \pi^{-1}(x) \) is a \( \pi \)-fiber (for some \( x \in \pi(B) \)). Set \( T_i := S_{i+d} \cap F \) for \( 0 \leq i \leq n - d \). Then \( (T_i)_{i \leq n - d} \) is a t-stratification of \( F \subseteq B \) reflecting \( \chi|_{T_i \cap B} \).

**Proof.** By Lemma 3.10, \( \dim S_{i+d} \leq i + d \) implies \( \dim T_i \leq i \), so it remains to show the translatability condition. Consider a ball \( B' \subseteq B \) with \( B' \cap F \neq \emptyset \) and suppose that \( j \) is minimal with \( B' \cap F \subseteq T_{j+1} \). We have to show \( j \)-translatability on \( B' \cap F \).

Set \( V := \text{tsp}_B((S_i)) \) and \( V' := \text{tsp}_B((S_i)) \). By \( V \)-translatability of \( S_{j+d} \), \( B' \cap F \subseteq S_{j+d} \) implies \( B' \subseteq S_{j+d} \), so \( \dim V' = j + d \). Since \( V \subseteq V' \), we have \( \pi(V') = k^d \), so Lemma 3.9 implies \( (V' \cap \ker \pi)-\text{translatability of } ((S_i), \chi) \) on \( B' \cap F \). Now, we are done since \( \dim(V' \cap \ker \pi) = j \).

Here are some ‘global’ properties of t-stratifications.

**Lemma 3.17.** Let \( (S_i) \) be a t-stratification of \( B_0 \subseteq K^n \). Then the following holds.

1. For each \( d \) and each \( x \in S_{d+1} \), there exists a maximal ball \( B \) containing \( x \) such that \( B \cap S_{d+1} = \emptyset \). Moreover, if \( B \neq B_0 \) then \( B \) is open. In particular, the sets \( S_{d+1} \) are topologically closed.

2. Each stratum \( S_d \) has dimension exactly \( d \) locally at each point \( x \in S_d \). In particular, either \( \dim S_d = d \) or \( S_d = \emptyset \).


Proof. (1) For \( d = 0 \), this is clear since \( S_0 \) is finite; now suppose \( d > 0 \). By induction, there is a maximal ball \( B \) containing \( x \) with \( B \cap S_{\leq d-1} = \emptyset \). If \( B \cap S_d = \emptyset \), then \( B \) is the ball we are looking for, so suppose now that \( B \cap S_d \neq \emptyset \). Then \( V := tsp_B((S_j)_i) \) is \( d \)-dimensional; let \( \pi : B \to K^d \) be an exhibition of \( V \) and let \( F \subseteq B \) be the \( \pi \)-fiber containing \( x \). Since \( F \cap S_d \) is finite and non-empty, we find a maximal open ball \( B' \subseteq B \) such that \( B' \cap F \cap S_d = \emptyset \). Now \( V \)-translatability implies \( B' \cap S_d = \emptyset \), so \( B' \) is the ball we were looking for.

(2) Let \( x \in S_d \) be given. By (1), there exists a ball \( B \) containing \( x \) with \( B \subseteq S_{\geq d} \), hence on any sub-ball \( B' \subseteq B \), we have \( d \)-translatability. Now \( \dim(S_d \cap B') < d \) would contradict Lemma 3.10.

\( \Box \)

3.3. Families of sets up to risometry

Given a definable family \((\chi_q)_{q \in Q}\) of maps \( K^n \to RV^{eq} \), whether two maps \( \chi_q, \chi_{q'} \) are definably risometric defines an equivalence relation on \( Q \). This equivalence relation is in general not definable. (For the family \((X_a)_{a \in \mathbb{N}}\) from Example 3.15, \( \mathbb{N} \) is one of the equivalence classes.) The main result of this subsection is that it does become definable if we equip each \( \chi_q \) with a \( t \)-stratification. Moreover, for each equivalence class, we can find a definable family of risometries which are compatible under composition.

Under an additional assumption, we can even get some more information about these risometries; this will be needed in the proof of the main theorem. To formulate that assumption we temporarily introduce the following, very weak variant of translatability; this definition will only be used in this subsection and in the application in the proof of Lemma 4.8.

**Definition 3.18.** Suppose that \( B \subseteq K^n \) is a definable subset (usually a ball), \( \chi : B \to RV^{eq} \) is a definable map, and \( V \subseteq k^n \) is a vector space exhibited by \( \pi : B \to K^d \). We say that \( \chi \) is **pointwise translatable** on \( B \) in direction \( V \) with respect to \( \pi \) (or simply **\( V \cdot \pi \)-pointwise translatable**) if for any \( y \in B \) and any \( x' \in \pi(B) \), there exists an \( y' \in \pi^{-1}(x') \) with \( \chi(y') = \chi(y) \) and \( \text{dir}(y - y') \in V \). We also use Convention 3.2.

Note the similarity of pointwise translatable to Condition (4) in Lemma 3.7 (the definition of translator).

**Proposition 3.19.** Suppose that \( Q \) is a \( \emptyset \)-definable set (in any sort), \((S_i)_{i \leq n}\) is a \( \emptyset \)-definable partition of \( Q \times K^n \) and \( \chi : Q \times K^n \to RV^{eq} \) is a \( \emptyset \)-definable map. Write \( \pi \) for the projection \( Q \times K^n \to Q \). For \( q \in Q \), set \( S_{i,q} := S_i \cap \pi^{-1}(q) \) and \( \chi_q := \chi|_{\pi^{-1}(q)} \). Then we have the following.

1. The set \( Q' \subseteq Q \) of those \( q \) for which \((S_{i,q})_{i \leq n}\) is a \( t \)-stratification of \( \{q\} \times K^n \) reflecting \( \chi_q \) is \( \emptyset \)-definable.

2. There exists a \( \emptyset \)-definable map \( \chi' : Q' \to RV^{eq} \) such that for all \( q, q' \in Q' \), \( \chi'(q) = \chi'(q') \) if and only if there exists a definable isometry \( \phi : \{q\} \times K^n \to \{q'\} \times K^n \) respecting \((S_{i,q})_i, \chi_q \).

3. (For each \( \chi' \)-fiber \( C \subseteq Q' \), there exists a compatible \( 'C^n \)-definable family \( \alpha_{q,q'} : ((S_{i,q})_i, \chi_q) \to ((S_{i,q'})_i, \chi_{q'}) \) of risometries, where \( q, q' \) run through \( C \). Compatible means: \( \alpha_{q,q''} \circ \alpha_{q,q'} = \alpha_{q,q''} \) for \( q, q', q'' \in C \).

Suppose now that \( Q \subseteq K^m \) and that \( V \subseteq k^{m+n} \) is exhibited by the projection \( \pi : Q \times K^n \to Q \). Then we also have the following variant of (3).

3' If, for some \( \chi' \)-fiber \( C \subseteq Q' \), \((S_{i,q})_i \) is \( V \cdot \pi \)-pointwise translatable on \( C \times K^n \) and moreover \( S_0 \cap (C \times K^n) \neq \emptyset \), then the family \( \alpha_{q,q'} \) can be chosen such that additionally, \( \text{dir}(\alpha_{q,q'}(x) - x) \in V \) for all \( q, q' \in C \) and all \( x \in \{q\} \times K^n \).
All of the above works uniformly for all models $K$ of our theory $T$, that is, given formulas defining $Q$, $S_i$ and $\chi$, we can find formulas defining $Q'$, $\chi'$ and $\alpha_{q,q'}$ not depending on $K$.

**Remark 3.20.** By taking $Q := \{0,1\}$, in particular, we obtain: if there exists a definable risometry between two $\emptyset$-definable t-stratifications, then there already exists a $\emptyset$-definable one.

Before we prove the proposition, let us consider the following corollary, which shows how statement (3') can be used to deduce translatability.

**Corollary 3.21.** Suppose that $B \subseteq K^n$ is a ball, $\chi : B \to \text{RV}^{eq}$ is a definable map, $V \subseteq k^n$ is a subspace exhibited by $\pi : B \to K^d$, and $(S_i)_{0 \leq i \leq n-d}$ is a definable partition of $B$ such that for each $\pi$-fiber $F \subseteq B$, $(S_i \cap F)_i$ is a t-stratification reflecting $\chi | F$. Suppose, moreover, that $S_0$ is non-empty, that for any two $\pi$-fibers $F,F'$ there exists a definable risometry $\phi : F \to F'$ respecting ($(S_i)_i,\chi$), and that $(S_i)_i$ is $V$-$\pi$-pointwise translatable on $B$. Then $((S_i)_i,\chi)$ is $V$-translatable on $B$.

**Proof.** Without loss, $\pi$ is the projection to the first $d$ coordinates; set $Q := \pi(B)$. We extend the domains of $\pi$, $\chi$, and $(S_i)_i$ from $B$ to $Q \times K^{n-d}$ trivially. More precisely, for $\chi$, we send all of $(Q \times K^{n-d}) \setminus B$ to a single new element in $\text{RV}^{eq} \setminus \chi(B)$, and for $(S_i)_i$, we simply enlarge $S_{n-d}$ (and keep all $S_i$ for $i < n - d$). Then for each $q \in Q$, $(S_i \cap \pi^{-1}(q))_i$ is a t-stratification of $\pi^{-1}(q)$ reflecting $\chi | \pi^{-1}(q)$ (this uses $S_0 \cap \pi^{-1}(q) \neq \emptyset$; cf. Remark 3.13). When applying Proposition 3.19 to this data, the map $\chi'$ we obtain is constant on $Q$ and $Q$ satisfies the prerequisites of (3'), hence we obtain a family of risometries $\alpha_{q,q'} : \pi^{-1}(q_1) \to \pi^{-1}(q_2)$ as in (3') for $q_1, q_2 \in Q$. Define a family $(\beta_q)_{q \in Q - Q}$ of maps $B \to B$ by $\beta_q(x) := \alpha_{\pi(x),\pi(x)+q}(x)$. We claim that this family is a translator proving $V$-translatability of $((S_i)_i,\chi)$ on $B$.

It is clear that these $\beta_q$ satisfy Conditions (1)–(4) of Lemma 3.7 (by definition of $\beta_q$ and by the properties of $\alpha_{q_1,q_2}$), so it remains to check that each $\beta_q$ is a risometry. To see this, choose $x_1, x_2 \in B$ and set $q_i := \pi(x_i)$ and $x_3 := \alpha_{q_2,q_1}(x_2)$. Then $\beta_q$ preserves both, $\hat{rv}(x_1 - x_3)$ (since $\alpha_{q_1,q_1+q}$ is a risometry) and $\hat{rv}(x_3 - x_2)$ (by Lemma 2.10(3), since $\pi(x_3 - x_2)$ and $\text{dir}(x_3 - x_2) = V$ are preserved), and these two values together determine $\hat{rv}(x_1 - x_2)$ by Lemma 2.10(1).

**Proof of Proposition 3.19.** The whole proof is by induction on $n$, that is, we assume that the proposition holds for smaller $n$.

(1) Here is an informal formula defining $Q'$ (where $q$ is the variable):

$$\bigwedge_{d=0}^n \dim S_{d,q} \leq d \land \bigwedge_{d=1}^n \forall \text{ balls } B \subseteq S_{d,q} \text{ with } B \cap S_{d,q} \neq \emptyset :$$

$$\exists V \subseteq k^n \text{ subspace :}$$

$$\rho : B \to \{q\} \times K^d \text{ coordinate projection}$$

$$\rho \text{ exhibits } V$$

$$\land (S_{i,q})_i \text{ is } V-\rho\text{-pointwise translatable on } B$$

[For $x \in \rho(B)$, set $T_{i,x} := S_{i+d,q} \cap \rho^{-1}(x)$ and $\chi_x := \chi_q|_{\rho^{-1}(x)}$]

$$\land (T_{i,x})_{i \leq n-d} \text{ is a t-stratification reflecting } \chi_x \text{ for all } x \in \rho(B)$$

$$\land \text{ all } ((T_{i,x})_i,\chi_x) \text{ are definably risometric for } x \in \rho(B)$$

This is first order: in the first line, we use that dimension is definable; in the last two lines, we use (1) and (2) of the induction hypothesis.
If \((S_{i,q})_i\) is a t-stratification reflecting \(\chi_q\), then it is clear that the formula holds. For the other direction, note that by the induction hypothesis and Corollary 3.21, the last four lines of the formula together with \(B \cap S_{i,q} \neq \emptyset\) imply that \((\langle S_{i,q}\rangle)_i, \chi_q\) is \(V\)-translatable on \(B\).

(2), (3) Without loss, \(Q = Q'\). Moreover, if we have a definable map \(\chi': Q \rightarrow RV^{eq}\) such that the existence of a risometry \(\{q\} \times K^n \rightarrow \{q'\} \times K^n\) respecting \((\langle S_{i}\rangle_i, \chi)\) implies \(\chi'(q) = \chi'(q')\), we can consider each \(\chi'\)-fiber separately for the remainder of the proof (adding the image of the fiber to the language). We will do this several times; at the end, we will obtain a definable compatible family of risometries on the whole of \(Q\), thus proving both (2) and (3).

By Lemma 2.15(1), there is at most one risometry sending \(S_{0,q}\) to \(S_{0,q'}\). Whether such a risometry exists can definably be tested by choosing an enumeration \((x_\mu)\) of \(S_{0,q}\) and comparing the matrix \((\vartheta \nu(x_\mu - x_\nu))_{\mu, \nu}\) to a corresponding matrix for \(S_{0,q'}\). (Note that the cardinality \(|S_{0,q}|\) is bounded.) Thus, we can suppose that for each \(q, q' \in Q\), a risometry \(\beta_{q,q'}: S_{0,q} \rightarrow S_{0,q'}\) exists and that \(\beta_{q,q'}\) respects \(\chi|_{S_0}\). Moreover (again by uniqueness of this risometry), \(\beta_{q,q'}\) is \((q, q')\)-definable and the family \((\beta_{q,q'})_{q,q'}\) is compatible with composition (as required in (3)).

Consider a set \(R \subseteq RV^{eq}(\ell)\) such that \(B_{R,q} := \{x \in \{q\} \times K^n | \vartheta \nu(x - S_{0,q}) = R\}\) is non-empty. By Lemma 2.15, this non-emptiness condition does not depend on \(q\), \(B_{R,q}\) is a maximal ball not intersecting \(S_{0,q}\) (possibly equal to \(\{q\} \times K^n\)), and any risometry \(\{q\} \times K^n \rightarrow \{q'\} \times K^n\) respecting \(S_0\) sends \(B_{R,q}\) to \(B_{R,q'}\). This means that we can treat each family \((B_{R,q})_q\) separately as follows. For each \(R\) as above, we will construct an \(\ell R\)-definable map \(\chi'_R: Q \rightarrow RV^{eq}\) and an \(\ell R\)-definable compatible family of risometries \(\alpha_{R,q,q'}: B_{R,q} \rightarrow B_{R,q'}\) such that (2) and (3) hold for the restricted \((\ell R\)-definable) family \((\langle S_{i,q} \cap B_{R,q}\rangle_i, \chi_q|_{B_{R,q}, q \in Q}\).

By compactness, we can assume that the definitions of \(\chi_R\) and \(\alpha_{R,q,q'}\) are uniform in \(\ell R\). Using stable embeddedness of RV (Hypothesis 2.21(1)), we define the aggregate map \(\chi': Q \rightarrow RV^{eq}\) and an \(\ell R\)-definable compatible family of risometries \(\alpha_{R,q,q'}: B_{R,q} \rightarrow B_{R,q'}\) respecting (2) and (3) hold for the restricted \((\ell R\)-definable) family \((\langle S_{i,q} \cap B_{R,q}\rangle_i, \chi_q|_{B_{R,q}, q \in Q}\).

Thus from now on, we fix \(R\), and \(\ell R\) to the language, and to simplify notation, we write \(B_q\) instead of \(B_{R,q}\). Moreover, we set \(B := \bigcup_{q \in Q} B_q \subseteq Q \times K^n\).

For some \(q \in Q\), let \(W := tsp_{B_q}(\langle S_{i,q}\rangle_i)\). By Lemma 3.14, \(W\) is \(q\)-definable, so using the map \(q \rightarrow \langle W \gamma \rangle \in RV^{eq}\), we may assume that \(W\) does not depend on \(q\). Set \(d := \dim W\), choose an \(\rho': K^n \rightarrow K^d\) of \(W\), and set \(p: B \rightarrow Q \times K^d, (q, y) \mapsto (q, \rho'(y))\). Note that \(d \geq 1\), since \(B_q \cap S_{0,q} = \emptyset\). By Lemma 3.16, we get a family of t-stratifications of the fibers of \(\rho\), parameterized by \(Q := \rho(B)\). Applying induction (2) to this family (after using Remark 3.13) yields a map \(\tilde{\chi}: Q \rightarrow RV^{eq}\) depending only on the \(Q\)-coordinate. By Lemma 3.6, if there exists a risometry \(B_q \rightarrow B_{q'}\) respecting \((\langle S_{i}\rangle_i, \chi)\), then there also exist such risometries between any two \(\rho\)-fibers contained in \(B_q\) or \(B_{q'}\), so we can assume that \(\tilde{\chi}\) is constant on \(Q\). Now induction (3) yields a definable compatible family of risometries \(\tilde{\alpha}_{\tilde{q},q'}: \rho^{-1}(\tilde{q}) \rightarrow \rho^{-1}(\tilde{q}')\) for \(\tilde{q}, \tilde{q}' \in Q\).

To finish the construction of a definable compatible family of risometries \(\alpha_{q,q'}: B_q \rightarrow B_{q'}\), it remains to find a definable compatible family of risometries \(\gamma_{q,q'}: \rho(B_q) \rightarrow \rho(B_{q'})\) (which does not need to respect anything); after that, we can set \(\alpha_{q,q'}(y) := \tilde{\alpha}_{\tilde{q},q'}(y),\) where \(\tilde{q} := \rho(y)\) and \(\tilde{q}' := \gamma_{q,q'}(\tilde{q})\).

If \(S_0\) is empty, then \(B_q = \{q\} \times K^n\) and we can set \(\gamma_{q,q'}(q, z) := (q', z)\) for every \(q, q' \in Q, z \in K^n\), so suppose now that \(S_0\) is non-empty. For each \(q\), let \(N_q\) consist of those elements of \(S_{0,q}\) that are closest to \(B_q\), that is, \(N_q = S_{0,q} \cap B_q\), where \(B_q\) is the unique closed ball containing \(B_q\) with \(\text{rad}_d(B_q) = \text{rad}_d(B_q)\). Define \(c_q := (1/|N_q|) \sum_{z \in N_q} z\) to be the barycenter of \(N_q\). The translation \(\tilde{\gamma}_{q,q'}: y \mapsto y - c_q + c_{q'}\) sends \(B_q\) to \(B_{q'}\), so we can define \(\gamma_{q,q'}: \rho(B_q) \rightarrow \rho(B_{q'})\) to be the induced translation on the projections.

(3') Let us say that a map \(\phi\) between two subsets of \(Q \times K^n\) moves in direction \(V\) if \(\text{dir}(\phi(x) - x) \in V\) for all \(x\) in the domain. We claim that under the additional assumptions of
(3'), the maps $\alpha_{q,q'}$ constructed in the proof of (3) do already move in direction $V$; so let us go through the construction of $\alpha_{q,q'}$.

First, we have to check that the risometries $\beta_{q,q'}: S_{0,q} \to S_{0,q'}$ move in direction $V$. Set $\delta := \hat{v}(q - q')$, and let us say that $T \subseteq S_{0,q'}$ is a set of $\delta$-representatives (of $S_{0,q'}$) if for each $s \in S_{0,q'}$ there exists exactly one $t \in T$ with $\hat{v}(s - t) > \delta$. Choose any set of $\delta$-representatives $T \subseteq S_{0,q'}$. For each $t \in T$, using pointwise translatable of $S_0$, we can choose an element $\phi(t) \in S_{0,q}$ with $\text{dir}(\phi(t) - t) \in V$. Using that $\hat{v}(t - t') \leq \hat{v}(q - q')$ for any two different $t, t' \in T$, we get that $\phi: T \to \phi(T)$ is a risometry. Composing with $\beta_{q,q'}$ yields a risometry from $T$ to $T' := \beta_{q,q'}(\phi(T))$, and $T'$ is also a set of $\delta$-representatives of $S_{0,q'}$. The bijection from $T$ to $T'$ sending $t$ to the unique $t' \in T'$ with $\hat{v}(t - t') > \delta$ is also a risometry, so by Lemma 2.15, it is equal to $\beta_{q,q'} \circ \phi$. This implies that $\text{dir}(y - \beta_{q,q'}(y)) \in V$, first for $y = \phi(T)$ and then also for all other $y \in S_{0,q}$.

To get that the maps $\alpha_{q,q'}: B_q \to B_{q'}$ move in direction $V$, it remains to check that both, the maps $\tilde{\alpha}_{q,q'}$ and the maps $\tilde{\gamma}_{q,q'}$ move in direction $V$. Let us first consider the maps $\tilde{\alpha}_{q,q'}$. By assumption, $S_0 \neq \emptyset$, so $\tilde{\gamma}_{q,q'}(y) = y - c_q + c_{q'}$, which moves in direction $V$ since $\beta_{q,q'}(N_q) = N_{q'}$ and $\beta_{q,q'}$ moves in direction $V$.

To obtain that the maps $\tilde{\alpha}_{q,q'}$ move in direction $V$, it suffices to check that we can apply (3') instead of (3) in the induction. For this, we take $V := V + (\{0\}^m \times W) \subseteq k^{m+n}$. Since $B_q \cap S_{1,q} \neq \emptyset$, the 0-dimensional stratum of the induction is non-empty, and it remains to check pointwise translatable: for given $(q,x),(q',x') \in Q$, $i \leq n$, and $y \in \rho^{-1}(q,x) \cap S_i$, we need to find an element $y' \in \rho^{-1}(q',x') \cap S_i$ such that $\text{dir}(y' - y) \in \hat{V}$.

Set $\delta := \hat{v}(q - q')$. If $\delta \leq \text{rad}_d B_q$, then any $y' \in B_{q'}$ satisfies $\text{dir}(y' - y) \in \hat{V}$, since $\tilde{\gamma}_{q,q'}$ moves in direction $V$ and sends $B_q$ to $B_{q'}$, so the risometry $B_q \to B_{q'}$ from the proof of (3) yields an $y'$ with the desired properties.

If $\delta > \text{rad}_d B_q$, then let $y'' \in \pi^{-1}(q') \cap S_i$ be a point obtained from the $V$-pointwise translatable in the assumptions. Using again that $\tilde{\gamma}_{q,q'}$ moves in direction $V$, we get $\hat{v}(y'' - \tilde{\gamma}_{q,q'}(y)) > \delta$ and thus $y'' \in B_{q'}$. Now we use $W$-translatability of $B_{q'}$ to move $y''$ to the fiber $\rho^{-1}(q',x')$.

\[ \text{A priori}, \text{ being a t-stratification is not first order, since there might be no bound on how complicated the straighteners in a single t-stratification are. (Recall that the straighteners are the risometries making things translation invariant; see Definition 3.1.) However, Proposition 3.19(1) says that after all, being a t-stratification is first order; from this, we can deduce a posteriori that all straighteners appearing in a single t-stratification can be defined uniformly. (In fact, these uniformly defined straighteners can also directly be extracted from the proof of Proposition 3.19.)} \]

\[ \text{COROLLARY 3.22. If } (S_i)_i \text{ is a t-stratification reflecting a definable map } \chi: B_0 \to RV^q, \text{ then the straighteners on all balls can be defined uniformly, that is, there is a formula } \eta(x,x',y), \text{ where } x,x' \text{ are } n \text{-tuples of valued field variables and } y \text{ is an arbitrary tuple of variables, such that for any ball } B \subseteq S_{2,d}, \text{ there exists an element } b \text{ such that } \eta(x,x',b) \text{ defines the graph of a straightener of } ((S_i)_i,\chi) \text{ on } B \text{ witnessing } d\text{-translatability. If } (S_i)_i \text{ and } \chi \text{ are given by formulas, then } \eta(x,x',y) \text{ can be chosen to work in all } K \models T \text{ where } (S_i)_i \text{ is a t-stratification reflecting } \chi. \]

\[ \text{Proof. For any formula } \eta(x,x',y), \text{ let } \text{str}_\eta(b,\{^rB\}) \text{ be a formula expressing that } \eta(x,x',b) \text{ defines a straightener which witnesses } d\text{-translatability of } ((S_i)_i,\chi) \text{ on the ball } B, \text{ where } d \text{ is minimal with } B \cap S_i \neq \emptyset. \text{ Applying Proposition 3.19(1) to } ((S_i)_i,\chi), \text{ where } Q \text{ is a one-point-set, yields a sentence } \psi \text{ such that a model } K \models T \text{ satisfies } \psi \text{ if and only if for every ball } B \subseteq K^n, \text{ a straightener exists. In particular, for every } K \models (T,\psi) \text{ and every } B, \text{ there exists an } \eta(x,x',y) \text{ such that } K \models \exists b \text{ str}_\eta(b,\{^rB\}). \text{ By compactness, there is a single } \eta(x,x',y) \text{ such that } T \cup \{\psi\} \implies \forall B \exists b \text{ str}_\eta(b,\{^rB\}); \text{ this } \eta \text{ uniformly defines all straighteners.} \]
4. Proof of the existence of $t$-stratifications

We now come to the proof of the main theorem about existence of $t$-stratifications: Theorem 4.12. If not specified otherwise, we will only assume Hypothesis 2.21$_0$ everywhere. In fact, the only places where we will need more than that are Proposition 4.5 and the main theorem itself.

Here is a very rough sketch of the proof (omitting many technicalities). Suppose we have a definable map $\chi: K^n \to \textrm{RV}^{eq}$. The overall idea is to construct the sets $S_d$ one after the other, starting with $S_n$. Suppose that $S_n, \ldots, S_{d+1}$ are already constructed and let $X := K^n \setminus S_{d+1}$ be the remainder, which we suppose to be of dimension at most $d$. To obtain $S_d$, we only have to find a set $X' \subseteq X$ which is at most $(d-1)$-dimensional such that on any ball not intersecting $X'$, we have (at least) $d$-translatability; then we can set $S_d := X \setminus X'$. However, to be able to obtain such an $X'$ in a definable way, we have to drop the condition $X' \subseteq X$. This is not a problem; we simply shrink the sets $S_i$ we already constructed before (removing $X'$ from them).

To prove $d$-translatability on many balls $B$ (where ‘many’ means: outside of a $(d-1)$-dimensional set), we roughly proceed as follows. First note that we can always ‘refine’ $\chi$, that is, we can replace it by a map $\chi'$ such that each $\chi'$-fiber is entirely contained in a $\chi$-fiber. We use Proposition 4.5 (which in turn uses the Jacobian property) to refine $\chi$ in such a way that each $\chi$-fiber $C$ separately becomes $(\dim C)$-translatable on suitable balls. The biggest difficulty then consists in showing that on many balls $B$, these individual translatabilities fit together well enough to yield translatability of the whole map $\chi$; this is done using Lemma 4.8. A main prerequisite for that lemma is that there exists a coordinate projection $\pi: B \to K^d$ such that between any two $\pi$-fibers, there exists a risometry respecting $\chi$.

One strategy to obtain the required risometries between $\pi$-fibers is as follows. Consider a fiber $F \subseteq K^n$ of a coordinate projection $\rho: K^n \to K^{d-1}$. If $d \geq 2$, then $\dim F < n$, so by induction, we may assume that there exists a $t$-stratification reflecting $\chi|_F$. This implies that in many cases, risometries between the $\pi$-fibers inside $F$ exist. (‘In many cases’ means: for many balls $B$ and many projections $\pi$.) By doing this for different $\rho$, we finally obtain that in many cases, we have risometries between any two $\pi$-fibers (Lemma 4.9). More precisely, we indeed find a $(d-1)$-dimensional ‘bad’ set $X'$ such that for any ball $B$ not intersecting $X'$, there exists a $\pi: B \to K^d$ such that between any two $\pi$-fibers, there exists a risometry respecting $\chi$.

In the case $d = 1$, the method from the previous paragraph does not work, since we cannot apply induction. In that case, we apply Proposition 3.19 to the family of all $\pi$-fibers, which implies that between many $\pi$-fibers, we find risometries respecting $\chi$. More precisely, by doing this for all $\pi$ (and using $d = 1$), we can confine the bad set $X'$ to a union of finitely many $(n-1)$-dimensional hyperplanes. This argument can be repeated and each time, the dimension of the hyperplanes drops by one. In this way, we finally obtain that $X'$ is contained in a finite set; then we set $S_0 := X'$ and we are done.

4.1. Subaffine pieces

**Definition 4.1.** For two maps $\chi, \chi': K^n \to \textrm{RV}^{eq}$, we say that $\chi'$ refines $\chi$ (or: $\chi'$ is a refinement of $\chi$) if $\chi = f \circ \chi'$ for a suitable map $f$ (or, equivalently, if the partition of $K^n$ given by the fibers of $\chi'$ refines the one for $\chi$). Using Convention 3.2, we also speak of maps $\chi'$ refining tuples of maps and sets.

To get translatability of a definable map $\chi: K^n \to \textrm{RV}^{eq}$ on certain balls, the first step is to refine it such that each fiber $C$ is, up to a risometry, a subset of a subvector space of $K^n$ of the same dimension as $C$. More precisely, we will require each $C$ to have the following property.
DEFINITION 4.2. For any subset $C \subseteq K^n$, we define the affine direction space of $C$ to be the subspace $\text{affdir}(C) \subseteq k^n$ generated by $\text{dir}(x - x')$, where $x, x'$ run through $C$ (and $x \neq x'$). We call $C$ subaffine (in direction $\text{affdir}(C)$) if for every $x \in C$, $\dim_x(C) = \dim(\text{affdir}(C))$.

We use $\dim_x(C)$ and not $\dim(C)$ in the definition to ensure that the intersection of a subaffine set with a ball is again subaffine. Note that by (1) of the next lemma, $\dim(\text{affdir}(C))$ cannot be less than $\dim(C)$.

**Lemma 4.3.** Suppose that $C \subseteq K^n$ is a definable set and that $\pi: K^n \to K^d$ exhibits $V = \text{affdir}(C)$. Then we have the following.

1. Each $\pi$-fiber contains at most one point of $C$; in particular, $C$ is the graph of a map $c: \pi(C) \to K^{n-d}$ and $\dim C \leq d = \dim V$.
2. Suppose that $c$ is as in (1) and to simplify notation, suppose that $\pi$ is the projection to the first $d$ coordinates. Then the map

$$\phi: \pi(C) \times K^{n-d} \to \pi(C) \times K^{n-d}, \quad (x, y) \mapsto (x, y + c(x))$$

can be written as $\phi = \psi \circ M$, where $\psi$ is a risometry, $M \in \text{GL}_n(O_K)$, and $\pi \circ \psi = \pi \circ M = \pi$.

3. For any coordinate projection $\rho: K^n \to K^d$, we have $\hat{\rho}(\text{affdir}(C)) \subseteq \text{affdir}(\rho(C))$.

**Proof.** (1) Two different elements $x, x'$ in the same $\pi$-fiber would have $\hat{\pi}(\text{dir}(x - x')) = 0$, contradicting that $\pi$ exhibits $V$.

(2) Choose any lift $\tilde{V} \subseteq K^n$ of $V$ and define $M$ to be the linear map sending $(x, y) \in K^d \times K^{n-d}$ to $(x, y + b(x))$, where $(x, b(x)) \in \tilde{V}$. Since $\pi$ exhibits $V$, we have $\hat{\nu}(b(x)) \geq \hat{\nu}(x)$ and hence $M \in \text{GL}_n(O_K)$; it remains to check that the map $\tilde{\psi} = \phi \circ M^{-1}$, which sends $(x, y)$ to $(x, y - b(x) + c(x))$, is a risometry. For $x, x' \in \pi(C)$, Lemma 2.10(3) implies $\hat{\nu}(x - x', c(x) - c(x')) = \hat{\nu}(x - x', b(x) - b(x'))$; from this, the claim follows using Lemma 2.10(1).

(3) For any pair of points $x, x' \in C$, we have to check that $\hat{\rho}(\text{dir}(x - x')) \in \text{affdir}(\rho(C))$. If $\hat{\rho}(\text{dir}(x - x')) = 0$, there is nothing to prove; otherwise Lemma 2.10(3) implies $\hat{\rho}(\text{dir}(x - x')) = \text{dir}(\hat{\rho}(x) - \hat{\rho}(x'))$.

Being subaffine is closely related to translatability.

**Lemma 4.4.** Let $B \subseteq K^n$ be a ball and $C \subseteq B$ a definable subset.

1. If $C$ is $V$-translatable on $B$ for some $V \subseteq k^n$, then $V \subseteq \text{affdir}(C)$.
2. If there is an exhibition $\pi: B \to K^d$ of $V := \text{affdir}(C)$ with $\pi(C) = \pi(B)$, then $C$ is $V$-translatable on $B$.

**Proof.** (1) Clear.

(2) Assume without loss $0 \in B$. Then the risometry $\psi$ obtained from Lemma 4.3(2) sends $B$ to itself and it is a straightener.

Now we can formulate the main result of this subsection. Its proof is the only place in the proof of Theorem 4.12 where the Jacobian property is needed.

**Proposition 4.5.** Assume Hypothesis 2.21_{n-1} and let $\chi: K^n \to RV^{eq}$ be a $\emptyset$-definable map. Then there exists a $\emptyset$-definable refinement $\chi'$ of $\chi$ such that each $\chi'$-fiber $C' \subseteq K^n$ is subaffine.
In the proof, we will need the following lemma.

**Lemma 4.6.** Suppose that $X \subseteq K^d$ is a definable set of dimension $d$ and that $Z \subseteq K^{d+1}$ is the graph of a definable function $f: X \to K$ that has the Jacobian property (Definition 2.19). Then $Z$ is subaffine.

**Proof of Lemma 4.6.** We may assume that $f$ is not constant; let $z \in K^n \setminus \{0\}$ be as in Definition 2.19 and write $\pi: K^{d+1} \to K^d$ for the projection to the first $d$ coordinates.

Suppose that $Z$ is not subaffine. Since $\dim Z = d$, this implies $\text{aff}\text{dir}(Z) = k^{d+1}$, so by definition, there exist $d + 1$ pairs of points $x'_i, x''_i \in X$ with $x'_i \neq x''_i$ such that $(\text{dir}(x'_i - x''_i), f(x'_i) - f(x''_i))$, is a basis of $k^{d+1}$. Set $x_i := x'_i - x''_i$ and $y_i := f(x'_i) - f(x''_i)$. Using this notation, the inequality in Definition 2.19 becomes

$$v(y_i - \langle z, x_i \rangle) > \hat{v}(z) + \hat{v}(x_i).$$

(+) Suppose first that $\hat{v}(z) < 0$. Choose $i$ with $\langle \text{dir}(z), \pi(\text{dir}(x_i, y_i)) \rangle \neq 0$. Then, by Lemma 2.10(3), we have $\pi(\text{dir}(x_i, y_i)) = \text{dir}(x_i)$ and $v(y_i) \geq \hat{v}(x_i) > \hat{v}(z) + \hat{v}(x_i)$; moreover, Lemma 2.10(5) implies $v(\langle z, x_i \rangle) = \hat{v}(z) + \hat{v}(x_i)$, contradicting (+).

Now suppose $\hat{v}(z) \geq 0$. Then (+) implies $\hat{v}(\langle x_i, y_i \rangle - \langle x_i, \langle z, x_i \rangle \rangle) > \hat{v}(x_i, y_i)$ and hence $\text{dir}(x_i, y_i) = \text{dir}(x_i, \langle z, x_i \rangle)$. Now $V := \{(x, \langle z, x \rangle) | x \in K^d\}$ is a $d$-dimensional subspace of $K^{d+1}$, so each $\text{dir}(x_i, \langle z, x_i \rangle)$ lies in the $d$-dimensional subspace $\text{res}(V) \subseteq k^{d+1}$ (by Lemma 2.10(4)), contradicting that the $\text{dir}(x_i, y_i)$ form a basis of $k^{d+1}$.

**Proof of Proposition 4.5.** We will prove the following claim: For any $\emptyset$-definable set $C \subseteq K^n$ of dimension $d$, there is a $\emptyset$-definable map $\tilde{\chi}: C \to \text{RV}^{\text{eq}}$ such that each $\tilde{\chi}$-fiber $\tilde{C} \subseteq C$ of dimension $d$ satisfies $\dim(\text{affdir}(\tilde{C})) = d$.

Once we have this, we can finish the proof of the proposition as follows. We do an induction over the maximum of the dimensions of $\chi$-fibers that are not subaffine. Denote this maximum by $d$. On each $\chi$-fiber $C$ of dimension $d$ which is not subaffine, we refine $\chi$ as follows.

First, we apply the claim to $C$ (with $\text{affdir}^C$ added to the language), which yields a $\text{affdir}^C$-definable map $\tilde{\chi}: C \to \text{RV}^{\text{eq}}$. Now consider a $\chi$-fiber $\tilde{C}$ of dimension $d$. By Lemma 2.30, the set $D := \{x \in C | \dim_\rho C < d\}$ has dimension less than $d$, so in particular $\tilde{C} \setminus D$ is subaffine. Refine $\chi$ such that each $\tilde{C} \setminus D$ becomes a separate fiber. The result is a refinement of $\chi$ whose fibers of dimension $d$ are subaffine, so then we are done by the induction hypothesis.

It remains to prove the claim. For $d = n$, there is nothing to show, so we assume $d < n$. We define $\tilde{\chi}(x)$ (for $x \in C$) to be the tuple $(\chi_\pi(x))_\pi$, where $\pi$ runs over all coordinate projections $\pi: K^n \to K^d$ and where $\chi_\pi: C \to \text{RV}^{\text{eq}}$ is defined as follows.

1. Let $C_0$ be the union of all $\pi$-fibers $F = \pi^{-1}(y) \cap C$ (for $y \in \pi(C)$) satisfying $\dim F = 0$ and let $C_1 := C \setminus C_0$ be the remainder. We set $\chi_\pi(x) = 1_\pi$ for all $x \in C_1$, where $1_\pi \in \text{RV}^{\text{eq}}$ is any element that is not used again (that is, $C_1$ is one fiber of $\chi_\pi$).

2. By Lemma 2.15(2) and using stable embeddedness of RV, we get a $\emptyset$-definable map $\chi_0: C_0 \to \text{RV}^{\text{eq}}$ that is injective on each $\pi$-fiber (namely $\chi_0(x) = \text{affdir}(x - (\sigma^{-1}(\pi(x)) \cap C_0))$).

3. A $\chi_0$-fiber $C' = \chi_0^{-1}(\sigma) \subseteq C_0$ can be seen as the graph of a $\sigma$-definable function $f: \pi(C') \to K^{n-d}$. Since the theory has the Jacobian property up to dimension $n-1 \geq d$, we find a $\sigma$-definable map $\chi_\sigma: \pi(C') \to \text{RV}^{\text{eq}}$ such that on each $\chi_\sigma$-fiber of dimension $d$, each coordinate of $f$ has the Jacobian property (Definition 2.19). For $x \in C_0$, set $\chi_\pi(x) := (\chi_0(x), \chi_{\chi_0^{-1}(\sigma)}(\pi(x)))$.

We now have to check that if $\tilde{C}$ is a $d$-dimensional $\tilde{\chi}$-fiber, then $\dim(\text{affdir}(\tilde{C})) = d$, so assume for contradiction that $d' := \dim(\text{affdir}(\tilde{C})) > d$. Choose an exhibition $\rho: K^n \to K^{d'}$ of
affdir(\tilde{C}). Then \(\dim \rho(\tilde{C}) = d\) since otherwise, there would be \(\rho\)-fibers containing several points of \(\tilde{C}\), contradicting that \(\rho\) exhibits affdir(\(\tilde{C}\)). Next choose a coordinate projection \(\pi' : \Omega^{d} \to \Omega^{d}\) such that for \(\pi := \pi' \circ \rho\), we still have \(\dim(\tilde{C}) = d\), and choose an arbitrary decomposition of \(\pi'\) into two coordinate projections \(\rho' : \Omega^{d} \to \Omega^{d+1}\) and \(\pi'' : \Omega^{d+1} \to \Omega^{d}\). Note that by the choice of \(\rho\), we have \(\rho'(\rho(\text{affdir}(\tilde{C}))) = k^{d+1}\).

Since \(\dim(\pi(\tilde{C})) = \dim(\tilde{C})\), we have \(\chi_{\pi}(\tilde{C}) \neq 1_{\pi}\), so by definition of \(\chi_{\pi}\), \(Z := \rho'(\rho(\tilde{C}))\) is the graph of a function \(f : \pi(\tilde{C}) \to \Kbar\) that has the Jacobian property; hence \(Z\) is subaffine by Lemma 4.6. However, this contradicts that by Lemma 4.3(3) we have affdir(\(Z\)) \(\supseteq \rho'(\rho(\text{affdir}(\tilde{C})))\) \(\to k^{d+1}\).

\[\Box\]

4.2. Merging translatability

In the previous subsection, we obtained some first translatability separately for each fiber of a definable map \(\chi : \Kbar^{n} \to \Omega^{eq}\). Now we will show how this can be merged to translatability of the whole map \(\chi\) (under a lot of technical assumptions). We start with a lemma that allows us to relate affine direction spaces of different fibers.

**Lemma 4.7.** Let \(B \subseteq \Kbar^{n}\) be a ball and let \(C, C' \subseteq B\) be non-empty definable subsets that are subaffine in directions \(\Vbar\) and \(\Vbar'\), respectively. Suppose that \(\pi : B \to \Omega^{d}\) exhibits \(\Vbar\) and that for any two elements \(y_{1}, y_{2} \in \pi(B)\), there exists a risometry \(\pi^{-1}(y_{1}) \to \pi^{-1}(y_{2})\) respecting \((C, C')\). Then \(V \subseteq \Vbar'\).

**Proof.** It suffices to find \(x_{1}', x_{2}' \in C'\) with \(\text{dir}(x_{1}' - x_{2}') = v\) for any given \(v \in \Vbar \setminus \{0\}\), so let such a \(v\) be given.

Choose any \(x_{1}' \in C'\) and set \(y_{1} := \pi(x_{1}')\). Any fiber of \(\pi\) contains exactly one element of \(C\); let \(x_{1}\) be this unique element of \(C \cap \pi^{-1}(y_{1})\). Choose \(y_{2} \in \pi(B)\) such that \(\text{dir}(y_{1} - y_{2}) = \pi(v)\) and \(\tilde{v}(y_{1} - y_{2}) = \tilde{v}(x_{1}' - x_{1})\). Now let \(x_{2}\) and \(x_{2}'\) be the images of \(x_{1}\) and \(x_{1}'\) under a risometry \(\phi : \pi^{-1}(y_{1}) \to \pi^{-1}(y_{2})\).

We have \(x_{2} \in C\), so \(\text{dir}(x_{1} - x_{2}) = v\). Since \(\phi\) is a risometry, we have \(\tilde{v}(x_{1}' - x_{1}) = \tilde{v}(x_{2}' - x_{2})\), so \(\tilde{v}((x_{1}' - x_{2}') - (x_{1} - x_{2})) \geq \tilde{v}(x_{1}' - x_{1}) = \tilde{v}(x_{1} - x_{2})\), which implies \(\tilde{v}(x_{1}' - x_{2}') = \tilde{v}(x_{1} - x_{2})\) and thus \(\text{dir}(x_{1}' - x_{2}') = v\).

\[\Box\]

The following lemma is the main tool to prove \(V\)-translatability of a map \(\chi : B \to \Omega^{eq}\), where \(B \subseteq \Kbar^{n}\) is a ball and \(V \subseteq \Kbar^{n}\) is a \(d\)-dimensional vector space. Let \(\pi : B \to \Omega^{d}\) exhibit \(V\). The prerequisites are that (i) there exist risometries between the \(\pi\)-fibers respecting \(\chi\), (ii) the fibers of \(\chi\) are subaffine, (iii) there exists a \(\chi\)-fiber \(C\) with affdir \(C = V\) and (iv) we already have a partial \(t\)-stratification which works outside of a \(d\)-dimensional set. However, in applications of the lemma, we will not be able to ensure (iv) simultaneously with (i)–(iii); therefore, we allow (i)–(iii) to apply to a refinement \(\chi'\) of \(\chi\), which is enough to get the result.

**Lemma 4.8.** Suppose that we have definable sets \(C \subseteq B \subseteq \Kbar^{n}\), definable maps \(\chi, \chi' : B \to \Omega^{eq}\), an integer \(d \in \{1, \ldots, n\}\), a definable partition \((S_{i})_{d \leq i \leq n}\) of \(B\) and a coordinate projection \(\pi : B \to \Omega^{d}\), with the following properties:

1. \(B\) is a ball;
2. \(\dim S_{i} \leq i\);
3. for any ball \(B' \subseteq B \setminus S_{d}\), \((S_{i})_{i} = j\)-translatable on \(B'\), where \(j\) is minimal with \(B' \cap S_{j} \neq \emptyset\);
4. \(\chi'\) is a refinement of \((S_{i})_{i}, \chi\) (in the sense of Definition 4.1 and Convention 3.2);
(5) each fiber of $\chi'$ is subaffine;
(6) for each pair of points $x, x' \in \pi(B)$, there exists a definable risometry $\pi^{-1}(x) \rightarrow \pi^{-1}(x')$ respecting $\chi'$;
(7) $C$ is a $\chi'$-fiber whose affine direction space $V := \text{affdir}(C)$ is exhibited by $\pi$ (in particular, $\dim C = \dim V = d$).

Then $((S_i), \chi)$ is $V$-translatable.

Proof. By Lemma 4.7, for any $\chi'$-fiber $C'$ we have $V \subseteq \text{affdir}(C')$. In particular, if $C' \subseteq S_d$, then $\text{affdir}(C') = V$ and $C'$ is $V$-translatable on $B$ by Lemma 4.4(2).

Claim 1. If $B' \subseteq B$ is a ball with $B' \cap S_d = \emptyset$, then $V \subseteq W := \text{tsp}_{B'}((S_i)_i)$.

Proof of Claim 1. Set $d' := \dim W$ and let $\pi': B' \rightarrow K^{d'}$ be an exhibition of $W$. The $\pi'$-fibers of $S_{d'}$ are finite but non-empty. Choose a sub-ball $B'' \subseteq B'$ such that $B'' \cap (\pi')^{-1}(x) \cap S_{d'}$ is a singleton for each $x \in \pi'(B'')$. Then $\text{affdir}(S_{d'} \cap B'') = \text{tsp}_{B''}((S_i)_i) = W$. Now choose any $\chi'$-fiber $C' \subseteq S_{d'}$ with $\dim(C' \cap B'') = d'$. Then $W \subseteq \text{affdir} C'$ and $\dim(\text{affdir} C') = \dim C' = d'$ together imply $W = \text{affdir} C' \supseteq V$.

If $S_d = \emptyset$, then we are done using $B' = B$, so from now on suppose $S_d \neq \emptyset$.

Claim 2. Fix $x \in \pi(B)$, let $F = \pi^{-1}(x)$ be the fiber over $x$, and set $T_i := S_{i+d} \cap F$ for $i \leq n - d$. Then $(T_i)_{i \leq n-d}$ is a $t$-stratification of $F$ reflecting $\chi|_F$.

Proof of Claim 2. Using that the risometries from (6) between the $\pi$-fibers respecting $\chi'$ in particular respect $S_{j+d}$, we obtain $\dim T_j \leq j$. Consider a ball $B' \subseteq B$ intersecting $F$ non-trivially and set $B'_j := B' \cap F$. We have to show that if $B'_j \subseteq T_{j+1}$, then $((T_i), \chi)$ is $j$-translatable on $B'_j$. For $j = 0$, there is nothing to do, so suppose $j \geq 1$. Then $B'_j \cap S_d = \emptyset$, and using that any $\chi'$-fiber $C'$ contained in $S_{j}$ is $V$-translatable, we get $B' \cap S_d = \emptyset$. Now Claim 1 together with Lemma 3.9 implies $j$-translatability on $B'_x$.

Claim 3. $((S_i), \chi)$ is $V$-$\pi$-pointwise translatable (see Definition 3.18).

Proof of Claim 3. Let $x, x' \in \pi(B)$ and $y \in \pi^{-1}(x)$ be given; we need to find $y' \in \pi^{-1}(x')$ with $\text{dir}(y - y') \in V$ such that $y$ and $y'$ are elements of the same set $S_i$. Set $\delta := \hat{v}(x - x')$. If $B(y, \geq \delta) \cap S_d = \emptyset$, then $y'$ is obtained using Claim 1. Otherwise, let $C' \subseteq S_d$ be a $\chi'$-fiber intersecting $B(y, \geq \delta)$ non-trivially, let $z, z'$ be the unique elements of $C' \cap \pi^{-1}(x)$ and $C' \cap \pi^{-1}(x')$, respectively, and let $y'$ be the image of $y$ under a risometry $\pi^{-1}(x) \rightarrow \pi^{-1}(x')$ respecting $\chi'$. Now $\hat{v}(y - z) \geq \delta$ and $\hat{v}(y - z) = \hat{v}(y' - z')$ together imply $\hat{v}((y - y') - (z - z')) > \delta$, and thus $\text{dir}(y - y') = \text{dir}(z - z') \in V$.

Now, Claims 2 and 3 (together with $S_d \neq \emptyset$) are all we need to apply Corollary 3.21, which yields the desired $V$-translatability.

The next lemma will be useful to prove the prerequisites of the previous lemma. More precisely, given $\pi$ and a $\chi'$ as above, it yields a way to prove that there exist risometries respecting $\chi'$ between any to $\pi$-fibers.
Lemma 4.9. Let the following be given.

(i) A definable map $\chi : B \to \text{RV}^{eq}$, where $B \subseteq K^n$ is a ball.
(ii) A vector space $V \subseteq k^n$ of dimension $d \geq 1$ exhibited by $\pi : B \to K^d$.
(iii) A $\chi$-fiber $C \subseteq B$ with $\text{affdir}(C) \subseteq V$.

Suppose that for each coordinate projection $\rho : K^d \to K^{d-1}$ and each $y \in \rho(\pi(B))$, $\chi$ is $1$-translatable on the fiber $\pi^{-1}(\rho^{-1}(y))$.

Then for any $x_1, x_2 \in \pi(B)$, there exists a definable isometry $\pi^{-1}(x_1) \to \pi^{-1}(x_2)$ respecting $\chi$.

Remark 4.10. A posteriori, this implies $\dim C = d$, so $\text{affdir}(C) = V$ and $C$ is subaffine.

Proof of Lemma 4.9. It is enough to find such isometries $\pi^{-1}(x_1) \to \pi^{-1}(x_2)$ under the assumptions that $x_1$ and $x_2$ differ in only one coordinate and that $\pi^{-1}(x_1) \cap C \neq \emptyset$. Indeed, the existence of such a isometry implies that we also have $\pi^{-1}(x_2) \cap C \neq \emptyset$, so by repeatedly applying this (starting with a fiber intersecting $C$ non-trivially and modifying coordinates one by one), we first get that every $\pi$-fiber intersects $C$ non-trivially, and then we obtain isometries between any two fibers by composition.

So suppose now that $x_1$ and $x_2$ differ in only one coordinate and let $\rho : K^d \to K^{d-1}$ be the coordinate projection satisfying $\rho(x_1) = \rho(x_2) = y$; let $F := \pi^{-1}(\rho^{-1}(y)) \subseteq B$ be the corresponding fiber. By assumption, there exists a one-dimensional $W \subseteq \ker(\rho \circ \pi) \subseteq k^n$ such that $\chi$ is $W$-translatable on $F$. In particular, the non-empty set $C \cap F$ is $W$-translatable, so $W \subseteq V$ by Lemma 4.4(1). Since $\dim(\ker(\rho \circ \pi) \cap V) = 1$, $W$ is equal to this intersection, so $\pi|_F$ exhibits $W$. From a translator $(\alpha_x)_{x \in \pi(F - F)}$ of $\chi|_F$ with respect to $\pi$ (see Definition 3.8), we obtain a isometry $\phi : \pi^{-1}(x_1) \to \pi^{-1}(x_2)$ by restricting $\alpha_{x_2 - x_1}$ to $\pi^{-1}(x_1)$. \qed

4.3. The big induction

This subsection contains the actual proof of the main theorem. We first prove the case $n = 1$ separately.

Lemma 4.11. For every $\emptyset$-definable map $\chi : K \to Q \subseteq \text{RV}^{eq}$, there exists a finite $\emptyset$-definable set $T_0 \subseteq K$ such that $\chi$ is constant on each ball $B \subseteq K \setminus T_0$.

Proof. By Hypothesis 2.21(3), for each $q \in Q$, there exists a finite set $S_q$ such that each ball $B \subseteq K \setminus S_q$ is either disjoint from $\chi^{-1}(q)$ or contained in $\chi^{-1}(q)$. By Lemma 2.29(1), the union $T_0 := \bigcup_q S_q$ is $0$-dimensional, so by Remark 2.28, it is finite. The construction of $T_0$ ensures that $\chi$ is constant on each ball $B \subseteq K \setminus T_0$. \qed

Now we are ready for the main theorem. For readers who jumped directly to this point, let us recall: the basic notation is fixed in Subsection 2.1, the language $\mathcal{L}_{\text{Hen}}$ is introduced in Definition 2.16, and t-stratifications are introduced in Definition 3.12. Recall also Remark 2.22, by which we can replace each occurence of ‘$\emptyset$-definable’ in the theorem by ‘$A$-definable’ for some fixed parameter set $A$ (which, using compactness, yields that the theorem works uniformly in families).

Theorem 4.12. Fix $n \in \mathbb{N}$. Let $\mathcal{L}$ be an expansion of the valued field language $\mathcal{L}_{\text{Hen}}$ and let $K$ be an $\mathcal{L}$-structure whose theory satisfies Hypothesis 2.21$_{n-1}$. (In particular, $K$ is a Henselian-valued field of equi-characteristic 0.) Then, for every $\emptyset$-definable ball $B_0 \subseteq K^n$ and every $\emptyset$-definable map $\chi : B_0 \to \text{RV}^{eq}$, there exists a $\emptyset$-definable t-stratification $(S_i)_{i \leq n}$ of $B_0$ reflecting $\chi$. 
Proof. The case $n = 1$ is exactly Lemma 4.11. For $n \geq 2$, we do a big induction on $n$, that is, we assume that the theorem holds for all smaller $n$.

By extending $\chi$ trivially outside of $B_0$, we may suppose $B_0 = K^n$.

By decreasing induction on $d$, we prove the following.

($\ast_1$) There exists a $0$-definable partition $(S_i)_{d \leq i \leq n}$ of $K^n$ with $\dim S_i \leq i$ such that for any ball $B \subseteq S_{d} \cap i$, $((S_i), \chi)$ is $j$-translatable on $B$, where $j$ is minimal with $B \cap S_j \neq \emptyset$.

Note that ($\ast_0$) implies the theorem. (For balls intersecting $S_0$, there is nothing to prove.)

The start of induction ($\ast_n$) is trivial (set $S_n = K^n$). Now suppose that $(S_i)_{d \leq i \leq n}$ is given such that ($\ast_d$) holds (for some $d \geq 1$). It suffices to find a set $S_{d-1}$ of dimension at most $d - 1$ such that on any ball $B \subseteq K^n \setminus S_{d-1}$, $((S_i), \chi)$ is $d$-translatable; after that, we obtain ($\ast_{d-1}$) using the partition $S_{d-1}, (S_i \setminus S_{d-1})_{i \geq d}$. Moreover, it is enough to check $d$-translatability on balls $B$ with $B \cap S_d \neq \emptyset$.

We have to do the case $d = 1$ separately.

The case $d \geq 2$:

First, we choose a refinement $\chi'$ of $((S_i), \chi)$ whose fibers are subaffine (using Proposition 4.5). Now consider a coordinate projection $\pi: K^n \rightarrow K^{d-1}$. By induction on $n$ (and using $d \geq 2$), we can find $t$-stratifications of the fibers of $\pi$ reflecting $\chi'$ on the fibers. Taking the union of corresponding strata of different fibers yields a $0$-definable partition $(T_i)_{i \leq n-d+1}$ of $K^n$ with $\dim T_i \leq i + d - 1$. Define $S_{d-1}$ to be the union of the sets $T_0$ for all coordinate projections $\pi: K^n \rightarrow K^{d-1}$.

Now let a ball $B \subseteq K^n \setminus S_{d-1}$ with $B \cap S_d \neq \emptyset$ be given; we have to prove that $((S_i), \chi)$ is $d$-translatable on $B$. We will do this by applying Lemma 4.8 to $B$, $\chi$, $\chi'$, and $(S_i)$; so let us produce the remaining ingredients.

Let $C \subseteq S_d$ be any $\chi'$-fiber intersecting $B$ non-trivially, let $V \subseteq K^n$ be $d$-dimensional such that affdir($C$) $\subseteq V$ (which exists since $\dim C \leq d$), and choose an exhibition $\rho: B \rightarrow K^d$ of $V$. The only missing ingredient for Lemma 4.8 is that between any two $\pi$-fibers, there exists a definable isometry respecting $\chi'$; this then also implies $\dim C = d$ and hence affdir($C$) $= V$.

To get the isometries between the fibers, we apply Lemma 4.9 to $\chi'$, $V$, and $C$. Suppose that $\rho: K^d \rightarrow K^{d-1}$ is a coordinate projection and $F$ is a fiber of $\pi' := \rho \circ \pi$. Consider the partition $(T_i)_{i \leq n-d+1}$ of $K^n$ obtained from $t$-stratifications of the fibers of $\pi'$ in the above definition of $S_{d-1}$. Since $T_0 \subseteq S_{d-1}$ we have $B \cap T_0 = \emptyset$, so in particular, $\chi'|_F$ is 1-translatable on $B \cap F$, which is what we need for Lemma 4.9.

The case $d = 1$:

Recall that we do already have a partition $(S_i)_{i \geq 1}$ which is good outside of $S_1$. We will now carry out an additional induction, during which the ‘bad set’ will become ‘more and more 0-dimensional’.

(*$\ast_n$) There exists a family of definable sets $X_\rho$ parameterized by the coordinate projections $\rho: K^n \rightarrow K^e$, such that $\rho(X_\rho)$ is finite, $\dim X_\rho \leq 1$, and for any ball $B \subseteq K^n \setminus \bigcup \rho X_\rho$, $((S_i), \chi)$ is 1-translatable on $B$.

Write $X := \bigcup \rho X_\rho$ for the union. The statement ($\ast_0$) follows from ($\ast_1$), since we can take $X = X_\rho = S_1$ (where $\rho: K^n \rightarrow K^0$). The statement ($\ast_n$) is what we want to prove; in that case, $\rho = 1_{K^n}$ implies that $X$ itself is finite, so we can set $S_0 = X$ (and replace $S_i$ by $S_i \setminus S_0$ for $i \geq 1$); then ($\ast_1$) implies 1-translatability on balls $B \subseteq S_{d+1}$ and ($\ast_1$) implies $d$-translatability on balls $B \subseteq S_{d} \cap d \geq 2$.

Thus it remains to prove ‘($\ast_n$) $\Rightarrow$ ($\ast_{n+1}$)’ for $0 \leq e < n$. Let $X = \bigcup \rho X_\rho$ be given for $e$, and let us construct a set $X'$ for $e + 1$. We start by choosing a refinement $\chi'$ of $((S_i), \chi, (X_\rho)_\rho)$ whose fibers are subaffine.

Let $\rho: K^n \rightarrow K^e$ and $\pi: K^n \rightarrow K$ be coordinate projections ‘projecting to different coordinates’, that is, such that $(\rho, \pi): K^n \rightarrow K^e \times K$ is surjective.
By the main induction on $n$, we can find t-stratifications of the fibers of $\pi$ reflecting $\chi'$ on the fibers. By Proposition 3.19(2), there exists a definable map $\chi_0: K \to \text{RV}^{\text{eq}}$ such that for any $\chi_0$-fiber $C_0 \subseteq K$ and any $x, x' \in C_0$, we have a definable isometry $\pi^{-1}(x) \to \pi^{-1}(x')$ respecting $\chi'$. Lemma 4.11 yields a finite subset $T_0 \subseteq K$ such that $\chi_0$ is constant on each ball $B' \subseteq K \setminus T_0$. Recall that $\pi^\chi: K^n \to K^{n-1}$ denotes the 'complement' of $\pi$ and define the set $X_{\rho, \pi}$ as follows:

$$X_{\rho, \pi} := \{x \in \pi^{-1}(T_0) \mid \pi^\chi(x) \in \pi^\chi(X_\rho)\}.$$ 

We define $X'$ to be the union of all such $X_{\rho, \pi}$ (for all $\rho, \pi$ as above).

Since $T_0$ is finite, $\dim X_{\rho, \pi} \leq \dim X_\rho \leq 1$ and $(\rho, \pi)(X_{\rho, \pi})$ is finite, so it remains to check that on a ball $B \subseteq K^n \setminus X'$, $((S_i)_{i}, \chi)$ is 1-translatable. If $B \cap X = \emptyset$, then we know this by induction on $e$, so suppose that $B \cap X_\rho \neq \emptyset$ for some $\rho: K^n \to K^e$.

Let $C \subseteq X_\rho$ be a $\chi'$-fiber with $C \cap B \neq \emptyset$; note that $\dim C \leq \dim X_\rho \leq 1$. If $\dim(C \cap B) = 0$, then let $\pi: K^n \to K$ be any coordinate projection projecting to a different coordinate than $\rho$. Otherwise, set $V := \text{affdir}(C)$ and let $\psi: K^n \to K$ be an exhibition of $V$. Since $\rho(C)$ is finite, we have $V \subseteq \ker \rho$, so in this case too, $\rho$ and $\pi$ project to different coordinates.

Let $\chi_0$, $T_0$ be as in the construction of $X_{\rho, \pi}$. Then $\pi(B) \cap T_0 = \emptyset$, since otherwise, for $x \in \pi(B) \cap T_0$ and $y \in B \cap X_\rho$, the point $y' \in K^n$ with $\pi(y') = x$ and $\pi^\chi(y') = \pi^\chi(y)$ lies both in $B$ and in $X_{\rho, \pi}$, contradicting $B \cap X_{\rho, \pi} = \emptyset$. By our choice of $T_0$, this implies that $\chi_0$ is constant on $\pi(B)$ and thus there are risometries respecting $\chi'$ between any two fibers $\pi^{-1}(x)$ (for $x \in \pi(B)$). In particular, $C$ intersects every fiber non-trivially and thus $\dim(C \cap B) = 1$.

Now we can apply Lemma 4.8 to $C \cap B$, $B$, $\chi$, $\chi'$, $\pi$, and the partition $(S_i \cup X, (S_i \setminus X)_{i \geq 2})$ (restricted to $B$); this yields that $((S_i)_{i}, \chi)$ is $V$-translatable on $B$, which is what we had to show.

4.4. Corollaries

Using compactness, we can deduce a version of the main theorem which works uniformly for all models of a theory $T$ satisfying Hypothesis 2.21, and also for all models of a finite subset of $T$, provided that the notion of t-stratification makes sense. In particular, we get t-stratifications in all Henselian-valued fields of sufficiently big residue characteristic (both, in the equi-characteristic and the mixed characteristic case). Note that in equi-characteristic, there is no good notion of dimension of a definable set; there, ‘$\dim S_i = i$’ means that we naively apply Definition 2.27. However, in the case of the pure valued field language, this problem will be solved by Corollary 6.5, which says that we can choose the t-stratification such that each set $S_{\leq i}$ is Zariski closed and has dimension $i$ in the algebraic sense.

**Corollary 4.13.** Suppose that $T$ is an $\mathcal{L}$-theory satisfying Hypothesis 2.21. Let $\chi$ be an $\mathcal{L}$-formula defining a map $\chi_K: K^n \to \text{RV}^{\text{eq}}$ (for any model $K \models T$). Then there exist $\mathcal{L}$-formulas $\psi_0, \ldots, \psi_n$ and a finite subset $T_0 \subseteq T$ such that for each model $K$ of $T_0$, $(\psi_i(K))$, is a t-stratification of $K^n$ reflecting $\chi_K$. (For this to make sense, we assume that $T_0$ in particular says that $K$ is a valued field.)

**Proof.** By Theorem 4.12, we find formulas $(\psi_i)$, defining a t-stratification for any fixed model $K \models T$. Moreover, by Corollary 3.22, we also find a formula $\eta$ (depending on $(\psi_i)$) defining the corresponding straighteners on all balls $B \subseteq K^n$ (using parameters). This allows us to formulate a first-order sentence which holds in an $\mathcal{L}$-structure $K'$ if and only if $(\psi_i(K'))$, is a t-stratification reflecting $\chi_K$, namely:

$(\Delta)$ for each $i$, $\psi_i(K')$ is either empty or has dimension $i$ in the sense of Definition 2.27, for each ball $B \subseteq (K')^n$, there exists a parameter $b$ such that $\eta(K', b)$ defines a
straightener on $B$ which witnesses that $((\psi_i(K'))_i, \chi_{K'})$ is $j$-translatable on $B$, where $j$ is minimal with $B \cap S_j \neq \emptyset$.

By compactness, $\psi_i$ and $\eta$ can be chosen such that $(\Delta)$ holds in all models of $T$. Moreover, $(\Delta)$ then follows already from a finite subset of $T$.

The next corollary says that in some sense, the risometry type of a definable subset of $K^n$, or, more generally, of a definable map $K^n \to \text{RV}^\text{eq}$, can be encoded using only $\text{RV}$-data.

**Corollary 4.14.** We assume Hypothesis 2.21. Let $\chi_q: K^n \to \text{RV}^\text{eq}$ be a $\emptyset$-definable family of maps, parameterized by $q \in Q$ (for some definable set $Q$ in any sort). Then there exists a $\emptyset$-definable map $\chi': Q \to \text{RV}^\text{eq}$ such that $\chi'(q_1) = \chi'(q_2)$ implies that there exists a $(q_1, q_2)$-definable risometry $\phi: K^n \to K^n$ with $\chi_{q_1} \circ \phi = \chi_{q_2}$. This also works uniformly for all models $K$ of a finite subset of $T$.

**Proof.** If we add a constant symbol for $q$ to the language, then Corollary 4.13 yields uniformly defined t-stratifications reflecting $\chi_q$ in each model of a finite subset of $T$ and for each $q \in Q$. Now $\chi'$ is obtained from Proposition 3.19(2).

In Section 6 (where we prove an algebraic version of the main result), we will give an algebraic version of this corollary (Corollary 6.7). (That version follows directly from Corollary 4.14, but thematically, it fits better into Section 6.)

4.5. **Characterizations of reflection**

For applications of the main theorem, it will be useful to understand more precisely what it means that a t-stratification $(S_i)_i$ reflects a map. Proposition 4.17 gives different equivalent conditions for this; in particular, there exists a finest map reflected by $(S_i)_i$, the ‘rainbow’ of $(S_i)_i$. (If one thinks of the fibers of this map as having different colors, then it indeed looks a bit like a rainbow, in particular near $S_1$.)

A simple consequence of Proposition 4.17 (which was not clear from the definition of reflection) is that if $(S_i)_i$ reflects both $\chi$ and $\chi'$, then it also reflects $(\chi, \chi')$ (Remark 4.18). Other consequences are Corollary 4.19 and Lemma 4.20, which together will allow us to ‘enhance’ t-stratifications in the following sense. Given $(S_i)_i$, we will find $(S'_i)_i$ which reflects at least as much as $(S_i)_i$ and which has additional good properties.

The last results of this subsection give some more information about the fibers $C$ of a rainbow. These will be needed in Lemma 5.9, which, as a side result, yields an even more precise description of these $C$; see Remark 5.11.

**Definition 4.15.** Let $(S_i)_i$ be a t-stratification of a ball $B_0 \subseteq K^n$. We define the rainbow of $(S_i)_i$ to be the map $\rho: B_0 \to \text{RV}^\text{eq}, x \mapsto \gamma(\hat{rv}(x - S_i))_{i \leq n}$, where $\hat{rv}(x - S_i) = \{\hat{rv}(x - y) \mid y \in S_i\}$. (Recall that such a code exists in $\text{RV}^\text{eq}$ by stable embeddedness of $\text{RV}$.)

The rainbow is not uniquely determined, since we have to choose a code; however, this choice will never matter; therefore, we take the freedom to speak of ‘the’ rainbow of a t-stratification.

**Remark 4.16.** It is clear that the rainbow $\rho$ of $(S_i)_i$, refines $(S_i)_i$ in the sense of Definition 4.1 (and Convention 3.2), so any risometry $\phi: B_0 \to B_0$ respecting $\rho$ also respects $(S_i)_i$. Vice versa, if $\phi$ is a risometry respecting $(S_i)_i$, then for any $x \in B_0$ we have $\hat{rv}(x - S_i) = \hat{rv}(\phi(x) - S_i)$ and hence $\phi$ respects $\rho$. 
Proposition 4.17. Let $(S_i)_i$ be a t-stratification of $B_0 \subseteq K^n$ and let $\chi: B_0 \rightarrow RV^{eq}$ be a definable map. Then the following are equivalent.

1. The t-stratification $(S_i)_i$ reflects $\chi$.
2. The rainbow of $(S_i)_i$ is a refinement of $\chi$.
3. Any definable risometry $\phi: B_0 \rightarrow B_0$ respecting $(S_i)_i$ also respects $\chi$.

Proof. (2) $\Rightarrow$ (3) follows from Remark 4.16.

(3) $\Rightarrow$ (1): For every ball $B \subseteq B_0$, we have to show that $\text{tsp}_B((S_i)_i, \chi) = \text{tsp}_B((S_i)_i)$. Let $(\alpha_x)_x$ be a translataler of $(S_i)_i$ on $B$, with respect to any exhibition of $\text{tsp}_B((S_i)_i)$ (see Definition 3.8). Extending each $\alpha_x$ by the identity on $B \setminus B$ yields risometries $\alpha_x: B_0 \rightarrow B_0$ respecting $(S_i)_i$. By (3), these risometries also respect $\chi$, hence $(\alpha_x)_x$ is also a translataler for $(S_i)_i$.

Choose an exhibition $\pi: B \rightarrow K^d$ of $V$ and a corresponding translataler $(\alpha_x)_{x \in \pi(B - B)}$ of $(S_i)_i$. Let $x_j := \pi(y_j)$ and let $F_j := \pi^{-1}(x_j)$ be the fiber containing $y_j$. Then for $y'_1 := \alpha_{x_2-x_1}(y_1) \in F_2$, we have $\chi(y'_1) = \chi(y_1) \neq \chi(y_2)$. Moreover, since $\alpha_{x_2-x_1}$ respects $(S_i)_i$, it also respects its rainbow (by Remark 4.16), that is, $\rho(y'_1) = \rho(y_1) = \rho(y_2)$.

Now set $B' := B(y'^1_1, \geq \delta(y'_1 - y_2)) \subseteq B$. It remains to show that $B' \cap S_d = \emptyset$ to get a contradiction to the maximality of $d$. The set $T := S_d \cap F_2$ is finite but non-empty. For any $y \in F_2$, we have $\Delta(v(y - T) = \Delta(v(y - S_d) \cap \Delta(v(F_2 - F_2))$ by $V$-translatability on $B$, thus $\rho(y'_1) = \rho(y_2)$ implies $\Delta(v(y'_1 - T) = \Delta(v(y_2 - T)$. Now, Lemma 2.15 implies $(B' \cap F_2) \cap T = \emptyset$, which in turn implies $B' \cap S_d = \emptyset$.

Remark 4.18. The equivalence (1) $\Leftrightarrow$ (2) implies that for any two definable maps $\chi_1, \chi_2: B_0 \rightarrow RV^{eq}$, $(S_i)_i$ reflects the product $(\chi_1, \chi_2)$ if and only if it reflects $\chi_1$ and $\chi_2$ separately. In particular, in the remainder of the article, we will use these two statements interchangeably.

As mentioned at the beginning of this subsection, the following two results will be useful to ‘enhance’ a given t-stratification; see the proofs of Lemma 5.9 and Proposition 6.2 for applications.

Corollary 4.19. Let $(S_i)_i$ and $(S'_i)_i$ be two t-stratifications. Then the following are equivalent.

1. The rainbow of $(S'_i)_i$ refines the rainbow of $(S_i)_i$.
2. Any definable map into $RV^{eq}$ reflected by $(S'_i)_i$ is also reflected by $(S_i)_i$.
3. The t-stratification $(S'_i)_i$ reflects $S_j$ for each $j \leq n$.

Proof. (1) $\Leftrightarrow$ (2) follows from Proposition 4.17(1) $\Leftrightarrow$ (2).

(3) $\Rightarrow$ (2) is obtained by using Proposition 4.17(1) $\Leftrightarrow$ (3) to translate everything into statements about which risometries respect what.

For (1) $\Rightarrow$ (3), note that (1) implies that the rainbow of $(S'_i)_i$ refines $(S_i)_i$, so (3) follows from Proposition 4.17(2) $\Rightarrow$ (1).

If the above conditions hold, then $S_{\leq i} \subseteq S'_{\leq i}$ for all $i$ by Lemma 3.10. However, requiring $S_{\leq i} \subseteq S'_{\leq i}$ for all $i$ is not enough to imply the conditions of the corollary.
Lemma 4.20. Suppose that $X \subseteq K^n$ is a definable set of dimension $d$, that $\chi: X \to RV^\text{eq}$ is a definable map, and that $(S_i)_i$, $(T_i)_i$ are two $t$-stratifications of $K^n$, where $(T_i)_i$ reflects $(S_i)_i$ and $\chi$. (Here, we extend the domain of $\chi$ to $K^n \setminus X$ to a single new element.) Then the following defines a $t$-stratification which reflects $(S_i)_i$ and $\chi$ and which agrees with $(S_i)_i$ outside of $X \cup T_{\leq d-1}$:

$$S'_{\leq i} := \begin{cases} T_{\leq i} & \text{for } i < d, \\ S_{\leq i} \cup X \cup T_{\leq d-1} & \text{for } i \geq d. \end{cases}$$

Proof. It is clear that $\dim S'_{\leq i} \leq i$, so now consider a ball $B \subseteq S'_{\geq j}$; we have to show that $((S'_{\leq i}), (S_i)_i, \chi)$ is $j$-translatable on $B$. If $j < d$, then we have $j$-translatability since $B \subseteq S'_{\geq j} = T_{\geq j}$ and $(T_i)_i$ reflects $((S_i)_i, \chi)$; if $j \geq d + 1$, then $B \cap (X \cup T_{\leq d-1}) = \emptyset$ and $S'_i \cap B = S_i \cap B$ for every $i$, so $j$-translatability follows from $j$-translatability of $(S_i)_i$. □

To finish this subsection, we give some more properties of the fibers of a rainbow.

Lemma 4.21. Let $C$ be a fiber of the rainbow of a $t$-stratification $(S_i)_i$. Then either $C$ consists of a single element of $S_0$ or it is entirely contained in a ball $B \subseteq S_{\geq 1}$ and moreover, for any exhibition $\pi: B \to K^d$ of $\text{tsp}_B((S_i)_i)$ and any fiber $F = \pi^{-1}(y)$ (where $y \in \pi(B)$), $C \cap F$ is exactly one fiber of the rainbow of the induced $t$-stratification $(S_i+d \cap F)_{i\leq n-d}$ of $F$.

Proof. Suppose that $x, x' \in C$. Since $\hat{rv}(x - S_0) = \hat{rv}(x' - S_0)$, Lemma 2.15 implies that there exists a isometry $K^n \to K^n$ fixing $S_0$ pointwise and sending $x$ to $x'$. If $x \in S_0$, we obtain $x' = x$, so $C = \{x\} \subseteq S_0$; otherwise, by Lemma 2.15(2) (b), there exists a ball $B \subseteq S_{\geq 1}$ containing both $x$ and $x'$. By doing this for all pairs $x, x' \in C$, we obtain a single ball $B \subseteq S_{\geq 1}$ with $C \subseteq B$.

Let $\pi: B \to K^d$ be an exhibition of $V := \text{tsp}_B((S_i)_i)$ and let $F = \pi^{-1}(y)$ be a fiber (where $y \in \pi(B)$); denote the induced $t$-stratification of $F$ by $(S'_i)_{i\leq n-d}$ and consider $x_1, x_2 \in F$. We have to check that $x_1$ and $x_2$ have the same image under the rainbow of $(S_i)_i$ if and only if they have the same image under the rainbow of $(S'_i)_i$. It is clear that for any $i$, we have $\hat{rv}(x_1 - (S_i \setminus B)) = \hat{rv}(x_2 - (S_i \setminus B))$. This implies $\hat{rv}(x_1 - S_i) = \hat{rv}(x_2 - S_i)$ for $i < d$ and it remains to verify that for $i \geq d$, we have $\hat{rv}(x_1 - (S_i \cap B)) = \hat{rv}(x_2 - (S_i \cap B))$ if and only if $\hat{rv}(x_1 - S'_{i-d}) = \hat{rv}(x_2 - S'_{i-d})$. This equivalence follows from $V$-translatability of $S_i$ on $B$. Indeed, translatability, on the one hand, implies $\hat{rv}(x_j - S'_{i-d}) = \hat{rv}(x_j - S_i) \cap \hat{rv}(F \cap B)$ and hence ‘⇒’; on the other hand, we obtain $\hat{rv}(x_j - (S_i \cap B)) = \hat{rv}(x_j - S'_{i-d}) + \hat{rv}(S_i \cap B) \cap F)$ for any lift $\hat{V} \subseteq K^n$ of $V$ (the sum on the right-hand side is well-defined by Lemma 2.10(1), and this implies ‘⇐’. □

Lemma 4.22. Suppose that $(S_i)_i$ is a $t$-stratification and that $C \subseteq S_d$ is a fiber of the rainbow of $(S_i)_i$. Then $C$ is subaffine (cf. Definition 4.2) with $\text{affdir}(C) = \text{tsp}_B((S_i)_i)$ for any $B \subseteq S_{\geq d}$ with $B \cap C = \emptyset$.

Proof. Lemma 3.10 implies $\dim C = d$, hence it suffices to prove that $\text{affdir}(C) \subseteq \text{tsp}_B((S_i)_i)$. We do an induction on $d$. If $d = 0$, then $C$ consists of a single element by Lemma 4.21 and the claim is clear. Otherwise, choose a ball $B' \subseteq S_{\geq 1}$ containing $C$ and choose an exhibition $\pi: B' \to K^d$ of $V' := \text{tsp}_{B'}((S_i)_i)$. By induction, for each $\pi$-fiber $F \subseteq B'$ and each suitable $B'' \subseteq F$, we have $V'' := \text{affdir}(C \cap F) = \text{tsp}_{B''}((S_i \cap F)_i)$. By $V'$-translatability, $V''$ does not depend on the choice of $F$, and by choosing $F$ and $B''$ such that $B'' = B \cap F \neq \emptyset$, the claim follows.
we obtain tsp$_B((S_i)_i) = V' + V''$. Now choose any $x, x' \in C$, denote the corresponding $\pi$-fibers by $F$ and $F'$, respectively, and let $x''$ be the image of $x'$ in $F$ under a translator sending $F'$ to $F$. Then we have $\text{dir}(x - x'') \in V''$ and $\text{dir}(x'' - x') \in V'$ and hence $\text{dir}(x - x') \in V' + V''$ by Lemma 2.10(2).

5. Fields with analytic structure satisfy Hypothesis 2.21

Now it is time to prove that there do exist theories $T$ satisfying Hypothesis 2.21. In [3], Cluckers and Lipshitz introduce a notion of ‘Henselian-valued field with analytic structure’, which generalizes many older notions of analytic structures. We will prove that any Henselian-valued field of residue characteristic 0 with analytic structure in that sense satisfies Hypothesis 2.21.

Hypothesis 2.21$_0$ follows directly from the results of [3]; we will give the arguments in Subsection 5.2. Proving the Jacobian property needs more work. In dimension one, this is also done in [3] (up to some more minor differences in the definitions), but one essential ingredient to that proof is that any definable map $K \rightarrow \text{RV}^{eq}$ can be refined in such a way that each fiber becomes a ball or a point. In higher dimensions, we will instead use our main theorem inductively to refine any definable map $K^n \rightarrow \text{RV}^{eq}$ to the rainbow $\rho$ of a t-stratification $(S_i)_i$. We will see that each $\rho$-fiber is, up to a linear map and a risometry, a product of balls, and if we are careful with the choice of $(S_i)_i$, we can moreover ensure that these risometries are analytic (Lemma 5.9). This will allow the arguments from [3] proving the Jacobian property in dimension one to go through also in higher dimension (Proposition 5.12).

Since $T = T_{\text{Hen}}$ is a special case of a theory of fields with analytic structure (see Example 5.1), the proofs of this section in particular apply to that case, that is, $T_{\text{Hen}}$ satisfies Hypothesis 2.21.

5.1. The setting

Let us fix the setting for the whole of Section 5. Concerning the notion of fields with analytic structure, I will only repeat those properties from [3] that are relevant to us, since the complete definition is somewhat technical.

Given a ‘separated Weierstraß system’ $\mathcal{A}$ (see [3, Definition 4.1.5]), one obtains a language $\mathcal{L}_{\text{Hen},\mathcal{A}}$ (see [3, beginning of Section 6.2]) and there is a notion of a ‘separated analytic $\mathcal{A}$-structure’ on a valued field $K$ [3, Definition 4.1.6] which turns $K$ into an $\mathcal{L}_{\text{Hen},\mathcal{A}}$-structure. The sorts of $\mathcal{L}_{\text{Hen},\mathcal{A}}$ are $K$ and $\text{RV}$. (More precisely, there are several $\text{RV}$-like sorts, but these are all the same when the residue characteristic is zero.) We let $\mathcal{L}$ be the union of $\mathcal{L}_{\text{Hen},\mathcal{A}}$ and the remaining sorts of $\text{RV}^{eq}$ (together with the canonical maps), and we let $T$ be the $\mathcal{L}$-theory of all Henselian-valued fields of equi-characteristic 0 with separated analytic $\mathcal{A}$-structure. (This is the same as the $\mathcal{L}_{\text{Hen},\mathcal{A}}$-theory $T_{\text{Hen},\mathcal{A}}$ defined at the beginning of [3, Section 6.2], except for the additional sorts in $\mathcal{L}$ and for the fact that $T_{\text{Hen},\mathcal{A}}$ does not require the residue characteristic to be 0.)

As always, $K$ will denote a model of $T$.

Example 5.1. By [3, Section 4.4, Example (13)], on any Henselian-valued field there exists an analytic structure whose definable sets are exactly the $\mathcal{L}_{\text{Hen}}$-definable sets. Since we care about the language only up to interdefinability, this implies that the results of this section apply to $\mathcal{L}_{\text{Hen}}$ and $T_{\text{Hen}}$.

Here is another concrete example of fields with analytic structure (a special case of [3, Section 4.4(1)]); a lot of more general examples are given in [3, Section 4.4].
Example 5.2. Let $A := \mathbb{Z}[\lbrack \! \lbrack t \rbrack \! \rbrack]$ be equipped with the $t$-adic valuation (which we denote by $v$), let

$$T_m := A(\xi_1, \ldots, \xi_m) = \left\{ \sum_{\nu \in \mathbb{N}^m} c_{\nu} \xi^\nu \mid c_{\nu} \in A, \lim_{|\nu| \to \infty} v(c_{\nu}) = \infty \right\}$$

be the algebra of restricted power series (here, we use multi-index notation), and set

$$S_{m,n} := T_m[[\rho_1, \ldots, \rho_n]].$$

As a language, take $L := \mathcal{L}_{\text{Hen}} \cup \bigcup_{m,n} S_{m,n}$, where each element of $S_{m,n}$ is a symbol for an $(m+n)$-ary function.

Now suppose that $K$ is a complete valued field of rank 1 and of residue characteristic 0 (for example, $K = \mathbb{C}(\lbrack \! \lbrack t \rbrack \! \rbrack)$). Suppose moreover that $K$ extends $A$ as a valued ring; in particular, we identify $t \in A$ with an element of $\mathcal{M}_K$. Then each element of $S_{m,n}$ naturally defines a function $O^n_K \times \mathcal{M}_K \to O_K$. This turns $K$ into an $L$-structure (after extending these functions trivially to $K^{m+n}$) and as such, $K$ is a valued field with analytic structure in the sense of [3].

5.2. Fields with analytic structure satisfy Hypothesis 2.210

We recall those definitions and results of [3] that we will need.

The following definition of ‘b-minimality with centers’ is the conjunction of [3, Definitions 6.3.1 and 6.3.2] (slightly simplified, since we work in equi-characteristic 0). It differs from the notion of b-minimality from Definition 2.25 in two aspects. First, ‘with centers’ is a strengthening of Condition (1), and second, it only uses the sorts $K$ and $RV$, whereas in Subsection 2.5, we also allowed the other sorts of $RV^{eq}$. (The latter seems to be a purely technical difference, but I don’t know a general proof that adding imaginaries to the auxiliary sorts is harmless.)

Definition 5.3. An expansion $T$ of $T_{\text{Hen}}$ is b-minimal with centers over $RV$ if for every model $K \models T$ and every set $A \subseteq K \cup RV$, the following holds.

1. For every $A$-definable set $X \subseteq K$, there exists an $A$-definable map $\chi : X \to Q \subseteq RV^\ell$ (for some $\ell$) and an $A$-definable map $c : Q \to K$ such that for each $q \in Q$, the fiber $\chi^{-1}(q)$ is of the form $c(q) + v^{-1}(\xi)$ for some $\xi \in RV$ (depending on $q$).
2. There exists no surjective definable map from a subset of $RV^\ell$ to an open ball $B \subseteq K$ (for any $\ell$).
3. For every $A$-definable $X \subseteq K$ and $f : X \to K$, there exists an $A$-definable map $\chi : X \to Q \subseteq RV^\ell$ (for some $\ell$) such that for each $q \in Q$, $f|_{\chi^{-1}(q)}$ is either injective or constant.

Lemma 5.4 ([3, Theorem 6.3.7]). Every $L$-formula is, modulo $T$, equivalent to an $L$-formula without valued field quantifiers. Moreover, $T$ is b-minimal with centers over $RV$.

In [3], the first statement of that lemma is formulated with $\mathcal{L}_{\text{Hen},A}$ instead of $L$, but it is easy to deduce the $L$-version from the $\mathcal{L}_{\text{Hen},A}$-version.

The next result uses a definitory expansion $\mathcal{L}^*_{\text{Hen},A}$ of the language $\mathcal{L}_{\text{Hen},A}$ by certain functions $h_{m,n}$ which yield zeros of polynomials (cf. [3, Definition 6.1.7]); we set $\mathcal{L}^* := \mathcal{L} \cup \mathcal{L}^*_{\text{Hen},A}$. This language $\mathcal{L}^*_{\text{Hen},A}$ (and hence also $\mathcal{L}^*$) has the following useful property: for any function symbol defining a function $f : K^m \times RV^\ell \to K$ and for any fixed $a \in K^m$, the function $f(a, \cdot) : RV^\ell \to K$ has finite image.
LEMMA 5.5 [3, Theorem 6.3.8]. For any parameter set $A \subseteq K \cup \RV^{eq}$, any $A$-definable function $K^n \to K$ can be written as $t(x, g(x))$, where $t$ is an $\mathcal{L}^*(A)$-term and $g$ is an $A$-definable function from $K^n$ to $\RV^\ell$ for some $\ell$.

Again, Cluckers and Lipshitz [3] prove this for $\mathcal{L}_{\text{Hen},A}$ (and in particular $A \subseteq K \cup \RV$), but the proof can easily be adapted to $\mathcal{L}^*$.

PROPOSITION 5.6. Hypothesis 2.21$_0$ holds in the setting described in Subsection 5.1.

Proof. Hypothesis 2.21(1) follows from Lemma 5.4 using that in $\mathcal{L}$, the only connection between $K$ and $\RV^{eq}$ is the map $\text{rv}$. Hypothesis 2.21(2) follows from Lemma 5.5 for $n = 0$, using the property of $\mathcal{L}^*$ described above that lemma.

Concerning Hypothesis 2.21(3), let $K \models T$, $A \subseteq K \cup \RV^{eq}$ and $X \subseteq K$ be given and set $A_0 := A \cap K$. Then $X$ is $(A_0 \cup \{b\})$-definable for some tuple $b \in \RV^{m}$. Applying Definition 5.3(1) yields $(A_0 \cup \{b\})$-definable maps $\chi : X \to Q \subseteq \RV^\ell$ and $c : Q \to K$. By writing $c(q) = c'(q, b)$ for some $A_0$-definable map $c'$ and applying Hypothesis 2.21(2) to $c'$, we find that the image of $c$ is contained in a finite, $A_0$-definable set $S_0$. Now suppose that for some ball $B \subseteq K \setminus S_0$, we have $B \cap X \neq \emptyset$, say $x_0 \in B \cap X$; we have to verify that $B \subseteq X$. Using that $c(\chi(x_0)) \in S_0$, we obtain that the map $x \mapsto \text{rv}(x - c(\chi(x_0)))$ is constant on $B$. By the description of the fibers of $\chi$, this implies $B \subseteq \chi^{-1}(\chi(x_0)) \subseteq X$.

To obtain Hypothesis 2.21$_0$(4$''$), we only have to generalize Definition 5.3(3) from parameter sets in $K \cup \RV$ to parameter sets in $K \cup \RV^{eq}$. Thus, let $K$, $A$, $X$ and $f$ be given (where $A \subseteq K \cup \RV^{eq}$). As before, set $A_0 := A \cap K$ and choose $b \in \RV^{m}$ such that $X$ and $f$ are $(A_0 \cup \{b\})$-definable. We find an $(A_0 \cup \{b\})$-definable map $\chi' : X \to \RV^{eq}$ that is as desired, and it remains to modify it to make it $A$-definable. Again, we have $\chi'(x) = \chi''(x, b)$ for some $A_0$-definable map $\chi'' : X \times \RV^{m} \to \RV^{eq}$; we define $\chi(x) := \gamma(y \mapsto \chi''(x, y))$. This map is $A$-definable, we may suppose that its range lies in $\RV^{eq}$ by Hypothesis 2.21(1), and since $\chi$ refines $\chi'$, $f$ has the desired property on each $\chi$-fiber.

5.3. Fields with analytic structure have the higher-dimensional Jacobian property

As described at the beginning of this section, the strategy to obtain the Jacobian property is to use our main result inductively and to describe fibers of rainbows more precisely. Before doing that, we will adapt two lemmas of [3]. Throughout, we will use the abstract notion of analytic functions introduced in [3]. This notion works more smoothly when the field $K$ is algebraically closed, so in most of this subsection, we will restrict to that case. We will see at the end that this is enough to obtain the Jacobian property also for non-algebraically closed $K$.

So, from now on assume that $K$ is algebraically closed. Then we have the following notions and results from [3].

(i) A domain is a particular kind of definable subset of $K^n$ [3, Definition 5.2.2]; in particular, products of balls $B_i \subseteq K$ are domains. (Strictly speaking, according to [3, Definition 5.2.2], only subsets of $\mathcal{O}^*_K$ can be domains; however, via scaling one easily generalizes the notion to include bigger domains; cf. also [3, Remark 5.2.15].) There are also notions of open and closed domains. (A closed domain is essentially an abstract version of a rational domain in the sense of rigid geometry.)

(ii) If $X$ is either an open domain or a closed domain, then we have a well-defined ring of analytic functions $A(X)$ consisting of certain definable maps from $X$ to $K$; see [3, Definition 5.2.2 and Corollary 5.2.14]. (In [3], the ring $A(X)$ is denoted by $\mathcal{O}^*_K(X)$.)

We start by proving a higher-dimensional version of [3, Lemma 6.3.15]; the idea of the proof is the same, with balls replaced by subsets of domains.
Lemma 5.7. Suppose that $K$ is algebraically closed and let $f: K^n \to K$ be an $A$-definable function (for some set of parameters $A \subseteq K \cup RV^{eq}$). Then there exists an $A$-definable map $\chi: K^n \to RV^{eq}$ such that for each $\chi$-fiber $C \subseteq K^n$, there exists an open domain $X \subseteq K^n$ containing $C$ such that $f|_C$ is equal to $\tilde{f}|_C$ for some analytic function $\tilde{f} \in \mathcal{A}(X)$.

Proof. By Lemma 5.5, $f$ can be written as $f(x) = t(x, g(x))$, where $t$ is an $\mathcal{L}^*(A)$-term and $g$ is an $A$-definable function with range in $RV^{eq}$. By compactness, it therefore suffices to prove the lemma for the maps $x \mapsto t(x, \xi)$, where $\xi$ is a fixed element of the image of $g$. We may assume that all subterms of $t$ are $K$-valued and we will omit $\xi$ from the notation.

The cases $t = x_i$ (for $i_0 \leq n$) and $t = a$ (for $a \in A$) are trivial (we only need to set $\chi(x_1, \ldots, x_n) := (rv(x_1), \ldots, rv(x_n))$ to ensure that each $\chi$-fiber is contained in a domain), so suppose that $t = h(t_1', \ldots, t'_i)$ where $h$ is a function symbol in $\mathcal{L}^*$. Induction yields definable maps $\chi'_j: K^n \to RV^{eq}$ for the terms $t_j'$. We set $\chi(x) := (\chi'_1(x), \ldots, \chi'_i(x), rv(t_1'(x)), \ldots, rv(t_i'(x)))$.

Fix $q_j \in im \chi'_j$ and $\xi_j \in RV$ for each $j$ and consider $C := \chi^{-1}(q_1, \ldots, q_i, \xi_1, \ldots, \xi_i)$. For each $j$, we have $C \subseteq C_j' := (\chi_j')^{-1}(q_j)$ and induction yields a domain $X_j' \supseteq C_j'$ and an analytic function $t_j' \in \mathcal{A}(X_j')$ with $t_j'|_{C_j'} = \tilde{t_j}|_{C_j'}$.

We define $X := \bigcap_j X_j'$, where $X_j = \{x \in X_j' \mid rv(t_j'(x)) = \xi_j\}$ if $\xi_j \neq 0$ and $X_j'' = X_j'$ if $\xi_j = 0$. This is an open domain and it contains $C$. Next, we define $t := h(t_1'', \ldots, t_i'')$, where $t_j'' := \tilde{t_j}$ if $\xi_j \neq 0$ and $t_j'' := 0$ if $\xi_j = 0$. With this definition, we have $t|_{C} = \tilde{t}|_{C}$, and it remains to check that $\tilde{t}$ is analytic on $X$. If $h$ is $y_1 + y_2$ or $y_1 \cdot y_2$, then this is clear; otherwise, this follows from [3, Lemma 6.3.11], since our construction ensures that $rv(t_j'')$ is constant on $X$ for each $j$. (Note that Lemma 6.3.11 also covers the case $h(y_1) = y_1^{-1} = h_{1,1}(1, -y_1, rv(1/y_1))$.)

The next lemma is a variant of [3, Lemma 6.3.9].

Lemma 5.8. Suppose that $K$ is algebraically closed and that $g \in \mathcal{A}(\mathcal{O}_K)$ is an analytic function such that $g'(x) \in \mathcal{O}_K$ for every $x \in \mathcal{O}_K$ and $res(g'(x))$ is constant (where $g'$ denotes the derivative of $g$). Then, for any $x_1, x_2 \in \mathcal{O}_K$ with $x_1 \neq x_2$, we have

$$v(g(x_1) - g(x_2) - g'(0) \cdot (x_1 - x_2)) > v(x_1 - x_2).$$

Proof. First of all, note that we may suppose that $g'(0) = 0$; otherwise, replace $g(x)$ by $g(x) - g'(0)x$. Thus, the assumption becomes that $g'(x) \in \mathcal{M}_K$ for all $x$ and we want to prove that $v(g(x_1) - g(x_2)) > v(x_1 - x_2)$.

Recall how analytic functions on $\mathcal{O}_K^n$ are defined in [3]: an analytic structure on the field $K$ yields a ring of abstract power series $A_{1,0}(K) \subseteq \mathcal{O}_K[[\xi_1, \ldots, \xi_n]] \otimes \mathcal{O}_K$ and a ring homomorphism $f \mapsto f^\sigma$ from $A_{1,0}(K)$ to the ring of maps from $\mathcal{O}_K^n$ to $K$: $\mathcal{A}(\mathcal{O}_K^n)$ is the image of $A_{1,0}(K)$ under $\sigma$. Let $f = \sum_i a_i \xi_i \in A_{1,0}(K)$ be a preimage of $g$ under $\sigma$, that is, $g = f^\sigma$.

The assumption $g'(\mathcal{O}_K) \subseteq \mathcal{M}_K$ implies $a_i \in \mathcal{M}_K$ for all $i \geq 1$. Indeed, otherwise, choose any $r \in K$ with $v(r) = -\min_{i \geq 1} v(a_i)$ (the minimum exists by [3, Remark 4.1.10]) and consider the series $r \cdot f^\sigma \in \mathcal{O}_K[[\xi]]$. By [3, Definition 4.1.2(iv)], $res(r \cdot f^\sigma)$ is a polynomial and since $g'(\mathcal{O}_K) \subseteq \mathcal{M}_K$ and $v(r) \geq 0$, it is 0 everywhere on $k$. However, this implies that all its coefficients are 0 and hence $v(ra_i) > 0$ for all $i \geq 1$, contradicting the choice of $r$.

Using Weierstraß division [3, Definition 4.1.3], we find a power series $h(x, y) \in A_{2,0}(K)$ such that $f(x) - f(x + y) = h(x, y) \cdot y$. Since the constant term of the left-hand side cancels, all its coefficients lie in $\mathcal{M}_K$. This then also holds for $h$ and hence (using [3, Remark 4.1.10]
again) the range of $h^\sigma$ is contained in $\mathcal{M}_K$. This implies the lemma, since $g(x_1) - g(x_2) = h^\sigma(x_1, x_2 - x_1) \cdot (x_2 - x_1)$.

Recall that if a t-stratification $(S'_i)_i$ reflects another t-stratification $(S_i)_i$, then it also reflects any definable map into $RV^{eq}$ reflected by $(S_i)_i$ (Corollary 4.19). The following lemma morally consists of two separate statements, namely (1) given any $(S_i)_i$, one can find $(S'_i)_i$ that reflects $(S_i)_i$ and that is ‘piecewise analytic’; and (2) any rainbow of any t-stratification has fibers of a simple form. In the lemma, the piecewise analyticity is formulated in terms of the fibers of the rainbow of $(S'_i)_i$ (which is what we really need in the application). In Remark 5.11, we will give a separate formulation of statement (2).

We call a map from an open or closed domain to $K^d$ analytic if each of its coordinates is analytic.

**Lemma 5.9.** Suppose that we are in the setting of Subsection 5.1 and that $T$ additionally satisfies Hypothesis 2.21_{n-1}. Suppose moreover that $K$ is algebraically closed and that $(S_i)_i$ is an $A$-definable t-stratification of $K^n$ for some set of parameters $A \subseteq K \cup RV^{eq}$. Then there exists an $A$-definable t-stratification $(S'_i)_i$ reflecting $(S_i)_i$ such that for any fiber $C$ of the rainbow of $(S'_i)_i$ with $C \subseteq S'_n$, we have the following.

1. The fiber $C$ is an open domain.
2. There exist open balls $B_1, \ldots, B_n \subseteq K$ and an analytic bijection $\phi: B_1 \times \cdots \times B_n \to C$ that can be written as the composition of a risometry and a matrix from $GL_n(\mathcal{O}_K)$.

**Remark 5.10.** By Remark 2.12, the order of the composition in (2) does not matter, and the composition of several such maps can again be written as a composition of a single risometry and a single matrix from $GL_n(\mathcal{O}_K)$.

**Proof of Lemma 5.9.** The proof consists of two parts. First, we will construct $(S'_i)_i$ in such a way that each fiber of the rainbow is (essentially) the graph of an analytic function. (This will be needed for all fibers and not just for the ones contained in $S'_n$). In the second part, we will show that this is enough to imply the lemma.

**Part 1:**
Recall that if $C$ is a fiber of the rainbow of a t-stratification, then $C$ is subaffine by Lemma 4.22, and by Lemma 4.3, if $\pi: K^{n} \twoheadrightarrow K^{j}$ exhibits affdir$(C)$ (with $j = \dim C$), then $C$ is the graph of a function $c: \pi(C) \rightarrow K^{n-j}$.

By downwards induction on $d$, we will prove that we can find an $A$-definable t-stratification $(S'_i)_i$ reflecting $(S_i)_i$ with the following property. If $C \subseteq S'_j$ is a fiber of the rainbow of $(S'_i)_i$, for some $j \geq d$ and $\pi, c$ are as above, then there exists an open domain $X \subseteq K^j$ containing $\pi(C)$ such that $c$ is the restriction to $\pi(C)$ of an analytic function $X \rightarrow K^{n-j}$.

By induction, we may assume that $(S_i)_i$ itself has the above property for $d + 1$. To obtain a t-stratification $(S'_i)_i$, which has the property for $d$, we proceed as follows. For any rainbow fiber $C \subseteq S_d$ and for $\pi, c$ as above, we apply Lemma 5.7 to find a $C^{eq}$-definable map $\tilde{\chi}: \pi(C) \rightarrow RV^{eq}$ such that each $\tilde{\chi}$-fiber $F \subseteq \pi(C)$ is contained in an open domain $X \subseteq K^d$ such that $c|_{F}$ is the restriction of an analytic function on $X$. Composing $\tilde{\chi}$ with the projection $C \rightarrow \pi(C)$ yields a definable map $C \rightarrow RV^{eq}$; we take the product of these maps for all exhibitions $\pi$ of affdir$(C)$ to obtain a single map $C \rightarrow RV^{eq}$ and we do this for all $C \subseteq S_d$; using compactness, this yields an $A$-definable map $\chi: S_d \rightarrow RV^{eq}$.

Let $(T_i)_i$ be a t-stratification reflecting $(S_i)_i$ and $\chi$ (this exists by Theorem 4.12, since we are assuming Hypothesis 2.21_{n-1}). Applying Lemma 4.20 to $(S_i)_i$, $(T_i)_i$, $X := S_d$, and $\chi$, we
obtain a t-stratification \((S'_i)_i\), reflecting both \((S_i)_i\) and \(\chi\) with \(S'_i \subseteq S_i\) for \(i \geq d\). We claim that this \((S'_i)_i\) has the desired properties.

Let \(C' \subseteq S'_j\) be a fiber of the rainbow of \((S'_i)_i\), where \(j \geq d\). By Corollary 4.19, there exists a fiber \(C\) of the rainbow of \((S_i)_i\), entirely containing \(C'\) and since \(S'_j \subseteq S_j\), we have \(C \subseteq S_j\) and hence (using Lemma 4.22) \(\text{affdir}(C') = \text{affdir}(C)\). Let \(\pi:\ K^n \to K^d\) be an exhibition of \( \text{affdir}(C)\) and let \(c: \pi(C) \to K^{n-j}\) and \(c': \pi(C') \to K^{n-j}\) be the functions whose graphs are \(C\) and \(C'\), respectively; note that \(c'\) is simply the restriction of \(c\) to \(\pi(C')\).

If \(j > d\), then by induction \(c\) is the restriction of an analytic function to \(\pi(C)\), so \(c'\) is the restriction of the same function to \(\pi(C')\). If \(j = d\), then the fact that \((S'_i)_i\) reflects the map \(\chi\) implies that \(\chi\) is constant on \(C'\) and hence that the corresponding map \(\tilde{\chi}: \pi(C) \to \text{RV}^{\text{eq}}\) is constant on \(\pi(C')\). By construction of \(\tilde{\chi}\), this implies the claim.

Part 2:

To avoid some special cases, we assume that \(S'_0 \neq \emptyset\) (if \(S'_0 = \emptyset\), we can replace it by \(\{0\}\)). Fix a fiber \(C \subseteq S'_n\) of the rainbow of \((S'_i)_i\). For \(d \leq n\), consider the following statement. There exists a coordinate projection \(\pi:\ K^n \to K^d\) and a \(\lambda \in \Gamma\) such that the following conditions hold:

1. For each \(q \in \pi(C)\), \(C \cap \pi^{-1}(q)\) is contained in an open ball \(B_q \subseteq \pi^{-1}(q)\) of radius \(\lambda\).
2. For each \(q_1,q_2 \in \pi(C)\), there exists a isometry \(B_q \to B_{q'}\) respecting the rainbow of \((S'_i)_i\).
3. For each \(q \in \pi(C)\), \(\text{tsp}_B((S'_i)_i)\) is exhibited by \(\pi\), where \(B\) is the open sub-ball of \(K^n\) of radius \(\lambda\) containing \(B_q\) (in particular \(\dim \text{tsp}_B((S'_i)_i) = d\) and hence \(B_q \cap S'_d \neq \emptyset\)).
4. The projection \(\pi(C)\) is an open domain.
5. There exists a set \(R = B_1 \times \cdots \times B_{d'}\), where each \(B_i \subseteq K\) is an open ball, and an analytic bijection \(\phi: R \to \pi(C)\) which can be written as the composition of a isometry and a matrix from \(\text{GL}_{d'}(\mathbb{C})\).

In the case \(d = n\), (4) and (5) together imply the lemma. (In that case, we are not interested in (1)–(3), which are a bit pathological since \(B_q\) is supposed to be a zero-dimensional ball. However, everything makes sense even in that case, using that a ball of any radius in \(K^0\) is equal to \(\{0\}\).)

For \(d = 0\), this statement follows from \(S'_0 \neq \emptyset\). Indeed, the latter implies that \(C\) is contained in a ball \(B \subseteq K^n\) (which is needed for (1)) and that for \(B\) sufficiently big, \(\text{tsp}_B((S'_i)_i) = \{0\}\) (which is needed for (3)).

To finish the proof, we will show that if the statement holds for some \(d < n\), then there also exists a \(d' > d\) for which it holds.

To simplify notation, we assume that \(\pi\) projects to the first \(d\) coordinates. By (3), \((S'_i)_i\) induces a t-stratification of \(B_q\) for each \(q \in \pi(C)\) and by Lemma 4.21, \(B_q \cap C\) is a fiber of the rainbow of that t-stratification and is contained in a ball \(B'_q \subseteq B_q \setminus S'_d\). We may assume this \(B'_q\) to be a maximal open ball in \(B_q \setminus S'_d\). Since \(B_q \cap S'_d \neq \emptyset\) (which follows from (3)), we have \(B'_q = s_q + \{(0)^d \times \rho \cdot v^{-1}(\xi_q)\}\) for some \(s_q \in S'_d \cap B_q\) and some \(\xi_q \in \text{RV}^{(n-d)}\). We first fix choices of \(s_q\) and \(\xi_q\) for one single \(q \in \pi(C)\) and then use isometries from (2) to obtain analogous elements \(s_{q'}\) and \(\xi_{q'}\) for all other \(q' \in \pi(C)\); this ensures that \(\xi := \xi_q\) does not depend on \(q\) and that all the \(s_q\) are contained in a single fiber \(C \subseteq S'_d\) of the rainbow of \((S'_i)_i\).

Set \(\lambda' := v_{\text{RV}}(\xi)\).

By (2), \(V := \text{tsp}_{B'_q}((S'_i \cap B_q)_i) \subseteq K^{n-d}\) does not depend on \(q\). Since \(B'_q \cap S'_d = \emptyset\), we have \(\dim V \geq 1\). Set \(d' := d + \dim V\), let \(\rho: K^n \to K^{d'}\) be an exhibition of \(V\), assume that \(\rho\) is the projection to the first \(d' - d\) coordinates of \(K^{n-d}\), and let \(\pi' := \rho \circ \pi: K^n \to K^{d'}\) be the projection to the first \(d'\) coordinates. We will now verify that the above statement holds for these choices of \(\pi'\), \(\lambda'\), \(\pi:\); we denote the corresponding conditions by \((1')\) to \((5')\).

\((1')\), \((2')\) and \((3')\) are clear. By (3) and Lemmas 4.22 and 4.3, \(\tilde{C}\) is the graph of a function \(c: \pi(C) \to K^{n-d}\) which, by part 1 of the proof, is equal to the restriction of an analytic function; from this, we will now easily deduce \((4')\) and \((5')\). First note that since \(\pi(C)\) is an open domain,
also $e|_{\pi(C)}$ is analytic. (In fact, one could also check that $\pi(\tilde{C}) = \pi(C)$.) Define $e: \pi(C) \to K^{d-d}, q \mapsto \rho(e(q))$.

By $V$-translatability of $C \cap B_q$ on $B_q'$, we have $\pi'(C \cap B_q) = \pi'(B_q')$, which in turn is equal to $\{q\} \times \langle e(q) + B\rangle$, where $B := \rho(R^{-1}(x)) \subseteq K^{d-d}$. This implies that $\pi'(C)$ is an open domain, and for $R, \phi$ as in (5) and $R' := R \times B$, we obtain an analytic bijection $\phi': R' \to \pi'(C), (r, x) \mapsto (\phi(r), e(\phi(r)) + x)$. By Lemma 4.3, the map $\psi: \pi(C) \times \tilde{B} \to \pi'(C), (q, x) \mapsto (q, e(q) + x)$ is the composition of a risometry and an element of $\text{GL}_{d-d}(O_K)$, so using Remark 5.10, the same holds for $\phi' = \psi \circ (\phi \times \text{id}_B)$, which finishes the proof.

**Remark 5.11.** By skipping the whole first part of this proof, we obtain that for any fiber $C \subseteq S_n$ of the rainbow of any $t$-stratification $(S_t)_t$ (in any valued field satisfying Hypothesis 2.21), there exists a definable bijection $\phi: B_1 \times \cdots \times B_n \to C$ that can be written as the composition of a risometry and a matrix in $\text{GL}_n(O_K)$ (and where each $B_i \subseteq K$ is an open ball). Moreover, it is not difficult to modify the proof to obtain a similar statement for fibers $C \subseteq S_d$ with $d < n$.

Now, we can finally prove that fields with analytic structure have the Jacobian property in any dimension.

**Proposition 5.12.** In the setting of Subsection 5.1, $T$ has the Jacobian property in the sense of Definition 2.19; in particular, Hypothesis 2.21 holds.

**Proof.** We first reduce the general case to the case where $K$ is algebraically closed.

Let $K$ be a model of $T$ (as usual) and let $f: K^n \to K$ be $A$-definable. As in the proof of Lemma 5.7, using Lemma 5.5 we may assume that $f(x) = t(x, \xi)$, where $t$ is an $L^s(A)$-term and $\xi \in \text{RV}^{eq}$ is $A$-definable. By [3, Theorem 4.5.11(i)], the analytic structure on $K$ has a (unique) extension to an analytic structure on the algebraic closure $\bar{K}$ of $K$ (in the same language). In particular, our term $t(x, \xi)$ also defines a function $\bar{K}^n \to \bar{K}$. Assuming that the proposition holds in $\bar{K}$, we find an $A$-definable map $\tilde{\chi}: \bar{K}^n \to \text{RV}^{eq}$ such that $t$ has the Jacobian property on each $n$-dimensional $\tilde{\chi}$-fiber; here $\text{RV}^{eq}$ denotes the $\text{RV}^{eq}$-sorts corresponding to $\bar{K}$.

By elimination of valued field quantifiers in the language $L$ (Lemma 5.4), we may assume that $\chi(x) = (\text{rv}(t_1(x)), \ldots, \text{rv}(t_k(x)))$ for some $L(A)$-terms $t_i$. Indeed, after elimination of the quantifiers, the map $x \mapsto (\text{rv}(t_i(x)))_i$ is a refinement of $\tilde{\chi}$, where $t_i$ runs over all $\bar{K}$-valued $L(A)$-terms appearing in $\tilde{\chi}$.

Now $\chi$ is given by a tuple of terms, so it restricts to a map $\chi: K^n \to \text{RV}^{eq}$. If a fiber $\chi^{-1}(q)$ has dimension $n$, then so has $\tilde{\chi}^{-1}(q)$, so $t$ has the Jacobian property on $\tilde{\chi}^{-1}(q)$ and in particular on $\chi^{-1}(q)$.

We now have to prove the proposition in the case where $K$ is algebraically closed. As announced, we prove the ‘Jacobian property up to dimension $n$’ by induction over $n$. For $n = 0$, there is nothing to show, so now suppose $n \geq 1$ and suppose that $f: K^n \to K$ is an $A$-definable map.

Lemma 5.7 yields an $A$-definable map $\chi: K^n \to \text{RV}^{eq}$ such that for each $\chi$-fiber $C$, there exists an open domain $X \supseteq C$ and an analytic function $g \in \mathfrak{A}(X)$ such that $f|_C = g|_C$. Recall that such an analytic function $g$ has a well-defined Jacobian at every $x \in X$: $(\text{Jac } g)(x) = (\partial g/\partial x_1, \ldots, \partial g/\partial x_n) \in K^n$. Suppose that $\dim C = n$. By refining $\chi$ (and using Lemma 2.30), we may assume that $\dim_x C = n$ for every $x \in C$. This implies that for $x \in C'$, $(\text{Jac } g)(x)$ is determined by $f|_C$, so we can further refine $\chi$ in such a way that $\text{rv}((\text{Jac } f|_C)(x)) \in \text{RV}^{(n)}$ is constant on each $\chi$-fiber $C$ of dimension $n$.

We apply Theorem 4.12 to $\chi$ and obtain an $A$-definable $t$-stratification $(S_t)_t$ of $K^n$ reflecting $\chi$ (the prerequisites of Theorem 4.12 hold by Proposition 5.6 and the induction hypothesis).
Then we apply Lemma 5.9 (using the induction hypothesis again); the resulting t-stratification \((S_i')_i\), still reflects \(\chi\) by Corollary 4.19. Let \(\rho\) be the rainbow of \((S_i')_i\) and let \(C\) be an \(n\)-dimensional \(\rho\)-fiber. Then \(C\) is an open domain, \(f\) is analytic on \(C\) (since \(f|_C\) is the restriction of an analytic function on a larger domain), \(\hat{v}(\text{Jac } f|_C(x))\) is constant on \(C\), and there exists an analytic bijection \(\phi = \psi \circ M : R \to C\), where \(R = B_1 \times \cdots \times B_n\) for some open balls \(B_i \subseteq K\), where \(M \in \text{GL}_n(O_K)\).

To finish the proof, we will check that \(f\) has the Jacobian property on \(C\), that is, either \(f|_C\) is constant or for any \(x, x' \in C\) with \(x \neq x'\), we have

\[
v(f(x) - f(x') - (z, x - x')) > \hat{v}(z) + \hat{v}(x - x')
\]

for some \(z \in K^n\) not depending on \(x, x'\). In fact, we will prove \((*)\) for \(z = (\text{Jac } f|_C(x'))\), which is enough by Remark 2.20 and since \(\hat{v}(\text{Jac } f|_C)\) is constant.

We may assume that \(\hat{v}(\text{Jac } f|_C) \neq 0\), since otherwise, \(f|_C\) is constant and we are done. Let \(x, x' \in C\), \(x \neq x'\) be given and define \(\eta : O_K \to R, y \mapsto y \cdot \phi^{-1}(x) + (1 - y) \cdot \phi^{-1}(x')\). The remainder of the proof consists in pulling back the problem on \(C\) to a problem on \(O_K\) using \(\theta := \phi \circ \eta : O_K \to C\); the problem on \(O_K\) then follows from Lemma 5.8. All this is straightforward, but we give some details.

The map \(\theta\) is obviously analytic and an easy computation shows that \(\hat{v}((\theta(y) - \theta(y'))/(y - y')) \in \text{RV}^n\) is constant for \(y, y' \in O_K\), \(y \neq y'\). Indeed, we have \(\hat{v}(\theta(y) - \theta(y')) = \hat{v}(M(\eta(y)) - M(\eta(y')))\) since \(\psi\) is a risometry, and \(\hat{v}((M(\eta(y)) - M(\eta(y')))/(y - y'))\) is constant since \(M \circ \eta\) is linear. Moreover, constantness of \(\hat{v}((\theta(y) - \theta(y'))/(y - y'))\) implies that \(\hat{v}(\text{Jac } \theta)\) is also constant and that these two values are equal. In particular, by plugging \(y = 1\) and \(y' = 0\) (the preimages of \(x\) and \(x'\)) into this equality, we obtain

\[
\hat{v}(x - x' - (\text{Jac } \theta)(0)) > \hat{v}(x - x').
\]

Set \(g := f \circ \theta : O_K \to K\); it is clear that \(g\) is analytic. Using the chain rule \(g' = \langle \text{Jac } f|_C, \text{Jac } \theta \rangle\) and that \(\hat{v}(\text{Jac } f|_C)\) and \(\hat{v}(\text{Jac } \theta)\) are constant, we obtain \(v(g'(y)) \geq \hat{v}(\text{Jac } f|_C) + \hat{v}(\text{Jac } \theta)\) for any \(y, y' \in O_K\) with \(y \neq y'\). Therefore, we can apply Lemma 5.8 to the function \(y \mapsto g(y)/r\), where \(r \in K\) is any element with \(v(r) = \hat{v}(\text{Jac } f|_C) + \hat{v}(\text{Jac } \theta)\); this yields

\[
v(g(y) - g(y') - g'(0) \cdot (y - y')) \geq \hat{v}(\text{Jac } f|_C) + \hat{v}(\text{Jac } \theta) + v(y - y').
\]

For \(y = 1\), \(y' = 0\), and \(z = (\text{Jac } f|_C)(x')\), this becomes

\[
v(f(x) - f(x') - g'(0)) > \hat{v}(z) + \hat{v}(x - x').
\]

Now \((***)\) already implies that \((*)\) and \((***)\) have the same right-hand side, and it remains to verify that

\[
v((z, x - x') - g'(0)) > \hat{v}(z) + \hat{v}(x - x').
\]

Indeed, \((***)\) implies

\[
v((z, x - x') - (z, (\text{Jac } \theta)(0))) > \hat{v}(z) + \hat{v}(x - x'),
\]

which is what we want, since \(g'(0) = (z, (\text{Jac } \theta)(0))\).

\[\Box\]

6. Algebraic results

Up to now, for a t-stratification \((S_i)_i\), we know that the sets \(S_{<i}\) are closed in the valued field topology. However, in a purely algebraic setting, it would be natural to require the sets \(S_{<i}\) to be Zariski closed. We will now show that indeed this can be achieved (Corollary 6.5). In fact, we will first prove a more general result (Proposition 6.2) in arbitrary theories satisfying Hypothesis 2.21.
6.1. Getting closed sets $S_{\leq i}$

In this subsection, we assume that $T$ satisfies Hypothesis 2.21.

In Proposition 6.2, we introduce a set $\Delta$ of formulas which can be thought of as defining the closed sets of a topology (although the conditions on $\Delta$ will be weaker) and we prove that any t-stratification $(S_i)_i$ can be enhanced to a t-stratification $(S'_i)_i$ in such a way that each $S'_{\leq i}$ is closed in this sense. For this to be possible, we only need that taking the closure of a definable set does not increase its dimension. Here, ‘enhancing’ means that any map into $\text{RV}^{eq}$ reflected by $(S_i)_i$ is also reflected by $(S'_i)_i$. By Corollary 4.19, this is equivalent to: $(S'_i)_i$ reflects $S_j$ for each $j$.

To be able to work uniformly for all models of $T$, we introduce a uniform notion of dimension.

**Definition 6.1.** For an $\mathcal{L}$-formula $\phi$ whose free variables live in the valued field sort, set $\dim \phi := \max_{K \models T} \dim(\phi(K))$.

**Proposition 6.2.** In the following, all $\mathcal{L}$-formulas have $n$ free valued field variables, and $\phi \rightarrow \psi$ means ‘$T \models \forall X (\phi(x) \rightarrow \psi(x))$’.

Suppose that we have a family $\Delta$ of $\mathcal{L}$-formulas with the following properties.

1. The family $\Delta$ is closed under disjunctions and contains $\bot$.
2. For each $\mathcal{L}$-formula $\phi$, there exists a minimal formula $\phi^\Delta \in \Delta$ with $\phi \rightarrow \phi^\Delta$; minimal means: for any other $\psi \in \Delta$ with $\phi \rightarrow \psi$, we have $\phi^\Delta \rightarrow \psi$.
3. For each $\mathcal{L}$-formula $\phi$, we have $\dim \phi^\Delta = \dim \phi$.

Suppose moreover that $(\phi_i)_{0 \leq i \leq n}$ is a tuple of formulas defining a t-stratification in every model of $T$. Then we can find a tuple of formulas $(\phi'_i)_i$ which, in every model, defines a t-stratification reflecting the sets defined by the formulas $\phi_i$ and such that for each $i$, $\phi'_0 \lor \cdots \lor \phi'_i$ is equivalent to a formula in $\Delta$.

**Proof.** For any formula $\phi$, we set $\partial \phi := \phi^\Delta \land \neg \phi$. Note that using $\phi^\Delta \lor \psi^\Delta \in \Delta$, one obtains $(\phi \lor \psi)^\Delta \rightarrow (\phi^\Delta \lor \psi^\Delta)$ and hence $\partial(\phi \lor \psi) \rightarrow (\partial \phi \lor \partial \psi)$.

We write $\phi_{\leq i}$ for $\phi_0 \lor \cdots \lor \phi_i$, and similarly for $\phi'_i$ and $\psi_i$ (which will be introduced below).

Suppose that for some given $d \in \{0, \ldots, n\}$, $(\phi_i)_i$ satisfies $\dim \partial \phi_{\leq i} \leq d$ for all $i$. From this, we construct a t-stratification $(\phi'_i)_i$ reflecting $(\phi_i)_i$ and satisfying $\dim \partial \phi'_{\leq i} \leq d - 1$. Applying this repeatedly yields the proposition (where $\dim \phi \leq -1$ will mean $T \models \neg \forall x \phi(x)$). So, let $d$ be given as above.

For $i$ from $n$ to $0$, recursively define $\delta_i := \partial(\phi_{\leq i} \lor \delta_{i+1} \lor \cdots \lor \delta_n)$. Inductively, we get $\dim \delta_i \leq d$. Set $\delta := \bigvee_{i=0}^n \delta_i$, choose any t-stratification $(\psi_i)_i$ reflecting $((\phi_i)_i, \delta)$, and apply Lemma 4.20 to $X = \delta(K)$, a constant map $\chi: X \rightarrow \text{RV}^{eq}$, $(S_i)_i = (\phi_i(K))_i$ and $(T_i)_i = (\psi_i(K))_i$. We claim that for the resulting t-stratification $(\phi'_i(K))_i := (S'_i)_i$, we have $\partial \phi'_{\leq i} \rightarrow (\phi'_{\leq i-d-1})^\Delta$ (where $\phi'_{\leq -1} = \bot$); in particular, we obtain $\dim \partial \phi'_{\leq i} \leq d - 1$.

For $i \leq d - 1$, the claim is clear, so suppose $i \geq d$. The formula $\phi'_{\leq i} = \phi_{\leq i} \lor \delta \lor \psi_{\leq d-1}$ is equivalent to $\bigvee_{j \leq i}(\phi_{\leq j} \lor \delta_j \lor \cdots \lor \delta_n) \lor \psi_{\leq d-1}$. Each formula $\phi_{\leq j} \lor \delta_j \lor \cdots \lor \delta_n$ is equivalent to a formula in $\Delta$ by definition of $\delta_j$, so we get $\partial \phi'_{\leq i} \rightarrow \partial \psi_{\leq d-1}$, and $\partial \psi_{\leq d-1}$ is equal to $\partial \phi'_{\leq d-1}$. \hfill \Box

6.2. Algebraic strata

In the pure valued field language $\mathcal{L}_{\text{Hen}}$, we can apply Proposition 6.2 to the family of formulas that are conjunctions of polynomial equations and in this way obtain t-stratifications where the sets $S_{\leq i}$ are Zariski closed. This yields a version of the main theorem that can almost be
formulated in a purely algebraic language; only almost, since in the definition of t-stratification, we require the straighteners to be definable. (Of course, this condition can simply be omitted, but this weakens the result.) We take the opportunity to present a setting that is as algebraic as possible.

Fix a Noetherian integral domain $A$ of characteristic 0. We set $\mathcal{L} := \mathcal{L}_{\text{Hen}}(A)$ and $\mathcal{T} := \mathcal{T}_{\text{Hen}} \cup \{\text{positive atomic diagram of } A \in \mathcal{L}_{\text{ring}}\}$; in other words, models $K$ of $\mathcal{T}$ are Henselian-valued fields of equi-characteristic 0 together with a ring homomorphism $A \to K$. We fix $n \in \mathbb{N}$ and let $\Delta$ be the set of conjunctions of polynomial equations in $n$ variables with coefficients in $A$. For any model $K \models \mathcal{T}$, by considering the sets $\phi(K), \phi \in \Delta$ as closed, we obtain the Zariski topology (‘over $A$’) on $K^n$. More precisely, formulas in $\Delta$ correspond to Zariski closed subsets of the scheme $\mathbb{A}^n_A$, and our topology on $K^n$ is the one which the Zariski topology on $\mathbb{A}^n_A$ induces on the $K$-valued points $\mathbb{A}^n_A(K)$.

Note that for $\phi \in \Delta$, we have two notions of dimension: the one given in Definition 6.1 and the algebraic one, where we consider $\phi$ as a variety over $A$. However, by considering an algebraically closed model of $\mathcal{T}$, we see that the two notions of dimension coincide.

Given an arbitrary $\mathcal{L}$-formula $\phi$, it is clear that there exists a well-defined ‘Zariski closure’ of $\phi$, that is, a minimal formula $\phi^\Delta \in \Delta$ implied by $\phi$. To be able to apply Proposition 6.2, it remains to check that $\phi^\Delta$ has the same dimension as $\phi$. This has been proved in [13] or [5] for example, but in slightly different contexts than ours, so let us quickly repeat the proof from [13]. We first work in a fixed model $K$.

**Lemma 6.3.** For every $\emptyset$-definable set $X \subseteq K^n$, there exists a formula $\psi \in \Delta$ such that $X \subseteq \psi(K)$ and $\dim \psi = \dim X$.

**Proof.** In this proof, we will write $rv^\ell : K^\ell \to RV^\ell$ for the cartesian power of $rv$ (in contrast to the map $\hat{rv} : K^\ell \to RV^{\ell()}$ mainly used in the remainder of the article).

By quantifier elimination (see, for example, [7, Proposition 4.3]), $X$ is of the form
\[
X = \{ x \in K^n \mid (rv(f_1(x)), \ldots, rv(f_\ell(x))) \in \Xi \} = f^{-1}((rv^\ell)^{-1}(\Xi)),
\]
where $f = (f_1, \ldots, f_\ell)$ is an $\ell$-tuple of polynomials with coefficients in $A$ and $\Xi \subseteq RV^\ell$ is $\emptyset$-definable. The statement of the lemma is preserved by finite unions, so we can do a case distinction on whether $f_i(x) = 0$ or not for each $i$; in other words, $X$ is of the form
\[
X = \psi(K) \cap f^{-1}((rv^\ell)^{-1}(\Xi))
\]
for $\psi \in \Delta$, $f$ as above and $\Xi \subseteq (RV \setminus \{0\})^\ell$.

Write $\bar{K}$ for the algebraic closure of $K$. We may assume that $\psi(\bar{K})$ is the Zariski closure of $X$ in $\bar{K}^n$. In particular, $X$ contains a regular point $x$ of $\psi(\bar{K})$, that is, on a Zariski-neighborhood of $x$, $\psi(\bar{K})$ is defined by $n - \dim \psi$ polynomials and the Jacobian matrix at $x$ of this tuple of polynomials has maximal rank.

Now $(rv^\ell)^{-1}(\Xi)$ is open in the valuation topology, so in that topology, there is a neighborhood $U \subseteq \psi(K)$ of $x$ which is contained in $X$. Using the implicit function theorem and regularity at $x$, we find a coordinate projection $\pi : K^n \to K^{\dim \psi}$ such that $\pi(U)$ contains a ball in $K^{\dim \psi}$. This implies $\dim X = \dim \psi$.

Now we make the result uniform for all models of $\mathcal{T}$.

**Lemma 6.4.** For every $\mathcal{L}$-formula $\phi$ in $n$ valued field variables, there exists a formula $\psi \in \Delta$ with $\dim \psi = \dim \phi$ and $\phi(K) \subseteq \psi(K)$ for all models $K \models \mathcal{T}$. 
**Proof.** For each $K$ separately, Lemma 6.3 yields a formula $\psi_K \in \Delta$ with $\phi(K) \subseteq \psi_K(K)$ and $\dim \psi_K = \dim \phi(K)$. By compactness, there exists a finite disjunction $\psi$ of some of the $\psi_K$ such that $\phi(K) \subseteq \psi(K)$ for all $K$. Since
\[ \dim \psi \leq \max_{K} \dim \psi_K = \max_{K} \dim \phi(K) = \dim \phi, \]
we are done. \qed

Now Proposition 6.2 can be applied to the Zariski topology and we get t-stratifications such that each set $S_{\leq i}$ is defined by a conjunction of polynomials (uniformly for all models). Moreover, using that being a t-stratification is first order in the sense of Corollary 4.13, the same t-stratification also works in models of a finite subset of $T$. Here is the precise result.

**Corollary 6.5.** Let $A$ be a Noetherian integral domain of characteristic 0, $L = L_{\text{Hen}}(A)$, and $T$ the theory of Henselian-valued fields $K$ of equi-characteristic 0 together with a ring homomorphism $A \to K$. For every $L$-formula $\chi$ defining a map $\chi_K : K^n \to \text{RV}_\text{eq}$ (for any $K \models T$), there exists a finite subset $T_0 \subseteq T$ and formulas $(\phi_i)_i$ such that:

(i) either $\dim \phi_i = i$ (in the sense of Definition 6.1) or $\phi_i = \perp$;
(ii) the disjunction $\phi_0 \lor \cdots \lor \phi_d$ is equivalent to a conjunction of polynomial equations with coefficients in $A$;
(iii) for every model $K \models T_0$, $(\phi_i(K))_i$ is a t-stratification reflecting $\chi_K$.

(As in Corollary 4.13, we assume that models of $T_0$ are valued fields for the statements to make sense.)

Here is a an algebraic formulation of Corollary 6.5: by a ‘subvariety of $A^n$’, we simply mean a reduced (not necessarily irreducible) subscheme. Since the notion of a definable map to $\text{RV}_{\text{eq}}$ is not so algebraic, we instead formulate the theorem for a finite family $(X_\nu)_\nu$ of subvarieties of $A^n$.

**Theorem 6.6.** Let $A$ be a Noetherian integral domain of characteristic 0 and let $X_\nu$ be subvarieties of $A^n$ for $\nu = 1, \ldots, \ell$. Then there exists an integer $N \in \mathbb{N}$ and a partition of $A^n$ into subvarieties $S_i$ with the following properties.

(1) For each $i$, $\dim S_i = i$ or $S_i = \emptyset$.
(2) Each $S_{\leq i}$ is a closed subvariety of $A^n$.
(3) For every Henselian-valued field $K$ over $A$ of residue characteristic either 0 or at least $N$, $(S_i(K))_i$ is a t-stratification of $K^n$ reflecting the family of sets $(X_\nu(K))_\nu$, that is, for every $d \leq n$ and every ball $B \subseteq S_{\geq d}(K)$, the tuple $(S_d(K), \ldots, S_n(K), X_1(K), \ldots, X_\ell(K))$ is $d$-translatable on $B$ (see Definition 3.1 or 3.12 and Convention 3.2).

In the next section, we will prove that a t-stratification $(S_i)_i$ as in Theorem 6.6 induces a classical Whitney stratification $(S_i(\mathbb{C}))_i$ of $\mathbb{C}^n$ (for any ring homomorphism $A \to \mathbb{C}$), and similarly for $\mathbb{C}$ replaced by $\mathbb{R}$ (see Theorem 7.11). In particular, this implies that each $S_i$ is smooth over the fraction field of $A$.

We conclude this section with an algebraic formulation of Corollary 4.14 about how the risometry type can vary in a uniform family.

**Corollary 6.7.** Let $A$ be a Noetherian integral domain of characteristic 0, let $Q$ be any affine variety over $A$, and let $X_\nu$ be subvarieties of $A^n_Q$ for $\nu = 1, \ldots, \ell$. Then
there exists an integer $N \in \mathbb{N}$ and algebraic maps $f_1, \ldots, f_m : Q \to \mathbb{A}^m_k$ such that for every Henselian-valued field $K$ over $A$ of residue characteristic either 0 or at least $N$, we have the following.

Given $q \in Q(K)$, write $X_{\nu,q} = X_{\nu} \times Q$ spec $K$ for the fiber of $X_{\nu}$ over $q$ and consider $X_{\nu,q}(K)$ as a subset of $K^n$. If two elements $q, q' \in Q(K)$ satisfy $rv(f_{\mu}(q)) = rv(f_{\mu}(q'))$ for all $\mu$, then there exists a risometry $\phi_{q,q'} : K^n \to K^n$ such that for each $\nu$, we have $\phi_{q,q'}(X_{\nu,q}(K)) = X_{\nu,q'}(K)$.

**Proof.** We fix an embedding $Q \hookrightarrow \mathbb{A}^m_k$. Applying Corollary 4.14 yields an integer $N$ and a formula $\phi$ such that for every $K$ as above, $\phi$ defines a map $\phi_K : Q(K) \to RV^{\sq}$ such that $\phi_K(q) = \phi_K(q')$ implies existence of a risometry as above for $q, q' \in Q(K)$. By quantifier elimination [7, Proposition 4.3], we may refine the map defined by $\phi$ to a map of the form $q \mapsto (rv(f_1(q)), \ldots, rv(f_m(q)))$ for some polynomials $f_i$; this implies the claim.

\[ \square \]

7. Obtaining classical Whitney stratifications

The main result of this section is that the existence of t-stratifications implies the existence of classical Whitney stratifications. More precisely, a non-standard model of $\mathbb{R}$ or $\mathbb{C}$ can be considered as a valued field, and we will see that any definable partition in the standard model that induces a t-stratification in the non-standard model is already a Whitney stratification. We will start by proving that t-stratifications satisfy a kind of analogue of Whitney’s Condition (b) (Corollary 7.6). This needs the following additional natural Hypothesis on the language $\mathcal{L}$ (and the theory $T$).

**HYPOTHESIS 7.1.** In this section, we require that the residue field is orthogonal to the value group, that is, in any model of $T$, any definable set $X \subseteq k^n \times \Gamma^m$ is a finite union of sets of the form $Y_i \times Z_i$, for some definable sets $Y_i \subseteq k^n$ and $Z_i \subseteq \Gamma^m$.

**PROPOSITION 7.2.** The theory of any Henselian-valued field $K$ with analytic structure in the sense of [3] satisfies Hypothesis 7.1.

**Proof.** We work with the language $\mathcal{L}$ introduced in Subsection 5.1. By quantifier elimination (Lemma 5.4), any definable subset of $RV^n$ can be defined in the restriction to RV of $\mathcal{L}$. To that restricted language, add the sorts $k$ and $\Gamma$ and a splitting $RV \setminus \{0\} \to k^\times$ of the sequence $k^\times \hookrightarrow RV \setminus \{0\} \to \Gamma$ (such a splitting corresponds to an angular component map $K \to k$ of the valued field); then it becomes interdefinable with the language $\mathcal{L}'$ consisting of $k$ with the ring language and $\Gamma$ with the language $\{0, +, -, <\}$ of ordered abelian groups (where RV is identified with $k \times \Gamma$). In particular, any set $X \subseteq k^n \times \Gamma^m$ definable in our original language is also $\mathcal{L}'$-definable. Since $\mathcal{L}'$ contains no connection between $k$ and $\Gamma$, the proposition follows.

\[ \square \]

At some point, we will use the following easy consequence of the above hypothesis.

**REMARK 7.3.** For any parameter set $A \subseteq k$, we have $acl(A) \cap \Gamma = acl(\emptyset) \cap \Gamma$ (where acl is the algebraic closure in the model theoretic sense). Using the order on $\Gamma$, we get the same with acl replaced by the definable closure. In particular, if $\mathcal{L} = \mathcal{L}_{\text{Hen}}$ and $K$ is either real closed or algebraically closed, then $\Gamma$ is a pure divisible ordered abelian group and the only finite, $A$-definable subsets of $\Gamma$ are $\emptyset$ and $\{0\}$. 

7.1. An analogue of Whitney’s Condition (b)

Our main theorem about the existence of t-stratifications only speaks about the dimension of translatability spaces. The following theorem additionally (partially) specifies their direction. The analogue of Whitney’s Condition (b) will then be a corollary. (Recall from Definition 2.7 that \( \text{dir}_{RV} : RV(n) \to \mathbb{P}^{n-1}k \) is the map induced by \( \text{dir} : K^n \to \mathbb{P}^{n-1}k \).)

**Theorem 7.4.** Suppose that the language \( \mathcal{L} \) and the \( \mathcal{L} \)-theory \( T \) satisfy Hypotheses 2.21 and 7.1 and that \( K \) is a model of \( T \). Let \( \chi : K^n \to RV^eq \) be a definable map and let \( x \in K^n \) be any point. Let \( \Xi \subseteq RV(n) \setminus \{0\} \) be the set of those \( \xi \) such that \( \chi \) is not \( \text{dir}_{RV}(\xi) \)-translatable on the ball \( B := x + rv^{-1}(\xi) \). Then \( \hat{v}_{RV}(\Xi) \) is finite.

**Remark 7.5.** In general, \( \Xi \) is not definable. However, we can choose a t-stratification \( (S_i)_{i} \), reflecting \( \chi \) and refine \( \chi \) to \( ((S_i)_{i}, \chi) \); after this modification, \( \Xi \) is definable (over \( x \) and the parameters of the original \( \chi \)) by Lemma 3.14.

**Proof of Theorem 7.4.** By Remark 7.5, we may assume that \( \Xi \) is definable. Without loss, fix \( x = 0 \) and suppose for contradiction that \( \hat{v}_{RV}(\Xi) \) is infinite. By orthogonality of the value group and the residue field, there exists a one-dimensional \( V \subseteq K^n \) such that the subset \( \Xi_0 := \{ \xi \in \Xi \mid \text{dir}_{RV}(\xi) \in V \} \) is already infinite. Choose a lift \( \tilde{V} \subseteq K^n \) of \( V \) and consider the map \( \chi' \) obtained from \( (\chi, \tilde{V}) \) via Convention 3.2. For \( \xi \in \Xi_0 \), \( \chi \) is not \( V \)-translatable on the ball \( B := rv^{-1}(\xi) \); on the other hand, \( V \cap B \neq \emptyset \), so \( tsp_B(V) = V \), which implies that \( \chi' \) is not translatable at all on \( rv^{-1}(\xi) \). In particular, if \( (S_i)_{i} \) is a t-stratification reflecting \( \chi' \) (which exists by Theorem 4.12), then we have \( rv^{-1}(\xi) \cap S_0 \neq \emptyset \) for all \( \xi \in \Xi_0 \), which contradicts \( S_0 \) being finite.

Note that the only way we used Hypothesis 2.21 in this proof is to apply Theorem 4.12 to \( \chi \) and \( \chi' \).

In the classical version of Whitney’s Condition (b), one has two sequences of points in two different strata \( S_d \) and \( S_j \) with \( d < j \), and both sequences converge to the same point in \( S_d \). In the valued field version, each sequence is replaced by a single point, and ‘converging to the same point in \( S_d \)’ is replaced by ‘lying in a common ball \( B \subseteq S_{\geq d} \).’ In the proof of Proposition 7.10, we will see how this implies the classical Condition (b) via non-standard analysis.

**Corollary 7.6.** Assume Hypotheses 2.21 and 7.1. Let \( (S_i)_{i} \) be a \( \emptyset \)-definable t-stratification of a \( \emptyset \)-definable ball \( B_0 \subseteq K^n \), let \( B \subseteq B_0 \) be a sub-ball, and let \( d \) be maximal with \( B \subseteq S_{\geq d} \). Then there exists a finite \( \Gamma B' \)-definable set \( M \subseteq \Gamma \) such that the following holds. For any \( j > d \), any \( x' \in B \cap S_d \), and any \( y' \in B \cap S_j \), if \( \nu(x' - y') \notin M \), then \( \text{dir}(x' - y') \in tsp_B((S_i)_{i}) \), where \( B' \subseteq S_{\geq j} \) is a ball containing \( y' \).

**Proof.** Let \( \pi : B \to K^d \) be an exhibition of \( tsp_B((S_i)_{i}) \). Choose any \( z \in \pi(B) \) and any \( x \in \pi^{-1}(z) \cap S_d \), and apply Theorem 7.4 to \( (S_i)_{i} \) and \( x \). This yields a finite set \( \hat{v}_{RV}(\Xi) \subseteq \Gamma \), which is \( x \)-definable by Remark 7.5. Doing this for all \( x \in \pi^{-1}(z) \cap S_d \) and taking the union of the (finitely many) corresponding sets \( \hat{v}_{RV}(\Xi) \) yields a finite, \( \Gamma B' \)-definable set which we denote by \( M_z \). For any other \( z' \in \pi(B) \), Lemma 3.7 yields a risometry \( \alpha : B \to B \) sending \( \pi^{-1}(z) \) to \( \pi^{-1}(z') \); extending \( \alpha \) by the identity on \( B_0 \setminus B \), we get a risometry which shows that \( M_z = M_{z'} \); hence \( M := M_z \) is \( \Gamma B' \)-definable.

Now let \( x' \in B \cap S_d \) and \( y' \in B \cap S_j \) be given with \( \nu(x' - y') \notin M \) and set \( z = \pi(x') \). Since \( \lambda := \nu(y' - x') \notin M_z \), \((S_i)_{i} \) is \( \text{dir}(x' - y') \)-translatable on \( B_1 := B(y', > \lambda) \) and hence also on \( B' \subseteq B_1 \). □
7.2. The classical Whitney conditions

We now recall the definition of Whitney stratifications; see, for example, [1] for more details. We will consider Whitney stratifications both over \( k = \mathbb{R} \) and \( k = \mathbb{C} \), in a semi-algebraic resp. algebraic setting. A Whitney stratification is a partition of \( k^n \) into certain kinds of manifolds. In the case \( k = \mathbb{R} \), we will work with Nash manifolds and also with a weakening of that notion.

**Definition 7.7.** A Nash manifold is a \( C^\infty \)-submanifold of \( \mathbb{R}^n \) (for some \( n \)), which is \( \mathcal{L}_{\text{ring}} \)-definable (or, equivalently, which is semi-algebraic). By a \( C^1 \)-Nash manifolds, we mean a \( C^1 \)-submanifold of \( \mathbb{R}^n \) which is \( \mathcal{L}_{\text{ring}} \)-definable.

Note that by ‘\( M \) is a submanifold of \( k^n \)', we mean that also the inclusion map \( M \hookrightarrow k^n \) is in the corresponding category, that is, either \( C^1 \) or \( C^\infty \) (but we do not require \( M \) to be closed in \( k^n \)). All our manifolds will be submanifolds of some \( k^n \) in this sense (for \( k \) either \( \mathbb{R} \) or \( \mathbb{C} \)); this will not always be written explicitly.

In the case \( k = \mathbb{C} \), we will only have one notion of manifolds, namely algebraic submanifolds of \( \mathbb{C}^n \). Note that this is in perfect analogy to the case \( k = \mathbb{R} \); if we simply replace \( \mathbb{R} \) by \( \mathbb{C} \) in Definition 7.7, then ‘definable’ means constructible instead of semi-algebraic; moreover ‘differentiable’ should now be read as ‘complex differentiable’. Thus in that case, both kinds of manifolds introduced in Definition 7.7 simply become algebraic manifolds.

In the remainder of the section, we will treat \( k = \mathbb{R} \) and \( k = \mathbb{C} \) simultaneously, and we will write ‘Nash/algebraic manifolds’ or ‘\( C^1 \)-Nash/algebraic manifolds’ (depending on the notion of manifold we want to consider in the case \( k = \mathbb{R} \)).

We will not require our manifolds to be connected, but if they are not, then each connected component has to have the same dimension.

For a \( C^1 \)-Nash/algebraic manifold \( M \subseteq k^n \) and a point \( x \in M \), there is a well-defined notion of tangent space \( T_x M \subseteq k^n \) of \( M \) at \( x \). Such a space can be seen as an element of the corresponding Grassmanian \( G_{n, \dim M}(k) \) and as such, it makes sense to speak of limits of sequences of such spaces.

**Definition 7.8.** Let \( k \) be either \( \mathbb{R} \) or \( \mathbb{C} \). A Whitney stratification of \( k^n \) is a partition of \( k^n \) into Nash/algebraic manifolds \( (S_i)_{0 \leq i \leq n} \) with the following properties. (As always, we write \( S_{i,j} \) for \( S_i \cup \cdots \cup S_j \)).

(1) For each \( i \), either \( \dim S_i = i \) or \( S_i = \emptyset \).

(2) Each set \( S_{i,j} \) is topologically closed in the analytic topology.

(3) Each pair \( S_i, S_j \) with \( i < j \) satisfies Whitney’s Condition (a), that is, for any element \( u \in S_i \) and any sequence \( v_\mu \in S_j \) converging to \( u \), if \( \lim_{\mu \to \infty} T_{v_\mu} S_j \) exists, then

\[
T_u S_i \subseteq \lim_{\mu \to \infty} T_{v_\mu} S_j.
\]

(4) Each pair \( S_i, S_j \) with \( i < j \) satisfies Whitney’s Condition (b), that is, for any two sequences \( u_\mu \in S_i \), \( v_\mu \in S_j \) both converging to the same element \( u \in S_i \), if both \( \lim_{\mu \to \infty} T_{v_\mu} S_j \) and \( \lim_{\mu \to \infty} k \cdot (u_\mu - v_\mu) \) exist, then

\[
\lim_{\mu \to \infty} k \cdot (u_\mu - v_\mu) \subseteq \lim_{\mu \to \infty} T_{v_\mu} S_j.
\]

We will say that \( (S_i)_i \) is a \( C^1 \)-Whitney stratification if it is a partition of \( k^n \) into \( C^1 \)-Nash/algebraic manifolds satisfying the above conditions (1)–(4).
In fact, it is known that anyway (4) implies (3); we will prove (3) separately nevertheless, since the argument is short and elegant.

Often, one additionally requires that the topological closure of any connected component of any \( S_j \) is the union of some connected components of some of the \( S_i, \ i < j \). However, once one knows how to obtain Whitney stratifications in our sense, it is also easy to obtain this additional condition.

7.3. Transfer to the Archimedean case

Let \( k \) be either \( \mathbb{R} \) or \( \mathbb{C} \). We will consider \( k \) as a structure in the language \( \mathcal{L}_{\text{aberring}} := \mathcal{L}_{\text{ring}} \cup \{ \cdot, | \cdot | \} \), where \( | \cdot | : k \to \mathbb{R}_{\geq 0} \subseteq k \) is the absolute value. (Of course, in the case \( k = \mathbb{R} \), \(| \cdot | \) is already \( \mathcal{L}_{\text{ring}} \)-definable.) Fix \( K \) to be a (non-principal) ultra-power of \( k \) with index set \( \mathbb{N} \); this will be the non-standard model of \( k \) we will be working in. (In fact, any \( \mathbb{N}_1 \)-saturated elementary extension of the \( \mathcal{L}_{\text{aberring}} \)-structure \( k \) would do.)

The image in \( K \) of any \( u \in k \) (under the canonical embedding) is denoted by \( ^u \). Similarly, for any set \( X \subseteq k^n \), the ultra-power of \( X \), considered as a subset of \( K^n \), will be denoted by \( ^X \). (In particular, \( ^k = K \) and \( ^\mathbb{R} \subseteq K \).)

Define
\[
\mathcal{O}_K := \{ x \in K \mid \exists (u \in \mathbb{R}) |x| < u \};
\]
this is a valuation ring, turning \( K \) into a valued field which is Henselian and of equicharacteristic 0. The maximal ideal is
\[
\mathcal{M}_K = \{ x \in K \mid \forall (u \in \mathbb{R}_{>0}) |x| < u \},
\]
the residue field is \( k \), and \( \text{res} : \mathcal{O}_K \to k \) is simply the standard part map.

Using the absolute value on \( k \), we can define the Euclidean norm on \( k^n \), which we denote by \(| \cdot |_2 : k^n \to \mathbb{R}_{\geq 0} \); this also induces an ‘Euclidean norm’ \(| \cdot |_2 : K^n \to \mathbb{R}_{\geq 0} \). and this Euclidean norm induces a topology on \( K^n \), given by the sub-base \( \{ x \in K^n \mid |x - a|_2 < r \}, a \in K^n, r \in \mathbb{R}_{>0} \). This topology is the same as the valuation topology on \( K^n \), since for any \( a \in K^n \), any \( \lambda \in \Gamma \), and any \( r \in \mathbb{R}_{>0} \) with \( v(r) > \lambda \), we have
\[
B(a, > v(r)) \subseteq \{ x \in K^n \mid |x - a|_2 < r \} \subseteq B(a, > \lambda);
\]
note that we continue to use the notation \( B(a, > \lambda), B(a, \geq \lambda) \) for balls in the valuative sense.

Let \( X \subseteq k^n \) be any definable set. Any sequence \( (u_\mu)_{\mu \in \mathbb{N}} \) with \( u_\mu \in X \) and \( \lim_{\mu \to \infty} u_\mu = u \in k^n \) represents an element of \( \text{res}^{-1}(u) \cap ^X \) in the ultra-product; vice versa, any element of \( \text{res}^{-1}(u) \cap ^X \) can be represented by a sequence in \( X \) converging to \( u \). If \( (u_\mu)_{\mu} \) is such a converging sequence, we will write \( [u_\mu] \) for the corresponding element of \( \text{res}^{-1}(\lim_{\mu} u_\mu) \). We will also use this notation with more complicated expressions inside the square brackets; the index variable of the sequence will always be \( \mu \). Note that square brackets commute with definable maps as follows. If, in addition to \( X \) and \( u_\mu \) as above, we have definable \( Y \subseteq k^m \) and \( f : X \to Y \), then \( [f(u_\mu)] = f([u_\mu]) \), where \( f \) denotes the corresponding map \( ^X \to ^Y \).

The following lemma is almost trivial, but it is the main tool which makes the transfer between \( k \) and \( K \) work. (Note that we implicitly use that the two different topologies on \( K \) coincide.)

**Lemma 7.9.** For any definable set \( X \subseteq k^n \) and any element \( u \in k^n \), the following are equivalent:

1. \( u \) lies in the topological closure of \( X \);
2. \( ^u \) lies in the topological closure of \( ^X \);
3. \( ^X \cap \text{res}^{-1}(u) \) is non-empty.

Proof. (1) $\iff (2)$ follows from definability of being in the topological closure. (1) $\iff (3)$: If $u$ lies in the closure of $X$, then any sequence $v_\mu \in X$ converging to $u$ yields an element $[v_\mu] \in X \cap \text{res}^{-1}(u)$. Vice versa, if $[v_\mu] \in X \cap \text{res}^{-1}(u)$, then we may assume $v_\mu \in X$ for all $\mu$ and thus $u = \lim_\mu v_\mu$ lies in the closure of $X$. \hfill $\square$

Note that the equivalence $(2) \iff (3)$ does not hold if one replaces "$X$" by an arbitrary definable subset of $K^n$; the point is that "$X$" is $\mathcal{L}_{\text{absring}}$-definable and using only parameters from the image of $k$ in $K$.

On the other hand, Lemma 7.9 also applies to definable subsets of varieties (instead of subsets of $k^n$); in particular, we will apply it in the Grassmanians $\mathbb{G}_{n,d}$.

Now, we can formulate the main proposition of this section.

**Proposition 7.10.** Let $k$ be either $\mathbb{R}$ or $\mathbb{C}$ and let $K$ be as in the beginning of Subsection 7.3. Suppose that $(S_i)_i$ are $\mathcal{L}_{\text{ring}}$-definable subsets of $k^n$ such that $(\ast(S_i))_i$ is a $t$-stratification of $K^n$. Then $(S_i)_i$ is a $C^1$-Whitney stratification of $k^n$ (in the sense of Definition 7.8).

**Proof.** In this proof, we will use the letters $u, v$ for elements of $k^n$ and $x, x'$ for elements of $K^n$. We have to prove conditions (1)–(4) of Definition 7.8 and that each $S_i$ is a $C^1$-Nash/algebraic manifold.

Since dimension is definable, (1) follows from the corresponding property of $\ast S_i$. (To obtain a definition of dimension which works both in $k^n$ and in $K^n$, we can replace the valuative ball in Definition 2.27 by an Euclidean ball.)

Using Lemma 7.9, closedness of $\ast S_{\leq i}$ implies (2) and moreover that for any $u \in S_d$, $\text{res}^{-1}(u)$ is a subset of $\ast S_{\geq d}$; in particular, $(\ast S_i)_i$ is $d$-translatable on $\text{res}^{-1}(u)$.

Fix $u \in S_d$ and set $B := \text{res}^{-1}(u)$ and $V_u := \text{tsp}_B((\ast S_i)_i)$. We claim that $\text{affdir}(B \cap \ast S_d) = V_u$ (cf. Definition 4.2). For dimension reasons, it suffices to verify `$\subseteq$', that is, for any $x, x' \in B \cap \ast S_d$, we have $\text{dir}(x - x') \in V_u$. To prove this, choose an exhibition $\pi: K^n \to K^d$ of $V_u$, set $F := \pi^{-1}(\pi(u)) \cap \ast S_d$, and apply Lemma 7.9 to $F \setminus \{u\}$. Since $u$ does not lie in the closure of $F \setminus \{u\}$, we obtain $F \cap B = \{u\}$, that is, $\pi$-fibers of $\ast S_d$ in $B$ consist of a single element.

Now $V_u$-translatability of $\ast S_d$ on $B$ implies the claim.

Next, we prove that $S_d$ is a $C^1$-manifold and that $V_u$ is the tangent space at $u$ for every $u \in S_d$ and $V_u$ as above. First note that each point $u \in S_d$ has a neighborhood $U \subseteq k^n$ such that for a suitable coordinate projection $\pi: k^n \to k^d$, $\pi$ induces a bijection $U \cap S_d \to \pi(U)$. (Indeed, this is a first-order statement and it holds in $K$.) We will use $\pi$ as a chart of $S_d$ around $u$. To prove that its inverse $(\pi|_{U \cap S_d})^{-1}$ is $C^1$ and that $V_u$ is the tangent space at $u$ (for every $u \in S_d$), it suffices to verify the following. For any $u \in S_d$ and any two sequences $v_\mu, v'_\mu \in S_d$ with $\lim_\mu v_\mu = \lim_\mu v'_\mu = u$ and $v_\mu \neq v'_\mu$, if $\lim_\mu k \cdot (v_\mu - v'_\mu)$ exists (in $\mathbb{G}_{n,1}(k)$), then $\lim_\mu k \cdot (v_\mu - v'_\mu) \subseteq V_u$. So, suppose that such $u, v_\mu, v'_\mu$ are given. Working in $\mathbb{G}_{n,1}$, we have $\lim_\mu k \cdot (v_\mu - v'_\mu) = \text{res}(k \cdot (v_\mu - v'_\mu)) = \text{res}(K \cdot ([v_\mu] - [v'_\mu]))$.

Now $[v_\mu], [v'_\mu] \in \text{res}^{-1}(u) \cap \ast S_d$ implies $\text{dir}([v_\mu] - [v'_\mu]) \in V_u$ by (1) and hence $\text{res}(K \cdot ([v_\mu] - [v'_\mu])) \subseteq V_u$.

In the case $k = \mathbb{C}$, we just proved that $S_d$ is $C^1$ in the sense of complex differentiation, so in that case, we obtain that $S_d$ is an algebraic manifold.

Sending a point $u \in S_d$ to its tangent space $T_u S_d$ is a definable map $S_d \to \mathbb{G}_{n,d}(k)$; transferring this to $K$ yields a notion of tangent space of $\ast S_d$ at any $x \in \ast S_d$; we denote that...
tangent space (which is a subspace of $K^n$) by $T_x^*S_d$. Fix $x \in S_d$. By definition, if $x' \in S_d \setminus \{x\}$ is close to $x$, then $K \cdot (x' - x)$ is close to a space contained in $T_x^*S_d$. In particular and more precisely, there exists a ball $B' \subseteq K^n$ containing $x$ such that for any $x' \in B' \cap S_d \setminus \{x\}$, we have $\text{res}(K \cdot (x' - x)) \subseteq \text{res}(T_x^*S_d)$. After possibly further shrinking $B'$, $(S_i)_i$ becomes $d$-translatable on $B'$ and then, any one-dimensional subspace of $\text{tsp}_{B'}((S_i)_i)$ is of the form $\text{res}(K \cdot (x' - x))$ for some $x' \in B' \cap S_d \setminus \{x\}$. For dimension reasons, this implies

$$\text{res}(T_x^*S_d) = \text{tsp}_{B'}((S_i)_i).$$

Now consider Whitney’s Condition (a), that is, suppose we are given a point $u \in S_d$ and a sequence $v_\mu \in S_j$ ($j > d$) as in Definition 7.8(3). Set $B := \text{res}^{-1}(u)$ and let $B' \subseteq B \cap S_{\geq j}$ be a ball containing $[v_\mu]$. Then

$$\lim_{\mu} T_{v_\mu}S_j = \text{res}(T_{[v_\mu]}^*S_j) = \text{tsp}_{B'}((S_i)_i) \supseteq \text{tsp}_{B}((S_i)_i) = T_uS_d.$$}

For Whitney’s Condition (b), suppose we are given $u \in S_d$ and sequences $u_\mu \in S_d$ and $v_\mu \in S_j$ ($j > d$) as in Definition 7.8(4). Again set $B := \text{res}^{-1}(u)$. Since $[u_\mu], [v_\mu] \in B \subseteq S_{\geq d}$, we can apply Corollary 7.6 to $[u_\mu] \in S_d$ and $[v_\mu] \in S_j$ and obtain a finite, $\mathcal{L}_{\text{Hens}}(\mathbb{B})$-definable set $M \subseteq \Gamma$ such that $\mathcal{V}([u_\mu] - [v_\mu]) \notin M$ implies $\text{dir}([u_\mu] - [v_\mu]) \notin \text{tsp}_{B'}((S_i)_i)$ for a ball $B' \subseteq S_{\geq j}$ containing $[v_\mu]$. In particular, $M$ is $\mathcal{L}_{\text{Hens}}(u)$-definable (viewing $u$ as an element of the residue field), so $M \subseteq \{0\}$ by Remark 7.3 and thus indeed $\mathcal{V}([u_\mu] - [v_\mu]) \notin M$. Therefore, we obtain

$$\lim_{\mu} k \cdot (u_\mu - v_\mu) = \text{res}(k \cdot (u_\mu - v_\mu)) = \text{res}(K \cdot ([u_\mu] - [v_\mu]))$$

$$\subseteq \text{tsp}_{B'}((S_i)_i) = \text{res}(T_{[v_\mu]}^*S_j) = \text{res}([T_{v_\mu}S_j]) = \lim_{\mu} T_{v_\mu}S_j,$$

which finishes the proof.

Using Proposition 7.10, it is now easy to deduce that t-stratifications ‘are’ also classical Whitney stratifications. To be consistent with Subsection 6.2, we fix a Noetherian integral domain $A$ of characteristic 0, we set $\mathcal{L} := \mathcal{L}_{\text{Hens}}(A)$, and we let $T$ be the theory of Henselian-valued fields $K$ of equi-characteristic 0 with ring homomorphism $A \rightarrow K$.

**Theorem 7.11.** Let $A$, $\mathcal{L}$ and $T$ be as defined right above. Suppose that $\phi_\nu$, $(\nu = 1, \ldots, \ell)$ and $\psi_i$, $(i = 0, \ldots, \ell)$ are $\mathcal{L}_{\text{ring}}(A)$-formulas in $n$ free variables such that for any model $K \models T$, $(\psi_i(K))_i$ is a t-stratification of $K^n$ reflecting $(\phi_\nu(K))_\nu$. Suppose moreover that the formulas $\psi_i$ are quantifier free. Then for both $k = \mathbb{R}$ and $k = \mathbb{C}$ and for any ring homomorphism $A \rightarrow k$, we have the following, where $X_\nu := \phi_\nu(k)$ and $S_i := \psi_i(k)$.

1. $(S_i)_i$ is a Whitney stratification of $k^n$ (see Definition 7.8).
2. Each $X_\nu$ is a union of some of the connected components of the sets $S_i$ (in the analytic topology).

In particular, each $\psi_i$ is an algebraic variety which is smooth over the fraction field of $A$.

Note that by Corollary 6.5, for any $(\phi_\nu)_\nu$ as above we can find $(\psi_i)_i$ defining a t-stratification as above, so indeed we obtain a new proof of the existence of Whitney stratifications (for $\mathcal{L}_{\text{ring}}(A)$-definable subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$).

**Proof of Theorem 7.11.** Let $K$ be the non-standard model of $k$ used in Proposition 7.10; we consider it as an $\mathcal{L}$-structure using the ring homomorphism $A \rightarrow k \hookrightarrow K$. Then the conclusion of Proposition 7.10 is that $(S_i)_i$ is a $C^\infty$-Whitney stratification. To finish the proof of (1), we have to get rid of this ‘$C^\infty$’. By taking $k = \mathbb{C}$, we obtain that each $\psi_i(\mathbb{C})$ is an algebraic
manifold; since $\psi_i$ is quantifier free, it can be viewed as a variety which is smooth over $\mathbb{C}$; in particular $\psi_i(\mathbb{R})$ is a $C^\infty$-submanifold of $\mathbb{R}^n$.

It remains to verify (2). We have to show that for each $d \leq n$ and each $\nu \leq \ell$, both $S_d \cap X_\nu$ and $S_d \setminus X_\nu$ are open in $S_d$. Since this is first order, we can instead prove the corresponding statement in $K^n$, that is, that $\nu S_d \cap X_\nu$ and $\nu S_d \setminus X_\nu$ are open in $\nu S_d$.

Let $x \in \nu S_d$ be given. We choose a ball $B \subseteq \nu S_{\geq d}$ containing $x$, we choose an exhibition $\pi : B \rightarrow k^d$ of $V := tsp_B((\nu S)_i)$, and we shrink $B$ such that each $\pi$-fiber intersects $B \cap \nu S_d$ in a single point. Then $V$-translatability implies that the set $B \cap \nu X_\nu$ is either disjoint from $\nu X_\nu$ or entirely contained in $\nu X_\nu$. \hfill \Box

8. Sets up to isometry in $\mathbb{Q}_p$ 

The main conjecture of [9] essentially is a classification of definable subsets of $\mathbb{Z}_p^n$ up to isometry. More precisely, it classifies certain trees associated to definable sets, which are closely related to isometry types. The original motivation for the present article was to prove that conjecture for $p$ sufficiently big. This is achieved with Theorem 8.3. Let us recall the trees considered in [9].

**Definition 8.1.** For a set $X \subseteq \mathbb{Z}_p^n$, the tree $T(X)$ associated to $X$ is the partially ordered set of those balls $B \subseteq \mathbb{Z}_p^n$ which intersect $X$ non-trivially; the ordering is given by inclusion. (If $X \neq \emptyset$, then $T(X)$ is indeed a rooted tree, with root $\mathbb{Z}_p^n$.)

It is not difficult to check that for topologically closed sets $X$, a tree encodes exactly the isometry type of $X$. In general, we have the following (see [9, Lemma 3.1]).

**Lemma 8.2.** Let $X, X' \subseteq \mathbb{Z}_p^n$ be any sets and write $\bar{X}, \bar{X}'$ for their topological closures. Then there is a natural bijection between the set of isometries $X \rightarrow \bar{X}'$ and the set of isomorphisms of partially ordered sets $T(X) \rightarrow T(X')$.

If $X \subseteq \mathbb{Z}_p^n$ is a definable set of dimension $d$, then according to [9, Conjecture 1.1] $T(X)$ should be a ‘tree of level $d$’. We will more or less recall this definition below, but instead of speaking about a tree $T$ itself, we will formulate it in terms of sets $X$ with $T(X) \cong T$. More precisely, we will introduce the notion of a subset $X \subseteq \mathbb{Z}_p^n$ being ‘of level at most $d$ ’; this will be slightly stronger than $T(X)$ being of level $d$. Our main result will then be that in sufficiently big residue characteristic, every definable set of dimension at most $d$ is also a set of level at most $d$. (Note that instead of calling the corresponding trees ‘of level $d$’, they should better also have been called ‘of level at most $d$ ’. This better terminology is used in [10], and we will also use it below.)

The differences between $T(X)$ being of level at most $d$ and $X$ being of level at most $d$ are the following.

(i) Trees classify the topological closures of definable sets up to isometry. The notion of a set of level at most $d$ captures the isometry type of the set itself.

(ii) Some definable sets have a more complicated isometry type in small residue characteristic. Since our present result only speaks about sufficiently big residue characteristic, we omit these from the notion of sets of level at most $d$.

We also take the opportunity to generalize the conjecture from [9] as follows.

(i) In [9], only subsets of $\mathbb{Z}_p^n$ are considered. we allow subsets of $\mathbb{Q}_p^n$. 


(ii) Instead of working only in \( \mathbb{Q}_p \), we work in any Henselian-valued field whose residue field is finite and has sufficiently big characteristic. In particular, this includes finite extensions of \( \mathbb{Q}_p \) and function fields \( \mathbb{F}_{p^r}(t) \). Non-discrete value groups are also allowed.

Here is the precise formulation of the main result of this section.

**Theorem 8.3.** Suppose that \( \phi(x, y) \) is an \( \mathcal{L}_{\text{Hen}} \)-formula (see Definition 2.16), where \( x \) is a tuple of valued field variables and \( y \) is a tuple of arbitrary variables. Then there exists an \( N \in \mathbb{N} \) with the following property. If \( K \) is a Henselian-valued field whose residue field is finite and has characteristic at least \( N \) and if moreover \( b \) is any tuple in \( K \) of the same sort as \( y \), then \( X := \phi(K, b) \) is a set of level at most \( d \) in the sense of Definition 8.5.

For this to (almost) classify definable subsets of \( \mathbb{Z}_p^n \) up to isometry, one also needs a converse. Indeed, by [9, Theorem 1.2], for any tree \( T \) of level at most \( d \), there exists a definable set \( X \subseteq \mathbb{Z}_p^n \) of dimension at most \( d \) with \( T(X) \cong T \).

Now we introduce the notion of sets of level at most \( d \). The translation between this and trees of level at most \( d \) in the sense of [9, Definition 4.1] is pretty straightforward: a part of this translation is written down in detail in the claim below Definition 4.3 of [10]. (The precise relation is: if \( X \) is a subset of \( \mathbb{Z}_p^n \) and it is of level at most \( d \), then \( T(X) \) is of level at most \( d \).)

**Notation 8.4.** We write \( \mathcal{L}_{\text{oag}} = \{0, +, -, \leq\} \) for the language of ordered abelian groups; by ‘\( \mathcal{L}_{\text{oag}}(\text{par}) \)-definable’, we mean definable in the language \( \mathcal{L}_{\text{oag}} \), where parameters are allowed.

**Definition 8.5.** Suppose that \( K \) is a valued field with finite residue field and \( d, n \in \mathbb{N} \). A subset \( X \subseteq K^n \) is a set of level at most \( d \) if it can be obtained as follows.

Choose any \( m \in \mathbb{N} \), any \( s_1, \ldots, s_m \in K^n \), and set \( S_0 := \{s_1, \ldots, s_m\} \). In the case \( d = 0 \), we (only) require \( X \subseteq S_0 \).

In the case \( d \geq 1 \), additionally choose, for each \( \ell \leq m \) and each \( \lambda \in \Gamma \), an enumeration \( B_{\ell,1,\lambda}, \ldots, B_{\ell,|k|\lambda,\lambda} \) of the maximal strict sub-balls of \( B(s_{\ell}, \geq \lambda) \) (that is, \( \text{rad}_o(B_{\ell,j,\lambda}) = \lambda \) and \( B(s_{\ell}, \geq \lambda) = \bigcup B_{\ell,j,\lambda} \)). Finally choose, for each \( \ell \leq m, j \leq |k|, \lambda \in \Gamma \), a set \( Y_{\ell,j,\lambda} \subseteq K^{n-1} \) of level at most \( d - 1 \). We require the following.

1. For each \( \ell, j, \lambda \) as above, if \( B_{\ell,j,\lambda} \cap S_0 = \emptyset \), then \( X \cap B_{\ell,j,\lambda} \) is isometric to \( Y_{\ell,j,\lambda} \times B(0, > \lambda) \), where by \( B(0, > \lambda) \) we mean a one-dimensional ball.
2. For each fixed \( \ell, j \), the family \( (Y_{\ell,j,\lambda})_{\lambda \in \Gamma} \) is of level at most \( d - 1 \) uniformly in \( \lambda \), in the sense described below.

A family of sets \( X_\kappa \subseteq K^n \) parameterized some \( \kappa \in M \subseteq \Gamma^v \) is of level at most \( d \) uniformly in \( \kappa \) (where \( M \) is \( \mathcal{L}_{\text{oag}}(\text{par}) \)-definable), if all of the above choices can be made for all \( \kappa \) such that moreover the following holds. (We use the notation from above, without indices \( \kappa \).)

3. For each \( m_0 \in \mathbb{N} \) and each \( I_0 \subseteq \{1, \ldots, m_0\} \), the set \( M' \subseteq M \) of those \( \kappa \) such that \( m = m_0 \) and \( X \cap S_0 = \{s_\ell \mid \ell \in I_0\} \) is \( \mathcal{L}_{\text{oag}}(\text{par}) \)-definable.
4. For each \( M' \) as in (3) and each \( \ell, \ell' \leq m_0, \tilde{v}(s_{\ell} - s_{\ell'}) \) is an \( \mathcal{L}_{\text{oag}}(\text{par}) \)-definable function of \( \kappa \in M' \).
5. For each \( M' \) as in (3) and each \( \ell \leq m_0, j \leq |k|_n \), the sets \( Y_{\ell,j,\lambda} \) are of level at most \( d - 1 \) uniformly in \( (\lambda, \kappa) \in \Gamma \times M' \) (and not just uniformly in \( \lambda \)).

Note that whether a set is of level at most \( d \) only depends on the isometry type of the set.
Proof of Theorem 8.3. Let \( L \) be a language consisting of \( \mathcal{L}_{\text{Hen}} \), any set \( C \) of constant symbols in any sorts, an angular component map \( \overline{\alpha}: K \to k \), and Skolem functions inside \( k \), and let \( T \) be the corresponding expansion of \( \mathcal{T}_{\text{Hen}} \). Elimination of valued field quantifiers implies that every \( L \)-formula \( \psi(z) \), where \( z \) is a tuple of \( \Gamma \)-variables, is already equivalent (modulo \( T \)) to an \( \mathcal{L}_{\text{ang}}(C') \)-formula for some suitable set of constants \( C' \) (namely, \( C' = \text{dcl}_L(\emptyset) \cap \Gamma \)).

For \( N \in \mathbb{N} \), let \( C_N \) be the class of all Henselian-valued fields with finite residue field of characteristic more than \( N \), considered as \( L \)-structures. (Note that by [11, Corollary 1.6], valued fields with finite residue field always admit an angular component map.)

In the following, we will work uniformly in all models \( K \models T \). In particular, unless specified otherwise, by a ‘definable set \( X \) in \( K \)’, we mean an \( L \)-formula \( X \) and, abusing notation, we write \( X \) instead of \( X(K) \). We will prove the following by induction.

**Claim.** Suppose that we have \( n, d, \nu \in \mathbb{N} \), a \( \emptyset \)-definable set \( Q \), \( \emptyset \)-definable families \( (S_{i,q})_{q \in Q} \) and \( (X_q)_{q \in Q} \), and a \( \emptyset \)-definable map \( \chi: Q \to M \subseteq \Gamma' \) such that for every model \( K \models T \), the following holds.

(i) For every \( q \in Q, (S_{i,q}), \) is a t-stratification of a sub-ball \( B_q \subseteq K^n \) and \( X_q \subseteq B_q \) is a subset of dimension at most \( d \) reflected by \( (S_{i,q}) \).

(ii) For every \( q, q' \in Q \) with \( \chi(q) = \chi(q') \), there exists a risometry \( \alpha_{q,q'}: B_q \to B_{q'} \) which sends \( ((S_{i,q}), X_q) \) to \( ((S_{i,q'})_{q'}, X_{q'}) \) and which is definable (with parameters) separately for each \( q, q' \in Q \) and \( K \models T \).

Then there exists an \( N \in \mathbb{N} \) such that in every \( K \in C_N \), we have the following. For every (not necessarily definable) cross section \( M \to Q, \kappa \mapsto q_\kappa \in \chi^{-1}(\kappa) \), the family \((X_{q_\kappa})_{\kappa \in M} \) is uniformly of level at most \( d \).

The claim implies the theorem using a singleton for \( Q \). Indeed, set \( X := \phi(K, c) \), where \( \phi(x, y) \) is the formula given in the theorem and \( c \) is a tuple of constants. Then, working in the language \( \mathcal{L}_{\text{Hen}} \cup \{c\} \), we can apply Corollary 4.13 to uniformly obtain t-stratifications \((S_i)\), reflecting \( X \) in each \( K \models \mathcal{T}_{\text{Hen}} \). After that, we enlarge the language to \( L \) and apply the claim.

**Proof of the Claim.** First note that we may suppose that \( m := |S_{0,q}| \) is constant (for all models \( K \models T \) and all \( q \in Q \)). Indeed, we can partition \( Q \) according to the cardinality of \( S_{0,q} \), and the existence of the risomtries \( \alpha_{q,q'} \) implies that this partition induces a partition of \( M \). The partition of \( M \) can be defined by \( \mathcal{L}_{\text{ang}}(C') \)-formulas, so that it also induces a finite, \( \mathcal{L}_{\text{ang}}(\text{par}) \)-definable partition of \( M \) in any \( K \in C_N \) for \( N \gg 1 \); the notion of being uniformly of level at most \( d \) is not affected by such a partition.

Next, we choose an enumeration \( s_{1,q}, \ldots, s_{m,q} \) of \( S_{0,q} \) which is definable uniformly in \( q \) (and uniformly in all models \( K \models T \)) and which satisfies

\[
s_{\ell,q'} = \alpha_{q,q'}(s_{\ell,q}),
\]

whenever \( \chi(q) = \chi(q') \). To see that such an enumeration exists, first note that since we have Skolem functions for finite subsets of \( \Gamma \) and \( k \), we also have Skolem functions for finite subsets of \( \text{RV}^{(n)} \) (the angular component map yields a definable bijection \( \text{RV}^{(n)} \setminus \{0\} \to (k^n \setminus \{0\}) \times \Gamma \)). By Lemma 2.15, the map \( \rho_q: x \mapsto \text{rv}(x - S_{0,q}) \) is injective on \( S_{0,q} \), and the Skolem functions can be used to enumerate the image \( \text{im}(\rho_q) \) in a way depending definably on \( \text{im}(\rho_q)^3 \). Then we automatically obtain \((\circ)\), since we have \( \rho_q \circ \alpha = \rho_q \) for any risometry \( \alpha: S_{0,q} \to S_{0,q'} \).

For \( N \gg 1 \), \((s_{\ell,q})_{\ell} \) is an enumeration of \( S_{0,q} \) in any \( K \in C_N \). With this enumeration, Definitions 8.5(3) and (4) are satisfied. Indeed, \((\circ)\) implies that the set \( I_q := \{ \ell \leq m \mid s_{\ell,q} \in X_q \} \) only depends on \( \chi(q) \), so fixing \( I_q \) yields a definable subset \( M' \subseteq M \), which then yields the required \( \mathcal{L}_{\text{ang}}(\text{par}) \)-definability of \( M' \) in any \( K \in C_N \) for \( N \gg 0 \) (independently of the cross section \( \kappa \mapsto q_\kappa \)). Similarly, for each \( \ell, \ell' \leq m \), \((\circ)\) implies that \( \text{rv}(s_{\ell,q} - s_{\ell',q}) \) only depends on \( \chi(q) \), which yields (4).
If \( d = 0 \), then \( \dim X_q = 0 \) implies \( X_q \subseteq S_{0,q} \) for all \( q \) and we are already done, so now assume \( d \geq 1 \). Also fix \( \ell \leq m \) for the remainder of the proof.

For every \( \lambda \in \Gamma \) and \( q \in Q \), let \( (B_{u,\lambda,q})_{u \in k^n} \) be the family of maximal strict sub-balls of \( B(s_{\ell,q}, \geq \lambda) \). We assume that for \( q, q' \in Q \) with \( \chi(q) = \chi(q') \), we have \( \alpha_{q,q'}(B_{u,\lambda,q}) = B_{u,\lambda,q'} \) and (using the map \( \alpha \)) that \( B_{u,\lambda,q} \) is definable uniformly in \( u, \lambda, \) and \( q \).

Now also fix \( u, \lambda, q \) for the moment, set \( B := B_{u,\lambda,q} \), and suppose that \( B \cap S_{0,q} = \emptyset \). Choose a one-dimensional subspace \( V \subseteq \tsp B ((S_{i+1,q})_i) \) and an exhibition \( \pi : K^n \to K \) of \( V \). Using that straighteners are definable uniformly for all \( K \models T \) (Corollary 3.22), we also have a straightener \( \lambda, q, \lambda, q \in C \lambda, q, \lambda \in C = \Gamma \) if \( \lambda, q, \lambda \) is given (for \( \lambda, q, \lambda \in C \)).

Next, we define \( \chi : Q^\prime \rightarrow M' := \Gamma \times M, (\lambda, q, x, \lambda, q) \mapsto (\lambda, \chi(q)) \); it remains to find suitable risometries

\[
\alpha'_{(\lambda, q, x, \lambda, q')}: \pi^{-1}(x) \cap B_{u,\lambda, q} \rightarrow \pi^{-1}(x') \cap B_{u,\lambda, q'}
\]

for \( \lambda \in \Gamma, q, q' \in Q \) with \( \chi(q) = \chi(q'), x \in \pi(B_{u,\lambda, q}) \), and \( x' \in \pi(B_{u,\lambda, q'}) \). These are obtained by applying Lemma 3.6(2) to \( \alpha_{q,q'}(B_{u,\lambda, q}) = B_{u,\lambda, q'} \).

Now the conclusion of the induction hypothesis yields exactly Definition 8.5(5). Indeed, suppose that \( K \models C \lambda \) is given (for \( N \geq 1 \)) and that in \( K \), we have a cross section \( M \rightarrow Q, \kappa \rightarrow q_\kappa \). Fix \( u \in k^n \); this turns \( K \) into an \( L' \)-structure. (Note that our \( u \) here corresponds to the \( j \) in Definition 8.5.) We choose a cross section \( M' \rightarrow Q' \) of the form \( (\lambda, \kappa) \mapsto (\lambda, q_\kappa, x_{\lambda, \kappa}) \) with \( x_{\lambda, \kappa} \in \pi(B_{u,\lambda, q_\kappa}) \) arbitrary. By induction, \( X'_{(\lambda, q_\kappa, x_{\lambda, \kappa})} \) is of level at most \( d - 1 \) uniformly in \( \lambda \) and \( \kappa \), which is what we had to prove. \( \square \)

In [9, Section 7], several strengthenings of the conjecture about the trees have been proposed; to conclude this section, let me comment on these strengthenings.

(i) The tree \( T(Z^p) \) can be considered as an imaginary sort; then, for any definable \( X \subseteq Z^p \), the tree \( T(X) \) is a definable subsets of \( T(Z^p) \). Conjecture 7.1 of [9] describes arbitrary definable subsets \( Y \subseteq T(Z^p) \) instead of only those of the form \( T(X) \). For big \( p \), it should also be possible to prove that conjecture, using a t-stratification reflecting the map

\[
\chi(x) := \{ \gamma \in \Gamma \mid B(x, \geq \gamma) \in Y \}^\gamma
\]

(ii) In [9, Section 7.2], a version of the conjecture has been proposed for arbitrary Henselian-valued fields of characteristic \((0,0)\) (without giving much details). However, as noted there, the conjecture has far less meaning when the residue field is infinite (since then, too many isometries exist), so instead of considering pure abstract trees, one should consider trees with some additional residue field data. Driving this idea further is what finally led to the definition of t-stratifications.

According to [9], these ‘trees in characteristic \((0,0)\)’ should imply the conjecture in \( \mathbb{Q}_p \) for big \( p \) and should even yield some kind of uniformly in \( p \). Our proof of Theorem 8.3 indeed yields uniformity in the following sense. Given a formula \( \phi(x,y) \) as in the theorem, the \( L_{\text{org}} (\text{par}) \)-definable objects that we construct to prove that \( \phi(K,b) \) is of level at most \( d \) (for Definition 8.5(3), (4)) can be defined by \( L \)-formulas not depending on \( K \) and \( b \) (but taking
b as a parameter). Moreover, since in this uniform setting, the cardinality of the residue field grows, it becomes worthwhile to note that our proof moreover yields that in Definition 8.5(5), $Y_{i,j,\lambda}$ is uniform also in $j$ and not just in $\lambda$ and $\kappa$.

9. Open questions

There are several ways in which it might be possible to enhance the results of this article.

9.1. A stronger notion of t-stratifications

Recall from the introduction that t-stratifications do not satisfy the straightforward translation of Whitney’s Condition (a): for two strata $S_d$ and $S_j$ with $d < j$, $x \in S_d$ and $y \in S_j$, we have that $T_y S_j$ is close to containing $T_x S_d$ when $y$ is close to $x$, whereas Condition (a) requires that $T_y S_j$ converges to a space containing $T_x S_d$. Also recall that in the p-adics, the existence of Whitney stratifications in this more classical sense has been proved in [2]. It seems plausible that there exists a common generalization of both kinds of stratifications (at least in equicharacteristic 0 and if the value group is of rank 1). Such a generalization might be defined as follows.

Let us define ‘stronger risometries’: maps $\phi$ such that $\hat{v}((\phi(x) - \phi(x')) - (x - x')) > \hat{v}(x - x') + \delta$ for some given $\delta \geq 0$. (For $\delta = 0$, this is just a usual risometry.) This yields corresponding notions of translatability which we call ‘$\delta$-strong translatability’.

Using this, a ‘strong t-stratification’ should roughly require that for any $\delta \geq 0$ and any ball $B$ ‘sufficiently far away from $S_{\leq d-1}$’, we have $\delta$-strong $d$-translatability on $B$. More precisely, it seems plausible that we can obtain $\delta$-strong $d$-translatability on any ball $B$ which is contained in a ball $B' \subseteq S_{d'}$ with $\text{rad}_d B \geq \text{rad}_d B' + \delta$.

Note that this indeed implies Condition (a). For any $x \in S_d$ and any $\delta \geq 0$, there exists a ball $B$ around $x$ which is sufficiently far away from $S_{\leq d-1}$ in the above sense, and $\delta$-strong translatability on $B$ then implies that for $y \in B \cap S_j$, $T_y S_j$ is $\delta$-close to a space containing $T_x S_d$.

9.2. Mixed characteristic

It should be possible to prove the existence of a variant of t-stratifications in mixed characteristic, but again, it is not entirely clear how this variant has to be formulated. For a ball $B \subseteq S_{d'}$, even 0-strong (that is, usual) $d$-translatability can only be expected on sub-balls $B'$ of $B$ with $\text{rad}_d B' \geq \text{rad}_d B + \delta$ for some fixed $\delta$ (depending only on the t-stratification). This can be seen, for example, at the cusp curve in characteristic 2 (see [9, Section 3.3] or [10, Section 5.4] for a detailed computation). In terms of the description of the trees of [9], this $\delta$ would be exactly the maximal length of the finite trees appearing at the beginning of side branches.

When the valuation of the residue characteristic $p$ is finite (that is, when there are only finitely many $\Gamma$ between 0 and $v(p)$), then in the previous paragraph, it should also be possible to require $\delta$ to be finite, and the resulting notion of t-stratification might be the ‘right one’. However, if $v(p)$ is not finite, we are forced to allow finite multiples of $v(p)$ for $\delta$. But then, I am afraid that then the notion of t-stratification becomes too weak, for example, to imply Proposition 3.19; in particular, the induction in the proof of Theorem 4.12 would fail.

9.3. Getting classical Whitney stratifications more generally

The fact that the existence of t-stratifications implies the existence of Whitney stratifications should also work in languages other than the pure (semi-)algebraic one. For this to work, we need the existence of t-stratifications ($S_i$), which are defined without using the valuation.
Probably Proposition 6.2 can be applied to prove such a result, but I did not check it. In the algebraic language, we used this to deduce a posteriori that each $S_i$ is smooth. This too, should work more generally, again with an argument that manifolds in $C^n$ which are $C^1$ in the sense of complex differentiation are automatically smooth.

9.4. Minimal t-stratifications

It would be nice if, for every definable set $X \subseteq K^n$, there would be a ‘minimal’ t-stratification $(S_i)_i$, reflecting $X$. ‘Minimal’ could mean that for any other t-stratification $(S'_i)_i$ reflecting $X$, we have $S_{\leq i} \subseteq S'_{\leq i}$ for all $i$. Moreover (or alternatively), one might hope that for a minimal $(S_i)_i$, a definable risometry $K^n \to K^n$ preserves $X$ if and only if it preserves $(S_i)_i$ (in general, there are less risometries preserving $(S_i)_i$). In the case of Whitney stratifications of complex analytic spaces, minimal stratifications in the first sense have indeed been constructed by Teissier; see [12].

There are (at least) two reasons for minimal t-stratifications not to exist, but for both of them, all hope is not lost. The first obstacle is the non-canonicity of t-stratifications explained in Example 1.2. This might be overcome as follows. Instead of letting $S_{\leq i}$ be a subset of $K^n$, we let it be a subset of the set of sub-balls of $K^n$, where points are also considered as balls. Then we require $d$-translatability on a ball $B \subseteq K^n$ if no ball of $S_{\leq d-1}$ is (strictly?) contained in $B$. At least for Examples 1.2 and 1.3, this seems to solve the problem.

A second problem is that one can construct a set $X$ such that whether $X$ is $d$-translatable on some ball $B$ does not depend definably on $B$ (see Example 3.15). Since for t-stratifications, $d$-translatability is always definable (Lemma 3.14), any t-stratification reflecting $X$ will necessarily have less risometries than $X$ preserving it. I do not think that it is possible to solve this problem in general, but it might be possible to find a good condition on the residue field which avoids the problem. A candidate which at least destroys Example 3.15 is the following. For any definable function $f : k^n \to k$, we require that there exists a definable function $\tilde{f} : \mathcal{O}_K^n \to \mathcal{O}_K$ such that $\text{res } f = \tilde{f} \circ \text{res}$.

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