Stable Hydrogen burning limits in rapidly rotating brown dwarfs

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ABSTRACT

We consider the effects of uniform rapid stellar rotation on the minimum mass of stable hydrogen burning. To focus on the effects of rotation, we use a polytropic model of the star, and employ a suitable numerical scheme, relaxing the assumption of spherical symmetry. We obtain an analytical formula for the minimum mass of hydrogen burning as a function of the stellar rotation speed. Further, we show the possibility of a maximum mass of stable hydrogen burning in such stars, which is purely an artefact of rotation. The existence of this extremum in mass results in a maximum admissible value of the stellar rotation speed, beyond which no brown dwarfs can exist, within the ambits of our model. For a given angular speed, we predict a mass range beyond which no brown dwarf will evolve into a main sequence star.

1. INTRODUCTION

Brown dwarfs (BDs), which were theoretically predicted by Kumar (1963), and Hayashi and Nakano (1963), are substellar objects whose masses range between those of Jupiter ($\sim 10^{-3} M_\odot$) and stars at the bottom of the main sequence ($\sim 10^{-1} M_\odot$). During their lifetimes, these “failed stars” do not attain sustained nuclear fusion of Hydrogen into Helium, as their masses are less than a certain minimum value, dubbed as the minimum mass of hydrogen burning ($M_{\text{mmhb}}$) or sometimes as the minimum main sequence mass. Burrows and Liebert (1993) review the works in the area from the mid 60’s to the early 90’s (see also D’Antona and Mazzitelli (1985), Burrows, Hubbard and Lunine (1989)), and provide analytical models of BDs, although their observational aspects were still in the nascent stages at that time, given that BDs are particularly difficult to detect, due to their typically low luminosities. Later, Rebolo, Zapatero-Osorio and Martin (1995) announced the first observation of a brown dwarf in the Pleiades cluster, and this was closely followed by the similar discovery of Nakajima et. al. (1995). The plethora of activities that followed immediately thereafter, are well documented in the review articles by Chabrier and Baraffe (2000), Basri (2000), Burrows et. al. (2001), and the textbook by Rebolo and Zapatero-Osorio (2000) (see also D’Antona and Mazzitelli (1997), Burrows et. al. (1997), Chabrier and Baraffe (1997)). The research carried out in the area in the next decade is outlined in the more recent textbook by Joergens (2014) (see also Allard, Homeier, and Fryetag (2012), Chabrier et. al. (2014), Marley and Robinson (2015)).

What distinguishes BDs from main sequence stars is the $M_{\text{mmhb}}$, with the currently accepted value of $\sim 0.08 M_\odot$, assuming a static scenario (the recent review of Audby, Basu and Valhuri (2016) quotes the range $0.064 - 0.087 M_\odot$, based on some modifications of earlier analytical models). However, it is by now well known that various factors may affect the $M_{\text{mmhb}}$, one example being accretion in binary systems (Salpeter (1992)). In this context, we show here that the $M_{\text{mmhb}}$ can also be enhanced from its accepted value, via stellar rotation (the physics of rotating stars are described in sufficient details in the older literature, e.g. Kippenhahn and Thomas (1970) and in the recent monographs by Tassoul (2000) and Maeder (2009)). Indeed, more than five decades ago, Kippenhan (1970) showed a possible increase in the $M_{\text{mmhb}}$ due to rotational effects. The basic physics may seem simple. Namely, that with centrifugal forces reducing gravity inside a stellar object, a rotating star can in some sense accommodate more mass than its non-rotating cousin. Here one has to keep in mind that in rotating stellar objects, all stellar parameters like the density, temperature etc., depend on the rotational speed $\Omega$, and that there are several competing effects involving the degeneracy as well. One of the results in this paper is an analytic formula of $M_{\text{mmhb}}$ as a function of $\Omega$.

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In particular, we consider rapid rotation, where the approximation of spherical symmetry needs to be abandoned. By rapid, we mean a rotation period much smaller than that of Jupiter, which has a period of ~ 10 h. Such rapid rotations in cool dwarfs have been abundantly reported in the recent past. Clarke et al. (2008) presented photometric observations of a T6 dwarf with a rotation period of 1.41 h. Metchev, Heinze and Apai (2015) presented data on a T7 dwarf with a rotation period of 1.55 h. The analysis of Route and Wolszczan (2016) obtains a dramatically shorter period of ~ 17 min for a T6 dwarf, although the authors point out that this might be a subharmonic of a longer period. Followup observations of the same object by Williams, Gizis and Berger (2017) however indicated that this period might in fact be closer to 1.93 h, although these authors also mention the need for more data to confirm this. The most recent analysis appears in Tannock et al. (2021), who reported on the observation of photometric periods ranging from 1.08 to 1.23 h. Clearly then, the latest available data on the rapidly rotating brown dwarfs point to the smallest period of 1.08 h, and Tannock et al. (2021) claim that these are “unlikely” to rotate much faster, given the clustering of the BDs having the smallest rotation periods.

With this status of observational signatures, the question we ask here is, if there are any constraints on brown dwarf rotations set by theory. This is important and interesting for two reasons. First, it is not difficult to imagine that this provides a hitherto unknown dependence of $M_{\text{mmhb}}$ with the (rapid) rotational speed $\Omega$, and as we show in sequel, provides quantitative evidence for over-massive BDs, i.e., BDs with mass greater than $M_{\text{mmhb}}$. Second, as we will see, it sets an upper limit of the angular speed $\Omega_{\text{max}}$, beyond which BDs do not exist. Importantly, this is not the well known break up angular speed of a star which defines the limit for its disruption in Newtonian gravity via centrifugal forces. The latter is given in a standard fashion for a star of mass $M$ and radius $R$ by $(GM)^{1/2}/R^{3/2}$, where $G$ is the gravitational constant and one assumes spherical symmetry. Here, we arrive at $\Omega_{\text{max}}$ via completely different physical considerations involving the luminosity. For a given $\Omega$, we obtain a transition mass range $M_{\text{mmhb}} \leq M < M_{\text{max}}$ (with $M_{\text{max}}$ being a maximal mass) for BDs to evolve into main sequence stars. Further, we provide an analytical formula for the luminosity of a BD when it reaches the main sequence, as a function of its mass and angular speed.

Here, we adapt the analytical polytropic model of Burrows and Liebert (1993) via a suitable numerical scheme to accommodate rapid rotation. Indeed, rotating polytropes have been studied for almost a century, starting with Jeans (1928), Chandrasekhar (1933), and later by Roberts (1963a), Roberts (1963b), Hurley and Roberts (1964), James (1964), Stoeckly (1965). The novelty of our work is the implementation of the physics of brown dwarfs in a rapidly rotating scenario. The organisation of this paper is as follows. In the next section 2, we recall the basic features of non-rotating BDs, and set up the analytical model. This is then used in section 3 to include rotation, and we present our main results in the subsequent section 4. The paper ends with a summary in section 5.

2. NON-ROTATING BROWN DWARFS

The basic assumptions that we use here are as follows. Firstly, the BD is assumed to be fully convective, containing an ionised hydrogen/helium mixture, with partially degenerate electrons. The pressure, which arises due to both thermal effects (ions) and degeneracy (electrons) is considered to be non-relativistic. Importantly, these last two assumptions imply that we can safely use a polytropic equation of state (EOS) (polytropic approximations are discussed in the textbook of Chandrasekhar (1939). For more discussions on the applicability of this approximation to BDs, see Nelson, Rappaport and Joss (1986), Rappaport and Joss (1984)). The entropy is assumed to be constant throughout the star. Further, it is assumed that the core temperature in BD is not sufficient to produce $He^4$. Hence the truncated $p - p$ chain thermonuclear reactions takes place in the stellar interior. Note that the original model of Burrows and Liebert (1993) model assumes spherical symmetry, which we will relax when we consider rapid stellar rotation.

To set the stage, and to develop the notations used in the rest of this paper, we will now recall some known facts in the evolution of BDs. After its formation, during the initial stages, a BD keeps contracting owing to its self gravity. In the process, it keeps radiating energy from its surface, which is referred to as surface luminosity $L_S$. Now, $L_S$ keeps decreasing as the BD contracts with time. The contraction, however, initially leads to an increase in its core temperature and density. It is known that rates of thermonuclear reactions are dependent on both of these. Hence, at some stage, if the attained core temperature and density are sufficient, then thermonuclear reactions start taking place. The energy generated within the star due to this is referred to as hydrogen burning luminosity $L_{HB}$. With further contraction of BD, $L_{HB}$ starts increasing. A star is said to undergo stable/sustained hydrogen burning if the amount of energy liberated from surface is balanced by that produced from thermonuclear reaction within the star i.e., $L_S = L_{HB}$. Hence, at some point during the BD’s contracting phase, if stable hydrogen burning is attained, then further contraction ceases and the BD is said to become a main sequence star (MSS). However, if considerable
amount of degeneracy sets in before the BD attains stability, a part of the thermal energy of the star is used up in
accommodating a large number of degenerate electrons in a smaller volume. This forbids the core temperature from
rising further. The core temperature thus starts falling with further contraction. This eventually leads to a decrease
in the \( L_{HB} \) too, and hence the star does not stabilise thermally.

Now, if the initial mass of the BD, after formation, happens to be greater than a certain minimum value, then the BD
eventually stabilises before the onset of considerable electron degeneracy. It is commonly believed that this minimum
value, the \( M_{mmhb} \) (in the non-rotating case), sets the boundary between a MSS and a BD. Recently, however, Forbes
and Loeb (2019) have shown that theoretically over-massive BDs (mass \( \gtrsim M_{mmhb} \)) are possible, via accretion effects.
According to this analysis, \( M_{mmhb} \) should no longer demarcate between MSS and BD. However, it is still the minimum
value below which a BD can never reach the main sequence.

![Figure 1. \( L_{ratio} \) vs \( \eta \) for non-rotating BD : The red curve corresponds to \( M_{mmhb}(0.075M_\odot) \), the green one for mass = 0.077\( M_\odot \)
and the black one for 0.074\( M_\odot \). The dashed portion of the curves, after attaining stable hydrogen burning, are unphysical.](image)

The continuous contraction of a BD, after its formation, leads to increase in its degeneracy. Hence one simulates
the time-evolution of BD of a given mass, by varying the degeneracy parameter (called \( \eta \) in sequel), instead of time.
The reason behind this will be clear from their relationship, as will be explained in the next section 3. For a given
mass of BD, we compute \( L_{HB} \) and \( L_S \) at every instant of its contracting phase (i.e., lower \( \eta \) value to higher ones). In
the process, for every \( \eta \), we get the value of the ratio of the two luminosities, \( L_{ratio} = \frac{L_{HB}}{L_S} \). We then plot \( L_{ratio} \) vs \( \eta \) for the given mass. From the above discussions, we see that \( L_{ratio} \) first increases, and then starts descending after
attaining a maxima. If the maximum value is less than unity, then it indicates that the BD can never reach the stable
hydrogen burning condition \( L_{HB} = L_S \). Hence, we repeat the above numerical procedure for a mass higher than the
one previously chosen and repeat the numerical procedure, till the maxima of the plot attains unity. This mass is then
the \( M_{mmhb} \).

At the point where a BD of a given mass attains stable hydrogen burning (i.e. \( L_{ratio} = 1 \)), it has evolved into a
MSS. Thus, from that point onwards, the model for BDs ceases to be valid for subsequent evolution. One needs to
consider a MSS model to further the evolutionary process. Hence, in Fig. 1, the dashed portion of the curves, after
attaining stable hydrogen burning, are unphysical. One can see from Fig. 1, that BDs having masses greater than
\( M_{mmhb} \) attains stability (i.e. \( L_{ratio} = 1 \)) at lower \( \eta \) values.

### 3. EFFECT OF ROTATION IN THE BD’S EVOLUTIONARY PROCESS

We first consider the star to be centered at the origin of a Cartesian coordinate system \( \{x^1, x^2, x^3\} \). Now we consider
uniform rotation of the star along the \( x^2 \)-axis. Owing to centrifugal forces, the star bulges near the equatorial plane
(i.e. \( x^2 = 0 \) plane), deforming it into an oblate spheroid. Hence, a star under rapid rotation loses spherical symmetry.
Then, the essential features of the BD model, chosen for studying effects of rapid rotation, are same as that of Burrows
and Liebert (1993), excepting for the assumption of spherical symmetry there, which will lead to crucial modifications
as we discuss below.

#### 3.1. The stellar equations and numerical recipe
We begin with the polytropic equation of state, Poisson’s equation and the Euler equation corresponding to momentum conservation. The polytropic equation reads

\[ P = \kappa \rho^{(1+\frac{1}{n})}, \quad (1) \]

where \( P \) is the pressure, \( \kappa \) is the polytropic constant and \( n \) the polytropic index, which is to be taken as 1.5, as appropriate for a BD. The Poisson’s equation reads

\[ \nabla^2 \phi = 4\pi G \rho, \quad (2) \]

where \( \rho \) is the density and \( \phi \) is the gravitational potential. Finally, the Euler equation corresponding to momentum conservation is

\[ \rho \frac{\partial v^i}{\partial t} + \rho v^j \frac{\partial v^i}{\partial x^j} = -\frac{\partial P}{\partial x^i} - \rho \frac{\partial \phi}{\partial x^i}, \quad (3) \]

where \( t \) is the temporal coordinate and \( v^i = \Omega \{x^3, 0, -x^1\} \) is the velocity field of the star with \( \Omega \) being its uniform angular speed.

We get the deformed equilibrium configuration of the rotating star of a fixed mass \( M \), by numerically solving Eqs. (2) and (3), for a given \( \kappa \). Initially we solve the Lane-Emden equation to obtain the spherically symmetric density profile, which is then used in Eq. (2), to obtain the gravitational potential \( \phi \). Then using this \( \phi \) in Eq. (3), we obtain the updated density profile \( \rho \). We feed this updated \( \rho \) back into Eq. (2), to obtain an updated \( \phi \), which in turn yields an updated \( \rho \) from Eq. (3). This iteration is repeated until a desired convergence is achieved for a given tuple \( \{M, \Omega, \kappa\} \). For further details of the numerical procedure, the reader is referred to Ishii, Shibata and Mino (2005), Banerjee et al. (2021).

From the converged solution, we also obtain the central density \( \rho_c \) of the deformed star in equilibrium. Now, the polytropic constant for a BD is related to the degeneracy parameter (\( \eta \)) of the star as follows

\[ \kappa = \frac{(3\pi^2)^{\frac{2}{3}}h^2}{5m_e m_H \mu_e^2} \left(1 + \frac{\alpha}{\eta} \right), \quad \eta = \frac{(3\pi^2)^{\frac{2}{3}}h^2}{2m_e m_H \mu_e^2 \frac{\mu}{k_B}} \frac{\rho_c^\frac{2}{3}}{T_c}, \quad (4) \]

where \( m_e \) and \( m_H \) denote the electron and hydrogen mass respectively, and \( \alpha = 5\mu_e/2\mu \). Here, \( \mu_e \) is the number of baryons per electron and \( \mu \) is the mean molecular weight of the hydrogen/helium mixture [\( \mu = 0.593 \mu_e = 1.143 \)]. Also, \( \eta \) is defined to be the ratio of the Fermi energy to \( k_B T \), where \( T \) is the temperature, \( k_B \) being the Boltzmann constant. Thus fixing \( \eta \) inherently determines the corresponding \( \kappa \). We then compute the central temperature \( T_c \) from Eq. (4), using the obtained value of \( \rho_c \) for the converged equilibrium configuration of the deformed star. Using these, we numerically calculate the hydrogen burning luminosity of the star,

\[ L_{HB} = \int_V \rho \epsilon \ dx^1 \ dx^2 \ dx^3, \quad (5) \]

where \( \rho \) and \( \epsilon \) are functions of the spatial coordinates \( \{x^1, x^2, x^3\} \), with \( \epsilon \) being the energy generation rate per unit mass. The integration is performed numerically over the deformed volume \( V \). For a typical BD, we approximate

\[ \epsilon = \epsilon_e \left( \frac{T}{T_e} \right)^{\frac{s}{2}} \left( \frac{\rho}{\rho_e} \right)^{u-1}, \quad \epsilon_e = \epsilon_0 T_e^s \rho_e^{u-1}, \quad (6) \]

where \( s \simeq 6.31, u \approx 2.28 \) and \( \epsilon_0 = 1.66 \times 10^{-46} \ ergs \ g^{-1} s^{-1} \). Next, we compute the luminosity at the photosphere (i.e., surface luminosity) of the deformed star. At any point near the surface, the atmosphere can be locally approximated to be plane parallel, irrespective of rotation. Thus we use the following definition of optical depth (\( \tau (z) \)) for a planar atmosphere to determine the location of photosphere in a deformed star :

\[ \tau (z) = \int_z^\infty \kappa_R \rho \ dz, \quad (7) \]

where \( z \) is the local vertical depth of the atmosphere and \( \kappa_R \) is the Rosseland mean opacity. The photosphere is then defined to be located at \( z_e \) for which \( \tau (z_e) = 2/3 \). The temperature \( T_e \) and the density \( \rho_e \) at the photosphere are related through

\[ \frac{T_e}{K} = 1.8 \times 10^6 \left( \frac{\rho_e}{g/cm^3} \right)^{0.42}, \quad (8) \]
Now using Eq. (8) and the ideal gas law, and assuming approximate constancy of the acceleration due to gravity near the surface, we obtain a local expression for temperature at the photosphere.

\[ T_e = \left( \frac{2g \mu m \mu}{3K_B k_B} \right)^{\frac{1}{4+2}} \left( \frac{1.8 \times 10^6}{\eta^{1.545}} \right)^{1/7}. \]  

(9)

For a deformed star, the relative position of the photosphere with respect to the surface of the star does not remain constant throughout, i.e., it varies from one surface point to another, unlike the case for a spherically symmetric star. This would also be the case with \( T_e \). Now applying Stefan-Boltzmann law, we compute the total surface luminosity as

\[ L_S = \int_S \sigma T_e^4 dA \]

(10)

where \( dA \) is the elemental surface area and \( \sigma \) is the Stefan-Boltzmann constant. The integration is performed over the entire surface of the deformed star. Finally, we compute the \( L_{\text{ratio}} \) for the given \( \{ M, \Omega, \eta \} \), which would be of fundamental importance to decide upon the fate of a rotating BD’s evolution.

4. RESULTS AND ANALYSIS

We now present the main results obtained from our computational scheme discussed above.

4.1. \( M_{\text{mmhb}} \) as a function of \( \Omega \)

Rotation tends to reduce the strength of gravity inside a star. This effectively makes a rotating star, with a given mass and degeneracy, achieve hydrostatic equilibrium at a lower core temperature and density. As a consequence, we find that the \( M_{\text{mmhb}} \), in presence of rotation to be larger than non rotating case. In order to find \( M_{\text{mmhb}} \) corresponding to a given stellar rotation \( \Omega \), we carry out a similar algorithm described in section 2, using the numerical prescription mentioned in section 3.1. We perform this numerical procedure for different \( \Omega \) values, to obtain a fitted formula for \( M_{\text{mmhb}} \) as a function of \( \Omega \). We find,

\[ M_{\text{mmhb}}(\Omega) = 0.075 + 0.22\Omega + 288.38\Omega^2 + 81149\Omega^3, \]

(11)

where \( \Omega \) is in \( s^{-1} \) and the formula gives \( M_{\text{mmhb}}(\Omega) \) in units of the solar mass \( M_\odot \). From Eq. (11), we find that \( M_{\text{mmhb}} \) increases monotonically with \( \Omega \) as depicted by the red curve in Figure 2, but that for small \( \Omega \), the change from the case \( \Omega = 0 \) is maximally by a few percents. For example, for the smallest observed period of 1.08 h of Tannock et al. (2021), the increase in \( M_{\text{mmhb}} \) is by 1.9%. Faster rotations can however significantly change the result. Using the period of 17 min of Route and Wolszczan (2016), the increase in \( M_{\text{mmhb}} \) is almost 42%. However, such a small value of the period is ruled out of our analysis, as we momentarily see. We will find that the minimum period of a BD can be \( \sim 23 \) min, and hence the maximal increase in \( M_{\text{mmhb}} \) is \( \sim 19\% \). Importantly, we have demonstrated that rotation can lead to the existence of BDs beyond the \( M_{\text{mmhb}} \).

4.2. The Existence of \( M_{\text{max}} \)

If a rotating star (of a given degeneracy) has a central density lower than a certain minimum, then the centrifugal force dominates over self-gravity. This forbids the star from attaining hydrostatic equilibrium, and the star breaks under its own rotation. This minimum value of the central density is dubbed the critical density \( \rho_{\text{crit}} \), and we label the corresponding stellar configuration as a critical configuration.

For a given \( \Omega \), we find the critical configuration for each value of the degeneracy \( \eta \). For each of these critical stellar configurations, we record the critical density \( \rho_{\text{crit}} \) and compute the corresponding \( L_{\text{ratio}} \) and mass, which we call the critical \( L_{\text{ratio}} \) and \( M_{\text{crit}} \), respectively. While we find that \( \rho_{\text{crit}} \) remains constant with \( \eta \), Figure 3 indicates that both critical \( L_{\text{ratio}} \) and \( M_{\text{crit}} \) falls with increasing degeneracy, as is not difficult to justify physically. Now, a BD of a given mass after its formation, keeps contracting owing to self-gravity. This leads to a rise in its central density with \( \eta \). Unless a BD of given mass \( M \) and rotation \( \Omega \), attains sufficient degeneracy, say \( \eta_{\text{start}} \), an equilibrium configuration will not exist. This is because, only for degeneracy \( \eta \geq \eta_{\text{start}} \), the central stellar density \( \rho_c \) is greater than or equal to corresponding \( \rho_{\text{crit}} \).

Now, the \( L_{\text{ratio}} \) value corresponding to \( \eta_{\text{start}} \) of the specified BD reveals the relative magnitude of hydrogen burning luminosity and surface luminosity, at the initial stage of its evolution. At \( \eta_{\text{start}} \) there can be three situations. Firstly,
we can have \( L_{\text{ratio}} < 1.0 \), in which case the BD will attain stability with further evolution, only if \( M \geq M_{\text{mmbb}}(\Omega) \). Secondly, for \( L_{\text{ratio}} = 1 \), the BD has turned into a MSS at the very beginning, and our model ceases to be valid hereafter. Finally, a similar situation arises for the case \( L_{\text{ratio}} > 1.0 \). Here, the hydrogen burning luminosity exceeds surface luminosity. We will come back to this case in a moment.

For a given \( \Omega \), let us label the \( \eta_{\text{start}} \), corresponding to which the critical \( L_{\text{ratio}} \) is unity, as \( \eta_{\text{start}}^* \). Let us label the corresponding \( M_{\text{crit}} \) as \( M_{\text{crit}}^* \). In Figure 3, the point on the critical \( L_{\text{ratio}} \) curve, which corresponds to unity, is the top black dot. The vertical line from that point intersects the \( \eta \) axis at \( \eta_{\text{start}}^* \). The point of intersection between the same vertical line and the \( M_{\text{crit}} \) curve, corresponds to \( M_{\text{crit}}^* \). From Figure 3, we now see that the above three cases correspond to \( M < M_{\text{crit}}^* \), \( M = M_{\text{crit}}^* \) and \( M > M_{\text{crit}}^* \), respectively. Hence, for a given rotation \( \Omega \), the valid range of mass \( M \), for which the BD can eventually evolve into a MSS, is \( M_{\text{mmbb}} \leq M < M_{\text{crit}}^* \). We call this \( M_{\text{crit}}^* \) as the maximum mass \( M_{\text{max}} \) for the given \( \Omega \). We shall refer to this mass range as the transition mass range.

Let us now comment upon the case \( L_{\text{ratio}} > 1.0 \). To understand this, we use the standard stellar energy equation \( \dot{\epsilon} - \partial L/\partial M = T dS/dt \), where \( L \) denotes the luminosity, \( M \) the mass, \( T \) the temperature and \( S \) is the entropy per unit mass. Integrating this equation, we get after a little bit of algebra,

\[
L_s(1 - L_{\text{ratio}}) = \frac{K d\eta}{\eta^2} dt \int \rho^{5/3} dV, \quad K = 4.81 \times 10^5 \frac{N_A k_B}{\mu_e^{2/3}},
\]

with \( N_A \) being Avogadro’s number and \( k_B \) the Boltzmann’s constant. Clearly then, at \( \eta_{\text{start}} \) if \( L_{\text{ratio}} > 1 \), Eq. (12) dictates that \( d\eta/dt < 0 \), and the star ceases to exist in that case, as we know that the star cannot be supported below the minimum degeneracy \( \eta_{\text{start}} \).

### 4.3. \( M_{\text{max}} \) as a function of \( \Omega \)

The concept of a critical density is only valid for a rotating star. Hence, for a non-rotating stellar object, \( M_{\text{max}} \) is not defined. This means that for a non-rotating BD, the valid range of \( M \), for which it can evolve into a MSS is theoretically \( M_{\text{mmbb}} \leq M < \infty \). In order to find \( M_{\text{max}} \) corresponding to a given non-zero stellar rotation \( \Omega \), we numerically compute the \( M_{\text{crit}}^* \), defined above. We perform this numerical procedure for different \( \Omega \) values, to obtain a fitted formula for \( M_{\text{max}} \) as a function of \( \Omega \). We find in units of \( M_\odot \),

\[
M_{\text{max}}(\Omega) = -1.99 + 0.037\Omega^{-1} - 2.5440 \times 10^{-4}\Omega^{-2} + 8.69 \times 10^{-7}\Omega^{-3} + O(\Omega^{-4}),
\]

where \( \Omega \) is in \( \text{s}^{-1} \) and we have shown terms up to \( O(\Omega^{-3}) \). From Figure 2, which depicts this behaviour, one can see that with increase in \( \Omega \), \( M_{\text{max}} \) decreases. This can be explained as follows. We know that for a given \( \Omega \),
$M_{\text{max}}$ corresponds to a particular critical configuration, for which stable hydrogen burning takes place from the outset. Now, corresponding to a given degeneracy, the critical configuration maintains hydrostatic equilibrium at higher central density and temperature for higher values of $\Omega$. As a consequence, $M_{\text{max}}$ decreases with increase in $\Omega$.

From the behavior of $M_{\text{mmhb}}$ and $M_{\text{max}}$ with $\Omega$, we see that there exists a certain $\Omega$, where these two are equal. We shall call this angular speed $\Omega_{\text{max}} = 0.0045 \text{ s}^{-1}$. Beyond $\Omega_{\text{max}}$, the transition mass range mentioned above ceases to exist, which forbids the existence of BDs. Thus, for stellar rotations with periods less than $\sim 23 \text{ min}$, there exist no BDs. According to our model, this is the lower limit of the time period of a rotating BD.

4.4. Stable Luminosity Formula

Finally, we deduce a formula for the stellar Luminosity ($L_{\text{HB}}$ due to H-burning), at the point when a BD reaches the main sequence, after initial evolution. This is denoted by $\tilde{L}_{\text{HB}}$ and is a function of both the stellar mass and the stellar rotation. The lowest order polynomial which best fits our generated data is represented as $\tilde{L}_{\text{HB}}(M, \Omega)/L_\odot = \sum_{\alpha,\beta} C_{\alpha \beta} (M/M_\odot)^{\alpha} (\Omega/\text{s}^{-1})^\beta$, and the coefficients $C_{\alpha \beta}$ are listed in Table 1. Figure 4 represents the contour plot of $\tilde{L}_{\text{HB}}$. All the BDs, having particular masses and rotations $(M, \Omega)$ tuples, constituting any given contour, will end up in the main sequence, with the same luminosity. The results from this plot are not to be extrapolated beyond the valid range of BD masses $(M_{\text{mmhb}}(\Omega) \leq M < M_{\text{max}}(\Omega))$, where $\Omega \in (0, \Omega_{\text{max}}))$, since beyond this range, a BD never evolves into a MSS.

| $\alpha \backslash \beta$ | 0     | 1     | 2     | 3     |
|-------------------------|-------|-------|-------|-------|
| 0                       | -0.0016 | 1.69  | -428.94 | 26469.60 |
| 1                       | 0.066  | -64.98 | 16534.30 | $-1.07 \times 10^6$ |
| 2                       | -0.88  | 827.27 | -211061.0 | $1.41 \times 10^7$ |
| 3                       | 3.91   | -3506.89 | 891398.0 | $-6.12 \times 10^7$ |

Figure 4. $\tilde{L}_{\text{HB}}$ Contours. Any particular contour of $\tilde{L}_{\text{HB}}$ specifies all admissible values of mass $M$ and rotation $\Omega$ of BDs, that acquire the same hydrogen burning luminosity when they reach the main sequence.

Figure 5. Schematic diagram of the results obtained in this paper.

5. DISCUSSIONS

In this paper, we have used an analytical BD model due to Burrows and Liebert (1993) to study the effects of rapid rotation on the $M_{\text{mmhb}}$. We have obtained an analytical formula of the $M_{\text{mmhb}}$ as a function of the angular speed $\Omega$. Using the $L_{\text{ratio}} < 1$ criterion of brown dwarf formation, we have obtained an upper bound $\Omega = 0.0045 \text{s}^{-1}$
beyond which a BD will not exist. For a given $\Omega$, we have obtained the mass range $M_{\text{mmhb}} \leq M < M_{\text{max}}$ for BDs to evolve into main sequence stars. Finally, we obtained the luminosity of a BD at the point where it reaches the main sequence, as a function of $M$ and $\Omega$. A schematic diagram of the main results in the paper is given in Figure 5. Here, the red curve AB represents $M_{\text{mmhb}} (\Omega)$, while the blue curve FB represents $M_{\text{max}} (\Omega)$. The two curves intersect at point B. Point A corresponds to the $M_{\text{mmhb}}$ value for a non rotating BD, while point F is theoretically at infinity. The horizontal line AC denotes a constant mass curve corresponding to the $M_{\text{mmhb}}$ for the non-rotating case. Also, 0 corresponds to origin, while E denotes $\Omega_{\text{max}}$. Over-massive BDs (in the region ABC), have been shown to exist purely due to uniform stellar rotation. Outside the region FBE0, no BDs exist.

We have used a toy model, with a number of assumptions. First, all the thermodynamic relations have been assumed to remain unaltered in the presence of rotation. This can be justified, as the rotational kinetic energy $I\Omega^2/2$ with $I$ being the moment of inertia of the deformed star computed numerically, can always be shown to be two orders of magnitude lower than the gravitational potential energy. Second, we have considered the effect of uniform stellar rotation on BD’s evolution. A more realistic situation with differential and time-varying rotation is left for a future study. Finally, our model is polytropic, and does not take account of atmospheric corrections and other details.

However, this simplistic toy model has successfully been able to decode the underlying physics of a rotating BD and has revealed several important limits.

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