1 Introduction

In these past decades, vector soliton equations have received so much attention in mathematical physics and nonlinear physics \[1,2,3,4\]. Recently, we derived the following system in a two-layer fluid using reductive perturbation method, which was motivated by a paper by Onorato et. al. \[5,6\]:

\[
\begin{align*}
    i(S_t^{(1)} + S_y^{(1)} - S_{xx}^{(1)} + LS^{(1)}) &= 0, \\
    i(S_t^{(2)} - S_y^{(2)} - S_{xx}^{(2)} + LS^{(2)}) &= 0, \\
    L_t &= 2(|S^{(1)}|^2 + |S^{(2)}|^2)_x.
\end{align*}
\]

This system is an extension of the two-dimensional long wave-short wave resonance interaction system\[7,8\] and describes the two-dimensional resonant interaction between an interfacial gravity wave and two surface gravity packets propagating in directions symmetric about the propagation direction of the interfacial wave in a two-layer fluid.
In this paper, we will study this system and its integrable modification,
\[ i(\psi_1^{(1)} + \psi_2^{(1)}) - S_{xx}^{(1)} + LS_1^{(1)} = 2i S_1^{(2)*} Q, \]
\[ i(\psi_1^{(2)} - \psi_2^{(2)}) - S_{xx}^{(2)} + LS_1^{(2)} = 2i S_1^{(1)*} Q, \]
\[ L_t = 2(|S_1^{(1)}|^2 + |S_1^{(2)}|^2)_x, \quad Q_x = S_1^{(1)} S_1^{(2)}. \quad (2) \]
where * means complex conjugate. In our recent paper [9], we studied
\[ i(\psi_1^{(1)} + \psi_2^{(1)}) - S_{xx}^{(1)} + LS_1^{(1)} = 0, \quad i(\psi_1^{(2)} + \psi_2^{(2)}) - S_{xx}^{(2)} + LS_1^{(2)} = 0, \]
\[ L_t = 2(|S_1^{(1)}|^2 + |S_1^{(2)}|^2)_x. \quad (3) \]
Note that this system is different from the system (1) only in the sign of y-derivative term \( S_1^{(2)}. \)

2 Bilinear Forms and Wronskian Solutions

Consider a two-component analogue of two-dimensional long wave-short wave resonance interaction (2c-2d-LSRI) system [2]. Using the dependent variable transformation \( L = -2(2 \log F)_xx, S_1^{(1)} = G/F, S_2^{(2)} = H/F, Q = -K^* / F, \) we obtain
\[ (D_2^2 - i(D_1 + D_2)) G \cdot F = 2iH^* K^*, \quad D_2 D_1 F \cdot F = -2(GG^* + HH^*), \]
\[ (D_2^2 - i(D_1 - D_2)) H \cdot F = 2iG^* K^*, \quad D_2 K \cdot F = -G^* H^*. \quad (4) \]
These bilinear forms have the three-component Wronskian solution [10,11,12].
Consider the following three-component Wronskian:
\[ \tau_{NML} = |\varphi \quad \psi \quad \chi|, \]
where \( \varphi, \psi \) and \( \chi \) are \( (N + M + L) \times N, (N + M + L) \times M \) and \( (N + M + L) \times L \) matrices, respectively: \( \varphi = (\partial_{\psi_1} \varphi_1)_{1 \leq j \leq N}, \psi = (\partial_{\varphi_1} \psi_1)_{1 \leq j \leq M} \) and \( \chi = (\partial_{\varphi_1} \chi_1)_{1 \leq j \leq L} \) and \( \varphi_1 \) is an arbitrary function of \( x_1 \) and \( x_2 \) satisfying \( \partial_{\varphi_1} \varphi_1 = \partial_{\psi_1} \psi_1 \) and \( \psi_1 \) and \( \chi_1 \) are arbitrary functions of \( y_1 \) and \( z_1 \), respectively. The above Wronskan satisfies
\[ (D_{11} - D_{22}) \tau_{N+1,M-1,L} \cdot \tau_{NML} = 0, \quad (D_{11}^2 - D_{22}^2) \tau_{N+1,M,L} \cdot \tau_{NML} = 0, \]
\[ D_{11} \tau_{NML} \cdot \tau_{NML} = 2\tau_{N+1,M-1,L} \tau_{N-1,M+1,L}, \quad D_{22} \tau_{NML} \cdot \tau_{NML} = 2\tau_{N+1,M,L-1} \tau_{N-1,M,L+1}, \]
\[ D_{11} \tau_{N,M+1,L-1} \cdot \tau_{NML} = -\tau_{N-1,M+1,L} \tau_{N+1,M,L-1}, \quad D_{22} \tau_{N,M-1,L+1} \cdot \tau_{NML} = -\tau_{N+1,M-1,L} \tau_{N-1,M,L+1}, \]
\[ D_{11} \tau_{N+1,M-1,L} \cdot \tau_{NML} = -\tau_{N+1,M,L} \tau_{N-1,M,L-1}, \quad D_{22} \tau_{N+1,M,L+1} \cdot \tau_{NML} = -\tau_{N+1,M,L-1} \tau_{N-1,M,L+1}. \]
Setting
\[ f = \tau_{NML}, \quad g = \tau_{N+1,M-1,L}, \quad h = \tau_{N-1,M,L+1}, \quad k = \tau_{N,M+1,L-1}; \]
\[ \tilde{f} = \tau_{N+1,M-1,L}, \quad \tilde{g} = \tau_{N-1,M+1,L}, \quad \tilde{h} = \tau_{N+1,M,L-1}, \quad \tilde{k} = \tau_{N,M-1,L+1}; \]
we have the following bilinear forms:

\[
(D_x^2 - D_y^2)\bar{g} \cdot f = 0, \quad (D_x^2 + D_y^2)\bar{g} \cdot f = 0, \quad D_x D_y f \cdot f = 2g\bar{g},
\]

\[
(D_x^2 + D_y^2)h \cdot f = 0, \quad (D_x^2 - D_y^2)\bar{h} \cdot f = 0, \quad D_x D_y f \cdot f = 2h\bar{h},
\]

\[
D_x k \cdot f = -\bar{g}\bar{h}, \quad D_y h \cdot f = -\bar{g}k, \quad D_z g \cdot f = -\bar{h}k,
\]

\[
D_x \bar{k} \cdot f = gh, \quad D_y \bar{h} \cdot f = gk, \quad D_z \bar{g} \cdot f = hk.
\]

By the change of independent variables \(x_1 = x, x_2 = -iy, y_1 = y - t, z_1 = -y - t \) \((x, y, t : \text{real})\), we have \(\partial_{x_1} = \partial_{x_1}, \partial_{y_1} = -i\partial_{x_2} + \partial_{z_1}, \partial_t = -\partial_{y_1} - \partial_{z_1}\). Thus we obtain

\[
(D_x^2 - i(D_x + D_y))g \cdot f = -2i\bar{h}k, \quad (D_x^2 + i(D_x + D_y))\bar{g} \cdot f = -2i\bar{h}k,
\]

\[
(D_x^2 - i(D_x - D_y))h \cdot f = -2i\bar{g}k, \quad (D_x^2 + i(D_x - D_y))\bar{h} \cdot f = -2i\bar{g}k,
\]

\[
D_x D_y f \cdot f = -2g\bar{g} + h\bar{h}, \quad D_y k \cdot f = -\bar{g}h, \quad D_x \bar{k} \cdot f = gh.
\]

Consider solutions satisfying the following condition

\[ g\bar{G} = (gG)^*, \quad \bar{h}G = (hG)^*, \quad \bar{k}G = -(kG)^*, \quad f\bar{G} : \text{real}, \quad (5) \]

where \(G\) is a gauge factor. Then, for \(F = f\bar{G}, G = g\bar{G}, H = h\bar{G}, K = kG\), we will obtain the bilinear equations of the 2c-2d-LSRI system \([4]\). Thus the 2c-2d-LSRI system has a three-component Wronskian solution.

To satisfy the condition \((5)\), we consider the following constrained case: \(N = M + L, \psi_i = 0\) for \(2M + 1 \leq i \leq 2M + 2L, \chi_i = 0\) for \(1 \leq i \leq 2M\) and

\[
\psi_i = e^{\psi_i}, \quad \psi_{M+i} = e^{-\psi_i}, \quad \bar{\psi}_i = p_i x_1 + p_i^2 x_2,
\]

\[
\psi_i = a_i e^{\eta_i}, \quad \psi_{M+i} = a_{M+i} e^{-\eta_i}, \quad \eta_i = q_i y_1 + \eta_i \bar{\eta},
\]

for \(i = 1, 2, \ldots, M\), and

\[
\varphi_{2M+i} = e^{\theta_i}, \quad \varphi_{2M+L+i} = e^{-\theta_i}, \quad \theta_i = s_i x_1 + s_i^2 x_2,
\]

\[
\chi_{2M+i} = b_i e^{\zeta_i}, \quad \chi_{2M+L+i} = b_{L+i} e^{-\zeta_i}, \quad \zeta_i = r_i z_1 + \zeta_i \bar{\zeta},
\]

for \(i = 1, 2, \ldots, L\), where \(p_i, s_i, q_i, r_i\) are wave numbers and \(\eta_i \bar{\eta}, \zeta_i \bar{\zeta}\) are phase constants. The parameters \(a_i\) and \(b_i\) must be determined from the condition of complex conjugacy. By using the standard technique \([13]\), \(a_i\) and \(b_i\) are determined as

\[ a_i = \prod_{k=1}^{M} \frac{p_k - p_i}{q_k - q_i} \prod_{k=1}^{M} \frac{p_k^* + p_i}{q_k^* + q_i}, \quad a_{M+i} = \prod_{k=1}^{L} (s_k + p_i^*) (s_k^* - p_i^*), \quad 1 \leq i \leq M,
\]

\[ b_i = \prod_{k=1}^{L} \frac{s_k - s_i}{r_k - r_i} \prod_{k=1}^{L} \frac{s_k^* + s_i}{r_k^* + r_i}, \quad b_{L+i} = \prod_{k=1}^{M} (p_k + s_i^*) (p_k^* - s_i^*), \quad 1 \leq i \leq L.
\]
Figure 1: Single line soliton of eqs. (2), which is obtained by tau-functions of (6). (a) $-L$, (b) $|S^{(1)}|$, (c) $|S^{(2)}|$, (d) Re $[S^{(1)}]$, (e) Re $[S^{(2)}]$. The parameters are $p = 1 + i, q = -1 + 2i, r = -2 + i$.

and the condition (5) is satisfied for the gauge factor,

$$G = \prod_{1 \leq i < j \leq M} (p_j^* - p_i^*)(q_i - q_j) \prod_{1 \leq i < j \leq L} (s_j^* - s_i^*)(r_i - r_j) \prod_{i=1}^{M} \prod_{j=1}^{L} (p_i - s_j)$$

$$\times e^{\frac{\sum_{i=1}^{M} (\xi_i - \eta_i) + \sum_{j=1}^{L} (\theta_j - \zeta_j)}{2}}.$$ 

This solution represents the $(M + L)$-soliton, i.e., $M$ solitons propagate on the first component of short wave $S^{(1)}$ whose complex wave numbers are given by $p_i, q_i$ and complex phase constants are $\eta_0$, and $L$ solitons propagate on the second one $S^{(2)}$ whose complex wave numbers and phase constants are $s_i, r_i$ and $\zeta_0$.

For instance by taking $M = L = 1$, (1+1)-soliton solution is given as

$$Gf = c \left( \frac{p + p^*}{q + q^*} \frac{s + s^*}{r + r^*} \frac{1}{|p + s|^2} - \frac{s + s^*}{r + r^*} e^{\xi - \eta - \xi^* - \eta^*} - \frac{p + p^*}{q + q^*} e^{\theta + \theta^* - \zeta - \zeta^*} + |p - s|^2 e^{\xi - \eta - \xi^*} \right),$$
\[ G_g = c(p + p^*)e^{\xi - \eta}(\frac{s + s^*}{p + p^*} - (p - s))e^{\theta - \zeta - \zeta^*}, \]
\[ G_h = -c(s + s^*)e^{\theta - \zeta^*} \left( \frac{p + p^*}{q + q^*} \right) + (p - s) e^{\xi + \xi^* - \eta - \eta^*}, \]
\[ G_k = c \left( \frac{p + p^*}{p + s^*} \right) e^{\xi - \eta + \theta - \zeta}, \]

where \( c = -|\frac{(p - s)(p + s^*)|^2 \) and we dropped the index 1 for simplicity. In order to satisfy the regularity condition \( f \neq 0 \), we can take Re \( p > 0 \), Re \( s > 0 \), Re \( q < 0 \) and Re \( r < 0 \). After removing the gauge and constant factors, by choosing the same wave number in \( x \) direction for the above two solitons, i.e., \( s = p \), we obtain the single soliton solution,

\[ f = \frac{1}{p + p^*} - e^{\xi + \xi^*} \left( (q + q^*) e^{-\eta - \eta^*} + (r + r^*) e^{-\xi - \xi^*} \right), \]
\[ g = (q + q^*) e^{\xi - \eta}, \quad h = -(r + r^*) e^{\xi - \xi^*}, \quad k = (q + q^*) (r + r^*) e^{\xi + \xi^* - \eta - \xi^*}. \tag{6} \]

where \( \xi^* = px - iq^2y \), \( \eta = q(y - t) + \eta_0 \) and \( \zeta = -r(y + t) + \xi_0 \). Figure 14 shows the plots of this single soliton solution. L shows V-shape soliton, \(|S^{(1)}|\) and \(|S^{(2)}|\) shows solitoff behaviour [14].

3 Solutions in the case without \( Q \)

We consider the 2c-2d-LSRI system (11) without the fourth field \( Q \) in (2). This system (11) describes waves in the two-layer fluid. Setting \( L = -(2 \log F)_{xx}, S^{(1)} = G/F, S^{(2)} = H/F \), we have

\[ [i(D_x + D_y) - D^2_{xx}] G \cdot F = 0, \quad [i(D_x - D_y) - D^2_{xx}] H \cdot F = 0, \]
\[ -(D_y D_x - 2c)F \cdot F = 2GG^* + 2HH^*. \]

Here we consider the case of \( c = 0 \).

Using the procedure of the Hirota bilinear method, we obtain the single soliton solution

\[ F = 1 + A_{11} \exp(\eta_1 + \eta_1^*), \quad G = a_1 \exp(\eta_1), \quad H = b_1 \exp(\xi_1), \]
\[ \eta_j = p_j x + iq_j y + \lambda_j t + \eta_j^{(0)}, \quad \xi_j = p_j x - iq_j y + \lambda_j t + \eta_j^{(0)}, \]
\[ A_{11} = -\frac{a_1 a_1^* + b_1 b_1^*}{(p_1 + p_1^*) (\lambda_1 + \lambda_1^*)}, \quad \lambda_1 = -ip_1^2 - iq_1. \]

Here \( q_j \) is a real number. We can rewrite \( A_{11} \) as

\[ A_{11} = -\frac{a_1 a_1^* + b_1 b_1^*}{(p_1 + p_1^*) (ip_1^2 - ip_1)}. \]
Thus we have

\[ S^{(1)} = \frac{a_1 \exp(\eta_1)}{1 + A_{11} \exp(\eta_1 + \eta_1^*)}, \quad S^{(2)} = \frac{b_1 \exp(\xi_1)}{1 + A_{11} \exp(\eta_1 + \eta_1^*)}, \]

\[ L = -2 \frac{q^2}{dx^2} \log(1 + A_{11} \exp(\eta_1 + \eta_1^*)). \]

Since \( |S^{(1)}|^2 = GG^*/F^2, |S^{(2)}|^2 = HH^*/F^2, L = -2 \frac{q^2}{dx^2} \log F \) do not include \( y \), all solitons propagate in the \( x \) direction.

There is an exact solution depending on \( y \)-variable,

\[ S^{(1)} = \frac{A_1 \exp(px + qy + rt)}{1 + \exp(2(px + qy + rt))} \exp(i(k_1x + l_1y + m_1t)), \]

\[ S^{(2)} = \frac{A_2 \exp(px + qy + rt)}{1 + \exp(2(px + qy + rt))} \exp(i(k_2x + l_2y + m_2t)), \]

\[ L = \frac{A \exp(2(px + qy + rt))}{(1 + \exp(2(px + qy + rt)))^2}, \]

where \( p, q, r, k_1, l_1, m_1, k_2, l_2, m_2, A_1, A_2, A \) satisfy the relations \( r = (k_1 + k_2)p, q = (k_1 - k_2)p, m_1 = k_1^2 - l_1 - p^2, m_2 = k_2^2 + l_2 - p^2, A = -8p^2, A_1^2 + A_2^2 = -4(k_1 + k_3)p^2 \), and \( p, q, k_1, l_1, k_2, l_2 \) are arbitrary parameters. In figure 2 we see that waves in \( S^{(1)} \) and \( S^{(2)} \) have different modulation property, i.e., carrier waves in \( S^{(1)} \) and \( S^{(2)} \) have different directions of propagation. Note that the solutions of equations (2) also have this property.

It seems that eqs. (1) are nonintegrable and do not admit general \( N \)-soliton solution. Similar system (2) has an \( N \)-soliton solution, but its physical derivation has not been done yet.

4 Concluding Remarks

We have studied solutions of a new integrable two-component two-dimensional long wave-short wave resonant interaction (2c-2d LSRI) system (2). We presented a Wronskian formula for 2c-2d LSRI system (2) with complex conjugacy condition. We have also presented solutions of the system (1) in the case of two-layer fluid, i.e. the 2c-2d LSRI system without \( Q \). In this case, the system (1) seems to be non-integrable, i.e. the system (1) does not have multi-soliton solutions. We have found that waves in \( S^{(1)} \) and \( S^{(2)} \) in both systems have different modulation property, i.e., carrier waves in \( S^{(1)} \) and \( S^{(2)} \) have different directions of propagation.

But the system (2) has much more interesting solutions such as the V-shape soliton and solitoff because of integrability.

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Figure 2: Line soliton of eqs. (1). (a) $-L$, (b) $|S^{(1)}|$, (c) $|S^{(2)}|$, (d) Re $[S^{(1)}]$, (e) Re $[S^{(2)}]$. The parameters are $k_1 = -1, k_2 = -2, A_1 = 1, A_2 = 2, l_1 = 3, l_2 = 4$.

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