METAMORPHISM—AN INTEGRAL TRANSFORM
REDUCING THE ORDER OF A DIFFERENTIAL EQUATION

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ABSTRACT. We propose an integral transform, called metamorphism, which allows us to reduce the order of a differential equation. For example, the second order Helmholtz equation is transformed into a first order equation, which can be solved by the method of characteristics.

1. Introduction

Partial differential equations (PDEs) provide a fundamental language describing laws of nature. To tackle intrinsic complexity of PDEs one often tries to transform an equation to a simpler one. For example:

1. Separation of variables [14] allows one to replace a given PDE by several other differential equations, each with a smaller number of variables. Ideally, one wants to obtain a system of ordinary differential equations (ODEs), which are much more accessible.

2. Integral transformations [17, § 15.2] which map some derivatives to simpler objects, e.g. multiplication by variables. The Fourier and Laplace transforms are the most common choice, with the Mellin, Hankel, etc. transforms to follow in more specialised cases.

3. Special functions [13, 20] (including orthogonal polynomials) are a blend of the two previous techniques. A decomposition over a suitable family of special functions is a sort of integral transform and level curves of the special functions define separating coordinates for PDEs, cf. [7].

4. Transmutations relate solutions of a differential equation with variable coefficients to solutions of an equation with constant coefficients. The latter admits a better understanding, thus such a reduction is very helpful starting from second order ODEs [12].

This paper adds to this list a generic method, called metamorphism, to study PDEs. It is an integral transform (aka coherent state transform) [1] for a certain subgroup of the Jacobi–Schrödinger group [9 § 4.4] [8 § 8.5]. This was implemented in the working Jupyter notebook [11] and is spelled in [4]. Here, all required properties...
of the metamorphism will be verified by direct arguments without a reference to the representation theory.

2.1. The Integral Transform. In the following presentation the parameter $\hbar$ shall be associated with the wave length (or the Planck constant) and it is a fixed parameter most of the time. The metamorphism is an integral operator defined by:

$$f(x, y, b, r) = \sqrt{\pi r} \int f(u) \exp \left( -\pi \hbar \left( \frac{r^2 - ib}{2} \right) (u - y)^2 + 2i(u - y)x \right) \, du.$$  \hspace{1cm} (2.1)

Here $x, y, b$ are reals and $r$ is a positive real, their collection is denoted by $\mathbb{R}_+^4$. The integral (2.1) is meaningful for $f(u)$ from many linear spaces, e.g. the Schwartz space, $\mathcal{L}_p(\mathbb{R})$ with $1 \leq p \leq \infty$. Being based on the Gaussian, metamorphism (2.1) is also well-defined for tempered distributions from $\mathcal{S}'(\mathbb{R})$ and maps them to smooth functions. This transformation is a covariant (aka coherent states) transform [1], namely it has the form

$$\tilde{f}(x, y, b, r) = \langle f, \Phi_{x,y,b,r}\rangle, \quad \text{for} \quad \Phi_{x,y,b,r} = \rho_{x,y,b,r} \Phi,$$

where $\Phi(u) = e^{-\pi \hbar u^2}$ is the Gaussian and $\rho_{x,y,b,r}$ is a unitary irreducible representation of the SSR group [11]—the semidirect product of the Heisenberg group $\mathbb{H}_p^1$ with the $a x + b$ group acting by symplectic automorphisms of $\mathbb{H}_p^1$ [9 § 1.2]. Alternatively, the metamorphism can be stated as the convolution

$$\tilde{f}(x, y, b, r) = [f * K_{x,b,r}](y)$$

with the Gauss-type kernel

$$K_{x,b,r}(u) = \sqrt{\pi} e^{-\pi \hbar ((r^2 - ib)u^2 - 2iux)} = \rho_{x,0,-b,r} \Phi(u)$$

parametrised by $x, b \in \mathbb{R}$ and $r \in \mathbb{R}_+$. Note the reverse sign of $b$ in the last formula.

Remark 1. Metamorphism (2.1) incorporates many known integral transformations:

- For $r \to 0$ and $y = b = 0$ it reduces to the Fourier transform.
- For $x = b = 0$ we get the Gauss–Weierstrass(–Hille) transform [21] which solves the Cauchy problem for the heat equation. Its imaginary cousins with $x = 0$ and $r \to 0$ solve the time evolution of a free quantum particle [14 Ch.1 (1.25)].
- For $r = 1$, $b = 0$ this is the Fock–Segal–Bargmann (FSB) transform [5, 19], see also [9 § 1.6].
- For $b = 0$ (with variable $r$) it is the Fourier–Bros–Iagolnitzer (FBI) transform [9 § 3.3].
- For $b = 0$ and $r = (1 + |x|)^{\alpha}$ for some $\alpha$ the metamorphism asymptotics for $x \to \infty$ is equivalent to the Cõrdoba–Fefferman treatment of wavefronts [8 § 3.1].
- With $r = 1$ it was considered in [12] to treat the Schrödinger equation.
- The metamorphism is also connected to the wavelet transform for the affine group [11 Ch. 12], which is a subgroup of the SSR group.

Integral transforms of this type were extensively studied [15, 16] and applied in many areas [9, 10]. Although metamorphism can be considered as a special type of a complex linear canonical transform (LCT) [10], the particular form (2.1) still has a large unexplored potential in applications.

Metamorphisms can be computed explicitly for some important functions.
Example 2. (1) We start from the wave packet \( P_{\sigma, \lambda}(u) = e^{-\pi \sigma u^2 - 2\pi i \lambda u} \in L_2(\mathbb{R}) \) for a complex \( \sigma \) such that \( \Re \sigma > 0 \):

\[
\bar{P}_{\sigma, \lambda}(x, y, b, r) = \frac{\sqrt{\pi}}{\sqrt{r^2 + \sigma/h - ib}} \exp(-\pi \sigma y^2 - 2\pi i \lambda y) \times \exp\left(-\frac{\pi h(x + (\lambda - i\sigma y)/h)^2}{r^2 + \sigma/h - ib}\right)
\]

(2.3)

This follows from the well-known formula \([9, \text{App. A}, (1)]\):

\[
\int_{\mathbb{R}} e^{-\pi \sigma u^2 - 2\pi i u v} du = \frac{1}{\sqrt{\sigma}} e^{-\pi u^2/\sigma},
\]

where \( \Re \sigma > 0 \) and \( z \in \mathbb{C} \) with the usual agreement on the branch of \( \sqrt{\sigma} \).

(2) For the exponent \( E_k(u) = e^{-iku} = e^{-2\pi i ku} \) (with \( k = k/(2\pi) \)) representing a wave with wave number \( k \) we have:

\[
\bar{E}_k(x, y, b, r) = \frac{\sqrt{\pi}}{\sqrt{r^2 + \sigma/h - ib}} \exp(-iky) \exp(-\pi h(k/(2\pi h) + x)^2/(r^2 - ib))
\]

(2.4)

\[
= \frac{\sqrt{\pi}}{\sqrt{r^2 - ib}} \exp(-2\pi i ky) \exp(-\pi h(k/h + x)^2/(r^2 - ib)).
\]

It formally coincides with the transform of the wave packet \( P_{\sigma, \lambda} \) with \( \sigma = 0 \) and \( \lambda = k \).

(3) For the Dirac delta function \( \delta \) and its derivative \( \delta_1 \) we have:

\[
\bar{\delta}(x, y, b, r) = \sqrt{\pi} e^{-\pi h(r^2 - ib)y^2} e^{2\pi i h y x},
\]

\[
\bar{\delta}_1(x, y, b, r) = -2\pi h \sqrt{\pi} (x + (b + i r^2)y) e^{-\pi h(r^2 - ib)y^2} e^{2\pi i h y x}.
\]

It is shown in \([2, 3]\) that the repeated pattern \( x + (b + i r^2)y \) in the above formulae is not accidental.

2.2. Sesqui-unitarity and inverse metamorphism. The known sesqui-unitarity property of Fourier–Wigner transform \([9, \text{§ I.4}]\) implies that:

\[
\langle f_1, f_2 \rangle = \int_{\mathbb{R}^2} \overline{f_1(x, y, b_0, r_0)} f_2(x, y, b_0, r_0) \frac{h \, dx \, dy}{\sqrt{2\pi} r_0},
\]

(2.5)

\[
=: \langle \bar{f}_1, \bar{f}_2 \rangle \quad \text{(b_0, r_0)} ,
\]

for any fixed \((b_0, r_0) \in \mathbb{R}^2_+ := \mathbb{R} \times \mathbb{R}_+\).

As an immediate consequence we obtain the (phase-space) reproducing property for the metamorphism, cf \([2, 2]\):

\[
\bar{f}(x, y, b, r) = \langle f, \Phi_{x,y,b,r} \rangle \quad \mathbb{R},
\]

(2.6)

with the reproducing kernel

\[
\Phi_{x,y,b,r}(x_0, y_0, b_0, r_0) = \langle \Phi_{x_0,y_0,b_0,r_0}, \Phi_{x,y,b,r} \rangle \mathbb{R} = \overline{\Phi_{x_0,y_0,b_0,r_0}(x, y, b, r)}. 
\]
Utilising formula (2.3) with \( \sigma = \hbar (r_0^2 + i b_0) \) and \( \lambda = - \hbar (x_0 + (b_0 - i r_0^2) y_0) \) we get:

\[
\tilde{\varphi}_{x,y,b,r}(x_0, y_0, b_0, r_0) = \sqrt{\frac{\pi \hbar}{r_0^2 + r^2 + i(b - b_0)}} \times \exp \left( -\frac{\pi \hbar (x - x_0 + (b_0 + i r_0^2)/(y - y_0))^2}{r_0^2 + r^2 + i(b - b_0)} \right) \times \exp \left( \pi \hbar (2(y_0 - y)x_0 + (b_0 + i r_0^2)(y_0^2 - y^2)) \right).
\]

The significance of (2.6) is that we are able to restore \( \tilde{f}(x, y, b, r) \) for any values of \((b, r)\) if \( \tilde{f}(x, y, b_0, r_0) \) is initially known only for particular \((b_0, r_0)\) but any \((x, y)\).

The metamorphism \( \tilde{f} \) in (2.1) contains an abundance of information on the function \( f \). Therefore, a function can be recovered from its metamorphosis in many different ways. For example, we can work from the reconstruction formula for the FSB space: sesqui-unitarity (2.5) implies that the inverse metamorphosis is provided by the adjoint operator for the pairing \( \langle \cdot, \cdot \rangle_{(b_0, r_0)} \). Thus, the original function \( f(u) \) can be recovered from its transformation \( \tilde{f}(x, y, b_0, r_0) \) for arbitrary fixed values \( b_0 \in \mathbb{R} \) and \( r_0 \in \mathbb{R}_+ \) as follows:

\[
f(u) = \sqrt{\pi} \int_\mathbb{R} \tilde{f}(x, y, b_0, r_0) e^{-\pi \hbar((r_0^2 + i b_0)(u - y)^2 - 2i(u - y)x)} \, dx \, dy
\]

with integration over the phase space.

### 2.3. Characterisation of the image space

The metamorphism integral kernel and thus the image space is annihilated by the following differential operators:

\[
\begin{align*}
\mathcal{C}_1 &= \frac{1}{r} \left( (r^2 - i b) \partial_x + i \partial_y + 2x \hbar \pi i \right); \\
\mathcal{C}_2 &= 2r^2 \partial_b + i r \partial_x - \frac{1}{2} i \hbar.
\end{align*}
\]

It is convenient to view these operators as the Cauchy–Riemann-type operators for complex variables:

\[
w = b + i r^2 \quad \text{and} \quad z = x + (b + i r^2) y = x + w y.
\]

The generic solution of two differential operators (2.8)–(2.9) is:

\[
(\mathcal{G} f_2)(x, y, b, r) = \sqrt{\pi} e^{-\pi \hbar x^2 / (i r^2 - i b)} f_2(x + (b + i r^2) y, b + i r^2) = \sqrt{\pi} e^{-\pi \hbar x^2 / w} f_2(z, w),
\]

where \( f_2 \) is a holomorphic function of two complex variables. Clearly, with \( b = 0 \) and \( r = 1 \) the function \( (\mathcal{G} f_2)(x, y, 1, 0) \) is an element of the (pre-)FSB space on \( \mathbb{C} \).

Furthermore, the integral kernel and metamorphism image space are annihilated by second-order differential operators:

\[
\begin{align*}
\delta_1 &= r^2 (4 \pi \hbar \partial_b - \partial_{xx}) ; \\
\delta_2 &= -2 \pi i \hbar \partial_b - b \partial_{xx} + \partial_{xy} - 2 \pi i \hbar \partial_x - i \hbar \pi i.
\end{align*}
\]

Of course, the list of annihilators is not exhausted and the above conditions are not independent. If \( \delta_1[\mathcal{G} f_2] = 0 \) for the generic solution (2.11) then the function \( f_2 \) has to satisfy the second-order differential equation:

\[
w \partial_z^2 f_2(z, w) - 4 \pi \hbar \partial_z \partial_w f_2(z, w) - 4 \pi \hbar \omega \partial_w f_2(z, w) - 2 \pi \hbar f_2(z, w) = 0.
\]

Equivalently:

\[
\partial_w f_2 = \frac{1}{4 \pi \hbar} \partial_z^2 f_2 - \frac{z}{w} \partial_z f_2 - \frac{1}{2w} f_2.
\]
This equation can be reduced to the standard Schrödinger equation of a free particle on the line by the change of variables \[17\] §3.8.3.4:

\[(2.15) \quad (z, w, f_2) \rightarrow \left(\frac{z}{w}, \frac{1}{w}, \frac{1}{\sqrt{w}} f_2\right).\]

The same equation (2.14) appears from the condition \(S_2[f_2] = 0\). Thus, we need only operators \(C_1, C_2\) and \(S_1\) to specify \(f_2\) (and therefore \(Sf_2 (2.11)\)). In the following we call (2.14) the structural condition.

We can check the above characterisation of the image space on the metamorphisms from Example 2.

**Example 3.**

1. For the metamorphism of the wave packet \(\tilde{P}_{\sigma, \lambda}\) in (2.3) we find the respective form \(\tilde{P}_{\sigma, \lambda} = Sf_2\) in (2.11) as follows:

\[(2.16) \quad f_2(z, w) = \frac{1}{\sqrt{-iw + \sigma/h}} \exp\left(\frac{\pi}{i} \left(\frac{\lambda^2 w^2 + 2\lambda z w h - iz^2 \sigma h}{(ihw - \sigma)w}\right)\right).\]

Here, variables (2.15) are substituted into the particular solution \(e^{i(\lambda^2 t - 2k_1v)}\) of the operator \((4\pi i t)\partial_t - \partial^2_{x,v}\). Also it is a particular case of (2.16) for \(\sigma = 0\) and \(\lambda = k/(2\pi)\).

2. **Multidimensional metamorphism.** It is straightforward to extend metamorphism for functions \(f(u_1, \ldots, u_n)\) on \(\mathbb{R}^n\), making \(n\) copies of metamorphism (2.1) in each variable \(u_1, \ldots, u_n\) and receive the function on \((\mathbb{R}^4)^n\). As in the case of the FSB transform, such tensorial approach allow us to immediately extend properties of the one-dimensional metamorphism to an arbitrary finite dimension \(n\). In particular, for two dimensions the function in the image space shall have the form, cf. (2.11):

\[(2.18) \quad |Sf_4(x_1, y_1, b_1, r_1; x_2, y_2, b_2, r_2) = \sqrt{r_1 r_2} e^{i\pi h(x_1^2/w_1 + x_2^2/w_2)} f_4(z_1, w_1; z_2, w_2),\]

for some function \(f_4\) holomorphic in four complex variables cf. (2.10):

\[w_k = b_k + i\tau_k^2\quad \text{and} \quad z_k = x_k + (b_k + i\tau_k) y_k = x_k + w_k y_k \quad \text{for} \quad k = 1, 2.\]

Furthermore, \(f_4\) needs to satisfy to the structural condition (2.14) in each pair \((z_1, w_1)\) and \((z_2, w_2)\) of its variables.

Thus, to reduce technicalities we stick in the following to the one-dimensional metamorphism whenever possible.

**Example 4.** For the plane wave \(E_{k_1k_2}(u) = e^{-i(k_1 u_1 + k_2 u_2)}\) we have, cf. (2.17):

\[(2.19) \quad \tilde{E}_{k_1k_2}(x_1, y_1, b_1, r_1; x_2, y_2, b_2, r_2) = \sqrt{|r_1 r_2|} e^{i\pi h(x_1^2/w_1 + x_2^2/w_2)} \times \frac{1}{\sqrt{-w_1 w_2}} \exp\left(-i k_1 \frac{z_1}{w_1} - i k_2 \frac{z_2}{w_2} + \frac{1}{4\pi i h} \left(\frac{k_1^2}{w_1} + \frac{k_2^2}{w_2}\right)\right),\]

where the last line represents the function \(f_4\) in terms of equation (2.18).

3. **Metamorphism of differential operators.**

We proceed with an application of the metamorphism to differential equations.
3.1. **Reduction of an order of derivations.** A simple calculation shows the intertwining property for the derivative:
\[
(f')^\circ = \partial_y f \quad \text{and thus} \quad (f'')^\circ = \partial_y^2 f.
\]

Using the annihilation operators (2.8) and (2.12) we can reduce the second derivative to the first order operator using the expression
\[
\partial_y^2 = D_0 - ir \left( (b + ir^2) \partial_x + \partial_y \right) S_1 - \frac{1}{r^2} (b + ir^2)^2 S_1,
\]
where the first order differential operator \(D_0\) is
\[
D_0 = 2\pi i h \left( (b + ir^2)x \partial_x + x \partial_y + 2(b + ir^2)^2 \partial_y + (b + ir^2)l \right).
\]
Thus we obtain the following lemma

**Lemma 5.** Metamorphism (2.1) intertwines the second-order derivative to the first-order differential operator:
\[
(f'')^\circ = D_0 f.
\]

The operator \(D_0\) becomes explicitly transparent if we use the general solution \([Sf_2]\) in (2.14).

**Proposition 6.** For a function \(f(u)\), let its metamorphism (2.1) be \(\tilde{f} = Sf_2\) of (2.14) for some holomorphic function \(f_2\) of two complex variables. Then:
\[
(f'')^\circ = S(Df_2),
\]
where for \(f_2(z, w)\):
\[
(3.1)\quad Df_2 = 2\pi i h w (2z \partial_z f_2 + 2w \partial_w f_2 + f_2).
\]

3.2. **Application to the Helmholtz equation.** As mentioned in §2.4 for a function \(f(u_1, u_2)\) on \(\mathbb{R}^2\) we can repeatedly apply metamorphism (2.1) in each variable \(u_1\) and \(u_2\). All the above calculations remain valid for the doubled set of variables, of course. The straightforward application of Prop. 6 implies the following.

**Corollary 7.** Let \(f(u_1, u_2)\) be a solution of the Helmholtz equation:
\[
(3.2)\quad \partial_1^2 f + \partial_2^2 f - k^2 f = 0.
\]

Then \(\tilde{f} = Sf_4\) for a function \(f_4\) holomorphic in four complex variables, which satisfies the first-order differential equation:
\[
(3.3)\quad 2w_1 (z_1 \partial_z f_4 + w_1 \partial_w f_4) + 2w_2 (z_2 \partial_z f_4 + w_2 \partial_w f_4) + \left( w_1 + w_2 + \frac{k^2}{2\pi i h} \right) f_4 = 0.
\]

The presence of the Euler operators for pairs \((z_1, w_1)\) and \((z_2, w_2)\) in (3.3) suggests to look for a solution of (3.3) in the form:
\[
f_4(z_1, w_1; z_2, w_2) = \phi \left( w_1, w_2, \frac{z_1}{w_1}, \frac{z_2}{w_2} \right).
\]

Then, equation (3.3) reduces to the following differential equation for \(\phi\):
\[
2w_1^2 \partial_1 \phi + 2w_2^2 \partial_2 \phi + \left( w_1 + w_2 + \frac{k^2}{2\pi i h} \right) \phi = 0.
\]
The last equation gives the following generic solution of (3.3), cf. [18] §4.8.2.4:
\[
(3.4)\quad f_4(z_1, w_1; z_2, w_2) = \frac{e^{k^2(w_1^{-1} + w_2^{-1})/(8\pi i h)}}{\sqrt{w_1 w_2}} f_3 \left( \frac{1}{w_1} - \frac{z_1}{w_1}, \frac{z_2}{w_1}, \frac{1}{w_2} - \frac{z_1}{w_2}, \frac{z_2}{w_2} \right),
\]
with a holomorphic function \( f_3 \) of three complex variables. Taking in account (2.18) the full metamorphism is

\[
\tilde{f}(z_1, w_1; z_2, w_2) = \sqrt{\frac{\tau_1 \tau_2}{w_1 w_2}} \exp \left( -\pi \hbar \left( \frac{k_1^2}{2w_1^2} + \frac{k_2^2}{2w_2^2} \right) \right) \times f_3 \left( \frac{1}{w_1} - \frac{1}{w_2}, \frac{z_1}{w_1}, \frac{z_2}{w_2} \right).
\]

(3.5)

Finally, we check the structural conditions (2.12) on the last expression. An application of the operator \( S_1 \) in variables \((z_1, w_1)\) to \( \mathcal{G} f_4 \) from (3.4) produces the equation:

\[
4\pi \hbar \partial_{1} f_3 - \partial_{22} f_3 - \frac{1}{2} k^2 f_3 = 0.
\]

(3.6)

Similarly an application of the operator \( S_1 \) in variables \((z_2, w_2)\) produces:

\[
4\pi \hbar \partial_1 f_3 + \partial_{33} f_3 + \frac{1}{2} k^2 f_3 = 0.
\]

(3.7)

These are Schrödinger equations of a free particle with one degree of freedom. Note the opposite flow of time in them.

Also, a direct calculation shows that an application of the Helmholtz operator to \( \mathcal{G} f_4 \) for some \( f_4 \) from (3.4) reduces to

\[
\partial_{22} f_3 + \partial_{33} f_3 + k^2 f_3 = 0,
\]

which is the difference of (3.6) and (3.7), and thus follows from them.

**Theorem 8.** Let \( f(u_1, u_2) \) be a solution to the Helmholtz equation (3.2), then \( \tilde{f} = \mathcal{G} f_4 \), where \( f_4 \) is a function of the form (3.4) for some holomorphic \( f_3 \), which satisfies two structural conditions (3.6)–(3.7).

Therefore, one can construct a particular solution of the Helmholtz equation in the following steps:

1. Take some particular solutions of the Schrödinger equation and blend them into a joint solution \( f_3 \) to the system of (3.6)–(3.7).
2. Use the obtained function \( f_3 \) to construct solution \( f_4 \) from (3.4) of the transmuted Helmholtz equation (3.3).
3. Use (2.18) to re-create the full solution \( \mathcal{G} f_4 \) on \( \mathbb{R}^8 \) from the above \( f_4 \).
4. Then, either
   - employ \( \mathcal{G} \) to analyse the problem in terms of coordinates on \( \mathbb{R}^5 \); or
   - use integral transform (2.7) to recover the solution in terms of original coordinates \((u_1, u_2)\).

**Example 9.** Many partial solutions of the heat/Schrödinger equation [17, § 3.1.3] are variations of two main themes: the plane wave-type and the fundamental solution (the Gaussian wave packet). We will treat both of them now.

1. Let us start from the partial solutions [17, § 3.1.3]

\[
\phi_j(z_j, w_j) = e^{(k_j^2 - k^2)/2 \hbar} e^{i k_j \cdot z_j}/\sqrt{w_j}, \quad \text{for } j = 1, 2.
\]

(3.8)

of a plane wave-type of the structural equations (3.6)–(3.7). Then we have a representation

\[
\phi_1(z_1, w_1) \phi_2(z_2, w_2) = f_3 \left( \frac{1}{w_2} - \frac{1}{w_1}, \frac{z_1}{w_1}, \frac{z_2}{w_2} \right)
\]
Then equation (3.10) reduces to the following differential equation for $f_4(z_1, w_1; z_2, w_2)$.

$$f_4(z_1, w_1; z_2, w_2) = \frac{e^{(k_1^2 + k_2^2)(w_1^{-1} + w_2^{-1})/8\pi i\hbar}}{\sqrt{w_1 w_2}} \times \exp \left( \frac{k_1^2 - k_2^2}{8\pi i\hbar} \left( \frac{1}{w_1} - \frac{1}{w_2} \right) - i k_1 \frac{z_1}{w_1} - i k_2 \frac{z_2}{w_2} \right)$$

$$= \frac{1}{\sqrt{w_1 w_2}} \exp \left( \frac{k_1^2}{4\pi i\hbar} \frac{1}{w_1} - i k_1 \frac{z_1}{w_1} \right) \exp \left( \frac{k_2^2}{4\pi i\hbar} \frac{1}{w_2} - i k_2 \frac{z_2}{w_2} \right).$$

(3.9)

In other words, we have recovered the metamorphism of a plane wave in the plane $(w_1, w_2)$ along the direction $(k_1, k_2)$ (2.19).

(2) Consider the fundamental solutions [7] § 3.1.3]

$$\psi_j(z_j, w_j) = \sqrt{\frac{1}{w_j}} \exp \left( \pi i \hbar \frac{z_j^2}{w_j} + \frac{k^2}{8\pi i\hbar} \frac{1}{w_j} \right), \quad \text{for } j = 1, 2.$$ of the structural conditions (3.6)–(3.7). We note that each $\psi_j$ is a superposition of the plane waves (3.8) over the wavenumber $k_j$ with a Gaussian density. Thus, we can build the respective function as similar superposition of the plane waves (3.9) considering the allowed range $k_j \in [-k, k]$ as follows:

$$f_3 \left( \frac{1}{w_2} - \frac{1}{w_1}, \frac{z_1}{w_1}, \frac{z_2}{w_2} \right) = \int_{-k}^{k} \exp \left( \frac{2k_1^2 - k^2}{8\pi i\hbar} \left( \frac{1}{w_1} - \frac{1}{w_2} \right) - i k_1 \frac{z_1}{w_1} \pm i \sqrt{k_1^2 - k_2^2} \frac{z_2}{w_2} \right) e^{-a k_1} \, d k_1.$$

Illustration of this beam for two different values of the parameter $a$ is given in Fig. [7]

3.3. Helmholtz operator in three dimensions. Solution of the Helmholtz equation in higher dimensions follows the same steps which were used in two dimensions. Using the three copies of transformation (2.1) we obtain the following analog of Cor. [7]

**Corollary 10.** Let $f(u_1, u_2)$ be a solution of the Helmholtz equation:

$$\partial_1^2 f + \partial_2^2 f + \partial_3^2 f - k^2 f = 0.$$

Then $\tilde{f} = \mathcal{G} f$, for a function $f_6$ holomorphic in four variables which satisfies the first-order differential equation:

$$2w_1(z_1 z_2 f_6 + w_1 \partial_{w_1} f_6) + 2w_2(z_2 \partial_{w_2} f_6 + w_2 \partial_{w_2} f_6) + 2w_3(z_3 \partial_{w_3} f_6 + w_3 \partial_{w_3} f_6) + \left( w_1 + w_2 + w_3 + \frac{k^2}{2\pi i\hbar} \right) f_6 = 0.$$

(3.10)

As before, the presence of the Euler operators for pairs $(z_j, w_j)$, $j = 1, 2, 3$ in (3.10) suggests to look for a solution of (3.10) of the form:

$$f_6(z_1, w_1; z_2, w_2; z_3, w_3) = \phi \left( w_1, w_2, w_3, \frac{z_1}{w_1}, \frac{z_2}{w_2}, \frac{z_3}{w_3} \right).$$

Then, equation (3.10) reduces to the following differential equation for $\phi$:

$$2w_1^2 \partial_1 \phi + 2w_2^2 \partial_2 \phi + 2w_3^2 \partial_3 \phi + \left( w_1 + w_2 + w_3 + \frac{k^2}{2\pi i\hbar} \right) \phi = 0.$$
It has the generic solution [18, § 8.8.2.1] of the form:

\[ f_6(z_1, w_1; z_2, w_2; z_3, w_3) = \frac{e^{k^2(w_1^{-1} + w_2^{-1} + w_3^{-1})/(12\pi\text{i}h)}}{\sqrt{w_1w_2w_3}} \times f_5\left(1 - \frac{1}{w_1}, 1 - \frac{1}{w_2}, 1 - \frac{1}{w_3}, \frac{z_1}{w_1}, \frac{z_2}{w_2}, \frac{z_3}{w_3}\right), \]  

(3.11)

with a holomorphic function \( f_5 \) of five complex variables. Furthermore, we again need to satisfy the following three structural conditions:

(3.12) \[ 4\pi\text{i}h\partial_1 f_5 + \partial_3^2 f_5 + \frac{1}{4}k^2 f_5 = 0, \]

(3.13) \[ 4\pi\text{i}h\partial_2 f_5 + \partial_4^2 f_5 + \frac{1}{4}k^2 f_5 = 0, \]

(3.14) \[ 4\pi\text{i}h\partial_1 f_5 + 4\pi\text{i}h\partial_2 f_5 - \partial_5^2 f_5 - \frac{1}{4}k^2 f_5 = 0. \]

A similar form of solutions for the Helmholtz equation can be obtained in any dimension.
4. Discussion and conclusions

We have presented integral transform (2.1) acting $L_2(\mathbb{R}) \to L_2(\mathbb{R}^4_+)$). The image space of the metamorphism consists of functions satisfying certain holomorphic conditions (2.12)–(2.13) and Schrödinger-type structural equations (2.12) (which is equivalent to (2.13)). These restrictions imply that the metamorphism intertwines the second derivative with the first order differential operator (3.1). Reduction of the order by 1 replaces a second-order PDE by a first-order equation (in higher dimensions), which can be solved by the method of characteristics. These techniques applied to the Helmholtz equations produce the metamorphism of a generic solution in the form (3.5).

The metamorphism is a realisation of the coherent state transform [1, Ch. 8] for a unitary irreducible representation of the semi-direct product of the Heisenberg group $H^1$ and the $ax + b$ group acting on $H^1$ by symplectomorphisms. Restricting the representation to various subgroups we obtain many familiar integral transforms, cf. Rem. [1] There combined power is naturally inherited by the metamorphism. The group-theoretic aspects of metamorphism will be presented elsewhere.

Summing up: metamorphism presents a new general method of solving and studying various PDEs and deserves further thoughtful investigation.

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