Schubert classes in the equivariant cohomology of
the Lagrangian Grassmannian

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To Ryoshi Hotta on the occasion of his 65th birthday

Abstract

Let $LG_n$ denote the Lagrangian Grassmannian parametrizing maximal isotropic
(Lagrangian) subspaces of a fixed symplectic vector space of dimension $2n$. For each
strict partition $\lambda = (\lambda_1,\ldots,\lambda_k)$ with $\lambda_1 \leq n$ there is a Schubert variety $X(\lambda)$. Let
$T$ denote a maximal torus of the symplectic group acting on $LG_n$. Consider the
$T$-equivariant cohomology of $LG_n$ and the $T$-equivariant fundamental class $\sigma(\lambda)$ of
$X(\lambda)$. The main result of the present paper is an explicit formula for the restriction
of the class $\sigma(\lambda)$ to any torus fixed point. The formula is written in terms of factorial
analogue of the Schur $Q$-function, introduced by Ivanov. As a corollary to the
restriction formula, we obtain an equivariant version of the Giambelli-type formula
for $LG_n$. As another consequence of the main result, we obtained a presentation of
the ring $H^*_{T}(LG_n)$.

1 Introduction

Let $LG_n$ denote the Lagrangian Grassmannian parametrizing $n$-dimensional isotropic
subspaces of a fixed $2n$-dimensional symplectic vector space. The Schubert classes give a
linear basis for the integral cohomology ring of $LG_n$. These classes can be parametrized
by sequences $\lambda = (\lambda_1,\ldots,\lambda_k)$ of integers such that $n \geq \lambda_1 > \cdots > \lambda_k \geq 1$. The same
set of sequences also parametrizes the $T$-fixed points in $LG_n$, where $T$ denotes a maximal
torus of the symplectic group acting on $LG_n$. Here we consider the $T$-equivariant integral
cohomology ring $H^*_{T}(LG_n)$. We are interested in the $T$-equivariant Schubert classes $\sigma(\lambda)$ in
$H^*_{T}(LG_n)$. The classes $\sigma(\lambda)$ form a free basis of $H^*_{T}(LG_n)$ over the ring $S$ of $T$-equivariant
cohomology of a point. It is known that $S$ is naturally identified with the polynomial ring
$\mathbb{Z}[\varepsilon_1,\ldots,\varepsilon_n]$ (for the definition of $\varepsilon_i$ see [11]). Let $e(\mu)$ be the $T$-fixed point corresponding
to an index $\mu$. The inclusion map $i_{\mu} : \{e(\mu)\} \hookrightarrow LG_n$ induces the restriction morphism
$i^*_{\mu} : H^*_{T}(LG_n) \to H^*_{T}(\{e(\mu)\}) \cong S$. The main result of the present paper (Theorem 6.2) is
an explicit formula for $i^*_{\mu}\sigma(\lambda)$, the restriction of the equivariant Schubert class $\sigma(\lambda)$ to a
$T$-fixed point $e(\mu)$, as a polynomial in $\mathbb{Z}[\varepsilon_1,\ldots,\varepsilon_n]$. For the classical Grassmannian $G_{d,n}$ of $d$-dimensional subspaces of $\mathbb{C}^n$, Knutson
and Tao [21] derived a formula for the restriction of the equivariant Schubert classes to a torus
fixed point. This formula is written in terms of the factorial analogue of the Schur function introduced by Biedenharn and Louck \[3\] and studied by other authors (see Section 5). Recently, Lakshmibai, Raghavan, and Sankaran \[28\] also proved the restriction formula by a different method. They also derived equivariant Giambelli formulas, determinantal formulas in the global ring $H^*_T(G_{d,n})$ of equivariant cohomology, that express the equivariant Schubert classes.

As for the ordinary cohomology of the Lagrangian Grassmannian, Hiller and Boe \[16\] proved Pieri-type formulas. Using the result in \[16\], Pragacz \[38\] proved Giambelli-type formulas that express each Schubert class in a Schur-type Pfaffian form. The derivation of the Pfaffian formula is based on a comparison of the formula in \[16\] and the Pieri formula for the Schur $Q$-function. Our formula (Theorem 6.2) is written in terms of the factorial analogue of the Schur $Q$-function, introduced by Ivanov \[18, 19\]. This leads to an equivariant Giambelli-type formula \[6.4\] analogous to Pragacz’s result.

Here is a brief summary of the paper. In Section 2 we recall some results on the Weyl group of type $C_n$ and fix some standard notation. The set of minimal coset representatives introduced in this section will be used as an index set that parametrize the main objects of this paper, the Schubert varieties, $T$-fixed points etc. We also present several combinatorial descriptions of this set. Section 3 is devoted to basic geometric settings, where we introduce the Schubert varieties. We proceed by studying the $T$-equivariant cohomology of $LG_n$ in Section 4. A recurrence relation \[4.8\] arising from the equivariant Pieri-Chevalley formula \[4.6\] plays the central role. In Section 5 we give a definition of the factorial analogue of the Schur $Q$-function and present some properties. Finally, in Section 6 we prove our main theorem. The proof is performed by a comparison of Ivanov’s Pieri-type formula \[5.2\] for factorial $Q$-function and the recurrence relation for the restricted Schubert class. Using the restriction formula, we prove the Giambelli-type formula \[6.4\] for the equivariant Schubert classes of $LG_n$. In Section 7 we also give a supplementary discussion on expressing the Schubert classes of two-row type diagrams in terms of a polynomial in the special Schubert classes $\sigma(i)$ ($1 \leq i \leq n$). A formula in Prop. 7.1 seems to be new. It provides an explicit form of any two-row type factorial $Q$-function. In Section 8 we prove a ring presentation for $H^*_T(LG_n)$ as a quotient of the polynomial ring over $S$. In Appendix (Section 9), we present a brief introduction to Ivanov’s functions and prove a vanishing property crucial to the main body of the paper.

It is well-known that the cohomology ring of the Grassmannian of orthogonal type is very similar to that of $LG_n$. It is natural to expect the equivariant Schubert classes for the orthogonal Grassmannian is described by factorial Schur $P$-functions (cf. \[38, 20\]). Indeed, after the first version of this paper is written, the author and H. Naruse succeed in deriving such formulas. We will discuss the subject in a separate paper \[17\] which focus on some combinatorial aspects of the equivariant Schubert calculus.

It should be mentioned that a recent result due to Ghorpade and Raghavan \[10\] provides an alternative combinatorial approach to our formula \[6.2\]. They presented the coordinate ring of the tangent cone of $X(\lambda)$ at $e(\mu)$ as a Stanley-Reisner ring. Note that a preceding result of Conca \[8\] corresponds to the case with the fixed point is the identity coset. This description immediately leads to a combinatorial formula for $\sigma(\lambda)|_{\mu}$, which is quite similar to a tableau type formula for $Q_\lambda(x|a)$ given in \[19\]. Details of these issues
will be discussed in [17]. Recently, the result, namely a combinatorial expression for the restriction of a Schubert class in $LG_n$ to a $T$-fixed point, was independently proved by Kreiman [25].

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2 Preliminaries

We first recall some basic notions about the Weyl group of type $C_n$, in order to fix our notation. The purpose of this section is to introduce an index set for the main ingredients of this paper, Schubert varieties, torus fixed points, etc. References for this section are [4], [15], and [16].

2.1 Weyl group of type $C_n$

Let $S_{2n}$ be the symmetric group of all permutations of $2n$ letters $\{1, \ldots, 2n\}$. Set $\bar{i} = 2n - i + 1$. Let $W$ be the subgroup of $w \in S_{2n}$ such that $w(i) = j \iff w(\bar{i}) = \bar{j}$. Then $w \in W$ can be determined by $w(1), \ldots, w(n)$. A standard set of generators of $W$ is given by $s_i = (i, i + 1)(i + 1, \bar{i})$ $(1 \leq i \leq n - 1)$ and $s_n = (n, \bar{n})$, where we denote by $(i, j)$ the transposition. The length $\ell(w)$ of an element $w$ in $W$ is the smallest number of the generators $s_1, \ldots, s_n$ (the simple reflections) whose product is $w$.

Let $W_P$ denote the parabolic subgroup of $W$ consisting of the element $w$ such that $w(\{1, \ldots, n\}) \subset \{1, \ldots, n\}$. Clearly $W_P$ is isomorphic to $S_n$. Let $W^P$ denote the set of $w \in W$ such that $w(1) < \cdots < w(n)$. Let $u \in W$. The coset $uW_P$ contains a unique element $w$ in $W^P$. Actually $w$ is the unique element in the coset $uW_P$ of minimal length. The longest element in $W$ is denoted by $w_0$. If the coset $uW_P \subset W/W_P$ is represented by $w \in W^P$, then $w_0 u W_P$ is represented by $w^\vee = (w(n), \ldots, w(1)) \in W^P$.

2.2 Combinatorial description of $W_P$

By a symmetric Young diagram, we mean a sequence $D = (d_1, \ldots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$ such that

$$d_i = \sharp\{j \mid d_j \geq i\} \quad (1 \leq i \leq n).$$

By $\mathcal{Y}_{sym}^n$ we denote the set of symmetric Young diagrams $D = (d_1, \ldots, d_n)$ contained in the square $n \times n$, where by the last condition we mean $d_1 \leq n$. 

3
Let \( w \in W^P \). Then the sequence
\[
D(w) = (n + 1 - w(1), n + 2 - w(2), \ldots, 2n - w(n)),
\]
is an element of \( \mathfrak{Y}_n^{sym} \). Note that here we consider \( w(i) \) simply as an element of \( \{1, \ldots, 2n\} \) without “bar”. For example if \( n = 5 \) and \( w = (1, 3, 4, \overline{5}, \overline{2}) = (1, 3, 4, 6, 9) \), then the corresponding Young diagram is \( D(w) = (5, 4, 4, 3, 1) \). See Fig. 1.

![Fig.1](image1)

For any symmetric Young diagram \( D = (d_1, \ldots, d_n) \) in \( \mathfrak{Y}_n^{sym} \), its upper shifted diagram \( S(D) \) is obtained from \( D \) by discarding the boxes strictly lower than the diagonal, i.e.
\[
S(D) = \{(i, j) \in \mathbb{Z}^2 | 1 \leq i \leq j \leq d_i\},
\]
which we regard as an array of boxes in the plane with matrix-style coordinates. For example if \( D = (5, 4, 4, 3, 1) \), its upper shifted diagram \( S(D) \) is depicted as Fig.2 above.

Let \( \lambda_i \) be the number of boxes in the \( i \)-th row of \( S(D) \). Then the sequence \( \lambda = (\lambda_1, \ldots, \lambda_n) \) is a strict partition. Namely there is \( k \) such that \( \lambda_1 > \cdots > \lambda_k > 0 \) and \( \lambda_j = 0 \) for \( j > k \). Let \( \mathcal{SP}_n \) denote the set of strict partitions \( \lambda = (\lambda_1, \ldots, \lambda_n) \) contained in the staircase \( \rho(n) = (n, n - 1, \ldots, 1) \), namely \( \lambda_1 \leq n \). For \( D \) in \( \mathfrak{Y}_n^{sym} \), its upper shifted diagram \( S(D) \) is thus considered to be a strict partition in \( \mathcal{SP}_n \). For example, the diagram of Fig. 2 is considered to be the strict partition \( \lambda = (5, 3, 2) \).

We let \( \mathcal{M}_n \) denote the set \( \{0, 1\}^n \). We use \( \delta = (\delta_1, \ldots, \delta_n) \) to denote an element in \( \mathcal{M}_n \). For \( w \in W^P \), we set \( \delta_i = 1 \) if \( i \in \{w(1), \ldots, w(n)\} \) and \( \delta_i = 0 \) if \( i \notin \{w(1), \ldots, w(n)\} \). Then we associate \( \delta = (\delta_1, \ldots, \delta_n) \in \mathcal{M}_n \) to \( w \in W^P \).

**Proposition 2.1** By the above correspondences, we have bijections between the following sets:

(i) The coset representatives \( W^P \);
(ii) The set \( \mathfrak{Y}_n^{sym} \) of symmetric Young diagrams contained in the square \( n \times n \);
(iii) The set \( \mathcal{M}_n \) of sequences \( \delta = (\delta_1, \ldots, \delta_n) \) with \( \delta_i \in \{0, 1\} \);
(iv) The set \( \mathcal{SP}_n \) of strict partitions \( \lambda \) contained in \( \rho(n) \).

**Proof.** It is clear that each correspondence is one to one. Also it is easy to see that the cardinality of each of the sets \( W^P, \mathfrak{Y}_n^{sym}, \mathcal{M}_n, \mathcal{SP}_n \) is \( 2^n \). Hence the claim follows. \( \square \)

The following result is well known (see e.g. [13]).

**Lemma 2.2** For \( w \in W^P \), the length \( \ell(w^\vee) \) is equal to \( |\lambda| = \sum_{i=1}^{k} \lambda_i \), where \( \lambda \in \mathcal{SP}_n \) corresponds to \( w \in W^P \).
3 Lagrangian Grassmannians and Schubert varieties

This section is devoted to the set up of geometric objects.

3.1 Lagrangian Grassmannians

Let $V$ be a vector space spanned by the basis $e_1, \ldots, e_n, e_{\bar{n}}, \ldots, e_1$. Introduce a symplectic form by $(e_i, e_j) = (e_i, e_j) = 0$ and $(e_i, e_j) = -(e_j, e_i) = \delta_{ij}$. Let $V_i$ denote the subspace spanned by the first $i$ vectors in $e_1, \ldots, e_n, e_{\bar{n}}, \ldots, e_1$. A subspace $W$ in $V$ is isotropic if $(u, v) = 0$ for all $u, v \in W$. Note that $V_i$ ($1 \leq i \leq n$) are isotropic of dimension $i$ and $V_{n+i} = (V_{n-i})^\perp$ ($1 \leq i \leq n$). Denote by $LG_n$ the set of $n$-dimensional isotropic subspaces of $V$. Then $LG_n$ is a closed subvariety of the Grassmannian of $n$-dimensional subspaces of $V$, and is called the Lagrangian Grassmannian.

The group $G = Sp(V)$ of linear automorphisms of $V$ preserving $(\ ,\ )$ acts transitively on $LG_n$. So $LG_n$ is identified with the quotient of $G$ by the stabilizer of any point. We identify $LG_n$ with the homogeneous space $G/P$, where $P$ denotes the stabilizer of the point $V_n$, the span of $e_1, \ldots, e_n$. The elements of $G$ that are diagonal with respect to the basis we took form a maximal torus $T$ of $G$. The elements of $G$ that are upper triangular matrices form a Borel subgroup $B$ of $G$.

3.2 Schubert varieties

The $T$-fixed points of $LG_n$ are parametrized by $W^P$ : for $w$ in $W^P$, the corresponding $T$-fixed point, denoted by $e(w)$, is the span of $e_{w(1)}, \ldots, e_{w(n)}$. Let $X(w)^\circ$ denote the $B$-orbit of $e(w)$. It is known that $X(w)^\circ$ is an affine space of dimension $\ell(w)$, called a Schubert cell. The Zariski closures $X(w) = \overline{X(w)^\circ}$ are called the Schubert varieties.

We have the following description:

$$X(w) = \{ L \in LG_n \mid \dim(L \cap V_{w(i)}) \geq i \quad \text{for} \quad 1 \leq i \leq n \}. $$

Let $\lambda = (\lambda_1 > \cdots > \lambda_k > 0) \in SP_n$ be a strict partition corresponding to $w \in W^P$. Then we also denote the variety $X(w)$ by $X(\lambda)$. We have

$$X(\lambda) = \{ L \in LG_n \mid \dim(L \cap V_{n+1-\lambda_i}) \geq i \quad \text{for} \quad 1 \leq i \leq k \},$$

whose codimension is given as $|\lambda| = \sum_{i=1}^{k} \lambda_i$ by Lemma 2.2.

3.3 Bruhat-Chevalley order

For $w, v \in W^P$, we say $w \geq v$ if the torus fixed point $e(v)$ belongs to $X(w)$. This is a partial order called the Chevalley-Bruhat order. The condition $w \geq v$ is given by the following two equivalent forms: (1) $w(i) \geq v(i)$ ($1 \leq i \leq n$), (2) $\lambda \subseteq \mu$, namely $\lambda_i \leq \mu_i$ ($1 \leq i \leq n$), where $\lambda, \mu \in SP_n$ correspond to $w, v$ respectively. The Schubert variety $X(\lambda)$ admits a canonical partition into Schubert cells $X(\mu)^\circ$ with $\mu \leq \lambda$. 

5
4 Equivariant cohomology

We are interested in the $T$-equivariant integral cohomology ring $H^*_T(LG_n)$. For general facts on the equivariant cohomology, we refer to Brion [11], Goresky, Kottwitz, and MacPherson [11] and references therein.

4.1 Equivariant Schubert classes

Let $S$ denote the $T$-equivariant integral cohomology ring of a point $\{pt\}$ (namely the ordinary integral cohomology ring of the classifying space of $T$). The natural map $LG_n \to \{pt\}$ induce an $S$-algebra structure on $H^*_T(LG_n)$. Given $w \in WP$, denote by $\sigma(w)$ the $T$-equivariant fundamental class of $X(w)$, called the equivariant Schubert class. We also denote $\sigma(w)$ by $\sigma(\lambda)$, where $\lambda \in SP_n$ corresponds to $w \in WP$. It is known that $H^*_T(LG_n)$ is a free $S$-module with the basis $\sigma(w)$ ($w \in WP$):

$$H^*_T(LG_n) = \bigoplus_{w \in WP} S \cdot \sigma(w).$$

For each $T$-fixed point $e(v)$, $v \in WP$, we have an embedding $i_v : \{e(v)\} \hookrightarrow LG_n$. This yields a homomorphism $i^*_v : H^*_T(LG_n) \to H^*_T(\{e(v)\}) \cong S$. The direct product of these is an injection of rings:

$$\prod_v i^*_v : H^*_T(LG_n) \to \prod_v H^*_T(\{e(v)\}). \quad (4.1)$$

The injectivity is a consequence of “equivariant formality” (cf. [11]) of the $T$-variety $LG_n$. For $w, v \in WP$, denote by $\sigma(w)|_v$ the image $i^*_v \sigma(w)$. The goal of this paper is to give an explicit formula for $\sigma(w)|_v \in S$.

**Remark.** A remarkable characterization of the image of the morphism $\prod_v i^*_v$ has been obtained by [11]. However we shall not use the result in the present paper.

Let $t = \text{Lie}(T)$ be the Lie algebra of the torus $T$. An element of $t$ takes the form $h = \text{diag}(h_1, \ldots, h_n, -h_n, \ldots, -h_1)$. Define linear functionals $\varepsilon_i \in t^*$ by $\varepsilon_i(h) = h_i$ for $1 \leq i \leq n$. Let $\hat{T}$ be the free abelian group generated by $\varepsilon_1, \ldots, \varepsilon_n$. Each element $\sum m_i \varepsilon_i$ in $\hat{T}$ determines a character of $T$ via $T \ni \exp(h) \mapsto e^{\sum m_i \varepsilon_i(h)} \in \mathbb{C}^\times$. By this correspondence we can identify $\hat{T}$ with the character group of $T$. There is a canonical map $\hat{T} \to S$ that extends to an isomorphism of the symmetric algebra $\text{Sym}(\hat{T})$ onto $S$ (see e.g. [15], §1). Thus we identify $S$ with the polynomial ring $\mathbb{Z}[\varepsilon_1, \ldots, \varepsilon_n]$. We shall also use the variables $x_i = -\varepsilon_i$ for $1 \leq i \leq n$. They are convenient for positivity reasons (cf. §4.6).

4.2 $T$-stable affine neighborhood of $e(v)$

Let $U(v)$ denote the set of points of $LG_n$ whose matrix representatives $\xi = (\xi_{i,j})_{2n \times n}$ satisfy $\xi_{w(i),j} = \delta_{i,j}$. This is a $T$-stable affine space isomorphic to $\mathbb{A}^{n(n+1)/2}$ containing the point $e(v)$ as the origin. The coordinate function on $U(v)$ determined by the matrix entry...
ξ_{i,j} with \( i \notin \{v(1), \ldots, v(n)\} \) is an eigenvector for \( T \) of the weight \(-(\varepsilon_i - \varepsilon_{v(j)})\), here we understand \( \varepsilon_1 = -\varepsilon_i \). Let \( \Xi \) be the square matrix \((\xi_{v^\vee(i),v(j)})_{1 \leq i,j \leq n}\). Not all the entries of \( \Xi \) are independent, since the column vectors of \( \xi \) should span an isotropic subspace. As a set of free parameters on the affine space \( U(v) \), we can take the set of entries of weakly upper triangular part of \( \Xi \) with respect to the anti-diagonal. Thus the coordinate ring of \( U(v) \) is \( R(v) = \mathbb{C}[\xi_{v^\vee(i),v(j)} \mid v^\vee(i) \geq v(j)] \).

It is convenient to consider \( v^\vee \) and \( v \) as the (ordered) index sets corresponding to the rows and columns of \( \Xi \) respectively. In this notation we write \( \Xi = (\xi_{r,c})_{r \in v^\vee, c \in v} \). Note that the weight of the coordinate function \( \xi_{r,c} \) is given by \(-(\varepsilon_r - \varepsilon_c)\).

### 4.3 A product formula

We shall prove a formula that expresses \( \sigma(w)|_w \) as a product of negative roots. This is a special case of the main result (Theorem 6.2). In the subsequent of the paper, we need only the fact that \( \sigma(w)|_w \) is a non-zero polynomial (see the Proof of Lemma 4.7).

**Lemma 4.1** We have the following formula:

\[
\sigma(w)|_w = \prod_{(i,j) \in \lambda} (x_{w(i)} - x_{w(j)}), \tag{4.2}
\]

where \( \lambda \in SP_n \) is the upper shifted diagram corresponding to \( w \in WP \).

**Proof.** The variety \( U(w) \cap X(w) \) is just a “coordinate subspace” in \( U(w) \) defined by \( \xi_{r,c} = 0 \) for \( r \in w^\vee, c \in w, \) and \( r > c \). We denote by \( I(w) \) the set of such pairs. For each \((r, c) \in I(w)\), we associate \((i, j)\) by \( w(i) = c \) and \( w(j) = r \). Then \((i, j)\) is a box in the upper shifted diagram \( \lambda \) corresponding to \( w \). This establishes a bijection from \( I(w) \) to the set of boxes of \( \lambda \). Recall that \( \xi_{r,c} \) has the weight \(-(\varepsilon_r - \varepsilon_c)\). Then by using Theorem 3 in [28] we obtain the formula. \(\square\)

For example, let \( w = (1, 3, 5, 4, 2) \). The corresponding strict partition is \( \lambda = (5, 3) \). Then \( \sigma(w)|_w = 2x_1(x_1 + x_3)(x_1 - x_5)(x_1 - x_4)(x_1 - x_2) \times 2x_3(x_3 - x_5)(x_3 - x_4) \).
4.4 The divisor class

Let \( \text{div} = (n, n-1, \ldots, 2, 1) \). The corresponding \( \sigma(\text{div}) \) is the unique Schubert class of codimension one. So we call it the divisor class. We know the following explicit form of this class restricted to any \( T \)-fixed point \( e(v) \).

Lemma 4.2 The restriction of the divisor class \( \sigma(\text{div}) \) to a \( T \)-fixed point \( e(v) \) is given by

\[
\sigma(\text{div})|_v = 2 \sum_{i=1}^n \delta_i x_i \quad (v \in W^P),
\]

where \( \delta = (\delta_1, \ldots, \delta_n) \in \mathcal{M}_n \) corresponds to \( v \in W^P \).

Proof. Consider the closed subvariety \( U(v) \cap X(\text{div}) \) of \( U(v) \cong \mathbb{A}^{n(n+1)/2} \). Let \( \xi = [\xi_{v(1)}, \ldots, \xi_{v(n)}] \) be a matrix representative of a point \( L \) in \( U(v) \). The condition for \( L \) to be in \( X(\text{div}) \) is equivalent to \( \dim(V_n + L) \leq 2n-1 \). If we define the \( n \times n \) matrix \( X \) by

\[
[e_1, \ldots, e_n, \xi_{v(1)}, \ldots, \xi_{v(n)}] = \begin{bmatrix} 1_n & * \\ 0 & X \end{bmatrix},
\]

then the last condition says that \( \det X \) should vanish. Let \( k \) be such a number that \( v(k) \leq n \) and \( v(k+1) > n \). Then by elementary manipulations of a determinant, we have \( \det X = \pm \det Y \) where we denote by \( Y = (\xi_{r,c}) \) the \( k \times k \) submatrix of \( X \) with \( c \in \{v(1), \ldots, v(k)\} \), and \( r \in \{v(k), \ldots, v(1)\} \). Introducing a suitable monomial order on \( R(v) \) such that the initial term of \( \det Y \) is the product of anti-diagonal entries \( \pm \prod_{i=1}^k \xi_{v(i), v(i)} \).

Now applying Theorem 3 in [28] we have \( \sigma(\text{div})|_v = -\sum_{i=1}^k 2 \varepsilon_{v(i)} \). Hence the claim follows. \( \square \)

Remark. To the flag variety of the Kac-Moody groups, Kostant and Kumar derived the corresponding formula (23, Prop. 4.24 (c), see also 27 §11).

4.5 Chevalley’s multiplicities

Let us recall the Chevalley multiplicites [7]. Let \( w, w' \in W^P \), such that \( X(w') \) is a Schubert divisor of \( X(w) \), i.e., \( X(w') \) is a codimension one subvariety in \( X(w) \). Then there is a positive root \( \beta \) such that \( w' = ws_\beta \) and \( \ell(w') = \ell(w) - 1 \), where \( s_\beta \) is the reflection corresponding to \( \beta \). Let \( (\cdot, \cdot) \) be the inner product on \( \widehat{T} \otimes_{\mathbb{Z}} \mathbb{R} = \bigoplus_{i=1}^n \mathbb{R} \varepsilon_i \) such that \( (\varepsilon_i, \varepsilon_j) = \delta_{ij} \), and \( \beta^\vee \) be \( 2\beta/\langle \beta, \beta \rangle \). Then the Chevalley multiplicity \( c(w, w') \) is defined

\[
c(w, w') = (\varpi_n, \beta^\vee),
\]

where \( \varpi_n = \sum_{i=1}^n \varepsilon_i \), the \( n \)-th fundamental weight. We can describe \( c(w, w) \) in a combinatorial way.

Lemma 4.3 (10) Let \( X(w') \) be a Schubert divisor in \( X(w) \). Let \( D(w), D(w') \) be the corresponding symmetric diagrams. Exactly one of the following holds.
(1) $D(w')$ is obtained from $D(w)$ adding two boxes at the positions $(i, j)$ and $(j, i)$ ($i \neq j$). Then the corresponding positive root $\beta$ is $\varepsilon_i + \varepsilon_j$, and we have $c(w, w') = 2$.

(2) $D(w')$ is obtained from $D(w)$ adding a box at the diagonal position $(i, i)$, then the corresponding positive root $\beta$ is $2\varepsilon_i$, and we have $c(w, w') = 1$.

In the following figure, the numbers indicate the Chevalley multiplicities, where $n = 3$:

![Chevalley multiplicities diagram]

One can easily verify the following rule:

**Lemma 4.4** We assume $w' \rightarrow w$ ($w, w' \in W^P$). Let $\lambda, \lambda' \in SP_n$ be correspond to $w, w'$ respectively. Let $k, k'$ be the numbers of non-zero parts of $\lambda, \lambda'$ respectively. Then $c(w, w') = 1$ if $k' = k + 1$, and $c(w, w') = 2$ if $k' = k$.

### 4.6 Equivariant Pieri-Chevalley formula

Since $\{\sigma(w)\}_{w \in W^P}$ forms a basis of $H^*_T(LG_n)$ over the ring $S$, we can define the structure constants $c_{w,v}^u \in S$ for all $w, v, u \in W^P$ by the formula

$$\sigma(w) \cdot \sigma(v) = \sum_u c_{w,v}^u \sigma(u). \quad (4.5)$$

The structure constant $c_{w,v}^u$ has degree $\ell(w) + \ell(v) - \ell(u)$ and vanishes unless $u \geq w, v$ and $\ell(u) \leq \ell(w) + \ell(v)$. It should be remarked that $c_{w,v}^u$ has a remarkable positivity property conjectured by D. Peterson and proved by Graham [14]. Namely each $c_{w,v}^u$ can be written as a linear combination of monomials in the negative roots with nonnegative integer coefficients.
Lemma 4.5 (The equivariant Pieri-Chevalley formula) Let \( w \in W^P \). Then the following formula holds:
\[
\sigma(\text{div}) \cdot \sigma(w) = c_{\text{div},w}^w \sigma(w) + \sum_{w' : w' \to w} c(w, w') \sigma(w'),
\]
(4.6)
where \( w' \to w \) means that \( X(w') \) is a Schubert divisor of \( X(w) \).

Proof. By the same argument of [21], Prop. 2, the claim follows from the Pieri-Chevalley-type formulas for the ordinary integral cohomology, which were proved by Fulton and Woodward [9], Lemma 8.1. □

For the flag variety of an arbitrary Kac-Moody group, the corresponding formula of Lemma 4.5 has been appeared in the context of the nil-Hecke algebra by Kostant and Kumar [23]. Later Arabia [1] established the fact that the equivariant cohomology is isomorphic to the dual of the nil-Hecke algebra. The parabolic analogue is also studied in [27], §11. See also Robinson [39], Andersen, Jantzen, and Sorgel [2], Appendix D.

4.7 Recurrence relation

In this section, we prove a Key lemma (Lemma 4.7) to the proof of our main result. First we need a simple lemma on structure constants.

Lemma 4.6 The structure constant \( c_{\text{div},w}^w \) is given by
\[
c_{\text{div},w}^w = \sigma(\text{div})|_w.
\]
(4.7)

Proof. If we restrict (4.6) to \( e(w) \), we have
\[
\sigma(\text{div})|_w \cdot \sigma(w)|_w = c_{\text{div},w}^w \sigma(w)|_w + \sum_{w' : w' \to w} c(w, w') \sigma(w')|_w.
\]
For \( w' \) such that \( w' \to w \), we have \( \sigma(w')|_w = 0 \) since \( w \not\leq w' \). Hence the sum in the right hand side vanishes. The claim follows since \( \sigma(w)|_w \) is non-zero as we see from Lemma 4.2. □

Now the equivariant Pieri-Chevalley formula (4.6) gives directly the following recurrence relation on the family of restricted classes \( \sigma(w)|_v (w \in W^P) \) for any fixed \( v \in W^P \).

Lemma 4.7 Let \( e(v) \) be any \( T \)-fixed point. The polynomials \( \sigma(w)|_v (w \in W^P) \) satisfy the following recurrence relation:
\[
d(w, v) \cdot \sigma(w)|_v = \sum_{w' : w' \to w} c(w, w') \sigma(w')|_v,
\]
(4.8)
where \( d(w, v) = \sigma(\text{div})|_v - \sigma(\text{div})|_w \).
Since \( d(w, v) \) is non-zero if \( w \geq v \) and \( w \neq v \), the recurrence relation (4.8) and the initial condition \( \sigma(\varphi)|_v = 1 \) determine the polynomials \( \sigma(w)|_v \) \((w \in W^P)\) uniquely. An analogous recurrence relation was used by Rosenthal and Zelevinsky [10] to prove a determinantal formula of the multiplicity of a \( T \)-fixed point in a Schubert variety in the Grassmannian.

**Remark.** The ordinary-cohomology version of Lemma 4.7 has been obtained by Lakshmibai and Weyman [29], and Hiller [15].

### 5 The factorial Schur \( Q \)-functions

Let \( x = (x_1, \ldots, x_n) \) be a finite sequence of variables and let \( a = (a_i)_{i \geq 1} \) be any sequence such that \( a_1 = 0 \). Let

\[
(x|a)^k = \prod_{i=1}^{k} (x - a_i)
\]

for each \( k \geq 1 \) and \((x|a)^0 = 1\). The **factorial Schur \( Q \)-function** for a strict partition \( \lambda = (\lambda_1 > \cdots > \lambda_k > 0) \) of length \( k \leq n \) is defined as follows [19].

**Definition 5.1** Let \( A(x) \) denote the skew-symmetric \( n \times n \) matrix \(((x_i - x_j)/(x_i + x_j))_{1 \leq i,j \leq n} \) and let \( B_\lambda(x|a) \) denote the \( n \times k \) matrix \(((x_i|a)^{\lambda_k-j+1})_{1 \leq i \leq n} \). Let

\[
A_\lambda(x|a) = \begin{bmatrix}
A(x) & B_\lambda(x|a) \\
-B_\lambda(x|a) & 0
\end{bmatrix}
\]

which is a skew-symmetric \((n+k) \times (n+k)\) matrix. Put

\[
Pf_\lambda(x|a) = \begin{cases}
Pf(A_\lambda(x_1, \ldots, x_n|a)) & \text{if } n+k \text{ is even;} \\
Pf(A_\lambda(x_1, \ldots, x_n, 0|a)) & \text{if } n+k \text{ is odd.}
\end{cases}
\]

Then put

\[
P_\lambda(x|a) = \frac{Pf_\lambda(x|a)}{D_n(x)}, \quad Q_\lambda(x|a) = 2^k P_\lambda(x|a), \tag{5.1}
\]

where \( D_n(x) = \prod_{1 \leq i<j \leq n} (x_i - x_j)/(x_i + x_j) \).

**Remark.** The above definition is a factorial analogue of Nimmo’s formula [35] (see also [31], Ch. III, 8, Example 13) for the Schur \( Q \)-functions. The reader can find other expressions for \( Q_\lambda(x|a) \) in [19].

The functions \( Q_\lambda(x|a) \) were introduced by Ivanov\(^1\) [18, 19]. He established some fundamental properties of the functions (combinatorial presentations, Schur-type Pfaffian formulas, vanishing and characterization properties etc.). In particular, a Pieri-type formula is available, which is crucial to our consideration. Note that \( P_{(1)}(x|a) \) does not

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\(^1\)According to Ivanov [18, 19], A.Okounkov defined them for the special parameter \( a \) with \( a_i = i - 1 \).
depend on the parameter $a = (a_i)$ and actually we have $P_{(1)}(x|a) = \sum_{i=1}^{n} x_i$. So we simply denote $P_{(1)}(x|a)$ by $P_{(1)}(x)$. Let $\lambda$ and $\lambda'$ be strict partitions of length $\leq n$. We will write $\lambda' \rightarrow \lambda$ if $\lambda \subset \lambda'$ and $|\lambda'| = |\lambda| + 1$.

**Proposition 5.2** ([19]) (A Pieri-type formula) For any strict partition $\lambda = (\lambda_1 > \cdots > \lambda_k > 0)$ of length $k \leq n$, we have

$$
\left( P_{(1)}(x) - \sum_{j=1}^{k} a_{\lambda_j+1} \right) \cdot P_{\lambda}(x|a) = \sum_{\lambda' : \lambda' \rightarrow \lambda} P_{\lambda'}(x|a),
$$

(5.2)

where $\lambda'$ runs for all strict partitions of length less than or equal to $n$ such that $\lambda' \rightarrow \lambda$.

**Proof.** The above formula corresponds to Theorem 6.2 in [19]. The only difference is that we use $n$-variables $x = (x_1, \ldots, x_n)$ here. Then we can consistently set $P_{\mu}(x|a)$ to be zero for any strict partition $\mu$ of length strictly greater than $n$ (see [19], Definition 2.10). □.

Factorial analogues of the Schur $S$-functions were introduced by Biedenharn and Louck [3] and further studied by Chen and Louck [6], Goulden and Greene [13], Goulden and Hamel [12], Macdonald [32], and Molev and Sagan [34] (see also Macdonald [31], Ch. I, 3, Example 20-21). In these works it was shown that several important facts about the Schur $S$-functions (combinatorial presentations, Jacobi-Trudi identities, Pieri-type formulas, Littlewood-Richetcharleod rules etc.) can be transferred to the factorial Schur $S$-functions. The factorial Schur $S$-functions also play a central role in the study of the center of the universal enveloping algebra of $\mathfrak{gl}_n$ (see Okounkov and Olshanski [37], Okounkov [36] and references therein).

In a geometric context, the factorial Schur functions appeared in [21], [28]. They present the restriction to torus fixed points of the Schubert classes in the equivariant cohomology of the Grassmannian. Recently Mihalcea [33] obtained a presentation by generators and relations for the equivariant quantum cohomology ring of the Grassmannian. In this work the factorial Schur $S$-functions appeared as the polynomial representatives of the equivariant quantum Schubert classes. A similar presentation for the quantum cohomology ring of the Lagrangian Grassmannian was given by Kresch and Tamvakis [26]. It will be an interesting problem to extend their result to the quantum equivariant cohomology ring.

## 6 Restriction and Giambelli-type formulas

### 6.1 Restriction formula

Let us take the following particular parameters:

$$
a_1 = 0, \quad a_i = x_{n-i+2} \quad (2 \leq i \leq n + 1), \quad a_i = 0 \quad (i > n + 1).
$$

(6.1)

\[\text{It has come to my knowledge via [33] (§5, Remark 2) that the factorial Schur } S\text{-functions coincide with the double Schubert polynomials by Lascoux and Schützenberger [30] when indexed by a Grassmann permutation. However, the details of this connection seems to be missing from the literature.}\]
We denote by \( x_{(n)} \) the specialization of \( a = (a_i)_{i \geq 1} \) given by (6.1). Let \( \mu \in W_P \) and \( \delta = (\delta_1, \ldots, \delta_n) \in \mathcal{M}_n \) be correspond to \( \mu \). Then we set \( x_{\mu} = (\delta_1 x_1, \ldots, \delta_n x_n) \).

**Definition 6.1** A specialization \( Q_\lambda(x_{\mu}|x_{(n)}) \) of \( Q_\lambda(x|a) \) is given as follows: First we substitute \( x_{\mu} \) for \( x = (x_1, \ldots, x_n) \) to obtain \( Q_\lambda(x_{\mu}|a) \), then we specialize \( a_i \)'s as in (6.1) to get \( Q_\lambda(x_{\mu}|x_{(n)}) \).

**Theorem 6.2** For strict partitions \( \lambda, \mu \in SP_n \), we have

\[
\sigma(\lambda)|_{\mu} = Q_\lambda(x_{\mu}|x_{(n)}).
\]  

**Proof.** It suffices to show that the right hand side of (6.2) satisfy the recurrence relation (4.8) and the initial condition \( \sigma(\phi)|_{\mu} = 1 \). As for the initial condition, we have \( Q_\phi(x|a) = 1 \) by definition. Hence the proof is completed by a comparison of (4.8) and (6.2). We first specializes \( P_\lambda(x|a) \) to \( P_\lambda(x_{\mu}|a) \) and then to \( P_\lambda(x_{\mu}|x_{(n)}) \). By applying this specialization to (6.2), we have

\[
(P_{(1)}(x_{\mu}) - \sum_{j=1}^{k} x_{n-\lambda_j+1}) \cdot P_{\lambda'}(x_{\mu}|x_{(n)}) = \sum_{\lambda'} P_{\lambda'}(x_{\mu}|x_{(n)}),
\]

where the sum is taken over those \( \lambda' \in SP_n \) such that \( \lambda' \to \lambda \) because \( P_{\lambda'}(x_{\mu}|x_{(n)}) \) vanishes unless \( \lambda' \in SP_n \) (Proposition 9.1). Now we multiply the both hand sides of (6.3) by \( 2^{k+1} \), where \( k \) is the number of non-zero parts of \( \lambda \). By Lemma 4.2 we have \( \sigma(\text{div})|_w = 2 \sum_{j=1}^{k} x_{n-\lambda_j+1} \) and \( \sigma(\text{div})|_v = 2 P_{(1)}(x_{\mu}) \). Therefore we have

\[
d(w, v) = 2 P_{(1)}(x_{\mu}) - 2 \sum_{i=1}^{k} x_{n-\lambda_i+1}.
\]

Now let \( \lambda' \in SP_n \) be such that \( \lambda' \to \lambda \) and \( k' \) be the number of non-zero parts of \( \lambda' \). From Lemmas 13.3 and 13.4 we can see that \( 2^{k+1} P_{\lambda'}(x|a) = c(w, w') Q_{\lambda'}(x|a) \). Note also \( 2^k P_{\lambda}(x|a) = Q_{\lambda}(x|a) \). Thus we proved that \( Q_{\lambda}(x_{\mu}|x_{(n)}) (\lambda \in SP_n) \) satisfy (4.8). \( \square \)

### 6.2 Giambelli-type formula

Now we can prove an equivariant analogue of Pragacz’ Giambelli-type formula. Let \( \lambda \in SP_n \). We write \( \lambda = (\lambda_1, \ldots, \lambda_{2r}) \) with \( \lambda_1 > \cdots > \lambda_{2r} \geq 0 \).

**Theorem 6.3** The equivariant Schubert class \( \sigma(\lambda) \) is expressed as a Pfaffian of the following form

\[
\sigma(\lambda) = Pf \left( \sigma(\lambda_i, \lambda_j) \right)_{1 \leq i, j \leq 2r}.
\]  

**Proof.** Because of the injection \( \{\lambda\} \to SP_n \), it is enough to show that, for arbitrary \( \mu \) in \( SP_n \), the restrictions to \( e(\mu) \) of the both hand sides of (6.4) coincide. We have

\[
\sigma(\lambda)|_{\mu} = Q_{\lambda}(x_{\mu}|x_{(n)}) = Pf \left( Q_{\lambda_i, \lambda_j}(x_{\mu}|x_{(n)}) \right)_{1 \leq i, j \leq 2r} = Pf \left( \sigma(\lambda_i, \lambda_j)|_{\mu} \right)_{1 \leq i, j \leq 2r}.
\]

13
In the second equality, we use the Pfaffian formula for factorial $Q$-functions \((6.1)\). Since the restriction $i^*_{\mu}$ is a ring homomorphism and we are done. \(\square\).

The above formula has a striking character in contrast to the ordinary Grassmannian case \([28]\), where the equivariant Giambelli formula is given in a Jacobi-Trudi type determinant, with matrix entries of linear combinations of (equivariant) special Schubert classes. In our formula, each matrix entry of the Pfaffian is itself an equivariant Schubert class. In spite of this simplicity, if we wish to express the equivariant Schubert class as a polynomial of the special Schubert classes $\sigma(k)$ \((1 \leq k \leq n)\), we need some work to be done. We will treat the problem in the next subsection.

7 On the two-row type classes

The formula \((6.1)\) looks the same as the classical one shown by Pragacz \((38\), Prop. 6.6, see also Józefiak \([20]\)) where the $Q$-functions $Q_\lambda(x)$ \((\lambda \in S^\infty P_n)\) represent the Schubert classes in the ordinary cohomology ring of $LG_n$. Recall that we have the following formula for $r > s \geq 0$:

\[
Q_{r,s}(x) = Q_r(x)Q_s(x) + 2\sum_{i=1}^{s}(-1)^iQ_{r+i}(x)Q_{s-i}(x). \tag{7.1}
\]

Therefore the Pragacz’ formula gives an expression for each Schubert class as a polynomial in the special Schubert classes.

Now in our setting of equivariant cohomology ring, equation \((6.4)\) actually provides an expression for each $\sigma(\lambda)$ as a polynomial in $\sigma(\lambda_i, \lambda_j)$. If $\lambda_j = 0$ then $\sigma(\lambda_i, \lambda_j) = \sigma(\lambda_i)$ is a special class. For the two-row type classes, i.e. $\sigma(\lambda_i, \lambda_j)$ with $\lambda_j > 0$, we want to express them as a polynomial in the special classes $\sigma(k)$ \((1 \leq k \leq n)\). In fact, we have the following expression for two-row type classes $\sigma(k,1)$ in $H^*_T(LG_n)$:

\[
\sigma(k,1) = \sigma(k)\sigma(1) - 2\sigma(k+1) - 2x_{n-k+1}\sigma(k) \quad (2 \leq k \leq n), \tag{7.2}
\]

where $\sigma(j) = 0$ for $j > n$. The above expression is a consequence of the following formula for Ivanov’s functions

\[
Q_{k,1}(x|a) = Q_k(x|a)Q_1(x|a) - 2Q_{k+1}(x|a) - 2a_{k+1}Q_k(x|a). \tag{7.3}
\]

As illustrated by this example, we need a correction term to classical formula \((7.1)\).

To generalize \((7.2)\), we prove the next proposition, which is also interesting from a purely combinatorial point of view. In this section, $x = (x_1, x_2, \ldots)$ and $a = (a_2, a_3, \ldots)$ are two sequences of infinite variables. We can define $Q_\lambda(x|a)$ for any strict partition $\lambda$. They are in the ring $\mathbb{Z}[a_2, a_3, a_4, \ldots] \otimes \mathbb{Z} \Gamma$, where $\Gamma$ denote a distinguished subring spanned by the Schur’s $Q$-functions in “the ring of symmetric functions $\Lambda$” \((31)\). For the detail of definition for $Q_\lambda(x|a)$, see \([19]\) and also Section \(\S\). Note that, if we substitute $x_j = 0$ \((j > n)\) for $Q_\lambda(x|a)$ \((\lambda \in S^\infty P_n)\) we can recover the polynomial introduced by Definition \((5.1)\). Let $h_r$ (resp. $e_r$) denote the $r$-th complete (resp. elementary) symmetric function.
Proposition 7.1 Let \( k > \ell > 0 \). We have

\[
Q_{k,\ell}(x|a) = Q_k(x|a)Q_\ell(x|a) + 2 \sum_{i=1}^{\ell} (-1)^i Q_{k+i}(x|a)Q_{\ell-i}(x|a) + G_{k,\ell}(x|a),
\]  
where

\[
G_{k,\ell}(x|a) = \sum_{r=k}^{k+\ell-1} \sum_{s=0}^{k+\ell-1-r} f_{k,\ell}^{r,s}(a)Q_r(x|a)Q_s(x|a),
\]

and the coefficient \( f_{k,\ell}^{r,s}(a) \) is given by

\[
f_{k,\ell}^{r,s}(a) = (-1)^{\ell-s} \sum_{j=0}^{k+\ell-r-s} 2h_{k+\ell-r-s-j}(a_{k+1}, a_{k+2}, \ldots, a_{r+1}) e_j(a_{s+2}, \ldots, a_{\ell-1}, a_{\ell}).
\]

Proof. We use the equation (8.2) of [19] that reads

\[
Q_{k+1,\ell} + Q_{k,\ell+1} + (a_{k+1} + a_{\ell+1})Q_{k,\ell} = Q_kQ_{\ell+1} - Q_{k+1}Q_\ell + (a_{\ell+1} - a_{k+1})Q_kQ_\ell,
\]

for \( k > \ell > 0 \), where we denote \( Q_{r,s}(x|a) \) simply by \( Q_{r,s} \). By this equation, it is easy to see that each function \( Q_{k,\ell} \) is a linear combination of the functions \( Q_rQ_s \) \((r > s \geq 0)\). Note that the functions \( Q_rQ_s \) \((r > s \geq 0)\) are linearly independent over the ring \( \mathbb{Z}[a_2, a_3, \ldots] \). (This fact can be seen from Prop. 2.11 in [19] and [31], III, (8.9)).

We shall prove the proposition by induction on \( \ell \). The case \( \ell = 1 \) is true by (7.3). Let \( \ell > 1 \). Suppose the proposition holds for \( \ell \). We have an expansion

\[
Q_{k,\ell+1} = \sum_{r>s \geq 0} g_{k,\ell+1}^{r,s}(a)Q_rQ_s,
\]

with coefficients \( g_{k,\ell+1}^{r,s}(a) \in \mathbb{Z}[a_2, a_3, \ldots] \). Our task is to show \( g_{k,\ell+1}^{r,s} = f_{k,\ell}^{r,s} \). By extracting the coefficient of \( Q_kQ_\ell \) in both hand sides of (7.4), we have

\[
g_{k,\ell+1}^{r,s}(a_{\ell+1} + a_{k+1}) = a_{\ell+1} - a_{k+1}.
\]

Hence we have \( g_{k,\ell+1}^{k,\ell} = -2a_{k+1} = f_{k,\ell}^{k,\ell} \). Let \((r, s) \neq (k, \ell)\) with \( r + s < k + \ell \). By comparing the coefficients of \( Q_rQ_s \) in both hand sides of (7.7), we have

\[
f_{k+1,\ell}^{r,s} + g_{k,\ell+1}^{r,s} + (a_{k+1} + a_{\ell+1})f_{k,\ell}^{r,s} = 0.
\]

We shall prove \( g_{k,\ell+1}^{r,s} = f_{k,\ell+1}^{r,s} \) by showing

\[
f_{k+1,\ell}^{r,s} + f_{k,\ell+1}^{r,s} + (a_{k+1} + a_{\ell+1})f_{k,\ell}^{r,s} = 0.
\]

This follows from the following equality:

\[
\frac{\prod_{\alpha=s+2}^{\ell}(1 + a_\alpha z)}{\prod_{\beta=k+2}^{r+1}(1 - a_\beta z)} + z(a_{k+1} + a_{\ell+1}) \frac{\prod_{\alpha=s+2}^{\ell}(1 + a_\alpha z)}{\prod_{\beta=k+1}^{r+1}(1 - a_\beta z)} = \frac{\prod_{\alpha=s+2}^{\ell+1}(1 + a_\alpha z)}{\prod_{\beta=k+1}^{r+1}(1 - a_\beta z)}.
\]
\[ G_{k,1} = -2a_{k+1}Q_k, \]
\[ G_{k,2} = 2(a_{k+1} + a_{k+2} + a_2)Q_{k+1} - 2a_{k+1}Q_k + 2(a_{k+1}^2 + a_2a_{k+1}^2)Q_k, \]
\[ G_{k,3} = -2(a_{k+1} + a_{k+2} + a_{k+3} + a_2 + a_3)Q_{k+2} + 2(a_{k+1} + a_{k+2} + a_3)Q_{k+1}Q_k + 2\left( \frac{a_{k+1}^2 + a_{k+2}^2 + (a_{k+1} + a_{k+2})(a_2 + a_3) + a_2a_3}{Q_{k+1}} \right)Q_k + 2a_{k+1}Q_kQ_2 + 2(a_{k+1} + a_{k+3})Q_kQ_1 - 2\left( a_{k+1}^3 + a_{k+1}^2(a_2 + a_3) + a_{k+1}a_2a_3 \right)Q_k, \]

where we denote \( Q_{r,s}(x|a) \) simply by \( Q_{r,s} \).

Proposition 7.1 combined with Theorem 6.2 give rise to a polynomial expression for \( \sigma(r,s) \) with \( n \geq r > s > 0 \) in terms of the special classes \( \sigma(k) \) (\( 1 \leq k \leq n \)). For example, we have

\[ \sigma(k,2) = \sigma(k)\sigma(2) - 2\sigma(k+1)\sigma(1) + 2\sigma(k+2) - 2x_{n-k+1}\sigma(k)\sigma(1) + 2(x_{n-k+1} + x_{n-k} + x_n)\sigma(k+1) + 2(x_{n-k+1} + x_{n-k+1}x_n)\sigma(k) \]

for \( 2 \leq k \leq n \), with \( \sigma(j) = 0 \) for \( j > n \) (cf. Proposition 9.1).

The next proposition will be used in Section 8.

**Proposition 7.2** For \( k \geq 1 \), we have

\[ Q_k(x|a)^2 + 2 \sum_{i=1}^{k} (-1)^i Q_{k+i}(x|a)Q_{k-i}(x|a) + \sum_{r=k}^{2k-1} \sum_{s=0}^{2k-1-r} f_{r,s}^{r,s}(a)Q_r(x|a)Q_s(x|a) = 0. \quad \text{(8.11)} \]

**Proof.** The proof Lemma 7.1 is valid also for \( k = \ell \) with \( Q_{k,k}(x|a) = 0 \) for \( k \geq 1 \). \( \square \)

## 8 Presentation of the ring \( H_T^*(LG_n) \)

As an application of Theorems 6.2 and 6.3 we obtain a presentation of the ring \( H_T^*(LG_n) \) in terms of generators and relations. Consider the ring \( \mathbb{Z}[a] = \mathbb{Z}[a_2, a_3, \ldots, a_{n+1}] \). Throughout the section, we identify \( S = \mathbb{Z}[x_1, \ldots, x_n] \) and \( \mathbb{Z}[a] \) by the isomorphism \( \iota_n : \mathbb{Z}[a] \rightarrow S \) of rings given by

\[ \iota_n(a_j) = x_{n-j+2} \quad (2 \leq i \leq n). \quad \text{(8.1)} \]

### 8.1 Statement of the result

Let \( X_1, \ldots, X_n \) denote a set of indeterminates. Set \( X_0 = 1 \) and \( X_j = 0 \) for \( j > n \) (cf. Proposition 9.1). Let \( k, \ell \) be \( n \geq k \geq \ell \geq 0 \). Consider the following elements of the polynomial ring \( S[X_1, \ldots, X_n] \):

\[ X_{k,\ell} = X_kX_{\ell} + 2 \sum_{i=1}^{\min(n-k, \ell)} (-1)^i X_{k+i}X_{\ell-i} + \sum_{r=k}^{\min(n,k+\ell-1)} \sum_{s=0}^{\min(n,k+\ell-1-r)} f_{r,s}^{r,s}(a) X_rX_s, \quad \text{(8.2)} \]
where $f^{r,s}_{k,k}(a)$ is given by the right hand side of (7.10). Since we restrict $r \leq n$, we can consider $f^{r,s}_{k,k}(a)$ to be in $\mathcal{S}$ via the isomorphism $t_n$. Note also that we also consider the case of $\ell = k$. Define an ideal $\mathcal{I}_n = \langle X_{1,1}, \ldots, X_{n,n} \rangle$ and consider the quotient ring

$$
\mathcal{R}_n = \mathcal{S}[X_1, \ldots, X_n]/\mathcal{I}_n.
$$

We shall define a morphism of $\mathcal{S}$-algebras $\phi : \mathcal{R}_n \rightarrow H^*_T(LG_n)$ by setting $X_i$ to $\sigma(i)$ ($1 \leq i \leq n$).

**Lemma 8.1** The map $\phi$ is well-defined.

**Proof.** Define a morphism of $\mathcal{S}$-algebras $\tilde{\phi} : \mathcal{S}[X_1, \ldots, X_n] \rightarrow H^*_T(LG_n)$ by $\tilde{\phi}(X_i) = \sigma(i)$. For $k$ with $1 \leq k \leq n$, and $\mu \in W^P$, we have

$$
\tilde{\phi}(X_{k,k})|_\mu = \sigma(k)^2|_\mu + 2 \sum_{i=1}^{\min(n-k,k)} (-1)^i \sigma(k+i)|_\mu \sigma(k-i)|_\mu + \sum_{r=k}^{\min(n,2k-1)} \sum_{s=0}^{2k-1-r} \tau_n(f^{r,s}_{k,k}(a)) \sigma(r)|_\mu \sigma(s)|_\mu
$$

$$
= Q_k(x_\mu|x_{(n)})^2 + 2 \sum_{i=1}^{k} (-1)^i Q_{k+i}(x_\mu|x_{(n)}) Q_{k-i}(x_\mu|x_{(n)})
$$

$$
+ \sum_{r=k}^{2k-1} \sum_{s=0}^{2k-1-r} f^{r,s}_{k,k}(x_{(n)}) Q_r(x_\mu|x_{(n)}) Q_s(x_\mu|x_{(n)})
$$

where in the second equality, we used Theorem 6.2 and a vanishing property (Proposition 9.1). We can see the last expression is zero by specializing (7.11) (see Definition 6.1). Thus we have $\tilde{\phi}(X_{k,k})|_\mu = 0$ for all $\mu \in W^P$, and hence $\tilde{\phi}(X_{k,k}) = 0$. So $\tilde{\phi}$ induces $\phi : \mathcal{R}_n \rightarrow H^*_T(LG_n)$ such that $\phi(X_i) = \sigma(i)$ ($1 \leq i \leq n$). □.

**Definition 8.2** Let $\lambda = (\lambda_1 > \cdots > \lambda_{2r} \geq 0)$ be in $\mathcal{S}\mathcal{P}_n$. We introduce the following Schur-type Pfaffian

$$
X_\lambda = \text{Pf}(X_{\lambda_i,\lambda_j})_{1 \leq i, j \leq 2r}.
$$

**Theorem 8.3** There exists an isomorphism of $\mathcal{S}$-algebras:

$$
\phi : \mathcal{R}_n \rightarrow H^*_T(LG_n)
$$

sending $X_i$ to $\sigma(i)$ ($1 \leq i \leq n$) and the Pfaffian $X_\lambda$ to the equivariant Schubert class $\sigma(\lambda)$.

By definition of $\phi$ and Giambelli formula (6.2), we have $\phi(X_\lambda) = \sigma(\lambda)$. Moreover, since $\sigma(\lambda)$ ($\lambda \in \mathcal{S}\mathcal{P}_n$) generates $H^*_T(LG_n)$ as an $\mathcal{S}$-module, $\phi$ is surjective. The rest of this section is devoted to the proof of injectivity of $\phi$. 

17
8.2 A monomial ordering

Here we give a preliminary discussion to prove Theorem 8.3. The argument below is quite similar to the one in Macdonald [31] (III, 8), however a different ordering on the partitions will be used, which proves to be useful in our situation.

For any partition \(\lambda = (1^{e_1} e_2 \cdots n^{e_n})\), we set
\[ X_\lambda = X_1^{e_1} \cdots X_n^{e_n}. \]

By \(\text{deg}(\lambda)\) we denote the degree \(\sum_{i=1}^n e_i\) of the monomial \(X_\lambda\). Let \(\mu = (1^{e'_1} \cdots n^{e'_n})\) be another partition. We write \(\lambda \succ \mu\) if \(\text{deg}(\lambda) > \text{deg}(\mu)\), or \(\text{deg}(\lambda) = \text{deg}(\mu)\) and there is \(k\) such that \(e_1 = e'_1, \ldots, e_k = e'_k\) and \(e_{k+1} < e'_{k+1}\).

Then we also write \(X_\lambda \succ X_\mu\). This is a monomial ordering called the *grevlex order* with \(X_1 \prec X_2 \prec \cdots \prec X_n\). In particular, if we have \(\lambda \succ \mu\), then \(\lambda + \nu \succ \mu + \nu\) for any partition \(\nu\).

Lemma 8.4 Let \(\lambda = (1^{e_1} 2^{e_2} \cdots n^{e_n})\) be a partition. If \(\lambda\) is not strict, then \(X_\lambda\) is an \(S\)-linear combination of the \(X_\mu\) with \(\mu \in \mathcal{SP}_n\), and \(\mu \prec \lambda\). In particular, the monomials \(X_\lambda (\lambda \in \mathcal{SP}_n)\) generate \(R_n\) as an \(S\)-module.

Proof. First note that if \(\lambda\) is strict then we have \(\lambda \in \mathcal{SP}_n\). We prove the first statement by induction, assuming the claim for all partition \(\mu\) such that \(\mu \prec \lambda\).

We can see that the monomials appearing in the right hand side of the above equation is strictly lower than \(X_k^2\) in the grevlex order \(<\). Replacing the factor \(X_k^2\) in \(X_\lambda\) by the right hand side of (8.3), we can express \(X_\lambda\) as an \(S\)-linear combination of the \(X_\mu\)’s where each \(\mu\) is a partition such that \(\mu \prec \lambda\). By the inductive hypothesis the claim is true for each \(X_\mu\), and the proof completes. \(\square\).

Lemma 8.5 Let \(\lambda \in \mathcal{SP}_n\). The Pfaffian \(X_\lambda\) is written in the form
\[ X_\lambda = X_\lambda + \sum_{\mu} b_{\lambda \mu}(a) X_\mu \]
with coefficients \(b_{\lambda \mu}(a) \in S\), where the sum is over \(\mu \in \mathcal{SP}_n\) such that \(\mu \prec \lambda\).

Proof. Let \(\lambda = (\lambda_1 > \cdots > \lambda_2r, \geq 0)\) be a strict partition in \(\mathcal{SP}_n\). We proceed by induction on \(r\). Let \(r = 1\). If \(\lambda = (i)\) with \(1 \leq i \leq n\) the Lemma is clear. For two-row type the Lemma is true by (8.2). Let \(r \geq 2\) and assume the Lemma holds for all
\( \mu = (\mu_1 > \cdots > \mu_{2s} \geq 0) \in S\mathcal{P}_n \) with \( s < r \). From the definition of the Pfaffian it follows that

\[
X_{\lambda} = \sum_{j=2}^{2r} (-1)^j X_{\lambda_1, \lambda_j} X_{\lambda_2, \cdots, \lambda_j, \cdots, \lambda_{2r}}.
\]

By the inductive hypothesis, we have

\[
X_{\lambda_2, \cdots, \lambda_j, \cdots, \lambda_{2r}} = X_{\lambda_2} \cdots \hat{X}_{\lambda_j} \cdots X_{\lambda_{2r}} + F_j
\]

where \( F_j \) is a \( S \)-linear combination of \( X^\mu \)'s with \( \mu \in S\mathcal{P}_n \) such that \( \mu < (\lambda_2, \cdots, \lambda_j, \cdots, \lambda_{2r}) \). Then it is easy to see that the Lemma holds for \( \lambda \). \( \square \).

From Lemmas 8.4 and 8.5, we have the following.

**Lemma 8.6** The Pfaffians \( X_{\lambda} (\lambda \in S\mathcal{P}_n) \) generate \( R_n \) as an \( S \)-module.

### 8.3 Completion of the proof of Theorem 8.3.

It remains to prove the injectivity of \( \phi \). Let \( F \) be in \( \text{Ker}(\phi) \). By Lemma 8.6 we have

\[
F = \sum_{\lambda \in S\mathcal{P}_n} c_{\lambda}(a) X_{\lambda}
\]

with coefficients \( c_{\lambda}(a) \in S \). We know \( \phi(X_{\lambda}) = \sigma(\lambda) \). So we have \( 0 = \sum_{\lambda} t_n(c_{\lambda}(a)) \sigma(\lambda) \). Since \( \sigma(\lambda) \) are linearly independent over \( S \), \( t_n(c_{\lambda}(a)) = 0 \) for all \( \lambda \in S\mathcal{P}_n \). Hence we have \( c_{\lambda}(a) = 0 \) (\( \lambda \in S\mathcal{P}_n \)) and \( F = 0 \).

### 9 Appendix

For the reader’s convenience, we provides a summary of some properties of \( Q_{\lambda}(x|a) \). We also prove a vanishing property (Proposition 9.1) essentially used in the main body of the paper.

We use standard notation for symmetric functions as in Macdonald’s book [31]. Let \( \Lambda \) denote the *ring of symmetric functions* in infinitely many indeterminates \( x = (x_1, x_2, \ldots) \). The ring \( \Lambda \) is graded as \( \Lambda = \bigoplus_{k=0}^{\infty} \Lambda^k \) and each graded part \( \Lambda^k \) has a \( \mathbb{Z} \)-basis consisting of the *monomial symmetric functions* \( m_\lambda = m_\lambda(x) \) (for all partitions \( \lambda \) of \( k \)).

Recall an expression for the \( Q_k(x) \) the Schur’s \( Q \)-functions for the one-row partition

\[
Q_k(x) = \sum_{\lambda} 2^{\ell(\lambda)} m_\lambda(x),
\]

where the sum runs over all partitions \( \lambda \) of \( k \) and \( \ell(\lambda) \) is the length of \( \lambda \), the number of nonzero parts of \( \lambda \). Let \( \Gamma \) be the subring of \( \Lambda \) generated by \( Q_k \):

\[
\Gamma = \mathbb{Z}[Q_1, Q_2, Q_3, \ldots].
\]
We have a gradation $\Gamma = \bigoplus_{k=0}^{\infty} \Gamma^k$ where $\Gamma^k = \Gamma \cap \Lambda^k$. The Schur’s $Q$-functions $Q_\lambda(x)$, with $\lambda$ strict partition of $k$, form a distinguished $\mathbb{Z}$-basis of $\Gamma^k$.

Let $a_2, a_3, a_4, \ldots$ be an infinite sequence of independent variables. We set $a_1 = 0$.

Ivanov introduced a factorial analogue of $Q$-functions $Q_\lambda(x|a)$ defined for any strict partition $\lambda$. Each $Q_\lambda(x|a)$ is an element of the ring $\mathbb{Z}[a_2, a_3, \ldots] \otimes_{\mathbb{Z}} \Gamma$. In particular, we have, by Ivanov [19], Theorem 8.2,

$$ Q_k(x|a) = \sum_{j=0}^{k-1} (-1)^j e_j(a_2, a_3, \ldots, a_k) Q_{k-j}(x). $$

For $k > \ell > 0$, we can define $Q_{k,\ell}(x|a)$ by Proposition [7.1]. Moreover, for arbitrary strict partition $\lambda$, we have

$$ Q_\lambda(x|a) = \text{Pf}(Q_{\lambda_i,\lambda_j}(x|a))_{1 \leq i < j \leq 2r}, \quad (9.1) $$

where we write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2r})$ with $\lambda_1 > \cdots > \lambda_{2r} \geq 0$.

The following result is very important. Recall Definition [6.1] for the meaning of $Q_\lambda(x|\mu|x(n))$.

**Proposition 9.1** $Q_\lambda(x|\mu|x(n))$ vanishes identically unless $\lambda \in S\mathcal{P}_n$.

**Proof.** We prove the Proposition for $\lambda = (k)$. In [19], Ivanov derived the following equation (Theorem 8.2 in [19])

$$ \sum_{k=0}^{\infty} \frac{Q_k(x|a)z^k}{\prod_{j=1}^{k}(1 - a_{j+1}z)} = \prod_{i=1}^{\infty} \frac{1 + x_i z}{1 - x_i z}. \quad (9.2) $$

If we specialize the variables as in the statement of the Proposition, we have

$$ \sum_{k=0}^{n} \frac{Q_k(x|\mu|x(n))z^k}{\prod_{j=1}^{k}(1 - x_{n+1-j}z)} + \sum_{k>n} Q_k(x|\mu|x(n))z^k \prod_{j=1}^{n}(1 - x_j z) = \prod_{1 \leq i \leq n, \delta_i = 1}^{\infty} \frac{1 + x_i z}{1 - x_i z}, $$

where $(\delta_1, \ldots, \delta_n) \in \mathcal{M}_n$ corresponds to $\mu \in W^P$. Multiplying $\prod_{1 \leq i \leq n}(1 - x_i z)$ to both hand sides, we have

$$ \sum_{k>n} Q_k(x|\mu|x(n))z^k = - \sum_{k=0}^{n} Q_k(x|\mu|x(n))z^k \prod_{j=1}^{n-k}(1 - x_j z) + \prod_{i=1}^{n} (1 + (-1)^{\delta_i+1} x_i z). $$

The right hand side of the equation is a polynomial in $z$ of degree lower than $n$ and we are done. For general $\lambda$, the Proposition follows from Proposition [7.1] and the Pfaffian formula (9.1) for $Q_\lambda(x|a)$. $\square$. 

20
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