Hadronic light-by-light corrections to the muon $g - 2$:
The pion-pole contribution

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Abstract

The correction to the muon anomalous magnetic moment from the pion-pole contribution to the hadronic light-by-light scattering is considered using a description of the $\pi^0 - \gamma^* - \gamma^*$ transition form factor based on the large-$N_C$ and short-distance properties of QCD. The resulting two-loop integrals are treated by first performing the angular integration analytically, using the method of Gegenbauer polynomials, followed by a numerical evaluation of the remaining two-dimensional integration over the moduli of the Euclidean loop momenta. The value obtained, $a_{\mu}^{LbyL,\pi^0} = +5.8 \times 10^{-10}$, disagrees with other recent calculations. In the case of the vector meson dominance form factor, the result obtained by following the same procedure reads $a_{\mu}^{LbyL,\pi^0}|_{VMD} = +5.6 \times 10^{-10}$, and differs only by its overall sign from the value obtained by previous authors. The inclusion of the $\eta$ and $\eta'$ poles gives a total value $a_{\mu}^{LbyL,PS} = +8.3 \times 10^{-10}$ for the three pseudoscalar states. This result substantially reduces the difference between the experimental value of $a_{\mu}$ and its theoretical counterpart in the standard model.

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1 Introduction

A high-precision measurement of the muon anomalous magnetic moment is presently being performed at the Brookhaven National Laboratory by the Muon ($g - 2$) Collaboration [1]. The value released recently [2],

$$a_{\mu^+} = 11 659 202 (14) (6) \times 10^{-10},$$

(1.1)

already improves the precision by a factor of 8 as compared to the previous determination at CERN [3]. The final aim of the experiment is to reach a precision of $\pm 4 \times 10^{-10}$, after completion of the analysis including the full set of data collected during recent years.

If only contributions of (multiflavored) QED are taken into account, the theoretical accuracy is still much better than the present experimental one [4, 5]. However, hadronic effects, which certainly cannot be ignored in the case of the muon $g - 2$ factor, are unfortunately more difficult to estimate, and the corresponding uncertainties substantially increase, and actually completely dominate, the error of the total theoretical value. Nevertheless, the value (1.1) disagrees with various theoretical calculations in a way that could become very significant as the BNL E821 experiment further decreases its statistical error. Seen from this perspective, it is not surprising that the announcement of the value (1.1) has triggered a wealth of theoretical activity (a complete bibliography of the recent work in this direction would represent a tedious task [6]). It is, however, our feeling that, rather than calling for hasty conclusions concerning possible evidence for physics beyond the standard model, the present situation first requires that the hadronic contributions be scrutinized anew and with greater care.

The hadronic contributions to the anomalous magnetic moment of the muon considered so far naturally fall into three categories: vacuum polarization, light-by-light scattering, and higher-order electroweak contributions, as represented in Fig. 1. Hadronic effects in two-loop electroweak corrections are estimated in Refs. [7, 8, 9]. They are small, of the order of the expected experimental error, and the associated theoretical uncertainties, of the order of 10%, can be brought under safe control [9]. Hadronic vacuum polarization effects benefit from the fact that they can be related to the total cross section for $e^+e^- \rightarrow \text{hadrons}$ [10]. However, existing data come from different sources, and do not always meet the required accuracy, so that they are supplemented with theoretical input. There exist therefore different estimates for the hadronic vacuum polarization, which either strengthen the difference between theory and the experimental value (1.1), or make it appear only marginal. Our purpose here is not to intervene in this ongoing debate [11, 12, 13, 14, 15, 16, 17], but rather to have a closer look at the remaining item on the list, namely, the hadronic effects in the so-called light-by-light scattering contribution. There exist several estimates of the latter [18, 19, 20, 21, 22, 23]. The latest to date, obtained by two different groups (see also [24]), give values that are consistent within the quoted errors,

$$a_{\mu}^{\text{LbyL, had}} = -7.9 (1.5) \times 10^{-10} \quad [20, 21],$$

(1.2)

$$a_{\mu}^{\text{LbyL, had}} = -9.2 (3.2) \times 10^{-10} \quad [22].$$

(1.3)

On the other hand, both groups have changed their value of $a_{\mu}^{\text{LbyL, had}}$, and even its sign, at some stage, and for reasons that are not very easy to follow – partly because both analyses
the theoretical knowledge of point of view. Resort to theory is therefore unavoidable. Now, from the QCD perspective, the other hand, the values (1.2) and (1.3) are both dominated by a well-identified component, $\Pi_{\mu\nu\lambda\rho}$ hadronic vacuum polarization tensor $\Pi_{\mu\nu\lambda\rho}$.

In each channel, $\Pi_{\mu\nu\lambda\rho}$ can be expressed as a sum of pion-exchange terms, $\Pi_{\mu\nu\lambda\rho}^{\pi\pi\gamma\gamma}$, followed by the one-pion-exchange component of $\Pi_{\mu\nu\lambda\rho}$. Complete experimental information on this quantity is available at the time being, so that even the double off-shell pion-photon-photon transition form factor $F_{\pi^{0}\pi^{0}\gamma\gamma}$ becomes large, the spectrum of QCD reduces, in each channel with given quantum numbers, to an infinite tower of zero-width resonances. As a consequence, the analytic structure of the form factor $F_{\pi^{0}\pi^{0}\gamma\gamma}$ becomes simpler: it consists of a succession of simple poles, due to the contributions of zero-width $J^{PC} = 1^{--}$ states (e.g., the $\rho$ meson and its radial excitations) in each channel. Dealing with an infinite number of resonances remains cumbersome, and to a large extent illusory, since the characteristics of these states (masses, couplings, etc.) are rapidly leave the realm of analytical work and resort to numerical techniques. This aspect of the present theoretical situation provided the motivation to improve from the analytical side the study of the hadronic light-by-light contributions.

A complete discussion of the hadronic light-by-light contributions involves the full rank-four hadronic vacuum polarization tensor $\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3)$, which is a rather involved object. On the other hand, the values (1.2) and (1.3) are both dominated by a well-identified component, $a_{\mu LbyL; had} = a_{\mu LbyL; \pi^0}^{\pi^0} + \cdots$,

\begin{align}
  a_{\mu LbyL; \pi^0} &= -5.7 \times 10^{-10} \quad [21], \\
  a_{\mu LbyL; \pi^0} &= -5.9 \times 10^{-10} \quad [22],
\end{align}

arising from the one-particle reducible pion-exchange pieces of $\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3)$. In the present paper, we focus on the latter, leaving for future work a complete discussion of $\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3)$ within a theoretical framework close to the one adopted here (and discussed below) for the pion-pole contribution.

The main ingredient in the determination of the pion-exchange graphs depicted in Fig. 2 is the double off-shell pion-photon-photon transition form factor $F_{\pi^{0}\pi^{0}\gamma\gamma}(q_1^2, q_2^2)$. No sufficiently complete experimental information on this quantity is available at the time being, so that even the one-pion-exchange component of $\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3)$ is beyond reach from an experimental point of view. Resort to theory is therefore unavoidable. Now, from the QCD perspective, the theoretical knowledge of $F_{\pi^{0}\pi^{0}\gamma\gamma}$ is sparse. There exists a well-defined limit of QCD where the states (e.g., the $\rho$ meson and its radial excitations) are

Figure 1: The three topologies involving hadronic contributions to the anomalous magnetic moment of the muon: from left to right, vacuum polarization insertion in the vertex, light-by-light scattering, and two-loop electroweak contributions. The cross indicates an insertion of the electromagnetic current, the shaded areas correspond to hadronic subgraphs, while the lines exchanged in the rightmost graph correspond to a photon and a neutral gauge boson.
Figure 2: The pion-pole contributions to light-by-light scattering. The shaded blobs represent the form factor $F_{\pi^0 \gamma^* \gamma^*}$. The first and second graphs give rise to identical contributions, involving the function $T_1(q_1, q_2; p)$ in Eq. (3.4), whereas the third graph gives the contribution involving $T_2(q_1, q_2; p)$.

in general not known. It has, however, been shown in several instances (see, e.g., the review [26] and references therein) that keeping, in each channel, only a finite number of resonances, supplemented with information on the QCD short-distance properties coming from the operator product expansion [27, 28], already gives a good description of quantities like form factors or correlation functions in the Euclidean region, especially when they occur in weighted integrals over the whole range of momenta. In particular, the form factor $F_{\pi^0 \gamma^* \gamma^*}$ has recently been studied [29] from the point of view of this lowest meson dominance (LMD) or minimal hadronic Ansatz (MHA) approximation to large-$N_C$ QCD. Since the analyses carried out in Refs. [21, 22] rely on models of $F_{\pi^0 \gamma^* \gamma^*}$, vector meson dominance (VMD) or extended Nambu–Jona-Lasinio (ENJL) (see [22] and references therein), that do not reproduce the correct QCD short-distance properties, a second motivation for the present study was to compare the results (1.4) and (1.5) with those derived from a representation of $F_{\pi^0 \gamma^* \gamma^*}$ that complies with these QCD constraints. Finally, let us mention for completeness that the pion-pole contribution we are interested in corresponds to the lowest-mass part of $\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3)$ that is leading in the large-$N_C$ limit [30], which might provide an explanation as to why it happens to constitute the dominant fraction of the light-by-light scattering correction to $a_\mu$.

The remaining material of this paper is organized as follows. Section 2 recalls a few definitions that are relevant for the light-by-light contribution to $a_\mu$, and also serves the purpose of introducing our notation. The expression for the two-loop integral which gives the pion-pole contribution $a^{LbyL,0}_\mu$ to the anomalous magnetic moment of the muon in terms of $F_{\pi^0 \gamma^* \gamma^*}$ is derived in Sec. 3. In Sec. 4 we discuss a generic class of form factors to which those inspired from large-$N_C$ QCD and considered here belong, but that also covers other cases, like the constant form factor, given by the Wess-Zumino-Witten term [31], or the vector meson dominance form factor. The method of Gegenbauer polynomials is presented in Sec. 5 and used in order to perform the angular integrations. This then leads to a two-dimensional integral representation for $a^{LbyL,0}_\mu$ in terms of $F_{\pi^0 \gamma^* \gamma^*}$ and of several weight functions (Sec. 6). Numerical results for $a^{LbyL,0}_\mu$ are presented in Sec. 7, where the contributions from the $\eta$ and $\eta'$ poles are also briefly
discussed. A discussion and conclusions make up the content of Sec. 8. For the reader’s convenience, the properties of the form factor $F_{\pi^0\gamma\gamma^*}$ that form the basis of the representations we use here have been gathered in an Appendix.

## 2 Definitions

The interaction of the photon field with the standard model fermionic degrees of freedom is described by the Lagrangian ($e$ denotes the electric charge of the electron)

$$\mathcal{L}_{\text{int}}(x) = -e A_\mu(x) \left[ \sum_\ell \bar{\psi}_\ell \gamma^\mu \psi_\ell(x) - \sum_q e_q (\bar{q} \gamma^\mu q)(x) \right]. \quad (2.1)$$

The first sum runs over the charged lepton flavors $\ell = e^-, \mu^-, \tau^-$, while the second sum runs over the quark flavors, with $e_q$ standing for the corresponding electric charges expressed in units of $|e|$. Our interest hereafter is restricted to the contributions of the light quarks $q = u, d, s$. The muon gyromagnetic ratio is obtained from the muon proper vertex function $\Gamma_\rho(p', p)$, defined as $(p^2 = p'^2 = m^2$, $m$ denotes the muon mass)

$$(-ie) \bar{u}(p') \Gamma_\rho(p', p) u(p) = \langle \mu^-(p') | (-ie) \sum_\ell \bar{\psi}_\ell \gamma_\rho \psi_\ell(0) + (ie) \sum_q e_q (\bar{q} \gamma_\rho q)(0) | \mu^-(p) \rangle, \quad (2.2)$$

at vanishing momentum transfer.

The contribution to $\Gamma_\rho(p', p)$ of relevance here is the matrix element, at lowest nonvanishing order in the fine structure constant $\alpha = e^2/(4\pi)$, of the light quark electromagnetic current

$$j_\rho(x) = \frac{2}{3} (\bar{u} \gamma_\rho u)(x) - \frac{1}{3} (\bar{d} \gamma_\rho d)(x) - \frac{1}{3} (\bar{s} \gamma_\rho s)(x) \quad (2.3)$$

between $\mu^-$ states,

$$(-ie) \bar{u}(p') \Gamma_\rho(p', p) u(p) = \langle \mu^-(p') | (ie) j_\rho(0) | \mu^-(p) \rangle = \frac{i}{(2\pi)^4} \int \frac{d^4q_1}{q_1^2} \frac{d^4q_2}{q_2^2} \frac{(-i)^3}{(p' - q_1)^2 - m^2 (p' - q_1 - q_2)^2 - m^2} \times (-ie)^3 \bar{u}(p') \gamma^\mu (p' - q_1 + m) \gamma^\nu (p' - q_1 - q_2 + m) \gamma^\lambda u(p) \times (ie)^4 \Pi_{\mu\nu\lambda\rho}(q_1, q_2, k - q_1 - q_2), \quad (2.4)$$

with $k_\mu = (p' - p)_\mu$ and

$$\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3) = \int d^4x_1 \int d^4x_2 \int d^4x_3 e^{i(q_1 x_1 + q_2 x_2 + q_3 x_3)} \times \langle \Omega | T\{j_\mu(x_1) j_\nu(x_2) j_\lambda(x_3) j_\rho(0)\} | \Omega \rangle \quad (2.5)$$

the fourth-rank light quark hadronic vacuum polarization tensor, $|\Omega\rangle$ denoting the QCD vacuum.
Since the flavor diagonal current $j_\mu(x)$ is conserved, the tensor $\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3)$ satisfies the Ward identities
\[ \{q_1^\mu; q_2^\nu; q_3^\lambda; (q_1 + q_2 + q_3)^\rho\} \Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3) = 0. \] (2.6)
This entails that $\Pi_{\mu\nu\lambda\rho}(q_1, q_2, k - q_1 - q_2) = -k^\sigma (\partial/\partial k^\rho) \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2)$ and
\[ \bar{u}(p')\hat{\Gamma}_\rho(p', p)u(p) = \bar{u}(p')\left[\gamma_\rho \hat{F}_1(k^2) + \frac{i}{2m} \sigma_{\rho\tau} k^\tau \hat{F}_2(k^2)\right]u(p), \] (2.7)
as well as $\hat{\Gamma}_\rho(p', p) = k^\sigma \hat{\Gamma}_{\rho\sigma}(p', p)$ with
\[ \bar{u}(p')\hat{\Gamma}_{\rho\sigma}(p', p)u(p) = -ie^6 \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1 + q_2 - k)^2} \]
\[ \times \frac{1}{(p' - q_1)^2 - m^2} \frac{1}{(p' - q_1 - q_2)^2 - m^2} \]
\[ \times \bar{u}(p') \gamma^\mu (p' - \not{q}_1 + m) \gamma^\nu (p' - \not{q}_1 - \not{q}_2 + m) \gamma^\lambda u(p) \]
\[ \times \frac{\partial}{\partial k^\rho} \Pi(\mu\nu\lambda\sigma)(q_1, q_2, k - q_1 - q_2). \] (2.8)

Following Ref. [32] and using the property $k^\rho k^\sigma \bar{u}(p')\hat{\Gamma}_{\rho\sigma}(p', p)u(p) = 0$, one deduces that $\hat{F}_1(0) = 0$ and that the hadronic light-by-light contribution to the muon anomalous magnetic moment is equal to
\[ \hat{F}_2(0) = \frac{1}{48m} \text{tr} \left\{ (\not{p} + m)[\gamma_\rho, \gamma_\sigma](\not{p} + m)\hat{\Gamma}_{\rho\sigma}(p, p) \right\}. \] (2.9)
Equivalently, one may project out $\hat{F}_2(k^2)$ from $\hat{\Gamma}_\rho(p', p)$,
\[ \hat{F}_2(k^2) = \text{tr} \left\{ (\not{p} + m)\Lambda^{(2)}_{\rho}(p', p)(\not{p}' + m)\hat{\Gamma}_{\rho}(p', p) \right\}, \] (2.10)
with the help of the projector [33]
\[ \Lambda^{(2)}_{\rho}(p', p) = \frac{m^2}{k^2(4m^2 - k^2)} \left[ \gamma_\rho + \frac{k^2 + 2m^2}{m(k^2 - 4m^2)} (p' + p)_\rho \right]. \] (2.11)
One then uses $\hat{\Gamma}_{\rho}(p', p) = k_\sigma \hat{\Gamma}^{\sigma\rho}(p', p)$ and employs the identity (for $p'^2 = p^2 = m^2$)
\[ (\not{p} + m) \gamma_\rho (\not{p}' + m) = (\not{p} + m) \left[ \frac{1}{2m} (p + p')_\rho + \frac{i}{2m} \sigma_{\rho\sigma} (p - p')^\sigma \right] (\not{p}' + m) \] (2.12)
to simplify $\Lambda^{(2)}_{\rho}(p', p)$. Averaging over the directions of the four-vector $k_\mu$ with the constraints $k \cdot p = -k^2/2$, $k \cdot p' = k'^2/2$ and keeping the leading terms in $k$ in the resulting expression then gives back Eq. (2.9). [34] [35]

\[ ^1 \text{We use the following conventions for Dirac's } \gamma \text{ matrices: } \{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}, \text{ with } \eta_{\mu\nu} \text{ the flat Minkowski space metric of signature } (+ - - - ), \sigma_{\mu\nu} = (i/2)[\gamma_\mu, \gamma_\nu], \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \text{ whereas the totally antisymmetric tensor } \varepsilon_{\mu\nu\rho\sigma} \text{ is chosen such that } \varepsilon_{0123} = +1. \]
3 Pion-pole contribution

From now on, we concentrate on the contributions to $\Pi_{\mu \nu \lambda \rho}(q_1, q_2, q_3)$ arising from single neutral pion exchanges (see Fig. 4), which read

$$\Pi^{(\pi^0)}_{\mu \nu \lambda \rho}(q_1, q_2, q_3) = \frac{i}{(q_1 + q_2)^2 - M_\pi^2} \frac{\varepsilon_{\mu \nu \rho \sigma} q_1^\rho q_2^\sigma (q_1 + q_2)^\tau}{(q_1 + q_2)^2 - M_\pi^2}$$

+ $i \frac{\mathcal{F}_{\pi^0 \gamma^* \gamma^*}(q_1^2, (q_1 + q_2 + q_3)^2)}{(q_2 + q_3)^2 - M_\pi^2} \frac{\varepsilon_{\mu \nu \rho \sigma} q_1^\rho q_2^\sigma (q_1 + q_2)^\tau}{(q_2 + q_3)^2 - M_\pi^2}$

+ $i \frac{\mathcal{F}_{\pi^0 \gamma^* \gamma^*}(q_1^2, (q_1 + q_2 + q_3)^2)}{(q_1 + q_2)^2 - M_\pi^2} \frac{\varepsilon_{\mu \nu \rho \sigma} q_1^\rho q_2^\sigma (q_1 + q_2)^\tau}{(q_1 + q_2)^2 - M_\pi^2}$

$= \frac{i}{d^4 q} e^{iq \cdot x} \langle \Omega | T \{ j_\mu(x) j_\nu(0) \} | \pi^0(p) \rangle = \varepsilon_{\mu \nu \rho \sigma} q^\rho F_{\pi^0 \gamma^* \gamma^*}(q_1^2, (q_2^2, (p - q)^2)$

The form factor $F_{\pi^0 \gamma^* \gamma^*}(q_1^2, q_2^2)$ is defined as (using the same convention as in Ref. [29])

$$F_{\pi^0 \gamma^* \gamma^*}(q_1^2, q_2^2) = F_{\pi^0 \gamma^* \gamma^*}(q_1^2, q_2^2).$$

For the computation of the corresponding value of $a_{\mu}^{L_{\text{bary}} \pi^0} \equiv \bar{F}_2(0)_{\text{pion pole}}$, we need

$$\frac{\partial}{\partial q^\rho} \Pi^{(\pi^0)}_{\mu \nu \lambda \rho}(q_1, q_2, k - q_1 - q_2)$$

$$= \frac{i}{d^4 q} e^{iq \cdot x} \langle \Omega | T \{ j_\mu(x) j_\nu(0) \} | \pi^0(p) \rangle = \varepsilon_{\mu \nu \rho \sigma} q^\rho F_{\pi^0 \gamma^* \gamma^*}(q_1^2, (q_2^2, (p - q)^2)$

Inserting this last expression into Eq. (2.8) and computing the corresponding Dirac traces (we used REDUCE [30] in Eq. (2.9), we obtain

$$a_{\mu}^{L_{\text{bary}} \pi^0} = -c^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1 + q_2)^2 (p + q_1)^2 - m^2}$$

$$\times \left[ \frac{\mathcal{F}_{\pi^0 \gamma^* \gamma^*}(q_1^2, (q_1 + q_2)^2) \mathcal{F}_{\pi^0 \gamma^* \gamma^*}(q_1^2, (q_1 + q_2)^2)}{(q_1 + q_2)^2 - M_\pi^2} T_1(q_1, q_2; p) \right]$$

$$+ \frac{\mathcal{F}_{\pi^0 \gamma^* \gamma^*}(q_1^2, q_2^2) \mathcal{F}_{\pi^0 \gamma^* \gamma^*}(q_1^2, (q_1 + q_2)^2)}{(q_1 + q_2)^2 - M_\pi^2} T_2(q_1, q_2; p) \right]$$,

with

$$T_1(q_1, q_2; p) = \frac{16}{3} (p \cdot q_1)(p \cdot q_2) (q_1 \cdot q_2) - \frac{16}{3} (p \cdot q_2)^2 q_1^2.$$
few features of Eq. (3.4) that are independent of the precise form of the two-loop integral (2.8), the two first terms of Eq. (3.3) lead to identical contributions. Furthermore, in deriving Eq. (3.4), we have used the fact that, upon a trivial change of variables in the two-dimensional integral, since the form factors depend on $q_1$ and $q_2$, whereby $p \cdot q_1$ only occur in the fermion propagators, not in the form factors. It might therefore be possible to perform these two integrations over $p \cdot q_1$ and $p \cdot q_2$ for general form factors. We would then be left with a three-dimensional integral, since the form factors depend on $q_1 \cdot q_2$. Therefore, this function could be extracted, over a finite energy range, from the data collected, e.g., by the CLEO collaboration [37]. This would reduce the model dependence.

Before discussing the pion-photon-photon transition form factor, let us briefly mention a few features of Eq. (3.4) that are independent of the precise form of $F_{\pi^0 \gamma^* \gamma^*}$. One observes that there are five independent variables $p \cdot q_1, p \cdot q_2, q_1^2, q_2$, and $q_1 \cdot q_2$, whereby $p \cdot q_1$ only occur in the fermion propagators, not in the form factors. It might therefore be possible to perform these two integrations over $p \cdot q_1$ and $p \cdot q_2$ for general form factors. We would then be left with a three-dimensional integral, since the form factors depend on $q_1 \cdot q_2$. Note that in both contributions in Eq. (3.4) there is one factor $F_{\pi^0 \gamma^* \gamma^*}(q^2, 0)$. In principle, this function could be extracted, over a finite energy range, from the data collected, e.g., by the CLEO collaboration [37]. This would reduce the model dependence.

4 The pion-photon-photon transition form factor

In the present work, we shall consider a representation of the form factor $F_{\pi^0 \gamma^* \gamma^*}$, based on the large-$N_C$ approximation to QCD, that takes into account constraints from chiral symmetry at low energies, and from the operator product expansion at short distances. Our numerical estimates for $g = 2$ below will be mainly based on expressions for the form factor $F_{\pi^0 \gamma^* \gamma^*}$ that involve either one vector resonance (lowest meson dominance, LMD) or two vector resonances (LMD+V). These Ansätze have been thoroughly discussed in Ref. [29], and a short summary can be found in the Appendix. Furthermore, as a reference point we shall also consider the simplest model for the form factor that follows from the Wess-Zumino-Witten (WZW) term [31] that describes the Adler-Bell-Jackiw anomaly [38] in chiral perturbation theory. Since in this case the form factor is constant, one needs an ultraviolet cutoff, at least in the contribution to Eq. (3.4) involving $T_1$. Therefore, this model cannot be used for a reliable estimate, but serves only illustrative purposes and as a check of our calculation. Finally we shall use the usual vector meson dominance form factor. Essentially, this corresponds to the form factor that has been employed in previous calculations for the pion-exchange contribution to $g - 2$ of the muon [19, 21, 22]; see also Ref. [39]. This will allow us to check the numerics. The expressions for the form factor $F_{\pi^0 \gamma^* \gamma^*}$ based on the ENJL model that have been used in Ref. [22] do not allow a straightforward analytical calculation of the loop integrals. However, compared with

\[- \frac{8}{3} (p \cdot q_1) (q_1 \cdot q_2) q_2^2 + 8(p \cdot q_2) q_1^2 q_2^2 - \frac{16}{3} (p \cdot q_2) (q_1 \cdot q_2)^2 \]

\[+ \frac{16}{3} m^2 q_1^2 q_2^2 - \frac{16}{3} m^2 (q_1 \cdot q_2)^2, \] (3.5)

\[T_2(q_1, q_2; p) = \frac{16}{3} (p \cdot q_1) (p \cdot q_2) (q_1 \cdot q_2) - \frac{16}{3} (p \cdot q_1)^2 q_2^2 \]

\[+ \frac{8}{3} (p \cdot q_1) (q_1 \cdot q_2) q_2^2 + \frac{8}{3} (p \cdot q_1) q_1^2 q_2^2 \]

\[+ \frac{8}{3} m^2 q_1^2 q_2^2 - \frac{8}{3} m^2 (q_1 \cdot q_2)^2. \] (3.6)
the results obtained in Refs. [20, 21], the corresponding numerical estimates are rather close to the VMD case (within the error attributed to the model dependence). We would like to stress again that both the VMD and ENJL form factors fail to correctly reproduce the QCD short-distance constraints discussed in [29]. As noted earlier, the question of how sensitive the results are to the VMD case (within the error attributed to the model dependence). We would like to reproduce the experimental rate.

For the four cases mentioned above, the form factors are given by [23, 39] (different choices for the global sign in the normalization of the form factors are of no relevance, cf. Eq. (3.4)]

\[
\mathcal{F}_{WZW}^{π^0\gamma\gamma^*}(q_1^2, q_2^2) = -\frac{N_C}{12\pi^2 F_π}, \quad (4.1)
\]

\[
\mathcal{F}_{VMD}^{π^0\gamma\gamma^*}(q_1^2, q_2^2) = -\frac{N_C}{12\pi^2 F_π (q_1^2 - M_V^2)(q_2^2 - M_V^2)}, \quad (4.2)
\]

\[
\mathcal{F}_{LMD}^{π^0\gamma\gamma^*}(q_1^2, q_2^2) = \frac{F_π}{3} \frac{q_1^2 + q_2^2 - c_V}{(q_1^2 - M_V^2)(q_2^2 - M_V^2)}, \quad (4.3)
\]

\[
\mathcal{F}_{LMD+V}^{π^0\gamma\gamma^*}(q_1^2, q_2^2) = \frac{F_π}{3} \frac{q_1^2 q_2^2 (q_1^2 + q_2^2) + h_1(q_1^2 + q_2^2)^2 + h_2 q_1^2 q_2^2 + h_5(q_1^2 + q_2^2) + h_7}{(q_1^2 - M_V^2)(q_2^2 - M_V^2)}, \quad (4.4)
\]

with (see the Appendix)

\[
c_V = \frac{N_C M^4}{4\pi^2 F_π^2}, \quad h_7 = -\frac{N_C M^4}{4\pi^2 F_π^2} M^4_{V_1} M^4_{V_2}. \quad (4.5)
\]

According to Ref. [10], the form factor \(\mathcal{F}_{π^0\gamma\gamma^*}(-Q^2, 0)\) with one photon on shell behaves like \(1/Q^2\) for large spacelike momenta, \(Q^2 = -q^2\). The experimental data are compatible with this behavior [37]. Whereas the LMD form factor does not have such a behavior, it can be reproduced with the LMD+V Ansatz, provided that \(h_1 = 0\). In Ref. [29], a fit of the LMD+V form factor to the CLEO data yielded \(h_1 = -0.01 \pm 0.16\) GeV\(^2\), \(h_5 = 6.88 \pm 0.61\) GeV\(^4\), or, if we fit \(h_5\) with \(h_1 = 0\), \(h_5 = 6.93 \pm 0.26\) GeV\(^4\). On the other hand, the parameter \(h_2\) that appears in Eq. (4.4) is not known. If we compare, in the context of Ref. [11], the expression for the \(\pi^0 \rightarrow e^+e^-\) partial width based on the LMD+V form factor to the experimental values [12, 13], we obtain a rather loose constraint on \(h_2\), \(|h_2| \lesssim 20\) GeV\(^2\), although slightly positive values for \(h_2\) seem to be preferred in order to reproduce the experimental rate.

A crucial observation at this stage is that all dependences on \(q_1 \cdot q_2\) in the numerators in \(\mathcal{F}_{π^0\gamma\gamma^*}^{LMD+V}(q_1^2, (q_1 + q_2)^2)\) can be canceled by the ones in the resonance propagators. Therefore, we can write

\[
\mathcal{F}_{π^0\gamma\gamma^*}^{π^0\gamma\gamma^*}(q_1^2, q_2^2) = \frac{F_π}{3} \left[ f(q_1^2) - \sum_{M_{V_i}} \frac{1}{q_2^2 - M^2_{V_i}} g_{M_{V_i}}(q_1^2) \right]. \quad (4.6)
\]

This representation, which seems to hold quite generally in the large-\(N_C\) limit of QCD (see the Appendix), obviously extends to the VMD and WZW form factors. The corresponding functions \(f(q^2)\) and \(g_{M_{V_i}}(q^2)\) can easily be worked out from the explicit expressions (4.1)–(4.4). They are displayed in Table 1. For the VMD and LMD form factors, the sum in Eq. (4.6) reduces to a single term, and we call the corresponding function \(g_{M_{V_i}}(q^2)\). In the case of the
Table 1: The functions \( f(q^2) \), \( g_{M_V}(q^2) \), and \( g(q^2; M^2) \) of Eqs. (4.6) and (4.7) for the different form factors in Eqs. (4.1)–(4.4).

|          | \( f(q^2) \)                                                                 | \( g_{M_V}(q^2) \)                                                                 |
|----------|-------------------------------------------------------------------------------|-------------------------------------------------------------------------------|
| WZW      | \(-\frac{N_C}{4\pi^2 F_\pi^2}\)                                             | 0                                                                             |
| VMD      | 0                                                                             | \( \frac{N_C}{4\pi^2 F_\pi^2} \frac{M^4_V}{q^2 - M^2_V} \)                  |
| LMD      | \( \frac{1}{q^2 - M^2_V} \)                                                  | \( -\frac{q^2 + M^2_V + c_V}{q^2 - M^2_V} \)                                |
| LMD + V  | \( \frac{q^2 + h_1}{(q^2 - M^2_{V_1})(q^2 - M^2_{V_2})} \)                  | \( g(q^2; M^2) = q^2 M^2(q^2 + M^2) + h_1(q^2 + M^2)^2 \)                     |
|          |                                                                               | \( + h_2 q^2 M^2 + h_5(q^2 + M^2)^2 + h_7 \)                                  |

LMD+V form factor, we have written

\[
\begin{align*}
g_{M_{V_1}}(q^2) &= \frac{1}{M^2_{V_2} - M^2_{V_1}} \frac{g(q^2; M^2_{V_1})}{(q^2 - M^2_{V_1})(q^2 - M^2_{V_2})}, \\
g_{M_{V_2}}(q^2) &= \frac{1}{M^2_{V_1} - M^2_{V_2}} \frac{g(q^2; M^2_{V_2})}{(q^2 - M^2_{V_1})(q^2 - M^2_{V_2})}. 
\end{align*}
\]

(4.7)

with the function \( g(q^2; M^2) \) given in the table. The expression (4.6) also holds for \( q^2_1 = 0 \), which implies that \( f(0) = 0 \) in order to reproduce the asymptotic \( 1/Q^2 \) behavior of \( F_\pi^0 \gamma^+ \gamma^- (-Q^2, 0) \).

As noted above, this fails to be the case for the LMD Ansatz \( f_{LMD}(0) \neq 0 \), and holds for the LMD+V Ansatz provided \( h_1 = 0 \). Note that the limit \( Q^2_2 \to \infty \) with \( Q^2_1 \) fixed can actually not be studied with the operator product expansion (see the discussion in Ref. [29]).

Thus, using partial fractions, we can write the product of the form factors in the two contributions to the right-hand side of Eq. (3.4) as follows:

\[
\begin{align*}
\mathcal{F}_{\pi^0 \gamma^+ \gamma^-}(q^2_1, (q_1 + q_2)^2) \mathcal{F}_{\pi^0 \gamma^+ \gamma^-}(q^2_2, 0) &= \frac{F_\pi}{3} \mathcal{F}_{\pi^0 \gamma^+ \gamma^-}(q^2_2, 0) \left( f(q^2_1) \right) \\
&- \sum_{M_{V_i}} g_{M_{V_i}}(q^2_1) \left( (q_1 + q_2)^2 - M^2_{V_i} \right) 
\end{align*}
\]

(4.8)

and

\[
\begin{align*}
\mathcal{F}_{\pi^0 \gamma^+ \gamma^-}(q^2_1, q^2_2) \mathcal{F}_{\pi^0 \gamma^+ \gamma^-}((q_1 + q_2)^2, 0) &= \frac{F_\pi}{3} \mathcal{F}_{\pi^0 \gamma^+ \gamma^-}(q^2_1, q^2_2) 
\end{align*}
\]
\[
\times \left( \frac{1}{(q_1 + q_2)^2 - M^2} \left[ f(0) + \sum_{M V_i} \frac{g_{M V_i}(0)}{(M^2 - M^2 V_i)} \right] + \sum_{M V_i} \left[ \frac{g_{M V_i}(0)}{(q_1 + q_2)^2 - M^2 V_i} (M^2 V_i - M^2) \right] \right). \tag{4.9}
\]

Note that in Eq. (4.9) all terms contain a propagator \(1/[(q_1 + q_2)^2 - M^2]\), \(M = M_\pi\) or \(M_V\), in contrast to the prefactor of \(T_1\) shown in Eq. (4.8).

Since the form factors discussed above go to a constant for small momenta, the integrals in Eq. (3.4) are infrared finite. The behavior at large momenta also ensures the ultraviolet convergence of these integrals, except, of course, for the constant form factor \(F_{W^0 Z W}^\gamma\). However, the integral involving \(T_2(q_1, q_2; p)\) is finite in the latter case also (this has been observed before [21], and we have confirmed it numerically as well).

5 Angular integrations

As we shall show in this section, for form factors \(F_{\pi^0 \gamma^* \gamma^*}(q_1^2, q_2^2)\) that have the general form given by Eq. (4.6), it is possible to perform all angular integrations in the two-loop integral of Eq. (3.4) using the technique of Gegenbauer polynomials (hyperspherical approach); see Refs. [44, 45, 35]. In order to do so, we perform a Wick rotation of the momenta \(q^\mu_1, q^\mu_2, \text{ and } p^\mu\), denoting by capital letters the rotated Euclidean momenta, with \(Q^2_1 = -q^2_1, P^2 = -m^2, \text{ etc.}\)

Let us mention that in the language of Refs. [45, 37] the loop integrals we have to deal with are planar. We also note that these integrals are of the two-loop self-energy type. Other techniques that lead to either one-dimensional dispersive or two-dimensional integral representations [46] could a priori also be relevant in the present context, although probably not without further specifying the functions \(f(q^2)\) and \(g_{M V_i}(q^2)\) in Eq. (4.6).

5.1 Properties of Gegenbauer polynomials

Let us briefly summarize some basic properties of the Gegenbauer polynomials; see also Refs. [44, 48]. We write the measure of the four-dimensional sphere as follows (Euclidean momenta):

\[
d^4 K = K^3 dK d\Omega(\hat{K}), \quad \int d\Omega(\hat{K}) = 2\pi^2. \tag{5.1}
\]

The generating function of the Gegenbauer polynomials\(^2\) \(C_n(x)\) is given by

\[
\frac{1}{z^2 - 2xz + 1} = \sum_{n=0}^{\infty} z^n C_n(x), \quad -1 \leq x \leq 1, \quad |z| < 1. \tag{5.2}
\]

From Eq. (5.2) we immediately obtain the following property under parity transformations \(C_n(-x) = (-1)^n C_n(x)\). Furthermore we get \(C_n(1) = n + 1.\) The Gegenbauer polynomials obey

\footnote{We shall need only the special case \(C_n(x) \equiv C_n^{(1)}(x)\).}
the orthogonality conditions

\[
\int d\Omega(\hat{K}) \, C_n(\hat{Q}_1 \cdot \hat{K}) \, C_m(\hat{K} \cdot \hat{Q}_2) = 2\pi^2 \frac{\delta_{nm}}{n+1} C_n(\hat{Q}_1 \cdot \hat{Q}_2), \\
\int d\Omega(\hat{K}) \, C_n(\hat{Q} \cdot \hat{K}) \, C_m(\hat{K} \cdot \hat{Q}) = 2\pi^2 \delta_{nm}.
\]

(5.3)

where, for instance, \( \hat{Q}_1 \cdot \hat{K} \) is the cosine of the angle between the four-dimensional vectors \( Q_1 \) and \( K \). Some low-order cases of the polynomials are given by \( C_0(x) = 1 \), \( C_1(x) = 2x \), \( C_2(x) = 4x^2 - 1 \) and therefore \( x = C_1(x)/2 \), \( x^2 = [C_2(x) + C_0(x)]/4 \).

From the generating function, we obtain the following representation of the propagators in Euclidean space:

\[
\frac{1}{(K-L)^2 + M^2} = \frac{Z_{KL}^M}{|K||L|} \sum_{n=0}^{\infty} (Z_{KL}^M)^n C_n(\hat{K} \cdot \hat{L}), \\
\frac{1}{(K+L)^2 + M^2} = \frac{Z_{KL}^M}{|K||L|} \sum_{n=0}^{\infty} (-Z_{KL}^M)^n C_n(\hat{K} \cdot \hat{L}), \\
Z_{KL}^M = \frac{K^2 + L^2 + M^2 - \sqrt{(K^2 + L^2 + M^2)^2 - 4K^2L^2}}{2|K||L|}.
\]

(5.4)

(5.5)

(5.6)

Note that we have to choose the negative sign in front of the square root in \( Z_{KL}^M \) in order that \( |Z_{KL}^M| < 1 \). For a massless propagator these expressions simplify as follows:

\[
\frac{1}{(K-L)^2} = \theta \left( 1 - \frac{|L|}{|K|} \right) \frac{1}{K^2} \sum_{n=0}^{\infty} \left( \frac{|L|}{|K|} \right)^n C_n(\hat{K} \cdot \hat{L}) \\
+ \theta \left( 1 - \frac{|K|}{|L|} \right) \frac{1}{L^2} \sum_{n=0}^{\infty} \left( \frac{|K|}{|L|} \right)^n C_n(\hat{K} \cdot \hat{L}).
\]

(5.7)

### 5.2 Basic angular integrals

Let us introduce the following abbreviations for the propagators in the loop integral in Eq. (3.4):

\[
D_1 = Q_1^2, \quad D_2 = Q_2^2, \quad D_3 = (Q_1 + Q_2)^2, \quad D_4 = (P + Q_1)^2 + m^2, \\
D_5 = (P - Q_2)^2 + m^2, \quad D_M = (Q_1 + Q_2)^2 + M^2.
\]

(5.8)

where \( M \) denotes either the pion mass or the mass of some vector resonance. For \( M \to 0 \) we recover the photon propagator \( D_3 \).

By rewriting the scalar products \( P \cdot Q_1, P \cdot Q_2, \) and \( Q_1 \cdot Q_2 \) in terms of the propagators \( D_4, D_5, \) and \( D_3 \), respectively, we can remove some of the terms in the numerator of \( T_n/\langle D_1 D_2 D_3 D_4 D_5 \rangle \), \( n = 1, 2, \) in Eq. (3.3). If we multiply by the form factors, taking into account their general form from Eqs. (4.8) and (4.9), and use a partial fraction decomposition, we finally obtain the following basic angular integrals [apart from the trivial one \( \int d\Omega(\hat{Q}_1)d\Omega(\hat{Q}_2) = 4\pi^4 \)], which
can all be performed using the method of Gegenbauer polynomials (from now on, we write $Q_1$ instead of $|Q_1|$, etc.):

\[ I_1^M = \int \frac{d\Omega(\hat{Q}_1) \, d\Omega(\hat{Q}_2)}{2\pi^2} \frac{1}{D_M D_4 D_5} = -\frac{1}{P^2 Q_1^2 Q_2^2} \ln \left( 1 - Z_{Q_1}^M Z_{PQ_1}^m Z_{PQ_2}^m \right), \]

\[ I_2 = \int \frac{d\Omega(\hat{Q}_1) \, d\Omega(\hat{Q}_2)}{2\pi^2} \frac{1}{D_4 D_5} Z_{PQ_1}^m Z_{PQ_2}^m \frac{Q_1^2 - R_1^m}{Q_2^2 - R_2^m}, \]

\[ I_3^M = \int \frac{d\Omega(\hat{Q}_1) \, d\Omega(\hat{Q}_2)}{2\pi^2} \frac{1}{D_M D_4} Z_{Q_1}^M Z_{PQ_1}^m \frac{(M^2 + Q_1^2 + Q_2^2 - R^M)(Q_1^2 - R_1^m)}{4m^2 Q_1^2 Q_2^2}, \]

\[ I_4^M = \int \frac{d\Omega(\hat{Q}_1) \, d\Omega(\hat{Q}_2)}{2\pi^2} \frac{1}{D_M D_5} Z_{Q_1}^M Z_{PQ_2}^m \frac{(M^2 + Q_1^2 + Q_2^2 - R^M)(Q_2^2 - R_2^m)}{4m^2 Q_1^2 Q_2^2}, \]

\[ I_5 = \int \frac{d\Omega(\hat{Q}_1) \, d\Omega(\hat{Q}_2)}{2\pi^2} \frac{1}{D_4} \frac{Q_1^2 - R_1^m}{P Q_1}, \]

\[ I_6 = \int \frac{d\Omega(\hat{Q}_1) \, d\Omega(\hat{Q}_2)}{2\pi^2} \frac{1}{D_5} \frac{Q_2^2 - R_2^m}{P Q_2}, \]

\[ I_7^M = \int \frac{d\Omega(\hat{Q}_1) \, d\Omega(\hat{Q}_2)}{2\pi^2} \frac{1}{D_M} \frac{Z_{Q_1}^M}{Q_1 Q_2} \frac{M^2 + Q_1^2 + Q_2^2 - R^M}{2Q_1^2 Q_2^2}, \]

(5.9)

where

\[ R^M = \sqrt{(M^2 + Q_1^2 + Q_2^2)^2 - 4Q_1^2 Q_2^2}, \]

\[ R_i^m = \sqrt{Q_i^4 + 4m^2 Q_i^2}, \quad i = 1, 2. \]

(5.10)

We shall also need the angular integrals where the massive propagator $D_M$ is replaced by the massless one $D_3$. We have set $P^2 = -m^2$ in the explicit expressions involving the square roots. Since the external momentum $P$ flows only through the massive fermion propagators,
we do not need to deform the integration contour for the radial integrals over $Q_1^2$ and $Q_2^2$ (see the discussion in Refs. [45, 35]). Note that $I_5$ and $I_6$ depend only on one variable, $Q_1^2$ and $Q_2^2$, respectively, whereas the expression for $I_2$ factorizes into a product of two single-variable functions.

Other angular integrals that occur during the calculation, involving the factors $D_3/(D_4D_5)$, $D_4/(D_3D_5)$, $D_5/(D_3D_4)$, and $D_3/D_4$, can be reduced to the basic ones, by noting that $D_3 = Q_1^2 + Q_2^2 + 2Q_1 \cdot Q_2$, $D_4 = Q_1^2 + 2P \cdot Q_1$, and $D_5 = Q_2^2 - 2P \cdot Q_2$, and with the help of the identities

\[
\begin{align*}
(Z_{Q_1Q_2}^M)^2 &= \frac{M^2 + Q_1^2 + Q_2^2}{Q_1Q_2} Z_{Q_1Q_2}^M - 1, \\
(Z_{PQq}^m)^2 &= \frac{m^2 + P^2 + Q_i^2}{PQ_i} Z_{PQq}^m - 1. 
\end{align*}
\]

In this way one obtains, for $P^2 = -m^2$,

\[
\begin{align*}
\int \frac{d\Omega(\hat{Q}_1) d\Omega(\hat{Q}_2)}{2\pi^2} \frac{Q_1 \cdot Q_2}{D_4D_5} &= -\frac{1}{4} \left( \frac{Z_{PQq}^m}{PQ_i} \right)^2 \left( \frac{Z_{PQq}^m}{PQ_i} \right)^2, \\
&= \frac{Q_2^2 I_2 - Q_1^2 I_5 - Q_2^2 I_6 + 1}{4m^2}, \\
\int \frac{d\Omega(\hat{Q}_1) d\Omega(\hat{Q}_2)}{2\pi^2} \frac{P \cdot Q_1}{D_3D_5} &= -\frac{1}{4} \left( \frac{Z_{Q_1Q_2}^M}{Q_2^2} \right)^2, \\
&= -\frac{[M^2 + Q_1^2 + Q_2^2]Q_2^2 I_4^M + (M^2 + Q_1^2 + Q_2^2) I_1^M + Q_2^2 I_6 - 1}{4Q_2^2}, \\
\int \frac{d\Omega(\hat{Q}_1) d\Omega(\hat{Q}_2)}{2\pi^2} \frac{P \cdot Q_2}{D_3D_4} &= \frac{1}{4} \left( \frac{Z_{Q_1Q_2}^M}{Q_1^2} \right)^2, \\
&= \frac{[M^2 + Q_1^2 + Q_2^2]Q_2^2 I_3^M - (M^2 + Q_1^2 + Q_2^2) I_1^M - Q_1^2 I_5 + 1}{4Q_1^2}, \\
\int \frac{d\Omega(\hat{Q}_1) d\Omega(\hat{Q}_2)}{2\pi^2} \frac{Q_1 \cdot Q_2}{D_4} &= 0. 
\end{align*}
\]

(5.12)

### 6 Two-dimensional integral representation for $a_{\mu}^{\text{LbyL}}; \pi^0$

After having performed the angular integrations, the pion-exchange contribution to $g - 2$ can be written in a two-dimensional integral representation as follows:

\[
\begin{align*}
a_{\mu}^{\text{LbyL}; \pi^0} &= \left( \frac{\alpha}{\pi} \right)^3 \left[ a_{\mu}^{\text{LbyL}; \pi^0(1)} + a_{\mu}^{\text{LbyL}; \pi^0(2)} \right], \\
a_{\mu}^{\text{LbyL}; \pi^0(1)} &= \int_0^\infty dQ_1 \int_0^\infty dQ_2 \left[ w_{f_1}(Q_1, Q_2) f^{(1)}(Q_1^2, Q_2^2) \right], \\
\end{align*}
\]

(6.1)
Note (see Table 1) that in the WZW model we obtain
\[ g_{\mu}^{(1)}(Q_1, Q_2) \] given by
\[ g_{\mu}^{(2)}(Q_1, Q_2) \]
with [see Eqs. (1.8)–(1.9)]
\[ f^{(1)}(Q_1^2, Q_2^2) = \frac{F_\pi}{3} f(-Q_1^2) \mathcal{F}_{\pi^0,\gamma^*}(-Q_2^2, 0), \]
\[ g_{M\nu}^{(1)}(Q_1^2, Q_2^2) = \frac{F_\pi}{3} \frac{g_{M\nu}(-Q_1^2)}{M_{\nu}^2} \mathcal{F}_{\pi^0,\gamma^*}(-Q_2^2, 0), \]
\[ g_{M\nu}^{(2)}(Q_1^2, Q_2^2) = \frac{F_\pi}{3} \mathcal{F}_{\pi^0,\gamma^*}(-Q_1^2, -Q_2^2) \left( f(0) + \sum_{M_{\nu}} \frac{g_{M\nu}(0)}{M_{\nu}^2 - M_\pi^2} \right), \]
\[ g_{M\nu}^{(2)}(Q_1^2, Q_2^2) = \frac{F_\pi}{3} \mathcal{F}_{\pi^0,\gamma^*}(-Q_1^2, -Q_2^2) \frac{g_{M\nu}(0)}{M_\pi^2 - M_{\nu}^2}. \]

Note (see Table 1) that in the WZW model we obtain \( g_{M\nu}^{(1)}(Q_1^2, Q_2^2) \equiv 0, g_{M\nu}^{(2)}(Q_1^2, Q_2^2) \equiv 0, \) since \( g^{WZW}(-Q_2^2) \equiv 0. \) On the other hand, for the VMD model we have the simplifications \( f^{(1)}(Q_1^2, Q_2^2) \equiv 0, g_{M\nu}^{(2)}(Q_1^2, Q_2^2) = -g_{M\nu}^{(2)}(Q_1^2, Q_2^2), \) since \( f^{VMD}(-Q_1^2) \equiv 0. \)

The universal [for the class of form factors that have a representation of the type (1.6)] weight functions in Eqs. (6.2) and (6.3) are given by
\[ w_{f_1}(Q_1, Q_2) = \frac{\pi^2}{Q_1^2 + M_\pi^2} \frac{1}{6m^2Q_1Q_2} \left\{ -4(2m^2 - Q_2^2)(Q_1^2 - Q_2^2)^2 \right. \]
\[ \times \ln \left[ 1 + \frac{(Q_1^2 + Q_3^2 - R^0)(Q_1^2 - R_1^m)(Q_2^2 - R_2^m)}{8m^2Q_1^2Q_2^2} \right] \]
\[ -4m^2Q_1^2Q_2^2 - \frac{1}{m^2} Q_1^2 Q_4^4 + \frac{1}{2m^4} Q_1^4 Q_2^4 + Q_1^6 \]
\[ -3Q_1^2 Q_2^2 - Q_1^2 Q_2^4 + Q_6^2 \left( 1 - \frac{R_1^m}{Q_1^2} \right) \]
\[ -(Q_1^2 - Q_2^2)^2 R_1^m \left( 1 - \frac{R_1^m}{Q_1^2} \right) \]
\[ + \frac{1}{m^2} Q_1^2 (Q_2^4 - 4m^4) R_1^m - \frac{1}{2m^4} Q_1^4 (Q_2^2 - 2m^2)^2 R_2^m \]
\[ + \frac{1}{2m^4} (Q_2^2 - 2m^2)(Q_2^4 Q_2^2 - 2m^2 Q_2^2 - 2m^2 Q_2^2) R_2^m R_2^m \right \}, \]
\[ w_{g_1}(M, Q_1, Q_2) = \frac{\pi^2}{Q_1^2 + M_\pi^2} \frac{1}{6m^2Q_1Q_2} \left\{ -4(2m^2 - Q_2^2)(Q_1^2 - Q_2^2)^2 \right. \]
\[ \times \ln \left[ 1 + \frac{(Q_1^2 + Q_3^2 - R^0)(Q_1^2 - R_1^m)(Q_2^2 - R_2^m)}{8m^2Q_1^2Q_2^2} \right] \]
\[+4(2m^2 - Q_2^2)[M^4 + (Q_1^2 - Q_2^2)^2 + 2M^2(Q_1^4 + Q_2^4)]\]
\[\times \ln \left[ 1 + \frac{(M^2 + Q_1^2 + Q_2^2 - R^M)(Q_1^2 - R_1^m)(Q_2^2 - R_2^m)}{8m^2 Q_1^2 Q_2^2} \right]^{-}\]
\[-M^6 + 3M^4Q_1^2 + 3M^4Q_2^2 + 2M^2Q_1^2Q_2^2 + 3M^2Q_1^2\]
\[-\frac{M^2}{m^2} Q_1^2 Q_2^2 \left(1 - \frac{R_1^m}{Q_1^2} \right)\]
\[-Q_1^2 - Q_2^2)^2 R^0 \left(1 - \frac{R_1^m}{Q_1^2} \right) - \frac{M^2}{m^2} Q_1^2 Q_2^2 \left(1 - \frac{R_1^m}{Q_1^2} \right)\]
\[+M^4 + 2M^2Q_1^2 + 2M^2Q_2^2 + Q_1^2 + Q_2^2 - 2Q_1^2Q_2^2\]
\[\times R^M \left(1 - \frac{R_1^m}{Q_1^2} \right) \right\} \tag{6.6}\]
\[w_{g_2}(M, Q_1, Q_2) = \pi^2 \frac{1}{6m^2 M^2 Q_1 Q_2} \left\{ 4[m^2(Q_1^2 - Q_2^2) + 2Q_1^2 Q_2^2(Q_1^2 - Q_2^2)]\right\}
\[\times \ln \left[ 1 + \frac{(Q_1^2 + Q_2^2 - R^0)(Q_1^2 - R_1^m)(Q_2^2 - R_2^m)}{8m^2 Q_1^2 Q_2^2} \right]^{-}\]
\[+4[M^4m^2 + (Q_1^2 - Q_2^2)(-2Q_1^2Q_2^2 + m^2(Q_1^2 - Q_2^2)\]
\[+2M^2[Q_1^2Q_2^2 + m^2(Q_1^2 + Q_2^2)\]
\[\times \ln \left[ 1 + \frac{(M^2 + Q_1^2 + Q_2^2 - R^M)(Q_1^2 - R_1^m)(Q_2^2 - R_2^m)}{8m^2 Q_1^2 Q_2^2} \right]^{-}\]
\[+M^4Q_1^2 + 2M^2Q_1^4 + M^4Q_2^2 - M^2Q_1^2Q_2^2 + 2M^2Q_2^4\]
\[+Q_1^2(Q_1^2 + Q_2^2) R^0 \left(1 - \frac{R_1^m}{Q_1^2} \right) + Q_2^2 Q_2^2 R^0 \left(1 - \frac{R_2^m}{Q_2^2} \right)\]
\[-Q_1^2(M^2 + Q_1^2 + Q_2^2) R^M \left(1 - \frac{R_1^m}{Q_1^2} \right) - M^2(M^2 + 2Q_1^2 + Q_2^2) R_1^m\]
\[-Q_2^2(M^2 - 3Q_1^2 + Q_2^2) R^M \left(1 - \frac{R_2^m}{Q_2^2} \right) - M^2(M^2 - 3Q_1^2 + 2Q_2^2) R_2^m\]
\[-M^2 R_1^m R_2^m \right\} \tag{6.7}\]

where
\[R^0 \equiv R^M=0 = \sqrt{(Q_1^2 + Q_2^2)^2 - 4Q_1^2 Q_2^2} \tag{6.8}\]

Although the analytical expressions for the weight functions look quite complicated and involve terms with different signs and of different sizes, the sum of all terms leads to rather smooth functions of the two variables \(Q_1\) and \(Q_2\), as can be seen from the plots in Fig. 3. We have not shown the corresponding plots for \(w_{g_1}(M_{V_2}, Q_1, Q_2)\) and \(w_{g_2}(M_{V_2}, Q_1, Q_2)\), since they look qualitatively similar. The contribution of the former is suppressed in the integral by the factor \(1/M_{V_2}^2\) in \(g_{M_{V_2}}^{(1)}\), whereas the latter weight function scales as \(1/M_{V_2}^2\) for large \(M_{V_2}\).
Figure 3: The weight functions of Eqs. (6.5)–(6.7). Note the different ranges of $Q_i$ in the subplots. The functions $w_{f_1}$ and $w_{g_1}$ are positive definite and peaked in the region $Q_1 \sim Q_2 \sim 0.5$ GeV. Note, however, the tail in $w_{f_1}$ in the $Q_1$ direction for $Q_2 \sim 0.2$ GeV. The functions $w_{g_2}(M_\pi, Q_1, Q_2)$ and $w_{g_2}(M_V, Q_1, Q_2)$ take both signs, but their magnitudes remain small as compared to $w_{f_1}(Q_1, Q_2)$ and $w_{g_1}(M_V, Q_1, Q_2)$. We have used $M_V = M_\rho = 769$ MeV.
The functions $w_{f_1}$ and $w_{g_1}$ are positive and concentrated around momenta of the order of 0.5 GeV. This behavior was already observed in Ref. [22] by varying the upper bound of the integrals (an analogous analysis is contained in Ref. [20]). Note, however, the tail in $w_{f_1}$ in the $Q_1$ direction for $Q_2 \sim 0.2$ GeV. On the other hand, the function $w_{g_2}$ has positive and negative contributions in that region, which will lead to a strong cancellation in the corresponding integrals, provided they are multiplied by a positive function composed of the form factors (see the numerical results below).

As can be seen from the plots, and checked analytically, the weight functions vanish for small momenta. Therefore, as already noted before, the integrals are infrared finite. For large momenta the weight function $w_{f_1}(Q_1, Q_2)$ and $w_{g_1}(M, Q_1, Q_2)$ have the following behavior:

$$
\lim_{Q_1 \to \infty} w_{f_1}(Q_1, Q_2) = \pi^2 \frac{[Q_2^5 - Q_2(Q_2^2 - 2m^2)R_{1m}]}{m^2(Q_2^2 + M_\pi^2)Q_1} + O\left(\frac{1}{Q_1^4}\right), \quad Q_2 \text{ fixed},
$$

$$
\lim_{Q_2 \to \infty} w_{f_1}(Q_1, Q_2) = \pi^2 \frac{2Q_1^2(Q_2^2 - 12m^2) - 2Q_1(Q_2^2 - 14m^2)R_{1m}}{9m^2Q_2^3} + O\left(\frac{1}{Q_2^5}\right), \quad Q_1 \text{ fixed},
$$

$$
\lim_{Q \to \infty} w_{f_1}(Q, Q) = \pi^2 \frac{13m^2}{3Q^2} + O\left(\frac{1}{Q^4}\right), \quad (6.9)
$$

and

$$
\lim_{Q_1 \to \infty} w_{g_1}(M, Q_1, Q_2) = \pi^2 \frac{M^2[Q_2^5 - Q_2(Q_2^2 - 2m^2)R_{1m}]}{m^2(Q_2^2 + M_\pi^2)Q_1^4} + O\left(\frac{1}{Q_1^4}\right), \quad Q_2 \text{ fixed},
$$

$$
\lim_{Q_2 \to \infty} w_{g_1}(M, Q_1, Q_2) = \pi^2 \frac{M^2[-18m^2Q_1^3 + 5Q_1^5 + Q_1(28m^2 - 5Q_1^2)R_{1m}]}{9m^2Q_2^5}
+ O\left(\frac{1}{Q_2^5}\right), \quad Q_1 \text{ fixed},
$$

$$
\lim_{Q \to \infty} w_{g_1}(M, Q, Q) = \pi^2 \frac{62M^2m^2}{3Q^4} + O\left(\frac{1}{Q^5}\right). \quad (6.10)
$$

Finally, the weight function $w_{g_2}(M, Q_1, Q_2)$ behaves as follows:

$$
\lim_{Q_1 \to \infty} w_{g_2}(M, Q_1, Q_2) = \pi^2 \frac{[-Q_2^3(Q_2^2 + m^2) + Q_2(Q_2^2 - m^2)R_{2m}]}{3m^2Q_1^4} + O\left(\frac{1}{Q_1^4}\right), \quad Q_2 \text{ fixed},
$$

$$
\lim_{Q_2 \to \infty} w_{g_2}(M, Q_1, Q_2) = \pi^2 \frac{[-Q_1^3(Q_1^2 + 5m^2) + Q_1(Q_1^2 + 3m^2)R_{1m}]}{3m^2Q_2^4}
+ O\left(\frac{1}{Q_2^5}\right), \quad Q_1 \text{ fixed},
$$

$$
\lim_{Q \to \infty} w_{g_2}(M, Q, Q) = \pi^2 \frac{16m^4}{9Q^4} + O\left(\frac{1}{Q^5}\right). \quad (6.11)
$$

Since $g_{M_\pi}^{(1)}(Q_1, Q_2) \equiv 0$ in the WZW model, the corresponding integral $a^{\mu}_{\pi}(\pi, \pi)_{M_\pi}$ involves only $w_{f_1}$ and diverges as $\mathcal{C} \ln^2 \Lambda$ for some ultraviolet cutoff $\Lambda$ [49]. Varying the cutoff $\Lambda$ in our numerical integration between 2 and 50 GeV, we obtain $\mathcal{C} \sim 0.025$. Since all form factors tend to the WZW model for $M_V \to \infty$, the results for $a^{\mu}_{\pi}(\pi, \pi)_{M_\pi}$ in the different models should
all scale as $C \ln^2 M_V$ for large resonance masses, with the same coefficient $C$ in front of the log-squared term as in the WZW model. We have numerically confirmed this observation for the VMD and LMD form factors. In fact, for the WZW form factor, the region $Q_1 > Q_2$ in the integral produces the $\ln^2 \Lambda$ term. Using the asymptotic expansion for large $Q_1$ of $w_f(Q_1, Q_2)$ in Eq. (6.9), one can extract the coefficient of the leading-logarithm term from the integral, with the result

$$C = 3 \left( \frac{N_C}{12\pi} \right)^2 \left( \frac{m}{F_\pi} \right)^2. \quad (6.12)$$

The corresponding value $C = 0.0248$ (for $N_C = 3$ and the input values for $m$ and $F_\pi$ given in the next section) is in good agreement with the numerical determination quoted above. On the other hand, $a_{\mu}^{\text{w.l.,\pi^0}}$ must vanish as the lepton mass $m$ tends to zero. Numerically, we find that it is indeed the case; see the next section.

Before we present our estimate for the pion-exchange contribution to the muon anomalous magnetic moment in the next section, we would like to stress again that the two-dimensional integral representation (6.1)–(6.3) is valid for any form factor that can be written as shown in Eq. (4.6). If in the future experimental data for the off-shell form factor $F_\pi^\rho\gamma\gamma\gamma(\gamma)_{Q_1, Q_2}$ become available, the region below 1 GeV should also be measured with high precision. Provided the data can be fitted with a representation of the form factor belonging to the class we have discussed, Eqs. (6.1)–(6.3) can then be used to improve the present estimates. In this sense, our integral representation is similar to the familiar one that connects the vacuum polarization contribution to $g - 2$ and the experimental cross section $\sigma(e^+e^- \rightarrow \text{hadrons})$ [10].

7 Numerical results

The weight functions in Eqs. (6.3)–(6.7) are composed of rational functions, square roots and logarithms. The combinations of form factors from Eq. (6.4) that enter in the integrals are rational functions as well. It might therefore be possible to perform one or even both integrations over $Q_1$ and $Q_2$ analytically for some parts of our expressions. On the other hand, the numerical evaluation of the two-dimensional integral representation for $a_{\mu}^{\text{w.l.,\pi^0}}$ in Eqs. (6.2) and (6.3) poses no real problems. The corresponding results for the different form factors discussed earlier can be found in Table 2. Apart from the masses of the muon $m = m_\mu = 105.66$ MeV and the pion $M_\pi = 134.98$ MeV, we have used the following input values: $\alpha = 1/137.03599976, F_\pi = 92.4$ MeV, $M_{V_1} = M_\rho = 769$ MeV, and $M_{V_2} = M_\rho' = 1465$ MeV [43]. The errors from the numerical integration are much smaller than the last digits given in the table.

As already announced in the Introduction we obtain a different sign for $a_{\mu}^{\text{w.l.,\pi^0}}$ as compared to the latest two calculations [20, 21, 22]. This is immediately visible from the plots of the weight functions $w_{f_1}$ and $w_{g_1}$, which are positive and which are multiplied by positive functions $f^{(1)}_{M_V}$ and $g^{(1)}_{M_V}$, at least for the WZW, VMD, and LMD form factors. We shall further discuss this important point in the next section. On the other hand the contribution $a_{\mu}^{\text{w.l.,\pi^0}(2)}$ is highly

\[3\text{For the analytical integration it might be convenient to extract the factor } Q_1 Q_2 \text{ from the weight functions and to rewrite the integrals in terms of } Q_1^2 \text{ and } Q_2^2.\]
Table 2: Results for the terms $a_{\mu}^{\text{LbyL}, \pi^0(1)}$, $a_{\mu}^{\text{LbyL}, \pi^0(2)}$ and for the pion-exchange contribution to the anomalous magnetic moment $a_{\mu}^{\text{LbyL}, \pi^0}$ according to Eq. (6.1) for the different form factors considered. In the WZW model we used a cutoff of 1 GeV in the first contribution, whereas the second term is ultraviolet finite. In the LMD+V Ansatz we used $h_1 = 0$ GeV$^2$ and $h_5 = 6.93$ GeV$^4$.

| Form factor      | $a_{\mu}^{\text{LbyL}, \pi^0(1)}$ | $a_{\mu}^{\text{LbyL}, \pi^0(2)}$ | $a_{\mu}^{\text{LbyL}, \pi^0} \times 10^{10}$ |
|------------------|----------------------------------|----------------------------------|-----------------------------------------------|
| WZW              | 0.095                            | 0.0020                           | 12.2                                          |
| VMD              | 0.044                            | 0.0013                           | 5.6                                           |
| LMD              | 0.057                            | 0.0014                           | 7.3                                           |
| LMD+V ($h_2 = -10$ GeV$^2$) | 0.049                            | 0.0013                           | 6.3                                           |
| LMD+V ($h_2 = 0$ GeV$^2$) | 0.045                            | 0.0013                           | 5.8                                           |
| LMD+V ($h_2 = 10$ GeV$^2$) | 0.041                            | 0.0013                           | 5.3                                           |

suppressed and very stable with respect to the various form factors (with the obvious exception of the WZW case). As noted earlier, the corresponding integral converges even for a constant form factor as in the WZW model. As far as the numerics is concerned, the absolute value for $a_{\mu}^{\text{LbyL}, \pi^0}$ for the VMD form factor is identical to the results quoted in Refs. [19, 20, 21, 22], whereas the one for the LMD form factor agrees, up to the sign, with the result for the form factor number 4 given in Ref. [39].

Since the results for the cases of the LMD+V and VMD form factors are very similar, one might conclude that imposing the asymptotic $1/Q^2$ behavior of $F_{\pi^0, \gamma, \gamma}(-Q^2, 0)$ for large spacelike momenta is more important than reproducing the QCD short-distance behavior for $Q_1^2 \to \infty$ and $Q_2^2 \to \infty$, as does the LMD form factor. However, the slightly higher result for $a_{\mu}^{\text{LbyL}, \pi^0}$ in the case of the LMD form factor might also be due to the fact that in this instance the slope of the form factor at the origin, as obtained, for instance, by the CELLO Collaboration [50], is not well reproduced, in contrast to the LMD+V and VMD cases (see the discussion in Ref. [23]). As noted in connection with the plots of the weight functions, it is mainly the region $Q_i \lesssim 1$ GeV that contributes in the integrals.

For illustration, we decompose, in the cases of the LMD and LMD+V form factors, the total numerical result of Table 2 according to the integrals corresponding to the different weight functions appearing in the sums in Eqs. (6.2) and (6.3),

\[
\begin{align*}
    a_{\mu}^{\text{LbyL}, \pi^0(1)} &= 0.015 + 0.042 + 0 = 0.057 \\
    a_{\mu}^{\text{LbyL}, \pi^0(2)} &= 0.0016 - 0.0002 + 0 = 0.0014 \\
    a_{\mu}^{\text{LbyL}, \pi^0(1)} &= 0.0026 + 0.0448 - 0.0026 = 0.045 \\
    a_{\mu}^{\text{LbyL}, \pi^0(2)} &= 0.0015 - 0.0002 - 1 \times 10^{-6} = 0.0013 \\
\end{align*}
\]

LMD

\[
\begin{align*}
    a_{\mu}^{\text{LbyL}, \pi^0(1)} &= 0.015 + 0.042 + 0 = 0.057 \\
    a_{\mu}^{\text{LbyL}, \pi^0(2)} &= 0.0016 - 0.0002 + 0 = 0.0014 \\
    a_{\mu}^{\text{LbyL}, \pi^0(1)} &= 0.0026 + 0.0448 - 0.0026 = 0.045 \\
    a_{\mu}^{\text{LbyL}, \pi^0(2)} &= 0.0015 - 0.0002 - 1 \times 10^{-6} = 0.0013 \\
\end{align*}
\]

LMD+V ($h_2 = 0$)

One observes that in both cases the term involving the weight function $w_{\gamma_1}(M_{\gamma_1}, Q_1, Q_2)$ gives the main contribution to the final result.

For a fixed value of $h_2$ in the LMD+V form factor, our results are rather stable under
the variation of the other parameters. For instance, varying \( M_{V_1}, M_{V_2}, \) and \( h_5 \) by \( \pm 20 \text{ MeV}, \pm 25 \text{ MeV}, \) and \( \pm 0.5 \text{ GeV}^4 \), respectively, the result for \( a_{\mu}^{\text{LbyL,}\pi^0} \) changes by \( \pm 0.2 \times 10^{-10} \). On
the other hand, if all other parameters are kept fixed, our result depends almost linearly on \( h_2 \), at least for \( |h_2| < 20 \text{ GeV}^2 \). In this range \( a_{\mu}^{\text{LbyL,}\pi^0} \) changes by \( \pm 0.9 \times 10^{-10} \) from the central value for \( h_2 = 0 \). As noted earlier, the experimental data for \( \pi^0 \rightarrow e^+e^- \) \cite{12,13} seem to favor slightly positive values for \( h_2 \), although the bounds are rather loose. A better experimental determination of this decay rate would therefore be highly welcome.

Thus, using the LMD+V form factor, our estimate for the muon anomalous magnetic moment from the pion-pole contribution in hadronic light-by-light scattering reads

\[
a_{\mu}^{\text{LbyL,}\pi^0} = +5.8 \ (1.0) \times 10^{-10},
\]

where the error includes the variation of the parameters and the intrinsic model dependence.

As far as the contribution to \( a_{\mu} \) from the exchange of the \( \eta \) or of the \( \eta' \) is concerned, a detailed short-distance analysis of the corresponding form factor in large-\( N_C \) QCD for nonzero quark masses as done for the pion is beyond the scope of the present work. Furthermore, as noted above, the integrals in Eqs. (6.2) and (6.3) do not seem to be very sensitive to the correct asymptotic behavior for large momenta. It is more important to have a good description at small and intermediate energies, e.g., by reproducing the slope of the form factor \( \mathcal{F}_{\eta\gamma^*\gamma^*}(−Q^2,0) \), \( \mathcal{F}_{\eta'\gamma^*\gamma^*}(−Q^2,0) \), normalized to the corresponding experimental width \( \Gamma(\eta \to \gamma\gamma) \), using a VMD Ansatz with an adjustable vector meson mass \( \Lambda_{\eta} \).

The CLEO Collaboration \cite{37} has made a fit of the form factors \( \mathcal{F}_{\eta\gamma^*\gamma^*}(−Q^2,0) \) and \( \mathcal{F}_{\eta'\gamma^*\gamma^*}(−Q^2,0) \), normalized to the corresponding experimental width \( \Gamma(\eta \to \gamma\gamma) \), using a VMD Ansatz with an adjustable vector meson mass \( \Lambda_{\eta} \). Taking their values \( \Lambda_{\eta} = 774\pm29 \text{ MeV} \) or \( \Lambda_{\eta'} = 859\pm28 \text{ MeV} \) as the vector meson mass \( M_{V}\) in the expression of the VMD form factor,

\[
\begin{align*}
a_{\mu}^{\text{LbyL,}\eta}\big|_{\text{VMD}} & = +1.3 \ (0.1) \times 10^{-10}, \\
a_{\mu}^{\text{LbyL,}\eta'}\big|_{\text{VMD}} & = +1.2 \ (0.1) \times 10^{-10}.
\end{align*}
\]

The error reflects only the corresponding variation in \( \Lambda_{\eta} \). Again, apart from the sign, these values are comparable with the numbers quoted in Refs. \cite{20,21,22,39}. We do not expect the results with the LMD+V form factor to differ much from the numbers quoted above. Thus, adding up all contributions from pseudoscalar exchange, we obtain

\[
\begin{align*}
a_{\mu}^{\text{LbyL,PS}} \equiv a_{\mu}^{\text{LbyL,}\pi^0} + a_{\mu}^{\text{LbyL,}\eta}\big|_{\text{VMD}} + a_{\mu}^{\text{LbyL,}\eta'}\big|_{\text{VMD}} & = +8.3 \ (1.2) \times 10^{-10}.
\end{align*}
\]

Finally, for illustration and as a check, the result for the pion exchange contribution to the anomalous magnetic moment of the electron reads

\[
\begin{align*}
a_{e}^{\text{LbyL,}\pi^0(1)} & = 2.0 \times 10^{-6}, \\
a_{e}^{\text{LbyL,}\pi^0(2)} & = 2.0 \times 10^{-6}, \\
a_{e}^{\text{LbyL,}\pi^0} & = 5.1 \times 10^{-14},
\end{align*}
\]

for the LMD+V form factor (with \( h_2 = 0 \)). Despite the various factors \( 1/m^2 \) in the weight functions, the final result is much smaller than for the muon, as it should be. The results for the
other form factors are of similar size (LMD: $a^{\text{LbyL},e^0}_e = 7.7 \times 10^{-14}$, VMD: $a^{\text{LbyL},e^0}_e = 2.6 \times 10^{-14}$), except for the WZW case. Note that for the electron the second contribution $a^{\text{LbyL},e^0}_e(2)$ is in general not smaller than the first one, in contrast to the muon case.

8 Discussion and conclusions

In this article, we have evaluated the correction to $a_\mu$ induced by the pion-pole contribution to hadronic light-by-light scattering. From the methodological point of view, the present approach differs from previous ones in two important points. First, we use a representation of the form factor $F_{\pi^0,\gamma^*\gamma^*}$ which incorporates short-distance properties of QCD within an approximation to the large-$N_C$ limit involving a finite number of resonances. It is important to stress that this approximation can in principle be improved by considering additional vector meson states, if additional information, subleading short-distance corrections, phenomenological and/or experimental input, is provided. Second, for the corresponding class of form factors, we have performed the angular integration in the two-loop expression in an analytical way, using the method of Gegenbauer polynomials. This leads to an expression for $a^{\text{LbyL},e^0}_e$ in terms of weighted two-dimensional integrals over the moduli of the Euclidean loop momenta. These integrals were then evaluated numerically, and the results displayed in Table 2.

Whereas the size of $a^{\text{LbyL},e^0}_e$ obtained in this way is quite comparable to the existing determinations, the sign comes out opposite. In view of the consequences implied by this result, it is clear that this discrepancy needs to be discussed. We first notice that our result is not an artifact due to the representation of the form factor we use. Indeed, repeating the same analysis with the VMD form factor, we exactly reproduce the result quoted by previous authors, but again with the opposite global sign. Now, one might observe that the VMD form factor contributes only to the second term in Eq. (6.2), involving the positive definite (see Fig. 3) weight function $w_{g_1}(M_V,Q_1,Q_2)$, so that our result in this case could be ascribed to the fact that we obtain the wrong sign for that function. In addition to the various checks that we have performed, we shall however put forward a few arguments that support the correctness of our result. In the case of the constant form factor $F^{WZW}_{\pi^0,\gamma^*\gamma^*}$, only the first term in Eq. (6.3), involving the positive definite weight function $w_{f_1}(Q_1,Q_2)$, contributes. This integral is divergent, and behaves as

$$\lim_{\Lambda \to \infty} \int_0^\Lambda dQ_1 \int_0^\Lambda dQ_2 \left( \frac{N_C}{12\pi^2 F_{\pi^0}} \right)^2 w_{f_1}(Q_1,Q_2) = C \ln^2 \Omega + \cdots$$

for a large ultraviolet cutoff $\Omega$ (the ellipsis stands for subleading $\ln \Omega$ divergences, and for finite terms). The coefficient $C$ of this log-squared dependence can be computed analytically in a way that is largely independent of the methods used here. It is positive and agrees with the value we obtain from a numerical analysis of our formulas and from the asymptotic behavior for large momenta of the weight function $w_{f_1}(Q_1,Q_2)$; see Eq. (6.12). Next, as $M_V$ becomes very large, the VMD form factor approaches the WZW form factor and therefore should reproduce the previous result,

$$\lim_{M_V \to \infty} \int_0^\infty dQ_1 \int_0^\infty dQ_2 w_{g_1}(M_V,Q_1,Q_2) g^{(1)}_{M_V}(Q_1^2,Q_2^2) = C \ln^2 M_V + \cdots$$
with the same constant $C$ as in Eq. (8.1). We have checked numerically that this indeed happens with Eq. (6.2) and the expression of $w_{g_1}(M_V, Q_1, Q_2)$ given in Eq. (6.4). Thus, the signs of the contributions involving $w_{f_1}$ and $w_{g_1}$ must be correct.

Finally, we mention that with the convention for the external momenta used in the third article quoted under Ref. [22], the right-hand side of Eq. (2.7) in that paper, which corresponds to our Eq. (2.9), should come with a global minus sign. We also stress that our Eq. (2.9) agrees with Eq. (2.9) of [32]. We have not been able to find a similar point of disagreement with Refs. [20, 21]. However, the lack of analytical expressions in intermediate steps precludes a more detailed comparison. Clearly, this matter needs to be investigated further in the future.

Concluding this study, we find that the pseudoscalar-pole contribution to the hadronic light-by-light scattering amounts to

$$a^\text{LbyL,PS}_\mu = + 8.3 \times 10^{-10},$$

(8.3)

where we have used the LMD+V representation (4.4) of the form factor $F_{\pi^0\gamma^*\gamma^*}$ and the VMD form factor for the $\eta$ and $\eta'$, both supplemented with experimental information. The error given in Eq. (8.3) includes our estimate of the intrinsic model dependence and the variation of the model parameters. If we assume that the other contributions to the hadronic light-by-light scattering remain unchanged, our result implies that the difference between theory and experiment reduces to about $25 (16) \times 10^{-10}$, in the “least favorable” case, i.e., the vacuum polarization analysis of Ref. [11], which has the smallest value and the smallest error as compared to other recent evaluations [12, 13, 14, 15, 16]. We intend to use the same approach to study the full four-point function $\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3)$ in order to obtain an evaluation of the complete contribution $a^\text{LbyL, had}_\mu$. We also plan to investigate in more detail the $\eta$ and $\eta'$ pole contributions, although we do not expect the numerical values to change significantly as compared to the ones obtained in Eq. (7.4) with the VMD form factor.

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*Note added:* After the submission of this work, both groups [12] found the sign error in their calculations in Refs. [20, 21, 22]. Furthermore, in the case of the VMD form factor, the analytical result was derived in Ref. [13], in the form of a double series in $(M_{\pi^0}^2 - m^2)/m^2$ and $m^2/M_{V,\pi}^2$. The numerical value agrees exactly with our result for this form factor. Independently, Bardeen and de Gouvea [54] have reproduced the coefficient of the log-squared term from Eq. (6.12) by an analytical calculation.
Appendix

This Appendix lists the properties of the form factor $F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ that are reproduced by the LMD and LMD+V representations.

At the low-energy end, the form factor is normalized by the $\pi^0 \to \gamma\gamma$ amplitude

$$e^2 F_{\pi^0\gamma^*\gamma^*}(0, 0) = A(\pi^0 \to \gamma\gamma). \quad (A1)$$

In the chiral limit $m_q \to 0$, $q = u, d, s$, this amplitude is fixed by the WZW anomaly to read

$$A(0)(\pi^0 \to \gamma\gamma) = -\frac{e^2}{2}\frac{N_C}{\pi F_0} \left(1 + O(m_q)\right). \quad (A2)$$

For massive light quarks, this expression receives corrections. In particular, the pion decay constant in the chiral limit, $F_0$, is replaced by its physical counterpart $F_{\pi^0} = F_0[1 + O(m_q)]$,

$$A(\pi^0 \to \gamma\gamma) = -\frac{e^2 N_C}{12\pi^2 F_{\pi^0}} \left[1 + O(m_q)\right]. \quad (A3)$$

It turns out that the additional $O(m_q)$ corrections in this relation are numerically small [55], so that one may drop them to a good approximation.

The behavior of $F_{\pi^0\gamma^*\gamma^*}$ at short distances was recently studied in Ref. [29] (see also the references therein) in the chiral limit. It turns out that the properties relevant for our present purposes still hold for massive light quarks, again upon replacing $F_0$ by $F_{\pi^0}$,

$$\lim_{\lambda \to \infty} F_{\pi^0\gamma^*\gamma^*}(\lambda^2 q^2, (p - \lambda q)^2) = 2\frac{F_{\pi^0}}{3q^2} \left\{ \frac{1}{\lambda^2} + \frac{1}{\lambda^3} \frac{q \cdot p}{q^2} + O\left(\frac{1}{\lambda^4}\right) \right\}. \quad (A4)$$

This expression holds up to $O(\alpha_s)$ corrections. Note, however, that the Wilson coefficients corresponding to the two first terms of the short-distance expansion shown here are free of anomalous dimensions.

In the large-$N_C$ limit of QCD, the singularities of $F_{\pi^0\gamma^*\gamma^*}$ are restricted to single poles in each channel,

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)\big|_{N_C \to \infty} = \sum_{ij} \frac{c_{ij}(q_1^2, q_2^2)}{(q_1^2 - M_{V_i}^2)(q_2^2 - M_{V_j}^2)}, \quad (A5)$$

where the sum runs over the infinite tower of zero-width $J^{PC} = 1^{-+}$ vector resonances of large-$N_C$ QCD. Beyond the fact that they must reproduce the short-distance behavior (A4) and be free of singularities, the functions $c_{ij}(q_1^2, q_2^2)$ are not further restricted by the (known) properties of large-$N_C$ QCD. The representations $F_{\pi^0\gamma^*\gamma^*}^{LMD}$ and $F_{\pi^0\gamma^*\gamma^*}^{LMD+V}$ are obtained by truncation of the infinite sum (A5) to one, respectively two, vector resonances per channel. As noted in the text, as $M_V \to \infty$, the form factor $F_{\pi^0\gamma^*\gamma^*}^{LMD}$ reduces to a constant given by the Wess-Zumino-Witten term. In the case of $F_{\pi^0\gamma^*\gamma^*}^{LMD+V}$ one has to proceed in two steps. First the heavier resonance $V_2$ is decoupled,

$$\lim_{M_{V_2}^2 \to \infty} F_{\pi^0\gamma^*\gamma^*}^{LMD+V}(q_1^2, q_2^2) = F_{\pi^0\gamma^*\gamma^*}^{LMD}(q_1^2, q_2^2), \quad (A6)$$

where the sum runs over the infinite tower of zero-width $J^{PC} = 1^{-+}$ vector resonances of large-$N_C$ QCD.
with
\[ \lim_{M_{V_2}^2 \to \infty} \frac{h_7}{M_{V_2}^4} = -c_V, \quad \lim_{M_{V_2}^2 \to \infty} \frac{h_5}{M_{V_2}^4} = 1, \quad \lim_{M_{V_2}^2 \to \infty} \frac{h_i}{M_{V_2}^4} = 0, \quad i = 1, 2. \] (A7)

Then one lets \( M_{V_1} \to \infty \) as before.

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