PARAMETERIZATIONS OF 1-BRIDGE TORUS KNOTS

DOO HO CHOI AND KI HYOUNG KO

Abstract. A 1-bridge torus knot in a 3-manifold of genus \( \leq 1 \) is a knot drawn on a Heegaard torus with one bridge. We give two types of normal forms to parameterize the family of 1-bridge torus knots that are similar to the Schubert’s normal form and the Conway’s normal form for 2-bridge knots. For a given Schubert’s normal form we give algorithms to determine the number of components and to compute the fundamental group of the complement when the normal form determines a knot. We also give a description of the double branched cover of an ambient 3-manifold branched along a 1-bridge torus knot by using its Conway’s normal form and obtain an explicit formula for the first homology of the double cover.

1. Introduction

One of traditions in knot theory is to study a family of knots satisfying a certain condition. Examples of such families include the family of torus knots studied by Dehn and Schreier and the family of 2-bridge knots studied by Schubert, Montesinos and Conway. These classes can be referred as the classes of knots and links indexed by the pairs \((g, b)\) of non-negative integers as defined in [1]. A knot \( K \) in a 3-manifold \( M \) has a \((g, b)\)-decomposition or is called a \((g, b)\)-knot if for some Heegaard splitting \( M = U \cup V \) of genus \( g \), each of \( K \cap U \) and \( K \cap V \) is consisted of trivial \( b \) arcs. A collection of properly embedded arcs in a 3-manifold \( W \) with boundary is trivial if arcs in the collection together with arcs on \( \partial W \) joining the two ends of the arcs bound mutually disjoint disks, called cancelling disks, in \( W \). A \((g, b)\)-knot can be embedded in a Heegaard surface of genus \( g \) in \( M \) except at \( b \) over(or under)-bridges and vice versa. Torus knots are \((1, 0)\)-knots and 2-bridge knots are \((0, 2)\)-knots. Clearly the family of \((g, b)\)-knots becomes strictly larger as \( g \) or \( b \) increases.

Since an over-bridge can be removed by adding a handle and by embedding the over-bridge into the added handle, \((g, b)\)-knots are contained in the family of \((g + 1, b - 1)\)-knots.

In this paper we study 1-bridge torus knots, that is, \((1, 1)\)-knots in a 3-manifold. An ambient 3-manifold necessarily has the Heegaard genus \( \leq 1 \) and so it can be either \( S^3 \) or a lens space. The family of 1-bridge torus knots in \( S^3 \) contains torus knots and 2-bridge knots and is contained in the family of double torus knots, that is, \((2, 0)\)-knots. Hill and Murasugi studied the family of double torus knots in [14, 15] and parametrized the family. Non-trivial knots with the trivial Alexander polynomial was found in the subfamily of double torus knots that separate the double torus. They also considered non-separating double torus knots and a subfamily of 1-bridge torus knots and found various double torus knots that are fibered.

AMS Subject Classifications: 57M25, 57M12.
Every 1-bridge torus knot has the tunnel number one, but not all tunnel-number-one knots are 1-bridge torus knots. In [20], Morimoto, Sakuma and Yokota found tunnel-number-one knots that are not 1-bridge torus knots as confirmed by a condition on the Jones polynomial for a knot to admit a \((g, b)\)-decomposition in [23]. In [21], they gave another criteria to determine whether a given knot has the tunnel number one and whether it is a 1-bridge torus knot.

Besides torus knots and 2-bridge knots, the family of 1-bridge torus knots includes Berge’s double-primitive knots, 1-bridge braids that classified by Gabai in [11], and satellite knots of tunnel number one. Morimoto and Sakuma studied satellite knots of tunnel number one and classified their unknotting tunnels in [19].

Fujii showed that any Alexander polynomial can be realized by a 1-bridge torus knot in [10]. He also found a family of non-trivial 1-bridge torus knots with trivial Alexander polynomial. In [7], a subfamily of 1-bridge torus knots is completely classified using their genera and Jones polynomial. Also cyclic branched covers of ambient spaces along 1-bridge torus knots are known as Dunwoody 3-manifolds and are studied in [24] and [13].

![1-bridge torus knot](image)

**Figure 1.** 1-bridge torus knot

We will parameterize the family of 1-bridge torus knots using two kinds of normal forms as done for the family of 2-bridge knots. Schubert described a 2-bridge knots by a pair of integers of a certain condition from its top view. In the top view a 2-bridge knots is embedded in a plane except the two bridges. He in fact completely classified 2-bridge knots using this normal form [23]. Since a 1-bridge torus knot can be embedded in a standard torus except the bridge (See Figure 1), we will describe it by a 4-tuple of integers from this top view. Section 2 will be devoted to simplify an embedded curve with two fixed ends on a torus up to surface isotopies that disturb neither bridges nor knot types of 1-bridge torus knots. We show that a 4-tuple of integers is enough to describe such a curve and give an algorithm to count the number of component of the curve given by a 4-tuple. We will call such a 4-tuple the *Schubert’s normal form* of the 1-bridge torus knot determined by a 4-tuple. The Schubert’s normal form is useful to compute the fundamental group of the exterior of a 1-bridge torus knot. In Section 3, we give an algorithm to compute the knot group from the Schubert’s normal form. As a corollary, we determine when the exterior of a 1-bridge torus knot in a lens space has a double cover.

On the other hand, a 2-bridge knot can also be viewed as a 4-plats as studied first in [3]. This corresponds to a side view and the composition of homeomorphisms of a
four-punctured sphere determines the 2-bridge knot. Using this description, Conway constructed a bijection between 2-bridge knots and lens spaces via double branched covers. A similar description using the composition of homeomorphisms on a two-punctured torus is possible for 1-bridge torus knots and this will be called the Conway’s normal form. In Section 4, we describe a Conway’s normal form of a 1-bridge torus knot and we show it is well-defined in the sense that a Conway’s normal form belongs to a free subgroup generated by three homeomorphisms in the mapping class group of a two-punctured torus. Finally we construct the double branched cover of an ambient space branched along a 1-bridge torus knot given by the Conway’s normal form and give a formula for the first homology of the double branched cover.

2. An embedded curve in a torus

Since 1-bridge torus knots can be embedded in a standard torus $T$ except the bridge connecting two points $x, y$ in $T$, we will try to classify embedded curves with two fixed ends $x, y$ in $T$ up to isotopies of two-punctured torus that preserves the bridge and the knot type of a given 1-bridge torus knot and extends to ambient isotopies on $S^3$. Let $m$ be a meridian curve on $T$ containing the points $x, y$. Consider the following two types of isotopies on $T$.

(I) An isotopy $h_t$ fixing $x, y$ pointwise such that $h_0$ is an identity map;

(II) An isotopy $h_t$ such that $h_0$ is an identity map and $h_1$ is a homeomorphism exchanging $x$ and $y$ counterclockwise or clockwise and fixed on $T - D$ for a small disk $D$ containing $x, y$ as illustrated in Figure 2.

![Figure 2](image.png)

Figure 2.

2.1. A normal form of an embedded curve. We parameterize the torus $T$ as $S^1 \times [0, 1]/x \times 1 \sim x \times 0$ for $x \in S^1 = [-4, 4]/4 \sim -4$. Let $x, y$ be $[1 \times 0], [-1 \times 0]$, respectively. Let $\gamma$ be an embedded curve on $T$ with two ends $x, y$. We suppose that $\gamma$ is not isotopic to an arc on $S^1 \times 0 \subset T$ and $|m \cap \gamma|$ is minimal up to isotopies of type I and II. If we cut the torus $T$ along the meridian circle $m = S^1 \times 0$, we obtain a collection $\Gamma$ of arcs from $\gamma$ on the cylinder $C = S^1 \times I$. We may assume that $\Gamma$ contains no closed curves since we are not interested in trivial components that splits. Then each arc $\alpha \in \Gamma$ is one of the following two types (See Figure 3):

1. An arc $\alpha$ is called a rainbow if either $\partial \alpha \cap (S^1 \times 0) = \emptyset$ or $\partial \alpha \cap (S^1 \times 1) = \emptyset$.
2. An arc $\alpha$ is called a stripe otherwise.
Let \( x_0 = (1, 0), y_0 = (-1, 0), x_1 = (1, 1), \) and \( y_1 = (-1, 1) \) in \( C \). If \( \alpha \) is a rainbow in \( C \), then \( C - \alpha \) has two components, one is a disk and the other is a cylinder. We denote the disk by \( D_\alpha \). If \( D_\alpha \) contains a point \( t \) in \( S^1 \times 0 \) or \( S^1 \times 1 \), we say that \( \alpha \) contains \( t \).

Important properties of the set \( \Gamma \) of arcs are collected in the following lemma.

**Lemma 2.1.** Under the above assumption on \( \gamma \) and \( \Gamma \), the following holds:

1. A rainbow contains one and only one of \( x_0, y_0, x_1, y_1 \).
2. There are no two arcs starting from both \( x_0 \) and \( y_0 \) nor both \( x_1 \) and \( y_1 \).
3. An arc starting from one of \( x_0, x_1, y_0, y_1 \) is a stripe.
4. If there is a stripe starting from \( x_0 \) (resp. \( x_1 \)) then there is also a stripe starting from \( y_1 \) (resp. \( y_0 \)).
5. Each end of a rainbow is joined to an end of a stripe.

**Proof.**

(1) If there is a rainbow containing either both \( x_0 \) and \( y_0 \) or both \( x_0 \) and \( y_0 \), \( \gamma \) joining \( x \) and \( y \) can be isotoped into the meridian \( m \) and this violates our assumption. If there is a rainbow containing none of \( x_0, x_1, y_0, y_1 \), then the rainbow can be remove from \( \Gamma \) by isotopies of type I and this contradicts the minimality of \( |m \cap \gamma| \).

(2) Suppose that \( \Gamma \) has two stripes starting from both \( x_0 \) and \( y_0 \), respectively. Since \( \gamma \) joins \( x \) and \( y \) on \( T \) the two stripes are connected via since \( \gamma \) joins \( x \) and \( y \) and then \( \Gamma \) must have a rainbow whose ends belong to \( S^1 \times 0 \). Since the rainbow contains none of \( x_0 \) and \( y_0 \), \( |m \cap l| \) can be reduced.

(3) Let \( \alpha \) be an arc in \( \Gamma \) starting from \( x_0 \). Suppose \( \alpha \) is a rainbow. Then \( \alpha \) must contain \( y_0 \). Otherwise \( |m \cap \gamma| \) can be reduced by an isotopy of type (I). By an isotopy of type II
as illustrated in Figure 4, $\alpha$ can be removed from $\Gamma$ and $|m \cap \gamma|$ was not minimal.

(4) Since $x$ and $y$ are connected via $\gamma$, this immediately follows from (2) and (3).

(5) Suppose that there is a rainbow whose one end is connected with that of the other rainbow. Then we have the two cases as in Figure 5. For each case of Figure 5 we can reduce $|m \cap \gamma|$ by isotopies of type I and II as shown in Figure 6.

If $L$ has a stripe starting from $x_0$ (resp. $y_0$), $\gamma$ is called $+1$-type (resp. $-1$-type).

**Lemma 2.2.** Any rainbow contains $x_1$ or $y_0$ (resp. $x_0$ or $y_1$) if $\gamma$ is $+1$-type (resp. $-1$-type). And the number of rainbows containing $x_0$ (resp. $y_0$) is equal to that of rainbows containing $y_1$ (resp. $x_1$).

**Proof.** Suppose $\gamma$ is $+1$-type. Then a rainbow can not contain $x_0$ nor $y_1$ and if it contains none of $x_1$, $y_0$ then it can be removed by an isotopy of type I but it is impossible since $|m \cap \gamma|$ is minimal. Therefore it must contain $x_1$ or $y_0$. And since $\gamma$ is embedded, the number of rainbows containing $x_1$ is equal to that of rainbows containing $y_0$. □

**Theorem 2.3.** Let $\gamma$ be an embedded curve on the torus $T$ with $\partial \gamma = \{x, y\}$ and $|m \cap \gamma|$ be minimal up to isotopies of type I and II where $m$ is a meridian circle containing $x, y$. Then $\gamma$ is represented by a 4-tuple $(r, s, t, \rho)\epsilon$, where $r, s, t$ are non-negative integers, $\rho$ is an integer and $\epsilon$ is a sign $\pm 1$. If $s \neq 0 \neq t$ then $(r, s, t, \rho)\epsilon$ is unique and if $s$ (resp. $t$) is 0 then a 4-tuple $(r, 0, t, \rho)_{-\epsilon}$ (resp. $(r, s, 0, \epsilon 2(2r + 1) + \rho)_{-\epsilon}$) also represents $\gamma$.

**Proof.** Let $\gamma$ be an embedded curve of $\epsilon$-type. From the view of Lemma 2.1 and Lemma 2.2, $\gamma$ must look like Figure 6 on a neighborhood $A$ of the meridian $m$. □
If we identify two ends of $A$ by the suitable $\frac{2\pi \rho}{n}$-rotation map on $S^1$ then we have an embedded curve isotopic to $\gamma$ on $T$ where $n = 2r + 1 + s + t$. Therefore $(r, s, t, \rho)_\epsilon$ represents $\gamma$ where $r, s, t$ are nonnegative integers, $\rho$ is an integer and $\epsilon = \pm 1$.

The 4-tuple $(r, s, t, \rho)_\epsilon$ is unique up to isotopies of type I and II if $s \neq 0 \neq t$ and $|m \cap \gamma|$ is minimal. And if $s$ or $t$ is 0 then the last statement of the theorem holds by exchanging rainbows via isotopies of type II.

2.2. Components counting algorithm. Given non-negative integers $r, s, t$, integer $\rho$ and a sign $\epsilon$, we will try to find a condition that makes a 4-tuple $(r, s, t, \rho)_\epsilon$ represent a simple arc on $T$ with ends $x, y$. We will in fact give an algorithm to count the number of components of the curve determined by a 4-tuple. We consider a set of arcs on a cylinder such as Figure 7 determined by $r, s, t$ and $\epsilon$. Let $n = 2r + 1 + s + t$. If we identify two ends of the cylinder by a $\frac{2\pi \rho}{n}$ rotation, then we obtain an embedded curve on a torus $T$. Denote the number of components of this curve by $|(r, s, t, \rho)_\epsilon|$. It is easy to see that $|(r, s, t, \rho)_\epsilon + 1|$ is equal to the number of components of the curve depicted in Figure 7. We first remark that

1. $|(r, s, t, \rho)_{\epsilon - 1}| = |(r, s, t, n - \rho + 1)|$.
2. $|(r, s, t, \rho)_\epsilon| = |(r, s, t, \bar{\rho})_\epsilon|$, where $\bar{\rho} \equiv \rho \mod n$.

Thus we assume that $\epsilon = +1$ and $0 \leq \rho < n = 2r + 1 + s + t$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7}
\caption{Figure 7.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8}
\caption{Figure 8.}
\end{figure}
Proposition 2.4. \(|(r, s, t, \rho)_{+1}| is equal to the number of components in the curve depicted in Figure 8, where if \(0 \leq \rho < n - t\) then \(\beta \equiv t \mod \rho\) with \(0 \leq \beta < \rho\), \(\alpha = \rho - \beta\), \(\gamma = n - \rho - t\), and \(\delta = s\), or if \(n - t \leq \rho < n\) then \(\beta \equiv t \mod (n - \rho)\) with \(0 \leq \beta < n - \rho\), \(\alpha = \rho - t\), \(\gamma = n - \rho - \beta\), and \(\delta = s\).

Proof. In the figures of the proof, two curves being “=” mean that the curves have the same number of components.

(i) If \(t < \rho\) and \(t < n - \rho\), we may isotope the upper rainbow with \(t\) strands as in Figure 9 to remove it.

(ii) If \(t \geq \rho\), we may isotope parts the upper rainbow consisted of \(\rho\) strands at a time until it becomes the case (i) as in Figure 10 where \(\beta\) is the remainder when \(t\) is divided by \(\rho\).

(iii) If \(t \geq n - \rho\), we similarly isotope parts of the upper rainbow consisted of \(n - \rho\) strands at a time until it becomes the case (i) where \(\beta\) is the remainder in the division of \(t\) by \(n - \rho\).

In the all three cases \(|(r, s, t, \rho)_{+1}| is equal to the number of components of the curve on the left in Figure 11 where if \(0 \leq \rho < n - t\) then \(\beta \equiv t \mod \rho\) with \(0 \leq \beta < \rho\), \(\alpha = \rho - \beta\), and \(\gamma = n - \rho - t\), or if \(n - t \leq \rho < n\) then \(\beta \equiv t \mod (n - \rho)\) with \(0 \leq \beta < n - \rho\), \(\alpha = \rho - t\), and \(\gamma = n - \rho - \beta\). In Figure 11, the curve in the middle is obtained from the
one on the left by creating a large curl involving \( s \) strands that maintains the connections on strands involved. The curve on the right is obtained from the one in the middle by unfolding.

\[
\begin{align*}
\text{Figure 11.} \\
\end{align*}
\]

In Figure 11, the curve in the middle is obtained from the one on the left by doubling each components, that is, taking the boundaries of a two dimensional untwisted regular neighborhood of the curve on the left. Since there is one arc component (joining two pictures) in the curve on the left, the curve in the middle contains \( 2\left| (r, s, t, \rho)_{+1} \right| - 1 \) components. Finally the curve in the right is the quotient of the curve in the middle by the \( \mathbb{Z}/2\mathbb{Z} \)-action given by the \( \pi \)-rotation about the axis indicated in the figure. Each component obtained by doubling a closed component has the linking number zero with the axis and the component obtained by doubling the arc component has the linking number 1 with the axis. Thus the number of components in the curve on the right exactly equals \( \left| (r, s, t, \rho)_{+1} \right| \).

\[
\begin{align*}
\text{Figure 12.} \\
\end{align*}
\]

Lemma 2.5. In Figure 12, the numbers of components in the curves (a) and (b) are \( \gcd(\alpha, \beta) \) and \( \gcd(\left| \alpha - \gamma \right|, \beta + \gamma) \), respectively.

\[
\begin{align*}
\text{□} \\
\end{align*}
\]
Proof. It is easy to see that Euclidean algorithm determines the number of components in the curve (a). The curve (b) can be modified to a curve like (a) as in the following figure:

We now consider the curve given in Figure 14.

We may assume $\alpha \geq \delta$ by flipping the curve if not. If $\alpha = \delta$, then it is easy to see that there are $\alpha + \gcd(\beta, \gamma)$ components in the curve by Lemma 2.5. If $\alpha > \delta$, then we let $\bar{\beta}$ be the remainder in the division of $\delta$ by $\alpha - \delta$, $\bar{\alpha} = \alpha - \delta - \bar{\beta}$, $\bar{\gamma} = \beta$, and $\bar{\delta} = \gamma$. Via the modifications in the following figure
we have

\[
\bar{\alpha} + \bar{\beta} + \bar{\gamma} + \bar{\delta} < \alpha + \beta + \gamma + \delta. 
\]

By repeating this process, we eventually reach the situation in which at least one of four numbers becomes zero and so Lemma 2.5 can be applied. Thus we obtain an algorithm to compute \(|(r, s, t, \rho)_{+1}|\) in Proposition 2.4 that equals the number of components of the curve in Figure 8. We call this algorithm the component counting algorithm.

We conclude this section by summarizing our result in a theorem.

**Theorem 2.6.** Let \((r, s, t, \rho)\) be a 4-tuple of integers such that \(r, s, t \geq 0\) and \(\epsilon = \pm 1\). And set \(n = 2r + 1 + s + t\) and \(\bar{\rho} \equiv \epsilon \rho \mod n\), \(0 \leq \bar{\rho} < n\). If \(0 \leq \rho < n - t\) then we let \(\beta \equiv t \mod \rho\) with \(0 \leq \beta < \rho\), \(\alpha = \rho - \beta\), \(\gamma = n - \rho - t\), and \(\delta = s\), or if \(n - t \leq \rho < n\) then we let \(\beta \equiv t \mod (n - \rho)\) with \(0 \leq \beta < n - \rho\), \(\alpha = \rho - t\), \(\gamma = n - \rho - \beta\), and \(\delta = s\). If the components counting algorithm returns "1" for the input \((\alpha, \beta, \gamma, \delta)\) then there exists a simple arc with ends \(x, y\) on \(T\) which is determined by \((r, s, t, \rho)\).

### 3. Schubert’s normal forms and knot groups

#### 3.1. Schubert’s normal forms of 1-bridge torus knots

Let \((r, s, t, \rho)\) be integers satisfying the assumption in Theorem 2.6. Then we obtain a simple arc \(\alpha\) on \(T\) with ends \(x, y\). Let \(\beta\) be an arc connecting \(x, y\) in a meridian disk of the solid torus \(V_1\) bounded by \(T\) and let \(h : \partial V_2 \rightarrow \partial V_1\) be a homeomorphism for another solid torus \(V_2\). Then \(\alpha \cup \beta\) forms a 1-bridge torus knot in the 3-manifold \(M_h\) where \(M_h = V_1 \cup_h V_2\). We note that a 1-bridge torus knot is embedded in \(V_1\) in our convention. Figure 15 contains an example of 1-bridge torus knots in \(S^3\).

![Figure 15. 1-bridge torus knot](image)

Conversely let \((V_1, t_1) \cup_h (V_2, t_2)\) be a (1,1)-decomposition of a 1-bridge torus knot \(K\). If we isotope the trivial arc \(t_2\) on the \(\partial V_2 = \partial V_1\) along a cancelling disk for \(t_2\) then it is a
simple arc in the torus and so it is determined by a 4-tuple \((r, s, t, \rho)\) by Theorem 2.3 since the isotopies of type (I) and (II) do not change the knot type and the given 3-manifold.

Thus a 4-tuple \((r, s, t, \rho)\) together with a homeomorphism \(h\) uniquely determines a 1-bridge torus knot in \(M_h\). This representation will be called a Schubert’s normal form and the 1-bridge torus knot in \(M_h\) represented by a Schubert’s normal form \((r, s, t, \rho)\) will be denoted by \(S(r, s, t, \rho)\).

The following facts can be observed immediately.

1. \(S(r, s, t, \rho) + 1\) is equivalent to \(S(r, t, s, \rho + (2r + 1)) - 1\) (See Figure 15).
2. Two 1-bridge torus knots \(S(r, s, t, \rho) + 1\) and \(S(r, s, t, -\rho) - 1\) are mirror images each other.
3. \(S(0, s, t, \rho) + 1\) in \(S^3\) is a 1-bridge braid studied in [11].
4. A \((p, q)\)-torus knot is a 1-bridge torus knot \(S(0, 0, p-1, -q) + 1\) or \(S(0, p-1, 0, -q+1) - 1\) in \(S^3\).
5. Any 2-bridge knot in \(S^3\) represented by an “original” Schubert’s normal form \(B(\alpha, \epsilon \beta)\) is a 1-bridge torus knot \(S(\beta - 1, \alpha - 2\beta + 1, 0, \epsilon)\) (See Figure 16), where \(\alpha\) and \(\beta\) are positive odd integers with \(\beta < \alpha\) and are relatively prime (For example, see Chapter 3 of [1]).

![Figure 16.](image)

6. Morimoto and Sakuma showed in [19] that any satellite knot which admits an unknotting tunnel is equivalent to a knot \(K(\alpha, \epsilon \beta; p, q)\) represented by 4 integers such that \(\alpha\) is even and \(p, q\) are positive and \(0 < \beta < \alpha/2\). The knot \(K(\alpha, \epsilon \beta; p, q)\) is a 1-bridge torus knot \(S(\beta - 1, \alpha - 2\beta + 1, 0, \epsilon)\).

In [14], Goda and Hayashi studied the intersection of two Heegaard surfaces of \((2, 0)\)-knots \((M, K)\) that give \((1, 1)\) and \((2, 0)\) decomposition, respectively. Using [17], they characterized the intersection when \(M\) admits a double cover branched along \(K\). In the next section we will give a necessary and sufficient condition for the existence of this double cover in terms of parameters in a Schubert’s normal form.

3.2. **Exteriors of 1-bridge torus knots.** In this section we compute the fundamental groups of exteriors of 1-bridge torus knots in a 3-manifold. As a corollary, we give a necessary and sufficient condition that a lens space has a double cover branched along a given 1-bridge torus knot in terms of parameters of its Schubert’s normal form.
Consider a 1-bridge torus knot $K = S(r, s, t, \rho)_{+1}$ in a 3-manifold $M_h$ with a Heegaard decomposition $V_1 \cup_h V_2$ of genus one. We may assume that the homeomorphism $h : \partial V_2 \rightarrow \partial V_1$ send the meridian of $\partial V_2$ to the $(p,q)$-curve of $\partial V_1$ and either $(p,q) = (0,1)$ or $(p,q) = (1,0)$ or $p$ and $q$ are coprime such that $2 \leq q < p$. From now on, the 3-manifold $M_h$ will be denoted by $L(p,q)$. Thus $L(1,0) = S^3$, $L(0,1) = S^1 \times S^2$, and $L(p,q)$ for $1 \leq q < p$ is a usual lens space. Let $K$ be expressed as a union of arcs $\alpha$ and $\beta$ such that $\beta$ is an insignificant part of $K$ drawn in Figure 17. We may assume $\alpha$ and $\beta$ are properly embedded in $V_1$ and $V_2$, respectively. A tubular neighborhood $N(K)$ of $K$ is also the union $N(\alpha) \cup N(\beta)$ of tubular neighborhoods of $\alpha$ and $\beta$. Then the exterior $E(K) = L(p,q) - \text{int} N(K)$ of $K$ is homeomorphic to a 3-manifold obtained from the handle body $(V_1 - \text{int} N(\alpha))$ by attaching a 2-handle along the $(p,q)$-curve determined by $h$(see Figure 18).

![Figure 17.](image)

In order to help to understand our algorithm that computes the fundamental group of $E(K)$, we give an example first. Let $K$ be the 1-bridge torus knot $S(1,3,0,2)_{+1}$ in $S^3$ shown in Figure 14 which is a mirror image of the knot 9$_{42}$ in the knot table of the Rolfsen’s book [22]. Since $V_1 - \text{int} N(\alpha)$ is a handle body of genus 2, its fundamental group is the free group of two generators $x$ and $y$ where $x$, $y$ represent a meridian of $K$ and the core of the torus as shown in Figure 18 and the base point is placed on the core. To present the free group $\pi_1(V_1 - \text{int} N(\alpha))$, we introduce 6 extra generators $x_1, \ldots, x_6$ as in Figure 18 that are freely homotopic to either $x$ or $x^{-1}$ and introduce 6 extra relations so that

$$\pi_1(V_1 - \text{int} N(\alpha)) = \langle x, y, x_1, x_2, x_3, x_4, x_5, x_6 \mid x_3 = W_1^{-1}xW_1, x_1 = W_2^{-1}x_3^{-1}W_2, x_6 = W_3^{-1}x_1W_3, x_5 = W_4^{-1}x_6W_4, x_4 = W_5^{-1}x_5W_5, x_2 = W_6^{-1}x_4^{-1}W_6 \rangle$$

where $W_1 = y^{-1}$, $W_2 = x^{-1}$, $W_3 = xy$, $W_4 = yx$, $W_5 = yx$ and $W_6 = yx^{-1}y^{-1}$. These relations are obtained by sliding $x_i$ toward $x$ along the arc $\alpha$ and can be written down without looking at a knot diagram. We note that $W_i$’s are always words on $x$, $y$.

The attaching curve $\gamma$ for the 2-handle can be easily described by a word $R$ on $x_1, \ldots, x_6$ and $y$. The word $R$ can also be generated by an algorithm. In our example $R = x_6^{-1}x_5^{-1}y$. After applying Tietze transformations that eliminate these extra generators and relations, we obtain a presentation of $\pi_1(E(K))$ with two generators $x$, $y$ and one relation $R$ as follows:

$$\pi_1(E(K)) = \langle x, y \mid (W_1W_2W_3)^{-1}x(W_1W_2W_3)(W_1W_2W_3W_4)^{-1}x(W_1W_2W_3W_4)y \rangle.$$  

Since a 1-bridge torus knot group is always presented by two generators and one relation, it is easy to compute its Alexander polynomial via free differential calculus. In our example,
the image of derivative of $R$ with respect to $x$ under the linear extension: $\mathbb{Z}(\pi_1(E(K))) \to \mathbb{Z}[t^{\pm 1}]$ of the abelization map is given by $-t^{-1}(1 - t - t^2 - t^3 - t^4 + t^5)$ and the image of derivative of $R$ with respect to $y$ is given by $-t(1 - 2t + t^2 - 2t^3 + t^4)$. Thus the Alexander polynomial of $K$ is their greatest common divisor $1 - 2t + t^2 - 2t^3 + t^4$.

We now describe an algorithm to compute the fundamental group of the exterior of a 1-bridge torus knot $K$ given by its Schubert’s normal form $S(r, s, t, \rho)_+$. Let $n = 2r + 1 + s + t$ and $\rho = mn + \bar{\rho}$ for $0 \leq \bar{\rho} < n$. As we did in the previous example, we choose small loops $x_1, \ldots, x_n$ that are freely homotopic to $x$ or $x^{-1}$ so that the product $x_1 \cdots x_n$ represents the $(0,1)$-curve on the torus. The free group $\pi_1(V_1 - \text{int } N(\alpha))$ generated by $x$, $y$ is presented by adding $n$ extra generators $x_1, \ldots, x_n$ and $n$ extra relations $x_1 = W_i^{-1}x_i^{\epsilon_i}W_i$ for $i = 1, \ldots, n$ where $x_{k_0} = x$ and $W_i$ is a word on $x$ and $y$. The permutation $\{k_1, \ldots, k_n\}$ of $\{1, \ldots, n\}$ is obtained by following the arc $\alpha$ starting from the $x$ end. The integer $k_i$ should be taken modulo $n$ in the following algorithm that computes the triple $(k_i, W_i, \epsilon_i)$.

(I) $(k_1, W_1, \epsilon_1) = (s + r + 1 - \bar{\rho}, y^{-1}, +1)$.

(II) $(k_i, W_i, \epsilon_i)$ is computed inductively as follows:

(i) If $\epsilon_1 \cdots \epsilon_{i-1} = +1$, then

$$
(k_i, W_i, \epsilon_i) = \begin{cases}
(2(r + 1) - k_{i-1}, x, -1) & 1 \leq k_{i-1} \leq r \\
(2(r + 1) - k_{i-1}, x^{-1}, -1) & r + 1 < k_{i-1} \leq 2r + 1 \\
(k_{i-1} - (2r + 1) - \bar{\rho}, (yx)^{-1}, +1) & 2r + 1 < k_{i-1} \leq n - t \\
(k_{i-1} - \bar{\rho}, y^{-1}, +1) & n - t < k_{i-1} \leq n
\end{cases}
$$

(ii) If $\epsilon_1 \cdots \epsilon_{i-1} = -1$, then for $j \equiv k_{i-1} + \bar{\rho} \mod n$ and $1 \leq j \leq n$,

$$
(k_i, W_i, \epsilon_i) = \begin{cases}
(j + 2r + 1, yx, +1) & 1 \leq j \leq s \\
(2(r + s + 1) - j - \bar{\rho}, yxy^{-1}, -1) & s < j \leq s + r \\
(2(r + s + 1) - j - \bar{\rho}, xy^{-1}y^{-1}, -1) & s + r + 1 < j \leq n - t \\
(j, y, +1) & n - t < j \leq n
\end{cases}
$$

We now find the word $R(p, q)$ in $\pi_1(V_1 - \text{int } N(\alpha))$ that represents the $(p,q)$-curve $\gamma$ on $\partial V_1$. Recall our restriction that either $(p, q) = (0, 1)$ or $(p, q) = (1, 0)$ or $p$ and $q$ are
Let Corollary 3.1. Notice that it is obviously infinitely cyclic if the ambient 3-manifold is $L_{p, q}$ of $\gcd(p, k)$ where $k$ is coprime with $0 < q < p$. it is clear that $R(0, 1) = x_1x_2 \cdots x_n$ and

$$R(1, 0) = \begin{cases} 
    y 
    \frac{R(0, 1)^{-1} \cdots R(0, 1)^{-1} x_n^{-1} x_{n-1}^{-1} \cdots x_{n-p+1}^{-1}}{m \text{ times}}, & \rho = 0 \\
    \frac{R(0, 1) \cdots R(0, 1) x_n^{-1} x_{n-1}^{-1} \cdots x_{n-p+1}^{-1}}{m \text{ times}}, & \rho > 0 \\
    \frac{R(0, 1) \cdots R(0, 1) x_n^{-1} x_{n-1}^{-1} \cdots x_{n-p+1}^{-1}}{m \text{ times}}, & \rho < 0
  \end{cases}$$

For a given $q \geq 1$, let $R(p, q) = R(p)$ for $p \geq 1$ and $R(p)$ can be computed inductively as follows:

(1) $R(1) = R(1, 0)$
(2) $R(i) = \begin{cases} 
    R(i - 1)R(1, 0) & \text{if } 1 \leq j \leq p - q \\
    R(i - 1)R(0, 1)R(1, 0) & \text{otherwise}
  \end{cases}$

where $j \equiv 1 + (i - 1)q \mod p$ and $1 \leq j \leq p$.

Therefore a presentation of $\pi_1(E(K))$ is given by $$\langle x, y, x_1, \ldots, x_n | R(p, q), x_{k_i} = W_i^{-1}x_{k_{i-1}}^{-1}W_i, i = 1, \ldots, n \rangle$$

Using the substitutions $x_{k_i} = W_i^{-1}W_1^{-1}x_{k_{i-1}}^{-1}W_1 \cdots W_i$, the word $R(p, q)$ is written only on $x$ and $y$ and consequently $\pi_1(E(K)) = \langle x, y | R(p, q) \rangle$ by Tietze transformations.

We consider the first homology of the exterior $E(K)$ of a 1-bridge torus knot $K$ in $L(p, q)$. Notice that it is obviously infinitely cyclic if the ambient 3-manifold is $L(1, 0) = S^3$.

**Corollary 3.1.** Let $K$ be a 1-bridge torus knot in $L(p, q)$.

$$H_1(E(K)) \cong \begin{cases} 
    \mathbb{Z} \oplus \mathbb{Z}_{\gcd(p, \ell)}, & p \geq 1 \\
    \mathbb{Z} \oplus \mathbb{Z}_{\ell}, & p = 0
  \end{cases}$$

where $\ell = \sum_{j=1}^n (\prod_{i=1}^j \epsilon_i)$.

**Proof.** Let $n_x(p, q)$ (or $n_y(p, q)$) be the exponent sum of $x$ (or $y$, respectively) in the relation $R(p, q)$ above. Then $n_x(0, 1) = \ell$, $n_y(0, 1) = 0$, and $n_y(1, 0) = 1$. Let $n_x(1, 0) = m$. Then $n_x(p, q) = q\ell + pm$ and $n_y(p, q) = p$. Since $H_1(E(K))$ is the abelianization of $\pi_1(E(K))$, it has an abelian presentation $\langle x, y | (q\ell + pm)x + py \rangle$. Since $p$ and $q$ are coprime, $H_1(E(K))$ is isomorphic to $\langle x', y' | \ell x' + py' \rangle \cong \mathbb{Z} \oplus \mathbb{Z}_{\gcd(p, \ell)}$. If $p = 0$, then $q = 1$ and so the result follows. \hfill \Box

Using the first homology of the exterior, it is now easy to determine when a lens space admits a $k$-fold cyclic branched cover branched along a 1-bridge torus knot $K$. In the proof of the above corollary, $H_1(E(K)) = \langle x, y | (q\ell + pm)x + py \rangle$ without changing generators and $x$ represents a meridian of $K$. A $k$-fold cyclic branched cover exits if and only if there is an epimorphism $\phi : H_1(E(K)) \to \mathbb{Z}_k$ sending $x$ to 1. This is equivalent to find $\phi(y) \in \mathbb{Z}$ satisfying $q\ell + pm + p\phi(y) \equiv 0 \mod k$. Such a $\phi(y)$ exists if and only if $q\ell + pm$ is a multiple of $\gcd(p, k)$. Since the exponent sum $m$ of $x$ in the word $R(1, 0)$ is not hard to compute, the last condition is readily verified. If $(p, q) = (0, 1)$, then $H_1(E(K)) = \langle x, y | \ell x \rangle$ and so
we have the condition that ℓ is a multiple of k instead. When k = 2, the last condition is simple to describe in term of parameters in a Schubert’s normal form.

**Corollary 3.2.** Let a 1-bridge torus knot K in a lens space L(p,q) have a Schubert’s normal form S(r,s,t,p). L(p,q) has a double branched cover branched along K if and only if either p is odd or p is even and s + t is odd.

**Proof.** Since k = 2, gcd(p,k) = 1 or 2. If p is odd, gcd(p,k) = 1 and so the last condition above always holds. If p is even, gcd(p,k) = 2 and the last condition holds iff ℓ is even since q must be odd. Since K intersects the meridian disk of V₁ geometrically 2r + s + t + 1 times, 2r + s + t + 1 ≡ ℓ mod 2 and so s + t is odd. This proof includes the case when (p,q) = (0,1).

In [3], the first homology of the exterior of a 1-bridge torus knot was given in terms of an abstract description of a 1-bridge torus knot via a mapping class of twice punctured torus and also a necessary and sufficient condition for the existence of k-fold branched cover of a lens space along a 1-bridge torus knot was given. However it is hard to apply their result to a 1-bridge torus knot given by a diagram, for example.

4. **Conway’s normal forms and Double branched covers**

In this section, we describe double branched covers of a 3-manifolds L(p,q) branched along 1-bridge torus knots and compute their first homologies as invariants of 1-bridge torus knots. In order to do this, we introduce another parameterization of 1-bridge torus knots that is an analogue of Conway’s normal forms of 2-bridge knots.

4.1. **Conway’s normal forms of 1-bridge torus knots.** Let M(1,0) and M(1,2) denote the mapping class groups of a torus and a two-punctured torus, respectively. By ignoring punctures, we obtain a homomorphism jₜ: M(1,2) → M(1,0). Let h be a mapping class in M(1,2) that sends a (0,1)-curve of ∂V₂ to a (p,q)-curve of ∂V₁. Then (V₁, t₁)∪₁(V₂, t₂) is a (1,1)-decomposition of a 1-bridge torus knot K in a 3-manifold L(p,q). Let ¯t₁ be an arc on ∂V₁ such that t₁ ∪ ¯t₁ bounds a cancelling disk in V₁ for i = 1, 2. Then a knot diagram of K can be recovered by either t₁ ∪ h(¯t₁) on the side of V₁ or t₂ ∪ h⁻¹(¯t₁) on the side of V₂. Let h₀ be a mapping class in M(1,2) such that jₜ(h₀) = jₜ(h) and h₀ fixes a small disk containing two punctures. Then h = h ∘ h₀⁻¹ is in ker jₜ. For a fixed ambient space L(p,q), a 1-bridge torus knot is essentially determined by an element of the subgroup ker jₜ. Using results in [3, 4] by Joan Birman, we will give a presentation of the group ker jₜ in the following lemma. Let σ be a homeomorphism exchanging two punctures as illustrated in Figure 2 and τₘ (resp. τₜ) be a homeomorphism sliding one of punctures along the meridian (resp. longitude) as illustrated in Figure 19.

**Lemma 4.1.** The group ker jₜ is generated by τₜ, τₘ and σ under the following defining relations:

1. τₜσ⁻¹ = στₜ⁻¹ and τₘσ = σ⁻¹τₘ⁻¹,
2. σ² = τₘ⁻¹τₜ⁻¹τₘτₜ.
Figure 19.

Proof. The group $\ker j_*$ is generated by $\tau_\ell$, $\tau_m$ and $\sigma$ by Theorem 9 in [3] and it is easy to check the above defining relation from the relations of Theorem 9 in [3] and Corollary 1.3 in [4].

Since a composition of homeomorphisms (or mapping classes) can be also regarded as a group multiplication, some confusion may arises. For example their written orders are opposite and so homeomorphisms act from the right as an element of a group. In this paper we will distinguish two operations by placing “$\circ$” between homeomorphisms when they are composed.

Consider the homeomorphisms $h_\ell = \tau_\ell \sigma^{-1} \tau_\ell^{-1}$ and $h_m = \tau_m \sigma \tau_m^{-1}$. Figure 20 shows their effect on a knot by giving the action by $h_\ell^{-1}$ and $h_m^{-1}$ on the arc $t_1$ in $V_1$.

Figure 20.

Lemma 4.2. Let $H$ be the subgroup of $\ker j_*$ generated by $\sigma$, $h_\ell$, and $h_m$. Then

1. A word in $\ker j_*$ belongs to $H$ if and only if both the exponent sum of $\tau_\ell$ and the exponent sum of $\tau_m$ in the word are even.

2. The subgroup $H$ is a free group.

Proof. (1) is clear since $\sigma \tau_\ell^{-2} = h_\ell$ and $\sigma^{-1} \tau_m^{-2} = h_m$ by the relation (1) of Lemma 4.1.

We will show that $H$ is a free group using the Schreier-Reidemeister method (See [18]). By Lemma 4.1, $X = \{\sigma, \tau_\ell, \tau_m\}$ is a set of generators and $R = \{(\tau_\ell \sigma^{-1})^2, (\tau_m \sigma)^2,$
\(\sigma^2\tau^{-1}_m \tau^{-1}_m \tau^{-1}_m \) is a set of defining relations for the group \(\ker j_*\). The subgroup \(H\) is normal in \(\ker j_*\) since conjugates of generators of \(H\) by generator of \(\ker j_*\) are again in \(H\) as one can see in the list: \(\tau_h \tau^{-1}_m = h^{-1}_1 \sigma^{-1}_1 h_t, \tau^{-1}_m h \sigma^{-1}_m = \sigma h \sigma h^{-1}_m, \tau h \tau^{-1}_m = h^{-1}_1 \sigma h_m, \tau h \tau^{-1}_m = \sigma^{-1}_1 h \sigma h^{-1}_m, \) and \(\tau h \tau^{-1}_m = h^{-1}_1 \sigma^{-1}_1 h\). In fact, \(\ker j_* / H\) is isomorphic to \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\) and so we can choose \(T = \{1, \tau_m, \tau_t, \tau_m \tau_t\}\) as a Schreier representative system. By Schreier-Reidemeister method, the subgroup \(H\) is generated by \(s\)-symbols \(s_{k,g} = kgk^{-1}\) where \(k \in T, g \in X\) and \(kg\) is a coset representative of \(kg\) in \(\ker j_* / H\). And \(H\) has the defining relations \((tkr^{-1})\) where \(k \in T, r \in R\) and \(t(\cdot)\) is a rewriting process. Hence this presentation of \(H\) has the following generators:

\[
s_{1,\sigma}, s_{1,\tau_t} = 1, s_{1,\tau_m} = 1, s_{\tau_m,\sigma}, s_{\tau_m,\tau_t} = 1, s_{\tau_m,\tau_m},
\]

and defining relations:

\[
t(\tau_1 \sigma^{-1} \tau_1 \sigma^{-1}) = s_{1,\tau_1} s_{\tau_1,\sigma} s_{\tau_1,\tau_1} s_{\tau_1,\tau_1} = s_{\tau_1,\sigma} s_{\tau_1,\tau_1} s_{1,\sigma},
\]

(1)

\[
t(\tau_1 \sigma \tau_1 \sigma) = s_{1,\tau_1} s_{\tau_1,\sigma} s_{\tau_1,\tau_1} s_{\tau_1,\tau_1} = s_{\tau_1,\sigma} s_{\tau_1,\tau_1} s_{1,\sigma},
\]

(2)

\[
t(\sigma^2 \tau^{-1}_m \tau^{-1}_m \tau^{-1}_m) = s_{2,\tau_1} s_{\tau_1,\tau_1} s_{\tau_1,\tau_1} s_{\tau_1,\tau_1} = s_{2,\tau_1} s_{\tau_1,\tau_1} s_{1,\tau_1},
\]

(3)

\[
t(\tau_1 \tau^{-1}_m \tau^{-1}_m \tau^{-1}_m) = s_{1,\tau_1} s_{\tau_1,\tau_1} s_{\tau_1,\tau_1} s_{\tau_1,\tau_1} = s_{1,\tau_1} s_{\tau_1,\tau_1} s_{1,\tau_1},
\]

(4)

\[
t(\tau_1 \sigma \tau^{-1}_m \tau^{-1}_m \tau^{-1}_m) = s_{1,\tau_1} s_{\tau_1,\sigma} s_{\tau_1,\tau_1} s_{\tau_1,\tau_1} = s_{1,\tau_1} s_{\tau_1,\tau_1} s_{1,\tau_1},
\]

(5)

\[
t(\tau_1 \sigma^2 \tau^{-1}_m \tau^{-1}_m \tau^{-1}_m) = s_{1,\tau_1} s_{\tau_1,\tau_1} s_{\tau_1,\tau_1} s_{\tau_1,\tau_1} = s_{1,\tau_1} s_{\tau_1,\tau_1} s_{1,\tau_1},
\]

(6)

\[
t(\tau_1 \sigma^{-1} \tau^{-1}_m \tau^{-1}_m \tau^{-1}_m) = s_{1,\tau_1} s_{\tau_1,\sigma} s_{\tau_1,\tau_1} s_{\tau_1,\tau_1} = s_{1,\tau_1} s_{\tau_1,\tau_1} s_{1,\tau_1},
\]

(7)

\[
t(\tau_1 \sigma \tau_1 \sigma \tau_1 \sigma) = s_{1,\tau_1} s_{\tau_1,\sigma} s_{\tau_1,\tau_1} s_{\tau_1,\tau_1} = s_{1,\tau_1} s_{\tau_1,\tau_1} s_{1,\tau_1},
\]

(8)

\[
t(\tau_1 \sigma \tau_1 \sigma \tau_1 \sigma) = s_{1,\tau_1} s_{\tau_1,\sigma} s_{\tau_1,\tau_1} s_{\tau_1,\tau_1} = s_{1,\tau_1} s_{\tau_1,\tau_1} s_{1,\tau_1},
\]

(9)

\[
t(\tau_1 \sigma \tau_1 \sigma \tau_1 \sigma) = s_{1,\tau_1} s_{\tau_1,\sigma} s_{\tau_1,\tau_1} s_{\tau_1,\tau_1} = s_{1,\tau_1} s_{\tau_1,\tau_1} s_{1,\tau_1},
\]

(10)

\[
t(\tau_1 \sigma \tau_1 \sigma \tau_1 \sigma) = s_{1,\tau_1} s_{\tau_1,\sigma} s_{\tau_1,\tau_1} s_{\tau_1,\tau_1} = s_{1,\tau_1} s_{\tau_1,\tau_1} s_{1,\tau_1},
\]

(11)

\[
t(\tau_1 \sigma \tau_1 \sigma \tau_1 \sigma) = s_{1,\tau_1} s_{\tau_1,\sigma} s_{\tau_1,\tau_1} s_{\tau_1,\tau_1} = s_{1,\tau_1} s_{\tau_1,\tau_1} s_{1,\tau_1},
\]

(12)
The defining relators (1) and (2) induce the same relation, \( s_{r_m, r_m} = s_{r_m, s_1, s} \), (3) and (4) induce \( s_{r_m, r_m} = s_{r_m, s_1, s}^{-1} \), (4) and (10) induce \( s_{r_m, r_m} = s_{r_m, s_1, s} \), (12) induces \( s_{r_m, r_m} = s_{r_m, s_1, s} \), and finally (8) and (11) induce \( s_{r_m, r_m} = s_{r_m, s_1, s}^{-1} \).

Thus \( s \)-symbols \( s_{r_m, r_m}, s_{r_m, r_m}, s_{r_m, r_m}, s_{r_m, r_m} \) and \( s_{r_m, r_m} \) together with these defining relators can be deleted from the presentation for \( H \) by Tietze transformations. Furthermore, the remaining defining relators are:

\[
\begin{align*}
(3) &= s_{1, s}^2 (s_{1, s}^2)^{-1}, (s_{r_m, s_1, s}^2 s_{r_m, s_1, s}^{-1}) (s_{r_m, s_1, s}^2) (s_{r_m, s_1, s}^2) = 1 \\
(4) &= s_{r_m, s_1, s}^2 (s_{r_m, s_1, s}^2 s_{r_m, s_1, s}^{-1}) (s_{r_m, s_1, s}^2) (s_{r_m, s_1, s}^2) = s_{r_m, s_1, s} s_{r_m, s_1, s} s_{r_m, s_1, s} \\
(5) &= s_{r_m, s_1, s}^2 (s_{r_m, s_1, s}^2 s_{r_m, s_1, s}^{-1}) (s_{r_m, s_1, s}^2) (s_{r_m, s_1, s}^2) = s_{r_m, s_1, s} s_{r_m, s_1, s} s_{r_m, s_1, s}.
\end{align*}
\]

Since the defining relator \( (14) \) induces the relation \( s_{r_m, r_m} = s_{r_m, s_1, s}^{-1} s_{r_m, s_1, s}^{-1} \), the \( s \)-symbol \( s_{r_m, r_m} \) and the defining relator \( (14) \) can be deleted. Finally

\[
(13) = s_{r_m, s_1, s} (s_{r_m, s_1, s}^2 s_{r_m, s_1, s}^{-1} s_{r_m, s_1, s}^{-1}) = 1
\]

Consequently no defining relators remain and so \( H \) is a free group generated by three \( s \)-symbols \( s_{1, s}, s_{r_m, s_1, s} \), and \( s_{r_m, r_m} \) that correspond to \( \sigma, \tau_m \), and \( \tau_m^{-1} \), respectively.

For integers \( a_1, \ldots, a_m, b_1, \ldots, b_m \) and \( \delta = 0 \) or 1, let \( \tilde{h} \) be a homeomorphism in \( H \) defined by

\[
\tilde{h} = \tau_\delta^i (\sigma^{-b_1} h_{a_m}^{-b_2} h_{a_2}^{-b_2} \cdots (\sigma^{-b_m} h_{a_m}^{-b_m} h_{a_m}^{-b_m})
\]

Recall that \( h_0 \) is a homeomorphism of a torus that fixes a neighborhood of two punctures and produces a 3-manifold \( L(p, q) \). Then \( \tilde{h} \) determines a 1-bridge torus knot \( K \) in \( L(p, q) \) via a \((1,1)\)-decomposition \((V_1, t_1) \cup_h (V_2, t_2)\) for \( h = \tilde{h} \circ h_0 \). Here the expression

\[
[(a_m, b_m, a_{m-1}, b_{m-1}), \ldots, (a_4, b_4, a_3, b_3), (a_2, b_2, a_1, b_1)]^\delta.
\]

will be called \( \text{Conway's normal form} \) of the 1-bridge torus knot \( K \) in \( L(p, q) \). When \( \delta = 0 \), \( \delta \) will be omitted in a Conway’s normal form.

**Theorem 4.3.** Every 1-bridge torus knot in a 3-manifold \( L(p, q) \) of genus \( \leq 1 \) has a Conway’s normal form and it is unique as an element of \( H \). Furthermore, if \( p \) is odd then a 1-bridge torus knot in \( L(p, q) \) has a Conway’s normal form with \( \delta = 0 \).

**Proof.** An element of \( \ker j_* \) belongs to the one of the subsets \( H, H \tau_m, \tau_\delta H, \) and \( \tau_H \tau_m \) as we have seen in the proof of Lemma 4.2. Thus for a homeomorphism \( \tilde{h} \) in \( \ker j_* \), \( \tilde{h} = \tau_\delta^i g_{r_m} \)

for some \( g \in H \) and \( \epsilon, \delta = 0 \) or 1. Two homeomorphisms \( \tilde{h} \) and \( \tilde{h} \tau_m^{-\epsilon} \) determine the same 1-bridge torus knot and so any 1-bridge torus knot is represented by an element of \( H \cup \tau_H \), that is, it has a Conway’s normal form, and since by Lemma 4.2 \( H \) is a free group generated by \( \sigma, h_{\epsilon} \) and \( h_m \), a Conway’s normal form is unique as an element of \( H \) if it is freely reduced.

Suppose that \( p \) is odd. Let \( f \) be a homeomorphism in \( \ker j_* \) that slides a puncture once around a \((p, q)\)-curve passing through the puncture. Then for any element \( \tilde{h} \) in \( \ker j_* \),
$\tilde{h}$ and $\tilde{h} \circ f$ represent the same 1-bridge torus knots in the lens space $L(p, q)$ since the effect of the action by $f$ on the arc $t_1$ is nullified by the attached 2-handle of $L(p, q)$. Since the word $f$ as an element of ker $j_*$ must contain $p$ $\tau_\ell$’s and $p$ is odd, either $\tilde{h}$ or $\tilde{h} \circ f$ contains an even number of $\tau_\ell$’s. This completes the proof because any word in $H \cup \tau_\ell H$ that contains an even number of $\tau_\ell$’s must be in $H$. \hfill \Box

Consider a 2-bridge knot that has the Conway’s normal form $[2a_1, 2a_2, \ldots, 2a_m]$ as a 2-bridge knot. Then we obtain its (1,1)-decomposition using an unknotting tunnel $\rho$ as illustrated in Figure 21 and the homeomorphism $\tilde{h}$ in ker $j_*$ of this (1,1)-decomposition can be read as $(\tau_\ell^{-1} \circ \sigma) \circ (\sigma^{2a_m} \circ \tau_m^{a_m-1}) \circ \cdots \circ (\sigma^{2a_2} \circ \tau_m^{a_1})$. By converting $\tilde{h}$ into an element of $H$ using the relations in Lemma 4.1, we may obtain a Conway’s normal form of a 2-bridge knot as a 1-bridge torus knot.

Figure 21.

Now we describe how to obtain a Conway’s normal form from a given Schubert’s normal form of a 1-bridge torus knot in $S^3$. Let $S(r, s, t, mn + \bar{\rho})_\epsilon$ be a Schubert’s normal form of a 1-bridge torus knot, where $r, s, t$ nonnegative integers, $-n < \bar{\rho} < n$, $m$ is an integer, $\epsilon = \pm 1$ and $n = 2r + 1 + s + t$. If $n$ is even, we may assume that $m \equiv \bar{\rho} \mod 2$ since $mn + \bar{\rho} = (m + 1)n + (\bar{\rho} - n)$. If $n$ is odd, we may assume that $\bar{\rho}$ is even by changing $\epsilon$ since $S(r, s, t, \rho)_{+1}$ is equivalent to $S(r, t, s, \rho + (2r + 1))_{-1}$.

$-1$ full twist

Figure 22. $S(2, 2, 0, 7 - 2)_{+1}$

In order to achieve a symmetry in a knot $S(r, s, t, mn + \bar{\rho})_\epsilon$, we slightly modify the knot diagram $K$ by moving $m$ full twists in the front as in Figure 22. Then $K$ intersects the $(1,0)$-curve $|m|(s+t)+|\bar{\rho}|$ times. Under our assumption, $|m|(s+t)+|\bar{\rho}|$ is always even. Thus an even number of $\tau_m$’s must appear in any word $\tilde{h}$ in ker $j_*$ that undoes the part $\alpha$ lying on the torus until the number of intersections between $\alpha$ and the $(0,1)$-curve becomes 0 or 1.
like Figure 23. This means $h$ belongs to the subgroup $H$. Thus we start to simplify the arc $\alpha$ on the torus by applying $h_\ell, h_m$ and $\sigma$ as in Figure 24 until the diagram becomes one of three arcs as in Figure 23, all of which form the trivial knot in $S^3$. For example, the knot in Figure 23 has a Conway’s normal form $[(1, 0, -1, 1), (0, 0, -1, 1)]_1$. If the ambient space is $S^3$ then $[(1, 0, -1, 1), (0, 0, -1, 1)]_1$ is isotopic to $[(1, 0, -1, 1), (0, 0, -1, 1)]$ by Theorem 4.3 and it is isotopic to the knot in Figure 25.

4.2. Double branched covers along 1-bridge torus knots. Let $K$ be a 1-bridge torus knot in a 3-manifold $L(p, q)$ with a Conway’s normal form

$$[(a_m, b_m, a_{m-1}, b_{m-1}), \ldots, (a_4, b_4, a_3, b_3), (a_2, b_2, a_1, b_1)]_\delta.$$ 

We now describe a Heegaard splitting of the double branched cover of $L(p, q)$ branched along $K$ and compute its first homology.

By Corollary 3.2, there exists a double branched cover branched along $K$ if and only if either $p$ is odd or $p$ is even and the geometric intersection of $K$ with the meridian disk of
Figure 25. Conway’s normal form \([(1, 0, -1, 1), (0, 0, -1, 1)] \) in \( S^3 \)

\( V_1 \) is even. Since this geometric intersection is exactly contributed by the action of each \( \tau_\ell \) on \( \tilde{t}_2 \), the number of \( \tau_\ell \)'s in \( \tilde{h} \) is even and so \( \delta = 0 \). When \( p \) is odd, we may assume \( \delta = 0 \) by Theorem 4.3. Thus we omit \( \delta \) in Conway’s normal forms from now on.

Let a genus two handlebody \( \Sigma \) denote the double branched cover of a solid torus \( V \) branched along a trivial arc properly embedded in \( V \). Let \( \ell_1, \ell_2, m_1, m_2, c_1, c_2, \) and \( c_3 \) be simple closed curves on \( \partial \Sigma \) as in Figure 26. Let \( d_c \) denote the Dehn twist along a simple closed curve \( c \). Then the following are evident from Figure 26 and Figure 27:

1. The lifting \( \tilde{h}_0 \) of \( h_0 \) is a homeomorphism of \( \partial \Sigma \) such that \( \tilde{h}_0(m_i) \) (or \( \tilde{h}_0(\ell_i) \)) represents \( q[m_i] + p[\ell_i] \) (\( q'[m_i] + p'[\ell_i] \), respectively) on \( H_1(\partial \Sigma) \) for \( i = 1, 2 \).
2. The lifting \( \tilde{\sigma} \) of \( \sigma \) is \( d_{c_2} \).
3. The lifting \( \tilde{h}_\ell \) of \( h_\ell \) is \( d_{c_1}^{-1}d_1^2d_2^2 \).
4. The lifting \( \tilde{h}_m \) of \( h_m \) is \( d_{c_3}^{-1}d_{m_1}^2d_{m_1}^2 \).

Thus the following theorem is immediate.

**Theorem 4.4.** If a 1-bridge torus knot \( K \) in a 3-manifold \( L(p, q) \) has a Conway’s normal form

\[
[(a_m, b_m, a_{m-1}, b_{m-1}), \ldots, (a_4, b_4, a_3, b_3), (a_2, b_2, a_1, b_1)]
\]

then the double branched cover \( X \) of \( L(p, q) \) branched along \( K \) has a genus-two Heegaard splitting \( \Sigma_1 \cup \tilde{h} \Sigma_2 \) where

\[
\tilde{h} = (\tilde{h}_\ell^a \circ \tilde{\sigma}^{-b} \circ \tilde{h}_m^{a_{m-1}} \circ \tilde{\sigma}^{-b_{m-1}}) \circ \cdots \circ (\tilde{h}_\ell^a \circ \tilde{\sigma}^{-b_2} \circ \tilde{h}_m^{a_1} \circ \tilde{\sigma}^{-b_1}) \circ \tilde{h}_0.
\]

To save notations, \( H_1(\partial \Sigma_1) \) and \( H_1(\partial \Sigma_2) \) are identified and denoted by \( H_1(\partial \Sigma) \). Then \( H_1(\partial \Sigma) \) is generated by \([m_1], [m_2], [\ell_1], \) and \([\ell_2] \). The homeomorphism \( \tilde{\sigma} \) induces the
Figure 27. The lifting of $\sigma$ and the homoemorphisms $h_\ell$, $h_m$

The identity map on $H_1(\partial \Sigma)$. The induced homomorphisms $(\tilde{h}_\ell^a)_*$ and $(\tilde{h}_m^b)_*$ are isomorphisms such that

$$(\tilde{h}_\ell^a)_*([m_1]) = [m_1] + a[\ell_1] - a[\ell_2], \quad (\tilde{h}_m^b)_*([\ell_1]) = [\ell_1],$$

$$(\tilde{h}_\ell^a)_*([m_2]) = -a[\ell_1] + [m_2] + a[\ell_2], \quad (\tilde{h}_m^b)_*([\ell_2]) = [\ell_2],$$

$$(\tilde{h}_m^b)_*([m_1]) = [m_1], \quad (\tilde{h}_m^b)_*([\ell_1]) = b[m_1] + [\ell_1] - b[m_2],$$

$$(\tilde{h}_m^b)_*([m_2]) = [m_2], \quad (\tilde{h}_m^b)_*([\ell_2]) = -b[m_1] + b[m_2] + [\ell_2].$$

Therefore $(\tilde{h}_\ell^a)_*$ and $(\tilde{h}_m^b)_*$ are represented by matrices

$$\begin{bmatrix}
1 & 0 & 0 & 0 \\
a & 1 & -a & 0 \\
0 & 0 & 1 & 0 \\
-a & 0 & a & 1
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
1 & b & 0 & -b \\
0 & 1 & 0 & 0 \\
0 & -b & 1 & b \\
0 & 0 & 0 & 1
\end{bmatrix},$$

respectively.

Hence the induced homomorphism $\tilde{h}_s$ is represented by a matrix $A$ where

$$A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
an_m & 1 & -a_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_m & 0 & a_m & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \cdots \begin{bmatrix}
1 & a_1 & 0 & -a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -a_1 & 1 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q & q' & 0 & 0 & 0 & 0 & 0 & 0 \\
p & p' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.$$

Lemma 4.5.

$$\tilde{h}_s([m_1]) = \sum_{i=1}^{2} (-1)^{i-1} \frac{1}{2} (z_m + (-1)^{i-1}q)[m_i] + (-1)^{i-1} \frac{1}{2} (z_{m+1} + (-1)^{i-1}p)[\ell_i],$$

where $z_m$ is a sequence such that $z_m = 2a_{m-1}z_{m-1} + z_{m-2}$, $z_0 = q$ and $z_1 = p$. 
Proof. Let

\[ \tilde{h}_*([m_1]) = \sum_{i=1}^{2} (x_i[m_i] + y_i[l_i]) \]  

and \( A \) be a matrix representing \( \tilde{h}_* \), then

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
\text{a}_1 & 1 & -\text{a}_1 & 0 \\
0 & 0 & 1 & 0 \\
-\text{a}_1 & 0 & \text{a}_1 & 1
\end{bmatrix}
\begin{bmatrix}
\text{r}_m \\
\text{r}_{m-1} \\
\text{s}_m \\
\text{s}_{m-1}
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
\text{r}_m \\
\text{r}_{m-1} \\
\text{s}_m \\
\text{s}_{m-1}
\end{bmatrix}
\begin{bmatrix}
1 & \text{a}_{m-1} & 0 & -\text{a}_{m-1} \\
0 & 1 & 0 & 0 \\
0 & -\text{a}_{m-1} & 1 & \text{a}_{m-1} \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\text{r}_{m-2} \\
\text{r}_{m-1} \\
\text{s}_{m-2} \\
\text{s}_{m-1}
\end{bmatrix}
\]

Therefore

\[
x_1 = \text{r}_m, \quad y_1 = \text{a}_m(\text{r}_m - \text{s}_m) + \text{r}_{m-1}, \quad (18)
\]
\[
x_2 = \text{s}_m, \quad y_2 = -\text{a}_m(\text{r}_m - \text{s}_m) + \text{s}_{m-1}, \quad (19)
\]
\[
\text{r}_m = \text{a}_{m-1}(\text{r}_{m-1} - \text{s}_{m-1}) + \text{r}_{m-2}, \quad (20)
\]
\[
\text{s}_m = -\text{a}_{m-1}(\text{r}_{m-1} - \text{s}_{m-1}) + \text{s}_{m-2}, \quad (21)
\]

and \( \text{r}_0 = \text{q}, \text{r}_1 = \text{p} \) and \( \text{s}_0 = \text{s}_1 = 0 \) since

\[
\begin{bmatrix}
\text{r}_0 \\
\text{r}_1 \\
\text{s}_0 \\
\text{s}_1
\end{bmatrix}
\begin{bmatrix}
\text{q} & \text{q'} & 0 & 0 \\
\text{p} & \text{p'} & 0 & 0 \\
0 & 0 & \text{q} & \text{q'} \\
0 & 0 & \text{p} & \text{p'}
\end{bmatrix}
\]

Let \( \text{z}_m = \text{r}_m - \text{s}_m \) and \( \tilde{\text{z}}_m = \text{r}_m + \text{s}_m \). Then by the relations \((20)\) and \((21)\)

\[
\text{z}_m = 2\text{a}_{m-1}\text{z}_{m-1} + \text{z}_{m-2}, \quad \text{z}_0 = \text{q} \quad \text{and} \quad \text{z}_1 = \text{p}.
\]
\[
\tilde{\text{z}}_m = \tilde{\text{z}}_{m-2}, \quad \tilde{\text{z}}_0 = \text{q} \quad \text{and} \quad \tilde{\text{z}}_1 = \text{p}.
\]

So \( \tilde{\text{z}}_m = \text{q} \) if \( m \) is even, \( \tilde{\text{z}}_m = \text{p} \) otherwise. Since \( m \) is even,

\[
\text{r}_m = 1/2(\text{z}_m + \tilde{\text{z}}_m) = 1/2(\text{z}_m + \text{q}), \quad \text{r}_{m-1} = 1/2(\text{z}_{m-1} + \text{p}),
\]
\[
\text{s}_m = -1/2(\text{z}_m - \tilde{\text{z}}_m) = -1/2(\text{z}_m - \text{q}), \quad \text{s}_{m-1} = -1/2(\text{z}_{m-1} - \text{p}).
\]
Therefore by the relations (18) and (19)

\[ x_1 = r_m = 1/2(z_m + q), \]
\[ y_1 = a_m z_m + r_{m-1} = a_m z_m + 1/2(z_{m-1} + p) = 1/2(z_{m+1} + p), \]
\[ x_2 = s_m = -1/2(z_m - q), \]
\[ y_2 = -a_m z_m + s_{m-1} = -a_m z_m - 1/2(z_{m-1} - p) = -1/2(z_{m+1} - p). \]

Hence the proof is completed by the equation (17). \[ \square \]

For a sequence \( \{a_m\} \) and \( i = 1, 2 \), define

\[ C_{i,m}^t = \{(j_1, \ldots, j_t) \in \mathbb{N}^t \mid i \leq j_1 < \cdots < j_t < m, \ j_k - j_{k-1} \geq 2, \ k = 1, \ldots, t \}, \]

and

\[ A_{i}^m(j_1, j_2, \ldots, j_t) = (a_{i}a_{i+1} \cdots a_{j_{t-1}})(a_{j_{t+2}} \cdots a_{j_{t-1}}) \cdots (a_{j_{t+2}} \cdots a_{m}), \]

Then in particular \( A_{1}^m(1, 3, \cdots, m - 1) = 1 \) if \( m \) is even and \( A_{2}^m(2, 4, \cdots, m - 1) = 1 \) if \( m \) is odd. Using these notations, we give a solution \( z_m \) to the recursive formula in Lemma 4.5. But we will omit the proof because it can be easily observed by an inspection rather than a written proof.

**Proposition 4.6.** Let \( z_m \) be a sequence satisfying the recursive formula

\[ z_{m+1} = 2a_m z_m + z_{m-1}, \ z_0 = q \text{ and } z_1 = p, \]

for some sequence \( a_m \). Then

\[ z_{m+1} = p z_{m+1}^{(1)} + q z_{m+1}^{(2)}, \]  \[ (22) \]

where for \( i = 1, 2 \)

\[ z_{m+1}^{(i)} = 2^{(m+1)-i} (a_i a_{i+1} \cdots a_m) + \sum_{t=1}^{(m+1)-1} 2^{(m+1)-2t} \sum_{(j_1, \ldots, j_t) \in C_{i,m}^t} A_{i}^m(j_1, j_2, \ldots, j_t). \]  \[ (23) \]

From Lemma 4.5 and Proposition 4.6, we calculate the first homology of \( X \).

**Theorem 4.7.** Let \( K \) be a 1-bridge torus knot in a 3-manifold \( L(p, q) \) with a Conway’s normal form

\[ [(a_m, b_m, a_{m-1}, b_{m-1}), \ldots, (a_4, b_4, a_3, b_3), (a_2, b_2, a_1, b_1)], \]

and \( X \) be a double branched cover of \( L(p, q) \) branched along \( K \). Then \( H_1(X) \cong \mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2} \) where \( k_1 = \gcd(p, 1/2 z_{m+1}, z_{m+1}), k_2 = \lfloor pz_{m+1}/k_1 \rfloor, z_{m+1} \) is a sequence in (22), and \( z_{m+1}^{(2)} \) is the formula given in (23).

**Proof.** \( X \) is obtained from \( \Sigma_1 \) by attaching two 2-handles along \( \tilde{h}(m_1) \) and \( \tilde{h}(m_1) \) and filling in a 3-ball. Thus \( H_1(X) \) is isomorphic to \( H_1(\Sigma_1) \) modulo the image of \( i_* \tilde{h}_* \) for the inclusion.
i: ∂Σ_1 → Σ_1. \( H_1(\Sigma_1) \) is a free abelian group generated by \([\ell_1]\) and \([\ell_2]\). Lemma 4.3 and

\[
i_\ast \tilde{h}_\ast([m_1]) = \frac{1}{2}(z_{m+1} + p)[\ell_1] - \frac{1}{2}(z_{m+1} - p)[\ell_2],
\]

\[
i_\ast \tilde{h}_\ast([m_2]) = \frac{1}{2}(z_{m+1} + p)[\ell_2] - \frac{1}{2}(z_{m+1} - p)[\ell_1].
\]

Thus \( H_1(X) \) is an abelian group with a presentation matrix

\[
R = \begin{bmatrix}
\frac{1}{2}(z_{m+1} + p) & -\frac{1}{2}(z_{m+1} - p) \\
-\frac{1}{2}(z_{m+1} - p) & \frac{1}{2}(z_{m+1} + p)
\end{bmatrix} \sim \begin{bmatrix}
-\frac{1}{2}(z_{m+1} - p) & 0 \\
0 & z_{m+1}
\end{bmatrix}.
\]

Since \( z_{m+1} = p z_{m+1}^{(1)} + q z_{m+1}^{(2)} \) by Proposition 4.6,

\[
R \sim \begin{bmatrix}
\frac{1}{2}(p (z_{m+1}^{(1)} - 1) + q z_{m+1}^{(2)}) & 0 \\
0 & q\frac{1}{2} z_{m+1}^{(2)}
\end{bmatrix} \sim \begin{bmatrix}
p & 0 \\
q\frac{1}{2} z_{m+1}^{(2)} & z_{m+1}
\end{bmatrix}.
\]

Consequently \( H_1(X) \cong \mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2} \) where

\[
k_1 = \gcd(p, q\frac{1}{2} z_{m+1}^{(2)}, z_{m+1}) = \gcd(p, z_{m+1}^{(2)}, z_{m+1}) \text{ and } k_2 = |pz_{m+1}/k_1|.
\]

\[\square\]

**Corollary 4.8.** If the ambient space is \( S^3 \), then \( H_1(X) \cong \mathbb{Z}_{z_{m+1}^{(1)}} \) and so \( |z_{m+1}^{(1)}| = |\Delta_K(-1)| \) where \( \Delta_K(t) \) is the Alexander polynomial of \( K \).

The 1-bridge torus knot \( K = S(2, 2, 1, 8 + 1) \) in \( S^3 \) has a Conway’s normal form \([3, 0, 1, 0], (-1, 0, 1, 0)\) as in Figure 28. Therefore \( z_{5}^{(1)} = 2^4(-3) + 2^2(-1 + 3 + 3) + 1 = -27 \).

By Corollary 4.8, we have \( H_1(X) = \mathbb{Z}_{27} \). Furthermore \( |\Delta_K(-1)| = 27 \).

\[\text{Figure 28. Conway’s normal form } [(3, 0, 1, 0), (-1, 0, 1, 0)]\]

**Corollary 4.9.** Suppose \( K \) is a 1-bridge torus knots in \( S^3 \) with Conway’s normal form

\[
[(a_m, b_m, a_{m-1}, b_{m-1}), \ldots, (a_4, b_4, a_3, b_3), (a_2, b_2, a_1, b_1)].
\]

1. If \( a_{2i} = 0 \) or \( a_{2i-1} = 0 \) for \( i = 1, \ldots, m/2 \) then \( K \) is a trivial knot.
2. If \( a_i \)'s are all positive or all negative then \( K \) is a nontrivial knot.
Proof. (1) is evident from the knot diagram of $K$. By Corollary 1.8 the first homology of the double branched cover is $\mathbb{Z}_{|z_{m+1}^{(1)}|}$. If either $a_i > 0$ or $a_i < 0$ for all $i$ then by the formula (23), $z_{m+1}^{(1)} > 2^m(a_1a_2\cdots a_m) > 1$ since $m$ is even. Therefore (2) holds.

References

[1] G. Burde and H. Zieschang, Knots, Berlin-New York, Walter de Gruyter (1985)
[2] C. Bankwitz and H. G. Schumann, Über Viergeflechte, Abh. Math. Sem. Univ. Hamburg, 10 (1934) 263–284
[3] Joan S. Birman, On braid groups, Comm. Pure Appl. Math., 22 (1969) 41–72
[4] Joan S. Birman, Mapping class groups and their relationship to braid groups, Comm. Pure Appl. Math., 22 (1969) 213–238
[5] Joan S. Birman, Braids, links, and mapping class groups, Princeton Univ. Press and Univ. of Tokyo Press (1975)
[6] A. Cattabriga and M. Mulazzani, Strongly-cyclic branched coverings of (1,1)-knots and cyclic presentations of groups, preprint
[7] D. H. Choi and K. H. Ko, Subfamily of 1-bridge torus knots, preprint
[8] J. H. Conway, An enumeration of knots and links, and some of their algebraic properties, Computational Problems in Abstract Algebra, Pergamon Press, New York (1970) 329–358
[9] H. Doll, A generalized bridge number for links in 3-manifolds, Math. Ann. 294, (1992) 701–717
[10] H. Fujii, Geometric indices and the Alexander polynomial of a knot, Proc. of AMS., vol. 128, no. 9 (1996) 2923–2933
[11] D. Gabai, 1-bridge braids in solid tori, Topology and its Appl. 37 (1990) 221–235
[12] H. Goda and C. Hayashi, Genus two Heegaard splittings of exteriors of 1-genus 1-bridge knots, preprint
[13] L. Grasselli and M. Mulazzani, Genus one 1-bridge knots and Dunwoody manifolds, preprint
[14] P. Hill, On Double-torus Knots. I, J. Knot Theory Ram. 8, no. 8 (1999) 1009–1048
[15] P. Hill and K. Murasugi, On Double-torus Knots. II, J. Knot Theory Ram. 9, no. 5 (2000) 617-667
[16] H. Kim, Classification of simple arcs with fixed ends in the punctured plane, Thesis M.S., KAIST, (1994)
[17] T. Kobayashi and O. Saeki, Rubinstein-Scharlemann graphic of 3-manifolds as the discriminant set of a stable map, Pac. J. Math., 195 (2000) 101–156
[18] W. Magnus, A. Karrass and D. Solitar, Combinatorial group theory, Dover Publications, Inc. (1976)
[19] K. Morimoto and M. Sakuma, On unknotting tunnels for knots, Math. Ann., 289 (1991) 143–167
[20] K. Morimoto, M. Sakuma and Y. Yokota, Examples of tunnel number one knots which have the property ‘$1 + 1 = 3$’, Math. Proc. Camb. Phil. Soc., 119 (1996) 113–118
[21] K. Morimoto, M. Sakuma and Y. Yokota, Identifying tunnel number one knots, J. Math. Soc. Japan, vol. 48, no 4 (1996) 667-688
[22] D. Rolfsen, Knots and Links, Publish or Perish, Inc. (1976)
[23] H. Schubert, Knoten mit zwei brüchen, Math. Z., 65 (1956) 133–170
[24] H. J. Song and S. H. Kim, Dunwoody 3-manifolds and (1,1)-decomposible knots, preprint
[25] Y. Yokota, On quantum SU(2) invariants and generalized bridge numbers of knots, Math. Proc. Camb. Phil. Soc., 117 (1995) 545–557

E-mail address: dhchoi@knot.kaist.ac.kr

Department of Mathematics, Korea Advanced Institute of Science and Technology, Taejon, 305-701, Korea

E-mail address: knot@knot.kaist.ac.kr
PARAMETERIZATIONS OF 1-BRIDGE TORUS KNOTS

DEPARTMENT OF MATHEMATICS, KOREA ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY, TAEJON, 305-701, KOREA