On a surface formed by randomly gluing together polygonal discs

Sergei Chmutov and Boris Pittel
Department of Mathematics, Ohio State University,
231 West 18th Avenue, Columbus, OH 43210, USA
chmutov@math.ohio-state.edu and bgp@math.ohio-state.edu

September 16, 2015

Mathematics Subject Classifications: 05C80, 05C30, 05A16, 05E10, 34E05, 60C05

Keywords: surfaces, polygonal discs, random permutations, irreducible characters, Euler characteristic, genus, limit distributions

Abstract

Starting with a collection of \( n \) oriented polygonal discs, with an even number \( N \) of sides in total, we generate a random oriented surface by randomly matching the sides of discs and properly gluing them together. Encoding the surface by a random permutation \( \gamma \) of \( [N] \), we use the Fourier transform on \( S_N \) to show that \( \gamma \) is asymptotic to the permutation distributed uniformly on the alternating group \( A_N \) (\( A_N^c \) resp.) if \( N - n \) and \( N/2 \) are of the same (opposite resp.) parity. We use this to prove a local central limit theorem for the number of vertices on the surface, whence also for its Euler characteristic \( \chi \). We also show that with high probability (as \( N \to \infty \), uniformly in \( n \)) the random surface consists of a single component, and thus has a well-defined genus \( g = 1 - \chi/2 \), which is asymptotic to a Gaussian random variable, with mean \( (N/2 - n - \log N)/2 \) and variance \( (\log N)/4 \).
1 Introduction and main results

In this paper we study random surfaces obtained by gluing, uniformly at random, sides of \( n \) polygons with various (not necessarily equal) numbers of sides. We call this scheme of generating a surface the map model. (A model dual to the map model is very important for algebraic geometry [15]. It can be generalized to hypermaps; in [4] it is called the \( \sigma \)-model.) In the map model the interiors of polygons represent countries (faces); the glued sides represent boundaries between countries (edges). Thus the map model can be considered as a graph embedded into the surface such that the faces correspond to the original polygons.

This model generalizes the random map model, motivated by studies in quantum gravity, of Pippenger and Schleich [22], in which all polygons are triangles. For the Euler characteristic \( \chi \) of the randomly triangulated surface they proved that \( \mathbb{E}[\chi] = n/2 - \log n + O(1) \) and \( \text{Var}(\chi) = \log n + O(1) \), as well as made startlingly sharp conjectures regarding the remainder terms \( O(1) \) based on simulations and results for similar models. The case when the numbers of sides of all polygons are equal, gluings of \( k \)-gons (\( k \geq 3 \)), was considered by Gamburd in [12]. His breakthrough result was that (for \( 2\text{lcm}(2, k) \mid kn \)) the random permutation of polygon sides, that determines the map, was asymptotically uniform on the alternating group \( A_{kn} \). This implies, for instance, that \( \chi \) was asymptotic, in distribution, to \( n/2 \text{ minus } \mathcal{N}(\log n, \log n) \), the Gaussian variable with mean and variance equal to \( \log n \).

Fleming and Pippenger [9] used Gamburd’s result to prove sharp asymptotic formulas for the first four moments of the Euler characteristic \( \chi \), in particular confirming the earlier conjectures for \( k = 3 \) in [22]. Another special case of this model is when there is only one polygon whose sides are glued in pairs. This case is well studied, popular, and important in combinatorics and the theory of moduli spaces of algebraic curves. The classic paper of Harer and Zagier [13] solved the difficult problem of enumerating the resulting surfaces by genus. Their result was used in [5] to determine the limiting genus distribution for the surface chosen uniformly at random from all such surfaces.

Getting back to the \( n \) polygons, their sides are glued in pairs. So the total number of sides \( N \) of all polygons must be even, and the resulting map will have \( N/2 \) edges. We also assume that all polygons are oriented and that in each glued pair the edges are oriented opposite-wise. Thus the resulting surface will be oriented.
The map model can be described in terms of permutations. Label the oriented sides \( e \)'s (edges) of all polygons by numbers from \([N] := \{1, 2, \ldots, N\}\); \( e_\ell \) will denote the edge labeled \( \ell \). Let \( n_j \) be the number of polygons with \( j \) sides, \( j \)-gons, and let \( J \) stand for the set of all possible numbers of sides of our \( n = \sum_j j n_j \) polygons, so that \( \sum_{j \in J} j n_j = N \) and each map will have \( n \) faces. We define the permutation \( \alpha \) of \([N]\) as follows: \( \alpha(e_\ell) = e_k \) if \( e_k \) follows immediately after \( e_\ell \) in one of the \( n \) oriented polygons. Thus \( \alpha \) has \( n \) cycles, each cycle consisting of the edges of the corresponding polygon listed according to the polygon (clockwise) orientation. The set of all such \( \alpha \)'s is the conjugacy class \( C_n, \ n := \{n_j\}_{j \in J} \), of all permutations of \([N]\) with \( n_j \) cycles of length \( j \). A gluing itself is encoded by a permutation \( \beta \) which is a product of transpositions of edges that are glued to each other; those \( \beta \)'s are all \((N - 1)!!\) elements of the conjugacy class \( C_{N/2} \) of all permutations of \([N]\) with cycles of length 2 only.

Here is how a given pair of permutations \( \alpha, \beta \) induces the corresponding surface. The first edge \( e_1 \) is glued to the edge \( \beta(e_1) \); the edge \( \beta(e_1) \) is followed by the edge \( e_2 = \alpha(\beta(e_1)) = (\alpha \beta)(e_1) \) in the directed polygon that contains \( \beta(e_1) \). Next \( e_2 \) is glued to \( \beta(e_2) \) followed by \( e_3 = \alpha(\beta(e_2)) = (\alpha \beta)(e_2) \) in the cycle that contains \( \beta(e_2) \). Repeating this procedure produces a sequence of edges \( e_1, e_2, \ldots \), whose tails collapse into a single vertex. Since \( \alpha \beta \) is a permutation of \([N]\), the sequence \( e_1, e_2, \ldots \) eventually loops back on the starting edge \( e_1 \), forming a cycle \( e_1 \rightarrow e_2 \rightarrow \cdots \rightarrow e_m \rightarrow e_1 \) of \( \alpha \beta \), see the picture:

Note that the arrow arc around the vertex on the right figure merely indicates the orientation of the resulting surface, while the edges incident to the vertex are interchanged in the opposite (counterclockwise) direction by the permutation \( \gamma := \alpha \beta \).

Likewise, starting from the first edge not in this cycle, i.e., distinct from \( e_1, \ldots, e_m \), we obtain a cycle with disjoint support, containing this edge,
that determines another vertex of the map. Proceeding in this fashion, we eventually partition the edge set into disjoint subsets, each associated with its own vertex of the map. Clearly, the number of those subsets, i. e. the number of vertices $V_n$, equals the number of cycles of $\gamma = \alpha \beta$.

Also it is obvious that the connected components of the resulting surface, without boundary, correspond to the orbits of the subgroup generated by permutations $\alpha$ and $\beta$. For each such orbit we know the number of faces (which is equal to the number of cycles of $\alpha$ restricted to the orbit), the number of edges (which is equal to the number of cycles of $\beta$ restricted to the orbit), and the number of vertices (which is equal to the number of cycles of $\gamma$ restricted to the orbit). Thus we know the Euler characteristic of each connected component. So the number of cycles of $\gamma = \alpha \beta$ completely determines the topology of the surface.

**Example 1.1.** Consider two oriented squares with labeled sides:

In this case $N = 8$. The labeling fixes the permutation $\alpha = (1234)(5678)$. For the permutation $\beta$ there are $7!! = 105$ possibilities, that is the number of elements in the conjugacy class $C_{N/2}$. So totally we have 105 gluings. However, there are only 5 topological types of the resulting surface possible: 1) a sphere $S^2$, 2) a torus $T^2$, 3) two tori $2T^2$, 4) torus and a sphere $T^2 + S^2$, and 5) two spheres $2S^2$. All 105 gluings are distributed between these results according to the table:

| surface   | $S^2$ | $T^2$ | $2T^2$ | $T^2 + S^2$ | $2S^2$ |
|-----------|-------|-------|--------|-------------|--------|
| # gluings | 36    | 60    | 1      | 4           | 4      |

Below we exemplify only one gluing for each possible resulting surface.

1) $\alpha = (1234)(5678)$
$\beta = (15)(28)(37)(46)$
$\gamma = (16)(25)(38)(47)$
$\alpha = (1234)(5678)$
$\beta = (15)(24)(37)(68)$
$\gamma = (1652)(3874)$

$\alpha = (1234)(5678)$
$\beta = (13)(24)(57)(68)$
$\gamma = (1432)(5876)$

$\alpha = (1234)(5678)$
$\beta = (13)(24)(56)(78)$
$\gamma = (1432)(57)(6)(8)$
We define a *random surface* as the surface obtained by gluing via the permutations $\alpha$ and $\beta$ that are independently chosen uniformly at random from the conjugacy classes $C_n$ and $C_{N/2}$ respectively.

In Section 2 we show, Theorem 2.2, that the probability distribution of the permutation $\gamma$ is asymptotically uniform (for $N \to \infty$ uniformly in $n$) on $A_N$ ($A_N := S_N \setminus A_N$ resp.) if $C_{N/2}$ and $C_n$ are of the same (opposite resp.) parity. This generalizes the result from [12, Theorem 4.1] and improves the guaranteed rate of convergence from $N^{-1/12}$ to $N^{-1}$. By and large, we follow [12] to reduce the problem to Fourier-based analysis of the total variation distance between two probability measures on $S_N$, the main tool being a fundamental general bound due to Diaconis and Shahshahani [7]. At the crucial point, when we need to estimate a character value of an irreducible representation on a general conjugacy class of $S_N$ (rather than the classes $C_{N/\ell}$ in [12] treated via a bound discovered by Fomin and Lulov [10] in 1997), we use a bound proved recently by Larsen and Shalev [16].

In Section 3, as a corollary of Theorem 2.2, we state that the total variation distance between the distribution of $V_n$, the number of vertices on the random surface, and the distribution of the number of cycles $C_N$ ($C_N^o$ resp.) in the uniformly random even (odd resp.) permutation of $[N]$ is of order $O(N^{-1})$. Our main result, a local central limit theorem (LCLT) for $V_n$, follows then from a LCLT for $C_N$, the number of cycles in the permutation distributed uniformly on $S_N$, due to Kolchin [14]. The LCLT for the Euler characteristic $\chi_n$ of the random surface follows immediately.

In Section 4 we discuss the distribution of the number of connected components of the surface. Generalizing the result of Pippenger and Schleich for $J = \{3\}$, [22], we prove in Theorem 4.1 that the resulting surface is connected with probability $1 - O(N^{-1})$. Thus, with high probability, the genus
$g_n$ of the random surface is well defined, and using $g_n = 1 - \chi_n/2$ we obtain a LCLT for $g_n$. For a very special case of one polygon, $J = \{N\}$, this proves a slightly weaker version of our earlier result in [5].

2 Limiting uniformity

Given an even $N$, let $J = J(N)$ be a subset of $\{3, 4, \ldots\}$; the notation $J(N)$ underscores the possibility that this set may vary with $N$. Let $n := \{n_j\}_{j \in J}$ be a collection of nonnegative integers such that $\sum_{j \in J} jn_j = N$. Consider the set of all unordered partitions of $[N]$ into $n = \sum_{j \in J} n_j$ disjoint, directed, cycles, with $n_j$ cycles of lengths $j \in J$. This set can be viewed as the conjugacy class $C_n$ of all permutations $\alpha \in S_N$ with $n_j$ cycles of length $j \in J$. Let $C_{N/2}$ denote the conjugacy class of $S_N$ consisting of the fixed-point free involutions, i.e. permutations that are products of $N/2$ disjoint 2-cycles. More generally, for $t | N$, we will use $C_{N/t}$ to denote the conjugacy class of all permutations that are the products of $N/t$ cycles of length $t$.

Let $\alpha$ and $\beta$ be chosen independently, uniformly at random from $C_n$ and $C_{N/2}$ respectively, and let $\gamma = \alpha\beta$. Our focus is on the distribution of the number of cycles in $\gamma$. As noticed in [9], this number and the number of cycles in $\gamma$ for a fixed $\alpha$ are equidistributed.

**Theorem 2.1.** (Gamburd [12]) Let $J$ be a singleton $\{k\}$, $(k \geq 3)$. Suppose that $N \to \infty$ through values divisible by $2 \text{lcm}(2, k)$. Then the random permutations $\alpha, \beta$, whence $\gamma$ are all even. Let $P_\gamma$ be the probability distribution of $\gamma$ and let $U$ be the uniform probability measure on the alternating group $A_N$. Let $\|P_\gamma - U\| = \|P_\gamma - U\|_{TV} := (1/2) \sum_{s \in S_N} |P_\gamma(s) - U(s)|$ denote the total variation distance between $P_\gamma$ and $U$. Then

$$\|P_\gamma - U\| = O(N^{-1/12}). \quad (2.1)$$

As noted by Fleming and Pippenger [9], the original condition $\text{lcm}(2, k)|N$ in [12] does not guarantee that both $\alpha$ and $\beta$ are even, implying evenness of $\gamma$. Namely (assuming $\text{lcm}(2, k)|N$):

1. $\beta$ is even (odd resp.), if $4|N$ (4 \not| N resp.);
2. if $k$ is even and $2k|N$ (2k \not| N resp.), then $\alpha$ is even (odd resp.);
3. if $k$ is odd then $\alpha$ is even.

Thus $\alpha, \beta, \gamma$ are all even iff $2 \text{lcm}(2, k)|N$; by itself, $\gamma$ is even iff $\alpha$ and $\beta$ are of the same parity, i.e. iff $2k | N(k - 2)$.
Gamburd proved (2.1) by applying a character-based bound, due to Diaconis and Shahshahani \[7\], for the total variation distance between two probability measures (one being uniform) on a general finite group \(G\), to the special case when \(G\) was the alternating group \(A_N\). We found that Gamburd’s argument can be modified to prove a far more general, and stronger, result by using a modification of the bound in \[7\] for the group \(S_N\) itself, when the “uniform” measure is supported either by \(A_N\) or its complement \(A_N^c\), the choice being dependent upon parity of \(\gamma\).

**Theorem 2.2.** Let \(J = J(N) \subset \{3,4,\ldots\}\). Let \(n := \{n_j\}_{j \in J}\) be such that \(\sum_j jn_j = N\). For \(N \to \infty\), uniformly over \(n\), \(\gamma = \alpha \beta\) is asymptotically uniform over \(A_N\) (over \(A_N^c\) resp.) if \(C_n \ni C_{N/2}\) are of the same parity (of opposite parity resp.), and more precisely

\[
\|P \gamma - U_A\| = O(N^{-1}), \quad (\|P \gamma - U_{A_N^c}\| = O(N^{-1}) \text{ resp.}),
\]

(2.2)

\(U_A\), \(U_{A_N^c}\) being the probability measures uniform on \(A_N\) and \(A_N^c\) respectively.

For \(|J| = 1\), and even \(\alpha, \beta\), the (first) bound in (2.2) improves the bound (2.1).

**Proof.** Like in the proof of Theorem 2.1 in \[12\], the starting point is the Diaconis-Shahshahani’s bound. Let \(G\) be a finite group and \(P, U\) be two probability measures on \(G\), \(U\) being uniform, i.e. \(U(g) = 1/|G|\) for every \(g \in G\). Then

\[
\|P - U\|^2 \leq \frac{1}{4} \sum_{\rho \in \hat{G}, \rho \neq \text{id}} \dim(\rho) \text{tr}(\hat{P}(\rho)\hat{P}(\rho)^*)\;
\]

(2.3)

here \(\hat{G}\) denotes the set of all irreducible representations \(\rho\) of \(G\), “id” denotes the trivial representation, \(\dim(\rho)\) is the dimension of \(\rho\), and \(\hat{P}(\rho)\) is the matrix value of the Fourier transform of \(P\) at \(\rho\), \(\hat{P}(\rho) := \sum_{g \in G} \rho(g)P(g)\).

This bound follows from the Cauchy-Schwartz inequality

\[
4\|P - U\|^2 \leq |G| \sum_{s \in G} |P(s) - U(s)|^2,
\]

(2.4)

combined with the Plancherel Theorem

\[
|G| \sum_{s \in G} |P(s) - U(s)|^2 = \sum_{\rho \in \hat{G}} \dim(\rho) \text{tr}[(\hat{P}(\rho) - \hat{U}(\rho))(\hat{P}(\rho) - \hat{U}(\rho))^*],
\]

(2.5)
and the observation that (i) \( \hat{P}(\rho) = \hat{U}(\rho) = 1 \) for \( \rho = \text{id} \), and (ii) \( \hat{U}(\rho) = 0 \) for \( \rho \neq \text{id} \).

Now, (2.4)-(2.5) hold for any two measures on \( G \), whence for two probability measures \( P_H \) and \( U_H \) supported by the same subset \( H \subseteq G \). In this case, the condition (i) still holds, and we get

\[
\|P_H - U_H\|^2 \leq \frac{1}{4} \sum_{\rho \neq \text{id}} d(\rho) \text{tr}[(\hat{P}_H(\rho) - \hat{U}_H(\rho))(\hat{P}_H(\rho) - \hat{U}_H(\rho))^*].
\] (2.6)

To specify (2.6) for \( G = S_N \), we use some standard facts about the representations of \( S_N \), (see (Fulton and Harris [11], Lecture 5; Stanley [24], Ch. 7). First of all, the irreducible representations \( \rho \) are indexed by \( \lambda \)'s, where each \( \lambda \) is a partition \( \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots) \) of \( N \), \( \lambda \vdash N \) in short. It is customary to visualize \( \lambda \) as the plane array of \( |\lambda| := N \) unit squares, with left-aligned, top-to-bottom rows of length \( \lambda_1, \lambda_2, \ldots \), called the Young diagram \( \lambda \). The dimension \( \dim(\rho^\lambda) = f^\lambda \) is given by the hook length formula,

\[
f^\lambda = \frac{N!}{\prod_{u \in \lambda} h(u)}.
\]

Here \( u \) is a generic square of the Young diagram \( \lambda \), and \( h(u) \) is the length of the hook with corner at \( u \). Here is an example of the Young diagram of \( \lambda = (6, 3, 3, 2, 1) \) with \( |\lambda| = 15 \) and with a hook \( u \) of length \( h(u) = 4 \) shaded.

\[
\begin{array}{cccccc}
10 & 8 & 6 & 3 & 2 & 1 \\
6 & 4 & 2 \\
5 & 3 & 1 \\
3 & 3 \\
1 & \\
\end{array}
\]

The lengths of other hooks are indicated in the corresponding squares on the rights picture. The hook length formula gives \( f^\lambda = 70070 \) in this case.

Furthermore one-row partition \( \lambda = (N) \) is the identifying label of the trivial representation “id”, and one-column partition \( \lambda = (1^N) \) is the label of the second one-dimensional representation “sign”, with value 1 on \( A_N \) and value \(-1\) on \( A_N^c \). In our case \( H \) is either \( A_N \) or \( A_N^c \), so \( \text{sign}(\sigma) \) is the same, \( \text{sign}(H) \), for all permutations \( \sigma \in H \). Consequently, for \( \rho = \text{sign} \),

\[
\hat{P}_H(\rho) = \sum_{\sigma \in H} \text{sign}(\sigma) P_H(\sigma) = \text{sign}(H) \sum_{\sigma \in H} P_H(\sigma) = \text{sign}(H),
\]

9
and likewise $\hat{U}_H(\rho) = \text{sign}(H)$. Therefore

$$\hat{P}_H(\rho) - \hat{U}_H(\rho) = 0, \quad (\rho = \text{sign}). \quad (2.7)$$

Consider $\lambda \neq (N), (1^N)$. Let $\lambda'$ denote the conjugate partition ($\lambda'_1 \geq \lambda'_2, \ldots$) of $N$, whose diagram is transpose of the diagram for $\lambda$: $\lambda'_j$ is the length of the $j$-th column of $\lambda$. If $\lambda$ is not self-dual, i.e. $\lambda \neq \lambda'$, then the $\rho^\lambda$ restricted to $A_N$ is a nontrivial irreducible representation $\rho$ of $A_N$, whence $\sum_{\sigma \in A_N} \rho^\lambda(\sigma) = 0$ (Diaconis [6], Ch. 2B, Exer. 3). Of course $\sum_{\sigma \in S_N} \rho^\lambda(\sigma) = 0$ too, whence for both $H = A_N$ and $H = A_N^c$ we have

$$\hat{U}_H(\rho^\lambda) = \frac{1}{|H|} \sum_{\sigma \in H} \rho^\lambda(\sigma) = 0, \quad (\lambda \neq \lambda'). \quad (2.8)$$

If $\lambda = \lambda'$ then $\rho^\lambda$ restricted to $A_N$ is a direct sum of two irreducible representations, each of dimension $f^\lambda/2$, which exceeds 1 for $N \geq 5$, because $f^\lambda \geq 6$ for the self-dual $\lambda$ with $|\lambda| = N \geq 5$. Therefore again we have: for $N \geq 5$,

$$\hat{U}_H(\rho^\lambda) = \frac{1}{|H|} \sum_{\sigma \in H} \rho^\lambda(\sigma) = 0, \quad (\lambda = \lambda', |\lambda| \geq 5). \quad (2.9)$$

Putting together (2.6), (2.7), (2.8) and (2.9), we obtain: for $N \geq 5$ and $H = A_N$ or $H = A_N^c$,

$$\|P_H - U_H\|^2 \leq \frac{1}{4} \sum_{\lambda \neq (N), (1^N)} f^\lambda \text{tr}[\hat{P}_H(\rho^\lambda) \hat{P}_H(\rho^\lambda)^\ast]. \quad (2.10)$$

Once (2.10) is proved, the next step is essentially the same as in Gamburd’s argument for the special case $J = \{k\}, C_{N/k}, C_{N/2}$ are both even, when $H = A_N$. In our case $P_H = P_\gamma = U_{C_n} \ast U_{C_{N/2}}$, the convolution of $U_{C_n}$ and $U_{C_{N/2}}$. So, by multiplicativity of the Fourier transform for convolutions of the probability measures on the finite groups,

$$\hat{P}_H(\rho^\lambda) = \hat{U}_{C_n}(\rho^\lambda) \cdot \hat{U}_{C_{N/2}}(\rho^\lambda).$$

Since $U_{C_n} = |C_n|^{-1} 1_{C_n}, U_{C_{N/2}} = |C_{N/2}|^{-1} 1_{C_{N/2}}$ are class functions, each supported by a single conjugacy class,

$$\hat{U}_{C_n}(\rho^\lambda) = \frac{\chi^\lambda(C_n)}{f^\lambda} I_{f^\lambda}, \quad \hat{U}_{C_{N/2}}(\rho^\lambda) = \frac{\chi^\lambda(C_{N/2})}{f^\lambda} I_{f^\lambda};$$
here $\chi^\lambda$ is the character of $\rho^\lambda$, and $I_f^\lambda$ is the identity matrix for the representation $\rho^\lambda$. So

$$\hat{P}_H(\rho^\lambda) = \frac{\chi^\lambda(C_n)\chi^\lambda(C_{N/2})}{(f^\lambda)^2} I_f^\lambda,$$

and therefore (2.10) becomes

$$\|P_\gamma - U_H\|^2 \leq \frac{1}{4} \sum_{\lambda \neq (N), (1^N)} \left( \frac{\chi^\lambda(C_n)\chi^\lambda(C_{N/2})}{f^\lambda} \right)^2.$$

(2.11)

With $1/2$ instead of $1/4$ and $C_{N/k}$ instead of the general $C_n$, the right-hand side of (2.11) is Gamburd’s upper bound for his case. To make use of his bound, Gamburd applied the following estimate due to Fomin and Lulov [10]: for $N = tn$,

$$|\chi^\lambda(C_{N/t})| = O(N^{1/2-1/(2t)}) (f^\lambda)^{1/t},$$

(2.12)

uniformly for all $N$ and $\lambda$. He used (2.12) for both $t = 2$ and $t > 2$. For $|J| > 1$ a similar bound for $|\chi^\lambda(C_n)|$, $n = \{n_j\}_{j \in J}$, was not available at that time. More recently Larsen and Shalev [16] proved a remarkable extension of the Fomin-Lulov bound: given $m$, uniformly for all permutations $\sigma$ without cycles of length below $m$, and partitions $\lambda$,

$$|\chi^\lambda(\sigma)| \leq (f^\lambda)^{1/m+o(1)}, \quad N \to \infty.$$

(2.13)

(For $m = 2$, i.e., for fixed-point-free permutations, this is very similar to a bound conjectured earlier by Fomin and Lulov.) With this bound applied to both $\chi^\lambda(C_{N/2})$ and $\chi^\lambda(C_{n})$, the remaining proof of Theorem 2.2 largely, but not entirely, follows the original Gamburd’s argument.

Applying (2.11) in combination with $|\chi^\lambda(s)| = |\chi^\lambda(s)|$, $(s \in S_N)$ (implied by $\chi^\lambda(s) = \text{sign}(s) \chi^\lambda(s)$), Macdonald [20], Ch. 1, Ex. 2), we write

$$\|P_\gamma - U_H\|^2 \leq \frac{1}{4} \sum_{\substack{\lambda \neq (N) \leq N-7 \lambda_1 \sum_{\lambda \neq (N) \leq N-7}} \left( \frac{\chi^\lambda(C_n)\chi^\lambda(C_{N/2})}{f^\lambda} \right)^2 + \frac{1}{2} \sum_{\substack{\lambda \neq (N) \leq N-7 \lambda_1 \sum_{\lambda \neq (N) \leq N-7}}} \left( \frac{\chi^\lambda(C_n)\chi^\lambda(C_{N/2})}{f^\lambda} \right)^2$$

$$=: \Sigma_1 + \Sigma_2,$$

as $\lambda \neq (N)$ implies that $\lambda_1 \leq N - 1$. Consider $\Sigma_1$. By (2.13),

$$\left( \frac{\chi^\lambda(C_n)\chi^\lambda(C_{N/2})}{f^\lambda} \right)^2 \leq \left( \frac{(f^\lambda)^{1/3+1/2+o(1)}}{f^\lambda} \right)^2 = (f^\lambda)^{-1/3+o(1)};$$

11
so using Proposition 4.2 in Gamburd [12],

\[ \Sigma_1 = O\left(N^{-7t}\right)_{t=1/3-o(1)} = o(N^{-2}). \] (2.14)

To handle \( \Sigma_2 \), we use the following bounds. If \( a > 0 \) is fixed, then uniformly for \( \lambda \) such that \( \lambda_1 = N - a \), and \( C_n \),

\[ f^\lambda \geq \binom{N - a}{a} \geq \frac{N^a}{2a!}, \]

\[ |\chi^\lambda(C_{N/2})| = O\left(N^{\lfloor a/2 \rfloor}\right), \quad |\chi^\lambda(C_n)| = O\left(N^{\lfloor a/3 \rfloor}\right). \] (2.15)

For the first line bound see [12] equation (4.17). Let us prove the second line bounds. Consider \( |\chi^\lambda(C_n)| \), for example. \( C_n \) is a set of all permutations \( s \in S_N \), whose cycles are of lengths from \( J \), with given counts \( n_j \) of cycles of each admissible length \( j \in J \). Let \( c = (c_1, c_2, \ldots) \) be an arbitrary composition of \( N \) formed by the cycle lengths of a permutation \( s \in C_n \). From Murnaghan-Nakayama rule,

\[ |\chi^\lambda(C_n)| \leq g^\lambda(c), \]

where \( g^\lambda(c) \) is the total number of ways to empty the diagram \( \lambda \) by successive deletion of rim hooks, one hook at a time, of lengths \( c_1, c_2, \ldots \), so that we get a nested sequence of subdiagrams \( \lambda(i) \), from the initial \( \lambda(0) = \lambda \) to the empty one, (Stanley [24, Section 7.17, Equation (7.75)] ). Here is an example of a realization of the deletion process for \( \lambda = (7, 5, 5, 5) \) and \( c = (7, 5, 3, 2, 5) \) borrowed from [24]:

\[
\begin{array}{cccccccc}
5 & 5 & 5 & 5 & 1 & 1 & 1 \\
5 & 4 & 4 & 2 & 1 \\
3 & 3 & 2 & 2 & 1 \\
3 & 2 & 2 & 1 & 1
\end{array}
\]

c_i unit squares labeled \( i \) form a rim hook of \( \lambda(i - 1) \).

Let us show that

\[ g^\lambda(c) = O\left(N^{\lfloor a/3 \rfloor}\right). \]

Each of the \( g^\lambda(c) \) ways to empty \( \lambda \) consists of an ordered sequence of hook deletions not touching any of the first \( a \) cells in the first row, concatenated with an ordered sequence of hook deletions, the first of which deletes at least the cell \((1, a)\) from among those \( a \) cells, with the remaining deletions taking place entirely in a remaining corner-subdiagram \( \mu \), with \( |\mu| \leq 2a - 1 \).
So the number of ways to empty the residual diagram $\mu$ is at most some $S_1(a) = O(1)$, as $a$ is fixed. As for the first batch of hook deletions, they are deletions of horizontal rim hooks from the first row, possibly interspersed with deletions of rim hooks from the subdiagram $\nu$ formed by all the other rows of $\lambda$. Since $|\nu| = a$, the length of the subsequence formed by these hook deletions is, very crudely, $\lfloor a/3 \rfloor$ at most, as $\min J \geq 3$, and the total number of those subsequences is at most some $S_2(a) = O(1)$. So $g^\lambda(c)$, the overall number of ways to empty $\lambda$, is bounded by the number of $\lfloor N/3 \rfloor$-long $\{0,1\}$-sequences, with at most $\lfloor a/3 \rfloor$ 1’s, corresponding to the deletions of rim hooks from the bottom subdiagram $\nu$, multiplied by $S_1(a)S_2(a)$, whence

$$g^\lambda(c) = O \left( \left( \lfloor N/3 \rfloor \right) \left( \lfloor a/3 \rfloor \right) \right) = O \left( N^{\lfloor a/3 \rfloor} \right).$$

Similarly

$$g^\lambda(2,2,\ldots) = O \left( N^{\lfloor a/2 \rfloor} \right).$$

Consequently, for the summands in $\Sigma_2$ from $\lambda_1 = N - a$ with $1 \leq a \leq 6$,

$$\frac{\chi^\lambda(c_0)\chi^\lambda(c_{N/2})}{f^\lambda} = O \left( \frac{N^{\lfloor a/2 \rfloor + \lfloor a/3 \rfloor}}{N^a} \right).$$

Since

$$\min_{a \in [1,6]} [a - (\lfloor a/2 \rfloor + \lfloor a/3 \rfloor)] = 1,$$

and the total number of summands in $\Sigma_2$ is fixed as $N \to \infty$, we obtain then

$$\Sigma_2 = O(N^{-2}). \quad (2.16)$$

Combining (2.14) and (2.16), we have

$$\|P_\gamma - U_H\|^2 = O(N^{-2}).$$

The proof of Theorem 2.2 is complete. \qed

3 Number of vertices and Euler characteristic

The next claim is directly implied by Theorem 2.2.
Theorem 3.1. Let $V_n$ denote the number of vertices on the surface formed by randomly gluing polygons with the numbers of sides from $J = J(N)$, $\min J \geq 3$, such that the counts $n_j$ of polygons with $j$ sides satisfy $\sum_{j \in J} jn_j = N$. Let $C^e_N$ ($C^o_N$ resp.) denote the total number of cycles of the permutation chosen uniformly at random from all even permutations (all odd permutations resp.) of $[N]$. If $\alpha$ and $\beta$ are of the same parity (the opposite parity resp.), then

$$\|P_{V_n} - P_{C^e_N}\| = O(N^{-1}), \quad (\|P_{V_n} - P_{C^o_N}\| = O(N^{-1}) \text{ resp.}),$$

uniformly for all admissible $n = \{n_j\}_{j \in J}$.

Note. Fleming and Pippenger [9] used Gamburd’s Theorem 2.1 to evaluate the $\ell$-th central moment of $V_n$ for $J = \{k\}$, $k \geq 3$, within an additive error term $O(N^{-1/12} \log N)$, ($\ell \geq 1$), for $J = \{k\}$, $k \geq 3$, and $N$ divisible by $2 \text{lcm}(2, k)$. With Theorem 3.1 at hand, the estimates in [9] can be extended to all $N$ divisible by $\text{lcm}(2, k)$ with a smaller error term $O(N^{-1} \log N)$.

Let us have a look at $P_{C^e,o_N}$. Let $s(N, \ell)$ be the signless Stirling number of first kind, i.e., the number of permutations of $[N]$ with $\ell$ cycles. Then

$$P(C^e_N = \ell) = \frac{2s(N, \ell)}{N!}, \text{ if } N - \ell \text{ even; else } P(C^e_N = \ell) = 0, \quad (3.1)$$

and

$$P(C^o_N = \ell) = \frac{2s(N, \ell)}{N!}, \text{ if } N - \ell \text{ odd; else } P(C^o_N = \ell) = 0, \quad (3.2)$$

see, for instance, Sachkov and Vatutin [23]. (The equation (3.1) is implicit in Fleming and Pippenger [9], Equation (2.2).) Thus the ranges of $C^e_N$ and $C^o_N$ interlace each other. Now $\{s(N, \ell)/N!\}_{\ell \leq N}$ is distribution of $C^e_N$, the number of cycles in the random permutation of $[N]$, and it is well known that $C^e_N$ is asymptotically normal with mean and variance given by

$$\text{E}[C^e_N] = \sum_{j=1}^{N} \frac{1}{j} = \log N + O(1),$$

$$\text{Var}(C^e_N) = \sum_{j=1}^{N} \frac{1}{j} \left(1 - \frac{1}{j}\right) = \log N + O(1).$$
The standard proof of this central limit theorem (CLT) is based on the observation that $C_N$ has the same distribution as $\sum_{j=1}^{N} Y_j$, where $Y_j \in \{0, 1\}$ are independent with $P(Y_j = 1) = 1/j$, see Feller [8] (Ch. X.6), for instance. This CLT is too weak to lead directly to a CLT for $C_N^e, C_N^o$. However, Kolchin [14] had proved a stronger, local central limit, theorem (LCLT) for $C_N$:

$$P(C_N = \ell) = \frac{(1 + o(1)) \exp\left(-\frac{(\ell - E[C_N])^2}{2\text{Var}(C_N)}\right)}{\sqrt{2\pi \text{Var}(C_N)}},$$

(3.3)

uniformly for $\ell$ such that

$$\frac{\ell - E[C_N]}{\sqrt{\text{Var}(C_N)}} \in [-a, a], \quad a > 0 \text{ fixed}. \quad (3.4)$$

Note that the probability generating function of $C_N$ has only real roots $0, -1, \ldots, -(n - 1)$, so that, by Menon’s theorem [21], the distribution of $C_N$ is log-concave. Using Canfield’s quantified version of Bender’s CLLT for log-concave distributions ([1], [3]), one can show that in (3.3) $o(1) = O(\text{Var}(C_N)^{-1/4})$. Applying a CLLT proved recently by Lebowitz et al [17] this bound can be further improved to $O(\text{Var}(C_N)^{-1/2})$. Combining (3.3) with (3.1)-(3.2), we obtain: uniformly for $\ell$ satisfying (3.4),

$$P(C_N^e, C_N^o = \ell) = \frac{(2 + O(\text{Var}(C_N)^{-1/2})) \exp\left(-\frac{(\ell - E[C_N])^2}{2\text{Var}(C_N)}\right)}{\sqrt{2\pi \text{Var}(C_N)}},$$

(3.5)

here $N - \ell$ is even (odd resp.) for $C_N^e$ ($C_N^o$ resp.) The equation (3.5) and Theorem 3.1 taken together imply a strong CLLT for $V_n$.

**Theorem 3.2.** Uniformly for all admissible $\ell$ satisfying (3.4),

$$P(V_n = \ell) = \frac{(2 + O(\text{log}^{-1/2} N)) \exp\left(-\frac{(\ell - E[C_N])^2}{2\text{Var}(C_N)}\right)}{\sqrt{2\pi \text{Var}(C_N)}};$$

(3.6)

admissibility means that $N - \ell$ is even (odd resp.) when $\alpha$ and $\beta$ are of the same parity (the opposite parity resp.). Consequently $V_n$ is asymptotically normal with mean and variance $\log N$ both, $V_n \sim \mathcal{N}(\log N, \log N)$ in short.

**Note.** Gamburd used his Theorem 2.1 to prove that $V_n$ is asymptotic in distribution (i.e. integrally) to $\mathcal{N}(\log N, \log N)$ for $J = \{k\}$ and $N$ divisible
by $2 \text{lcm}(2, k)$.

Since the surface has $V_n$ vertices, $N/2$ edges and $n = \sum_j n_j$ faces, its Euler characteristic $\chi_n$ is

$$\chi_n = V_n - N/2 + n.$$ 

Using Theorem 3.2, we obtain then

**Theorem 3.3.**

$$P(\chi_n = -N/2 + n + \ell) = \frac{(2 + O(\log^{-1/2} N)) \exp\left( -\frac{(\ell - E[C_N])^2}{2\Var(C_N)} \right)}{\sqrt{2\pi \Var(C_N)}}, \quad (3.7)$$

uniformly for all admissible $\ell$, satisfying (3.4).

**Note.** In effect, the equation (3.7) gives an asymptotic formula for the fraction of surfaces with a given value of the Euler characteristic in the case when the absolute-value difference between the number of vertices and $\log N$ is of order $O(\log^{1/2} N)$.

### 4 Number of components

Let $X_n$ denote the total number of components of the random surface.

**Theorem 4.1.**

$$P(X_n = 1) = 1 - O(N^{-1}).$$

**Notes.** (1) This estimate is qualitatively best in general, since Pippenger and Schleich [22] proved that $P(X_n = 1) = 1 - 5/(6N) + O(N^{-2})$ for $J = \{3\}$.

(2) $X_n$ can be viewed as the number of components in a random *multigraph* $MG$ on $n = \sum_j n_j$ vertices, with the given vertex-degree sequence, such that $n_j$ vertices have degree $j$. (Each of the vertices $j$ is represented by a set $S_j$ of cardinality $j$, and two vertices $j$ and $j'$ are joined by an edge iff in the uniformly random matching $M$ on $S = \cup_j S_j$ there are points $u \in S_j$ and $u' \in S_{j'}$ such that $(u, u') \in M$. This model was introduced by Bollobás [2].) Theorem 4.1 asserts that $MG$ is connected with probability $1 - O(N^{-1})$, uniformly over all degree sequences bounded by 3 from below.
For the maximum degree \( \leq n^{0.02} \) this claim is implicit in Łuczak [19], its focus being on graphs, rather than multigraphs; see also an earlier result by Wormald [25] for the case of bounded maxdegree case.

Proof. If \( X_n > 1 \) then there exists a partition of the \( n = \sum_{j \in J} n_j \) cycles into two groups such that no two sides of a pair of cycles belonging to different groups are glued together; call it “no-match” condition. For a generic partition into two groups of cycles let \( n'_j \) and \( n''_j \) denote the number of cycles of size \( j \) in the first group and the second group respectively; so \( n'_j + n''_j = n_j, j \in J \).

Introduce \( N'_0 = \sum_{j \in J} j n'_j, N''_0 = \sum_{j \in J} j n''_j \); so \( N = N'_0 + N''_0 \). For an admissible partition, both \( N'_0 \) and \( N''_0 \) must be even. Consequently \( N'_0 \geq m \), where \( m = \min J \) if \( \min J \) is even, and \( m = 2 \min J \) otherwise. The probability of no-match in this partition is

\[
P(N'_0, N''_0) := \frac{(N'_0 - 1)!!(N''_0 - 1)!!}{(N - 1)!!}.
\]

Using \((2a - 1)!! = \frac{(2a)!}{2^a a!}\) and Stirling formula for factorials, we obtain: uniformly for \( N'_0 \geq m, N''_0 \geq m \),

\[
P(N'_0, N''_0) = O(P^*(N'_0, N''_0)), \quad P^*(N'_0, N''_0) = \frac{(N'_0)^N/2(N''_0)^N/2}{N^N/2}.
\] (4.1)

Furthermore, the total number of \( \{n'_j, n''_j\}_{j \in J} \) with parameters \( N'_0, N''_0 \) is given by

\[
Q(N'_0, N''_0) = \sum_{\sum_{j \in J} j n'_j = N'_0, \sum_{j \in J} j n''_j = N''_0} \prod_{j \in J} \frac{n'_j!}{n'_j! n''_j!}.
\]

\[
= \left[ x'_1 x''_1 \right] \prod_{j \in J} \sum_{n'_j + n''_j = n_j} \frac{n'_j!}{n'_j! n''_j!} x'_1 x''_1^j
\]

\[
= \left[ x'_1 x''_1 \right] \prod_{j \in J} (x'_1 + x''_1^j)^{n_j}.
\] (4.2)

Now

\[
P(X_N > 1) \leq \sum_{\substack{N'_0, N''_0 \geq m \\
N'_0 + N''_0 = N}} P(N'_0, N''_0)Q(N'_0, N''_0);
\]
so, by (4.1), we need to bound $P^*(N', N'')Q(N', N'')$ for the generic $N', N''$. By symmetry, it suffices to consider $N' \leq N''$. By (4.2) and $N' + N'' = \sum_j jn_j$, we have: for all $x_1 > 0$, $x_2 > 0$,

$$Q(N', N'') \leq x_1^{-N'} x_2^{-N''} \prod_{j \in J} (x_1^j + x_2^j)^{n_j} \leq y^{-N'} \prod_{j \in J} (y^j + 1)^{n_j} = \exp(K(y, N'));$$

(4.3)

$$K(y, N') := \sum_j n_j \log(y^j + 1) - N' \log y, \quad y := \frac{x_1}{x_2}.$$  

The best value of $y$ minimizes $K(y, N')$, and so it is a root of $K'_y(y, N') = 0$, which is equivalent to

$$\sum_j n_j \frac{jy^j}{y^j + 1} = N'.$$  

(4.4)

The left-hand size of (4.4) strictly increases with $y$, equals 0 at $y = 0$ and equals $\sum_j jn_j/2 = N/2 \geq N'$ at $y = 1$. So $K(y, N')$ does attain its minimum at a unique point $y(N') \in (0, 1]$ for all $N' \leq N/2$. $y(N')$ is strictly increasing with $N'$, and if it is considered as a function of the continuous parameter $N'$, $y(N')$ is continuously differentiable for $N' > 0$, as

$$\frac{d}{dy} \left( \sum_j n_j \frac{jy^j}{y^j + 1} \right) > 0, \quad \forall y > 0.$$  

Thus $Q(N', N'') \leq \exp(K(y(N'), N'))$, and so

$$P^*(N', N'')Q(N', N'') = O(\exp(K(y(N'), N'))),$$

$$K(y(N'), N')) : = K(y(N'), N') + (N'/2) \log N' + (N''/2) \log N'' - (N/2) \log N.$$  

(4.5)

We want to show that $K(y(N'), N')$ is strictly decreasing with $N'$. Since $K'_y(y, N')|_{y=y(N')} = 0$, and $N'' = N - N'$, we have

$$\frac{d}{dN'} K(y(N'), N')) = \frac{\partial}{\partial N'} K(y(N'), N'))|_{y=y(N')}$$

$$= - \log y(N') + (1/2) \log N' - (1/2) \log N''$$

$$= \log \left( \sqrt{\frac{N'}{N''}} \cdot \frac{1}{y(N')} \right).$$

18
Therefore we need to show that $y(N') > y_1 = y_1(N') := \sqrt{\frac{N'}{N''}}$ for $N' < N/2$, or equivalently by (4.4), that

$$\sum_j n_j \frac{y_1^j}{y_1^j + 1} < N'.$$

By concavity of $z/(1 + z)$ for $z \geq 0$,

$$\sum_j n_j \frac{y_1^j}{y_1^j + 1} \leq N \frac{\sum_j y_1^j(n_j)/N}{\sum_j y_1^j(n_j)/N + 1} \leq N \frac{y_1^3 \sum_j (n_j)/N}{y_1^3 \sum_j (n_j)/N + 1} = N \frac{y_1^3}{y_1^3 + 1}.$$

Consequently

$$\sum_j n_j \frac{y_1^j}{y_1^j + 1} - N' \leq (N' + N'') \frac{y_1^3}{y_1^3 + 1} - N'$$

$$= N'' \left[ (y_1^2 + 1) \frac{y_1^3}{y_1^3 + 1} - y_1^2 \right]$$

$$= -N'' \frac{y_1^2(1 - y_1)}{y_1^3 + 1} < 0,$$

for $N' < N/2$. Thus indeed $y(N') > y_1 = y_1(N')$, whence $K(y(N'), N')$ is strictly decreasing for $N' \in (0, N/2]$. Consider $N' \in [\nu, N/2], \nu = [6 \log N]$. As $K(y, N') - K(y, N')$ depends on $N'$ only, $y(N')$ is also the minimizer of $K(y, \nu)$. So, using $n = \sum_j n_j \leq N/3$, we have

$$\sum_j n_j \frac{y_1^j}{y_1^j + 1} < N'.$$
we have
\[
\mathcal{K}(y(N'), N') \leq \mathcal{K}(y(\nu), \nu) \leq \mathcal{K}(y_1(\nu), \nu)
\]
\[
= \sum_j n_j \log(1 + y_1(\nu)^j) - \nu \log \sqrt{\frac{\nu}{N - \nu}}
\]
\[
+ (\nu/2) \log \nu + [(N - \nu)/2] \log(N - \nu) - (N/2) \log N
\]
\[
= \sum_j n_j \log(1 + y_1(\nu)) + (N/2) \log \frac{N - \nu}{N}
\]
\[
\leq n \left( \frac{6 \log N}{N - 7 \log N} \right)^{3/2} - \frac{\nu}{2} = O(\sqrt{N^{-1} \log^3 N}) - \frac{\nu}{2}
\]
\[
\leq -2 \log N.
\]

From this bound and (4.5) it follows then that
\[
\sum_{N' = N'' = N} P^*(N', N'') Q(N', N'') = O\left( N \exp(-2 \log N) \right) = O(N^{-1}). \tag{4.6}
\]

It remains to consider \( m \leq N' \leq 6 \log N \). We will use the bound (4.3) again, but this time we are content with a suboptimal \( \hat{y} = \hat{y}(N') := (N'/N'')^{1/j_1} \), \( j_1 := \min J \geq 3 \). Using the resulting bound for \( P^*(N', N'') Q(N', N'') \), i.e. (4.5) with \( \hat{y}(N') \) instead of \( y(N') \), and also
\[
\sum_j n_j \log(1 + \hat{y}^j) \leq \sum_j \hat{y}^j n_j \leq \frac{N'}{N''} \sum_j n_j = \frac{n N'}{N''},
\]
we obtain
\[
P^*(N', N'') Q(N', N'') = O(\exp(\hat{\mathcal{K}}(N'))),
\]
where
\[
\hat{\mathcal{K}}(N') := \frac{n N'}{N''} + \left( \frac{1}{2} - \frac{1}{j_1} \right) N' \log N'
\]
\[
+ \left( \frac{N'}{j_1} + \frac{N''}{2} \right) \log N'' - \frac{N}{2} \log N.
\]

In particular, using \( nm \leq N, m \leq 6 \log N \), we have
\[
\hat{\mathcal{K}}(m) \leq 2 + \left( \frac{1}{2} - \frac{1}{j_1} \right) m (\log m - \log N).
\]
If \( j_1 \geq 3 \) is even then \( m \geq j_1 \geq 4 \) and, since the function \( x(\log x - \log N) \) decreases for \( x \leq eN \), we have then

\[
\hat{K}(m) \leq 2 + \log 4 - \log N; \tag{4.7}
\]

if \( j_1 \) is odd then \( m \geq 2j_1 \geq 6 \) and so

\[
\hat{K}(m) \leq 2 + \log 6 - \log N. \tag{4.8}
\]

Next, considering \( N' \) as a continuously varying parameter,

\[
\frac{d\hat{K}}{dN'} = \frac{nN}{(N')^2} - \left( \frac{1}{2} - \frac{1}{j_1} \right) \log \frac{N''}{N'} - \frac{1}{j_1} \frac{N'}{j_1 N''} \leq -0.5 \left( \frac{1}{2} - \frac{1}{j_1} \right) \log N,
\]

uniformly for \( m \leq N' \leq 6 \log N \). Therefore, using this bound and (4.7)-(4.8), we have

\[
\sum_{m \leq N' \leq 6 \log N} \exp(\hat{K}(N')) \leq \exp(\hat{K}(m)) \sum_{s \geq 0} \left[ \exp(-0.5(1/2 - 1/j_1) \log N) \right]^s \leq 2 \exp(\hat{K}(m)) = O(N^{-1}). \tag{4.9}
\]

Combining (4.9) with (4.6), we complete the proof of Theorem 4.1. \( \square \)

On the event \( \{ X_n = 1 \} \), the genus \( g_n \) is given by \( g_n = 1 - \chi_n/2 \). Thus \( g_n \) is defined with probability \( 1 - O(N^{-1}) \), and so by Theorem 3.3 we have our main result.

**Theorem 4.2.** For all admissible \( \ell \), and \( N \) large enough,

\[
P\left( g_n = 1 + \frac{N}{4} - \frac{n}{2} - \frac{\ell}{2} \right) = \frac{(2 + O(\log^{-1/2} N)) \exp\left(\frac{-(\ell - E[C_N])^2}{2\text{Var}(C_N)}\right)}{\sqrt{2\pi \text{Var}(C_N)}}, \tag{4.10}
\]

(\( n = \sum_j n_j \)), uniformly for all admissible \( \ell \), satisfying \( \frac{\ell - E[C_N]}{\sqrt{\text{Var}(C_N)}} \in [-a, a] \).

**Note.** In particular, for \( J = \{ k \}, k \in [3, N] \),

\[
1 + \frac{N}{4} - \frac{n}{2} = 1 + \frac{N(k - 2)}{4k}.
\]
so that for a single disc with \(N\) sides, \((N\text{ even})\), the genus \(g\) is asymptotic, integrally and locally, to \(N/4 - \frac{1}{2}N(\log N, \log N)\). We proved this result in Chmutov and Pittel [5] by using the Harer-Zagier [13] formula for the generating function of chord diagrams enumerated by the genus of the corresponding surface. That study was prompted by an earlier result of Linial and Nowik [18], who proved, using the H-Z formula, that \(E[g] = N/4 - 0.5 \log N + O(1)\). They also proved that \(E[g] = N/2 - \Theta(\log N)\) for a different random surface induced by an oriented chord diagram, for which a counterpart of the H-Z formula is unknown.

**Acknowledgment.** We owe a debt of gratitude to Michael Chmutov and Elena Yudovina for reading the original manuscript with painstaking care and sending us a very extensive list of helpful, stylistic and mathematical comments. It is our pleasure to thank the referees for an extensive, constructive criticism of the original version of the manuscript.

**References**

[1] E. A. Bender, Central and local limit theorems applied to asymptotic enumeration, J. Combin. Theory Ser. A, 15 (1973) 91–111.

[2] B. Bollobás, A probabilistic proof of an asymptotic formula for the number of labelled regular graphs, European J. Combin., 1 (1980) 311–316.

[3] E. R. Canfield, Application of the Berry-Esseen inequality to combinatorial estimates, J. Combin. Theory Ser. A, 28 (1980) 17–25.

[4] S. Chmutov, F. Vignes-Tourneret, Partial Duality of Hypermaps, Preprint arXiv:1409.0632 [math.CO].

[5] S. Chmutov, B. Pittel, The genus of a random chord diagram is asymptotically normal, J. Combin. Theory Ser. A, 120 (2013) 102–110.

[6] P. Diaconis, Group Representations in Probability and Statistics, IMS Lecture Notes–Monograph Series, 11 (1988).

[7] P. Diaconis, M. Shahshahani, Generating a random permutation with random tranpositions, Z. Wahr. Verw. Gebiete, 57 (1981) 159–179.
[8] W. Feller, An Introduction to Probability Theory and Its Applications, Vol. 1, 3rd Edition, Wiley & Sons, Inc., New York, 1970.

[9] K. Fleming, N. Pippenger, Large deviations and moments for the Euler characteristic of a random surface, Random Struct. Algorithms, 37 (2010) 465–476.

[10] S. V. Fomin, N. Lulov, On the number of rim hook tableaux, J. Math. Sciences, 87 (1997) 4118–4123.

[11] W. Fulton, J. Harris, Representation Theory, (Graduate Texts in Mathematics, volume 129), Springer-Verlag, Berlin, 1991.

[12] A. Gamburd, Poisson-Dirichlet distribution for random Belyi surfaces, The Ann. Probability., 34 (2006) 1827–1848.

[13] J. Harer, D. Zagier, The Euler characteristic of the moduli space of curves, Invent. Math., 85 (1986) 457–485.

[14] V. F. Kolchin, Random Graphs, Encyclopedia of Mathematics and Its Applications, Cambridge University Press, Cambridge, 1999.

[15] S. K. Lando, A. K. Zvonkin, Graphs on Surfaces and Their Applications, Encyclopaedia of Mathematical Sciences, volume 141, Springer-Verlag, Berlin, 2004.

[16] M. Larsen, A. Shalev, Characters of symmetric groups: sharp bounds and applications, Invent. Math., 174 (2008) 645–687.

[17] J. L. Lebowitz, B. Pittel, D. Ruelle, E. R. Speer, Central limit theorems, Lee-Yang zeros, and graph-counting polynomials, JCTA (submitted).

[18] N. Linial, T. Nowik, The expected genus of a random chord diagram, Discrete Comput. Geom., 45 (2011) 161–180.

[19] T. Łuczak, Sparse random graphs with a given degree sequence, in: A. Frieze, T. Łuczak (Eds.), Random Graphs, Vol. 2, Wiley, New York, (1992) 165–182.

[20] I. G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd Edition, Clarendon Press, Oxford, 1995.
[21] K. V. Menon, On the convolution of logarithmically concave sequences, Proc. Am. Math. Soc., 23 (1969) 439–441.

[22] N. Pippenger, K. Schleich, Topological characteristics of random triangulated surfaces, Random Struct. Algorithms, 28 (2006) 247–288.

[23] V. N. Sachkov, V. A. Vatutin, Probabilistic Methods in Combinatorial Analysis, Cambridge University Press, Cambridge, 2010.

[24] R. P. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge University Press, Cambridge, 1999.

[25] N. C. Wormald, The asymptotic connectivity of labelled regular graphs, J. Combin. Theory, Ser. B, 31 (1981) 156–167.