What is Holography in the Plane-Wave Limit of the AdS$_5$/SYM$_4$ Correspondence?

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The issue of holographic principle in the PP-wave limit of the AdS/CFT correspondence is discussed, in the hope of clarifying some confusions in the literature. We show that, in the plane-wave limit, the relation between the partition function in the bulk and the gauge-invariant correlation functions on the boundary should be interpreted on the basis of a tunneling picture in the semi-classical approximation which is appropriate for the plane-wave limit. This leads to a natural relation between Euclidean S-matrix in the bulk and the short-distance operator-product expansion of the so-called BMN operators on the boundary.

§1. Introduction

About a year ago, a remarkable conjecture, extending the AdS/CFT correspondence such that a class of non-BPS stringy states are included in the correspondence in a particular limit of plane-wave geometries, has been put forward by Berenstein, Maldacena, and Nastase (BMN)\(^2\). In spite of many interesting works done along this conjecture, however, the formulation of holographic principle\(^4\) in the plane-wave limit remains still quite elusive. It is important to understand this problem, since the holographic principle is believed to govern such correspondences between bulk quantum gravity and (gauge) field theories on the boundary. We have discussed this issue in our previous work\(^3\) appeared in hep-th archive half a year ago. In the present article, we would like to revisit this issue, since it seems unfortunately that our previous work has not been appreciated sufficiently by most workers in this field. I hope to make further clarification of our main arguments with some corrections to a part of discussions in the original version of our work.

We point out some crucial puzzles, which I believe are lying in the heart of the problem, associated with the original proposal on the correspondence of a class of gauge-invariant local operators (BMN operators) on the Yang-Mills side with the string states defined along a null trajectory in the bulk. Then a simple tunneling picture is introduced, solving all of the puzzles. This leads us to a proposal of a natural holographic correspondence, namely, the direct relation between the operator-product expansion (OPE) of the BMN operators defined on the boundary of the AdS geometry and the Euclidean infinite-time transition amplitudes (‘Euclidean S-matrix’) defined along a tunneling null geodesic connecting two-points on the boundary.

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\section{Basic holographic relation and the BMN limit}

Let us first briefly recall the basic relation\(^5\) between the partition function in the bulk and the generating functional for correlation functions on the boundary, which has been conjectured to be valid in the supergravity limit of the AdS/CFT correspondence,

\begin{equation}
Z[\phi_0]_{\text{gravity}} = \langle \exp(\int d^4x \sum_i \phi_i^0(x) O_i(x)) \rangle_{\text{ym}}. \tag{2.1}
\end{equation}

The left-hand side is the partition function of supergravity with boundary conditions on the independent set of fluctuating fields \(\{\phi_i\}\) in the bulk,

\begin{equation}
\lim_{z \to 0} \phi^i(z, x) = z^{4-\Delta_i} \phi_0^i(x), \tag{2.2}
\end{equation}

where the variable \(z\) is defined to be vanishing at the boundary, using the Poincaré coordinates of the AdS\(_5\) geometry,

\begin{equation}
ds^2_P = \frac{R^2dz^2}{z^2} + \frac{d\vec{x}^2}{R^2z^2}. \tag{2.3}
\end{equation}

The right-hand side of (2.1) is the generating functional for an appropriate set of gauge-invariant operators \(\{O_i\}\) with definite conformal dimensions \(\Delta_i\), which couple to the set of fields \(\{\phi^0_i(x)\}\) at the boundary. The set \(\{O_i\}\) must have a one-to-one correspondence to the set \(\{\phi^0_i\}\).

Although we do not have any rigorous derivation of the above relation (2.1), there is a natural physical picture justifying it\(^6\): This relation can be interpreted as two apparently different but equivalent descriptions of the low-energy behavior of a source-probe system of D3-branes. Suppose that probe D3-branes are put somewhere around the conformal boundary of the AdS\(_5\) geometry which describes the near horizon region of the background space-time of a large number \(N\) of source D3-branes. From the viewpoint of closed-string theory or supergravity as its low-energy approximation, the information of the probe D3-branes can be encoded into the boundary condition of the fluctuating fields in the bulk. On the other hand, from the viewpoint of open-string theory or effective super Yang-Mills theory as its low-energy approximation, the information of the probe D3-branes can be encoded into the set of external fields coupled to appropriate gauge-invariant operators which corresponds to closed-string vertex operators for the bulk fields \(\{\phi_i\}\). The Yang-Mills description is justified if we restrict ourselves in the regime where the approximation in which only the lowest modes of open strings attached to D3-branes are explicitly taken into account is effective. This is natural at least in the extreme near horizon limit where the typical length scale of open strings are assumed much shorter than the string scale \(\ell_s\). On the other hand, the distance scale of the AdS geometry is \(R \equiv (4\pi g_s N)^{1/4}\) in the string unit \(\ell_s = 1\). Hence, to justify the low-energy approximation in the bulk in the weak string-coupling region \(g_s \ll 1\), we have to take the large \(N\) limit such

\(^{6}\) For further discussions on this picture, we invite the reader to the reference\(^6\)
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that \( R \gg 1 \). One point, which does not seem so natural in this conjecture but is actually of crucial importance, is that the description in terms of the lowest open string states must be justified in the whole near-horizon region where the length scale of open strings is small compared with the length scale \( R \ell_s \) of curvature of the target space-time. It is not clear whether the typical length scale of open strings really remain small compared to the string scale when \( R \) is large. In the original notation of ref.\(^1\), a typical energy scale \( U = r/\ell_s^2 \) is fixed, but it is tacitly assumed that the description is valid for arbitrarily large \( U \), after we take the near horizon approximation \( 1 + R^4/\ell_s^4 \approx 1 + R^4/(U\ell_s)^4 \). We expect that the superconformal symmetry and the large \( N \) limit is responsible for valid justification. To keep this point in our mind, we always use the string unit \( \ell_s = 1 \) without taking the zero-slope limit.

In the supergravity limit, we are usually interested in the Kaluza-Klein modes of supergravity fields with respect to \( S^5 \), since the energies of stringy excited states of closed strings are much higher than them and hence are decoupled. If the magnitude of angular momentum along \( S^5 \) is \( J \), the typical energy of KK modes is \( J/R \), while the typical energy of the string excitations is of order 1 in the string unit. Therefore, for finite \( J \) the string excitations can indeed be ignored in the limit of large \( R \). However, if we consider sufficiently large \( J \), the string excitation energies can become comparatively smaller than those of KK modes, and we are not allowed to neglect them even if \( R \) is taken to be large. Because of a large momentum associated with large \( J \), we are no more in the naive low-energy regime. Since the masses of KK modes are related to the conformal dimensions, we expect that stringy excitation energies contribute to anomalous dimensions of generic (non BPS) string states. For example, for chiral supergravity modes, the conformal dimension is equal to \( J \), \( \Delta = J \). Naively, the excitation energies \( |n| \sim O(1) \) contribute to the mass as \( M^2 \sim (J/R)^2 + |n| \). A possible identification of conformal dimension would then be \( \Delta \sim M^2/(J/R^2) \sim J + R^2|n|/J \). This suggests that the anomalous dimension can be finitely detected in the large \( R \) regime if one takes a double limit \( R, J \to \infty \) by keeping the ratio \( R^2/J \) finite.

Remarkably, BMN showed that these qualitative considerations can be elevated to a much more precise theory. Combining this with the earlier observation due to Metsaev\(^7\) that the world-sheet formulation of string theory in the so-called PP-wave limit which just corresponds to the above double limit is exactly solvable as a free massive 2-dimensional field theory, they argued that the light-cone energy of the excitations of the form \( \sqrt{1 + R^4n^2/J^2} \) is reproduced to all orders with respect to small \( R^4/J^2 \) expansion, using the large \( N \) perturbation theory in the planar limit, if one identifies the gauge invariant operators corresponding to string oscillations appropriately. Note that the above naive expectation for the form of \( \Delta \) is the first order approximation with respect to large \( R^2/J \) expansion corresponding to a flat-space limit which is opposite to large \( N \) perturbation theory. In their interpretation, the longitudinal light-cone momentum is nothing but the rescaled angular momentum \( P^+ \sim J/R^2 \) and the light-cone Hamiltonian is identified with \( P^- \sim \Delta - J \). Schematically, the proposed correspondence is as follows: First choose two-directions, say, 5 and 6, along \( S^5 \) in the 6 dimensional space which is orthogonal to D3-branes.
and is represented by 6 scalar fields on the Yang-Mills side. The angular momentum \( J \) is associated with the rotation in the 5-6 plane. The chiral supergravity state, the ground state in the light cone, is then identified with the local operator \( \text{Tr}(Z^J) \), with \( Z = (\phi_5 + i\phi_6)/\sqrt{2} \), which indeed has the protected conformal dimension \( \Delta = J \). The bosonic stringy excitation modes are assumed to correspond to operators with various insertions of other 4 scalar fields \( \phi_i \) \((i = 1, \ldots, 4)\) and of 4 derivatives \( D_i Z \) \((i = 1, \ldots, 4)\), each being accompanied by phase factor \( \exp(2\pi n \ell/J) \) where \( \ell \) is the position of insertion and \( n \) is the world-sheet momentum along the (closed) string. Thus the excited states with \( n = 0 \) correspond to non-chiral (with respect to \( J \)) supergravity modes and 8 physical transverse directions of string oscillations correspond to fields \( \{\phi_i, D_i Z : i = 1, \ldots, 4\} \). The fermionic modes are treated in a similar manner.

Perhaps, one of curious features of this proposal would be that all of the string excitations which are of course extended in bulk space-time correspond to local operators on the boundary, unlike naively expected Wilson-loop like operators. But this is not so surprising if one recalls that, on the Yang-Mills side, the bulk space-time actually corresponds to the configuration space of world-volume fields, and hence the extendedness in the bulk enters through the fluctuations of fields themselves, not as the extendedness with respect to the base-space coordinates of D3-branes. This phenomenon can also be regarded as a manifestation of the stringy uncertainty principle\(^8\) of space-time as applied to D3-branes or its macroscopic version, the UV/IR relation\(^9\). It would be very interesting to investigate the whole picture of holography and its plane-wave limit using entirely the language of open strings before going to the effective SYM description. Such an approach might lead to a more direct and systematic derivation of the basic holographic relation on the basis of open-closed string duality. I hope to report progress along this line in a forthcoming work.

### §3. Puzzles and resolution

Let us now reconsider the BMN proposal from the viewpoint of the basic holographic relation (2.1). If we first restrict ourselves to supergravity approximation, a supergravity state with large angular momentum can be approximated by a semiclassical particle picture. Consider therefore the simplest scalar field equation in the AdS geometry using the WKB approximation;

\[
\left( z^2 \partial_z^2 - 3z \partial_z + R^4 z^2 \omega^2 - J(J+4) \right) \phi(z) = 0, \tag{3.1}
\]

where we have factorized the dependencies on the angular momentum \( J \) and the (Minkowski) energy \( \omega \). Assuming the standard WKB form \( \phi(z) \sim N A(z) \exp iS(z) \), we find the WKB phase \( S(z) \) satisfies

\[
z^2 \left( \frac{dS}{dz} \right)^2 - R^4 z^2 \omega^2 + J^2 = 0. \tag{3.2}
\]
If we consider usual propagating solutions with real $S$, we have the condition
\[ z^2 \geq J^2/(\omega^2 R^4). \tag{3.3} \]
This inequality shows that for any finite energy $\omega$ and nonzero $J$, the particle trajectories (null geodesics) with finite ratio $J/R^2$ can never reach the boundary $z \to 0$. In the usual discussion of null geodesics in AdS space-times, it is convenient to use the global coordinates. The above property then corresponds to the fact that the null trajectory traversing a great circle along $S^5$ never reaches the conformal boundary and goes inside the horizon of the Poincaré patch in a finite interval with respect to the global time coordinate.

This is puzzling since according to (2.1) the identification of bulk states with the Yang-Mills operators are made using boundary conditions near the conformal boundary $z = 0$ of the AdS geometry. In fact, the behavior $z^{4-\Delta}$ in (2.2) comes from the choice of the non-normalizable wave function in the classically forbidden region. In other words, the correspondence between the bulk and the boundary actually takes place as a tunneling phenomenon in the semi-classical picture. Indeed, by assuming purely imaginary action function $S \to iS_E$, $\phi(r) \to NA(z) \exp -S_E(z)$, we can easily check that there exist tunneling trajectories connecting two points on the boundary as solutions for Hamilton-Jacobi equation
\[ z^2 \left( \frac{dS_E}{dz} \right)^2 = J^2(1 - \frac{z^2 \omega^2 R^4}{J^2}) \quad \to \quad S_E(z) \sim \pm J \log z, \quad A(z) \sim z^{2+\frac{2}{\Delta}}, \tag{3.4} \]
as $z \to 0$ giving $\Delta = J + 4$ or $-J$. This gives the well known mass-dimension relation $m^2 = J(J + 4) = \Delta(\Delta - 4)$ for scalar field. The former positive solution $\Delta = J + 4$ represents the non-normalizable wave function which is responsible for the relation (2.1) and (2.2).\footnote{Note that nonnormalizability is with respect to the ‘inside’ region corresponding to large $z$. If we take the viewpoint from the probe (‘outside’ region), namely, from $z = 0$, the nonnormalizable solutions actually correspond to normalizable solutions.} The explicit form of the tunneling trajectory is
\[ z = \frac{J}{R^2 \omega \cosh \tau}, \tag{3.5} \]
which is obtained by replacing $dS_E/dz$ by the $z$-momentum $P_z = J \omega^2 dz/d\tau$ and similarly by $J = P_\psi = J d\psi/d\tau$, $\omega = -P_t = J(R^4 z^2)^{-1} dt/d\tau$, with $\tau$ being the affine parameter along the trajectory. See Fig. 1.

Now we emphasize that our tunneling picture solves other puzzles related to the holographic interpretation. First, the 8 transverse directions in the BMN proposal includes the 4 base-space directions ($\vec{x}_3, t$) of D3-branes. If we use the ordinary real null geodesics, the direction of the trajectory in the region (corresponding to the turning point $z = J/\omega R^2$) where it becomes closest to the boundary goes parallel to the base space. This seems contradictory to the identification of transverse directions along the trajectory of strings. On the contrary, in the tunneling picture, the trajectory becomes manifestly orthogonal to the base space directions of D3-branes, as we
approach to the boundary. The above tunneling trajectory connects two points on the boundary. The coordinate distance between the two points at the boundary is asymptotically equal to $|r| = 2\frac{J}{\omega}$. Also, the natural affine time $\tau$ along the tunneling trajectory is related to the coordinate $z$, $z \rightarrow \frac{2J}{R^2 \omega} e^{-|\tau|}$ which gives a direct justification for the identification of energy with the conformal dimensions (minus $J$), in conformity with the conformal transformation law of the Poincaré coordinates.

Furthermore, the tunneling picture gives a natural rationale for another subtlety involved in the original proposal. To have the real null geodesics, it is absolutely essential to use Minkowskian signature for the background space-time. But then the 4 dimensional world volumes of D3-branes have also Minkowskian signature. We thus encounter again a contradiction to the identification of base-space directions with the transverse directions of light-cone formulation of strings. In our tunneling picture, however, the use of purely imaginary affine time, comparing with the case of propagating region, necessarily demands us to Wick-rotate the target time and angular coordinate simultaneously in order to keep $J$ and $\omega$ real in the WKB equation describing the tunneling region. Thus the 4 base space directions must be treated as Euclidean, while the angular variable $\psi$ along $S^5$ now replaces the role of target time coordinate. Incidentally, this ‘double’ Wick rotation of target space-coordinates required by the tunneling picture fits nicely with a similar formal prescription, as
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adopted in\textsuperscript{11),} which is technically demanded in the boundary state formalism for D-branes in the light-cone gauge.

Our simple consideration on the nature of the semi-classical picture behind the holographic relation (2.1) clearly suggests that we should be able to extend the relation to include stringy states by replacing the null trajectory in the usual derivation of the plane-wave limit of string theory by the tunneling null trajectories which directly connect points on the conformal boundary. This picture originates from the general fact that, for nonzero $J$, probing D3-brane from the conformal boundary is inevitably a tunneling phenomenon, though our proposal does not exclude other possible approaches to holography.\textsuperscript{*}

In connection with this, our viewpoint also seems to resolve the question of introducing physical observables in terms of bulk string theory. If we insist that the theory is defined around the ordinary real null geodesic, the background trajectory goes inside the horizon of D3-branes metric in a finite light-cone time $\tau$. Remember that the real null trajectory is $z = J/(\omega R^2 \cos \tau)$ in terms of the Poincaré coordinates. Then, it seems difficult to associate scattering events occurring in the bulk with physical observables of the Yang-Mills theory in any systematic way, since we do not know how to use the global coordinates of AdS geometry in defining string theory observables associated with D-branes. The situation is very different according to our proposal: Boundary-to-boundary transitions involve infinite time duration with respect to natural affine (Wick-rotated in the above sense) time $x^+ \propto \tau$, in the limit that the boundary, on which we deal with the Yang-Mills theory observables coupled to external sources corresponding to probe branes, approaches to the conformal boundary of AdS geometry. This is natural since only known manner of extracting physical observables in string theory is through scattering amplitudes corresponding to infinite time duration. In section 5, we propose a direct relation between such amplitudes, Euclidean S-matrix, and the short-distance structure of SYM correlators.

\section*{§4. String theory along a tunneling null geodesic}

Let us now briefly indicate how the string theory expanded around the tunneling null geodesic looks like. Since the details are given in our previous paper, we only present the final results correcting some errors in the original version of ref.\textsuperscript{3).} What we do is just the semi-classical expansion of the Green-Schwarz action around the tunneling trajectory (3.5) with the Wick rotations discussed above. To simplify the expressions below, we choose the parameters of the trajectory such that $J = R^2 \omega$ and define the longitudinal momentum as $\alpha = J/R^2 (>0)$. In the ‘standard’ notation commonly used in the recent literature, our convention corresponds to set $\mu = 1$. In the scaling limit, we can truncate the expansion to the second order after eliminating the negative metric associated with the Wick-rotated angle direction $\psi$ using the

\textsuperscript{*} A representative work which is very different from our viewpoint is\textsuperscript{12).} For a (necessarily, partial) list of other references, we refer the reader to the bibliography of\textsuperscript{3).}
Virasoro constraint. The bosonic action
\[ S_b = \frac{R^2}{2\pi} \int d\tau \int_0^{2\pi\alpha} d\sigma \frac{1}{2} \left[ z^{-2}(\partial z)^2 + z^{-2}(\partial \vec{x}_4)^2 - \cos^2\theta(\partial \psi)^2 + (\partial \theta)^2 + \sin^2\theta(\partial \Omega_3)^2 \right] \]
then reduces to that of free massive theory as
\[ S_b^{(2)} = \frac{1}{4\pi} \int d\tau \int_0^{2\pi\alpha} d\sigma \left[ (\partial \vec{x}_4)^2 + \vec{x}_4^2 + (\partial \vec{y}_4)^2 + \vec{y}_4^2 \right], \]
performing field redefinition and rescalings appropriately in the large \( R \) limit, where \( \vec{x}_4 \) is the redefined 4-vector originating from the fluctuations along the base space directions of D3-branes with suitable mixing with the \( z \)-direction inside the bulk \( (z \neq 0) \), and \( \vec{y}_4 \) is the 4-vector corresponding to the fluctuations along \( S^5 \) in the directions to orthogonal to the trajectory. This is the same action as we obtain in the case of real trajectory, except for the difference that 2-dimensional metric is Euclidean \((\partial^2 = \partial^2_\tau + \partial^2_\sigma)\).

Similarly, the final form of the fermionic action is found to be
\[ S_f^{(2)} = \frac{i}{2\pi} \int d\tau \int_0^{2\pi\alpha} d\sigma \left[ \theta^I \Gamma_0 \Gamma_- \partial_\tau \theta^I - i s^{IJ} \theta^I \Gamma_0 \Gamma_- \partial_\sigma \theta^J - i e^{IJ} \theta^I \Gamma_0 \Gamma_- \Pi \theta^J \right], \]
with
\[ \Pi = i\Gamma_{0123}, \quad \Pi^2 = 1, \quad [\Pi, \Gamma_\pm] = 0 = \{\Pi, \Gamma_0\}, \quad \Pi^T = -\Pi, \quad \Pi^\dagger = \Pi, \]
and with \( I, J = 1, 2 \) and \( s^{IJ} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). The fermionic coordinates satisfy
\[ \Gamma_+ \theta^I = 0, \quad \Gamma_+ \equiv (\Gamma_Z - i\Gamma_\psi)/\sqrt{2} = \Gamma_\dagger, \]
in addition to the Weyl condition of type IIB theory. Because of double Wick rotation, the fermionic coordinates \( \theta^I \) become complex with the same number of independent degrees of freedom \((16=8+8)\) as the usual Majorana-Weyl spinors in the ordinary Minkowskian 10 dimensional space-time.

The manifest global symmetry of the fermionic action is \( SO(4) \times SO(4) \), due to the presence of the mass term containing the Gamma-matrix factor \( \Pi \), while the bosonic action has manifest \( SO(8) \) symmetry. However, we can trivially eliminate \( \Pi \) by making the redefinition
\[ \theta^1 \rightarrow \theta^1, \theta^2 \rightarrow \Pi \theta^2, \]
since the kinetic term is invariant under this transformation. This makes manifest that the fermionic action has also the \( SO(8) \) symmetry. Formally this redefinition is the same as the one associated with T-duality transformation in flat space-time along the four base-space directions, since in the flat limit \( \theta^{1,2} \) reduce to holomorphic and anti-holomorphic coordinates, respectively. Then, the enhancement of global symmetry seems natural because a D3-brane turns into a D-instanton. Though
suggestive, this argument is incomplete, since the bosonic part is not T-dualized.\footnote{For a related discussion in flat (Minkowski) limit, see\textsuperscript{13}.} Naively, such an enhancement of symmetry may seem very strange, if we recall that the background RR-gauge field itself which is responsible to the fermionic mass term is obviously not invariant under the transformations mixing two different $SO(4)$ directions of AdS$_5$ and S$^5$, respectively. The situation is that in the plane-wave limit, the information on the direction of the RR-gauge field is lost and only its magnitude is detected by the strings at least in the particular case of the AdS$_5 \times$ S$^5$ geometry.\footnote{As a simple analogy, consider a set of four $O(2)$ vector fields $(\phi_1, \phi_2), (\phi_3, \phi_4), (\psi_1, \psi_2), (\psi_3, \psi_4)$ and the potential $V(\phi, \psi) = a \sum_{i=1}^{4} (\phi_i^2 + \psi_i^2) + b(\phi_1 \psi_1 + \phi_2 \psi_2 - \phi_3 \psi_3 - \phi_4 \psi_4) + c(\phi_1 \psi_1 + \phi_2 \psi_2)^2 + d(\phi_3 \psi_3 + \phi_4 \psi_4)^2$, which is symmetric under $O(2) \times O(2)$. If we take the limit of weak fields by neglecting the quartic terms, the symmetry is trivially enhanced to $O(4)$ by reinterpreting $(\phi_1, \phi_2, \phi_3, \phi_4)$ and $(\psi_1, \psi_2, -\psi_3, -\psi_4)$ as two $O(4)$ vectors. If the quartic terms are not totally ignored, we can never have enhanced symmetry.} In the following, we use the convention in which the factor $\Pi$ is eliminated for spinor coordinates and $\gamma$-vector notation $(\vec{x}_4, \vec{y}_4) \rightarrow (x^i; i = 1, 2, \ldots, 8)$ for vector coordinates. It would be very interesting if we could see this enhancement of symmetry directly on the Yang-Mills side. Remember that the appearance of the $SO(8)$ is quite mysterious on the Yang-Mills side even if we restrict ourselves to purely bosonic excitations, in view of very different origin of 4+4 transverse modes in the BMN proposal. Note also that we can never see the $SO(8)$ symmetry generators by simply contracting the symmetry algebra of the AdS$_5 \times$ S$^5$ geometry. We have to redefine the generators, correspondingly to the field redefinition (4.6).

To quantize the system, it is convenient to use manifestly $SO(8)$ conventions. Using the standard notation of $SO(8)$-$\gamma$ matrices ($8 \times 8$), Hamiltonian, and CCR are given as

\[
H = \frac{1}{2} \int_0^{2\pi} d\sigma : \left[ 2\pi p^2 + \frac{1}{2\pi} (x')^2 + \frac{1}{2\pi} x^2 - i \frac{1}{2\pi} (\theta \theta' + (2\pi)^2 \lambda \lambda') + 2\theta \lambda \right], \quad (4.7)
\]

\[
\theta = \theta^1 + i\theta^2, \quad \lambda = \frac{1}{2\pi} \gamma = \frac{1}{2\pi} (\theta^1 - i\theta^2), \quad (4.8)
\]

\[
[x^i(\sigma), p^j(\sigma')] = i\delta^{ij} \delta(\sigma - \sigma'), \quad \{\theta_a(\sigma), \lambda_b(\sigma')\} = \delta_{ab} \delta(\sigma - \sigma'). \quad (4.9)
\]

General world-sheet fields $O(\tau, \sigma)$ are defined by $O(\tau, \sigma) = e^{H\tau} O(\sigma) e^{-H\tau}$ and hence adjoint operation is $O(\tau, \sigma) \rightarrow O(-\tau, \sigma)^\dagger$. In this sense, the Euclidean theory satisfies physical positivity condition (or ‘reflection’ positivity).

The Hamiltonian are diagonalized by the following mode expansions, suppressing vector and spinor indices,

\[
x(\sigma) = x_0 + \sqrt{2} \sum_{n=1}^{\infty} (x_n \cos \frac{n\sigma}{\alpha} + x_{-n} \sin \frac{n\sigma}{\alpha}), \quad (4.10)
\]

\[
p(\sigma) = \frac{1}{2\pi\alpha} \left[ p_0 + \sqrt{2} \sum_{n=1}^{\infty} (p_n \cos \frac{n\sigma}{\alpha} + p_{-n} \sin \frac{n\sigma}{\alpha}) \right]. \quad (4.11)
\]
\[ \theta(\sigma) = \theta_0 + \sqrt{2} \sum_{n=1}^{\infty} \left( \theta_n \cos \frac{n\sigma}{\alpha} + \theta_{-n} \sin \frac{n\sigma}{\alpha} \right), \quad (4.12) \]

\[ \lambda(\sigma) = \frac{1}{2\pi\alpha} \left[ \lambda_0 + \sqrt{2} \sum_{n=1}^{\infty} \left( \lambda_n \cos \frac{n\sigma}{\alpha} + \lambda_{-n} \sin \frac{n\sigma}{\alpha} \right) \right], \quad (4.13) \]

\[ x_n = \frac{i}{\sqrt{2\alpha E_n}} (a_n - a_n^\dagger), \quad p_n = \frac{\alpha E_n}{2} (a_n + a_n^\dagger), \quad (4.14) \]

\[ \theta_n = \frac{1}{2\sqrt{\alpha E_n}} (\ell_n^+ b_n - i\ell_n^- b_{-n}^\dagger), \quad \lambda_n = \frac{1}{2\sqrt{\alpha E_n}} (i\ell_n^- b_{-n} + \ell_n^+ b_n), \quad (4.15) \]

where

\[ [a_n, a_m^\dagger] = \delta_{nm}, \quad \{b_n, b_m^\dagger\} = \delta_{nm}, \quad (4.17) \]

for all (positive, negative and zero) integers \( n \). We adopted the sine-cosine basis instead of the exponential basis of ref. 3. The Hamiltonian, to be identified with \( P^- = E - J \), is simply

\[ H = \sum_{n=-\infty}^{\infty} E_n (a_n^\dagger a_n + b_n^\dagger b_n). \]

With the vacuum being defined by \( a_n|0\rangle = 0 = b_n|0\rangle \), the \( SO(8) \) symmetry is completely manifest. In a forthcoming work, we hope to present an explicit construction of string-field theory using our manifest \( SO(8) \) formalism.

The supersymmetry in this formalism is rather subtle. The following fermionic generators, which we propose to call ‘pseudo’ susy generators, commute with the Hamiltonian,

\[ R_\alpha^a = \int_0^{2\pi\alpha} d\sigma \left[ (p + \frac{i}{2\pi} x) \cdot \gamma_{\dot{a}\dot{b}} \theta_{\dot{b}} + \frac{1}{2\pi} x' \cdot \gamma_{\dot{a}\dot{b}} \theta_{\dot{b}} \right], \quad \overline{R_\alpha^a} = \left( R_\alpha^{-\dagger} \right)^a, \quad (4.18) \]

and satisfy the algebra, in the Hilbert space of translation invariant states with respect to \( \sigma \to \sigma + \text{const.} \),

\[ \{R_\alpha^a, R_\beta^b\} = 2\delta_{\dot{a}\dot{b}} H - i\gamma_{\dot{a}\dot{b}}^ij L^{ij}, \quad \{R_\alpha^a, R_\beta^b\} = 0, \quad (4.19) \]

with

\[ L^{ij} = \int_0^{2\pi\alpha} d\sigma \left[ x^i p^j - x^j p^i + \frac{i}{4\pi} \theta \gamma^{ij} \theta \right]. \quad (4.20) \]

This algebra respects of course the \( SO(8) \) symmetry. However, the would-be \( SO(8) \) generator \( L_{ij} \) has a wrong sign for the fermionic contribution. This means that the algebra actually does not close with a finite number of generators. Namely, we are necessarily lead to an infinite dimensional algebra, if we require that the algebra is covariant under \( SO(8) \). In particular, we have to define an infinite number of fermionic generators to close the algebra.

\(^\ast\) We note that our \( SO(8) \)-vacuum is slightly different from the one discussed in ref. 10 in the Minkowski formalism, because of our redefinition of the spinor coordinates.
What is holography in the plane-wave limit?

The standard susy generator which is covariant only under $SO(4) \times SO(4)$, can be conveniently expressed by making a canonical transformation to the spinor coordinates as

$$\psi = \frac{1}{2}(1 + \Pi)\theta + \frac{1}{2}(1 - \Pi)\bar{\theta}, \quad \bar{\psi} = \frac{1}{2}(1 - \Pi)\theta + \frac{1}{2}(1 + \Pi)\bar{\theta} \quad (4.21)$$

where, in the standard $(8 \times 8)$ SO(8) spinor notation for gamma matrices

$$(\gamma_i^T \gamma_j + \gamma_j^T \gamma_i)_{\dot{a}\dot{b}} = 2\delta_{\dot{a}\dot{b}}, \quad (\gamma_i^T \gamma_j + \gamma_j^T \gamma_i)_{ab} = 2\delta_{ab}$$

$$\Pi_{ab} = \Pi_{ba} = (\gamma_1\gamma_2^T \gamma_3\gamma_4^T)_{ab}, \quad \Pi_{\dot{a}\dot{b}} = \Pi_{\dot{b}\dot{a}} = (\gamma_1^T \gamma_2\gamma_3^T \gamma_4)_{\dot{a}\dot{b}}. \quad (4.22)$$

$$Q_{\dot{a}} = \int_0^{2\pi\alpha} d\sigma \left( (p \cdot \gamma - \frac{i}{2\pi}x \cdot \gamma\Pi)\psi - \frac{1}{2\pi}x' \cdot \gamma\bar{\psi} \right)_{\dot{a}}, \quad (4.23)$$

$$\bar{Q}_{\dot{a}} = \int_0^{2\pi\alpha} d\sigma \left( (p \cdot \gamma + \frac{i}{2\pi}x \cdot \gamma\Pi)\bar{\psi} - \frac{1}{2\pi}x' \cdot \gamma\psi \right)_{\dot{a}}, \quad (4.24)$$

in terms of the spinor coordinates $\psi, \bar{\psi}$. Nontrivial part of the supersymmetry algebra is

$$\{Q_{\dot{a}}, \bar{Q}_{\dot{b}}\} = 2H\delta_{\dot{a}\dot{b}} + \sum_{(i,j)\in(1,2,3,4)} i(\gamma_{ij}\Pi)_{\dot{a}\dot{b}}J_{ij} - \sum_{(i,j)\in(5,6,7,8)} i(\gamma_{ij}\Pi)_{\dot{a}\dot{b}}J_{ij}, \quad (4.25)$$

$$J_{ij} = \int_0^{2\pi\alpha} d\sigma \left( x_ip_j - x_jp_i - \frac{1}{4\pi}iv_{ij}\psi \bar{\psi} \right). \quad (4.26)$$

If we combine these standard susy generators with our SO(8), the algebra is again extended to an infinite dimensional algebra, corresponding to the above phenomenon. This suggests that we can have much stronger constraints on the dynamics of the system by combining SO(8) and susy than taking into account only the standard supersymmetry and its smaller global symmetry $SO(4) \times SO(4)$. It is certainly reasonable to expect that the dynamics respects the SO(8) symmetry, which is hidden in the standard susy algebra but is exhibited in the world-sheet action. In any case, it seems very important to further clarify the role of the hidden SO(8) symmetry.

§5. Euclidean S-matrix and OPE

We have obtained the above theory as a limit from the string theory on the AdS space-time such that the background trajectory of semi-classical expansion connects directly from conformal boundary to conformal boundary. Therefore, we are bound to have a natural extension of the basic holographic relation (2.1). An obvious guess for the left-hand side (namely bulk partition function with boundary conditions for physical fluctuating modes) of (2.1) for general string states is the S-matrix in the Euclidean sense describing infinite propagation of states from $\tau = -T$ to $\tau = T$ along the trajectory in the limit $T \to \infty$. What is the natural correspondent for the right-hand side? In terms of the string language, putting boundary condition amounts to preparing initial and final multi-string states appropriately near the ends of the background trajectory. On the Yang-Mills side, they must be associated with
some products of BMN operators in an appropriate basis. Since the background trajectory meets the boundary only at two points, we expect that they are some sort of short distance products at each meeting points, initial and final points. However, it is not clear \textit{a priori} how to define such short-distance products, fitting for the present expectation.

Let us therefore first examine the structure of the Euclidean S-matrix in perturbation theory. Since the two-point S-matrix describing just the propagation of single string states can always be normalized to be identity, it is natural to define the perturbative expansion of the Euclidean S-matrix in analogy with the usual Minkowski case as

\[
\langle b | (S - 1) | a \rangle = \lim_{T \to \infty} \left[ -\langle b | V | a \rangle \int_{-T}^{T} d\tau e^{(E_b - E_a)\tau} + \sum_c \langle b | V | c \rangle \langle c | V | a \rangle \int_{-T}^{T} d\tau e^{(E_c - E_b)\tau} \int_{-T}^{\tau_1} d\tau_1 e^{(E_a - E_c)\tau_1} - \sum_c \sum_d \langle b | V | c \rangle \langle c | V | d \rangle \langle d | V | a \rangle \int_{-T}^{T} d\tau e^{(E_c - E_b)\tau} \int_{-T}^{\tau_1} d\tau_1 e^{(E_a - E_d)\tau_1} \times \int_{-T}^{\tau_2} d\tau_2 e^{(E_d - E_a)\tau_2} + \cdots \right],
\]

where \( V \) is the interaction part of the Hamiltonian of string field theory.

Take the simplest nontrivial case of 3-point matrix elements, corresponding to \( 1 \to 2 \) or \( 2 \to 1 \) scattering in the tree approximation. Then only the first term contributes

\[
\frac{e^{(E_b - E_a)T} - e^{-(E_b - E_a)T}}{E_a - E_b} \langle b | H^{(1)} | a \rangle
\]

with \( V \to H^{(1)} \) being 3-point interaction vertex of string fields. The singularity exhibited in this expression takes the following form, because of the conservation of angular momentum \( J_k - J_i - J_j = 0 \),

\[
\frac{V_{ijk}}{(\Delta_k - \Delta_i - \Delta_j)} e^{\pm (\Delta_i + \Delta_j - \Delta_k)T},
\]

where \((i, j)\) and \(k\) correspond to 2 and 1 string states, respectively, and the \( V_{ijk} \) is the matrix element of the 3-point vertex. We can derive this same form on the basis of the standard LSZ formalism using Euclidean Green functions. Only difference from the Minkowski case is that the factor in front of the on-shell matrix element \( \langle b | H^{(1)} | a \rangle \) replaces the \( \delta \)-function of energy conservation. One might wonder why the nonconservation of energy is allowed in time-translation invariant theory. However, for scattering events, invariance under time translation comes only after integration

\(^*)\) If we wish to consider more general \(n\)-point correlators, the semi-classical expansion around any single background trajectory is not sufficient: We have to necessarily combine \(n\) different tunneling null geodesics with vertices describing branching of such trajectories. But that almost amounts to dealing with string field theory in the AdS background without relying on the plane-wave limit. Such general vertices cannot be described in light-cone gauge since the light-cone times are in general different for branching trajectories.
over interaction times. In Minkowski case, the integration over the interaction times yields the energy-conserving $\delta$-function. But in the Euclidean case, that gives the singularity exhibited by this factor.

Now, the above form should be compared with the standard form of the OPE,

$$O_i(0)O_j(x) \sim \sum_k \frac{1}{|x|^{\Delta_i + \Delta_j - 2\Delta_k}} C_{ijk} O_k(0), \quad x \to 0. \quad (5.4)$$

Remembering that two-point amplitudes are normalized to be one, this strongly suggests us to identify the short distance cutoff as

$$|x| = e^{-T} \quad (5.5)$$

We suppose that the standard OPE form (5.4) is valid on the Yang-Mills side for an appropriate basis of the BMN operators. In general, we should expect certain operator mixing of the original BMN operators. Then provided that we can choose only the term with appropriate sign on the exponential (5.2) in the prefactor, we can identify the bulk S-matrix and correlation functions by setting

$$V_{ijk} = (\Delta_k - \Delta_i - \Delta_j) C_{ijk}. \quad (5.6)$$

The relation (5.5) is natural from what we have discussed in section 2 with respect to the UV/IR relation. See Fig. 2.

Let us now consider the meaning of the special choice of the sign for the exponentials in the singularities appearing in (5.2). In the case of ordinary S-matrix in Minkowski space-times, such a choice of signs on the exponentials would amount to the familiar $i\epsilon$ prescription. In our Euclidean case, we have no such formal arguments. The above choice of sign on the exponent amounts to assuming that the
initial or final states corresponding to two strings have always positive exponent $e^{ET}$ with $E$ being the total energy $E = E_i + E_j$. Here we recall that the perturbative expansion (5.1) of S-matrix comes from the formal definition

$$\lim_{T \to \infty} \langle b | e^{\mathcal{H}(0)T} e^{-2HT} e^{\mathcal{H}(0)T} | a \rangle$$

(5.7)

where $\mathcal{H}$ and $\mathcal{H}^{(0)}$ are full and free string field Hamiltonian. The factors $e^{\mathcal{H}(0)T}$ sandwiching transition operator $e^{-2HT}$ correspond to amputating the propagators of external lines in the language of LSZ formalism. The choice of the positive exponent for two-string states means that, the initial or final states, $e^{\mathcal{H}(0)T} | a \rangle$ or $\langle b | e^{\mathcal{H}(0)T}$, with two strings must be prepared such that we observe the interactions of two strings into a single string (or reversed one) occurring near the conformal boundary associated with the initial (or final) region of the trajectory, respectively. Thus we arrive at the following correspondence between the processes in string theory and in Yang-Mills theory, respectively;

[interaction of multi-string states near the boundary]

\[\hat{\mathcal{I}}\]

[short distance product of multiple BMN operators]

Preparation of the initial and final multi-string states as required above is possible by using wave packet basis along the trajectory appropriately. Note that the scattering we are dealing with is essentially of 1+1 dimensional, since the wave functions are bounded in all the transverse directions by harmonic potentials. Therefore the wave packet picture can be formulated in an elementary way, with a caveat that we are actually using it in the sense of Euclidean theory with the steepest descent approximation for momentum ($J$) integrations in performing superposition of plane-‘wave’ functions. We expect that the tunneling picture would be more naturally formulated if the near horizon limit is not assumed, but we avoid such technical refinements in the present exposition.

In terms of an obvious symbolic notation $\mathcal{C}$ for the string-field theoretic expression for the coefficients $C_{ijk}$, we can express the equation (5.6) symbolically as

$$\mathcal{H}^{(1)} = \pm [\mathcal{H}^{(0)}, \mathcal{C}].$$

(5.8)

for $2 \to 1$ or $1 \to 2$ matrix elements, respectively. Here and in what follows, the products of symbolic string-field operators denoted by calligraphic letters are to be interpreted within the restriction of semiclassical tree approximation of string field theory. We have assumed that the CFT coefficients are consistent with the kinematical symmetries such as our $SO(8)$ or, if one wishes, smaller $SO(4)$, $[J, \mathcal{C}] = 0$. In terms of the similar symbolic notation, we can easily see that the dynamical susy anticommutation relation is preserved by choosing the first order interaction correction to the dynamical susy generator as

$$\mathcal{Q}^{-(1)} = \pm [\mathcal{Q}^{-(0)}, \mathcal{C}]$$

(5.9)
for the same matrix elements $2 \rightarrow 1$ or $1 \rightarrow 2$ as for those of the interaction Hamiltonian.

In fact, we can reverse the above arguments, starting from supersymmetry: Assume a general form of the first order interaction Hamiltonian as a square of a component $Q^{(0)}$ of the lowest order susy generator,

$$
\mathcal{H}^{(0)} = \left( Q^{(0)} \right)^2. \quad (5.10)
$$

In our case, $Q^{(0)}$ can be any component of the second-quantized version of $Q_1^- = (Q^- + \bar{Q}^-)/\sqrt{2}$ or of $Q_2^- = (Q^- - \bar{Q}^-)/\sqrt{2}i$. Using only the relation (5.10), we can derive

$$
[Q^{(0)}, \mathcal{H}^{(1)}] = [\mathcal{H}^{(0)}, Q^{(1)}], \quad \mathcal{H}^{(1)} = \{Q^{(0)}, Q^{(1)}\}. \quad (5.11)
$$

For any matrix elements which do not conserve energy ($E_i + E_j - E_k \neq 0$), this is rewritten as

$$
\langle k | Q^{(1)} | i, j \rangle = \langle k | [Q^{(0)}, C] | i, j \rangle, \quad (5.12)
$$

with

$$
\langle \ell | C | i, j \rangle = \frac{\langle \ell | \mathcal{H}^{(1)} | i, j \rangle}{E_\ell - E_i - E_j} \quad (5.13)
$$

for arbitrary $2 \rightarrow 1$ matrix elements, since $Q^{(0)}$ has nonzero matrix elements only when energy is conserved, $[Q^{(0)}, \mathcal{H}^{(0)}] = 0$. This implies that the holographic relation (5.8) is inevitable, supposing that there is no nontrivial energy-conserving matrix elements for the interaction Hamiltonian, since then we can adopt (5.13) as the definition of $C$. The last condition seems indeed to be satisfied at least in the supergravity approximation\textsuperscript{15}, and also to be consistent with all the known perturbative results\textsuperscript{16} on the gauge theory side in the small $R^2/J$ limit.\textsuperscript{*} Conversely, our discussion shows that holographic principle requires the vanishing of $\langle \ell | \mathcal{H}^{(1)} | i, j \rangle$, when energy is conserved. We also note that the above argument is valid if we replace the susy generators by our pseudo-susy generators.

These are main messages derived from our considerations on how the basic holographic relation (2.1) of supergravity is extended to string theory in the plane-wave limit. Our result is directly based on our interpretation of holography and is completely independent of the arguments given in ref.\textsuperscript{14} where the relation (5.6) of the same form as ours were first discussed in a context which is nothing to do with our arguments. It should also be emphasized that our Euclidean prescription is deeply motivated by the tunneling picture and is not just a formal device for computations of Minkowskian amplitudes. For more detailed discussions on our results, including the limitation of our methods, we refer the reader to the paper\textsuperscript{3}. For example, by assuming that the BMN operators form a complete set of gauge-theory operators

\textsuperscript{*} One might wonder if this argument is applied to simpler systems, such as (first quantized) supersymmetric quantum mechanics. In the latter case, there are in general an infinite number of nonzero energy-conserving matrix elements for the interaction part. Hence, we would not have a well-defined candidate for the operator $C$.\textsuperscript{*}
with respect to OPE (5.4), we can extend above consideration to a more general $1 \to n + 1 (n + 1 \to 1)$ matrix elements, and the result is

$$\mathcal{H} = \frac{1}{1 - C} \mathcal{H}^{(0)} (1 - C), \quad \mathcal{Q}^- = \frac{1}{1 - C} \mathcal{Q}^{-(0)} (1 - C),$$

(5.14)

for $1 \to n + 1$ and their transpose for $n + 1 \to 1$, respectively.

§6. Concluding remarks

To avoid confusion, we note that the simple structure we found above does not necessarily mean that the interaction can completely be eliminated by a unitary transformation in full fledged quantum theory of string fields. In particular, our arguments are not sufficient for fixing the matrix elements of more general types than those discussed above. It should, however, be stressed that the above form clearly shows that the implementation of susy algebra alone in the sense of classical string field theory is not sufficient to fix the interaction vertices uniquely, since we can always construct the 3-point interaction vertex formally such that it satisfies the first order form of the susy algebra, once a $C$ preserving kinematical symmetries is given. Of course, for the result to be well behaved, the CFT coefficients should not diverge for the special cases when the energy is conserved. In other words, the 3-point vertex must vanish for such matrix elements. In the case of supergravity approximation, this is indeed satisfied before taking the plane-wave limit, as has been emphasized before. It seems that all the known results for CFT coefficients from perturbative computations\textsuperscript{16) on the Yang-Mills side are also consistent with this property. Another remark is that the relation (5.6) does not necessarily require that the so-called prefactor of the string-field theory is equal to the energy difference, contrary to some earlier works done using a wrong expression for a possible form of string field theory vertex. For references to these earlier works, we refer the reader to the bibliography of our paper\textsuperscript{3). Finally, we would like to mention a more recent proposal\textsuperscript{17), appeared after the Symposium, of a prefactor which is different from the standard one\textsuperscript{18) which leads to 3-point vertex consistent with our general prediction and with the known perturbative results on the SYM side.\textsuperscript{*} In any case, however, our consideration seems to indicate that the logic of light-cone string field theory \textit{alone} does not involve the first principles in order to fix the interacting string field theory in general curved space-times. At the same time, it is also important to have an explicit construction of string field theory which meets our criterion as holographic string field theory. We hope to report progress along such direction in a forthcoming work.

\textsuperscript{*} In preparing the present manuscript, I came to know a more recent and alternative discussion\textsuperscript{19) related to this issue. They maintain the relation (5.6) without taking into account the mixing effect. It is unfortunate that many authors in this area do not seem aware of our work. The reference of the present exposition itself is of course very incomplete, since the purpose of this article is not to give a review. I would like to apologize any authors whose works are overlooked here.
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