Comparing Entropies in Portfolio Diversification with Fuzzy Value at Risk and Higher-Order Moment

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ABSTRACT
Credibility fuzzy value at risk in portfolio models is a reasonable solution when investors may face ambiguity, lack of historical data, and when avoiding normally or symmetrically distributed assumptions is needed. Credibility fuzzy membership function because having a self-duality axiom is analogous to measurable function on a probability space for the random variable. The primary aim of this paper is to solve the portfolio problem by using the third and fourth credibility moments in multi-objective higher-order moment portfolio models with different entropies. Firstly, Minkowski, Shannon, Yager, Renyi, and Gini-Simpson entropies are presented as new objective functions to solve corner solutions of conventional fuzzy multi-objective weighted credibility higher-order moment portfolio selection models. Secondly, because of the non-linearity nature of multi-objective models, a genetic algorithm is used to solve the models. Finally, proposed models are tested – using the data of Tehran Stock Exchange – and then evaluated – applying adjusted Sharpe Ratio as a portfolio performance technique.

1. Introduction
The Markowitz mean-variance [1] model as a modern portfolio has been widely received attention from researchers in academia and applied by investors in the real world. However, the assumption that is financial asset returns follow normal distribution has been violated by real data and experience [2,3]; therefore, for the purpose of relaxation to the normality assumption, the investors should consider higher-order moments in their investment [4]. Some researchers have made efforts to shed some light into portfolio problems in a three-moment or four-moment framework. For instance, Araciolu et al. [5] and Harvey et al. [6] indicated that involving skewness in the optimization process as an object can help investors to get better returns. Aksarayli and Palain [7] and Yue and Wang in [8] maintained that returns distributions have fatter tails compared to normal distributions. One step further, Adcock [9] argued that a higher probability of very high and very low returns is more

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likely to have occurred in the case of having kurtosis in a portfolio model compared to normal distribution. However, these higher-order moments are problematic as they make portfolio models less diverse, resulting in an unexpected gain for investors [10–13]. To tackle this issue, the entropy function has been examined by some researchers [14,15]. In their study, M. Aksarayl, O. Pala [7] compare Gini- Simpson with Shannon entropies in a multi-objective higher-order model. Their findings show that the former entropy outperforms the later in terms of the average return. W. Yue, Y. Wang [8] illustrated that the mean-variance-skewness-kurtosis- Minkowski entropy (MVKSE) model can provide investors with more diversity so lotions compared to the mean-variance-skewness-kurtosis-Shannon entropy (MVSKE) and mean-variance-skewness-kurtosis-Yager entropy (MVKSYE) models.

Uncertainty is another significant factor when investors face a lack of historical data [16]. For example, Li et al. [17] proposed a genetic algorithm to solve the possibilistic fuzzy portfolio, selection model. Chen [18] proposed a weighted lower and upper possibilistic means and variances based on trapezoidal fuzzy numbers. Linear programming was the method used to solve this model. Kocadagli et al. [19] used numerical examples in order to show the effectiveness of the fuzzy goal programming portfolio selection model. To solve the fuzzy mean-variance portfolio model, Mashayekhi et al. [20] proposed the Non-dominated Sorting Genetic Algorithm II (NSGA-II). Despite the wide acceptance of possible measures, it has some limitations. The most severe shortcoming of possibility measure is the absence of self-dual axiom [21]. Some researchers [18,22,23] proposed a credibility measure that is self-dual and thus overcomes the limitations inherent in the possibility measure.

For constructing a reasonable portfolio selection model, it is highly recommended that appropriate risk measures should be considered together with higher-order moments, lower diversity of the portfolio, and uncertainty. One of the serious limitations of the analyses relied on variance as a risk measure is related to the complexity of calculating the variance-covariance matrix for all assets [24]. Value-at-risk is proposed as a new risk measure to overcome this limitation. According to JP Morgan (1996), value at risk (\(\text{VaR}\)) is’ a measure of the maximum potential change in the value of a portfolio of financial instruments with a given probability over a pre-set horizon’ [25]. The Basel Committee on Banking Supervision urged commercial banks to take \(\text{VaR}\) into consideration when making decisions about allocating resources to market risk exposure. Amongst various risk measurement techniques, mean-variance, mean semivariance, and mean entropy has been fairly well-studied. To just mention a few, Watada [26] proposed Markowitz’s mean-variance for the fuzzy set theory. Huang [27] define the semivariance for the fuzzy variable and proposed two fuzzy mean-semi variance portfolio models. Moreover, he [28] used entropy as a risk and proposed mean-entropy fuzzy portfolio selection; considering the fact that: The smaller the entropy, the portfolio will be better. These two models in [27] and [28] are solved by the genetic algorithm. Because the conventional stochastic \(\text{VaR}\) theory is not useful in order to compute \(\text{VaR}\) in fuzzy space, there is a few research involved \(\text{VaR}\) as a risk measure. On the other hand, fuzzy credibility appears to be new to the literature and has been surprisingly understudied. Wang et al. [29] are pioneered because of a new definition of credibility fuzzy \(\text{VaR}\).

It can be perceived that comparing different entropies simultaneously under the credibilistic fuzzy portfolio model has been neglected. This paper deliberately developed five credibilistic multi-objective higher-order moment models each based on different
entropies. This has been done to examine which one would better diversify the portfolio. A further contribution of this study is to present credibilistic VaR instead of variance in order to estimate portfolio risk. In this research, in order to create diversity, we compared and examined five different types of entropy measures: Shannon, Yager, Minkowski, Renyi, and Gini-Simpson entropies. Portfolio optimization can be defined as a non-linear, multi-functional approach to the issue of the optimization comprised of conflicting objectives. A further contribution is to adapt five entropies to a higher moment portfolio framework and obtain five credibilistic multi-objective, mean-VaR – skewness–kurtosis entropy, (MVarSKEN) models in order to maximize expected return, skewness and entropy and minimize VaR and kurtosis. A combination of simple additive weighting (SAW) method, with equal weight, and genetic algorithm are used to solve the models. SAW method is a value function that is established based on a simple addition of scores that represent the goal achievement under each criterion, multiplied by the particular weights; it has the ability to compensate among criteria. It is also intuitive to decision-makers. Finally, in order to test model performances of portfolios, we used adjusted Sharpe Ratio (ASR) criteria. The rest of this paper is organized as follows:

In Section 2, we review some preliminary knowledge on the fuzzy variable, credibility measure, different entropies, and genetic algorithm as a solving method of the model. In Section 3, we describe the model of fuzzy credibilistic mean, fuzzy credibilistic value at risk, fuzzy credibilistic skewness, fuzzy credibilistic kurtosis with different entropies and method of solving portfolio selection model. In Section 4, a more precise description of the problem-solving method is presented, the proposed model is tested by Tehran Exchange data and used ASR as performance technique to evaluate the performance of the portfolio model and finally, we will present the results and provide future research proposals.

2. Preliminaries

In this section, some useful definitions and concepts will be introduced.

Definition 1: If $\Theta$ be a non-empty set and $P(\Theta)$ is a power set, i.e. the largest algebra over $\Theta$. Each element in $P(\Theta)$ is called an event. The set function $Cr$ is the credibility measure if [23]:

- **(Axiom 1) (Normality)** $Cr(\Theta) = 1$.
- **(Axiom 2) (Monotonicity)** $A \subset B \rightarrow Cr(A) \subset Cr(B)$.
- **(Axiom 3) (Self-duality)** $Cr(A) + Cr(A^c) = 1$ for any event $A$.
- **(Axiom 4) (Maximality)** $\sup_i Cr(A_i) < 0.5 \rightarrow Cr(\bigcup_i A_i) = \sup_i Cr(A_i)$ for any events $\{A_i\}$ with $\sup_i Cr(A_i) < 0.5$. The value of $Cr(A)$ indicates the level that the event $A$ will occur [23].

Definition 2: If $\Theta$ be a non-empty set, $P(\Theta)$ a power set of $\Theta$ and $Cr$ a credibility measure then the triplet $(\Theta, P(\Theta), Cr)$ is called credibility space.

Definition 3: A fuzzy variable is defined as a function from credibility space $(\Theta, P(\Theta), Cr)$ to the set of a real number.
**Definition 4:** Let $\xi$ be a fuzzy variable defined on the credibility space $(\Theta, P(\Theta), Cr)$. Then the membership function is derived from the credibility measure by \[ \mu(t) = (2Cr(\xi = t)) \land 1, t \in \mathbb{R}. \] \hspace{1cm} (1)

**Definition 5:** The fuzzy variables $\xi_1, \xi_2, \ldots, \xi_n$ are independent if and only if for any sets $B_1, B_2, \ldots, B_n$ of $\forall i$, we have:

\[
\begin{align*}
Cr\left(\bigcup_{i=1}^{n} \{\xi_i \in B_i\} \right) &= \max Cr(\xi_i \in B_i) , \quad 1 \leq i \leq n. \tag{2}\n
Cr\left(\bigcap_{i=1}^{n} \{\xi_i \in B_i^c\} \right) &= 1 - Cr\left(\bigcup_{i=1}^{n} \{\xi_i \in B_i^c\} \right) = 1 - \min Cr(\xi_i \in B_i^c) \\
&= \max Cr(\xi_i \in B_i) , \quad 1 \leq i \leq n. \tag{3}
\end{align*}
\]

**Definition 6:** A fuzzy variable $\xi = (a, b, c, d)$ be a trapezoidal fuzzy number with $a \leq b \leq c \leq d$ if it has the following membership function:

\[
Cr(\{\xi \leq r\}) = \begin{cases} 
0, & r < a, \\
b - a, & a \leq r < b, \\
\frac{r - a}{2(b - a)}, & b \leq r < c, \\
\frac{1}{2}, & c \leq r < d, \\
1 - \frac{r - c}{2(c - d)}, & d \leq r. 
\end{cases}
\]

\[
Cr(\{\xi \geq r\}) = \begin{cases} 
1, & r < a, \\
1 - \frac{r - a}{2(b - a)}, & a \leq r < b, \\
\frac{1}{2}, & b \leq r < c, \\
\frac{r - c}{2(c - d)}, & c \leq r < d, \\
0, & d \leq r. 
\end{cases}
\]

**Definition 7:** For a trapezoidal fuzzy variable $\xi = (a, b, c, d)$ such that $a \neq b, c \neq d$, $supp(\xi) = [a, d]$ its support, $cor(\xi) = [b, c]$ its core, $l_s$ the length of $supp(\xi)$ and $l_c$ the length of $cor(\xi)$. We set: \[ l_s(\xi) = d - a, l_c(\xi) = c - b, \beta = d - b, \text{ and } \alpha = b - a. \] \hspace{1cm} (6)

### 2.1. Credibility Moments of Portfolio

Let $(\xi_k = (a_k, b_k, c_k, d_k))_{k=1,2,\ldots,n}$ be a family of $n$ independent trapezoidal fuzzy variables and $x = (x_1, x_2, \ldots, x_n)$ a family of $n$ positive reals. The portfolio return $\xi(x) = \sum_{k=1}^{n} \xi_k$ defined by $\xi(x) = \sum_{k=1}^{n} x_k \xi_k = (\sum_{k=1}^{n} x_k a_k, \sum_{k=1}^{n} x_k b_k, \sum_{k=1}^{n} x_k c_k, \sum_{k=1}^{n} x_k d_k)$ is a fuzzy variable and its expectation is \[ e(x) = E[\xi(x)] = \frac{\sum_{k=1}^{n} (a_k + b_k + c_k + d_k)x_k}{4}. \] \hspace{1cm} (7)
Definition 8: The variance of $\xi$ is [30,33]:

$$V(\xi) = -\frac{\left[\sum_{k=1}^{n} x_k(l_s(\xi_k) + l_c(\xi_k))\right]^3}{192 \sum_{k=1}^{n} \sum_{l=1}^{l} x_k \alpha_k \beta_l} \left[\sum_{k=1}^{n} x_k(\alpha_k - \beta_k)\right] + \frac{\left[\sum_{k=1}^{n} x_k(l_s(\xi_k) + l_c(\xi_k))\right]^2}{32 \sum_{k=1}^{n} \sum_{l=1}^{l} x_k \alpha_k \beta_l} \left[\sum_{k=1}^{n} x_k(\alpha_k - \beta_k)\right]$$

$$\times \left[\sum_{k=1}^{n} x_k(2l_s(\xi_k) - (\alpha_k + \beta_k))\right] + \frac{\left[\sum_{k=1}^{n} x_k(\alpha_k - \beta_k)\right]}{4} + \frac{\sum_{k=1}^{n} x_k l_s(\xi_k)}{2} \right]^3 \sum_{k=1}^{n} x_k(\alpha_k + \beta_k + |\alpha_k - \beta_k|)$$

$$- \frac{\left[\sum_{k=1}^{n} x_k(\alpha_k - \beta_k)\right]}{2} + \frac{\sum_{k=1}^{n} x_k l_c(\xi_k)}{2} \right]^3 + \left[\sum_{k=1}^{n} x_k(\alpha_k - \beta_k)\right] - \frac{\sum_{k=1}^{n} x_k l_c(\xi_k)}{2} \right]^3 \sum_{k=1}^{n} x_k(\alpha_k + \beta_k + |\alpha_k - \beta_k|)$$

$$\times 6 \sum_{k=1}^{n} x_k(\alpha_k + \beta_k + |\alpha_k - \beta_k|)$$

In the following subsection, we will introduce the credibility of higher order moments.

2.1.1. Higher-order Moments of Fuzzy Variables

Definition 9: The skewness of $\xi$ is [30]:

$$\text{Skew}[\xi] = \frac{\left[\sum_{k=1}^{n} x_k(b_k - e_k)\right]^4}{8 \sum_{k=1}^{n} x_k(b_k - a_k)} - \frac{\left[\sum_{k=1}^{n} x_k(a_k - e_k)\right]^4}{8 \sum_{k=1}^{n} x_k(a_k - b_k)}$$

$$+ \frac{\left[\sum_{k=1}^{n} x_k(c_k - e_k)\right]^4}{8 \sum_{k=1}^{n} x_k(c_k - d_k)} - \frac{\sum_{k=1}^{n} x_k(d_k - e_k)}{4} \right]^4}$$

$$\left[\sum_{k=1}^{n} x_k(d_k - e_k)\right] \right]^4}$$

Definition 10: The kurtosis of $\xi$ is [30]:

$$\text{Kur}[\xi] = -\frac{\left[\sum_{k=1}^{n} x_k(l_s(\xi_k) + l_c(\xi_k))\right]^5}{5120 \sum_{k=1}^{n} \sum_{l=1}^{l} x_k \alpha_k \beta_l} \left[\sum_{k=1}^{n} x_k(\alpha_k - \beta_k)\right]$$

$$+ \frac{\left[\sum_{k=1}^{n} x_k(l_s(\xi_k) + l_c(\xi_k))\right]^4}{2048 \sum_{k=1}^{n} \sum_{l=1}^{l} x_k \alpha_k \beta_l} \left[\sum_{k=1}^{n} x_k(\alpha_k - \beta_k)\right]$$

$$\times \left(\sum_{k=1}^{n} x_k(2l_s(\xi_k) - (\alpha_k + \beta_k))\right) + \frac{\left[\sum_{k=1}^{n} x_k(\alpha_k - \beta_k)\right]}{5} + \frac{\sum_{k=1}^{n} x_k l_s(\xi_k)}{2} \right]^5 \sum_{k=1}^{n} x_k(\alpha_k + \beta_k + |\alpha_k - \beta_k|)$$

$$\times \left[\sum_{k=1}^{n} x_k(\alpha_k - \beta_k)\right] - \frac{\sum_{k=1}^{n} x_k l_c(\xi_k)}{2} \right]^5 + \left[\sum_{k=1}^{n} x_k(\alpha_k - \beta_k)\right] - \frac{\sum_{k=1}^{n} x_k l_c(\xi_k)}{2} \right]^5 \sum_{k=1}^{n} x_k(\alpha_k + \beta_k + |\alpha_k - \beta_k|)$$

$$+ \left[\sum_{k=1}^{n} x_k(\alpha_k - \beta_k)\right] - \frac{\sum_{k=1}^{n} x_k l_c(\xi_k)}{2} \right]^5 \sum_{k=1}^{n} x_k(\alpha_k + \beta_k + |\alpha_k - \beta_k|)$$

$$\times 10 \sum_{k=1}^{n} x_k(\alpha_k + \beta_k + |\alpha_k - \beta_k|)$$

\text{.} (10)
In the section, we will introduce weighted credibility Value-at-Risk for the fuzzy variable.

### 2.2. Value-at-risk of Fuzzy Variables

Let \( \xi_1, \xi_2, \ldots, \xi_n \) be identically distributed independent random variables with a stochastic process that representing the financial return [1], \( \Omega_{t-1} \) is \( \sigma \)-algebra that represented all information is available at the time \( t-1 \) and \( F(\xi) \) denote the cumulative distribution function represented as follow:

\[
F(\xi) = \Pr(\xi < \xi | \Omega_{t-1}), \xi_t = \mu + \epsilon_t, \epsilon_t = z_t \sigma_t, \quad (11)
\]

\( z_t \) has a conditional distribution function \( V(z) \). The VaR with a given probability denoted by is defined as the quantile of the probability financial return distribution.

**Definition 1.** The value of \( \text{VaR}_{1-\beta} \) indicates the largest loss with the \( 1-\beta \) confidence level and \( S \) is the largest loss that one investor can accept. \( L \) is the loss function:

\[
L = -(x_1 \xi_1 + x_2 \xi_2 + \ldots + x_n \xi_n)
\]

\( \text{VaR}_{1-\beta} = \sup \{ \lambda : \Pr(L \geq \lambda) \geq \gamma \geq \beta \} \leq S. \) \( (12) \)

If the portfolio returns \( \xi = \sum_{k=1}^{n} \xi_k \) defined by \( \xi(x) = \sum_{k=1}^{n} x_k \xi_k = (\sum_{k=1}^{n} x_k a_k, \sum_{k=1}^{n} x_k b_k, \sum_{k=1}^{n} x_k c_k, \sum_{k=1}^{n} x_k d_k) \) and fuzzy variable \( \xi = (a, b, c, d) \) be a trapezoidal fuzzy number with \( a \leq b \leq c \leq d \) the VaR function of \( L = -(x_1 \xi_1 + x_2 \xi_2 + \ldots + x_n \xi_n) \) can be described as:

\[
\text{VaR}_{1-\beta} = \begin{cases} 
\sum_{i=1}^{n} [a_i(2\beta - 1) - 2\beta b_i]x_i, & 0 < \beta \leq 0.5, \\
\sum_{i=1}^{n} [(2\beta - 2)c_i - (2\beta - 1)d_i]x_i, & 0.5 < \beta \leq 1.
\end{cases} \quad (13)
\]

In order to decentralized investment, we use entropies, in the following section, we will introduce the entropies that are used in research.

### 2.3. Entropies

Entropy is useful to model a least biased distribution from the partial information represented by certain restriction.

#### 2.3.1. Shannon Entropy

Entropy was first proposed for a random variable by Shannon in the form of logarithm in 1948, it was first use in communication problem then use in finance to diversification the portfolio. The following is Shannon entropy [8,35]:

\[
SE = - \sum_{i=1}^{n} x_i \ln x_i, \quad (14)
\]
\( x_i \) is the weight of risk asset \( i, i \in \{1, \ldots, n\} \), \( n \) is the number of an investment asset. When \( x_i = \frac{1}{n} \) SE has a maximum and minimum value when \( x_i = 1 \) for one \( i \) and \( x_i = 0 \) for rest.

### 2.3.2. Renyi Entropy

In (1961) Renyi proposed a generalization of Shannon entropy with the help of a parameter \( \alpha \in R^+ \). \( \alpha \) is the order of entropy of probability density \( f(x) \) on a variable \( x \in R^d \) where normalization to unity as given by \( \int f(x)dx = 1 \). The idea was to consider a more general(non-linear) averaging of \( \ln f_X \). Starting from axioms associated with randomness measures. The following is Renyi entropy [36]:

\[
H_\alpha = \frac{\ln E[f_X^{\alpha - 1}(X)]}{1 - \alpha} = \frac{\ln \int (f_X(x))^{\alpha} dx}{1 - \alpha},
\]

(15)

### 2.3.3. Yager Entropy

Yager entropy aims to minimize the distance between the weight of investment assets \( x_i \) and \( 1/n \) in terms of portfolio selection. The following is the definition of Yager entropy [8]:

\[
YE = -z \left( \sqrt[n-1]{\sum_{i=1}^{n-1} |x_i - \frac{1}{n}|^z} \right).
\]

(16)

The maximum value of Yager entropy occurs when \( x_i = 1/n \). Where \( z \) is a constant and \( z \geq 1 \) while \( z = 1 \) Yager entropy can be transferred into linear type. The following is Yager entropy as \( z = 1 \):

\[
YE = -\sum_{i=1}^{n-1} |x_i - \frac{1}{n}|.
\]

(17)

\( x_i \) is the proportion of total investment devoted to risk asset \( i, i \in \{1, \ldots, n-1\} \), \( n \) is the number of investment asset

### 2.3.4. Minkowski Entropy

Minkowski entropy is based on Minkowski metric which is calculated the sum of Minkowski distance between the weight of investment which is a definition as follow [8]:

\[
Pn(x) = -\left( z \left( \sum_{i=1}^{n-1} |x_i - x_{i-1}|^z \right) \right).
\]

(18)

\( x_i \) is the proportion of total investment devoted to risk assets \( i, i \in \{1, \ldots, n-1\} \). When \( z = 1 \) it transferred to linear Minkowski entropy which is a definition as follow [8]:

\[
ME(x) = -\left( \sum_{i=1}^{n-1} |x_i - x_{i-1}| \right).
\]

(19)
2.3.5. Gini-Simpson Entropies
The following is Simi-Simpson entropy [7]:

\[
G - S \text{ Entropy} = 1 - \sum_{i=1}^{n} x_i^2.
\]  

(20)

\( x_i \) is the weight of risk asset \( i, i \in \{1, \ldots, n - 1\} \), \( n \) is the number of an investment asset.

2.4. Genetic Algorithm
Genetic Algorithms (GAs) are founded by Holland [37] and used to solve optimization problems based on biological organisms. GAs use a direct analogy of natural behavior. In nature, individuals in a population compete with each other and those individuals which are most successful in surviving and attracting mates will have relatively larger numbers of offspring. Poorly performing individuals will produce few of even no offspring at all. The combination of good characteristics from different ancestors can sometimes produce super fit offspring, whose fitness is greater than that of either parent. In this way, species evolve to become better and better suited to their environment. The genetic algorithm begins with an initial population that is usually generated randomly and members are evaluated based on the objective function and the initial population. After that, it’s time to do the crossover and mutation, in crossover process two chromosomes selected from the initial population and create a new generation. In a mutation, a chromosome is randomly selected and used to create a new generation. This method continues to reach the number of steps the researcher intends to accomplish.

In this paper, the solution of proposed portfolio selection problems (MOPs) can be formulated as following [8]:

\[
\begin{align*}
\text{Max} \quad & Z_0 = \sum_{j=1}^{k} w_j f_i(x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, k, \\
\text{s.t} \quad & g_j(x_1, x_2, \ldots, x_n) \leq b_j, \quad j = 1, 2, \ldots, m, \\
& h_j(x_1, x_2, \ldots, x_n) \leq c_j, \quad j = 1, 2, \ldots, m.
\end{align*}
\]  

(21)

\( w_i \) is the weight of the object in the model, in this paper we consider the equal weight for all objects.

3. Proposed Models
In this section, we present the portfolio model based on the information obtained from the stock exchange, we have five models containing five objectives, including the fuzzy mean, fuzzy value at risk, fuzzy skewness and fuzzy kurtosis are similar among models and used one entropy in each model. In order to set up a model tackling all the above-mentioned issues, we maximize return, skewness, and entropies while minimizing \( \text{VaR} \) and kurtosis, simultaneously. Let us consider a multi-objective fuzzy portfolio selection problem with \( n \) risk asset. The return rates of the risk asset are denoted as trapezoidal fuzzy numbers. For the notation convenience, we introduce the following notations.

\( x_i \): is the weight of risk asset \( i, i \in \{1, \ldots, n - 1\} \), \( n \) is the number of investment asset,
$\tilde{r}_i$: Fuzzy rate of return on the risk asset $i$, $i \in \{1, \ldots, n - 1\}$.

Entropies are the only difference among the model therefore we describe one model and in other models, we change the entropy which uses in the model.

### 3.1. Fuzzy Mean- VaR-skewness- Kurtosis- Entropy

Mode1(M-VaR-S-K-SE): The fuzzy multi-objective mean, Value-at-Risk, skewness, kurtosis, and entropy portfolio selection model is as follows:

$$
\begin{align*}
\max f_1(x) &= M_f \left[ \sum_{i=1}^{n} \tilde{r}_i x_i \right], \\
\min f_2(x) &= \text{VaR}_f \left[ \sum_{i=1}^{n} \tilde{r}_i x_i \right], \\
\max f_3(x) &= \text{Skew}_f \left[ \sum_{i=1}^{n} \tilde{r}_i x_i \right], \\
\min f_4(x) &= \text{Kur}_f \left[ \sum_{i=1}^{n} \tilde{r}_i x_i \right], \\
\sum_{i=1}^{n} x_i &= 1, x_i \geq 0, i = 1, 2, \ldots, n.
\end{align*}
$$

When we added the entropies for diversification in model 20 as the fifth objective the model change Mode1, the entropies are the only difference among the models, therefore we describe one model and for other models, we change the entropies.

Mode1(M-VaR-S-K-SE): The fuzzy multi-objective mean, Value-at-Risk, skewness, kurtosis, and Shannon entropy portfolio selection model are as follows:

$$
\begin{align*}
\max f_1(x) &= M_f \left[ \sum_{i=1}^{n} \tilde{\xi}_i x_i \right], \\
\min f_2(x) &= \text{VaR}_f \left[ \sum_{i=1}^{n} \tilde{\xi}_i x_i \right], \\
\max f_3(x) &= \text{Skew}_f \left[ \sum_{i=1}^{n} \tilde{\xi}_i x_i \right], \\
\min f_4(x) &= \text{Kur}_f \left[ \sum_{i=1}^{n} \tilde{\xi}_i x_i \right], \\
\max f_5(x) &= -\sum_{i=1}^{n} x_i \ln x_i, \\
\sum_{i=1}^{n} x_i &= 1, x_i \geq 0, i = 1, 2, \ldots, n.
\end{align*}
$$

In order to solve the model (21), we use a multi-objective optimization problem (MOPs).
3.1.1. Mops

In this paper, the solution of proposed portfolio selection problems (MOPs) can be formulated as following [19]:

\[
\begin{align*}
\max & \quad Z_0 = \sum_{i=1}^{k} w_i f_i(x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, k, \\
\text{s.t.} & \quad g_j(x_1, x_2, \ldots, x_n) \leq b_j, \quad j = 1, 2, \ldots, m, \\
& \quad h_j(x_1, x_2, \ldots, x_n) = 0, \quad j = 1, 2, \ldots, m.
\end{align*}
\]

(24)

\(w_i\) is the weight of the object in the model, in this paper we consider the equal weight for all objects. To solve five MOPs models, we use the SAW method with equal weight, the weight of objective depends on investor preference. The basic concept of the SAW method is to find a weighted sum of performance rating on each alternative. In each model, we consider first, third, and fifth objectives with positive marks and second and fourth objectives with a negative mark.

4. Numerical examples and analysis

In this section, we demonstrate the idea of our model and the algorithm to solve the models taken from daily historical data of Tehran Stock Exchange Market.

4.1. Data Processing

In this section, we intended to solve the models with a numerical example of Tehran Exchange. Initially, we choose 12 companies close price data from the beginning of the working day of 2016 to the first day of September 2017 which are included 365 working days data. These companies have chosen among the best companies in Tehran Exchange by using Data Envelopment Analysis (DEA). The exchange code of 12 assets are 2643, 1924, 5718, 7865, 7440, 3903, 3851, 4990, 3736, 6688, 2165 and 1873 we denote this asset by number 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. Table 1 provides the summary statistics of fuzzy return values. For fuzzy return rates of the models, we use the sample percentiles, the core of percentiles \([b, c]\) includes 40–60 percentiles of fuzzy return, and the left side contains 5–40 percentiles of fuzzy return and the right side includes 60–95 percentiles of fuzzy return.

To solve models with different entropies, we need to know the length of support, the length of the core. Table 2 provides the \(l_s, l_c, \alpha, \beta\) of fuzzy return values.

\(l_s, l_c, \alpha, \beta\) of fuzzy return is used in computed kurtosis of fuzzy return which \(l_s\) is the length of support, \(l_c\) is the length of the core, \(\alpha = b - a, \beta = d - b\).

To solve the model, we use the genetic algorithm, the general process involves a series of parent chromosomes in which the new generation is created by choosing a set of elites. Here are the exact steps we will take.

(1) we have five models which are included five objectives, the first, second, third, and fourth objectives, which represent the fuzzy mean, fuzzy Value at Risk, skewness, and kurtosis. This is the same in each model, and the only difference between these five
models is the fifth objective, which is the Shannon, Renyi, Yager, Minkowski’s, and Gini-Simpson entropies.

(2) In order to create the initial space matrix for a genetic algorithm that solves the model, we first created a random matrix consisting of 12 columns (stocks) which equals the number of companies selected and 8192 rows (parents). The numbers of each row produced by random function in MATLAB and the sum of the numbers of each row is equal to one.

(3) At each stage of the process of numbers, based on the weights, the objective function is selected and half of the worst cases is eliminated.

(4) In the first step mutation method for improvement of weight, the mutation rate is 0.25 in each period which means that in each period we choose 0.25 of the new generation to mutation process use new values for crossover and mutations. In this algorithm, the Crossover probability rate is 0.75. it means that in each stage 0.75 of the best possible answers are selected.

(5) In the end, the entropy is calculated for each portfolio for new weights. This process is performed 30 times until the best entropy is selected. To a better comparison, we compared entropies with each in Figure 1.

**Table 1.** Sample statistics for the daily returns on the asset (2016–2017) from historical data.

| Returns | 5th percentile | 40th percentile | 60th percentile | 95th percentile |
|---------|----------------|----------------|----------------|----------------|
| Asset1  | −0.00987       | −0.0003152     | 0.000195       | 0.010024       |
| Asset2  | −0.00985       | 0              | 0.000471       | 0.040525       |
| Asset3  | −0.04246       | −0.00731       | 0.001679       | 0.048391       |
| Asset4  | −0.03795       | −0.00303       | 0.001679       | 0.042836       |
| Asset5  | −0.03101       | −0.00471       | 0              | 0.039383       |
| Asset6  | −0.04522       | −0.00165       | 0.00435        | 0.047828       |
| Asset7  | −0.02015       | −0.00286       | −0.00075       | 0.037942       |
| Asset8  | −0.01301       | −0.00166       | 0              | 0.022621       |
| Asset9  | −0.02145       | −0.0013        | 0              | 0.035474       |
| Asset10 | −0.03177       | −0.00372       | 0              | 0.040392       |
| Asset11 | −0.04584       | −0.00409       | 0.006486       | 0.044521       |
| Asset12 | −0.0209        | 0              | 0              | 0.015236       |

**Table 2.** $I_s$, $I_c$, $\alpha$, $\beta$ of fuzzy return asset (2016-2017) from historical data.

| Return  | $I_s$          | $I_c$          | $\alpha$      | $\beta$      |
|---------|----------------|----------------|---------------|--------------|
| Asset1  | −0.00987       | −0.0003152     | 0.000195      | 0.010024     |
| Asset2  | −0.00985       | 0              | 0.000471      | 0.040525     |
| Asset3  | −0.04246       | −0.00731       | 0.001679      | 0.048391     |
| Asset4  | −0.03795       | −0.00303       | 0.001679      | 0.042836     |
| Asset5  | −0.03101       | −0.00471       | 0              | 0.039383     |
| Asset6  | −0.04522       | −0.00165       | 0.00435       | 0.047828     |
| Asset7  | −0.02015       | −0.00286       | −0.00075      | 0.037942     |
| Asset8  | −0.01301       | −0.00166       | 0              | 0.022621     |
| Asset9  | −0.02145       | −0.0013        | 0              | 0.035474     |
| Asset10 | −0.03177       | −0.00372       | 0              | 0.040392     |
| Asset11 | −0.04584       | −0.00409       | 0.006486      | 0.044521     |
| Asset12 | −0.0209        | 0              | 0              | 0.015236     |
4.2. Portfolio Performance Evaluation by Using the Adjusted Sharpe Ratio

Portfolio Sharpe Ratio is used as a criterion for portfolio performance evaluation, its formula is given by the following general form:

\[
SR = \frac{E[r(x)]}{\delta^2(x)} = \frac{E[r(x)]}{\delta(x)}.
\]  (25)

Which \(r(x)\) represents portfolio return, \(E(r(x))\) represents mean portfolio return, is portfolio variance, and is the standard deviation of the portfolio return. This model is based on the theory of mean-variance and is valid if the function is normal or quadratic. This criterion may result in misleading results when the yield function is skewed or has kurtosis. [8]

In order to overcome the problem related to the Sharpe Ratio, Pezier et al. [38] proposed an adjusted Sharpe Ratio that can measure portfolio performance in terms of mean-variance-skewness-kurtosis [38,39]. The mathematical form of the Adjusted Sharpe Ratio formula is given by the following form:

\[
ASR = SR \left(1 + \left(\frac{5}{6}\right)SR - \left(\frac{K}{24}\right)SR^2\right).
\]  (26)

In which \(S\) expresses skewness and \(K\) and \(SR\) represent respectively kurtosis and Sharpe Ratio. To an easier comparison in Table 3, the amounts of the mean, maximum, and minimum, and standard deviation of Adjusted Sharpe Ratio criterion are given for each model. Table 3 presents the statistical result of ASR for each model, including min, max, a mean and standard deviation of the ASR.

We can see that various statistical indicator of ASR of MVaRSK-GE model are larger than other. After MVaRSK-GE, MVaRSK-RE model, MVaRSK-YE model, MVaRSK-SE model MVaRSK-ME model have larger ASR respectively.
4.3. The Comparison of the Total Return among the Models MVARSK-ME, MVARSK-SE, MVARSK-YE, MVARSK-RE and MVARSK-GE

In real investment circumstances, investors pay more attention to the profitability of executing these asset strategies. Table 4 presents the statistical result of the absolute value of mean return for each model, including min, max, the mean and standard deviation of the return.

We can see that various statistical indicator of the absolute value of the mean return of MVARSK-GE model is larger than other. After MVARSK-GE, MVARSK-RE model, MVARSK-YE model, MVARSK-SE model MVARSK-ME model have a larger absolute value of mean return respectively.

4.4. Parameter Setting

The model is executed on Genuine Intel R Core(TM)i53230M and 4.00 GB RAM personal computer with MATLAB software. The parameter setting in this paper are as follows.

(1) Mutation probability is 0.25 and Crossover probability is 0.75.
(2) The population size \( N = 4096 \), the model is run 30 times independently for each model. The algorithm stops after 100 generations.
(3) \( K = 2 \) and \( M = 12 \), \( (N = K^M) \).

Markowitz frontier is an investment portfolio which occupies the efficient parts of the risk-return spectrum. This frontier shows minimum risk at expected level of return or maximum return at expected level of risk.

The entropy is diffuse this point in all Markowitz frontier, in order to comparing the entropies, we demonstrate each entropy with one color. This point shows the balance

| Return | Mean    | Max    | Min    | SD     |
|--------|---------|--------|--------|--------|
| MVARSK-ME model | 0.00018823 | 0.00051942 | 0.0000034125 | 0.00015845 |
| MVARSK-SE model | 0.00031684 | 0.00086596 | 0.0000034130 | 0.00022077 |
| MVARSK-YE model | 0.00036916 | 0.0012 | 0.0000056795 | 0.00031096 |
| MVARSK-RE model | 0.00037368 | 0.0013 | 0.0000011924 | 0.00032490 |
| MVARSK-GE model | 0.00045921 | 0.0014 | 0.000012563 | 0.00033843 |
between risk and return and entropies are diffuse along the Markowitz frontier and give opportunities to investor to have better choice between risk and return.

5. Conclusion

In this article, there are five multi-objective models for choosing a portfolio as described in the article. The proposed model includes five objectives including the weighted credibility mean, weighted credibility value at risk, weighted credibility skewness, and weighted credibility kurtosis which are the same in each case. The only difference between these five models is the fifth objective, which is the Minkowski, Shannon, Yager, Renyi, and Gini-Simpson entropies. To solve this model, we used the genetic algorithm to calculate the best output. Based on the results obtained from the statistical result of ASR and an absolute value of mean return measurement Gini-Simpson entropy, Renyi entropy, Yager entropy, Shannon entropy, and Minkowski entropy have the best diversification performance respectively.

For future studies, the model presented in this method can be conducted for several periods, or in combination with bonds. Also, other risk measures such as Credibilistic Average Value at Risk that have coherent risk properties can be used.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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