ROMAN {2}-DOMINATION IN GRAPHS AND GRAPH PRODUCTS

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Abstract. For a graph $G = (V,E)$ of order $n$, a Roman {2}-dominating function $f : V \to \{0,1,2\}$ has the property that for every vertex $v \in V$ with $f(v) = 0$, either $v$ is adjacent to a vertex assigned 2 under $f$, or $v$ is adjacent to least two vertices assigned 1 under $f$. In this paper, we classify all graphs with Roman {2}-domination number belonging to the set $\{2,3,4,n-2,n-1,n\}$. Furthermore, we obtain some results about Roman {2}-domination number of some graph operations.

1. Introduction

We study Roman {2}-dominating functions defined in [3]. We first present some necessary terminology and notation. Let $G = (V,E)$ be a graph with vertex set $V = V(G)$ and edge set $E(G)$. The open neighborhood $N(v)$ of a vertex $v$ consists of the vertices adjacent to $v$, and its closed neighborhood is $N[v] = N(v) \cup \{v\}$. The degree of $v$ is the cardinality of its open neighborhood. Let $\Delta(G)$ be the maximum degree of the graph $G$. If $S$ is a subset of $V$, then $N(S) = \bigcup_{x \in S} N(x)$, $N[S] = \bigcup_{x \in S} N[x]$, and the subgraph induced by $S$ in $G$ is denoted $G[S]$.

A dominating set of $G$ is a subset $S$ of $V$ such that every vertex in $V - S$ has at least one neighbor in $S$, in other words, $N[S] = V$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. By [6], a subset $S \subseteq V$ is a 2-dominating set if every vertex of $V - S$ has at least two neighbors in $S$. The 2-domination number $\gamma_2(G)$ is the minimum cardinality of a 2-dominating set of $G$.

Motivated by Stewart’s [10] article on defending the Roman Empire, Cockayne et al. introduced Roman dominating functions in [4]. For Roman domination, each vertex in the graph model corresponds to a location in the Roman Empire, and for protection, legions (armies) are stationed at various locations. A location is protected by a legion stationed there. A location having no legion can be protected by a legion sent from a neighboring location. However, this presents the problem of leaving a location unprotected (without a legion) when its legion is dispatched to a neighboring location. In order to prevent such problems, Emperor Constantine the Great [4] decreed that a legion cannot be sent to a neighboring location if it leaves its original station unprotected. In other words, every location with no legion must be adjacent to

2010 Mathematics Subject Classification. Primary:05C69; Secondary: 05C76.
Key words and phrases. Roman {2}-domination; Cartesian product; Grid graph.
a location that has at least two legions. This defense strategy prompted the following definition in [3].

A function \( f : V(G) \to \{0, 1, 2\} \) is a Roman dominating function (RDF) on \( G \) if every vertex \( u \in V \) for which \( f(u) = 0 \) is adjacent to at least one vertex \( v \) for which \( f(v) = 2 \). The weight of an RDF is the value \( f(V(G)) = \sum_{u \in V(G)} f(u) \). The Roman domination number \( \gamma_R(G) \) is the minimum weight of an RDF on \( G \). A vertex \( v \) with \( f(v) = 0 \) is said to be undefended with respect to \( f \) if it is not adjacent to a vertex \( w \) with \( f(w) > 0 \).

In this paper, we study Roman \( \{2\} \)-dominating functions. These functions are closely related to \( \{2\} \)-dominating functions introduced in [5] as follows. For a graph \( G \), a \( \{2\} \)-dominating function is a function \( f : V \to \{0, 1, 2\} \) having the property that for every vertex \( u \in V \), \( f(N[u]) \geq 2 \). The weight of a \( \{2\} \)-dominating function is the sum \( f(V) = \sum_{v \in V} f(v) \), and the minimum weight of a \( \{2\} \)-dominating function \( f \) is the \( \{2\} \)-domination number, denoted by \( \gamma_{\{2\}}(G) \).

A Roman \( \{2\} \)-dominating function \( f \) relaxes the restriction that for every vertex \( u \in V \), \( f(N[u]) = \sum_{v \in N[u]} f(v) \geq 2 \) to only requiring that this property holds for every vertex assigned 0 under \( f \). Formally, a Roman \( \{2\} \)-dominating function \( f : V \to \{0, 1, 2\} \) has the property that for every vertex \( v \in V \) with \( f(v) = 0 \), \( f(N(u)) \geq 2 \), that is, either there is a vertex \( u \in N(v) \), with \( f(u) = 2 \), or at least two vertices \( x, y \in N(u) \) with \( f(x) = f(y) = 1 \). In terms of the Roman Empire, this defense strategy requires that every location with no legion has a neighboring location with two legions, or at least two neighboring locations with one legion each. Note that for a Roman \( \{2\} \)-dominating function \( f \), it is possible that \( f(N[v]) = 1 \) for some vertex \( v \) with \( f(v) = 1 \). The weight of a Roman \( \{2\} \)-dominating function is the sum \( f(V) = \sum_{v \in V} f(v) \), and the minimum weight of a Roman \( \{2\} \)-dominating function \( f \) is the Roman \( \{2\} \)-domination number, denoted by \( \gamma_{\{R2\}}(G) \).

**Lemma 1.1.** [3] Corollary 10] for a cycle \( C_n \) and a path \( P_n \) we have
\[
\gamma_{\{R2\}}(C_n) = \lceil \frac{n}{2} \rceil, \quad \gamma_{\{R2\}}(P_n) = \lceil \frac{n+1}{2} \rceil.
\]

**Proposition 1.2.** [3] Proposition 5] For every graph \( G \); \( \gamma_{\{R2\}}(G) \leq \gamma_2(G) \).

For graphs \( G \) and \( H \), The join of graphs \( G \) and \( H \) is the graph \( G \vee H \) with the vertex set \( V = V(G) \cup V(H) \) where two vertices \( u \) and \( v \) are adjacent if
\[
\triangleright u, v \in V(G) \text{ and } uv \in E(G) \text{ or } u, v \in V(H) \text{ and } uv \in E(H) \text{ or } u \in V(G) \text{ and } v \in V(H).
\]

The Corona \( G[H] \) of \( G \) and \( H \) is constructed as follows:
Choose a labeling of the vertices of \( G \) with labels 1, 2, \ldots, \( n \). Take one copy of \( G \) and \( n \) disjoint copies of \( H \), labeled \( H_1, \ldots, H_n \), and connect each vertex of \( H_i \) to vertex \( i \) of \( G \).

The Cartesian product of two graphs \( G \) and \( H \), denoted by \( G \Box H \), has vertex set \( V(G \Box H) = V(G) \times V(H) \), where two distinct vertices \((u, v)\) and \((x, y)\) of \( G \Box H \) are
adjacent if either

\[ u = x \text{ and } vy \in E(H) \text{ or } v = y \text{ and } ux \in E(G). \]

The grid graph \( G_{m,n} \) is the Cartesian product of \( P_m \) and \( P_n \). In 1983, Jacobson and Kinch [9] established the exact values of \( \gamma(G_{m,n}) \) for \( 2 \leq m \leq 4 \) which are the first results on the domination number of grids. Also, In 1993, Chang and Clark [2] found those of \( \gamma(G_{m,n}) \) for \( m = 5 \) and 6. Fischer found those of \( \gamma(G_{m,n}) \) for \( m \leq 21 \) (see Goncalves et al. [7]). Recently, Goncalves et al. [7] finished the computation of \( \gamma(G_{m,n}) \) when \( 24 \leq m \leq n \). In [11], the authors have obtained the values of \( \gamma_2(G_{m,n}) \) for \( 2 \leq m \leq 4 \). In this paper, we will give some boundaries for \( \gamma_{(R2)}(G_{m,n}) \) for \( 2 \leq m \leq 4 \).

2. Graphs with small or large Roman \( \{2\} \)-domination number

In this section we provide a characterization of all connected graphs \( G \) of order \( n \) with Roman \( \{2\} \)-domination number belonging to \( \{2, 3, 4, n-2, n-1, n\} \).

**Proposition 2.1.** Let \( G \) be a graph. \( \gamma_{(R2)}(G) = 2 \) if and only if \( G = K_n \lor H \) for \( n = 1, 2 \) and for some graph \( H \).

**Proof.** Let \( f = (V_0, V_1, V_2) \) be a \( \gamma_{(R2)}(G) \)-function with weight 2. Hence, we have two cases. If there exists a vertex \( z \) with \( z \in V_2 \), then all other vertices of \( G \) are adjacent to \( z \). Therefore, \( G = K_1 \lor H \) for some graph \( H \). If there are two vertices \( u \) and \( v \) are in \( V_1 \), then all other vertices of \( G \) are adjacent to both vertices \( u \) and \( v \). If \( u \) and \( v \) are adjacent, then \( G = K_1 \lor H \) for some graph \( H \), and if \( u \) and \( v \) are not adjacent, then \( G = K_2 \lor H \) for some induced subgraph \( H \) of \( G \). Conversely, it is not hard to see the result. \( \square \)

For a graph \( G \), define \( N_i(G) \) for \( i = 1, \ldots, n - 1 \) as follows,

\[ N_i(G) = \{v \in V : \text{deg}(v) = i\}. \]

**Proposition 2.2.** Let \( G \) be a graph. \( \gamma_{(R2)}(G) = 3 \) if and only if one of the following holds:

(i) \( \Delta(G) = n - 2 \) and \( N_{n-2}(G) \) is a clique,

(ii) \( \Delta(G) < n - 2 \) and \( \gamma_2(G) = 3 \).

**Proof.** Let \( f = (V_0, V_1, V_2) \) be a \( \gamma_{(R2)}(G) \)-function with weight 3. By Proposition 2.1, \( \Delta(G) \leq n - 2 \). At first, suppose that \( \Delta(G) = n - 2 \). We consider two vertices \( u \) and \( v \) in \( N_{n-2}(G) \). If \( u \) and \( v \) are not adjacent, then \( u \) and \( v \) are adjacent to all other vertices of \( G \), and hence \( G = K_2 \lor H \), which is a contradiction by Proposition 2.1. Thus, \( N_{n-2}(G) \) is a clique.

If \( \Delta(G) < n - 2 \), then there are three vertices \( u, v \) and \( w \) in \( V_1 \). Hence, \( f \) is a 2-dominating function on \( G \), and then \( \gamma_2(G) \leq 3 \). Since \( \gamma_{(R2)}(G) = 3 \), we have \( \gamma_2(G) \geq 3 \). So, \( \gamma_2(G) = 3 \). Moreover, the converse proof can be easily checked. \( \square \)

**Proposition 2.3.** Let \( G \) be a graph. \( \gamma_{(R2)}(G) = 4 \) if and only if \( \Delta(G) \leq n - 3 \) and \( \gamma_2(G) \geq 4 \) as well as \( G \) satisfies one of the following conditions,
(i) \( \gamma(G) = 2 \),
(ii) \( \gamma_2(G) = 4 \),
(iii) There exists a vertex \( v \in V(G) \) such that \( \gamma_2(G[V(G) - N[v]]) = 2 \).

**Proof.** Suppose that \( \gamma_{(R2)}(G) = 4 \). By Propositions 2.1 and 2.2, we have \( \Delta(G) \leq n-3 \) and \( \gamma_{(R2)}(G) \geq 4 \). Let \( f = (V_0, V_1, V_2) \) be a \( \gamma_{(R2)}(G) \)-function. We consider three cases. First case, if \( |V_2| = 2 \), then \( \gamma(G) = 2 \). Second case, \( |V_1| = 4 \), so \( \gamma_2(G) = 4 \). Finally, \( |V_1| = 2 \) and \( |V_2| = 1 \). Suppose that \( V_1 = \{u, w\} \) and \( V_2 = \{v\} \). Obviously, each vertex in \( (V(G) - \{u, w\}) - N[v] \) must be connected to both \( u \) and \( w \). Hence, \( \gamma_2(G[V(G) - N[v]]) = 2 \). Conversely, the result is obvious if we have (i) or (ii). Now, suppose that \( G \) satisfies (iii). Since \( \Delta(G) \leq n-3 \) and \( \gamma_2(G) \geq 4 \), by Propositions 2.1 and 2.2, \( \gamma_{(R2)}(G) \geq 4 \). On the other hand, assume that \( \{u, w\} \) is a 2-dominating set for \( G[V(G) - N[v]] \). If we assign a 2 to \( v \) and a 1 to \( u \) and \( w \), we can show that \( \gamma_{(R2)}(G) \leq 4 \). Thus, \( \gamma_{(R2)}(G) = 4 \). \( \Box \)

**Proposition 2.4.** Let \( G \) be a connected graph with order \( n \). The following conditions are true,

(a) \( \gamma_{(R2)}(G) = n \) if and only if \( G = K_n \) for \( n = 1, 2 \).
(b) \( \gamma_{(R2)}(G) = n - 1 \) if and only if \( G \) is a \( C_3, P_3 \) or \( P_4 \).

**Proof.** For (a) it is clear that \( \Delta(G) \leq 1 \). For (b), if \( G \) is one of the \( C_3, P_3 \) or \( P_4 \), then the claim is true. Conversely, assume that \( \gamma_{(R2)}(G) = n - 1 \). Obviously \( \Delta(G) = 2 \). Among all \( \gamma_{(R2)}(G) \)-functions, let \( f = (V_0, V_1, V_2) \) be one with \( |V_2| \) as small as possible. It is easy to see that \( V_2 = \emptyset \) and \( |V_0| = 1 \). Suppose that \( v \in V_0 \) for some vertex \( v \in V(G) \), so \( \deg(v) = 2 \). Also, each vertex except \( v \) can be adjacent to at most one vertex in \( V_1 \). Hence, the vertices which have the degree 2 are at most \( v \) and \( N(v) \). Therefore, we have just three graphs, \( C_3, P_3 \) or \( P_4 \). \( \Box \)

Now, we need the following graphs in Proposition 2.5. \( \hat{E}_6 \) is a tree obtained from \( K_{1,3} \) by subdividing each edge exactly once. \( D_7 \) is also a tree obtained from \( K_{1,3} \) by subdividing one edge three times, (see [1]). We define the graph \( H_2 \) such that it is a graph with a 4-cycle and a path of order 2 joined to one of the vertices of the 4-cycle.

**Proposition 2.5.** Let \( G \) be a connected graph with order \( n \). Then \( \gamma_{(R2)}(G) = n - 2 \) if and only if \( G \) is one of the figures listed in Figure 7.

**Proof.** Suppose that \( \gamma_{(R2)}(G) = n - 2 \), then the following conditions hold,

(i) \( \Delta(G) \leq 3 \),
(ii) each non-adjacent pair of vertices with degree 3 has exactly two common neighbours,
(iii) \( G \) does not have one of the graphs \( P_7, C_6, \hat{E}_6, D_7 \), and \( H_2 \) as subgraph.

If there exists a vertex \( v \in V(G) \) with degree at least 4, then \( \gamma_{(R2)}(G) \leq n - 3 \). Also, if there exists a pair of nonadjacent vertices with degree 3 having zero, one or three common neighbours, then we obtain \( \gamma_{(R2)}(G) \leq n - 3 \). Moreover, Roman \( \{2\} \)-domination number of each of graphs \( P_7, C_6, \hat{E}_6, D_7 \), and \( H_2 \) is \( n - 3 \). Thus, they
cannot be as a subgraph of $G$. It is not hard to see that all graphs which have the above three properties are listed in Figure 1. Conversely, it is easy to verify that for all graphs $G$ listed in Figure 1, we have $\gamma_{\{R2\}}(G) = n - 2$. □

3. Graph products

In this section we study Roman $\{2\}$-domination on some graph products. Also, in the following theorems we classify Roman $\{2\}$-domination for join of two graphs.

**Theorem 3.1.** Let $G$ and $H$ be two graphs. Then $\gamma_{\{R2\}}(G \vee H) \leq 4$. Moreover, if $k = \gamma_{\{R2\}}(G) \leq \gamma_{\{R2\}}(H)$, then we have
(a) $k \leq 2$ if and only if $\gamma_{\{R_2\}}(G \vee H) = 2$,
(b) $k = 3$ or $k = 4$ and $\gamma(G) = 2$ if and only if $\gamma_{\{R_2\}}(G \vee H) = 3$.

Proof. The first assertion is obvious because for each graph $G$, $\gamma_{\{R_2\}}(G) \leq 2\gamma(G)$. For (a), assume that $k = 1$, then $G = K_1$. It is sufficient to use Proposition 2.1. Now, suppose $k = 2$. By Proposition 2.2 we have $G \vee H = \overline{K_n} \vee F$ for $n = 1, 2$ and for some graph $F$. Conversely, let $\gamma_{\{R_2\}}(G \vee H) = 2$. By Proposition 2.1 there exists a graph $L$ such that $G \vee H = \overline{K_n} \vee L$ for $n = 1, 2$. It is not hard to see that the vertices of $\overline{K_n}$ for $n = 1, 2$ together belong to $G$ or $H$. Anyway, $\gamma_{\{R_2\}}(G) \leq 2$.

For (b), if $k = 3$, then $\gamma_{\{R_2\}}(G \vee H) \leq 3$. By (a), $\gamma_{\{R_2\}}(G \vee H) \geq 3$. For the second claim, let $\{u, v\} \subseteq V(G)$ be a minimum dominating set for $G$ and $w$ be an arbitrary vertex in $V(H)$. It is seen that $\{u, v, w\}$ is a $1$-dominating set for $G \vee H$. Using Proposition 2.2 we have $\gamma_{\{R_2\}}(G \vee H) = 3$. Conversely, let $\gamma_{\{R_2\}}(G \vee H) = 3$. By (a), $k \geq 3$. First assume that $\gamma_2(G \vee H) = 3$. Let $\{u, v, w\} \subseteq V(G \vee H)$ be a $2$-dominating set on $G \vee H$. Without loss of generality, we consider two subcases,

(i) If $\{u, v, w\} \subseteq V(G)$, then by (a), $\gamma_{\{R_2\}}(G) = 3$.
(ii) If $\{u, v\} \subseteq V(G)$ and $w \in V(H)$, then $\gamma(G) = 2$. So by (a), $3 \leq \gamma_{\{R_2\}}(G) \leq 4$.

For the next case, there exist two vertices $u, v \in V(G \vee H)$ with label $1$ and $2$, respectively. It is not hard to see that $u, v \in V(G)$ or $u, v \in V(H)$. Therefore, $k = 3$.

In the following theorem we obtain Roman $\{2\}$-domination number for the Corona product of two graphs.

Theorem 3.2. Let $G$ and $H$ be two graphs such that the order of $G$ is $n$. If $H = K_1$, then $\gamma_{\{R_2\}}(G[H]) = n + \gamma(G)$, otherwise $\gamma_{\{R_2\}}(G[H]) = 2n$.

Proof. Let $H = K_1$. Easily we can show that for every graph $G$, $\gamma_{\{R_2\}}(G[K_1]) \leq n + \gamma(G)$. On the other hand, assume that $f = (V_0, V_1, V_2)$ is a $\gamma_{\{R_2\}}(G[K_1])$-function. Without loss of generality, suppose that $nK_1 \subseteq V_0 \cup V_1$. Also, let $\ell K_1 \subseteq V_1$ and $(n - \ell)K_1 \subseteq V_0$. Thus, $V_2 \subseteq V(G)$. Moreover, $(V_1 \cap V(G)) \cup V_2$ forms a dominating set for $G$.

$$wt(f) = \ell + |V_1 \cap V(G)| + 2|V_2|$$
$$\geq \ell + \gamma(G) + |V_2|$$
$$\geq n + \gamma(G).$$

For the second assertion, let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $f$ be a $\gamma_{\{R_2\}}(G[H])$-function. Then,

$$wt(f) = wt(f|_{H_1}) + wt(f|_{H_2}) + \ldots + wt(f|_{H_n}) \geq 2n,$$
where $H_i = v_i \vee H$ for $i = 1, 2, \ldots, n$. So, $\gamma_{\{R_2\}}(G[H]) = 2n$.

Moreover, we state a bound and some results about Cartesian product of graphs. Let $G$ and $H$ be two graphs with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{u_1, u_2, \ldots, u_m\}$. 

In $G \Box H$, we define $G^i$ and $H^j$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, as $i$th layer and $j$th layer of $G$ and $H$, respectively as follows,

$$G^i = \{(v, u_i) : v \in V(G)\}, \quad H^j = \{(v_j, u) : u \in V(H)\}.$$ 

**Theorem 3.3.** $\gamma_{\{R2\}}(G \Box H) \leq \min\{\gamma_{\{R2\}}(G)|V(H)|, \gamma_{\{R2\}}(H)|V(G)|\}$. Also, this bound is sharp.

**Proof.** Let $f$ be a $\gamma_{\{R2\}}$-function for $H$. Consider each copy of $H$ with $\gamma_{\{R2\}}$-function $f$ in cartesian product $G \Box H$. Since we have $|V(G)|$ copies of $H$, it is easy to see that $\gamma_{\{R2\}}(G \Box H) \leq \gamma_{\{R2\}}(H)|V(G)|$. By a similar way, we have $\gamma_{\{R2\}}(G \Box H) \leq \gamma_{\{R2\}}(G)|V(H)|$. In order to prove this bound is sharp, consider $\gamma_{\{R2\}}(K_1,n \Box P_2) = 2\gamma_{\{R2\}}(K_1,n) = 4$, for $n \geq 3$, see Proposition [2.3].

**Theorem 3.4.** Let $m$ and $n$ be two positive integers with $n \leq m$. Then

$$\gamma_{\{R2\}}(K_n \Box K_m) = \min\{m, 2n\}.$$ 

**Proof.** Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and $V(K_m) = \{u_1, u_2, \ldots, u_m\}$. Suppose that $\gamma_{\{R2\}}(K_n \Box K_m) < \min\{m, 2n\}$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_{\{R2\}}(K_n \Box K_m)$-function. Thus, we can say that there exists the layer $K_n^i$ for some $1 \leq i \leq m$, such that $wt(f|_{K_n^i}) = 0$. On the other hand, we can find a layer $K_m^j$ for some $1 \leq j \leq n$, with $wt(f|_{K_m^j}) \leq 1$. It is easy to see that $(v_i, u_j) \in V_0$ and $f(N(v_i, u_j)) \leq 1$. Therefore, we achieve a contradic. Now to get the equality, consider a Roman $\{2\}$-dominating function on $K_n \Box K_m$ that assigns to $(v_i, u_j)$ and $(v_1, u_j)$ a 1 for every $i$ and for every $j$ belonging to $\{n + 1, \ldots, m\}$, and a 0 to the remaining vertices of the graph.

We know that $\gamma_{\{R2\}}(G_{m,n}) \leq \gamma_2(G_{m,n})$ for all positive integers $m$ and $n$. Moreover, this bound is sharp for $G_{2,n}$ for each $n$ and $G_{3,n}$ for $n \leq 13$ as well as $G_{4,4}$. We recall the following results of [1].

**Theorem 3.5.** Let $n$ be a positive integer. Then the following equalities hold:

(i) $\gamma_2(G_{2,n}) = n$,

(ii) $\gamma_2(G_{3,n}) = \left\lfloor \frac{2n}{3} \right\rfloor$,

(iii) $\gamma_2(G_{4,n}) = \left\lfloor \frac{7n+3}{4} \right\rfloor$, for $n \geq 3$.

**Proposition 3.6.** $\gamma_{\{R2\}}(G_{2,n}) = n$.

**Proof.** We claim that the weight of each layer of $P_2$ is at least 1. Assume that there exists a layer with weight 0. To have a Roman $\{2\}$-dominating set for $G_{2,n}$, the weight of the adjacent layers will be 4. The obtained Roman $\{2\}$-domination number is not optimal because its weight is larger than $\gamma_2(G_{2,n})$.

**Proposition 3.7.**

(a) For $n = 2, 3, 6$, $\gamma_{\{R2\}}(G_{3,n}) \leq \left\lfloor \frac{5n+3}{4} \right\rfloor$. Otherwise, $\gamma_{\{R2\}}(G_{3,n}) \leq \left\lfloor \frac{5n+3}{4} \right\rfloor$.

(b) For $n = 2, 3, 5, 6, 9$, $\gamma_{\{R2\}}(G_{4,n}) \leq \left\lfloor \frac{5n+4}{3} \right\rfloor$. Otherwise, $\gamma_{\{R2\}}(G_{4,n}) \leq \left\lfloor \frac{5n+4}{3} \right\rfloor$. 

Proof. Suppose that \( v_{ij} \) is the vertex in the row \( i \) and column \( j \) for \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \) in \( G_{m,n} \). In each part we give a complete explanation about a basic case of the product and then we can obtain upper cases using it. For (a), let \( n = 4k - 1 \) for some positive integer \( k \geq 2 \). We define a Roman \( \{2\} \)-dominating function \( f = (V_0, V_1, V_2) \) such that \( v_{ij} \in V_2 \) for \( j = 4t \) for some positive integer \( 1 \leq t \leq k - 1 \), such that \( i = 1 \) if \( t \) is odd, otherwise \( i = 3 \). Also, \( V_0 = \{ v_{ij} : d(v_{ij}, v) = 1, 2, 4, \text{ for some } v \in V_2, \text{ and } 1 \leq i \leq 3, 1 \leq j \leq n \} \), where \( d(v_{ij}, v) \) is the length of shortest path between two vertices \( v_{ij} \) and \( v \). The label of other vertices is 1. Hence, \( wt(f) = 5k \). For \( n \neq 4k - 1 \) we obtain the result by adding at most 3 columns to the case \( n = 4k - 1 \). For (b), in figures A, B and C in Figure 2, a star, a black circle and a white circle denote a vertex with label 2, 1 and 0, respectively. We want to construct \( G_{4,n} \) for \( n \geq 7 \) by merging a number of figures A, B and C. Suppose that \( n(A), n(B) \) and \( n(C) \) are the number of used A, B and C in \( G_{4,n} \), respectively. Consider \( n = 3k + i \) for some positive integers \( k \) and \( i \) such that \( 1 \leq i \leq 3 \). For \( G_{4,n} \) assign \( n(A) = k - i, n(C) = i - 1 \) and \( n(B) = 1 \) except for \( n = 9, n(B) = 0 \). \( \square \)

**Figure 2.** A, B and C, respectively

**ACKNOWLEDGMENT**

We would like to thank Mustapha Chellali for his useful comments.

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