Coupling continuous neural networks to the electromagnetic field in nervous tissue

Peter beim Graben\textsuperscript{1,2,*} and Serafim Rodrigues\textsuperscript{3}

\textsuperscript{1}Department of German Language and Linguistics, Humboldt-Universität zu Berlin, Germany
\textsuperscript{2}Bernstein Center for Computational Neuroscience, Berlin, Germany
\textsuperscript{3}Department of Engineering Mathematics, University of Bristol, United Kingdom.

(Dated: October 25, 2010)

Abstract

We present a microscopic approach for the coupling of cortical activity, as resulting from proper dipole currents of pyramidal neurons, to the electromagnetic field in extracellular fluid in presence of diffusion and Ohmic conduction. As a result, neural activity becomes represented by a continuous neural field equation, while an observation model for electric field potentials is obtained from the interaction of cortical dipole currents with charge density in non-resistive extracellular space as described by the Nernst-Planck equation.

PACS numbers: 87.19.ld, 87.19.le, 87.19.lj, 87.50.-a, 87.50.C, 87.50.cf, 82.45.Gj

Keywords: biophysics, neural field theory, integro-differential equations, current dipoles, extracellular medium, diffusion

\*Electronic address: peter.beim.graben@hu-berlin.de
One of the most important problems in the biophysics of neural systems is understanding the coupling of complex neural network dynamics to the electromagnetic field, that is macroscopically measurable (as a feedforward effect) as neural mass potentials, such as local field potential (LFP) or electroencephalogram (EEG). Feedback effects, on the other hand, have recently been demonstrated via experiments \[1\]. Thus a theoretical framework for describing that coupling — outlined in this RC — is mandatory in clinical, computational and cognitive neurosciences, e.g. for treatment of epilepsy \[2, 3\] or modeling cognition-related brain potentials \[4\].

The generators of neural mass potentials are cortical pyramidal neurons (sketched in Fig. 1). They exhibit a long dendritic trunk separating mainly excitatory synapses at the apical dendritic tree from mainly inhibitory synapses at the basal dendritic tree. When both kinds of synapses are simultaneously active, inhibitory synapses generate current sources and excitatory synapses current sinks in extracellular space, causing the pyramidal cell to behave as a microscopic dipole surrounded by its characteristic electrical field, the dendritic field potential (DFP). The densely packed and parallel aligned pyramidal cells form a dipole layer whose superimposed currents give rise to LFP and EEG \[5, 6\].

Neural mass potentials are most realistically simulated by means of compartmental models \[7\]. However, because compartmental models are computationally extremely expansive, large-scale neural networks preferentially employ point models, based either on conductance \[8\] or population models \[9\], where mass potentials are estimated either through sums of postsynaptic potentials or postsynaptic currents.

Yet another difficulty is the coupling of the activity of discrete neural networks to the continuous electromagnetic field since neural network topology is not embedded into physical space as an underlying metric manifold. This problem could be avoided by continuous neural networks investigated in neural field theory (NFT) \[3, 10–12\], and in fact, \[12\] gave first accounts for such couplings in NFT population models.

However, both approaches \[12\] are not concerned with microscopic pyramidal dipole currents. Furthermore, extracellular space was assumed to be purely resistive. On the other hand, recent research has revealed that diffusion currents, represented by their corresponding Warburg impedances \[13\], cannot be neglected in extracellular space as they substantially contribute to the characteristic power spectra of neural mass potentials \[14, 15\].

In this RC, we propose a theoretical framework for the microscopic coupling of continuous...
neural networks (i.e. neural fields) to the electromagnetic field, properly described by dipole currents of cortical pyramidal neurons and diffusion effects in extracellular space. As a starting point we use a reduced compartment model for a single pyramidal cell and derive the evolution law for the activity of a neural network. Additionally, we obtain an expression for the dipole current as an observation model. Performing the continuum limit for the network yields a neural field equation coupled to the Maxwell equations in extracellular fluid.

We consider $N$ populations of neurons, arranged in layers $\Gamma_i$ ($i = 1, \ldots, N$). Neurons in layers 1 to $M$ should be excitatory, neurons in layers $M + 1$ to $N$ should be inhibitory and layer one contains the cortical pyramidal cells. For a continuous neural network, this arrangement is described by the Amari equation \[10\].
\[
\tau_i \partial_t u_i(x, t) + u_i(x, t) = \sum_{k=1}^{N} \int_{\Gamma_k} dx' \int_{-\infty}^{t} dt' w_{ik}(x, x') s_i(t - t') f_\theta(u_k(x', t')) + h_i(x, t),
\]

where \(\tau_i\) is the characteristic time constant of population \(i\), \(u_i(x, t)\) is the neural field activity in layer \(i\) at time \(t\), \(w_{ik}(x, x')\) is the synaptic weight kernel between sites \(x \in \Gamma_i, x' \in \Gamma_k\), \(s_i(t)\) is the postsynaptic impulse response function for layer \(i\), and \(h_i(x, t)\) is external input delivered to the neuron at \(x\) in layer \(\Gamma_i\). Wave-to-spike conversion is described by the sigmoidal activation function \(f_\theta(u) = 1/[1 + e^{-\frac{u}{\theta}}]\) with activation threshold \(\theta > 0\).

Herein, we derive an analogous equation for the neural field coupled to the electromagnetic field in extracellular space, where the input to layer one turns out as diffusion current \(h_1 = -\kappa A D \rho \partial_x \rho\); with \(\kappa\) as some coupling constant, \(A\) the current cross section, \(D\) Einstein’s diffusion constant [16], and \(\rho\) the extracellular charge density. This neural field equation will be complemented by an observation model for the extracellular dipole current and the resulting DFP.

We describe the \(i\)th cortical pyramidal neuron [Fig. 1] via an electronic equivalent three-compartmental model Fig. 2 [17], which is parsimonious to derive a scalar field: one compartment for the apical dendritic tree, another one for the basal dendritic tree, and the third for the axon hillock where membrane potential is converted into spike trains.

Excitatory synapses are represented by the left-most branch, where excitatory postsynaptic potentials (EPSP) at a synapse between a neuron \(j\) from layers 1 to \(M\) and neuron \(i\) act as electromotoric forces \(E_{ij}\). These potentials drive excitatory postsynaptic currents (EPSC) \(I_{ij}\), essentially consisting of sodium ions, through the cell plasma with resistance \(R_{ij}\) from the synapse towards the axon hillock.

The middle branch describes the inhibitory synapses between a neuron \(k\) from layers \(M + 1\) to \(N\) and neuron \(i\). Here, inhibitory postsynaptic potentials (IPSP) \(E_{ik}\) provide a shortcut between the excitatory branch and the trigger zone, where inhibitory postsynaptic currents (IPSC) \(I_{ik}\) (essentially chloride ions) close the loop between the apical and basal dendritic trees. The resistivity of the current paths along the cell plasma is given by \(R_{ik}\).

The cell membrane at the axon hillock itself is represented by the branch at the right hand side. Here, a capacitor \(C_i\) reflects the temporary storage capacity of the membrane. The serial circuit consisting of a battery \(E^M\) and a resistor \(R^M\) denotes the Nernst resting potential and the leakage conductance of the membrane, respectively [18]. Finally, a gener-
ator of Hodgkin-Huxley spikes [8] (indicated by a “black box”) is regarded of having infinite input impedance. Both, EPSP and IPSP result from the interaction of postsynaptic receptor kinetics with membrane capacitance of compartments one and two, respectively [19]. Hence these capacitances, omitted in Fig. 2, are already taken into account by $E_{ij}^E, E_{ik}^I$ (to what we refer as “reduced compartmental model” here).

The three compartments are coupled through longitudinal resistors, $R_i^A, R_i^B, R_i^C, R_i^D$ where $R_i^A, R_i^B$ denote the resistivity of the cell plasma [20] and $R_i^C, R_i^D$ that of extracellular space. Yet, in extracellular space not only Ohmic but also diffusion currents are present [14, 15]. These are considered by the current source $J_i^D$ connected in parallel to $R_i^D$. On the other hand, diffusion currents along the somatic resistor $R_i^C$ are disregarded in adiabatic approximation.

Finally, the membrane voltage at the axon hillock $U_i$ as the dynamical variable and the DFP $V_i$ are indicated. The latter drops along the extracellular resistor $R_i^D$. For the aim of calculation, the mesh currents $I_i^D$ (the dendritic current), $I_i^B$ (the basal current) and $I_i^{HH}$ (the Hodgkin-Huxley current) are indicated.

FIG. 2: Electronic equivalent circuit for a pyramidal neuron.
The circuit in Fig. 2 obeys the following equations:

\[ I_D^i = \sum_{j=1}^{p} I_{ij}^E \]  
(2)

\[ I_B^i = \sum_{k=1}^{q} I_{ik}^I \]  
(3)

\[ I_{HH}^i = I_D^i - I_B^i \]  
(4)

\[ I_{HH}^i = C_i \dot{U}_i + (U_i - E^M)/R^M \]  
(5)

\[ E_{ij}^E = R_{ij}^E I_{ij}^E + R_A^i I_B^i + (R_B^i + R_C^i) I_{HH}^i + U_i + \]
\[ + R_D^i (I_D^i - J_D^i), 1 \leq j \leq p \]  
(6)

\[ E_{ik}^I = R_{ik}^I I_{ik}^I + (R_B^i + R_C^i) I_{HH}^i + U_i, 1 \leq k \leq q \]  
(7)

Here, \( p \) is the number of excitatory and \( q \) is the number of inhibitory synapses connected to neuron \( i \).

The circuit described by Eqs. (2 – 7) shows that the neuron \( i \) is likely to fire when the excitatory synapses are activated. Then, the Hodgkin-Huxley current \( I_{HH}^i \) equals the dendritic current \( I_D^i \). If, by contrast, also the inhibitory synapses are active, the dendritic current \( I_D^i \) follows the shortcut between the apical and basal dendritic trees and only a portion could evoke spikes at the trigger zone [Eq. (5)]. On the other hand, the large dendritic current \( I_D^i \), diminished by some diffusion current \( J_D^i \), flowing through the extracellular space of resistance \( R_D^i \), gives rise to a large DFP \( V_i \).

In order to simplify the following derivations, we gauge the resting potential to \( E^M = 0 \), yielding

\[ I_{HH}^i = C_i \dot{U}_i + U_i/R^M. \]  
(8)

From Eqs. (2), (3) and (8) we algebraically derive an equation for the extracellular current

\[ I_D^i = \sum_{j=1}^{p} \alpha_{ij} E_{ij}^E - \beta_i \dot{U}_i - \gamma_i U_i + \delta_i J_D^i, \]  
(9)

with the following electrotonic parameters \( \alpha_{ij} = \{R_{ij}^E[1 + g_i^E(R_A^i + R_D^i)]\}^{-1} \), \( \beta_i = C_i g_i^E(R_B^i + R_C^i/[1 + g_i^E(R_A^i + R_D^i)]) \), \( \gamma_i = g_i^E(R_M^i + R_B^i + R_C^i/[R^M[1 + g_i^E(R_A^i + R_D^i)]]) \), \( \delta_i = g_i^E R_D^i/[1 + g_i^E(R_A^i + R_D^i)] \), and \( g_i^E = \sum_{j=1}^{p} 1/R_{ij}^E \).

Correspondingly, Eqs. (3), (4), (7 – 9) lead to the dynamical law for the membrane
potential at axon hillock

$$\tau_i \dot{U}_i + U_i = \frac{1}{p} \sum_{j=1}^{p} w_{ij}^E E_{ij} - \frac{1}{q} \sum_{k=1}^{q} w_{ik}^I E_{ik} + \kappa_i J_i^D ,$$

(10)

where we have introduced the following parameters: \textit{time constants:} \(\tau_i = r_i C_i [1 - g_i^E (R_i^B + R_i^C) + \beta_i]\), \textit{excitatory synaptic weights:} \(w_{ij}^E = r_i \alpha_{ij}\), \textit{inhibitory synaptic weights:} \(w_{ik}^I = r_i / R_{ik}^I\), \textit{dendritic diffusion resistance:} \(\kappa_i = r_i \delta_i\), with \(r_i = R_i^M / [1 - g_i^L (R_i^B + R_i^C + R_i^M) + \gamma_i R_i^M] \) and \(g_i^L = \sum_{k=1}^{q} 1/R_{ik}^L\).

By means of (10) we eliminate the temporal derivative in (9), yielding

$$I_i^D = \frac{1}{p} \sum_{j=1}^{p} \tilde{w}_{ij}^E E_{ij} + \frac{1}{q} \sum_{k=1}^{q} \tilde{w}_{ik}^I E_{ik} + \xi_i U_i + \eta_i J_i^D ,$$

(11)

with parameters \(\tilde{w}_{ij}^E = \alpha_{ij} - \beta_i / \tau_i w_{ij}^E, \tilde{w}_{ik}^I = \beta_i / \tau_i w_{ik}^I, \xi_i = \beta_i / \tau_i - \gamma_i, \) and \(\eta_i = \delta_i - \beta_i / \tau_i \kappa_i\).

The change in sign of the inhibitory contribution from Eq. (10) to Eq. (11) has an obvious physical interpretation: In (10), the change of membrane potential \(U_i\) and therefore the spike rate is enhanced by EPSPs but diminished by IPSPs. On the other hand, the dendritic shortcut current \(I_i^D\) in (11) is large for both, large EPSPs and large IPSPs.

From Eq. (10) we eventually get the neural network’s dynamics by taking into account that postsynaptic potentials are obtained from presynaptic spike rates through temporal convolution with postsynaptic impulse response functions, i.e.

$$E_{ij}^{EI}(t) = \int_{-\infty}^{t} s_{i}^{EI}(t - t') R_j(t') dt'$$

(12)

where \(s_i^{EI}(t)\) are excitatory and inhibitory synaptic impulse response functions, respectively, and \(R_j\) is the spike rate

$$R_j(t) = f_\theta(U_j(t)) .$$

(13)

Inserting (12) and (13) into (10) and performing a continuum limit \(U_i(t) \rightarrow u_i(x,t)\) where \(x \in \Gamma_i\) replaces the neuron index \(i\), that thereby becomes a population index, entails the Amari equation (1), with \(h_1 = \kappa A j^D\) and \(j^D = J^D/A\) as diffusion current density through cross section \(A\) [Fig. 1]. The synaptic weight kernels \(w_{ik}(x,x')\) are obtained from the excitatory and inhibitory synaptic weights by absorbing their signs after rearrangement to the Amari layers.

In a linear, isotropic medium, currents and gradients are all aligned parallel to the main dendritic trunk, indicated by the \(x\)-axis in Fig. 1. Therefore, a one-dimensional analysis of
neural electrodynamics is sufficient. As further simplification we consider currents of only one ionic species (sodium) here, by treating IPSC’s as “sodium hole” currents in analogy to semiconductor physics. The total current through extracellular fluid is given by the Nernst-Planck equation [14, 17, 18]

\[ j = -D_E \partial_x \rho + \sigma E, \quad (14) \]

with Einstein’s diffusion constant \( D_E \) [16], charge density \( \rho \), conductivity \( \sigma \) and electric field \( E \); the diffusion current in (14), \( j^D = -D_E \partial_x \rho \), replaces the input to the first layer of the Amari equation (1):

\[ \tau_i \partial_t u_i(x, t) + u_i(x, t) = \sum_{k=1}^{N} \int_{\Gamma_k} dx' \int_{-\infty}^{t} dt' w_{ik}(x, x') s_i(t - t') f_{\theta}(u_k(x', t')) - \delta_{1,i} \kappa A D_E \partial_x \rho, \quad (15) \]

where \( \delta_{1,i} = 1(0) \) if \( i = 1(i \neq 1) \) is the Kronecker symbol. Correspondingly, the continuum limit of Eq. (11) becomes

\[ j(x, t) = \sum_{k=1}^{N} \int_{\Gamma_k} dx' \int_{-\infty}^{t} dt' \bar{w}_{ik}(x, x') s_1(t - t') f_{\theta}(u_k(x', t')) + \xi u_1(x, t) - \eta A D_E \partial_x \rho. \quad (16) \]

Moreover, the electric field \( E \) is given by the gradient of the DFP \( V \), \( E = -\partial_x V \) and the conductivity of the extracellular electrolyte relates to its mobility \( \mu \) by \( \sigma = \mu \rho \) [18]. In addition to the Nernst-Planck equation (14), we have the first Maxwell equation \( \partial_x(\varepsilon E) = \rho \), with permittivity \( \varepsilon \), and the continuity equation \( \partial_x j + \partial_t \rho = 0 \) reflecting the conservation of charge as a result of the Maxwell equations. Computing the divergence of the Nernst-Planck equation (14) by taking the continuity equation into account, yields, after eliminating \( E \) by means of (14), \( \partial_x \rho \) by means of the first Maxwell equation and \( \sigma \),

\[ \partial_t \rho = D_E \partial_x^2 \rho - j \partial_x (\ln \rho) - D_E \left( \frac{(\partial_x \rho)^2}{\rho} - \frac{\mu}{\varepsilon} \rho^2 \right). \quad (17) \]

Finally, the DFP is obtained as a solution of (14),

\[ \partial_x V = -\frac{1}{\mu \rho} \left( j + D_E \partial_x \rho \right). \quad (18) \]

Altogether, Eqs. (15) – (18) describe the coupling of a layered neural field to the electromagnetic field in extracellular space, where the pyramidal layer described by the Amari
equation (15) is driven by diffusion currents. Moreover, Eqs. (16) – (18) constitute an observation model for currents, charge distribution and DFP in extracellular space. The system (15) – (18) of nonlinear coupled partial integro-differential equations has to be solved in a self-consistent manner, e.g. by means of mean-field techniques. We leave this analysis for subsequent research.

Our model of neural fields coupled to the electromagnetic field can be straightforwardly generalized in several directions: (1) Introducing anisotropic media described by conductivity and permittivity tensors, respectively, requires full-fledged 3-dimensional calculus. (2) LFP and EEG could be obtained as mean-fields from appropriate spatial coarse grainings. (3) Neural field effects such as ephaptic interactions [1, 3, 20, 21] could be phenomenologically modeled by voltage-modulation of activation thresholds \( \theta \) in the activation function of (15) [17].

Since (17) is a generalized diffusion equation, the present approach is consistent with the Warburg impedance approach to neural mass potentials [14, 15]. We expect that our theory for the coupling of continuous neural networks to the electromagnetic field in non-resistive nervous tissue is significantly important for applications in the clinical and cognitive neurosciences, such as research on epilepsy [2, 3] or cognition-related brain potentials [4].

We thank Michelle Lilith, Claude Bédard, Alain Destexhe, and Jürgen Kurths for fruitful discussion. This research was supported by a DFG Heisenberg grant awarded to PbG (GR 3711/1-1).

[1] F. Fröhlich and D. A. McCormick, Neuron 67, 129 (2010).
[2] B. J. Gluckman et al., J. Neurosci. 21, 590 (2001).
[3] K. A. Richardson et al., Phys. Rev. Lett. 94, 028103 (2005).
[4] P. Hagoort, NeuroImage 20, S18 (2003); P. beim Graben et al., Cogn. Neurodyn. 2, 229 (2008).
[5] C. Bédard et al., Biophys. J. 86, 1829 (2004).
[6] P. L. Nunez and R. Srinivasan, Electric Fields of the Brain: The Neurophysics of EEG (Oxford University Press, New York, 2006).
[7] A. Protopapas et al., in [22], pp. 461 – 498; A. R. Sargsyan et al., J. Neurosci. Meth. 104,
143 (2001).

[8] A. L. Hodgkin and A. F. Huxley, J. Physiol. 117, 500 (1952); A. Mazzoni et al., PLoS Comput. Biol. 4, e1000239 (2009).

[9] O. David and K. J. Friston, NeuroImage 20, 1743 (2003); W. J. Freeman, Biol. Cybern. 56, 139 (1987); P. beim Graben and J. Kurths, Neurocomputing 71, 999 (2008); B. H. Jansen and V. G. Rit, Biol. Cybern. 73, 357 (1995); F. H. Lopes da Silva et al., Kybernetik 15, 27 (1974); F. Wendling et al., Biol. Cybern. 83, 367 (2000); H. R. Wilson and J. D. Cowan, Biophys. J. 12, 1 (1972).

[10] S.-I. Amari, Biol. Cybern. 27, 77 (1977).

[11] M. Breakspear et al., Cereb. Cortex 16, 1296 (2006); S. Coombes et al., Physica D 178, 219 (2003); V. K. Jirsa and H. Haken, Phys. Rev. Lett. 77, 960 (1996); R. Potthast and P. beim Graben, Math. Meth. Appl. Sci. 33, 935 (2009); P. A. Robinson et al. Phys. Rev. E 56, 826 (1997); H. R. Wilson and J. D. Cowan, Kybernetik 13, 55 (1973).

[12] V. K. Jirsa, K. J. Jantzen, A. Fuchs, and J. A. S. Kelso, IEEE Transactions on Medical Imaging 21, 493 (2002); D. T. J. Liley, P. J. Cadusch, and M. P. Dafilis, Network: Computation in Neural Systems 13, 67 (2002).

[13] S. Skale et al., Corrosion Sci. 49, 1045 (2007).

[14] C. Bédard and A. Destexhe, Biophys. J. 96, 2589 (2009).

[15] C. Bédard et al., J. Comput. Neurosci. (2010); S. Gabriel et al. Phys. Med. Biol. 41 (1996); D. Nima et al., J. Comput. Neurosci. (2010).

[16] A. Einstein, Annal. Physik 19, 289 (1906).

[17] P. beim Graben, in Lectures in Supercomputational Neuroscience, edited by P. b. Graben et al., (Springer, Berlin, 2008), pp. 3 – 48.

[18] D. Johnston and S. M.-S. Wu, Foundations of Cellular Neurophysiology (MIT Press, Cambridge (MA), 1997).

[19] A. Destexhe et al., in [22], pp. 1 – 25.

[20] G. R. Holt and C. Koch, J. Comput. Neurosci. 6, 169 (1999).

[21] J. G. R. Jeffreys, Physiol. Rev. 75, 689 – 723 (1995).

[22] C. Koch and I. Segev, eds., Methods in Neuronal Modelling., (MIT Press, Cambridge (MA), 1998).