Displacement prediction and optimization of a non-circular planetary gear hydraulic motor

Biao Zhang, Shikuan Song, Chenghu Jing and Dong Xiang

Abstract
The non-circular planetary gear hydraulic motor is a low-speed and high-torque hydraulic motor with excellent performance. It has the characteristics of a wide speed range, low weight and is widely used in various fields. Aiming to solve the problem of there being no intuitive formula for calculating the displacement of the non-circular planetary gear hydraulic motor at present, based on the analysis of the effects of structural parameters on the displacement of the motor, this paper proposes a formula for calculating the displacement of a non-circular planetary gear hydraulic motor when the pitch curve of the sun wheel is a high-order ellipse. The formula allows the direct calculation and prediction of the displacement of the motor. To improve the unit volume displacement of the hydraulic motor (which determines the power density of the motor), based on the analysis of the unit volume displacement constraints, an optimization equation is proposed by adding an optimization factor to the original equation of the pitch curve of the sun wheel. It is seen that the addition of the new optimization factor eliminates the self-interlacing of the pitch curve of inner ring gear. This elimination increases the unit volume displacement of the motor.

Keywords
Non-circular planetary gear hydraulic motor, displacement calculation formula, power density, unit volume displacement optimization, non-circular gear pitch curve

Introduction
The hydraulic motor is a hydraulic actuator commonly used in engineering. Specifically, it is widely used for metal cutting machines, load simulators, manipulator arms, vehicles, and motion simulators owing to its characteristics of easy operation and control, a large power-to-weight ratio, reliable operation and large transfer load. Owing to their compact structure and high transmission torque, non-circular planetary gears are widely used in a variety of equipment, such as the wheel stacker and retractor, rice transplanting mechanism, gravity balancing instrument and differential vane pump. The non-circular planetary gear hydraulic motor (also known as a satellite hydraulic motor) is a new type of low-speed and high-torque hydraulic motor, which combines a non-circular planetary gear train and hydraulic motor technology. It has the obvious advantages of a high anti-pollution capacity, a large displacement, a simple structure and low weight.

In 1974, Sieniawski applied the non-circular planetary gear mechanism with a variable center distance to the hydraulic motor and developed the non-circular planetary gear hydraulic motor. Scholars have

School of Mechatronics Engineering, Harbin Institute of Technology, Harbin, China

Corresponding author:
Biao Zhang, School of Mechatronics Engineering, Harbin Institute of Technology, Harbin 150001, China.
Email: zhangbiao@hit.edu.cn

Creative Commons CC BY: This article is distributed under the terms of the Creative Commons Attribution 4.0 License (https://creativecommons.org/licenses/by/4.0/) which permits any use, reproduction and distribution of the work without further permission provided the original work is attributed as specified on the SAGE and Open Access pages (https://us.sagepub.com/en-us/nam/open-access-at-sage).
subsequently further developed and studied this type of hydraulic motor. In recent years, Sliwinski carried out many studies on the non-circular planetary gear hydraulic motor. He proposed a method of determining the theoretical and actual displacement of the motor by experiment and studied the effects of water and mineral oil on the motor’s mechanical loss and displacement loss and the leakage of the reversing device. The reversing device and the axial clearance compensation device of the hydraulic motor have been designed. In addition, there has been much research on this type of motor. Cao et al. studied the main leakage modes of the motor and established a mathematical model of the clearance leakage. Szkodo et al. found that bending fatigue and surface contact fatigue of the planetary gear train were the main factors of the failure of hydraulic motors. Volkov et al. designed a non-circular planetary gear hydraulic motor with the same number of waves of the solar wheel and the inner gear ring and demonstrated its feasibility.

In research on non-circular gears, Li and Volkov proposed the method of designing the tooth profile of non-circular gears. Riaza et al. studied the pitch curve of non-circular gears. Xu et al. studied a method of generating the corresponding non-circular gear with a given two-dimensional shape. Volkov et al. compared the quality of different types of planetary gear trains with non-circular gears and concluded that the 2–4, 4–6, and 6–8 types performed best. Furthermore, Volkov et al. completed a comprehensive geometric design of non-circular gear mechanisms. Lin et al. proposed a method based on the envelope principle to obtain an accurate tooth profile of non-circular gears. Mundo proposed a method for the geometric synthesis of an epicyclical gear train able to generate a variable gear ratio law.

Progress has been made in solving the pitch curve of non-circular gears and determining the tooth profile and the kinematic relationship between non-circular gears. The structural design, durability and leakage of non-circular planetary gear hydraulic motors have also been studied. As the core of the hydraulic motor, the non-circular gear directly determines the displacement of the motor. However, at present, there is no clear formula of the relationship between the design parameters of non-circular gears and the displacement of hydraulic motors. The calculation of the existing displacement method (which will be introduced in Section 3.3) is cumbersome, and the quantitative relationship between the design parameters of non-circular gears and the displacement of motors cannot be given. Therefore, the design parameters of the non-circular gear can only be selected by trial and error, which increases the difficulty of hydraulic motor design.

In view of the above problems and on the basis of the study of the relationship between the design parameters of the non-circular gear and the motor displacement, the formula for quantitatively calculating the motor displacement is derived in section 3. This formula plays a guiding role in the selection of non-circular gear parameters and is helpful in understanding the influencing factors of the motor displacement. To improve the power density of the hydraulic motor, the relationship between the design parameters of the non-circular gear and the displacement per unit volume of the motor is further studied. In section 4, a method of optimizing the pitch curve of the non-circular gear such that the optimized hydraulic motor has higher power density is proposed.

**Structure and types of non-circular planetary gear hydraulic motors**

The structure of the non-circular planetary gear hydraulic motor is shown in Figure 1. The sun wheel is a non-circular gear that is connected to a shaft by a steel ball or spline and rotates with the shaft. The inner ring gear is another non-circular gear, which is connected to the motor housing by a locating pin and remains stationary. The planetary gear is a circular gear that engages with both the solar gear and the inner ring gear. When the sun wheel rotates, the volume of the working volume of the planetary wheel, the sun wheel and the inner gear ring changes, and oil absorption and discharge can be realized through the oil distribution device located at the end face of the gear.

The motor is named after the number of waves of the sun wheel and inner ring gear. In the case that there are four waves of the sun wheel and six waves of the inner ring gear, the hydraulic motor is called a 4–6 type motor. Figure 1 shows a 4–6 type hydraulic motor as considered in this study. The 4–6 type hydraulic motor

![Figure 1. Hydraulic motor structure: 1–Inner ring gear, 2–Positioning pin, 3–Planetary gear, 4–Working volume, 5–Sun wheel, 6–Steel ball, and 7–Shaft.](image-url)
has a center symmetry, such that the hydraulic radial force of the oil on the shaft is balanced and the motor can thus be used under high pressure.

**Formulation of the displacement of the non-circular planetary gear hydraulic motor**

This section derives a formula for the displacement of the hydraulic motor. Section 3.1 introduces a method of solving the pitch curve of the non-circular gear and section 3.2 introduces a method of solving the position of the planetary wheel. On the basis of the theory presented in sections 3.1 and 3.2, the method of calculating the displacement introduced in section 3.3 can be used to obtain the corresponding hydraulic motor displacement under different design parameters of the non-circular gears. Using the obtained data, section 3.4 analyzes and summarizes the relationship between the design parameters of the non-circular gears and the motor displacement and finally presents a formula for quantitatively calculating the motor displacement.

**Solution of the gear pitch curve**

The essence of the design of the non-circular planetary gear hydraulic motor lies in the design of each gear. In the design of a gear, we should first find the solution of the gear pitch curve. In this paper, the pitch curve of the inner gear ring is obtained using the given pitch curve equation of the sun wheel. The known initial conditions are (1) the number of waves of the sun wheel, (2) the number of waves of the inner ring gear, (3) the gear module, (4) the number of planetary gear teeth, (5) the number of teeth of the sun wheel and the inner gear ring in one cycle, and (6) the equation for the solar wheel pitch curve in polar coordinates. In general, the curve of the solar wheel node has the form of a high-order ellipse:

\[
r_1 = \frac{A(1 - B^2)}{(1 + B \cos(n_1 \theta_1 - \pi))}
\]

where \( A \) is the radius of the long axis of the ellipse and \( B \) is the eccentricity of the ellipse. Both \( A \) and \( B \) are unknowns to be solved.

In addition to the high-order elliptic form, the solar pitch curve can be in the form of a high-order Pascal spiral:

\[
r_1(\theta_1) = A \cos(n_1 \theta_1) + B
\]

Theoretically, the solar pitch curve can be of any form as long as the solar pitch curve is a periodic function with period \( T = (2\pi) / n_1 \). According to the initial conditions given above, the number of teeth of the sun wheel \( Z_1 = n_1 Z_c \), the number of teeth of the inner gear ring \( Z_3 = n_3 Z_c \), and the number of teeth of the planetary wheel is an integer slightly less than \((Z_3 - Z_1)/2\).

The relative positions of the sun gear, planetary gear and inner gear ring in the planetary gear train with non-circular gears is shown in Figure 2. In the figure, \( O_1X_1, O_2X_2, \) and \( O_3X_3 \) are the polar axes of the pitch curves of the solar, planetary and inner ring gear, respectively. For the initial positioning, \( O_1X_1 \) is coincident with \( O_3X_3 \), and \( O_2X_2 \) is collinear with \( O_2X_3 \) but in the opposite direction. The pitch curve of the solar gear rotates clockwise whereas the pitch curve of the inner gear is fixed. The pitch curve of the planetary gear rolls without sliding with the pitch curve of the solar gear and the pitch curve of the inner ring gear. \( P_{21} \) is the contact point of the solar and planetary gear pitch curves, \( P_{23} \) is the contact point of the inner gear and planetary gear pitch curves, and the three points \( P_{21}, P_{23}, \) and \( O_1 \) are collinear.

The equation of the pitch curve of the inner ring gear in polar coordinates is

\[
\begin{aligned}
 r_3 &= r_1 + 2r_2 \sin(\mu) \\
 \theta_3 &= \frac{\theta_1 + 2r_3 \sin(\mu)}{r_1} d\theta_1
\end{aligned}
\]

where \( r_2 = (mZ_2)/2 \) is the radius of the curve of the planetary wheel pitch, \( \mu \) is the included angle between the forward tangent of the curve of the solar wheel \( r_1(\theta_1) \) pitch and the radial vector \( r_1, \sin(\mu) \), and \( d\mu/d\theta_1 \) can be further expressed as.

![Figure 2. Pitch curve positional relationship of the planetary gear train with non-circular gears.](image-url)
\[
\sin(\mu) = \frac{r_1}{\sqrt{(r_1)^2 + \left(\frac{dr_1}{d\theta_1}\right)^2}} \tag{4}
\]
\[
\frac{d\mu}{d\theta_1} = \frac{\left(\frac{dr_1}{d\theta_1}\right)^2 - r_1 \frac{d^2 r_1}{d\theta_1^2}}{(r_1)^2 + \left(\frac{dr_1}{d\theta_1}\right)^2} \tag{5}
\]

In this way, the polar coordinate equations of the curve of the solar wheel pitch and the curve of the inner gear ring pitch are expressed, and both contain unknowns \(A\) and \(B\). The following restrictions are needed to solve \(A\) and \(B\).

**Closure conditions for the pitch curve of the inner ring gear.** The pitch curve of the ring gear pitch must be periodic. The initial conditions have been given above. The number of waves of the inner ring gear is \(n_3\). Now suppose that the angular velocity of revolution of the planetary gear is zero, such that the transformation mechanism of the planetary gear system is obtained. The inner gear ring rotates counterclockwise when the sun wheel rotates clockwise. The sun wheel is periodic and the planetary wheel is a circular gear, and the pitch curve of the inner gear ring must therefore also be periodic. If \(\theta_1\) changes by a period \((2\pi)/n_1\), \(\theta_3\) will change by a period \((2\pi)/n_3\), such that we have

\[
2\pi \frac{n_1}{n_3} = \int_0^{2\pi} \frac{r_1 + 2r_2 \sin(\mu)\left(\frac{d\mu}{d\theta_1}\right)}{r_3} d\theta_1 \tag{6}
\]

**Conditions for a uniform tooth distribution.** The gears on the sun wheel should be evenly distributed, such that

\[
\pi mZ_1 = \int_0^{2\pi} \sqrt{(r_1)^2 + \left(\frac{dr_1}{d\theta_1}\right)^2} d\theta_1 \tag{7}
\]

The arc length of the inner gear ring is \(n_3/n_1\) times that of the sun wheel, and the number of teeth of the inner gear ring is also \(n_3/n_1\) times that of the sun wheel. Therefore, when the teeth of the sun wheel are evenly distributed, the inner gear ring also meets the condition of a uniform distribution.

By solving equations (6) and (7) simultaneously, the unknowns \(A\) and \(B\) can be solved, and the pitch curve equations of the solar wheel and the inner gear ring can then be obtained.

**Solution of the planetary wheel position**

There are two typical positions of planetary wheels in motion, as shown in Figures 3 and 4.

When \(\mu = \pi/2\) (i.e. typical position I), where \(\mu\) is the angle between the forward tangent line of the solar
wheel $r_1(\theta_1)$ pitch curve and the radial vector $r_1$, which can be obtained by solving

$$\cos(\mu) = \frac{dr_1}{d\theta_1} \sqrt{(r_1)^2 + \left(\frac{dr_1}{d\theta_1}\right)^2}$$  (8)

The position of the planetary wheel is deduced as\textsuperscript{32}

$$\theta_H = \theta_3 + \gamma_1$$  (9)

$$LH = O_1O_2 = \sin\left(\frac{3\pi}{2} - \mu\right) r_2/\sin(\gamma_1)$$  (10)

$$\tan(\gamma_1) = \frac{r_2\sin\left(\frac{\pi}{2} - \mu\right)}{r_1 + r_2\cos\left(\frac{\pi}{2} - \mu\right)}$$  (11)

where $LH$ is the polar radius of the center of the planetary wheel.

Similarly, the position of the planetary wheel when $\mu > \pi/2$ (i.e. typical position II) is deduced as\textsuperscript{32}

$$\theta_H = \theta_3 - \gamma_1$$  (12)

$$LH = O_1O_2 = \sin\left(\frac{3\pi}{2} - \mu\right) r_2/\sin(\gamma_1)$$  (13)

We assume that the rotation angle of the solar wheel at a certain moment is $\theta_m$ (where $\theta_m$ is positive in the clockwise direction). Using equation (3), $\theta_3$ can be expressed by $\theta_1$, and the sum of $\theta_1$ and $\theta_3$ is equal to $\theta_m$, that is,

$$\theta_1 + \theta_3 - \theta_m = 0$$  (14)

By solving equation (14), $\theta_1$ and $\theta_3$ are obtained when the rotation angle of the sun wheel is $\theta_m$. The position of the planetary wheel can be obtained by substituting $\theta_1$ and $\theta_3$ into the equation for the position of the planetary wheel to obtain the polar angle and polar radius of the center of the planetary wheel.

Assuming that the planetary wheel at the initial position ($\theta_1 = 0$ and $\theta_3 = 0$) is denoted 1 and the sun wheel rotates clockwise, the corresponding $\theta_1$, $\theta_3$, $\mu$, $\gamma_1$, $\theta_H$, and $LH$ of the first planetary wheel can be determined when the sun wheel rotates through any angle $\theta_m$, and the position of the second planetary wheel is the position of the first planetary wheel when the sun wheel rotates through $(2\pi)/n_1$. The position of the third planetary wheel is the position of the second planetary wheel when the sun wheel rotates through $(2\pi)/n_1$ and so on, such that the positions of all the planetary wheels can be found.

There are $(n_1 + n_3)$ planetary wheels in total. Taking 4–6 hydraulic motors as an example, the serial numbers of planetary wheels are shown in Figure 5.

Solving the hydraulic motor displacement

To calculate the displacement of the planetary gear train, we should first establish a method of calculating the volume of the enclosed cavity. Figure 6 shows the section of the enclosed cavity enclosed by two planetary wheels. The volume of the enclosed cavity can be obtained by multiplying the area enclosed by the curve abcfea by the width of the gear $B$. Using equations (3)
and (14), the angles \( \theta_{31}, \theta_{11}, \theta_{32}, \) and \( \theta_{12} \) corresponding to the two planetary wheels can be obtained. Taking \( \theta_{31} \) as the lower limit and \( \theta_{32} \) as the upper limit of the integral, the area integral of \( r_1 \) and \( r_2 \) can be obtained from the area \( A_1 \) enclosed by the curve cofc and the area \( A_2 \) enclosed by the curve aoda. The area enclosed by the curve aobe is

\[
A_3 = (r_2)^2 \left[ 2\pi - 2\mu + \sin(2\mu_1) \right] / 2 \tag{15}
\]

The area enclosed by curve defd is

\[
A_4 = (r_2)^2 \left[ 2\mu - \sin(2\mu_2) \right] / 2 \tag{16}
\]

The area of the section of the closed cavity is

\[
A = A_2 - A_1 - A_3 - A_4 \tag{17}
\]

The volume of the cavity enclosed by neighboring planetary wheels can be calculated for each rotation angle of the sun wheel. There are 10 such cavities for a 4–6 type of non-circular planetary gear train. When the sun wheel continues to rotate, the cavity will absorb/ discharge oil if the volume of the cavity increases/ decreases. The instantaneous displacement of the planetary gear train can be obtained by summing the volume increase in the oil absorption cavity and then dividing by the increasing angle of the sun wheel. The average displacement and instantaneous displacement pulsation rate of the planetary gear train are thus obtained.

**Formulaic processing of the hydraulic motor displacement**

The method of calculating the hydraulic motor displacement was given in section 3.3. However, the calculation is complex, and the relationship between the design parameters of the non-circular gear and the displacement of the hydraulic motor cannot be shown directly, which introduces difficulty to the selection of the parameters of the non-circular gear. It is therefore necessary to study in depth the effect of each parameter on the hydraulic motor displacement.

The determination of the pitch curve of the non-circular gear and the thickness \( B \) of the gear allows the displacement of the hydraulic motor to be determined. It is seen from section 3.1 that the pitch curve of the non-circular gear is determined by six parameters. In this paper \( n_1 = 4 \) and \( n_3 = 6 \). When the form of the solar pitch curve equation \( r_1 = r_1(\theta_1) \) is determined, the displacement of the hydraulic motor is related to the four parameters \( Z_2, Z_c, m, \) and \( B \).

To facilitate the analysis, the gear thickness \( B = 20 \) mm and gear module \( m = 1 \) are uniformly taken to analyze the relationships between \( Z_2 \) and \( Z_c \), and the hydraulic motor displacement.

**Displacement per unit volume \( V_m \).** Here, the unit volume displacement \( V_m \) is defined and taken as the evaluation index of the motor power density. In obtaining the data table for the motor displacement \( V \) and non-circular gear design parameters, the unit volume displacement \( V_m \) is also calculated to facilitate the analysis of the motor power density in section 4:

\[
V_m = \frac{V}{\pi r_{3m} B} \tag{18}
\]

Here, \( V \) is the displacement of the hydraulic motor and \( r_{3m} \) is the maximum radius of the pitch curve of the inner gear ring.

\( V_m \) is a quantity with dimension \( r^{-1} \), representing the ratio of the hydraulic motor displacement to the volume of the non-circular planetary gear train. A larger value of \( V_m \) corresponds to better displacement performance and a more compact structure of the non-circular planetary gear train.

**Selection of teeth.** The number of teeth of the planetary wheel \( Z_2 < (Z_3 - Z_1)/2 = (n_1 - n_2)Z_c/2 \). In this paper, \( n_1 = 4 \) and \( n_3 = 6 \), and therefore, \( Z_2 < Z_c \). According to \( n_c = Z_c - Z_2 \), non-circular planetary gear trains can be classified into the less 1 tooth type, less 2 teeth type, less 3 teeth type and so on. As examples, \( Z_2 = 12 \) and \( Z_c = 13 \) correspond to the less 1 tooth type, \( Z_2 = 12 \) and \( Z_c = 14 \) to the less 2 teeth type, and \( Z_2 = 12 \) to the less 3 teeth type.

From this point of view, there are many combinations of the number of teeth of the planetary gear and the number of teeth of the sun wheel, which makes it difficult to analyze the effect of the number of teeth on the displacement. It is therefore necessary to find the connection between different types of tooth number combinations. The gear module is certain, and the number of teeth on the gears thus affects the circumference of the pitch curve directly in a proportional relationship. We speculate that when the number of teeth of all gears is expanded \( x \) times, the circumference of the gear’s pitch curve will also expand \( x \) times and the shape remains unchanged. There is a squared relationship between the area and length, and the displacement of the motor is therefore \( x^2 \) times the original. The discharge per unit volume is a quantity of dimension \( r^{-1} \) and should be related only to the shape of the pitch curve and not to the size, such that \( V_m \) should remain constant. The data in Table 1 are obtained for the example of the form of the solar pitch curve being a higher-order ellipse.

It is seen that the above speculation is correct; that is, when \( Z_2 \) and \( Z_c \) in the less 1 tooth type simultaneously expand \( x \) times to the less \( x \) teeth type, the displacement is \( x^2 \) times the original, while the displacement per unit volume remains unchanged. In this way, if the...
It is noted that the number of teeth $Z_2$ of the planetary gear in Table 2 starts at seven. Considering the gear undercut, there is not necessarily such a gear in practice. However, this does not affect its presence in the table because we can choose the less 2 teeth type planetary gear train; that is, $Z_2 = 14$ and $Z_c = 16$. The gear with 14 teeth can be manufactured after modification. Here, the displacement of the hydraulic motor is $V = 110.68\, \text{ml/r}$ and the displacement per unit volume is $V_m = 0.7577\, \text{ml/r}$.

### Derivation of the displacement calculation formula

Taking the high-order ellipse as an example, Table 2 gives 21 sets of data for the number of teeth and displacement of the one-tooth hydraulic motor. The curve fitting method is now used to fit these 21 sets of data. The tooth number $Z_2$ and displacement $V$ are fitted by a power function in Figure 7. The approximate formula of displacement is thus obtained as

$$V = 1.433Z_2^{-1.481} + 2.144$$

(19)

When $Z_2 = 50$, the displacement calculated using formula (19) is 472.49 ml/r whereas that obtained using the area integral method presented in section 3.3 is 473.93 ml/r. The small difference of less than 0.3% verifies that the relationship between $Z_2$ and $V$ can be well expressed using this formula.

The formula is extended to other kinds of non-circular planetary gear hydraulic motors that have different values of the planetary gear number, solar gear number, gear thickness and gear module. The gear module and the number of teeth have a linear relationship with the length of the pitch curve, and the displacement thus has a squared relationship with the module whereas the thickness of the gear has a linear relationship with the volume. The general formula of the displacement is thus obtained as

$$V = \frac{Bm^2n_c^2[1.433(Z_2/n_c)^{1.481} + 2.144]}{20}$$

(20)

Formula (20) is verified as follows. We first take $B = 30\, \text{mm}$, $m = 1.5$, $Z_2 = 16$, $Z_c = 18$, and $n_c = 2$. Substituting these values into formula (20) gives $V = 449.70\, \text{ml/r}$ whereas $V = 449.55\, \text{ml/r}$ is calculated using the method presented in Section 3.3. We next substitute $B = 12.5\, \text{mm}$, $m = 1.8$, $Z_2 = 48$, $Z_c = 51$, and $n_c = 3$ into formula (20) to get $V = 1624.75\, \text{ml/r}$, whereas $V = 1624.64\, \text{ml/r}$ is calculated using the method presented in Section 3.3. It is thus seen that formula (20) well represents the relationship between the design parameters of the non-circular gear and the displacement of the hydraulic motor. In this way, the complicated area integration described in Section 3.3 is eliminated. When the design parameters $B$, $m$, $Z_2$, and
nc of non-circular gears are given, the corresponding hydraulic motor displacement can be directly calculated using formula (20). At the same time, formula (20) reveals the effect of the non-circular gear design parameters $B$, $m$, $Z_2$, and $n_c$ on the displacement of the hydraulic motor. In this way, when designing a hydraulic motor with a certain displacement, the size of each parameter can be adjusted to obtain the desired displacement. Formula (20) therefore brings great convenience to the selection of non-circular gear parameters when designing hydraulic motors.

It is noted that formula (20) only applies to the high-order ellipses of the solar pitch curve for the hydraulic motor of type 4–6. However, the derivations of the displacement calculation formula for other types of hydraulic motor follow the derivation in this section. Furthermore, the forms of the derived formulas are consistent, with there only being changes in parameters.

**Improvement in the hydraulic motor power density**

The unit volume displacement $V_m$ is an important index that reflects the compact structure of the hydraulic motor. A larger value of $V_m$ indicates a larger power density of the hydraulic motor. For the convenience of analysis, the module $m = 1$ and the gear tooth thickness $B = 20$ mm are uniformly taken in this section.

**Constraints on unit volume displacement**

Table 2 shows that for the hydraulic motor of less 1 tooth type, $V_m$ is greater for a smaller number of teeth of the planetary gear. However, after drawing the pitch curves of the sun wheel and inner ring gear, it is seen that when there are fewer than a certain number of teeth of the planetary gear, the pitch curve of the inner ring gear is self-intersecting and obviously cannot be used. We take the high-order Pascal spiral $r_1(\theta_1) = A \cos(n_1\theta_1) + B$ for the solar wheel pitch curve as an example. Figure 8 shows the pitch curves of the sun wheel and inner gear ring for different planetary gear numbers.

Figure 8 shows that as $Z_2$ decreases, the pitch curves of the sun wheel and inner gear ring gradually become concave at the minimum radius. Thus, the volume enclosed by the two planetary wheels changes more dramatically in the process of motion, and $V_m$ increases. However, when $Z_2$ decreases to a certain extent, the pitch curve of the inner gear ring cannot be used as a practical pitch curve because it is too sharp and self-intersecting at the minimum radius. Therefore, the self-intersecting phenomenon at the minimum radius of the pitch curve of the inner ring gear limits the increase in $V_m$. It is for this reason that the value of $Z_2$ is at least 7 in Table 2.

As described in section 3.1, the pitch curve is determined by six initial conditions. Different equation forms of the solar pitch curve can be considered in studying the method of eliminating the self-intersection phenomenon of the pitch curve of the inner gear ring when the number of planetary gear teeth $Z_2$ is small (e.g. $Z_2 = 5$).

**Optimization equation of the solar pitch curve**

There are presently two common equation forms of the solar pitch curve, namely the higher-order ellipse form and the high-order Pascal spiral form, as expressed by equations (1) and (2). However, in theory, as long as
the solar pitch curve equation is of a periodic function with period $T = (2\pi)/n_1$, the solar pitch curve equation will satisfy the requirement. There are thus many options for the form of the equation of the solar pitch curve, and it is important to find a general form of the equation.

We know that a periodic function can be expanded as a Fourier series. A periodic function can be expanded as a sine series if the function is odd and as a cosine series if the function is even. The solar wheel needs to ensure consistent performance in forward and reverse rotation, and the solar wheel pitch curve thus needs to be an even function. The solar pitch curve is thus expressed generally as

$$r_1(\theta_1) = A\cos(n_1\theta_1) + C\cos(2n_1\theta_1) + D\cos(3n_1\theta_1) + \sum_{i=4}^\infty a_i\cos(in_1\theta_1) + B$$ (21)

The high-order Pascal spiral $r_1(\theta_1) = A\cos(n_1\theta_1) + B$ mentioned in Section 3.1 is a special case of this form of expression. To improve the form of the pitch curve and further increase the motor displacement under the same conditions, an optimization factor $C\cos(2n_1\theta_1)$ is added to the formula (2) to obtain a new optimization equation of the solar pitch curve:

$$r_1(\theta_1) = A\cos(n_1\theta_1) + C\cos(2n_1\theta_1) + B$$ (22)

The effect of coefficient $C$ in equation (22) on the pitch curve is studied by taking $Z_2 = 5$ and $Z_2 = 6$. Figure 9 shows the pitch curves of the sun wheel and inner gear ring for different values of the coefficient $C$.

Figure 9 shows that a higher coefficient $C$ eliminates the self-intersection phenomenon of the inner gear pitch curve. In this way, by selecting a reasonable coefficient $C$, the number of planetary gear teeth can be further reduced and the unit volume displacement can be increased. We take $C = 0.2$. Figure 10 shows the relationship between the number of planetary gear teeth and the displacement of the motor and the displacement of the unit volume for the hydraulic motor of less 1 tooth type. $V_{m1}$ and $V_1$ are respectively the unit volume displacement and displacement of the hydraulic motor when the solar pitch curve is a high-order elliptic as equation (1). Here, the minimum value of $Z_2$ is 7 and the maximum value of $V_{m1}$ is $0.7577/r$. $V_{m2}$ and $V_2$ are

![Pitch curves of the solar wheel and inner gear ring](image)
respectively the unit volume displacement and displacement of the hydraulic motor when the solar pitch curve is
\[ r_1(\theta_1) = A \cos(n_1 \theta_1) + 0.2 \cos(2n_1 \theta_1) + B. \]
Here, the minimum value of \( Z_2 \) is 4 and the maximum value of \( V_{m2} \) is 0.9753/r.

Figure 10 shows that, in contrast with the commonly used high-order elliptic equation (1), the proposed solar pitch curve equation (22) provides little difference in the unit volume displacement and displacement for the same \( Z_2 \). However, when equation (22) is used for the solar pitch curve, a smaller \( Z_2 \) can be taken to obtain a larger displacement per unit volume. In Figure 10, the maximum value of \( V_{m2} \) is 0.9753/r, which means that when the sun wheel of the non-circular planetary gear train rotates through one revolution, the volume of oil discharged is close to that of the entire gear train. Obviously, the space utilization rate of the non-circular planetary gear train for this value of \( V_{m2} \) is extremely high. Additionally, because of the simple structure of the motor, the overall volume is based on the volume

Figure 9. Pitch curves of the solar wheel and inner gear ring: (a) \( C = 0 \), (b) \( C = 0.1 \), (c) \( C = 0.2 \), and (d) \( C = 0.3 \).

Figure 10. Comparison of the relationships between the tooth number and displacement before and after optimization.
of the non-circular planetary gear train, and therefore, the power density ratio of the manufactured hydraulic motor is also high.

Limitation of the selection of the pitch curve by the tooth profile

Section 4.2 judged whether the inner ring gear pitch curve is feasible simply by considering whether the inner ring gear pitch curve is self-interlacing. However, we also need to consider the inner ring gear tooth profile. Some inner gear ring pitch curves are not self-interlacing, but when the pitch curve is too sharp at the minimum radius, there may be an undercut gear. Therefore, when selecting the pitch curve, the tooth profile should be drawn to judge whether the curve is feasible.

We select the less 3 teeth type of hydraulic motor taking \( Z_2 = 4 \) in Figure 10 as an example. In this case, \( Z_2 = 12, Z_c = 15, \) and \( c = 0.6, \) and the form of the solar wheel pitch curve is \( r_1(\theta_1) = A\cos(n_1\theta_1) + 0.6\cos(2n_1\theta_1) + B. \) Figure 11 shows the pitch curves of the sun wheel and inner ring gear whereas Figure 12 shows the tooth profile of the inner ring gear.

Figure 11 shows that when \( Z_2 = 12 \) and \( Z_c = 15, \) there is no self-interlacing of the gear pitch curve of the inner ring, but the tooth profile of the inner ring tooth shows the undercutting phenomenon, as shown in Figure 12. Therefore, when selecting the pitch curve, it is also necessary to draw the tooth profile of each gear at the same time to judge its feasibility. The next problem to solve is determining how to select gear parameters such as the pressure angle and addendum coefficient in solving the undercut problem of the inner gear ring so that \( Z_2 \) is small enough for the less 1 tooth type of hydraulic motor.

**Example verification**

The solar gear pitch curve form is \( r_1(\theta_1) = A\cos(n_1\theta_1) + 0.5\cos(2n_1\theta_1) + B, \) the teeth is of less 2 teeth type, the number of teeth of the planetary gear \( Z_2 = 12, \) and the number of teeth of the solar gear per unit period \( Z_c = 14. \) The gear module \( m = 1 \) and the gear thickness \( B = 20 \text{ mm}. \) The pitch curves of the sun wheel, inner gear ring and planetary wheel are drawn in Figure 13. From calculation, the displacement of the hydraulic motor \( V = 91.34 \text{ ml/r} \) and the unit volume displacement \( V_m = 0.8193. \) Here, the unit volume displacement of the motor is higher than the highest

![Figure 11. Pitch curves of the sun wheel and inner ring gear for \( Z_2 = 12 \) and \( Z_c = 15. \)](image)

![Figure 12. Tooth profile of the inner ring gear.](image)
unit volume displacement of 0.7577 given in Table 2. (There is a self-intersection of the pitch curve of the inner ring gear when \(Z_2\) is further reduced in Table 2.)

The tooth profile is involute when using the profiling method to write a program to draw the tooth profile of each gear. The coefficient of the tooth tip height of the planetary gear is 1, the coefficient of the tooth tip height of the copying rack is 1.2, the pressure angle of the copying rack is 25°, the modification coefficient of the planetary gear is 0.2, and the coefficient of the tooth tip height of the forming gear of the solar gear and inner ring gear is 1.15. The degree of coincidence of the planetary wheel and the sun wheel and the inner ring gear is 1.2. According to the above parameters, the program written by the author can be used to draw the tooth profile of each gear, as shown in Figure 14. The tooth profile of each gear can be used in the design of a non-circular planetary gear hydraulic motor.

**Conclusion**

This paper investigated the displacement and unit volume displacement of the 4–6 type of non-circular planetary gear hydraulic motor. By establishing equations for the pitch curve of the sun wheel and the inner gear ring in polar coordinates, listing the closing conditions of the pitch curve of the inner gear ring and the condition of a uniform distribution of teeth, and solving the positioning of the planetary wheel, the following innovations were proposed.

The effects of the planetary gear number, solar gear number, gear thickness and gear module on the hydraulic motor displacement were analyzed. On this basis, a formula for calculating the displacement of the non-circular planetary gear hydraulic motor when the solar pitch curve is a high-order ellipse was put forward. The formula for calculating the displacement of the hydraulic motor can be used to calculate and predict the motor displacement easily.

The limiting factors of the unit volume displacement of the non-circular planetary gear hydraulic motor were analyzed, and an optimization equation was put forward by adding an optimization factor to the original equation of the solar pitch curve. It was shown that the newly added optimization factor eliminated the self-intersection phenomenon of the pitch curve of the inner ring gear and increased the unit volume displacement of the non-circular planetary-gear hydraulic motor.

**Declaration of conflicting interests**

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

**Funding**

The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: The authors would like to give their acknowledgment to the National Key Research and Development Plan (2018YFB200902) for the financial support to this study.

**ORCID iD**

Biao Zhang https://orcid.org/0000-0001-5882-0069

**References**

1. Tran NH, Le C and Ngo AD. An investigation on speed control of a spindle cluster driven by hydraulic motor: application to metal cutting machines. *Int J Rotating Machinery* 2019; 2019: 1–17.
2. Jing C, Xu H and Jiang J. Dynamic surface disturbance rejection control for electro-hydraulic load simulator. *Mech Syst Signal Process* 2019; 134: 1–14.
3. Lee W, Yoo S, Nam S, et al. Passivity-based robust compliance control of electro-hydraulic robot manipulators with joint angle limit. *IEEE Robot Autom Lett* 2020; 5: 3190–3197.

4. Zhou S, Walker P, Tian Y, et al. Comparison on energy economy and vibration characteristics of electric and hydraulic in-wheel drive vehicles. *Energies* 2021; 14: 2290.

5. Songshan H, Zongxia J, Chengwen W, et al. Fuzzy robust nonlinear control approach for electro-hydraulic flight motion simulator. *Chin J Aeronaut* 2015; 28: 294–304.

6. Yuan Y, Song X, Sun W, et al. Modeling and dynamic analysis of the non-circular gear system of a bucket wheel stacker/reclaimer. *AIP Adv* 2018; 8: 065318.

7. Zhao X, Chu M, Ma X, et al. Research on design method of non-circular planetary gear train transplanting mechanism based on precise poses and trajectory optimization. *Adv Mech Eng* 2018; 10: 1–12.

8. Bijlsma BG, Radaelli G and Herder JL. Design of a compact gravity equilibrator with an unlimited range of motion. *J Mech Robot* 2016; 9: 061003.

9. Xu G, Chen J and Zhao H. Numerical calculation and experiment of coupled dynamics of the differential velocity vane pump driven by the hybrid higher-order Fourier non-circular gears. *Adv Mater Sci* 2018; 27: 285–293.

10. Sieniawski B. Gyrating-cam engine, particularly as a hydraulic engine. U.S. Patent, 3852002, 1974.

11. Yao Q and Yao F. Planetary rotary type fluid motor or engine and compressor or pump. U.S. Patent 9322272, 2016.

12. Ding H. Application of non-circular planetary Gear mechanism in the Gear Pump. *Adv Mater Res* 2012; 591-593: 2139–2142.

13. Śliwiński P. Determination of the theoretical and actual working volume of a hydraulic motor. *Energies* 2020; 13: 5933.

14. Śliwiński P. Determination of the theoretical and actual working volume of a hydraulic motor—part II (the method based on the characteristics of effective absorbency of the motor). *Energies* 2021; 14: 1648.

15. Śliwiński P. The influence of water and mineral oil on mechanical losses in a hydraulic motor for offshore and marine applications. *Pol Marit Res* 2020; 27: 125–135.

16. Śliwiński P. The influence of water and mineral oil on volumetric losses in a hydraulic motor. *Pol Marit Res* 2017; 24: 213–223.

17. Śliwiński P. Influence of water and mineral oil on the leaks in satellite motor commutation unit clearances. *Pol Marit Res* 2017; 24: 58–67.

18. Śliwiński P. The basics of design and experimental tests of the commutation unit of a hydraulic satellite motor. *Arch Civil Mech Eng* 2016; 16: 634–644.

19. Śliwiński P. The methodology of design of axial clearances compensation unit in hydraulic satellite displacement machine and their experimental verification. *Arch Civil Mech Eng* 2019; 19: 1163–1182.

20. Cao W, Liu Y, Niu Z, et al. Leakage characteristics analysis of an end face clearance compensated non-circular planetary gear motor. In: 2019 IEEE 8th international conference on fluid power and mechatronics (FPM), Wuhan, China, 10–13 April 2019, pp.1082–1088. New York: IEEE.

21. Szkodo M, Stanisławska A and Śliwiński P. On the durability of the hydraulic satellite motor working mechanism in overload condition. *Adv Mater Sci* 2016; 16: 35–46.

22. Volkov G and Kurasov D. Planetary rotor hydraulic machine with two central gearwheels having similar tooth number. *Advanced Gear Engineering* 2018; 51: 435–446.

23. Li D, Liu Y, Gong J, et al. Design of a noncircular planetary gear mechanism for hydraulic motor. *Math Probl Eng* 2021; 2021: 1–9.

24. Yu Volkov G and Fadyushin DV. Improvement of the method of geometric design of gear segments of a planetary rotary hydraulic machine. *J Phys Conf Ser* 2021; 1889: 042052.

25. Raza HFQ, Cardona i Foix S and Nebot LJ. Study of the base curve and formation of singular points on the tooth profile of noncircular Gears. *J Mech Des* 2007; 129: 538–545.

26. Xu H, Fu T, Song P, et al. Computational design and optimization of non-circular Gears. *Comput Graph Forum* 2020; 39: 399–409.

27. Volkov G and Smirnov V. Systematization and comparative scheme analysis of mechanisms of planetary rotary hydraulic machines. *MATEC Web Conf* 2018; 224: 02083.

28. Volkov GY, Kurasov DA and Gorbunov MV. Geometric synthesis of the planetary mechanism for a rotary hydraulic machine. *Russ Eng Res* 2018; 38: 1–6.

29. Lin C, Xia X and Li P. Geometric design and kinematics analysis of coplanar double internal meshing non-circular planetary gear train. *Adv Mech Eng* 2018; 10: 1–12.

30. MUNDO D. Geometric design of a planetary gear train with non-circular gears. *Mech Mach Theory* 2006; 41: 456–472.

31. Wang C, Lan Z and Gao W. Design of pitch curve of internal-curved planet gear pump strain in type N-G-W based on three-order Ellipse. *Adv Mat Res* 2013; 787: 567–571.

32. Xu Y. Design system for hydraulic motor with non-circular planetary gears. Master Thesis, Hefei University of Technology, China, 2012 (in Chinese).