Total edge irregularity strength of triple book graphs

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Abstract. Let \( G(V, E) \) be a finite, simple and undirected graph with a vertex set \( V \) and an edge set \( E \). An edge irregular total \( k \)-labelling is a map \( f : V \cup E \rightarrow \{1, 2, \ldots, k\} \) such that for any two different edges \( xy \) and \( x'y' \) in \( E \), \( \omega(xy) \neq \omega(x'y') \) where \( \omega(xy) = f(x) + f(y) + f(xy) \). The minimum \( k \) for which the graph \( G \) admits an edge irregular total \( k \)-labelling is called the total edge irregularity strength of \( G \), denoted by \( \text{tes}(G) \). We have constructed the formula of an edge irregular total \( k \)-labelling and determined the total edge irregularity strength of book graphs and double book graphs. In this paper, we construct an edge irregular total \( k \)-labeling that can be used for book graphs, double book graphs, and triple book graphs. We also show the exact value of the total edge irregularity strength of triple book graphs.

1. Introduction

Let \( G(V, E) \) be a finite, simple and undirected graph with a vertex set \( V \) and an edge set \( E \). Labelling of graph \( G \) is a function that assigns elements on the graph (vertices, edges or both) to numbers (usually positive integer and called labels) that satisfy certain conditions [1]. There are various labelling of graphs that pay attention to the number of labels of elements on the graph. In [2] Chartrand et al. introduced an irregular edge \( k \)-labelling as a function \( f \) from the set of edge to the set of number from \( 1 \) until \( k \). The labelling \( f \) has different weights. Let \( v \) is a vertex in \( G \), the weight of vertex \( v \) is the sum of all labels of edges that are incident to vertex \( v \) and denote by \( \omega_f(v) \). The smallest \( k \) value is called irregular strength of \( G \) and is denoted by \( s(G) \) if the graph \( G \) admits an irregular edge \( k \)-labelling.

Furthermore in [3] an edge irregular total \( k \)-labelling on graph \( G(V, E) \) was introduced by Baˇca et al. as a function \( f \) from the set of vertex union the set of edge to the set of positive integer from \( 1 \) up to \( k \) such that for any two different edges \( xy \) and \( x'y' \) in \( G \) have distinct weights. Let \( xy \) is an edge in \( G \), the weight of \( xy \) denoted by \( \omega_f(xy) \) is defined \( \omega_f(xy) = f(x) + f(y) + f(xy) \). If the graph \( G \) can be labelled with an edge irregular total \( k \)-labelling then the smallest \( k \) is called the total edge irregularity strength of \( G \) and is denoted by \( \text{tes}(G) \). Baˇca et al. also give a lower bound of \( \text{tes}(G) \) which is \( \text{tes}(G) \geq \max\{\lceil\frac{|E|+2}{3}\rceil,\lceil\frac{\Delta(G)+1}{2}\rceil\} \) where \( \Delta(G) \) is the maximum vertex degree of \( G \).

For \( \text{tes} \) of trees, Ivanco and Jendrol [4] have determined it. Meanwhile, research on the \( \text{tes} \) cyclic graphs for various graph classes is still being done. Some results of the investigation of
tes in some cyclic graphs, include some book graphs, have determined and can be seen in [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15].

In [16], an edge irregular total \( k \)-labelling has been constructed on book graphs and double book graphs. However, the formula does not appropriate for the third part of triple book graphs.

In this paper, we construct an edge irregular total \( k \)-labelling which can be used to label book graphs, double book graphs and triple book graphs and determine the total irregularity strength of triple book graphs.

2. Method
The total edge irregularity strength of graph \( G \) (tes\((G)\)) is the smallest \( k \) value such that the graph \( G \) can be labelled with an irregular edge total \( k \)-labelling. To determine the tes triple book graphs, the procedure is as follows:

i. To determine the lower bound tes of triple book graphs, we use the lower bound given by [3] i.e. \( \text{tes}(G) \geq \max\{\lceil \frac{|E|+2}{3}\rceil,\lceil \frac{\Delta(G)+1}{2}\rceil\} \) where \( \Delta(G) \) is the maximum vertex degree of \( G \).

ii. To determine the upper bound tes of triple book graphs, we construc an edge irregular total \( k \)-labeling for triple book graphs with \( k = \max\{\lceil \frac{|E|+2}{3}\rceil,\lceil \frac{\Delta(G)+1}{2}\rceil\} \).

iii. From points (i) and (ii) it is obtained that the tes of triple book graphs is equal to \( \max\{\lceil \frac{|E|+2}{3}\rceil,\lceil \frac{\Delta(G)+1}{2}\rceil\} \).

3. Result and Discussion
3.1. Total Edge Irregularity Strength of Triple Book Graphs
Before we investigate tes of triple book graphs, we give definitions of book graphs and triple book graphs.

Definition 3.1 Let \( C_m^n, i = 1, 2, ..., n \) be cycle graphs with the vertex set \( V(C_m^n) = \{u, v\} \cup \{x_{i,j} : j = 1, ..., m – 2\} \) and the edge set \( E(C_m^n) = \{uw, ux_{i,1}, x_{i,j}x_{i,j+1}, x_{i,m-2}v \} : j = 1, ..., m – 3\}. \)

A book graph with \( m \) sides and \( n \) sheets denoted by \( B_n(C_m) \) is the graph obtained from cycle graphs \( C_m^n \), \( i = 1, ..., n \) by merging edge \( uv \) from each cycle. Thus the vertex set of \( B_n(C_m) \) is \( V(B_n(C_m)) = \{u, v\} \cup \{x_{i,j} : i = 1, ..., n, j = 1, ..., m – 2\} \) and the edge set of \( B_n(C_m) \) is \( E(G) = \{uw\} \cup \{ux_{i,1}, x_{i,j}x_{i,j+1}, x_{i,m-2}v \} : i = 1, ..., n, j = 1, ..., m – 2\} \).

Definition 3.2 Let \( B_n^q(C_m) \), \( 1 < q < 3 \) be the \( q \)-th copy of book graph \( B_n(C_m) \) as defined at the Definition 3.1. Let the vertices of \( B_n^q(C_m) \) be \( V(B_n^q(C_m)) = \{u^q, v^q\} \cup \{x_{i,j}^q : i = 1, ..., n, j = 1, ..., m – 2\} \).

A triple book graph is a graph obtained from three copies of book graph \( B_n^q(C_m) \) by identifying vertex \( x^q \) from book graph \( B_n^q(C_m) \) with vertex \( x^{q+1} \) from book graph \( B_n^{q+1}(C_m) \) and renaming this vertex by \( x^q \), \( 1 < q < 2 \). Thus the vertex set of \( 3B_n(C_m) \) is \( V(3B_n(C_m)) = \{u^1, w^1, w^2, v^3\} \cup \{x_{i,j}^1, x_{i,j}^2 : i = 1, ..., n, j = 1, ..., m – 2\} \cup \{x_{i,j}^2, x_{i,j}^3 : i = 1, ..., n, j = 1, ..., m – 2\} \) and the edge set of \( 3B_n(C_m) \) is \( E(3B_n(C_m)) = \{u^1w^1, w^2, w^2v^3\} \cup \{u^1w^1, x_{i,j}^1x_{i,j+1}, x_{i,m-2}w^1, w^1x_{i,j}^2x_{i,j+1}, x_{i,m-2}w^2, w^2x_{i,j}^3x_{i,j+1}, x_{i,m-2}v^3\} \).

The construction of an edge irregular total \( k \)-labelling for the first book graph, the second book graph and the third book graphs is shown in Theorem 3.3 as below.
Theorem 3.3 Let $3B_n(C_m)$ be a triple book graph. Then $tes(3B_n(C_m)) = \left\lceil \frac{3(3(m-1)n+1)+2}{3} \right\rceil$.

Proof: A triple book graph $3B_n(C_m)$ is obtained from three book graphs $B^1_n(C_m)$, $B^2_n(C_m)$ and $B^3_n(C_m)$ by identifying vertex $v_i^1 \in V(B^1_n(C_m))$ with vertex $v^2 \in V(B^2_n(C_m))$ and vertex $v^3 \in V(B^3_n(C_m))$ with vertex $w_i^3 \in V(B^3_n(C_m))$ hence any triple book graph $3B_n(C_m)$ has maximum degree $\Delta(3B_n(C_m)) = 2n + 2$. From the Definition 2.1 we know that a triple book graph has $m$ sides and $3n$ sheets so that it is obtained $|E(3B_n(C_m))| = (3m - 1)n + 3$. By using the lower bound given by Baća i.e. $tes(G) \geq \max\{|E(G)|, \left\lceil \frac{|E(G)|}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \}$ we obtain $tes(3B_n(C_m)) \geq \left\lceil \frac{3(3(m-1)n+1)+2}{3} \right\rceil$. Meanwhile, the upper bound is shown by constructing an edge irregular total $k_3$-labeling with $k_3 = \left\lceil \frac{3(3(m-1)n+1)+2}{3} \right\rceil$ as below.

Based on Definition 2.2, it is clear that $u^3 = v^3 = w^3$ and $u^3 = v^3 = w^3$.

We label the vertices of triple book graph as below:

\[
\begin{align*}
  f(u^1) &= k_0, \\
  f(v^q) &= k_q, \\
  f(x^q_{i,j}) &= k_q - i + j - 2, & 1 \leq q \leq 3 \\
  f(x^q_{i,j}) &= k_q - i + j - 2, & 1 \leq i \leq n, \quad 1 \leq j \leq y \\
  f(x^q_{i,j}) &= k_q - i + j - 2, & 1 \leq i \leq n, \quad j = y + p, p = 1, 3, \ldots, 2y - 5 \\
  f(x^q_{i,j}) &= k_q - i + j - 2, & 1 \leq i \leq n, \quad j = y + p, p = 2, 4, \ldots, 2y - e \\
  f(x^q_{i,j}) &= k_q, & if \quad m = 0(\text{mod} \ 3), 1 \leq i \leq r_q, \quad j = m - 3 \\
  f(x^q_{i,j}) &= k_q, & if \quad m = 0(\text{mod} \ 3), r_q + 1 \leq i \leq n, \quad j = m - 3 \\
  with k_0 = 1, \quad k_q = \left\lceil \frac{3(3(m-1)n+1)+2}{3} \right\rceil, \quad r_q = k_q - \frac{5-3}{3}n + k_{q-1} and q = 1 for the first book, q = 2 \quad for the second book, q = 3 \quad for the third book, and \\
  (i) a = 3, \quad b = 5, \quad c = 0, \quad d = 2, \quad e = 4, \quad y = \frac{m}{3} for m = 0(\text{mod} \ 3). \\
  (ii) a = 1, \quad b = 5, \quad c = 2, \quad d = 2, \quad e = 6, \quad y = \frac{m+2}{3} for m = 1(\text{mod} \ 3). \\
  (iii) a = 2, \quad b = 6, \quad c = 1, \quad d = 3, \quad e = 6, \quad y = \frac{m+1}{3} for m = 2(\text{mod} \ 3). \\

For the edge labeling $f$ the proof is divided into 3 cases.

1. For $m \equiv 1 \text{ mod } 3$.

The edge labeling $f$ for the first book $B^1_n(C_m)$ is defined as below:

\[
\begin{align*}
  f(u^1w^1) &= k_0 + n, \\
  f(u^1x^1_{i,j}) &= k_0, \\
  f(x^1_{i,j}x^1_{i,j+1}) &= k_0 + n - j + 2, & 1 \leq j \leq \frac{m-1}{3} \\
  f(x^1_{i,j}x^1_{i,j+1}) &= k_0 + \frac{m-1}{3}n - m + j + 4 - i, & \frac{m+2}{3} \leq j \leq m - 4 \\
  f(x^1_{i,j}x^1_{i,j+1}) &= k_0 + \frac{m-4}{3}n + 1, & j = m - 3 \\
  f(x^1_{i,j}w^1) &= k_0 + \frac{m-4}{3}n + i, & j = m - 2 \\

with 1 \leq i \leq n.

By using labelling $f$, we obtained the edge weights as below:
\[ \omega(u^1x^1_{i,j}) = i + 2, \]
\[ \omega(x^1_{i,j}x^1_{i,j+1}) = jn + i + 2, \quad 1 \leq j \leq \frac{m-1}{3} \]
\[ \omega(u^1w^1) = 2 + k_1 + n \]
\[ \omega(x^1_{i,j}x^1_{i,j+1}) = jn + i + 3, \quad \frac{m+2}{3} \leq j \leq m - 4 \]
\[ \omega(x^1_{i,j}x^1_{i,j+1}) = k_1 + 2\frac{m-1}{3}n + i + 2, \quad j = m - 3 \]
\[ \omega(x^1_{i,m-2}w^1) = 2k_1 + \frac{m-1}{3}n + i + 1, \quad 1 \leq i \leq n. \]

The edge labeling \( f \) for the second book \( B^2_n(C_m) \) is defined as below:

\[ f(w^1w^2) = k_1 + n, \]
\[ f(w^1x^1_{i,j}) = k_1 + 1, \]
\[ f(x^2_{i,j}x^1_{i,j+1}) = k_1 + jn - 2j - i + 3, \quad 1 \leq j \leq \frac{m-1}{3} \]
\[ f(x^2_{i,j}x^2_{i,j+1}) = k_1 + \frac{m-1}{3}n - m + j + 5 - i, \quad \frac{m+2}{3} \leq j \leq m - 4 \]
\[ f(x^2_{i,j}x^1_{i,j+1}) = k_1 + \frac{m-4}{3}n + 1, \quad j = m - 3 \]
\[ f(x^1_{i,j}w^2) = k_1 + \frac{m-4}{3}n + i - 1, \quad j = m - 2 \]
with \( 1 \leq i \leq n. \)

By using labeling \( f \), we obtained the edge weights as below:

\[ \omega(w_1x^2_{i,j}) = 3k_1 + i, \]
\[ \omega(x^2_{i,1}x^2_{i,2}) = 3k_1 + jn + i, \quad 1 \leq j \leq \frac{m-1}{3} \]
\[ \omega(w^1w^2) = 2k_1 + n + k_2 \]
\[ \omega(x^1_{i,j}x^1_{i,j+1}) = 3k_1 + jn + i + 1, \quad \frac{m+2}{3} \leq j \leq m - 4 \]
\[ \omega(x^2_{i,j}x^1_{i,j+1}) = 2k_1 + 2\frac{m-4}{3}n + k_2 + i, \quad j = m - 3 \]
\[ \omega(x^2_{i,j}w^2) = k_1 + 2k_2 + \frac{m-4}{3}n + i - 1, \quad j = m - 2 \]
with \( 1 \leq i \leq n. \)

The edge labeling \( f \) for the third book \( B^3_n(C_m) \) is defined as below:

\[ f(w^2w^3) = k_2 + n - 1, \]
\[ f(w^2x^3_{i,j}) = k_2 + 1, \]
\[ f(x^3_{i,j}x^3_{i,j+1}) = k_2 + jn - 2j - i + 1, \quad 1 \leq j \leq \frac{m-1}{3} \]
\[ f(x^3_{i,j}x^3_{i,j+1}) = k_2 + \frac{m-1}{3}n - m + j + 3 - i, \quad \frac{m+2}{3} \leq j \leq m - 4 \]
\[ f(x^3_{i,j}x^3_{i,j+1}) = k_2 + \frac{m-4}{3}n, \quad j = m - 3 \]
\[ f(x^3_{i,j}w^3) = k_2 + \frac{m-4}{3}n + i - 1, \quad j = m - 2 \]
with \( 1 \leq i \leq n. \)

By using labeling \( f \), we obtained the edge weights as below:

\[ \omega(w^2x^3_{i,j}) = 3k_2 + i - 2, \]
\[ \omega(x^3_{i,1}x^3_{i,2}) = 3k_2 + jn + i - 2, \quad 1 \leq j \leq \frac{m-1}{3} \]
\[ \omega(w^2w^3) = 2k_2 + n + k_3 - 1 \]
\[ \omega(x^3_{i,j}x^3_{i,j+1}) = 3k_2 + jn + i - 1, \quad \frac{m+2}{3} \leq j \leq m - 4 \]
\[ \omega(x^3_{i,j}x^3_{i,j+1}) = 2k_2 + 2\frac{m-4}{3}n + k_2 + i - 1, \quad j = m - 3 \]
\[ \omega(x^3_{i,j}w^3) = k_2 + 2k_3 + \frac{m-4}{3}n + i - 1, \quad j = m - 2 \]
with \( 1 \leq i \leq n. \)

We found that the weight of the edges of \( 3B_n(C_m) \) for \( m = 1(mod\ 3) \) by using the labeling \( f \).
form the set \( \{3, 4, ..., 3(m-1)n+5\} \).

2. For \( m \equiv 2 \mod 3 \).

The edge labeling \( f \) for the first book \( B^1_n(C_m) \) is defined as below:

\[
\begin{align*}
    f(u_1, w_1) &= \begin{cases} 
        k_0 + n - \frac{n+1}{3} & \text{if } n = 2(\mod 3) \\
        k_0 + n - \frac{n}{3} & \text{if } n = 0(\mod 3) \\
        k_0 + n - \frac{n+2}{3} & \text{if } n = 1(\mod 3)
    \end{cases} \\
    f(u^1_i, x^1_{i,j}) &= k_0, \quad 1 \leq i \leq n \\
    f(x^1_{i,j}, x^1_{i,j+1}) &= k_0 + jn - 2j - i + 2, \quad 1 \leq j \leq \frac{m-2}{3} \\
    f(x^1_{i,j}, x^1_{i+1,j}) &= k_0 + \frac{m-2}{3}n - m + j + 5 - i, \quad \frac{m+1}{3} \leq j \leq m - 4 \\
    f(x^1_{i,j}, x_{i,j+1}) &= \begin{cases} 
        k_0 + \frac{m-2}{3}n - \frac{n+1}{3} & \text{if } j = m - 3, \quad n = 2(\mod 3) \\
        k_0 + \frac{m-2}{3}n - \frac{n}{3} & \text{if } j = m - 3, \quad n = 0(\mod 3) \\
        k_0 + \frac{m-2}{3}n - \frac{n+2}{3} & \text{if } j = m - 3, \quad n = 1(\mod 3)
    \end{cases} \\
    f(x^1_{i,j}, x^1_{i,j+1}) &= \begin{cases} 
        k_0 + \frac{m-2}{3}n - 2 \left(\frac{n+1}{3}\right) + i & \text{if } j = m - 2, \quad n = 2(\mod 3) \\
        k_0 + \frac{m-2}{3}n - 2 \left(\frac{n}{3}\right) + i & \text{if } j = m - 2, \quad n = 0(\mod 3) \\
        k_0 + \frac{m-2}{3}n - 2 \left(\frac{n+2}{3}\right) + i & \text{if } j = m - 2, \quad n = 1(\mod 3)
    \end{cases}
\]

with \( 1 \leq i \leq n \).

By using labelling \( f \), we obtained the weight of edges as follows:

\[
\begin{align*}
    \omega(u^1 w^1) &= \begin{cases} 
        2k_0 + k_1 + n - \frac{n+1}{3} & \text{if } n = 2(\mod 3) \\
        2k_0 + k_1 + n - \frac{n}{3} & \text{if } n = 0(\mod 3) \\
        2k_0 + k_1 + n - \frac{n+2}{3} & \text{if } n = 1(\mod 3)
    \end{cases} \\
    \omega(u^1_i, x^1_{i,j}) &= 3k_0 + i - 1, \\
    \omega(x^1_{i,j}, x^1_{i,j+1}) &= 3k_0 + jn + i - 1, \quad 1 \leq j \leq \frac{m-2}{3} \\
    \omega(x^1_{i,j}, x^1_{i+1,j}) &= 3k_0 + jn + i, \quad \frac{m+1}{3} \leq j \leq m - 4 \\
    \omega(x^1_{i,j}, x_{i,j+1}) &= \begin{cases} 
        2k_0 + k_1 + i + \frac{(2m-8)n-1}{3} & \text{if } j = m - 3, \quad n = 2(\mod 3) \\
        2k_0 + k_1 + i + \frac{(2m-8)n}{3} & \text{if } j = m - 3, \quad n = 0(\mod 3) \\
        2k_0 + k_1 + i + \frac{(2m-8)n-2}{3} & \text{if } j = m - 3, \quad n = 1(\mod 3)
    \end{cases} \\
    \omega(x^1_{i,m-2}, w^1) &= \begin{cases} 
        2k_1 + k_0 + i + \frac{(m-4)n-2}{3} & \text{if } n = 2(\mod 3) \\
        2k_1 + k_0 + i + \frac{(m-4)n}{3} & \text{if } n = 0(\mod 3) \\
        2k_1 + k_0 + i + \frac{(m-4)n-4}{3} & \text{if } n = 1(\mod 3)
    \end{cases}
\]

with \( 1 \leq i \leq n \).

The edge labeling \( f \) for the second book \( B^2_n(C_m) \) is defined as below:

\[
\begin{align*}
    f(w^1, w^2) &= \begin{cases} 
        k_1 + n - \frac{n+1}{3} & \text{if } n = 2(\mod 3) \\
        k_1 + n - \frac{n}{3} & \text{if } n = 0(\mod 3) \\
        k_1 + n - \frac{n+2}{3} & \text{if } n = 1(\mod 3)
    \end{cases} \\
    f(w^1_i, x^1_{i,1}) &= \begin{cases} 
        k_1 + 1 & \text{if } n = 0(\mod 3) \\
        k_1 - 1 & \text{if } n = 1(\mod 3)
    \end{cases}
\]

\]

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By using labelling $f$, we obtained the edge weights as below:

$$
\omega(w^1w^2) = \begin{cases} 
2k_1 + k_2 + n - \frac{n+1}{3} & \text{if } n = 2(\text{mod} 3) \\
2k_1 + k_2 + n - \frac{n}{3} & \text{if } n = 0(\text{mod} 3) \\
2k_1 + k_2 + n - \frac{n+2}{3} & \text{if } n = 1(\text{mod} 3)
\end{cases}
$$

$$
\omega(w^1x^2_{i,1}) = \begin{cases} 
3k_1 + i - 1 & \text{if } n = 2(\text{mod} 3) \\
3k_1 + i & \text{if } n = 0(\text{mod} 3) \\
3k_1 + i - 2 & \text{if } n = 1(\text{mod} 3)
\end{cases}
$$

$$
\omega(x^2_{i,j}, x^2_{i,j+1}) = \begin{cases} 
3k_1 + jn + i - 1 & \text{if } 1 \leq j \leq \frac{m-2}{3}, \ n = 2(\text{mod} 3) \\
3k_1 + jn + i & \text{if } 1 \leq j \leq \frac{m-2}{3}, \ n = 0(\text{mod} 3) \\
3k_1 + jn + i - 2 & \text{if } 1 \leq j \leq \frac{m-2}{3}, \ n = 1(\text{mod} 3)
\end{cases}
$$

$$
\omega(x^2_{i,j}, x^2_{i,j+1}) = \begin{cases} 
3k_1 + jn + i & \text{if } \frac{m+1}{3} \leq j \leq \frac{m-2}{3}, \ n = 2(\text{mod} 3) \\
3k_1 + jn + i + 1 & \text{if } \frac{m+1}{3} \leq j \leq \frac{m-2}{3}, \ n = 0(\text{mod} 3) \\
3k_1 + jn + i - 1 & \text{if } \frac{m+1}{3} \leq j \leq \frac{m-2}{3}, \ n = 1(\text{mod} 3)
\end{cases}
$$

$$
\omega(x^2_{i,j}, x^2_{i,j+1}) = \begin{cases} 
2k_1 + k_2 + i + \frac{(2m-8)n-1}{3} & \text{if } j = m - 3, \ n = 2(\text{mod} 3) \\
2k_1 + k_2 + i + \frac{(2m-8)n}{3} & \text{if } j = m - 3, \ n = 0(\text{mod} 3) \\
2k_1 + k_2 + i + \frac{(2m-8)n-2}{3} & \text{if } j = m - 3, \ n = 1(\text{mod} 3)
\end{cases}
$$

$$
\omega(x_{i,m-2}, w^2) = \begin{cases} 
2k_2 + k_1 + i + \frac{(m-4)n-2}{3} & \text{if } n = 2(\text{mod} 3) \\
2k_2 + k_1 + i + \frac{(m-4)n}{3} & \text{if } n = 0(\text{mod} 3) \\
2k_2 + k_1 + i + \frac{(m-4)n-4}{3} & \text{if } n = 1(\text{mod} 3)
\end{cases}
$$

with $1 \leq i \leq n$.

The edge labeling $f$ for the third book $B^3_n(C_m)$ is defined in the following way:

$$
f(w^2, w^3) = \begin{cases} 
k_2 + n - \frac{n+1}{3} & \text{if } n = 2(\text{mod} 3) \\
k_2 + n - \frac{n}{3} & \text{if } n = 0(\text{mod} 3) \\
k_2 + n - \frac{n+2}{3} & \text{if } n = 1(\text{mod} 3)
\end{cases}
$$

$$
f(w^2, x^3_{i,1}) = \begin{cases} 
k_2 - 1 & \text{if } n = 0(\text{mod} 3) \\
k_2 + 2 & \text{if } n = 1(\text{mod} 3)
\end{cases}
$$
We found that the weight of the edges of $3B_n(C_m)$ form the set $B$ for $m = 2(\text{mod } 3)$.

By using labelling $f$, we obtained the weight of edges as follows:

- For $m = 2(\text{mod } 3)$:
  
  \[ f(x^3_{i,j}, x^3_{i,j+1}) = \begin{cases} 
  k_2 + jn - 2j - i + 2, & 1 \leq j \leq \frac{m-2}{3}, \ n = 2(\text{mod } 3) \\
  k_2 + jn - 2j - i + 1, & 1 \leq j \leq \frac{m-2}{3}, \ n = 0(\text{mod } 3) \\
  k_2 + jn - 2j - i + 3, & 1 \leq j \leq \frac{m-2}{3}, \ n = 1(\text{mod } 3) 
  \end{cases} \]

- For $m = 0(\text{mod } 3)$:
  
  \[ f(x^3_{i,j}, x^3_{i,j+1}) = \begin{cases} 
  k_2 + \frac{m-2}{3} - m + j + 5 - i & \text{if } \frac{m+1}{3} \leq j \leq m - 4, \ n = 2(\text{mod } 3) \\
  k_2 + \frac{m-2}{3} - m + j + 4 - i & \text{if } \frac{m+1}{3} \leq j \leq m - 4, \ n = 0(\text{mod } 3) \\
  k_2 + \frac{m-2}{3} - m + j + 6 - i & \text{if } \frac{m+1}{3} \leq j \leq m - 4, \ n = 1(\text{mod } 3) 
  \end{cases} \]

- For $m = 1(\text{mod } 3)$:
  
  \[ f(x^3_{i,j}, x^3_{i,j+1}) = \begin{cases} 
  k_2 + \frac{m-2}{3} - \left(\frac{n+1}{3}\right) & \text{if } j = m - 3, \ n = 2(\text{mod } 3) \\
  k_2 + \frac{m-2}{3} - \left(\frac{n+2}{3}\right) + 1 & \text{if } j = m - 3, \ n = 1(\text{mod } 3) 
  \end{cases} \]

where $1 \leq i \leq n$.

We found that the weight of the edges of $3B_n(C_m)$ for $m = 2(\text{mod } 3)$ by using the labeling $f$ form the set \{3, 4, ..., 3(m - 1)n + 5\}.

3. For $m = 0 \text{ mod } 3$.

The edge labeling $f$ for the first book $B^1_n(C_m)$ is defined as below:

\[ f(u^1, w^1) = \begin{cases} 
  k_0 + n - \frac{2n+2}{3} & \text{if } n = 2(\text{mod } 3) \\
  k_0 + n - \frac{2n}{3} & \text{if } n = 0(\text{mod } 3) \\
  k_0 + n - \frac{2n+1}{3} & \text{if } n = 1(\text{mod } 3) 
  \end{cases} \]
By using labelling $f$, we obtained the weight of edges as follows:

$$\omega(u^1 w^1) = \begin{cases} 2k_0 + k_1 + \frac{n+2}{3} & \text{if } 1 \leq i \leq n, \ n = 2(\text{mod } 3) \\ 2k_0 + k_1 + \frac{n}{3} & \text{if } 1 \leq i \leq n, \ n = 0(\text{mod } 3) \\ 2k_0 + k_1 + \frac{n-1}{3} & \text{if } 1 \leq i \leq n, \ n = 1(\text{mod } 3) \end{cases}$$

$$\omega(u^1, x^1_{i,j}) = \begin{cases} 3k_0 + i - 1 & \text{if } 1 \leq i \leq n \end{cases}$$

$$\omega(x^1_{i,j}, x^1_{i,j+1}) = \begin{cases} 3k_0 + j + i - 1 & \text{if } 1 \leq i \leq n, \ 1 \leq j \leq \frac{m-3}{3} \end{cases}$$

$$\omega(x^1_{i,j}, x^1_{i+1,j}) = \begin{cases} 3k_0 + \frac{2m}{3}n + i & \text{if } 1 \leq i \leq n, j = \frac{m}{3} \end{cases}$$

$$\omega(x^1_{i,j}, x^1_{i,j+1}) = \begin{cases} 3k_0 + jn + i & \text{if } 1 \leq i \leq n, \ \frac{m+3}{3} \leq j \leq m - 4 \end{cases}$$

$$\omega(x^1_{i,j}, x^1_{i,j+1}) = \begin{cases} 2k_0 + k_1 + i + \frac{(2m-8)n-2}{3} & \text{if } j = m - 3, \ n = 2(\text{mod } 3) \\ 2k_0 + k_1 + i + \frac{(2m-8)n}{3} & \text{if } j = m - 3, \ n = 0(\text{mod } 3) \\ 2k_0 + k_1 + i + \frac{(2m-8)n-1}{3} & \text{if } j = m - 3, \ n = 1(\text{mod } 3) \end{cases}$$

$$\omega(x^1_{i,j}, x^1_{i,j+1}) = \begin{cases} k_0 + 2k_1 + p + \frac{(m-5)n-2}{3} & \text{if } j = m - 3, \ n = 2(\text{mod } 3) \\ k_0 + 2k_1 + p + \frac{(m-5)n}{3} & \text{if } j = m - 3, \ n = 0(\text{mod } 3) \\ k_0 + 2k_1 + p + \frac{(m-5)n-1}{3} & \text{if } j = m - 3, \ n = 1(\text{mod } 3) \end{cases}$$

$$\omega(x^1_{i,j}, x^1_{i,j+1}) = \begin{cases} k_0 + 2k_1 + i + \frac{(m-4)n-4}{3} & \text{if } 1 \leq i \leq n, \ n = 2(\text{mod } 3) \\ k_0 + 2k_1 + i + \frac{(m-4)n}{3} & \text{if } 1 \leq i \leq n, \ n = 0(\text{mod } 3) \end{cases}$$

The edge labeling $f$ for the second book $B^2_n(C_m)$ is defined as below:
\[f(w^1, w^2) = \begin{cases} k_1 + n - \frac{2n+2}{3} & \text{if } n = 2 \text{ (mod 3)} \\ k_1 + n - \frac{2n}{3} & \text{if } n = 0 \text{ (mod 3)} \\ k_1 + n - \frac{2n+1}{3} & \text{if } n = 1 \text{ (mod 3)} \end{cases}\]

\[f(w^1, x^2_{i,1}) = \begin{cases} k_1 - 1 & \text{if } n = 2 \text{ (mod 3)} \\ k_1 + 1 & \text{if } n = 0 \text{ (mod 3)} \\ k_1 & \text{if } n = 1 \text{ (mod 3)} \end{cases}\]

\[f(x^2_{i,j}, x^2_{i,j+1}) = \begin{cases} k_1 + jn - 2j - i + 1, & 1 \leq j \leq \frac{m-3}{3}, n = 2 \text{ (mod 3)} \\ k_1 + jn - 2j - i + 3, & 1 \leq j \leq \frac{m-3}{3}, n = 0 \text{ (mod 3)} \\ k_1 + jn - 2j - i + 2, & 1 \leq j \leq \frac{m-3}{3}, n = 1 \text{ (mod 3)} \end{cases}\]

\[f(w^1) = \begin{cases} 2k_1 + k_2 + \frac{2n+2}{3} & \text{if } 1 \leq i \leq n, \ n = 2 \text{ (mod 3)} \\ 2k_1 + k_2 + \frac{n+2}{3} & \text{if } 1 \leq i \leq n, \ n = 0 \text{ (mod 3)} \\ 2k_1 + k_2 + \frac{n-1}{3} & \text{if } 1 \leq i \leq n, \ n = 1 \text{ (mod 3)} \end{cases}\]

By using labelling \( f \), the edge weights as follows:

\[\omega(w^1 w^2) = \begin{cases} 2k_1 + k_2 + \frac{n}{3} & \text{if } 1 \leq i \leq n, \ n = 2 \text{ (mod 3)} \\ 2k_1 + k_2 + \frac{n+2}{3} & \text{if } 1 \leq i \leq n, \ n = 0 \text{ (mod 3)} \\ 2k_1 + k_2 + \frac{n-1}{3} & \text{if } 1 \leq i \leq n, \ n = 1 \text{ (mod 3)} \end{cases}\]
\[\omega(w^1, x^2_{i,j}) = \begin{cases} 
3k_1 + i - 2 & \text{if } 1 \leq i \leq n, \quad n = 2(\text{mod } 3) \\
3k_1 + i & \text{if } 1 \leq i \leq n, \quad n = 0(\text{mod } 3) \\
3k_1 + i - 1 & \text{if } 1 \leq i \leq n, \quad n = 1(\text{mod } 3) 
\end{cases} \]

\[\omega(x^2_{i,j}, x^2_{i,j+1}) = \begin{cases} 
3k_1 + jn + i - 2 & \text{if } 1 \leq i \leq n, \quad 1 \leq j \leq \frac{m-3}{3}, \quad n = 2(\text{mod } 3) \\
3k_1 + jn + i & \text{if } 1 \leq i \leq r_1, \quad 1 \leq j \leq \frac{m-3}{3}, \quad n = 0(\text{mod } 3) \\
3k_1 + jn + i - 1 & \text{if } 1 \leq i \leq r_1, \quad 1 \leq j \leq \frac{m-3}{3}, \quad n = 1(\text{mod } 3) 
\end{cases} \]

\[\omega(x^2_{i,j}, x^2_{i,j+1}) = \begin{cases} 
3k_1 + \frac{2m}{n} + i - 1 & \text{if } 1 \leq i \leq n, \quad j = \frac{m}{n}, \quad n = 2(\text{mod } 3) \\
3k_1 + \frac{2m}{n} + i & \text{if } 1 \leq i \leq r_1, \quad j = \frac{m}{n}, \quad n = 0(\text{mod } 3) \\
3k_1 + \frac{2m}{n} + i & \text{if } 1 \leq i \leq r_1, \quad j = \frac{m}{n}, \quad n = 1(\text{mod } 3) 
\end{cases} \]

\[\omega(x^2_{i,j}, x^2_{i,j+1}) = \begin{cases} 
3k_1 + i + jn - 1 & \text{if } 1 \leq i \leq n, \quad \frac{m+3}{3} \leq j \leq m - 4, \quad n = 2(\text{mod } 3) \\
3k_1 + i + jn + 1 & \text{if } 1 \leq i \leq n, \quad \frac{m+3}{3} \leq j \leq m - 4, \quad n = 0(\text{mod } 3) \\
3k_1 + i + jn & \text{if } 1 \leq i \leq n, \quad \frac{m+3}{3} \leq j \leq m - 4, \quad n = 1(\text{mod } 3) 
\end{cases} \]

\[\omega(x^2_{i,j}, x^2_{i,j+1}) = \begin{cases} 
3k_1 + i & \text{if } 1 \leq i \leq n, \quad \frac{m+3}{3} \leq j \leq m - 4 
\end{cases} \]

\[\omega(x^1_{i,m-2}, w^1) = \begin{cases} 
k_1 + 2k_2 + i + \frac{(m-4)n-4}{3} & \text{if } 1 \leq i \leq n, \quad n = 2(\text{mod } 3) \\
k_1 + 2k_2 + i + \frac{(m-4)n-2}{3} & \text{if } 1 \leq i \leq n, \quad n = 0(\text{mod } 3) \\
k_1 + 2k_2 + i + \frac{(m-4)n}{3} & \text{if } 1 \leq i \leq n, \quad n = 1(\text{mod } 3) 
\end{cases} \]

The edge labeling \(f\) for the third book \(B^3_n(C_m)\) is defined in the following way:

\[f(w^2, v^3) = \begin{cases} 
k_2 + n - \frac{2n-1}{3} & \text{if } n = 2(\text{mod } 3) \\
k_2 + n - \frac{2n+3}{3} & \text{if } n = 0(\text{mod } 3) \\
k_2 + n - \frac{2n+1}{3} & \text{if } n = 1(\text{mod } 3) 
\end{cases} \]

\[f(w^2, x^3_{1,1}) = \begin{cases} 
k_2 + 1 & \text{if } n = 2(\text{mod } 3) \\
k_2 - 1 & \text{if } n = 0(\text{mod } 3) \\
k_2 & \text{if } n = 1(\text{mod } 3) 
\end{cases} \]

\[f(x^3_{1,j}, x^3_{i,j+1}) = \begin{cases} 
k_2 + jn - 2j - j + 3, & 1 \leq j \leq \frac{m-3}{3}, \quad n = 2(\text{mod } 3) \\
k_2 + jn - 2j - i + 1, & 1 \leq j \leq \frac{m-3}{3}, \quad n = 0(\text{mod } 3) \\
k_2 + jn - 2j - i + 2, & 1 \leq j \leq \frac{m-3}{3}, \quad n = 1(\text{mod } 3) 
\end{cases} \]

\[f(x^3_{i,j}, x^3_{i,j+1}) = \begin{cases} 
k_2 + \frac{m-3}{3}n - m + j + 6 - i & \text{if } j = \frac{m}{3}, \quad n = 2(\text{mod } 3) \\
k_2 + \frac{m-3}{3}n - m + j + 4 - i & \text{if } j = \frac{m}{3}, \quad n = 0(\text{mod } 3) \\
k_2 + \frac{m-3}{3}n - m + j + 5 - i & \text{if } j = \frac{m}{3}, \quad n = 1(\text{mod } 3) 
\end{cases} \]

\[f(x^3_{i,j}, x^3_{i,j+1}) = \begin{cases} 
k_2 + \frac{m+3}{3}n - m + j + 5 - i & \text{if } \frac{m+3}{3} \leq j \leq m - 5, \quad n = 2(\text{mod } 3) \\
k_2 + \frac{m+3}{3}n - m + j + 3 - i & \text{if } \frac{m+3}{3} \leq j \leq m - 5, \quad n = 0(\text{mod } 3) \\
k_2 + \frac{m+3}{3}n - m + j + 4 - i & \text{if } \frac{m+3}{3} \leq j \leq m - 5, \quad n = 1(\text{mod } 3) 
\end{cases} \]

\[f(x^3_{i,j}, x^3_{i,j+1}) = \begin{cases} 
k_2 + \frac{m+3}{3}n - m + j + 5 - i & \text{if } \frac{m+3}{3} \leq j \leq m - 5, \quad n = 2(\text{mod } 3) \\
k_2 + \frac{m+3}{3}n - m + j + 3 - i & \text{if } \frac{m+3}{3} \leq j \leq m - 5, \quad n = 0(\text{mod } 3) \\
k_2 + \frac{m+3}{3}n - m + j + 4 - i & \text{if } \frac{m+3}{3} \leq j \leq m - 5, \quad n = 1(\text{mod } 3) 
\end{cases} \]
\[ f(x_{i,j}^3, x_{i,j+1}^3) = \begin{cases} 
 2k_2 - k_3 + \frac{m-3}{3}n - i & \text{if } 1 \leq i \leq r_3, \quad j = m - 4, \quad n = 2(\text{mod } 3) \\
 2k_2 - k_3 + \frac{m-3}{3}n - (2n+1) & \text{if } 1 \leq i \leq r_3, \quad j = m - 3, \quad n = 1(\text{mod } 3) \\
 \end{cases} \]

\[ f(x_{i,j}^3, x_{i,j+1}^3) = \begin{cases} 
 2k_2 - k_3 + \frac{m-3}{3}n & \text{if } r_3 + 1 \leq i \leq n, \quad j = m - 4, \quad n = 2(\text{mod } 3) \\
 2k_2 - k_3 + \frac{m-3}{3}n - 2 & \text{if } r_3 + 1 \leq i \leq n, \quad j = m - 4, \quad n = 0(\text{mod } 3) \\
 2k_2 - k_3 + \frac{m-3}{3}n - 1 & \text{if } r_3 + 1 \leq i \leq n, \quad j = m - 4, \quad n = 1(\text{mod } 3) \\
 \end{cases} \]

\[ f(x_{i,j}^3, x_{i,j+1}^3) = \begin{cases} 
 k_2 + \frac{m-3}{3}n - \frac{2n+1}{3} & \text{if } 1 \leq i \leq r_3, \quad j = m - 3, \quad n = 0(\text{mod } 3) \\
 k_2 + \frac{m-3}{3}n - \frac{2n+1}{3} & \text{if } 1 \leq i \leq r_3, \quad j = m - 3, \quad n = 1(\text{mod } 3) \\
 \end{cases} \]

\[ f(x_{i,j}^3, x_{i,j+1}^3) = \begin{cases} 
 k_2 + \frac{m-3}{3}n - \frac{2n+1}{3} & \text{if } r_3 + 1 \leq i \leq n, \quad j = m - 3, \quad n = 0(\text{mod } 3) \\
 k_2 + \frac{m-3}{3}n - \frac{2n+1}{3} & \text{if } r_3 + 1 \leq i \leq n, \quad j = m - 3, \quad n = 1(\text{mod } 3) \\
 k_2 + \frac{m-3}{3}n - \frac{2n+1}{3} & \text{if } r_3 + 1 \leq i \leq n, \quad j = m - 3, \quad n = 2(\text{mod } 3) \\
 \end{cases} \]

\[ f(x_{i,j}^3, x_{i,j+1}^3) = \begin{cases} 
 k_2 + \frac{m-3}{3}n - \frac{2n+1}{3} & \text{if } r_3 + 1 \leq i \leq n, \quad j = m - 3, \quad n = 1(\text{mod } 3) \\
 k_2 + \frac{m-3}{3}n - \frac{2n+1}{3} & \text{if } r_3 + 1 \leq i \leq n, \quad j = m - 3, \quad n = 2(\text{mod } 3) \\
 k_2 + \frac{m-3}{3}n - \frac{2n+1}{3} & \text{if } r_3 + 1 \leq i \leq n, \quad j = m - 3, \quad n = 0(\text{mod } 3) \\
 \end{cases} \]

\[ f(x_{i,m-2}^3) = \begin{cases} 
 k_2 + \frac{m-3}{3}n + 2 \frac{n+4}{3} + i - 1 & \text{if } 1 \leq i \leq n, \quad n = 2(\text{mod } 3) \\
 k_2 + \frac{m-3}{3}n + 2 \frac{n+4}{3} + i - 3 & \text{if } 1 \leq i \leq n, \quad n = 0(\text{mod } 3) \\
 k_2 + \frac{m-3}{3}n + 2 \frac{n+4}{3} + i - 2 & \text{if } 1 \leq i \leq n, \quad n = 1(\text{mod } 3) \\
 \end{cases} \]

By using labelling \( f \), we obtained the weight of edges as follows:

\[ \omega(w^2v^3) = \begin{cases} 
 2k_2 + k_3 + \frac{n+1}{3} & \text{if } 1 \leq i \leq n, \quad n = 2(\text{mod } 3) \\
 2k_2 + k_3 + \frac{n+1}{3} & \text{if } 1 \leq i \leq n, \quad n = 0(\text{mod } 3) \\
 2k_2 + k_3 + \frac{n+1}{3} & \text{if } 1 \leq i \leq n, \quad n = 1(\text{mod } 3) \\
 \end{cases} \]

\[ \omega(w^2, x_{i,j}) = \begin{cases} 
 3k_2 + i & \text{if } 1 \leq i \leq n, \quad n = 2(\text{mod } 3) \\
 3k_2 + i - 2 & \text{if } 1 \leq i \leq n, \quad n = 0(\text{mod } 3) \\
 3k_2 + i - 1 & \text{if } 1 \leq i \leq n, \quad n = 1(\text{mod } 3) \\
 \end{cases} \]

\[ \omega(x_{i,j}^3, x_{i,j+1}^3) = \begin{cases} 
 3k_2 + jn + i & \text{if } 1 \leq i \leq n, \quad 1 \leq j \leq \frac{m-3}{3}, \quad n = 2(\text{mod } 3) \\
 3k_2 + jn + i - 2 & \text{if } 1 \leq i \leq r_1, \quad 1 \leq j \leq \frac{m-3}{3}, \quad n = 0(\text{mod } 3) \\
 3k_2 + jn + i - 1 & \text{if } 1 \leq i \leq r_1, \quad 1 \leq j \leq \frac{m-3}{3}, \quad n = 1(\text{mod } 3) \\
 \end{cases} \]

\[ \omega(x_{i,j}^3, x_{i,j+1}^3) = \begin{cases} 
 3k_2 + \frac{2m}{3} + i + 1 & \text{if } 1 \leq i \leq n, \quad j = \frac{m}{3}, \quad n = 2(\text{mod } 3) \\
 3k_2 + \frac{2m}{3} + i - 1 & \text{if } 1 \leq i \leq r_1, \quad j = \frac{m}{3}, \quad n = 0(\text{mod } 3) \\
 3k_2 + \frac{2m}{3} + i & \text{if } 1 \leq i \leq r_1, \quad j = \frac{m}{3}, \quad n = 1(\text{mod } 3) \\
 \end{cases} \]

\[ \omega(x_{i,j}^3, x_{i,j+1}^3) = \begin{cases} 
 3k_2 + \frac{2m}{3} + i + 1 & \text{if } 1 \leq i \leq n, \quad j = \frac{m}{3}, \quad n = 2(\text{mod } 3) \\
 3k_2 + \frac{2m}{3} + i - 1 & \text{if } 1 \leq i \leq r_1, \quad j = \frac{m}{3}, \quad n = 0(\text{mod } 3) \\
 3k_2 + \frac{2m}{3} + i & \text{if } 1 \leq i \leq r_1, \quad j = \frac{m}{3}, \quad n = 1(\text{mod } 3) \\
 \end{cases} \]

\[ \omega(x_{i,j}^3, x_{i,j+1}^3) = \begin{cases} 
 3k_2 + \frac{2m}{3} + i + 1 & \text{if } 1 \leq i \leq n, \quad j = \frac{m}{3}, \quad n = 2(\text{mod } 3) \\
 3k_2 + \frac{2m}{3} + i - 1 & \text{if } 1 \leq i \leq r_1, \quad j = \frac{m}{3}, \quad n = 0(\text{mod } 3) \\
 3k_2 + \frac{2m}{3} + i & \text{if } 1 \leq i \leq r_1, \quad j = \frac{m}{3}, \quad n = 1(\text{mod } 3) \\
 \end{cases} \]

\[ \omega(x_{i,j}^3, x_{i,j+1}^3) = \begin{cases} 
 3k_2 + \frac{2m}{3} + i + 1 & \text{if } 1 \leq i \leq n, \quad j = \frac{m}{3}, \quad n = 2(\text{mod } 3) \\
 3k_2 + \frac{2m}{3} + i - 1 & \text{if } 1 \leq i \leq r_1, \quad j = \frac{m}{3}, \quad n = 0(\text{mod } 3) \\
 3k_2 + \frac{2m}{3} + i & \text{if } 1 \leq i \leq r_1, \quad j = \frac{m}{3}, \quad n = 1(\text{mod } 3) \\
 \end{cases} \]
We found that the weight of the edges of $3B_n(C_m)$ for $m = 0(\text{mod} 3)$ by using the labeling $f$ form the set $\{3, 4, ..., 3(m - 1)n + 5\}$.

From the vertex labelling $f$ and the edge labelling $f$ which are defined in case 1, case 2, and case 3, it is obtained that the weights of edges form the set of integer from 3 up to $3(m - 1)n + 5$. It shows that the weights of all edges in triple book graph $3B_n(C_m)$ are all different. Therefore $f$ is an edge irregular total $k_3$-labelling with $k_3 = \lceil \frac{3(m-1)n+5}{3} \rceil$. We obtain $\text{tes}(3B_n(C_m)) = \lceil \frac{3(m-1)n+3}{3} \rceil$.

4. Conclusion
We studied the construction of edge irregular total $k$-labeling of triple book graphs $3B_n(C_m)$ and we found that the total edge irregularity strength of triple book graphs $B_n(C_m)$ is equal to $\lceil \frac{3(m-1)n+3}{3} \rceil$.

Using this construction, we found also that the total edge irregularity strength of book graphs $qB_n(C_m)$ is equal to $\lceil \frac{3(q(m-1)n+3)}{4} \rceil$, with $q = 1$ for book graphs, $q = 2$ for double book graphs and $q = 3$ for triple books graphs.

It is interesting to determine the generalized for $q$-tuple book graphs.

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