Attributing sense to some integrals in Regge calculus

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Abstract

Regge calculus minisuperspace action in the connection representation has the form in which each term is linear over some field variable (scale of area-type variable with sign). We are interested in the result of performing integration over connections in the path integral (now usual multiple integral) as function of area tensors even in larger region considered as independent variables. To find this function (or distribution), we compute its moments, i.e. integrals with monomials over area tensors. Calculation proceeds through intermediate appearance of \( \delta \)-functions and integrating them out. Up to a singular part with support on some discrete set of physically unattainable points, the function of interest has finite moments. This function in physical region should therefore exponentially decay at large areas and it really does being restored from moments. This gives for gravity a way of defining such nonabsolutely convergent integral as path integral.

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1. Introduction

Strict definition of the functional integral is possible for Gaussian case; for small deviations from this case it is considered to be definable perturbatively. For general relativity system perturbative expansion is poorly defined due to nonrenormalizability of gravity, and we have non-Gaussian path integral. The action is essentially nonlinear, but in the Cartan-Weyl form, in terms of tetrad and connections, the action can be viewed as linear in some field variable which is bilinear in the tetrad. This issuing feature of gravity inherent in some form also in minisuperspace formulations is important for what follows.

Since we do not possess exact definition of non-Gaussian functional integral, we need its finite dimensional realization on minisuperspace system. Piecewise flat manifold or simplicial complex provides such framework known as Regge calculus\textsuperscript{13}. Invoking the notion of discrete tetrad and connection first considered in Ref. \textsuperscript{5} we have suggested in Ref. \textsuperscript{10} representation of the minisuperspace Regge action in terms of area tensors and finite rotation $\text{SO}(4)$ ($\text{SO}(3,1)$ in the Minkowsky case) matrices, and also in terms of (anti-)selfdual parts of finite rotation matrices. For the latter we write

$$
\pm S = \sum_{\sigma^2} \sqrt{\pm v_{\sigma^2}^2} \arcsin \frac{\pm v_{\sigma^2} \ast \pm R_{\sigma^2}(\Omega)}{\sqrt{\pm v_{\sigma^2}^2}}.
$$

(1)

Here $\pm v_{\sigma^2}$ are vectors parameterizing (anti-)selfdual parts $\pm v_{\sigma^2}^{ab}$ of the bivector $v_{\sigma^2}^{ab}$ of the triangle $\sigma^2$ ($v^{ab} = \frac{1}{2} \epsilon^{abcd} l_1^c l_2^d$ for some two 4-vectors $l_1^c, l_2^d$ which span the triangle), $\sqrt{\pm v_{\sigma^2}^2}$ is area of the triangle, in the Minkowsky case $\Omega_{\sigma^3}$ is rotation $\text{SO}(3,1)$ matrix on the tetrahedron $\sigma^3$ which we call simply connection, $R_{\sigma^2}$ is curvature matrix on the triangle $\sigma^2$ (holonomy of $\Omega$’s). For a 3-vector $v$ and a $3 \times 3$ matrix $R$ we have denoted $v \ast R \equiv \frac{1}{2} v^a R^{bc} \epsilon_{abc}$, and for $\pm R_{\sigma^2}$, the (anti-)selfdual part of $R_{\sigma^2}$, we have used adjoint, $\text{SO}(3)$ representation (to be precise, $\text{SO}(3,C)$ matrix).

The sense of the considered representations is that upon excluding rotation matrices by classical equations of motion these result in the same Regge action (that is, on-shell). Taking into account that in the Minkowsky case $+S = (-S)^*$ we can write out the most general combination of $+S$, $-S$ which i) reduces to Regge action on-shell and ii) is real, as $S = C + S + C^* - S$ where $C + C^* = 1$, that is $C = 1/2 + i/2$ (real parameter). At the same time, in the continuum theory the Holst action which generalizes the Cartan-Weyl form of the Einstein action\textsuperscript{8} is easily seen to have the form $(1 + i/\gamma) + S_{\text{cont}} + (1 - i/\gamma) - S_{\text{cont}}$ where $\pm S_{\text{cont}}$ are (anti-)selfdual parts of the Cartan-Weyl continuum action, $\gamma$ is known
as Barbero-Immirzi parameter \(^{11,12}\). Therefore we can write \(C = (1 + i/\gamma)/2\) where the discrete analog of \(\gamma\) is denoted by the same letter. We assume \(0 < \gamma < \infty\).

Consider such the action \(S\) and discretized functional integral \(\int \exp(iS)Dq\), \(q\) are field variables (some factors of the type of Jacobians could also be present). Functional integral approach in Regge calculus was earlier developed, see, e. g., Refs. \(^{5,6,7}\).

Suppose we have performed integration over rotation matrices and are interested in the dependence of the intermediate result on area tensors. Of course, different area tensors are not independent, but nothing prevent us from studying analytical properties in the extended region of varying these area tensors as if these were independent variables. Namely, consider integral

\[
N = \int \exp \left( \frac{i}{2} \sum_{\sigma^2} \left[ \left(1 + \frac{i}{\gamma}\right) \sqrt{v_{\sigma^2}} \arcsin \frac{v_{\sigma^2} \ast R_{\sigma^2}(\Omega)}{\sqrt{v_{\sigma^2}}^2} \right. \right.
\]

\[
\left. \left. + \left(1 - \frac{i}{\gamma}\right) \sqrt{-v_{\sigma^2}} \arcsin \frac{-v_{\sigma^2} \ast -R_{\sigma^2}(\Omega)}{\sqrt{-v_{\sigma^2}}^2} \right] \prod_{\sigma^3} \mathcal{D}\Omega_{\sigma^3}. \right) \tag{2}
\]

Matrices \(\pm \Omega, \pm R\) can be parameterized by complex vector angles \(\pm \phi = \varphi \mp i\psi\) (rotation by the angle \(\sqrt{\pm \phi^2}\) around the unit vector \(\pm \phi/\sqrt{\pm \phi^2}\)).

We regard (2) as function of arbitrary \(\pm v, -v = (\pm v)^*\) which we redenote as \(v, v^*\) in the main body of the paper. To be specific, we study the following integrals,

\[
N_{\gamma\delta}(v, v^*) \equiv \int \exp \left[ \frac{i}{2} v h(nr) + \frac{i}{2} v^* h(nr)^* \right] r_{c_1}...r_{c_\lambda} (r_{d_1}...r_{d_\mu})^* \mathcal{D}R. \tag{3}
\]

Here \(\gamma = (c_1...c_\lambda), \delta = (d_1...d_\mu)\) are multiindices; the dot on an index has the only sense that corresponding vector component enters complex conjugated. The \(h(z)\) is analytical at \(z = 0\) odd function \(h(z) = -h(-z)\). Principal value \(\arcsin z\) or simply \(z\) are examples of \(h(z)\). Besides that, \(v = \sqrt{\nu^2}, n = v/v, n^2 = 1, r_a = \epsilon_{abc} \pm R^{bc}/2 = \phi_a(\sin \phi)/\phi, \phi = \sqrt{\phi^2}, \phi = \varphi - i\psi\),

\[
\mathcal{D}R = \left(\frac{1}{\sqrt{1 - r^2}} - 1 \right) \left(\frac{1}{\sqrt{1 - r^*2}} - 1 \right) \frac{d^3r d^3r^*}{(8\pi^2)^2 r^2 r^*2}. \tag{4}
\]

Here \(d^3r d^3r^* \equiv d^3d^3\text{Re} r d^3\text{Im} r \equiv d^3d^6r^*\). The monomial \(r_{c_1}...r_{c_\lambda} (r_{d_1}...r_{d_\mu})^*\) originates as a term in Taylor expansion of possible dependence on \(R\) of the factors provided by \(R_{\sigma^2}\) in other triangles due to the Bianchi identities. As usual, functional integral is not absolutely convergent. The additional complications could be connected with growth (exponential) of the Haar measure on Lorentz boosts. Then the result of integration over the latter might be defined as a generalized function, or distribution, rather than...
an ordinary function. For example, the following integral diverges but could be defined as distribution,

\[ \int \exp(iwr) r^n dr = 2\pi(-i)^n \delta^{(n)}(v), \]  

\( (5) \)

namely, as Fourier transform of \( r^n \) treated as another distribution. Then it is appropriate to study instead of equation \( (5) \) the result of integrating both parts of it with suitable probe functions. For the latter we chose those ones for which corresponding integrals could be easily defined. Let us issue from the integral of \( N_{\gamma \delta} \) with powers of \( v, v^* \),

\[ \mathcal{M}^{\alpha \beta}_{\gamma \delta}(l, m) = \int N_{\gamma \delta}(v, v^*)(v^2) ^l (v^{*2}) ^m v^{a_1} ... v^{a_j} (v^{b_1} ... v^{b_k}) ^* d^6 v, \]

\( (6) \)

and change overall integration order: first integrate over \( d^6 v \), then over \( d^6 r \). Here \( d^6 v \equiv d^3 \text{Re} v d^3 \text{Im} v, \) etc. The \( \alpha, \beta \) are multiindices. The only sense of distinguishing between superscripts and subscripts is that the former refer to \( v, v^* \), the latter refer to \( r, r^* \). Call \( (6) \) the moment of \( N_{\gamma \delta} \) (specified by \( \alpha, \beta, l, m \)).

Note that the case \( h(z) \propto z \) can be of significance as well as arcsin \( z \), and probably even more related to canonical approach to constructing the functional integral measure. Namely, we could ask whether form of the full discrete path integral (i.e. simply many-fold integral) exists which results in the canonical integral form \( \int \exp(iS(p, q)) \prod_t dp(t)dq(t) \) in the continuous time limit when we shrink the edges along any direction chosen as time \( t \) and pass to the canonical formalism with conjugate pairs \( p, q \). More generally, gravity action is of the form \( S(p, q, \lambda) \) with non-dynamical \( \lambda \). Then standard path integral derivation gives the form \( \int \exp(iS(p, q, \lambda)) \prod_t dp(t)dq(t)d\lambda(t) \) on condition that \( S \) is linear in \( \lambda, S = \int (p\dot{q} - \sum_\alpha \lambda_\alpha \Phi_\alpha(p, q)) dt, \) and \( \Phi_\alpha(p, q) \) mutually commute w.r.t. Poisson brackets. The latter just takes place in the 3 dimensional case, in accordance with Waelbroeck’s derivation of the commuting constraints in general discrete 3-dimensional gravity system\(^[15]\). This allows us to define the discrete path integral form of interest\(^[11]\). A particular point in this derivation is that both genuine Regge action \( S \) and that one \( \tilde{S} \) differing from \( S \) by omitting the ‘arcsin’ functions (i.e. by replacing \( h(z) \propto \arcsin z \rightarrow z \)) are equivalent on-shell due to the local triviality of the 3 dimensional gravity. Of these namely \( \tilde{S} \) in the continuous time limit has the above form \( S(p, q, \lambda) \) linear in \( \lambda \) and therefore just appears in the exponential.

In the reminder of the present paper we define the moments of the integrals over connections of interest, show that support for the singular distributional part of these
integrals restored from moments lays outside the physical region \( \text{Im} v^2 = 0 \), separate out the regular part and present it for the simplest integral.

2. Defining moments of the path integral distribution

At \( h(z) \propto z \) when calculating \( \mathcal{M}^{\alpha \beta}_{\gamma \delta}(l, m) \) we get derivatives of \( \delta \)-functions \( \delta^\beta(r) \equiv \delta^3(\text{Re } r) \cdot \delta^3(\text{Im } r) \) which are then integrated over \( DR \). Finiteness is provided by analyticity of this measure at \( r, r^* \to 0 \) w.r.t. \( r, r^* \) viewed as independent complex variables, \( DR = |c_0 + c_1 r^2 + c_2 (r^2)^2 + \ldots|^2 d^3 r d^3 r^* \).

In general case \( h(z) \neq \text{const} \cdot z \) integral also can be defined. Again, consideration goes through intermediate appearance of \( \delta \)-functions. For that we make use of special structure of the exponential in (6) and temporarily pass to components of \( v \) which remind spherical ones, but are modified for complex case,

\[
v = vn, \quad v = u + iw, \quad n = e_1 \chi \rho + i e_2 s h \rho, \quad e_1^2 = 1 = e_2^2, \quad e_1 e_2 = 0. \tag{7}
\]

The orthogonal pair \( e_1, e_2 \) is specified by three angles, e. g. by azimuthal \( \theta_1 \) and polar \( \varphi_1 \) angles of \( e_1 \) and polar angle \( \varphi_2 \) of \( e_2 \) (in the plane orthogonal to \( e_1 \)). The integration measure in the coordinates \( u, w, \rho, \theta_1, \varphi_1, \varphi_2 \)

\[
d^6 v \equiv d^3 \text{Re } v d^3 \text{Im } v = (u^2 + w^2) dudw d^4 n, \quad d^4 n = ch \rho sh \rho d \rho \sin \theta_1 d \theta_1 d \varphi_1 d \varphi_2. \tag{8}
\]

Unlike the Euclidean case, \( n \) varies in the noncompact region. Whenever this might violate convergence of some intermediate integrals over \( d^4 n \) below, we could imply some intermediate regularization being applied to these, e. g. \( |\rho| < \Lambda \) at some large but finite \( \Lambda \). The \( u, w \) are defined via \( (u + iw)^2 = v^2 \), i. e. region of variation for \( u + iw \) is a half of the complex plane. For example, for the standard choice of the cut for square root function \( u \geq 0 \). However, integration over \( du \) in (6) can be extended to the full real axis \( (-\infty, +\infty) \). This is only possible because formal putting \( u \to -u, w \to -w \) is equivalent to \( e_1 \to -e_1, e_2 \to -e_2 \) in (6) due to the oddness of \( h(z) \). Such identity of integration points leads to \( \delta \)-functions of \( h \),

\[
\int \exp \left[ \frac{i}{2} v h(nr) + \frac{i}{2} v^* h(nr)^* \right] (v^2)^l (v^*)^m \chi^{a_1} \ldots \chi^{a_j} (v^{b_1} \ldots v^{b_k})^* d^6 v
\]

\[
= \frac{1}{2} \int n^{a_1} \ldots n^{a_j} (n^{b_1} \ldots n^{b_k})^* d^4 n \int du \int dwe^{i [u f(nr) + w g(nr)]} (u + iw)^{2l + 2} (u - iw)^{k + 2m + 2}
\]

\[
= \frac{1}{2} \int n^{a_1} \ldots n^{a_j} (n^{b_1} \ldots n^{b_k})^* d^4 n (2\pi)^2 \int \frac{\partial}{i \partial f} + \frac{\partial}{i \partial g} \delta^{k + 2m + 2} \delta^2(h) \tag{9}
\]
Derivatives of $\delta^2(h)$ expand into combinations of the derivatives of $\delta^2(z) \equiv \delta(x)\delta(y)$, $z \equiv nr = x - iy$. These combinations can be found by applying to probe functions $\varphi(z, z^*)$,

$$
\int \left[ \left( \frac{d}{dh} \right)^{j+2l+2} \left( \frac{d}{dh^*} \right)^{k+2m+2} \delta^2(h) \right] \varphi(z, z^*) d^2z = (-1)^{j+k} \left( \frac{d}{dh} \right)^{j+2l+2} \left( \frac{d}{dh^*} \right)^{k+2m+2} \frac{\varphi(z(h), z(h)^*) dz}{dh dh^*} \bigg|_{h, h^* = 0}, \; d^2z = dx dy. \tag{11}
$$

Let us choose

$$
\varphi(z, z^*) = \frac{\varphi^{(n,p)}(0, 0)}{n!p!} z^n \bar{z}^p,
$$

thus we find coefficient of $\delta^{(n)}(z)\delta^{(p)}(z^*) \equiv (d/dz)^n(d/dz^*)^p \delta(z)\delta(z^*)$ (here $2\delta(z)\delta(z^*) \equiv \delta^2(z)$) in $\delta^{(j+2l+2)}(h)\delta^{(k+2m+2)}(h^*)$,

$$
\delta^{(j+2l+2)}(h)\delta^{(k+2m+2)}(h^*) = ... + (-1)^{j+n+k+p} \delta^{(n)}(z)\delta^{(p)}(z^*)
\cdot \left( \frac{d}{dh} \right)^{j+2l+3} \left( \frac{d}{dh^*} \right)^{k+2m+3} \frac{z(h)^{n+1} \bar{z}(h)^{p+1}}{(n+1)! (p+1)!} \bigg|_{h, h^* = 0} + ... \tag{13}
$$

(This term is nonzero at $(j - n)(\text{mod}2) = 0$, $(k - p)(\text{mod}2) = 0$, $j + 2l + 2 \geq n$, $k + 2m + 2 \geq p$.) Let us apply formula [9] read from right to left to appearing here $\delta^{(n)}(z)\delta^{(p)}(z^*)$, now for $h(z) = z$,

$$
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} du \, dw e^{i[w \text{Re}(nr) - w \text{Im}(nr)]} (u + iw)^n (u - iw)^p = \int_{-\infty}^{+\infty} \exp \left( \frac{i}{2} v^* r^* + \frac{i}{2} v^* r \right) v^{n-j} \bar{v}^{p-k} v^{a_1} \bar{v}^{b_1} ... v^{a_j} \bar{v}^{b_j} \cdot \delta^2 v \bigg|_{v^2 = v^{*2}}. \tag{14}
$$

Substitute (13) and (14) to (9) and integrate (9) over $r_{c_1}...r_{c_{\alpha}}(r_{d_1}...r_{d_{\mu}})^* DR$. We get for the moment

$$
\mathcal{M}^{\delta^2}_{\gamma} \langle l, m \rangle = \int r_{c_1}...r_{c_{\alpha}}(r_{d_1}...r_{d_{\mu}})^* DR \int \exp \left[ \frac{i}{2} vh(nr) + \frac{i}{2} v^* h(nr)^* \right]
\cdot (v^2)^n v^{a_1} ... v^{a_{\alpha}} (v^{b_1} ... v^{b_{\mu}})^* d^6 v = \int r_{c_1}...r_{c_{\alpha}}(r_{d_1}...r_{d_{\mu}})^* DR
\cdot (-1)^{j+k} \left( \frac{2}{i} \frac{d}{dh} \right)^{j+2l+2} \left( \frac{2}{i} \frac{d}{dh^*} \right)^{k+2m+2} \frac{dz}{dh} \frac{dz^*}{dh^*} \int \exp \left( \frac{i}{2} vr + \frac{i}{2} v^* r^* \right)
\cdot v^{a_1} ... v^{a_{\alpha}} (v^{b_1} ... v^{b_{\mu}})^* d^6 v \bigg\{ ... + \frac{[vz/(2i)]^n [v^* z^*/(2i)]^p}{n! p!} + ... \bigg\} \bigg|_{h, h^* = 0}. \tag{15}
$$
We can extend summation over \( n, p \) to that over infinite set of nonnegative integers of which only finite number of terms at the given finite \( j, k, l, m \) (pointed out after formula (18)) are active. Thus we have

\[
\mathcal{M}^\alpha_\gamma (l, m) = \left( 2i \frac{d}{dh} \right)^{j+2l+2} \left( 2i \frac{d}{dh^*} \right)^{k+2m+2} \left[ \frac{dz}{dh} \frac{dz^*}{dh^*} T^\alpha_\gamma (z, z^*) \right]_{h, h^* = 0}
\]  

where

\[
T^\alpha_\gamma (z, z^*) = \int r_c ... r_c (r_{d_1} ... r_{d_j})^* DR \int \exp \left( \frac{i}{2} v r + \frac{i}{2} v^* r^* \right) \frac{v^{a_1} ... v^{a_j} (v^{b_1} ... v^{b_k})^*}{v^j v^* k} \frac{1}{2} \exp \frac{i}{2} \left( v^* z + \frac{(-1)^k}{2} \exp \frac{i}{2} v^* z \right) \frac{d^6 v}{v^2 v^* 2}
\]

is "generating function".

3. Factorization into (anti-)selfdual parts

At \( h(z) \propto z \) or \( \text{arcsin} \, z \) such factorization for \( \mathcal{N}_\gamma \) defined from moments (16) can be proven. (Although the case \( h(z) \propto z \) could be treated in more simple way via intermediate appearance of vector \( \delta \)-functions \( \delta^3(\text{Re} \, r)\delta^3(\text{Im} \, r) \) as mentioned in the beginning of paragraph 2.)

Consider the terms like \( z^n C_n(z^*) \) or \( C_n(z) z^{*n} \) in \( T^\alpha_\gamma (z, z^*) \) with holomorphic \( C_n(z) \).

**Lemma 1.** Adding \( z^n C_n(z^*) \) or \( C_n(z) z^{*n} \) at a nonnegative integer \( n \) with \( C_n(z) \) holomorphic at \( z = 0 \) to \( T^\alpha_\gamma (z, z^*) \) does not contribute to \( \mathcal{N}_\gamma (v, v^*) \) at \( h(z) \propto \text{arcsin} \, z \) in the region with points \( v^2 = 4\tilde{n}^2 (1 + i/\gamma)^{-2} \), \( \tilde{n} = n + 1, n - 1, ..., n(\text{mod}2) + 1 \) excluded.

**Proof.** Consider adding the term \( z^n C_n(z^*) \). The dependence of contribution to \( \mathcal{M}^\alpha_\gamma (l, m) \) on \( l \) decouples as

\[
\left( 2i \frac{d}{dh} \right)^{j+2l+2} \left( \frac{dz}{dh} \right)^n \bigg|_{h=0} = \left( 2i \frac{d}{dh} \right)^{j+2l+3} \frac{z^{n+1}}{2i(n+1)} \bigg|_{h=0}
\]

At \( z = \sin \frac{h}{1+i/\gamma} \) the power \( z^{n+1} \) contains harmonics \( \sin \frac{n h}{1+i/\gamma} \) or \( \cos \frac{n h}{1+i/\gamma} \), \( \tilde{n} = n + 1, n - 1, ..., n(\text{mod2}) + 1 \), for even or odd \( n \), respectively. Contribution to the moment from harmonic \( \sin \frac{n h}{1+i/\gamma} \) or \( \cos \frac{n h}{1+i/\gamma} \) is proportional to \( \frac{1}{2i(n+1)} \left( 2i \frac{\tilde{n} \, d}{1+i/\gamma} \right)^{j+2l+3} (\sin h) \) or \( \frac{1}{2i(n+1)} \left( 2\tilde{n} \frac{d}{1+i/\gamma} \right)^{j+2l+3} (\cos h) \), respectively. Nonzero contribution to the moment follows for even or odd \( j \), respectively, and the dependence on \( l \) is proportional to \( [2\tilde{n}/(1 + i/\gamma)]^{2l} \). This corresponds to the singular term in the functional

\[
\mathcal{M}^\alpha_\gamma (f(v^2) g(v^2)^*) = \mathcal{N}_\gamma (v, v^*) f(v^2) g(v^2)^* a_{i1} ... a_{ij} v^{b_1} ... v^{b_k} \]

(19)
This generating function defines moments of some \( \tilde{N}_{\gamma\delta}(v, v^*) \) proportional to \( f(4\tilde{n}^2(1 + i/\gamma)^{-2}) \), that is, to probe function \( f(v^2) \) taken at the point \( 4\tilde{n}^2(1 + i/\gamma)^{-2} \). We may define set of probe (anti-)holomorphic functions \( f(v^2), g(v^2)^* \) to vanish at \( v^2 = 4\tilde{n}^2(1 + i/\gamma)^{-2}, \tilde{n} = n + 1, n - 1, ..., n(\text{mod} 2) + 1 \) for a given finite \( n \). These functions probe \( \mathcal{N}_{\gamma\delta}(v, v^*) \) in the region with points \( v^2 = 4\tilde{n}^2(1 + i/\gamma)^{-2}, \tilde{n} = n + 1, n - 1, ..., n(\text{mod} 2) + 1 \) excluded. The functional \( \mathcal{M}_{\gamma\delta}^{\alpha\beta} \) defined on these functions does not change upon adding the terms \( z^n C_n(z^*) \) and \( C_n(z) z^{*n} \) to \( \mathcal{I}_{\gamma\delta}^{\alpha\beta} \), and \( \mathcal{N}_{\gamma\delta}(v, v^*) \) recovered from \( \mathcal{M}_{\gamma\delta}^{\alpha\beta} \) defined on these functions, i. e. in the region with points \( v^2 = 4\tilde{n}^2(1 + i/\gamma)^{-2}, \tilde{n} = n + 1, n - 1, ..., n(\text{mod} 2) + 1 \) excluded, does not change too.

Now we would like to prove factorization of \( \mathcal{N}_{\gamma\delta}(v, v^*) \) at \( h(z) \propto z \) or \( \arcsin z \) into holomorphic and antiholomorphic parts in the region with the above nonphysical points (at \( h(z) \propto \arcsin z \)) excluded. To find \( \mathcal{N}_{\gamma\delta}(v, v^*) \) with certain multiindices \( \gamma, \delta \) it is sufficient to know \( \mathcal{M}_{\gamma\delta}^{\alpha\beta}(l, m) \) at certain \( \alpha, \beta \) with certain lengths of these multiindices \( j, k \) (normally the same as lengths \( \lambda, \mu \) of \( \gamma, \delta \), respectively).

**LEMMA 2.** The \( \mathcal{N}_{\gamma\delta}(v, v^*) \) at \( h(z) \propto z \) or \( \arcsin z \) if recovered from moments factorizes into holomorphic and antiholomorphic parts in the region of its definition and with nonphysical points (at \( h(z) \propto \arcsin z \)) \( v^2 = 4\tilde{j}^2(1 + i/\gamma)^{-2}, \tilde{j} = j + 1, j - 1, ..., j(\text{mod} 2) + 1 \) and \( v^2 = 4\tilde{k}^2(1 + i/\gamma)^{-2}, \tilde{k} = k + 1, k - 1, ..., k(\text{mod} 2) + 1 \) excluded.

**Proof.** Let us subtract from \( \exp(\pm ivz/2) \) and from \( \exp(\pm iv^*z^*/2) \) in square brackets in the formula (17) for \( \mathcal{I}_{\gamma\delta}^{\alpha\beta}(z, z^*) \) the first up to \( \propto (vz)^j \) inclusive and the first up to \( \propto (v^*z^*)^k \) inclusive terms of the Taylor expansions of these functions over \( z \) and over \( z^* \), respectively. At \( h(z) \propto \arcsin z \) use LEMMA 1. At \( h(z) \propto z \) note that, e. g., the term \( z^n C_n(z^*) \) in \( \mathcal{I}_{\gamma\delta}^{\alpha\beta} \) at \( n \leq j \) does not contribute to the moments (13) vanishes. The \( \mathcal{I}_{\gamma\delta}^{\alpha\beta}(z, z^*) \) is replaced by

\[
\tilde{\mathcal{I}}_{\gamma\delta}^{\alpha\beta}(z, z^*) = \int r_{c_1}...r_{c_n}(r_{d_1}...r_{d_n})^{*} \mathcal{D}R \left[ \exp \left( \frac{i}{2} vr + \frac{i}{2} v^*r^* \right) \right] v^{a_1}...v^{a_j} (v^{b_1}...v^{b_k})^* \]

\[
\cdot \left[ \frac{1}{2} \exp \frac{vz}{2i} + \frac{(-1)^j}{2} \exp \frac{i vz}{2} - \sum_{n=0}^{[j/2]} \frac{1}{(j - 2n)!} \left( \frac{vz}{2i} \right)^{j - 2n} \cdot \right]
\cdot \left[ \frac{1}{2} \exp \frac{v^*z^*}{2i} + \frac{(-1)^k}{2} \exp \frac{i v^*z^*}{2} - \sum_{p=0}^{[k/2]} \frac{1}{(k - 2p)!} \left( \frac{v^*z^*}{2i} \right)^{k - 2p} \cdot \right] \frac{d^6v}{v^2v^{*2}}. \tag{20}
\]

This generating function defines moments of some \( \tilde{N}_{\gamma\delta}(v, v^*) \) which coincides with \( \mathcal{N}_{\gamma\delta}(v, v^*) \) in the region with nonphysical points (at \( h(z) \propto \arcsin z \)) \( v^2 = 4\tilde{j}^2(1 + i/\gamma)^{-2}, \tilde{j} = j + 1, j - 1, ..., j(\text{mod} 2) + 1 \) and \( v^2 = 4\tilde{k}^2(1 - i/\gamma)^{-2} \) or \( v^2 = 4\tilde{k}^2(1 + i/\gamma)^{-2} \).
\[ \hat{k} = k + 1, k - 1, \ldots, k(\text{mod}2) + 1 \text{ excluded. Expansion over } z, z^* \text{ gives nonnegative powers of } v^2, v^*2, \]

\[
\hat{I}_{\gamma \delta}^{\alpha \beta}(z, z^*) = \left( \frac{z}{2i} \right)^{j+2} \left( \frac{z^*}{2i} \right)^{k+2} \int r_{c_1} \ldots r_{c_3} (r_{d_1} \ldots r_{d_4}) \, dR \int \exp \left( \frac{i}{2} \mathbf{v} \mathbf{r} + \frac{i}{2} v^* \mathbf{r}^* \right)
\]

\[
\cdot \left[ \sum_{n=0}^{\infty} \left( \frac{vz}{2i} \right)^{2n} \right] \left[ \sum_{p=0}^{\infty} \left( \frac{v^* z^*}{2i} \right)^{2p} \right] \nu^{a_1} \ldots v^{a_j} (v^{b_1} \ldots v^{b_k}) \, d^3 \mathbf{v}.
\]

Upon separating real and imaginary parts \( \mathbf{v} = \mathbf{u} + i \mathbf{w}, \mathbf{r} = \mathbf{s} - i \mathbf{q} \) each term transforms through intermediate appearance of \( \delta \)-functions,

\[
\int r_{c_1} \ldots r_{c_3} (r_{d_1} \ldots r_{d_4}) \, dR \int \exp \left( \frac{i}{2} \mathbf{u} \mathbf{s} + \frac{i}{2} \mathbf{v} \mathbf{q} \right) [(\mathbf{u} + i \mathbf{w})^2]^{n} [(\mathbf{u} - i \mathbf{w})^2]^{p}
\]

\[
\cdot (u^{a_1} + i w^{a_1})(u^{a_j} + i w^{a_j})(u^{b_1} - i w^{b_1})(u^{b_k} - i w^{b_k}) d^3 \mathbf{u} d^3 \mathbf{w}
\]

\[
= \int r_{c_1} \ldots r_{c_3} (r_{d_1} \ldots r_{d_4}) \, dR \ (2\pi)^6 \left[ \left( \frac{\partial}{i \partial s} + \frac{\partial}{\partial q} \right) \right]^2 \left[ \left( \frac{\partial}{i \partial s} - \frac{\partial}{\partial q} \right) \right]^2 \delta^3(s) \delta^3(q)
\]

\[
= 8\pi^2 \left\{ (2i)^{j+2n} \frac{\partial}{\partial r_{a_1}} \ldots \frac{\partial}{\partial r_{a_j}} \left[ \left( \frac{\partial}{\partial \mathbf{r}} \right)^2 \right]^n \frac{r_{c_1} \ldots r_{c_3}}{r^2} \left( \frac{1}{\sqrt{1 - r^2}} - 1 \right) \right\}
\]

Here

\[
\frac{\partial}{i \partial s_a} + \frac{\partial}{\partial q_a} \equiv \frac{\partial}{i \partial r_{a}} \quad \text{and} \quad \frac{\partial}{i \partial s_a} \equiv \frac{\partial}{i \partial r_{a}^*}
\]

and

\[
\frac{\partial}{\partial r_{a}} \left( \frac{1}{\sqrt{1 - r^2}} - 1 \right) = 0, \quad \frac{\partial}{\partial r_{a}^*} \left( \frac{1}{\sqrt{1 - r^{*2}}} - 1 \right) = 0
\]

due to analyticity (Cauchy-Riemann conditions). This is key point for the factorization to occur. Complex dummy variables \( \mathbf{r}, \mathbf{r}^* \) in the RHS of (22) can equally be viewed as real independent variables, the result being the same. This looks as possibility to replace integration over SO(3,1) by integration over SO(4). Eventually we trace back to (20) where now \( \mathbf{v}, \mathbf{r}, z \) on one hand and \( \mathbf{v}^*, \mathbf{r}^*, z^* \) on another hand can be taken as independent real variables. Then \( z, z^* \) on which the result \( \hat{I}_{\gamma \delta}^{\alpha \beta} \) depends can be continued to the desired region. Therefore \( 2^3 \hat{I}_{\gamma \delta}^{\alpha \beta}(z, z^*) = \hat{I}_{\gamma \delta}^{\alpha \beta}(z) \hat{I}_{\gamma \delta}^{\alpha \beta}(z^*) \) where

\[
\hat{I}_{\gamma \delta}^{\alpha \beta}(z) = \int r_{c_1} \ldots r_{c_3} \left( \frac{1}{\sqrt{1 - r^2}} - 1 \right) \frac{d^3 \mathbf{r}}{8\pi^2 r^2} \int \exp \left( \frac{i}{2} \mathbf{v} \mathbf{r} \right)
\]

\[
\cdot \frac{v^{a_1} \ldots v^{a_j}}{v^j} \left[ \frac{1}{2} \exp \left( \frac{vz}{2i} \right)(-1)^j \frac{ivz}{2} \sum_{n=0}^{j/2} \frac{1}{(j - 2n)!} \left( \frac{vz}{2i} \right)^{j-2n} \right] \frac{d^3 \mathbf{v}}{v^2}.
\]
Here integration is performed over real SO(3), $\text{Im} \, \mathbf{r} = 0$, $r^2 \leq 1$, and over real $\mathbf{v}$. Evidently, $\mathcal{N}_{\gamma\delta}(\mathbf{v}, \mathbf{v}^*)$ (that is $\mathcal{N}_{\gamma}(\mathbf{v})$ outside singularity points) restored from these $\tilde{T}_{\gamma\delta}(z, z^*)$ should factorize too, $\tilde{\mathcal{N}}_{\gamma\delta}(\mathbf{v}, \mathbf{v}^*) = \tilde{\mathcal{N}}_{\gamma}(\mathbf{v}) \tilde{\mathcal{N}}_{\delta}(\mathbf{v}^*)$.

A pleasant feature arising in this proof is correspondence with SO(4) (Euclidean) case.

4. The simplest (basic) integral

Consider important particular case when $\mathcal{N}_{\gamma\delta} = \mathcal{N}$ is scalar (indices $\gamma, \dot{\delta}$ are empty), the more general expressions have similar features. Then nontrivial $\tilde{I}_\gamma(z) \equiv \tilde{I}_\gamma(z), \tilde{I}_\delta(z^*) \equiv \tilde{I}_\delta(z^*)$ are expressible in the simplest way in terms of metric tensor $g^{ab}$ and scalars, i.e. it is sufficient to consider empty $\alpha, \beta$ as well. This also means that $\tilde{\mathcal{N}}(\mathbf{v}, \mathbf{v}^*)$ can be considered as function of $v^2, v^{*2}$ only (although this is evident in this simple case from the very beginning since there is no singled out vector(s) on which structures over $\alpha, \beta$ might depend). It will not cause confusion if we shall denote this function by the same symbol $\tilde{\mathcal{N}}(v^2, v^{*2})$. We have $\tilde{I}_\alpha(z) \equiv \tilde{I}(z), \tilde{I}_\delta(z^*) \equiv \tilde{I}(z^*)$,

$$
\tilde{I}(z) = \int \left( \frac{1}{\sqrt{1 - r^2}} - 1 \right) \frac{d^3 \mathbf{r}}{8\pi^2 r^2} \int \exp \left( \frac{i}{2} \mathbf{v} \mathbf{r} \right) \left( \cos \frac{v z}{2} - 1 \right) \frac{d^3 \mathbf{v}}{v^2} 
= 2\pi \ln \frac{1 + \sqrt{1 - z^2}}{2} 
$$

and

$$
\int \tilde{\mathcal{N}}(v^2, v^{*2}) v^{2l} v^{*2m} d^6 \mathbf{v} = 2^{-3} \tilde{\mathcal{N}}(v^2) \tilde{\mathcal{N}}(v^{2m})^*, 
$$

$$
\tilde{\mathcal{N}}(v^2) = \pi (-1)^l \left[ \frac{1}{\gamma} \left( \frac{d}{dh} \right)^{2l+2} \left( \frac{dz}{dh} \ln \frac{1 + \sqrt{1 - z^2}}{2} \right) \right]_{h=0},
$$

at $z = \sin \frac{h}{1 + i/\gamma}$ (rescaling $h \to (1 + i/\gamma) h$ is made). Let us use the value of the following table integral,

$$
\int_0^\infty \frac{l}{l^2 + 1} \sinh l dl = \frac{h}{2} \sin h - \frac{1}{2} + \frac{1}{2} \cos h \ln [2(1 + \cos h)],
$$

to express RHS of (27) in terms of it and thus map differentiation over $h$ to operation of multiplication. The terms $\cos h$ and $h \sin h$ lead to appearance of the terms $\propto f(v^2) v^{2(1+i/\gamma)-2}$ and $\propto f'(v^2) v^{2(1+i/\gamma)-2}$ in the functional $\tilde{\mathcal{N}}(f(v^2))$. Situation is analogous to that one appeared in LEMMA 1, and the set of probe functions $f(v^2)$ as already chosen in LEMMA 2 for the considered scalar case $j = 0, k = 0$ is vanishing.
at the nonphysical point $v^2 = 4(1 + i/\gamma)^{-2}$. Additional requirement is that also first derivatives of these functions be vanishing at this point. Upon simple transformation of integration contours the functional in question reads

$$2^3 \tilde{M}(f(v^2)g(v^2)^*) = \int \tilde{N}(v^2, v'^2) f(v^2)g(v^2)^*2^3d^6v$$

$$= \frac{i}{2} \int \frac{(1/\gamma - i)v/2}{(1/\gamma - i)^2v^2/4 + 1 \text{sh}[\pi(1/\gamma - i)v/2]} f(v^2)\text{d}^3v$$

$$- \frac{i}{2} \int \frac{(1/\gamma + i)v/2}{(1/\gamma + i)^2\bar{v}/4 + 1 \text{sh}[\pi(1/\gamma + i)\bar{v}/2]} \text{g}^*(v^2)\text{d}^3\bar{v}$$

(29)

at $f(4(1 + i/\gamma)^{-2}) = 0$, $f'(4(1 + i/\gamma)^{-2}) = 0$, $g(4(1 + i/\gamma)^{-2}) = 0$, $g'(4(1 + i/\gamma)^{-2}) = 0$; integrals in the RHS are over independent real $v, \bar{v}$. We have introduced notation $g^*(v^2) = g(v^{2*})(\equiv [g(v^{2*})]^*)$ having in view generalization to complex $v^2$. To pass to integration over complex $v, v^*$ we consider integrals in the RHS of (29) as single 6-fold integral over $\text{d}^3v\text{d}^3\bar{v}$, redenote $v = u + w, \bar{v} = u - w$, and, for $w = w_n$, $n^2 = 1$ rotate integration interval $w \in (0, \infty)$ in the plane of complex $w$ according to $w \rightarrow iw$. Thus we arrive at the desired form modulo pole contribution of the terms proportional to $f(v^2)$ or/and $g(v^2)^*$ taken at nonphysical points $v^2 = 4n^2(1 + i/\gamma)^{-2}$. So we get

$$\mathcal{N}(v^2, v'^2) = \left| \frac{1}{\frac{1}{4} \left( \frac{1}{\gamma} - i \right)^2 v^2 + 1} \cdot \frac{1}{4} \left( \frac{1}{\gamma} - i \right) v \right|^2$$

(30)

in the region with the points $v^2 = 4n^2(1 + i/\gamma)^{-2}, n = 1, 2, ...$ excluded.

The form of dependence providing singularity of $\mathcal{N}(v^2, v'^2)$ (30) at $v^2 = 4n^2(1 + i/\gamma)^{-2}, n = 1, 2, ...$ looks as result of a summation in the path integral over branches of the ‘arcsin’ function, as if we had substituted arcsin $\rightarrow$ arcsin $+2\pi n$ in the exponential and summed over $n$ for each of the two ‘arcsin’ functions. (This would just result in the hyperbolic or trigonometric function in the denominator of (30).) Thus, when calculating any moment of distribution $\mathcal{N}$ we have dealt with only the values of a finite number of derivatives i. e. with local properties of the principal value of arcsin $z$ at $z = 0$. Nevertheless when restoring $\mathcal{N}$ from the moments we have recovered full non-perturbative picture.

Effect of the considered singular points on behavior of $\mathcal{N}$ in physical region grows especially at $\gamma \ll 1$ or at $\gamma \gg 1$, when these points $v^2 = 4(1 + i/\gamma)^{-2}n^2, n = 1, 2, ...$

a) At $h(z) \propto z$ we reproduce (the module squared of) our result (11) on 3 dimensional SO(3) gravity $K_{i_1}(l)(2\pi l)^{-1}$ continued to complex argument $l = \sqrt{(1/\gamma - i)^2v^2/2}$ (the branch of square root is chosen in standard way such that $\text{Re} \sqrt{(1/\gamma - i)^2v^2} \geq 0$), with modified integral Bessel function $K_{i_1}(l)$ which exponentially decays at $\text{Re} l \rightarrow \infty$. 
approach physical region $\text{Im} v^2 = 0$. This displays as appearance of a set of local maxima of $N$ approximately at $v^2 = -4\gamma^2 n^2$, $\gamma \ll 1$ or at $v^2 = 4n^2$, $\gamma \gg 1$ where $n = 1, 2, \ldots$, see Figure 1. There $-n^2$ at $\gamma \ll 1$ or $n^2$ at $\gamma \gg 1$, $n = 1, 2, \ldots$, are

![Figure 1: Dependence of $N$ on $v^2$ for small and large values of $\gamma$.](image)

approximate values of $v^2\gamma^{-2}/4$ or $v^2/4$, respectively, corresponding to the position of local maxima. Taking into account that $v$ is triangle area $A$, the maxima are located at $|A| = 2\gamma n$ or at $|A| = 2n$ in the spacelike or timelike region, respectively.

As for asymptotic behavior in physical region at $|A| \to \infty$, $N$ decays as $\exp(-\pi|A|)$ in spacelike region or as $\exp(-\pi|A|/\gamma)$ in timelike region (at $h(z) \propto \arcsin z$; at $h(z) \propto z$ the exponents are $\exp(-|A|)$ and $\exp(-|A|/\gamma)$, respectively).

5.Conclusion

The above considered moments could have transparent physical sense in a theory with independent area tensors $v_{\sigma^2}$. The theory with independent scalar areas is known as area Regge calculus\textsuperscript{2} \textsuperscript{13}. Now we can speak of the area tensor Regge calculus. It is in many respects analogous to the 3 dimensional Regge calculus, and we can find the form of the full discrete path integral\textsuperscript{12} which becomes true canonical one in the formal continuous time limit irrespectively of the coordinate chosen as time just as
in the 3 dimensional case mentioned at the end of **Introduction.** Then the vacuum expectation values of the area tensor monomials are just the considered moments \( \langle v_{\sigma^2}, v_{\sigma^2}^* \rangle \) for the whole set of 2-simplices \( \sigma^2 \). Important point in this consideration is that the set of holonomies \( \{ R_{\sigma^2} : \sigma^2 \supset \sigma^1 \} \) for a given link \( \sigma^1 \) obey Bianchi identities [13]. Then integration over \( \prod_{\sigma^2 \supset \sigma^1} d^6 v_{\sigma^2} \) in the path integral will result in the singularity of the type of \( (\delta^6(r_{\sigma^2}))^2 \) for some \( \sigma^2 \). Rather, integration over certain subset of area tensors \( F \) should be omitted (a kind of gauge fixing). Correspondingly, the moments in general case can be defined as integrals of \( N \) with area tensor monomials over \( d^6 v_{\sigma^2} \), \( \sigma^2 \notin F \). At \( h(z) \propto \arcsin z \) restrictions on the set of area tensor components to integrate over are relaxed, and this set might be larger than \( \{ v_{\sigma^2} : \sigma^2 \notin F \} \); this point requires further studying.

To resume, direct definition of (nonabsolutely convergent) path integral in a theory with finite SO(3,1) rotations should be made with care (mainly because of exponential growth of the Haar measure on Lorentz boosts). In the particular case of such theory, Regge calculus in terms of rotation matrices in Minkowsky spacetime, path integral can be well defined. This definition respects correspondence with Euclidean version. Upon integrating out connections, probability distribution turns out to decay exponentially at large areas. Thus vertices do not go away to infinity and in this sense the minisuperspace system described by elementary lengths/areas is self-consistent.

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