Electron-hole symmetry, integrability, and a generalized Richardson model

W. V. Pogosov¹,², L. V. Bork¹,³

¹Center for Fundamental and Applied Research, N. L. Dukhov All-Russia Research Institute of Automatics, Sushchevskaya 22, 127055 Moscow, Russia
²Institute for Theoretical and Applied Electrodynamics, Russian Academy of Sciences, Izhorskaya 13, 125412, Moscow, Russia and
³Institute for Theoretical and Experimental Physics, B. Cheremushkinskaya 25, 117218, Moscow, Russia

We address a generalized Richardson model (Russian doll BCS model), which is known to be exactly solvable and integrable. We point out that this model, on the level of Hamiltonian, also contains the electron-hole pairing symmetry. The quantum invariants in both the electron and hole representations are analyzed. We then derive novel exact relations between the rapidities in these two representations. Together with initial Bethe equations, they form an overdetermined set of equations, which take the most simple form in the case of a usual Richardson model.

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I. INTRODUCTION

Basic properties of conventional superconductors can be described by the microscopic Bardeen-Cooper-Schrieffer (BCS) theory, which assumes that pairing between electrons is due to their interaction through phonons. The simplest possible Hamiltonian, known as a reduced BCS Hamiltonian, accounts only couplings between the spin up and spin down electrons having opposite momenta; moreover, these couplings are supposed to be constant. Within the BCS theory, this Hamiltonian is solved approximately by using a mean-field treatment.

It was shown by Richardson long time ago that the same Hamiltonian can be solved exactly. The approach to the problem, developed by Richardson, resembles a coordinate Bethe ansatz method. The Hamiltonian eigen states and eigen values are expressed through the set of energy-like quantities (rapidities). They satisfy the set of nonlinear algebraic equations, which are now widely referred to as Richardson equations. The system described by the reduced BCS Hamiltonian is closely related to the so-called Gaudin ferromagnet, for which the exact solution is also known.

Unfortunately, the resolution of Richardson equations is a formidable task, so that very few explicit results have been obtained so far. However, in the case of a system, which contains a limited number of pairs, the equations can be solved numerically. Nowadays, this approach is used to investigate pairing correlations in ultrasmall metallic grains at low temperatures.

In addition, as shown in Ref., the reduced BCS Hamiltonian is integrable, so that one can construct a complete set of operators, which commute with the Hamiltonian. Later on, the same model was addressed within the framework of the quantum inverse scattering method (see also). Richardson equations have been rederived by means of the algebraic Bethe ansatz (ABA) method. These equations can be treated as Bethe equations (BE). Quantum inverse scattering method significantly simplifies the problem of computation of correlations functions, but one anyway has to know the solution of BE.

In Ref., the conformal field theory interpretation of the reduced BCS Hamiltonian was developed, which suggests unexpected links between BCS theory and Laughlin states relevant for the fractional Hall effect. There also exist connections with the random-matrix models and growth problems. Eigen values of quantum invariants have been expressed through rapidities for the first time by using this approach (see also Ref.).

Recently, one of us proposed a method to solve Richardson equations, which is based on the analytical evaluations of integrals, similar to Selberg integrals appearing in conformal field theories. In this approach, when considering certain situations, a special trick was used, which enabled us to switch from the electron representation to the hole representation of the Hamiltonian (the existence of this symmetry in the solutions of Richardson equations were noted before in Ref.). The Hamiltonian in the hole representation is also exactly solvable. If Richardson equations provide a complete set of Hamiltonian eigen states, the same state can be expressed using either the electron representation or the hole representation. This rather trivial observation can, however, result in rather nontrivial consequences, since numbers of rapidities, in general case, differ from each other in the two representations. In Ref. the same idea was used for the analysis of the ground state energy of small-sized systems. In Ref., it was applied to investigate highly-excited states in the thermodynamical limit.

The aim of the present paper is to explore the electron-hole symmetry of the pairing Hamiltonian in the view of its integrability. We wish to analyze quantum invariants as well as their eigen values in both the electron and hole representations. This approach allows us to obtain a set of exact relations for the rapidities in the two representations,
which can hardly be derived staying on the level of BE. Together with the two initial sets of BE in these representations, they form an overdetermined system of nonlinear equations. Apart from the general interest, our results might be useful for the resolution of BE or computing correlations functions, since they provide additional tools to tackle these complex problems.

In addition, we do not restrict ourselves to the usual Richardson model, but instead we consider a more general model, known as a generalized Richardson model or Russian doll BCS model. This model is also integrable by means of ABA. It enables one to get rid of the strict requirement that all pair scattering couplings must be the same, since it contains one free parameter. Moreover, the generalized Richardson model was shown to have other quite remarkable properties. For example, the behaviour of spectrum of model special limits (when one energy level decouples) can be described by means of cyclical renormalization group.

This paper is organized as follows. In section II, we briefly discuss a derivation of generalized Richardson model and its solution by means of ABA. In section III, we derive new relations between solutions of Bethe equations as a consequence of electron-hole symmetry of the model. In section IV, we discuss these relations in more details in the semiclassical limit of the generalized Richardson model. In particular, we suggest that in the case of so called equally-spaced model these relations can be considered as equivalent (complementary) set of equations for Bethe roots, which may be in some cases more simple to deal with. We conclude in Section V.

II. GENERALIZED RICHARDSON MODEL

A. Preliminaries

Cooper pairing in disordered metallic grains occurs between time-reversed states\textsuperscript{15}. The appropriate Hamiltonian responsible for the interaction in Cooper channel is

\[
H = \sum_{n=1}^{L} \varepsilon_n (b_{n+}^\dagger b_{n+} + b_{n-}^\dagger b_{n-}) - \sum_{n,n'=1}^{L} g_{nn'} b_{n+}^\dagger b_{n'}^\dagger b_{n'} b_{n},
\]

where \(n\) labels \(L\) doubly-degenerate energy levels, \(\pm\) refers to pairs of time-reversed states, while \(b_{n,\pm}^\dagger\) are creation operators for fermions at level \(n\).

Eigen states of this Hamiltonian can be classified in accordance with the collection of blocked states. The state with the energy \(\varepsilon_n\) is blocked provided that it is occupied by a single electron. This state then does not contribute to the interaction energy. Hereafter we consider only the subspace without singly-occupied levels.

We introduce pair creation \(B_n^\dagger\) and destruction \(B_n\) operators, defined as \(B_n^\dagger = b_{n+}^\dagger b_{n-}^\dagger\) and \(B_n = b_{n-} - b_{n+}\). Their commutator reads

\[
[B_n, B_n^\dagger] = \delta_{n,n'}(1 - b_{n+}^\dagger b_{n+} - b_{n-}^\dagger b_{n-}).
\]

For the subspace without singly-occupied levels, \(b_{n+}^\dagger b_{n+} + b_{n-}^\dagger b_{n-}\) can be replaced by \(2B_n^\dagger B_n = 2N_n\). In particular, the ground state always belongs to this subspace.

The model with constant \(g_{nn'} = g\), known as Richardson model, is exactly solvable\textsuperscript{16} and integrable\textsuperscript{17}. Actually, the same Hamiltonian, taken in the thermodynamical limit, is also used in the BCS theory of superconductivity, in which it is solved by the mean-field approach.

There also exists a more general integrable model\textsuperscript{18}, which is known as a generalized Richardson model (Russian doll BCS model). Its Hamiltonian reads as

\[
H = 2 \sum_{n=1}^{L} \varepsilon_n N_n - g \sum_{n<n'}^{L} (e^{i\beta} B_n^\dagger B_n + e^{-i\beta} B_n^\dagger B_n'),
\]

where \(\beta\) is an arbitrary angle. At \(\beta = 0\), the Richardson Hamiltonian reduces to Eq. \(3\) up to the multiple of a number operator \(\sum_n N_n\). The generalized Richardson model is well studied in the literature\textsuperscript{16}. In particular, it is known that this model can be solved by means of ABA. Here we are going to repeat crucial parts of this solution.
B. Integrability

Let us introduce pseudo-spin operators $\hat{S}_+ = B^\dagger$, $\hat{S}_- = B$ and $\hat{B}_z = \hat{N} - 1/2$. We then consider the $R$-matrix of the form

$$R(u) = \frac{1}{u + i\eta}(uI \otimes I + i\eta P).$$

Here, as usual, $R$ acts on the tensor product of two linear spaces $V_1 \otimes V_1$, $P$ is permutation operator, which acts on the tensor product $x \otimes y$ of two elements form $V_1$ and $V_2$ as $P(x \otimes y) = y \otimes x$. This $R$-matrix satisfies Yang-Baxter (YB) equation

$$R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v).$$

Using $SU(2)$ spin 1/2 representation of operators $\hat{S}_z$ and $\hat{S}_\pm$ we can write $R$ matrix in the form:

$$R(u) = \frac{1}{u + i\eta}\begin{pmatrix} uI + i\eta(\hat{S}_z + 1/2I) & i\eta\hat{S}_- \\ i\eta\hat{S}_+ & uI + i\eta(-\hat{S}_z + 1/2I) \end{pmatrix}.$$

Using this $R$-matrices we represent monodromy matrix $T(u)$ as:

$$T(u) = \Omega_0R_{0L}(u - \varepsilon_L)...R_{02}(u - \varepsilon_2)R_{01}(u - \varepsilon_1),$$

where $\Omega_{00}$ is so called twist matrix $\Omega = \exp(i\beta\sigma), \sigma = \text{diag}(1, -1)$. $R_{0i}$ acts on $V_0 \otimes V_i$, where $V_0$ is a so called auxiliary subspace, which is $\mathbb{C}_2$ and $V_i$ is physical subspace associated with $i$'th site which is also $\mathbb{C}_2$ in the case of spin 1/2 representation. $\{\varepsilon_i\}_{i=1}^L$ is a given set of real parameters.

Using $T(u)$ and taking trace $Tr_0$ with respect to the auxiliary subspace we can define the transfer matrix $t(u)$

$$t(u) = Tr_0[\Omega_0R_{0L}(u - \varepsilon_L)...R_{02}(u - \varepsilon_2)R_{01}(u - \varepsilon_1)],$$

which can be considered as an operator acting on $V_L = \bigotimes_{i=1}^L V_i$, $V_i = \mathbb{C}_2$. $T(u)$ satisfies another variation of YB equations:

$$R_{12}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u - v),$$

from which one can see that

$$[t(u), t(v)] = 0, \forall u, v \in \mathbb{C}.\quad (10)$$

The problem of finding eigenvectors and eigenvalues for $t(u)$

$$t(u)|t, \{E_i\} \rangle = \Lambda(u, \{E_i\})|t, \{E_i\} \rangle,$$

can be solved. Eigenvectors $|t, \{E_i\} \rangle$ and eigenvalues $\Lambda(u, \{E_i\})$ of $t(u)$ are parameterized by the set of parameters $\{E_i\}_{i=1}^M, M \leq L$. It is convenient to rescale $t(u)$ as

$$t(u) \rightarrow \prod_{i=1}^L \frac{u - \varepsilon_i + i\eta}{u - \varepsilon_i + i\eta/2} t(u),$$

then the explicit expression for eigenvalues of $t(u)$ can be written as

$$\Lambda(u, \{E_i\}) = e^{-i\beta} \prod_{k=1}^L \frac{u - \varepsilon_k}{u - \varepsilon_k + i\eta/2} \prod_{j=1}^M \frac{u - E_j/2 + 3i\eta/2}{u - E_j/2 + i\eta/2} + e^{i\beta} \prod_{k=1}^L \frac{u - \varepsilon_k + i\eta}{u - \varepsilon_k + i\eta/2} \prod_{j=1}^M \frac{u - E_j/2 - i\eta/2}{u - E_j/2 + i\eta/2},$$

where $\{E_i\}_{i=1}^M$ satisfies the set of equations (Bethe equations)

$$e^{2i\beta} \prod_{k=1}^L \frac{E_i/2 - \varepsilon_k - i\eta/2}{E_i/2 - \varepsilon_k + i\eta/2} = \prod_{j=1}^M \frac{E_i/2 - E_j/2 - i\eta}{E_i/2 - E_j/2 + i\eta}.$$

(14)
Using $t(u)$ one can construct an infinite set of commuting operators $t_k$, which can be obtained expanding $t(u)$ in the powers of $u$ at infinity:

$$t(u) = \sum_{k=1}^{\infty} u^{-k} t_k, \; [t_i, t_k] = 0, \; \forall \; i, k. \quad (15)$$

Eigen values of $t_k$ can be obtained using the explicit expression for $\Lambda(u, \{E_i\})$ and expanding it also in powers of $\eta$ at infinity. The form of $t_2$ is

$$t_2 \sim \left( \eta \sin(\beta) \sum_{i=1}^{L} \frac{\epsilon_i}{\epsilon_p + i\eta/2} \sum_{i<k}^L (e^{i\beta} \hat{S}_{+i} \hat{S}_{-k} + e^{-i\beta} \hat{S}_{-i} \hat{S}_{+k}) \right) + CI,$$

where $C$ is some constant. Dropping out a trivial contribution proportional to the constant and dividing by $\eta \sin(\beta)/2$ we see that $t_2 = H$ of the generalized Richardson model with $q = \eta / \sin(\beta)$. Expanding $\Lambda(u, \{E_i\})$ and extracting a term proportional to $u^{-2}$ after comparison with $t_2$ we conclude that eigenvalues $E$ of $H$ (spectrum) are equal to

$$E = \sum_{i=1}^{M} E_i + gM \cos(\beta),$$

where $\{E_i\}$ satisfies Bethe equations.

There is also another set of integrals of motion for $H$. They can be obtained as values of $t(u)$ for special values of spectral parameter.

$$R_m = t(\varepsilon_m). \quad (18)$$

Using explicit expressions for the eigenvalues of $\Lambda$ of $t(u)$ one can write eigenvalues $\lambda_m$ of $R_m$:

$$\lambda_m = 2e^{i\beta} \prod_{p=1, j \neq m}^{L} \frac{\varepsilon_m - \varepsilon_p + i\eta/2}{\varepsilon_m - \varepsilon_p - i\eta/2} \prod_{j=1}^{M} \frac{\varepsilon_m - E_j/2 - i\eta/2}{\varepsilon_m - E_j/2 + i\eta/2}. \quad (19)$$

**C. Quasiclassical limit**

It is of interest that in the $\eta \rightarrow 0$ limit generalized Richardson model can be reduced to the Richardson model. On the level of BE this limit resembles a quasiclassical one. Indeed, considering $\beta \sim \eta/V$, (here $V$ is some real constant) and taking $\eta \rightarrow 0$, BE are reduced to:

$$1 = \sum_{n=1}^{L} \frac{V}{2\varepsilon_n - E_j} + \sum_{l=1, (\neq j)}^{M} \frac{2V}{E_j - E_l} \quad (20)$$

while eigenvalues of the Hamiltonian are given by Eq. (17) at $\beta = 0$.

**III. ELECTRON-HOLE SYMMETRY IN GENERALIZED RICHARDSON MODEL**

It is remarkable that on the level of Hamiltonian $H$ one can use a dual description in terms of "hole" operators. We will call the initial representation of $H$ as "electron representation" and will use index "(e)".

Let us introduce creation operators for holes as $b_{n+}^{(h)\dagger} = b_{n+}^{(e)}$ and $b_{n-}^{(h)\dagger} = b_{n-}^{(e)}$. Holes represent empty one-electron states. The creation $B_n^{(h)\dagger}$ and destruction $B_n^{(h)}$ operators for pairs of holes are $B_n^{(h)\dagger} \equiv b_{n+}^{(h)\dagger} b_{n-}^{(h)\dagger}$ and $B_n \equiv b_{n+}^{(h)} b_{n-}^{(h)}$.

We also introduce an operator defined as $N_n^{(h)} = B_n^{(h)\dagger} B_n^{(h)}$, which is a number operator for the pairs of holes. Within the subspace without singly-occupied levels, this operator is connected to $N_n^{(e)\dagger}$ through the simple relation:

$$N_n^{(e)} + N_n^{(h)} = 1. \quad (21)$$

Next, we define a set of energies as

$$\varepsilon_n^{(h)} = -\varepsilon_n^{(e)}. \quad (22)$$
Using these quantities and replacing $\beta \mapsto -\beta$ we can rewrite $H^{(e)}$ as $H^{(h)}$ up to the constant term. Both Hamiltonians describe the same system.

It is assumed that there is one to one correspondence between sets of eigenvectors and eigenvalues in both representations. Hamiltonian $H^{(e)}$ can be obtained from the transfer matrix $t^{(e)}(u)$, which was described above, while $H^{(h)}$ can be obtained from the transfer matrix $t^{(h)}(u)$, where one has to use operators and "inhomogeneities" $\varepsilon$ in $(h)$ representation.

When the number of "electrons" is $M$, the number of holes is $L - M$. Thus, there must be two equivalent representations for the same eigenvector of $H$ (provided that the ABA yields a complete set of solutions):

$$|t^{(e)}, \{E\}_M\rangle \sim |t^{(h)}, \{K\}_{L-M}\rangle. \tag{23}$$

Here the set of $\{E\}_M$ satisfies the system of BE for $H^{(e)}$, while $\{K\}_{L-M}$ satisfies the system of BE for $H^{(h)}$.

One can see that the following identity holds ($\mathbb{P}$ is a reflection (parity) operator, which "inverts z axis" and acts on the pseudo-spin operators)

$$Tr_0[R^{(e)}_{0L}(u)...R^{(e)}_{02}(u)R^{(e)}_{01}(u)] = Tr_0[\mathbb{P}R^{(h)}_{0L}(-u^*)\mathbb{P}...\mathbb{P}R^{(h)}_{02}(-u^*)\mathbb{P}\mathbb{P}R^{(h)}_{01}(-u^*)\mathbb{P}], \tag{24}$$

where $R^{(e)}_{0i}$ and $R^{(h)}_{0i}$ are $R$ matrices written in terms of the electron and hole operators correspondingly. This identity also implies that

$$t^{(e)}(\varepsilon^{(e)}_k) = t^{(h)}(\mathbb{P}(\varepsilon^{(h)}_k)). \tag{25}$$

Note that one can consider these identities as a manifestation of CP symmetry of some kind.

Using different representations for eigenvectors of $H$ we see that this gives the following relations between eigenvalues of $t^{(e)}$ and $t^{(h)}$:

$$\Lambda(\varepsilon^{(e)}_k, \{E\}_M) = \Lambda^*(\varepsilon^{(h)}_k, \{K\}_{L-M}). \tag{26}$$

They can be considered as nontrivial relations between solutions of different sets of BE.

In other words, using explicit form of $\Lambda(u, \{E\}_M)$ we can write

$$e^{i2\beta} \prod_{j=1}^M \left( \varepsilon^{(e)}_j - E_j/2 - i\eta/2 \right) = \left( \prod_{j=1}^{L-M} \left( \varepsilon^{(h)}_j - K_j - i\eta/2 \right) \right)^*, \tag{27}$$

where $\{E_i\}$ satisfies $M$ equations

$$e^{i2\beta} \prod_{k=1}^L \left( E_i/2 - \varepsilon^{(e)}_k - i\eta/2 \right) = \prod_{j=1}^M \left( E_i/2 - E_j/2 - i\eta \right) \tag{28}$$

and $\{K_i\}$ satisfies $L - M$ equations

$$e^{i2\beta} \prod_{k=1}^L \left( K_i - \varepsilon^{(h)}_k - i\eta/2 \right) = \prod_{j=1}^{L-M} \left( K_i - K_j - i\eta \right). \tag{29}$$

$\{\varepsilon^{(e)}_i\}_{i=1}^L$ is a given set and $\varepsilon^{(h)}_i = -\varepsilon^{(e)}_i$ for any $i$.

One can also think that these conditions on BE solutions can be considered as self-consistency conditions. Indeed. We can see by explicit computations that $t^{(e)}(\varepsilon^{(e)}_k) = t^{(h)}(\mathbb{P}(\varepsilon^{(h)}_k))$. Eigenvectors

$$|t^{(e)}, \{E\}_M\rangle = B^{(e)}(E_1)...B^{(e)}(E_M)|\text{vac} e\rangle$$

and

$$|t^{(h)}, \{K\}_{L-M}\rangle = B^{(h)}(K_1)...B^{(h)}(K_{L-M})|\text{vac} h\rangle$$

will be proportional to each other if $\Lambda(\varepsilon^{(e)}_k, \{E\}_M) = \Lambda^*(\varepsilon^{(h)}_k, \{K\}_{L-M})$ will be satisfied. Here $B(E)$ is a standard parametrisation of a monodromy matrix:

$$t(u) = Tr_0[\Omega_0 R_{0L}(u - \varepsilon_L)...R_{02}(u - \varepsilon_2)R_{01}(u - \varepsilon_1)] = Tr \left( A(u) \begin{array}{cc} B(u) \\ C(u) \\ D(u) \end{array} \right). \tag{30}$$
IV. ELECTRON-HOLE SYMMETRY IN RICHARDSON MODEL

A. Integrals of motion for Richardson model

Now we consider in a more detail the Richardson model, which is of a particular interest for superconductors at nanoscale. Let us present the Hamiltonian in the usual form, in which electron pair creation and destruction operators with coincident indices are included explicitly into the interaction term:

\[ H = 2 \sum_{n=1}^{L} \varepsilon^{(e)}_{n} N^{(e)}_{n} - V \sum_{n,n'=}^{L} B^{(e)\dagger}_{n} B^{(e)}_{n}. \] 

(31)

The Richardson equation for each rapidity \( E^{(e)}_{j} \) in the electron representation is given by Eq. (20). Eigen energies are given simply by the sum of the rapidities.

The Hamiltonian (31) is integrable. There exists a set of operators \( R^{(e)}_{m} \), which commute with each other as well as with the Hamiltonian. Each operator is associated with its own one-electron energy level. As follows from Eq. (18) (expanding it in powers of \( \eta \) and extracting term \( \sim \eta \), see also (6,7), the expression of \( R^{(e)}_{m} \) can be written as

\[ R^{(e)}_{m} = \left( \frac{1}{2} - N^{(e)}_{m} \right) - 2V \sum_{l=1(\neq m)}^{L} \frac{(\frac{1}{2} - N^{(e)}_{m})(\frac{1}{2} - N^{(e)}_{l}) + \frac{1}{2}(B^{(e)\dagger}_{m} B^{(e)}_{l} + B^{(e)\dagger}_{l} B^{(e)}_{m})}{\varepsilon^{(e)}_{m} - \varepsilon^{(e)}_{l}} \] 

(32)

The eigenvalue \( \lambda^{(e)}_{m} \) corresponding to \( R^{(e)}_{m} \) reads as

\[ \lambda^{(e)}_{m} = -\frac{1}{2} + V \left( \sum_{j=1}^{M} \frac{1}{2 \varepsilon^{(e)}_{j} - E^{(e)}_{j}} - \frac{1}{4} \sum_{l=1(\neq m)}^{L} \frac{1}{\varepsilon^{(e)}_{m} - \varepsilon^{(e)}_{l}} \right) \] 

(33)

B. Electron quantum invariants in terms of hole operators

Next, we introduce a set of energies defined as

\[ \varepsilon^{(h)}_{n} = -\left( \varepsilon^{(e)}_{n} - V \right), \] 

(34)

which are shifted with respect to (22) due to the inclusion of the terms with coincident indices into the interaction term of the Hamiltonian.

Now, we rewrite \( R^{(e)}_{m} \) in terms of hole operators. After some simple algebra, we obtain:

\[ R^{(e)}_{m} = \left( \frac{1}{2} - N^{(h)}_{m} \right) + 2V \sum_{l=1(\neq m)}^{L} \frac{(\frac{1}{2} - N^{(h)}_{m})(\frac{1}{2} - N^{(h)}_{l}) + \frac{1}{2}(B^{(h)\dagger}_{m} B^{(h)}_{l} + B^{(h)\dagger}_{l} B^{(h)}_{m})}{\varepsilon^{(h)}_{m} - \varepsilon^{(h)}_{l}} \] 

(35)

C. Hamiltonian in terms of hole operators

Let us now turn to the Hamiltonian. By using commutation relations for fermionic operators, it is easy to rewrite the Hamiltonian in terms of holes:

\[ H = \sum_{n} (2\varepsilon^{(c)}_{n} - V) + 2 \sum_{n=1}^{L} \varepsilon^{(h)}_{n} N^{(h)}_{n} - V \sum_{n,n'=}^{L} B^{(h)\dagger}_{n} B^{(h)}_{n}. \] 

(36)

This expression coincides with the Hamiltonian in terms of electrons up to the constant term and the replacement of \( \varepsilon^{(c)}_{n} \) by \( \varepsilon^{(h)}_{n} \). Thus, this Hamiltonian is also integrable. Hence, one can introduce a set of quantum invariants \( R^{(h)}_{m} \) for it. The expression for \( R^{(h)}_{m} \) is given by Eq. (33), in which all electron operators are replaced by hole operators, while \( \varepsilon^{(e)}_{n} \) is replaced by \( \varepsilon^{(h)}_{n} \). We immediately find that \( R^{(h)}_{m} \) coincides with the expression of \( -R^{(e)}_{m} \) in terms of hole operators, as seen from Eq. (33):

\[ R^{(h)}_{m} = -R^{(e)}_{m}. \] 

(37)
D. Exact relations for rapidities

The number of rapidities in the hole representation is \( L - M \). We can express the eigenvalue \( \lambda_m^{(h)} \) of \( R_m^{(h)} \) in terms of these rapidities as

\[
\lambda_m^{(h)} = -\frac{1}{2} + V \left( \sum_{j=1}^{L-M} \frac{1}{2\varepsilon_m^{(h)} - E_j^{(h)}} - \frac{1}{4} \sum_{i=1(\neq j)}^{L} \frac{1}{\varepsilon_m^{(h)} - \varepsilon_i^{(h)}} \right),
\]

while \( E_j^{(h)} \) satisfy the system of \( L - M \) equations

\[
1 = \sum_{n=1}^{L-M} \frac{V}{2\varepsilon_m^{(h)} - E_j^{(h)}} + \sum_{i=1(\neq j)}^{L-M} \frac{2V}{E_j^{(h)} - E_i^{(h)}}.
\]

Taking into account Eq. (37), we obtain a simple relation \( (m = 1, 2, \ldots, L) \)

\[
\lambda_m^{(e)} + \lambda_m^{(h)} = 0,
\]

which can be further rewritten as

\[
\sum_{j=1}^{M} \frac{V}{2\varepsilon_m^{(e)} - E_j^{(e)}} + \sum_{j=1}^{L-M} \frac{V}{2\varepsilon_m^{(h)} - E_j^{(h)}} = 1.
\]

Formula (41) provides a highly nontrivial set of \( L \) exact relations between the solutions for the two systems of Richardson equations, (20) (where the index "(e)" must be used for both \( E_j^{(e)} \) and \( \varepsilon_m^{(e)} \)) and (39), supplemented by Eq. (34). As known, the system of equations (20) has \( \binom{L}{2} \) solutions, which correspond to different states of the initial quantum problem and form an energy spectrum (for the given set of singly-occupied levels). The system of equations (39) has \( \binom{L}{L-M} \) solutions, this number being the same. This fact indicates that the relation (40) must hold not only for some single solution of (20) and another single solution of (39), but for all existing couples of solutions. In other words, (41) is valid for ground states of electron system and hole system, for first excited states of these systems, etc.

Let us stress that we do not know how relations (41) can be derived staying on the level of Richardson equations, as well as a much simpler condition for the sum of rapidities, obtained recently in Ref. 12, where only energies of the same state in electron and hole representation have been related. This latter condition, dictated by Eq. (36), can be written as

\[
\sum_{j=1}^{M} E_j^{(e)} = \sum_{m=1}^{L} (2\varepsilon_m^{(e)} - V) + \sum_{j=1}^{L-M} E_j^{(h)}.
\]

It is also valid for the whole spectrum of the two systems, not only for their ground states. Note that it can be shown that this condition follows from Eq. (41). Note also that treating two systems of Richardson equations for electrons and holes, (20) and (39), as a single system of equations, and summing right-hand and left-hand sides of these equations, we obtain the same result, as by summing right-hand and left-hand sides of (41).

E. Electrostatic picture in the hole representation

It is known that Richardson equations can be formally written as stationary conditions for an energy of free classical particles with electrical charges \( 2\sqrt{V} \) located on the plane with coordinates given by \((\text{Re} E_j^{(e)}, \text{Im} E_j^{(e)})\). These particles are subjected into an external uniform force directed along the axis of abscissa with the strength \(-2\). They are attracted to fixed particles each having a charge \(-\sqrt{V}\) and located at \(2\varepsilon_m^{(e)}\). Free charges repeal each other. The interaction between the particles is logarithmic.

Richardson equations for holes (39) correspond to the electrostatic system, for which the distribution of one-energy levels is mirror-imaged with respect to the zero energy and also shifted, according to Eq. (41). In addition, the number of free charges is also different, while a direction of the external force acting on each charge is the same. Thus, we here deal with the inverted with respect to the bottom of the interaction ‘window’ distribution of energy levels. Low-energy states of the electron system correspond to rapidities tending to concentrate near the bottom of the ‘window’, while, for the hole system, they correspond to different distribution of levels, which applies to the top of the ‘window’. It is quite remarkable that these states are interconnected by various exact relations valid for the rapidities.
F. Equally-spaced model: arbitrary filling

Let us now focus on the so-called equally-spaced model, for which electron and hole electrostatic pictures are characterized by the same distribution of energy levels, as already have been noted in Ref. This model assumes that energy levels \( \varepsilon_n \) are located equidistantly between two cutoffs, \( \varepsilon_{F_0} \) and \( \varepsilon_{F_0} + (L-1)d \), so that \( \varepsilon_n = \varepsilon_{F_0} + (n-1)d \), where \( n \) runs from 1 to \( L \), \( d \) being a distance between two neighboring energy levels.

We analyze a situation, when the total number of electron pairs \( M \) in the “window” is arbitrary. In the usual BCS theory, \( M = L/2 \) (half filling).

Let us represent \( E_j^{(e)} \) as

\[
E_j^{(e)} = 2\varepsilon_{F_0} + \varepsilon_j^{(M,L)} ,
\]

where the notation \( (M,L) \) shows that \( \varepsilon_j^{(M,L)} \) corresponds to \( M \) pairs and \( L \) available states. These quantities satisfy the set of Richardson equations. The \( j \)-th equation \( (j = 1, \ldots, M) \) can be written as:

\[
1 = \sum_{m=0}^{L-1} \frac{V}{2md - \varepsilon_j^{(M,L)}} + \sum_{l=1(l\neq j)}^{M} \frac{2V}{\varepsilon_l^{(M,L)} - \varepsilon_j^{(M,L)}} .
\]

(44)

Eq. (41) now takes a simple form \( (m = 0, 1, \ldots, L-1) \)

\[
\sum_{j=1}^{M} \frac{V}{2md - \varepsilon_j^{(M,L)}} + \sum_{j=1}^{L-M} \frac{V}{2[(L-1) - m]d - \varepsilon_j^{(M,L-M)}} = 1 .
\]

(45)

When deriving this equation, we replaced a summation over \( m \) in Richardson equations \( (39) \) by a summation over \( [(L-1) - m] \). This trick enabled us to preserve a universal form of Richardson equations and to relate \( \{\varepsilon_j^{(M,L)}\} \) \( (j = 1, \ldots, M) \) and \( \{\varepsilon_j^{(M,L-M)}\} \) \( (j = 1, \ldots, L-M) \), which are solutions of the two systems of equations differing from each other only by their number, the form of this system \( (44) \) being universal. This simplification is a direct consequence of the equally-spaced energy-level distribution. Of course, it also works for any other distribution, symmetric with respect to the middle of the interaction window.

Note that in Ref. the Hamiltonian electron-hole symmetry was used to derive exact relation between the ground state energy of \( M \) and \( L-M \) pairs, which are given by the sums of rapidities. In Ref. this idea was used to study highly-excited states within the Richardson approach. Integrability provides further insights into the problem, leading to other exact relations.

It is of interest to use the electron-hole symmetry in order to explore singularities, appearing in Richardson solution, when some of rapidities coincide.

G. Equally-spaced model: half-filling

Perhaps, the most interesting (and most physical) situation is a half-filling, when \( M = L-M \), so that sets \( \{\varepsilon_j^{(M,L)}\} \) \( (j = 1, \ldots, M) \) and \( \{\varepsilon_j^{(M,L-M)}\} \) \( (j = 1, \ldots, L-M) \) are supposed to be the same. In this case, the system of equations \( (45) \) is reduced to

\[
\sum_{j=1}^{L/2} \frac{V}{2md - \varepsilon_j^{(L/2,L)}} + \sum_{j=1}^{L/2} \frac{V}{2[(L-1) - m]d - \varepsilon_j^{(L/2,L)}} = 1 .
\]

(46)

This is the system of \( L \) equations. It can be readily seen that not all of them are independent. Actually, the equations for \( m \) and \( L-1-m \) are identical. Therefore, the appropriate values of \( m \) are \( m = 0, 1, \ldots, L/2-1 \). Note that in the most elementary case, \( L = 2 \), Eq. \( (40) \) is correctly reduced to the single Richardson equation. For \( L = 4 \), we analyzed ground-state solutions of \( (40) \) in two limits, \( V \to 0 \) and \( V \to \infty \), and found that they do coincide with the solutions of the original system of two Richardson equations in the same limits. A detailed analysis of the derived relations is, however, beyond the scope of the this paper.

Thus, relations \( (40) \) form a system of equations, which are expected to be complementary and equivalent to the system of Richardson equations for the equally-spaced model at half-filling. Each equation is symmetric under the
reduction of any two rapidities, in contrast to Richardson equations. By shifting rapidities, the equations can be also written in the form, completely symmetric with respect to the middle of the interaction 'window'. Note that the condition (42) is automatically fulfilled at half filling.

Together with Richardson equations, the relations (46) form an overdetermined set of equations. Hopefully, this may provide additional tools for the resolution of the problem or computing correlation functions. Notice that relations (46) lack singular terms of the form $1/(E_1 - E_2)$ typical for Richardson equations (and some other Bethe equations).

Our results might be also of interest for pure mathematics, since they deal with somehow hidden relations between solutions of two sets of polynomial equations.

V. CONCLUSIONS

A generalized Richardson model, on the level of Hamiltonian, is characterized by the electron-hole pairing symmetry. We examined consequences of this symmetry on the solution of the model by the algebraic Bethe ansatz. We pointed out that, within both representations, the model is integrable, but the number of rapidities in these two pictures are, in general case, different. In addition, the bare kinetic energies of electron pairs and hole pairs are inverted with respect to each other. Nevertheless, both representations should provide the same eigenvalues for quantum invariants.

By analyzing these quantum invariants, we obtained a set of quite nontrivial relations between the rapidities in the electron and hole representations. Together with the two initial sets of Bethe equations, they form an overdetermined system of equations. This system takes the most simple form for the usual Richardson model and at some special realizations of one-body energy-level distributions (for example, an equally-spaced distribution and half-filling).

Our results may be helpful for the problems of resolution of Bethe (Richardson) equations and/or computations of correlation functions.

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