INVERTING THE COUPLING OF THE SIGNED GAUSSIAN FREE FIELD WITH A LOOP SOUP

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Abstract. Lupu introduced a coupling between a random walk loop-soup and a Gaussian free field, where the sign of the field is constant on each cluster of loops. This coupling is a signed version of isomorphism theorems relating the square of the GFF to the occupation field of Markovian trajectories. His construction starts with a loop-soup, and by adding additional randomness samples a GFF out of it. In this article we provide the inverse construction: starting from a signed free field and using a self-interacting random walk related to this field, we construct a random walk loop-soup. Our construction relies on the previous work by Sabot and Tarres, which inverts the coupling from the square of the GFF rather than the signed GFF itself. As a consequence, we also deduce an inversion of the coupling between the random current and the FK-Ising random cluster models introduced by Lupu and Werner.

1. Introduction

Let \( G = (V, E) \) be a connected undirected graph, with \( V \) at most countable and each vertex \( x \in V \) of finite degree. We do not allow self-loops, however the edges might be multiple. Given \( e \in E \) an edge, we will denote \( e^+ \) and \( e^- \) its end-vertices, even though \( e \) is non-oriented and one can interchange \( e^+ \) and \( e^- \). Each edge \( e \in E \) is endowed with a conductance \( W_e > 0 \).

There may be a killing measure \( \kappa = (\kappa_x)_{x \in V} \) on vertices.

We consider \((X_t)_{t \geq 0}\) the Markov jump processes on \( V \) which being in \( x \in V \), jumps along an adjacent edge \( e \) with rate \( W_e \). Moreover if \( \kappa_x \neq 0 \), the process is killed at \( x \) with rate \( \kappa_x \) (the process is not defined after that time). \( \zeta \) will denote the time up to which \( X_t \) is defined.

If \( \zeta < +\infty \), then either the process has been killed by the killing measure \( \kappa \) (and \( \kappa \neq 0 \)) or it has gone off to infinity in finite time (and \( V \) infinite). We will assume that the process \( X \) is transient, which means, if \( V \) is finite, that \( \kappa \neq 0 \). \( P_x \) will denote the law of \( X \) started from \( x \).

Let \((G(x, y))_{x,y \in V}\) be the Green function of \( X_t \):

\[ G(x, y) = G(y, x) = E_x \left[ \int_0^\zeta 1_{\{X_t=y\}} dt \right]. \]

Let \( \mathcal{E} \) be the Dirichlet form defined on functions \( f \) on \( V \) with finite support:

\[ \mathcal{E}(f, g) = \sum_{x \in V} \kappa_x f(x)^2 + \sum_{e \in E} W_e (f(e^+) - f(e^-))^2. \]

\( P_\varphi \) will be the law of \((\varphi_x)_{x \in V}\) the centred Gaussian free field (GFF) on \( V \) with covariance \( E_{\varphi}[\varphi_x \varphi_y] = G(x, y) \). In case \( V \) is finite, the density of \( P_\varphi \) is

\[ \frac{1}{(2\pi)^{|V|/2} \sqrt{\det G}} \exp \left( -\frac{1}{2} \mathcal{E}(f, f) \right) \prod_{x \in V} df_x. \]
Given $U$ a finite subset of $V$, and $f$ a function on $U$, $P_{\varphi}^{U,f}$ will denote the law of the GFF $\varphi$ conditioned to equal $f$ on $U$. $(\ell_x(t))_{x \in V, t \in [0,\zeta]}$ will denote the family of local times of $X$:

$$\ell_x(t) = \int_0^t 1_{\{X_s = x\}} ds.$$ 

For all $x \in V$, $u > 0$, let

$$\tau_x^u = \inf\{t \geq 0; \ell_x(t) > u\}.$$  

Recall the generalized second Ray-Knight theorem on discrete graphs by Eisenbaum, Kaspi, Marcus, Rosen and Shi [2] (see also [8, 10]):

**Generalized second Ray-Knight theorem.** For any $u > 0$ and $x_0 \in V$,

$$(\ell_x(\tau_{x_0}^u) + \frac{1}{2} \varphi_x^2)_{x \in V} \text{ under } \mathbb{P}_{x_0}(\cdot | \tau_{x_0}^u < \zeta) \otimes P_{\varphi}^{\{x_0\},0}$$

has the same law as

$$(\frac{1}{2} \varphi_x^2)_{x \in V} \text{ under } P_{\varphi}^{\{x_0\}}, \sqrt{2u}.$$  

Sabot and Tarrès showed in [9] that the so-called “magnetized” reverse Vertex-Reinforced Jump Process provides an inversion of the generalized second Ray-Knight theorem, in the sense that it enables to retrieve the law of $(\ell_x(\tau_{x_0}^u), \varphi_x^2)_{x \in V}$ conditioned on $(\ell_x(\tau_{x_0}^u) + \frac{1}{2} \varphi_x^2)_{x \in V}$. The jump rates of that latter process can be interpreted as the two-point functions of the Ising model associated to the time-evolving weights.

However in [9] the link with the Ising model is only implicit, and a natural question is whether Ray-Knight inversion can be described in a simpler form if we enlarge the state space of the dynamics, and in particular include the “hidden” spin variables.

The answer is positive, and goes through an extension of the Ray-Knight isomorphism introduced by Lupu [6], which couples the sign of the GFF to the path of the Markov chain. The Ray-Knight inversion will turn out to take a rather simple form in Theorem 3 of the present paper, where it will be defined not only through the spin variables but also random currents associated to the field through an extra Poisson Point Process.

The paper is organised as follows.  

In Section 2 we recall some background on loop soup isomorphisms and on related couplings and state and prove a signed version of generalized second Ray-Knight theorem. We begin in Section 2.1 by a statement of Le Jan’s isomorphism which couples the square of the Gaussian Free Field to the loop soups, and recall how the generalized second Ray-Knight theorem can be seen as its Corollary: for more details see [4]. In Subsection 2.2 we state Lupu’s isomorphism which extends Le Jan’s isomorphism and couples the sign of the GFF to the path of the Markov chain. Lupu’s isomorphism yields an interesting realisation of the well-known FK-Ising coupling, and provides as well a “Current+Bernoulli=FK” coupling lemma [7], which occur in the relationship between the discrete and cable graph versions. We briefly recall those couplings in Sections 2.3 and 2.4 as they are implicit in this paper. In Section 2.5 we state and prove the generalized second Ray-Knight “version” of Lupu’s isomorphism, which we aim to invert.

Section 3 is devoted to the statements of inversions of those isomorphisms. We state in Section 3.1 a signed version of the inversion of the generalized second Ray-Knight theorem through an extra Poisson Point Process, namely Theorem 3. In Section 3.2 we provide a discrete-time description of the process, whereas in Section 3.3 we yield an alternative version of that process through jump rates, which can be seen as an annealed version of the first one.
We deduce a signed inversion of Le Jan’s isomorphism for loop soups in Section 3.4 and an
inversion of the coupling of random current with FK-Ising in Section 5.2.
Finally Section 4 is devoted to the proof of Theorem 4.9. Section 4.4 deals with the case of a
finite graph without killing measure, and Section 4.2 deduces the proof in the general case.

2. Le Jan’s and Lupu’s isomorphisms

2.1. Loop soups and Le Jan’s isomorphism. The loop measure associated to the Markov
jump process \((X_t)_{0 \leq t < \zeta}\) is defined as follows. Let \(P_{x,y}^t\) be the bridge probability
measure from \(x\) to \(y\) in time \(t\) (conditioned on \(t < \zeta\)). Let \(p_t(x,y)\) be the transition probabilities of \((X_t)_{0 \leq t < \zeta}\).

Let \(\mu_{\text{loop}}\) be the measure on time-parametrised nearest-neighbour based loops (i.e. loops
with a starting point)

\[
\mu_{\text{loop}} = \sum_{x \in V} \int_{t > 0} P_{x,x}^t p_t(x,x) \frac{dt}{t}.
\]

The loops will be considered here up to a rotation of parametrisation (with the corresponding
pushforward measure induced by \(\mu_{\text{loop}}\), that is to say a loop \((\gamma(t))_{0 \leq t \leq t_\gamma}\) will be the same as
\((\gamma(T + t))_{0 \leq t \leq t_\gamma} \circ (\gamma(T + t - t_\gamma))_{t_\gamma - t \leq t \leq t_\gamma},\) where \(\circ\) denotes the concatenation of paths. A
loop soup of intensity \(\alpha > 0\), denoted \(L_\alpha\), is a Poisson random measure of intensity \(\alpha \mu_{\text{loop}}\). We
see it as a random collection of loops in \(G\). Observe that a.s. above each vertex \(x \in V\), \(L_\alpha\)
contains infinitely many trivial “loops” reduced to the vertex \(x\). There are also with positive
probability non-trivial loop that visit several vertices.

Let \(L_x(L_\alpha)\) be the occupation field of \(L_\alpha\) on \(V\) i.e., for all \(x \in V\),

\[
L_x(L_\alpha) = \sum_{(\gamma(t))_{0 \leq t \leq t_\gamma} \in L_\alpha} \int_0^{t_\gamma} 1\{\gamma(t) = x\} dt.
\]

In [3] Le Jan shows that for transient Markov jump processes, \(L_x(L_\alpha) < +\infty\) for all \(x \in V\) a.s.
For \(\alpha = \frac{1}{2}\) he identifies the law of \(L_x(L_\alpha)\):

Le Jan’s isomorphism. \(L_x(L_{1/2}) = (L_x(L_{1/2}))_{x \in V}\) has the same law as
\(\frac{1}{2} \varphi^2 = \left(\frac{1}{2} \varphi_{x_0}^2\right)_{x \in V}\)
under \(P_\varphi\).

Let us briefly recall how Le Jan’s isomorphism enables one to retrieve the generalized second
Ray-Knight theorem stated in Section 3.1 for more details, see for instance [4]. We assume that
\(\kappa\) is supported by \(x_0\): the general case can be dealt with by an argument similar to the proof of
Proposition 4.6. Let \(D = V \setminus \{x_0\}\), and note that the isomorphism in particular implies that
\(L_x(L_{1/2})\) conditionally on \(L_{x_0}(L_{1/2}) = u\) has the same law as \(\varphi^2/2\) conditionally on \(\varphi_{x_0}^2/2 = u\).

On the one hand, given the classical energy decomposition, we have \(\varphi = \varphi^D + \varphi_{x_0}\), with \(\varphi^D\)
the GFF associated to the restriction of \(E\) to \(D\), where \(\varphi^D\) and \(\varphi_{x_0}\) are independent. Now \(\varphi^2/2\)
conditionally on \(\varphi_{x_0}^2/2 = u\) has the law of \((\varphi^D + \eta \sqrt{2u})^2/2\), where \(\eta\) is the sign of \(\varphi_{x_0}\), which
is independent of \(\varphi^D\). But \(\varphi^D\) is symmetric, so that the latter also has the law of \((\varphi^D + \sqrt{2u})^2/2\).

On the other hand, the loop soup \(L_{1/2}\) can be decomposed into the two independent loop
soups \(L_{1/2}^D\) contained in \(D\) and \(L_{1/2}^{(x_0)}\) hitting \(x_0\). Now \(L_x(L_{1/2}^D)\) has the law of \((\varphi^D)^2/2\) and
\(L_x(L_{1/2}^{(x_0)})\) conditionally on \(L_{x_0}(L_{1/2}^{(x_0)}) = u\) has the law of the occupation field of the Markov
chain \(\ell(\tau_u^{x_0})\) under \(P_{x_0}(\cdot | \tau_u^{x_0} < \zeta)\), which enables us to conclude.
2.2. Lupu’s isomorphism. As in [8], we consider the metric graph $\tilde{G}$ associated to $G$. Each edge $e$ is replaced by a continuous line of length $\frac{1}{2}W_e^{-1}$.

The GFF $\varphi$ on $G$ with law $P_\varphi$ can be extended to a GFF $\tilde{\varphi}$ on $\tilde{G}$ as follows. Given $e \in E$, one considers inside $e$ a conditionally independent Brownian bridge, actually a bridge of a $\sqrt{2} \times$ standard Brownian motion, of length $\frac{1}{2}W_e^{-1}$, with end-values $\varphi_{e_-}$ and $\varphi_{e_+}$. This provides a continuous field on the metric graph which satisfies the spatial Markov property.

Similarly one can define a standard Brownian motion $(B_t^e)_{0 \leq t \leq \zeta}$ on $\tilde{G}$, whose trace on $G$ indexed by the local times at $V$ has the same law as the Markov process $(X_t)_{t \geq 0}$ on $V$ with jump rate $W_e$ to an adjacent edge $e$ up to time $\zeta$, as explained in Section 2 of [8]. One can associate a measure on time-parametrized continuous loops $\mu$, and let $\tilde{L}_\frac{1}{2}$ be the Poisson Point Process of loops of intensity $\mu/2$: the discrete-time loops $L_{\frac{1}{2}}$ can be obtained from $\tilde{L}_{\frac{1}{2}}$ by taking the print of the latter on $V$.

Lupu introduced in [6] an isomorphism linking the GFF $\tilde{\varphi}$ and the loop soup $\tilde{L}_{\frac{1}{2}}$ on $\tilde{G}$.

**Theorem 1** (Lupu’s Isomorphism,[8]). There is a coupling between the Poisson ensemble of loops $\tilde{L}_{\frac{1}{2}}$ and $(\tilde{\varphi}_y)_{y \in \tilde{G}}$ defined above, such that the two following constraints hold:

- For all $y \in \tilde{G}$, $L_y(\tilde{L}_{\frac{1}{2}}) = \frac{1}{2}\tilde{\varphi}_y^2$
- The clusters of loops of $\tilde{L}_{\frac{1}{2}}$ are exactly the sign clusters of $(\tilde{\varphi}_y)_{y \in \tilde{G}}$.

Conditionally on $(|\tilde{\varphi}_y|)_{y \in \tilde{G}}$, the sign of $\tilde{\varphi}$ on each of its connected components is distributed independently and uniformly in $\{-1,+1\}$.

Lupu’s isomorphism and the idea of using metric graphs were applied in [5] to show that on the discrete half-plane $\mathbb{Z} \times \mathbb{N}$, the scaling limits of outermost boundaries of clusters of loops in loop soups are the Conformal Loop Ensembles CLE.

Let $\mathcal{O}(\varphi)$ (resp. $\mathcal{O}(\tilde{L}_{\frac{1}{2}})$) be the set of edges $e \in E$ such that $\tilde{\varphi}$ (resp. $\tilde{L}_{\frac{1}{2}}$) does not touch 0 on $e$, in other words such that all the edge $e$ remains in the same sign cluster of $\tilde{\varphi}$ (resp. $\tilde{L}_{\frac{1}{2}}$). Let $\mathcal{O}(L_{\frac{1}{2}})$ be the set of edges $e \in E$ that are crossed (i.e. visited consecutively) by the trace of the loops $L_{\frac{1}{2}}$ on $V$.

In order to translate Lupu’s isomorphism back onto the initial graph $G$, one needs to describe on one hand the distribution of $\mathcal{O}(\varphi)$ conditionally on the values of $\varphi$, and on the other hand the distribution of $\mathcal{O}(\tilde{L}_{\frac{1}{2}})$ conditionally on $L_{\frac{1}{2}}$ and the cluster of loops $\mathcal{O}(L_{\frac{1}{2}})$ on the discrete graph $G$. These two distributions are described respectively in Subsections 2.3 and 2.4 and provide realisations of the FK-Ising coupling and the “Current+Bernoulli=FK” coupling lemma [7].

2.3. The FK-Ising distribution of $\mathcal{O}(\tilde{\varphi})$ conditionally on $|\varphi|$.

**Lemma 2.1.** Conditionally on $(\varphi_x)_{x \in V}$, $(1_{e \in \mathcal{O}(\tilde{\varphi})})_{e \in E}$ is a family of independent random variables and

$$P(e \notin \mathcal{O}(\tilde{\varphi})|\varphi) = \begin{cases} 1 & \text{if } \varphi_{e_-} \varphi_{e_+} < 0, \\ \exp \left( -2W_e \varphi_{e_-} \varphi_{e_+} \right) & \text{if } \varphi_{e_-} \varphi_{e_+} > 0. \end{cases}$$

**Proof:** Conditionally on $(\varphi_x)_{x \in V}$, are constructed as independent Brownian bridges on each edge, so that $(1_{e \in \mathcal{O}(\tilde{\varphi})})_{e \in E}$ are independent random variables, and it follows from the reflection principle that, if $\varphi_{e_-} \varphi_{e_+} > 0$, then

$$P(e \notin \mathcal{O}(\tilde{\varphi})|\varphi) = \frac{\exp \left( -\frac{1}{2}W_e (\varphi_{e_-} + \varphi_{e_+})^2 \right)}{\exp \left( -\frac{1}{2}W_e (\varphi_{e_-} - \varphi_{e_+})^2 \right)} = \exp \left( -2W_e \varphi_{e_-} \varphi_{e_+} \right).$$

□
Let us now recall how the conditional probability in Lemma 2.1 yields a realisation of the FK-Ising coupling.

Assume \( V \) is finite. Let \((J_e)_{e \in E}\) be a family of positive weights. An Ising model on \( V \) with interaction constants \((J_e)_{e \in E}\) is a probability on configuration of spins \((\sigma_x)_{x \in V} \in \{+1, -1\}^V\) such that

\[
P^\text{Ising}_J((\sigma_x)_{x \in V}) = \frac{1}{Z^\text{Ising}_J} \exp \left( \sum_{e \in E} J_e \sigma_{e^+} \sigma_{e^-} \right).
\]

An FK-Ising random cluster model with weights \((1 - e^{-2J_e})_{e \in E}\) is a random configuration of open (value 1) and closed edges (value 0) such that

\[
P^\text{FK-Ising}_J((\sigma_x)_{x \in V}) = \frac{1}{Z^\text{FK-Ising}_J} \times 2^\sharp \text{clusters} \prod_{e \in E} (1 - e^{-2J_e})^{\omega_e} (e^{-2J_e})^{1-\omega_e},
\]

where \(\sharp\) clusters\) denotes the number of clusters created by open edges.

The well-known FK-Ising and Ising coupling reads as follows.

**Proposition 2.2** (FK-Ising and Ising coupling). Given an FK-Ising model, sample on each cluster an independent uniformly distributed spin. The spins are then distributed according to the Ising model. Conversely, given a spins configuration \(\sigma\) following the Ising distribution, consider each edge \(e\), such that \(\sigma_{e^+}, \sigma_{e^-} < 0\), closed, and each edge \(e\), such that \(\sigma_{e^+}, \sigma_{e^-} > 0\) open with probability \(1 - e^{-2J_e}\). Then the open edges are distributed according to the FK-Ising model. The two coupleings between FK-Ising and Ising are the same.

Consider the GFF \(\varphi\) on \(G\) distributed according to \(P\). Let \(J_e(\varphi)\) be the random interaction constants

\[J_e(\varphi) = W_e|\varphi_{e^-} - \varphi_{e^+}|.\]

Conditionally on \(\varphi\), \((\text{sign}(\varphi_x))_{x \in V}\) follows an Ising distribution with interaction constants \((J_e(\varphi))_{e \in E}\). Indeed, the Dirichlet form (1.1) can be written as

\[
\mathcal{E}(\varphi, \varphi) = \sum_{x \in V} \kappa_x \varphi(x)^2 + \sum_{x \in V} (\varphi(x))^2 \sum_{y \sim x} W_{x,y} - 2 \sum_{e \in E} J_e(\varphi) \text{sign}(\varphi(e^-)) \text{sign}(\varphi(e^+)).
\]

Similarly, when \(\varphi \sim P_{1/\sqrt{2u}}\) has boundary condition \(\sqrt{2u} \geq 0\) on \(v_0\), then \((\text{sign}(\varphi_x))_{x \in V}\) has an Ising distribution with interaction \((J_e(\varphi))_{e \in E}\) and conditioned on \(\sigma_{v_0} = +1\).

Now, conditionally on \(\varphi\), \(\mathcal{O}(\varphi)\) has FK-Ising distribution with weights \((1 - e^{-2J_e(\varphi)})_{e \in E}\). Indeed, the probability for \(e \in \mathcal{O}(\varphi)\) conditionally on \(\varphi\) is \(1 - e^{-2J_e(\varphi)}\), by Lemma 2.1 as in Proposition 2.2.

Note that, given that \(\mathcal{O}(\varphi)\) has FK-Ising distribution, the fact that the sign of on its connected components is distributed independently and uniformly in \(-1, 1\) can be seen either as a consequence of Proposition 2.2 or from Theorem 1.

Given \(\varphi = (\varphi_x)_{x \in V}\) on the discrete graph \(G\), we introduce in Definition 2.1 as the random set of edges which has the distribution of \(\mathcal{O}(\varphi)\) conditionally on \(\varphi = (\varphi_x)_{x \in V}\) given by Lemma 2.2.

**Definition 2.1.** We let \(\mathcal{O}(\varphi)\) be a random set of edges which has the distribution of \(\mathcal{O}(\varphi)\) conditionally on \(\varphi = (\varphi_x)_{x \in V}\) given by Lemma 2.2.

2.4. Distribution of \(\mathcal{O}(\hat{L}_x)\) conditionally on \(L_x\). The distribution of \(\mathcal{O}(\hat{L}_x)\) conditionally on \(L_x\) can be retrieved by Corollary 3.6 in [6], which reads as follows.
Lemma 2.3 (Corollary 3.6 in [6]). Conditionally on $\mathcal{L}_2$, the events $e \notin \mathcal{O}(\mathcal{L}_2)$, $e \in E \setminus \mathcal{O}(\mathcal{L}_2)$, are independent and have probability
\begin{equation}
\exp \left(-2W_e \sqrt{L_{e+}(\mathcal{L}_2)L_{e-}(\mathcal{L}_2)}\right).
\end{equation}

This result gives rise, together with Theorem 1 to the following discrete version of Lupu’s isomorphism, which is stated without any recourse to the cable graph induced by $G$.

Definition 2.2. Let $(\omega_e)_{e \in E} \in \{0,1\}^E$ be a percolation defined as follows: conditionally on $\mathcal{L}_2$, the random variables $(\omega_e)_{e \in E}$ are independent, and $\omega_e$ equals 0 with conditional probability given by (2.1).

Let $\mathcal{O}_+(\mathcal{L}_2)$ the set of edges:
$$\mathcal{O}_+(\mathcal{L}_2) = \mathcal{O}(\mathcal{L}_2) \cup \{e \in E | \omega_e = 1\}.$$ 

Proposition 2.4 (Discrete version of Lupu’s isomorphism, Theorem 1 bis in [6]). Given a loop soup $\mathcal{L}_2$, let $\mathcal{O}_+(\mathcal{L}_2)$ be as in Definition 2.2. Let $(\sigma_x)_{x \in V} \in \{-1,+1\}^V$ be random spins taking constant values on clusters induced by $\mathcal{O}_+(\mathcal{L}_2)$ ($\sigma_{e-} = \sigma_{e+}$ if $e \in \mathcal{O}_+(\mathcal{L}_2)$) and such that the values on each cluster, conditional on $\mathcal{L}_2$ and $\mathcal{O}_+(\mathcal{L}_2)$, are independent and uniformly distributed. Then
$$\left(\sigma_x \sqrt{2L_x(\mathcal{L}_2)}\right)_{x \in V}$$
is a Gaussian free field distributed according to $P_\varphi$.

Proposition 2.4 induces the following coupling between FK-Ising and random currents.

If $V$ is finite, a random current model on $G$ with weights $(J_e)_{e \in E}$ is a random assignment to each edge $e$ of a non-negative integer $\hat{n}_e$ such that for all $x \in V$,
$$\sum_{e \text{ adjacent to } x} \hat{n}_e$$
is even, which is called the parity condition. The probability of a configuration $(n_e)_{e \in E}$ satisfying the parity condition is
$$P_{RC}(\forall e \in E, \hat{n}_e = n_e) = \frac{1}{Z_{RC}} \prod_{e \in E} \frac{(J_e)^{n_e}}{n_e!},$$
where actually $Z_{RC} = Z_{RC}^{Ising}$. Let
$$\mathcal{O}(\hat{n}) = \{e \in E | \hat{n}_e > 0\}.$$ 

The open edges in $\mathcal{O}(\hat{n})$ induce clusters on the graph $G$.

Given a loop soup $\mathcal{L}_2$, we denote by $N_e(\mathcal{L}_2)$ the number of times the loops in $\mathcal{L}_2$ cross the nonoriented edge $e \in E$. The transience of the Markov jump process $X$ implies that $N_e(\mathcal{L}_2)$ is a.s. finite for all $e \in E$. If $\alpha = \frac{1}{2}$, we have the following identity (see for instance [11]):

Loop soup and random current. Assume $V$ is finite and consider the loop soup $\mathcal{L}_2$. Conditionally on the occupation field $(L_x(\mathcal{L}_2))_{x \in V}$, $(N_e(\mathcal{L}_2))_{e \in E}$ is distributed as a random current model with weights $\left(2W_e \sqrt{L_{e-}(\mathcal{L}_2)L_{e+}(\mathcal{L}_2)}\right)_{e \in E}$. If $\varphi$ is the GFF on $G$ given by Le Jan’s or Lupu’s isomorphism, then these weights are $(J_e(|\varphi|))$.
Conditionally on the occupation field \((L_x(\mathcal{L}_\downarrow))_{x \in V}, \mathcal{O}(\mathcal{L}_\downarrow)\) are the edges occupied by a random current and \(\mathcal{O}_+(\mathcal{L}_\downarrow)\) the edges occupied by FK-Ising. Lemma 2.1 and Proposition 2.4 imply the following coupling, as noted by Lupu and Werner in [7].

**Proposition 2.5** (Random current and FK-Ising coupling, [7]). Assume \(V\) is finite. Let \(\hat{n}\) be a random current on \(G\) with weights \((J_e)_{e \in E}\). Let \((\omega_e)_{e \in E} \in \{0,1\}^E\) be an independent percolation, each edge being opened (value 1) independently with probability \(1 - e^{-J_e}\). Then

\[
\mathcal{O}(\hat{n}) \cup \{e \in E|\omega_e = 1\}
\]
is distributed like the open edges in an FK-Ising with weights \((1 - e^{-2J_e})_{e \in E}\).

2.5. **Generalized second Ray-Knight “version” of Lupu’s isomorphism.** We are now in a position to state the coupled version of the second Ray-Knight theorem.

**Theorem 2.** Let \(x_0 \in V\). Let \((\varphi_x^{(0)})_{x \in V}\) with distribution \(P_{\varphi}^{(x_0), 0}\), and define \(\mathcal{O}(\varphi^{(0)})\) as in Definition 2.1. Let \(X\) be an independent Markov jump process started from \(x_0\).

Fix \(u > 0\). If \(\tau_u^{x_0} < \zeta\), we let \(\mathcal{O}_u\) be the random subset of \(E\) which contains \(\mathcal{O}(\varphi^{(0)}),\) the edges used by the path \((X_t)_{0 \leq t \leq \tau_u^{x_0}}\), and additional edges \(e\) opened conditionally independently with probability

\[
1 - e^{W_{e}|\varphi_x^{(0)}|} - W_e \sqrt{\left(\varphi_x^{(0)} + 4\varphi_x^{(0)}(\tau_u^{x_0})\right)}.
\]

We let \(\sigma \in \{-1, +1\}\) be random spins sampled uniformly independently on each cluster induced by \(\mathcal{O}_u\), pinned at \(x_0\), i.e. \(\sigma_{x_0} = 1\), and define

\[
\varphi_x^{(u)} := \sigma_x \sqrt{\varphi_x^{(0)} + 2\varphi_x^{(0)}(\tau_u^{x_0})}.
\]

Then, conditionally on \(\tau_u^{x_0} < \zeta\), \(\varphi^{(u)}\) has distribution \(P_{\varphi}^{(x_0), \sqrt{2u}}\), and \(\mathcal{O}_u\) has distribution \(\mathcal{O}(\varphi^{(u)})\) conditionally on \(\varphi^{(u)}\).

**Remark 2.6.** One consequence of that coupling is that the path \((X_s)_{s \leq \tau_u^{x_0}}\) stays in the positive connected component of \(x_0\) for \(\varphi^{(u)}\). This yields a coupling between the range of the Markov chain and the sign component of \(x_0\) inside a GFF \(P_{\varphi}^{(x_0), \sqrt{2u}}\).

**Proof of Theorem 2.** The proof is based on [6]. Let \(D = V \setminus \{x_0\}\), and let \(\hat{\mathcal{L}}_D^{\downarrow}\) be the loop soup of intensity 1/2 on the graph \(\hat{G}\), which we decompose into \(\hat{\mathcal{L}}_D^{(x_0)}\) (resp. \(\hat{\mathcal{L}}_D^{\downarrow}\)) the loop soup hitting (resp. not hitting) \(x_0\), which are independent. We let \(\mathcal{L}_D^{\downarrow}\) and \(\mathcal{L}_D^{(x_0)}\) (resp. \(\hat{\mathcal{L}}_D^{\downarrow}\)) be the prints of these loop soups on \(V\) (resp. on \(D = V \setminus \{x_0\}\)). We condition on \(L_{x_0}(\mathcal{L}_D^{\downarrow}) = u\).

Theorem 1 implies (recall also Definition 2.1) that we can couple \(\hat{\mathcal{L}}_D^{\downarrow}\) with \(\varphi^{(0)}\) so that

\[
L_x(\mathcal{L}_D^{\downarrow}) = \varphi_x^{(0)} / 2 \text{ for all } x \in V, \text{ and } \mathcal{O}(\hat{\mathcal{L}}^{\downarrow}) = \mathcal{O}(\varphi^{(0)}).
\]

Define \(\varphi^{(u)} = (\varphi_x^{(u)})_{x \in V}\) from \(\hat{\mathcal{L}}^{\downarrow}\) by, for all \(x \in V\),

\[
|\varphi_x^{(u)}| = \sqrt{2L_x(\mathcal{L}_D^{\downarrow})}
\]

and \(\varphi_x^{(u)} = \sigma_x|\varphi_x^{(u)}|\), where \(\sigma \in \{-1, +1\}\) are random spins sampled uniformly independently on each cluster induced by \(\mathcal{O}(\hat{\mathcal{L}}^{\downarrow})\), pinned at \(x_0\), i.e. \(\sigma_{x_0} = 1\). Then, by Theorem 1, \(\varphi^{(u)}\) has distribution \(P_{\varphi}^{(x_0), \sqrt{2u}}\).
For all \( x \in V \), we have
\[
L_x(\hat{L}_\frac{1}{2}) = \frac{\zeta_x^2}{2} + L_x(\mathcal{L}^{(x_0)}_\frac{1}{2}).
\]

On the other hand, conditionally on \( L_{x_0}(\mathcal{L}^{(x_0)}_\frac{1}{2}) \),
\[
\mathbb{P}(e \notin O(\hat{L}_\frac{1}{2}) \mid e \notin O(\mathcal{L}^D_\frac{1}{2}) \cup O(\mathcal{L}^{(x_0)}_\frac{1}{2})) = \frac{\mathbb{P}(e \notin O(\hat{L}_\frac{1}{2}))}{\mathbb{P}(e \notin O(\hat{L}_\frac{1}{2}) \cup O(\mathcal{L}^{(x_0)}_\frac{1}{2}))} = \frac{\mathbb{P}(e \notin O(\hat{L}_\frac{1}{2}))}{\mathbb{P}(e \notin O(\mathcal{L}^D_\frac{1}{2}) \mid e \notin O(\mathcal{L}^{(x_0)}_\frac{1}{2}))}
\]
where we use in the third equality that the event \( e \notin O(\mathcal{L}^D_\frac{1}{2}) \) is measurable with respect to the \( \sigma \)-field generated by \( \hat{L}_\frac{1}{2} \), which is independent of \( \hat{L}^{(x_0)}_\frac{1}{2} \), and where we use Lemma 2.3 in the fourth equality, for \( \hat{L}_\frac{1}{2} \) and for \( \mathcal{L}^D_\frac{1}{2} \).

We conclude the proof by observing that \( \mathcal{L}^{(x_0)}_\frac{1}{2} \) conditionally on \( L_{x_0}(\mathcal{L}^{(x_0)}_\frac{1}{2}) = u \) has the law of the occupation field of the Markov chain \( \ell(\tau_u^{x_0}) \) under \( \mathbb{P}_{x_0}(\cdot | \tau_u^{x_0} < \zeta) \).

\[\square\]

### 3. Inversion of the Signed Isomorphism

In [9], Sabot and Tarrès give a new proof of the generalized second Ray-Knight theorem together with a construction that inverts the coupling between the square of a GFF conditioned by its value at a vertex \( x_0 \) and the excursions of the jump process \( X \) from and to \( x_0 \). In this paper we are interested in inverting the coupling of Theorem 2 with the signed GFF: more precisely, we want to describe the law of \( (X_t)_{0 \leq t \leq \tau_u^{x_0}} \) conditionally on \( \varphi(u) \).

We present in Section 3.1 an inversion involving an extra Poisson process. We provide in Section 3.2 a discrete-time description of the process and in Section 3.3 an alternative description via jump rates. Sections 3.4 and 3.5 are respectively dedicated to a signed inversion of Le Jan’s isomorphism for loop soups, and to an inversion of the coupling of random current with FK-Ising.

#### 3.1. A description via an Extra Poisson point process

Let \( (\tilde{\varphi}_x)_{x \in V} \) be a real function on \( V \) such that \( \tilde{\varphi}_{x_0} = +\sqrt{2u} \) for some \( u > 0 \). Set
\[
\tilde{\Phi}_x = |\tilde{\varphi}_x|, \quad \sigma_x = \text{sign}(\tilde{\varphi}_x).
\]

We define a self-interacting process \( (\tilde{\tilde{X}}_t, (\tilde{n}_e(t))_{e \in E}) \) living on \( V \times \mathbb{N}^E \) as follows. The process \( \tilde{\tilde{X}} \) starts at \( \tilde{\tilde{X}}(0) = x_0 \). For \( t \geq 0 \), we set
\[
\tilde{\tilde{\Phi}}_x(t) = \sqrt{(\tilde{\Phi}_x)^2 - 2\tilde{\ell}_x(t)}, \quad \forall x \in V, \quad \text{and} \quad J_e(\tilde{\Phi}(t)) = W_e\tilde{\Phi}_{e^-}(t)\tilde{\Phi}_{e^+}(t), \quad \forall e \in E.
\]

where \( \tilde{\ell}_x(t) = \int_0^t 1_{\{\tilde{\tilde{X}}_s = x\}} ds \) is the local time of the process \( \tilde{\tilde{X}} \) up to time \( t \). Let \( (N_e(u))_{u \geq 0} \) be an independent Poisson Point Processes on \( \mathbb{R}_+ \) with intensity 1, for each edge \( e \in E \). We set
\[
\tilde{n}_e(t) = \begin{cases} N_e(2J_e(t)), & \text{if } \sigma_e - \sigma_{e^+} = +1, \\ 0, & \text{if } \sigma_e - \sigma_{e^+} = -1. \end{cases}
\]
We also denote by $\hat{C}(t) \subset E$ the configuration of edges such that $\hat{n}_e(t) > 0$. As time increases, the interaction parameters $J_e(\hat{\Phi}(t))$ decreases for the edges neighboring $\hat{X}_t$, and at some random times $\hat{n}_e(t)$ may drop by 1. The process $(\hat{X}_t)_{t \geq 0}$ is defined as the process that jumps only at the times when one of the $\hat{n}_e(t)$ drops by 1, as follows:

- if $\hat{n}_e(t)$ decreases by 1 at time $t$, but does not create a new cluster in $\hat{C}_t$, then $\hat{X}_t$ crosses the edge $e$ with probability $1/2$ or does not move with probability $1/2$,
- if $\hat{n}_e(t)$ decreases by 1 at time $t$, and does create a new cluster in $\hat{C}_t$, then $\hat{X}_t$ moves/or stays with probability 1 on the unique extremity of $e$ which is in the cluster of the origin $x_0$ in the new configuration.

We set

$$\hat{T} := \inf\{t \geq 0, \exists x \in V, \text{s. t. } \hat{\Phi}_x(t) = 0\},$$

Clearly, the process is well-defined up to time $\hat{T}$.

**Proposition 3.1.** For all $0 \leq t \leq \hat{T}$, $\hat{X}_t$ is in the connected component of $x_0$ of the configuration $\hat{C}(t)$. If $V$ is finite, the process ends at $x_0$, i.e. $\hat{X}_{\hat{T}} = x_0$.

**Theorem 3.** Assume that $V$ is finite. With the notation of Theorem 2, conditioned on $\varphi^{(u)} = \hat{\varphi}$, $(X_t)_{t \leq t_{\hat{T}}} = \hat{\varphi}$ has the law of $(\hat{X}_{\hat{T}^{-1}})_{0 \leq t \leq \hat{T}}$.

Moreover, conditioned on $\varphi^{(u)} = \hat{\varphi}$, $(\varphi^{(0)}, O(\varphi^{(0)}))$ has the law of $(\sigma'_x \hat{\Phi}_x(\hat{T}), \hat{C}(\hat{T}))$ where $(\sigma'_x)_{x \in V} \in \{-1,+1\}^V$ are random spins sampled uniformly independently on each cluster induced by $\hat{C}(T)$, with the condition that $\sigma'_x = +1$.

If $V$ is infinite, then $P_{\hat{\varphi}}^{\{x_0\}\setminus E}$ a.s., $\hat{X}_t$ (with the initial condition $\hat{\varphi} = \varphi^{(u)}$) ends at $x_0$, i.e. $\hat{T} < +\infty$ and $\hat{X}_{\hat{T}} = x_0$. All previous conclusions for the finite case still hold.

### 3.2. Discrete time description of the process

We give a discrete time description of the process $(\hat{X}_t, (\hat{n}_e(t))_{e \in E})$ that appears in the previous section. Let $t_0 = 0$ and $0 < t_1 < \cdots < t_j$ be the stopping times when one of the stacks $\hat{n}_e(t)$ decreases by 1, where $t_j$ is the time when one of the stacks is completely depleted. It is elementary to check the following:

**Proposition 3.2.** The discrete time process $(\hat{X}_t, (\hat{n}_e(t))_{e \in E})_{0 \leq t \leq j}$ is a stopped Markov process. The transition from time $i - 1$ to $i$ is the following:

- first chose $e$ an edge adjacent to the vertex $\hat{X}_{t_{i-1}}$ according to a probability proportional to $\hat{n}_e(t_{i-1})$,
- decrease the stack $\hat{n}_e(t_{i-1})$ by 1,
- if decreasing $\hat{n}_e(t_{i-1})$ by 1 does not create a new cluster in $\hat{C}_{t_{i-1}}$, then $\hat{X}_{t_{i-1}}$ crosses the edge $e$ with probability $1/2$ or does not move with probability $1/2$,
- if decreasing $\hat{n}_e(t_{i-1})$ by 1 does create a new cluster in $\hat{C}_{t_{i-1}}$, then $\hat{X}_{t_{i-1}}$ moves/or stays with probability 1 on the unique extremity of $e$ which is in the cluster of the origin $x_0$ in the new configuration.

### 3.3. An alternative description via jump rates

We provide an alternative description of the process $(\hat{X}_t, \hat{C}(t))$ that appears in Section 3.1.

**Proposition 3.3.** The process $(\hat{X}_t, \hat{C}(t))$ defined in section 3.1 can be alternatively described by its jump rates: conditionally on its past at time $t$, if $\hat{X}_t = x, y \sim x$ and $(x, y) \in \hat{C}(t)$, then

1. $\hat{X}$ jumps to $y$ without modification of $\hat{C}(t)$ at rate

$$W_{x,y} \frac{\hat{\Phi}_y(t)}{\hat{\Phi}_x(t)}$$
Proposition 3.3 as follows. Consider the representation of Proposition 3.3 on the graph where Remark 3.5. One can also retrieve the process in Section 3.1 from the representation in Remark 3.4. It is clear from this description that the joint process of the coupling in Theorem 2. Similarly, (2) follows from the following computation:

\[ \frac{2W_{x,y}}{\Phi_x(t)} \left( e^{2W_{x,y} \tilde{\Phi}_x(t) \Phi_y(t)} - 1 \right)^{-1} \]

and, conditionally on that last event:
- if \( y \) is connected to \( x \) in the configuration \( \check{C}(t) \setminus \{x, y\} \), then \( \check{X} \) simultaneously jumps to \( y \) with probability \( 1/2 \) and stays at \( x \) with probability \( 1/2 \).
- otherwise \( \check{X} \) moves/or stays with probability 1 on the unique extremity of \( \{x, y\} \) which is in the cluster of the origin \( x_0 \) in the new configuration.

**Remark 3.4.** It is clear from this description that the joint process \( (\check{X}_t, \check{C}(t), \check{\Phi}(t)) \) is Markov process, and well defined up to the time \( \check{T} := \inf\{t \geq 0 : \exists x \in V, \ s.t. \ \check{\Phi}_x(t) = 0\} \).

**Remark 3.5.** One can also retrieve the process in Section 3.1 from the representation in Proposition 3.3 as follows. Consider the representation of Proposition 3.3 on the graph where each edge \( e \) is replaced by a large number \( N \) of parallel edges with conductance \( W_e/N \). Consider now \( n_e(N) = 1 \) the number of parallel edges that are open in the configuration \( \check{C}(t) \) between \( x \) and \( y \). Then, when \( N \to \infty \), \( (n_e(N))_{t \geq 0} \), converges in law to \( (n_e(t))_{t \geq 0} \), defined in section 3.1.

**Proof of Proposition 3.3.** Assume \( \check{X}_t = x \), fix \( y \sim x \) and let \( e = \{x, y\} \). Recall that \( \{x, y\} \in \check{C}(t) \) iff \( n_e(t) \geq 1 \).

Let us first prove (1):

\[ \mathbb{P} (\check{X} \text{ jumps to } y \text{ on time interval } [t, t + \Delta t] \text{ without modification of } \check{C}(t) \mid \{x, y\} \in \check{C}(t)) = \frac{1}{2} \mathbb{P} (n_e(t) - n_e(t + \Delta t) = 1, n_e(t + \Delta t) \geq 1 \mid n_e(t) \geq 1) \]

\[ = \frac{1}{2} (2J_e(t) - 2J_e(t + \Delta t)) + o(\Delta t) = W_{x,y} \frac{\Phi_y(t)}{\Phi_x(t)} \Delta t + o(\Delta t). \]

Similarly, (2) follows from the following computation:

\[ \mathbb{P} (\{x, y\} \text{ closed in } \check{C}(t + \Delta t) \mid \{x, y\} \in \check{C}(t)) = \mathbb{P} (n_e(t + \Delta t) = 0 \mid n_e(t) \geq 1) \]

\[ = \frac{\mathbb{P} (n_e(t) = 1, n_e(t + \Delta t) = 0)}{\mathbb{P} (n_e(t) \geq 1)} = \frac{e^{-2J_e(t)} 2J_e(t)}{1 - e^{-2J_e(t)}} (J_e(t) - J_e(t + \Delta t)) + o(\Delta t) \]

We easily deduce from the Proposition 3.3 and Theorem 2 the following alternative inversion of the coupling in Theorem 4.

**Theorem 4.** With the notation of Theorem 2, conditionally on \( (\varphi^{(0)}, O_u) \), \( (\check{X}_t)_{t \leq T_{x_0}} \) has the law of self-interacting process \( (\check{X}_{T-t})_{0 \leq t \leq T} \) defined by jump rates of Proposition 3.3 starting with \( \check{\Phi}_x = \sqrt{\varphi^{(0)}(x)^2 + 2\ell_x(x_0)} \) and \( \check{C}(0) = O_u \).

Moreover \( (\varphi^{(0)}, O(\varphi^{(0)})) \) has the same law as \( (\sigma^\varphi(T), \check{C}(T)) \) where \( (\sigma^\varphi)_{x \in V} \) is a configuration of signs obtained by picking a sign at random independently on each connected component of \( \check{C}(T) \), with the condition that the component of \( x_0 \) has a + sign.
3.4. A signed version of Le Jan’s isomorphism for loop soup. Let us first recall how the loops in \( L_\alpha \) are connected to the excursions of the jump process \( X \).

**Proposition 3.6** (From excursions to loops). Let \( \alpha > 0 \) and \( x_0 \in V \). \( L_{x_0}(L_\alpha) \) is distributed according to a Gamma \( \Gamma(\alpha, G(x_0, x_0)) \) law, where \( G \) is the Green’s function. Let \( u > 0 \), and consider the path \((X_t)_{0 \leq t \leq \tau^u_{x_0}}\) conditioned on \( \tau^u_{x_0} < \zeta \). Let \((Y_j)_{j \geq 1}\) be an independent Poisson-Dirichlet partition \( PD(0,\alpha) \) of \([0,1]\). Let \( S_0 = 0 \) and

\[
S_j = \sum_{i=1}^j Y_i.
\]

Let

\[
\tau_j = \tau^u_{x_0} S_j.
\]

Consider the family of paths

\[
((X_{\tau_{j-1}+t})_{0 \leq t \leq \tau_j - \tau_{j-1}})_{j \geq 1}.
\]

It is a countable family of loops rooted in \( x_0 \). It has the same law as the family of all the loops in \( L_\alpha \) that visit \( x_0 \), conditioned on \( L_{x_0}(L_\alpha) = u \).

Next we describe how to invert the discrete version of Lupu’s isomorphism Proposition 2.4 for the loop-soup in the same way as in Theorem 3.

Let \((\tilde{\phi}_x)_{x \in V}\) be a real function on \( V \) such that \( \tilde{\phi}_{x_0} = +\sqrt{2u} \) for some \( u > 0 \). Set

\[
\hat{\Phi}_x = |\tilde{\phi}_x|, \quad \sigma_x = \text{sign}(\tilde{\phi}_x).
\]

Let \((x_i)_{1 \leq i \leq |V|}\) be an enumeration of \( V \) (which may be infinite). We define by induction the self interacting processes \((\tilde{X}_{i,t})_{1 \leq i \leq |V|}, (\tilde{n}_e(t))_{e \in E})\). \( \tilde{T}_i \) will denote the end-time for \( \tilde{X}_{i,t} \), and \( \tilde{T}^+_i = \sum_{1 \leq j \leq i} \tilde{T}_j \). By definition, \( \tilde{T}^+_0 = 0 \). \( L(t) \) will denote

\[
L_x(t) := \sum_{1 \leq i \leq |V|} \tilde{\ell}_x(i, 0 \vee (t - \tilde{T}^+_i)),
\]

where \( \tilde{\ell}_x(i, t) \) are the occupation times for \( \tilde{X}_{i,t} \). For \( t \geq 0 \), we set

\[
\tilde{\Phi}_x(t) = \sqrt{(\hat{\Phi}_x)^2 - 2L_x(t)}, \quad \forall x \in V, \quad \text{and} \quad J_e(\tilde{\Phi}(t)) = W_e \tilde{\Phi}_e(t) \tilde{\Phi}_e^+(t), \quad \forall e \in E.
\]

The end-times \( \tilde{T}_i \) are defined by inductions as

\[
\tilde{T}_i = \inf\{t \geq 0 | (\tilde{\Phi}_{X_{i,t}}(t + \tilde{T}^+_{i-1}) = 0}\}.
\]

Let \((N_e(u))_{u \geq 0}\) be independent Poisson Point Processes on \( \mathbb{R}_+ \) with intensity 1, for each edge \( e \in E \). We set

\[
\tilde{n}_e(t) = \begin{cases} N_e(2J_e(t)), & \text{if } \sigma_e - \sigma_{e^+} = +1, \\ 0, & \text{if } \sigma_e - \sigma_{e^+} = -1. \end{cases}
\]

We also denote by \( \tilde{C}(t) \subset E \) the configuration of edges such that \( \tilde{n}_e(t) > 0 \). \( \tilde{X}_{i,t} \) starts at \( x_i \). For \( t \in [\tilde{T}_{i-1}^+, \tilde{T}_{i+}^+] \),

- if \( \tilde{n}_e(t) \) decreases by 1 at time \( t \), but does not create a new cluster in \( \tilde{C}_t \), then \( \tilde{X}_{i,t} - \tilde{T}_{i-1}^+ \) crosses the edge \( e \) with probability 1/2 or does not move with probability 1/2,
- if \( \tilde{n}_e(t) \) decreases by 1 at time \( t \), and does create a new cluster in \( \tilde{C}_t \), then \( \tilde{X}_{i,t} - \tilde{T}_{i-1}^+ \) moves/or stays with probability 1 on the unique extremity of \( e \) which is in the cluster of the origin \( x_i \) in the new configuration.

By induction, using Theorem 3 we deduce the following:
Theorem 5. Let \( \varphi \) be a GFF on \( \mathcal{G} \) with the law \( P_\varphi \). If one sets \( \hat{\varphi} = \varphi \) in the preceding construction, then for all \( i \in \{1, \ldots, |V|\} \), \( \hat{T}_i < +\infty \), \( \mathcal{X}_i, \hat{T}_i = x_i \) and the path \((\mathcal{X}_i, t)_{t \leq \hat{T}_i}\) has the same law as a concatenation in \( x_i \) of all the loops in a loop-soup \( \mathcal{L}_{1/2} \) that visit \( x_i \), but none of the \( x_1, \ldots, x_{i-1} \). To retrieve the loops out of each path \((\mathcal{X}_i, t)_{t \leq \hat{T}_i}\) on has to partition it according to a Poisson-Dirichlet partition as in Proposition 3.6. The coupling between the GFF \( \varphi \) and the loop-soup obtained from \((\mathcal{X}_i, t)_{t \leq \hat{T}_i}, (\hat{n}_e(t))_{e \in E}\) is the same as in Proposition 2.4.

3.5. Inverting the coupling of random current with FK-Ising. By combining Theorem 3 and the discrete time description of Section 3.2 and by conditioning on the occupation field of the loop-soup, one deduces an inversion of the coupling of Proposition 2.5 between the random current and FK-Ising.

We consider that the graph \( \mathcal{G} = (V, E) \) and that the edges are endowed with weights \((J_e)_{e \in E}\). Let \((x_i)_{1 \leq i \leq |V|}\) be an enumeration of \( V \). Let \( \mathcal{C}(0) \) be a subset of open edges of \( E \). Let \((\hat{n}_e(0))_{e \in E}\) be a family of random integers such that \( \hat{n}_e(0) = 0 \) if \( e \notin \mathcal{C}(0) \), and \((\hat{n}_e(0) - 1)_{e \in \mathcal{C}(0)}\) are independent Poisson random variables, where \( E[\hat{n}_e(0)] - 1 = 2J_e \).

We will consider a family of discrete time self-interacting processes \((\mathcal{X}_{i,j}^{(1)}, \ldots, \mathcal{X}_{i,j}^{(|V|)}; (\hat{n}_e(j))_{e \in E})\). \( \mathcal{X}_{i,j} \) starts at \( j = 0 \) at \( x_i \) and is defined up to an integer time \( \hat{T}_i \). Let \( \hat{T}_i = \sum_{1 \leq k \leq i} \hat{T}_k \), with \( \hat{T}_0^+ = 0 \). The end-times \( \hat{T}_i \) are defined by induction as

\[
\hat{T}_i = \inf \left\{ j \geq 0 \left| e \text{ edge adjacent to } \mathcal{X}_{i,j} \right. \sum_{1 \leq k \leq i} \hat{n}_e(j + \hat{T}_i^{k+1}) = 0 \right\}.
\]

For \( j \geq 1 \), \( \hat{C}(j) \) will denote

\[
\hat{C}(j) = \{ e \in E | \hat{n}_e(j) \geq 1 \},
\]

which is consistent with the notation \( \hat{C}(0) \).

The evolution is the following. For \( j \in \{\hat{T}_i + 1, \ldots, \hat{T}_i^+\} \), the transition from time \( j - 1 \) to time \( j \) is the following:

- first chose an edge \( e \) adjacent to the vertex \( \mathcal{X}_{i,j-1-j}^+ \) with probability proportional to \( \hat{n}_e(j - 1) \),
- decrease the stack \( \hat{n}_e(j - 1) \) by 1,
- if decreasing \( \hat{n}_e(j - 1) \) by 1 does not create a new cluster in \( \hat{C}(j - 1) \), then \( \mathcal{X}_{i,j} \) crosses \( e \) with probability \( 1/2 \) and does not move with probability \( 1/2 \),
- if decreasing \( \hat{n}_e(j - 1) \) by 1 does create a new cluster in \( \hat{C}(j - 1) \), then \( \mathcal{X}_{i,j} \) moves/or stays with probability 1 on the unique extremity of \( e \) which is in the cluster of the origin \( x_i \) in the new configuration.

Denote \( \hat{n}_e \) the number of times the edge \( e \) has been crossed, in both directions, by all the walks \((\mathcal{X}_{i,j})_{0 \leq j \leq \hat{T}_i} \).

Proposition 3.7. A.s., for all \( i \in \{1, \ldots, |V|\} \), \( \hat{T}_i < +\infty \) and \( \mathcal{X}_i, \hat{T}_i = x_i \). If the initial configuration of open edges \( \hat{C}(0) \) is random and follows an FK-Ising distribution with weights \((1 - e^{-2J_e})_{e \in E}\), then the family of integers \((\hat{n}_e)_{e \in E}\) is distributed like a random current with weights \((J_e)_{e \in E}\). Moreover, the coupling between the random current and the FK-Ising obtained this way is the same as the one given by Proposition 2.5.

4. Proof of Theorem 3

4.1. Case of finite graph without killing measure. Here we will assume that \( V \) is finite and that the killing measure \( \kappa \equiv 0 \).
In order to prove Theorem 3 we first enlarge the state space of the process \((X_t)_{t \geq 0}\). We define a process \((X, (n_x(t)))_{t \geq 0}\) living on the space \(V \times \mathbb{N}^E\) as follows. Let \(\varphi^{(0)} \sim P_\varphi^{(x_0)}, 0\) be a GFF pinned at \(x_0\). Let \(\sigma_x = \text{sign}(\varphi^{(0)}_x)\) be the signs of the GFF with the convention that \(\sigma_{x_0} = +1\). The process \((X_t)_{t \geq 0}\) is as usual the Markov Jump process starting at \(x_0\) with jump rates \((W_e)\). We set
\[
\Phi_x = \varphi^{(0)}_x, \quad \Phi(t) = \sqrt{\Phi_x^2 + 2t\Phi_e(t)} \quad \forall x \in V, \quad J_e(\Phi(t)) = W_e(\Phi_e(t)) \Phi_e(t), \quad \forall e \in E.
\]
The initial values \((n_x(0))\) are chosen independently on each edge with distribution
\[
n_x(0) \sim \begin{cases} 0, & \text{if } \sigma_e - \sigma_e^+ = -1 \vspace{1mm} \\
P(2J_e(\Phi)), & \text{if } \sigma_e - \sigma_e^+ = +1
\end{cases}
\]
where \(P(2J_e(\Phi))\) is a Poisson random variable with parameter \(2J_e(\Phi)\). Let \(\{(N_e(u))_{u \geq 0}\}_{e \in E}\) be independent Poisson point processes on \(\mathbb{R}_+\) with intensity 1. We define the process \((n_e(t))\) by
\[
n_e(t) = n_x(0) + N_e(J_e(\Phi(t))) - N_e(J_e(\Phi)) + K_e(t),
\]
where \(K_e(t)\) is the number of crossings of the edge \(e\) by the Markov jump process \(X\) before time \(t\).

**Remark 4.1.** Note that compared to the process defined in Section 3.1, the speed of the Poisson process is related to \(J_e(\Phi(t))\) and not \(2J_e(\Phi(t))\).

We will use the following notation
\[
\mathcal{C}(t) = \{e \in E \mid n_e(t) > 0\}.
\]
Recall that \(\tau_{x_0} = \inf\{t \geq 0, \ell_{x_0}(t) = u\}\) for \(u > 0\). To simplify notation, we will write \(\tau_u\) for \(\tau_{x_0}\) in the sequel. We define \(\varphi^{(u)}\) by
\[
\varphi^{(u)}_x = \sigma_x \Phi(\tau_u), \quad \forall x \in V,
\]
where \((\sigma_x)_{x \in V} \in \{-1, +1\}^V\) are random spins sampled uniformly independently on each cluster induced by \(\mathcal{C}(T)\) with the condition that \(\sigma_{x_0} = +1\).

**Lemma 4.2.** The random vector \((\varphi^{(0)}, \mathcal{C}(0), \varphi^{(u)}, \mathcal{C}(\tau_u))\) thus defined has the same distribution as \((\varphi^{(0)}, \mathcal{O}(\varphi^{(0)}), \varphi^{(u)}, \mathcal{O}_u)\) defined in Theorem 2.

**Proof.** It is clear from construction, that \(\mathcal{C}(0)\) has the same law as \(\mathcal{O}(\varphi^{(0)})\) (cf Definition 2.1), the FK-Ising configuration coupled with the signs of \(\varphi^{(0)}\) as in Proposition 2.2. Indeed, for each edge \(e \in E\) such that \(\varphi_{e^-} - \varphi_{e^+} > 0\), the probability that \(n_e(0) > 0\) is \(1 - e^{-2J_e(\Phi)}\). Moreover, conditionally on \(\mathcal{C}(0) = \mathcal{O}(\varphi^{(0)})\), \(\mathcal{C}(\tau_u)\) has the same law as \(\mathcal{O}_u\) defined in Theorem 2. Indeed, \(\mathcal{C}(\tau_u)\) is the union of the set \(\mathcal{C}(0)\), the set of edges crossed by the process \((X_u(t))_{u \leq \tau_u}\), and the additional edges such that \(N_e(J_e(\tau_u)) - N_e(J_e(\Phi)) > 0\). Clearly \(N_e(J_e(\tau_u)) - N_e(J_e(\Phi)) > 0\) independently with probability \(1 - e^{-(J_e(\Phi(\tau_u)) - J_e(\Phi))}\) which coincides with the probability given in Theorem 2.

We will prove the following theorem that, together with Lemma 4.2 contains the statements of both Theorem 2 and 3.

**Theorem 6.** The random vector \(\varphi^{(u)}_x\) is a GFF distributed according to \(P_{\varphi^{(x_0)}, \sqrt{2\sigma}}\). Moreover, conditionally on \(\varphi^{(u)}_x = \varphi\), the process
\[
(X,t, (n_e(t))_{e \in E})_{t \leq \tau_{x_0}}
\]
has the law of the process \((\tilde{X}_{T-t}, (n_e(T-t))_{e \in E})_{t \leq T}\) described in section 3.1.
Proof. **Step 1**: We start by a simple lemma.

**Lemma 4.3.** The distribution of $(\Phi := |\varphi(0)|, n_c(0))$ is given by the following formula for any bounded measurable test function $h$

\[
\mathbb{E}(h(\Phi, n(0))) = \sum_{(n_c)\in \mathbb{N}^E} \int_{\mathbb{R}^+_V \setminus \{x_0\}} d\Phi h(\Phi, n) e^{-\frac{1}{2} \sum_{x \in V} W_x(\Phi)^2 - \sum_{e \in E} J_e(\Phi) \left( \prod_{e \in E} \frac{(2J_e(\Phi))^{n_e}}{n_e!} \right)} 2^{\#C(n_c)-1},
\]

where the integral is on the set $\{(\Phi_x)_{x \in V}, \Phi_x > 0 \forall x \neq x_0, \Phi_{x_0} = 0\}$ and $d\Phi = \prod_{x \in V \setminus \{x_0\}} d\Phi_x$ and $\#C(n)$ is the number of clusters induced by the edges such that $n_c > 0$.

**Proof.** Indeed, by construction, summing on possible signs of $\varphi(0)$, we have

\[
\mathbb{E}(h(\Phi, n(0))) = \sum_{\sigma_x \in \{+1, -1\}^V} \sum_{n_c \in \sigma_x} \int_{\mathbb{R}^+_V \setminus \{x_0\}} d\Phi h(\Phi, n) e^{-\frac{1}{2} \sum_{x \in V} W_x(\Phi)^2 - \sum_{e \in E} J_e(\Phi) \left( \prod_{e \in E} \frac{e^{-2J_e(\Phi)} (2J_e(\Phi))^{n_e}}{n_e!} \right)}. \tag{4.2}
\]

where the first sum is on the set $\{\sigma_x \in \{+1, -1\}^V, \sigma_{x_0} = +1\}$ and the second sum is on the set of $\{(n_c) \in \mathbb{N}^E, n_c = 0\text{ if } \sigma_{e^+} - \sigma_{e^-} = 1\}$ (we write $n \ll \sigma$ to mean that $n$ vanishes on the edges such that $\sigma_{e^+} - \sigma_{e^-} = 1$). Since

\[
\frac{1}{2} \mathcal{E}(\sigma \Phi) = \frac{1}{2} \sum_{x \in V} W_x(\Phi_x)^2 - \sum_{e \in E} J_e(\Phi) \sigma_{e^+} - \sigma_{e^-}. \tag{4.2}
\]

we deduce that the integrand in (4.2) is equal to

\[
h(\Phi, n) e^{-\frac{1}{2} \mathcal{E}(\sigma \Phi)} \left( \prod_{e \in E, \sigma_{e^+} - \sigma_{e^-} = +1} \frac{e^{-2J_e(\Phi)} (2J_e(\Phi))^{n_e}}{n_e!} \right).
\]

where we used in the first equality that $n_c = 0$ on the edges such that $\sigma_{e^+} - \sigma_{e^-} = 1$. Thus, \[
\mathbb{E}(h(\Phi, n(0))) = \sum_{\sigma_x} \sum_{n_c \in \sigma_x} \int_{\mathbb{R}^+_V \setminus \{x_0\}} d\Phi h(\Phi, n) e^{-\frac{1}{2} \sum_{x \in V} W_x(\Phi)^2 - \sum_{e \in E} J_e(\Phi) \left( \prod_{e \in E} \frac{(2J_e(\Phi))^{n_e}}{n_e!} \right)}.
\]

Inverting the sum on $\sigma$ and $n$ and summing on the number of possible signs which are constant on clusters induced by the configuration of edges $\{e \in E, n_e > 0\}$, we deduce Lemma 4.3. \hfill \square

**Step 2**: We denote by $Z_t = (X_t, \Phi(t), n_c(t))$ the process defined previously and by $E_{x_0, \Phi, n_0}$ its law with initial condition $(x_0, \Phi, n_0)$.
We now introduce a process $\tilde{Z}_t$, which is a "time reversal" of the process $Z_t$. This process will be related to the process defined in section 3.1 in Step 4, Lemma 4.5.

For $(\tilde{n}_e) \in \mathbb{N}^E$ and $(\tilde{\Phi}_x)_{x \in V}$ such that

$$\tilde{\Phi}_{x_0} = u, \quad \tilde{\Phi}_x > 0, \quad \forall x \neq x_0,$$

we define the process $\tilde{Z}_t = (\tilde{X}_t, \tilde{\Phi}(t), \tilde{n}_e(t))$ with values in $V \times \mathbb{R}_{+}^V \times \mathbb{Z}^E$ as follows. The process $(\tilde{X}_t)$ is a Markov jump process with jump rates ($W_e$) (so that $\tilde{X} \text{ law} \mathcal{X}$), and $\tilde{\Phi}(t), \tilde{n}_e(t)$ are defined by

\begin{equation}
\tilde{\Phi}_x(t) = \sqrt{\tilde{\Phi}_x^2 - 2\tilde{\ell}_x(t)}, \quad \forall x \in V,
\end{equation}

where $\tilde{\ell}_x(t)$ is the local time of the process $\tilde{X}$ up to time $t$,

\begin{equation}
\tilde{n}_e(t) = \tilde{n}_e - \left(N_e(J_e(\tilde{\Phi})) - N_e(J_e(\tilde{\Phi}(t)))\right) - \tilde{K}_e(t)
\end{equation}

where $((N_e(u))_{u \geq 0})_{e \in E}$ is independent Poisson point process on $\mathbb{R}_{+}$ with intensity 1 for each edge $e$, and $\tilde{K}_e(t)$ is the number of crossings of the edge $e$ by the process $\tilde{X}$ before time $t$. We set

\begin{equation}
\tilde{Z}_t = (\tilde{X}_t, (\tilde{\Phi}_x(t), (\tilde{n}_e(t))法律规定
\end{equation}

This process is well-defined up to time

$$\tilde{T} = \inf \left\{ t \geq 0, \exists x \in V : \tilde{\Phi}_x(t) = 0 \right\}.$$

We denote by $E_{x_0, \tilde{\Phi}, \tilde{n}}$ its law. Clearly $\tilde{Z}_t = (\tilde{X}_t, \tilde{\Phi}(t), \tilde{n}_e(t))$ is a Markov process, we will later on make explicit its generator.

We have the following change of variable lemma.

**Lemma 4.4.** For all bounded measurable test functions $F, G, H$

$$\sum_{(n_e) \in \mathbb{N}^E} \int d\Phi F(\Phi, n)E_{x_0, \Phi, n} \left(G((Z_{\tau_u - t})_{0 \leq t \leq \tau_u})H(\Phi(\tau_u), n(\tau_u))\right) =$$

$$\sum_{(n_e) \in \mathbb{N}^E} \int d\Phi H(\Phi, \tilde{n})\tilde{E}_{x_0, \tilde{\Phi}, \tilde{n}} \left(\tilde{F}_{\tilde{X}_t \tilde{\Phi}, \tilde{n}} \left\{ \{X_{\tau_u - t} \right\}_{0 \leq \tau_u} \right\} G((\tilde{Z}_t)_{0 \leq \tilde{T}})F(\tilde{\Phi}(\tilde{T}), \tilde{n}(\tilde{T})) \prod_{x \in V \setminus \{x_0\}} \frac{\tilde{\Phi}_x}{\tilde{\Phi}_x(t)}\right),$$

where the integral on the l.h.s. is on the set $\left\{ (\Phi_x) \in \mathbb{R}_{+}^V, \Phi_{x_0} = 0 \right\}$ with $d\Phi = \frac{\prod_{x \in V \setminus \{x_0\}} d\Phi_x}{\sqrt{2\pi}^{V \setminus \{x_0\}}}$

and the integral on the r.h.s. is on the set $\left\{ (\tilde{\Phi}_x) \in \mathbb{R}_{+}^V, \tilde{\Phi}_{x_0} = u \right\}$ with $d\tilde{\Phi} = \frac{\prod_{x \in V \setminus \{x_0\}} d\tilde{\Phi}_x}{\sqrt{2\pi}^{V \setminus \{x_0\}}}$

**Proof.** We start from the left-hand side, i.e. the process, $(X_t, n_e(t))_{0 \leq \tau_u}$. We define

$$\tilde{X}_t = X_{\tau_u - t}, \quad \tilde{n}_e(t) = n_e(\tau_u - t),$$

and

$$\tilde{\Phi}_x = \Phi_x(\tau_u), \quad \tilde{\Phi}_x(t) = \Phi_x(\tau_u - t),$$

The law of the processes such defined will later be identified with the law of the processes $(\tilde{X}_t, \tilde{\Phi}(t), \tilde{n}(t))$ defined at the beginning of step 2, cf (4.3) and (4.4). We also set

$$\tilde{K}_e(t) = K_e(\tau_u) - K_e(t),$$
where \( \hat{\Phi}(x) = \sqrt{\hat{\Phi}_x^2 - 2\hat{\ell}_x(t)} \),

where \( \hat{\ell}_x(t) = \int_0^t \mathbb{1}_{\{\hat{X}_u = x\}} \, du \) is the local time of \( \hat{X} \) at time \( t \), and

\[
\hat{n}_e(t) = \hat{n}_e(0) + (N_e(J_e(\Phi(t))) - N_e(J_e(\Phi(0)))) - \hat{K}_e(t).
\]

By time reversal, the law of \((\hat{X}_t)_{0 \leq t \leq \tau_u}\) is the same as the law of the Markov Jump process \((X_t)_{0 \leq t \leq \tau_u}\), where \( \tau_u = \inf\{t \geq 0, \hat{\ell}_x(t) = u\} \). Hence, we see that up to the time \( \hat{T} = \inf\{t \geq 0, \exists x \Phi(x(t)) = 0\} \), the process \((\hat{X}_t, (\Phi(x(t)))_{x \in V}, (\hat{n}_e(t)))_{t \leq \hat{T}}\) has the same law as the process defined at the beginning of step 2.

Then, following [9], we make the following change of variables conditionally on the processes \((X_t, (N_e(t)))\)

\[
(\mathbb{R}_+^V \times \mathbb{N}^E) \mapsto (\mathbb{R}_+^V \times \mathbb{N}^E) \\
((\Phi(x), (n_e)_{e \in E}) \mapsto ((\hat{\Phi}x), (\hat{n}_e)_{e \in E})
\]

which is bijective onto the set

\[
\{(\hat{\Phi}_x, \hat{\Phi}_{x_0} = \sqrt{2u}, \hat{\Phi}_x > \sqrt{2\hat{\ell}_x(\tau_u^0)} \, \forall x \neq x_0) \times \{(\hat{n}_e), \hat{n}_e \geq K_e(\tau_u) + (N_e(J_e(\hat{\Phi}(\tau_u))) - N_e(J_e(\Phi)))\}
\]

(Note that we always have \( \hat{\Phi}_{x_0} = \sqrt{2u} \)). The last conditions on \( \hat{\Phi} \) and \( \hat{n}_e \) are equivalent to the conditions \( \hat{X}_\hat{T} = x_0 \) and \( \hat{n}_e(\hat{T}) \geq 0 \). The Jacobian of the change of variable is given by

\[
\prod_{x \in V \setminus \{x_0\}} d\Phi_x = \left( \prod_{x \in V \setminus \{x_0\}} \frac{\hat{\Phi}_x}{\Phi_x} \right) \prod_{x \in V \setminus \{x_0\}} d\Phi_x.
\]

\[\Box\]

**Step 3:** With the notations of Theorem [9], we consider the following expectation for \( g \) and \( h \) bounded measurable test functions

\[
\mathbb{E}\left(g\left((X_{\tau_u-t}, n_e(\tau_u-t))_{0 \leq t \leq \tau_u}\right) h(\varphi^{(u)})\right)
\]

By definition, we have

\[
\varphi^{(u)} = \sigma \Phi(\tau_u),
\]

where \( \sigma \in \{\pm 1\}^V \) are random signs sampled uniformly independently on clusters induced by \( \{e \in E, n_e(\tau_u) > 0\} \) and conditioned on the fact that \( \sigma_{x_0} = +1 \). Hence, we define for \( (\Phi_x) \in \mathbb{R}_+^V \) and \( (n_e) \in \mathbb{N}^E \)

\[
H(\Phi, n) = 2^{-\#C(n)+1} \sum_{\sigma \leq n} h(\sigma \Phi),
\]

where \( \sigma \leq n \) means that the signs \( (\sigma_e) \) are constant on clusters of \( \{e \in E, n_e > 0\} \) and such that \( \sigma_{x_0} = +1 \). Hence, setting

\[
F(\Phi, n) = e^{-\frac{1}{2}} \sum_{x \in V} W_x(\Phi_x)^2 - \sum_{e \in E} J_e(\Phi) \left( \prod_{e \in E} \frac{(2J_e(\Phi))^{n_e}}{n_e!} \right) 2^{\#C(n_e)-1},
\]

\[
G\left((Z_{\tau_u-t})_{t \leq \tau_u}\right) = g\left((X_{\tau_u-t}, n_e(\tau_u-t))_{t \leq \tau_u}\right),
\]
using lemma 4.3 in the first equality and lemma 4.4 in the second equality, we deduce that (4.6) is equal to

\[(4.8) \quad \mathbb{E} \left( G ((Z_{\tau_t} - t)_{0 \leq t \leq \tau_n}) H (\Phi (\tau_t), n(\tau_t)) \right) = \sum_{(n_e) \in \mathcal{N}_E} \int d\Phi F(\Phi, n) E_{x_0, \Phi, n} \left( G ((Z_{\tau_t} - t)_{t \leq \tau_t}) H (\Phi (\tau_t), n(\tau_t)) \right) d\Phi = \sum_{(\tilde{n}_e) \in \mathcal{N}_E} \int d\Phi H \left( \tilde{\Phi}, \tilde{n} \right) \tilde{E}_{x_0, \Phi, \tilde{n}} \left( \mathbb{1}_{\{ \tilde{X}_t = x_0, \tilde{n}_e(T) \geq 0 \forall e \in E \}} F \left( \tilde{\Phi}(T), \tilde{n}(T) \right) G \left( \tilde{Z}_t \leq T \right) \prod_{x \in V \setminus \{x_0\}} \tilde{\Phi}_x(T) \right) \]

with notations of Lemma 4.3.

Let $\tilde{\mathcal{F}}_t = \sigma \{ \tilde{X}_s, s \leq t \}$ be the filtration generated by $\tilde{X}$. We define the $\tilde{\mathcal{F}}$-adapted process $\tilde{M}_t$, defined up to time $T$ by

\[(4.9) \quad \tilde{M}_t = \frac{F(\tilde{\Phi}(t), \tilde{n}(t))}{\prod_{x \in V \setminus \{x_0\}} \tilde{\Phi}_x(t)} \mathbb{1}_{\{ \tilde{X}_t \in \mathcal{C}(x_0, \tilde{n}) \}} \mathbb{1}_{\{ \tilde{n}_e(t) \geq 0 \forall e \in E \}} = e^{-\frac{1}{2} \sum_{x \in V} W_x(\tilde{\Phi}(t))^2 - \sum_{e \in E} J_e(\tilde{\Phi}(t)) \left( \prod_{e \in E} \left( \frac{2J_e(\tilde{\Phi}(t)))^{\tilde{n}_e(t)}}{\tilde{n}_e(t)!} \right) \frac{2\# \mathcal{C}(\tilde{n}_e(t)) - 1}{\prod_{x \in V \setminus \{x_0\}} \tilde{\Phi}_x(t)} \right) \mathbb{1}_{\{ \tilde{X}_t \in \mathcal{C}(x_0, \tilde{n}(t)), \tilde{n}_e(t) \geq 0 \forall e \in E \}} \]

where $\mathcal{C}(x_0, \tilde{n}(t))$ denotes the cluster of the origin $x_0$ induced by the configuration $\mathcal{C}(\tilde{n}(t))$.

Note that at time $t = T$, we also have

\[(4.10) \quad \tilde{M}_T = \frac{F(\tilde{\Phi}(T), \tilde{n}(T))}{\prod_{x \in V \setminus \{x_0\}} \tilde{\Phi}_x(T)} \mathbb{1}_{\{ \tilde{X}_T = x_0, \tilde{n}_e(t) \geq 0 \forall e \in E \}} \]

since $\tilde{M}_T$ vanishes on the event where $\{ \tilde{X}_T = x \}$, with $x \neq x_0$. Indeed, if $\tilde{X}_T = x \neq x_0$, then $\tilde{\Phi}_x(T) = 0$ and $J_e(\tilde{\Phi}(T)) = 0$ for $e \in E$ such that $x \in e$. It means that $\tilde{M}_T$ is equal to 0 if $\tilde{n}_e(T) > 0$ for some edge $e$ neighboring $x$. Thus, $\tilde{M}_T$ is null unless $\{x\}$ is a cluster in $\mathcal{C}(\tilde{n}(T))$. Hence, $\tilde{M}_T = 0$ if $x \neq x_0$ since $\tilde{M}_T$ contains the indicator of the event that $\tilde{X}_T$ and $x_0$ are in the same cluster.

Hence, using identities (4.8) and (4.10) we deduce that (4.6) is equal to

\[(4.11) \quad \mathbb{E} \left( G ((Z_{\tau_t} - t)_{0 \leq t \leq \tau_n}) H (\Phi (\tau_t), n(\tau_t)) \right) = \sum_{(\tilde{n}_e) \in \mathcal{N}_E} \int d\Phi H \left( \tilde{\Phi}, \tilde{n} \right) \tilde{E}_{x_0, \Phi, \tilde{n}} \left( \tilde{M}_T \frac{\tilde{M}_T}{\tilde{M}_0} G \left( \tilde{Z}_t \leq T \right) \right) \]

Step 4: We denote by $\tilde{Z}_t = (\tilde{X}_t, \tilde{\Phi}_t, \tilde{n}(t))$ the process defined in section 3.1 which is well defined up to stopping time $\tilde{T}$, and $\tilde{Z}_t^\infty = \tilde{Z}_{t \wedge \tilde{T}}$. We denote by $\tilde{E}_{x_0, \Phi, \tilde{n}}$ the law of the process $\tilde{Z}$ conditionally on the initial value $\tilde{n}(0)$, i.e. conditionally on $(N_e(2J(\tilde{\Phi}))) = (\tilde{n}_e)$. The last step of the proof goes through the following lemma.

Lemma 4.5. i) Under $\tilde{E}_{x_0, \Phi, \tilde{n}}$, $\tilde{X}$ ends at $X_T = x_0$ a.s. and $\tilde{n}_e(T) \geq 0$ for all $e \in E$.

ii) Let $\tilde{P}_{x_0, \Phi, \tilde{n}}^\leq t$ and $\tilde{P}_{x_0, \Phi, \tilde{n}}^\leq t$ be the law of the process $(\tilde{Z}_t^\infty)_{s \leq t}$ and $(\tilde{Z}_t^T)_{s \leq t}$, then

\[
\frac{d\tilde{P}_{x_0, \Phi, \tilde{n}}^\leq t}{d\tilde{P}_{x_0, \Phi, \tilde{n}}^\leq t} = \frac{\tilde{M}_{\tilde{T}}}{\tilde{M}_0}.
\]

Using this lemma we obtain that in the right-hand side of (4.11)

\[
\tilde{E}_{x_0, \Phi, \tilde{n}} \left( \frac{\tilde{M}_{\tilde{T}}}{\tilde{M}_0} G \left( \tilde{Z}_t \leq T \right) \right) = \tilde{E}_{x_0, \Phi, \tilde{n}} \left( G \left( \tilde{Z}_t \leq T \right) \right)
\]
Hence, we deduce, using formula (4.7) and proceeding as in lemma 4.3 that (4.6) is equal to

$$\int_{\mathbb{R}^d \setminus \{x_0\}} d\tilde{\varphi} e^{-\frac{1}{2}\mathcal{L}(\tilde{\varphi})} h(\tilde{\varphi}) \sum_{(\tilde{n}_e) \ll (\varphi_x)} \left( \prod_{e \in E, \tilde{\varphi}_e \leq \varphi_x+} \frac{e^{-2J_e(|\tilde{\varphi}|)}(2J_e(|\tilde{\varphi}|))_{\tilde{n}_e}}{\tilde{n}_e!} \right)$$

$$\tilde{E}_{x_0,|\tilde{\varphi}|,\tilde{n}} \left( \tilde{M}_T G((\tilde{Z}_t)_{t \leq T}) \right),$$

where the last integral is on the set \{(\varphi_x) \in \mathbb{R}^V, \ varphi_{x_0} = u\}, \ d\tilde{\varphi} = \frac{\prod_{e \in E \setminus \{x_0\}} d\tilde{\varphi}_e}{\sqrt{2\pi} |\tilde{\varphi}|^{-1}}$, and where \((n_e) \ll (\varphi_x)\) means that \((\tilde{n}_e) \in \mathbb{N}^E\) and \(\tilde{n}_e = 0\) if \(\varphi_e - \varphi_{e+} \leq 0\). Finally, we conclude that

$$\mathbb{E} \left[ g \left( \left( X_{\tau_{x_0} - t}, n_e(\tau_{x_0} - t) \right)_{0 \leq t \leq \tau_{x_0}} \right) h(\varphi^u) \right] = \mathbb{E} \left[ g \left( \left( X_t, n_e(t) \right)_{0 \leq t \leq T} \right) h(\tilde{\varphi}) \right]$$

where in the right-hand side \(\tilde{\varphi} \sim P_{\varphi_{x_0}}^{(x_0),\sqrt{2u}}\) is a GFF and \((X_t, n(t))\) is the process defined in section 3.1 from the GFF \(\varphi\). This exactly means that \(\varphi^u \sim P_{\varphi_{x_0}}^{(x_0),\sqrt{2u}}\) and that

$$\mathcal{L} \left( \left( X_{\tau_{x_0} - t}, n_e(\tau_{x_0} - t) \right)_{0 \leq t \leq \tau_{x_0}} \bigg| \varphi^u = \tilde{\varphi} \right) = \mathcal{L} \left( \left( X_t, n(t) \right)_{t \leq T} \right).$$

This concludes the proof of Theorem 1.

**Proof of lemma 4.7** The generator of the process \(\tilde{Z}_t\) defined in (4.5) is given, for any bounded and \(C^1\) for the second component test function \(f\), by

$$\left( \mathcal{L} f \right)(x, \tilde{\Phi}, \tilde{n}) = -\frac{1}{\tilde{\Phi}_x} \left( \frac{\partial}{\partial \tilde{\Phi}_x} f \right)(x, \tilde{\Phi}, \tilde{n}) +$$

$$\sum_{y, y \neq x} \left( W_{x,y} f(y, \tilde{\Phi}, \tilde{n} - \delta_{\{x,y\}}) - f(x, \tilde{\Phi}, n) \right) + W_{x,y} \tilde{\Phi}_y \frac{\partial}{\partial \tilde{\Phi}_x} \left( f(x, \tilde{\Phi}, n - \delta_{\{x,y\}}) - f(x, \tilde{\Phi}, n) \right)$$

where \(n - \delta_{\{x,y\}}\) is the value obtained by removing 1 from \(n\) at edge \(\{x,y\}\). Indeed, since \(\tilde{\Phi}_x(t) = \sqrt{\tilde{\Phi}_x(0)^2 - 2\tilde{\Phi}_x(t)}\), we have

$$\frac{\partial}{\partial t} \tilde{\Phi}_x(t) = -\mathbb{1}_{\{X_t = x\}} \frac{1}{\tilde{\Phi}_x(t)},$$

which is explains the first term in the expression. The second term is obvious from the definition of \(\tilde{Z}_t\), and corresponding to the term induced by jumps of the Markov process \(\tilde{X}_t\). The last term corresponds to the decrease of \(\tilde{n}\) due to the increase in the process \(\tilde{N}_e(\tilde{\Phi}) - \tilde{N}_e(\tilde{\Phi}(t))\). Indeed, on the interval \([t, t + dt]\), the probability that \(\tilde{N}_e(\tilde{\Phi}(t)) - \tilde{N}_e(\tilde{\Phi}(t + dt))\) is equal to 1 is of order

$$-\frac{\partial}{\partial t} \tilde{N}_e(\tilde{\Phi}(t)) dt = \mathbb{1}_{\{X_t \in e\}} \frac{W_{x,e} \tilde{\Phi}_x(t) \tilde{\Phi}_e(t)}{\tilde{\Phi}_x(t)^2} dt$$

using identity (4.13).
Let $\tilde{L}$ be the generator of the Markov jump process $\tilde{Z}_t = (\tilde{X}_t, (\tilde{\Phi}_t(t)), (\tilde{n}_e(t)))$. We have that the generator is equal, for any smooth test function $f$, to

\[
(\tilde{L} f) (x, \Phi, n) = -\frac{1}{\Phi_x} \left( \frac{\partial}{\partial \Phi_x} f \right) (x, \Phi, n) + \frac{1}{2} \sum_{y, y \sim x} \frac{n_{x,y}}{\Phi^2_x} [A_1(x,y) f(y, \tilde{\Phi}, n - \delta_{x,y}) + f(x, \tilde{\Phi}, n - \delta_{x,y}) - 2 f(x, \tilde{\Phi}, n)] + \sum_{y, y \sim x} \frac{n_{x,y}}{\Phi^2_x} [A_2(x,y) f(y, \tilde{\Phi}, n - \delta_{x,y}) - f(x, \tilde{\Phi}, n)] + \sum_{y, y \sim x} \frac{n_{x,y}}{\Phi^2_x} [A_3(x,y) f(x, \tilde{\Phi}, n - \delta_{x,y}) - f(x, \tilde{\Phi}, n)]
\]

where $A_i(x,y)$ correspond to the following disjoint events

- $A_1(x,y)$ if the numbers of connected clusters induced by $n - \delta_{x,y}$ is the same as that of $\tilde{n}$.
- $A_2(x,y)$ if a new cluster is created in $n - \delta_{x,y}$ compared with $\tilde{n}$ and if $y$ is in the connected component of $x_0$ in the cluster induced by $n - \delta_{x,y}$.
- $A_3(x,y)$ if a new cluster is created in $n - \delta_{x,y}$ compared with $n$ and if $x$ is in the connected component of $x_0$ in the cluster induced by $n - \delta_{x,y}$.

Indeed, conditionally on the value of $\tilde{n}_e(t) = N_e(2J_e(\tilde{\Phi}(t)))$ at time $t$, the point process $N_e$ on the interval $[0, 2J_e(\tilde{\Phi}(t))]$ has the law of $n_e(t)$ independent points with uniform distribution on $[0, 2J_e(\tilde{\Phi}(t))]$. Hence, the probability that a point lies in the interval $[2J_e(\tilde{\Phi}(t) + dt)), 2J_e(\tilde{\Phi}(t))]$ is of order

\[
-\tilde{n}_e(t) \frac{1}{J_e(\tilde{\Phi}(t))} \frac{\partial}{\partial t} J_e(\tilde{\Phi}(t)) dt = \mathbb{1}_{\{x \in C\}} \tilde{n}_e(t) \frac{1}{\Phi_x t^2} dt.
\]

We define the function

\[
\Theta(x, (\Phi_x), (n_e)) = e^{-\frac{1}{2} \sum_{x \in V} W_x(\Phi_x)^2 - \sum_{x \in E} J_e(\Phi) \left( \prod_{x \in E} \frac{(2J_e(\Phi))^{n_e}}{n_e!} \right)^{2 \#C(n_e)-1} \prod_{x \in V \setminus \{x\}} \mathbb{1}_{\{x \in C\}} \mathbb{1}_{\{n_e \geq 0 \forall e \in E\}}},
\]

so that

\[
\tilde{M}_{t \wedge \bar{T}} = \Theta(\tilde{Z}_{t \wedge \bar{T}}).
\]

To prove the lemma it is sufficient to prove (11, Chapter 11) that for any bounded smooth test function $f$

(4.14)

\[
\frac{1}{\Theta} \tilde{L} (\Theta f) = \tilde{L} (f)
\]

Let us first consider the first term in (11.12). Direct computation gives

\[
\left( \frac{1}{\Theta} \frac{1}{\Phi_x} \left( \frac{\partial}{\partial \Phi_x} \Theta \right) \right) (x, \Phi, n) = -W_x + \sum_{y \sim x} \left( -W_{x,y} \frac{\Phi_y}{\Phi_x} + n_{x,y} \frac{1}{\Phi^2_y} \right).
\]

For the second part, remark that the indicators $\mathbb{1}_{\{x \in C(x_0, n)\}}$ and $\mathbb{1}_{\{n_e \geq 0 \forall e \in E\}}$ imply that $\Theta(\Phi, n - \delta_{x,y})$ vanishes if $n_{x,y} = 0$ or if $y \notin C(x_0, n - \delta_{x,y})$. By inspection of the expression
of $\Theta$, we obtain for $x \sim y$,
\[
\Theta(y, \Phi, n - \delta_{x,y}) = \left(1_{\{n_{x,y} > 0\}}(1_A + 21_{A_2}) \frac{n_{x,y}}{2f_{x,y}(\Phi)\Phi_x} \right) \Theta(x, \Phi, n)
\]
\[
= \left(1_A + 21_{A_2} \right) \frac{n_{x,y}}{2W_{x,y} \Phi_x^2} \Theta(x, \Phi, n).
\]
Similarly, for $x \sim y$,
\[
\Theta(x, \Phi, n - \delta_{x,y}) = \left(1_{\{n_{x,y} > 0\}}(1_A + 21_{A_3}) \frac{n_{x,y}}{2f_{x,y}(\Phi)\Phi_x} \right) \Theta(x, \Phi, n)
\]
\[
= \left(1_A + 21_{A_3} \right) \frac{n_{x,y}}{2W_{x,y} \Phi_x \Phi_y} \Theta(x, \Phi, n).
\]
Combining these three identities with the expression (4.12) we deduce
\[
\frac{1}{\Theta} \partial \frac{\partial}{\partial \Phi} f(x, \Phi, n) - \sum_{y \sim x} \left(1_{\{n_{x,y} > 0\}}(1_A + 21_{A_2}) \frac{n_{x,y}}{2f_{x,y}(\Phi)\Phi_x} \right) f(x, \Phi, n)
\]
\[
+ \sum_{y \sim x} \left(1_A + 21_{A_2} \right)n_{x,y} \frac{1}{2f_{x,y}^2} f(y, n - \delta_{x,y}, \Phi) + \sum_{y \sim x} \left(1_A + 21_{A_3} \right) \frac{1}{2f_{x,y}^2} f(x, n - \delta_{x,y}, \Phi).
\]
It exactly coincides with the expression for $\partial$ since $1 = 1_A + 1_{A_2} + 1_{A_3}$. \hfill \Box

4.2. General case.

**Proposition 4.6.** The conclusion of Theorem 3 still holds if the graph $G = (V, E)$ is finite and the killing measure is non-zero ($\kappa \neq 0$).

**Proof.** Let $h$ be the function on $V$ defined as
\[
h(x) = \mathbb{P}_x(X \text{ hits } x_0 \text{ before } \zeta).
\]
By definition $h(x_0) = 1$. Moreover, for all $x \in V \setminus \{x_0\}$,
\[
-\kappa_x h(x) + \sum_{y \sim x} W_{x,y} (h(y) - h(x)) = 0.
\]
Define the conductances $W_{x,y}^h := W_{x,y} h(x) h(y)$, and the corresponding jump process $X^h$, and the GFF $\varphi_h^{(0)}$ and $\varphi_h^{(a)}$ with conditions $0$ respectively $\sqrt{2u}$ at $x_0$. The Theorem 3 holds for the graph $G$ with conductances $(W_{x,y})_{x \in E}$ and with zero killing measure. But the process $(X^h_s)_{s \leq \tau_{x_0}^h}$ has the same law as the process $(X_s)_{s \leq \tau_{x_0}^h}$, conditioned on $\tau_{x_0}^h < \zeta$, after the change of time
\[
dt = h(X_s)^{-2} ds.
\]
This means in particular that for the occupation times,
\[
(4.15) \quad \ell_x(t) = h(X_s)^{-2} \ell_x(s).
\]
Moreover, we have the equalities in law
\[
\varphi_h^{(0)} \text{ law } h^{-1} \varphi^{(0)}, \quad \varphi_h^{(a)} \text{ law } h^{-1} \varphi^{(a)}.
\]
Indeed, at the level of energy functions, we have:

\[
\mathcal{E}(hf, hf) = \sum_{x \in V} \kappa_x h(x)^2 f(x)^2 + \sum_e W_e (h(e_+) f(e_+) - h(e_-) f(e_-))^2
\]

= \sum_{x \in V} [\kappa_x h(x)^2 f(x)^2 + \sum_{y \sim x} W_{x,y} h(y) f(y) (h(y) f(y) - h(x) f(x))]

= \sum_{x \in V} [\kappa_x h(x)^2 f(x)^2 - \sum_{y \sim x} W_{x,y} (h(y) - h(x)) h(x) f(x)]^2 - \sum_{x \in V} W_{x,y} h(x)(y) (f(y) - f(x)) f(x)

= [\kappa_{x_0} - \sum_{y \sim x_0} W_{x_0,y} (h(y) - 1)] f(x_0)^2 + \sum_e W_e^h (h(e_+) f(e_+) - h(e_-) f(e_-))^2

= \text{Cste}(f(x_0)) + \varepsilon^h (f, f),
\]

where \(\text{Cste}(f(x_0))\) means that this term does not depend of \(f\) once the value of the function at \(x_0\) fixed.

Let \(\tilde{X}^u_t\) be the inverse process for the conductances \(W_e^h\) and the initial condition for the field \(\varphi^u_h\), given by Theorem 3. By applying the time change \(4.15\) to the process \(\tilde{X}^u_t\), we obtain an inverse process for the conductances \(W_e\) and the field \(\varphi^u\).

**Proposition 4.7.** Assume that the graph \(G = (V, E)\) is infinite. The killing measure \(\kappa\) may be non-zero. Then the conclusion of Theorem 3 holds.

*Proof.* Consider an increasing sequence of connected sub-graphs \(G_i = (V_i, E_i)\) of \(G\) which converges to the whole graph. We assume that \(V_0\) contains \(x_0\). Let \(G^*_i = (V^*_i, E^*_i)\) be the graph obtained by adding to \(G_i\) an abstract vertex \(x_*\), and for every edge \(\{x, y\}\), where \(x \in V_i\) and \(y \in V \setminus V_i\), adding an edge \(\{x, x_*\}\), with the equality of conductances \(W_{x,x_*} = W_{x,y}\). \((X^u, t)\geq 0\) will denote the Markov jump process on \(G^*_i\), started from \(x_0\). Let \(\zeta_i\) be the first hitting time of \(x_*\) or the first killing time by the measure \(\kappa \mathbb{1}_{V_i}\). Let \(\varphi_i^0\), \(\varphi_i^u\) will denote the GFFs on \(G^*_i\) with condition 0 respectively \(\sqrt{2u}\) at \(x_0\), with condition 0 at \(x_*\), and taking in account the possible killing measure \(\kappa \mathbb{1}_{V_i}\). The limits in law of \(\varphi_i^0\) respectively \(\varphi_i^u\) are \(\varphi^0\) respectively \(\varphi^u\).

We consider the process \((\tilde{X}_{i,t}, (\tilde{n}_{i,e}(t))_{e \in E^*_i})_{0 \leq t \leq T_i}\), for \(i \to +\infty\), \(\tilde{X}_{i, t_{i,0}}\) be the inverse process on \(G_i^*\), with initial field \(\varphi_i^u\), \((X_{i,t})_{t_{i,0} \leq t \leq T_i}\), conditional on \(\tau_{i,0}^u\), has the same law as \((\tilde{X}_{i, t_{i,0}})_{t \leq T_i}\). Taking the limit in law as \(i\) tends to infinity, we conclude that \((X_t)_{t \leq t_{0}}\), conditional on \(\tau_u^0 < +\infty\), has the same law as \((\tilde{X}_{T_{-1}})_{t \leq T}\) on the infinite graph \(G\). The same for the clusters. In particular,

\[
\mathbb{P}(\tilde{T} \leq t, \tilde{X}_{[0, \tilde{T}]}) \text{ stays in } V_j = \lim_{i \to +\infty} \mathbb{P}(\tilde{T}_i \leq t, X_{i,[0, \tilde{T}_i]} \text{ stays in } V_j)
\]

= \[\lim_{i \to +\infty} \mathbb{P}(\tau_u^{x_0} \leq t, X_{i,[0, \tau_u^{x_0}]} \text{ stays in } V_j) \mid \tau_u^{x_0} < \zeta_i) = \mathbb{P}(\tau_u^{x_0} \leq t, X_{[0, \tau_u^{x_0}]} \text{ stays in } V_j) \mid \tau_u^{x_0} < \zeta,\]

where in the first two probabilities we also average by the values of the free fields. Hence

\[
\mathbb{P}(\tilde{T} = +\infty \text{ or } \tilde{X}_{\tilde{T}} \neq x_0) = 1 - \lim_{j \to +\infty} \mathbb{P}(\tau_u^{x_0} \leq t, X_{[0, \tau_u^{x_0}]} \text{ stays in } V_j | \tau_u^{x_0} < \zeta) = 0.
\]

\[\square\]

**Acknowledgements**

TL acknowledges the support of Dr. Max Rössler, the Walter Haefner Foundation and the ETH Zurich Foundation.
References

[1] K.L. Chung and J.B. Walsh. Markov Processes, Brownian Motion, and Time Symmetry, volume 249 of Grundlehren der mathematischen Wissenschaften. 2005.
[2] Nathalie Eisenbaum, Haya Kaspi, Michael B. Marcus, Jay Rosen, and Zhan Shi. A Ray-Knight theorem for symmetric Markov processes. Ann. Probab., 28(4):1781–1796, 2000.
[3] Y. Le Jan. Markov paths, loops and fields. In 2008 St-Flour summer school, L.N. Math., volume 2026. Springer, 2011.
[4] Yves Le Jan. Markov loops, free field and Eulerian networks. J. Math. Soc. Japan, 67(4):1671–1680, 2015.
[5] T. Lupu. Convergence of the two-dimensional random walk loop soup clusters to CLE. arXiv:1502.06827, 2015.
[6] T. Lupu. From loop clusters and random interlacements to the free field. Ann. Probab., 44(3):2117–2146, 2016.
[7] Titus Lupu and Wendelin Werner. A note on Ising random currents, Ising-FK, loop-soups and the Gaussian free field. Electron. Commun. Probab., 21:Paper No. 13, 7, 2016.
[8] M.B. Marcus and J. Rosen. Markov processes, Gaussian processes and local times, volume 100 of Cambridge Stud. Adv. Math. Cambridge University Press, 1st edition, 2006.
[9] C. Sabot and P. Tarrès. Inverting Ray-Knight identity. Probab. Theory Related Fields, 165(3):559–580, 2015.
[10] A.S. Sznitman. Topics in occupation times and Gaussian free field. Zurich lectures in advanced mathematics. European Mathematical Society, 2012.
[11] W. Werner. On the spatial Markov property of soups of unoriented and oriented loops. In Séminaire de Probabilités XLVIII, volume 2168 of Lecture Notes in Mathematics, pages 481–503. Springer, 2016.

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