Erratum: Prog. Theor. Exp. Phys. 2015, 113B02 (2015)

Ising model on a twisted lattice with holographic renormalization-group flow

So Matsuura* and Norisuke Sakai*

Department of Physics, and Research and Education Center for Natural Sciences, Keio University, Hiyoshi 4-1-1, Yokohama, Kanagawa 223-8521, Japan
*E-mail: s.matsu@phys-h.keio.ac.jp, norisuke.sakai@gmail.com

Received February 9, 2016; Accepted February 9, 2016; Published 22, April 2016

In the paper PTEP 2015, 113B02, the partition function of the 2D Ising model is exactly obtained on a lattice with a twisted boundary condition by introducing the so-called shift matrix. Although the strategy of the computation is correct, we found that we need a phase factor in the definition of the shift matrix in the fundamental representation of $\text{Spin}(2n;\mathbb{C})$ in order that all the elements in the spin representation become real. In addition, there are several typos in the article. We correct them and refine the derivation of the partition function in this erratum.

(1) On page 5, there is a typo in equation (2.19). We replace the paragraph including this equation by the following:

In evaluating (2.17), the following fact is useful: Suppose that $\exists A \in \text{Spin}(2n;\mathbb{C})$ in the fundamental representation is transformed into the "canonical form" by $T \in O(2n)$ as

$$ T^T A T = \bigoplus_{k=1}^{n} e^{\theta_k \hat{J}_{2k-1,2k}} = \bigoplus_{k=1}^{n} R(-i\theta_k), \quad (\theta_k \in \mathbb{C}) \quad (2.18) $$

where $\hat{J}_{\mu\nu}$ are the generators of $\text{Spin}(2n;\mathbb{C})$ in the fundamental representation and $R(\theta)$ is the two-dimensional rotation matrix with the (complex) angle $\theta$ given by (A4). Then we can write $\text{Tr}_\pm(A)$ as

$$ \text{Tr}_\pm(A) = \frac{1}{2} \left( \prod_{k=1}^{n} 2 \cosh \frac{\theta_k}{2} \pm \text{det}(T) \prod_{k=1}^{n} 2 \sinh \frac{\theta_k}{2} \right). \quad (2.19) $$

Thus, we first express $H^m \Sigma^P_\pm$ in the fundamental representation and then transform them into the canonical form using appropriate matrices $T_\pm \in O(2n)$.

(2) Corresponding to the change of the definition of $T_\pm$ mentioned below, $\text{det} T_\pm = \pm 1$ written three lines above equation (2.22) on page 6 is replaced by $\text{det} T_\pm = 1$.

(3) On page 6, the text from the equations (2.22) to (2.24) is replaced by:

$$ Z = \frac{1}{2} (2 \sinh 2a)^{\frac{m^2}{2}} \sum_{i=1}^{4} Z_i, \quad (2.22) $$
with

\[
Z_1 = R_1 \prod_{r=1}^{[\frac{n}{2}]} 2 \cosh \left( \frac{m}{2} \gamma_{2r-1} \right) \right|^2 = \left( \prod_{k=1}^{n} e^{\frac{m}{2} \gamma_{2k-1}} \right) P_1, \\
Z_2 = R_2 \prod_{r=1}^{[\frac{n}{2}]} 2 \sinh \left( \frac{m}{2} \gamma_{2r-1} \right) \right|^2 = \left( \prod_{k=1}^{n} e^{\frac{m}{2} \gamma_{2k-1}} \right) P_2, \\
Z_3 = 2 \cosh \left( \frac{m}{2} \gamma_0 \right) R_3 \prod_{r=1}^{\frac{n-1}{2}} 2 \cosh \left( \frac{m}{2} \gamma_{2r} \right) \right|^2 = \left( \prod_{k=1}^{n} e^{\frac{m}{2} \gamma_{2k}} \right) (1 + e^{-m \gamma_0}) P_3, \\
Z_4 = 2 \sinh \left( \frac{m}{2} \gamma_0 \right) R_4 \prod_{r=1}^{\frac{n-1}{2}} 2 \sinh \left( \frac{m}{2} \gamma_{2r} \right) \right|^2 = \left( \prod_{k=1}^{n} e^{\frac{m}{2} \gamma_{2k}} \right) (1 - e^{-m \gamma_0}) P_4.
\]

(2.23)

where we have used the reflection property \( e^{m \gamma_{2n-1}} = e^{m \gamma_n^*} \) and introduced

\[
R_1 = \begin{cases} 
1 & (n : \text{even}) \\
2 \cosh \left( \frac{m}{2} \gamma_n \right) & (n : \text{odd, } p : \text{even}) \\
2 \sinh \left( \frac{m}{2} \gamma_n \right) & (n : \text{odd, } p : \text{odd}) \\
1 & (n : \text{odd}) 
\end{cases} \quad R_2 = \begin{cases} 
1 & (n : \text{even}) \\
2 \sinh \left( \frac{m}{2} \gamma_n \right) & (n : \text{odd, } p : \text{even}) \\
2 \cosh \left( \frac{m}{2} \gamma_n \right) & (n : \text{odd, } p : \text{odd}) \\
1 & (n : \text{odd}) 
\end{cases} \quad R_3 = \begin{cases} 
1 & (n : \text{even}) \\
2 \cosh \left( \frac{m}{2} \gamma_n \right) & (n : \text{even, } p : \text{even}) \\
2 \sinh \left( \frac{m}{2} \gamma_n \right) & (n : \text{even, } p : \text{odd}) \\
1 & (n : \text{odd}) 
\end{cases} \quad R_4 = \begin{cases} 
1 & (n : \text{even}) \\
2 \cosh \left( \frac{m}{2} \gamma_n \right) & (n : \text{even, } p : \text{even}) \\
2 \sinh \left( \frac{m}{2} \gamma_n \right) & (n : \text{even, } p : \text{odd}) \\
1 & (n : \text{odd}) 
\end{cases}
\]

and

\[
P_1 = \left( \prod_{r=1}^{[\frac{n}{2}]} \left| 1 + e^{-m \gamma_{2r-1}} \right|^2 \right) (1 + (-1)^p \delta_{(-1)^n,-1} e^{-m \gamma_n}), \\
P_2 = \left( \prod_{r=1}^{[\frac{n}{2}]} \left| 1 - e^{-m \gamma_{2r-1}} \right|^2 \right) (1 - (-1)^p \delta_{(-1)^n,-1} e^{-m \gamma_n}), \\
P_3 = \left( \prod_{r=1}^{\frac{n-1}{2}} \left| 1 + e^{-m \gamma_{2r}} \right|^2 \right) (1 + (-1)^p \delta_{(-1)^n,1} e^{-m \gamma_n}), \\
P_4 = \left( \prod_{r=1}^{\frac{n-1}{2}} \left| 1 - e^{-m \gamma_{2r}} \right|^2 \right) (1 - (-1)^p \delta_{(-1)^n,1} e^{-m \gamma_n}).
\]

(2.24)

(4) On page 7, above equation (3.3), we replace “Then \( \gamma_1 \) can be” with “Then \( \gamma_1 \) for \( I \ll n \) can be.”
(5) Equation (A1) is replaced by:
\[
\left( \hat{J}_{\mu\nu} \right)_{\rho\sigma} = \left( \delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho} \right). \quad (1 \leq \mu < \nu \leq 2n)
\] (A1)

Correspondingly, equations (A3) and (A10) are replaced by:
\[
x = \begin{pmatrix} \cosh 2\tilde{a} \cosh 2b & i \sinh 2\tilde{a} \cosh 2b \\ -i \sinh 2\tilde{a} \cosh 2b & \cosh 2\tilde{a} \cosh 2b \end{pmatrix}, \quad y = \begin{pmatrix} -\frac{1}{2} \sinh 2\tilde{a} \sinh 2b & -i \sinh^2 \tilde{a} \sinh 2b \\ i \cosh^2 \tilde{a} \sinh 2b & -\frac{1}{2} \sinh 2\tilde{a} \sinh 2b \end{pmatrix},
\] (A3)

and
\[
M_I = \begin{pmatrix} A_I & iB_I \\ -iB_I & A_I \end{pmatrix}, \quad N_I = \begin{pmatrix} iC_I & 0 \\ 0 & -iC_I \end{pmatrix},
\] (A10)
respectively.

(6) We replace equation (A2) with the following equation and comments:
\[
\hat{H}_{\pm} = \begin{pmatrix} x & y & 0 & \cdots & 0 & \mp y^\dagger \\ y^\dagger & x & y & 0 & \cdots & 0 \\ 0 & y^\dagger & x & y & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \mp y & 0 & \cdots & y^\dagger & x \end{pmatrix}, \quad \hat{\Sigma}_{\pm} = \alpha_{\pm} \begin{pmatrix} 0 & 0 & 0 & \cdots & \mp 1_2 \\ 1_2 & 0 & 0 & \cdots & 0 \\ 0 & 1_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 1_2 & 0 \end{pmatrix},
\] (A2)

where the overall factor \( \alpha_{\pm} \) in the definition of \( \hat{\Sigma}_{\pm} \) is given by
\[
\alpha_{+} = \begin{cases} 1 & (n : \text{even}) \\ -i & (n : \text{odd}) \end{cases}, \quad \alpha_{-} = \begin{cases} -i & (n : \text{even}) \\ 1 & (n : \text{odd}) \end{cases},
\]
which are necessary in order to make all the nonzero elements of the shift matrix in the spin representation unity.

(7) We replace the text from equation (A13) to the end of Appendix A with the following:

Since \( A_I, B_I, \) and \( C_I \) satisfy
\[
A_I^2 - B_I^2 - C_I^2 = 1,
\] (A13)

we can uniquely determine the parameters \( \gamma_I > 0, \theta_I \in [0, \frac{\pi}{2}] \), and \( \epsilon_r = \pm 1 \) by
\[
A_I \equiv \cosh \gamma_I, \quad B_I \equiv \epsilon_I \sinh \gamma_I \cos \theta_I, \quad C_I \equiv \pm \sinh \gamma_I \sin \theta_I,
\] (A14)

where the sign in the definition of \( C_I \) takes + for \( 1 \leq I \leq n \) and − for \( n + 1 \leq I \leq 2n \). Note that the \( \gamma_I \) appearing in (A14) is the same one defined in (2.20). We also note that \( N_0 = N_n = 0 \) and \( M_0 \) and \( M_n \) are given by
\[
M_0 = R(\pm i\gamma_0), \quad M_n = R(-i\gamma_n),
\] (A15)

where the sign appearing in the expression of \( M_0 \) takes + in the disordered phase (\( \tilde{a} > b \)) and − in the ordered phase (\( \tilde{a} < b \)).
We can rearrange the matrices (A6) and (A7) by permuting the elements properly. To this end, we introduce the matrices which generate the transportations,

$$T_{i,j} \equiv\begin{pmatrix} 1_{2(i-1)} & 0_2 & \cdots & 1_2 \\
\vdots & 1_{2(j-i-1)} & \vdots \\
1_2 & \cdots & 0_2 \\
1_{2(n-j)} \end{pmatrix}, \quad (1 \leq i < j \leq n) \quad (A16)$$

and the cyclic rotations,

$$C_{i,j} = T_{j-1,i} T_{j-2,i-1} \cdots T_{i,i+1}. \quad (A17)$$

Then by defining

$$S_+ = \begin{cases} C_{2,n} C_{4,n} \cdots C_{n-2,n} & (n \text{ : even}) \\
C_{-\frac{1}{2},n} C_{2,n-1} C_{4,n-1} \cdots C_{n-3,n-1} & (n \text{ : odd}) \end{cases},$$

$$S_- = \begin{cases} C_{-\frac{1}{2},n-1} C_{2,n-2} C_{4,n-2} \cdots C_{n-4,n-2} & (n \text{ : even}) \\
C_{2,n-1} C_{4,n-1} \cdots C_{n-3,n-1} & (n \text{ : odd}) \end{cases}, \quad (A18)$$

we obtain

$$\begin{aligned}
(\Omega_+ S_+)^T \hat{H}_+ (\Omega_+ S_+) &= \begin{cases} \bigoplus_{r=1}^{n-1} X_{2r-1} & (n \text{ : even}) \\
\bigoplus_{r=1}^{n-1} X_{2r-1} \oplus M_n & (n \text{ : odd}) \end{cases} \\
(\Omega_+ S_+)^T \hat{\Sigma}_+ (\Omega_+ S_+) &= \begin{cases} \bigoplus_{r=1}^{n-1} R_4 \left( \frac{2r-1}{n} \pi \right) & (n \text{ : even}) \\
-\bigoplus_{r=1}^{n-1} R_4 \left( \frac{2r-1}{n} \pi \right) \oplus (-1_2) & (n \text{ : odd}) \end{cases} \quad (A19)
\end{aligned}$$

and

$$\begin{aligned}
(\Omega_- S_-)^T \hat{H}_- (\Omega_- S_-) &= \begin{cases} \bigoplus_{r=1}^{n-1} X_{2r} \oplus M_n \oplus M_0 & (n \text{ : even}) \\
\bigoplus_{r=1}^{n-1} X_{2r} \oplus M_0 & (n \text{ : odd}) \end{cases} \\
(\Omega_- S_-)^T \hat{\Sigma}_- (\Omega_- S_-) &= \begin{cases} \bigoplus_{r=1}^{n-1} R_4 \left( \frac{2r}{n} \pi \right) \oplus (-1_2) \oplus 1_2 & (n \text{ : even}) \\
\bigoplus_{r=1}^{n-1} R_4 \left( \frac{2r-1}{n} \pi \right) \oplus 1_2 & (n \text{ : odd}) \end{cases} \quad (A20)
\end{aligned}$$
where \( X_I \) and \( R_4(\theta) \) are \( 4 \times 4 \) matrices defined by
\[
X_I \equiv \begin{pmatrix} M_I & -N_I \\ N_I & M_I \end{pmatrix}, \quad R_4(\phi) \equiv R(-\phi) \oplus R(\phi).
\]
(A21)

The matrices \( X_I \) and \( R_4(\phi) \) are simultaneously transformed into the canonical forms,
\[
P_I^T X_I P_I = R(-i\epsilon_I \gamma_1) \oplus R(-i\epsilon_I \gamma_1), \quad P_I^T R_4(\phi) P_I = R(-\phi) \oplus R(\phi),
\]
(A22)

using the \( 4 \times 4 \) matrices
\[
P_I \equiv \begin{pmatrix} \cos \frac{\theta_I}{2} & 0 & 0 & \epsilon_I \sin \frac{\theta_I}{2} \\ 0 & \cos \frac{\theta_I}{2} & \epsilon_I \sin \frac{\theta_I}{2} & 0 \\ 0 & -\epsilon_I \sin \frac{\theta_I}{2} & \cos \frac{\theta_I}{2} & 0 \\ -\epsilon_I \sin \frac{\theta_I}{2} & 0 & 0 & \cos \frac{\theta_I}{2} \end{pmatrix}.
\]
(A23)

Thus, we define the matrices
\[
P_+ \equiv \bigoplus_{r=1}^{n/2} P_{2r-1} \oplus 1 (n : even), \quad P_- \equiv \bigoplus_{r=1}^{n/2} P_{2r} \oplus 1 (n : odd).
\]
(A24)

We finally consider the combinations
\[
T_{\pm} \equiv \Omega_{\pm} S_{\pm} P_{\pm},
\]
(A25)

which transform \( H_{\pm}^m \Sigma_{\pm}^p \) in the fundamental representation into the canonical forms, respectively:

\[\text{n: even}\]
\[
T_+^T \left( \hat{H}_{\pm}^m \hat{\Sigma}_{\pm}^p \right) T_+ = \bigoplus_{r=1}^{n/2} \left( R \left( -i m \epsilon_{2r-1} \gamma_{2r-1} - \frac{2r-1}{n} p \pi \right) \right.
\]
\[
\quad \oplus R \left( -i m \epsilon_{2r-1} \gamma_{2r-1} + \frac{2r-1}{n} p \pi \right),
\]
\[
T_-^T \left( \hat{H}_{-m} \hat{\Sigma}_{-}^p \right) T_- = (-i)^p \bigoplus_{r=1}^{n-2} \left( R \left( -i m \epsilon_{2r} \gamma_{2r} - \frac{2r}{n} p \pi \right) \oplus R \left( -i m \epsilon_{2r} \gamma_{2r} + \frac{2r}{n} p \pi \right) \right.
\]
\[
\quad \oplus R \left( -i m \gamma_n + p \pi \right) \oplus R \left( \pm i m \gamma_n \right) \right),
\]
(A26)
\( n: \text{ odd} \)

\[
T_+^T \left( \hat{H}_+^m \hat{\Sigma}_+^p \right) T_+ = (-i)^p \left\{ \bigoplus_{r=1}^{n-1} \left( R \left( -i m \epsilon_{2r-1} \gamma_{2r-1} - \frac{2r - 1}{n} p\pi \right) \right) \oplus R \left( -i m \epsilon_{2r-1} \gamma_{2r-1} + \frac{2r - 1}{n} p\pi \right) \right\}.
\]

\[
T_-^T \left( \hat{H}_-^m \hat{\Sigma}_-^p \right) T_- = \bigoplus_{r=1}^{n-1} \left( R \left( -i m \epsilon_{2r} \gamma_{2r} - \frac{2r}{n} p\pi \right) \oplus R \left( -i m \epsilon_{2r} \gamma_{2r} + \frac{2r}{n} p\pi \right) \right) \oplus R(\pm im\gamma_0),
\]

(A27)

where we have used (A15). We can easily see \( \det T_\pm = 1 \). These results motivate the introduction of (2.21). Note that, when we consider the continuum limit, we should approach the critical temperature from the ordered phase. Thus the sign appearing in \( R(\pm im\gamma_0) \) is chosen as +.