Linear-Time Algorithms for the Paired-Domination Problem in Interval Graphs and Circular-Arc Graphs

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Abstract

In a graph $G$, a vertex subset $S \subseteq V(G)$ is said to be a dominating set of $G$ if every vertex not in $S$ is adjacent to a vertex in $S$. A dominating set $S$ of a graph $G$ is called a paired-dominating set if the induced subgraph $G[S]$ contains a perfect matching. The paired-domination problem involves finding a smallest paired-dominating set of $G$. Given an intersection model of an interval graph $G$ with sorted endpoints, Cheng et al. [9] designed an $O(m + n)$-time algorithm for interval graphs and an $O(m(m + n))$-time algorithm for circular-arc graphs. In this paper, to solve the paired-domination problem in interval graphs, we propose an $O(n)$-time algorithm that searches for a minimum paired-dominating set of $G$ incrementally in a greedy manner. Then, we extend the results to design an algorithm for circular-arc graphs that also runs in $O(n)$ time.

Keywords: paired-domination problem, perfect matching, interval graph, circular-arc graph.
1 Introduction

The museum protection problem can be accurately represented by a graph $G = (V(G), E(G))$. The vertex set of $G$, denoted by $V(G)$, represents the sites to be protected; and the edge set of $G$, denoted by $E(G)$, represents the set of protection capabilities. There exists an edge $xy$ connecting vertices $x$ and $y$ if a guard at site $x$ is capable of protecting site $y$ and vice versa. In the classical domination problem, it is necessary to minimize the number of guards such that each site has a guard or is in the protection range of some guard. For the paired-domination problem, in addition to protecting the sites, the guards must be able to back each other up [11]. Throughout this paper, we let $n = |V(G)|$ and $m = |E(G)|$.

In a graph $G$, a vertex subset $S \subseteq V(G)$ is said to be a dominating set of $G$ if every vertex not in $S$ is adjacent to a vertex in $S$. A dominating set $S$ of a graph $G$ is called a paired-dominating set if the induced subgraph $G[S]$ contains a perfect matching. The paired-domination problem involves finding a smallest paired-dominating set of $G$. Haynes and Slater [11] defined the paired-domination problem and showed that it is NP-complete in general graphs. More recently, Chen et al. [7] demonstrated that the problem is also NP-complete in bipartite graphs, chordal graphs, and split graphs. Panda and Pradhan [22] strengthened the above results by showing that the problem is NP-complete for perfect elimination bipartite graphs. In addition, McCoy and Henning [21] investigated variants of the paired-domination problem in graphs.

Meanwhile, several polynomial-time algorithms have been developed for some special classes of graphs such as tree graphs, interval graphs, strongly chordal graphs, and circular-arc graphs. Qiao et al. [23] proposed an $O(n)$-time algorithm for tree graphs; Kang et al. [17] presented an $O(n)$-time algorithm for inflated trees; Chen et al. [6] designed an $O(m + n)$-time algorithm for strongly chordal graphs; and Cheng et al. [8] developed an $O(mn)$-time algorithm for permutation graphs. To improve the results in [8], Lappas et al. [18] introduced an $O(n)$-time algorithm. In addition, Hung [16] described an $O(n)$-time algorithm for convex...
bipartite graphs; Panda and Pradhan [22] proposed an $O(m+n)$-time algorithm for chordal bipartite graphs; Chen et al. [7] introduced $O(m + n)$-time algorithms for block graphs and interval graphs; and Cheng et al. [9] designed an $O(m + n)$-time algorithm for interval graphs and an $O(m(m + n))$-time algorithm for circular-arc graphs. In this paper, given an intersection model of interval graph $G$ with sorted endpoints, we improve the above results with time complexity $O(n)$ for interval graphs and circular-arc graphs.

Several variants of the classic domination problem, such as the weighted domination, edge domination, independent domination, connected domination, locating domination, and total domination problems, have generated a great deal of research interest in recent decades [3, 12–14]. It has been proved that the above problems are NP-complete in general graphs but they yield polynomial-time results in some special classes of graphs [1, 4, 5, 10, 19, 24]. In particular, these variants have been studied intensively in interval and circular-arc graphs [5, 24].

For weighted interval graphs, Ramalingam and Rangan [24] proposed a unified approach to solve the independent domination, domination, total domination and connected domination problems in $O(n + m)$ time. Subsequently, Chang [5] developed an $O(n)$-time algorithm for the independent domination problem, an $O(n)$-time algorithm for the connected domination problem and an $O(n \log \log n)$-time algorithm for the total domination problem in weighted interval graphs. The author also extended the results to derive $O(m + n)$-time algorithms for the same problems in circular-arc graphs. Moreover, Hsu and Tsai [15] developed an $O(n)$-time algorithm for the classic domination problem in circular-arc graphs. The algorithm, which utilizes a greedy strategy, motivated the algorithms proposed in this paper. Note that all the above algorithms assume that an intersection model of $G$ with sorted endpoints is given.

In this paper, we show that the paired-domination in interval graphs and circular-arc graphs is solvable in linear time. More precisely, given an intersection model of a circular-arc graph $G$ with sorted endpoints, we propose an $O(n)$-time algorithm that produces a
minimum paired-dominating set of $G$. Moreover, because circular-arc graphs are a natural generalization of interval graphs, the paired-domination problem in the latter can also be solved in $O(n)$ time.

The remainder of this paper is organized as follows. In Section 2, we introduce an $O(n)$-time algorithm for interval graphs; and in Section 3, we extend the result to derive an $O(n)$-time algorithm for circular-arc graphs. Section 4 contains some concluding remarks.

## 2 The Proposed Algorithm for Interval Graphs

To find a minimum paired-dominating set of an interval graph $G$, we designed an $O(n)$-time algorithm that derives the set incrementally in a greedy manner. Before describing the approach in detail, we introduce some preliminaries for interval graphs.

![Figure 1: (a) A family of intervals on a real line. (b) The corresponding interval graph $G$ for the family of intervals in (a).](image)

A graph $G$ is deemed an *interval graph* if there is a one-to-one correspondence between its vertices and a family of intervals, $I$, on a real line, such that two vertices in the graph have an edge between them if and only if their corresponding intervals overlap. Interval graphs have received considerable attention because of their application in the real world. Booth and Lueker [2] designed an algorithm that can recognize interval graphs in $O(n + m)$ time. As a byproduct, an intersection model $I$ of an interval path graph $G$ can be constructed in $O(n + m)$ time. In the remainder of this section, we assume that $G$ is an interval graph.
with $V(G) = \{v_1, v_2, \ldots, v_n\}$, where $n \geq 3$. We also assume that an intersection model $I$ is available to $G$, as shown by the examle in Figure 1, where Figure 1(b) depicts the corresponding interval graph $G$ for the family of intervals in Figure 1(a).

The neighborhood $N_G(v)$ of a vertex $v$ is the set of all vertices adjacent to $v$ in $G$; and the closed neighborhood $N_G[v] = \{v\} \cup N_G(v)$. For each $v \in V(G)$, let $a(v)$ denote the corresponding interval of $v$ in $I$. Each interval is represented by $[\ell(v), r(v)]$, where $\ell(v)$ and $r(v)$ are, respectively, the left endpoint and the right endpoint of $a(v)$. It is assumed that the left endpoint $\ell(v_i)$ is on the left of the left endpoint $\ell(v_{i+1})$ for all $1 \leq i \leq n - 1$. Without loss of generality, we assume that all interval endpoints (i.e., $\ell(v)$ and $r(v)$) are distinct. In addition, each interval endpoint is assigned a positive integer between 1 and $2n$ in ascending order in a left-to-right traversal. For any two endpoints $x$ and $y$, $x$ is said to be lower than $y$, denoted by “$x < y$”, if its label is lower than the label of $y$, i.e., $x$ lies on the left of $y$ in $I$; otherwise, $x$ is said to be greater than or equal to $y$, denoted by “$x \geq y$”.

### 2.1 The algorithm

As mentioned earlier, our algorithm for finding a minimum paired-dominating set of an interval graph utilizes a greedy strategy. With $O(n)$-time preprocessing, the algorithm traverses the intersection mode $I$ of an interval graph $G$ from left to right exactly once. For a subset $S \subseteq V(G)$, a vertex $v$ in $G$ is said to be the next undominated vertex with respect to $S$ if $a(v)$ has the leftmost right endpoint among the corresponding intervals of the vertices that are not in $S$ or adjacent to any vertex in $S$. For each $v \in V(G)$, let the partner of $v$, denoted by $p(v)$, be the neighbor of $v$ such that $a(p(v))$ has the rightmost right endpoint in $I$ among the corresponding intervals of $N_G(v)$. For the example of Figure 1, we have $p(v_3) = v_4$, $p(v_5) = v_7$, and $p(v_7) = v_5$. Initially, we set $S = \emptyset$. Then, the algorithm iteratively finds the next undominated vertex $v$ with respect to $S$ and adds a pair $(p(v), p(p(v)))$ to $S$ until every vertex not in $S$ is adjacent to a vertex in $S$. The steps of the algorithm are detailed in Algorithm 1.
Algorithm 1 Finding a minimum paired-dominating set in an interval graph

Input: An intersection model $I$ of an interval graph $G$ with sorted endpoints.
Output: A minimum paired-dominating set $S$ of $G$.

1: let $S \leftarrow \emptyset$;
2: repeat
3: find the next undominated vertex $v$ with respect to $S$;
4: let $S \leftarrow S \cup \{p(v), p(p(v))\}$;
5: until every vertex not in $S$ is adjacent to a vertex in $S$
6: return $S$;

For the example in Figure 1, the algorithm generates a minimum paired-dominating set $S = \{v_3, v_4, v_5, v_7\}$ of $G$. In the following, we demonstrate the correctness of the algorithm; and then describe an $O(n)$-time implementation of the algorithm in the next subsection.

First, we introduce some necessary notations. Let $G[S]$ denote the subgraph of $G$ induced by a subset $S$ of $V(G)$; and let $\tilde{N}_{G_i}(v_x)$ represent the set of all vertices adjacent to $v_x$ in $G[\{v_x, v_i, v_i+1, \ldots, v_h\}]$. In addition, let $v_*$ be the vertex in $V(G)$ such that $\ell(v_*)$ is the first left endpoint encountered in a left-to-right traversal from $r(v_i)$ in $I$. For the example in Figure 1, we have $v_2* = v_4$, $v_3* = v_5$, $\tilde{N}_{G_2*}(v_3) = \{v_4\}$ and $\tilde{N}_{G_3*}(v_4) = \{v_5, v_6\}$.

**Lemma 1** If $v_x \in N_G(v_i)$ and $v_y \in N_G(v_x)$, we have

$$\{\tilde{N}_{G_i*}(v_x) \cup \tilde{N}_{G_i*}(v_y)\} \subseteq \{\tilde{N}_{G_i*}(p(v_i)) \cup \tilde{N}_{G_i*}(p(p(v_i)))\} \quad \text{for } 1 \leq i \leq n.$$  

**Proof.** By the definition of $p(v_i)$, we have $\max\{r(p(v_i)), r(p(p(v_i)))\} \geq \max\{r(v_x), r(v_y)\}$. For the case where $r(p(p(v_i))) > r(p(v_i))$, the segment $[r(v_i), r(p(p(v_i)))$ is contained in $a(p(v_i)) \cup a(p(p(v_i)))$. This implies that if $v_j \in \tilde{N}_{G_i*}(v_x) \cup \tilde{N}_{G_i*}(v_y)$, we also have $v_j \in \tilde{N}_{G_i*}(p(v_i)) \cup \tilde{N}_{G_i*}(p(p(v_i)))$. The arguments are similar in the case where $r(p(p(v_i))) < r(p(v_i))$. \hfill $\Box$

**Lemma 2** Given an intersection model $I$ of an interval graph $G$ with sorted endpoints, Algorithm 1 outputs a minimum paired-dominating set $S$ of $G$.

**Proof.** Suppose that the algorithm outputs $S = \{v_{s_1}, v_{s_1'}, \ldots, v_{s_x}, v_{s_x'}\}$, where $v_{s_i}$ and $v_{s_i'}$ are added to $S$ in the $i$th iteration. Clearly, $S$ is a paired-dominating set of $G$. We prove that
to

Next, we consider the case where \( v \) we have \( A \) nated vertex and \( x_i \). Consider the case where \( v \) we have \( v \in \mathbb{N}_G \) and \( z_i < z_{i+1} \) for \( 1 \leq i \leq y-1 \). In addition, let \( S_i = \{v_{s_1}, v_{s_1'}, \ldots, v_{s_k}, v_{s_k'}\} \) and \( Z_i = \{v_{z_1}, v_{z_1'}, \ldots, v_{z_i}, v_{z_i'}\} \). For a subset \( R \subseteq V(G) \), we define that \( A(R) = \max\{i \mid v_i \in R \) or \( v_i \) is adjacent to a vertex \( v_j \in R \). To prove that \( S \) is a minimum paired-dominating set of \( G \), it is sufficient to show that \( A(S_i) \geq A(Z_i) \) for \( 1 \leq i \leq y \).

We prove the above statement by induction on \( i \). By the definitions of the next undominated vertex and \( p(v) \), the statement holds for \( i = 1 \), and we assume the statement holds for \( i = k \). Consider the case where \( i = k+1 \). Let \( v_j \) be the next undominated vertex with respect to \( \{v_{s_1}, v_{s_1'}, \ldots, v_{s_k}, v_{s_k'}\} \). Clearly, \( j > A(S_k) \geq A(Z_k) \). First, we consider the case where \( v_j \not\in N_G(v_{z_{(k+1)}}) \cup N_G(v_{z_{(k+1)'}}) \). By the definition of \( v_j \), we have \( A(S_{k+1}) \geq j > A(Z_{k+1}) \).

Next, we consider the case where \( v_j \in N_G(v_{z_{(k+1)}}) \cup N_G(v_{z_{(k+1)'}}) \). According to Lemma 1, we have \( \{N_{G,j}(v_{z_{(k+1)}}) \cup N_{G,j}(v_{z_{(k+1)'}})\} \subseteq \{N_{G,j}(v_{s_{(k+1)}}) \cup N_{G,j}(v_{s_{(k+1)'}})\} \). It follows that \( A(S_{k+1}) \geq A(Z_{k+1}) \) in both cases, so \( x \leq y \). By the definition of \( Z \), we have \( y \leq x \); thus, \( x = y \). The lemma then follows.

2.2 An \( O(n) \)-time implementation of Algorithm 1

For each \( v \in V(G) \), let \( \text{next}(v) \) denote the vertex in \( V(G) - N_G[v] \) whose corresponding right endpoint is the first right endpoint encountered in a left-to-right traversal from \( r(v) \) in \( I \). The algorithm traverses the intersection mode \( I \) of an interval graph \( G \) from left to right exactly once. Therefore, to prove that the algorithm runs in \( O(n) \) time, it suffices to show that, with preprocessing in \( O(n) \) time, the algorithm takes \( O(1) \) time to determine \( p(v_i) \) and \( \text{next}(v_i) \) for \( 1 \leq i \leq n \). In the following, we describe two \( O(n) \)-time preprocessing procedures used to determine all \( p(v_i) \) and \( \text{next}(v_i) \), respectively.

To obtain \( p(v_i) \), we traverse the endpoints of \( I \) from left to right and maintain a variable \( \text{rightmost} \) that represents the interval containing the rightmost endpoint in the current stage. Initially, we set \( \text{rightmost} = v_1 \). When a left endpoint \( \ell(v_i) \) is visited, we compare
If \( r(v_i) > r(rightmost) \), we set \( rightmost = v_i \); otherwise, we do nothing. When a right endpoint \( r(v_i) \) is visited, we set \( p(v_i) = rightmost \). Because there are \( 2n \) endpoints in \( I \), the procedure can be completed in \( O(n) \) time.

Next, we describe the \( O(n) \)-time procedure used to determine \( next(v_i) \). Let \( f(v_i) \) be the vertex in \( V(G) \) such that \( \ell(f(v_i)) \) is the first left endpoint encountered in a left-to-right traversal from \( r(v_i) \) in \( I \). In addition, let \( c(v_i) = v_j \) be the vertex in \( V(G) \) such that \( r(v_j) \) is the first right endpoint encountered in a left-to-right traversal from \( \ell(v_i) \) in \( I \) with \( j \geq i \). Then, \( next(v_i) \) can be seen as the composite function of \( c \) and \( f \), i.e., \( next(v_i) = c(f(v_i)) \). The following discussion shows that, with \( O(n) \)-time preprocessing, \( c(v_i) \) and \( f(v_i) \) can be determined in \( O(1) \) time.

First, we describe an \( O(n) \)-time procedure to determine \( f(v_i) \) for each \( v_i \in V(G) \). Again, we traverse the endpoints of \( I \) from left to right and maintain a set \( P \). Initially, we set \( P = \emptyset \). When a right endpoint \( r(v_i) \) is visited, we add \( v_i \) to \( P \); and when a left endpoint \( \ell(v_i) \) is visited, we set \( f(v_j) = v_i \) for each vertex \( v_j \) in \( P \) and set \( P \) to be empty. The arguments for determining \( c(v_i) \) are similar. If we reach a left endpoint \( \ell(v_i) \), we insert \( v_i \) into the queue \( Q \). When we find a right endpoint \( r(v_i) \), we set \( c(v_j) = v_i \) and remove \( v_j \) from the queue for all \( v_j \) with \( j \leq i \). Because there are \( 2n \) endpoints in \( I \), the above two procedures can also be completed in \( O(n) \) time.

Combining Lemma 2 and above discussion, we have the following theorem, which is one of the key results presented in this paper.

**Theorem 3** Given an intersection model \( I \) of an interval graph \( G \) with sorted endpoints, Algorithm 1 outputs a minimum paired-dominating set \( S \) of \( G \) in \( O(n) \) time.

### 3 The Algorithm for Circular-arc Graphs

In this section, we extend the previous results to derive an \( O(n) \)-time algorithm for finding a minimum paired-dominating set in a circular-arc graph. The algorithm also exploits a greedy
strategy. A graph $G$ is deemed a circular-arc graph if there is a one-to-one correspondence between $V(G)$ and a set of arcs on a circle such that $(u, v) \in E(G)$ if and only if the corresponding arc of $u$ overlaps with the corresponding arc of $v$. An intersection model of $G$ is a circular ordering of its corresponding arc endpoints when moving in a counterclockwise direction around the circle. McConnell [20] proposed an $O(n + m)$-time algorithm that recognizes a circular-arc graph $G$, and simultaneously obtains an intersection model of $G$ as a byproduct. In the following discussion, we assume that (1) $G$ is a circular-arc graph such that $V(G) = \{v_1, v_2, \ldots, v_n\}$ with $n \geq 3$; and (2) the intersection model $F$ of $G$ is available.

![Figure 2: (a) A family of arcs on a circle. (b) The corresponding circular-arc graph $G$ for the family of arcs in (a).](image)

For each $v \in V(G)$, let $a(v)$ denote the corresponding arc of $v$ in $F$. Each arc is represented by $[h(v), t(v)]$, where $h(v)$ is the head of $a(v)$, $t(v)$ is the tail of $a(v)$, and $h(v)$ precedes $t(v)$ in a clockwise direction. Moreover, for any subset $W$ of $V(G)$, we define $a(W) = \{a(v) \mid v \in W\}$. It is assumed that all $h(v)$ and $t(v)$ are distinct and no single arc in $F$ covers the whole circle. All endpoints are assigned positive integers between 1 and $2n$ in ascending order in a clockwise direction.

In addition, we assume that $h(v_1) = 1$. We also assume that $a(v_1)$ is chosen arbitrarily.
from $F$; and we let $a(v_2), a(v_3), \ldots, a(v_n)$ be the ordering of arcs in $F - \{a(v_1)\}$ such that $h(v_i)$ is encountered before $h(v_j)$ in a clockwise direction from $h(v_1)$ if $i < j$. Figure 2 shows an illustrative example, in which Figure 2(b) depicts the corresponding circular-arc graph $G$ for the family of arcs in Figure 2(a). An ordering of the family of arcs is also provided. For each $v \in V(G)$, let the tail partner of $v$, denoted by $p_t(v)$, be the neighbor of $v$ such that $a(p_t(v))$ contains $t(v)$; and $t(p_t(v))$ is the last tail encountered in clockwise direction from $t(v)$ in $F$. Similarly, let the head partner of $v$, denoted by $p_h(v)$, be the neighbor of $v$ such that $a(p_h(v))$ contains $h(v)$; and $h(p_h(v))$ is the last head encountered in a counterclockwise direction from $h(v)$ in $F$. For the example in Figure 2, we have $p_h(v_1) = v_{11}$, $p_t(v_1) = v_2$, $p_h(v_2) = v_{11}$, and $p_t(v_2) = v_3$.

3.1 The Algorithm

The algorithm for finding a minimum paired-dominating set of a circular-arc graph $G$ is similar to the algorithm for interval graphs. An arc is maximal if it is not contained in any other arc of $F$. Suppose $W$ is the set of neighbors, $u$, of $v_1$ such that $a(u)$ is a maximal arc in $F$, i.e., $W = \{u \mid u \in N[v_1] \text{ and } a(u) \text{ is a maximal arc in } F\}$. Then, we can show that there exists a minimum paired-dominating set $S$ of $G$ such that $W \cap S \neq \emptyset$. If $W \cap S \neq \emptyset$, we are done. Otherwise, let $v_x$ be a vertex in $N[v_1] \cap S$ and $v_y$ be a vertex in $W$ such that $a(v_y)$ contains $a(v_x)$ in $F$. Clearly, $(S - \{v_x\}) \cup \{v_y\}$ is also a minimum paired-dominating set of $G$.

Based on the above observation, we designed a two-step algorithm for circular-arc graphs. First, the algorithm computes a paired-dominating set $S_i$ for each vertex $w_i \in W$, where $S_i$ is a minimum paired-dominating set among all paired-dominating sets that contain $w_i$. Then, a minimum paired-dominating set $S$ of $G$ is chosen from $S_1, \ldots, S_k$ with $k = |W|$. To find $S_i$, the algorithm traverses the intersection mode $F$ of a circular-arc graph $G$ in a clockwise direction. Lemma 4 below proves that there exists a minimum paired-dominating set $S_i$ such that we have $p_t(w_i) \in S_i$ or $p_h(w_i) \in S_i$. With the aid of the lemma, the algorithm first
computes two paired-dominating sets $S_i^t$ and $S_i^h$ that contain the vertices $\{w_i, p_i(w_i)\}$ and 
$\{w_i, p_h(w_i)\}$ respectively. If $|S_i^t| \leq |S_i^h|$, we have $S_i = S_i^t$; otherwise, we have $S_i = S_i^h$.

To explain the algorithm, we define some notations. Let $w_1, w_2, \ldots, w_k$ be an ordering of 
v vertices in $W$ such that $h(w_1)$ is the last head encountered in a counterclockwise direction 
from $h(v_1)$ in $F$ and $h(w_{i+1})$ immediately succeeds $h(w_i)$ in a clockwise direction for $1 \leq i \leq 
k - 1$. For a subset $S \subseteq V(G)$, we define $N[S] = \{v \mid v \in N_G[u] \text{ and } u \in S\}$. In addition, a 
vertex $v$ in $V(G) - N[S]$ is said to be the next undominated vertex with respect to $S$ if $t(v)$ 
is the first tail encountered in a clockwise direction from $t(v_1)$.

Initially, the algorithm sets $S_i^t = \{w_i, p_i(w_i)\}$ and $S_i^h = \{w_i, p_h(w_i)\}$. Then, it iteratively 
finds the next undominated vertex $v$ with respect to $S_i^t$ ($S_i^h$) and adds two vertices to $S_i^t$ 
($S_i^h$) until every vertex not in $S_i^t$ ($S_i^h$) is adjacent to a vertex in $S_i^t$ ($S_i^h$). If $|S_i^t| \leq |S_i^h|$, 
we have $S_i = S_i^t$; otherwise, we have $S_i = S_i^h$. Finally, the algorithm selects a minimum 
paired-dominating set $S$ of $G$ from $S_1, \ldots, S_k$ such that the cardinality of $S$ is the minimum. 
The steps of the algorithm are detailed below.

Algorithm 2 Finding a minimum paired-dominating set in a circular-arc graph

Input: An intersection model $F$ of a circular-arc graph $G$ with sorted endpoints.
Output: A minimum paired-dominating set $S$ of $G$.

1: let $W \leftarrow \{w \mid w \in N[v_1] \text{ and } a(w) \text{ is maximal}\}$;
2: let $k \leftarrow |W|$;
3: for each vertex $w_i \in W$ do
4: let $S_i^t \leftarrow \{w_i, p_i(w_i)\}$ and $S_i^h \leftarrow \{w_i, p_h(w_i)\}$;
5: repeat
6: find the next undominated vertex $v$ with respect to $S_i^t$;
7: if $p_i(p_i(v)) \in S_i^t$, then let $S_i^t \leftarrow S_i^t \cup \{v, p_i(v)\}$;
8: otherwise, let $S_i^t \leftarrow S_i^t \cup \{p_i(v), p_i(p_i(v))\}$;
9: until every vertex not in $S_i^t$ is adjacent to a vertex in $S_i^t$
10: repeat steps 5 to 9 to obtain $S_i^h$ by replacing $S_i^t$ with $S_i^h$;
11: if $|S_i^t| \leq |S_i^h|$, then let $S_i \leftarrow S_i^t$; otherwise, let $S_i \leftarrow S_i^h$;
12: end for
13: choose $S$ from $S_1, \ldots, S_k$ such that the cardinality of $S$ is the minimum;
14: return $S$

For the example in Figure 2, the closed neighborhood of $v_1$ is $N_G[v_1] = \{v_1, v_2, v_{11}\}$ and
we have \((w_1, w_2, w_3) = (v_2, v_{11}, v_1)\). Then, by the rules for finding \(S_1^h\) and \(S_1^t\) in Steps 3 to 12, we have \(S_1^h = \{v_2, v_{11}, v_5, v_6, v_9, v_{10}\}\), \(S_1^t = \{v_2, v_3, v_7, v_9\}\), \(S_2^h = \{v_{11}, v_{10}, v_5, v_6, v_8, v_9\}\), \(S_2^t = \{v_{11}, v_2, v_5, v_6, v_9, v_{10}\}\), \(S_3^h = \{v_1, v_{11}, v_5, v_6, v_9, v_{10}\}\), and \(S_3^t = \{v_1, v_2, v_5, v_6, v_9, v_{10}\}\). Consequently, Step 11 determines the sets \(S_1 = \{v_2, v_3, v_7, v_9\}\), \(S_2 = \{v_{11}, v_2, v_5, v_6, v_9, v_{10}\}\), and \(S_3 = \{v_1, v_2, v_5, v_6, v_9, v_{10}\}\). Finally, Step 13 generates \(S = \{v_2, v_3, v_7, v_9\}\), which is a minimum paired-dominating set of \(G\).

The properties of the following lemma are useful for finding a minimum paired-dominating set and help us prove the correctness of the algorithm.

**Lemma 4** Suppose \(a(v)\) is a maximal arc in \(F\) and \(S_v\) is a minimum paired-dominating set of \(G\) among all the paired-dominating sets that contains \(v\). Then, there exists a minimum paired-dominating set \(S_v\) such that we have \(p_t(v) \in S_v\) or \(p_h(v) \in S_v\).

**Proof.** If \(p_t(v) \in S_v\) or \(p_h(v) \in S_v\), we are done; otherwise, we assume that neither \(p_t(v) \in S_v\) nor \(p_h(v) \in S_v\). Furthermore, let \(v'\) be a vertex in \(S_v\) such that a perfect matching in \(G[S_v]\) contains the edge \((v, v')\). Note that \(a(v)\) is a maximal arc in \(F\). Hence, for each vertex \(u \in N_G(v)\), we have \(N_G[v] \cup N_G[u] \subseteq N_G[v] \cup N_G[p_t(v)]\) or \(N_G[v] \cup N_G[u] \subseteq N_G[v] \cup N_G[p_h(v)]\).

For the case where \(N_G[v] \cup N_G[v'] \subseteq N_G[v] \cup N_G[p_t(v)]\), it is clear that \((S_v - \{v'\}) \cup \{p_t(v)\}\) is a minimum paired-dominating set of \(G\). Similarly, for the case where \(N_G[v] \cup N_G[v'] \subseteq N_G[v] \cup N_G[p_h(v)]\), it is clear that \((S_v - \{v'\}) \cup \{p_h(v)\}\) is a minimum paired-dominating set of \(G\). The lemma then follows.

Based on Lemma 4, we are ready to prove the following lemma, which provides the correctness of the algorithm.

**Lemma 5** Given an intersection model \(F\) of a circular-arc graph \(G\) with sorted endpoints, Algorithm 2 outputs a minimum paired-dominating set \(S\) of \(G\).

**Proof.** Clearly, \(S\) is a paired-dominating set of \(G\). To prove that \(S\) is a minimum paired-dominating set of \(G\), it suffices to show that, for each vertex \(w_i \in W\), \(S_i\) is a minimum
paired-dominating set of $G$ among all the paired-dominating sets that contain $w_i$. According to Lemma 4, there exists a minimum paired-dominating set $Z_i$ of $G$ among all the paired-dominating sets containing $w_i$ such that we have $p_t(w_i) \in Z_i$ or $p_h(w_i) \in Z_i$. Below, we only show that the cardinality of $S_i$ is the minimum when $p_t(w_i) \in Z_i$. The proof for the case where $p_h(w_i) \in Z_i$ is similar.

Let $S^t_i = \{v_{s_0}, v_{s_0'}, \ldots, v_{s_x}, v_{s_x'}\}$ be a paired-dominating set of $G$ such that $(v_{s_0}, p_t(w_i)) = (w_i, p_t(w_i))$, and let the vertices $v_{s_j}$ and $v_{s_j'}$ be added to $S^t_i$ in the $j$th iteration of the repeat-loop for $1 \leq j \leq x$. In addition, let $Z_i = \{v_{z_0}, v_{z_0'}, \ldots, v_{z_y}, v_{z_y'}\}$ be a minimum paired-dominating set of $G$ such that $(v_{z_0}, v_{z_0'}), \ldots, (v_{z_y}, v_{z_y'})$ is a perfect matching of $G[Z_i]$ and $z_{j-1} < z_j$ for $1 \leq i \leq y$; let $S^t_{ij} = \{v_{s_0}, v_{s_0'}, \ldots, v_{s_j}, v_{s_j'}\}$; and $Z_{ij} = \{v_{z_0}, v_{z_0'}, \ldots, v_{z_j}, v_{z_j'}\}$. For a subset $R \subseteq V(G)$ and a vertex $w_i \in W$, we define that $A(R) = \max \{g \mid (1) \ v_g \in R; \text{ or } (2) \ v_g \text{ is adjacent to a vertex } v_j \in R \text{ and } a(v_g) \text{ does not contain } h(w_i)\}$. Hence, to prove that $S_i$ is a minimum paired-dominating set of $G$ among all the paired-dominating sets contain $w_i$, it suffices to show that $A(S^t_{ij}) \geq A(Z_{ij})$ for $0 \leq j \leq y$. Clearly, the above statement can be proved by induction on $j$. We omit the details of the proof because they are similar to the arguments used to derive Lemma 2.

## 3.2 An $O(n)$-time implementation of Algorithm 2

The procedure for finding $S_i$ can be implemented in $O(n)$ time by modifying Algorithm 1 for interval graphs. Therefore, a naive implementation of algorithm 2 has a time complexity of $O(kn)$ with $k = |W|$. However, by exploiting the elegant properties of circular-arc graphs, we can design a useful data structure that helps us find all paired-dominating sets $S_1, S_2, \ldots, S_k$ in $O(n)$ time. Hence, the time complexity of the algorithm for finding a minimum paired-dominating set of a circular-arc graph can be improved to $O(n)$.

To find a minimum paired-dominating set $S$ in $G$, algorithm 2 first obtains the paired-dominating sets $S^t_1, S^t_2, \ldots, S^t_k$ and $S^h_1, S^h_2, \ldots, S^h_k$. Then, the set $S$ with the minimum cardinality is chosen from the sets. In the following, we only consider an $O(n)$-time implemen-
tation for finding all the paired-dominating sets \( S_1, S_2, \ldots, S_k \). Using a similar method, it can be shown that \( S_1, S_2, \ldots, S_k \) can also be obtained in \( O(n) \) time. For simplicity, we denote \( S_i \) and \( p_i(v) \) by \( S_i \) and \( p(v) \) respectively, in the remainder of this section.

For each vertex \( w_i \in W \), the algorithm constructs a paired-dominating set \( S_i \) containing \( w_i \) and \( p(w_i) \). Suppose that \( S_i = \{w_{i_1}, w_{i_2}, \ldots, w_{i_k}, w_{i_0}\} \), where \( (w_{i_0}, w_{i_0}) = (w_i, p(w_i)) \), and that the vertices \( w_{i_j} \) and \( w_{i_j'} \) are added to \( S_i \) in the \( j \)th iteration of the repeat-loop for \( 1 \leq j \leq \ell \). In addition, let \( S_x = \{w_{x_0}, w_{x_1}, \ldots, w_{x_{g-2}}, w_{x_{g+1}}, w_{x_{g+2}}, \ldots, w_{x_{g-1}}\} \) and \( S_y = \{w_{y_0}, w_{y_1}, \ldots, w_{y_{g}}, w_{y_{g+1}}\} \) be two paired-dominating sets of \( G \) such that \( S_x, S_y \in \{S_1, S_2, \ldots, S_k\} \). It is clear that if \( (w_{x_e}, w_{x_e'}) = (w_{y_f}, w_{y_f'}) \), then \( (w_{x_{e+1}}, w_{x_{e+1}'}) = (w_{y_{f+1}}, w_{y_{f+1}'}) \), \( (w_{x_{e+2}}, w_{x_{e+2}'}) = (w_{y_{f+2}}, w_{y_{f+2}'}) \), \( \ldots \), \( (w_{x_{e+g}}, w_{x_{e+g}'}) = (w_{y_{f+g}}, w_{y_{f+g}'}) \), where \( g = \min\{a - e, b - f\} - 1 \).

Based on the above observation, we define a digraph \( D \) to improve the complexity of the algorithm from \( O(kn) \) to \( O(n) \).

![Figure 3: The corresponding digraph \( D \) for the circular-arc graph \( G \) in Figure 2(b).](image)

Recall that a vertex \( v \) in \( V(G) - N[S] \) is deemed the next undominated vertex with respect to a subset \( S \subseteq V(G) \) if \( t(v) \) is the first tail encountered in a clockwise direction from \( t(v_1) \). We define \( \text{succ}(v, p(v)) = (p(u), p(p(u))) \) for \( v \in V(G) \), where \( u \) is the next undominated vertex with respect to \( \{v, p(v)\} \). Let \( D = (V(D), E(D)) \) be a digraph such that

\[
V(D) = \{(v, p(v)) \mid v \in V(G)\}; \quad \text{and}
\]

\[
E(D) = \{((v, p(v)) \rightarrow (u, p(u)) \mid (u, p(u)) = \text{succ}(v, p(v)) \text{ and } u, p(u) \notin N[v_1]\}\}.
\]
Figure 3 shows an example of the corresponding digraph $D$ for the circular-arc graph $G$ in Figure 2(b).

Next, we show that the graph $D$ is a directed forest graph that can be constructed in $O(n)$ time. By the definition of $E(D)$, there exists no vertex $(v, p(v))$ in $D$ such that the in-degree of $(v, p(v))$ is greater than or equal to 1; and \( \{v, p(v)\} \cap N[v_1] \neq \emptyset \). Meanwhile, because every cycle $C = (c_1, c_2, \ldots , c_q)$ in $D$ must contain such a vertex $c_i = (v, p(v))$, $D$ is a directed forest graph. The $O(n)$-time procedure used to construct $D$ is as follows. Using similar arguments to those presented in Section 2.2, it can be shown that, with $O(n)$-time preprocessing, $p(v)$ and $\text{succ}(v, p(v))$ can be determined in $O(1)$ time for each vertex $v \in V(G)$. Moreover, because the out-degree of each vertex in $V(D)$ is at most one and $|V(D)| = n$, the digraph $D$ can be constructed in $O(n)$ time.

Let $P_i$ denote the maximal directed path in $D$ starting from $(w_i, p(w_i))$ for each vertex $w_i \in W$, and let $\bar{P}_i = \{v, u \mid (v, u) \in V(P_i)\}$. The next lemma provides an important property that can be used to derive the paired-domination sets $S_i$ from maximal directed path $P_i$ for $1 \leq i \leq k$.

**Lemma 6** Suppose that $P_i$ is the maximal directed path in $D$ starting from $(w_i, p(w_i))$ and $\bar{P}_i = \{v, u \mid (v, u) \in V(P_i)\}$. Then, we have $\bar{P}_i \subseteq S_i$ and $|S_i| - |\bar{P}_i| \leq 4$ for $1 \leq i \leq k$.

**Proof.** Because $D$ is a directed forest graph, it is clear from the definitions of $S_i$ and $P_i$ that $\bar{P}_i \subseteq S_i$. Suppose $\bar{P}_i = \{w_{i_0}, w_{i_0'}, \ldots , w_{i_\ell}, w_{i_\ell'}\}$ such that $(w_{i_0}, w_{i_0'}) = (w_i, p(w_i))$ and $(w_{i_j}, w_{i_{j'}}) = \text{succ}((w_{i(j-1)}', w_{i(j-1)})$) for $1 \leq j \leq \ell$. Then, by the definition of $P_i$, we can verify that the vertices in $V(G) - N[v_1]$ are dominated by $\bar{P}_i \cup \{w_{i(\ell+1)}, w_{i(\ell+1)}\}$, where $(w_{i(\ell+1)}, w_{i(\ell+1)}) = \text{succ}((w_{i_{\ell}}), w_{i_{\ell'}}))$. This implies that we have $|S_i| - |\bar{P}_i| \leq 4$ as desired. \[\square\]

In the following, we show the paired-domination sets $S_1, S_2, \ldots , S_k$ can be obtained in $O(n)$ time by exploiting Lemma 6. First, we determine whether or not the set $\bar{P}_i$ is a paired-domination set of $G$ for $1 \leq i \leq k$. If the answer is positive, we set $S_i = \bar{P}_i$. Otherwise, we set $S_i = \bar{P}_i$ and augment $S_i$ with the vertices $p(u)$ and $p(p((u))$ until $S_i$ becomes a paired-
domination set of $G$, where $u$ is the next undominated vertex with respect to $S_i$. According to Lemma 6, the augmentation will occur twice at most. This implies that the augmentation can be completed in $O(1)$ time for each vertex $w_i \in W$. Furthermore, because $\bar{P}_i \subseteq S_i$ and the digraph $D$ can be constructed in $O(n)$ time, the paired-dominating sets $S_1, S_2, \ldots, S_k$ can be obtained in $O(n)$ time. Now that $D$ is a directed forest, the length of $P_i$ in $D$ can be determined in $O(n)$ time by running a depth first search algorithm on all vertices $(v, p(v))$ in $\bar{D} = (V(\bar{D}), E(\bar{D}))$ such that the in-degree of $(v, p(v))$ is equal to 0, where $V(\bar{D}) = V(D)$ and $E(\bar{D}) = \{(v, p(v)) \rightarrow (u, p(u)) \mid ((u, p(u)) \rightarrow (v, p(v)) \in E(D)\}$. It follows that a minimum paired-dominating set $S$ of $G$ can be chosen from $S_1, \ldots, S_k$ in $O(n)$ time.

Combining Lemma 5 and above discussion, we have the following theorem.

**Theorem 7** Given an intersection model $F$ of a circular-arc graph $G$ with sorted endpoints, algorithm 2 outputs a minimum paired-dominating set $S$ of $G$ in $O(n)$ time.

### 4 Concluding Remarks

We have proposed two algorithms for the paired-domination problem in interval graphs and circular-arc graphs respectively. The algorithm for interval graphs produces a minimum paired-dominating set incrementally in a greedy manner. We extended the results to design the algorithm for circular-arc graphs. If the input graph is comprised of a family of $n$ arcs, both algorithms can be implemented in $O(n \log n)$ time. However, if the endpoints of the arcs are sorted, both algorithms only require $O(n)$ time. These results are optimal within a constant factor.

Finally, we consider some open questions related to the paired-domination problem. It would be interesting to investigate the weighted analogue of this problem, i.e., to compute a minimum weight paired-dominating set in which each vertex is associated with a weight. Furthermore, many optimization problems are NP-complete if they are defined on general graphs; and they are solvable in polynomial time if they are defined on some special classes of
graphs, such as bounded treewidth graphs, co-comparability graphs, and distance-hereditary graphs. Therefore, it would also be interesting to design polynomial-time algorithms for these graph classes. In addition, it would be useful if we could develop a polynomial-time approximation algorithm for general graphs; or prove that the problem remains NP-complete in planar graphs and devise a polynomial-time approximation scheme for it.

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