Structure Constant of the Yang-Lee Edge Singularity

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Abstract. This presentation gives a review of the conformal field theory (CFT) for the Yang-Lee edge singularity of the 2D Ising model and describes a numerical test of that CFT’s prediction of a universal amplitude. The review explains why the \( (\mathbf{A}_4, \mathbf{A}_1) \) minimal CFT was identified with the Yang-Lee edge singularity. The numerical test uses transfer matrix and finite-size scaling techniques to measure the structure constant at the Yang-Lee edge singularity and compares the measured value to the prediction from the \( (\mathbf{A}_4, \mathbf{A}_1) \) minimal CFT.

1. Review of Conformal Field Theory for Yang-Lee edge Singularity

In 1952, Yang and Lee showed that the partition function of the Ising spin model could have zeros at imaginary values of the magnetic field [1]. Yang and Lee also showed that the zeros have accumulation points and that the magnetization has a singular behavior near such points. The accumulation points have become known as Yang-Lee edge singularities.

In 1978, Fisher suggested that correlations should scale at Yang-Lee edge singularities in a manner similar to how correlations scale at ordinary critical points [2]. Fisher described the Yang-Lee edge singularity of the Ising model with a \( \Phi^3 \) Landau-Ginzburg model having a single real scalar field. Then, Fisher used various techniques to predict the exponent, \( \sigma \), which describes the magnetization near such singularities in Ising models.

In 1985, Cardy attempted to find a CFT to describe the Yang-Lee edge singularity of the 2D Ising [3]. Cardy assumed that the Yang-Lee singularity would be described by one of the minimal CFTs [4], i.e., CFTs with finite numbers of order parameters. Cardy also assumed that the CFT would have a single relevant field and would have a nontrivial three-point coupling as in Fisher’s \( \Phi^3 \) Landau-Ginzburg model [2].

In a minimal CFT, each field has a dimension \( X_{p,q} \), which is identified by a pair \( (p, q) \) of positive integers. In particular, each field dimension satisfies the Kac formula:

\[
X_{p,q} = \frac{1}{2}[(p \cdot \alpha_+ + q \cdot \alpha_-)^2 - (\alpha_+ + \alpha_-)^2].
\]  

(1)

Here, the constants \( \alpha_{\pm} \) are equal to \( \alpha_0 \pm \sqrt{1 + (\alpha_0)^2} \) where \( \alpha_0 \) defines the central charge, \( c \), of the CFT, i.e., \( c = 1 - 24(\alpha_0)^2 \). To be relevant, such a field must also have a dimension of less...
than two in 2D. For that reason, eq. (1) implies that a relevant field corresponds to a pair \((p, q)\) satisfying:

\[
\alpha_- - \alpha_+ < (p \cdot \alpha_+ + q \cdot \alpha_-) < \alpha_+ - \alpha_-.
\]  

(2)

A graphical illustration of these constraints is shown in Fig. [1]. There, the heavy lines indicate boundaries of the region where \(X_{p,q} < 2\), and \(L(p, q) = (p \cdot \alpha_+ + q \cdot \alpha_-)\).

From Fig. [1], one can see that there is always one or more nontrivial relevant conformal fields. In particular, eq. (1) implies that the line \(L(p, q) = (\alpha_+ + \alpha_-)\) is in the region where \(X_{p,q} < 2\). From the intercepts of this line and the region’s boundary lines, one can easily show that fields for \((p, q) = (1, 1), (1, 2),\) and \((1, 3)\) are always in the region where \(X_{p,q} < 2\). Since the field for \((p, q) = (1, 1)\) is a simple constant, the pairs \((p, q) = (1, 2)\) and \((p, q) = (1, 3)\) correspond to potential relevant fields of a CFT for the Yang-Lee edge singularity of the 2D Ising model.

The non-vanishing of the 3-point correlation also restricts the CFT for the Yang-Lee edge singularity. In particular, due to the CFT fusion rules, the 3-point correlation:

\[
\langle \Phi_{p,q}(z_1, \bar{z}_1)\Phi_{p,q}(z_2, \bar{z}_2)\Phi_{p',q'}(z_3, \bar{z}_3) \rangle
\]  

(3)

will vanish unless \(1 \leq p' \leq (2p - 1), 1 \leq q' \leq (2q - 1),\) and \(p'\) and \(q'\) are odd integers. For the nontrivial relevant fields, the 3-point correlation will only be nonzero if \((p, q) = (1, 2)\) and \((p', q') = (1, 3)\). Thus, to have a nonzero 3-point correlation, Cardy was also led to impose the condition that \(X_{1,3} = X_{1,2}\). This condition causes \((p, q) = (1, 2)\) and \((p, q) = (1, 3)\) to correspond to the same physical field. The solution of \(X_{1,3} = X_{1,2}\) fixes \(c_0\) and the central charge, \(c\). In particular, the central charge, \(c\), turns out to be -22/5. This minimal CFT is known as the \((A_4, A_1)\) model of the ADE classification [6, 5].
Cardy’s identification of the \((A_4, A_1)\) minimal CFT with the Yang-Lee edge singularity of the 2D Ising model has provoked tests to confirm the resulting CFT predictions. The CFT predictions for the central charge, \(c\), and the spin field dimension, \(X_{1,2}\), were quickly confirmed at the Yang-Lee edge singularity [7]. The CFT prediction for the low energy excitation spectrum has also been confirmed at the Yang-Lee edge singularity [8].

2. Structure Constant of the \((A_4, A_1)\) minimal CFT

Cardy also determined the forms of 2-point and 3-point correlations of the single relevant field in the \((A_4, A_1)\) minimal CFT [3]. These correlations define a universal amplitude known as a structure constant [4, 9]. This article reports on a test of this CFT prediction.

The \((A_4, A_1)\) minimal CFT has a single primary field, \(\phi(z, \bar{z})\), whose left and right conformal weights are \(-1/5\). Thus, the 2-point and 3-point correlations, i.e., \(G\) and \(G'\), of the primary field \(\phi(z, \bar{z})\) have the respective forms:

\[
G(z_1, \bar{z}_1, z_2, \bar{z}_2) = \left| (z_1 - z_2) \right|^{4/5}
\]

and

\[
G'(z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3) = C \left| (z_1 - z_2)(z_2 - z_3)(z_3 - z_1) \right|^{2/5}.
\]

Based on the Coulomb gas formalism [9], Cardy showed that the structure constant, \(C\), of eq. (5) has the value [3]:

\[
C = \sqrt{- \frac{\Gamma(6/5)^2 \Gamma(1/5) \Gamma(2/5)}{\Gamma(3/5) \Gamma(4/5)^3}}
\]

This article numerically checks this CFT prediction.

3. Numerical Evaluation of Structure Constant

Our measurements were performed on the 2D ferromagnetic Ising model, which has a Hamiltonian, \(H\), given by:

\[
H = \sum_{j=1}^{M} \sum_{i=1}^{N} \left[ -J (S_{i,j} S_{i,j+1} + S_{i,j} S_{i+1,j}) + h S_{i,j} \right].
\]

The Yang-Lee edge singularity occurs at temperatures above the critical temperature and at purely imaginary values of the magnetic field, \(h\), i.e., \(h = iB\) with \(B\) real [1]. For that reason, we measured spin correlations at a temperature, \(T\), for which \(J/K_B T = 0.1\).

We used transfer matrix techniques to evaluate the spin correlations on torii of length, \(M\), and of various diameters, \(N\). In these evaluations, \(M\) was always much larger than \(N\), i.e., \(M = 512\) and \(N = 4 - 7\). For that reason, measured correlations had the same distance behavior as correlations on an infinitely long cylinder when distances between spin fields were small compared to \(M\).

Finite-size scaling provided the tool for extracting values of physical properties in the thermodynamic limit [11]. In particular, the correlations are measured at special values of the magnetic field, \(h(N) = iB_{Y L}(N)\). Each special value satisfies the phenomenological renormalization group (PRG) equation on the infinite cylinder:

\[
\frac{\xi(iB_{Y L}(N), N - 1)}{N - 1} = \frac{\xi(iB_{Y L}(N), N)}{N}.
\]

In the PRG equation, \(\xi(iB, N)\) is the spin-spin correlation length on the infinite cylinder of diameter \(N\) when the magnetic field is \(iB\). The PRG equation imposes that the spin-spin
correlation length scale linearly as $N \to \infty$. When evaluated at PRG values of the magnetic field, other physical quantities will scale with the width, $N$, to their values in the thermodynamic limit near the Yang-Lee edge singularity [12, 7].

On a cylinder of width $N$, CFT predicts that correlations will depend exponentially on distances between fields when the distances are large compared to the cylinder’s diameter [13]. In particular, the 2-point correlation has the form $\exp(-2\pi X(y_1 - y_2)/N)$ when $|y_1 - y_2| >> N$. Here, $y_1$ and $y_2$ are positions of the two fields of the correlation along the axis of the infinite cylinder and $X$ is the scaling dimension of the fields. In the 3-point correlation, the exponential behavior is again fixed by the scaling dimensions, i.e., $X$‘s, of the fields therein.

At the Yang-Lee edge singularity, we measured amplitudes of 2-spin and 3-spin correlations, i.e., $A_{ss}$ and $A_{sss}$, respectively, to evaluate the 3-spin structure constant. The measured values of the 3-spin structure constant, $C(N)$, were obtained from the relation:

$$C(N) = \frac{A_{sss}(iB_{YL}(N))}{[A_{ss}(iB_{YL}(N))]^{3/2}}.$$  \hfill (9)

Here, $A_{ss}(iB_{YL}(N))$ and $A_{sss}(iB_{YL}(N))$, are the amplitudes of the 2-spin and 3-spin correlations, respectively, at the PRG values of the magnetic field. We used the PRG measurements of the correlation length $\xi(N)$ to also measure the conformal dimension, $X$, of the spin field, i.e., $X(N) = N/[2\pi \xi(iB_{YL}(N))]$. The behavior of these physical quantities as $N \to \infty$ was used to obtain values in thermodynamic limit.

Table 1 summarizes our transfer matrix results for $M = 512$ and $K = 0.1$. In the measurements, adjacent spin fields were separated by distances large compared to $N$ and small compared to $M$ to ensure an exponential dependence on distances between fields therein. In Table 1, the $\infty$ line shows values obtained by extrapolating our measurements to the thermodynamic limit. The extrapolated values were obtained via fits of the measured values of $X(N)$ and $|C(N)|$ to functions of form $f(N) = f(\infty) + f_1 N^{-\alpha}$ to account for leading finite-size corrections. In each such extrapolation, the coefficient, $f_1$, and the scaling power $\alpha$ were determined by finding the best fit to the measured data points. In Table 1, the last line shows Cardy’s CFT predictions for $X$ and $|C|$ from the $(A_4, A_1)$ CFT.

To conclude, our finite-size scaling measurements at the Yang-Lee edge singularity of the 2D Ising model produce a structure constant whose value agrees well with that of the $(A_4, A_1)$ minimal CFT. This confirms the correctness of the CFT prediction of this universal amplitude at the Yang-Lee edge singularity.
Figure 2. The measured structure constant for cylinders of diameter $N$ plotted against nonlinear fit for $N = 4 - 7$ (squares). The dot shows the structure constant predicted by CFT.

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