TWO SUPERCONGRUENCES RELATED TO MULTIPLE HARMONIC SUMS

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Abstract

Let \( p \) be a prime and let \( x \) be a \( p \)-adic integer. We prove two supercongruences for truncated series of the form

\[
\sum_{k=1}^{p-1} \frac{\psi_k}{k} \sum_{1 \leq \cdots \leq \sum_{j_1, \ldots, j_r \leq k}} \frac{1}{j_1 \cdots j_r}
\]

and

\[
\sum_{k=1}^{p-1} \frac{(\psi_k - x)_k}{k} \sum_{1 \leq \cdots \leq \sum_{j_1, \ldots, j_r \leq k}} \frac{1}{j_1^2 \cdots j_r^2}
\]

which generalise previous results. We also establish \( q \)-analogues of two binomial identities.

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1. Introduction and main result

In [9, Theorem 1.1] and [10, Theorem 7], we showed that for any prime \( p \neq 2 \),

\[
\sum_{k=1}^{p-1} \frac{(2k)_k}{k4^k} \equiv_p p^3 - H_{(p-1)/2} \quad \text{and} \quad \sum_{k=1}^{p-1} \frac{(2k)^2_k}{k16^k} \equiv_p 2H_{(p-1)/2},
\]

where \( H_n = \sum_{j=1}^n 1/j \) is the \( n \)th harmonic number and the notation \( a \equiv_m b \) means \( a \equiv b \pmod{m} \). Here we present two extensions of such congruences which involve the (nonstrict) multiple harmonic sums

\[
S_n(t_1, \ldots, t_r) = \sum_{1 \leq \cdots \leq \sum_{j_1, \ldots, j_r \leq n}} \frac{1}{j_1^t_1 \cdots j_r^t_r}
\]

with \( t_1, t_2, \ldots, t_r \) positive integers. For the sake of brevity, if \( t_1 = t_2 = \cdots = t_r = t \), we write \( S_n(t)^r \). Note that \( S_n(t) \) is simply the \( n \)th harmonic number of order \( t \), that is, \( H_n^{(t)} = \sum_{j=1}^n 1/j^t \).

Let \( (x)_n = x(x+1) \cdots (x+n-1) \) be the Pochhammer symbol and let \( B_n(x) \) be the \( n \)th Bernoulli polynomial. For any prime \( p \), \( \mathbb{Z}_p \) denotes the ring of all \( p \)-adic integers and \( (\cdot)_p \) is the least nonnegative residue modulo \( p \) of the \( p \)-integral argument.

Our main result is the following theorem.

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\textbf{Theorem 1.1.} Let \( p \) be a prime, \( x \in \mathbb{Z}_p \) and \( r \in \mathbb{N} \). Let \( s = (x + (-x)_p)/p \).

(i) If \( p > r + 3 \), then
\[
\sum_{k=1}^{p-1} \frac{(x)_k}{(1)_k} \cdot \frac{S_k([1]^r)}{k} \equiv_p p^2 H^{(r+1)}_{(-x)_p} - (-1)^r spB_{p-r-2}(x).
\] (1.2)

(ii) If \( p > 2r + 3 \), then
\[
\sum_{k=1}^{p-1} \frac{(x)_k(1-x)_k}{(1)_k^2} \cdot \frac{S_k([2]^r)}{k} \equiv_p p^3 -2H^{(2r+1)}_{(-x)_p} - 2(2r + 1)spH^{(2r+2)}_{(-x)_p} + \frac{2x (1 + 3sr + 2sr^2)}{2r + 3} p^2 B_{p-2r-3}(x).
\] (1.3)

When \( r = 0 \), both (1.2) and (1.3) have been established by Sun in [7]. For the special value \( x = 1/2 \), (1.2) and (1.3) yield
\[
\sum_{k=1}^{p-1} \frac{(2k)}{k} \cdot S_k([2]^r) \equiv_p \begin{cases} -H^{(r+1)}_{(p-1)/2} & \text{if } r \equiv_2 0, \\ 2^{r+2} - 1 & \text{if } r \equiv_2 1 \end{cases}
\] (1.4)

and
\[
\sum_{k=1}^{p-1} \frac{(2k)^2}{k16^k} \cdot S_k([2]^r) \equiv_p p^3 -2H^{(2r+1)}_{(p-1)/2} \frac{r(2^{2r+3} - 1)}{2} p^2 B_{p-2r-3}.
\] (1.5)

By letting \( r = 0 \) in (1.4) and (1.5), we deduce the congruences (1.1) mentioned at the beginning of the paper, whereas for \( r = 1 \), congruence (1.5) proves the conjecture in [8, Conjecture 5.3].

The next two sections are devoted to proving (1.2) and (1.3). In the last section we establish \( q \)-analogues of the two key binomial identities which play a crucial role in the proofs of the main result.

\section{2. Proof of (1.2) in Theorem 1.1}

By taking the partial fraction expansion of the rational function
\[
x \rightarrow \frac{(x)_k}{(x)_n}
\]
with \( 0 \leq k < n \),
\[
\sum_{k=0}^{n-1} \frac{(x)_k}{(1)_k} \cdot a_k = (x)_n \sum_{j=0}^{n-1} (-1)^j T_j \frac{(n-1-j)!}{j! (n-1-j)!} \cdot \frac{1}{x+j},
\] (2.1)
where $T_j$ is the binomial transform of the sequence $a_k$,

$$T_j := \sum_{k=0}^{j} (-1)^k \binom{j}{k} \cdot a_k.$$ 

It is easy to see from (2.1) that

$$\sum_{k=0}^{p-1} \frac{(x)_k}{(1)_k} \cdot a_k \equiv_p T_{(-x)_p} \quad (2.2)$$

when $a_0, \ldots, a_{p-1}, x \in \mathbb{Z}_p$. In order to show (1.2), we introduce the function

$$G_n^{(r)}(x) := \sum_{k=1}^{n} \frac{(x)_k}{(1)_k} \cdot S_k([1]^r).$$

Then

$$G_n^{(0)}(x) = \frac{(1+x)_n}{(1)_n} - 1$$

and $S_k([1]^r) = S_{k-1}([1]^r) + S_k([1]^{r-1})/k$ leads to

$$G_n^{(r)}(x) = \frac{(1+x)_n}{(1)_n} \cdot S_n([1]^r) - \frac{G_n^{(r-1)}(x)}{x}. \quad (2.3)$$

Moreover,

$$F_n^{(r)}(x+1) - F_n^{(r)}(x) = \frac{G_n^{(r)}(x)}{x},$$

where

$$F_n^{(r)}(x) := \sum_{k=1}^{n} \frac{(x)_k}{(1)_k} \cdot \frac{S_k([1]^r)}{k}.$$ 

For any positive integer $m$,

$$F_n^{(r)}(x+m) - F_n^{(r)}(x) = \sum_{j=0}^{m-1} \frac{G_n^{(r)}(x+j)}{x+j}. \quad (2.4)$$

By (2.3), for $u = 1, \ldots, n$,

$$G_n^{(r)}(-u) = \frac{G_n^{(r-1)}(-u)}{u} = \cdots = \frac{G_n^{(0)}(-u)}{u^r} = -\frac{1}{u^r}.$$ 

Hence, by letting $x = -n$ and $m = n$ in (2.4), we deduce the known identity (see [1])

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} S_k([1]^r) \frac{k}{k} = -H_n^{(r+1)} \quad (2.5)$$

Thus, for $a_k = S_k([1]^r)/k$, we have $T_j = -H_j^{(r+1)}$ and, by (2.2), we already obtain the modulo $p$ version of (1.2).
Proof of (1.2) in Theorem 1.1. Since \( sp = x + (-x)_p \),

\[
G^{(0)}_{p-1}(x) = \frac{(1 + x)_{p-1}}{(1)_p} - 1 \equiv_p \frac{sp}{x} - 1.
\]

According to [11, Theorem 1.6], \( S_{p-1}\{1\}' \equiv_p 0 \) and therefore

\[
G^{(r)}_{p-1}(x) \equiv_p \frac{G^{(r-1)}_{p-1}(x)}{x} \equiv_p \cdots \equiv_p (-1)^{r-1} \frac{G^{(0)}_{p-1}(x)}{x^{r}} \equiv_p \frac{(-1)^r sp}{x^{r+1}} - \frac{(-1)^r}{x^{r+1}}.
\]

Moreover,

\[
F^{(r)}_{p-1}(sp) = \sum_{k=1}^{p-1} \frac{(sp)_k}{(1)_k} \cdot \frac{S_k\{1\}'}{k} \equiv_p \sum_{k=1}^{p-1} \frac{sp}{k} \cdot \frac{S_k\{1\}'}{k}
\]

\[
= spS_{p-1}\{1\}'(2) \equiv_p spB_{p-r-2},
\]

where we applied \( S_{p-1}\{1\}', 2 \equiv_p B_{p-r-2} \) (see [2, Theorem 4.5]). Finally, by (2.4),

\[
F^{(r)}_{p-1}(x) \equiv_p \sum_{j=0}^{(r)-1} (-1)^r \left( \frac{(-1)^r}{x + j} \right) \frac{sp}{(x + j)^{r+1}} + spB_{p-r-2}
\]

\[
\equiv_p \sum_{j=1}^{(r)} \frac{1}{j - sp} - \frac{1}{j^{r+2}} + spB_{p-r-2}
\]

\[
\equiv_p \frac{-B^{(r+1)}_{(-x)} - (r + 2)spH^{(r+2)}_{(-x)} + spB_{p-r-2}}{spB_{p-r-2}}
\]

In the last step we used the following congruence: for \( 2 \leq t < p - 1 \),

\[
H^{(t)}_{(-x)} \equiv_p \sum_{j=1}^{(x)} j^{p-1-t} = \frac{B_{p-t}((-x)_p + 1) - B_{p-t}}{p - t} \equiv_p \frac{(-1)^t B_{p-t}(x) - B_{p-t}}{x^{t}},
\]

which is an immediate consequence of [5, Lemma 3.2].

\[\square\]

3. Proof of (1.3) in Theorem 1.1

We follow a similar strategy as outlined in the previous section. We start by considering the partial fraction decomposition of the rational function

\[
x \rightarrow \frac{(x)_k(1 - x)_k}{(x)_n(1 - x)_n}
\]

with \( 0 \leq k < n \). We have

\[
\sum_{k=0}^{n-1} \frac{(x)_k(1 - x)_k}{(1)_k^2} \cdot a_k = \sum_{k=0}^{n-1} \frac{(-1)^j A_j}{(n + j)! (n - 1 - j)!} \left( \frac{1}{x + j} + \frac{1}{1 - x + j} \right).
\]

(3.1)
where
\[ A_j := \sum_{k=0}^{j} (-1)^k \binom{j}{k} \binom{j+k}{k} \cdot a_k. \]

For \( n \to \infty \), in case the series is convergent, the identity (3.1) becomes
\[ \sum_{k=0}^{\infty} \frac{(x)_k(1-x)_k}{(1)_k^2} \cdot a_k = \frac{\sin(\pi x)}{\pi} \sum_{j=0}^{\infty} (-1)^j A_j \left( \frac{1}{x+j} + \frac{1}{1-x+j} \right). \]

In many cases the transformed sequence \( A_j \) has a nice formula. For example, if \( a_k = 1/(k+z) \), then
\[ A_j = \frac{(1-z)_j}{(z)_{j+1}} \]
and, for \( x = z = 1/2 \), we recover the series representations of the Catalan constant \( G = \sum_{j=0}^{\infty} (-1)^j/(2j+1)^2 \):
\[ \sum_{k=0}^{\infty} \frac{(2k)^2}{(2k+1)16^k} = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(1/2)_k^2}{(1)_k^2(k+1/2)} = \frac{1}{2\pi} \sum_{j=0}^{\infty} (-1)^j \frac{4}{(1/2+j)^2} = \frac{8G}{\pi}. \]

As regards congruences, we have the following result.

**Theorem 3.1.** Let \( p \) be a prime and \( a_0, \ldots, a_{p-1}, x \in \mathbb{Z}_p \). Then
\[ \sum_{k=0}^{p-1} \frac{(x)_k(1-x)_k}{(1)_k^2} \cdot a_k \equiv_p p^2 A_{(-x)} + s(A_{p-1-(-x)p} - A_{(-x)p}). \quad (3.2) \]

For \( x = 1/2 \) and \( p > 2 \),
\[ \sum_{k=0}^{p-1} \frac{(2k)^2}{16^k} \cdot a_k \equiv_p A_{(p-1)/2}. \]

**Proof.** Rearranging (3.1) in a convenient way,
\[ \sum_{k=0}^{p-1} \frac{(x)_k(1-x)_k}{(1)_k^2} \cdot a_k = \frac{(x)_p(1-x)_p}{(1)_p^2} \frac{2p-1}{p-1} \sum_{j=0}^{p-1} (-1)^j \binom{2p-1}{p+j} A_j \left( \frac{p}{x+j} + \frac{p}{1-x+j} \right). \]

If \( 0 \leq k \leq j \leq p-1 \), then \( A_{p-1-j} \equiv_p A_j \) because
\[ \binom{p-1-j}{k} \binom{p-1-j+k}{k} = \frac{(p-1-j) \cdots (p-j-k)(p-1-j+k) \cdots (p-j)}{(k)!^2} \equiv_p \frac{(j+1) \cdots (j+k)(j-k+1) \cdots j}{(k)!^2} = \binom{j+k}{k}. \]
Therefore, \( \sum_{j=0}^{p-1} (-1)^j \left( \frac{2p-1}{p+j} \right) pA_j \equiv p^2 \sum_{j=0}^{p-1} \frac{pA_j}{x+j} + (-1)^{(-x)_p} \left( \frac{2p-1}{p + (-x)_p} \right) \frac{A^{(-x)_p}}{s} + \sum_{j=0}^{p-1} \frac{pA_{p-j}}{1-x+j} \).

Therefore,

\[
\sum_{j=0}^{p-1} (-1)^j \left( \frac{2p-1}{p+j} \right) A_j \left( \frac{p}{x+j} + \frac{p}{1-x+j} \right) \equiv p^2 (-1)^{(-x)_p} \left( \frac{2p-1}{p + (-x)_p} \right) \frac{A^{(-x)_p}}{s} + \frac{A^{(-1-x)_p}}{1-s}.
\]

Finally, by using

\[
\left( \frac{2p-1}{p-1} \right) \equiv_p 1,
\]

\[
\left( \frac{2p-1}{p+j} \right) \equiv_p (-1)^j (1 - 2pH_j),
\]

\[
\frac{(x)_p(1-x)_p}{(1)_p^2} \equiv_p s(1-s)(1 + 2pH_{(-x)_p}),
\]

the proof of (3.2) is complete. For \( x = 1/2 \), it suffices to recall that

\[
\langle -x \rangle_p = \frac{p-1}{2} = p - 1 - \langle -x \rangle_p. \quad \Box
\]

As an application of the previous theorem, we notice that when \( a_k = 1 \) we have \( A_j = (-1)^j \) and, by (3.2),

\[
\sum_{k=1}^{p-1} \frac{(x)_k(1-x)_k}{(1)_k^2} \equiv_p (-1)^{(-x)_p},
\]

which has been shown in [6, Corollary 2.1].

Another noteworthy example is \( a_k = 1/k' \) for \( k \geq 1 \) (and \( a_0 = 0 \)). Then, by [4, Theorem 1],

\[
A_j = - \sum_{1:k_1+3:k_3+\cdots=r} \frac{2^{k_1+k_3+\cdots}(H^{(1)}_j)^{k_1}(H^{(3)}_j)^{k_3}}{1^{k_1}3^{k_3}\cdots k_1!k_3!}\ldots.
\]

Now we consider the case \( a_k = S_k([2]/k) \). Let

\[
G_n^{(r)}(x) : = \sum_{k=1}^{n} \frac{(x)_k(-x)_k}{(1)_k^2} \cdot S_k([2]/k).
\]
Then
\[ G_n^{(0)}(x) = \frac{(1 + x)_n(1 - x)_n}{(1)_n^2} - 1 \]
and \( S_k([2]^r) = S_{k-1}([2]^r) + S_k([2]^{r-1})/k^2 \) implies that
\[ G_n^{(r)}(x) = \frac{(1 + x)_n(1 - x)_n}{(1)_n^2} \cdot S_n([2]^r) + \frac{G_n^{(r-1)}(x)}{x^2}. \] (3.3)
Moreover,
\[ F_n^{(r)}(x + 1) - F_n^{(r)}(x) = 2G_n^{(r)}(x), \]
where
\[ F_n^{(r)}(x) := \sum_{k=1}^{n} \frac{(x)_k(1 - x)_k}{(1)_k^2} \cdot \frac{S_k([2]^r)}{k}. \]
Thus,
\[ F_n^{(r)}(x + m) - F_n^{(r)}(x) = 2 \sum_{j=0}^{m-1} \frac{G_n^{(r)}(x + j)}{x + j}. \] (3.4)
The next identity is a variation of (2.5) and it appears to be new.

**Theorem 3.2.** For any integers \( n \geq 1 \) and \( r \geq 0 \),
\[ \sum_{k=1}^{n} (-1)^k \binom{n}{k}(n + k) \binom{S_k([2]^r)}{k} = -2H_n^{2r+1}. \] (3.5)

**Proof.** By (3.3), for \( u = 1, \ldots, n \),
\[ G_n^{(r)}(-u) = \frac{G_n^{(r-1)}(-u)}{u^2} = \cdots = \frac{G_n^{(0)}(-u)}{u^{2r}} = -\frac{1}{u^{2r}}. \]
By letting \( x = -n \) and \( m = n \) in (3.4),
\[ \sum_{k=1}^{n} (-1)^k \binom{n}{k}(n + k) \binom{S_k([2]^r)}{k} = F_n^{(r)}(-n) = F_n^{(r)}(0) - 2 \sum_{j=0}^{n-1} \frac{G_n^{(r)}(-n + j)}{-n + j} \]
\[ = 2 \sum_{j=0}^{n-1} \frac{1}{(-n + j)^{2r+1}} = -2H_n^{2r+1}. \]
Thus, by applying (3.2), we find a modulo \( p^2 \) version of (1.3). A more refined reasoning will lead us to the modulo \( p^3 \) congruence.

**Proof of (1.3) in Theorem 1.1.** Since \( sp = x + \langle -x \rangle_p \),
\[ G_{p-1}^{(0)}(x) = \frac{(1 + x)_{p-1}(1 - x)_{p-1}}{(1)_{p-1}^2} - 1 \equiv_p s(1 - s)p^2 \cdot \frac{x^2}{x^2} = 1. \]
By [11, Theorem 1.6], $S_{p-1}([2]'') \equiv_p 0$ and therefore

$$G^{(r)}_{p-1}(x) \equiv_p \frac{G^{(r-1)}_{p-1}(x)}{x^2} \equiv_p \cdots \equiv_p \frac{G^{(0)}_{p-1}(x)}{x^{2r}} \equiv_p \frac{s(1-s)p^2}{x^{2r+2}} - \frac{1}{x^{2r}}.$$  

It follows that

$$F^{(r)}_{p-1}(sp) - F^{(r)}_{p-1}(x) \equiv_p 2 \sum_{j=0}^{\langle x \rangle_{p}-1} \frac{G^{(0)}_{p-1}(x+j)}{(x+j)^{2r+1}} \equiv_p -2s(1-s)p^2 \sum_{j=1}^{\langle x \rangle_{p}} \frac{1}{j^{2r+3}} - 2 \sum_{j=0}^{\langle x \rangle_{p}-1} \frac{1}{(x+j)^{2r+1}}.$$  

By (2.6),

$$\sum_{j=1}^{\langle x \rangle_{p}} \frac{1}{j^{2r+3}} = H^{(2r+3)}_{\langle x \rangle_{p}} \equiv_p \frac{B_{p-2r-3}(x)-B_{p-2r-3}}{2r+3}.$$  

Moreover,

$$F^{(r)}_{p-1}(sp) = \sum_{k=1}^{p-1} \frac{(sp)_k(1-sp)_k}{(1)_k^2} \cdot \frac{S_k([2]''\rangle}{k} \equiv_p \sum_{k=1}^{p-1} \frac{sp(k-sp)}{k^2} \cdot \frac{S_k([2]''\rangle}{k}$$

$$= sp \sum_{k=1}^{p-1} \frac{S_k([2]''\rangle}{k^2} - p^2 s^2 \sum_{k=1}^{p-1} \frac{S_k([2]''\rangle}{k^3}$$

$$= spS_{p-1}([2]'') + p^2 s^2 S_{p-1}([2]'', 3)$$

$$\equiv_p \frac{2pB_{p-2r-3}}{2r+3} + p^2 s^2 2rB_{p-2r-3}$$

$$\equiv_p \frac{2sp^2(1 + sr(2r + 3))B_{p-2r-3}}{2r+3},$$

where we applied

$$\frac{(sp)_k(1-sp)_k}{(1)_k^2} = \frac{sp(k-sp)}{k^2} \cdot \frac{(1+s(p)_{k-1}(1-sp)_{k-1}}{(1)_{k-1}^2} \equiv_p \frac{sp(k-sp)}{k^2}$$

and the congruences

$$S_{p-1}([2]'') \equiv_p \frac{2pB_{p-2r-1}}{2r+1} \text{ and } S_{p-1}([2]'', 3) \equiv_p -2rB_{p-2r-3},$$
which have been established in [11, Theorem 1.6] and [2, Theorem 4.1], respectively. Therefore,
\[
F_p(x) = \frac{2sp^2(1 + sr(2r + 3))B_{p-2r-3}}{2r + 3} - \frac{2s(s - 1)p^2(B_{p-2r-3}(x) - B_{p-2r-3})}{2r + 3} \\
+ 2 \sum_{j=0}^{(-x)p-1} \frac{1}{(x+j)^{2r+1}} \\
\equiv p^3 \sum_{j=0}^{(-x)p-1} \frac{1}{(x+j)^{2r+1}} + \frac{2s(1 - s)}{2r + 3} p^2 B_{p-2r-3}(x) \\
+ \frac{2s^2(r + 1)(2r + 1)}{2r + 3} p^2 B_{p-2r-3}. \hspace{1cm} \square
\]

We observe that congruence (1.5) follows directly by letting \( x = 1/2 \) in (1.3). Then \( (-x)_p - 1 = (p - 1)/2, \) \( B_{2n}(1/2) = (2^{1-2n} - 1)B_{2n} \) and, for \( p - 4 > t > 1, \) by [5, Theorem 5.2],
\[
H^{(t)}_{(p-1)/2} \equiv \begin{cases} 
\frac{t(2t+1) - 1}{2(t+1)} pB_{p-t-1} \pmod{p^2} & \text{if } t \equiv 2 \pmod{0}, \\
- \frac{(2^t - 2)}{t} B_{p-t} \pmod{p} & \text{if } t \equiv 2 \pmod{1}.
\end{cases}
\]

4. Final remarks: \( q \)-analogues of (2.5) and (3.5)

Each of the identities (2.5) and (3.5) has an elegant \( q \)-version (which means that they return the original identities as \( q \rightarrow 1 \)).

As regards (2.5), a \( q \)-analogue was found by Prodinger (see [3]):
\[
\sum_{k=1}^{n} (-1)^k \binom{n}{k}_q q^{(1)k} = - \sum_{k=1}^{n} \frac{q^k}{1 - q^k},
\]

where \( \binom{m}{k}_q \) is the Gaussian binomial coefficient
\[
\binom{m}{k}_q = \begin{cases} 
(1 - q^m)(1 - q^{m-1}) \cdots (1 - q^{m-k+1}) & \text{if } 0 \leq k \leq m, \\
0 & \text{otherwise}
\end{cases}
\]

and
\[
S_n(t_1, \ldots, t_r; q) = \sum_{1 \leq j_1 \leq \cdots \leq j_r \leq n} \frac{q^{j_1 + \cdots + j_r}}{(1 - q^{t_1})^{j_1} \cdots (1 - q^{t_r})^{j_r}}.
\]

A \( q \)-analogue of (3.5) is given in the next statement as (4.2). We encourage the interested reader to show that another proof of (4.1) can be obtained along the same lines.
**Theorem 4.1.** For any integers \( n \geq 1 \) and \( r \geq 0 \),
\[
\sum_{k=1}^{n} (-1)^k \binom{n+k}{k} q^{(2^r)k} \cdot S_k(\{2\}^r; q) \cdot \frac{1}{1-q^k} = - \sum_{k=1}^{n} \frac{(1+q^k)q^ru}{(1-q^u)^{2r+1}}.
\]

**Proof.** The procedure is quite similar to the one described earlier for the corresponding ordinary identity (3.5). Let
\[
G_n^{(r)}(u) := \sum_{k=1}^{n} (-1)^k \binom{u+k}{k} q^{(2^r)k} \cdot S_k(\{2\}^r; q).
\]

Then, for \( u = 1, \ldots, n \), \( G_n^{(0)}(u) = -1 \) and
\[
G_n^{(r)}(u) = \frac{q^u G_n^{(r-1)}(u)}{(1-q^u)^2} = \cdots = \frac{q^u G_n^{(0)}(u)}{(1-q^u)^{2r}} = - \frac{q^u}{(1-q^u)^{2r}}.
\]

Moreover,
\[
F_n^{(r)}(u) - F_n^{(r)}(u-1) = \frac{(1+q^u)G_n^{(r)}(u)}{(1-q^u)} = - \frac{(1+q^u)q^ru}{(1-q^u)^{2r+1}},
\]
\[\text{where}\]
\[
F_n^{(r)}(u) := \sum_{k=1}^{n} (-1)^k \binom{u+k}{k} q^{(2^r)k} \cdot S_k(\{2\}^r; q).
\]

Thus, since \( F_n^{(0)}(n) = 0 \),
\[
F_n^{(r)}(n) = \sum_{u=1}^{n} \frac{(1+q^u)G_n^{(r)}(u)}{(1-q^u)} + F_n^{(0)}(n) = - \sum_{u=1}^{n} \frac{(1+q^u)q^ru}{(1-q^u)^{2r+1}}
\]
and the proof is complete. \(\square\)

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