On Shell Renormalization Scheme From the Loopwise Expansion of the Pole Mass

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(Dated: March 19, 2019)

Abstract

We introduce an on shell renormalization scheme in which the mass parameter of minimal MS scheme is replaced with the pole mass obtained from the loop order expansion of the pole mass in the MS scheme. As a consequence, the quartic coupling constant remains same as that of the MS scheme and the vacuum expectation value gets contributions from the one-particle-irreducible diagrams. We also show the renormalization group invariance of the pole mass in this scheme.

PACS numbers: 11.15.Bt, 12.38.Bx

Keywords: On Shell Scheme, Loop Order Expansion

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The pole mass plays an important role in the process where the characteristic scale is close to the mass shell\[^1\]. It was shown that the pole mass is infrared finite and gauge invariant\[^2\] and is also invariant under the renormalization group(RG)\[^3\]. The relation between the bare mass \(M_B^2\) and the pole mass \(M^2\) in the on shell scheme\[^4\] is given by

\[
M_B^2 = M^2 + \delta M^2. 
\]

The mass counterterm \(\delta M^2\) is given by

\[
\delta M^2 = -\left[\Pi(p^2, m_B^2, \lambda_B)\right]_{p^2 = -M^2},
\]

in Euclidean space-time. Here the self energy \(\Pi(p^2, m_B^2, \lambda_B)\) is either given by the one-particle-irreducible diagrams where the contributions from the vacuum expectation value(VEV) \(v^2\) are included in the pole mass as in \[^5\] or given by the sum of the one-particle-irreducible and one-particle-reducible diagrams as in \[^6,7\] where only the contributions from the tree level vacuum expectation values \(v^2_0\) are included in the pole mass and those from the higher order vacuum expectation values such as \(\lambda v_0 v_1 + \frac{1}{2} v_1^2 + \cdots\) contribute one-particle-reducible diagrams to \(\Pi(p^2, m_B^2, \lambda_B)\). However, the RG running of the Higgs quartic coupling constant which is used in the vacuum stability analysis\[^8\] is given in MS renormalization scheme\[^9\] whereas in the former case, the Higgs quartic coupling constant does not coincide with that of minimal subtraction(MS) renormalization scheme. Hence, although the RG running of the Higgs coupling constant appear to coincide at one loop, the coincidence to all orders is not guaranteed and needs further investigation. The latter renormalization scheme, known as Fleischer and Jegerlehner(FJ) scheme, is widely used in the two-Higgs-doublet models\[^10,11\] recently. In this paper, we will introduce a procedure for an on shell renormalization scheme in which the mass parameter of the MS scheme is replaced with the pole mass obtained from the loop order expansion of the pole mass in the MS scheme. In order to do this, we first obtain the pole mass as a function of mass and the Higgs quartic coupling constant of the MS scheme in a loop order. Then, by inverting this series, we obtain the mass parameter of the MS scheme as a function of the pole mass and the Higgs quartic coupling constant of the MS scheme. It turns out that the resulting vacuum expectation value(VEV) contains not only the tadpole diagrams but the one-particle-irreducible(1PI) self energy diagrams. Since this is a finite transformation between the mass parameters, the renormalization constant of the Higgs quartic coupling constant remains same as the one in the MS scheme and hence have the same RG running.
The bare Lagrangian for the neutral scalar field theory with spontaneous symmetry breaking in the Euclidean space-time is given by
\[
L_B = \frac{1}{2}(\partial \Phi_B)^2 - \frac{1}{2}m_B^2(\Phi_B + v_B)^2 + \frac{1}{24}\lambda_B(\Phi_B + v_B)^4. \tag{3}
\]
The VEV is obtained from the minimum condition for the renormalized effective potential \(V_{\text{eff}}(\phi)\) where \(\phi\) is the classical field:
\[
0 = \frac{1}{v_B} \left[ \frac{\partial V_{\text{eff}}(\phi)}{\partial \phi} \right]_{\phi=0} = -m_B^2 + \frac{1}{6}\lambda_B v_B^2 + \Pi^{\text{ad}}(m_B^2, \lambda_B). \tag{4}
\]
where \(\Pi^{\text{ad}}(m_B^2, \lambda_B)\) is the unrenormalized tadpole. The bare quantities are related to the renormalized quantities as
\[
\Phi_B = \sqrt{Z_\Phi} \Phi = \sqrt{1 + \delta Z_2 + \cdots} \Phi, \quad v_B = \sqrt{Z_\lambda v} = \sqrt{1 + \delta v_2 + \cdots} v,
\]
\[
m_B^2 = m^2(1 + \delta m_1^2 + \cdots) \quad \text{and} \quad \lambda_B = \lambda (1 + \delta \lambda_1 + \cdots), \tag{5}
\]
where we have used the fact that \(\Phi_B\) and \(v_B\) have same renormalization constants which vanishes at one-loop in neutral scalar theory (\(\delta Z_1 = 0\)). By solving Eq.(4), we obtain the VEV as a series in the loop order expansion
\[
v = v_0 + v_1 + v_2 + \cdots \quad (v_0^2 = \frac{6m^2}{\lambda}). \tag{6}
\]
The tree level relation between the MS mass \(m\) and the pole mass \(M\) is given by
\[
M^2 = -m^2 + \frac{1}{2}\lambda v_0^2 = 2m^2, \tag{7}
\]
Then, in order to obtain the one-loop counterterms in the broken symmetric phase, let us consider the one-loop effective potential \([12]\) including the one-loop terms obtained by substituting Eqs.(5) and (6) into Eq.(3) as
\[
\frac{1}{2} \int \frac{d^Dp}{(2\pi)^D} \log[p^2 + 2m^2 + \lambda v_0 \phi + \frac{1}{2}\lambda \phi^2] + \{2m^2v_1 + v_0m^2(-\delta m_1^2 + \delta \lambda_1)\} \phi
\]
\[
+ \frac{1}{2}\{\lambda v_0 v_1 + m^2(-\delta m_1^2 + 3\delta \lambda_1)\} \phi^2 + \frac{1}{6}(\lambda v_1 + \delta \lambda_1 v_0) \phi^3 + \frac{1}{24}\delta \lambda_1 \phi^4
\]
\[
= -\frac{1}{4}(2m^2 + \lambda v_0 \phi + \frac{1}{2}\lambda \phi^2)\left(\frac{1}{\varepsilon} - \log \frac{2m^2 + \lambda v_0 \phi + \frac{1}{2}\lambda \phi^2}{\mu^2} + \frac{3}{2}\right)
\]
\[
+ \{2m^2v_1 + v_0m^2(-\delta m_1^2 + \delta \lambda_1)\} \phi + \frac{1}{2}\{\lambda v_0 v_1 + m^2(-\delta m_1^2 + 3\delta \lambda_1)\} \phi^2
\]
\[
+ \frac{1}{6}(\lambda v_1 + \delta \lambda_1 v_0) \phi^3 + \frac{1}{24}\delta \lambda_1 \phi^4, \tag{8}
\]
where we have used the $D = 4 - 2\varepsilon$ dimensional regularization. By noting that the one-loop renormalization constant of $Z\phi$ is zero in the neutral scalar theory and using the one-loop counterterms in symmetric phase given by [13]

$$
\delta m^2_1 = \frac{\lambda}{2\varepsilon}m^2 \text{ and } \delta \lambda_1 = \frac{3}{2\varepsilon}\lambda.
$$

(9)

we can check that the $\frac{1}{\varepsilon}$ poles of the effective potential in the broken symmetry phase given in Eq.(8) can be removed by the one-loop counterterms of the symmetric phase in the loop order expansion. The one-loop VEV term $v_1$ can be obtained from the vanishing condition of the one-point function. Now, let us introduce a procedure in which the mass parameter of minimal subtraction(MS) scheme is replaced with the pole mass obtained from the loop order expansion pole mass in the MS scheme. The pole mass $M^2$ is defined as the pole of the renormalized inverse two point function $\Gamma^{1PI}(p^2, m^2, \lambda)$ obtained from the unrenormalized 1PI self-energy $\Pi^{1PI}(p^2, m_B^2, \lambda_B)$ as

$$
\Gamma^{1PI}(p^2) = Z\phi \left[ p^2 - m_B^2 + \frac{\lambda_B}{2} v_B^2 + \Pi^{1PI}(p^2, m_B^2, \lambda_B) \right] = 0 \text{ when } p^2 = -M^2.
$$

(10)

The tree level bare mass term of Eq.(3) becomes the bare mass in the on shell scheme as

$$
M_B^2 = M^2 + \delta M^2 = -m_B^2 + \frac{\lambda_B}{2} v_B^2,
$$

(11)

and hence Eq.(10) gives the renormalization condition for $\delta M^2$ as

$$
\delta M^2 + \left[ \Pi^{1PI}(p^2, M_B^2, \lambda_B) \right]_{p^2=-M^2} = 0.
$$

(12)

We can check that the $\frac{1}{\varepsilon}$ pole the of the one-loop two-point function in the broken symmetric phase given by

$$
\Pi^{1PI}(p^2, m^2, \lambda) = \ldots \ldots + \ldots \ldots
$$

$$
= \frac{\lambda}{2} A(2m^2) - \frac{1}{2} \lambda^2 v_0^2 B(p^2, 2m^2),
$$

(13)

can be removed by using the one-loop counterterms in the symmetric phase given by $\lambda v_0 v_1 + m^2 (-\delta m^2_1 + 3\delta \lambda_1)$ (see Eqs.(8) and (9)) in the loop order expansion. Here the one-loop function $A(m^2)$ and $B(p^2, m^2)$ is given by [14]

$$
A(m^2) = \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2},
$$

(14)

and

$$
B(p^2, m^2) = \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + m^2)((p + q)^2 + m^2)}.
$$

(15)
Then we can obtain the loop order expansion of the pole mass $M^2$ as

$$M^2 = 2m^2 + \lambda v_0 v_1 + \Pi_1^{\text{1PI}}(-2m^2, m^2, \lambda) + \left[ \frac{d\Pi_1^{\text{1PI}}(p^2, m^2, \lambda)}{dp^2} \right]_{p^2 = -2m^2} (\lambda v_0 v_1$$

$$+ \Pi_1^{\text{1PI}}(-2m^2, m^2, \lambda)) + \Pi_2^{\text{1PI}}(-2m^2, m^2, \lambda) + \lambda v_0 v_2 + \frac{\lambda}{2} v_1^2 + \cdots, \quad (16)$$

where $\Pi_1^{\text{1PI}}(p^2, m^2, \lambda)$ is the renormalized $l$-loop 1PI self-energy obtained in the MS scheme.

By solving Eq.(4), we can obtain $v_i$ as a function of $m^2$ and $\lambda$ and hence Eq.(16) determines the pole mass $M^2$ as function of $m^2$ and $\lambda$. At tree level, we obtain the mass relation $M^2 = 2m^2$ as in Eq.(7). Then, by inverting Eq.(18), we can obtain the loop order expansion of $m^2$ as function of $M^2$ as

$$m^2 = \frac{1}{2} M^2 - M_1^2(M^2) - M_2^2(M^2) - \cdots, \quad (17)$$

where $M_2^2(M^2)$ is the $l$–loop terms in the expansion. Moreover, if we write the series expansion of the VEV where the pole mass $M^2$ is the tree level mass parameter as in Eq.(17), the order of the VEV changes also. For example, $v_0$ given in Eq.(6) becomes infinite series of function of $M^2$ which consists of not only tadpole diagrams but 1PI diagrams

$$v_0(m^2) = \sqrt{\frac{6m^2}{\lambda}} \simeq \sqrt{\frac{3M^2}{\lambda}} (1 - \frac{M_1^2}{M^2} + \cdots), \quad (18)$$

and let us write the resulting series expansion of the VEV where the pole mass $M^2$ is the tree level mass parameter as

$$v = \overline{v}_0 + \overline{v}_1 + \overline{v}_2 + \cdots, \quad (19)$$

where the relation between $v$ and $\overline{v}$ is

$$\overline{v}_0 = [v_0]_{m^2 = \frac{1}{2}M^2} = \sqrt{\frac{3M^2}{\lambda}} \quad \text{and} \quad \overline{v}_1 = -\frac{M_2^2}{M^2} \overline{v}_0 + [v_1]_{m^2 = \frac{1}{2}M^2} \quad \text{etc.} \quad (20)$$

Now, $M_1^2(M^2)$ and $\overline{v}_i$ can be determined if we substitute Eqs.(17) and (19) into the two conditions given in Eqs.(4) and (15). At one-loop, Eqs.(4) and (10) gives

$$-\frac{1}{2} M^2 \delta m_1^2 + M_1^2 + \frac{\delta \lambda_1}{6} \overline{v}_0 \overline{v}_1 + \frac{\lambda}{3} \overline{v}_0 \overline{v}_1 + \Pi_1^{\text{1ad}}(m^2, \lambda) = 0, \quad (21)$$

and

$$-\frac{1}{2} M^2 \delta m_1^2 + M_1^2 + \frac{\delta \lambda_1}{2} \overline{v}_0^2 + \lambda \overline{v}_0 \overline{v}_1 + \Pi_1^{\text{1PI}}(-M^2, m^2, \lambda) = 0, \quad (22)$$

where $\delta m_1^2$ and $\delta \lambda_1$ is the one-loop counterterms for $m^2$ and $\lambda$ and we have used the fact that $Z_\phi = 1$ up to one-loop order. Since Eq.(17) is a finite transformation of the mass parameters
from $m^2$ to the pole mass $M^2$, all the $\frac{1}{\varepsilon}$ poles will be removed by the renormalization constants of the MS scheme. Actually, by using $\Pi_1^{PI}(-M^2)$ given in Eq.(13) and the one-loop renormalization constants for the neutral scalar theory in the MS scheme given in Eq.(9), we can see that the $\frac{1}{\varepsilon}$ pole in the one-loop functions vanishes in Eqs. (21) and (22) as expected. Then, by solving the remaining finite equations, we can determine $M_1^2$ and $v_1$ as

$$M_1^2 = \frac{1}{2} \Pi_1^{PI}(-M^2, M^2, \lambda) - \frac{3}{2} \Pi_1^{tad}(M^2, \lambda) = \left[ -\frac{1}{2} \lambda A(M^2) - \frac{3}{4} M^2 B(-M^2, M^2) \right]_{\text{finite}},$$

and

$$\lambda v_0 v_1 = -\frac{3}{2} \Pi_1^{PI}(-M^2, M^2, \lambda) + \frac{3}{2} \Pi_1^{tad}(M^2, \lambda) = \left[ \frac{9}{4} M^2 B(-M^2, M^2) \right]_{\text{finite}},$$

where we have used Eq.(13) and the renormalized one-loop tadpole $\Pi_1^{tad}(p^2, m^2, \lambda)$ which can be obtained from Eq.(4) as

$$\Pi_1^{tad}(M^2, \lambda) = \frac{1}{2} \lambda A(M^2),$$

at one-loop order. $[X]_{\text{finite}}$ means the finite part of $X$ obtained by removing the $\frac{1}{\varepsilon}$ pole of $X$. Now let us consider the RG invariance of the pole mass in this scheme. Since the bare mass $M_B^2$ is RG invariant, we can see from Eq.(11) that the RG invariance of the pole mass requires the RG invariance of $\delta M^2$ so that

$$\mu \frac{d}{d\mu} \delta M^2 = (\mu \frac{\partial}{\partial \mu} + \beta_\lambda \frac{\partial}{\partial \lambda} + \beta_{m^2} \frac{\partial}{\partial m^2}) \delta M^2 = 0.$$  

(26)

By substituting Eq.(13) into (12), we obtain one-loop mass counterterm in the on shell scheme as

$$\delta M_1^2 = -\frac{1}{2} \lambda A(M^2) + \frac{3}{2} M^2 B(-M^2, M^2)$$

(27)

and by noting that RG function $\beta_\lambda$ is given by

$$\beta_\lambda = -2\varepsilon \lambda + 3\lambda^2 + \cdots,$$

(28)

we can see that the one loop counterterm given in Eq.(28) satisfies Eq.(26) and hence pole mass is RG invariant up to one loop. The VEV can be obtained directly from Eqs.(4),(11) and (12). In order to see this, let us eliminate $m_B^2$ in Eq.(4) by using Eq.(11) to obtain

$$M^2 + \delta M^2 - \frac{1}{3} \lambda_B v_B^2 + \Pi^{tad}(M_B^2, \lambda_B) = 0,$$

(29)
and then, by using Eq.(12) we obtain
\[ \lambda_B v_B^2 = 3[M^2 - \Pi^{PI}(p^2, M_B^2, \lambda_B) + \Pi^{tad}(M_B^2, \lambda_B)]. \]  
(30)

After eliminating $\frac{1}{\epsilon}$ pole with the MS renormalization constants, we obtain the formula for the renormalized VEV as
\[ \lambda v^2 = 3 \left[ M^2 - \Pi^{PI}(-M^2, M^2, \lambda) + \Pi^{tad}(M^2, \lambda) \right]_{\text{finite}}, \]  
(31)
which agrees with the results given in Eqs.(20) and (24). In this way, we can determine $\delta M^2$ and $\tau$ from Eqs.(12) and (29) without need to calculate $M_2^2(M^2)$.

In this paper, we have introduced a procedure for an on shell renormalization scheme in which the mass parameter of minimal MS scheme is replaced with the pole mass obtained from the loop order expansion of the pole mass in the MS scheme. In order to see the difference between the on shell renormalization scheme based on the loopwise expansion introduced in this paper and previously known schemes, let us consider the case of the Sirlin and Zucchini(SZ) scheme[5] which is used most frequently. First, in SZ scheme, the Higgs mass $M_H$ is related to running mass $m$ as
\[ M_H^2 = \frac{1}{3} \lambda_{\text{Sirlin}} v_0^2 = 2m^2 \]  
(32)
Note the different normalization for $\lambda$ between Ref.[5] and our paper. In our scheme, the Higgs pole mass is obtained by loopwise inversion of the defining equation of the Higgs pole mass given in Eq.(10) and as a result the corresponding relation becomes
\[ M_H^2 - 2M_1^2 - 2M_1^2 - \cdots = \frac{1}{3} \lambda_{\text{MS}} v_0^2 = 2m^2 \]  
(33)
as in Eq.(16). By noting that $\beta_{m^2} = \lambda m^2$ and by using $M_1^2$ given in Eq.(23), we can see that both sides of Eq.(33) have same RG running. If we extend the on shell renormalization scheme based on loopwise expansion to the electroweak sector of the standard model, we can choose the parameter set as $\{G_\mu, M_H^2, M_W^2, \lambda_{\text{MS}}\}$ instead of $\{G_\mu, M_H^2, M_W^2, \lambda_{\text{Sirlin}}\}$ so that the RG evolution of the Higgs quartic coupling constant $\lambda$ which is important to determine the vacuum stability condition can be obtained by the RG functions of the MS scheme. Second, the vacuum expectation value $v$ that emerges at the triple scalar vertex including the Higgs (HHH, HGG etc.) have a loopwise expansion and should be determined order by order from Eq.(30). This equation shows that the vacuum expectation value gets contributions
not only from the tadpoles as in previous schemes but also from the one-particle-irreducible diagrams in the loopwise expansion scheme. We have investigated in the neutral scalar field theory which is most simple model and the extension to more complicated models like Standard Model is under investigation.

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