NONLINEAR STABILITY OF GARDNER BREATHERS

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ABSTRACT. We show that breather solutions of the Gardner equation, a natural generalization of the KdV and mKdV equations, are $H^2(\mathbb{R})$ stable. Through a variational approach, we characterize Gardner breathers as minimizers of a new Lyapunov functional and we study the associated spectral problem, through (i) the analysis of the spectrum of explicit linear systems (spectral stability), and (ii) controlling degenerated directions by using low regularity conservation laws.

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1. Introduction

1.1. Preliminaries. In this paper we consider the nonlinear stability of breathers of the Gardner equation

$$w_t + (w_{xx} + 3\mu w^2 + w^3)_x = 0, \quad \mu \in \mathbb{R}\setminus\{0\}, \quad w(t, x) \in \mathbb{R}, \quad (t, x) \in \mathbb{R}^2.$$ (1.1)

Specifically, we present here a proof on the stability in $H^2(\mathbb{R})$ of Gardner breathers, showing that this stability is independent of the value of the parameter $\mu$, which controls the strength of the quadratic nonlinear part or KdV term $w^2$, in its existence interval for real Gardner breathers.

The Gardner equation (1.1) is a well-known completely integrable model [14, 11, 10], with infinitely many conservation laws and well-known (long-time) asymptotic behavior of its solutions obtained with the help of the inverse scattering transform [15]. As a physical model, (1.1) describes large-amplitude internal solitary waves, showing a dynamics which can look rather different from the KdV form. On the other hand, solutions of (1.1) are invariant under space and time translations. Indeed, for any $t_0, x_0 \in \mathbb{R}$, $w(t - t_0, x - x_0)$ is also a solution. Note that (1.1) is not
scaling invariant. Moreover, (1.1) is closely related to the modified Korteweg-de Vries (mKdV) equation
\[ u_t + (u_{xx} + u^3)_x = 0, \quad u(t, x) \in \mathbb{R}, \quad (t, x) \in \mathbb{R}^2, \] (1.2)
through the search of \( L^\infty \)-solutions. In fact, it is easy to see by substitution, that the following holds:

**Proposition 1.1.** Let \( u \) be a solution of the mKdV equation (1.2) with a nonvanishing boundary value or condition (NVBC) \( \mu \in \mathbb{R}\setminus\{0\} \) at \( \pm \infty \). Then \( w(t, x) := u(t, x + 3\mu^2 t) - \mu \) is a solution of the Gardner equation (1.1).

The key characteristic of the Gardner equation (1.1) is that it contains a nonlinear part composed of a Korteweg-de Vries (KdV) quadratic term \((w^2)_x\) and a positive modified KdV (mKdV) cubic term \((w^3)_x\). The competition between this nonlinear part and the linear dispersive term \( w_{xxx} \) allows the existence of intricate soliton, multisolitons as well as exact real-valued breather solutions (see (1.12)). A soliton is a localized, moving or stationary solution which maintains its form for all time. Similarly, a multi-soliton is a (not necessarily) explicit solution describing the interaction of several solitons [20].

In the case of the Gardner equation (1.1), the profile of the soliton solution is slightly cumbersome, but it is still given explicitly by the formula
\[ w(t, x) := Q_{c,\mu}(x - ct), \quad Q_{c,\mu}(s) := \frac{c}{\mu + \sqrt{\mu^2 + c^2 \cosh(\sqrt{c}s)}}, \] (1.3)
By substituting (1.3) into (1.1), one has that \( Q_{c,\mu} > 0 \) satisfies the nonlinear elliptic equation
\[ Q''_{c,\mu} - c Q_{c,\mu} + 3\mu Q^2_{c,\mu} + Q^3_{c,\mu} = 0, \quad Q_{c,\mu} > 0, \quad Q_{c,\mu} \in H^1(\mathbb{R}). \] (1.4)
This second order, elliptic equation is deeply related to the so-called variational structure of the soliton solution. To be more precise, it is well-known that for the Gardner equation the standard conservation laws at the \( H^1 \)-level are the mass
\[ M[w](t) := \frac{1}{2} \int_\mathbb{R} w^2(t, x) dx = M[w](0), \] (1.5)
and energy
\[ E_\mu[w](t) := \frac{1}{2} \int_\mathbb{R} w^2_x - \mu \int_\mathbb{R} w^3 - \frac{1}{4} \int_\mathbb{R} w^4 = E_\mu[w](0), \] (1.6)
which is \( H^1 \)-subcritical. For the Gardner equation (1.1), the Cauchy problem is globally well-posed at such a level of regularity or even better, see e.g. Alejo and Kenig-Ponce-Vega [4, 27]. Note that these results are not trivial since (1.1) is not scaling invariant.

Moreover, for a mKdV solution \( u \) with NVBC \( \mu \) we also have the natural conservation laws
\[ M_{nv}[u](t) := \frac{1}{2} \int_\mathbb{R} (u^2 - \mu^2) dx = M_{nv}[u](0), \] (1.7)
and energy
\[ E[u](t) := \frac{1}{2} \int_\mathbb{R} u^2_x - \frac{1}{4} \int_\mathbb{R} (u^4 - \mu^4) = E[u](0), \] (1.8)
which is \( H^1 \)-subcritical. Using these conserved quantities, the variational structure of any Gardner soliton can be characterized as follows: there exists a suitable Lyapunov functional, invariant in time and such that the soliton \( Q_{c,\mu} \) is a corresponding extremal point. Moreover, it is a global minimizer under fixed mass. For the Gardner case, this functional is given by (see [9] for the mKdV case)
\[ \mathcal{H}_0[w](t) = E_\mu[w](t) + c M[w](t), \] (1.9)
where \( c > 0 \) is the scaling of the solitary wave, and \( M[w], E_\mu[w] \) are given in (1.5) and (1.6). Indeed, it is easy to see that for any \( z(t) \in H^1(\mathbb{R}) \) small,
\[ \mathcal{H}_0[Q_{c,\mu} + z](t) = \mathcal{H}_0[Q_{c,\mu}] + \int_\mathbb{R} z(Q''_{c,\mu} - c Q_{c,\mu} + 3\mu Q^2_{c,\mu} + Q^3_{c,\mu}) + O(\|z(t)\|_{H^1}^2). \] (1.10)
The zero order term above is independent of time, while the first order term in \( z \) is zero from (1.11), proving the critical character of \( Q_{c,\mu} \).

1.2. Breathers and their stability. Besides these soliton solutions of the Gardner equation (1.1), it is possible to find another big set of explicit and oscillatory solutions, known in the physical and mathematical literature as the breather solution, and which is a periodic in time, spatially localized real function. Although there is no universal definition for a breather, we will adopt the following convention, that will match the Gardner, mKdV and also sine-Gordon cases (see [7]).

**Definition 1.2** (Aperiodic breather). We say that \( B = B(t, x) \) is a breather solution for a particular one-dimensional dispersive equation if there are \( T > 0 \) and \( L = L(T) \in \mathbb{R} \) such that, for all \( t \in \mathbb{R} \) and \( x \in \mathbb{R} \), one has

\[
B(t + T, x) = B(t, x - L),
\]

and moreover, the infimum among times \( T > 0 \) such that property (1.11) is satisfied for such a time \( T \) is uniformly positive in space.

**Remark 1.1.** Observe that the last condition ensures that solitons (and multisolitons) are not breathers, since e.g. \( Q_{c,\mu}(x - (c(t + T))) = Q_{c,\mu}(x - L - ct) \) for \( L := cT \) but \( T \) can be any real-valued time.

For the Gardner equation (1.1), the breather solution can be obtained by using different methods (e.g. Inverse Scattering, Hirota method, etc), and its expression is characterized by the introduction of the parameter \( \mu \) which controls the quadratic nonlinearity in (1.1).

**Definition 1.3** (Gardner breather). Let \( \alpha, \beta, \mu \in \mathbb{R} \setminus \{0\} \) such that \( \Delta = \alpha^2 + \beta^2 - 2\mu^2 > 0 \), and \( x_1, x_2 \in \mathbb{R} \). The real-valued breather solution of the Gardner equation (1.1) is given explicitly by the formula

\[
B_{\mu} \equiv B_{\alpha,\beta,\mu}(t, x; x_1, x_2) := 2\sqrt{2}\beta\left[ \arctan\left( \frac{G_{\alpha,\beta,\mu}(t, x)}{F_{\alpha,\beta,\mu}(t, x)} \right) \right],
\]

with \( y_1 \) and \( y_2 \)

\[
y_1 = x + \delta t + x_1, \quad y_2 = x + \gamma t + x_2, \quad \delta := \alpha^2 - 2\beta^2, \quad \gamma := 3\alpha^2 - \beta^2,
\]

and

\[
G_{\mu} \equiv G_{\alpha,\beta,\mu}(t, x) := \frac{\beta \sqrt{\alpha^2 + \beta^2}}{\alpha \sqrt{\Delta}} \sin(\alpha y_1) - \sqrt{2}\mu \frac{[\cosh(\beta y_2) + \sinh(\beta y_2)]}{\Delta},
\]

\[
F_{\mu} \equiv F_{\alpha,\beta,\mu}(t, x) := \cosh(\beta y_2) - \sqrt{2}\mu \frac{[\alpha \cos(\alpha y_1) - \beta \sin(\alpha y_1)]}{\alpha \sqrt{\Delta}}.
\]

**Remark 1.2.** This is a four-parametric solution, with two scalings \((\alpha, \beta)\) and two shift translations \((x_1, x_2)\). Note that we impose \( \alpha^2 + \beta^2 - 2\mu^2 > 0 \), since we deal with real-valued solutions. Therefore \( \mu \in (0, \mu_{\max}) \), where

\[
\mu_{\max} := \sqrt{\frac{\alpha^2 + \beta^2}{2}}.
\]

Moreover from (1.12) one has, for any \( k \in \mathbb{Z} \),

\[
B_{\alpha,\beta,(-1)^k\mu}(t, x; x_1 + \frac{k\pi}{\alpha}, x_2) = (-1)^k B_{\alpha,\beta,\mu}(t, x; x_1, x_2),
\]

which are also solutions of (1.1). This identity reveals the periodic character of the first translation parameter \( x_1 \), coupled this time to the parameter \( (-1)^k \mu, \ k \in \mathbb{Z} \).

\(^1\)In the case of NLS equations and their solitons, Definition 1.2 include them because of the \( U(1) \) invariance.
Remark 1.3. Note that we can take the limit when $\alpha \to 0$ in (1.12), obtaining the so call double pole solution for the Gardner equation (1.1), which it is a natural generalization of the well-known double pole solution of mKdV:

**Definition 1.4.** Let $\beta, \mu \in \mathbb{R}\backslash \{0\}$ such that $\Delta_0 = \beta^2 - 2\mu^2 > 0$, and $x_1, x_2 \in \mathbb{R}$. The real-valued double pole solution of the Gardner equation (1.1) is given explicitly by the formula

$$B_{\beta, \mu}(t,x) := \partial_x \hat{B} := 2\sqrt{2}\partial_x \left[ \arctan \left( \frac{G_{\beta, \mu}(t,x)}{F_{\beta, \mu}(t,x)} \right) \right],$$

(1.16)

with

$$G_{\beta, \mu}(t,x) := \frac{\beta^2 y_1}{\sqrt{\Delta_0}} - \sqrt{2}\beta \frac{\cosh(\beta y_2) \sinh(\beta y_2)}{\Delta_0},$$

$$F_{\beta, \mu}(t,x) := \cosh(\beta y_2) - \frac{\sqrt{2}\mu [1 - \beta y_1]}{\sqrt{\Delta_0}}.$$

and where now $y_1 = x - 3\beta^2 t + x_1$ and $y_2 = x - \beta^2 t + x_2$.

Note that Gardner breather solutions (1.12) are periodic in time, but not in space. Additionally, the variables $y$ and $\beta$, corresponding to the velocity of the breather solution, since it corresponds to the velocity of the carried hump in the breather profile. Note additionally that breathers have to be considered as bound states, since they do not decouple into simple solitons as time evolves.

For the Gardner equation, the breather solution was discovered by [43, 2], using the IST. These solutions have become a canonical example of complexity in nonlinear integrable systems [31, 1]. Moreover, it is interesting to point out that mKdV and Gardner breather solutions have also been considered by Kenig, Ponce and Vega and Alejo respectively, in their proofs of the non-uniform stability of mKdV and Gardner flows in the Sobolev spaces $H^s$, $s < \frac{1}{2}$ [28, 3].

If one studies perturbations of solitons in (1.11), (1.12) and more general equations, the concepts of orbital, and asymptotic stability emerge naturally. In particular, since energy and mass are conserved quantities, it is natural to expect that solitons are stable in a suitable energy space. Indeed, $H^1$-stability of mKdV and more general solitons and multi-solitons has been considered e.g. in Benjamin [9], Bona-Souganidis-Strauss [10], Weinstein [46], Maddocks-Sachs [33], Martel-Merle-Tsai [34], Martel-Merle [35] and Muñoz [40]. Moreover, asymptotic stability properties for gKdV equations have been studied by Pego-Weinstein [42] and Martel-Merle [36, 37], among many other authors.

The underlying question is then the study of the corresponding stability of these breather solutions. A first step in that direction was already done in [5] (see also [6, 7, 8]), where the nonlinear stability of mKdV breathers was presented. Recent studies about the stability-instability of these breather structures are [31] and [11]. Other references dealing with similar problems on stability/instability of coherent structures are Kowalczyk-Martel-Muñoz [29] and [30], Comech-Cuccagna-Pelinovsky [12], Cuccagna-Pelinovsky-Vougalter [13], Grillakis-Shatah-Strauss [17], Howard-Zumbrum [19], Kapitula [22, 23], Kapitula-Kevrekidis-Sandstede [24], Kapitula-Promislov [25], Kaup-Yang [26], Sandstede [24], Zumbrum [49], Yang [47] and [48].

1.3. Main Results. In this paper, we show a positive answer to the question of the stability of Gardner breathers. In fact, our main result is stated in short as follows:

**Theorem 1.5.** Gardner breathers (1.12) are orbitally stable for $H^2$-perturbations, whenever the parameter $\mu \in (0, \mu_{\text{max}})$, with $\mu_{\text{max}}$ as in (1.13).
A more precise version of this theorem is given in Theorem 6.1. We will see from the variational characterization of Gardner breathers (see Sect. 4) that the Sobolev space $H^2$ will arise naturally, since the Lyapunov functional that will have Gardner breathers as minimizers, will be defined in that space.

For the proof of the previous theorem, we will follow some steps introduced in [5] for mKdV breathers. This approach in the mKdV setting is far to be trivial when we deal with Gardner breathers, since many of at hand proofs presented for mKdV breathers, are no longer practical as a consequence of the complicated functional form of breather solutions like (1.12). Therefore, we have to resort to the deep integrability structure of the (1.1) in order to perform these proofs, by using adapted identities for the Gardner breather and new nonlinear identities (see Lemma 2.5). Moreover, we are able to compute explicitly the mass, the energy and the third conserved quantity defined in $H^2$ of any Gardner breather solution (1.12), showing explicitly the nonlinear dependence on the parameter $\mu$.

The main steps of the proof are the following. Firstly, we show that Gardner breathers (1.12) satisfy a fourth-order, nonlinear ODE. More precisely, every Gardner breather verifies

$$J_{\mu}[B_{\mu}] := B_{\mu,xx} - 2(\beta^2 - \alpha^2)(B_{\mu,xx} + \mu B_{\mu}^2 + B_{\mu}^3) + (\alpha^2 + \beta^2)B_{\mu} + 5\mu B_{\mu}B_{\mu,xx} + 5B_{\mu}^2B_{\mu,xx} + 3\frac{3}{2}B_{\mu}^5 + 5\mu B_{\mu}^2B_{\mu,xx} + 10\mu B_\mu B_{\mu,xx} + 10\mu^2 B_{\mu}^3 + \frac{15}{2}\mu B_{\mu}^4 = 0.$$ (1.17)

(cf. equation (3.9)). This result for Gardner breathers is, as far as we know, not present in the literature, although similar ones were obtained in [5, 7, 8] for the case of mKdV breathers in the line and periodic case and also for sine-Gordon breathers respectively. The proof of this identity is based on the close connection between Gardner and mKdV with NVBC equations (see Proposition 1.1). That is, we are going to prove that the above identity (1.17) for Gardner breathers is related with an equivalent identity (3.4) for mKdV breathers with NVBC, that will be easier to be proved, by using the explicit form of the breather with NVBC of mKdV, and several new identities related to the structure of the breather.

It seems that this equation cannot be obtained from the original approach by Lax [32], since the dynamics do not decouple in time. As far as we know, this is the first time that the previous equation is proved for breathers of the Gardner (and also for mKdV with NVBC (1.2)) equation (1.1).

As second step, we show the variational structure of Gardner breathers (1.12), with the introduction of a new Lyapunov functional (see (1.6) for further details)

$$H_{\mu}[w](t) := F_{\mu}[w](t) + 2(\beta^2 - \alpha^2)E_{\mu}[w](t) + (\alpha^2 + \beta^2)^2 M[w](t),$$ (1.18)

well-defined in the $H^2$-topology, and for which Gardner breathers are not only extremal points, but also local minimizers, up to symmetries. This functional will control the perturbative terms and the instability directions arising during of the dynamics, the latter as a consequence of the symmetries satisfied by (1.1). We will study spectrally the linearized operator around Gardner breather solutions and we will discover, as in the mKdV case, that it has only one negative eigenvalue. We will prove that by using new identities which simplify the computation of the Wronskian in the kernel elements of this operator (see Greenberg [16] for further details of the theory dealing with fourth order eigenvalue problems). This strategy was used first by Weinstein in [45].

1.4. Organization of this paper. In Sect 2 we study generalized Weinstein conditions and we present some nonlinear identities and stability tests satisfied by Gardner breathers. In Sect 3 we prove that any Gardner and mKdV (with NVBC) breather solutions satisfy a fourth order nonlinear ODE, which characterizes them. In Sect 4 we present a variational characterization of the Gardner breather, by introducing a new Lyapunov functional which controls the dynamics of small perturbations in the stability problem. In Sect 5 we focus on the spectral properties of a
linear self-adjoint operator related to the Gardner breather solution. Finally in Sect. 6 we present a detailed version of Theorem 1.5 and a sketch of its proof.

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2. Nonlinear identities and Weinstein conditions

The aim of this section is to get stability tests by computing generalized Weinstein conditions for any Gardner breather \( B_\mu \) \(^{(1.12)}\). We begin with the simpler case of the Gardner soliton \(^{(1.3)}\), where the mass \(^{(1.5)}\) and the energy \(^{(1.6)}\) are given by the quantities:

\[
M[Q_{c,\mu}] := 2\sqrt{c} - 2\sqrt{2}\mu \arctan \left( \frac{\sqrt{c}}{\sqrt{2}\mu} \right),
\]

and

\[
E_\mu[Q_{c,\mu}] := -\frac{2}{3}\beta^{3/2} + 4\mu^2\sqrt{c} - 4\sqrt{2}\mu^3 \arctan \left( \frac{\sqrt{c}}{\sqrt{2}\mu} \right),
\]

which reduce, when \( \mu = 0 \), to the well known mass and energy of the mKdV soliton solution \(^{(5, eqn. (2.1))}\). From these expressions we see explicitly the coupling between the soliton scaling \( c \) and the parameter \( \mu \) in the computed mass and the energy. Note that the Weinstein condition \(^{(46)}\) is now, for \( c > 0 \), \( \mu \in \mathbb{R} \\{0\} \),

\[
\partial_c M[Q_{c,\mu}] = \frac{\sqrt{c}}{c + 2\mu^2} > 0.
\]

This inequality guarantees the nonlinear stability of the Gardner soliton \(^{(1.3)}\). Moreover note that the same condition for the energy of the Gardner soliton \( \partial_c E_\mu[Q_{c,\mu}] = -\frac{\beta^{3/2}}{c + 2\mu^2} \) does not vanish either.

Now, we approach the case of Gardner breathers. Firstly we present the following identity for solutions of the Gardner equation and which will be useful when computing the mass and energy of the breather solution (see appendix A for a proof):

\textbf{Lemma 2.1.} Let \( w(t, x) := \sqrt{2}i\partial_x \log \left( \frac{F_\mu - iG_\mu}{F_\mu + iG_\mu} \right) \) be any Gardner solution of \(^{(1.1)}\). Then

\[
w^2 = 2\frac{\partial^2}{\partial x^2} \log (G_\mu^2 + F_\mu^2) - 2\mu w.
\]

This result allows us to compute explicitly the mass of any Gardner breather:

\textbf{Lemma 2.2.} Let \( B_\mu = B_{\alpha,\beta,\mu} \) be any Gardner breather, for \( \alpha, \beta, \mu \) as in definition \(^{(1.12)}\). Then, the mass of \( B_\mu \) is

\[
M[B_\mu] := 4\beta + 2\sqrt{2}\mu \arctan \left( \frac{2\sqrt{2}\mu\beta}{\Delta} \right).
\]

\textbf{Proof.} It follows directly by using the above identity \(^{(2.4)}\) and substitution in the definition \(^{(1.5)}\).

In fact, we obtain:

\[
M[B_\mu] = \frac{1}{2} \int \mathbb{R} B_\mu^2 = 4\beta + 2\sqrt{2}\mu \arctan \left( \frac{2\sqrt{2}\mu\beta}{\Delta} \right).
\]
Remark 2.1. Note that as we could expect, after the Gardner soliton case, the mass of any Gardner breather depends on the scalings $\alpha, \beta$, and the parameter $\mu$. This dependence slightly differs from the exclusive dependence of the mass of any mKdV breather on the second scaling parameter $\beta$.

From the involved integral in (2.5), we can define the partial mass of any Gardner breather as follows:

**Definition 2.3.** Let $B_\mu = B_{\alpha, \beta, \mu}$ be any Gardner breather, for $\alpha, \beta, \mu$ as in definition (1.12) (such that $\Delta = \alpha^2 + \beta^2 - 2\mu^2 > 0$). Then, we define the partial mass associated to any Gardner breather as:

$$M_\mu(t, x) \equiv M_{\alpha, \beta, \mu}(t, x) := \frac{1}{2} \int_{-\infty}^{x} B^2_\mu(t, s; x_1, x_2) ds = 2\beta + \partial_x \log(G^2_\mu + F^2_\mu)(t, x) - 2\sqrt{2}\mu \arctan\left(\frac{G_\mu(t, x)}{F_\mu(t, x)}\right). \quad (2.6)$$

A direct consequence of the above results are the following generalized Weinstein conditions:

**Corollary 2.4.** Let $B_\mu = B_{\alpha, \beta, \mu}$ be any Gardner breather of the form (1.12). Given $t \in \mathbb{R}$ fixed, let

$$\Lambda_\alpha B_\mu := \partial_\alpha B_\mu, \quad \text{and} \quad \Lambda_\beta B_\mu := \partial_\beta B_\mu. \quad (2.7)$$

Then $\forall \mu \in (0, \mu_{\text{max}})$ both functions $\Lambda_\alpha B_\mu$ and $\Lambda_\beta B_\mu$ are in the Schwartz class for the spatial variable and they satisfy the identities

$$\partial_\alpha M[B_\mu] = \int_{\mathbb{R}} B_\mu \Lambda_\alpha B_\mu = -\frac{16\mu^2\beta\alpha}{\Delta^2 + 8\mu^2\beta^2} < 0, \quad (2.8)$$

and

$$\partial_\beta M[B_\mu] = \int_{\mathbb{R}} B_\mu \Lambda_\beta B_\mu = 4\left(\frac{\Delta^2 + 2\mu^2\Delta + 4\mu^2\beta^2}{\Delta^2 + 8\mu^2\beta^2}\right) > 0, \quad (2.9)$$

independently of time.

**Proof.** By simple inspection, one can see that, given $t$ fixed, $\Lambda_\alpha B_\mu$ and $\Lambda_\beta B_\mu$ are well-defined Schwartz functions. The proof of (2.8) and (2.9) is consequence of (2.5). \qed

Consider now the two directions associated to spatial translations. Let $B_{\alpha, \beta, \mu}$ as introduced in (1.12). We define

$$B_1(t; x_1, x_2) := \partial_{x_1} B_{\alpha, \beta, \mu}(t; x_1, x_2), \quad \text{and} \quad B_2(t; x_1, x_2) := \partial_{x_2} B_{\alpha, \beta, \mu}(t; x_1, x_2). \quad (2.10)$$

It is clear that, for all $t \in \mathbb{R}$, and $\alpha, \beta, \mu$ as in definition (1.3) and $x_1, x_2 \in \mathbb{R}$, both $B_1$ and $B_2$ are real-valued functions in the Schwartz class, exponentially decreasing in space. Moreover, it is not difficult to see that they are linearly independent as functions of the $x$-variable, for all time $t$ fixed.

**Lemma 2.5.** Let $B_\mu = B_{\alpha, \beta, \mu}$ be any Gardner breather of the form (1.12). Then we have

1. $B_\mu = \tilde{B}_{\mu, x}$, with $\tilde{B}_\mu = B_{\alpha, \beta, \mu}$ given by the smooth $L^\infty$-function

$$\tilde{B}_\mu(t, x) := 2\sqrt{2} \arctan\left(\frac{G_\mu}{F_\mu}\right). \quad (2.11)$$

2. For any fixed $t \in \mathbb{R}$, we have $(\tilde{B}_\mu)_t$ well-defined in the Schwartz class, satisfying

$$B_{\mu, xx} + \tilde{B}_{\mu, t} + 3\mu B^2_\mu + B^3_\mu = 0. \quad (2.12)$$

3. Let $\mathcal{M}_\mu$ be defined by (2.5). Then

$$B^2_{\mu, x} + \frac{1}{2} B^4_\mu + 2\mu^2 B_\mu^2 + 2\mu \tilde{B}_{\mu, t} - 2(\mathcal{M}_\mu)_t = 0. \quad (2.13)$$
(4) Finally, let $B_1$ and $B_2$ as in (2.10) and $\tilde{B}_i \equiv \partial_{x_i} \tilde{B}_\mu$, $\tilde{B}_{ij} \equiv \partial_{x_i} \partial_{x_j} \tilde{B}_\mu$, $i, j = 1, 2$. Then
\[
\int_{-\infty}^{x} (\tilde{B}_{12}^2 - \tilde{B}_{11} \tilde{B}_{22}) = - (\mu + B_\mu) \tilde{B}_{11} + \partial_{x}^2 \partial_{x} \log \left( G_\mu^2 + F_\mu^2 \right). \tag{2.14}
\]

Proof. The first item above is a direct consequence of the definition of $B_\mu = B_{\alpha, \beta, \mu}$ in (1.12). On the other hand, (2.12) is a consequence of (2.11) and integration in space (from $-\infty$ to $x$) of (1.1). To obtain (2.13) we multiply (2.12) by $B_{\mu, x}$ and integrate in space. Finally to prove (2.14), since we are working with smooth functions, one has $B_{\mu} = B_1 + B_2$, and also

\[
B_1 = \tilde{B}_{11} + \tilde{B}_{12}, \quad \text{and} \quad B_2 = \tilde{B}_{12} + \tilde{B}_{22}.
\]

Now, since $\tilde{B}_{12}^2 = B_{12}^2 + \tilde{B}_{11} - 2B_1 \tilde{B}_{11}$ and $\tilde{B}_{22} = B_2 - B_1 + \tilde{B}_{11}$, we get

\[
\begin{align*}
(\tilde{B}_{12}^2 - \tilde{B}_{11} \tilde{B}_{22}) &= B_{12}^2 + \tilde{B}_{11} - 2B_1 \tilde{B}_{11} - \tilde{B}_{11} (B_2 - B_1 + \tilde{B}_{11}) \\
&= B_{12}^2 - \tilde{B}_{11} (B_1 + B_2) = B_{12}^2 - \tilde{B}_{11} \partial_{x} (\tilde{B}_1 + \tilde{B}_2) = B_{12}^2 - \tilde{B}_{11} B_{\mu, x}.
\end{align*}
\tag{2.15}
\]

Now integrating (2.15) in $x$, we obtain:

\[
\int_{-\infty}^{x} (\tilde{B}_{12}^2 - \tilde{B}_{11} \tilde{B}_{22}) = \int_{-\infty}^{x} (B_{12}^2 - \tilde{B}_{11} B_{\mu, x}) \tag{2.16}
\]

\[
= \int_{-\infty}^{x} B_{12}^2 - \left[ B_{\mu} \tilde{B}_{11} \right]_{-\infty}^{x} - \int_{-\infty}^{x} B_{\mu} B_{11} \right]. \tag{2.17}
\]

Using that $B_{\mu} B_{11} = \partial_{x}^2 (\frac{1}{2} B_{\mu}^2) - B_{12}^2$, the last integral above simplifies as follows:

\[
\begin{align*}
\int_{-\infty}^{x} B_{12}^2 &= \left[ B_{\mu} \tilde{B}_{11} \right]_{-\infty}^{x} - \int_{-\infty}^{x} B_{\mu} B_{11} \\
&= \int_{-\infty}^{x} B_{12}^2 - \left[ B_{\mu} \tilde{B}_{11} \right]_{-\infty}^{x} - \int_{-\infty}^{x} (\partial_{x}^2 \left( \frac{1}{2} B_{\mu}^2 \right) - B_{12}^2) \\
&= - B_{\mu} \tilde{B}_{11} \big|_{-\infty}^{x} + \partial_{x}^2 \int_{-\infty}^{x} \frac{1}{2} B_{\mu}^2 = - B_{\mu} \tilde{B}_{11} + \partial_{x}^2 \mathcal{M}_{\mu}[t, x]. \tag{2.18}
\end{align*}
\]

Now, remembering (2.3), we get

\[
\partial_{x} \mathcal{M}_{\mu} = \partial_{x}^2 \log \left( G_{\mu}^2 + F_{\mu}^2 \right) - \mu \tilde{B}_{11}. \tag{2.19}
\]

Finally, substituting this derivative in the last equation above, we obtain the desired simplification. The proof is complete. \hfill \Box

Remark 2.2. The reader may compare (2.12)-(2.14) with the well known identities for the Gardner soliton solution (1.3):

\[
Q_{c, \mu}''' - c Q_{c, \mu} + 3 \mu Q_{c, \mu}^2 + Q_{c, \mu}^3 = 0, \quad (Q_{c, \mu}^2)' - c Q_{c, \mu}^2 + 2 \mu Q_{c, \mu}^3 + \frac{1}{2} Q_{c, \mu}^4 = 0.
\]

We compute now the energy (1.6) of any Gardner breather solution.

Lemma 2.6. Let $B_{\mu} = B_{\alpha, \beta, \mu}$ be any Gardner breather, for $\alpha, \beta, \mu$ as in definition (1.12). Then the energy of $B_{\mu}$ is

\[
E_{\mu}[B_{\mu}] := \frac{4}{3} \beta \gamma + 8 \beta \mu^2 + 4 \sqrt{2} \mu^3 \text{arctan} \left[ \frac{2 \sqrt{2} \mu \beta}{\Delta} \right]. \tag{2.21}
\]

Remark 2.3. Note that in the Gardner case, in comparison with the mKdV case ($\mu = 0$), the sign of the energy is dictated by a nonlinear balance among the velocity $\gamma$ and the $\mu$ terms.
Proof. First of all, let us prove the following reduction

$$E_\mu[B_\mu](t) = \frac{1}{3} \int_\mathbb{R} (\mathcal{M}_\mu(t) - \mu B_\mu^3(t)) dx. \quad (2.22)$$

Indeed, we multiply (2.12) by $B_\mu$ and integrate in space: we get

$$\int_\mathbb{R} B_{\mu,x}^2 = \int_\mathbb{R} B_\mu \hat{B}_{\mu,t} + 3\mu \int_\mathbb{R} B_\mu^4 + \int_\mathbb{R} B_\mu^3.$$

On the other hand, integrating (2.13),

$$\int_\mathbb{R} B_{\mu,x}^2 + \frac{1}{2} \int_\mathbb{R} B_{\mu,t}^4 + 2\mu \int_\mathbb{R} B_\mu^3 + 2 \int_\mathbb{R} B_\mu \hat{B}_{\mu,t} - 2 \int_\mathbb{R} (\mathcal{M}_\mu)_t = 0.$$

From these two identities, we get

$$\int_\mathbb{R} B_{\mu,x}^2 = \frac{4}{3} \int_\mathbb{R} (\mathcal{M}_\mu)_t - \frac{10}{3} \mu \int_\mathbb{R} B_\mu^3,$$

and therefore

$$\int_\mathbb{R} B_{\mu,x}^2 = \frac{4}{3} \int_\mathbb{R} (\mathcal{M}_\mu)_t - \int_\mathbb{R} B_\mu \hat{B}_{\mu,t} - \frac{\mu}{3} \int_\mathbb{R} B_\mu^3.$$

Finally, substituting the last two identities into (1.6), we get (2.22), as desired.

Now we prove (2.21). From (2.3), we have that

$$\mathcal{M}_\mu(t, x) = 2\beta + \partial_x \log(G^2 + F^2) - \mu \hat{B}_\mu,$$

and hence,

$$(\mathcal{M}_\mu)_t(t, x) = \partial_x \partial_t \log(G^2 + F^2) - \mu \hat{B}_{\mu,t}.$$ 

Now substituting in the energy (2.22), remembering the identity (2.12) and the explicit expression for $\mathcal{M}_\mu[B_\mu]$ in (2.5), we get

$$E_\mu[B_\mu](t) = \frac{1}{3} \int_\mathbb{R} (\mathcal{M}_\mu)_t(t, x) + \mu B_\mu^3) dx = \frac{1}{3} \int_\mathbb{R} \left( \partial_x \partial_t \log(G^2 + F^2) + \mu B_\mu^3 \right) dx$$

$$= \frac{1}{3} \int_\mathbb{R} \partial_x \partial_t \log(G^2 + F^2) + 3\mu B_\mu^3 dx$$

$$= \frac{1}{3} \int_\mathbb{R} \partial_t \log(G^2 + F^2) + \frac{\mu}{3} B_{\mu,x}^2 + 3\mu B_\mu^3 dx$$

$$= \frac{1}{3} \partial_t \log(G^2 + F^2) + 3\mu B_\mu^3 dx$$

$$= \frac{1}{3} \partial_t \log(G^2 + F^2) + 4\mu B_{\mu,x}^2 + 2\mu^2 M[B_\mu]$$

$$= \frac{1}{3} \partial_t \log(G^2 + F^2) + 4\mu B_{\mu,x}^2 + 2\mu^2 (4\beta + 4\sqrt{2}\mu \arctan \left( \frac{\sqrt{2}\mu \beta}{\Delta} \right))$$

$$= \frac{4}{3} \beta \gamma + 8\mu^2 \gamma + 4\sqrt{2}\mu^3 \arctan \left( \frac{\sqrt{2}\mu \beta}{\Delta} \right).$$

\[ \square \]

Corollary 2.7. Let $B_\mu = B_{\alpha,\beta,\mu}$ be any Gardner breather. Then

$$\partial_\alpha E_\mu[B_\mu] = 8\alpha \beta (1 - \frac{4\mu^4}{\Delta^2 + 8\mu^2 \beta^2}), \quad \partial_\beta E_\mu[B_\mu] = 4(\alpha^2 - \beta^2) + 8\mu^2 \left( \frac{\Delta^2 + 2\mu^2 \Delta + 4\mu^2 \beta^2}{\Delta^2 + 8\mu^2 \beta^2} \right). \quad (2.23)$$

Remark 2.4. Note that the condition $\alpha = \beta$ is equivalent to the identity

$$\partial_\beta E_\mu[B_\mu] = \frac{8\mu^2 \beta^4}{\beta^4 + \mu^4},$$

which is always positive, on the contrary to the mKdV breather case ($\mu = 0$). On the other hand, the similar identity (2.3) for the energy of Gardner solitons can not vanish for any $\mu$. 

3. Elliptic equations for breathers

3.1. Nonlinear stationary equation for mKdV breathers with NVBC $\mu$. The objective of this section is to prove that any mKdV breather solution with NVBC $\mu$ satisfies a suitable stationary, elliptic equation. Indeed, this elliptic equation will be a key step in the proof of an equivalent stationary elliptic equation for any Gardner breather, which it is a direct consequence of the close connection between mKdV and Gardner solutions, as showed in Proposition 1.1. First and for the sake of completeness, we present these kind of mKdV breathers with NVBC $\mu$.

**Definition 3.1.** Let $\alpha, \beta, \mu \in \mathbb{R}\setminus\{0\}$ such that $\Delta = \alpha^2 + \beta^2 - 2\mu^2 > 0$, and $x_1, x_2 \in \mathbb{R}$. The real-valued breather solution of the mKdV equation (1.2) with NVBC $\mu$ at $\pm\infty$ is given explicitly by the formula

$$B = B_{\alpha,\beta,\mu}^{nv}(t, x; x_1, x_2) := \mu + 2\sqrt{2} \arctan \left( \frac{g(t, x)}{f(t, x)} \right), \quad (3.1)$$

with

$$g(t, x) := G_\mu(t, x - 3\mu^2 t),$$

$$f(t, x) := F_\mu(t, x - 3\mu^2 t),$$

and $G_\mu, F_\mu$ defined in (1.12).

**Remark 3.1.** Note that every mKdV breather (with or without NVBC) satisfies Definition 1.2 with the same selection for $L$ and $T$ parameters that Gardner breathers $B^\mu_\mu$ (1.12). Let $B$ any mKdV breather with NVBC. When we compute its mass (1.7) (see appendix A for a complementary proof of this identity), we obtain

$$M^{nv}[B] := \frac{1}{2} \int_{\mathbb{R}} (B^2 - \mu^2) dx = \partial_x \log(f^2 + g^2)|^{+\infty}_{-\infty} = 4\beta. \quad (3.2)$$

We can also define the partial mass associated to a mKdV breather with NVBC in the following way

$$M^{nv}[B] := \frac{1}{2} \int_{-\infty}^{x} (B^2 - \mu^2) dx = 2\beta + \partial_x \log(f^2 + g^2)(t, x). \quad (3.3)$$

Note that the mass of any mKdV breather with NVBC $\mu$ is indeed equal to the mass of the well known mKdV breather solution (see [5]). Now, we show the following identities for mKdV breathers with NVBC:

**Lemma 3.2.** Let $B = B_{\alpha,\beta,\mu}^{nv}$ be any mKdV breather with NVBC of the form (3.1). Then we have

1. $B = \tilde{B}_x$, with $\tilde{B} = \tilde{B}_{\alpha,\beta,\mu}^{nv}$ given by

$$\tilde{B}(t, x) := \mu x + 2\sqrt{2} \arctan \left( \frac{g}{f} \right), \quad (3.4)$$

2. For any fixed $t \in \mathbb{R}$, we have $(\tilde{B})_t$ well-defined in the Schwartz class, satisfying

$$B_{xx} + \tilde{B}_t + B^3 - \mu^3 = 0. \quad (3.5)$$

3. Finally, let $\mathcal{M}^{nv}$ be defined by (3.3). Then

$$B^2_x + \frac{1}{2} B^4 + 2B\tilde{B}_t - 2(\mathcal{M}^{nv})_t - 2\mu^3 B + \frac{3}{2} \mu^4 = 0. \quad (3.6)$$

**Proof.** The first item above is a direct consequence of the definition of $B$ in (3.1). On the other hand, (3.3) is a consequence of (3.4) and integration in space (from $-\infty$ to $x$) of (3.1). Finally, to obtain (3.6) we multiply (3.5) by $B_x$ and integrate in space taking into account the NVBC $\mu$ at $\pm\infty$. 

---

4See [2] for further details.
The next nontrivial identity for mKdV breathers with NVBC (3.1) will be useful in the proof of the nonlinear stationary equation that they satisfy.

**Lemma 3.3.** Let $B = B_{\alpha, \beta, \mu}^n$ be any mKdV breather with NVBC (3.1). Then, for all $t \in \mathbb{R}$,

$$B_{xt} + 2(M_{nv})_t B = \left(2(\beta^2 - \alpha^2) + 5\mu^2\right)\tilde{B}_t + \left((\alpha^2 + \beta^2)^2 + 6\mu^2(\beta^2 - \alpha^2 + \frac{5}{2}\mu^2)\right)(B - \mu). \quad (3.7)$$

**Proof.** See appendix [3] for a detailed proof of this nonlinear identity. □

**Proposition 3.4.** Let $B = B_{\alpha, \beta, \mu}^n$ be any mKdV breather with NVBC (3.1). Then, for any fixed $t \in \mathbb{R}$, $B$ satisfies the nonlinear stationary equation

$$J[B] := B(4x) + \left(2(\beta^2 - \alpha^2) + 5\mu^2\right)(B_{xx} + B^3) + \left((\alpha^2 + \beta^2)^2 + 6\mu^2(\beta^2 - \alpha^2 + \frac{5}{4}\mu^2)\right)B$$

$$+ 5BB_x^2 + 5B^2B_\mu + \frac{3}{2}B^5 - 4(\beta^2 - \alpha^2 + \mu^2) \equiv 0. \quad (3.8)$$

**Proof.** From (3.5) and (3.6), one has

$$J[B] = -(\tilde{B}_t + B^3 - \mu^3)_{xx} + (2(\beta^2 - \alpha^2) + 5\mu^2)(\tilde{B}_t - \mu^3) + ((\alpha^2 + \beta^2)^2 + 6\mu^2(\beta^2 - \alpha^2 + \frac{5}{4}\mu^2)B$$

$$+ 5BB_x^2 + 5B^2B_\mu + \frac{3}{2}B^5 - 4(\beta^2 - \alpha^2 + \mu^2) - (\alpha^2 + \beta^2)^2 \mu$$

$$= -B_{tx} + B\left[\frac{1}{2}B^4 + 2B\tilde{B}_t - 2(M_{nv})_t - 2\mu^3 B + \frac{3}{2}\mu^4 \right] - 2B^2(\tilde{B}_t + B^3 - \mu^3) + \frac{3}{2}B^5$$

$$- 4(\beta^2 - \alpha^2 + \mu^2) - (\alpha^2 + \beta^2)^2 \mu$$

$$+ (2(\beta^2 - \alpha^2) + 5\mu^2)(\tilde{B}_t - \mu^3) + ((\alpha^2 + \beta^2)^2 + 6\mu^2(\beta^2 - \alpha^2 + \frac{5}{4}\mu^2)B$$

$$- 6(\beta^2 - \alpha^2 + 3/2\mu^2)\mu_3 - (\alpha^2 + \beta^2)^2 \mu$$

$$= -B_{tx} + 2(M_{nv})_t B + [2(\beta^2 - \alpha^2) + 5\mu^2]\tilde{B}_t + [(\alpha^2 + \beta^2)^2 + 6\mu^2(\beta^2 - \alpha^2 + \frac{3}{4}\mu^2)]B$$

$$- 6(\beta^2 - \alpha^2 + 3/2\mu^2)\mu_3 - (\alpha^2 + \beta^2)^2 \mu$$

$$= -B_{tx} + 2(M_{nv})_t B + [2(\beta^2 - \alpha^2) + 5\mu^2]\tilde{B}_t + [(\alpha^2 + \beta^2)^2 + 6\mu^2(\beta^2 - \alpha^2 + \frac{3}{4}\mu^2)](B - \mu) = 0.$$  

In the last line we have used (3.7). □

### 3.2. Nonlinear stationary equation for Gardner breathers.

Our aim in this section is to prove that any Gardner breather solution $B_{\mu}$ satisfies a suitable stationary, elliptic equation, by using the close relation with mKdV breathers with NVBC, as presented in Proposition (1.1).

**Theorem 3.5.** Let $B_{\mu} = B_{\alpha, \beta, \mu}$ be any Gardner breather (1.12). Then, for any fixed $t \in \mathbb{R}$, $B_{\mu}$ satisfies the nonlinear stationary equation

$$J_{\mu}[B_{\mu}] := \frac{\partial}{\partial x} B_{\mu, xx} - 2(\beta^2 - \alpha^2)(B_{\mu, xx} + 3\mu B_{\mu}^2 + B_{\mu}^3) + (\alpha^2 + \beta^2)^2 B_{\mu} + 5B_{\mu} B_{\mu, xx} + 5B_{\mu}^2 B_{\mu, xx}$$

$$+ \frac{3}{2} B_{\mu}^5 + 5B_{\mu, xx} + 10\mu B_{\mu} B_{\mu, xx} + 10\mu^2 B_{\mu}^3 + \frac{15}{2} \mu B_{\mu}^4 = 0. \quad (3.9)$$

**Remark 3.2.** This identity can be seen as the nonlinear stationary equation satisfied by any Gardner breather profile (1.12), and therefore it is independent of time and translation parameters $x_1, x_2 \in \mathbb{R}$. One can compare with the Gardner soliton solution $Q_{\alpha, \mu}(x - ct - x_0)$, which satisfies the standard elliptic equation (1.4), obtained as the first variation of the $H^1$ Weinstein functional (1.9). Moreover note that (3.9) and (1.12) reduces to [3] eqn. (5.2)] and (1.2) (up to translations) when $\mu = 0.$
Proof. From Proposition 3.6 we first rewrite our Gardner breather \( B_\mu \) as \( B_\mu = B - \mu \), where here \( B \) is a mKdV breather solution with NVBC \( \mu \) at \( \pm \infty \) as we presented in (3.11). Hence, substituting \( B_\mu = B - \mu \) in (3.9) we obtain:

\[
J_\mu[B - \mu] = B_{4x} - 2(\beta^2 - \alpha^2)(B_{xx} + 3\mu(B - \mu)^2 + (B - \mu)^3) + (\alpha^2 + \beta^2)(B - \mu) + 5(B - \mu)B_x^2
+ 5(B - \mu)^2B_{xx} + \frac{3}{2}(B - \mu)^3 + 5\mu B_x^2 + 10\mu(B - \mu)B_{xx} + 10\mu^2(B - \mu)^3 + \frac{15}{2} \mu(B - \mu)^4
\]

where in the last line we have used (3.8). \( \square \)

Although the shift parameters \( x_1, x_2 \) are chosen independently of time, a simple argument ensures that the previous Theorem still holds under time dependent, translation parameters \( x_1(t) \) and \( x_2(t) \).

Corollary 3.6. Let \( B_{0,\alpha,\beta}^0 = B_{0,\alpha,\beta}(t; \alpha, \beta, \mu) \) be any Gardner breather as in (1.12), and \( x_1(t), x_2(t) \in \mathbb{R} \) two continuous functions, defined for all \( t \) in a given interval. Consider the modified breather

\[
B_{\alpha,\beta}(t, x) := B_{0,\alpha,\beta}(t; x_1(t), x_2(t)), \quad (\text{cf. } 1.12).
\]

Then \( B_{\alpha,\beta} \) satisfies (3.9), for all \( t \) in the considered interval.

Proof. A direct consequence of the invariance of the equation (3.9) under spatial translations. \( \square \)

4. Variational characterization of Gardner breathers

In this section we introduce a new \( H^2 \)-Lyapunov functional for the Gardner equation (1.1). Consider \( w_0 \in H^2(\mathbb{R}) \) and let \( w = w(t) \in H^2(\mathbb{R}) \) be the associated local in time solution of the Cauchy problem associated to (1.1), with initial condition \( w(0) = w_0 \) (cf. (27)). We first define the \( H^2 \)-functional

\[
F_\mu[w](t) := \frac{1}{2} \int_{\mathbb{R}} w_{xx}^2 \, dx - 5\mu \int_{\mathbb{R}} w^2 w_x^2 \, dx + \frac{5}{2} \mu^2 \int_{\mathbb{R}} w^4 \, dx - \frac{5}{2} \mu \int_{\mathbb{R}} w^2 w_x^2 \, dx + \frac{3}{2} \mu \int_{\mathbb{R}} w^6 \, dx + \frac{1}{4} \int_{\mathbb{R}} w^8 \, dx.
\]

(4.1)

Lemma 4.1. Let \( w \) be a local (in time) \( H^2 \)-solution of the the Gardner equation (1.1) with initial data \( w_0 \). Then the functional \( F_\mu[w](t) \) (4.1) is a conserved quantity. Moreover, \( w \) is a global (in time) \( H^2 \)-solution.
The existence of this last conserved quantity is a deep consequence of the *integrability property* for the Gardner equation. In particular, it is not present in a general, non-integrable gKdV equation. The proof of Lemma 4.1 is an easy computation. Moreover, as we did for the mass and the energy (1.5)-(1.6), we are also able to get an explicit formula of $F_{\mu}$ at the Gardner breather $B_{\mu}$, by using the following relations between breathers and solitons which are valid for the Gardner (and mKdV) equation

\[ M[B_{\mu}] = 2\text{Re}\left[ M[Q_{c,\mu}] |_{c = \beta + i\alpha} \right] \quad \text{and} \quad E_{\mu}[B_{\mu}] = 2\text{Re}\left[ E_{\mu}[Q_{c,\mu}] |_{c = \beta + i\alpha} \right]. \tag{4.2} \]

**Lemma 4.2.** Let $B_{\mu} = B_{\alpha,\beta,\mu}$ be any Gardner breather, for $\alpha, \beta, \mu$ as in definition (1.12). Then we have that

\[ F_{\mu}[B_{\mu}] := 4 \frac{1}{15} \left( 3(\beta^4 - 10\beta^2\alpha^2 + 5\alpha^4) - 10\mu^2\beta(\beta^2 - 3\alpha^2 - 6\mu^2) \right) + 8\sqrt{2}\mu^5 \arctan\left( \frac{\sqrt{c}}{\sqrt{2}\mu} \right). \tag{4.3} \]

**Proof.** Firstly, integrating directly (4.1) we get an expression for $F_{\mu}$ at $Q_{c,\mu}$:

\[ F_{\mu}[Q_{c,\mu}] = 2 \frac{1}{15} \sqrt{c} \left( 3c^2 - 10\mu^2 c + 60\mu^4 \right) - 8\sqrt{2}\mu^5 \arctan\left( \frac{\sqrt{c}}{\sqrt{2}\mu} \right). \tag{4.4} \]

Now, applying the same strategy used for relations (4.2), we get

\[ F_{\mu}[B_{\mu}] = 2\text{Re}\left[ F_{\mu}[Q_{c,\mu}] |_{c = \beta + i\alpha} \right] = (4.3). \tag{4.5} \]

\[ \square \]

Using the functional $F_{\mu}[w]$ (4.1), we build a new $H^2$-Lyapunov functional specifically associated to the breather solution. Let $B_{\mu} = B_{\alpha,\beta,\mu}$ be any Gardner breather, and $t \in \mathbb{R}$, and $M[w]$ and $E_{\mu}[w]$ given in (1.5) and (1.6) respectively. We define

\[ \mathcal{H}_{\mu}[w](t) := F_{\mu}[w](t) + 2(\beta^2 - \alpha^2)E_{\mu}[w](t) + (\alpha^2 + \beta^2)M[w](t). \tag{4.6} \]

Therefore, $\mathcal{H}_{\mu}[w]$ is a real-valued conserved quantity, well-defined for $H^2$-solutions of (1.1). Note additionally that the functionals $\mathcal{H}_{\mu=0}$ and $\mathcal{H}_{\mu}$ for the mKdV and Gardner equations are surprisingly the same.

**Lemma 4.3.** Let $B_{\mu} = B_{\alpha,\beta,\mu}$ be any Gardner breather, for $\alpha, \beta, \mu$ as in definition (1.12). Then we have that

\[ \mathcal{H}_{\mu}[B_{\mu}] := h_1 + h_2 2\sqrt{2}\mu \arctan\left( \frac{2\sqrt{2}\mu\beta}{\Delta} \right), \tag{4.7} \]

\[ h_1 = \frac{8\beta}{15} \left( 4\beta^4 + 20\alpha^2\beta^2 + 5\mu^2(5\beta^2 - 3\alpha^2 + 6\mu^2) \right), \quad h_2 = \left( (\alpha^2 + \beta^2)^2 + 4\mu^2(\beta^2 - \alpha^2 + \mu^2) \right). \tag{4.8} \]

**Proof.** Collecting the mass (2.5), the energy (2.21) and $F_{\mu}$ at $B_{\mu}$ (4.3), and substituting at (4.6), we get (4.8).

\[ \square \]

Moreover, one has the following
Lemma 4.4. Gardner breathers $B_\mu$ (1.12) are critical points of the Lyapunov functional $\mathcal{H}_\mu$ (1.10). In fact, for any $z \in H^2(\mathbb{R})$ with sufficiently small $H^2$-norm, and $B_\mu = B_{\alpha,\beta,\mu}$ any Gardner breather solution, then, for all $t \in \mathbb{R}$, one has

$$\mathcal{H}_\mu[B_\mu + z] - \mathcal{H}_\mu[B_\mu] = \frac{1}{2} Q_\mu[z] + N_\mu[z],$$

(4.9)

with $Q_\mu$ being the quadratic form defined in (4.10), and $N_\mu[z]$ satisfying $|N_\mu[z]| \leq K \|z\|_{H^2(\mathbb{R})}^3$.

Proof. We compute:

$$\mathcal{H}_\mu[B_\mu + z] = \frac{1}{2} \int_{\mathbb{R}} (B_\mu + z)^2_{xx} - \frac{5}{2} \int_{\mathbb{R}} (B_\mu + z)^2 (B_\mu + z)^2 + \frac{1}{4} \int_{\mathbb{R}} (B_\mu + z)^6$$

$$- 5\mu \int_{\mathbb{R}} (B_\mu + z)(B_\mu + z)^2 + \frac{3}{2} \mu \int_{\mathbb{R}} (B_\mu + z)^5 + \frac{5}{2} \mu^2 \int_{\mathbb{R}} (B_\mu + z)^4 + (\beta^2 - \alpha^2) \int_{\mathbb{R}} (B_\mu + z)^2$$

$$- \frac{1}{2} (\beta^2 - \alpha^2) \int_{\mathbb{R}} (B_\mu + z)^4 - 2\mu(\beta^2 - \alpha^2) \int_{\mathbb{R}} (B_\mu + z)^3 + \frac{1}{2} (\alpha^2 + \beta^2)^2 \int_{\mathbb{R}} (B_\mu + z)^2$$

$$= \frac{1}{2} \int_{\mathbb{R}} B_{\mu,xx}^2 - \frac{5}{2} \int_{\mathbb{R}} B_{\mu}^2 B_{\mu,xx} + \frac{1}{4} \int_{\mathbb{R}} B_{\mu}^6 - 5\mu \int_{\mathbb{R}} B_{\mu} B_{\mu,xx}^2 + \frac{3}{2} \mu \int_{\mathbb{R}} B_{\mu}^5 + \frac{5}{2} \mu^2 \int_{\mathbb{R}} B_{\mu}^4$$

$$+ (\beta^2 - \alpha^2) \int_{\mathbb{R}} B_{\mu,xx}^2 - \frac{1}{2} (\beta^2 - \alpha^2) \int_{\mathbb{R}} B_{\mu,xx}^4 - 2(\beta^2 - \alpha^2) \mu \int_{\mathbb{R}} B_{\mu}^3 + \frac{1}{2} (\alpha^2 + \beta^2)^2 \int_{\mathbb{R}} B_{\mu}^2$$

$$+ \int_{\mathbb{R}} [B_{\mu,4x} - 2(\beta^2 - \alpha^2) (B_{\mu,xx} + 3\mu B_{\mu}^2 B_{\mu,xx}^2 + (\alpha^2 + \beta^2)^2 B_{\mu} + 5\mu B_{\mu,xx}^2$$

$$+ 5\mu B_{\mu,xx} + \frac{3}{2} B_{\mu}^3 + 5\mu^2 B_{\mu,xx} + 10\mu B_{\mu} B_{\mu,xx} + 10\mu^2 B_{\mu}^3 + \frac{15}{2} \mu B_{\mu}^4] z$$

$$+ \frac{1}{2} \left[ \int_{\mathbb{R}} z_{xx}^2 + 2(\beta^2 - \alpha^2) \int_{\mathbb{R}} z_{x}^2 + (\alpha^2 + \beta^2)^2 \int_{\mathbb{R}} z^2 - 5 \int_{\mathbb{R}} B_{\mu,xx}^2$$

$$- 10\mu \int_{\mathbb{R}} B_{\mu} z_{xx}^2 + 10 \int_{\mathbb{R}} B_{\mu} B_{\mu,xx} z_{xx} + \int_{\mathbb{R}} (5B_{\mu,xx}^2 + 10B_{\mu} B_{\mu,xx} + \frac{15}{2} B_{\mu}^4$$

$$- 6(\beta^2 - \alpha^2) B_{\mu}^2 z^2 + 3\mu \int_{\mathbb{R}} (10B_{\mu,xx}^3 - 4(\beta^2 - \alpha^2) B_{\mu} + \frac{10}{3} B_{\mu,xx} + 10\mu B_{\mu}^2) z^2$$

$$- \frac{5}{2} \int_{\mathbb{R}} (z_{xx}^2 + 2B_{\mu,xx} z_{xx} + 2B_{\mu} z_{xx}) + \int_{\mathbb{R}} 5B_{\mu,xx}^3 + \frac{15}{4} B_{\mu}^6 z^2 + \frac{5}{2} \int_{\mathbb{R}} B_{\mu} z^6 + \frac{1}{4} \int_{\mathbb{R}} z^6$$

$$- 5\mu \int_{\mathbb{R}} z_{xx}^2 + 15\mu \int_{\mathbb{R}} B_{\mu,xx}^2 + \frac{15}{2} \mu \int_{\mathbb{R}} B_{\mu} z^4 + \frac{3\mu}{2} \int_{\mathbb{R}} z^5 + 10\mu \int_{\mathbb{R}} B_{\mu}^3$$

$$+ \frac{5}{2} \mu^2 \int_{\mathbb{R}} z^4 - 2(\beta^2 - \alpha^2) \int_{\mathbb{R}} B_{\mu} z^3 - \frac{1}{2} (\beta^2 - \alpha^2) \int_{\mathbb{R}} z^4 - 2\mu(\beta^2 - \alpha^2) \int_{\mathbb{R}} z^3.$$

We finally obtain:

$$\mathcal{H}_\mu[B_\mu + z] = \mathcal{H}_\mu[B_\mu] + \int_{\mathbb{R}} J_\mu[B_\mu] z(t) + \frac{1}{2} Q_\mu[z] + N_\mu[z],$$

where the quadratic form $Q_\mu$, associated to the linearized operator $L_\mu$ (5.1), is defined in the following way:

$$Q_\mu[z] := \int_{\mathbb{R}} z L_\mu[z] = \int_{\mathbb{R}} z_{xx}^2 + 2(\beta^2 - \alpha^2) \int_{\mathbb{R}} z_{xx}^2 + (\alpha^2 + \beta^2)^2 \int_{\mathbb{R}} z^2 - 5 \int_{\mathbb{R}} B_{\mu,xx}^2 - 10\mu \int_{\mathbb{R}} B_{\mu} z_{xx}^2$$

$$+ 10 \int_{\mathbb{R}} B_{\mu} B_{\mu,xx} z_{xx} + \int_{\mathbb{R}} (5B_{\mu,xx}^2 + 10B_{\mu} B_{\mu,xx} + \frac{15}{2} B_{\mu}^4 - 6(\beta^2 - \alpha^2) B_{\mu}^2) z^2$$

$$+ 10\mu \int_{\mathbb{R}} B_{\mu,xx} z_{xx} + 3\mu \int_{\mathbb{R}} (10B_{\mu,xx}^3 - 4(\beta^2 - \alpha^2) B_{\mu} + \frac{10}{3} B_{\mu,xx} + 10\mu B_{\mu}^2) z^2.$$
Note that, from Theorem 3.5, one has $J_\mu[B_\mu] \equiv 0$. Finally, the term $N_\mu[z]$ is given by

$$N_\mu[z] := -\frac{5}{2} \int_\mathbb{R} \left( z^2 z_x^2 + 2B_\mu,z z_x^2 z + 2B_\mu z z_x z_x \right) + \int_\mathbb{R} 5B_\mu^3 z^3 + \frac{15}{4} B_\mu^2 z^4$$

$$+ \frac{3}{2} \int_\mathbb{R} B_\mu z^5 + \frac{1}{4} \int_\mathbb{R} z^6 - 5\mu \int_\mathbb{R} z z_x^2 + 15\mu \int_\mathbb{R} 2B_\mu z^3 + \frac{15}{2} \mu \int_\mathbb{R} B_\mu z^4$$

$$+ \frac{3\mu}{2} \int_\mathbb{R} z^5 + 10\mu^2 \int_\mathbb{R} B_\mu z_x^3 + \frac{5}{2} \mu^2 \int_\mathbb{R} z^4 - 2(\beta^2 - \alpha^2) \int_\mathbb{R} B_\mu z^3$$

$$- \frac{1}{2} (\beta^2 - \alpha^2) \int_\mathbb{R} z^4 - 2\mu(\beta^2 - \alpha^2) \int_\mathbb{R} z^3.$$

Therefore, from direct estimates one has $N_\mu[z] = O(\|z\|^3_{H^2(\mathbb{R})})$, as desired. \qed

5. Spectral properties around Gardner breathers

Let $z \in H^4(\mathbb{R})$, and $B_\mu$ be any Gardner breather, with shift parameters $x_1, x_2$. We define $L_\mu$ as the linearized operator associated to $B_\mu$, i.e. the bilinear operator obtained after a linearization of the Lyapunov functional (4.6) at the Gardner breather $B_\mu$, as follows:

$$L_\mu[z] := z(4x) - 2(\beta^2 - \alpha^2)z_{xx} + (\alpha^2 + \beta^2)^2 z$$

$$+ 5B_\mu^2 z_{xx} + 10B_\mu B_{\mu,x} z_x + (5B_{\mu,x}^2 + 10B_\mu B_{\mu,xx} + \frac{15}{2} B_\mu^4 - 6(\beta^2 - \alpha^2)B_\mu^2) z$$

$$+ 10\mu B_\mu z_{xx} + 10\mu B_{\mu,x} z_x + 3\mu \left[ 10B_\mu^3 - 4(\beta^2 - \alpha^2)B_\mu + \frac{10}{3} B_{\mu,xx} + 10B_\mu^2 \right] z.$$

The following concept, associated to a bilinear operator like $Q_\mu$ (4.10), is standard and it will be useful for us.

**Definition 5.1.** Any nonzero $z \in H^2$ is said to be a positive (null, negative) direction for $Q_\mu$ if we have $Q_\mu[z] > 0$ ($= 0$, $< 0$).

Essential for the proof of the main result of this work, Theorem 1.5, is the spectral study of the associated linear operator $L_\mu$ appearing from Theorem 3.5 and particularly, equation (3.9). Hence, in this section we describe the spectrum of this operator. More precisely, our main purpose is to find a suitable coercivity property, independently of the nature of scaling parameters. The main result of this section is contained in Proposition 5.11. Part of the analysis carried out in this section has been previously introduced for solitons by Lax [32], Maddocks-Sachs [33] and for mKdV breathers by Alejo-Muñoz [5], so we follow their arguments adapted to the Gardner breather case, sketching several proofs.

**Lemma 5.2.** $L_\mu$ is a linear, unbounded operator in $L^2(\mathbb{R})$, with dense domain $H^4(\mathbb{R})$. Moreover, $L_\mu$ is self-adjoint.

From standard spectral theory of unbounded operators with rapidly decaying coefficients, it is enough to prove that $L_\mu^* = L_\mu$ in $H^4(\mathbb{R})$. 
Proof. Let $z, w \in H^4(\mathbb{R})$. Integrating by parts, one has
\[
\int_{\mathbb{R}} w \mathcal{L}_\mu[z] = \int_{\mathbb{R}} w \left[ z(4x) - 2(\beta^2 - \alpha^2)z_{xx} + (\alpha^2 + \beta^2)^2 z + 5B_\mu^2 z_{xx} + 10B_\mu B_{\mu,xx}z \right]
\]
\[
+ \int_{\mathbb{R}} \left[ 5B_\mu^2 + 10B_\mu B_{\mu,xx} + \frac{15}{2} B_\mu^4 - 6(\beta^2 - \alpha^2)B_\mu^2 \right] z w
\]
\[
+ \int_{\mathbb{R}} \left[ 10\mu B_\mu z_{xx} + 10\mu B_{\mu,xx}z + 3\mu(10B_\mu^3 - 4(\beta^2 - \alpha^2)B_\mu + \frac{10}{3} B_{\mu,xx} + 10\mu B_\mu^2)z \right] w
\]
\[
= \int_{\mathbb{R}} z \left[ w(4x) - 2(\beta^2 - \alpha^2)w_{xx} + (\alpha^2 + \beta^2)^2 w + 5B_\mu^2 w_{xx} + 10B_\mu B_{\mu,xx}w \right]
\]
\[
+ \int_{\mathbb{R}} z \left[ 5B_\mu^2 + 10B_\mu B_{\mu,xx}w + \frac{15}{2} B_\mu^4 w - 6(\beta^2 - \alpha^2)B_\mu^2 w \right]
\]
\[
+ \int_{\mathbb{R}} z \left[ 10\mu B_\mu w_{xx} + 10\mu B_{\mu,xx}w + 3\mu(10B_\mu^3 - 4(\beta^2 - \alpha^2)B_\mu + \frac{10}{3} B_{\mu,xx} + 10\mu B_\mu^2)w \right]
\]
\[
= \int_{\mathbb{R}} z \mathcal{L}_\mu[w].
\]
Finally, it is clear that $D(\mathcal{L}_\mu^*)$ can be identified with $D(\mathcal{L}_\mu) = H^4(\mathbb{R})$. □

A consequence of the previous result is the fact that the spectrum of $\mathcal{L}_\mu$ is real-valued. Furthermore, the following Lemma describes the continuous spectrum of $\mathcal{L}_\mu$.

Lemma 5.3. Let $\alpha, \beta, \mu$ as in definition (1.12). The operator $\mathcal{L}_\mu$ is a compact perturbation of the constant coefficients operator
\[
\mathcal{L}_0[z] := z(4x) - 2(\beta^2 - \alpha^2)z_{xx} + (\alpha^2 + \beta^2)^2 z.
\]
In particular, the continuous spectrum of $\mathcal{L}_\mu$ is the closed interval $[(\alpha^2 + \beta^2)^2, +\infty)$ in the case $\beta \geq \alpha$, and $[4\alpha^2 \beta^2, +\infty)$ in the case $\beta < \alpha$. No embedded eigenvalues are contained in this region. The eigenvalue zero is isolated.

Proof. This result is a consequence of the Weyl Theorem on continuous spectrum. Let us note that the nonexistence of embedded eigenvalues is consequence of the rapidly decreasing character of the potentials involved in the definition of $\mathcal{L}_\mu$.

The isolatedness of the zero eigenvalue is a direct consequence of standard elliptic estimates for the eigenvalue problem associated to $\mathcal{L}_\mu$, corresponding uniform convergence on compact subsets of $\mathbb{R}$, and the non degeneracy of the kernel associated to $\mathcal{L}_\mu$.

□

Now, remembering the definition (2.11) of the two directions associated to spatial translations $B_1, B_2$, it is easy to see the following:

Lemma 5.4. For each $t \in \mathbb{R}$, one has
\[
\ker \mathcal{L}_\mu = \text{span}\{B_1(t; x_1, x_2), B_2(t; x_1, x_2)\}.
\]

Proof. From Theorem 3.3 one has that $\partial_{x_1} J_{\mu}[B_\mu] = \partial_{x_2} J_{\mu}[B_\mu] \equiv 0$. Writing down these identities, we obtain
\[
\mathcal{L}_\mu[B_1](t; x_1, x_2) = \mathcal{L}_\mu[B_2](t; x_1, x_2) = 0,
\]
with $\mathcal{L}_\mu$ the linearized operator defined in (5.1) and $B_1, B_2$ defined in (2.11). A direct analysis involving ordinary differential equations shows that the null space of $\mathcal{L}_0$ is spanned by functions of the type
\[
e^{\pm \beta x} \cos(\alpha x), \quad e^{\pm \beta x} \sin(\alpha x), \quad \alpha, \beta > 0,
\]
(note that this set is linearly independent). Among these four functions, there are only two $L^2$-integrable ones in the semi-infinite line $[0, +\infty)$. Therefore, the null space of $\mathcal{L}_\mu|_{H^4(\mathbb{R})}$ is spanned by at most two $L^2$-functions. Finally, comparing with (5.2), we have the desired conclusion. □
We consider now the natural modes associated to the scaling parameters, which are the best candidates to generate negative directions for the related quadratic form defined by \( L_\mu \). Recall the definitions of \( \Lambda_\alpha B_\mu \) and \( \Lambda_\beta B_\mu \) introduced in (2.7). For these two directions, one has the following

**Lemma 5.5.** Let \( B_\mu = B_{\alpha,\beta,\mu} \) be any Gardner breather. Consider the scaling directions \( \Lambda_\alpha B_\mu \) and \( \Lambda_\beta B_\mu \) introduced in (2.7). Then, given \( \alpha, \beta > 0 \) and \( \forall \mu \in (0, \mu_{\text{max}}) \), we have

\[
\int_{\mathbb{R}} \Lambda_\alpha B_\mu \mathcal{L}_\mu[\Lambda_\alpha B_\mu] = 32\alpha^2\beta\left[1 + \frac{2\mu^2\Delta}{\Delta^2 + 8\mu^2\beta^2}\right] > 0, \tag{5.3}
\]

and

\[
\int_{\mathbb{R}} \Lambda_\beta B_\mu \mathcal{L}_\mu[\Lambda_\beta B_\mu] = -16\beta \left[(\alpha^2 - \beta^2) + (\alpha^2 + \beta^2 + 2\mu^2)\left(\frac{\Delta^2 + 2\mu^2\Delta + 4\mu^2\beta^2}{\Delta^2 + 8\mu^2\beta^2}\right)\right] < 0. \tag{5.4}
\]

**Proof.** From (5.9), we get after derivation with respect to \( \alpha \) and \( \beta \),

\[
\mathcal{L}_\mu[\Lambda_\alpha B_\mu] = -4\alpha [B_{\mu,xx} + B_\mu^3 + 3\mu B_\mu^2 + (\alpha^2 + \beta^2)B_\mu],
\]

\[
\mathcal{L}_\mu[\Lambda_\beta B_\mu] = 4\beta [B_{\mu,xx} + B_\mu^3 + 3\mu B_\mu^2 - (\alpha^2 + \beta^2)B_\mu].
\]

We deal with the first identity (5.3). Note that from (2.8), (1.6) and (2.23),

\[
\int_{\mathbb{R}} \Lambda_\alpha B_\mu \mathcal{L}_\mu[\Lambda_\alpha B_\mu] = -4\alpha \int_{\mathbb{R}} [B_{\mu,xx} + B_\mu^3 + 3\mu B_\mu^2 + (\alpha^2 + \beta^2)B_\mu] \Lambda_\alpha B_\mu - 4\alpha \partial_\alpha E_\mu[B_\mu] - 4\alpha(\alpha^2 + \beta^2)\partial_\alpha M[B_\mu] = 32\alpha^2\beta \left[1 + \frac{4\mu^2}{\Delta^2 + 8\mu^2\beta^2}\right] \geq 0, \tag{5.3}
\]

Following a similar analysis, we have

\[
\int_{\mathbb{R}} \Lambda_\beta B_\mu \mathcal{L}_\mu[\Lambda_\beta B_\mu] = 4\beta \int_{\mathbb{R}} [B_{\mu,xx} + B_\mu^3 + 3\mu B_\mu^2 + (\alpha^2 + \beta^2)B_\mu] \Lambda_\beta B_\mu - 4\beta \partial_\beta E_\mu[B_\mu] - 4\beta(\alpha^2 + \beta^2)\partial_\beta M[B_\mu] = 0, \tag{5.4}
\]

A direct consequence of the previous identities and Corollary 2.4 is the following:

**Corollary 5.6.** With the notation of Lemma 5.5, let

\[
B_{0,\mu} := \frac{\alpha \Lambda_\alpha B_\mu + \beta \Lambda_\beta B_\mu}{8\alpha\beta(\alpha^2 + \beta^2)}, \tag{5.5}
\]

Then \( B_{0,\mu} \) is Schwartz, satisfies \( \mathcal{L}_\mu[B_{0,\mu}] = -B_\mu \) and \( \forall \mu \in (0, \mu_{\text{max}}) \)

\[
\int_{\mathbb{R}} B_{0,\mu} B_\mu = \frac{1}{2\beta(\alpha^2 + \beta^2)} \left(\frac{\Delta^2 + 2\mu^2\Delta}{\Delta^2 + 8\mu^2\beta^2}\right) > 0. \tag{5.6}
\]

Moreover,

\[
\frac{1}{2} \int_{\mathbb{R}} B_{0,\mu} \mathcal{L}_\mu[B_{0,\mu}] < 0. \tag{5.7}
\]
Remark 5.1. In other words, from Definition 5.1, we can see $B_{0, \mu}$ as a negative direction of $Q_\mu$ for all $\mu \in (0, \mu_{\text{max}})$. Besides that, $B_{0, \mu}$ is not orthogonal to the breather itself. Note additionally that constants involved in (5.6) are independent of time.

Proof. Using (5.5), we are lead to understanding the sign of the function

$$\int_\mathbb{R} B_{0, \mu} B_{\mu} = \frac{1}{8\alpha \beta (\alpha^2 + \beta^2)} \int_\mathbb{R} (\alpha \Lambda B_{\mu} + \beta \Lambda_B B_{\mu})$$

$$= \frac{1}{8\alpha \beta (\alpha^2 + \beta^2)} (\alpha \partial_\beta M[B_{\mu}] + \beta \partial_\alpha M[B_{\mu}])$$

$$= \frac{1}{8\alpha \beta (\alpha^2 + \beta^2)} (4\alpha [1 + 2\mu^2 (\Delta - 2\beta^2) / (\Delta^2 + 8\mu^2 \beta^2)] - 16\mu^2 \alpha \beta^2 / (\Delta^2 + 8\mu^2 \beta^2)) \ (5.8)$$

where $\partial_\beta M[B_{\mu}]$ was computed in (2.9).

Now, in order to prove that $L_\mu$ possesses, for all time, only one negative eigenvalue, we follow the Greenberg and Maddocks-Sachs strategy [16, 33], applied this time to the linear, oscillatory operator $L_\mu$. More specifically, we will use the following

**Lemma 5.7** (Uniqueness criterium, see also [16, 33]). Let $B = B_{\mu}$ be any Gardner breather, and $B_1, B_2$ the corresponding kernel of the operator $L_\mu$. Then $L_\mu$ has

$$\sum_{x \in \mathbb{R}} \dim \ker W[B_1, B_2](t; x)$$

negative eigenvalues, counting multiplicity. Here, $W$ is the Wronskian matrix of the functions $B_1$ and $B_2$,

$$W[B_1, B_2](t; x) := \begin{bmatrix} B_1 & B_2 \\ (B_1)_x & (B_2)_x \end{bmatrix} (t, x). \ (5.9)$$

Proof. This result is essentially contained in [16, Theorem 2.2], where the finite interval case was considered. As shown in several articles (see e.g. [33, 21]), the extension to the real line is direct and does not require additional efforts. We skip the details. \hfill \Box

In what follows, we compute the Wronskian (5.9). The surprising fact is the following greatly simplified new expression for the determinant of (5.8) and which generalizes the Wronskian for the mKdV’s breather case ($\mu = 0$) (see [5, Lemma 4.7]):

**Lemma 5.8.** Let $B_{\mu} = B_{\alpha, \beta, \mu}$ be any Gardner breather, $B_1, B_2$ the corresponding kernel elements defined in (2.11) and $D_{\mu} = F_{\mu}^2 + G_{\mu}^2$. Then

$$\det W[B_1, B_2](t; x) := \frac{4\beta^3 (\alpha^2 + \beta^2)^2 ((\alpha^2 + \beta^2)^2 - 4\mu^2 (\alpha^2 - \mu^2))}{\Delta^3 D_{\mu}^2} \left[ \sinh(2\beta y) + \frac{4\beta^2 \mu^2 \cosh(2\beta y)}{(\alpha^2 + \beta^2)^2 - 4\mu^2 (\alpha^2 - \mu^2)} - \frac{\beta \Delta ((\alpha^2 + \beta^2)^2 - 2\mu^2 (\alpha^2 - \beta^2)) \sin(2\alpha y)}{\alpha (\alpha^2 + \beta^2) ((\alpha^2 + \beta^2)^2 - 4\mu^2 (\alpha^2 - \mu^2))} \right] \ (5.10)$$

Proof. We start with a very useful simplification. We claim that

$$\det W[B_1, B_2](x) = -2(\alpha^2 + \beta^2) \left[-(\mu + B_{\mu}) \tilde{B}_{i1} + \partial_{x_1} \partial_x \left(G_{\mu}^2 + F_{\mu}^2\right)\right], \ (5.11)$$

with $\tilde{B}_{\mu} = \tilde{B}_{\mu}(t, x; x_1, x_2)$ defined in (2.11), and $\tilde{B}_j, \tilde{B}_{ij}, i, j = 1, 2$, as in (2.13). In order to prove the above simplification, we start from (2.12), and taking derivative with respect to $x_1$ and $x_2$, we get

$$(B_1)_{xx} + (\tilde{B}_1)_{t} + 3B_{\mu}^2 B_1 + 6\mu B_{\mu} B_1 = 0, \quad (B_2)_{xx} + (\tilde{B}_2)_{t} + 3B_{\mu}^2 B_2 + 6\mu B_{\mu} B_2 = 0. \ (5.12)$$
Multiplying the first equation above by \( B_2 \) and the second by \(-B_1\), and adding both equations, we obtain
\[
(B_1)_{x_2} B_2 - (B_2)_{x_2} B_1 + (\tilde{B}_1)_{t} B_2 - (\tilde{B}_2)_{t} B_1 = 0,
\]
that is,
\[
((B_1)_x B_2 - (B_2)_x B_1)_x = (\tilde{B}_2)_t B_1 - (\tilde{B}_1)_t B_2.
\] (5.13)

On the other hand, since we are working with smooth functions, one has \( B_{\mu} = \tilde{B}_1 + \tilde{B}_2\),
\[ B_1 = \tilde{B}_{11} + \tilde{B}_{12}, \quad B_2 = \tilde{B}_{12} + \tilde{B}_{22}, \]
and finally
\[
(\tilde{B}_1)_t = \delta \tilde{B}_{11} + \gamma \tilde{B}_{12}, \quad (\tilde{B}_2)_t = \delta \tilde{B}_{12} + \gamma \tilde{B}_{22}.
\]

Substituting into (5.13), we get
\[
((B_1)_x B_2 - (B_2)_x B_1)_x = (\delta - \gamma)(\tilde{B}_{12} - \tilde{B}_{11})\tilde{B}_{22}.
\]

Now, integrating in \( x \) and using the nonlinear identity (2.14) we get
\[
\det W[B_1, B_2](x) = -2(\alpha^2 + \beta^2)\left[-(\mu + B_{\mu})\tilde{B}_{11} + \partial_{x_1}^2 \partial_x \log \left(G_{\mu}^2 + F_{\mu}^2\right)\right].
\]

Finally to prove (5.10), we write explicitly the two terms involved at the r.h.s. of equation above. We will follow notation of Appendix B, but this time changing \( \partial_t \) by \( \partial_{x_1} \), and therefore defining \( G := G_{\mu} \), \( G_1 := G_x \), \( G_2 := G_{x_1} \), \( G_3 := G_{x_2} \), \( G_4 := G_{xx_1} \), \( G_5 := G_{xx_2} \) and \( F := F_{\mu} \), \( F_1 := F_x \), \( F_2 := F_{x_1} \), \( F_3 := F_{x_2} \), \( F_4 := F_{xx_1} \), \( F_5 := F_{xx_2} \). Hence we get,
\[
-(\mu + B_{\mu}) \tilde{B}_{11} = -\frac{\mu D_{\mu} - 2\sqrt{2}(F_{1}G + FG_{1})}{D_{\mu}},
\]
and finally
\[
-(\mu + B_{\mu}) \tilde{B}_{11} = -2\sqrt{2} \frac{M_1}{D_{\mu}}, \quad(5.14)
\]
with
\[
M_1 := \left[\mu D_{\mu} - 2\sqrt{2}(F_{1}G + FG_{1})\right]\left[-G^2(F_{1}G - 2F_{2}G_{2}) - F^2(F_{4}G + 2F_{2}G_{2}) + F^2G_{4} + FG(2F_{2}^2 - 2G_{2}^2 + GG_{4})\right].
\] (5.15)

Similarly, for the second term in (5.11) at the r.h.s. we have
\[
\partial_{x_2}^2 \partial_x \log \left(G_{\mu}^2 + F_{\mu}^2\right) = \frac{M_2}{D_{\mu}},
\] (5.16)
with
\[
M_2 := 16(F_{1}G + GG_{1})(FF_{2} + GG_{2})^2 - 4D_{\mu}(F_{1}G + GG_{1})(F_{2}^2 + FF_{4} + G_{2}^2 + GG_{4})
- 8D_{\mu}(FF_{2} + GG_{2})(F_{1}F_{2} + FF_{3} + G_{1}G_{2} + GG_{3})
+ 2D_{\mu}(2F_{2}F_{3} + F_{1}F_{4} + FF_{5} + 2G_{2}G_{3} + G_{1}G_{4} + GG_{5}).
\] (5.17)

Therefore putting together (5.14) and (5.16), we get
\[
-(\mu + B_{\mu}) \tilde{B}_{11} + \partial_{x_1}^2 \partial_x \log \left(G_{\mu}^2 + F_{\mu}^2\right) = \frac{-2\sqrt{2}M_1 + M_2}{D_{\mu}}.
\] (5.18)

Indeed, it is possible to see that the above numerator reduces to
\[
-2\sqrt{2}M_1 + M_2 = 2D_{\mu}M_3,
\] (5.19)
where \( M_3 \) is given by
\[M_3 := F^2[-2F_3F_3 - F_1F_4 + 2G_2G_3 - 3G_1G_4 + GG_5 + \sqrt{2\mu}GF_4 + 2\sqrt{2\mu}F_2G_2] + F^3(F_5 - \sqrt{2\mu}G_4) + F[F_3G^2 + 2F_2G_3G_1 + 4F_6G_2 + 4F_1G_1G_2 - 4F_2G_3 - \sqrt{2\mu}(2F_3G - 2GG_2 + G^2G_4) + 2F_1(F_3 - G_3 + G_4)] + G[-3F_1G - 2E_3G_1 + 2G_1G_2 - 2GG_3 - GG_1G_4 + G^2G_5 + \sqrt{2\mu}F_3G^2 + 2F_2(F_3G + 2F_1G_2 - \sqrt{2\mu}GG_2)].\]

We verify, using the symbolic software Mathematica, that after substituting \(F's\) and \(G's\) terms explicitly in \(M_3\) and lengthy rearrangements, \((5.20)\) simplifies as follows:

\[
2M_3 = -2\beta^3(\alpha^2 + \beta^2)((\alpha^2 + \beta^2)^2 - 4\mu^2(\alpha^2 - \mu^2)) \frac{\Delta^4_\mu}{\Delta_\mu} \left[ \sinh(2\beta y_2) + \frac{4\beta^2\mu^2 \cosh(2\beta y_2)}{(\alpha^2 + \beta^2)^2 - 4\mu^2(\alpha^2 - \mu^2)} - \frac{\beta \Delta_\mu((\alpha^2 + \beta^2)^2 - 2\mu^2(\alpha^2 - \beta^2)) \sin(2\alpha y_1)}{\alpha(\alpha^2 + \beta^2)((\alpha^2 + \beta^2)^2 - 4\mu^2(\alpha^2 - \mu^2))} \right] \frac{2\beta^2\mu^2 \Delta_\mu \cos(2\alpha y_1)}{(\alpha^2 + \beta^2)((\alpha^2 + \beta^2)^2 - 4\mu^2(\alpha^2 - \mu^2))}.
\]

Finally we get

\[-(\mu + B_\mu)\dot{B}_{11} + \partial_{x_1} \partial_x \log \left( G_\mu^2 + F_\mu^2 \right) = \frac{-2\sqrt{2M_1} + M_2}{D_\mu^3} = \frac{2D_\mu M_3}{D_\mu^3} = \frac{-2(\alpha^2 + \beta^2)}{(5.10)}.
\]

Proposition 5.9. The operator \(L_\mu\) defined in \((5.1)\) and for every \(\mu \in (0, \mu_{\text{max}})\) has a unique negative eigenvalue \(-\lambda_0^0 < 0\), of multiplicity one, and \(\lambda_0^0 = \lambda_0(\alpha, \beta, \mu, x_1, x_2, t)\).

Proof. We compute the determinant \((5.9)\) required by Lemma \(5.7\). From Lemma \(5.8\) after a standard translation argument, we will denote \(\tilde{y}_2 = y_2 + (\delta - \gamma)t + \tilde{x}_2\), and we just need to consider the behavior of the function

\[
f_\mu(y_2) = f_{i, \alpha, \beta, \mu, \tilde{x}_2}(y_2) := \sinh(2\beta y_2) + \frac{4\beta^2\mu^2 \cosh(2\beta y_2)}{(\alpha^2 + \beta^2)^2 - 4\mu^2(\alpha^2 - \mu^2)} - \frac{\beta \Delta_\mu((\alpha^2 + \beta^2)^2 - 2\mu^2(\alpha^2 - \beta^2)) \sin(2\alpha y_1)}{\alpha(\alpha^2 + \beta^2)((\alpha^2 + \beta^2)^2 - 4\mu^2(\alpha^2 - \mu^2))} \frac{4\beta^2\mu^2 \Delta_\mu \cos(2\alpha y_1)}{(\alpha^2 + \beta^2)((\alpha^2 + \beta^2)^2 - 4\mu^2(\alpha^2 - \mu^2))}.
\]

for \(\tilde{x}_2 := x_1 - x_2 \in \mathbb{R}\), and \(\delta - \gamma = -2(\alpha^2 + \beta^2)\).

A simple argument shows that for \(y_2 \in \mathbb{R}\) such that

\[
|\sinh(2\beta y_2)| > \frac{\beta \Delta((\alpha^2 + \beta^2)^2 - 2\mu^2(\alpha^2 - \beta^2)) + 4\beta^2\mu^2 \Delta}{\alpha(\alpha^2 + \beta^2)((\alpha^2 + \beta^2)^2 - 4\mu^2(\alpha^2 - \mu^2))}.
\]

\(f_\mu\) has no root. Moreover, there exists \(R_0 = R_0(\alpha, \beta, \mu) > 0\) such that, for all \(y_2 > R_0\) one has \(f_\mu(y_2) > 0\) and for all \(y_2 < -R_0\), \(f_\mu(y_2) < 0\). Therefore, since \(f_\mu\) is continuous, there is a root \(y_0 = y_0(\alpha, \beta, \mu, \tilde{x}_2) \in [-R_0, R_0]\) for \(f_\mu\). Additionally, we have that

\[
f'_\mu(y_2) = 2\beta \cosh(2\beta y_2) + \frac{4\beta^2\mu^2 \sinh(2\beta y_2)}{(\alpha^2 + \beta^2)^2 - 4\mu^2(\alpha^2 - \mu^2)} - \frac{\Delta((\alpha^2 + \beta^2)^2 - 2\mu^2(\alpha^2 - \beta^2)) \cos(2\alpha y_1)}{(\alpha^2 + \beta^2)((\alpha^2 + \beta^2)^2 - 4\mu^2(\alpha^2 - \mu^2))} - \frac{4\beta^2\mu^2 \Delta \sin(2\alpha y_1)}{(\alpha^2 + \beta^2)((\alpha^2 + \beta^2)^2 - 4\mu^2(\alpha^2 - \mu^2))},
\]

hence, two cases have to be considered. We remember here that \(\mu \in (0, \mu_{\text{max}})\). In the first case, when \(\mu \to 0^+ \equiv \epsilon\), we have that \(f'_\mu(y_2) \approx \cosh(2\beta y_2) - \cos(2\alpha y_2 - 2(\alpha^2 + \beta^2)t + \tilde{x}_2) + O(\epsilon) > 0\) if \(y_2 \neq 0\). On the other side, when \(\mu^2 \to \mu_{\text{max}}^- \equiv \mu_{\text{max}}^- \frac{\epsilon}{2}\), with \(\epsilon \ll 1\), we have that
f_\mu(y_2) = 2\beta(\cosh(2\beta y_2) + \sinh(2\beta y_2)) - \frac{\epsilon^2 \sinh(2\beta y_2)}{2\beta^2(\alpha^2 + \beta^2 - \epsilon + \epsilon^2)} - \frac{\epsilon(2\beta^4 - \epsilon\beta^2 + \alpha^2(2\beta^2 + \epsilon)) \cos(2\alpha y_2)}{(\alpha^2 + \beta^2)(2\alpha^2\beta^2 + 2\beta^4 - 2\epsilon\beta^2 + \epsilon^2)} - \frac{2\alpha\beta(\alpha^2 + \beta^2 - \epsilon) \sin(2\alpha y_2)}{(\alpha^2 + \beta^2)(2\alpha^2\beta^2 + 2\beta^4 - 2\epsilon\beta^2 + \epsilon^2)} > 0,

by simple inspection. Therefore, if \( y_0 \neq 0 \) then it is unique and then

\[ \sum_{x \in \mathbb{R}} \dim \ker W[B_1, B_2](t; x) = \dim \ker W[B_1, B_2](t; y_0 - \gamma t - x_2) = 1, \]

since \( B_1 \) or \( (B_1)_x \) are not zero at that time. Indeed, it is enough to show that \( W[B_1, B_2](t, x) \) is not identically zero, then \( \dim \ker W[B_1, B_2] < 2 \). In order to prove this fact, note that from (1.12) \( B_1 \) solves, for \( t, x_1, x_2 \in \mathbb{R} \) fixed, a second order linear ODE with source term \(-\tilde{B}_\mu(t)\). Therefore, by standard well-posedness results, both \( B_1 \) and \( (B_1)_x \) cannot be identically zero at the same point.

We consider now some standard remarks. We can reduce the spectral problem to another independent of time. Indeed, from (1.15) and after translation and redefinition of the shift parameters \( x_1 \) and \( x_2 \) we can assume that

\[ B_\mu = B_{\alpha, \beta, \mu}(0, x; x_1, 0), \quad x_1 \in [0, \frac{2\pi}{\alpha}). \]

In what follows we assume that \( B_\mu \) is given by the previous formula.

Remark 5.2. Let \( z \in H^2(\mathbb{R}) \), and \( B_\mu = B_{\alpha, \beta, \mu} \) be any Gardner breather. Now remembering the quadratic form (4.10) associated to \( L_\mu \), \( Q_\mu[z] = \int_\mathbb{R} z L_\mu[z] \) and from Lemma 5.4 it is easy to see that \( Q_\mu[B_1] = Q_\mu[B_2] = 0 \). Moreover, inequality (5.3) means that \( \Lambda_\mu B_\mu \) is a positive direction for \( Q_\mu \) when \( \mu \in (0, \mu_{\max}) \). Additionally, from (4.10) \( Q_\mu \) is bounded below, namely

\[ Q_\mu[z] \geq -c_{\alpha, \beta, \mu} \|z\|^2_{H^2(\mathbb{R})}, \]

Let \( B_{-1} \in S \setminus \{0\} \) be an eigenfunction associated to the unique negative eigenvalue of the operator \( L_\mu \), as stated in Proposition 5.9. We assume that \( B_{-1} \) has unit \( L^2 \)-norm, so \( B_{-1} \) is now unique. In particular, one has \( L_\mu[B_{-1}] = -\lambda_0^2 B_{-1} \). It is clear from Proposition 5.9 and Lemma 5.3 that the following result holds.

Lemma 5.10. There exists a continuous function \( v_0 = v_0(\alpha, \beta, \mu) \), well-defined and positive for all \( \alpha, \beta > 0 \), with \( \mu \in (0, \mu_{\max}) \), and such that, for all \( z_0 \in H^2(\mathbb{R}) \) satisfying

\[ \int_\mathbb{R} z_0 B_{-1} = \int_\mathbb{R} z_0 B_1 = \int_\mathbb{R} z_0 B_2 = 0, \quad (5.24) \]

then

\[ Q_\mu[z_0] \geq v_0 \|z_0\|^2_{H^2(\mathbb{R})}. \quad (5.25) \]

Proof. The existence of a positive constant \( v_0 = v_0(\alpha, \beta, \mu, x_1) \) such that (5.25) is satisfied is now clear from Remark (5.2) and the three orthogonality conditions. Moreover, thanks to the periodic character of the variable \( x_1 \), and the nondegeneracy of the kernel, we obtain a uniform, positive bound independent of \( x_1, x_2 \) and \( t \), still denoted \( v_0 \). The proof is complete.

Since \( B_{-1} \) is difficult to work with, we look for an easier version of the previous result. We can easily prove, as in [5, Proposition (4.11)], that the eigenfunction \( B_{-1} \) associated to the negative eigenvalue of \( L_\mu \) can be replaced by the breather itself, which has better behavior in terms of error controlling, unlike the first eigenfunction. This simple fact allows us to prove the nonlinear stability result as in the standard approach, without using scaling modulations. Recall that using the first eigenfunction as orthogonality condition does not guarantee a suitable control on the scaling modulation parameter, because the control given by this direction might be not good enough to
close the stability estimates. However, the breather can be used as an alternative direction, and all these previous arguments remain valid, exactly as in [3], provided the Weinstein’s sign condition

\[ \int_{\mathbb{R}} B_{0,\mu} B_\mu > 0 \quad \text{(or equivalently } \int_{\mathbb{R}} B_{0,\mu} \mathcal{L}_\mu [B_{0,\mu}] < 0)

(5.26)
do hold. Note that this precisely is what happens in (5.7) when \( \mu \in (0, \mu_{\text{max}}) \).

**Proposition 5.11.** Let \( B_\mu = B_{\alpha,\beta}\mu \) be any Gardner breather, and \( B_1, B_2 \) the corresponding kernel of the associated operator \( \mathcal{L}_\mu \). Let \( \alpha, \beta > 0 \) and \( \mu \in (0, \mu_{\text{max}}) \). There exists \( \sigma_0 > 0 \), depending on \( \alpha, \beta, \mu \) only, such that, for any \( z \in H^2(\mathbb{R}) \) satisfying

\[ \int_{\mathbb{R}} B_1 z = \int_{\mathbb{R}} B_2 z = 0, \]

(5.27)
one has

\[ Q_\mu[z] \geq \sigma_0 \|z\|_{H^2(\mathbb{R})}^2 - \frac{1}{\sigma_0} \left( \int_{\mathbb{R}} z B_\mu \right)^2. \]

(5.28)

**Proof.** We follow a similar strategy as stated in [3], and we include here for the sake of completeness. Indeed, it is enough to prove that, under hypothesis \( \mu \in (0, \mu_{\text{max}}) \), the conditions (5.27) and the additional orthogonality condition \( \int_{\mathbb{R}} z B_\mu = 0 \), one has

\[ Q_\mu[z] \geq \sigma_0 \|z\|_{H^2(\mathbb{R})}^2. \]

In what follows we prove that we can replace \( B_{-1} \) by the breather \( B_\mu \) in Lemma 5.10 and the result essentially does not change. Indeed, note that from (5.5), the function \( B_{0,\mu} \) satisfies \( \mathcal{L}_\mu[B_{0,\mu}] = -B_\mu \), and from (5.6) and hypothesis \( \mu \in (0, \mu_{\text{max}}) \),

\[ \int_{\mathbb{R}} B_{0,\mu} B_\mu = - \int_{\mathbb{R}} B_{0,\mu} \mathcal{L}_\mu[B_{0,\mu}] = -Q_\mu[B_{0,\mu}] > 0. \]

(5.29)
The next step is to decompose \( z \) and \( B_{0,\mu} \) in \( \text{span}(B_{-1}, B_1, B_2) \) and the corresponding orthogonal subspace. Using the same notation that in [3] Prop 4.13, one has

\[ z = \tilde{z} + mB_{-1}, \quad B_{0,\mu} = b_0 + nB_{-1} + p_1 B_1 + p_2 B_2, \]

where

\[ \int_{\mathbb{R}} \tilde{z} B_{-1} = \int_{\mathbb{R}} \tilde{z} B_1 = \int_{\mathbb{R}} \tilde{z} B_2 = \int_{\mathbb{R}} B_{0,\mu} B_{-1} = \int_{\mathbb{R}} B_{0,\mu} B_1 = \int_{\mathbb{R}} B_{0,\mu} B_2 = 0. \]

Note in addition that

\[ \int_{\mathbb{R}} B_{-1} B_1 = \int_{\mathbb{R}} B_{-1} B_2 = 0. \]

From here and the previous identities we have

\[ Q_\mu[z] = \int_{\mathbb{R}} (\mathcal{L}_\mu[\tilde{z}] - m\lambda_0^2 B_{-1})(\tilde{z} + mB_{-1}) = Q_\mu[\tilde{z}] - m^2\lambda_0^2. \]

(5.30)

Now, since \( \mathcal{L}_\mu[B_{0,\mu}] = -B_\mu \), one has

\[ 0 = \int_{\mathbb{R}} z B_\mu = - \int_{\mathbb{R}} z \mathcal{L}_\mu[B_{0,\mu}] = \int_{\mathbb{R}} \mathcal{L}_\mu[\tilde{z} + mB_{-1}] B_{0,\mu} \]

\[ = \int_{\mathbb{R}} (\mathcal{L}_\mu[\tilde{z}] - m\lambda_0^2 B_{-1})(b_0 + nB_{-1} + p_1 B_1 + p_2 B_2) = \int_{\mathbb{R}} \mathcal{L}_\mu[\tilde{z}] b_0 - mn\lambda_0^2. \]

(5.31)

On the other hand, from Corollary 5.6

\[ \int_{\mathbb{R}} B_{0,\mu} B_\mu = - \int_{\mathbb{R}} B_{0,\mu} \mathcal{L}_\mu[B_{0,\mu}] = - \int_{\mathbb{R}} (b_0 + nB_{-1})(\mathcal{L}_\mu[b_0] - n\lambda_0^2 B_{-1}) = -Q_\mu[b_0] + n^2\lambda_0^2. \]

(5.32)

Substituting (5.31) and (5.32) into (5.30), we get

\[ Q_\mu[z] = Q_\mu[\tilde{z}] - \frac{\left( \int_{\mathbb{R}} \mathcal{L}_\mu[\tilde{z}] b_0 \right)^2}{\int_{\mathbb{R}} B_{0,\mu} B_\mu + Q_\mu[b_0]}. \]

(5.33)
Note that from (5.29) and (5.25) both quantities in the denominator are positive. Additionally, note that if \( \tilde{z} = \lambda b_0 \), with \( \lambda \neq 0 \), then

\[
\left( \int_\mathbb{R} \mathcal{L}_\mu[\tilde{z}]b_0 \right)^2 = Q_\mu[\tilde{z}]Q_\mu[b_0].
\]

In particular, if \( \tilde{z} = \lambda b_0 \),

\[
\frac{\left( \int_\mathbb{R} \mathcal{L}_\mu[\tilde{z}]b_0 \right)^2}{\int_\mathbb{R} B_{0,\mu} + Q_\mu[b_0]} \leq a Q_\mu[\tilde{z}], \quad 0 < a < 1.
\]

(5.34)

In the general case, we get the same conclusion as before. Namely, choosing

\[
z = \tilde{z} + mB_{-1} + q_1B_1 + q_2B_2, \quad m, q_1, q_2 \in \mathbb{R},
\]

and using the orthogonal decomposition induced by the scalar product \( (\mathcal{L}_\mu \cdot \cdot)_{L^2} \) on span(\( B_{-1}, B_1, B_2 \)), we get

\[
Q_\mu[\tilde{z}] = \int_\mathbb{R} z\mathcal{L}_\mu[z] = \int_\mathbb{R} (\tilde{z} + mB_{-1} + q_1B_1 + q_2B_2)(\mathcal{L}_\mu[\tilde{z}] - m^2\lambda_0^2B_{-1})
\]

\[
= Q_\mu[\tilde{z}] - m^2\lambda_0^2,
\]

and following the same steps as above, we conclude. Therefore, we have proved (5.34) for all possible \( \tilde{z} \). Finally, substituting into (5.33) and (5.30), \( Q_\mu[z] \geq (1-a)Q_\mu[\tilde{z}] \geq 0 \), and \( Q_\mu[z] \geq m^2\lambda_0^2 \). We have, for some \( C > 0 \),

\[
Q_\mu[z] \geq (1-a)Q_\mu[\tilde{z}] \geq \frac{1}{2}(1-a)Q_\mu[\tilde{z}] + (1-a)m^2\lambda_0^2
\]

\[
\geq \frac{1}{C}(2\|\tilde{z}\|_{H^2(\mathbb{R})}^2 + 2m^2\|B_{-1}\|_{H^2(\mathbb{R})}^2) \geq \frac{1}{C}\|z\|_{H^2(\mathbb{R})}^2.
\]

\[\square\]

6. PROOF OF THE MAIN THEOREM

In this section we prove a detailed version of Theorem 1.5.

**Theorem 6.1** (\( H^2 \)-stability of Gardner breathers). Let \( \alpha, \beta \in \mathbb{R} \setminus \{0\} \) and \( \mu \in (0, \mu_{\text{max}}) \). There exist positive parameters \( \eta_0, A_0 \), depending on \( \alpha, \beta \) and \( \mu \), such that the following holds. Consider \( w_0 \in H^2(\mathbb{R}) \), and assume that there exists \( \eta \in (0, \eta_0) \) such that

\[
\|w_0 - B_\mu(t = 0; 0, 0)\|_{H^2(\mathbb{R})} \leq \eta.
\]

(6.1)

Then there exist \( x_1(t), x_2(t) \in \mathbb{R} \) such that the solution \( w(t) \) of the Cauchy problem for the Gardner equation (1.11), with initial data \( w_0 \), satisfies

\[
\sup_{t \in \mathbb{R}} \|w(t) - B_\mu(t; x_1(t), x_2(t))\|_{H^2(\mathbb{R})} \leq A_0\eta,
\]

(6.2)

with

\[
\sup_{t \in \mathbb{R}} |x_1'(t)| + |x_2'(t)| \leq KA_0\eta,
\]

(6.3)

for a constant \( K > 0 \).

**Remark 6.1**. The initial condition (6.1) can be replaced by any initial breather profile of the form \( \tilde{B}_\mu := B_{\alpha,\beta,\mu}(t_0; x_1^0, x_2^0) \), with \( t_0, x_1^0, x_2^0 \in \mathbb{R} \), thanks to the invariance of the equation under translations in time and space. In addition, a similar result is available for the negative breather \( -B_{\alpha,\beta,-|\mu|} \) which is also a solution of (1.11).
Proof of Theorem 6.1. The proof of this result is completely similar to the proof of the $H^2$-stability of mKdV breathers [5, Theorem (6.1)], after following the same steps, and once we guarantee coercivity of the bilinear form $Q_\mu$, (see Proposition 5.11 and Lemma 5.10), and the nonvanishing of the denominator $\int_\mathbb{R} B_{0,\mu} B_\mu + Q_\mu[0]$, appearing in (5.33). For the sake of completeness, we include here a sketch with some steps.

1. From the continuity of the Gardner flow for $H^2(\mathbb{R})$ data, there exists a time $T_0 > 0$ and continuous parameters $x_1(t), x_2(t) \in \mathbb{R}$, defined for all $t \in [0, T_0]$, and such that the solution $w(t)$ of the Cauchy problem for the Gardner equation (1.2), with initial data of mKdV breathers [5, Theorem (6.1)], after following the same steps, and once we guarantee co-

2. After that, we apply a well known theory of modulation for the solution $w(t)$.

Lemma 6.2 (Modulation). There exists $\eta_0 > 0$ such that, for all $\eta \in (0, \eta_0)$, the following holds. There exist $C^1$ functions $x_1(t), x_2(t) \in \mathbb{R}$, defined for all $t \in [0, T^*]$, and such that

\[
(6.4) \quad T^* := \sup \left\{ T > 0 \mid \text{for all } t \in [0, T], \text{ there exist } \tilde{x}_1(t), \tilde{x}_2(t) \in \mathbb{R} \text{ such that} \right. \]

\[
\sup_{t \in [0, T]} \left\| w(t) - B_\mu(t; \tilde{x}_1(t), \tilde{x}_2(t)) \right\|_{H^2(\mathbb{R})} \leq K^* \eta.
\]

It is clear from (6.4) that $T^*$ is a well-defined quantity. Our idea is to find a suitable contradiction to the assumption $T^* < +\infty$.

Proof of Lemma 6.2. We have

\[
(6.5) \quad \sup_{t \in [0, T]} \left\| w(t) - B_\mu(t; \tilde{x}_1(t), \tilde{x}_2(t)) \right\|_{H^2(\mathbb{R})} \leq K^* \eta.
\]

Moreover, one has

\[
(6.6) \quad \int_\mathbb{R} B_1(t; x_1(t), x_2(t)) z(t) = \int_\mathbb{R} B_2(t; x_1(t), x_2(t)) z(t) = 0.
\]

3. Now, we apply Lemma 4.4 to the function $w(t)$. Since $z(t)$ defined by (6.6) is small, we get from (4.4) and Corollary 3.10

\[
\mathcal{H}_\mu[w](t) = \mathcal{H}_\mu[B_\mu](t) + \frac{1}{2} Q_\mu[z](t) + N_\mu[z](t).
\]

Note that $|N_\mu[z](t)| \leq K \|z(t)\|_{H^1(\mathbb{R})}$. On the other hand, by the translation invariance in space,

\[
\mathcal{H}_\mu[B_\mu](t) = \mathcal{H}_\mu[B_\mu](0) = \text{constant}.
\]

\[
\mathcal{H}_\mu[B_\mu(t; \tilde{x}_1(t), \tilde{x}_2(t))] = \mathcal{H}_\mu[B_\mu(t - t_0(t), \cdot; \tilde{x}_1(t), \tilde{x}_2(t)) + x_0(t)\mathcal{H}_\mu[B_\mu(t-t_0(t), \cdot; 0, 0)].
\]

Finally, $\mathcal{H}_\mu[B_\mu(t - t_0(t), \cdot; 0, 0)] = \mathcal{H}_\mu[B_\mu(t - t_0(t), \cdot; 0, 0)](t - t_0(t))$. Taking time derivative,

\[
\partial_t \mathcal{H}_\mu[B_\mu(t; x_1(t), x_2(t))] = \mathcal{H}_\mu[B_\mu(t; x_1(t), x_2(t))](t - t_0(t)) \times (1 - t_0'(t)) \equiv 0,
\]
Substituting this last identity into (6.10), we get stability property (6.2) holds true. Finally, (6.3) is a consequence of (6.8).

Additionally, from (5.27)-(5.28) applied this time to the time-dependent function \( K \) by taking \( s \) satisfies (6.7), we get

\[
H \text{ hence we obtain:} \quad 4.1 \text{ and (4.4) we have}
\]

4. Conclusion of the proof. Using the conservation of mass (1.3), we have, after expanding \( w = B_\mu + z \),

\[
\left| \int_{\mathbb{R}} B_\mu(t)z(t) \right| \leq K \left| \int_{\mathbb{R}} B_\mu(0)z(0) \right| + K \| z(0) \|_{H^2(\mathbb{R})}^2 + K \| z(t) \|_{H^2(\mathbb{R})}^2 \\
\leq K(\eta + (K^*)^2 \eta^2), \quad \text{for each } t \in [0, T^*].
\]

Substituting this last identity into (6.10), we get

\[
\| z(t) \|_{H^2(\mathbb{R})}^2 \leq K \eta^2(1 + (K^*)^2 \eta^2) \leq \frac{1}{2}(K^*)^2 \eta^2,
\]

by taking \( K^* \) large enough. This last fact contradicts the definition of \( T^* \) and therefore the stability property (5.2) holds true. Finally, (6.3) is a consequence of (6.8).

\[\square\]

APPENDIX A. PROOF OF LEMMA 2.1

From the Gardner equation (1.1), we select an ansatz for \( w \) as

\[
w(t, x) := \phi_x, \quad \phi(t, x) := \sqrt{2}t \log(\frac{G(t, x)}{F(t, x)}), \quad \text{where } F := F_\mu + iG_\mu, \quad G = F_\mu - iG_\mu = F^*.
\]

First note that here \( F_\mu \) and \( G_\mu \) are not necessarily the same functions introduced in (1.12) but generic ones for this ansatz. Then, substituting the above expression in (1.1), and using Hirota’s bilinear operators, we arrive to the following conditions on \( G \) and \( F \):

\[
D_t(GF) + D_x^2(GF) = 0, \\
D_x^2(GF) - i\sqrt{2} \mu D_x(GF) = 0.
\]

(A.1)

Then, dividing by \( GF \) the second equation in (A.1), and taking into account the following identity

\[
D_x^2(GF) = \partial_x^2 \log(GF) + (\partial_x \log(GF))^2,
\]

we obtain:

\[
\frac{D_x^2(GF)}{GF} - i\sqrt{2} \mu \frac{D_x(GF)}{GF} = \partial_x^2 \log(GF) + (\partial_x \log(GF))^2 - i\sqrt{2} \mu \partial_x \log(GF) \leq 0.
\]

Hence,

\[
w^2 = 2 \partial_x^2 \log(G \cdot F) - 2\mu w.
\]

\[\text{See [38] p.152 for further reading.}\]
and the proof is complete. Indeed, with the same steps, it can be proved a similar result for the mKdV equation with NVBC. In fact, we arrive to the following relations for any solution of the mKdV with NVBC $\mu$:

$$D_t(GF) + D_{xx}(GF) + 3\mu^2 D_x(GF) = 0, \quad D_t^2(GF) = i\sqrt{2}\mu D_x(GF) = 0. \quad (A.2)$$

And, with the same steps than in the Gardner case, we obtain that any mKdV solution $u$ with NVBC $\mu$ satisfies:

$$u^2 = \mu^2 + 2 \frac{\partial^2}{\partial x^2} \log(G \cdot F).$$

**APPENDIX B. PROOF OF LEMMA 3.3**

Firstly and for the sake of simplicity, we will use the following notation:

$$A_1 := (2(\beta^2 - \alpha^2) + 5\mu^2), \quad A_2 := ((\alpha^2 + \beta^2)^2 + 6\mu^2(\beta^2 - \alpha^2 + \frac{3}{2}\mu^2)), \quad \Delta = \alpha^2 + \beta^2 - 2\mu^2, \quad e^2 = \cosh(z) + \sinh(z), \quad D := f^2 + g^2,$$

where $f, g$ and derivatives are given by:

$$f = \frac{\beta\sqrt{\alpha^2 + \beta^2}}{\alpha\sqrt{\Delta}} \sin(\alpha y_1) - \frac{\sqrt{2}\beta\mu e^{\beta y_2}}{\Delta}, \quad (B.1)$$

$$g = \frac{\beta\sqrt{\alpha^2 + \beta^2}}{\alpha\sqrt{\Delta}} \sin(\alpha y_1) - \frac{\sqrt{2}\beta\mu e^{\beta y_2}}{\Delta}, \quad (B.2)$$

$$g_1 := g_x = \frac{\beta\sqrt{\alpha^2 + \beta^2}}{\sqrt{\Delta}} \cos(\alpha y_1) - \frac{\sqrt{2}\beta\mu e^{\beta y_2}}{\Delta}, \quad (B.3)$$

$$g_2 := g_t = \frac{\beta\delta \sqrt{\alpha^2 + \beta^2}}{\sqrt{\Delta}} \cos(\alpha y_1) - \frac{\sqrt{2}\beta^2 \gamma \mu e^{\beta y_2}}{\Delta}, \quad (B.4)$$

$$g_3 := g_{xt} = -\frac{\alpha\beta\delta \sqrt{\alpha^2 + \beta^2}}{\sqrt{\Delta}} \sin(\alpha y_1) - \frac{\sqrt{2}\beta^3 \gamma \mu e^{\beta y_2}}{\Delta}, \quad (B.5)$$

$$g_4 := g_{xx} = -\frac{\alpha\beta\sqrt{\alpha^2 + \beta^2}}{\sqrt{\Delta}} \sin(\alpha y_1) - \frac{\sqrt{2}\beta^3 \mu e^{\beta y_2}}{\Delta}, \quad (B.6)$$

$$g_5 := g_{xxt} = -\frac{\alpha^2\beta\delta \sqrt{\alpha^2 + \beta^2}}{\sqrt{\Delta}} \cos(\alpha y_1) - \frac{\sqrt{2}\beta^4 \gamma \mu e^{\beta y_2}}{\Delta}, \quad (B.7)$$

$$f = \cosh(\beta y_2) - \frac{\sqrt{2}\beta \mu}{\alpha \sqrt{\Delta}} \cos(\alpha y_1 + \arctan(\beta/\alpha)), \quad (B.8)$$

$$f_1 := f_x = \beta \sinh(\beta y_2) + \frac{\sqrt{2}\beta \mu}{\sqrt{\Delta}} \sin(\alpha y_1 + \arctan(\beta/\alpha)), \quad (B.9)$$
\[ f_2 := f_t = \beta \gamma \sinh(\beta y_2) + \frac{\sqrt{2} \beta \delta \mu}{\sqrt{\Delta}} \sin(\alpha y_1 + \arctan(\beta/\alpha)), \] (B.10)

\[ f_3 := f_{xt} = \beta^2 \gamma \cosh(\beta y_2) + \frac{\sqrt{2} \alpha \beta \delta \mu}{\sqrt{\Delta}} \cos(\alpha y_1 + \arctan(\beta/\alpha)), \] (B.11)

\[ f_4 := f_{xx} = \beta^2 \cosh(\beta y_2) + \frac{\sqrt{2} \alpha \beta \delta \mu}{\sqrt{\Delta}} \cos(\alpha y_1 + \arctan(\beta/\alpha)), \] (B.12)

\[ f_5 := f_{xxt} = \beta^3 \gamma \sinh(\beta y_2) - \frac{\sqrt{2} \alpha^2 \beta \delta \mu}{\sqrt{\Delta}} \sin(\alpha y_1 + \arctan(\beta/\alpha)). \] (B.13)

From the explicit expression of the mKdV breather with NVBC (3.1) but now written in terms of the above derivatives (B.2)-(B.13), we obtain that:

\[ B = \mu + 2\sqrt{2} \frac{f g_1 - f_1 g}{D} \quad \text{and} \quad \tilde{B}_t = 2\sqrt{2} \frac{f_2 g - f_2 g}{D}. \] (B.14)

Moreover, from (8.2), we also have an equivalent expression for the quantity \((M_{\alpha, \beta})_t\)

\[ (M_{\alpha, \beta})_t = -\frac{4}{D^2} (g g_1 + f f_1)(g g_2 + f f_2) + \frac{2}{D} (g g_3 + g_1 g_2 + f f_3 + f_1 f_2). \] (B.15)

and therefore from (B.14) and (B.15),

\[ 2(M_{\alpha, \beta})_t B = \frac{N_2}{f^2 D^3}; \] (B.16)

where

\[ N_2 := -f^2 \left( 16\sqrt{2}(g_1 f - g f_1)(f f_1 + g g_1)(f f_2 + g g_2) - 8\sqrt{2} D(g_1 f - g f_1)(g g_3 + g_1 g_2 + f f_3 + f_1 f_2) + 8\mu D(f_1 f + g g_1)(f f_2 + g g_2) - 4 D^2 \mu (g g_3 + g_1 g_2 + f f_3 + f_1 f_2) \right). \] (B.17)

Now, we compute \(B_{xt}\). First we get

\[ B_x = -\frac{4\sqrt{2}}{f D^2} \left( (g f g_1 - g^2 f_1)(f g_1 - f f_1) \right) + \frac{2\sqrt{2}}{f D} \left( f^2 g_4 - 2 g_1 f f_1 + 2 g f^2_1 - g f f_1 \right), \] (B.18)

and then

\[ B_{xt} = 4\sqrt{2} \frac{N_1}{f^2 D^3}; \] (B.19)

where

\[ N_1 := \left( 4 g^2 (g_1 f - g f_1)^2 (g_2 f - g f_2) - D(g_1 f - g f_1)(g_3 f^2 - 2 g_2 f_1 - g_1 f f_2 + 2 g_1 f f_2 - g f f_3) - D(g_1 f - g f_1)(g_1 g_2 f^2 + g g_3 f^2 - 2 g g_2 f_1 - 2 g_1 f f_2 + 3 g^2 f_1 f_2 - g^2 f_3) - D(g_2 f - g f_2)(g_4 f^2 - 2 g_1 f f_1 + 2 g f^1_1 - g f f_3) + D^2 \left( \frac{1}{2}(g_3 f^3 - 4 g f^2 f_2 - g_2 f^2 f_2 - g f^2 f_5 - 2 g_3 f^2 f_1) + g_2 f f^1_1 + 2 g_1 f f_1 f_2 - 3 g f^1_1 f_2 - g f^1_2 f_3 + 2 g f f_1 f_3 + g f f_2 f_4 \right) \right). \] (B.20)

Now, we verify by using the symbolic software Mathematica that, after expanding \(f’s\) and \(g’s\) terms (B.2)-(B.13) and lengthy rearrangements, the sum \(N_1 + N_2\) simplifies as follows:
\[ N_1 + N_2 = f^2 D^2 \cdot A_1 2\sqrt{2}(g_2 f - g f_2) + f^2 D^2 \cdot A_2 2\sqrt{2}(g_1 f - g f_1). \]  
(B.21)

Finally, remembering (B.14), we have that:

\[ B_{xt} + 2(M_{\alpha,\beta})_t B = \frac{N_1 + N_2}{f^2 D^3} \]
\[ = \frac{f^2 D^2 \cdot A_1 2\sqrt{2}(g_2 f - g f_2)}{f^2 D^3} + \frac{f^2 D^2 \cdot A_2 2\sqrt{2}(g_1 f - g f_1)}{f^2 D^3} \]
\[ = A_1 2\sqrt{2} \frac{(g_2 f - g f_2)}{D} + A_2 2\sqrt{2} \frac{(g_1 f - g f_1)}{D} = A_1 \tilde{B}_t + A_2 (B - \mu). \]  
(B.23)

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