A DIFFERENT PERSPECTIVE ON H-LIKE LIE ALGEBRAS

CATHY KRILOFF AND TRACY PAYNE

Abstract. We characterize H-like Lie algebras in terms of subspaces of cones over conjugacy classes in $\mathfrak{so}(\mathbb{R}^q)$, translating the classification problem for H-like Lie algebras to an equivalent problem in linear algebra. We study properties of H-like Lie algebras, present new methods for constructing them, including tensor products and central sums, and classify H-like Lie algebras whose associated $J_Z$-maps have rank two for all nonzero $Z$.

1. Introduction

A nilmanifold is a simply connected nilpotent Lie group endowed with a left-invariant Riemannian metric. Kaplan defined the class of nilmanifolds of Heisenberg type (also called H-type) [Kap80]. Nilmanifolds in this class have many extraordinary symmetry properties ([Ebe94b, BTV95]). Most notably, they admit solvable extensions, Damek-Ricci spaces, which are harmonic spaces but not necessarily symmetric spaces ([DR92] [BTV95]).

It is natural to consider defining conditions for nilmanifolds weaker than those in the definition of H-type, with the hope that some nice properties of H-type nilmanifolds may be preserved in the larger class. Many generalizations of the H-type property have been studied. The nonsingular nilpotent Lie algebras were first studied by Métivier and later studied by others from both geometric and analytic perspectives under various names ([Mét80] [Ebe94a, MS04, KT13, LO14]). For pseudo H-type or generalized H-type nilmanifolds, the inner product is assumed to be nondegenerate instead of positive definite, and even possibly sub-semi-Riemannian ([Cia00, GMKM13]). Lauret generalized the notion of H-type to modified H-type in [Lau99], and this was further generalized in [JLP08].

In analyzing properties of the length spectrum of two-step nilmanifolds, Gornet and Mast defined the notion of an H-like nilmanifold ([GM00]) as a generalization of an H-type nilmanifold. Some of the remarkable geometric properties possessed by H-type nilmanifolds have natural analogs held by H-like nilmanifolds ([GM00] [DDM11]). In this work we study the metric Lie algebras associated to H-like nilmanifolds and examine their properties. (We will give a precise definition of $H$-like in Definition 1.1.)
Quite a few low-dimensional examples of H-like nilmanifolds, including continuous families of two-step nilmanifolds with center of dimension greater than one, have been found ([GM00, DDM11, Sch15]). There is a one-to-one correspondence between nilmanifolds and metric nilpotent Lie algebras. Two nilmanifolds are isometric if and only if the corresponding metric nilpotent Lie algebras are metrically isomorphic ([Wil82]). In [DDM11], a general method for constructing H-like Lie algebras using representations of \(\mathfrak{su}(2)\) was given, and in [DDM] H-like Lie algebras defined by undirected, uncolored graphs were classified.

Fix a two-step metric nilpotent Lie algebra \((n, Q)\). Let \(\mathfrak{z} = [n, n]\) and let \(v = \mathfrak{z}^\perp\). (Note that some authors use \(z\) to mean the center of \(n\) instead of the commutator.) If \(\dim \mathfrak{z} = p\) and \(\dim v = q\), we say that \(n\) is of type \((p, q)\). Define \(J : \mathfrak{z} \to \text{End}(v)\) by

\[
\langle J(Z)X, Y \rangle = \langle Z, [X, Y] \rangle \quad \text{for} \quad X, Y \in v, Z \in \mathfrak{z}.
\]

The map \(J\) is linear and \(J_Z : v \to v\) is skew-symmetric for all \(Z\). Furthermore, \(J(\mathfrak{z})\) is a \(p\)-dimensional subspace of \(\mathfrak{so}(v)\) (see Lemma 3.8).

Conversely, given finite-dimensional inner product spaces \(\mathbb{R}^p\) and \(\mathbb{R}^q\) and a nonzero linear map \(J : \mathbb{R}^p \to \mathfrak{so}(\mathbb{R}^q)\), there is a two-step metric nilpotent Lie algebra \((n, Q)\) defined by \(J\). The underlying vector space is \(n = v \oplus \mathfrak{z} = \mathbb{R}^q \oplus \mathbb{R}^p\) and the inner product \(Q\) on \(n\) is the orthogonal sum of the inner products on \(v\) and \(\mathfrak{z}\). The Lie bracket is defined by \([\cdot, \cdot] = 0\) and the assumptions that \([v, v]\) is contained in \(\mathfrak{z}\) and \(\mathfrak{z}\) is central. Two such nilpotent Lie algebras defined by \(J : \mathbb{R}^p \to \mathfrak{so}(\mathbb{R}^q)\) and \(J' : \mathbb{R}^p \to \mathfrak{so}(\mathbb{R}^q)\), define metrically isomorphic metric Lie algebras if and only if there are \(A \in O(q)\) and \(B \in O(p)\) so that \(J'(BZ) = A \circ J(Z) \circ A^T\) for all \(Z \in \mathfrak{z}\) (see [GW97]).

Because the maps \(J(Z)\) are skew-symmetric, any nonzero eigenvalues of \(J(Z)\) are purely imaginary, and the eigenvalues of \(J(Z)\) completely determine \(J(Z)\) up to conjugacy by an element of \(GL(v)\). We use a multiset \(S\) to record the eigenvalues of \(J(Z)\). (A multiset is a set \(S\) endowed with a multiplicity function from \(S\) to \(\mathbb{N}\).) Conjugacy classes of nonzero skew-symmetric maps in \(\text{End}(v)\) are indexed by multisets \(S\) of eigenvalues that are purely imaginary with the multiplicities of \(bi\) and \(-bi\) the same for all possible real numbers \(b\). We will call such a multiset admissible.

Gornet and Mast originally defined H-like Lie algebras in terms of totally geodesic submanifolds. There are several equivalent characterizations of H-like Lie algebras (see [GM00] and [DDM11]); we take one of these as part (3) in the following.

**Definition 1.1.** Let \((n, Q)\) be a two-step nilpotent Lie algebra endowed with inner product \(Q\). Let \(S\) be an admissible multiset.

1. The metric Lie algebra \((n, Q)\) is \(H\)-type if for all \(Z \in \mathfrak{z}\), \(J_Z^2 = -\|Z\|^2 \text{Id}_v\), where \(\text{Id}_v\) is the identity map on \(v\).
2. The metric Lie algebra \((n, Q)\) has constant J-spectrum \(S\) if for all unit \(Z \in \mathfrak{z}\), the map \(J_Z\) has constant spectrum \(S\).
3. The metric Lie algebra \((n, Q)\) is \(H\)-like if \((n, Q)\) has constant J-spectrum and \(n\) has no nontrivial abelian factors.

When \((n, Q)\) has constant J-spectrum, the rank of \(J_Z\) is constant for nonzero \(Z\); this number is called the \(J\)-rank of \((n, Q)\).
Examples are given in Section 2. It follows from the definitions that

\[ \text{H-type} \Rightarrow \text{H-like} \Rightarrow \text{constant } J\text{-spectrum}. \]

In Gornet and Mast’s definition of H-like they take \( z \) to be the center of \( n \) rather than the commutator, so that any abelian factor of \( n \) lies in \( z \). We take \( z \) to be the commutator so that any abelian factor is contained in \( v \).

A \( p \)-dimensional subspace \( W \) of \( \mathfrak{so}(\mathbb{R}^q) \) defines a two-step metric nilpotent Lie algebra of type \((p, q)\).

**Definition 1.2.** Let \( W \) be a \( p \)-dimensional subspace of \( \mathfrak{so}(\mathbb{R}^q) \). Define the metric nilpotent Lie algebra \((n, Q)\) by taking as the underlying vector space \( n = W \oplus \mathbb{R}^q \), and endowing this vector space with the inner product \( \langle , \rangle^* \) so that \( W \) and \( \mathbb{R}^q \) are orthogonal, the restriction of \( \langle , \rangle^* \) to \( W \) is given by the restriction of the Frobenius inner product to \( W \), and \( \langle , \rangle^* \) is the standard inner product on \( \mathbb{R}^q \); i.e., the one with respect to which the standard basis is orthonormal. The Lie bracket for \( n \) is defined by assuming that \( W \) is central and letting \( J : W \to \mathfrak{so}(\mathbb{R}^q) \) be the inclusion map:

\[
\langle [X, Y], Z \rangle^* = \langle Z(X), Y \rangle_{\mathbb{R}^q}.
\]

Such a metric nilpotent Lie algebra is called the standard two-step metric nilpotent Lie algebra defined by \( W \).

Eberlein showed that every two-step metric nilpotent Lie algebra is metrically isomorphic to a standard two-step metric nilpotent Lie algebra of type \((p, q)\) (Proposition 3.1.2, [Ebe04]). The standard metric nilpotent Lie algebras defined by two \( p \)-dimensional subspaces \( W_1 \) and \( W_2 \) of \( \mathfrak{so}(\mathbb{R}^q) \) are isomorphic if and only if there exists \( g \in GL_q(\mathbb{R}) \) so that \( W_2 = gW_1g^T \) (Proposition 3.1.3, [Ebe04]).

Eberlein used this point of view to study a variety of problems involving two-step nilpotent Lie algebras, including moduli spaces, optimal metrics, metrics with geodesic flow invariant Ricci tensors, and Riemannian submersions and lattices ([Ebe04], [Ebe03a], [Ebe03b]). Gordon and Kerr used the correspondence between standard metric two-step nilpotent Lie algebras and subspaces of \( \mathfrak{so}(\mathbb{R}^q) \) to characterize Carnot Einstein solvmanifolds in terms of subspaces of \( \mathfrak{so}(\mathbb{R}^q) \) that they called “orthogonal uniform subspaces” ([GK01]).

Let \( v \) be a \( q \)-dimensional vector space, where \( q \geq 2 \), let \( S \) be an admissible multiset of size \( q \), and let \( \mathcal{C}_S \) denote the conjugacy class of matrices in \( \mathfrak{so}(v) \) with eigenvalues given by the multiset \( S \). Let \( \mathbb{R}\mathcal{C}_S = \{ rA : A \in \mathcal{C}_S, r \in \mathbb{R} \} \) be the cone over the conjugacy class \( \mathcal{C}_S \). It is a subset of \( \mathfrak{so}(v) \). The main innovation of our work is to take the perspective of Eberlein, using standard nilpotent Lie algebras, and to re-interpret the definition of H-like using cones over conjugacy class in \( \mathfrak{so}(v) \). This makes some of the proofs we present here much more simple than they would be otherwise.

Our main theorem is that for any admissible multiset \( S \), there is a correspondence between \( \text{H-like metric nilpotent Lie algebras of type } (p, q) \) whose spectrum is a multiple of \( S \) and standard two-step metric nilpotent Lie algebras defined by \( p \)-dimensional subspaces of \( \mathbb{R}\mathcal{C}_S \) in \( \mathfrak{so}(v) \).

**Theorem 1.3.** Let \( S \) be an admissible multiset with size \( q \geq 2 \). Let \( \mathbb{R}\mathcal{C}_S \) be the cone over the conjugacy class of elements of \( \mathfrak{so}(\mathbb{R}^q) \) with spectrum \( S \).
(1) If $W$ is a $p$-dimensional subspace of $\mathbb{R}C_S$, then the standard two-step metric nilpotent Lie algebra of type $(p, q)$ defined by $W$ has constant $J$-spectrum $S$.

(2) If $(n = v \oplus z, Q)$ is a two-step metric nilpotent Lie algebra of type $(p, q)$ which has constant $J$-spectrum $S$, then $J(z)$ is a $p$-dimensional subspace of $\mathbb{R}C_S$, and $(n, Q)$ is homothetic to a standard two-step metric nilpotent Lie algebra.

Therefore the problem of classifying $H$-like Lie algebras is equivalent to the problem of classifying linear subspaces of $so(\mathbb{R}^q)$ that are contained in subsets of the form $\mathbb{R}C_S$. This kind of problem—finding sets of matrices so that nonlinear properties such as rank or spectrum are preserved under linear combinations—has been studied since at least as early as Hurwitz and Radon ([Hur22, Rad22]). For example, see [EM16, IL99, EH88, Wes87, Bea81, Fla62] for subspaces of matrices having fixed or bounded rank, see [BM15, FMI1, MM05] for subspaces of skew-symmetric matrices, and see [Skr02] for subspaces of cones over conjugacy classes.

Vector spaces of matrices of maximal rank two were first classified in [Atk83]. Early classification results are reviewed and obtained as special cases of a more general theorem in [EH88] (see Theorem 1.1).

**Theorem 1.4.** [Atk83] A vector space of matrices of rank $\leq 2$ is equivalent to one of the following:

(1) a subspace of the space of matrices of the form

\[
\begin{bmatrix}
0 & 0 & \cdots & 0 & * \\
0 & 0 & \cdots & 0 & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & * \\
* & * & \cdots & 0 & * \\
\end{bmatrix}
\]

(2) $so(\mathbb{R}^3)$.

Even a general classification of skew-symmetric matrices of rank 4 analogous to the classification in Theorem 1.4 remains unknown.

In Definition 3.2, we define a class of metric Lie algebras by taking the tensor product of a $J$-map for a metric nilpotent Lie algebra with another linear map, and we show in Proposition 3.3 that if the original metric nilpotent Lie algebra is $H$-like and the linear map is symmetric, then the resulting Lie algebra is $H$-like. We show in Proposition 3.4 that if $(n, Q)$ is $H$-like, then the $J$ map is unitary. In Propositions 3.4, 4.4 and 4.6 we give new methods for constructing $H$-like Lie algebras using subspaces of cones over conjugacy classes; these correspond to families of block diagonal matrices, Riemannian submersions with fibers in the center and central sums.

We classify $H$-like Lie algebras such that rank$(J_Z) \leq 2$ for all $Z \in n$:

**Theorem 1.5.** Suppose that $(n, Q)$ is an $H$-like metric nilpotent Lie algebra and rank$(J_Z) \leq 2$ for all $Z \in n$. Then $(n, Q)$ is homothetically isomorphic to one of the following:

(1) $(n, Q) = (f_{3,2}, Q)$ as in Example 2.5
(2) an almost abelian metric Lie algebra as in Example 2.6.
In the first theorem of [DDM], the authors classify two-step nilpotent Lie algebras defined by graphs which admit H-like inner products and obtain the H-like Lie algebras listed in Theorem 1.5. All Lie algebras defined by a graph have rank$(J_Z) \leq 2$ for all $Z \in \mathfrak{n}$, so the classification in [DDM] follows from Theorem 1.5. In higher dimensions, there are continuous families of nonhomothetic metric Lie algebras with rank$(J_Z) \leq 2$ for all $Z \in \mathfrak{n}$, while there are only countably many Lie algebras defined by graphs. Hence, the classification in Theorem 1.5 is more general than that in [DDM].

Theorem 1.5 can be proved using Theorem 1.3 and Theorem 1.4. Instead we provide a new, self-contained proof which uses the language and properties of $J$-maps and uses the metric throughout the argument.

The paper is organized as follows. In Section 2 we review properties of multisets and present some examples of H-like Lie algebras. We discuss properties of H-like Lie algebras in Section 3. In Section 4, we prove Theorem 1.3 and use it to prove Propositions 4.1, 4.4 and 4.6. In Section 5, we prove Theorem 1.5.

### 2. Background and examples

We will denote the spectra of matrices as multisets with elements from $\mathbb{C}$. Recall that a multiset from $\mathbb{C}$ is a subset of $\mathbb{C}$ with multiplicities. For example, the set $\{i, -i, 0\}$ endowed with multiplicity function $m : S \to \mathbb{N}$ given by $m(i) = m(-i) = 2$, $m(0) = 1$, and $m(z) = 0$ for all other $z \in \mathbb{C}$ is a multiset. We say that the sum of the multiplicities of a multiset $S$ is its size and we denote the size of $S$ by $|S|$. The sum $S_1 \cup S_2$ of multisets $S_1$ and $S_2$ is the multiset determined by summing the multiplicity functions for $S_1$ and $S_2$. For a multiset $S$ and nonzero scalar $k$ we define the multiset $kS$ so that the multiplicity for $kz$ in $kS$ is always equal to the multiplicity of $z$ in $S$.

Let $S$ be an admissible multiset of size $q$. The Frobenius norm on $\mathfrak{so}(\mathbb{R}^q)$ is given by $\|A\|^2 = \text{trace}(AA^T)$. All elements $A$ of $\mathfrak{so}(\mathbb{R}^q)$ with spectrum $S$ have the same Frobenius norm, so we define the norm of the multiset $S$ by

$$N(S) = \|A\| = \sqrt{\sum_{a \in S} |a|^2 \cdot m(a)},$$

where $m(a)$ is the multiplicity of $a$. For example, the multiset $S = \{i, -i, 3i, -3i\}$ with $m(i) = m(-i) = 2$ and $m(3i) = m(-3i) = 1$ has $N(S) = \sqrt{22}$.

**Example 2.1.** Suppose that $(\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}, Q)$ is H-type. If $\|Z\| = 1$, then $J_Z$ is nondegenerate with eigenvalues in the multiset $S$ defined over $\{i, -i\}$, where $i$ and $-i$ have multiplicity ${1 \over 2} \dim \mathfrak{v}$. Then $N(S) = \sqrt{\dim \mathfrak{v}}$.

Deformations of inner products on H-type metric Lie algebras may give metrics on the same underlying Lie algebra that are H-like but not H-type. This occurs in the next example.
Example 2.2. Let $v \cong \mathbb{R}^4$ and $\mathfrak{z} \cong \mathbb{R}$ with $Z$ a basis vector for $\mathfrak{z}$ and $\{X_1, Y_1, X_2, Y_2\}$ a basis for $v$. Let $a$ and $b$ be nonzero real numbers. Define $J : \mathfrak{z} \to \text{End}(v)$ by

$$J(Z) = \begin{bmatrix} 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & -b \\ 0 & 0 & b & 0 \end{bmatrix},$$

and let $(n, Q)$ be the resulting metric nilpotent Lie algebra with orthonormal basis $\{Z, X_1, Y_1, X_2, Y_2\}$. Clearly $(n, Q)$ has constant $J$-spectrum equal to the multiset $\{ai, -ai, bi, -bi\}$. When $|a| = |b| = 1$, we get the H-type inner product on the five-dimensional Heisenberg Lie algebra and when $|a| \neq |b|$ and both are nonzero, we get an H-like inner product that is not H-type. If we allow exactly one of the parameters to be zero, then $n$ is isomorphic to $\mathfrak{h}_3 \oplus \mathbb{R}^2$, and $(n, Q)$ has constant $J$-spectrum but is not H-like.

Gornet and Mast gave the following families of nonisometric H-like Lie algebras of type $(2, 4)$ \([GM00]\).

Example 2.3. Let $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, and assume that $(c, d) \in \{\pm(-b, a), \pm(-a, b)\}$. Define for orthonormal $Z_1, Z_2$ in $\mathbb{R}^2$,

$$J(Z_1) = \begin{bmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{bmatrix}, \quad \text{and} \quad J(Z_2) = \begin{bmatrix} 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \\ -c & 0 & 0 & 0 \\ 0 & -d & 0 & 0 \end{bmatrix}. $$

The spectrum of $J(Z)$ for unit $Z$ in $\mathbb{R}^2$ is $\{ai, -ai, bi, -bi\}$. When $a = \pm b$, the resulting metric Lie algebra is H-type. When $a$ and $b$ are both nonzero and $|a| \neq |b|$, it is H-like but not H-type.

Remark 2.4. The previous two examples show that H-like metric nilpotent Lie algebras are not necessarily soliton: Soliton inner products on a fixed nilpotent Lie algebra are unique up to scaling, but there may be two nonhomothetic inner products on a fixed Lie algebra which are both H-like.

In the next example we consider the free two-step nilpotent Lie algebra on three generators.

Example 2.5. Let $\mathfrak{f}_{3,2}$ be the free two-step nilpotent Lie algebra on three generators. It has basis $\{E_1, E_2, E_3\} \cup \{F_1, F_2, F_3\}$, and the Lie bracket is determined by the relations

$$[E_1, E_2] = F_1, \quad [E_2, E_3] = F_2, \quad \text{and} \quad [E_1, E_3] = F_3.$$ 

Let $Q_0$ be the inner product that makes the basis orthonormal. Then $v = \text{span}\{E_i\}_{i=1}^3$ and $\mathfrak{z} = \text{span}\{F_i\}_{i=1}^3$. With respect to the basis $\mathcal{B} = \{E_1, E_2, E_3\}$ of $v$, the endomorphism $J_{a_1 F_1 + a_2 F_2 + a_3 F_3}$ is given by

$$[J_{a_1 F_1 + a_2 F_2 + a_3 F_3}]_{\mathcal{B}} = \begin{bmatrix} 0 & -a_1 & -a_3 \\ a_1 & 0 & -a_2 \\ a_3 & a_2 & 0 \end{bmatrix}. $$
The square of this mapping has eigenvalues \(-(a_1^2 + a_2^2 + a_3^2)\) and 0, with multiplicities 2 and 1 respectively. Hence \((f_{3,2}, Q_0)\) has \(J\)-rank two and is H-like. Because the \(J_Z\) maps are always singular, it is not H-type.

The following family of metric Lie algebras was shown to be H-like in [DDM] (Example 6).

**Example 2.6.** Let \((n^n, Q_0)\) be the metric nilpotent Lie algebra with orthonormal basis \(\{E_0, E_1, E_2, \ldots, E_m\} \cup \{F_1, \ldots, F_m\}\) with Lie bracket determined by relations

\[
[E_0, E_k] = F_k \quad \text{for } k = 1, \ldots, m.
\]

Note that \(\text{span}(\{E_1, E_2, \ldots, E_m\} \cup \{F_1, \ldots, F_m\})\) is a codimension one abelian ideal and that all bracket relations are determined by \(\text{ad}_{E_0}\). Such algebras are called *almost abelian*. If \(Z = a_1 F_1 + \cdots + a_k F_k\), the map \(J_Z^2\) has eigenvalues \(-(a_1^2 + \cdots + a_m^2)\) and 0, with multiplicities 2 and \(m - 1 = \dim(\mathfrak{v}) - 2\) respectively. Therefore, if \(\|Z\| = 1\), the spectrum of \(J_Z\) is \(\{i, -i, 0\}\). Hence \((n^n, Q_0)\) has \(J\)-rank 2 and is H-like.

**Example 2.7.** Let \((n = \mathfrak{v} \oplus \mathfrak{z}, Q)\) be a two-step metric nilpotent Lie algebra so that the isometry group of \(Q\) acts transitively on the unit sphere in \(\mathfrak{z}\).

For any \(Z \in \mathfrak{z}\), \(J_{\phi(Z)} = \phi \circ J_Z \circ \phi^{-1}\), so \(J_Z\) and \(J_{\phi(Z)}\) have the same spectrum. Hence \((n, Q)\) is H-like.

### 3. Properties of H-like Lie Algebras

The next proposition describes how the property of having constant \(J\)-spectrum behaves under direct sums.

**Proposition 3.1.** Let \((n_1, Q_1)\) and \((n_2, Q_2)\) be metric nilpotent Lie algebras which are abelian or two-step. Define the metric Lie algebra \((n_1 \oplus n_2, Q)\) with \(Q\) so that \(Q|_{n_1} = Q_1, Q|_{n_2} = Q_2\) and \(n_1 \perp n_2\). Assume \(n_1 \oplus n_2\) is non-abelian. The following are equivalent:

- The direct sum \((n_1 \oplus n_2, Q)\) has constant \(J\)-spectrum.
- One of \((n_1, Q_1)\) or \((n_2, Q_2)\) is abelian, and the other has constant \(J\)-spectrum.

**Proof.** Write \(n_1 = \mathfrak{v}_1 \oplus \mathfrak{z}_1\) and \(n_2 = \mathfrak{v}_2 \oplus \mathfrak{z}_2\).

Suppose that \((n_1 \oplus n_2, Q)\) has constant \(J\)-spectrum. If neither \(n_1\) nor \(n_2\) is abelian, then \(\mathfrak{z}_1\) and \(\mathfrak{z}_2\) are nontrivial, so there exist nonzero \(Z_1 \in \mathfrak{z}_1 = [\mathfrak{v}_1, \mathfrak{v}_1]\) and nonzero \(Z_2 \in \mathfrak{z}_2 = [\mathfrak{v}_2, \mathfrak{v}_2]\). By the constant spectrum hypothesis, the ranks of \(J(Z_1 + Z_2)\), \(J(Z_1)\), and \(J(Z_2)\) are the same. But because \(\mathfrak{v}_1 \cap \mathfrak{v}_2 = \{0\}\) in \(n_1 \oplus n_2\), the rank of \(J(Z_1 + Z_2)\) is the sum of the ranks of \(J(Z_1)\) and \(J(Z_2)\). Therefore one of them has rank zero, a contradiction to \(Z_1\) and \(Z_2\) being in the commutator. If \(n_2\) is abelian, then clearly \(n_1\) must have constant spectrum.

For the converse, suppose that \((n_1, Q_1)\) is two-step with constant spectrum \(S\) and \(n_0\) is abelian. Then \((n_1, Q_1)\) has nontrivial commutator and \(J^n = J^{n_1} \oplus 0^{n_2}\) where \(0^{n_2}\) is the zero map on \(\mathfrak{v}_2\). The spectrum of \(J^n\) is \(S \oplus T\), where \(T\) is \(\{0\}\) with multiplicity \(\dim(\mathfrak{v}_2)\). \(\square\)
Proposition 3.4. Suppose that \((\mathfrak{n}, Q)\) is a two-step metric nilpotent Lie algebra with \(\mathfrak{n} = v \oplus \mathfrak{j}\). Let \(J: \mathfrak{j} \to \text{End}(v)\) be the \(J\)-map for \((\mathfrak{n}, Q)\). Let \(S: \mathbb{R}^m \to \mathbb{R}^m\), where \(m > 1\), be a linear map which is symmetric with respect to the standard inner product \(\langle \cdot, \cdot \rangle\) on \(\mathbb{R}^m\). Then the map \(S: \mathfrak{j} \to \mathfrak{n}\), where \(\mathfrak{j}\) is an eigenvector of \(S\), is isometrically isomorphic to the one in Example 2.2.

Definition 3.2. Suppose that \((\mathfrak{n}, Q)\) is a two-step metric nilpotent Lie algebra with \(\mathfrak{n} = v \oplus \mathfrak{j}\). Let \(J: \mathfrak{j} \to \text{End}(v)\) be the \(J\)-map for \((\mathfrak{n}, Q)\). Let \(S: \mathbb{R}^m \to \mathbb{R}^m\), where \(m > 1\), be a linear map which is symmetric with respect to the standard inner product \(\langle \cdot, \cdot \rangle\) on \(\mathbb{R}^m\).

The resulting Lie algebra \(\mathfrak{n} \otimes S\) as in Definition 3.2 is isometrically isomorphic to the one in Example 2.2.

Example 3.3. Let \((\mathfrak{h}_3, Q)\) be the Heisenberg metric Lie algebra with orthonormal basis \(\{X, Y, Z\}\) where \([X, Y] = Z\). Take orthonormal basis \(\{E_1, E_2\}\) for \(\mathbb{R}^2\) and let \(S: \mathbb{R}^2 \to \mathbb{R}^2\) be the \(\mathfrak{j}\)-map with \(S(E_1) = aE_1\) and \(S(E_2) = bE_2\), where \(a, b\) are nonzero. Then the map \(J^S: \mathfrak{j} \to \text{End}(\mathfrak{n})\) by \(J^S(Z) = J_Z \otimes S\). Let \(\mathfrak{n} \otimes S = \mathfrak{j} \oplus (v \otimes \mathbb{R}^m)\). Make \(\mathfrak{n} \otimes S\) into a Lie algebra by defining the bracket using the \(J\)-map through Equation (3).

Proposition 3.4. Suppose that \((\mathfrak{n} = v \oplus \mathfrak{j}, Q)\) is a two-step metric nilpotent Lie algebra. Let \(S: \mathbb{R}^m \to \mathbb{R}^m\) be a nonsingular linear map which is symmetric with respect to the standard inner product on \(\mathbb{R}^m\), where \(m > 1\).

Then, as a multiset, the spectrum of \(J^S(Z)\) is the multiset consisting of all products \(\lambda \mu\), where \(\lambda \in \text{Spec}(J_Z)\) and \(\mu \in \text{Spec}(S)\), and the multiplicity of \(\gamma \in \text{Spec}(J^S(Z))\) is the sum of products of multiplicities, \(m(\lambda_i)m(\mu_i)\) where \(\lambda_i \mu_i = \gamma\). Hence, if \((\mathfrak{n}, Q)\) is \(H\)-like, then \(\mathfrak{n} \otimes S\) is \(H\)-like.

Proof. First we show that \([\mathfrak{n} \otimes S, \mathfrak{n} \otimes S] = \mathfrak{j}\). From the definition of \(\mathfrak{n} \otimes S\), the commutator for \(\mathfrak{n} \otimes S\) is contained in the domain of \(J^S\), that is \(\mathfrak{j}\). Fix unit \(Z \in \mathfrak{j} \subseteq \mathfrak{n}\). Then there exist orthogonal \(X\) and \(Y\) in \(v\) so that \([X, Y] = Z\) in \(\mathfrak{n}\). Choose unit \(U\) to be an eigenvector of \(S\) with (nonzero) eigenvalue \(\lambda\). Then

\[
\langle Z, [X \otimes U, Y \otimes U] \rangle = \langle J_Z^S(X \otimes U), Y \otimes U \rangle = \langle J_ZX \otimes S(U), Y \otimes U \rangle = \langle J_ZX \otimes \lambda U, Y \otimes U \rangle = \langle Z, Z \rangle \cdot \langle \lambda U, U \rangle = \lambda \|Z\|^2 \neq 0.
\]

Hence \(Z \in [\mathfrak{n} \otimes S, \mathfrak{n} \otimes S]\). Therefore \(\mathfrak{j} \subseteq [\mathfrak{n} \otimes S, \mathfrak{n} \otimes S]\).

Because the eigenvalues of the tensor product of maps is the set of products of eigenvalues of each, it is immediate that if \((\mathfrak{n}, Q)\) has constant \(J\)-spectrum, then \(\mathfrak{n} \otimes S\) has constant \(J\)-spectrum. \(\square\)
Example 3.5. The generalized Heisenberg groups defined by Goze and Haraguchi in [GH82] arise from tensor products. Their Lie algebras, which we will call generalized Heisenberg Lie algebras, are of the form $m^r \otimes I_p$, where $m^r$ is the Lie algebra of dimension $2r + 1$ from Example 2.6 and $I_p$ is the $p \times p$ identity matrix. Taking $r = 1$ yields the $(2p + 1)$-dimensional Heisenberg algebra, and taking $p = 1$ yields $m^r$. Because the Lie algebras $m^r$ support H-like inner products, Proposition 3.4 implies that the generalized Heisenberg Lie algebras of Goze and Haraguchi admit H-like inner products.

We show that if $(n, Q)$ is an H-like metric Lie algebra, then the map $J$ for $(n, Q)$ is unitary with respect to a rescaled Frobenius norm on the image space.

Proposition 3.6. Suppose that $(n = v \oplus \mathfrak{z}, Q)$ is an H-like Lie algebra with spectrum $S$. Endow $\mathfrak{z}$ with the inner product that is the restriction of the inner product $Q$, and give $\text{End}(v)$ the rescaled Frobenius inner product
\[
\langle A, B \rangle_{\text{End}(v)} = \frac{1}{N(S)^2} \text{trace} AB^T,
\]
where $N(S)$ is as in Equation (2), and the trace and transpose are taken with respect to the restriction of the inner product $Q$ to $v$. Then
\[
\langle J_Y, J_Z \rangle_{\text{End}(v)} = \langle Y, Z \rangle_Q
\]
for all $Y, Z \in \mathfrak{z}$.

Proof. Let $Z \in \mathfrak{z}$ be a unit vector. Because $(n, Q)$ is H-like, $\text{trace } J_Z^2 = -N(S)^2$. This and skew-symmetry of $J_Z$ give
\[
\langle J_Z, J_Z \rangle_{\text{End}(v)} = \frac{1}{N(S)^2} \text{trace} J_Z J_Z^T = \frac{1}{N(S)^2} \text{trace} J_Z^2 = 1.
\]
Therefore $J$ maps the unit sphere in $\mathfrak{z}$ into the unit sphere in $\text{End}(v)$ so it is unitary.

As a consequence, with respect to the unscaled Frobenius norm $\| \cdot \|_{\text{Frobenius}}$, if $(n, Q)$ is H-like, and $\| Z \| = \| W \|$, then $\| J_Z \|_{\text{Frobenius}} = \| J_W \|_{\text{Frobenius}}$.

The next proposition describes how the spectrum of a vector in a subspace of the cone over a conjugacy class depends on its norm.

Lemma 3.7. Let $S$ be an admissible multiset of size $q$. Let $\mathbb{R} C_S$ be the cone over the conjugacy class for $S$ in $\mathfrak{so}(\mathbb{R}^q)$. Then any $A \in \mathbb{R} C_S$ has spectrum $\frac{\| A \|_{\text{Frobenius}}}{N(S)} S$, where $\| A \|_{\text{Frobenius}}^2 = \text{trace}(AA^T)$.

Proof. Clearly the statement holds when $A = 0$. Let $A \in \mathbb{R} C_S$ be nonzero. Because $A$ is in $\mathbb{R} C_S$, its spectrum is a multiple of $S$, and we can rescale $A$ by $\lambda$ so that $\lambda A$ has spectrum $S$. As $A$ is skew-symmetric, we may assume that $\lambda > 0$.

Choose $B \in \mathbb{R} C_S$ with spectrum $S$. Then $\| B \| = N(S)$, where $N(S)$ is as in Equation (2). Because they have the same spectrum, $\lambda A$ and $B$ are conjugate. Conjugation preserves the Frobenius norm, so $\lambda \| A \| = \| B \|$. But $\lambda \text{Spec}(A) = \text{Spec}(B) = S$. Hence
\[
\text{Spec}(A) = \frac{1}{\lambda} S = \frac{\| A \|}{\| B \|} S = \frac{\| A \|}{N(S)} S.
\]
We will use the following simple yet crucial lemma in the proof of the main theorem. It lies behind the correspondence between nilpotent Lie algebras of type \((p, q)\) and \(p\)-dimensional subspaces of \(\mathfrak{so}(\mathbb{R}^q)\) in [Ebe03a].

**Lemma 3.8.** Suppose that \((n = v \oplus \mathfrak{z}, Q)\) is a two-step metric nilpotent Lie algebra of type \((p, q)\). Then \(J(\mathfrak{z})\) is a \(p\)-dimensional subspace of \(\text{End}(v)\).

**Proof.** Let \(\{Z_1, \ldots, Z_p\}\) be a basis of \(\mathfrak{z}\). We want to show that \(J(Z_1), \ldots, J(Z_p)\) are independent. Suppose that there are real numbers \(a_1, \ldots, a_p\) so that \(\sum_{i=1}^{p} a_i J(Z_i) = 0\). But then \(J(\sum_{i=1}^{p} a_i Z_i) = 0\). Hence \(J(Z) \equiv 0\) for \(Z = \sum_{i=1}^{p} a_i Z_i\). But for \(Z \in [n, n]\), \(J_Z \equiv 0\) if and only if \(Z = 0\). Because \(\{Z_1, \ldots, Z_p\}\) is independent, \(a_i = 0\) for all \(i\). \(\square\)

**Remark 3.9.** It can be shown that if \((n, Q)\) is H-like, then the restriction of the Ricci endomorphism to \(\mathfrak{z}\) is a constant times the identity (see [Pay10, Lemma 1]).

### 4. The Main Theorem and some of its consequences

To prove Theorem 1.3, we show that the bijection between two-step metric nilpotent Lie algebras \((n, Q)\) of type \((p, q)\) and \(p\)-dimensional subspaces of \(\mathfrak{so}(\mathbb{R}^q)\) endowed with the natural inner product is a bijection between algebras with spectrum a multiple of \(S\) and \(p\)-dimensional subspaces of the cone \(\mathbb{R}C_S\) in \(\mathfrak{so}(\mathbb{R}^q)\).

**Proof of Theorem 1.3.** Fix \((p, q)\). Let \(S\) be an admissible multiset of size \(q \geq 2\). Let \(W\) be a \(p\)-dimensional subspace of \(\mathbb{R}C_S \subseteq \mathfrak{so}(\mathbb{R}^q)\) and let \((n, Q)\) be the standard two-step metric nilpotent Lie algebra defined by \(W\) as in Definition 1.2. Choose \(Z \in \mathfrak{z}\) with \(\|Z\| = 1\). Then \(J(Z) \in W \subseteq \mathbb{R}C_S\), so its spectrum is a scalar multiple of \(S\).

Suppose that \((n = v \oplus \mathfrak{z}, Q)\) is a two-step metric nilpotent Lie algebra of type \((p, q)\) with constant \(J\)-spectrum \(S\). By Lemma 3.8, \(J(\mathfrak{z})\) is a \(p\)-dimensional subspace of \(\mathfrak{so}(v)\). By definition of constant \(J\)-spectrum, \(J(Z)\) has spectrum \(S\) for all unit \(Z\). Therefore the image of the unit sphere in \(\mathfrak{z}\) is contained in \(C_S\). Because rescaling an element of \(\mathfrak{so}(v)\) by \(\lambda \in \mathbb{R}\) rescales the spectrum by \(\lambda\), all elements in \(J(\mathfrak{z})\) are in the cone \(\mathbb{R}C_S\). \(\square\)

The next proposition describes how to explicitly construct H-like Lie algebras using bases with a particular form. Recall that to form the sum \(S = S_1 \oplus S_2 \oplus \cdots \oplus S_k\) of multisets \(S_1, \ldots, S_k\), we add the multiplicity functions of \(S_1, \ldots, S_k\).

**Proposition 4.1.** For \(i = 1, \ldots, k\), let \(v_i\) be a vector space of dimension at least two, and let \(S_i\) be an admissible multiset with size equal to \(\dim(v_i)\).

Suppose that \(W\) is a \(p\)-dimensional subspace of \(\mathfrak{so}(\oplus_{i=1}^{k} v_i)\) having a basis \(B = \{B_1, \ldots, B_p\}\) consisting of vectors \(B_j = \oplus_{i=1}^{k} A_i^j\), \(j = 1, \ldots, p\), where for each \(i\),

1. \(A_i^j \in \mathfrak{so}(v_i)\) for each \(j\),
2. for all \(j\) the spectrum of \(A_i^j\) is \(S_i\),
3. \(\text{span}\{A_i^j\}_{j=1}^{p} \subseteq \mathbb{R}C_{S_i}\), and
4. \(\{A_i^j : j = 1, \ldots, p\}\) is orthogonal with respect to the Frobenius inner product.
Then $W$ is a subspace of $\mathbb{R}C_S$, where $S = S_1 \cup S_2 \cup \cdots \cup S_k$ and $\mathbb{R}C_S$ is the cone over the conjugacy class $C_S$ in $\mathfrak{so}(\bigoplus_{i=1}^{k} v_i)$.

Proof. Let $B = \{B_1, \ldots, B_p\}$ be a basis for $W$ as in the statement of the proposition. Property (1) forces $B$ to be orthogonal, and by Property (3), all of the vectors in $B$ have spectrum $S = S_1 \cup S_2 \cup \cdots \cup S_k$ and Frobenius norm $N(S)$, where $N$ is the function associated to the multiset $S$ as in Equation (2). Let $C$ in $W$ be nonzero.

We may write $C$ as $C = \sum_{j=1}^{p} c_j (\bigoplus_{i=1}^{k} A_i^j) = \bigoplus_{i=1}^{k} \left( \sum_{j=1}^{p} c_j A_i^j \right)$. Then the norm of $C$ is given by

$$\|C\|^2 = \sum_{i=1}^{k} \sum_{j=1}^{p} \|c_j A_i^j\|^2 = \sum_{j=1}^{p} c_j^2 \sum_{i=1}^{k} N(S_i)^2 = N(S)^2 \sum_{j=1}^{p} c_j^2,$$

which shows that $\sum_{j=1}^{p} c_j^2 = \|C\|^2/N(S)^2$.

The restriction of $C$ to $v_i$ for fixed $i$ is $C_i = \sum_{j=1}^{p} c_j A_i^j$. By the same reasoning as above, $\sqrt{\sum_{j=1}^{p} c_j^2} = \|C_i\|/N(S_i)$ for all $i$. Now we apply Lemma 3.7 to $C_i$ in $\mathbb{R}C_{S_i}$ and see that its spectrum is $\lambda S_i$, where

$$\lambda = \frac{\|C_i\|}{N(S_i)} = \sqrt{\sum_{j=1}^{p} c_j^2}$$

is independent of $i$. Summing over $i$, we deduce that the spectrum of $C$ is

$$\lambda S_1 \cup \cdots \cup \lambda S_k = \lambda (S_1 \cup S_2 \cup \cdots \cup S_k) = \lambda S.$$

Thus, $W \subseteq \mathbb{R}C_S$.

Example 4.2. Take $v_1 = \mathbb{R}^2$ and $v_2 = \mathbb{R}^2$. Let $a$ and $b$ be nonzero. Take

$$A_1^1 = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} \in \mathfrak{so}(v_1), \quad A_1^2 = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \in \mathfrak{so}(v_2), \quad \text{and} \quad B_1 = A_1^1 \oplus A_1^2 \subseteq \mathfrak{so}(v_1 \oplus v_2).$$

Let $W$ be the subspace of $\mathfrak{so}(v_1 \oplus v_2)$ spanned by $A_1^1 \oplus A_1^2$. The resulting metric Lie algebra is isometrically isomorphic to $\mathfrak{so}_5$ endowed with an H-like inner product as in Example 2.2. Here $S_1 = \{ai, -ai\}$, $S_2 = \{bi, -bi\}$, and $S_1 \cup S_2 = \{ai, -ai, bi, -bi\}$.

Example 4.3. Let $(f_3, Q_0)$ and $J_{F_1}, J_{F_2}, J_{F_3} \in \mathfrak{so}(\mathbb{R}^3)$ be as in Example 2.5. By Proposition 4.1 any basis selected from the linearly dependent set

$$\{ J_{F_k} \oplus J_{F_l} : k, l = 1, 2, 3 \} \subseteq \mathfrak{so}(\mathbb{R}^3) \oplus \mathfrak{so}(\mathbb{R}^3) \subseteq \mathfrak{so}(\mathbb{R}^6)$$

defines a subspace of the cone over the conjugacy class for $\{i, i, -i, -i, 0, 0\}$. An example of this type appears in (4.7) of [FMIII].

The next proposition shows that Riemannian submersions in a certain class map Lie algebras with constant spectrum onto Lie algebras with constant spectrum. In fact, by [Ebe03b], the Riemannian submersion of the corresponding nilmanifolds has simply connected, flat, totally geodesic fibers.
Proposition 4.4. Suppose that \((n_1 = v_1 \oplus z_1, Q_1)\) and \((n_2 = v_2 \oplus z_2, Q_2)\) are metric Lie algebras, and \(\phi : n_1 \to n_2\) is a surjective homomorphism with \(\ker \phi \subseteq z_1\) such that \(\langle X, Y \rangle = \langle \phi(X), \phi(Y) \rangle\) for all \(X, Y \in (\ker \phi)^\perp\). If \((n_1, Q_1)\) has constant spectrum \(S\), then \((n_2, Q_2)\) has constant spectrum \(S\).

Proof. Let \(Z_2\) be an arbitrary unit vector in \(z_2\). Since the restriction \(\phi_0\) of \(\phi\) to \((\ker \phi)^\perp\) is a bijection, it is invertible. Hence \(Z_2\) has a unique pre-image \(Z_1 = \phi_0^{-1}(Z_2)\) in \((\ker \phi)^\perp\) and this pre-image has length one. Because \((n_1, Q_1)\) has spectrum \(S\), \(J_{Z_1}\) has spectrum \(S\).

For any \(Z \in (\ker \phi)^\perp\), \(J_{\phi_0(Z)} = \phi_0 \circ J_Z \circ \phi_0^{-1}\). Hence \(J_{Z_2} = \phi_0 \circ J_{Z_1} \circ \phi_0^{-1}\). Since \(J_{Z_2}\) is conjugate to \(J_{Z_1}\), it also has spectrum \(S\).

Thus, all vectors in the unit sphere in \(z_2\) have spectrum \(S\) and \(J(z_2) \subseteq \mathbb{R}C_S\). 

The central sum construction glues two groups together along a subgroup; when we glue two nilpotent Lie algebras together along their centers we call this a central sum. This was called concatenation by Jablonski in [Jab11].

Definition 4.5. Let \(n_1\) and \(n_2\) be Lie algebras with centers \(Z(n_1)\) and \(Z(n_2)\) respectively. Let \(\phi : Z(n_1) \to Z(n_2)\) be a bijective linear map.

1. Define the Lie algebra \(n_1 +_\phi n_2\) as \((n_1 \oplus n_2)/i\) where \(i\) is the ideal \(\{(W, -\phi(W)) : W \in Z(n_1)\}\). This Lie algebra is called the central sum of \(n_1\) and \(n_2\) defined by \(\phi\).

2. Suppose that \(Q_1\) and \(Q_2\) are inner products on \(n_1\) and \(n_2\) respectively, and that \(\phi\) is an isometry with respect to \(Q_1\) and \(Q_2\). Let \(\pi : n_1 \oplus n_2 \to n_1 +_\phi n_2\) be the natural projection map. Define the inner product \(Q\) on \(n_1 +_\phi n_2\) so that the projections \(\pi|_{n_1}\) and \(\pi|_{n_2}\) are isometries. Then the metric nilpotent Lie algebra \((n_1 +_\phi n_2, Q)\) defined by \(J\) is called the (metric) central sum of \((n_1, Q_1)\) and \((n_2, Q_2)\) defined by \(\phi\).

It is not hard to check that the inner product \(Q\) in Definition 4.5 (2) is well-defined.

This construction may be iterated. For example the \((2k+1)\)-dimensional Heisenberg algebra may be viewed as a \((k - 1)\)-fold central sum of 3-dimensional Heisenberg algebras. In Section 6 of [DDM11], the authors present families of H-like Lie algebras which are central sums. We show that in general, central sums of Lie algebras with constant spectrum again have constant spectrum.

An elementary way to define a subspace of the cone over a conjugacy class \(\mathbb{R}C_S\) is by taking the direct sum of subspaces of \(\mathbb{R}C_{S_1}\) and \(\mathbb{R}C_{S_2}\), where \(S = S_1 \sqcup S_2\). Taking direct sums of subspaces translates to a natural construction of metric Lie algebra called the metric central sum. Before we state the proposition we make some simple observations about the structure of metric central sums. Write \(n = n_1 +_\phi n_2\). Because \(\pi\) is a surjective homomorphism,

\[
[n, n] = \pi([n_1 \oplus n_2, n_1 \oplus n_2]) = \pi(z_1 \oplus z_2) = \pi(z_1).
\]

The restriction of the canonical projection map \(\pi : n_1 \oplus n_2 \to n\) to \(v_1 \oplus v_2\) is an isometry, and the restriction of \(\pi\) to \(z_1 \oplus \{0\}\) or \(\{0\} \oplus z_2\) is an isometry to \(\pi(z_1 \oplus z_2)\). We thus can identify \(z = [n, n]\) with \(z_1\) or \(z_2\). We let \(v = z^\perp\), so \(n = v \oplus z\). After using
the three isomorphisms to make appropriate identifications we can view the \( J \) map for \( n \) as a map from \( \mathfrak{z}_1 \) to \( \mathfrak{so}(v_1 \oplus v_2) \).

**Proposition 4.6.** Suppose that \((n_1 = v_1 \oplus \mathfrak{z}_1, Q_1)\) and \((n_2 = v_2 \oplus \mathfrak{z}_2, Q_2)\) are metric nilpotent Lie algebras with centers \( Z(n_1) \) and \( Z(n_2) \) respectively. Let \( \phi : Z(n_1) \to Z(n_2) \) be an isometry.

1. Denote the \( J \) map for the metric central sum \( n = n_1 + \mathfrak{z} n_2 \) by

\[
J^n : \mathfrak{z} \cong \mathfrak{z}_1 \to \mathfrak{so}(v) \cong \mathfrak{so}(v_1 \oplus v_2)
\]

For \( Z = \pi(Z_1) \in \mathfrak{z} \),

\[
J^n(Z) = J^{n_1}(Z_1) \oplus J^{n_2}(\phi(Z_1)) \subseteq \mathfrak{so}(v_1) \oplus \mathfrak{so}(v_2) \subseteq \mathfrak{so}(v_1 \oplus v_2).
\]

In particular, if \((n_1, Q_1)\) and \((n_2, Q_2)\) have constant spectra \( S_1 \) and \( S_2 \) respectively, then \((n_1 + _{\phi} n_2, Q)\) has constant spectrum \( S_1 \uplus S_2 \).

2. The subspace of \( \mathfrak{so}(v) \) corresponding to \((n_1 + _{\phi} n_2, Q)\) is \( J^{n_1}(\mathfrak{z}_1) \oplus J^{n_2}(\mathfrak{z}_2) \subseteq \mathfrak{so}(v_1) \oplus \mathfrak{so}(v_2) \subseteq \mathfrak{so}(v) \).

**Proof.** Suppose \( n_1 \) is type \((p_1, q_1)\) and \( n_2 \) is type \((p_2, q_2)\). Because \((n_1, Q_1)\) and \((n_2, Q_2)\) are \( H \)-like, \( Z(n_1) = \mathfrak{z}_1 \) and \( Z(n_2) = \mathfrak{z}_2 \). Because \( \phi \) is an isometry, \( p_1 = p_2 \).

Let \( S_1 \) and \( S_2 \) be the spectra for \((n_1, Q_1)\) and \((n_2, Q_2)\) respectively. From Theorem 1.3, \( J(\mathfrak{z}_1) \) is a \( p_1 \)-dimensional subspace of \( \mathbb{R} C_{S_1} \) contained in the cone over the conjugacy class \( C_{S_1} \), and \( J(\mathfrak{z}_2) \) is a \( p_2 \)-dimensional subspace of \( \mathbb{R} C_{S_2} \) contained in the cone over the conjugacy class \( C_{S_2} \).

We write the map \( J^n \) for \((n_1 + _{\phi} n_2, Q)\) in terms of \( J^{n_1} : \mathfrak{z}_1 \to \mathfrak{so}(v_1) \) and \( J^{n_2} : \mathfrak{z}_2 \to \mathfrak{so}(v_2) \). Let \( X_1, Y_1 \in v_1, X_2, Y_2 \in v_2, \) and \( Z_1 \in \mathfrak{z}_1 \). Denote their images under \( \pi \) with bars. Then

\[
\langle J^n_{Z_1}(X_1 + X_2), (Y_1 + Y_2) \rangle = \langle Z_1, [X_1, Y_1] \rangle + \langle \phi(Z_1), [X_2, Y_2] \rangle
\]

\[
= \langle Z_1, [X_1, Y_1] \rangle + \langle \phi(Z_1), [X_2, Y_2] \rangle
\]

\[
= \langle J^{n_1}_{Z_1}(X_1, Y_1) \rangle + \langle J^{n_2}_{\phi(Z_1)}(X_2, Y_2) \rangle.
\]

Thus, Equation (1) holds.

Let \( Z = \pi(Z_1) \) be a unit vector in \( \mathfrak{z} \). Because \( \pi \) maps \( \mathfrak{z}_1 \) onto \( \mathfrak{z} \) isometrically, \( Z_1 \) is a unit vector in \( \mathfrak{z}_1 \). Hence \( J(Z_1) \) has spectrum \( S_1 \). Because \( \phi \) is an isometry, \( \phi(Z_1) \) has norm one. Therefore, \( J(\phi(Z_1)) \) has spectrum \( S_2 \). It follows that \( J^n(Z) \) as in Equation (1) has spectrum \( S_1 \uplus S_2 \). By Theorem 1.3, \((n_1 + _{\phi} n_2, Q)\) is \( H \)-like with constant spectrum \( S_1 \uplus S_2 \).

Statement (2) follows from Equation (1). \( \square \)

Proposition 3.3 could have been used to complete the first part of the proof, along with the fact that \( \{ J(Z_i) \}^{n_1}_{i=1} \) and \( \{ J(\phi(Z_i)) \}^{n_1}_{i=1} \) are orthogonal bases for \( \mathfrak{so}(v_1) \) and \( \mathfrak{so}(v_2) \) respectively.

The tensor product construction of \( n \otimes S \) in Definition 3.2 is a central sum, with each summand corresponding to an element of a linearly independent set of eigenvectors for the symmetric map \( S \).

A DIFFERENT PERSPECTIVE ON H-LIKE LIE ALGEBRAS 13
5. Classification of H-like Lie algebras with $J$-rank two

Throughout this section we let $E(Z)$ denote the eigenspace for the nonzero eigenvalue of $J^2 Z$, where $J$ is associated to a two-step metric nilpotent Lie algebra of $J$-rank two. Note that if $(n, Q)$ is a non-zero metric nilpotent Lie algebra, $J^2 Z \neq 0$ for all non-zero $Z \in \mathfrak{z}$. Note that when $(n = \mathfrak{v} \oplus \mathfrak{z}, Q)$ has no abelian factors (as in the H-like case), $\oplus_i E(Z_i) = \mathfrak{v}$, where $\{Z_i\}$ is a basis for $\mathfrak{z}$.

**Lemma 5.1.** Let $(n = \mathfrak{v} \oplus \mathfrak{z}, Q)$ be a metric nilpotent Lie algebra. If $(n, Q)$ is H-like and has $J$-rank two, then for any independent vectors $Z_1$ and $Z_2$ in $\mathfrak{z}$, 

1. $E(Z_1) \neq E(Z_2)$
2. $E(Z_1) \cap E(Z_2)$ is one-dimensional, and
3. if $Z_1$ and $Z_2$ are orthogonal, and nonzero $X$ is in $E(Z_1) \cap E(Z_2)$, the vectors $J_{Z_1} X$ and $J_{Z_2} X$ are nonzero and orthogonal.

Part (3) of the lemma is a special case of Theorem 3.8b of [DDM11], which in turn relies on Lemma 3.3 of [GM00]. We give an alternate proof.

**Proof.** Assume that $(n, Q)$ is H-like and $Z_1$ and $Z_2$ are independent vectors in $\mathfrak{z}$. Then $Z_1, Z_2 \neq 0$. If $E(Z_1) = E(Z_2)$, then because $(n, Q)$ is H-like, $\lambda_1 J_{Z_1} = \lambda_2 J_{Z_2}$ for nonzero $\lambda_1, \lambda_2 \in \mathbb{R}$. It follows that $Z = \lambda_1 Z_1 - \lambda_2 Z_2$ has $J_Z \equiv 0$. Hence $Z = 0$, contradicting the independence of $Z_1, Z_2$.

Now assume $(n, Q)$ has $J$-rank two. We claim $E(Z_1) \cap E(Z_2) \neq \{0\}$. Otherwise $E(Z_1) + E(Z_2)$ is four-dimensional and $J_{Z_1 + Z_2}$ will have rank four, a contradiction. Also, $E(Z_1) \cap E(Z_2)$ is not two-dimensional, because that would lead to the contradiction $E(Z_1) = E(Z_2)$.

Suppose that $Z_1$ and $Z_2$ above are orthogonal and $X \in E(Z_1) \cap E(Z_2)$ is nonzero. Then certainly $J_{Z_1} X$ and $J_{Z_2} X$ are nonzero. Since $J_{Z_1}$ has rank two, there exists an orthogonal basis $\{X_1, X_2, \ldots, X_p\}$ for $\mathfrak{v}$ so that $X = X_1$, and with respect to this basis, $J_{Z_1} = X_1 \wedge X_2$ in $\mathfrak{so}(\mathfrak{v})$. Proposition 3.6 forces $J_{Z_1}$ and $J_{Z_2}$ to be orthogonal in $\mathfrak{so}(\mathfrak{v})$. Hence, 

$$J_{Z_2} = \sum_{i<j \neq (1,2)} a_{ij} X_i \wedge X_j.$$ 

Evaluating $J_{Z_2}$ at $X$ gives $J_{Z_2} X = J_{Z_2} X_1 = \sum_{j=3}^q a_{1j} X_j$. Therefore $J_{Z_1} X$ and $J_{Z_2} X$ are orthogonal. \qed

We have seen in Examples 2.5 and 2.6 that the free two-step nilpotent Lie algebra on three generators and certain two-step nilpotent Lie algebras with codimension one abelian ideals admit H-like inner products. In fact, these are all the H-like Lie algebras with $J$-rank two.

**Theorem 5.2.** Suppose that $(n, Q)$ is H-like and has $J$-rank two.

- If $\cap_{Z \neq 0} E(Z)$ is nontrivial, then it is one-dimensional, and $(n, Q)$ is homothetic to $(n^p, Q_0)$, the metric Lie algebra with $p$-dimensional center as in Example 2.6.
• If $\cap_{Z \neq 0} E(Z) = \{0\}$, then $(n, Q)$ is homothetic to $(\mathfrak{f}_{3,2}, Q_0)$, the free two-step nilpotent Lie algebra endowed with the inner product $Q_0$ as in Example 2.5.

Proof. We are classifying up to homothety, so we assume the nonzero elements of the spectrum are $i$ and $-i$. We break our argument into cases depending on the value of $p = \dim \mathfrak{z}$.

First suppose that $p = 1$ and $Z$ is a unit vector in $\mathfrak{z}$. If $J_Z$ has any zero eigenvalues, then $n$ has an abelian factor, a contradiction. Thus, for unit $Z$, $J_Z$ must be conjugate to a multiple of $X_1 \cap X_2$. Hence $n$ is isomorphic to $\mathfrak{h}_3$. But $\mathfrak{h}_3$ is isomorphic to $n^1$, and after rescaling the metric, $(n, Q)$ is isomorphic to $(n^p, Q_0)$ with $p = 1$.

Now suppose that $p \geq 2$ and that $\cap_{Z \neq 0} E(Z)$ is one-dimensional. Let $\{Z_1, \ldots, Z_p\}$ be an orthonormal basis for $\mathfrak{z}$. Let $X_1$ be a unit vector spanning $\cap_{Z \neq 0} E(Z)$. Because of our assumption that the nonzero eigenvalues are $\pm i$, $\{X_1, J_Z X_1\}$ is an orthonormal basis for $E(Z_i)$ for all $j$. By Lemma 5.1 part (3), the set $\{J_Z X_1, \ldots, J_Z^p X_1\}$ is independent and orthogonal. Hence $\{X_1, J_Z X_1, \ldots, J_Z^p X_1\}$ is an orthonormal basis for $v = \oplus_{i=1}^p E(Z_i)$. It is easy to check that $(n, Q)$ is isometrically isomorphic to $(n^p, Q_0)$.

Next suppose that $p \geq 2$ and $\dim \cap_{Z \neq 0} E(Z) \neq 1$. Part (1) of Lemma 5.1 implies $\cap_{Z \neq 0} E(Z) \subseteq E(Z_1) \cap E(Z_2)$ is at most one-dimensional and hence $\cap_{Z \neq 0} E(Z) = \{0\}$. If $p = 2$ then $E(Z_1) \cap E(Z_2) = \{0\}$ would contradict $(n, Q)$ having $J$-rank two. Hence we assume from now on that $p \geq 3$.

Let $p = 3$ and $\{Z_1, Z_2, Z_3\}$ be an orthonormal basis for $\mathfrak{z}$. Again by Lemma 5.1 we know that $E(Z_1) \cap E(Z_2), E(Z_2) \cap E(Z_3)$ and $E(Z_3) \cap E(Z_1)$ are one-dimensional. Choose unit vectors $X_1, X_2, X_3$ with

$X_1 \in E(Z_1) \cap E(Z_2), \quad X_2 \in E(Z_2) \cap E(Z_3), \quad \text{and} \quad X_3 \in E(Z_3) \cap E(Z_1)$.

The vectors $X_1, X_2,$ and $X_3$ are linearly independent because $J_{Z_1}, J_{Z_2},$ and $J_{Z_3}$ are independent. Thus we have

$[X_1, X_2] = \pm Z_2, \quad [X_2, X_3] = \pm Z_3, \quad [X_3, X_1] = \pm Z_1,$

which describes $\mathfrak{f}_{3,2}$.

Now suppose $p > 3$. Let $\{Z_1, Z_2, Z_3, \ldots, Z_p\}$ be an orthonormal basis for $\mathfrak{z}$ and choose independent unit vectors $X_1, X_2, X_3$ with

$X_1 \in E(Z_1) \cap E(Z_2), \quad X_2 \in E(Z_2) \cap E(Z_3), \quad \text{and} \quad X_3 \in E(Z_3) \cap E(Z_1)$

as when $p = 3$. Consider $E(Z_4)$. By Lemma 5.1, $E(Z_4)$ intersects $E(Z_1)$ in a one-dimensional subspace spanned by some unit $Y_1 \in v$. Since the map $\mathbb{R}Z \mapsto E(Z)$ from the set of lines in $\text{span}\{Z_1, Z_2, Z_3\}$ to the set of two-planes in $\text{span}\{X_1, X_2, X_3\}$ is surjective and $Y_1 \in \text{span}\{X_1, X_3\}$, there exists unit $Z$ in $\text{span}\{Z_1, Z_2, Z_3\}$ so that $E(Z) \perp Y_1$. Note that $Z$ and $Z_4$ are independent.

Now consider the subspace $F = E(Z_4) \cap E(Z)$. By Lemma 5.1, it is one-dimensional. Because $J_{Z_1}, J_{Z_2}, J_{Z_3}$ and $J_{Z_4}$ are independent, the subspace $E(Z_4) \cap \text{span}\{X_1, X_2, X_3\}$ is one-dimensional. Hence it is spanned by the common element $Y_1$. But then

$F = E(Z_4) \cap E(Z) \subseteq E(Z_4) \cap \text{span}\{X_1, X_2, X_3\} = \text{span}\{Y_1\}$.
Since $F$ is one-dimensional, it must be spanned by $Y_1$. But $Y_1$ is orthogonal to $E(Z)$, a contradiction. □

**References**

[Atk83] M. D. Atkinson. Primitive spaces of matrices of bounded rank. II. *J. Austral. Math. Soc. Ser. A.*, 34(3):306–315, 1983.

[Bea81] LeRoy B. Beasley. Spaces of matrices of equal rank. *Linear Algebra Appl.*, 38:227–237, 1981.

[BM15] Ada Boralevi and Emilia Mezzetti. Planes of matrices of constant rank and globally generated vector bundles. *Ann. Inst. Fourier (Grenoble)*, 65(5):2069–2089, 2015.

[BTV95] J. Berndt, F. Tricerri, and L. Vanhecke. Generalized Heisenberg groups and Damek-Ricci harmonic spaces, volume 1598 of *Lecture Notes in Math.* Springer-Verlag, 1995.

[Cia00] Paolo Ciatti. Scalar products on Clifford modules and pseudo-$H$-type Lie algebras. *Ann. Mat. Pura Appl. (4)*, 178:1–31, 2000.

[DDM] Rachelle DeCoste, Lisa DeMeyer, and Meera Mainkar. Graphs and metric 2-step nilpotent Lie algebras. arXiv:math.DG/1512.07944. To appear in Adv. Geom.

[DDM11] Rachelle C. DeCoste, Lisa DeMeyer, and Maura B. Mast. Characterizations of Heisenberg-like Lie algebras. *J. Lie Theory*, 21(3):711–727, 2011.

[DR92] E. Damek and F. Ricci. A class of nonsymmetric harmonic Riemannian spaces. *Bull. Amer. Math. Soc.*, 27(1):139–142, 1992.

[Ebe94a] Patrick Eberlein. Geometry of 2-step nilpotent groups with a left invariant metric. *Ann. Sci. École Norm. Sup. (4)*, 27(5):611–660, 1994.

[Ebe94b] Patrick Eberlein. Geometry of 2-step nilpotent groups with a left invariant metric. II. *Trans. Amer. Math. Soc.*, 343(2):805–828, 1994.

[Ebe03a] Patrick Eberlein. The moduli space of 2-step nilpotent Lie algebras of type $(p, q)$. In *Explorations in complex and Riemannian geometry*, volume 332 of *Contemp. Math.*, pages 37–72. Amer. Math. Soc., Providence, RI, 2003.

[Ebe03b] Patrick Eberlein. Riemannian submersions and lattices in 2-step nilpotent Lie groups. *Comm. Anal. Geom.*, 11(3):441–488, 2003.

[Ebe04] Patrick Eberlein. Geometry of 2-step nilpotent Lie groups. In *Modern dynamical systems and applications*, pages 67–101. Cambridge Univ. Press, Cambridge, 2004.

[EM16] Ph. Ellia and P. Menegatti. Spaces of matrices of constant rank and uniform vector bundles. *Linear Algebra Appl.*, 507:474–485, 2016.

[Fla62] H. Flanders. On spaces of linear transformations with bounded rank. *J. London Math. Soc.*, 37:10–16, 1962.

[FM11] Maria Lucia Fania and Emilia Mezzetti. Vector spaces of skew-symmetric matrices of constant rank. *Linear Algebra Appl.*, 434(12):2383–2403, 2011.

[GH82] Michel Goze and Yuri Haraguchi. Sur les $r$-systèmes de contact. *C. R. Acad. Sci. Paris Sér. I Math.*, 294(2):95–97, 1982.

[GK01] Carolyn S. Gordon and Megan M. Kerr. New homogeneous Einstein metrics of negative Ricci curvature. *Ann. Global Anal. Geom.*, 19(1):75–101, 2001.

[GM00] Ruth Gornet and Maura B. Mast. The length spectrum of Riemannian two-step nilmanifolds. *Ann. Sci. École Norm. Sup. (4)*, 33(2):181–209, 2000.

[GMKM13] Mauricio Godoy Molina, Anna Korolko, and Irina Markina. Sub-semi-Riemannian geometry of general $H$-type groups. *Bull. Sci. Math.*, 137(6):805–833, 2013.

[GW97] Carolyn S. Gordon and Edward N. Wilson. Continuous families of isospectral Riemannian metrics which are not locally isometric. *J. Differential Geom.*, 47(3):504–529, 1997.
[Hur22] A. Hurwitz. Über die Komposition der quadratischen Formen. Math. Ann., 88(1-2):1–25, 1922.

[IL99] Bo Ilic and J. M. Landsberg. On symmetric degeneracy loci, spaces of symmetric matrices of constant rank and dual varieties. Math. Ann., 314(1):159–174, 1999.

[Jab11] Michael Jablonski. Moduli of Einstein and non-Einstein nilradicals. Geom. Dedicata, 152(1):63–84, 2011.

[JLP08] Changrim Jang, Taehoon Lee, and Keun Park. Conjugate loci of 2-step nilpotent Lie groups satisfying $J^2 = ⟨Sz, z⟩$. A. J. Korean Math. Soc., 45(6):1705–1723, 2008.

[Kap80] A. Kaplan. Fundamental solutions for a class of hypoelliptic PDE generated by compositions of quadratic forms. Trans. Amer. Math. Soc., 258(1):147–153, 1980.

[KT13] Aroldo Kaplan and Alejandro Tiraboschi. Automorphisms of non-singular nilpotent Lie algebras. J. Lie Theory, 23(4):1085–1100, 2013.

[Lau99] Jorge Lauret. Modified $H$-type groups and symmetric-like Riemannian spaces. Differential Geom. Appl., 10(2):121–143, 1999.

[LO14] Jorge Lauret and David Oscari. On non-singular 2-step nilpotent Lie algebras. Math. Res. Lett., 21(3):553–583, 2014.

[Mét80] Guy Métrivier. Hypoellipticité analytique sur des groupes nilpotents de rang 2. Duke Math. J., 47(1):195–221, 1980.

[MM05] L. Manivel and E. Mezzetti. On linear spaces of skew-symmetric matrices of constant rank. Manuscripta Math., 117(3):319–331, 2005.

[MS04] Detlef Müller and Andreas Seeger. Singular spherical maximal operators on a class of two step nilpotent Lie groups. Israel J. Math., 141:315–340, 2004.

[Pay10] Tracy L. Payne. The existence of soliton metrics for nilpotent Lie groups. Geom. Dedicata, 145:71–88, 2010.

[Rad22] J. Radon. Lineare Scharen orthogonaler Matrizen. Abh. Math. Sem. Univ. Hamburg, 1(1):1–14, 1922.

[Sch15] Matthew Schroeder. Metric nilpotent Lie algebras defined by graphs. Master’s thesis, Idaho State University, 2015.

[Skr02] Marcin Skrzysiński. On the linear capacity of algebraic cones. Math. Bohem., 127(3):453–462, 2002.

[Wes87] R. Westwick. Spaces of matrices of fixed rank. Linear and Multilinear Algebra, 20(2):171–174, 1987.

[Wil82] Edward N. Wilson. Isometry groups on homogeneous nilmanifolds. Geom. Dedicata, 12(3):337–346, 1982.