LOW MACH NUMBER LIMIT FOR THE COMPRESSIBLE INERTIAL QIAN-SHENG MODEL OF LIQUID CRYSTALS: CONVERGENCE FOR CLASSICAL SOLUTIONS

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Abstract. In this paper we study the incompressible limit of the compressible inertial Qian-Sheng model for liquid crystal flow. We first derive the uniform energy estimates on the Mach number $\epsilon$ for both the compressible system and its differential system with respect to time under uniformly in $\epsilon$ small initial data. Then, based on these uniform estimates, we pass to the limit in the compressible system as $\epsilon \to 0$, so that we establish the global classical solution of the incompressible system by compactness arguments. We emphasize that, on global in time existence of the incompressible inertial Qian-Sheng model under small size of initial data, the range of our assumptions on the coefficients are significantly enlarged, comparing to the results of De Anna and Zarnescu’s work [6]. Moreover, we also obtain the convergence rates associated with $L^2$-norm with well-prepared initial data.

1. Introduction.

1.1. Compressible inertial Qian-Sheng model. In this paper, we consider the hydrodynamics of nematic liquid crystal flow of compressible inertial Qian-Sheng model, whose mathematical theory was first studied in [18] and which corresponds to the incompressible one proposed by T. Qian and P. Sheng in [28]. The model provides an extension of the classical compressible hyperbolic Ericksen-Leslie model established in [22] and particularly captures the biaxial alignment of the molecules, a feature not available in the classical compressible Ericksen-Leslie model.

In order to clearly describe the system, we introduce some terminology. The local orientation of the molecules is described through a function $Q$ taking values from $[0,1]$. The degree of orientation is measured by the function $d$, which is defined by the following expression:

$$d = \frac{1}{|Q|} \sqrt{Q^T I Q},$$

where $I$ is the identity matrix. The function $d$ ranges from 0 to 1, where 0 indicates no orientation and 1 indicates perfect orientation.

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\( \mathbb{R}^d \) \( (d = 2, 3) \) into the set of so-called \( d \)-dimensional \( Q \)-tensors, that is symmetric and traceless \( d \times d \) matrices:

\[
S_0^{(d)} := \{ Q \in \mathbb{R}^{d \times d}, \ Q_{ij} = Q_{ji}, \ \text{tr}(Q) = 0, i, j = 1, \ldots, d \}.
\]

The evolution of \( Q \) is driven by the free energy of the molecules, as well as the transport distortion and alignment effects caused by the flow. The density of the liquid crystal materials subjects to the continuity equation of mass conservation law. The bulk velocity of the centers of masses of molecules obeys a forced compressible Navier-Stokes system, with an additional stress tensor, a forcing term modeling the effect that the interaction of the molecules has on the dynamics of the center of masses. More precisely, the equation can be explicitly written as:

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla p &= \text{div}(\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4), \\
\rho \dot{Q} + \mu_1 \dot{Q} &= L \Delta Q - aQ + bQ^2 - \frac{1}{2} \|Q^2 I_d\| - cQ|Q|^2 \\
&+ \frac{\mu_2}{2} (A - \frac{1}{3} \text{div} u I_d) + \mu_1 \Omega, \\
\end{aligned}
\]

with the forms of the notations \( \Sigma_i \) \( (i = 1, 2, 3, 4) \)

\[
\begin{aligned}
\Sigma_1 &= \frac{\mu_2}{4} (\nabla u + \nabla^\top u) + \xi \text{div} u I_d, \\
\Sigma_2 &= L \left( \frac{1}{2} |Q|^2 I_d - \nabla Q \otimes \nabla Q \right) + \psi_B(Q) I_d, \\
\Sigma_3 &= \beta_1 Q \text{tr}(QA) + \beta_5 AQ + \beta_6 QA, \\
\Sigma_4 &= \frac{\mu_2}{2} (\dot{Q} - [\Omega, Q]) + \mu_1 (\dot{Q} - [\Omega, Q]).
\end{aligned}
\]

Here

\[
\begin{aligned}
\psi_B(Q) &= \frac{\sigma}{2} \text{tr}(Q^2) - \frac{1}{4} \text{tr}(Q^3) + \frac{1}{2} \left( \text{tr}(Q^2) \right)^2, \\
A &= \frac{1}{2} (\nabla u + \nabla^\top u), \quad \Omega = \frac{1}{2} (\nabla u - \nabla^\top u), \\
\dot{Q} &= \partial_t Q + u \cdot \nabla Q.
\end{aligned}
\]

The symbol \( \dot{f} = (\partial_t + u \cdot \nabla) f \) denotes the material derivative and for any two \( d \times d \) matrices \( M, N \), we denote their commutator by \([M, N] := MN - NM\). We also denote the inner product on the space of matrices as \( M : N = \text{tr}(MN^\top) \), \(|M|\) denotes the Frobenius norm of the matrix, i.e. \(|M| = \sqrt{M : M}^\top\). Furthermore, we denote \( A_{ij} := \frac{1}{2} (\partial_j u_i + \partial_i u_j), \quad \Omega_{ij} := \frac{1}{2} (\partial_j u_i - \partial_i u_j) \) for \( i, j = 1, \ldots, d \) and the Ericksen tensor \( (\nabla Q \otimes \nabla Q)_{ij} = \sum_{k=1}^d \partial_i Q_{kk} \partial_j Q_{kl} \), where we employ the notation \( \partial_i f = \frac{\partial f}{\partial x_i} \) for a scalar function \( f \). The \( I_d \) denotes the \( d \times d \) identity matrix. For simplicity, we assume that the pressure \( p \) obeys the \( \gamma \)-law, i.e. \( p(\rho) = e \rho^\gamma \) with constants \( \gamma > 1, \ e > 0 \).

The constants

\[
a > 0, \quad b, c \in \mathbb{R}
\]

are the phenomenological material constants and

\[
L > 0
\]

is the positive diffusion coefficient. The \( \beta_1, \beta_4, \beta_5, \beta_6, \xi, \mu_1, \mu_2 \) and \( \bar{\rho}_2 \) are the viscosity coefficients subject to the well-known Parodi’s relation

\[
\beta_5 - \beta_6 = \mu_2,
\]

which is derived from the Onsagar theorem \cite{7}.
Since the main concern of this paper is on the asymptotic behavior of system (1), we now have to write the system (1) into the dimensionless form. First we set the units for the different physical quantities. Let \( l_*, t_* \) and \( u_* \) be the units for (macroscopic) length, time and bulk velocity, respectively, where \( u_* = l_*/t_* \). Let \( \rho_*, s_* \) and \( \beta_* \) be the units of density, sound speed and fluid viscosity, and let \( L_* \) be the unit of the elastic constant \( L \). We also use \( \ell \) to represent the objective microscopic unit scale. To nondimensionalize system (1), we set

\[
x = l_* \tilde{x}, \quad t = t_* \tilde{t}, \quad u = u_* \tilde{u}, \quad \rho = \rho_* \tilde{\rho}, \quad p = \rho_* s_*^2 \tilde{\rho}, \quad Q = \ell \tilde{Q},
\]

\[
L = L_* \tilde{L}, \quad \beta_1 = \beta_* \tilde{\beta}_1, \quad \xi = \beta_* \tilde{\xi}, \quad \beta_1 = \frac{\beta_*}{\ell} \tilde{\beta}_1, \quad \beta_5 = \frac{\beta_*}{\ell} \tilde{\beta}_5, \quad \beta_6 = \frac{\beta_*}{\ell} \tilde{\beta}_6,
\]

\[
\mu_1 = \frac{\beta_*}{\ell} \tilde{\mu}_1, \quad \mu_2 = \frac{\beta_*}{\ell} \tilde{\mu}_2, \quad \tilde{\mu}_2 = \frac{\beta_*}{\ell} \tilde{\mu}_2, \quad a = \frac{\mu_*}{\ell^2} \tilde{a}, \quad b = \frac{\mu_*}{\ell^2} \tilde{b}, \quad c = \frac{\mu_*}{\ell^2} \tilde{c}.
\]

The above constants are called Mach number, Reynolds number, Ericksen number and the inertial constant, respectively. We remark that the inertial constant \( J \) measures the inertial effect of the liquid crystal molecules. It is usually quite small in physical experiments. By (8) and (9), one directly writes the system (1) into dimensionless form (drop all the hats):

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \text{div}(\rho u) &= 0, \\
\frac{\partial (pu)}{\partial t} + \text{div}(pu \otimes u) + \frac{1}{\ell} \nabla p &= 0, \\
\frac{J}{\ell^2} \rho \tilde{Q} + \mu_1 \tilde{Q} &= \frac{\ell}{\mu_*} \Delta \tilde{Q} - \frac{\ell}{\mu_*} \tilde{Q} + \frac{\ell}{\mu_*} (\tilde{Q}^2 - \frac{1}{2} |\tilde{Q}|^2 I_d) - \frac{\ell}{\mu_*} \tilde{Q} |\tilde{Q}|^2 \\
&+ \frac{\tilde{\mu}_2}{2} (A - \frac{1}{2} \text{div} u I_d) + \mu_1 [\Omega, \tilde{Q}].
\end{align*}
\]

Based on the dimensionless form (10), many asymptotic behaviors as the dimensionless numbers vary could be investigated. For example, as the inertial number \( J \) goes to zero, the system (10) will formally convergence to the non-inertial Qian-Sheng model. As shown in Wu-Xu-Zarnescu’s work [34] associated with the Beris-Edwards system, one could also justify the limit letting the Reynolds and Ericksen numbers go to infinity.

Since we focus on the low Mach number limit in this paper, we set the coefficients \( \text{Re} \), \( \text{Er} \) as 1, the inertial constant

\[
J > 0
\]

is fixed and set Mach number \( \text{Ma} = \varepsilon \). Then the system (10) reads as

\[
\begin{align*}
\frac{\partial \rho^*}{\partial t} + \text{div}(\rho^* u^*) &= 0, \\
\frac{\partial (\rho^* u^*)}{\partial t} + \text{div}(\rho^* u^* \otimes u^*) + \frac{1}{\ell} \nabla p(\rho^*) &= \text{div}(\Sigma^1 + \Sigma^2 + \Sigma^3 + \Sigma^4), \\
J^* \rho^* \tilde{Q}^* + \mu_1 \tilde{Q}^* &= L^* \Delta \tilde{Q}^* - a \tilde{Q}^* + b (Q^* Q^* - \frac{1}{2} |Q^*|^2 I_d) - c Q^* |Q^*|^2 \\
&+ \frac{\tilde{\mu}_2}{2} (A^* - \frac{1}{2} \text{div} u^* I_d) + \mu_1 [\Omega^*, \tilde{Q}^*],
\end{align*}
\]

where the tensors \( \Sigma^1, \Sigma^2, \Sigma^3 \) and \( \Sigma^4 \) have the same forms of \( \Sigma_1, \Sigma_2, \Sigma_3 \) and \( \Sigma_4 \) defined in (2), just replacing the unknown functions \( \rho, u \) and \( Q \) are replaced by the scaled unknown functions \( \rho^*, u^* \) and \( Q^* \), which is to emphasize that the unknown functions depend on the small Mach number \( \varepsilon > 0 \).
In this paper, the density will be perturbed around the equilibrium state \( \rho = 1 \), hence \( \rho^\varepsilon \) can be rewritten as
\[
\rho^\varepsilon = 1 + \varepsilon \phi^\varepsilon.
\]
Then the system (12) reads
\[
\begin{cases}
\partial_t \phi^\varepsilon + u^\varepsilon \cdot \nabla \phi^\varepsilon + \phi^\varepsilon \text{div}u^\varepsilon + \frac{1}{\varepsilon} \text{div}u^\varepsilon = 0, \\
\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \frac{1}{\varepsilon^2} \rho^\varepsilon \psi^\varepsilon \nabla \phi^\varepsilon = \frac{1}{\rho^\varepsilon} \text{div}(\Sigma_1^\varepsilon + \Sigma_2^\varepsilon + \Sigma_3^\varepsilon + \Sigma_4^\varepsilon), \\
J \dot{Q}^\varepsilon + \frac{\mu_2}{\rho^\varepsilon} \dot{Q}^\varepsilon = \frac{b}{\rho^\varepsilon} \Delta Q^\varepsilon - \frac{a}{\rho^\varepsilon} Q^\varepsilon + \frac{b}{\rho^\varepsilon} (Q^\varepsilon Q^\varepsilon - \frac{1}{2} |Q^\varepsilon|^2 I_d) - \frac{1}{\rho^\varepsilon} Q^\varepsilon |Q^\varepsilon|^2 \\
\quad + \frac{\mu_2}{\rho^\varepsilon} (A^\varepsilon - \frac{1}{d} \text{div}u^\varepsilon I_d) + \frac{\mu_2}{\rho^\varepsilon} [\Omega^\varepsilon, Q^\varepsilon].
\end{cases}
\] (13)

We endow with the initial conditions of the system (13) as follows:

\[
(\phi^0, u^0, Q^0, \dot{Q}^0)(0, x) = (\phi^0_0, u^0_0, Q^0_0, \dot{Q}^0_0) \in \mathbb{R} \times \mathbb{R}^d \times S_0^{(d)} \times S_0^{(d)}
\] (14)

with compatibility \( \text{div}u^0_0 = 0 \). We formally set \( \rho^\varepsilon = 1 + \varepsilon \phi^\varepsilon \rightarrow 1 \), \( u^\varepsilon \rightarrow u \) and \( Q^\varepsilon \rightarrow Q \) and initially \( \rho_0^\varepsilon = 1 + \varepsilon \phi_0^\varepsilon \rightarrow 1 \), \( u_0^\varepsilon \rightarrow u_0 \) and \( Q_0^\varepsilon \rightarrow Q_0 \) and as \( \varepsilon \rightarrow 0 \). Then \((u, Q)(0, x) = (u_0, Q_0, \dot{Q}_0) \in \mathbb{R}^d \times S_0^{(d)} \times S_0^{(d)} \), we have

\[
(\nabla \phi^\varepsilon, u^\varepsilon, Q^\varepsilon, \dot{Q}^\varepsilon)(0, x) = (\phi^0_0, u^0_0, Q^0_0, \dot{Q}^0_0) \in \mathbb{R} \times \mathbb{R}^d \times S_0^{(d)} \times S_0^{(d)}
\] (16)

with initial data

The main goal of this paper is to rigorously justify the above limit in the framework of classical solutions regime. One notices that this problem is a singular limit problem. Usually, there are two directions to deal with the singular limit problems:

1. to obtain a uniform (in small parameter \( \varepsilon \)) bounds on the solutions to the original scaled singular equations and then to extract a convergent subsequence converging to the solutions of the target (limit) equations as \( \varepsilon \rightarrow 0 \).

2. to obtain the solutions for the limiting equations and then to construct a sequence of special solutions of the original scaled singular equations for small parameter \( \varepsilon \), which is often named the Hilbert expansion method.

We remark that the first direction is usually much harder than the second way, since the uniform bounds are very difficult to be sought in most situations. In this paper, we will employ the first way to reach our goal.

1.2. Notations and main results. For the sake of convenience, we first introduce some notations throughout this paper. We denote by \( A \lesssim B \) if there exists a constant \( C > 0 \) such that \( A \leq CB \) and denote by \( A \sim B \) if there are two constants \( C_1, C_2 > 0 \), independent of \( \varepsilon \), such that \( C_1 A \leq B \leq C_2 A \). For convenience, we also denote \( L^p := L^p(\mathbb{R}^d) \) by the standard \( L^p \) space for all \( p \in [1, +\infty] \). For \( p = 2 \), we use the notation \( \langle \cdot, \cdot \rangle \) to represent the inner product on the Hilbert space \( L^2 \). For \( p \in (1, +\infty) \) and any given positive scalar function \( \phi : \mathbb{R}^d \rightarrow \mathbb{R}^+ \), we define the weighted space \( L^p_{\phi} := L^p_{\phi}(\mathbb{R}^d) \) endowed with the norm \( \| \cdot \|_{L^p_{\phi}} = \| \cdot \phi^{\frac{1}{2}} \|_{L^p} \).
For any multi-index \( k = (k_1, k_2, \cdots, k_d) \) in \( \mathbb{N}^d \), we denote the \( k^{th} \) partial derivative operator by
\[
\partial^k = \partial^k_{x_1} \partial^k_{x_2} \cdots \partial^k_{x_d}.
\]
We employ the notation \( k \leq k' \) to represent that every component of \( k \in \mathbb{N}^d \) is not greater than that of \( k' \in \mathbb{N}^d \). Moreover, \( k < k' \) means that \( k \leq k' \) and \( |k| < |k'| \), where \( |k| = k_1 + k_2 + \cdots + k_d \in \mathbb{N} \). We now define the following two Sobolev weighted-norms (with weight \( \phi \)) as
\[
\|f\|_{H^s_\phi} = \left( \sum_{|k|=0}^s \|\partial^k f\|_{L^2_\phi}^2 \right)^{1/2}, \quad \|f\|_{\dot{H}^s_\phi} = \left( \sum_{|k|=1}^s \|\partial^k f\|_{L^2_\phi}^2 \right)^{1/2}.
\]
If \( \phi \equiv 1 \), we denote by \( H^s = H^s_\phi \) and \( \dot{H}^s = \dot{H}^s_\phi \).

Besides the coefficients’ relations (4)-(5)-(6)-(7) and (11), i.e.,
\[
J > 0, \quad \gamma > 1, \quad e > 0, \quad L > 0, \quad a > 0, \quad b, c \in \mathbb{R}, \quad \beta_5 - \beta_6 = \mu_2, \quad (17)
\]
the following coefficients constraints are also required:
\[
\beta_1 \geq 0, \quad \frac{1}{2}\beta_4 + \xi \geq 0, \quad \beta_4 \geq 0, \quad \mu_1 > 0 \quad (18)
\]
and
\[
(\bar{\mu}_2 - \mu_2)^2 < 8\beta_4\mu_1, \quad (19)
\]
where the inequality (19) is equivalent to the fact: there are constants \( \delta_0, \delta_1 \in (0, 1) \) such that
\[
F(X, Y) := \frac{1}{2}\beta_4|X|^2 + \mu_1|Y|^2 + \frac{1}{2}(\bar{\mu}_2 - \mu_2)X : Y \geq \delta_0\beta_4|X|^2 + \delta_1\mu_1|Y|^2 \quad (20)
\]
for all matrices \( X, Y \in \mathbb{R}^{d \times d} \). We remark that the coefficients’ constraints (18) and (19) come from the entropy inequality satisfied by the system (1) (see Jiang-Luo-Ma-Tang’s work [18]).

Now we state our main results as follows.

**Theorem 1.1.** Let integer \( s > \frac{d}{2} + 1(d = 2, 3) \) and \( 0 < \varepsilon \leq 1 \). Assume the coefficients are forced to (17), (18) and (19). Let the initial date \((\phi_0, u_0, \tilde{Q}_0, \tilde{Q}_0) \in \mathbb{R} \times \mathbb{R}^{d} \times S^{(d)}_0 \times S^{(d)}_0 \) satisfy
\[
\|\phi_0\|_{L^\infty} \leq \frac{1}{2}, \quad \phi_0 \in H^{s}(\rho_0), \quad u_0 \in H^{s}_{\rho_0}, \quad \tilde{Q}_0 \in H^{s}_{\rho_0}, \quad Q_0 \in H^{s+1}_{\rho_0},
\]
where \( \rho_0 = 1 + \varepsilon \phi_0 \in \left[ \frac{1}{2}, \frac{3}{2} \right] \). Then there is a small \( \lambda_0 > 0 \), independent of \( \varepsilon > 0 \), such that if
\[
\mathcal{E}^\in := \|\phi_0\|_{L^\infty}^2 + \|u_0\|_{L^\infty}^2 + J\|\tilde{Q}_0\|_{L^\infty}^2 + L\|\nabla Q_0\|_{L^2}^2 + a\|Q_0\|_{L^2}^2 \leq \lambda_0 \quad (21)
\]
for all \( \varepsilon \in (0, 1) \), then the scaled system (12)-(14) admits a unique solution \((\phi^\varepsilon, u^\varepsilon, Q^\varepsilon) \) satisfying
\[
\phi^\varepsilon \in L^\infty(\mathbb{R}^+; H^{s}_{\rho^\varepsilon}(\mathbb{R}^d)), \quad u^\varepsilon \in L^\infty(\mathbb{R}^+; H^{s}_{\rho^\varepsilon}(\mathbb{R}^d)),
\]
\[
Q^\varepsilon \in L^\infty(\mathbb{R}^+; H^{s+1}_{\rho^\varepsilon}(\mathbb{R}^d)), \quad \nabla u^\varepsilon \in L^2(\mathbb{R}^+; H^s(\mathbb{R}^d)),
\]
with the following uniform energy bounds
\[
\sup_{t \geq 0} \left( \|\phi^\varepsilon\|_{L^\infty(\rho^\varepsilon)}^2 + \|u^\varepsilon\|_{L^\infty(\rho^\varepsilon)}^2 + \|\tilde{Q}^\varepsilon\|_{L^\infty(\rho^\varepsilon)}^2 + \|Q^\varepsilon\|_{L^2(\rho^\varepsilon)}^2 \right)
\]
\[
+ \int_0^\infty \|\nabla u^\varepsilon\|_{L^2}^2 dt \leq \mathcal{E}^\in, \quad (22)
\]
\[
\rho^\varepsilon(t, x) = 1 + \varepsilon \phi^\varepsilon(t, x) \sim 1 \quad \text{for all} \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \quad (23)
\]
and

\[ \| \partial_t \dot{Q}^\epsilon \|_{L^\infty (\mathbb{R}^+; H^{s-1})}^2 + \| \partial_t Q^\epsilon \|_{L^\infty (\mathbb{R}^+; H^s)}^2 \leq C_Q \]  

(24)

for some positive constant \( C_Q > 0 \) independent of \( \epsilon \).

If the following “well-prepared” initial conditions

\[ \| \nabla \phi^\epsilon_0 \|_{H^{s-2}} \leq C_0 \epsilon \quad \text{and} \quad \| \text{div} u_0^\epsilon \|_{H^{s-2}} \leq C_u \epsilon \]  

(25)

are further assumed for some \( \epsilon \)-independent positive constant \( C_0, C_u > 0 \), then

\[ \| \partial_t \dot{Q}^\epsilon \|_{L^\infty (0,T; H^{s-2}_p)}^2 + \| \partial_t u^\epsilon \|_{L^\infty (\mathbb{R}^+; H^{s-2}_p)}^2 \leq C_{\phi u}(T) \]  

(26)

and

\[ \frac{1}{2} \| \text{div} u^\epsilon \|_{L^\infty (0,T; H^{s-2})} + \frac{1}{2} \| p' (\rho^\epsilon) \nabla \phi^\epsilon \|_{L^\infty (0,T; H^{s-2})} \leq C_{\phi u}(T) \]  

(27)

hold for any fixed \( T > 0 \), where the positive constants \( C_{\phi u}(T), C_{\phi u}(T) > 0 \) are both independent of \( \epsilon \).

**Remark 1.** While setting \( \epsilon = 1 \) in the above theorem, we automatically give the global existence results in [18]. However, Theorem 1.1 here provides more precise estimates, such as the uniform in \( \epsilon \) bounds on the perturbations with size \( \epsilon \) in the first part of Theorem 1.1.

The next theorem is about the limit from the compressible inertial Qian-Sheng model to the corresponding incompressible flow.

**Theorem 1.2.** Under the same assumptions in Theorem 1.1 (including (25)), we further assume that there are functions \( u_0(x), \dot{Q}_0(x) \in H^s \) and \( Q_0(x) \in H^{s+1} \) with \( \text{div} u_0 = 0 \), such that as \( \epsilon \to 0 \),

\[ (u_0^\epsilon, \dot{Q}_0^\epsilon, \nabla Q_0^\epsilon, Q_0^\epsilon) \to (u_0, \dot{Q}_0, \nabla Q_0, Q_0) \text{ strongly in } H^s. \]  

(28)

Let \( (\rho^\epsilon, u^\epsilon, Q^\epsilon)(t,x) \) be the family of solutions to the system (13) constructed in Theorem 1.1. Then there exist functions \( u(t,x), \pi(t,x) \) and \( Q(t,x) \) with

\[ u, \dot{Q}, \nabla Q, Q \in \mathcal{L}^\infty (\mathbb{R}^+; H^s) \cap C(\mathbb{R}^+; H^{s+1}_{\text{loc}}), \]

\[ \nabla u \in L^2(\mathbb{R}^+; H^s), \ \pi \in \mathcal{L}^\infty (\mathbb{R}^+; H^{s-1}), \]

such that as \( \epsilon \to 0 \) (in the sense of subsequence),

\[ \rho^\epsilon - 1 \to 0 \text{ strongly in } L^\infty (\mathbb{R}^+; H^s), \]

\[ \frac{1}{\epsilon} \nabla p(\rho^\epsilon) - \frac{1}{2} \nabla |\nabla Q^\epsilon|^2 - \nabla \psi_B(Q^\epsilon) \to \nabla \pi \text{ weakly-* for all } t \geq 0, \]

\[ \text{weakly in } H^{s-2}, \]

and

\[ (u^\epsilon, \dot{Q}^\epsilon, \nabla Q^\epsilon, Q^\epsilon) \to (u, \dot{Q}, \nabla Q, Q) \]

weakly-* for \( t \geq 0 \), weakly in \( H^s \) and strongly in \( C(\mathbb{R}^+; H^{s+1}_{\text{loc}}) \). Here \((u, \pi, Q)\) is the solution of the limit system (15)-(16) with the global energy bound

\[
\sup_{\epsilon \geq 0} \left( \| u \|_{H^s}^2 + \| \dot{Q} \|_{H^s}^2 + \| \nabla Q \|_{H^s}^2 + \| Q \|_{H^s}^2 + \| \pi \|_{H^{s-1}}^2 \right) \\
+ \int_0^\infty \| \nabla u \|_{H^s}^2 \, dt \lesssim \lambda_0,
\]

(29)

where \( \lambda_0 \) is given in Theorem 1.1.
Remark 2. In fact, Theorem 1.2 here shows the global existence results of the incompressible inertial Qian-Sheng model (15)-(16) under small size of the initial data in the space $H^s$. We remark that the small global existence of the solution to the system (15)-(16) has been proved by F. De Anna and A. Zarnescu in [6]. However, they further required the following coefficients' constraints:

$$\beta_5 + \beta_6 = 0, \quad \tilde{\mu}_2 = -\mu_2 \quad (\text{or } \tilde{\mu}_2 = \mu_2 = 0), \quad J < J_0, \quad \mu_1 > \tilde{\mu}_1, \quad \beta_4 > \tilde{C}_d$$

(30)

for some computable positive constant $J_0 = J_0(\mu_1, a, b, c) > 0$, $\tilde{\mu}_1 = \tilde{\mu}_1(a, b, c) > 0$ and $\tilde{C}_d = \tilde{C}_d(\tilde{\mu}_2, \beta_5, \beta_6, \mu_2) > 0$. In Theorem 1.2 here, the corresponding coefficients' relations are

$$\beta_5 + \beta_6 \in \mathbb{R}, \quad \tilde{\mu}_2, \mu_2 \in \mathbb{R}, \quad 8\beta_4 \mu_1 > (\tilde{\mu}_2 - \mu_2)^2.$$  

(31)

Namely, when $\tilde{\mu}_2 \neq \mu_2$, the positive lower bound $\beta_4 \mu_1 > \frac{1}{8}(\tilde{\mu}_2 - \mu_2)^2 > 0$ is required. While $\tilde{\mu}_2 = \mu_2$, the positive lower bounds of the viscosities $\beta_1$ and $\mu_1$ in (30) are not required. One notices that employing the same arguments on the entropy inequalities and energy dissipation in [18] also can imply the small global existence results to the incompressible inertial Qian-Sheng model (15)-(16) with the coefficients' constraints (31).

The last theorem is about the convergence rate of the limit constructed in Theorem 1.2.

Theorem 1.3. Under the same assumptions of the Theorem 1.2, if we further assume that

$$\|\sqrt{\rho^0}u_0 - u_0\|_{L^2} + J\|\sqrt{\rho^0}Q_0^0 - Q_0\|_{L^2} + L\|\nabla Q_0^0 - \nabla Q_0\|_{L^2}$$

$$+ a\|Q_0^0 - Q_0\|_{L^2} + \langle \Pi^0, 1 \rangle \lesssim \varepsilon^{\theta_0}$$

(32)

for some positive constant $\theta_0$, where $\Pi^0 = \frac{1}{\gamma - 1}(\rho^0)^{\gamma} - \gamma(\rho^0 - 1) - 1$ satisfies

$$\|\sqrt{\rho^0} - 1\|^2 \lesssim |\rho^0 - 1|^21_{|\rho^0 - 1| \leq \frac{1}{2}} + |\rho^0 - 1|^\gamma 1_{|\rho^0 - 1| \geq \frac{1}{2}} \lesssim \Pi^0,$$

(33)

then, for any fixed $T > 0$,

$$\|\sqrt{\rho^T}u^T - u\|_{L^2} + \|\sqrt{\rho^T}Q^T - Q\|_{L^2} + L\|\nabla Q^T - \nabla Q\|_{L^2}$$

$$+ a\|Q^T - Q\|_{L^2} + \langle \Pi^T, 1 \rangle \leq C_T \varepsilon^{\theta_0}$$

(34)

for all $t \in (0, T)$, where $\Pi^T = \frac{1}{\gamma - 1}(\rho^T)^{\gamma} - \gamma(\rho^T - 1) - 1$, the constant $\theta_0 = \min\{2, \theta_0, 1 + \frac{\theta_0}{2}\} > 0$ and $C_T = C(1 + T)\exp(CT) > 0$ for some positive $\varepsilon$-independent constant $C > 0$.

1.3. Technical ideas and sketch of proofs. In this paper, we mainly concern with deriving the uniform in $\varepsilon$ energy bounds for the unknown functions $(\phi^0, u^0, Q^0, Q^c)$ and their time derivatives $(\partial_t \phi^0, \partial_t u^0, \partial_t Q^0, \partial_t Q^c)$ of the system (13)-(14). The key point is to control the singular terms $\frac{1}{\varepsilon}\text{div}u^c$ in the first equation of (13) and $\frac{1}{\varepsilon}p'(\rho^c)\nabla \phi^c$ in the second equation of (13) when deriving the uniform global energy bounds. In order to reach this goal, under the equality $\rho^c = 1 + \varepsilon \phi^c$, we seek the following key cancellation

$$\frac{1}{\varepsilon}(\text{div}u^c, p'(\rho^c)\phi^c) + \frac{1}{\varepsilon}(p'(\rho^c)\nabla \phi^c, \rho^c u^c) = (p''(\rho^c)\nabla \phi^c, u^c),$$

(35)

which will be employed when multiplying the first equation by $p'(\rho^c)\phi^c$ and the second equation by $\rho^c u^c$ in the system (13), respectively. Moreover, the corresponding cancellations associated with the higher order derivatives are also required.
Then, based on the local existence results to the system (13) with small initial data, we seek some more dissipative structures on the density fluctuation of $\phi^\varepsilon$ and the direction field $Q^\varepsilon$, so that we can globally extend the small local solution and obtain uniform in $\varepsilon$ global energy bounds.

We also need to prove the uniform in $\varepsilon$ bounds (24) and (26), associated with the time derivatives $\partial_t u^\varepsilon$, $\partial_t \phi^\varepsilon$, $\partial_t Q^\varepsilon$ and $\partial_t \dot{Q}^\varepsilon$. The two uniform bounds are important to obtain some strong convergences when applying the Aubin-Lions-Simon Theorem given in Lemma 3.1. The key point here is still to deal with the singularity. Noticing that the equalities $\partial_t Q^\varepsilon = \dot{Q}^\varepsilon - u^\varepsilon \cdot \nabla Q^\varepsilon$ and $\partial_t \dot{Q}^\varepsilon = \ddot{Q}^\varepsilon - u^\varepsilon \cdot \nabla \dot{Q}^\varepsilon$ do not involve singular terms, we can directly deduce (24) from the bound (22) and (23). However, the singular terms $\frac{1}{\varepsilon} \text{div} u^\varepsilon$ and $\frac{1}{\varepsilon} p'(\rho^\varepsilon) \nabla \phi^\varepsilon$ in the evolutions of $\phi^\varepsilon$ and $u^\varepsilon$, respectively, are still uncontrolled, which are canceled in proving the bounds (22). When deriving the uniform bounds of $\partial_t \phi^\varepsilon$ and $\partial_t u^\varepsilon$, we thereby need to eliminate these singular effects under the following cancellation:

$$\frac{1}{\varepsilon} \langle \text{div} \partial_t u^\varepsilon, \rho' \partial_t \phi^\varepsilon \rangle + \frac{1}{\varepsilon} \langle \rho' \nabla \partial_t \phi^\varepsilon, \rho' \partial_t u^\varepsilon \rangle = \langle p''(\rho^\varepsilon) \nabla \phi^\varepsilon, \partial_t \phi^\varepsilon, \partial_t u^\varepsilon \rangle. \quad (36)$$

Moreover, the higher order derivatives version of the above cancellation are also required. Therefore, we can derive the energy inequality (94), i.e.,

$$E(\partial_t \phi^\varepsilon, \partial_t u^\varepsilon)(t) \leq (1 + E(\partial_t \phi^\varepsilon, \partial_t u^\varepsilon)(0)) \exp(C_5 T) \quad (37)$$

for all $t \in [0, T]$, where $T > 0$ is any fixed number and the energy functional defined in (93). In order to obtain the uniform bound of $E(\partial_t \phi^\varepsilon, \partial_t u^\varepsilon)(t)$, we only need to ensure the uniform boundedness of the initial energy $E(\partial_t \phi^\varepsilon, \partial_t u^\varepsilon)(0)$. Consequently, the additional initial conditions (25) is required. Based on the uniform bounds (22), (23), (24) and (26), we then easily show the uniform bound (27) of the singular terms $\frac{1}{\varepsilon} \text{div} u^\varepsilon$ and $\frac{1}{\varepsilon} p'(\rho^\varepsilon) \nabla \phi^\varepsilon$ from the equations (13).

When proving Theorem 1.2, we employ the Aubin-Lions-Simon Theorem in Lemma 3.1 to obtain some strong convergence. With the strong convergence, we can prove that the global classical solutions of the compressible inertial Qian-Sheng mode (13) constructed in Theorem 1.1 converge to solution of incompressible equations (12). Finally, we employ the modulated energy method (motivated by [14] or [11], for instance) to justify the Theorem 1.3.

1.4. **Historical remarks.**

1.4.1. **Low Mach number limits.** In fluid (and related) models, Mach number measures the compressibility of fluids. As Mach number goes to zero, the compressible models behave asymptotically as the incompressible models. The rigorous mathematical justifications were initialized from Klainerman and Majda [20, 21] in the framework of classical solutions, which addressed the limit from compressible Euler to incompressible Euler equations. After then, there has been vast of literatures on this topic in different models (Navier-Stokes, MHD, liquid crystal models, etc.), different contexts solutions (classical solutions, weak solutions, etc.) and different domains (whole space, periodic domain, bounded domain, etc.). Among them, we mention Ukai [33] (initial layer in $\mathbb{R}^d$), Schochet [30] and Grenier [10] (fast acoustic waves on torus), Lions and Masmoudi [24] (incompressible limit for global in time weak solutions of isentropic compressible Navier-Stokes equations), Métivier and Schochet [27, 29] (incompressible limit for non-isentropic Euler equations), Alazard [2] (low Mach number limit of the full Navier-Stokes equations in $\mathbb{R}^d$), Jiang-Ju-Li [15] (incompressible limit for non-isentropic MHD equations in...
Here we describe the incompressible limit for the compressible inertial Qian-Sheng model.

1.4.2. The inertial Qian-Sheng models. The analytical study of the incompressible inertial Qian-Sheng model of liquid crystal flow started only very recently. The second order material derivative in the inertial model brought tremendous difficulties, so its analytical study is much harder than the corresponding non-inertial case. The first result in this direction was obtained by De Anna and Zarnescu [6]. They proved the global well-posedness under the assumptions that the initial data are small and the coefficients satisfy some further damping property. However, only the case \( \beta_5 + \beta_6 = 0 \) was considered. As a consequence, only small data local and global in time could be obtained. In [6], they also provided an example of twist-wave solutions, which are solutions of the coupled system for which the flow vanishes for all times. Furthermore, for the inviscid version of the inertial Qian-Sheng model, in [9], Feireisl et al. proved a global existence of the dissipative solution which is inspired from that of incompressible Euler equation defined by P.-L. Lions [23]. We also mention the recent work of the second named author of this paper on the well-posedness of non-inertial Qian-Sheng model [25]. Comparing to other \( Q \)-tensor model, such as Beris-Edwards system, one of the main difficulty of Qian-Sheng model is it is hard to find Lyapunov functional, and the energy estimate does not close in \( L^2 \) sense.

Recently, Jiang-Luo-Ma-Tang [18] proved the well-posedness of the compressible inertial Qian-Sheng model. They introduced a novel Condition (H) which ensured the energy dissipation. It was the first time that the roles played by the entropy inequality, Condition (H) in the well-posedness of the compressible inertial Qian-Sheng model (1) are illustrated. Comparing to De Anna and Zarnescu’s work [6], for local in time well-posedness, they made clarification that the large and small data were quite different. More importantly, the criterion was directly based on entropy inequality and energy dissipation: to obtain large data local in time well-posedness, both the entropy inequality and the Condition (H) must be obeyed. If any one of these two conditions was not satisfied, only small data local existence can be obtained. When both the entropy inequality and Condition (H) were broken, and furthermore \( \tilde{\mu}_2 \neq \mu_2 \), the local small local in time solutions could not be extended globally. Furthermore, in both global and local in time cases, the range of their assumptions on the coefficients were significantly enlarged, comparing to the results [6].
1.5. **Organizations of this paper.** In the next section, we derive the global energy bounds of the compressible inertial system (13) uniformly in $\varepsilon$. In Section 3, we rigorously show the limit process between (12) and (13) by employing the compactness arguments. Finally, based on Theorem 1.2, we prove the convergence rates in $L^2$ space with well-prepared initial data in Section 4.

2. **Global uniform energy bounds: proof of theorem 1.1.** In this section, we mainly aim at deriving the uniform in $\varepsilon$ global-in-time energy bounds of the system (13) with the small initial data (14), namely justifying Theorem 1.1. We further prove the uniform bounds (24) and (26), which are concerned on the derivatives $\partial_t u^\varepsilon$, $\partial_t \phi^\varepsilon$, $\partial_t Q^\varepsilon$ and $\partial_t \dot{Q}^\varepsilon$. Finally, based on the uniform bounds (22), (23), (24) and (26), we easily derive the uniform bound (27) of the singular terms $\frac{1}{\varepsilon}\text{div}u^\varepsilon$ and $\frac{1}{\varepsilon}p'(\rho^\varepsilon)\nabla\phi^\varepsilon$ from the structures of system (13).

2.1. **Preliminaries.** In this subsection, we first present some useful lemmas in order to prove the main results.

**Lemma 2.1** (Lemma 3.1 of [19]). Assume that $\rho \leq \rho^n(x) \leq \bar{\rho}$ for some constants $\rho, \bar{\rho} > 0$, and the density $\rho(t, x) \geq 0$ satisfies the following equation

$$
\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0, \\
\rho(0, x) = \rho^n,
\end{cases}
$$

for some given velocity $u(t, x)$. Then the following inequalities holds

$$
\rho \exp\left\{-\frac{1}{\varepsilon} \int_0^t \|\text{div}u\|_{L^\infty}(\tau) d\tau\right\} \leq \rho(t, x) \leq \bar{\rho} \exp\left\{\frac{1}{\varepsilon} \int_0^t \|\text{div}u\|_{L^\infty}(\tau) d\tau\right\}.
$$

**Lemma 2.2** (Lemma 3.2 of [19]). Let $f(\rho)$ be a smooth function. Then for any multi-index $k \in \mathbb{N}^d$ with $|k| \geq 1$ and $\rho \in H^{|k|} \cap L^\infty$, we have

$$
\partial^k f(\rho) = \sum_{i=1}^k f^{(i)}(\rho) \sum_{\sum_{j=1}^{|k|} \mid j_l \geq 1} \prod_{l=1}^i \partial^{k_j} \rho.
$$

In particular, when $\rho$ satisfies the assumption stated in Lemma 2.1 and $f(\rho) = \frac{1}{\rho}$, we have

$$
\|\partial^k f(\rho)\|_{L^2} \leq C(\rho, k) \exp\left\{\frac{i+1}{\varepsilon} \int_0^t \|\text{div}u\|_{L^\infty}(\tau) d\tau\right\} P_k(\|\rho\|_{H^s}) \leq Q(u^\varepsilon) P_k(\|\rho\|_{H^s}),
$$

where

$$
Q(u^\varepsilon) = \kappa_1 \exp\left\{\kappa_2 \int_0^t \|\text{div}u\|_{L^\infty}(\tau) d\tau\right\}
$$

with generic constants $\kappa_1, \kappa_2$, and $P_k(x) = \sum_{j=1}^k x^j$.

**Lemma 2.3** (Moser-type inequality, [26]). For functions $f, g \in H^s \cap L^\infty, s \in \mathbb{Z}_+ \cup \{0\}$ and for any multi-index $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d) \in \mathbb{N}^d$ and $1 \leq |\alpha| \leq s$, we have

$$
\|\partial^\alpha f \cdot g\|_{L^2} \lesssim \|\nabla f\|_{L^\infty} \|g\|_{H^s} + \|f\|_{H^s} \|g\|_{L^\infty},
$$

$$
\|\partial^\alpha (fg)\|_{L^2} \lesssim \|f\|_{L^\infty} \|g\|_{H^s} + \|f\|_{H^s} \|g\|_{L^\infty}.
$$

In particular, if $s > \frac{d}{2}$, then $fg \in H^s$ and

$$
\|fg\|_{H^s} \lesssim \|f\|_{H^s} \|g\|_{H^s}.
$$
From the similar arguments on the proof of Theorem 1.2 in [18], we obtain the following local existence results under small initial data. The readers can also be referred to Proposition 2.1 of [11].

**Proposition 1** (Small local well-posedness). Under the same coefficients’ constraints in Theorem 1.1, there is a positive numbers $\varepsilon_0$, independent of $\varepsilon$, such that if $E^{in} \leq \varepsilon_0$, then there exists a $T > 0$, independent of $\varepsilon$, such that the Cauchy problem (13)-(14) admits a unique solution $(\phi^\varepsilon, u^\varepsilon, Q^\varepsilon)$ on $[0, T]$, satisfying

$$
\begin{align*}
\phi^\varepsilon \in & L^\infty(0, T; H^s_{\rho''(\rho^\varepsilon)}(\mathbb{R}^d)), u^\varepsilon \in L^\infty(0, T; H^s_{\rho''}(\mathbb{R}^d)) \cap L^2(0, T; \dot{H}^{s+1}(\mathbb{R}^d)), \\
Q^\varepsilon \in & L^\infty(0, T; H^s_{\rho''}(\mathbb{R}^d)) \cap L^2(0, T; H^s(\mathbb{R}^d)), Q^\varepsilon \in L^\infty(0, T; H^{s+1}(\mathbb{R}^d)),
\end{align*}

$$

where $\rho^\varepsilon = 1 + \varepsilon \phi^\varepsilon$ satisfies

$$
0 < r_1 \leq \rho^\varepsilon(t, x) = 1 + \varepsilon \phi^\varepsilon(t, x) \leq r_2
$$

for all $(t, x) \in [0, T] \times \mathbb{R}^3$, $0 < \varepsilon \leq 1$ and some $r_1, r_2 > 0$, independent of $\varepsilon$.

**Remark 3.** From the bounds (40), one easily knows that for any continuous function $f : [r_1, r_2] \to \mathbb{R}^+$,

$$
\|f(\rho^\varepsilon)\|_{L^\infty(0, T; L^\infty(\mathbb{R}^3))} \leq C(r_1, r_2).
$$

2.2. A priori estimate of the system (13). In this subsection, we derive the a priori estimate of the system (13)-(14). There are two singular terms $\frac{1}{\varepsilon} \text{div} u^\varepsilon$ and $\varepsilon \frac{\rho''(\rho^\varepsilon)}{\rho^\varepsilon} \nabla \phi^\varepsilon$ occurred in the system (13) should be controlled.

We now define the following energy functional $E(t)$:

$$
E(t) = \|\phi^\varepsilon\|^2_{H^s_{\rho''}(\mathbb{R}^d)} + \|u^\varepsilon\|^2_{H^s_{\rho''}} + J\|Q^\varepsilon\|^2_{H^s} + L\|\nabla Q^\varepsilon\|^2_{H^s} + a\|Q^\varepsilon\|^2_{H^s},
$$

and the energy dissipation $D(t)$:

$$
D(t) = \|\nabla u^\varepsilon\|^2_{H^s} + \|\dot{Q}^\varepsilon\|^2_{H^s} + \beta_1 \sum_{|k|=0} \|Q^\varepsilon : \partial^k A^\varepsilon\|^2_{H^s},
$$

We first give the following lemma about the a priori estimate of the system (13)-(14).

**Lemma 2.4.** Assume $(\phi^\varepsilon, u^\varepsilon, Q^\varepsilon)$ is the solution to system (13)-(14) on $[0, T]$ given in Proposition 1. Then there exists a constant $c_0 > 0$, independent of $\varepsilon$, such that for all $t \in [0, T]$,

$$
\frac{1}{2} \frac{d}{dt} E(t) + c_0 D(t) \leq (1 + \varepsilon^2) E^\frac{1}{2}(t) D^\frac{1}{2}(t) \left( D^\frac{1}{2}(t) + A^\frac{1}{2}(t) \right),
$$

where the functional $A(t)$ is defined as

$$
A(t) = \|\phi^\varepsilon\|^2_{H^s} + \|\nabla Q^\varepsilon\|^2_{H^s} + \|Q^\varepsilon\|^2_{H^s}.
$$

**Proof.** We divide the proof into three steps.

*Step 1. $L^2$-estimate.*

First, from taking $L^2$-inner product with $p'(\rho^\varepsilon)\phi^\varepsilon$ in the first equation of system (13), we obtain

$$
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\phi^\varepsilon\|^2_{L^2_{\rho''}(\mathbb{R}^d)} + \frac{1}{\varepsilon} \langle \text{div} u^\varepsilon, p'(\rho^\varepsilon)\phi^\varepsilon \rangle - \langle \frac{1}{\varepsilon} |\phi^\varepsilon|^2, \partial_t p'(\rho^\varepsilon) \rangle \\
= -\langle u^\varepsilon \cdot \nabla \phi^\varepsilon, p'(\rho^\varepsilon)\phi^\varepsilon \rangle - \langle p'(\rho^\varepsilon) |\phi^\varepsilon|^2, \text{div} u^\varepsilon \rangle.
\end{align*}
$$
Second, taking $L^2$-inner product with $\rho' u^r$ in the second equation of system (13), we obtain

$$\frac{1}{2} \frac{d}{dt} \|u^r\|_{L^2}^2 + \frac{\alpha_2}{2} \|
abla u^r\|_{L^2}^2 + \left( \frac{1}{2} \beta_4 + \xi \right) \|\text{div} u^r\|_{L^2}^2$$

$$= - \frac{1}{\epsilon} \langle p'(\rho') \nabla \phi^r, u^r \rangle + \langle \text{div} \Sigma^2, u^r \rangle + \langle \text{div} \Sigma^3, u^r \rangle + \langle \text{div} \Sigma^4, u^r \rangle. \tag{47}$$

Finally, taking $L^2$-inner product with $\rho' \dot{Q}^r$ in the third equation of system (13), we can deduce that

$$\frac{1}{2} \frac{d}{dt} \|Q^r\|_{L^2}^2 + L \|
abla Q^r\|_{L^2}^2 + a \|Q^r\|_{L^2}^2 + \mu_4 \|Q^r\|_{L^2}^2$$

$$= L (\Delta Q^r, u^r \cdot \nabla Q^r) - a(Q^r, u^r \cdot \nabla Q^r) + b((Q^r Q^r - \frac{1}{4} |Q^r|^2 I_d), \dot{Q}^r)$$

$$- c(Q^r |Q^r|^2, \dot{Q}^r) + \frac{\alpha_2}{2} ((A^r - \frac{1}{4} \text{div} u^r I_d), \dot{Q}^r) + \mu_4 \langle [\Omega^r, Q^r], \dot{Q}^r \rangle. \tag{48}$$

We therefore deduce from combining the above equalities (46), (47) and (48) that

$$\frac{1}{2} \frac{d}{dt} \|\phi^r\|_{L^2}^2 + \frac{\alpha_2}{2} \|\nabla \phi^r\|_{L^2}^2 + \frac{\alpha_2}{2} \|\text{div} \phi^r\|_{L^2}^2$$

$$= - \frac{1}{\epsilon} \langle \text{div} u^r, p'(\rho') \phi^r \rangle - \langle p'(\rho') \nabla \phi^r, u^r \rangle$$

$$+ \langle \text{div} Q^r, u^r \cdot \nabla Q^r \rangle + L (\text{div} \frac{1}{2} |\nabla Q^r|^2 I_d - \nabla Q^r \cdot \nabla Q^r, u^r) + L (\Delta Q^r, u^r \cdot \nabla Q^r) \right.$$  

$$\left. + \langle \text{div} \Sigma^3, u^r \rangle + \frac{\alpha_2}{2} \langle \text{div} \Sigma^4, u^r \rangle - \frac{\alpha_2}{2} \langle \text{div} \Sigma^4, u^r \rangle \right) \right) \tag{49}$$

Now we compute the terms on the right-hand side of the equality (49) one by one. For the singularity term $H_1$, we have the following cancellation relations:

$$- \frac{1}{\epsilon} \langle \text{div} u^r, p'(\rho') \phi^r \rangle - \langle p'(\rho') \nabla \phi^r, u^r \rangle$$

$$= - \frac{1}{\epsilon} \langle \text{div}(u^r p'(\rho') \phi^r), 1 \rangle + \langle (\nabla(p'(\rho') \phi^r), u^r \rangle - \frac{1}{\epsilon} \langle p'(\rho') \nabla \phi^r, u^r \rangle$$

$$= - \frac{1}{\epsilon} \langle \nabla p'(\rho') \phi^r, u^r \rangle = \langle p''(\rho') \nabla \phi^r, u^r \rangle.$$

Then, by using Hölder inequality and Sobolev embedding theory, we have

$$H_1 \lesssim \|p''(\rho')\|_{L^\infty} \|\nabla \phi^r\|_{L^2} \|u^r\|_{L^4} \|\phi^r\|_{L^4}$$

$$\lesssim \|\nabla \phi^r\|_{L^2} \|\nabla u^r\|_{L^2} \|\phi^r\|_{H^1} \lesssim \mathcal{E}^\frac{1}{2}(t \mathcal{D}^\frac{1}{2}(t)) \|\nabla \phi^r\|_{L^2},$$

where we make use of

$$\|p'(\rho')\|_{L^\infty} + \|p''(\rho')\|_{L^\infty} \leq C(r_1, r_2) \tag{50}$$

derived from (41) by letting $f(y) = p'(y)$ and $f(y) = p''(y)$.

Recalling that $\partial_r \rho^r + \text{div}(\rho' u^r) = 0$, we have

$$\partial_r p'(\rho') + ep''(\rho') u^r \cdot \nabla \phi^r + p'(\rho') \rho' \text{div} u^r = 0. \tag{51}$$
We then can dominate the term \( \langle \frac{1}{2} |\phi'|^2, \partial_x p'(\rho') \rangle \) as
\[
\langle \frac{1}{2} |\phi'|^2, \partial_x p'(\rho') \rangle = -\epsilon \langle \frac{1}{2} |\phi'|^2, p''(\rho') u^\approx \cdot \nabla \phi' \rangle - \langle \frac{1}{2} |\phi'|^2, p''(\rho') \rho' \text{div} u^\approx \rangle \leq \epsilon \|p''(\rho')\|_{L^\infty} \|u^\approx\|_{L^2} \|\nabla \phi'\|_{L^2} \|\phi'\|_{L^2}^2 + \|p''(\rho')\|_{L^\infty} \|\rho'\|_{L^\infty} \|\text{div} u^\approx\|_{L^2} \|\phi'\|_{L^2}^2 \tag{52}
\]
\[
\lesssim \|\nabla u^\approx\|_{L^2} \|\nabla \phi'\|_{L^2} \|\phi'\|_{L^2}^2 + \|\phi'\|_{H^1} \lesssim \left(1 + \mathcal{E}^\frac{1}{2}(t)\right) \mathcal{E}^\frac{1}{2}(t) \mathcal{D}^\frac{1}{2}(t) \|\nabla \phi'\|_{L^2},
\]
where we make use of (40) and (50). The H"older inequality, Sobolev embedding theory and the bound (40), (50) imply that
\[
- \langle u^\approx \cdot \nabla \phi', p'(\rho') \phi' \rangle - \langle p'(\rho') |\phi'|^2, \text{div} u^\approx \rangle \leq \|p'(\rho')\|_{L^\infty} \|\nabla \phi'\|_{L^2} \|u^\approx\|_{L^2} \|\phi'\|_{L^2}^2 + \|\text{div} u^\approx\|_{L^2} \|\phi'\|_{L^2}^2 \tag{53}
\]
Combining the inequalities (52) and (53) deduces that
\[
H_2 \lesssim \|\nabla u^\approx\|_{L^2} \|\nabla \phi'\|_{L^2} \|\phi'\|_{L^2}^2 + \|\phi'\|_{H^1} \lesssim \left(1 + \mathcal{E}^\frac{1}{2}(t)\right) \mathcal{E}^\frac{1}{2}(t) \mathcal{D}^\frac{1}{2}(t) \|\nabla \phi'\|_{L^2}.
\]
As the same calculations in Section 2 of [18], one easily derives the cancellation relations \(H_3 = 0\) and
\[
H_4 - \mu_1 \|\hat{Q}'\|^{2}_{L^2} - \frac{\beta_2}{2} \|\nabla u^\approx\|^{2}_{L^2} = -\beta_1 \|Q' : A'\|^{2}_{L^2}
\]
- \( \int_{\mathbb{R}^d} F(\nabla u^\approx, \hat{Q}') dx - (\beta_5 + \beta_6) \langle A'Q', A' \rangle + \mu_2 \langle A', [\Omega', Q'] \rangle, \)
\[
\tag{54}
\]
where the functional \(F(\cdot, \cdot)\) is defined in (20). Thanks to H"older inequality and Sobolev embedding theory, we can bound the last two terms on the right-hand side of the equality (54) by
\[
- (\beta_5 + \beta_6) \langle A'Q', A' \rangle + \mu_2 \langle A', [\Omega', Q'] \rangle \lesssim \|\nabla u^\approx\|_{L^2} \|u^\approx\|_{H^1} \|Q'\|_{H^1} \leq \mathcal{E}^\frac{1}{2}(t) \mathcal{D}(t).
\]
Then we have
\[
H_4 - \mu_1 \|\hat{Q}'\|^{2}_{L^2} - \frac{\beta_2}{2} \|\nabla u^\approx\|^{2}_{L^2}
\]
\[
\leq -\beta_1 \|Q' : A'\|^{2}_{L^2} - \int_{\mathbb{R}^d} F(\nabla u^\approx, \hat{Q}') dx + C \mathcal{E}^\frac{1}{2}(t) \mathcal{D}(t).
\]
For the term \(H_5\), from the H"older inequality and Sobolev embedding theory imply that
\[
H_5 \lesssim \|\nabla Q'\|^{2}_{L^2} \|\nabla u^\approx\|_{L^2} + \|\nabla Q'\|^{2}_{L^2} \|\hat{Q}'\|_{L^2} + \|\nabla Q'\|^{2}_{L^2} \|\hat{Q}'\|_{L^2}
\]
+ (\|Q'\|^2_{L^2} + \|Q'\|^2_{L^2} + \|Q'\|^2_{L^2} + \|Q'\|^2_{H^2} + \|Q'\|^2_{H^1} \|\hat{Q}'\|_{H^1} \|\nabla u^\approx\|_{L^2}
\]
\[
+ \|\nabla u^\approx\|_{L^2} \|u^\approx\|_{H^1} \|Q'\|_{H^1} \lesssim \left(1 + \mathcal{E}^\frac{1}{2}(t)\right) \mathcal{E}^\frac{1}{2}(t) \mathcal{D}(t) + (1 + \mathcal{E}(t)) \mathcal{E}^\frac{1}{2}(t) \mathcal{D}^\frac{1}{2}(t) \|Q'\|_{H^1}.
\]
Summarizing all the previous estimate, we get
\[
\frac{1}{2} \|\phi'\|^{2}_{L^2} (\rho') + \|u^\approx\|^{2}_{L^2} + J(\|\hat{Q}'\|^{2}_{L^2} + \|\nabla Q'\|^{2}_{L^2} + a \|Q'\|^{2}_{L^2})
\]
\[
+ (\frac{\alpha_4 + \xi}{2}) \|\text{div} u^\approx\|^{2}_{L^2} + \beta_1 \|Q' : A'\|^{2}_{L^2} + \int_{\mathbb{R}^d} F(\nabla u^\approx, \hat{Q}') dx \tag{55}
\]
\[
\lesssim \left(1 + \mathcal{E}^\frac{1}{2}(t)\right) \mathcal{E}^\frac{1}{2}(t) \mathcal{D}(t) + (1 + \mathcal{E}(t)) \mathcal{E}^\frac{1}{2}(t) \mathcal{D}^\frac{1}{2}(t) \|\nabla \phi'\|_{L^2} + \|Q'\|_{H^1}.
\]
**Step 2. Higher order derivative estimates.**

For all multi-index \(k \in \mathbb{N}^d\) with \(1 \leq |k| \leq s(s > \frac{d}{2} + 1)\), we act \(\partial^k\) on the first equation of (13) and taking \(L^2\)-inner product with \(p'(\rho') \partial^k \phi^\approx\) in the resulted
equation. We then yield

\[ \frac{1}{2} \frac{d}{dt} \| \partial^k \phi' \|^2_{L^2_{t,v}} + \frac{1}{\epsilon} \langle \text{div} \partial^k u^e, \rho' (\rho') \partial^k \phi' \rangle - \left( \frac{1}{2} |\partial^k \phi'|^2, \partial_t p'(\rho') \right) \]

\[ = - \langle \partial^k (u^e \cdot \nabla \phi'), p'(\rho') \partial^k \phi' \rangle - \langle \partial^k (\phi' \text{div} u^e), p'(\rho') \partial^k \phi' \rangle. \]  

(56)

Then, applying multi-derivative operator \( \partial^k \) to the second equation of (13), and taking \( L^2 \)-inner product with \( \rho' \partial^k u^e \) in the resulted equation yield

\[ \frac{1}{2} \frac{d}{dt} \| \partial^k u^e \|^2_{L^2_{t,v}} + \frac{1}{\epsilon} \| \nabla \partial^k u^e \|^2_{L^2_{t,v}} + \left( \frac{1}{2} \beta_4 + \xi \right) \| \text{div} \partial^k u^e \|^2_{L^2_{t,v}} \]

\[ = - \langle [\partial^k, u^e \cdot \nabla] \partial^k \phi', \rho' \partial^k u^e \rangle - \frac{1}{\epsilon} \langle [\partial^k, \frac{1}{\rho} \partial_t \phi'], \rho' \partial^k u^e \rangle \]

\[ + \langle [\partial^k, \frac{1}{\rho} \text{div}] \Sigma_1, \rho' \partial^k u^e \rangle + \langle [\partial^k, \frac{1}{\rho} \text{div}] \Sigma_2, \rho' \partial^k u^e \rangle \]

\[ + \langle [\partial^k, \frac{1}{\rho} \text{div}] \Sigma_3, \rho' \partial^k u^e \rangle + \langle [\partial^k, \frac{1}{\rho} \text{div}] \Sigma_4, \rho' \partial^k u^e \rangle \]

\[ + \langle \partial^k \text{div} \Sigma_2, \partial^k u^e \rangle + \langle \partial^k \text{div} \Sigma_3, \partial^k u^e \rangle + \langle \partial^k \text{div} \Sigma_4, \partial^k u^e \rangle. \]  

(57)

Furthermore, from acting \( \partial^k \) on the third equation of (13), and taking \( L^2 \)-inner product with \( \rho' \partial^k \hat{Q}' \) in the resulted equation, we deduce that

\[ \frac{1}{2} \frac{d}{dt} \| \partial^k \hat{Q}' \|^2_{L^2_{t,v}} + L \| \nabla \partial^k \hat{Q}' \|^2 + a \| \partial^k \hat{Q}' \|^2 \]

\[ = - J \langle [\partial^k, u^e \cdot \nabla] \hat{Q}', \rho' \partial^k \hat{Q}' \rangle - \mu_1 \langle [\partial^k, \frac{1}{\rho} \hat{Q}'], \rho' \partial^k \hat{Q}' \rangle \]

\[ + J \langle [\Delta \partial^k Q', \partial^k (u^e \cdot \nabla \hat{Q}')] + L \langle [\partial^k, \frac{1}{\rho} \Delta \hat{Q}', \rho' \partial^k \hat{Q}' \rangle \]

\[ - a \langle \partial^k \hat{Q}', \partial^k (u^e \cdot \nabla \hat{Q}') \rangle - a \langle [\partial^k, \frac{1}{\rho} \hat{Q}'], \rho' \partial^k \hat{Q}' \rangle \]

\[ + b \langle [\partial^k, \frac{1}{\rho} (Q' \hat{Q}' - \frac{1}{4} |Q'|^2 I_d)], \rho' \partial^k \hat{Q}' \rangle \]

\[ + \frac{1}{\epsilon} \langle [\partial^k, \frac{1}{\rho} (A^e - \frac{1}{d} \text{div} u^e I_d)], \rho' \partial^k \hat{Q}' \rangle \]

\[ + \mu_1 \langle [\partial^k, \frac{1}{\rho} ([\Omega^e, \hat{Q}']), \rho' \partial^k \hat{Q}' \rangle \]

\[ + \frac{1}{2 \epsilon} \langle \partial^k (\frac{1}{\rho} (Q' \hat{Q}' - \frac{1}{4} |Q'|^2 I_d)), \partial^k \hat{Q}' \rangle \]

\[ + \mu_1 \langle [\partial^k, \frac{1}{\rho} ([\Omega^e, \hat{Q}']), \rho' \partial^k \hat{Q}' \rangle. \]  

(58)

Therefore, for all \( 1 \leq |k| \leq s \), combining the above equalities (56), (57) and (58) leads to

\[ \frac{1}{2} \frac{d}{dt} \left( \| \partial^k \phi' \|^2_{L^2_{t,v}} + \| \partial^k u^e \|^2_{L^2_{t,v}} + J \| \partial^k \hat{Q}' \|^2_{L^2_{t,v}} + L \| \nabla \partial^k \hat{Q}' \|^2 + a \| \partial^k \hat{Q}' \|^2 \right) \]

\[ + \frac{1}{\epsilon} \| \nabla \partial^k u^e \|^2_{L^2_{t,v}} + J_1 \left( \| \partial^k \phi' \|^2_{L^2_{t,v}} + \| \partial^k u^e \|^2_{L^2_{t,v}} + J \| \partial^k \hat{Q}' \|^2_{L^2_{t,v}} + L \| \nabla \partial^k \hat{Q}' \|^2 + a \| \partial^k \hat{Q}' \|^2 \right) \]

\[ = \mathcal{I} + \mathcal{J}, \]  

(59)

where

\[ \mathcal{I} = - \frac{1}{\epsilon} \langle \text{div} \partial^k u^e, p'(\rho') \partial^k \phi' \rangle - \frac{1}{\epsilon} \langle \partial^k \left( \frac{1}{\rho} \hat{Q}' \right) \nabla \phi', \rho' \partial^k u^e \rangle \]

\[ + \langle \partial^k \text{div} \Sigma_5, \partial^k u^e \rangle + \frac{1}{\epsilon} \langle \text{div} \partial^k \hat{Q}', \partial^k u^e \rangle \]

\[ + \frac{1}{\epsilon} \langle \text{div} \partial^k ([\Omega^e, \hat{Q}']), \partial^k u^e \rangle + \frac{1}{\epsilon} \langle \partial^k (A^e - \frac{1}{d} \text{div} u^e I_d), \partial^k \hat{Q}' \rangle \]

\[ - \frac{1}{\epsilon} \langle \text{div} \partial^k ([\Omega^e, \hat{Q}']), \partial^k u^e \rangle + \frac{1}{\epsilon} \langle \partial^k (A^e - \frac{1}{d} \text{div} u^e I_d), \partial^k \hat{Q}' \rangle \]  

\[ \left\{ \right\} \mathcal{I}_3 \]
We remark that the last formally singular term in (60) is not a real singularity, due to the following key cancellation relation, namely,

\[
\mathcal{J} = \langle \frac{1}{2} |\partial^k \phi|^2, \partial^k u'(\rho') \rangle - \langle \partial^k (u^e \cdot \nabla \phi^e), \rho'(\rho') \partial^k \phi^e \rangle - \langle \partial^k (\phi^e \text{div} u^e), \rho'(\rho') \partial^k \phi^e \rangle - \langle \partial^k (\rho^e \partial^k \phi^e), \rho'(\rho') \partial^k \phi^e \rangle - \langle \partial^k (\rho^e \partial^k \phi^e), \rho'(\rho') \partial^k \phi^e \rangle - \langle \partial^k (\rho^e \partial^k \phi^e), \rho'(\rho') \partial^k \phi^e \rangle
\]

We now turn to deal with the terms in (50) and (51). We derived that

\[
\mathcal{J} = \langle \frac{1}{2} |\partial^k \phi|^2, \partial^k u'(\rho') \rangle - \langle \partial^k (u^e \cdot \nabla \phi^e), \rho'(\rho') \partial^k \phi^e \rangle - \langle \partial^k (\phi^e \text{div} u^e), \rho'(\rho') \partial^k \phi^e \rangle - \langle \partial^k (\rho^e \partial^k \phi^e), \rho'(\rho') \partial^k \phi^e \rangle - \langle \partial^k (\rho^e \partial^k \phi^e), \rho'(\rho') \partial^k \phi^e \rangle - \langle \partial^k (\rho^e \partial^k \phi^e), \rho'(\rho') \partial^k \phi^e \rangle
\]

We remark that the last formally singular term in (60) is not a real singularity, due to the commutator operator \( [\partial^k, \frac{\rho'}{\rho} \nabla] \) will generate a small coefficient \( \epsilon \). From employing the Hölder inequality, Sobolev embedding theory the bounds (40) and (50), we derived that

\[
I_1 \lesssim -\frac{1}{2} \langle \partial^k (\frac{\rho'}{\rho} \nabla \phi^e), \rho' \partial^k \phi^e \rangle - \langle \partial^k (\frac{\rho'}{\rho} \nabla \phi^e), \rho' \partial^k \phi^e \rangle - \langle \partial^k (\frac{\rho'}{\rho} \nabla \phi^e), \rho' \partial^k \phi^e \rangle - \langle \partial^k (\frac{\rho'}{\rho} \nabla \phi^e), \rho' \partial^k \phi^e \rangle - \langle \partial^k (\frac{\rho'}{\rho} \nabla \phi^e), \rho' \partial^k \phi^e \rangle - \langle \partial^k (\frac{\rho'}{\rho} \nabla \phi^e), \rho' \partial^k \phi^e \rangle
\]

Moreover, it follows from the similar estimates \( I_3 \) and \( I_4 \) in [18] that

\[
I_2 \lesssim \| \nabla \phi^e \|_{L^2} \| \nabla u^e \|_{H^s} \lesssim (1 + \mathcal{E}_I(t) \mathcal{D}_{\frac{1}{2}}(t)) \| \nabla \phi^e \|_{H^s},
\]

and

\[
I_3 - \mu_1 \| \partial^k \phi^e \|_{L^2}^2 - \frac{1}{2} \| \nabla \partial^k \phi^e \|_{L^2}^2 
\]

\[
\leq - \beta_1 \| \partial^k \phi^e \|_{L^2}^2 - \int_{\mathbb{R}^d} F(\nabla \partial^k u^e, \partial^k \phi^e) dx + C \| \nabla u^e \|_{H^s} \| \phi^e \|_{H^s}
\]

\[
\leq - \beta_1 \| \partial^k \phi^e \|_{L^2}^2 - \int_{\mathbb{R}^d} F(\nabla \partial^k u^e, \partial^k \phi^e) dx + C (1 + \mathcal{E}_I(t) \mathcal{D}_{\frac{1}{2}}(t)) \| \nabla \phi^e \|_{H^s} \| \nabla u^e \|_{H^s}.
\]
for all $0 < \epsilon \leq 1$ and $|k| \leq s$. As a result, we have

$$I \leq -\beta_1 \|Q' \cdot \partial^k A'\|^2_{L_2} - \int_{\mathbb{R}^d} F(\nabla \partial^k u', \partial^k \hat{Q}') dx + \mu_1 \|\partial^k \hat{Q}'\|^2_{L_2} + 2\epsilon \|\nabla \partial^k u'\|^2_{L_2} + C(1 + \mathcal{E}^{\frac{\epsilon}{2}}(t))\mathcal{E}^{\frac{1}{2}}(t)$$

(61)

\[ \times D^{\frac{1}{2}}(t)(\|\partial^k \phi\|_{H^s} + \|\nabla Q'\|_{H^s} + D^{\frac{1}{2}}(t)). \]

We next turn to deal with $J$ in (59). From the similar estimate (52), it is easy to get that

$$J_1 = -\frac{\epsilon}{2}(p''(\rho') u \cdot \nabla \phi', |\partial^k \phi'|^2) - \frac{\epsilon}{2}(p''(\rho') \rho \div u', |\partial^k \phi'|^2)$$

$$\lesssim \|p''(\rho')\|_{L^\infty}(1 + \|\rho''\|_{L^\infty})\|u\|_{L^\infty}\|\nabla \phi'\|_{L^\infty} + \|\div u'\|_{L^\infty})\|\partial^k \phi'\|^2_{L_2}$$

(62)

holds for all $0 < \epsilon \leq 1$ and for all $|k| \leq s$. Thanks to Lemma 2.3, Hölder inequality and Sobolev embedding theory, we imply that for $|k| \leq s$

$$J_2 = -\|u'' \cdot \nabla \phi', p''(\rho') \partial^k \phi'\|_{L^2} - \|p''(\rho') \partial^k \phi'\|_{L^2} + \|u'' \cdot \nabla \phi', p''(\rho') \partial^k \phi'\|_{L^2}$$

$$\lesssim \|u''\|_{L^2} + \|\nabla \phi'\|_{H^s} + \|\partial^k \phi'\|_{L_2}$$

(63)

and

$$J_3 \lesssim \|\partial^k \phi' \div u'\|_{L^2} + \|p''(\rho')\|_{L^2} \|\partial^k \phi'\|_{L^2} + \|\nabla u'\|_{H^s} \lesssim \mathcal{E}^{\frac{1}{2}}(t)D^{\frac{1}{2}}(t)\|\partial^k \phi'\|_{H^s}$$

(64)

hold for all $0 < \epsilon \leq 1$.

From employing the same technical arguments and process in deriving the a priori estimates in Lemma 2.4 that

$$J = \sum_{i=1}^{3} J_i - J_{21}$$

$$\lesssim \|Q'\|_{H^s} + \|\nabla Q'\|_{H^s} + \|Q'\|_{H^s} + \|\nabla Q'\|_{H^s} + \|Q'\|_{H^s} + \|Q'\|_{H^s} + \|Q'\|_{H^s} + \|\nabla Q'\|_{H^s}$$

$$\times \|\nabla u'\|_{H^s} + \|\nabla \phi'\|_{H^s} + \|\nabla \phi'\|_{H^s} + \|\nabla \phi'\|_{H^s} + \|\nabla \phi'\|_{H^s} + \|\nabla \phi'\|_{H^s} + \|\nabla \phi'\|_{H^s}$$

$$\lesssim (1 + \mathcal{E}^{\frac{\epsilon}{2}}(t))\mathcal{E}^{\frac{1}{2}}(t)D^{\frac{1}{2}}(t) + \|\partial^k \phi'\|_{H^s} + \|Q'\|_{H^s} + \|\nabla Q'\|_{H^s}.$$  

(65)

For simplicity, we omit the details here.
Moreover, we have
\[ -\mu_1 \langle \mathrm{div}^k ([Q^c, [\Omega^c, Q^c]]), \partial^k u^c \rangle = -\mu_1 \langle \mathrm{div}([Q^c, [\partial^k \Omega^c, Q^c]]), \partial^k u^c \rangle \]
\[ -\mu_1 \sum_{m_1 + m_2 + m_3 = k \atop 0 \leq m_2 \leq k - 1} \langle \mathrm{div}([\partial^{m_1} Q^c, [\partial^{m_2} \Omega^c, \partial^{m_3} Q^c]]), \partial^k u^c \rangle \]
As to the first term of the right hand of \( J_{21} \), we take advantage of the Hölder inequality and the Sobolev embedding inequality to get that
\[ -\mu_1 \langle \mathrm{div}([Q^c, [\partial^k \Omega^c, Q^c]]), \partial^k u^c \rangle \lesssim \| \nabla u^c \|_{H^r}^2 \| Q^c \|_{H^r}^2. \]
Moreover, we have
\[ -\mu_1 \sum_{m_1 + m_2 + m_3 = k \atop 0 \leq m_2 \leq k - 1} \langle \mathrm{div}([\partial^{m_1} Q^c, [\partial^{m_2} \Omega^c, \partial^{m_3} Q^c]]), \partial^k u^c \rangle \lesssim \| \partial^k Q^c \|_{L^2} \| Q^c \|_{L^\infty} \| \nabla u^c \|_{L^\infty} \| \nabla \partial^k u^c \|_{L^2} \]
\[ + \sum_{m_1 + m_2 + m_3 = k \atop 1 \leq m_2, m_3 \leq k - 1} \| \partial^{m_1} Q^c \|_{L^\infty} \| \nabla \partial^{m_2} \Omega^c \|_{L^2} \| \partial^{m_3} Q^c \|_{L^\infty} \| \nabla \partial^k u^c \|_{L^2} \]
\[ \lesssim \| \nabla u^c \|_{H^r}^2 \| Q^c \|_{H^r}^2 + \| \nabla Q^c \|_{H^r}^2 \lesssim \mathcal{E}(t) \mathcal{D}(t). \]
We therefore have
\[ J_{21} \lesssim \mathcal{E}(t) \mathcal{D}(t). \] (66)

From plugging the inequalities (61), (62), (63), (64), (65) and (66) into (59), summing up for all \( 1 \leq |k| \leq s \) and combining the \( L^2 \)-estimate (55), we deduce that
\[ \frac{1}{2} \frac{d}{dt} \mathcal{E}(t) + \beta_1 \sum_{|k|=0}^{s} \| Q^c : \partial^k A^c \|_{L^2}^2 + \sum_{|k|=0}^{s} \int_{\mathbb{R}^d} F(\nabla \partial^k u^c, \partial^k \dot{Q}^c) dx \]
\[ + (\frac{1}{2} \beta_4 + \xi) \| \mathrm{div} u^c \|_{H^r}^2 \]
\[ \lesssim (1 + \mathcal{E}^{\frac{a+2}{2}}(t)) \mathcal{E}^{\frac{1}{2}}(t) \mathcal{D}^{\frac{1}{2}}(t) \mathcal{D}^{\frac{1}{2}}(t) + \| \phi^c \|_{H^r} + \| Q^c \|_{H^r} + \| \nabla Q^c \|_{H^r}, \]
where the Young’s inequality is utilized. Since (19) holds, namely, (20) holds, there are two constants \( \delta_0, \delta_1 \in (0, 1) \) such that
\[ \sum_{|k|=0}^{s} \int_{\mathbb{R}^d} F(\nabla \partial^m u^c, \partial^k \dot{Q}^c) dx \geq \delta_0 \frac{1}{2} \beta_4 \| \nabla u^c \|_{H^r}^2 + \delta_1 \mu_1 \| \dot{Q}^c \|_{H^r}^2. \]
Then there is a constant \( c_0 > 0 \) such that
\[ \beta_1 \sum_{|k|=0}^{s} \| Q^c : \partial^k A^c \|_{L^2}^2 + \sum_{|k|=0}^{s} \int_{\mathbb{R}^d} F(\nabla \partial^k u^c, \partial^k \dot{Q}^c) dx \]
\[ + (\frac{1}{2} \beta_4 + \xi) \| \mathrm{div} u^c \|_{H^r}^2 \geq c_0 \mathcal{D}(t). \] (68)
We therefore deduce from the inequalities (67) and (68) that
\[ \frac{1}{2} \frac{d}{dt} \mathcal{E}(t) + c_0 \mathcal{D}(t) \lesssim (1 + \mathcal{E}^{\frac{a+2}{2}}(t)) \mathcal{E}^{\frac{1}{2}}(t) \mathcal{D}^{\frac{1}{2}}(t) \mathcal{D}^{\frac{1}{2}}(t) \]
\[ + \| \phi^c \|_{H^r} + \| Q^c \|_{H^r} + \| \nabla Q^c \|_{H^r}. \]
Consequently, the proof of Lemma 2.4 is completed. \( \Box \)
One notices that the bound (44) in Lemma 2.4 is not enough to prove the global energy bounds uniformly in Theorem 1.1. The more dissipative structures should be sought in the system (13). More precisely, we need to construct the dissipations on $\phi^r$ and $Q^r$.

For some fixed constant $\eta > 0$ to be determined, we first introduce the following energy functional

$$E_{\eta}(t) = \mathcal{E}(t) + \eta \|u^r + \nabla \phi^r\|_{H_{t=1}}^2 + \eta J\|\dot{Q}^r + Q^r\|_{H_{t=1}}^2,$$

and the energy dissipative rate

$$\mathcal{D}(t) = c_0 \mathcal{D}(t) - 2C\eta (\|\nabla u\|^2_{H^r} + \|\dot{Q}^r\|^2_{H^r}) + \frac{\eta}{2}\|\phi^r\|^2_{H^r} + \frac{\eta}{2}a\|Q\|^2_{H^r},$$

where $\mathcal{E}(t)$ and $\mathcal{D}(t)$ are defined in (42) and (43), respectively, and $c_0 > 0$ is given in Lemma 2.4.

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For some fixed constant $\eta > 0$ to be determined, we first introduce the following energy functional

$$E_{\eta}(t) = \mathcal{E}(t) + \eta \|u^r + \nabla \phi^r\|_{H_{t=1}}^2 + \eta J\|\dot{Q}^r + Q^r\|_{H_{t=1}}^2,$$

and the energy dissipative rate

$$\mathcal{D}(t) = c_0 \mathcal{D}(t) - 2C\eta (\|\nabla u\|^2_{H^r} + \|\dot{Q}^r\|^2_{H^r}) + \frac{\eta}{2}\|\phi^r\|^2_{H^r} + \frac{\eta}{2}a\|Q\|^2_{H^r},$$

where $\mathcal{E}(t)$ and $\mathcal{D}(t)$ are defined in (42) and (43), respectively, and $c_0 > 0$ is given in Lemma 2.4.

One notices that the energy $E_{\eta}$ and the energy dissipative rate $\mathcal{D}(t)$ may not be nonnegative for all $\eta > 0$. However, if the positive constant $\eta$ is sufficiently small, the functionals $E_{\eta}$ and $\mathcal{E}(t)$ will be both nonnegative. More precisely, we derive the following lemma.

**Lemma 2.5.** There exists a small constant $\eta_0 > 0$, independent of $\varepsilon \in (0, 1]$, such that the energy $E_{\eta_0}(t)$ and the energy dissipative rate $\mathcal{D}(t)$ are both nonnegative. Moreover, we have

$$E_{\eta_0}(t) \sim \mathcal{E}(t),$$

and

$$\mathcal{D}(t) \sim \mathcal{D}(t) + \eta \|\phi^r\|^2_{H^r} + ||\nabla Q||^2_{H^r} + ||Q||^2_{H^r}.$$
We then give the following lemma to construct the global solution to (13), of which main aim is to seek two further dissipative structures on the perturbed density \( \phi' \) and the order parameter \( Q' \).

**Lemma 2.6.** Assume \((\phi', u', Q')\) is the solution to system (13)-(14) on \([0,T]\) given in Proposition 1. Then there is \( C > 0 \), independent of \( \epsilon \), such that for all \( \epsilon \in (0,1] \) and \( t \in [0,T] \),

\[
\frac{1}{2} \frac{d}{dt} \mathcal{E}_0(t) + \mathcal{P}_0(t) \leq C \left( 1 + \mathcal{E}_{\eta_0}^{\varepsilon} (t) \right) \mathcal{E}_0^{\varepsilon} (t) \mathcal{P}_0 (t),
\]

where \( \eta_0 \) is given in Lemma 2.5.

**Proof.** We divide the proof into three steps.

Step 1. The dissipation of \( \phi' \).

Since \( p'(\rho') > 0 \), we can use the term \( \frac{E'(\rho')}{\rho''} \nabla \phi' \) in the \( u' \)-equation of (13) to construct the dissipation of \( \phi' \). For all \( 0 \leq |k| \leq s - 1 \), acting \( \partial^k \) on the second equation of (13) and taking \( L^2 \)-inner product with \( \epsilon \nabla \partial^k \phi' \), we obtain

\[
e \langle \partial_t \partial^k u', \nabla \partial^k \phi' \rangle + \langle \frac{E'(\rho')}{\rho''} \nabla \partial^k \phi', \nabla \partial^k \phi' \rangle
\]

\[
= -e \langle \partial^k (u' \cdot \nabla u'), \nabla \partial^k \phi' \rangle - \langle \partial^k (\frac{\rho'(\rho')}{\rho''}) \nabla \phi', \nabla \partial^k \phi' \rangle
\]

\[
+ \langle \partial^k \left( \frac{1}{\rho''} \text{div} \Sigma^k \right), \nabla \partial^k \phi' \rangle + \langle \partial^k \left( \frac{1}{\rho''} \text{div} \Sigma^k \right), \nabla \partial^k \phi' \rangle
\]

\[
+ \langle \partial^k \left( \frac{1}{\rho''} \text{div} \Sigma^k \right), \nabla \partial^k \phi' \rangle + \langle \partial^k \left( \frac{1}{\rho''} \text{div} \Sigma^k \right), \nabla \partial^k \phi' \rangle
\]

\[
(73)
\]

For the first term of the left-hand side of (73), it is easy to get by using the first equation of (13) that

\[
e \langle \partial_t \partial^k u', \nabla \partial^k \phi' \rangle = \frac{d}{dt} \langle \partial^k u', \epsilon \nabla \partial^k \phi' \rangle - \langle \partial^k u', \epsilon \nabla \partial^k \partial_t \phi' \rangle
\]

\[
= \frac{d}{dt} \langle \partial^k u', \epsilon \nabla \partial^k \phi' \rangle - \langle \text{div} \partial^k u', \epsilon \nabla \partial^k (u' \cdot \nabla \phi') \rangle - \langle \text{div} \partial^k u', \epsilon \nabla \partial^k (\phi' \text{div} u') + \text{div} \partial^k u' \rangle.
\]

Direct computation tells us that

\[
-\langle \text{div} \partial^k u', \epsilon \nabla \partial^k (u' \cdot \nabla \phi') \rangle \lesssim \| \nabla u' \|_{H^s} \| \nabla \phi' \|_{H^{s-1}} \| u' \|_{H^{s}}
\]

\[
\lesssim \| \nabla u' \|_{H^s} \| \nabla \phi' \|_{H^{s-1}} \| u' \|_{H^{s}} \lesssim \mathcal{E}_{\eta_0}^{\varepsilon} (t) \mathcal{P}_0 (t),
\]

and

\[
-\langle \text{div} \partial^k u', \epsilon \nabla \partial^k (\phi' \text{div} u') \rangle + \text{div} \partial^k u' \lesssim \| \nabla u' \|_{H^s}^2 + \| \nabla u' \|_{H^{s-1}}^2 \| \phi' \|_{H^{s-1} (\rho'' \rho'')} \]

\[
\lesssim \| \nabla u' \|_{H^s}^2 + \mathcal{E}_{\eta_0}^{\varepsilon} (t) \mathcal{P}_0 (t).
\]

Consequently, we have

\[
e \langle \partial_t \partial^k u', \nabla \partial^k \phi' \rangle \geq \frac{d}{dt} \langle \partial^k u', \epsilon \nabla \partial^k \phi' \rangle - C \| \nabla u' \|_{H^s}^2 - C \mathcal{E}_{\eta_0}^{\varepsilon} (t) \mathcal{P}_0 (t)
\]

\[
(74)
\]

for some constant \( C > 0 \) and for all \( 0 < \epsilon \leq 1 \).

We now estimate terms \( R_i (1 \leq i \leq 6) \) in (73). By using Hölder inequality, Sobolev embedding theory and Moser-type inequality in Lemma 2.3, we can infer
that
\[
R_1 \lesssim \sum_{m_1 + m_2 = k} \|\partial^{m_1} u^*\|_{L^4} \|\nabla \partial^{m_2} u^*\|_{L^4} \|\nabla \partial^k \phi^*\|_{L^2}
\]
\[
\leq \|u^*\|_{H^1_{\rho}} \|\nabla u^*\|_{H^s} \|\nabla \phi^*\|_{H^{s-1}_{\rho}(\rho^{1/2})} \lesssim \varepsilon_{h_0}^{\frac{1}{2}}(t) \mathcal{Q}_{h_0}(t),
\]
(75)
and
\[
R_2 \lesssim (\|\nabla (p^{(\rho^*)}/\rho^*\|_{L^\infty} \|\partial^k \phi^*\|_{L^2} + \|\partial^k (p^{(\rho^*)}/p^*)\|_{L^2} \|\nabla \phi^*\|_{L^\infty}) \|\nabla \partial^k \phi^*\|_{L^2}
\]
\[
\lesssim (\|\phi^*\|_{H^s} + \|\phi^*\|_{H^{s-1}}^2) \|\nabla \phi^*\|_{H^{s-1}_{\rho}(\rho^{1/2})} \lesssim (1 + \varepsilon_{h_0}^{\frac{\alpha+2}{2}}(t)) \varepsilon_{h_0}^{\frac{1}{2}}(t) \mathcal{Q}_{h_0}(t)
\]
(76)
for all $0 < \varepsilon \leq 1$.

For the terms $R_3, R_4, R_5, R_6$, it follows the standard estimates in the previous that
\[
R_3 \leq C(1 + \|\phi^*\|_{H^1_{\rho}}^{s-1}) \|\nabla u^*\|_{H^s} \|\nabla \phi^*\|_{H^{s-1}_{\rho}(\rho^{1/2})}^2
\]
\[
\leq \frac{1}{2} \|\nabla \phi^*\|_{H^{s-1}_{\rho}(\rho^{1/2})}^2 + C \|\nabla u^*\|_{H^s} + C(1 + \varepsilon_{h_0}^{\frac{s-2}{2}}(t)) \varepsilon_{h_0}^{\frac{1}{2}}(t) \mathcal{Q}_{h_0}(t),
\]
(77)
and
\[
R_4 \lesssim (\|\nabla Q^r\|_{H^s}^2 + \|\nabla Q^r\|_{H^s}^2 + \|\nabla Q^s\|_{H^s}^2 + \|Q^r\|_{H^s}) \|\nabla \phi^*\|_{H^{s-1}_{\rho}(\rho^{1/2})}^2
\]
\[
+ (\|\phi^*\|_{H^s} + \|\phi^*\|_{H^{s-1}}^2) (\|\nabla Q^r\|_{H^s}^2 + \|\nabla Q^s\|_{H^s}^2 + \|Q^s\|_{H^s}^2 + \|Q^r\|_{H^s}^2)
\]
\[
\lesssim (1 + \varepsilon_{h_0}^{\frac{s-1}{2}}(t)) \varepsilon_{h_0}^{\frac{1}{2}}(t) \mathcal{Q}_{h_0}(t),
\]
(78)
and
\[
R_5 \leq (1 + \|\phi^*\|_{H^1_{\rho}}^{s-1}) \|\nabla u^*\|_{H^s} \|\nabla \phi^*\|_{H^{s-1}_{\rho}(\rho^{1/2})}^2
\]
\[
\times (\|Q^r\|_{H^s} + \|\nabla Q^r\|_{H^s} + \|Q^r\|_{H^{s+1}}^2)
\]
\[
\lesssim (1 + \varepsilon_{h_0}^{\frac{s}{2}}(t)) \varepsilon_{h_0}^{\frac{1}{2}}(t) \mathcal{Q}_{h_0}(t),
\]
(79)
and
\[
R_6 \leq C(\|Q^r\|_{H^s} \|\nabla \phi^*\|_{H^{s-1}_{\rho}(\rho^{1/2})} + C \|\nabla u^*\|_{H^s} \|\nabla \phi^*\|_{H^{s-1}_{\rho}(\rho^{1/2})}^2
\]
\[
\times (\|Q^r\|_{H^s} + \|\nabla Q^r\|_{H^s} + \|Q^r\|_{H^{s+1}}^2 + \|\nabla Q^r\|_{H^s}^2)
\]
\[
+ C(\|Q^r\|_{H^s} \|Q^r\|_{H^s} + \|\nabla Q^r\|_{H^s}) \|\nabla \phi^*\|_{H^{s-1}_{\rho}(\rho^{1/2})}^2
\]
\[
+ C(\|\phi^*\|_{H^s} + \|\phi^*\|_{H^{s-1}}^2) \left( \|Q^r\|_{H^s} + \|Q^r\|_{H^s} \|Q^r\|_{H^s} + \|\nabla Q^r\|_{H^s} \|Q^r\|_{H^s} \right)
\]
\[
\lesssim (1 + \varepsilon_{h_0}^{\frac{s}{2}}(t)) \varepsilon_{h_0}^{\frac{1}{2}}(t) \mathcal{Q}_{h_0}(t),
\]
(80)
for all $0 < \varepsilon \leq 1$, where the constant $C > 0$ is independent of $\varepsilon$.

Plugging the bounds (74), (75), (76), (77), (78), (79) and (80) into the equality (73), and summing up for all $|k| \leq s - 1$, we obtain
\[
\left[ \frac{1}{2} \frac{d}{dt} (\varepsilon \|u^*\|_{H^{s-1}}^2 - \varepsilon \|u^*\|_{H^s}^2 - \varepsilon \|\nabla \phi^*\|_{H^{s-1}}^2 - \varepsilon \|\nabla \phi^*\|_{H^s}^2) \right]
\]
\[
+ \frac{1}{2} \|\nabla \phi^*\|_{H^{s-1}_{\rho}(\rho^{1/2})}^2 - C(\|\nabla u^*\|_{H^s}^2 + \|\dot{Q}^r\|_{H^s}^2)
\]
\[
\leq (1 + \varepsilon_{h_0}^{\frac{s+2}{2}}(t)) \varepsilon_{h_0}^{\frac{1}{2}}(t) \mathcal{Q}_{h_0}(t)
\]
(81)
for some positive constant $C > 0$ and for all $0 < \epsilon \leq 1$.

**Step 2. The dissipation of $Q^r$.**

Employing the similar as the arguments (2.61) in step 1 of proof of Proposition 2.2 in [11], we can find a dissipative structure of $Q^r$. We only sketch the process of derivations. More precisely, for all $0 \leq |k| \leq s$, acting $\partial^k$ on the third equation of the system (13), and taking $L^2$-inner product with $\partial^k Q^r$ yield that

$$
\frac{1}{2} \frac{d}{dt} (J\|\partial^k \dot{Q}^r + \partial^k Q^r\|^2_{\mathcal{H}^*} - J\|\partial^k \dot{Q}^r\|^2_{\mathcal{H}^*} - J\|\partial^k Q^r\|^2_{\mathcal{H}^*}) + L||\nabla \partial^k Q^r\|^2_{\mathcal{H}^*} + \frac{a}{2} \|\partial^k Q^r\|^2_{\mathcal{H}^*} - J\|\partial^k \dot{Q}^r\|^2_{\mathcal{H}^*}
$$

$$
= - J\langle \partial^k (u^r \cdot \nabla Q^r), \partial^k \dot{Q}^r \rangle - J\langle \partial^k (u^r \cdot \nabla \dot{Q}^r), \partial^k Q^r \rangle - \mu_1 \langle \partial^k (\frac{1}{2} \dot{Q}^r), \partial^k Q^r \rangle
$$

$$
- L\langle (\partial^k, \frac{1}{\rho} \Delta) Q^r, \partial^k Q^r \rangle - c \langle \partial^k (\frac{1}{\rho} (Q^r \|Q^r\|^2)), \partial^k Q^r \rangle
$$

$$
+ b \langle \partial^k (\frac{1}{\rho} (Q^r \|Q^r\|^2 I_d)), \partial^k Q^r \rangle + \frac{\mu_2}{2} \langle \partial^k (\frac{1}{\rho} \text{div} u^r I_d)), \partial^k Q^r \rangle
$$

$$
+ \mu_1 \langle \partial^k (\frac{1}{\rho} (\Omega^r, Q^r)), \partial^k Q^r \rangle.
$$

Following the standard estimates in the previous and summing up with $0 \leq |k| \leq s$ infer that

$$
\frac{1}{2} \frac{d}{dt} (J\|\dot{Q}^r + Q^r\|^2_{\mathcal{H}^*} - J\|\dot{Q}^r\|^2_{\mathcal{H}^*} - J\|Q^r\|^2_{\mathcal{H}^*}) + L||\nabla Q^r\|^2_{\mathcal{H}^*} + \frac{a}{2} \|Q^r\|^2_{\mathcal{H}^*} - C\langle \dot{Q}^r, \nabla u^r \rangle
$$

$$
\lesssim (\|u^r\|_{\mathcal{H}^*} + \|\nabla u^r\|_{\mathcal{H}^*} + \|\nabla Q^r\|_{\mathcal{H}^*})\|\dot{Q}^r\|_{\mathcal{H}^*} + \|\phi^r\|_{\mathcal{H}^*} \|\nabla Q^r\|^2_{\mathcal{H}^*} + \|Q^r\|^3_{\mathcal{H}^*}
$$

$$
+ (\|Q^r\|^3_{\mathcal{H}^*} + \|\nabla Q^r\|^3_{\mathcal{H}^*})\|Q^r\|_{\mathcal{H}^*} + \|\nabla u^r\|_{\mathcal{H}^*} \|Q^r\|^2_{\mathcal{H}^*}
$$

$$
+ \|\nabla u^r\|^2_{\mathcal{H}^*} \|Q^r\|^2_{\mathcal{H}^*} + (\|\phi^r\|_{\mathcal{H}^*} \|\phi^r\|_{\mathcal{H}^*}^{-1})\|Q^r\|_{\mathcal{H}^*}
$$

$$
\times (\|\dot{Q}^r\|^3_{\mathcal{H}^*} + \|\nabla u^r\|_{\mathcal{H}^*} \|Q^r\|_{\mathcal{H}^*} + \|\nabla u^r\|_{\mathcal{H}^*} + \|\nabla Q^r\|_{\mathcal{H}^*} + \|\dot{\phi}^r\|_{\mathcal{H}^*} + \|Q^r\|^3_{\mathcal{H}^*} + \|\dot{Q}^r\|_{\mathcal{H}^*} + \|Q^r\|^4_{\mathcal{H}^*}),
$$

where the constant $C > 0$ is independent of $\epsilon$. From the bounds (40), we therefore have

$$
\|u^r\|_{\mathcal{H}^*} \lesssim \|u^r\|_{\mathcal{H}^*}^\prime, \quad \|\nabla Q^r\|_{\mathcal{H}^*} \lesssim \|\nabla Q^r\|_{\mathcal{H}^*}^\prime, \quad \|Q^r\|_{\mathcal{H}^*} \lesssim \|Q^r\|_{\mathcal{H}^*}^\prime,
$$

$$
\|\phi^r\|_{\mathcal{H}^*} \lesssim \|\phi^r\|_{\mathcal{H}^*}^\prime, \quad P_s(\|\rho^r\|^2) \lesssim (1 + \|\phi^r\|_{\mathcal{H}^*}^\prime)^{-1}\|\nabla \phi^r\|_{\mathcal{H}^*}^{-1}\|\rho^r\|_{\mathcal{H}^*}^\prime
$$

for all $t \in [0, T]$ and $0 < \epsilon \leq 1$. Then, from substituting the inequalities (84) into (83), we infer that

$$
\frac{1}{2} \frac{d}{dt} (J\|\dot{Q}^r + Q^r\|^2_{\mathcal{H}^*} - J\|\dot{Q}^r\|^2_{\mathcal{H}^*} - J\|Q^r\|^2_{\mathcal{H}^*}) + L||\nabla Q^r\|^2_{\mathcal{H}^*} + \frac{a}{2} \|Q^r\|^2_{\mathcal{H}^*} - C\langle \dot{Q}^r, \nabla u^r \rangle
$$

$$
\lesssim (1 + \delta_{\rho^r}(t))\|\phi^r\|_{\mathcal{H}^*}^\prime(\|\rho^r\|^2)\|\dot{Q}^r\|_{\mathcal{H}^*}^\prime(\|\rho^r\|^2)
$$

for all $t \in [0, T]$ and $0 < \epsilon \leq 1$.

**Step 3. Close the uniform energy estimates.**

Recalling the definition of $\mathcal{A}(t)$ in (45) in Lemma 2.4, it is easily derived from the relations (84) and the bound (40) that

$$
\mathcal{A}(t) \lesssim \|\phi^r\|_{\mathcal{H}^*}^2 + ||\nabla Q^r\|_{\mathcal{H}^*}^2 + \|Q^r\|^2_{\mathcal{H}^*}.
$$
Furthermore, we add the $\eta_0$ times of the inequalities (81) and (85) into the inequality (44) in Lemma 2.4, where $\eta_0 > 0$ is given in Lemma 2.5. We therefore have
\[
\frac{1}{2} \frac{d}{dt} (E(t) + \eta_0 ||\nu^e||^2_{H^s} + \eta_0 J ||\dot{Q}^e + Q^e||^2_{H^s} - c_0 \eta_0 ||\nu^e||_{H^s} + ||\nabla \phi^e||^2_{H^s})
\]
\[
- \eta_0 (||Q^e||^2_{H^s} + ||Q^e||_{L^2}) + c_0 D(t) - 2C\eta_0 (||\nu||^2_{H^s} + ||\dot{Q}^e||^2_{H^s})
\]  
\[
+ \frac{1}{2} \eta_0 ||\phi^e||_{H^s} + \eta_0 L ||\nabla \phi||^2_{H^s} + \frac{1}{2} \eta_0 a ||Q||^2_{H^s}
\]
\[
\lesssim (1 + \mathcal{E}^{\eta_0} + \mathcal{P}_0) \lesssim \left(1 + \mathcal{E}^{\eta_0} + \mathcal{P}_0 \right) \mathcal{E}_0(t) \mathcal{P}_0(t)
\]
for all $t \in [0, T]$ and $0 < \epsilon \leq 1$. From the definition of $\mathcal{E}_0(t)$ and $\mathcal{P}_0(t)$ in (69) and (70), respectively, we deduce that
\[
\frac{1}{2} \frac{d}{dt} \mathcal{E}_0 + \mathcal{P}_0 \lesssim \left(1 + \mathcal{E}^{\eta_0} + \mathcal{P}_0 \right) \mathcal{E}_0(t) \mathcal{P}_0(t)
\]
for all $t \in [0, T]$ and $0 < \epsilon \leq 1$. Then the proof of Lemma 2.6 is finished. 

\[\square\]

2.3. Proof of global uniform energy bounds (22) and (23) in Theorem 1.1.

From Proposition 2.6, we have
\[
\frac{1}{2} \frac{d}{dt} \mathcal{E}_0 + \mathcal{P}_0 \lesssim \left(1 + \mathcal{E}^{\eta_0} + \mathcal{P}_0 \right) \mathcal{E}_0(t) \mathcal{P}_0(t)
\]
for all $t \in [0, T]$ and $0 < \epsilon \leq 1$. It follows from the relation (71) in Lemma 2.5 that there exist constants $C_1, C_2 > 0$, independent of $\epsilon \in (0, 1]$, such that
\[
C_1 \mathcal{E}^{\text{ini}} \leq \mathcal{E}_0(0) \leq C_2 \mathcal{E}^{\text{ini}},
\]
where the initial energy $\mathcal{E}^{\text{ini}}$ is defined in (21). We choose
\[
\lambda_0 = \min \left\{1, \epsilon_0, \frac{1}{16C_2(1+C_2)^2} \right\} \in (0, 1]
\]
such that if the initial energy $\mathcal{E}^{\text{ini}} \leq \lambda_0$, then
\[
C \left(1 + \mathcal{E}^{\eta_0} \right) \mathcal{E}_0(t) \leq \frac{1}{2} \leq \frac{1}{2},
\]
where $\epsilon_0 > 0$ is mentioned in Proposition 1 and the constant $C > 0$ is given in Lemma 2.6.

We introduce a number
\[
T^* = \sup \{\tau \in (0, T] \mid \sup_{t \in [0, \tau]} \left(1 + \mathcal{E}^{\eta_0}\right) \mathcal{E}_0(t) \leq \frac{1}{2} \} \geq 0,
\]
where $T > 0$ is given in Proposition 1. By taking advantage of the continuity of the energy functional $\mathcal{E}_0(t)$, one derives that $T^* > 0$. Thus,
\[
\frac{1}{2} \frac{d}{dt} \mathcal{E}_0(t) + \frac{1}{2} \mathcal{P}_0(t) \leq \frac{1}{2} \frac{d}{dt} \mathcal{E}_0(t) + \left[1 - C \left(1 + \mathcal{E}^{\eta_0} \right) \mathcal{E}_0(t) \right] \mathcal{P}_0(t) \leq 0
\]
holds for all $t \in [0, T^*)$, which means that $\mathcal{E}_0(t) \leq \mathcal{E}_0(0) \leq C_2 \mathcal{E}^{\text{ini}} \leq C_2 \lambda_0$ for all $t \in [0, T^*)$. Then we can derive that
\[
\sup_{t \in [0, T^*)} \left\{ \left(1 + \mathcal{E}^{\eta_0} \right) \mathcal{E}_0(t) \right\} \leq \frac{1}{2}.
\]
We therefore claim that $T^* = T$. Otherwise, $T^* < T$. The continuity of the energy $\mathcal{E}_0(t)$ implies that there exists a sufficiently small positive $\gamma > 0$ such that
\[
\sup_{t \in [0, T^* + \gamma]} \left\{ \left(1 + \mathcal{E}^{\eta_0} \right) \mathcal{E}_0(t) \right\} \leq \frac{1}{2},
\]
which contradicts to the definition of $T^*$. As a consequence, it holds
\[
\sup_{t \in [0, T]} \mathcal{E}_m(t) + \int_0^T \mathcal{P}_m(t) dt \leq C_2 \mathcal{E}^{in} \leq C_2 \lambda_0.
\]

Then we extend the solution constructed in Proposition 1 from the $t = T$. It follows from the previous arguments that $\sup_{t \in [0, 2T]} \mathcal{E}_m(t) + \int_0^{2T} \mathcal{P}_m(t) dt \leq C_2 \mathcal{E}^{in} \leq C_2 \lambda_0$. Repeating the above process, we finally know that
\[
\sup_{t \geq 0} \mathcal{E}_m(t) + \int_0^\infty \mathcal{P}_m(t) dt \leq C_2 \mathcal{E}^{in} \leq C_2 \lambda_0.
\]

Hence the bound (22) holds. Finally, the bound (23) is implied by the bound (40).

2.4. **Proof of bounds** (24), (26) and (27) in Theorem 1.1. In this subsection, based on the uniform bounds (22) and (23), we will show the uniform estimates for the time derivative of the system (13).

2.4.1. **Uniform estimates of $\partial_t \bar{Q}'$ and $\nabla \partial_t Q'$.** By the third equation of system (13) and notice that $\bar{Q}' = \partial_t \bar{Q}' + u' \cdot \nabla \bar{Q}'$, we have
\[
J \partial_t \bar{Q}' = -t\partial_t \bar{Q}' \cdot \nabla \bar{Q}' - \frac{b}{\rho'} \bar{Q}' \rho' \Delta \bar{Q}' - \frac{\mu}{\rho'} \bar{Q}' - \frac{1}{\rho'} (Q' \bar{Q}' - \frac{1}{4} |Q'|^2 I_d)
- \frac{b}{\rho'} Q' \rho' \nabla \bar{Q}' \cdot \nabla |Q'|^2 dI_d + \frac{\mu}{\rho'} [\Omega', Q'].
\]

Applying the derivative operator $\partial^k (0 \leq |k| \leq s - 1)$ to the above equation (86), we can deduce that
\[
J \partial_t \partial^k \bar{Q}' = -t \partial^k (u' \cdot \nabla \bar{Q}') - \mu_i \partial^k (\frac{1}{\rho'} \bar{Q}') + L \partial^k (\frac{1}{\rho'} \Delta \bar{Q}') - a \partial^k (\frac{1}{\rho'} Q')
+ b \partial^k (\frac{1}{\rho'} (Q' \bar{Q}' - \frac{1}{4} |Q'|^2 I_d)) - c \partial^k (\frac{1}{\rho'} Q' |Q'|^2)
+ \partial^k \partial^k (\frac{1}{\rho'} (A' - \frac{1}{3} \text{div} u') I_d) + \mu_i \partial^k (\frac{1}{\rho'} [\Omega', Q']).
\]

We next deal with the right-hand side of (87) term by term. By Moser-type inequality in Lemma 2.3 and Sobolev embedding theory, we have
\[
\| \partial^k (u' \cdot \nabla \bar{Q}') \|_{L^2} \lesssim \left( \| u' \|_{L^\infty} \| \nabla \partial^k \bar{Q}' \|_{L^2} + \| \partial^k u' \|_{L^2} \| \nabla \bar{Q}' \|_{L^\infty} \right)
\lesssim \| u' \|_{H^{s}} \| \bar{Q}' \|_{H^{s}},
\]
\[
\| \partial^k (\frac{1}{\rho'} \bar{Q}') \|_{L^2} \lesssim \left( \| \frac{1}{\rho'} \|_{L^\infty} \| \nabla \partial^k \bar{Q}' \|_{L^2} + \| \partial^k \frac{1}{\rho'} \|_{L^2} \| \bar{Q}' \|_{L^\infty} \right)
\lesssim (1 + \| \phi' \|_{H^{s}}) \| \bar{Q}' \|_{H^{s}},
\]
\[
\| \partial^k (\frac{1}{\rho'} \Delta \bar{Q}') \|_{L^2} \lesssim \left( \| \frac{1}{\rho'} \|_{L^\infty} \| \nabla \partial^k \bar{Q}' \|_{L^2} + \| \partial^k \frac{1}{\rho'} \|_{L^2} \| \Delta \bar{Q}' \|_{L^\infty} \right)
\lesssim (1 + \| \phi' \|_{H^{s}}) \| \nabla \bar{Q}' \|_{H^{s}},
\]
\[
\| \partial^k (\frac{1}{\rho'} Q') \|_{L^2} \lesssim \left( \| \frac{1}{\rho'} \|_{L^\infty} \| \nabla \partial^k Q' \|_{L^2} + \| \partial^k \frac{1}{\rho'} \|_{L^2} \| Q' \|_{L^\infty} \right)
\lesssim (1 + \| \phi' \|_{H^{s}}) \| Q' \|_{H^{s}},
\]
\[
\| \partial^k (\frac{1}{\rho'} (Q' |Q'|^2)) \|_{L^2} \lesssim \left( \| \frac{1}{\rho'} \|_{L^\infty} \| \nabla \partial^k (Q' |Q'|^2) \|_{L^2} + \| \partial^k \frac{1}{\rho'} \|_{L^2} \| Q' |Q'|^2 \|_{L^\infty} \right)
\lesssim (1 + \| \phi' \|_{H^{s}}) (\| \nabla Q' \|_{H^{s}}^3 + \| Q' \|_{H^{s}}^3),
\]
\[
\| \partial^k (\frac{1}{\rho'} [\Omega', Q']) \|_{L^2} \lesssim \left( \| \frac{1}{\rho'} \|_{L^\infty} \| \nabla \partial^k ([\Omega', Q']) \|_{L^2} + \| \partial^k \frac{1}{\rho'} \|_{L^2} \| [\Omega', Q'] \|_{L^\infty} \right)
\lesssim (1 + \| \phi' \|_{H^{s}}) \| u' \|_{H^{s}} \| Q' \|_{H^{s}}.
\]
The term $b \partial^k \left( \frac{1}{\rho'} (Q' Q' - \frac{1}{2} |Q'|^2 I_d) \right)$ can be bounded by
\[
||b \partial^k \left( \frac{1}{\rho'} (Q' Q' - \frac{1}{2} |Q'|^2 I_d) \right)||_{L^2} \\
\lesssim \left( \frac{1}{\rho'} ||\partial^k (Q' Q')||_{L^2} + ||\partial^k \frac{1}{\rho'} ||_{L^2} ||Q' Q'||_{L^\infty} \right) \\
\lesssim \left( 1 + \|\phi'^s\|_{H^s} \right) \left( ||\nabla Q'||_{H^s}^2 + ||Q'||_{H^s}^2 \right).
\]

Similarly, we have
\[
||\bar{b} \frac{1}{\rho} \partial^k \left( \frac{1}{\rho'} (A' - \frac{1}{\rho} \text{div} u' I_d) \right)||_{L^2} \lesssim \left( 1 + \|\phi'^s\|_{H^s} \right) ||u'||_{H^s}.
\]

Therefore, together with the definition of $\mathcal{E}_{\eta_0}(t)$ in (69), we deduce from combining the above inequalities that
\[
||\partial_t Q'||_{H^{-s-1}} \leq \left( 1 + \|\phi'^s\|_{H^s} \right) \left( ||\hat{Q}'||_{H^s} + ||u'||_{H^s} + ||Q'||_{H^s} + ||\nabla Q'||_{H^s} \right) \\
+ ||Q'||_{H^s}^2 + ||\nabla Q'||_{H^s}^2 + ||u'||_{H^s} ||\hat{Q}'|| + ||u'||_{H^s} ||Q'|| \\
+ ||Q'||_{H^s}^2 + ||\nabla Q'||_{H^s}^2 \quad \text{(88)}
\]

Furthermore, we have
\[
||\partial_t Q'||_{H^s} = ||\hat{Q}' - u' \cdot \nabla Q'||_{H^s} \lesssim ||\hat{Q}'||_{H^s} + ||u'||_{H^s} ||\nabla Q'||_{H^s} \\
\lesssim \left( 1 + \|\phi'^s\|_{H^s} \right) \mathcal{E}_{\eta_0}^\frac{1}{2}(t). \quad \text{(89)}
\]

Consequently, the bounds (22), (88) and (89) imply the uniform bound (24).

2.4.2. **Uniform estimates of $\partial_t \phi'$ and $\partial_t u'$**, Differentiating the first and second equations in the system (13) with respect to $t$, we obtain
\[
\left\{ \begin{aligned}
\partial_t \phi' + \partial_t u' \cdot \nabla \phi' + u' \cdot \nabla \partial_t \phi' + \partial_t \phi' \text{div} u' \\
+ \phi' \text{div} \partial_t u' + \frac{1}{\rho} \text{div} \partial_t u' = 0,
\end{aligned} \right. \quad \text{(90)}
\]

where $\rho' = 1 + \epsilon \phi'$.

For all multi-index $k(|k| \leq s - 2)$, we apply the derivative operator $\partial^k$ to the system (90), multiply the first equation by $p'(\rho') \partial_t \partial^k \phi'$ and the second equation by $\rho' \partial_t \partial^k u'$ and add the results together. We thereby have
\[
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left( ||\partial_t \partial^k \phi'||_{L^2(\epsilon, \rho})^2 \right) + ||\partial_t \partial^k u'||_{L^2(\epsilon, \rho})^2 \right) + \frac{1}{2} \left( ||\partial_t \partial^k u'||_{L^2}^2 + \frac{1}{2} ||\text{div} \partial_t \partial^k u'||_{L^2}^2 \right) \\
= &-\frac{1}{2} \langle \text{div} \partial_t \partial^k u', p'(\rho') \partial_t \partial^k \phi' \rangle - \frac{1}{2} \langle p'(\rho') \nabla \partial_t \partial^k \phi', \partial_t \partial^k u' \rangle + \frac{1}{2} \langle ||\partial_t \partial^k \phi'||^2, \partial_t p'(\rho') \rangle \\
- &\langle \partial^k (\partial_t u' \cdot \nabla \phi'), p'(\rho') \partial_t \partial^k \phi' \rangle - \langle \partial^k (u' \cdot \nabla \partial_t \phi'), p'(\rho') \partial_t \partial^k \phi' \rangle \\
- &\langle \partial^k (\partial_t \phi' \text{div} u'), p'(\rho') \partial_t \partial^k \phi' \rangle - \langle \partial^k (\phi' \text{div} \partial_t u'), p'(\rho') \partial_t \partial^k \phi' \rangle \\
- &\langle \partial^k (\partial_t u' \cdot \nabla u'), p'(\rho') \partial_t \partial^k \phi' \rangle - \langle ||\partial^k, u' \cdot \nabla \partial_t u', \rho' \partial_t \partial^k u' \rangle \\
\end{aligned}
\]
\(-\frac{1}{\tau} (\partial_k (\frac{p'}{\rho'}) \nabla \phi'), \rho' \partial_t \partial_k u') - \frac{1}{\tau} (\partial_k \cdot \frac{p'(\rho')}{\rho'} \nabla \partial_t \phi', \rho' \partial_t \partial_k u')\)

\[+ (\text{div} \partial_t \partial^k \Sigma_2, \partial_t \partial^k u') + (\text{div} \partial_t \partial^k \Sigma_3, \partial_t \partial^k u') + (\text{div} \partial_t \partial^k \Sigma_4, \partial_t \partial^k u')\]

\[\text{div} \partial_t \partial^k \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4\), \rho' \partial_t \partial^k u')\]

\[\text{div} \partial_t \partial^k \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4\), \rho' \partial_t \partial^k u')\].

The key point is to control the singular terms

\[\frac{1}{\tau} (\text{div} \partial_t \partial^k u', p'(\rho') \partial_t \partial^k \phi') \text{ and } \frac{1}{\tau} (p'(\rho') \nabla \partial_t \partial^k \phi', \partial_t \partial^k u')\).

The other terms without singularity \(\frac{1}{\tau}\) will be carefully controlled by utilizing the Moser-type calculus inequalities in Lemma 2.3, Sobolev embedding theory, as the similar estimates to the a priori estimates in the proof of Lemma 2.4. We emphasize that the terms \(-\frac{1}{\tau} (\partial_k (\frac{p'(\rho')}{\rho'}) \nabla \phi'), \rho' \partial_t \partial^k u')\) and \(-\frac{1}{2} (\partial_k (\frac{p'(\rho')}{\rho'}) \nabla \partial_t \phi', \rho' \partial_t \partial^k u')\) are not a real singular term, although there is a coefficient \(\frac{1}{\tau}\) in the front of it. Actually, \(\rho' = 1 + \epsilon \rho\) will generate an \(\epsilon\) after operate the derivative, so that the singularity will be canceled. Fortunately, for the singular terms \(\frac{1}{\tau} (\text{div} \partial_t \partial^k u', p'(\rho') \partial_t \partial^k \phi')\) and \(\frac{1}{\tau} (p'(\rho') \nabla \partial_t \partial^k \phi', \partial_t \partial^k u')\), we have the following cancellation

\[-\frac{1}{\tau} (\text{div} \partial_t \partial^k u', p'(\rho') \partial_t \partial^k \phi') - \frac{1}{\tau} (p'(\rho') \nabla \partial_t \partial^k \phi', \partial_t \partial^k u')\]

\[= -\frac{1}{\tau} (\text{div} \partial_t \partial^k u', p'(\rho') \partial_t \partial^k \phi') + \frac{1}{\tau} (p'(\rho') \partial_t \partial^k \phi', \partial_t \partial^k u')\]

\[+ \frac{1}{\tau} (p'(\rho') \nabla \partial_t \partial^k \phi', \partial_t \partial^k u')\]

Form employing the almost similar arguments in deriving the a priori estimates in Lemma 2.4 and repeatedly using the uniform bound (22), the terms \(\mathcal{L}_i\) (1 \(\leq i \leq 13\)) without singularity can be directly estimated as follows: for all \(0 < \epsilon \leq 1\) and \(|k| \leq s - 2\),

\[\mathcal{L}_1 = \langle p''(\rho') \nabla \phi', \partial_t \partial^k \phi', \partial_t \partial^k u' \rangle \lesssim \|\partial_t \phi'\|_{H^s_{\rho'(\rho')}} \|\phi'\|_{H^s_{\rho'(\rho')}} \|\partial_t u'\|_{H^s_{\rho'}} ;
\]

\[\mathcal{L}_2 + \mathcal{L}_3 \lesssim \|\partial_t \phi'\|_{H^s_{\rho'(\rho')}} \|\phi'\|_{H^s_{\rho'(\rho')}} + \|\partial_t u'\|_{H^s_{\rho'}} \|\partial_t \phi'\|_{H^s_{\rho'}} ;
\]

\[\mathcal{L}_4 + \mathcal{L}_5 \lesssim \|\partial_t \phi'\|_{H^s_{\rho'(\rho')}} \|\phi'\|_{H^s_{\rho'(\rho')}} + \|\partial_t u'\|_{H^s_{\rho'}} \|\partial_t \phi'\|_{H^s_{\rho'}} ;
\]

\[\mathcal{L}_6 + \mathcal{L}_7 \lesssim \|\partial_t \phi'\|_{H^s_{\rho'(\rho')}} \|\partial_t u'\|_{H^s_{\rho'}} + \|\partial_t \phi'\|_{H^s_{\rho'}} \|\partial_t u'\|_{H^s_{\rho'}} ;
\]

\[\mathcal{L}_8 + \mathcal{L}_9 \lesssim \|\partial_t \phi'\|_{H^s_{\rho'(\rho')}} \|\partial_t u'\|_{H^s_{\rho'}} + \|\partial_t \phi'\|_{H^s_{\rho'}} \|\partial_t u'\|_{H^s_{\rho'}} ;
\]

\[\mathcal{L}_{10} \lesssim (1 + \|\phi'\|_{H^s_{\rho'(\rho')}}) \|\partial_t \phi'\|_{H^s_{\rho'(\rho')}} \|\partial_t u'\|_{H^s_{\rho'}} ;
\]

\[\mathcal{L}_{11} + \beta_1 \|Q : \partial_t \partial^k A\|_{L^2} + \mu_1 \|\partial_t \partial^k \Omega, \nabla \partial_t u'\|_{L^2} ;
\]

\[+ (\beta_5 + \beta_6) (\partial_t \partial^k \Omega' \partial_t \partial^k A') - \mu_2 (\partial_t \partial^k \Omega', \partial_t \partial^k A') \lesssim (1 + \|u'\|_{H^s_{\rho'}}) \|\phi'\|_{H^s_{\rho'}} \|\partial_t \phi'\|_{H^s_{\rho'}} \|\partial_t u'\|_{H^s_{\rho'}} .\]
\[ L_{12} \lesssim (\| \nabla \partial_t u^\varepsilon \|_{H^{-s-2}} + \| \partial_t u^\varepsilon \|_{H^{-s-2}}^2)(1 + \| \phi^\varepsilon \|_{H^{-s-2}}(\mu'))(1 + \| Q^\varepsilon \|_{H^s}^2 + \| u^\varepsilon \|_{H^s}) \]
\[ \times \| \partial_t \phi^\varepsilon \|_{H^{-s-2}}(\| \dot{Q}^\varepsilon \|_{H^{s-1}} + \| Q^\varepsilon \|_{H^s} + \| \nabla Q^\varepsilon \|_{H^s}), \]
\[ L_{13} \lesssim (1 + \| \phi^\varepsilon \|_{H^{-s-2}}(\mu'))(\| \nabla \partial_t u^\varepsilon \|_{H^{-s-2}} + \| \partial_t u^\varepsilon \|_{H^{-s-2}})(1 + \| \partial_t \dot{Q}^\varepsilon \|_{H^{-s-1}}) \]
\[ + \| \partial_t Q^\varepsilon \|_{H^s} \times (1 + \| u^\varepsilon \|_{H^s}(\| Q^\varepsilon \|_{H^s} + \| \nabla Q^\varepsilon \|_{H^s} + \| Q^\varepsilon \|_{H^s}^3) \]
\[ + \| \nabla Q^\varepsilon \|_{H^s}(\| Q^\varepsilon \|_{H^s}^2 + \| \partial_t \dot{Q}^\varepsilon \|_{H^s}), \quad (92) \]

For simplicity, we omit the details here. From substituting the bounds (92) (i.e., the bounds of \( L_i \), \( 1 \leq i \leq 13 \)) into the equality (91), summing up for all \( |k| \leq s - 2 \) and combining the bound (88), (89), we deduce that
\[ \frac{1}{2} \frac{d}{dt} E(\partial_t \phi^\varepsilon, \partial_t u^\varepsilon)(t) + \beta_1 \sum_{|k|=0}^{s-2} \| Q : \partial_t \partial^k A \|_{L^2}^2 + \mu_1 \sum_{|k|=0}^{s-2} \| [\partial_t \partial^k Q^\varepsilon, Q^\varepsilon] \|_{L^2}^2 \]
\[ - \mu_2 \sum_{|k|=0}^{s-2} \langle [\partial_t \partial^k \Omega^\varepsilon, Q^\varepsilon], \partial_t \partial^k A^\varepsilon \rangle + (\beta_5 + \beta_6) \sum_{|k|=0}^{s-2} \langle [\partial_t \partial^k A^\varepsilon Q^\varepsilon, \partial_t \partial^k A^\varepsilon] \rangle + D(\partial_t u^\varepsilon) \]
\[ \lesssim \mathcal{E}^\frac{1}{2}(t) E(\partial_t \phi^\varepsilon, \partial_t u^\varepsilon) + (D(\partial_t u^\varepsilon)(t)) \frac{1}{2} \left( 1 + \mathcal{E}^{\frac{1}{4}}(t) \right) \left( 1 + E^\frac{1}{2}(t) \right) \mathcal{E}^\frac{1}{2}(t), \]
for all \( t \in \mathbb{R}^+ \) and \( 0 < \varepsilon \leq 1 \), where
\[ E(\partial_t \phi^\varepsilon, \partial_t u^\varepsilon)(t) = \| \partial_t \phi^\varepsilon \|_{H^{-s-2}}^2 + \| \partial_t u^\varepsilon \|_{H^{-s-2}}^2, \]
\[ D(\partial_t u^\varepsilon)(t) = \frac{k}{2} \| \nabla \partial_t u^\varepsilon \|_{H^{-s-2}}^2 + \left( \frac{1}{2} \beta_4 + \xi \right) \| \text{div} \partial_t u^\varepsilon \|_{H^{-s-2}}^2, \quad (93) \]
and \( \mathcal{E}(t) \) is defined in (42). Moreover, it follows from Sobolev embedding theory and the uniform bound (22) that
\[ - (\beta_5 + \beta_6) \sum_{|k|=0}^{s-2} \langle [\partial_t \partial^k A^\varepsilon Q^\varepsilon, \partial_t \partial^k A^\varepsilon] \rangle \lesssim \| Q^\varepsilon \|_{H^s} \| \nabla \partial_t u^\varepsilon \|_{H^{-s-2}}^2 \lesssim \sqrt{\lambda_0} D(\partial_t u^\varepsilon)(t), \]
\[ \mu_2 \sum_{|k|=0}^{s-2} \langle [\partial_t \partial^k \Omega^\varepsilon, Q^\varepsilon], \partial_t \partial^k A^\varepsilon \rangle \lesssim \| Q^\varepsilon \|_{H^s} \| \nabla \partial_t u^\varepsilon \|_{H^{-s-2}}^2 \lesssim \sqrt{\lambda_0} D(\partial_t u^\varepsilon)(t). \]

Since \( \beta_1 \geq 0, \mu_1 > 0 \) and the \( \lambda_0 \) given in Theorem 1.1 is such that \( C \sqrt{\lambda_0} < \frac{1}{2} \), it holds that there is a constant \( C_3 > 0 \), independent of \( \varepsilon \in (0, 1) \), such that
\[ \frac{d}{dt} E(\partial_t \phi^\varepsilon, \partial_t u^\varepsilon)(t) + D(\partial_t u^\varepsilon)(t) \leq C_3 \left( 1 + E(\partial_t \phi^\varepsilon, \partial_t u^\varepsilon)(t) \right) \]
holds for all \( t \in \mathbb{R}^+ \) and \( 0 < \varepsilon \leq 1 \). From the above differential inequality we know that for any fixed \( T > 0 \),
\[ E(\partial_t \phi^\varepsilon, \partial_t u^\varepsilon)(t) \leq (1 + E(\partial_t \phi^\varepsilon, \partial_t u^\varepsilon)(0)) \exp(C_3 T), \quad (94) \]
holds for all \( t \in (0, T] \) and \( 0 < \varepsilon \leq 1 \). From the structures of the first two equation of (13), we deduce that
\[ \partial_t \phi^\varepsilon(0) = -u_0^\varepsilon \cdot \nabla \phi_0^\varepsilon + \phi_0^\varepsilon \text{div} u_0^\varepsilon + \frac{1}{\rho_0} \text{div} u_0^\varepsilon, \]
\[ \partial_t u^\varepsilon(0) = -u_0^\varepsilon \cdot \nabla u_0^\varepsilon + \frac{1}{\rho_0} p'(\rho_0) \nabla \phi_0^\varepsilon + \frac{1}{\rho_0} \text{div} (\Sigma_1^\varepsilon(0) + \Sigma_2^\varepsilon(0) + \Sigma_3^\varepsilon(0) + \Sigma_4^\varepsilon(0)), \]
where
\[ \Sigma_1'(0) = \frac{\beta_1}{2} \left( \nabla u_0^\epsilon + \nabla^\top u_0^\epsilon \right) + \xi \text{div} u_0^\epsilon I_d, \]
\[ \Sigma_2'(0) = L \left( \frac{1}{2} \nabla Q_0^\epsilon \right)^2 I_d - \nabla Q_0^\epsilon \otimes \nabla Q_0^\epsilon + \psi_B(Q_0^\epsilon) I_d, \]
\[ \Sigma_3'(0) = \beta_1 Q_0^\epsilon \text{tr}(Q_0^\epsilon A_0^\epsilon) + \beta_5 A_0^\epsilon Q_0^\epsilon + \beta_6 Q_0^\epsilon A_0^\epsilon, \]
\[ \Sigma_4'(0) = \frac{\mu_1}{2} (Q_0^\epsilon - [Q_0^\epsilon, Q_0^\epsilon]) + \mu_1 [Q_0^\epsilon, (Q_0^\epsilon - [Q_0^\epsilon, Q_0^\epsilon])], \]
\[ \psi_B(Q_0^\epsilon) = \frac{1}{2} \text{tr}(Q_0^\epsilon Q_0^\epsilon) - \frac{1}{2} \text{tr}(Q_0^\epsilon Q_0^\epsilon Q_0^\epsilon) + \frac{\xi}{4} (\text{tr}(Q_0^\epsilon Q_0^\epsilon))^2. \]

Form the initial data (21) and (25), one easily deduces that
\[
\| \partial_t \phi'(0) \|_{H^{r-1}} \lesssim \| u_0 \|_{H^r} \| \phi_0 \|_{H^{r-1}} + \frac{1}{\epsilon} \| \text{div} u_0 \|_{H^{r-2}} \lesssim \lambda_0 + C u \leq +\infty,
\]
\[
\| \partial_t u'(0) \|_{H^{r-1}} \lesssim (\| Q_0 \|_{H^r} + \| u_0 \|_{H^r} + \| Q_0^\epsilon \|_{H^{r-1}} + \| \nabla Q_0 \|_{H^{r-1}}) \times (1 + \| \phi_0 \|_{H^{r-1}} + \frac{1}{\epsilon} \| \phi_0 \|_{H^{r-1}}),
\]
\[
\leq C \phi (1 + \lambda_0^2 + \lambda_0^2) + (1 + \lambda_0^2) \lambda_0^2 < +\infty
\]
for all \( 0 < \epsilon \leq 1 \), which means that \( E(\partial_t \phi', \partial_t u')(0) \leq C \). Consequently, the inequality (94) reduces to the bound (26).

Finally, we justify the uniform bound (27). From the first two equations of (13), Moser-type inequalities and the uniform bound (23), we derive that
\[
\frac{1}{\epsilon} \| \text{div} u' \|_{H^{r-2}} \leq \| \partial_t \phi' \|_{H^{r-2}} + \| u' \cdot \nabla \phi' \|_{H^{r-2}} + \| \phi' \|_{H^{r-1}} \leq \frac{1}{\epsilon} \| \rho' (\rho') \nabla \phi' \|_{H^{r-2}}
\]
\[ \lesssim (\| \partial_t u' \|_{H^r} + \| u' \|_{H^r} + \| Q' \|_{H^r} + \| \nabla Q' \|_{H^r}) (1 + \| \phi' \|_{H^{r-1}}), \]
which, combining with the uniform bounds (22), (23) and (26), implies that
\[
\frac{1}{\epsilon} \| \text{div} u' \|_{L^\infty(0,T;H^{r-2})} + \frac{1}{\epsilon} \| \rho' (\rho') \nabla \phi' \|_{L^\infty(0,T;H^{r-2})} \leq C_{\text{op}}(T) < +\infty
\]
for any fixed \( T > 0 \) and for all \( 0 < \epsilon \leq 1 \). Thus we complete the proof of Theorem 1.1.

3. Limit to incompressible system: proof of Theorem 1.2. In this section, based on the uniform global energy bounds (22), (23), (24), (26) and (27) in Theorem 1.1, we aim at deriving the incompressible inertial Qian-Sheng model (15) from the corresponding compressible system (13) as \( \epsilon \to 0 \).

3.1. Limits from the global energy estimates. We first introduce the following Aubin-Lions-Simon Theorem, a fundamental result of compactness in the study of nonlinear evolution problems, which can be referred to Theorem II.5.16 of [4] or [32], for instance.

**Lemma 3.1** (Aubin-Lions-Simon Theorem). Let \( B_0 \subset B_1 \subset B_2 \) be three Banach space. We assume that the embedding of \( B_1 \) in \( B_2 \) is continuous and that the embedding of \( B_0 \) in \( B_1 \) is compact. Let \( 1 \leq p, q \leq +\infty \). For \( T > 0 \), we define
\[ E_{p,q} = \{ u \in L^p(0,T;B_0), \partial_t u \in L^q(0,T;B_2) \}. \]

1. If \( p \leq +\infty \), the embedding of \( E_{p,q} \) in \( L^p(0,T;B_1) \) is compact.
2. If \( p = +\infty \) and \( q > 1 \), the embedding of \( E_{p,q} \) in \( C(0,T;B_1) \) is compact.

From the Theorem 1.1, we deduce that the Cauchy problem (13) and initial condition (14) admits a family of global solutions \((\phi^\epsilon, u^\epsilon, Q^\epsilon)\) with

\[
\phi^\epsilon \in L^\infty(\mathbb{R}^+; H^s_{loc}(\mathbb{R}^d)), \quad u^\epsilon, \dot{Q}^\epsilon \in L^\infty(\mathbb{R}^+; H^s(\mathbb{R}^d)),
\]

\[
Q^\epsilon \in L^\infty(\mathbb{R}^+; H^{s+1}(\mathbb{R}^d)), \quad \nabla u^\epsilon \in L^2(\mathbb{R}^+; H^s(\mathbb{R}^d)),
\]

which subject to the uniform global energy estimates (22), (23), (24), (26) and (27). Namely, there is a positive constant \( C \), independent of \( \epsilon \), such that

\[
\|\phi^\epsilon\|_{L^\infty(\mathbb{R}^+; H^s)}^2 + \|u^\epsilon\|_{L^\infty(\mathbb{R}^+; H^{s+1})}^2 + \|\dot{Q}^\epsilon\|_{L^\infty(\mathbb{R}^+; H^s)}^2 + \|
abla Q^\epsilon\|_{L^\infty(\mathbb{R}^+; H^s)}^2 + \|
abla u^\epsilon\|_{L^2(\mathbb{R}^+; H^s)}^2 \leq C, \tag{95}
\]

and

\[
\frac{1}{2} \left\| \text{div} u^\epsilon \right\|_{L^2(0,T;H^{s+2})}^2 \leq C, \tag{96}
\]

for any fixed \( T > 0 \) and \( 0 < \epsilon < 1 \).

From the uniform bounds (95), there are functions

\[
\phi(t,x), u(t,x) \in L^\infty(\mathbb{R}^+; H^s), \quad Q(t,x) \in L^\infty(\mathbb{R}^+; H^{s+1}),
\]

\[
u(t,x) \in L^\infty(\mathbb{R}^+; H^s) \cap L^2(\mathbb{R}^+; H^{s+1}), \quad \pi_1 \in L^\infty(0,T; H^{s-1}),
\]

such that in the sense of subsequence,

\[
(\phi^\epsilon, u^\epsilon, Q^\epsilon) \to (\phi, u, w) \quad \text{weakly-}^* \quad \text{for } t > 0, \quad \text{weakly in } H^s,
\]

\[
Q^\epsilon \to Q \quad \text{weakly-}^* \quad \text{for } t \geq 0, \quad \text{weakly in } H^{s+1},
\]

\[
\nabla u^\epsilon \to \nabla u \quad \text{weakly in } L^2(\mathbb{R}^+, H^s),
\]

\[
\frac{1}{2} p'(\rho^\epsilon) \nabla \phi^\epsilon \to \nabla \pi_1 \quad \text{weakly-}^* \quad \text{for } t \geq 0, \quad \text{weakly in } H^{s-2},
\]

as \( \epsilon \to 0 \).

One notices that

\[
H^s \hookrightarrow H^s_{loc} \hookrightarrow H^{s-1}_{loc} \quad \text{(or } \hookrightarrow H^{s-2}_{loc}), \tag{100}
\]

where the embedding of \( H^s \) in \( H^s_{loc} \) is compact derived from Rellich-Kondrangov Theorem (see [1], for instance) and the embedding of \( H^{s-1}_{loc} \) in \( H^{s-1}_{loc} \) (or \( H^{s-2}_{loc} \) is naturally continuous. Then, from Aubin-Lions-Simon Theorem in Lemma 3.1, the bounds (95) , (96), (97) and the embedding (100), we deduce that

\[
(\phi^\epsilon, u^\epsilon, Q^\epsilon, \nabla Q^\epsilon, Q^\epsilon) \to (\phi, u, w, \nabla Q, Q) \tag{101}
\]

strongly in \( C(\mathbb{R}^+; H^{s-1}_{loc}) \) as \( \epsilon \to 0 \). We immediately know that

\[
u^\epsilon \cdot \nabla Q^\epsilon \to u \cdot \nabla Q \tag{102}
\]

strongly in \( C(\mathbb{R}^+; H^{s-1}_{loc}) \) as \( \epsilon \to 0 \). Moreover, from the convergences (99) and uniform bound (96), one easily deduces that

\[
\partial_t Q^\epsilon \to \partial_t w \quad \text{weakly-}^* \quad \text{for } t \geq 0, \quad \text{weakly in } H^{s-1},
\]

\[
\partial_t Q^\epsilon \to \partial_t Q \quad \text{weakly-}^* \quad \text{for } t \geq 0, \quad \text{weakly in } H^s \tag{103}
\]

as \( \epsilon \to 0 \). Combining with \( \dot{Q}^\epsilon = \partial_t Q^\epsilon + u^\epsilon \cdot \nabla Q^\epsilon \), we have

\[
w = \partial_t Q + u \cdot \nabla Q = \dot{Q}. \tag{104}
\]
3.2. Convergence to limit equations. In this subsection, we will derive the incompressible inertial Qian-Sheng model (15) from (13) by using the convergences obtained in the previous subsection.

3.2.1. Equation of $u$. First, from the uniform bound (98), we know that $\text{div} u^\varepsilon \to 0$ strongly in $L^\infty(\mathbb{R}^+;H^{s-2})$ as $\varepsilon \to 0$, which, combining with the convergence (99) or (101), implies the incompressibility

$$\text{div} u = 0. \quad (105)$$

Next, we derive the $u$-equation of (15) from the second equation of (13), namely

$$\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \varepsilon \phi^\varepsilon \partial_t u^\varepsilon + \varepsilon \phi^\varepsilon u^\varepsilon \cdot \nabla u^\varepsilon + \frac{1}{\varepsilon} p'(\rho^\varepsilon)\nabla \phi^\varepsilon = \text{div}(\Sigma_1^\varepsilon + \Sigma_2^\varepsilon + \Sigma_3^\varepsilon + \Sigma_4^\varepsilon).$$

The convergence (101) implies that

$$u^\varepsilon \cdot \nabla u^\varepsilon \to u \cdot \nabla u \quad (106)$$

strongly in $C(\mathbb{R}^+;H^{s-2})$ as $\varepsilon \to 0$. From the uniform bounds (95), (97) and Moser-type calculus inequalities in Lemma 2.3, we have

$$\|\phi^\varepsilon \partial_t u^\varepsilon\|_{L^\infty(0,T;H^{s-2})} + \|\phi^\varepsilon u^\varepsilon \cdot \nabla u^\varepsilon\|_{L^\infty(\mathbb{R}^+;H^{s-1})}$$

$$\lesssim \|\phi^\varepsilon\|_{L^\infty(\mathbb{R}^+;H^{s})} \|\partial_t u^\varepsilon\|_{L^\infty(\mathbb{R}^+;H^{s-2})} + \|\phi^\varepsilon\|_{L^\infty(\mathbb{R}^+;H^{s})} \|u^\varepsilon\|^2_{L^\infty(\mathbb{R}^+;H^{s})} \leq C$$

for any fixed $T > 0$, which implies that for any fixed $T > 0$,

$$\varepsilon \phi^\varepsilon \partial_t u^\varepsilon \to 0 \quad \text{strongly in } L^\infty(0,T;H^{s-2}),$$

$$\varepsilon \phi^\varepsilon u^\varepsilon \cdot \nabla u^\varepsilon \to 0 \quad \text{strongly in } L^\infty(0,T;H^{s-1}) \quad (107)$$

as $\varepsilon \to 0$. The convergence (99) show that

$$\frac{1}{\varepsilon} p'(\rho^\varepsilon)\nabla \phi^\varepsilon \to \nabla \pi_1 \quad (108)$$

weakly-$\ast$ for $t \geq 0$ and weakly in $H^{s-2}$ as $\varepsilon \to 0$.

Recalling that $\Sigma_1^\varepsilon = \frac{\beta_5}{2} (\nabla u^\varepsilon + \nabla^\top u^\varepsilon) + \xi \text{div} u_I d$, we deduce from the convergence (101) and the incompressibility (105) that

$$\text{div} \Sigma_1^\varepsilon \to \text{div}(\frac{\beta_5}{2} (\nabla u + \nabla^\top u) + \xi \text{div} u_I d) = \frac{\beta_5}{2} \Delta u \quad (109)$$

strongly in $C(\mathbb{R}^+;H^{s-3})_{\text{loc}}$ as $\varepsilon \to 0$. Since $\Sigma_2^\varepsilon = L(\frac{1}{2} |\nabla Q|^2 I_d - \nabla Q^\top \odot \nabla Q^\top) + \psi_B(Q^\top) I_d$, the convergence (101) yields that

$$\text{div} \Sigma_2^\varepsilon \to \text{div}(L(\frac{1}{2} |\nabla Q|^2 I_d - \nabla Q \odot \nabla Q) + \psi_B(Q) I_d)$$

$$= - L \text{div}(\nabla Q \odot \nabla Q) + \nabla (\frac{L}{2} |\nabla Q|^2 + \psi_B(Q)) \quad (110)$$

strongly in $C(\mathbb{R}^+;H^{s-2})_{\text{loc}}$ as $\varepsilon \to 0$.

For the notation $\Sigma_3^\varepsilon = \beta_1 Q^\top \text{tr}(Q^\top A^\top) + \beta_5 A^\top Q^\top + \beta_6 Q^\top A^\top$. We write the term $\beta_1 Q^\top \text{tr}(Q^\top A^\top)$ as following

$$\beta_1 Q^\top \text{tr}(Q^\top A^\top) = V_\varepsilon + \beta_1 \text{tr}(QA^\top)$$

where

$$V_\varepsilon = \beta_1 (Q^\top - Q) \text{tr}(Q^\top A^\top) + \beta_1 Q^\top \text{tr}((Q^\top - Q) A^\top).$$

It follows from the Moser-type calculus inequalities, the uniform bound (95), and the convergences (101) that

$$\|V_\varepsilon\|_{L^\infty(\mathbb{R}^+;H^{s-1})_{\text{loc}}} \lesssim \|Q^\top - Q\|_{L^\infty(\mathbb{R}^+;H^s)} \|Q^\top\|_{L^\infty(\mathbb{R}^+;H^s)} \|u^\varepsilon\|_{L^\infty(\mathbb{R}^+;H^s)}$$

$$\lesssim \|Q^\top - Q\|_{L^\infty(\mathbb{R}^+;H^s)} \to 0$$
as \( \epsilon \to 0 \), which means that \( \mathcal{V}_\epsilon \to 0 \) strongly in \( L^\infty(\mathbb{R}^+; H^{s-1}_{loc}) \) as \( \epsilon \to 0 \). We denote by \( A = \frac{1}{2}(\nabla u + \nabla u^\top) \) and \( \Omega = \frac{1}{2}(\nabla u - \nabla u^\top) \). Then, form the convergence (101), we derive that

\[
\beta_1 Q^\epsilon tr(QA^\epsilon) \to \beta_1 Qtr(QA)
\]

strongly in \( C(\mathbb{R}^+; H^{s-2}_{loc}) \) as \( \epsilon \to 0 \). In summary, we have

\[
\beta_1 Q^\epsilon tr(Q'A^\epsilon) \to \beta_1 Qtr(QA)
\]

(111)

strongly in \( C(\mathbb{R}^+; H^{s-2}_{loc}) \) as \( \epsilon \to 0 \). By the similar argument in (111), one can easily derive form the convergences (101) and the bound (22) that

\[
\beta_5 A^\epsilon Q^\epsilon + \beta_6 Q^\epsilon A^\epsilon \to \beta_5 AQ + \beta_6 QA
\]

(112)

strongly in \( C(\mathbb{R}^+; H^{s-2}_{loc}) \) as \( \epsilon \to 0 \). Consequently, the limits (111) and (112) give

\[
\Sigma_5 \to \beta_1 Qtr(QA) + \beta_5 AQ + \beta_6 QA
\]

(113)

strongly in \( C(\mathbb{R}^+; H^{s-2}_{loc}) \) as \( \epsilon \to 0 \).

Recall \( \Sigma_4 = \frac{d}{\Vert} Q^\epsilon - [\Omega', Q^\epsilon] + \mu_1 [Q^\epsilon, (\dot{Q}^\epsilon - [\Omega', Q^\epsilon])] \). For the term \( \dot{Q}^\epsilon \), we have

\[
\dot{Q}^\epsilon \to \dot{Q}
\]

(114)

strongly in \( C(\mathbb{R}^+; H^{s-1}_{loc}) \) as \( \epsilon \to 0 \). Then, for the term \( [\Omega', Q^\epsilon] \), we rewrite it as

\[
[\Omega', Q^\epsilon] = [\Omega', Q^\epsilon - Q] + [\Omega', Q].
\]

It follows from the Moser-type calculus inequalities, the uniform bound (22), and the convergences (101) that

\[
\| [\Omega', Q^\epsilon - Q] \|_{L^\infty(\mathbb{R}^+; H^{s-1}_{loc})} \lesssim \| u^\epsilon \|_{L^\infty(\mathbb{R}^+; H^{s}_{loc})} \| Q^\epsilon - Q \|_{L^\infty(\mathbb{R}^+; H^{s}_{loc})} \to 0
\]

as \( \epsilon \to 0 \), which means that \( [\Omega', Q^\epsilon - Q] \to 0 \) strongly in \( L^\infty(\mathbb{R}^+; H^{s-1}_{loc}) \) as \( \epsilon \to 0 \).

Then from the convergence (101), we derive that

\[
[\Omega', Q] \to [\Omega, Q]
\]

(115)

strongly in \( L^\infty(\mathbb{R}^+; H^{s-2}_{loc}) \) as \( \epsilon \to 0 \). In summary, we have

\[
[\Omega', Q^\epsilon] \to [\Omega, Q]
\]

(115)

strongly in \( C(\mathbb{R}^+; H^{s-2}_{loc}) \) as \( \epsilon \to 0 \). Similar arguments in above estimate, we can easily derive from the bound (22) and the convergences (101) that

\[
\mu_1 [Q^\epsilon, (\dot{Q}^\epsilon - [\Omega', Q^\epsilon])] \to \mu_1 [Q, (\dot{Q} - [\Omega, Q])]
\]

(115)

strongly in \( C(\mathbb{R}^+; H^{s-2}_{loc}) \) as \( \epsilon \to 0 \). Consequently, the limits (114), (115) and (116) give

\[
\Sigma_4 \to \frac{d}{\Vert} Q - [\Omega, Q] + \mu_1 [Q, (\dot{Q} - [\Omega, Q])]
\]

(117)

strongly in \( C(\mathbb{R}^+; H^{s-2}_{loc}) \) as \( \epsilon \to 0 \).

For any \( T > 0 \), let a vector-valued test function \( \psi(t, x) \in C^1(0, T; C^\infty_{c}(\mathbb{R}^d)) \) with \( \psi(0, x) = \psi_0(x) \in C_c(\mathbb{R}^d) \) and \( \psi(t, x) = 0 \) for \( t \geq T' \), where \( T' < T \). Then we
deduce from the initial conditions in Theorem 1.2 and the convergence (99) that

\[
\int_0^T \int_{\mathbb{R}^d} \partial_t u^\epsilon \cdot \psi(t, x) dx dt = - \int_{\mathbb{R}^d} u_0^\epsilon(x) \cdot \psi_0(x) dx - \int_0^T \int_{\mathbb{R}^d} u^\epsilon \cdot \partial_t \psi(t, x) dx dt \tag{118}
\]

\[
\to - \int_{\mathbb{R}^d} u_0(x) \cdot \psi_0(x) dx - \int_0^T \int_{\mathbb{R}^d} u \cdot \partial_t \psi(t, x) dx dt
\]

as \( \epsilon \to 0 \). As a consequence, the limits (106), (107), (108), (109), (110), (113), (117) and (118) imply that \( u(t, x) \in L^\infty(\mathbb{R}^+; H^s) \cap L^2(\mathbb{R}^+; \dot{H}^{s+1}) \) subjects to the evolution

\[
\partial_t u + u \cdot \nabla u + \nabla \pi - \frac{\beta_2}{\mu} \Delta u = \text{div}\left(-L \nabla Q \circ \nabla Q + \beta_1 Q \text{tr}(QA) + \beta_5 A Q + \beta_6 Q A\right) + \text{div}\left(\frac{\mu_2}{\mu}(\hat{Q} - [\Omega, Q]) + \mu_1 [Q, (\hat{Q} - [\Omega, Q])]\right),
\]

\[
\text{div} u = 0,
\]

with the initial data

\[
u|_{t=0} = u_0(x),
\]

where \( \pi = \pi_1 - \frac{1}{2} |\nabla Q|^2 - \psi_B(Q) \in L^\infty(\mathbb{R}^+; H^{s-1}) \).

**3.2.2. Equation of \( Q \).** Based on the convergences obtained in the previous subsection, we now derive the \( Q \)-equation in (13) from the last equation of (15), i.e.,

\[
J \partial_t \hat{Q}^\epsilon + J u^\epsilon \cdot \nabla \hat{Q}^\epsilon + \epsilon J \phi^\epsilon \partial_t \hat{Q}^\epsilon + \epsilon J \phi^\epsilon u^\epsilon \cdot \nabla \hat{Q}^\epsilon + \mu_1 \hat{Q}^\epsilon = L \Delta Q^\epsilon - a Q^\epsilon + b(Q^\epsilon Q^\epsilon - \frac{1}{3} |Q^\epsilon|^2 I_d) - c Q^\epsilon |Q^\epsilon|^2 + \frac{\mu_2}{\mu}(A^\epsilon - \frac{1}{3} \text{div} u^\epsilon I_d) + \mu_1 [\Omega^\epsilon, Q^\epsilon].
\tag{119}
\]

From the convergence (101) and the relation (104), we have

\[
J u^\epsilon \cdot \nabla \hat{Q}^\epsilon \to J u \cdot \nabla \hat{Q}
\tag{120}
\]

strongly in \( C(\mathbb{R}^+, H^{s-2}_{loc}) \) as \( \epsilon \to 0 \). Furthermore, from the Moser-type calculus inequalities in Lemma 2.3 and the uniform bounds (95) and (96), we deduce that

\[
\|\phi^\epsilon \partial_t \hat{Q}^\epsilon\|_{L^\infty(\mathbb{R}^+, H^s)} + \|\phi^\epsilon u^\epsilon \cdot \nabla \hat{Q}^\epsilon\|_{L^\infty(\mathbb{R}^+, H^{s-1})} \lesssim \|\phi^\epsilon\|_{L^\infty(\mathbb{R}^+, H^s)} \|\partial_t \hat{Q}^\epsilon\|_{L^\infty(\mathbb{R}^+, H^{s-1})} + \|u^\epsilon\|_{L^\infty(\mathbb{R}^+, H^s)} \|\nabla \hat{Q}^\epsilon\|_{L^\infty(\mathbb{R}^+, H^s)} \lesssim C
\]

which means that

\[
\epsilon J \phi^\epsilon \partial_t \hat{Q}^\epsilon, \epsilon J \phi^\epsilon u^\epsilon \cdot \nabla \hat{Q}^\epsilon \to 0
\tag{121}
\]

strongly in \( L^\infty(\mathbb{R}^+, H^{s-1}) \) as \( \epsilon \to 0 \). From convergence (101), it is easily to derive that

\[
\mu_1 \hat{Q}^\epsilon \to \mu_1 Q \quad \text{and} \quad a Q^\epsilon \to a Q
\tag{122}
\]

strongly in \( C(\mathbb{R}^+, H^{s-1}_{loc}) \) as \( \epsilon \to 0 \). The convergence (101) also tells us

\[
L \Delta Q^\epsilon \to L \Delta Q
\tag{123}
\]

strongly in \( C(\mathbb{R}^+, H^{s-2}_{loc}) \) as \( \epsilon \to 0 \). It follows from the convergence (101), the similar arguments in the limit (111) and the analogous derivations of (122) that

\[
b(Q^\epsilon Q^\epsilon - \frac{1}{3} |Q^\epsilon|^2 I_d) \to b(Q Q - \frac{1}{3} |Q|^2 I_d)
\]

\[
c Q^\epsilon |Q^\epsilon|^2 \to c Q |Q|^2
\]

\[
\frac{\mu_2}{\mu}(A^\epsilon - \frac{1}{3} \text{div} u^\epsilon I_d) \to \frac{\mu_2}{\mu} A
\]

\[
\mu_1 [\Omega^\epsilon, Q^\epsilon] \to \mu_1 [\Omega, Q]
\tag{124}
\]
strongly in strongly in $C(\mathbb{R}^+, H^{s-2}_{loc})$ as $\epsilon \to 0$. Finally, for any $T > 0$, by letting $\eta(t, x) \in C^3(0, T; C^\infty(\mathbb{R}^3))$ with $\eta(0, x) = \eta_0(x) \in C^\infty(\mathbb{R}^3)$ and $\eta(t, x) = 0$ for $t \geq T'$ ($T' < T$), we deduce from the initial conditions in Theorem 1.2 and the convergence (99) that

$$\int_0^T \int_{\mathbb{R}^d} J_0 \dot{Q}^t \cdot \eta(t, x) dx dt$$

$$= \int_0^T \int_{\mathbb{R}^d} \dot{Q}^t \cdot \eta_0(t, x) dx - \int_0^T \int_{\mathbb{R}^d} J_0 \cdot \partial_t \eta_0(t, x) dx dt$$

$$\to - \int_0^T \int_{\mathbb{R}^d} \dot{Q} \cdot \eta_0(t, x) dx - \int_0^T \int_{\mathbb{R}^d} J_0 \cdot \partial_t \eta_0(t, x) dx dt$$

as $\epsilon \to 0$. We summarize the limits (120), (121), (122), (123), (124) and (125) that $Q$ obey the evolution

$$J_0 \partial_t \dot{Q} + J_0 \cdot \nabla \dot{Q} + \mu_1 \dot{Q} = L\Delta Q - aQ + b(QQ - \frac{1}{4}|Q|^2 I_d)$$

$$- c|Q|^2 + \frac{\mu}{2} A + \mu_1 [Q, \Omega]$$

with initial data

$$\dot{Q}|_{t=0} = \dot{Q}_0(x).$$

Consequently, the limit functions $(u, Q)$ satisfy the following system

$$\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla \pi - \frac{\beta_1}{2} \Delta u \\
= \text{div} \left( -L
\partial_t \dot{Q} + \frac{1}{4} \Delta \dot{Q} + \beta_1 Q \text{tr}(QA) + \beta_2 AQ + \beta_3 QA \right) \\
+ \text{div} (\mu_0(Q - [\Omega, Q]) + \mu_1 [Q, (\dot{Q} - [\Omega, Q])]),
\end{cases}$$

$$\text{div} u = 0,$$

$$J\dot{Q} + \mu_1 \dot{Q} = L\Delta Q - aQ + b(Q^2 - \frac{1}{4}|Q|^2 I_d)$$

$$- c|Q|^2 + \frac{\mu}{2} A + \mu_1 [\Omega, Q],$$

with the initial data $(u, Q, \dot{Q})|_{t=0} = (u_0(x), Q_0(x), \dot{Q}_0(x))$. Moreover, the uniform bound (95) gives

$$\sup_{t \geq 0} (\|u\|_{H^s}^2 + \|\dot{Q}\|_{H^s}^2 + \|\nabla Q\|_{H^s}^2 + \|Q\|_{H^s}^2 + \|\pi\|_{H^s-1}^2)$$

$$+ \int_0^\infty \|\nabla u\|_{H^s}^2 dt \lesssim \lambda_0.$$ 

Then the proof of Theorem 1.2 is completed.

4. Convergence rate: Proof of Theorem 1.3. In this section, we aim at proving the convergence rate (in $L^2$ norms) of the limit process in Theorem 1.2 by employing the modulated energy method. We first give the following Lemma, which will be play an important role in our calculation.

Lemma 4.1 (Lemma 4.1 of [11]). Under the same assumptions in Theorem 1.3, we have

$$\|\sqrt{\rho} - 1\|_{L^2} \lesssim \epsilon (\Pi^r, 1)^{\frac{3}{2}}, \quad \|\sqrt{\rho_0} - 1\|_{L^2} \lesssim \epsilon (\Pi^r_0, 1)^{\frac{3}{2}} \lesssim \epsilon^{1 + \frac{\Theta}{2}},$$

where $\Pi^r = \frac{1}{r} \eta_r^{\gamma}[(\rho^r)\gamma - \gamma(r^r - 1)]$ and $\Pi^r_0$ has the same form as $\Pi^r$, just replacing $\rho^r$ by $\rho_0^r$ in $\Pi^r$. 

We now turn to prove the Theorem 1.3. First, multiplying (13)_2 by \( \mu^c \) and (13)_3 by \( \dot{Q}^c \), and using (13)_1 and integration by parts, we infer that

\[
\frac{1}{2} \frac{d}{dt} (\|\sqrt{\rho^c} u^c\|^2_{L^2} + J \|\sqrt{\rho^c} \dot{Q}^c\|^2_{L^2} + L \|\nabla Q^c\|^2_{L^2} + a \|Q^c\|^2_{L^2} + 2(\Pi^c, 1)) \\
+ \int_{\Omega^c} F(\nabla u^c, \dot{Q}^c) dx + (\beta_5 + \beta_6) \langle A^c Q^c, A^c \rangle - \mu_2 \langle A^c, [\Omega^c, Q^c] \rangle \\
+ (\frac{1}{2} \beta_4 + \xi) \|\text{div}(u^c)\|^2_{L^2} + \gamma_1 \|Q^c : A^c\|^2_{L^2}\]

\[
= -a \int t \langle Q^c, u^c \cdot \nabla Q^c \rangle dt + \int \langle b(Q^c Q^c - \frac{1}{2} |Q^c|^2 I_d) - cQ^c |Q^c|^2, \dot{Q}^c \rangle dt \\
- \mu_1 \int \langle |\Omega^c, Q^c|, \dot{Q}^c \rangle dt + \int \langle \text{div}(Q^c, [\Omega^c, Q^c]), u^c \rangle dt \\
+ \mu_1 \int \langle \text{div}(Q^c, \dot{Q}^c), u^c \rangle dt + \langle [\Omega^c, Q^c], \dot{Q}^c \rangle dt \\
+ \frac{1}{2} \langle \sqrt{\rho^c} u_0^c \|^2_{L^2} + J \|\sqrt{\rho^c} \dot{Q}_0^c\|^2_{L^2} + L \|\nabla Q_0^c\|^2_{L^2} + a \|Q_0^c\|^2_{L^2} + 2(\Pi_0^c, 1), \rangle 
\]

where the functional \( F(\cdot, \cdot) \) is defined in (20). For any fixed \( T > 0 \), integrating (127) over \((0, t) \subseteq (0, T)\), we get

\[
\frac{1}{2} \int (\|\sqrt{\rho^c} u^c\|^2_{L^2} + J \|\sqrt{\rho^c} \dot{Q}^c\|^2_{L^2} + L \|\nabla Q^c\|^2_{L^2} + a \|Q^c\|^2_{L^2} + 2(\Pi^c, 1)) \\
+ \int_0^t \int_{\Omega^c} F(\nabla u^c, \dot{Q}^c) dx dt + (\beta_5 + \beta_6) \int_0^t \langle A^c Q^c, A^c \rangle dt \\
- \mu_2 \int_0^t \langle A^c, [\Omega^c, Q^c] \rangle dt + (\frac{1}{2} \beta_4 + \xi) \int_0^t \|\text{div}(u^c)\|^2_{L^2} dt + \gamma_1 \int_0^t \|Q^c : A^c\|^2_{L^2} dt
\]

\[
= -a \int \langle Q^c, u^c \cdot \nabla Q^c \rangle dt + \int \langle b(Q^c Q^c - \frac{1}{2} |Q^c|^2 I_d) - cQ^c |Q^c|^2, \dot{Q}^c \rangle dt \\
- \mu_1 \int \langle \text{div}(Q^c, [\Omega^c, Q^c]), u^c \rangle dt + \int \langle \text{div}(Q^c, \dot{Q}^c), u^c \rangle dt \\
+ \mu_1 \int \langle \text{div}(Q^c, \dot{Q}^c), u^c \rangle dt + \langle [\Omega^c, Q^c], \dot{Q}^c \rangle dt \\
+ \frac{1}{2} \langle \sqrt{\rho^c} u_0^c \|^2_{L^2} + J \|\sqrt{\rho^c} \dot{Q}_0^c\|^2_{L^2} + L \|\nabla Q_0^c\|^2_{L^2} + a \|Q_0^c\|^2_{L^2} + 2(\Pi_0^c, 1), \rangle
\]

where \( \Pi^c_0 \) is from replacing the \( \rho^c \) by \( \rho_0^c \) in the quantity \( \Pi^c \).

From the similar calculations in Section 2 of [6], we can derive the following \( L^2 \)-estimate of the incompressible system (15):

\[
\frac{1}{2} \int \|u\|^2_{L^2} + J \|\dot{Q}\|^2_{L^2} + L \|\nabla Q\|^2_{L^2} + a \|Q\|^2_{L^2} + \beta_1 \int \|A\|^2_{L^2} dt \\
+ \int_0^t \int_{\Omega^c} F(\nabla u, \dot{Q}) dx dt + (\beta_5 + \beta_6) \int_0^t \langle A, Q \rangle dt - \mu_2 \int_0^t \langle A, [\Omega, Q] \rangle dt \\
= \int \langle b(QQ - \frac{1}{2} |Q|^2 I_d) - cQ |Q|^2, \dot{Q} \rangle dt + \mu_1 \int \langle \text{div}(Q, \dot{Q}), u \rangle + \langle [\Omega, Q], \dot{Q} \rangle dt \\
- \mu_1 \int \langle \text{div}(Q, [\Omega, Q]), u \rangle dt \\
+ \frac{1}{2} \langle \|u_0\|^2_{L^2} + J \|\dot{Q}_0\|^2_{L^2} + L \|\nabla Q_0\|^2_{L^2} + a \|Q_0\|^2_{L^2} \rangle
\]

holds for all \( t \in [0, T] \), where \( T > 0 \) is an arbitrary fixed number and the functional \( F(\cdot, \cdot) \) is defined in (20).
It follows from taking the inner product of the second equation of (13) with \( u \), and using the second equation of (15) that
\[
\langle \rho' u', u \rangle - \langle \rho'_0 u'_0, u \rangle = \int_0^t \langle \partial_t (\rho' u'), u \rangle + \langle \rho' u', \partial_t u \rangle d\tau
\]
\[
= -L \int_0^t \langle \rho' u', \text{div}(\nabla Q \otimes \nabla Q) \rangle d\tau - \int_0^t \langle \rho' u', u \cdot \nabla u + \nabla \pi - \frac{\beta_4}{2} \Delta u \rangle d\tau
\]
\[
+ \int_0^t \langle \rho' u', \text{div}(\beta_1 \text{tr}(QA) + \beta_5 AQ + \beta_6 QA) \rangle d\tau
\]
\[
+ \int_0^t \langle \rho' u', \text{div}(\frac{\mu_2}{2} (\hat{Q} - [\Omega, Q]) + \mu_1 [Q, (\dot{Q} - [\Omega, Q])] \rangle d\tau
\]
\[
+ \int_0^t \langle \text{div}(\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4) \rangle d\tau + \int_0^t \langle \rho' u' \otimes u' : \nabla u \rangle d\tau,
\]
where we use the fact that \( \text{div} u = 0 \). Observe that the divergence-free property of \( u \) implies
\[
\langle \text{div}(\Sigma_1 + \div(\Sigma_3), u) \rangle = -\frac{\mu_4}{2} \langle \nabla u', \nabla u \rangle + L \langle \nabla Q^c \otimes \nabla Q^c, \nabla u \rangle.
\]
Then the equality (130) can be further rewritten as
\[
-\langle \sqrt{\rho'} u', u \rangle - \beta_4 \int_0^t \langle \nabla u', \nabla u \rangle d\tau
\]
\[
= \mathcal{R}^e_u - \langle \rho'_0 u'_0, u \rangle - \int_0^t \langle \text{div}(\Sigma_3 + \Sigma_4), u \rangle d\tau - \beta_1 \int_0^t \langle u', \text{div}(\text{tr}(QA)) \rangle d\tau
\]
\[
- \int_0^t \langle u', \text{div} (\beta_5 AQ + \beta_6 QA) \rangle d\tau - \frac{\mu_2}{2} \int_0^t \langle u', \text{div} Q \rangle d\tau
\]
\[
+ \frac{\mu_2}{2} \int_0^t \langle u', \text{div} ([\Omega, Q]) \rangle d\tau + \mu_1 \int_0^t \langle u', \text{div} ([Q, [\Omega, Q]]) - \text{div} ([Q, \dot{Q}]) \rangle d\tau
\]
\[
+ L \int_0^t \langle \text{div}(\nabla Q^c \otimes \nabla Q^c), u \rangle d\tau + L \int_0^t \langle u', \text{div}(\nabla Q \otimes \nabla Q) \rangle d\tau, \tag{131}
\]
where
\[
\mathcal{R}^e_u = \langle (\rho' - \sqrt{\rho'}) u', u \rangle - \int_0^t \langle \rho' u' \otimes (u' - u) : \nabla u \rangle d\tau - \frac{\beta_4}{2} \int_0^t \langle (\rho' - 1) u', \Delta u \rangle d\tau
\]
\[
+ \int_0^t \langle \rho' u', \nabla \pi \rangle d\tau + L \int_0^t \langle (\rho' - 1) u', \text{div}(\nabla Q \otimes \nabla Q) \rangle d\tau
\]
\[
- \int_0^t \langle (\rho' - 1) u', \text{div} (\beta_1 \text{tr}(QA) + \beta_5 AQ + \beta_6 QA) \rangle d\tau
\]
\[
- \int_0^t \langle (\rho' - 1) u', \text{div} (\frac{\mu_2}{2} (\hat{Q} - [\Omega, Q]) + \mu_1 [Q, (\dot{Q} - [\Omega, Q])] \rangle d\tau. \tag{132}
\]
From taking the inner product of third equation of (13) with \( \dot{Q} \), and using the third equation of (15), it infer that
\[
J(\sqrt{\rho'} \dot{Q}', \dot{Q}) - L \langle \nabla Q^c, \nabla Q \rangle - a(Q^c, Q)
\]
\[
= J(\langle \rho' - \sqrt{\rho'} \rangle Q', \dot{Q}) - J(\rho'_0 \dot{Q}'_0, \dot{Q}_0) - L(\nabla Q_0, \nabla Q_0) - a(Q'_0, Q_0)
\]
\[
- \int_0^t \langle \rho' \dot{Q}' \otimes u', \nabla Q \rangle d\tau + L \int_0^t \langle \dot{Q}', \Delta Q \rangle d\tau - L \int_0^t \langle u' \cdot \nabla Q^c, \Delta Q \rangle d\tau.\]
\[-L \int_0^t \langle \Delta Q^\epsilon, u \cdot \nabla Q \rangle d\tau - a \int_0^t \langle \dot{Q}^\epsilon, Q \rangle d\tau + a \int_0^t \langle u^\epsilon \cdot \nabla Q^\epsilon, Q \rangle d\tau \]
\[+ \int_0^t \langle \rho^\epsilon \dot{Q}^\epsilon, u \cdot \nabla \dot{Q} + \mu_1 \dot{Q} - L \Delta Q + aQ - b(Q^2 - \frac{1}{2}|Q|^2I_d) \rangle d\tau \]
\[+ a \int_0^t \langle Q^\epsilon, u \cdot \nabla Q \rangle d\tau + \int_0^t \langle \rho^\epsilon \dot{Q}^\epsilon, cQ|Q|^2 - \frac{b_1}{2} A - \mu_1 [\Omega, Q] \rangle d\tau \]
\[- \int_0^t \langle \dot{Q}, -\mu_1 \dot{Q}^\epsilon + b(Q^\epsilon Q^\epsilon - \frac{1}{3}|Q|^2I_d) \rangle d\tau \]
\[- \int_0^t \langle \dot{Q}, cQ^\epsilon|Q|^2 + \frac{b_1}{2} (A^\epsilon - \frac{1}{3}\text{div}u^\epsilon I_d) + \mu_1 [\Omega^\epsilon, Q^\epsilon] \rangle d\tau ,\]

which can be equivalently rewritten as

\[-J \langle \sqrt{\rho} \dot{Q}^\epsilon, \dot{Q} \rangle - L \langle \nabla Q^\epsilon, \nabla Q \rangle - a \langle Q^\epsilon, Q \rangle - 2\mu_1 \int_0^t \langle \dot{Q}, \dot{Q}^\epsilon \rangle d\tau \]
\[= \mathcal{R}_Q + \langle \rho^\epsilon \dot{Q}_0^\epsilon, \dot{Q}_0 \rangle - L \langle \nabla Q_0^\epsilon, \nabla Q_0 \rangle - a \langle Q_0^\epsilon, Q_0 \rangle - L \int_0^t \langle u^\epsilon \cdot \nabla Q^\epsilon, \Delta Q \rangle d\tau \]
\[-L \int_0^t \langle \Delta Q^\epsilon, u \cdot \nabla Q \rangle d\tau + a \int_0^t \langle u^\epsilon \cdot \nabla Q^\epsilon, Q \rangle d\tau + a \int_0^t \langle Q^\epsilon, u \cdot \nabla Q \rangle d\tau \]
\[- \frac{b_1}{2} \int_0^t \langle \dot{Q}, (A^\epsilon - \frac{1}{3}\text{div}u^\epsilon I_d) \rangle d\tau - \mu_1 \int_0^t \langle \dot{Q}, [\Omega, Q] \rangle d\tau \quad (133) \]
\[- \mu_1 \int_0^t \langle \dot{Q}, [\Omega^\epsilon, Q^\epsilon] \rangle d\tau + \int_0^t \langle \dot{Q}, -b(Q^2 - \frac{1}{3}|Q|^2I_d) + cQ|Q|^2 \rangle d\tau \]
\[- \int_0^t \langle \dot{Q}, \dot{Q}^\epsilon Q^\epsilon - \frac{1}{3}|Q|^2I_d \rangle - cQ^\epsilon|Q|^2 \rangle d\tau ,\]

where

\[\mathcal{R}_Q = J \langle \sqrt{\rho} (\sqrt{\rho} - 1) \dot{Q}^\epsilon, \dot{Q} \rangle - \int_0^t \langle \rho^\epsilon \dot{Q}^\epsilon \otimes (u^\epsilon - u), \nabla \dot{Q} \rangle d\tau \]
\[- L \int_0^t \langle (\rho^\epsilon - 1) \dot{Q}^\epsilon, \Delta Q \rangle d\tau + a \int_0^t \langle (\rho^\epsilon - 1) \dot{Q}^\epsilon, Q \rangle d\tau \]
\[+ \mu_1 \int_0^t \langle (\rho^\epsilon - 1) \dot{Q}^\epsilon, [\Omega, Q] \rangle d\tau \quad (134) \]
\[+ \int_0^t \langle (\rho^\epsilon - 1) \dot{Q}^\epsilon, -b(Q^2 - \frac{1}{3}|Q|^2I_d) + cQ|Q|^2 \rangle d\tau - \frac{b_1}{2} \int_0^t \langle (\rho^\epsilon - 1) \dot{Q}^\epsilon, A \rangle d\tau .\]

Direct calculations give

\[\langle \text{div}(\nabla Q^\epsilon \otimes \nabla Q^\epsilon), u \rangle = \langle \Delta Q^\epsilon, u \cdot \nabla Q^\epsilon \rangle + \langle \nabla(\frac{1}{2}|\nabla Q^\epsilon|^2), u \rangle \]
\[= \langle \Delta Q^\epsilon, u \cdot \nabla Q^\epsilon \rangle \quad (135)\]

and

\[\langle \text{div}(\nabla Q \otimes \nabla Q), u^\epsilon \rangle \]
\[= \langle \Delta Q, u^\epsilon \cdot \nabla Q \rangle - \langle \nabla(\frac{1}{2}|\nabla Q|^2), (\rho^\epsilon - 1)u^\epsilon \rangle + \langle \nabla(\frac{1}{2}|\nabla Q|^2), \rho^\epsilon u^\epsilon \rangle , \quad (136)\]

where we using the fact \( \text{div}u = 0 \). Then we can derive

\[L \int_0^t \langle \text{div}(\nabla Q^\epsilon \otimes \nabla Q^\epsilon), u \rangle + L \int_0^t \langle \text{div}(\nabla Q \otimes \nabla Q), u^\epsilon \rangle \]
\[-L \int_0^t \langle u^e \cdot \nabla Q^e, \Delta Q \rangle d\tau - L \int_0^t \langle \Delta Q^e, u \cdot \nabla Q \rangle d\tau \]

\[= L \int_0^t \langle u \cdot \nabla (Q^e - Q), \Delta (Q^e - Q) \rangle + \langle (u^e - u) \cdot \nabla (Q^e - Q), \Delta Q \rangle d\tau \]

\[-L \int_0^t \langle \nabla (\frac{1}{2} |\nabla Q|^2), (\rho^e - 1)u^e \rangle - \langle \nabla (\frac{1}{2} |\nabla Q|^2), \rho^e u^e \rangle \rangle d\tau . \]

We can also get the following equalities

\[-\mu_1 \int_0^t \langle \text{div}([Q^e, [Q^e]], u^e) \rangle d\tau - \mu_1 \int_0^t \langle \text{div}([Q, [\Omega, Q]]), u \rangle d\tau \]

\[+ \mu_1 \int_0^t \langle \text{div}([Q^e, [\Omega', Q^e]]), u \rangle d\tau + \mu_1 \int_0^t \langle \text{div}([Q, [\Omega, Q]]), u^e \rangle d\tau \]

\[= \mu_1 \int_0^t \langle [Q^e - Q, [\Omega, Q']], \nabla u^e - \nabla u \rangle d\tau \]

\[+ \mu_1 \int_0^t \langle [Q, [\Omega, Q' - Q]], \nabla u^e - \nabla u \rangle d\tau \]

\[+ \mu_1 \int_0^t \langle [Q^e, [\Omega^e - \Omega, Q^e]], \nabla u^e - \nabla u \rangle d\tau , \] (138)

and

\[\mu_1 \int_0^t \langle \text{div}([Q^e, [\Omega', Q^e]], u^e) \rangle + \langle [[\Omega^e, Q^e]], \dot{Q}^e \rangle d\tau \]

\[+ \mu_1 \int_0^t \langle \text{div}([Q, \dot{\Omega}]), u \rangle + \langle [\Omega, Q], \dot{Q} \rangle d\tau \]

\[- \mu_1 \int_0^t \langle u^e, \text{div}([Q, \dot{Q}]) \rangle + \langle \text{div}([\Omega', Q^e]), u \rangle d\tau \]

\[- \mu_1 \int_0^t \langle \dot{Q}^e, [\Omega, Q] \rangle + \langle \dot{Q}, [\Omega', Q^e] \rangle d\tau \]

\[= \mu_1 \int_0^t \langle \text{div}([Q^e, (\dot{Q}^e - \dot{Q})]), \text{div}([Q^e - Q]), \dot{Q}^e - \dot{Q} \rangle d\tau \]

\[+ \mu_1 \int_0^t \langle ([\Omega^e - \Omega], Q^e) + [\Omega, (Q^e - Q)], \dot{Q}^e - \dot{Q} \rangle d\tau , \] (139)

and

\[-a \int_0^t \langle Q^e, u^e \cdot \nabla Q^e \rangle d\tau + a \int_0^t \langle u^e \cdot \nabla Q^e, Q \rangle d\tau + a \int_0^t \langle Q^e, u \cdot \nabla Q \rangle d\tau \]

\[= -a \int_0^t \langle Q^e - Q, (u^e - u) \cdot \nabla Q^e \rangle d\tau - a \int_0^t \langle Q^e - Q, u \cdot (\nabla Q^e - \nabla Q) \rangle d\tau , \] (140)

and

\[\int_0^t \langle b(Q^e Q^e - \frac{1}{2} |Q^e|^2) I_d \rangle - c \langle Q^e |Q^e|^2, \dot{Q}^e \rangle d\tau \]

\[+ \int_0^t \langle b(QQ - \frac{1}{2} |Q|^2) I_d \rangle - c \langle Q |Q|^2, \dot{Q} \rangle d\tau \]

\[+ \int_0^t \langle \dot{Q}^e, -b(Q^2 - \frac{1}{2} |Q|^2) I_d \rangle + cQ |Q|^2 \rangle d\tau \]
\[- \int_0^t \langle \dot{Q}, b(Q^eQ^e - \frac{1}{2}|Q^e|^2 I_d) - cQ^e|Q^e|^2 \rangle d\tau \]
\[= b \int_0^t \langle (Q^e - Q)Q^e - \frac{1}{2} \text{tr}((Q^e - Q)Q^e)I_d, \dot{Q}^e - \dot{Q} \rangle d\tau \]
\[- c \int_0^t \langle (Q^e - Q)\text{tr}(Q^eQ^e), \dot{Q}^e - \dot{Q} \rangle d\tau \]
\[+ b \int_0^t \langle Q(Q^e - Q) - \frac{1}{2} \text{tr}(Q(Q^e - Q))I_d, \dot{Q}^e - \dot{Q} \rangle d\tau \]
\[- c \int_0^t \langle \text{tr}(Q^eQ^e), \dot{Q}^e - \dot{Q} \rangle d\tau \]
\[- c \int_0^t \langle \text{tr}(Q(Q^e - Q)), \dot{Q}^e - \dot{Q} \rangle d\tau. \tag{141} \]

From summing up for (128), (129), (131) and (133) and using the cancellation (137), (138), (139), (140) and (141), we deduce that
\[
\frac{1}{2} (\| \sqrt{\mu} u^e - u \|_{L^2}^2 + J\| \sqrt{\beta} \dot{Q}^e - \dot{Q} \|_{L^2}^2 + L \| \nabla Q^e - \nabla Q \|_{L^2}^2
\]
\[+ a \|Q^e - Q\|_{L^2}^2 + 2(\Omega, 1) + \int_0^t \int_{\mathbb{R}^d} F(\nabla u^e - \nabla u, \dot{Q}^e - \dot{Q}) dx d\tau \]
\[+ (\beta_5 + \beta_6) \int_0^t \langle (A^e - A)Q^e, (A^e - A) \rangle d\tau \]
\[+ \left( \frac{1}{2} \beta_4 + \xi \right) \int_0^t \| \text{div} u^e \|_{L^2}^2 d\tau + \beta_1 \int_0^t \| Q^e : (A^e - A) \|_{L^2}^2 d\tau \]
\[- \mu_2 \int_0^t \langle A^e - A, [\Omega^e - \Omega, Q^e] \rangle d\tau + \mu_1 \int_0^t \| \Omega^e - \Omega, Q^e \|_{L^2}^2 d\tau \]
\[= \mathcal{R}_0^e + \mathcal{R}_u^e + \mathcal{R}_Q^e + \mathcal{R}_{\text{sum}}, \tag{142} \]

where $F(\cdot, \cdot) : \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d} \rightarrow \mathbb{R}$ is given in (20), $\mathcal{R}_u^e$ and $\mathcal{R}_Q^e$ are defined in (132) and (134), respectively, $\mathcal{R}_0^e$ and $\mathcal{R}_{\text{sum}}$ are given as
\[
\mathcal{R}_0^e = \frac{1}{2} \| \sqrt{\mu} u^e - u \|_{L^2}^2 + \frac{1}{2} \| \sqrt{\beta} \dot{Q}^e - \dot{Q} \|_{L^2}^2 + \frac{1}{2} L \| \nabla Q^e - \nabla Q \|_{L^2}^2
\]
\[+ \frac{1}{2} a \|Q^e - Q\|_{L^2}^2 + (\Omega_0, 1) - \langle (\rho^e_0 - \sqrt{\rho_0})u^e_0, u_0 \rangle - \langle (\rho^e_0 - \sqrt{\rho_0})\dot{Q}_0^e, \dot{Q}_0 \rangle\]

and
\[
\mathcal{R}_{\text{sum}} = - \beta_1 \int_0^t \langle (Q^e - Q)\text{tr}(Q^eA^e) + Q\text{tr}((Q^e - Q)A^e), \nabla (u^e - u) \rangle d\tau
\]
\[- \beta_5 \int_0^t \langle (Q^e - Q)A, \nabla (u^e - u) \rangle d\tau - \beta_6 \int_0^t \langle A(Q^e - Q), \nabla (u^e - u) \rangle d\tau
\][+ \frac{\mu_2}{2} \int_0^t \langle \Omega, (Q^e - Q) \rangle, \nabla (u^e - u) \rangle d\tau + \int_0^t \langle \text{div} \psi_B(Q^e)I_d, u^e \rangle d\tau
\]+ \mu_1 \int_0^t \langle [(Q^e - Q), [\Omega, Q^e]] + [\Omega, (Q^e - Q)], \nabla (u^e - u) \rangle d\tau
\[+ L \int_0^t \langle u \cdot \nabla (Q^e - Q), \Delta (Q^e - Q) \rangle - \langle (u^e - u) \cdot \nabla (Q^e - Q), \Delta Q \rangle d\tau
\[- L \int_0^t \langle \nabla (\frac{1}{2}|\nabla Q|^2), (\rho^e - 1)u^e \rangle - \langle \nabla (\frac{1}{2}|\nabla Q|^2), \rho^e u^e \rangle d\tau. \]
We emphasize that, in what follows, the difference forms are the most important terms, and the other terms without difference forms, like $u^\epsilon$, $u$, $\rho^\epsilon$, $Q$, $\nabla Q^\epsilon$, $\nabla Q$, $Q^\epsilon$, $Q$ etc., can be bounded by the norms $\|u^\epsilon\|_H$, $\|\rho^\epsilon\|_H$, $\|\nabla Q^\epsilon\|_H$, $\|\nabla Q\|_H$, $\|Q^\epsilon\|_H$, $\|Q\|_H$ via utilizing the Moser-type calculus inequalities in Lemma 2.3. From the uniformly bounds (22) and (23) in Theorem 1.1 and the energy bound (29) in Theorem 1.2, these norms will be bounded by some constants. Consequently, for simplicity, we will focus on the difference forms and control the other terms by some harmless constants in the estimates later.

**Step 1. Estimates for $R^\epsilon_{sum}$.**

First, it is derived from Lemma 4.1 and the Moser-type calculus inequalities in Lemma 2.3 that

\[
- \langle (\rho_0^\epsilon - \sqrt{\rho_0^\epsilon}) u_0^0, u_0 \rangle \langle (\rho_0^\epsilon - \sqrt{\rho_0^\epsilon}) Q_0^\epsilon, Q_0 \rangle = \|\sqrt{\rho_0^\epsilon} - 1\|_{L^2} \|u_0\|_H \|u_0\|_H + \|Q_0^\epsilon\|_H \|Q_0\|_H \lesssim \epsilon^{1+\frac{\delta_0}{2}},
\]

which combining with the initial data given in Theorem 1.3, implies

\[
R^\epsilon_{sum} \lesssim \epsilon^{\delta_0} + \epsilon^{1+\frac{\delta_0}{2}}.
\]  

(143)

Thanks to the Hölder inequality and Young’s inequality, one has

\[
- \beta_1 \int_0^t \langle (Q^\epsilon - Q) \text{tr}(Q^\epsilon A^\epsilon) + Q \text{tr}((Q^\epsilon - Q)A^\epsilon), \nabla (u^\epsilon - u) \rangle d\tau
\]

\[
- \beta_5 \int_0^t \langle (Q^\epsilon - Q)A, \nabla (u^\epsilon - u) \rangle d\tau - \beta_6 \int_0^t \langle A(Q^\epsilon - Q), \nabla (u^\epsilon - u) \rangle d\tau
\]
estimates, we deduce that
\[ \chi \int_0^t \| \nabla u' - \nabla u \|^2_{L^2} d\tau + \int_0^t \| Q' - Q \|^2_{L^2} d\tau. \]
(144)

where \( \chi > 0 \) is small to be determined. From the divergence free property of \( u \) implies
\[
\int_0^t \langle \text{div}_B (Q') I_d, u' \rangle d\tau = \int_0^t \langle \text{div}_B (Q') I_d, u' - u \rangle d\tau
= - \int_0^t \langle \psi_B (Q') I_d, \nabla (u' - u) \rangle d\tau \lesssim \chi \int_0^t \| \nabla u' - \nabla u \|^2_{L^2} d\tau,
\]
(145)

where \( \chi > 0 \) is small to be determined. For the first part of the term \( \mu_1 \int_0^t \langle (Q' - Q), [\Omega, (Q' - Q)] \rangle, \nabla (u' - u) \rangle d\tau \), we have
\[
\mu_1 \int_0^t \langle (Q' - Q), [\Omega, (Q')] + \langle [Q, \Omega, (Q' - Q)], \nabla (u' - u) \rangle d\tau
\lesssim \chi \int_0^t \| \nabla u' - \nabla u \|^2_{L^2} d\tau + \int_0^t \| Q' - Q \|^2_{L^2} d\tau.
\]
(146)

where \( \chi > 0 \) is small to be determined. For the first part of the term \( L \int_0^t \langle u \cdot \nabla (Q' - Q), \Delta (Q' - Q) \rangle - \langle (u' - u) \cdot \nabla (Q' - Q), \Delta Q \rangle d\tau \), we have
\[
L \int_0^t \langle u \cdot \nabla (Q' - Q), \Delta (Q' - Q) \rangle d\tau = - L \int_0^t \langle \nabla u \nabla (Q' - Q), \nabla (Q' - Q) \rangle d\tau
\lesssim \int_0^t \| \nabla Q' - \nabla Q \|^2_{L^2} d\tau
\]
where we use the fact \( \text{div} u = 0 \). Furthermore, for the other parts, we have
\[
- L \int_0^t \langle (u' - u) \cdot \nabla (Q' - Q), \Delta Q \rangle d\tau
= - L \int_0^t \langle ((\sqrt{\rho} u' - u) - (\sqrt{\rho} - 1) u'), \nabla (Q' - Q), \Delta Q \rangle d\tau
\lesssim \int_0^t \| \sqrt{\rho} u' - u \|_{L^2} + \| \sqrt{\rho} - 1 \|_{L^2} \| \nabla Q' - \nabla Q \|_{L^2} d\tau
\lesssim \int_0^t \epsilon^2 (\Pi', 1) + \| \sqrt{\rho} u' - u \|^2_{L^2} + \| \nabla Q' - \nabla Q \|^2_{L^2} d\tau,
\]
where the last inequality is implied by Lemma 4.1. From collecting above two estimates, we deduce that
\[
L \int_0^t \langle u \cdot \nabla (Q' - Q), \Delta (Q' - Q) \rangle - \langle (u' - u) \cdot \nabla (Q' - Q), \Delta Q \rangle d\tau
\lesssim \int_0^t \epsilon^2 (\Pi', 1) + \| \sqrt{\rho} u' - u \|^2_{L^2} + \| \nabla Q' - \nabla Q \|^2_{L^2} d\tau
\]
(147)
For the term $-L \int_0^t \langle \nabla \left( \frac{1}{2} |\nabla Q|^2 \right), (\rho^\varepsilon - 1)u^\varepsilon \rangle - \langle \nabla \left( \frac{1}{2} |\nabla Q|^2 \right), \rho^\varepsilon u^\varepsilon \rangle d\tau$, we estimate that

$$\begin{align*}
-L \int_0^t \langle \nabla \left( \frac{1}{2} |\nabla Q|^2 \right), (\rho^\varepsilon - 1)u^\varepsilon \rangle d\tau \\
= -L \int_0^t \langle \nabla \left( \frac{1}{2} |\nabla Q|^2 \right), (\sqrt{\rho^\varepsilon} - 1)(\sqrt{\rho^\varepsilon} + 1)u^\varepsilon \rangle \\
\lesssim \int_0^t \| \sqrt{\rho^\varepsilon} - 1 \|_{L^2} d\tau \lesssim \int_0^t \epsilon (\Pi', 1) \frac{1}{2} d\tau \lesssim \epsilon^2 T + \int_0^t (\Pi', 1) d\tau
\end{align*}$$

(148)

for all $t \in [0, T]$, where the last inequality is implied by Lemma 4.1. Furthermore, we have

$$\begin{align*}
L \int_0^t \langle \nabla \left( \frac{1}{2} |\nabla Q|^2 \right), \rho^\varepsilon u^\varepsilon \rangle d\tau &= -\frac{1}{2} L \int_0^t \langle |\nabla Q|^2, \text{div}(\rho^\varepsilon u^\varepsilon) \rangle d\tau \\
= &\frac{1}{2} L \int_0^t \langle |\nabla Q|^2, \partial_t (\rho^\varepsilon) \rangle d\tau = \frac{1}{2} L \int_0^t \langle |\nabla Q|^2, \partial_t (\rho^\varepsilon - 1) \rangle d\tau \\
= &\frac{1}{2} L \| |\nabla Q|^2, (\rho^\varepsilon - 1) \| - \frac{1}{2} L \| (\rho^\varepsilon_0 - 1) \| - \frac{1}{2} L \int_0^t \langle |\nabla Q|^2, (\rho^\varepsilon - 1) \rangle d\tau \\
\lesssim &\| \sqrt{\rho^\varepsilon} - 1 \|_{L^2} + \| \sqrt{\rho^\varepsilon_0} - 1 \|_{L^2} + \int_0^t \| \sqrt{\rho^\varepsilon} - 1 \|_{L^2} d\tau \\
\lesssim &\epsilon (\Pi', 1)^{\frac{1}{2}} + \epsilon^{1 + \frac{\gamma}{2}} + \int_0^t \epsilon (\Pi', 1) d\tau \\
\lesssim & (1 + T) \epsilon^2 + \epsilon^{1 + \frac{\gamma}{2}} + \chi (\Pi', 1) + \int_0^t (\Pi', 1) d\tau,
\end{align*}$$

(149)

for all $t \in [0, T]$, where $\chi > 0$ is small to be determined and the Lemma 4.1 are also utilized. For the term $-a \int_0^t \langle Q^\varepsilon - Q, (u^\varepsilon - u) \cdot \nabla Q^\varepsilon \rangle$, we have

$$\begin{align*}
-a \int_0^t \langle Q^\varepsilon - Q, (u^\varepsilon - u) \cdot \nabla Q^\varepsilon \rangle \\
= -a \int_0^t \langle Q^\varepsilon - Q, ((\sqrt{\rho^\varepsilon}) u^\varepsilon - u) - (\sqrt{\rho^\varepsilon} - 1) u^\varepsilon \rangle \cdot \nabla Q^\varepsilon \\
\lesssim \int_0^t (|\sqrt{\rho^\varepsilon} u^\varepsilon - u|_{L^2} + \| \sqrt{\rho^\varepsilon} - 1 \|_{L^2}) \| Q^\varepsilon - Q \|_{L^2} d\tau \\
\lesssim \int_0^t \epsilon^2 (\Pi', 1) + \| \sqrt{\rho^\varepsilon} u^\varepsilon - u \|_{L^2}^2 + \| \nabla Q^\varepsilon - \nabla Q \|_{L^2} d\tau
\end{align*}$$

(150)

As for the estimate the term $-a \int_0^t \langle Q^\varepsilon - Q, u \cdot (\nabla Q^\varepsilon - \nabla Q) \rangle$, we have

$$\begin{align*}
-a \int_0^t \langle Q^\varepsilon - Q, u \cdot (\nabla Q^\varepsilon - \nabla Q) \rangle \lesssim \int_0^t \| Q^\varepsilon - Q \|_{L^2}^2 + \| \nabla Q^\varepsilon - \nabla Q \|_{L^2}^2 d\tau
\end{align*}$$

(151)

For the term $b \int_0^t \langle (Q^\varepsilon - Q)Q^\varepsilon - \frac{1}{2} \text{tr}((Q^\varepsilon - Q)Q^\varepsilon) I_d, \dot{Q}^\varepsilon - \dot{Q} \rangle d\tau$, we have

$$\begin{align*}
b \int_0^t \langle (Q^\varepsilon - Q)Q^\varepsilon - \frac{1}{2} \text{tr}((Q^\varepsilon - Q)Q^\varepsilon) I_d, \dot{Q}^\varepsilon - \dot{Q} \rangle d\tau \\
= b \int_0^t \langle (Q^\varepsilon - Q)Q^\varepsilon - \frac{1}{2} \text{tr}((Q^\varepsilon - Q)Q^\varepsilon) I_d, (\sqrt{\rho^\varepsilon} \dot{Q}^\varepsilon - \dot{Q}) - (\sqrt{\rho^\varepsilon} - 1) \dot{Q}^\varepsilon \rangle d\tau
\end{align*}$$
\[
\begin{align*}
\leq & \int_0^t (\|\sqrt{\rho} \dot{Q}^e - \dot{Q}\|_{L^2} + \|\sqrt{\rho} - 1\|_{L^2}) \|Q^e - Q\|_{L^2} d\tau \\
\leq & \int_0^t c^2 \langle \Pi^e, 1 \rangle + \|\sqrt{\rho} \dot{Q}^e - \dot{Q}\|_{L^2}^2 + \|Q^e - Q\|_{L^2}^2 d\tau.
\end{align*}
\]

By the similar arguments in (152), one has

\[
\begin{align*}
& b \int_0^t \langle Q(Q^e - Q) - \frac{1}{h} \text{tr}(Q(Q^e - Q)) I_d, \dot{Q}^e - \dot{Q} \rangle d\tau \\
\leq & \int_0^t c^2 \langle \Pi^e, 1 \rangle + \|\sqrt{\rho} \dot{Q}^e - \dot{Q}\|_{L^2}^2 + \|Q^e - Q\|_{L^2}^2 d\tau, \\
& - c \int_0^t \langle (Q^e - Q) \text{tr}(Q^e Q^e) + Q \text{tr}((Q^e - Q) Q^e) + Q \text{tr}(Q(Q^e - Q)), \dot{Q}^e - \dot{Q} \rangle d\tau \\
\leq & \int_0^t c^2 \langle \Pi^e, 1 \rangle + \|\sqrt{\rho} \dot{Q}^e - \dot{Q}\|_{L^2}^2 + \|Q^e - Q\|_{L^2}^2 d\tau, \\
& \mu_1 \int_0^t \langle \langle Q^e - \Omega^e, Q^e \rangle + [\Omega^e, (Q^e - Q)], \dot{Q}^e - \dot{Q} \rangle d\tau \\
\leq & \int_0^t \langle \|\sqrt{\rho} \dot{Q}^e - \dot{Q}\|_{L^2}^2 + \|\sqrt{\rho} - 1\|_{L^2} \|Q^e - Q\|_{L^2}^2 + \|\nabla u^e - \nabla u\|_{L^2} d\tau \\
\leq & \int_0^t c^2 \langle \Pi^e, 1 \rangle + \|Q^e - Q\|_{L^2}^2 + \|\sqrt{\rho} \dot{Q}^e - \dot{Q}\|_{L^2}^2 d\tau + \chi \int_0^t \|\nabla u^e - \nabla u\|_{L^2}^2 d\tau.
\end{align*}
\]

It remains to control the term \(\mu_1 \int_0^t \langle \text{div}[Q^e, (\dot{Q}^e - \dot{Q})] + \langle \text{div}([Q^e - Q], \dot{Q}], u^e - u)\rangle d\tau\). It is easy to know that

\[
\begin{align*}
& \mu_1 \int_0^t \langle \text{div}[Q^e, (\dot{Q}^e - \dot{Q})] + \langle \text{div}([Q^e - Q], \dot{Q}], u^e - u)\rangle d\tau \\
= & - \mu_1 \int_0^t [Q^e, (\dot{Q}^e - \dot{Q})] + [[Q^e - Q], \dot{Q}], u^e - \nabla u) \\
\leq & \int_0^t c^2 \langle \Pi^e, 1 \rangle + \|Q^e - Q\|_{L^2}^2 + \|\sqrt{\rho} \dot{Q}^e - \dot{Q}\|_{L^2}^2 d\tau + \chi \int_0^t \|\nabla u^e - \nabla u\|_{L^2}^2 d\tau
\end{align*}
\]

where \(\chi > 0\) is small to be determined.

Consequently, the bounds (144), (145), (146), (147), (148), (149), (150), (151), (152), (153) and (154) tell us

\[
\mathcal{R}^e_{\text{sum}} \leq (1 + T)c^2 + \epsilon^{1 + \frac{\alpha_2}{2}} + \chi \langle \Pi^e, 1 \rangle + \chi \int_0^t \|\nabla u^e - \nabla u\|_{L^2}^2 d\tau + \int_0^t \langle \Pi^e, 1 \rangle d\tau
\]

\[
+ \int_0^t \|\sqrt{\rho} \dot{Q}^e - \dot{Q}\|_{L^2}^2 + \|\sqrt{\rho} u^e - u\|_{L^2}^2 + \|Q^e - Q\|_{L^2}^2 + \|\nabla Q^e - \nabla Q\|_{L^2}^2 d\tau.
\]

Step 2. Estimates for \(\mathcal{R}^e_u\).

First, from Lemma 4.1, we have

\[
\langle \rho^e - \sqrt{\rho^e} u^e, u \rangle \leq \|\sqrt{\rho^e} - 1\|_{L^2} \|\rho^e - \sqrt{\rho^e} \|_{L^2} \leq \epsilon \langle \Pi^e, 1 \rangle^\frac{3}{2} \leq \epsilon^2 + \chi \langle \Pi^e, 1 \rangle
\]
and
\[- \int_0^t \langle \rho^\ast u^\ast \otimes (u^\ast - u) : \nabla u \rangle d\tau \]
\[= - \int_0^t \langle u, \rho^\ast u^\ast \cdot \nabla u \rangle d\tau - \int_0^t \langle \sqrt{\rho^\ast} u^\ast - u, (\sqrt{\rho^\ast} u^\ast - u) \cdot \nabla u \rangle d\tau \]
\[+ \int_0^t \langle u \otimes (\sqrt{\rho^\ast} - 1)\sqrt{\rho^\ast} u^\ast + (\sqrt{\rho^\ast} - 1)\sqrt{\rho^\ast} u^\ast \otimes u, \nabla u \rangle d\tau .\]

From the similar arguments in (149), we have
\[- \int_0^t \langle u, \rho^\ast u^\ast \cdot \nabla u \rangle d\tau \lesssim (1 + T)\epsilon^2 + \epsilon^{1+\frac{\alpha}{2}} + \chi \langle \Pi^r, 1 \rangle + \int_0^t \langle \Pi^r, 1 \rangle d\tau \]
for some small \( \chi > 0 \) to be determined. Furthermore, we estimate
\[- \int_0^t \langle u \otimes (\sqrt{\rho^\ast} - 1)\sqrt{\rho^\ast} u^\ast + (\sqrt{\rho^\ast} - 1)\sqrt{\rho^\ast} u^\ast \otimes u, \nabla u \rangle d\tau \]
\[\lesssim \int_0^t \|\sqrt{\rho^\ast} u^\ast - u\|^2_{L^2} + \|\sqrt{\rho^\ast} - 1\|^2_{L^2} d\tau \]
\[\lesssim T\epsilon^2 + \int_0^t \langle \Pi^r, 1 \rangle + \|\sqrt{\rho^\ast} u^\ast - u\|^2_{L^2} d\tau \]
Consequently, we have
\[- \int_0^t \langle \rho^\ast u^\ast \otimes (u^\ast - u) : \nabla u \rangle d\tau \]
\[\lesssim (1 + T)\epsilon^2 + \epsilon^{1+\frac{\alpha}{2}} + \chi \langle \Pi^r, 1 \rangle + \int_0^t \langle \Pi^r, 1 \rangle + \|\sqrt{\rho^\ast} u^\ast - u\|^2_{L^2} d\tau \quad (157) \]
for all \( t \in [0, T] \). Moreover, Lemma 4.1 tells us
\[- \frac{\beta_3}{2} \int_0^t \langle (\rho^\ast - 1)u^\ast, \Delta u \rangle d\tau = - \frac{\beta_3}{2} \int_0^t \langle \sqrt{\rho^\ast} (\sqrt{\rho^\ast} - 1)u^\ast, \Delta u \rangle d\tau \]
\[\lesssim \int_0^t \|\sqrt{\rho^\ast} - 1\|^2_{L^2} \lesssim \int_0^t \epsilon \langle \Pi^r, 1 \rangle d\tau \lesssim T\epsilon^2 + \int_0^t \langle \Pi^r, 1 \rangle d\tau \quad (158) \]
and similarly
\[- \int_0^t \langle (\rho^\ast - 1)u^\ast, \div(\beta_1 Q \tr(Q^A) + \beta_5 A \div Q + \beta_6 QA) \rangle d\tau \]
\[- \int_0^t \langle (\rho^\ast - 1)u^\ast, \div((\frac{\mu_2}{2} (\mathbf{Q}^\ast - [\Omega, Q])) + \mu_1 [Q, (\mathbf{Q}^\ast - [\Omega, Q])] ) \rangle d\tau \]
\[+ \int_0^t \langle (\rho^\ast - 1)u^\ast, \Ldiv(Q \otimes \nabla Q) \rangle d\tau \lesssim T\epsilon^2 + \int_0^t \langle \Pi^r, 1 \rangle d\tau \quad (159) \]
It is also deduced from the analogous arguments in (149) that
\[\int_0^t \langle \rho^\ast u^\ast, \nabla \pi \rangle d\tau \lesssim (1 + T)\epsilon^2 + \epsilon^{1+\frac{\alpha}{2}} + \chi \langle \Pi^r, 1 \rangle + \int_0^t \langle \Pi^r, 1 \rangle d\tau \quad (160) \]
for all $t \in [0, T]$ and $\chi > 0$ is small to be determined. As a result, from the collecting the bounds (156), (157), (158), (159) and (160), we deduce
\[
\mathcal{R}_u \lesssim (1 + T)c^2 + e^{1 + \frac{a}{2T}} + \chi \langle \Pi^r, 1 \rangle + \int_0^t \langle \Pi^r, 1 \rangle + \|\sqrt{\nu}u^r - u\|_{L^2}^2 d\tau \tag{161}
\]

**Step 3. Estimates for $\mathcal{R}_Q$.**

From the estimates in (156) and (157), we can derive
\[
\langle \sqrt{\nu'}(\sqrt{\nu} - 1)\dot{Q}', \dot{Q}' \rangle \lesssim c^2 + \chi \langle \Pi^r, 1 \rangle \tag{162}
\]
and
\[
- \int_0^t \langle \rho' \dot{Q}' \otimes (u^r - u), \nabla \dot{Q}' \rangle d\tau \\
\lesssim (1 + T)c^2 + e^{1 + \frac{a}{2T}} + \chi \langle \Pi^r, 1 \rangle \\
+ \int_0^t \langle \Pi^r, 1 \rangle + \|\sqrt{\nu}u^r - u\|_{L^2}^2 + \|\sqrt{\rho} \dot{Q}' - \dot{Q}\|_{L^2}^2 d\tau \tag{163}
\]
for all $t \in [0, T]$, where $\chi > 0$ is small to be determined. For the term $-L \int_0^t \langle (\rho^r - 1)\dot{Q}', \Delta Q \rangle d\tau$, we have
\[
- L \int_0^t \langle (\rho^r - 1)\dot{Q}', \Delta Q \rangle d\tau = -L \int_0^t \langle (\sqrt{\nu} - 1)\dot{Q}', \Delta Q \rangle d\tau \\
\lesssim \int_0^t \|\sqrt{\nu} - 1\|_{L^2}^2 d\tau \lesssim \int_0^t \epsilon \langle \Pi^r, 1 \rangle^\frac{1}{2} d\tau \lesssim Tc^2 + \int_0^t \langle \Pi^r, 1 \rangle d\tau. \tag{164}
\]
Similarly in (164), we have
\[
a \int_0^t \langle (\rho^r - 1)\dot{Q}', Q \rangle d\tau + \mu_1 \int_0^t \langle (\rho^r - 1)\dot{Q}', \dot{Q} \rangle d\tau \lesssim Tc^2 + \int_0^t \langle \Pi^r, 1 \rangle d\tau \\
- \mu_1 \int_0^t \langle (\rho^r - 1)\dot{Q}', [\Omega, Q] \rangle d\tau \lesssim Tc^2 + \int_0^t \langle \Pi^r, 1 \rangle d\tau \\
\int_0^t \langle (\rho^r - 1)\dot{Q}', -b(Q^2 - \frac{1}{4}|Q|^2 I_d) + cQ|Q|^2 \rangle d\tau \lesssim Tc^2 + \int_0^t \langle \Pi^r, 1 \rangle d\tau \\
- \frac{\tilde{a}_2}{2} \int_0^t \langle (\rho^r - 1)\dot{Q}', A \rangle d\tau \lesssim Tc^2 + \int_0^t \langle \Pi^r, 1 \rangle d\tau. \tag{165}
\]
Finally, from the inequalities (162), (163), (164) and (165), we have
\[
\mathcal{R}_Q \lesssim (1 + T)c^2 + e^{1 + \frac{a}{2T}} + \chi \langle \Pi^r, 1 \rangle + \int_0^t \langle \Pi^r, 1 \rangle d\tau \\
+ \int_0^t \|\sqrt{\nu}u^r - u\|_{L^2}^2 + \|\sqrt{\rho} \dot{Q}' - \dot{Q}\|_{L^2}^2 d\tau \tag{166}
\]
It therefore easily follows from substituting the inequalities (155), (161) and (166) into (142) that for and fixed $T > 0$,
\[
\frac{1}{2} \left(\|\sqrt{\nu}u^r - u\|_{L^2}^2 + J\|\sqrt{\nu} \dot{Q}' - \dot{Q}\|_{L^2}^2 + L\|\nabla Q^r - \nabla Q\|_{L^2}^2 + a\|Q' - Q\|_{L^2}^2 + 2\langle \Pi^r, 1 \rangle \right) \\
+ \int_0^t \int_{\mathbb{R}^d} F(\nabla u^r - \nabla u, \dot{Q}' - \dot{Q}) dx d\tau + (\beta_5 + \beta_6) \int_0^t \langle (A' - A)Q^r, (A' - A) \rangle d\tau
\]
\[ + \left( \frac{1}{2} \beta_4 + \xi \right) \int_0^t \| \text{div} u' \|^2 L^2 d\tau + \beta_1 \int_0^t \| Q' : (A' - A) \|^2 L^2 d\tau \]
\[ - \mu_2 \int_0^t \langle A' - A, [\Omega' - \Omega, Q'] \rangle d\tau + \mu_1 \int_0^t \| \Omega' - \Omega, Q' \|^2 L^2 d\tau \]
\[ \lesssim \epsilon^{\theta_0} + \epsilon^{1 + \frac{2 \theta}{\nu}} + (1 + T) \epsilon^2 + \chi \langle \Pi', 1 \rangle + \chi \int_0^t \| \nabla u' - \nabla u \|^2 L^2 d\tau + \int_0^t \langle \Pi', 1 \rangle d\tau \]
\[ + \int_0^t \| \sqrt{\rho} u' - u \|^2 L^2 + \| \sqrt{\rho} \dot{Q}' - \dot{Q} \|^2 L^2 + \| \nabla Q' - \nabla Q \|^2 L^2 + \| Q' - Q \|^2 L^2 d\tau \]
holds for all \( t \in [0, T] \) and \( 0 < \epsilon \leq 1 \), where \( \chi > 0 \) is small to be determined. We then assume \( 0 < \chi \leq \frac{1}{2} \) such that (167) reduces to
\[ f(t) + g(t) \leq C(1 + T) \epsilon^{\theta_0} + C \int_0^t f(\tau) d\tau + C \chi \int_0^t \| \nabla u' - \nabla u \|^2 L^2 d\tau \]
for all \( t \in [0, T] \), where \( \theta_0 = \min \{ 2, \theta_0, 1 + \frac{2 \theta}{\nu} \} > 0 \), \( C > 0 \) is an \( \epsilon \)-independent constant, the function \( f(t) \) is given as
\[ f(t) = \| \sqrt{\rho} u' - u \|^2 L^2 + J \| \sqrt{\rho} \dot{Q}' - \dot{Q} \|^2 L^2 + L \| \nabla Q' - \nabla Q \|^2 L^2 \]
\[ + a \| Q' - Q \|^2 L^2 + \langle \Pi', 1 \rangle \]
and the functions \( g(t) \) reads
\[ g(t) = \int_0^t \int_{\mathbb{R}^d} F(\nabla u' - \nabla u, \dot{Q}' - \dot{Q}) dx d\tau \]
\[ + (\beta_5 + \beta_6) \int_0^t \langle (A' - A) Q', (A' - A) \rangle d\tau - \mu_2 \int_0^t \langle A' - A, [\Omega' - \Omega, Q'] \rangle d\tau . \]
We remark that the coefficients’ relations \( \frac{1}{2} \beta_4 + \xi \geq 0, \beta_1 \geq 0 \) and \( \mu_1 > 0 \) are also used here.

It follows from the properties of \( F(\cdot, \cdot) \) in (20) that
\[ \int_0^t \int_{\mathbb{R}^d} F(\nabla u' - \nabla u, \dot{Q}' - \dot{Q}) dx d\tau \]
\[ \geq \delta_0 \frac{1}{2} \beta_4 \int_0^t \| \nabla u' - \nabla u \|^2 L^2 d\tau + \delta_1 \mu_1 \int_0^t \| \dot{Q}' - \dot{Q} \|^2 L^2 d\tau \]
(168)
for some \( \delta_0, \delta_1 \in (0, 1) \). Moreover, we derive from the uniform bound (22) that
\[ (\beta_5 + \beta_6) \int_0^t \langle (A' - A) Q', (A' - A) \rangle d\tau - \mu_2 \int_0^t \langle A' - A, [\Omega' - \Omega, Q'] \rangle d\tau \]
\[ \leq C\| Q' \|_{H^s} \int_0^t \| \nabla u' - \nabla u \|^2 L^2 d\tau \leq C' \sqrt{\lambda_0} \int_0^t \| \nabla u' - \nabla u \|^2 L^2 d\tau \]
(169)
where the small \( \lambda_0 > 0 \) is given in Theorem 1.1. Actually, following the proof of Theorem 1.1, we know that the small \( \lambda_0 > 0 \) is such that \( \delta_0 \frac{1}{2} \beta_4 \beta_1 - C' \sqrt{\lambda_0} > 0 \). Consequently, we have
\[ g(t) \geq c_0 \int_0^t \| \nabla u' - \nabla u \|^2 L^2 + \| \dot{Q}' - \dot{Q} \|^2 L^2 d\tau , \]
where \( c_0 = \min \{ \delta_0 \frac{1}{2} \beta_4 \beta_1 - C' \sqrt{\lambda_0}, \delta_1 \mu_1 \} > 0 \). We now choose \( \chi = \min \{ \frac{1}{2}, \frac{1}{2c_0} \} > 0 \). Then we know that \( f(t) \leq C(1 + T) \epsilon^{\theta_0} + C \int_0^t f(\tau) d\tau \), which implies by the Grönwall
inequality that

$$f(t) \leq C(1 + T) \exp(CT)e^{\theta_0}$$

for all $t \in [0, T]$. This concludes the Theorem 1.3.

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