Monotonicity and rigidity of the $\mathcal{W}$-entropy on RCD$(0, N)$ spaces

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Abstract. By means of a space-time Wasserstein control, we show the monotonicity of the $\mathcal{W}$-entropy functional in time along heat flows on possibly singular metric measure spaces with non-negative Ricci curvature and a finite upper bound of dimension in an appropriate sense. The associated rigidity result on the rate of dissipation of the $\mathcal{W}$-entropy is also proved. These extend known results even on weighted Riemannian manifolds in some respects. In addition, we reveal that some singular spaces will exhibit the rigidity models while only the Euclidean space does in the class of smooth weighted Riemannian manifolds.

1. Introduction

The $\mathcal{W}$-entropy functional was first introduced on Ricci flow in the celebrating work of G. Perelman [44] for the resolution of the Poincaré conjecture. Since then, it has played important role in the various topics in the study of geometric analysis and stochastic analysis. Among others, the $\mathcal{W}$-entropy exhibits two specific properties: monotonicity in time along conjugate heat equations and rigidity. The latter means that the time derivative of the $\mathcal{W}$-entropy along a conjugate heat equation vanishes if and only if the Ricci flow is a gradient shrinking Ricci soliton. As one of possible extensions in connection with these properties, the notion of $\mathcal{W}$-entropy is exported in several different situations. Ni [42,43] brought the notion of $\mathcal{W}$-entropy to the heat equation $\partial_t u = \Delta u$ on static Riemannian manifolds, where static means that the Riemannian metric does not depend on time. He proved the same sort of monotonicity and rigidity for the $\mathcal{W}$-entropy for the heat equation $\partial_t u = \Delta u$ under...
non-negative Ricci curvature with an additional bounded geometry assumption. In a series of works by the second named author [37–40], the $W$-entropy formula as well as its monotonicity and rigidity have been extended to the heat equation $\partial_t u = Lu$ associated with the Witten Laplacian $L = \Delta - \nabla \phi \cdot \nabla$ on complete Riemannian manifolds with weighted volume measure $d\mu = e^{-\phi} dv$, where $v$ is the Riemannian volume measure. Furthermore, such extensions have been carried out for the heat equation associated with the time-dependent Witten Laplacian on the so-called $(K, m)$ super-Ricci flows by the second named author and S. Li [33–36]. Here the $(K, m)$ super-Ricci flow means that the Riemannian metric $g_t$ is time-dependent and evolves along the following differential inequality
\[
\frac{1}{2} \frac{\partial g_t}{\partial t} + \text{Ric}_{m,n} \geq Kg_t,
\]
where $\text{Ric}_{m,n} = \text{Ric}(g_t) + \nabla^2 \phi_t - \frac{\nabla \phi_t \otimes \nabla \phi_t}{m-n}$ is the $m$-dimensional Bakry–Émery Ricci curvature on $n$-dimensional complete Riemannian manifolds $(M, g_t)$ with fixed weighted volume measure $d\mu = e^{-\phi} dv_{g_t}$. Note that the notion of the $(K, m)$-super Ricci flow can be regarded as an extension of the super Ricci flow in geometric analysis
\[
\frac{\partial g_t}{\partial t} \geq -2 \text{Ric},
\]
which includes the Ricci flows as the case of equality, and also an extension of static Riemannian manifolds with Ricci curvature bounded from below by a constant (i.e., $\text{Ric} \geq K$).

In this article, we study the same sort of problem on more singular spaces than differentiable manifolds. The notion of spaces with a lower Ricci curvature bound has been extended from Riemannian manifolds to metric measure spaces by means of optimal transport [41,48]. Thus it seems natural to consider this problem on such spaces with non-negative Ricci curvature in a generalized sense. As an initial work to this direction, we only consider the static case, though it seems possible to consider the corresponding problem on time-dependent metric measure spaces on the basis of [28,47].

To state our result in comparison with the one in the smooth case, let us begin with reviewing the result on $n$-dimensional weighted Riemannian manifolds as mentioned above according to [33–35,38,39]. To begin with, let us remark that, for $K \in \mathbb{R}$, Bakry–Émery’s curvature-dimension condition for $L$
\[
\frac{1}{2} L|\nabla f|^2 - \langle \nabla f, \nabla Lf \rangle \geq K |\nabla f|^2 + \frac{1}{m} |Lf|^2 \tag{1.1}
\]
holds for any $f \in C^3(M)$ if and only if
\[
\text{Ric}_{m,n} \geq K. \tag{1.2}
\]
Roughly speaking, $m$ plays the role of an upper bound of the dimension of the space. Indeed, (1.1) can be used as an abstract generalization of the condition “$\text{Ric} \geq K$
and dim ≤ m” in Bakry–Émery theory (see [10,11]). Suppose n < m < ∞. Then we define the \( W \)-entropy for \( f \in C^1(M) \) and \( t > 0 \) as follows:

\[
W(f, t) := \int_M \left[ t|\nabla f|^2 + m - f \right] \frac{e^{-f}}{(4\pi t)^{m/2}} \, dm. \tag{1.3}
\]

It coincides with the one introduced in [43] when \( m = n \) and \( \phi = 0 \). Suppose that \( f \) depends also on \( t \) and that \( u := e^{-f}/(4\pi t)^{m/2} \) solves the heat equation \( \partial_t u = Lu \).

Then, the monotonicity of \( W \)

\[
\frac{d}{dt} W(f, t) \leq 0 \tag{1.4}
\]

holds under Ric\(_{m,n} \geq 0 \) and a bounded geometry assumption. Suppose additionally that \( u \) is a heat kernel. That is, \( u \to \delta_y \) for some \( y \in M \) weakly. Then the rigidity of \( W \) states that the equality holds in (1.4) at some \( t > 0 \) only when \( (M, g) \) is isometric to \( \mathbb{R}^m \) and \( \phi \) is constant. The proof is based on establishing the so-called “entropy formula” which explicitly describes the time derivative of \( W \). For details, see [33–35,38,39].

A natural class of metric measure spaces where we consider our problem is RCD(0, N) (or equivalently RCD\(^\ast\)(0, N)) spaces. Intuitively, RCD\(^\ast\)(K, N) means that the space satisfies “Ric ≥ K and dim ≤ N” and the canonical heat flow given by the metric measure structure is linear in initial data. There are several different characterizations of RCD\(^\ast\)(K, N) spaces, and the Bochner inequality like (1.1) is one of them. See [1,5,6] for \( N = \infty \) and [7,15] for \( N < \infty \). Note that the study of RCD(K, N) spaces for finite \( N \) are initiated in [19,20] and connection with the Bakry–Émery condition is established in [7,15]. See the next section for more details and additional references. Recall that any \( n \)-dimensional weighted Riemannian manifold \((M, g, \mu)\) with Ric\(_{N,n} \geq K\) is an RCD\(^\ast\)(K, N) space by regarding it as a metric measure space by the Riemannian distance and the weighted measure \( \mu \).

Our main theorems are the monotonicity of \( W \) (Theorem 3.3) and the associated rigidity (Theorem 4.1) on RCD(0, N) spaces. Note that our rigidity theorem improves the previous result even on weighted Riemannian manifolds in the following three respects. First, we do not require a differentiability of \( W \) in time along the heat flow but we consider the right upper derivative instead. Second, we do not need to assume the initial data to be the Dirac mass. Actually, it follows as a consequence of the rigidity: If the right upper derivative vanishes, then the initial data must be Dirac. Third, we do not require any assumption corresponding to the “bounded geometry”. Moreover, we find that not only Euclidean spaces but the Euclidean cones enjoys the vanishing time derivative of \( W \), where the vertex of the cone coincides with the point where the initial mass is located. The Euclidean cones appearing in our rigidity have singularity at vertex if it is not a Euclidean space, and in general it is even not a manifold. In this sense, our result is compatible with the previous ones and we succeed in finding new examples as a result of expanding the class of spaces we consider.

For the proof of our main results, we relies on an approach from optimal transport which is not used in previous results [33,37,38,40]. A naive approach to
our problem is to establish the entropy formula on $\text{RCD}(0, N)$ spaces. However, it does not seem to be straightforward by the following two reasons. First, the entropy formula involves the Ricci curvature $\text{Ric}$, the Hessian $\text{Hess }f$ and some second order tensor calculus is required. Although such objects have been introduced on $\text{RCD}$ spaces in recent development [18], the lack of smoothness can be an obstacle. Second, it seems that some assumption like the bounded geometry is required to obtain the entropy formula. It is not clear how we formulate such an assumption with keeping non-trivial examples since the bounded geometry assumption involves the Riemann curvature tensor and its derivatives. Because of them, we prove the monotonicity and the rigidity without entropy formulae. The idea of our proof comes from [49], where P. Topping studies the monotonicity of the $\mathcal{W}$-entropy on a Ricci flow on a compact manifold by proving an estimate of a transportation cost between two heat distributions where the cost function is given by Perelman’s $\mathcal{L}$-distance. In the static case, his estimate reduces to a so-called space-time $L^2$-Wasserstein control for heat distributions (2.8), and it is indeed one of characterizing properties of $\text{RCD}(0, N)$ space [15, 29]. On the one hand, heat flow can be regarded as a gradient flow of the relative entropy functional (or the Boltzmann-Shannon entropy with an opposite sign) on $L^2$-Wasserstein space, and thus there is a strong connection between heat flows and the relative entropy by means of optimal transport. On the other hand, we can write the $\mathcal{W}$-entropy by using the relative entropy and its dissipation along heat flow (or the Fisher information) [37–40]. The proof of the monotonicity follows from combining these two observations with the idea in [49]. Note that the monotonicity of $\mathcal{W}$ is already studied on $\text{RCD}(0, N)$ spaces by different means. This problem is considered first in [26] when the underlying space is compact. The noncompact case is discussed in [30] by following an argument in [13] on Riemannian manifolds. However, it seems that some technical details are not well described in the latter case. The proof of the rigidity also relies on the space-time Wasserstein control in its first step, but we use the condition in a more subtle way. It implies an identity for the Fisher information, and the final conclusion is reduced to the recent result on the volume rigidity [21] (See Theorem 2.2 below) by using recently developed analytic tools such as the Li-Yau inequality [24] and the Varadhan type short time asymptotic for the heat kernel following from [25]. Roughly speaking, the assumption of the rigidity of the $\mathcal{W}$-entropy implies the equality in the Laplacian comparison theorem (See Proposition 4.5 and Remark 4.6). From geometric viewpoint, it is almost equivalent to the equality in Bishop-Gromov inequality studied in [21]. From probabilistic viewpoint, the transition probability becomes the Gaussian kernel if it starts from the reference point (see Proposition 4.10). A typical, elementary but non-Euclidean example for the latter one is the $N$-dimensional Bessel process with $N \in [1, \infty)$ starting from the endpoint of the interval. In this case, we take a weighted measure $m(dr) = r^{N-1}dr$ to make $[0, \infty)$ equipped with the Euclidean distance to be a metric measure space.

The structure of this article is as follows. In the next section, we introduce several notions concerning with metric measure spaces and $\text{RCD}$ spaces. Known properties of $\text{RCD}(0, N)$ spaces we will use in the sequel are also prepared there. The monotonicity of the $\mathcal{W}$-entropy and the rigidity are shown in Sects. 3 and 4 respec-
tively. For the rigidity, we first argue that the case when the initial data is Dirac, after showing that it certainly happens at some point (Lemma 4.4). By using the consequence of it, we show that the initial data must be Dirac (Lemma 4.12). Some results related with our main theorem are gathered in Sect. 5. We deal with four different topics there. First, we show that the heat flow becomes also \( L^2 \)-Wasserstein geodesic if the assumption of the rigidity holds (Proposition 5.1). Indeed, as already observed by the second named author and S. Li in [32,35], there is some similarity in the study of \( \mathcal{W} \)-entropy for the heat flow on the underlying manifolds and the geodesic flow on the \( L^2 \)-Wasserstein space. To understand such a similarity better, the second named author and S. Li [32,35] introduced the Langevin deformation of flows over weighted Riemannian manifolds, which can be regarded as a natural interpolation between the heat flow on underlying manifolds and the geodesic flow on the Wasserstein space equipped with Otto’s infinite dimensional Riemannian metric. Moreover, Perelman’s \( \mathcal{W} \)-entropy formula has been extended in [32,35] to the geodesic flow and the Langevin deformation of flows on the Wasserstein space on Riemannian manifolds with non-negative Ricci curvature and on weighted Riemannian manifolds with non-negative \( m \)-dimensional Bakry–Emery Ricci curvature. From this point of view, it seems meaningful that the same property holds even in the framework of \( \text{RCD}(K, N) \) spaces. Second, we discuss some relations between the (logarithmic) Sobolev inequality and the \( \mathcal{W} \)-entropy in our framework. It is already pointed out in [44] that \( \mathcal{W} \)-entropy is related with the logarithmic Sobolev inequality. See also [33,34,37,38]. Third, we consider a stronger rigidity result under a stronger assumption (Theorem 5.6). In this case, the conclusion becomes the same as the rigidity theorem on weighted Riemannian manifolds which was proved previously in [33,37,39]. Fourth, we consider the almost rigidity. Here “almost rigidity” asserts that the conclusion of the rigidity almost holds if the assumption of the rigidity is almost satisfied. That is, a weaker assumption implies a weaker conclusion. The most famous almost rigidity result would be an extension of the Cheeger-Gromoll splitting theorem. See e.g. [16] for such an extension on weighted Riemannian manifolds with non-negative finite dimensional or infinite dimensional Bakry–Emery Ricci curvature. Now the most general “almost splitting theorem” is formulated in the framework of \( \text{RCD}(0, N) \) spaces [19] (See references therein also). The key property for the almost rigidity in [19] is that the \( \text{RCD}(0, N) \) condition is stable under a (pointed measured Gromov–Hausdorff) convergence of metric measure spaces. Though our assumption on the rigidity of the \( \mathcal{W} \)-entropy seems less stable under convergence of spaces, we are somehow able to formulate an almost rigidity.

2. Framework

Let \((X, d)\) be a complete and separable geodesic metric space. Here “geodesic” means that for any \(x_0, x_1 \in X\), there exists \(\gamma : [0, 1] \to X\) such that \(\gamma_i = x_i\) (\(i = 0, 1\)) and \(d(\gamma_s, \gamma_t) = |s - t|d(x_0, x_1)\). We call such \(\gamma\) a (minimal) geodesic joining \(x_0\) and \(x_1\). Let \(m\) be \(\sigma\)-finite Borel measure on \(X\). Suppose that \(m(B_r(x)) \in (0, \infty)\) for any metric ball \(B_r(x)\) of radius \(r > 0\) centered at \(x \in X\). In particular,
supp \( m = X \) holds. We call the triplet \((X, d, m)\) a metric measure space in this article. A typical example of metric measure space we should have in mind is the weighted Riemannian manifold as reviewed in the introduction.

Both for defining RCD spaces and for considering the canonical heat flow on \((X, d, m)\), we require the notion of the \((L^2)\)-Cheeger energy functional. Let \(\text{Lip}(X)\) be the set of all Lipschitz continuous functions on \(X\) and \(\text{Lip}_b(X) = \text{Lip}(X) \cap L^\infty(m)\). For \(f \in \text{Lip}(X)\), we define the local Lipschitz constant \(\text{lip}(f)(x)\) of \(f\) at \(x \in X\) by

\[
\text{lip}(f)(x) := \lim_{y \to x} \frac{|f(y) - f(x)|}{d(x, y)}.
\]

We regard \(\text{lip}(f)\) as a function on \(X\). By means of local Lipschitz constant, we define the Cheeger energy \(\text{Ch}\) as follows: for \(f \in L^2(m)\),

\[
\text{Ch}(f) := \frac{1}{2} \inf \left\{ \liminf_{n \to \infty} \int_X \text{lip}(f_n)^2 \, dm \bigg| f_n \in \text{Lip}(X) \cap L^2(m), \ f_n \to f \text{ in } L^2(m) \right\}.
\]

We say \(f \in \mathcal{D}(\text{Ch})\) if \(f \in L^2(m)\) and \(\text{Ch}(f) < \infty\). Note that, for \(f \in \mathcal{D}(\text{Ch})\), there exists \(\langle Df \rangle : X \to [0, \infty]\), which is called a minimal weak upper gradient of \(f\). See [3] for a more precise definition and its equivalence with the minimal relaxed gradient [3, Theorem 6.2]. It plays the role of the modulus of gradient of \(f\) in the theory of Sobolev spaces. For instance, it satisfies

\[
\text{Ch}(f) = \frac{1}{2} \int_X |Df|^2 \, dm.
\]

We call \((X, d, m)\) infinitesimally Hilbertian if \(\text{Ch}\) is quadratic form. That is, \(\text{Ch}\) satisfies the parallelogram law (see [5]). It implies that \(f \mapsto |Df|^2\) also becomes an quadratic form. That is, there exists a bilinear form \(\langle D\cdot, D\cdot \rangle : \mathcal{D}(\text{Ch}) \times \mathcal{D}(\text{Ch}) \to L^1(m)\) such that \(\langle Df, Df \rangle = |Df|^2\). On an infinitesimally Hilbertian \((X, d, m)\), we denote the bilinear form corresponding to \(2\text{Ch}\) by \(\mathcal{E}\) with \(\mathcal{D}(\mathcal{E}) = \mathcal{D}(\text{Ch})\): That is,

\[
\mathcal{E}(f, g) = \int_X \langle Df, Dg \rangle \, dm
\]

and hence \(\mathcal{E}(f, f) = 2\text{Ch}(f)\) and \(\langle Df, Dg \rangle\) becomes the carré du champ associated with \(\mathcal{E}\). We denote the (linear) self-adjoint operator on \(L^2(m)\) associated with \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) by \(\Delta\) and the (linear) semigroup of contractions generated by \(\Delta\) by \(P_t : L^2(m) \to L^2(m)\). Note that we can define \(\Delta\) and \(P_t\) as a non-linear operator even in absence of infinitesimal Hilbertianity; see [3, Section 4].

We call \((X, d, m)\) an RCD\(^n(K, N)\) space for \(K \in \mathbb{R}\) and \(N \in (1, \infty)\) if \((X, d, m)\) is infinitesimally Hilbertian and \((X, d, m)\) enjoys the reduced curvature-dimension condition CD\(^n(K, N)\) introduced in [9]. In [7, 15], it is shown that, for infinitesimally Hilbertian \((X, d, m)\), CD\(^n(K, N)\) condition is equivalent to the following three conditions:

- There exists \(C > 0\) and \(x_0 \in X\) such that

\[
\int_X e^{-Cd(x_0, x)^2} \, dm(x) < \infty.
\]

(2.1)
• For $f \in \mathcal{D}(\text{Ch})$ with $|Df| \leq 1$ m-a.e., $f$ has a 1-Lipschitz representative.

• For all $f \in \mathcal{D}(\Delta)$ with $\Delta f \in \mathcal{D}(\text{Ch})$ and $g \in \mathcal{D}(\Delta) \cap L^\infty(m)$ with $g \geq 0$ and $\Delta g \in L^\infty(m)$,

$$\frac{1}{2} \int_X |Df|^2 \Delta g \, dm - \int_X \langle Df, D\Delta f \rangle g \, dm \geq K \int_X |Df|^2 g \, dm + \frac{1}{N} \int_X (\Delta f)^2 g \, dm.$$  

(2.2)

The last one is nothing but a weak formulation of Bochner inequality or Bakry–Émery’s curvature-dimension condition. See [1,3,5,6] for the case $N = \infty$. We omit the precise definition of $\text{CD}^*(K, N)$ here since it is not directly used in this article. When $K = 0$, by definition, $\text{CD}^*(0, N)$ is equivalent to the original curvature-dimension condition $\text{CD}(0, N)$ in [48]. Thus we denote “$\text{RCD}^*(0, N)$” by “$\text{RCD}(0, N)$” alternatively. More generally, it is recently proved in [14] that $\text{CD}(K, N)$ is equivalent to $\text{CD}^*(K, N)$ for infinitesimally Hilbertian $(X, d, m)$ (Indeed, this equivalence is proved under a weaker assumption). As mentioned in Sect. 1, a basic class of $\text{RCD}^*(K, N)$ spaces consists of weighted Riemannian manifolds satisfying (1.2). It is also known that the $\text{RCD}^*(K, N)$ spaces are stable under the pointed measured Gromov convergence of metric measure spaces (See [23] and references therein; see Sect. 5 also). In particular, the limit of a sequence of weighted Riemannian manifolds satisfying (1.2) is an $\text{RCD}^*(K, N)$ space.

In the rest of this section, we review several notions and properties on metric measure spaces, optimal transports and $\text{RCD}$ spaces which will be used in the sequel. We begin with properties on minimal weak upper gradient and the Cheeger energy. If $f \in \text{Lip}(X) \cap L^2(m)$ and $\text{lip}(f) \in L^2(m)$, then $f \in \mathcal{D}(\text{Ch})$ and $|Df| \leq \text{lip}(f)$. $f \mapsto |Df|$ is convex in the following sense: For $f, g \in \mathcal{D}(\text{Ch})$ and $\alpha, \beta \in \mathbb{R}$,

$$|D(\alpha f + \beta g)| \leq |\alpha||Df| + |\beta||Dg|.$$  

It indeed implies that $\text{Ch}$ is convex on $L^2(m)$. In addition, $\text{Ch}$ is lower semi-continuous on $L^2(m)$ (See [3, Theorem 4.5]). If $f$ is constant on a measurable set $A \subset X$, then $|Df| = 0$ m-a.e. on $A$. Moreover, $|Df| = |Dg|$ m-a.e. on $\{f = g\}$ for $f, g \in \mathcal{D}(\text{Ch})$ (See [3, Proposition 4.8 (a)(b)]). We call these properties the locality of the minimal weak upper gradient in this article. By using the locality, we can define $|Df|$ in the extended sense for those measurable $f$ which satisfies $(-n) \vee (f \wedge n) \in \mathcal{D}(\text{Ch})$ for each $n > 0$. See [3, Section 4]. Suppose that $(X, d, m)$ is infinitesimally Hilbertian. Then we have the Leibniz rule: For $f, g, h \in \mathcal{D}(\mathcal{E}) \cap L^\infty(m)$, we have $gh \in \mathcal{D}(\mathcal{E})$ and

$$\langle Df, D(gh) \rangle = \langle Df, Dg \rangle h + \langle Df, Dh \rangle g \quad \text{m-a.e.}$$  

(See [20, (4.16)] for instance). Note that

$$|Dd(x_0, \cdot)| = 1 \quad \text{m-a.e.}$$  

(2.3)

holds, where the left hand side is in the extended sense (see [20, Proof of Corollary 5.15]). We refer to [3,5,20] for other basic properties.
In order to review some properties of the heat flow, we first recall several notations in optimal transport and metric geometry. Let \( \mathcal{P}_2(X) \subset \mathcal{P}(X) \) be the set of probability measure with finite second moment. That is, \( \mu \in \mathcal{P}_2(X) \) means that \( \|d(x_0, \cdot)\|_{L^2(\mu)} < \infty \) holds for some (and hence all) \( x_0 \in X \). For \( \mu, \nu \in \mathcal{P}(X) \), we call \( \pi \in \mathcal{P}(X \times X) \) a coupling of \( \mu \) and \( \nu \) if the first and second marginal of \( \pi \) are \( \mu \) and \( \nu \) respectively. That is, for any Borel measurable \( A \subset X \), we have \( \pi(A \times X) = \mu(A) \) and \( \pi(X \times A) = \nu(A) \). We define the \( L^2 \)-Wasserstein distance \( W_2(\mu, \nu) \in [0, \infty] \) as follows:

\[
W_2(\mu, \nu) := \inf \{ \|d\|_{L^2(\pi)} | \pi: a coupling of \mu and \nu \}.
\]

Note that \( (\mathcal{P}_2(X), W_2) \) becomes a complete separable geodesic metric space. Indeed, these properties are inherited from \( (X, d, m) \). Recall that the convergence in \( W_2 \) is equivalent to the weak convergence and the convergence of the second moment (See [50, Theorem 7.12] for instance). We define the relative entropy functional \( \text{Ent}: \mathcal{P}_2(X) \to (-\infty, \infty] \) by

\[
\text{Ent}(\mu) := \begin{cases} 
\int_X \rho \log \rho \, dm & \text{if } \mu = \rho m, \\
\infty & \text{if } \mu \not\ll m.
\end{cases}
\]

With the aid of (2.1), \( \text{Ent} \) is well-defined as a map as mentioned above. In addition, \( \text{Ent} \) is lower semi-continuous on \( (\mathcal{P}_2(X), W_2) \) (See [3, Section 7]). Let \( \mathcal{D}(\text{Ent}) := \{ \mu \in \mathcal{P}_2(X) | \text{Ent}(\mu) < \infty \} \). Let \( I : \mathcal{P}_2(X) \to [0, \infty] \) be the Fisher information given by

\[
I(\mu) := \begin{cases} 
4 \int_X |D\sqrt{\rho}|^2 \, dm & \text{if } \mu = \rho m, \sqrt{\rho} \in \mathcal{D}(\mathcal{E}), \\
\infty & \text{otherwise}.
\end{cases}
\]

Note that we have

\[
I(\rho m) = \int_X \frac{|D\rho|^2}{\rho} \, dm
\]

(2.4)

when \( I(\rho m) < \infty \) (See [3, Lemma 4.10]), where \( |D\rho| \) in the right hand side of (2.4) is taken to be an extended sense. By [3, Lemma 4.10] again, \( I : \mathcal{P}_2(X) \to [0, \infty] \) is convex with respect to convex combinations of elements in \( \mathcal{P}_2(X) \) and

\[
\lim_{n \to \infty} I(\rho_n m) \geq I(\rho m)
\]

(2.5)

if probability densities \( \rho_n \) converges to \( \rho \) weakly in \( L^1(m) \) as \( n \to \infty \). We call a curve \( (\gamma_t)_{t \in J} \) indexed by an interval \( J \subset \mathbb{R} \) on a metric space \( (Y, d_Y) \) absolutely continuous if there exists \( g \in L^1_{\text{loc}}(J) \) such that

\[
d_Y(\gamma_s, \gamma_t) \leq \int_s^t g(r) \, dr
\]
for any $s, t \in J$ with $s < t$. For an absolutely continuous curve $(\gamma_t)_{t \in J}$, the metric speed $|\dot{\gamma}_t|$ at $t$ is given by

$$|\dot{\gamma}_t| := \lim_{s \downarrow 0} \frac{d\gamma(t, \gamma(s, t))}{s}$$

Note that we can take $g(s) = |\dot{\gamma}_s|$ in the definition of absolutely continuous curve if $\gamma$ is absolutely continuous (See [2, Theorem 1.1.2]).

In the rest of the article, we always assume $(X, d, m)$ to be an RCD$(0, N)$ space with $N \in [1, \infty)$. Note that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ becomes a strongly local regular Dirichlet form in the sense of [17] in this framework. Indeed, it is quasi-regular Dirichlet form by [46, Theorem 4.1] and $X$ is locally compact by the Bishop-Gromov inequality (2.12). The regularity comes from the fact that Lip$(X) \cap L^2(m)$ is dense in $\mathcal{D}(\mathcal{E})$ (see [4]; see [5, Proposition 4.10] also). The chain rule for $E$ [17, Theorem 3.2.2] says that, for $f, g \in \mathcal{D}(\mathcal{E}) \mathcal{L}^\infty(m)$ and $\varphi \in C^1(\mathbb{R})$ with $\varphi(0) = 0$, we have $\varphi(f) \in \mathcal{D}(\mathcal{E}) \cap L^\infty(m)$ and

$$\langle D\varphi(f), Dg \rangle = \varphi'(f)\langle Df, Dg \rangle \text{ m.a.e.}$$

We now turn to review some properties of the heat semigroup $P_t$ which we use in this article. Since $P_t$ is symmetric and Markovian, there exists an extension of $P_t$ as an linear contraction from $L^p(m)$ to itself for $1 \leq p \leq \infty$ (cf. [3, Theorem 4.16]). In addition, $P_t$ preserves the total mass (or $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is conservative). That is, for $f \in L^1(m)$ with $f \geq 0$, $\|P_t f\|_{L^1(m)} = \|f\|_{L^1(m)}$ holds for $t > 0$. This is a consequence of (2.1) (See [3, Theorem 4.20]). Thus $\mu_t = P_t f m \in \mathcal{P}(X)$ when $f$ is a probability density with respect to $m$ and it can be regarded as a curve in $\mathcal{P}(X)$ parametrized by $t$. As a very crucial property of $P_t$ on RCD spaces, the curve $(\mu_{t})_{t \geq 0}$ as given by $P_t$ in the last sentence becomes a gradient flow on $\mathcal{P}(X)$ (See [1,3,5,15]). For any $\mu = \rho m \in \mathcal{P}_2(X)$ with $\rho \in L^2(m)$, $\mu_t := P_t \rho m$ is a gradient flow of Ent on $(\mathcal{P}_2(X), W_2)$ in the sense that $(\mu_t)_{t \geq 0}$ solves $(0, N)$-evolution variational inequality starting from $\mu$ by [15, Theorem 3.17] and its proof. Indeed, the existence of the solution is one of characterizing properties of RCD$(0, N)$ space. Note that this sort of result is obtained first when $N = \infty$ (See [1,5]). For the definition of $(0, N)$-evolution variational inequality, see [15, Definition 3.16]. We omit the definition but exhibit some properties obtained from it instead. First of all, we can extend $P_t$ to be an operator from $\mathcal{P}_2(X)$ to itself in the sense that $P_t \mu = P_t \rho m$ holds if $\mu \in \mathcal{P}_2(X)$ and $\mu = \rho m$ (See [1, Theorems 6.1 and 6.2 for instance). As an immediate consequence of the definition of the evolution variational inequality, $(P_t \mu)_{t \geq 0}$ is an absolutely continuous curve in $(\mathcal{P}_2(X), W_2)$, $W_2(P_t \mu, \rho m) \rightarrow 0$ (t ↓ 0) and Ent$(P_t \mu) < \infty$ for $\mu \in \mathcal{P}_2(X)$ and $t > 0$. By [15, Remark 3.19] and [5, Proposition 2.22 (i)], for $\mu \in \mathcal{P}_2(X)$, $t \mapsto \text{Ent}(P_t \mu)$ is absolutely continuous on $(0, \infty)$ and $\mu_t = P_t \mu$ solves the energy dissipation identity, i.e. $\mu_t \rightarrow \mu_0$ as $t \rightarrow 0$ and for $0 < s < t$,

$$\text{Ent}(\mu_s) = \text{Ent}(\mu_t) + \frac{1}{2} \int_s^t |\dot{\mu}_r|^2 dr + \frac{1}{2} \int_s^t I(\mu_r) dr. \text{ a.e. } t$$ (2.6)
(For instance, see [3, Definition 2.14] and comments after it). Since we are on \( \text{RCD}(0, N) \) spaces, (2.6) is equivalent to the following:

\[
- \frac{d}{dt} \text{Ent}(\mu_t) = |\dot{\mu}_t|^2 = I(\mu_t) < \infty \quad \text{a.e. } t.
\] (2.7)

Here the finiteness follows from [5, (2.37)]. Note that (2.6) holds even when \( s = 0 \) if \( \mu_0 \in \mathcal{D}(\text{Ent}) \).

The key property for the proof of the main theorem of this article is the following space-time \( W_2 \)-control for heat distributions: For \( \mu, \nu \in \mathcal{P}_2(X) \) and \( t, s > 0 \),

\[
W_2(P_t \mu, P_t \nu)^2 \leq W_2(\mu, \nu)^2 + 2N(\sqrt{t} - \sqrt{s})^2.
\] (2.8)

It follows from either the \((0, N)\)-evolution variational inequality (See [15, Theorem 2.19]) or (2.2) (See [29]).

We also review some analytic properties of \( P_t \). As a regularization property, \( P_t f \in \text{Lip}_b(X) \) holds for \( f \in L^\infty(m) \) [1, Theorem 7.1]. One of important tools is the following Bakry–Émery’s \( L^1 \)-gradient estimate: For \( f \in D(\mathcal{E}) \cap L^\infty(m) \), we have

\[
|DP_t f| \leq e^{-Kt} P_t(|Df|).
\] (2.9)

We obtain this bound from the self-improvement of (2.2) with neglecting the term involving \( N \) (See [46] and references therein). As a technical tool, we recall the following mollification of \( P_t \) (See [6, Section 2.1] for instance). Let \( \kappa \in C_0^\infty((0, \infty)) \) with \( \kappa \geq 0 \) and \( \int_0^\infty \kappa(r) \, dr = 1 \). For \( \eta > 0 \) and \( f \in L^p(\mu) \) with \( p \in [1, \infty) \), we define \( h_\eta f \) by

\[
h_\eta f := \frac{1}{\eta} \int_0^\infty P_r f \kappa \left( \frac{r}{\eta} \right) \, dr.
\]

It is immediate that \( \mathcal{E}(h_\eta f - f, h_\eta f - f) \to 0 \) and \( \|h_\eta f - f\|_{L^2(m)} \to 0 \) as \( \eta \to 0 \) for \( f \in \mathcal{D}(\mathcal{E}) \). Moreover, for \( f \in L^2(m) \cap L^\infty(m) \), \( h_\eta f, \Delta(h_\eta f) \in \mathcal{D}(\Delta) \cap \text{Lip}_b(X) \). Here the latter one comes from the following representation:

\[
\Delta h_\eta f = -\frac{1}{\eta^2} \int_0^\infty P_r f \kappa' \left( \frac{r}{\eta} \right) \, dr.
\]

As an additional result from the evolution variational inequality, \( P_t \) admits a symmetric heat kernel \( p_t(x, y) \) [1, Theorem 7.1]. That is,

\[
P_t f(x) = \int_X p_t(x, y) f(y) m(dy)
\]

for \( f \in L^p(m), p \in [1, \infty] \). Moreover, \( p_t(x, y) \) admits a sharp two-sided Gaussian heat kernel estimate [25]: For any \( \delta > 0 \), there exists \( C(\delta) > 0 \) such that

\[
\frac{1}{C(\delta) m(B_{\sqrt{t}}(x))} \exp \left( -\frac{d(x, y)^2}{(4 - \delta)t} \right) \leq p_t(x, y) \leq \frac{C(\delta)}{m(B_{\sqrt{t}}(x))} \exp \left( -\frac{d(x, y)^2}{(4 + \delta)t} \right)
\] (2.10)
for \( t > 0 \) and \( x, y \in X \). Indeed, this is a consequence of the following Li-Yau inequality \([24, \text{Theorem 1.1}]\): For \( t > 0 \) and \( f \in \bigcup_{p \in [1, \infty)} L^p(m) \) with \( f \geq 0 \) and \( f \neq 0 \),

\[
\frac{|DP_t f|^2}{(P_t f)^2} - \frac{\Delta P_t f}{P_t f} \leq \frac{N}{2t} \text{ m-a.e.} \quad (2.11)
\]

Note that we can replace \( P_t f \) by \( p_t(x, \cdot) \), \( x \in X \) \([24, \text{Corollary 1.1}]\).

\( \text{RCD}^*(K, N) \) spaces satisfy several geometric properties corresponding to ones on a Riemannian manifold with a lower Ricci curvature bound. Among them, we recall the Bishop-Gromov inequality on \( \text{RCD}(0, N) \) spaces. It says that, for \( 0 < r < R \) and \( x \in X \),

\[
\frac{m(B_R(x))}{m(B_r(x))} \leq \left( \frac{R}{r} \right)^N. \quad (2.12)
\]

Moreover, \([21]\) studies the case when equality holds in (2.12). To state their result, we require the following definition:

**Definition 2.1.** (\([27, \text{Definition 5.1}]\)) A metric measure space \((X', d', m')\) is a \((0, N-1)\)-cone built over a metric measure space \((Y, d_Y, m_Y)\) if the following holds:

(i) \( X' = [0, \infty) \times Y/[0] \times Y \),

(ii) \( d'((r, y), (s, z)) = r^2 + s^2 - 2rs \cos(d_Y(y, z) \wedge \pi) \),

(iii) \( m'(drdy) = r^{N-1}drm_Y(dy) \).

**Theorem 2.2.** (Special case of \([21, \text{Theorem 1.1}]\)) Let \( x \in X \). Suppose that the equality holds in (2.12) for any \( 0 < r < R \).

(i) If \( N \geq 2 \), then \((X, d, m)\) is \((0, N-1)\)-cone over an \( \text{RCD}^*(N-2, N-1) \) space and \( x \) is the vertex of the cone.

(ii) If \( N < 2 \), then \((X, d, m)\) is isomorphic to either \((\mathbb{R}, d_{\text{Eucl}}, |x|^{N-1}dx)\) or \((\{0, \infty\}, d_{\text{Eucl}}, x^{N-1}dx)\), where \( d_{\text{Eucl}} \) is the canonical Euclidean distance. In both cases, \( x \in X \) corresponds to 0 by the isomorphism.

This result will be used in a crucial way in the proof of our main theorem.

Before closing this section, we re-define of the \( \mathcal{W} \)-entropy in a different manner from (1.3). Indeed we define it as a functional on \( \mathcal{P}_2(X) \).

**Definition 2.3.** Let us define the \( \mathcal{W} \)-entropy \( \mathcal{W} : \mathcal{D}(\text{Ent}) \times (0, \infty) \to (\infty, \infty] \) by

\[
\mathcal{W}(\mu, t) := tI(\mu) - \text{Ent}(\mu) - \frac{N}{2} \log t.
\]

If \( \mu = ud\mu \in \mathcal{P}_2(X) \) and \( f \) satisfies \( u = e^{-f} / (4\pi t)^{N/2} \), \( \mathcal{W}(\mu, t) \) coincides with the right hand side of (1.3) up to additive constant (See [37–39] for instance). The choice of additive constant in (1.3) can be regarded as a normalizing constant so that \( \mathcal{W}(\mu_t, t) = 0 \) on \( \mathbb{R}^N \) when \( N \in \mathbb{N} \) and \( \mu_t \) is a Gaussian kernel. Since the constant plays no role in monotonicity and rigidity, we neglect it for brevity.
3. Monotonicity of $\mathcal{W}$-entropy

The goal of this section is to show the monotonicity of the $\mathcal{W}$-entropy along heat distributions (Theorem 3.3). For the proof, we show a monotonicity of a rescaled Fisher information in Proposition 3.2. Though it is more general than what we use in the proof of Theorem 3.3, we require this general form in the next section. We begin with the following auxiliary lemma.

**Lemma 3.1.** For $\mu \in \mathcal{P}_2(X)$, $t \mapsto I(P_t \mu)$ is right continuous and non-increasing on $[0, \infty)$.

**Proof.** We first claim that $t \mapsto I(P_t \mu)$ is non-increasing on $[0, \infty)$. Let $0 < s < t$ and we denote the density of $P_s \mu$ with respect to $\mu$ by $\rho$. By (2.10), $\rho$ is bounded. Then, by Bakry–Émery’s $L^1$-gradient estimate (2.9) and the Schwarz inequality for the heat semigroup,

$$I(P_t \mu) = \int_X \frac{|DP_{t-s} \rho|^2}{P_{t-s} \rho} \, d\mu \leq \int_X \frac{(P_{t-s}(|D\rho|))^2}{P_{t-s} \rho} \, d\mu \leq \int P_{t-s} \left( \frac{|D\rho|^2}{\rho} \right) \, d\mu = \int_X \frac{|D\rho|^2}{\rho} \, d\mu = I(P_s \mu). \quad (3.1)$$

Thus the claim holds on $(0, \infty)$. Let us consider the case $s = 0$. We may suppose $I(\mu) < \infty$ without loss of generality. As above, we denote $\mu = \rho \cdot \mu$. For $n \in \mathbb{N}$, let $\tilde{\rho}_n := z_n^{-1} \rho_n$, where $\rho_n := \rho \wedge n$ and $z_n := \int_X \rho_n \, d\mu$. By the locality of the minimal weak upper gradient, we have $|D\rho_n| = |D\rho|1_{\{|\rho|<n\}}$ a.e. Thus we have

$$I(\rho \cdot \mu) \geq \int_X \frac{|D\rho|^2}{\rho} 1_{\{|\rho|<n\}} \, d\mu \geq \frac{1}{n} \int_X |D\rho|^2 1_{\{|\rho|<n\}} \, d\mu = \frac{1}{n} \int_X |D\rho_n|^2 \, d\mu.$$ 

Hence $\tilde{\rho}_n \in \mathcal{D}(\mathcal{E})$ holds since $I(\rho \cdot \mu) < \infty$ and $\rho_n \in L^2(\mu)$. Thus, by the same argument as in (3.1),

$$I((P_t \tilde{\rho}_n) \cdot \mu) \leq \int_X \frac{|D\tilde{\rho}_n|^2}{\tilde{\rho}_n} \, d\mu = \frac{1}{z_n} \int_X \frac{|D\rho|^2}{\rho} 1_{\{|\rho|<n\}} \, d\mu. \quad (3.2)$$

Since we have (2.5), $I(P_t \mu) \leq I(\mu)$ holds by letting $n \to \infty$ in (3.2).

For a probability density $\rho$ on $X$, $t \mapsto P_t \rho$ is continuous in $L^1(\mu)$. Thus (2.5) yields that $t \mapsto I(P_t \mu)$ is lower semi-continuous. It implies the desired right continuity since $t \mapsto I(P_t \mu)$ is non-increasing. \hfill $\Box$

We next show the following monotonicity formula, which is closely related with the monotonicity of $\mathcal{W}$-entropy.

**Proposition 3.2.** (i) For $t > s \geq 0$, $t' \geq 0$ and $\alpha \in \mathbb{R}$,

$$(t + t')^{2\alpha} I(P_t \mu) \leq (s + t')^{2\alpha} I(P_{t'} \mu) + \frac{N}{2} \cdot \frac{(t + t')^\alpha - (s + t')^\alpha}{t - s} \cdot \frac{(t + t')^2 - (s + t')^2}{t - s}, \quad (3.3)$$

where we regard $(s + t')^{2\alpha} I(P_{s'} \mu)$ as 0 if $s = t' = 0$. 

(ii) \((t + t')^2 I(P_t \mu) - Nt/2\) is non-increasing in \(t \in (0, \infty)\).

**Proof.** (i) Let \(J \subset (0, \infty)\) be the set of \(t \in (0, \infty)\) satisfying (2.7). For \(t, s \in J\) with \(t > s\) and \(\delta > 0\), (2.8) yields

\[
\frac{W_2(P_t \mu, P_{t+(t+t')\alpha\delta} \mu)^2}{\delta^2} \leq \frac{W_2(P_s \mu, P_{s+(s+t')\alpha\delta} \mu)^2}{\delta^2} + 2N \left(\frac{\sqrt{(t-s)} + ((t + t')\alpha - (s + t')\alpha\delta)}{\delta}\right)^2.
\]

By letting \(\delta \to 0\), and using the fact

\[
\lim_{\delta \downarrow 0} \frac{W_2(P_t \mu, P_{t+\delta} \mu)}{\delta} = |\dot{\mu}_t| = \sqrt{I(P_t \mu)} \quad a.e. \ t,
\]

we obtain (3.3).

Then (3.3) holds for any \(0 \leq s < t\) since \(J\) is dense in \((0, \infty)\) and \(t \mapsto I(P_t \mu)\) is right continuous by Lemma 3.1.

(ii) The assertion immediately follows by applying (i) with \(\alpha = 1\). \(\square\)

Let us turn to consider the monotonicity of the \(\mathcal{W}\)-entropy. For later use in Sect. 5 (Theorem 5.2), we make our assertion to be slightly stronger than the usual form, by inserting an additional parameter \(t'\).

**Theorem 3.3.** For any \(\mu \in \mathcal{P}_2(X)\) and \(t' \geq 0\), \(\mathcal{W}(P_t \mu, t + t')\) is non-increasing in \(t \in (0, \infty)\). In addition, the same monotonicity holds for \(t \in [0, \infty)\) if \(\mu \in \mathcal{D}(\text{Ent})\) and \(t' > 0\).

**Proof.** By Proposition 3.2, for \(0 \leq s < t\),

\[
(t + t')I(P_t \mu) - (s + t')I(P_s \mu) = \left(\frac{1}{t + t'} - \frac{1}{s + t'}\right)(s + t')^2 I(P_s \mu) + \frac{1}{t + t'}((t + t')^2 I(P_t \mu) - (s + t')^2 I(P_s \mu)) \leq \frac{(s - t)(s + t')}{t + t'} I(P_s \mu) + \frac{N}{2(t + t')}(t - s).
\]

Suppose \(s > 0\) if \(\mu \notin \mathcal{D}(\text{Ent})\) or \(t' = 0\). Suppose \(I(\mu) < \infty\) if \(s = 0\). Indeed, when \(s = 0\) and \(I(\mu) = \infty\), the conclusion obviously holds. By (2.6) and (2.7) (see the comment after (2.7) if \(s = 0\)), it yields

\[
\mathcal{W}(P_t \mu, t + t') - \mathcal{W}(P_s \mu, s + t') = -\frac{s + t'}{t + t'} I(P_s \mu)(t - s) + \frac{N}{2(t + t')}(t - s) - \text{Ent}(P_t \mu) + \text{Ent}(P_s \mu) - \frac{N}{2} \int_{s+t'}^{t+t'} \frac{du}{u}
\]

\[
= \int_s^t \left( I(P_u \mu) - \frac{s + t'}{t + t'} I(P_s \mu) \right) \, du + \frac{N}{2} \int_s^t \left( \frac{1}{t + t'} - \frac{1}{u + t'} \right) \, du
\]

\[
\leq \frac{s + t'}{t + t'}\left( I(P_s \mu)(t - s) - \frac{N}{2(t + t')}(t - s) \right) - \text{Ent}(P_t \mu) + \text{Ent}(P_s \mu) - \frac{N}{2} \int_{s+t'}^{t+t'} \frac{du}{u}.
\]
\[ \leq \int_s^t (I(P_t \mu) - I(P_s \mu)) \, du + \frac{(t-s)^2}{t+t'} I(P_s \mu). \]

Fix \( s \in [0, t) \) (let \( s > 0 \) if \( I(\mu) = \infty \)). By Lemma 3.1,

\[ \mathcal{W}(P_t \mu, t + t') - \mathcal{W}(P_s \mu, s + t') \leq \frac{(t-s)^2}{t+t'} I(P_s \mu). \]  

(3.4)

Let \( n \in \mathbb{N} \) and \( t_k = s+k(t-s)/n \) (\( k = 0, 1, \ldots, n \)). By applying (3.4) for \((t_k, t_{k-1})\) instead of \((t, s)\), we obtain

\[ \mathcal{W}(P_t \mu, t + t') - \mathcal{W}(P_s \mu, s + t') = \sum_{k=1}^{n} \mathcal{W}(P_{t_k} \mu, t_k + t') - \mathcal{W}(P_{t_{k-1}} \mu, t_{k-1} + t') \leq \frac{(t-s)^2 I(P_s \mu)}{n(s + t')} . \]

Hence the conclusion follows by letting \( n \to \infty \).

\[ \square \]

4. Rigidity of \( \mathcal{W} \)-entropy

Our goal in this section is to show the following theorem.

**Theorem 4.1.** Suppose that the right upper derivative of \( \mathcal{W}(P_t \mu, t) \) is 0 at \( t = t_\ast \in (0, \infty) \), that is,

\[ \lim_{t \downarrow t_\ast} \frac{\mathcal{W}(P_t \mu, t) - \mathcal{W}(P_{t_\ast} \mu, t_\ast)}{t - t_\ast} = 0 \]

for some \( \mu \in \mathcal{P}_2(X) \) and \( t_\ast \in (0, \infty) \). Then \( \mu = \delta_{x_0} \) for some \( x_0 \in X \). Moreover, we have the following:

(i) If \( N \geq 2 \), then \((X, d, m)\) is \((0, N - 1)\)-cone built over an \( \text{RCD}^* (N-2, N-1) \) space and \( x_0 \) is the vertex of the cone.

(ii) If \( N < 2 \), then \((X, d, m)\) is isomorphic to either \((\mathbb{R}, d_{\text{Eucl}}, |x|^{N-1} \, dx)\) or \(([0, \infty), d_{\text{Eucl}}, x^{N-1} \, dx)\), where \( d_{\text{Eucl}} \) is the canonical Euclidean distance. In both cases, \( x_0 \in X \) corresponds to 0 by the isomorphism.

In each of these cases, \( \mathcal{W}(P_t \mu, t) \) is a constant function of \( t \in (0, \infty) \).

Note that the conclusion in Theorem 4.1 completely corresponds to the one in Theorem 2.2. Indeed, we will reduce the proof of Theorem 4.1 to verification of the assumption of Theorem 2.2.

We begin with the following two lemmas concerning with the Fisher information.

**Lemma 4.2.** For any \( t > 0 \) and \( \mu \in \mathcal{P}_2(X) \),

\[ I(P_t \mu) \leq \frac{N}{2t} . \]  

(4.1)
Monotonicity and rigidity of the \( W \)-entropy

This lemma immediately follows by applying Proposition 3.2 (i) with \( s = 0 \) and \( \alpha = 1 \). Alternatively, we can show Lemma 4.2 by using the Li-Yau inequality (2.11) (See the proof of Proposition 4.5).

**Lemma 4.3.** (i) \( \mu \mapsto I(P_s \mu) \) is lower semi-continuous on \( \mathcal{P}_2(X) \) for \( s > 0 \).

(ii) Let \((J, \mathcal{J}, \nu)\) be a probability space. Let \((\mu_j)_{j \in J} \subset \mathcal{P}_2(X)\) be a family of probability measures such that \( f \mapsto \mu_j(A) \) is measurable for each measurable \( A \subset X \). Let \( \mu_* = \int_J \mu_j \nu(dj) \). Suppose \( \mu_* \in \mathcal{P}_2(X) \). Then, for \( s > 0 \), we have

\[
I(P_s \mu_*) \leq \int J I(P_s \nu_j) \nu(dj).
\]

**Proof.** (i) By the Lipschitz regularization property of \( P_{s/2} \), \( P_{s/2} \) holds for \( f \in L^\infty(m) \). Take \( \mu, \mu^{(n)} \in \mathcal{P}_2(X) \) \((n \in \mathbb{N})\) and suppose \( W_2(\mu^{(n)}, \mu) \) tends to 0 as \( n \to \infty \). By (2.8), we have \( W_2(P_{s/2} \mu^{(n)}, P_{s/2} \mu) \to 0 \) and thus \( P_{s/2} \mu^{(n)} \) converges to \( P_{s/2} \mu \) weakly. Hence, for each \( f \in L^\infty(m) \),\n
\[
\int_X f \ dP_{s/2} \mu^{(n)} = \int_X P_{s/2} f \ dP_{s/2} \mu^{(n)} \to \int_X P_{s/2} f \ dP_{s/2} \mu = \int_X f \ dP_{s} \mu.
\]

It implies that the density of \( P_{s/2} \mu^{(n)} \) converges to that of \( P_{s} \mu \) weakly in \( L^1(m) \). Then (2.5) yields the assertion.

(ii) Let \((Z_k)_{k \in \mathbb{N}}\) be \( J \)-valued, independent and identically distributed random variables with the law \( \nu \). By the law of large numbers, we have

\[
\lim_{n \to \infty} W_2 \left( \frac{1}{n} \sum_{k=1}^n \mu_{Z_k}, \mu_* \right) = 0
\]

almost surely. Since \( I \) is convex in the sense as mentioned in Sect. 2, the assertion (i) yields

\[
I(P_s \mu_*) \leq \lim_{n \to \infty} I \left( \frac{1}{n} \sum_{k=1}^n P_s \mu_{Z_k} \right) \leq \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n I \left( P_s \mu_{Z_k} \right)
\]

almost surely. By taking the expectation and applying the Fatou lemma in the last inequality, we obtain the desired inequality. \( \square \)

The next lemma shows that the assumption of Theorem 4.1 implies an identity for the Fisher information.

**Lemma 4.4.** Suppose that the assumption of Theorem 4.1 is satisfied. Then the following holds:

(i) The equality holds in (4.1) for any \( t \in (0, t_*] \).

(ii) For \( \mu \)-a.e. \( x_0 \in X \),

\[
I(P_t \delta_{x_0}) = \frac{N}{2t} \tag{4.2}
\]

holds for \( t \in (0, t_*] \). In particular, there exists \( x_0 \in X \) satisfying (4.2).
Proof. (i) Let \( h(r) := N/2 - r I(P_r \mu) \). Note that \( h(r) \geq 0 \) holds by Lemma 4.2. By the definition of \( \mathcal{W} \) and (2.7), for \( t > t_* \),

\[
\mathcal{W}(P_t \mu, t) - \mathcal{W}(P_{t_*} \mu, t) = t I(P_t \mu) - t_* I(P_{t_*} \mu) + \int_{t_*}^t \left( I(P_r \mu) - \frac{N}{2r} \right) dr
\]

Thus the assumption yields the conclusion.

Thus all the inequality in the last line must be equality. It immediately implies the conclusion. By Proposition 3.2 (i), for \( \alpha \in (0, 1) \),

\[
t_*^{2\alpha-1}(h(t_*)) - h(t) + (t^{2\alpha-1} - t_*^{2\alpha-1})t I(P_t \mu) = t^{2\alpha} I(P_t \mu) - t_*^{2\alpha} I(P_{t_*} \mu) \leq \frac{N}{2} \cdot \left( \frac{r - t_*}{t - t_*} \right)^2.
\]

Then, combining this with (4.3) with Lemma 3.1 in mind, we obtain

\[
t_*^{2\alpha-2} h(t_*) + (2\alpha - 1)t_*^{2\alpha-1} I(P_{t_*} \mu) \leq \frac{N}{2} \cdot \alpha^2 t_*^{2\alpha-2}.
\]

By using \( I(P_{t_*} \mu) = t_*^{-1}[N/2 - h(t_*)] \), this inequality can be simplified to \( h(t_*) \leq N(1 - \alpha)/4 \). Since \( \alpha \in (0, 1) \) is arbitrary, we obtain \( h(t_*) = 0 \). By Proposition 3.2 and Lemma 4.2, for \( 0 \leq t < t_* \), we have \( 0 \leq t h(t) \leq t_* h(t_*) \). Therefore, \( h \equiv 0 \) on \( (0, t_*] \) and this is nothing but the assertion.

(ii) It suffices to show that \( \mu \text{-a.e.} x_0 \in X \) satisfies \( I(P_{t_*} \delta_{x_0}) = N/(2t_*) \). Indeed, the conclusion follows from this by the same argument as in the proof of the assertion (i). By Lemmas 4.3 (ii) and 4.2, we obtain

\[
\frac{N}{2t_*} = I(P_{t_*} \mu) \leq \int_X I(P_{t_*} \delta_x) \mu(dx) \leq \frac{N}{2t_*}.
\]

Thus all the inequality in the last line must be equality. It immediately implies the conclusion.

For \( y \in X \), we define \( d_y : X \to [0, \infty) \) by \( d_y(x) := d(y, x) \). By employing analytic tools for the heat flow, the conclusion of the last lemma is transformed into the crucial identity for the distance function.

**Proposition 4.5.** Suppose that there exist \( t_* > 0 \) and \( x_0 \in X \) such that (4.2) holds for \( t \in (0, t_*] \). Then, for \( f \in \mathcal{D}(\mathcal{E}) \cap L^1(m) \) with \( d_{x_0} f, d_{x_0} |Df| \in L^1(m) \), we have

\[
-\int_X \left( Dd_{x_0}^2, Df \right) dm = 2N \int_X f dm.
\]

**Remark 4.6.** Proposition 4.5 asserts \( \Delta d_{x_0}^2 = 2N \) in a distributional sense. This means that the equality is attained in the Laplacian comparison theorem on spaces with \( \text{Ric} \geq 0 \) and \( \text{dim} \leq N \). If we are on a Riemannian manifold, this identity already implies \( X \simeq \mathbb{R}^N \).
Proof. Take a cut-off function $g_R$ for $R > 1$ by $g_R = 1 \wedge (R - d_{x_0})_+$. By definition, $0 \leq g_R \leq 1$, $g_R = 0$ on $B_R(x_0)^c$ and $g_R \to 1$ as $R \to \infty$ pointwisely. By definition, $g_R \in L^1(m) \cap L^\infty(m) \subset L^2(m)$. Let $\rho_t$ be the density of $P_t\delta_{x_0}$ with respect to $m$ That is, $\rho_t(x) = \rho_1(x_0, x)$. We claim that the equality holds in the Li-Yau inequality (2.11) for $\rho_t$. Let $[t, t'] \subset (0, t_s]$ and set

$$A_{\delta} := \{(x, s) \in [t, t'] \times X \mid \frac{|D\rho_s|}{\rho_t^2} - \frac{\Delta \rho_s}{\rho_t} \leq \frac{N}{2s} - \delta \}$$

for $\delta > 0$. By integrating (2.11) for $\rho_s$ by $g_R dP_s \delta_{x_0} \otimes ds$ on $X \times [t, t']$, we obtain

$$\int_t^{t'} \int_X \frac{|D\rho_s|^2}{\rho_t} g_R \, dm \, ds - \int_t^{t'} \int_X \Delta \rho_s g_R \, dm \, ds \leq \frac{N}{2} \int_{X \times [t, t'] \setminus A_\delta} g_R \, ds + \int_{A_\delta} \left( \frac{N}{2s} - \delta \right) g_R \, ds \, ds.$$

Since we have

$$\lim_{R \to \infty} \int_t^{t'} \int_X \Delta \rho_s g_R \, dm \, ds = \lim_{R \to \infty} \left( \int_X \rho_t' g_R \, dm - \int_X \rho_t g_R \, dm \right) = 0,$$

the last inequality implies

$$\int_t^{t'} I(\rho_s) \, ds = \int_t^{t'} \int_X \frac{|D\rho_s|^2}{\rho_t} \, dm \, ds \leq \frac{N}{2} \int_t^{t'} \frac{ds}{s} - \delta \int_{A_\delta} \rho_t g_R \, ds$$

by letting $R \to \infty$. Since we have (4.2), (2.10) implies that $A_\delta$ is of null measure with respect to $dm \otimes ds$. Hence the equality holds in (2.11) for $\rho_t$ a.e. $(x, t)$. Let $J \subset (0, t_s^\ast]$ be the set of $t$ such that the equality holds in (2.11) for $\rho_t$ m.a.e. By the Fubini theorem, $(0, t_s^\ast \setminus J$ is of null Lebesgue measure. We show the assertion by combining this identity with a short time asymptotic of $\rho_t$. Let $f_0 \in \mathcal{D}(\Delta) \cap \text{Lip}_b(X)$ with a bounded support satisfying $\Delta f_0 \in L^1(m) \cap L^\infty(m)$. By (2.10), there exists $c > 0$ such that $\rho_t \geq c$ holds on supp $f_0$. On the basis of this fact, the derivation property and the integration by parts formula yields

$$\int_X \log(\rho_t) \Delta f_0 \, dm = - \int_X \left( D\rho_t, D f_0 \right) \, dm$$

$$= - \int_X \left( D\rho_t, D \left( \frac{f_0}{\rho_t} \right) \right) \, dm - \int_X \frac{|D\rho_t|^2}{\rho_t^2} f_0 \, dm$$

$$= \int_X \left( \frac{\Delta \rho_t}{\rho_t} - \frac{|D\rho_t|^2}{\rho_t^2} \right) f_0 \, dm.$$

By plugging the m.a.e. equality in (2.11) for $t \in J$ into the last identity, we obtain

$$\int_X \log(\rho_t) \Delta f_0 \, dm = -\frac{N}{2t} \int_X f_0 \, dm.$$
By (2.10), $\log \rho_t$ is bounded on $\text{supp} \Delta f_0$. Moreover (2.10) and (2.12) yield the Varadhan type short time asymptotic for the heat kernel:

$$4t \log \rho_t(x) \to -d(x_0, x)^2$$ as $t \downarrow 0$ uniformly on each bounded set.

Thus we obtain

$$-\frac{N}{2} \int_X f_0 \, dm = \lim_{t \downarrow 0} \int_X (t \log \rho_t) \Delta f_0 \, dm = -\frac{1}{4} \int_X d^2_{x_0} \Delta f_0 \, dm = \frac{1}{4} \int_X \langle Dd^2_{x_0}, Df_0 \rangle \, dm.$$

By combining them, we obtain (4.4) for $f_0$.

To prove the assertion, we employ an approximation argument. Let $f \in D(\mathcal{E}) \cap L^1(m)$ and $f_\eta := h_\eta(f \wedge \eta^{-1})$ for $\eta > 0$. By the Lipschitz regularization property of $P_t$, we have $f_\eta \in \text{Lip}_b(X)$ with $0 \leq r \leq \frac{1}{2}, \frac{1}{2} \leq r \leq 0, g_R \in \mathcal{D}(\Delta)$ and $\Delta g_R \in L^\infty(m)$. Note that $g_R f_\eta \in \mathcal{D}(\Delta)$. Indeed, by applying the Leibniz rule and the integration by parts formula, for $h \in D(\mathcal{E}) \cap L^\infty(m)$, we have

$$\mathcal{E}(h, g_R f_\eta) = -\int_X h \left( f_\eta \Delta g_R + 2 \langle Df_\eta, Dg_R \rangle + g_R \Delta f_\eta \right) \, dm$$

(cf. [20, Theorem 4.29]). Thus the same holds for $h \in \mathcal{D}(\mathcal{E})$ by a truncation argument. It implies $g_R f_\eta \in \mathcal{D}(\Delta)$ and $\Delta(g_R f_\eta) = f_\eta \Delta g_R + 2 \langle Df_\eta, Dg_R \rangle + g_R \Delta f_\eta$. From this expression, we can easily verify $\Delta(g_R f_\eta) \in L^1(m) \cap L^\infty(m)$. Thus $g_R f_\eta$ satisfies all assumptions for $f_0$ and hence again by the Leibniz rule,

$$\int_X \langle Dd^2_{x_0}, Dg_R \rangle f_\eta \, dm + \int_X \langle Dd^2_{x_0}, Df_\eta \rangle g_R \, dm = 2N \int_X g_R f_\eta \, dm.$$

By virtue of the fact that $\text{supp} \, g \subset B_{R+1}(x_0)$, the last identity implies

$$\int_X \langle Dd^2_{x_0}, Dg_R \rangle f \, dm + \int_X \langle Dd^2_{x_0}, Df \rangle g_R \, dm = 2N \int_X g_R f \, dm$$

by letting $\eta \to 0$. Thus, for those $f$ in the assertion, the conclusion follows by letting $R \to \infty$ in the last identity with the aid of the locality of minimal weak upper gradient and (2.3).

From now on, we study the consequence of the equality (4.4). We set $V(r) := m(B_r(x_0))$. We begin with the following two auxiliary lemmas (Lemmas 4.7 and 4.8).

**Lemma 4.7.** $\int_X d^p_{x_0} \exp \left( -cd^2_{x_0} \right) \, dm < \infty$ for any $p \geq 0$ and $c > 0$. 
Proof. It suffices to show \( \exp(-cd_{x_0}^2) \in L^1(m) \). By the Fubini theorem,
\[
\int_X \exp(-cd_{x_0}^2) \, dm = \int_X \left( \int_{d_{x_0}}^{\infty} 2c r e^{-cr^2} \, dr \right) \, dm = 2c \int_0^{\infty} r V(r)e^{-cr^2} \, dr.
\]
By the Bishop-Gromov inequality (2.12), there exists \( C > 0 \) such that \( V(r) \leq V(1) \vee Cr^N \). Thus the desired result holds.

**Lemma 4.8.** Let \( f : [0, \infty) \to \mathbb{R} \) be a measurable function such that
\[
\int_0^{\infty} r^n |f(r)| e^{-r/2} \, dr < \infty
\]
for all \( n \in \mathbb{N} \). Suppose
\[
\frac{N+2}{2} \int_0^{\infty} f(r)e^{-\xi r} \, dr = \xi \int_0^{\infty} r f(r)e^{-\xi r} \, dr \tag{4.5}
\]
holds for \( \xi \in (1/2, 2) \). Then there exists \( c_1 \in \mathbb{R} \) such that \( f(r) = c_1 r^{N/2} \) for a.e. \( r \in [0, \infty) \).

Proof. Let us choose \( c_1 \in \mathbb{R} \) so that
\[
\int_0^{\infty} f(r)e^{-r} \, dr = c_1 \int_0^{\infty} r^{N/2} e^{-r} \, dr
\]
and set \( g(r) := f(r) - c_1 r^N \). We can easily verify that (4.5) holds for \( g \) instead of \( f \). That is,
\[
\frac{N+2}{2} \int_0^{\infty} g(r)e^{-\xi r} \, dr = \xi \int_0^{\infty} r g(r)e^{-\xi r} \, dr. \tag{4.6}
\]
We will show
\[
\int_0^{\infty} r^n g(r)e^{-r} \, dr = 0 \tag{4.7}
\]
by induction for \( n \in \mathbb{N} \cup \{0\} \). The assertion for \( n = 0 \) holds by the definition of \( g \). Suppose the claim to be true for \( n \in \mathbb{N} \). By differentiating (4.6) \( n \) times with respect to \( \xi \) at \( \xi = 1 \) with taking the assumption of \( f \) into account. Then, by the assumption of the induction, we immediately obtain (4.7) for \( n + 1 \). Thus (4.7) holds for any \( n \in \mathbb{N} \cup \{0\} \). It means that \( g(r) \) is orthogonal to any polynomial in \( L^2(e^{-r} \, dr) \). Hence \( g = 0 \) a.e. and the conclusion follows.

With keeping Lemma 4.7 in mind, let us define \( \hat{\rho}^{x_0} : (0, \infty) \times X \to \mathbb{R} \) by
\[
\hat{\rho}_t^{x_0}(x) := \frac{1}{Z(t)} \exp \left( -\frac{d_{x_0}(x)^2}{4t} \right),
\]
where \( Z(t) \) is a normalizing constant so that \( \|\hat{\rho}^{x_0}\|_{L^1(m)} = 1 \). For simplicity of notations we denote \( \hat{\rho}^{x_0} \) by \( \hat{\rho} \) if there is no possibility of confusions. Note that \( \hat{\mu}_t := \hat{\rho}_t m \in \mathcal{P}_2(X) \) holds by Lemma 4.7.
Lemma 4.9. Suppose (4.4). Then \( I(\hat{\mu}_t) = N/(2t) \) for \( t > 0 \) and there exists \( c_*, c_{**} > 0 \) such that \( Z(t) = c_*/t^{N/2} = c_{**}V(\sqrt{t}) \) for \( t > 0 \). In particular, \( X \) is non-compact.

Proof. Since \( \hat{\rho}_t \in \text{Lip}(X) \), we have \( \text{lip}(\hat{\rho}_t) \leq d_{x_0}^2 \hat{\rho}_t/(2t) \). Since \( d_{x_0}^2 \hat{\rho}_t, d_{x_0}^2 \hat{\rho}_t \in L^2(m) \) by Lemma 4.7, \( \hat{\mu}_t \in \mathcal{D}(I) \) and

\[
\left| D\hat{\rho}_t \right| \leq \frac{d_{x_0}^2 \hat{\rho}_t}{2t} \tag{4.8}
\]

holds. For \( R > 0 \), take \( g_R \in \text{Lip}_b(X) \) satisfying \( 0 \leq g_R \leq 1 \), \( g_R\mid_{B(x_0,R)} \equiv 1 \), \( g_R\mid_{B(x_0,R+1)} \equiv 0 \). Then, by the Leibniz rule, the locality of minimal weak upper gradient and the chain rule, we have

\[
\int_X \left( \frac{|D\hat{\rho}_t|^2 g_R}{\hat{\rho}_t} + \langle D\hat{\rho}_t, Dg_R \rangle \right) \, dm = \int_X \left( \frac{|D\hat{\rho}_t|^2}{\hat{\rho}_t} - \frac{1}{4t} \right) \, dm = \frac{1}{4t} \int_X \left( \langle Dd_{x_0}^2 \hat{\rho}_t, D\hat{\rho}_t \rangle g_R + \langle Dd_{x_0}^2 \hat{\rho}_t, Dg_R \rangle \hat{\rho}_t \right) \, dm.
\]

By virtue of (2.3), (4.8), Lemma 4.7 and the locality of minimal weak upper gradient, letting \( R \to \infty \) in (4.9) implies

\[
I(\hat{\mu}_t) = \int_X \frac{|D\hat{\rho}_t|^2}{\hat{\rho}_t} \, dm = -\frac{1}{4t} \int_X \left( \langle Dd_{x_0}^2 \hat{\rho}_t, D\hat{\rho}_t \rangle g_R + \langle Dd_{x_0}^2 \hat{\rho}_t, Dg_R \rangle \hat{\rho}_t \right) \, dm = \frac{N}{2t} \int_X \hat{\rho}_t \, dm = \frac{N}{2t},
\]

where the third inequality comes from Proposition 4.5. Thus the first assertion holds. For the second assertion, we compute \( I(\hat{\mu}_t) \) in a different manner. By the Leibniz rule, the chain rule and (2.3),

\[
\int_X \left( \langle Dd_{x_0}^2, D\hat{\rho}_t g_R \rangle \right) \, dm = \int_X \left( \frac{1}{4t} \right) \left( \langle Dd_{x_0}^2, D\hat{\rho}_t g_R \rangle + \langle Dd_{x_0}^2, Dg_R \rangle \hat{\rho}_t \right) \, dm = \int_X \left( \frac{1}{4t} \right) \left( \langle Dd_{x_0}^2, Dg_R \rangle \hat{\rho}_t \right) \, dm.
\]

By substituting this identity into (4.9) and letting \( R \to \infty \), we obtain

\[
I(\hat{\mu}_t) = \frac{1}{4t^2} \int_X d_{x_0}^2 \hat{\rho}_t \, dm. \tag{4.11}
\]

As in the proof of Lemma 4.7, we have

\[
\int_X d_{x_0}^{2p} \exp \left( -\frac{d_{x_0}^2}{4t} \right) \, dm = \int_0^\infty V(r) \left( \frac{r^{2p+1}}{2t} - 2pr^{2p-1} \right) \exp \left( -\frac{r^2}{4t} \right) \, dr
\]

for \( p \geq 0 \). By combining this with \( \hat{\mu}_t(X) = 1 \), (4.10) and (4.11), we obtain

\[
Z(t) = \frac{1}{2t} \int_0^\infty r V(r) \exp \left( -\frac{r^2}{4t} \right) \, dr
\]
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\[
= \frac{1}{2Nt} \int_0^\infty V(r) \left( \frac{r^3}{2t} - 2r \right) \exp \left( -\frac{r^2}{4t} \right) \, dr.
\]  \hspace{1cm} (4.12)

Thus we have

\[
\frac{N + 2}{2} \int_0^\infty r V(r) \exp \left( -\frac{r^2}{4t} \right) \, dr = \int_0^\infty \frac{r^3 V(r)}{4t} \exp \left( -\frac{r^2}{4t} \right) \, dr.
\]

After the change of variable \(\tilde{r} = r^2\), we can apply Lemma 4.8 to conclude that there exists \(c_1 \in \mathbb{R}\) such that \(V(r) = c_1 r^N\) for a.e. \(r\). Note that \(c_1 > 0\) holds by the definition of \(V\). Since \(V\) is left-continuous, the last identity for \(V\) holds for all \(r \geq 0\). Then the assertion for \(Z(t)\) follows from the first identity in (4.12). Finally, \(X\) must be non-compact since \(V(r) \to \infty\) as \(r \to \infty\).

As we see in Theorem 2.2, the conclusion of Lemma 4.9 is already sufficient to specify \((X, d, m)\) as in Theorem 4.1 by Lemma 4.4 and Proposition 4.5. To show \(\mu\) to be a Dirac measure, we need the following additional arguments.

**Proposition 4.10.** Suppose (4.4). Then \(\hat{\mu}_t = P_t \delta_{x_0}\) for \(t > 0\). In particular, \(\hat{\mu}_t \to \delta_{x_0}\) as \(t \to 0\).

**Proof.** Let \(\delta > 0\) and \(\hat{\mu}_{t+\delta} := \hat{\mu}_t + \hat{\rho}_t\). Note that \(\hat{\mu}_{t+\delta} = \hat{\mu}_0 \in D(\text{Ent})\) and \(\hat{\rho}_t \in L^2(m)\). We first show \(\hat{\mu}_t = P_t \hat{\mu}_{t+\delta}\). By [3, Theorem 9.3], \((P_t \hat{\mu}_0)_{t \geq 0}\) is a unique gradient flow of \(\text{Ent}\) starting from \(\hat{\mu}_0\) in the sense of the energy dissipation identity (2.6). Thus it suffices to show that \((\hat{\mu}_t)_{t \geq 0}\) is also a gradient flow of \(\text{Ent}\) in the same sense. We show it by following a strategy in [3].

Since we are on \(\text{RCD}(0, N)\) space, we already know that the descending slope \(|D^- \text{Ent}|\) of \(\text{Ent}\) is an upper gradient of \(\text{Ent}\) and \(|D^- \text{Ent}|^2 = I\) by [3, Theorem 9.3]. In this case, the inequality “\(\leq\)” in (2.6) holds automatically if \((\mu_t)_{t \geq 0}\) is absolutely continuous curve in \((\mathcal{P}_2(X), W_2)\). Thus the proof is reduced to show the absolute continuity of \((\hat{\mu}_t)_{t > 0}\) and the following:

\[
\frac{d}{dt} \text{Ent} \left( \hat{\mu}_t^\delta \right) = -I \left( \hat{\mu}_t^\delta \right) \quad \text{for a.e. } t, \hspace{1cm} (4.13)
\]

\[
\lim_{s \downarrow 0} \frac{W_2 \left( \hat{\mu}_t^\delta, \hat{\mu}_{t+s}^\delta \right)^2}{s} \leq I \left( \hat{\mu}_t^\delta \right) \quad \text{for a.e. } t. \hspace{1cm} (4.14)
\]

By the definition of \(\hat{\mu}_t^\delta\), (4.11), (4.10) and Lemma 4.9,

\[
\text{Ent} \left( \hat{\mu}_t^\delta \right) = -\frac{1}{4(t + \delta)} \int_X d_{x_0}^2 \hat{\rho}_{t+\delta} \, dm - \log Z(t + \delta)
\]

\[
= -\frac{N}{2} - \log c_* - \frac{N}{2} \log(t + \delta).
\]

Thus, again by Lemma 4.9,

\[
\frac{d}{dt} \text{Ent} \left( \hat{\mu}_t^\delta \right) = -\frac{N}{2(t + \delta)} = -I \left( \hat{\mu}_t^\delta \right).
\]
Hence we obtain \((4.13)\). For \((4.14)\), the Kantorovich duality yields
\[
W_2(\hat{\mu}_t, \hat{\mu}_{t+s})^2 = \frac{2}{s} \sup_{\varphi \in \text{Lip}_b(X)} \left[ \int_X Q_s \varphi \, d\hat{\mu}_{t+s} - \int_X \varphi \, d\hat{\mu}_t \right],
\]
where \(Q_s\) is the Hopf-Lax semigroup defined by
\[
Q_s \varphi(x) := \inf_{y \in X} \left[ \varphi(y) + \frac{d(x, y)^2}{2s} \right].
\]
It is known that \(Q_s \varphi \in \text{Lip}_b(X)\) for \(\varphi \in \text{Lip}_b(X)\) (see [12] for instance). It is not difficult to verify that we can suppose \(\varphi\) to be of bounded support in the range of the supremum in \((4.15)\). Let \(\varphi \in \text{Lip}_b(X)\) with a bounded support. It is easy to see \(\varphi \in \mathcal{D}(\mathcal{E})\). Note that \(\hat{\rho}_{t+\delta}\) is differentiable in \(t \in (0, \infty)\) and possesses a sufficiently good integrability by Lemma 4.7. Thus, by the dominated convergence theorem,
\[
\int_X Q_r \varphi \, d\hat{\rho}_{t+\delta+r} \, dm \text{ is a.e. differentiable in } r \text{ and }
\]
\[
\int_X Q_s \varphi \, d\hat{\mu}_{t+s} - \int_X \varphi \, d\hat{\mu}_t = \int_0^s \left( \frac{d}{dr} \int_X Q_r \varphi \, d\hat{\rho}_{t+\delta+r} \, dm \right) \, dr.
\]
Since \(Q_r \varphi\) satisfies the following Hamilton-Jacobi equation
\[
\frac{\partial}{\partial s} Q_s \varphi + \frac{1}{2} \text{lip}(Q_s \varphi)^2 = 0
\]
(see [3, Theorem 3.6]), by Lemma 4.9 and the fact \(|Df| \leq \text{lip}(f)\), we have
\[
\frac{d}{dr} \int_X Q_r \varphi \, d\hat{\rho}_{t+\delta+r} \, dm \leq -\frac{1}{2} \int_X |DQ_r \varphi|^2 \hat{\rho}_{t+\delta+r} \, dm
\]
\[
+ \frac{1}{4(t + \delta + r)^2} \int_X d_{x_0}^2 Q_r \varphi \, d\hat{\rho}_{t+\delta+r} \, dm
\]
\[
- \frac{N}{2(t + \delta + r)} \int_X Q_r \varphi \, d\hat{\rho}_{t+\delta+r} \, dm.
\]
By \((4.4)\), the Leibniz rule, the chain rule and \((2.3)\) together with a localization argument as we did in the proof of Lemma 4.9,
\[
- \frac{N}{2(t + \delta + r)} \int_X Q_r \varphi \, d\hat{\rho}_{t+\delta+r} \, dm
\]
\[
= -\frac{1}{4(t + \delta + r)} \int_X \left( D_{x_0}^2, D(Q_r \varphi \hat{\rho}_{t+\delta+r}) \right) \, dm
\]
\[
= \frac{1}{4(t + \delta + r)} \left( \int_X \left( D_{x_0}^2, D Q_r \varphi \right) \, d\hat{\rho}_{t+\delta+r} \, dm
\]
\[
+ \int_X Q_r \varphi \left( D_{x_0}^2, D \hat{\rho}_{t+\delta+r} \right) \, dm \right)
\]
\[
= -\int_X \left( D\hat{\rho}_{t+\delta+r}, D Q_r \varphi \right) \, dm - \frac{1}{4(t + \delta + r)^2} \int_X d_{x_0}^2 Q_r \varphi \, d\hat{\rho}_{t+\delta+r} \, dm.
\]
Thus, by substituting this identity into (4.17), we obtain
\[
\frac{d}{dr} \int_X Q_r \varphi \hat{\mu}_{t+r} \, dm \leq -\frac{1}{2} \int_X |D Q_r \varphi|^2 \, d\mu_{t+r}^\delta \\
- \int_X \frac{1}{\varphi(t+\delta+r)} \left\langle D \hat{\rho}_{t+\delta+r}, D Q_r \varphi \right\rangle \, d\hat{\mu}_{t+r}^\delta \\
\leq \frac{1}{2} I(\hat{\mu}_{t+r}^\delta) = \frac{N}{4(t+\delta+r)}.
\]

By applying this inequality to (4.16), (4.15) yields
\[
W_2^2(\hat{\mu}_t^\delta, \hat{\mu}_{t+s}^\delta) \leq \frac{1}{s^2} \int_0^s \frac{N}{(t+\delta+r)} \, dr.
\]
This inequality together with Lemma 4.9 yields (4.14). It is immediate that the last inequality implies the absolute continuity of $\hat{\mu}_{t+r}^\delta$ for $t \geq 0$.

Note that we have
\[
W_2(\hat{\mu}_t, \hat{\mu}_{t+r}) = \int_X d^2 x_0 \hat{\rho}_t \, dm.
\]
Here the right hand side goes to 0 as $\delta \to 0$ by virtue of the explicit expression of $Z(t)$ in Lemma 4.9. This fact implies the last assertion. In addition, by (2.8),
\[
W_2^2(\hat{\mu}_{t+\delta}, P_t \delta_{x_0}) = W_2^2(P_t \hat{\mu}_t, P_t \delta_{x_0}) \leq W_2^2(\hat{\mu}_{t+r}^\delta, \delta_{x_0}).
\]
Then the assertion for $t > 0$ holds by letting $\delta \downarrow 0$ since $(\hat{\mu}_t)_{t \geq 0}$ is a continuous curve in $(P_2(X), W_2)$. \qed

Remark 4.11. A converse of Proposition 4.10 holds in the following sense. Let $\hat{\mu}_t$ be as above and suppose $Z(t) = c_* t^{N/2}$ for some constant $c_* > 0$. If $\hat{\mu}_t$ is a solution to the heat equation, then $\mathcal{W}(\hat{\mu}_t, t)$ is constant in $t$ and hence $\mathcal{W}(\hat{\mu}_t, t)$ has vanishing $t$-derivatives.

Lemma 4.12. Suppose that the equality holds in (4.1). Then $\mu$ is a Dirac measure.

Proof. Let $\Phi : (0, \infty) \times [0, \infty)$ be given by $\Phi(u, v) := v^2/u$. It is verified in a straightforward way that $\Phi$ is convex and that
\[
\Phi((1-\lambda)(u_1, v_1) + \lambda(u_2, v_2)) = (1-\lambda)\Phi(u_1, v_1) + \lambda\Phi(u_2, v_2)
\]
holds for some $\lambda \in (0, 1)$ if and only if $v_1/u_1 = v_2/u_2$. Note that we have
\[
P_t \mu = \frac{1}{2} \left( \int_X P_t \delta_x \mu(dx) + \int_X P_t \delta_y \mu(dy) \right)
\]
\[
= \int_{X \times X} \frac{1}{2} (P_t \delta_x + P_t \delta_y) \mu \otimes^2 (dx \, dy).
\]

By combining Lemma 4.3 (ii) with the convexity of minimal weak upper gradient and the convexity of \(\Phi\), we have

\[
\frac{N}{2t} = I(P_t \mu)
\leq \int_X \left( \int_{X^2} \Phi \left( \frac{P_t \delta_x + P_t \delta_y}{2}, \frac{|DP_t \delta_x| + |DP_t \delta_y|}{2} \right) \mu \otimes^2 (dx \, dy) \right) \, dm
\leq \int_X \left( \int_X \Phi \left( P_t \delta_x, |DP_t \delta_x| \right) \mu(dx) \right) \, dm
= \int_X I(P_t \delta_x) \mu(dx) \leq \frac{N}{2t},
\]

where the last inequality follows from Lemma 4.2. Hence all the inequalities are indeed equality. By the property of \(\Phi\) mentioned at the beginning of the proof together with the Fubini theorem, for \(\mu \otimes^2\)-a.e. \((x, y)\), we have

\[
\frac{|DP_t \delta_x|}{P_t \delta_x} = \frac{|DP_t \delta_y|}{P_t \delta_y} \text{ m-a.e.} \quad (4.18)
\]

By virtue of (4.2), Propositions 4.5 and 4.10 ensure \(P_t \delta_x = \hat{\rho}_t^x m\) \(\mu\)-a.e. \(x\). This representation of \(P_t \delta_x\) together with (2.3) implies

\[
\frac{|DP_t \delta_x|}{P_t \delta_x} = \frac{|D \hat{\rho}_t^x|}{\hat{\rho}_t^x} = \frac{d_x}{2t} \quad \text{m-a.e.} \quad (4.19)
\]

By combining (4.18) and (4.19), we conclude that \(d_x = d_y\) m-a.e. for \(\mu \otimes^2\)-a.e. \((x, y)\). Hence \(\mu\) must be a Dirac measure. \(\square\)

Proof of Theorem 4.1. By Lemma 4.12, we know \(\mu = \delta_{x_0}\) for some \(x_0 \in X\). Then, by Lemma 4.9, there exists \(c > 0\) such that \(m(B_r(x_0)) = cr^N\) holds for all \(r > 0\). It yields the equality in (2.12). Then the conclusion follows from Theorem 2.2. \(\square\)

5. Related results

Here we gather some results related with our main theorem. First we show that the heat flow coincides with the \(W_2\)-geodesic in the rigidity case. According to Proposition 4.10, we define \(\hat{\mu}_0 := \delta_{x_0}\).

Proposition 5.1. Suppose (4.4). Then \((\hat{\mu}_{t^2/(2N)})_{t \geq 0}\) is a unit-speed minimal geodesic in \(W_2\).

Proof. Let \(\bar{\mu}_t := \hat{\mu}_{t^2/(2N)}\). By (4.11) and Lemma 4.9,

\[
W_2(\bar{\mu}_0, \bar{\mu}_t) = \left( \int_X d_{x_0}^2 d \bar{\mu}_t \right)^{1/2} = t.
\]
On the other hand, by Proposition 4.10 and Lemma 4.9,
\[
\lim_{s \downarrow 0} \frac{W_2(\tilde{\mu}_t, \tilde{\mu}_{t+s})}{s} = t N \sqrt{I(\tilde{\mu}_t)} = 1
\]
(See [3, Definition 2.14]). We denote the left hand side of the last identity by $|\dot{\tilde{\mu}}_t|$. This is the metric speed for $\tilde{\mu}_t$ in $(\mathcal{P}_2(X), W_2)$. Thus, as remarked in Sect. 2, we have

\[
W_2(\tilde{\mu}_s, \tilde{\mu}_t) \leq \int_s^t |\dot{\tilde{\mu}}_r| \, dr
\]
for $0 < s < t$ and hence
\[
t = W_2(\tilde{\mu}_0, \tilde{\mu}_t) \leq W_2(\tilde{\mu}_0, \tilde{\mu}_s) + W_2(\tilde{\mu}_s, \tilde{\mu}_t) \leq s + \int_s^t |\dot{\tilde{\mu}}_r| \, dr = t.
\]
Thus all the last inequality must be equality. In particular, $W_2(\tilde{\mu}_s, \tilde{\mu}_t) = t - s$. Therefore the assertion holds.

The second result asserts monotonicity in time of the infimum of $W$-entropy. It partially extends [38, Theorem 2.4] to our framework. It is related with the (logarithmic) Sobolev inequality. See Remarks 5.3 and 5.4 below.

**Theorem 5.2.** Let $c(t) := \inf \{ W(\mu, t) \mid \mu \in \mathcal{D}(\text{Ent}) \}$ for $t \in (0, \infty)$. Then $c$ is non-increasing.

**Proof.** Let $0 < s < t$. Note that $P_r \mu \in \mathcal{D}(\text{Ent})$ holds for each $\mu \in \mathcal{P}_2(X)$ and $r > 0$. Thus, for any $\mu \in \mathcal{D}(\text{Ent})$, we have

\[
W(\mu, s) \geq W(P_{t-s} \mu, t) \geq c(t)
\]
by Theorem 3.3. Hence the conclusion holds by taking infimum over all $\mu \in \mathcal{D}(\text{Ent})$ in the left hand side of the last inequality.

**Remark 5.3.** By the definition of $c(t)$ in Theorem 5.2, we have

\[
\text{Ent}(\mu) \leq t I(\mu) - \frac{N}{2} \log t - c(t)
\]
for all $\mu \in \mathcal{D}(\text{Ent})$. That is, if $c(t) > -\infty$ for some $t$, then we have a defective log-Sobolev inequality.

**Remark 5.4.** Let $N > 2$. Then $c := \inf_{t>0} c(t) > -\infty$ if and only if there exists a constant $C > 0$ depending on $c$ and $N$ such that

\[
\exp \left( \frac{2}{N} \text{Ent}(\mu) \right) \leq C I(\mu)
\]
holds for any $\mu \in \mathcal{D}(\text{Ent})$. By [11, Theorem 6.2.3], this inequality implies the Sobolev inequality:

\[
\| f \|_{2N/(N-2)}^2 \leq C \int_X |Df|^2 \, dm.
\]
Indeed, weaker forms of these inequalities are equivalent. Note that $c = \lim_{t \to \infty} c(t)$ by Theorem 5.2. Moreover, by [11, Theorem 6.3.1], A weaker form of the Sobolev inequality is equivalent to the ultracontractivity:

$$\|P_t\|_{1 \to \infty} \leq \frac{C'}{t^{N/2}}.$$ 

Combined with the heat kernel lower bound (2.10), $c > -\infty$ implies that there exists $C'' > 0$ such that $m(B_r(x)) \geq C''r^N$ for each $r > 0$ and $x \in X$. Note that the Bishop-Gromov inequality implies a similar bound but it is local in the sense it holds for $r < R$ for each fixed $R$, and the constant corresponding to $C''$ depends on $R$ and $x$. Note that a similar result is obtained in [38, Theorem 6.1]. It is shown on weighted Riemannian manifolds but the same argument works even in our framework.

**Remark 5.5.** In [38, Theorem 2.4], it was proved that, in the case $(M, g)$ is a compact Riemannian manifold, and $\phi \in C^4(M)$, there exists a positive and smooth function $u = e^{-f/2}$ such that $f$ achieves

$$\mu(t) := \inf \left\{ \mathcal{W}_m(g, \phi, f, \tau) : \int_M e^{-f} (4\pi \tau)^{-m/2} d\mu = 1 \right\},$$

where $d\mu = e^{-\phi} dv$, $m \geq n$, and

$$\mathcal{W}_m(g, \phi, f, \tau) = \int_M \left( \tau |\nabla f|^2 + f - m \right) \frac{e^{-f}}{(4\pi \tau)^{m/2}} d\mu$$

is the $\mathcal{W}_m$-entropy on the weighted Riemannian manifold $(M, g, \mu)$. Indeed, by a similar argument as used in Perelman [44], it is shown in [38] that the minimization problem (5.1) has a non-negative minimizer $u \in H^1(M, \mu)$, which satisfies the Euler-Lagrange equation

$$-4\tau Lu - 2u \log u - mu = \mu(\tau)u,$$

where $L = \Delta - \nabla \phi \cdot \nabla$ is the Witten Laplacian on the weighted Riemannian manifold $(M, g, \mu)$. By the regularity theory of elliptic PDEs on Riemannian manifolds, we have $u \in C^{1,\alpha}(M)$ for $\alpha \in (0, 1)$. Then, an argument due to Rothaus [45] allows us to prove that $u$ is strictly positive and smooth. Hence $f = -2 \log u$ is also smooth. In [40], this result has been extended to weighted complete Riemannian manifolds satisfying the $L^2$-Sobolev inequality. See also [33, 34] for further extension. It will be interesting to study the similar question and to see what happens on RCD$(0, N)$-spaces.

The third result asserts a stronger rigidity under a stronger assumption.

**Theorem 5.6.** Suppose that the assumption of Theorem 4.1 holds for $\mu = \delta_x$ for any $x \in X$. Then $N \in \mathbb{N}$ and $(X, d, m)$ is isomorphic to the Euclidean space $\mathbb{R}^N$ with the canonical metric measure structure, up to positive multiplicative constant on the measure.
Proof. Since $X$ is non-compact by Lemma 4.9, $X$ must contain more than two points. Let $x, y \in X$. Then Theorem 4.1 yields that $X$ is $(0, N)$-cone with vertex at $x$. Thus the unique minimal (unit speed) geodesic from $x$ to $y$ can be extended to a geodesic ray. By the same reason, the unique minimal (unit speed) geodesic from $y$ to $x$ can be extended to a geodesic ray. By concatenating these two geodesic rays, we obtain a line in $X$. Thus Gigli’s splitting theorem [19] yields that there exists an $\text{RCD}(0, N - 1)$ space $(Y, d_Y, \mu_Y)$ such that $(X, d, m)$ is isomorphic to $\mathbb{R} \times Y$ if $N \geq 2$. When $N < 2$, $Y$ consists of one point and hence $X \simeq \mathbb{R}$. In the latter case, $N = 1$ must hold since there is $c > 0$ such that $m(B_r(x)) = cr^N$ by Lemma 4.9. Suppose that the former case happens. By the same reason as in the last argument, $Y$ must contain more than two points since $N \geq 2$. Let $x_1, y_1 \in Y$. Then, by applying the same argument as above to $(s, x_1)$ and $(s, y_1)$ instead of $x$ and $y$, we obtain a line in $X$ passing through $(s, x_1)$ and $(s, y_1)$. Then we obtain the corresponding line in $Y$ containing $x_1$ and $y_1$. Then we can apply Gigli’s splitting theorem to $Y$. Accordingly, we can repeat the same argument to obtain the conclusion. \(\Box\)

Remark 5.7. We have a simpler proof of Theorem 5.6 based on the notion of tangent cones. We just give an outline of the proof here. By [22, Theorem 1.1], $\text{m}$-a.e. points in $X$ has a Euclidean tangent cone and the dimension $k$ of the cone satisfies $k \leq N$. We choose such a point $x \in X$. As a consequence of Theorem 4.1 applied to $\mu = \delta_x$, tangent cones at $x$ of $(X, d, m)$ is unique and isomorphic to $(X, d, m)$ itself (more precisely, isomorphic to $(X, d, m, x)$ as pointed metric measure space). Thus, by the choice of $x$, $(X, d, m)$ is isomorphic to $\mathbb{R}^k$ for some $k \in \mathbb{N}$ with $k \leq N$. Then $k = N$ must hold by Lemma 4.9.

On the one hand, we can see from this alternative proof that it is sufficient to assume that there exists a measurable $A \subset X$ with $m(A) > 0$ such that the assumption of Theorem 4.1 holds for $\mu = \delta_x$ for any $x \in A$. On the other hand, the first proof requires essentially just $(N + 1)$-points satisfying the assumption which are located so that “they span the whole space”.

Finally, as a fourth result of this section, we discuss an almost rigidity. To state it, we borrow the notion of pointed Gromov weak distance $p_{\mathcal{G}}\text{w}$ from [23, Definition 3.13] between pointed metric measure spaces. In our situation below, a convergence in $p_{\mathcal{G}}\text{w}$ (pointed measured Gromov convergence in the terminology of [23]) is equivalent to a convergence in the pointed measured Gromov–Hausdorff topology (See [23, Theorem 3.30 and Theorem 3.33]). We choose $p_{\mathcal{G}}\text{w}$ just for simplicity of the statement. We omit the definition of $p_{\mathcal{G}}\text{w}$ here, but use properties of it instead. For brevity of presentation, we state only the case $N \geq 2$, but the corresponding assertion holds for $N \in [1, 2]$.

Theorem 5.8. (Almost rigidity) Suppose $N \geq 2$. Let $T > 0$ and $r_n : (0, T] \to (-\infty, 0)$ ($n \in \mathbb{N}$) a series of non-increasing functions such that $(r_n(t))_{n \in \mathbb{N}}$ is non-decreasing for each $t \in (0, T]$ with

$$\lim_{t \downarrow 0} \lim_{n \to \infty} r_n(t) = \sup_{n,t} r_n(t) = 0.$$
Fix \( s > 0 \). Let \( \mathcal{M}_I \) (\( l \in \mathbb{N} \)) be the set of all pointed \( \text{RCD}(0, N) \) spaces \((X, d, m, x^*)\) satisfying

\[ \mathcal{W}(P_{t^*+s} \delta_{x^*}, t' + s) - \mathcal{W}(P_s \delta_{x^*}, s) \geq r_I(t) t' \text{ for all } t' \in (0, t], \quad (5.4) \]

and let \( \hat{\mathcal{M}} \) be as follows:

\[ \hat{\mathcal{M}} = \left\{ (\hat{X}, \hat{d}, \hat{m}, \hat{x}^*) \mid \text{There exists an } \text{RCD}^*(N - 2, N - 1) \text{ space } (Y, dy, m_Y) \text{ such that } (X, d, m) \text{ is } (0, N - 1) \text{-cone built over } (Y, dy, m_Y) \text{ with vertex } \hat{x}^* \right\}. \]

Then we have

\[ \lim_{l \to \infty} \sup_{(X, d, m, x^*) \in \mathcal{M}_l} \inf_{(\hat{X}, \hat{d}, \hat{m}, \hat{x}^*) \in \hat{\mathcal{M}}} p_{GW} \left( (X, d, m^*, x^*), (\hat{X}, \hat{d}, \hat{m}^*, \hat{x}^*) \right) = 0, \]

where \( m^* \) (resp. \( \hat{m}^* \)) is a normalization of \( m \) (resp. \( \hat{m} \)) so that \( m^*(B_1(x^*)) = 1 \) (resp. \( \hat{m}^*(B_1(\hat{x}^*)) = 1 \)).

**Proof.** We first recall that the family of pointed normalized \( \text{RCD}(0, N) \) spaces \((X, d, m^*, x^*)\) (where “normalized” means \( m^*(B_1(x^*)) = 1 \)) are compact with respect to \( p_{GW} \). It follows by combining Lemma 3.32, Theorem 3.30, Remark 3.29, comments at the beginning of Subsection 4.2 and Theorem 7.2 of [23].

Suppose that the conclusion does not hold. Then there are \( \varepsilon > 0 \) and an increasing sequence \( l_n \in \mathbb{N} \) \( (n \in \mathbb{N}) \) such that there exists \((X_n, d_n, m_n, x_n^*) \in \mathcal{M}_{l_n}\) satisfying

\[ \inf_{(\hat{X}, \hat{d}, \hat{m}, \hat{x}^*) \in \hat{\mathcal{M}}} p_{GW} \left( (X_n, d_n, m_n^*, x_n^*), (\hat{X}, \hat{d}, \hat{m}^*, \hat{x}^*) \right) \geq \varepsilon \quad (5.5) \]

for each \( n \in \mathbb{N} \). Then there exists a convergent subsequence of \((X_{n_k}, d_{n_k}, m_{n_k}^*, x_{n_k}^*)_{n_k \in \mathbb{N}}\) with respect to \( p_{GW} \). We may assume that \((X_{n_k}, d_{n_k}, m_{n_k}^*, x_{n_k}^*)_{n_k \in \mathbb{N}}\) itself converges without loss of generality. We denote the limit by \((X, d, m^*, x^*)\) and the remark at the beginning of this proof tells us that \((X, d, m^*, x^*)\) is a (normalized) \( \text{RCD}(0, N) \) space. Let \( P_{t}^{(n)} \) (resp. \( P_t \)) be the heat semigroup on \((X_{n_k}, d_{n_k}, m_{n_k}^*)\) (resp. \((X, d, m^*)\)). By [23, Theorem 7.7], \( \text{Ent}(P_{t^*+s}^{(n)} \delta_{x_n^*}) \to \text{Ent}(P_{t^*+s} \delta_{x^*}) \) for \( t \in [0, T] \) and \( I(P_{t^*+s}^{(n)} \delta_{x_n^*}) \to I(P_{t^*+s} \delta_{x^*}) \) for a.e. \( t \in [0, T] \). Let \( J \subset [0, T] \) be the set of points where the latter convergence occurs. Take \( t \in (0, T] \) and \( t', t'' \in J \) with \( t'' < t' < t \). Then, by Lemma 3.1, we have

\[ r_{l_n}(t) t' \leq \mathcal{W}(P_{s+t}^{(n)} \delta_{x_n^*}, s + t') - \mathcal{W}(P_s \delta_{x^*}, s) \]

\[ \leq \mathcal{W}(P_{s+t}^{(n)} \delta_{x_n^*}, s + t') - s I(P_{s+t''}^{(n)} \delta_{x_n^*}) + \text{Ent}(P_s^{(n)} \delta_{x_n^*}) + \frac{N}{2} \log s. \]

Let \( r_{\infty} := \lim_{n \to \infty} r_{l_n} \). Then, by taking \( n \to \infty \) in the last inequality together with Lemma 3.1, we have

\[ r_{\infty}(t) t' \leq \mathcal{W}(P_{s+t'} \delta_{x^*}, s + t') - \mathcal{W}(P_s \delta_{x^*}, s) \]
for all $t' \in [0, t]$. Since $\lim_{t \downarrow 0} r_\infty(t) = 0$, the last inequality easily yields

$$\lim_{t \downarrow 0} \frac{\mathcal{W}(P_t \delta_{x_\infty}, t + s) - \mathcal{W}(P_s \delta_{x_\infty}, s)}{t} = 0$$

with the aid of Theorem 3.3. Thus, by Theorem 4.1, we obtain $(X, d, m^*, x^*) \in \hat{\mathcal{M}}$. It contradicts with (5.5) and hence the conclusion follows. \hfill \Box

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