ON THE TIGHTNESS OF $G_\delta$-MODIFICATIONS

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Abstract. The $G_\delta$-modification $X_\delta$ of a topological space $X$ is the space on the same underlying set generated by, i.e. having as a basis, the collection of all $G_\delta$ subsets of $X$. Bella and Spadaro recently investigated in [2] the connection between the values of various cardinal functions taken on $X$ and $X_\delta$, respectively. In their paper, as Question 2, they raised the following problem: Is $t(X_\delta) \leq 2^{t(X)}$ true for every (compact) $T_2$ space $X$? Note that this is actually two questions.

In this note we answer both questions: In the compact case affirmatively and in the non-compact case negatively. In fact, in the latter case we even show that it is consistent with ZFC that no upper bound exists for the tightness of the $G_\delta$-modifications of countably tight, even Frechet spaces.

1. Introduction

The $G_\delta$-modification $X_\delta$ of a topological space $X$ is the space on the same underlying set generated by, i.e. having as a basis, the collection of all $G_\delta$ subsets of $X$. Bella and Spadaro recently investigated in [2] the connection between the values of various cardinal functions taken on $X$ and $X_\delta$, respectively. In their paper, as Question 2, they raised the following problem: Is $t(X_\delta) \leq 2^{t(X)}$ true for every (compact) $T_2$ space $X$? Note that this is actually two questions. In this note we answer both questions: In the compact case affirmatively and in the non-compact case negatively. Actually, for the compact case we prove something stronger: We show that for every regular Lindelöf space $X$ we have $t(X_\delta) \leq 2^{t(X)}$. In the non-compact case we shall show that it is consistent with ZFC that no upper bound exists for...
the tightness of the $G_δ$-modifications of countably tight, even Frechet spaces.

We shall use standard notation and terminology from set theory and general topology. In particular, concerning cardinal functions, we follow the notation and terminology of [4]. It will be useful to denote by $G_δ(X)$ the family of all $G_δ$ subsets of a space $X$. So, as we said, $G_δ(X)$ is a basis for $X_δ$. For any $A ⊂ X$ we shall use $\overline{A}$ to denote the closure of $A$ in $X_δ$.

Just like in [2], our proofs will often use elementary submodels of appropriate "initial segments" of the form $H(λ)$ of the universe. Most readers will be familiar enough with these notions and for those who are not they are surveyed e.g. in [3].

2. Bounds for the tightness of $G_δ$-modifications

Theorem 2.1. If $X$ is a regular Lindelőf space then $t(X_δ) ≤ 2^{t(X)}$.

Proof. Assume that $X$ is a regular Lindelőf space, $p ∈ X$ and $A ⊂ X$ are such that $p ∈ \overline{A}$. Let us put $κ = 2^{t(X)}$ and choose an elementary submodel $M$ of an appropriate $H(λ)$ such that $|M| = κ$, $M^{t(X)} ⊂ M$, moreover $\{X, A, p\} ⊂ M$. We shall show that then $p ∈ A ∩ M_δ$.

To see this, assume that $p ∈ H ∈ G_δ(X)$, i.e. $H = \bigcap_{n < ω} V_n$ where each $V_n$ is open in $X$. We have to prove that $H \cap A \cap M ≠ \emptyset$.

For every point $x ∈ A \cap M$ (note that this is the closure in $X$) there is a subset $B_x ⊂ A \cap M$ with $|B_x| ≤ t(X)$ such that $x ∈ \overline{B_x}$. Then $M^{t(X)} ⊂ M$ implies $B_x ∈ M$. Clearly, if $x ≠ p$ then $B_x$ can be chosen so that $p ∉ \overline{B_x}$. In this case, by the regularity of $X$, there are open sets $U_x ⊃ B_x$ and $W_x$ with $p ∈ W_x$ such that $U_x \cap W_x = \emptyset$, moreover as both $\overline{B_x}$ and $p$ belong to $M$, we may assume that $U_x$ and $W_x$ also belong to $M$.

Now, the closed subspace $\overline{A ∩ M} \setminus V_n$ of $X$ is Lindelőf for each $n < ω$, hence there is a countable subset $C_n ⊂ \overline{A ∩ M} \setminus V_n$ such that $\overline{A ∩ M} \setminus V_n ⊂ \bigcup_{x ∈ C_n} U_x$. This clearly implies that if we put $W_n = \bigcap_{x ∈ C_n} W_x$ then $A ∩ M \cap W_n ⊂ V_n$.

Although we do not know if $C_n ∈ M$, we do know that $W_n ∈ M$ because $W_x ∈ M$ for each $x ∈ C_n$ and $M$ is countably closed.
It follows then that \( W = \bigcap_{n < \omega} W_n \in M \cap G_\delta(X) \), hence \( p \in \overline{A} \)
and \( p \in W \) imply \( W \cap A \neq \emptyset \) and, by elementarity, \( W \cap A \cap M \neq \emptyset \) as well. But then we have \( \emptyset \neq W \cap A \cap M \subset H \cap A \cap M \), which completes our proof.

The one-point compactification of an uncountable discrete space has countable tightness, it is even Frechet, and its \( G_\delta \)-modification clearly has tightness \( \omega_1 \). This, of course, shows that Theorem 2.1 is sharp for countably tight compact spaces under CH. But what happens if the continuum \( c \) is large? Actually, we do not know the full answer to this question.

However, we happen to have a ready made consistent answer in [5] where a weakening of CH called CH* was introduced. It was shown there that CH* holds in any model obtained by adding any number of Cohen reals to a ground model that satisfies CH. Thus CH* is consistent with \( c \) being anything it can be.

Let us denote by \( \ell_{\omega_1}(A) \) the set of all points obtainable as the limit of a converging \( \omega_1 \)-sequence of points of \( A \) in a space \( X \). We permit constant sequences, hence \( A \subset \ell_{\omega_1}(A) \). It is obvious that we always have \( \ell_{\omega_1}(A) \subset \overline{A} \). Now, for countably tight compacta, the proof of Theorem 3.2 of [5] actually establishes the following converse of this.

**Theorem 2.2.** CH* implies that if \( X \) is any countably tight compactum and \( A \subset X \) then \( \ell_{\omega_1}(A) \supset \overline{A} \), hence \( \ell_{\omega_1}(A) = \overline{A} \). Consequently, if \( X_\delta \) is non-discrete then \( t(X_\delta) = \omega_1 \).

Although the statement of Theorem 3.2 in [5] is slightly weaker than this, the reader may easily check that actually this is proved there.

This result then leads us to the following natural and intriguing question.

**PROBLEM 1.** Is it consistent to have a countably tight compactum \( X \) for which \( t(X_\delta) > \omega_1 \)?

It turns out by our next two ZFC results that the, somewhat surprising, equality \( \ell_{\omega_1}(A) = \overline{A} \) may occur in other situations as well. It will be useful to introduce the following notation: If \( X \) is a space and \( \kappa \) is a cardinal then we write

\[
D_\kappa(X) = \{ D \in [X]^{<\kappa} : D \text{ is discrete} \}.
\]
A countably tight compact, or just Lindelöf space $X$ contains no uncountable free sequences, i.e. satisfies $F(X) = \omega$. This makes the assumption $F(X) = \omega$ in our following result fitting with the topic of this paper.

**Theorem 2.3.** Let $X$ be a regular space such that $F(X) = \omega$ and for every $D \in D_\omega(X)$ we have $\psi(D) \leq \omega$. Then for every $A \subset X$ we have $\ell_\omega(A) = A^\delta$.

**Proof.** Assume that $p \in X$ and $A \subset X$ are such that $p \in \overline{A}^\delta \setminus A$. By induction on $\alpha < \omega_1$ we shall define closed $G_\delta$ sets $H_\alpha$ containing $p$ and points $x_\alpha \in A \cap H_\alpha$ as follows.

Assume that $\{H_\beta : \beta < \alpha\}$ and $Y_\alpha = \{x_\beta : \beta < \alpha\}$ have been defined, moreover $Y_\alpha$ is a free sequence in $X \setminus \{p\}$, hence $Y_\alpha \in D_\omega(X)$. Then either $p \not\in \overline{Y_\alpha}$ or $\psi(p, \overline{Y_\alpha}) \leq \omega$. But in both cases there is a closed $G_\delta$ set $H$ containing $p$ such that $H \cap (\{p\} \cup \overline{Y_\alpha}) = \{p\}$. We then let $H_\alpha = H \cap \bigcap_{\beta < \alpha} H_\beta \in G_\delta(X)$ and use $p \in \overline{A}^\delta$ to pick the point $x_\alpha \in A \cap H_\alpha$.

Thus we have constructed $\{x_\beta : \beta < \omega_1\} \subset A$. The sequence $\{x_\beta : \beta < \omega_1\}$ is free in $X \setminus \{p\}$ because for every $\alpha < \omega_1$ we have $\overline{Y_\alpha} \cap H_\alpha \subset \{p\}$ and $\overline{Y_\alpha} \cap H_\alpha \subset \{p\}$ and $\{x_\beta : \beta \geq \alpha\} \subset H_\alpha$.

If $U$ is any open set containing $p$ then $\{x_\alpha : x_\alpha \not\in U\}$ is free in $X$, hence it is countable. In other words, $U$ contains a tail of $\{x_\alpha : \alpha < \omega_1\}$, i.e. the $\omega_1$-sequence $\{x_\alpha : \alpha < \omega_1\} \subset A$ indeed converges to $x$. \hfill \Box

Clearly, the condition $\psi(D) \leq w(D) = \omega$ is satisfied for any countable subset $D$ of the $\Sigma$-product $\Sigma(\kappa)$ taken inside the Tychonov cube of weight $\kappa$. Also, compact subspaces of such $\Sigma$-products, i.e. Corson-compacta are Frechet, hence do not contain uncountable free sequences. Thus we immediately obtain the following corollary of Theorem 2.3.

**Corollary 2.4.** For every subset $A$ of a Corson-compact space $X$ we have $\ell_\omega(A) = \overline{A}^\delta$.

To facilitate the formulation of our next result, we introduce the notation $CAP(\kappa)$ to denote the class of all spaces in which every subset of cardinality $\kappa$ has a complete accumulation point.
Theorem 2.5. Assume that $X \in \text{CAP}(\omega_1)$ is a countably tight regular space such that $\psi(S) \leq \omega_1$ for every countable subset $S \subset X$. Then for every $A \subset X$ we have $\ell_{\omega_1}(A) = \overline{A}^\delta$.

Proof. Consider any point $p \in \overline{A}^\delta \setminus A$ and then choose an $\omega_1$-chain $\langle N_\alpha : \alpha < \omega_1 \rangle$ of countable elementary submodels of an appropriate $H(\lambda)$ such that

(i) $\{X, A, p\} \subset N_0$;
(ii) for every $\beta < \omega_1$ we have $\langle N_\alpha : \alpha < \beta \rangle \in N_\beta$. Let $N = \bigcup_{\alpha < \omega_1} N_\alpha$.

Since $X$ has countable tightness, for every $\alpha < \omega_1$ we have $p \in N_\alpha \cap A$, hence $\psi(p, N_\alpha \cap A) \leq \omega_1$. It follows that there is a family $U_\alpha \in N_\alpha + 1$ of open sets with $|U_\alpha| \leq \omega_1$ such that $\{p\} = \bigcap_{\alpha < \omega_1} U_\alpha \cap N_\alpha \cap A$.

Let us now put $H_\alpha = \bigcap_{\alpha < \omega_1} U_\alpha \cap N_\alpha \cap A$. Then $H_\alpha$ is a $G_\delta$ set that we claim is closed in $X$. Indeed, this is because for every $U \in N_{\alpha} \cap \tau(X)$ with $p \in U$ there is, by the regularity of $X$ and by elementarity, some $V \in N_\alpha \cap \tau(X)$ with $p \in V$ such that $V \subset U$.

Since $N \cap \tau(X) = \bigcup_{\alpha < \omega_1} N_\alpha \cap \tau(X)$, it follows that $\{p\} = \bigcap_{\alpha < \omega_1} H_\alpha \cap N \cap A$.

Now $p \in \overline{A}^\delta$ and $p \in H_\alpha \in N$ imply that for every $\alpha < \omega_1$ we can pick a point $x_\alpha \in N \cap A \cap H_\alpha$.

We claim that the sequence $S = \{x_\alpha : \alpha < \omega_1\} \subset A$ converges to $p$. Indeed, if $q \neq p$ then there is a $\beta < \omega_1$ with $q \notin H_\beta \cap N \cap A$. Then
$X \setminus H_\beta \cap N \cap A$ is a neighborhood of $q$ that misses the final segment \( \{ x_\alpha : \beta \leq \alpha < \omega \} \), hence $q$ is not a complete accumulation point of $S$. But $X \in \text{CAP}(\omega_1)$ then implies that $p$ is the unique complete accumulation point of $S$, and hence $S$ indeed converges to $p$.

\[ \square \]

3. A LARGE CARDINAL BOUND FOR THE TIGHTNESS OF THE $G_\delta$-MODIFICATIONS

The first result of this section shows not only that the answer to the original question of Bella and Spadaro in the non-compact (or non-Lindelöf) case is negative, in fact it shows that no reasonable bound exists, at least in ZFC.

**Theorem 3.1.** Assume that $S$ is a non-reflecting stationary set of $\omega$-limits in an uncountable regular cardinal $\kappa$. Then there is a 0-dimensional Frechet topology $\tau$ on $\kappa + 1 = \kappa \cup \{ \kappa \}$ such that for the space $X = (\kappa + 1, \tau)$ we have $t(X_\delta) = \kappa$.

**Proof.** Let us denote by $V$ the family of all subsets $V$ of $\kappa$ having the property that for every $\alpha \in S$ there is $\beta < \alpha$ with $(\beta, \alpha) = \alpha \setminus \beta \subset V$. We define $\tau$ to be the topology on $\kappa + 1$ for which all points in $\kappa$ are isolated and $\{ V \cup \{ \kappa \} : V \in V \}$ forms a neighborhood base for the point $\kappa$. More precisely, $\tau = P(\kappa) \cup \{ V \cup \{ \kappa \} : V \in V \}$. It is simple to verify that $\tau$ is indeed a 0-dimensional $T_2$ topology.

To see that $\tau$ is Frechet, observe that if $A \subset \kappa$ accumulates to the point $\kappa$ then there is an $\alpha \in S$ such that $\sup(A \cap \alpha) = \alpha$. But then there is an increasing $\omega$-sequence $B = \{ \beta_n : n < \omega \} \subset A \cap \alpha$ with $\sup B = \alpha$ and clearly $B$ converges to the point $\kappa$.

Since $S$ is stationary, it is an immediate consequence of Fodor’s theorem that every set $V \in V$ includes a final segment of $\kappa$. By $\text{cf}(\kappa) > \omega$ it follows then that every $G_\delta$ set containing the point $\kappa$ also includes a final segment of $\kappa$, hence the set $\kappa$ accumulates to the point $\kappa$ in the space $X_\delta$. Thus, since $\kappa$ is regular, to prove $t(X_\delta) = \kappa$ it will suffice to show that no proper initial segment of $\kappa$ accumulates to the point $\kappa$ in the space $X_\delta$.

This, of course, is equivalent to showing that for each $\eta < \kappa$ the initial segment $\eta$ is an $F_\sigma$ set in $X$. We do this by transfinite induction on $\eta < \kappa$. Thus assume that $\eta < \kappa$ and we know that for every $\zeta < \eta$ the initial segment $\zeta$ is an $F_\sigma$ set $X$, i.e. $\zeta = \cup_{n<\omega} F_{\zeta,n}$ with each $F_{\zeta,n}$ closed. Of course, if $\text{cf}(\eta) \leq \omega$ then it is trivial that $\eta$ is also an $F_\sigma$ set.
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So, assume that \( cf(\eta) > \omega \) and, using that \( S \) is non-reflecting, fix a closed unbounded subset \( C \) in \( \eta \) with \( 0 \in C \) that is disjoint from \( S \). For every \( \gamma \in C \) let \( \gamma^+ \) denote the least member of \( C \) above \( \gamma \).

Let us now define for each \( n < \omega \) the set \( F_{\eta,n} \) as follows:

\[
F_{\eta,n} = \bigcup \{ F_{\gamma^+,n} \cap [\gamma,\gamma^+) : \gamma \in C \}.
\]

Then it is obvious that every \( F_{\eta,n} \) is closed in \( X \), i.e. does not contain the point \( \kappa \) in its closure, and the union of the \( F_{\eta,n} \)'s equals \( \eta \), hence it is indeed an \( F_\sigma \).

\( \square \)

Consistently, Theorem 3.1 yields a very strong negative answer to the question of Bella and Spadaro in the non-compact case. Indeed, for example, in the constructible universe \( L \), or in fact in any set generic extension of \( L \), there is a proper class of regular cardinals that contain non-reflecting stationary sets of \( \omega \)-limits. On the other hand, we do not know the answer to the following natural question:

**PROBLEM 2.** Is there a ZFC example of a countably tight Hausdorff (or regular, or Tychonov) space \( X \) for which \( t(X_\delta) > 2^\omega \)?

It is known that if there is a non-reflecting stationary set of \( \omega \)-limits in a regular cardinal \( \kappa \), then \( \kappa \) is less than the first strongly compact cardinal \( \lambda \), provided that it exists. On the other hand, modulo some large cardinals, it is also known that there may be ZFC models in which such a \( \lambda \) exists and the cardinals \( \kappa \) admitting a non-reflecting stationary set of \( \omega \)-limits are cofinal in \( \lambda \), see e.g. [1]. Consequently, our second result implies that the first one is in some sense sharp.

In this we shall use the following characterization of strongly compact cardinals, see [6]: The cardinal \( \lambda \) is strongly compact iff for every set \( A \) having cardinality at least \( \lambda \) there is a \( \lambda \)-complete and fine free ultrafilter \( U \) on the set \( [A]^{<\lambda} \). That \( U \) is fine means that for every element \( a \in A \) we have

\[
\{ B \in [A]^{<\lambda} : a \in B \} \in U.
\]

**Theorem 3.2.** Let \( \lambda \) be a strongly compact cardinal. Then for every topological space \( X \) satisfying \( t(p,X) < \lambda \) for every \( p \in X \) we have \( t(X_\delta) \leq \lambda \).

**Proof.** Assume that \( A \subset X \) and \( p \in X \) is a point such that for every \( B \in [A]^{<\lambda} \) we have \( p \notin B' \). We claim that then \( p \notin A' \) as well. This clearly implies \( t(p, X_\delta) \leq \lambda \), hence as \( p \in X \) was arbitrary, \( t(X_\delta) \leq \lambda \).
To prove our claim, let us fix for each set \( B \in \mathcal{A}^{<\lambda} \) a countable collection \( \{ V_{B,n} : n < \omega \} \) of open neighborhoods of \( p \) such that \( \bigcap \{ V_{B,n} : n < \omega \} \cap B = \emptyset \). This allows us to define the function \( f_B : A \to \omega \) with the stipulation

\[
    f_B(x) = \min \{ n : x \notin V_{B,n} \}.
\]

Since \( A \) clearly can be assumed to have cardinality \( \geq \lambda \), we may next fix a \( \lambda \)-complete and fine free ultrafilter on \( U \). Then we may define the function \( f : A \to \omega \) with the stipulation

\[
    f(x) = n \Leftrightarrow \{ B \in \mathcal{A}^{<\lambda} : x \in B \text{ and } f_B(x) = n \} \in U.
\]

This makes sense because \( U \) is \( \lambda \)-complete and for every \( x \in A \) we have

\[
    \{ B \in \mathcal{A}^{<\lambda} : x \in B \} \in U.
\]

Next we show that for every \( n < \omega \) the closure in \( X \) of the set \( A_n = f^{-1}(n) \) misses the point \( p \). This clearly will imply that \( p \notin \mathcal{A} \).

Now let \( \mu = t(p,X) < \lambda \) and hence it will suffice to show that for every \( J \in [A_n]^{\mu} \) we have \( p \notin J \). As \( U \) is fine and \( \lambda \)-complete, we have

\[
    U = \{ B \in \mathcal{A}^{<\lambda} : J \subset B \} \in U.
\]

Moreover, this clearly implies that we also have

\[
    W = \{ B \in U : \forall x \in J \left( f(x) = f_B(x) \right) \} \in U.
\]

But for any \( B \in W \) we then have \( J \cap V_{B,n} = \emptyset \), hence \( p \notin J \). This completes our proof.

\[ \square \]

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