ON THE NON-HYPERCYCLICITY
OF SCALAR TYPE SPECTRAL OPERATORS
AND COLLECTIONS OF THEIR EXPONENTIALS

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Abstract. We give a straightforward proof of the non-hypercyclicity of an arbitrary scalar type spectral operator $A$ (bounded or not) in a complex Banach space as well as of the collection $\{e^{\lambda A}\}_{\lambda \geq 0}$ of its exponentials, which, under a certain condition on the spectrum of $A$, coincides with the $C_0$-semigroup generated by it. The spectrum of $A$ lying on the imaginary axis, it is shown that non-hypercyclic is also the generated by it strongly continuous group $\{e^{\lambda A}\}_{\lambda \in \mathbb{R}}$ of bounded linear operators. As an important particular case, we immediately obtain that of a normal operator $A$ in a complex Hilbert space. From the general results, we infer that, in the complex Hilbert space $L_2(\mathbb{R})$, the anti-self-adjoint differentiation operator $A := \frac{d}{dx}$ with the domain $D(A) := \mathcal{W}_{1}^2(\mathbb{R})$ is not hypercyclic and neither is the left-translation group generated by it.

Everything should be made as simple as possible, but not simpler.

Albert Einstein

1. Introduction

In this paper, we give a straightforward proof of the non-hypercyclicity of an arbitrary scalar type spectral operator $A$ (bounded or not) in a complex Banach space as well as of the collection $\{e^{\lambda A}\}_{\lambda \geq 0}$ of its exponentials (see, e.g., [4, 5, 8]), which, provided the spectrum of $A$ is located in a left half-plane

$$\{\lambda \in \mathbb{C} \mid \text{Re}\lambda \leq \omega\}$$

with some $\omega \in \mathbb{R}$, coincides with the $C_0$-semigroup generated by $A$ [14] (see also [3, 20]). The spectrum of $A$ lying on the imaginary axis $i\mathbb{R}$ ($i$ is the imaginary unit), it is shown that non-hypercyclic is also the generated by it strongly continuous group $\{e^{\lambda A}\}_{\lambda \in \mathbb{R}}$ of bounded linear operators.

As an important particular case, we immediately obtain that of a normal operator $A$ in a complex Hilbert space.

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From the general results, we infer that, in the complex Hilbert space $L_2(\mathbb{R})$, the anti-self-adjoint differentiation operator $A := \frac{d}{dx}$ with the domain

$$W_2^1(\mathbb{R}) := \{ f \in L_2(\mathbb{R}) | f(\cdot) \text{ is absolutely continuous on } \mathbb{R} \text{ with } f' \in L_2(\mathbb{R}) \}$$

is not hypercyclic and neither is the left-translation strongly continuous unitary group generated by it [9, 23].

First, we are going to extend the definitions of hypercyclicity, traditionally given for bounded linear operators (see, e.g., [10, 11]), to unbounded ones. The reason for such a shortcoming appears to stem out of the fact that, for an unbounded linear operator

$$A : X \supseteq D(A) \to X$$

(where $D(\cdot)$ is the domain of an operator) in a normed vector space $(X, \| \cdot \|)$, the subspace

$$(1.1) \quad C^\infty(A) := \bigcap_{n=0}^{\infty} D(A^n)$$

($A^0 := I$, $I$ is the identity operator on $X$) of all those vectors $f$ in $X$, whose orbit

$$\{ A^n f \}_{n \in \mathbb{Z}_+}$$

($\mathbb{Z}_+ := \{0, 1, 2, \ldots \}$ is the set of nonnegative integers) is well defined, can be meager and even degenerate to $\{0\}$.

However, this not being the case for many important unbounded operators, including the scalar type spectral in a complex Banach space $(X, \| \cdot \|)$, for which the subspace of all permissible orbit starters defined by (1.1) is dense in $X$ (see Preliminaries), without further reservations, we naturally extend the known definition of hypercyclicity (see, e.g., [10, 11]) as follows.

**Definition 1.1 (Hypercyclicity).**

Let

$$A : X \supseteq D(A) \to X$$

be a linear operator in a (real or complex) normed vector space $(X, \| \cdot \|)$. A vector $f \in C^\infty(A)$ is called hypercyclic if its orbit

$$\{ A^n f \}_{n \in \mathbb{Z}_+}$$

is dense in $X$.

Operators possessing hypercyclic vectors are called hypercyclic.

More generally, a collection $\{ T(t) \}_{t \in I}$ ($I$ is a nonempty indexing set) of linear operators in $X$ is called hypercyclic if it possesses hypercyclic vectors, i.e., such vectors $f \in \bigcap_{t \in I} D(T(t))$, whose orbit

$$\{ T(t) f \}_{t \in I}$$

is dense in $X$. 
Remarks 1.1.

• It is quite obvious that, in the definition of hypercyclicity for an operator, the underlying space must necessarily be separable. Generally, for a collection of operators, this need not be so.

• According to [10, Observation 2.17], an operator has no nontrivial invariant closed subsets iff every nonzero vector is hypercyclic. Since nontrivial scalar type spectral operators are certain to have nontrivial invariant closed subspaces [4,5,8], our quest to prove their non-hypercyclicity appears to be amply justified.

2. Preliminaries

Here, for the reader’s convenience, we outline certain essential preliminaries.

Henceforth, unless specified otherwise, \( A \) is supposed to be a scalar type spectral operator in a complex Banach space \( (X, \| \cdot \|) \) with strongly \( \sigma \)-additive spectral measure (the resolution of the identity) \( E_A(\cdot) \) assigning to each Borel set \( \delta \) of the complex plane \( \mathbb{C} \) a projection operator \( E_A(\delta) \) on \( X \) and having the operator’s spectrum \( \sigma(A) \) as its support [4,5,8].

Observe that, in a complex finite-dimensional space, the scalar type spectral operators are all linear operators on the space, for which there is an eigenbasis (see, e.g., [4,5,8]) and, in a complex Hilbert space, the scalar type spectral operators are precisely all those that are similar to the normal ones [24].

Associated with a scalar type spectral operator in a complex Banach space is the Borel operational calculus analogous to that for a normal operator in a complex Hilbert space [7,21], which assigns to any Borel measurable function \( F : \sigma(A) \to \mathbb{C} \) a scalar type spectral operator

\[
F(A) := \int_{\sigma(A)} F(\lambda) \, dE_A(\lambda)
\]

(see [5,8]).

In particular,

\[
A^n = \int_{\sigma(A)} \lambda^n \, dE_A(\lambda), \quad n \in \mathbb{Z}_+,
\]

and

\[
e^{tA} := \int_{\sigma(A)} e^{t\lambda} \, dE_A(\lambda), \quad t \in \mathbb{R}.
\]

Provided

\[
\sigma(A) \subseteq \{ \lambda \in \mathbb{C} | \text{Re } \lambda \leq \omega \},
\]

with some \( \omega \in \mathbb{R} \), the collection of exponentials \( \{ e^{tA} \}_{t \geq 0} \) coincides with the \( C_0 \)-semigroup generated by \( A \) [14, Proposition 3.1] (cf. also [3,20]), and hence, if

\[
\sigma(A) \subseteq \{ \lambda \in \mathbb{C} | -\omega \leq \text{Re } \lambda \leq \omega \},
\]
with some $\omega \geq 0$, the collection of exponentials $\{e^{tA}\}_{t \in \mathbb{R}}$ coincides with the strongly continuous group of bounded linear operators generated by $A$.

**Remarks 2.1.**

- By [13, Theorem 4.2], the orbits
  \begin{equation}
  y(t) = e^{tA}f, \quad t \geq 0, \quad f \in \bigcap_{t \geq 0} D(e^{tA}),
  \end{equation}
  describe all weak/mild solutions of the abstract evolution equation
  \begin{equation}
  y'(t) = Ay(t), \quad t \geq 0,
  \end{equation}
  whereas, by [19, Theorem 7], the orbits
  \begin{equation}
  y(t) = e^{tA}f, \quad t \in \mathbb{R}, \quad f \in \bigcap_{t \in \mathbb{R}} D(e^{tA}),
  \end{equation}
  describe all weak/mild solutions of the abstract evolution equation
  \begin{equation}
  y'(t) = Ay(t), \quad t \in \mathbb{R},
  \end{equation}
  which need not be differentiable in the strong sense and encompass the classical ones, strongly differentiable and satisfying the equations in the traditional plug-in sense, [2] (cf. [9, Ch. II, Definition 6.3], see also [18, Preliminaries]).

- The operator $A$ generating a $C_0$-semigroup or a strongly continuous group of bounded linear operators (see, e.g., [9,12]), the associated abstract Cauchy problem (ACP)
  \begin{equation}
  \begin{cases}
  y'(t) = Ay(t), & t \in I, \\
  y(0) = f
  \end{cases}
  \end{equation}
  with $I := [0, \infty)$ or $I := \mathbb{R}$, respectively, is well-posed (cf. [9, Ch. II, Definition 6.8]).

- Observe that all three subspaces
  \[
  C^\infty(A), \bigcap_{t \geq 0} D(e^{tA}), \text{ and } \bigcap_{t \in \mathbb{R}} D(e^{tA})
  \]
  are dense in $X$ since they contain the dense in $X$ subspace \[ \bigcup_{\alpha > 0} E_A(\Delta_\alpha)X, \]
  where
  \[
  \Delta_\alpha := \{ \lambda \in \mathbb{C} \mid |\lambda| \leq \alpha \}, \quad \alpha > 0,
  \]
  which coincides with the class $\mathcal{E}^{(0)}(A)$ of entire vectors of $A$ of exponential type [17,22].

The properties of the spectral measure and operational calculus, exhaustively delineated in [5,8], underlie the subsequent discourse. Here, we touch upon a few facts of particular importance.

Due to its strong countable additivity, the spectral measure $E_A(\cdot)$ is bounded [6,8], i.e., there is such an $M \geq 1$ that, for any Borel set $\delta \subseteq \mathbb{C}$,

\begin{equation}
\|E_A(\delta)\| \leq M.
\end{equation}
Observe that the notation \( \| \cdot \| \) is used here to designate the norm in the space \( L(X) \) of all bounded linear operators on \( X \). We adhere to this rather conventional economy of symbols in what follows also adopting the same notation for the norm in the dual space \( X^* \).

For any \( f \in X \) and \( g^* \in X^* \), the total variation measure \( v(f, g^*, \cdot) \) of the complex-valued Borel measure \( \langle E_A(\cdot) f, g^* \rangle \) (\( \langle \cdot, \cdot \rangle \) is the pairing between the space \( X \) and its dual \( X^* \)) is a finite positive Borel measure with

\[
(2.10) \quad v(f, g^*, C) = v(f, g^*, \sigma(A)) \leq 4M\|f\|\|g^*\|
\]

(see, e.g., [15, 16]).

Also (Ibid.), for a Borel measurable function \( F: \mathbb{C} \to \mathbb{C} \), \( f \in \mathcal{D}(F(A)) \), \( g^* \in X^* \), and a Borel set \( \delta \subseteq \mathbb{C} \),

\[
(2.11) \quad \int_{\delta} |F(\lambda)| \, dv(f, g^*, \lambda) \leq 4M\|E_A(\delta) F(A) f\|\|g^*\|.
\]

In particular, for \( \delta = \sigma(A) \),

\[
(2.12) \quad \int_{\sigma(A)} |F(\lambda)| \, dv(f, g^*, \lambda) \leq 4M\|F(A) f\|\|g^*\|.
\]

Observe that the constant \( M \geq 1 \) in \((2.10)-(2.12)\) is from \((2.9)\).

### 3. Main Results

**Theorem 3.1.** An arbitrary scalar type spectral operator \( A \) in a complex Banach space \((X, \| \cdot \|)\) with spectral measure \( E_A(\cdot) \) is not hypercyclic and neither is the collection \( \{e^{tA}\}_{t \geq 0} \) of its exponentials, which, provided the spectrum of \( A \) is located in a left half-plane

\[
\{ \lambda \in \mathbb{C} \mid \text{Re } \lambda \leq \omega \}
\]

with some \( \omega \in \mathbb{R} \), coincides with the \( C_0 \)–semigroup generated by \( A \).

**Proof.** Let \( f \in C^\infty(A) \setminus \{0\} \) be arbitrary.

There are two possibilities: either

\[
E_A(\{ \lambda \in \sigma(A) \mid |\lambda| > 1 \}) f \neq 0
\]

or

\[
E_A(\{ \lambda \in \sigma(A) \mid |\lambda| > 1 \}) f = 0.
\]

In the first case, as follows from the Hahn-Banach Theorem (see, e.g., [6]), there is a \( g^* \in X^* \setminus \{0\} \) such that

\[
\langle E_A(\{ \lambda \in \sigma(A) \mid |\lambda| > 1 \}) f, g^* \rangle \neq 0
\]

and hence, for any \( n \in \mathbb{Z}_+ \),

\[
\|A^n f\| \geq [4M\|g^*\|]^{-1} \int_{\sigma(A)} |\lambda|^n \, dv(f, g^*, \lambda) \geq [4M\|g^*\|]^{-1} \int_{\{\lambda \in \sigma(A) \mid |\lambda| > 1\}} |\lambda|^n \, dv(f, g^*, \lambda)
\]

by \((2.12)\).
\[
\geq [4M\|g^*\|]^{-1}v(f, g^*, \{\lambda \in \sigma(A)\|\lambda\| > 1\})
\]
\[
\geq [4M\|g^*\|]^{-1}|E_A(\{\lambda \in \sigma(A)\|\lambda\| > 1\}) f, g^*)| > 0,
\]
which immediately implies that the orbit \(\{A^n f\}_{n \in \mathbb{Z}_+}\) is not dense in \(X\).

In the second case,
\[
f = E_A(\{\lambda \in \sigma(A)\|\lambda\| \leq 1\}) f \neq 0
\]
and, for any \(n \in \mathbb{Z}_+\),
\[
\|A^n f\| = \left\| \int_{\{\lambda \in \sigma(A)\|\lambda\| \leq 1\}} \lambda^n dE_A(\lambda)f \right\| \quad \text{by the properties of the operational calculus;}
\]
as follows from the Hahn-Banach Theorem;
\[
= \sup_{\{g^* \in X^*\|\|g^*\| = 1\}} \left\| \left\langle \int_{\{\lambda \in \sigma(A)\|\lambda\| \leq 1\}} \lambda^n dE_A(\lambda)f, g^* \right\rangle \right\|
\]
by the properties of the operational calculus;
\[
\leq \sup_{\{g^* \in X^*\|\|g^*\| = 1\}} \int_{\{\lambda \in \sigma(A)\|\lambda\| \leq 1\}} |\lambda|^n dv(f, g^*, \lambda)
\]
\[
\leq \sup_{\{g^* \in X^*\|\|g^*\| = 1\}} \int_{\{\lambda \in \sigma(A)\|\lambda\| \leq 1\}} 4M \|E_A(\{\lambda \in \sigma(A)\|\lambda\| \leq 1\}) f\| \|g^*\|
\]
\[
\leq 4M \|E_A(\{\lambda \in \sigma(A)\|\lambda\| \leq 1\}) f\|,
\]
which also implies that the orbit \(\{A^n f\}_{n \in \mathbb{Z}_+}\), being bounded, is not dense in \(X\) and completes the proof for the operator case.

Now, let us consider the case of the exponential collection \(\{e^{tA}\}_{t \geq 0}\) assuming that \(f \in \bigcap_{t \geq 0} D(e^{tA}) \setminus \{0\}\) is arbitrary.

There are two possibilities: either
\[
E_A \left(\{\lambda \in \sigma(A)\|\Re \lambda > 0\}\right) f \neq 0
\]
or
\[
E_A \left(\{\lambda \in \sigma(A)\|\Re \lambda > 0\}\right) f = 0.
\]
In the first case, as follows from the Hahn-Banach Theorem, there is a \(g^* \in X^* \setminus \{0\}\) such that
\[
\langle E_A \left(\{\lambda \in \sigma(A)\|\Re \lambda > 0\}\right) f, g^* \rangle \neq 0
\]
and hence, for any $t \geq 0$,

$$
\|e^{tA}f\| \quad \text{by (2.12)};
$$

$$
\geq \left[4M\|g^*\|\right]^{-1} \int_{\sigma(A)} |e^{t\lambda}| \, dv(f, g^*, \lambda)
$$

$$
\geq \left[4M\|g^*\|\right]^{-1} \int_{\{\lambda \in \sigma(A) \mid \Re \lambda > 0\}} e^{t\Re \lambda} \, dv(f, g^*, \lambda)
$$

$$
\geq \left[4M\|g^*\|\right]^{-1} v(f, g^*, \{\lambda \in \sigma(A) \mid \Re \lambda > 0\})
$$

$$
\geq \left[4M\|g^*\|\right]^{-1} |\langle E_A(\{\lambda \in \sigma(A) \mid \Re \lambda > 0\}) f, g^* \rangle| > 0,
$$

which immediately implies that the orbit $\{e^{tA}f\}_{t \geq 0}$ is not dense in $X$.

In the second case,

$$
f = E_A(\{\lambda \in \sigma(A) \mid \Re \lambda \leq 0\}) f \neq 0
$$

and, for any $t \geq 0$,

$$
\|e^{tA}f\| \quad \text{by the properties of the operational calculus};
$$

$$
= \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \left| \int_{\{\lambda \in \sigma(A) \mid \Re \lambda \leq 0\}} e^{t\lambda} \, dE_A(\lambda)f, g^* \right|
$$

as follows from the Hahn-Banach Theorem;

$$
= \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \left| \int_{\{\lambda \in \sigma(A) \mid \Re \lambda \leq 0\}} e^{t\lambda} \, d\langle E_A(\lambda)f, g^* \rangle \right|
$$

by the properties of the operational calculus;

$$
\leq \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \left| \int_{\{\lambda \in \sigma(A) \mid \Re \lambda \leq 0\}} |e^{t\lambda}| \, dv(f, g^*, \lambda) \right|
$$

$$
= \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \left| \int_{\{\lambda \in \sigma(A) \mid \Re \lambda \leq 0\}} e^{t\Re \lambda} \, dv(f, g^*, \lambda) \right|
$$

$$
\leq \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \int_{\{\lambda \in \sigma(A) \mid \Re \lambda \leq 0\}} 1 \, dv(f, g^*, \lambda)
$$

by (2.11) with $F(\lambda) \equiv 1$;

$$
\leq \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} 4M \|E_A(\{\lambda \in \sigma(A) \mid \Re \lambda \leq 0\}) f\| \|g^*\|
$$

$$
\leq 4M \|E_A(\{\lambda \in \sigma(A) \mid \Re \lambda \leq 0\}) f\|,
$$
which also implies that the orbit \( \{ e^{tA}f \}_{t \geq 0} \) being bounded, is not dense on \( X \) and completes the entire proof. \(\square\)

If further \( \sigma(A) \subseteq i\mathbb{R} \), by [8, Theorem XVIII.2.11 (c)], for any \( t \in \mathbb{R} \)

\[
\| e^{tA} \| = \left\| \int_{\sigma(A)} e^{\lambda t} dE_A(\lambda) f \right\| \leq 4M \sup_{\lambda \in \sigma(A)} |e^{\lambda t}| = 4M \sup_{\lambda \in \sigma(A)} e^{t \text{Re} \lambda} = 4M,
\]

where the constant \( M \geq 1 \) is from (2.9). Therefore, the generated by \( A \) strongly continuous group \( \{ e^{tA} \}_{t \in \mathbb{R}} \) is bounded (cf. [3]), which implies that every orbit \( \{ e^{tA}f \}_{t \in \mathbb{R}} \), \( f \in X \), is bounded, and hence, cannot be dense in \( X \), and we arrive at the following

**Proposition 3.1.** For a scalar type spectral operator \( A \) in a complex Banach space \( (X, \| \cdot \|) \) with \( \sigma(A) \subseteq i\mathbb{R} \), the generated by it strongly continuous group \( \{ e^{tA} \}_{t \in \mathbb{R}} \) is bounded, and hence, non-hypercyclic.

4. **The Case of a Normal Operator**

As an important particular case of Theorem 3.1, we obtain

**Corollary 4.1** (The Case of a Normal Operator).

An arbitrary normal operator \( A \) in a complex Hilbert space is not hypercyclic and neither is the collection \( \{ e^{tA} \}_{t \geq 0} \) of its exponentials, which, provided the spectrum of \( A \) is located in a left half-plane

\[ \{ \lambda \in \mathbb{C} \mid \text{Re} \lambda \leq \omega \} \]

with some \( \omega \in \mathbb{R} \), coincides with the \( C_0 \)-semigroup generated by \( A \).

As is known [23], for an anti-self-adjoint operator \( A \) in a complex Hilbert space, \( \sigma(A) \subseteq i\mathbb{R} \) and the generated by it strongly continuous group \( \{ e^{tA} \}_{t \in \mathbb{R}} \) is unitary, which, in particular, implies that

\[ \| e^{tA} \| = 1, \ t \in \mathbb{R}. \]

Hence, in this case, Proposition 3.1 acquires the following form.

**Corollary 4.2** (The Case of an Anti-Self-Adjoint Operator).

For an anti-self-adjoint operator \( A \) in a complex Hilbert space, the generated by it strongly continuous group \( \{ e^{tA} \}_{t \in \mathbb{R}} \) is unitary, and hence, non-hypercyclic.

5. **An Application**

Since, in the complex Hilbert space \( L_2(\mathbb{R}) \), the differentiation operator \( A := \frac{d}{dx} \) with the domain

\[ W^1_2(\mathbb{R}) := \{ f \in L_2(\mathbb{R}) \mid f(\cdot) \text{ is absolutely continuous on } \mathbb{R} \text{ with } f'(\cdot) \in L_2(\mathbb{R}) \} \]

is anti-self-adjoint (see, e.g., [1]), and hence, the generated by it left-translation group is strongly continuous and unitary [9, 23], by the Corollaries 4.1 and 4.2, we obtain
Corollary 5.1 (The Case of Differentiation Operator).

In the complex Hilbert space $L_2(\mathbb{R})$, the differentiation operator $A := \frac{d}{dx}$ with the domain

$$D(A) := W^1_2(\mathbb{R}) := \{ f \in L_2(\mathbb{R}) | f(\cdot) \text{ is absolutely continuous on } \mathbb{R} \text{ with } f' \in L_2(\mathbb{R}) \}$$

is not hypercyclic and neither is the left-translation group generated by it.

Remark 5.1. In a different setting, the situation with the differentiation operator can be vastly different (Cf. MacLanes operator [10, Example 2.21]).

6. Concluding Remark

The exponentials given by (2.4) describing all weak/mild solutions of equation (2.5) (see Remarks 2.1, Theorem 3.1, in particular, implies that the latter is void of chaos (cf. [10]). The same, by Proposition 3.1, is true also for equation (2.7) provided $\sigma(A) \subseteq i\mathbb{R}$.

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References

[1] N.I. Akhiezer and I.M. Glazman, Theory of Linear Operators in Hilbert Space, Dover Publications, Inc., New York, 1993.
[2] J.M. Ball, Strongly continuous semigroups, weak solutions, and the variation of constants formula, Proc. Amer. Math. Soc. 63 (1977), no. 2, 101–107.
[3] E. Berkson, Semi-groups of scalar type operators and a theorem of Stone, Illinois J. Math. 10 (1966), no. 2, 345-352.
[4] N. Dunford, Spectral operators, Pacific J. Math. 4 (1954), 321–354.
[5] , A survey of the theory of spectral operators, Bull. Amer. Math. Soc. 64 (1958), 217–274.
[6] N. Dunford and J.T. Schwartz with the assistance of W.G. Bade and R.G. Bartle, Linear Operators. Part I: General Theory, Interscience Publishers, New York, 1958.
[7] , Linear Operators. Part II: Spectral Theory. Self Adjoint Operators in Hilbert Space, Interscience Publishers, New York, 1963.
[8] , Linear Operators. Part III: Spectral Operators, Interscience Publishers, New York, 1971.
[9] K.-J. Engel and R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000.
[10] K.-G. Grosse-Erdmann and A.P. Manguillon, Linear Chaos, Universitext, Springer-Verlag, London, 2011.
[11] A.J. Guirao, V. Montesinos, and V. Zizler, Open Problems in the Geometry and Analysis of Banach Spaces, Springer International Publishing, Switzerland, 2016.
[12] E. Hille and R.S. Phillips, Functional Analysis and Semi-groups, American Mathematical Society Colloquium Publications, vol. 31, Amer. Math. Soc., Providence, RI, 1957.
[13] M.V. Markin, On an abstract evolution equation with a spectral operator of scalar type, Int. J. Math. Math. Sci. 32 (2002), no. 9, 555–563.
[14] , A note on the spectral operators of scalar type and semigroups of bounded linear operators, Ibid. 32 (2002), no. 10, 635–640.
[15] On scalar type spectral operators, infinite differentiable and Gevrey ultradifferentiable $C_0$-semigroups, Ibid. 2004 (2004), no. 45, 2401–2422.

[16] On the Carleman classes of vectors of a scalar type spectral operator, Ibid. 2004 (2004), no. 60, 3219–3235.

[17] On the Carleman ultradifferentiable vectors of a scalar type spectral operator, Methods Funct. Anal. Topology 21 (2015), no. 4, 361–369.

[18] On the mean ergodicity of weak solutions of an abstract evolution equation, Ibid. 24 (2018), no. 1, 53–70.

[19] On the differentiability of weak solutions of an abstract evolution equation with a scalar type spectral operator on the real axis, Int. J. Math. Math. Sci. 2018 (2018), Article ID 4168609, 14 pp.

[20] T.V. Panchapagesan, Semi-groups of scalar type operators in Banach spaces, Pacific J. Math. 30 (1969), no. 2, 489–517.

[21] A.I. Plesner, Spectral Theory of Linear Operators, Nauka, Moscow, 1965 (Russian).

[22] Ya.V. Radyno, The space of vectors of exponential type, Dokl. Akad. Nauk BSSR 27 (1983), no. 9, 791–793 (Russian with English summary).

[23] M.H. Stone, On one-parameter unitary groups in Hilbert space, Ann. of Math. (2) 33 (1932), no. 3, 643–648.

[24] J. Wermer, Commuting spectral measures on Hilbert space, Pacific J. Math. 4 (1954), 355–361.