Pettis integrability in $L^{1}_{E'}[E]$ related to the truncation

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Abstract. We study the Pettis integrability in terms of truncation. We focus our study particularly on space $L^{1}_{E'}[E]$. 

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1. Introduction

Several authors studied the Pettis integrability of Banach space valued functions (see for example [1],[10],[11],[13],[14],[12],[18] and references therein) and especially of dual Banach space valued functions ([2],[17],[19]). Similarly, the study of Pettis integrability for multifunctions has been the focus of various papers (for example [9],[15] and [22]). In this note, we are interested in Pettis integrability for scalarly integrable functions of $L^{1}_{E'},[E]$.

Our study is based on the truncation technique that has been adopted in ([5],[6]) to state some Komlós type theorems for Bochner integrable functions and in [16] to provide a Komlós type theorem in $L^{1}_{E'}[E]$. It is well known that a strongly measurable and scalarly integrable function $f : \Omega \to E$ is Pettis integrable if and only if the set $\{(x', f) : \|x'\| \leq 1\}$ is uniformly integrable in $L^{1}_{\mathbb{R}}(\mu)$ ([14] Theorem 5.2). We give a characterization of Pettis integrability for scalarly integrable function (non-necessary strongly measurable) with norm measurable function (Proposition 1) and, when $E$ is a separable Banach space, we establish that a function $f \in L^{1}_{E'}[E]$ is Pettis integrable if and only if its truncated function $1_{\{\|f\| \leq n\}}f$ is Pettis integrable for all $n \geq 1$ (Corollary 1). We also give some criteria that guarantee the Pettis integrability of the limit of a Pettis integrable $L^{1}_{E'}[E]$-convergent sequence. More precisely, we show that if a sequence of Pettis integrable functions bounded in $L^{1}_{E'}[E]$ converges weakly a.e. in $E'$ (resp. converges pointwise in $L^{\infty}_{\mathbb{R}}(\mu) \otimes E''$) to a scalarly integrable function $f$, then $f$ is Pettis integrable Theorem 2 (resp. Theorem 4). It is important to note that a bounded scalarly integrable function is not in general Pettis integrable, one can find some examples in [2],[19]. We note that the results in [16] will play an important role for the development of this work and a version of Theorem 4 in [16] with Pettis integrable functions is given (Theorem 6).

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2. Notations and Preliminaries

Let \((\Omega, \mathcal{F}, \mu)\) be a complete probability space, \(E\) a Banach space and \(E'\) its topological dual. The weak topology \(\sigma(E, E')\) on \(E\) (resp. the weak* topology \(\sigma(E', E)\) on \(E'\)) will be referred to by the symbol "w" (resp. "w*"). A function \(f : \Omega \to E\) (resp \(f : \Omega \to E'\)) is w-measurable (resp w*-measurable), if for any \(x' \in E'\), (resp \(x \in E\)) the function \(\langle f, x' \rangle : \Omega \to \mathbb{R}\) (resp \(\langle f, x \rangle : \Omega \to \mathbb{R}\)) is measurable. Two functions \(f, g : \Omega \to E\) (resp \(f, g : \Omega \to E'\)) are w-equivalent (resp w*-equivalent), if \(\langle f, x' \rangle = \langle g, x' \rangle\) \(\mu\)-a.e. for every \(x' \in E'\), (resp \(\langle f, x \rangle = \langle g, x \rangle\) \(\mu\)-a.e. for every \(x \in E\)). A function \(f : \Omega \to E\) (resp \(f : \Omega \to E'\)) is scalarly integrable (resp w*-scalarly integrable) if for every \(x' \in E'\) the function \(\langle f, x' \rangle\) (resp for every \(x \in E\) the function \(\langle f, x \rangle\)) is \(\mu\)-integrable. If \(f : \Omega \to E\) is scalarly integrable, then ([7] Lemma 1. p. 52) for every \(A \in \mathcal{F}\) there exists \(x'_f(A)\) in \(E''\) such that, for every \(x' \in E'\)

\[ \langle x'_f(A), x' \rangle = \int_A \langle f, x' \rangle \, d\mu, \]

the element \(x'_f(A)\) is called the Dunford integral of \(f\) over \(A\) and denoted by \((D) - \int_A f \, d\mu\). By definition, \(f\) is Pettis integrable if \((D) - \int_A f \, d\mu \in E\) for all \(A \in \mathcal{F}\) and we write \((P) - \int_A f \, d\mu\) instead of \((D) - \int_A f \, d\mu\). Also, ([7] p. 53) if \(f : \Omega \to E'\) is w*-scalarly integrable then for every \(A \in \mathcal{F}\) there exists \(x'_f(A)\) in \(E''\) such that, for every \(x \in E\)

\[ \langle x'_f(A), x \rangle = \int_A \langle f, x \rangle \, d\mu, \]

the element \(x'_f(A)\) is called the weak* integral (or Gelfand integral) of \(f\) over \(A\) and denoted by \((w^*) - \int_A f \, d\mu\). A sequence \((f_n)\) of \(E\)-valued scalarly integrable functions converges pointwise on \(L^\infty_{R} (\mu) \otimes E'\) to an \(E\)-valued scalarly integrable function \(f\) if

\[ \forall h \in L^\infty_{R} (\mu), \forall x' \in E', \int_{\Omega} h(\langle f_n, x' \rangle) \, d\mu \to \int_{\Omega} h(\langle f, x' \rangle) \, d\mu, \]

or equivalently ([8] Theorem 7. p. 291) for every \(x' \in E'\), the sequence \((\langle f_n, x' \rangle)\) is bounded in \(L^1_{R}(\mu)\) and

\[ \forall A \in \mathcal{F}, \int_A \langle f_n, x' \rangle \, d\mu \to \int_A \langle f, x' \rangle \, d\mu. \]

Let \(P^1_E(\mu)\) denote the (quotient) space of Pettis integrable \(E\)-valued functions. The weak topology on \(P^1_E(\mu)\) is the weak topology induced by the duality \((P^1_E(\mu), L^\infty_{R} (\mu) \otimes E')\). If \(E\) is separable and \(f : \Omega \to E'\) is w*-measurable, the function \(\|f(.)\|\) is measurable [20] however, this is not always the case if \(E\) is a general Banach space ([14] Example 3.3). With \(E\) being separable, the Banach space \((L^1(E), \overline{N}_1)\) ([3],[21],[16]) is simply the (quotient) space of w*-scalarly integrable functions \(f : \Omega \to E'\) such that \(\|f(.)\|\) is \(\mu\)-integrable, and

\[ \overline{N}_1(f) = \int_{\Omega} \|f(\omega)\| \, d\mu(\omega), \quad f \in L^1_{E'}(E). \]
Finally, we recall that a set $H$ of $L^1_{\mathbb{R}}(\mu)$ is uniformly integrable (briefly UI) if it is bounded and
\[
\lim_{\mu(A)\to 0} \sup_{f \in H} \int_A |f| \, d\mu = 0.
\]
A set $K$ of $L^1_{\mathbb{E}'}[E]$ is UI if the set $\{\|f(.)\| : f \in K\}$ is UI in $L^1_{\mathbb{R}}(\mu)$, and we say that a set $H$ of $E$-valued scalarly integrable functions is scalarly uniformly integrable briefly SUI (resp w-scalarly uniformly integrable briefly WSUI), if the set $\{\langle x', f \rangle : \|x'\| \leq 1, f \in H\}$ (resp for each $x' \in E'$, the set $\{\langle x', f \rangle : f \in H\}$) is UI in $L^1_{\mathbb{R}}(\mu)$.

3. Pettis integrability and truncation

By ([10] p.82), if $f : \Omega \to E$ is Pettis integrable then $\{f\}$ is SUI and the converse remains true if $f$ is strongly measurable ([14] Theorem 5.2). For the instance of $L^1_{\mathbb{E}'}[E]$, we give some characterizations of the Pettis integrability by the mean of the associated truncated functions. Our work build on the following ([4], Theorem 3.1):

**Theorem 1.** Let $E$ be a Banach space, $(f_n)$ a sequence of $E$-valued Pettis integrable functions and $f : \Omega \to E$ a scalarly integrable function satisfying:

(i) $\{f\}$ is SUI,

(ii) $(f_n)$ converges pointwise on $L^\infty_{\mathbb{R}}(\mu) \otimes E'$ to $f$.

Then $f$ is Pettis integrable.

The next lemma is useful.

**Lemma 1.** If $f : \Omega \to E$ is scalarly integrable and $\|f(.)\|$ is measurable, then the sequence $(1_{\{\|f\| \leq n\}} f)_n$ converges pointwise on $L^\infty_{\mathbb{R}}(\mu) \otimes E'$ to $f$.

**Proof.** Let $h \in L^\infty_{\mathbb{R}}(\mu)$ and $x' \in E'$. We have
\[
h(\omega)\langle 1_{\{\|f\| \leq n\}} f(\omega), x' \rangle \to h(\omega)\langle f(\omega), x' \rangle \quad \forall \omega \in \Omega,
\]
and
\[
|h(\omega)\langle 1_{\{\|f\| \leq n\}} f(\omega), x' \rangle| \leq \|h\|_{\infty}\|\langle f(\omega), x' \rangle\| \quad a.e.,
\]
then by the Lebesgue dominated convergence theorem
\[
\int_\Omega |(h(\omega)1_{\{\|f\| \leq n\}} f(\omega) - f(\omega), x')| \, d\mu(\omega) \to 0,
\]
and therefore
\[
\int_\Omega h(\omega)\langle 1_{\{\|f\| \leq n\}} f(\omega), x' \rangle \, d\mu(\omega) \to \int_\Omega h(\omega)\langle f(\omega), x' \rangle \, d\mu(\omega).
\]
Proposition 1. If $f : \Omega \to E$ is scalarly integrable and $\|f(\cdot)\|$ is measurable, then $f$ is Pettis integrable if and only if

(i) $\{f\}$ is SUI, and

(ii) $1_{\{\|f\| \leq n\}}f$ is Pettis integrable for all $n \geq 1$.

Proof. If $f$ is Pettis integrable then $\{f\}$ is SUI and $1_{\{\|f\| \leq n\}}f$ is Pettis integrable for all $n \geq 1$. The converse follows from Theorem 1 and Lemma 1.

The above result gives a characterization of Pettis integrability for scalarly integrable function with measurable norm function (compare with Theorem 5.2 in [14]) and it can be seen as a generalization for the case of strongly measurable functions since, if $f : \Omega \to E$ is strongly measurable then $\|f(\cdot)\|$ is measurable and hence $1_{\{\|f\| \leq n\}}f$ is Bochner then Pettis integrable. We obtain the following characterization of Pettis integrability in $L^1_{E'}[E]$.

Corollary 1. Let $E$ be a separable Banach space and $f \in L^1_{E'}[E]$. Then $f$ is Pettis integrable iff $1_{\{\|f\| \leq n\}}f$ is Pettis integrable for all $n \geq 1$.

Proof. As $E$ is separable then $\|f(\cdot)\|$ is measurable. The direct implication is immediate we show the converse. For every $x'' \in E''$ the function $(f(\cdot), x'')$ is measurable a simple limit of $(1_{\{\|f\| \leq n\}}f(\cdot), x'')_n$. For all $\omega \in \Omega$ and $x'' \in B_{E''}$, we have $|\langle f(\omega), x'' \rangle| \leq \|f(\omega)\|$. As $\|f(\cdot)\| \in L^1_{E'}(\mu)$ then $\{f\}$ is SUI. Therefore we apply Proposition 1.

From now, we suppose that $E$ is separable. If $(f_n)$ is a convergent sequence of Pettis integrable functions of $L^1_{E'}[E]$, when does $(f_n)$ have a Pettis integrable limit? Here the convergence is taken in the sense of weak convergence a.e. or the pointwise convergence on $L^\infty_{E'}(\mu) \otimes E''$. The following result is an analogue of Vitali’s convergence theorem for Pettis integrable functions.

Lemma 2. Let $f \in L^1_{E'}[E]$ be a scalarly integrable function. Suppose that there exists a sequence of Pettis integrable functions $(f_n)$ such that

(i) $(f_n)$ is WSUI, and

(ii) for each $x'' \in E''$, $\lim_{n \to \infty} \langle f_n, x'' \rangle = \langle f, x'' \rangle$ a.e.

Then $f$ is Pettis integrable and $(f_n)$ converges weakly to $f$ in $P^1_{E'}(\mu)$.

Proof. As $\|f(\cdot)\|$ is integrable then $\{f\}$ is SUI. We apply Theorem 1 and Vitali’s theorem in $L^1_{E'}(\mu)$.

Theorem 2. Let $(f_n)_{n \in \mathbb{N}}$ a bounded sequence in $L^1_{E'}[E]$. If $(f_n)$ $w^*$-converges a.e. to a function $f : \Omega \to E'$ then $f \in L^1_{E'}[E]$. If $f_n$ is Pettis integrable for all $n$ and $(f_n)$ $w$-converges a.e. to $f$, then $f$ is Pettis integrable.
Proof. As \((f_n(\omega))_n\) w*-converges a.e. to \(f(\omega)\) we have
\[
\|f(\omega)\| \leq \liminf_n \|f_n(\omega)\| \quad \text{a.e.}
\]
By Fatou’s lemma and the boundedness of \((f_n)\) in \(L^1_{\mathbb{E}'} [E]\) we get
\[
\int_{\Omega} \|f\| d\mu \leq \liminf_n \int_{\Omega} \|f_n\| d\mu < \infty,
\]
thus \(f \in L^1_{\mathbb{E}'} [E]\).

Now suppose that \(f_n\) is Pettis integrable for all \(n\) and \((f_n)\) w-converges a.e. to \(f\). Then \(f\) is w-measurable with \(\|f(.)\|\) is integrable, so that \(f\) is scalarly integrable. By Lemma 2 in [16] there exists a subsequence \((g_n)\) of \((f_n)\) such that \((1_{\{\|g_n\| < n\}} g_n)\) is UI and \((g_n - 1_{\{\|g_n\| < n\}} g_n)\) converges a.e. to 0 in \(E'\), hence \((1_{\{\|g_n\| < n\}} g_n)\) is WSUI and weakly converges a.e. to \(f\). It remains to use Lemma 2 to conclude that \(f\) is Pettis integrable.

Now we give a criterion of the \(\sigma(L^1_{E'} [E], L^\infty_{\mathbb{R}} (\mu) \otimes E)\)-compactness for \(L^1_{E'} [E]\)-bounded subsets.

**Theorem 3.** Let \(H\) a bounded subset of \(L^1_{E'} [E]\). Then \(H\) is \(\sigma(L^1_{E'} [E], L^\infty_{\mathbb{R}} (\mu) \otimes E)\)-sequentially relatively compact if and only if for each \(x \in E\) the set \(H_x = \{\langle f, x \rangle : f \in H\}\) is UI in \(L^1_{\mathbb{R}}(\mu)\).

**Proof.** If \(H\) is \(\sigma(L^1_{E'} [E], L^\infty_{\mathbb{R}} (\mu) \otimes E)\)-sequentially relatively compact then for each \(x \in E\), \(H_x\) is \(\sigma(L^1_{E'} [E], L^\infty_{\mathbb{R}} (\mu))\)-sequentially relatively compact and equivalently is UI in \(L^1_{\mathbb{R}}(\mu)\). Conversely, let \((f_n)\) a sequence of \(H\). By Theorem 2 in [16] there exists a subsequence \((f'_n)\) of \((f_n)\) and a function \(f \in L^1_{E'} [E]\) such that, for every subsequence \((h_n)\) of \((f'_n)\)
\[
\langle \frac{1}{n} \sum_{i=1}^{n} h_i(\omega) \rangle\] w*-converges a.e. to \(f(\omega)\).

For every \(x \in E\), the sequence \((\frac{1}{n} \sum_{i=1}^{n} h_i, x)\) is UI in \(L^1_{\mathbb{R}}(\mu)\) since \(H_x\) it is, then by the Vitali’s theorem in \(L^1_{\mathbb{R}}(\mu)\)
\[
\forall A \in \mathcal{F}, \quad \int_{A} \langle \frac{1}{n} \sum_{i=1}^{n} h_i, x \rangle d\mu \rightarrow \int_{A} \langle f, x \rangle d\mu. \quad (1)
\]
As (1) is valid for every subsequence \((h_n)\) of \((f'_n)\), by an elementary property of Cesàro convergence in \(\mathbb{R}\) we get
\[
\forall A \in \mathcal{F}, \forall x \in E, \quad \int_{A} \langle f'_n, x \rangle d\mu \rightarrow \int_{A} \langle f, x \rangle d\mu.
\]
It follows by the boundedness of \((f_n)\) in \(L^1_{E'} [E]\) that
\[
\forall h \in L^\infty_{\mathbb{R}}(\mu), \forall x \in E, \quad \int_{\Omega} h(\langle f'_n, x \rangle) d\mu \rightarrow \int_{\Omega} h(\langle f, x \rangle) d\mu.
\]
Thus \((f_n)\) is \(\sigma(L^1_{E'}, E, L^\infty_R(\mu) \otimes E)\)-sequentially relatively compact.

The next result show that if a sequence of Pettis integrable functions bounded in \(L^1_{E'}[E]\) converges pointwise in \(L^\infty_R(\mu) \otimes E''\) to a scalarly integrable function \(f\), then \(f\) is Pettis integrable.

**Theorem 4.** Let \(f : \Omega \to E'\) and \((f_n)\) a bounded sequence of \(L^1_{E'}[E]\).

1. If \(f\) is \(w^*\)-scalarly integrable and 
   \[
   \forall A \in F, \forall x \in E, \quad \int_A \langle f_n, x \rangle d\mu \to \int_A \langle f, x \rangle d\mu, 
   \]
   then \(f \in L^1_{E'}[E]\).

2. If \(f\) is scalarly integrable, \(f_n\) is Pettis integrable for all \(n\) and 
   \[
   \forall A \in F, \forall x'' \in E'', \quad \int_A \langle f_n, x'' \rangle d\mu \to \int_A \langle f, x'' \rangle d\mu, 
   \]
   then \(f\) is Pettis integrable.

**Proof.** (1) By the Vitali-Hahn-Saks theorem ([7] Corollary I.4.10) for each \(x \in E\) the sequence \(\langle f_n, x \rangle\) is UI in \(L^1_{B}(\mu)\). Applying Theorem 3 to \((f_n)\) we have that \((f_n)\) is \(\sigma(L^1_{E'}, E, L^\infty_R(\mu) \otimes E)\)-sequentially relatively compact, so there exists a subsequence \((f'_n)\) converging \(\sigma(L^1_{E'}[E], L^\infty_R(\mu) \otimes E)\) to a \(g \in L^1_{E'}[E]\). Then we have

\[
\forall A \in F, \forall x \in E, \quad \int_A \langle f'_n, x \rangle d\mu \to \int_A \langle g, x \rangle d\mu. 
\]

By (2) and (4) we get

\[
\forall A \in F, \forall x \in E, \quad \int_A \langle g, x \rangle d\mu = \int_A \langle f, x \rangle d\mu, 
\]

hence

\[
\forall x \in E, \quad \langle g, x \rangle = \langle f, x \rangle. \quad a.e.
\]

It follows by the separability of \(E\) that \(\|g\| = \|f\| \ a.e.\) and therefore \(f \in L^1_{E'}[E]\).

(2) Now suppose that \(f\) is scalarly integrable, \(f_n\) is Pettis integrable for each \(n\) and (3) is satisfied and let us prove that \(f\) is Pettis integrable. By Theorem 1 it is enough to check that \(\{f\}\) is SUI, which is the case since \(\|f(.)\|\) is integrable.

The next result is an immediate application of the above theorem.

**Corollary 2.** The subset of \(L^1_{E'}[E]\) of Pettis integrable functions is norm closed.
Proof. Let \((f_n)\) a norm convergent sequence of Pettis integrable functions of \(L_{E'}^1[E]\) and \(f\) its limit in \(L_{E'}^1[E]\), there exists a subsequence \((f'_n)\) of \((f_n)\) such that
\[
\lim_n \|f'_n(\omega) - f(\omega)\| = 0 \quad \text{a.e.}
\]
So \(f\) is w-measurable and
\[
\forall A \in \mathcal{F}, \forall x'' \in E'', \int_A \langle f_n, x'' \rangle d\mu \to \int_A \langle f, x'' \rangle d\mu.
\]
By Theorem 4 (2) \(f\) is Pettis integrable.

By combining Theorem 2 and Theorem 4 we have the following.

**Theorem 5.** Let \((f_n)\) a bounded sequence of \(L_{E'}^1[E]\). Suppose that the following hold:

1. \(f_n\) w*-converges a.e. to a function \(f\),
2. \(f_n\) is Pettis integrable for each \(n\),
3. for each \(k \in \mathbb{N}^*\) there is a scalarly integrable function \(v_k\) such that
\[
\forall A \in \mathcal{F}, \forall x'' \in E'', \int_A \langle 1_{\{\|f_n\| \leq k\}} f_n, x'' \rangle d\mu \to \int_A \langle v_k, x'' \rangle d\mu.
\]
4. \(\exists A \in \mathcal{F}, \forall x \in E, \int_A \langle 1_{\{\|f_n\| \leq k\}} f_n, x \rangle d\mu \to \int_A \langle f, x \rangle d\mu.\)

Then \(f\) is Pettis integrable.

**Proof.** By (1) and Theorem 2 we get \(f \in L_{E'}^1[E]\), so by Corollary 1 we have to prove that \(1_{\{\|f\| \leq k\}} f\) is Pettis integrable for every \(k \in \mathbb{N}^*\). Fix \(k \in \mathbb{N}^*\) and applying Theorem 4 (2) to \(v_k\) and \((1_{\{\|f_n\| \leq k\}} f_n)\) we get that \(v_k\) is Pettis integrable. As \((1_{\{\|f_n\| \leq k\}} f_n)\) is WSUI and by (1) is w*-converges a.e. to \(1_{\{\|f\| \leq k\}} f\), it follows by the Vitali’s theorem in \(L_{\mathbb{R}}^1(\mu)\) that
\[
\forall A \in \mathcal{F}, \forall x \in E, \int_A \langle 1_{\{\|f_n\| \leq k\}} f_n, x \rangle d\mu \to \int_A \langle 1_{\{\|f\| \leq k\}} f, x \rangle d\mu.\]
By (5) and (6) we get
\[
\forall x \in E, \quad \langle 1_{\{\|f\| \leq k\}}, x \rangle = \langle v_k, x \rangle \quad \text{a.e.}
\]
Being \(E\) separable, it follows that \(v_k = 1_{\{\|f\| \leq k\}} f\) a.e. and therefore \(1_{\{\|f\| \leq k\}} f\) is Pettis integrable.

We finish this work by the following version of Theorem 4 in [16] with Pettis integrable functions. Recall that \(\mathcal{Rwc}(E')\) denoted the set of nonempty convex ball weakly compact subsets of \(E'\).

**Theorem 6.** Let \((f_n)\) be a bounded sequence in \(L_{E'}^1[E]\). Suppose that \(f_n\) is Pettis integrable for all \(n \in \mathbb{N}\) and there exist a \(\mathcal{Rwc}(E')\)-valued multifunction \(\Gamma\) such that \(f_n(\omega) \in \Gamma(\omega)\) for a.e. \(\omega \in \Omega\) and for all \(n \in \mathbb{N}\). Then there exists a Pettis integrable function \(f \in L_{E'}^1[E]\) and a subsequence \((g_n)\) of \((f_n)\) such for every subsequence \((h_n)\) of \((g_n)\) the following holds
(j) \( \left( \frac{1}{n} \sum_{i=1}^{n} h_i \right) \) weakly converges a.e. to \( f \).

(ii) \( \left( \frac{1}{\|h_n\|} h_n \right) \) converges \( \sigma(L^1_{E'}, [E]), (L^1_{E'} [E])' \) (weakly) to \( f \) in \( L^1_{E'} [E] \) and \( (h_n - 1_{\{\|h_n\|<n\}} h_n) \) converges a.e. to 0 in \( E' \).

\textbf{Proof.} By Theorem 4 in [16] there exists a function \( f \in L^1_{E'} [E] \) and a subsequence \( (g_n) \) of \( (f_n) \) such that (j) and (jj) hold. Now since \( \left( \frac{1}{n} \sum_{i=1}^{n} h_i \right) \) is bounded in \( L^1_{E'} [E] \) and weakly converges a.e. to \( f \), it follows by Theorem 2 that \( f \) is Pettis integrable.

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