Exact augmented Lagrangians for constrained optimization problems in Hilbert spaces I: theory

M. V. Dolgopolik

Institute for Problems in Mechanical Engineering of the Russian Academy of Sciences, Saint Petersburg, Russia

ABSTRACT
In this two-part study, we develop a general theory of the so-called exact augmented Lagrangians for constrained optimization problems in Hilbert spaces. In contrast to traditional nonsmooth exact penalty functions, these augmented Lagrangians are continuously differentiable for smooth problems and do not suffer from the Maratos effect, which makes them especially appealing for applications in numerical optimization. Our aim is to present a detailed study of various theoretical properties of exact augmented Lagrangians and discuss several applications of these functions to constrained variational problems, problems with PDE constraints, and optimal control problems. The first paper is devoted to a theoretical analysis of an exact augmented Lagrangian for optimization problems in Hilbert spaces. We obtain several useful estimates of this augmented Lagrangian and its gradient, and present several types of sufficient conditions for KKT-points of a constrained problem corresponding to locally/globally optimal solutions to be local/global minimizers of the exact augmented Lagrangian.

1. Introduction
The concept of exactness of a penalty function was first introduced by Eremin [1] and Zangwill [2] in the mid-1960s. The penalty function $F_c(x) = f(x) + c\|h(x)\|$ for the constrained optimization problem

$$\min f(x) \quad \text{subject to} \quad h(x) = 0$$

is called exact, if for any sufficiently large (but finite) value of the penalty parameter $c > 0$ its points of global/local minimum coincide with globally/locally optimal solutions of the constrained problem. It turned out that penalty functions for many convex and nonconvex constrained optimization problems are exact under relatively mild assumptions [3–10], which made them one of the cornerstones
of constrained optimization for several decades. However, exact penalty functions have several drawbacks. Firstly, they are inherently nonsmooth (see, e.g. [10, Remark 3] for details), which means that one either has to develop specific numerical methods for minimizing such functions (see, e.g. [11,12]) or use them as merely an auxiliary tool for stepsize evaluation, e.g. in SQP methods [13–15]. Secondly, numerical methods based on nonsmooth exact penalty functions often suffer from the Maratos effect, which makes them significantly less appealing for applications (see [13, Example 15.4] and [14, Example 17.6] for simple particular examples of the Maratos effect, and [16] for a discussion of ways to counter this effect).

In 1970, Fletcher [17] introduced a new penalty function, which overcomes drawbacks of traditional nonsmooth penalty functions. Namely, under natural assumptions Fletcher’s penalty function is exact, continuously differentiable, and robust with respect to the Maratos effect (see [13,15]). This penalty function is, in essence, the Hestenes-Powell-Rockafellar augmented Lagrangian $\mathcal{L}(x, \lambda, c) = f(x) + \langle \lambda, h(x) \rangle + c\|h(x)\|^2$ (see [18–21]), in which the Lagrange multipliers $\lambda$ are replaced by their estimates $\lambda(x)$, computed as a solution of a system of linear equations related to the KKT optimality conditions. Fletcher’s penalty function and its various modifications were studied in details in [22–27]. Despite its merits, Fletcher’s penalty function has been traditionally considered impractical due to the fact that each evaluation of this function (and especially its gradient) is very computationally expensive. Nevertheless, an efficient implementation of a constrained optimization method based on Fletcher’s penalty function was recently proposed in [28,29].

In 1979, Di Pillo and Grippo [30] introduced a new augmented Lagrangian for equality constrained optimization problems, which can be viewed as a modification of Fletcher’s penalty function. Instead of replacing the Lagrange multipliers $\lambda$ in the augmented Lagrangian $\mathcal{L}(x, \lambda, c) = f(x) + \langle \lambda, h(x) \rangle + c\|h(x)\|^2$ with their estimates, Di Pillo and Grippo proposed to add an auxiliary term to this function that penalizes the violation of the KKT optimality conditions and is directly connected to the system of linear equations for computing the estimate $\lambda(x)$ for Fletcher’s penalty function. In [30], it was shown that under some additional assumptions, local/global minimizers of the Di Pillo-Grippo augmented Lagrangian jointly in primal and dual variables are precisely KKT-points of the equality constrained problem corresponding to its locally/globally optimal solutions. This analogy with exact penalty functions led to the fact that Di Pillo-Grippo augmented Lagrangians were later on called exact. Furthermore, just like Fletcher’s penalty function, Di Pillo-Grippo augmented Lagrangian does not suffer from the Maratos effect.

The exact augmented Lagrangian from [30] was extended to the case of inequality constrained problems in [31]. The augmented Lagrangian from [31] was further modified, analysed, and applied to various inequality constrained optimization problems in [31–37]. Exact augmented Lagrangians for equality
constrained problems were further studied in [38,39], while such augmented Lagrangians for problems with equality and two-sided (box) constraints were discussed in [40,41]. Numerical methods for solving constrained optimization problems based on exact augmented Lagrangians were studied in [42–44].

Fukuda and Lourenço [45] extended the theory of exact augmented Lagrangians to the case of nonlinear semidefinite programming problems. Finally, a general theory of exact augmented Lagrangians for cone constrained optimization problems was developed by the author in [46]. In particular, in [46] it was shown that one can construct an exact augmented Lagrangian from many other augmented Lagrangians apart from the Hestenes-Powell-Rockafellar augmented Lagrangian. However, to the best of the author’s knowledge, all existing results on exact augmented Lagrangians were obtained only in the finite dimensional case.

The main goal of this study is to develop a general theory of exact augmented Lagrangians for optimization problems in infinite dimensional spaces and to develop new exact augmented Lagrangian methods for solving these problems. The motivation behind such extension is based on the fact that exact augmented Lagrangians have been used to develop efficient superlinearly convergent optimization methods for nonlinear programming problems that are robust with respect to the Maratos effect (see [34–36,40,41]). Our goal is to develop a theory that would allow one to extend these methods to optimization problems in infinite dimensional problems, such as optimal control problems and problems with PDE constraints.

The first part of our study is devoted to a theoretical analysis of exact augmented Lagrangians for optimization problems in Hilbert spaces. We restrict our consideration to the Hilbert space setting, since it is unclear whether exact augmented Lagrangians can be defined in a more general case of optimization problems in Banach spaces. Nevertheless, in the second paper we will show that in many particular cases, numerical methods based on exact augmented Lagrangians work well for problems in Banach spaces, although our theoretical results do not permit such a general problem setting.

In this paper, we introduce an exact augmented Lagrangian for optimization problems in real Hilbert spaces with inequality and nonlinear operator equality constraints. We study some properties of this augmented Lagrangian and obtain some useful estimates of this function and its gradient, which play a crucial role in the analysis of its exactness. We also obtain sufficient conditions for the exact augmented Lagrangian to have bounded sublevel sets and study its exact penalty properties with the use of a nonlocal constraint qualification, which is closely related to the linear independence constraint qualification and conditions on nonlocal metric regularity of constraints. In particular, we show that under some general assumptions, local/global minimizers of the exact augmented Lagrangian jointly in primal and dual variables are precisely KKT-points of the original problem corresponding to its locally/globally optimal solutions, provided the penalty
parameter is sufficiently large. Various applications of the theory developed in this paper to particular classes of constrained variational problems, problems with PDE constraints, and optimal control problems will be presented in the second part of our study.

The paper is organized as follows. The problem statement and the definition of the exact augmented Lagrangian for optimization problems in Hilbert spaces are discussed in Section 2. Various properties of the augmented Lagrangian are studied in Section 3. In particular, this section contains sufficient conditions for sublevel sets of the exact augmented Lagrangian to be bounded. Some useful estimates of the gradient of the augmented Lagrangian are collected in Section 4. Finally, Section 5 is devoted to various sufficient conditions for the local/global exactness of the augmented Lagrangian introduced in this paper.

2. The definition of exact augmented Lagrangian

Let $X$ and $H$ be real Hilbert spaces. Throughout this article we study the following constrained optimization problem:

$$
\min f(x) \quad \text{subject to} \quad F(x) = 0, \quad g_i(x) \leq 0, \quad i \in M. \quad (P)
$$

Here $f, g_i: X \to \mathbb{R}$ and $F: X \to H$ are given functions, and $M = \{1, \ldots, m\}$. Below, we suppose that there exists a globally optimal solution of the problem $(P)$. Our aim is to reduce the problem $(P)$ to a completely equivalent unconstrained problem of minimizing a certain augmented Lagrangian in primal and dual variables simultaneously. Following Di Pillo, Grippo, and Lucidi [30–34], we call such functions exact augmented Lagrangians.

Let $\langle \cdot, \cdot \rangle$ be the inner product in $X$, $H$ or $\mathbb{R}^n$, depending on the context, and $g(\cdot) = (g_1(\cdot), \ldots, g_m(\cdot))$. In the case when the functions $g_i$ are differentiable, we denote by $\nabla g(x)y \in \mathbb{R}^m$ the vector whose $i$-th coordinate is $\langle \nabla g_i(x), y \rangle$, where $y \in X$. Let

$$
L(x, \lambda, \mu) = f(x) + \langle \lambda, F(x) \rangle + \langle \mu, g(x) \rangle, \quad \lambda \in H, \quad \mu \in \mathbb{R}^m,
$$

be the classical Lagrangian for the problem $(P)$.

To include several particular cases into a general theory, choose a convex non-decreasing lower semicontinuous (l.s.c.) function $\phi: [0, +\infty) \to [0, +\infty]$ such that $\phi(t) = 0$ if and only if $t = 0$, and $\text{dom}\phi \neq \{0\}$. In particular, one can define

$$
\phi(t) \equiv t \quad \text{or} \quad \phi(t) = \begin{cases} 
t/(\alpha - t), & \text{if } t \in [0, \alpha), \\
+\infty, & \text{if } t \geq \alpha,
\end{cases}
$$

(here $\alpha > 0$ is fixed). From the assumptions on the function $\phi$ (in particular, its convexity) it follows that $\phi$ is continuous on its effective domain, and either $\text{dom}\phi = [0, +\infty)$ and $\phi(t) \to +\infty$ as $t \to \infty$ or there exists $\alpha > 0$ such that $\text{dom}\phi = [0, \alpha)$ and $\phi(t) \to +\infty$ as $t \to \alpha$. 
Choose also a continuously differentiable concave function $\psi : [0, +\infty)^m \to \mathbb{R}$ such that $\psi(0) > 0$, zero is a point of global maximum of $\psi$, and
\[
\frac{\partial \psi}{\partial y_i}(y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_m) = 0 \quad \forall y \in \mathbb{R}^m, \quad i \in M.
\]
The equality on partial derivatives ensures that the function $\psi(\max\{g(x), 0\})$ is continuously differentiable, provided the functions $g_i$ are differentiable. Note that one can define
\[
\psi(y) \equiv 1 \quad \text{or} \quad \psi(y) = \beta - \sum_{i=1}^m y_i^s
\]
for some $\beta > 0$ and $s > 1$.

Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^m$. Introduce the functions
\[
b(x) = \psi(\max\{g(x), 0\}), \quad p(x, \mu) = \frac{b(x)}{1 + |\mu|^2} \quad \forall x \in X, \quad \mu \in \mathbb{R}^m.
\]
Denote $\Omega = \{x \in X \mid b(x) > 0, \phi(\|F(x)\|^2) < +\infty\}$. It should be noted that the set $\Omega$ is open, provided the functions $F$ and $g_i$ are continuous. For any vectors $y, z \in \mathbb{R}^m$, let $\max\{y, z\} \in \mathbb{R}^m$ be the vector whose $i$-th coordinate is $\max\{y_i, z_i\}$. The vector $\min\{y, z\}$ is defined in the same way.

Finally, suppose that the functions $f, F$, and $g_i, i \in M$, are continuously Fréchet differentiable and introduce the following augmented Lagrangian:
\[
L(x, \lambda, \mu, c) = f(x) + \langle \lambda, F(x) \rangle + \frac{c}{2}(1 + \|\lambda\|^2)\phi(\|F(x)\|^2)
\]
\[
+ \left\langle \mu, \max \left\{ g(x), -\frac{1}{c}p(x, \mu) \right\} \right\rangle
\]
\[
+ \frac{c}{2p(x, \mu)} \left| \max \left\{ g(x), -\frac{1}{c}p(x, \mu) \right\} \right|^2
\]
\[
+ \eta(x, \lambda, \mu), \quad (1)
\]
if $x \in \Omega$, and $L(x, \lambda, \mu, c) = +\infty$, otherwise. Here $\lambda \in H$ and $\mu \in \mathbb{R}^m$ are Lagrange multipliers, $c > 0$ is the penalty parameter,
\[
\eta(x, \lambda, \mu) = \frac{1}{2} \left\| DF(x) \left[ \nabla_x L(x, \lambda, \mu) \right] \right\|^2
\]
\[
+ \frac{1}{2} \sum_{i=1}^m \left( \langle \nabla g_i(x), \nabla_x L(x, \lambda, \mu) \rangle + g_i(x)^2 \mu_i \right)^2, \quad (2)
\]
and $DF(x)[\cdot] : X \to H$ is the Fréchet derivative of the nonlinear operator $F$ at $x$, while $\nabla_x L(x, \lambda, \mu)$ is the gradient of the function $x \mapsto L(x, \lambda, \mu)$ (recall that $X$ is a Hilbert space, which implies that the gradient of any real-valued differentiable function on $X$ is correctly defined). Augmented Lagrangian (1) is a
natural extension of the definition of exact augmented Lagrangian for mathematical programming problems from [30,32–34,36,46] to the infinite dimensional case.

**Remark 2.1:** As one can readily verify, the following equality holds true:

\[ L(x, \lambda, \mu, c) = L(x, \lambda, \mu) + \frac{c}{2}(1 + \|\lambda\|^2)\phi (\|F(x)\|^2) \]

\[ + \frac{c}{2p(x, \mu)}(|g(x)|^2 - |\min\{0, g(x) + c^{-1}p(x, \mu)\}|^2) \]

\[ + \eta(x, \lambda, \mu), \]

if \( x \in \Omega \). Observe that the augmented Lagrangian \( L(x, \lambda, \mu, c) \) consists of three terms. The first one is just the standard Lagrangian \( L(x, \lambda, \mu) \) for the problem \( (P) \). The second term, roughly speaking, penalizes the violation of the constraints of the problem \( (P) \) and resists an excessive increase of the norm of the Lagrange multipliers \( \lambda \) and \( \mu \). Finally, the term \( \eta(x, \lambda, \mu) \), in a sense, penalizes the violation of the Karush-Kuhn-Tucker (KKT) optimality conditions.

### 3. Properties of the augmented Lagrangian

Let us present some auxiliary results of the augmented Lagrangian \( L(x, \lambda, \mu, c) \), which will be used in the following sections. First, we point out continuity and differentiability properties of this function, which can be readily verified directly.

**Proposition 3.1:** For any \( c > 0 \) the function \((x, \lambda, \mu) \mapsto L(x, \lambda, \mu, c)\) is lower semicontinuous on \( X \times H \times \mathbb{R}^d \) and continuous on its effective domain \( \Omega \times H \times \mathbb{R}^d \). Moreover, this function is continuously Fréchet differentiable on \( \Omega \times H \times \mathbb{R}^d \), provided the functions \( f, F, \) and \( g_i, i \in M \), are twice continuously Fréchet differentiable on \( \Omega \) and the function \( \phi \) is continuously differentiable on its effective domain.

Under these conditions for any \( x \in \Omega, \lambda \in H, \mu \in \mathbb{R}^m \), and \( c > 0 \) one has

\[ \nabla_x L(x, \lambda, \mu, c) = \nabla_x L(x, \lambda, \mu) + c(1 + \|\lambda\|^2)\phi' (\|F(x)\|^2) DF(x)^* [F(x)] \]

\[ + \frac{c}{p(x, \mu)} \sum_{i=1}^m \max \left\{ g_i(x), -\frac{1}{c}p(x, \mu)\mu_i \right\} \nabla g_i(x) \]

\[ - \frac{c}{b(x)p(x, \mu)} \left| \max \left\{ g(x), -\frac{1}{c}p(x, \mu)\mu \right\} \right|^2 \sum_{i=1}^m \frac{\partial \psi}{\partial y_i} (\max\{g(x), 0\}) \nabla g_i(x) \]

\[ + \left( D^2 F(x)[\nabla_x L(x, \lambda, \mu), :] \right)^* [DF(x)[\nabla_x L(x, \lambda, \mu)]] \]

\[ + \left( D_x (\nabla_x L(x, \lambda, \mu)) \right)^* [DF(x)^* [DF(x)[\nabla_x L(x, \lambda, \mu)]]] \]
+ \sum_{i=1}^{m} \left( \langle \nabla x L(x, \lambda, \mu), \nabla g_i(x) \rangle + g_i(x)^2 \mu_i \right) D(\nabla g_i(x))^* [\nabla x L(x, \lambda, \mu)] \\
+ \left( D_x(\nabla x L(x, \lambda, \mu)) \right)^* [\nabla g_i(x)] + 2g_i(x) \mu_i \nabla g_i(x),

and

\nabla_{x, \mu} \mathcal{L}(x, \lambda, \mu, c) = F(x) + c\phi(\|F(x)\|^2) = \nabla_{x, \mu} \mathcal{L}(x, \lambda, \mu, c)

\nabla_{x, \mu} \mathcal{L}(x, \lambda, \mu, c)
= \max \left\{ g(x), -\frac{1}{c}p(x, \mu) \mu \right\}
+ \frac{c}{b(x)} \left( \max \left\{ g(x), -\frac{1}{c}p(x, \mu) \mu \right\} \right)^2 \mu
+ \nabla g(x) \left( DF(x)^* \left( DF(x) [\nabla x L(x, \lambda, \mu)] \right) \right)
+ \left( Gr(x) + \text{diag}(g_i(x)^2) \right) [\nabla g(x) \nabla x L(x, \lambda, \mu) + \text{diag}(g_i(x)^2) \mu],

where \( D^2 F(x) \langle \cdot, \cdot \rangle \) is the second order Fréchet derivative of the nonlinear operator \( F \),
\( A^* \) is the adjoint operator of a bounded linear operator \( A \) mapping between Hilbert spaces,
\( D_x \nabla L(x, \lambda, \mu) \) is the Fréchet derivative of the function \( x \mapsto \nabla L(x, \lambda, \mu) \),
and \( Gr(x) = \{ \langle \nabla g_i(x), \nabla g_j(x) \rangle \}_{i,j \in M} \) is the Gram matrix of the vectors \( \nabla g_i(x) \), \( i \in M \).

**Remark 3.1:** (i) In the case when there are no inequality constraints, the augmented Lagrangian \( \mathcal{L}(x, \lambda, c) \) is \( k \) times continuously Fréchet differentiable in \((x, \lambda)\) on its effective domain for an arbitrary \( k \in \mathbb{N} \), provided the functions \( f \) and \( F \) are \( k + 1 \) times continuously Fréchet differentiable, while the function \( \phi \) is \( k \) times continuously differentiable on its effective domain.

(ii) With the use of the previous proposition one can readily verify that if \((x, \lambda, \mu)\) is a KKT point of the problem \((\mathcal{P})\) (i.e. \( x \) is feasible for this problem, \( \nabla x L(x, \lambda, \mu) = 0 \), and for all \( i \in M \) one has \( \mu_i g_i(x) = 0 \) and \( \mu_i \geq 0 \), then \( \nabla x \mathcal{L}(x, \lambda, \mu, c) = 0 \), \( \nabla_{x, \mu} \mathcal{L}(x, \lambda, \mu, c) = 0 \), and \( \nabla_{\mu} \mathcal{L}(x, \lambda, \mu, c) = 0 \) for all \( c > 0 \). Thus, KKT points of the problem \((\mathcal{P})\) are stationary points of the augmented Lagrangian \( \mathcal{L}(x, \lambda, \mu, c) \). Below, we will show that under some additional assumptions the converse statement holds true, that is, stationary points of the augmented Lagrangian are, in fact, KKT points of the problem \((\mathcal{P})\).
Let us also obtain a simple, yet useful lower estimate of the augmented Lagrangian.

**Lemma 3.2:** Let there exist \( \phi_0 > 0 \) such that \( \phi(t) \geq \phi_0 t \) for all \( t \geq 0 \) (or, equivalently, \( \liminf_{t \to +0} \phi(t)/t > 0 \)). Then

\[
\mathcal{L}(x, \lambda, \mu, c) \geq f(x) + \frac{c}{2} \phi(\|F(x)\|^2) - \frac{1}{2c\phi_0} \\
+ \frac{c}{2b(x)} \left| \max \left\{ g(x), -\frac{1}{c} p(x, \mu) \mu \right\} \right|^2 - \frac{\psi(0)}{2c} + \eta(x, \lambda, \mu) \\
\geq f(x) + \frac{c}{2} \phi(\|F(x)\|^2) - \frac{1}{2c \phi_0} \\
+ \frac{c}{2\psi(0)} \left| \max \{ g(x), 0 \} \right|^2 - \frac{(1 + m) \psi(0)}{2c} + \eta(x, \lambda, \mu)
\]

for all \( x \in X, \lambda \in H, \mu \in \mathbb{R}^m, \) and \( c > 0. \)

**Proof:** With the use of the inequality \( \phi(t) \geq \phi_0 t \) and the Cauchy-Bunyakovsky-Schwarz inequality one obtains that

\[
\mathcal{L}(x, \lambda, \mu, c) \geq f(x) + \frac{c}{2} \phi(\|F(x)\|^2) - \|\lambda\| \|F(x)\| + \frac{c\phi_0}{2} \|\lambda\|^2 \|F(x)\|^2 \\
+ \frac{c}{2b(x)} \left| \max \left\{ g(x), -\frac{1}{c} p(x, \mu) \mu \right\} \right|^2 \\
- |\mu| \left| \max \left\{ g(x), -\frac{1}{c} p(x, \mu) \mu \right\} \right|^2 \\
+ \frac{c|\mu|^2}{2\psi(0)} \left| \max \left\{ g(x), -\frac{1}{c} p(x, \mu) \mu \right\} \right|^2 + \eta(x, \lambda, \mu)
\]

for all \( x \in \Omega, \lambda \in H, \mu \in \mathbb{R}^m, \) and \( c > 0. \) Hence applying the following obvious lower estimates

\[
-t + \frac{c\phi_0}{2} t^2 \geq -\frac{1}{2c\phi_0}, \quad -t + \frac{c}{2\psi(0)} t^2 \geq -\frac{\psi(0)}{2c} \quad \forall t \in \mathbb{R},
\]

one obtains that

\[
\mathcal{L}(x, \lambda, \mu, c) \geq f(x) + \frac{c}{2} \phi(\|F(x)\|^2) - \frac{1}{2c\phi_0} \\
+ \frac{c}{2b(x)} \left| \max \left\{ g(x), -\frac{1}{c} p(x, \mu) \mu \right\} \right|^2 - \frac{\psi(0)}{2c} + \eta(x, \lambda, \mu),
\]

for all \( x \in \Omega \) and \( \mu \in \mathbb{R}^m. \)
Hence taking into account the fact that case Combining this estimate and inequality (3), we arrive at the required result in the provided the penalty parameter $L$ on the augmented Lagrangian finite dimensional case and play a key role in the derivation of most results is reduced to the linear independence constraint qualification (LICQ) in the cal (uniform) constraint qualification holds true. This constraint qualification dimensional case LICQ plays a similar role (cf. [30–34, 36]).

Denote

$$M_+(x, \mu, c) = \left\{ i \in M \mid g_i(x) \geq p(x, \mu) \frac{\mu_i}{c} \right\} , \quad M_-(x, \mu, c) = M \setminus M_+(x, \mu, c)$$

and define $M_+(x) = \{ i \in M \mid g_i(x) \geq 0 \}$, $M_-(x) = M \setminus M_+(x)$. Observe that

$$\left| \max \left\{ g(x), -\frac{p(x, \mu)}{c} \mu \right\} \right|^2 = \sum_{i \in M_+(x, \mu, c)} g_i(x)^2 + \sum_{i \in M_-(x, \mu, c)} p(x, \mu)^2 \frac{\mu_i^2}{c^2}$$

$$\geq \sum_{i \in M_+(x, \mu, c)} g_i(x)^2 = | \max[g(x), 0]|^2$$

$$- \sum_{i \in M_-(x, \mu, c) \cap M_+(x)} g_i(x)^2 + \sum_{i \in M_+(x, \mu, c) \cap M_-(x)} g_i(x)^2 .$$

Hence taking into account the fact that $p(x, \mu) |\mu_i|/c \leq \psi(0)/c$ for any $i \in M$, that is, $|g_i(x)| \leq \psi(0)/c$ for any $i \in M_-(x, \mu, c) \cap M_+(x)$, one obtains that

$$\left| \max \left\{ g(x), -\frac{1}{c} p(x, \mu) \mu \right\} \right|^2 \geq | \max[g(x), 0]|^2 - m \frac{\psi(0)^2}{c^2} .$$

Combining this estimate and inequality (3), we arrive at the required result in the case $x \in \Omega$. The validity of the lemma in the case $x \notin \Omega$ is obvious. ■

Our next goal is to show that under some natural assumptions the function $\mathcal{L}(\cdot, c)$ has bounded sublevel sets

$$S_c(\gamma) := \left\{ (x, \lambda, \mu) \in X \times H \times \mathbb{R}^m \mid \mathcal{L}(x, \lambda, \mu, c) \leq \gamma \right\} , \quad \gamma \in \mathbb{R} , \quad (4)$$

provided the penalty parameter $c > 0$ is sufficient large and a certain nonlocal (uniform) constraint qualification holds true. This constraint qualification is reduced to the linear independence constraint qualification (LICQ) in the finite dimensional case and plays a key role in the derivation of most results on the augmented Lagrangian $\mathcal{L}(x, \lambda, \mu, c)$ in this article. Note that in the finite dimensional case LICQ plays a similar role (cf. [30–34, 36]).

To define the required constraint qualification, introduce the function

$$Q(x)[\lambda, \mu] = \frac{1}{2} \left\| DF(x) \left[ DF(x)^*[\lambda] + \sum_{i=1}^m \mu_i \nabla g_i(x) \right] \right\|^2$$

$$+ \frac{1}{2} \left\| \nabla g(x) \left( DF(x)^*[\lambda] + \sum_{i=1}^m \mu_i \nabla g_i(x) \right) + \text{diag}(g_i(x)^2) \mu \right\|^2 . \quad (5)$$
This function is obviously quadratic with respect to \((\lambda, \mu)\). We say that \(Q(x)[\cdot]\) is positive definite at a point \(x\), if there exists \(a > 0\) such that
\[
Q(x)[\lambda, \mu] \geq a(\|\lambda\|^2 + |\mu|^2) \quad \forall \lambda \in H, \quad \mu \in \mathbb{R}^m.
\] (6)

In this case one says that \(Q(x)[\cdot]\) is positive definite with constant \(a > 0\).

Let us show how the positive definiteness of the function \(Q(x)[\cdot]\) is connected with a well-known constraint qualification for optimization problems in infinite dimensional spaces. For any \(x \in X\) denote \(M(x) = \{i \in M \mid g_i(x) = 0\}\), and let \(m(x) = |M(x)|\) be the cardinality of the set \(M(x)\). For any Hilbert spaces \(Y\) and \(Z\), we endow the product space \(Y \times Z\) with the inner product
\[
\langle (y_1, z_1), (y_2, z_2) \rangle = \langle y_1, y_2 \rangle + \langle z_1, z_2 \rangle \quad \forall y_1, y_2 \in Y, \quad z_1, z_2 \in Z
\]
and the corresponding norm.

**Lemma 3.3:** Let \(x \in X\) be fixed. The function \(Q(x)[\cdot]\) is positive definite if and only if the linear operator \(T : X \rightarrow H \times \mathbb{R}^{m(x)}\), defined as
\[
T z = \{DF(x)[z]\} \times \prod_{i \in M(x)} \{\langle \nabla g_i(x), z \rangle\} \quad \forall z \in X,
\]
is surjective.

**Proof:** To simplify the notation, without loss of generality we assume that \(M(x) = \{1, \ldots, m_0\}\) for some \(m_0 \in \mathbb{N}, \ m_0 \leq m\).

**Part 1.** Let \(Q(x)[\cdot]\) be positive definite. Then there exists \(a > 0\) such that inequality (6) holds true. Observe that
\[
TT^*(\lambda, \nu) = DF(x)^*[\lambda] + \sum_{i \in M(x)} \nu_i \nabla g_i(x) \quad \forall \lambda \in H, \quad \nu \in \mathbb{R}^{m(x)}.
\]
Consequently, for any \(\lambda \in H\) and \(\mu \in \mathbb{R}^m\) such that \(\mu_i = 0\) for all \(i \notin M(x)\) one has
\[
\frac{1}{2} \|TT^*(\lambda, \nu)\|^2 = Q(x)[\lambda, \mu] \geq a(\|\lambda\|^2 + |\mu|^2),
\]
where the vector \(\nu \in \mathbb{R}^{m(x)}\) is obtained from \(\mu\) by deleting all those coordinates that correspond to \(i \notin M(x)\). Thus, one has
\[
\|TT^*(\lambda, \nu)\| \geq \sqrt{2a}\|\lambda, \nu\| \quad \forall (\lambda, \nu) \in H \times \mathbb{R}^{m(x)}.
\]
The operator \(TT^*\) is obviously self-adjoint. Therefore, by [47,Theorem 4.13] the inequality above implies that this operator is surjective. Consequently, the operator \(T\) is surjective as well.
**Part 2.** Let us prove the converse statement. Suppose that the operator $T$ is surjective. Define linear operator $\mathcal{E} : X \times \mathbb{R}^m \to H \times \mathbb{R}^m$ as follows:

$$
\mathcal{E} \begin{pmatrix} z \\ \xi \end{pmatrix} = \begin{pmatrix} DF(x)[z] \\ \nabla g(x)z + \text{diag}(g_i(x))\xi \end{pmatrix} \quad \forall z \in X, \quad \xi \in \mathbb{R}^m.
$$

It is easily seen that

$$
\mathcal{E}^* \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} DF(x)^*[\lambda] + \sum_{i=1}^m \mu_i \nabla g_i(x) \\ \text{diag}(g_i(x))\mu \end{pmatrix} \quad \forall \lambda \in H, \quad \mu \in \mathbb{R}^m,
$$

which implies that

$$
Q(x)[\lambda, \mu] = \frac{1}{2} \| \mathcal{E}^*(\lambda, \mu) \|^2 \quad \forall \lambda \in H, \quad \mu \in \mathbb{R}^m.
$$

Hence taking into account [47, Theorem 4.13] and the fact that the operator $\mathcal{E} \mathcal{E}^*$ is obviously self-adjoint, one obtains that the function $Q(x)[\cdot]$ is positive definite if and only if the operator $\mathcal{E} \mathcal{E}^*$ is surjective. Let us verify that this operator is surjective, provided the operator $\mathcal{E}$ is surjective.

Indeed, let $\mathcal{E}$ be surjective. With the use of the identity

$$
\langle (\lambda, \mu), \mathcal{E}(z, \xi) \rangle = \langle \mathcal{E}^*(\lambda, \mu), (z, \xi) \rangle \quad \forall z \in X, \quad \lambda \in H, \quad \mu, \xi \in \mathbb{R}^m
$$

one gets that $\text{Ker} \mathcal{E} = (\text{Im} \mathcal{E}^*)^\perp$, where $\text{Ker} \mathcal{E}$ is the kernel of $\mathcal{E}$ and $(\text{Im} \mathcal{E}^*)^\perp$ is the orthogonal complement of the image of $\mathcal{E}^*$. Hence bearing in mind the fact that the image $\text{Im} \mathcal{E}^*$ is closed by [47, Theorem 4.13], one obtains that $(\text{Ker} \mathcal{E})^\perp = \text{Im} \mathcal{E}^*$.

Fix any $(z, \xi) \in H \times \mathbb{R}^m$. From the fact that $\mathcal{E}$ is surjective it follows that there exists $y \in (\text{Ker} \mathcal{E})^\perp = \text{Im} \mathcal{E}^*$ such that $\mathcal{E}y = (z, \xi)$. By the definition of image, there exists $(\lambda, \mu) \in H \times \mathbb{R}^m$ such that $\mathcal{E}^*(\lambda, \mu) = y$, which implies that $\mathcal{E} \mathcal{E}^*(\lambda, \mu) = (z, \xi)$. Thus, the operator $\mathcal{E} \mathcal{E}^*$ is surjective.

To conclude the proof of the lemma, we need to check that the operator $\mathcal{E}$ is surjective, if the operator $T$ is surjective. Indeed, fix any $\lambda \in H$ and $\mu \in \mathbb{R}^m$. Our aim is to find $z \in X$ and $\xi \in \mathbb{R}^m$ such that $\mathcal{E}(z, \xi) = (\lambda, \mu)$, that is,

$$
DF(x)[z] = \lambda, \quad \nabla g(x)z + \text{diag}(g_i(x))\xi = \mu.
$$

Denote by $v \in \mathbb{R}^{m(x)}$ the vector obtained from $\mu$ by removing all those coordinates that correspond to $i \notin M(x)$. Since the operator $T$ is surjective by our assumption, one can find $z \in X$ such that $Tz = (\lambda, v)$, that is,

$$
DF(x)[z] = \lambda, \quad \langle \nabla g_i(x), z \rangle = v_i \quad \forall i \in M(x).
$$

Define $\xi_i = 0$ for any $i \in M(x)$, and put

$$
\xi_i = \frac{1}{g_i(x)} \left( \mu_i - \langle \nabla g_i(x), z \rangle \right) \quad \forall i \notin M(x).
$$

Then by definition $\mathcal{E}(z, \xi) = (\lambda, \mu)$. Thus, the operator $\mathcal{E}$ is surjective, which implies that the quadratic function $Q(x)[\cdot]$ is positive definite. \hfill \blacksquare
Corollary 3.4: Let for any \( x \in \Omega \) linear operator \( \mathcal{E}(x) : X \times \mathbb{R}^m \rightarrow H \times \mathbb{R}^m \) be defined as follows:

\[
\mathcal{E}(x) \left( \begin{array}{c} z \\ \xi \end{array} \right) = \left( DF(x) [z] \nabla g(x) z + \text{diag}(g_i(x)) \xi \right) \quad \forall z \in X, \quad \xi \in \mathbb{R}^m.
\]

For any \( x \in \Omega \) the following statements are equivalent:

(1) the quadratic function \( Q(x)[\cdot] \) is positive definite;
(2) the operator \( \mathcal{E}(x) \) is surjective;
(3) there exists a \( > 0 \) such that \( \| \mathcal{E}(x) \mathcal{E}(x)^*(\lambda, \mu)\| \geq a\| (\lambda, \mu)\| \) for all \( \lambda \in H \) and \( \mu \in \mathbb{R}^m; \)
(4) the operator \( \mathcal{E}(x) \mathcal{E}(x)^* \) is invertible.

Moreover, the third statement holds true if and only if the function \( Q(x)[\cdot] \) is positive definite with constant \( a^2/2 \).

Proof: 1 \( \Rightarrow \) 2. If the quadratic function \( Q(x)[\cdot] \) is positive definite, then by the previous lemma the corresponding operator \( T \) is surjective. Hence arguing in the same way as in the proof of Lemma 3.3 one obtains that the operator \( \mathcal{E}(x) \) is surjective.

2 \( \Rightarrow \) 3. If the operator \( \mathcal{E}(x) \) is surjective, then, as was shown in the proof of Lemma 3.3, the operator \( \mathcal{E}(x) \mathcal{E}(x)^* \) is surjective as well. Therefore by [47,Theorem 4.13] the third statement of the corollary holds true.

The implication 3 \( \Rightarrow \) 4 follows directly from [47,Theorem 4.13] and the fact that the operator \( \mathcal{E}(x) \mathcal{E}(x)^* \) is obviously self-adjoint.

Finally, from the obvious implication 4 \( \Rightarrow \) 3 and equality (7) it follows that the last statement of the corollary implies the first one.

It remains to note that the third statement is satisfied if and only if \( Q(x)[\cdot] \) is positive definite with constant \( a^2/2 \) due to equality (7). ■

Remark 3.2: (i) Recall that the surjectivity of the Fréchet derivative of a non-linear operator is a central assumption of the Lusternik-Graves theorem (see, e.g. [48]), which by this theorem is equivalent to the metric regularity of the corresponding operator. In particular, in the context of Lemma 3.3, the surjectivity assumption is equivalent to the metric regularity of the mapping \( W_x(\cdot) = \{ F(\cdot) \} \times \prod_{i \in M(x)} \{ g_i(\cdot) \} \) near the point \( x \). Thus, by Lemma 3.3 the quadratic function \( Q(x)[\cdot] \) is positive definite if and only if the mapping \( W_x(\cdot) \) is metrically regular near \( x \).

(ii) Suppose that the space \( H \) is finite dimensional and the constraint \( F(x) = 0 \) is rewritten as a finite number of equality constraints \( f_j(x) = 0 \) for some functions \( f_j : X \rightarrow \mathbb{R}, j \in \{1, \ldots, \ell\} \). Then, as one can readily verify, the operator \( T \) from Lemma 3.3 is surjective if and only if the gradients \( \nabla f_j(x), j \in \{1, \ldots, \ell\}, \) and \( \nabla g_i(x), i \in M(x) \), are linearly independent, i.e. LICQ holds true at \( x \). Thus,
the positive definiteness of the quadratic function $Q(x)[·]$ is equivalent to the validity of LICQ. Below, we will use the assumption that the function $Q(x)[·]$ is uniformly positive definite on certain sets, that is, there exists $a > 0$ such that $Q(x)[λ, μ] ≥ a(‖λ‖^2 + |μ|^2)$ for all $(λ, μ) ∈ H × ℝ^m$ and for any $x$ from a given set. In the light of this remark, one can interpret this assumption as nonlocal LICQ or as an assumption on nonlocal metric regularity of the constraints of the problem ($P$). Let us also note that nonlocal CQ and nonlocal metric regularity play a central role in the theory of exact penalty functions in the infinite dimensional case [6–10,49,50].

For any $γ ∈ ℝ$ and $c > 0$ introduce the set

$$Ω_c(γ) := \left\{ x ∈ Ω \mid f(x) + c(\|F(x)\|^2 + \max\{g(x), 0\})^2 ≤ γ \right\}.$$  

We are finally ready to obtain sufficient conditions for the boundedness of the sublevel set $S_c(γ)$ defined in (4).

**Theorem 3.5:** Let $γ ∈ ℝ$ be fixed and the following assumptions be valid:

1. there exist $φ_0 > 0$ such that $φ(t) ≥ φ_0 t$ for all $t ≥ 0$;
2. the set $Ω_r(γ + ε)$ is bounded for some $r > 0$ and $ε > 0$;
3. $f$ is bounded below on the set $Ω_r(γ + ε)$, and $g$ is bounded on this set;
4. the gradients $∇f(x), ∇g_i(x), i ∈ M$, and the Fréchet derivative $DF(x)$ are bounded on the set $Ω_r(γ + ε)$;
5. there exists $a > 0$ such that for all $x ∈ Ω_r(γ + ε)$ one has

$$Q(x)[λ, μ] ≥ a(‖λ‖^2 + |μ|^2) \quad ∀λ ∈ H, \quad μ ∈ ℝ^m.$$  

Then there exists $c_0 > 0$ such that for all $c ≥ c_0$ the sublevel set $S_c(γ)$ is bounded.

**Proof:** Note that the function $L(x, λ, μ, c)$ is non-decreasing in $c$. Therefore, $S_{c_1}(γ) ⊆ S_{c_2}(γ)$ for all $c_1 ≥ c_2 > 0$, and it is sufficient to prove that the sublevel set $S_c(γ)$ is bounded only for some $c > 0$.

From Lemma 3.2 it follows that for any $(x, λ, μ) ∈ S_c(γ)$ one has

$$f(x) + \frac{cφ_0}{2} \|F(x)\|^2 + \frac{c}{2\psi(0)} \max\{g(x), 0\})^2 ≤ γ + \frac{1}{2cφ_0} + \frac{(1 + m)ψ(0)}{2c}.$$  

Consequently, for any

$$c ≥ \tilde{c} := \max \left\{ \frac{2r}{φ_0}, \frac{1}{2} \left( \frac{1}{2\psi(0)} + \frac{(1 + m)ψ(0)}{2c} \right) \right\}$$  

and for all $(x, λ, μ) ∈ S_c(γ)$ one has $x ∈ Ω_r(γ + ε)$.

Arguing by reductio ad absurdum, suppose that for any $c > 0$ the set $S_c(γ)$ is unbounded. Then for any $n ∈ ℕ$ there exists $(x_n, λ_n, μ_n) ∈ S_n(γ)$ such that
\[ \|x_n\| + \|\lambda_n\| + |\mu_n| \geq n. \] As was noted above, for any \( n \geq \hat{c} \) one has \( x_n \in \Omega_r(\gamma + \varepsilon) \). Therefore by Assumption 2 the sequence \( \{x_n\} \) is bounded, which implies that \( \|\lambda_n\| + |\mu_n| \to +\infty \) as \( n \to \infty \). Moreover, by Assumption 5 one has

\[ Q(x_n)[\lambda, \mu] \geq a(\|\lambda\|^2 + |\mu|^2) \quad \forall \lambda \in H, \quad \mu \in \mathbb{R}^m \]  

for all \( n \geq \hat{c} \)

From the definition of the function \( \eta \) (see (2)) it follows that this function is quadratic with respect to \((\lambda, \mu)\) and has the form

\[ \eta(x, \lambda, \mu) = Q(x)[\lambda, \mu] + \langle Q_{1,\lambda}(x), \lambda \rangle + \langle Q_{1,\mu}(x), \mu \rangle + Q_0(x), \]

where

\[ Q_{1,\lambda}(x) = DF(x) \left[ DF(x)^* \left( DF(x)[\nabla f(x)] \right) \right] \]

\[ + \sum_{i=1}^{m} \langle \nabla g_i(x), \nabla f(x) \rangle DF(x)[\nabla g_i(x)], \]

and \( Q_{1,\mu}(x) = (Q_{1,\mu}(x)_1, \ldots, Q_{1,\mu}(x)_m) \in \mathbb{R}^m \) with

\[ Q_{1,\mu}(x)_i = \langle DF(x)[\nabla f(x)], DF(x)[\nabla g_i(x)] \rangle \]

\[ + \sum_{j=1}^{m} \langle \nabla g_j(x), \nabla f(x) \rangle \langle \nabla g_i(x), \nabla g_j(x) \rangle + \langle \nabla g_i(x), \nabla f(x) \rangle \nabla g_i(x)^2, \]

and

\[ Q_0(x) = \frac{1}{2} \|DF(x)[\nabla f(x)]\|^2 + \frac{1}{2} \sum_{i=1}^{m} (\nabla g_i(x), \nabla f(x))^2. \]

From our assumption on the boundedness of the function \( g \) and all first order derivatives it follows that there exists \( C > 0 \) such that

\[ \|Q_{1,\lambda}(x_n)\| \leq C, \quad |Q_{1,\mu}(x_n)| \leq C, \quad Q_0(x_n) \leq C \quad \forall n \geq \hat{c}. \]

Hence with the use of inequality (9) one obtains that

\[ \eta(x_n, \lambda_n, \mu_n) \geq a(\|\lambda_n\|^2 + |\mu_n|^2) - C(\|\lambda_n\| + |\mu_n|) - C \quad \forall n \geq \hat{c}. \]

Consequently, applying Lemma 3.2 one gets that

\[ \mathcal{L}(x_n, \lambda_n, \mu_n, n) \geq f(x_n) + \frac{n}{2} \phi(\|F(x_n)\|^2) \]

\[ + \frac{n}{2\psi(0)} \left[ \max \{g(x_n), 0\} \right]^2 - \frac{1}{2n\phi_0} - \frac{1}{2} (1 + m) \frac{\psi(0)}{n} \]

\[ + a(\|\lambda_n\|^2 + |\mu_n|^2) - C(\|\lambda_n\| + |\mu_n|) - C \]
for all \( n \geq \widehat{c} \). Therefore \( \mathcal{L}(x_n, \lambda_n, \mu_n, n) \to +\infty \) as \( n \to \infty \), since by our assumptions \( \|\lambda_n\| + |\mu_n| \to +\infty \) as \( n \to \infty \), the function \( f \) is bounded below on the set \( \Omega\_r(\gamma + \varepsilon) \), and \( x_n \in \Omega\_r(\gamma + \varepsilon) \) for all \( n \geq \widehat{c} \). On the other hand, by definition \( \mathcal{L}(x_n, \lambda_n, \mu_n, n) \leq \gamma \) for all \( n \in \mathbb{N} \), which leads to the obvious contradiction. Thus, there exists \( c > 0 \) such that the set \( S_c(\gamma) \) is bounded, and the proof is complete. □

**Corollary 3.6:** Let the following assumptions be valid:

1. there exist \( \phi_0 > 0 \) such that \( \phi(t) \geq \phi_0 t \) for all \( t \geq 0 \);
2. \( f \) is bounded below and \( g \) is bounded on bounded subsets of \( \Omega \);
3. the gradients \( \nabla f(x) \), \( \nabla g_i(x) \), \( i \in M \), and the Fréchet derivative \( DF(x) \) are bounded on bounded subsets of \( \Omega \);
4. either the penalty function \( \Psi_c(x) = f(x) + c(\|F(x)\|^2 + |\max\{g(x), 0\}|^2) \) is coercive on \( \Omega \) for some \( c > 0 \) or the set \( \Omega \) is bounded;
5. for any bounded set \( V \subset \Omega \) there exists \( a > 0 \) such that for all \( x \in V \) one has

\[
Q(x)[\lambda, \mu] \geq a(\|\lambda\|^2 + |\mu|^2) \quad \forall \lambda \in H, \quad \mu \in \mathbb{R}^m.
\]

Then for any \( \gamma \in \mathbb{R} \) there exists \( c(\gamma) > 0 \) such that for all \( c \geq c(\gamma) \) the sublevel set \( S_c(\gamma) \) is bounded.

**Proof:** Fix any \( \gamma \in \mathbb{R} \). From Assumption 4 it follows that for any \( \epsilon > 0 \) the set

\[
\Omega\_c(\gamma + \epsilon) = \left\{ x \in \Omega \mid \Psi_c(x) \leq \gamma + \epsilon \right\}
\]

is bounded. Hence applying Theorem 3.5, we arrive at the required result. □

### 4. Properties of the gradient of \( \mathcal{L}(x, \lambda, \mu, c) \)

In this section, we prove an auxiliary result, describing an important property of the gradient of the augmented Lagrangian \( \mathcal{L}(x, \lambda, \mu, c) \). Namely, our aim is to show that the norm of the gradient of the function \( (x, \lambda, \mu) \mapsto \mathcal{L}(x, \lambda, \mu, c) \), denoted by \( \nabla \mathcal{L}(x, \lambda, \mu, c) \), can be estimated from below via the infeasibility measure

\[
\|F(x)\| + \left| \max \left\{ g(x), -\frac{p(x, \mu)}{c} \right\} \right|,
\]

provided the nonlocal constraint qualification from Theorem 3.5 holds true. With the use of such estimate one can readily verify that critical points of the augmented Lagrangian are, in fact, KKT-points of the original problem. In other
words, this estimate is instrumental in the proof of the exactness of \( \mathcal{L}(x, \lambda, \mu, c) \). Moreover, it plays an important role in the design and analysis of numerical methods based on the use of this augmented Lagrangian (cf. [34]).

**Theorem 4.1:** Let the following assumptions be valid:

1. \( f, g_i, i \in M, \text{ and } F \) are twice continuously Fréchet differentiable on \( \Omega, \phi \) is continuously differentiable on its effective domain, and \( \phi'(0) > 0 \);
2. for some bounded set \( V \subseteq \Omega \) there exists \( a > 0 \) such that for all \( x \in V \) one has
   \[
   Q[x](\lambda, \mu) \geq a(\|\lambda\|^2 + |\mu|)^2 \quad \forall \lambda \in H, \quad \mu \in \mathbb{R}^m.
   \]
3. the functions \( f, g_i, i \in M, \text{ and } F \), as well as their first and second order Fréchet derivatives, are bounded on \( V \).

Then for all \( K > 0 \) and \( \gamma \in \mathbb{R} \), and any bounded set \( \Lambda \subseteq H \times \mathbb{R}^m \) there exists \( c_\gamma > 0 \) such that for all \( c \geq c_\gamma \) and \( (x, \lambda, \mu) \in (V \times \Lambda) \cap S_c(\gamma) \) the following inequality holds true:

\[
\| \nabla \mathcal{L}(x, \lambda, \mu, c) \| \geq K \left( \| F(x) \| + \max \left\{ g(x), -\frac{p(x, \mu)}{c} \mu \right\} \right). \tag{10}
\]

In particular, if the assumptions of the theorem are satisfied for \( V = \Omega_r(\gamma + \varepsilon) \) with some \( r > 0, \varepsilon > 0 \), and \( \gamma \in \mathbb{R} \), then for all \( K > 0 \) there exists \( c_\gamma > 0 \) such that inequality (10) holds true for all \( c \geq c_\gamma \) and \( (x, \lambda, \mu) \in S_c(\gamma) \).

We divide the proof of this theorem into three lemmas. We start with a somewhat cumbersome technical lemma, which is the core part of the proof of Theorem 4.1.

**Lemma 4.2:** Under the assumptions of Theorem 4.1 for all \( K > 0 \) and \( \gamma \in \mathbb{R} \), and any bounded set \( \Lambda \subseteq H \times \mathbb{R}^m \) there exist \( \varkappa > 0 \) and \( c_\gamma > 0 \) such that for any \( c \geq c_\gamma \) and all \( \xi = (x, \lambda, \mu) \in (V \times \Lambda) \cap S_c(\gamma) \) satisfying the inequality

\[
\left\| \left( \nabla_\lambda \mathcal{L}(\xi, c), \nabla_\mu \mathcal{L}(\xi, c) \right) \right\| \leq K \left( \| F(x) \| + \max \left\{ g(x), -\frac{p(x, \mu)}{c} \mu \right\} \right) \quad \tag{11}
\]

one has

\[
\| \nabla_x \mathcal{L}(x, \lambda, \mu, c) \| \geq \varkappa \left( \| F(x) \| + \max \left\{ g(x), -\frac{p(x, \mu)}{c} \mu \right\} \right). 
\]
\textbf{Proof:} Fix any $K > 0$, $\gamma \in \mathbb{R}$, and bounded set $\Lambda \subset H \times \mathbb{R}^m$. By Corollary 3.4 and the second assumption of Theorem 4.1, for any $x \in V$ one has
\begin{equation}
\|\mathcal{E}(x)\mathcal{E}^*(\lambda, \mu)\| \geq \sqrt{2a}\|\lambda, \mu\| \quad \forall \lambda \in H, \quad \mu \in \mathbb{R}^m.
\end{equation}
With the use of Proposition 3.1 and the definition of $\mathcal{E}(x)$ from Corollary 3.4 one can readily verify that
\begin{equation}
\begin{pmatrix}
\nabla_{\lambda} \mathcal{L}(\xi, c) \\
\nabla_{\mu} \mathcal{L}(\xi, c)
\end{pmatrix}
= \begin{pmatrix}
\max \left\{ g(x), -\frac{p(x, \mu)}{c} - \mu \right\} + \frac{c}{b(x)} \max \left\{ g(x), -\frac{p(x, \mu)}{c} - \mu \right\}^2 \\
\mathcal{E}(x)\mathcal{E}^*(\lambda, \mu) \nabla g(x)\nabla x L(\lambda, \mu) + \text{diag}(g_i(x)^2) \mu
\end{pmatrix}
\end{equation}
for any $\xi = (x, \lambda, \mu) \in \Omega \times H \times \mathbb{R}^m$ and $c > 0$. Recall that by our assumption the function $\phi$ is continuously differentiable, $\phi(0) = 0$, and the nonlinear operator $F$ is bounded on $V$. Consequently, there exists $\phi_{\max} > 0$ such that for all $x \in V$ one has $\phi(\|F(x)\|^2) \leq \phi_{\max}\|F(x)\|^2$. Hence applying (13), (12), and the definition of $\eta$ (see (2)), one obtains that
\begin{equation}
\sqrt{2\eta(x, \lambda, \mu)}
\leq \frac{1}{\sqrt{2a}} \left[ \left( K + 1 + c\phi_{\max}\|F(x)\|\|\lambda\| \right) \|F(x)\| \\
+ \left( K + 1 + \frac{c|\mu|}{b(x)} \max \left\{ g(x), -\frac{p(x, \mu)}{c} - \mu \right\} \right) \right]
\end{equation}
for any $x \in V$, $\lambda \in H$, $\mu \in \mathbb{R}^m$, and $c > 0$ satisfying inequality (11).
Let us now estimate the norm of $\nabla_{\lambda} \mathcal{L}(x, \lambda, \mu, c)$. To this end, fix any $\xi = (x, \lambda, \mu) \in \Omega \times H \times \mathbb{R}^m$ and $c > 0$, and consider the functions
\begin{equation}
DF(x)[\nabla_{\lambda} \mathcal{L}(x, \lambda, \mu, c)], \quad \nabla g(x)[\nabla_{\lambda} \mathcal{L}(x, \lambda, \mu, c)].
\end{equation}
Applying Proposition 3.1, and adding and subtracting
\begin{equation}
w(x, \mu, c) := \text{diag}(g_i(x)^2) \left( \mu + \frac{c}{p(x, \mu)} \max \left\{ g(x), -\frac{1}{c}p(x, \mu) \mu \right\} \right)
\end{equation}
in the second row, one obtains that
\begin{equation}
\begin{pmatrix}
DF(x)[\nabla_{\lambda} \mathcal{L}(\xi, c)] \\
\nabla g(x)[\nabla_{\lambda} \mathcal{L}(\xi, c)]
\end{pmatrix}
= \begin{pmatrix}
DF(x)[\nabla_{\lambda} L(\xi)] \\
\nabla g(x)[\nabla_{\lambda} L(\xi)] + \text{diag}(g_i(x)^2) \mu
\end{pmatrix}
\end{equation}
where the vector \( A_0(x, \mu) \), and the linear operators \( A_1(x, \lambda, \mu) \) and \( A_2(x, \lambda, \mu) \)
are defined via the vectors \( \lambda \) and \( \mu \), the functions \( f, F \), and \( g_i, i \in M \), as well as their first and second order derivatives, and the functions \( \psi(\max\{0, g(x)\}) \) and
\( \nabla \psi(\max\{g(x), 0\}) \).

Let
\[
q(x, \mu, c) = \max\{g(x), -c^{-1}p(x, \mu)\mu\}
\]
As was pointed out in [34] (see equality (A.3)), the following equality holds true:

\[
\text{diag}(g_i(x))\mu = \text{diag}(\mu_i)q(x, \mu, c)
+ \frac{c}{p(x, \mu)} \left( \text{diag} \left( \max \left\{ g_i(x), -\frac{p(x, \mu)}{c} \mu_i \right\} \right) - \text{diag}(g_i(x)) \right) q(x, \mu, c)
\]

(the validity of this equality can be easily verified by considering two cases:
\( g_i(x) \geq c^{-1}p(x, \mu)\mu_i \) and \( g_i(x) < c^{-1}p(x, \mu)\mu_i \)). Therefore

\[
w(x, \mu, c) = \text{diag}(g_i(x)) \left[ \text{diag}(\mu_i)q(x, \mu, c)
+ \frac{c}{p(x, \mu)} \text{diag} \left( \max \left\{ g_i(x), -\frac{1}{c} \frac{p(x, \mu)}{} \mu_i \right\} \right) q(x, \mu, c) \right].
\]

Hence with the use of (15) and (12) one gets that

\[
\left( \|DF(x)\| + \|\nabla g(x)\| \right) \| \nabla_x \mathcal{L}(x, \lambda, \mu, c) \| \geq -\sqrt{2a} \left( 1 + \|\lambda\|^2 \phi'(\|F(x)\|^2) \right) \|F(x)\| + \frac{1}{p(x, \mu)} \|q(x, \mu, c)\|
\]

\[
\left[ (1 + \|\lambda\|^2) \phi'(\|F(x)\|^2) \|F(x)\| + \frac{1}{p(x, \mu)} \|q(x, \mu, c)\| \right]
\]

\[
- |\mu| |g(x)| |q(x, \mu, c)| - c \left( \frac{|g(x)|}{p(x, \mu)} + \|A_0(x, \mu)\| \right) |q(x, \mu, c)|^2
\]

\[
- \|A_1(x, \lambda, \mu)\| \|DF(x)[\nabla_x L(x, \lambda, \mu)]\|
\]

\[
- \|A_2(x, \lambda, \mu)\| \|\nabla g(x)[\nabla_x L(x, \lambda, \mu)] + \text{diag}(g_i(x)^2)\mu\|
\]

for all \( x \in V, \lambda \in H, \mu \in \mathbb{R}^m \), and \( c > 0 \).

By our assumptions the functions \( f, g_i, i \in M \), and \( F \), as well as their first
and second order derivatives are bounded on the set \( V \). Consequently, one can
find $S_0, S_1, S_2, S_D, S_\mu, S_g > 0$ such that for any $x \in V$ and for any $(\lambda, \mu)$ from the bounded set $\Lambda \subset H \times \mathbb{R}^m$ one has

$$\|A_0(x, \mu)\| \leq S_0, \quad \|A_1(x, \lambda, \mu)\| \leq S_1, \quad \|A_2(x, \lambda, \mu)\| \leq S_2$$

$$\|DF(x)\| + \|\nabla g(x)\| \leq S_D, \quad |\mu| \leq S_\mu, \quad |g(x)| \leq S_g.$$ 

Moreover, from the convexity of $\phi$ it follows that $\phi'(\|F(x)\|^2) \geq \phi'(0)$ for any $x \in X$. Hence for any $(x, \lambda, \mu) \in V \times \Lambda$ and for all $c > 0$ one has

$$S_D \|\nabla_x \mathcal{L}(x, \lambda, \mu, c)\| \geq \frac{c \sqrt{2a}}{\max\{\psi(0), 1/\phi'(0)\}} \left(\|F(x)\| + |q(x, \mu, c)|\right) - S_\mu S_g |q(x, \mu, c)| - c \left(\frac{S_g (1 + S_\mu^2)}{b(x)} + S_0\right) |q(x, \mu, c)|^2 - (1 + S_1 + S_2) \sqrt{2\eta(x, \lambda, \mu)},$$

which with the use of (14) implies that

$$\|\nabla_x \mathcal{L}(x, \lambda, \mu, c)\| \geq \frac{c}{S_D} t(x, \mu, c) \left(\|F(x)\| + \max \left\{ g(x), -\frac{1}{c} p(x, \mu) \mu \right\} \right),$$

where

$$t(x, \mu, \lambda, c) = \frac{\sqrt{2a}}{\max\{\psi(0), 1/\phi'(0)\}} - \frac{S_\mu S_g}{c} - \left(\frac{S_g (1 + S_\mu^2)}{b(x)} + S_0\right) |q(x, \mu, c)| - \frac{(1 + S_1 + S_2)}{\sqrt{2a}} \times \left(\frac{K + 1}{c} + \phi_{\max} \|F(x)\| \|\lambda\| + \frac{S_\mu}{b(x)} |q(x, \mu, c)| \right).$$

Let us check that there exist $c_0 > 0$ and $t_0 > 0$ such that for all $c \geq c_0$ and $(x, \lambda, \mu) \in (V \times \Lambda) \cap S_c(\gamma)$ one has $t(x, \mu, \lambda, c) \geq t_0$. Then putting $\kappa = t_0/S_D$ one obtains the required result.

To prove the existence of $c_0$ and $t_0$, it is sufficient to show that for any $\varepsilon > 0$ there exists $\widehat{\varepsilon} > 0$ such that for all $c \geq \widehat{\varepsilon}$ and $(x, \lambda, \mu) \in S_c(\gamma)$ with $x \in V$ the following inequalities hold true:

$$\|F(x)\| < \varepsilon, \quad \max \left\{ g(x), -\frac{1}{c} p(x, \mu) \mu \right\} < \varepsilon, \quad b(x) \geq \frac{\psi(0)}{2}.$$ 

Let us prove the existence of such $\widehat{\varepsilon}$.

Fix any $\varepsilon > 0$. By our assumption the function $f$ is bounded on $V$. Consequently, by Lemma 3.2 for any $c > 0$ and $(x, \lambda, \mu) \in S_c(\gamma)$ with $x \in V$ one
has
\[
\gamma \geq \mathcal{L}(x, \lambda, \mu, c) \geq f_0 + \frac{c\phi'(0)}{2} \|F(x)\|^2 + \frac{c}{2\psi(0)} \max \left\{ g(x), -\frac{1}{c}p(x, \mu) \right\}^2 - \frac{1}{2c} \left( \frac{1}{\phi'(0)} + \psi(0) \right),
\]
where \( f_0 = \inf \{ f(x) \mid x \in V \} \). Therefore, for any
\[
c \geq \max \left\{ \frac{1}{2} \left( \frac{1}{\phi'(0)} + \psi(0) \right), \frac{2(\gamma + 1 - f_0)}{\varepsilon \phi'(0)}, \frac{2(\gamma + 1 - f_0)\psi(0)}{\varepsilon^2} \right\}
\]
and for all \((x, \lambda, \mu) \in S_c(\gamma)\) with \(x \in V\) the first two inequalities in (16) hold true.

Note that from the second inequality in (16) it follows that \( \max \{ g_i(x), 0 \} < \varepsilon \) for all \( i \in M \). Consequently, decreasing \( \varepsilon > 0 \), if necessary, one can suppose that \( b(x) := \psi(\max \{ g(x), 0 \}) \geq \psi(0)/2 \), since by definition zero is a point of global maximum of the function \( \psi \).

**Lemma 4.3:** Under the assumptions of Theorem 4.1 for all \( K > 0 \) and \( \gamma > 0 \), and any bounded set \( \Lambda \subset H \times \mathbb{R}^m \) one can find \( c_\ast > 0 \) such that inequality (10) is satisfied for all \( c \geq c_\ast \) and \((x, \lambda, \mu) \in (V \times \Lambda) \cap S_c(\gamma)\).

**Proof:** Arguing by reductio ad absurdum, suppose that there exist \( K > 0 \), \( \gamma \in \mathbb{R} \), and a bounded set \( \Lambda \subset H \times \mathbb{R}^m \) such that for any \( c > 0 \) one can find \( \xi_c = (x_c, \lambda_c, \mu_c) \in (V \times \Lambda) \cap S_c(\gamma) \) satisfying the inequality
\[
\| \nabla \mathcal{L}(\xi_c, c) \| < K_c := K \left( \| F(x_c) \| + \left\| \max \left\{ g(x_c), -\frac{1}{c}p(x_c, \mu_c) \right\} \right\| \right)
\]
Then for any \( c > 0 \) one has
\[
\| \nabla_x \mathcal{L}(\xi_c, c) \| < K_c, \quad \| \nabla_{\lambda, \mu} \mathcal{L}(\xi_c, c) \| < K_c.
\]
From the second inequality and Lemma 4.2 it follows that for any sufficiently large \( c > 0 \) one has
\[
\| \nabla_x \mathcal{L}(\xi_c, c) \| \geq c\varkappa \left( \| F(x_c) \| + \max \left\{ g(x_c), -\frac{1}{c}p(x_c, \mu_c) \right\} \right)
\]
for some \( \varkappa > 0 \) independent of \( c \). However, for any \( c > K/\varkappa \) this inequality contradicts the first inequality in (17). Thus, the statement of Theorem 4.1 is true.

**Lemma 4.4:** Let the assumptions of Theorem 4.1 be satisfied for \( V = \Omega_r(\gamma + \varepsilon) \) with some \( r > 0, \varepsilon > 0 \), and \( \gamma \in \mathbb{R} \). Then for all \( K > 0 \) there exists \( c_\ast > 0 \) such that inequality (10) holds true for all \( c \geq c_\ast \) and \((x, \lambda, \mu) \in S_c(\gamma)\).
Proof: If the assumptions of Theorem 4.1 are satisfied for $V = \Omega_{r}(\gamma + \epsilon)$, then by Theorem 3.5 there exists $c_0 > 0$ such that the set $S_c(\gamma)$ is bounded for all $c \geq c_0$. Moreover, as was shown in the proof of Theorem 3.5, in this case there exists $\hat{c} > 0$ such that for all $c \geq \hat{c}$ and $(x, \lambda, \mu) \in S_c(\gamma)$ one has $x \in \Omega_{r}(\gamma + \epsilon)$. Therefore, one can find a bounded set $\Lambda_1 \subset H \times \mathbb{R}^m$ such that $S_c(\gamma) \subseteq V \times \Lambda$ for all $c \geq \max\{c_0, \hat{c}\}$, which by the previous lemma implies that for all $K > 0$ there exists $c_* > 0$ such that inequality (10) holds true for all $c \geq \max\{c_*, c_0, \hat{c}\}$ and $(x, \lambda, \mu) \in S_c(\gamma)$. ■

5. Exactness of the augmented Lagrangian

This section is devoted to an analysis of several concepts of exactness of the augmented Lagrangian $L(x, \lambda, \mu, c)$. Namely, we present various types of sufficient conditions for this augmented Lagrangian to be locally, globally or completely exact. These conditions are based either on the nonlocal constraint qualification introduced in the previous sections and the use of the gradient estimate from Theorem 4.1 or second order sufficient optimality conditions.

5.1. Global exactness

To give a precise definition of what is meant by ‘exactness’ of the augmented Lagrangian $L(x, \lambda, \mu, c)$, consider the following auxiliary unconstrained optimization problem:

$$\min_{(x, \lambda, \mu)} L(x, \lambda, \mu, c). \tag{18}$$

Under the assumptions of Theorem 3.5, the sublevel set $S_c(\gamma)$ of the augmented Lagrangian is bounded, which implies that auxiliary problem (18) has globally optimal solutions, provided the function $(x, \lambda, \mu) \mapsto L(x, \lambda, \mu, c)$ is weakly sequentially lower semicontinuous and $c > 0$ is sufficiently large. We would like to know how these optimal solutions are connected with globally optimal solutions of the original problem $(P)$.

Suppose that for any globally optimal solution $x_*$ of the problem $(P)$ there exist $\lambda_* \in H$ and $\mu_* \in \mathbb{R}^m$ such that the triplet $(x_*, \lambda_*, \mu_*)$ satisfies the KKT optimality conditions:

$$\nabla_x L(x_*, \lambda_*, \mu_*) = 0, \quad F(x_*) = 0, \quad \max\{g(x_*), -\mu_*\} = 0.$$ 

Any triplet $(x_*, \lambda_*, \mu_*)$ satisfying these equalities is called a KKT-point of the problem $(P)$.

Definition 5.1: One says that the augmented Lagrangian $L(x, \lambda, \mu, c)$ is globally exact, if there exists $c_* > 0$ such that for all $c \geq c_*$ a triplet $(x_*, \lambda_*, \mu_*)$ is a globally
optimal solution of problem (18) if and only if \( x^* \) is a globally optimal solution of the problem \((P)\) and \((x^*, \lambda^*, \mu^*)\) is a KKT-point of this problem.

Thus, if the augmented Lagrangian \( \mathcal{L}(x, \lambda, \mu, c) \) is globally exact, then globally optimal solutions of auxiliary problem (18) with a sufficiently large value of the penalty parameter \( c \) are precisely KKT-points corresponding to globally optimal solutions of the problem \((P)\). Furthermore, observe that if \((x^*, \lambda^*, \mu^*)\) is a KKT-point of the problem \((P)\), then \( \mathcal{L}(x^*, \lambda^*, \mu^*, c) = f(x^*) \) for all \( c > 0 \) (see (1)). Therefore, if the augmented Lagrangian \( L(x^*, \lambda^*, \mu^*, c) \) is globally exact, then for any \( c > 0 \) large enough optimal value of problem (18) coincides with the optimal value of the problem \((P)\), which we denote by \( f_* \). Recall that by our assumption there exists a globally optimal solution of the problem \((P)\), which implies that \( f_* \) is finite.

Note that in the general case

\[
\inf_{(x,\lambda,\mu)} \mathcal{L}(x, \lambda, \mu, c) \leq f_* \quad \forall c > 0,
\]

since \( \mathcal{L}(x^*, \lambda^*, \mu^*, c) = f_* \) for any globally optimal solution \( x^* \) of the problem \((P)\) and the corresponding Lagrange multipliers \( \lambda^* \) and \( \mu^* \). Let us show that this inequality turns into equality precisely when the augmented Lagrangian \( \mathcal{L}(x, \lambda, \mu, c) \) is globally exact.

Lemma 5.2: Let \( Q(x^*)[\cdot] \) be positive definite at every globally optimal solution \( x^* \) of the problem \((P)\). Then the augmented Lagrangian \( \mathcal{L}(x, \lambda, \mu, c) \) is globally exact if and only if the optimal value of problem (18) is equal to \( f_* \) for some \( c > 0 \).

Proof: Suppose that the optimal value of problem (18) is equal to \( f_* \) for some \( c > 0 \). Our aim is to show that for any \( r > c \) points of global minimum of the augmented Lagrangian \( \mathcal{L} \) are precisely KKT-points of the problem \((P)\) corresponding to its globally optimal solutions. Then by definition \( \mathcal{L}(x, \lambda, \mu, c) \) is globally exact.

Fix any \( r > c \). From (19) and the fact that the function \( \mathcal{L}(x, \lambda, \mu, c) \) is non-decreasing in \( c \) it follows that \( \inf_{(x,\lambda,\mu)} \mathcal{L}(x, \lambda, \mu, r) = f_* \). Consequently, for any globally optimal solution \( x^* \) of the problem \((P)\) and the corresponding Lagrange multipliers \( \lambda^* \) and \( \mu^* \) (which exist, since \( Q(x^*) \) is positive definite by our assumption; see, e.g. [51, Theorem 1.1.4] and Lemma 3.3), the triplet \((x^*, \lambda^*, \mu^*)\) is a point of global minimum of \( \mathcal{L} \) by virtue of the fact that \( \mathcal{L}(x^*, \lambda^*, \mu^*, r) = f(x^*) = f_* \).

Suppose now that \((x^*, \lambda^*, \mu^*)\) is a point of global minimum of \( \mathcal{L} \). Let us check that \((x^*, \lambda^*, \mu^*)\) is a KKT-point and \( x^* \) is a globally optimal solution of the
problem \((\mathcal{P})\). Indeed, observe that the function

\[
\begin{aligned}
\left\{ \mu, \max \left\{ g(x), - \frac{p(x, \mu)}{c - \mu} \right\} \right\} + \frac{c}{2p(x, \mu)} \left| \max \left\{ g(x), - \frac{1}{c} p(x, \mu) \right\} \right| \right|^{2}
\end{aligned}
\]

(20)
is nondecreasing in \(c\). Consequently, if \(F(x_*) \neq 0\), then

\[
\mathcal{L}(x_*, \lambda_*, \mu_*, r)
\]

\[
= f(x_*) + \langle \lambda_*, F(x_*) \rangle + \frac{r}{2} (1 + \| \lambda_* \|^2) \phi(\| F(x_*) \|^2)
\]

\[
+ \left\{ \mu_*, \max \left\{ g(x_*), - \frac{p(x_*, \mu_*)}{r - \mu_*} \right\} \right\}
\]

\[
+ \frac{r}{2p(x_*, \mu_*)} \left| \max \left\{ g(x_*), - \frac{p(x_*, \mu_*)}{r - \mu_*} \right\} \right| \right|^{2} + \eta(x_*, \lambda_*, \mu_*)
\]

\[
\geq \mathcal{L}(x_*, \lambda_*, \mu_*, c) + \frac{(r - c)}{2} (1 + \| \lambda_* \|^2) \phi(\| F(x_*) \|^2)
\]

\[
\geq f_* + \frac{(r - c)}{2} (1 + \| \lambda_* \|^2) \phi(\| F(x_*) \|^2) > f_* = \inf_{(x, \lambda, \mu)} \mathcal{L}(x, \lambda, \mu, r).
\]

Therefore \(F(x_*) = 0\). Arguing in a similar way and applying the fact that the function (20) is strictly increasing in \(c\), if \(\max\{g(x), -\mu\} \neq 0\), one can easily check that \(\max\{g(x_*), -\mu_*\} = 0\), which implies that \(x_*\) is a feasible point of the problem \((\mathcal{P})\). Hence one has

\[
f_* = \inf_{(x, \lambda, \mu)} \mathcal{L}(x, \lambda, \mu, r) = \mathcal{L}(x_*, \lambda_*, \mu_*, r) = f(x_*) + \eta(x_*, \lambda_*, \mu_*)
\]

which implies that \(f(x_*) = f_*\) and \(\eta(x_*, \lambda_*, \mu_*) = 0\), since the function \(\eta\) is non-negative. Thus, \(x_*\) is a globally optimal solution of the problem \((\mathcal{P})\), and it remains to check that \((x_*, \lambda_*, \mu_*)\) is a KKT-point of the problem \((\mathcal{P})\).

Observe that for any \(\lambda \in H\) and \(\mu \in \mathbb{R}^m\) one has

\[
\eta(x_*, \lambda, \mu) = \frac{1}{2} \left\| \mathcal{E}(x_*) \mathcal{E}(x_*)^* \left( \frac{\lambda}{\mu} \right) + \frac{DF(x_*)}{\nabla g(x_*)} \left[ \nabla f(x_*) \right] \right\|^{2}
\]

(see Corollary 3.4). Therefore, \(\eta(x_*, \lambda, \mu) = 0\) if and only if

\[
\mathcal{E}(x_*) \mathcal{E}(x_*)^* \left( \frac{\lambda}{\mu} \right) = - \frac{DF(x_*)}{\nabla g(x_*)} \left[ \nabla f(x_*) \right].
\]

By Corollary 3.4 the operator \(\mathcal{E}(x_*) \mathcal{E}(x_*)^*\) is invertible, which implies that the equation above has a unique solution. Therefore, \(\eta(x_*, \lambda, \mu) = 0\) if and only if \(\lambda = \lambda_*\) and \(\mu = \mu_*\).

As was noted above, from the fact that \(Q(x_*)[\cdot]\) is positive definite it follows that there exists Lagrange multipliers \(\lambda_0 \in H\) and \(\mu_0 \in \mathbb{R}^m\) such that \((x_*, \lambda_0, \mu_0)\)
is a KKT-point. By the definition of \( \eta \) (see (2)) one has \( \eta(x_*, \lambda_0, \mu_0) = 0 \), which implies that \( \lambda_0 = \lambda_*, \mu_0 = \mu_* \), and \( (x_*, \lambda_*, \mu_*) \) is a KKT-point.

Let us obtain several types of sufficient conditions for the global exactness of the augmented Lagrangian \( \mathcal{L}(x, \lambda, \mu, c) \). We start with, perhaps, the most general conditions based on the direct usage of Theorem 4.1. For the sake of completeness, we will explicitly formulate all assumptions of the following theorem, although most of them coincide with the assumptions of Theorem 4.1 with \( V = \Omega_r(f_* + \varepsilon) \).

**Theorem 5.3:** Let the following assumptions be valid:

1. \( f, F, \) and \( g_i, i \in M \), are twice continuously Fréchet differentiable on \( \Omega \), \( \phi \) is continuously differentiable on its effective domain, and \( \phi'(0) > 0 \);
2. the function \( \mathcal{L}(\cdot, c) \) is weakly sequentially l.s.c. for all \( c > 0 \);
3. the set \( \Omega_r(f_* + \varepsilon) = \{ x \in \Omega \mid f(x) + r(\|F(x)\|^2 + |\max[g(x), 0]|^2) \leq f_* + \varepsilon \} \) is bounded for some \( r > 0 \) and \( \varepsilon > 0 \);
4. the functions \( f, g_i, i \in M, \) and \( F \), as well as their first and second order Fréchet derivatives, are bounded on \( \Omega_r(f_* + \varepsilon) \);
5. there exists \( a > 0 \) such that for all \( x \in \Omega_r(f_* + \varepsilon) \) one has
   \[
   Q(x)[\lambda, \mu] \geq a(\|\lambda\|^2 + |\mu|^2) \quad \forall \lambda \in H, \quad \mu \in \mathbb{R}^m.
   \]

Then the augmented Lagrangian \( \mathcal{L}(x, \lambda, \mu, c) \) is globally exact.

**Proof:** By Theorems 3.5 and 4.1 there exists \( c_* > 0 \) such that for all \( c \geq c_* \) the set \( S_c(f_*) \) is bounded and

\[
\|\nabla \mathcal{L}(x, \lambda, \mu, c)\| \geq \|F(x)\| + \max \left\{ g(x), -\frac{1}{c} \rho(x, \mu) \right\}
\]

(21)

for all \( (x, \lambda, \mu) \in S_c(f_*) \).

Taking into account inequality (19) and the facts that \( \mathcal{L}(\cdot, c) \) is weakly sequentially l.s.c., and \( X \) and \( H \) are Hilbert spaces, one can conclude that for any \( c \geq c_* \) the function \( \mathcal{L}(\cdot, c) \) attains a global minimum at a point \( (x(c), \lambda(c), \mu(c)) \in S_c(f_*) \). From the fact that the augmented Lagrangian is Fréchet differentiable on its effective domain by Proposition 3.1 it follows that \( \nabla \mathcal{L}(x(c), \lambda(c), \mu(c), c) = 0 \) for all \( c \geq c_* \). Therefore, by (21) the point \( x(c) \) is feasible for the problem (P) and

\[
\mathcal{L}(x(c), \lambda(c), \mu(c), c) = f(x(c)) + \eta(x(c), \lambda(c), \mu(c)) \geq f(x(c)) \geq f_*
\]

for any \( c \geq c_* \). Hence bearing in mind (19) and Lemma 5.2 one can conclude that the augmented Lagrangian \( \mathcal{L}(x, \lambda, \mu, c) \) is globally exact.

The most restrictive assumption of the previous theorem is, of course, the assumption on the uniform positive definiteness of the quadratic function \( Q(x)[\cdot] \).
on the set $\Omega_r(f_* + \varepsilon)$, which can be viewed as a uniform nonlocal constraint qualification or an assumption on the nonlocal metric regularity of constraints. In various particular cases, one can significantly relax this assumption and replace it with a local constraint qualification. Here we present two simple results of this kind, merely to illustrate how one can use a particular structure/properties of the problem under consideration to relax the assumptions of Theorem 5.3.

Being inspired by the ideas of Zaslavsky [7–9] on the theory of exact penalty functions, first we strengthen Theorem 5.3 with the use of the Palais-Smale condition [52]. To introduce a suitable version of this condition, denote by $a_{\text{max}}(Q(x))$ the supremum of all $a \geq 0$ such that $Q(x)[\cdot]$ is positive definite with constant $a$. As is easily seen,

$$a_{\text{max}}(Q(x)) = \inf \left\{ Q(x)[\lambda, \mu] \left| \| (\lambda, \mu) \| = 1 \right. \right\},$$

and in the finite dimensional case $a_{\text{max}}(Q(x))$ is simply the smallest eigenvalue of the matrix of the quadratic form $Q(x)[\cdot]$. Moreover, with the use of the equality above and the definition of $Q(x)[\cdot]$ one can check that the value $a_{\text{max}}(Q(x))$ continuously depends on $x$.

**Definition 5.4:** One says that the constraints of the problem $(\mathcal{P})$ satisfy the Palais-Smale condition, if every bounded sequence $\{x_n\} \subset X$, such that

1. $\|F(x_n)\| + \max\{g(x_n), 0\} \to 0$ as $n \to \infty$,
2. $a_{\text{max}}(Q(x_n)) \to 0$ as $n \to \infty$,

has a convergent subsequence.

To understand how the above definition is connected with the traditional Palais-Smale condition, consider the simplest case when there are no inequality constraints and $H = \mathbb{R}$. Then, as one can readily verify,

$$Q(x)[\lambda] = \frac{1}{2} |\nabla F(x)|^4 \lambda^2, \quad a_{\text{max}}(Q(x)) = \frac{1}{2} |\nabla F(x)|^4.$$

Therefore, in this case the constraint $F(x) = 0$ satisfies the Palais-Smale condition from Definition 5.4, if every bounded sequence $\{x_n\}$, such that $F(x_n) \to 0$ and $\|\nabla F(x_n)\| \to 0$ as $n \to \infty$, has a convergent subsequence. This is the well-known localized Palais-Smale condition [52].

With the use of the Palais-Smale condition from Definition 5.4 we can significantly relax the nonlocal constraint qualification from Theorem 5.3.

**Theorem 5.5:** Let Assumption 1–4 of Theorem 5.3 be valid, and suppose that the constraints of the problem $(\mathcal{P})$ satisfy the Palais-Smale condition and the function $Q(x)[\cdot]$ is positive definite at every globally optimal solution of the problem $(\mathcal{P})$. Then the augmented Lagrangian $\mathcal{L}(x, \lambda, \mu, c)$ is globally exact.
**Proof:** We divide the proof of the theorem into two parts. First we show that the sublevel set $S_c(f_\ast)$ is bounded for any sufficiently large $c$, and then with the use of Theorem 4.1 and the Palais-Smale conditions we will prove the global exactness of the augmented Lagrangian.

**Part 1.** Arguing by reductio ad absurdum, suppose that the set $S_c(f_\ast)$ is unbounded for any $c > 0$. Then for any increasing unbounded sequence $\{c_n\} \subset (0, +\infty)$ one can find $(x_n, \lambda_n, \mu_n) \in S_{c_n}(f_\ast)$, $n \in \mathbb{N}$, such that $\|x_n\| + \|\lambda_n\| + |\mu_n| \geq n$.

As was shown in the proof of Theorem 3.5, there exists $\hat{c} > 0$ such that for any $c \geq \hat{c}$ and $(x, \lambda, \mu) \in S_c(f_\ast)$ one has $x \in \Omega_r(f_\ast + \varepsilon)$. Therefore, the sequence $\{x_n\}$ is bounded and $\|\lambda_n\| + |\mu_n| \to +\infty$ as $n \to \infty$.

Suppose that there exist $a > 0$ and a subsequence $\{x_{n_k}\}$ such that for all $k \in \mathbb{N}$ one has $a_{\text{max}}(Q(x_{n_k})) \geq a$. Then replacing the sequence $\{x_n\}$ with this subsequence and almost literally repeating the proof of Theorem 3.5 one check that the condition $\|\lambda_n\| + |\mu_n| \to +\infty$ as $n \to \infty$ contradicts the assumption that $(x_n, \lambda_n, \mu_n) \in S_c(f_\ast)$.

Thus, without loss of generality one can suppose that $a_{\text{max}}(Q(x_n)) \to 0$ as $n \to \infty$. By Lemma 3.2 and the definition of $(x_n, \lambda_n, \mu_n)$ one has

$$f(x_n) + \frac{c_n}{2} \phi(\|F(x_n)\|^2) + \frac{c_n}{2\psi(0)} \max\{|g(x_n), 0|\}^2 \leq f_\ast + \frac{1}{2c_n\phi_0} + \frac{(1 + m)\psi(0)}{2c_n}.$$ 

Therefore, as one can readily verify, one has

$$\lim_{n \to \infty} \left( \|F(x_n)\| + |\max\{g(x_n), 0\}| \right) = 0, \quad \limsup_{n \to \infty} f(x_n) \leq f_\ast. \quad (22)$$

Hence by the Palais-Smale condition one can extract a subsequence $\{x_{n_k}\}$ converging to some point $x_\ast$, which is obviously feasible and satisfies the inequality $f(x_\ast) \leq f_\ast$. Consequently, $x_\ast$ is a globally optimal solution of the problem (P), which by our assumption implies that $a_\ast := a_{\text{max}}(Q(x_\ast)) > 0$. As was noted above, the function $a_{\text{max}}(Q(\cdot))$ is continuous. Therefore, $a_{\text{max}}(Q(x_{n_k})) \geq a_\ast/2$ for all sufficiently large $k$, which contradicts the fact that $a_{\text{max}}(Q(x_n)) \to 0$ as $n \to \infty$. Thus, there exists $c_0 > 0$ such that the set $S_c(f_\ast)$ is bounded for all $c \geq c_0$.

**Part 2.** Bearing in mind inequality (19), and the facts that $\mathcal{L}(\cdot, c)$ is weakly sequentially l.s.c. and the set $S_c(f_\ast)$ is bounded for all $c \geq c_0$, one can conclude that the augmented Lagrangian $\mathcal{L}(\cdot, c)$ attains a global minimum at a point $(x(c), \lambda(c), \mu(c)) \in S_c(f_\ast)$ for any $c \geq c_0$.

Choose an increasing unbounded sequence $\{c_n\} \subset [c_0, +\infty)$, and denote $\xi_n = (x_n, \lambda_n, \mu_n) = (x(c_n), \lambda(c_n), \mu(c_n))$. Observe that the sequence $\{(x_n, \lambda_n, \mu_n)\} \subset S_{c_n}(f_\ast)$ is bounded, since $S_c(f_\ast) \subseteq S_{c_0}(f_\ast)$ for any $c \geq c_0$ due to the fact that the function $\mathcal{L}(x, \lambda, \mu, c)$ is nondecreasing in $c$. Note that increasing $c_0$, if necessary, one can suppose that $\{x_n\} \subset \Omega_r(f_\ast + \varepsilon)$ for all $n \in \mathbb{N}$, thanks to Lemma 3.2.
Suppose, at first, that there exist \( a > 0 \) and a subsequence \( \{x_{n_k}\} \) such that \( a_{\text{max}}(Q(x_{n_k})) \geq a \) for all \( k \in \mathbb{N} \). Then setting \( V = \{x_{n_k}\} \), \( \Lambda = \{(\lambda_{n_k}, \mu_{n_k})\} \), \( \gamma = f_s \), and \( K = 1 \) in Theorem 4.1, one obtains that there exists \( k_0 \in \mathbb{N} \) such that for any \( k \geq k_0 \) the following inequality holds true:

\[
0 = \| \nabla \mathcal{L}(\xi_{n_k}, c_{n_k}) \| \geq \| F(x_{n_k}) \| + \max \left\{ g(x_{n_k}), -\frac{p(x_{n_k}, \mu_{n_k})}{c_{n_k}} - \mu_{n_k} \right\}
\]

(the first equality follows from the fact that \( \xi_{n_k} = (x_{n_k}, \lambda_{n_k}, \mu_{n_k}) \) is a point of global minimum of \( \mathcal{L}(\cdot, c_{n_k}) \) by definition). Hence arguing in the same way as in the proof of Theorem 5.3, one can conclude that the augmented Lagrangian is globally exact.

Thus, one can suppose that the sequence \( \{a_{\text{max}}(Q(x_n))\} \) does not contain a subsequence that is bounded away from zero. Hence without loss of generality one can assume that \( a_{\text{max}}(Q(x_n)) \to 0 \) as \( n \to \infty \).

Applying the lower estimate of the augmented Lagrangian from Lemma 3.2 and the fact that \( (x_n, \lambda_n, \mu_n) \in S_{c_n}(f_*) \), one can easily check that

\[
\lim_{n \to \infty} \left( \| F(x_n) \| + \max \{ g(x_n), 0 \} \right) = 0, \quad \limsup_{n \to \infty} f(x_n) \leq f_*.
\]

Consequently, by the Palais-Smale condition there exists a subsequence \( \{x_{n_k}\} \) converging to a point \( x_* \), which is obviously a globally optimal solution of the problem \((P)\). Therefore, by our assumption \( Q(x_*)[\cdot] \) is positive definite, and due to the continuity of the function \( a_{\text{max}}(Q(\cdot)) \) there exist \( a_* > 0 \) and \( k_* \in \mathbb{N} \) such that \( a_{\text{max}}(Q(x_{n_k})) \geq a_* \) for all \( k \geq k_* \), which contradicts our assumption that \( a_{\text{max}}(Q(x_n)) \to 0 \) as \( n \to \infty \).

Let us also consider another way one can relax the nonlocal constraint qualification from Theorem 5.3 with the use of a particular structure of the constraints, which can be applied, e.g. to variational problems with nonlinear constraints at the boundary of the domain. Namely, let \( Y \) be a real Hilbert space, and suppose that the constraints of the problem \((P)\) have the form

\[
F(x) = F_0(\mathcal{A}x), \quad g(x) = g_0(\mathcal{A}x) \quad \forall x \in X, \tag{24}
\]

where \( F_0 : Y \to H \) and \( g_0 : Y \to \mathbb{R}^m \) are continuously differentiable nonlinear maps, while \( \mathcal{A} : X \to Y \) is a compact linear operator. Thus, the constraints are defined via a compact embedding of the space \( X \) into another Hilbert space \( Y \).

For any \( y \) introduce the function

\[
Q_0(y)[\lambda, \mu] = \frac{1}{2} \left\| DF_0(y) \left[ DF_0(y)^* [\lambda] + \sum_{i=1}^{m} \mu_i \nabla g_0(y) \right] \right\|^2
\]

\[
+ \frac{1}{2} \left\| \nabla g_0(y) \left( DF_0(y)^* [\lambda] + \sum_{i=1}^{m} \mu_i \nabla g_0(y) \right) + \text{diag}(g_0(y)^2) \mu \right\|^2,
\]
which is a modification of the function $Q(x)[\cdot]$ to the case of the constraints

$$F_0(y) = 0, \quad g_{0i}(y) \leq 0, \quad i \in M.$$  

It is convenient to formulate sufficient conditions for the global exactness of the augmented Lagrangian for the problem under consideration in terms of the function $Q_0(y)$.

**Theorem 5.6:** Let the following assumptions be valid:

1. $f, F_0,$ and $g_0$ are twice continuously Fréchet differentiable, $\phi$ is continuously differentiable on its effective domain, and $\phi'(0) > 0$;
2. the functions $f$ and $\mathcal{L}(\cdot, c)$, $c > 0$, are weakly sequentially l.s.c.;
3. the set $\Omega_r(f_\ast + \varepsilon)$ is bounded for some $r > 0$ and $\varepsilon > 0$;
4. the functions $f, g_i, i \in M$, and $F$, as well as their first and second order Fréchet derivatives, are bounded on $\Omega_r(f_\ast + \varepsilon)$;
5. the operator $A A^*$ is the identity map;
6. for any globally optimal solution $x_\ast$ of the problem (P) the function $Q_0(y)[\cdot]$ is positive definite at the point $y = A x_\ast$.

Then the augmented Lagrangian $\mathcal{L}(x, \lambda, \mu, c)$ is globally exact.

**Proof:** We split the proof of this theorem into two parts, in precisely the same way as the proof of Theorem 5.5.

**Part 1.** Let us prove that under the assumptions of the theorem the sublevel set $S_c(f_\ast)$ is bounded for any sufficiently large $c$. Arguing by reductio ad absurdum, suppose that this claim is false. Then, just like in the proofs of Theorems 3.5 and 5.5, one can show that for any $n \in \mathbb{N}$ there exists $(x_n, \lambda_n, \mu_n) \in S_n(f_\ast)$ such that the sequence $\{x_n\}$ is bounded, but $\|\lambda_n\| + |\mu_n| \to +\infty$ as $n \to \infty$.

From the facts that $X$ is a Hilbert space and the sequence $\{x_n\}$ is bounded it follows that one can extract a subsequence $\{x_{nk}\}$ that weakly converges to some point $x_\ast$. Since the operator $A$ is compact, the sequence $\{A x_{nk}\}$ converges to $A x_\ast$ in the norm topology.

With the use of Lemma 3.2 one can readily verify that

$$\lim_{n \to \infty} \left( \|F(x_n)\| + \left| \max[g(x_n), 0] \right| \right) = 0, \quad \limsup_{n \to \infty} f(x_n) \leq f_\ast.$$  

Hence taking into account the facts that $f$ is weakly sequentially l.s.c., and $A x_{nk}$ converges to $A x_\ast$ in the norm topology one can conclude that $x_\ast$ is a globally optimal solution of the problem (P). Thus, $Q_0(A x_\ast)$ is positive definite. Therefore, there exists $k_0 \in \mathbb{N}$ such that

$$a_{\max}(Q_0(A x_{nk})) \geq \frac{a_\ast}{2} \quad \forall k \geq k_0, \quad a_\ast := a_{\max}(Q_0(A x_\ast)) > 0,$$  

due to the facts that the function $a_{\max}(Q_0(\cdot))$ is continuous in the norm topology and $A x_{nk}$ strongly converges to $A x_\ast$. 

From the definition of $Q(x)[\cdot]$ (see equality (5) on page 10), equalities (24), and the fact that $\mathcal{A}^*$ is the identity map it follows that

$$
Q(x)[\lambda, \mu] = \frac{1}{2} \left\| DF_0(Ax) \left( \mathcal{A}^* \left[ DF_0(Ax)^* [\lambda] + \sum_{i=1}^{m} \mu_i \nabla g_{0i}(Ax) \right] \right) \right\|^2 
+ \frac{1}{2} \left\| \nabla g_0(Ax) \left( \mathcal{A}^* \left( DF_0(Ax)^* [\lambda] + \sum_{i=1}^{m} \mu_i \nabla g_{0i}(Ax) \right) \right) \right\|^2 
+ \text{diag}(g_{0i}(Ax_*^2) \mu)^2 
= Q_0(Ax)[\lambda, \mu]
$$

for all $x \in X, \lambda \in H$ and $\mu \in \mathbb{R}^m$. Consequently, $a_{\text{max}}(Q(x_{n_k})) \geq a_*/2$ for all $k \geq k_0$. Hence applying Lemma 3.2 one obtains that

$$
\mathcal{L}(x_{n_k}, \lambda_{n_k}, \mu_{n_k}, n_k) \geq f(x_{n_k}) - \frac{1}{2n_k} - \frac{(1 + m) \psi(0)}{2n_k} + \eta(x_{n_k}, \lambda_{n_k}, \mu_{n_k}) 
\geq f(x_{n_k}) - \frac{1}{2n_k} - \frac{(1 + m) \psi(0)}{2n_k} + \frac{a_*}{2} \left\| (\lambda_{n_k}, \mu_{n_k}) \right\|^2 
- \|Q_{1,\lambda}(x_{n_k})\|\|\lambda_{n_k}\| - |Q_{1,\mu}(x_{n_k})|\|\mu_{n_k}\| + Q_0(x_{n_k})
$$

for all $k \geq k_0$ (here we used the same notation as in the proof of Theorem 3.5). As was noted multiple times above, $x_n \in \Omega_r(f_* + \varepsilon)$ for any sufficient large $n$. Consequently, the quantities $\|Q_{1,\lambda}(x_{n_k})\|$, $|Q_{1,\mu}(x_{n_k})|$, and $Q_0(x_{n_k})$ are bounded due to our assumption on the boundedness of all functions and their derivatives on $\Omega_r(f_* + \varepsilon)$. Hence one gets that $\mathcal{L}(x_{n_k}, \lambda_{n_k}, \mu_{n_k}, n_k) \to +\infty$ as $k \to \infty$, which contradicts the fact that by definition $\mathcal{L}(x_n, \lambda_n, \mu_n, n) \leq f_*$ for all $n \in \mathbb{N}$. Thus, there exists $c_0 > 0$ such that the sublevel set $S_c(f_*)$ is bounded for all $c \geq c_0$.

**Part 2.** Let us now prove the global exactness of the augmented Lagrangian. Choose an increasing unbounded sequence $\{c_n\} \subseteq [c_0, +\infty)$. From the fact that the sublevel set $S_c(f_*)$ is bounded for all $c \geq c_0$ and the augmented Lagrangian is weakly sequentially l.s.c. it follows that for any $n \in \mathbb{N}$ the function $\mathcal{L}(\cdot, c_n)$ attains a global minimum at a point $(x_n, \lambda_n, \mu_n) \in S_{c_n}(f_*)$.

Note that the sequence $\{(x_n, \lambda_n, \mu_n)\}$ is bounded, due to the fact that $S_{c_n}(f_*) \subseteq S_{c_0}(f_*)$ for all $c \geq c_0$. Therefore, replacing, if necessary, this sequence with a subsequence, one can suppose that the sequence $\{x_n\}$ weakly converges to some point $x_*$. The corresponding sequence $\{Ax_n\}$ strongly converges to $Ax_*$ due to the fact that the operator $\mathcal{A}$ is compact. Hence with the use of the lower estimate from Lemma 3.2 and the fact that $(x_n, \lambda_n, \mu_n) \in S_{c_n}(f_*)$ one can easily verify that $x_*$ is a globally optimal solution of the problem $(P)$. Consequently, the function $Q_0(Ax_*)[\cdot] = Q(x_*)[\cdot]$ is positive definite, and one can find $a > 0$ and $n_0 \in \mathbb{N}$
such that

\[ a_{\max}(Q(x_n)) = a_{\max}(Q_0(Ax_n)) \geq a \quad \forall n \geq n_0, \]

due to the facts that the map \( a_{\max}(Q_0(\cdot)) \) is continuous and \( Ax_n \) strongly converges to \( Ax_\ast \).

Now, applying Theorem 4.1 with \( V = \{x_n\}_{n \geq n_0} \) (one can obviously suppose that \( \{x_n\}_{n \geq n_0} \subseteq \Omega_{\ast}(f_\ast + \epsilon) \), \( \Lambda = \{(\lambda_n, \mu_n)\}_{n \geq n_0}, y = f_\ast \), and \( K = 1 \) one obtains that there exists \( N \geq n_0 \) such that for any \( n \geq N \) one has

\[ 0 = \| \nabla L(x_n, \lambda_n, \mu_n, c_n) \| \geq \| F(x_n) \| + \max \left\{ g(x_n), -\frac{p(x_n, \mu_n)}{c_n} \mu_n \right\} . \]

Therefore, for any \( n \geq N \) the point \( x_n \) is feasible for the problem (\( P \)) and

\[ f_\ast \geq L(x_n, \lambda_n, \mu_n, c_n) = f(x_n) + \eta(x_n, \lambda_n, \mu_n) \geq f(x_n) \geq f_\ast. \]

Hence by Lemma 5.2 the augmented Lagrangian is globally exact. \( \blacksquare \)

**Remark 5.1:** Let us point out one important particular case in which the assumption of the previous theorem that \( AA^* \) is the identity map holds true. Namely, let \( X \) be the Sobolev space \( H^1([a, b]; \mathbb{R}^d) \) of vector-valued functions \( x: [a, b] \to \mathbb{R}^d \) endowed with the inner product

\[ \langle x, y \rangle_X = \langle x(a), y(a) \rangle + \langle x(a) + x(b), y(a) + y(b) \rangle + (b - a) \int_a^b \langle \dot{x}(t), \dot{y}(t) \rangle \, dt \]

and the corresponding norm, which is equivalent to the standard norm on \( H^1([a, b]; \mathbb{R}^d) \). Suppose also that \( Y \) is the space \( \mathbb{R}^d \times \mathbb{R}^d \) endowed with the inner product

\[ \langle (x_1, x_2), (y_1, y_2) \rangle_Y = 3\langle x_1, y_1 \rangle + 2\langle x_2, y_2 \rangle \quad \forall (x_1, x_2), (y_1, y_2) \in Y. \]

Let \( Ax = (x(a), x(b)) \). In this case, the constraints \( F(x) = 0 \) and \( g(x) \leq 0 \) restrict the values of the function \( x \) at the boundary points \( t = a \) and \( t = b \). As is easy seen, one has

\[ (A^*y_1, y_2)(t) = y_1 + (y_2 - y_1) \frac{t - a}{b - a} \quad \forall t \in [a, b], \quad (y_1, y_2) \in Y, \]

since for all \( x \in X \) and \( y = (y_1, y_2) \in Y \) the following equalities hold true:

\[ \langle A^*y, x \rangle_X = \langle y_1, x(a) \rangle + \langle y_1 + y_2, x(a) + x(b) \rangle + \int_a^b \langle y_2 - y_1, \dot{x}(t) \rangle \, dt \]

\[ = \langle y_1, x(a) \rangle + \langle y_1 + y_2, x(a) + x(b) \rangle + \langle y_2 - y_1, x(b) - x(a) \rangle \]

\[ = 3\langle y_1, x(a) \rangle + 2\langle y_2, x(b) \rangle = \langle y, Ax \rangle_Y. \]

It remains to note that in this case \( AA^* \) is indeed the identity map. Moreover, the operator \( A \) is obviously compact, which allows one to apply the previous theorem to corresponding problems.
5.2. Complete exactness

In many cases, optimization methods can find only points of local minimum or even only stationary (critical) points of a nonconvex function. Therefore, apart from global exactness, it is important to have conditions ensuring that not only points of global minimum of the augmented Lagrangian \( L(x, \lambda, \mu, c) \) correspond to points of global minimum of the original problem \((P)\), but also points of local minimum/stationary points of the augmented Lagrangian correspond to points of local minimum/KKT-points of the problem \((P)\). Under such conditions the problem of unconstrained minimization of the augmented Lagrangian \( L(x, \lambda, \mu, c) \) is, in a sense, completely equivalent to the original problem \((P)\). In this case it is natural to call \( L(x, \lambda, \mu, c) \) completely exact (cf. completely exact penalty functions in [49,50]).

The following theorem contains natural sufficient conditions for the complete exactness of the augmented Lagrangian on the sublevel set \( S_c(\gamma) \). The question of whether the complete exactness of this augmented Lagrangian on the entire space \( X \times H \times \mathbb{R}^d \) can be proved under some additional assumptions remains an interesting open problem.

**Theorem 5.7:** Let the assumptions of Theorem 4.1 be satisfied for \( V = \Omega_r(\gamma + \varepsilon) \) with some \( r > 0, \varepsilon > 0, \) and \( \gamma > f_\ast \), and suppose that the augmented Lagrangian \( L(\cdot, c) \) is weakly sequentially l.s.c. for all \( c > 0 \). Then there exists \( c_\ast > 0 \) such that for all \( c \geq c_\ast \) the augmented Lagrangian \( L(x, \lambda, \mu, c) \) is completely exact on the set \( S_c(\gamma) \) in the following sense:

1. the optimal values of the problem \((P)\) and problem (18) coincide;
2. \((x_\ast, \lambda_\ast, \mu_\ast)\) is point of global minimum of \( L(x, \lambda, \mu, c) \) if and only if \( x_\ast \) is a globally optimal solution of the problem \((P)\) and \((x_\ast, \lambda_\ast, \mu_\ast)\) is a KKT-point of this problem;
3. \((x_\ast, \lambda_\ast, \mu_\ast) \in S_c(\gamma)\) is a stationary point of \( L(x, \lambda, \mu, c) \) if and only if \((x_\ast, \lambda_\ast, \mu_\ast)\) is a KKT-point of the problem \((P)\) and \( f(x_\ast) \leq \gamma \);
4. if \((x_\ast, \lambda_\ast, \mu_\ast) \in S_c(\gamma)\) is a point of local minimum of \( L(x, \lambda, \mu, c) \), then \( x_\ast \) is a locally optimal solution of the problem \((P)\), \( f(x_\ast) \leq \gamma \), and \((x_\ast, \lambda_\ast, \mu_\ast)\) is a KKT-point of this problem.

**Proof:** Note that by the definition of global exactness the validity of the first two statements of the theorem follows directly from Theorem 5.3. Let us prove the last two statements of the theorem. We prove the statement about stationary points first, since its proof is simpler than the proof of the statement on locally optimal solutions.

**Part 1.** By Theorem 4.1 there exists \( c_\ast > 0 \) such that for all \( c \geq c_\ast \) and \((x, \lambda, \mu) \in S_c(\gamma)\) the lower estimate of the gradient (10) holds true. Consequently, for any \( c \geq c_\ast \) and any stationary point \( \xi_\ast = (x_\ast, \lambda_\ast, \mu_\ast) \in S_c(\gamma) \) of \( L(x, \lambda, \mu, c) \)
one has
\[ F(x^*) = 0, \quad \max \left\{ g(x^*), -\frac{p(x^*, \mu^*)}{c} - \mu^* \right\} = 0, \tag{25} \]
which implies that
\[ f(x^*) \leq f(x^*) + \eta(x^*, \lambda^*, \mu^*) = \mathcal{L}(x^*, \lambda^*, \mu^*, c) \leq \gamma, \]
and \( x^* \in \Omega_f(\gamma + \varepsilon) \) for any \( c > 0 \). Hence the quadratic function \( Q(x^*)[\cdot] \) is positive definite by our assumption.

Observe that from (25), the equality \( \nabla \mathcal{L}(x^*, \lambda^*, \mu^*, c) = 0 \), and Proposition 3.1 (see also (13)) it follows that
\[ 0 = \begin{pmatrix} \nabla_{\lambda} \mathcal{L}(\xi^*, c) \\ \nabla_{\mu} \mathcal{L}(\xi^*, c) \end{pmatrix} = \mathcal{E}(x^*) \mathcal{E}(x^*)^* \begin{pmatrix} DF(x^*)[\nabla_{x} L(\xi^*)] \\ \nabla g(x^*) \nabla_{x} L(\xi^*) + \text{diag}(g(x^*)^2) \mu^* \end{pmatrix}. \]
By Corollary 3.4 the operator \( \mathcal{E}(x^*) \mathcal{E}(x^*)^* \) is invertible, which yields
\[ DF(x^*)[\nabla_{x} L(\xi^*)] = 0, \quad \nabla g(x^*) \nabla_{x} L(\xi^*) + \text{diag}(g(x^*)^2) \mu^* = 0. \]
Hence applying (25) and Proposition 3.1 once again one gets that
\[ 0 = \nabla_{x} \mathcal{L}(x^*, \lambda^*, \mu^*, c) = \nabla_{x} L(x^*, \lambda^*, \mu^*), \]
that is, \((x^*, \lambda^*, \mu^*)\) is a KKT-point of the problem \((P)\).

Conversely, let \((x^*, \lambda^*, \mu^*)\) be a KKT-point of the problem \((P)\) such that \( f(x^*) \leq \gamma \). Then by definition
\[ \nabla_{x} L(x^*, \lambda^*, \mu^*) = 0, \quad F(x^*) = 0, \quad \max \{g(x^*), \mu^*\} = 0. \]
Therefore, \( \mathcal{L}(x^*, \lambda^*, \mu^*) = f(x^*) \leq \gamma \), i.e. \((x^*, \lambda^*, \mu^*) \in S_c(\gamma)\) for any \( c > 0 \). Furthermore, with the use of Proposition 3.1 one obtains that for any \( c > 0 \) the equality \( \nabla_{x} \mathcal{L}(x^*, \lambda^*, \mu^*, c) = 0 \) holds true, that is, \((x^*, \lambda^*, \mu^*)\) is a stationary point of \( \mathcal{L}(x, \lambda, \mu, c) \) for all \( c > 0 \).

**Part 2.** Let \( c_* > 0 \) be as above. Let us now show that for all \( c \geq c_* \) and for any point of local minimum \((x^*, \lambda^*, \mu^*) \in S_c(\gamma)\) of \( \mathcal{L}(x, \lambda, \mu, c) \) the point \( x^* \) is a locally optimal solution of the problem \((P)\). The fact that \((x^*, \lambda^*, \mu^*)\) is a KKT-point follows directly from the previous part of the proof.

Indeed, fix any \( c \geq c_* \) and a point of local minimum \((x^*, \lambda^*, \mu^*) \in S_c(\gamma)\) of \( \mathcal{L}(x, \lambda, \mu, c) \). Note that \( \nabla_{x} \mathcal{L}(x^*, \lambda^*, \mu^*, c) = 0 \) by the necessary optimality condition. Therefore, equalities (25) hold true due to our choice of \( c_* \). With the use of these equalities one gets that
\[ f(x^*) \leq f(x^*) + \eta(x^*, \lambda^*, \mu^*) = \mathcal{L}(x^*, \lambda^*, \mu^*, c) \leq \gamma \]
and \( x^* \in \Omega_f(\gamma) \) for any \( t > 0 \). Hence the quadratic form \( Q(x^*)[\cdot] \) is positive definite by our assumption. Furthermore, from the previous part of the proof it
follows that \((x_*, \lambda_*, \mu_*)\) is a KKT-point of the problem \((P)\), which implies that \(\eta(x_*, \lambda_*, \mu_*) = 0\) and \(\mathcal{L}'(x_*, \lambda_*, \mu_*, c) = f(x_*)\).

By the definition of local minimum there exist neighbourhoods \(U_\epsilon\) of \(x_*\), \(U_\lambda\) of \(\lambda_*\), and \(U_\mu\) of \(\mu_*\) such that
\[
f(x_*) = \mathcal{L}(x_*, \lambda_*, \mu_*, c) \leq \mathcal{L}(x, \lambda, \mu, c) \quad \forall (x, \lambda, \mu) \in U := U_\epsilon \times U_\lambda \times U_\mu.
\]

Note that
\[
\left\{ \mu, \max\left\{ g(x), \frac{p(x, \mu)}{c} \mu \right\} \right\} + \frac{c}{2p(x, \mu)} \max\left\{ g(x), \frac{p(x, \mu)}{c} \mu \right\} \leq 0
\]
for any \(x\) such that \(g(x) \leq 0\) (see the proof of [34, Prp. 3.1, part (b)]). Therefore by the definition of augmented Lagrangian \((1)\), for any \((x, \lambda, \mu) \in U\) such that \(F(x) = 0\) and \(g(x) \leq 0\) one has
\[
f(x_*) \leq \mathcal{L}(x, \lambda, \mu, c) \leq f(x) + \eta(x, \lambda, \mu).
\]

As was noted above, the quadratic form \(Q(x_*)[\cdot]\) is positive definite, which by Corollary 3.4 implies that the linear operator \(\mathcal{E}(x_*)\mathcal{E}(x_*)^*\) is invertible. It is easily seen that under our assumptions the operator \(\mathcal{E}(x)\mathcal{E}(x)^*\) continuously depends on \(x\). Hence taking into account the fact that the set of invertible operators is open and the inversion is continuous in the uniform operator topology (see, e.g. [47, Theorem 10.12]), one obtains that there exists a neighbourhood \(V_x \subseteq U_x\) of \(x_*\) such that for any \(x \in V_x\) the operator \(\mathcal{E}(x)\mathcal{E}(x)^*\) is invertible and the corresponding inverse operator continuously depends on \(x\).

For any \(x \in V_x\) define \(\lambda(x) \in H\) and \(\mu(x) \in \mathbb{R}^m\) as a unique solution of the following equation:
\[
\mathcal{E}(x)\mathcal{E}(x)^* \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = -\begin{pmatrix} DF(x)[\nabla f(x)] \\ \nabla g(x)[\nabla f(x)] \end{pmatrix}.
\]

By the definition of \(\mathcal{E}(x)\) (see Corollary 3.4) one has
\[
DF(x)[\nabla_x L(x, \lambda(x), \mu(x))] = 0,
\]
\[
\nabla g(x)\nabla_x L(x, \lambda(x), \mu(x)) + \text{diag}(g(x))^2 \mu(x) = 0
\]
for all \(x \in V_x\), which implies that \(\eta(x, \lambda(x), \mu(x)) = 0\) and, thanks to the uniqueness of solution of \((27)\), \(\lambda(x_*) = \lambda_*\) and \(\mu(x_*) = \mu_*\). Furthermore, \(\lambda(x)\) and \(\mu(x)\) continuously depend on \(x\), since the inverse operator of \(\mathcal{E}(x)\mathcal{E}(x)^*\) and the right-hand side of \((27)\) continuously depend on \(x \in V_x\). Consequently, replacing, if necessary, the neighbourhood \(V_x\) with a smaller one, we can suppose that \((x, \lambda(x), \mu(x)) \in U\) for all \(x \in V_x\). Hence with the use of \((26)\) one obtains that
\[
f(x_*) \leq \mathcal{L}(x, \lambda(x), \mu(x), c) \leq f(x) + \eta(x, \lambda(x), \mu(x)) = f(x)
\]
for any \(x \in V_x\) that is feasible for the problem \((P)\). In other words, \(x_*\) is a locally optimal solution of this problem.
5.3. Local exactness

Although Theorem 5.7 somewhat completely describes an intimate relation between optimal solutions/KKT-point of the problem (P) and minimizers/stationary points of the augmented Lagrangian \( \mathcal{L}(x, \lambda, \mu, c) \) for any sufficiently large \( c > 0 \), it does not tell one whether locally optimal solutions of the problem (P) correspond to the points of local minimum of the augmented Lagrangian. It is possible that some KKT-points \((x_\ast, \lambda_\ast, \mu_\ast)\), corresponding to locally optimal solutions \( x_\ast \) of the problem (P), are only stationary points of \( \mathcal{L}(x, \lambda, \mu, c) \), but not its points of local minimum. The aim of this section is to provide simple sufficient conditions for such KKT-points to be points of local minimum of the augmented Lagrangian for any \( c > 0 \) large enough.

Let \((x_\ast, \lambda_\ast, \mu_\ast)\) be a KKT-point of the problem (P), and \( f, F, \) and \( g \) be twice Fréchet differentiable at \( x_\ast \). One says that the second order sufficient optimality conditions hold true at \( x_\ast \), if there exists \( \rho > 0 \) such that

\[
D^2_{xx} L(x_\ast, \lambda_\ast, \mu_\ast)[z, z] \geq \rho \|z\|^2 \quad \forall z \in \mathcal{C}(x_\ast),
\]

where

\[
\mathcal{C}(x_\ast) = \left\{ z \in X \mid DF(x_\ast)[z] = 0, \langle \nabla g_i(x_\ast), z \rangle = 0, \quad i \in M(x_\ast) \right\}
\]

is the critical cone at the point \( x_\ast \). We say that the strict complementarity condition is satisfied for the KKT-point \((x_\ast, \lambda_\ast, \mu_\ast)\), if \( \mu_\ast > 0 \) for any index \( i \in M(x_\ast) \).

**Theorem 5.8:** Let \((x_\ast, \lambda_\ast, \mu_\ast)\) be a KKT-point of the problem (P) satisfying the strict complementarity condition, the functions \( f, F, \) and \( g \) be twice Fréchet differentiable at \( x_\ast \), \( \phi \) be differentiable at zero, and \( \phi'(0) > 0 \). Suppose also that \( Q(x_\ast) \cdot \cdot \cdot \) is positive definite and the second order sufficient optimality conditions hold true at \( x_\ast \). Then there exist \( c_\ast > 0 \) and \( \theta > 0 \) such that for any \( c \geq c_\ast \) the triplet \((x_\ast, \lambda_\ast, \mu_\ast)\) is point of isolated local minimum of \( \mathcal{L}(\cdot, c) \) and

\[
\mathcal{L}(x, \lambda, \mu, c) \geq \mathcal{L}(x_\ast, \lambda_\ast, \mu_\ast, c) + \theta \left( \|x - x_\ast\|^2 + \|\lambda - \lambda_\ast\|^2 + |\mu - \mu_\ast|^2 \right).
\]

for any \((x, \lambda, \mu)\) in a neighbourhood of \((x_\ast, \lambda_\ast, \mu_\ast)\).

**Proof:** Fix some \( \theta > 0 \) and denote \( \xi = (x, \lambda, \mu), \xi_\ast = (x_\ast, \lambda_\ast, \mu_\ast) \), and

\[
\omega_c(\xi) = \mathcal{L}(\xi, c) - \theta \left( \|x - x_\ast\|^2 + \|\lambda - \lambda_\ast\|^2 + |\mu - \mu_\ast|^2 \right).
\]

Our aim is to compute a second order expansion of the function \( \omega_c \) in a neighbourhood of \( \xi_\ast \) and utilize it to prove the theorem.

Firstly, note that from Proposition 3.1 and the fact that \((x_\ast, \lambda_\ast, \mu_\ast)\) is a KKT-point it follows that \( \nabla \mathcal{L}(x_\ast, \lambda_\ast, \mu_\ast, c) = 0 \) for all \( c > 0 \). Hence \( \nabla \omega_c(\xi_\ast) = 0 \).
For any \( i \in I \) denote

\[
G_i(\xi, c) = \mu_i \max \left\{ g_i(x), -\frac{p(x, \mu)}{c} \mu_i \right\} + \frac{c}{2p(x, \mu)} \left\{ g_i(x), -\frac{p(x, \mu)}{c} \mu_i \right\}^2.
\]

For any \( i \in M_1 = M \setminus M(x_*) = \{ i \in M : g_i(x_*) < 0 \} \) one has \( (\mu_*)_i = 0 \) and

\[
G_i(\xi, c) = 0, \quad G_i(\xi, c) = -\frac{p(x, \mu)}{2c} \mu_i^2
\]

in a neighbourhood of \( \xi_0 \), which implies that

\[
G_i(\xi + \Delta \xi, c) = -\frac{\psi(0)}{2c} \Delta \mu_i^2 + o(\|\Delta \xi\|^2).
\]

In turn, for any \( i \in M_2 := M(x_*) \) one has \( (\mu_*)_i > 0 \), thanks to the strict complementarity condition, and

\[
G_i(\xi, c) = 0, \quad G_i(\xi, c) = \mu_i g_i(x) + \frac{c}{2p(x, \mu)} g_i(x)^2
\]

in a neighbourhood of \( \xi_0 \), which yields the expansion:

\[
G_i(\xi + \Delta \xi, c) = (\mu_*)_i \left( \langle \nabla g_i(x_*) , \Delta x \rangle + \frac{1}{2} D^2 g_i(x_*)[\Delta x , \Delta x] \right)
+ \Delta \mu_i \langle \nabla g_i(x_*) , \Delta x \rangle + \frac{c}{2\psi(0)} \langle \nabla g_i(x_*) , \Delta x \rangle^2 + o(\|\Delta \xi\|^2).
\]

Hence taking into account the definition of the augmented Lagrangian one gets that the function \( \omega_*(\cdot) \) admits the following second order expansion in a neighbourhood of \( \xi_0 \):

\[
\omega_c(\xi + \Delta \xi) - \omega_c(\xi_0) = H_c(\Delta x) + Q(x_*)[\Delta \lambda , \Delta \mu] + R_c(\Delta \xi) - \theta \|\Delta \xi\|^2 + o(\|\Delta \xi\|^2). \tag{29}
\]

Here

\[
H_c(\Delta x) = \frac{1}{2} D^2_{xx} L(x_*, \lambda_*, \mu_*)[\Delta x , \Delta x]
+ \frac{c}{2} \left( 1 + \|\lambda_*\|^2 \right) \phi'(0) \|DF(x)[\Delta x]\|^2 + \frac{c}{2\psi(0)} \sum_{i \in M_2} \langle \nabla g_i(x_*) , \Delta x \rangle^2 \tag{30}
\]

and

\[
R_c(\Delta \xi) = \langle \Delta \lambda , DF(x_*)[\Delta x] \rangle - \sum_{i \in M_1} \frac{\psi(0)}{2c} \Delta \mu_i^2 + \sum_{i \in M_2} \Delta \mu_i \langle \nabla g_i(x_*) , \Delta x \rangle
+ \frac{1}{2} \|DF(x_*)[P[\Delta x]]\|^2 + \frac{1}{2} \sum_{i=1}^m \langle \nabla g_i(x_*) , P[\Delta x] \rangle^2.
\]
+ \left( DF(x_\ast)\left[P[\Delta x]\right], DF(x_\ast)\left[Z[\Delta \lambda, \Delta \mu]\right]\right) \\
+ \sum_{i=1}^{m} \left\langle \nabla g_i(x_\ast), P[\Delta x]\right\rangle \left( \left\langle \nabla g_i(x_\ast), Z[\Delta \lambda, \Delta \mu]\right\rangle + g_i(x)^2 \Delta \mu_i \right) ,

where

P[\Delta x] = D_x(\nabla_x L(\xi_\ast))[\Delta x], \quad Z[\Delta \lambda, \Delta \mu] = DF(x_\ast)^\ast[\Delta \lambda] + \sum_{i=1}^{m} \Delta \mu_i \nabla g_i(x_\ast).

Let us estimate the function \( R_c(\cdot) \) from below. Fix some \( \varepsilon \in (0, 1) \). Applying the inequality

\[ \langle x, y \rangle \geq -\|x\|\|y\| \geq -(1 + \varepsilon) \frac{1}{2} \|x\|^2 - \frac{1}{2(1 + \varepsilon)} \|y\|^2 \quad \forall x, y \in X \]

to the last two terms of \( R_c(\cdot) \) and taking into account the definition of \( Q(x)[\cdot] \) (see (5)), one obtains that

\[ R_c(\Delta \xi) \geq \langle \Delta \lambda, DF(x_\ast)[\Delta x]\rangle - \sum_{i \in M_1} \frac{\psi(0)}{2\varepsilon} \Delta \mu_i^2 + \sum_{i \in M_2} \Delta \mu_i \langle \nabla g_i(x_\ast), \Delta x \rangle \\
- \frac{\varepsilon}{2} \| DF(x_\ast)\left[P[\Delta x]\right]\|^2 - \frac{\varepsilon}{2} \sum_{i=1}^{m} \left\langle \nabla g_i(x_\ast), P[\Delta x]\right\rangle^2 \\
- \frac{1}{1 + \varepsilon} Q(x_\ast)[\Delta \lambda, \Delta \mu]. \]

Clearly, there exists \( K > 0 \) such that

\[ \frac{1}{2} \| DF(x_\ast)\left[P[\Delta x]\right]\|^2 + \frac{1}{2} \sum_{i=1}^{m} \left\langle \nabla g_i(x_\ast), P[\Delta x]\right\rangle^2 \leq K \| \Delta x \|^2. \]

Denote \( a = a_{\text{max}}(Q(x_\ast)) \). Applying the inequality

\[ \langle x, y \rangle \geq -\frac{\varepsilon a}{4} \|x\|^2 - \frac{1}{\varepsilon a} \|y\|^2 \quad \forall x, y \in X \]

to the first and third terms of \( R_c(\cdot) \), one finally gets that for any \( c > 2\psi(0)/\varepsilon a \) the following inequality holds true:

\[ R_c(\Delta \xi) \geq - \frac{1}{\varepsilon a} \| DF(x_\ast)[\Delta x]\|^2 - \frac{1}{\varepsilon a} \sum_{i \in M_2} \langle \nabla g_i(x_\ast), \Delta x \rangle^2 - K \varepsilon \| \Delta x \|^2 \\
- \frac{\varepsilon a}{4} \| (\Delta \lambda, \Delta \mu) \|^2 - \frac{1}{1 + \varepsilon} Q(x_\ast)[\Delta \lambda, \Delta \mu]. \]
Hence taking into account (29) and the definition of $H_c(\Delta x)$ one obtains that

$$\omega_c(\xi^* + \Delta \xi) - \omega_c(\xi^*) \geq H_{c-c_0}(\Delta x) + \frac{\varepsilon a}{4} \| (\Delta \lambda, \Delta \mu) \|^2$$

$$- K \varepsilon \| \Delta x \|^2 - \theta \| \Delta \xi \|^2 + o(\| \Delta \xi \|^2),$$

for any $\varepsilon \in (0, 1)$ and $c > c_0 := 2 \max\{\psi(0), 1/\phi'(0)\}/\varepsilon a$.

Let us check that there exist $c^* > 0$ and $\beta > 0$ such that

$$H_c(\Delta x) \geq \beta \| \Delta x \|^2 \quad \forall \Delta x \in X, \quad c > c^*. \quad (31)$$

Then choosing any $0 < \varepsilon < \beta / 3K$ and $0 < \theta < \min\{\varepsilon a/8, \beta / 3\}$ one obtains that

$$\omega_c(\xi^* + \Delta \xi) - \omega_c(\xi^*) \geq \min\left\{\frac{\varepsilon a}{8}, \frac{\beta}{3}\right\} \| \Delta \xi \|^2 + o(\| \Delta \xi \|^2),$$

for any $c > c^* + c_0$. Hence, as one can easily check, $\omega_c(\xi) \geq \omega_c(\xi^*)$ for any $\xi$ from a sufficiently small neighbourhood of $\xi^*$, which implies the required result.

Thus, it remains to prove inequality (31). To this end, introduce the linear operator $T : X \rightarrow H \times \mathbb{R}^{m(x^*)}$, defined as

$$T z = \{DF(x_*)(z)\} \times \prod_{i \in M(x_*)} \{\langle \nabla g_i(x_*), z \rangle\} \quad \forall z \in X.$$

Note that the kernel of this operator coincides with the critical cone $\mathcal{C}(x_*)$. For any $z \in X$, below we denote by $z_1$ the orthogonal projection of $z$ onto $\mathcal{C}(x_*)$ and by $z_2$ the orthogonal projection of $z$ onto the orthogonal complement of $\mathcal{C}(x_*)$. Then $z = z_1 + z_2$ for any $z \in X$.

Let $\Theta > 0$ be such that

$$\left| D_{xx}^2 L(\xi^*)[x, y]\right| \leq \Theta \| x \| \| y \| \quad \forall x, y \in X.$$

Then with the use of the second order sufficient optimality conditions (28) one gets that

$$\frac{1}{2}D_{xx}^2 L(\xi^*)[z, z] = \frac{1}{2}D_{xx}^2 L(\xi^*)[z_1, z_1] + D_{xx}^2 L(\xi^*)[z_1, z_2] + \frac{1}{2}D_{xx}^2 L(\xi^*)[z_2, z_2]$$

$$\geq \rho \| z_1 \|^2 - \Theta \| z_1 \| \| z_2 \| - \Theta \| z_2 \|^2$$

$$\geq \frac{\rho}{2} \| z_1 \|^2 - \left( \Theta + \frac{\Theta^2}{2\rho} \right) \| z_2 \|^2,$$

for any $z \in X$.

By Lemma 3.3 the operator $T$ is surjective due to our assumption on the positive definiteness of $Q(x_*)[\cdot]$. Consequently, by the open mapping theorem there
exists $\tau > 0$ such that
\[ \|Tz\| \geq \tau \|z_2\| \quad \forall z \in X \]
(see, e.g. [48])). Hence taking into account the definition of $H_c$ (see (30)) one obtains that
\[
H_c(z) = \frac{1}{2} D_{xx}^2 L(x_*, \lambda_*, \mu_*)[z, z] + \frac{c}{2} \left(1 + \|\lambda_*\|^2\right) \phi'(0) \|DF(x)[z]\|^2
+ \frac{c}{2} \psi(0) \sum_{i \in M(x_*)} \langle \nabla g_i(x_*), z \rangle^2
\geq \frac{\rho}{2} \|z_1\|^2 - \left(\Theta + \frac{\Theta^2}{2\rho}\right) \|z_2\|^2 + \frac{c}{2} \min \left\{\phi'(0), \frac{1}{\psi(0)}\right\} \|Tz\|^2
\geq \frac{\rho}{2} \|z_1\|^2 + \left(\frac{c\tau}{2} \min \left\{\phi'(0), \frac{1}{\psi(0)}\right\} - \Theta - \frac{\Theta^2}{2\rho}\right) \|z_2\|^2 \geq \frac{\rho}{2} \|z\|^2
\]
for any $z \in X$ and
\[ c > \frac{\rho^2 + 2\Theta \rho + \Theta^2}{\rho \tau \min \{\phi'(0), 1/\psi(0)\}} , \]
which completes the proof of the theorem.

6. Conclusions

In this paper, we developed a general theory of exact augmented Lagrangians for constrained optimization problems in Hilbert spaces with inequality and nonlinear operator equality constraints. The core result of this theory is the lower estimate of the gradient of the augmented Lagrangian via the infeasibility measure from Theorem 4.1, which allows one to obtain several types of sufficient conditions for the global or complete exactness of the augmented Lagrangian. These conditions ensure that local/global minimizers or critical points of the augmented Lagrangian correspond to locally/globally optimal solutions or KKT-points of the constrained optimization problem. Main results of the paper are obtained with the use of a nonlocal constraint qualification, which is reduced to LICQ in this finite dimensional case, and is closely related to assumptions on nonlocal metric regularity of constraints.

Various applications of the theoretical results from this paper to constrained variational problems, problems with PDE constraints, and optimal control problems, as well as several numerical examples, will be presented in the second part of our study.

Disclosure statement

No potential conflict of interest was reported by the author(s).
**Funding**

This work was performed in IPME RAS and supported by the Russian Science Foundation [grant number 20-71-10032].

**ORCID**

M. V. Dolgopolik http://orcid.org/0000-0003-0677-0797

**References**

[1] Eremin II. The penalty method in convex programming. Soviet Math Dokl. **1966**;8:459–462.

[2] Zangwill WI. Nonlinear programming via penalty functions. Manag Sci. **1967**;13:344–358.

[3] Han SP, Mangasarian OL. Exact penalty functions in nonlinear programming. Math Program. **1979**;17:251–269.

[4] Di Pillo G, Grippo L. On the exactness of a class of nondifferentiable penalty functions. J Optim Theory Appl. **1988**;57:399–410.

[5] Di Pillo G, Facchinei F. Exact barrier function methods for Lipschitz programs. Appl Math Optim. **1995**;32:1–31.

[6] Demyanov VF. Nonsmooth optimization. In: Di Pillo G, Schoen F, editors. Nonlinear optimization. lecture notes in mathematics, Vol. 1989, Berlin-Heidelberg: Springer-Verlag; 2010. p. 55–164.

[7] Zaslavski AJ. Optimization on metric and normed spaces. New York: Springer Science + Business Media; 2010.

[8] Zaslavski AJ. A sufficient condition for exact penalty functions. Optim Lett. **2009**;3:593–602.

[9] Zaslavski AJ. Exact penalty property in optimization with mixed constraints via variational analysis. SIAM J Optim. **2013**;23:1–31.

[10] Dolgopolik MV. A unifying theory of exactness of linear penalty functions. Optim. **2016**;65:1167–1202.

[11] Maratos N. Exact penalty function algorithms for finite dimensional and control optimization problems [dissertation]. London: University of London; 1978.

[12] Polak E. Optimization: algorithms and consistent approximations. New York: Springer-Verlag; 1997.

[13] Nocedal J, Wright SJ. Numerical optimization. New York: Springer; 2006.

[14] Bonnans JF, Gilbert JC, Lemaréchal C, et al. Numerical optimization. Berlin: Springer-Verlag; 2006.

[15] Conn AR, Gould NIM, Toint PL. Trust-region methods. Philadelphia: SIAM; 2000.

[16] Coope ID. The Maratos effect in sequential quadratic programming algorithms using the $L_1$ exact penalty function. Technical Report No. CS-85-32. University of Waterloo; 1985. Available from: https://cs.uwaterloo.ca/research/tr/1985/CS-85-32.pdf.

[17] Fletcher R. A class of methods for nonlinear programming with termination and convergence properties. In: Abadie J, editor. Integer and nonlinear programming. Amsterdam: North-Holland; 1970. p. 157–175.

[18] Birgin EG, Martinez JM. Practical augmented Lagrangian methods for constrained optimization. Philadelphia: SIAM; 2014.

[19] Rockafellar RT. Augmented Lagrange multiplier functions and duality in nonconvex multiplier functions and duality in nonconvex programming. SIAM J Control Optim. **1974**;12:268–285.
[20] Rockafellar RT. Lagrange multipliers and optimality. SIAM Rev. 1993;35:183–238.
[21] Ito K, Kunisch K. The augmented Lagrangian method for equality and inequality constraints in Hilbert spaces. Math Program. 1990;46:341–360.
[22] Fletcher R. An exact penalty function for nonlinear programming with inequalities. Math Program. 1973;5:129–150.
[23] Han SP, Mangasarian OL. A dual differentiable exact penalty function. Math Program. 1983;25:293–306.
[24] Di Pillo G, Grippo L. A continuously differentiable exact penalty function for nonlinear programming problems with inequality constraints. SIAM J Control Optim. 1985;23:72–84.
[25] Lucidi S. New results on a continuously differentiable exact penalty function. SIAM J Optim. 1992;2:558–574.
[26] Contaldi G, Di Pillo G, Lucidi S. A continuously differentiable exact penalty function for nonlinear programming problems with unbounded feasible set. Oper Res Lett. 1993;14:153–161.
[27] Fukuda EH, Silva PJS, Fukushima M. Differentiable exact penalty functions for nonlinear second-order cone programs. SIAM J Optim. 2012;22:1607–1633.
[28] Estrin R, Friedlander MP, Orban D, et al. Implementing a smooth exact penalty function for equality-constrained nonlinear optimization. SIAM J Sci Comput. 2020;42:A1809–A1835.
[29] Estrin R, Friedlander MP, Orban D, et al. Implementing a smooth exact penalty function for general constrained nonlinear optimization. SIAM J Sci Comput. 2020;42:A1836–A1859.
[30] Di Pillo G, Grippo L. A new class of augmented Lagrangians in nonlinear programming. SIAM J Control Optim. 1979;17:618–628.
[31] Di Pillo G, Grippo L. A new augmented Lagrangian function for inequality constraints in nonlinear programming problems. J Optim Theory Appl. 1982;36:495–519.
[32] Lucidi S. New results on a class of exact augmented Lagrangians. J Optim Theory Appl. 1988;58:259–282.
[33] Di Pillo G, Lucidi S. On exact augmented Lagrangian functions in nonlinear programming. In: Di Pillo G, Giannessi F, editors. Nonlinear optimization and applications. New York: Plenum Press; 1996. p. 85–100.
[34] Di Pillo G, Lucidi S. An augmented Lagrangian function with improved exactness properties. SIAM J Optim. 2001;12:376–406.
[35] Di Pillo G, Liuazzi G, Lucidi S, et al. Fruitful uses of smooth exact merit functions in constrained optimization. In: Di Pillo G, Murli A, editors. High performance algorithms and software for nonlinear optimization. Dordrecht: Kluwer Academic Publishers; 2003. p. 201–225.
[36] Di Pillo G, Liuazzi G, Lucidi S, et al. An exact augmented Lagrangian function for nonlinear programming with two-sided constraints. Comput Optim Appl. 2003;25:57–83.
[37] Luo H, Wu H, Liu J. Some results on augmented Lagrangians in constrained global optimization via image space analysis. J Optim Theory Appl. 2013;159:360–385.
[38] Du X, Zhang L, Gao Y. A class of augmented Lagrangians for equality constraints in nonlinear programming problems. Appl Math Comput. 2006;172:644–663.
[39] Du X, Liang Y, Zhang L. Further study on a class of augmented Lagrangians of Di Pillo and Grippo in nonlinear programming. J Shanghai Univ (Engl Ed). 2006;10:293–298.
[40] Di Pillo G, Lucidi S, Palagi L. An exact penalty-Lagrangian approach for a class of constrained optimization problems with bounded variables. Optim. 1993;28:129–148.
[41] Di Pillo G, Liuazzi G, Lucidi S. An exact penalty-Lagrangian approach for large-scale nonlinear programming. Optim. 2011;60:223–252.
[42] Di Pillo G, Lucidi S, Palagi L. A truncated Newton method for constrained optimization. In: Di Pillo G, Giannessi F, editors. Nonlinear optimization and related topics. Dordrecht: Kluwer Academic Publishers; 2000. p. 79–103.

[43] Di Pillo G, Lucidi S, Palagi L. Convergence to second-order stationary points of a primal-dual algorithm model for nonlinear programming. Math Oper Res. 2005;30:897–915.

[44] Di Pillo G, Liuzzi G, Lucidi S, et al. A truncated Newton method in an augmented Lagrangian framework for nonlinear programming. Comput Optim Appl. 2010;45:311–352.

[45] Fukuda E, Lourenço BF. Exact augmented Lagrangian functions for nonlinear semidefinite programming. Comput Optim Appl. 2018;71:457–482.

[46] Dolgopolik MV. Augmented Lagrangian functions for cone constrained optimization: the existence of global saddle points and exact penalty property. J Glob Optim. 2018;71:237–296.

[47] Rudin W. Functional analysis. Singapore: McGraw-Hill; 1991.

[48] Borwein JM, Dontchev AL. On the Bartle-Graves theorem. Proc Amer Math Soc. 2003;131:2553–2560.

[49] Dolgopolik MV, Fominyh AV. Exact penalty functions for optimal control problems I: main theorem and free-endpoint problems. Optim Control Appl Methods. 2019;40:1018–1044.

[50] Dolgopolik MV. Exact penalty functions for optimal control problems II: exact penalization of terminal and pointwise state constraints. Optim Control Appl Methods. 2020;41:898–947.

[51] Ioffe AD, Tihomirov VM. Theory of extremal problems. Amsterdam: North-Holland; 1979.

[52] Mawhin J, Willem M. Origin and evolution of the Palais-Smale condition in critical point theory. J Fixed Point Theory Appl. 2010;7:265–290.