A novel method for nonlinear singular oscillators

Ali Akgül¹, Hijaz Ahmad²,³, Yu-Ming Chu⁴,⁵ and Phatiphat Thounthong⁶

Abstract
The present work deals with a study of a nonlinear singular oscillator. To approximate the frequency–amplitude relationship of the singular oscillator, reproducing kernel method is employed. The approximate solution is compared with the exact solution as well as the results obtained by the He’s frequency–amplitude formulation, to show the effectiveness of the proposed technique for solving the problem.

Keywords
Frequency–amplitude relationship, nonlinear singular oscillator, reproducing kernel method

Introduction
Mathematical modeling of various physical and natural phenomena enables us to deduce useful results about behavior of undergoing processes and help to get better future predictions. Almost every field of study is surrounded by models used to serve different kinds of purposes. In recent times, various standard classical mathematical models from different areas of studies have been investigated via newly proposed mathematical tools. The field of nonlinear singular oscillator analysis in engineering is no exception. Therefore, it is necessary to use an effective and simple method to approximate the frequency–amplitude relationship of nonlinear singular oscillators. The amplitude-period relationship plays a vital role in engineering and sciences, and it is the fundamental property of the nonlinear singular oscillator.¹⁻⁶ Some useful amplitude-period formulae found in the literature are Garcia’s amplitude–frequency formula,⁷ He’s amplitude–frequency formula,⁸⁻¹⁰ and Suarez–Antola’s amplitude-frequency formula.¹¹ Khatami et al.¹² have obtained the efficient solution of nonlinear duffing oscillator. Big-Alabo and Ossia¹³ have investigated the analysis of the coupled nonlinear vibration of a two-mass system. Johannes and Peter¹⁴ have worked on the high-frequency vibrations in the contact of brake systems. For more details see literature.¹⁵⁻¹⁷

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In this article, we apply the reproducing kernel method to a nonlinear singular oscillator. Consider a nonlinear singular oscillator

\[
\begin{align*}
x''(m) + f(x) &= 0 \\
x(0) &= A, \quad x'(0) = 0
\end{align*}
\]

where \(-f(x)\) is a conservative force and \(A\) is a constant. This study will investigate the oscillator when \(f(x) = \frac{1}{x^{(m)}}\).

**Reproducing kernel spaces and functions**

We construct the reproducing kernel spaces for domain and range in this section.

**Definition 2.1.** We describe the reproducing kernel Hilbert space \(0_{T_2}^3[0, 1]\) as: The space of functions such as \(0_{T_2}^3[0, 1] = \{p[p^{(k)}(t) \text{ are absolutely continuous functions on } [0, 1], p^{(3)}(t) \epsilon L^2[0, 1], k = 0, 1, 2, p(0) = p'(0)\}\). We have its inner product and norm by

\[
\langle p, q \rangle_{0_{T_2}^3} = \sum_{k=0}^{2} p^{(k)}(0)q^{(k)}(0) + \int_{0}^{1} p^{(3)}(t)q^{(3)}(t)dt
\]

and

\[
\|p\|_{0_{T_2}^3} = \sqrt{\langle p, p \rangle_{0_{T_2}^3}}
\]

in which \(p, q \in 0_{T_2}^3[0, 1]\).

**Theorem 2.1** We acquire the reproducing kernel function \(H_n(m)\) as

\[
H_n(m) = \begin{cases} 
\frac{m^2n^2}{4} + \frac{m^3n^2}{12} - \frac{m^4n}{24} + \frac{m^5}{120}, & m \leq n \\
\frac{n^2m^2}{4} + \frac{n^3m^2}{12} - \frac{n^4m}{24} + \frac{n^5}{120}, & m > n
\end{cases}
\]

**Proof.** We have to prove that

\[
\langle p, H_n \rangle_{0_{T_2}^3} = p(n)
\]

We know that

\[
\langle p, H_n \rangle_{0_{T_2}^3} = \sum_{k=0}^{2} p^{(k)}(0)H_n^{(k)}(0) + \int_{0}^{1} p^{(3)}(t)H_n^{(3)}(t)dt
\]

\[
= p(0)H_n(0) + p'(0)H_n'(0) + p''(0)H_n''(0)
\]

\[
+ \int_{0}^{1} p^{(3)}(t)H_n^{(3)}(t)dt
\]

We use the initial conditions and obtain \(p(0) = 0 = p'(0)\). Therefore, we reach

\[
\langle p, H_n \rangle_{0_{T_2}^3} = p''(0)H_n''(0) + \int_{0}^{1} p^{(3)}(t)H_n^{(3)}(t)dt
\]
Implementing two times integration by parts gives

\[ \langle p, H_n \rangle_{\mathcal{O}T_2^3} = p''(0)H_n'(0) + \int_0^1 p^{(3)}(t)H_n^{(3)}(t)dt \]  

(10)

\[ = p''(0)H_n'(0) + p''(1)H_n''(1) - p''(0)H_n'''(0) \]  

(11)

\[ -p'(1)H_n^{(4)}(1) + p'(0)H_n^{(4)}(0) + \int_0^1 p'(t)H_n^{(5)}(t)dt \]  

(12)

Then, we get

\[ \langle p, H_n \rangle_{\mathcal{O}T_2^3} = p''(0)(H_n'(0) - H_n''(0)) \]  

(13)

\[ + p''(1)H_n''(1) - p'(1)H_n^{(4)}(1) \]  

(14)

Now we should find \( H_n''(0), H_n'''(0), H_n^{(4)}(0), H_n''''(1), H_n^{(5)}(1) \) and \( H_n^{(5)}(t) \). Then, we get

\[ H_n''(0) = \frac{n^2}{2} \]  

(15)

\[ H_n'''(0) = \frac{n^2}{2} \]  

(16)

\[ H_n^{(4)}(0) = -n \]  

(17)

\[ H_n''''(1) = 0 \]  

(18)

\[ H_n^{(4)}(1) = 0 \]  

(19)

and

\[ H_n^{(5)}(m) = \begin{cases} 
1, & m < n \\
0, & m > n 
\end{cases} \]  

(20)

Using the above equations gives

\[ \langle p, H_n \rangle_{\mathcal{O}T_2^3} = p''(0)\left(\frac{n^2}{2} - \frac{n^2}{2}\right) \]  

(21)

\[ + p''(1)0 - p'(1)0 + \int_0^n p'(t)H_n^{(5)}(t)dt + \int_0^1 p'(t)H_n^{(5)}(t)dt \]  

(22)

\[ = \int_0^n p'(t)dt \]  

(23)

\[ = p(n) - p(0) \]  

(24)
This completes the proof.

Definition 2.2. We describe the reproducing kernel Hilbert space $T_3^2[0, 1]$ as: The space of functions such as $T_3^2[0, 1] = \{p|p^{(k)}(t)\text{ are absolutely continuous functions on } [0, 1], p^{(3)}(t) \in L^2[0, 1], k = 0, 1, 2.\}$.

We have its inner product and norm by

\[ (p, q)_{T_3^2} = \sum_{k=0}^{2} p^{(k)}(0)q^{(k)}(0) + \int_{0}^{1} p^{(3)}(t)q^{(3)}(t)dt \]  

and

\[ \|p\|_{T_3^2} = \sqrt{(p, p)_{T_3^2}} \]

in which $p, q \in T_3^2[0, 1]$.

Theorem 2.2. We acquire the reproducing kernel function $G_n(m)$ as

\[ G_n(m) = \begin{cases} 1 + nm + \frac{m^2n^2}{4} + \frac{m^3n^2}{12} - \frac{m^4n}{24} + \frac{m^5}{120}, & m \leq n \\ 1 + nm + \frac{n^2m^2}{4} + \frac{n^3m^2}{12} - \frac{n^4m}{24} + \frac{n^5}{120}, & m > n \end{cases} \]

Proof. The proof of the Theorem 2.2 is similar to the proof of the Theorem 2.1. Therefore, we omitted the proof.

Solution of the problem

We take into consideration

\[ \begin{cases} x''(m) + \frac{1}{x(m)} = 0 \\ x(0) = A, \quad x'(0) = 0 \end{cases} \]

(29)

We use the following transformation to homogenize the initial conditions

\[ v(m) = x(m) - A \]

(30)

Then, we obtain

\[ \begin{cases} v''(m) = -\frac{1 + v(m)v'(m)}{A} \\ v(0) = 0, \quad v'(0) = 0 \end{cases} \]

(31)

We construct the operator $M$ as $M: T_3^3[0, 1] \rightarrow T_3^3[0, 1]$ such that $Mv(m) = -\frac{1 + v(m)v'(m)}{A}$. Then, we obtain

\[ Mv(m) = K(m) \]

(32)

where $K(m) = -\frac{1 + v(m)v'(m)}{A}$.

We use the reproducing kernel method based on the construction of the orthogonal function system of $T_3^3[0, 1]$. Let $\sigma_k(m) = G_m(m)$ and $\varphi_k(m) = M' \sigma_k(m)$, where the countable set $\{m_k\}_{k=1}^\infty$ is dense in the interval
[0, 1]. The operator $M^*$ is the adjoint operator of $M$ and $G_{mk}(m)$ defines the reproducing kernel function associated to the space $T^2_3[0, 1]$. We define the orthonormal system \{$\tilde{\psi}_k$\}_k=1 in $0 T^2_2[0, 1]$ by using the process of Gram-Schmidt as follows

$$\tilde{\psi}_j(m) = \sum_{k=1}^{\infty} \sigma_{jk} \psi_k(m), \quad \sigma_{jj} > 0, \ j = 1, 2, \ldots$$

(33)

Theorem 3.1. We assume that \{$m_k$\}_k=1 is a dense set on [0, 1] and the solution is unique on $0 T^2_2[0, 1]$. Then, we obtain the solution as

$$v(m) = \sum_{j=1}^{\infty} \sum_{k=1}^{j} \sigma_{jk} G(m_k) \tilde{\psi}_j(m)$$

(34)

Proof. We obtain

$$v(m) = \sum_{j=1}^{\infty} \langle v(m), \tilde{\psi}_j(m) \rangle_{0 T^2_2} \tilde{\psi}_j(m)$$

(35)

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{j} \sigma_{jk} \psi_k(m) \tilde{\psi}_j(m)$$

(36)

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{j} \sigma_{jk} \langle v(m), \psi_k(m) \rangle_{0 T^2_2} \tilde{\psi}_j(m)$$

(37)

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{j} \sigma_{jk} \langle v(m), M^* \varphi_k(m) \rangle_{0 T^2_2} \tilde{\psi}_j(m)$$

(38)

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{j} \sigma_{jk} \langle M v(m), \sigma_k(m_k) \rangle_{T^2_2} \tilde{\psi}_j(m)$$

(39)

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{j} \sigma_{jk} \langle M v(m), G_{mk} \rangle_{T^2_2} \tilde{\psi}_j(m)$$

(40)

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{j} \sigma_{jk} G(m_k) \tilde{\psi}_j(m)$$

(41)

by properties of the adjoint operator and the reproducing property. This completes the proof.

We obtain the approximate solution as

$$v_n(m) = \sum_{j=1}^{n} \sum_{k=1}^{j} \sigma_{jk} G(m_k) \tilde{\psi}_j(m)$$

(42)

**Numerical experiments**

In this section, we present the comparison of the obtained approximate solutions with the exact solutions for $A = 1$ in Figures 1 and 6 for different values of $x$. Approximate solution is plotted for different values of the constant $A$. In Figure 2, we choose $A = 0.6$. In Figure 3, we select $A = 0.7$. In Figure 4, we choose $A = 0.8$.
Figure 1. Exact solution (blue) and approximate solution (green) for $A = 1$.

Figure 2. Numerical simulation for $A = 0.6$.

Figure 3. Numerical simulation for $A = 0.7$. 
Figure 4. Numerical simulation for $A = 0.8$.

Figure 5. Numerical simulation for $A = 0.9$.

Figure 6. Exact solution (blue) and approximate solution (green) for $A = 1$. 
In Figure 5, we select $A = 0.9$. Figure 6 shows the comparison of exact solution, approximate solution obtained by reproducing kernel method, which reveals that our suggested procedure gives accurate results close to the exact solution and is useful for singular oscillators. In Zhang and Qin, the authors chose $A = 1$ and $x = 1.224744871$.

In this paper, we demonstrate the exact and approximate solutions for same values by Figures 7 and 8.

Conclusion

The present research findings reveal that the reproducing kernel method is an effective and useful tool for the approximate solutions of the nonlinear oscillators. The proposed convergence analysis confirmed the exactness or analyticity of the approximate solution. In addition, approximate solution of the singular oscillator system is obtained with some arbitrary parameters and graphical representation of the solution may be important in the study of dynamical behavior of the equations of the singular oscillator.

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References

1. He JH. Some asymptotic methods for strongly nonlinear equations. Int J Mod Phys B 2006; 20: 1141–1199.
2. He JH. Comment on He’s frequency formulation for nonlinear oscillators. Eur J Phys 2008; 29: 19–22.
3. He JH. A tutorial review on fractal spacetime and fractional calculus. Int J Theor Phys 2014; 53: 3698–3718.
4. He JH. Max-min approach to nonlinear oscillators. Int J Nonlin Sci Numer Simul 2008; 9: 207–210.
5. Liu ZJ, Adamu MY, Suleiman E, et al. Hybridization of homotopy perturbation method and Laplace transformation for the partial differential equations. Therm Sci 2017; 21: 1843–1846.
6. El-Dib YO. Multiple scales homotopy perturbation method for nonlinear oscillators. Nonlin Sci Lett A 2017; 8: 352–364.
7. Garcia A. An amplitude-period formula for a second order nonlinear oscillator. Nonlin Sci Lett A 2017; 8: 340–347.
8. He JH. An improved amplitude-frequency formulation for nonlinear oscillators. Int J Nonlin Sci Numer Simul 2008; 9: 211–212.
9. He JH. An approximate amplitude-frequency relationship for a nonlinear oscillator with discontinuity. Nonlin Sci Lett A 2016; 7: 77–85.
10. He JH. Amplitude-frequency relationship for conservative nonlinear oscillators with odd nonlinearities. Int J Appl Comput Math 2017; 3: 1557–1560.
11. Suarez AR. Remarks on an approximate formula for the period of conservative oscillations in nonlinear second order ordinary differential equations. Nonlin Sci Lett A 2017; 8: 348–351.
12. Khatami I, Zahedi E and Zahedi M. Efficient solution of nonlinear duffing oscillator. J Appl Comput Mech 2020; 6: 219–234.
13. Big-Alabo A and Ossia C. Analysis of the coupled nonlinear vibration of a two-mass system. J Appl Comput Mech 2019; 5: 935–950.
14. Johannes O and Peter G. O, High-frequency vibrations in the contact of brake systems. Facta Univ Ser Mech Eng 2019; 17: 103–112.

Figure 7. Exact solution for $A = 1$.

Figure 8. Approximate solution for $A = 1$. 
In Figure 5, we select $A = 0.9$. Figure 6 shows the comparison of exact solution, approximate solution obtained by reproducing kernel method, which reveals that our suggested procedure gives accurate results close to the exact solution and is useful for singular oscillators. In Zhang and Qin, the authors chose $A = 1$ and $\omega = 1.224744871$. In this paper, we demonstrate the exact and approximate solutions for same values by Figures 7 and 8.

**Conclusion**

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**References**

1. He JH. Some asymptotic methods for strongly nonlinear equations. *Int J Mod Phys B* 2006; 20: 1141–1199.
2. He JH. Comment on He’s frequency formulation for nonlinear oscillators. *Eur J Phys* 2008; 29: 19–22.
3. He JH. A tutorial review on fractal spacetime and fractional calculus. *Int J Theor Phys* 2014; 53: 3698–3718.
4. He JH. Max-min approach to nonlinear oscillators. *Int J Nonlinear Sci Numer Simul* 2008; 9: 207–210.
5. Liu ZJ, Adamu MY, Suleiman E, et al. Hybridization of homotopy perturbation method and Laplace transformation for the partial differential equations. *Therm Sci* 2017; 21: 1843–1846.
6. El-Dib YO. Multiple scales homotopy perturbation method for nonlinear oscillators. *Nonlin Sci Lett A* 2017; 8: 352–364.
7. Garcia A. An amplitude-period formula for a second order nonlinear oscillator. *Nonlin Sci Lett A* 2017; 8: 340–347.
8. He JH. An improved amplitude-frequency formulation for nonlinear oscillators. *Int J Nonlin Sci Numer Simul* 2008; 9: 211–212.
9. He JH. An approximate amplitude-frequency relationship for a nonlinear oscillator with discontinuity. *Nonlin Sci Lett A* 2016; 7: 77–85.
10. He JH. Amplitude-frequency relationship for conservative nonlinear oscillators with odd nonlinearities. *Int J Appl Comput Math* 2017; 3: 1557–1560.
11. Suarez AR. Remarks on an approximate formula for the period of conservative oscillations in nonlinear second order ordinary differential equations. *Nonlin Sci Lett A* 2017; 8: 348–351.
12. Khatami I, Zahedi E and Zahedi M. Efficient solution of nonlinear duffing oscillator. *J Appl Comput Mech* 2020; 6: 219–234.
13. Big-Alabo A and Ossia C. Analysis of the coupled nonlinear vibration of a two-mass system. *J Appl Comput Mech* 2019; 5: 935–950.
14. Johannes O and Peter G. O, High-frequency vibrations in the contact of brake systems. *Facta Univ Ser Mech Eng* 2019; 17: 103–112.
15. Akgül A and Ahmad H. A novel application of the reproducing kernel method for the solution of nonlinear oscillator produced in nanotechnology for water collection from air. *Math Meth Appl Sci.*

16. He CH, He JH and Sedighi HM. Fangzhu: an ancient Chinese nanotechnology for water collection from air: history, mathematical insight, promises, and challenges. *Math Methods Appl Sci.*

17. He JH. Homotopy perturbation method for Fangzhu oscillator. *Math Methods Appl Sci.*

18. Zhang HL and Qin LJ. An ancient Chinese mathematical algorithm and its application to nonlinear oscillators. *Comput Math Appl* 2011; 61: 2071–2075.

19. Cui M and Lin Y. *Nonlinear numerical analysis in the reproducing kernel space.* New York, NY: Nova Science Publishers, Inc., 2009.

20. Zhang W, Wang Y and Wang M. The reproducing kernel for the reaction-diffusion model with a time variable fractional order. *Therm Sci* 2020; 24: 2553–2559.