Holomorphic Morse Inequalities on Covering Manifolds

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Abstract

The goal of this paper is to generalize Demailly’s asymptotic holomorphic Morse inequalities to the case of a covering manifold of a compact manifold. We shall obtain estimates which involve Atiyah’s “normalized dimension” of the square integrable harmonic spaces. The techniques used are those of Shubin who gave a proof for the usual Morse inequalities in the presence of a group action relying on Witten ideas. As a consequence we obtain estimates for the dimension of the square integrable holomorphic sections of the pull-back of a line bundle on the base manifold under some mild hypothesis for the curvature.

1 Introduction

To emphasize the meaning of our result let us consider a compact projective manifold $X$, $F$ an ample line bundle on $X$, $\tilde{X}$ the universal covering of $X$ and $\tilde{F}$ the pull-back of $F$ on $\tilde{X}$. Then Kollár’s Theorem 6.4 from [3] shows that

$$\dim_{\pi_1(X)} H^0_{(2)}(\tilde{X}, K_{\tilde{X}} \otimes \tilde{F}) = \dim H^0(X, K_X \otimes F)$$

where we denote as usual $K_N$ the canonical bundle of a manifold $N$. It follows from Theorem 0.1 in Demailly [4] that $\dim_{\pi_1(X)} H^0_{(2)}(\tilde{X}, K_{\tilde{X}} \otimes \tilde{F}^k)$ has polynomial growth of order the dimension of $X$ as $k \to \infty$. We shall generalize this result to the case of a complex analytic manifold $M$ on which a discrete group $\Gamma$ acts freely and properly discontinuous such that $X = M/\Gamma$ is compact and carries a line bundle $F$ satisfying Demailly’s condition:

$$\int_{X(\leq 1)} (ic(F))^n > 0$$

Let us mention that holomorphic Morse inequalities on non-compact manifolds have been obtained before on $q$-concave and $q$-convex manifolds (see Marinescu [3] and Bouche [4]).

The cohomology groups of $M$ are usually infinite-dimensional and we cannot use the usual dimension. The dimension we shall use is the $\Gamma$-dimension introduced by Atiyah in [2]. Let us define the $\Gamma$-dimension of certain subspaces of $L^2(M, \tilde{E})$, where $E \to X$ is a hermitian vector bundle, $\tilde{E} \to M$ is the pull-back of $E$, $X$ (and hence $M$) is endowed with a Riemannian metric and

$$L^2(M, \tilde{E}) = \{ s : M \to \tilde{E} \mid s is a measurable section, \int_M |s|^2 dV < \infty \}$$

Let $G$ be a closed subspace in $L^2(M, \tilde{E})$ such that $L_\gamma G \subset G$ where $L_\gamma$ is the action of $\Gamma$ on $L^2(M, \tilde{E})$. Let $U$ be a fundamental domain for the action of $\Gamma$ and $(\varphi_m)_m$
The main theorem of this paper is:

\[ \text{dim}_r G := \sum_m \int_U |\varphi_m(x)|^2 dV(x) \]  

(2)

It can be shown that the definition of \( \text{dim}_r G \) does not depend on the orthonormal base \( (\varphi_m)_m \) or on the fundamental domain \( U \) (see Atiyah [3]).

Let \( M \) be a complex analytic manifold of complex dimension \( n \) on which a discrete group \( \Gamma \) acts freely and properly discontinuous such that \( X = M/\Gamma \) is compact. Since \( M/\Gamma \) is compact one can easily see that the pull-back of a hermitian metric on \( X \) is a complete metric on \( M \) which we consider fixed from now on. Let \( E \) be a hermitian holomorphic vector bundle on \( X \) and \( \tilde{E} = \pi^* E \) its pull-back, where \( \pi : M \to X \) is the projection.

Let \( \bar{\partial}_q : C^\infty_0(M, \tilde{E}) \to C^\infty_0(M, \tilde{E}) \)

be the well-known Cauchy–Riemann operator and

\[ \delta_q : C^\infty_0(M, \tilde{E}) \to C^\infty_0(M, \tilde{E}) \]

the formal adjoint of \( \bar{\partial}_q \). Then \( \Delta''_q = \bar{\partial}_q \delta_q + \delta_q \bar{\partial}_q \) is an elliptic differential operator.

Let \( \bar{\partial}_q : L^2_{0,q}(M, \tilde{E}) \to L^2_{0,q+1}(M, \tilde{E}) \) be the weak maximal extension of \( \bar{\partial}_q \) and likewise we denote by the same letter the weak maximal extensions of \( \delta_q \) and \( \Delta''_q \). Let us denote by \( N^q(\bar{\partial}) \) the kernel of \( \bar{\partial}_q \), by \( R^{\bar{\partial}^{-1}}(\bar{\partial}) \) the range of \( \bar{\partial}_q^{-1} \), \( N^q(\delta) \) the kernel of \( \delta_q^{-1} : L^2_{0,q}(M, \tilde{E}) \to L^2_{0,q-1}(M, \tilde{E}) \) and by \( N^q(\Delta'') \) the kernel of \( \Delta''_q : L^2_{0,q}(M, \tilde{E}) \to L^2_{0,q}(M, \tilde{E}) \).

By basic results of Andreotti and Vesentini [4] the hilbertian adjoint of \( \bar{\partial} \) coincides with \( \delta \), \( \Delta''_q \) is self-adjoint and

\[ \mathcal{H}^q_{(2)}(M, \tilde{E}) := N^q(\Delta'') = N^q(\delta) \cap N^q(\bar{\partial}) \]

where the first equality is the definition of the space of \( L^2 \) harmonic forms.

Let \( E \) and \( F \) be hermitian holomorphic fibre bundles on \( X \) of rank \( 1 \) and \( r \) respectively, \( \tilde{F} = \pi^* F \), \( \tilde{E} = \pi^* E \). Let us denote \( D = D' + \bar{\partial} \) the canonical connection of \( E \) and \( c(E) = D^2 = D'\bar{\partial} + \bar{\partial}D' \) its curvature form. Also, let

\[ X(q) = \{ x \in X \mid ic(E) \text{ has } q \text{ negative eigenvalues and } n - q \text{ positive eigenvalues} \} \]

and

\[ X(\leq q) = X(0) \cup X(1) \cup \ldots \cup X(q). \]

The main theorem of this paper is:

**Theorem 1.1.** As \( k \to \infty \), the following inequalities hold for every \( q = 0, 1, \ldots, n \):

i) The weak Morse inequalities:

\[ \dim_r \mathcal{H}^q_{(2)}(M, \tilde{E}^k \otimes \tilde{F}) \leq r \frac{k^n}{n!} \int_{X(q)} (-1)^q \left( \frac{i}{2\pi} c(E) \right)^n + o(k^n). \]

ii) The strong Morse inequalities:

\[ \sum_{j=0}^q (-1)^{q-j} \dim_r \mathcal{H}^j_{(2)}(M, \tilde{E}^k \otimes \tilde{F}) \leq r \frac{k^n}{n!} \int_{X(\leq q)} (-1)^q \left( \frac{i}{2\pi} c(E) \right)^n + o(k^n). \]
iii) The asymptotic Riemann-Roch formula:

$$\sum_{j=0}^{n} (-1)^j \dim_r H^j_{(2)}(M, \mathcal{E}^k \otimes \mathcal{F}) = r \frac{k^n}{n!} \int_X \left( \frac{i}{2\pi c(E)} \right)^n + o(k^n).$$

It follows easily

**Corollary 1.2.** Let $E$ and $F$ be as above and suppose that $E$ satisfies (1). Then the space of $L^2$ holomorphic sections satisfies

$$\dim_r H^0_{(2)}(M, \mathcal{E}^k \otimes \mathcal{F}) \approx k^n$$

as $k \to \infty$. In particular the usual dimension of the space of $L^2$ holomorphic sections of $E^k$ has the same cardinal as $|\Gamma|$ for large $k$.

This generalizes the result for the covering of a projective manifold by T. Napier [7].

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2 \; \Gamma–dimension and Estimates

Let $M$ be a real Riemann manifold of dimension $n$, $\Gamma$ a discrete group acting freely and properly discontinuous on $M$ such that $X = M/\Gamma$ is compact, $F \to X$ a hermitian vector bundle of rank $r$ and $\mathcal{F} \to M$ is the pull–back of $F$. Let $U$ be a fundamental domain for the action of $\Gamma$. We identify $L^2(M, \mathcal{F}) \sim = L^2(\Gamma) \otimes L^2(U, \mathcal{F}) \sim = L^2(\Gamma) \otimes L^2(X, \mathcal{F})$.

Let us consider $A_{\Gamma}$ the von Neumann algebra of bounded operators on $L^2(M, \mathcal{F})$ which commute with $\Gamma$. If $A \in A_{\Gamma}$, then let $K_A \in D'(M \times M, \mathcal{F} \otimes_{M \times M} \mathcal{F})$ its kernel. As $A$ is $\Gamma$–invariant, it follows that $K_A \in D'(M \times M/\Gamma, \mathcal{F} \otimes_{M \times M} \mathcal{F}/\Gamma)$ where the action of $\Gamma$ on $M \times M$ is $(x, y) \to (\gamma x, \gamma y)$.

$A \in A_{\Gamma}$ is said to be $\Gamma$–Hilbert–Schmidt if $K_A \in L^2(M \times M/\Gamma, \mathcal{F} \otimes_{M \times M} \mathcal{F}/\Gamma)$ and of $\Gamma$–trace class if $A = A_1 A_2$ with $A_1, A_2$ being $\Gamma$–Hilbert–Schmidt. If $A \in A_{\Gamma}$ is of $\Gamma$–trace–class, one can define

$$\text{Tr}_\Gamma A := \text{Tr}(\varphi A \psi)$$

where $\varphi, \psi \in L^\infty_{\text{comp}}(M)$ such that $\sum_{\gamma \in \Gamma} (\varphi \psi) \circ \gamma = 1$. If $L \subset L^2(M, \mathcal{F})$ is a closed, $\Gamma$–invariant subspace, that is $L$ is a $\Gamma$–module, and $P_L$ is the ortogonal projection onto $L$, then

$$\dim_r L := \text{Tr}_\Gamma P_L \in [0, \infty]$$

This is in short the theory of $\Gamma$–traces. For more results see Atiyah [2] and Shubin [8]. We shall use the following three results; for the proofs see Shubin [8].

**Proposition 2.1.** Let

$$0 \to L_0 \to L_1 \to ... \to L_q \to L_{q+1} \to ... \to L_n \to 0$$

be a complex of $\Gamma$–modules ($d_q$ commutes with the action of $\Gamma$ and $d_{q+1}d_q = 0$). If $l_q = \dim_r L_q < \infty$ and $\bar{h}_q = \dim_r H_q(L)$ where

$$H_q(L) = N(d_q)/R(d_{q-1})$$

Then...
then
\[ \sum_{j=1}^{q} (-1)^{q-j} h_{ij} \leq \sum_{j=1}^{q} (-1)^{q-j} l_{ij} \]  
(5)

for every \( q = 0, 1, \ldots, n \) and for \( q = n \) the inequality becomes equality.

Let \( H = H^* \) be a linear operator in \( L^2(M, \tilde{F}) \) which commutes with the action of \( \Gamma \), that is \( E\lambda \in \mathcal{A} \), where \((E\lambda)\) is the spectral family of \( H \). Let us denote \( N_r(\lambda, H) = \dim_r R(E\lambda) \) and \( h \) the quadratic form of \( H \).

**Proposition 2.2.** If \( H \geq 0 \), then
\[
N_r(\lambda, H) = \sup \{ \dim_r L \mid L \text{ is a } \Gamma - \text{module } \subset \text{Dom}(h), \\
h(f, f) \leq \lambda((f, f)), \forall f \in L \}. 
\]  
(6)

**Proposition 2.3.** If there is \( T : L^2(M, \tilde{F}) \rightarrow L^2(M, \tilde{F}) \) a \( \Gamma \)-endomorphism (i.e. \( T \) commutes with the action of \( \Gamma \)) such that \((|H + T||f, f|) \geq \mu(|(f, f)|), f \in \text{Dom}(H) \) and \( \text{rank}_r T = \dim_r R(T) \leq p \), then
\[
N_r(\mu - \varepsilon, H) \leq p, \forall \varepsilon > 0. 
\]  
(7)

Let \( H \) be an elliptic differential operator, formally self-adjoint of order \( 2m \) on \( \tilde{F} \), which commutes with the action of \( \Gamma \). We shall denote by the same letter \( H \) the weak maximal extension of \( H \). If \( H \) is strongly elliptic, then \( H \) is bounded from below (see Shubin [8]).

**Theorem 2.4.** Let \( H_0 \) be the self-adjoint operator in \( L^2(U, \tilde{F} \mid_U) \) defined by the restriction of \( H \) to \( U \) with Dirichlet boundary conditions. \((H_0 \text{ is bounded from below and has compact resolvent}) \). Then
\[
N_r(\lambda, H) \geq N(\lambda, H_0), \forall \lambda \in \mathbb{R} 
\]  
(8)

where \( N(\lambda, H) = \dim R(F_\lambda) \) if \((F_\lambda)\) is the spectral family of \( H_0 \).

**Proof.** Let \((e_i)\) be an orthonormal basis of \( L^2(U, \tilde{F}) \) which consists of eigenfunctions of \( H_0 \) corresponding to the eigenvalues \((\lambda_i)\); if we let \( e_i = 0 \) on \( M \setminus U \) and \( e_i = e_i \) on \( U \), then \( \tilde{e}_i \in \text{Dom}(h) \) and \((L, \tilde{e}_i)\) is an orthonormal basis of \( L^2(M, \tilde{F}) \) and \( \tilde{e}_i = L, e_i \in \text{Dom}(h) \). We have \( h(e_i, e_j) = \delta_{ij} \delta_{i, j}, \lambda_i \). Let \( \Phi_0^0 \) be the subspace spanned by \((\tilde{e}_i)_{\lambda_i \leq \lambda} \) and \( \Phi_\lambda \) the closed subspace spanned by \((\tilde{e}_i)_{\lambda_i \leq \lambda} \). Then
\[
\dim_r \Phi_\lambda = \sum_{\lambda_i \leq \lambda} \dim ((P\mu, \tilde{e}_i, \tilde{e}_i)) = \sum_{\lambda_i \leq \lambda} \dim \Phi_0^0 = N(\lambda, H_0). 
\]

If \( f \) is a linear combination of \((e_i)_{\lambda_i \leq \lambda} \), then \( h(f, f) \leq \lambda \| f \|^2 \) and, as \( \text{Dom}(h) \) is complete, we obtain that \( \Phi_\lambda \subset \text{Dom}(h) \) and \( h(f, f) \leq \lambda \| f \|^2, f \in \Phi_\lambda \). From Proposition 2.2 it follows that \( N_r(\lambda, H) \geq N(\lambda, H_0) \).

Let \( s > 0 \), \( U_s = \{ x \in M \mid d(x, U) < s \} \) where \( d \) is the distance on \( M \) associated to the Riemann metric on \( M \) and \( U_{s, \gamma} := \gamma U_s \). Let \( \varphi(\gamma) \in C_0^\infty(M) \), \( \varphi(\gamma) \geq 0 \), \( \varphi(\gamma) = 1 \) on \( \tilde{U} \) and \( \text{supp} \varphi(\gamma) \subset U_s \), \( \varphi(\gamma) = \varphi(\gamma) \circ \gamma^{-1} \). Put
\[
C_\gamma^{(s)}(\gamma) = \frac{\varphi(\gamma)}{\left( \sum_{\gamma} \varphi(\gamma)^2 \right)^{1/22}} \in C_0^\infty(M) 
\]
so that \( \sum_{\gamma \in \Gamma} (C^{(s)}_\gamma)^2 = 1 \). If \( m = 1 \) (that is \( H \) is of order 2) then

\[
H = \sum_{\gamma \in \Gamma} C^{(s)}_\gamma HC^{(s)}_\gamma - \sum_{\gamma \in \Gamma} \sigma_0(H)(dC^{(s)}_\gamma)
\]

(9)

where \( \sigma_0 \) is the principal symbol of \( H \) (see Shubin\[8\]).

Let us assume that \( X \) is a complex analytic manifold, \( \pi : M \to X \), \( E \) and \( F \) hermitian holomorphic vector bundles on \( X \), \( \bar{E} = \pi^*F \), \( \bar{F} = \pi^*E \). Let \( \Delta''_{k,q} \) be the Laplace–Beltrami operator on \( \Lambda^0 \otimes T^*M \otimes \bar{E}^k \otimes \bar{F} \). If \( s = k^{-\frac{1}{4}} \), \( H = \frac{1}{k} \Delta''_{k,q} \) in \([\]\) then it follows that there is a constant \( C \) such that

\[
\frac{1}{k} \Delta''_{k,q} \geq \frac{1}{k} \sum_{\gamma \in \Gamma} \frac{1}{k} J^{(k)}_{\gamma} \Delta''_{k,q} J^{(k)}_{\gamma} - \frac{C}{\sqrt{k}} \text{Id}
\]

(10)

where \( J^{(k)}_{\gamma} = C^{(k-\frac{1}{4})}_\gamma \), \( k \in \mathbb{N}^* \). We have used that \( \sigma_0(\Delta''_{k,q})(dJ) = |\partial J|^2 \text{Id} \) if \( J \in C^\infty(M,\mathbb{R}) \).

Let us denote by \( V = \Lambda^0 \otimes T^*M \otimes \bar{E}^k \otimes \bar{F} \), \( H = \frac{1}{k} \Delta''_{k,q} \) and \( H^{(k)}_0 = \frac{1}{k} \Delta''_{k,q} | U_{k^{-\frac{1}{4}}} \) the operator defined in \( L^2(U_{k^{-\frac{1}{4}}},V) \) by the restriction of \( \frac{1}{k} \Delta''_{k,q} \) to \( U_{k^{-\frac{1}{4}}} \) with Dirichlet boundary conditions. Let \( (E^{(k)}_\lambda)_\lambda \) be the spectral family of \( H^{(k)}_0 \). In the sequel we fix \( \lambda \) and consider \( M^{(k)} \) a real number such that \( M^{(k)} \geq \lambda - \inf \text{spec}(H^{(k)}_0) \), where \( \text{spec}(H^{(k)}_0) \) is the spectrum of \( H^{(k)}_0 \) to the effect that

\[
H^{(k)}_0 + M^{(k)} E^{(k)}_\lambda \geq \lambda \text{Id}.
\]

Define

\[
G^{(k)}_\gamma : L^2(M,V) \to L^2(M,V) \quad G^{(k)}_\gamma = J^{(k)}_{\gamma} L^{-1} M^{(k)} E^{(k)}_\lambda L^{-1} J^{(k)}_{\gamma},
\]

(i.e. we trunk the section over \( U_{s,\gamma} \), transport it on \( U_s \), apply the spectral projection and then send it back to \( U_{s,\gamma} \)) and

\[
G^{(k)} = \sum_{\gamma \in \Gamma} G^{(k)}_\gamma.
\]

We have

\[
H + G^{(k)} \geq \sum_{\gamma \in \Gamma} \left( J^{(k)}_{\gamma} H J^{(k)}_{\gamma} + J^{(k)}_{\gamma} L^{-1} M^{(k)} E^{(k)}_\lambda L^{-1} J^{(k)}_{\gamma} \right) - \frac{C}{\sqrt{k}} \text{Id}
\]

\[
= \sum_{\gamma \in \Gamma} J^{(k)}_{\gamma} L^{-1} (H^{(k)}_0 + M^{(k)} E^{(k)}_\lambda) L^{-1} J^{(k)}_{\gamma} - \frac{C}{\sqrt{k}} \text{Id}
\]

\[
\geq \sum_{\gamma \in \Gamma} J^{(k)}_{\gamma} L^{-1} M^{(k)} L^{-1} J^{(k)}_{\gamma} - \frac{C}{\sqrt{k}} \text{Id}
\]

(11)

\[
= \left( \lambda - \frac{C}{\sqrt{k}} \right) \text{Id}.
\]

**Lemma 2.5.** \( \text{rank} H^{(k)} \leq N(\lambda, H^{(k)}_0) \).
Proof. The operator

$$L^{(1)}_{\alpha} : \bigoplus_{\gamma \in \Gamma} L^2(U_{s,\gamma}, V) \rightarrow \bigoplus_{\gamma \in \Gamma} L^2(U_{s,\gamma}, V),$$

$$L^{(1)}_{\alpha}(\gamma) = (w_{\alpha-\gamma})_{\gamma}$$

is a unitary operator for any \( \alpha \in \Gamma \). Consider \( i : L^2(M, V) \rightarrow \bigoplus_{\gamma \in \Gamma} L^2(U_{s,\gamma}, V) \), \( i(u) = (u | U_{s,\gamma})_{\gamma} \). Then \( \|u\| \leq \|i(u)\| \leq C_1 \|u\| \) and hence \( i \) is into and bounded. Moreover \( L^{(1)}_{\alpha} i = i L_{\alpha} \), for \( \alpha \in \Gamma \). Let

$$F : \bigoplus_{\gamma \in \Gamma} L^2(U_{s,\gamma}, V) \rightarrow L^2(M, V),$$

$$F((w_{\gamma})_{\gamma}) = \sum_{\gamma \in \Gamma} w_{\gamma}.$$

\( F \) is onto, bounded and \( F L^{(1)}_{\alpha} = L_{\alpha} F \), \( \alpha \in \Gamma \). We define

$$\tilde{G}^{(k)} : \bigoplus_{\gamma \in \Gamma} L^2(U_{s,\gamma}, V) \rightarrow \bigoplus_{\gamma \in \Gamma} L^2(U_{s,\gamma}, V),$$

$$\tilde{G}^{(k)}((w_{\gamma})_{\gamma}) = (j^{(k)} \lambda^{(k)} L^{-1} M^{(k)} E^{(k)}_{\lambda} L \lambda H^{(k)}_{w_{\gamma}})_{\gamma}.$$

Then \( \tilde{G}^{(k)} \) is bounded, commutes with \( L^{(1)}_{\alpha} \) and \( G^{(k)} = F \tilde{G}^{(k)} i \). We define also the operator

$$K : \bigoplus_{\gamma \in \Gamma} L^2(U_{s,e}, V) \rightarrow \bigoplus_{\gamma \in \Gamma} L^2(U_{s,\gamma}, V),$$

$$K((w_{\gamma})_{\gamma}) = (L_\gamma w_{\gamma})_{\gamma},$$

which is unitary and \( K L^{(2)}_{\alpha} = L^{(1)}_{\alpha} K \), \( \alpha \in \Gamma \) where

$$L^{(2)}_{\alpha} : \bigoplus_{\gamma \in \Gamma} L^2(U_{s,e}, V) \rightarrow \bigoplus_{\gamma \in \Gamma} L^2(U_{s,\gamma}, V),$$

$$L^{(2)}_{\alpha}(\gamma) = (w_{\alpha-\gamma})_{\gamma}.$$

Finally, let

$$\bar{G}^{(k)} : \bigoplus_{\gamma \in \Gamma} L^2(U_{s,e}, V) \rightarrow \bigoplus_{\gamma \in \Gamma} L^2(U_{s,\gamma}, V),$$

$$\bar{G}^{(k)}((w_{\gamma})_{\gamma}) = (E^{(k)}_{\lambda} M^{(k)} H^{(k)}_{w_{\gamma}})_{\gamma}.$$

Then

$$K \bar{G}^{(k)} = \bar{G}^{(k)} K.$$

As \( G^{(k)} = F \bar{G}^{(k)} i \), we have that \( \text{rank}_r G^{(k)} \leq \text{rank}_r \bar{G}^{(k)} \). The operator \( K \) being unitary it follows from (12) that \( \text{rank}_r \bar{G}^{(k)} = \text{rank}_r \tilde{G}^{(k)} \). But \( R(\tilde{G}^{(k)}) \) is closed because \( R(\bar{G}^{(k)}_{e}) \) is closed \( (\bar{G}^{(k)}_{e} \) is the component of \( \tilde{G}^{(k)} \) on \( L^2(U_{s,e}, V) \) and has finite rank). If we identify \( \bigoplus_{\gamma \in \Gamma} L^2(U_{s,e}, V) \) with \( L^2 \Gamma \otimes L^2(U_{s,e}, V) \) and consider \( \tilde{G}^{(k)} \) as an operator in \( L^2 \Gamma \otimes L^2(U_{s,e}, V) \), then \( R(\tilde{G}^{(k)}) \) corresponds to \( L^2 \Gamma \otimes R(\bar{G}^{(k)}_{e}) \) and hence

$$\text{rank}_r \tilde{G}^{(k)} = \text{rank}_r \bar{G}^{(k)} \leq \text{rank}_r E^{(k)}_{\lambda} = N(\lambda, H^{(k)}_{0}).$$

Now the conclusion follows from the inequality

$$\text{rank}_r G^{(k)} \leq \text{rank}_r \bar{G}^{(k)} = \text{rank}_r \tilde{G}^{(k)}.$$

\( \square \)
Proposition 2.6. There is a constant $C \geq 0$ such that

$$N_\Gamma \left( \lambda, \frac{1}{k} \Delta''_{k,q} \right) \leq N \left( \lambda + \frac{C}{\sqrt{k}}, \frac{1}{k} \Delta''_{k,q} | U_{k^{-\frac{1}{2}}} \right) \quad \lambda \in \mathbb{R}, \ k \in \mathbb{N}^*$$

Proof. Proposition 2.2 with $\mu = \lambda - \frac{C}{\sqrt{k}}$ and $p = N \left( \lambda, \frac{1}{k} \Delta''_{k,q} | U_{k^{-\frac{1}{2}}} \right)$, (11) and Lemma 2.3 entail

$$N_\Gamma \left( \lambda - \frac{C}{\sqrt{k}} - \varepsilon, 1 \right) \leq N \left( \lambda + \frac{C}{\sqrt{k}} + \varepsilon, 1 \right)$$

Replacing $\lambda$ with $\lambda + \frac{C}{\sqrt{k}} + \varepsilon$, we obtain

$$N_\Gamma \left( \lambda, \frac{1}{k} \Delta''_{k,q} \right) \leq N \left( \lambda + \frac{C}{\sqrt{k}} + \varepsilon, \frac{1}{k} \Delta''_{k,q} | U_{k^{-\frac{1}{2}}} \right)$$

When $\varepsilon \to 0$ it follows

$$N_\Gamma \left( \lambda, \frac{1}{k} \Delta''_{k,q} \right) \leq N \left( \lambda + \frac{C}{\sqrt{k}}, \frac{1}{k} \Delta''_{k,q} | U_{k^{-\frac{1}{2}}} \right)$$

\[\square\]

3 Holomorphic Morse Inequalities

Let $M$ be a Riemannian manifold of dimension $n$ with volume element $d\sigma$. Let $E$ and $F$ be hermitian vector bundles on $M$, rank $E = 1$, rank $F = r$, with $D$ and $\nabla$ the canonical connections, $S$ a continuous section in $\Lambda^1 M \otimes \mathbb{R}$ Hom$_\mathbb{C}(F,F)$ and $V$ a continuous section in Herm$(F)$. Let $\nabla_k$ be the connection in $E^k \otimes F$. We denote the endomorphisms $\text{Id}_{E^k} \otimes S$ and $\text{Id}_{E^k} \otimes V$ by $S$ and $V$. Given $\Omega \subset M$, let

$$Q_{\Omega,k}(u) = \int_{\Omega} \left( \frac{1}{k} | \nabla_k u + Su|^2 - (Vu, u) \right) d\sigma,$$

$$\text{Dom}(Q_{\Omega,k}) = W^1_0(\Omega, E^k \otimes F)$$

where by $W^1_0$ we denote the Sobolev space. Let $V_1(x) \leq ... \leq V_r(x)$ be the eigenvalues of $V(x)$. We shall use the following

Theorem 3.1 (Demailly [4]). The counting function of the eigenvalues of $Q_{\Omega,k}$ satisfies for every $\lambda \in \mathbb{R}$ the following asymptotic estimates as $k \to \infty$:

$$\sum_{j=1}^r \int_{\Omega} \nu_B(V_j + \lambda) d\sigma \leq \liminf k^{-\frac{n}{2}} N(\lambda, Q_{\Omega,k}) \leq \limsup k^{-\frac{n}{2}} N(\lambda, Q_{\Omega,k}) \leq \sum_{j=1}^r \int_{\Omega} \tilde{\nu}_B(V_j + \lambda) d\sigma$$

where $B$ is the magnetic field of the connection $D$ and

$$\nu_B(\lambda) = \frac{2^{s-n-\frac{n}{2}}}{\Gamma(\frac{n}{2} - s + 1)} B_1 \cdots B_s \sum_{(p_1, ..., p_n) \in \mathbb{N}^n} \left[ \lambda - \sum_{j=1}^s (2p_j + 1) B_j \right]^{\frac{n}{2} - s}$$

if $B_1(x) \geq ... \geq B_s(x)$ are the absolute values of the non–zero eigenvalues of $B$, $[\lambda]_+^0 = 0$ for $\lambda \leq 0$, $[\lambda]_+^0 = 1$ for $\lambda > 0$ and $\tilde{\nu}_B = \lim \nu_B(\lambda + \varepsilon), \varepsilon \searrow 0$. 

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We assume now that $M$ is a hermitian complex analytic manifold of complex dimension $n$, $E$ and $F$ hermitian holomorphic vector bundles, $F_k = E^k \otimes F$ and $\Delta''_{k,q}$ the Laplace–Beltrami operator on $F_k$. If $\alpha_1(x), \ldots, \alpha_n(x)$ are the eigenvalues of $ic(E)(x)$ with respect to the metric on $M$ then, as in Demailly [4], we deduce that there is a countable set $A \subset \mathbb{R}$ such that

$$\lim k^{-n}N\left(\lambda, \frac{1}{k} \Delta''_{k,q} \mid \Omega \right) = r \sum_{|J|=q} \int_{\Omega} \nu_B(2\lambda + \alpha_{C(J)} - \alpha_J) \, d\sigma$$

(17)

for $\lambda \in \mathbb{R} \setminus A$, where $\alpha_J = \sum_{j \in J} \alpha_j$, $C(J) = \{1, \ldots, n\} \setminus J$.

Let $M$ be an analytic complex manifold of dimension $n$ and $\Gamma$ a discrete group which acts freely and properly discontinuous on $M$ such that $X = M/\Gamma$ is compact.

We choose a hermitian metric on $X$ and we lift it on $M$. Let $E$ and $F$ be hermitian holomorphic vector bundles on $X$, rank $E = 1$, rank $F = r$ and $\bar{E} = \pi^*E$, $\bar{F} = \pi^*F$.

Let $E(\cdot, \Delta''_{k,q})$ the spectral family of the self-adjoint operator $\Delta''_{k,q}$ in $L^2_{\bar{\Delta}}(M, \bar{E}^k \otimes \bar{F})$ and $\bar{L}^\lambda_k = R\left(E \left([0, \lambda], \frac{1}{k} \Delta''_{k,q} \right)\right)$. Then

$$E([0, \lambda], \Delta''_{k,q}) \bar{\Delta}_{q-1} = \bar{\Delta}_{q-1} E([0, \lambda], \Delta''_{k,q-1})$$

on $\bar{L}^\lambda_{q-1}$ and it follows that $\bar{\Delta}_{q-1} \bar{L}^\lambda_{q-1} \subset \bar{L}^\lambda_k$. If $\bar{\Delta}_{q}^\lambda$ denotes the restriction

$$\bar{\Delta}_{q} : \bar{L}^\lambda_k \to \bar{L}^\lambda_{k+1}$$

then

$$\{u \in \bar{L}^\lambda_{q-1} \mid \bar{\Delta}_{q-1}^\lambda u = 0, (\bar{\Delta}_{q-1}^\lambda)^* u = 0\} = H^q(M, \bar{E}^k \otimes \bar{F})$$

(18)

(see Shubin [8]). By definition $N_r(\lambda, \frac{1}{k} \Delta''_{k,q}) = \text{dim}_r \bar{L}^\lambda_k$. There is a complex

$$0 \to \bar{L}^{\lambda,1}_0 \to \bar{L}^{\lambda,1}_1 \to \cdots \to \bar{L}^{\lambda,1}_n \to 0$$

(19)

From Proposition 2.1 we get

$$\sum_{j=1}^q (-1)^{q-j} \text{dim}_r (N(\bar{\Delta}^\lambda_{q-j})/R(\bar{\Delta}^\lambda_{q-1})) \leq \sum_{j=1}^q (-1)^q N_r\left(\lambda, \frac{1}{k} \Delta''_{k,q} \right)$$

for $q = 0, 1, \ldots, n$ and for $q = n$ the inequality becomes equality.

From \cite{13}, $N(\bar{\Delta}^\lambda_{q-j})/R(\bar{\Delta}^\lambda_{q-1}) \simeq H^q_{\bar{\omega}}(M, \bar{E}^k \otimes \bar{F})$. From Theorem 2.4 and Proposition 2.6 it follows that there is a constant $C$ such that

$$N(\lambda, \frac{1}{k} \Delta''_{k,q} \mid U) \leq N_r(\lambda, \frac{1}{k} \Delta''_{k,q}) \leq N\left(\lambda + \frac{C}{\sqrt{k}}, \frac{1}{k} \Delta''_{k,q} \mid U \frac{|U|}{n}\right)$$

(20)

But

$$N(\lambda, \frac{1}{k} \Delta''_{k,q} \mid U) = r k^n \sum_{|J|=q} \int_U \nu_B(2\lambda + \alpha_{C(J)} - \alpha_J) \, d\sigma + o(k^n)$$

and

$$\lim sup k^{-n} N\left(\lambda + \frac{C}{\sqrt{k}}, \frac{1}{k} \Delta''_{k,q} \mid U \frac{|U|}{n}\right) \leq \lim k^{-n} N(\lambda, \frac{1}{k} \Delta''_{k,q} \mid \bar{U}_e)$$

$$= r \sum_{|J|=q} \int_{\bar{U}_e} \nu_B(2\lambda + 2\varepsilon + \alpha_{C(J)} - \alpha_J) \, d\sigma$$

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so when $\varepsilon \to 0$ we get
\[
\lim k^{-n} N \left( \lambda + \frac{C}{k} \Delta''_{k,q} | U_{k^{-\frac{1}{2}}} \right) = r \sum_{|J|=q} \nu_B (2\lambda + \alpha_{C(J)} - \alpha_J) d\sigma
\]
for every $\lambda \in \mathbb{R} \setminus A$. As $\partial U = \tilde{U} \setminus U$ is of measure zero because $U$ is a fundamental domain, it follows that
\[
N_r(\lambda, \frac{1}{k} \Delta''_{k,q}) = r k^n \sum_{|J|=q} \nu_B (2\lambda + \alpha_{C(J)} - \alpha_J) d\sigma + o(k^n)
\]
for $\lambda \in \mathbb{R} \setminus A$. Hence for $\lambda \to 0$, $\lambda \in \mathbb{R} \setminus A$ we obtain
\[
\sum_{j=0}^{q} (-1)^{q-j} \dim \mathcal{H}^2_j (M, \mathcal{E}^k \otimes \tilde{F}) \leq k^n \sum_{j=0}^{q} (-1)^{q-j} I^j + o(k^n) \quad (21)
\]
where
\[
I^j = \frac{r}{n!} \int_{M(j) \cap U} (-1)^j \left( \frac{i}{2\pi} c(\tilde{E}) \right)^n = \frac{r}{n!} \int_{X(j)} (-1)^j \left( \frac{i}{2\pi} c(E) \right)^n \quad (22)
\]
with $M(j) = \{ x \in M | ic(\tilde{E})(x) \text{ has } j \text{ negative eigenvalues and } n-j \text{ positive ones} \}$. We have used that $c(\tilde{E})$ is the lifting of $c(E)$. Theorem 1.1 now follows from (21) and (22).

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