Topological Resonances on Quantum Graphs

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April 7, 2016

Introduction

We will consider metric graphs $G$ which consist of a finite graph $\Gamma$ with some leads attached to some vertices. To this metric graph is associated a Laplacian using the Kirchhoff conditions. Resonances on such Quantum graphs are introduced in the book [BK13] and studied in several papers (see [EL10, DP11, LZ16]). In the paper [DP11], the following result is proved:

**Theorem 0.1** All resonances $k_j = \sigma_j + i\tau_j$, $j \in \mathbb{N}$, lie in a band $-M \leq \tau_j \leq 0$ and they have the large $K$ asymptotic

$$\# \{ j \mid |\sigma_j| \leq K \} = \frac{2V}{\pi} K + O(1),$$

with $0 < V \leq |L|$ where $|L|$ is the total length of the finite graph $\Gamma$.

In this paper, we will be interested in resonances close to the real axis which in physics are the most important. They are linked to compactly supported eigenfunctions as anticipated in [EL10]. The goal of this paper is to describe some asymptotic properties of these resonances, called “topological resonances” in the paper [GSS13]. See also the paper [LZ16] for the explicit calculations of the related “Fermi golden rule”. We show that there is a dichotomy between graphs which can have eigenfunctions with compact support for some particular metrics and the other ones which are some specific trees, namely those with at most one vertex of degree one. In the first case, there are many resonances close to the real axis and we are able to say something on their asymptotics, while in the second one, there is a gap which is an invariant associated to the graph.

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We will follow the notations of the paper [CdV15] of the first author. 

Acknowledgements: the first author thanks Uzy Smilansky for motivating him to study the topological resonances while visiting our Institut. This work has been partially supported by the LabEx PERSYVAL-Lab (ANR–11-LABX-0025).

1 Main results and conjectures

We consider a finite graph $\Gamma = (V, E)$ and fix a subset $V_0 \subset V$. To each vertex $w$ of $V_0$, we attach a non zero number $n_w$ of infinite half-lines, called the “leads” in [GSS13]. The total number of leads is denoted by $N$ with $N = \sum_{w \in V_0} n_w$.

Allowing loops and multiple edges, we can (and will) always assume that

$$\forall v \in V, \text{deg}_G(v) \neq 2.$$ 

The metric graph denoted by $G$ is the union of $\Gamma$ with some lengths $l_e > 0$ for $e \in E$, and the attached infinite half-lines. Let us denote by $\vec{l}$ the vector in $\mathbb{R}^n$ with coordinates $l_e$, $e \in E$. To this set of data, we associate, using the usual Kirchhoff conditions at the vertices, a non negative self-adjoint Laplacian $\Delta_{\vec{l}}^G$ acting on $L^2(|G|, |dx|)$ where $|G|$ is the 1D singular manifold associated to $G$ and $|dx|$ is the Riemannian measure. Let us specify that, at any vertex of degree 1, we impose Neumann boundary conditions. This Laplacian has a discrete sequence of non negative eigenvalues (possibly empty) and a continuous spectrum $[0, +\infty[$ of multiplicity $N$. The Schwartz kernel of the resolvent $\left(\frac{k^2 - \Delta_{\vec{l}}^G}{\sigma + i\tau}\right)^{-1}$ defined for $\Im k > 0$ extends to the lower half plane in a meromorphic way. The poles of this extension in $\Im k \leq 0$ are called the resonances. We denote by $\text{Res}_{\vec{l}}^G$ the set of resonances of $\Delta_{\vec{l}}^G$. Our goal is to study $\text{Res}_{\vec{l}}^G$, mainly in the case where the lengths $(l_e)_{e \in E}$ are independent over the integers (we will say that $\vec{l}$ is “irrational”).

There are two mutually disjoint families of graphs $G$:

- **Type I**: the trees with at most one vertex of degree 1
- **Type II**: all other graphs.

The Type I graphs are the graphs $G$ for which, for any choice of $\vec{l}$, the operator $\Delta_{\vec{l}}^G$ has no $L^2$ eigenfunctions or equivalently $\Delta_{\vec{l}}^\Gamma$ has no eigenfunctions vanishing on $V_0$.

The main results of this note are

**Theorem 1.1** If the graph $G$ is a tree with at most one vertex of degree 1 ($G$ is of type I), there exists a minimal finite number $h(G) > 0$ so that, for any choice of the lengths $l_e > 0$ with $|L| := \sum l_e$, we have

$$\text{Res}_{\vec{l}}^G \subset \{\sigma + i\tau \mid \tau |L| \leq -h(G)\}.$$

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The optimal constant $h(G)$ is an interesting graph parameter. It follows from the next results that $h(G) = 0$ for type II graphs. Moreover we have that

**Theorem 1.2** If $G'$ is obtained from $G$ by contracting some edges of $\Gamma$ while keeping the leads in a natural way, then $h(G') \geq h(G)$.

It would be interesting to say more on this graph parameter.

In order to state the second main result, we need the

**Definition 1.1** For a graph $G$ as before, $W_G$ is the sub-set of the torus $T^E := (e^{it})_{e \in E}$ so that $\Delta^\Gamma$ admits 1 as an eigenvalue with a non zero eigenfunction vanishing on $V_0$; we define also $W^o_G$ as the subset of $W_G$ where the eigenfunction of $\Delta^\Gamma$ vanishing on $V_0$ is unique, up to scaling.

Note that $W_G$ depends only on the choice of $V_0$. If $G$ is of type II, the sets $W_G$ and $W^o_G$ are non-empty semi-algebraic sets (see [CdV15] and Appendices B and C). It follows from Theorem 4.1 that $W^o_G$ is a smooth (non closed in general) submanifold of $T^E$ which is an union of connected components (called the strata) of various dimensions. The maximal dimension of these strata is called the dimension of $W^o_G$. We will need the

**Definition 1.2** The number $d(G)$ is defined by

$$d(G) := \#E - 1 - \dim W^o_G.$$ 

We will be interested in the following quantity:

**Definition 1.3** For $\varepsilon \geq 0$, let us define $N_{G,\vec{l}}(\varepsilon)$ as follows:

$$N_{G,\vec{l}}(\varepsilon) := \liminf_{K \to +\infty} \frac{1}{K} \# \{ \sigma_j + i\tau_j \in \text{Res}^\vec{l}_G : 0 \leq \sigma_j \leq K, -\varepsilon \leq \tau_j \leq 0 \}.$$ 

Finally let us define a combinatorial invariant $g(G)$ of the graph $G$:

**Definition 1.4** If $G$ is of type I, $g(G) = +\infty$, otherwise $g(G)$ is the smallest number so that there exists either a simple cycle in $G$ with $g(G)$ vertices or a path in $G$ joining two vertices of degree 1 with $g(G) + 1$ vertices (and $g(G)$ edges).

The number $g(G)$ can be smaller than the girth of $G$, for example if $G$ is a tree of type II.

We have the

**Theorem 1.3** 1. If the graph $G$ is of type II and $\vec{l}$ is irrational, there exist $\varepsilon_0 > 0$ and $C > 0$, depending on $\vec{l}$, so that, for $0 < \varepsilon \leq \varepsilon_0$, we have

$$N_{G,\vec{l}}(\varepsilon) \geq C \varepsilon^{d(G)/2}.$$
2. Moreover, we have

- If there exists \( a \in V_0 \) so that there is no loops at the vertex \( a \), we have \( d(G) \geq 1 \).
- \( d(G) \leq g(G) - 1 \).
- If \( \gamma \) is a simple cycle of \( \Gamma \), then
  - either \( d(G) \leq \#V_0 \) for all \( V_0 \subset V(\gamma) \)
  - or there exists a vertex \( a \) of \( \gamma \) so that \( d(G) \leq \#V_0 \) if \( V_0 \subset V(\gamma) \) and \( a \notin V_0 \).
- If \( \Gamma \) is 2-connected,
  - either \( d(G) = 1 \) for all sets \( V_0 \) with \( \#V_0 = 1 \)
  - or there exists a unique vertex \( a \) of \( \Gamma \) so that \( d(G) = 1 \) for all sets \( V_0 \) with \( \#V_0 = 1 \) and \( V_0 \neq \{a\} \).
- If \( V_0 = V \), then \( d(G) = g(G) - 1 \).

We are not able to derive an upper bound in general, but we conjecture, following [GSS13], the following estimates:

**Conjecture 1.1** As \( \varepsilon \to 0^+ \), there exists \( C > 0 \) so that

\[
N_{G,\vec{l}}(\varepsilon) \sim C\varepsilon^{d(G)/2} ,
\]

and

\[
d(G) = \min(g(G) - 1, \#V_0) .
\]

**Remark 1.1** How is \( N_{G,\vec{l}}(\varepsilon) \) related to the Gnutzmann-Schanz-Smilansky paper [GSS13]? Let us choose the resonant state \( u_j \) associated to \( k_j = \sigma_j + i\tau_j \) so that \( \sum_{m=1}^{N} |t_m|^2 = 1 \), then we look at its energy in \( \Gamma \) defined by

\[
\mathcal{E}(u) := \int_{|\Gamma|} |u|^2 |dx|_{|\Gamma|} .
\]

**Proposition 2.1** gives that \( \mathcal{E}(u) = 1/(2|\tau|) \). The previous authors look at the asymptotic behaviour, as \( \alpha \to \infty \), of

\[
P_{G,\vec{l}}(\alpha) := \lim_{K \to \infty} \frac{1}{K} \# \{ k_j = \sigma_j + i\tau_j | 0 \leq \sigma_j \leq K, \mathcal{E}(u_j) \geq \alpha \} ,
\]

which is the same as \( N_{G,\vec{l}}(1/2\alpha) \) at which we are looking in the present paper.
2 Finding resonances

We will use the following notations: if \( z \) is a vector in \( \mathbb{C}^n \), \((z)\) will be the diagonal matrix with entries the coordinates of \( z \) and \((z)_2\) will denote the diagonal matrix of size 2n

\[
(z)_2 = \begin{pmatrix}
(z) & 0 \\
0 & (z)
\end{pmatrix}.
\]

We choose an orientation of each edge of \( \Gamma \) and parametrize the edge \( e \) by a real parameter \( x_e \) with \( 0 \leq x_e \leq l_e \) according to the orientation. The leads are parametrized by \( x_m, m = 1, \ldots, N \), with \( 0 \leq x_m < \infty \). We will denote by \(|dx|_G^2 = \sum_{e \in E} |dx_e| + \sum_{m=1}^N |dx_m|\) the Riemannian measure on \(|G|\) where \(|G|\) is the 1D singular topological space associated to \( G \).

**Definition 2.1** A complex number \( k = \sigma + i\tau \) with \( \tau \leq 0 \) is a “resonance” of \( \Delta_G^{\vec{l}} \) if and only if there exists a non zero “resonant state” \( u \) which satisfies \( (\Delta_G^{\vec{l}} - k^2) u = 0 \) and \( \forall m = 1, \ldots, N, \exists t_m \in \mathbb{C}, u(x_m) = t_m \exp(ikx_m) \).

Equivalent definitions of resonances for Quantum graphs are given in [EL07].

Following [BK13] sec. 5.4., we can describe all solutions of \((\Delta_G^{\vec{l}} - k^2) u = 0\) in the following way: writing \( u(x_e) = a_e \exp(ikx_e) + b_e \exp(-ikx_e) \) and \( u(x_m) = t_m^{\text{out}} \exp(ikx_m) + t_m^{\text{in}} \exp(-ikx_m) \), and denoting by \( e^{ik\vec{l}} \) the point in the \( \mathbb{C}^E \) of coordinates \( e^{ikl_e}, e \in E \), the Kirchhoff conditions at the vertices express as

\[
\begin{pmatrix}
t^{\text{out}} \\
a \\
b
\end{pmatrix} =
\begin{pmatrix}
R & T_o \\
T_i & U(e^{ik\vec{l}})_2
\end{pmatrix}
\begin{pmatrix}
t^{\text{in}} \\
a \\
b
\end{pmatrix}.
\]

The \( 2n + N \) square matrix

\[
S = \begin{pmatrix}
R & T_o \\
T_i & U(e^{ik\vec{l}})_2
\end{pmatrix}
\]

is unitary for real \( k \)'s. The resonances are the value of \( k \) for which there exists a non trivial solution of Equation (1) with \( t^{\text{in}} = 0 \) and are hence given by the equation

\[
\mathcal{R}_G(\exp(ikl_1), \ldots, \exp(ikl_n)) = 0
\]

where

\[
\mathcal{R}_G(z) = \det(\text{Id} - U(z)_2).
\]

The associated resonant state is given by

\[
U(e^{ik\vec{l}})_2 \begin{pmatrix}
a \\
b
\end{pmatrix} = \begin{pmatrix}
a \\
b
\end{pmatrix},
\]

\( t^{\text{in}} = 0, t^{\text{out}} = T_o \begin{pmatrix}
a \\
b
\end{pmatrix} \).

We will need the
Proposition 2.1 If $u$ is a resonant state for the resonance $k = \sigma + i\tau$ of $\Delta^\vec{l}_G$ with $\sigma \neq 0$, we have

$$-2\tau \int_{|\Gamma|} |u|^2 |dx|_l = \sum_{m=1}^{N} |t_m|^2.$$  

Proof. – Let us evaluate the integral

$$I = \int_{|\Gamma|} (u\Delta \bar{u} - \bar{u}\Delta u) |dx|_l$$

in two ways: first, using the differential equation $\Delta u = k^2 u$, we have

$$I = -4i\sigma \tau \int_{|\Gamma|} |u|^2 |dx|_l,$$

whereas using the Green-Riemann formula, we get:

$$I = 2i\sigma \sum_{k=1}^{N} |t_m|^2.$$

□

Corollary 2.1 The resonant states with $\tau = 0$ are eigenstates of $\Delta^\vec{l}_G$ with support in $|\Gamma|$. And conversely, each such eigenfunction of $\Delta^\vec{l}_G$ is a resonant state of $\Delta^\vec{l}_G$.

Proposition 2.1 implies that all $t_m$’s vanish, hence the conclusions.

3 Type I: trees with at most one vertex of degree one

Let us prove Theorem 1.1. We start with the following Lemma:

Lemma 3.1 If $G$ is a tree with at most one vertex of degree 1 and $\vec{l}$ is given, then $\Delta^\vec{l}_G$ has no non vanishing $L^2$ eigenfunction.

Proof. – Let us assume that the eigenvalue is $k^2 > 0$. It is clear that the corresponding eigenfunction $u$ has to vanish on the leads because $u$ is of the form $a_i \cos{kx_i} + b_i \sin{kx_i}$ on $l_i$ and is square integrable. Let us assume that $u$ does not vanish identically along an edge $e_1 = [v_1, v_2]$, then, either $v_2$ is of degree 1, or there exists an edge $e_2 = [v_2, v_3]$ with $v_3 \neq v_1$, so that $u$ does not vanish identically on $e_2$. Iterating, and possibly going in the other direction, we get that there exists either a cycle, or a path joining two vertices of degree 1 so that $u$ has a finite number of zeroes on them. By assumption, $G$ has no cycles, and all
paths starting from a vertex of degree 1 will end into a lead where it has to vanish identically. □

Let us now give the proof of Theorem 1.1.

Proof.– By rescaling the lengths $\ell_v$, we can assume that $|L| = 1$. By contradiction, there exists a sequence of vectors $\vec{l}_n$ so that $u_n$ is a resonant state of $\Delta_n := \Delta^\vec{l}_n_G$ of resonance $k_n = \sigma_n + i\tau_n$, with $\int_{|\Gamma|} |u_n|^2 dx |n| = 1$, and $\tau_n \to 0$. We have the equations

$$(U(z_n)_2 - \text{Id}) \begin{pmatrix} a_n \\ b_n \end{pmatrix} = 0, \quad t_n^{\text{out}} = T_0 \begin{pmatrix} a_n \\ b_n \end{pmatrix},$$

with

$$(z_n)_e = e^{i k_n(l_n)_e}.$$  

From Proposition 2.1 we already know that $t_n^{\text{out}} \to 0$. We can extract converging sub-sequences of the vectors $\exp(i \sigma_n l_n^e)$ going to $\exp(i l_\infty^e e)$. It is not possible that the vectors $(a_n, b_n)$ tend to 0, because it would imply that the $L^2$ norms of $u_n$ on $|\Gamma|$ tend to 0. On the other hand, if we denote by $M^2_n = \max_e (|a_n^e|^2 + |b_n^e|^2)$, the sequence $(a_n, b_n)/M_n$ converges (up to extraction of a subsequence) to the coefficients of an eigenfunction with compact support of $\Delta^\vec{l}_\infty_G$ with eigenvalue 1 and the contradiction follows from Lemma 3.1. □

Example 3.1 We compute $h(G)$ defined in Theorem 1.1 in some simple examples:

- $G$ is a star graph where all edges are leads: there is no resonances.
- $G$ is a star graph with only one edge of finite length $l$ and $N > 1$ leads: the resonances are
  $$k_j = \frac{1}{2l} \left( (1 + 2j) \pi - i \log \frac{N + 1}{N - 1} \right), \quad j \in \mathbb{Z},$$
  so that $h(G) = \frac{1}{2} \log \frac{N+1}{N-1}$.
- $\Gamma$ is an interval $[v, v']$ of length $l > 0$ and $G_{N,N'}$, with $N, N' \geq 2$, is obtained from $\Gamma$ by attaching $N$ leads to $v$ and $N'$ leads to $v'$. The set of resonances is
  $$\left\{ \frac{1}{2l} \left( 2\pi j - i \log \frac{(N+1)(N'+1)}{(N-1)(N'-1)} \right) \ | \ j \in \mathbb{Z} \right\}$$
  and
  $$h(G) = \frac{1}{2} \log \frac{(N+1)(N'+1)}{(N-1)(N'-1)}.$$
4 Graphs of Type II

The goal of this section is to describe in a precise way the asymptotic behaviour of the resonances in the type II case.

4.1 Geometric preliminaries

Let us start introducing some algebraic sets:

- If \( \vec{l} \) is given,
  \[
  R_{\vec{l}}^G := \{ (\varepsilon e^{-\tau l_e}) e \in E, \ \alpha_e, \ \tau \in \mathbb{R} \} \subset T^E \times \mathbb{R}.
  \]
  We denote by \( Y_{\vec{l}}^G \) the projection of \( R_{\vec{l}}^G \) onto the torus \( T^E \):
  \[
  Y_{\vec{l}}^G := \{ (e^{i\alpha_e}) e \in E \mid \exists \ \tau \in \mathbb{R} \text{ with } (e^{i\alpha_e^{-\tau l_e}}) e \in R_{\vec{l}}^G \}.
  \]

- \( Z_{\Gamma} \) is the determinant manifold of \( \Gamma \) defined by \((e^{i\varepsilon l_e}) e \in E \in Z_{\Gamma} \) if and only if \( 1 \) is an eigenvalue of \( \Delta_{\vec{l}} \Gamma \).

It follows from Section 4 of [CdV15] that

**Proposition 4.1** The graph \( G \) is of type II if and only if \( W^o_G \) is non empty.

For type I, \( W_G \) is empty. For type II, except for circles, the first author constructed, in Section 4 of [CdV15], non zero eigenstates associated to a non degenerate eigenvalue of some \( \Delta_{\vec{l}} \Gamma \) which vanish on all vertices of \( \Gamma \), except may be two vertices of degree 1: in these cases, \( W^o_G \) is non empty. In the case of the circular graphs (i.e. \( |\Gamma| \) is a circle), the eigenspaces of \( \Delta_{\vec{l}} \Gamma \) are degenerate, but the resonant states are still non degenerate.

We have the

**Theorem 4.1** If \( w_0 \in W^o_G \), then \( R_{\vec{l}}^G \) is smooth near \( (w_0, 0) \) and, if \( u_0 \) is a corresponding eigenfunction with coefficient \( (a_e, b_e) e \in E \), then

\[
T_{w_0,0}R_{\vec{l}}^G = \left\{ \sum_e (|a_e|^2 + |b_e|^2) d\alpha_e = 0, \ d\tau = 0 \right\}.
\]

Similarly, \( Y_{\vec{l}}^G \) is smooth near \( w_0 \) and

\[
T_{w_0}Y_{\vec{l}}^G = \left\{ \sum_e (|a_e|^2 + |b_e|^2) d\alpha_e = 0 \right\}.
\]

If \( u_0 \) is also non degenerate as an eigenfunction of \( \Delta_{\vec{l}} \Gamma \), then

\[
T_{w_0,0}R_{\vec{l}}^G = T_{w_0}Z_{\Gamma} \oplus 0.
\]
and
\[ T_{w_0}Y_G^\iota = T_{w_0} Z_{\Gamma}. \]

The set $R^\iota_G$ is near $w_0$ a graph $\tau = x(\imath \alpha e)$ with $(\imath \alpha e) \in Y_G^\iota$.

The differential of $\tau$ vanishes at $w_0$, so that the Hessian of $\tau$ is well defined on $T_{w_0}Y_G^\iota$.

First proof.- Let $w_0 = (\imath \alpha e)$ be a point in $W_G^0$. The eigenvalue 1 of $U(w_0)_2$ is non degenerate and we choose an eigenvector $u_0 \in \mathbb{C}^E$ of norm 1 so that $U(w_0)_2 u_0 = u_0$. If $w \in \mathbb{C}^E$ is close to $w_0$, $U(w)_2$ admits an unique eigenvalue $\lambda(w)$ close to 1 which is non degenerate, and the associated eigenvector of norm 1, $u(w)$, is smooth w.r. to $w$. Let us assume that $w(t)$ depends smoothly of $t$ with $w(0) = w_0$ and compute the derivative $\dot{\lambda}$ of $\lambda(w(t))$ at $t = 0$. Taking the derivative of the equation $U(w(t))_2 u(w(t)) = \lambda(t) u(w(t))$ w.r. to $t$ at $t = 0$, we get with natural notations

\[ U(w_0)_2 u_0 + U(w_0)_2 \dot{u}_0 = \dot{\lambda} u_0 + \dot{u}_0. \]

Taking the scalar product of both sides of Equation [2] with $U(w_0)_2 u_0 = u_0$, we get, after simplifications and using the fact that we can choose $u(t)$ so that $\langle \dot{u}_0 | u_0 \rangle = 0$,

\[ \dot{\lambda}_0 = \langle U((w_0)(w_0)^{-1}\dot{w}_0)_2 u_0| U(w_0)_2 u_0 \rangle + \langle U(w_0)_2 \dot{u}_0| U(w_0)_2 u_0 \rangle. \]

Now we can use the identity

\[ \langle S \begin{pmatrix} 0 \\ v \end{pmatrix} | S \begin{pmatrix} 0 \\ u_0 \end{pmatrix} \rangle = \langle U(w_0)_2 v| U(w_0)_2 u_0 \rangle. \]

Because $S$ is unitary at $w_0$, we get

\[ \langle U(w_0)_2 v| U(w_0)_2 u_0 \rangle = \langle v| u_0 \rangle. \]

Using this, we get

\[ \dot{\lambda}_0 = \langle (w_0)^{-1}\dot{w}_0)_2 u_0| u_0 \rangle. \]

Now let us take, with $\iota$ fixed $w(\alpha, \tau) = (\imath \alpha e - \imath \tau e)$, and compute the derivatives of $\lambda$ w.r. to $\alpha$ and $\tau$ at $w_0$; from Equation [3], we get

\[ \frac{\partial \lambda}{\partial \tau} = - \sum l_e m_e \]

and

\[ \frac{\partial \lambda}{\partial \alpha} = i m_e \]

with $m_e = |a_e|^2 + |b_e|^2$. Now using the fact that the determinant is the product of all eigenvalues, we get that the set $\det(U(w)_2 - \text{Id}) = 0$ is also defined by $\lambda(w) = 1$ with the same non degeneracy properties. This gives the fact that $R^\iota_G$ is near $(w_0, 0)$ a submanifold of codimension 2 of $T^E \oplus \mathbb{R}_\tau$ whose tangent space is what is given in the Theorem 4.1.
Theorem 4.2  For almost all choices of $\vec{l}$, $\Delta^\vec{l}_G$ has no $L^2$ eigenfunctions; in particular $N_{G,\vec{l}}(\varepsilon = 0) = 0$.

This is simply because $W^o_G$ is of codimension at least 1 in $Z_G$; hence the line $\sigma \to \exp(i\sigma \ell_v)$ does not cross $W^o_G$ for a generic choice of the lengths.

4.2 Asymptotics of $N^\vec{l}_{G,\vec{l}}(\varepsilon)$

We now prove the first part of Theorem 1.3.

Proof. - The proof follows from the same kind of ergodicity argument than in the paper [CdV15]. More precisely, let $w_0 \in W^o_G$ so that $W^o_G$ is of codimension $d(G)$ in $Y^G_G$ near $w_0$. Thanks to Theorem 4.1 there exists a compact neighborhood $D$ of $w_0$ in $Y^G_G$ with smooth boundary, so that $\tau(\alpha)$ is well defined and smooth on $D$. Then we have, following the argument in [CdV15],

$$N^\vec{l}_{G,\vec{l}}(\varepsilon) \geq \text{vol}(\{\tau \geq -\varepsilon\} \cap D)$$

where the volume is computed w.r. to the Barra-Gaspard measure $\mu_{\vec{l}} = |\nu(\vec{l})d\alpha|$ which is smooth non negative on $D$. Because $\tau$ vanishes on $W^o_G$, which is of codimension $d(G)$ in $D$, this volume is greater than $C\varepsilon^{d(G)/2}$ with $C > 0$ (equality holds if the Hessian of $\tau$ at $w_0$ is transversally non degenerate). \(\square\)

5 Bounds for $d(G)$

5.1 The bound $d(G) \geq 1$

Let us assume that there is no loops at the vertex $a \in V_0$. It follows from [BL16] Theorem 3.6 that for a generic choice of lengths the eigenvalue 1 is non degenerate and the corresponding eigenfunction does not vanish at $a$. Knowing that $W^o_G$ is semi-algebraic, this implies that $d(G) \geq 1$.

5.2 Upper bounds for $d(G)$

In the following sections, we will prove the upper bounds for $d(G)$ given in the second part of Theorem 1.3. If $\gamma$ is a simple cycle of $\Gamma$, we will decompose the vector $\vec{l}$ as $\vec{l} = (\vec{l}_\gamma, \vec{l}')$ where $\vec{l}_\gamma$ is the set of lengths of the edges of the cycle $\gamma$. Let us recall from [CdV15], proof of Theorem 4.1, p. 356 (see also Appendix B) that, if $\vec{l}_\gamma = (2\pi, \cdots, 2\pi)$, we can always choose $\vec{l}'$ so that 1 is a non degenerate eigenvalue of $\Delta^\vec{l}_G$. The point $w_0 = (1, \cdots, 1, e^{i\ell_v})$ belongs then to $W^o_G$ and we will study the set $W^o_G$ near such a point $w_0$. A similar study can be done for paths joining two vertices of degree 1, but we will omit it.
5.3 The case of circular graphs

Let us assume that $|\Gamma|$ is a circle. We can always assume that $V_0 = V$. Then $W_0^\circ$ is a finite set of points of the form $(\pm 1)$ with the number of $-1$ even. Hence $d(G) = #V_0 - 1 = g(G) - 1$.

From now, we will assume the $|\Gamma|$ is not a circle and $w_0 \in W_0^\circ$ is choosen as described before.

5.4 The bound $d(G) \leq g(G) - 1$

Let us now discuss the upper bounds for $d(G)$ in terms of the girth (part 2 of Theorem 1.3).

Let $\gamma$ be a simple cycle in $\Gamma$ with $g(G)$ vertices and consider the submanifold (not closed in general) $W_0^\gamma$ of $W_0^\circ$ defined by $\vec{l}_\gamma = (2\pi, \cdots, 2\pi)$ and $\vec{l}$ so that $1$ is a non degenerate eigenvalue of $\Delta_{\vec{l}\Gamma}$. Then $\Delta_{\vec{l}\Gamma}$ admits an eigenfunction with eigenvalue $1$ supported on $\gamma$ and vanishing on all vertices of $G$. This defines a stratum $W_0^\gamma$ of $W_0^\circ$ of dimension $#E - g(G)$. The codimension of $W_0^\gamma$ in $Y_{\vec{l}\Gamma}$ is $g(G) - 1$.

Similar estimates hold by using the smallest path ending at 2 vertices of degree $1$.

5.5 The bound $d(G) \leq #V_0$

Let us now discuss the upper bounds for $d(G)$ in terms of $#V_0$ (part 2 of Theorem 1.3). Let $\gamma$ be a simple cycle of $\Gamma$ of length $k$. Let us choose $\vec{l}_0 = (2\pi, \cdots, 2\pi, \vec{l}')$ (the lengths $2\pi$ occur $k$ times as lengths of the edges of $\gamma$), so that $1$ is a non degenerate eigenvalue of $\Delta_{\vec{l}_0\Gamma}$ with an eigenfunction which restricts to each edge of $\gamma$ to $\sin x$, and which vanishes outside $\gamma$. For $\vec{l}$ in some neighborhood $U_0$ of $\vec{l}_0$ with $e^{i\vec{l}} \in Z_\Gamma$ we can choose some eigenfunction $u(\vec{l})$ smoothly dependent of $\vec{l}$.

Let us denote by $F$ the map from $\{w = e^{i\vec{l}} | \vec{l} \in U_0\} \cap Z_\Gamma$ to $\mathbb{R}^{V(\gamma)}$ which associates to $w$ the restriction to $V(\gamma)$ of $u(\vec{l})$, and by $E$ the subspace of $T_{w_0} Z_\Gamma$ of variations $\delta \vec{l}$ of lengths supported by $\gamma$ so that $\sum_{e \in \gamma} \delta l_e = 0$. The dimension of $E$ is clearly $k - 1$. Let us prove that the differential $L : E \to \mathbb{R}^{V(\gamma)}$ of $F$ is injective. Denoting by $\vec{l}(t)$ a variation of $\vec{l}_0$ and denoting derivatives at $t = 0$ by dots, we get

$$\Delta u + (\Delta - 1) \dot{u} = 0.$$  \hfill (4)

We have to prove that $\dot{u}$ cannot vanish on all vertices of $V_0$ unless $\delta \vec{l}$ vanishes. Let us assume, by contradiction, that $\dot{u}$ vanishes on $V_0$. Using Lemma [1.1] and ordering the vertices of $\gamma$ using the orientation of $\gamma$ as $V(\gamma) = \{v_1, \cdots, v_k\}$, we get $\dot{u}(v_{i+1}) = \dot{u}(v_i) + \delta l_{e_i}$ with $e_i = (v_i, v_{i+1})$. We get that $\dot{u}$ cannot vanish identically on $V_0$ unless $\delta \vec{l}$ does. The final result follows from linear algebra: If
\( \mathcal{F} = F'(w_0)(\mathcal{E}) \), \( \dim \mathcal{F} = k - 1 \) and hence there is at most one \( a \in V(\gamma) \) so that \( \mathcal{F} \subset \{v|v(a) = 0\} \). If \( V_0 \subset V(\gamma) \) with \( \#V_0 \leq k - 1 \) and \( a \notin V_0 \), the map \( \pi \circ F'(w_0) \) is surjective where \( \pi \) is the canonical projection from \( \mathbb{R}^{V(\gamma)} \) onto \( \mathbb{R}^{V_0} \).

5.6 The case \( \#V_0 = 1 \)

Let us say that \( a \in V(\Gamma) \) is irregular if, for all simple cycles \( \gamma \) with \( a \in V(\gamma) \), the previous space \( \mathcal{F} \) is included in \( \{v|v(a) = 0\} \). Using the 2-connectivity of \( \gamma \), we see that there is at most one such vertex \( a \). We conjecture that such a vertex never exists. In any case it does not exist if the graph is homogeneous, ie for any vertex \( a \) and \( b \), there exists an automorphism of \( \Gamma \) sending \( a \) onto \( b \).

5.7 The case \( V_0 = V \)

If \( X \subset E \), we define

\[ W_X := \{[\vec{l}] \in W_G \mid \forall e \in X, \exists u \in \ker(\Delta^{\vec{l}} - 1), u|_e \neq 0 \}. \]

Then the family of sets \( W_X, X \subset E \) is a partition of \( W_G \). If \( [\vec{l}] \in W_X \), all lengths of \( e \in X \) are multiple of \( \pi \). The eigenvalue 1 is of multiplicity \( b_1(\Gamma_X) \) where \( \Gamma_X \) is the subgraph of \( \Gamma \) whose edge set is \( X \). Moreover, if \( b_1(\Gamma_X) = 1 \), \( \Gamma_X \) reduces to that simple cycle. Hence \( W_X \cap W_G^0 \neq \emptyset \) if and only if \( X \) is a simple cycle. The dimension of the corresponding stratum is \( \#E - \#X \) and the maximal dimension of such a stratum is obtained by taking the cycle of minimal length. Hence \( d(G) = \min_{X \text{ simple cycle}}(\#X - 1) = g(G) - 1 \).

6 Examples

6.1 Homogeneous 2-connected graphs

If the graph \( \Gamma \) is homogeneous and 2-connected, there is no exceptional vertex, hence \( d(G) = 1 \) if \( \#V_0 = 1 \).

In the next examples, all subsets \( V_0 \) of \( V(\Gamma) \) with \( \#V_0 < g(\Gamma) - 1 \) are contained in a simple cycle. It is known (see [Be73]) that this follows from the fact that \( \Gamma \) is \( (g(\Gamma) - 1) \)-connected.

In fact, we can do better using the

Lemma 6.1 If \( g \) is an automorphism of \( \Gamma \) leaving the simple cycle \( \gamma \) invariant and with no vertex of \( \gamma \) fixed by \( g \), then there is no exceptional vertex on \( \gamma \), ie \( d(G) \leq \#V_0 \) for all \( V_0 \subset V(\gamma) \).

Proof. – We can choose \( \vec{l}_0 \) invariant by \( g \). It follows that the space \( \mathcal{F} \subset \mathbb{R}^{V(\gamma)} \) defined in Section 5.5 is invariant by \( g \) and cannot be a coordinate hyperplane because \( g \) acts without fixed points on \( \gamma \). \( \square \)
6.1.1 Complete graphs

If $\Gamma$ is the complete graph

\[ d(G) \leq \min(2, \#V_0) . \]

6.1.2 Cubes

All sets of 2 vertices are part vertices of a simple cycle which satisfies the assumptions of Lemma 6.1 hence

\[ d(G) \leq \min(3, \#V_0) . \]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{cube.png}
\caption{cube with a simple cycle on $\Gamma$ with a symmetry of order 2 without fixed vertices on $\gamma$. Any set of 2 vertices of $\Gamma$ is part of such a cycle.}
\end{figure}

6.1.3 Dodecahedron

All sets of 3 vertices are part vertices of a simple cycle which satisfies the assumptions of Lemma 6.1 hence

\[ d(G) \leq \min(4, \#V_0) . \]

6.1.4 Petersen graph

All sets of 3 vertices are part vertices of a simple cycle which satisfies the assumptions of Lemma 6.1 hence

\[ d(G) \leq \min(4, \#V_0) . \]
Figure 2: Dodecahedron with a simple cycle on $\Gamma$ with a symmetry of order 5 without fixed vertices on $\gamma$.

Figure 3: Petersen graph with a simple cycle on $\Gamma$ with a symmetry of order 2 without fixed vertices on $\gamma$. Any set of 3 vertices of $\Gamma$ is part of such a cycle.
6.1.5  Tetrahedron

The goal of this section is to compute the dimension \( d(G) \) in the cases where \( \Gamma \) is a tetrahedron. Instead of making direct computations, we present geometrical arguments which could be extended to other graphs. The link with the minor relation between graphs is implicit in our construction of reduced graphs.

We will show the

**Theorem 6.1**  If \( \Gamma \) is the tetrahedron, i.e. the clique with four vertices, we have \( d(G) = \min(\# V_0, 2) \) where 2 is the girth of \( \Gamma \) minus 1.

The result follows from Theorem 1.3 for \( \# V_0 = 1 \) and \( \# V_0 = 4 \). It remains to prove that \( d(G) = 2 \) if \( \# V_0 = 2 \) or 3. The bound \( d(G) \leq 2 \) comes from Theorem 1.3, because the girth of \( \Gamma \) is 3.

If \([\bar{l}]\) belongs to \( W_0^G \) and \( u \) is the unique associated eigenfunction vanishing on \( V_0 \), let us denote by \( n_{\bar{l}} \) the number of edges of \( \Gamma \) on which \( u \) vanishes identically. We decompose each stratum of \( W_0^G \) following the values of \( n_{\bar{l}} \) into a finite number of sub-strata. The maximal dimension of these sub-strata is the same as the dimension of \( W_0^G \). If \([\bar{l}]\) \( \in \ W_0^G \), we have to show that the dimension, denoted \( \text{dim}(\bar{l}) \), of the sub-stratum of \( W_0^G \) containing \([\bar{l}]\) is smaller than 3.

Let us discuss the different possibilities:

- \( n_{\bar{l}} = 0 \):

  - \( \# V_0 = 2 \): let us assume that \( V_0 = \{a, b\} \). The length of the edge \( e = \{a, b\} \) is equal to 0 modulo \( \pi \). In these cases the function \( u \) on \( e \) is of the form \( \mu \sin x_e \) with \( \mu \neq 0 \). Let us consider the graph \( \Gamma_{c}^{\text{red}} \).
obtained by identifying the vertices \( a \) and \( b \) (contracting the edge \( e \)). This reduced graph has 5 edges. Denote by \( A \) the vertex of degree four obtained by identifying \( a \) and \( b \).

* \( l_e = \pi \) modulo \( 2\pi \): the Kirchhoff conditions at the vertex \( A \) are \( u'_1 + u'_2 + u'_3 + u'_4 = 0 \). We have to look at \( W^e_{Gr_{\text{red}}} \) where \( V_0 = \{ A \} \). The dimension of this manifold is less than \((5 - 1) - 1 = 3\): we apply Theorem 1.3 to \( G_{\text{red}}^e \) with \( V_0 = \{ A \} \) and clearly there is no loop at the vertex \( A \). Hence \( \dim(\vec{l}) \leq 3 \).

* \( l_e = 2\pi \) modulo \( 2\pi \): the Kirchhoff conditions at the vertex \( A \) are \( u'_1 + u'_2 = u'_3 + u'_4 \). Similar construction gives the same conclusion.

– \#\( V_0 = 3 \): in this case the lengths of the cycle whose vertices are \( V_0 \) are equal to 0 modulo \( \pi \). This implies that the dimension \( \dim(\vec{l}) \) of the corresponding sub-stratum is at most 3.

- \( n_{\vec{l}} = 1 \): let us assume that \( u \) vanishes identically on the edge \( e = \{ a, b \} \). Let us consider the reduced graph \( \Gamma_{\text{red}} \) obtained by removing the edge \( e \). Topologically, this graph has two vertices \( c \) and \( d \) of degree 3 and and three edges joingning them. The dimension of \( Z_{\Gamma_{\text{red}}} \) is 2. If \( [\vec{l}] \in Z^o_{\Gamma_{\text{red}}} \), then the position of the zeroes of the eigenfunction \( u \) (unique up to rescaling) on the two edges obtained by removing \( a \) and \( b \) determines all the remaining lengths except \( l_e \). This implies that \( \dim(\vec{l}) = 2 + 1 \).

- If \( n_{\vec{l}} \geq 2 \), one has two cases
  
  – If the two edges where \( u \) vanishes identically have a commun vertex, say \( a \), it follows from the Kirchoff conditions at the vertex \( a \) that \( u \) vanishes also identically on the third edge with vertex \( a \). Hence the support of \( u \) is the remaining cycle (otherwise \( u \equiv 0 \)). This forces the lengths of the edges of this cycle to be congruent to 0 modulo \( \pi \) and hence \( \dim(\vec{l}) = 3 \).

  – If the two edges where \( u \) vanishes identically have no commun vertex, then \( u \) vanishes at all vertices and the four remaining edges have lengths congruent to 0 modulo \( \pi \). This implies that \( \dim(\vec{l}) = 2 \).

6.2 A simple example where we compute the asymptotics of \( N_G^\vec{l}(\varepsilon) \): the Y-graph

Let us consider a \( Y \) graph with two edges of length \( l \) and \( L \) and an infinite edge. If \( z = \exp(ikl) \), \( w = \exp(ikL) \), we get that

\[
\mathcal{R}_G(z, w) = z^2w^2 - z^2 - w^2 - 3 ,
\]
whereas $Z_{\Gamma}$ is defined by $z = e^{i\alpha}, w = e^{i\beta}$ with $\alpha + \beta = 0 \mod \pi$. Since $W_G = Z_{\Gamma} \cap R_{\Gamma}^l$, with $\hat{l} = (l, L)$, it follows that $W_G = \{(\pm i, \pm i)\}$, and that the tangent space to $R_{\Gamma}^l$ is $d\alpha + d\beta = 0$ (due to theorem 4.1). Let us consider a neighborhood $D$ in $R_{\Gamma}^l$ of $w_0 = (i, i)$, namely points $(z = \exp(i\pi/2 + i\alpha - \tau l), w = \exp(i\pi/2 + i\beta - \tau L))$ satisfying the resonance equation, with $\alpha = +u, \beta = -u$ and $\tau, u$ small. We get that

$$e^{2i(\alpha + \beta)}e^{-\tau(l+L)} + e^{2ia}e^{-\tau l} + e^{2i\beta}e^{-\tau L} = 3,$$

which yields to the following asymptotics

$$\tau \sim -u^2/4(l + L),$$

as $\tau$ tends to 0. Then the Barra-Gaspard volume $V_{\epsilon}$ of the set $D_{\epsilon} = D \cap \{\tau \geq -\epsilon\}$ is given by

$$V_{\epsilon} = \frac{1}{2\pi^2} \int_{D_{\epsilon}} |Ld\beta - ld\alpha| = \frac{1}{2\pi^2} \int_{u^2/4(l+L) \leq \epsilon} (l + L)|du|,$$

and thus we get the following asymptotics

$$N(\epsilon) \sim \frac{4(l + L)^{3/2}}{\pi^2} e^{1/2}.$$

### 6.3 Circular graphs

A graph $G$ is called circular if $|\Gamma|$ is homeomorphic to a circle. A general circular graph is denoted $C_{N_1, \ldots, N_p}$ with $N_i > 0$: this is a subset of $p$ points in circular order $\{v_1, \ldots, v_p\}$ where $N_1$ leads are attached to $v_1$, ...

#### 6.3.1 $C_1$

The resonances of $C_1$ are easily shown to be the spectrum of $\Gamma$:

$$\text{Res}_{C_1}^{l} = \{2\pi j/L \mid j \in \mathbb{Z}\}.$$  

#### 6.3.2 $C_{1,1}$

In this case, there are two lengths $l, L$ and putting $z = e^{ikl}$ and $w = e^{ikL}$, we get

$$\mathcal{R}_{C_{1,1}}(z, w) = (zw - w - z - 3)(zw + z + w - 3).$$

Then $W_{C_{1,1}} = \{(-1, -1), (1, 1)\}$

#### 6.3.3 $C_{1,1,1}$

It is still possible to compute

$$\mathcal{R}_{C_{1,1,1}}(z_1, z_2, z_3) = z_1^2 z_2^2 z_3^2 - (z_1^2 z_2^2 + z_2^2 z_3^2 + z_3^2 z_1^2) - 3(z_1^2 + z_2^2 + z_3^2) - 16z_1 z_2 z_3 + 27,$$

with $z_i = e^{ikl}$. In this case $W_{C_{1,1,1}}$ consists of four points: $w_1 = (1, -1, -1), w_2 = (-1, 1, -1), w_3 = (-1, -1, 1), w_4 = (1, 1, 1).$
6.3.4 \( C_{1,\ldots,1} \)

For all values of \( N \), \( W_{C_{1,\ldots,1}} \) is the finite set of vectors \( X \) in \( \{-1,+1\}^N \) so that the number of negative components of \( X \) is even.

6.3.5 \( C_2 \)

This example is studied in [DPII]. The resonances of \( C_2 \) are the same as for \( C_1 \):

\[
\text{Res}_{C_2}^L = \{2\pi j/L \mid j \in \mathbb{Z}\}.
\]

7 Open questions

The main open question is clearly to get upper bounds of \( N_G^L(\varepsilon) \) as in the conjecture [1.1].

Two other questions seem interesting:

- Is it possible to get an estimate of the minimal imaginary part of resonances given by Theorem [1.1] of the form \( M = M(G)/|L| \) with \( M(G) \) depending only of the combinatorics of \( G \)?

- Is it true that if \( \hat{l} \) is irrational there is no (or at most a finite number) of embedded eigenvalues? Compare this to Theorem [4.2]

- Is the constant \( h(G) \) in Theorem [1.1] minor monotonic?

A Calculation of a derivative

The goal of this section is to prove the

**Lemma A.1** Let \( \lambda = 1 \) be a simple eigenvalue of \( \Delta_{\hat{l}_0} \) with an eigenfunction \( u_0 \). Let us assume that \( u_0 \) restricts to some edge \( e = [a, b] \) of length \( 2\pi \) to \( \sin x \) where \( x \in [0, 2\pi] \) is an arc-length parametrization of \( e \) with origine \( a \) and end \( b \). If \( \hat{l}(t) \) is a smooth deformation of \( \hat{l}_0 \) so that the eigenvalue \( \lambda(t) \) is constant equal to 1, we have

\[
\left(\frac{d}{dt}\right)_{t=0} (u(b) - u(a)) = \left(\frac{dl_e}{dt}\right)_{t=0}.
\]

**Proof.**—We denote by dots the derivative at \( t = 0 \) and get

\[
(\Delta - 1)\dot{u} + \hat{\Delta}u_0 = 0.
\]

The restriction of this equation to the edge \( e \) gives putting \( v = \dot{u} \):

\[
v''(x) + v(x) = \hat{\Delta}\sin x.
\]
Taking the metric \( g(t) = (1 + 2t)dx^2 \), gives \( \dot{l}_e = 2\pi \), while \( \dot{\Delta} = 2 \frac{d^2}{dx^2} \). The derivative \( v \) satisfies

\[
v''(x) + v(x) = -2 \sin x .
\]

Using the method of variation of constants, we get:

\[
v(x) = \alpha \cos x + \beta \sin x + x \cos x ,
\]

with some constants \( \alpha \) and \( \beta \), and hence

\[
v(2\pi) - v(0) = 2\pi .
\]

\[\Box\]

**B Non degenerated eigenfunctions supported by simple cycles**

Let \( \gamma \) be a simple cycle of \( \Gamma \) with ordered vertices \( (x_0, x_1, x_2, \ldots, x_n = x_0) \). Let us consider lengths \( \vec{l} = (2\pi, \ldots, 2\pi, \vec{l}') \) where the \( 2\pi \)'s are the lengths of the \( n \) edges of \( \gamma \) and \( \vec{l}' \) the other lengths. Let us assume in what follows that \( \Gamma \) is not homeomorphic to a circle, ie not reduced to \( \gamma \). Let us reprove the following

**Lemma B.1** There exists an open dense subset \( \Omega \) of \( (\mathbb{R}^+)^{E(\Gamma)} \setminus E(\gamma) \) so that if \( \vec{l}' \) belongs to \( \Omega \), then \( 1 \) is a non degenerate eigenvalue of \( \Delta_{\Gamma} l' \).

The fact that \( \Omega \) is open is clear from general perturbation theory. Let us start now with \( u_0 \in \mathcal{E} := \ker(\Delta_{\Gamma}^{\vec{l}} - 1) \) the function which restricts to \( \sin t_e \) on each edge of \( \gamma \) and vanishes outside. Let us assume that \( \dim \mathcal{E} > 1 \) and look at the degenerate perturbation theory of the eigenvalue \( 1 \) while moving the vector \( \vec{l}' \).

The first order perturbation of the eigenvalues if \( \vec{l} \) depends on a parameter \( t \) is given by the eigenvalues of the quadratic form

\[
Q := -\sum_{e \in E} m_e \frac{\partial l_e}{\partial t} \big|_{t=0}
\]

If the variation of lengths is given by

\[
\vec{l}(t) = (2\pi, \cdots, 2\pi, \vec{l}'_0 + t \mathbf{1}) ,
\]

we get

\[
Q = -\sum_{e \in E} m_e
\]

which is \(< 0\) on the orthogonal of \( u_0 \) in \( \mathcal{E} \). Hence, for \( u \) small, the eigenvalue \( 1 \) is non degenerate.
C Semi-algebraic sets

A semi-algebraic subset of $\mathbb{R}^N$ is a set defined by a finite number of equations and inequations with polynomial entries. A fundamental result is the Tarski-Seidenberg Theorem which says

**Theorem C.1** The image of a semi-algebraic set by a linear map from $\mathbb{R}^N$ onto $\mathbb{R}^n$ is still semi-algebraic.

One of the main properties of semi-algebraic sets is that they admit a stratification: such a set is a union of a finite number of sub-manifold of $\mathbb{R}^N$, called the strata, so that the closure of each of them is the union of a finite number of strata. The dimension of a semi-algebraic set is then the maximal dimension of these strata. More details can be found in the classical book [BCR98].

Let us show as a typical example that $W_G$ is semi-algebraic as well as $W^o_G$. The equation for eigenfunctions of $\Delta^T_\Gamma$ with eigenvalue 1 is of the form $M(z)(\vec{a},\vec{b}) = 0$ with $M(z)$ polynomial in $z = \exp(i\vec{l})$. The corresponding eigenfunction vanishes at the vertex $x_0$ if $a_e = 0$ or $a_e \cos l_e + b_e \sin l_e = 0$, depending of the orientation of $e = \{x_0, x_1\}$. All these equations are polynomial in the coordinates of $z$ and the vectors $\vec{a}, \vec{b}$. It is then enough to add $|z_e|^2 = 1$. All of this gives an algebraic sub-set of $\mathbb{R}^{4\#E}$. The set $W_G$ is the image of this set by the projection on the $2\#E$ first factors. Concerning $W^o_G$, the uniqueness of the solution of a system of homogeneous linear equations up to scaling reduces to the non-vanishing of some minors. Algebraicity follows then by taking the sum of the squares of these minors.

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