The Neutrino Mass Matrix –
From $A_4$ to $Z_3$

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Abstract

With the recent experimental advance in our precise knowledge of the neutrino oscillation parameters, the correct form of the $3 \times 3$ neutrino mass matrix is now approximately known. I discuss how this may be obtained from symmetry principles, using as examples the finite groups $A_4$ and $Z_3$, in two complete theories of leptons (and quarks).

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1 Introduction

After the new experimental results of KamLAND [1] on top of those of SNO [2] and SuperKamiokande [3], etc. [4], we now have very good knowledge of 5 parameters:

\[ \Delta m^2_{\text{atm}} \simeq 2.5 \times 10^{-3} \text{ eV}^2, \]
\[ \Delta m^2_{\text{sol}} \simeq 6.9 \times 10^{-5} \text{ eV}^2, \]
\[ \sin^2 2\theta_{\text{atm}} \simeq 1, \]
\[ \tan^2 \theta_{\text{sol}} \simeq 0.46, \]
\[ |U_{e3}| < 0.16. \]

The last 3 numbers tell us that the neutrino mixing matrix is rather well-known, and to a very good first approximation, it is given by

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix}
= \begin{pmatrix}
c & -s & 0 \\
\frac{s}{\sqrt{2}} & \frac{c}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{s}{\sqrt{2}} & \frac{c}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix},
\]

where \( \sin^2 2\theta_{\text{atm}} = 1 \) and \( U_{e3} = 0 \) have been assumed, with \( s \equiv \sin \theta_{\text{sol}}, c \equiv \cos \theta_{\text{sol}} \).

2 Approximate Generic Form of the Neutrino Mass Matrix

Assuming three Majorana neutrino mass eigenstates with real eigenvalues \( m_1, m_2, m_3 \), the neutrino mass matrix in the basis \( (\nu_e, \nu_\mu, \nu_\tau) \) is then of the form [5]

\[
\mathcal{M}_\nu = \begin{pmatrix}
a + 2b + 2c & d & d \\
d & b & a + b \\
d & a + b & b
\end{pmatrix}.
\]

Note that \( \mathcal{M}_\nu \) is invariant under the discrete \( \mathbb{Z}_2 \) symmetry: \( \nu_e \leftrightarrow \nu_e, \nu_\mu \leftrightarrow \nu_\tau \). Depending on the relative magnitudes of the 4 parameters \( a, b, c, d \), this matrix has 7 possible limits:
have the normal hierarchy, 2 have the inverted hierarchy, and 2 have 3 nearly degenerate masses.

In neutrinoless double beta decay, the effective mass is \( m_0 = |a + 2b + 2c| \). In the 2 cases of inverted hierarchy, we have

\[
m_0 \simeq \sqrt{\Delta m_{\text{atm}}^2} \simeq 0.05 \text{ eV}, \tag{8}
\]

\[
m_0 \simeq \cos 2\theta_{\text{sol}} \sqrt{\Delta m_{\text{atm}}^2}, \tag{9}
\]

respectively for \( m_1/m_2 = \pm 1 \), i.e. for their relative \( CP \) being even or odd. In the 2 degenerate cases,

\[
m_0 \simeq |m_{1,2,3}|, \tag{10}
\]

\[
m_0 \simeq \cos 2\theta_{\text{sol}} |m_{1,2,3}|. \tag{11}
\]

With \( M_\nu \) of Eq. (7), \( U_{e3} \) is zero necessarily, in which case there can be no \( CP \) violation in neutrino oscillations. However, suppose we consider instead \[5, 6\]

\[
M_\nu = \begin{pmatrix}
a + 2b + 2c & d & d^* \\
d & b & a + b \\
d^* & a + b & b
\end{pmatrix}, \tag{12}
\]

where \( d \) is now complex, then the \( Z_2 \) symmetry of Eq. (7) is broken and \( U_{e3} \) becomes nonzero. In fact, it is proportional to \( iImd \), thus predicting maximal \( CP \) violation in neutrino oscillations.

### 3 Nearlly Degenerate Majorana Neutrino Masses

Suppose that at some high energy scale, the charged lepton mass matrix and the Majorana neutrino mass matrix are such that after diagonalizing the former, i.e.

\[
M_l = \begin{pmatrix}
m_e & 0 & 0 \\
0 & m_\mu & 0 \\
0 & 0 & m_\tau
\end{pmatrix}, \tag{13}
\]
the latter is of the form

$$M_\nu = \begin{pmatrix} m_0 & 0 & 0 \\ 0 & 0 & m_0 \\ 0 & m_0 & 0 \end{pmatrix}. \quad (14)$$

From the high scale to the electroweak scale, one-loop radiative corrections will change $M_\nu$ as follows:

$$(M_\nu)_{ij} \rightarrow (M_\nu)_{ij} + R_{ik}(M_\nu)_{kj} + (M_\nu)_{ik}R^T_{kj}, \quad (15)$$

where the radiative correction matrix is assumed to be of the most general form, i.e.

$$R = \begin{pmatrix} r_{ee} & r_{e\mu} & r_{e\tau} \\ r^*_{e\mu} & r_{\mu\mu} & r_{\mu\tau} \\ r^*_{e\tau} & r^*_{\mu\tau} & r_{\tau\tau} \end{pmatrix}. \quad (16)$$

Thus the observed neutrino mass matrix is given by

$$M_\nu = m_0 \begin{pmatrix} 1 + 2r_{ee} & r_{e\tau} + r^*_{e\mu} & r_{e\mu} + r^*_{e\tau} \\ r_{e\mu} + r^*_{e\tau} & 2r_{\mu\tau} & 1 + r_{\mu\mu} + r_{\tau\tau} \\ r^*_{e\tau} + r_{e\mu} & 1 + r_{\mu\mu} + r_{\tau\tau} & 2r^*_{\mu\tau} \end{pmatrix}. \quad (17)$$

Let us rephase $\nu_\mu$ and $\nu_\tau$ to make $r_{\mu\tau}$ real, then the above $M_\nu$ is exactly in the form of Eq. (12), with of course $a$ as the dominant term. In other words, we have obtained a desirable description of all present data on neutrino oscillations including $CP$ violation, starting from almost nothing.

4 Plato’s Fire

The successful derivation of Eq. (17) depends on having Eqs. (13) and (14). To be sensible theoretically, they should be maintained by a symmetry. At first sight, it appears impossible that there can be a symmetry which allows them to coexist. The solution turns out to be the non-Abelian discrete symmetry $A_4$ [7,8]. What is $A_4$ and why is it special?

Around the year 390 BCE, the Greek mathematician Theaetetus proved that there are five and only five perfect geometric solids. The Greeks already knew that there are four basic
elements: fire, air, water, and earth. Plato could not resist matching them to the five perfect geometric solids and for that to work, he invented the fifth element, i.e. quintessence, which is supposed to hold the cosmos together. His assignments are shown in Table 1.

| solid     | faces | vertices | Plato   | Group |
|-----------|-------|----------|---------|-------|
| tetrahedron | 4     | 4        | fire    | $A_4$ |
| octahedron | 8     | 6        | air     | $S_4$ |
| icosahedron | 20    | 12       | water   | $A_5$ |
| hexahedron | 6     | 8        | earth   | $S_4$ |
| dodecahedron | 12    | 20       | ?       | $A_5$ |

The group theory of these solids was established in the early 19th century. Since a cube (hexahedron) can be imbedded perfectly inside an octahedron and the latter inside the former, they have the same symmetry group. The same holds for the icosahedron and dodecahedron. The tetrahedron (Plato’s “fire”) is special because it is self-dual. It has the symmetry group $A_4$, i.e. the finite group of the even permutation of 4 objects. The reason that it is special for the neutrino mass matrix is because it has three inequivalent one-dimensional irreducible representations and one three-dimensional irreducible representation exactly. Its character table is given below.

| class | n | h | $\chi_1$ | $\chi_2$ | $\chi_3$ | $\chi_4$ |
|-------|---|---|----------|----------|----------|----------|
| $C_1$  | 1 | 1 | 1        | 1        | 1        | 3        |
| $C_2$  | 4 | 3 | 1        | $\omega$ | $\omega^2$ | 0        |
| $C_3$  | 4 | 3 | 1        | $\omega^2$ | $\omega$ | 0        |
| $C_4$  | 3 | 2 | 1        | 1        | 1        | $-1$     |
In the above, \( n \) is the number of elements, \( h \) is the order of each element, and

\[
\omega = e^{2\pi i/3} \tag{18}
\]
is the cube root of unity. The group multiplication rule is

\[
\mathbf{3} \times \mathbf{3} = \mathbf{1} + \mathbf{1}' + \mathbf{1}'' + \mathbf{3} + \mathbf{3}'. \tag{19}
\]

5 Details of the \( A_4 \) Model

The fact that \( A_4 \) has three inequivalent one-dimensional representations \( \mathbf{1}, \mathbf{1}', \mathbf{1}'', \) and one three-dimensional representation \( \mathbf{3}, \) with the decomposition given by Eq. (19) leads naturally to the following assignments of quarks and leptons:

\[
(u_i, d_i)_L, \quad (\nu_i, e_i)_L \sim \mathbf{3}. \tag{20}
\]
\[
u_{1R}, \quad d_{1R}, \quad e_{1R} \sim \mathbf{1}. \tag{21}
\]
\[
u_{2R}, \quad d_{2R}, \quad e_{2R} \sim \mathbf{1}'. \tag{22}
\]
\[
u_{3R}, \quad d_{3R}, \quad e_{3R} \sim \mathbf{1}''. \tag{23}
\]

Heavy fermion singlets are then added:

\[
U_{iL(R)}, \quad D_{iL(R)}, \quad E_{iL(R)}, \quad N_{iR} \sim \mathbf{3}. \tag{24}
\]

together with the usual Higgs doublet and new heavy singlets:

\[
(\phi^+, \phi^0) \sim \mathbf{1}, \quad \chi_i^0 \sim \mathbf{3}. \tag{25}
\]

With this structure, charged leptons acquire an effective Yukawa coupling matrix \( \bar{e}_i L e_j R \phi^0 \) which has 3 arbitrary eigenvalues (because of the 3 independent couplings to the 3 inequivalent one-dimensional representations) and for the case of equal vacuum expectation values of \( \chi_i, \) i.e.

\[
\langle \chi_1 \rangle = \langle \chi_2 \rangle = \langle \chi_3 \rangle = u, \tag{26}
\]
which occurs naturally in the supersymmetric version of this model [8], the unitary transformation $U_L$ which diagonalizes $\mathcal{M}_l$ is given by

$$U_L = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}. \tag{27}$$

This implies that the effective neutrino mass operator, i.e. $\nu_i \nu_j \phi^0 \phi^0$, is proportional to

$$U_L^T U_L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{28}$$

exactly as desired.

6 New Flavor-Changing Radiative Mechanism

The original $A_4$ model [7] was conceived to be a symmetry at the electroweak scale, in which case the splitting of the neutrino mass degeneracy is put in by hand and any mixing matrix is possible. Subsequently, it was proposed [8] as a symmetry at a high scale, in which case the mixing matrix is determined completely by flavor-changing radiative corrections and the only possible result happens to be Eq. (17). This is a remarkable convergence in that Eq. (17) is in the form of Eq. (12), i.e. the phenomenologically preferred neutrino mixing matrix based on the most recent data from neutrino oscillations.

We should now consider the new physics responsible for the $r_{ij}$’s of Eq. (16). Previously [8], arbitrary soft supersymmetry breaking in the scalar sector was invoked. It is certainly a phenomenologically viable scenario, but lacks theoretical motivation and is somewhat complicated. Here a new and much simpler mechanism is proposed [9], using a triplet of charged scalars under $A_4$, i.e. $\eta^+_i \sim \mathbf{3}$. Their relevant contributions to the Lagrangian of this model is then

$$\mathcal{L} = f \epsilon_{ijk} (\nu_i e_j - e_i \nu_j) \eta^+_k + m^2_{ij} \eta^+_i \eta^-_j. \tag{29}$$
Whereas the first term is invariant under $A_4$ as it should be, the second term is a soft term which is allowed to break $A_4$, from which the flavor-changing radiative corrections will be calculated.

Let

$$\begin{pmatrix} \eta_e \\ \eta_\mu \\ \eta_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix},$$

(30)

where $\eta_{1,2,3}$ are mass eigenstates with masses $m_{1,2,3}$. The resulting radiative corrections are given by

$$r_{\alpha\beta} = -\frac{f^2}{8\pi^2} \sum_{i=1}^{3} U_{\alpha i}^* U_{\beta i} \ln m_i^2.$$ 

(31)

To the extent that $r_{\mu\tau}$ should not be larger than about $10^{-2}$, the common mass $m_0$ of the three degenerate neutrinos should not be less than about 0.2 eV in this model. This is consistent with the recent WMAP upper bound [10] of 0.23 eV and the range 0.11 to 0.56 eV indicated by neutrinoless double beta decay [11].

7 Models based on $S_3$ and $D_4$

Two other examples of the application of non-Abelian discrete symmetries to the neutrino mass matrix have recently been proposed. One [12] is based on the symmetry group of the equilateral triangle $S_3$, which has 6 elements and the irreducible representations $\mathbf{1}$, $\mathbf{1}'$, and $\mathbf{2}$. The 3 families of leptons as well as 3 Higgs doublets transform as $\mathbf{1} + \mathbf{2}$ under $S_3$. An additional $Z_2$ is introduced where $\nu_R(\mathbf{1})$ and $H(\mathbf{2})$ are odd, while all other fields are even. After a detailed analysis, the mixing matrix of Eq. (6) is obtained with $U_{e3} \simeq -3.4 \times 10^{-3}$ and $0.4 < \tan \theta_{sol} < 0.8$. The neutrino masses are predicted to have an inverted hierarchy satisfying Eq. (8).

Another example [13] is based on the symmetry group of the square $D_4$, which has 8 elements and the irreducible representations $\mathbf{1}^{++}$, $\mathbf{1}^{+-}$, $\mathbf{1}^{-+}$, $\mathbf{1}^{--}$, and $\mathbf{2}$. The 3 families of
leptons transform as $1^{++} + 2$. The Higgs sector has 3 doublets with $\phi_3 \sim 1^{+-}$ and 2 singlets $\chi \sim 2$. Under an extra $Z_2$, $\nu_R$, $e_R$, $\phi_1$ are odd, while all other fields are even, including $\phi_2$. This results in the neutrino mass matrix of Eq. (7) with an additional constraint, i.e. $m_1 < m_2 < m_3$ such that the $m_0$ of neutrinoless double beta decay is equal to $m_1 m_2 / m_3$.

8  **Form Invariance of the Neutrino Mass Matrix**

Consider a specific $3 \times 3$ unitary matrix $U$ and impose the condition

$$U M_\nu U^T = M_\nu$$

(32)

on the neutrino mass matrix $M_\nu$ in the ($\nu_e, \nu_\mu, \nu_\tau$) basis. Iteration of the above yields

$$U^n M_\nu (U^T)^n = M_\nu.$$  

(33)

Therefore, unless $U^n = 1$ for some finite $n$, the only solution for $M_\nu$ would be a multiple of the identity matrix. Take for example $n = 2$, then the choice

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(34)

leads to Eq. (7). In other words, the present neutrino oscillation data may be understood as a manifestation of the discrete symmetry $\nu_e \rightarrow \nu_e$ and $\nu_\mu \leftrightarrow \nu_\tau$.

Suppose instead that $n = 4$, with $U^2$ given by Eq. (34), then one possible solution for its square root is

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1-i)/\sqrt{2} & (1+i)/\sqrt{2} \\ 0 & (1+i)/\sqrt{2} & (1-i)/\sqrt{2} \end{pmatrix},$$

(35)

which leads to

$$M_1 = \begin{pmatrix} 2b + 2c & d & d \\ d & b & b \\ d & b & b \end{pmatrix},$$

(36)

9
i.e. the 4 parameters of Eq. (7) have been reduced to 3 by setting $a = 0$.

Another solution is

$$U_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

which leads to

$$\mathcal{M}_2 = \begin{pmatrix} 2b + 2d & d & d \\ d & b & b \\ d & b & b \end{pmatrix},$$

i.e. $\mathcal{M}_1$ has been reduced by setting $c = d$. The 3 mass eigenvalues are then $m_{1,2} = 2b \mp \sqrt{2}d$ and $m_3 = 0$, i.e. an inverted hierarchy, with $\tan^2 \theta_{\text{sol}}$ predicted to be $2 - \sqrt{3} = 0.27$, as compared to the allowed range $[0.29$ to $0.86$ from fitting all present data.

9 New $Z_3$ Model of Neutrino Masses

Very recently, two new complete models of lepton masses have been obtained, one based on $Z_4$ \cite{16} and the other on $Z_3$ \cite{17}. The former does not fix the solar mixing angle, whereas the latter predicts $\tan^2 \theta_{\text{sol}} = 0.5$. Here I will discuss only the $Z_3$ case. Let $\mathcal{M}_\nu$ be given by

$$\mathcal{M}_\nu = \mathcal{M}_A + \mathcal{M}_B + \mathcal{M}_C,$$

where

$$\mathcal{M}_A = A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{M}_B = B \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathcal{M}_C = C \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (40)$$

Since the invariance of $\mathcal{M}_A$ requires only $U_A U_A^T = 1$, $U_A$ can be any orthogonal matrix. As for $\mathcal{M}_B$ and $\mathcal{M}_C$, they are both invariant under the $Z_2$ transformation of Eq. (34) and each is invariant under a $Z_3$ transformation, i.e. $U_B^3 = 1$ and $U_C^3 = 1$, but $U_B \neq U_C$. Specifically,

$$U_B = \begin{pmatrix} -1/2 & -\sqrt{3}/8 & -\sqrt{3/8} \\ \sqrt{3/8} & 1/4 & -3/4 \\ \sqrt{3/8} & -3/4 & 1/4 \end{pmatrix}, \quad U_C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$
Note that \( U_B \) commutes with \( U_2 \), but \( U_C \) does not. If \( U_C \) is combined with \( U_2 \), then the non-Abelian discrete symmetry \( S_3 \) is generated.

First consider \( C = 0 \). Then \( \mathcal{M}_\nu = \mathcal{M}_A + \mathcal{M}_B \) is the most general solution of

\[
U_B \mathcal{M}_\nu U_B^T = \mathcal{M}_\nu, \tag{42}
\]

and the eigenvectors of \( \mathcal{M}_\nu \) are \( \nu_e, (\nu_\mu + \nu_\tau)/\sqrt{2}, \) and \( (\nu_\mu - \nu_\tau)/\sqrt{2} \) with eigenvalues \( A - B, \ A - B, \) and \( A + B \) respectively. This explains atmospheric neutrino oscillations with \( \sin^2 2\theta_{atm} = 1 \) and

\[
(\Delta m^2)_{atm} = (A + B)^2 - (A - B)^2 = 4BA. \tag{43}
\]

Now consider \( C \neq 0 \). Then in the basis spanned by \( \nu_e, (\nu_\mu + \nu_\tau)/\sqrt{2}, \) and \( (\nu_\mu - \nu_\tau)/\sqrt{2}, \)

\[
\mathcal{M}_\nu = \begin{pmatrix} A - B + C & \sqrt{2}C & 0 \\ \sqrt{2}C & A - B + 2C & 0 \\ 0 & 0 & A + B \end{pmatrix}. \tag{44}
\]

The eigenvectors and eigenvalues become

\[
\nu_1 = \frac{1}{\sqrt{6}}(2\nu_e - \nu_\mu - \nu_\tau), \quad m_1 = A - B, \tag{45}
\]
\[
\nu_2 = \frac{1}{\sqrt{3}}(\nu_e + \nu_\mu + \nu_\tau), \quad m_2 = A - B + 3C, \tag{46}
\]
\[
\nu_3 = \frac{1}{\sqrt{2}}(\nu_\mu - \nu_\tau), \quad m_3 = A + B. \tag{47}
\]

This explains solar neutrino oscillations as well with \( \tan^2 \theta_{sol} = 1/2 \) and

\[
(\Delta m^2)_{sol} = (A - B + 3C)^2 - (A - B)^2 = 3C(2A - 2B + 3C). \tag{48}
\]

Whereas the mixing angles are fixed, the proposed \( \mathcal{M}_\nu \) has the flexibility to accommodate the three patterns of neutrino masses often mentioned, i.e.

(I) the hierarchical solution, e.g. \( B = A \) and \( C << A \);

(II) the inverted hierarchical solution, e.g. \( B = -A \) and \( C << A \);
(III) the degenerate solution, e.g. $C << B << A$.

In all cases, $C$ must be small. Therefore $\mathcal{M}_\nu$ of Eq. (39) satisfies Eq. (42) to a very good approximation, and $Z_2 \times Z_3$ as generated by $U_2$ and $U_B$ should be taken as the underlying symmetry of this model.

Since $\mathcal{M}_C$ is small and breaks the symmetry of $\mathcal{M}_A + \mathcal{M}_B$, it is natural to think of its origin in terms of the well-known dimension-five operator [18]

$$L_{\text{eff}} = \frac{f_{ij}}{2\Lambda} (\nu_i \phi^0 - \nu_j \phi^0) (\nu_j \phi^0 - \nu_i \phi^0) + H.c.,$$

(49)

where $(\phi^+, \phi^0)$ is the usual Higgs doublet of the Standard Model and $\Lambda$ is a very high scale. As $\phi^0$ picks up a nonzero vacuum expectation value $v$, neutrino masses are generated, and if $f_{ij} v^2 / \Lambda = C$ for all $i, j$, $\mathcal{M}_C$ is obtained. Since $\Lambda$ is presumably of order $10^{16}$ to $10^{18}$ GeV, $C$ is of order $10^{-3}$ to $10^{-5}$ eV, and $A - B + 3C/2$ is of order $10^{-2}$ to 1 eV. This range of values is just right to encompass all three solutions mentioned above.

To justify the assumption that $U_B$ operates in the basis $(\nu_e, \nu_\mu, \nu_\tau)$, the complete theory of leptons must be discussed. Under the assumed $Z_3$ symmetry, the leptons transform as follows:

$$(\nu, l)_i \to (U_B)_{ij} (\nu, l)_j, \quad l^c_k \to l^c_k,$$

(50)

implemented by 3 Higgs doublets and 1 Higgs triplet:

$$(\phi^0, \phi^-)_i \to (U_B)_{ij} (\phi^0, \phi^-)_j, \quad (\xi^{++}, \xi^+, \xi^0) \to (\xi^{++}, \xi^+ , \xi^0).$$

(51)

The Yukawa interactions of this model are then given by

$$L_Y = h_{ij} [\xi^0 \nu_i \nu_j - \xi^+ (\nu_i l_j + l_i \nu_j) / \sqrt{2} + \xi^{++} l_i l_j]$$

$$+ f_{ij} (l_i \phi^0_j - \nu_i \phi^-_j) l^c_k + H.c.$$ (52)

with

$$h = \begin{pmatrix} a - b & 0 & 0 \\ 0 & a & -b \\ 0 & -b & a \end{pmatrix}, \quad \mathcal{M}_\nu = 2h \langle \xi^0 \rangle,$$ (53)

12
and

\[ f^k = \begin{pmatrix} a_k - b_k & d_k & d_k \\ -d_k & a_k & -b_k \\ -d_k & -b_k & a_k \end{pmatrix} . \quad (54) \]

Note that the \( d \) terms are absent in \( h \) because it has to be symmetric. Assume \( v_{1,2} << v_3 \), and \( d_k << b_k << a_k \), then \( V_L M_{ij} M_i^\dagger V_L^\dagger = \text{diagonal} \) implies that \( V_L \) is nearly diagonal. This justifies the original choice of basis for \( M_\nu \).

Any model of neutrino mixing implies the presence of lepton flavor violation at some level. In this case, \( \phi_1^0 \) couples dominantly to \( e\tau^c \) and \( \phi_2^0 \) to \( \mu\tau^c \). Taking into account also the other couplings, the branching fractions for \( \mu \to eee \) and \( \mu \to e\gamma \) are estimated to be of order \( 10^{-12} \) and \( 10^{-11} \) respectively for a Higgs mass of 100 GeV. Both are at the level of present experimental upper bounds.

10 Conclusions

The correct form of \( M_\nu \) is now approximately known. In the \( (\nu_e, \nu_\mu, \nu_\tau) \) basis, it obeys the discrete symmetry of Eq. (34). Using Eq. (32), the phenomenologically successful Eq. (7) is obtained, which has 7 possible limits for \( M_\nu \).

Assuming some additional symmetry, such as \( A_4 \) or \( Z_3 \), with possible flavor changing radiative corrections, complete theories of leptons (and quarks) may be constructed with the prediction of specific neutrino mass patterns and other experimentally verifiable consequences.

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Appendix

It is amusing to note the parallel between the 5 perfect geometric solids and the 5 anomaly-free superstring theories in 10 dimensions. Whereas the former are related among themselves by geometric dualities, the latter are related by $S, T, U$ dualities: Type I $\leftrightarrow$ SO(32), Type IIA $\leftrightarrow$ $E_8 \times E_8$, and Type IIB is self-dual. Whereas the 5 geometric solids may be embedded in a sphere, the 5 superstring theories are believed to be different limits of a single underlying $M$ Theory.

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