Analytic Approximate for the Plasma Sheath Potential

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Abstract. Here a new analytic approximation for the Bhom Sheath Potential is presented, which is valid for any value of the characteristic parameter \(K\), measuring the mean ion velocity. The procedure to obtain this approximation is different to those used by previous authors, because now, the characteristic exponential parameter \(\lambda\), depends on the parameter \(K\), as well as the wall potential \(\phi_w\). In previous works that parameter used to be function of \(K\) only, and all the approximation used to fail for \(K \leq \frac{1}{2}\), which is not the case now.

1. Introduction

The plasmas in physics are characterized by the quasi-neutrality, which means that the number of ions and electrons are in nearly equal numbers. This property is lost near the walls containing the plasma. The classic treatment for this region, called Bhom Sheath model [1-3] leads to the idea that there are two regions a long one denoted as presheath, where the quasi-neutrality is still preserved, and the sheath region, where there are a few electrons, and the ions are those feded by the plasma, whose velocity are determined by the wall potential and plasma density. There the electron density is determined by a Boltzman factor.

In this way the plasma potential near the wall is determined by Poisson equation, which can be written in dimensionless variables as

\[
\frac{\partial^2 \phi}{\partial y^2} = -\frac{1}{2} \left[ e^{-\phi} - \frac{1}{\sqrt{1 + \phi}} \right].
\]

here the dimensionless potential \(\phi\), is defined in terms of the wall potential \(\varphi\) as

\[
\phi = \frac{e\varphi}{k_BT}
\]

where \((-e)\) is the electron charge, \(k_B\) the Boltzmann constant and \(T_e\) the electron and ion temperature of the plasma. The dimensionless distance to the wall \(y\) is measured in Debye length units \(\lambda_D\), that is, \(y = x/\lambda_D\); \(\lambda_D^2 = \varepsilon_0 k_B T/2n e^2\), where \(x\) is the actual distance to the wall.
wall, and \( n \) is the ion and electron plasma density. The parameter \( K \) introduced in Eq. (1) is also a dimensionless quantity, measuring the characteristic ion plasma velocity \( v \)

\[
    K = \frac{\frac{1}{2} m v^2}{k_B T}
\]

2. Previous Approximations to \( \phi \)

The main problem with the Eq. (1) is that no analytic solution is known, and it has to be solved numerically for each value of \( K \). Furthermore, from the equation is much easier, to obtain the distance \( x \) as a function of \( \phi \), than \( \phi \) as function of \( x \), which is usually needed it. For this reason several approximated solutions have been found for this equations [4,5].

The simplest and most usual one is

\[
    \phi_2(y) = \phi_4 \exp \left[ -\frac{1}{2} \sqrt{\frac{2K-1}{K}} y \right].
\]

This is obtained by keeping the first term of the non-linear first order differential equation

\[
    \frac{d\phi}{dy} = -\sqrt{F(\phi)},
\]

where

\[
    F(\phi) = 2K \sqrt{1 + \frac{\phi}{K}} - 2K + e^{-\phi} - 1. \quad (3)
\]

This equation is obtained by a first integration of Eq. (1), with the boundary condition that \( \frac{d\phi}{dy} \) is zero at the end of the sheath, which is also the beginning of the presheath, that is, when \( \phi \) is zero.

The Taylor expansion for \( F(\phi) \) can be written as

\[
    F(\phi) = \left( \frac{2K-1}{4K} \right) \phi^2 \left[ 1 + \phi \left( \frac{3 - 4K^2}{12K^2 - 6K} \right) + \frac{\phi^2}{96K^3 - 48K^2} \right] + \ldots \quad (4)
\]

Now \( F(\phi) \) is approached by

\[
    F(\phi) \simeq \lambda^2 \phi^2 \left( 1 + \frac{1}{2} \phi \right)^2 ; \quad \lambda^2 = \frac{2K - 1}{4K}.
\]

A better approximation for \( \phi \) is obtained [5], in the following way

\[
    \tilde{\phi}(y) = \frac{\phi_w}{e^{\lambda y} \left( 1 + \frac{1}{2} \phi_w \right) - \frac{\alpha}{2} \phi_w}, \quad (6)
\]

with

\[
    \alpha = \frac{2}{\phi_w^2} \left[ \frac{4K \sqrt{1 + \frac{\phi_w}{K}} + e^{-\phi_w} - (2K + 1)}{(2K - 1)} - \phi_w \right], \quad (7)
\]

where \( \alpha \) has been determined, imposing the condition that the slope at \( y = 0 \), must coincide with that of the exact function \( \phi \).
3. New approximate solution

The previous approximate solutions have the problem that they fail for values of $K$ equal to $\frac{1}{2}$, or nearly that value. Furthermore, in the case of the first approximation, the slope at the wall become independent of the potential wall and depend only of the value of $K$. However numerical integration of the main differential equation shows that the slope is also depending of $\phi_w$. On the other hand, through the slope in the second approximation was chosen to coincide with the right one, however the coefficient of the exponential $\lambda$ is the same than that in the first approximation. This means, that it is also independent of the potential $\phi_w$. It seems that it will much better if that coefficient were also dependent of $\phi_w$. With these ideas in mind we are presenting here, a way to obtain approximations, where all the parameters of the approximation will be depending of $K$ and $\phi_w$, that is, ion velocity and wall potential.

Here, the simplest approximation with those ideas in mind will be presented, and more elaborated ones will be also found in future works.

The simplest new approximation has the form

$$\tilde{\Phi}_1(y) = \phi_w e^{-\tilde{\lambda}y}. \quad (8)$$

Now, the parameter $\tilde{\lambda}$ is determined by the condition that the slope at $y = 0$ must be equal to the exact one, that is,

$$\left. \frac{d\phi}{dy} \right|_{y=0} = -\sqrt{F(\phi_w)}, \quad (9)$$

$$\left. \frac{d\tilde{\Phi}_1(y)}{dy} \right|_{y=0} = -\tilde{\lambda} \phi_w e^{-\tilde{\lambda}y}, \quad (10)$$

Thus

$$-\tilde{\lambda} \phi_w = -\sqrt{F(\phi_w)}. \quad (11)$$

Finally, it is obtained

$$\tilde{\lambda} = \frac{1}{\phi_w} \sqrt{F(\phi_w)} = \frac{1}{\phi_w} \left( 2K \sqrt{1 + \frac{\phi_w}{K} - 2K + e^{-\phi_w}} - 1. \right) \quad (13)$$

Now the parameter in the exponential depends not only of $K$, but also of $\phi_w$, as it should be. As it is shown in Figures 1 and 2, the accuracy of the approximation is only a little better than $\phi_2$ and $\tilde{\phi}$, but the most important advantage is that, it is good for $K = \frac{1}{2}$, as well as, values of $K$ smaller than $\frac{1}{2}$. This is very important, because of all previous approximations failed near to $\frac{1}{2}$ or lower values.

In the Figures 1a, 1b, 1c and 1d, the exact potential $\phi$ is compared with the usual exponential approximation $\phi_2$, the most complicated approximation $\tilde{\phi}$ and the new approximation $\tilde{\Phi}_1$. In the figures, four values of $K$ has been chosen: $K = 1, 0.8, 0.6$ and 0.51, which are shown respectively in Figures 1a, 1b, 1c and 1d. The Figure 2a, the absolute error of each approximation is presented.
Figure 1. In the plots (1a), (1b), (1c) and (1d), the exact potential $\phi$ is compared with the usual exponential approximation $\phi_2$, the best approximation in previous paper $\tilde{\phi}$, and the approximation here obtained $\Phi_1$, for different values of $K = 1.0; 0.8; 0.6$ and $0.51$. 
Figure 2. In the figure (2a), the absolute error of each approximation is presented, $\Delta \phi_2$ (plain line), $\Delta \tilde{\phi}$ (point line) and $\Delta \tilde{\Phi}_1$ (dash line).

Conclusion
New analytic approximation for the Bohm sheath potential has been found. This new approximation, as previous one, allows the direct calculation of the sheath potential as a function of the distance to the wall. The errors of the new approximation and the previous one have been determined for four different values of $K$. Previous approximations use to fail for values smaller than $\frac{1}{2}$ or nearly to $\frac{1}{2}$. The new approximation is good also for any value of $K$, including $K = 1/2$, which is a very important advantage compared with the previous ones. Furthermore it is also very simple, since it is an exponential as $\phi_2$, notwithstanding that its accuracy is higher.

Acknowledgments
Work supported by: (1) Decanatura de la Facultad de Ciencias Básicas, Universidad de Antofagasta, Antofagasta, Chile; (2) Grant G-22 (P. Martin). Decanato de Investigaciones, Universidad Simón Bolívar, Caracas, Venezuela, and Grant FONDECYT (L. Cortés-Vega) N° 1121103, Chile.

†The oral presentation was performed by F. Maass in SOCHIFI-2014 meeting.

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