Topological Borsuk problem
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To the memory of my wife Tanya with love

1 Introduction

This paper is based on my lecture at the US-Canada Mathcamp-2002 in Colorado Springs. The text is very elementary (it was designed for advanced high school students). At the same time the main problem about the topological Borsuk number for \( \mathbb{R}^n \), \( n > 2 \) remains open since 1977.

I thank the organizers of the Mathcamp-2002 for inviting me to give lectures.

2 Borsuk conjecture and topological Borsuk number

Famous Borsuk conjecture says that any compact \( F \) in \( \mathbb{R}^n \) can be partitioned into \( n + 1 \) closed subsets of smaller diameter. The conjecture is true for \( n = 2 \) and \( n = 3 \) as well as for all \( F \) having smooth boundary. The latter result is topological. It is based on the following Borsuk theorem: if the standard sphere \( S^{n-1} \subset \mathbb{R}^n \) is represented as a union \( F_1 \cup F_2 \cup \ldots \cup F_n \) of \( n \) closed subsets then at least one \( F_i \) contains a pair of antipodal points \( x \) and \( -x \).

General Borsuk conjecture was disproved in [KK]. Let us call \( b_{\mathbb{R}^n}(F) \) the minimal number of parts of smaller diameter necessary to partition \( F \). Then Kahn and Kalai constructed such \( F \) that

\[ b_{\mathbb{R}^n}(F) > (1.2)^{\sqrt{n}}. \]
The counterexample (as well as many of its simplifications) has combinatorial nature and deals with specific properties of the Euclidean metric in $\mathbb{R}^n$.

Topological version of the Borsuk problem was formulated around 1977 (see [So]). We recall it below.

Let $(X, \rho_0)$ be a metric space ($\rho_0$ is the metric). For any compact $F \subset X$ we denote by $b_{(X, \rho_0)}(F) := b_X(F)$ its Borsuk number, i.e. the minimal number of parts of smaller diameter necessary to partition $F$. We denote by $B(X, \rho_0)$ the Borsuk number of the metric space $X$. By definition, any compact in $X$ can be partitioned into $B(X, \rho_0)$ parts of smaller diameter, but there exist compacts which cannot be partitioned into $B(X, \rho_0) - 1$ such parts.

Let $\Omega(\rho_0)$ be the set of metrics on $X$ which define the topology equivalent to the one given by $\rho_0$. We will call elements of this set $\rho_0$-equivalent metrics.

The number $B(X, \rho)$ can change as $\rho$ varies inside of $\Omega(\rho_0)$.

**Example 1** Let $\rho_0$ be the standard Euclidean metric in $\mathbb{R}^2$ and $\rho$ be the Minkowski metric, i.e. $\rho((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$ in coordinates. Clearly $\rho \in \Omega(\rho_0)$. Then the classical result, which goes back to 1950’s, says that $B(X, \rho) = 4$. On the other hand $B(X, \rho_0) = 3$.

The following definition was given in [So].

**Definition 1** Topological Borsuk number of $(X, \rho_0)$ is defined as

$$B(X) = \min_{\rho \in \Omega(\rho_0)} B(X, \rho).$$

**Topological Borsuk Problem.** Estimate $B(X)$ for the Euclidean space $X = \mathbb{R}^n$.

In particular, is it true that $B(\mathbb{R}^n) \geq n + 1$? More speculatively, is it true that $B(\mathbb{R}^n)$ is bounded from below by $B(\mathbb{R}^n, \rho_0)$, where $\rho_0$ is the standard Euclidean metric?

### 3 2-dimensional case

In order to prove that $B(X) \geq m$ it suffices to prove that for any metric $\rho \in \Omega(\rho_0)$ there exists $c > 0$ and a finite subset $I \subset X$ consisting of $m$ elements such that $\rho(i, j) = c$ for all $i \neq j$. If $X = \mathbb{R}^n$ and $\rho = \rho_0$ is the standard Euclidean metric then one can take $c = 1, m = n + 1$ and $I$ be the
set of vertices of the regular simplex. It is not obvious that such $I$ exists for other $\rho_0$-equivalent metrics. In the case $n=2$ the answer is positive due to the following theorem.

**Theorem 1 ([So]) Topological Borsuk number of the Euclidean $\mathbb{R}^2$ is equal to 3.**

The proof presented below is basically the same as in [So].

**Proof.** Let $\rho$ be a metric on $X = \mathbb{R}^2$ which defines the topology equivalent to the standard one. Let us consider the map $f : X^3 \to \mathbb{R}^3$ such that

$$f(x_1, x_2, x_3) = (\rho_{12}, \rho_{23}, \rho_{13}),$$

where $\rho_{ij} = \rho(x_i, x_j), i \neq j$.

By our assumption the map $f$ is continuous. It suffices to prove that the image of $f$ intersects the line $l = \{(\rho_{12}, \rho_{23}, \rho_{13})|\rho_{12} = \rho_{23} = \rho_{13}\}$ besides the obvious point $(0, 0, 0)$ which is the image of the diagonal $x_1 = x_2 = x_3$.

Notice that $f$ is equivariant with respect to the natural actions of the cyclic group $\mathbb{Z}/3$ on $X^3$ and $\mathbb{R}^3$ (view $\mathbb{Z}/3$ as the subgroup of the symmetric group $\Sigma_3$ acting by permutations of coordinates).

It suffices to prove that there is no continuous $\mathbb{Z}/3$-equivariant map between $X^3 \setminus f^{-1}(0)$ and $\mathbb{R}^3 \setminus l$.

Since $X^3 = \mathbb{R}^6$ and $f^{-1}(0) = \{(x_1, x_2, x_3)|x_1 = x_2 = x_3\}$ the set $X^3 \setminus f^{-1}(0)$ is equivariantly homotopic to the 3-dimensional sphere $S^3$ (which can be considered as a $\mathbb{Z}/3$-invariant subset of the plane $x_1 + x_2 + x_3 = 0$). Similarly, $\mathbb{R}^3 \setminus l$ is equivariantly homotopic to the 1-dimensional sphere $S^1$ considered as a $\mathbb{Z}/3$-invariant subset of the plane $\rho_{12} + \rho_{23} + \rho_{13} = 0$. Notice that the natural action of $\mathbb{Z}/3$ is free on both spheres. It is enough to prove that there is no $\mathbb{Z}/3$-equivariant map between $S^3$ and $S^1$. It was done in [So] by using the notion of the category (genus) of a topological space. Here we will use its modern version called the $G$-index of a topological space where $G$ is a group acting on the space (see [M]). In our case $G = \mathbb{Z}/3$.

**Definition 2** Let $G$ be a non-trivial finite group. An $E_n G$-space is a $G$-space $Y$ such that

a) $G$ acts freely on $Y$;

b) $Y$ is $n$-dimensional;

c) $Y$ is $(n-1)$-connected.
If $X$ is a $G$-space then $\text{ind}_G(X)$ (the $G$-index of $X$) is the minimal $n$ such that there exists a $G$-equivariant map $X \to E_nG$ (the index can be infinite). It is easy to see that $\text{ind}_G(E_nG) = n$. One can prove (see [M], 6.2.5) that there is no $G$-equivariant continuous map $E_nG \to E_{n-1}G$ (Borsuk-Ulam type theorem). More generally, there is no $G$-equivariant continuous map $f : X \to Y$ between $G$-spaces $X$ and $Y$ such that $\text{ind}_G(X) > \text{ind}_G(Y)$.

On the other hand, if $p$ is a prime number then any odd-dimensional sphere is a $\mathbb{Z}/p$-space. Indeed, the group $\mathbb{Z}/p$ acts on $S^{2n-1} = \{(z_1, ..., z_n) \in \mathbb{C}^n | \sum_i |z_i|^2 = 1\}$ via $(z_1, ..., z_n) \mapsto (z_1 \exp(2\pi i/p), ..., z_n \exp(2\pi i/p))$. Taking $p = 3$ we finish the proof of the theorem. ■

4 Conclusion

The proof above uses only elementary algebraic topology and very little information about the metric. The proof does not work when we replace $\mathbb{R}^2$ by $\mathbb{R}^n, n > 2$. In that case we have a $\Sigma_{n+1}$-equivariant continuous map $(\mathbb{R}^n)^{n+1} \to R^{n(n-1)/2}$. Although arising topological spaces are still spheres, the natural actions of various subgroups of the symmetric group $\Sigma_{n+1}$ are not free on the target. Perhaps one needs new ideas in order to estimate the topological Borsuk number of $\mathbb{R}^n$.

References

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