Duality in Long-Range Ising Ferromagnets

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ABSTRACT

It is proved that for a system of spins $\sigma_i = \pm 1$ having an interaction energy $-\sum K_{ij}\sigma_i\sigma_j$ with all the $K_{ij}$ strictly positive, one can construct a dual formulation by associating a dual spin $S_{ijk} = \pm 1$ to each triplet of distinct sites $i, j$ and $k$. The dual interaction energy reads $-\sum_{(ij)} D_{ij} \prod_{k \neq i, j} S_{ijk}$ with $\tanh(K_{ij}) = \exp(-2D_{ij})$, and it is invariant under local symmetries. We discuss the gauge-fixing procedure, identities relating averages of order and disorder variables and representations of various quantities as integrals over Grassmann variables. The relevance of these results for Polyakov’s approach of the 3D Ising model is briefly discussed.
1. Introduction, Basic Ideas and Main Results

The order-disorder duality \cite{1,2,3} has been a useful tool to approach statistical or field theory models. In particular, the self-duality \cite{7} of the two-dimensional Ising model with nearest neighbor interactions on a square lattice provides an easy way to obtain the critical temperature and suggests a convenient way to write the partition function and the specific heat. The model dual to the three-dimensional nearest neighbor Ising model is a gauge theory \cite{13} and interesting loop equations, combining the order and disorder variables, have been obtained by Polyakov.\cite{11} These equations suggest that the model near criticality can be described in terms of a string model. Despite interesting results,\cite{3} several aspects of this idea await clarification.

Recently, we have proposed\cite{9} to study related problems in the case of the hierarchical model. For this model, the renormalization group procedure can be reduced to a local recursion relation which has been studied in detail\cite{2} and rigorous results concerning its fixed points and the spectrum of its tangent map have been obtained. This recursion relation is closely related to the the “approximate recursion formula” used by Wilson\cite{14} to estimate (rather well) the critical exponents of the three-dimensional Ising model. This motivated us to try to express the renormalization group transformations in various reformulations of the hierarchical model. In particular, we constructed the dual formulation and the several representations of the partition function in terms of Grassmann variables. Due to the non-locality of the model - the price to pay for the locality of the renormalization group transformation - appropriate methods are necessary to enumerate the terms appearing in the high (or low) temperature expansion. The methods we found apply to a larger class of Ising models than the one considered initially and are reported here.

In this paper, we consider the class of systems of spins \( \sigma_i = \pm 1 \), where \( i \) runs from 1 to \( N \), with an interaction energy

\[
\beta H = - \sum_{i<j} K_{ij} \sigma_i \sigma_j \tag{1}
\]

and all the \( N(N - 1)/2 \) couplings \( K_{ij} \) strictly positive but otherwise unspec-
This class of models includes for instance, the hierarchical model and one-dimensional Ising models with $|i - j|^{-\alpha}$ interactions.

As usual, we define

$$Z = \sum_{\{\sigma_i = \pm 1\}} e^{-\beta H}$$

and

$$<\sigma_1 \ldots \sigma_{i_2n}> = Z^{-1} \sum_{\{\sigma_i = \pm 1\}} \sigma_1 \ldots \sigma_{i_2n} e^{-\beta H}.$$  

In the following, a sequence of indices is understood as strictly ordered unless specified. For instance, the notation $K_{ij}$ implies $i < j$. We use parenthesis when the ordering needs to be performed. For instance, $K_{(ij)}$ is equal to $K_{ij}$ if $i < j$ and to $K_{ji}$ if $j < i$. This ordering is a matter of notational convenience and has no intrinsic significance in the problem. We also define the dual couplings by the relation

$$tanhK_{ij} = e^{-2D_{ij}}.$$ 

It is clear that the strict positivity of $K_{ij}$ implies the strict positivity of $D_{ij}$.

One of the main results obtained here is a dual formulation of the class of models defined above. The disorder variables are attached to each triple of distinct sites $ijk$ and denoted $S_{(ijk)}$. The interaction energy can be constructed out of products of $S_{(ijk)}$ with $i$ and $j$ fixed and $k$ running over all its values except for $i$ and $j$. More precisely:

**Theorem 1.** The class of Ising models defined in Eqs. (1-4) admits the following equivalent (dual) representation

$$Z = (\prod_{i<j} \cosh K_{ij}) 2^{N-(N-1)} \sum_{\{S_{ijk} = \pm 1\}} e^{+\sum_{i<j}(D_{ij}(\prod_{k,k \neq i,j} S_{ijk}^{-1}))}.$$
and if all the pairs $i_1i_2, \ldots, i_{2n-1}i_n$ are distinct

$$Z[D_{i_1i_2} \to -D_{i_1i_2}, \ldots, D_{i_{2n-1}i_{2n}} \to -D_{i_{2n-1}i_{2n}}] =$$

$$Ze^{+2(D_{i_1i_2} + \ldots + D_{i_{2n-1}i_{2n}})} <\sigma_{i_1}\sigma_{i_2}\ldots\ldots\sigma_{i_{2n-1}}\sigma_{i_{2n}}>.$$ (6)

In addition, for any four distinct sites $i,j,k$ and $l$, the dual interaction energy is invariant under the simultaneous changes of sign of $S_{(ijk)}, S_{(ijl)}, S_{(ikl)}$ and $S_{(jkl)}$.

This theorem is proven in three steps in the next section. The first is a standard high-temperature expansion. The second establishes a connection between the terms of this expansion and some elements of a vector space. The third uses the fact that the homology of a certain boundary operator over this vector space is trivial to rewrite Eqs. (2) and (3) under the form given in Eqs. (5) and (6). This proof makes the local invariance mentioned at the end of the above theorem rather transparent. It is possible to eliminate this degeneracy by imposing an appropriate gauge-fixing condition. An example is given in section 3.

Theorem 1 gives a procedure to express average values of the order variables in terms of average values of the disorder variables. This procedure can be inverted using the standard character expansion for Ising spins. Simple identities obtained this way are shown in section 4. In the case of short range Ising models, identities relating appropriately chosen products of order and disorder variables can be interpreted as a discrete version of some kind of Dirac (or Ramond) equation.$^{[11,3]}$ These equations are closely related to the Schwinger-Dyson equations associated with a representation of the partition function as an integral$^{[12]}$ over (anticommuting) Grassmann variables.$^{[1]}$ In section 5, we rewrite the partition function of Eq. (2) as an integral over Grassmann variables (Theorem 2). This formulation is very compact and the Schwinger-Dyson equations follow straightforwardly. Note that the integral representation is not minimal, it involves twice as many variables as we normally need. The elimination of these auxiliary variables is discussed at the end of the section 6 where Theorem 2 is proven. In the conclusions, we briefly explain how these results can be used in a treatment of the hierarchical model inspired by the work of Polyakov, Dotsenko and other authors.
2. Proof of Theorem 1

The first step in the construction of the dual formulation is a standard high-temperature expansion of the partition function and the correlation functions. We shall recall it briefly in order to introduce useful notation. We use the character expansion \( \exp(K_{ij}\sigma_i\sigma_j) = \cosh(K_{ij}) + \sinh(K_{ij})\sigma_i\sigma_j \) and we introduce at each link \( ij \) a new variable \( n_{ij} \) which takes the values 0 or 1. These can be used to rewrite Eqs. (2) and (3) through the identity

\[
\prod_{i<j}(1 + A_{ij}) = \sum_{\{n_{ij}=0,1\}} \prod_{i<j} A_{ij}^{n_{ij}} \tag{7}
\]

The non-zero contributions come from the terms where all the \( \sigma_i \) appear an even number of times. The high temperature expansion can thus be written as a sum over paths were the links \( ij \) appear at most once and the number of “visits” at the site \( i \)

\[
N_i = \sum_{j=1}^{i-1} n_{ji} + \sum_{j=i+1}^{N} n_{ij} = \sum_{j:j\neq i} n_{ij} \tag{8}
\]

is odd if \( i \) is one of sites of the spins appearing in the correlation functions and even otherwise. In the following we shall use \( a = b(2) \) as a short notation for \( a = b \mod 2 \). The above arguments can be summarized by the first lemma.

**Lemma 1** For the class of Ising models defined in Eqs. (1-4)

\[
\sum_{\{\sigma_i=\pm1\}} \sigma_{i_1}....\sigma_{i_{2n}} e^{-\beta H} = (\prod_{i<j} \cosh K_{ij}) 2^N \sum_{\{n_{ij}=0,1\}} e^{\sum_{i<j} -2D_{ij}n_{ij}} \tag{9}
\]

with \( m_i = 1 \) if \( i \in \{i_1, \ldots, i_{2m}\} \) and 0 otherwise.

The second step consists in establishing a correspondence between the terms of the above expansion and the elements of a vector space over the field of integers modulo 2. Our goal is to express the condition \( N_i = m_i(2) \) appearing in Lemma 1 in terms of a boundary operator having a trivial homology. For this purpose we introduce the following definitions.
Definitions. Let $V$ be a $2^N$ dimensional vector space over the field of integers modulo 2. To any sequence of integers $1 \leq i_1 < i_2 < \ldots < i_n \leq N$, we associate an element of a basis of $V$ denoted $|i_1i_2\ldots i_n>$. The elements of this basis corresponding to sequences with exactly $n$ indices span a subspace denoted $V_n$. We define the boundary operator $\partial_n : V_n \to V_{n-1}$ for $1 \leq n \leq N$ such that

$$\partial_n|i_1i_2\ldots i_n> = |\hat{i_1}i_2\ldots i_n> + |i_1\hat{i_2}\ldots i_n> + \ldots + |i_1i_2\ldots \hat{i_n}>$$

(10)

where the hat designates a skipped index. We also define $D_n : V_n \to V_{n+1}$ for $0 \leq n \leq N-1$ such that

$$D_n|i_1i_2\ldots i_n> = \begin{cases} 
|1\,i_1i_2\ldots i_n> & \text{if } i_1 > 1 \\
0 & \text{otherwise}.
\end{cases}$$

(11)

The choice of the first index is as inessential as the ordering itself.

We can now establish a one-to-one correspondence between any choice of configuration $\{n_{ij}\}$ appearing in the high temperature expansion (9) and the vector of $V_2$

$$|\{n_{ij}\} >= \sum_{i<j} n_{ij} |ij> .$$

(12a)

Similarly, we can denote the elements of $V_1$

$$|\{m_i\} >= \sum_i m_i |i> .$$

(12b)

From (10), $\partial_2|i j> = |i> + |j>$. This relates in a clear way the link $ij$ and the two sites “visited” $i$ and $j$. Collecting all the terms and using Eq.(7) we obtain the following.

**Lemma 2.** The condition $N_i = m_i(2)$ can be rewritten as

$$\partial_2|\{n_{ij}\} > = |\{m_i\} > .$$

(13)

In order to find the general solution of Eq.(13), it is crucial to note that the homology of $\partial_n$ is trivial. In order to establish this result, we first prove the following.
Proposition 1. If $1 \leq n \leq N - 1$, then

\begin{align*}
a) \ & \partial_n \partial_{n+1} = 0 \\
b) \ & D_{n-1} \partial_n + \partial_{n+1} D_n = 1 \\
c) \ & D_{n+1} D_n = 0 \ .
\end{align*}

This follows in a elementary way from the definitions (Eqs. (10)-(11)). First, $\partial_{n+1}$ deletes one index in all the possible ways and $\partial_n \partial_{n+1}$ deletes all possible pairs of indices twice. Since $V$ is a vector space over the integer modulo 2, this proves $a)$. Similarly, we obtain $b)$ and $c)$ as a mere application of the definitions by discussing separately the cases where the first index is 1 or is not 1.

Corollary 1. If $1 \leq n \leq N - 1$, then $\text{Im} \ \partial_{n+1} = \text{Ker} \ \partial_n$.

Part $a)$ of the Proposition 1 implies that $\text{Im} \ \partial_{n+1} \subseteq \text{Ker} \ \partial_n$. If $|v_n> \in \text{Ker} \ \partial_n$ then part $b)$ implies that $|v_n> = \partial_{n+1} D_n |v_n>$ which proves that $\text{Im} \ \partial_{n+1} \supseteq \text{Ker} \ \partial_n$.

Corollary 2. $\text{Dim} \ \text{Im} \ \partial_{n+1} = \binom{N-1}{n}$.

Using that $\text{Dim} \ V_n = \binom{N}{n} = \text{Dim} \ \text{Im} \ \partial_n + \text{Dim} \ \text{Ker} \ \partial_n$ and Corollary 1, we find $\text{Dim} \ \text{Im} \ \partial_{n+1} = \binom{N}{n} - \text{Dim} \ \text{Im} \ \partial_n$. Repeating $n$ times we obtain,

$$
\text{Dim} \ \text{Im} \ \partial_{n+1} = \sum_{m=0}^{n} \binom{N}{n-m} (-1)^m \tag{15}
$$

which amounts to the second corollary.

We are now ready to rewrite restricted sums like those appearing in Eq (9) as unrestricted sums.

Lemma 3. If $f$ is a function $V_2 \to \mathbb{R}$ and $|\{m_i\} > \in V_1$ then

$$
\sum_{\substack{|n_{ij}> \in V_2 \\
\partial_2(n_{ij}) = |\{m_i\}>}} f(|\{n_{ij}\}> = 2^{-(N-3)} \sum_{|v_3> \in V_3} f(|\{n_{ij}^0\}> + \partial_3 |v_3>) \tag{16}
$$
where \(|\{n_{ij}^0\}\rangle\>\) is any solution of

\[
\partial_2|\{n_{ij}^0\}\rangle\=>|\{m_i\}\rangle\>
\]

Two arbitrary solutions of Eq.(13) differ by an element of \(\text{Ker} \ \partial_2\) which is equal to \(\text{Im} \ \partial_3\) by Corollary 1. Consequently, we can write any solution as the sum of as particular solution \(|\{n_{ij}^0\}\rangle\>\) plus \(\partial_3|v_3\rangle\>\) for some \(|v_3\rangle\>\in V_3\). This element of \(V_3\) is determined up to an element of \(\text{Ker} \ \partial_3 = \text{Im} \ \partial_4\), a subspace having dimension \(\binom{N-1}{3}\) by Corollary 2. In other words, if we consider now \(f\) to be a function over \(V_3\), it is invariant under the “gauge transformation”

\[
|v_3\rangle\>\rightarrow |v_3\rangle\> + \partial_4|v_4\rangle\>
\]

with \(|v_4\rangle\>\in V_4\). When taking the unrestricted sum over \(V_3\), we need to divide by the “gauge multiplicity” which from Corollary 2 is \(2\binom{N-1}{3}\). This concludes the proof of Lemma 3.

We are now in position to prove Theorem 1. Using Lemma 3, we can replace the restricted sum over \(V_2\) (i.e., over the \(n_{ij}\)) appearing in Lemma 1, by an unrestricted sum over \(V_3\). More precisely, for any triples of distinct ordered sites \(i,j\) and \(k\), we introduce a new variable \(d_{ijk}\) taking the values 0 or 1. A given choice of \(\{d_{ijk}\}\) defines an element of \(V_3\) in an obvious generalization of Eqs. (12). We need to replace \(n_{ij}\) by \((n_{ij}^0 + \sum_{k: k \neq i, j} d_{ijk})\) modulo 2 considered as a real number taking the values 0 or 1. This can be conveniently done with the substitution

\[
2n_{ij} \rightarrow 1 + (-1)^{n_{ij}^0 + \sum_{k: k \neq i, j} d_{ijk}}.
\]

If the \(m_i\) are like in Lemma 1, we can choose for instance, \(n_{i_1i_2}^0 = ... = n_{i_{2n-1}i_{2n}}^0 = 1\) the others being zero, as in the particular solution appearing in Lemma 3, provided that all the pairs \(i_1i_2,..., i_{2n-1}i_n\) are distinct. The local invariance of the new formulation has been made clear in Eq.(18). Defining for convenience \(S_{ijk} = (-1)^{d_{ijk}}\), we obtain Theorem 1 as stated in the introduction.
It is instructive to check that the original model can be recovered if the duality transformation is applied to the dual model. Starting from Eq. (5) one can proceed as above and prove that the cohomology of a coboundary operator (algebraically dual to the boundary operator considered above) is trivial. The original model is then recovered easily. Since no new concepts or results are involved in this proof, it has been left as an exercise.

3. An Example of Gauge-Fixing in the Dual Formulation

The proof of Lemma 3 has made clear the fact that the sum over $V_3$ can indeed be rewritten as a sum over $V_3/\text{Im} \, \partial_4$. In this section, we show that the gauge condition

$$D_3 |v_3> = 0 \quad (20)$$

is sufficient to fix a unique representative for each class of equivalence of $V_3/\text{Im} \, \partial_4$. This choice is motivated by the fact that we have studied the properties of $D_3$ in the previous section. It is not a “covariant” choice since the first site plays a special role. When studying particular models, existing symmetries may suggest more appropriate choices.

In order to show that the condition (20) specifies one and only one element of $V_3$ among those differing by a gauge transformation (18), we introduce a projection operator $P$ defined as

$$P = 1 + \partial_4 D_3 \quad (21)$$

From Proposition 1, it is immediate that $P^2 = P$ and more importantly that

$$D_3 P = P \partial_4 = 0 \quad (22)$$

Given any element $|v_3>$ of $V_3$, this implies that $P|v_3>$ satisfies the condition (20) and that if two elements differ by a transformation (18) their images under $P$ are identical. This concludes the proof of the statement made above.
In terms of the coefficients $d_{ijk}$ in the $|ijk>$ basis, the condition (20) amounts to
\[ d_{ijk} = 0 \text{ if } 1 < i < j < k \] (23)
the remaining ones ($d_{1jk}$) being unspecified by the condition. It is easy to check that a similar result may be obtained starting with the high-temperature expansion (9) and eliminating the $n_{1j}$ using the $N - 1$ independent restrictions on the $N_i$. (It is clear that $\sum_{i=1}^{N} N_i = 0(2).$)

In general, a gauge condition of the form $G|v_3>=0$ specifies uniquely the representatives and can be enforced by a projection operator $P_G = 1 + \partial_4 H_3$, where $H_3$ is an operator from $V_3$ to $V_4$ depending on $G$, provided that we can satisfy the consistency conditions $G = G\partial_4 H_3$ and $\partial_4 = \partial_4 H_3\partial_4$.

4. Simple Applications of Theorem 1

In this section, we work out simple applications of Theorem 1. We explain how to introduce order variables in the dual formulation and disorder variables in the original formulation. We define the average value of a function of the dual spins $S_{ijk}$ as the r.h.s of Eq.(5) with this function inserted in the sum over the dual spins, divided by $Z$. We shall use the same brackets as in Eq.(3), their content indicating unambiguously which kind of average is considered.

We know from Theorem 1 that an insertion of $\sigma_i \sigma_j$ in an average value amounts to a change $D_{ij} \rightarrow -D_{ij}$ in the dual formulation. This change can be implemented by an insertion of $exp(-2D_{ij} \prod_{k; k \neq i,j} S_{(ijk)})$ provided it occurs only once. The simplest application of this procedure is
\[ <\sigma_i \sigma_j> = <e^{-2D_{ij} \prod_{k; k \neq i,j} S_{(ijk)}}> \geq 0 \] (24)
The positivity of the average follows from the positivity of the exponential and yields the first Griffiths inequality.$^{[4]}$ The generalization to an arbitrary number of
distinct pairs is straightforward. An interesting example is

$$
< (\sigma_i \sigma_j) (\sigma_j \sigma_k) (\sigma_k \sigma_i) > = < e^{-2(D_{ij} \prod_{l \neq i,j} S_{(ijl)}) + (D_{jk} \prod_{m \neq j,k} S_{(jkm)}) + (D_{ik} \prod_{n \neq i,k} S_{(ikn)})} >
$$

$$
= 1 .
$$

(25)

The second equality can be seen either from the fact that the square of the sigmas is 1 or from the invariance of $Z$, as written in Eq.(5), under the change of sign of $S_{ijk}$.

The above examples show how to express products of an even number of order variables in the dual formulation. The procedure can easily be inverted in order to express gauge-invariant products of the disorder variables within the original formulation. The practical implementation is that $\exp(-2K_{ij} \sigma_i \sigma_j)$ corresponds to $\prod_{k:k \neq i,j} S_{(ijk)}$ in the dual formulation. This follows from a character expansion of the exponential, the expression of $\sigma_i \sigma_j$ in the dual formulation discussed above, another character expansion and the identities $\sinh 2D_{ij} \sinh K_{ij} = 1$ and $\cosh 2D_{ij} \sinh 2K_{ij} = \cosh 2K_{ij}$. Obviously, this implies

$$
< \prod_{k:k \neq i,j} S_{(ijk)} > = < e^{-2K_{ij} \sigma_i \sigma_j} >
$$

$$
\geq 0 .
$$

(26)

The analog of Eq.(25) reads

$$
< \prod_{j:j \neq i} ( \prod_{k:k \neq i,j} S_{(ijk)}) > = < e^{-2 \sum_{j:j \neq i} K_{ij} \sigma_i \sigma_j} >
$$

$$
= 1 .
$$

(27)

Note that each $S_{(ijk)}$ appears twice in the above products. Using similar methods, we obtain easily

$$
< \prod_{m:m \neq i,j} S_{(ijm)} \prod_{n:n \neq k,m} S_{(klm)} > - < \prod_{m:m \neq i,j} S_{(ijm)} < \prod_{n:n \neq k,m} S_{(klm)} > =
$$

$$
\sinh K_{ij} \sinh K_{kl} ( < \sigma_i \sigma_j \sigma_k \sigma_l > - < \sigma_i \sigma_j > < \sigma_k \sigma_l > ) .
$$

(28)
The positivity of the r.h.s is the second Griffiths inequality.\cite{4}

The above procedures can be used to write products of order and disorder variables in both formulations. In the case of short range Ising models, it is possible to write suggestive identities relating average values of suitably chosen products of this type. A better insight concerning these equations may be obtained by reexpressing the partition function as an integral over Grassmann variables. Note that this representation seems also well suited for the renormalization group method.\cite{10}

5. A Representation of $Z$ as an Integral over Grassmann Variables

In this section, we describe a representation of the partition function as an integral over Grassmann variables (Theorem 2). We use this integral representation to express average values of the type considered in the previous section and to obtain Schwinger-Dyson equations. The proof of Theorem 2 is given in the next section.

The basic properties of a set $\{\Psi_a\}$ of Grassmann variables can be summarized as follows.

$$\Psi_a^2 = 0$$
$$\Psi_a \Psi_b = -\Psi_b \Psi_a$$
$$d\Psi_a \Psi_b = -\Psi_b d\Psi_b$$
$$\int d\Psi_a = 0$$
$$\int d\Psi_a \Psi_a = 1.$$  \hfill \text{(29)}

A detailed presentation on this subject can be found in Berezin’s book.\cite{1} This technique has been used previously for short-range Ising models.\cite{12} In the following, Grassmann variables are denoted $\psi$ or $\chi$ with various set of indices.

For notational convenience, we define

$$z(x) = \begin{cases} \cosh(x) & \text{if } N \text{ is odd;} \\ \sinh(x) & \text{if } N \text{ is even.} \end{cases}$$  \hfill \text{(30)}
\[ f(x) = \frac{d}{dx} \ln(z(x)) \]  
(31)

\[ [d\psi d\chi] = \prod_{i<j} d\chi^j_i d\psi^j_i d\psi^i_j \]  
(32)

The following theorem will be proven in the next section

**Theorem 2.** The partition function of the Ising models defined in Eqs. (1-4) admits the integral representation

\[
Z = \left(\prod_{i<j} \cosh K_{ij}\right)2^N \int [d\psi d\chi] \ e^{\sum_{i<j}((\th K_{ij})\psi^i_j \psi^j_i - \chi^i_i \chi^j_j)} \prod_{i=1}^N z(\sum_{j, j \neq i} \psi^j_i \chi^j_i) \]  
(33)

As in the previous section, we define the average of a function of the Grassmann variables by inserting this function in the integral representation of \( Z \) given in Theorem 2 and dividing by \( Z \). Again we use the same brackets, their content raising the ambiguity. It is easy to express the average value of products of order or disorder variables considered in the previous section in terms of average values of functions of \( \psi^j_i \psi^i_j \). In particular, comparing the derivatives of \( Z \) with respect to \( K_{ij} \) in Eq. (2) and Theorem 2, we obtain for \( i < j \)

\[
< \sigma_i \sigma_j > = \th K_{ij} + (\cosh K_{ij})^{-2} < \psi^i_j \psi^j_i > . \]  
(34)

On the other hand, by changing the sign of \( K_{ij} \) in these two expressions of \( Z \), it follows that

\[
< e^{-2K_{ij} \sigma_i \sigma_j} > = 1 - \th K_{ij} < \psi^i_j \psi^j_i > . \]  
(35)

These operations can be repeated in an obvious way, in order to obtain more involved products, provided that distinct pairs of indices are used.

It is also possible to use the well-known fact that a constant shift does not change an integral over a Grassmann variables to obtain useful identities. This type of identities are often called equations of motions or Schwinger-Dyson equations.
Let $A$ be an arbitrary function of the Grassmann variables and $\Psi$ either $\psi^j_i$ or $\chi^j_i$ with $i < j$. Using the invariance of $<A>$ under a shift in $\Psi$, we obtain the identity

$$<\frac{\partial}{\partial \Psi}[(thK_{ij})\psi^j_i\psi^j_i - \chi^j_i\chi^j_i] + f\left(\sum_{j:j \neq i} \psi^j_i\chi^j_i\right)\frac{\partial}{\partial \Psi}\left[\sum_{j:j \neq i} \psi^j_i\chi^j_i\right]>A + \frac{\partial}{\partial \Psi}A> = 0 \quad (36)$$

The function $f(x)$ has been defined in Eq.(31). In the case $N$ even, coth$(x)$ must be multiplied the factor sinh$(x)$ appearing in the integral before being Taylor expanded. We have used the usual convention that the derivative with respect to $\Psi$ anticommutes with the other Grassmann variables. As a simple example, for $A = \psi^j_i$ and $\Psi = \psi^j_i$ the identity reads

$$<\psi^j_i\psi^j_i> = (thK_{ij})^{-1}(1+<f(\sum_{k:k \neq i} \psi^k_i\chi^k_i)\chi^j_i\psi^j_i>) \quad (37)$$

Similarly, if $\Psi$ takes the values $\psi^j_i$ or $\chi^j_i$ with $i < j$, we obtain

$$<\frac{\partial}{\partial \Psi}[(thK_{ij})\psi^j_i\psi^j_i - \chi^j_i\chi^j_i] + f\left(\sum_{i:i \neq j} \psi^j_i\chi^j_i\right)\frac{\partial}{\partial \Psi}\left[\sum_{i:i \neq j} \psi^j_i\chi^j_i\right]>A + \frac{\partial}{\partial \Psi}A> = 0 \quad (38)$$

6. Proof of Theorem 2 and Comments

In this section, we prove Theorem 2, we discuss the extension of the method to the dual case and we discuss the elimination of the “auxiliary” variables $\chi$.

The proof of Theorem 2 is based on the following identity

$$1 + thK_{ij}\sigma_i\sigma_j =$$

$$\int d\chi^j_i d\psi^j_i d\chi^j_i d\psi^j_i e^{(thK_{ij}\psi^j_i\psi^j_i - \chi^j_i\chi^j_i)}$$

$$\sum_{n_{ij}=0,1} (\sigma_i)^{n_{ij}}(\psi^j_i\chi^j_i)^{(1-n_{ij})} \sum_{m_{ij}=0,1} (\sigma_j)^{m_{ij}}(\psi^j_i\chi^j_i)^{(1-m_{ij})}. \quad (39)$$

After using this identity for each link, we can rearrange inside the integral all the terms having a common subindex $i$, i.e., a common $\sigma_i$. Summing over the $\sigma_i$, we
obtain a “local” condition namely that

\[
\sum_{j=1}^{i-1} m_{ji} + \sum_{j=i+1}^{N} n_{ij} = 0 \tag{40}
\]

By construction, the \( m_{ji} \) and \( n_{ij} \) do not appear at other sites and their sum can be performed locally. When \( N \) is odd (even), we obtain the sum over all products of an even (odd) number of distinct \( \psi_i^j \chi^j_i \). Using the fact that the square of a Grassmann number is zero, we can express this sum as the \( \cosh (\sinh) \) of the sum over \( j \) of the \( \psi_i^j \chi^j_i \). This concludes the proof of Theorem 2.

The dual version of this construction is easy to obtain. To each link \( ij \) we assign the Grassmann variables \( \psi_{ij}^k \) and \( \chi_{ij}^k \) with \( k \) running over all possible values but \( i \) and \( j \). The non-local part of the integrand can be chosen as

\[
e e_{i<j} \sum_{k,k\neq i,j} \psi_{ij}^k \prod_{i<j,k,k\neq i,j} \chi_{ij}^k \tag{41}
\]

where the products are ordered in the natural order and \( \gamma = (-1)^{(N-2)(N-3)} \). Proceeding as in the above proof, we obtain the local part of the integrand

\[
\prod_{i<j<k} \sinh(\psi_{ij}^k \chi_{ij}^k + \psi_{ik}^j \chi_{ik}^j + \psi_{jk}^i \chi_{jk}^i) \tag{42}
\]

The simplicity of the proof of Theorem 2 comes from the fact that we were able to rearrange the \( \psi_i^j \chi_i^j \) into local products. This was the reason to introduce the auxiliary variables \( \chi_{ij}^j \) which “screen” the anticommuting nature of \( \psi_i^j \). Had we started with the minimal identity

\[
1 + \theta K_{ij} \sigma_i \sigma_j = \\
\int d\psi_i^j d\psi_j^i e^{(\theta K_{ij} \psi_i^j \psi_j^i)} \sum_{n_{ij}=0,1} \sum_{m_{ij}=0,1} \sigma_i^{n_{ij}} \sigma_j^{m_{ij}} (\psi_i^j)^{(1-n_{ij})} (\psi_j^i)^{(1-m_{ij})} \tag{43}
\]

we would have had to specify an ordering and to keep a detailed bookkeeping of the minus signs resulting from the rearrangement of the \( \psi_i^j \). In this process, we
obtain non-local factors of the type \((-1)^{n_{ij}n_{kl}}\) where all the indices \(i, j, k\) and \(l\) are distinct (or similar expressions with \(m_{ij}\) or \(m_{kl}\)). This prevents us from performing the sums locally as in the above proof. However, we can use the decomposition

\[
(-1)^{n_{ij}n_{kl}} = \frac{1}{2}(1 + (-1)^{n_{ij}} + (-1)^{n_{kl}} - (-1)^{n_{ij}+n_{kl}}) .
\] (44)

Each of the terms of the r.h.s can be decomposed into local factors. Each time this decomposition is used, the number of terms in the partition function is multiplied by four. We found that the number of terms in the partition function is at least \(2^{\frac{(N-3)(N-4)}{6}}\). This result is in agreement with a remark of Kasteleyn [8] and its proof in the present formulation will only be outlined.

The lower bound on the number of terms can be obtained by establishing a correspondence between the non-local factors and the edge-crossings of a planar projection (depending on the ordering) of the complete graph [5] with \(N\) vertices. We consider a surface on which the complete graph can be embedded without edge-crossing. From Euler theorem, the genus of this surface is at least \(\frac{(N-3)(N-4)}{12}\), the inequality being saturated if all the faces were triangles. We then note that if the \(n_{ij}\) and \(n_{kl}\) in (43) were replaced by sums, the identity would still be valid. At least one non-local factor of this type corresponds to the projection of each handle of the surface considered. This concludes the outlined proof.

7. Conclusions

The duality transformation and the representations in terms of Grassmann variables have enlightened our understanding of the nearest neighbor Ising models in various dimensions. We have extended these reformulations and their immediate applications to the case where an arbitrary number of Ising spins have a strictly negative interaction energy associated to any pair of spin. No other assumption has been made on the couplings.
These results apply to the hierarchical models. For these models, the renormalization group transformation is very simple and can be handled satisfactorily with several methods \cite{2} when the order variables are used. The study of the continuum limit of the reformulations presented here using the renormalization group method is a challenging problem \cite{10}. This study is expected to shed a new light on questions related to the 3-dimensional nearest neighbors Ising model where the hierarchical approximation is rather good, as far as the critical behavior is concerned. In particular, we would like to understand the continuum limit of the reformulation as “sums over equipped surfaces” proposed by Dotsenko and Polyakov.\cite{3}

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