SYMMETRIES AND CONSERVATION LAWS OF A KDV6 EQUATION

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ABSTRACT. In the present work we make an analysis of the Korteweg-de Vries of sixth order. We apply the classical Lie method of infinitesimals and the nonclassical method, due to Bluman and Cole, to deduce new symmetries of the equation which cannot be obtained by Lie classical method. Moreover, we obtain ten different conservation laws depending on the parameters and we conclude that potential symmetries project on the infinitesimals corresponding to the classical symmetries.

1. Introduction. We have studied the Korteweg-de Vries of sixth order, called KdV6, a nonlinear partial differential equation (PDE) with (1+1) dimensions, which models the dynamics of two-layered shallow-water flow

\[
\Delta \equiv u_{6x} + au_x u_{4x} + bu_{xx} u_{xxx} + cu_x^2 u_{xx} + du_{tt} + eu_{xxx} + fu_{x} u_{xt} + hu_{xx} = 0 \quad (1)
\]

where subscripts denote partial derivatives and \( a, b, c, d, e, f \) and \( h \) are arbitrary parameters.

Karasu-Kalkanli et al. [18], extending the well known Painlevé analysis [23] to the class of sixth-order nonlinear wave equations, have found that the general solution of (1) is free from movable critical singularity manifolds and admits sufficient number of arbitrary functions for the following values of parameters, which lead to the well known Sawada-Kotera-Caudrey-Dodd-Gibbon [10, 22], Kaup-Kupershmidt [14, 19] and Drinfeld-Sokolov-Satsuma-Hirota system of coupled Korteweg-de Vries equations [13, 21].

On the other hand, symmetry reductions theory and method have continuously been in focus of research of many well-known mathematicians and physicists due to their important applications in the context of differential equations [9, 16]. For example, solutions of partial differential equations, obtained by symmetry reduction, tend to solutions of lower-dimensional equations and some of these special solutions will illustrate important physical phenomena.

Moreover, the method also allow us to find exact solutions. In particular, exact solutions arising from symmetry methods can often be used effectively to study properties such as asymptotics and “blow-up”. By the way explicit solutions, like those found by symmetry methods, provide an important practical check on the accuracy and reliability of integrators [12]. So they can also be used in the design and testing of numerical integrators.

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Additionally and motivated by the fact that symmetry reductions for many partial differential equations are known that are not obtained by using the classical Lie method (in \cite{8, 15} authors got new reductions which cannot be obtained by Lie classical symmetries), we have studied another generalization of the classical Lie group method too. Bluman and Cole \cite{5} proposed the so-called nonclassical method of group-invariant solutions in which although the number of determining equations is smaller, the set of solutions is larger than for the classical method.

Conservation laws have several important uses in the study of PDEs, especially for determining conserved quantities and constants of motion. They are also useful detecting integrability and linearizations, finding potentials and nonlocally-related systems, as well as checking the accuracy of numerical solution methods. The conservation laws admitted by a system can be obtained by the direct method of Anco and Bluman \cite{1, 2, 3}. In \cite{7} Bluman et al. have introduced the concept of potential symmetry for any differential equation which can be written as a conservation law and this type of symmetries are nonlocal symmetries.

Besides, due to the large amount of algebra and auxiliary calculations needed for the results in this paper we have used symbolic manipulation programs as wxMaxima - symmgrp2009.max \cite{17} and the Maple software.

To sum up, the structure of the work is as follows: in Sec. 2 we study the classical Lie symmetries of equation (1), we find different cases depending on the constants for which we obtain the Lie group of point transformations admitted by the corresponding equation. Next, for Sec. 3 we obtain the nonclassical symmetries for the equation, showing that there are more symmetries applying the nonclassical method than the classical one. In the following, for Sec. 4 and Sec. 5 we study conservation laws and potential symmetries, respectively. We have obtained ten conservation laws divided in six different cases depending on the constants. Finally, conclusions are presented in Sec. 6.

2. Classical symmetries. To apply the Lie classical method to equation (1) we have considered the one-parameter Lie group of infinitesimal transformations in \((x, t, u)\) given by

\[
\begin{align*}
x^* &= x + \epsilon \xi(x, t, u) + O(\epsilon^2), \\
t^* &= t + \epsilon \tau(x, t, u) + O(\epsilon^2), \\
u^* &= u + \epsilon \eta(x, t, u) + O(\epsilon^2),
\end{align*}
\]

where \(\epsilon\) is the group parameter. We have required that this transformation leaves invariant the set of solutions of equation (1). This yields to an overdetermined, linear system of equations for the infinitesimals \(\xi(x, t, u), \tau(x, t, u)\) and \(\eta(x, t, u)\). The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

\[
V = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + \eta(x, t, u) \frac{\partial}{\partial u}.
\]

The functions \(u = u(x, t)\), which are invariant under the infinitesimals \(V\), are, in essence, solutions to an equation arising as the “invariant surface condition”:

\[
\eta(x, t, u) - \xi(x, t, u) \frac{\partial u}{\partial x} - \tau(x, t, u) \frac{\partial u}{\partial t} = 0.
\]

The symmetry variables are found by solving the invariant surface condition. The reduction transforms the PDE into ODEs.
So applying the classical Lie group symmetry analysis to equation (1). The set of solutions of the equation is invariant under the transformation (2)-(4) provided
\[ \text{pr}^{(6)}V(\Delta) = 0 \quad \text{whenever} \quad \Delta = 0, \]
where \( \text{pr}^{(6)}V \) is the sixth prolongation of the vector field (5) and yields a system of equations for the infinitesimals \( \xi(x,t,u), \tau(x,t,u) \) and \( \eta(x,t,u) \). By solving the system we have got that \( \xi = \xi(x,t), \tau = \tau(t), \eta = \alpha(x,t)u + \beta(x,t) \) where \( \xi(x,t), \tau(t), \alpha(t,x) \) and \( \beta(x,t) \) must satisfy the following sixteen determining equations for the infinitesimals, when the constants that multiply to the equation are not null:
\[
\begin{align*}
\tau_t - 3\xi_x &= 0, \\
\alpha_x - \xi_{xx} &= 0, \\
2\alpha_x - \xi_{xx} &= 0, \\
\alpha + \xi_x &= 0, \\
2\alpha_x - 5\xi_{xx} &= 0, \\
\alpha - \tau_t + 4\xi &= 0, \\
2h\alpha_x + f\alpha_x - \xi_x &= 0, \\
\alpha_{xxxxxx}u + d\alpha_tu + e\alpha_{txxx}u + \beta_{xxxxxx} + d\beta_{tt} + e\beta_{txxx} &= 0, \\
f\alpha_x + f\beta_x + 3e\alpha_x - e\xi_x - 2d\xi_t, h\alpha_xu + h\beta_x + e\alpha_{xxx} + 2d\alpha_t - d\tau_{tt} &= 0, \\
2bo_x + 4a\alpha_x - b\xi_{xx} - 6a\xi_x &= 0, \\
a\alpha_xu + a\beta_x + 15\alpha_{xx} - 20\xi_{xxx} + e\xi_t &= 0, \\
c\alpha_{xx}u + c\beta_{xx} + 4a\alpha_{xxx} + f\alpha_t + \xi_{tt}f - a\xi_{xxx} &= 0, \\
bo_{xx}u + b\beta_{xx} + 20\alpha_{xxx} + e\alpha_t - 15\xi_{xxxx} - 3e\xi_x &= 0, \\
b\alpha_{xxx}u + h\alpha_tu + b\beta_{xxx} + h\beta_t + 15\alpha_{xxxx} + 3e\alpha_{tx} - 6\xi_{xxxx} - 3e\xi_{txxx} &= 0, \\
2bo_{xx}u + 2e\beta_{xx} + 3b\alpha_{xx} + 6a\alpha_{xx} - \xi_xh - \xi_{tt}f - b\xi_{xx} - 4a\xi_x &= 0, \\
2a\alpha_{xxx}u + f\alpha_{tx}u + a\beta_{xxxx} + f\beta_{tx} + 6a\alpha_{xxxx} + 3e\alpha_{txx} - \xi_{xxxxxx} - d\xi_t - e\xi_{txxx} &= 0.
\end{align*}
\]

The solutions of this system depend on the constants of the equation and solving it we have obtained five cases which classify the Lie symmetries.

- **Case 1.** If \( a, b, c, d, e, f \) and \( h \) are arbitrary constants, the infinitesimal generators are
  \[
  V_1 = \partial x, \\
  V_2 = \partial t, \\
  V_3 = \partial u, \\
  V_4 = x\partial x + 3t\partial t - u\partial u
  \]

- **Case 2.** If \( h = 0 \) and \( a, b, c, d, e \) and \( f \) are arbitrary constants, the infinitesimal generators are \( V_1, V_2, V_3, V_4 \) and
  \[
  V_5^1 = t\partial u
  \]

- **Case 3.** If \( c = a(1 + \frac{h}{f}), d = \frac{1}{2}(f + h), c = \frac{1}{2}f, d, f \neq 0, h, a \) are arbitrary constants, the infinitesimal generators are \( V_1, V_2, V_3, V_4 \) and
  \[
  V_5^2 = t\partial x + \left( 1 + \frac{h}{f} \right) x\partial u
  \]

- **Case 4.** If \( d = 0 \) and \( h = 0 \), the infinitesimal generators are \( V_1, V_2, V_3, V_4, V_5^1 \) and
  \[
  V_5^\infty = \beta\partial u
  \]
where \( \beta \) is a function depending on \( t \).
• Case 5. If \( d = 0, \ h = 0 \) and \( af = 2ce \) the symmetries admitted by (1) are
\( V_1, \ V_2, \ V_3, \ V_4, \ V_5^1, \ V_5^\infty \) and
\[
V_6^\infty = \alpha \partial x + \frac{e}{a} x \partial u,
\]
where \( \alpha \) is a function depending on \( t \).

3. Nonclassical symmetries. The basic idea of the method is that the PDE (1)
is augmented with the invariance surface condition
\[
\Phi \equiv \xi \partial_x + \tau \partial_t - \eta = 0,
\]
which is associated to the vector field (5).

By requiring that both, (1) and (8), are invariant under the transformation with
infinitesimal generator (5), an overdetermined nonlinear system of equations for the
infinitesimals \( \xi(x,t,u) \), \( \tau(x,t,u) \) and \( \eta(x,t,u) \) is obtained.

The number of determining equations arising in the nonclassical method is smaller
than for the classical method. Consequently the set of solutions is, in general, larger
than for the classical method. However, the associated vector fields do not form a
vector space.

To obtain nonclassical symmetries of (1) we have applied the algorithm described
in [11] for calculating the determining equations. We distinguished two different
cases:

I. In the case \( \tau \neq 0 \), without loss of generality, we may set \( \tau(x,t,u) = 1 \).

Then, from (8) we get \( u_t = \eta - \xi u_x \) and we apply the chain rule to obtain
the successive derivatives. We have obtained a set of fourteen determining
equations for the infinitesimals \( \xi(x,t,u) \) and \( \eta(x,t,u) \).
Solving the fourteen determining equations we have obtained the following four cases:

- **Case I.1.** If \( a, c, d, e, f, h \) are arbitrary parameters, the generators are \( V_2 \) and \( V_4 \).
- **Case I.2.** If \( a, b, c, d, e, f \) are arbitrary parameters and \( h = 0 \), the generators are \( V_2, V_4 \) and \( V_0^1 \).
- **Case I.3.** If \( a, c, e, f \) are arbitrary parameters, \( d = 0 \) and \( h = 0 \), the generators are \( V_2, V_4, V_0^1, V_0^∞ \).
- **Case I.4.** If \( b, e \) are arbitrary parameters, \( a \neq 0 \), \( d = h = 0 \), \( e f = 0 \) and \( af = 2ce \) the nonclassical symmetries admitted by (1) are \( V_2, V_4, V_0^1, V_0^∞ \) and \( V_0^∞ \).

So, by comparing these symmetries with the symmetries obtained by the classical method in Section 2, we have observed that the nonclassical method with \( \tau \neq 0 \) applied to (1) gives only rise to the classical symmetries.

II. In the case \( \tau = 0 \), without loss of generality we assume that \( \xi = 1 \) and that way from (8) we get \( u_x = \eta \). Then, applying the classical Lie method with these conditions to equations (1) and (8) we have obtained an overdetermined system for the infinitesimal \( \eta \).

\[
\begin{align*}
\eta_{uu} & = 0 \\
\eta_{ux} & = 0 \\
h\eta_{xx} + h\eta_{u}\eta_{x} + f\eta_{u}\eta_{x} + h\eta(\eta_{u})^{2} + f\eta(\eta_{u})^{2} + 2d\eta_{u} & = 0 \\
h\eta_{xxx} - \eta_{u}\eta_{xxxx} + \eta_{u}\eta_{xxxx} + a\eta_{xxxx} + b\eta_{xxx} + 2b\eta_{u}\eta_{xx} + a\eta_{u}\eta_{xx} + 2b\eta_{u}\eta_{xx} + c\eta_{xx} + c\eta_{xx} + b\eta_{u}\eta_{xx} + a\eta_{u}\eta_{xx} + 2b\eta_{u}\eta_{xx} + c\eta_{xx} + c\eta_{xx} + d\eta_{x} & = 0 \\
b(\eta_{u})^{2}(\eta_{u})^{2} + a(\eta_{u})^{2}(\eta_{u})^{2} + 2c(\eta_{u})^{2} + 2b(\eta_{u})^{2} + b(\eta_{u})^{2} + c\eta_{x} + c\eta_{x} + d\eta_{x} + e\eta_{x} + f\eta_{x} + b\eta_{x} + c\eta_{x} + d\eta_{x} + e\eta_{x} + f\eta_{x} + b\eta_{x} + c\eta_{x} + d\eta_{x} + e\eta_{x} + f\eta_{x} & = 0
\end{align*}
\]

The system is quite complex and it seems impossible to find the general solution, however we have obtained that equation (1) admits the following infinitesimals:

- **Case II.1.** If \( a, c, d, e, f, h \) are arbitrary parameters, the infinitesimal admitted is
  \[
  \eta = k_1 t + k_2
  \]
- **Case II.2.** If \( d = 0 \) and \( a, c, e, f, h \) are arbitrary parameters,
  \[
  \eta = \eta(t)
  \]
- **Case II.3.** If \( a = c = f = 0 \) and \( e, h \) are arbitrary parameters, then
  \[
  \eta = k_1 x + k_2
  \]
- **Case II.4.** If \( a = 0, b = 0, c = 0, h = 0 \),
  \[
  \eta = k_4 x^4 + k_3 x^3 + k_2 x^2 + k_1 x + k_0
  \]

Therefore, comparing these symmetries with the symmetries obtained by the classical method in Section 2 we can observe that the nonclassical method applied to (1) is more general that the classical Lie method.
4. Conservation laws. In order to obtain conservation laws for equation (1) we
have applied the multiplier method [1, 2, 3, 4, 20]. Anco and Bluman gave a general
treatment of a direct conservation law method for partial differential equations
expressed in a standard Cauchy-Kovalevskaya form. We have observed that equation
(1) has a Cauchy-Kovalevskaya form, so all non-trivial conservation laws arise from
multipliers [2, 3, 20].

Let us give an $s$th-order partial differential equation
\[ F(x, u, u(1), \ldots, u(s)) = 0 \]  
with independent variables $x = (x^1, \ldots, x^n)$ and a dependent variable $u$, where
$u(1) = \{u_i\}$, $u(2) = \{u_{ij}\}$, \ldots denote the sets of the partial derivatives of the first,
second, etc. orders, $u_i = \partial u/\partial x^i$, $u_{ij} = \partial^2 u/\partial x^i \partial x^j$. The Euler-Lagrange operator
is defined by the formal sum
\[ \frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1} \cdots i_s}, \]  
where, for every $s$, the summation is presupposed over the repeated indices $i_1 \cdots i_s$
running from 1 to $n$ and
\[ D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots \]
the total differentiation with respect to $x^i$.

Noether’s theorem states that if the variational integral with the Lagrangian $\mathcal{L}(x, u, u(1))$ is invariant under a group $G$ with a generator
\[ X = \xi^i(x, u, u(1), \ldots) \frac{\partial}{\partial x^i} + \eta(x, u, u(1), \ldots) \frac{\partial}{\partial u} \]  
then the vector field $C = (C^1, \ldots, C^n)$ defined by
\[ C^i = \xi^i \mathcal{L} + (\eta^a - \xi^j u_j^a) \frac{\partial \mathcal{L}}{\partial u^a_i}, \quad i = 1, \ldots, n, \]
provides a conservation law for the Euler-Lagrange equations (10), i.e. it obeys the
equation $\text{div} \: C \equiv D_i(C^i) = 0$ for all solutions of (10).

A function $\Lambda(x, u, u(1), \ldots)$ is called multiplier if it verifies that
\[ D_i \mathcal{C}^i = \Lambda F \]
is a divergence expression. The determining equation for the multiplier is
\[ \frac{\delta (\Lambda F)}{\delta u} = 0. \]
Once the multipliers are obtained the conservation laws are calculated by using an
homotopy formula [3].

Then, we have applied the multiplier method to equation (1), with $\Lambda = \Lambda(x, t, u)$
and $a, b, c \neq 0$, and we have obtained:

1. For $c, d, f, h$ arbitrary constants the multiplier is $\Lambda_1 = 1$ and the conservation
law is
\[ C^1 = \frac{1}{2} (-h + f) u_x^2 + du_t, \]
\[ C^2 = \frac{1}{3} cu_x^3 + (au_{xxx} + hu_t) u_x + \frac{1}{2} (b - a) u_{xxx}^2 + u_{xxxxx} + eu_{txt} \]
2. For c, d arbitrary constants and $f = h$, besides $\Lambda_1$, we have obtained the multiplier $\Lambda_2 = t$ and the conservation law

\[ C^1 = d(tu_t - u), \]
\[ C^2 = t(u_{xxxx} + au_2u_{xxx} + ku_{txx} - \frac{1}{2} u_{xx}^2 a + \frac{1}{2} u_{xx}^2 b + \frac{1}{3} u_x^3 c + hu_t u_x) \]

3. For c arbitrary constant, $d = 0$ and $f = h$, we have obtained a new multiplier $\Lambda_3 = f_1(t)$ and the conservation law

\[ C^1 = 0, \]
\[ C^2 = f_1(t) (u_{xxxx} + au_2u_{xxx} + ku_{txx} - \frac{1}{2} u_{xx}^2 a + \frac{1}{2} u_{xx}^2 b + \frac{1}{3} u_x^3 c + hu_t u_x) \]

4. For $d, f$ arbitrary constants, $c = h = 0$ and $a = \frac{1}{3} b$, we have obtained the following multipliers and conservation laws:

(a) $\Lambda_4 = x$ and the conservation law

\[ C^1 = \frac{1}{2} f x u_x^2 + x d u_t, \]
\[ C^2 = x u_{xxxx} - u_{xxx} + \frac{1}{3} x b u_x u_{xxx} + x k u_{txx} + \frac{1}{3} x b u_x^2 - k u_{tx} - \frac{1}{3} b u_t u_{xx} \]

(b) $\Lambda_5 = \frac{1}{2} x^2$ and the conservation law

\[ C^1 = \frac{1}{4} (u_x f + 2 d u_t) x^2 + k u_x, \]
\[ C^2 = \frac{1}{6} \left( (u_x u_{xxx} + u_{xx}^2) b + 3 k u_{txx} + 3 u_{xxxxx} \right) x^2 + \frac{1}{6} u_x^2 b + u_{xx} + \frac{1}{6} \left( -2 b u_x u_{xx} - 6 k u_{tx} - 6 u_{xxxx} \right) x \]

(c) $\Lambda_6 = t - \frac{1}{2} \frac{f x^3}{b}$ and the conservation law

\[ C^1 = \frac{1}{10} \left[ - f^2 x^3 u_x^2 + \left( -2 d x^3 u_t + 2 b t u_x^2 - 12 k x u_x \right) f + 4 b d (tu_t - u) \right], \]
\[ C^2 = \frac{1}{6} \left( (u_x u_{xxx} - u_{xx}^2) x^3 + 3 x^2 u_x u_{xx} - 3 x u_x^2 b + (-3 k u_{xx} - 3 u_{xxxx}) x^3 + (9 k u_{xx} + 9 u_{xxxx}) x^2 \right) - 18 x u_{xx} + 18 u_x f + 2 b \left( (u_x u_{xxx} + u_{xx}^2) b + 3 k u_{txx} + 3 u_{xxxx} \right) \]

(d) $\Lambda_7 = t x - \frac{f x^4}{8 b}$ and the conservation law

\[ C^1 = \frac{1}{5} \left( \frac{17}{10} f^2 x^3 u_x^2 + \left( \frac{1}{2} b t u_x^2 - \frac{1}{8} x (x^2 d u_t + 12 k u_x) \right) f + b d (tu_t - u) \right), \]
\[ C^2 = \frac{1}{24} \left( - f \left( (u_x u_{xxx} + u_{xx}^2) b + 3 k u_{txx} + 3 u_{xxxx} \right) x^4 + 4 f \left( b u_x u_{xx} + 3 k u_{tx} + 3 u_{xxxx} \right) x^3 - 6 f \left( u_x^2 b + 6 u_{xx} \right) x^2 + (8 t (u_x u_{xxx} + u_{xx}^2) b + 24 t (k u_{xx} + u_{xxxx}) b + 72 f u_{xx}) x - 8 b^2 u_x u_{xx} - 24 t (k u_{xx} + u_{xxxx}) b - 72 f u_x \right) \]


(e) \( \Lambda_8 = \frac{1}{4} t x^2 - \frac{f x^6}{60} \) and the conservation law

\[
C^1 = \frac{1}{800} \left( -f^2 u_x u_{xx} - 2 f d u_t \right) x^5 - 40 f x^3 k u_x + 40 b (1/2 f u_x^2 \\
+ d (t u_t - u)) x^2 + 80 k t u_x b),
\]

\[
C^2 = \frac{1}{240} \left( f \left( (u_x u_{xxx} + u_{xx}^2) b + 3 k u_{txx} + 3 u_{xxxxx} \right) x^4 \\
+ 4 f \left( b u_x u_{xx} + 3 k u_{tx} + 3 u_{xxxx} \right) x^3 - 6 f \left( u_x^2 b + 6 u_{xxx} \right) x^2 \\
+ (8 t (u_x u_{xxx} + u_{xx}^2)) b^2 + 24 t (k u_{txx} + u_{xxxx}) b + 72 f u_x \right) x \\
- 8 b u_x u_{xx} - 24 t (k u_x + u_{xxx}) b - 72 f u_x
\]

5. For \( b \) arbitrary constant, \( c = d = f = h = 0 \) and \( a = \frac{1}{b} \), we have obtained the multiplier \( \Lambda_9 = \frac{1}{2} f_1(t) x^2 + f_2(t) x + f_3(t) \) and the conservation law

\[
C^1 = 0,
\]

\[
C^2 = \frac{1}{6} \left( (u_x u_{xxx} + u_{xx}^2) b + 3 k u_{txx} + 3 u_{xxxxx} \right) x^2 \\
+ (-2 b u_x u_{xx} - 6 k u_{tx} - 6 u_{xxxx}) x + u_x^2 b + 6 k u_x + 6 u_{xxx} \right) f_1(t) \\
+ \frac{1}{6} \left( (2 u_x u_{xxx} + 2 u_{xx}^2) b + 6 k u_{txx} + 6 u_{xxxxx} \right) x - 2 b u_x u_{xx} \\
- 6 k u_{tx} - 6 u_{xxx} \right) f_2(t) + \frac{1}{3} \left( (u_x u_{xxx} + u_{xx}^2) b + 3 k u_{txx} \\
+ 3 u_{xxxxx} \right) f_3(t)
\]

6. For \( b \) arbitrary constant, \( c = h = 0, d = -3 f \frac{(b k + f)}{b^2} \) and \( a = \frac{1}{b} \), we have obtained the multiplier \( \Lambda_{10} = h(t,x) \), where \( h \) must satisfy \( h_{xxx} + \frac{3 f}{b} h_t = 0 \), and the conservation law

\[
C^1 = \frac{1}{36} \left( 2 \left( \frac{\partial^2}{\partial x^2} h(t,x) \right) \right) k u_x b^2 + f \left( -6 u (b k - 3 f) \frac{\partial}{\partial x} h(t,x) \\
+ h(t,x) \left( b^2 u_x^2 + 6 b k u_t - 18 f u_t \right) \right),
\]

\[
C^2 = \left( \frac{\partial^2}{\partial x^2} h(t,x) \right) u_x + \left( \frac{\partial^3}{\partial x^3} h(t,x) \right) u_x - \left( \frac{\partial^3}{\partial x^3} h(t,x) \right) k u \\
- \left( \frac{\partial^3}{\partial x^3} h(t,x) \right) u_{xx} + \frac{1}{6} \left( \partial^4 / \partial x^4 \right) h(t,x) \\
- \left( \frac{\partial^3}{\partial x^3} h(t,x) \right) u_{xx} + \frac{1}{6} \left( \partial^4 / \partial x^4 \right) h(t,x) \\
+ \frac{1}{6} \left( -2 b u_x u_{xx} - 6 k u_{tx} - 6 u_{xxx} \right) \frac{\partial}{\partial x} h(t,x) \\
+ \frac{1}{6} \left( -2 b u_x u_{xx} - 6 k u_{tx} - 6 u_{xxx} \right) \frac{\partial}{\partial x} h(t,x)
\]

5. **Classical potential symmetries.** In [6, 7] Bluman et al. introduced a method to find a new class of symmetries for a PDE. Suppose a given scalar PDE of second order

\[
F(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = 0,
\]

where the subscripts denote the partial derivatives of \( u \), it can be written as a conservation law

\[
\frac{D}{D_t} f(x,t,u,u_x,u_t) - \frac{D}{D_x} g(x,t,u,u_x,u_t) = 0,
\]

for some functions \( f \) and \( g \) of the indicated arguments. Here \( \frac{D}{D_x} \) and \( \frac{D}{D_t} \) are total derivative operators defined by

\[
\frac{D}{D_x} = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \ldots,
\]

\[
\frac{D}{D_t} = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + \ldots
\]
Through the conservation law (13) one can introduce an auxiliary potential variable \( v \) and form an auxiliary potential system (system approach) called \( S \)

\[
\begin{align*}
v_x &= f(x, t, u, u_x, u_t), \\
v_t &= g(x, t, u, u_x, u_t).
\end{align*}
\]

For many physical equations one can eliminate \( u \) from the potential system (14) and form an auxiliary integrated or potential equation (integrated equation approach)

\[
G(x, t, v, v_x, v_t, v_{xx}, v_{xt}, v_{tt}) = 0,
\]

for some function \( G \) of the indicated arguments. Any Lie group of point transformations

\[
X_S = \xi(x, t, u, v) \partial_x + \tau(x, t, u, v) \partial_t + \eta(x, t, u, v) \partial_u + \psi(x, t, u, v) \partial_v
\]

admitted by (14) yields a nonlocal symmetry potential symmetry of the given PDE (13) if and only if the following condition is satisfied

\[
\xi_v^2 + \tau_v^2 + \eta_v^2 \neq 0.
\]

In order to find potential symmetries of (1) \( a, b, c, d, e \) arbitrary constants and \( f, h \neq 0 \), from conservation law (1) the associated auxiliary system is given by

\[
\begin{align*}
v_x &= \frac{1}{2} (-h + f) u_x^2 + du_t, \\
v_t &= \frac{1}{3} cu_x^3 + (au_{xxx} + hu_t) u_x + \frac{1}{2} (b - a) u_{xx}^2 + u_{xxxx} + eu_{txx}.
\end{align*}
\]

A Lie point symmetry admitted by \( S(x, t, u, v) \) is a symmetry characterized by an infinitesimal transformation of the form

\[
\begin{align*}
x^* &= x + \epsilon \xi(x, t, u, v) + \mathcal{O}(\epsilon^2), \\
t^* &= t + \epsilon \tau(x, t, u, v) + \mathcal{O}(\epsilon^2), \\
u^* &= u + \epsilon \eta(x, t, u, v) + \mathcal{O}(\epsilon^2), \\
v^* &= v + \epsilon \psi(x, t, u, v) + \mathcal{O}(\epsilon^2)
\end{align*}
\]

admitted by system (18).

In the current work, we have presented the point symmetries of (18) and we have studied that those symmetries induce potential symmetries of equation (1). These symmetries are such that the condition (17)

\[
\xi_v^2 + \tau_v^2 + \eta_v^2 \neq 0
\]

is satisfied. If the above relation does not hold, then the point symmetries of (18) project into point symmetries of (1).

System (18) admits Lie symmetries if and only if

\[
\begin{align*}
\text{pr}^{(5)} X_S (v_x - \frac{1}{2} (-h + f) u_x^2 - du_t) &= 0, \\
\text{pr}^{(5)} X_S (v_t - \frac{1}{3} cu_x^3 + (au_{xxx} + hu_t) u_x - \frac{1}{2} (b - a) u_{xx}^2 - u_{xxxx} - eu_{txx}) &= 0,
\end{align*}
\]

where \( \text{pr}^{(5)} X_S \) is the fifth extended generator of (16). In other words, we require that the infinitesimal generator leaves invariant the set of solutions of (18). This yields to an overdetermined system of equations for the infinitesimals \( \xi(x, t, u, v), \tau(x, t, u, v), \psi(x, t, u, v) \) and \( \phi(x, t, u, v) \).
From this system we have obtained that
\[ \xi = \frac{1}{3} k_1 x + k_3, \quad \tau = k_1 t + k_2, \quad \eta = -\frac{1}{3} k_1 u + k_3, \quad \varphi = -k_1 v + k_4 \]
And it is not a potential symmetry of the equation (1) because the condition (17) is not satisfied.

6. Conclusions. In this paper, we have considered the Korteweg-de Vries of sixth order which is a nonlinear partial differential equation with (1+1) dimensions. We have looked for the Lie classical symmetries and, depending on the parameters, we have found the classification in five different cases of them and its Lie algebra.

On the other hand, we have studied the nonclassical Lie symmetries for the two main cases: considering \( \tau \neq 0 \) and \( \tau = 0 \), and we have determined some nonclassical symmetries. Moreover, we prove that the nonclassical method applied to (1) is more general that the classical Lie method.

In next section we studied the conservation laws of the (1) equation and we have found ten conservation laws divided in six different cases because of the constants.

Finally, for \( c, d, f, h \) arbitrary constants, and from the conservation laws, we have determined the auxiliary potential system and we have studied classical potential symmetries. So, we conclude that potential symmetries project on the infinitesimals corresponding to the classical symmetries.

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