MATRIX REPRESENTATIONS FOR SOME SELF-SIMILAR MEASURES ON $\mathbb{R}^d$

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Abstract. We establish matrix representations for self-similar measures on $\mathbb{R}^d$ generated by equicontractive IFSs satisfying the finite type condition. As an application, we prove that the $L^q$-spectrum of every such self-similar measure is differentiable on $(0, \infty)$. This extends an earlier result of Feng (J. Lond. Math. Soc. (2) 68(1):102–118, 2003) to higher dimensions.

1. Introduction

In this paper, we study self-similar measures on $\mathbb{R}^d$ generated by iterated function systems satisfying the finite type condition. By an iterated function system (IFS) (of similitudes) on $\mathbb{R}^d$, we mean a finite family of contracting similitudes on $\mathbb{R}^d$. According to a result of Hutchinson [20], given an IFS $\Phi = \{S_i\}_{i=1}^m$ on $\mathbb{R}^d$, there is a unique non-empty compact set $K \subset \mathbb{R}^d$ satisfying $K = \bigcup_{i=1}^m S_i(K)$, which is called the self-similar set generated by $\Phi$. Moreover, given a probability vector $p = (p_1, \ldots, p_m)$, i.e. each $p_i > 0$ and $\sum_{i=1}^m p_i = 1$, there exists a unique Borel probability measure $\mu$ supported on $K$ such that

$$\mu = \sum_{i=1}^m p_i \mu \circ S_i^{-1}. \tag{1.1}$$

We call $\mu$ the self-similar measure generated by $\Phi$ and $p$.

Given an IFS $\Phi = \{S_i\}_{i=1}^m$ on $\mathbb{R}^d$, let $\Sigma = \{1, \ldots, m\}$ be the alphabet associated to $\Phi$. For each $n \in \mathbb{N}$, let $\Sigma_n = \{i_1 \ldots i_n : i_k \in \Sigma, 1 \leq k \leq n\}$. Moreover, set $\Sigma_0 = \{\varepsilon\}$, where $\varepsilon$ denotes the empty word. Let $\Sigma_* = \bigcup_{n=0}^\infty \Sigma_n$ and let $\Sigma^\infty$ be the collection of infinite words over $\Sigma$. For $I = i_1 \ldots i_n \in \Sigma_*$, write $S_I = S_{i_1} \circ \cdots \circ S_{i_n}$. In particular, set $S_\varepsilon = \text{id}$, the identity map on $\mathbb{R}^d$. For a similitude $S$ on $\mathbb{R}^d$, we let $\rho_S > 0$ be the similarity ratio of $S$. We say that $\Phi$ is equicontractive if $\rho_{S_1} = \cdots = \rho_{S_m}$.

This paper is motivated by the study of equicontractive self-similar measures on $\mathbb{R}$ by Feng [9]. More precisely, let $\mu$ be a self-similar measure on $\mathbb{R}$ generated by an IFS $\Phi = \{S_i\}_{i=1}^m$ on $\mathbb{R}$ of the form

$$S_i(x) = \lambda x + b_i, \quad i = 1, \ldots, m, \tag{1.2}$$

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where $\lambda \in (0,1)$ and $b_1, \ldots, b_m \in \mathbb{R}$. In [9], Feng investigated the case when $\Phi$ satisfies the so-called finite type condition, i.e., there exists a finite set $\Gamma_0$ such that for any $n \geq 1$ and any $I, J \in \Sigma_n$,

\begin{equation}
\text{either } \lambda^{-n}|S_I(0) - S_J(0)| \geq c \text{ or } \lambda^{-n}|S_I(0) - S_J(0)| \in \Gamma_0,
\end{equation}

where $c = (1 - \lambda)^{-1}(\max_{1 \leq i \leq m} b_i - \min_{1 \leq i \leq m} b_i)$. Feng established the matrix representations for $\mu$ on the so-called basic net intervals and proved that the $L^q$-spectrum of $\mu$ (see (1.7) for the definition) is differentiable on $(0, \infty)$. The result of matrix representations for $\mu$ is also used to give a checkable criterion for the absolute continuity of $\mu$ (cf. [14, Theorem 6.2]), and it is also applied to study the topological structure of the set of local dimensions of $\mu$ in a series of papers [15, 16, 18].

The purpose of this paper is to extend the results of [9] to higher dimensions. We will consider self-similar measures on $\mathbb{R}^d$ generated by equicontractive IFSs satisfying the following version of the finite type condition.

**Definition 1.1.** Let $\Phi = \{S_j\}_{j=1}^m$ be an equicontractive IFS on $\mathbb{R}^d$ which generates a self-similar set $K$. We say that $\Phi$ satisfies the finite type condition (FTC) if there exists a finite set $\Gamma$ such that for any $n \geq 1$ and $I, J \in \Sigma_n$,

\begin{equation}
\text{either } S_I(K) \cap S_J(K) = \emptyset \text{ or } S_I^{-1} \circ S_J \in \Gamma.
\end{equation}

The concept of the FTC was first introduced by Ngai and Wang [27] in a more general setting. It is easily seen that for an IFS on $\mathbb{R}$ of the form (1.2), the conditions (1.3) and (1.4) are equivalent.

To state our main results of this paper, we first introduce some notation. First we define the (canonical) Borel partitions of a self-similar set generated by an equicontractive IFS.

Let $K \subset \mathbb{R}^d$ be the self-similar set generated by an equicontractive IFS $\Phi = \{S_j\}_{j=1}^m$. Let $\mathcal{S}$ denote the set of similitudes on $\mathbb{R}^d$ and $2^\mathcal{S}$ be the collection of all the subsets of $\mathcal{S}$. For each $n \geq 0$, we define a mapping $\Lambda_n : K \to 2^\mathcal{S}$ by

\begin{equation}
\Lambda_n(x) = \{S_I : I \in \Sigma_n \text{ with } x \in S_I(K)\} \text{ for } x \in K.
\end{equation}

Let $\Lambda_n(K)$ be the image of $K$ under $\Lambda_n$, i.e. $\Lambda_n(K) = \{\Lambda_n(x) : x \in K\}$. Then define

\begin{equation}
\xi_n = \{\Lambda_n^{-1}(\mathcal{U}) : \mathcal{U} \in \Lambda_n(K)\}.
\end{equation}

It is easy to see that $\xi_n$ is a finite partition of $K$ whose elements are all Borel sets. We call $\xi_n$ the $n$-th (canonical) Borel partition of $K$.

Also we need the notion of $L^q$-spectrum of measures. Let $\nu$ be a finite Borel measure on $\mathbb{R}^d$ with compact support. For $q \in \mathbb{R}$, the $L^q$-spectrum of $\nu$ is defined by

\begin{equation}
\tau(q) = \tau(\nu, q) = \liminf_{\delta \to 0} \frac{\log \left(\sup_{\delta > 0} \sum_i \nu(B(x_i, \delta))^q\right)}{\log \delta},
\end{equation}

where the supremum is taken over all families of disjoint closed balls $B(x_i, \delta)$ of radius $\delta$ and centres $x_i \in \text{supp}\nu$.

The main results of this paper are the following two results, which extend [9, Theorem 1.1] from $\mathbb{R}$ to $\mathbb{R}^d$. 

Theorem 1.2. Let $\Phi = \{S_i\}_{i=1}^m$ be an equicontractive IFS on $\mathbb{R}^d$ and $K$ be the self-similar set generated by $\Phi$. Suppose that $\Phi$ satisfies the FTC. Let $\mu$ be the self-similar measure generated by $\Phi$ and a probability vector $(p_1, \ldots, p_m)$. Then there exist $s, N \in \mathbb{N}, N \times N$ non-negative matrices $M_1, \ldots, M_s$, and $N$-dimensional positive row vectors $w_1, \ldots, w_s$ such that for any $n \in \mathbb{N}$ and $\Delta \in \xi_n$ with $\mu(\Delta) > 0$, we have
\[
\mu(\Delta) = e_1 M_{i_1} \cdots M_{i_n} w_{i_n}^T,
\]
where $n_1n_2 \cdots n_s$ is the symbolic expression of $\Delta$ (see Section 2 for the definition), $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N$ and $a^T$ denotes the transpose of $a$.

Theorem 1.3. Under the assumptions of Theorem 1.2, the $L^q$-spectrum $\tau(q)$ of $\mu$ is differentiable on $(0, \infty)$.

We present two examples to which Theorems 1.2-1.3 apply directly. Recall that an algebraic integer $\beta > 1$ is called a Pisot number if all its Galois conjugates have modulus less than 1. Similarly, we call an algebraic integer $\zeta \in \mathbb{C} \setminus \mathbb{R}$ a complex Pisot number if $|\zeta| > 1$ and all its Galois conjugates, except $\bar{\zeta}$, have modulus less than 1.

Example 1.4. [27, Theorem 2.5] Let $\Phi = \{\rho x + a_i\}_{i=1}^m$ be an IFS on $\mathbb{R}^d$, where $\rho^{-1}$ is a Pisot number and $a_i \in \mathbb{Q}(\rho)^d$ for $1 \leq i \leq m$. Then $\Phi$ satisfies the FTC.

Example 1.5. Let $\Phi = \{\rho x + a_i\}_{i=1}^m$ be an IFS on $\mathbb{C}$ in the complex form, where $\rho^{-1}$ is a complex Pisot number and $a_i \in \mathbb{Q}(\rho)$ for $1 \leq i \leq m$. Then $\Phi$ satisfies the FTC.

The proof of Example 1.5 is similar to that of Example 1.4, which we omit.

Theorems 1.2-1.3 can also be applied to self-similar measures on the so-called golden gaskets studied in [3]. Indeed, more generally, let $\Phi = \{S_i(x) = \lambda x + (1 - \lambda)a_i\}_{i=1}^3$ be an IFS on $\mathbb{R}^2$, where $\lambda^{-1}$ is a Pisot number, and $a_1, a_2, a_3 \in \mathbb{R}^2$ are non-collinear points. Then there exists an invertible affine transformation $f$ on $\mathbb{R}^2$ such that
\[
\{f \circ S_i \circ f^{-1}\}_{i=1}^3 = \{\lambda x, \lambda x + (1, 0), \lambda x + (0, 1)\},
\]
see [3, Section 8, Remark (1)]. Then by Example 1.4 the IFS $\{f \circ S_i \circ f^{-1}\}_{i=1}^3$ satisfies the FTC. It then follows easily from the definition that $\Phi$ itself satisfies the FTC.

The $L^q$-spectrum is one of the basic ingredients in multifractal analysis. There is a well-known heuristic relation between the $L^q$-spectrum and the dimension spectrum of a measure called the multifractal formalism; see [8, 32] for the definitions and detailed properties of these notions. There have been a lot of studies on the multifractal formalism for self-conformal (including self-similar) and self-affine measures, see e.g. [4, 30, 23, 11, 13, 1, 2].

\[1\]The self-similar set generated by $\Phi$ is called a golden gasket in [3] if $\lambda^{-1}$ is a multinacci number and $a_1, a_2, a_3$ are vertices of an equilateral triangle.
For a self-similar measure generated by an IFS satisfying the open set condition (OSC) [20], it is well-known that its $L^q$-spectrum $\tau(q)$ is given by a precise formula and is analytic on $\mathbb{R}$ (cf. [4]). For a general self-similar measure, it is known [31] that the limit in (1.7) always exists for $q \geq 0$. However, without the OSC, it is generally difficult to obtain a formula for $\tau(q)$; see [21, 22, 24, 9, 10, 13, 17] for related works. For some self-similar measures, $\tau$ may not be differentiable at some points. For instance, Feng [10] showed that for the Bernoulli convolution associated with the golden ratio, $\tau$ is differentiable on $\mathbb{R}$ except at one point $q_0 < 0$. Moreover, Barral and Feng [1] showed that for any $q \in (1, 2)$, there exists a self-similar measure on $\mathbb{R}$ for which $\tau$ is not differentiable at $q$. According to a recent result of Shmerkin [34, Theorem 6.6], the above result also extends to $q \in (1, \infty)$. It remains an interesting question to determine for what classes of self-similar measures, $\tau$ is differentiable on $(0, \infty)$. Only a few results have been obtained on this question. As we have mentioned, Feng [9] proved the differentiability of $\tau$ on $(0, \infty)$ for the self-similar measures on $\mathbb{R}$ generated by the IFS of the form (1.2) satisfying (1.3). Recently, in [5, 28] the same conclusion was proved for certain self-similar measures on $\mathbb{R}^d$ that satisfy the generalized finite type condition (GFTC) and that are of essentially finite type (EFT), respectively. Below we make some comments on how the assumptions in [5, 28] are related to that in this paper.

It is known that IFSs satisfying our definition of the FTC satisfy the GFTC; see [6, Theorem 1.1]. However, the key of the approach used in [5] is to construct an infinite graph-directed IFS satisfying the OSC from a finite IFS, which relies heavily on the particular structure of the corresponding self-similar set; see [5, Subsection 4.2]. Thus (and as mentioned in [5, Section 9]) it is not clear whether the method in [5] can be extended to cover other IFSs satisfying the GFTC, including those studied by Feng [9]. As for the condition EFT used in [28], it is known that for the special case of Example 1.4 when $\mu$ is the Bernoulli convolution associated with the golden ratio, the EFT is satisfied (cf. [26, Example 3.2]). As shown in the proof of [26, Example 3.2], this relies on the fact that $\mu$ satisfies the so-called second-order identities (cf. [21, Equation 1.5]). However, to the best of the author’s knowledge, for Bernoulli convolutions associated with Pisot numbers other than the golden ratio, no second-order identities have been proved (this situation is also mentioned in the end of the second page of [26]), and the EFT has not been verified. Thus it is not clear whether the main result of [28] (i.e. [28, Theorem 1.1]) applies to this case, which is also stated as an unsolved problem in [28, Section 6].

In the study of self-similar sets and measures, extending results from $\mathbb{R}$ to $\mathbb{R}^d$ can often be difficult when the OSC fails. One reason is that unlike the case in $\mathbb{R}$, in $\mathbb{R}^d$ when $d \geq 2$ the orthogonal matrices in the linear parts of the similitudes in the IFS may cause obstacles; see for instance [19]. In the following, we give a description of our method used in this paper in extending Feng’s work [9] from $\mathbb{R}$ to $\mathbb{R}^d$.

Our general strategy to prove the main results is analogous to that of [9, Theorem 1.1], but several new ideas play key roles in our situation. First, for a self-similar set $K$ on $\mathbb{R}^d$ generated by an equicontractive IFS satisfying the FTC, we introduce the notion of Borel partitions of $K$, which is analogous to the notion of basic net
intervals in [9]. Second, we introduce a linear order on the set of the compositions of
the IFS and use it to define the characteristic vectors and the symbolic expressions
for the elements of the Borel partitions (cf. Section 2). This linear order enables us
to avoid possible difficulties caused by the rotations in the IFS. Third, in the proof of
Theorem 1.3, as in [9] our strategy is to connect \( \tau(q) \) with the pressure function \( P(q) \)
for a certain family of squared matrices (cf. Theorem 4.8); see Subsection 4.3 for
the definition of \( P(q) \). Then Theorem 1.3 follows from a result of Feng and Lau [12]
which states that \( P(q) \) is differentiable on \((0, \infty)\) under the condition that the sum
of these matrices is irreducible. A key difference is that, to verify the irreducibility
condition, we make use of the Borel density lemma (cf. Lemma 4.4), which is not
necessary in the one dimensional case studied in [9] due to the fact that the interior
of each basic net interval intersects \( K \).

Our method can be slightly extended to the more general case that the IFS is
commensurable and satisfies a more general form of the FTC; see Theorem 5.1.
Since the proof of Theorem 5.1 uses essentially the same ideas of that of Theorems
1.2-1.3, we will first prove Theorems 1.2-1.3, and then we point out in Section 5 the
modifications needed in the proofs of Theorems 1.2-1.3 to prove Theorem 5.1.

As we have mentioned, for a self-similar measure \( \mu \) on \( \mathbb{R} \) satisfying the FTC,
Feng’s result in [9] has been applied in [14] to give a checkable criterion for the
absolute continuity of \( \mu \), and in [15, 16, 18] to study the topological structure of
the set of local dimensions of \( \mu \). It would be interesting to see if our method in
this paper can be applied to these topics, which however is beyond the scope of the
present paper.

The rest of this paper is organized as follows. Throughout Sections 2-4, we let
\( \Phi = \{S_i\}_{i=1}^m \) be an equicontractive IFS on \( \mathbb{R}^d \) satisfying the FTC, \( K \) be the self-
similar set generated by \( \Phi \) and \( \mu \) be the self-similar measure generated by \( \Phi \) and a
probability vector \((p_1, \ldots, p_m)\). In Section 2, we define the characteristic vector and
symbolic expression for each element of \( \xi_n \) \((n \geq 0)\). In Section 3, we prove Theorem
1.2. Section 4 is devoted to the proof of Theorem 1.3. In Section 5, we show how
to modify the developments in Sections 2-4 to prove Theorem 5.1.

2. The characteristic vectors and symbolic expressions of \( \Delta \in \xi_n \)

For \( n \geq 0 \), let the mapping \( \Lambda_n \), and the \( n \)-th Borel partition \( \xi_n \) of \( K \) be defined
as in (1.5) and (1.6), respectively. It is clear that \( \Lambda_n \) takes a constant value on each
\( \Delta \in \xi_n \), which we denote by \( \Lambda_n(\Delta) \). Define the “neighbor” of \( \Lambda_n(\Delta) \) by

\[
N_n(\Delta) = \left\{ S_I : I \in \Sigma_n \text{ with } S_I(K) \cap \left( \bigcap_{f \in \Lambda_n(\Delta)} f(K) \right) \neq \emptyset \right\}.
\]
Clearly, $\Lambda_n(\Delta) \subseteq N_n(\Delta)$. Moreover, $\Delta$ is determined by $\Lambda_n(\Delta)$ and $N_n(\Delta)$, since
\[(2.1) \quad \Delta = \left( \bigcap_{f \in \Lambda_n(\Delta)} f(K) \right) \setminus \left( \bigcup_{I \in \Sigma_n: S_I \notin \Lambda_n(\Delta)} S_I(K) \right) \]

\[(2.2) \quad = \left( \bigcap_{f \in \Lambda_n(\Delta)} f(K) \right) \setminus \left( \bigcup_{g \in N_n(\Delta) \setminus \Lambda_n(\Delta)} g(K) \right). \]

In the following, we present some basic properties of $\xi_n$, $\Lambda_n(\Delta)$ and $N_n(\Delta)$. First, notice that the elements of $\xi_n$ are precisely the atoms of the algebra generated by $\{S_I(K) : I \in \Sigma_n\}$ (see e.g. [7, p.86, p.115] for the definitions of an algebra of sets and its atoms). Hence we have the following two lemmas.

**Lemma 2.1.** Let $n \geq 0$, $\mathcal{U} \subseteq \{S_I : I \in \Sigma_n\}$ and
\[ \mathcal{V} = \left\{ S_J : J \in \Sigma_n \text{ with } S_J(K) \cap \left( \bigcap_{f \in \mathcal{U}} f(K) \right) \neq \emptyset \right\}. \]

Set $\Delta = \left( \bigcap_{f \in \mathcal{U}} f(K) \right) \setminus \left( \bigcup_{g \in \mathcal{V} \setminus \mathcal{U}} g(K) \right)$. Then $\Delta \in \xi_n$ if and only if $\Delta \neq \emptyset$. In the case when $\Delta \neq \emptyset$, we have $\Lambda_n(\Delta) = \mathcal{U}$ and $N_n(\Delta) = \mathcal{V}$.

**Lemma 2.2.** Let $n \geq 0$ and $\mathcal{U}, \mathcal{V} \subseteq \{S_I : I \in \Sigma_n\}$. If $\left( \bigcap_{f \in \mathcal{U}} f(K) \right) \setminus \left( \bigcup_{g \in \mathcal{V}} g(K) \right)$ is non-empty, then it is a union of some elements of $\xi_n$.

The following lemma shows that $\{\xi_n\}_{n=0}^\infty$ has a net structure.

**Lemma 2.3.** (i) For any $n \geq 0$, $\xi_{n+1}$ refines $\xi_n$. That is, each element of $\xi_{n+1}$ is a subset of an element of $\xi_n$.

(ii) Moreover, given $\Delta \in \xi_{n+1}$ let $\hat{\Delta} \in \xi_n$ such that $\Delta \subseteq \hat{\Delta}$, then
\[ \Lambda_n(\hat{\Delta}) = \{S_I : I \in \Sigma_n, \exists S \in \Phi \text{ such that } S_I \circ S \in \Lambda_{n+1}(\Delta)\}. \]

**Proof.** Let $n \geq 0$ and $\Delta \in \xi_{n+1}$. Set
\[(2.3) \quad \mathcal{U} = \{S_I : I \in \Sigma_n, \exists S \in \Phi \text{ such that } S_I \circ S \in \Lambda_{n+1}(\Delta)\}, \]
\[ \mathcal{V} = \left\{ S_J : J \in \Sigma_n \text{ with } S_J(K) \cap \left( \bigcap_{f \in \mathcal{U}} f(K) \right) \neq \emptyset \right\}. \]

Define
\[(2.4) \quad \hat{\Delta} = \left( \bigcap_{f \in \mathcal{U}} f(K) \right) \setminus \left( \bigcup_{g \in \mathcal{V}} g(K) \right). \]

Below we show that $\hat{\Delta} \in \xi_n$, $\Delta \subseteq \hat{\Delta}$ and $\Lambda_n(\hat{\Delta}) = \mathcal{U}$, which will prove the lemma.
For every $h \in \Lambda_{n+1}(\Delta)$, we have $h = S_I \circ S$ for some $I \in \Sigma_n$ and $S \in \Phi$, in which case $h(K) = S_I \circ S(K) \subseteq S_I(K)$. By this fact and (2.3) we see that

$$(2.5) \quad \bigcap_{h \in \Lambda_{n+1}(\Delta)} h(K) \subseteq \bigcap_{f \in U} f(K).$$

On the other hand, (2.3) also implies that $g \circ S \notin \Lambda_{n+1}(\Delta)$ for all $g \in \mathcal{V} \setminus \mathcal{U}$ and $S \in \Phi$. Since $g(K) = \bigcup_{i=1}^{m} g \circ S_I(K)$ for every $g \in \mathcal{V} \setminus \mathcal{U}$, it follows that

$$(2.6) \quad \bigcup_{I \in \Sigma_{n+1}: S_I \notin \Lambda_{n+1}(\Delta)} S_I(K) \supseteq \bigcup_{g \in \mathcal{V} \setminus \mathcal{U}} g(K).$$

Recall that

$$(2.7) \quad \Delta = \left( \bigcap_{h \in \Lambda_{n+1}(\Delta)} h(K) \right) \setminus \left( \bigcup_{I \in \Sigma_{n+1}: S_I \notin \Lambda_{n+1}(\Delta)} S_I(K) \right).$$

Now by (2.4)-(2.7) we see that $\Delta \subseteq \tilde{\Delta}$. Hence $\tilde{\Delta} \neq \emptyset$ as $\Delta \neq \emptyset$. It then follows from Lemma 2.1 that $\hat{\Delta} \in \xi_n$ and $\Lambda_n(\hat{\Delta}) = \mathcal{U}$. This proves the lemma. \hspace{1cm} \Box

Recall that $\mathcal{S}$ is the set of similitudes on $\mathbb{R}^d$. Here and afterwards, for $g \in \mathcal{S}$, $A \subseteq \mathcal{S}$ and a vector $V = (f_1, \ldots, f_n)$ with all $f_i \in \mathcal{S}$, we write $g \circ A = \{ g \circ f : f \in A \}$ and $g \circ V = (g \circ f_1, \ldots, g \circ f_n)$.

Let $\#A$ denote the cardinality of a set $A$. The following result is a direct consequence of the FTC.

**Lemma 2.4.** For any $n \in \mathbb{N}$ and $\Delta \in \xi_n$, $\#\Lambda_n(\Delta) \leq \#N_n(\Delta) \leq \#\Gamma$, where $\Gamma$ is given as in Definition 1.1.

**Proof.** Let $n \in \mathbb{N}$ and $\Delta \in \xi_n$. The first inequality is clear as $\Lambda_n(\Delta) \subseteq N_n(\Delta)$. To see the second inequality, fix $f \in \Lambda_n(\Delta)$. Then by (1.4) and the definition of $N_n(\Delta)$, we see that $f^{-1} \circ N_n(\Delta) \subseteq \Gamma$. Hence $\#N_n(\Delta) \leq \#\Gamma$. \hspace{1cm} \Box

In the rest of this section, we define for each $\Delta \in \xi_n$ ($n \geq 0$) its characteristic vector and symbolic expression. To this end, we first introduce a linear order on $\Phi_* := \{ S_I : I \in \Sigma_* \}$, which enables us to rewrite $\Lambda_n(\Delta)$ and $N_n(\Delta)$ as ordered vectors and is important for our further analysis (cf. Lemma 2.10).

Let $\leq_{\text{lex}}$ be the lexicographic order on $\Sigma_*$. That is, for $I = i_1 \ldots i_k, J = j_1 \ldots j_\ell \in \Sigma_*$, $I \leq_{\text{lex}} J$ if and only if either $I$ is a prefix of $J$, or there exists $1 \leq s \leq \min\{k, \ell\}$ such that $i_1 = j_1, \ldots, i_{s-1} = j_{s-1}$ and $i_s < j_s$. Write $I <_{\text{lex}} J$ if $I \leq_{\text{lex}} J$ and $I \neq J$. It is easy to check that $\leq_{\text{lex}}$ is a linear order on $\Sigma_*$. Moreover, $\leq_{\text{lex}}$ satisfies the following property, which is obvious from the definition.

**Lemma 2.5.** Let $n \geq 0$ and $I = i_1 \ldots i_{n+1}, J = j_1 \ldots j_{n+1} \in \Sigma_{n+1}$. Then $I <_{\text{lex}} J$ if and only if either $i_1 \ldots i_n <_{\text{lex}} j_1 \ldots j_n$, or $i_1 \ldots i_n = j_1 \ldots j_n$ and $i_{n+1} < j_{n+1}$.

Using $\leq_{\text{lex}}$ on $\Sigma_*$ we define a linear order $\preceq$ on $\Phi_*$ as follows: Let $\omega : \Phi_* \rightarrow \Sigma_*$ be the mapping which assigns each $f \in \Phi_*$ the minimal element of $\{ I \in \Sigma_* : S_I = f \}$.
under $\leq_{\text{lex}}$. Notice that $\omega$ is well-defined since $\leq_{\text{lex}}$ is linear. For $f, g \in \Phi_*$, write $f \prec g$ ($f \prec g$) if $\omega(f) \leq_{\text{lex}} \omega(g)$ ($\omega(f) \leq_{\text{lex}} \omega(g)$, respectively). It is clear that $\prec$ is a linear order on $\Phi_*$.

From now on, for $n \geq 0$ and $\Delta \in \xi_n$ with $\Lambda_n(\Delta) = \{f_i\}_{i=1}^k$ and $N_n(\Delta) = \{g_j\}_{j=1}^\ell$, without loss of generality we assume that the elements of $\Lambda_n(\Delta)$ and $N_n(\Delta)$ are ranked increasingly in the order $\prec$, i.e., $f_1 \prec \cdots \prec f_k$ and $g_1 \prec \cdots g_\ell$. Then without causing confusion we will view $\Lambda_n(\Delta)$ and $N_n(\Delta)$ as ordered vectors

$$\Lambda_n(\Delta) = (f_1, \ldots, f_k), \quad N_n(\Delta) = (g_1, \ldots, g_\ell).$$

We define two vectors $V_n(\Delta)$ and $U_n(\Delta)$ by

$$V_n(\Delta) = (\varphi_1, \ldots, \varphi_k), \quad U_n(\Delta) = (\psi_1, \ldots, \psi_\ell),$$

where $\varphi_i = f_i^{-1} \circ f_i$ for $1 \leq i \leq k$ and $\psi_j = f_j^{-1} \circ g_j$ for $1 \leq j \leq \ell$. By Lemma 2.4 and its proof, we see that $k \leq \ell \leq \#\Gamma$, and all the entries of $V_n(\Delta)$ and $U_n(\Delta)$ are contained in $\Gamma$. Hence $\{(V_n(\Delta), U_n(\Delta)) : \Delta \in \xi_n, n \geq 0\}$ is a finite set.

Next we define for $\Delta$ a similitude $r_n(\Delta)$ on $\mathbb{R}^d$. Let $r_0(\Delta) := id$, the identity map on $\mathbb{R}^d$. If $n \geq 1$, let $\hat{\Delta}$ denote the unique element of $\xi_{n-1}$ which contains $\Delta$. Assume that $\Lambda_{n-1}(\hat{\Delta}) = (h_1, \ldots, h_\ell)$. Then we define $r_n(\Delta) = h_i^{-1} \circ f_i$. By Lemma 2.3(ii), there exist $S \in \Phi$ and $j \in \{1, \ldots, k\}$ so that $h_1 \circ S = f_j$. Since $f_j(K) \cap f_i(K) \neq \emptyset$, the FTC (1.4) implies that $f_j^{-1} \circ f_i \in \Gamma$. Hence we have

$$r_n(\Delta) = h_i^{-1} \circ f_i = S \circ f_j^{-1} \circ f_i \in \{S_i \circ f : 1 \leq i \leq m, f \in \Gamma\}.$$ 

As a consequence, we see that the set $\{r_n(\Delta) : \Delta \in \xi_n, n \geq 0\}$ is finite.

Finally, we define a triple

$$C_n(\Delta) = (V_n(\Delta), U_n(\Delta), r_n(\Delta)),$$

and call it the characteristic vector of $\Delta$. Let

$$\Omega = \{C_n(\Delta) : \Delta \in \xi_n, n \geq 0\}.$$

Then by the above argument, $\Omega$ is a finite set.

For $n \geq 0$ and $\Delta \in \xi_n$, since $\Delta$ is determined by $\Lambda_n(\Delta)$ and $N_n(\Delta)$ (cf. (2.2)), $V_n(\Delta)$ and $U_n(\Delta)$ are used to record the shape of $\Delta$. The reason to introduce the term $r_n(\Delta)$ in $C_n(\Delta)$ is to guarantee that $C_{n+1}(\Delta_1) \neq C_{n+1}(\Delta_2)$ whenever $\Delta_1, \Delta_2 \in \xi_{n+1}$ are contained in $\Delta$ and $\Delta_1 \neq \Delta_2$, as shown in the following lemma.

**Lemma 2.6.** Given $n \geq 0$ and $\Delta \in \xi_n$ let $\Delta_1, \Delta_2$ be two distinct elements of $\xi_{n+1}$ that are contained in $\Delta$. Then $C_{n+1}(\Delta_1) \neq C_{n+1}(\Delta_2)$.

**Proof.** Since $\Delta_1 \neq \Delta_2$ and they are determined by $\Lambda_n(\Delta_1)$ and $\Lambda_n(\Delta_2)$ respectively (cf. (2.1)), we have $\Lambda_{n+1}(\Delta_1) \neq \Lambda_{n+1}(\Delta_2)$. Let $f, g, h$ be the first entries of $\Lambda_n(\Delta_1), \Lambda_{n+1}(\Delta_1)$ and $\Lambda_{n+1}(\Delta_2)$, respectively. If $g \neq h$, then we have $r_{n+1}(\Delta_1) \neq r_{n+1}(\Delta_2)$, since $r_{n+1}(\Delta_1) = f^{-1} \circ g$ and $r_{n+1}(\Delta_2) = f^{-1} \circ h$. If $g = h$, then since $V_{n+1}(\Delta_1) = g^{-1} \circ \Lambda_n(\Delta_1)$, $V_{n+1}(\Delta_2) = h^{-1} \circ \Lambda_n(\Delta_2)$ and $\Lambda_{n+1}(\Delta_1) \neq \Lambda_{n+1}(\Delta_2)$, we see that $V_{n+1}(\Delta_1) \neq V_{n+1}(\Delta_2)$. Hence we have shown
that either $r_{n+1}(\Delta_1) \neq r_{n+1}(\Delta_2)$ or $V_{n+1}(\Delta_1) \neq V_{n+1}(\Delta_2)$. This implies that $C_{n+1}(\Delta_1) \neq C_{n+1}(\Delta_2)$, as desired. \hfill \Box \Box

Now we proceed to introduce the symbolic expression for each element in \{\Delta \in \xi_n : n \in \mathbb{N}\}. For this purpose, we need establish the following result.

**Lemma 2.7.** Let $k, \ell \in \mathbb{N}$, $\Delta_1 \in \xi_k$ and $\Delta_2 \in \xi_\ell$. If $C_k(\Delta_1) = C_\ell(\Delta_2)$, then

$$\{C_{k+1}(\Delta) : \Delta \in \xi_{k+1}, \Delta \subseteq \Delta_1\} = \{C_{\ell+1}(\Delta) : \Delta \in \xi_{\ell+1}, \Delta \subseteq \Delta_2\}.$$ 

To prove Lemma 2.7, we first give several lemmas.

**Lemma 2.8.** Let $n \geq 0$, $\Delta \in \xi_n$, $\Delta_1 \in \xi_{n+1}$ with $\Delta_1 \subseteq \Delta$ and assume that $\Lambda_n(\Delta) = (f_1, \ldots, f_k)$. Given $h \in \Lambda_{n+1}(\Delta_1)$ let $i$ be the smallest integer in $\{1, \ldots, k\}$ so that $h = f_i \circ S_j$ for some $j \in \{1, \ldots, m\}$. Then $\omega(h) = \omega(f_i)j$.

**Proof.** Let $h \in \Lambda_{n+1}(\Delta_1)$. First, from Lemma 2.3(ii) we see that the above $i, j$ exist. Clearly, $\omega(f_i) \in \Sigma_n$ and $\omega(h) \in \Sigma_{n+1}$. Moreover, since $h = f_i \circ S_j = S_{\omega(f_i)}j$, we have $\omega(h) \leq \omega(f_i)j$. Below we prove that $\omega(h) = \omega(f_i)j$ by contradiction.

Suppose on the contrary that $\omega(h) < \omega(f_i)j$. Write $\omega(h) = i_1 \ldots i_{n+1}$. Then by Lemma 2.5, either $i_1 \ldots i_n < \omega(f_i)$ or $i_1 \ldots i_n = \omega(f_i)$ and $i_{n+1} < j$. Notice that $S_{i_1 \ldots i_n} \in \{f_1, \ldots, f_k\}$ by Lemma 2.3(ii). If $i_1 \ldots i_n < \omega(f_i)$, then $\omega(S_{i_1 \ldots i_n}) \leq i_1 \ldots i_n < \omega(f_i)$, which implies that $S_{i_1 \ldots i_n} < f_i$. Therefore $S_{i_1 \ldots i_n} = f_\ell$ and so $h = f_\ell \circ S_{i_1 \ldots i_n}$ for some $\ell < i$, contradicting the minimality of $i$. If the other case occurs, i.e. $i_1 \ldots i_n = \omega(f_i)$ and $i_{n+1} < j$, then $S_{i_1 \ldots i_n} = S_j$. This contradicts our assumption that $S_1, \ldots, S_n \in \Phi$ are distinct. Hence we have $\omega(h) = \omega(f_i)j$. \hfill \Box \Box

**Remark 2.9.** The conclusion of Lemma 2.8 also holds if we replace $\Lambda_n(\Delta)$ by $N_n(\Delta)$ and $\Lambda_{n+1}(\Delta_1)$ by $N_{n+1}(\Delta_1)$. Indeed, by the definitions of $N_n(\Delta)$ and $N_{n+1}(\Delta_1)$, it is easily seen that for every $i_1 \ldots i_{n+1} \in \Sigma_{n+1}$ with $S_{i_1 \ldots i_{n+1}} \in N_{n+1}(\Delta_1)$, we have $S_{i_1 \ldots i_n} \in N_n(\Delta)$. Then the above assertion follows by the same proof of Lemma 2.8.

An essential part to prove Lemma 2.7 is the following result, which says that if $\phi$ is a similitude which maps $\Lambda_k(\Delta_1)$ to $\Lambda_\ell(\Delta_2)$ and $N_k(\Delta_1)$ to $N_\ell(\Delta_2)$ and preserves the order of the elements of $\Lambda_k(\Delta_1)$ and $N_k(\Delta_1)$, then $\phi$ also preserves the order of the elements of $\Lambda_{k+1}(\Delta)$ and $N_{k+1}(\Delta)$ whenever $\Delta$ is an offspring of $\Delta_1$ in $\xi_{k+1}$.

**Lemma 2.10.** Let $k, \ell \in \mathbb{N}$, $\Delta_1 \in \xi_k$ and $\Delta_2 \in \xi_\ell$. Suppose that $\Lambda_k(\Delta_1) = (f_1, \ldots, f_p)$, $N_k(\Delta_1) = (g_1, \ldots, g_q)$, $\Lambda_\ell(\Delta_2) = (h_1, \ldots, h_p)$ and $N_\ell(\Delta_2) = (u_1, \ldots, u_q)$, and there is a similitude $\phi$ such that $\phi \circ f_i = h_i$, $\phi \circ g_j = u_j$ for $1 \leq i \leq p$ and $1 \leq j \leq q$. Let $\Delta \in \xi_{k+1}$ with $\Delta \subseteq \Delta_1$, and $v, \check{v} \in \Lambda_{k+1}(\Delta)$ with $v \neq \check{v}$. Then

(i) $\phi(\Delta_1) = \Delta_2$. 


(ii) \( v \prec \tilde{v} \) if and only if \( \phi \circ v \prec \phi \circ \tilde{v} \), and the same conclusion holds if \( \Lambda_{k+1}(\Delta) \) is replaced by \( N_{k+1}(\Delta) \).

**Proof.** Part (i) of the lemma follows directly from the definition of the elements of \( \xi_n(n \geq 0) \) (cf. (2.2)) and the assumption that \( \phi \circ f_i = h_i, \phi \circ g_j = u_j \) for \( 1 \leq i \leq p \) and \( 1 \leq j \leq q \). Below we prove (ii).

Let \( s \) be the smallest integer in \( \{1, \ldots, p\} \) so that \( v = f_s \circ S_t \) for some \( t \in \{1, \ldots, m\} \), and \( \tilde{s} \) be the smallest integer in \( \{1, \ldots, p\} \) so that \( \tilde{v} = f_{\tilde{s}} \circ S_{\tilde{t}} \) for some \( \tilde{t} \in \{1, \ldots, m\} \). Since \( \phi \circ f_i = h_i \) for \( 1 \leq i \leq p \), it is easily seen from the minimality of \( s \) that \( s \) is also the smallest integer in \( \{1, \ldots, p\} \) such that \( \phi \circ v = h_s \circ S_t \). Similarly, \( \tilde{s} \) is the smallest integer in \( \{1, \ldots, p\} \) such that \( \phi \circ \tilde{v} = h_{\tilde{s}} \circ S_{\tilde{t}} \). Then by Lemma 2.8, we have

\[
\omega(v) = \omega(f_s)t, \quad \omega(\phi \circ v) = \omega(h_s)t, \quad \omega(\tilde{v}) = \omega(f_{\tilde{s}})\tilde{t}, \quad \omega(\phi \circ \tilde{v}) = \omega(h_{\tilde{s}})\tilde{t}.
\]

Now by (2.8) and Lemma 2.5, we have

\[
v \prec \tilde{v} \iff \omega(v) \prec_{\text{lex}} \omega(\tilde{v})
\]

\[
\iff \omega(f_s)t \prec_{\text{lex}} \omega(f_{\tilde{s}})\tilde{t} \quad \text{(by (2.8))}
\]

\[
\iff \omega(f_s) \prec_{\text{lex}} \omega(f_{\tilde{s}}), \text{ or } \omega(f_s) = \omega(f_{\tilde{s}}) \text{ and } t < \tilde{t} \quad \text{(by Lemma 2.5)}
\]

\[
\iff s < \tilde{s}, \text{ or } s = \tilde{s} \text{ and } t < \tilde{t}.
\]

A similar argument yields that

\[
\phi \circ v \prec \phi \circ \tilde{v} \iff s < \tilde{s}, \text{ or } s = \tilde{s} \text{ and } t < \tilde{t}.
\]

Therefore, \( v \prec \tilde{v} \iff \phi \circ v \prec \phi \circ \tilde{v} \). In view of Remark 2.9, it is easy to see that the same conclusion holds if \( \Lambda_{k+1}(\Delta) \) is replaced by \( N_{k+1}(\Delta) \). \( \square \) \( \square \)

**Lemma 2.11.** Under the assumptions of Lemma 2.10, we have \( \phi(\Delta) \subseteq \Delta_2, \phi(\Delta) \in \xi_{\ell+1}, \Lambda_{\ell+1}(\phi(\Delta)) = \phi \circ \Lambda_{k+1}(\Delta) \) and \( N_{\ell+1}(\phi(\Delta)) = \phi \circ N_{k+1}(\Delta) \).

**Proof.** By Lemma 2.10(i) and \( \Delta \subseteq \Delta_1 \), we have \( \phi(\Delta) \subseteq \phi(\Delta_1) = \Delta_2 \). To prove the remaining statements of the lemma, we first show that for every \( x \in \Delta_1 \),

\[
(2.9) \quad \phi \circ \Lambda_{k+1}(x) = \Lambda_{\ell+1}(\phi(x)).
\]

To see this, let \( x \in \Delta_1 \). Notice that from Lemma 2.3(ii) we see that

\[
(2.10) \quad \Lambda_{k+1}(x) = \{f_i \circ S : 1 \leq i \leq p, S \in \Phi, x \in f_i \circ S(K)\},
\]

\[
(2.11) \quad \Lambda_{\ell+1}(\phi(x)) = \{h_i \circ S : 1 \leq i \leq p, S \in \Phi, \phi(x) \in h_i \circ S(K)\}.
\]
we see that the notion of \( \subseteq \) is independent of the choice of the sequence \( \Delta_k \geq \xi \).

Definition that

\[
\Lambda_{k+1}(2.14) \Lambda
\]

Proof of Lemma

Now the lemma follows easily from (2.9), (2.12) and Lemma 2.10. \( \square \) \( \square \)

We are ready to prove Lemma 2.7.

Proof of Lemma 2.7. By Lemma 2.6, it suffices to show that for any \( \Delta \in \xi_{k+1} \) with \( \Delta \subseteq \Delta_1 \), we can find \( \Delta' \in \xi_{\ell+1} \) with \( \Delta' \subseteq \Delta_2 \) such that \( C_{k+1}(\Delta) = C_{\ell+1}(\Delta') \).

Write \( \Lambda_k(\Delta_1) = (f_1, \ldots, f_p) \), \( N_k(\Delta_1) = (g_1, \ldots, g_q) \), \( \Lambda_\ell(\Delta_2) = (h_1, \ldots, h_{p'}) \) and \( N_\ell(\Delta_2) = (u_1, \ldots, u_{q'}) \). Let \( \phi = h_1 \circ f_1^{-1} \). Then since \( C_k(\Delta_1) = C_\ell(\Delta_2) \), we have by definition that \( p = p', q = q' \),

\[
\phi \circ f_i = h_i \quad \text{and} \quad \phi \circ g_j = u_j \quad \text{for} \ 1 \leq i \leq p, 1 \leq j \leq q.
\]

Take \( \Delta \in \xi_{k+1} \) with \( \Delta \subseteq \Delta_1 \) and assume that

\[
\Lambda_{k+1}(\Delta) = (v_1, \ldots, v_s), \quad N_{k+1}(\Delta) = (w_1, \ldots, w_t).
\]

Set \( \Delta' = \phi(\Delta_1) \). Then by Lemma 2.11, we have \( \Delta' \subseteq \Delta_2 \), \( \Delta' \in \xi_{\ell+1} \) and

\[
\Lambda_{\ell+1}(\Delta') = (\phi \circ v_1, \ldots, \phi \circ v_s), \quad N_{\ell+1}(\Delta') = (\phi \circ w_1, \ldots, \phi \circ w_t).
\]

It is straightforward to see from (2.13)-(2.15) that \( C_{k+1}(\Delta) = C_{\ell+1}(\Delta') \). Since \( \Delta \in \xi_{k+1} \) with \( \Delta \subseteq \Delta_1 \) is arbitrary, we complete the proof of the lemma. \( \square \) \( \square \)

Recall that \( \Omega = \{ C_n(\Delta) : \Delta \in \xi_n, n \geq 0 \} \) is a finite set, which in what follows we view as an alphabet. We say that a word \( \alpha_1 \alpha_2 \ldots \alpha_\ell \) over \( \Omega \) is \textit{admissible} if there exist \( k \geq 0 \) and \( \Delta_i \in \xi_{k+i-1} \) (\( i = 1, \ldots, \ell \)) such that \( \Delta_1 \supseteq \cdots \supseteq \Delta_\ell \) and \( C_{k+i-1}(\Delta_i) = \alpha_i \) for \( 1 \leq i \leq \ell \). From Lemma 2.7 we see that the notion of \( \alpha_1 \alpha_2 \ldots \alpha_\ell \) being admissible is independent of the choice of the sequence \( \Delta_1, \Delta_2, \ldots, \Delta_\ell \).

Finally, we define the symbolic expression for each element of \( \{ \Delta \in \xi_n : n \geq 0 \} \). Given \( n \geq 0 \) and \( \Delta \in \xi_n \), let \( \Delta_0, \Delta_1, \ldots, \Delta_n \) be the unique sequence of sets satisfying

\[
K = \Delta_0 \supseteq \Delta_1 \supseteq \cdots \supseteq \Delta_n = \Delta
\]

and \( \Delta_i \in \xi_i \) for \( i = 0, 1, \ldots, n \). We call the sequence of characteristic vectors

\[
C_0(\Delta_0), C_1(\Delta_1), \ldots, C_n(\Delta_n)
\]

the \textit{symbolic expression} of \( \Delta \).

By Lemma 2.6, we immediately have the following.
Lemma 2.12. For any \( n \geq 1 \) and \( \Delta_1, \Delta_2 \in \xi_n \) with \( \Delta_1 \neq \Delta_2 \), the symbolic expressions of \( \Delta_1 \) and \( \Delta_2 \) are different. Consequently, for any \( n \in \mathbb{N} \) and \( \Delta \in \xi_n \), \( \Delta \) can be identified as an admissible word of length \( n + 1 \) with initial letter \( C_0(K) \).

3. Matrix representation of \( \mu \): proof of Theorem 1.2

In this section we study the distribution of \( \mu \) on each element of \( \{ \Delta \in \xi_n : n \geq 0 \} \).

For \( n \geq 0 \) and \( \Delta \in \xi_n \) with \( \mu(\Delta) > 0 \), we will express \( \mu(\Delta) \) as an inner product \( u_{n,\Delta} \cdot v_{n,\Delta} \), where \( u_{n,\Delta}, v_{n,\Delta} \) are positive row vectors with \( v_{n,\Delta} \) being determined by \( C_n(\Delta) \). If \( n \geq 1 \), we let \( \hat{\Delta} \) be the unique element of \( \xi_{n-1} \) which contains \( \Delta \). Then we will construct a transitive matrix \( T_{\hat{\Delta},\Delta} = T(C_{n-1}(\hat{\Delta}), C_n(\Delta)) \) which depends only on \( C_{n-1}(\hat{\Delta}) \) and \( C_n(\Delta) \) such that \( u_{n,\Delta} = u_{n-1,\hat{\Delta}} T_{\hat{\Delta},\Delta} \). Below we will give the detailed definitions of \( u_{n,\Delta}, v_{n,\Delta} \) and \( T_{\hat{\Delta},\Delta} \).

For \( n \geq 0 \), set
\[
(3.1) \quad \mathcal{F}_n = \{ \Delta \in \xi_n : \mu(\Delta) > 0 \}.
\]

Let \( \Delta \in \mathcal{F}_n \) and assume that
\[
\Lambda_n(\Delta) = (f_1, \ldots, f_k), \quad N_n(\Delta) = (g_1, \ldots, g_\ell).
\]

Then by definition
\[
V_n(\Delta) = (\varphi_1, \ldots, \varphi_k), \quad U_n(\Delta) = (\psi_1, \ldots, \psi_\ell),
\]
where \( \varphi_i = f_i^{-1} \circ f_i, \psi_j = f_i^{-1} \circ g_j \) for \( 1 \leq i \leq k \) and \( 1 \leq j \leq \ell \). Then we define
\[
\Lambda_n^*(\Delta) = (h_1, \ldots, h_{\tilde{k}}), \quad V_n^*(\Delta) = (\phi_1, \ldots, \phi_{\tilde{k}}),
\]
where \( h_1, \ldots, h_{\tilde{k}} \) (ranked increasingly in the order \( \prec \)) are those \( f \in \{ f_i \}_{i=1}^k \) satisfying \( \mu(f^{-1}\Delta) > 0 \), and \( \phi_i := f_i^{-1} \circ h_i \) for \( 1 \leq i \leq \tilde{k} \). Let \( v_n^*(\Delta) \) denote the dimension of \( V_n^*(\Delta) \), i.e. \( v_n^*(\Delta) = \tilde{k} \).

We point out that \( V_n^*(\Delta) \) is determined by \( C_n(\Delta) \). To see this, first observe that \( V_n^*(\Delta) \) is obtained by removing those entries \( \varphi \) of \( V_n(\Delta) \) which satisfy \( \mu(\varphi^{-1} \circ f_i^{-1}\Delta) = 0 \) and keeping the relative positions of the other entries of \( V_n(\Delta) \) unchanged. Meanwhile, we note that \( f_i^{-1}(\Delta) \) is determined by \( V_n(\Delta) \) and \( U_n(\Delta) \), as
\[
f_i^{-1}(\Delta) = \left( \bigcap_{i=1}^k \varphi_i(K) \right) \setminus \left( \bigcup_{\psi \in \{ \psi_j \}_{j=1}^\ell \setminus \{ \varphi_i \}_{i=1}^k} \psi(K) \right).
\]

Hence \( V_n^*(\Delta) \) is determined by \( V_n(\Delta) \) and \( U_n(\Delta) \), and thus by \( C_n(\Delta) \).

For \( I = i_1 \ldots i_n \in \Sigma_n \), write \( p_I = p_{i_1} \cdots p_{i_n} \). Iterating (1.1) for \( n \) times gives
\[
(3.2) \quad \mu = \sum_{I \in \Sigma_n} p_I \mu \circ S_I^{-1}.
\]
Then by (3.2) and the definition of $\Lambda_n^*(\Delta)$, we have

$$\mu(\Delta) = \sum_{i \in \Sigma_n} p_i \mu(S_i^{-1} \Delta)$$

$$= \sum_{i \in \Sigma_n; \mu(S_i^{-1} \Delta) > 0} p_i \mu(S_i^{-1} \Delta)$$

$$= \sum_{i=1}^{\tilde{k}} \left( \sum_{i \in \Sigma_n; S_i = \phi_i} p_i \right) \mu(h_i^{-1} \Delta)$$

$$= \mathbf{u}_{n,\Delta} \cdot \mathbf{v}_{n,\Delta},$$

(3.3)

where $\mathbf{u}_{n,\Delta} = \left( \sum_{i \in \Sigma_n; S_i = \phi_i} p_i \right)_{i=1}^{\tilde{k}}$ and $\mathbf{v}_{n,\Delta} = \left( \mu(h_i^{-1} \Delta) \right)_{i=1}^{\tilde{k}}$. Notice that both $\mathbf{u}_{n,\Delta}$ and $\mathbf{v}_{n,\Delta}$ are positive vectors. Since $h_i^{-1} \Delta = \phi_i^{-1} \circ f_i^{-1} \Delta$ for $i = 1, \ldots, \tilde{k}$, by our argument in the preceding paragraph, we see that $\mathbf{v}_{n,\Delta}$ is determined by $\mathcal{C}_n(\Delta)$.

For $n \geq 1$ and $\Delta \in \mathcal{F}_n$, let $\hat{\Delta}$ be the unique element of $\mathcal{F}_{n-1}$ which contains $\Delta$. Assume that

$$\Lambda_{n-1}(\hat{\Delta}) = (u_1, \ldots, u_{\tilde{k}}), \quad \Lambda_{n-1}^*(\hat{\Delta}) = (v_1, \ldots, v_{\tilde{k}}), \quad V_{n-1}^*(\hat{\Delta}) = (w_1, \ldots, w_{\tilde{k}}).$$

Then we define a $v_{n-1}^*(\hat{\Delta}) \times \Lambda_{n-1}^*(\hat{\Delta})$ matrix $T_{\Delta,\hat{\Delta}} = (t_{j,i})_{1 \leq j \leq \tilde{k}, 1 \leq i \leq \tilde{k}}$ by setting

$$t_{j,i} = \begin{cases} p_r & \text{if } \exists r \in \{1, \ldots, m\} \text{ such that } S_r = v_j^{-1} \circ h_i, \\ 0 & \text{otherwise}, \end{cases}$$

(3.4)

for $1 \leq j \leq \tilde{k}$ and $1 \leq i \leq \tilde{k}$.

We claim that

(3.5) $$\mathbf{u}_{n,\Delta} = \mathbf{u}_{n-1,\hat{\Delta}} T_{\Delta,\hat{\Delta}}.$$  

To see this, for each $i \in \{1, \ldots, \tilde{k}\}$, let $J \in \Sigma_{n-1}$ and $r \in \{1, \ldots, m\}$ be such that $S_J \circ S_r = h_i$. Notice that $S_J \in \{u_j\}_{j=1}^{\tilde{k}}$ by Lemma 2.3(ii). Moreover, by $\Delta \subseteq \hat{\Delta}$ and (1.1), $\mu(S_J^{-1} \hat{\Delta}) \geq \mu(S_J^{-1} \Delta) \geq p_r \mu(S_J^{-1} \circ S_J^{-1} \Delta) = p_r \mu(h_i^{-1} \Delta) > 0$. Hence $S_J \in \{v_j\}_{j=1}^{\tilde{k}}$ by the definition of $\Lambda_{n-1}^*(\hat{\Delta})$. By this fact and (3.4),

$$\sum_{i \in \Sigma_n; S_i = h_i} p_i = \sum_{J \in \Sigma_{n-1}; r \in \{1, \ldots, m\}; S_J \circ S_r = h_i} p_J p_r = \sum_{j=1}^{\tilde{k}} t_{j,i} \sum_{J \in \Sigma_{n-1}; S_J = v_j} p_J.$$  

Since $i \in \{1, \ldots, \tilde{k}\}$ is arbitrary, this proves (3.5).

Notice that for $i \in \{1, \ldots, \tilde{k}\}$ and $j \in \{1, \ldots, \tilde{k}\}$, we have

(3.6) $$v_j^{-1} \circ h_i = w_j^{-1} \circ u_j^{-1} \circ f_i \circ \phi_i = w_j^{-1} \circ r_n(\Delta) \circ \phi_i.$$  

Since we have shown that $V_n^*(\Delta), V_{n-1}^*(\hat{\Delta})$ are determined by $\mathcal{C}_n(\Delta)$ and $\mathcal{C}_{n-1}(\hat{\Delta})$ respectively, by (3.6) and the definition of $t_{j,i}$ (cf. (3.4)) we see that $T_{\Delta,\hat{\Delta}}$ is determined by $\mathcal{C}_n(\Delta)$ and $\mathcal{C}_{n-1}(\hat{\Delta})$. So we write $T_{\Delta,\hat{\Delta}} = T(\mathcal{C}_{n-1}(\hat{\Delta}), \mathcal{C}_n(\Delta)).$
Let 
\[ \tilde{\Omega} = \{ C_n(\Delta) : \Delta \in \mathcal{F}_n, n \geq 0 \}, \]
where \( \mathcal{F}_n \) is defined as in (3.1). For \( \alpha \in \tilde{\Omega} \), pick \( n \geq 0 \) and \( \Delta \in \mathcal{F}_n \) with \( C_n(\Delta) = \alpha \). We have shown that \( V^*_n(\Delta) \) and \( v^*_{n,\Delta} \) are determined by \( \alpha \) (independent of the choice of \( n \) and \( \Delta \)). So we can write \( v'(\alpha) = v^*_{n,\Delta} \) and \( v^*(\alpha) = v^*_n(\Delta) \); recall that \( v^*_n(\Delta) \) is the dimension of \( V^*_n(\Delta) \) and \( v^*_{n,\Delta} \).

For any \( \alpha, \beta \in \tilde{\Omega} \) with \( \alpha \beta \) being admissible, we have constructed a \( v^*(\alpha) \times v^*(\beta) \) dimensional non-negative matrix \( T(\alpha, \beta) \). Since \( \tilde{\Omega} \) is finite, we see that

\[ \left\{ T(\alpha, \beta) : \alpha, \beta \in \tilde{\Omega} \text{ and } \alpha \beta \text{ is admissible} \right\} \]
is a finite family of non-negative matrices.

Recall that \( u_{0,K} = 1 \). Applying (3.5) repeatedly and (3.3), we obtain the following.

**Lemma 3.1.** For any \( k \geq 1 \) and \( \Delta \in \mathcal{F}_k \), we have

\[ u_{k,\Delta}^* = T(\gamma_0, \gamma_1) \cdots T(\gamma_{k-1}, \gamma_k), \quad \mu(\Delta) = T(\gamma_0, \gamma_1) \cdots T(\gamma_{k-1}, \gamma_k) v(\gamma_k)^T, \]
where \( \gamma_0 \gamma_1 \cdots \gamma_k \) is the symbolic expression of \( \Delta \) and \( v(\gamma_k)^T \) is the transpose of \( v(\gamma_k) \).

Let \( \alpha_1 \ldots \alpha_n \) be an admissible word with \( \alpha_i \in \tilde{\Omega} \) for \( i = 1, \ldots, n \). By the same proof of [9, Corollary 3.4],

\[ (3.7) \quad \vec{e}(\alpha_1) T(\alpha_1, \alpha_2) \cdots T(\alpha_{n-1}, \alpha_n) \]
is a \( v^*(\alpha_n) \)-dimensional positive row vector, where \( \vec{e}(\alpha_1) \) is the row vector consisting of \( v^*(\alpha_1) \) many 1’s.

Pick \( k \geq 0 \) and \( \Delta_i \in \xi_{k+i-1} \) \((i = 1, \ldots, n)\) such that \( \Delta_1 \supseteq \cdots \supseteq \Delta_n \) and \( C_{k+i-1}(\Delta_i) = \alpha_i \) for \( 1 \leq i \leq n \). Assume that \( \Lambda^*_k(\Delta_1) = (h_1, \ldots, h_{v^*(\alpha_1)}) \) and \( \Lambda^*_{k+n-1}(\Delta_n) = \left( h_1', \ldots, h_{v^*(\alpha_n)}' \right) \). By (3.4) and induction, we have the following.

**Lemma 3.2.** For \( i \in \{1, \ldots, v^*(\alpha_1)\} \) and \( j \in \{1, \ldots, v^*(\alpha_n)\} \), the \((i,j)\)-entry of the matrix

\[ T(\alpha_1, \alpha_2) \cdots T(\alpha_{n-1}, \alpha_n) \]
is given by

\[ \sum_{I \in \Sigma_{n-1}} p_I. \]

In the rest of this section, we prove Theorem 1.2 by using Lemma 3.1 and a strategy employed in the proof of [9, Lemma 4.1].

Write \( \tilde{\Omega} = \{ \eta_1, \ldots, \eta_s \} \) and let \( N = \sum_{i=1}^s v^*(\eta_i) \). Without loss of generality, assume \( \eta_1 = C_0(K) \). In the following, we construct a family of \( N \times N \) non-negative matrices \( \{ M_i \}_{i=1}^s \) and a family of \( N \)-dimensional positive row vectors \( \{ w_i \}_{i=1}^s \).

For \( i \in \{1, \ldots, s\} \), we define \( M_i \) to be the partitioned matrix

\[ M_i = (U_{k,j})_{1 \leq k, j \leq s}, \]
where for $k, j \in \{1, \ldots, s\}$, $U^i_{k,j}$ is a $v^*(\eta_k) \times v^*(\eta_j)$ matrix defined by

$$U^i_{k,j} = \begin{cases} T(\eta_k, \eta_j) & \text{if } j = i \text{ and } \eta_k \eta_j \text{ is admissible,} \\ 0 & \text{otherwise.} \end{cases}$$

We define a partitioned row vector $w_i = (W^i_j)_{1 \leq j \leq s}$, where for $j \in \{1, \ldots, s\}$,

$$W^i_j = \begin{cases} v(\eta_j) & \text{if } j = i, \\ e(\eta_j) & \text{otherwise,} \end{cases}$$

where $e(\eta_j)$ denotes the row vector consisting of $v^*(\eta_j)$ many 1’s. It is clear that $w_i$ is an $N$-dimensional positive row vector.

**Proof of Theorem 1.2.** It follows directly from Lemma 3.1, the definitions of $\{M^i\}_{i=1}^s$ and $\{w^i\}_{i=1}^s$, and the product formula of partitioned matrices. □ □

### 4. Application to $L^q$-spectrum: proof of Theorem 1.3

To prove Theorem 1.3, we adopt the same strategy as in [9]. Our main target in this section is to show that $	au(q) = \frac{P(q)}{\log \rho}$ for $q > 0$ (cf. Theorem 4.8), where $P(q)$ is the pressure function for a certain family of squared matrices and $\rho \in (0, 1)$ is the common similarity ratio of the similitudes in the IFS. Then Theorem 1.3 follows from a result of Feng and Lau [12] on the differentiability of $P(q)$ on $(0, \infty)$ under an irreducibility condition.

For this purpose, in Subsection 4.1 we construct a family of squared matrices from an essential class $\hat{\Omega}$ of $\tilde{\Omega}$ and prove Lemma 4.1. Then we prove in Subsection 4.2 that the sum of these matrices is irreducible, which is needed in applying the result of Feng and Lau. Finally, in the last subsection of this section, we prove Theorem 4.8 and completes the proof of Theorem 1.3.

#### 4.1. Essential class $\hat{\Omega}$ of $\tilde{\Omega}$.

Recall that

$$\tilde{\Omega} = \{C_n(\Delta) : \Delta \in \mathcal{F}_n, n \geq 0\} = \{\eta_1, \ldots, \eta_s\},$$

where $\mathcal{F}_n = \{\Delta \in \xi_n : \mu(\Delta) > 0\}$. We call a non-empty subset $\hat{\Omega}$ of $\tilde{\Omega}$ an essential class of $\tilde{\Omega}$ if $\hat{\Omega}$ satisfies: (i) $\{\beta \in \hat{\Omega} : \alpha \beta \text{ is admissible}\} \subseteq \hat{\Omega}$ whenever $\alpha \in \hat{\Omega}$; (ii) for any $\alpha, \beta \in \hat{\Omega}$, there exists an admissible word $\alpha_1 \ldots \alpha_n$ such that $\alpha_1 = \alpha$, $\alpha_n = \beta$ and $\alpha_i \in \hat{\Omega}$ for $1 \leq i \leq n$. Such $\hat{\Omega}$ always exists, see e.g. [33, Lemma 1.1]. From now on, we fix an essential class $\hat{\Omega}$ of $\tilde{\Omega}$.

Without loss of generality, we assume that $\hat{\Omega} = \{\eta_1, \ldots, \eta_t\}$ for some $1 \leq t \leq s$. Let $L = \sum_{i=1}^t v^*(\eta_i)$. Using the same method as in Section 3, we construct a family of $L \times L$ non-negative matrices $\{M^i\}_{i=1}^t$ and a family of $L$-dimensional positive row vectors $\{w^i\}_{i=1}^t$.

Pick $n_0 \geq 1$ and $\Delta_0 \in \mathcal{F}_{n_0}$ so that $C_{n_0}(\Delta_0) = \eta_1$. In the following, we will consider the distribution of $\mu$ on the elements of $\mathcal{F}_n$ for $n \geq n_0$ which are contained in $\Delta_0$.  

For \( n \geq n_0 \) and \( \Delta \in \mathcal{F}_n \) with \( \Delta \subseteq \Delta_0 \), we define a partitioned vector
\[
\hat{u}_{n,\Delta} = (U_1, \ldots, U_t),
\]
where for \( i \in \{1, \ldots, t\} \), \( U_i \) is a \( v^*(\eta_i) \)-dimensional row vector defined by
\[
U_i = \begin{cases} 
\hat{u}_{n,\Delta} & \text{if } C_n(\Delta) = \eta_i, \\
0 & \text{otherwise}.
\end{cases}
\]
Clearly, \( \hat{u}_{n,\Delta} \) is of \( L \)-dimensional.

Let \( \Theta \) denote the symbolic expression of \( \Delta_0 \). By Lemma 3.1, the definitions of \( \{M_i\}_{i=1}^t \) and \( \{w_i\}_{i=1}^t \), and the product formula of partitioned matrices, we have

**Lemma 4.1.** (i) Let \( n \geq 0 \) and \( \Delta \in \mathcal{F}_{n+n_0} \) with \( \Delta \subseteq \Delta_0 \). Then
\[
\mu(\Delta) = \hat{u}_{n_0,\Delta_0} M_{i_1} \cdots M_{i_n} w_{i_n}^T,
\]
where \( \Theta \eta_{i_1} \cdots \eta_{i_n} \) is the symbolic expression of \( \Delta \),

(ii) A word \( \eta_{j_1} \cdots \eta_{j_k} \) over \( \hat{\Omega} \) is admissible if and only if \( M_{j_1} \cdots M_{j_k} \neq 0 \).

### 4.2. Irreducibility of \( \sum_{i=1}^t M_i \)

This subsection is devoted to proving the following result, which is a key step to prove Theorem 1.3.

**Proposition 4.2.** Let \( H = \sum_{i=1}^t M_i \). Then \( H \) is irreducible. That is, there exists a positive integer \( r \) such that all the entries of \( \sum_{i=1}^r H^i \) are positive.

To prove Proposition 4.2, we first give several lemmas. The following result is a consequence of the net structure of \( \{\xi_n\}_{n=0}^\infty \) and the FTC.

**Lemma 4.3.** \( \# \{\mu(S_1^{-1} \circ f^{-1} \Delta) : I \in \Sigma_s, f \in \Lambda_n(\Delta), \Delta \in \xi_n, n \in \mathbb{N}\} < \infty \).

**Proof.** First, fix \( n \in \mathbb{N}, \Delta \in \xi_n \) and \( f \in \Lambda_n(\Delta) \). Let \( k \in \mathbb{N} \) and \( I \in \Sigma_k \). By Lemma 2.3, we have
\[
\Delta = \bigcup_{\Delta' \in \xi_{n+k} : \Delta' \subseteq \Delta} \Delta',
\]
and the sets in the above union are mutually disjoint.

If \( \mu(S_1^{-1} \circ f^{-1} \Delta) > 0 \), then we have
\[
\mu(S_1^{-1} \circ f^{-1} \Delta) = \sum_{\Delta' \in \xi_{n+k} : \Delta' \subseteq \Delta} \mu(S_1^{-1} \circ f^{-1} \Delta') \quad \text{(by (4.1))}
\]
\[
= \sum_{\Delta' \in \xi_{n+k} : \Delta' \subseteq \Delta, \mu(S_1^{-1} \circ f^{-1} \Delta') > 0} \mu(S_1^{-1} \circ f^{-1} \Delta')
\]
\[
\mu(S_1^{-1} \circ f^{-1} \Delta') \quad \text{(by (4.2))}
\]
where (4.2) is due to the definitions of \( \Lambda_{n+k}^*(\Delta') \) for \( \Delta' \in \mathcal{F}_{n+k} \). By the FTC (1.4), \( f \circ S_I(K) \) intersects at most \( \#\Gamma \) different elements of \( \{S_J(K) : J \in \Sigma_{m+k}\} \). Hence there are at most \( 2\#\Gamma \) many \( \Delta' \in \mathcal{F}_{n+k} \) with \( f \circ S_I \in \Lambda_{n+k}^*(\Delta') \). This implies that
the number of terms in the sum (4.2) is at most $2^{|\Gamma|}$. Meanwhile, notice that each term in the sum (4.2) belongs to the set \(\{\mu(g^{-1}\Delta') : g \in \Lambda'_n(\Delta'), \Delta' \in \mathcal{F}_\ell, \ell \in \mathbb{N}\}\), which is easily seen to be finite by (1.4) and Lemma 2.4. Therefore, \(\mu(S_I^{-1} \circ f^{-1}\Delta)\) is contained in a finite set independent of \(n, \Delta, f, k\) and \(I\). This completes the proof of the lemma. \hfill \Box \Box \\

The following result plays a key role in the proof of Proposition 4.2, whose proof is an application of the Borel density lemma.

**Lemma 4.4.** Suppose \(E \subseteq \mathbb{R}^d\) is a Borel set such that \(\mu(S_I^{-1}E) = \mu(E) > 0\) for all \(I \in \Sigma_x\). Then \(\mu(E) = 1\).

**Proof.** For \(I \in \Sigma_x\), let \(|I|\) be the length of \(I\). Write

\[ [I] = \{(x_k)_{k=1}^\infty \in \Sigma^N : x_1 \ldots x_{|I|} = I\}. \]

Let \(\eta\) be the infinite Bernoulli product measure on \(\Sigma^N\) generated by the weight \((p_1, \ldots, p_m)\), i.e. \(\eta([I]) = p_{i_1} \cdots p_{i_n}\) for \(I = i_1 \ldots i_n \in \Sigma_x\). Let \(\pi : \Sigma^N \rightarrow K\) be the projection map defined by

\[ \pi((x_k)_{k=1}^\infty) = \lim_{n \to \infty} S_{x_1} \circ \cdots \circ S_{x_n}(0) \quad \text{for} \quad (x_k)_{k=1}^\infty \in \Sigma^N. \]

It is well-known that \(\mu = \eta \circ \pi^{-1}\).

Set \(A = \pi^{-1}(E)\) and \(A_I = \pi^{-1}(S_I^{-1}E)\) for \(I \in \Sigma_x\). It is clear that \(A\) and \(A_I\) are Borel subsets of \(\Sigma^N\). Moreover, by our assumption we have \(\eta(A) = \eta(A_I) > 0\).

We claim that

\[ (4.3) \quad [I] \cap A = [I] \cap \sigma^{-|I|}(A_I) \quad \text{for all} \quad I \in \Sigma_x. \]

To see this, notice that for any \(I = i_1 \ldots i_n \in \Sigma_x\) and \(x = (x_k)_{k=1}^\infty \in \Sigma^N,\)

\[ x \in A_I \iff x \in \pi^{-1}(S_I^{-1}E) \]

\[ \iff S_I \circ \pi(x) \in E \]

\[ \iff \pi(Ix) \in E \quad (Ix := i_1 \ldots i_n x_1 \ldots) \]

\[ (4.4) \quad \iff Ix \in A. \]

Hence,

\[ y \in [I] \cap \sigma^{-|I|}(A_I) \iff y \in [I], \sigma^{|I|}y \in A_I \]

\[ \iff y \in [I], I\sigma^{|I|}y \in A \quad \text{(by (4.4))} \]

\[ \iff y \in [I] \cap A, \]

from which (4.3) follows.

Next we show that \(\eta(A) = 1\), which implies that \(\mu(E) = 1\). Suppose on the contrary that \(0 < \eta(A) < 1\). By the Borel density lemma (see e.g. [25, Corollary 2.14]), we have for \(\eta\)-a.e. \(x = (x_k)_{k=1}^\infty \in A,\)

\[ \lim_{n \to \infty} \frac{\eta(A \cap [x_1 \ldots x_n])}{\eta([x_1 \ldots x_n])} = 1. \]
Hence we can find \( n \in \mathbb{N} \) and \( J \in \Sigma_n \) such that
\[
\frac{\eta(A \cap [J])}{\eta([J])} > \eta(A).
\]
However, by (4.3) we have
\[
\frac{\eta(A \cap [J])}{\eta([J])} = \frac{\eta(\sigma^{-[J]}A \cap [J])}{\eta([J])} = \frac{\eta(A \sigma)\eta([J])}{\eta([J])} = \eta(A),
\]
where the second equality is due to the product property of \( \eta \). Thus (4.6) contradicts (4.5). Hence \( \eta(A) = 1 \) and we are done. \( \square \)

Lemmas 4.3-4.4 have the following consequence, which is important in the proof of Proposition 4.2 (indeed Lemma 4.7).

**Lemma 4.5.** For any \( n \in \mathbb{N} \), \( \Delta \in \mathcal{F}_n \) and \( f \in \Lambda^*_n(\Delta) \), there exists \( I \in \Sigma_* \) such that \( \mu(S_I^{-1} \circ f^{-1}\Delta) = 1 \).

**Proof.** Fix \( n \in \mathbb{N} \), \( \Delta \in \mathcal{F}_n \) and \( f \in \Lambda^*_n(\Delta) \). By Lemma 4.3, the set
\[ \{ \mu(S_I^{-1} \circ f^{-1}\Delta) : J \in \Sigma_* \} \]
is finite. So we can find \( I \in \Sigma_* \) such that
\[
\mu(S_I^{-1} \circ f^{-1}\Delta) = \max \{ \mu(S_J^{-1} \circ f^{-1}\Delta) : J \in \Sigma_* \}. \tag{4.7}
\]
Clearly, \( \mu(S_I^{-1} \circ f^{-1}\Delta) > 0 \) as \( \mu(f^{-1}\Delta) > 0 \).

Let \( k \in \mathbb{N} \). By (3.2) (in which we take \( n = k \)), we have
\[
\mu(S_I^{-1} \circ f^{-1}\Delta) = \sum_{J \in \Sigma_k} p_J \mu(S_J^{-1} \circ S_I^{-1} \circ f^{-1}\Delta) = \sum_{J \in \Sigma_k} p_J \mu(S_J^{-1} \circ f^{-1}\Delta). \tag{4.8}
\]
Then by (4.7)-(4.8) and the fact that \( \sum_{J \in \Sigma_k} p_J = 1 \), we easily deduce that
\[
\mu(S_J^{-1} \circ f^{-1}\Delta) = \mu(S_I^{-1} \circ S_J^{-1} \circ f^{-1}\Delta) \quad \text{for all } J \in \Sigma_k. \tag{4.9}
\]
Since \( k \in \mathbb{N} \) is arbitrary, (4.9) holds for all \( J \in \Sigma_* \). Now it follows from Lemma 4.4 that \( \mu(S_I^{-1} \circ f^{-1}\Delta) = 1 \), completing the proof of the lemma. \( \square \)

The following result follows easily from the definitions of \( M_1, \ldots, M_t \), the product formula of partitioned matrices and induction, whose proof we omit.

**Lemma 4.6.** Given an admissible word \( \eta_1 \ldots \eta_n \) with \( n \geq 2 \), write the matrix \( M_{i_2} \cdots M_{i_n} \) in the form of partitioned matrix \( (U_{i,j})_{i \leq i,j \leq t} \), where \( U_{i,j} \) is a \( v^*(\eta_i) \times v^*(\eta_j) \) matrix. Then we have
\[
U_{i_1,i_n} = T(\eta_{i_1}, \eta_{i_2}) \cdots T(\eta_{i_{n-1}}, \eta_{i_n}).
\]

An essential part to prove Proposition 4.2 is the following result.

**Lemma 4.7.** For any \( i, j \in \{1, \ldots, t\} \) and \( k \in \{1, \ldots, v^*(\eta_i)\} \), there exists an admissible word \( \eta_1 \ldots \eta_n \) with \( \eta_1 = \eta_i \) and \( \eta_n = \eta_j \) such that each entry of the \( k \)-th row of the matrix
\[
T(\eta_{i_1}, \eta_{i_2}) \cdots T(\eta_{i_{n-1}}, \eta_{i_n})
\]
is positive.
Proof. Recall that \( \bar{\Omega} = \{ \eta_i \}_{i=1}^\ell \) is an essential class of \( \Xi = \{ C_n(\Delta) : \Delta \in \mathcal{F}_n, n \geq 0 \} \), and \( n_0 \in \mathbb{N}, \Delta_0 \in \mathcal{F}_{n_0} \) are chosen so that \( C_n(\Delta_0) = \eta_1 \). Let \( i, j \in \{1, \ldots, \ell\} \) and \( k \in \{1, \ldots, v^*(\eta_i)\} \) be fixed. Pick \( n_1 \in \mathbb{N} \) and \( \Delta_1 \in \mathcal{F}_{n_1} \) so that \( \Delta_1 \subseteq \Delta_0 \) and \( C_{n_1}(\Delta_1) = \eta_i \). Assume \( \Lambda^*_n(\Delta_1) = (h_1, \ldots, h_{v^*(\eta_i)}) \). By Lemma 4.5, there exist \( n' \in \mathbb{N} \) and \( I \in \Sigma_{n'} \) such that

\[
\mu(S^{-1}_I \circ h^{-1}_k \Delta_1) = 1.
\]

Pick \( n_2 \in \mathbb{N} \) and \( \Delta_2 \in \mathcal{F}_{n_2} \) with \( \Delta_2 \subseteq \Delta_0 \) such that

\[
\#\Lambda_{n_2}(\Delta_2) = \max\{ \#\Lambda_\ell(\Delta') : \Delta' \in \mathcal{F}_\ell, \Delta' \subseteq \Delta_0, \ell \geq n_0 \} := u,
\]

\[
\#N_{n_2}(\Delta_2) = \max\{ \#N_\ell(\Delta') : \Delta' \in \mathcal{F}_\ell, \Delta' \subseteq \Delta_0, \Lambda_\ell(\Delta') = u, \ell \geq n_0 \} := v.
\]

Assume that \( \Lambda_{n_2}(\Delta_2) = (f_1, \ldots, f_u) \) and \( N_{n_2}(\Delta_2) = (g_1, \ldots, g_v) \). Set \( \Delta = h_k \circ S_I(\Delta_2) \). Then

\[
\Delta = \left( \bigcap_{p=1}^u h_k \circ S_I \circ f_p(K) \right) \setminus \left( \bigcup_{g \in \{g_p\}_{p=1}^v \setminus \{f_p\}_{p=1}^u} h_k \circ S_I \circ g(K) \right).
\]

We assert that \( \Delta \in \mathcal{F}_{n' + n_1 + n_2} \). To see this, first notice that by the similarity of \( \mu \) (see (3.2), in which we take \( n = n' + n_1 \)) and (4.10), we have

\[
\mu(\Delta \cap \Delta_0) \geq \mu(\Delta \cap \Delta_1)
= \mu(h_k \circ S_I(\Delta_2) \cap \Delta_1)
\geq \left( \sum_{J \in \Sigma_{n' + n_1} : S_J = h_k \circ S_I} p_J \right) \mu(\Delta_2 \cap S^{-1}_I \circ h^{-1}_k \Delta_1)
= \left( \sum_{J \in \Sigma_{n' + n_1} : S_J = h_k \circ S_I} p_J \right) \mu(\Delta_2) > 0 \quad \text{(by (4.10))}.
\]

Since \( \mathcal{F}_{n' + n_1 + n_2} \) is a partition of \( K \) in measure, it follows that

\[
\mu(\Delta_0 \cap \Delta') \geq \mu(\Delta \cap \Delta_0 \cap \Delta' ) > 0
\]

for some \( \Delta' \in \mathcal{F}_{n' + n_1 + n_2} \). In particular, \( \Delta_0 \cap \Delta' \) and \( \Delta \cap \Delta' \) are both non-empty. Hence \( \Delta' \subseteq \Delta_0 \) by the net structure of \( \{ \xi_n \}_{n \geq 0} \) (cf. Lemma 2.3). Moreover, by (4.13) and the definition of the mapping \( \Lambda_{n' + n_1 + n_2} \) on \( K \) (cf. (1.5)), we see that

\[
\Lambda_{n' + n_1 + n_2}(x) \supseteq \{ h_k \circ S_I \circ f_p \}_{p=1}^u \quad \text{for all } x \in \Delta \cap \Delta'.
\]

It then follows from the definition of \( \Lambda_{n' + n_1 + n_2}(\Delta') \) that

\[
\Lambda_{n' + n_1 + n_2}(\Delta') \supseteq \{ h_k \circ S_I \circ f_p \}_{p=1}^u.
\]

This combining with (4.11) yields that indeed

\[
\Lambda_{n' + n_1 + n_2}(\Delta') = \{ h_k \circ S_I \circ f_p \}_{p=1}^u.
\]
By the definition of \( N_{n_2}(\Delta_2) \),
\[
g_q(K) \cap \left( \bigcap_{p=1}^{u} f_p(K) \right) \neq \emptyset, \quad \forall 1 \leq q \leq v,
\]
which implies that
\[
h_k \circ S_I \circ g_q(K) \cap \left( \bigcap_{p=1}^{u} h_k \circ S_I \circ f_p(K) \right) \neq \emptyset, \quad \forall 1 \leq q \leq v.
\]
By this fact, the definition of \( N'_{n_1+n_2}(\Delta') \) and (4.12), we see that
\[
N'_{n_1+n_2}(\Delta') = \{ h_k \circ S_I \circ g_q \}_{q=1}^{v}.
\]
Now (4.15)-(4.16) imply that \( \Delta = \Delta' \), and so \( \Delta \in \mathcal{F}_{n_1+n_2} \). This proves the above assertion.

Since \( \Delta_1 \in \mathcal{F}_{n_1} \), \( \Delta \in \mathcal{F}_{n_1+n_2} \) and \( \mu(\Delta_1 \cap \Delta) > 0 \) (cf. (4.14)), we have \( \Delta \subseteq \Delta_1 \).
Assume that \( C_{n_1+n_2}(\Delta) = \eta' \) for some \( t' \in \{1, \ldots, t\} \). Write \( \Lambda^*_{n_1+n_2}(\Delta) = (h_1', \ldots, h_{\nu'}(\eta')) \). Then by (4.15) and the fact that \( \Delta = \Delta' \), we have
\[
\{ h_1', \ldots, h_{\nu'}(\eta') \} \subseteq \{ h_k \circ S_I \circ f_p \}_{p=1}^{u}.
\]
Let \( \gamma_0 \gamma_1 \cdots \gamma_{n_1-1} \eta_k \) be the symbolic expression of \( \Delta_1 \) and
\[
\gamma_0 \gamma_1 \cdots \gamma_{n_1-1} \eta \eta_2 \cdots \eta_{n_1+n_2} \eta'
\]
be that of \( \Delta \). By Lemma 3.2, for any \( 1 \leq \ell \leq \nu'(\eta') \), the \((k, \ell)\)-entry of the matrix
\[
T(\eta_1, \eta_2) \cdots T(\eta_{n_1+n_2}, \eta')
\]
is given by
\[
\sum_{J \in \Sigma_{n_1+n_2}: S_J = h_k^{-1} \circ h_\ell'} p_J,
\]
which is positive by (4.17). Hence each entry of the \( k \)-th row of (4.18) is positive.

To finish the proof, pick \( n_3 \in \mathbb{N} \) and \( \Delta_3 \in \mathcal{F}_{n_1+n_2+n_3} \) so that \( \Delta_3 \subseteq \Delta \) with \( C_{n_1+n_2+n_3}(\Delta_3) = \eta_j \). Let \( \gamma_0 \gamma_1 \cdots \gamma_{n_1-1} \eta \eta_2 \cdots \eta_{n_1+n_2} \eta \eta_1 \cdots \eta_{n_3-1} \eta_j \) be the symbolic expression of \( \Delta_3 \). Then by the above argument and (3.7), each entry of the \( k \)-th row of the matrix
\[
T(\eta_1, \eta_2) \cdots T(\eta_{n_1+n_2}, \eta') T(\eta', \eta_j) T(\eta_j, \eta_{j_2}) \cdots T(\eta_{j_{n_3-1}}, \eta_j)
\]
is positive. This completes the proof of the lemma. \( \square \)

**Proof of Proposition 4.2.** With Lemmas 4.6-4.7 in hand, the proof of Proposition 4.2 is identical to that of [9, Proposition 4.2]. We omit the repetition here. \( \square \)
4.3. **Proof of Theorem 1.3.** Let \(M_1, \ldots, M_t\) be the \(L \times L\) matrices that we have constructed in the beginning of Section 4. For \(q \in \mathbb{R}\), define

\[
P(q) = \lim_{n \to \infty} \frac{1}{n} \log \left( \sum \|M_{i_1} \cdots M_{i_n}\|^q \right),
\]

where the summation is taken over all words \(i_1 \ldots i_n \in \{1, \ldots, t\}^n\) with \(M_{i_1} \cdots M_{i_n} \neq 0\), and \(\|A\| = \sum_{i,j} a_{i,j}\) for any non-negative matrix \(A = (a_{i,j})_{1\leq i,j \leq N}\). The function \(P\) is called the *pressure function* of \(M_1, \ldots, M_t\). Since \(\sum_{i=1}^t M_i\) is irreducible (cf. Proposition 4.2), the limit in (4.19) exists and \(P(q)\) is differentiable on \((0, \infty)\) (see [12, Theorem 3.3 and Proposition 4.4]). This combining with the following result immediately yields Theorem 1.3, where \(\rho \in (0, 1)\) is the common similarity ratio of the similitudes in the IFS.

**Theorem 4.8.** For \(q > 0\), \(\tau(q) = \frac{P(q)}{\log \rho}\).

The rest of this subsection is devoted to the proof of Theorem 4.8. We will use the following equivalent definition of \(L^q\)-spectrum for \(q > 0\); see [23, Proposition 3.1]. For \(q > 0\), the \(L^q\)-spectrum \(\tau(q)\) of \(\mu\) can be given by

\[
\tau(q) = \liminf_{n \to \infty} \frac{1}{n \log 2} \log \sum_{D \in \mathcal{D}_n} \mu(D)^q,
\]

where for each \(n \in \mathbb{N}\), \(\mathcal{D}_n = \left\{ \prod_{i=1}^d \left[ \frac{k_i}{2^n}, \frac{k_i+1}{2^n} \right) : k_i \in \mathbb{Z} \text{ for } 1 \leq i \leq d \right\} \).

Let \(n_0 \in \mathbb{N}\) and \(\Delta_0 \in \mathcal{F}_{n_0}\) be as in the beginning of Section 4. Define \(\mu_0 = \mu_{\Delta_0}\), i.e. \(\mu_0(A) = \mu(\Delta_0 \cap A)\) for any Borel set \(A \subset \mathbb{R}^d\).

**Proposition 4.9.** For \(q > 0\), we have

\[
\tau(q) = \liminf_{n \to \infty} \frac{1}{n \log \rho} \log \sum_{\Delta \in \mathcal{F}_n} \mu(\Delta)^q,
\]

\[
\tau(\mu_0, q) = \liminf_{n \to \infty} \frac{1}{n \log \rho} \log \sum_{\Delta \in \mathcal{F}_n : \Delta \subseteq \Delta_0} \mu(\Delta)^q.
\]

**Proof.** Fix \(q > 0\). For each \(n \in \mathbb{N}\), let \(k_n \in \mathbb{N}\) be such that

\[
\rho^{k_n} \leq 2^{-n} < \rho^{k_n-1}.
\]

Then it is clear that there is a constant \(N_1\) (independent of \(n\)) such that each \(\Delta \in \xi_{k_n}\) can intersect at most \(N_1\) elements of \(\mathcal{D}_n\). On the other hand, since \(\Phi\) satisfies the FTC, it is well-known that \(\Phi\) satisfies the weak separation condition [29]. That is,

\[
\sup_{x \in \mathbb{R}^d, n \in \mathbb{N}} \# \{ S_I : S_I(K) \cap B(x, \rho^n) \neq \emptyset, I \in \Sigma_n \} < \infty.
\]

By this fact and (4.23), it is not hard to see that there is a constant \(N_2\) (independent of \(n\)) such that each \(D \in \mathcal{D}_n\) can intersect at most \(N_2\) elements of \(\xi_{k_n}\).
By the above argument, for any $D \in \mathcal{D}_n$, we have
\[
\mu(D)^q = \left( \sum_{\Delta \in \xi_n : \Delta \cap D \neq \emptyset} \mu(\Delta) \right)^q \leq N_2^q \sum_{\Delta \in \xi_n : \Delta \cap D \neq \emptyset} \mu(\Delta)^q.
\]
Hence
\[
\sum_{D \in \mathcal{D}_n} \mu(D)^q \leq N_2^q \sum_{D \in \mathcal{D}_n} \sum_{\Delta \in \xi_n : \Delta \cap D \neq \emptyset} \mu(\Delta)^q \leq N_1^q \sum_{\Delta \in \xi_n} \mu(\Delta)^q.
\]
Then it follows from (4.20) that
\[
\tau(q) \geq \liminf_{n \to \infty} \frac{\log \sum_{\Delta \in \xi_n} \mu(\Delta)^q}{n \log \rho} = \liminf_{n \to \infty} \frac{\log \sum_{\Delta \in F_n} \mu(\Delta)^q}{n \log \rho}.
\]
The ‘\(\leq\)’ part of (4.21) can be proved analogously, whose details we omit. Hence we have proved (4.21). To prove (4.22), we simply notice that the above argument still works with slight modifications if we replace \(\mu\) by \(\mu_0\).

**Lemma 4.10.** For \(q > 0\), \(\tau(q) = \tau(\mu_0, q)\).

**Proof.** Let \(q > 0\) be fixed. It is clear from Proposition 4.9 that \(\tau(q) \leq \tau(\mu_0, q)\). Below we prove that \(\tau(q) \geq \tau(\mu_0, q)\).

By Lemma 4.5, there exist \(n' \in \mathbb{N}\) and \(J \in \Sigma_{n'}\) so that \(\mu(S_J^{-1}\Delta_0) = 1\). Let \(n \geq 1\) and \(\Delta \in \mathcal{F}_n\). Then we have
\[
(4.24) \quad \mu_0(S_J(\Delta))^q = \mu(S_J(\Delta) \cap \Delta_0)^q \geq p_J^q \mu(\Delta \cap S_J^{-1}(\Delta_0))^q = p_J^q \mu(\Delta)^q,
\]
where the second inequality follows from (3.2) (in which we take \(n = n'\)) and the last equality holds since \(\mu(S_J^{-1}\Delta_0) = 1\). On the other hand, by Lemma 2.2, \(S_J(\Delta)\) is a union of some elements of \(\xi_{n+n'}\). Moreover, by the FTC (1.4), it is not hard to see that \(S_J(\Delta)\) contains at most \(N_0 := 2^n\#F\) elements of \(\mathcal{F}_{n+n'}\).

By the above argument, for any \(n \geq 1\) and \(\Delta \in \mathcal{F}_n\),
\[
\mu_0(S_J(\Delta))^q = \left( \sum_{\Delta' \in \mathcal{F}_{n+n'} : \Delta' \subseteq S_J(\Delta)} \mu_0(\Delta') \right)^q \leq N_0^q \sum_{\Delta' \in \mathcal{F}_{n+n'} : \Delta' \subseteq S_J(\Delta)} \mu_0(\Delta')^q.
\]
Therefore,
\[
(4.25) \quad \sum_{\Delta \in \mathcal{F}_n} \mu_0(S_J(\Delta))^q \leq N_0^q \sum_{\Delta \in \mathcal{F}_n} \sum_{\Delta' \in \mathcal{F}_{n+n'} : \Delta' \subseteq S_J(\Delta)} \mu_0(\Delta')^q \leq N_0^q \sum_{\Delta' \in \mathcal{F}_{n+n'}} \mu_0(\Delta')^q.
\]
It follows from (4.24)-(4.25) and Proposition 4.9 that \(\tau(q) \geq \tau(\mu_0, q)\), as desired. \(\square\)

**Proof of Theorem 4.8.** By Lemma 4.10, it suffices to prove that for \(q > 0\), \(\tau(\mu_0, q) = P(q)/\log \rho\). The proof is a slight modification of that of [9, Proposition 5.7].

For two vectors \(a = (a_1, \ldots, a_L), b = (b_1, \ldots, b_L)\), write \(a \approx b\) if there is a constant \(C \geq 1\) such that \(C^{-1}a_i \leq b_i \leq Ca_i\) for all \(i\). Let \(e\) be the row vector
consisting of \( L \) 1’s. By the definitions of \( \hat{u}_{n_0,\Delta_0} \), \( M_1 \) and \( \{w_i\}_{i=1}^t \), it is direct to see that

\[
\hat{u}_{n_0,\Delta_0} \approx eM_1 \quad \text{and} \quad w_i \approx e \quad \text{for } 1 \leq i \leq t.
\]

Let \( q > 0 \) be fixed. By Proposition 4.9 and Lemma 4.1, we have

\[
\tau(\mu_0, q) = \liminf_{n \to \infty} \frac{1}{n \log \rho} \log \sum (\hat{u}_{n_0,\Delta_0}M_{i_1} \cdots M_{i_n}w_i^T)^q,
\]

where the summation is taken over all admissible words \( \eta_1 \eta_2 \cdots \eta_n \). Then by \((4.26)\)-(\(4.27)\) and Lemma 4.1(ii), we easily see that

\[
\tau(\mu_0, q) = \liminf_{n \to \infty} \frac{1}{n \log \rho} \log \sum \|M_1M_{i_1} \cdots M_{i_n}\|^q.
\]

Now the remaining part of the proof is exactly the same as that of [9, Proposition 5.7], so we omit the repetition here. \( \square \) \( \square \)

5. THE CASE THAT \( \Phi \) IS COMMENSURABLE

As mentioned in the introduction, Theorems 1.2-1.3 can be extended to the case that the IFS is commensurable and satisfies a more general form of the FTC.

We say that an IFS \( \Phi = \{S_i\}_{i=1}^m \) on \( \mathbb{R}^d \) is **commensurable** if there exist \( r \in (0, 1) \) and positive integers \( k_1, \ldots, k_m \) such that \( \rho_i = r^{k_i} \) for \( 1 \leq i \leq m \), where \( \rho_i \) is the similarity ratio of \( S_i \). Let \( K \) be the self-similar set generated by \( \Phi \).

Let \( \rho = \min_{1 \leq i \leq m} \rho_i \). Let \( A_0 = \{\varepsilon\} \), where \( \varepsilon \) is the empty word. For \( n \geq 1 \), set

\[
A_n = \{i_1 \cdots i_k \in \Sigma_n : \rho_i_1 \cdots \rho_i_k \leq \rho^n < \rho_i_1 \cdots \rho_i_{k-1}\}.
\]

We say that \( \Phi \) satisfies the finite type condition if there exists a finite set \( \Gamma \) such that for any \( n \geq 1 \) and \( I, J \in A_n \),

\[
\begin{align*}
either \quad S_I(K) \cap S_J(K) = \emptyset \quad \text{or} \quad S_I^{-1} \circ S_J \in \Gamma.
\end{align*}
\]

We define for each \( n \geq 0 \), a Borel partition \( \xi_n \) of \( K \) by using \( A_n \) instead of \( \Sigma_n \). Precisely, for \( n \geq 0 \), define \( \Lambda_n : K \to 2^S \) by

\[
\Lambda_n(x) = \{S_I : I \in A_n \text{ with } x \in S_I(K)\} \quad \text{for } x \in K.
\]

Then set

\[
\xi_n = \{\Lambda_n^{-1}(U) : U \in \Lambda_n(K)\}.
\]

It is clear that \( \xi_n \) is a finite Borel partition of \( K \). For \( \Delta \in \xi_n \), let \( \Lambda_n(\Delta) \) be the value of \( \Lambda_n \) on \( \Delta \). Then define \( N_n(\Delta) \) by

\[
N_n(\Delta) = \left\{ S_I : I \in A_n \text{ with } S_I(K) \cap \left( \bigcap_{f \in \Lambda_n(\Delta)} f(K) \right) \neq \emptyset \right\}.
\]

Our result in this section is the following.
Theorem 5.1. Let $\Phi = \{S_i\}_{i=1}^m$ be a commensurable IFS on $\mathbb{R}^d$ satisfying (5.1). Let $\mu$ be the self-similar measure generated by $\Phi$ and a probability vector $(p_1, \ldots, p_m)$. Then the conclusions of Theorems 1.2-1.3 hold.

The idea to prove Theorem 5.1 is essentially the same as that of the equicontractive case but many details are different. In the following, we point out the major modifications of Sections 2-4 needed to prove Theorem 5.1.

5.1. The characteristic vectors of $\Delta \in \xi_n$. Let the two linear orders $\leq_{\text{lex}}$ on $\Sigma_*$ and $\preceq$ on $\Phi_*$, and the mapping $\omega : \Phi_* \rightarrow \Sigma_*$ be the same as in Section 2.

For $n \geq 0$ and $\Delta \in \xi_n$, we define the characteristic vector
\[ \mathcal{C}_n(\Delta) = (V_n(\Delta), U_n(\Delta), r_n(\Delta)) \]
of $\Delta$ in following way: Write $\Lambda_n(\Delta)$ and $N_n(\Delta)$ as ordered vectors
\[ \Lambda_n(\Delta) = (f_1, \ldots, f_k), \quad N_n(\Delta) = (g_1, \ldots, g_\ell), \]
where $f_1 \prec \cdots \prec f_k$ and $g_1 \prec \cdots \prec g_\ell$ in $\Phi_*$. Then define $V_n(\Delta)$ and $U_n(\Delta)$ by
\[ V_n(\Delta) = ((\varphi_1, s_1), \ldots, (\varphi_k, s_k)), \quad U_n(\Delta) = ((\psi_1, t_1), \ldots, (\psi_\ell, t_\ell)), \]
where $\varphi_i = f_i^{-1} \circ f_i$, $s_i = \rho^{-n} \rho f_i$ for $1 \leq i \leq k$, and $\psi_j = f_j^{-1} \circ g_j$, $t_j = \rho^{-n} \rho g_j$ for $1 \leq j \leq \ell$. Define $r_n(\Delta)$ as in Section 2. By (5.1) and the assumption that $\Phi$ is commensurable, it is easily checked that the set $\{\mathcal{C}_n(\Delta) : \Delta \in \xi_n, n \geq 0\}$ is finite.

It is easy to see that all the results in Section 2 have analogous statements which hold in the setting in this section.

5.2. The transitive matrices $T_{\hat{\Delta}, \Delta}$. We need some modifications in the definition of the transitive matrices.

For $n \geq 1$, let $\mathcal{F}_n = \{\Delta \in \xi_n : \mu(\Delta) > 0\}$. Let $\Delta \in \mathcal{F}_n$ and assume $\Lambda_n(\Delta) = (f_1, \ldots, f_k)$. Then define
\[ \Lambda_n^*(\Delta) = (h_1, \ldots, h_{\hat{k}}), \quad V_n^*(\Delta) = ((\phi_1, \tau_1), \ldots, (\phi_{\hat{k}}, \tau_{\hat{k}})), \]
where $h_1, \ldots, h_{\hat{k}}$ (ranked increasingly in the order $\prec$) are those $f \in \{f_i\}_{i=1}^k$ satisfying $\mu(f^{-1} \Delta) > 0$, and $\phi_i := f_i^{-1} \circ h_i$, $\tau_i := \rho^{-n} \rho h_i$ for $i = 1, \ldots, \hat{k}$. Let $v_n^*(\Delta)$ denote the dimension of $V_n^*(\Delta)$, i.e. $v_n^*(\Delta) = \hat{k}$. Similar to Section 3, we can show that $V_n^*(\Delta)$ is determined by $\mathcal{C}_n(\Delta)$.

Let $\hat{\Delta} \in \mathcal{F}_{n-1}$ be such that $\Delta \subseteq \hat{\Delta}$. Assume that $\Lambda_n(\hat{\Delta}) = (h'_1, \ldots, h'_{\hat{k}})$. For $1 \leq i \leq \hat{k}$ and $1 \leq j \leq \hat{k}$, we define a number $t_{j,i}$ by
\[ t_{j,i} = \sum_{pW} p_W, \]
where the summation is taken over all words $W \in \Sigma_*$ satisfying that there exists $J \in \mathcal{A}_{n-1}$ such that $JW \in \mathcal{A}_n$, $S_J = h'_j$ and $S_{JW} = h_i$. Define $t_{j,i} = 0$ if such $W$ does not exist. Then we define a $v_n^*(\hat{\Delta}) \times v_n^*(\Delta)$ matrix by
\[ T_{\hat{\Delta}, \Delta} = (t_{j,i})_{1 \leq j \leq \hat{k}, 1 \leq i \leq \hat{k}}. \]
Similar to Section 3, we can see that \( T_{\Delta, \Delta} \) is determined by \( C_{n-1}(\Delta) \) and \( C_n(\Delta) \).

Using the transitive matrices constructed above, all the results in Sections 3-4 can be extended to the setting in this section, without using new ideas. As a consequence, Theorem 5.1 is proved. We omit the details.

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