RICH QUASI-LINEAR SYSTEM FOR INTEGRABLE GEODESIC FLOWS ON 2-TORUS

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Abstract. Consider a Riemannian metric on two-torus. We prove that the question of existence of polynomial first integrals leads naturally to a remarkable system of quasi-linear equations which turns out to be a Rich system of conservation laws. This reduces the question of integrability to the question of existence of smooth (quasi-) periodic solutions for this Rich quasi-linear system.

1. Introduction

In this paper we study the problem of existence of integrals which are homogeneous polynomials with respect to momenta for geodesic flows on 2-torus. We show that this problem is equivalent to the problem of finding periodic solutions for a remarkable quasi-linear system of equations. We shall prove that in a domain of hyperbolicity this system can be represented in Riemann invariants, and also have a form of conservation laws. Such systems are called the rich systems (or semi-hamiltonian). It is a challenging problem to show that a given system of size greater than 2 do not have smooth solutions. For Rich systems, the classical analysis by Lax along characteristics can be used to establish the shock formation provided the so-called genuine non-linearity of eigenvalues is satisfied (see [15] and also [2] where this analysis was performed for a problem of integrability for a system of 1,5 degrees of freedom). However in our case this genuine nonlinearity condition does not necessarily holds. This leaves a hope to find new examples of non-trivial integrals for geodesic flows on 2-torus. It would be interesting to study if the generalized hodograph method in the region of hyperbolicity [16] could give a non-trivial information on smooth solutions for this system. We hope to come back to these questions in subsequent paper.

Consider a geodesic flow on the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let

$$ds^2 = \sum_{i,j=1}^{2} g_{ij}(q) dq^i dq^j, H = \sum_{i,j=1}^{2} g^{ij}(q) p_i p_j$$

be a Riemannian metric and the corresponding Hamiltonian function of the geodesic flow. The geodesic flow is called integrable, if the Hamiltonian

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system
\[ \dot{q}^j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q^j}, \quad j = 1, 2, \tag{1} \]
on \mathbb{T}^2 \simeq \mathbb{T}^2 \times \mathbb{R}^2, admits a smooth function \( F(q, p) \) (called a first integral) (functionally independent with \( H \) almost everywhere) which has constant values along the trajectories of the geodesic flow, i.e.
\[ \{ F, H \} = \sum_{j=1}^2 \left( \frac{\partial F}{\partial q^j} \frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q^j} \frac{\partial F}{\partial p_j} \right) = 0. \]

We shall denote by \( \tau \) the time variable along the trajectories and \( \frac{d}{d\tau} \) will be denoted (untraditionally) by dot while the variable \( t \) will be used later as a coordinate on 2-torus. In this paper we shall deal with the case when \( F \) is a homogeneous polynomial of certain degree with \( C^3 \) coefficients.

There are two classically known classes of metrics with integrable geodesic flows. They are written in a conformal coordinates as follows:

1. \( ds^2 = f(x)(dx^2 + dy^2) \),
2. \( ds^2 = (f(x) + g(y))(dx^2 + dy^2) \)

In the first case the Hamiltonian system admits a one parametric group of symmetries and has the integral which is a polynomial of the first degree with respect to momenta. While in the second case the integral appears to be of the second degree and is related to separation of variables in Hamilton-Jacobi equation. The existence of the metrics on \( \mathbb{T}^2 \) with the integrable geodesic flows having polynomial integrals of the degree higher than 2 and non-reducible to the integrals of the 1 and 2 degree, is not known. Amazingly there exist non-trivial examples of geodesic flows on 2-sphere with integrals which are homogeneous polynomials of degrees 3 and 4. These examples (see [14] and [8]) were inspired by integrable cases of Goryachev-Chaplygin and Kovalevskaya in rigid body dynamics (see [6]).

It is important to mention that in the classical examples 1,2 as well as in the examples on the sphere mentioned above the metric is presented in conformal coordinates. This approach goes back at least to Darboux, and was studied extensively (see for example [11], [12], [9]) We the reader refer to a nice exposition [13].

The main idea of our approach is to work in different coordinates which are angle coordinates associated to an invariant torus of the geodesic flow. The advantage of these coordinates is a possibility to write the quasi-linear equations on the coefficients of the unknown integral and the metric in the form of evolution equations as it is described in the following:

**Theorem 1.1.** Suppose that the Hamiltonian system (1) has an integral \( F \), which is a homogeneous polynomial of degree \( n \). Then on the covering plane \( \mathbb{R}^2 \) there exist the global coordinates \((t, x)\), where the metric has the following form
\[ ds^2 = g^2(t, x)dt^2 + dx^2, \]
and the integral \( F \) can be written in the form
\[ F = \sum_{k=0}^{n} a_k(t, x) \frac{p_1^{n-k} p_2^k}{g^{n-k}}, \]
Where the last two coefficients can be normalized to be \(a_{n-1} \equiv g\) and \(a_n \equiv 1\). Then the commutation relation \(\{F,H\} = 0\) is equivalent to the system of \(n\) quasi-linear equations on the unknown \(U = (a_0, \ldots, a_{n-2}, a_{n-1})^T\) (remember \(a_{n-1} \equiv g\) and \(a_n \equiv 1\)):

\[
U_t + A(U)U_x = 0, \tag{2}
\]

where the matrix \(A\) has the form:

\[
A = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & a_1 \\
a_{n-1} & 0 & \cdots & 0 & 0 & 2a_2 - na_0 \\
0 & a_{n-1} & \cdots & 0 & 0 & 3a_3 - (n-1)a_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & a_{n-1} & 0 & (n-1)a_{n-1} - 3a_{n-3} \\
0 & 0 & \cdots & 0 & a_{n-1} & na_n - 2a_{n-2}
\end{pmatrix}. \tag{3}
\]

The functions \(a_i, g\) are periodic on the variable \(x\), and quasi-periodic on the variable \(t\).

Let us recall that a hyperbolic diagonal system

\[(r_i)_t + \lambda_i(r_1, \ldots, r_n)(r_i)_x = 0, \quad i = 1, \ldots, n\]

is called Rich if the eigenvalues satisfy the following conditions

\[
\partial_{r_k} \left( \frac{\partial_{r_i} \lambda_j}{\lambda_i - \lambda_j} \right) = \partial_{r_i} \left( \frac{\partial_{r_k} \lambda_j}{\lambda_k - \lambda_j} \right).
\]

It can be proved that a diagonal system is Rich if and only if it can be written in some coordinates as a system of conservation laws (we refer to [15] for a nice exposition of rich systems including the blow up analysis and to [7], [16] for Hamiltonian formalism for system of Hydrodynamics type).

For our system the following theorem holds

**Theorem 1.2.** The system (2) is the Rich quasi-linear system. More precisely:

1. In the region of hyperbolicity (all eigenvalues are real and distinct) there exists a change of variables (Riemann invariants) \((a_0, \ldots, a_n) \rightarrow (r_1, \ldots, r_n)\) transforming the system to a diagonal form:

\[(r_i)_t + \lambda_i(r_1, \ldots, r_n)(r_i)_x = 0, \quad i = 1, \ldots, n.\]

2. There exists a regular change of variables \((a_0, \ldots, a_n) \rightarrow (G_1, \ldots, G_n)\) such that \(G_i, i = 1, \ldots, n\) are conservation laws:

\[(G_i(a_0, \ldots, a_n))_t + (H_i(a_0, \ldots, a_n))_x = 0, \quad i = 1, \ldots, n.\]

We shall see that both Riemann invariants and Conservation laws of the theorem are ultimately related to the phase portrait of the geodesic flow.

There are no general methods for proof of existence or non-existence of smooth periodic solutions of the system of quasi-linear equations where the eigenvalues can collide and become complex. However in [2], [3] non-existence of nontrivial periodic solutions for the so-called Benney chain is proved. It relies heavily on the property of genuine non-linearity of smallest and largest eigenvalues. It seems however that for our system (2) the genuine non-linearity condition doesn’t necessarily holds. This gives a hope to find smooth (quasi-) periodic solutions for the system.
Let us mention that, starting at $t = 0$ with a periodic in $x$ initial data for the Cauchy problem which is hyperbolic. Then one can "solve" the initial value problem and thus to obtain a metric $g$ and the integral $F$ of the geodesic flow on a cylinder $S^1(x) \times \{|t| < \epsilon\}$. Of course this is local result in $t$ and this metric is not necessarily complete. It would be very interesting to understand this "catastrophe" in more details.

In the Section 1 we prove the first theorem. We split the proof of the second theorem: in the Sections 2 and 3 we prove respectively, that the system (2) can be represented in the Riemann invariants and in the form of conservation laws.

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2. Fermi coordinates. Proof of Theorem 1.1

The metric of the form $ds^2 = g^2(t,x)dt^2 + dx^2$ is of course very well known in Riemannian geometry. It can always be introduced locally near a given geodesic. In such a case the coordinates $(t,x)$ are called usually Fermi coordinates or sometimes equi-distant coordinates. It turns out that for integrable geodesic flow these coordinates can be chosen globally. This is done with the help of a regular invariant torus of the flow which projects diffeomorphically on the base. The existence of such a torus is rather deep fact. More precisely, it is proved in [4] (Theorem 1.6) that if the geodesic flow on 2-torus has a non-constant polynomial integral, then there exists an invariant Lagrangian torus $L \subset T^*\mathbb{T}^2$, such, that the mapping

$$\pi|_L : L \to \mathbb{T}^2$$

is a diffeomorphism, where $\pi$ is the canonical projection of the cotangent bundle. Moreover this torus lies in a regular level $F = f$ and it consists of periodic trajectories. One can assume that $f \neq 0$ by perturbing a little the level if it happened to be zero. Multiplying $F$ by a constant we can achieve $f = 1$, this we shall assume in the sequel. Let $\varphi_1, \varphi_2$ denote the "angle" part of the action-angle coordinates on $L$ (see [1]). In these coordinates both Hamiltonian flows on $L$, $f^\tau$ of $F$ and $h^\tau$ of $H$ are linearized. Therefore, we can assume without loss of generality that the Hamiltonian fields of $H,F$ on $L$ equal in these coordinates to:

$$sgradH = (0,1), \quad sgradF = (\omega_1, \omega_2),$$

where $\omega_1, \omega_2$ are some frequencies. Since $\pi|_L : L \to \mathbb{T}^2$ is a diffeomorphism, we take $\varphi_1, \varphi_2$ as coordinates on the base $\mathbb{T}^2$. We shall keep for them the same notations. The following lemma holds

**Lemma 2.1.** In the coordinates $\varphi_1, \varphi_2$ the metric $ds^2$ has the form

$$ds^2 = a(\varphi_1, \varphi_2)d\varphi_1^2 + 2bd\varphi_1d\varphi_2 + d\varphi_2^2,$$

where $b \in \mathbb{R}$ is a constant and $a(\varphi_1, \varphi_2)$ is a positive periodic function.
Proof. Let us assume that in the coordinates $\varphi_1, \varphi_2$ the metric $ds^2$ has the form

$$ds^2 = a(\varphi_1, \varphi_2)d\varphi_1^2 + 2b(\varphi_1, \varphi_2)d\varphi_1d\varphi_2 + c(\varphi_1, \varphi_2)d\varphi_2^2.$$ 

According to our choice of coordinates $(\varphi_1, \varphi_2)$ the curves, defined in the parametric form $(\varphi_1 = \varphi_{10}, \varphi_2 = \varphi_{20} + \tau)$, are geodesics. Since the length of a velocity vector of these geodesics is one, we get $c(\varphi_1, \varphi_2) \equiv 1$. Moreover, we have

$$< \pi_*(s\text{grad}H), \pi_*(s\text{grad}F) > = b\omega_1 + \omega_2,$$

on the other hand it can be easily checked using the homogeneity of $F$, that on $L$ the following holds:

$$< \pi_*(s\text{grad}H), \pi_*(s\text{grad}F) > = \pi_*(\text{dq} (s\text{grad}F)) = p_1\frac{\partial F}{\partial p_1} + p_2\frac{\partial F}{\partial p_2} = nF = nf = n.$$

Here $\text{dq}$ stands for the canonical one form in the cotangent bundle. Thus $b\omega_1 + \omega_2 = nf = n$, and consequently, $b = (n - \omega_2)/\omega_1$ is a constant.

Let’s introduce new coordinates $(t, x)$:

$$\varphi_1 = t, \varphi_2 = x - bt.$$ 

Therefore, in the new coordinates the metric takes the form

$$ds^2 = g^2 dt^2 + dx^2,$$

where $g^2 = a - b^2$. In the coordinates $t, x$ the vector field $\pi_*(s\text{grad}F)$ becomes equal to $(\omega_1, \omega_2 + b\omega_1) = (\omega_1, n)$.

Write the integral $F$ with respect to these coordinates:

$$F = \sum_{k=0}^{n} \frac{a_k(t, x)}{g^{n-k}} p_1^{n-k} p_2^k,$$

where $p_1, p_2$ are the conjugate momenta to $\dot{t}, \dot{x}$ respectively. Then obviously for the torus $L$ we have $L = \{p_1 = 0, p_2 = 1\}$ and so $F|_L = a_n \equiv 1$. Moreover, partial derivatives of $F$ are easily computed on $L$ to be:

$$\frac{\partial F}{\partial p_1} |_L = \frac{a_{n-1}}{g}, \quad \frac{\partial F}{\partial p_2} |_L = na_n = n.$$

Compare these values with $\pi_*(s\text{grad}F) = (\omega_1, n)$, we conclude

$$\frac{a_{n-1}}{g} = \omega_1.$$ 

In addition, we shall assume that $\omega_1 = 1$. Indeed, this can be achieved easily by a rescaling the $t$ variable, $t \to rt$ for some constant $r$. In order to complete the proof of Theorem 1.1 one has to write the condition $\{F, H\} = 0$ explicitly and in this way to get system (2). We omit this computation. Notice that by the very construction of the coordinates $t, x$ it follows that for any function $f$ on the torus $\mathbb{T}^2$ it can be written as a periodic function $f(\varphi_1, \varphi_2)$, which equals $f(t, x - bt)$. Therefore, $f$ becomes periodic in $x$ and ($\text{qua}$) periodic in $t$. Theorem 1.1 is proved.
3. Riemann invariants

In this Section we show that the system (2) can be represented in Riemann invariants in the hyperbolic domain by change of field variables \((a_0, \ldots, a_n) \rightarrow (r_1, \ldots, r_n)\).

Let’s fix the energy level of the Hamiltonian \(H = \frac{1}{2}(p_1^2 + p_2^2) = \frac{1}{2}\). Assume that \(p_1 = g \cos \varphi, p_2 = \sin \varphi\), where \(\varphi\) is an angular coordinate in the fibre. Then \(F = F(t, x, \varphi)\) becomes a trigonometric polynomial. We have

\[
F = F(t, x, \varphi) = \sum_{k=0}^{n} a_k \cos^{n-k} \varphi \sin^k \varphi; \quad a_{n-1} = g, a_n = 1. \tag{4}
\]

\[
\frac{dF}{d\tau} = F_t \dot{t} + F_x \dot{x} + F_\varphi \dot{\varphi} = F_t \frac{\cos \varphi}{g} + F_x \sin \varphi + F_\varphi \dot{\varphi} = 0. \tag{5}
\]

**Lemma 3.1.** The following equality holds

\[
\chi_A(\lambda) = -g^{n-1} \frac{\cos^n \varphi}{\cos^n \varphi} F_\varphi(\varphi),
\]

where \(\chi_A\) is the characteristic polynomial of the matrix \(A\) with the relation \(\lambda = g \tan \varphi\).

The proof can be easily obtained by a direct verification. Remarkably this lemma gives a very clear geometric meaning of the eigenvalues of the matrix \(A(U)\) of our system. They correspond precisely to those \(\varphi\) where the derivative \(F_\varphi\), i.e where the invariant tori of the geodesic flow are tangent to the fibres.

With the help of the lemma and the equations (4),(5) one can easily write the system (2) in the form of Riemann invariants. Let \(\lambda_i, i = 1, \ldots, n\) be \(n\) distinct real eigenvalues of \(A(U)\). Define

\[
r_i = F(\varphi_i), \quad \varphi_i = \arctan(\lambda_i/g), \quad \varphi_i \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ for } i = 1, \ldots, n.
\]

It follows from the lemma that \(F'_\varphi(\varphi_i)\) vanishes and so by the equation (5) it follows that

\[
(r_i)_t + \lambda_i (r_i)_x = 0, \quad i = 1, \ldots, n.
\]

Moreover formula (4) implies that \((r_1, \ldots, r_n)\) are regular coordinates, since the Jacobian matrix for this change of variables equals

\[
\left(\frac{\partial r_i}{\partial a_k}\right) = \left(\cos^{n-k} \varphi_i \sin^k \varphi_i\right), \quad i = 1, \ldots, n; \quad k = 0, \ldots, n - 1
\]

which is non-degenerate matrix if \(\varphi_i \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\) are distinct.

4. Conservation laws

In this Section we show that the system (2) can be represented in the form of the conservation laws. The method of proof of this statement is based on the work \[5\]. Let us first rewrite Hamiltonian system of the geodesic flow as a system with 1,5 degrees of freedom away from the invariant torus \(L\). This can be done as follows. Any geodesic \(\gamma(\tau)\) different from the "vertical" ones
\{t = \text{const}\} has to have \(\frac{dt}{d\tau} \neq 0\) and so can be written as a graph \(\{x = x(t)\}\).

The length functional can be rewritten in the following way:

\[
S(\gamma) = \int |\dot{\gamma}| d\tau = \int \sqrt{g^2 \left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2} d\tau = \int \sqrt{(x')^2 + g^2} = \int L dt,
\]

where \(L(x', x, t) = \sqrt{(x')^2 + g^2}\) where we write (untraditionally) \(x' = \frac{dx}{dt}, \dot{\gamma} = \frac{dx}{d\tau}\). Then the momenta variable for \(x'\) is

\[
p = \frac{\partial L}{\partial x'} = \frac{\dot{x'}}{\sqrt{(x')^2 + g^2}}.
\]

Then Legendre transform for \(L\) is

\[
H(p, x, t) = px' - L = -g \sqrt{1 - p^2}.
\]

So Hamiltonian equations for geodesic flow get the form:

\[
x' = \frac{\partial H}{\partial p} = \frac{gp}{\sqrt{1 - p^2}},
\]

\[
p' = \frac{\partial H}{\partial x} = g x \sqrt{1 - p^2}.
\]

Notice that in coordinates \((p, x, t)\) formula (4) of the previous section for \(F\) gets the form

\[
F(p, x, t) = \sum_{k=0}^{n} a_k (1 - p^2)^{\frac{k}{2}} p^{n-k},
\]

where the dependence on \((t, x)\) enters through the coefficients only.

Assume \(L_1\) is any torus which is invariant under the flow given as a graph of a function \(p = f(t, x)\). Then the invariance condition means that the 1-form \(f_t dt + f_x dx - dp\) vanishes on the Hamiltonian vector field \(\text{grad} H\). Therefore we have:

\[
f_t + f_x \frac{\partial H}{\partial p} + \frac{\partial H}{\partial x} = 0,
\]

which is equivalent to

\[
\frac{\partial f}{\partial t} + \frac{\partial}{\partial x} H(f(t, x), x, t) = 0. \tag{6}
\]

Equation (6) implies that every invariant torus which projects diffeomorphically to the base produces in fact a conservation law for our quasi-linear system (2). More precisely, let us choose \(n\) disjoint tori \(L_1, \ldots, L_n\) in the neighborhood of the torus \(L\), lying in regular levels \(\{F = c_i\}\) such that their projections to \(\mathbb{T}^2\) are diffeomorphisms.

Each of these torii \(L_i\) is a graph of a function \(f_i\), \(p = f_i(a_0, \ldots, a_{n-1})\). They satisfy the equation

\[
F(a_0, \ldots, a_{n-1}, p) = c_i. \tag{7}
\]

Notice that \(f_i\) depend on \((t, x)\) implicitly through the coefficients. Using (6) for \(f_i\) we get the conservation laws:

\[
\frac{\partial f_i}{\partial t} + \frac{\partial}{\partial x} (H(f_i, x, t)) = 0.
\]

Let’s show that the transition from the variables \(a_0, \ldots, a_{n-1}\) to the variables \(f_1, \ldots, f_n\) is a diffeomorphism. Introduce for convenience

\[
u_1 = a_0, \ldots, u_{n-1} = a_{n-2}, u_n = a_n = g.
\]
We have

\[ F(u_1, \ldots, u_n, f_i) = c_i, \]

Therefore

\[ \frac{\partial F}{\partial u_j} + \frac{\partial F}{\partial p} \bigg|_{p=f_i, \partial u_j} = 0 \]

or equivalently in the matrix form we have:

\[
A + B \begin{pmatrix}
\frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_n}
\end{pmatrix} = 0.
\]

where

\[
A = \begin{pmatrix}
\frac{\partial F}{\partial u_1}(f_1) & \cdots & \frac{\partial F}{\partial u_n}(f_1) \\
\vdots & \ddots & \vdots \\
\frac{\partial F}{\partial u_1}(f_n) & \cdots & \frac{\partial F}{\partial u_n}(f_n)
\end{pmatrix},
B = \begin{pmatrix}
\frac{\partial F}{\partial p} \bigg|_{p=f_1} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \frac{\partial F}{\partial p} \bigg|_{p=f_n}
\end{pmatrix}.
\]

Note that the matrix \( A = \left( \frac{\partial F}{\partial u_j}(f_i) \right) \) is essentially Vandermonde matrix, in particular it is non-degenerate since the torii \( L_i \) are disjoint. The diagonal matrix \( B \) in the formula is non-degenerate either, because the torii \( L_i \) were chosen to be regular graphs. Therefore,

\[ \frac{\partial (f_1, \ldots, f_n)}{\partial (u_1, \ldots, u_n)} \neq 0. \]

This completes the proof of Theorem 2.

In practice it’s difficult to reverse the equation (7) explicitly. Therefore, instead of solving of the equation (7) we act in the following way. Torus \( L \) has the the equation \( p = 1 \) in the energy level. The value of the integral \( F \) equals 1 on \( L \). For a small parameter \( \varepsilon \) solve the equation

\[ F = \sum_{k=0}^{n} a_k (1 - p^2)^k p^{n-k} = 1 + \varepsilon \]

by substituting the power series

\[ p = 1 - G_2 \varepsilon^2 - G_3 \varepsilon^3 \ldots. \]

Then it follows from (6) that all the coefficients \( G_i(a_0, \ldots, a_{n-1}) \) are conservation laws.

**Example 1.** Let \( n = 3 \). In this case \( U = (a_0, a_1, a_2)^T \), with \( a_2 \equiv g, a_3 \equiv 1 \). The matrix \( A(U) \) has the form:

\[
A(U) = \begin{pmatrix}
0 & 0 & a_1 \\
a_2 & 0 & 2a_2 - 3a_0 \\
0 & a_2 & 3a_3 - 2a_1
\end{pmatrix}
\]

The integral has the form:

\[ F = a_0(1 - p^2)^{\frac{1}{3}} + a_1(1 - p^2)p + a_2(1 - p^2)^{\frac{1}{3}} + a_3 p^3. \]

Write

\[ p = 1 - G_2 \varepsilon^2 - G_3 \varepsilon^3 - G_4 \varepsilon^4 \ldots \]
and substitute into the expression for $F$ in (8), and equate to $1 + \varepsilon$. Computing coefficients for $\varepsilon, \varepsilon^2, \varepsilon^3$ and equating them to $1, 0, 0$ respectively one gets the following conservation laws:

$$G_2 = \frac{1}{2g^2}, G_3 = \frac{3a_3 - 2a_1}{2g^4}, G_4 = \frac{9}{8g^4} + \frac{5(3a_3 - 2a_1)^2}{8g^6} - \frac{a_0}{g^5}.$$ 

One can prove in general for any $n$, that all the conservation laws obtained in this way are rational functions of the field variables. It is not an easy exercise to verify by hands that these are in fact conservation laws.

5. Concluding remarks

1. An interesting direction for further study would be to understand algebraic properties of the system. Notice that the matrix $A(U)$ depends linearly on the components of $U$, this case was studied first by Gelfand, Dorfman see [10].

2. It is important to understand what the analysis along characteristics can give for smooth solutions of the system.

3. It is not clear to us if the generalized hodograph method can give a nontrivial information for system (2).

4. Let us mention an interesting connection between real eigenvalues of the matrix $A(U)$ and the so called separatrix chains introduced in [4]. The geometric language seems to be of tight relation with analytic properties of our system.

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