Structure of Helicity and Global Solutions of Incompressible Navier–Stokes Equation

ZHEN LEI, FANG-HUA LIN & YI ZHOU

Abstract

In this paper we derive a new energy identity for the three-dimensional incompressible Navier–Stokes equations by a special structure of helicity. The new energy functional is critical with respect to the natural scalings of the Navier–Stokes equations. Moreover, it is conditionally coercive. As an application we construct a family of finite energy smooth solutions to the Navier–Stokes equations whose critical norms can be arbitrarily large.

1. Introduction

The question of whether a solution of the three-dimensional incompressible Navier–Stokes equations can develop a finite time singularity from smooth initial data with finite energy is one of the Millennium Prize problems [3]. The only known coercive a priori estimate is the Leray–Hopf energy estimate which implies that the three-dimensional Navier–Stokes equations are supercritical with respect to its natural scalings. The latter may capture the essence of difficulties of this long standing open problem.

In this paper, by virtue of a special structure of Helicity, we derive a new a priori energy estimate which is critical with respect to the natural scalings for the three-dimensional Navier–Stokes equations. This new energy functional is coercive for a class of initial data. Based on this a priori estimate, a family of finite energy global smooth and large solutions can then be constructed. Current known examples of large smooth solutions to the three-dimensional Navier–Stokes equations often assume both the axial symmetricity and the vanishing of the swirl component of the velocity, see [4, 5, 7].

Let us recall that the incompressible Navier–Stokes equations in $\mathbb{R}_+ \times \mathbb{R}^3$ are:

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p &= \nu \Delta u, \quad t > 0, x \in \mathbb{R}^3, \\
\nabla \cdot u &= 0, \quad t > 0, x \in \mathbb{R}^3,
\end{aligned}
$$

(1.1)
where \( u \) is the velocity field of the fluid, \( p \) is the scalar pressure and the constant \( v \) is the viscosity. To solve the Navier–Stokes equations (1.1) in \( \mathbb{R}_+ \times \mathbb{R}^3 \), one assumes that the initial data

\[
   u(0, x) = u_0(x)
\]

are divergence-free and possess certain regularity.

The known \textit{a priori} Leray–Hopf energy estimate satisfied by classical solutions of (1.1) is as follows:

\[
   \sup_{t>0} \|u(t, \cdot)\|_{L^2} \leq \|u_0\|_{L^2}, \quad \nu \int_0^\infty \|\nabla u(t, \cdot)\|_{L^2}^2 \, dt \leq \frac{1}{2} \|u_0\|_{L^2}^2. \tag{1.2}
\]

Recall the natural scalings of the Navier–Stokes equations: if \((u, p)\) solves (1.1), so does \((u^\lambda, p^\lambda)\) for any \( \lambda > 0 \), where

\[
   u^\lambda(t, x) = \lambda u(\lambda t, \lambda x), \quad p^\lambda(t, x) = \lambda^2 p(\lambda t, \lambda x). \tag{1.3}
\]

As usual, we assign each \( x_i \) a positive dimension 1, \( t \) a positive dimension 2, \( u \) a negative dimension \(-1\) and \( p \) a negative dimension \(-2\). A simple dimensional analysis shows that all energy norms in (1.2) have positive dimensions, and thus the Navier–Stokes equations are \textit{supercritical} with respect to the natural scalings. An example of a dimensionless norm is \( L^\infty_t(H^1) \), and it will be related to discussions below.

Denote

\[
   D = \sqrt{-\Delta}.
\]

Our starting point is the following new energy identity.

\textbf{Theorem 1.1.} (Structure of Helicity) Let \( u \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)) \) solve the incompressible Navier–Stokes equation (1.1). For each \( t \in [0, T) \), decompose \( u(t, \cdot) \) as in (2.1). Then one has

\[
   E^c(u_+) = E^c(u_-) + c_0, \quad \forall \ t \in [0, T), \tag{1.4}
\]

where the constant \( c_0 \) is given by

\[
   c_0 = \frac{1}{2} \left( \|u_0+\|_{H^1}^2 - \|u_0-\|_{H^1}^2 \right),
\]

and the critical energy \( E^c(u) \) is defined as

\[
   E^c(u) = \frac{1}{2} \left\| D^\frac{3}{2} u(t, \cdot) \right\|_{L^2}^2 + \nu \int_0^t \left\| D^\frac{3}{2} \nabla u(s, \cdot) \right\|_{L^2}^2 \, ds.
\]