 Blow up analysis for Boltzmann-Poisson equation in Onsager’s theory for point vortices with multi-intensities

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Abstract

In this paper we consider the minimizing sequence for some energy functional of an elliptic equation associated with the mean field limit of the point vortex distribution one-sided Borel probability measure. If such a sequence blows up, we derive some estimate which is related to the behavior of solution near the blow-up point. Moreover, we study the two-intensities case to consider the sufficient condition for this estimate. Our main results are new for the standard mean field equation as well.

1 Introduction

Motivated by several mean field equations recently derived in the context of Onsager’s statistical mechanics description of turbulence [13], we consider the Boltzmann-Poisson equation:

\[-\Delta v = \lambda \int_{I^+} \frac{ae^{\alpha v}}{\int_{\Omega} e^{\alpha v}} P(\alpha) \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega,\]

where \(\Omega \subset \mathbb{R}^2\) is a smooth bounded domain, \(v\) denotes the stream function, \(\lambda > 0\) is a constant related to the inverse temperature and \(P(\alpha)\) is a Borel probability measure on \(I^+ = [0, +1]\) denoting the distribution of the circulations. A formal derivation of \((1)\) is provided in [5, 21].

If \(P(\alpha) = \delta_{+1}(\alpha)\), corresponding to the case where all vortices have the same intensity and orientation, equation \((1)\) reduces to the Liouville type equation

\[-\Delta v = \lambda \frac{e^{v}}{\int_{\Omega} e^{v}} \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega.\]

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Equation (2) is mathematically justified by the minimizing free energy method in the canonical formulation [2, 9], and its mathematical analysis has revealed the quantized blow-up mechanism of sequences of solutions, see, e.g., [1, 11, 12, 24, 25, 26].

Especially, the Y. Y. Li type estimate which is the behavior of blow-up solutions for (2) near the blow-up points has been studied [7, 10]. Let \( \Omega \) be a unit ball and \((\lambda_k, v_k)\) satisfy (2) without boundary condition and

\[
\lambda_k \to \lambda_0 \geq 0, \quad \|v_k\|_{\infty} = v_k(x_k) \to +\infty, \quad x_k \to 0 \in \Omega
\]

as \( k \to +\infty \) where \( x_k \) is the maximizer of \( v_k \) and 0 is the only blow-up point of \( v_k \). Then the following result holds:

**Theorem 1.1.** ([7], Theorem 0.3) Under the blow-up case (3), suppose that there exists a constant \( C > 0 \) such that

\[
\max_{\partial \Omega} v_k - \min_{\partial \Omega} v_k \leq C. \tag{4}
\]

Then it holds that

\[
v_k(x) - v_k(x_k) = -2 \log \left( 1 + \frac{\lambda_k}{8} \frac{e^{v_k(x_k)}}{\int_{\Omega} e^{v_k}} |x - x_k|^2 \right) + O(1) \tag{5}
\]

as \( k \to \infty \) uniformly \( x \in B_r(0) \) with some \( 0 < r < 1 \).

**Remark 1.1.** We can understand (4) as boundary condition in Theorem 1.1 and there are no need to suppose the zero Dirichlet boundary condition for Theorem 1.1.

Y. Y. Li type estimate of (5) is valid for the computation of the Leray-Schauder degree for (2), asymptotic non-degeneracy of multi-point blowup solutions to the Liouville Gel’fand problem and the Trudinger-Moser inequality with the extremal case, see [7, 13, 23].

It is known that there are two proofs for Theorem 1.1. The first one which is the original way of Y. Y. Li, is the combination with some conformal transformation and the moving plane argument [7]. The other one is the argument of C. S. Lin [10]. In [10], we can control the mass of bubble in the quantized blow-up argument thanks to the boundary condition (4). By such a information of mass and a result of [3], we obtain the mass identity which is described precisely later, and this identity plays an essential role in the proof of Theorem 1.1.

Comparing with the case \( P(da) = \delta_{+1}(da) \), however, there are no works of describing the Y. Y. Li type estimate for mean field equation in the multi-intensities case. Our aim in this paper is to derive the variant of Y. Y. Li type estimate in the multi-intensities case [1]. To achieve this, we shall employ the argument of [23]. Here, we introduce some notations and assumptions to describe our results.

Setting

\[
J_\lambda(v) = \frac{1}{2} \|\nabla v\|^2_2 - \lambda \int_{I^+} \log \left( \int_{\Omega} e^{\alpha v} dx \right) P(da), \quad v \in H^1_0(\Omega),
\]
then equation (1) is the Euler-Lagrange equation of this functional. The extremal value of $\lambda$ for $\inf_{v \in H^1_0(\Omega)} J_\lambda(v) > -\infty$ is defined by

$$\overline{\lambda} := \sup \left\{ \lambda > 0 \mid \inf_{v \in H^1_0(\Omega)} J_\lambda(v) > -\infty \right\}. \quad (6)$$

This extremal value is actually given by [16], that is,

$$\overline{\lambda} = \inf \left\{ \frac{8\pi \mathcal{P}(K)}{\left( \int_K \alpha \mathcal{P}(d\alpha) \right)^2} \mid K \subset \text{supp } \mathcal{P} \right\}. \quad (7)$$

where $\text{supp } \mathcal{P} = \{ \alpha \in I_+ \mid \mathcal{P}(N) > 0 \text{ for any open neighborhood } N \text{ of } \alpha \}$. Then it holds that

$$\lambda < \overline{\lambda} \Rightarrow \inf_{v \in H^1_0(\Omega)} J_\lambda(v) > -\infty,$$

$$\lambda > \overline{\lambda} \Rightarrow \inf_{v \in H^1_0(\Omega)} J_\lambda(v) = -\infty.$$

Therefore, given $\lambda_k \uparrow \overline{\lambda}$, we have a minimizer $v_k \in H^1_0(\Omega)$ of $J_{\lambda_k}$, and $(\lambda_k, v_k)$ satisfy (1). For the solution sequence to (1), the following Brezis-Merle type blow-up alternatives holds [12, 17, 19]:

**Proposition 1.1.** Let $(\lambda_k, v_k)$ be a solution sequence of (1) with $\lambda_k > 0$ and $\lambda_k \to \lambda_0$. Assume that $S \cap \partial \Omega = \emptyset$ holds, where $S = \{ x_0 \in \Omega \mid \text{there exists } x_k \in \Omega \text{ such that } x_k \to x_0 \text{ and } v_k(x_k) \to \infty \}$. Then, passing to a subsequence, we have the following alternatives.

(I) Compactness: $\limsup_{k \to \infty} \| v_k \|_\infty < +\infty$, that is, $S = \emptyset$.

Then, there exists $v \in H^1_0(\Omega)$ such that $v_k \to v$ in $H^1_0(\Omega)$ and $v$ is a solution of (1).

(II) Concentration: $\limsup_{k \to \infty} \| v_k \|_\infty = +\infty$, that is, $S \neq \emptyset$.

Then, $S$ is finite and there exists $0 \leq s(x) \in L^1(\Omega) \cap L^\infty_{\text{loc}}(\Omega \setminus S)$ such that

$$\mu_k(dx) \equiv \lambda_k \int_{I_+} \frac{\alpha e^{\alpha v} \mathcal{P}(d\alpha)}{e^{\alpha v} dx} \to s(x) dx + \sum_{x_0 \in S} m(x_0) \delta_{x_0}(dx) \quad \text{in } \mathcal{M}(\Omega),$$

with $m(x_0) \geq 4\pi$ where $\delta_{x_0}$ denotes the Dirac measure centered at $x_0$ and $\mathcal{M}(\Omega)$ is the space of measures identified with the dual space of $C_0(\Omega)$.

**Remark 1.2.** If we apply Proposition 1.1 to the solution $(\lambda_k, v_k)$ of (3), then it is known that we get the more detail of the blow-up information. For example, $s(x) \equiv 0$ in $\Omega$, which we call residual vanishing and $m(x_0) \in 8\pi \mathbb{N}$ for $x_0 \in S$ [11, 17]. Since the minimizing sequence $(\lambda_k, v_k)$ satisfies (1), we can apply Proposition 1.1 to it if we get the condition [16]. In general, thanks to a result in [6], p.223, (8) follows for solution sequence to (1). It is enough to check the following statement:
Lemma 1.1. Let \((\lambda_k, v_k)\) be a solution sequence to (1) with \(\lambda_k > 0\) and \(\lambda_k \to \lambda_0\). There exists a tubular neighborhood \(\Omega_\delta\) of \(\partial \Omega\) and a constant \(C > 0\) such that \(\|v_k\|_{L^\infty(\Omega_\delta)} \leq C\) for any \(k \in \mathbb{N}\).

The proof of Lemma 1.1 is almost the same as in [20], Lemma 2.5. Therefore, we have
\[
S \cap \partial \Omega = \emptyset, \quad \#S < \infty
\]
for minimizing sequence \((\lambda_k, v_k)\). In the following, we consider the minimizing sequence \((\lambda_k, v_k)\) for \(J_{\lambda_k}\) in the Concentration case, that is,
\[
\lambda_k \to \bar{\lambda}, \quad \|v_k\|_\infty = v_k(x_k) \to \infty \quad \text{as} \quad k \to \infty
\]
where \(x_k\) is the maximizer of \(v_k\). Indeed, if \(\mathcal{P}\) is the one-intensity or two-intensity case and \(\Omega\) is a ball then (11) is justified [2, 18]. By (10), up to a subsequence, \(x_k \to x_0 \in \Omega\).

Next, we define
\[
w_{k, \alpha}(x) := \alpha v_k(x + x_k) - \log \int_\Omega e^{\alpha v_k}, \quad k \in \mathbb{N}, \quad \alpha \in I_+ \setminus \{0\} \quad \text{and} \quad w_k(x) := w_{k, 1}(x).
\]
Then we have
\[
-\Delta w_k = \lambda_k \int_{I_+} \alpha e^{w_{k, \alpha}} \mathcal{P}(d\alpha) \quad \text{in} \quad \Omega, \quad \int_\Omega e^{w_{k, \alpha}} = 1, \quad (12)
\]
and we shall show that for \(\alpha \in I_+ \setminus \{0\}, \quad w_k(0) \geq w_{k, \alpha}(0) \to +\infty, \quad k \to \infty.\)

Furthermore, setting
\[
\tilde{w}_{k, \alpha}(x) := w_{k, \alpha}(\sigma_k x) + 2 \log \sigma_k, \quad \sigma_k = e^{-w_k(0)/2} \to 0, \quad \tilde{w}_k := \tilde{w}_{k, 1}
\]
then, we obtain
\[
-\Delta \tilde{w}_k = \tilde{f}_k, \quad \tilde{w}_k(x) \leq \tilde{w}_k(0) = 0 \quad \text{in} \quad B_{4R_0 \sigma_k^{-1}},
\]
where \(\tilde{f}_k := \lambda_k \int_{I_+} \alpha e^{\tilde{w}_{k, \alpha}} \mathcal{P}(d\alpha), \quad 4R_0 = dist(x_0, \partial \Omega).\) By elliptic regularity arguments, we can show that there exists \(\tilde{w}, \tilde{f} \in C^2(\mathbb{R}^2)\) such that
\[
\tilde{w}_k \to \tilde{w}, \quad \tilde{f}_k \to \tilde{f} \quad \text{in} \quad C^2_{\text{loc}}(\mathbb{R}^2),
\]
and
\[
-\Delta \tilde{w} = \tilde{f} \neq 0, \quad \tilde{w} \leq \tilde{w}(0) = 0, \quad 0 \leq \tilde{f} \leq \bar{\lambda} \int_{I_+} \alpha \mathcal{P}(d\alpha) \quad \text{in} \quad \mathbb{R}^2,
\]
\[
\int_{\mathbb{R}^2} e^{\tilde{w}} \leq 1, \quad \int_{\mathbb{R}^2} \tilde{f} \leq \bar{\lambda} \int_{I_+} \alpha \mathcal{P}(d\alpha).
\]
Then we assume that
\[ \beta_0 := \int_{\mathbb{R}^2} \tilde{f} \, dx = m(x_0), \]
where \( m(x_0) \) is as in (9).

**Remark 1.3.** Since (13) means that the total mass of scaling limit coincides with the local mass of bubble, we call (13) mass identity. Indeed, in Theorem 1.1, the both sides of (13) coincides with \( 8\pi \) by a result of [3, 10].

In addition to (11) and (13), we also assume
\[ \alpha_{\min} > 0 \quad \text{and} \quad \mathcal{P}(\{\alpha_{\min}\}) > 0, \]
where \( \mathcal{P} = \{\alpha \in I_+ \mid \mathcal{P}(N) > 0 \text{ for any open neighborhood } N \text{ of } \alpha\} \) and \( \alpha_{\min} = \inf_{\alpha \in \text{supp}\mathcal{P}} \alpha \). Then the variant of Y. Y. Li type estimate holds:

**Theorem 1.2.** Suppose (11), (13), (14) and \( s(x) \equiv 0 \) as in (9) then it holds that
\[ v_k(x) - v_k(x_k) = -\left( \frac{\beta_0}{2\pi} + o(1) \right) \log \left( 1 + \left( \frac{e^{v_k(x_k)}}{\int_{\Omega} e^{v_k}} \right)^{\frac{1}{2}} |x - x_k| \right) + O(1) \]
as \( k \to \infty \) uniformly in \( B_{R/2}(x_0) \) where \( \beta_0 = \int_{\mathbb{R}^2} \tilde{f} \, dx \).

**Remark 1.4.** \( s(x) \equiv 0 \) which we call residual vanishing, occurs under the suitable assumptions on \( \mathcal{P} \). Indeed, if \( \alpha_{\min} > 1/2 \) then the residual vanishing occurs to the \( (\lambda_k, v_k) \) in (11) ([22], Theorem 3). Moreover, if the residual vanishing occurs to the above \( (\lambda_k, v_k) \) then it follows that
\[ \#S = 1, \quad \lambda = \frac{8\pi}{\left( \int_{I_+} \alpha \mathcal{P}(d\alpha) \right)^2}, \]
see [22], Lemma 3.

**Remark 1.5.** The estimate (15) is weaker than (5). Indeed, if \( \mathcal{P}(d\alpha) = \delta_1(d\alpha) \) then \( \beta_0 = 8\pi \) by Chen-Li [3] and (15) does not correspond to (5). However, by a direct calculation, (15) leads to (9) with the case \( \mathcal{P}(d\alpha) = \delta_1(d\alpha) \) in the meaning of the log function term. Indeed, suppose \( (\lambda_k, v_k) \) satisfy (11) then it holds that
\[ (i) \quad \left( 1 + \left( \frac{e^{v_k(x_k)}}{\int_{\Omega} e^{v_k}} \right)^{\frac{1}{2}} |x - x_k| \right)^2 = \left( 1 + \frac{e^{v_k(x_k)}}{\int_{\Omega} e^{v_k}} |x - x_k|^{\frac{1}{2}} \right) (1 + o(1)) \quad \text{as} \quad k \to +\infty, \]
\[ (ii) \quad \log \left( 1 + \frac{e^{v_k(x_k)}}{\int_{\Omega} e^{v_k}} |x - x_k|^{\frac{1}{2}} \right) = \log \left( 1 + \lambda_k \frac{e^{v_k(x_k)}}{\int_{\Omega} e^{v_k}} |x - x_k|^{\frac{1}{2}} \right) + O(1) \quad \text{as} \quad k \to +\infty, \]
(iii)

\[1 + \frac{\lambda_k e^{v_k(x_k)}}{8} \int_{B_{R_0/2}(x_0)} |x - x_k|^2 = \left(1 + \frac{\lambda_k e^{v_k(x_k)}}{8} \int_{B_{R_0/2}(x_0)} |x - x_k|^2\right) \cdot O(1) \text{ as } k \to +\infty,\]

uniformly \(B_{R_0/2}(x_0)\) as in Theorem 1.2. Applying (i), (ii) and (iii) to (15), we have the form of (5) as \(k \to \infty\).

For the sufficient conditions of Theorem 1.2, we consider the following identity:

\[\int_{\mathbb{R}^2} \hat{f} dx = \lambda \int_{I_+} \alpha \mathcal{P}(d\alpha),\]  

(16)

The above identity implies the following Proposition.

**Proposition 1.2.** Under the assumption of \((\lambda_k, v_k)\) in (11), (16) holds if and only if the residual vanishing occurs and mass identity (13) holds.

Lastly, we derive the identity (16) in the minimizing problem with \(\mathcal{P}(d\alpha)\) two-intensities, that is,

\[\mathcal{P}(d\alpha) = \tau \delta_1(d\alpha) + (1 - \tau) \delta_\gamma(d\alpha),\]  

(17)

where \(\tau, \gamma \in (0, 1)\) and note that

\[\lambda = \begin{cases} \frac{8\pi}{\tau}, & \gamma \leq \frac{\sqrt{\tau}}{1 + \sqrt{\tau}}; \\ \frac{8\pi}{\tau + (1 - \tau)\gamma^2}; & \gamma > \frac{\sqrt{\tau}}{1 + \sqrt{\tau}}. \end{cases}\]  

(18)

The following statements hold under the assumption of \((\lambda_k, v_k)\) in (11):

**Theorem 1.3.** (i) If \(\mathcal{P}(d\alpha)\) is as in (17) and \(\gamma \in (\sqrt{\tau}/(1 + \sqrt{\tau}), 1)\) then the identity (16) holds and the Y. Y. Li type estimate as in (15) also holds.

(ii) If \(\mathcal{P}(d\alpha)\) is as in (17) and \(\gamma \in (0, \sqrt{\tau}/(1 + \sqrt{\tau}))\) then the identity (16) does not hold.

**Remark 1.6.** Proposition 1.2 and Lemma 1.1 follow for the general solution sequence \((\lambda_k, v_k)\), while our main results Theorem 1.2-1.3 describe just for minimizing sequence \((\lambda_k, v_k)\). In particular, to obtain the estimate (15) for the general blow-up solution sequence, we have to assume the identity like (16). In such a case, however, we do not know this identity holds or not. For the proof of Theorem 1.3, we need the property of \(\lambda\). This detail shall be mentioned as Remark 1.4.2 in Section 4.

Our paper is composed of four sections and Appendix. First, we shall discuss the blow-up argument for general \(\mathcal{P}\) as Preliminary in Section 2. Next, we show Theorem 1.2 in Section 3. Lastly, we prove Theorem 1.3 and Proposition 1.2 in Section 4. An auxiliary lemma of Section 2 in Appendix.
2 Preliminary

In this section, we discuss the blow-up argument for \((\lambda_k, v_k)\) in \([11]\) without residual vanishing.

Lemma 2.1. For \(\alpha \in I_+\), we have
\[
\frac{d}{d\alpha} w_{k,\alpha}(0) \geq 0,
\]
where \(w_{k,\alpha}(x) = \alpha v_k(x + x_k) - \log \int_{\Omega} e^{\alpha v_k} \).

Proof. For \(k \) and \(\alpha \in I_+\), we have
\[
\frac{d}{d\alpha} w_{k,\alpha}(0) = v_k(x_k) - \int_{\Omega} v_k e^{\alpha v_k} \geq v_k(x_k) \left( 1 - \frac{\int_{\Omega} e^{\alpha v_k}}{\int_{\Omega} e^{\alpha v_k}} \right) = 0,
\]
recalling that \(x_k\) is the maximizer of \(v_k\).

Henceforth, we put \(w_{k}(x) = w_{k,1}(x)\).

It follows from (19) that
\[
w_{k,1}(0) = \max_{\alpha \in I_+} w_{k,\alpha}(0).
\]

The following Lemma is the starting point of our blow-up analysis.

Lemma 2.2. For every \(\alpha \in I_+ \setminus \{0\}\), it holds that
\[
w_{k,\alpha}(0) = \max_{\alpha \in I_+} w_{k,\alpha}(0) \to +\infty \quad \text{as} \quad k \to \infty.
\]

Proof. Since \(e^{w_{k,\alpha}(0)} = e^{\alpha v_k} \int_{\Omega} e^{\alpha v_k} \geq |\Omega|^{\alpha-1} e^{\alpha w_{k,1}(0)}\) for \(\alpha \in I_+ \setminus \{0\}\), it suffices to show that \(w_k(0) = w_{k,1}(0) \to +\infty\) as \(k \to +\infty\). Suppose \(w_k(0) = O(1)\) as \(k \to +\infty\), from (19) we have \(w_{k,\alpha}(0) = O(1)\) as \(k \to +\infty\) for all \(\alpha \in I_+ \setminus \{0\}\). Therefore the right-hand side on the equation (1) is uniformly bounded. This contradicts to (11) from elliptic regularity arguments.

Putting
\[
\tilde{w}_{k,\alpha}(x) := w_{k,\alpha}(\sigma_k x) + 2 \log \sigma_k, \quad \sigma_k = e^{-w_k(0)/2} \to 0, \quad \tilde{w}_k := \tilde{w}_{k,1}.
\]

Then, we have
\[
-\Delta \tilde{w}_k = \tilde{f}_k, \quad \tilde{w}_k(x) \leq \tilde{w}_k(0) = 0 \quad \text{in} \quad B_{R_0 \sigma_k^{-1}(0)},
\]
\[
\int_{B_{R_0 \sigma_k^{-1}(0)}} e^{\tilde{w}_{k,\beta}} \leq 1, \quad \int_{B_{R_0 \sigma_k^{-1}(0)}} \tilde{f}_k \leq \lambda_k \int_{I_+} \beta \mathcal{P}(d\beta),
\]
where
\[
\tilde{f}_k := \lambda_k \int_{I_+} \beta e^{\tilde{w}_{k,\beta}} \mathcal{P}(d\alpha), \quad 4R_0 = \text{dist}(x_0, \partial \Omega).
\]

We shall use a fundamental fact of which proof is provided in Appendix.
Lemma 2.3. Given \( f \in L^1 \cap L^\infty(\mathbb{R}^2) \), let
\[
z(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(y) \log \frac{|x - y|}{1 + |y|} dy.
\]
Then, it holds that
\[
\lim_{|x| \to +\infty} \frac{z(x)}{\log |x|} = \frac{1}{2\pi} \int_{\mathbb{R}^2} f.
\]
The following lemma is also classical (see [15] p. 130).

Lemma 2.4. If \( \phi = \phi(x) \) is a harmonic function on the whole space \( \mathbb{R}^2 \) such that
\[
\phi(x) \leq C_1 (1 + \log |x|), \quad x \in \mathbb{R}^2 \setminus B_1
\]
then it is a constant function.

Proposition 2.1. There exists \( \tilde{w}, \tilde{f} \in C^2(\mathbb{R}^2) \) such that
\[
\tilde{w}_k \to \tilde{w}, \quad \tilde{f}_k \to \tilde{f} \quad \text{in} \quad C^2_{\text{loc}}(\mathbb{R}^2), \quad (25)
\]
and
\[
-\Delta \tilde{w} = \tilde{f} \neq 0, \quad \tilde{w} \leq \tilde{w}(0) = 0, \quad 0 \leq \tilde{f} \leq \tilde{\lambda} \int_{I_+} \beta \mathcal{P}(d\alpha) \quad \text{in} \quad \mathbb{R}^N, \quad (26)
\]
\[
\int_{\mathbb{R}^2} e^{\tilde{w}} \leq 1, \quad \int_{\mathbb{R}^2} \tilde{f} \leq \tilde{\lambda} \int_{I_+} \beta \mathcal{P}(d\beta).
\]
In addition, for \( x \in \mathbb{R}^2 \),
\[
\tilde{w}(x) \geq -\frac{\beta_0}{2\pi} \log (|x| + 1) + \frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{f}(y) \log \frac{|y|}{1 + |y|} \quad (27)
\]
where \( \beta_0 = \int_{\mathbb{R}^2} \tilde{f}(y) dy \).

Proof. We have
\[
\tilde{w}_{k,\beta}(x) = \beta \tilde{w}_k(x) + (w_{k,\beta}(0) - w_k(0)) \quad (28)
\]
for any \( \beta \in I_+ \setminus \{0\} \), and also
\[
\tilde{w}_k \leq \tilde{w}_k(0) = 0, \quad w_{k,\beta}(0) \leq w_k(0), \quad \beta \in I_+ \setminus \{0\} \quad (29)
\]
by \([19]\). Hence \( \tilde{f}_k = \tilde{f}_k(x) \) satisfies
\[
0 \leq \tilde{f}_k(x) \leq \lambda_k \int_{I_+} \beta \mathcal{P}(d\beta) \quad \text{in} \quad B_{R_0 \sigma_k^{-1}}(0) \quad (30)
\]
Fix \( L > 0 \) and decompose \( \tilde{w}_k, k \gg 1 \), as \( \tilde{w}_k = \tilde{w}_{1,k} + \tilde{w}_{2,k} \) where \( \tilde{w}_{j,k}, j = 1, 2 \), are the solutions to
\[
-\Delta \tilde{w}_{1,k} = \tilde{f}_k \quad \text{in} \quad B_L, \quad \tilde{w}_{1,k} = 0 \quad \text{on} \quad \partial B_L,
\]
\[
-\Delta \tilde{w}_{2,k} = 0 \quad \text{in} \quad B_L, \quad \tilde{w}_{2,k} = \tilde{w}_k \quad \text{on} \quad \partial B_L.
\]
First, by (30) and elliptic regularity arguments, there exists $C_{1,L} > 0$ such that
\[ 0 \leq \tilde{w}_{1,k} \leq C_{1,L} \quad \text{on} \quad B_L. \]
Next it follows from $\tilde{w}_k \leq 0$ that
\[ \tilde{w}_{2,k} \leq 0 \quad \text{on} \quad B_L. \]
Hence $\tilde{w}_{2,k} = \tilde{w}_{2,k}(x)$ is a negative harmonic function in $B_L$. Then the Harnack inequality yield $C_{2,L} > 0$ such that
\[ \tilde{w}_{2,k} \geq -C_{2,L} \quad \text{in} \quad B_{L/2}. \]
We thus end up with
\[ -C_{2,L} \leq \tilde{w}_k \leq \tilde{w}_k(0) = 0 \quad \text{in} \quad B_{L/2}, \tag{31} \]
and then standard elliptic regularity arguments assure the limit (25) and (26) thanks to (30) and (31).
If $\tilde{f} \equiv 0$ then
\[ -\Delta \tilde{w} = 0, \quad \tilde{w} \leq \tilde{w}(0) = 0 \quad \text{in} \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^{\tilde{w}} \leq 1, \]
which is impossible by the Liouville theorem, and hence $\tilde{f} \neq 0$.
Since $\tilde{f} \in L^1 \cap L^\infty(\mathbb{R}^2)$, the function
\[ \tilde{z}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{f}(y) \log \frac{|x - y|}{1 + |y|} \, dy \tag{32} \]
is well-defined, and satisfies
\[ \frac{\tilde{z}(x)}{\log |x|} \rightarrow \frac{\beta_0}{2\pi} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{f} \quad \text{as} \quad |x| \to \infty \tag{33} \]
by Lemma 2.3. Also (33) implies
\[ -\Delta \tilde{w} = \tilde{f}, \quad -\Delta \tilde{z} = -\tilde{f}, \quad \tilde{w} \leq \tilde{w}(0) = 0 \quad \text{in} \quad \mathbb{R}^2, \]
\[ \tilde{z}(x) \leq \left( \frac{\beta_0}{2\pi} + 1 \right) \log |x|, \quad x \in \mathbb{R}^2 \setminus B_r \]
for some $r > 0$ by (33). Hence we obtain $\tilde{w} \equiv \tilde{w} + \tilde{z} \equiv$ constant by Lemma 2.4. Since $\tilde{w}(0) = 0$ it holds that
\[ \tilde{w}(x) = -\tilde{z} + \tilde{z}(0). \tag{34} \]
Now we note
\[ \tilde{z}(x) \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{f} \log \frac{|x|}{1 + |y|} \, dy \]
\[ \leq \log(1 + |x|) \cdot \frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{f} = \frac{\beta_0}{2\pi} \log(1 + |x|) \]
by $\tilde{f} \geq 0$. Hence, $\tilde{w}(x) \geq -\frac{\beta_0}{2\pi} \log(1 + |x|) + \tilde{z}(0)$, and the proof is completed. \(\square\)
Next we focus on the quantity $\beta_0 = \int_{\mathbb{R}^2} \tilde{f}$.

**Lemma 2.5.** For any bounded open set $\omega \subset \mathbb{R}^2$, there exists $\tilde{\zeta}^\omega = \tilde{\zeta}^\omega(d\beta) \in \mathcal{M}(I_+)$ such that

$$\left( \int_\omega e^{\tilde{w}_{k,\beta}} \, dx \right) P(d\beta) \rightharpoonup \tilde{\zeta}^\omega(d\beta) \quad \text{in} \quad \mathcal{M}(I_+). \quad (35)$$

Furthermore, there exists $\tilde{\psi}^\omega \in L^1(I_+, P)$ such that $0 \leq \tilde{\psi}^\omega \leq 1$ $P$-a.e. on $I_+$ and

$$\tilde{\zeta}^\omega(\eta) = \int_\eta \tilde{\psi}^\omega(\beta) P(d\beta)$$

for any Borel set $\eta \subset I_+$.

**Proof.** Given bounded open set $\omega \subset \mathbb{R}^2$, we have

$$\int_{I_+} \left( \int_\omega e^{\tilde{w}_{k,\beta}} \, dx \right) P(d\beta) \leq 1.$$

Hence it holds that

$$\left( \int_\omega e^{\tilde{w}_{k,\beta}} \, dx \right) P(d\beta) \rightharpoonup \tilde{\zeta}^\omega(d\beta) \quad \text{in} \quad \mathcal{M}(I_+). \quad (36)$$

Now we shall show that the limit measure $\tilde{\zeta}^\omega = \tilde{\zeta}^\omega(d\beta) \in \mathcal{M}(I_+)$ is absolutely continuous with respect to $P$.

Let $\eta \subset I_+$ be a Borel set and $\epsilon > 0$. Then each compact set $K \subset \eta$ admits an open set $J \subset I_+$ such that

$$K \subset \eta \subset J, \quad P(J) \leq \epsilon + P(K).$$

Now we take $\varphi \in C(I_+)$ satisfying

$$\varphi = 1 \quad \text{on} \ K, \quad 0 \leq \varphi \leq 1 \quad \text{on} \ I_+, \quad \text{supp} \varphi \subset J.$$

Then (36) implies

$$\tilde{\zeta}^\omega(K) = \int_K \tilde{\zeta}^\omega(d\beta) \leq \int_{I_+} \varphi(\beta) \tilde{\zeta}^\omega(d\beta)
= \lim_{k \to \infty} \int_{I_+} \varphi(\beta) \left( \int_\omega e^{\tilde{w}_{k,\beta}} \right) P(d\beta) \leq \int_{I_+} \varphi(\beta) P(d\beta)
\leq \int_J P(d\beta) = P(J) \leq \epsilon + P(\eta),$$

and therefore

$$0 \leq \tilde{\zeta}^\omega(\eta) = \sup \{ \tilde{\zeta}^\omega(K) \mid K \subset \eta : \text{compact} \} \leq \epsilon + P(\eta).$$
This shows the absolute continuity of \( \hat{\omega} \) with respect to \( P \). Therefore, by the Radon-Nikodym theorem, there exists \( \hat{\psi} \in L^1(I_+, \mathcal{P}) \) such that \( 0 \leq \hat{\psi} \leq 1 \) \( \mathcal{P} \)-a.e. on \( I_+ \) and
\[
\hat{\zeta}(\eta) = \int_{\eta} \hat{\psi}(\beta) \mathcal{P}(d\beta)
\]
for any Borel set \( \eta \subset I_+ \).

**Proposition 2.2.** There exists \( \hat{\psi} \in L^1(I_+, \mathcal{P}) \) and \( 0 \leq \hat{\psi}(\beta) \leq 1 \) \( \mathcal{P} \)-a.e. \( \beta \) such that
\[
\int_{\mathbb{R}^2} \hat{f} dy = \lambda \int_{I_+} \beta \hat{\psi}(\beta) \mathcal{P}(d\beta).
\]

**Proof.** \( \omega \) and \( \hat{\psi} \) as in Lemma 2.5. Taking \( R_j \uparrow +\infty \) and \( \omega_j = B_{R_j} \), by the monotonicity of \( \hat{\psi} \) with respect to \( \omega \), there exists \( \hat{\zeta} \in M(I_+) \) and \( \hat{\psi} \in L^1(I_+, \mathcal{P}) \) such that
\[
0 \leq \hat{\psi}(\beta) \leq 1, \quad \mathcal{P} \text{-a.e. } \beta \leq \hat{\psi}(\beta) \leq \hat{\psi}(\beta) \leq \cdots \rightarrow \hat{\psi}(\beta), \quad \mathcal{P} \text{-a.e. } \beta
\]
\[
\hat{\zeta}(\eta) = \int_{\eta} \hat{\psi}(\beta) \mathcal{P}(d\beta) \quad \text{for any Borel set } \eta \subset I_+.
\]
First, (25) implies
\[
\lambda \int_{I_+} \beta \hat{\psi}(\beta) \mathcal{P}(d\beta) = \lim_{k \to \infty} \lambda_k \int_{I_+} \beta \left( \int_{\omega_j} e^{\hat{w}_{k,\beta}} dx \right) \mathcal{P}(d\beta) = \int_{\omega_j} \hat{f}.
\]
Then we obtain
\[
\beta_0 := \int_{\mathbb{R}^2} \hat{f} = \lambda \int_{I_+} \beta \hat{\psi}(\beta) \mathcal{P}(d\beta)
\]
by the monotone convergence theorem.

Let
\[
\mathcal{B} = \{ \beta \in \text{supp} \mathcal{P} | \limsup_{k \to \infty} (w_{k,\beta}(0) - w_k(0)) > -\infty \}.
\]
From the proof of Proposition 2.1 it follows that if \( \mathcal{P}(\mathcal{B}) = 0 \) then \( \hat{f} \equiv 0 \), a contradiction. Hence \( \mathcal{P}(\mathcal{B}) > 0 \), and the value
\[
\beta_{inf} = \inf_{\beta \in \mathcal{B}} \beta
\]
is well-defined. Then we find
\[
\mathcal{B} = I_{inf} \cap \text{supp} \mathcal{P}
\]
by the monotonicity (19), where
\[
I_{inf} = \begin{cases} 
[\beta_{inf}, 1] & \text{if } \beta_{inf} \in \mathcal{B}, \\
(\beta_{inf}, 1) & \text{if } \beta_{inf} \notin \mathcal{B}.
\end{cases}
\]
Lemma 2.6. For any $\beta \in I_{\text{inf}}$, it holds that

$$\beta > \frac{4\pi}{\beta_0}.$$  

Proof. By the definition, every $\beta \in \mathcal{B}$ admits a subsequence such that $\tilde{w}_{k,\beta}(0) = w_{k,\beta}(0) - w_k(0) = O(1)$. From (28), $\tilde{w}_{k,\beta}$ satisfies

$$-\Delta \tilde{w}_{k,\beta} = \beta (-\Delta \tilde{w}_k) = \beta \tilde{f}_k.$$  

By the argument developed for the proof of (25)-(27), we have $\tilde{w}_\beta = \tilde{w}_\beta(x) \in C^2(\mathbb{R}^2)$ such that

$$\tilde{w}_{k,\beta} \rightarrow \tilde{w}_\beta \text{ in } C^2_{\text{loc}}(\mathbb{R}^2).$$  

The limit $\tilde{w}_\beta$ satisfies

$$-\Delta \tilde{w}_\beta = \beta \tilde{f}, \quad \tilde{w}_\beta(0) = 0 \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^{\tilde{w}_\beta} \leq 1$$  

and

$$\tilde{w}_\beta(x) \geq -\beta_0 \frac{2\pi}{\beta} \log(1 + |x|) + \frac{\beta}{2\pi} \int_{\mathbb{R}^2} \tilde{f}(y) \log \frac{|y|}{1 + |y|} \quad (42)$$  

with $\tilde{f} = \tilde{f}(x)$ given in Proposition 2.1.

Since $\tilde{f} \in L^1 \cap L^\infty(\mathbb{R}^2)$ and $\int_{\mathbb{R}^2} e^{\tilde{w}_\beta} < +\infty$ for any $\beta \in I_{\text{inf}}$, we obtain $\beta > 4\pi/\beta_0$. \hfill \Box

Similarly to [4], on the other hand, we have the following lemma, where $(r, \theta)$ denotes the polar coordinate in $\mathbb{R}^2$.

Lemma 2.7. We have

$$\lim_{r \to +\infty} r \tilde{w}_r = -\frac{\beta_0}{2\pi}, \quad \lim_{r \to +\infty} \tilde{w}_\theta = 0$$  

uniformly in $\theta$.

Proof. From (32) and (34), it follows that

$$r \tilde{w}_r(x) = -\frac{\beta_0}{2\pi} - \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y \cdot (x - y)}{|x - y|^2} \tilde{f}(y)dy,$$

$$\tilde{w}_\theta(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\tilde{y} \cdot (x - y)}{|x - y|^2} \tilde{f}(y)dy, \quad \tilde{y} = (y_2, -y_1).$$

Hence it suffices to show

$$\lim_{|x| \to +\infty} I_1(x) = \lim_{|x| \to +\infty} I_2(x) = 0,$$

where

$$I_1(x) = \int_{|x-y|>|x|/2} \frac{|y|}{|x-y|^2} \tilde{f}(y)dy, \quad I_2(x) = \int_{|x-y|\leq|x|/2} \frac{|y|}{|x-y|^2} \tilde{f}(y)dy.$$  

12
Since \( \tilde{f} \in L^1(\mathbb{R}^2) \), we have \( \lim_{|x| \to +\infty} I_1(x) = 0 \) by the dominated convergence theorem.

Next, (25) implies

\[
I_2(x) = \lim_{k \to \infty} \int_{|x-y| \leq |x|/2} \frac{|y|}{|x-y|} \left( \lambda_k \int_{I_+} \beta e^{\tilde{w}_{k,\beta}(y)} \mathcal{P}(d\beta) \right) dy
\]

\[
= \lim_{k \to \infty} \int_{\beta_{\min,1}} \beta \left( \int_{|x-y| \leq |x|/2} \frac{|y|}{|x-y|} e^{\tilde{w}_{k,\beta}(y)} \mathcal{P}(d\beta) \right) dy,
\]

recalling (39) and (40). Now we use (28), (29) and (25) with (34), to confirm

\[
\tilde{w}_{k,\beta}(x) \leq \beta \tilde{w}_{k}(x) = \beta (-\tilde{z}(x) + \tilde{z}(0)) + o(1) \tag{43}
\]

as \( k \to \infty \), locally uniformly in \( x \in \mathbb{R}^2 \). Hence it holds that

\[
0 \leq I_2(x) \leq C_4 \int_{|x-y| \leq |x|/2} \frac{|y|}{|x-y|} \int_{\beta_{\min,1}} e^{-\beta \tilde{z}(y)} \mathcal{P}(d\beta) dy.
\]

Then (33) and Lemma 2.6 imply

\[
0 \leq I_2(x) \leq C_5 |x|^{-(1+\epsilon_0)} \int_{|x-y| \leq |x|/2} \frac{dy}{|x-y|} \leq C_6 |x|^{-(2+\epsilon_0)}
\]

with some \( \epsilon_0 > 0 \), where we have used

\[
|x-y| \leq \frac{|x|}{2} \Rightarrow \frac{1}{2} \leq |y| \leq \frac{3}{2}
\]

Hence \( \lim_{|x| \to \infty} I_2(x) = 0 \) follows.

The Pohozaev identity

\[
R \int_{\partial B_R} \frac{1}{2} \nabla u^2 - u^2 ds = R \int_{\partial B_R} A(x)F(u) ds
- \int_{B_R} 2A(x)F(u) + F(u)(x \cdot \nabla A(x)) dx \tag{44}
\]

is valid to \( u = u(x) \in C^2(\overline{B_R}) \) satisfying

\[-\Delta u = A(x)F'(u) \quad \text{in} \quad B_R, \tag{45}\]

where \( F \in C^1(\mathbb{R}) \), \( A \in C^1(\overline{B_R}) \), and \( ds \) denote the surface element on the boundary. By this identity and Lemma 2.7, we obtain the following fact.

**Lemma 2.8.** It holds that

\[
\int_{I_+} \tilde{\psi}(\beta) \mathcal{P}(d\beta) = \frac{\lambda}{8\pi} \left( \int_{I_+} \beta \tilde{\psi}(\beta) \mathcal{P}(d\beta) \right)^2. \tag{46}
\]
Proof. We apply (44) for (45) to (22) where \( u = \tilde{w}_k \) and

\[
F(\tilde{w}_k) = \lambda_k \int_{I_+} e^{\tilde{w}_k, \beta} \mathcal{P}(d\beta), \quad A(x) \equiv 1.
\]

It follows that

\[
R \int_{\partial B_R} \frac{1}{2} |\nabla \tilde{w}_k|^2 - (\tilde{w}_k)_+^2 ds = -2\lambda_k \int_{I_+} \left( \int_{B_R} e^{\tilde{w}_k, \beta} dx \right) \mathcal{P}(d\beta) + R\lambda_k \int_{I_+} \left( \int_{\partial B_R} e^{\tilde{w}_k, \beta} \right) \mathcal{P}(d\beta). \tag{47}
\]

By Lemma 2.7 we have

\[
[L.H.S of (47)] \rightarrow -\pi \left( \frac{\beta_0}{2\pi} \right)^2 \text{ as } k \to \infty \text{ and } R \to \infty.
\]

The second term of right hand side of (47) tends to 0 as \( k \to \infty \) and \( R \to \infty \). Indeed, we have

\[
\int_{I_+} \left( \int_{\partial B_R} e^{\tilde{w}_k, \beta} \right) \mathcal{P}(d\beta) = \int_{I_{inf}} \left( \int_{\partial B_R} e^{\tilde{w}_k, \beta} \right) \mathcal{P}(d\beta) + \int_{I_+ \setminus I_{inf}} \left( \int_{\partial B_R} e^{\tilde{w}_k, \beta} \right) \mathcal{P}(d\beta). \tag{48}
\]

Thanks to Lemma 2.6 and (33), the first term of the right hand side of (48) tends to 0. And the second term also so because of the definition of \( I_{inf} \). Therefore, we have

\[
-\pi \left( \frac{\beta_0}{2\pi} \right)^2 = -2\lambda \int_{I_+} \tilde{\psi}(\beta) \mathcal{P}(d\beta),
\]

and imply that (46) holds.

\[\square\]

3 Proof of Theorem 1.2

By our assumption, residual vanishing occurs and we have

\[
\int_{\mathbb{R}^2} \tilde{f} dx = m(x_0) = \lambda \int_{I_+} \alpha \mathcal{P}(d\alpha). \tag{49}
\]

Moreover from (49), (38) and \( 0 \leq \tilde{\psi}(\beta) \leq 1 \ \mathcal{P}-\text{a.e. on } I_+ \), it follows that

\[
\tilde{\psi}(\beta) = 1 \ \mathcal{P}-\text{a.e. on } I_+. \tag{50}
\]

Proposition 3.1. Under the assumptions of Theorem 1.2 it follows that

\[
\alpha_{min} = \beta_{inf} \in B, \quad \alpha_{min} > \frac{4\pi}{\beta_0}. \tag{51}
\]
Proof. First of all, we have
\[ \alpha_{\text{min}} = \beta_{\text{inf}}. \] (52)
Indeed, \( \beta_{\text{inf}} \geq \alpha_{\text{min}} \) is obvious, we assume the contrary, \( \beta_{\text{inf}} > \alpha_{\text{min}} \). Then it holds that supp \( \tilde{\psi} \subset [\beta_{\text{inf}}, 1] \) by the definition of \( \beta_{\text{inf}} \) and \( \tilde{\psi} \), thus we obtain \( \mathcal{P}([\alpha_{\text{min}}, (\beta_{\text{inf}} + \alpha_{\text{min}})/2]) > 0 \) and \( \tilde{\psi} = 0 \) \( \mathcal{P} \)-a.e. on \([\alpha_{\text{min}}, (\beta_{\text{inf}} + \alpha_{\text{min}})/2]\). However, this is impossible by (50).

Next, it holds that \( \alpha_{\text{min}} \in \mathcal{B} \). (53)
If not, \( \alpha_{\text{min}} \not\in \mathcal{B} \), by our assumption of \( \mathcal{P}([\alpha_{\text{min}}]) > 0 \),
\[ \hat{f}_k(x) = \lambda_k \left( \int_{I_+ \setminus \{\alpha_{\text{min}}\}} \alpha e^{\tilde{w}_{k, \alpha}(x)} \mathcal{P}(d\alpha) + \alpha_{\text{min}} e^{\tilde{w}_{k, \alpha_{\text{min}}}(x)} \mathcal{P}(\{\alpha_{\text{min}}\}) \right), \]
and then passing to a subsequence, by Proposition 2.1 and the definition of \( \mathcal{B} \),
\[ \hat{f}(x) = \tilde{\lambda} \int_{I_+ \setminus \{\alpha_{\text{min}}\}} \alpha e^{\tilde{w}_o(x)} \mathcal{P}(d\alpha) \] (54)
From (54) and the integral condition \( \int_{\mathbb{R}^2} e^{\tilde{w}_o} \leq 1 \), we have
\[ \int_{\mathbb{R}^2} \hat{f}(x) dx = \tilde{\lambda} \int_{I_+ \setminus \{\alpha_{\text{min}}\}} \alpha \int_{\mathbb{R}^2} e^{\tilde{w}_o(x)} dx \mathcal{P}(d\alpha) \leq \tilde{\lambda} \int_{I_+ \setminus \{\alpha_{\text{min}}\}} \alpha \mathcal{P}(d\alpha) < \tilde{\lambda} \int_{I_+} \alpha \mathcal{P}(d\alpha), \]
which is a contradiction to (49). Therefore (53) holds.

Finally, (53) and Lemma 2.6 imply that \( \alpha_{\text{min}} > 4\pi/\beta_0 \) holds. Now we complete the proof of Proposition 3.1. \( \square \)

Let \( G = G(x, y) \) be the Green function:
\[ -\Delta_x G(x, y) = \delta_y \quad \text{in} \quad \Omega, \quad G(\cdot, y) = 0 \quad \text{on} \quad \partial\Omega. \]

Proposition 3.2. Under the assumption of Theorem 1.2, it follows that
\[ \lim_{k \to \infty} \int_{\Omega} e^{\alpha_{\text{min}} v_k} dx = +\infty. \]

Proof. By an argument of [1], we shall establish the desired result.

Indeed, setting \( \omega = B_{R_0}(x_0) \) and note that \( v_k \geq C_0 \) on \( \partial\omega \) and \( z_k \) be a solution of
\[ -\Delta z_k = \mu_k \quad \text{in} \quad \omega, \quad z_k = C_0 \quad \text{on} \quad \partial\omega \]
where \( \mu_k \) in (9). By the maximum principle, we have
\[ v_k \geq z_k \quad \text{in} \quad \omega. \] (55)
On the other hand $z_k \to z$ locally uniformly in $\omega \setminus \{x_0\}$ and

$$\mu_k(dy) \xrightarrow{\ast} s'(y)dy + m(x_0)\delta_{x_0}(dy) \quad \text{in } \mathcal{M}(\mathbb{R}),$$

where $s' \in L^1(B_{R_0})$ is nonnegative. Therefore,

$$-\Delta z = \mu \quad \text{in } \omega, \quad z = C_0 \quad \text{on } \partial \omega,$$

and

$$z(x) \geq \frac{m(x_0)}{2\pi} \log \frac{1}{|x-x_0|} - C, \quad x \in \mathbb{R} \setminus \{x_0\}. \quad (56)$$

By Proposition 3.1, we have

$$\alpha_{min} = \beta_{inf} > \frac{4\pi}{\beta_0}. \quad (57)$$

By (55)-(57) and (13) we obtain

$$\alpha_{min} \left( \liminf_{k \to \infty} v_k(x) \right) \geq \log \frac{1}{|x-x_0|^2} - C', \quad x \in \mathbb{R} \setminus \{x_0\}. \quad (58)$$

Therefore from Fatou’s lemma and the definition of $\alpha_{min},$

$$\liminf_{k \to \infty} \int_{\Omega} e^{\alpha_{min} v_k} \geq C'' \int_{\omega} \frac{1}{|x-x_0|^2} dx = +\infty. \quad (59)$$

Lemma 3.1. It holds that

$$w_k(x) + \log \int_{\Omega} e^{v_k} \to \overline{X} \int_{\alpha} \alpha P(d\alpha)G(\cdot + x_0, x_0) \quad \text{in } C^2_{loc}(B_{3R_0} \setminus \{0\}), \quad (60)$$

as $k \to \infty.$ For every $\omega \subset \subset B_{3R_0} \setminus \{0\},$ there exists $C_{1,\omega} > 0$ such that

$$\text{osc}_{\omega} w_k \equiv \sup_{\omega} w_k - \inf_{\omega} w_k \leq C_{1,\omega} \quad (61)$$

for large $k \in \mathbb{N}.$

Proof. Without loss of generality, we may assume that $B_{3R_0} \subset \Omega_k \equiv \Omega \setminus \{x_k\}$ for large $k \in \mathbb{N}.$ By the definition of $w_k,$ we have

$$w_k(x) + \log \int_{\Omega} e^{v_k} = \int_{\Omega} G(x+x_k, y)\mu_k(dy),$$

for $x \in B_{3R_0}$ where $\mu_k(dy)$ as in Proposition 1.1. Note that

$$w_k(x) + \log \int_{\Omega} e^{v_k} = G(x+x_k, x_0) \int_{\Omega} \mu_k(dy) \quad (62)$$

+ \int_{\Omega} [G(x+x_k, y) - G(x+x_k, x_0)]\mu_k(dy).$$

16
From the fact that \( [G(x+x_k, y) - G(x+x_k, x_0)] \to 0 \) as \( y \to x_0 \) locally uniformly for \( x \in B_{3R_0} \setminus \{0\} \) and \([59]\), the second term of right hand side in the above relation tends to 0 as \( k \to \infty \). Indeed, we set for any \( r \in (0, R_0) \),

\[
\int_\Omega [G(x + x_k, y) - G(x + x_k, x_0)] \mu_k(dy)
= \int_{B_r(x_0)} [G(x + x_k, y) - G(x + x_k, x_0)] \mu_k(dy)
+ \int_{\Omega \setminus B_r(x_0)} [G(x + x_k, y) - G(x + x_k, x_0)] \mu_k(dy)
=: I_1^k(x) + I_2^k(x).
\]

By the direct calculation and \([59]\), we have for \( r > 0 \),

\[
|I_2^k(x)| \leq \int_{\Omega \setminus B_r(x_0)} |G(x + x_k, y) - G(x + x_k, x_0)| \lambda_k \int_{I_r} \frac{\alpha e^{\alpha v_k}}{e^{\alpha v_k} d\alpha} \mathcal{P}(d\alpha) dy
\leq \frac{\lambda_k}{\int_{\Omega} e^{\alpha v_k} d\alpha} \int_{\Omega \setminus B_r(x_0)} |G(x + x_k, y) - G(x + x_k, x_0)| e^{v_k} dy
\to 0
\]

as \( k \to \infty \) locally uniformly in \( x \in B_{3R_0} \setminus \{0\} \). On the other hands, for large \( k \in \mathbb{N} \),

\[
|I_1^k(x)| \leq \int_{B_r(x_0)} |G(x + x_k, y) - G(x + x_k, x_0)| \mu_k(dy)
\leq \sup_{B_r(x_0)} |G(x + x_k, y) - G(x + x_k, x_0)|
\to 0
\]

as \( r \to 0 \) locally uniformly in \( x \in B_{3R_0} \setminus \{0\} \). From \([62]-[64]\), it follows that

\[
w_k(x) + \log \int_\Omega e^{v_k} \to \lambda_0 \int_{I_r} \alpha \mathcal{P}(d\alpha) G(\cdot + x_0, x_0),
\]

as \( k \to \infty \) locally uniformly in \( x \in B_{3R_0} \setminus \{0\} \). Furthermore, we have

\[
\frac{\partial}{\partial x_{ij}} (w_k(x) + \log \int_\Omega e^{v_k}) = \int_\Omega \frac{\partial}{\partial x_{ij}} G(x + x_k, y) \mu_k(dy),
\]

for \( i, j = 1, 2 \) so that by the same argument here we obtain \([60]\). \([61]\) is the direct consequence of \([60]\). \( \square \)

Note that \( B_{2R_0} \subset \Omega - \{x_k\} \) for large \( k \in \mathbb{N} \). We decompose \( w_k \) as \( w_k = w_k^{(1)} + w_k^{(2)} \), using the solutions \( w_k^{(1)} \) and \( w_k^{(2)} \) to

\[
-\Delta w_k^{(1)} = g_k \quad \text{in} \quad B_{2R_0}, \quad w_k^{(1)} = 0 \quad \text{on} \quad \partial B_{2R_0},
\]

\[
-\Delta w_k^{(2)} = 0 \quad \text{in} \quad B_{2R_0}, \quad w_k^{(2)} = w_k \quad \text{on} \quad \partial B_{2R_0}.
\]
where
\[ g_k = g_k(x) = \lambda_k \int_{I_+} \alpha e^{w_{k,\alpha}(y)} \mathcal{P}(d\alpha), \]
and \( R_0 > 0 \) as in [24]. By the maximum principle and Lemma 3.1 we also have
\( C_2 > 0 \) independent of \( k \) such that
\[
\text{osc}_{B_2R_0} w_k^{(2)} \leq C_2.
\]
Thus it holds that
\[
w_k(x) - w_k(0) = w_k^{(1)}(x) - w_k^{(1)}(0) + O(1) \tag{65}
\]
as \( k \to \infty \) uniformly in \( x \in B_{2R_0} \).

Let \( G_0 = G_0(x,y) \) be the another Green function defined by
\[
-\Delta_x G_0(\cdot,y) = \delta_y \quad \text{in} \quad B_{2R_0}, \quad G_0(\cdot,y) = 0 \quad \text{on} \quad \partial B_{2R_0}.
\]
Then it holds that
\[
w_k^{(1)}(x) - w_k^{(1)}(0) = \int_{B_{2R_0}} (G_0(x,y) - G_0(0,y)) g_k(y) dy \tag{66}
\]
for \( x \in B_{2R_0} \). We have, more precisely,
\[
G_0(x,y) = \begin{cases} \Gamma(|x - y|) - \Gamma\left(\frac{|y|}{2R_0} |x - y|\right), & \text{if } y \neq 0, \ y \neq x, \\ \Gamma(|x|) - \Gamma(2R_0) & \text{if } y = 0, \ y \neq x, \end{cases}
\]
using the fundamental solution and the Kelvin transformation:
\[
\Gamma(|x|) = \frac{1}{2\pi} \log \frac{1}{|x|}, \quad \overline{y} = \left(\frac{2R_0}{|y|}\right)^2 y,
\]
which implies
\[
G_0(x,y) - G_0(0,y) = \frac{1}{2\pi} \log \frac{|y|}{|x - y|} - \frac{1}{2\pi} \log \frac{|\overline{y}|}{|x - \overline{y}|}
\]
for \( y \in B_{2R_0} \), satisfying \( y \neq x \) and \( y \neq 0 \).

By
\[
\frac{2}{3} \leq \frac{|\overline{y}|}{|x - \overline{y}|} \leq 2, \quad x \in B_{R_0}, \quad y \in B_{2R_0} \setminus \{0\},
\]
and
\[
0 \leq \int_{B_{2R_0}} g_k \leq \lambda_k \int_{I_+} \alpha \mathcal{P}(d\alpha) = O(1),
\]
we end up with
\[
\int_{B_{2R_0}} (G_0(x,y) - G_0(0,y)) g_k(y) dy = \frac{1}{2\pi} \int_{B_{2R_0}} g_k(y) \log \frac{|y|}{|x - y|} dy + O(1) \tag{67}
\]
as $k \to \infty$ uniformly in $x \in B_{R_0}$.
Consequently, (65)-(67) yield
\[
 w_k(x) - w_k(0) = \frac{1}{2\pi} \int_{B_{2R_0}} g_k(y) \log \frac{|y|}{|x - y|} dy + O(1)
\]
as $k \to \infty$ uniformly in $x \in B_{R_0}$. This means
\[
 \tilde{w}_k(x) = \frac{1}{2\pi} \int_{B_{2R_0}} g_k(y) \log \frac{|y|}{|\sigma_k x - y|} dy + O(1)
\]
as $k \to \infty$ uniformly in $x \in B_{R_0}$. Let $\beta_k$ be as in (38), and put
\[
 \beta_k := \int_{B_{2R_0} \setminus B_{L_\epsilon}} \tilde{f}_k dx.
\]
To employ the argument of [10], we prepare the following lemma.

**Lemma 3.2.** For any $\epsilon > 0$, there exists $k_\epsilon \in \mathbb{N}$ and $L_\epsilon > 0$ such that
\[
 \int_{B_{2R_0} \setminus B_{L_\epsilon}} \tilde{f}_k dx \leq \epsilon
\]
for $k \geq k_\epsilon$.

**Proof.** For any $r > 0$, setting
\[
 \beta_k = \int_{B_r} \tilde{f}_k dx + \int_{B_{2R_0} \setminus B_r} \tilde{f}_k dx =: I_{1}^{k,r} + I_{2}^{k,r}.
\]
By (25), we have
\[
 \lim_{r \to \infty} \lim_{k \to \infty} I_{1}^{k,r} = \int_{\mathbb{R}^2} \tilde{f} dx.
\]
Moreover for any $k$, it holds that
\[
 \beta_k \leq \lambda_k \int_{I_+} \alpha \mathcal{P}(d\alpha).
\]
Therefore from \((71), (72) and (13)\), we obtain
\[
 I_{2}^{k,r} = o(1) \quad \text{as } k \to \infty, \quad r \to \infty
\]
and get the desired result. \qed
By the result of Lemma 3.2, we have
\[
\lim_{k \to \infty} \beta_k = \beta_0.
\] (73)

**Lemma 3.3.** For every \(0 < \epsilon \ll 1\), there exists \(R_\epsilon > 0\) and \(C_{4,\epsilon} > 0\) such that
\[
\tilde{w}_k(x) \leq -\left(\frac{\beta_k}{2\pi} - \epsilon\right) \log |x| + C_{4,\epsilon}
\] for \(k \gg 1\) and \(x \in B_{R_0 \sigma_k^{-1}} \setminus B_{R_\epsilon}\).

**Proof.** By (70), given \(0 < \epsilon \ll 1\), there exists \(R_\epsilon > 0\), \(k_\epsilon \in \mathbb{N}\) such that
\[
\frac{1}{2\pi} \int_{B_{R_\epsilon / 2}} \tilde{f}_k dx \geq \frac{\beta_k}{2\pi} - \frac{\epsilon}{4}
\] (75) for \(k \geq k_\epsilon\). It follows from (68) that
\[
\tilde{w}_k(x) = K_1^k(x) + K_2^k(x) + K_3^k(x) + O(1), \quad k \to \infty,
\] (76) uniformly in \(x \in B_{R_0 \sigma_k^{-1}} \setminus B_{R_\epsilon}\), where
\[
K_1^k(x) = \frac{1}{2\pi} \int_{B_{R_\epsilon / 2}} \tilde{f}_k(y) \log \frac{|y|}{|x - y|} dy,
\]
\[
K_2^k(x) = \frac{1}{2\pi} \int_{B_{2|x|/2}(0)} \tilde{f}_k(y) \log \frac{|y|}{|x - y|} dy,
\]
\[
K_3^k(x) = \frac{1}{2\pi} \int_{B'(x)} \tilde{f}_k(y) \log \frac{|y|}{|x - y|} dy,
\]
for \(B'(x) = B_{R_0 \sigma_k^{-1}} \setminus (B_{R_\epsilon / 2} \cap B_{2|x|/2}(x))\).

Since
\[
\frac{|y|}{|x - y|} \leq 2 \frac{|y|}{|x|} \leq \frac{R_\epsilon}{|x|}, \quad y \in B_{R_\epsilon / 2}, \quad x \in B_{R_0 \sigma_k^{-1}} \setminus B_{R_\epsilon},
\]
there exists \(C_{5,\epsilon} > 0\) such that
\[
K_1^k(x) \leq \frac{1}{2\pi} (\log R_\epsilon - \log |x|) \int_{B_{R_\epsilon / 2}} \tilde{f}_k(y) \log \frac{|y|}{|x - y|} \leq C_{5,\epsilon} - \left(\frac{\beta_k}{2\pi} - \frac{\epsilon}{4}\right) \log |x|
\] (77) for \(k \geq k_\epsilon\) and \(x \in B_{R_0 \sigma_k^{-1}} \setminus B_{R_\epsilon}\) by (75). We also have
\[
\frac{|y|}{|x - y|} \leq 3, \quad y \in B_{R_0 \sigma_k^{-1}} \setminus B_{2|x|/2}(x),
\]
and hence
\[
K^3_k(x) \leq \frac{\log 3}{2\pi} \int_{B'(x)} \tilde{f}_k \leq \frac{\log 3}{2\pi} \|\tilde{f}_k\|_{L^1(B_{R_0\sigma_k^{-1}})} \\
\leq \frac{\lambda_k \log 3}{2\pi} \int_{L^+} \alpha P(d\alpha),
\]
for large \(k\) and \(x \in B_{R_0\sigma_k^{-1}} \setminus B_\epsilon\).

Now we take
\[
D_1(x) = B_{|x|-1}(x), \quad D_2(x) = B_{|x|/2}(x) \setminus B_{|x|-1}(x)
\]
for \(|x| > R_\epsilon \geq \sqrt{2}\). Since
\[
|y| < |x| + 1/|x|, \quad y \in D_1(x)
\]
and
\[
\frac{|y|}{|x-y|} \leq \frac{3}{2} |x|^2, \quad y \in D_2(x), \quad x \in B_{R_0\sigma_k^{-1}} \setminus B_\epsilon,
\]
we have
\[
\frac{1}{2\pi} \int_{D_1(x)} \tilde{f}_k(y) \log \frac{|y|}{|x-y|} dy \\
\leq \frac{1}{2\pi} \int_{\frac{1}{2} |x| \leq |y| \leq \frac{3}{2} |x|} (2 \log |x| + \log \frac{3}{2}) \tilde{f}_k(y) dy \\
\leq \frac{1}{2\pi} \int_{\frac{1}{2} R_\epsilon \leq |y| \leq \frac{3}{2} R_0\sigma_k^{-1}} (2 \log |x| + \log \frac{3}{2}) \tilde{f}_k(y) dy \\
\leq \frac{1}{2\epsilon} \log |x| + O(1)
\]
(79)
on the other hand,
\[
\frac{1}{2\pi} \int_{D_1(x)} \tilde{f}_k(y) \log \frac{|y|}{|x-y|} dy \\
= \|\tilde{f}_k\|_{L^\infty(D_1)} \int_{D_1(x)} \left| \log \frac{1}{|x-y|} \right| dy + \frac{1}{2\pi} \int_{D_1(x)} \tilde{f}_k(y) (\log |x| + C) dy \\
\leq C \int_0^{|x|-1} r |\log r| dr + \frac{1}{4} \epsilon \log |x| + O(1) \\
= \frac{1}{4} \epsilon \log |x| + O(1)
\]
(80)
for \(k \gg 1, x \in B_{R_0\sigma_k^{-1}} \setminus B_\epsilon\). From (79) and (80), we get
\[
K^2_k(x) \leq \frac{3}{4} \epsilon \log |x| + C
\]
(81)
for \(k \gg 1, x \in B_{R_0\sigma_k^{-1}} \setminus B_\epsilon\). From (77), (78), and (81), we get the desired result.
Lemma 3.4. It holds that
\[ \int_{B_{R_0^{\sigma_k-1}}} \hat{f}_k(y) \log |y| dy = O(1) \quad \text{as } k \to \infty. \]  
(82)

Proof. By \( \lim_{k \to \infty} \beta_k = \beta_0 \) and (51), there exists \( \epsilon_0 > 0 \) and \( \delta_0 > 0 \) such that
\[ -\alpha_{\min}(\frac{\beta_k}{2\pi} - \frac{\epsilon_0}{2}) \leq -(2 + 3\delta_0) \]  
(83)
for \( k \gg 1 \). Let
\[ R'_0 = R_{\epsilon_0/2} \]
for \( R_\epsilon \) as in Lemma 3.3 with \( \epsilon = \epsilon_0/2 \). Then, by (25)-(26), (74) and (83) we obtain \( C_{7,\epsilon_0} \) such that
\[ \tilde{w}_k(x) = -\frac{\beta_k}{2\pi} \log |x| + O(1) \quad \text{as } k \to \infty \]
uniformly in \( x \in B_{R_0^{\sigma_k^{-1}} \setminus B_{R_0^{\delta_k}}} \).

Lemma 3.5. There exists \( \delta_0 > 0 \) such that
\[ \tilde{\psi}_k(x) = -\frac{\beta_k}{2\pi} \log |x| + O(1) \quad \text{as } k \to \infty \]
uniformly in \( x \in B_{R_0^{\sigma_k^{-1}}} \setminus B_{(\log \sigma_k^{-1})^{1/\delta_0}} \).

Proof. Let \( \epsilon_0 > 0 \) and \( \delta_0 > 0 \) as in (83) and consider
\[ \beta'_k(x) = \int_{B_{|x|/2}} \hat{f}_k \]  
(85)
for $x \in B_{R_0 \sigma_k^{-1}} \setminus B_{(\log \sigma_k^{-1})^{1/\delta_0}}$ and $k \gg 1$.

First of all,
\[
\left| \tilde{w}_k(x) + \frac{\beta_k}{2\pi} \log |x| \right| \leq \frac{\beta_k}{2\pi} - \frac{\beta_k}{2\pi} \log |x| + \left| \tilde{w}_k(x) + \frac{\beta_k(x)}{2\pi} \log |x| \right| \tag{86}
\]
for $x \in B_{R_0 \sigma_k^{-1}} \setminus B_{(\log \sigma_k^{-1})^{1/\delta_0}}$, $k \gg 1$. To get a estimate of right hand side of \[86], we divide this proof two steps.

Step1. Since \[84] and \[70] hold, there exists $C_{9, 6, 0, 0} > 0$ such that
\[
0 \leq \beta_k - \beta_k'(x) \leq \int_{B_{R_0 \sigma_k^{-1}} \setminus B_{(\log \sigma_k^{-1})^{1/\delta_0}}} \tilde{f}_k
\]
\[
\leq C_{7, 6, 0} \int_{B_{R_0 \sigma_k^{-1}} \setminus B_{(\log \sigma_k^{-1})^{1/\delta_0}}} |y|^{-2+3\delta_0} dy + \int_{B_{2R_0 \sigma_k^{-1}} \setminus B_{R_0 \sigma_k^{-1}}} \tilde{f}_k
\]
\[
\leq C_{9, 6, 0, 0} \sigma_k^{-1} + o(1) \tag{87}
\]
for $x \in B_{R_0 \sigma_k^{-1}} \setminus B_{(\log \sigma_k^{-1})^{1/\delta_0}}$, $k \gg 1$.

Step2. By \[88],
\[
\left| \tilde{w}_k(x) + \frac{\beta_k'(x)}{2\pi} \log |x| \right| \leq \frac{1}{2\pi} \int_{|y| < 2R_0 \sigma_k^{-1}} \tilde{f}_k(y) \log \frac{|y|}{|x-y|} \tag{88}
\]
\[
+ \frac{1}{2\pi} \int_{B_{\frac{|x|}{2}}} \tilde{f}_k(y) \log \frac{|x|}{|x-y|} + O(1) \tag{89}
\]
as $k \to \infty$.

Note that if $z := x/|y|$, $z_0 := y/|y|$ and $|z| < 1/2$ then $1/2 < |z - z_0| < 3/2$, we have
\[
\left| \int_{\frac{|y|}{2} < |y| < 2R_0 \sigma_k^{-1}} \tilde{f}_k(y) \log \frac{|y|}{|x-y|} \right|
\]
\[
\leq \int_{\frac{|y|}{2} < |y| < 2R_0 \sigma_k^{-1}} \tilde{f}_k(y) \log \frac{1}{|y| - \frac{2R_0 \sigma_k^{-1}}{|y|}}
\]
\[
= \left( \int_{\frac{|y|}{2} < |y| < 2R_0 \sigma_k^{-1}} + \int_{0 < \frac{|y|}{2} < \frac{1}{2}, |y| < 2R_0 \sigma_k^{-1}} \tilde{f}_k(y) \log \frac{1}{|y| - \frac{2R_0 \sigma_k^{-1}}{|y|}} \right)
\]
\[
\leq \int_{\frac{|y|}{2} < |y| < 2R_0 \sigma_k^{-1}} \tilde{f}_k(y) \log \frac{1}{|y| - \frac{2R_0 \sigma_k^{-1}}{|y|}} + \log 2 \int_{2|x| < |y| < 2R_0 \sigma_k^{-1}} \tilde{f}_k(y).
\]

Moreover, by \[84] and \[70],
\[
\int_{2|x| < |y| < 2R_0 \sigma_k^{-1}} \tilde{f}_k(y) \leq \int_{2(\log \sigma_k^{-1})^{1/2} < |y| < 2R_0 \sigma_k^{-1}} \tilde{f}_k(y)
\]
\[
\leq \int_{2(\log \sigma_k^{-1})^{1/2} < |y| < R_0 \sigma_k^{-1}} \tilde{f}_k(y) + \int_{R_0 \sigma_k^{-1} < |y| < 2R_0 \sigma_k^{-1}} \tilde{f}_k(y) = o(1).
\]

23
On the other hand, if \( z := y/|x| \) and \( z_0 := x/|x| \) then we have

\[
\int_{\frac{1}{2} < |z| < 2, |y| < 2R_0 \sigma_k^{-1}} \tilde{f}_k(y) \left| \log \frac{1}{|x|} - \frac{y}{|y|} \right| dy
\]

\[
= \int_{\frac{1}{2} < |z| < 2, |y| < 2R_0 \sigma_k^{-1}} \tilde{f}_k(|x|z) \left| \log \frac{|z|}{|z - z_0|} \right| |x|^2 dz
\]

\[
\leq \int_{\frac{1}{2} < |z| < 2} C_7 \epsilon_0 (|z|)^{-2(3\delta_0)} \left( \log \frac{|z|}{|z - z_0|} \right)^2 |x|^2 dz
\]

\[
+ \int_{\frac{1}{2} < |z| < 2R_0 \sigma_k^{-1} < |y| < 2R_0 \sigma_k^{-1}} \tilde{f}_k(|x|z) \left| \log \frac{|z|}{|z - z_0|} \right| |x|^2 dz
\]

\[
\leq C_7 \epsilon_0 (\log \sigma_k^{-1})^{-3} + 5 \int_{R_0 \sigma_k^{-1} < |y| < 2R_0 \sigma_k^{-1}} \tilde{f}_k(y) dy
\]

\[= o(1) \quad (92)\]

for \( x \in B_{R_0 \sigma_k^{-1}} \setminus B_{(\log \sigma_k^{-1})^{1/\delta_0}}, \ k \gg 1 \) by (84) and (70).

Therefore, from (90)-(92), we obtain (88) = o(1) as \( k \to \infty \).

Lastly, since \( |y| < |x|/2, 1/(2|y|) \leq |x - y|/(|x||y|) \leq 3/(2|y|) \) and (82), it follows that

\[
\frac{1}{2\pi} \int_{|y| \leq |x|/2} \tilde{f}_k(y) \left| \log \frac{|x|/|y|}{|x - y|} \right| dy
\]

\[
\leq \frac{1}{2\pi} \int_{|y| \leq R_0 \sigma_k^{-1}} \tilde{f}_k(y) (\log 2 + \log |y|) dy
\]

\[= O(1) \quad (93)\]

as \( k \to \infty \). Hence, (89) = O(1) as \( k \to \infty \).

From Step1, Step2 and (86), we complete the proof of Lemma 3.5. \( \square \)

**Proof of Theorem 1.2** We take \( \delta_0 \) and \( R'_0 \) as in Lemma 3.4. First, (25), (67), and (69) imply

\[
|\tilde{w}_k(x) + \frac{\beta_k}{2\pi} \log (1 + |x|)| \leq |\tilde{w}_k(x)| + \frac{\beta_k}{2\pi} \log (1 + |x|) \leq C_{12} \]

(93)

for \( x \in B_{R'_0}, \) while Lemma 3.5 means

\[
|\tilde{w}_k(x) + \frac{\beta_k}{2\pi} \log (1 + |x|)| \leq C_{13}, \quad x \in B_{R_0 \sigma_k^{-1}} \setminus B_{(\log \sigma_k^{-1})^{1/\delta_0}}, \quad (94)\]

where \( k \gg 1 \).

Now we put

\[
\tilde{w}^+_k(x) = -\frac{\beta_k}{2\pi} \log |x| + C_{14} + \frac{C_7 \epsilon_0}{9\delta_0} |x|^{3\delta_0}
\]

\[
\tilde{w}^-_k(x) = -\frac{\beta_k}{2\pi} \log |x| - C_{14} - \frac{1}{2} |\sigma_k x|^2
\]
for $C_{14} = 1 + \max\{C_{12}, C_{13}\}$ and $k \gg 1$, recalling (19), and let
\[
A_k = B_{(\log \sigma_k^{-1})^{1/4}} \setminus B_{R_0},
\]
Then (84) implies
\[
-\Delta \tilde{w}_k^+(x) = C_{7,\epsilon_0} |x|^{-(2+3\delta_0)} \geq \tilde{f}_k \quad \text{in} \quad A_k, \\
\tilde{w}_k^+ \geq \tilde{w}_k \quad \text{on} \quad \partial A_k.
\]
Next, we have
\[
-\Delta \tilde{w}_k^-(x) = -\sigma_k \leq \tilde{f}_k \quad \text{in} \quad A_k, \\
\tilde{w}_k^- \leq \tilde{w}_k \quad \text{on} \quad \partial A_k.
\]
Since $-\Delta \tilde{w}_k = \tilde{f}_k$ in $A_k$, it follows from the maximum principle that
\[
\tilde{w}_k^- \leq \tilde{w}_k \leq \tilde{w}_k^+ \quad \text{in} \quad A_k.
\]
Using
\[
\left| \frac{1}{2} \sigma_k x^2 \right| \leq C_{15}, \quad x \in B_{R_0} \sigma_k^{-1},
\]
and
\[
\left| C_{7,\epsilon_0} |x|^{-3\delta_0} \right| \leq C_{16}, \quad x \in A_k,
\]
we obtain
\[
|\tilde{w}_k(x) + \frac{\beta_k}{2\pi} \log |x|| \leq C_{14} + \max\{C_{15}, C_{16}\}, \quad x \in A_k
\]
for $k \gg 1$.

Properties (93)-(95) and (21) imply that
\[
w_k(x) - w_k(0) = -\left(\frac{\beta_0}{2\pi} + o(1)\right) \log(1 + e^{w_k(0)/2}|x|) + O(1)
\]
as $k \to \infty$ uniformly in $x \in B_{R_0}$. We complete the proof of Theorem 1.2. \qed

4 Proof of Theorem 1.3

First, we prove Proposition 1.2.

Proof of Proposition 1.2. If (16) holds then we get $S = \{x_0\}$ and
\[
\int_{\mathbb{R}^2} \tilde{f} dx = m(x_0) = \int_{I_+} \alpha P(d\alpha),
\]
which is the mass identity. Furthermore, the above mass identity and (37) imply that
\[
\psi(\beta) = 1 \quad \mathcal{P}-a.e \text{ on } I_+, \quad \alpha_{min} = \beta_{inf} \geq \frac{4\pi}{\beta_0}
\]
(97)
by Proposition 3.1 and Lemma 2.6. By the argument of Proposition 3.2 it follows that
\[
\lim_{k \to \infty} \int_{\Omega} e^{\alpha v_k} dx = +\infty
\]
for any \( \alpha \in \text{supp} P \). This relation implies \( s \equiv 0 \) in (9), that is, the residual vanishing occurs, see [22], Lemma 4. The inverse is clearly true.

Before proving Theorem 1.3 we need to prepare some facts with the case general \( P(d\alpha) \). It holds that
\[
\int_{I_+} \tilde{\psi}(\beta) \mathcal{P}(d\beta) = \left( \int_{I_+} \phi_0(\beta) \tilde{\psi}(\beta) \mathcal{P}(d\beta) \right)^2,
\]
where
\[
\phi_0(\beta) = \sqrt{\frac{\lambda}{8\pi \beta}}.
\]

Let
\[
\mathcal{L}^0(\psi) = \int_{I_+} \phi_0(\beta) \psi(\beta) \mathcal{P}(d\beta)
\]
\[
C_d = \{ \psi \mid 0 \leq \psi \leq 1 \text{ P-a.e. on } I_+ \text{ and } \int_{I_+} \psi(\beta) \mathcal{P}(d\beta) = d \}
\]
and \( \chi_A \) be the characteristic function of the set \( A \). The following lemma is a variant of the result of [8].

**Lemma 4.1.** For each \( 0 < d \leq 1 \), the value \( \sup_{\psi \in C_d} \mathcal{L}^0(\psi) \) is attained by
\[
\psi_d(\beta) = \chi_{\phi_0 > s_d}(\beta) + c_d \chi_{\phi_0 = s_d}(\beta)
\]
with \( s_d \) and \( c_d \) defined by
\[
s_d = \inf \{ t \mid \mathcal{P}(\{ \phi_0 > t \}) \leq d \},
\]
\[
c_d \mathcal{P}(\{ \phi_0 = s_d \}) = d - \mathcal{P}(\{ \phi_0 > s_d \}), \quad 0 \leq c_d \leq 1.
\]
Furthermore, the maximizer is unique in the sense that \( \psi_m = \psi_d \text{ P-a.e. on } I_+ \) for any maximizer \( \psi_m \in C_d \).
Proof. Fix $0 < d \leq 1$. Given $\psi \in C_d$, we compute

$$
\int_{I_+} \phi_0(\psi_d - \psi)P(d\beta) = \int_{\{\phi_0 > s_d\}} \phi_0(\psi_d - \psi)P(d\beta) + s_d \int_{\{\phi_0 = s_d\}} (\psi_d - \psi)P(d\beta) - \int_{\{\phi_0 < s_d\}} \phi_0P(d\beta)
$$

\begin{align*}
&\geq s_d \int_{\{\phi_0 > s_d\}} (\psi_d - \psi)P(d\beta) + s_d \int_{\{\phi_0 = s_d\}} (\psi_d - \psi)P(d\beta) - \int_{\{\phi_0 < s_d\}} \phi_0P(d\beta) \\
&\geq s_d \left( \int_{\{\phi_0 > s_d\}} (\psi_d - \psi)P(d\beta) + \int_{\{\phi_0 = s_d\}} (\psi_d - \psi)P(d\beta) \right) - \int_{\{\phi_0 < s_d\}} \psi P(d\beta) \\
&= s_d \int_{I_+} (\psi_d - \psi)P(d\beta) = 0,
\end{align*}

(102)

which means that $\psi_d$ is the maximizer.

The equalities hold in (102) and (103) if and only if $\psi$ is the maximizer, and so we shall derive the two conditions. The first condition is that $(\phi_0 - s_d)(\psi_d - \psi) = 0$ $P$-a.e. on $\{\phi_0 > s_d\}$, so that

$$
\psi = \psi_d \quad P\text{-a.e. on } \{\phi_0 > s_d\}
$$

(104)

by the monotonicity of $\phi_0$ and $\psi_d \geq \psi$ on $\{\phi_0 > s_d\}$. The second one is that $(s_d - \phi_0)\psi = 0$ $P$-a.e. on $\{\phi_0 < s_d\}$, or

$$
\psi = 0 \quad P\text{-a.e. on } \{\phi_0 < s_d\}
$$

(105)

by the monotonicity of $\phi_0$ and $\psi \geq 0$. The uniqueness follows from (104) and (105) and $\psi_d$, $\psi \in C_d$.\hfill \Box

Let $d \in (0, 1]$ such that $\tilde{\psi} \in C_d$ and from (98), it holds that

$$
d = \int_{I_+} \tilde{\psi}(\beta)P(d\beta) = \left( \int_{I_+} \phi_0(\beta)\tilde{\psi}(\beta)P(d\beta) \right)^2.
$$

Lemma 2.8 and (99) yield

$$
d = P(\{\phi_0 > s_d\}) + c_d P(\{\phi_0 = s_d\}) \leq \left( \sqrt{\frac{\lambda}{8\pi}} \int_{I_+} \psi_d(\beta)P(d\beta) \right)^2
$$

(106)

for $\psi_d = \psi_d(\beta)$ defined by (100) and (101). By the monotonicity of $\phi_0 = \phi_0(\beta)$, there exists the unique element $\beta_d \in I_+$ such that

$$
\phi_0(\beta_d) = s_d,
$$

27
and then (106) leads
\[ d = \mathcal{P}(\beta_d, 1) + c_d \mathcal{P}(\{\beta_d\}) \leq \frac{X}{8\pi} \left( \int_{[\beta_d, 1]} \beta \mathcal{P}(d\beta) + c_d \beta_d \mathcal{P}(\{\beta_d\}) \right)^2. \] (107)

Here we introduce
\[ H(\tau) = \mathcal{P}(\beta_d, 1) + \tau \mathcal{P}(\{\beta_d\}) - \frac{X}{8\pi} \left( \int_{[\beta_d, 1]} \beta \mathcal{P}(d\beta) + \tau \beta_d \mathcal{P}(\{\beta_d\}) \right)^2. \] (108)

It follows from (7) that
\[ H(0) \geq 0, \quad H(1) \geq 0. \] (109)

**Remark 4.1.** Here we use the property of \( \lambda \) for (109).

Moreover, we have either \( c_d = 0 \) or \( c_d = 1 \) if \( \mathcal{P}(\{\beta_d\}) > 0 \). In fact, since
\[ H''(\tau) = -\frac{X}{4\pi} \left( \beta_d \mathcal{P}(\{\beta_d\}) \right)^2 < 0 \]
by \( \mathcal{P}(\{\beta_d\}) > 0 \), it holds that \( H(\tau) > 0 \) for \( 0 < \tau < 1 \) by (109). On the other hand, \( H(c_d) \leq 0 \) by (108).

We now claim
\[ \tilde{\psi} = \psi_d = \chi_{I_d} \text{ P-a.e. on } I_+ \] (110)
where
\[ I_d = \begin{cases} [\beta_d, 1] & \text{if } \mathcal{P}(\{\beta_d\}) > 0 \text{ and } c_d = 1, \\ (\beta_d, 1] & \text{if } \mathcal{P}(\{\beta_d\}) = 0 \text{ or } \mathcal{P}(\{\beta_d\}) > 0 \text{ and } c_d = 0. \end{cases} \]

First, we assume that \( \mathcal{P}(\{\beta_d\}) = 0 \). Then, \( H(\tau) = H(0) \) for \( \tau \in [0, 1] \). In this case, the equality holds in (108) by (109), and thus
\[ d = \left( \int_{I_+} \phi_0(\beta) \psi_d(\beta) \mathcal{P}(d\beta) \right)^2 = \left( \int_{I_+} \phi_0(\beta) \tilde{\psi}(\beta) \mathcal{P}(d\beta) \right)^2, \]
which means \( \tilde{\psi} = \psi_d \) P-a.e. on \( I_+ \) by the uniqueness of Lemma 4.1. Note that the integrations are non-negative. It is clear that \( \psi_d = \chi_{I_d} \) P-a.e. on \( I_+ \). Next we assume that \( \mathcal{P}(\{\beta_d\}) > 0 \). Then we use (108) and (109) to obtain \( H(c_d) = 0 \), which again implies that the equality holds in (108), and hence
\[ \tilde{\psi} = \psi_d = \begin{cases} \chi_{[\beta_d, 1]} & \text{if } c_d = 1, \\ \chi_{(\beta_d, 1]} & \text{if } c_d = 0. \end{cases} \]

The claim (110) is established.
Here we divide two cases as $\beta_d > \beta_{inf}$ and $\beta_d \leq \beta_{inf}$.

First, we consider the case $\beta_d > \beta_{inf}$. Then, we have,

$$\mathcal{P}(I_{inf} \setminus I_d) = 0.$$ 

Indeed, assume $\mathcal{P}(I_{inf} \setminus I_d) > 0$. Then,

$$\tilde{\psi}(\beta) = 0 \quad \text{for } \mathcal{P}\text{-a.e. } \beta \in I_{inf} \setminus I_d \quad (111)$$

by (110). On the other hand, $\tilde{\psi}(\beta) > 0$ for any $\beta \in I_{inf} \setminus I_d$ by the definition of $I_{inf}$ and $\tilde{\psi}$, and by the convergence (25), which contradicts (111).

In the case of $\beta_d \leq \beta_{inf}$, we obtain the following result:

**Proposition 4.1.** Suppose $\beta_d \leq \beta_{inf}$ then it holds that

$$\tilde{\psi}(\beta) = \chi_{I_{inf}}(\beta) \quad \mathcal{P}\text{-a.e } \beta,$$

where

$$I_{inf} = \begin{cases} [\beta_{inf}, 1] & \text{if } \beta_{inf} \in \mathcal{B}, \\ (\beta_{inf}, 1] & \text{if } \beta_{inf} \notin \mathcal{B}. \end{cases}$$

**Proof.** There are the following five possibilities:

(i) $\beta_d < \beta_{inf},$
(ii) $\beta_d = \beta_{inf}, \ I_d = (\beta_d, 1]$ and $\beta_{inf} \in I_{inf},$
(iii) $\beta_d = \beta_{inf}, \ I_d = [\beta_d, 1)$ and $\beta_{inf} \notin I_{inf},$
(iv) $\beta_d = \beta_{inf}, \ I_d = [\beta_d, 1]$ and $\beta_{inf} \in I_{inf},$
(v) $\beta_d = \beta_{inf}, \ I_d = (\beta_d, 1)$ and $\beta_{inf} \notin I_{inf}.$

The result is clearly true for the cases (iv)-(v), and thus it suffices to prove $\mathcal{P}(I_d \setminus I_{inf}) = 0$, and $\mathcal{P}(\{\beta_d\}) = \mathcal{P}(\{\beta_{inf}\}) = 0$ for the cases (i) and (ii), (iii), respectively.

(i) Assume $\mathcal{P}(I_d \setminus I_{inf}) > 0$. Then

$$\tilde{\psi}(\beta) = 0 \quad \text{for } \beta \in I_d \setminus I_{inf} \quad (112)$$

by the definitions of $I_{inf}$ and $\tilde{\psi}$. Note that $\tilde{w}_{k, \beta} \to -\infty$ locally uniformly in $\mathbb{R}^2$ for $\beta \in I_d \setminus I_{inf}$. On the other hand, $\tilde{\psi}(\beta) = 1$ for some $\beta \in I_d \setminus I_{inf}$ by (110), which contradicts (112).

(ii) If $\mathcal{P}(\{\beta_d\}) = \mathcal{P}(\{\beta_{inf}\}) > 0$ then $\tilde{\psi}(\beta_d) = \tilde{\psi}(\beta_{inf}) = 0$ by (110) and $I_d = (\beta_d, 1]$. On the other hand, $\tilde{\psi}(\beta_d) = \tilde{\psi}(\beta_{inf}) > 0$ by $\beta_{inf} \in I_{inf}$ as shown for the case $\beta_d > \beta_{inf}$ above, a contradiction.

(iii) If $\mathcal{P}(\{\beta_d\}) = \mathcal{P}(\{\beta_{inf}\}) > 0$ then $\tilde{\psi}(\beta_d) = \tilde{\psi}(\beta_{inf}) = 1$ by (110) and $I_d = [\beta_d, 1]$. On the other hand, $\tilde{\psi}(\beta_d) = \tilde{\psi}(\beta_{inf}) = 0$ by $\beta_{inf} \notin I_{inf}$ as shown for the case (i) above, a contradiction. \qed
Proof of Theorem 1.3. Let $P(\alpha)$ be as in [17]. First, we consider the case $\sqrt{\tau}/(1 + \sqrt{\tau}) < \gamma < 1$. Now, we divide this proof as two cases:

\[ \beta_d > \beta_{inf} \quad \text{and} \quad \beta_d \leq \beta_{inf}. \]

First, we consider the case $\beta_d > \beta_{inf}$. In this case, by Lemma 2.8 and (110), we have,

\[ \tau = \frac{\tau^2}{(\tau + (1 - \tau)\gamma)^2}, \]

that is, $\gamma = \sqrt{\tau}/(1 + \sqrt{\tau})$ which is a contradiction to $\gamma > \sqrt{\tau}/(1 + \sqrt{\tau})$. Therefore we just consider the case $\beta_d \leq \beta_{inf}$. Note that we have $\gamma = \beta_{inf}$ and $\gamma \in \mathcal{B}$. Indeed, if $\gamma < \beta_{inf}$ or $\gamma \notin \mathcal{B}$ holds then we can lead a contradiction by the same argument of the case $\beta_d > \beta_{inf}$ thanks to Proposition 4.1. Since $\gamma = \beta_{inf}$ and $\gamma \in \mathcal{B}$ holds, by Proposition 4.1 we have

\[ \tilde{\psi}(\beta) = \chi_{(\gamma,1)}(\beta) \quad P-a.e \beta. \quad (113) \]

By (113) and Proposition 2.2, we obtain the following identity:

\[ \int_{\mathbb{R}^2} \tilde{f} dy = \chi \int_{I_u} \beta \chi_{(\gamma,1)}(\beta) P(d\beta) = \chi(\tau + (1 - \tau)\gamma). \]

By Proposition 1.2, the above identity implies the estimate (15).

Next, in the case $0 < \gamma < \sqrt{\tau}/(1 + \sqrt{\tau})$ we suppose the identity (16) holds. From this assumption and Proposition 1.2, the residual vanishing occurs and we have

\[ \chi = \frac{8\pi}{(\tau + (1 - \tau)\gamma)^2}, \]

which is a contradiction to $\chi = 8\pi/\tau < 8\pi/(\tau + (1 - \tau)\gamma)^2$.

Remark 4.2. In the case of $0 < \gamma < \sqrt{\tau}/(1 + \sqrt{\tau})$, we use the property of $\chi$ again for the contradiction.

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Appendix  Proof of Lemma 2.3

Given $K > 0$, we put

$$I_1(x) = \int_{D_1} \frac{\log |x - y| - \log(1 + |y|) - \log |x|}{\log |x|} f(y) dy,$$

$$I_{2,K}(x) = \int_{D_{2,K}} \frac{\log |x - y| - \log(1 + |y|) - \log |x|}{\log |x|} f(y) dy,$$

$$I_{3,K}(x) = \int_{D_{3,K}} \frac{\log |x - y| - \log(1 + |y|) - \log |x|}{\log |x|} f(y) dy,$$

where,

$$D_1 = D_1(x) = \{ y \in \mathbb{R}^2 \, | \, |y - x| < 1 \},$$

$$D_{2,K} = D_{2,K}(x) = \{ y \in \mathbb{R}^2 \, | \, |y - x| > 1, |y| \leq K \},$$

$$D_{3,K} = D_{3,K}(x) = \{ y \in \mathbb{R}^2 \, | \, |y - x| > 1, |y| > K \}.$$

Then it holds that

$$\frac{z(x)}{\log |x|} = \frac{\beta_0}{2\pi} = \frac{1}{2\pi} (I_1(x) + I_{2,K}(x) + I_{3,K}(x)).$$

We have only to show that each $\epsilon > 0$ admits $K_\epsilon$ and $L_\epsilon$ such that

$$|I_1(x)| + |I_{2,K}(x)| + |I_{3,K}| \leq \epsilon$$

(114)

for all $x \in \mathbb{R}^2 \setminus B_{L_\epsilon}$.

Since

$$\frac{\log(1 + |y|) + \log |x|}{\log |x|} \leq \frac{\log(2 + |x|) + \log |x|}{\log |x|} \leq 3, \quad x \in \mathbb{R}^2 \setminus B_2, \quad y \in D_1(x),$$

we have,

$$|I_1(x)| \leq 3 \int_{D_1} f(y) dy + \frac{1}{\log |x|} \int_{D_1} f(y) \log |x - y| dy$$

$$\leq 3 \int_{D_1} f(y) dy + \frac{\|f\|_{\infty}}{\log |x|} \int_{D_1} f(y) \log |y| dy \to 0$$

(115)

uniformly as $|x| \to +\infty$, recalling $f \in L^1 \cap L^\infty(\mathbb{R}^2)$.

Next, we have

$$\left| \frac{\log |x - y| - \log(1 + |y|) - \log |x|}{\log |x|} \right| \leq \frac{1}{\log |x|} \left\{ \log(1 + K) + \log \frac{|x - y|}{|x|} \right\}$$

31
for \( x \in \mathbb{R}^2 \setminus B_2 \) and \( y \in D_{2,K}(x) \), and thus

\[
|I_{2,K}(x)| \leq \int_{D_{2,K}(x)} \left\{ \log(1 + K) + \log \frac{|x - y|}{|x|} \right\} f(y) dy \tag{116}
\]

for \( x \in \mathbb{R}^2 \setminus B_2 \). From

\[
\frac{1}{2 + |x|} \leq \frac{|x - y|}{1 + |y|} \leq 1 + |x|, \quad x \in \mathbb{R}^2, \quad |y - x| \geq 1,
\]

we derive

\[
\left| \frac{\log |x - y| - \log(1 + |y|) - \log |x|}{\log |x|} \right| \leq 3, \quad x \in \mathbb{R}^2 \setminus B_2, \quad |y - x| \geq 1,
\]

to obtain

\[
|I_{3,K}(x)| \leq 3 \int_{D_{3,K}(x)} f(y) dy \leq 3 \int_{\mathbb{R}^2 \setminus B_K} f(y) dy \tag{117}
\]

for \( x \in \mathbb{R}^2 \setminus B_2 \).

Recalling \( 0 \leq f \in L^1(\mathbb{R}^2) \), let \( \epsilon_0 > 0 \) be given. From \( \text{(117)} \), there exists \( K_0 > 0 \) such that

\[
|I_{3,K}(x)| \leq \epsilon_0
\]

for all \( K \geq K_0 \) and \( x \in \mathbb{R}^2 \setminus B_2 \). Next, by \( \text{(115)} \) any \( K > 0 \) admits \( L_K > 0 \) such that

\[
|I_{2,K}(x)| \leq \epsilon_0
\]

for all \( x \in \mathbb{R}^2 \setminus B_{L_K} \), and therefore

\[
|I_{2,K_0}(x)| + |I_{3,K_0}(x)| \leq 2\epsilon_0 \tag{118}
\]

for all \( x \in \mathbb{R}^2 \setminus B_{L_K} \).

Thus we obtain \( \text{(114)} \) by \( \text{(115)} \) and \( \text{(118)} \).

\[\square\]

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