Stochastic evolution of finite level systems: classical versus quantum

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Abstract

Quantum dynamics of the density operator in the framework of a single probability vector is analyzed. In this framework, quantum states define a proper convex quantum subset in an appropriate simplex. It is shown that the corresponding dynamical map preserving a quantum subset need not be stochastic, contrary to the classical evolution that preserves the entire simplex. Therefore, violation of stochasticity witnesses the quantumness of evolution.

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1. Introduction

The basic idea of quantum tomography is to encode the information of a quantum state in a family of probability distributions. This idea goes back to Pauli [1], who considered the possibility of associating quantum states with marginal probability distributions in the phase space, both for positions and momenta, as in classical statistical mechanics. In spite of the fact that these marginal distributions do not determine the quantum state wave function [2], that idea was realized by considering an infinite family of marginal distributions (tomograms).

The tomographic representation is based on the Radon transform of the Wigner function which relates the measurable optical tomographic probability to reconstruct the Wigner function of the quantum state (see [3] for a recent review). A similar tomographic approach may be constructed for finite level quantum systems (spin systems) corresponding to finite-dimensional Hilbert space $\mathcal{H}$. In this case a spin (or qudit) density operator $\rho$ can be described [7] by probability distributions of random spin projection (spin tomogram) depending on the direction of the quantization axis. Such a family of probability distribution encodes the complete information about the qudit state and hence a density operator may be uniquely recovered out of a spin tomogram.

Quantum tomography enables one to encode the information about $\rho$ into the finite family of probability distributions (see the next section) and eventually to represent $\rho$ as a single probability vector living in an appropriate simplex. It is clear that only a convex quantum subset within this simplex corresponds to legitimate density operators. Using this stochastic representation $\rho \rightarrow P$ one may translate a quantum evolution of $\rho(t)$ into a stochastic evolution of $P(t)$. The problem of stochastic evolution and dynamical maps of density matrices was studied and reviewed recently [8–10]. In this way, both classical and quantum evolution of finite level systems may be described within the same framework. Now comes a natural question: suppose that one is given the evolution of a stochastic vector $P(t) = T(t)P$ represented by a semigroup of maps $T(t)$ satisfying $T(t+u) = T(t)T(u)$ for all $t, u \geq 0$. Is it possible to discriminate whether $T(t)$ describes the purely classical evolution of $P$ or a quantum evolution of $\rho$ encoded in $P$? It is clear that classical dynamics is governed by stochastic matrices $T(t)$. Interestingly, it need not be true any longer for quantum dynamics. Some preliminary aspects of this phenomenon were mentioned earlier in [11, 12]. Note that in the quantum case, one requires that $T(t)$ maps probability vectors $P$ within the quantum subset of the simplex. In particular, it turns out that if the dynamics of $\rho(t)$ is unitary, then $T(t)$ is never a stochastic family of maps. This way, violation of stochasticity witnesses the quantumness of evolution.

This paper is organized as follows. In section 2, we review the tomographic approach to quantum mechanics. In
section 3, we show how to encode the information about a density operator $\rho$ into a single probability vector $P$ living in an appropriate simplex. Vectors corresponding to legitimate quantum states form a convex quantum subset within a simplex. It is shown how the evolution of $\rho(t)$ induces the evolution of $P(t)$. Finally, section 4 provides several examples of qubit dynamics in the framework of six-dimensional probability vector $P$. The conclusions are presented in section 5.

2. Tomography of spin states

In this section, we review the approach [3] to describing the quantum states of a spin (qudit) by means of the probability distributions called quantum tomograms. The tomogram $W(m|U)$ corresponding to a qudit state with the density operator $\rho$ is given by the diagonal matrix elements of the unitarily rotated density operator in the standard computational basis $|m\rangle$, where $-j \leq m \leq j$ and $J_z|m\rangle = m|m\rangle$, with $J_z$ being a spin projection operator on the $z$-axis. Thus, one has

$$W(m, U) = |m\rangle U^{-1} \rho U |m\rangle,$$

(1)

where $U$ is a $(2j+1) \times (2j+1)$ unitary matrix. In the case where the matrix $U$ is a matrix of irreducible representations of the group $SU(2)$, the tomogram $W(m|\mathbf{n})$, called a spin tomogram, introduced in [4, 5] depends on the variable $\mathbf{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$. Spherical angles $(\theta, \varphi)$ determine the point on the Poincaré sphere. Both tomograms $W(m|U)$ and $W(m|\mathbf{n})$ define probability distributions, that is, $W(m, U) \geq 0$, $W(m, \mathbf{n}) \geq 0$, and the normalization condition

$$\sum_{m=-j}^{j} W(m, U) = 1, \quad \sum_{m=-j}^{j} W(m, \mathbf{n}) = 1.$$

(2)

These properties mean that the tomograms can be interpreted as conditional probability distributions $W(m, U) \equiv W(m|U)$ and $W(m, \mathbf{n}) \equiv W(m|\mathbf{n})$ which can be associated with a joint probability distribution $P(m|U)$ or $P(m, \mathbf{n})$. The construction of the probability distribution $P(m, \mathbf{n})$ in the form $P(m, \mathbf{n}) = W(m, \mathbf{n}) \Pi(\mathbf{n})$, where $\Pi(\mathbf{n})$ is a probability distribution on the sphere $S^2$, was discussed for particular cases in [13, 14] and the probability vector with components $W(m, U_j) (j = 1, \ldots, N + 1)$ for qudit states was considered in [15]. An important property of this probability vector is that for the corresponding density operator $\rho$ describing the quantum states, the vector components occupy only a subset of the corresponding simplex. To see this, let us consider the conditional probability

$$p(m|\alpha) = \langle m|U_\alpha \rho U_\alpha^\dagger |m\rangle$$

(3)

for $m = 1, \ldots, N$ and $\alpha = 1, \ldots, N + 1$ (with $N = 2j + 1$). One assumes that the set of unitaries $\{U_1, \ldots, U_{N+1}\}$ defines a quorum, that is, one may reconstruct $\rho$ out of $p(m|\alpha)$. Note that $p(m|\alpha)$ encodes $(N + 1) \times (N - 1) = N^2 - 1$ real parameters, that is, exactly the same number as the density operator $\rho$ in the $N$-dimensional complex Hilbert space. Fixing the $(N + 1)$-dimensional probability distribution $\pi_\alpha$, that is, we take into account the measurement of $U_\alpha$ with a weight $\pi_\alpha > 0$, one defines

$$p_k^{(\alpha)} := p(k|\alpha)\pi_\alpha,$$

(4)

and finally, the $(N(N + 1))$-dimensional probability vector

$$P = (p_1^{(1)}, \ldots, p_N^{(1)}, \ldots, p_1^{(N+1)}, \ldots, p_N^{(N+1)})^\dagger.$$

(5)

It is clear that only very special vectors $P$ correspond to legitimate quantum states on the $N$-level quantum system. Any such vector has to respect $(N + 1)$ linear constraints

$$p_1^{(\alpha)} + \cdots + p_N^{(\alpha)} = \pi_\alpha$$

(6)

for each $\alpha = 1, 2, \ldots, N + 1$. It is clear that a proper set of $U_\alpha$ enables one to reconstruct $\rho$ out of $p^{(\alpha)}$. Note that positivity of $\rho$, that is, $\rho \geq 0$, provides a highly non-trivial constraint for a set of probabilities $p^{(\alpha)}$ and hence for a set of admissible vectors $P$. Hence, probability vectors $P$ corresponding to legitimate quantum states define only a proper $(N^2 - 1)$-dimensional convex subset $Q_N$ of $(N^2 + N - 1)$-dimensional simplex $S_{N(N+1)}$.

To summarize: using a tomographic approach with a fixed quorum $U_\alpha$, any quantum state $\rho$ of an $N$-level quantum system may be attributed a probability vector $P$ in $N(N + 1)$-simplex. Vectors compatible with legitimate quantum states define a proper convex subspace $Q_N$.

As an illustration consider a qubit state $\rho$. Using the Bloch representation, one has

$$\rho = \frac{1}{2} \left( |0\rangle \langle 0| + x |0\rangle \langle 1| + |1\rangle \langle 0| + y |1\rangle \langle 1| + z |1\rangle \langle 1| \right).$$

(7)

The requirement $\rho \geq 0$ provides the following constraint for the Bloch vector $r = (x, y, z)$:

$$|r|^2 = x^2 + y^2 + z^2 \leq 1.$$

(8)

The corresponding conditional probability $p(m|\alpha)$ reads

$$p(1|x) = \frac{1 + x}{2}, \quad p(2|x) = \frac{1 - x}{2},$$

$$p(1|y) = \frac{1 + y}{2}, \quad p(2|y) = \frac{1 - y}{2},$$

$$p(1|z) = \frac{1 + z}{2}, \quad p(2|z) = \frac{1 - z}{2},$$

where as a quorum one takes $\{U_1 = \sigma_x, U_2 = \sigma_y, U_3 = \sigma_z\}$. Hence, fixing a probability distribution $(\pi_x, \pi_y, \pi_z)$, one defines

$$P = \begin{pmatrix} p_1^{(1)} & p_2^{(1)} & p_1^{(2)} & p_2^{(2)} \\ p_1^{(1)} & p_2^{(1)} & p_1^{(2)} & p_2^{(2)} \\ p_1^{(1)} & p_2^{(1)} & p_1^{(2)} & p_2^{(2)} \end{pmatrix}.$$

(9)

One easily finds the following relations:

$$p_1^{(1)} = \frac{\pi_x}{2} (1 + [\rho_{12} + \rho_{21}]), \quad p_1^{(2)} = \frac{\pi_y}{2} (1 - i[\rho_{12} - \rho_{21}]),$$

$$p_1^{(3)} = \pi_z \rho_{11}, \quad p_2^{(3)} = \frac{\pi_x}{2} (1 - [\rho_{12} + \rho_{21}]),$$

$$p_2^{(3)} = \frac{\pi_y}{2} (1 + i[\rho_{12} - \rho_{21}]), \quad p_2^{(2)} = \pi_z \rho_{22}.$$
Vectors $p_k^{(a)}$ satisfy three linear constraints
\[
p_1^{(x)} + p_2^{(x)} = \pi_x, \quad p_1^{(y)} + p_2^{(y)} = \pi_y, \quad p_1^{(z)} + p_2^{(z)} = \pi_z.
\] (10)

The quadratic constraint (8) yields
\[
\frac{a_s}{\pi_x} + \frac{a_s}{\pi_y} + \frac{a_s}{\pi_z} \leq 1,
\] (11)

where
\[
a_s = \frac{\pi_x}{2}, \quad a_s = \frac{\pi_y}{2}, \quad a_s = \frac{\pi_z}{2}.
\] (12)

Hence a Bloch ball (8) is transformed into an ellipsoid (11). Note that for a uniform distribution $\pi_x = \pi_y = \pi_z = \frac{1}{3}$ the above ellipsoid reduces to a ball.

3. Classical versus quantum dynamics of probability vectors

Both classical and quantum states of finite level systems can be encoded as probability vectors in the corresponding simplex. Now comes a crucial question: suppose we are given a probability vector $P_0$ and its evolution $P(t)$ for $t \geq 0$. Can we discriminate between classical and quantum evolution?

Let us briefly recall a standard description of classical and quantum stochastic dynamics. Consider first $N$-level classical systems described by the following master equation:
\[
\frac{d}{dt} P(t) = MP(t), \quad P(0) = P_0,
\] (13)

where $P = (p_1, \ldots, p_N)$ denotes a probability distribution, and $M$ stands for the time-independent generator. The solution defines a classical dynamical map $T(t)$ such that $P(t) = T(t)P_0$. It is clear that $T(t)$ is a stochastic $N \times N$ matrix such that $T(0) = I_N$. Let us recall that a real $n \times n$ matrix $T$ is stochastic iff: (i) $T_{ij} \geq 0$ and (ii) $\sum_j T_{ij} = 1$ for all $j = 1, \ldots, n$. The master equation (14) rewritten in terms of $T(t)$ reads
\[
\frac{d}{dt} T(t) = M T(t), \quad T(0) = I_N
\] (14)

and hence the solution is given by $T(t) = e^{MT}$. It is well known [16] that $e^{MT}$ provides a stochastic matrix for all $t \geq 0$ if and only if $M$ satisfies the following Kolmogorov conditions: (i) $M_{ij} \geq 0$ for $i \neq j$ and (ii) $\sum_j M_{ij} = 0$ for all $j = 1, \ldots, N$. In this case $M$ possesses the following representation:
\[
M_{ij} = \pi_{ij} - \delta_{ij} \sum_k \pi_{kj}
\] (15)

with $\pi_{ij} \geq 0$ for $i \neq j$. Actually, the diagonal elements $\pi_{ii}$ cancel out and hence they are completely arbitrary. Putting this form into (14), one arrives at the classical Pauli rate equation
\[
\frac{d}{dt} p_i(t) = \sum_{j=1}^N \left[ \pi_{ij} p_j(t) - \pi_{ji} p_i(t) \right],
\] (16)

where $\pi_{ij}$ play the role of transition rates.

Consider now the dynamics of a quantum $n$-level system. The corresponding quantum master equation reads
\[
\frac{d}{dt} \rho(t) = L\rho(t), \quad \rho(0) = \rho_0,
\] (17)

where $\rho$ denotes a density operator and $L$ stands for the time-independent (quantum) generator. The corresponding quantum evolution gives rise to the family of dynamical maps $\Lambda(t) : \mathcal{T}(H) \rightarrow \mathcal{T}(H)$ such that $\rho(t) = \Lambda(t)\rho_0$. For any $t \geq 0$ a map $\Lambda(t)$ is completely positive and trace preserving (i.e. defines a quantum channel). The formal solution $\Lambda(t) = e^{tL}$ defines a legitimate dynamical map if and only if the corresponding generator has a well-known Gorini–Kossakowski–Sudarshan–Lindblad form [17, 18] (see also [19])
\[
L\rho = -i[H, \rho] + \frac{1}{2} \sum_k \left( [V_k, \rho, V_k^\dagger] + [V_k, \rho V_k^\dagger] \right),
\] (18)

where $H^\dagger = H \in \mathfrak{B}(H)$ and $V_k \in \mathfrak{B}(H)$ are arbitrary. Interestingly, if one fixes an orthonormal basis $|e_1, \ldots, e_n\rangle$ in $H$, then the following matrix:
\[
M_{ij} = Tr\{P_i L(P_j)\},
\] (19)

where $P_i = |e_i\rangle\langle e_i|$, satisfies the Kolmogorov conditions, and
\[
T_{ij}(t) = Tr\{P_i \Lambda(t)(P_j)\}
\] (20)

is stochastic for all $t \geq 0$.

Up to now, we have described classical and quantum evolutions using different frameworks: stochastic evolution of a probability vector and completely positive evolution of a density operator. To compare how classical and quantum systems evolve, let us encode $\rho$ as an $n(n+1)$-dimensional probability vector $P \in Q_n$. It is clear that the quantum master equation (17) rewritten in terms of $P$ gives rise to the following linear equation:
\[
\frac{d}{dt} P(t) = \mathbb{T}(P(t))
\] (21)

and the corresponding solution has the following form: $P(t) = \mathbb{T}(t)P_0$, with $\mathbb{T}(t) = e^{t\mathbb{M}}$.

Now comes the question: suppose one is given a master equation (21) for a probability vector $P \in Q_n \subset \mathbb{C}^{n(n+1)}$. Is it possible to discriminate whether $P(t)$ represents classical dynamics of an $n(n+1)$-level system or quantum dynamics of an $n$-level system encoded in $P$? Any $\mathbb{M}$ satisfying Kolmogorov conditions gives rise to legitimate classical dynamics. It means that in the classical case, $\mathbb{T}(t) = e^{t\mathbb{M}}$ is always a stochastic matrix for $t \geq 0$. Interestingly, for quantum dynamics it is no longer the case. It means that if $\mathbb{T}(t)$ represents quantum evolution it need not be a family of stochastic maps. Equivalently, the corresponding generator need not have a Kolmogorov form. Hence, whenever $\mathbb{T}(t)$ violates stochasticity it proves the quantumness of the corresponding evolution.
4. Examples: qubit dynamics

In this section, we provide simple examples of qubit dynamics giving rise to the evolution $T(t)$ which need not be stochastic.

**Proposition 1.** If $L\rho = -i[H, \rho(t)]$, then the corresponding $M$ is never of a Kolmogorov form.

Indeed, note that in this case the dynamics is invertible, that is, $L$ and $-L$ share the same property. Now, if $M$ were of a Kolmogorov form, then $-M$ is not and hence the symmetry is broken. Note that $M_{ij} = \text{tr}[PL(P_j)] \equiv 0$.

**Example 1.** Consider a Schrödinger evolution of a qubit state

\begin{equation}
\frac{d}{dt} \rho(t) = -i[H, \rho(t)]
\end{equation}

with $H = \omega \sigma_z$. One finds that

\[
M = \omega \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2\mu & 2\mu \\ 0 & 2\mu & 0 & -2\mu \\ 0 & -2\mu & 0 & 0 \end{pmatrix}
\]

(23)

with $\mu = \pi_\gamma / \pi_z$. It is clear that $M$ is not a Kolmogorov generator and hence the evolution $T(t) = e^{\omega t}M$ is not stochastic. We stress that a sign of $\omega$ does not play any role.

One might be tempted to expect that purely dissipative quantum dynamics leads to legitimate stochastic evolution $T(t)$. To show that this is not the case, let us consider an elementary generator constructed out of a single operator $V$:

\begin{equation}
L\rho = V\rho V^\dagger - \frac{1}{2}(V^\dagger V\rho + \rho V^\dagger V).
\end{equation}

(24)

One finds for the diagonal elements

\[
\dot{\rho}_{11} = -\gamma \rho_{11} + \gamma \rho_{22} + \kappa \rho_{12} + \kappa \rho_{21},
\]

\[
\dot{\rho}_{22} = -\gamma \rho_{22} - \kappa \rho_{12} - \kappa \rho_{21},
\]

where

\[
\gamma = |V_{12}|^2, \quad \kappa = V_{12}^\dagger V_{21} - \frac{1}{2} (2|V^\dagger V|1) = \frac{1}{2} (V_{11}^\dagger V_{21} - V_{21}^\dagger V_{12})
\]

and hence

\[
\frac{d}{dt} p_{11}^{(x)} = -\gamma p_{11}^{(x)} + \gamma p_{11}^{(x)} X_1 p_{11}^{(x)} + X_2 p_{11}^{(x)} + Y_1 p_{11}^{(y)} + Y_2 p_{11}^{(y)},
\]

\[
\frac{d}{dt} p_{22}^{(x)} = \gamma p_{11}^{(x)} + \gamma p_{11}^{(x)} - X_1 p_{11}^{(x)} - X_2 p_{11}^{(x)} - Y_1 p_{11}^{(y)} - Y_2 p_{11}^{(y)}
\]

for real parameters $X_1, X_2, Y_1, Y_2$ (they are easily calculatable in terms of $\kappa$ and $\gamma$). It is therefore clear that unless $X_1 = X_2 = Y_1 = Y_2 = 0$ the corresponding dynamics of a probability vector $P$ cannot be stochastic; indeed if $X_1 > 0$, then $-X_1 < 0$ and hence the generator does not satisfy the Kolmogorov conditions $M_{ij} \geq 0$ for $i \neq j$. The above analysis gives rise to the following.

**Theorem 1.** A legitimate quantum evolution $\Lambda(t)$ gives rise to legitimate stochastic evolution $T(t)$ if and only if $T(t)$ is block diagonal, that is,

\[
T(t) = \bigoplus_a T^{(a)}(t)
\]

and each $T^{(a)}(t)$ defines stochastic evolution of the probability distribution $p(m|\alpha) = p^{(m)}_a / \pi_a$. Equivalently, the corresponding generator

\[
M = \bigoplus_a M^{(a)}
\]

(26)

and each $M^{(a)}$ satisfies Kolmogorov conditions.

Interestingly, several well-known quantum generators lead to a legitimate Kolmogorov form of $M$.

**Example 2.** Consider a dissipative evolution governed by the following master equation:

\[
\frac{d}{dt} \rho(t) = -i\frac{\omega}{2} [\sigma_z, \rho(t)] + L_D \rho(t),
\]

(27)

where the dissipative part $L_D$ is defined by

\[
L_D \rho = \frac{1}{2} \gamma_1 (|\sigma_+ \sigma_-\rangle + |\sigma_- \sigma_+\rangle) + \gamma_2 (|\sigma_\pi \sigma_\pi\rangle - |\sigma_\pi \sigma_\pi\rangle + |\sigma_\pi \sigma_\pi\rangle)
\]

(28)

with $\sigma_+ = |2\rangle\langle 1|$ and $\sigma_- = |1\rangle\langle 2|$, the standard raising and lowering operators. One easily finds the corresponding generator

\[
M = \frac{1}{2} \begin{pmatrix} -\Gamma & \Gamma & -\omega v & \omega v & 0 & 0 \\ \Gamma & -\Gamma & -\omega v & \omega v & 0 & 0 \\ -\omega v & -\omega v & -\Gamma & \Gamma & 0 & 0 \\ \omega v & \omega v & \Gamma & -\Gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\gamma_1 & 2\gamma_2 \\ 0 & 0 & 0 & 0 & 2\gamma_1 & -2\gamma_2 \end{pmatrix}
\]

(29)

where $\Gamma = \frac{1}{2} (\gamma_1 + \gamma_2) + \gamma_1$ and $\nu = \pi_\gamma / \pi_z$. Interestingly, block-diagonal blocks satisfy separately Kolmogorov conditions. Note that if $\omega = 0$, i.e. dynamics is purely dissipative, then $M$ is block diagonal and hence perfectly of the Kolmogorov form.

**Example 3.** Consider the following generator:

\[
L \rho = \sum_{k=1}^3 \gamma_k (\sigma_k \rho \sigma_k - \rho)
\]

(30)

with $\gamma_k \geq 0$. The evolution generated by the above generator is called random unitary since it is defined by the following formula:

\[
\Lambda(t) \rho = \sum_{m=0}^3 p_\mu(t) \sigma_\mu \rho \sigma_\mu,
\]

(31)
where \( p_j(t) \) is a probability vector for \( t \geq 0 \). One finds that
\[
\begin{align*}
p_0(t) &= \frac{1}{4} \left[ 1 + A_{12}(t) + A_{13}(t) + A_{23}(t) \right], \\
p_1(t) &= \frac{1}{4} \left[ 1 - A_{12}(t) - A_{13}(t) + A_{23}(t) \right], \\
p_2(t) &= \frac{1}{4} \left[ 1 - A_{12}(t) + A_{13}(t) - A_{23}(t) \right], \\
p_3(t) &= \frac{1}{4} \left[ 1 + A_{12}(t) - A_{13}(t) - A_{23}(t) \right],
\end{align*}
\]
where \( A_{ij}(t) = e^{-2((\gamma_i + \gamma_j) t)} \). Using the following matrix representation:
\[
L \rho = \begin{pmatrix}
(\gamma_1 + \gamma_2)(\rho_{22} - \rho_{11}) & -\rho_{12}(\gamma_1 + \gamma_2 + 2\gamma_3) \\
-\rho_{21}(\gamma_1 + \gamma_2 + 2\gamma_3) & (\gamma_1 + \gamma_2)(\rho_{11} - \rho_{22})
\end{pmatrix},
\]
one easily finds that
\[
\mathcal{M} = M^{(x)} \oplus M^{(y)} \oplus M^{(z)}
\]
with
\[
M^{(x)} = \begin{pmatrix}
-\gamma_x & \gamma_x \\
\gamma_x & -\gamma_x
\end{pmatrix}, \quad M^{(y)} = \begin{pmatrix}
-\gamma_y & \gamma_y \\
\gamma_y & -\gamma_y
\end{pmatrix}, \quad M^{(z)} = \begin{pmatrix}
-\gamma_z & \gamma_z \\
\gamma_z & -\gamma_z
\end{pmatrix}
\]
and
\[
\gamma_x = \gamma_2 + \gamma_3, \quad \gamma_y = \gamma_1 + \gamma_3, \quad \gamma_z = \gamma_1 + \gamma_2.
\]
Hence random unitary dynamics of a qubit provides perfectly stochastic dynamics of the six-dimensional probability vectors \( \mathcal{P} \). Interestingly, one has three two-dimensional stochastic evolutions
\[
\frac{d}{dt} p^{(a)}(t) = M^{(a)} p^{(a)}(t)
\]
with \( a = x, y, z \) and \( M^{(a)} = \gamma_a (\sigma_a - \mathbb{I}_2) \). Note that asymptotically \( p^{(a)}(t) \to p^{(a)}(\frac{1}{2}, \frac{1}{2})^T \), i.e., each two-dimensional probability vector \( p^{(a)}/\pi_a \) becomes maximally mixed.

5. Conclusions

To summarize, we point out our main results. The dynamics of random classical systems is usually described by the time dependence of a probability vector which is identified with the system state. The linear dynamics is associated with maps realized by stochastic matrices forming a semigroup. In the framework of the tomographic approach, the state of quantum systems are also identified with probability vectors. But these vectors are located in a subdomain of the simplex, which is the usual region for classical probability vectors. The quantum subdomain is determined by constraints that all the probability vectors determining the quantum system states adhere to. The linear maps of this subdomain form a semigroup. The matrices realizing the linear maps are not only stochastic matrices, as they may contain negative matrix elements (an analogue of quantum quasi-distributions like the Wigner function).

We analyzed the quantum dynamics of the density operator in the framework of a single probability vector \( \mathcal{P} \). Interestingly, the corresponding dynamical map \( \mathcal{T}(t) \) need not be stochastic, contrary to the classical evolution of \( \mathcal{P} \). Therefore, violation of stochasticity witnesses the quantumness of evolution. It turns out that unitary dynamics always violates stochasticity. We showed that the purely dissipative dynamics of a qubit analyzed in examples 2 and 3 is perfectly stochastic being a direct sum of three stochastic evolutions. It seems that this feature is responsible for the solvability of these models.

We will present in a future paper other examples of dynamical quantum maps belonging to semigroups of matrices, with both non-negative and negative matrix elements.

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References

[1] Pauli W 1933 Handbuch der Physik vol XXIV, Part I ed H Geiger and K Scheel (Berlin: Springer) Reprinted in Pauli W 1958 Encyclopedia of Physics vol V, Teil I (Berlin: Springer)
[2] Reichenbach H 1942 Philosophy of Quantum Mechanics (Los Angeles, CA: University of California Press)
[3] Ibort A, Man’ko V I, Marmo G, Simoni A and Ventriglia F 2009 Phys. Scr. 79 065013
[4] Dodonov V V and Man’ko V I 1997 Phys. Lett. A 229 335
[5] Man’ko V I and Man’ko O V 1997 J. Exp. Theor. Phys. 85 430
[6] Man’ko O V, Man’ko V I and Marmo G 2000 Phys. Scr. 62 446
[7] Man’ko V I, Marmo G, Simoni A and Ventriglia F 2008 Phys. Lett. A 372 6490
[8] Chruściński D and Kossakowski A 2012 J. Phys. B: At. Mol. Opt. Phys. 45 154002
[9] Chruściński D, Kossakowski A, Aniello P, Marmo G and Ventriglia F 2010 Open Syst. Inform. Dyn. 17 255
[10] Chruściński D and Kossakowski A 2012 Europhys. Lett. 97 20005
[11] Filippov S N and Man’ko V I 2010 J. Russ. Laser Res. 31 32
[12] Belousov Y M, Filippov S N, Man’ko V I and Traskunov I V 2011 J. Russ. Laser Res. 32 584
[13] Albini P, De Vito E and Toigo A 2009 J. Phys. A: Math. Theor. 42 295302
[14] Bellini M, Coelho A S, Filippov S N, Manko V I and Zavatta A 2012 Phys. Rev. A 85 052129
[15] Man’ko M A and Man’ko V I 2012 AIP Conf. Proc. 1488 110
[16] van Kampen N 1992 Stochastic Processes in Physics and Chemistry (Amsterdam: North-Holland)
[17] Gorini V, Kossakowski A and Sudarshan E C G 1976 J. Math. Phys. 17 821
[18] Lindblad G 1976 Commun. Math. Phys. 48 119
[19] Alicki R and Lendi K 1987 Quantum Dynamical Semigroups and Applications (Berlin: Springer)