Dickson algebras are atomic at $p$

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Abstract. The notion of atomicity defined by Cohen, Moore and Neisendorfer is studied for the Dickson algebras. Not any ring of invariants respects this property. It depends on the property of the Dickson algebra that given any monomial $d$ there exists a sequence of Steenrod operations $P(\Gamma, d)$ such that $P(\Gamma, d) d$ becomes a $p$-th power of the top Dickson algebra generator.

1. Statement of results

The term atomic was introduced by Cohen, Moore and Neisendorfer in ([1]) to answer the question of whether a given space admits a non-trivial product decomposition up to homotopy. We consider the analogue question for the Dickson algebra. Let $V^n$ denote an $n$-dimensional vector space over $\mathbb{Z}/p\mathbb{Z}$, then

$$H^*(BV^n; \mathbb{Z}/p\mathbb{Z}) \cong E(x_1, \cdots, x_n) \otimes P[y_1, \cdots, y_n].$$

$P[y_1, \cdots, y_n]_{GL_n} = \mathbb{Z}/p\mathbb{Z}[d_{n,1}, \cdots, d_{n,n-1}, d_{n,n}]$ denotes the classical Dickson algebra which is a graded polynomial algebra and $D_n := (E(x_1, \cdots, x_n) \otimes P[y_1, \cdots, y_n])_{GL_n}$ the extended Dickson algebra studied by Mui ([II]).

In a series of papers, Campbell, Cohen, Peterson and Selick studied self maps on certain important spaces in topology. In order to prove homotopy equivalence at $p$, they considered the corresponding $mod - p$ homology homomorphisms. So they had to use the Dyer-Lashof algebra and (or) quotients of it as the main ingredient. It is well known that the Dyer-Lashof algebra is closely related with Dickson algebras. Hence, they studied and used properties of the Dickson algebras, especially

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papers [2] and [3]. The advantage is that the Steenrod algebra action on Dickson algebras is better understood than the Nishida relations on the Dyer-Lashof algebra.

Motivated by topological questions regarding the cohomology of an infinite loop space and strongly influenced by the work of Campbell, Cohen, Peterson and Selick in [2] and [3] we study the problem under which conditions is a degree preserving $A$-endomorphism of $D_n$ an isomorphism. Here $A$ stands for the Steenrod algebra.

**Theorem.** [27] The extended Dickson algebra $D_n$ is atomic as a Steenrod algebra module.

The proof depends on a remarkable property that $D_n$ satisfies with respect to its Steenrod algebra action.

**Theorem.** [11] Let $d^K = \prod_{1}^{n} d_{n,i}^{p(i)}$. There exists a sequence of Steenrod operations $P(\Gamma, K)$ such that

$$P(\Gamma, K) d^K = u d_{n,n}^p,$$

for some natural number $m$ and unit $u$.

In particular,

**Corollary.** [29] $D_n$ is not directly decomposable as an $A$-module.

Here $D_n$ denotes the augmentation ideal of $D_n$ and corollary implies that the only direct summands are 0 and $D_n$.

Finally we apply Theorem [27] to the study of self maps between $Q_0 S^0$.

**Theorem.** [30] Let $f : Q_0 S^0 \to Q_0 S^0$ be an $H$-map which induces an isomorphism on $H_{2p-3}(Q_0 S^0; \mathbb{Z}/p\mathbb{Z})$. Let $p > 2$ and

$$f_*(Q^{(p,1)}[1]) = u Q^{(p,1)}[1] + \text{others}$$

for some $u \in (\mathbb{Z}/p\mathbb{Z})^*$. Then $f_*$ is an isomorphism.

Last Theorem has been proved by Campbell, Cohen, Peterson and Selick in [3] for $p = 2$.

We recently informed that Pengelley and Williams have studied similar properties of the Dickson algebra in [13].

2. Classical Dickson algebras are atomic at $p$

The term atomic was defined by Cohen, Moore and Neisendorfer. Let us recall from [2] that a CW complex $X$ is atomic at $p$, if given any map $f : X \to X$ such that $f$ induces an isomorphism on $H_r(X, \mathbb{Z}/p\mathbb{Z})$, then $f_{(p)}$ is a homotopy equivalence. Here $r$ is the lowest degree such that $H_r(X, \mathbb{Z}/p\mathbb{Z}) \neq 0$.

In this section we study the action of the Steenrod algebra on monomials of the Dickson algebra. We prove that its augmentation ideal is
directly indecomposable. First, we recall definitions and basic properties for the benefit of the reader. The main result of this section is Theorem 1.

The following theorem is well known:

**Theorem 1.** [5] The classical Dickson algebra is

\[ P[y_1, \cdots, y_n]^{GL_n} = \mathbb{Z}/p\mathbb{Z}[d_{n,1}, \cdots, d_{n,n-1}, d_{n,n}] . \]

It is a polynomial algebra and \(|d_{n,i}| = 2(p^n - p^{n-i})\) (\(|d_{n,i}| = 2^n - 2^{n-i}\) for \(p = 2\)).

As \(n\) varies, we get Dickson algebras of various length. In the last section, Dickson algebras of different lengths will be considered in connection with the \(mod - p\) cohomology of the base point of \(QS^0 = \lim \Omega^n \Sigma^n S^0\).

We shall recall some well known results concerning the action of the Steenrod algebra on Dickson algebra generators.

**Proposition 2.** [7] (Th. 30, p. 169)

\[
P^p (d_{n,i}^{p^j}) = \begin{cases} 
  d_{n,i+1}^{p^j}, & \text{if } k = n + j - i - 1 \text{ and } i < n \\
  -d_{n,i}^{p^j}d_{n,1}^{p^j}, & \text{if } k = j + n - 1 \\
  0, & \text{otherwise}
\end{cases}
\]

**Definition 3.** Let \(c\) and \(j\) be natural numbers such that \(j \leq c+1\). Let \(P(c,j)\) stand for the Steenrod iterated operation

\[ P(c,j) = P^{p^{c-j+1}} \cdots P^{p^{c-j+j}}. \]

**Lemma 4.** Let \(i = 1, \ldots, n\) and \(k(i)\) a natural number. Let \(c = k(i) + n - i - 1\) for \(i < n\) and \(c = k(n) + n - 1\). Then

\[
P(c,j) (d_{n,i}^{p^{k(i)}}) = \begin{cases} 
  ud_{n,n}^{p^{k(i)}}, & \text{if } 0 < n - i = j \text{ and } i < n \\
  ud_{n,n}^{p^{k(i)}}, & \text{if } i = j = n \\
  d_{n,i+j}^{p^{k(i)}}, & \text{if } i + j < n \\
  0, & \text{otherwise}
\end{cases}
\]

Here \(u\) is a unit.

**Proof.** This is a repeated application of last proposition. If \(k(i) + n - 1 = k(n) + n - 1\), then \(k(i) = k(n)\) and \(k(i) + n - i - 1 < k(n) + n - 1\) for \(1 \leq i < n\). Hence case (2) of last proposition applies only for \(i = n\). \(\square\)

**Lemma 5.** Let \(i = 1, \ldots, n\) and \(k(i)\) natural numbers such that \(c = k(i) + n - i - 1\) and \(c = k(n) + n - 1\). Let \(m(i) = a_ip^{k(i)}\) with
0 \leq a_i \leq p - 1 \text{ and } j = \max \{n, n - i \mid a_n, a_i \neq 0\}. \text{ Let } d = \prod_{i \leq n} d_{n,i}^{a_i}.

Then

\[ P(c, j)(d) = \begin{cases} \ u d_{n,n}^{k(n)} d, & \text{if } j = n \\ u d_{n,n}^{k(n-j)} d d_{n,n-j}^{-p(n-j)}, & \text{if } j < n \end{cases}. \]

Here \( u \) is a unit. Moreover, \( P(c, t)(d) = 0 \), if \( j < t \).

**Proof.** The proof depends on the Cartan formula and last lemma. Two cases shall be considered: \( j = n \) and \( j < n \).

First case: \( j = n \).

\[ P^{r \in} d = u_n d_{n,1}^{p(n)} d + \sum_{i=1}^{n-1} u_i d_{n,i+1}^{p(i)} d d_{n,i}^{-p(i)}. \]

Here \( u_i = 0 \), if \( a_i = 0 \). Since \( k(n) < k(1) < \ldots < k(n-1) \), we have

\[ P^{r \in-1} \left( d_{n,1}^{p(n)} d \right) = u_n' d_{n,n}^{p(n)} d. \]

And

\[ P(c-1, n-1) \left( d_{n,1}^{p(n)} d \right) = u_n' d_{n,n}^{p(n)} d. \]

For the rest of the terms we have

\[ P^{r \in-1} \left( d_{n,i+1}^{p(i)} d d_{n,i}^{-p(i)} \right) = u_i' d_{n,i+2}^{p(i)} d d_{n,i}^{-p(i)}. \]

Since \( n > i \), last lemma implies

\[ P(c-1, n-1) \left( d_{n,i+1}^{p(i)} d d_{n,i}^{-p(i)} \right) = 0. \]

The proof of the case \( j < n \) follows a similar pattern.

For the last statement of the lemma, proposition 2 is applied: For \( j < t \leq n \) and \( c - j = k(n - j) - 1 < k(n - j) + j - 1 = k(i) + n - i - 1 \), we have

\[ P^{r \in-j} \left( d_{n,n}^{p(n-j)} d d_{n,n-j}^{-p(n-j)} \right) = 0. \]

\[ \square \]

We are ready to proceed to the main Theorem of this section which is the building block to construct an algorithm turning a monomial \( d \) to \( d_{n,n}^{t} \). Let us firstly demonstrate our method.

**Example 6.** Let \( p = 2 \) and \( n = 3 \). Instead of \( d_{3,1}, d_{3,2} \) and \( d_{3,3} \) we write \( d_1, d_2 \) and \( d_3 \) respectively. Let \( K = (k_1 = 2^2 + 2^3, k_2 = 2^3 + 2^4, k_3 = 2^1 + 2^2) \) and

\[ d^K = d_1^{2^2 + 2^3} d_2^{2^3 + 2^4} d_3^{2^1 + 2^2}. \]

Let \( J \) be the sequence consisting of the first terms of those of \( K \).

\[ J = (2^3, 2^2, 2^1). \]
For each exponent \( m_i \) in \( J \) consider the minimum of \( m_n + n - 1 \) and \( m_i + n - i - 1 \) for \( 1 \leq i \leq n - 1 \):

\[
\min(J) = \{3 + 0, 2 + 1, 1 + 2\} = 3.
\]

Let \( i(J) \) be the maximum of the following set

\[
\{n - i, n| m_i + n - i - 1 = \min(J) \text{ and/or } m_n + n - 1 = \min(J)\}.
\]

Which is 3 in this case.

We apply \( i(J) = 3 \) squaring operations, namely:

\[
S^\min(J), S^\min(J)-1, \text{ and } S^\min(J)-2.
\]

\[
S^\min(J) d^K = d_3^{2+2^2} d_1^{2+2^3} d_1^{2+2^3} + d_3^{2+2^2} d_3^{2+2^3} + d_3^{2+2^2} d_2^{2+2^3} d_1^{2^3}.
\]

Finally,

\[
S^3 d^K = d_3^{2+2} d_2^{2+4} d_1^{2+1}.
\]

Let \( K = (2^3 + 2^4, 2^2 + 2^3, 2^3) \). Then \( K = (2^3, 2^2, 2^3) \), \( \min(J) = 3 \), and \( i(J) = 2 \).

Applying the procedure described above five more times we get:

\[
S^6 S^4 S^2 \text{ be the following operation}
\]

\[
S^6 S^4 S^2, S^6 S^4, S^6 S^2, S^6, S^4, S^2, S^3, S^3, S^3, S^3, S^3,
\]

then

\[
S^6 S^4 S^2 d^K = d_3^{2+2^2} d_2^{2+4} d_1^{2^3} = d_3^{26}.
\]

**Definition 7.** Let \( \Gamma = (k(1), \ldots, k(n)) \) and \( d^K = \prod_{i=1}^{n} d_{n,i}^{k(i)} \). For each \( k(i) \), let \( a_i p^{m(k(i))} \) be its lowest non-zero term in its \( p \)-adic expansion. Here \( a_i = 0 \), if \( k(i) = 0 \). Let

\[
J = (a_1 p^{m(k(1))}, \ldots, a_n p^{m(k(n))}),
\]

and \( \min(J) \):

\[
\min\{m(k(n)) + n - 1, m(k(i)) + n - i - 1 | 1 \leq i < n \text{ and } a_n, a_i \neq 0\}.
\]

Let \( i(J) \) stand for the maximum of the \( n - i \)'s and/or \( n \) such that \( m(k(i)) + n - i - 1 = \min(J) \) and/or \( m(k(n)) + n - 1 = \min(J) \).
Theorem 8. Let $d^K$, $\min (J)$ and $i_{(J)}$ as in the definition above, then

$$P \left( \min (J), i_{(J)} \right) (d^K) = \begin{cases} u d_{n,n}^{p m(k(n))} d^K, & \text{if } i_{(J)} = n \\ u d_{n,n}^{p m(k(n-i_{(J)}))} d^K d_{n,n-i_{(J)}}^{p m(k(n-i_{(J)}))}, & \text{if } i_{(J)} < n \end{cases}.$$ 

Here $u$ is a unit.

**Proof.** Let $b_{n-i_{(J)}} = a_{n-i_{(J)}}$, and $b_t = 0$, otherwise. Let $B = (b_1, ..., b_n)$. The second statement of lemma 5 implies that

$$P \left( \min (J), i_{(J)} \right) (d^K) = \left( P \left( \min (J), i_{(J)} \right) (d^B) \right) d^{K-B}.$$ 

The first claim of lemma 5 provides the claim. □

Remark 9. Let $d^{K'} = P \left( \min (J), i_{(J)} \right) (d^K)$, where $K$, $\min (J)$, and $i_{(J)}$ as in the Theorem above. Let $K' = (k'(1), ..., k'(n))$, then $k(n) < k'(n)$ and $k(i) \geq k'(i)$ for $i < n$.

Corollary 10. Let $d^K = \prod_1^n d_{n,i}^{k(i)}$ such that $\sum_1^{n-1} k(i) > 0$. Then there exists a sequence $P(K)$ of Steenrod operations such that 

$$P(K) d^K = d^L$$

Here $d^L = \prod_1^n d_{n,i}^{d(i)}$ satisfies $k(n) < l(n)$, $k(i) > l(i)$ for some $i < n$ and $k(t) = l(t)$ for $t \neq i, n$.

**Proof.** The hypothesis $\sum_1^{n-1} k(i) > 0$ implies that case (2) of Theorem above will be applied at some stage of the procedure. So, for some $i$, the corresponding exponent will be smaller in the new monomial. □

Theorem 11. Let $d^K = \prod_1^n d_{n,i}^{k(i)}$. There exists a sequence of Steenrod operations $P(\Gamma, K)$ such that 

$$P(\Gamma, K) d^K = u d_{n,n}^{p m},$$

for some natural number $m$ and unit $u$.

**Proof.** Last corollary is applied repeatedly, so the sequence of the exponents of the resulting monomial will be $(0, ..., 0, l(n))$. If $l(n)$ is not a $p$-th power, applying case (1) of last Theorem repeatedly the exponent of $d_{n,n}$ will become a $p$-th power. □

Before proceeding to the proof that the classical Dickson algebra is atomic, let us consider an example which illuminates the key ingredient for the proof.
Example 12. Let $p = 2$ and $n = 3$. Let $d^K = d_1^{2^2+2^2}d_2^{2+2^4}d_3^{2+2^2}$ and $d^{K'} = d_1^{2^2+2^4}d_3^{2+2^2}$. Then $|d^K| = |d^{K'}|$. As in the last example there exist sequences of Steenrod operations $Sq(\Gamma, K)$ and $Sq(\Gamma, K')$ such that

$$Sq(\Gamma, K)d^K = d_3^{2^6} = Sq(\Gamma', K')d^{K'}.$$  

We recall that $Sq(\Gamma, K) = Sq(2^6, 3)Sq(2^4, 1)Sq(2^4, 2)Sq(2^3, 1)Sq(2^3, 2)Sq(2^3, 3)$ and

$$Sq(\Gamma', K') = Sq(2^6, 3)Sq(2^5, 2)Sq(2^4, 3)Sq(2^3, 2)Sq(2^3, 3).$$

But

$$Sq(\Gamma, K)d^{K'} = 0.$$  

Definition 13. A graded module $M$ is called atomic, if given any degree preserving module map $f : M \to M$ which is an isomorphism on the lowest positive degree, then $f$ is an isomorphism.

It turns out that the classical Dickson algebra is atomic as a Steenrod algebra module.

Corollary 14. Let $f : D_n \to D_n$ be an $A$-linear map of degree 0 such that $f(d_{n,1}) \neq 0$. Then $f$ is an isomorphism.

Proof. By hypothesis and proposition 2, $f(d_{n,i}) = \lambda d_{n,i}$ for $i = 1, ..., n$ after applying a suitable Steenrod operation.

Let $d^K$ and $f(d^K) = 0$, then according to last Theorem there exists a sequence of Steenrod operations such that $P(\Gamma, K)d^K = ud_{n,n}^{m_n}$. Thus $P(\Gamma, K)f(d^K) = uf(d_{n,n}^{m_n})$ and $0 = ud_{n,n}^{m_n}$.

Let homogeneous monomials $d^{K(t)}$ for $1 \leq t \leq l$ and $f(\sum a_t d^{K(t)}) = 0$. Let $P(\Gamma(t), K(t))$ be the appropriate corresponding sequences of Steenrod operations as in last Theorem:

$$P(\Gamma(t), K(t)) = \prod_{i=1}^{m_t} P(c_s(t), k_s(t)).$$

Without lost of generality, we suppose that at least one of the $P(c_s(1), k_s(1))$’s is different. Otherwise, the common part is applied on the $d^{K'}$’s and we consider the new terms. Let $P(\Gamma(1), K(1)) = \prod_{i=1}^{m_t} P(c_s(1), k_s(1))$ satisfy the properties $c_1(1) = \min \{c_1(t) \mid 1 \leq t \leq l\}$ and if there are more than one, then $k_1(1)$ is the biggest among the equal ones.
If $c_1 (1) < c_1 (t)$, then

$$P (\Gamma (1), K (1)) f (\sum a_i d^{K (i)}) = a_1 P (\Gamma (1), K (1)) d^{K (1)}$$

and the first part of the proof provides a contradiction.

If $c_1 (1) = c_1 (2)$, then $k_1 (1) > k_1 (2)$ according to our assumption. Thus (please consider last example)

$$P (c_1 (1), k_1 (1)) d^{K (2)} = 0 \neq P (c_1 (1), k_1 (1)) d^{K (1)}.$$

This is because

$$P (c_1 (1), k_1 (1)) = P (c_1 (1) - k_1 (2) + 1, k_1 (1) - k_1 (2)) P (c_1 (2), k_1 (2))$$

and $c_1 (1) - k_1 (2) + 1 < c_2 (2)$. By lemma 3

$$P (c_1 (1) - k_1 (2) + 1, k_1 (1) - k_1 (2)) P (c_1 (2), k_1 (2)) d^{K (2)} = 0.$$

And the first part of the proof provides a contradiction. \qed

Next we show that the ring of upper triangular invariants does not have this property.

**Example 15.** Let $p = 2$ and $H_2 = P[y_1, y_2]^U_2$ the ring of upper triangular invariants. Here $U_2 = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a = \mathbb{Z}/2\mathbb{Z} \right\}$. It is known that it is a polynomial algebra on $h_1 = y_1$ and $h_2 = y_1^2 + y_2 y_1$ (\cite{11}). Let $f : H_2 \to H_2$ be an $A$-linear map such that $f(h_1) = h_1$.

Since $Sq^1 h_1 = h_1^2 \neq h_2$, $f(h_2)$ can be defined independently of $h_1$: $f(h_2) = ah_2 + bh_1^2$ with $a, b \in \mathbb{Z}/2\mathbb{Z}$. Even if $f(h_2) = h_2^2$, $f$ is not an isomorphism: $f(d_{2,1}) = f(h_2 + h_1^2) = 0 = f(d_{2,0}) = f(Sq^1 d_{2,1})$.

3. Dickson algebras are atomic

In this section the previous results are extended to the extended Dickson algebra.

Mui gave an invariant theoretic description of the cohomology algebra of the symmetric group and calculated rings of invariants involving the exterior algebra $E(x_1, \ldots, x_n)$ as well in \cite{11}. We recall that $|x_i| = 1$ and $\beta x_i = y_i$.

**Theorem 16.** \cite{11} The extended Dickson algebra

$$D_n := (E(x_1, \ldots, x_n) \otimes P[y_1, \ldots, y_n])^{GL_n}$$

is a tensor product of the polynomial algebra $P[y_1, \ldots, y_n]^{GL_n}$ and the $\mathbb{Z}/p\mathbb{Z}$-module spanned by the set of elements consisting of the following monomials:

$$M_{n; s_1, \ldots, s_m} I_n^{p-2}; \quad 1 \leq m \leq n, \text{ and } 0 \leq s_1 < \cdots < s_m \leq n - 1.$$
Its algebra structure is determined by the following relations:
a) \((M_{n; s_1, \ldots, s_m} L_n^{p-2})^2 = 0\) for \(1 \leq m \leq n\), and \(0 \leq s_1 < \cdots < s_m \leq n - 1\).
b) \(M_{n; s_1, \ldots, s_m} L_n^{(p-2)} d_{m,1}^{m-1} = (-1)^{m(m-1)/2} M_{n; s_1} L_n^{p-2} \cdots M_{n; s_m} L_n^{p-2}\).
Here \(1 \leq m \leq n\), and \(0 \leq s_1 < \cdots < s_m \leq n - 1\).

The elements \(M_{n; s_1, \ldots, s_m}\) above have been defined by Mui in \([11]\) as follows:

\[
M_{n; s_1, \ldots, s_m} = \frac{1}{m!} \begin{vmatrix}
  x_1 & \cdots & x_1 \\
  \vdots & & \vdots \\
  y_1 & \cdots & y_n \\
  y_1^{p-1} & \cdots & y_n^{p-1}
\end{vmatrix},
\]

where there are \(m\) rows of \(x_i\)'s and the \(s_i\)-th's powers are omitted, where \(0 \leq s_1 < \cdots < s_m \leq n - 1\) in the first determinant.

The degrees of elements above are \(|M_{n; s_1, \ldots, s_m} (L_n)^{p-2}| = m+2((p^n - 1) - (p^{s_1} + \cdots + p^{s_m}))\).

Next, some important subalgebras of \(D_n\) are defined.

**Definition 17.** Let \(SD_n\) be the subalgebra of \(D_n\) generated by:

\[d_{n,s+1}, M_{n,s}(L_n)^{p-2} \text{ and } M_{n,s_1,s_2}(L_n)^{p-2}\]

Here \(0 \leq s \leq n - 1\). \(0 \leq s_1 < s_2 \leq n - 1\).

\(D_n\) and \(SD_n\) are \(A\)-algebras. It is known that \(SD_n\) is related with the hom-dual of the length \(n\) coalgebra \(R[n]\) of the Dyer-Lashof algebra \(R\) \((8)\).

**Definition 18.** Let \(I_n\) stand for the ideal of \(SD_n\) generated by

\[\{d_{n,n}, M_{n,i}(L_n)^{p-2} \text{ and } M_{n,0,i}(L_n)^{p-2} \mid 0 \leq i \leq n - 1\}\].

The ideal \(I_n\) is related with the hom-dual of the length \(n\) module of indecomposable elements of \(H_*(Q_0S^0; \mathbb{Z}/\mathbb{Z}_p)\) \((2)\).

First, we recall the next proposition concerning the action of the Steenrod algebra generators on exterior generators on the extended Dickson algebra. For the benefit of the reader we also recall that \(p^{p^k} L_n^{p-2} = 0\) for \(0 \leq k \leq n - 2\).

**Proposition 19.** \([8]\)

\[
\begin{align*}
\beta M_{n;0} L_n^{n-2} &= d_{n,n}; \\
\beta M_{n;0,s} L_n^{n-2} &= -M_{n,s} L_n^{n-2}, \text{ for } n - 1 \geq s > 0; \\
p^{p^k} M_{n,s} L_n^{n-2} &= M_{n,s-1} L_n^{n-2}, \text{ for } n - 1 \geq s > 0.
\end{align*}
\]
Remark 20. Propositions 2 and the last one imply that $I_n$ is closed under the Steenrod algebra action.

In the next lemmata we explain step by step how a monomial is transformed in to a power of $d_{n,n}$. We start with an application of the proposition above.

Lemma 21.

$$\beta P(i - 1, i) M_{n;i,s_1,...,s_l} L_{p-2}^n = - M_{n;i,s_1,...,s_l} L_{p-2}^n.$$ 

Here $0 \leq i < s_1 < ... < s_l \leq n - 1$.

$$\beta P(t - 1, t) M_{n;i} L_{p-2}^n = \begin{cases} d_{n,n} & \text{for } t = i \\ 0 & \text{for } t \neq i \end{cases}.$$ 

$$\beta P^0 \beta ... P^{l-2} ... P^0 \beta M_{n;0,1,...,l-1} L_{p-2}^n = ud_{n,n}.$$ 

Definition 22. Let $M = M_{n;s_1,...,s_l} L_{p-2}^n$ and $(t_1,...,t_{n-1})$ be the support of $(s_1,...,s_l)$ in $\{0, 1,..., n-1\}$. Let $P(B(M)) :=$

$$\beta P^0 \beta ... P^{l-2} ... P^0 \beta P^{l-1} ... P^0 ... P^{l+k-2} ... P^{l+k} ... P^{n-2} ... P^{n-l}.$$ 

Lemma 23. Let $M = M_{n;s_1,...,s_l} L_{p-2}^n$, then

$$P(B(M)) M = ud_{n,n}.$$ 

Proof. We recall that $M_{n;s_1,...,s_l}$ is a sum of monomials of the form $x_1 ... x_l y_{t_1+1}^{p_1} ... y_{t_{n-l}}^{p_{n-l}}$. Let $s_l = n - 1$. Then

$$P^{p-2} ... P^{t_{n-l} - 2} y_{t_{l+1}}^{p_{l+1}} ... y_{t_{n-l}}^{p_{n-l}} = y_{t_{l+1}}^{p_{l+1}} ... y_{t_{n-l} - 1}^{p_{n-l} - 1} y_{n-l}^{p-2}.$$ 

Hence $P^{p-2} ... P^{t_{n-l} - 2} M = M_{n;s_1,...,s_{l-1},t_{n-l}} L_{p-2}^n$.

Let $s_l < n - 1$ and $k$ maximal such that $t_k < t_{k+1} - 1$ and $k < n - l$. In this case

$$P^{t_{k+1} - 2} ... P^{t_{k}} y_{t_{l+1}+1}^{p_{l+1}} ... y_{t_{n-l}}^{p_{n-l}} = y_{t_{l+1}+1}^{p_{l+1}} ... y_{t_{n-l} - 1}^{p_{n-l} - 1} y_{k}^{p_{k} - 1}.$$ 

Hence $P^{t_{k+1} - 2} ... P^{t_{k}} M = M_{n;s_1,...,t_{k},t_{k+1}-1,...,s_l} L_{p-2}^n$. Here $t_{k+1} - 1$ means that the index $t_{k+1} - 1$ is missing. Now the claim follows.

The main point of this section is to prove that, if $M \neq M'$, then $P(B(M)) M' = 0$. The next example demonstrates the idea.

Example 24. 1) $P^{10} P^{7} M_{10;4,7,8} L_{10}^{p-2} = M_{10;4,6,7} L_{10}^{p-2}$.

$P^{10} P^{7} M_{10;3,5,7} L_{10}^{p-2} = P^{10} M_{10;3,5,6} L_{10}^{p-2} = 0.$

2) $P^{7} M_{10;3,5,9} L_{10}^{p-2} = M_{10;3,5,8} L_{10}^{p-2}$.

$P^{7} M_{10;3,5,7} L_{10}^{p-2} = 0.$
Proposition 25. Each element $M = M_{n,s_1,\ldots,s_k}L_n^{p-2}$ uniquely determines $P(B(M))$ such that $P(B(M))M = ud_n$ and $P(B(M))M' = 0$ for $M' = M_{n,s'_1,\ldots,s'_k}L_n^{p-2} \neq M$.

Proof. Let $(t_1, \ldots, t_{n-l})$ and $(t'_1, \ldots, t'_{n-l})$ be the corresponding supports. We recall that $M_{n,s_1,\ldots,s_l}$ is a sum of monomials of the form $x_1\cdots x_1y_{l+1}\cdots y_{n-l}$.

Let $s_l = n - 1 > s'_l$. Then $P^{p-2} \cdots P^{p_1}y_{l+1}^t \cdots y_{n-l}^t$ is either zero (if $t_{n-l} \neq t'_{n-l}$) or contains two identical powers $p^{m-1}$ and in either case the determinant $P^{p-2} \cdots P^{p_1}M'$ is zero.

Let $s'_l < s_l < n - 1$ and $k$ maximal such that $t_k < t_{k+1} - 1$ and $k < n - l$.

If $t_k \notin (t'_1, \ldots, t'_{n-l})$, then $P^{p_k}M' = 0$ (please recall the proof of the last lemma). Let $t_k \in (t'_1, \ldots, t'_{n-l})$ and $t'_k = t_k$. If $t'_{k+1} = t'_k + 1$, then $P^{p_k}y_{l+1}^t \cdots y_{n-l}^t$ contains two identical powers $p^{t_k+1}$ and the determinant $P^{p_k}M'$ is zero.

If $t'_{k+1} > t'_k + 1$, then $t_{k+1} - t_k > t'_{k+1} - t'_k$. Again for the same reason $P^{p_{k+1}} \cdots P^{p_k}M' = 0$.

Let $s_l < s'_l$, then

$$P(B(M)) = (P^{p_0} \cdots P^{p_1})P^{p_0} \cdots P^{p_1}$$

and

$$P(B(M')) = (P^{p_0} \cdots P^{p_1})P^{p_0} \cdots P^{p_1}$$

such that $i_1 < i'_1$ (please see lemma 23). Now

$$P^{p_0} \cdots P^{p_1}M = M_{n;0,1,\ldots,l-1}L_n^{p-2} = P^{p_0} \cdots P^{p_1}M'$$

If $P^{p_0} \cdots P^{p_1}M' \neq 0$, then $P^{p_0} \cdots P^{p_1}M' \neq M_{n;0,1,\ldots,l-1}L_n^{p-2}$, because $i_1 < i'_1$. Thus $P(B(M))M' = 0$.

If $s_l = s'_l$, then by applying a suitable sequence of Steenrod operations the case is reduced to one of the previous ones.

Corollary 26. Let $M = M_{n,s_1,\ldots,s_l}L_n^{p-2}d^K$ be a monomial in $D_n$, then $P(B(M))M = ud_n d^K$. Let $M' = M_{n,s'_1,\ldots,s'_l}L_n^{p-2}d^{K'}$ such that $s_l \neq s'_l$ for some $t$, then $P(B(M))M' = 0$.

Proof. For the first statement, last lemma implies that

$$P(B(M))M = ud_n d^K + (other)$$

Last proposition implies that $(other) = 0$. The second statement is an application of last proposition.
Now we are ready to proceed to our main results of this section.

**Theorem 27.** The extended Dickson algebra $D_n$ is atomic.

**Proof.** Let $g : D_n \to D_n$ be an $A$-linear map such that $g(M) \neq 0$. We will prove that $g$ is an isomorphism. Here $M = \prod_{i=1}^{n} x_i L_n^{p-2}$. Applying corollaries 26 and 14, the claim is obtained. □

**Proposition 28.**

a) $SD_n$ is atomic as a Steenrod algebra module.

b) $I_n$ is atomic as a Steenrod algebra module.

**Proof.**

a) Let $f : SD_n \to SD_n$ satisfy $f(M_{n;n-2,n-1} L_n^{p-2}) = u M_{n;n-2,n-1} L_n^{p-2}$. Applying corollaries 26 and 14, the claim is obtained.

b) Let us recall that $I_n$ is the ideal generated by \( \{d_{n,s_1} M_{n,n-2} L_n^{p-2}, M_{n,0}s'_1 L_n^{p-2}\} \). Here $0 \leq s_1 < n$ and $0 < s'_1 \leq n - 1$. Let $f$ satisfy $f(M_{n,0,n-1} L_n^{p-2}) = u M_{n,0,n-1} L_n^{p-2}$. The proof follows the same pattern as above. □

Proposition b) above is a reformulation of Theorem 4.1 in [2].

We close this section by observing a property of the Steenrod algebra. We recall that an $A$-module is indecomposable, if it is not a non-trivial direct sum.

Let $\overline{D}_n$ denote the augmentation ideal of $D_n$.

**Corollary 29.** $\overline{D}_n$ is not directly decomposable as an $A$-module.

**Proof.** Assume $\overline{D}_n = \bigoplus_{i \in I} (D_n)_i$ such that $(D_n)_i \neq 0$. If $d(i)$ and $d(j)$ are homogeneous polynomials in $(D_n)_i$ and $(D_n)_j$ respectively, then there exist $P^r$ and $P^{r'}$ such that $a_i P^r d(i) = b_i^{0} = b_j P^{r'} d(j)$. □

4. $Q_0 S^0$ is $H$-atomic at $p$

We close this work by applying the main result in the mod $p$ homology of $Q_0 S^0$. $H_*(Q_0 S^0; \mathbb{Z}/\mathbb{Z}_p)$ is described in terms of Dyer-Lashof operations. For their properties please see May [4].

Iterates of the Dyer-Lashof operations are of the form $Q^{(I, \varepsilon)} = \beta^{\epsilon_1} Q^{i_1} \cdots \beta^{\epsilon_k} Q^{i_k}$ where $(I, \varepsilon) = (i_k, \ldots, i_1), (\varepsilon_k, \ldots, \varepsilon_1)$ with $\varepsilon_j = 0$ or $1$ and $i_j$ a non-negative integer for $j = 1, \ldots, k$. If $p = 2$, $\varepsilon_j = 0$ for all $j$. 
The degree is defined by \(| (I, \varepsilon) | := |Q^{(I,\varepsilon)}| = 2(p-1) \left( \sum_{t=1}^{k} i_t \right) - \left( \sum_{t=1}^{k} \varepsilon_t \right)
\[ |Q^{I,\varepsilon}| = \left( \sum_{t=1}^{k} i_t \right), \text{ for } p = 2 \]. Let \( l(I, \varepsilon) = k \) denote the length of \((I, \varepsilon)\) or \(Q^{I,\varepsilon}\) and let the excess of \((I, \varepsilon)\) or \(Q^{I,\varepsilon}\), denoted by \(exc(Q^{I,\varepsilon}) = i_k - \varepsilon_k - |Q^{I(k-1),\varepsilon(k-1)}|\), where \((I(t), \varepsilon(t)) = ((i_1, \ldots, i_1), (\varepsilon_1, \ldots, \varepsilon_1))\).

\[ exc(Q^{I,\varepsilon}) = i_k - \varepsilon_k - 2(p-1) \sum_{t=1}^{k-1} i_t, \quad [exc(Q^I)] = i_k - \sum_{t=1}^{k-1} i_t. \]

The excess is defined \(\infty\), if \(I = \emptyset\) and we omit the sequence \((\varepsilon_1, \ldots, \varepsilon_k)\), if all \(\varepsilon_i = 0\). We refer to elements \(Q^I\) as having non-negative excess, if \(exc(Q^{I(t),\varepsilon(t)})\) is non-negative for all \(t\).

There are relations among the iterated operations called Adem relations, so an operation can be reduced to a sum of admissible operations after applying Adem relations. A sequence \((I, \varepsilon)\) is called admissible, if \(p\delta_j - \varepsilon_j \geq i_{j-1} (2i_j \geq i_{j-1})\) for \(2 \leq j \leq k\).

The Kronecker pairing and the left \(\mathcal{A}\)-module on \(H^*(Q_0S^0; \mathbb{Z}/\mathbb{Z}p)\) induces a right \(\mathcal{A}\)-module structure on \(H_*(Q_0S^0; \mathbb{Z}/\mathbb{Z}p)\). We follow Cohen and May in writing the Steenrod operations on the left.

The Dyer-Lashof algebra can be decomposed as opposite Steenrod coalgebras with respect to length

\[ R = \bigoplus_{k \geq 0} R[k]. \]

Let \(R^+\) be the positive degree elements of \(R\) and \(R_0\) be the ideal generated by positive degree elements of excess zero

\[ R_0 = < Q^{(I,\varepsilon)} \mid exc(I, \varepsilon) = 0 >. \]

Let \((I, \varepsilon)\) be an admissible sequence such that \(|Q^{(I,\varepsilon)}| > 0\), then \(Q^{(I,\varepsilon)}[1]\) corresponds to

\[ [1]_{(I,\varepsilon)} := Q^{(I,\varepsilon)}[1] \ast Q^{(I,\varepsilon)}[1]^{-1} \in H_*(Q_0S^0). \]

Here \(<Q^{(I,\varepsilon)}[1]]^{-1} = [-p^{(I)}]\).

According to Madsen and May, \(H_*(Q_0S^0; \mathbb{Z}/\mathbb{Z}p)\) is the free commutative graded algebra generated by \(Q^{(I,\varepsilon)}[1] \ast [-p^{(I)}]\). Here \((I, \varepsilon)\) are admissible sequences of positive excess and \(\ast\) denotes Pontryagin multiplication. There exists an \(\mathcal{A}\)-module isomorphism between the generators of \(H_*(Q_0S^0; \mathbb{Z}/\mathbb{Z}p)\) and the quotient \(R/Q_0R\) where \(Q_0R = \{Q^{I,\varepsilon} | exc(I, \varepsilon) = 0\}\). It is known that \(R[k]^* \cong SD_k\) as Steenrod algebras \([12, 8]\) and \((R/Q_0R)[k]^* \cong I_k\) as Steenrod modules \([2]\).
Next Theorem has been given in [3] as Theorem 2.3 for \( p = 2 \) by a similar method.

**Theorem 30.** Let \( f : Q_0 S^0 \to Q_0 S^0 \) be an \( H \)-map which induces an isomorphism on \( H_{2p-3}(Q_0 S^0; \mathbb{Z}/\mathbb{Z} p) \). Let \( p > 2 \) and 
\[
f_*(Q^{(p,1)}[1]) = uQ^{(p,1)}[1] + \text{others}
\]
for some \( u \in (\mathbb{Z}/\mathbb{Z} p)^* \). Then \( f_* \) is an isomorphism.

**Proof.** The case \( p > 2 \) shall be considered. We shall prove that \( f_* \) is an isomorphism on \( Q(H_*(Q_0 S^0; \mathbb{Z}/\mathbb{Z} p)) \), the module of indecomposable elements. There is an \( \mathcal{A}_* \) module isomorphism between the previous module and \( R/Q_0 R \). The last isomorphism provides an \( \mathcal{A} \)-isomorphism between \( (R/Q_0 R)^* \) and \( I = \oplus I_k \). Let \( f_k \) be the induced map on \( f_* \) in \( k \). It suffices to prove that \( f_k \) is an isomorphism for each \( k \) and this is true as long as \( f_k(d) = ud \) for \( d = M_{k,0,s}(I_p^{k-2} \oplus k_{k,0}) \) and \( 0 < s \leq k - 1 \) according to proposition 28. Here \( u \) is a unit.

Given \( f_*(\beta Q^1[1]) = u\beta Q^1[1] \) we have 
\[
\beta f_*(Q^1[1]) = f_*(\beta Q^1[1]) = u\beta Q^1[1].
\]
Thus \( f_*(Q^1[1]) = uQ^1[1] \), for degree reasons. Moreover, 
\[
f_*(Q^1[1])p^m = u(Q^1[1])p^m \approx uQ^{(p^m-1,\ldots,1)}[1].
\]
Dually (9), \( f_1(d_{1,1}) = ud_{1,1} \). Given \( f_*(Q^{(p,1)}[1]) = u'Q^{(p,1)}[1] + \text{others} \), we have \( f_2(d_{2,2}) = u'd_{2,2} + \text{others} \). Induction on the length \( k \) is applied. Suppose that 
\[
f_*(Q^{(p^{k-1},\ldots,p,1)}[1]) = uQ^{(p^{k-1},\ldots,p,1)}[1] + \text{others}.
\]
Now, \( f_*(Q^{(p^{k-1},\ldots,p,1)}[1]) = u(Q^{(p^{k-1},\ldots,p,1)}[1])p + \text{others} \). And 
\[
f_*(Q^{(p^{k-1},p^{k-1},\ldots,p,1)}[1]) = uQ^{(p^{k-1},p^{k-1},\ldots,p,1)}[1] + \text{others}.
\]
Since \( k > 2 \), \( P^1_*(Q^{(p^{k-1},p^{k-1},\ldots,p,1)}[1]) = Q^{(p^{k-1},p^{k-1},\ldots,p,1)}[1] \) uniquely by Nishida relations. Thus 
\[
f_*(Q^{(p^{k},p^{k-1},\ldots,p,1)}[1]) = uQ^{(p^{k},p^{k-1},\ldots,p,1)}[1] + \text{others}.
\]
Dually (9), \( f_{k+1}(d_{k+1,k+1}) = u'd_{k+1,k+1} + \text{others} \). Proposition 28 implies that \( f_{k+1} \) is an isomorphism for all \( k \). Now the Theorem follows.

\[\square\]

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