ON A HEATED INCOMPRESSIBLE MAGNETIC FLUID MODEL

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Abstract. In this paper we study the equations describing the dynamics of heat transfer in an incompressible magnetic fluid under the action of an applied magnetic field. The system consists of the Navier-Stokes equations, the magnetostatic equations and the temperature equation. We prove global-in-time existence of weak solutions to the system posed in a bounded domain of \( \mathbb{R}^3 \) and equipped with initial and boundary conditions. The main difficulty comes from the singularity of the term representing the Kelvin force due to magnetization.

1. Introduction. The purpose of the present work is the mathematical analysis of a model describing heat transfer in an incompressible magnetic fluid heated from below. Magnetic fluids (also called ferrofluids) are colloidal suspensions of nanoscale magnetic particles in a carrier fluid. These fluids have found a wide variety of applications in technology, industry and medicine, see for instance [22, 35] and the references therein.

We consider an electrically non-conducting incompressible ferrofluid occupying a bounded domain \( D \subset \mathbb{R}^3 \) and heated from a part of its boundary \( \partial D \). We assume that the domain \( D \) is a cylinder \( D = \Omega \times (-d/2, d/2) \) of height \( d \), the cross section \( \Omega \) is a regular open bounded subset of \( \mathbb{R}^2 \). The generic point of \( D \) is denoted \( x = (\hat{x}, z) \) with \( \hat{x} = (x_1, x_2) \in \Omega \). The fluid is subjected to an external magnetic field and gravity acts in the negative \( z \)-direction \( g = -|g|e^3 \), with \( e^3 = (0, 0, 1) \). The resulting magnetization \( M \) is assumed to be parallel to the demagnetizing field \( H \), namely \( M = \chi H \) where \( \chi \), the total magnetic susceptibility, is a function depending on the temperature \( \tau \) of the fluid and the intensity \( H = |H| \) of the magnetic field \( H \). Thus, \( \chi = \chi(\tau, H) = M/H \) where \( M = |M| \) is the magnetization intensity. The magnetization intensity is then a function of both the temperature \( \tau \) and the magnetic intensity \( H \); we write \( M = M(\tau, H) \). The upper and lower boundaries of \( D \), \( z = d/2 \) and \( z = -d/2 \), are maintained at constant temperature \( \tau_+ \) and \( \tau_- \),

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The Kelvin force, represented by \( \mu \) of vacuum, can be rewritten as

\[
M(\tau, H) = M_a + k_1(H - H_a) - k(\tau - \tau_a)
\]

where \( M_a, H_a \) are the intensity of magnetization and magnetic field of reference, respectively, and \( k_1, k \) represent the susceptibility and pyromagnetic coefficients, respectively. Relation (1) is deduced by linearization of the Langevin magnetization law about an equilibrium state, see [9, 13] for example. Thus the magnetization reads

\[
M = \left( M_a + k_1(H - H_a) - k(\tau - \tau_a) \right) \frac{H}{H}.
\]

The state variables of the model are the fluid velocity \( U \) and the pressure \( p \) satisfying the Navier-Stokes equations (with the Kelvin force as external force), the magnetic field \( H \) satisfying the magnetostatic equations and the temperature \( \tau \) satisfying the heat equation. We assume that the Oberbeck-Boussinesq approximation is valid. Since we expect that the temperature \( \tau \) is less than \( \tau_* = \max(\tau_+, \tau_-) \) we rewrite (1) in the form

\[
M(\tau, H) = M_* + k_1 H + k(\tau_* - \tau)
\]

with \( M_* = M_a - k_1 H_a + k(\tau_a - \tau_a) \) and (2) becomes

\[
M = \left( M_* + k_1 H + k(\tau_* - \tau) \right) \frac{H}{H}.
\]

The Kelvin force, represented by \( \mu_0 M \nabla H \) where \( \mu_0 \) is the magnetic permeability of vacuum, can be rewritten as

\[
\mu_0 M \nabla H = \nabla p_m + \mu_0 k(\tau_* - \tau) \nabla H
\]

where \( p_m = \mu_0(M_a H + \frac{k_1}{2} H^2) \) is the magnetic pressure which we add to the hydrostatic pressure \( p \) in the Navier-Stokes equations. We assume that \( M_* \geq 0 \); this means that the reference parameters \( M_a \) and \( H_a \) are linked by the condition \( M_a \geq k_1 H_a + k(\tau_* - \tau_a) \). This condition seems to be necessary for the solvability of the magnetostatic equation, see Lemma 2.2 below.

Let us denote \( D_T = (0,T) \times D \) where \( T > 0 \) is fixed, \( \Gamma^\pm = \partial D, \Gamma^T = (0,T) \times \Gamma^\pm, \Sigma_T = \partial D, \Sigma = \partial \Omega \times (-d/2,d/2) \) is the lateral boundary of \( D \). The equations satisfied by \( (U, H, \tau) \) in \( D_T \) are, see [6, 8, 9, 20, 23, 26, 27],

\[
\text{div} \ U = 0,
\]

\[
\rho_0 (\partial_t U + (U \cdot \nabla)U) - \mu \Delta U + \nabla p = S,
\]

\[
\rho_0 c_p (\partial_t \tau + U \cdot \nabla \tau) - \kappa \Delta \tau = 0,
\]

\[
\text{curl} \ H = 0, \quad \nabla \cdot (H + M) = F,
\]

where the source term \( F \) is a given function defined in \( D_T \), representing the effect of the external magnetic field, and

\[
S = S(\tau, |H|) = \rho_0 (1 + \alpha(\tau_* - \tau)) g + \mu_0 k(\tau_* - \tau) \nabla |H|.
\]

System (3)–(7) is supplemented with the initial and boundary conditions

\[
U|_{t=0} = U_0, \quad \tau|_{t=0} = \tau_0 \quad \text{in} \ D,
\]

\[
U = 0, \quad (M + H) \cdot n = 0 \quad \text{on} \ (0,T) \times \partial D,
\]

\[
\tau = \tau_\pm \quad \text{on} \ \Gamma^\pm, \quad \kappa \frac{\partial \tau}{\partial n} = 0 \quad \text{on} \ \Sigma_T.
\]
Here $\mathbf{n} = (n_1, n_2, 0)$ is the outer unit normal on the lateral surface $\Sigma$, $\mu$ is the dynamical viscosity, $\kappa$ is the thermal conductivity coefficient of the fluid, $c_p$ is the specific heat at constant pressure, and $\rho_0$ is the reference density. Note that, in accordance with \((2), (6)\) and \((9)\), the magnetic field $H$ reads $H = \nabla \varphi$ where $\varphi$ solves the magnetostatic equation
\[
\text{div} \left( \nabla \varphi + \frac{M_s + k(t_s - \tau)}{1 + k_1} \frac{\nabla \varphi}{|\nabla \varphi|} \right) = \frac{F}{1 + k_1} \quad \text{in } D_r,
\]
\[
\left( \nabla \varphi + \frac{M_s + k(t_s - \tau)}{1 + k_1} \frac{\nabla \varphi}{|\nabla \varphi|} \right) \cdot n = 0 \quad \text{on } (0, T) \times \partial D. \tag{11}
\]
Since $\varphi$ is only determined up to a constant, we additionally require that $\varphi$ is normalized, i.e. $\int_D \varphi(t, x) \, dx = 0$ a.e. in $(0, T)$.

Equation \((3)\)–\((7)\) with the initial and boundary conditions \((8)\)–\((10)\) have been considered in many papers where the authors investigate the stability analysis of solutions with respect to various parameters, see for instance \([6, 7, 8, 23, 24, 27, 28]\).

Note that relation \((2)\) is valid in the framework of the quasi-stationary approximation of Neuringer and Rosensweig \([20]\). The magnetization $M$ is assumed to be independent of the fluid velocity $U$, and the relaxation time of magnetization is $0$, i.e. the magnetization is collinear with the magnetic field $H$. Collinearity is a good approximation when the internal rotation of colloidal particles can be neglected. In models of magnetic fluid, including the intrinsic rotation of colloidal particles, the magnetization is described by a dynamic equation. Neglecting the spin diffusion terms, the magnetization equation reads \([25]\):
\[
\partial_t M + (U \cdot \nabla)M = \Omega \times M - \frac{1}{t_m} (M - M_{eq}) \tag{12}
\]
where $\Omega$ is the ferrofluid spin velocity which is an average particle rotation angular velocity, $t_m$ is the Brownian relaxation time, and $M_{eq}$ is the equilibrium magnetization which is in the form \((2)\). When the spin diffusion terms are included, equation \((12)\), which is of Bloch type, is replaced by the equation \([33]\)
\[
\partial_t M + (U \cdot \nabla)M = \Omega \times M - \frac{1}{t_m} (M - M_{eq}) + \lambda \Delta M
\]
which is of Bloch-Torrey type, $\lambda > 0$ being a diffusion coefficient that carry spins. See \([13, 14]\) where the authors consider the convective instability problem in the horizontal layer of a magnetic fluid with internal rotation with vortex viscosity. Let us also mention some recent papers \([1, 2, 3, 4, 5, 30, 34]\) addressing the question of existence, uniqueness and stability of the equations describing the motion of magnetic fluids in the isothermal case, the magnetization obeys a dynamic equation. In a forthcoming work we investigate a model describing heat transfer in an incompressible magnetic fluid where the magnetization obeys a Bloch-Torrey equation.

Our aim here is to study the global-in-time existence of solutions to system \((3)\)–\((10)\), denoted problem \((\mathcal{P})\) in what follows. Let $L^q(D)$, $H^s(D)$ and $W^{s,q}(D)$ ($1 \leq q \leq \infty$, $s \in \mathbb{R}$) be the usual Lebesgue and Sobolev spaces of scalar-valued functions, respectively. We denote $L^q(D) = (L^q(D))^3$, $H^s(D) = (H^s(D))^3$, $W^{s,q}(D) = (W^{s,q}(D))^3$. We denote by $\|\cdot\|$ the $L^2$-norm on $D$. If $\mathcal{X}$ is a Banach space, the duality product between $\mathcal{X}'$ (the dual space of $\mathcal{X}$) and $\mathcal{X}$ is denoted by $\langle \cdot, \cdot \rangle_{\mathcal{X}' \times \mathcal{X}}$ or simply by $\langle \cdot, \cdot \rangle$ when there is no confusion of notation. We denote by $C([0,T]; \mathcal{X}, weak)$ the space of functions $v : [0,T] \rightarrow \mathcal{X}$ which are continuous with respect to the
We suppose that and recall that the Navier-Stokes equations, see [10, 11, 16, 18, 19, 31, 32],

\[ C_0^\infty(D) = \{ v \in (C_0^\infty(D))^3 : \text{div} v = 0 \text{ in } D \} , \]

\[ \mathcal{U} = \text{closure of } C_0^\infty(D) \text{ in } \mathbb{H}^1(D) , \]

\[ \mathcal{U}_0 = \text{closure of } C_0^\infty(D) \text{ in } L^2(D) , \]

and recall that

\[ \mathcal{U} = \{ v \in \mathbb{H}^1(D) : \text{div} v = 0 \text{ in } D \} , \]

\[ \mathcal{U}_0 = \{ v \in L^2(D) : \text{div} v = 0 \text{ in } D, v \cdot n = 0 \text{ on } \partial D \} , \]

\[ \mathcal{U} \subset \mathcal{U}_0 \subset \mathcal{U}' = \text{dual space of } \mathcal{U} \text{ when } \mathcal{U}_0 \text{ is identified with its dual.} \]

We suppose that

\[ U_0 \in \mathcal{U}_0, \quad \tau_0 \in L^\infty(D), \quad 0 \leq \tau_0 \leq \tau_* = \max(\tau_+, \tau_-) \text{ a.e. in } D , \]

\[ F \in L^\infty(0,T;L^q(D)) \text{ with } q \geq 3, \quad \int_D F \, dx = 0 \text{ in } (0,T). \]

Without lost of generality we also suppose that \( \rho_0 = c_p = 1 \), the parameter \( \frac{k}{1+k} \) will be replaced by \( k \) and \( \frac{F}{1+k} \) by \( F \).

**Definition 1.1.** We say that \((U,H,\tau)\) is a global weak solution of problem \((P)\) if the conditions (i)--(vii) below are satisfied.

(i) The velocity \( U \) belongs to \( L^\infty(0,T;\mathcal{U}_0) \cap L^2(0,T;\mathcal{U}) \cap C([0,T];\mathcal{U}_0 \text{ weak}) \); (ii) the magnetic field \( H \) is such that \( H = \nabla \varphi \) where \( \varphi \in L^\infty(0,T;W^{1,q}(D)) \) and \( \int_D \varphi \, dx = 0 \); (iii) the temperature \( \tau \) belongs to \( L^\infty(D_T) \cap L^2(0,T;H^1(D)) \) and satisfies \( 0 \leq \tau \leq \tau_* \) a.e. in \( D_T \); (iv) the external body force \( S \), defined by (7), belongs to \( L^2(0,T;\mathbb{H}^{-1}(D)) \); (v) the momentum equation (4) holds weakly, in the sense that, for every \( v \in \mathcal{U} \),

\[
\frac{d}{dt} \int_D U \cdot v \, dx + \int_D (U \cdot \nabla)U \cdot v \, dx + \mu \int_D \nabla U \cdot \nabla v \, dx = \int_D S \cdot v \, dx \quad \text{in } \mathcal{D}'(0,T],
\]

\[
U|_{t=0} = U_0; \tag{15}
\]

(vi) the function \( \varphi \) is a weak solution of problem (11), i.e. for a.e. \( t \in (0,T) \),

\[
\int_D \left( \nabla \varphi + (M_* + k(\tau_* - \tau)) \frac{\nabla \varphi}{|\nabla \varphi|} \right) \cdot \nabla v \, dx = -\int_D Fv \, dx, \quad \forall v \in V, \tag{17}
\]

where \( V = \{ v \in H^1(D) : \int_D v \, dx = 0 \} \); (vii) the function \( \tau \) is in the form \( \tau = \theta + \xi(z) \) with \( \xi(z) = \frac{2\beta}{\alpha} z + \tau_a, \beta = \frac{\tau_+ - \tau_-}{2} \), the function \( \theta \) belongs to \( L^2(0,T;H^1(D)) \), with \( \theta|_{t=0} = 0 \), and satisfies the integral identity

\[
-\int_{D_T} \theta (\partial_t \psi + U \cdot \nabla \psi) \, dx + \kappa \int_{D_T} \nabla \theta \cdot \nabla \psi \, dx dt \\
= \frac{2\beta}{d} \int_{D_T} U \cdot e^4 \psi \, dx dt + \int_D \tau_0 \psi(0) \, dx \tag{18}
\]
for any $\psi \in C^1([0, T] \times \overline{D})$ with compact support and such that $\psi(T) = 0$ and $\psi|_{\Gamma_T^\pm} = 0$.

Our main result is the following one.

**Theorem 1.2.** Under assumptions (13) and (14), problem (P) admits a global weak solution $(U, H, \tau)$, in the sense of Definition 1.1. Moreover, there exists a function $p \in W^{-1, \infty}(0, T; L^2(D))$ such that equation (4) holds in $\mathcal{D}'(D_T)$.

We organize the rest of the paper as follows. The next section deals with a priori estimates. We show that the temperature satisfies the uniform bounds $0 \leq \tau \leq \tau_* \ a.e. \ in \ D_T$; this allows to establish suitable a priori estimates satisfied by any smooth solution of problem (P). Section 3 is devoted to the proof of Theorem 1.2. We introduce a regularized problem ($P_\varepsilon$) depending on a small parameter $\varepsilon$ by regularizing the magnetic force in the momentum equation. We construct approximate solutions $(U_n, H_n, \tau_n)$ of problem ($P_\varepsilon$) by using the semi-Galerkin method, then show the existence of a weak solution to problem ($P_\varepsilon$) by passing to the limit, as $n \to \infty$, in the sequence of approximate solutions. Passing to the limit, as $\varepsilon \to 0$, in the sequence of solutions to problem ($P_\varepsilon$) we obtain the existence of a weak solution (in the sense of Definition 1.1) to problem (P). In Section 4 we consider the differential system (3)–(7) with other boundary conditions on $\Gamma_T^\pm$ for the temperature; we show the existence of a global weak solution to the corresponding boundary-value problem.

In the paper, $C$ indicates a generic constant, depending only on some bounds of the physical data, which can take different values in different occurrences.

2. A priori estimates. We assume in this section that the solutions $(U, H, \tau)$ of problem (P) are regular enough.

2.1. Bounds for the temperature. Consider the heat equation

$$\begin{align*}
\partial_t \tau + U \cdot \nabla \tau - \kappa \Delta \tau &= 0 \quad \text{in } D_T, \\
\tau &= \tau_\pm \quad \text{on } \Gamma_T^\pm, \\
\kappa \frac{\partial \tau}{\partial n} &= 0 \quad \text{on } \Sigma_T, \\
\tau(0) &= \tau_0 \quad \text{in } D,
\end{align*}$$

(19)

where $U$ is a given function. By a weak solution of problem (19) we mean a function $\tau$ in the form

$$
\tau = \theta + \xi(z) \quad \text{with } \xi(z) = \frac{2\beta}{d} z + \tau_0, \quad \beta = \frac{\tau_+ - \tau_-}{2},
$$

(20)

the function $\theta$ belongs to $L^2(0, T; H^1(D))$, with $\theta|_{\Gamma_T^\pm} = 0$, and satisfies the integral identity (18).

We have:

**Lemma 2.1.** Assume that $U \in L^\infty(0, T; \mathcal{U}_0) \cap L^2(0, T; \mathcal{U})$ and $\tau_0 \in L^\infty(D)$, $0 \leq \tau_0 \leq \tau_* = \max(\tau_+, \tau_-) \ a.e. \ in \ D$. Then, there is a unique weak solution $\tau$ of problem (19) satisfying

$$0 \leq \tau \leq \tau_* \quad a.e. \ in \ D_T.$$  

(21)

Moreover, the function $\theta$, linked with $\tau$ by (20), satisfies

$$-\xi(z) \leq \theta \leq \tau_* - \xi(z) \quad a.e. \ in \ D_T,$$

(22)

$$
\|\theta(t)\|^2 + \int_0^t \|\nabla \theta(s)\|^2 ds \leq C(\|\theta_0\|^2 + \beta^2 \int_0^t \|U(s)\|^2 ds) \quad a.e. \ in \ (0, T).$$

(23)
Proof. The existence and uniqueness of weak solutions to problem (19) can be proved as in [17] (Chapter III). Indeed, it suffices to show that any smooth solution of (19) satisfies estimates (21)–(23). Let us prove that that $\tau \geq 0$. Multiplying equation (19) by the negative part $\tau^-$ of $\tau$, integrating by parts and using the conditions $\tau^-=(\pm d/2)=0$, we deduce that
\[
\frac{1}{2} \frac{d}{dt} \|\tau^-\|^2 + \kappa \|\nabla \tau^-\|^2 = 0
\]
which implies $\tau^- = 0$. Hence $\tau \geq 0$ a.e. in $D_T$.

Let then $\psi = \tau - \tau_*$. Clearly $\psi$ satisfies
\[
\begin{align*}
&\partial_t \psi + U \cdot \nabla \psi - \kappa \Delta \psi = 0 \text{ in } D_T, \\
&\psi = \tau_{\pm} - \tau_* \leq 0 \text{ on } \Gamma_T^{\pm}, \kappa \frac{\partial \psi}{\partial n} = 0 \text{ on } \Sigma_T, \\
&\psi(0) = \tau_0 - \tau_* \leq 0 \text{ in } D.
\end{align*}
\]
(24)

Multiplying equation (24) by $\psi^+$, the positive part of $\psi$, and integrating by parts yields
\[
\frac{1}{2} \frac{d}{dt} \|\psi^+\|^2 + \kappa \|\nabla \psi^+\|^2 = 0
\]
which implies $\psi^+ = 0$ and then $\tau \leq \tau_*$ a.e. in $D_T$. Hence (21).

Consider now the function $\theta$ given by (20). Inequalities (22) follow directly from (21). We easily verify that $\theta$ satisfies
\[
\begin{align*}
&\partial_t \theta + U \cdot \nabla \theta - \kappa \Delta \theta = \frac{2\beta}{d} U \cdot e^3 \text{ in } D_T, \\
&\theta = 0 \text{ on } \Gamma_T^{\pm}, \kappa \frac{\partial \theta}{\partial n} = 0 \text{ on } \Sigma_T, \\
&\theta(0) = \tau_0 - \xi(z) \text{ in } D.
\end{align*}
\]
(25)

Multiplying equation (25) by $\theta$, integrating by parts and using the Poincaré and Young inequalities yields
\[
\frac{d}{dt} \|\theta(t)\|^2 + \kappa \|\nabla \theta(t)\|^2 \leq C \frac{\|\beta\|^2}{\kappa d^2} \|U(t)\|^2.
\]
Integrating the latter inequality from 0 to $t$ we obtain (23). Lemma 2.1 is proved.

2.2. Bounds for the magnetic field. Consider the magnetostatic equation
\[
\begin{align*}
&\text{div} \left( \nabla \varphi + (M_* + k(\tau_* - \tau)) \frac{\nabla \varphi}{|\nabla \varphi|} \right) = F \text{ in } D_T, \\
&\left( \nabla \varphi + (M_* + k(\tau_* - \tau)) \frac{\nabla \varphi}{|\nabla \varphi|} \right) \cdot n = 0 \text{ on } (0, T) \times \partial D, \\
&\int_D \varphi \, dx = 0 \text{ in } (0, T),
\end{align*}
\]
(26)

where $\tau$ is a given function in $L^\infty(D_T)$, satisfying $0 \leq \tau \leq \tau_*$ a.e. in $D_T$. Let $A(t, x, \xi)$ be the vector field defined by
\[
A(t, x, \xi) = \xi + (M_* + k(\tau_* - \tau)) \frac{\xi}{|\xi|}, \quad (t, x, \xi) \in D_T, \xi \in \mathbb{R}^3.
\]

Problem (26) reads
\[
\begin{align*}
&\text{div} (A(t, x, \nabla \varphi)) = F \text{ in } D_T, \\
&A(t, x, \nabla \varphi) \cdot n = 0 \text{ on } (0, T) \times \partial D, \quad \int_D \varphi \, dx = 0 \text{ in } (0, T).
\end{align*}
\]
(27)
By a weak solution of problem (27) we mean a function $\varphi \in L^\infty(0, T; H^1(D))$ with $\int_D \varphi dx = 0$ and satisfying, for a.e. $t \in (0, T)$,

$$\int_D A(t, x, \nabla \varphi) \cdot \nabla v dx = -\int_D F(t, x) v dx, \ \forall v \in V.$$ 

We have:

Lemma 2.2. Assume that $F$ satisfies assumption (14), and $\tau$ belongs to $L^\infty(D_T)$ and satisfies $0 \leq \tau \leq \tau_* \ a.e. \ in \ D_T$. Then, problem (27) admits a unique weak solution $\varphi \in L^\infty(0, T; W^{1,2}(D))$ and we have, for a.e. $t \in (0, T)$,

$$\|H(t)\|^2 + k \int_D (\tau_* - \tau) |H(t)| dx \leq \|F(t)\|^2,$$

$$\|H(t)\|_{L^\infty(D)} \leq C \left(1 + \|F(t)\|_{L^\infty(D)}\right),$$

where $H = \nabla \varphi$.

Proof. Since $M_* \geq 0$, the operator $A : V \to V'$ given by

$$A(\varphi)(v) = \int_D A(t, x, \nabla \varphi) \cdot \nabla v dx \ (\text{for a.e. } t)$$

is strictly monotone, i.e.

$$\langle A(\varphi) - A(\psi), \varphi - \psi \rangle \geq 0, \ \forall \varphi, \psi \in V,$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $V'$ and $V$, and

$$\langle A(\varphi) - A(\psi), \varphi - \psi \rangle = 0 \Rightarrow \varphi = \psi.$$ 

Moreover, it is easily seen that the operator $A$ is hemi-continuous, bounded, and satisfies

$$\frac{\langle A(\varphi), \varphi \rangle}{\|\varphi\|_V} \to +\infty, \ \text{as } \|\varphi\|_V \to +\infty.$$ 

We conclude, see e.g. [15] (Chapter III), [18] (Chapter 2, Section 2), that for a.e. $t \in (0, T)$, problem (27) admits one and only one weak solution $\varphi(t) \in V$. Taking $\varphi(t)$ as a test function in the variational formulation associated with (27) we obtain (28).

Let us rewrite problem (27) in the form

$$\Delta \varphi = -\text{div} \left( (M_* + k(\tau_* - \tau)) \frac{H}{|H|} \right) + F \quad \text{in } D_T,$$

$$A(t, x, \nabla \varphi) \cdot n = 0 \quad \text{on } (0, T) \times \partial D, \ \int_D \varphi dx = 0 \quad \text{in } (0, T).$$

We have $\left|(M_* + k(\tau_* - \tau)) \frac{H}{|H|}\right| \leq M_* + k\tau_*$ then, by a classical $L^q$-regularity for the solution of the homogeneous Neumann problem (30), e.g. [21] (Lemma 4.47.1), [12], we deduce that $\varphi \in L^\infty(0, T; W^{1,2}(D))$ and $H = \nabla \varphi$ satisfies (29). Lemma 2.2 is proved.

Lemma 2.3. Assume that $F$ satisfies (14), $\tau \in L^\infty(D_T) \cap L^2(0, T; H^1(D))$ and satisfies $0 \leq \tau \leq \tau_* \ a.e. \ in \ D_T$, and the function $\theta$ given by (20) satisfies (23). Assume also that $H \in L^\infty(0, T; L^3(D))$ and satisfies (28), (29). Then, for all $t \in (0, T)$,

$$\int_0^t \|\tau_* - \tau\|_V^2 dx \leq C \left(\|F\|^2_{L^2(0, T; L^2(D))} + C(1 + \|F\|^2_{L^\infty(0, T; L^3(D))})(\|\theta_0\|^2 + \beta^2 \int_0^t \|U(s)\|^2 ds)$$

$$+ C(1 + \|F\|^2_{L^\infty(0, T; L^3(D))})(\|\theta_0\|^2 + \beta^2 \int_0^t \|U(s)\|^2 ds)$$
Lemma 2.4. Assume that the hypotheses of Lemma 2.3 hold true. Then, any weak solution to problem (32) satisfies the estimates

\[ \|U(t)\|^2 + \int_0^t \|
abla U(s)\|^2 \, ds \leq C (\|U_0\|^2 + A) \exp (C\beta^2 Bt) \]  

where the constants \( A \) and \( B \) are given in Corollary 1.
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Proof. The usual energy estimate for the Navier-Stokes equations, together with the estimate in Corollary 1, implies that

\[ \|U(t)\|^2 + \int_0^t \|\nabla U(t)\|^2 \, ds \leq C(\|U_0\|^2 + A + \beta^2 B \int_0^t \|U(s)\|^2 \, ds). \]

Then Gronwall’s inequality gives

\[ \|U(t)\|^2 \leq C(\|U_0\|^2 + A) \exp(C\beta^2 Bt), \]

and then we easily deduce (33). The lemma is proved.

3. Proof of the main theorem. The proof consists in three steps:

– Step 1. We introduce a regularized problem \((P_\varepsilon)\) depending on a small parameter \(\varepsilon > 0\) by regularizing the magnetic force in the momentum equation.

– Step 2. We prove the existence of a weak solution \((U_\varepsilon, H_\varepsilon, \tau_\varepsilon)\) to problem \((P_\varepsilon)\) (see Proposition 3 below) by using the semi-Galerkin method and the Schauder fixed point theorem.

– Step 2. We prove the existence of a weak solution \((U, H, \tau)\) to problem \((P)\) by passing to the limit, as \(\varepsilon \to 0\), in the sequence \((U_\varepsilon, H_\varepsilon, \tau_\varepsilon)\) of solutions of problem \((P_\varepsilon)\).

3.1. The regularized problem. We introduce the standard smoothing operators based on convolution with a sequence of regularizing kernels. Let \(\sigma \in D(\mathbb{R}^3), \sigma \geq 0\) with support in the unit ball and with finite mass \(\int_{\mathbb{R}^3} \sigma(x) \, dx = 1\). For \(\varepsilon > 0\) small enough, we set \(\sigma_\varepsilon(x) = \frac{1}{\varepsilon^3} \sigma(\frac{x}{\varepsilon}).\) For a function \(v \in L^1(D)\) we set

\[ (\sigma_\varepsilon * v)(x) = \int_{\mathbb{R}^3} \sigma_\varepsilon(x-y) v(y) \, dy \]

where \(v\) has been extended by 0 outside \(D\).

The parameter \(\varepsilon > 0\) being fixed, we consider the system formed by the following coupled equations.

– Magnetostatic equation:

\[
\begin{align*}
\text{div} \left( \nabla \varphi + (M_s + k(\tau_s - \tau)) \frac{\nabla \varphi}{|\nabla \varphi|} \right) &= F & \text{in } D_T, \\
\left( \nabla \varphi + (M_s + k(\tau_s - \tau)) \frac{\nabla \varphi}{|\nabla \varphi|} \right) \cdot n &= 0 & \text{on } (0,T) \times \partial D, \\
\int_D \varphi \, dx &= 0 & \text{in } (0,T).
\end{align*}
\] (34)

We denote \(H = \nabla \varphi\) and

\[ S_\varepsilon = (1 + \alpha(\tau_s - \tau)) g + \mu_0 k(\tau_s - \tau) \nabla (\sigma_\varepsilon * |H|). \] (35)

– Momentum equation:

\[
\begin{align*}
\text{div} U &= 0 & \text{in } D_T, \\
\partial_t U + (U \cdot \nabla) U - \mu \Delta U + \nabla p &= S_\varepsilon & \text{in } D_T, \\
U &= 0 & \text{on } (0,T) \times \partial D, \quad U|_{t=0} = U_0 & \text{in } D.
\end{align*}
\] (36)

– Temperature equation:

\[
\begin{align*}
\partial_t \tau + U \cdot \nabla \tau - \kappa \Delta \tau &= 0 & \text{in } D_T, \\
\tau &= \tau_{\pm} & \text{on } \Gamma_T^\pm, \quad \kappa \frac{\partial \tau}{\partial n} &= 0 & \text{on } \Sigma_T, \\
\tau|_{t=0} &= \tau_0 & \text{in } D.
\end{align*}
\] (37)

In what follows problem (34)–(37) is referred as problem \((P_\varepsilon)\).
3.2. The semi-Galerkin approximation. In Sections 3.2 and 3.3, the parameter \( \varepsilon \) is fixed and we are concerned with the existence of a weak solution to problem \((P_\varepsilon)\). For notational convenience we omit to index by \( \varepsilon \) the variables which generally depend on this parameter (except \( \sigma \)).

To solve problem \((P_\varepsilon)\) we use the semi-Galerkin method. Let \((a_j)_{j \geq 1}\) be a smooth basis of the Hilbert space \( \mathcal{U} \). We define an approximate solution \((U_n, H_n, \tau_n)\) of problem \((P_\varepsilon)\) as follows. We look for \( U_n, H_n \) and \( \tau_n \) in the form

\[
U_n = \sum_{j=1}^{n} a_j^n(t) a_j, \quad H_n = \nabla \varphi_n, \quad \tau_n = \theta_n + \zeta(z) \quad (\text{where } \zeta(z) \text{ is given by } (20))
\]

and the functions \( \varphi_n, U_n \) and \( \theta_n \) are defined by the following scheme.

- For every \( t \in (0, T) \), the function \( \varphi_n \) solves the problem

\[
\text{div} (A_n(t, x, \nabla \varphi_n)) = F \quad \text{in } D_T, \\
A_n(t, x, \nabla \varphi_n) \cdot n = 0 \quad \text{on } (0, T) \times D, \\
\int_D \varphi_n \, dx = 0 \quad \text{in } (0, T),
\]

(38)

where \( A_n(t, x, \xi) = \xi + (M_\ast + k(\tau_\ast - \tau_n(t, x))) \xi \mathbb{I}_{\mathbb{R}^3}, \quad (t, x) \in D_T, \quad \xi \in \mathbb{R}^3 \). We then define \( H_n = \nabla \varphi_n \) and

\[
S_n = (1 + \alpha(\tau_\ast - \tau_n))g + \mu_0 k(\tau_\ast - \tau_n) \nabla (\sigma_\varepsilon \ast |H_n|).
\]

- The function \( U_n \) solves the equation

\[
\frac{d}{dt} \int_D U_n \cdot a_j \, dx + \int_D (U_n \cdot \nabla) U_n \cdot a_j \, dx + \mu \int_D \nabla U_n \cdot \nabla a_j \, dx \\
= \int_D S_n \cdot a_j \, dx \quad (j = 1, \ldots, n)
\]

and the initial condition

\[
U_n|_{t=0} = U_{0n}
\]

(40)

where \( U_{0n} \) is the orthogonal projection in \( \mathcal{U}_0 \) of \( U_0 \) onto the space spanned by \( a_1, \ldots, a_n \).

- The function \( \theta_n \) is the solution of

\[
\partial_t \theta_n + U_n \cdot \nabla \theta_n - \kappa \Delta \theta_n = \frac{2\beta}{d} U_n \cdot e^3 \quad \text{in } D_T, \\
\theta_n = 0 \quad \text{on } \Gamma_T^\pm, \quad \kappa \frac{\partial \theta_n}{\partial n} = 0 \quad \text{on } \Sigma_T, \\
\theta_n|_{t=0} = \theta_0 \quad \text{in } D.
\]

(41)

In what follows, problem \((38)\)–\((41)\) is referred as problem \((P_n)\). We denote by \( \mathcal{X}_n \) the space spanned by \( a_1, \ldots, a_n \).

3.2.1. Solving problem \((P_n)\). We shall solve problem \((P_n)\) by using the Schauder fixed point theorem. Introduce the closed bounded convex subset of \( L^2(D_T) \)

\[
\mathcal{H} = \{ \omega \in L^2(D), -\xi(z) \leq \omega(t, x) \leq \tau_\ast - \xi(z) \quad \text{for a.e. } (t, x) \in D_T \}.
\]
To each \( \omega \in \mathcal{H} \) we associate \( \tilde{\tau} = \omega + \xi(z) \) then consider the magnetostatic equation

\[
\text{div} \left( \nabla \varphi + (M_s + k(\tau - \tilde{\tau})) \frac{\nabla \varphi}{|\nabla \varphi|} \right) = F \quad \text{in} \ D_T,
\]

\[
\left( \nabla \varphi + (M_s + k(\tau - \tilde{\tau})) \frac{\nabla \varphi}{|\nabla \varphi|} \right) \cdot n = 0 \quad \text{on} \ (0, T) \times \partial D, \tag{42}
\]

\[
\int_D \varphi \, dx = 0 \quad \text{in} \ (0, T).
\]

We have:

**Lemma 3.1.** Assume that \( F \) satisfies (14). Then, problem (42) has a unique weak solution \( \varphi \in L^\infty(0, T; W^{1,q}(D)) \) and \( H = \nabla \varphi \) satisfies, for a.e. \( t \in (0, T) \), the estimates

\[
\|H(t)\| \leq \|F(t)\|, \tag{43}
\]

\[
\|H(t)\|_{L^\infty(D)} \leq C(1 + \|F\|_{L^q(D)}). \tag{44}
\]

Moreover, for any given pair \( (\omega_1, \omega_2) \in \mathcal{H} \times \mathcal{H} \) we have, a.e. in \( (0, T) \),

\[
\|H_1(t) - H_2(t)\| \leq k\|\omega_1(t) - \omega_2(t)\| \tag{45}
\]

where \( H_i = \nabla \varphi_i \) and \( \varphi_i \) is the solution of (42) associated with \( \omega_i, i = 1, 2 \).

**Proof.** Since \( 0 \leq \tau - \tilde{\tau} \leq \tau \) a.e. in \( D_T \), the first part of the lemma is given by Lemma 2.2.

Let us prove (45). Denote \( H = H_2 - H_1 = \nabla(\varphi_2 - \varphi_1) \) and \( a(\xi) = \frac{\xi}{||\xi||} (\xi \in \mathbb{R}^3) \).

One can verify that

\[
\text{div} \left( H + (M_s + k(\tau - \tilde{\tau})) \left( a(H_2) - a(H_1) \right) \right) = \text{div} \left( k(\omega_2 - \omega_1)a(H_2) \right).
\]

Taking \( \varphi_2 - \varphi_1 \) as a test function in the variational formulation associated with this equation and integrating over \( (0, T) \) yields

\[
\int_{D_T} |H|^2 \, dxdt + \int_{D_T} (M_s + k(\tau - \tilde{\tau})) \left( a(H_2) - a(H_1) \right) \cdot (H_2 - H_1) \, dxdt
\]

\[
= \int_{D_T} k(\omega_2 - \omega_1) \, a(H_2) \cdot H \, dxdt.
\]

Since \( 0 \leq \tau - \tilde{\tau}, M_s \geq 0 \), and \( a \) is a monotone vector field, the second term in the previous equality is nonnegative, we have

\[
\int_{D_T} |H|^2 \, dxdt \leq k \int_{D_T} |\omega_2 - \omega_1||H| \, dxdt
\]

and, using the Cauchy-Schwarz inequality we obtain

\[
\int_{D_T} |H|^2 \, dxdt \leq k^2 \int_{D_T} |\omega_2 - \omega_1|^2 \, dxdt,
\]

hence (45). The proof of the lemma is complete. \( \square \)

Consider

\[
\tilde{S}(\omega) = (1 + \alpha(\tau - \tilde{\tau}))g + \mu_0 k(\tau - \tilde{\tau}) \nabla (\sigma \ast |H|) \tag{46}
\]

where \( \tilde{\tau} = \omega + \xi(z), H = \nabla \varphi \) and \( \varphi \) is the solution of (42) associated with \( \omega \). We have:
Lemma 3.2. For any given $\omega \in \mathcal{H}$, the function $\tilde{S}(\omega)$, defined by (46), belongs to $L^2(0, T; L^2(D))$ and satisfies the estimate

$$\|\tilde{S}(\omega)\|_{L^2(0, T; L^2(D))} \leq C_\varepsilon (1 + \|F\|_{L^2(0, T; L^2(D))}).$$

(47)

Moreover, for any given pair $(\omega_1, \omega_2) \in \mathcal{H} \times \mathcal{H}$, we have

$$\|\tilde{S}(\omega_1) - \tilde{S}(\omega_2)\|_{L^2(0, T; L^2(D))} \leq C_\varepsilon \|\omega_1 - \omega_2\|_{L^2(0, T; L^2(D))}$$

(48)

where $H_\varepsilon = \nabla \varphi_\varepsilon$ and $\varphi_\varepsilon$ is the solution of (42) associated with $\omega_i$, $i = 1, 2$. Here the constant $C_\varepsilon > 0$ depends only on some bounds of the physical data and on the $L^2$-norm of $\nabla \sigma_\varepsilon$.

Proof. Clearly, the $L^2$-norm of the first term of $\tilde{S}(\omega)$ is bounded by $C\|g\|$. The $L^2$-norm of the second term is bounded by $C_\varepsilon \|H\|_{L^2(0, T; L^2(D))}$ which, according to (43), is bounded by $C_\varepsilon \|F\|_{L^2(D)}$. Hence (47).

To prove (48) we write

$$\tilde{S}(\omega_1) - \tilde{S}(\omega_2) = k\mu_0 (\tau_\varepsilon - \xi(z)) \nabla (\sigma_\varepsilon \ast (|H_1| - |H_2|))$$

$$- \alpha (\omega_1 - \omega_2) \eta + k\mu_0 (\omega_1 - \omega_2) \nabla (\sigma_\varepsilon \ast |H_1|) + k\mu_0 \omega_2 \nabla (\sigma_\varepsilon \ast (|H_1| - |H_2|))$$

$$\equiv I_1 + I_2 + I_3 + I_4.$$

Clearly,

$$\|I_1\| \leq C \|\nabla \sigma_\varepsilon\| \|H_1 - H_2\| \leq C_\varepsilon \|H_1 - H_2\|,$$

$$\|I_2\| \leq \alpha \|g\| \|\omega_1 - \omega_2\|.$$

Using the Hölder inequality and (44), we have, for a.e. $t \in (0, T)$,

$$\|I_3(t)\|_{L^{6/5}(D)} \leq C \|\omega_1(t) - \omega_2(t)\| \|\nabla \sigma_\varepsilon \ast |H_1(t)|\|_{L^{3}(D)}$$

$$\leq C_\varepsilon \|\omega_1(t) - \omega_2(t)\| \|H_1(t)\|_{L^3(D)}$$

$$\leq C_\varepsilon (1 + \|F(t)\|_{L^3(D)}) \|\omega_1(t) - \omega_2(t)\|$$

from which follows

$$\|I_3\|_{L^2(0, T; L^{6/5}(D))} \leq C_\varepsilon \left(1 + \left\|\frac{\partial F}{\partial t}\right\|_{L^\infty(0, T; L^2(\mathcal{D}))}\right) \|\omega_1 - \omega_2\|_{L^2(0, T; L^2(\mathcal{D}))}.$$

For the last term we have

$$\|I_4\| \leq C_\varepsilon \|\omega_1 - \omega_2\|.$$

We conclude that

$$\|\tilde{S}(\omega_1) - \tilde{S}(\omega_2)\|_{L^2(0, T; L^2(\mathcal{D}))} \leq C_\varepsilon \|\omega_1 - \omega_2\|_{L^2(0, T; L^2(D))},$$

and using the Sobolev embedding $L^{6/5}(D) \hookrightarrow \mathbb{H}^{-1}(D)$ we obtain (48). The lemma is proved.

Let us now consider problem (39), (40). We have:

Lemma 3.3. For any given $\omega \in \mathcal{H}$, let $\tilde{S}(\omega)$ denote the function defined by (46). Then, problem (39), (40), with $S_n = \tilde{S}(\omega)$, has a unique solution $U_n \in H^1(0, T; X_n)$ satisfying the estimates

$$\|U_n\|_{L^\infty(0, T; L^2(D))} + \|U_n\|_{L^2(0, T; \mathbb{H}^1(D))} \leq C(\varepsilon)$$

(49)

where the constant $C_\varepsilon > 0$ depends only on some bounds of the physical data and on the $L^2$-norm of $\nabla \sigma_\varepsilon$. Moreover, the map $F : L^2(0, T; \mathbb{H}^{-1}(D)) \to L^2(0, T; L^2(D))$ which associate with $S_n$ the solution $U_n$ of problem (39), (40), is continuous.
Proof. The nonlinear differential system (39), (40) has a unique solution $\alpha^\nu_j(t) \in H^1(0, T_n)$, $j = 1, \cdots, n$, where $T_n \leq T$. Multiplying (39) by $\alpha^\nu_j(t)$ and adding these equations for $j = 1, \cdots, n$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|U_n\|^2 + \mu \|\nabla U_n\|^2 = \int_D \bar{S}(\omega) \cdot U_n \, dx.$$  \hspace{1cm} (50)

Integrating over $(0, t)$, applying the Cauchy-Schwarz, the Young and Poincaré inequalities, and using (47) we get

$$\frac{1}{2} \|U_n(t)\|^2 + \frac{\mu}{2} \int_0^t \|\nabla U_n\|^2 \, ds \leq \frac{C_\varepsilon}{\mu} (1 + \|F\|_{L^2(0, T; L^2(D))})^2 + \frac{1}{2} \|U_{0n}\|^2 \leq C_\varepsilon$$

where $C_\varepsilon$ denotes a generic constant depending on $\varepsilon$ but independent of $n$ and $T_n \leq T$. From this inequality it follows, in particular, that $|\alpha^\nu_j(t)| \leq C_\varepsilon$ for all $j = 1, \cdots, n$, therefore $T_n = T$, for all $n \geq 1$. Then there is a unique solution $U_n \in H^1(0, T; \mathcal{X}_n)$ of problem (39), (40) satisfying (49).

Let us now prove the continuity of $\mathcal{F}$. Consider a sequence $(S^m)$ in the space $L^2(0, T; H^{-1}(D))$ converging to $S$ in $L^2(0, T; H^{-1}(D))$. Let $U^m, U \in H^1(0, T; \mathcal{X}_n)$ the solution of problem (39), (40), with $S_n = S^m$ and $S_n = S$, respectively. Arguing as above one shows that the sequence $(U^m)$ is bounded in $H^1(0, T; \mathcal{X}_n)$. Then one can extract a subsequence still indexed by $m$ and there is $W \in H^1(0, T; \mathcal{X}_n)$ such that $(U^m)$ converges to $W$ strongly in $L^2(0, T; L^2(D))$ and weakly in $L^2(0, T; H^1(D))$. Note that all norms are equivalent in $\mathcal{X}_n$. Passing to the limit, as $m \to \infty$, in the equation of $U^m$ we obtain that $W \in H^1(0, T; \mathcal{X}_n)$ is the solution of (39), (40), with $S_n = S$. Since the solution of this equation is unique we conclude that $W = U$ and that the whole sequence $(U^m)$ converges to $U$ strongly in $L^2(0, T; L^2(D))$. The function $\mathcal{F}$ is then continuous. The proof of the lemma is complete. \hfill $\Box$

Finally, let us consider problem (41). We have:

**Lemma 3.4.** Let $U_n \in H^1(0, T; \mathcal{X}_n)$ be the solution of problem (39), (40). Then, problem (41) admits a unique weak solution $\theta \in \mathcal{H} \cap L^2(0, T; H^1(D))$, with $\theta|_{\Gamma_T^\pm} = 0$, and satisfying the integral identity

$$- \int_{D_T} \theta (\partial_t \psi + U_n \cdot \nabla \psi) \, dxdt + \kappa \int_{D_T} \nabla \theta \cdot \nabla \psi \, dxdt = \frac{2\beta}{d} \int_{D_T} U_n \cdot e^3 \psi \, dxdt + \int_D \tau_\theta \psi(0) \, dx \hspace{1cm} (51)$$

for any $\psi \in C^1([0, T] \times \overline{D})$ with compact support and such that $\psi(T) = 0$ and $\psi|_{\Gamma_T^\pm} = 0$. The solution $\theta$ satisfies the uniform estimate

$$\|\theta(t)\|^2 + \int_0^t \|\nabla \theta(s)\|^2 \, ds \leq C_\varepsilon \text{ a.e. in } (0, T), \hspace{1cm} (52)$$

where the constant $C_\varepsilon > 0$ depends only on some bounds of the physical data and on the $L^2$-norm of $\nabla \sigma_\varepsilon$. Moreover, for any given pair $(U_1, U_2) \in (H^1(0, T; \mathcal{X}_n))^2$ we have

$$\|	heta_1(t) - \theta_2(t)\|^2 \leq \left( \frac{1}{\kappa} \|\theta_2\|_{L^\infty(D_T)} + \frac{2\beta^2}{d^2} \right) e^T \int_0^T \|U_1 - U_2(s)\|^2 \, ds \text{ a.e. in } (0, T), \hspace{1cm} (53)$$

where $\theta_i$ is the solution of problem (41) associated with $U_i$, $i = 1, 2$. \hfill $\Box$
Proof. Arguing as in the proof of Lemma 2.1, one shows the existence and uniqueness of a weak solution \( \theta \) of problem (41), \( \theta \in L^2(0,T;H^1(D)) \), with \( \theta|_{\Gamma_T^\pm} = 0 \), satisfying the integral identity (51) and the estimate
\[
\| \theta(t) \|^2 + \int_0^t \| \nabla \theta(s) \|^2 \, ds \leq C(\| \theta_0 \|^2 + \beta^2 \int_0^t \| U_n(s) \|^2 \, ds) \quad \text{a.e. in (0,T)}.
\]
Using (49) we then deduce (52). Moreover, \( -\xi(z) \leq \theta(t,x) \leq \tau_\ast - \xi(z) \) a.e. in \( D_T \), thus \( \theta \in \mathcal{H} \).

Let us prove (53). Denoting \( U = U_1 - U_2 \) and \( \theta = \theta_1 - \theta_2 \), it is easily seen that the function \( \theta \) satisfies
\[
\partial_t \theta + U_1 \cdot \nabla \theta - \kappa \Delta \theta = -U \cdot \nabla \theta_2 + \frac{2\beta}{d} U \cdot e_3 \quad \text{in } D_T,
\]
\[
\theta = 0 \quad \text{on } \Gamma_T^\pm, \quad \kappa \frac{\partial \theta}{\partial n} = 0 \quad \text{on } \Sigma_T,
\]
\[
|\theta|_{t=0} = \theta_0 \quad \text{in } D.
\]

Multiplying equation (54) by \( \theta \), integrating by parts and using the Young inequality we obtain
\[
\frac{d}{2} \| \theta(t) \|^2 + \kappa \| \nabla \theta(s) \|^2 \leq \left( \frac{1}{2\kappa} \| \theta_2 \|^2_{L^\infty(D_T)} + \frac{4\beta^2}{d^2} \| U \|^2 \right) + \frac{1}{2} \| \theta \|^2.
\]

Employing the Gronwall inequality we deduce (53). The proof of Lemma 3.4 is complete. \( \square \)

We can now prove the following result.

**Proposition 1.** For any fixed \( n \) and \( T \), there exist functions
\[
U_n \in H^1(0,T;\mathcal{X}_n), \quad H_n \in L^\infty(0,T;\mathbb{L}^3(D)),
\]
\[
\tau_n \in L^\infty(D_T) \cap L^2(0,T;H^1(D)),
\]
solving problem \( (P_n) \) on the time interval \([0,T]\). Moreover, \( 0 \leq \tau_n(t,x) = \theta_n(t,x) + \xi(z) \leq \tau_\ast \) a.e. in \( D_T \), with \( \theta_n|_{\Gamma_T^\pm} = 0 \).

**Proof.** Consider the map \( \mathcal{R} : \mathcal{H} \to \mathcal{H} \) defined by \( \mathcal{R}(\omega) = \theta \) where \( \theta \) is the weak solution of problem (41), as constructed above. By virtue of Lemmas 3.1–3.4, \( \mathcal{R} \) is a continuous map. By Lemma 3.4, \( \theta \in \mathcal{H} \cap L^2(0,T;H^1(D)) \), with \( \theta_n|_{\Gamma_T^\pm} = 0 \), and we deduce from equation (41) that \( \partial_t \theta \in L^2(0,T;H^{-1}(D)) \). Consequently, one can use the Aubin–Lions lemma to deduce that \( \mathcal{R} \) has a precompact image. Applying the Schauder fixed point theorem, we get the existence of a function \( \theta_n \in \mathcal{H} \cap L^2(0,T;H^1(D)) \), with \( \theta_n|_{\Gamma_T^\pm} = 0 \), such that \( \mathcal{R}(\theta_n) = \theta_n \). Considering \( \tau_n = \theta_n + \xi(z) \), \( H_n = \nabla \varphi_n \) (where \( \varphi_n \) is the solution of (42)) and \( U_n \) (the solution of (39), (40)), we obtain the result. The proof of Proposition 1 is complete. \( \square \)

### 3.2.2. Estimates independent of \( n \). We have:

**Proposition 2.** Assume (13) and (14). Then, the approximate solution \( (U_n,H_n,\tau_n) \) constructed above satisfies the estimates
\[
\| U_n \|_{L^\infty(0,T;L^2(D))} + \| U_n \|_{L^2(0,T;H^1(D))} \leq C,
\]
\[
\| H_n \|_{L^\infty(0,T;\mathbb{L}^3(D))} \leq C,
\]
\[
0 \leq \tau_n \leq \tau_\ast \quad \text{a.e. in } D_T,
\]
\[
\| \tau_n \|_{L^2(0,T;H^1(D))} \leq C.
\]

Here all constants are independent of \( n \) and \( \varepsilon \).
Proposition 3. Problem \((P_n)\) admits one weak solution in the following sense.

(i) The function \(U\) belongs to \(L^\infty(0,T;\mathcal{U}_0) \cap L^2(0,T;\mathcal{U}) \cap C([0,T];\mathcal{U}_0 \text{ weak})\);
(ii) the magnetic field \(H\) is such that \(H = \nabla \varphi\) where \(\varphi \in L^\infty(0,T;W^{1,q}(D))\) and \(
abla \varphi \in L^2(D)\);
(iii) the temperature \(\tau\) belongs to \(L^\infty(D_T) \cap L^2(0,T;H^1(D))\) and satisfies \(0 \leq \tau \leq \tau_*\) a.e. in \(D_T\);
(iv) the external body force \(S_\varepsilon\), defined by \((35)\), belongs to \(L^2(0,T;\mathbb{H}^{-1}(D))\);
(v) the momentum equation \((36)\) holds weakly, in the sense that, for every \(v \in \mathcal{U}\),
\[
\frac{d}{dt} \int_D U \cdot v \, dx + \int_D (U \cdot \nabla)U \cdot v \, dx + \mu \int_D \nabla U \cdot \nabla v \, dx = \int_D S_\varepsilon \cdot v \, dx \quad \text{in} \quad D'(\{0,T\}),
\]
\[
U|_{t=0} = U_0;
\]
(vi) the function \(\varphi\) is a weak solution of problem \((34)\), i.e. for a.e. \(t \in (0,T)\),
\[
\int_D (\nabla \varphi + (M_* + k(\tau_* - \tau)) \frac{\nabla \varphi}{|\nabla \varphi|}) \cdot \nabla v \, dx = -\int_D F v \, dx, \quad \forall v \in \mathcal{V};
\]
(vii) the function \(\tau\) is such that \(\tau = \theta + \xi(z)\) (where \(\xi(z)\) is given by \((20)\)), the function \(\theta\) belongs to \(L^2(0,T;H^1(D))\), with \(\theta|_{\Gamma_T^-} = 0\), and satisfies the integral identity
\[
-\int_{D_T} \theta (\partial_t \psi + U \cdot \nabla \psi) \, dxdt + \kappa \int_{D_T} \nabla \theta \cdot \nabla \psi \, dxdt = \frac{2\beta_1}{d} \int_{D_T} U \cdot \mathbf{e}_3 \psi \, dxdt + \int_D \theta_0 \psi(0) \, dx
\]
for any \(\psi \in C^1([0,T] \times \overline{D})\) with compact support and such that \(\psi(T) = 0\) and \(\psi|_{\Gamma_T^0} = 0\).

We will obtain the existence of a weak solution to problem \((P_n)\) by passing to the limit, as \(n \to \infty\), in the sequence of approximate solutions \((U_n, H_n, \tau_n)\) constructed.
Moreover, According to estimates (55)–(58), there are subsequences (still indexed by \( n \)) and functions \( U, H, \tau \) such that
\[
U_n \rightharpoonup U \quad \text{weakly in} \quad L^2(0,T; U) \quad \text{and in} \quad L^\infty(0,T; L^2(D)) \quad \text{weak-*},
\]
\[
H_n \rightharpoonup H \quad \text{in} \quad L^\infty(0,T; L^q(D)) \quad \text{weak-*} \quad (q \geq 3),
\]
\[
\tau_n \rightharpoonup \tau \quad \text{in} \quad L^\infty(D_T) \quad \text{weak-*} \quad \text{and in} \quad L^2(0,T; H^1(D)) \quad \text{weak}.
\]

Moreover,
\[
\tau = \theta + \xi(z), \quad \text{with} \quad \theta \in L^2(0,T; H^1(D)), \quad \theta|_{t=0} = 0, \quad \text{and} \quad 0 \leq \tau \leq \tau_* \quad \text{a.e. in} \quad D_T. \quad (66)
\]

3.3.1. \textit{Strong convergence of} \((\tau_n)\) \text{ and} \((H_n)\). It follows from equation (41) and the estimates obtained in Proposition 2 that \( \partial_t \theta_n \) is bounded in \( L^2(0,T; H^{-1}(D)) \). Consequently, one can use the Aubin-Lions lemma to deduce that the sequence \((\theta_n)\) contains a subsequence (still indexed by \( n \)) such that \( \theta_n \rightharpoonup \theta \) in \( L^2(D_T) \) and a.e. in \( D_T \), and then \( \tau_n \rightharpoonup \tau \) in \( L^2(D_T) \) and a.e. in \( D_T \).

Using Lemma 3.1 one shows that the sequence \( H_n = \nabla \varphi_n \) contains a subsequence which converges strongly in \( L^2(0,T; L^2(D)) \), and a.e. in \( D_T \), to \( H = \nabla \varphi \) where \( \varphi \) is the unique solution of problem (61).

Consider now
\[
S_n = (1 + \alpha(\tau_* - \tau_n)) g + \mu_0 k(\tau_* - \tau_n) \nabla(\sigma_\varepsilon \ast |H_n|).
\]

We have, for any \( v \in L^2(0,T; H^1_0(D)) \),
\[
\int_0^T \langle (\tau_* - \tau_n) \nabla(\sigma_\varepsilon \ast |H_n|), v \rangle \, dt = -\int_{D_T} (\tau_* - \tau_n) (\sigma_\varepsilon \ast |H_n|) \div v \, dx \, dt
\]
\[
+ \int_{D_T} (\sigma_\varepsilon \ast |H_n|) \nabla \theta_n \cdot v \, dx \, dt - \int_{D_T} (\sigma_\varepsilon \ast |H_n|) \nabla(\tau_* - \xi(z)) \cdot v \, dx \, dt.
\]

Here \( \langle , \rangle \) denotes the duality pairing between \( H^{-1}(D) \) and \( H^1_0(D) \). Using the strong convergence of \((\tau_n)\) and \((H_n)\), the convergence of \((\sigma_\varepsilon \ast |H_n|)\) to \( \sigma_\varepsilon \ast |H|\) in \( L^2(0,T; L^2(D)) \) and a.e. in \( D_T \), and the convergence in \( L^2(0,T; L^2(D)) \) weak of \((\nabla \theta_n)\), we obtain
\[
(\tau_* - \tau_n) \nabla(\sigma_\varepsilon \ast |H_n|) \rightharpoonup (\tau_* - \tau) \nabla(\sigma_\varepsilon \ast |H|) \quad \text{in} \quad L^2(0,T; H^{-1}(D)) \quad \text{weak}.
\]

We conclude that
\[
S_n \rightharpoonup S \quad \text{in} \quad L^2(0,T; H^{-1}(D)) \quad \text{weak},
\]
with
\[
S = (1 + \alpha(\tau_* - \tau)) g + \mu_0 k(\tau_* - \tau) \nabla(\sigma_\varepsilon \ast |H|).
\]

3.3.2. \textit{Strong convergence of} \((U_n)\). To prove the compactness of \((U_n)\) we need an estimate on the derivative with respect to \( t \) of \( U_n \). We use the method introduced in [18] (pp. 64–79) for the study of weak solutions to the Navier-Stokes equations.

\textbf{Lemma 3.5.} The sequence \((U_n)\) belongs to a compact set of \( L^2(0,T; L^2(D)) \).

\textit{Proof.} Let \( \hat{U}_n : \mathbb{R} \to \mathcal{U} \), the function defined by
\[
\hat{U}_n = U_n \quad \text{on} \quad [0,T], \quad 0 \quad \text{outside this interval}.
\]

We denote by \( \hat{U}_n \) the Fourier transform of \( \hat{U}_n \). Let us show that, for \( 0 < \gamma < 1/4 \),
\[
\int_{-\infty}^{+\infty} |k|^{2\gamma} ||\hat{U}_n(\tau)||^2_{L^2(D)} \, dk \leq C.
\]

(69)
We first rewrite (39) in the form
\[
\frac{d}{dt} \int_D \tilde{U}_n \cdot a_j \, dx = \int_D \tilde{G}_n \cdot a_j \, dx + \left( \int_D \tilde{U}_0 \cdot a_j \, dx \right) \delta_0 - \left( \int_D U_n(T) \cdot a_j \, dx \right) \delta_T
\]
where \( \delta_0 \) and \( \delta_T \) are Dirac distributions at 0 and \( T \) and \( \tilde{G}_n \) is defined by
\[
\tilde{G}_n = G_n \text{ on } [0, T], 0 \text{ outside this interval,}
\]
with
\[
G_n = -(U_n \cdot \nabla)U_n + \mu \Delta U_n + S_n.
\]
By the Fourier transform we have
\[
2i\pi \xi \int_D \tilde{U}_n(\xi) \cdot a_j \, dx = \int_D \tilde{G}_n(\xi) \cdot a_j \, dx + \int_D \tilde{U}_0(\xi) \cdot a_j \, dx - e^{-2i\pi \xi T} \int_D U_n(T) \cdot a_j \, dx \quad (70)
\]
where \( \tilde{G}_n \) denotes the Fourier transform of \( G_n \). Let \( \tilde{\alpha}_n^j(\xi) \) denote the Fourier transform of \( \alpha_n^j(t) \). Multiplying (70) by \( \tilde{\alpha}_n^j(\xi) \) and adding the resulting equations for \( j = 1, \ldots, n \), we get
\[
2i\pi \xi \| \tilde{U}_n(\xi) \|_{\mathbb{L}^2(D)}^2 = \int_D \tilde{G}_n(\xi) \cdot \tilde{U}_n(\xi) \, dx + \int_D \tilde{U}_0(\xi) \cdot \tilde{U}_n(\xi) \, dx - e^{-2i\pi \xi T} \int_D U_n(T) \cdot \tilde{U}_n(\xi) \, dx. \quad (71)
\]
Using the Sobolev embedding \( H^1(D) \subset L^6(D) \) and estimate (55) we have
\[
\int_0^T \| G_n(t) \|_{\mathcal{L}^6} \, dt \leq C \int_0^T \left( \| S_n \|_{\mathbb{H}^{-1}(D)} + \| U_n(t) \|_{\mathbb{H}^1(D)} + \| U_n(t) \|_{\mathbb{H}^1(D)}^2 \right) \, dt \leq C,
\]
hence
\[
\| \tilde{G}_n(\xi) \|_{\mathcal{L}^6} \leq C, \forall \xi \in \mathbb{R}.
\]
Using (55) and the bounds \( \| U_n(T) \|_{\mathbb{L}^2(D)} \leq C, \| U_0 \|_{\mathbb{L}^2(D)} \leq C \), we deduce from (71) that
\[
|\xi| \| \tilde{U}_n(\xi) \|_{\mathbb{L}^2(D)} \leq C \left( \| \tilde{U}_n(\xi) \|_{\mathcal{L}^6} + \| \tilde{U}_n(\xi) \|_{\mathbb{L}^2(D)} \right)
\]
and then, using the Poincaré inequality we get
\[
|\xi| \| \tilde{U}_n(\xi) \|_{\mathbb{L}^2(D)} \leq C \| \tilde{U}_n(\xi) \|_{\mathcal{L}^6}, \forall \xi \in \mathbb{R}.
\]
This inequality implies (69), as in [18] (pp. 77–79). Then, by a compactness theorem involving fractional derivatives, see [18] (pp. 60–62) we deduce that \( \{U_n\} \) contains a subsequence (still indexed by \( n \)) which converges to \( U \) strongly in \( L^2(0, T; \mathbb{L}^2(D)) \). Lemma 3.5 is proved.

3.3.3. Passing to the limit in problem \( (P_n) \) as \( n \to \infty \). The strong convergence of \( \{U_n\} \) in \( L^2(0, T; \mathbb{L}^2(D)) \) and (63) imply that
\[
(U_n \cdot \nabla)U_n \rightharpoonup (U \cdot \nabla)U \text{ weakly in } L^2(0, T; \mathbb{L}^{3/2}(D)).
\]
Using also (67), (68), we easily pass to the limit in equation (39). We obtain that the limit function \( U \) satisfies the integral identity (59). Moreover, \( U \in C([0, T]; \mathcal{U}_0 \text{ weak}) \) and satisfies the initial condition (60).

Using the strong convergence in \( L^2(0, T; \mathbb{L}^2(D)) \) of the sequence \( H_n = \nabla \varphi_n \) to \( H = \nabla \varphi \) we easily pass to the limit in problem (38) and obtain that \( \varphi \) satisfies (34).
Let us finally pass to the limit in problem (41). We know that
\[- \int_{D_T} \theta_n (\partial_t \psi + U_n \cdot \nabla \psi) \, dxdt + \kappa \int_{D_T} \nabla \theta_n \cdot \nabla \psi \, dxdt\]
\[= \frac{2\beta}{d} \int_{D_T} U_n \cdot e^3 \psi \, dxdt + \int_D \theta_0 \psi(0) \, dx \tag{72}\]
for any $\psi \in C^1([0,T] \times \mathcal{D})$ with compact support and such that $\psi(T) = 0$ and $\psi|_{\Gamma^T} = 0$. Using the convergence properties obtained above we easily pass to the limit in (72) and obtain (62). The proof of Proposition 3 is complete.

3.4. **Passing to the limit as $\varepsilon \to 0$.** Let $(U_\varepsilon, H_\varepsilon, \tau_\varepsilon)$ denote the weak solution of problem $(P_\varepsilon)$, the existence of which was stated in Proposition 3. Our goal now is to let $\varepsilon \to 0$. From inequalities (55)–(58), we deduce the estimates, independent of $\varepsilon$,
\[\|U_\varepsilon\|_{L^\infty(0,T;L^2(D))} + \|U\|_{L^2(0,T;\mathcal{H})} \leq C, \tag{73}\]
\[\|H_\varepsilon\|_{L^\infty(0,T;L^2(D))} \leq C, \tag{74}\]
\[0 \leq \tau_\varepsilon \leq \tau_* \text{ a.e. in } D_T, \tag{75}\]
\[\|\tau_\varepsilon\|_{L^2(0,T;H^1(D))} \leq C. \tag{76}\]

In accordance with (73)–(76), there are subsequences (still indexed by $\varepsilon$) and functions $U, H, \tau$ such that, as $\varepsilon \to 0$,
\[U_\varepsilon \rightharpoonup U \text{ weakly in } L^2(0,T;\mathcal{H}) \text{ and in } L^\infty(0,T;L^2(D) \text{ weak-\star}), \tag{77}\]
\[H_\varepsilon \rightharpoonup H \text{ in } L^\infty(0,T;L^q(D)) \text{ weak-\star (}q \geq 3\text{),} \tag{78}\]
\[\tau_\varepsilon \rightharpoonup \tau \text{ in } L^\infty(D_T) \text{ weak-\star and in } L^2(0,T;H^1(D)) \text{ weak.} \tag{79}\]

Moreover,
\[\tau = \theta + \xi(z), \text{ with } \theta \in L^2(0,T;H^1(D)), \theta|_{\Gamma^T} = 0, \text{ and } 0 \leq \tau \leq \tau_* \text{ a.e. in } D_T. \tag{80}\]

We also have
\[\|S_\varepsilon\|_{L^2(0,T;H^{-1}(D))} \leq C. \tag{81}\]

We deduce from equation (59) that $\|\partial_t U_\varepsilon\|_{L^1(0,T;\mathcal{H})} \leq C$. Consequently, one can use the Aubin–Lions lemma to deduce that the sequence $(U_\varepsilon)$ contains a subsequence (still indexed by $\varepsilon$) such that
\[U_\varepsilon \to U \text{ in } L^2(0,T;L^2(D)) \text{ strong.} \tag{82}\]
This together with (77) imply that
\[(U_\varepsilon \cdot \nabla) U_\varepsilon \rightharpoonup (U \cdot \nabla) U \text{ weakly in } L^2(0,T;L^{3/2}(D)). \tag{83}\]

We also deduce from the equation
\[\partial_t \theta_\varepsilon + U_\varepsilon \cdot \nabla \theta_\varepsilon - \kappa \Delta \theta_\varepsilon = \frac{2\beta}{d} U_\varepsilon \cdot e^3 \text{ in } D_T, \tag{84}\]
that $\|\partial_t \theta_\varepsilon\|_{L^2(0,T;H^{-1}(D))} \leq C$. Using the Aubin–Lions lemma we deduce that the sequence $(\theta_\varepsilon)$ contains a subsequence (still indexed by $\varepsilon$) such that $\theta_\varepsilon$ converges to $\theta$ in $L^2(0,T;L^2(D)) \text{ strong, then}$
\[\tau_\varepsilon \rightharpoonup \tau \text{ in } L^2(0,T;L^2(D)) \text{ strong.} \tag{85}\]
Using Lemma 3.1, one shows that the sequence \( H_\varepsilon = \nabla \varphi_\varepsilon \) contains a subsequence which converges strongly in \( L^2(0, T; L^2(D)) \), and a.e. in \( D_T \), to \( H = \nabla \varphi \) where \( \varphi \) is the unique solution of problem (17).

Consider the source term
\[
S_\varepsilon = (1 + \alpha(\tau_\varepsilon - \tau_\varepsilon))g + \mu_0 k(\tau_\varepsilon - \tau_\varepsilon) \nabla(|H_\varepsilon|).
\]
We have, for any \( v \in L^2(0, T; H^1_0(D)) \),
\[
\int_0^T \langle (\tau_\varepsilon - \tau_\varepsilon) \nabla(|H_\varepsilon|), v \rangle \, dt = - \int_{D_T} (\tau_\varepsilon - \tau_\varepsilon)(\sigma_\varepsilon \ast |H_\varepsilon|) \, \text{div} \, v \, dx \, dt \\
+ \int_{D_T} (\sigma_\varepsilon \ast |H_\varepsilon|) \nabla \theta_\varepsilon \cdot v \, dx \, dt - \int_{D_T} (\sigma_\varepsilon \ast |H_\varepsilon|) \nabla(\tau_\varepsilon - \xi(z)) \cdot v \, dx \, dt.
\]
Using the strong convergence of \( (\tau_\varepsilon) \) and \( (H_\varepsilon) \), the convergence in \( L^2(0, T; L^2(D)) \) weak of \( (\nabla \theta_\varepsilon) \) to \( \nabla \theta \) and the convergence of \( (\sigma_\varepsilon \ast |H_\varepsilon|) \) to \( |H| \) in \( L^2(0, T; L^2(D)) \) strong and a.e. in \( D_T \), we obtain
\[
(\tau_\varepsilon - \tau_\varepsilon) \nabla(|H_\varepsilon|) \rightharpoonup (\tau_\varepsilon - \tau) \nabla|H| \quad \text{in} \quad L^2(0, T; H^{-1}(D)) \text{ weak}.
\]
To pass to the limit, as \( \varepsilon \to 0 \), in the term \( \int_{D_T}(\sigma_\varepsilon \ast |H_\varepsilon|) \nabla \theta_\varepsilon \cdot v \, dx \, dt \) we observe that \( (\sigma_\varepsilon \ast |H_\varepsilon|) \nabla \theta_\varepsilon \) is uniformly bounded in \( L^2(0, T; \mathbb{L}^{6/5}(D)) \). Then there is a subsequence, still indexed by \( \varepsilon \), so that
\[
(\sigma_\varepsilon \ast |H_\varepsilon|) \nabla \theta_\varepsilon \rightharpoonup \xi \quad \text{weakly in} \quad L^2(0, T; \mathbb{L}^{6/5}(D)).
\]
To identify \( \xi \) we take \( v \in L^2(0, T; D(D)) \) and use the strong converge of \( (\sigma_\varepsilon \ast |H_\varepsilon|) \) to \( |H| \) in \( L^2(0, T; L^2(D)) \) and a.e. in \( D_T \). Thus
\[
\lim_{\varepsilon \to 0} \int_{D_T} (\sigma_\varepsilon \ast |H_\varepsilon|) \nabla \theta_\varepsilon \cdot v \, dx \, dt = \int_{D_T} \xi \cdot v \, dx \, dt = \int_{D_T} |H| \nabla \theta \cdot v \, dx \, dt
\]
We obtain \( \xi = |H| \nabla \theta \) and we conclude that
\[
S_\varepsilon \rightharpoonup S \quad \text{in} \quad L^2(0, T; H^{-1}(D)) \text{ weak},
\]
with
\[
S = (1 + \alpha(\tau_\varepsilon - \tau))g + \mu_0 k(\tau_\varepsilon - \tau) \nabla|H|.
\]
Using (77)–(85) we can pass to the limit, as \( \varepsilon \to 0 \), in (59)–(62) and obtain that the limit functions satisfy (15)–(18).

(ii) Existence of the pressure. We have \( \frac{\partial U}{\partial t} \in W^{-1,\infty}(0, T; \mathbb{L}^2(D)) \). Moreover, the functions \( (U \cdot \nabla)U, \Delta U, S \), belong to the space \( L^1(0, T; H^{-1}(D)) \) which is included in the space \( W^{-1,\infty}(0, T; H^{-1}(D)) \). Thus, the function \( \partial_t U + (U \cdot \nabla)U - \mu \Delta U - S \) belongs to \( W^{-1,\infty}(0, T; H^{-1}(D)) \). Then, using a result in [29], there exists \( p \in W^{-1,\infty}(0, T; L^2(D)) \) such that
\[
\partial_t U + (U \cdot \nabla)U - \mu \Delta U - S = -\nabla p.
\]
The proof of the main theorem is achieved.

4. Concluding remarks. In this section we consider the differential system (3)–(7) with other boundary conditions; we show the existence of a weak solution to the corresponding boundary-value problem. For notational convenience we continue to denote by \( (P) \) the corresponding boundary-value problem.
4.1. The problem with a Dirichlet-Neumann boundary condition on $\Gamma_T^\pm$.

Consider, instead of (10), the boundary conditions

$$\tau = \tau_- \quad \text{on } \Gamma_T^-, \quad \partial_\nu \tau = 0 \quad \text{on } \Gamma_T^+, \quad \kappa \frac{\partial \tau}{\partial n} = 0 \quad \text{on } \Sigma_T.$$  

One can adapt the proof of Theorem 1.2 to show the existence of a global weak solution $(U, H, \tau)$ of the corresponding problem $(\mathcal{P})$. The maximum principle for the temperature holds, we have $0 \leq \tau \leq \tau_-$ a.e. in $D_T$. The function $\theta = \tau - \tau_-$ belongs to $L^2(0, T; H^1(D))$, with $\theta|_{T^-} = 0$, and satisfies

$$\|\theta(t)\|^2 + 2\kappa \int_0^t \|\nabla \theta(s)\|^2 \, ds \leq \|\theta_0\|^2 \quad \text{a.e. in } (0, T).$$

Taking $\tau_* = \tau_-$, the estimates for the magnetic field $H$, the source term $S$ and the velocity $U$ remain unchanged.

4.2. The problem with a Dirichlet-Fourier boundary condition on $\Gamma_T^\pm$.

Consider, instead of (10), the boundary conditions

$$\tau = \tau_- \quad \text{on } \Gamma_T^-, \quad \kappa \frac{\partial \tau}{\partial z} + \sigma \tau = \sigma \tau_+ \quad \text{on } \Gamma_T^+, \quad \kappa \frac{\partial \tau}{\partial n} = 0 \quad \text{on } \Sigma_T,$$

where $\sigma$ is a positive constant. Here again, one can adapt the proof of Theorem 1.2 to show the existence of a global weak solution $(U, H, \tau)$ of the corresponding problem $(\mathcal{P})$. We only mention the main modifications to treat this case.

Consider the heat equation

$$\partial_t \tau + U \cdot \nabla \tau - \kappa \Delta \tau = 0 \quad \text{in } D_T,$$

$$\tau = \tau_- \quad \text{on } \Gamma_T^-, \quad \kappa \frac{\partial \tau}{\partial z} + \sigma \tau = \sigma \tau_+ \quad \text{on } \Gamma_T^+, \quad \kappa \frac{\partial \tau}{\partial n} = 0 \quad \text{on } \Sigma_T, \quad (86)$$

$$\tau(0) = \tau_0 \quad \text{in } D,$$

where $U$ is a given function. We have the analogue of Lemma 2.1.

**Lemma 4.1.** Assume that $U \in L^\infty(0, T; \mathcal{U}_0) \cap L^2(0, T; \mathcal{U})$ and $\tau_0 \in L^\infty(D)$, $0 \leq \tau_0 \leq \tau_* = \max(\tau_+, \tau_-)$ a.e. in $D$. Then, there exists a unique weak solution $\tau$ of problem (86) satisfying

$$0 \leq \tau \leq \tau_* \quad \text{a.e. in } D_T. \quad (87)$$

Moreover, the function $\theta = \tau - \tau_-$ satisfies, a.e. in $(0, T)$,

$$\|\theta(t)\|^2 + 2\kappa \int_0^t \|\nabla \theta(s)\|^2 \, ds + \sigma \int_0^t \|\theta(d/2)\|^2_{L^2(\Omega)} \, ds \leq \sigma |\Omega| (\tau_+ - \tau_-)^2 + \|\tau_0 - \tau_-\|^2. \quad (88)$$

**Proof.** The existence and uniqueness of weak solutions to problem (86) can be proved as in [17] (Chapter III). Indeed, it suffices to show that any smooth solution of problem (86) satisfies estimates (87), (88). Let us prove that $\tau \geq 0$. Multiplying equation (86) by the negative part $\tau^-$ of $\tau$, integrating by parts and using the conditions $\tau^-(d/2) = 0$, we easily deduce that

$$\frac{1}{2} \frac{d}{dt} \|\tau^-(t)\|^2 + \kappa \|\nabla \tau^-\|^2 + \sigma \int_{\Omega} (\tau^+_+ - \tau^- d/2) \tau^- d/2 \, d\vec{x} = 0.$$  

Since $(\tau_+ - \tau^- d/2)\tau^+(d/2) \geq 0$ we deduce that $\tau^- = 0$ and then $\tau \geq 0$ a.e. in $D_T$. Assume $\tau_+ \leq \tau_-$. We easily verify that the function $\theta$, which equals $\tau - \tau_-$,
satisfies
\[
\partial_t \theta + U \cdot \nabla \theta - \kappa \Delta \theta = 0 \quad \text{in } D_T,
\]
\[
\theta(-d/2) = 0, \quad \frac{\partial \theta}{\partial z}(d/2) = -\sigma \theta(d/2) + \sigma(\tau_+ - \tau_-), \quad \kappa \frac{\partial \theta}{\partial n} = 0 \quad \text{on } \Sigma_T, \tag{89}
\]
\[
\theta(0) = \tau_0 - \tau_- \quad \text{in } D.
\]
Multiplying equation (89) by \( \theta^+ \), the positive part of \( \theta \), we get
\[
\frac{d}{dt}\|\theta^+\|^2 + \kappa \|\nabla \theta^+\|^2 - \sigma \int_{\Omega} (-\theta^+(d/2) + (\tau_+ - \tau_-))\theta^+(d/2) \, d\hat{x} = 0
\]
where we used \( \theta^-(-d/2) = 0 \). We deduce that \( \theta^+ = 0 \) and then \( 0 \leq \tau \leq \tau_- \).

If \( \tau_- \leq \tau_+ \) we have \( \theta = \tau - \tau_+ \) and \( \theta \) satisfies the boundary condition
\[
\theta(-d/2) = \tau_- - \tau_+ \leq 0, \quad \kappa \frac{\partial \theta}{\partial z}(d/2) = -\sigma \theta(d/2),
\]
then multiplying equation (89) by \( \theta^+ \) and integrating by parts yields
\[
\frac{1}{2} \frac{d}{dt}\|\theta^+\|^2 + \kappa \|\nabla \theta^+\|^2 + \sigma \int_{\Omega} (\theta^+(d/2))|d\hat{x}| = 0.
\]
Since \( \theta^+(d/2) = 0 \) we obtain
\[
\frac{1}{2} \frac{d}{dt}\|\theta^+\|^2 + \kappa \|\nabla \theta^+\|^2 + \sigma \int_{\Omega} (\theta(d/2))^2 \, d\hat{x} = 0
\]
which implies \( \theta^+ = 0 \) and then \( 0 \leq \tau \leq \tau_+ \). We conclude that the solution \( \tau \) of problem (86) satisfies (87).

Let \( \theta = \tau - \tau_- \). Multiplying the equation of \( \theta \) by \( \theta \) and integrating by parts we obtain
\[
\frac{1}{2} \frac{d}{dt}\|\theta\|^2 + \kappa \|\nabla \theta\|^2 + \sigma \int_{\Omega} (\theta(d/2))^2 \, d\hat{x} = \sigma(\tau_+ - \tau_-) \int_{\Omega} (\theta(d/2)) \, d\hat{x}
\]
from which we deduce, since \( \Omega \) is bounded, the energy estimate (88). The proof of the lemma is complete. \( \square \)

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