N=4, d=1 tensor multiplet and hyper-Kähler σ-models

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Abstract

We demonstrate how hyper-Kähler manifolds arise from a sigma-model action for $N=4$, $d=1$ tensor supermultiplet after dualization of the auxiliary bosonic component into a physical bosonic one.

1 Introduction

One of the interesting features of the $N=4$ supersymmetric mechanics theories is the diversity of off-shell supermultiplets. It is shown in [1, 2] that the minimal supermultiplets, which contain four fermionic components, may have a different number of physical bosons: any natural number from zero till four. Moreover, in one dimension one may switch between different supermultiplets by expressing the auxiliary components through time-derivative of physical bosons, and vice versa. This procedure is a pure algebraic one and it may be used without any reference to actions. Basing on these peculiarities, in [3] the term “root supermultiplet” was proposed for $N=4$, $d=1$ supermultiplet with four physical bosons and four fermions. Such off-shell supermultiplets, which do not contain any auxiliary components exist only in one dimension. Many of known supermultiplets with a smaller number of physical bosons can be obtained from the “root” one by a proper reduction. The geometric nature of the relations between different supermultiplets was clarified in [4] where a wide set of off-shell $N=4$ supermultiplets was deduced by using nonlinear realizations of $N=4$, $d=1$ superconformal group. It turns out that the physical bosonic components of these supermultiplets parameterize different coset spaces of the same superconformal group. Finally, in [5] the relations between different supermultiplets were extended to the level of actions and the term “root” action was invented just for the most general action of the “root” supermultiplets. It was explicitly demonstrated that the general actions for the supermultiplets with a smaller number of physical bosons might be obtained via a proper reduction procedure from the “root” one.

The key observation, used in establishing the relations between different supermultiplets, is very simple. Roughly speaking, if the transformation properties of some bosonic auxiliary component $A$ of $N=4$ supermultiplet under supersymmetry read

$$\delta A \sim \text{parameter} \times \partial_t (\text{physical fermions}),$$

then one may introduce a new physical bosonic component $u$ as $\partial_t u = A$ with the transformations law

$$\delta u \sim \text{parameter} \times (\text{physical fermions}).$$

Then one may replace the auxiliary component $A$ by a new bosonic field $u$. After such a replacement in the action, the term quadratic in the auxiliary component $A$ turns into kinetic term for the field $u$. As a result,
we get the supermultiplet with one additional physical boson $u$. Clearly enough that if the field $u$ enters the action only through its time derivative and, therefore, the action possesses isometry with the Killing vector $\partial_u$, one may reverse the consideration above and replace $\partial_u u$ by the new auxiliary component $A$.

Although being intuitively transparent, the discussed procedure has a loophole. It may happen that for some supermultiplet there exists another expression $B$ which also starts with the auxiliary component and transforms as a full time derivative under supersymmetry but nevertheless is essentially different from $A$. This fact has no immediate consequences for the given supermultiplet, because $B$ is algebraically related to the former auxiliary component $A$. However, if we decide to dualize $B$ instead of $A$ into a physical component, the resulting supermultiplet will be completely different.

In the present Letter we demonstrate that for the $N = 4, d = 1$ tensor supermultiplet with three physical bosons, one auxiliary component, and four fermions there exists a functional freedom in choosing an auxiliary component with the transformation properties (1). After dualization of this new auxiliary component into a bosons, one auxiliary component, and four fermions there exists a functional freedom in choosing an auxiliary resulting supermultiplet will be completely different.

Hawking solution for four-dimensional hyper-Kähler metrics with one triholomorphic isometry [6].

the resulting bosonic action after dualization coincides with the one dimensional variant of the general Gibbons-Hawking solution for four-dimensional hyper-Kähler metrics with one triholomorphic isometry [8].

Summarizing, the $N = 4, d = 1$ tensor supermultiplet may be dualized into new nonlinear supermultiplets with four bosonic and four fermionic components. The proper constraints imposed on the metric of the general sigma-model type action for the tensor supermultiplet, the resulting bosonic action after dualization coincides with the one dimensional variant of the general Gibbons-Hawking solution in the bosonic sector of the dualized system.

2 Preliminaries: $N = 4, d = 1$ tensor supermultiplet

The simplest description of the $N = 4, d = 1$ tensor supermultiplet is achieved by introducing a real triplet of $N = 4$ superfields $V^{ab}$

$$V^{ab} = V^{ba}, \quad (V^{ab})^\dagger = V_{ab}, \quad a, b = 1, 2,$$

depending on the coordinates of the $N = 4, d = 1$ superspace $\mathbb{R}^{1|4}$

$$\mathbb{R}^{1|4} = (t, \theta_a, \bar{\theta}^b), \quad (\theta_a)^\dagger = \bar{\theta}^a,$$

where $a, b$ are doublet indices of $SU(2)$ group. The constraints which identify the multiplet are

$$D^{(a} V^{bc)} = 0, \quad \bar{D}^{(a} V^{bc)} = 0,$$  \hspace{1cm} (3)

where the covariant spinor derivatives $D^a, \bar{D}_a$ are defined by

$$D^a = \frac{\partial}{\partial \theta_a} + i\bar{\theta}^a \partial_t, \quad \bar{D}_a = \frac{\partial}{\partial \bar{\theta}^a} + i\theta_a \partial_t,$$

The constraints leave in $V^{ab}$ a real triplet of the dimensionless physical bosons $v_m$, four spinors $\lambda_a, \bar{\lambda}^b$ with dimensions $cm^{-1/2}$, and the auxiliary field $A$ with dimension $cm^{-1}$ that we define as

$$v_m = \frac{i}{2} \sigma^{m}_{ab} V^{(ab)}, \quad \lambda_a = \frac{1}{3} \bar{D}^b V_{ab}, \quad \bar{\lambda}^b = \frac{1}{3} D_b V^{ab}, \quad A = \frac{i}{6} D^a \bar{D}^b V_{ab},$$  \hspace{1cm} (4)

where $|$ means restriction to $\theta_a = \bar{\theta}^b = 0$ and $\sigma^{m}_{ab}$ are the Pauli matrices. Under $N = 4$ supersymmetry these components transform as follows:

$$\delta v_m = i \epsilon^a \sigma^{m}_{ab} \bar{\lambda}^b - i \lambda^a \sigma^{m}_{ab} \bar{v}_m, \quad \delta \lambda_a = i \epsilon_a A + \sigma^{m}_{ab} \bar{\epsilon}^b \dot{v}_m, \quad \delta \bar{\lambda}^a = -i \bar{\epsilon}^a \bar{A} + \sigma^{m}_{ab} \epsilon_b \dot{\bar{v}}_m, \quad \delta A = \epsilon^a \bar{\lambda}_a + \bar{\epsilon}^a \lambda_a.$$  \hspace{1cm} (5)

As regards the actions of the $d = 1$ tensor multiplet, the general off-shell action

$$S = \int dt d^2 \theta d^2 \bar{\theta} F(V^{ab}),$$  \hspace{1cm} (6)

\(^1\text{Observe, that one may use these new expressions as the new additional Fayet-Iliopoulos terms in the action(see e.g. [4]).}\)

\(^2\text{We use the following convention for the skew-symmetric tensor } \epsilon: \epsilon_{a b} \epsilon^{b c} = \delta^b_{a}, \epsilon_{1 2} = \epsilon^{2 1} = 1. \text{ As usual, the round brackets imply symmetrization of the enclosed indices with the normalizing factor } 1/2.\)
where $F(V^{ab})$ is an arbitrary real function of $V^{ab}$, is obviously invariant under $N = 4$ Poincaré supersymmetry. Being rewritten in terms of the components the action reads

$$S = \int dt \left[ g \left( \dot{v}_m \dot{v}_m + A^2 + i(\lambda^a \dot{\lambda}_a - \dot{\lambda}^a \lambda_a) \right) + \partial_k g \left( iA\delta_{mk} - \varepsilon_{mnk} \dot{v}_n \right) \lambda^a \sigma^m_{ab} \dot{\lambda}^b + \frac{\Delta g}{4} \lambda^a \lambda_a \dot{\lambda}^b \dot{\lambda}_b \right]. \quad (7)$$

Here, $\partial_m$ means differentiation with respect to $v_m$ and the metric $g$ is defined as

$$g \equiv \triangle F(v) = \frac{\partial^2 F(v)}{\partial v_m \partial v_n}. \quad (8)$$

The proper potential terms may be added to the action (6) (see e.g. [8]). But what is important for us here is that the bosonic sigma-model part of the action for the $N = 4$ tensor supermultiplet forms a conformally flat three-dimensional manifold with the metric given in eq. (3).

### 3 Dualization of the auxiliary component

As we have already mentioned in the introduction, in one dimension one may switch between different supermultiplets by a proper expression of auxiliary bosonic components through time-derivatives of physical bosons, and vice versa. The reduction from the “root” supermultiplet to the supermultiplets with a smaller number of the physical bosonic components was considered in detail in [5]. Here we proceed in the opposite direction. Our goal is to turn the auxiliary component $A$ which is present in the tensor supermultiplet $V^{ab}$ into a physical boson and then to analyze the geometry of the resulting four-dimensional manifold.

The heart of the possible “dualization” of the auxiliary component $A$ into physical boson lies in its transformation properties under supersymmetry [5]. In one dimension one may integrate [4] and pass to the physical bosonic component $u$ defined as

$$\partial_t u = A \quad (9)$$

with the supersymmetry transformation properties

$$\delta u = \varepsilon^a \dot{\lambda}_a + \bar{\varepsilon}^a \lambda_a. \quad (10)$$

The resulting supermultiplet contains four physical bosonic and four fermionic components. The corresponding action can be easily obtained from (7) using (4)

$$S = \int dt \left[ g \left( \dot{v}_m \dot{v}_m + A^2 + i(\lambda^a \dot{\lambda}_a - \dot{\lambda}^a \lambda_a) \right) + \partial_k g \left( iA\delta_{mk} - \varepsilon_{mnk} \dot{v}_n \right) \lambda^a \sigma^m_{ab} \dot{\lambda}^b + \frac{\Delta g}{4} \lambda^a \lambda_a \dot{\lambda}^b \dot{\lambda}_b \right]. \quad (11)$$

This action represents the particular type of the action for the so-called $N = 4, d = 1$ hypermultiplet [5][9]. The restriction with respect to the general case is that the metric $g$ depends on three bosonic fields $v_m$ and thus the action possesses one obvious isometry with the Killing vector $\partial_u$.

One should stress that any auxiliary bosonic component which transforms as a full time derivative under supersymmetry can be “dualized” into a physical field. So the question is whether the auxiliary component $A$ defined in (4) is unique? To answer this question let us consider the following easy-to-prove

**Statement** The most general real combination of dimension $cm^{-1}$ that is composed of the tensor supermultiplet components, linear in its auxiliary component and transforms as a total time derivative has the following form (modulo total time derivative terms):

$$B = a_m \dot{v}_m - f_m \lambda^a \sigma^m_{ab} \dot{\lambda}^b + f A, \quad (12)$$

with

$$\delta B = \varepsilon^a \frac{d}{dt} \left[ f \dot{\lambda}_a + ia_m \sigma^m_{ab} \dot{\lambda}^b \right] + \bar{\varepsilon}^a \frac{d}{dt} \left[ f \lambda_a + ia_m \sigma^m_{ab} \lambda^b \right], \quad (13)$$

where real dimensionless coefficients $a_m$ and $f$ being functions of the vector $v_m$ satisfy

$$\Delta f = 0, \quad \text{rot} \dot{a} = \nabla f. \quad (14)$$

3
Clearly enough, our previous choice of the auxiliary component \( A \) corresponds to the very particular case \( f = 1 \). The vector \( a_m \) is defined up to a gradient of a scalar function which is out of relevance here since this corresponds to adding the total time derivative of the scalar function to \( B \).

What is really important is that the newly defined auxiliary component \( B \) is linear in the former auxiliary component \( A \). This means that we can replace \( A \) by the proper combination which follows from

\[
A \rightarrow \frac{1}{f} \left[ B - a_m \dot{v}_m + f_{,m} \lambda^a \sigma_{ab} \bar{\lambda}^b \right].
\]

The immediate consequences of this replacement are nonlinear transformations of the components. Of course, while we are dealing with the \( N = 4 \) tensor supermultiplet this nonlinearity is fake: we may always come back to the auxiliary component \( A \). However, this equivalence will be broken if we dualize the new auxiliary component \( B \) as

\[
\partial_t \phi = B
\]

with the following transformation properties under supersymmetry:

\[
\delta \phi = \epsilon^a \left[ f \bar{\lambda}_a + ia_m \sigma_{ab} \bar{\lambda}^b \right] + \bar{\epsilon}^a \left[ f \lambda_a + ia_m \sigma_{ab} \lambda^b \right]
\]

The new supermultiplet with four bosonic components \( v_m, \phi \) and four fermions \( \lambda^a, \bar{\lambda}^b \) is essentially nonlinear. The key point is that to pass to the previous linear supermultiplet, one should solve the differential equation

\[
\dot{u} = \frac{1}{f} \left[ \dot{\phi} - a_m \dot{v}_m + f_{,m} \lambda^a \sigma_{ab} \bar{\lambda}^b \right].
\]

Thus, two equivalent formulations of the \( N = 4 \) tensor supermultiplet being dualized give rise to two different nonequivalent \( N = 4 \) supermultiplets with four bosonic and four fermionic components: one is linear, while the other supermultiplet is essentially nonlinear.

To clarify the role of the new physical bosonic field \( \phi \), let us plug expression (15) with the auxiliary component \( B \) dualized as in (16) into action (17) and choose the metric \( g = f \). The bosonic part of the resulting action reads

\[
S_{bosonic} = \int dt \left[ f \dot{v}_m \dot{v}_m + \frac{1}{f} \left( \dot{\phi} - a_m \dot{v}_m \right)^2 \right].
\]

Observe, this is just the one dimensional version of the general Gibbons-Hawking solution for four-dimensional hyper-Kähler metrics with one triholomorphic isometry (16). The corresponding Killing vector is obviously \( \partial_\phi \).

Thus, we explicitly demonstrate that starting from the sigma-model type action for the \( N = 4 \) tensor supermultiplet one may recover the hyper-Kähler sigma model by dualizing the proper auxiliary component. Let us remark that \( N = 4 \) supersymmetric four dimensional sigma model in one dimension does not oblige to have the hyper-Kähler sigma model in the bosonic sector. This fact is reflected in the arbitrariness of the metric \( g \). Only in the case \( g = f \) with harmonic function \( f \), the bosonic part of the \( N = 4 \) supersymmetric model becomes of hyper-Kähler type.

Summarizing, after dualization of the auxiliary component \( B \) in the \( N = 4, \ d = 1 \) tensor supermultiplet we get a nonlinear supermultiplet with four physical bosonic and four fermionic components which transform under \( N = 4 \) supersymmetry as follows:

\[
\delta v_m = ie^a \sigma_{ab} \bar{\lambda}^b - i \lambda^a \sigma_{ab} \epsilon^b, \quad \delta \phi = \epsilon^a \left( f \bar{\lambda}_a + i a_m \sigma_{ab} \bar{\lambda}^b \right) + \bar{\epsilon}^a \left( f \lambda_a + i a_m \sigma_{ab} \lambda^b \right),
\]

\[
\delta \lambda_a = i f \epsilon_a \left( \dot{\phi} - a_m \dot{v}_m + f_{,m} \lambda^b \sigma_{bc} \bar{\lambda}^c \right) + \sigma_{ab} \epsilon^b \dot{v}_m,
\]

\[
\delta \bar{\lambda}^a = - i f \bar{\epsilon}^a \left( \dot{\phi} - a_m \dot{v}_m + f_{,m} \lambda^b \sigma_{bc} \bar{\lambda}^c \right) + \sigma^{ab} \bar{\epsilon}_b \dot{v}_m,
\]

with the functions \( f(v^m) \) and \( a_m(v^m) \) obeying to constraints

\[
\Delta f = 0, \quad \text{rot} \, \vec{a} = \vec{\nabla} f.
\]
With the choice \( g = f \) in the general sigma-model type action for tensor supermultiplet it is dualized into the action

\[
S = \int dt \left[ f \dot{v}_m \dot{\bar{v}}_m + \frac{1}{f} \left( \dot{\phi} - a_m \dot{v}_m \right)^2 + i f \left( \lambda^a \dot{\chi}_a - \dot{\lambda}^a \chi_a \right) + \right.
\]

\[
+ \frac{3}{f} \left( \dot{\phi} - a_m \dot{v}_m \right) f_{\alpha m} \lambda^a \sigma^{ab}_{\alpha b} \tilde{\lambda}^b - \varepsilon_{\alpha \beta \gamma} f_{\alpha m} \lambda^a \sigma^{ab}_{\beta \gamma} \tilde{\lambda}^b + \frac{1}{f} f_{\alpha \beta \gamma} f_{\alpha m} \lambda^a \tilde{\lambda}^b \tilde{\lambda}_b \right],
\] (22)

which contains the hyper-Kähler sigma model in the bosonic sector.

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