THE BOUNDEDNESS OF CAUCHY INTEGRAL OPERATOR ON A
DOMAIN HAVING CLOSED ANALYTIC BOUNDARY

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Abstract. In this paper, we prove that the Cauchy integral operators (or Cauchy transforms) define
continuous linear operators on the Smirnov classes for some certain domain with closed analytic boundary.

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1. Introduction

As usual, we define the Hardy space $H^2 = H^2(\Delta)$ as the space of all functions $f : z \rightarrow \sum_{n=0}^{\infty} a_n z^n$
for which the norm $\left( \|f\| = \sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2}$ is finite. Here, $\Delta$ is the open unit disc. For a more general
simply-connected domain $\Omega$ in the complex plane $\mathbb{C}$ with at least two boundary points, and a conformal
mapping $\varphi$ from $\Omega$ onto $\Delta$ (that is, a Riemann mapping function), a function $g$ analytic in $\Omega$ is said to
belong to the Smirnov class $E^2(\Omega)$ if and only if $g = (f \circ \varphi) \varphi'_{1/2}$ for some $f \in H^2(\Delta)$ where $\varphi'_{1/2}$ is an
analytic branch of the square root of $\varphi'$. The reader is referred to [4], [5], [8], [10], and references therein for
a basic account of the subject.

$\partial\Delta$ and $\partial\Omega$ will be used to denote the boundary of open unit disc $\Delta$ and the
boundary of $\Omega$ respectively.

Suppose that $\Gamma$ is a simple $\sigma$-rectifiable arc (not necessarily closed). The notation $L^p(\Gamma)$ will denote the
$L^p$ space of normalized arc length measure on $\Gamma$. Let $\Omega$ denote the complement of $\Gamma$. The Cauchy Integral
of a function $\tilde{f}$ defined on $\Gamma$ and integrable relative to arc length is defined as:

\begin{equation}
C_\Omega \tilde{f}(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\tilde{f}(\zeta)}{\zeta - z} d\zeta \quad (z \in \Omega).
\end{equation}

$C_\Omega \tilde{f}$ is analytic at each point of $\Omega$. If $\Gamma$ is not closed, then (1.1) defines a single analytic function. If $\Gamma$ is
closed, then $\Omega$ has two components, the interior and the exterior of $\Gamma$. Then in each component of (1.1)
defines an analytic function.
Recall that a closed analytic curve is a curve $\gamma = k(\partial \Delta)$ where $k$ is analytic and conformal in a neighbourhood $U$ of $\partial \Delta$. If $\gamma$ is simple it is called an analytic Jordan curve.

In this paper we prove

**Theorem 1.1.** Suppose that that $D$ is a bounded simply connected domain and $\gamma = \partial D$ is a closed analytic curve (e.g. ellipse). Then the Cauchy Integral

$$C_D \tilde{f}(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{\tilde{f}(\zeta)}{\zeta - z} d\zeta$$

defines a continuous linear operator mapping $L^2(\partial D)$ into $E^2(D)$.

**Remark 1.2.** The result of the above Theorem is well known in the literature; see, for example, [3] and [6]. However, we give a basic and direct proof.

To prove Theorem 1.1 we need the following Lemma and Remark.

**Lemma 1.3 ([4], p.170).** Suppose that $\Omega$ is a simply connected and bounded domain and the boundary $\Gamma = \partial \Omega$ of $\Omega$ is a rectifiable Jordan curve. Then

i.: Each $f \in E^2(\Omega)$ has a nontangential limit function $\tilde{f} \in L^2(\partial \Omega)$ and

$$||f||^2_{E^2(\Omega)} = ||\tilde{f}||^2_{L^2(\partial \Omega)} = \frac{1}{2\pi} \int_{\partial \Omega} |\tilde{f}(z)|^2 |dz|.$$  

ii.: Each $f \in E^2(\Omega))$ has a Cauchy representation

$$f(z) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{\tilde{f}(\zeta)}{\zeta - z} \Omega \zeta$$  

($z \in \Omega$).

In this case, for equation (1.2) we say that Cauchy Integral Formula is valid.

A special case of the above theorem is the following Remark. In fact, we prove Theorem 1.1 using this Remark.

**Remark 1.4 ([11], p.423).** The Cauchy integral formula

$$C_\Delta \tilde{f}(z) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{\tilde{f}(\zeta)}{\zeta - z} d\zeta$$

defines a continuous linear operator $C_\Delta : L^2(\partial \Delta) \to E^2(\Delta)$ with $||C_\Delta|| = 1$.

There is another integral operator $C_\Gamma$ on the curve $\Gamma$, which is sometimes called also Cauchy integral, viewed as an operator-valued function of the curve. This operator is given by a principal value singular integral: if $\tilde{f}$ is a function on $\Gamma$, we define $C_\Gamma(\tilde{f})$ on $\Gamma$ by

$$C_\Gamma \tilde{f}(z) = \frac{1}{2\pi i} P.V. \int_{\Gamma} \frac{\tilde{f}(\zeta)}{\zeta - z} d\zeta$$  

($z \in \Gamma$).

The operator $C_\Gamma$ is probably less familiar than $C_\Omega$. These operators are very important in real and complex analysis, and have attracted many mathematicians to investigate them.
In fact, there are other types of Cauchy integrals and there have been extensive literature about them and their applications as papers and books. For the books concerning Cauchy type integrals and related subjects; see, for instance, [1], [2], [12], [14] and [15]. For the books concerning boundedness of Cauchy type integrals; see, for example, [7], [9] and [13].

2. Proof of Theorem 1.1

We are now ready to give the proof of Theorem 1.1.

Proof. Suppose that \( \varphi \) is a conformal map of \( D \) onto \( \Delta \). Let \( \psi = \varphi^{-1} : \Delta \rightarrow D \). Consider the maps \( C_{\Delta} : L^2(\partial \Delta) \rightarrow E^2(\Delta) \) is given by \( C_{\Delta} \tilde{f} = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{\tilde{f}(\zeta)}{\zeta - z} d\zeta \) and \( \tilde{U}_{\psi} : L^2(\partial \Delta) \rightarrow L^2(\partial \Delta) \) is given by \( \tilde{U}_{\psi} f(z) = f(\psi(z))\psi'(z)^{1/2} \) and \( U_{\varphi} : E^2(\Delta) \rightarrow E^2(D) \) is given by \( U_{\varphi} f(z) = f(\varphi(z))\varphi'(z)^{1/2} \). \( U_{\varphi} \) and \( \tilde{U}_{\psi} \) are unitary operators. The situation is illustrated in the Figure 1.

\[
\begin{array}{ccc}
L^2(\partial \Delta) & \xrightarrow{\tilde{U}_{\psi} = \tilde{U}_{\varphi}^*} & L^2(\partial D) \\
\downarrow C_{\Delta} & & \downarrow C_{D} \\
E^2(\Delta) & \xrightarrow{U_{\varphi}} & E^2(D)
\end{array}
\]

**Figure 1.** the maps \( U_{\varphi} \) and \( \tilde{U}_{\psi} \)

For \( \tilde{f} \in L^2(\partial \Delta) \), we have \( U_{\psi} C_{D} \tilde{U}_{\varphi} \tilde{f} = U_{\psi} C_{D}(\tilde{f} \circ \varphi)\varphi'(z)^{1/2} \) and

\[
C_{D}(\tilde{f} \circ \varphi(w))\varphi'(w)^{1/2} = \frac{1}{2\pi i} \int_{\partial D} \frac{\tilde{f}(\varphi(\zeta))\varphi'(\zeta)^{1/2}}{\zeta - w} d\zeta
\]

so that

\[
U_{\psi} C_{D} \tilde{U}_{\varphi} \tilde{f}(z) = \psi'(z)^{1/2} \frac{1}{2\pi i} \int_{\partial D} \frac{\tilde{f}(\varphi(\zeta))\varphi'(\zeta)^{1/2}}{\zeta - \psi(z)} d\zeta
\]

\[
= \frac{1}{2\pi i} \int_{\partial \Delta} \frac{\tilde{f}(w)\psi'(w)^{1/2}\psi'(z)^{1/2}}{\psi(w) - \psi(z)} dw \quad (z \in \Delta).
\]

Then we have kernel

\[
K(z, w) = \frac{1}{w - z} H(z, w)
\]
where
\[ H(z, w) = \frac{(w - z)\psi'(w)^{1/2}\psi'(z)^{1/2}}{\psi(w) - \psi(z)}. \]

For any \( 1 < r \), denote \( \Delta_r \) and \( \sigma_r \) by \( \Delta_r = \{ z : |z| < 1 \} \) and \( \sigma_r = \partial \Delta_r = \{ z : |z| = r \} \).

Since \( \partial D \) is analytic (closed) curve, \( \psi \) is analytic and conformal in a neighbourhood of \( \overline{\Delta} \). So without loss of generality, we may assume that \( \psi \) is analytic and conformal on \( \Delta_R \) for some \( R > 1 \). Then \( \psi \) is analytic and conformal on and inside \( \sigma_r \), where \( R > r' > 1 \).

Choose \( r, s \) such that \( 1 < r < s < R \). We shall show that \( H \) is analytic on \( \Delta_r \times \Delta_r \). Hence, because \( r \) is arbitrary, it will follow that \( H \) is analytic on \( \Delta_R \times \Delta_R \).

Fix \( z \in \Delta_R \). Then \( w \to F_z(w) = H(z, w) \) is analytic in \( \Delta_R \) except at \( w = z \). But since residue at \( w = z \) is 0, the singularities of \( H \) for \( w = z \) is removable. Hence \( w \to H(z, w) \) is analytic on \( \Delta_R \supseteq \overline{\Delta_s} \supseteq \overline{\Delta_r} \). We can thus apply Cauchy’s integral formula to it, giving
\[
H(z, w) = \frac{1}{2\pi i} \int_{\sigma_r} H(z, v) \frac{1}{v - w} \, dv \quad (w \in \Delta_r \text{ and fixed } z \in \Delta_s, v \in \sigma_r).
\]

Hence since \( z \in \Delta_s \) is arbitrary, for all \( z \in \Delta_s \) and \( w \in \Delta_r \), equation 2.1 is valid.

By symmetry, for every \( v \in \Delta_R \) the function \( z \to H(z, v) \) is analytic on \( \Delta_R \). Hence
\[
H(z, v) = \frac{1}{2\pi i} \int_{\sigma_s} H(u, v) \frac{1}{u - z} \, du \quad (z \in \Delta_s, v \in \Delta_R, u \in \sigma_s).
\]

\( H \) is separately continuous on \( \Delta_R \times \Delta_R \), so it is (jointly) continuous on \( \Delta_R \times \Delta_R \). Substitute the value of \( H(z, v) \) from 2.2 in the integrand of 2.1. Since the function \( H(u, v) \) is continuous, we obtain
\[
H(z, w) = \left( \frac{1}{2\pi i} \right)^2 \int_{\sigma_r} \int_{\sigma_s} H(u, v) \frac{1}{(u - z)(v - w)} \, du \, dv \quad (z \in \Delta_s, w \in \Delta_r, v \in \sigma_r, u \in \sigma_s)
\]

Since \( r \) is arbitrary, we will show that \( H(z, w) \) is analytic on \( \Delta_R \times \Delta_R \).

Now,
\[
\frac{1}{(u - z)(v - w)} = \sum_{m,n=0}^{\infty} \frac{z^m w^n}{u^{m+1} v^{n+1}} \quad (v \in \sigma_r, u \in \sigma_s)
\]

and this series is uniformly convergent for \( z \in \Delta_s, w \in \Delta_r, v \in \sigma_r, u \in \sigma_s \).

Hence
\[
H(z, w) = \left( \frac{1}{2\pi i} \right)^2 \int_{\sigma_r} \int_{\sigma_s} \sum_{m,n=0}^{\infty} \frac{z^m w^n}{u^{m+1} v^{n+1}} H(u, v) \, du \, dv \quad (z \in \Delta_s, w \in \Delta_r, v \in \sigma_r, u \in \sigma_s)
\]

Since \( H \) is bounded , the series \( \sum_{m,n=0}^{\infty} \frac{z^m w^n}{u^{m+1} v^{n+1}} H(u, v) \) is uniformly convergent for \( z \in \Delta_s, w \in \Delta_r, v \in \sigma_r, u \in \sigma_s \).

Because of uniformly convergence, we can integrate the series term-by-term and we obtain
\[
H(z, w) = \sum_{m,n=0}^{\infty} \frac{1}{2\pi i} \int_{\sigma_r} \int_{\sigma_s} \frac{H(u, v)}{u^{m+1} v^{n+1}} \, du \, dv \quad (z \in \Delta_s, w \in \Delta_r, v \in \sigma_r, u \in \sigma_s)
\]

From this we obtain
\[
H(z, w) = \sum_{m,n=0}^{\infty} a_{mn} z^m w^n
\]
where the coefficients $a_{mn}$ are given by the integral formula

$$a_{mn} = \left( \frac{1}{2\pi i} \right)^2 \int_{\sigma_r} \int_{\sigma_s} \frac{H(u, v)}{u^{m+1}v^{n+1}} \, du \, dv.$$  

Since $H$ is bounded, we obtain

$$|a_{mn}| = \left| \left( \frac{1}{2\pi i} \right)^2 \int_{\sigma_r} \int_{\sigma_s} \frac{H(u, v)}{u^{m+1}v^{n+1}} \, du \, dv \right| \leq \frac{1}{4\pi^2} (2\pi s)^2 \, \|H\| \sup_{s^{m+1}r^{n+1}} \sum_{m,n=0}^\infty |a_{mn}| \leq s^2 \|H\| \sup_{s^{m+1}r^{n+1}} \sum_{m,n=0}^\infty \frac{1}{s^{m+1}r^{n+1}}$$

where $\|H\|_\infty = \sup_{u,v \in \sigma'} |H(u, v)| < \infty$ and so

$$\sum_{m,n=0}^\infty |a_{mn}| < \infty$$

(i.e. the series $\sum_{m,n=0}^\infty a_{mn}$ is absolutely convergent). Hence $\sum_{m,n=0}^\infty a_{mn}z^m w^n$ is absolutely convergent on $\Delta_r \times \Delta_r$. Thus $H(z, w)$ is analytic on $\Delta_r(\subseteq \Delta_s) \times \Delta_r$, and so since $r$ is arbitrary it is analytic on $\Delta_R \times \Delta_R$.

Now the series $H(z, w) = \sum_{m,n=0}^\infty a_{mn}z^m w^n$ is uniformly convergent for $z \in \Delta_s, w \in \Delta_r$. If we set $A = U \psi C_D \tilde{U}_\varphi$, then we have

$$Af(z) = U \psi C_D \tilde{U}_\varphi f(z) \quad (f \in L^2(\partial \Delta), z \in \Delta)$$

$$= \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f(w)\psi(w)^{1/2}\psi'(z)^{1/2}}{\psi(w) - \psi(z)} \, dw \quad (w \in \partial \Delta)$$

$$= \frac{1}{2\pi i} \int_{\partial \Delta} \frac{1}{w - z} H(z, w) f(w) \, dw$$

(2.5)

$$= \frac{1}{2\pi i} \int_{\partial \Delta} \frac{1}{w - z} \sum_{m,n=0}^\infty a_{mn}z^m w^n f(w) \, dw.$$  

In fact, we will show that the sum and the integral in the equation (2.5) can be permutable. For $f \in L^2(\partial \Delta), z \in \Delta$, we have

$$\sum_{m,n=0}^\infty \int_{\partial \Delta} \left| \frac{1}{w - z} a_{mn}z^m w^n f(w) \, dw \right| \leq \sum_{m,n=0}^\infty \int_{\partial \Delta} \left| a_{mn} \right| \left| z \right|^m \left| w \right|^n \left| f(w) \right| \left| \frac{dw}{2\pi} \right|$$

$$\leq \sum_{m,n=0}^\infty \left| a_{mn} \right| \left| z \right|^m \left( \int_{\partial \Delta} \left| f(w) \right|^2 \left| \frac{dw}{2\pi} \right| \right)^{1/2} \left( \int_{\partial \Delta} \left| \frac{1}{w - z} \right| \left| \frac{dw}{2\pi} \right| \right)^{1/2}$$

$$\leq \sum_{m,n=0}^\infty \left| a_{mn} \right| \left| z \right|^m \|f\| \left( \frac{1}{(1 - |z|)^2} \right)^{1/2} < \infty$$

and by Tonelli Theorem

(2.6)  

$$\frac{1}{2\pi i} \int_{\partial \Delta} \frac{1}{w - z} \sum_{m,n=0}^\infty a_{mn}z^m w^n f(w) \, dw = \sum_{m,n=0}^\infty a_{mn}z^m \frac{1}{2\pi i} \int_{\partial \Delta} \frac{1}{w - z} w^n f(w) \, dw$$
and so
\[ Af(z) = \sum_{m,n=0}^{\infty} a_{mn} z^m \frac{1}{2\pi i} \int_{\partial \Delta} \frac{1}{w-z} w^n f(w) dw \quad (f \in L^2(\partial \Delta), z \in \Delta). \]

Now consider the following series of operators
\[ \sum_{m,n=0}^{\infty} a_{mn} M_m C_n f(z) \quad (f \in L^2(\partial \Delta), z \in \Delta) \]

where \( M_m : H^2(\Delta) \to H^2(\Delta) \) is defined by \( M_m f(z) = z^m f(z) \) so that \( \| M_m \| = \| z^m \|_\infty = 1 \), and \( N_n : L^2(\partial \Delta) \to L^2(\partial \Delta) \) is defined by \( N_n f(z) = z^n f(z) \) so that \( \| N_n \| = \| z^n \|_\infty = 1 \). So we have
\[ Af(z) = \sum_{m,n=0}^{\infty} a_{mn} M_m C_n N_n f(z) \quad (f \in L^2(\partial \Delta), z \in \Delta) \]

Then the series \( A = \sum_{m,n=0}^{\infty} a_{mn} M_m C_n N_n \) is absolutely convergent in operator norm in the space \( B(L^2(\partial \Delta), H^2(\Delta)) \), in fact,
\[ \| M_m C_n f \|_{H^2(\Delta)} \leq \| M_m \| \| C_n \| \| N_n \| \| f \| \quad (f \in L^2(\partial \Delta)) \]
and
\[ \| M_m C_n N_n \| \leq 1 \]
so that
\[ \sum_{m,n=0}^{\infty} \| a_{mn} M_m C_n N_n \| \leq \sum_{m,n=0}^{\infty} |a_{mn}| < \infty \]
(i.e. \( \sum_{m,n=0}^{\infty} a_{mn} M_m C_n N_n \) converges absolutely) and
\[ \| Af \|_{E^2(\Delta)} = \left\| \sum_{m,n=0}^{\infty} a_{mn} M_m C_n f \right\|_{E^2(\Delta)} \]
\[ \leq \sum_{m,n=0}^{\infty} \| a_{mn} M_m C_n f \| \quad \text{since} \quad \sum_{m,n=0}^{\infty} a_{mn} M_m C_n N_n \text{ converges absolutely} \]
\[ = \sum_{m,n=0}^{\infty} |a_{mn}| \| M_m \| \| C_n \| \| N_n \| \| f \| \leq \sum_{m,n=0}^{\infty} |a_{mn}| \cdot 1.1 \cdot \| f \| \]
\[ \leq \| f \| \sum_{m,n=0}^{\infty} |a_{mn}| < \infty \]
and so
\[ \| A \| = \left\| \sum_{m,n=0}^{\infty} a_{mn} M_m C_n N_n \right\| \leq \sum_{m,n=0}^{\infty} |a_{mn}|. \]

This shows that \( A = U_\psi C_D \bar{U}_\varphi \) is a continuous operator. It follows that \( C_D \) is a continuous operator.

**Second proof of the continuity of \( A \)**
Since in a Banach space $X$ (here $X = B(L^2(\partial \Delta), H^2(\Delta))$), every absolutely convergent series is convergent, in the norm of $X$, to an element of $X$, \( \sum_{m,n=0}^{\infty} a_{mn} M_{m\Delta} N_{n} \) converges to an element $B \in B(L^2(\partial \Delta), H^2(\Delta))$, in the sense that
\[
\lim_{m,n \to \infty} \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} M_{k\Delta} N_{l} = B
\]
i.e.
\[
\left\| B - \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} M_{k\Delta} N_{l} \right\| \to 0 \quad \text{as} \quad m, n \to \infty.
\]
Our aim is to show that $B = A$, (i.e. $Bf(z) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{H(z, w)f(w)dw}{w-z}$). Fix $z \in \Delta$ and $f \in L^2(\partial \Delta)$.

Then
\[
\left\| Bf - \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} M_{k\Delta} N_{l} f \right\| \to 0
\]
i.e.
\[
Bf = \lim_{m,n \to \infty} \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} M_{k\Delta} N_{l} f.
\]

Hence
\[
\sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} M_{k\Delta} N_{l} f(z) \to Bf(z)
\]

Now
\[
Bf(z) = \lim_{m,n \to \infty} \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} M_{k\Delta} N_{l} f(z)
\]
\[
= \lim_{m,n \to \infty} \frac{1}{2\pi i} \sum_{k=0}^{m} \sum_{l=0}^{n} \int_{\partial \Delta} a_{kl} z^k w^l f(w) \frac{dw}{w-z}
\]
\[
= \lim_{m,n \to \infty} \frac{1}{2\pi i} \int_{\partial \Delta} \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} z^k w^l f(w) \frac{dw}{w-z}
\]
\[
= \frac{1}{2\pi i} \int_{\partial \Delta} f(w) \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{mn} z^m w^n dw \quad \text{(from equation 2.6)}
\]
\[
= \frac{1}{2\pi i} \int_{\partial \Delta} f(w) H(z, w)dw
\]
\[
= Af(z).
\]
So for $z \in \Delta$ and $f \in L^2(\partial \Delta)$, $Bf(z) = Af(z)$. Hence $B = A$. Therefore, since $B$ is a continuous operator it follows that $A = U_{\phi} C_D \tilde{U}_{\phi}$ is a continuous operator. Therefore $C_D$ is a continuous operator.

References

[1] Bell, S.R., The Cauchy Transform, Potential Theory and Conformal Mapping, Chapman and Hall/CRC, (2nd Edition, 2015)
[2] Cima, J.A., Matheson, A.L., Ross W.T., The Cauchy Transform, American Mathematical Society (2006).
[3] Coifman, R.R., Jones, P.W., Semmes, S., Two elementary proofs of the $L^2$ boundedness of Cauchy integrals on Lipschitz curves. J. Amer. Math.Soc. 2 (1989), no.3, 353-364.
[4] Duren P.L., Theory of $H^p$ Spaces, (Academic Press, 1970).
[5] Duren P.L., Smirnov Domains, Journal of Mathematical Sciences, Volume 63, Number 2 / January, (1993), 167-170.

[6] Dyn’kin, E. M. Methods of the Theory of singular integrals: Littlewood-Paley theory and its applications, Commutative harmonic analysis IV, Volume 42 of the series Encyclopaedia of Mathematical Sciences pp 97-194, Springer, Berlin, (1992).

[7] Edmunds, D.E., Kokilashvili, V., Meskhi, A., Bounded and Compact Integral Operators (Mathematics and Its Applications), Springer, (2002).

[8] Goluzin, G.M., Functions of a Complex Variable, Amer. Math. Soc., Providence, RI, (translated from Russian, 1969).

[9] Halmos, P.R., Sunder, V.S., Bounded Integral Operators on $L^2$ Spaces, Springer, (1978).

[10] Khavinson, D., Factorization theorems for certain classes of analytic functions in multiply connected domains, Pacific. J. Math. 108 (1983) 295-318.

[11] G. Little, Equivalences of positive integral operators with rational kernels, Proc. London. Math. Soc. (3) 62 (1991) 403-426.

[12] Murai, T., A Real Variable Method for the Cauchy Transform, and Analytic Capacity, Springer, (2008).

[13] Okikiolu, G.O., Aspects of Bounded Integral Operators in Lp Spaces, Academic Press Inc (1971).

[14] Tolsa, X., Analytic Capacity, the Cauchy Transform, and Non-homogeneous Calderón-Zygmund Theory, Birkhäuser, (2016).

[15] Tolsa, X., Rectifiable Measures, Square Functions Involving Densities, and the Cauchy Transform, American Mathematical Society, (2017).