Robustness of capture radius with respect to Gromov–Hausdorff distance in Lion and Man game

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Abstract

We consider the Lion and Man game that is a two-person pursuit–evasion game with equal players’ top speeds. We assume that capture radius is positive and is chosen in advance. The main aim of the paper is describing pursuer’s winning strategies in compact metric spaces that are close to the given one in the sense of Gromov–Hausdorff distance. In this way, we cannot use smoothness, curvature, or any other properties. We prove that existence of an $\alpha$-capture by a time $T$ in one compact geodesic space implies existence of the $(\alpha + (20T + 8)\sqrt{\varepsilon})$-capture by this time $T$ in any compact geodesic space that is $\varepsilon$-close to the given space. It means that capture radii (in a nearby spaces) tends to the given one as the distance between spaces tends to zero. Thus, this result justifies calculations on graphs instead of complicated spaces.

Keywords: pursuit–evasion game, Lion and Man problem, guidance, robustness, Gromov–Hausdorff distance, capture radius, geodesic space, finite graph.

Introduction

Pursuit–evasion games are a popular class of dynamic games and have many applications. Some of their variants relate with the particular case of these games called Lion and Man game. This game is a two-person game the peculiarity of which is the same capacity (usually, the same top speed) of players and a positive value of capture radius. Namely, we assume that players are in a compact metric space and move along 1-Lipschitz curves only. These assumptions allows to expand our consideration to general metric spaces, beyond Euclidean cases. Such wide view is applicable, for instance, to Browian motions [4] and geometric characterisation of spaces [11].

We continue the studying of Lion and Man problem in general metric spaces by examining behaviour of ‘nearby’ spaces; more precisely, we are interested in studying the robustness of capture radius, i.e., of pursuit problem. In force of our assumptions, we cannot use smooth structure. This significantly differs from the conception of Krasovskii and Subbotin: both classical works [9, 10] and their stochastic generalization [3, 8] use derivations of Lyapunov functions. We use the similar constructions (as guidance), however, we apply the methods, that are usable for arbitrary compact metric spaces or even for the space of compact metric

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spaces. More exactly, we study the robustness of capture radius with respect to Gromov–Hausdorff distance between compact metric spaces.

Thus, here we prove the following theorem:

Let two compact geodesic spaces be such that the Gromov–Hausdorff or Hausdorff distance between them is not greater than \( \varepsilon \). Existence of \( \alpha \)-capture by a time \( T \) in one space implies existence of \( (\alpha + (20T + 8)\sqrt{\varepsilon}) \)-capture by the time \( T \) in the other space.

Let us take the next step and consider a sequence of spaces instead of a pair: it may be useful in the following case. Suppose that we have a complicated (compact and geodesic) space and aim to examine an \( \alpha \)-capture in it. Then, we can approximate this space by a sequence of finite metric graphs and consider the \( \alpha \)-capture in them. Indeed, this action allows us to assay the infimum of possible capture radii in the chosen space. Moreover, this argument justifies the papers dealing with graphs only, as, for instance, the work [7], which studies motion planning. We return to such applications in Section 2.

We should notice that this theorem uses the upper bound on the time, whereas the infinite horizon is usually considered (see e.g. [1, 11, 13]). Note that existence of the upper bound is proved in [4] for bounded CAT(0) domains with several extra restrictions. In addition, there are estimates for upper and low bounds on capture time for the disk in the paper [2]. However, as far as we know, there are no similar estimates in general metric spaces.

The paper is organised as follows. Section 1 introduces game and geometry definitions, Section 2 contains the theorem and a little discussion. Finally, Section 3 provides the proofs.

1 Preliminaries

1.1 Lion and Man game

We consider Lion and Man problem, where Lion is a pursue and Man is an evader. We assume that both players move in a metric space \((X, \rho)\) and denote by \(L(\cdot)\) and \(M(\cdot)\) the trajectories of Lion and Man respectively. Notice that \(L(\cdot)\) and \(M(\cdot)\) are 1-Lipschitz functions of time to the space \(X\). We consider the game from the standpoints of Lion. Hence, Lion uses non-anticipative (or even stepwise) strategies against arbitrary possible movements of Man. By non-anticipative and stepwise strategies we mean the following.

Let us denote by \(1L(X)\) the set of all 1-Lipschitz curves from \(\mathbb{R}_+\) to a set \(X\) with respect of a metric \(\rho\) on this set \(X\). It is the set of each player’s admissible trajectories. A map \(s_{L_0}: 1L(X) \to 1L(X)\) is called Lion’s non-anticipative strategy (with Lion’s initial position \(L_0\)) if it satisfies the equality \(s_{L_0}(M)(0) = L_0\) for all \(M \in 1L(X)\) and the following implication holds true: for any admissible trajectories \(M_1, M_2 \in 1L(X)\) and a number \(\tau \geq 0\) if

\[ M_1(t) = M_2(t) \quad \forall t \in [0, \tau], \]

then

\[ s_{L_0}(M_1)(t) = s_{L_0}(M_2)(t) \quad \forall t \in [0, \tau]. \]

A particular case of a non-anticipative strategy is a stepwise strategy. For a positive number \(\beta\), we say that a map \(s_{L_0}\) is called Lion’s \(\beta\)-stepwise strategy (with Lion’s initial position \(L_0\)) if it satisfies the equality \(s_{L_0}(M)(0) = L_0\) for all \(M \in 1L(X)\) and the following implication holds true: for any admissible trajectories \(M_1, M_2 \in 1L(X)\) and a number \(n \in \mathbb{N}\) if

\[ M_1(t) = M_2(t) \quad \forall t \in [0, n\beta], \]

then

\[ s_{L_0}(M_1)(t) = s_{L_0}(M_2)(t) \quad \forall t \in [0, n\beta]. \]
then
\[ s_{L_0}(M_1)(t) = s_{L_0}(M_2)(t) \quad \forall t \in [0, (n+1)\beta]. \]

**Definition 1.** If \((X, \rho)\) is a compact metric space, \(\alpha\) is a positive number, then by the phrase ‘there are \((X, \rho, \alpha, T)\)-winning strategies’ let us denote the fact that, for any Lion’s initial position \(L_0\), he has a non-anticipative strategy \(s_{L_0} : 1L(X) \to 1L(X)\) leading to the condition
\[ \inf_{t \in [0, T]} \rho(M(t), s_{L_0}(M)(t)) \leq \alpha \text{ for all Man’s movement } M(\cdot). \]

### 1.2 Basic notations

Let us provide definitions and notations under the assumption that \((X, \rho_X), (Y, \rho_Y)\) are metric spaces, map \(f\) is from \(X\) to \(Y\), and \(a\) is a positive number.

- \(\text{dom}(f)\) denotes the domain of the map \(f\);
- \(\mathbb{R}_+ = [0, +\infty)\);
- \(\lceil a \rceil\) is the ceil of the number \(a\);
- if \(S \subset X\), then \(f[S]\) denotes the set \(\{f(x) \mid x \in S\}\);
- a set \(S \subset X\) is called an \(\varepsilon\)-net of the metric space \((X, \rho)\) if for any point \(x \in X\) there is a point \(s \in S\) such that \(\rho(x, s) \leq \varepsilon\);
- \((X, \rho_X)\) is said to be a geodesic space if every two points \(x\) and \(y\) in \(X\) are joined by a geodesic, i.e., there is a map \(g\) from a closed interval \([0, l] \subset \mathbb{R}\) to \(X\) such that \(g(0) = x, g(l) = y\) and \(\rho_X(g(t), g(t')) = |t - t'|\) for all \(t, t' \in [0, l]\) (see [5, I.1.3]);
- if, for any pair of points \(x_1, x_2 \in X\), the distance \(\rho_X(x_1, x_2)\) is equal to the infimum of the length of rectifiable curves joining these points, then \((X, \rho_X)\) is called a length space, otherwise known as an inner metric space (see [5, I.3.3]).

### 1.3 Distance between metric spaces

Definition of the Gromov–Hausdorff distance is fundamental in this paper, but one can imagine the Hausdorff distance instead of the Gromov–Hausdorff one if all considering spaces are subspaces of one ambient space. The distance was proposed by Gromov and Edwards independently (see [12]). Let us give preference to Gromov’s approach and use the following definition of the Gromov–Hausdorff distance borrowed from [6, Def. 7.3.10.]:

**Definition 2.** Let \(X\) and \(Y\) be metric spaces. The Gromov–Hausdorff distance between them, denoted by \(d_{GH}(X, Y)\), is defined by the following relation. For an \(r > 0\) we have \(d_{GH}(X, Y) < r\) if and only if there exist a metric space \(Z\) and subspaces \(X'\) and \(Y'\) of it which are isometric to \(X\) and \(Y\) respectively and such that \(d_H(X', Y') < r\). In other words, \(d_{GH}(X, Y)\) is the infimum of positive \(r\) for which the above \(Z, X'\) and \(Y'\) exist. Here \(d_H\) denotes the Hausdorff distance between subsets of \(Z\).
There exist other equivalent definitions, they are helpful to prove that this distance is a metric on the (continual) set of compact metric spaces. But, these definitions do not reflect the main idea so good. One can find both historic remarks and the list of important properties of the Gromov–Hausdorff space in the paper [12]. Let us introduce some related definitions and properties.

**Definition 3.** let \((X, \rho_X)\) and \((Y, \rho_Y)\) be metric spaces and let \(f : X \rightarrow Y\) be an arbitrary map. The distortion of \(f\) (denoted by \(\text{dis} f\)) is defined as
\[
\text{dis} f = \sup_{x_1, x_2 \in X} |\rho_X(x_1, x_2) - \rho_Y(f(x_1), f(x_2))|.
\]

**Definition 4.** Let \(X\) and \(\tilde{X}\) be metric spaces and let \(\varepsilon > 0\). A map \(h : X \rightarrow \tilde{X}\) is called an \(\varepsilon\)-isometry from \(X\) to \(\tilde{X}\) if \(\text{dis} h \leq \varepsilon\) and \(h[X]\) is an \(\varepsilon\)-net in \(\tilde{X}\).

**Lemma 1** ([6] Cor.7.3.28). Let \(X\) and \(\tilde{X}\) be two metric spaces and let \(\varepsilon > 0\). Then
1. If \(d_{GH}(X, \tilde{X}) \leq \varepsilon\), then there exists a \(2\varepsilon\)-isometry from \(X\) to \(\tilde{X}\).
2. If there exists an \(\varepsilon\)-isometry from \(X\) to \(\tilde{X}\), then \(d_{GH}(X, \tilde{X}) \leq 2\varepsilon\).

Moreover, one more construction and its properties are in Subsection 3.2.

**2 Results**

Let us recall that, in virtue of Hopf–Rinow theorem, a compact geodesic space is a compact length space and vice versa. Hence, distance between every two points of a compact geodesic space is given by the infimum of the lengths of rectifiable paths joining these points. We deal with these spaces because of convenience and since any compact metric space can be reparametrized into a length space if players’ trajectories form a length structure (for details, see [6, Chapter 2]).

**Theorem 1.** Let \((\tilde{X}, \tilde{\rho})\) and \((X, \rho)\) be compact geodesic spaces, let \(T > 0\) and \(\alpha \in (0, 1)\). If \(d_{GH}(\tilde{X}, X) \leq \varepsilon \in (0, \alpha^2)\), then existence of \((\tilde{X}, \tilde{\rho}, \alpha, T)\)-winning strategies implies existence of \((X, \rho, \alpha + (20T + 8)\sqrt{\varepsilon}, T)\)-winning strategies.

**Remark 1.** Since the Gromov–Hausdorff distance is never greater than any Hausdorff distance (among possible isometric embeddings), then it suffices to check the Hausdorff distance inequality \(d_{H}(\tilde{X}, X) \leq \varepsilon\) instead of the Gromov–Hausdorff one.

**Remark 2.** Although we consider non-anticipative strategies, to prove the theorem, we construct \(\sqrt{\varepsilon}\)-stepwise strategy. Hence, all statements hold for the so-called discrete-time Lion and Man games.

The following corollary is trivial, however, it provides a necessary condition of capture; which is quite rare.

**Corollary 1** (Necessary condition). If \(d_{GH}(\tilde{X}, X) \leq \varepsilon \in (0, \alpha^2)\) and there are no \((\tilde{X}, \tilde{\rho}, \alpha + (20T + 8)\sqrt{\varepsilon}, T)\)-winning strategies, then there are no \((X, \rho, \alpha, T)\)-winning strategies.

Moreover, one can combine these results with the fact that, for any compact geodesic (as well as length) space \((X, \rho)\), there exists a sequence of finite metric graphs \((G_n, \rho_n)\) (i.e., metric graphs with finite number of vertices) such that \(d_{GH}(X, G_n) \rightarrow 0\) as \(n \rightarrow \infty\).
The notion of metric graphs is quite intuitive, while the strict definition is not short; one can find the definition in [6, Def. 3.2.11.]. This and Theorem 1 encourage one to try to check $\alpha$-capture in some approximating spaces before doing this in the approximated space. In this way, Theorem 1 yields the following corollary.

**Corollary 2.** Let $(X, \rho)$ be compact geodesic spaces, let $\{(G_n, \rho_n)\}_{n \in \mathbb{N}}$ be a sequence of finite metric graphs. If $d_{GH}(X, G_n) \to 0$ as $n \to \infty$, then the following statements are equivalent:

1. for all $\tilde{\alpha} > \alpha$ there exist $(X, \rho, \tilde{\alpha}, T)$-winning strategies;
2. for each $n \in \mathbb{N}$ there exist $(G_n, \rho_n, \alpha_n, T)$-winning strategies and $\alpha_n \to \alpha$ as $n \to \infty$.

**Remark 3.** There is a constructive method to build such graphs.

Thus, investigating the game in finite metric graphs should be useful. In particular, if we knew ‘good enough’ pursuer’s strategies in graphs, we would easily construct ‘good enough’ strategies in the any compact metric space.

# 3 Proofs

We begin by constructing $\beta$-pursuing curves (Subsection 3.1) and $\varepsilon$-chaining relation between spaces (Subsection 3.2). These subsections are auxiliary for the Subsection 3.3, which is directed towards the target proving.

## 3.1 Simple pursuit

**Definition 5.** Let $N$ be a natural number, let $\beta > 0$, and let $(X, \rho)$ be a compact geodesic space. We say that a 1-Lipschitz curve $p : [0, N\beta] \to X$ $\beta$-pursues a collection $\{g(i\beta)\}_{i = 0, \ldots, N}$ iff, for each $i = 0, \ldots, N - 1$,

1. the condition $\rho(p(i\beta), g(i\beta)) > \beta$ implies

   $$
   \begin{align*}
   \rho(p(i\beta), p((i + 1)\beta)) &= \beta, \\
   \rho(p(i\beta), p((i + 1)\beta)) + \rho(p((i + 1)\beta), g(i\beta)) &= \rho(p(i\beta), g(i\beta));
   \end{align*}
   $$

2. the opposite condition $\rho(p(i\beta), g(i\beta)) \leq \beta$ implies

   $$
   p((i + 1)\beta) = g(i\beta).
   $$

**Remark 4.** The equality

$$
\rho(p(i\beta), p((i + 1)\beta)) + \rho(p((i + 1)\beta), g(i\beta)) = \rho(p(i\beta), g(i\beta))
$$

(from Item 1 of Definition 5) means that $p((i + 1)\beta)$ is on a geodesic segment between $p(i\beta)$ and $g(i\beta)$. Since $(X, \rho)$ is a compact geodesic space, any two points are connected by a geodesic path, so whatever points $p(i\beta)$ and $g(i\beta)$ are, point $p((i + 1)\beta)$ can be construct.

**Observation 1.** Using stepwise procedure, one can construct a curve $p : [0, N\beta] \to X$ that satisfies $p(0) = P_0 \in X$ and $\beta$-pursues a curve $g : [0, N\beta] \to X$. In other words, Lion can generate his trajectory as such a curve, and formally his strategy would be $\beta$-stepwise.
Lemma 2. Let \((X, \rho)\) be a compact geodesic space. If a 1-Lipschitz curve \(p: [0, N\beta] \to X\) \(\beta\)-pursues a collection \(\{g(i\beta)\}_{i=0}^{N}\) satisfying
\[
\rho(g(i\beta), g((i+1)\beta)) \leq \beta + \delta \beta
\]
for all \(i = 0, ..., N-1\), then
\[
\rho(p(N\beta), g(N\beta)) \leq \beta + N\delta \beta + \rho(p(0), g(0)).
\]

Proof. Let us prove this by induction on \(N\). The base of induction is trivial: \(\rho(p(0), g(0)) \leq \beta + 0\delta \beta + \rho(p(0), g(0))\). Further, assume the inequality
\[
\rho(p(i\beta), g(i\beta)) \leq \beta + i\delta \beta + \rho(p(0), g(0))
\]
and show that it still holds for \((i + 1)\) instead of \(i\). Consider two cases. First, if we have \(\rho(p(i\beta), g(i\beta)) \leq \beta\), then \(p((i + 1)\beta) = g(i\beta)\) by Item 2 of Definition 5 and, as corollary,
\[
\rho(p((i + 1)\beta), g((i + 1)\beta)) \leq \beta + \delta \beta \leq \beta + (i + 1)\delta \beta + \rho(p(0), g(0)).
\]
Second, if we have \(\rho(p(i\beta), g(i\beta)) > \beta\), then the following holds
\[
\begin{align*}
\rho(p((i + 1)\beta), g((i + 1)\beta)) &\leq \rho(p((i + 1)\beta), g(i\beta)) + \rho(g(i\beta), g((i + 1)\beta)) \quad \text{by triangle equality} \\
&\leq \rho(p((i + 1)\beta), g(i\beta)) + \beta + \delta \beta \quad \text{by the assumption of this lemma} \\
&= \rho(p(i\beta), g(i\beta)) - \rho(p(i\beta), p((i + 1)\beta)) + \beta + \delta \beta \quad \text{by Item 1 of Definition 5} \\
&= \rho(p(i\beta), g(i\beta)) + \delta \beta \quad \text{by Item 1 of Definition 5} \\
&\leq \beta + i\delta \beta + \rho(p(0), g(0)) \quad \text{by the hypothesis (1)}.
\end{align*}
\]

\[\square\]

3.2 Chaining

Definition 6. Let \((X, \rho)\) and \((\tilde{X}, \tilde{\rho})\) be compact metric spaces and let \(d_{GH}(X, \tilde{X}) \leq \varepsilon/4\). We say a pair of maps \((f, \tilde{f})\) where
\[
f: \tilde{X} \to X, \quad \tilde{f}: X \to \tilde{X}
\]
is an \(\varepsilon\)-chaining between \(X\) and \(\tilde{X}\) if there exist

1. finite \(\varepsilon\)-nets \(R_X\) and \(R_{\tilde{X}}\) in \(X\) and \(\tilde{X}\) respectively;
2. bijections \(h: R_{\tilde{X}} \to R_X\) and \(\tilde{h} = h^{-1}: R_X \to R_{\tilde{X}}\) such that
\[
\text{dis } h \leq \varepsilon/2, \quad \text{dis } \tilde{h} \leq \varepsilon/2;
\]
3. maps \(g: X \to R_X\) and \(\tilde{g}: \tilde{X} \to R_{\tilde{X}}\) such that
\[
g(x) \in \arg\min_{y \in R_X} \rho(x, y), \quad \tilde{g}(\tilde{x}) \in \arg\min_{\tilde{y} \in R_{\tilde{X}}} \rho(\tilde{x}, \tilde{y}),
\]
\[
f = h \circ g, \quad \tilde{f} = \tilde{h} \circ \tilde{g}.
\]
Lemma 3. If \((X, \rho)\) and \((\hat{X}, \hat{\rho})\) are compact metric spaces and \(d_{GH}(X, \hat{X}) \leq \varepsilon\), then there exists a \(4\varepsilon\)-chaining \((f, \hat{f})\) between \(X\) and \(\hat{X}\).

Proof. We can pick a \(2\varepsilon\)-isometry \(\hat{h} : X \to \hat{X}\) in accordance to Lemma 1. This implies that the set \(\hat{h}[X]\) is a \(2\varepsilon\)-net, and in virtue of compactness of \(\hat{X}\), we can select a finite subset of this \(2\varepsilon\)-net which is a \(2\varepsilon\)-net too; let us denote it by \(R_{\hat{X}}\). Hence, for each \(y \in R_{\hat{X}}\) one can find \(x_y \in X\) such that \(\hat{h}(x_y) = y\). Then let \(R_X = \{x_y \in X \mid y \in R_{\hat{X}}\}\). Now we can show that such \(R_X\) is a \(4\varepsilon\)-net. Assume the converse, i.e. let there be a point \(x \in X\) such that \(\rho(x, x_y) > 4\varepsilon\) for all \(x_y \in R_X\), this yields that

\[\hat{\rho}(\hat{h}(x), y) > 4\varepsilon - \text{dis } \hat{h} = 2\varepsilon\quad \forall y \in R_{\hat{X}}\]

due to the definition of distortion. This inequality contradicts the definition of \(R_{\hat{X}}\), namely that it is a \(2\varepsilon\)-net. Thus, the sets \(R_X\) and \(R_{\hat{X}}\) are finite \(4\varepsilon\)-nets indeed.

Moreover, the restriction \(h = \hat{h}|_{R_X}\) is bijective and

\[\text{dis } h \leq \text{dis } \hat{h} \leq 2\varepsilon.\]

Then \(\hat{h} = h^{-1}\) is also bijective and satisfies \(\text{dis } \hat{h} \leq 2\varepsilon\). Since the sets \(R_X\) and \(R_{\hat{X}}\) are finite, the maps \(g\) and \(\hat{g}\) exist. Thus, the maps \(f = h \circ \hat{g}\), \(\hat{f} = \hat{h} \circ g\) are constructed. \(\square\)

Lemma 4. If a pair \((f, \hat{f})\) is a \(4\varepsilon\)-chaining between compact metric spaces \(X\) and \(\hat{X}\), then

1. \(\text{dis } f \leq 10\varepsilon\), \(\text{dis } \hat{f} \leq 10\varepsilon;\)
2. \(\hat{\rho}(\hat{f}(\hat{x}), \hat{x}) \leq 4\varepsilon\) holds for any \(\hat{x} \in \hat{X}\).

Proof. To prove the first statement, recall that \(\text{dis } \hat{f} = \sup_{x_1, x_2 \in X} |\rho(x_1, x_2) - \hat{\rho}(\hat{f}(x_1), \hat{f}(x_2))|\). Further, notice that, for all \(x_1, x_2 \in X\), the following inequality holds

\[\rho(x_1, x_2) \leq \rho(x_1, g(x_1)) + \rho(x_2, g(x_2)) + |\rho(g(x_1), g(x_2))|\]

and hence

\[|\rho(x_1, x_2) - \hat{\rho}(\hat{f}(x_1), \hat{f}(x_2))|\]

\[\leq |\rho(x_1, x_2) - \rho(g(x_1), g(x_2))| + |\rho(g(x_1), g(x_2)) - \hat{\rho}(\hat{f}(x_1), \hat{f}(x_2))|\]

\[\leq |\rho(x_1, g(x_1)) + \rho(x_2, g(x_2)) + |\rho(g(x_1), g(x_2)) - \hat{\rho}(\hat{h}g(x_1), \hat{h}g(x_2))|\]

\[\leq |\rho(x_1, g(x_1)) + \rho(x_2, g(x_2)) + \text{dis } \hat{h} \leq 4\varepsilon + 4\varepsilon + 2\varepsilon = 10\varepsilon.\]

Thus, \(\text{dis } \hat{f} \leq 10\varepsilon\) and the same for \(\text{dis } f\).

To prove the second statement, notice that, for any \(\hat{x} \in \hat{X}\), we have that \(f(\hat{x}) = h \circ \hat{g}(\hat{x}) \in R_X\) and that \(g\) is an identical map on \(R_X\); then

\[\hat{f} \circ f(\hat{x}) = \hat{h} \circ g \circ f(\hat{x}) = \hat{h} \circ f(\hat{x}) = \hat{h} \circ h \circ \hat{g}(\hat{x}) = h^{-1} \circ h \circ \hat{g}(\hat{x}) = \hat{g}(\hat{x}).\]

Consequently, \(\hat{\rho}(\hat{f}(\hat{x}), \hat{x}) = \hat{\rho}(\hat{g}(\hat{x}), \hat{x}) \leq 4\varepsilon.\) \(\square\)

3.3 Proof of Theorem 1

We will construct Lion’s \(\beta\)-stepwise strategy in \(X\) for \(\beta = \sqrt{\varepsilon}\). Since we are interested in capture by a time \(T\), it suffices to describe Lion’s strategy only for \(N = \left\lceil \frac{T}{\beta} \right\rceil\) steps.
A construction of Lion’s strategy. Using Lemma 3 let us fix a 4ε-chaining \((f, \tilde{f})\) between the spaces \(X\) and \(\tilde{X}\) and introduce Lion’s strategy relying on several auxiliary constructions and Observation 1.

- Let a curve \(\tilde{M}: [0, N\beta] \to \tilde{X}\) \(\beta\)-puruses the collection \(\{\tilde{f}(M(i\beta))\}_{i=0,\ldots,N}\) and satisfies the condition \(\tilde{M}(0) = \tilde{f}(M(0))\). On the one hand, since the collection satisfies property

\[
\rho(\tilde{f}(M(i\beta)), \tilde{f}(M((i+1)\beta))) \leq \rho(M(i\beta), M((i+1)\beta)) + \text{dis} \tilde{f} \leq \beta + \text{dis} \tilde{f}
\]

for all \(i = 0,\ldots,N-1\), then, applying Lemma 2 for \(\delta = \text{dis} \tilde{f}/\beta\), we obtain

\[
\rho(\tilde{f}(M(i\beta)), \tilde{M}(i\beta)) \leq \beta + i \text{dis} \tilde{f} + \text{dis} \tilde{f} = \beta + i \text{dis} \tilde{f}
\]  
(2)

for all \(i = 0,\ldots,N-1\). On the other hand, \(\tilde{M}\) is a 1-Lipschitz curve, i.e. its image belongs to the set of admissible Man’s trajectory in the space \(\tilde{X}\).

- By the hypothesis of this theorem, for Lion’s initial position \(\tilde{f}(L(0)) \in \tilde{X}\), there is Lion’s strategy \(\tilde{s}_{f(L(0))}\) leading to \(\alpha\)-capture by the time \(T\) in the space \(\tilde{X}\). Further, let a curve \(L: [0, N\beta] \to \tilde{X}\) equal to \(\tilde{s}_{f(L(0))}(\tilde{M})\) on the time interval \([0, T] \subset [0, N\beta]\) and satisfy the condition \(L(0) = f(L(0))\).

- Finally, let the Lion’s trajectory \(L(\cdot)\) \(\beta\)-pursue the collection \(\{f(\tilde{L}(i\beta))\}_{i=0,\ldots,N}\). As above, we have

\[
\rho(f(\tilde{L}(i\beta)), f(\tilde{L}((i+1)\beta))) \leq \rho(\tilde{L}(i\beta), \tilde{L}((i+1)\beta)) + \text{dis} f \leq \beta + \text{dis} f
\]

for all \(i = 0,\ldots,N-1\), and by Lemma 2 for \(\delta = \text{dis} f/\beta\), we get

\[
\rho(f(\tilde{L}(i\beta)), L(i\beta))) \leq \beta + i \text{dis} f + \rho(f(\tilde{L}(0)), L(0)) = \beta + i \text{dis} f
\]  
(3)

for all \(i = 0,\ldots,N-1\).

Estimates. It remains to estimate the guaranteed capture radius. Indeed, there exists time \(t^* \in [0, T]\) such that

\[
\rho(\tilde{L}(t^*), \tilde{M}(t^*)) = \rho(\tilde{s}_{f(L(0))}(\tilde{M}))(t^*), \tilde{M}(t^*)) \leq \alpha
\]

because the strategy \(\tilde{s}_{f(L(0))}\) was chosen to lead to the \(\alpha\)-capture. For convenience, let us choose the number \(\tau\) that is nearest to \(t^*\) among numbers \(i\beta\) for \(i = 1,\ldots,N-1\); therefore,

\[
\rho(\tilde{L}(\tau), \tilde{M}(\tau))) \leq \alpha + 2\beta.
\]  
(4)

Thus, we can estimate the distance \(\rho(L(\tau), M(\tau))\) in the following way:

\[
\rho(L(\tau), M(\tau)) \leq \rho(L(\tau), f(\tilde{L})(\tau)) + \rho(f(\tilde{L})(\tau), M(\tau)) \leq \beta + i \text{dis} f + \rho(f(\tilde{L})(\tau), M(\tau)) \leq \beta + i \text{dis} f + \text{dis} \tilde{f} + \rho(\tilde{f}(f(\tilde{L}(\tau))), \tilde{f}(M(\tau)))
\]
Further, let us evolve $\tilde{\rho}(\tilde{f}(f(L(\tau))), \tilde{f}(M(\tau)))$ as follows:

\[
\tilde{\rho}(\tilde{f}(f(L(\tau))), \tilde{f}(M(\tau))) \leq \tilde{\rho}(f(L(\tau)), f(M(\tau))) + \tilde{\rho}(\tilde{L}(\tau), \tilde{f}(M(\tau))) \\
\leq 4\varepsilon + \tilde{\rho}(\tilde{L}(\tau), \tilde{M}(\tau)) + \tilde{\rho}(\tilde{M}(\tau), \tilde{f}(M(\tau))) \\
\leq 4\varepsilon + \alpha + 2\beta + \tilde{\rho}(\tilde{M}(\tau), \tilde{f}(M(\tau))) \\
\leq 4\varepsilon + \alpha + 2\beta + \beta + i \text{dis} \tilde{f}
\]

by triangle equality

by Item 2 of Lemma 4

by triangle inequality

by (4)

by (2)

Then, recall that $\beta = \sqrt{\varepsilon}$ and

\[
i \leq N - 1 = \left\lceil \frac{T}{\beta} \right\rceil - 1 \leq \frac{T}{\beta} + 1 - 1 = \frac{T}{\beta}.
\]

Moreover, both dis $f$ and dis $\tilde{f}$ are not greater than $10\varepsilon$ by Item 1 of Lemma 4; hence

\[
i(\text{dis} \tilde{f} + \text{dis} f) \leq \frac{T}{\beta} 20\varepsilon = 20T\sqrt{\varepsilon}.
\]

Thus,

\[
\rho(L(\tau), M(\tau)) \\
\leq \alpha + 4\beta + 4\varepsilon + i(\text{dis} \tilde{f} + \text{dis} f) + \text{dis} \tilde{f} \\
\leq \alpha + (20T + 8)\sqrt{\varepsilon}
\]

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References

[1] S. Alexander, R. Bishop, and R. Ghrist. Pursuit and evasion in non-convex domains of arbitrary dimensions. In *Robotics: Science and Systems*, 2006.

[2] L. Alonso, A. S. Goldstein, and E. M. Reingold. “Lion and man”: Upper and lower bounds. *ORSA Journal on Computing*, 4(4):447–452, 1992.

[3] Y. Averboukh. Krasovskii-Subbotin approach to mean field type differential games. *arXiv preprint arXiv:1802.00487*, 2018.

[4] M. Bramson, K. Burdzy, and W. Kendall. Shy couplings, CAT(0) spaces, and the lion and man. *The Annals of Probability*, 41(2):744–784, 2013.

[5] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319. Springer Science & Business Media, 2013.
[6] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33. American Mathematical Soc., 2001.

[7] A. Dumitrescu, I. Suzuki, and P. Żyliński. Offline variants of the “lion and man problem”: - Some problems and techniques for measuring crowdedness and for safe path planning. *Theoretical Computer Science*, 399(3):220–235, 2008.

[8] N. N. Krasovskii and A. N. Kotel’nikova. An approach-evasion differential game: stochastic guide. *Proceedings of the Steklov Institute of Mathematics*, 269(1):191–213, 2010.

[9] N. N. Krasovskii and A. I. Subbotin. *Pozitsionnye differentsial’nye igry*. M.: Nauka, 1974.

[10] N. N. Krasovskii and A. I. Subbotin. *Game-theoretical control problems*. Springer-Verlag New York, Inc., 1987.

[11] G. López-Acedo, A. Nicolae, and B. Piątek. ‘Lion-Man’ and the fixed point property. *arXiv preprint arXiv:1712.04005*, 2017.

[12] A. A. Tuzhilin. Who invented the Gromov-Hausdorff distance? *arXiv preprint arXiv:1612.00728*, 2016.

[13] O. Yufereva. Lion and Man game in compact spaces. *Dynamic Games and Applications*, pages 1–12 (published online), 2018.