MFGs for Partially Reversible Investment

Haoyang Cao* Xin Guo* Joon Seok Lee †

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Abstract

This paper analyzes a class of MFGs with singular controls motivated from the partially reversible problem. It establishes the existence of the solution when controls are of bounded velocity, solves explicitly the game when controls are of finite variation, and presents sensitivity analysis to compare the single-player game with the MFG. Our analysis shows that MFGs, when appropriately formulated, can demonstrate genuine game effects even without heterogeneity among players and additional common noise.

1 Introduction

Pioneered by [LL07] and [HMC06], mean field game (MFG) provides an ingenious aggregation approach for analyzing the otherwise notoriously hard $N$-player stochastic game, and has recently experienced exponential growth in both theory and applications. See for instance, the works of [BFY13, CD18a, CD18b, GLL11, LL07], and the references therein.

Non-degeneracy of MFG. One of the main criticisms of MFG is that its aggregation approach often over-simplifies the stochastic game. In particular, the solution of the MFG tends to degenerate to the corresponding single-player game. For instance, in [CL15], the systemic risk vanishes to zero in MFGs with homogeneous players unless a common noise is added. Similar conclusion is seen in [LZ19] where the equilibrium strategy in MFGs degenerates to the single-player case, without the heterogeneity among players and the common noise. Recently, [GX19] studies the finite fuel problem and compares the $N$-player game and the MFG. It acknowledges that the MFG degenerates to the single-player game due to the aggregation and symmetry and differs in the solution structure from the general $N$-player game. Thus, a very natural question is if there exists any simple example where the MFG without the common noise does not degenerate to the single-player game and demonstrates the game effect of interaction among homogeneous players.

*Department of Industrial Engineering and Operations Research, University of California, Berkeley. Email: {hycao, xinguo}@berkeley.edu
†Laboratoire de Probabilites et Modeles Aleatoires, CNRS, UMR 7599, Universite Paris Diderot. Email: delinbetances@gmail.com
MFG with an infinite-time horizon. Another notable development in the theory of MFG is the increasing interest in the long term behavior of MFGs; see for instance [ADLT18, Bar11, CLLP12, CP19]. The analysis in [CP19] indicates that MFGs over an infinite-time horizon could be of critical interest for understanding the convergence of the $N$-player game to the MFG. However, MFGs over an infinite-time horizon have not been well studied in the existing literature, due to extra technical difficulties. For instance, for a non-stationary MFG, the time-dependent mean information process $\mu_t$ leads to a parabolic HJB equation instead of an elliptic type, even for an infinite-time horizon. Moreover, the probabilistic approach of forward-backward stochastic differential equations (FBSDEs) does not work for the infinite-time horizon.

Our work. Motivated by these two lines of research interests, we take the partially reversible investment model in [GP05], and consider its MFG counterpart.

In the model in [GP05], a company tries to adjust its production level of a single commodity according to the market demand fluctuation. The commodity yields a revenue which is a function of the production level. The objective of the company is to find an optimal investment strategy in order to maximize its overall expected net profit. This is a typical real option problem that originated in the classical work of [DDP94].

In the MFG framework, instead of one company, we consider a continuum of infinitely many indistinguishable companies reacting to the market, and assume that the production level of a company $i$ at time $t \geq s$ is

$$dx_i^t = b(x_i^t, \mu_t)dt + \sigma dW_i^t, \quad x_{s-}^i = x^i,$$

where the volatility $\sigma > 0$ is a constant, the mean information process $\mu_t$ denotes the distribution of the production level of all companies at time $t$, and the drift term $b(\cdot, \cdot)$ can be influenced by the aggregated effect reflected in $\mu_t$. Here $W_i$’s are one-dimensional independent standard Brownian motions. Company $i$ can either increase or decrease the production level such that

$$dx_i^t = b(x_i^t, \mu_t)dt + \sigma dW_i^t + d\xi_i^t = b(x_i^t, \mu_t)dt + \sigma dW_i^t + d\xi_i^{i,+} - d\xi_i^{i,-}, \quad \xi_{s-}^{i,\pm} = 0$$

where $\xi_i^{i,+}$ and $\xi_i^{i,-}$ are appropriate adapted cádlág process representing the total amount of increase and decrease, respectively. Company $i$ aims to choose the optimal production plan to maximize its expected profit over an infinite-time horizon

$$E \left[ \int_s^\infty e^{-r(t-s)} \left[ f(x_i^t, \mu_t)dt - \gamma^+ d\xi_i^{i,+} - \gamma^- d\xi_i^{i,-} \right] \right],$$

where $r > 0$ is the discount rate and $\gamma^+, \gamma^- \in \mathbb{R}$ are the proportional cost of control. For the well-posedness of the control problem, we assume that $\gamma^+ + \gamma^- > 0$. The revenue function $f$ is also affected by the aggregated effect of decisions made by all the companies on the market. In this case, all companies interact through both the revenue function $f$ and the underlying dynamics $x_i^t$.

We analyze this MFG from two aspects. First, we establish the existence of a solution to the MFG when controls are of bounded velocity. To overcome the aforementioned issue with
an infinite-time horizon, we construct a sequence of auxiliary control problems on finite-time horizons to approximate the original problem. Next, we analyze explicitly when the controls are of finite variation. We compare analytically in details the MFG and the single-player game when the revenue function is of the Cobb-Douglas type. We will see the distinction between the solutions both quantitatively and qualitatively. In particular, model parameters directly affect optimal strategies in the single-player game, whereas these parameters affect the equilibrium price in the MFG. More importantly, our analysis shows that MFGs, when appropriately formulated, can demonstrate genuine game effects even without heterogeneity among players and additional common noise.

**Outline of the paper.** Section 2 formally defines the MFG with singular control of bounded velocity, provides proper technical conditions and assumptions, and presents the main theoretical result; Section 3 presents a full derivation of explicit stationary solution to the MFG of finite variation, provides sensitivity analysis with respect to model parameters, and compares the MFG with the single-player game.

### 2 MFGs with singular control of bounded velocity

In this section, we present the mathematical framework of the MFG for the partially reversible investment problem.

#### 2.1 Problem formulation

**Notation and definitions.** To start, let \( \mathcal{P}(\mathbb{R}) \) denote the set of all probability measures on \( \mathbb{R} \). Define the set of all probability measures of \( p \)-th order on \( \mathbb{R} \) as

\[
\mathcal{P}_p(\mathbb{R}) = \left\{ \mu \in \mathcal{P}(\mathbb{R}) \left| \left( \int_{\mathbb{R}} |x|^p \mu(dx) \right)^{\frac{1}{p}} < \infty \right. \right\},
\]

on which the \( p \)-th order Wasserstein metric \( D^p(\mu, \nu) \) for any \( \mu, \nu \in \mathcal{P}_p(\mathbb{R}) \) is defined as

\[
D^p(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \left( \int_{\mathbb{R} \times \mathbb{R}} |x - x'|^p \pi(dx, dx') \right)^{\frac{1}{p}},
\]

where \( \Pi(\mu, \nu) \) is the set of all possible couplings of \( \mu \) and \( \nu \). To characterize the flow of probability measures \( \{\mu_t\}_{t \geq 0} \), let \( \mathcal{C}([0, \infty), \mathcal{P}_2(\mathbb{R})) \) be a collection of mappings from \([0, \infty)\) to \( \mathcal{P}_2(\mathbb{R}) \) that are continuous under the \( D^2 \) metric. Define \( \mathcal{M}_{[0, \infty)} \subset \mathcal{C}([0, \infty), \mathcal{P}_2(\mathbb{R})) \) such that

\[
\mathcal{M}_{[0, \infty)} = \left\{ \{\mu_t\}_{t \geq 0} \left| \forall T \geq 0, \exists c_T > 0 \text{ s.t. } \sup_{s \neq t} \frac{D^1(\mu_t, \mu_s)}{|t - s|^{\frac{1}{2}}} \leq c_T, \sup_{0 \leq t \leq T} \int_{\mathbb{R}} |x|^2 \mu_t(dx) \leq c_T \right. \right\},
\]

which is associated with the metric

\[
d^\beta_M (\{\mu_t\}_{t \geq 0}, \{\mu'_t\}_{t \geq 0}) = \int_0^\infty e^{-\beta t} D^2(\mu_t, \mu'_t) dt \tag{1}
\]

for sufficiently large \( \beta > 0 \).
Mathematical formulation. Let \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, P)\) be a probability space and \(\{W_t\}_{t \geq 0}\) be a one-dimensional standard Brownian motion adapted to \(\{\mathcal{F}_t\}_{t \geq 0}\). Under a given flow of probability measure \(\{\mu_t\}_{t \geq 0} \in \mathcal{M}_{[0,\infty)}\), a representative company in the MFG is allowed to adjust its own production level \(x_t\) such that
\[
   dx_t = b(x_t, \mu_t)dt + \sigma dW_t + d\xi_t, \quad x_s \sim \mu_s, \tag{2}
\]
with
\[
   d\xi_t = d\xi_t^+ - d\xi_t^- = \xi_t dt = (\xi_t^+ - \xi_t^-) dt. \tag{3}
\]
Here, \((\xi_t^+, \xi_t^-)\) are the pair of controls, representing the total amount of increase and decrease in the production level by time \(t\), respectively. \(\xi_t = \xi_t^+ - \xi_t^-\) is assume to be of bounded velocity. That is, there exists \(\theta > 0\) such that \(\dot{\xi}_t^\pm \in (0, \theta]\).

For any initial time \(s \geq 0\) and initial state \(x \in \mathbb{R}\), the company chooses an optimal strategy from an appropriate admissible control set \(U_\theta\) to maximize its discounted total profit,
\[
   \sup_{\{(\xi_t^+, \xi_t^-)\} \in U_\theta} \mathbb{E} \left[ \int_s^\infty e^{-r(t-s)} \left( f(x_t, \mu_t) - \gamma^+ \dot{\xi}_t^+ - \gamma^- \dot{\xi}_t^- \right) dt \bigg| x_s = x \right]. \tag{MFG}
\]
Here the set of admissible controls \(U_\theta\) is
\[
   U_\theta = \left\{ \{(\xi_t^+, \xi_t^-)\} \mid \{\mathcal{F}_t\}_{t \geq 0} \text{are } \{\mathcal{F}_t\}_{t \geq 0}\text{-adapted, càdlàg, nondecreasing,} \right. \\
   \left. \quad \xi_s^\pm = 0, 0 \leq \xi_t^\pm \leq \theta \text{ for } s \leq t < \infty \right\},
\]
where \(\{\mathcal{F}_t\}_{t \geq 0}\) is the natural filtration of \(\{x_t\}_{t \geq 0}\). The revenue function \(f : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}\) and the drift term \(b : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}\) satisfy some technical conditions to be specified later.

Solution to MFGs involves an optimal control and a flow of probability measure. There are several concepts related to optimal control, such as the open-loop control \([CFS15, \text{CZ16, PW18, DF19}]\), the closed-loop control \([HMC06]\), and the Markovian control \([EKP+97, \text{BLP09, CD14, CFS15, Lac15, GX19}]\). Here we shall focus on Markovian control and define the solution to MFGs accordingly. More precisely,

**Definition 2.1.** Given a fixed flow of probability measure \(\{\mu_t\}_{t \geq 0} \in \mathcal{M}_{[0,\infty)}\), a control of bounded velocity \(\xi_t\) is Markovian if \(d\xi_t = \dot{\xi}_t dt = \varphi(t, x)\{\mu_t\}_{t \geq 0} dt\), where
\[
   \varphi(\cdot, \cdot)\{\mu_t\}_{t \geq 0} : [0, \infty) \times \mathbb{R} \to \mathbb{R}
\]
is called a control function under \(\{\mu_t\}_{t \geq 0}\).

**Definition 2.2.** If a flow of probability measures \(\{\mu_t^*\}_{t \geq 0} \in \mathcal{M}_{[0,\infty)}\) and a Markovian control \(\{\xi_t^*\}_{t \geq 0}\) in the sense of Definition 2.1 satisfy the following conditions,

1. under \(\{\mu_t^*\}_{t \geq 0}\), for any \(s \geq 0\) and \(x \in \mathbb{R}\), \(\{\xi_t^*\}_{t \geq 0}\) is an optimal strategy for
\[
   v_\theta(s, x)\{\mu_t^*\}_{t \geq s} = \sup_{\{\xi_t^*\}_{t \geq s} \in U_\theta} J_\theta(s, x, \{\xi_t^*\}_{t \geq s}) \{\mu_t^*\}_{t \geq s} \tag{4}
\]
   \[
   = \sup_{\{\xi_t^*\}_{t \geq s} \in U_\theta} \mathbb{E} \left[ \int_s^\infty e^{-r(t-s)} \left( f(x_t, \mu_t^*) - \gamma^+ \dot{\xi}_t^+ - \gamma^- \dot{\xi}_t^- \right) dt \bigg| x_s = x \right]
\]
subject to
\[ dx_t = \left( b(x_t, \mu^*_t) + \dot{\xi}^+_t - \dot{\xi}^-_t \right) dt + \sigma dW_t, \quad x_s \sim \mu_s; \]

2. \( \mu^*_t \) is the probability distribution of \( x^*_t \) given by
\[ dx^*_t = \left( b(x^*_t, \mu^*_t) + \dot{\xi}^+_t - \dot{\xi}^-_t \right) dt + \sigma dW_t, \quad x^*_s \sim \mu^*_s, \]  \( (5) \)

then the control-mean pair \( (\{\xi^*_t\}_{t \geq 0}, \{\mu^*_t\}_{t \geq 0}) \) is said to be a solution to the (MFG).

2.2 Main assumptions

To analyze (MFG), we first specify some technical conditions, consistent with the existing literature on MFGs.

Assumptions.

\( \text{(A1).} \) \( b(x, \mu) \) and \( f(x, \mu) \) are Lipschitz continuous in \( x \) and \( \mu \). That is, \( |b(x_1, \mu_1) - b(x_2, \mu_2)| \leq \text{Lip}(b)(|x_1 - x_2| + D^1(\mu_1, \mu_2)) \) for some constant \( \text{Lip}(b) > 0 \), and \( |f(x_1, \mu_1) - f(x_2, \mu_2)| \leq \text{Lip}(f)(|x_1 - x_2| + D^1(\mu_1, \mu_2)) \) for some constant \( \text{Lip}(f) > 0 \).

\( \text{(A2).} \) For any fixed \( \mu \in \mathcal{P}_2(\mathbb{R}), f(x, \mu) \) is concave and nonlinear in \( x; f(x, \mu) \) has a first order derivative in \( x \), where \( f(x, \mu) \) and \( \partial_x f(x, \mu) \) satisfy the polynomial growth condition. Finally, there exists some constant \( c_f \) such that
\[ |f(x, \mu)| \leq c_f \left( 1 + |x|^2 + \int_\mathbb{R} y^2 \mu(dy) \right) \]
for any \( x \in \mathbb{R}, \mu \in \mathcal{P}_2(\mathbb{R}) \).

\( \text{(A3).} \) For any fixed \( \mu \in \mathcal{P}_2(\mathbb{R}), b(x, \mu) \) has uniformly continuous and bounded first and second order derivatives with respect to \( x \), and there exist \( b_1, b_2 \in \mathbb{R} \) such that \( b(x, \mu) \leq b_1 + b_2 x \) and \( b_2 < r \) for any \( x \in \mathbb{R} \). This guarantees \( v_\theta(\cdot, |\{\mu_t\}_{t \geq 0}|) \) to be finite as shown in [GP05].

\( \text{(A4).} \) (Monotonicity of the cost function) Either
i) \( f \) satisfies
\[ \int_\mathbb{R} (f(x, \mu^1) - f(x, \mu^2))((\mu^1 - \mu^2)(dx) \leq 0, \text{ for any } \mu^1, \mu^2 \in \mathcal{P}_1(\mathbb{R}), \]
and \( H(x, p) = \sup_{\xi^+, \xi^- \in [0, \theta]} \{|\xi^+ - \dot{\xi}^-|p - \gamma^+ \dot{\xi}^+ - \gamma^- \dot{\xi}^-\} \) satisfies the following condition for any \( x, p, q \in \mathbb{R}, \)
\[ \text{if } H(x, p + q) - H(x, p) - \partial_p H(x, p)q = 0, \text{ then } \partial_p H(x, p + q) = \partial_p H(x, p); \]
or
ii) \( f \) satisfies
\[ \int_\mathbb{R} (f(x, \mu^1) - f(x, \mu^2))((\mu^1 - \mu^2)(dx) < 0, \text{ for any } \mu^1 \neq \mu^2 \in \mathcal{P}_1(\mathbb{R}). \]

As in [LL07, Car10], Assumption (A5) is critical to ensure the solution of (MFG) is unique. This will be clear from the proof of Theorem 2.4.
(A5). (Rationality of players) For any control function $\varphi$, any fixed $t \in [0, T]$ and $\{\mu_t\}$,

$$(x - y) \left( \varphi(t, x|\{\mu_t\}) - \varphi(t, y|\{\mu_t\}) \right) \leq 0, \forall x, y \in \mathbb{R}.$$  

(A5) is satisfied, for example, when the revenue function $f$ is concave with respect to the distance between individual company and the mean information, see [BSW80, GX19]. Intuitively, it says that the better off the state of an individual company, the less likely the company exercises controls, in order to maximize its profit. This assumption first appeared in [EKP+97] in the analysis of BSDEs.

### 2.3 Main results

In this section, we shall establish the existence and uniqueness of the solution to (MFG) in the sense of Definition 2.2. We shall adopt a fixed-point approach as in [LL07].

**Control problems under fixed flow of probability measure.** It is clear that under a fixed flow of probability measures $\{\mu_t\}_{t \geq 0} \in \mathcal{M}_{[0, \infty)}$, (MFG) becomes the following control problem,

$$v_\theta(s, x|\{\mu_t\}) \triangleq \sup_{(\xi^+, \xi^-) \in \mathcal{U}_0} \mathbb{E} \left[ \int_s^T e^{-r(t-s)} \left( f(x_t, \mu_t) - \gamma^+ \xi^+_t - \gamma^- \xi^-_t \right) dt \bigg| x_s = x \right],$$  

(Control-BD)

subject to Eqn. (2).

As mentioned earlier, the HJB equation for (Control-BD) is parabolic instead of elliptic due to the time-dependent $\mu_t$ for non-stationary MFGs. To deal with the issue of parabolic fully non-linear PDE’s for an infinite-time control problem, we introduce the following auxiliary control problem.

**Auxiliary control problem of $T < \infty$.** Define a value function for the stochastic controls over a finite-time horizon for any $(s, x) \in [0, T] \times \mathbb{R}$

$$w(s, T, x) \triangleq \sup_{(\xi^+, \xi^-) \in \mathcal{U}_{T, \theta}} \mathbb{E} \left[ \int_s^T e^{-r(t-s)} \left( f(x_t, \mu_t) - \gamma^+ \xi^+_t - \gamma^- \xi^-_t \right) dt \bigg| x_s = x \right],$$  

(Control-AUX)

subject to Eqn. (2). Here, $\mathcal{U}_{T, \theta}$ is defined as

$$\mathcal{U}_{T, \theta} = \left\{ (\xi^+, \xi^-) \bigg| \xi^+_t \text{ and } \xi^-_t \text{ are } \mathcal{F}^x_t\text{-adapted, \cadlag, nondecreasing, } \xi^+_s = \xi^-_s = 0, \right. \left. 0 \leq \xi^+_s, \xi^-_s \leq \theta, \mathbb{E} \left[ \int_s^T d\xi^+_t \right] < \infty, \text{ and } \mathbb{E} \left[ \int_s^T d\xi^-_t \right] < \infty, \text{ for } 0 \leq s \leq t \leq T \right\}.$$  

The following result on (Control-BD) is based on its connection with (Control-AUX).

**Theorem 2.3.** Assume (A1)-(A5). For any $(s, x) \in [0, \infty) \times \mathbb{R}$,

$$\lim_{T \to \infty} e^{rs} w(s, T, x) = v_\theta(s, x|\{\mu_t\}) \text{ as } T \to \infty.$$  

Furthermore, there exists a unique optimal control to (Control-BD).
Proof of Theorem 2.3. The HJB equation associated with the value function \( w \) of (Control-AUX) is given by

\[
-\partial_t w(t, T, x) = b(x, \mu) \partial_x w(t, T, x) + \frac{\sigma^2}{2} \partial_{xx} w(t, T, x) + e^{-rt} f(x, \mu) \\
+ \sup_{\xi \in [0,1]} \left\{ \dot{\xi}^+ \left[ \partial_x w(t, T, x) - \gamma^+ e^{-rt} \right] - \dot{\xi}^- \left[ \partial_x w(t, T, x) + \gamma^- e^{-rt} \right] \right\},
\]

with the terminal condition \( w(T, T, x) = 0 \) for any \( x \in \mathbb{R} \). Notice that

\[
\sup_{\xi \in [0,1]} \left\{ \dot{\xi}^+ \left[ \partial_x w(t, T, x) - \gamma^+ e^{-rt} \right] - \dot{\xi}^- \left[ \partial_x w(t, T, x) + \gamma^- e^{-rt} \right] \right\} = \max_{\xi \in [0,1]} \left\{ \dot{\xi}^+ \left[ \partial_x w(t, T, x) - \gamma^+ e^{-rt} \right] - \dot{\xi}^- \left[ \partial_x w(t, T, x) + \gamma^- e^{-rt} \right] \right\},
\]

with

\[
\dot{\xi}^+ - \dot{\xi}^- = \begin{cases} 
\theta & \text{if } \partial_x e^{rt} w(t, T, x_t) \geq \gamma^+, \\
0 & \text{if } -\gamma^- < \partial_x e^{rt} w(t, T, x_t) < \gamma^+, \\
-\theta & \text{if } \partial_x e^{rt} w(t, T, x_t) \leq -\gamma^-,
\end{cases}
\]

\[
\in \arg \max_{\xi \in [0,1]} \left\{ \dot{\xi}^+ \left[ \partial_x w(t, T, x) - \gamma^+ e^{-rt} \right] - \dot{\xi}^- \left[ \partial_x w(t, T, x) + \gamma^- e^{-rt} \right] \right\}, t \in (s, T).
\]

Therefore (6) is equivalent to

\[
-\partial_t w(t, T, x) = b(x, \mu) \partial_x w(t, T, x) + \frac{\sigma^2}{2} \partial_{xx} w(t, T, x) + e^{-rt} f(x, \mu) \\
+ \max \left\{ \left[ \partial_x w(t, T, x) - e^{-rt} \gamma^+ \right] \theta, \left[ -\partial_x w(t, T, x) - e^{-rt} \gamma^- \right] \theta, 0 \right\}.
\]

By Theorem 6.2 in [FR75], the HJB Eqn. (7) has a unique \( C^1([0, T] \times \mathbb{R}) \) solution \( w(s, T, x) \), and it is the value function for problem (Control-AUX). Furthermore, the optimal control is also unique.

Next we shall show that \( w(s, T, x) \) converges as \( T \to \infty \), and \( \lim_{T \to \infty} e^{rs} w(s, T, x) = v_\theta(s, x|\{\mu_t\}) \).

To see the convergence of \( w(s, T, x) \), let \( \{T_n\}_{n \in \mathbb{N}} \) be any monotonic increasing positive sequence satisfying \( \lim_{n \to \infty} T_n = \infty \). By definition, \( w(s, T_n, x) \) are monotonic increasing as \( T_n \) increases. To prove that \( w(s, T_n, x) \) converges as \( n \to \infty \), it suffices to prove that for any \( \epsilon > 0 \), there exists an \( n_0 \) such that for any \( m > n \geq n_0 \), \( |w(s, T_m, x) - w(s, T_n, x)| < \epsilon \). This is clear: by the Dynamic Programming Principle, \( w(s, T_m, x) - w(s, T_n, x) = \sup_{(\xi^+, \xi^-) \in \mathcal{U}_{T_n, \theta}} \mathbb{E} \left[ w(T_n, T_m, x_{T_n}) \right] \) \( x_s = x \); moreover, by Assumptions (A3) and (A4), \( w(0, \infty, x) < \infty \), and \( w(T_n, T_m, x) \to 0 \) as \( T_n \) and \( T_m \) go to infinity (Chapter 3 in [Pha09]). Therefore, there exists \( n_0 \) such that for any \( m > n \geq n_0 \), \( |w(s, T_m, x) - w(s, T_n, x)| < \epsilon \), and \( w(s, T_n, x) \) converges pointwise to \( \lim_{T \to \infty} w(s, T, x) \) as \( T_n \to \infty \).

For any \( (s, x) \in [0, \infty) \times \mathbb{R} \), \( \lim_{T \to \infty} e^{rs} w(s, T, x) = v_\theta(s, x|\{\mu_t\}) \) can be established through the following two steps.
Step 1. Applying Itô’s formula to \( w(s, T, x) \), we can establish \( v_\theta(s, x \{ \mu_t \}) \leq \lim_{T \to \infty} e^{rs}w(s, T, x) \).

Let \((\xi^+, \xi^-) \in U_{T, \theta}\) be any admissible control and \(x_t\) be the dynamic under the control \((\xi^+, \xi^-)\). By Itô’s formula, for any \(s \in [0, T]\) and \(t \in [s, T]\),

\[
dw(t, T, x_t) = \left\{ \partial_t w(t, T, x_t) + \left[ b(x_t, \mu_t) + \hat{\xi}^+_t - \hat{\xi}^-_t \right] \partial_x w(t, T, x_t) + \frac{\sigma^2}{2} \partial_x^2 w(t, T, x_t) \right\} dt
+ \left[ b(x_t, \mu_t) + \hat{\xi}^+_t - \hat{\xi}^-_t \right] \partial_x w(t, T, x_t) dW_t.
\]

Fix \(T > 0\), \(w(s, T, x)\) is a classical \(C^{1,2}([0, T] \times \mathbb{R})\) solution to the HJB Eqn. (7). By the equivalency between (7) and (6), we have

\[
\partial_t w(t, T, x_t) + \left[ b(x_t, \mu_t) + \hat{\xi}^+_t - \hat{\xi}^-_t \right] \partial_x w(t, T, x_t) + \frac{\sigma^2}{2} \partial_x^2 w(t, T, x_t)
+ e^{-rt} \left( f(x_t, \mu_t) - \gamma^+ \hat{\xi}^+_t - \gamma^- \hat{\xi}^-_t \right) \leq 0.
\]

Thus,

\[
e^{rs} \mathbb{E} \left[ w(T, T, x_T) \big| x_s = x \right] = e^{rs}w(s, T, x) + \mathbb{E} \left[ \int_s^T \partial_t w(t, T, x_t) \right.
+ \left. \left[ b(x_t, \mu_t) + \hat{\xi}^+_t - \hat{\xi}^-_t \right] \partial_x w(t, T, x_t) + \frac{\sigma^2}{2} \partial_x^2 w(t, T, x_t) dt \big| x_s = x \right]
\leq e^{rs}w(s, T, x) - \mathbb{E} \left[ \int_s^T e^{-r(t-s)} \left( f(x_t, \mu_t) - \gamma^+ \hat{\xi}^+_t - \gamma^- \hat{\xi}^-_t \right) dt \big| x_s = x \right].
\]

Since \(w(T, T, x_T) = 0\),

\[
\mathbb{E} \left[ \int_s^T e^{-r(t-s)} \left( f(x_t, \mu_t) - \gamma^+ \hat{\xi}^+_t - \gamma^- \hat{\xi}^-_t \right) dt \big| x_s = x \right] \leq e^{rs}w(s, T, x).
\]

Hence, as \(T \to \infty\),

\[
\mathbb{E} \left[ \int_s^\infty e^{-r(t-s)} \left( f(x_t, \mu_t) - \gamma^+ \hat{\xi}^+_t - \gamma^- \hat{\xi}^-_t \right) dt \big| x_s = x \right] \leq \lim_{T \to \infty} e^{rs}w(s, T, x). \tag{8}
\]

Eqn. (8) holds for any \((\xi^+, \xi^-) \in U_\theta\). Thus,

\[
v_\theta(s, x \{ \mu_t \}) = \sup_{(\xi^+, \xi^-) \in U_\theta} \mathbb{E} \left[ \int_s^\infty e^{-r(t-s)} \left( f(x_t, \mu_t) - \gamma^+ \hat{\xi}^+_t - \gamma^- \hat{\xi}^-_t \right) dt \big| x_s = x \right] \leq \lim_{T \to \infty} e^{rs}w(s, T, x),
\]

for any \((s, x) \in [0, \infty) \times \mathbb{R}\).
Step 2. Applying Itô’s lemma to $w(s, T, x)$ again under the optimal control
\[
\dot{x}_t^{s,+} - \dot{x}_t^{s,-} = \begin{cases} 
0 & \text{if } -\gamma < \partial_x e^{rt}w(t, x_t) < \gamma, \quad t \in (s, T], \\
-\theta & \text{if } \partial_x e^{rt}w(t, x_t) \leq -\gamma, \\
\theta & \text{if } \partial_x e^{rt}w(t, x_t) \geq \gamma,
\end{cases}
\]
we can establish $v_\theta(s, x|\mu_t) \geq \lim_{T \to \infty} e^{rs}w(s, T, x)$.

Notice that
\[
\partial_t w(t, x_t) + \left[ b(x_t, \mu_t) + \dot{x}_t^{s,+} - \dot{x}_t^{s,-}\right] \partial_x w(t, x_t) + \frac{\sigma^2}{2} \partial_x^2 w(t, x_t) + e^{-rt} \left( f(x_t, \mu_t) - \gamma^+ \dot{x}_t^{s,+} - \gamma^- \dot{x}_t^{s,-} \right) = 0.
\]

Similar to Step 1, we have
\[
e^{rs}E \left[ w(T, T, x) \bigg| x_s = x \right] = e^{rs}w(s, T, x) + E \left[ \int_s^T \partial_t w(t, T, x_t) \right.
\]
\[
+ \left[ b(x_t, \mu_t) + \dot{x}_t^{s,+} - \dot{x}_t^{s,-}\right] \partial_x w(t, T, x_t) + \frac{\sigma^2}{2} \partial_x^2 w(t, T, x_t) dt \bigg| x_s = x \right]
\]
\[
= e^{rs}w(s, T, x) - E \left[ \int_s^T e^{-r(t-s)} \left( f(x_t, \mu_t) - \gamma^+ \dot{x}_t^{s,+} - \gamma^- \dot{x}_t^{s,-} \right) dt \bigg| x_s = x \right].
\]

Hence, for any $T \in (s, \infty)$,
\[
e^{rs}w(s, T, x) = E \left[ \int_s^T e^{-r(t-s)} \left( f(x_t, \mu_t) + \gamma^+ \dot{x}_t^{s,+} - \gamma^- \dot{x}_t^{s,-} \right) dt \bigg| x_s = x \right].
\]

Define for any $t > T$, $\dot{x}_t^{s,+} = \dot{x}_t^{s,-} = 0$.
\[
v_\theta(s, x|\mu_t) \geq E \left[ \int_s^\infty e^{-r(t-s)} \left( f(x_t, \mu_t) + \gamma^+ \dot{x}_t^{s,+} - \gamma^- \dot{x}_t^{s,-} \right) dt \bigg| x_s = x \right]
\]
\[
= E \left[ \int_s^T e^{-r(t-s)} \left( f(x_t, \mu_t) + \gamma^+ \dot{x}_t^{s,+} - \gamma^- \dot{x}_t^{s,-} \right) dt \right. + \left. \int_T^\infty e^{-r(t-s)} f(x_t, \mu_t) dt \bigg| x_s = x \right]
\]
\[
= e^{rs}w(s, T, x) + e^{-r(T-s)} E \left[ \int_T^\infty e^{-r(t-T)} f(x_t, \mu_t) dt \bigg| x_s = x \right].
\]

Let $T \to \infty$. Then, $e^{-r(T-s)} E \left[ \int_T^\infty e^{-r(t-T)} f(x_t, \mu_t) dt \bigg| x_s = x \right] \to 0$, and
\[
v_\theta(s, x|\mu_t) \geq \lim_{T \to \infty} e^{rs}w(s, T, x).
\]

From the previous steps, $v_\theta(s, x|\mu_t) = \lim_{T \to \infty} e^{rs}w(s, T, x)$.

Now we have

**Theorem 2.4** (Existence and uniqueness of MFG solution). Assume (A1)-(A5). If for any fixed $\{\mu_t\} \in \mathcal{M}_{[0, \infty)}$ the value function of (Control-BD) is in $C^{1,2}([0, \infty) \times \mathbb{R})$ and its first order derivative with respect to $x$ is uniformly bounded, then (MFG) has a unique solution $((\xi^+, \xi^-), \{\mu^*_t\})$. 

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**Proof of Theorem 2.4**  The proof consists of three steps.

**Step A.** First, by Theorem 2.3, there exists a unique control for any fixed \( \mu_t \). Let us define the corresponding optimal control function as \( \varphi_\theta(t, x|\{\mu_t\}) \) and define a mapping

\[
\Gamma_1(\{\mu_t\}) = \left( \varphi_\theta(t, x|\{\mu_t\}), \{\mu_t\} \right).
\]

Then according to Chapter V. 19 and V. 24 of [RW00], for a given fixed \( \{\mu_t\} \in M_{[0, \infty)} \) and \( \varphi_\theta(t, x|\{\mu_t\}) \),

\[
dx_{t, \theta} = \left( b(x_{t, \theta}, \mu_t) + \varphi_\theta(t, x_{t, \theta}|\{\mu_t\}) \right) dt + \sigma dW_t, \quad x_{s, \theta} = x,
\]

has a unique weak solution. Define a mapping

\[
\Gamma_2 \left( \varphi_\theta(t, x|\{\mu_t\}), \{\mu_t\} \right) = \{\tilde{\mu}_t\},
\]

where \( \tilde{\mu}_t \) is a probability measure of \( x_{t, \theta} \) for each \( t \in [0, \infty) \). Then, consider a mapping \( \Gamma : M_{[0, \infty)} \to M_{[0, \infty)} \) as

\[
\Gamma(\{\mu_t\}) = \Gamma_2 \circ \Gamma_1(\{\mu_t\}) = \{\tilde{\mu}_t\}.
\]

**Step B.** Next, we show that \( \Gamma \) is compact and continuous.

**Proposition 2.5.** Assume (A1)-(A3). \( \Gamma \) is a compact mapping from \( M_{[0, \infty)} \to M_{[0, \infty)} \).

**Proof of Proposition 2.5.** For any \( \{\mu_t\} \in M_{[0, \infty)} \), we shall prove that \( \{\tilde{\mu}_t\} = \Gamma(\{\mu_t\}) \) is also in \( M_{[0, \infty)} \). Fix a \( T > 0 \). Without loss of generality, suppose \( 0 \leq t < s \leq T \), and

\[
x_s = x_t + \int_t^s \left( b(x_r, \mu_r) + \varphi_\theta(r, x_r|\{\mu_t\}) \right) dr + \int_t^s \sigma dW_r.
\]

Since \( b(x, \mu) \) is Lipschitz, \( |\varphi_\theta(s, x_s|\{\mu_t\})| \leq \theta \), and

\[
\mathbb{E} \left| b(x_r, \mu_r) + \varphi_\theta(r, x_r|\{\mu_t\}) \right| \leq M \text{ for large } M \text{ and for any } r \in [0, T],
\]

\[
D^1(\tilde{\mu}_s, \tilde{\mu}_t) \leq \mathbb{E}|x_s - x_t| \leq \mathbb{E} \int_t^s \left| b(x_r, \mu_r) + \varphi_\theta(r, x_r|\{\mu_t\}) \right| dr + \sigma \mathbb{E} \sup_{r \in [t, s]} |W_r - W_t|
\]

\[
\leq M|s - t| + \sigma \mathbb{E} \sup_{r \in [t, s]} |W_r - W_t|
\]

\[
\leq M|s - t| + \sigma |s - t|^{\frac{1}{2}}.
\]

Therefore, there exists a constant \( c_T \) depending on \( T \) such that

\[
\sup_{s \neq t} \frac{D^1(\tilde{\mu}_t, \tilde{\mu}_s)}{|t - s|^{\frac{1}{2}}} \leq c_T,
\]

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For any \( t \in [0, T] \), since \( |b(x, \mu)| \leq c_1 \) is bounded,

\[
\int_{\mathbb{R}} |x|^2 \tilde{\mu}_t(dx) \leq 2 \mathbb{E} \left( \int_{\mathbb{R}} |x|^2d\tilde{\mu}_0 + c_2^2 t^2 + \sigma^2 t \right) \leq 2 \mathbb{E} \left( \int_{\mathbb{R}} |x|^2d\tilde{\mu}_0 + c_2^2 T^2 + \sigma^2 T \right),
\]

and \( \sup_{t \in [0,T]} \int_{\mathbb{R}} |x|^2 \tilde{\mu}_t(dx) \leq c_T \). By Theorem 7.2 of [Bil99], the range of \( \Gamma \) is relatively compact and \( \Gamma \) is a compact mapping.

\[\square\]

**Proposition 2.6.** Assume (A1)-(A6). If the value function \( v_\theta(s, x|\{\mu_t\}) \) to (Control-BD) with any \( \{\mu_t\} \in \mathcal{M}_{[0,\infty)} \) is in \( C^{1,2}([0, \infty) \times \mathbb{R}) \), then \( \Gamma : \mathcal{M}_{[0,\infty)} \rightarrow \mathcal{M}_{[0,\infty)} \) is continuous.

**Proof of Proposition 2.6.** For any fixed \( \{\mu_t\} \in \mathcal{M}_{[0,\infty)} \), consider a sequence \( \{\mu^n_t\}_{n=1}^\infty \) where each \( \mu^n_t \) is in \( \mathcal{M}_{[0,\infty)} \) and

\[
d^\beta_{\mathcal{M}_{[0,\infty)}} \left( \{\mu^n_t\}, \{\mu_t\} \right) \rightarrow 0 \text{ as } n \to \infty.
\]

For each \( \{\mu^n_t\} \), problem (Control-BD) has a value function \( v^n_\theta(s, x|\{\mu^n_t\}) \) with the optimal control function \( \varphi^n_\theta(t, x|\{\mu^n_t\}) \). Let \( \{x^n_t\} \) be the corresponding optimal controlled process,

\[
dx^n_t = \left( b(x^n_t, \mu^n_t) + \varphi^n_\theta(t, x^n_t|\{\mu^n_t\}) \right) dt + \sigma dW_t, \quad x^n_0 = x,
\]

and let \( \{\tilde{\mu}^n_t\} \) be the flow of probability measures of \( \{x^n_t\} \), then \( \Gamma(\{\mu^n_t\}) = \{\tilde{\mu}^n_t\} \).

Similarly, for a fixed \( \{\mu_t\} \), problem (Control-BD) has a value function \( v_\theta(s, x|\{\mu^n_t\}) \) with the optimal control function \( \varphi_\theta(t, x|\{\mu_t\}) \). Let \( \{x_t\} \) be the corresponding optimal controlled process,

\[
dx_t = \left( b(x_t, \mu_t) + \varphi(t, x_t) \right) dt + \sigma dW_t, \quad x_0 = x,
\]

and let \( \{\tilde{\mu}_t\} \) be the flow of probability measures of \( \{x_t\} \), then \( \Gamma(\{\mu_t\}) = \{\tilde{\mu}_t\} \).

Denote also \( \varphi_1(t, x) = \max\{\varphi(t, x), 0\} \), \( \varphi_2(t, x) = -\max\{-\varphi(t, x), 0\} \), \( \varphi^n_1(t, x) = \max\{\varphi^n_\theta(t, x|\{\mu^n_t\}), 0\} \) and \( \varphi^n_2(t, x) = -\max\{-\varphi^n_\theta(t, x|\{\mu^n_t\}), 0\} \). To prove the continuity of \( \Gamma \) is to prove

\[
d^\beta_{\mathcal{M}_{[0,\infty)}} \left( \{\tilde{\mu}^n_t\}, \{\tilde{\mu}_t\} \right) \rightarrow 0 \text{ as } n \to \infty.
\]

We shall accomplish this in several steps.

**Step 1.** Bound \( d^\beta_{\mathcal{M}_{[0,\infty)}} (\{\tilde{\mu}^n_t\}, \{\tilde{\mu}_t\}) \) with \( D^2(\{\mu^n_t\}, \{\mu_t\}) \). For arbitrary \( t \in [0, \infty) \), for any \( s \in [0, t] \),

\[
d(x_s - x^n_s) = \left( b(x_s, \mu_s) - b(x^n_s, \mu^n_s) + \varphi(s, x_s) - \varphi^n_\theta(s, x^n_s|\{\mu^n_t\}) \right) ds.
\]
Applying Itô's formula to \( f(t, x) = e^{-\beta t}x^2 \) yields

\[
e^{-\beta t}|x_t - x_t^n|^2 = -\beta \int_0^t e^{-\beta s}|x_s - x_s^n|^2 ds
\]

\[
+ 2 \int_0^t \left( b(x_s, \mu_s) - b(x_s^n, \mu_s^n) + \varphi(s, x_s) - \varphi^n(s, x_s^n | \mu^n_t) \right) e^{-\beta s}(x_s - x_s^n) ds
\]

\[
\leq -\beta \int_0^t e^{-\beta s}|x_s - x_s^n|^2 ds
\]

\[
+ 2 \int_0^t \text{Lip}(b) \left( |x_s - x_s^n| + D^1(\mu_s, \mu_s^n) \right) e^{-\beta s}|x_s - x_s^n| ds
\]

\[
+ 2 \int_0^t \left( \varphi(s, x_s) - \varphi^n(s, x_s^n | \mu^n_t) \right) e^{-\beta s}(x_s - x_s^n) ds.
\]

By Assumption (A6),

\[
(\varphi(s, x_s) - \varphi^n(s, x_s^n | \mu^n_t))(x_s - x_s^n)
\]

\[
\leq \left( \varphi(s, x_s) - \varphi^n(s, x_s^n | \mu^n_t) + \varphi^n(s, x_s^n | \mu^n_t) - \varphi^n(s, x_s^n | \mu^n_t) \right)(x_s - x_s^n)
\]

\[
\leq (\varphi(s, x_s) - \varphi^n(s, x_s^n | \mu^n_t))(x_s - x_s^n)
\]

\[
= (\varphi_1(s, x_s) - \varphi^n_1(s, x_s))(x_s - x_s^n) + (\varphi_2(s, x_s) - \varphi^n_2(s, x_s))(x_s - x_s^n)
\]

\[
\leq \frac{1}{2} \left( |\varphi_1(s, x_s) - \varphi^n_1(s, x_s)|^2 + |x_s - x_s^n|^2 \right) + \frac{1}{2} \left( |\varphi_2(s, x_s) - \varphi^n_2(s, x_s)|^2 + |x_s - x_s^n|^2 \right).
\]

Consequently,

\[
e^{-\beta t}|x_t - x_t^n|^2 \leq \int_0^t e^{-\beta s}(3\text{Lip}(b) + 2 - \beta)|x_s - x_s^n|^2 ds + \int_0^t e^{-\beta s}\text{Lip}(b)(D^1(\mu_s, \mu_s^n))^2 ds
\]

\[
+ \int_0^t e^{-\beta s} \left( |\varphi_1(s, x_s) - \varphi^n_1(s, x_s)|^2 + |\varphi_2(s, x_s^n) - \varphi^n_2(s, x_s^n)|^2 \right) ds.
\]

By Gronwall’s inequality,

\[
e^{-\beta t}|x_t - x_t^n|^2 \leq e^{(3\text{Lip}(b)+2-\beta)t} \left( \int_0^t e^{-\beta s}\text{Lip}(b)(D^1(\mu_s, \mu_s^n))^2 ds
\]

\[
+ \int_0^t e^{-\beta s} \left( |\varphi_1(s, x_s) - \varphi^n_1(s, x_s)|^2 + |\varphi_2(s, x_s^n) - \varphi^n_2(s, x_s^n)|^2 \right) ds \right),
\]

hence

\[
\int_0^\infty e^{-\beta t}|x_t - x_t^n|^2 dt
\]

\[
\leq \int_0^\infty e^{(3\text{Lip}(b)+2-\beta)t} \left\{ \int_0^t e^{-\beta s}\text{Lip}(b)(D^1(\mu_s, \mu_s^n))^2 ds
\]

\[
+ \int_0^t e^{-\beta s} \left( |\varphi_1(s, x_s) - \varphi^n_1(s, x_s)|^2 + |\varphi_2(s, x_s^n) - \varphi^n_2(s, x_s^n)|^2 \right) ds \right\} dt.
\]

(9)
Step 2. Show that \( \partial_x v^n(t, x|\{\mu^n\}) \to \partial_x v_\theta(t, x|\{\mu_1\}) \) as \( n \to \infty \) for any \( t, x \in [0, \infty) \times \mathbb{R} \).

Since \( \varphi^n_1, \varphi^n_2 \) are optimal controls, for any \( x \in \mathbb{R} \) and any \( n \in \mathbb{N} \),

\[
v^n_\theta(0, x|\{\mu_1\}) = \mathbb{E} \int_0^\infty e^{-rt} \left( f(x_i^n, \mu_i^n) - \gamma^+ \varphi^n_1(t, x_i^n) - \gamma^- \varphi^n_2(t, x_i^n) \right) dt.
\] (10)

Applying Itô’s formula to \( e^{-rt}v_\theta(t, y|\{\mu_1\}) \) yields

\[
\begin{align*}
\neg v_\theta(0, x|\{\mu_1\}) & = \int_0^\infty e^{-rt} \left( -rv_\theta(t, x^n) + \partial_t v_\theta(t, x^n|\{\mu_1\}) + \left( b(x^n, \mu_i^n) + \varphi^n_0(t, x^n|\{\mu_1\}) \right) \partial_x v_\theta(t, x^n|\{\mu_1\}) \\
& \quad + \frac{\sigma^2}{2} \partial_{xx} v_\theta(t, x^n|\{\mu_1\}) \right) dt + \int_0^\infty e^{-rt} \sigma \partial_x v_\theta(t, x^n|\{\mu_1\}) dW_t \\
& = \int_0^\infty e^{-rt} \left( -rv_\theta(t, x^n) + \partial_t v_\theta(t, x^n) + \left( b(x^n, \mu_i) + \varphi_\theta(t, x^n|\{\mu_1\}) \right) \partial_x v_\theta(t, x^n|\{\mu_1\}) \\
& \quad + \frac{\sigma^2}{2} \partial_{xx} v_\theta(t, x^n|\{\mu_1\}) \right) dt + \int_0^\infty \sigma \partial_x v_\theta(t, x^n|\{\mu_1\}) dW_t \\
& \quad - \int_0^\infty e^{-rt} \left( b(x^n, \mu_i) - b(x_i^n, \mu_i^n) + \varphi(t, x^n) - \varphi^n_\theta(t, x^n|\{\mu_1\}) \right) \partial_x v_\theta(t, x^n|\{\mu_1\}) dt \\
& = \int_0^\infty e^{-rt} \left( f(x_i^n, \mu_i) - \gamma^+ \varphi_1(t, x_i^n) + \gamma^- \varphi_2(t, x_i^n) \right) dt + \int_0^\infty e^{-rt} \sigma \partial_x v_\theta(t, x^n|\{\mu_1\}) dW_t \\
& \quad - \int_0^\infty e^{-rt} \left( b(x^n, \mu_i) - b(x_i^n, \mu_i^n) + \varphi(t, x^n) - \varphi^n_\theta(t, x^n|\{\mu_1\}) \right) \partial_x v_\theta(t, x^n|\{\mu_1\}) dt.
\end{align*}
\]

Hence,

\[
v^n_\theta(0, x|\{\mu_1^n\}) - v_\theta(0, x|\{\mu_1\}) = \mathbb{E} \left[ v^n(0, x|\{\mu^n\}) - v(0, x|\{\mu_1\}) \right]
\]

\[
= \mathbb{E} \left[ \int_0^\infty e^{-rt} \left( f(x_i^n, \mu_i^n) - f(x_i^n, \mu_i) - \gamma^+ (\varphi^n_1(t, x_i^n) - \varphi_1(t, x_i^n)) - \gamma^- (\varphi^n_2(t, x_i^n) - \varphi_2(t, x_i^n)) \right) dt \right] \\
+ \mathbb{E} \left[ \int_0^\infty e^{-rt} \left( b(x^n, \mu_i) - b(x_i^n, \mu_i^n) + \varphi^n_\theta(t, x^n|\{\mu_1^n\}) - \varphi_\theta(t, x^n|\{\mu_1\}) \right) \partial_x v_\theta(t, x^n|\{\mu_1^n\}) dt \right] \\
= \mathbb{E} \left[ \int_0^\infty e^{-rt} \left( f(x_i^n, \mu_i^n) - f(x_i^n, \mu_i) + (\gamma^+ \partial_x v_\theta(t, x_i^n|\{\mu_1^n\}))((\varphi^n_1(t, x_i^n) - \varphi_1(t, x_i^n)) \\
\quad + (\gamma^- - \partial_x v_\theta(t, x_i^n|\{\mu_1^n\}))((\varphi^n_2(t, x_i^n) - \varphi_2(t, x_i^n)) + b(x_i^n, \mu_i^n) - b(x_i^n, \mu_i)) \right) dt \right].
\]
Consequently,
\[
\begin{align*}
&\left| v_\theta(0, x|\{\mu_1\}) - v^n_\theta(0, x|\{\mu^n_1\}) \right| + E \left[ \int_0^\infty e^{-rt} \left( f(x^n_i, \mu^n_i) - f(x_i^*|\{\mu_i\}) + (b(x^n_i, \mu^n_i) - b(x^n_i, \mu_i)) \partial_x v_\theta(t, x_i^*|\{\mu_i\}) \right) dt \right] \\
&\geq \left| v_\theta(0, x|\{\mu_1\}) - v^n_\theta(0, x|\{\mu^n_1\}) \right| \\
&\quad - E \left[ \int_0^\infty e^{-rt} \left( f(x^n_i, \mu^n_i) - f(x_i^*|\{\mu_i\}) + (b(x^n_i, \mu^n_i) - b(x^n_i, \mu_i)) \partial_x v_\theta(t, x_i^*|\{\mu_i\}) \right) dt \right] \\
&= E \left[ \int_0^\infty e^{-rt} \left( (-\gamma^+ + \partial_x v_\theta(t, x_i^*|\{\mu_i\})) (\varphi^n_1(t, x_i^n) - \varphi_1(t, x_i^n)) + (-\gamma^- - \partial_x v_\theta(t, x_i^n|\{\mu_i\})) (\varphi^n_2(t, x_i^n) - \varphi_2(t, x_i^n)) \right) dt \right].
\end{align*}
\]

By the Markovian property of \( \varphi_1 \), if \( \varphi_1(t, x_i^n) = \theta \), then \(-\gamma^+ + \partial_x v_\theta(t, x_i^n|\{\mu_i\}) \geq 0 \), and if \( \varphi_1(t, x_i^n) = 0 \), then \(-\gamma^+ + \partial_x v_\theta(t, x_i^n|\{\mu_i\}) \leq 0 \). Hence,
\[
\left( \gamma^+ + \partial_x v_\theta(t, x_i^n|\{\mu_i\}) \right) \left( \varphi_1^n(t, x_i^n) - \varphi_1(t, x_i^n) \right) \leq 0.
\]

Similarly, by the Markovian property of \( \varphi_2 \), if \( \varphi_2(t, x_i^n) = \theta \), then \(-\gamma^- - \partial_x v_\theta(t, x_i^n|\{\mu_i\}) \geq 0 \), and if \( \varphi_2(t, x_i^n) = 0 \), then \(-\gamma^- - \partial_x v_\theta(t, x_i^n|\{\mu_i\}) \leq 0 \). Hence,
\[
\left( \gamma^- - \partial_x v_\theta(t, x_i^n|\{\mu_i\}) \right) \left( \varphi_2^n(t, x_i^n) - \varphi_2(t, x_i^n) \right) \leq 0.
\]

Therefore, by Lipschitz continuity of \( b, f \),
\[
\begin{align*}
E \left[ \int_0^\infty e^{-rt} \left( (-\gamma^+ + \partial_x v_\theta(t, x_i^n|\{\mu_i\})) (\varphi^n_1(t, x_i^n) - \varphi_1(t, x_i^n)) \right. \\
\left. + (-\gamma^- - \partial_x v_\theta(t, x_i^n|\{\mu_i\})) (\varphi^n_2(t, x_i^n) - \varphi_2(t, x_i^n)) \right) dt \right] \\
\leq \left| v_\theta(0, x|\{\mu_1\}) - v^n_\theta(0, x|\{\mu^n_1\}) \right| \\
+ E \left[ \int_0^\infty e^{-rt} \left| f(x^n_i, \mu^n_i) - f(x_i^*|\{\mu_i\}) \right| + e^{-rt} \left| (b(x^n_i, \mu^n_i) - b(x^n_i, \mu_i)) \partial_x v_\theta(t, x_i^*|\{\mu_i\}) \right| ds \right] \\
\leq \left| v_\theta(0, x|\{\mu_1\}) - v^n_\theta(0, x|\{\mu^n_1\}) \right| \\
+ E \left[ \int_0^\infty e^{-rt} \text{Lip}(f) D^1(\mu_i, \mu^n_i) + e^{-rt} \text{Lip}(b) D^1(\mu_i, \mu^n_i) \right| \partial_x v_\theta(t, x_i^*|\{\mu_i\})\right| dt \right].
\end{align*}
\]

(11)

Since for any \( s \in [t, T] \), \( D^1(\mu^n_s, \mu_s) \to 0 \) as \( n \to \infty \), and
\[
\left| b(x_s, \mu_s) - b(x_s, \mu^n_s) \right| \leq \text{Lip}(b) D^1(\mu_s, \mu^n_s) \to 0 \text{ as } n \to \infty.
\]
By Proposition 4.1 from Chapter 4 in [Zha12], \( v^n(t, x|\{\mu^n_t\}) \rightarrow v_0(t, x|\{\mu_t\}) \) for any \((t, x) \in [0, \infty) \times \mathbb{R}\) as \(n \to \infty\). Hence, as \(n \to \infty\), the last term in the inequality (11) converges to 0. Consequently, for any \(x \in \mathbb{R}\), as \(n \to \infty\)

\[
\begin{align*}
\mathbb{E} \left[ \int_0^\infty e^{-rt} & \left( (-\gamma^+ + \partial_x v_0(t, x^n_t|\{\mu_t\}))(\varphi_1(t, x^n_t) - \varphi^n_1(t, x^n_t)) \\
&+ (-\gamma^- - \partial_x v_0(t, x^n_t|\{\mu_t\}))(\varphi_2(t, x^n_t) - \varphi^n_2(t, x^n_t)) \right) \right] \to 0.
\end{align*}
\]  

(12)

Now, from (12), for any \(t \in [0, \infty)\),

\[
\mathbb{E} \left[ (-\gamma^+ + \partial_x v_0(t, x^n_t|\{\mu_t\}))(\varphi_1(t, x^n_t) - \varphi^n_1(t, x^n_t)) \right] \to 0.
\]

By definition of \(\varphi_1\), \(\mathbb{E} \left[ (-\gamma^+ + \partial_x v_0(t, x^n_t|\{\mu_t\}))(\varphi_1(t, x^n_t) - \varphi^n_1(t, x^n_t)) \right] = 0\). Consequently,

\[
\mathbb{E} \left[ (-\gamma^+ + \partial_x v_0(t, x^n_t|\{\mu_t\}))(\varphi^n_1(t, x^n_t) \right] \to 0 \text{ as } n \to \infty.
\]

For any \(n \in \mathbb{N}\) and \(t \in [0, \infty)\), denote

\[
\begin{align*}
x^n_{b_1}(t) &= \sup \left\{ y \mid -\gamma^+ + \partial_x v_0^n(t, y|\{\mu^n_t\}) < 0 \right\}, \\
x_{b_1}(t) &= \sup \left\{ y \mid -\gamma^+ + \partial_x v_0(t, y|\{\mu_t\}) < 0 \right\}, \\
x^n_{b_2}(t) &= \inf \left\{ y \mid \gamma^- + \partial_x v_0^n(t, y|\{\mu^n_t\}) > 0 \right\}, \\
x_{b_2}(t) &= \inf \left\{ y \mid \gamma^- + \partial_x v_0(t, y|\{\mu_t\}) > 0 \right\}.
\end{align*}
\]

Then, \(\varphi^n_1(t, x^n_{b_1}(t)) = \theta\). Thus, \(-\gamma^+ + \partial_x v_0(t, x^n_{b_1}(t)|\{\mu_t\}) \rightarrow 0 \text{ as } n \to \infty\). Furthermore, \(-\gamma^+ + \partial_x v_0(t, x_{b_1}(t)|\{\mu_t\}) = 0\). Note by the definition and the viscosity solution property of \(v_0(t, x|\{\mu_t\})\), \(v_0(t, x|\{\mu_t\})\) is concave and nonlinear when \(f(x, \mu)\) is nonlinear, hence \(\partial_x v_0(t, x|\{\mu_t\})\) is strictly decreasing and continuous in \(x\). Hence, \(\lim_{n \to \infty} x^n_{b_1}(t) = x_{b_1}(t)\) and \(\lim_{n \to \infty} x^n_{b_2}(t) = x_{b_2}(t)\) for each \(t \in [0, \infty)\). Therefore, for each \(t \in [0, \infty)\),

\[
\varphi^n_0(t, x_t|\{\mu^n_t\}) \rightarrow \varphi(t, x_t) \text{ a.s. as } n \to \infty.
\]

By the inequality (9) with \(\beta > 3\text{Lip}(b) + 2\) and the Dominated Convergence Theorem, \(\Gamma\) is continuous.

\[\Box\]

**Step C.** By Propositions 2.5 and 2.6, (MFG) has a fixed point solution by the Schauder fixed point theorem (Theorem 4.1.1. in [Sma74]), and it is unique by Assumption (A4).
3 MFGs with singular control of finite variation

In this section, we shall analyze (MFG) when the control is of finite variation, i.e., when \( \theta \to \infty \).

To allow for direct comparison with the single-player game in [GP05], we assume that the production level is a geometric Brownian motion, and that the revenue function is of a Cobb-Douglas type such that \( f(x, \mu) = F(\mu)x^\alpha \) for some \( \alpha \in (0, 1) \), where \( F \) is some functional of \( \mu \) to be specified later. That is, a representative company in the MFG adjusts its production level \( x_t \) according to a policy chosen from the set of admissible controls \( \mathcal{U} \) to maximize its discounted total profit over an infinite-time horizon,

\[
\sup_{\langle \xi^+, \xi^- \rangle \in \mathcal{U}} \mathbb{E} \left[ \int_0^{\infty} e^{-rt} \left[ f(x_t, \mu_t) dt - \gamma^+ d\xi_t^+ - \gamma^- d\xi_t^- \right] \right| x_{0^-} = x,
\]

subject to

\[ dx_t = x_t (\delta dt + \gamma dW_t) + d\xi_t^+ - d\xi_t^- , \]

and the set of admissible controls \( \mathcal{U} \) is

\[
\mathcal{U} = \{(\xi^+, \xi^-) : \xi^+, \xi^- \text{ nondecreasing processes adapted to } \mathcal{F}_t, \xi^+_{0^-} = \xi^-_{0^-} = 0, \quad E \left[ \int_0^{\infty} e^{-rt} d\xi_t^+ \right] < \infty, \quad x_t \geq 0, \quad \forall t \geq 0 \},
\]

where \( \mathcal{F}_t = \bigcap_{0 \leq s < t} \mathcal{F}_s \), for all \( t \geq 0 \). Here, adopting the notation in [GP05], \( \gamma^+ \) and \( \gamma^- \) take the form of \( \gamma^+ = p \) and \( \gamma^- = -p(1 - \lambda) \). That is to say, it takes \( p > 0 \) units of investment cost to increase one unit of production level; and to ensure no-arbitrage, the company can gain \( p(1 - \lambda) \) to reduce one unit of production level, with \( \lambda \in (0, 1) \).

Note that the form of the revenue function \( f \) here differs from that in [GP05]. The additional term \( \mu \) reflects the dependence on the game interaction among companies, and is consistent with the inverse demand function. To see this, take the inverse demand function so that the price under production level \( x \) is \( \hat{\rho}(x) = a_0 - a_1 (x^\alpha)^{1/\alpha} = a_0 - a_1 x^{1-\alpha} \) for some positive constant \( a_0 \) and \( a_1 \). If the limiting distribution of \( \mu_t \) exists, say \( \mu \), then the expected price \( \rho \) is \( \rho = \rho(\mu) = E[\hat{\rho}(X)] = \int (a_0 - a_1 y^{1-\alpha}) \mu(dy) \), a functional of \( \mu \). Now, if the revenue function is of a Cobb-Douglas type and depends on both the production level \( x \) and the price \( \rho \), then effectively one can write

\[
f(x, \mu) = cpx^\alpha
\]

for some constant \( c > 0 \). That is, the revenue function depends on both the product level and the price of the commodity. Clearly, \( f \) satisfies the Inada condition and thus Assumption (A2).

To find explicit solutions, we shall focus on the time invariant \( \mu_t \equiv \mu \). That is, we are essentially look for a stationary solution to (SMFG). The solution to SMFG consists of an optimal policy \( (\xi^+, \xi^-) \in \mathcal{U} \) and an equilibrium price \( \rho^* \). Formally,

**Definition 3.1.** Given a fixed \( \rho \in \mathbb{R} \), a control of finite variation \( \xi_t \) is Markovian if \( d\xi_t = d\varphi(t, x|\rho) \), where

\[
\varphi(\cdot, \cdot | \rho) : [0, \infty) \times \mathbb{R} \to \mathbb{R}
\]

is called a control function under \( \rho \).
Definition 3.2. If there exists a Markovian control \((\xi^{+,*}, \xi^{-,*}) \in \mathcal{U}\) and \(\rho^* \in \mathbb{R}\) satisfying the following conditions,

1. Under \(\rho^*, (\xi^{+,*}, \xi^{-,*})\) is an optimal control for

\[
\hat{v}(x) = \sup_{(\xi^{+,*}, \xi^{-,*}) \in \mathcal{U}} \mathbb{E} \left[ \int_0^\infty e^{-rt} [f(x_t, \mu) dt - pd\xi_t^+ + p(1 - \lambda) d\xi_t^-] \bigg| x_{0-} = x \right] \tag{SMFG}
\]

s.t. \(dx_t = x_t(\delta dt + \gamma dW_t) + d\xi_t^+ - d\xi_t^-;\)

2. \(\rho^* = \int (a_0 - a_1 y^{1-\alpha}) \mathbb{P}_{x_\infty} (dy),\) where \(x_t^*\) is given by

\[
dx_t^* = x_t^*(\delta dt + \gamma dW_t) + d\xi_t^{+,*} - d\xi_t^{-,*}; \tag{13}
\]

then the control-mean pair \((\xi^{+,*}, \xi^{-,*}, \rho^*)\) is said to be a solution to the SMFG.

To ensure the well-posedness of (SMFG), we assume \(\delta < r\) and \(\frac{2}{\gamma^2} \not\in \{\alpha, 1\}\).

3.1 Solution to (SMFG).

We shall now solve (SMFG).

Step 1. Control problem under fixed mean information. Fix \(\rho\), then (SMFG) is a singular control problem of finite variation, similar to that in [GP05]. The associated HJB equation to the SMFG model (SMFG) under a fixed \(\rho\) is

\[
0 = \min \{ r\hat{v} - cx^\alpha \rho - \delta x \partial_x \hat{v} - \frac{1}{2} \gamma^2 x^2 \partial_{xx} \hat{v}, p - \partial_x \hat{v}, \partial_x \hat{v} - p(1 - \lambda) \}. \tag{14}
\]

Following the argument in [GP05], we see that the optimal policy is a bang-bang type and is characterized by an expansion threshold \(\tilde{x}_b\) and a contraction threshold \(\tilde{x}_s\) so that \(x_t \in [\tilde{x}_b, \tilde{x}_s]\) almost surely. More precisely, at time \(t = 0\), if \(x \in (0, \tilde{x}_b)\), then \(\xi_0^+ = \tilde{x}_b - x\) and \(\xi_0^- = 0\); if \(x \in (\tilde{x}_s, \infty)\), then \(\xi_0^+ = 0\) and \(\xi_0^- = x - \tilde{x}_s\). Note that \(x_0 = x_{0-} + \xi_0^+ - \xi_0^- \in [\tilde{x}_b, \tilde{x}_s]\). For \(t > 0\), it is optimal to impose a minimum amount of adjustment so that \(x_t \in [\tilde{x}_b, \tilde{x}_s]\).

Accordingly, the solution \(\hat{v}\) is of the form

\[
\hat{v}(x) = \begin{cases} 
px + C_1, & x \leq \tilde{x}_b, \\
Ax^m + Bx^n + Hx^\alpha, & \tilde{x}_b < x < \tilde{x}_s, \\
p(1 - \lambda)x + C_2, & \tilde{x}_s \leq x,
\end{cases}
\]

where \(\tilde{x}_b = \inf \{ x : \partial_x \hat{v}(x) = p \}, \tilde{x}_s = \sup \{ x : \partial_x \hat{v}(x) = p(1 - \lambda) \}\) with \(0 < \tilde{x}_b \leq \tilde{x}_s\) (see Lemma 4.4 in [GP05]), and since \(\delta < r\),

\[
m = - \left( \frac{\delta}{\gamma^2} - \frac{1}{2} \right) - \sqrt{\left( \frac{\delta}{\gamma^2} - \frac{1}{2} \right)^2 + \frac{2r}{\gamma^2}} < 0, \quad n = - \left( \frac{\delta}{\gamma^2} - \frac{1}{2} \right) + \sqrt{\left( \frac{\delta}{\gamma^2} - \frac{1}{2} \right)^2 + \frac{2r}{\gamma^2}} > 1,
\]

\[
H = \frac{2c_\rho}{\gamma^2(n - \alpha)(\alpha - m)}.
\]
Moreover, by the smooth-fit principle, we have

\[
\begin{aligned}
A\tilde{x}_b^m + B\tilde{x}_b^n + H\tilde{x}_b^\alpha &= p\tilde{x}_b + C_1, \\
mA\tilde{x}_b^{m-1} + nB\tilde{x}_b^{n-1} + \alpha H\tilde{x}_b^{\alpha-1} &= p, \\
m(m-1)A\tilde{x}_b^{m-2} + n(n-1)B\tilde{x}_b^{n-2} + \alpha(\alpha-1)H\tilde{x}_b^{\alpha-2} &= 0, \\
A\tilde{x}_s^m + B\tilde{x}_s^n + H\tilde{x}_s^\alpha &= p(1-\lambda)\tilde{x}_s + C_2, \\
mA\tilde{x}_s^{m-1} + nB\tilde{x}_s^{n-1} + \alpha H\tilde{x}_s^{\alpha-1} &= p(1-\lambda), \\
m(m-1)A\tilde{x}_s^{m-2} + n(n-1)B\tilde{x}_s^{n-2} + \alpha(\alpha-1)H\tilde{x}_s^{\alpha-2} &= 0.
\end{aligned}
\]

Some algebraic manipulations yield

\[
A = \frac{p(n-1)\tilde{x}_b - \alpha(n-\alpha)H\tilde{x}_b^\alpha}{m(n-m)\tilde{x}_b^m} = \frac{p(1-\lambda)(n-1)\tilde{x}_s - \alpha(n-\alpha)H\tilde{x}_s^\alpha}{m(n-m)\tilde{x}_s^m}; \tag{16}
\]

and

\[
B = \frac{p(m-1)\tilde{x}_b - \alpha(m-\alpha)H\tilde{x}_b^\alpha}{n(m-n)\tilde{x}_b^n} = \frac{p(1-\lambda)(m-1)\tilde{x}_s - \alpha(m-\alpha)H\tilde{x}_s^\alpha}{n(m-n)\tilde{x}_s^n}. \tag{17}
\]

Furthermore, denote \( y_0 = \frac{\tilde{x}_s}{\tilde{x}_b} \) and \( y_0 \geq 1 \). By (16) and (17), we have

\[
\begin{cases}
p(n-1)\left[(1-\lambda)y_0 - y_0^m\right] = \alpha(n-\alpha)H\tilde{x}_b^{\alpha-1}(y_0^\alpha - y_0^m), \tag{18} \\
p(m-1)\left[(1-\lambda)y_0 - y_0^n\right] = \alpha(m-\alpha)H\tilde{x}_s^{\alpha-1}(y_0^\alpha - y_0^n). \tag{19}
\end{cases}
\]

and

\[
\frac{(n-1)(\alpha-m)y_0^{m-1}(y_0^\alpha - y_0^m) + (1-m)(n-\alpha)y_0^{n-1}(y_0^\alpha - y_0^m)}{(n-1)(\alpha-m)(y_0^\alpha - y_0^m) + (1-m)(n-\alpha)(y_0^m - y_0^\alpha)} = 1 - \lambda. \tag{20}
\]
Now, to show that there exists a $y_0$ for (20), for $y > 1$, define
\[
F(y) = \frac{(n-1)(\alpha-m)y^{m-1}(y^n-y^n) + (1-m)(n-\alpha)y^{m-1}(y^n-y^n)}{(n-1)(\alpha-m)(y^n-y^n) + (1-m)(n-\alpha)(y^n-y^n)}.
\]
Since $\lim_{y \to 1^+} F(y) = 1$, $\lim_{y \to \infty} F(y) = 0$, and $F$ is continuous, there exists a $y_0 > 1$ satisfying $F(y_0) = 1 - \lambda \in (0, 1)$ (see also Figure 1). Note that the function $F$ does not depend on $\rho$, therefore $y_0$ does not rely on $\rho$ either. From (19), we can conclude that
\[
\tilde{x}_b = \left\{ \frac{2ca(y_0^n - y_0^n)}{\gamma^2p(1-m)(n-\alpha)[y_0^n - (1-\lambda)y_0]} \right\}^{\frac{1}{\alpha}} \rho^{\frac{1}{\alpha}}.
\]
where $\left\{ \frac{2ca(y_0^n - y_0^n)}{\gamma^2p(1-m)(n-\alpha)[y_0^n - (1-\lambda)y_0]} \right\}^{\frac{1}{\alpha}}$ does not rely on $\rho$, and
\[
\tilde{x}_s = \tilde{x}_b y_0 = \left\{ \frac{2cay_0^{1-\alpha}(y_0^n - y_0^n)}{\gamma^2p(1-m)(n-\alpha)[y_0^n - (1-\lambda)y_0]} \right\}^{\frac{1}{\alpha}} \rho^{\frac{1}{\alpha}}.
\]
Consequently, $A$ and $B$ are given by (16) and (17), respectively, and $C_1 = A\tilde{x}_b^m + B\tilde{x}_b^n + H\tilde{x}_b^\alpha - p\tilde{x}_b$, $C_2 = A\tilde{x}_s^m + B\tilde{x}_s^n + H\tilde{x}_s^\alpha - p(1-\lambda)\tilde{x}_s$.

**Step 2. Updating the price $\rho$ and the locating the fixed point.** Under any fixed $\rho$, the optimal controlled process $x_t$ is a reflected Brownian motion within the interval $[\tilde{x}_b, \tilde{x}_s]$. By [BW95], for any $x \in [\tilde{x}_b, \tilde{x}_s]$, the scale density is given by
\[
s(x) = \exp \left\{ -\int_{\theta}^{x} \frac{2\delta}{\gamma^2}s dy \right\} = \theta^{\frac{2\delta}{\gamma^2}} x^{\frac{2\delta}{\gamma^2}}, \quad \forall \theta \in (\tilde{x}_b, \tilde{x}_s),
\]
the speed density is
\[
m(x) = \frac{2}{\gamma^2 x^2 s(x)} = \frac{2}{\gamma^2 \theta^{\frac{2\delta}{\gamma^2}}} x^{\frac{2\delta}{\gamma^2} - 2},
\]
and finally
\[
M(x) = \int_{\tilde{x}_b}^{x} m(y) dy = \frac{2}{\gamma^2 \theta^{\frac{2\delta}{\gamma^2}}} \frac{x^{\frac{2\delta}{\gamma^2} - 1} - \tilde{x}_b^{\frac{2\delta}{\gamma^2} - 1}}{\tilde{x}_s^{\frac{2\delta}{\gamma^2} - 1} - \tilde{x}_b^{\frac{2\delta}{\gamma^2} - 1}}.
\]
The density function of the limiting distribution of $x_t$, $P_{x_{\infty}}$, is thus
\[
f(x) = \frac{m(x)}{M(\tilde{x}_s)} = \frac{x^{\frac{2\delta}{\gamma^2} - 1} - \tilde{x}_b^{\frac{2\delta}{\gamma^2} - 1}}{\tilde{x}_s^{\frac{2\delta}{\gamma^2} - 1} - \tilde{x}_b^{\frac{2\delta}{\gamma^2} - 1} x^{\frac{2\delta}{\gamma^2} - 2}}, \quad \forall x \in [\tilde{x}_b, \tilde{x}_s].
\]
The updated price $\tilde{\rho}$ under the limiting distribution $\tilde{\mu} = P_{x_{\infty}}$ is
\[
\tilde{\rho} = \Gamma(\rho) = a_0 - a_1 \int_{\tilde{x}_b}^{\tilde{x}_s} x^{1-\alpha} f(x) dx = a_0 - a_1 \frac{2\delta - \gamma^2}{2\delta - \alpha \gamma^2} \frac{\tilde{x}_s^{\frac{2\delta}{\gamma^2} - 1} - \tilde{x}_b^{\frac{2\delta}{\gamma^2} - 1}}{\tilde{x}_s^{\frac{2\delta}{\gamma^2} - 1} - \tilde{x}_b^{\frac{2\delta}{\gamma^2} - 1} x^{\frac{2\delta}{\gamma^2} - 2}}
\]
\[= a_0 - \rho \cdot a_1 \frac{2\delta - \gamma^2}{2\delta - \alpha \gamma^2} \frac{\tilde{x}_s^{\frac{2\delta}{\gamma^2} - 1} - 1}{\tilde{x}_s^{\frac{2\delta}{\gamma^2} - 1} - 1} \frac{2ca(y_0^n - y_0^n)}{\gamma^2p(1-m)(n-\alpha)[y_0^n - (1-\lambda)y_0]},
\]
(23)
where the coefficient $a_1 \frac{2\delta - \gamma^2}{2\delta - \alpha \gamma^2} \frac{2\delta - \alpha}{2\delta - \alpha \gamma^2} \frac{1}{y_0^* - 1} \frac{2\alpha(y_0^* - y_0^m)}{\gamma^2 p(1-m)(n-\alpha)[y_0^m - (1-\lambda)y_0]}$ does not rely on $\rho$. Clearly, for $a_1 > 0$ such that

$$a_1 \frac{2\delta - \gamma^2}{2\delta - \alpha \gamma^2} \frac{2\delta - \alpha}{2\delta - \alpha \gamma^2} \frac{1}{y_0^* - 1} \frac{2\alpha(y_0^* - y_0^m)}{\gamma^2 p(1-m)(n-\alpha)[y_0^m - (1-\lambda)y_0]} < 1,$$

the mapping $\Gamma$ is a contraction and therefore admits a unique fixed point

$$\rho^* = \frac{a_0}{1 + a_1 \frac{2\delta - \gamma^2}{2\delta - \alpha \gamma^2} \frac{2\delta - \alpha}{2\delta - \alpha \gamma^2} \frac{1}{y_0^* - 1} \frac{2\alpha(y_0^* - y_0^m)}{\gamma^2 p(1-m)(n-\alpha)[y_0^m - (1-\lambda)y_0]} \cdot (24)}.$$

Substitute $\rho^*$ of (24) into (21) and (22), we can derive optimal action boundaries $\bar{x}_b^*$ and $\bar{x}_s^*$. Under Definition 3.2, $(\bar{x}_b^*, \bar{x}_s^*, \rho^*)$ is a solution to (SMFG).

### 3.2 Comparing single-player game and SMFG

As seen from (23), the iterations do not stop after the first round, indicating that (SMFG) demonstrates a genuine game effect from the weak interactions among the players. In fact, we can see that in the single-player game, model parameters $\lambda$, $\delta$, $\gamma$, $r$ and $\alpha$ directly affect the optimal strategy $(x_b, x_s)$, whereas in the MFG (SMFG) the parameters affect the equilibrium price $\rho^*$.

To illustrate, consider the following case where $\delta = 1$, $\gamma = 2$, $r = 3$, $\alpha = 0.6$, $\lambda = 0.6$, $p = 0.5$, $c = 1$, $a_0 = 1$ and $a_1 = 0.1$. Suppose the iterative process starts from a fixed value $\rho = 1$. In the single-player setting where the price $\rho = 1$ is seen as externally given and fixed, the optimal thresholds are given by $x_b = 0.053$ and $x_s = 0.264$. Figure 2 shows that both $x_b$ and $x_s$ increase along with the value of $\rho$ and the non-action region $[x_b, x_s]$ expands. In (SMFG), in contrast, the iteration leads to an equilibrium price $\rho^* = 0.96$ under which the optimal thresholds are $\bar{x}_b^* = 0.048$ and $\bar{x}_s^* = 0.239$. Figure 3 shows difference between the single-player game and the SMFG.

**Impact of $\lambda$.** $\lambda \in (0, 1)$ measures the irreversibility of the investment, that is, the closer $\lambda$ to 1, the more irreversible the investment. For the single-player game (Figures 4a and 4b), while the expansion threshold $x_b$ stays relatively insensitive with respect to an increasing $\lambda$, the contraction threshold $x_s$ increases dramatically along with $\lambda$. It means that for an individual company, as the investment is more irreversible, it becomes less profitable to frequently decrease the production level; correspondingly, the contraction threshold is raised to a higher level. Under the MFG setting, the irreversibility does not have an immediate impact on the optimal strategies (Figure 4c); instead, it drives down the equilibrium price (Figure 4d). Here is some intuition: as it becomes less profitable to reduce production when $\lambda$ approaches 1, companies in the MFG tend to keep a high production level and this tendency collectively reduces the price due to the risk-aversion implied by the Cobb-Douglas function.
Impact of $\delta$ and $\gamma$. The drift coefficient $\delta$ represents the expected growth rate of the production and $\gamma$ measures the volatility of the growth. The decision of whether or not to adjust the production level is the trade-off between the running payoff $cpx_t^\alpha$ and the profit from direct intervention $p(1 - \lambda)d\xi^- - pd\xi^+$, with $\alpha \in (0, 1)$. Without any intervention within the time interval $[t, t + \Delta t]$, $x_t^{\alpha+\Delta t}$ is given by

$$x_t^{\alpha} \exp \left\{ \left[ \alpha \delta - \frac{\gamma^2}{2} \alpha(1 - \alpha) \right] \Delta t \right\} \exp \left\{ \alpha \gamma (W_{t+\Delta t} - W_t) - \frac{\alpha^2 \gamma^2}{2} \Delta t \right\} , \quad (25)$$

therefore $\alpha \delta - \frac{\gamma^2}{2} \alpha(1 - \alpha)$ represents the expected growth rate of $x_t^{\alpha}$. Under the single-player setting, when $\delta$ increases, the revenue function grows faster, which causes the action boundaries to increase, as shown in Figures 5a and 5b. We can see that the growth in $\delta$ has larger impact on the contraction threshold $x_s$ compared to the the expansion threshold $x_b$. It also implies that each company tends to maintain a higher production level as $\delta$ grows. Under the MFG setting, this tendency on the individual level is aggregated, driving down the equilibrium price $\rho^*$, as shown in Figure 5d.

The impact of an increasing $\gamma$ can be seen from the following two perspectives. As $\gamma$ increases, the growth rate of the revenue function $\alpha \delta - \frac{\gamma^2}{2} \alpha(1 - \alpha)$ decreases, which can
potentially decrease both action boundaries. An increasing $\gamma$ indicates a larger volatility in the growth rate of the production level and the company can take advantage of the high volatility and reduce the frequency of intervention, which potentially decreases the expansion threshold and increases the contraction threshold.

Under both perspectives, the expansion threshold is expected to decrease when $\gamma$ increases. But an increase in $\gamma$ potentially has opposite effects on the contraction threshold. In the single-player game, the expansion threshold $x_b$ decreases as we expect, see Figure 6a; the contraction threshold $x_s$ first increases and then decreases, see Figure 6b. In MFG, the prevailing impact of a decreasing growth rate of $x_t^\alpha$ causes the equilibrium price $\rho^*$ to increase, as shown in Figure 6d.

**Impact of $r$.** In the single-player game, as the discount rate $r$ increases, the revenue decays faster as time goes by so it becomes more beneficial to decrease the production and convert into profit directly. Consequently, we can see a significant drop in the contraction threshold in Figure 7b. In MFG, the tendency of decreasing production for each company ultimately drives up the equilibrium price, as shown in Figure 7d.
Figure 5: Impact of $\delta$.

Impact of $\alpha$. $\alpha \in (0, 1)$ measures the elasticity of the profit with respect to the production. Under the single-player setting, both thresholds first increase and then decrease as $\alpha$ approaches 1. In MFG, the more responsive the profit with respect to production, the higher the equilibrium price, as shown in Figure 8d.

4 Conclusion and remarks

This paper analyzes a class of MFGs with singular controls motivated from the partially reversible problem. It establishes the existence of the solution when controls are of bounded velocity, solves explicitly the game when controls are of finite variation, and presents sensitivity analysis to compare the single-player game with the MFG.

Although the focus of the paper is the MFG, it is worth noting that one can establish that the optimal strategy in the solution to the MFG is an $\epsilon$-Nash equilibrium of the corresponding $N$-player game, for both cases of bounded velocity and finite variation. The proof is a slight modification of [GL18] which deals with a finite-time horizon MFG. The only added technical assumption is that the discount rate $r$ is sufficiently large. This assumption is standard in the control literature to ensure the well-posedness for an infinite-time horizon problem.

However, it seems more challenging to consider the reverse relation. Indeed, the issue
of convergence of the $N$-player game to the MFG with regular controls has been studied in [Lac17, Lac18, CR18, NST18]. It will be interesting to explore the case when controls are possibly discontinuous.

Finally, It is well-known that under proper technical conditions, singular controls of finite variation can be approximate by singular controls of bounded velocity. (See for instance the recent work [HHPY16]). More recently, in an $N$-player game setting [DF19] studies the convergence of singular control of bounded velocity to that of finite variation, assuming submodularity of the cost function and under the notion of weak Nash-equilibrium. A curious question is whether the convergence relation holds in a MFG framework, and if so, under what form of equilibrium. This is an intriguing question beyond the scope of this paper.

References

[ADLT18] Yves Achdou, Manh Khang Dao, Olivier Ley, and Nicoletta Tchou. A class of infinite horizon mean field games on networks. arXiv preprint arXiv:1805.11290, 2018.
(a) Expansion threshold under different values of $r$: single-player v.s. SMFG

(b) Contraction threshold under different values of $r$: single-player v.s. SMFG

(c) SMFG optimal thresholds versus $r$

(d) Equilibrium price versus $r$

Figure 7: Impact of $r$.

[Bar11] Martino Bardi. Explicit solutions of some linear-quadratic mean field games. *Networks and heterogeneous media*, 7(2):243–261, 2011.

[BFY13] Alain Bensoussan, Jens Frehse, and Phillip Yam. *Mean Field Games and Mean Field Type Control Theory*. SpringerBriefs in Mathematics. Springer New York, New York, NY, 2013.

[Bil99] Patrick Billingsley. *Convergence of Probability Measures*. John Wiley & Sons, 1999.

[BLP09] Rainer Buckdahn, Juan Li, and Shige Peng. Mean-field backward stochastic differential equations and related partial differential equations. *Stochastic Processes and their Applications*, 119:3133–3154, 2009.

[BSW80] Václav E. Beneš, Larry A. Shepp, and Hans S. Witsenhausen. Some solvable stochastic control problems. *Stochastics*, 4(1):39–83, 1980.

[BW95] Sid Browne and Ward Whitt. Piecewise-linear diffusion processes. In Jewgeni H. Dshalalow, editor, *Advances in Queueing: Theory, Methods, and Open Problems*, chapter 18, pages 463–480. CRC Press, 1995.
(a) Expansion threshold under different values of $\alpha$: single-player v.s. SMFG

(b) Contraction threshold under different values of $\alpha$: single-player v.s. SMFG

(c) SMFG optimal thresholds versus $\alpha$

(d) Equilibrium price versus $\alpha$

Figure 8: Impact of $\alpha$.

[Car10] Pierre Cardaliaguet. Notes on mean field games. Technical report, 2010.

[CD14] René Carmona and François Delarue. The master equation for large population equilibriums. In Dan Crisan, Ben Hambly, and Thaleia Zariphopoulou, editors, Stochastic Analysis and Applications 2014, pages 77–128. Springer International Publishing, 2014.

[CD18a] René Carmona and François Delarue. Probabilistic Theory of Mean Field Games with Applications I: Mean Field FBSDEs, Control, and Games. Springer, 2018.

[CD18b] René Carmona and François Delarue. Probabilistic Theory of Mean Field Games with Applications II: Mean Field Games with Common Noise and Master Equations. Springer, 2018.

[CFS15] René Carmona, Jean-Pierre Fouque, and Li-Hsien Sun. Mean field games and systemic risk. Communications in Mathematical Sciences, 13(4):911–933, 2015.

[CL15] René Carmona and Daniel Lacker. A probabilistic weak formulation of mean field games and applications. The Annals of Applied Probability, 25(3):1189–1231, 2015.
[CLLP12] Pierre Cardaliaguet, Jean-Michel Lasry, Pierre-Louis Lions, and Alessio Porretta. Long time average of mean field games. *Networks and Heterogeneous Media*, 7(2):279–301, 2012.

[CP19] Pierre Cardaliaguet and Alessio Porretta. Long time behavior of the master equation in mean-field game theory. *Analysis & PDE*, 12(6):1397–1453, 2019.

[CR18] Pierre Cardaliaguet and Catherine Rainer. On the (in)efficiency of MFG equilibria. *arXiv preprint arXiv:1802.06637*, 2018.

[CZ16] René Carmona and Xiueng Zhu. A probabilistic approach to mean field games with major and minor players. *The Annals of Applied Probability*, 26(3):1535–1580, 2016.

[DDP94] Avinash K. Dixit, Robert K. Dixit, and Robert S. Pindyck. *Investment under Uncertainty*. Princeton university press, 1994.

[DF19] Jodi Dianetti and Giorgio Ferrari. Nonzero-sum submodular monotone-follower games: existence and approximation of Nash equilibria. 2019.

[EKP+97] Nicole El Karoui, Christophe Kapoudjian, Étienne Pardoux, Shige Peng, and Marie-Claire Quenez. Reflected solutions of backward SDE’s, and related obstacle problems for PDE’s. *The Annals of Probability*, 25(2):702–737, 1997.

[FR75] Wendell H. Fleming and Raymond W. Rishel. *Deterministic and Stochastic Optimal Control*. Springer, 1975.

[GL18] Xin Guo and Joon Seok Lee. Stochastic games and mean field games with singular controls. *arXiv preprint arXiv:1703.04437*, 2018.

[GLL11] Olivier Guéant, Jean-Michel Lasry, and Pierre-Louis Lions. Mean field games and applications. In *Paris-Princeton lectures on mathematical finance 2010*, pages 205–266. Springer Berlin Heidelberg, 2011.

[GP05] Xin Guo and Huyên Pham. Optimal partially reversible investment with entry decision and general production function. *Stochastic Processes and their Applications*, 115(5):705–736, 2005.

[GX19] Xin Guo and Renyuan Xu. Stochastic games for fuel follower problem: N versus mean field game. *SIAM Journal on Control and Optimization*, 57(1):659–692, 2019.

[HHPY16] Daniel Hernández-Hernández, José Luis Pérez, and Kazutoshi Yamazaki. Optimal of refraction strategies for spectrally negative Lévy processes. *SIAM Journal on Control and Optimizations*, 54(3):1126–1156, 2016.

[HMC06] Minyi Huang, Roland P. Malha, and Peter E. Caines. Large population stochastic dynamic games: closed-loop Mckean-Vlasov systems and the Nash certainty equivalence principle. *Communications in Information & Systems*, 6(3):221–252, 2006.
[Lac15] Daniel Lacker. Mean field games via controlled martingale problems: existence of Markovian equilibria. *Stochastic Processes and their Applications*, 125:2856–2894, 2015.

[Lac17] Daniel Lacker. Limit theory for controlled McKean-Vlasov dynamics. *SIAM Journal on Control and Optimization*, 55(3):1641–1672, 2017.

[Lac18] Daniel Lacker. On the convergence of closed-loop Nash equilibria to the mean field game limit. *arXiv preprint arXiv:1808.02745*, 2018.

[LL07] Jean-Michel Lasry and Pierre-Louis Lions. Mean field games. *Japanese Journal of Mathematics*, 2(1):229–260, 2007.

[LZ19] Daniel Lacker and Thaleia Zariphopoulou. Mean field and N-agent games for optimal investment under relative performance criteria. *Mathematical Finance*, 0(0), 2019.

[NST18] Marcel Nutz, Jaime San Martin, and Xiaowei Tan. Convergence to the mean field game limit: a case study. *arXiv preprint arXiv:1806.00817*, 2018.

[Pha09] Huyê̂n Pham. *Continuous-time Stochastic Control and Optimization with Financial Applications*, volume 61. Springer Science & Business Media, 2009.

[PW18] Huyê̂n Pham and Xiaoli Wei. Bellman equation and viscosity solutions for mean-field stochastic control problem. *ESAIM: Control, Optimisation and Calculus of Variations*, 24:437–461, 2018.

[RW00] L. Chris G. Rogers and David Williams. *Diffusions, Markov Processes and Martingales Volume 2: Itô Calculus*. Cambridge University Press, 2000.

[Sma74] David Roger Smart. *Fixed Point Theorems*. Cambridge University Press, London, New York, 1974.

[Zha12] Liangquan Zhang. The relaxed stochastic maximum principle in the mean-field singular controls. *arXiv preprint arXiv:1202.4129*, 2012.