SPECTRAL MAPPING THEOREMS FOR ESSENTIAL SPECTRA AND
NATURAL FUNCTIONAL CALCULUS

JESÚS OLIVA-MAZA

Abstract. Gramsch and Lay [10] gave spectral mapping theorems for the Dunford-Taylor calculus of a closed linear operator $T$,
\[ \tilde{\sigma}_i(f(T)) = f(\tilde{\sigma}_i(T)), \]
for several extended essential spectra $\tilde{\sigma}_i$. In this work, we extend such theorems for the natural functional calculus introduced by Haase [12, 13]. We use the model case of bisectorial operators. The proofs presented here are generic, and are valid for similar functional calculi.

1. Introduction

Through the paper, $X$ will denote an infinite dimensional complex Banach space. Let $\mathcal{L}(X)$, $C(X)$ denote the sets of bounded operators and closed operators on $X$, respectively. For $T \in C(X)$, let $\mathcal{D}(T)$, $\mathcal{R}(T)$, $\mathcal{N}(T)$ denote the domain, range, null space of $T$, respectively. Moreover, we will denote the dimension of the null space or nullity of $T$ by $\text{null}(T)$, and the codimension of the range or defect of $T$ by $\text{def}(T)$. The ascent of $T$, $\alpha(T)$, is the smallest integer $n$ such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$, and the descent of $T$, $\delta(T)$, is the smallest integer $n$ such that $\mathcal{R}(T^n) = \mathcal{R}(T^{n+1})$. Different definitions of essential spectrum have been given in the bibliography in terms of the operator classes below, where we follow the notation and terminology of [9, 10].

\[ \Phi_0 := \{ T \in C(X) \mid \text{null}(T) = \text{def}(T) = 0 \}, \]
\[ \Phi_1 := \{ T \in C(X) \mid \text{null}(T), \text{def}(T) < \infty \}, \]
\[ \Phi_2 := \{ T \in C(X) \mid \text{null}(T) < \infty, \mathcal{R}(T) \text{ complemented} \}, \]
\[ \Phi_3 := \{ T \in C(X) \mid \text{def}(T) < \infty, \mathcal{N}(T) \text{ complemented} \}, \]
\[ \Phi_4 := \{ T \in C(X) \mid \text{null}(T) < \infty, \mathcal{R}(T) \text{ closed} \}, \]
\[ \Phi_5 := \{ T \in C(X) \mid \text{def}(T) < \infty \}, \]
\[ \Phi_6 := \Phi_4 \cup \Phi_5, \]
\[ \Phi_7 := \{ T \in C(X) \mid \text{null}(T) = \text{def}(T) < \infty \}, \]
\[ \Phi_8 := \{ T \in \Phi_7 \mid \alpha(T) = \delta(T) < \infty \}, \]
\[ \Phi_9 := \{ T \in C(X) \mid \alpha(T), \delta(T) < \infty \}, \]
\[ \Phi_{10} := \{ T \in C(X) \mid \mathcal{R}(T) \text{ closed} \}. \]

Respective spectra $\sigma_i(T)$ are defined in terms of the above families, that is $\sigma_i(T) := \{ \lambda \in \mathbb{C} \mid \lambda - T \notin \Phi_i \}$ for $i \in \{0, 1, ..., 10\}$. Note that $\sigma_0(T)$ is the usual spectrum $\sigma(T)$. The set $\sigma_1(T)$ has been called

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essential spectrum by Wolf [20]. The spectra $\sigma_2(T), \sigma_3(T)$ have been examined in the work of Yood [21]. The sets $\sigma_4(T), \sigma_5(T)$ have been referred as essential spectra by Gustafson and Weidmann [11].

The set $\sigma_6(T)$ is called essential spectrum by Kato [15], while Schechter [18] uses $\sigma_7(T)$ and Browder [3] uses $\sigma_8(T)$. The essential spectrum $\sigma_9(T)$ was considered by Gramsch and Lay [10]. Dunford and Schwartz [6] and Goldberg [8] defined the essential spectrum $\sigma_{10}(T)$.

Next we define the extended essential spectra $\overline{\sigma}_{i}(T)$.

**Definition 1.1.** Let $T \in C(X)$. We define

$$
\overline{\sigma}_{i}(T) := \begin{cases}
\sigma_{i}(T) \text{ if } & \begin{align*}
D(T) &= X, \text{ for } i \in \{0, 7, 8\}, \\
\text{codim}(D(T)) &< \infty, \text{ for } i \in \{1, 3, 5\}, \\
D(T) &\text{ closed, for } i \in \{4, 6, 10\}, \\
D(T) &\text{ complemented, for } i = 2, \\
D(T^n) &= D(T^{n+1}) \text{ for some } n \in \mathbb{N}, \text{ for } i = 9,
\end{align*}
\sigma_{i}(T) \cup \{\infty\}, \text{ otherwise.}
\end{cases}
$$

Note that $\overline{\sigma}_0(T)$ is the usual extended spectrum $\overline{\sigma}(T)$. If the resolvent set $\rho(T)$ is not empty, $\overline{\sigma}_i(T)$ coincides with the extended essential spectrum introduced by González and Onieva [9], which satisfies that $\infty \in \overline{\sigma}_i(T)$ if and only if $0 \in \sigma_i((\mu - T)^{-1})$ for any $\mu \in \rho(T)$. In particular, if $T$ has non-empty resolvent set, $\overline{\sigma}_i(T)$ are non-empty compact subsets of $\mathbb{C}_\infty$ except for $i \in \{9, 10\}$ (see [10]), where $\mathbb{C}_\infty$ denotes the Riemann sphere $\mathbb{C} \cup \{\infty\}$. If $T$ has empty resolvent set, $\sigma_i(T)$ is a closed subset of $\mathbb{C}$ for $i \in \{0, 1, 2, 4, 5, 6, 7\}$, see [7] Section I.3 and [21]. We do not know if $\overline{\sigma}_i(T)$ or $\sigma_i(T)$ are closed in the other cases.

Gramsch and Lay [10] and González and Onieva [9] have given spectral mapping theorems, for the Dunford-Taylor calculus of closed operators with non-empty resolvent set, for multiple of the above extended essential spectra, namely

$$
\overline{\sigma}_i(f(T)) = f(\overline{\sigma}_i(T)), \quad i \in \{0, 1, 2, 3, 4, 5, 8\}.
$$

To define $f(T)$ through the Dunford-Taylor calculus, the function $f$ has to be holomorphic in an open set containing the extended spectrum $\overline{\sigma}(T)$ of $T$. In particular, $f$ must be holomorphic in a neighbourhood of $\infty$ if $T$ is unbounded. Delaubenfels [5] and Cowling, Doust, McIntosh and Yagi [4] extended the Dunford-Taylor calculus to include functions $f$ which may not be holomorphic at a finite subset $M_T$ of $\overline{\sigma}(T) \cap \rho(T)$, as long as $f$ has ‘good enough’ behaviour at $M_T$ and the norm of the resolvent $\|(z - T)^{-1}\|$ satisfies suitable bounds in neighbourhoods of $M_T$. For instance, if $T$ is a sectorial operator, then $M_T = \{0, \infty\}$ and these calculi include functions such as $z^\beta$ ($\beta > 0$) or $\log z$. Furthermore, Haase [12] extended these functional calculi, via a regularization method, to a natural functional calculus which includes meromorphic functions $f$ such that their set of poles does not intersect the point spectra $\sigma_p(T)$ of $T$.

The contribution of this paper is to extend the spectral mapping theorems (1.1) to the setting of natural functional calculus. This was done by Haase [13] for $i = 0$, a suitable function $f$, and a sectorial operator $T$. In this work, we will use the model case for bisectorial operators for illustration. This is partly motivated by the fact that the holomorphic functional calculus for bisectorial operators has been used and studied by many authors due to its importance in the field of abstract inhomogeneous differential equations, see for instance [14, 2, 16, 17]. However, the proofs presented here are generic, and are valid for similar natural functional calculi, see the end of Section 2.

The paper is organized as follows. The natural functional calculus for bisectorial operators is detailed in Section 2. In Section 3 we give the spectral mapping theorems for a bisectorial operator $A$ with no singular points, i.e. $M_A = \emptyset$, see Proposition 3.5. We deal with the singular points in Section 4 where
we give the spectral mapping theorems for any bisectorial operator $A$ and suitable functions $f$, see Theorem 4.7. We give some final remarks in Section 5 such as a slight improvement of the spectral mapping theorem for sectorial operators given by Haase [13].

2. Function spaces and natural functional calculus

Given any $\varphi \in (0, \pi)$, we denote the sector $S_\varphi := \{ z \in \mathbb{C} : |\arg(z)| < \varphi \}$. For any $\omega \in (0, \pi/2]$ and $a \geq 0$, we set the bisector

$$BS_{\omega,a}(\varphi) = \left\{ (-a + S_{\pi-\omega}) \cap (a - S_{\pi-\omega}) : \omega, a \right\}$$

\text{if } \omega < \pi/2 \text{ or } a > 0,

\text{if } \omega = \pi/2 \text{ and } a = 0.

Definition 2.1. Let $(\omega, a) \in (0, \pi/2] \times [0, \infty)$ and let $A \in C(X)$. We will say that $A$ is a bisectorial operator of angle $\omega$ and half-width $a$ if the following conditions hold:

- $\sigma(A) \subset BS_{\omega,a}$.
- For all $\omega' \in (0, \omega)$, $A$ satisfies the resolvent bound

$$\sup \left\{ \min \{ |\lambda - a|, |\lambda + a| \} \| (\lambda - A)^{-1} \| : \lambda \notin BS_{\omega,a}(\omega') \right\} < \infty.$$

We also set $M_A := \sigma(A) \cap \{-a, a, \infty\}$. For the rest of the paper, $(\omega, a)$ will always denote a pair in $(0, \pi/2] \times [0, \infty)$.

Given a Banach space $X$, we will denote the set of all bisectorial operators on $X$ of angle $\omega$ and half-width $a$ in $X$ by $B\text{Sect}(\omega, a)$. We will omit an explicit mention to $X$ for the sake of simplicity. Notice that $A \in B\text{Sect}(\omega, a)$ if and only if both $a + A, -A$ are sectorial of angle $\pi - \omega$ in the sense of [14].

We denote by $O(\Omega), M(\Omega)$ the sets of holomorphic functions and meromorphic functions defined in an open subset $\Omega \subset \mathbb{C}$, respectively. For $A \in B\text{Sect}(\omega, a)$, let $U_A := \{-a, a, \infty\} \setminus M_A$. If $\sigma(A) \neq \emptyset$, set

$$r_d := \begin{cases} \operatorname{dist}\{d, \sigma(A)\}, & \text{if } d \in \{-a, a\}, \\ r(A)^{-1}, & \text{if } d = \infty, \end{cases} \quad d \in U_A,$$

where $\operatorname{dist}\{\cdot, \cdot\}$ denotes the distance between two sets, and $r(A)$ the spectral radius of $A$. If $\sigma(A) = \emptyset$ (so $\sigma(A) = \{ \infty \}$ and $\infty \notin U_A$), set $r_a = r_{-a} := \infty$.

For any $\varphi \in (0, \omega)$ and $\{(d, s_d)\}_{d \in U_A}$ with $0 < s_d < r_d$, set $\Omega(\varphi, \{(d, s_d)\}_{d \in U_A})$ as follows. If $U_A = \emptyset$ (i.e., $M_A = \{-a, a, \infty\}$), we set $\Omega(\varphi) := B\text{Sect}(\omega, a)$. Otherwise, for each $d \in U_A$, let $B(d, s_d)$ be a ball centred at $d$ of radius $s_d$, where $B_{\infty}(r_{\infty}) = \{ z \in \mathbb{C} : |z| > r_{\infty}^{-1} \}$. Then, we set $\Omega(\varphi, \{(d, s_d)\}_{d \in U_A}) := B\text{Sect}(\omega, a) \setminus \bigcup_{d \in U_A} B(d, s_d)$. Note that, if $\varphi < \varphi' < \omega$ and $s_d < s_d' < r_d$ for each $d \in U_A$, then the inclusion $\Omega(\varphi, \{(d, s_d)\}_{d \in U_A}) \subset \Omega(\varphi', \{(d, s_d')\}_{d \in U_A})$ holds. Thus we can form the inductive limits

$$O[\Omega_A] := \bigcup \left\{ O(\Omega(\varphi, \{(d, s_d)\}_{d \in U_A})) : \varphi < \omega, 0 < s_d < r_d \text{ for } d \in U_A \right\},$$

$$M[\Omega_A] := \bigcup \left\{ M(\Omega(\varphi, \{(d, s_d)\}_{d \in U_A})) : \varphi < \omega, 0 < s_d < r_d \text{ for } d \in U_A \right\}.$$

Hence, $O[\Omega_A], M[\Omega_A]$ are algebras of holomorphic functions and meromorphic functions (respectively) defined on an open set containing $\sigma(A) \subset M_A$. Next, we define the following notion of regularity at $M_A$. 

Definition 2.2. Let \( f \in \mathcal{M}[\Omega_A] \). We say that \( f \) is regular at \( d \in \{-a, a\} \cap M_A \) if \( \lim_{z \to d} f(z) =: c_d \in \mathbb{C} \) exists and, for some \( \varphi \in (0, \omega) \)
\[
\int_{\partial \text{BSect}(\varphi', \varphi(\omega), a)} \left| \frac{f(z) - c_d}{z - d} \right| |dz| < \infty, \quad \text{for some } \varepsilon > 0 \text{ and for all } \varphi' \in \left( \varphi, \frac{\pi}{2} \right).
\]
If \( \infty \in M_A \), we say that \( f \) is regular at \( \infty \) if \( \lim_{z \to \infty} f(z) =: c_\infty \in \mathbb{C} \) exists and
\[
\int_{\partial \text{BSect}(\varphi', \varphi(\omega), a)} \left| \frac{f(z) - c_\infty}{z} \right| |dz| < \infty, \quad \text{for some } R > 0 \text{ and for all } \varphi' \in \left( \varphi, \frac{\pi}{2} \right).
\]
We say that \( f \) is quasi-regular at \( d \in M_A \) if \( f \) or \( 1/f \) is regular at \( d \). Finally, we say that \( f \) is (quasi-)regular at \( M_A \) if \( f \) is (quasi-)regular at each point of \( M_A \).

Remark 2.3. Note that if \( f \) is regular at \( M_A \) with every limit being not equal to 0, then \( 1/f \) is also regular at \( M_A \). If \( f \) is quasi-regular at \( M_A \), then \( \mu - f \) and \( 1/f \) are also quasi-regular at \( M_A \) for each \( \mu \in \mathbb{C} \). A function \( f \) which is quasi-regular at \( M_A \) has well defined limits in \( \mathbb{C}_\infty \) as \( z \) tends to each point of \( M_A \).

Next, let \( \mathcal{E}(A) \) be the subset of functions of \( \mathcal{O}[\Omega_A] \) which are regular at \( M_A \). Then
\[
\mathcal{E}(A) = \mathcal{E}_0(A) + \mathbb{C} \frac{1}{b+z} + \mathbb{C} \frac{1}{b-z} + \mathbb{C} 1,
\]
for any \( b \in \mathbb{C} \setminus \text{BSect}(\varphi', \varphi(\omega), a) \), where \( 1 \) is the constant function with value 1, and
\[
\mathcal{E}_0(A) := \left\{ f \in \mathcal{O}[\Omega_A] : f \text{ is regular at } M_A \text{ with } \lim_{z \to d \in M_A} f(z) = 0 \right\}.
\]
Given a bisectorial operator \( A \in \text{B Sect}(\omega, a) \), we define the algebraic homomorphism \( \Phi : \mathcal{E}(A) \to \mathcal{L}(X) \) given by
\[
\Phi\left( \frac{1}{z-a} \right) = (b + A)^{-1}, \quad \Phi\left( \frac{1}{z-a} \right) = (b - A)^{-1}, \quad \Phi(1) = I,
\]
and
\[
\Phi(f) := f(A) := \frac{1}{2\pi i} \int_{\Gamma} f(z)(z-A)^{-1} \, dz, \quad f \in \mathcal{E}_0(A),
\]
where \( \Gamma \) is the positively oriented boundary of \( \Omega(\varphi', \{(d, s_d') \}_{d \in U_A}) \) with \( \varphi < \varphi' < \omega \) and \( s_d < s_d' < r_d \), where \( f \in \mathcal{O}(\Omega(\varphi', \{(d, s_d') \}_{d \in U_A})) \). Note that the above integral is well-defined in the Bochner sense since \( \int_{\Gamma} |f(z)||\left| (z-A)^{-1} \right||dz| < \infty \). It is readily seen that both \( \mathcal{E}(A) \) and \( f(A) \) are well defined (where \( f \in \mathcal{E}(A) \)) and that they do not depend on the election of \( \varphi', \{(d, s_d') \}_{d \in U_A} \) or \( b \).

Next, we follow the regularization method given in [12] to extend the functional calculus \( \Phi \) to a natural functional calculus (also denoted by \( \Phi \)), which involves meromorphic functions.

Definition 2.4. Let \( a \geq 0, 0 < \omega \leq \frac{\pi}{2} \) and \( A \in \text{B Sect}(\omega, a) \). Then, a function \( f \in \mathcal{M}[\Omega_A] \) is said regularizable by \( \mathcal{E}(A) \) if there exists \( e \in \mathcal{E}(A) \) such that
\[
\bullet \ e(A) \text{ is injective},
\]
\[
\bullet \ e f \in \mathcal{E}(A).
\]
For any regularizable \( f \in \mathcal{M}[\Omega_A] \) with regularizer \( e \in \mathcal{E}(A) \), we set
\[
\Phi(f) := f(A) := e(A)^{-1}(ef)(A).
\]

By [12 Lemma 3.2], one has that this definition is independent of the regularizer \( e \), and that \( f(A) \) is a well defined closed operator. We will denote by \( \mathcal{M}(A) \) to the subset of functions of \( \mathcal{M}[\Omega_A] \) which are regularizable by \( \mathcal{E}(A) \). This natural functional calculus satisfies the following properties.

Lemma 2.5. Let \( A \in \text{B Sect}(\omega, a) \) and \( f \in \mathcal{M}(A) \). Then
Moreover, setting
\( f \) and let
by making
\( \phi \),
\[ \{ g \} \]
Lemma 2.7. [13, Proposition 3.1] for the analogous result for sectorial operators.

Remark
identity, and [12, Section 3]. □

Proof. The statement follows by straightforward applications of the Cauchy’s theorem, the resolvent regularizer here since we will need it in the proof of Theorem 5.3.

Proof. The proof is the same as in the case of sectorial operators, see [13, Lemma 6.2]. We include it here since we will need it in the proof of Theorem 5.3.

Let \( f \in M[\Omega_A] \) be as required, so \( f \in M(\Omega(\varphi, \{ (d, s_d) \}_{d \in U_A})) \) for some \( 0 < \varphi < \omega \) and \( 0 < s_d < r_d \) for each \( d \in U_A \). Since \( f \) has finite limits at \( M_A \), we can assume that \( f \) has only finitely many poles by making \( \varphi, \{ s_d \}_{d \in U_A} \) bigger. Thus, let \( \lambda_j \) for \( j \in \{ 1, ..., N \} \) be an enumeration of those poles of \( f \) and let \( n_j \in \mathbb{N} \) be the order of pole of \( f \) located at \( \lambda_j \), for \( j \in \{ 1, ..., N \} \). Then, the function
\[ g(z) := f(z) \prod_{j=1}^{N} \frac{(\lambda_j - z)^{n_j}}{(b - z)^{n_j}} \]
has no poles, i.e. \( g \in O[\Omega_A] \), and is regular at \( M_A \). Hence \( g \in \mathcal{E}(A) \).

Moreover, setting \( r(z) := \prod_{j=1}^{N} \frac{(\lambda_j - z)^{n_j}}{(b - z)^{n_j}} \), one has that \( r(A) = \prod_{j=1}^{N} \frac{(\lambda_j - A)^{n_j}}{(b - A)^{n_j}} \) is bounded and injective, since by assumption \( \{ \lambda_1, ..., \lambda_n \} \subset \sigma_p(A) \). In short, \( f \) is regularized by \( r \), so \( f \in M(A) \).

Now, assume that the poles of \( f \) lie inside \( \rho(A) \). Then the operator \( r(A) \) is not only bounded and injective, but invertible too, from which follows that \( f(A) = r(A)^{-1}(rf(A)) \in \mathcal{L}(X) \).

Lemma 2.8. Remark 2.6, Lemma 2.7 and Lemma 2.8 are the only properties of the natural functional calculus of bisectorial operators that are used in the proofs given in Sections 3 and 4. Therefore, for any other natural functional calculus satisfying such properties, one can obtain the spectral mapping theorems for the essential spectra given in Theorem 4.7, see for instance Theorem 5.4 and Theorem 5.5.
3. Spectral mapping theorems for $M_A = \emptyset$

For $A \in \text{BSect}(\omega, a)$, the spectral mapping theorems \( (11) \) given in \cite{1} \cite{II} are applicable to any $f \in \mathcal{E}(A)$ whenever $M_A = \emptyset$. This section is devoted to extend these spectral mapping theorems for all $f \in \mathcal{M}(A)$, assuming that $M_A = \emptyset$.

First, we proceed to state the spectral mapping inclusion of the spectrum $\tilde{\sigma}$ for bisectorial operators.

**Proposition 3.1.** Let $A \in \text{BSect}(\omega, a)$, $f \in \mathcal{M}(A)$, and assume that $f$ is quasi-regular at $M_A$. Then

\[
\tilde{\sigma}(f(A)) \subset f(\tilde{\sigma}(A)).
\]

**Proof.** The proof runs the same as in the case for sectorial operators, see \cite{13} Proposition 6.3. As in Lemma 2.8, we include the proof here since it will be needed in the proof of Theorem 5.3.

The proof runs the same as in the case for sectorial operators, see \cite{13} Proposition 6.3. As in Lemma 2.8, we include the proof here since it will be needed in the proof of Theorem 5.3.

Next, we give some technical lemmas which will be useful to obtain the main results of this section.

**Lemma 3.2.** Let $A \in \text{BSect}(\omega, a)$, $\varepsilon \in \mathcal{M}(A)$ with $e(A) \in \mathcal{L}(X)$ injective, $\lambda, b \in \mathbb{C}$ with $b \in \rho(A)$. Assume that there is $c \in \mathbb{C} \setminus \{0\}$ such that

\[
f(z) := \frac{b - z}{\lambda - z} (e(z) - c) \in \mathcal{M}(A) \quad \text{with} \ f(A) \in \mathcal{L}(X).
\]

Then $\mathcal{R}(\lambda - A) = \mathcal{R} \left( \frac{\lambda - A}{b - A} e(A)^{-1} \right) = \mathcal{R} \left( e(A)^{-1} \frac{\lambda - A}{b - A} \right)$.

**Proof.** Note that $\mathcal{R}(\lambda - A) = \mathcal{R} \left( \frac{\lambda - A}{b - A} \right) = \mathcal{R} \left( \frac{\lambda - A}{b - A} e(A)^{-1} \right)$ since $e(A)^{-1}$ is surjective. The inclusion $\frac{\lambda - A}{b - A} e(A)^{-1} \subset e(A)^{-1} \frac{\lambda - A}{b - A}$ implies that $\mathcal{R} \left( \frac{\lambda - A}{b - A} e(A)^{-1} \right) \subset \mathcal{R} \left( e(A)^{-1} \frac{\lambda - A}{b - A} \right)$. Thus, all is left to prove is the reverse inclusion.

Let $u \in \mathcal{R} \left( e(A)^{-1} \frac{\lambda - A}{b - A} \right)$, so there is $x \in X$ such that $e(A) u = \frac{\lambda - A}{b - A} x$. Since $e(z) = \frac{\lambda - z}{b - z} f(z) + c$, one has that $u = \frac{1}{c} \frac{\lambda - A}{b - A} (x - f(A) u)$ so $u \in \mathcal{R}(\lambda - A) = \mathcal{R} \left( \frac{\lambda - A}{b - A} e(A)^{-1} \right)$, and the claim follows. \( \square \)

**Lemma 3.3.** Let $f \in \mathcal{M}(A)$ and $\lambda \in \sigma(A) \setminus M_A$ with $f(\lambda) = 0$, and let $b \in \rho(A)$. If $g(z) := \frac{b - z}{\lambda - z} f(z)$, then $g \in \mathcal{M}(A)$, $D(g(A)) = D(f(A))$ and

\[
f(A) = \frac{\lambda - A}{b - A} g(A) = g(A) \frac{\lambda - A}{b - A}.
\]

**Proof.** Let $e \in \mathcal{E}(A)$ be a regularizer for $f$ with $e(\lambda) \neq 0$, see Lemma 2.7. The fact that $eg$ has the same behaviour as $ef$ at $M_A$ implies that $eg \in \mathcal{E}(A)$, that is, $e$ is a regularizer for $g$ and $g \in \mathcal{M}(A)$, so $g(A)$ is well defined.

On one hand, it follows by Lemma 2.5 (3) that $f(A) = g(A) \frac{\lambda - A}{b - A} \supset \frac{\lambda - A}{b - A} g(A)$ with $D \left( \frac{\lambda - A}{b - A} g(A) \right) = D(f(A)) \cap D(g(A))$. But by the definition of composition of closed operators, $D \left( \frac{\lambda - A}{b - A} g(A) \right) = D(g(A)) \cap g^{-1} \left( D \left( \frac{\lambda - A}{b - A} \right) \right) = D(g(A))$ since $D \left( \frac{\lambda - A}{b - A} \right) = X$. As a consequence, $D(g(A)) \subset D(f(A))$. \( \square \)
Next, let \( x \in \mathcal{D}(f(A)) \) and set \( \bar{x} := (eg)(A)x \). One has

\[
f(A) = e(A)^{-1} \left( \frac{\lambda - z}{b - z} (eg)(z) \right)(A) = e(A)^{-1} \frac{\lambda - A}{b - A} (eg)(A),
\]
so \( \bar{x} \in \mathcal{D} \left( e(A)^{-1} \frac{\lambda - A}{b - A} \right) \). An application of Lemma 3.2 with \( c = e(\lambda) \neq 0 \) shows that there is \( v \in \mathcal{R}(e(A)) \) with \( \frac{\lambda - A}{b - A} e(A)^{-1} v = e(A)^{-1} \frac{\lambda - A}{b - A} \bar{x} \). By composing with \( e(A) \) one gets that \( \bar{x} - v \in \mathcal{N} \left( \frac{\lambda - A}{b - A} \right) = \mathcal{N}(\lambda - A) \). Moreover, \( \mathcal{N}(\lambda - A) \subset \mathcal{R}(e(A)) \) since \( y = \frac{1}{e(\lambda)} e(A)y \) for any \( y \in \mathcal{N}(\lambda - A) \), see Remark 2.6 Hence, \( \bar{x} = (\bar{x} - v) + v \in \mathcal{R}(e(A)) \), that is \( x \in \mathcal{D}(g(A)) \), so \( \mathcal{D}(g(A)) = \mathcal{D}(f(A)) \). Lemma \( 2.3 ) \) shows that

\[
f(A) = \frac{\lambda - A}{b - A} g(A) = g(A) \frac{\lambda - A}{b - A},
\]
and the commutativity property follows.

\[\square\]

**Remark 3.4.** Let \( T \in C(X) \) with non-empty resolvent set, and \( \alpha(T), \delta(T) < \infty \). Then \( \alpha(T) = \delta(T) := p_T \) and \( X = \mathcal{N}(T^p) \oplus \mathcal{R}(T^p) \), see for example [19] Theorem V.6.2].

Lemma below is inspired by [10] Lemma 5).

**Lemma 3.5.** Let \( S, T \in C(X) \) with non-empty resolvent set and such that \( ST = TS \). One has the following

(a) If \( S, T \in \Phi_9 \), then \( ST \in \Phi_9 \). If \( \mathcal{R}(T) \subset \mathcal{D}(S) \) and \( S, T \in \Phi_8 \), then \( ST \in \Phi_8 \).

(b) Assume that \( T \) is injective and \( \mathcal{D}(S) \subset \mathcal{R}(T) \). If \( ST \in \Phi_i \), then \( S \in \Phi_i \) for \( i = \{8, 9\} \).

**Proof.** (a) Let \( S, T \in \Phi_9 \). By Remark 3.4 \( \alpha(S) = \delta(S) = p_S \), \( \alpha(T) = \delta(T) = p_T \), and \( X = \mathcal{R}(S^p) \oplus \mathcal{N}(S^p) = \mathcal{R}(T^p) \oplus \mathcal{N}(T^p) \). Let \( P_S \) be the projection onto \( \mathcal{N}(S^p) \) along \( \mathcal{R}(S^p) \), \( Q_S := I - P_S \), and set the analogous projections \( P_T, Q_T \). Since \( ST = TS \), one has that \( \mathcal{N}(ST^n) \subset \mathcal{D}(S) \), \( \mathcal{N}(S^n) \subset \mathcal{D}(T) \) for any \( n \in \mathbb{N} \), and that \( P_S, Q_S, P_T, Q_T \) commute between themselves. Then \( Q := Q_T Q_S \) is a bounded projection onto \( \mathcal{R}(S^p) \cap \mathcal{R}(T^p) \), and it is readily seen that \( ST \) is a (possibly unbounded) invertible operator when restricted to \( Q(X) \). Since \( I - Q = P_S + P_T - P_S P_T \), it is clear that \( (I - Q)(X) \subset \mathcal{N}(ST)^{\text{max}(p_S, p_T)} \). Then, \( ST \in \Phi_9 \) with \( \alpha(ST) = \delta(ST) \leq \text{max}(p_S, p_T) \) \( < \infty \) by [19] Problem V.6.

If in addition, \( S, T \in \Phi_8 \subset \Phi_7 \), with \( \mathcal{R}(T) \subset \mathcal{D}(S) \), then \( ST \in \Phi_7 \), see [7] Theorem 3.16 (although this result is stated for bounded operators, its proof is purely algebraic). Hence, we conclude that \( ST \in \Phi_8 \).

(b) It follows by induction that \( \mathcal{D}(S^n) \subset \mathcal{R}(T^n) \) for \( n \in \mathbb{N} \). Since \( (ST)^n = S^n T^n = T^n S^n \) and \( T \) is injective, one has that \( \alpha(ST) = \alpha(S) \) and \( \delta(ST) = \delta(S) \), and the claim follows (note that \( ST \in \Phi_1 \) implies that \( S, T \in \Phi_1 \)).

**Remark 3.6.** Let \( T \in C(X) \) with non-empty resolvent set, and let \( \lambda \in \rho(T) \) and \( \lambda \in \mathbb{C} \). \( \lambda - T \in \Phi_i \) if and only if \( \frac{\lambda - T}{\lambda - T} \in \Phi_i \) for \( i = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \), see for example [19] Lemma 1).

**Lemma 3.7.** Let \( A \in \text{Bsect}(\omega, a) \), \( f, g \in \mathcal{M}(A) \) with \( f, g \) quasi-regular at \( M_A \), \( 0 \notin g(\overline{\sigma(A)}) \) and such that

\[
f(z) := g(z) \prod_{j=1}^{N} \left( \frac{\lambda_j - z}{b - z} \right)^{n_j},
\]
for some \( b \in \rho(A) \), \( \lambda_j \in \sigma(A) \setminus M_A \), and \( n_j \in \mathbb{N} \) for \( j = 1, ..., N \). Then

(a) If \( f(A) \in \Phi_i \), then \( \lambda_j - A \in \Phi_i \) for all \( j = 1, ..., N \) and for \( i = \{0, 1, 2, 3, 4, 5, 6, 8, 9\} \).
(b) If \( \lambda_j - A \in \Phi_i \) for all \( j = 1, ..., N \), then \( f(A) \in \Phi_i \) for \( i \in \{0, 1, 2, 3, 4, 5, 7, 8, 9\} \).

Proof. Set \( r(z) := \prod_{j=1}^{N} \left( \frac{\lambda_j - z}{b - z} \right)^{n_j} \), so \( r(A) \in L(X) \). Several applications of Lemma 3.3 imply that \( D(g(A)) = D(f(A)) \) and \( f(A) = r(A)g(A) \). Moreover, Proposition 3.1 yields that \( 0 \notin \bar{\sigma}(g(A)) \), so \( g(A) \) is surjective and injective. Thus \( g(A) : D(g(A)) \rightarrow X \) is an isomorphism when \( D(f(A)) \) is endowed with the graph norm given by \( f(A) \) (which is equivalent to the graph norm given by \( g(A) \)). Therefore, \( f(A) \in \Phi_i \) if and only if \( r(A) \). This follows by the very definition of \( \Phi_i \) for all \( i \in \{0, 1, 2, 3, 4, 5, 7, 8, 9\} \).

Since the bounded operators \( \frac{\lambda_j - A}{b - A} \) commute between themselves, we have that

1. If \( r(A) \in \Phi_i \), then \( \frac{\lambda_j - A}{b - A} \in \Phi_i \) for all \( j = 1, ..., N \), and for \( i \in \{0, 1, 2, 3, 4, 5, 6, 8, 9\} \),
2. If \( \frac{\lambda_j - A}{b - A} \in \Phi_i \) for all \( j = 1, ..., N \), then \( r(A) \in \Phi_i \), for \( i \in \{0, 1, 2, 3, 4, 5, 7, 8, 9\} \).

see for example [9] Lemma 3] and [10] Lemma 5(c)]. Hence, the claim follows from Remark 3.9. 

We give below the main result of this section.

**Proposition 3.8.** Let \( A \in B\text{Sect}(\omega, a) \), \( f \in M(A) \), where \( f \) is quasi-regular at \( M_A \). Then

(a) \( f(\bar{\sigma}_i(A)) \setminus f(M_A) \subset \bar{\sigma}_i(f(A)) \) for \( i \in \{0, 1, 2, 3, 4, 5, 6, 8, 9\} \).

(b) \( \bar{\sigma}_i(f(A)) \subset f(\bar{\sigma}_i(A)) \cup f(M_A) \) for \( i \in \{0, 1, 2, 3, 4, 5, 7, 8, 9\} \).

Proof. Take \( i \in \{0, 1, 2, 3, 4, 5, 6, 8, 9\} \) and let \( \mu \in \mathbb{C} \) be such that \( \mu \notin f(\bar{\sigma}_i(A)) \setminus f(M_A) \). By considering the function \( f - \mu \) instead of \( f \), we can assume without loss of generality that \( \mu = 0 \). As \( 0 \notin f(M_A) \), \( f^{-1}(0) \cap \bar{\sigma}(A) \) must be finite. Let \( \{\lambda_j\}_{j=1,...,N} \) be a numeration of \( f^{-1}(0) \cap \bar{\sigma}(A) \) (so \( \lambda_j \in \bar{\sigma}(A) \) for some \( j \in \{1, ..., N\} \)), and let \( n_j \) be the order of the zero of \( f \) at \( \lambda_j \). Let \( b \in \rho(A) \) and set

\[
g(z) := f(z) \prod_{j=1}^{N} \left( \frac{b - z}{\lambda_j - z} \right)^{n_j}.
\] (3.1)

Then \( 0 \notin g(\bar{\sigma}(A)) \) and is \( g \) quasi-regular at \( M_A \). Several applications of Lemma 3.3 imply that \( g \in M(A) \), and Lemma 3.7(a) yields that \( f(A) \notin \Phi_i \).

Take now \( i \in \{0, 1, 2, 3, 4, 5, 7, 8, 9\} \) and let \( \mu \in \mathbb{C} \) be such that \( \mu \notin f(\bar{\sigma}_i(A)) \cup f(M_A) \), and we will prove that \( \mu \notin \bar{\sigma}_i(f(A)) \). We can assume that \( \mu = 0 \). Again, \( f^{-1}(0) \cap \sigma(A) \) has finite cardinal, so let \( g \) be as given in (3.1). Since \( \lambda_j - A \in \Phi_i \) for all \( j = 1, ..., n \), applications of Lemma 3.3 and Lemma 3.7(b) yield that \( f(A) \in \Phi_i \), as we wanted to show.

Assume now that \( \mu = \infty \). If \( \rho(f(A)) \neq \emptyset \) take \( b \in \rho(f(A)) \). An application of what we have already proven to the function \( \frac{1}{b - f(z)} \) shows the claim, see the paragraph below Definition 3.1. Hence, all is left to prove is that we can assume without loss of generality that \( \rho(f(A)) \neq \emptyset \). Take \( \nu \in \mathbb{C} \setminus f(M_A) \), so \( f^{-1}(\nu) \cap \bar{\sigma}(A) \) has finite cardinal. Let \( \{\nu_j\}_{j=1,...,M} \) be a numeration of \( f^{-1}(\nu) \cap \sigma(A) \), and let \( m_j \) be the order of the zero of \( f - \nu \) at \( \nu_j \). Let \( b \in \rho(A) \) and set

\[
h(z) := (f(z) - \nu) \prod_{j=1}^{M} \left( \frac{b - z}{\nu_j - z} \right)^{m_j}.
\] (3.2)

Lemma 3.3 yields that \( h \in M(A) \) with \( D(f(A)) = D(h(A)) \), and using (3.2) it is readily seen that \( D(f(A)^n) = D(h(A)^n) \) for all \( n \in \mathbb{N} \). In particular, \( \infty \in \bar{\sigma}_i(f(A)) \) if and only if \( \infty \in \bar{\sigma}_i(h(A)) \). Since
which are contained in \( \rho \) and the interior of \( \Gamma \) respectively. It is easy to see that \( d \neq 0 \), then Remark 4.1(c) implies that \( \rho(f(A)) \neq 0 \), and the proof is done.

4. Complete spectral mapping theorems

In this section we deal with the case \( M_A \neq \emptyset \). The difficulty of this setting arises from the fact that \( f \) is not necessarily either holomorphic or meromorphic at \( M_A \), so the factorization techniques used in Section 3 do not apply here.

First, we give some remarks about \( M_A \) which will be the key for the proof of the spectral mapping theorems.

Remark 4.1. Let \( T \in C(X) \) with non-empty resolvent set, \( d \in \sigma(T) \) with \( d \) an accumulation point of \( \rho(T) \), and \( i \in \{1, 2, 3, 4, 5, 6, 7, 8\} \). The following statements about the essential spectrum are well-known, see for example [7] Sections I.3 & I.4, [15] Chapter 4S5 and [19] Section V.6.

(a) If \( d \) is also an accumulation point of \( \sigma(T) \), then \( d \in \sigma_i(T) \).

(b) If \( d \in \sigma_i(T) \) and \( d \) is not an accumulation point of \( \sigma_i(T) \), then \( d \) may either be an isolated point of \( \sigma_i(T) \), or an accumulation point of \( \sigma(T) \) of at most a countable set of eigenvalues with finite dimensional eigenspace, which are isolated between themselves.

(c) If \( d \notin \sigma_i(T) \), then \( d \) is an isolated point of \( \sigma(T) \). Moreover, \( d \in \sigma_\rho(T) \) with \( \operatorname{mul}(d - T) = \operatorname{def}(d - T) < \infty \), \( \alpha(d - T) = \delta(d - T) < \infty \), and \( \dim(\bigcup_{n \geq 1} N((d - T)^n)) < \infty \).

Lemma 4.2. Let \( A \in \operatorname{BSect}(\omega, a) \), \( d \in M_A \) and \( i, j \in \{1, 2, 3, 4, 5, 6, 7, 8\} \). Then

- \( d \in \sigma_i(A) \) if and only if \( d \in \sigma_j(A) \),
- \( i \in \sigma(A) \) then \( i \in \sigma_i(A) \).

Proof. If \( d \in \sigma_\omega(A) \), then \( d \in \sigma_i(A) \) since \( \sigma_\omega(A) \subset \sigma_i(A) \) for any \( i \in \{1, 2, 3, 4, 5, 6, 7, 8\} \). If \( d \notin \sigma_\omega(A) \), then Remark 4.1(c) implies that \( d \notin \sigma_i(A) \) for \( i \in \{1, 2, 3, 4, 5, 6, 7, 8\} \), and the claim follows.

We will need the following two lemmas. For a Jordan curve \( \Gamma \), let \( \operatorname{ext}(\Gamma) \), \( \operatorname{int}(\Gamma) \) denote the exterior and the interior of \( \Gamma \) respectively.

Lemma 4.3. Let \( A \in \operatorname{BSect}(\omega, a) \), let \( \Gamma \) be a finite collection of smooth non-intersecting Jordan curves which are contained in \( \rho(A) \). Set

\[
Q := \frac{1}{2\pi i} \int_{\Gamma} (z - A)^{-1} \, dz, \quad P := I - Q.
\]

Then \( P, Q \) are bounded projections on \( \mathcal{R}(P), \mathcal{R}(Q) \) respectively such that \( \mathcal{N}(\lambda - A) \subset \mathcal{R}(Q) \subset \mathcal{D}(A) \) and \( \mathcal{R}(P) \subset \mathcal{R}(\lambda - A) \) for any \( \lambda \in \operatorname{int}(\Gamma) \).

In addition, for \( Y \in \{\mathcal{R}(P), \mathcal{R}(Q)\} \) and \( B := A|_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y} \), one has the following

(a) \( B \in \operatorname{BSect}(\omega, a) \) with \( \mathcal{M}(A) \subset \mathcal{M}(B) \). Moreover,

\[
f(B) := f(A)|_{\mathcal{Y}}, \quad \sigma_i(f(B)) \subset \sigma_i(f(A)), \quad f \in \mathcal{M}(A), \ i \in \{0, 1, 2, 3, 4, 5, 6, 9, 10\}.
\]

(b) \( \operatorname{ext}(\Gamma) \subset \rho(B) \) if \( Y = \mathcal{R}(Q) \) and \( \operatorname{int}(\Gamma) \subset \rho(B) \) if \( Y = \mathcal{R}(P) \).

Proof. It is easy to see that \( Q, P \) are complementary bounded projections, see [19] Section V.9. Let \( Y \in \{\mathcal{R}(P), \mathcal{R}(Q)\} \) and set \( B := A|_{\mathcal{Y}} \). Since both \( P, Q \) commute with \( A \), then they commute with \( f(A) \) for any \( f \in \mathcal{M}(A) \). In particular, \( AY \subset Y \) and \( (z - B)^{-1} = (z - A)^{-1} |_{\mathcal{Y}} \) for any \( z \in \rho(A) \), so \( \rho(A) \subset \rho(B) \) and it easily follows that \( B \in \operatorname{BSect}(\omega, a) \) as an operator on \( Y \). Using again the commutativity property between \( f(A) \) and \( P, Q \), one shows that \( f(A)|_{\mathcal{Y}} = f(B) \) for any \( f \in \mathcal{E}(A) \). Thus, if \( e \in \mathcal{E}(A) = \mathcal{E}(B) \) is a regularizer for \( f \in \mathcal{M}(A) \), then \( e \) is also a regularizer for \( f \) with respect
to $B$, so $\mathcal{M}(A) \subset \mathcal{M}(B)$. By the commutativity property again, it is readily seen that $f(B) = f(A)|_Y$, $\mathcal{D}(f(B)) = \mathcal{D}(f(A)) \cap Y$, $\mathcal{N}(f(B)) = \mathcal{N}(f(A)) \cap Y$, $\mathcal{R}(f(B)) = \mathcal{R}(f(A)) \cap Y$ for any $f \in \mathcal{M}(A)$. Hence $\tilde{\sigma}_i(f(B)) \subset \tilde{\sigma}_i(f(A))$ for $i \in \{0, 1, 4, 5, 6, 9, 10\}$ by the very definition of $\Phi_i$ (note that $Y$ is a closed subspace since it is the image of a bounded projection).

For $i = 2$, assume that $f(A) \in \Phi_2$, i.e $\mathcal{R}(f(A)) \oplus T = X$ for some closed linear subspace $T$. Let $Z$ denote the complement of $Y$ through $P$ if $Y = \mathcal{R}(P)$ or through $Q$ if $Y = \mathcal{R}(Q)$. Then $\mathcal{R}(f(B)) \oplus (\mathcal{R}(f(A)) \cap Z) \oplus T = X$ since $\mathcal{R}(f(A)) = \mathcal{R}(f(B)) \oplus (\mathcal{R}(f(A)) \cap Z)$. Then there exists a bounded projection $P$ from $X$ onto $\mathcal{R}(f(B))$, so it suffices to restrict $P$ to $Y$ to obtain a bounded projection from $Y$ to $\mathcal{R}(f(B))$, hence $\mathcal{R}(f(B))$ is complemented in $Y$. Same reasoning with $\mathcal{D}(f(A))$, $\mathcal{D}(f(B))$ instead of $\mathcal{R}(f(A))$, $\mathcal{R}(f(B))$ shows that, if $\infty \in \tilde{\sigma}_2(f(B))$, then $\infty \in \tilde{\sigma}_2(f(A))$. Therefore, the inclusion $\tilde{\sigma}_2(f(B)) \subset \tilde{\sigma}_2(f(A))$ holds for any $f \in \mathcal{M}(A)$.

Similar reasoning proves the claim for $i = 3$. For $i = 9$, it follows from $\alpha(f(A)) = \max\{\alpha(f(B)), \alpha(f(A)|_Z)\}$ and $\delta(f(A)) = \max\{\delta(f(B)), \delta(f(A)|_Z)\}$, see [19 Problem V.6].

Part (b) is a well-known result of spectral projections, see for instance [19 Corollary V.9.3]. We include a proof of it for the sake of completeness. Set first $Y = \mathcal{R}(Q)$. It is readily seen that the integral of the definition of $Q$ can be regarded as a Bochner integral in the Banach space $\mathcal{D}(A)$ endowed with the graph norm $\|x\|_{\mathcal{D}(A)} = \|x\|_X + \|Ax\|_X$. Thus $Y \subset \mathcal{D}(A)$. It is straightforward that $Qx = x$ for any $x \in \mathcal{N}(\lambda - A)$ with $\lambda \in \text{int}(\Gamma)$. Let us prove that $\tilde{\sigma}(B) \subset \text{int}(\Gamma)$. First, $\infty \notin \tilde{\sigma}(B)$ since $Y \subset \mathcal{D}(A)$, so $\mathcal{D}(B) = Y$. Second, assume that there is a $\mu \in \sigma(B) \cap \text{ext}(\Gamma)$. Then, there is a continuous character $\Lambda$ defined on the bicommutant of $B$ such that $\Lambda(B) = \mu$. However, for any $R > \|B\|$ such that $\Gamma \subset B_{B(R)}(0)$, one has that

$$1 = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{1}{z - \mu} dz = \frac{1}{2\pi i} \int_{\Gamma_R \cup -\Gamma} \frac{1}{z - \mu} dz$$

$$= \frac{1}{2\pi i} \int_{\Gamma_R \cup -\Gamma} \Lambda((z - B)^{-1}) dz = \Lambda \left( \frac{1}{2\pi i} \int_{\Gamma_R \cup -\Gamma} (z - B)^{-1} dz \right)$$

$$= \mu(0) = 0,$$

where $\Gamma_R = \partial B_{B(R)}(0)$ and we used that $I|_Y = Q|_Y = \frac{1}{2\pi i} \int_{\Gamma_R} (z - B)^{-1} dz = \frac{1}{2\pi i} \int_{\Gamma_R} (z - B)^{-1} dz$. Since we get a contradiction, it follows that $\tilde{\sigma}(B) \subset \text{int}(\Gamma)$.

Let now $Y = \mathcal{R}(P)$. For each $\mu \in \text{int}(\Gamma)$, one can check that

$$(\mu - B)^{-1} = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{\mu - z} (z - A)^{-1} f|_Y dz.$$

As a consequence, $\tilde{\sigma}(B) \subset \text{ext}(\Gamma)$ and $\lambda - B$ is surjective for any $\lambda \in \text{int}(\Gamma)$, thus $Y \subset \mathcal{R}(\lambda - A)$, and the proof is finished.

Let $A \in \text{Bsect}(\omega, a)$. By Remark 4.1(c), one can construct a finite collection of non-intersecting Jordan curves $\Gamma$ such that $\text{int}(\Gamma) \cap \tilde{\sigma}(A) = M_A \setminus \tilde{\sigma}_i(A)$ for any $i \in \{1, 2, 3, 4, 5, 6\}$. Recall that $M_A \setminus \tilde{\sigma}_i(A) = M_A \setminus \tilde{\sigma}_j(A)$ for any $i, j \in \{1, 2, ..., 6\}$, see Lemma 4.2.

**Lemma 4.4.** Let $A, \Gamma$ be as above, and let $P, Q$ be the bounded projections associated to $\Gamma$ as in Lemma 4.3. Set $Y := \mathcal{R}(P)$ and $B := A|_Y$. Then $M_B \subset \tilde{\sigma}_i(B)$ for $i \in \{0, 1, 2, 3, 4, 5, 6\}$ and $\tilde{\sigma}_i(f(A)) = \tilde{\sigma}_i(f(B)), \quad f \in \mathcal{M}(A), \quad i \in \{1, 2, 3, 4, 5, 6, 10\}$.

**Proof.** The inclusions $\tilde{\sigma}_i(f(B)) \subset \tilde{\sigma}_i(f(A))$ are given in Lemma 4.3(a). We will prove the inclusions $\sigma_i(f(A)) \subset \sigma_i(f(B))$ if we prove the following claims for any $f \in \mathcal{M}(A)$,

1. If $\text{null}(f(B)) < \infty$, then $\text{null}(f(A)) < \infty$.
2. If $\text{def}(f(B)) < \infty$, then $\text{def}(f(A)) < \infty$.
(3) If $\mathcal{R}(f(B))$ is closed/complemented, then $\mathcal{R}(f(A))$ is closed/complemented.

(4) If $\mathcal{N}(f(B))$ is complemented, then $\mathcal{N}(f(A))$ is complemented.

Let $Z = \mathcal{R}(I - P) = \mathcal{R}(Q)$, so $X = Y \oplus Z$. An application of Lemma 4.3(b) implies that $\tilde{\sigma}_i(A|_Z) = \emptyset$ for any $i \in \{1, 2, 3, 4, 5, 6\}$, hence $Z$ is finite dimensional and $\text{codim}Y < \infty$. Thus, the claims (1) and (2) are straightforward since $\mathcal{N}(f(B)) = \mathcal{N}(f(A)) \cap Y$ and $\mathcal{R}(f(B)) = \mathcal{R}(f(A)) \cap Y$, see the proof of Lemma 4.3. For the claim about closeness in (3), assume that $\mathcal{R}(f(B))$ is closed in $Y$, so $\mathcal{R}(f(B))$ is closed in $X$ too. Since $\mathcal{R}(f(A)) = \mathcal{R}(f(B)) \oplus \mathcal{R}(f(A)|_Z)$, we have that $\mathcal{R}(f(A))/\mathcal{R}(f(B))$ is finite-dimensional in $X/\mathcal{R}(f(B))$, hence closed. Thus $\mathcal{R}(f(A))$ is closed in $X$. For the claim about complementation in (3), assume that $\mathcal{R}(f(B)) \oplus U = Y$ for some closed subspace $U$. We also have that $\mathcal{R}(f(A)|_Z) \oplus V = Z$ for some closed subspace since $\text{dim}Z < \infty$. Therefore $\mathcal{R}(f(A)) \oplus (U \oplus V) = X$, and the claim follows. An analogous reasoning proves the claim (4).

Similar steps replacing $\mathcal{R}(f(A)), \mathcal{R}(f(B))$ by $\mathcal{D}(f(A)), \mathcal{D}(f(B))$ show that, if $\infty \in \tilde{\sigma}_i(f(A))$, then $\infty \in \tilde{\sigma}_i(f(B))$ for $i \in \{1, 2, 3, 4, 5, 6, 10\}$. Therefore $\tilde{\sigma}_i(f(A)) \subset \tilde{\sigma}_i(f(B))$ as we wanted to show.

To finish the proof, note that $M_A \setminus \tilde{\sigma}_i(A) \subset \rho(B)$ by Lemma 4.3(b), so $M_B \subset \tilde{\sigma}_i(A)$ for $i \in \{1, 2, 3, 4, 5\}$. And by what we have already proven, $\tilde{\sigma}_i(A) = \tilde{\sigma}_i(A)$ for $i \in \{1, 2, 3, 4, 5, 6\}$. \hfill $\square$

We are now ready to prove the spectral mapping theorems for most of the extended essential spectra considered in the Introduction. For the sake of clarity, we separate the proof of each inclusion into two different propositions.

**Proposition 4.5.** Let $A \in \text{Bsect}(\omega, a)$ and let $f \in \mathcal{M}(A)$ be quasi-regular at $M_A$. Then

$$\tilde{\sigma}_i(f(A)) \subset f(\tilde{\sigma}_i(A)), \quad i \in \{0, 1, 2, 3, 4, 5, 7, 8\}.$$  

**Proof.** Let us show the claim for $i \in \{0, 1, 2, 3, 4, 5\}$ first. An application of Lemma 4.2 yields that we can assume that $M_A \subset \tilde{\sigma}_i(A)$ without loss of generality. In this case, if $\mu \notin f(\tilde{\sigma}_i(A))$, then $\mu \notin f(\tilde{\sigma}_i(A)) \cup f(M_A)$, so $\mu \notin f(\tilde{\sigma}_i(f(A))$ by Proposition 3.8(b). Thus $f(\tilde{\sigma}_i(f(A)) \subset f(\tilde{\sigma}_i(A))$ for $i \in \{0, 1, 2, 3, 4, 5\}$.

Next, let $i \in \{7, 8\}$ and $\mu \in \mathbb{C}$ with $\mu \notin f(\tilde{\sigma}_i(A))$. If $\mu \notin f(M_A)$, it suffices to apply Proposition 3.8(b) to obtain the claim. Thus, let $\mu \in f(M_A)$. We can assume without loss of generality that $\mu = 0$. As $0 \notin f(\tilde{\sigma}_i(A))$, an application of Lemma 4.2 together with Remark 4.1(c) yields that $f^{-1}(0) \cap \tilde{\sigma}_i(A)$ is a finite set. Let $\lambda_1, ..., \lambda_N$ be an enumeration of $(f^{-1}(0) \cap \tilde{\sigma}_i(A)) \setminus M_A$, and let $n_1, ..., n_N$ be their multiplicities. Set $r(z) := \sum_{j=1}^{N} \left( \frac{\lambda_j - z}{b - z} \right)^{n_j}$ for some $b \in \rho(A)$, and $g(z) = f(z)/r(z)$.

Several applications of Lemma 5.7 yield that $g \in \mathcal{M}(A)$ and $f(A) = r(A)g(A) = g(A)r(A)$ with $\mathcal{D}(f(A)) = \mathcal{D}(g(A))$. Then $g^{-1}(0) \cap \tilde{\sigma}_i(A) \subset M_A \setminus \tilde{\sigma}_i(A)$, so we can consider the spectral projections $P, Q$ as in Lemma 4.4 with $Y := \mathcal{R}(P), Z := \mathcal{R}(Q)$. As stated in the proof of Lemma 4.4, one has $\text{dim}Z < \infty$, so $f(A)|_Z \subset \Phi_i$. Moreover, since $M_A \setminus \tilde{\sigma}_i(A) \cap \tilde{\sigma}_i(A|_Y) = \emptyset$, then $0 \notin g(\tilde{\sigma}_i(A|_Y))$, so Proposition 3.1 yields that $g(A)|_Y$ is invertible. As a consequence, $g(A) \in \Phi_i$. Furthermore, $r(A)$ is a bounded operator and belongs to $\Phi_i$. Therefore, $f(A) = r(A)g(A) = g(A)r(A) \in \Phi_i$ by Theorem 3.16 for $i = 7$, and by Lemma 4.3 for $i = 8$, that is $0 \notin \tilde{\sigma}_i(f(A))$.

Next, let $i \in \{0, 1, 2, 3, 4, 5, 7, 8\}$ and assume that $\infty \notin f(\tilde{\sigma}_i(A))$. Reasoning as at the end of the proof of Proposition 3.8, we can assume without loss of generality that $\rho(f(A)) \neq \emptyset$, so let $\nu \in \rho(f(A))$. An application of what we have already proven to the function $1/\nu - f(z)$ shows that $0 \notin \tilde{\sigma}_i \left( \frac{1}{\nu - f(A)} \right)$, that is $\infty \notin \tilde{\sigma}_i(f(A))$ for $i \in \{0, 1, 2, 3, 4, 5, 7, 8\}$, and the proof is finished. \hfill $\square$

**Proposition 4.6.** Let $A \in \text{Bsect}(\omega, a)$ and let $f \in \mathcal{M}(A)$ be quasi-regular at $M_A$. Then

$$f(\tilde{\sigma}_i(A)) \subset f(\tilde{\sigma}_i(f(A))), \quad i \in \{0, 1, 2, 3, 4, 5, 6, 8\}.$$
Proof. Let $\mu \in f(\tilde{\sigma}(A))$ with $\mu \neq \infty$, so we take $\mu = 0$ without loss of generality. If $0 \in f(\tilde{\sigma}(A)) \setminus f(M_A)$, then the claim follows from Proposition 3.3(a). Then, assume that $0 \in f(\tilde{\sigma}(A))$ with $0 \in f(M_A)$. Since $\tilde{\sigma}_0(f(A)) \subset \tilde{\sigma}(f(A))$ for each $i \in \{0, 1, 2, 3, 4, 5, 6, 8\}$, an application of Lemma 3.2 yields that it is enough to prove the claim for $i \in \{0, 6\}$. Hence, we assume $i \in \{0, 6\}$ from now on. If any point in $f^{-1}(0) \cap \tilde{\sigma}(A)$ is an accumulation point of $\tilde{\sigma}(A)$ (and we rule out the trivial case where $f$ is constant), then $0$ is an accumulation point of $f(\tilde{\sigma}(A)) \setminus f(M_A) \subset \tilde{\sigma}(f(A))$ (see Proposition 3.3(a)), thus $0 \in \tilde{\sigma}(f(A))$ since $\sigma(T)$ is closed for any $T \in C(X)$. So assume that each point in $f^{-1}(0) \cap \tilde{\sigma}(A)$ is an isolated point in $\tilde{\sigma}(A)$, and set

$$V_A := \{d \in f^{-1}(0) \cap \tilde{\sigma}(A) \mid d \text{ is not an isolated point of } \tilde{\sigma}(A)\},$$

which is a finite set by Remark 4.1(c).

Assume first that $V_A$ is not empty (thus $i = 6$). One has that, for each $d \in V_A$, there is some neighbourhood $\Omega_d$ of $d$ such that $\Omega_d \setminus \tilde{\sigma}(A) = \{d, \lambda_{j_1}^d, \lambda_{j_2}^d, \ldots\}$, where $\lambda_{j_d}^d \in \sigma_p(A) \setminus \sigma_i(A)$, each $\lambda_{j_d}^d$ is an isolated point of $\sigma(A)$, and $\lambda_{j_d}^d \xrightarrow{j \to \infty} d$. Thus, the $(f^{-1}(0) \setminus \tilde{\sigma}(f(A))) \setminus (\cup_{d \in V_A} \Omega_d)$ is finite, namely $\{\kappa_j\}_{j=1,\ldots,N}$. Let $n_j$ be the multiplicity of the zero of $f$ at $\kappa_j$ for $j = 1, \ldots, N$, and set
g(z) := f(z) \prod_{j=1}^{N} \left(\frac{b - z}{\kappa_j - z}\right)^{n_j}. \tag{4.1}

Several applications of Lemma 3.3 yield that $g \in \mathcal{M}(A)$ with $\mathcal{D}(g(A)) = \mathcal{D}(f(A))$,

$$f(A) = \left(\prod_{j=1}^{N} \left(\frac{\kappa_j - A}{b - A}\right)^{n_j}\right)^* g(A) \left(\prod_{j=1}^{N} \left(\frac{\kappa_j - A}{b - A}\right)^{n_j}\right),$$

where in the last term we regard $\frac{\kappa_j - A}{b - A}$ as bounded operators on $\mathcal{D}(f(A))$. Let us show that $0 \in \tilde{\sigma}_0(g(A))$, which will imply that $0 \in \tilde{\sigma}_0(f(A))$, see for example [27, Theorem I.3.20]. Note that $g^{-1}(0) \cap \tilde{\sigma}(A) \subset \cup_{d \in V_A} \Omega_d$, which is a countable set. Hence, points in $g^{-1}(0) \cap \tilde{\sigma}_0(A)$ are either isolated points in $\tilde{\sigma}(A)$ or accumulation points of at most a countable set of points of $\tilde{\sigma}(A)$. Thus, the spectral mapping inclusion (Proposition 3.1) implies that $0$ is an accumulation point of $\rho(g(A))$. If $0$ is also an accumulation point of $\tilde{\sigma}(g(A))$, then $0 \in \tilde{\sigma}_0(g(A))$ by Remark 4.1. So assume that $0$ is not an accumulation point of $\tilde{\sigma}(g(A))$. Since $\sigma_p(g(A)) \subset g(\sigma_p(A))$ (Remark 2.6), and $\lambda_{j_d}^d \in \sigma_p(A)$ with $\lambda_{j_d}^d \xrightarrow{j \to \infty} d$ for each $d \in V_A$, it follows that $g(\lambda_{j_d}^d) = 0$ for all but finitely many pairs $(j, d) \in \mathbb{N} \times V_A$.

Hence, the set $g^{-1}(0) \cap \sigma_p(A)$ has infinite cardinal, so $\text{null}(g(A)) \geq \sum_{\lambda \in g^{-1}(0) \cap \sigma_p(A)} \text{null}(\lambda - A) = \infty$. Then Remark 4.1(c) yields that $0 \in \tilde{\sigma}_0(g(A))$, as we wanted to prove.

Now, assume that $V_A = \emptyset$, so each $d \in f^{-1}(0) \cap \tilde{\sigma}(A)$ is an isolated point of $\tilde{\sigma}(A)$, and let $d \in f^{-1}(0) \cap M_A$. If $d = \infty$, set $\Gamma = \partial B_0(R)$ for $R > 0$ big enough so $\sigma(A) \subset B_0(R)$, take the bounded projections $P, Q$ associated to $\Gamma$ as in Lemma 4.3 and set $Z := \mathcal{R}(P)$ and $B := A|Z$, so $Z$ is a closed subspace. Then $\dim Z \geq \text{codim}(\mathcal{D}(A))$, so $Z \neq \emptyset$ since $\infty \in \tilde{\sigma}(A)$. Moreover, $\text{codim}(\mathcal{D}(A)) = \infty$ if $i = 6$, so $\dim Z = \infty$ in this case. Another application of Lemma 4.3(b) yields that $\tilde{\sigma}(B) \subset \tilde{\sigma}(A) \cap \text{ext}(\Gamma) = \{\infty\}$, thus $\tilde{\sigma}(B) = \{\infty\}$ since $\tilde{\sigma}(B)$ cannot be empty. But then $\tilde{\sigma}(f(B)) \subset \{0\}$ by Proposition 3.1, thus $\tilde{\sigma}(f(B)) = \{0\}$ since $\tilde{\sigma}(f(B))$ cannot be empty either (at least for any operator with non-empty resolvent set, see for example [10]). Since by Lemma 4.3(a) $\tilde{\sigma}_i(f(B)) \subset \tilde{\sigma}(f(B))$, it follows that $0 \in \tilde{\sigma}_i(f(B))$, as we wanted to show.

Next, if $d \neq \infty$ then $d \in \{-a, a\}$, so there is $r > 0$ such that $\tilde{\sigma}(A) \cap B_r(d) = \{d\}$. Define now $\Gamma = \partial B_r(d)$ for any $\delta \in (0, r)$, take $P, Q$ be the bounded projections associated to $\Gamma$ as in Lemma 4.3 with $Z := \mathcal{R}(Q)$ and $B := A|Z$, so $Z$ is a closed subspace. Then $\dim Z \geq \max\{\text{null}(d-A), \text{def}(d-A)\}$,
Thus \( Z \neq \{0\} \) since \( d \in \bar{\sigma}_i(A) \). In particular, \( \dim Z = \infty \) if \( i = 6 \). Reasoning as in paragraph above, we conclude that \( 0 \in \bar{\sigma}_i(f(A)) \).

Finally, we deal with the case \( \mu = \infty \). Reasoning as at the end of the proof of Proposition 5.8 we can assume that \( \rho(f(A)) \neq \emptyset \). Take any \( \nu \in \rho(f(A)) \), so \( \infty \in \bar{\sigma}_i(f(A)) \) if and only if \( 0 \in \bar{\sigma}_i \left( \frac{1}{\nu - f(A)} \right) \).

But \( 0 \in \bar{\sigma}_i \left( \frac{1}{\nu - f(A)} \right) \) by applying what we have already proven to the function \( \frac{1}{\nu - f(z)} \), and the proof is finished.

\[ \square \]

As a consequence, we have the following

**Theorem 4.7.** Let \( A \in \text{BSect}(\omega, a) \) and \( f \in \mathcal{M}(A) \) quasi-regular at \( M_A \). Then

\[
\bar{\sigma}_i(f(A)) = f(\bar{\sigma}_i(A)), \quad i \in \{0, 1, 2, 3, 4, 5, 8\},
\]

\[
f(\bar{\sigma}_6(A)) \subset \bar{\sigma}_6(f(A)),
\]

\[
\bar{\sigma}_7(f(A)) \subset f(\bar{\sigma}_7(A)).
\]

**Proof.** Immediate consequence of Proposition 4.5 and Proposition 4.6. \( \square \)

It is well known that the spectral mapping theorem does not hold (in general) for \( \bar{\sigma}_9, \bar{\sigma}_7, \bar{\sigma}_10 \). However, we do not know if it holds for \( \bar{\sigma}_9 \) for the natural functional calculus. Indeed, it holds if \( M_A = \emptyset \), see Proposition 5.8.

5. **Final remarks**

5.1. **Operators with bounded functional calculus.** A natural question is whether the condition of quasi-regularity can be relaxed and still obtain the spectral mapping theorems given in Theorem 4.7. A possible candidate for this relaxed condition could be to ask for \( f \) to have well-defined limits at \( M_A \). To prove such a spectral mapping theorem, we ask the operator \( A \) to fulfil the following property, which is studied in [11] [14] [17].

**Definition 5.1.** Let \( A \in \text{BSect}(\omega, a) \). We say that the natural calculus of \( A \) is bounded if \( f(A) \in \mathcal{L}(X) \) for every bounded \( f \in \mathcal{M}(A) \).

**Lemma 5.2.** Let \( A \in \text{BSect}(\omega, a) \) and \( f \in \mathcal{M}(A) \). Then \( f \) is regular at \( \sigma_p(A) \cap M_A \).

**Proof.** The proof is analogous to the case of sectorial operators, see [12] Lemma 4.2. \( \square \)

**Theorem 5.3.** Let \( A \in \text{BSect}(\omega, a) \) such that the natural functional calculus of \( A \) is bounded, and let \( f \in \mathcal{M}(A) \) with (possibly \( \infty \)-valued) limits at \( M_A \). Then

\[
\bar{\sigma}_i(f(A)) = f(\bar{\sigma}_i(A)), \quad i \in \{0, 1, 2, 3, 4, 5, 8\},
\]

\[
f(\bar{\sigma}_6(A)) \subset \bar{\sigma}_6(f(A)),
\]

\[
\bar{\sigma}_7(f(A)) \subset f(\bar{\sigma}_7(A)).
\]

**Proof.** The proof of this claim is completely analogous to the path followed in this paper to prove Theorem 4.7. Indeed, the quasi-regularity notion is only explicitly needed in the proofs of Lemma 2.8 and Proposition 5.1. This creates a ‘cascade’ effect, and all following results need the quasi-regularity assumption in order to apply Proposition 5.1. Therefore, we will prove the claim once we prove the following version of Proposition 5.1.

“Let \( A \in \text{BSect}(\omega, a) \) such that the natural functional calculus of \( A \) is bounded, and let \( f \in \mathcal{M}(A) \) with (possibly \( \infty \)-valued) limits at \( M_A \). Then \( \bar{\sigma}(f(A)) \subset f(\bar{\sigma}(A)) \).”
We outline the proof of this claim. Let \( \mu \in \mathbb{C}_\infty \) with \( \mu \notin \mathcal{f}(\sigma) \), and set \( f_\mu = \frac{1}{\mu - f} \) if \( \mu \in \mathbb{C} \) or \( f_\mu = f \) if \( \mu = \infty \), and we will show that \( f_\mu \in \mathcal{M}(A) \) with \( f_\mu(A) \in \mathcal{L}(X) \). Note that \( f_\mu \) has finite limits at \( M_A \). Even more, \( f_\mu \) is regular at \( \sigma_p(A) \cap M_A \) by Lemma 5.2. Proceeding as in the proof of Lemma 2.8 we can assume that \( f_\mu \) has finitely many poles, all of them contained in \( \rho(A) \). Let \( r(z) := \prod_{j=1}^{n} \frac{(\lambda_j - z)^{n_j}}{(b - z)^{n_j}} \), where \( \lambda_j, n_j \) are the poles of \( f_\mu \) and their order, respectively. Hence, \( rf_\mu \) has no poles, is regular at \( \sigma_p(A) \cap M_A \) and has finite limits at \( M_A \), thus \( rf_\mu \) is bounded. For any \( b \in \rho(A) \), the function \( \lambda(z) := \frac{1}{b - z} \prod_{d \in \{ -a, a \} \setminus \sigma_p(A)} \frac{z - a}{b - z} \) regularizes \( rf_\mu \), so \( rf_\mu \in \mathcal{M}_A \). Since the NFC of \( A \) is bounded, then \( rf_\mu(A) \in \mathcal{L}(X) \). Moreover, \( r(A) \) is bounded and invertible. Therefore, \( rf \) regularizes \( f_\mu \) with \( f_\mu(A) = r(A)^{-1}(rf_\mu)(A) \in \mathcal{L}(X) \), and the claim follows.

5.2. Spectral mapping theorems for sectorial operators. As stated at the end of Section 2, the spectral mapping theorems for essential spectra given in Theorem 4.7 are valid for similar natural functional calculi. In particular, one has the following result for the natural functional calculus of sectorial operators and the natural functional calculus of strip operators considered in [13].

**Theorem 5.4.** Let \( A \) be a sectorial operator of angle \( \phi \in [0, 2\pi) \), and let \( f \) be a function in the natural functional calculus of \( A \) such that \( f \) is quasi-regular at \( \{0, \infty\} \cap \sigma(A) \). Then

\[
\begin{align*}
\overline{\sigma}_i(f(A)) &= f(\overline{\sigma}_i(A)), & i & \in \{0, 1, 2, 3, 4, 5, 8\}, \\
n f(\overline{\sigma}_0(A)) &\subset \overline{\sigma}_0(f(A)), \\
\overline{\sigma}_7(f(A)) &\subset f(\overline{\sigma}_7(A)).
\end{align*}
\]

**Theorem 5.5.** Let \( A \) be a strip operator of height \( h \geq 0 \), and let \( f \) be a function in the natural functional calculus of \( A \) such that \( f \) is quasi-regular at \( \{\infty\} \cap \sigma(A) \). Then

\[
\begin{align*}
\overline{\sigma}_i(f(A)) &= f(\overline{\sigma}_i(A)), & i & \in \{0, 1, 2, 3, 4, 5, 8\}, \\
n f(\overline{\sigma}_0(A)) &\subset \overline{\sigma}_0(f(A)), \\
\overline{\sigma}_7(f(A)) &\subset f(\overline{\sigma}_7(A)).
\end{align*}
\]

5.3. A spectral mapping theorem for the point spectrum. To finish this paper, we give a spectral mapping theorem for the point spectrum. To prove it, we need to restrict to a smaller class of functions. This is inspired by [13 Corollary 6.6].

**Proposition 5.6.** Let \( A \in \text{Bsect}(\omega, a) \) and \( f \in \mathcal{M}(A) \) such that \( f \) is quasi-regular at \( M_A \). Then

\[ f(\sigma_p(A)) \subset \sigma_p(f(A)) \subset f(\sigma_p(A)) \cup f(M_A). \]

Assume furthermore that, for any \( d \in M_A \) such that \( f(d) \notin f(\sigma_p(A)) \cup \{\infty\} \), there is some \( \beta > 0 \) for which

- if \( d \in \mathbb{C} \), then \( |f(z) - c_d| \gtrsim |z - d|^\beta \) as \( z \to d \), or
- if \( d = \infty \), then \( |f(z) - c_d| \gtrsim |z|^{-\beta} \) as \( z \to d \),

where \( c_d \) denotes the limit of \( f(z) \) as \( z \to d \). Then, one has that

\[ f(\sigma_p(A)) = \sigma_p(f(A)). \]

**Proof.** The proof of the inclusions \( f(\sigma_p(A)) \subset \sigma_p(f(A)) \subset f(\sigma_p(A)) \cup f(M_A) \) runs the same as for sectorial operators, see [13 Proposition 6.5]. Regarding the second statement, all that is left to prove is that if \( \mu \in f(M_A) \setminus f(\sigma_p(A)) \), then \( \mu \notin \sigma_p(f(A)) \). The statement is trivial if \( \mu = \infty \), so assume
that $\mu \in \mathbb{C}\setminus f(\sigma_p(A))$, and consider the function $g := \frac{1}{z-a}$, which is quasi-regular at $M_A$. Note that poles of $g$ are precisely $f^{-1}(\mu) \subset \mathbb{C}\setminus \sigma_p(A)$. Moreover, $g$ is regular at $M_A \cap \sigma_p(A)$, since by assumption $\mu \notin f(\sigma_p(A))$. Let now

$$
    h_{l,m,n}(z) := \frac{(z-a)^m(z+a)^n}{(b-z)^{l+m+n}}, \quad z \in \mathbb{C}, \quad l, m, n \in \mathbb{N}, \quad b > a.
$$

Then, by the assumptions made on $f$, $h_{l,m,n}g$ is regular for some $m, n, l$ large enough, and where $l, m, n = 0$ if $\infty, -a \notin \sigma(A) \setminus \sigma_p(A)$ respectively. Since $h_{l,m,n}(A) = (A-a)^m(A+a)^nR(b, A)^{m+n+l}$ is bounded and injective, $h_{l,m,n}$ regularizes $g$, so $g \in \mathcal{M}(A)$, which by Lemma 2.5(4) implies that $\mu - f$ is injective, as we wanted to show.

\[ \square \]

References

[1] W. Arendt and M. Duelli. Maximal $\ell^p$-regularity for parabolic and elliptic equations on the line. J. Evol. Equ., 6(4):773, 2006.
[2] W. Arendt and A. Zamboni. Decomposing and twisting bisectorial operators. Studia Math., 3(197):205–227, 2010.
[3] F.E. Browder. On the spectral theory of elliptic differential operators. I. Math. Ann., 142(1):22–130, 1961.
[4] M. Cowling, I. Doust, A. McIntosh, and A. Yagi. Banach space operators with a bounded polynomially bounded resolvents. J. Funct. Anal., 114(2):348–394, 1993.
[5] R. Delaubenfels. Unbounded holomorphic functional calculus and abstract Cauchy problems for operators with polynomially bounded resolvents. J. Funct. Anal., 114(2):348–394, 1993.
[6] N. Dunford and J.T. Schwartz. Linear operators. II. Interscience Publishers, New York, 1963.
[7] D.E. Edmunds and W.D. Evans. Spectral theory and differential operators. Oxford University Press, Oxford/New York, 1987.
[8] S. Goldberg. Unbounded linear operators: theory and applications. McGraw-Hill, New York, 1966.
[9] M. González and V.M. Onieva. On the spectral mapping theorem for essential spectra. Publ. Sec. Mat. Univ. Autònoma Barcelona, pages 105–110, 1985.
[10] B. Gramsch and D. Lay. Spectral mapping theorems for essential spectra. Math. Ann., 192(1):17–32, 1971.
[11] K. Gustafson and J. Weidmann. On the essential spectrum. J. Math. Anal. Appl., 25(1):121–127, 1969.
[12] M. Haase. A general framework for holomorphic functional calculi. Proc. Edinb. Math. Soc., 48(2):423–444, 2005.
[13] M. Haase. Spectral mapping theorems for holomorphic functional calculi. J. London Math. Soc., 71(3):723–739, 2005.
[14] M. Haase. The functional calculus for Sectorial operators, volume 169. Oper. Theory Adv. Appl., Birkhäuser, Basel, 2006.
[15] T. Kato. Perturbation theory for linear operators. Springer-Verlag, Berlin, 1966.
[16] Alexander Mielke. Über maximaler p-regularität für differentialgleichungen in banach-und hilbert-räumen. Math. Ann., 277(1):121–133, 1987.
[17] A.J. Morris. Local quadratic estimates and the holomorphic functional calculus. In The AMSI-ANU Workshop on Spectral Theory and Harmonic Analysis, volume 44, pages 211–231. Proc. Centre Math. Appl. Austral. Nat. Univ., 2010.
[18] M. Schecter. On the essential spectrum of an arbitrary operator. I. J. Math. Anal. Appl., 13(2):205–215, 1966.
[19] A.E. Taylor and D.C. Lay. Introduction to Functional Analysis. Wiley & sons, New York, 1958.
[20] F. Wolf. On the invariance of the essential spectrum under a change of the boundary conditions of partial differential operators. Indag. Math., 21:142–147, 1959.
[21] B. Yood. Properties of linear transformations preserved under addition of a completely continuous transformation. Duke Math. J., 18(3):599–612, 1951.

Departamento de Matemáticas, Instituto Universitario de Matemáticas y Aplicaciones, Universidad de Zaragoza, 50009 Zaragoza, Spain

Email address: joliva@unizar.es