General Methods of Quantum Field Theory

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We employ the polar form to investigate all possible classes spinor fields from a perspective that does not involve the use of plane waves. Results can be taken as the starting point for a procedure of field quantization that can be developed in the most general of circumstances.

I. INTRODUCTION

In the theoretical framework of Quantum Field Theory, one of the most important elements is the Dirac spinorial field, describing spin-1/2 charged massive particles.

Despite its rather ubiquitous presence in physics, the Dirac spinor represents only half of the full space of spinor fields that can be encompassed within the exhaustive categorization given by Lounesto [1]. In the Lounesto classification, which is based on the role of the spinor bi-linear covariant quantities (those containing the pure physical information), spinorial fields are divided into two major classes, whether they are regular (first three classes) or singular (last three classes). The Dirac field embeds half of the whole variety because it extinguishes the three regular classes. As for the three singular classes, they can be further split into the fourth, accounting for general singular spinors called flag-dipoles, with the fifth, accounting for the special case of singular spinors called flagpoles, and the sixth, accounting for the special case of singular spinors called dipoles. Elements of the fifth class are the self-conjugated Majorana spinors while elements of the sixth class are the single-handed Weyl spinor. For more details about spinor classification we refer to [2–9].

Following Weinberg [10], the construction of the Dirac theory (the Dirac spinors and later the associated quantum field) can be understood as the union of two irreducible representations (particles and anti-particles), invariant by transformations of the Lorentz group, specifically belonging to the proper orthochronous group, which are transformed by spatial inversion (in other words, parity) operation) as shown in [11]. In this context, Weinberg proves a no-go theorem stating the suppressed existence of specific types of spin-1/2 particles holding parity invariance. Although never stated as such initially, such a formulation is quite exhaustive. Recent developments showing its consistency were scrutinized in reference [12]. However, a judicious inspection of the spinorial dual structure suffices to circumvent Weinberg no-go theorem [13]. This mathematical program shows the possibility to construct a new variety of spin-1/2 spinors with mass-dimension 1 [13–21], ensuring local quantum fields.

In such a quantum field theoretical construction, the role played by plane waves is prominent. In fact, all computations are normally performed by taking fields to undergo the plane-wave condition \(i \nabla \psi = P_{\nu} \psi [22]\) as the result of the condition \([\sigma^\mu, P_{\nu}] = i \delta^\nu_\mu\) needed to quantize the system and eventually made manifestly covariant.

These conditions, however ubiquitous, are not encountered, nevertheless, in very general situations. The treatment of regular spinors in plane waves, for one thing, cannot encompass the study of the hydrogen atom, whose solutions are not planar waves in the least; singular spinors are even worse, with Majorana spinors that do not have a plane-wave expansion in the standard form. The role played by general spinors [23–25], flag-dipoles [26–30], or Majorana [31] can be better appreciated by employing the so-called polar form, the form in which the spinor complex components are written as product of modules times unitary phases: polar forms of general spinors have been studied in [32–35], of the case of flag-dipoles is studied in [36], and Majorana in [37, 38].

In this paper, we will employ the polar form to investigate the general structure of spinors in all their classes, exploiting the generality of the treatment for applications to the field quantization. Specifically, we will try to find a way in which field quantization can be performed in a way so general as to be applicable to all classes of spinors under all circumstances, but which of course will reduce to the usual form when plane waves are used.

II. PLANE WAVES PARTICLE SCATTERING

While QFT can be applied in various areas so far as to include condensed matter, the most direct application is particle physics. Here, the full phenomenology is based on the assumption that particles be material points, with no internal dynamics nor extension. Such an assumption is mathematically reflected in the use of plane waves.

In this way, fields can be expanded as a linear combination of plane-wave solutions with coefficients given by operators verifying specific commutation relations. Hence, the amplitude for an initial state \(\psi_i\) to scatter off to the final state \(\psi_f\) is given according to

\[
S_{fi} = \bar{\psi}_f U \psi_i
\]
The Hamiltonian of interaction is given by

\[ H_{\text{int}} = \int_V e \bar{\psi} \gamma^\mu \psi A_\mu dV \]  

in which \( \gamma^\mu \) is an element of the Clifford algebra needed to define the spinor \( \psi \) with charge \( e \) and \( A_\mu \) is the electromagnetic field. Thus, in QED, the elementary process has Feynman diagram

\[ \begin{align*}
&\overline{\pi}(p') \\
&\quad \overline{\nu}(p) \\
&\quad \alpha(\nu(p - p')) \\
&\alpha(\nu(p - p')) \overline{\pi}(p') \\
&\quad \nu(\overline{v}(k')) \\
&\quad \nu(u(p + k)) \\
&\quad \nu(u(p)) \\
&\quad \nu(p + k) \end{align*} \]

so that every internal line would merely be obtained as a hinge between two external legs having equal momenta. So for example, consider the combination

\[ \begin{align*}
&\overline{\pi}(k) \\
&\overline{\nu}(v(k')) \\
&\quad \alpha(\nu(p - p')) \\
&\quad \alpha(\nu(p - p')) \overline{\pi}(p') \\
&\quad \nu(\overline{\nu}(u(p + k)) \\
&\quad \nu(u(p))) \end{align*} \]

whose invariant matrix element would be given by

\[ M = \overline{\pi}(p')(ie\gamma^\nu)u(p)\alpha(\nu(p - p')) \cdot \alpha(\nu(p - p'))\overline{\pi}(k')(ie\gamma^\nu)\nu(k') \]  

(4)

with \( \alpha(\nu(p - p'))u(\nu(p - p')) \) to be evaluated. Employing plane waves one can always write any vector field so that

\[ \alpha(\nu(p))\alpha(\nu(p)) = \frac{1}{p^2 - m^2 + \not{p}m} \]  

(5)

as explained in [22] page 159.

This electron-positron scattering is thus given by the Feynman diagram

\[ \begin{align*}
&\overline{\pi}(p') \\
&\overline{\nu}(p) \\
&\quad \alpha(\nu(p - p')) \\
&\quad \alpha(\nu(p - p')) \overline{\pi}(p') \\
&\quad \nu(\overline{\nu}(u(p + k)) \\
&\quad \nu(u(p)) \\
&\quad \nu(p + k) \end{align*} \]

and its invariant matrix element is then

\[ M = \overline{\pi}(p')(ie\gamma^\nu)u(p)\frac{-g_{\mu\nu}}{(p - p')^2}\overline{\pi}(k')(ie\gamma^\nu)\nu(k') \]  

(6)

with \( u(p) \) plane-wave spinor corresponding to an electron and \( v(p') \) plane-wave spinor corresponding to a positron.

The interested reader may check with [22] page 131.

Next we consider

\[ \begin{align*}
&\overline{\pi}(p') \\
&\overline{\nu}(p) \\
&\quad \alpha(\nu(p + k)) \\
&\quad \alpha(\nu(p + k)) \overline{\pi}(p + k) \\
&\quad \nu(\overline{\nu}(u(p)) \\
&\quad \nu(u(p)) \end{align*} \]

so that the invariant matrix element

\[ M = \overline{\pi}(p')\alpha(\nu(k'))(ie\gamma^\nu)u(p + k) \cdot \alpha(\nu(p + k))\overline{\pi}(k')(ie\gamma^\nu)\nu(k') \]  

(7)

in which \( u(p + k)\overline{\pi}(p + k) \) must be calculated. Again, for plane waves it is possible to evaluate the spin average as

\[ u(p)\overline{\pi}(p) = \frac{1}{p^2 - m^2 + \not{p}m} \]  

(8)

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as is well known and can be checked in [22] page 49. Finally the electron-photon scattering is given by

\[ u(p') a_\mu(k') \frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2} \]

\[ a_\nu(k) u(p) a_\nu(k) \frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2} - m^2 \]

whose invariant matrix element is

\[ M = \pi(p') a_\mu(k') (ie\gamma^\mu) \frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2} (ie\gamma^\mu) a_\nu(k) u(p) \]

(9)

where \( a_\mu \) is the polarization vector of the photon.

And once more this can be checked from [22] page 158. In a perfectly similar manner we can re-construct all sorts of higher-order combinations.

For example the one-loop photon correction to electron vertex is given by

\[ \frac{-g_{\mu\nu}}{(k-p)^2} \frac{\not{k} + m}{k^2 - m^2} \]

\[ a_\mu(q) \]

with correction given by

\[ M = \frac{1}{(2\pi)^4} \int \frac{\pi(p')(ie\gamma^\nu)}{(k+q)^2 - m^2} (ie\gamma^\mu) a_\nu(q) - \frac{g_{\nu\rho}}{(k-p)^2} \frac{\not{k} + m}{k^2 - m^2} (ie\gamma^\rho) u(p) d^4k \]

(10)

where the integration is performed over all values of the unconstrained \( k \) in the 4-dimensional momentum space.

Again, the readers may check from [22] page 189.

Other one-loop corrections are found similarly.

And of course, any additional \( n \)-loop correction can be obtained in exactly the same manner.

In summary, we have the following Feynman rules:

1. Element assignment:
   for every elementary electron vertex

   \[ \pi(p') \]
   \[ a_\mu(p - p') \]

   assign fermionic and photonic legs and vertices as:
   - fermion legs of momentum \( p \) are associated to the spinor \( u(p) \)
   - photon legs of momentum \( p \) are associated to the polarization \( a_\mu(p) \)
   - vertices are associated to \(-ie\gamma^\mu\)

2. Legs hinging:
   for every pair of legs hinged together a polarization or spin sum give internal line propagators in terms of \( \phi^2 \) with \( \phi^2 \) assigned by normalization, so that:
   - internal fermion lines of momentum \( k \) are associated to the propagator
     \[ \frac{\gamma^\mu k_\mu + m\not{1}}{k^2 - m^2 + i\epsilon} \]
     (11)
   - internal photon lines of momentum \( k \) are associated to the propagator
     \[ \frac{-g_{\mu\nu}}{k^2 + i\epsilon} \]
     (12)

3. Loop integration:
   for every closed loop all of unconstrained momenta have to be integrated over all possible values:
   - the volume element is given by \( \int d^4k/(2\pi)^4 \)
   and the procedure has to be repeated for each term having the same order in the perturbative expansion. Hence, for all terms in the perturbative expansion, if possible.

The reader interested in details can find an exhaustive review of Feynman rules for electrodynamics in [22].

The establishment of this protocol for the computation of the invariant matrix element, and thus of the scattering amplitudes, leads to the most astonishing predictions of QFT, most of which experimentally corroborated (some predictions such as the value of the cosmological constant
are by converse remarkably wrong, but we are not going to be interested in them in this paper). However, despite its success, there are still issues that ought eventually to be addressed. For example, QFT is built on the use of plane waves, which generally do not exist. To name one case we can think at the Hydrogen Atom, for which the electron has a non-zero Yvon-Takabayashi angle [9] despite that a plane wave must have Yvon-Takabayashi angle identically zero. This means that strictly speaking, QFT could not quantify any electron that is a bound state of atoms. To an even more general perspective, Majorana spinors have no standard form of the plane waves at all [36–38]. Thus one may wonder whether it is possible to have a general description of quantum processes that reduces to the one we already have when we employ plane waves, but which can also be applied where no plane wave can be found.

This is what we are going to investigate next.

III. GENERAL METHODS IN QFT

In the previous section, we have recalled the technology of QFT as reported for plane waves in [22] from page 45.

Now it is time to distill the hypotheses of the previous section and see where exactly the assumption of planar waves was made and how we can generalize things. Hence we start by considering that the plane wave assumption was found in two places: evaluating polarization sums

\[ a_{\mu}(p)a_{\nu}(p) = \frac{1}{p^2}(-g_{\mu\nu}) \]  
(13)

and spin sums

\[ u(p)\bar{\nu}(p) = \frac{1}{p^2-m^2}(p+mp) \]  
(14)

as is easy to check. In QFT these are evaluated straightforwardly for plane waves, and it is now our goal to assess whether they can be defined more in general. Therefore, we must consider the most general form in which vectorial and spinorial fields can be written. To do that we are going to use the trick of writing fields in polar form.

The polar form of spinor fields is the one for which all components can be written as product of module times \(\alpha\) and with the scalars

\[ \Theta = 2\phi^2 \sin \beta \]  
(31)

\[ \Phi = 2\phi^2 \cos \beta \]  
(32)

\[ S^a = 2\phi^2 s^a \]  
(29)

\[ U^a = 2\phi^2 u^a \]  
(30)

being all real tensors and verifying in general

\[ \psi \bar{\psi} \equiv \frac{1}{4} \Phi \bar{I} + \frac{1}{2} U_4 \gamma^0 + \frac{1}{2} M_{ab} \sigma^{ab} = -\frac{1}{2} \Sigma_{ab} \sigma^{ab} \pi - \frac{1}{4} S^\alpha \pi - \frac{1}{4} \Theta \pi \]  
(24)

called Fierz identities: for singular spinor fields we have

\[ S^a = -\sin \alpha U^a \]  
(25)

with Fierz identities reducing to

\[ \bar{\psi} \psi \equiv \frac{1}{4} U_4 \gamma^0 (1+\sin \alpha \pi) + \frac{1}{2} M_{ab} \sigma^{ab} \]  
(26)

with \( U_4 U^4 = 0 \) as well as \( M_{ab} M^{ab} = \epsilon^{abij} M_{ab} M_{ij} = 0 \) and \( M_{ik} U^i = \epsilon_{ikab} M^{ab} U^i = 0 \) while for regular spinor fields

\[ \Sigma_{ab} = 2\phi^2 (\cos \beta u^a s^b - \sin \beta u^a s^b \epsilon^{kab}) \]  
(27)

\[ M_{ab} = 2\phi^2 (\cos \beta u^a s^b \epsilon^{kab} + \sin \beta u^a s^b \epsilon^{jkb}) \]  
(28)

in terms of the two vectors defined as

\[ \Phi = 2\phi^2 \sin \beta \]  
(31)

\[ \Theta = 2\phi^2 \cos \beta \]  
(32)

\[ S^a = 2\phi^2 s^a \]  
(29)

\[ U^a = 2\phi^2 u^a \]  
(30)

in chiral representation, with \( S \) being a specific complex Lorentz transformation, and in which \( \alpha \) is a pseudo-scalar and the only degree of freedom: in particular \( \alpha = 0 \) gives the Majorana spinor while \( \alpha \pm \pi/2 \) gives Weyl spinors in left-handed/right-handed chiral parts respectively.

We may define the spinorial bi-linear sums given by

\[ \Sigma_{ab} = 2\bar{\psi} \sigma^{ab} \pi \psi \]  
(18)

\[ M_{ab} = 2i \bar{\psi} \sigma^{ab} \psi \]  
(19)

\[ S^a = \bar{\psi} \gamma^a \pi \psi \]  
(20)

\[ U^a = \bar{\psi} \gamma^a \pi \psi \]  
(21)

\[ \Theta = i \bar{\psi} \gamma^a \pi \psi \]  
(22)

\[ \Phi = \bar{\psi} \psi \]  
(23)

\[ S^a = \bar{\psi} \gamma^a \pi \psi \]  
(18)

\[ M_{ab} = 2i \bar{\psi} \sigma^{ab} \psi \]  
(19)

\[ S^a = \bar{\psi} \gamma^a \pi \psi \]  
(20)

\[ U^a = \bar{\psi} \gamma^a \pi \psi \]  
(21)

\[ \Theta = i \bar{\psi} \gamma^a \pi \psi \]  
(22)

\[ \Phi = \bar{\psi} \psi \]  
(23)

in chiral representation, with \( S \) one specific complex Lorentz transformation, and in which \( \beta \) and \( \phi \) are a pseudo-scalar and scalar called Yvon-Takabayashi angle and module as the only degrees of freedom. Conversely, if the spinor field is singular \( \bar{\psi} \pi \psi \equiv 0 \) and so it is always possible to find a frame in which the spinor is given as

\[ \psi = e^{-i\beta \pi/2} S \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \]  
(16)

and \( \bar{\psi} \psi \) are not both equal to zero identically, then it is always possible to find a frame in which

\[ \psi = \phi e^{-i\beta \pi S} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \]  
(17)
having great importance for what we intend to do next. 

Finally, it is also possible to see that in general 

\[ S \partial_\mu S^{-1} = iq \partial_\nu \xi_\nu + \frac{1}{2} i \partial_\nu \xi_\nu \sigma^{ij} \]  

in the same way as 

\[ (L)_k \partial_\mu (L^{-1})_j = \partial_\mu \xi_j \]  

for some \( \xi \) and \( \xi_{ij} = -\xi_{ji} \), so that we can define 

\[ q(\partial_\mu \xi - A_\mu) \equiv P_\mu \]  

\[ \partial_\mu \xi_{ij} - \Omega_{ij} = R_{ij} \]  

which can be proven to be tensors and invariant under a gauge transformation simultaneously. With these objects it is possible to see that the covariant derivatives can be expressed for singular spinor fields as 

\[ \nabla_\mu \psi = \left[ -\frac{1}{2}(\tan \alpha + \pi \sec \alpha) \nabla_\mu \alpha - i P_\mu - \frac{1}{2} R_{ij} \sigma^{ij} \right] \psi \]  

or for regular spinorial fields as 

\[ \nabla_\mu \psi = \left( -\frac{1}{2} \nabla_\mu \beta \pi + \nabla_\mu \ln \phi - i P_\mu - \frac{1}{2} R_{ij} \sigma^{ij} \right) \psi \]  

while for vector fields 

\[ \nabla_\mu V = (\eta_{ab} \nabla_\mu \ln \phi - R_{ab} \mu) V^b \]  

as it is possible to see with a straightforward computation. It is instructive to recall that for the Dirac equations their polar form is equivalent to the pair of field equations 

\[ \nabla_\mu \beta - B_\mu - 2P_\mu u(s) = 2s \mu m \cos \beta = 0 \]  

\[ \nabla_\mu \ln \phi^2 + R_{\mu} - 2P_\mu u(s) = 2s \mu m \sin \beta = 0 \]  

with \( R_{\mu} \equiv R_{\mu} \) and \( \frac{1}{2} \xi_{\mu \nu} R^{\mu \nu} = B_\mu \) defined for the sake of compact notation. From these we can also get 

\[ P^\mu = m \cos \beta \nu^\nu - Y \cdot s \nu^\nu + Y \cdot t s \nu^\nu + Z \mu s \nu e^{\mu \nu \rho \sigma} = 0 \]  

\[ Y \mu u(s) e^{\mu \nu \rho \sigma} = 0 \]  

\[ Z \cdot s + m \sin \beta = 0 \]  

\[ Z \cdot u = 0 \]  

as again can be checked in [36], although it is no longer possible to express the momentum explicitly. For vector fields we can also obtain similar results as shown in [39].

We now have all the tools needed to answer the question asked at the beginning: how can we express (13) and (14) in general? As for (13) the answer comes from (15) written according to the form 

\[ V^\alpha = \phi \nu^\alpha \]  

for a generic polarization \( \nu^\alpha \) as we would have in the case of QFT. Then we can compute 

\[ V_\mu V_\nu = \phi^2 (-g_{\mu \nu}) \]  

and with the same argument used in section II we have 

\[ V_\mu V_\nu = \phi^2 (-g_{\mu \nu}) \]  

so long as it enters the invariant matrix element. Instead, for the spinorial fields we have (16) and (17) written as 

\[ \psi = \phi e^{-\frac{i}{2} \beta \pi u} \]  

\[ \psi = \frac{1}{\sqrt{2}} (\cos \frac{\pi}{2} - \pi \sin \frac{\pi}{2}) u \]  

for a generic spinor \( u \) that is what we would have in QFT although now we have to expect results that differ from those of QFT due to the presence of the \( \beta \) angle. In fact the Fierz identities (24) are given by 

\[ \psi \bar{\psi} = \frac{1}{2} \phi^2 (e^{i \beta \pi \sigma} - e^{-i \beta \pi \sigma} - s_\sigma \gamma^\alpha \pi) \]  

\[ \psi \bar{\psi} = \frac{1}{2} U_\alpha \gamma^\alpha (1 + \sin \alpha \pi) + \frac{1}{2} M_{\alpha \beta} \sigma^{\alpha \beta} \]  

as it is possible to verify employing a direct substitution. Let us now consider them one by one.

For the vectorial field solutions of the free equations in terms of plane waves are normalized as to have 

\[ \phi^2 = \frac{1}{p^2 - m^2} \]  

and thus (55) reduces to (13), although (55) can also be used without employing plane waves in general. However, apart from this they still have the same formal structure.

For the spinorial fields in regular case the plane wave condition \( i \nabla_\mu \psi = P_\mu \psi \) when compared against (41) gives that \( R_{\mu \nu} = 0 \) with all scalar fields constant. Because any constant pseudo-scalar must vanish and taking again the usual normalization of the module so to have 

\[ \psi^{-1} \psi = \frac{1}{2} \psi^{-1} (1 + u(s) \gamma^\alpha \pi) \]  

which are precisely the known Michel-Wightman identities [40]. Under the hypotheses \( R_{\mu \nu} = 0 \) with \( \beta = 0 \) and constant module we also get from (45-48) that 

\[ P^\mu = m u^\nu \]
as the only remaining equation. In spin average
\[ \sum_{\text{spin}} \psi \bar{\psi} = \frac{1}{p^2 - m^2} (m \gamma^a + P_a \gamma^a) \] (64)
showing that (58) reduces to (14), although (58) is more general. Notice that the most general expression and its plane wave form do not even have the same structure in general. For spinor fields in singular case the plane wave condition gives \( R_{\mu \nu} = 0 \) and \( \alpha = 0 \) and thus
\[ \psi \bar{\psi} \equiv \frac{i}{2} U_a \gamma^a + \frac{i}{4} M_{ab} \sigma^{ab} \] (65)
already coinciding with its expression in spin average.

Therefore, we have proven that it is in fact possible to implement the procedures necessary for the computation of scattering amplitudes in general circumstances reducing to the known form in the case of planar waves, and yet being applicable also if no plane wave can be defined.

Strong with such a formalism, in the following we will show how to apply it so to examine specific examples.

### IV. SPECIFIC COMPUTATION FOR COMPTON SCATTERING

In the previous section we have shown that the invariant matrix elements can be evaluated in general reducing to the usual form when plane waves are used, but not actually needing such an assumption. Indeed, plane wave solutions, having vanishing Yvon-Takabayashi angle and constant module, represent particles for which both the internal structure and the spatial extension are neglected.

Neither assumption is realistic, and so it may be wise to consider no plane wave solution. So in the following we will forget about plane waves and try to compute some radiative processes in very general circumstances.

One assumption that we shall still consider will be that of taking into account only the spin averages. In doing so Fierz identity (58) will reduce to the form
\[ \sum_{\text{spin}} \psi \bar{\psi} = \phi^2 (e^{-i \beta \pi} + u_a \gamma^a) \] (66)
and therefore
\[ \sum_{\text{spin}} \psi \bar{\psi} = \frac{\phi^2}{m} (m e^{-i \beta \pi} + \gamma^a) \] (67)
having defined \( p^a = m a^a \) as the kinematic momentum in terms of mass and velocity and that is different from \( P_c \) in (45) which is instead a dynamical quantity. Therefore
\[ \sum_{\text{spin}} \psi \bar{\psi} = \frac{1}{p^2 - m^2} (\hat{p} + m e^{-i \beta \pi}) \] (68)
is the form of our propagator for internal lines. External legs will be taken in the case of a free propagation instead.

In order for us to assess the effect of the presence of the Yvon-Takabayashi angle \( \beta \) in this propagator we have to apply it to a case in which it is spinors what constitutes the virtual particles represented by the internal lines.

As an illuminating example, we apply this propagator to the Compton scattering. The invariant matrix element (9) in this case [22] is given according to
\[ M = -e^2 \pi (p') a_\mu (k') \gamma^\mu \frac{\hat{p} + k + m e^{-i \beta \pi}}{(p + k)^2 - m^2} \gamma^\nu a_\nu (k) u(p) \] (69)
or more in detail
\[ M = -e^2 \pi (p') a_\mu (k') \gamma^\mu \frac{\hat{p} + k + m e^{-i \beta \pi}}{2 p \cdot k'} \gamma^\nu a_\nu (k) u(p) \] (70)
since \( k^2 = 0 \) and \( p^2 = m^2 \) for photon and electron.

The full treatment of the Compton scattering involves also the complementary diagram, which leads to
\[ M = +e^2 \pi (p') a_\mu (k') \gamma^\mu \frac{\hat{p} - k + m e^{-i \beta \pi}}{2 p \cdot k'} \gamma^\nu a_\nu (k) u(p) \] (71)
now with \( k'^2 = 0 \) but still \( p^2 = m^2 \) for photon and electron.

The full invariant matrix element is thus
\[ M = -e^2 \pi (p') a_\mu (k') [\gamma^\mu \frac{\hat{p} + k + m e^{-i \beta \pi}}{2 p \cdot k'} \gamma^\nu + + \gamma^\nu \frac{\hat{p} - k + m e^{-i \beta \pi}}{2 p \cdot k'} \gamma^\mu a_\nu (k) u(p) ] \] (72)
which now must be squared.

After the square is taken, its trace is given by
\[ \frac{1}{4} |M|^2 = \frac{e^4}{4} \text{tr} \left[ (\hat{p} + m) [\gamma^\mu \frac{\hat{p} + k + m e^{-i \beta \pi}}{2 p \cdot k'} \gamma^\nu + + \gamma^\nu \frac{\hat{p} - k + m e^{-i \beta \pi}}{2 p \cdot k'} \gamma^\mu ] \right] \] (73)
having used \( u \bar{u} = \hat{p} + m \) and \( a_\alpha a_\nu = -g_{\alpha \nu} \) for the external legs of both electron and photon. The whole computation now uses usual methods of QFT and the anticommuting properties of \( \pi \) and \( \gamma_\alpha \) matrices. Long but straightforward computations, involving Clifford algebras, and especially the traces of the Clifford matrices, lead us to the resulting expression that is given according to
express the very well-known Klein-Nishina formula. having used the Compton relation it is natural to assume it is described by the electronisinaboundstatewithitsnucleus, and therefore since to be applied to scattering of particles interacting with a practice however, the actual computation is cumbersome. We notice that to this order in the perturbative expansion -loop radiative processes, and therefore our result can be given in terms of Majoranaspinors. And specifically, to investigate ELKO. It is worth noticing that this method is general enough to be applied to scattering of particles interacting with a fixed external field, but it could also be used for scattering of particles having self-interactions. A first example may be the electron self-energy. Nevertheless, in these cases a form for the function is yet to be known.

In principle, with the Compton relation (80) one could express $\omega'$ as a function of $\omega$ and the cosine of the scattering angle $\cos \theta$ so that the integrated cross-section would be given in terms of $\omega$ alone, apart from the mass of the spinor field, and of course the fine-structure constant. In practice however, the actual computation is cumbersome. We notice that to this order in the perturbative expansion, the $\beta$ correction is comparable to the correction due to 2-loop radiative processes, and therefore our result can only be taken as proof of concept, as reliable calculations must take into account larger contributions first.

It is worth noticing that this method is general enough to be applied to scattering of particles interacting with a fixed external field, but it could also be used for scattering of particles having self-interactions. A first example may be the electron self-energy. Nevertheless, in these cases a form for the function is yet to be known.

As it is clear, the contribution of the Yvon-Takabayashi angle appears to modify the standard expression. For the first-order perturbative in $\beta$ it reduces to

$$\frac{1}{4} |M|^2 = 2e^4 \left[ \frac{\omega}{\omega'} + \frac{\omega'}{\omega} + 4m \left( \frac{1}{\omega} - \frac{1}{\omega'} \right) + 3m^2 \left( \frac{1}{\omega^2} + \frac{1}{\omega'^2} \right) - 2m \cos \beta \left[ 2m \left( \frac{1}{\omega \omega'} + \frac{1}{\omega^2} + \frac{1}{\omega'^2} \right) + \frac{1}{2} \right] + 2m^2 \cos (2\beta) \left( \frac{1}{\omega^2} + \frac{1}{\omega'^2} \right) \right]$$

(74)

where the standard expression is now obvious.

To evaluate the correction, we recall that the scattered electron is in a bound state with its nucleus, and therefore it is natural to assume it is described by the 1S orbitals of hydrogen-like atoms [9]. In this case we know what is the form of the Yvon-Takabayashi angle, as it is given by

$$\frac{1}{4} |M|^2 = 2e^4 \left[ \frac{\omega}{\omega'} - 2m \left( \frac{1}{\omega} - \frac{1}{\omega'} \right) + m^2 \left( \frac{1}{\omega^2} - \frac{1}{\omega'^2} \right) - \beta^2 m^2 \left[ 2 \left( \frac{1}{\omega \omega'} + \frac{1}{\omega^2} + \frac{1}{\omega'^2} \right) + \frac{1}{m} \left( \frac{1}{\omega} - \frac{1}{\omega'} \right) \right] \right]$$

(75)

and for small $e^2$ it is

$$\beta \approx - e^2 \cos \theta$$

(77)

where $\theta$ is the elevation in spherical coordinates. The fact that it is point dependent is not surprising for a function, but for better applicability it is more convenient to use a mean over all possible orientations, which is given by

$$\beta^2_{\text{mean}} = \frac{1}{4\pi} \int \beta^2 \sin \theta d\theta d\phi \approx \frac{e^4}{3}$$

(78)

of the same magnitude of $e^2$ itself. Hence plugging the result into (75), hence plugging it into the cross-section, furnishes

$$\frac{d\sigma}{d\cos \theta} \approx \frac{\pi \alpha^2}{m^2} \left( \frac{\omega'}{\omega} \right)^2 \left[ \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - |\sin \theta|^2 - \frac{16\pi^2 \alpha^2}{3} \left[ 2m^2 \left( \frac{1}{\omega \omega'} + \frac{1}{\omega^2} + \frac{1}{\omega'^2} \right) + 1 - \cos \theta \right] \right]$$

(79)

having used the Compton relation

$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m} (1 - \cos \theta)$$

(80)

and setting $e^2 = 4\pi \alpha$ so to obtain the beta correction to the very well-known Klein-Nishina formula.

V. CONCLUSION

In this work we have re-considered the prescriptions that are normally given in QFT to compute perturbative expansions and Feynman diagrams. We have however considered the mathematical tools in full generality, by
employing the polar form of fields. We have found the most general form of spin-sum relationships, and which in the limit of plane waves includes the usual expressions for the spin-sum as obtained in QFT. Nevertheless, without such a restriction, the most general structure of the scattering amplitudes has been eventually found.

This most general form allows to compute processes that cannot normally computed in QFT, such as for instance processes that involve interacting particles and not only freely moving point-particles. As an example we have computed for the Compton scattering the correction to first order in the fine structure constant.

The data presented in this paper can be obtained from the author directly upon reasonable request.

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