Information geometry under hierarchical quantum measurements

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In most quantum technologies, measurements need to be performed on the parametrized quantum states to transform the quantum information to classical information. The measurements, however, inevitably distort the information. The characterization of the discrepancy is an important subject in quantum information science, which plays a key role in understanding the difference between the structures of the quantum and classical information. Here we analyze the discrepancy in terms of the Fisher information metric and present a framework that can provide analytical bounds on the difference under hierarchical quantum measurements. Specifically, we present a set of analytical bounds on the difference between the quantum and classical Fisher information metric under hierarchical p-local quantum measurements, which are measurements that can be performed collectively on at most p copies of quantum states. The results can be directly transformed to the precision limit in multi-parameter quantum metrology, which leads to characterizations of the tradeoff among the precision of different parameters. The framework also provides a coherent picture for various existing results by including them as special cases.

Quantum measurement serves as the gateway between quantum information and classical information. In most quantum technologies, the information encoded in the parametrized quantum states needs to be extracted by the measurements. For example, in quantum metrology, the estimation of unknown parameters encoded in the quantum states is achieved through the measurements on the parametrized quantum states; in the variational quantum circuit certain information needs to be extracted via the measurements on the parametrized quantum states to update the circuit. Upon the measurements, however, certain properties of quantum information, such as noncommutativity, are lost. This inevitably induces distortions on the information structure. Understanding such distortion is an important subject in quantum information science which helps distinguish the structure of the quantum and classical information. It also helps to understand the maximal amount of information that can be extracted from quantum states.

In this Letter, we characterize the structure of the information in terms of the Fisher information metric1–10 and study the distortion of the metric under hierarchical p-local quantum measurements, which are the measurements that can be performed collectively on at most p copies of quantum states. We note that the Fisher information metric is the only Riemannian metric in information geometry that is invariant under sufficient statistics17, and has been employed in a broad range of applications13, such as quantum metrology1–2, quantum phase transition19–21, entanglement witness22, as well as the natural gradient and effective dimension in statistical learning17–22.

We first introduce the quantum and classical Fisher information metric together with the existing results, then present an approach to characterize the achievable classical Fisher information metric under hierarchical p-local quantum measurements. This framework provides a systematic way to generate various analytical bounds on the difference between the quantum and classical Fisher information metric under general p-local measurements. The framework also includes various existing results, which were seemingly disconnected from each other previously, as special cases thus put them into a coherent picture.

Given d-dimensional parametrized quantum state, \( \rho_x \), with n parameters as \( x = (x_1, \cdots, x_n) \), the \( jk \)-th entry of the quantum Fisher information matrix (QFIM), denoted as \( F_Q(x) \), is given by \[ F_Q(x)_{jk} = \frac{1}{2} \text{Tr}[\rho_x (L_j L_k + L_k L_j)] , \]
where \( L_j, L_k \) is the symmetric logarithmic derivative (SLD) with respect to the parameter \( x_j, x_k \) and can be obtained from the equation \[ \frac{\partial \rho_x}{\partial x_j} = \frac{1}{2} (\rho_x L_j + L_j \rho_x) . \] The QFIM quantifies the quantum Fisher information metric, which is related to the Bures metric as \[ D_B^2(\rho_x, \rho_{x+dx}) = \frac{1}{4} dx F_Q(x) dx^T , \] where \( dx = (dx_1, \cdots, dx_n) \) are infinitesimal changes of the parameters and \( D_B(\rho_1, \rho_2) = \sqrt{2 - 2 \text{Tr} \sqrt{\rho_1} \rho_2 \sqrt{\rho_1}} \) is the Bures distance24. The QFIM is additive with respect to the copies of the state, i.e., the QFIM of \( \rho_x^{\otimes p} \), which we denote as \( F_{Qp} \), equals to \( p F_Q \).

General measurements that transform the parametrized quantum states to classical data is the positive operator-valued measurement (POVM), which is a set of \( \{ M_\alpha \} \) with \( M_\alpha \geq 0 \) and \( \sum_\alpha M_\alpha = I \). The probability of obtaining the result \( \alpha \) is given by \( p(\alpha | x) = \text{Tr}(\rho_x M_\alpha) \). As illustrated in Fig.1, the measurement changes the parametrized quantum state to the parametrized probability distribution.

The Fisher information metric of the probability distribution is given by the classical Fisher information matrix (CFIM)25, denoted as \( F_C(x) \), whose \( jk \)-th entry is given by \[ F_C(x)_{jk} = \int_\alpha \frac{1}{p(\alpha | x)} \frac{\partial p(\alpha | x)}{\partial x_j} \frac{\partial p(\alpha | x)}{\partial x_k} d\alpha . \] The
classical Fisher information metric is related to the Euclidean distance between $\sqrt{p(\alpha|x)}$ and $\sqrt{p(\alpha|x + dx)}$ as

$$\int_\alpha (\sqrt{p(\alpha|x)} - \sqrt{p(\alpha|x + dx)})^2 d\alpha = \frac{1}{4} dx F_C(x) dx^T. \quad (1)$$

We note that for diagonal quantum states, where the diagonal entries can be regarded as the classical probability distribution, the Bures distance reduces to the Euclidean distance.

The QFIM and CFIM characterize the geometrical structure of the parametrized quantum states and the classical probability distribution respectively. As measurements can be regarded as a special set of quantum channels (known as the quantum-classical channel), the data process inequality tells that the distance can not increase after the measurements. This implies that under any measurement $F_C(x) \leq F_Q(x)$.

In classical estimation, the CFIM provides an asymptotically achievable lower bound on the covariance for locally unbiased estimators of the parameters, which is known as the Cramer-Rao bound [44] with $\text{Cov}(\hat{x}) \geq \frac{1}{2} F_C^{-1}(x)$, here $\hat{x}$ denotes the locally unbiased estimator of $x$ and $\text{Cov}(\hat{x})$ denotes the covariance matrix where the $jk$-th entry is given by $\text{Cov}(\hat{x})_{jk} = E[(\hat{x}_j - x_j)(\hat{x}_k - x_k)]$, $\nu$ is the number of sampled data. Since $F_C(x) \leq F_Q(x)$, the covariance matrix is further bounded by the QFIM as $\text{Cov}(\hat{x}) \geq \frac{1}{p} F_Q^{-1}(x)$, which is known as the quantum Cramer-Rao bound (QCRB) [1, 2].

If there exists a measurement such that $F_C(x) = F_Q(x)$, the measurement then preserves the local geometrical structure and the QCRB is saturable. It is known that if the quantum state is parametrized by a single parameter, i.e., $n = 1$, then there always exists a measurement that can preserve the local Fisher information structure [1]. Furthermore, the measurement that saturates the QCRB can be taken as 1-local measurement, collective measurements are not required. One such measurement is the projective measurement on the eigenvectors of the SLD [1]. When there are multiple parameters, the information structure becomes much more complicated. First, the SLDs for different parameters may not commute with each other, thus in general there does not exist a measurement that can make $F_C(x) = F_Q(x)$ [27–29, 56]. The distortion of the Fisher information structure is then typically inevitable. Second, the collective measurements matter. As illustrated in Fig. 2, if we repeat the 1-local measurement on two copies of quantum states, the maximal CFIM at most doubles. However, if collective measurements can be performed, it is possible to obtain larger CFIM due to more degrees of freedom in the measurements. In general under $p$-local measurements, the CFIM that can be extracted from $p$ copies of quantum states, which we denote as $F_{Cp}(x)$, can be larger than $pF_C(x)$. The CFIM is thus super-additive with respect to the number of copies of quantum states. This is related to the notion of the nonlocality without entanglement [63].

We use $Tr[F_Q^{-1}(x) F_{Cp}(x)]$ to quantify the difference between the QFIM and the CFIM that can be extracted by $p$-local measurements on $\rho_p \otimes \rho_p \otimes \rho_p$ [27, 29, 56]. Compared to other quantifiers of the difference, such as $\|F_{Qp}(x) - F_{Cp}(x)\|$, $\text{Tr}[F_{Qp}^{-1}(x) F_{Cp}(x)]$ has the advan-

FIG. 1. The measurement transforms the parametrized quantum states to classical probability distribution where the classical Fisher information metric is always upper bounded by the quantum Fisher information metric.

FIG. 2. The transformation of information geometry under 1-local and 2-local measurements respectively. Since there are more freedom in the 2-local measurements, the classical Fisher information metric under the optimal 2-local measurement is in general less distorted than the metric under the 1-local measurements, i.e., the classical Fisher information matrix under 2-local measurements can be closer to the quantum Fisher information matrix. The classical Fisher information matrix under $p$-local measurements can get even closer when $p$ increases.
tage that it is invariant under reparametrization. Since \( F_{C_{p}}(x) \leq F_{Q_{p}}(x) \), we have a trivial upper bound \( TR[F_{Q_{p}}^{-1}(x)F_{C_{p}}(x)] \leq TR(I_{n}) = n \) (here \( I_{n} \) is the \( n \times n \) identity matrix), and this upper bound is saturated when there exists a p-local measurement that makes \( F_{C_{p}}(x) = F_{Q_{p}}(x) \). In general \( TR[F_{Q_{p}}^{-1}(x)F_{C_{p}}(x)] < n \) and \( -TR[F_{Q_{p}}^{-1}(x)F_{C_{p}}(x)] \) quantifies the gap between the QFIM and the achievable CFIM under p-local measurements. We use \( \Gamma_{p} \) to denote the maximal \( TR[F_{Q_{p}}^{-1}(x)F_{C_{p}}(x)] \) over all p-local measurements, then \( \Gamma_{1} \leq \Gamma_{2} \leq \cdots \leq \Gamma_{\infty} \leq n \).

Previous results on achievable CFIM are mostly on the extreme cases with \( p = 1, 2 \) or \( p = \infty \), see [27, 29, 51]. Some of the previous results are stated in terms of the covariance matrix instead of the CFIM, we note that since the classical Cramér-Rao bound is achievable asymptotically, the covariance matrix and the inverse of the CFIM is interchangeable.

For 1-local measurement (\( p = 1 \)), Nagaoka provided a bound for the estimation of two parameters (\( n = 2 \)) as [51, 65]

\[
\nu TR[Cov(\hat{x})] \geq \min_{\{X_{1},X_{2}\}} TR(\rho_{x}X_{1}^{2}) + TR(\rho_{x}X_{2}^{2}) + \|\sqrt{\rho_{x}}[X_{1},X_{2}]\sqrt{\rho_{x}}\|_{1},
\]

(2)

Here \( \{X_{j}\} \) are Hermitian operators that satisfy the locally unbiased condition, \( TR(\rho_{x}X_{j}) = 0 \) and \( TR(\sqrt{\rho_{x}}X_{j}) = \delta_{jk} \) with \( \delta_{jk} \) as the Kronecker delta function. Recently Nagaoka bound has been generalized to n parameters [69], which in general can only be evaluated numerically. Gill and Massar provided an analytical bound under 1-local measurements as \( TR[F_{Q_{1}}^{-1}(x)F_{C}(x)] \leq d - 1 \) with \( d \) as the dimension of the Hilbert space for a single \( \rho_{x} \). The Gill-Massar bound is nontrivial only when \( n \geq d \).

A necessary condition for the saturation of the QCRB under 1-local measurements is the partial commutativity condition [59], which requires the SLDs commute on the support of \( \rho_{x} \). Specifically if we write \( \rho_{x} \) in the eigenvalue decomposition as \( \rho_{x} = \sum_{l=1}^{m} \lambda_{l} |\Psi_{l}\rangle \langle \Psi_{l}| \) with \( \lambda_{l} > 0 \), the partial commutativity condition is \( \Psi_{r} \{[L_{j},L_{k}],|\Psi_{s}\rangle\langle \Psi_{s}| \} = 0 \) for any \( j, k \in \{1, \cdots, n\} \) and \( r, s \in \{1, \cdots, m\} \).

For \( p = 2 \), Zhu and Hayashi provided an upper bound on \( \Gamma_{2} \) as \( TR[F_{Q_{2}}^{-1}(x)F_{C_{2}}(x)] \leq \frac{1}{2}(3d - 1) \) [51], which is nontrivial only when \( n \geq \frac{3}{2}(d - 1) \).

For \( p = \infty \), Holevo provided an achievable bound [2] in terms of the weighted covariance matrix as \( \nu TR[W Cov(x)] \geq \min_{\{X_{j}\}} \{ TR[W ReZ(X)] + \|\sqrt{W} ImZ(X)\sqrt{W}\|_{1} \} \), where \( W \geq 0 \) is a weighted matrix, \( Z(X) \) is a matrix with its \( jk \)-th entry given by \( Z(X)_{jk} = TR(\rho_{x}X_{j}X_{k}) \). Here \( \{X_{j}\} \) is a set of Hermitian operators that satisfy the local unbiased condition, \( ReZ(x) \) and \( ImZ(X) \) are the real and imaginary part of \( Z(x) \) respectively. The Holevo bound in general can only be evaluated numerically [62]. For pure states the Holevo bound can be saturated by 1-local measurements [52], while for mixed states the saturation of the Holevo bound in general requires collective measurements on infinite number of copies of the states. The necessary and sufficient condition for the Holevo bound to coincide with the QCRB is the weak commutative condition, which is \( TR(\rho_{x}[L_{j},L_{k}]) = 0 \) for all \( j, k \in \{1, \cdots, n\} \).

For general p-local measurements, there is little understanding on the achievable CFIM. We present an approach that can lead to various bounds on \( \Gamma_{p} \) for general \( p \). These bounds provide a necessary condition for the saturation of the QCRB under general p-local measurements, which recovers the partial commutative condition at \( p = 1 \) and the weak commutative condition at \( p \rightarrow \infty \).

For a state, \( \rho_{x} \) with \( x = (x_{1}, \cdots, x_{n}) \), given any POVM, \( \{M_{u}\} \), and any \( |u\rangle \), we define \( Cov_{u} \) as a \( n \times n \) matrix with the \( jk \)-th entry given by

\[
(Cov_{u})_{jk} = \sum_{\alpha} (\langle \hat{x}_{j}(\alpha) - x_{j} | \hat{x}_{k}(\alpha) - x_{k} \rangle |u\rangle \sqrt{\rho_{x}} M_{\alpha} \sqrt{\rho_{x}} |u\rangle),
\]

and \( A_{u} \) as a \( n \times n \) matrix with the \( jk \)-th entry given by

\[
(A_{u})_{jk} = \langle u| \sqrt{\rho_{x}} X_{j} X_{k} \sqrt{\rho_{x}} |u\rangle + \frac{1}{2i} (\langle u| \sqrt{\rho_{x}} X_{j} X_{k} \sqrt{\rho_{x}} |u\rangle - \langle u| \sqrt{\rho_{x}} X_{k} X_{j} \sqrt{\rho_{x}} |u\rangle),
\]

(4)

Here \( \{\hat{x}_{j}\} \) are locally unbiased estimators and \( \{X_{j} = \sum_{\alpha} |\Psi_{\alpha}\rangle \langle \Psi_{\alpha}| M_{\alpha}\} \) satisfy the locally unbiased condition. For any \( |u\rangle \) we can prove that \( Cov_{u} \geq A_{u} \) and \( Cov_{u} \geq A_{u}^{T} \). And for any set of \( \{|u_{q}\} \) that satisfies \( \sum_{q} \langle u_{q}|u_{q}\rangle = I \), it is easy to verify that \( Cov(\hat{x}) = \sum_{q} Cov_{u_{q}} \). Then for any choices of \( \tilde{A}_{u_{q}} \in \{A_{u_{q}}, A_{u_{q}}^{T}\} \), we have \( Cov(\hat{x}) = \sum_{q} Cov_{u_{q}} \geq \tilde{A} = \sum_{q} \tilde{A}_{u_{q}} \), here each \( \tilde{A}_{u_{q}} \) can be taken independently as either \( A_{u_{q}} \) or \( A_{u_{q}}^{T} \). By decompose \( \tilde{A} \) into the real and imaginary part as \( \tilde{A} = \tilde{A}_{Re} + i \tilde{A}_{Im} \), we obtain a bound on the weighted covariance matrix

\[
\nu TR[W Cov(x)] \geq \min_{\{X_{j}\}} TR[W \tilde{A}_{Re} + \sqrt{W} \tilde{A}_{Im} \sqrt{W}],
\]

(5)

where the number of repetition, \( \nu \), is included. Any choices of \( \{|u_{q}\} \) with \( \sum_{q} \langle u_{q}|u_{q}\rangle = I \) and any \( \tilde{A}_{u_{q}} \in \{A_{u_{q}}, A_{u_{q}}^{T}\} \) lead to a valid bound.

This provides a versatile tool to obtain many useful bounds by properly choosing \( \{|u_{q}\} \) and \( \{A_{u_{q}}\} \). In particular the Holevo bound [2] and the Nagaoka bound [64, 65] can be recovered from this general bound by making particular choices of \( \{|u_{q}\}\) and \( \{A_{u_{q}}\} \). Furthermore, by combining with an improved Robertson’s uncertainty relation for multiple observables [67, 69], we can obtain a set of analytical bounds on the gap between the QFIM and the CFIM under general p-local measurements. Specifically for pure states we have

\[
\Gamma \leq n - f(n)\|F_{Q_{1}}^{-\frac{1}{2}} F_{Im} F_{Q_{1}}^{-\frac{1}{2}}\|_{F},
\]

(6)
here $F_{lm}$ is the matrix with the entries given by $(F_{lm})_{jk} = \frac{1}{2} Tr(\rho_x [L_j, L_k])$ and $\| \cdot \|_F = \sqrt{\sum_{j,k} |(\cdot)_{jk}|^2}$ is the Frobenius norm, $f(n)$ can take \( \frac{1}{4(n-1)} \sum_{j,k} |(\cdot)_{jk}|^2 \) or $\frac{1}{5}$, which all lead to valid bounds. Since larger $f(n)$ leads to tighter bound, we can take $f(n) = \max\{ \frac{1}{4(n-1)}, \frac{n-2}{(n-1)^2}, \frac{1}{5} \}$. Note that here we use $\Gamma$ instead of $\Gamma_p$ since for pure states $\Gamma_1 = \Gamma_2 = \cdots = \Gamma_\infty$.

For mixed states under p-local measurements we have

$$\Gamma_p \leq n - f(n) \left\| \left( F_{\tilde{Q}} \right)^{-\frac{1}{2}} F_{\text{lmp}} F_{\tilde{Q}}^{-\frac{1}{2}} \right\|_F^2, \quad (7)$$

where $f(n) = \max\{ \frac{1}{4(n-1)}, \frac{n-2}{(n-1)^2}, \frac{1}{5} \}$, $\tilde{F}_{\text{lmp}}$ is the imaginary part of $\tilde{F} = \sum_q \tilde{F}_{u_q} u_q$ with each $\tilde{F}_{u_q}$ equal to either $F_{u_q}$ or $F_{u_q^T}$, here $u_q$ is a $n \times 1$ matrix with the $j$-th entry given by $(F_{u_q})_{jk} = \langle u_q | \sqrt{\rho^n_{\tilde{Q}} L_{j}} L_{k} \sqrt{\rho^n_{\tilde{Q}} | u_q \rangle}$. $L_{j}$ is the SLD of $\rho^n_{\tilde{Q}}$ corresponding to the parameter $\tilde{x}_j(\tilde{x}_k)$, where $\tilde{x} = F_{\tilde{Q}}^{-\frac{1}{2}} x$ is a reparametrization under which $F_{\tilde{Q}} = I$.

Note that we always have $\left\| \left( F_{\tilde{Q}} \right)^{-\frac{1}{2}} F_{\text{lmp}} F_{\tilde{Q}}^{-\frac{1}{2}} \right\|_F \geq \left\| \left( F_{\tilde{Q}} \right)^{-\frac{1}{2}} \tilde{F}_{\text{lmp}} \left( F_{\tilde{Q}} \right)^{-\frac{1}{2}} \right\|_F$, the bound in Eq.(7) is thus tighter than the bound in Eq.(7) when $f(n) = \frac{1}{4(n-1)}$, but it could be less tight when $f(n) = \frac{n-2}{(n-1)^2} \frac{1}{5}$.

The bound in Eq.(8) provides a necessary condition for the saturation of the QCRB under p-local measurements, which is $\frac{C_p}{p} = 0$. For $p = 1$, this condition is equivalent to the partial commutative condition. For $p \to \infty$, we prove that

$$\lim_{p \to \infty} \frac{(C_p)_{jk}}{p} = \frac{1}{2} Tr(\rho_x [L_j, L_k]) \quad (9)$$

At $p \to \infty$ the condition is thus equivalent to the weak commutative condition, $Tr(\rho_x [L_j, L_k]) = 0$, $\forall j, k$. This builds a bridge between the partial commutative condition and the weak commutative condition at the two extremes.

The bound in Eq.(5) involves operators on $p$ copies of quantum states, whose dimension grows exponentially with $p$. We provide another simpler bound which only uses operators on a single $\rho_x$ as

$$\Gamma_p \leq n - \frac{1}{4(n-1)} \left\| \frac{T_p}{p} \right\|_F^2, \quad (10)$$

where $T_p$ can be computed from the eigenvalues, eigenvectors and SLDs of a single $\rho_x$. Specifically, given $\rho_x = \sum_{i=1}^n \lambda_i |\Psi_i\rangle \langle |\Psi_i\rangle$ in the eigenvalue decomposition with $\lambda_i > 0$, the $j$-th entry of $T_p$ is given by

$$\langle T_p \rangle_{jk} = \frac{1}{2} E(\{ \sum_{r=1}^p \langle \Phi_r | [L_j, L_k] |\Phi_r\rangle \}), \quad (11)$$

here $E(\cdot)$ denotes the expected value, each $|\Phi_r\rangle$ is randomly and independently chosen from the eigenvectors of $\rho_x$ with the probability equal to the corresponding eigenvalue, i.e., each $|\Phi_r\rangle$ takes $|\Psi_i\rangle$ with probability $\lambda_i$, $i = \{1, \cdots, n\}$. $L_j(L_k)$ is the SLD for $\rho_x$ with respect to the parameter $\tilde{x}_j(\tilde{x}_k)$, where $\tilde{x} = F_{\tilde{Q}}^{-\frac{1}{2}} x$. The difference between this bound and the bound in Eq.(8) is at most $O(\frac{1}{\sqrt{p}})$. Thus when the bound in Eq.(8) is hard to compute at large $p$, we can use this bound instead which is almost as tight.

The upper bounds on $\Gamma_p$ can be directly transformed to the lower bounds on the covariance matrix. For example, suppose a p-local measurement is repeated with $\mu$ times (so total $\nu = \mu p$ copies of $\rho_x$), from the classical Cramér-Rao bound we have $Cov(\hat{x}) \geq \frac{1}{\nu} F_{\tilde{Q}}^{-\frac{1}{2}} F_{\tilde{Q}} \frac{1}{\mu} F_{\tilde{Q}}^{-\frac{1}{2}} F_{\tilde{Q}}\mu$ (here the equality is achievable since the classical Cramér-Rao bound is saturable). This implies that $\nu p Cov^{-1}(\hat{x}) \leq F_{\tilde{Q}}^{-\frac{1}{2}} F_{\tilde{Q}}\mu$. Any upper bound, $\Gamma_p \leq D$, then leads to an upper bound on $\frac{1}{\nu} Tr[F_{\tilde{Q}}^{-\frac{1}{2}} Cov^{-1}(\hat{x})]$ as $\frac{1}{\nu} Tr[F_{\tilde{Q}}^{-\frac{1}{2}} Cov^{-1}(\hat{x})] \leq Tr[F_{\tilde{Q}}^{-\frac{1}{2}} F_{\tilde{Q}}\mu] \leq D$ (note $F_{\tilde{Q}} = p F_{\tilde{Q}}$). From the Cauchy-Schwartz inequality,

$$Tr[W Cov(\hat{x})] Tr[F_{\tilde{Q}}^{-\frac{1}{2}} W F_{\tilde{Q}}^{-\frac{1}{2}}] \geq (Tr[F_{\tilde{Q}}^{-\frac{1}{2}} W F_{\tilde{Q}}^{-\frac{1}{2}}])^2, \quad (12)$$

we then obtain

$$\nu Tr[W Cov(\hat{x})] \geq \frac{(Tr[F_{\tilde{Q}}^{-\frac{1}{2}} W F_{\tilde{Q}}^{-\frac{1}{2}}])^2}{D}$$

By substituting $D$ with any of the upper bounds on $Tr[F_{\tilde{Q}}^{-\frac{1}{2}} W F_{\tilde{Q}}^{-\frac{1}{2}}]$ obtained above we then get analytical bounds on the weighted covariance matrix.

Summary We provided a framework to quantify the difference between the quantum and classical Fisher information metric under hierarchical quantum measurements. The framework provides a systematic way to generate bounds on the achievable CFIM for general quantum states under general measurements, which significantly improves our understandings on the Fisher information geometry under hierarchical quantum measurements. A necessary condition for the zero gap between the quantum and classical Fisher metric has also been identified, which is shown to recover the partial commutative condition at $p = 1$ and the weak commutative condition at $p \to \infty$. The result can be directly transformed to the precision limits in multi-parameter quantum metrology and have implications in various other fields [18, 33, 70]. The detailed derivation can be found in the companion paper [67] which also contains additional bounds that can be obtained with the framework.
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