COMPLEX BODIES WITH MEMORY: LINEARIZED SETTING

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Abstract. The mechanics of complex bodies with memory effects is discussed in linearized setting. The attention is focused on the characterization of free energies in terms of minimum work and maximum recoverable work in the bulk and along a discontinuity surface endowed with its own surface energy, a surface internal to the body. To this aim, use is made of techniques proposed by Del Piero. Consequences of the Clausius-Duhem inequality are investigated for complex bodies with instantaneous linear elastic response.

1. Introduction

Some materials display sensibility to the past history of their present state: they are called the materials with memory. A paradigmatic (although special) example of description of hereditary behavior is the standard viscoelasticity, even in small strain regime, for example the one described by the rheological models by Newton, Maxwell or Kelvin. The pioneering work by Volterra [42, 43] opened the way to the analysis of complicated constitutive structures built up on the histories of state variables (basically their graphs in time).

The topic received careful attention in the late 1950s, the subsequent decade and more. Researches were developed with the aim of establishing a theory of linear and non-linear behavior of materials with memory, especially in the case of fading memory [31, 9, 10, 5, 6]. All these works generated specific developments and general effort toward ordering of the existing results in a clear and systematic way. The articles on this topic are manifold and it is difficult to list all of them. Essential examples are [27, 7, 8, 11, 28, 32, 14, 15, 16, 17, 18, 19, 20, 11, 21, 22, 24, 25, 26, 36, 38], all dealing with the description of memory effects in Cauchy bodies that are those bodies the morphology of every material element of which is described by the sole place that its centre of mass occupies in space.

Essential questions have been tackled in the representation of the mechanical behavior of the materials with memory. Some of them have foundational nature: (i) the meaning to be given to the notion of state, (ii) the definition of appropriate free energies, (iii) the nature of the chain rule when functionals of histories are called upon, (iv) the correct use of Clausius-Duhem inequality to find a priori constitutive restrictions.

In particular, in linear viscoelasticity items (i) and (ii) have been successfully tackled in [21] and [22]: viscoelastic material elements of Cauchy bodies have been described as thermodynamic systems in the sense of [12].

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Here, techniques proposed by Del Piero in [21] are used extensively to address the mathematical modelling of the behavior of complex bodies with memory in the general model-building framework of the mechanics of complex bodies (different analyses on the specific case of micromorphic materials with memory are available in [23, 30]).

Bodies are called complex when changes in the molecular or crystalline texture at various microscopic scales (substructure) influence the macroscopic behavior through peculiar actions. Examples are manifold: liquid crystals, ferroelectrics, quasicrystals, nematic elastomers, magnetostrictive solids etc. Although all these examples are referred to a variety of phenomena, common essential features can be referred to a unique abstract model-building framework for the mechanics of complex bodies (see [2, 33, 34]). Such a framework unifies in a single format the existing models of special classes of complex bodies and is a flexible tool for analyzing new materials.

Some complex bodies, as for example relaxor ferroelectrics [4, 44], exhibit memory of substructural events. Such a circumstance motivates the analysis of the mechanics and thermodynamics of bodies with memory. The setting is the linear one for the sake of simplicity. The representation of the substructural morphology of bodies is maintained abstract in order to include a number of special cases as large as possible.

Free energies are characterized in terms of minimum work and maximum recoverable work in the bulk and along a discontinuity surface endowed with its own energy. Consequences of the Clausius-Duhem inequality are investigated for complex bodies with instantaneous linear elastic response.

2. Kinematics of complex bodies

2.1. Generalities. A regular region $\mathcal{B}$ in the ambient space $\mathbb{R}^3$ - regular in the sense that it is a ‘fit region’ or, more simply, an open set with Lipschitz boundary - is selected to host a body in its macroscopic reference configuration. Every point $x$ from $\mathcal{B}$ represents a material element. It is assumed that subsequent configurations are achieved by means of differentiable bijections (transplacements) from $\mathcal{B}$ to a copy $\hat{\mathbb{R}}^3$ of the ambient space, obtained by means of an isomorphism $i : \mathbb{R}^3 \to \hat{\mathbb{R}}^3$. Maps $x \mapsto y := y(x) \in \hat{\mathbb{R}}^3$, $x \in \mathcal{B}$, are then defined. Their spatial derivative is indicated by $F := Dy(x) \in \text{Hom}(T_x\mathcal{B}, T_y(x)\mathcal{B}_a)$, with $\mathcal{B}_a := y(\mathcal{B})$, and is such that $\det F > 0$. Comparison between the metric $g$ in the actual shape in $\mathcal{B}_a$ and the natural metric $\gamma$ in $\mathcal{B}$ allows one to measure crowding and shearing of material elements. The tensor $E := \frac{1}{2}(y^\#g - \gamma)$, where $y^\#g$ is the pull-back of $g$ through $y$ given in components by $(y^\#g)_{RS} = F^i_R g_{ij} F^j_S$, is then a true measure of deformation: it vanishes under global rigid transplacements. Motions are then time parametrized families of transplacements:

$$(x,t) \mapsto y := y(x,t) \in \hat{\mathbb{R}}^3, \quad x \in \mathcal{B}, \ t \in [0,d].$$

Sufficient smoothness in time $t$ is presumed. The macroscopic velocity is defined by $\dot{y} := \frac{d}{dt}y(x,t)$ in the referential description (that is as a field over the tube $\mathcal{B} \times [0,d]$). The standard kinematics of deformable bodies is represented this way. No geometrical information on the material texture at scales lower than the macroscopic one (substructure) is commonly added in the description of the morphology of the body under scrutiny.
When materials display acute sensibility to phenomena at minute scales, a representation of such phenomena, combined with the description of the macroscopic behavior is required. The standard kinematics is then enriched. A descriptor of the material texture (a morphological descriptor of the substructure) is assigned to each point. It is selected in a set $\mathcal{M}$. A field
\[
(x, t) \mapsto \nu := \nu(x, t) \in \mathcal{M}, \quad x \in \mathcal{B}, \quad t \in [0, d],
\]
is then introduced: differentiability in space and sufficient smoothness in time are presumed. The time rate of change of the morphological descriptor field in referential description is $\dot{\nu} := \frac{d}{dt} \nu(x, t) \in T_{\nu(x,t)} \mathcal{M}$, the spatial derivative is indicated by $N := D\nu(x) \in Hom(T_x \mathcal{B}, T_{\nu(x)} \mathcal{M})$.

In selecting a specific morphological descriptor $\nu$ of the material texture (a parameter also called a fabric tensor or an order parameter) one chooses the prominent geometrical features of the minute world inside the generic material element to be described, transferring their peculiarities at gross scale. However, the construction of the essential structures of the mechanics of complex bodies requires only that $\mathcal{M}$ be considered as an abstract finite-dimensional differentiable manifold. Specific geometrical property of $\mathcal{M}$ have often a clear physical meaning. A metric is the basic ingredient for the representation of the independent peculiar kinetic energy pertaining to the substructure, if such energy can exist in special cases (see [3]). A connection allows one to represent contact substructural interaction (microstresses) and to decompose in invariant way them from the self-actions occurring in each material element (see [39]). For these reasons, the specific nature of $\mathcal{M}$ is left unspecified in the developments below. The subsequent results then hold for a wide class of complex bodies.

2.2. A discontinuity surface. A surface
\[
\Sigma := \{x \in d\mathcal{B}, \ f(x) = 0\},
\]
with $f$ a smooth function, is selected in $\mathcal{B}$. At $x$ the normal $m$ to $\Sigma$ is defined by
\[
m = \frac{\nabla f(x)}{|\nabla f(x)|},
\]
and orients $\Sigma$ locally. Notice that by such a definition the normal is considered as a co-vector. The projector over $\Sigma$ is the second-rank tensor $(I - m \otimes m)$, with $I$ the identity. Let $x \mapsto a(x)$ be a differentiable field over $\mathcal{B}$. Its surface gradient at $x \in \Sigma$ is given by $\nabla_S a := \nabla a(I - m \otimes m)$. The trace of $\nabla_S a$ defines the surface divergence of $a$ at $x$, namely $Div_S a = tr\nabla_S a$.

Assume that $x \mapsto a(x)$ takes values in a linear space. If it is piecewise differentiable and suffers a bounded discontinuity over $\Sigma$, its jump $[a]$ across $\Sigma$ is defined by $[a] := a^+ - a^-$ at every $x \in \Sigma$, that is by the difference between the inner and the outer traces of $a$ at $\Sigma$, defined by the limits $a^\pm := \lim_{\varepsilon \to 0^\pm} a(x \pm \varepsilon m)$. The average $\langle a \rangle$ of $a$ across $\Sigma$ at every $x$ is defined by $2 \langle a \rangle := a^+ + a^-$. For every pair of fields $a_1$ and $a_2$ with the same properties of $a$, the relation $[a_1 a_2] = [a_1] \langle a_2 \rangle + [a_1] [a_2]$ holds if the product $a_1 a_2$ is defined in distributive way.

It is assumed here that both $x \mapsto F$ and $x \mapsto N$ are discontinuous across $\Sigma$, while the field $x \mapsto \nu$ is continuous there. The symbols $F$ and $N$ denote the surface gradients of deformation and the morphological descriptor, respectively. They are defined by
\[
F := \langle F \rangle (I - m \otimes m), \quad N := \langle N \rangle (I - m \otimes m).
\]
2.3. **Linearized kinematics.** Convenience suggests the introduction of the displacement field $(x, t) \longmapsto u := u(x, t) = y(x, t) - i(x), \quad (x, t) \in B \times [0, d].$

The spatial derivative of $x \longmapsto u$ is indicated here by $W := Du(x, t).$ One gets obviously $\dot{y}(x, t) = \dot{u}(x, t)$ and $F = I + W,$ with $I$ the second rank identity tensor. The condition $|W| \ll 1$ defines the infinitesimal deformation regime. The deformation tensor $E$ is then substituted by its linearized part $\varepsilon := \text{sym} W$. Moreover, no distinction is also made between $x \longmapsto \nu$ and $y \longmapsto \nu := \nu \circ y^{-1}.$ Along the discontinuity surface $\Sigma,$ the surface displacement gradient is indicated by $W := \langle W \rangle (I - \mathbb{I} \otimes \mathbb{I}).$

The linearized kinematical setting justifies the mixed use in the same context of symbols adopted elsewhere for distinct actual and referential measures of interactions, as it is made in the ensuing section.

3. **Power and balance of interactions**

3.1. **Classification of the actions.** Relative changes of places between neighboring material elements generate standard actions represented in the Lagrangian description (that is as a field over $\mathcal{B}$) by the first Piola-Kirchhoff stress $P \in \text{Hom} \left( T^*_\nu(x) \mathcal{B}, T^*_\nu(x) \mathcal{M} \right)$ and the vector of body forces $b \in \mathbb{R}^3$. Substructural events may occur within the material elements even when the material elements themselves are frozen in space. Inhomogeneous substructural changes in space generate new contact actions measured by the so-called microstress tensor $S \in \text{Hom} \left( T^*_\nu(x) \mathcal{M} \right)$. External bulk fields can act directly over the substructures (magnetic and/or electric fields, or some other radiative fields). They are represented by the co-vector $\beta \in T^*_\nu(x) \mathcal{M},$ at each $x.$ All these interactions contribute to the expression of the power of all external actions on a generic part of the body, namely on any subset $\mathcal{B}$ of $\mathcal{B}$ with non vanishing volume and the same regularity properties of $\mathcal{B}$ itself. It is said that a generic part $\mathcal{B}$ crosses $\Sigma$ - in this case it is indicated by $\mathcal{B}_\Sigma$ - when $\partial \mathcal{B}_\Sigma \cap \Sigma$ is a simple closed curve where the normal $n$ is defined as a vector from $T^*_\gamma \Sigma$ at all $x \in \partial \mathcal{B}_\Sigma \cap \Sigma$ where the normal to $\partial \mathcal{B}_\Sigma$ exists; in particular $n$ at a given $x$ belongs to the tangent plane to $\Sigma$ at the same point.

The surface $\Sigma$ can be considered as a model of a material layer with vanishing thickness. In this case it is called a structured surface and it is assumed that it can carry standard and substructural surface actions, the former represented by a surface stress $T,$ the latter by a surface microstress $S.$ For a generic part $\mathcal{B}_\Sigma$ crossing $\Sigma,$ the explicit expression of the power is the given by (see [33])

$$P^\text{ext} \left( \dot{y}, \dot{\nu} \right) : = \int_{\mathcal{B}} (b \cdot \dot{y} + \beta \cdot \dot{\nu}) \, dx \int_{\partial \mathcal{B}_\Sigma} (P \mathbf{n} \cdot \dot{y} + S \mathbf{n} \cdot \dot{\nu}) \, d\mathcal{H}^2 + \int_{\partial \mathcal{B}_\Sigma \cap \Sigma} (T \mathbf{n} \cdot \langle \dot{y} \rangle + S \mathbf{n} \cdot \langle \dot{\nu} \rangle ) \, d\mathcal{H}^1.$$
A Lagrangian representation is used in the earlier formula. The link with the Eulerian (actual) description - a link given by the standard Piola transform - is recalled later. Then everything is reduced to the linearized setting.

3.2. Observers. An observer is intrinsically a representation of all geometrical environments which are necessary to describe the morphology of a given body and its motion.

The setting discussed here then incudes the assignment of atlantes over the reference place $\mathcal{B}$, the ambient space $\mathbb{R}^3$, the interval of time and the manifold of substructural shapes $\mathcal{M}$. Changes in such atlantes are changes in observers. Amid them the interest is focused here on synchronous changes in observers - the ones leaving invariant the representation of the time scale - which evaluate the same reference place. In this sense only changes in $\mathbb{R}^3$ and $\mathcal{M}$ are accounted for.

The ambient space $\mathbb{R}^3$ is altered by the action of the group of diffeomorphisms onto itself, namely the group $Diff(\mathbb{R}^3, \mathbb{R}^3)$. Its action has infinitesimal generator coinciding with the vector field which assigns to each point the vector $\frac{df}{ds}(s=0)$, where $f_s$ is a point selected over a smooth curve $s \mapsto f_s$, $s \in \mathbb{R}^7$, in $Diff(\mathbb{R}^3, \mathbb{R}^3)$ such that $f_0 = identity$. The parameter $s$ can be identified with the time.

Since the material substructures are in fact placed in space, changes of frames in $\mathbb{R}^3$ alter in principle the geometry of the substructures and their consequent representation over $\mathcal{M}$. There is exception when $\nu$ represents only a generic property of the material substructure not associated with its geometry in space. Besides this circumstance, one may presume the existence of an homomorphism $h : Diff(\mathbb{R}^3, \mathbb{R}^3) \to G$, with $G$ the Lie group of diffeomorphisms of $\mathcal{M}$ onto itself and $h$ mapping the identity in $Diff(\mathbb{R}^3, \mathbb{R}^3)$ to the identity in $G$. The curve $s \mapsto f_s$ then generates a curve $s \mapsto g_s := h(f_s)$ over $G$, and the corresponding infinitesimal generator of the action of $G$ over $\mathcal{M}$ is then defined by $\xi_{\mathcal{M}}(\nu) := \frac{dg_s}{ds}|_{s=0} = \frac{dh(f_s)}{ds}|_{s=0}$ (see related discussions in \[31\] and \[35\]).

Changes in observers generated by the group $SO(3)$ of the proper rotations, a subgroup of $Diff(\mathbb{R}^3, \mathbb{R}^3)$ are specifically under scrutiny. For $\forall q$ an element of the Lie algebra $so(3)$, $q \in \mathbb{R}^3$, one writes the corresponding $\mathcal{M}$, obtained through $h$, as the product $A(\nu) q$ with $A(\nu) \in Hom(\mathbb{R}^3, T_\nu\mathcal{M})$.

By indicating by $\dot{y}^*$ and $\dot{\nu}^*$ the pull-back in the frame of the first observer of the rates evaluated by the second observer, one gets

$$\dot{y}^* = \dot{y} + q \wedge (y - y_0)$$

where $y_0$ is an arbitrarily fixed centre of rotation in the ambient space, and

$$\dot{\nu}^* = \dot{\nu} + A(\nu) q.$$

Here $s$ is identified with the time.

3.3. Invariance and its consequences.

Axiom 1. At (dynamic) equilibrium $\mathcal{P}^ext_{\Sigma}(\dot{y}, \dot{\nu})$ is invariant under rotational changes in observers.

Theorem 1. (i) If for every $b_\Sigma$ the vector fields assigning the values $\sigma n$ and $A^* S n$ are defined over $\partial b_\Sigma$ and are integrable there, the integral balances of actions on $b_\Sigma$

$$\int_{b_\Sigma} b \ dx + \int_{\partial b_\Sigma} P n \ d\mathcal{H}^2 + \int_{\partial b_\Sigma \cap \Sigma} T n \ d\mathcal{H}^1 = 0,$$
\[
\int_{b_\Sigma} ((x - x_0) \wedge b + A^* \beta) \, dx + \int_{\partial b_\Sigma} ((x - x_0) \wedge Pn + A^* Sn) \, d\mathcal{H}^2 +
\int_{\partial b_\Sigma \cap \Sigma} ((y - y_0) \wedge Tn + A^* Sn) \, d\mathcal{H}^1 = 0,
\]

(ii) Moreover, if the tensor fields \( x \mapsto P, S \) are of class \( C^1 (B \setminus \Sigma) \) and are also continuous over the boundary of the body, then
\[
\text{Div} P + b = 0,
\]
and there exist a co-vector field \( x \mapsto z \in T_{\nu(x)} M \) such that
\[
\text{skw} (PF^*) = \frac{1}{2} e (A^* z + (\nabla A^*) S)
\]
and
\[
\text{Div} S - z + \beta = 0,
\]
with \( z = z_1 + z_2, \; z_2 \in \text{Ker} A^* \) in the bulk. Additionally, if the tensor fields \( x \mapsto T, S \) are of class \( C^1 (\Sigma) \) along the surface \( \Sigma \) and are also continuous along its boundary, one gets
\[
\text{Div}_{\Sigma} T + [P] m = 0,
\]
and there exists a co-vector field \( x \mapsto \hat{z} \in T_{\nu(x)} M, \; \text{with} \; x \in \Sigma \), such that
\[
\text{skw} (T \hat{F}^*) = \frac{1}{2} e (A^* \hat{z} + (\nabla_{\Sigma} A^*) S)
\]
and
\[
\text{Div}_{\Sigma} S - \hat{z} + [S] m = 0.
\]
(iii) If the rate fields \( (x, t) \mapsto \dot{y}(x, t) \in \hat{\mathbb{R}}^3 \) and \( (x, t) \mapsto \dot{\nu}(x, t) \in T_{\nu(x)} M \) are differentiable in space, the local balances imply
\[
\mathcal{P}^\text{ext}_b (\dot{y}, \dot{\nu}) = \mathcal{P}^\text{int}_b (\dot{y}, \dot{\nu})
\]
where
\[
\mathcal{P}^\text{int}_b (\dot{y}, \dot{\nu}) := \int_B (P \cdot F + z \cdot \dot{\nu} + S \cdot \dot{N}) \, dx + \int_{\partial b_\Sigma \cap \Sigma} (T \cdot F + \hat{z} \cdot \dot{\nu} + S \cdot \dot{N}) \, d\mathcal{H}^2.
\]

\( \mathcal{P}^\text{int}_b (\dot{y}, \dot{\nu}) \) is called an inner (or internal) power. \( e \) indicates Ricci's alternating index.

If one enforces Axiom 1 with a requirement of invariance with respect to the action of the semi-direct product \( \hat{\mathbb{R}}^3 \ltimes \text{SO} (3) \), rather than calling upon only the action of \( \text{SO} (3) \), a proof of Theorem 1 can be found in \cite{33}. The weaker requirement here imposes a change in the proof. By using the Axiom 1, in fact, one first obtains only the integral balance of moments, then one first exploits the arbitrariness of the centre of rotation and substitutes \( y_0 \) with \( y_0 + w \), with \( w \) an arbitrary vector depending only on time. If one subtracts the integral balance of moments from the resulting equation (the one obtained by the substitution \( y_0 \mapsto y_0 + w \)) and the arbitrariness of \( w \) also imply the integral balance of forces. Pointwise balances follow by the standard use of Gauss theorem (see also remarks in \cite{34, 35}).

In the Eulerian representation the balance equations become
\[
div \sigma + b_a = 0,
\]
\[
\text{skw} (\sigma) = \frac{1}{2} e (A^* z_a + (\text{grad} A^*) S_a),
\]
\[
\text{div} S_a - z_a + \beta_a = 0,
\]
in the bulk and
\[
div \mathbb{T}_a + [\sigma] m_a = 0, \\
\text{skw}(\mathbb{T}_a) = \frac{1}{2} \varepsilon (A^* \mathbb{J}_a + (\text{grad}_\Sigma A^*) \mathbb{S}_a), \\
div \mathbb{S}_a - \mathbb{J}_a + [\mathbb{S}_a] m_a = 0,
\]
over \( \Sigma \). Moreover, by indicating by \( v \) and \( \nu \) the values the values of the actual (Eulerian) representation of the (differentiable) velocity fields \( (y, t) \mapsto v(y, t) \) and \( (y, t) \mapsto \nu(y, t) \), one also get the equation
\[
\int_{y(b)} (b_a \cdot v + \beta_a \cdot v) \ dy + \int_{\partial y(b)} (\sigma n_a \cdot v + \mathbb{S}_a n_a \cdot v) \ dH^2 + \\
\int_{\partial y(b) \cap y(\Sigma)} (\mathbb{T}_a n_a \cdot \langle v \rangle + \mathbb{S}_a n_a \cdot \langle v \rangle) \ dH^1 = \int_{y(b)} (\sigma \cdot \text{grad} v + z_a \cdot v + \mathbb{S}_a \cdot \text{grad} v) \ dy + \\
\int_{y(b) \cap y(\Sigma)} (\mathbb{T}_a \cdot \text{grad}_\Sigma v + \mathbb{J}_a \cdot v + \mathbb{S}_a \cdot \text{grad}_\Sigma v) \ dH^2.
\]
In previous formulas \( n_a \), \( m_a \) and \( n_a \) are the counterparts of \( n \), \( m \) and \( n \) in \( E_a \). The differential operators \( \text{div} \), \( \text{div}_\Sigma \), \( \text{grad} \) and \( \text{grad}_\Sigma \) involve derivatives with respect to the coordinates \( y \) in the current macroscopic placement. The index \( a \) means ‘actual’. Moreover, the actual measures of interactions are obtained by means of the standard Piola transform as
\[
b_a \equiv (\det F)^{-1} b, \quad \sigma \equiv (\det F)^{-1} PF^*, \\
z_a \equiv (\det F)^{-1} z, \quad \beta_a \equiv (\det F)^{-1} \beta, \quad \mathbb{S}_a \equiv (\det F)^{-1} SF^*, \\
\mathbb{T}_a \equiv (\det F)^{-1} TF^*, \quad \mathbb{J}_a \equiv (\det F)^{-1} \mathbb{J}, \quad \mathbb{S}_a \equiv (\det F)^{-1} SF^*.
\]
The proof of the Piola transform can be found on any textbook in nonlinear continuum mechanics. Less popular is its counterpart on surfaces embedded in a body: for the relevant proof see [29].

The Piola transform implies that, in infinitesimal deformation setting, referential and actual measures of interaction in the bulk and over the surface \( \Sigma \) coincide as \( |W| \) and \( |\mathbb{W}| \) tend to zero.

The ensuing sections are just restricted to the infinitesimal deformation setting. Thus, by taking into account the substantial coincidence of referential and actual measures of interactions in the linearized setting (as remarked in earlier comments), although the Cauchy stress \( \sigma \) is used, the index ”\( a \)” in the other actual measures of interaction is omitted for the sake of conciseness.

4. Linear constitutive structures

Constitutive structures of the type
\[
\sigma = \sigma(W, \nu, N), \quad z = z(W, \nu, N), \quad \mathbb{S} = \mathbb{S}(W, \nu, N),
\]
in the bulk and
\[
\mathbb{T} = \mathbb{T}(\mathbb{W}, \nu, N), \quad \mathbb{J} = \mathbb{J}(\mathbb{W}, \nu, N), \quad \mathbb{S} = \mathbb{S}(\mathbb{W}, \nu, N),
\]
on the surface \( \Sigma \) can be selected for elastic complex bodies. The entries of the previous constitutive structures are instantaneous value. Linearization of them about a pair \((\dot{u}, \dot{\nu})\) requires the embedding of \( \mathcal{M} \) in some linear space isomorphic to \( \mathbb{R}^k \) for some \( k \). This embedding allows one to consider the space of pairs of maps
(y, ν) as an infinite-dimensional manifold modelled over a Sobolev space: the use of the Frechet derivative in the linearization procedure then follows.

It is then assumed that \( M \) is endowed with a \( C^1 \) Riemannian metric and the relevant Levi-Civita parallel transport. Such structural assumption is constitutive in its essential nature. By Nash theorem an isometric embedding of \( M \) in a linear space is then always available but it is neither unique nor rigid. The selection of an embedding plays the role of a constitutive ingredient of every special model.

Under these conditions, constitutive equations expressed by linear operators \( L^{(\cdot)} \), namely
\[
\sigma = L^{(\sigma)} (W, \nu, N), \quad z = L^{(z)} (W, \nu, N), \quad S = L^{(S)} (W, \nu, N),
\]
in the bulk and
\[
T = L^{(T)} (\mathcal{W}, \nu, N), \quad \mathfrak{z} = L^{(\mathfrak{z})} (\mathcal{W}, \nu, N), \quad S = L^{(S)} (\mathcal{W}, \nu, N),
\]
on the discontinuity surface, make sense.

The procedure discussed here holds also when the measures of interaction depend not only on instantaneous values of the state variables but also on their entire history. In this case memory effects can be accounted for. Viscosity come into play.

5. Characterization of the bulk free energy in terms of work: variations on a Del Piero’s theme

5.1. Histories. At a point \( x \), a history is a \( BV \) right continuous map
\[
H : \mathbb{R}^+ \to M_{3 \times 3} \times \mathbb{R}^k \times M_{k \times 3}
\]
such that, for \( s \in \mathbb{R}^+ \),
\[
H(s) = (W(s), \nu(s), N(s)) .
\]
Its restriction \( K^r_p \) over an interval \([r, p)\) is called process and is defined by
\[
K^r_p(s) := (W(r + s), \nu(r + s), N(r + s)), \quad 0 \leq s < p - r .
\]
As shorthand notation, \( K_p \) indicates a process when it is of the type \( K^0_p \). The symbols \( \Gamma \) and \( \Pi \) denote the spaces of histories and that of processes, respectively. Of course, here \( \Pi \subseteq \Gamma \). Processes prolong histories. Given \( H \), the history
\[
(K_p \ast H)(s) := \begin{cases} 
K_p(s) & 0 \leq s < p \\
H(s - p) & s \geq p
\end{cases}
\]
is called prolongation of \( H \) by means of the process \( K_p \). It is assumed also that
\[
K_p (p)^- = H (0) ,
\]
where \( K_p (p)^- := \lim_{s \nearrow p} K_p (s) \), to assure differentiability in time, a property necessary for later use.

Along \( \Sigma \), a surface history
\[
s \mapsto \mathcal{H}(s) := (\mathcal{W}(s), \langle \nu \rangle (s), N(s)) ,
\]
can be defined when the map \( x \mapsto \nu(x) \) is continuous across the surface. Of course, since \( \mathcal{M} \) is embedded in a linear space, the average \( \langle \nu \rangle \) makes now sense so that one may consider also \( \mathcal{H}(s) \) to be coincident with \( (\mathcal{W}(s), \langle \nu \rangle (s), N(s)) \). The results collected below hold also in this case.
5.2. History dependent measures of interaction and equivalence of histories. In the earlier notes, the state variables have been indicated just formally. When memory effects are accounted for, the notion of state requires careful definition because different equivalence relations between pairs of histories exist so that the notion of state appears to be rather natural in terms of equivalence classes. Such a question has been tackled variously (see \cite{22, 21, 25}) on the basis of the abstract approach to thermodynamics proposed in \cite{12} (see also \cite{13, 40, 41}).

In what follows, the point of view developed by Del Piero in \cite{21} for linear simple bodies with memory is adapted to cover linear complex bodies displaying memory effects at macroscopic and microscopic scales. The aim is of (1) characterizing states and (2) deducing the main property of the free energy of such complex bodies. A concrete example of such bodies is the one of relaxor ferroelectrics. Butterfly loops in the diagrams of strain versus applied electric fields indicate the presence of memory effects \cite{41}.

Linear constitutive structures in the bulk are here assumed to be of the form

\[
\sigma (H) = G_{\sigma W} (0) W (0) + G_{\sigma \nu} (0) \nu (0) + G_{\sigma \nu} (0) N (0) + \int_0^{+\infty} \left( \dot{G}_{\sigma W} (s) W (s) + \dot{G}_{\sigma \nu} (s) \nu (s) + \dot{G}_{\sigma \nu} (s) N (s) \right) ds,
\]

\[
S (H) = G_{SW} (0) W (0) + G_{S \nu} (0) \nu (0) + G_{SN} (0) N (0) + \int_0^{+\infty} \left( \dot{G}_{SW} (s) W (s) + \dot{G}_{S \nu} (s) \nu (s) + \dot{G}_{SN} (s) N (s) \right) ds,
\]

\[
z (H) = G_{zW} (0) W (0) + G_{z \nu} (0) \nu (0) + G_{zN} (0) N (0) + \int_0^{+\infty} \left( \dot{G}_{zW} (s) W (s) + \dot{G}_{z \nu} (s) \nu (s) + \dot{G}_{zN} (s) N (s) \right) ds,
\]

where the \(G_{AB}\)'s are the so-called relaxation functions, tensor functions (taking values in different tensor spaces) that are assumed to be Lebesgue integrable in time: they are absolutely continuous and the limit \(G_{AB}(\infty) := \lim_{s \to +\infty} G_{AB}(s)\) exists. Of course, the indexes \(A\) and \(B\) run in \(\{\sigma, z, S\}\) and \(\{W, \nu, N\}\) respectively.

Notice that the integrals above are well defined because \(M\) is considered embedded in a linear space isomorphic to \(\mathbb{R}^k\) for some \(k\).

**Definition 1.** Two generic histories \(H\) and \(H'\), such that \(H(0) = H'(0)\), are said to be equivalent (in symbols \(H \sim H'\)) when for every process \(K_p\), with \(p \geq 0\),

\[
\sigma (K_p * H) = \sigma (K_p * H'), \quad z (K_p * H) = z (K_p * H'), \quad S (K_p * H) = S (K_p * H').
\]

As a consequence, for \(p \geq 0\), the condition of equivalence \(H \sim H'\) implies

\[
\int_0^{+\infty} \left( \dot{G}_{\sigma W} (s + p) W (s) + \dot{G}_{\sigma \nu} (s + p) \nu (s) + \dot{G}_{\sigma \nu} (s + p) N (s) \right) ds =
\]

\[
= \int_0^{+\infty} \left( \dot{G}_{\sigma W} (s + p) W' (s) + \dot{G}_{\sigma \nu} (s + p) \nu' (s) + \dot{G}_{\sigma \nu} (s + p) N' (s) \right) ds
\]

\[
= \int_0^{+\infty} \left( \dot{G}_{SW} (s + p) W (s) + \dot{G}_{S \nu} (s + p) \nu (s) + \dot{G}_{SN} (s + p) N (s) \right) ds
\]

\[
= \int_0^{+\infty} \left( \dot{G}_{SW} (s + p) W' (s) + \dot{G}_{S \nu} (s + p) \nu' (s) + \dot{G}_{SN} (s + p) N' (s) \right) ds.
\]
\[
\int_{0}^{+\infty} \left( \dot{G}_{zW} (s+p) W(s) + \dot{G}_{\sigma N} (s + p) N(s) \right) \, ds = \\
= \int_{0}^{+\infty} \left( \dot{G}_{zW} (s+p) W'(s) + \dot{G}_{\sigma N} (s + p) N'(s) \right) \, ds.
\]

Let the distance \(d(H, H')\) be defined by
\[
d(H, H') := \sup_{t \geq 0} \left\{ \int_{0}^{+\infty} \left( \dot{G}_{\sigma W} (s+t) W(s) + \dot{G}_{\sigma N} (s + t) N(s) \right) \, ds - \right.
\[
- \int_{0}^{+\infty} \left( \dot{G}_{zW} (s+t) W'(s) + \dot{G}_{\sigma N} (s + t) N'(s) \right) \, ds \bigg| + \\
+ \int_{0}^{+\infty} \left( \dot{G}_{zW} (s+t) W(s) + \dot{G}_{\sigma N} (s + t) N(s) \right) \, ds - \\
- \int_{0}^{+\infty} \left( \dot{G}_{zW} (s+t) W'(s) + \dot{G}_{\sigma N} (s + t) N'(s) \right) \, ds \bigg| + \\
+ \int_{0}^{+\infty} \left( \dot{G}_{SW} (s+t) W(s) + \dot{G}_{SN} (s + t) N(s) \right) \, ds - \\
- \int_{0}^{+\infty} \left( \dot{G}_{SW} (s+t) W'(s) + \dot{G}_{SN} (s + t) N'(s) \right) \, ds \bigg\}.
\]

It induces a pseudometric in the space \(\Gamma\) of histories and a metric in the quotient space \(\Gamma/\sim\). By taking the limit \(t \to 0\) in the expression above, one gets
\[
|\sigma(H) - \sigma(H')| + |z(H) - z(H')| + |S(H) - S(H')| \leq d(H, H') + \\
|\langle G_{\sigma W} (0) W(0) + G_{\sigma N} (0) \rangle \nu(0) + G_{\sigma N} (0) N(0) \rangle - \rangle G_{\sigma W} (0) W(0) + G_{\sigma N} (0) \rangle \nu'(0) + G_{\sigma N} (0) N'(0) \rangle | + \\
|\langle G_{zW} (0) W(0) + G_{zN} (0) \rangle \nu(0) + G_{zN} (0) N(0) \rangle - \rangle G_{zW} (0) W(0) + G_{zN} (0) \rangle \nu'(0) + G_{zN} (0) N'(0) \rangle | + \\
|\langle G_{SW} (0) W(0) + G_{SN} (0) \rangle \nu(0) + G_{SN} (0) N(0) \rangle - \rangle G_{SW} (0) W(0) + G_{SN} (0) \rangle \nu'(0) + G_{SN} (0) N'(0) \rangle |.
\]

Since two equivalent histories are characterized by identical initial values, namely \(W(0) = W'(0), \nu(0) = \nu'(0), N(0) = N'(0)\), from previous inequality it follows that two equivalent histories determine the same macroscopic stress, microstress and substructural self-action. The proof of the analogous property for simple bodies in [21] is based on the use of a seminorm.

**Proposition 1.** The pseudometric \(d(\cdot, \cdot)\) has the following properties:

- **Contraction:** for every \(p > 0\)
  \[
d(K_p * H, K_p * H') \leq d(H, H').
  \]

- **Fading memory:** for every \(\varepsilon > 0\) there exists \(r\) such that, for every \(p > r\), one gets
  \[
d(K_p * H, K_p * H') < \varepsilon.
  \]

- **Approachability:** if \(H_p(\cdot)^- = H'(0)\), then
  \[
  \lim_{d \to +\infty} d(H, H_p * H') = 0,
  \]
  with \(H_p\) the process generated by \(H\) over \([0, p)\).
Proof. Contraction arises directly from the definition. In fact, in writing explicitly \( d\left( K_p \ast H, K_p \ast H' \right) \) by taking into account (5.1) and the definition of the semimetric \( d\left( \cdot, \cdot \right) \), one manages integrals containing terms of the type \( \int_{A}^{B} (s + t) B(s - r) \) with \( A \) and \( B \) running in \( \{ \sigma, z, S \} \) and \( \{ W, \nu, N \} \) respectively. Each of these terms is also equal to \( \int_{A}^{B} (s + t + r) B(s) \) (see also [21]). Consequently, by taking the superenum over \( t, p > 0 \) one gets the contraction property straight away.

Let now define

\[
M := \max \left\{ \sup_{s > 0} |W'(s)|, \sup_{s > 0} |W(s)|, \sup_{s > 0} |v'(s)|, \sup_{s > 0} |v(s)|, \sup_{s > 0} |N'(s)|, \sup_{s > 0} |N(s)| \right\}.
\]

By taking into account that

\[
\left| \int_{a}^{b} \hat{G}_{\bar{A}}(s)B(s) \, ds \right| \leq \sup_{s \in [a,b]} |B(s)| \int_{a}^{b} |\hat{G}_{\bar{A}}(s)| \, ds,
\]

one obtains

\[
d\left( K_p \ast H, K_p \ast H' \right) \leq 2M \sup_{t > 0} \left\{ \int_{t}^{+\infty} |\hat{G}_{\sigma W} (s + t)| \, ds + \right.
\]
\[
+ \int_{t}^{+\infty} |\hat{G}_{\sigma v} (s + t)| \, ds + \int_{t}^{+\infty} |\hat{G}_{\sigma N} (s + t)| \, ds \bigg. + \int_{t}^{+\infty} |\hat{G}_{z W} (s + t)| \, ds + \int_{t}^{+\infty} |\hat{G}_{z v} (s + t)| \, ds + \int_{t}^{+\infty} |\hat{G}_{z N} (s + t)| \, ds \bigg.
\]
\[
+ \int_{t}^{+\infty} |\hat{G}_{s W} (s + t)| \, ds + \int_{t}^{+\infty} |\hat{G}_{s v} (s + t)| \, ds + \int_{t}^{+\infty} |\hat{G}_{s N} (s + t)| \, ds \bigg\}.
\]

However, since it has been assumed that the maps \( s \mapsto |\hat{G}_{\bar{A}}|(s) \) are integrable, there exists \( r \) such that, for \( r > m \), one may find \( \varepsilon > 0 \) such that the right-hand side of the previous relation is lesser or equal to \( \varepsilon (2M)^{-1} \). Fading memory then follows. It also implies the property of approachability. In fact, from

\[
d(H_p \ast H', H) = d(H_p \ast H', H_p \ast H^d)
\]

and fading memory, the approachability can be obtained by letting \( p \) to \( +\infty \). \( \Box \)

Previous theorem suggests the following definition:

**Definition 2.** \( H \) is said to be approachable from another history \( H' \) if there exists a family of processes \( p \mapsto K_p, p \in \mathbb{R}^+ \) prolonging \( H' \) and such that \( K_p \ast H' \) converges to \( H' \) with respect to the pseudometric \( d\left( \cdot, \cdot \right) \) as \( p \to +\infty \).

5.3. **States and actions.** The state space is identified here with the space of histories \( \Gamma \) endowed with the norm

\[
\|H\|_{\Gamma} = \|F\|_{L^2} + |\nu|_{W^{1,2}}.
\]

In this way each state is defined to within an equivalent history.

**Definition 3.** A function \( f : \Gamma \to \mathbb{R} \) is called a state function if \( H \sim H' \) implies \( f(H) = f(H') \).

**Definition 4.** A function \( a : \Gamma \times \Pi \to \mathbb{R} \) is called an action if

\[
(1): \text{a is additive with respect to prolongations, namely} \quad a(\bar{K} \ast K, H) = a(\bar{K}' \ast K, H) + a(K, H),
\]
(2): the map \( a(K,\cdot) : \Gamma \to \mathbb{R} \) is continuous.

**Definition 5.** The action \( a \) satisfies the **dissipation property** along \( H \) if, for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that
\[
d(K_p * H, H) < \delta \implies a(K_p, H) > -\varepsilon.
\]

**Definition 6.** Given a generic action \( a(\cdot,\cdot) \), a function \( f : \Gamma \to \mathbb{R} \) is called a **lower potential** for \( a(\cdot,\cdot) \) if for every \( H \) and \( H' \) belonging to \( \Gamma \) and for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[
f(H) - f(H') < a(K, H) + \varepsilon,
\]
for every \( K \) such that \( d(K * H', H) < \delta \).

The definition of state function comes from [21] - the difference is here only the extended meaning of the history - while the definitions of action and lower potential have been introduced in [12].

**5.4. Work density in the bulk.** At every \( x \in \mathcal{B} \) the work density \( w(H) \) is defined by
\[
w(H) := \int_{0}^{+\infty} \left[ \sigma(H^s) \cdot \dot{F}(s) + z(H^s) \cdot \dot{\nu}(s) + S(H^s) \cdot \dot{N}(s) \right] \, ds.
\]
Note the different algebraic sign in the analogous definition of \( w(H) \) given in [21] with reference to simple bodies, i.e. in absence of substructural interactions.

Given an history \( H \) and a prolongation \( K_p = (F_p, \nu_p, N_p) \) of it, the **work density over the prolongation** \( K_p \), indicated by \( w(K_p, H) \) is then defined by
\[
w(K_p, H) := w(K_p * H) - w(H)
\]
that is
\[
w(K_p, H) = \int_{0}^{p} \left[ \sigma(K^s_p * H) \cdot \dot{W}(s) + z(K^s_p * H) \cdot \dot{\nu}(s) + S(K^s_p * H) \cdot \dot{N}(s) \right] \, ds.
\]
An analogous power density over prolongations is defined in [21] but with reference to relative continuations (defined below). The use of relative prolongation implies the appearance of further terms in the explicit expression of \( w(K_d, H) \), terms due to the jump in \( H(0) \).

**Theorem 2.** The work density over a prolongation is an action and for any fixed \( K_p \) the map \( w(K_p,\cdot) \) is a state function.

**Proof.** To prove the latter property, first define
\[
\tilde{M} := \sup \left\{ \int_{0}^{p} |\dot{W}(s)| \, ds, \int_{0}^{p} |\dot{\nu}(s)| \, ds, \int_{0}^{p} |\dot{N}(s)| \, ds \right\}
\]
and remind that the manifold of substructural shapes is embedded in a linear space. It then follows that
\[
|w(K_p, H) - w(K_p, H')| \leq \tilde{M} d(H, H')
\]
from which one realizes that \( w(K_p,\cdot) \) is a state function. Previous inequality implies also that \( w(K_p,\cdot) \) is also Lipschitz continuous. The additivity of \( w \) with respect to the processes is implied by the definition.

The following lemma is a version of a proposition in [21] (see also [20]), the proof of which can be easily adapted to the present case.
Lemma 1. Assume that $\nu$ can be freely selected in the linear space in which the manifold of substructural shapes is isometrically embedded. Given two different histories $H = (W, \nu, N)$ and $H' = (W', \nu', N')$ and a process $L_p(r)$ defined by

$$L_p(r) := \left( \frac{p-r}{p} W(p) + \frac{r}{p} W'(0), \frac{p-r}{p} \nu(p) + \frac{r}{p} \nu'(0), \frac{p-r}{p} N(p) + \frac{r}{p} N'(0) \right),$$

one gets

$$\lim_{p \to +\infty} d(H_p * L_p * H', H) = 0$$

and

$$w(H_p * L_p * H') = w(H) + w(H') + \frac{1}{2} \left( \sum_{A,B} G_{AB}(\infty) B(\infty) \cdot B(\infty) - \sum_{A,B'} G_{AB'}(\infty) B'(0) \cdot B'(0) \right).$$

Definition 7. A history $H$ is said to be $w-$approachable from another history $H'$ if $H$ is approachable from $H'$ and the sequence $(p \mapsto K_p, p \in \mathbb{R}^+)$ is such that the sequence $p \mapsto w(K_p, H)$ converges too.

Theorem 3. If $\nu$ can be freely selected in the linear space in which the manifold of substructural shapes is isometrically embedded, the space $\Gamma$ is closed under $w-$approachability.

The proofs of both the previous lemma and the approachability theorem follow the same paths of the analogous results for simple bodies in [21]. The circumstance that $\nu$ is now selected in a linear space implies just that, in re-following the path of the proofs in [21], one needs only to consider the distance $d(\cdot, \cdot)$ and the presence of the substructural terms.

In order to prove the closure theorem (under $w-$approachability) the key point is the use of the last result of the Lemma. In fact, one replaces $H_p * L_p$ with

$$K_{2p} := \left( H_p - H'(0) \right),$$

where $H'(0)$ is the constant history of value $H'(0)$ and duration $p$, then one proves by Lemma that the work expended along the continuation $K_{2p}$, namely $w(K_{2p}, H)$ converges to $w(H)$ plus the work done in the extreme retardation (see [21]) from $H'(0)$ to $H(\infty)$.

The work $w$ helps also in characterizing the kernels in the constitutive expressions of the interaction measures.

Definition 8. The relaxation functions $s \mapsto G_{AB}(s)$ are said to be dissipative if

$$w(H) \geq 0$$

for any $H \in \Gamma$ such that $F(\infty) = 0$, $\nu(\infty) = 0$, $N(\infty) = 0$.

Note that the requirement of the positivity of the work is weaker with respect

5.5. Relaxed work.

Definition 9. For every pair of $w-$approachable histories $H$ and $H'$, the relaxed work $w_H^r(H')$ along $H$, starting from $H'$, is defined by

$$w_H^r(H') := \inf \left\{ \liminf_{p \to +\infty} w(K_p, H') \mid K_p \in \Pi, \lim_{p \to +\infty} d(K_p * H', H) = 0 \right\}.$$
Moreover, an estimate follows:

$$w_H^r(H) \leq w(H) + \frac{1}{2} \left( \sum_{A,B} G_{AB}(\infty) B(\infty) \cdot B(\infty) - \sum_{A,B'} G_{AB'}(\infty) B'(0) \cdot B'(0) \right).$$

It means that the relaxed power along $H$, starting from $H'$, is bounded from above by the power along $H$ plus the difference of the powers under extreme retardation from $H'(0)$ to $H(\infty)$. By restricting $w_H^r(\cdot)$ to the histories $K$ that are prolongations of $H$ itself, one gets an additional upper bound:

$$w_H^r(K^\dagger \ast H) \leq w(K^\dagger, H).$$

**Theorem 4.** The following statements hold:

1. Both $w_H^r(\cdot)$ and $w_{H'}(H)$ are state functions.
2. (Sub-additivity.) For $H$, $H'$ and $H''$ histories such that $w_H^r(H) > -\infty$ and $w_{H''}(H') > -\infty$ one gets the triangular inequality
   \[ w_H^r(H') \leq w_{H''}(H') + w_H^r(H). \]
3. (Lower semicontinuity.) If $w_H^r(H) > -\infty$, $w_H^r(\cdot)$ is lower semicontinuous.
4. (Dissipation inequality.) If $w_H^r(H) > -\infty$ and $K_p$ is a process such that $w_H^r(K_p \ast H) > -\infty$, then
   \[ w_H^r(K_d \ast H) - w_{H'}^r(H) \leq w(K_p \ast H), \]
   moreover, if $K_p$ is such that $w_{H'}^r(K_p \ast H) > -\infty$ and $w_{K_p \ast H'}^r(H) > -\infty$ then
   \[ w_H^r(H) - w_{K_p \ast H'}^r(H) \leq w(K_p, H'). \]

The proof of the theorem above is essentially independent of the explicit expression of the work. For this reason the proof of the analogous result in [21] applies providing one substitutes the norm used there with the distance defined above. Other results in [21] can be adapted here. Such results are listed below. Differences rest essentially on (i) the presence of substructural terms, (ii) the use of the distance $d(\cdot, \cdot)$ and (iii) the use of strict continuations of histories, not the relative continuations used in [21]. The latter are indicated by $R$ superposed to $\ast$ and are defined by

$$\left(K_p \ast H\right)(s) := \begin{cases} K_p(s) + H(0) & 0 \leq s < d \\ H(s - d) & s \geq d \end{cases}.$$

**Theorem 5.** The following statements are equivalent:

1. $w$ satisfies the dissipation property on all constant histories.
2. $w_{H^\dagger}^r(H^\dagger) = 0$ along every constant history $H^\dagger$.
3. $w_{H^\dagger}^r(H) \geq 0$ for every history $H$.

---

<sup>1</sup>Precisely, for any $\varepsilon > 0$ there exists $\delta > 0$ such that, given $H$, for any $H_1$ such that $d(H_1, H) < \delta$, one gets

$$w_{H_1}^r(H_1) \geq w_{H^\dagger}^r(H) - \varepsilon.$$
For any history $H$ one gets
\[ w(H) \geq \frac{1}{2} \left( \sum_{A,B} G_{AB}(\infty)B(0) \cdot B(0) - \sum_{A,B} G_{AB}(\infty)B(\infty) \cdot B(\infty) \right). \]

The relaxation functions $s \mapsto G_{AB}(s)$ are dissipative.

For every pair of histories $H$ and $H'$.

For every constant history $H^\dagger$, $w_{H^\dagger}(\cdot)$ and $w_{\{,\}}(H^\dagger)$ are determined respectively by $w_{0^\dagger}(\cdot)$ and $w_{\{,\}}(0^\dagger)$, namely
\[ w_{H^\dagger}(H) = w_{0^\dagger}(H) - \frac{1}{2} \sum_{A,B} G_{AB}(\infty)B_{H^\dagger} \cdot B_{H^\dagger}, \]
\[ w_{H^\dagger}(H^\dagger) = w_{H^\dagger}(0^\dagger) + \frac{1}{2} \sum_{A,B} G_{AB}(\infty)B_{H^\dagger} \cdot B_{H^\dagger}. \]

The restriction of $w_{\{,\}}(\cdot)$ to constant histories is determined: for every pair of constant histories $H_1^\dagger$ and $H_2^\dagger$ one gets
\[ w_{H_1^\dagger}(H_2^\dagger) = \frac{1}{2} \sum_{A,B} G_{AB}(\infty)B_{H_1^\dagger} \cdot B_{H_2^\dagger} - \sum_{A,B} G_{AB}(\infty)B_{H_1^\dagger} \cdot B_{H_1^\dagger}. \]

For every pair of histories $H$ and $H'$
\[ w_{H^\dagger}(H) \geq w_{H^\dagger}(H(0)^\dagger) = w_{H^\dagger}(0^\dagger) + \frac{1}{2} \sum_{A,B} G_{AB}(\infty)B(0) \cdot B(0). \]

For every $H$
\[ -w_0^H(0^\dagger) \geq \frac{1}{2} \sum_{A,B} G_{AB}(\infty)B_{H(0)} \cdot B_{H(0)}. \]

For every pair of histories $H$ and $H'$
\[ w_{H^\dagger}(H^\dagger) \geq -w_0^H(0^\dagger) + \frac{1}{2} \sum_{A,B} G_{AB}(\infty)B_{H(0)} \cdot B_{H(0)}. \]

For every pair of histories $H$ and $H'$
\[ w_{0^\dagger}(H) \geq w_{H^\dagger}(H). \]

For every $H$
\[ w_0^H(0^\dagger) = \inf_{K \in \Pi} w(K, H). \]

For every constant history $H^\dagger$, the functional $w_{\{,\}}(H^\dagger)$ is upper semicontinuous.

The symbols $B_{H^\dagger}$ and $B_{H(0)}$ used earlier indicate one of the entries of the list defining the state, evaluated along the constant history $H^\dagger$ or at $H(0)$, respectively.
5.6. **Free energies.** The free energy can be defined in terms of actions (see [12]). There are several possible free energies. Upper and lower bounds for their set can be determined.

**Definition 11.** Every lower potential of \( w \) is called a free energy.

The properties of the power discussed above allow one to prove the following theorem.

**Theorem 7.** The following assertions hold:

1. Every free energy \( \psi \) satisfies the dissipation inequality
   \[
   \psi(K * H) - \psi(H) < w(K, H)
   \]
   for every \( H \in \Gamma \) and every compatible \( K \in \Pi \). Moreover, every l.s.c. function \( \psi : \Gamma \to \mathbb{R} \) that satisfies the dissipation inequality is a free energy.

2. If the dissipation postulate is satisfied, then, for every \( H \) and \( H' \) belonging to \( \Gamma \), the maps \( w_H^r(\cdot) \) and \( -w_H^l(\cdot) \) are free energies. Moreover, for every free energy \( \psi \) one gets
   \[
   -w_H^r(H') \leq \psi(H) - \psi(H') \leq w_H^l(H')
   \]
   for arbitrary histories \( H \) and \( H' \). In particular, if there is \( H' \) and a family of free energies \( \psi_s \) such that \( \psi_s(H') = 0 \), the maps \( w_H^r(\cdot) \) and \( -w_H^l(\cdot) \) are the maximum and the minimum free energies in such a family.

3. Each free energy is a state function and, for every \( H \), it satisfies the inequality
   \[
   \psi(H(0)\dagger) \leq \psi(H).
   \]
   In particular, the restriction of the free energy to constant histories is given by
   \[
   \psi(H\dagger) - \psi(0\dagger) = \frac{1}{2} \sum_{A,B} G_{AB}(\infty) B_{H\dagger} \cdot B_{H\dagger}.
   \]

The technique of the proof is strictly analogous (modulo the variations associated with the use of the metric \( d(\cdot,\cdot) \)) to the one used in [21] for an analogous result for simple viscoelastic bodies (see also [12]), so the details of the proof are not reported here. In [21] a weaker condition is adopted: discontinuity is admitted at 0 under relative continuations. Here, the need of the use of the chain rule in an ensuing section suggests to avoid this discontinuity for the sake of simplicity.

6. **Characterization of the surface free energy in terms of the surface work**

The propositions presented so far can be extended in presence of structured discontinuity surfaces. In particular, it is assumed that across \( \Sigma \) the map \( \nu \) is continuous while the gradients \( W \) and \( N \) suffer bounded jumps.

The attention is focused on constitutive relations of the type

\[
T(H) = G_{TW}(0) W(0) + G_{TV}(0) \nu(0) + G_{TN}(0) N(0) +
\]
\[
+ \int_0^{+\infty} \left[ \dot{G}_{TW}(s) W(s) + \dot{G}_{TV}(s) \nu(s) + \dot{G}_{TN}(s) N(s) \right] ds,
\]
Definition 12. Two surface histories \( \mathcal{H} \) and \( \mathcal{H}' \), such that \( \mathcal{H}(0) = \mathcal{H}')(0) \), are said to be equivalent if

\[
\mathcal{T}(\mathcal{K}_p \ast \mathcal{H}) = \mathcal{T}(\mathcal{K}_p \ast \mathcal{H}'), \quad \mathcal{S}(\mathcal{K}_p \ast \mathcal{H}) = \mathcal{S}(\mathcal{K}_p \ast \mathcal{H}'), \quad \mathcal{J}(\mathcal{K}_p \ast \mathcal{H}) = \mathcal{J}(\mathcal{K}_p \ast \mathcal{H}'),
\]

for every prolongation \( \mathcal{K}_p \).

A distance \( d_\Sigma(\mathcal{H}, \mathcal{H}') \) between surface histories can be defined by

\[
d_\Sigma(\mathcal{H}, \mathcal{H}') := \sup_{t \geq 0} \left\{ \int_0^{+\infty} \left[ \dot{G}_{TW}(s+t)W(s) - \dot{G}_{TW}(s+t)W'(s) + \dot{G}_{\mathcal{K}p}(s+t)\nu(s) - G_{TW}(s+t)\nu'(s) + G_{\mathcal{K}p}(s+t)N(s) - G_{\mathcal{K}p}(s+t)N'(s) \right] ds \right\} +
\frac{1}{2} \left( \int_0^{+\infty} \left[ \dot{G}_{\mathcal{K}p}(s+t)\nu(s) - G_{\mathcal{K}p}(s+t)\nu'(s) + G_{\mathcal{K}p}(s+t)N(s) - G_{\mathcal{K}p}(s+t)N'(s) \right] ds \right) +
\frac{1}{2} \left( \int_0^{+\infty} \left[ G_{\mathcal{K}p}(s+t)\nu(s) - G_{\mathcal{K}p}(s+t)\nu'(s) + G_{\mathcal{K}p}(s+t)N(s) - G_{\mathcal{K}p}(s+t)N'(s) \right] ds \right) \right\}.
\]

It is a semimetric on the space of surface histories and a metric over the quotient space generated by the equivalence relation defined above.

Proposition 2. The distance \( d_\Sigma(\cdot, \cdot) \) has the following properties:

- **Contraction:** for every \( r > 0 \)
  \[
d_\Sigma(\mathcal{K}_r \ast \mathcal{H}', \mathcal{K}_p \ast \mathcal{H}) \leq d_\Sigma(\mathcal{H}', \mathcal{H}) \quad \forall r \geq 0.
\]

- **Fading memory:** for \( \varepsilon > 0 \) there exists \( \ell \) such that, for every \( p > \ell \), one gets
  \[
d_\Sigma(\mathcal{K}_p \ast \mathcal{H}', \mathcal{K}_p \ast \mathcal{H}) < \varepsilon.
\]

- **Approachability:** if \( \mathcal{H}_p^{-}(0) = \mathcal{H}')(0), then
  \[
  \lim_{p \to +\infty} d_\Sigma(\mathcal{H}_p \ast \mathcal{H}', \mathcal{H}) = 0,
\]
  with \( \mathcal{H}_p \) the process generated by \( \mathcal{H} \) over \([0, p)\).
The proof is analogous to the one of Proposition 1. In this case the constant $M$ is the maximum of the suprema of the surface histories.

Surface state functions and surface actions can be then defined.

From Theorem 1 one realizes that the surface work density $w^\Sigma$ is defined by

$$w^\Sigma := \int_0^{+\infty} \left( \sigma \cdot \dot{\nu} + \mathbf{S} \cdot \dot{N} \right) dt + \int_0^{+\infty} \left( \langle \mathbf{m} \cdot \nabla \mathbf{y} \rangle + \langle \mathbf{S} \rangle \mathbf{m} \cdot [\dot{\nu}] \right) dt.$$ 

It includes both peculiar surface interactions and traces of the bulk stresses at the discontinuity surface itself. Previous work on the instantaneous response of complex bodies with structured discontinuity surfaces [33] suggests that only a reduced surface work density is defined by

$$\hat{w}^\Sigma := \int_0^{+\infty} \left( \mathbf{T} \cdot \dot{\mathbf{W}} + \mathbf{S} \cdot \dot{\mathbf{N}} \right) dt.$$

The reduced surface work density over prolongations is defined by

$$\hat{w}^\Sigma (\mathbb{K}_p; \mathbb{H}) := \hat{w}^\Sigma (\mathbb{K}_p * \mathbb{H}) - \hat{w}^\Sigma (\mathbb{H}),$$

where $\mathbb{K}_p$ is the surface counterpart of $K_p$.

By making use of the technique leading to Theorem 2, one may prove that $\hat{w}^\Sigma$ is an action and the map $\mathbb{H} \mapsto \hat{w}^\Sigma (\mathbb{K}_p; \mathbb{H})$ is a state function.

A relaxed surface work $\hat{w}_{\mathbb{H}}^{\Sigma \mathbb{H}'}$ can be then defined by

$$\hat{w}_{\mathbb{H}}^{\Sigma \mathbb{H}'} (\mathbb{H}) := \inf \left\{ \liminf_{p \to -\infty} \hat{w}^\Sigma (\mathbb{K}_p; \mathbb{H}') \mid \mathbb{K}_p \in \mathbb{H}, \lim_{p \to -\infty} d(\mathbb{K}_p * \mathbb{H}', \mathbb{H}) = 0 \right\}.$$

As in the case of the bulk relaxed work, both $\hat{w}_{\mathbb{H}}^{\Sigma \mathbb{H}'} (\cdot)$ and $\hat{w}_{\mathbb{H}}^{\Sigma \mathbb{H}'} (\mathbb{H})$ are state functions. Moreover, $\hat{w}_{\mathbb{H}}^{\Sigma \mathbb{H}'} (\cdot)$ is subadditive in the sense that, for histories $\mathbb{H}$, $\mathbb{H}'$ and $\mathbb{H}''$ such that $\hat{w}_{\mathbb{H}}^{\Sigma \mathbb{H}'} (\mathbb{H}) > -\infty$ and $\hat{w}_{\mathbb{H}}^{\Sigma \mathbb{H}'} (\mathbb{H}') > -\infty$, the inequality

$$\hat{w}_{\mathbb{H}}^{\Sigma \mathbb{H}'} (\mathbb{H}) \leq \hat{w}_{\mathbb{H}}^{\Sigma \mathbb{H}'} (\mathbb{H}') + \hat{w}_{\mathbb{H}}^{\Sigma \mathbb{H}'} (\mathbb{H})$$

holds. If $\hat{w}_{\mathbb{H}}^{\Sigma \mathbb{H}'} (\mathbb{H}) > -\infty$, $\hat{w}_{\mathbb{H}}^{\Sigma \mathbb{H}'} (\cdot)$ is lower semicontinuous. Under the same hypothesis, for every process $\mathbb{K}_p$ such that $\hat{w}_{\mathbb{H}}^{\Sigma \mathbb{H}'} (\mathbb{K}_p * \mathbb{H}) > -\infty$, the surface dissipation inequality

$$\hat{w}_{\mathbb{H}}^{\Sigma \mathbb{H}'} (\mathbb{K}_p * \mathbb{H}) - \hat{w}_{\mathbb{H}}^{\Sigma \mathbb{H}'} (\mathbb{H}) \leq w^\Sigma (\mathbb{K}_p * \mathbb{H})$$

is verified.

In summary, all the properties of $w$ and $w^\tau$ hold also for $w^\Sigma$ and $\hat{w}^\Sigma$. The proofs can be constructed in the same way adopted in analyzing the power in the bulk.

**Definition 13.** Every lower potential of $w^\Sigma$ is called a surface free energy.

The theorem collecting the properties of the free energy in the bulk has its counterpart for the surface free energy. The proof is essentially the same.

**Theorem 8.** The following assertions hold:

1. Every free energy $\phi$ satisfies the dissipation inequality

$$\phi (\mathbb{K} * \mathbb{H}) - \phi (\mathbb{H}) < w^\Sigma (\mathbb{K}, \mathbb{H})$$

for every $\mathbb{H}$ and every compatible $\mathbb{K}$. Moreover, every l.s.c. function $\mathbb{H} \mapsto \phi (\mathbb{H}) \in \mathbb{R}$ that satisfies the dissipation inequality is a free energy.
If the dissipation postulate is satisfied, then, for every pair of histories $H$ and $H'$, the maps $\hat{w}_H^\Sigma (\cdot)$ and $-\hat{w}_H^\Sigma (H')$ are free energies. Moreover, for every free energy $\phi$ one gets

$$-\hat{w}_H^\Sigma (H') \leq \phi (H) - \phi (H') \leq \hat{w}_H^\Sigma (H')$$

for arbitrary histories $H$ and $H'$. In particular, if there is $H'$ and a family of free energies $\phi_s$ such that $\phi_s (H') = 0$, the maps $\hat{w}_H^\Sigma (\cdot)$ and $-\hat{w}_H^\Sigma (H')$ are the maximum and the minimum free energies in such a family.

Every free energy is a state function and, for every $H$, it satisfies the inequality

$$\phi (H (0)^\dagger) \leq \phi (H).$$

In particular, the restriction of the free energy to constant histories is given by

$$\phi (H^\dagger) - \phi (0^\dagger) = \frac{1}{2} \sum_{A,S} G_{ab} (\infty) \mathcal{B}_{bH^\dagger} \cdot \mathcal{B}_{bH^\dagger}.$$

$A$ ranges in $\{T, z, S\}$ and $B$ in $\{W, \nu, N\}$.

### 7. The mechanical dissipation inequality and its consequences

#### 7.1. Mechanical dissipation inequality.

In isothermal setting, the second law of thermodynamics in the form of Clausius-Duhem inequality reduces to a mechanical dissipation inequality. In Lagrangian representation, for any part $b$ it reads

$$\frac{d}{dt} \Psi (b, y, \nu) - P^\text{ext}_b (\dot{y}, \dot{\nu}) \leq 0.$$

The functional $\Psi$ is the overall free energy of $b$ along the fields $y$ and $\nu$: it is the integral over $b$ itself of the free energy density. If $b$ is selected to cross $\Sigma$, so it is indicated by $b_\Sigma$, both bulk and surface energy densities - the ones discussed previously - must be involved. Local forms of the mechanical dissipation inequality - local in the bulk and along $\Sigma$ - can be obtained by exploiting the arbitrariness of the part considered. They are reported in summary here, written with reference to the infinitesimal deformation setting discussed in the earlier sections. The local form of the mechanical dissipation inequality in the bulk then reads

$$\dot{\psi} - \sigma \cdot \dot{W} - z \cdot \dot{\nu} - S \cdot \dot{N} \leq 0,$$

while the one at points over $\Sigma$ is given by

$$\dot{\phi} - T \cdot \dot{W} - z \cdot \dot{\nu} - S \cdot \dot{N} \leq 0.$$

#### 7.2. Complex bodies with instantaneous elastic response.

By borrowing terms from the mechanics of simple bodies, here bodies with instantaneous elastic response are the ones admitting constitutive structures of the type

$$\psi = \psi (H (t), H^t),$$

$$\sigma = \sigma (H (t), H^t),$$

$$z = z (H (t), H^t),$$

$$S = S (H (t), H^t),$$

in the bulk and

$$\phi = \phi (H (t), H^t),$$

$$T = T (H (t), H^t),$$
\[ f(\mathbb{H}(t), H^t), \]
\[ S = \mathbb{S}(\mathbb{H}(t), H^t). \]

The symbols \( H \) and \( \mathbb{H} \) summarize the state. Precisely, \( H(t) = (W(t), \nu(t), N(t)) \in M_{3 \times 3} \times \mathbb{R}^k \times M_{k \times 3} \) is the state at the instant \( t \) while \( H^t \in \Gamma \) is the past history of the state up to the instant \( t \) (in the notation used here \( H^t \) is the graph of \( H \) from \( t \) to infinity, namely \( H^t(s) = H(t+s) \)). Analogous meaning can be attributed to \( \mathbb{H}(t) \) and \( \mathbb{H}^t \), namely \( \mathbb{H}(t) = (\mathbb{W}(t), \nu(t), \mathbb{N}(t)) \in M_{3 \times 3} \times \mathbb{R}^k \times M_{k \times 3} \) while \( \mathbb{H}^t \) is the past surface history and belongs to \( \Gamma \).

In the infinitesimal deformation setting treated here, it is assumed that the free energy density in the bulk is a quadratic form in the instantaneous values \( W(t), \nu(t) \) and \( N(t) \). It is also assumed that also the surface free energy density is a quadratic form in the instantaneous values \( \mathbb{W}(t), \nu(t) \) and \( \mathbb{N}(t) \).

### 7.3. Chain rule.

To exploit the local versions of the mechanical dissipation inequality a chain rule must be used in evaluating the time derivative of the energy. Appropriate chain rules have been obtained in [37] and [20], and can be adapted here.

Consider a functional
\[ F : M_{3 \times 3} \times \mathbb{R}^k \times M_{k \times 3} \times \Gamma \rightarrow \mathbb{R} \]
defined for every \( H(t) \) (or \( \mathbb{H}(t) \)) in \( M_{3 \times 3} \times \mathbb{R}^k \times M_{k \times 3} \) and for every \( H^t \) (or \( \mathbb{H}^t \)) in \( \Gamma \) such that \( H^t(s) \) (or \( \mathbb{H}^t(s) \)) is in the open and connected subset \( \mathcal{U} \) from \( M_{3 \times 3} \times \mathbb{R}^k \times M_{k \times 3} \), characterized by \( \det(I+K^t) > 0 \) (or \( \det(I+\mathbb{K}^t) > 0 \)), for almost \( s > 0 \) (the time parametrizing the history ‘prior’ \( t \) in the representation adopted here \( s > t \)). Assume that (i) \( F \) is continuously differentiable, (ii) the function \( t \mapsto H(t) \) (or \( \mathbb{H}(t) \)) with values in \( \mathcal{U} \) has two continuous derivatives \( t \mapsto \dot{H}(t) \) and \( t \mapsto \ddot{H}(t) \), and (iii) for every \( t \) the past histories \( \dot{H}^t \) and \( \ddot{H}^t \) are in \( \Gamma \). Under these conditions the function \( f(t) := F(H(t), H^t) \) (alternatively \( f(t) := F(\mathbb{H}(t), \mathbb{H}^t) \)) is continuously differentiable and its time derivative is
\[ \dot{f}(t) = D\mathcal{F}(H(t), H^t) \cdot \dot{H}(t) + \delta \mathcal{F}(H(t), H^t) \]
(alternatively \( \dot{f}(t) = D\mathcal{F}(\mathbb{H}(t), \mathbb{H}^t) \cdot \dot{\mathbb{H}}(t) + \delta \mathcal{F}(\mathbb{H}(t), \mathbb{H}^t) \))
where \( D\mathcal{F}(H(t), H^t) \) is a continuous functional taking values in \( T_{H(t)}^* \mathcal{U} \) for every fixed \( H(t) \) and \( H^t \), and \( \delta \mathcal{F}(H(t), H^t) \) is a continuous vector-valued functional depending linearly on \( K^t \) and defined on the closed subspace of \( \Gamma \) spanned by the functions \( K^t \) such that \( H^t(s) + K^t(s) \) is in \( \mathcal{U} \) for almost \( s > 0 \) (analogous remarks hold also for \( D\mathcal{F}(\mathbb{H}(t), \mathbb{H}^t) \) and \( \delta \mathcal{F}(\mathbb{H}(t), \mathbb{H}^t) \)). Fixed \( K^t \), an appropriate technical assumption is that \( \delta \mathcal{F}(H(t), H^t|K^t) \) is continuous in \( (H(t), H^t) \). The proof of this chain rule can be found in [20].

### 7.4. Consequences of the mechanical dissipation inequality.

As it is well known, to exploit the local version of the mechanical dissipation inequality one should have the possibility to select at will the instantaneous rate \( \dot{H}(t) \) of the state. If this point is straightforward in the mechanics of (simple or complex) bodies without memory effects, some additional problems appear in presence of memory effects, due to the dependence of the interaction measures on the whole history of the state variables. The technique discussed in [20] to avoid these difficulties can be adapted here and is summarized in the following paragraphs, then it is applied to the case of complex bodies.
For instrumental reasons, it is useful to introduce a $C^2$ function $f : \mathbb{R}^+ \to \mathbb{R}$ such that $f (s) = 0$ for $|s| \geq 1$, $f (0) = 0$, $\dot{f} (0) = 1$. By using $f$, for any $\alpha \in \mathbb{R}^+$ and a fixed $t \in \mathbb{R}^+$ one may define (see [Day], p. 91) varied histories

$$H_\alpha (s) := H (s) + \alpha f \left( \frac{s-t}{\alpha} \right) \left( \mathcal{H} - \dot{H} (t) \right),$$

where $\mathcal{H}$ is a generic element from $T_{H(t)} \mathcal{U}$, namely $\mathcal{H}$ is a triple $(V, \nu, \Upsilon)$ of virtual rates of $W$, $\nu$ and $N$. Essential properties of $H_\alpha (t)$ - in a sense the properties that suggest the definition of $H_\alpha (t)$ itself - are (i) $H_\alpha (t) = H (t)$, (ii) $\dot{H}_\alpha (t) = \mathcal{H}$, (iii) $H_\alpha (s) = H (s)$ for every $s \leq t - \alpha$ and $s \geq t + \alpha$ and (iv) for $\alpha$ sufficiently small $H_\alpha (\cdot)$ meets the hypotheses of the chain rule and both $H_\alpha (\cdot)$ and $\dot{H}_\alpha (\cdot)$ converge in norm respectively to $H (\cdot)$ and $\dot{H} (\cdot)$ as $\alpha \to 0$.

A similar definition can be adopted for the surface history $\mathbb{H}_\alpha (s)$ so that one gets

$$\mathbb{H}_\alpha (s) := \mathbb{H} (s) + \alpha f \left( \frac{s-t}{\alpha} \right) \left( \delta - \dot{\mathbb{H}} (t) \right),$$

where, now, $\delta$ is a generic element from $T_{\mathbb{H}(t)} \mathcal{U}$, namely $\delta$ is a triple $(\mathfrak{W}, \nu, \mathfrak{G})$ of virtual rates of $\mathfrak{W}$, $\nu$ and $N$.

By making use of the chain rule and substituting $H (s)$ with $H_\alpha (s)$, from the local mechanical dissipation inequality in the bulk one gets

$$(\partial_{W(t)} \psi) (H (t), H^t) - \sigma (H (t), H^t) \cdot V + (\partial_{\nu(t)} \psi) (H (t), H^t) - z (H (t), H^t) \cdot \nu +$$

$$+ (\partial_{\Upsilon(t)} \psi) (H (t), H^t) - \mathcal{S} (H (t), H^t) \cdot \Upsilon + \delta \psi \left( H (t), H^t \right) | H^t | \dot{H}^t \leq 0$$

as $\alpha \to 0$, an inequality holding for all choices of the triple $(V, \nu, \Upsilon)$. The arbitrariness of $(V, \nu, \Upsilon)$ implies that in the bulk

$$\sigma (H (t), H^t) = \partial_{W(t)} \psi \left( H (t), H^t \right),$$

$$z (H (t), H^t) = \partial_{\nu(t)} \psi \left( H (t), H^t \right),$$

$$\mathcal{S} (H (t), H^t) = \partial_{\Upsilon(t)} \psi \left( H (t), H^t \right),$$

$$\delta \psi \left( H (t), H^t \right) | H^t | \dot{H}^t \leq 0.$$

An analogous result hold along the surface $\Sigma$ where, locally one gets

$$(\partial_{W(t)} \phi) \left( \mathbb{H} (t), \mathbb{H}^t \right) - T \left( \mathbb{H} (t), \mathbb{H}^t \right) \cdot \mathfrak{W} + (\partial_{\nu(t)} \phi) \left( \mathbb{H} (t), \mathbb{H}^t \right) - \mathfrak{Z} \left( \mathbb{H} (t), \mathbb{H}^t \right) \cdot \mathfrak{G} + \delta \phi \left( \mathbb{H} (t), \mathbb{H}^t \right) \cdot \mathfrak{H} \leq 0$$

as $\alpha \to 0$, an inequality holding for all choices of the triple $(\mathfrak{W}, \nu, \mathfrak{G})$. The arbitrariness of $(\mathfrak{W}, \nu, \mathfrak{G})$ implies that, along the surface,

$$T \left( \mathbb{H} (t), \mathbb{H}^t \right) = \partial_{W(t)} \phi \left( \mathbb{H} (t), \mathbb{H}^t \right),$$

$$\mathfrak{Z} \left( \mathbb{H} (t), \mathbb{H}^t \right) = \partial_{\nu(t)} \phi \left( \mathbb{H} (t), \mathbb{H}^t \right),$$

$$\mathcal{S} (\mathbb{H} (t), \mathbb{H}^t) = \partial_{\Upsilon(t)} \phi \left( \mathbb{H} (t), \mathbb{H}^t \right),$$

$$\delta \phi \left( \mathbb{H} (t), \mathbb{H}^t \right) \cdot \mathfrak{H} \leq 0.$$

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