Kolmogorov - Sinaj entropy on MV-algebras

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Abstract

There is given a construction of the entropy of a dynamical system on arbitrary MV-algebra $M$. If $M$ is the MV-algebra of characteristic functions of a $\sigma$-algebra (isomorphic to the $\sigma$-algebra), then the construction leads to the Kolmogorov - Sinaj entropy. If $M$ is the MV-algebra (tribe) of fuzzy sets, then the construction coincides with the Maličký modification of the Kolmogorov - Sinaj entropy for fuzzy sets ([6], [14], [15]).

1 Introduction

If $(\Omega, S, P)$ is a probability space and $T: \Omega \rightarrow \Omega$ is a measure preserving transformation (i.e. $A \in S$ implies $T^{-1}(A) \in S$, and $P(T^{-1}(A)) = P(A)$), the entropy is defined as follows. If $A = \{A_1, ..., A_k\}$ is a measurable partition of $\Omega$, then

$$H(A) = \sum_{i} \phi(P(A_i)),$$

where $\phi(x) = -x \log x$, if $x > 0$, and $\phi(0) = 0$. Denote by $\bigvee_{i=0}^{n-1} T^{-i}(A)$ the common refinement of the partitions $A, T^{-1}(A) = \{T^{-1}(A_1), ..., T^{-1}(A_k)\}, ..., T^{-(n-1)}(A) = \{T^{-(n-1)}(A_1), ..., T^{-(n-1)}(A_k)\}$. Then there exists

$$h(A, T) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} H\left(\bigvee_{i=0}^{n-1} T^{-i}(A)\right)$$

and the Kolomogorov - Sinaj entropy is defined by the formula

$$h(T) = \sup\{h(A, T); A \text{ is a measurable partition of } \Omega\}. \quad (2)$$

Let us try to substitute partitions by fuzzy partitions $A = \{f_1, ..., f_k\}$, where $f_j$ are non-negative, measurable functions. $\sum_{i=1}^{k} f_i = 1_\Omega$. Put $m(f_i) = \int f_i \, dP$. The common refinement of $A$ and $B = \{g_1, ..., g_l\}$ is defined using products of functions: $A \vee B = \{f_i \cdot g_j; i = 1, ..., k, j = 1, ..., l\}$. Further $T^{-1}(A) = \{f_1 \circ T, ..., f_k \circ T\}$. Then the entropy can be defined by the formula

$$H(A) = \sum_{i=1}^{k} \phi(m(f_i)),$$
and then by (1) and (2). Of course, in this case \( h(T) = \infty \), if e.g. the supremum is taken over the all measurable partitions. Therefore P. Maličký suggested ([6], see also [14], [15]) the following modification: instead of \( H(\bigvee_{i=0}^{n-1} T^{-1}(A_i)) \) to use

\[
H_n(A, T) = \inf \{ H(C); C \geq A, C \geq T^{-1}(A_1), \ldots, C \geq T^{-(n-1)}(A) \}
\]

(3)

and then to define the entropy by the formula

\[
h(A, T) = \lim_{n \to \infty} \frac{1}{n} H_n(A, T).
\]

(4)

A natural algebraization of a tribe of functions is an MV-algebra. On a special type of MV-algebras - so-called MV-algebras with product ([8],[13]) there has been realized the construction of entropy in [11],[12]. In the paper, following the Maličký construction (3), (4), we construct entropy on any MV-algebra. Secondly, we need only additivity of considered states instead of usually demanded \( \sigma \)-additivity.

In Section 2 we present necessary informations about MV-algebras and some auxiliary results concerning partitions of unity. Then in Section 3 we prove the existence of entropy. Section 4 registers basic facts about isomorphism and entropy and Section 5 contains a counting formula.

2 MV-algebras

An MV-algebra \( M = (M, 0, 1, \neg, \oplus, \odot) \) is a system where \( \oplus \) is associative and commutative with neutral element 0, and, in addition, \( \neg 0 = 1, \neg 1 = 0, x \oplus 1 = 1, x \odot y = \neg(\neg x \oplus \neg y) \), and \( y \oplus \neg(\neg x \oplus y) = x \oplus \neg(y \oplus \neg y) \) for all \( x, y \in M \).

MV-algebras stand to the infinite-valued calculus of Lukasiewicz as boolean algebras stand to classical two-valued calculus.

An example of an MV-algebra is the real unit interval \([0,1]\) equipped with the operations

\[
\neg x = 1 - x, x \oplus y = \min(1, x + y), x \odot y = \max(0, x + y - 1)
\]

It is interesting that any MV-algebra has a similar structure. Let \( G \) be a lattice-ordered Abelian group (shortly \( l \)-group). Let \( u \in G \) be a strong unit of \( G \), i.e. for all \( g \in G \) there exists and integer \( n \geq 1 \) such that \( nu \geq g \). Let \( \Gamma(G, u) \) be the unit interval \([0,u] = \{h \in G; 0 \leq h \leq u\} \) equipped with the operations

\[
\neg g = u - g, g \oplus h = u \wedge (g + h), g \odot h = 0 \lor (g + h - u).
\]

Then \(([0,u], 0, u, \neg, \oplus, \odot) \) is an MV-algebra and by the Mundici theorem ([9]), up to isomorphism, every MV-algebra \( M \) can be identified with the unit interval of a unique \( l \)-group \( G \) with strong unit, \( M = \Gamma(G, u) \).

A partition of unity \( u \) in \( M \) is an \( k \)-tuple \( \mathcal{A} = (a_1, \ldots, a_k) \) of elements of \( M \) such that

\[
a_1 + \ldots + a_k = u.
\]
If $\mathcal{A} = \{a_1, \ldots, a_n\}$, and $\mathcal{B} = \{c_1, \ldots, c_k\}$ are partitions of unity, then their common refinement is any matrix $S = \{s_{ij}; i = 1, \ldots, k, j = 1, \ldots, l\}$ of elements of $M$ such that

$$a_i = \sum_{j=1}^{l} c_{ij}, i = 1, \ldots, k, b_j = \sum_{i=1}^{k} c_{ij}, j = 1, \ldots, l$$

Although in view of the Mundici theorem the next result is known ([5] Prop. 2.2 (c), see also [7]), we mention the proof.

**Lemma 1.** To any partitions $\mathcal{A}, \mathcal{B}$ there exists their common refinement.

Proof. Let $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}, \mathcal{B} = \{b_1, b_2, \ldots, b_m\}$. By the well known Riesz theorem, to any $a, b, c$ such that $0 \leq a \leq b + c, b \geq 0, c \geq 0$ there exist $d, e$ such that $0 \leq d \leq b, 0 \leq e \leq c$, and $a = d + e$. (Indeed, put $d = a \land b, e = a - a \land b$.) This property can be generalized by induction for a finite number of elements. Therefore, since $a_1 \leq b_1 + b_2 + \ldots + b_m = a$, there exist $c_{11}, c_{12}, \ldots, c_{1m}$ such that

$$c_{1i} \leq b_i(i = 1, \ldots, m),$$

and

$$a_1 = c_{11} + c_{12} + \ldots + c_{1m}.$$  

Since $a_2 \leq u - a_1 = (b_1 - c_{11}) + (b_2 - c_{12}) + \ldots + (b_m - c_{1m})$, there exist $c_{2i}(i = 1, 2, \ldots, m)$ such that

$$c_{2i} \leq b_i - c_{1i}(i = 1, 2, \ldots, m),$$

and

$$a_2 = c_{21} + c_{22} + \ldots + c_{2m}.$$  

Continuing the process we construct $c_{3i}(i = 1, 2, \ldots, m)$ such that

$$c_{3i} \leq b_i - c_{1i} - c_{2i}(i = 1, 2, \ldots, m),$$

and

$$a_3 = c_{31} + c_{32} + \ldots + c_{3m}.$$  

After constructing $c_{n-1,1}, c_{n-1,2}, \ldots, c_{n-1,m}$ with $c_{n-1,i} \leq b_i - \sum_{j=1}^{n-2} c_{ji}, a_{n-1} = \sum_{j=1}^{m} c_{n-1,j}$, we put

$$c_{ni} = b_i - \sum_{j=1}^{n-1} c_{ji}(i = 1, \ldots, m)$$

Then evidently

$$b_i = \sum_{j=1}^{n} c_{ji}(i = 1, \ldots, m),$$

and

$$c_{n1} + c_{n2} + \ldots + c_{nm} = b_1 + \ldots + b_m - \sum_{i=1}^{n-1} \sum_{j=1}^{m} c_{ji}.$$
Remark. Common refinements of partitions are not defined uniquely. E.g. let $M = [0, 1], A = \{0.5, 0.5\}, B = \{0.4, 0.6\}$. Then any matrix

\[
t - 0.1; 0.5 - t
0.6 - t; t
\]

$t \in [0.1, 0.5]$ represents a common refinement of $A, B$.

3 Entropy of dynamical systems

Definition 1. By a dynamical system on an MV-algebra we understand a couple of mappings $m : M \rightarrow [0, 1], \tau : M \rightarrow M$ satisfying the following conditions:

(i) if $a = b + c$, then $m(a) = m(b) + m(c), \tau(a) = \tau(b) + \tau(c)$

(ii) $\tau(u) = u, m(u) = 1$

(iii) $m(\tau(a)) = m(a), a \in M$

Definition 2. If $A = \{a_1, ..., a_n\}$ is a partition of unity, then its entropy is defined by the formula

\[
H(A) = \sum_{i=1}^{n} \varphi(m(a_i)),
\]

where $\varphi(x) = -x \log x$, if $x > 0, \varphi(0) = 0$.

Definition 3. If $A = \{a_1, ..., a_n\}, B = \{b_1, ..., b_k\}$ are two partitions of unity and $C = \{c_{ij}; i = 1, ..., n, j = 1, ..., m\}$ is a common refinement of $A$ and $B$, then we define

\[
H_C(A|B) = \sum_{i=1}^{n} \sum_{j=1}^{m} m(b_j) \varphi\left(\frac{m(c_{ij})}{m(b_j)}\right).
\]

Lemma 2. $H_C(A|B) \leq H(A)$.

Proof. Fix $i$, and put $\alpha_j = m(b_j), x_j = \frac{m(c_{ij})}{m(b_j)}, j = 1, ..., k$. Then $\sum_{j=1}^{k} \alpha_j = m(u) = 1$

\[
\sum_{j=1}^{k} \alpha_j x_j = \sum_{j=1}^{k} m(c_{ij}) = m\left(\sum_{j=1}^{k} c_{ij}\right) = m(a_i).
\]

Since $\varphi$ is concave, we have

\[
\sum_{j=1}^{k} \alpha_j \varphi(x_j) \leq \varphi\left(\sum_{j=1}^{k} \alpha_j x_j\right),
\]
hence

\[ H_C(A|B) = \sum_{i=1}^{n} \sum_{j=1}^{k} m(b_j) \frac{m(c_{ij})}{m(b_j)} = \sum_{i=1}^{n} \sum_{j=1}^{k} \alpha_i \phi(x_j) \leq \sum_{i=1}^{n} \phi(\sum_{j=1}^{k} \alpha_j x_j) = \sum_{i=1}^{n} \phi(m(a_i)) = H(A). \]

**Lemma 3.** \( H(C) = H(A) + H_C(B|A) \) for any common refinement \( C \) of \( A \) and \( B \).

Proof. It follows by the additivity of logarithms.

\[ H(C) = \sum_{i=1}^{n} \sum_{j=1}^{k} \phi(c_{ij}) = \sum_{i=1}^{n} \sum_{j=1}^{k} \phi(m(a_i)) \frac{m(c_{ij})}{m(a_i)} \]

\[ = - \sum_i \sum_j m(c_{ij}) \log(m(a_i)) - \sum_i \sum_j m(c_{ij}) \log \frac{m(c_{ij})}{m(a_i)} \]

\[ = - \sum_i \log(m(a_i)) \sum_j m(c_{ij}) + \sum_i \sum_j m(a_i) \phi \frac{m(c_{ij})}{m(a_i)} \]

\[ = - \sum_i m(a_i) \log(m(a_i)) + H_C(B|A) \]

\[ = H(A) + H_C(B|A) \]

**Corollary.** For any common refinement \( C \) of \( A \) and \( B \) there holds \( H(C) \leq H(A) + H(B) \).

**Lemma 4.** For any partition \( A = \{a_1, \ldots, a_n\} \) put \( \tau(A) = \{\tau(a_1), \ldots, \tau(a_n)\} \). Then \( \tau(A) \) is a partition of unity, too. Moreover, \( H(\tau(A)) = H(A) \).

Proof. By (i) and (iii) of the definition of dynamical system we have

\[ \tau(a_1) + \tau(a_2) + \ldots + \tau(a_n) = \tau(a_1 + \ldots + a_n) = \tau(u) = u. \]

Moreover, by (iii)

\[ H(\tau(A)) = \sum_{i=1}^{n} \phi(m(\tau(a_i))) = \sum_{i=1}^{n} \phi(m(a_i)) = H(A). \]

**Definition 3.** For any partition \( A \) of unity and any positive integer \( n \) we define \( H_n(A, \tau) = \inf\{H(C) ; C \text{ is a common refinement of } A, \tau(A), \ldots, \tau^{(n-1)}(A) \} \).

**Theorem 1.** There exists \( \lim_{n \to \infty} H_n(A, \tau) \).

Proof. It suffices to prove that \( H_{n+m}(A, \tau) \leq H_n(A, \tau) + H_m(A, \tau) \). We shall use the following notation: \( C \in A \vee B \), if \( C \) is a common refinement of \( A \) and \( B \). Let \( C, D \) be partitions of unity such that

\[ C \in A \vee \tau(A) \vee \ldots \vee \tau^{(n-1)}(A), \quad (5) \]

\[ D \in A \vee \tau(A) \vee \ldots \vee \tau^{(m-1)}(A). \quad (6) \]
By (6) we obtain
\[ \tau_n(D) \in \tau_n(A) \lor \tau^{n+1}(A) \lor \ldots \lor \tau^{n+m-1}(A). \] (7)

Finally consider any partition \( \mathcal{E} \) such that
\[ \mathcal{E} \in \mathcal{C} \lor \tau_n(D). \] (8)

By (8), (5) and (7) we obtain
\[ \mathcal{E} \in \mathcal{A} \lor \tau(A) \lor \ldots \lor \tau^{n+m-1}(A), \]

hence
\[ H_{n+m}(A) \leq H(\mathcal{E}). \]

Of course, by (8), Corollary and Lemma 4
\[ H(\mathcal{E}) \leq H(C) + H(\tau_n(D)) \]
\[ = H(C) + H(D), \]

hence
\[ H_{n+m}(A) \leq H(C) + H(D). \] (9)

Fix for a moment \( D \). Since (9) holds for any \( C \in \mathcal{A} \lor \tau(A) \lor \ldots \lor \tau^{(n-1)}(A) \), by the definition of \( H_n(A) \) we obtain
\[ H_{n+m}(A) - H(D) \leq H(A), \]

and by a similar argument
\[ H_{n+m}(A) - H_n(A) \leq H_m(A). \]

4 Isomorphism

The aim of the Kolmogorov - Sinaj entropy was to distinguish non-isomorphic dynamical systems. If \( T \sim T' \) implies \( h(T) = h(T') \), then dynamical systems \( T, T' \) with different entropies \( h(T) \neq h(T') \) cannot be isomorphic.

Definition 4. Entropy of an MV-algebra dynamical system \((M, m, \tau)\) is defined by the formula
\[ h(\tau) = \sup \{ h(A, \tau); A \text{ is a partition of unity} \} \]
Definition 5. Two MV-algebra dynamical systems \((M_1, m_1, \tau_1), (M_2, m_2, \tau_2)\) are equivalent, if there exists a mapping \(\psi : M_1 \rightarrow M_2\) satisfying the following conditions:

(i) \(\psi\) is a bijection
(ii) if \(a, b, c \in M_1, a = b + c\), then \(\psi(a) = \psi(b) + \psi(c)\).
(iii) \(\psi(u_1) = u_2\)
(iv) \(m_2(\psi(a)) = m_1(a)\) for any \(a \in M_1\)
(v) \(\tau_2(\psi(a)) = \psi(\tau_1(a))\) for any \(a \in M_1\)

Theorem 2. If \((M_1, m_1, \tau_1)\) and \((M_2, m_2, \tau_2)\) are equivalent, then \(h(\tau_1) = h(\tau_2)\).

Proof. If \(A\) is any partition of \(u_1\), then \(\psi(A)\) is a partition of \(u_2\), and \(H(A) = H(\psi(A))\). Let \(\varepsilon\) be an arbitrary positive number. Choose a common refinement \(C\) of \(A, \ldots, \tau_1^{n-1}(A)\) such that \(H_n + \varepsilon > H(C)\).

Evidently \(H(C) \geq H_n(\psi(A))\). Since
\[
H_n(A) + \varepsilon \geq H_n(\psi(A))
\]
holds for any \(\varepsilon > 0\), we have
\[
H_n(A) \geq H_n(\psi(A)),
\]
hence
\[
h_1(\tau_1, A) = \lim_{n \to \infty} \frac{1}{n} H_n(A) \geq \lim_{n \to \infty} \frac{1}{n} H_n(\psi(A)) = h_2(\tau_2, \psi(A)),
\]

\[
h_1(\tau_1) = \sup \{h_1(\tau_1, A); A \geq h_2(\tau_2, \psi(A))\}.
\]

Let \(B\) be any partition of \(u_2\). Then \(A = \psi^{-1}(B)\) is a partition of \(u_1\) and \(\psi(A) = B\).

Therefore
\[
h_1(\tau_1) \geq h_2(\tau_2, \psi(A)) = h_2(\tau_2, B)
\]
for any \(B\), hence \(h_1(\tau_1) \geq h_2(\tau_2)\).
5 MV-algebras with product

MV-algebra with product (see [8], [13], [14]) is a pair \((M, \cdot)\), where \(M\) is an MV-algebra, and \(\cdot\) is a commutative and associative operation on \(M\) satisfying the following conditions:

(i) \(u.a = a\);

(ii) if \(a, b, c \in M, a + b \leq u\), then \(c.a + c.b \leq u\), and \(c.(a + b) = c.a + c.b\).

In MV-algebras with product a suitable theory of entropy of the Kolmogorov type has been constructed in [12]. Of course, the construction is different. The main idea is the following. If \(A = \{a_1, \ldots, a_k\}, B = \{b_1, \ldots, b_l\}\) are two partitions of unity, then one defines 

\[ A \lor B = \{a_i \cdot b_j; i = 1, \ldots, k, j = 1, \ldots, l\}. \]

The entropy is defined by the formula

\[ h(A, \tau) = \lim_{n \to \infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} \tau^i(A)). \]

Evidently \(A \lor B\) is a common refinement of \(A\) and \(B\). Therefore

\[ H_n(A, \tau) \leq H(\bigvee_{i=0}^{n-1} \tau^i(A)), \]

and

\[ h(A, \tau) = \lim_{n \to \infty} \frac{1}{n} H_n(A, \tau) \leq \lim_{n \to \infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} \tau^i(A)) = H(A, \tau). \]

We want to study some relations between \(h(A, \tau)\) and \(H(A, \tau)\). Of course, we shall assume that the given MV-algebra is divisible, i.e. to any \(a \in M\) and any \(n \in N\) there exists \(b \in M\) such that \(a = n.b\). As it has been shown in [4] any divisible \(\sigma\)-complete MV-algebra \(M\) admits a product such that \((M, \cdot)\) is an MV-algebra with product.

**Theorem 3.** Let \(M\) be a divisible \(\sigma\)-complete MV-algebra, \(m\) be continuous, positive state on \(M\) (i.e. \(m(a) = 0 \Rightarrow a = 0\)). If \(A = \{a_1, \ldots, a_k\}, B = \{b_1, \ldots, b_l\}\) are partitions of unity, and \(A\) consists of idempotents (i.e. \(a_i \lor a_i = a_i\) for all \(i\)), then \(A \lor B\) is the unique common refinement of \(A\) and \(B\).

Proof. Our main tool is the Dvurečenskij and Mundici modification of the Loomis - Sikorski theorem ([2], [10]). By [3] Theorem 7.2.6 there exists a compact Hausdorff space \(\Omega\), a family \(\mathcal{T}\) of functions \(f : \Omega \to [0, 1]\) and a mapping \(\psi : \mathcal{T} \to M\) satisfying the following properties:

(i) \(\mathcal{T}\) is a tribe, i.e. \(1_\Omega \in \mathcal{T}\), \(1 - f \in \mathcal{T}\) whenever \(f \in \mathcal{T}\), and \(\min\{\sum_{i=1}^n 1\} \in \mathcal{T}\), whenever \(f_n \in \mathcal{T}(n = 1, 2, \ldots)\);

(ii) the natural product \(f.g \in \mathcal{T}\), whenever \(f \in \mathcal{T}, g \in \mathcal{T}\);
(iii) \( \psi \) is an MV-\( \sigma \)-homomorphism from \( T \) onto \( M \) preserving the product in \( T \).

Since \( M \) is divisible and \( \sigma \)-complete, the tribe \( T \) contains all constant functions. Indeed, by [3] theorem 7.1.22 \( \Omega \) consists of all functions \( m : M \to [0,1] \) satisfying the identity \( m(a + b) = \min(m(a) + m(b), 1) \), and the tribe \( T \) is generated by the family of all functions \( \hat{a} : \Omega \to [0,1] \) defined by \( \hat{a}(m) = m(a), a \in M \). Of course, divisibility of \( M \) implies for any \( q \in \mathbb{N} \) the existence of \( \frac{1}{q} u \) such that \( q m(\frac{1}{q} u) = 1 \). If \( r \in Q, r = \frac{q}{q} \), then \( q \mu(r u) = p m(\frac{1}{q} u) = q m(u) = r \), hence the constant function \( r \) belongs to \( T \). Finally \( \sigma \)-completeness of \( M \) implies that \( T \) contains all constant functions \( \alpha \), \( \alpha \in [0,1] \).

Since \( T \) contains all constant fuzzy sets, by the Butnariu - Klement theorem ([11] Prop. 3.3., [15] Theorem 7.1.7) \( T \) consists of all \( S \)-measurable functions, where \( S = \{ A \subset \Omega : \chi_A \in T \} \).

Define \( \mu : T \to [0,1] \) by \( \mu(f) = m(\psi(f)) \). Evidently \( \mu \) is a continuous state on the MV-algebra \( T \). There exists a probability measure \( P : S \to [0,1] \) such that \( \mu(f) = \int_\Omega f dP \) for any \( f \in T \).

Let \( \{ c_{ij} : i = 1, ..., k, j = 1, ..., l \} \) be any common refinement of \( \mathcal{A} \) and \( \mathcal{B} \). Since \( \psi \) is epimorphism, there are \( f_1, ..., f_k, g_1, ..., g_l, h_{ij} \in T \) such that \( \psi(f_i) = a_i, \psi(g_j) = b_j, \psi(h_{ij}) = c_{ij} \). Since \( a_i \) are idempotent, it is not difficult to see ([3] Prop. 7.1.20) that \( f_i = \chi_{A_i} \). We have \( P \)-almost everywhere

\[
\sum_{j=1}^k h_{ij} = \chi_{A_i} g_j, \quad i = 1, ..., k, j = 1, ..., l.
\]

Therefore

\[
h_{ij} = \chi_{A_i} g_j,
\]

\( P \)-a.e., hence

\[
c_{ij} = \psi(h_{ij}) = \psi(\chi_{A_i}), \psi(g_j) = a_i b_j.
\]

**Corollary.** Let the assumptions of Theorem 3 be satisfied. If \( \mathcal{A} \) consists of idempotent elements, then

\[ h(\mathcal{A}, \tau) = n(\mathcal{A}, \tau). \]

**Theorem 4.** Let the assumptions of Theorem 3 be satisfied. If \( \mathcal{A} \) consists of idempotent elements, then for any partition \( \mathcal{B} \) there holds the inequality

\[ h(\mathcal{B}, \tau) \leq h(\mathcal{A}, \tau) + H(\mathcal{B} \parallel \mathcal{A}) \]

where

\[
H(\mathcal{B} \parallel \mathcal{A}) = \sum_{i=1}^k \sum_{j=1}^l m(b_j)\Phi\left(\frac{m(a_i, b_j)}{m(b_j)}\right).
\]

Proof. It follows by [12] Prop. 8 and the above Corollary.

**Remark.** Theorem 4 presents a basic tool for counting entropy in some structures. For details see e.g. [15], Chapter 10.
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