Multiview Differential Geometry of Curves

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Abstract The field of multiple view geometry has seen tremendous progress in reconstruction and calibration due to methods for extracting reliable point features and key developments in projective geometry. Point features, however, are not available in certain applications and result in unstructured point cloud reconstructions. General image curves provide a complementary feature when keypoints are scarce, and result in 3D curve geometry, but face challenges not addressed by the usual projective geometry of points and algebraic curves. We address these challenges by laying the theoretical foundations of a framework based on the differential geometry of general curves, including stationary curves, occluding contours, and non-rigid curves, aiming at stereo correspondence, camera estimation (including calibration, pose, and multiview epipolar geometry), and 3D reconstruction given measured image curves. By gathering previous results into a cohesive theory, novel results were made possible, yielding three contributions. First, we derive the differential geometry of an image curve (tangent, curvature, curvature derivative) from that of the underlying space curve (tangent, curvature, curvature derivative, torsion). Second, we derive the differential geometry of a space curve from that of two corresponding image curves. Third, the differential motion of an image curve is derived from camera motion and the differential geometry and motion of the space curve. The availability of such a theory enables novel curve-based multiview reconstruction and camera estimation systems to augment existing point-based approaches. This theory has been used to reconstruct a “3D curve sketch”, to determine camera pose from local curve geometry, and tracking; other developments are underway.

Keywords Structure from motion · Multiview stereo · Torsion · Non-rigid space curves

1 Introduction The automated estimation of camera parameters and the 3D reconstruction from multiple views is a fundamental problem in computer vision. These tasks rely on correspondence of image structure commonly in the form of keypoints, although dense patches and curves have also been used. Keypoint-based methods extract point features designed be stable with view variations. They satisfy certain local conditions in the spatial and scale dimensions (Mikolajczyk and Schmid 2004; Harris and Stephens 1988; Moravec 1977; Lowe 2004), and are attributed with a local description of the image (Mikolajczyk and Schmid 2005). While many of these points are not stable with view changes in that they disappear/appear or vary abruptly, many are stable enough for matching with RANSAC (Fischler and Bolles 1981) to initialize camera models for bundle adjustment (Hartley and Zisserman 2000; Pollefeys et al. 2004; Agarwal et al. 2009; Heinly et al. 2015; Diskin and Asari 2015).

A major drawback of interest points is their sparsity compared to the curves and surfaces composing the scene, producing a cloud of 3D points where geometric structure is not explicit, Fig. 1a. This is less of a problem for camera estimation (Agarwal et al. 2009), but in applications
Keypoint-based approaches give a sparse point cloud reconstruction (Agarwal et al. 2009; Heinly et al. 2015) which can be made richer, sharper, and denser with curve structure. b Wide baseline views not sharing interest points often share curve structure. c, d Not enough points matching views of homogenous objects with sufficient curve structure. e Each moving object or non-rigid object requires its own set of features, but may not have enough texture such as architecture, industrial design, object recognition, and robotic manipulation, explicit 3D geometry is required. Moreover, meshing unstructured point clouds produces over-smoothing (Kazhdan et al. 2006; Furukawa and Ponce 2010). These techniques are therefore inadequate for man-made environments (Simoes et al. 2012) and objects such as cars (Shinozuka and Saito 2014), non-Lambertian surfaces such as that of the sea, appearance variation due to changing weather (Baatz et al. 2012), and wide baseline (Moreels and Perona 2007), Fig. 1b. We claim that by using even some curve information the 3D reconstruction can be made more structurally rich, sharper and less sparse.

The use of keypoints requires an abundance of features surviving the variations between views. While this occurs for many scenes, in many others this is not the case, such as (i) homogeneous regions from man-made objects, Fig. 1c, d; (ii) moving objects require their own set of features which are too few without sufficient texture, Fig. 1e; (iii) non-rigid objects require a rich set of features per local patch, Fig. 1f. While curve features, like keypoints, may not be abundant
the interior of highly homogeneous regions or in nonrigid or small moving objects, curves rely on texture to a much lesser extent, as shown in Fig. 1c, d, e; the boundary of homogeneous regions are good cues (often the only cues), enabling extracting information structure and motion. In all these situations, there may be sufficient curvilinear structure, motivating augmenting the use of interest points with curves.

**Pixel-based multiview stereo** relies on matching intensities across views, resulting in a dense point cloud or a mesh reconstruction. This produces detailed 3D reconstructions of objects imaged under controlled conditions by a large number of precisely calibrated cameras (Furukawa and Ponce 2007; Habbecke and Kobbelt 2007; Hernández Esteban and Schmitt 2004; Goesle et al. 2007; Seitz et al. 2006; Calakli et al. 2012; Restrepo et al. 2014). However, there are a number of limitations: they typically assume that the scene consists of a single object or that objects are of a specific type, such as a building; they often require accurate camera calibration and operate under controlled acquisition; and they need to be initialized by the visual hull of the object or a bounded 3D voxel volume, compromising applicability for general scenery.

**Curve-based multiview methods** can be divided into three categories: (i) convex hull construction, (ii) occluding contour reconstruction, and (iii) use of differential geometry in binocular and trinocular stereo. First, when many views are available around an object, a visual hull has been constructed from silhouette curves and then evolved to optimize photometric constraints while constraining the projection to the silhouettes. The drawbacks are similar to those of pixel-based multiview stereo. Second, the occluding contours extracted from frames of a video have been used to reconstruct a local surface model given the camera parameters for each frame. These methods require highly controlled acquisition and image curves that are easy to segment and track. In addition, since only silhouettes are used, internal surface variations which may not map to apparent contours in any view will not be captured, e.g., surface folds of a sculpture.

Third, some methods employ curve differential geometry in correlating structure across views. Complete 3D reconstruction pipelines based on straight lines (Lebeda et al. 2014; Zhang 2013; Fathi et al. 2015), algebraic and general curve features (Teney and Piater 2012; Litvinov et al. 2012; Fabbri and Kimia 2010; Fabbri et al. 2012; Pötsch and Pinz 2011) have been proposed. The compact curve-based 3D representation has found demand in several tasks: fast recognition of general 3D scenery (Pötsch and Pinz 2011), efficient transmission of general 3D scenes, scene understanding and modeling by reasoning at junctions (Mattingly et al. 2015), consistent non-photorealistic rendering from video (Chen and Klette 2014), modeling of branching structures, to name a few (Rao et al. 2012; Kowdle et al. 2012; Wang et al. 2014).

**Related work** Differential geometry does not provide hard constraints for matching in static binocular stereo, as known for tangents and curvatures (Robert and Faugeras 1991), and shown here for higher order. Heuristics have been employed in short baseline to limit orientation difference (Arnold and Binford 1980; Grimson 1981; Sherman and Peleg 1990), to match appearance via locally planar approximations (Schmid and Zisserman 2000), or to require 3D curve reconstructions arising from two putative correspondence pairs to have minumum torsion (Li and Zucker 2003). When each stereo camera provides a video and the scene (or stereo head) moves rigidly, differential geometry provides a hard constraint (Faugeras and Papadopoulo 1993; Papadopoulo 1996; Faugeras and Papadopoulo 1992). Differential geometry is more directly useful in trinocular and multiview stereo, as pioneered by Ayache and Lustman (1987), due to the constraint that corresponding pairs of points and tangents from two views uniquely determine a point and tangent in a third to match line segments obtained from edge linking (Ayache and Lustman 1987; Spetsakis and Aloimonos 1991; Shashua 1994; Hartley 1995). Robert and Faugeras (1991) extended this to include curvature: 3D curvature and normal can be reconstructed from 2D curvatures at two views, determining the curvature at a third. The use of curvature improved reconstruction precision and density, with heuristics such as the ordering constraint (Ohta and Kanade 1985). Schmid and Zisserman (2000) derived multiview curvature transfer when only the trifocal tensor is available, by a projective-geometric approach to the osculating circle as a conic.

Curves have also been employed for camera estimation using the concept of epipolar tangencies: corresponding epipolar lines are tangent to a curve at corresponding points (Cipolla and Giblin 1999; Astrom et al. 1999; Astrom and Kahl 1999; Kahl and Heyden 1998; Porrill and Pollard 1991; Kaminski and Shashua 2004; Berthilsson et al. 2001; Wong et al. 2001; Mendonça et al. 2001; Wong and Cipolla 2004; Furukawa et al. 2006; Hernandez et al. 2007; Cipolla et al. 1995; Reyes and Bayro Corrochano 2005; Sinha et al. 2004). This is used to capture epipolar geometry or relative pose.

**Curve-Based Multiview Geometry** What would be desirable is a generally applicable framework, e.g., a handheld video acquiring images around objects or a set of cameras monitoring a scene, where image curve structure can be used to estimate camera parameters and reconstruct a 3D curve sketch on which a surface can be tautly stretched like a tent on a metallic scaffold. This paper provides the mathematical foundation for this curve-based approach. Image curve fragments are attractive because they have good localization, have greater invariance than interest points to changes in illumination, are stable over a greater range of baselines, and are denser than interest points. Moreover, for the special case of occluding contours, dense 3D surface patch reconstructions are available. The notion that image curves contain much of the image information is supported by recent studies...
This paper develops the theoretical foundations for using the differential geometry of image curve structure as a complementary alternative to interest points. This paper is organized along the lines of these questions: (i) How does the differential geometry of a space curve map to the differential geometry of the image curve it projects to? (ii) How can the differential geometry of a space curve be reconstructed from that of two corresponding image curves? (iii) How does the differential geometry of an image curve evolve under camera motion? Section 2 establishes notation for image and space curves, camera projection and motion, and discusses the distinction between stationary and non-stationary 3D contours. Section 3 relates the differential geometry of image curves, i.e., tangent, curvature, and curvature derivative from the differential geometry of the space curves they arise from, i.e., tangent and normal, curvature, torsion, and curvature derivatives. Section 4 derives the differential geometry of a space curve at a point from that at two corresponding image curve points showing the key result that the ratio of parametrization speeds is an intrinsic quantity. The key new result is the reconstruction of torsion and curvature derivative, given corresponding differential geometry in two views. Section 5 considers differential camera motion and relates the differential geometry of a space curve to that of the image and camera motion. Results are provided concerning image velocities and accelerations with respect to time for different types of curves; in particular, distinguishing apparent and stationary contours requires second-order time derivatives (Cipolla and Giblin 1999). We study the spatial variation of the image velocity field along curves, which can be useful for exploiting neighborhood consistency of velocity fields along curves. The main new result generalizes a fundamental curve-based differential structure from motion equation (Papadopoulo and Faugeras 1996; Papadopoulo 1996) to occluding contours.

This paper integrates the above results under the umbrella of a unified formulation and completes missing relationships. As a generalized framework, it is expected to serve as reference for research relating local properties of general curves and surfaces to those of cameras and images. Much of this has already been done, but a considerable amount has not, as mentioned earlier, and most results are scattered in the literature. This theoretical paper has been the foundation of practical work already reported on reconstruction and camera estimation systems as follows. First, the pipeline for the reconstruction of a 3D Curve Sketch from image fragments in numerous views (Fabbri and Kimia 2010; Usumezbas et al. 2016) relies on the results on 3D reconstruction and projection of differential geometry reported in this paper, and a future extension of this pipeline would require most results in this paper. Second, a recent practical algorithm for pose estimation based on differential geometry of curves (Fabbri et al. 2012) relies on the theory reported in this paper, treating the camera pose as unknowns and using differential geometry to solve for them, cf. ensuing efforts by Kuang and Åström (2013) and Kuang et al. (2014). Third, work on differential camera motion estimation from families of curves based on the present work has been explored by Jain (2009), Jain et al. (2007a,b). These works are currently under intense development in order to build a complete structure from motion pipeline based on curves, which would use the majority of the results described in this paper, including analogous results for multiview surface differential geometry that are under development.

2 Notation and Formulation

2.1 Differential Geometry of Curves

For our purposes, a 3D space curve \( \Gamma \) is a smooth map \( S \mapsto \Gamma^w(s) \) of class \( C^\infty \) from an interval of \( \mathbb{R} \) to \( \mathbb{R}^3 \), where \( S \) is an arbitrary parameter, \( \hat{S} \) is the arc-length parameter, and the superscript \( w \) denotes the world coordinates. The local Frenet frame of \( \Gamma \) in world coordinates is defined by the unit vectors tangent \( T^w \), normal \( N^w \), binormal \( B^w \); \( G \) is speed of parametrization, curvature \( K \), and torsion \( \tau \). Similarly, a 2D curve \( \gamma \) is a map \( s \mapsto \gamma(s) \) of class \( C^\infty \) from an interval of \( \mathbb{R} \) to \( \mathbb{R}^2 \), where \( s \) is an arbitrary parameter, \( \hat{s} \) is arc-length, \( g \) is speed of parametrization, \( t \) is (unit) tangent, \( n \) is (unit) normal, \( k \) is curvature, and \( \kappa \) is curvature derivative, Table 1. We will be concerned with regular curves, so that \( G \neq 0 \) and \( g \neq 0 \) unless otherwise stated. By classical differential geometry (do Carmo 1976), we have

\[
\begin{align*}
G &= \|\Gamma^w\| \\
T^w &= \frac{\Gamma^w}{G} \\
N^w &= \frac{T^w}{\|T^w\|} \\
B^w &= T^w \times N^w \\
K &= \frac{\|T^w\|}{G} \\
\hat{K} &= \frac{K'}{G} \\
\tau &= -\frac{B^w \cdot N^w}{G} \\
\end{align*}
\]

(2.1)

and

\[
\begin{align*}
g &= \|\gamma'\| \\
t &= \frac{\gamma'}{g} \\
\nu &= t^\perp \\
k &= \frac{\mathbf{t} \cdot \mathbf{n}}{g} \\
\hat{k} &= \frac{\kappa}{g},
\end{align*}
\]

(2.2)

where prime "'" denotes differentiation with respect to an arbitrary spatial parameter (\( S \) or \( s \)). We use dot "·" to denote
differentiation with respect to arc-length \( \tilde{\xi} \) or \( \tilde{s} \) only when an entity clearly belongs to either a space or an image curve. The matrix equations on the right of (2.1) are the Frenet equations. Note that both the curvature derivatives \( K \) and \( \kappa \) are intrinsic quantities.

### 2.2 Perspective Projection

The projection of a 3D space curve \( \Gamma \) into a 2D image curve \( \gamma \) is illustrated by Fig. 2a, where the world coordinate system is centered at \( O \) with basis vectors \( \{e^w_1, e^w_2, e^w_3\} \). The **camera coordinate system** is centered at \( c \) with basis vectors \( \{e_1, e_2, e_3\} \). A generic way of referring to individual coordinates is by means of the specific subscripts \( x, y, z \) and \( i = 1, 2, 3 \) attached to a symbol, i.e., \( v = [v_x, v_y, v_z]^T \) for any vector \( v \); other subscripts denote partial differentiation, unless otherwise stated. When describing coordinates in the camera coordinate system we drop the \( w \) superscript, e.g., \( \Gamma \) versus \( \Gamma^w \), which are related by

\[
\Gamma = \mathcal{R}(\Gamma^w - c) = \mathcal{R}\Gamma^w + \mathcal{T},
\]

where \( \mathcal{R} \) is a rotation and \( \mathcal{T} = -\mathcal{R}c \) denotes the world coordinate origin in the camera coordinate system. The projection of a 3D point \( \Gamma = [x, y, z]^T \) onto the image plane at \( z = 1 \) is the point \( \gamma = [\xi, \eta, 1]^T \) related by

\[
\Gamma = \rho \gamma \quad \text{or} \quad [x, y, z]^T = [\rho \xi, \rho \eta, \rho]^T.
\]

where we say that \( \gamma \) is in normalized image coordinates (focal distance is normalized to 1), and the depth is \( \rho = z = e^w_3 \Gamma \) from the third coordinate equation. Observe that image points are treated as 3D points with \( z = 1 \). Thus, we can write

\[
\gamma = \frac{\Gamma}{\rho}.
\]

We note that \( e^w_3 \gamma^{(i)} = 0 \) and \( e^w_3 \Gamma^{(i)} = \rho^{(i)} \), where \( \gamma^{(i)} \) is the \( i^{th} \) derivative of \( \gamma \) with respect to an arbitrary parameter, for any positive integer \( i \). Specifically,

\[
\rho = z, \quad \rho' = GT_z, \quad \rho'' = G'T_z + G^2Kn_z.
\]

It is interesting to note that at near/far points of the curve, i.e., \( \rho' = 0, T_z = 0 \).

In practice, normalized image coordinates \( \gamma = [\xi, \eta, 1]^T \) are described in terms of image pixel coordinates \( \gamma_{im} = [x_{im}, y_{im}, 1]^T \) through the intrinsic parameter matrix \( \mathcal{K}_{im} \) according to

\[
\gamma_{im} = \mathcal{K}_{im}\gamma, \quad \mathcal{K}_{im} = \begin{bmatrix} \alpha_x & \tau \xi_o & 0 \\ \alpha_y & \tau \eta_o & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]
where as usual $\xi_o$ and $\eta_o$ are the principal points, $\sigma$ is skew, and $\alpha \xi$ and $\alpha \eta$ are given by the focal length divided by the width and height of a pixel in world units, respectively.

### 2.3 Discrete and Continuous Sets of Views

Two scenarios are considered. The first scenario consists of a discrete set of views where a set of $n$ pinhole cameras observe a scene as shown in Fig. 2b, with the last subscript in the symbols indentifying the camera, e.g., $\gamma_i$ denotes an image point in the $i$th camera, and $e_{3,i}$ denotes $e_3$ in the $i$th view. The second scenario consists of a continuous set of views from a continuously moving camera observing a space curve which may itself be moving, $\Gamma^w(S, t) = [x^w(S, t), y^w(S, t), z^w(S, t)]^T$, where $S$ is the parameter along the curve and $t$ is time, described in the camera coordinate system associated with time $t$ as $\Gamma(S, t) = [x(S, t), y(S, t), z(S, t)]^T$, Fig. 3. For simplicity, we often omit the parameters $S$ or $t$. Let the camera position over time (camera orbit) be described by the space curve $c(t)$ and the camera orientation by a rotation matrix $R(t)$. For simplicity, and without loss of generality, we take the camera coordinate system at $t = 0$ to be the world coordinate system, i.e., $c(0) = 0$, $T(0) = 0$, and $R(0) = I$, where $I$ is the identity matrix. Also, a stationary point can be modeled in this notation by making $\Gamma^w(t) = \Gamma^w(0) = \Gamma_0$.

A differential camera motion model using time derivatives of $R(t)$ and $T(t)$ can be used to relate frames in a small time interval. Since $R R^T = I$,

$$\frac{dR}{dt} R^T + R \frac{dR^T}{dt} = 0,$$

(2.8)
which implies that $\Omega_x \equiv \frac{d\mathcal{R}}{dt}\mathcal{R}^\top$ is a skew-symmetric matrix, explicitly written as

\[
\Omega_x = \begin{bmatrix}
0 & -\Omega_z & \Omega_y \\
\Omega_z & 0 & -\Omega_x \\
-\Omega_y & \Omega_x & 0
\end{bmatrix},
\tag{2.9}
\]

so that $\frac{d\mathcal{R}}{dt} = \Omega_x \mathcal{R}$. Denote $\Omega = [\Omega_x, \Omega_y, \Omega_z]^\top$ as a vector form characterization of $\Omega_x$. Similarly, the second-derivative of $\mathcal{R}(t)$ is represented by only three additional numbers $\frac{d\Omega_x}{dt}$, so that

\[
\frac{d^2\mathcal{R}}{dt^2} = \frac{d\Omega_x}{dt} \mathcal{R} + \Omega_x \frac{d\mathcal{R}}{dt} + \frac{d\Omega_x}{dt} \mathcal{R} + \Omega_x^2 \mathcal{R}.
\tag{2.10}
\]

Thus, a second-order Taylor approximation of the camera rotation matrix using $R(0) = I$ is

\[
\mathcal{R}(t) \approx I + \Omega_x(0)t + \frac{1}{2} \left( \frac{d\Omega_x}{dt}(0) + \Omega_x^2(0) \right)t^2.
\tag{2.11}
\]

Similarly, the camera translation can be described by a differential model

\[
\begin{align*}
V(t) & = \frac{d\mathcal{T}}{dt}(t) = -\Omega_x(t)\mathcal{R}(t)c(t) - \mathcal{R}(t)\frac{dc}{dt}(t), \\
V(0) & = -\frac{dc}{dt}(0),
\end{align*}
\tag{2.12}
\]

and

\[
\begin{align*}
\frac{dV}{dt}(t) & = \frac{d^2\mathcal{T}}{dt^2}(t) \\
& = -\frac{d^2\mathcal{R}}{dt^2}(t)c(t) - 2\frac{d\mathcal{R}}{dt}(t)\frac{dc}{dt}(t) - \mathcal{R}(t)\frac{d^2c}{dt^2}(t),
\end{align*}
\tag{2.13}
\]

which at $t = 0$ gives $\frac{dV}{dt}(0) = -2\Omega_x(0)\frac{dc}{dt}(0) - \frac{d^2c}{dt^2}(0)$.

The choice of whether to adopt the Taylor approximation of $c(t)$ or $\mathcal{T}(t)$ as primary is entirely dependent in which domain the higher derivatives are expected to diminish, giving

\[
\begin{align*}
\mathcal{T}(t) & \approx V(0)t + \frac{1}{2} V(t) t^2, \\
c(t) & \approx -V(0)t + \frac{1}{2} \left[ -V(t) + 2\Omega_x(0)V(0) \right] t^2.
\end{align*}
\tag{2.14}
\]

### 2.4 Relating World and Camera-Centric Derivatives.

**Proposition 1** The velocity of a 3D point $\Gamma(t)$ in camera coordinates, $\Gamma_i(t)$, is related to its velocity in the world coordinates $\Gamma^w(t)$ by

\[
\begin{align*}
\Gamma_t & = \Omega_x \mathcal{R}\Gamma^w + \mathcal{R}\Gamma^w_t + V \\
& = \Omega_x \Gamma + \mathcal{R}\Gamma^w - \mathcal{R}c_t, \\
\Gamma_t & = \Omega_x \mathcal{R}\Gamma_0 + V = \Omega_x \Gamma - \mathcal{R}c_t,
\end{align*}
\tag{2.15, 2.16}
\]

**Proof** Differentiating Eq. 2.3 with respect to time,

\[
\begin{align*}
\Gamma_t & = \mathcal{R}_t\Gamma^w + \mathcal{R}\Gamma^w_t + \mathcal{T}_t \\
& = \Omega_x \mathcal{R}\Gamma^w + \mathcal{R}\Gamma^w_t + V \\
& = \Omega_x (\Gamma - \mathcal{T}) + \mathcal{R}\Gamma^w_t + V \\
& = \Omega_x \Gamma + \mathcal{R}\Gamma^w + V - \Omega_x \mathcal{T}.
\end{align*}
\tag{2.17, 2.18, 2.19, 2.20}
\]

The result follows from using $\mathcal{T} = -\mathcal{R}c$,

\[
\begin{align*}
V & = \mathcal{T}_t = -\mathcal{R}_t c - \mathcal{R}c_t = -\Omega_x \mathcal{R}c - \mathcal{R}c_t, \\
& = \Omega_x \mathcal{T} - \mathcal{R}c_t.
\end{align*}
\tag{2.21}
\]

\[ \square \]
2.5 Stationary and Non-Stationary Contours

It is important to differentiate between image contours arising from a space curve that is changing at most with a rigid transform (stationary contours), e.g., reflectance contours and sharp ridges, and image curves arising from deforming space curves (non-stationary contours), e.g., occluding contours, the contour generators projecting to apparent contours. Stationary contours are characterized by $\Gamma^w = 0$ while for occluding contours the viewing direction $\Gamma(S,t)$ is tangent to the surface $\mathcal{M}$ with surface normal $N(N^w = R^T N)$

$$\Gamma^T N = 0, \quad \text{or} \quad (\Gamma^w - c)^T N^w = 0. \quad (2.22)$$

For the image curve $\gamma(s,t)$ arising from the occluding contour, Fig. 3, the normal $N$ to $\mathcal{M}$ at an occluding contour (Cipolla and Giblin 1999) can be consistently taken as $N = \frac{\gamma_s \times [u \times \gamma_s]}{\|u \times \gamma_s\|}$.

Unless otherwise stated, we assume that the parametrization $\Gamma^w(S,t)$ of $\mathcal{M}$ is regular for occluding contours, so that $\Gamma^w(S,t)$ and $\Gamma^w(S,t)$ span the tangent plane to $\mathcal{M}$ at $\Gamma^w(S,t)$, and $t$ can be seen as a spatial parameter (Giblin and Weiss 1995). The correlation of the parametrization $S$ of $\Gamma$ at time $t$ to that of nearby times is captured by $\Gamma^w(S,t)$, which is orthogonal to $N^w$ (since $N^w$ is orthogonal to the tangent plane), but is otherwise arbitrary as a one dimensional choice. It is common to require that $\Gamma^w(S,t)$ lay on the (infinitesimal) epipolar plane, spanned by $\Gamma_w(S,t), c(t)$, and $c(t)$, referred to as the epipolar parametrization (Cipolla and Giblin 1999; Giblin and Weiss 1995),

$$\Gamma^w \times (\Gamma^w - c) = 0, \quad \text{or} \quad \Gamma^w = \lambda (\Gamma^w - c) \quad \text{for some} \lambda. \quad (2.23)$$

3 Projecting Differential Geometry Onto a Single View

This section relates the intrinsic differential-geometric attributes of the space curve and those of its perspective image curves. Specifically, the derivatives $\Gamma', \Gamma''$, and $\Gamma'''$ are first expressed in terms of the differential geometry of $\Gamma$, namely $\{T, N, B, K, \hat{K}, \tau, G, G', G''\}$, and second, they are expressed in terms of the differential geometry of $\gamma$, namely $\{t, n, k, \hat{k}\}$ using $\Gamma = \rho \gamma$. Note that $\hat{K}$ and $\hat{k}$ are both intrinsic quantities. In equating these two expressions, we relate $\{T, N, B, K, \hat{K}, \tau\}$ to $\{t, n, k, \hat{k}\}$. Our purpose is to eliminate the dependence on the parametrizations $(g, G)$, and depth $\rho$, i.e., final expressions do not contain these unknowns nor their derivatives $(g, g', g''), (G, G', G'')$, or unknown depth and its derivatives $(\rho, \rho', \rho'', \rho''')$. Intrinsic camera parameters are dealt with in Sect. 3.1.

Proposition 2 $\{T, N, B, K, \hat{K}, \tau, G, G', G''\}$ are related to $\{\gamma, t, n, k, \hat{k}, g, g', g'', \rho, \rho', \rho'', \rho''\}$ by

$$\gamma - \rho \gamma$$

$$\begin{align*}
G_T &= \rho' \gamma + \rho \gamma' \\
G_T + G G_N &= \rho'' \gamma + (2 \rho' g + \rho g') \tau + \rho g^2 n \\
(G'' - G_3 K^2) T + (3 G G' K + G^3 \hat{K}) N + G^3 \hat{K} B &= \rho'' \gamma + [3 \rho'' g + 3 \rho' g' + \rho (g'' - g'') \gamma] N + (3 G G' K + G^3 \hat{K}) n + G^3 \hat{K} T.
\end{align*} \quad (3.1)$$

where $\Gamma$ and $\gamma$ are linked by a common parameter via projection $\Gamma = \rho \gamma$.

Proof First, writing $\Gamma', \Gamma''$, and $\Gamma'''$ in the Frenet frame of $\Gamma$ as

$$\begin{align*}
\Gamma' &= G T \\
\Gamma'' &= G T + G G N \\
\Gamma''' &= (G'' - G^3 K^2) T + (3 G G' K + G^3 \hat{K}) N + G^3 \hat{K} B.
\end{align*} \quad (3.4)$$

Note that when expressed with respect to the arc-length of $\Gamma$, i.e., $G = 1$, simple expressions result:

$$\begin{align*}
\hat{T} &= T \\
\hat{K} &= K N \\
\hat{K} &= -K^2 T + \hat{K} N + K \tau B.
\end{align*} \quad (3.7)$$

Second, differentiating $\Gamma = \rho \gamma$ gives

$$\begin{align*}
\Gamma' &= \rho' \gamma + \rho \gamma' \\
\Gamma'' &= \rho'' \gamma + 2 \rho' \gamma' + \rho \gamma'' \\
\Gamma''' &= \rho''' \gamma + 3 \rho'' \gamma' + 3 \rho' \gamma'' + \rho \gamma'''.
\end{align*} \quad (3.10)$$

This can be rewritten using expressions for the derivatives of $\gamma$, which are

$$\begin{align*}
\gamma' &= g t \\
\gamma'' &= g' t + g^2 \kappa n \\
\gamma''' &= (g'' - g^3 \kappa^2) t + (3 g g' \kappa + g^2 \kappa') n.
\end{align*} \quad (3.13)$$

Thus, $\Gamma', \Gamma''$, and $\Gamma'''$ can be written in terms of $\gamma, t, n, \kappa, \hat{k}, g, g', g'', \rho, \rho', \rho'', \rho'''$ as

$$\begin{align*}
\chi' &= g t \\
\chi'' &= g' t + g^2 \kappa n \\
\chi''' &= (g'' - g^3 \kappa^2) t + (3 g g' \kappa + g^2 \kappa') n.
\end{align*} \quad (3.15)$$
Theorem 1
Given the tangent relationship is expressed below.

\[ \begin{align*}
\Gamma' &= \rho \gamma + \rho g t \\
\Gamma'' &= \rho' \gamma + (2\rho' g + \rho') t + \rho g^2 \kappa n \\
\Gamma''' &= \rho'' \gamma + (3\rho'' g + 3\rho' g' + \rho (g'' - g^3 \kappa^2)) t \\
& \quad + [3\rho' g^2 \kappa + \rho (3gg' \kappa + g^3 \kappa)] n.
\end{align*} \tag{3.16} \tag{3.17} \tag{3.18} \]

Equating (3.4–3.6) and (3.16–3.18) proves the proposition. \hfill \Box

Corollary 1 Using the arc-length \( \tilde{S} \) of the space curve as the common parameter, i.e., when \( G \equiv 1 \), we have

\[ \begin{align*}
T &= \rho \gamma + \rho g t \\
K N &= \rho' \gamma + (2\rho' g + \rho') t + \rho g^2 \kappa n \\
- K^2 T + \hat{K} N + K \tau B &= \rho'' \gamma \\
& \quad + [3\rho'' g + 3\rho' g' + \rho (g'' - g^3 \kappa^2)] t \\
& \quad + [3\rho' g^2 \kappa + \rho (3gg' \kappa + g^3 \kappa)] n.
\end{align*} \tag{3.19} \tag{3.20} \tag{3.21} \]

First-Order Differential Geometry. We are now in a position to derive the first-order differential attributes of the image curve \( (g, t) \) from that of the space curve \( (G, T) \). Note from (3.1) or (3.19) that \( T \) lies on the plane spanned by \( t \) and \( \gamma \), i.e., \( T \) is a linear combination of these vectors. An exact relationship is expressed bellow.

Theorem 1 Given the tangent \( T \) at \( \Gamma \) when \( T \) is not aligned with \( \gamma \), then the corresponding tangent \( t \) and normal \( n \) at \( \gamma \) are determined by

\[ t = \frac{T - T_z \gamma}{\|T - T_z \gamma\|}, \quad n = t^\perp \equiv t \times e_3. \tag{3.22} \]

Proof Eq. 3.1 states that \( T, t, \) and \( \gamma \) are coplanar. Taking the dot product with \( e_3 \) and using \( e_3 \times \gamma = 1, e_3 \times t = 0, \) and \( \rho' = GT_z \) (Equation 2.6), isolate \( t \) in the original equation as

\[ t = \frac{1}{\rho g} [G T - \rho' \gamma] = \frac{G}{\rho g} [T - T_z \gamma] \tag{3.23} \]

and the result follows by normalizing. The formula for the normal comes from the fact that it lies in the image plane, therefore being orthogonal to both \( t \) and \( e_3 \). \hfill \Box

Observe that the depth scale factor \( \rho \) is not needed to find \( t \) from \( T \). Moreover, when \( \gamma \) and \( T \) are aligned for a point on \( \gamma \), Eq. 3.23 still holds, but implies that \( g = 0 \) and \( t \) is undefined, i.e., that the image curve will have stationary points and possibly corners or cusps. Stationary points are in principle not detectable from the trace of \( \gamma \) alone, but by the assumption of general position these do not concern us.

A crucial quantity in relating differential geometry along the space curve to that of the projected image curve is the ratio of speeds of parametrizations \( \frac{g}{G} \) (s). The following theorem derives the key result that this quantity is intrinsic in that it does not depend on the parametrization of \( \Gamma \) or of \( \gamma \), thus allowing a relationship between the differential geometry of the space and image curves.

Theorem 2 The ratio of speeds of the projected 2D curve \( g \) and of the 3D curve \( G \) at corresponding points is an intrinsic quantity given by

\[ \frac{g}{G} = \frac{\|T - T_z \gamma\|}{z} \quad \text{or} \quad g = \frac{\|GT - \rho' \gamma\|}{\rho}, \tag{3.24} \]

i.e., it does not depend on the parametrization of \( \Gamma \) or of \( \gamma \).

Proof Follows from a dot product of Eq. 3.23 with \( t \) and dividing by \( \rho = z \). \hfill \Box

Second-Order Differential Geometry. The curvature of an image curve can be derived from the curvature of the space curve, as shown by the next theorem.

Theorem 3 The curvature \( \kappa \) of a projected image curve is given by

\[ \kappa = \left[ \frac{N - N_z \gamma \cdot n}{\rho g^2} \right] K, \quad \text{when} \ G \equiv 1, \tag{3.25} \]

or

\[ \kappa = \left( \frac{G}{g} \right)^2 \left[ N^\top (\gamma \times t) \right] \frac{K}{\rho}, \tag{3.25} \]

where \( g \) and \( \frac{G}{g} \) are given by Eq. 3.24, and \( \rho = e_3 \times \Gamma \). The tangential acceleration of a projected curve with respect to the arc length of the space curve is given by

\[ \begin{align*}
\frac{dg}{d\tilde{S}} &= \frac{[N - N_z \gamma]^\top T K}{\rho} - 2g \frac{T_z}{\rho}, \\
\frac{d^2 g}{d\tilde{S}^2} &= -\frac{K N^\top (\gamma \times n)}{\rho} - 2g \frac{T_z}{\rho}.
\end{align*} \tag{3.26} \tag{3.27} \]

Proof Using Eq. 2.6 in Eq. 3.2 leads to

\[ G'T + G^2 K N (\gamma \times t) = (G'Tz + G^2 K N_z) \gamma + 2G Tz g t + \rho g^2 \kappa n. \tag{3.27} \]

First, in the case of \( G \equiv 1 \) curvature \( \kappa \) can be isolated by taking the dot product of the last equation with \( n \) which gives the curvature projection formula (3.25). By instead taking the dot product with \( \gamma \times t \) we arrive at the alternative formula, since the only remaining terms are those containing \( N \) or \( n \),

\[ G^2 K N^\top (\gamma \times t) = \rho g^2 \kappa n^\top (\gamma \times t). \tag{3.28} \]

Isolating \( \kappa \) and using \( n^\top (\gamma \times t) = \gamma^\top (t \times n) = \gamma^\top e_3 = 1 \) gives the desired equation. Second, the term \( g' \) can be isolated...
by taking the dot product with $t$ or with $y \times n$, giving the first and second formulas, respectively, noting that $t^\top (y \times n) = -1$.

Note that formulas for the projection of 3D tangent and curvatures onto 2D tangent and geodesic curvature appear in Cipolla and Zisserman (1992) and Cipolla and Giblin (1999, pp. 73–75), but an actual image curvature was not determined there. That the curvature of the space curve is related to the curvature of the projected curve was derived in previous work (Li and Zucker 2003; Robert and Faugeras 1991), but our proof is much simpler and more direct. Moreover, our proof methodology generalizes to relating higher order derivatives such as curvature derivative and torsion, as shown below.

**Theorem 4** The curvature derivative of a projected image curve $y$ is derived from the local third-order differential geometry of the space curve as follows

$$\kappa = \left[ \frac{\dot{K}N + K\tau B}{\rho g^3} (y \times t) \right] - 3\kappa \left( T_z \left( \frac{\kappa'}{\rho g} + \frac{g'}{g^2} \right) \right),$$  \hspace{1cm} (3.29)

assuming $G \equiv 1$.

**Proof** Taking the scalar product of Eq. 3.21 with $y \times t$, and using $T^\top (y \times t) = 0$ and $n^\top (y \times t) = y^\top t = 1$,

$$[\dot{K}N + K\tau B]^\top (y \times t) = 3\rho' g^2 \kappa' + \rho (3gg' + g^3 \dot{k}),$$  \hspace{1cm} (3.30)

which using $\rho' = T_z$ gives

$$3T_z g^2 \kappa' + \rho (3gg' + g^3 \dot{k}) = [\dot{K}N + K\tau B]^\top (y \times t).$$  \hspace{1cm} (3.31)

Isolating $\dot{k}$ gives the desired result. Since both $g$ and $g'$ are available from Eqs. 3.24 to 3.26, the theorem follows. □

### 3.1 Intrinsic Camera Parameters and Differential Geometry

This section derives the relationship between the intrinsic differential geometry $\{t, n, \kappa, \kappa\}$ of the curve in normalized image coordinates to those in image pixel coordinates, $\{t_{im}, n_{im}, \kappa_{im}, \kappa_{im}\}$. Using the intrinsic parameter matrix $K_{im}$ relating $y_{im} = K_{im} y$, by the linear Eq. 2.7.

**Theorem 5** The intrinsic quantities $\{t, n, \kappa, \kappa\}$ and $\{t_{im}, n_{im}, \kappa_{im}, \kappa_{im}\}$ under linear transformation $y_{im} = K_{im} y$ are related by

$$\begin{align*}
g_{im} &= \|K_{im} t\|, \quad t_{im} = \frac{K_{im} t}{\|K_{im} t\|}, \\
n_{im} &= t_{im} \times e_3, \\
g'_{im} &= \frac{\kappa t^\top K_{im}^\top K_{im} n}{g_{im}}, \quad \kappa_{im} = \frac{n_{im}^\top K_{im} \kappa_{im} n}{g_{im}^2}, \\
k'_{im} &= \frac{1}{g_{im}^2} n_{im}^\top K_{im} (-\kappa^2 t + \kappa n) - \frac{3g'_{im} \kappa_{im}}{2 g_{im}}.
\end{align*}$$  \hspace{1cm} (3.32-3.34)

where the speed $g_{im}$ is relative to unit speed at $y$.

**Proof** Differentiating (2.7) with respect to the arc-length $\dot{s}$ of $y$ and using (3.13), $y'_{im} = K_{im} \dot{y}$ gives

$$g_{im} t_{im} = K_{im} t.$$  \hspace{1cm} (3.35)

Differentiating (2.7) a second time with respect to $\dot{s}$, and using Eq. 3.14,

$$g'_{im} t_{im} + g_{im}^2 \kappa_{im} n_{im} = K_{im} \kappa_{im} n.$$  \hspace{1cm} (3.36)

Taking the dot product with $t_{im}$ gives the formula for $g'_{im}$, and taking the dot product with $n_{im}$ gives the formula for $\kappa_{im}$. Differentiating (2.7) a third time with respect to $\dot{s}$, and using Eq. 3.15, we have

$$(\kappa''_{im} - g_{im}^3 \kappa_{im}^3) t_{im} + \left( 3g_{im} g'_{im} \kappa_{im} + g_{im}^3 \kappa_{im} \right) n_{im} = K_{im} (-\kappa^2 t + \dot{k} n).$$  \hspace{1cm} (3.37)

Taking the dot product with $n_{im}$,

$$3g_{im} g'_{im} \kappa_{im} + g_{im}^3 \kappa_{im} = n_{im}^\top K_{im} (-\kappa^2 t + \kappa n),$$  \hspace{1cm} (3.38)

and isolating $\kappa_{im}$, the last result follows. □

The above theorem can also be used in its inverse form from $y_{im}$ to $y$ by substituting $K_{im}$ for $K_{im}^{-1}$, and trivially exchanging the sub-indices. Moreover, the theorem is generally valid for relating differential geometry under any linear transformation in place of $K_{im}$.

### 4 Reconstructing Differential Geometry from Multiple Views

In the previous section, we derived the differential geometry of a projected curve from a space curve. In this section, we derive the differential geometry of a space curve $F_i$ from that of its projected image curves in multiple views, namely $y_i$ for camera $i, i = 1, \ldots, N$. In order to simplify the equations, in this section all vectors are written in the common world coordinate basis, including $y_i$. Denote $F_i := \Gamma^w - c_i$.
namely \( \Gamma_i \) represents the vector from the \( i \)th camera center to the 3D point \( \Gamma^w \) in the world coordinate system.

The reconstruction of a point on the space curve \( \Gamma \) from two corresponding image curve points \( \gamma_1 \) and \( \gamma_2 \) can be obtained by equating the two expressions for \( \Gamma^w \) given by Eq. 2.4,

\[
\begin{align*}
\Gamma^w - c_1 &= \rho_1 \gamma_1, \\
\Gamma^w - c_2 &= \rho_2 \gamma_2, \\
\end{align*}
\]

\( \rho_1 \gamma_1 - \rho_2 \gamma_2 = c_2 - c_1. \)  \( (4.1) \)

Taking the dot product with \( \gamma_1, \gamma_2, \) and \( \gamma_1 \times \gamma_2 \) gives

\[
\begin{align*}
\rho_1 \gamma_1 \cdot \gamma_1 - \rho_2 \gamma_1 \cdot \gamma_2 &= (c_2 - c_1) \cdot \gamma_1 \\
\rho_1 \gamma_1 \cdot \gamma_2 - \rho_2 \gamma_2 \cdot \gamma_2 &= (c_2 - c_1) \cdot \gamma_2 \\
0 &= (c_2 - c_1) \cdot (\gamma_1 \times \gamma_2),
\end{align*}
\]

which gives

\[
\begin{align*}
\rho_1 &= \frac{(c_2 - c_1) \cdot \gamma_1 (\gamma_2 \cdot \gamma_2) - (c_2 - c_1) \cdot \gamma_2 (\gamma_1 \cdot \gamma_2)}{(\gamma_1 \cdot \gamma_1) (\gamma_2 \cdot \gamma_2) - (\gamma_1 \cdot \gamma_2)^2}, \\
\rho_2 &= \frac{(c_2 - c_1) \cdot \gamma_1 (\gamma_1 \cdot \gamma_2) - (c_2 - c_1) \cdot \gamma_2 (\gamma_1 \cdot \gamma_1)}{(\gamma_1 \cdot \gamma_1) (\gamma_2 \cdot \gamma_2) - (\gamma_1 \cdot \gamma_2)^2},
\end{align*}
\]

provided that \( (c_2 - c_1) \cdot (\gamma_1 \times \gamma_2) = 0. \) This is precisely the well-known fact that this system of three equations in two unknowns \( \rho_1 \) and \( \rho_2 \) can only be solved if the lines \( c_1 \gamma_1 \) and \( c_2 \gamma_2 \) intersect.

The crucial factor in relating the differential geometry of image curves in distinct views is the relationship between their parametrization in each view, given in the next theorem.

**Proposition 3** The ratio of parametrization speeds in two views of a space curve at corresponding points is given by

\[
\begin{align*}
g_1 &= \frac{\rho_2 \|T - (e_{3,1}^w T)\gamma_1\|}{\rho_1 \|T - (e_{3,2}^w T)\gamma_2\|}, \\
\end{align*}
\]

\( (4.4) \)

**Proof** Follows by dividing expressions for \( g_1 \) and \( g_2 \) from Eq. 3.24. \( \square \)

Next, note from Eq. 3.1 that the unit vector \( T \) can be written as

\[
T = \frac{\rho'}{G} \gamma + \frac{g}{G} t.
\]

\( (4.5) \)

Since \( T \) is a unit vector, it can be written as

\[
T = \cos \theta \frac{\gamma}{\|\gamma\|} + \sin \theta t,
\]

where \( \cos \theta = \frac{\rho'}{G} \|\gamma\|, \sin \theta = \frac{g}{G}. \)  \( (4.6) \)

Note that \( \rho > 0 \) implies that \( \sin \theta \geq 0 \) or \( \theta \in [0, \pi). \) Thus the reconstruction of \( T \) from \( t \) requires the discovery of the additional parameter \( \theta \) which can be provided from tangents at two corresponding points, as stated in the next result.

**Theorem 6** Two tangent vectors at a corresponding pair of points, namely \( t_1 \) at \( \gamma_1 \) and \( t_2 \) at \( \gamma_2 \), reconstruct the corresponding space tangent \( T \) at \( \Gamma \) as

\[
T = \cos \theta_1 \frac{\gamma_1}{\|\gamma_1\|} + \sin \theta_1 t_1 = \cos \theta_2 \frac{\gamma_2}{\|\gamma_2\|} + \sin \theta_2 t_2,
\]

\( (4.7) \)

and

\[
\begin{align*}
\rho_{1} &= \sin \theta_1 G, \\
\rho_{1} &= \frac{-t_1 \cdot (\gamma_2 \times t_2)}{\gamma_1 \cdot (\gamma_2 \times t_2)}, \\
\rho_{2} &= \sin \theta_2 G, \\
\rho_{2} &= \frac{-t_2 \cdot (\gamma_1 \times t_1)}{\gamma_2 \cdot (\gamma_1 \times t_1)},
\end{align*}
\]

\( (4.8) \)

where

\[
\begin{align*}
\tan \theta_1 &= -\frac{1}{\gamma_1 \cdot (\gamma_2 \times t_2) \gamma_1 \cdot (\gamma_2 \times t_2)}, \quad \theta_1 \in [0, \pi) \\
\tan \theta_2 &= -\frac{1}{\gamma_2 \cdot (\gamma_1 \times t_1) \gamma_2 \cdot (\gamma_1 \times t_1)}, \quad \theta_2 \in [0, \pi).
\end{align*}
\]

\( (4.9) \)

**Proof** Equating the two expressions for \( T \) from Eq. 4.6, one for each view, gives Eq. 4.7. Solving for \( \theta_1 \) by taking the dot product with \( \gamma_2 \times t_2 \) gives

\[
\cos \theta_1 \frac{\gamma_1}{\|\gamma_1\|} \cdot (\gamma_2 \times t_2) + \sin \theta_1 t_1 \cdot (\gamma_2 \times t_2) = 0,
\]

\( (4.10) \)

which leads to Eq. 4.9 and similarly for \( \theta_2. \) Eq. 4.8 follows from equating (4.10) and (4.5), then taking dot products with \( \gamma_2 \times t_2 \). \( \square \)

**Remark** Since \( T \) is orthogonal to both \( \gamma_1 \times t_1 \) and \( \gamma_2 \times t_2 \) we have the following reconstruction formula

\[
\varepsilon T = \frac{(t_1 \times \gamma_1) \times (t_2 \times \gamma_2)}{\|(t_1 \times \gamma_1) \times (t_2 \times \gamma_2)\|} \varepsilon = \pm 1,
\]

\( (4.11) \)

Fig. 4, where \( \varepsilon \) is determined from

\[
\begin{align*}
\varepsilon [T - (T \cdot e_{3,1}^w T) \gamma_1] \cdot t_1 > 0, \\
\varepsilon [T - (T \cdot e_{3,2}^w T) \gamma_2] \cdot t_2 > 0.
\end{align*}
\]

\( (4.12) \)

**Remark** This theorem implies that any two tangents at corresponding points can be consistent with at least one space tangent. Furthermore, the discovery of \( T \) does not require the \textit{a priori} solution of \( \rho_1 \) or \( \rho_2. \)
Remark An analogous tangent reconstruction expression under continuous motion may be derived, see Faugeras and Papadopoulo (1993).

Theorem 7 The normal vector \( \mathbf{N} \) and curvature \( \mathbf{K} \) of a point on a space curve \( \Gamma \) with point-tangent-curvature at projections in two views \( (\mathbf{y}_1, t_1, \kappa_1) \) and \( (\mathbf{y}_2, t_2, \kappa_2) \) are given by solving the system in the vector \( \mathbf{NK} \)

\[
\begin{align*}
G^2(\mathbf{y}_1 \times t_1)^\top \mathbf{NK} &= \rho_1 g_1^2 \kappa_1 \\
G^2(\mathbf{y}_2 \times t_2)^\top \mathbf{NK} &= \rho_2 g_2^2 \kappa_2 \\
\mathbf{T}^\top \mathbf{NK} &= 0,
\end{align*}
\]

(4.13)

where \( \mathbf{T} \) is given by Eq. 4.5, \( \rho_1 \) and \( \rho_2 \) by Eq. 4.3, and \( g_1 \) and \( g_2 \) by Eq. 3.24.

Proof Taking the dot product of (3.27) with \( \mathbf{y} \times t \), for each view, we arrive at the first two equations. The third equation imposes the solution \( \mathbf{NK} \) to be normal to \( \mathbf{T} \). \( \square \)

Remark An analogous curvature reconstruction expression under continuous motion can be derived, see Cipolla (1991). Theorem 7 is a variant form of a known result (Robert and Faugeras 1991; Li and Zucker 2003, 2006), using the proposed unified formulation which enables a more condensed and generalizable proof for the practical case of planar images. The next theorem leverages this effective theoretical framework to achieve the reconstruction of the torsion and curvature derivative of a space curve from two image curves, which is the central novel result of the present work.

Theorem 8 The torsion and curvature derivative at a point of a space curve can be obtained from up to third order differential geometry \( \kappa, \kappa' \) at a pair of corresponding points in two views by solving for an unknown vector \( \mathbf{Z} \) in the system

\[
\begin{align*}
(\mathbf{y}_1 \times t_1)^\top \mathbf{Z} &= 3g_1^2 \kappa_1 e_3^\top T + \rho_1 \left(3g_1 g_1' \kappa_1 + g_1^3 \kappa_1'\right) \\
(\mathbf{y}_2 \times t_2)^\top \mathbf{Z} &= 3g_2^2 \kappa_2 e_3^\top T + \rho_2 \left(3g_2 g_2' \kappa_2 + g_2^3 \kappa_2'\right) \\
\mathbf{T}^\top \mathbf{Z} &= 0,
\end{align*}
\]

(4.14)

and by solving for the torsion \( \tau \) and curvature derivative \( \dot{\mathbf{K}} \) from \( \mathbf{Z} = \dot{\mathbf{K}} \mathbf{N} + \mathbf{K} \tau \mathbf{B} \), i.e.,

\[
\begin{align*}
\tau &= \frac{\mathbf{Z}^\top \mathbf{B}}{\mathbf{K}} \\
\dot{\mathbf{K}} &= \mathbf{Z}^\top \mathbf{N},
\end{align*}
\]

(4.15)

(4.16)

with \( \mathbf{T}, \mathbf{N}, \mathbf{B}, \mathbf{K}, g_1, g_2, g_1', g_2', \rho_1, \) and \( \rho_2 \) determined from previous derivations, and assuming \( G \equiv 1 \).

Proof Apply Eq. (3.30) for two views, and let \( \mathbf{Z} := \dot{\mathbf{K}} \mathbf{N} + \mathbf{K} \tau \mathbf{B} \) to get the first two equations of (4.14). The last equation of (4.14) constrains \( \mathbf{Z} \) to be orthogonal to \( \mathbf{T} \). \( \square \)

5 Projecting Differential Geometry Under Differential Motion

The goal of this section is to relate differential observations in a series of images from a continuous video sequence to the differential geometry of the space curve. As this relationship is governed by the differential motion of the camera and its intrinsic parameters, we also aim to recover scene geometry and camera motion/pose from these observations and equations. An account of unknown intrinsic camera calibration in this setting is left for future work. We explore how differential scene properties are projected onto differential image properties for points and curves, and expect future work to apply this to surfaces.

Differential models of camera motion observing a rigid scene were studied in Longuet-Higgins and Prazdny (1980), Waxman and Ullman (1985), Maybank (1992), Ma et al. (2004), van den Hengel (2000), Triggs (1999), Astrom and Heyden (1996), Heyden (2006), Baumela et al. (2000), Doronaička and Sappa (2006), Kahl and Heyden (2001), Vieville and Faugeras (1996), van den Hengel et al. (2007), Brodsky et al. (2000), Brodsky and Fermüller (2002). These papers studied how the first and second-order motion of the image of fixed points relate to a differential camera motion model. They also envisioned recovering local 3D surface properties from the local behavior of the velocities of projected surface points in an image neighborhood. Differential models for nonrigid curves observed in a monocular video sequence were studied in Faugeras and Papadopoulo (1993), Papadopoulo and Faugeras (1996), Papadopoulo (1996), Faugeras (1990), where it was established that multiple
simultaneous video sequences would be needed. This led to practical work in the reconstruction of nonrigid curves from multiview video (Carceroni and Kutulakos 1999; Carceroni 2001), exploiting temporal consistency within each video frame, as well as consistency across the two video sequences. Differential models of occluding contours were studied mainly in Cipolla (1991), Cipolla and Blake (1992), Cipolla and Giblin (1999), relating the deformation of apparent contours under differential camera motion to scene properties such as occluding contours, and a differential-geometric model of the underlying 3D surface.

5.1 Differential Relations for a Point

**Theorem 9** (Moving 3D point) Let $\Gamma^w(t)$ be a moving point in space, projected onto a moving camera as $\gamma(t)$ with depth $\rho(t)$. Let the differential velocity and rotation of the camera be $V$ and $\Omega$, respectively, and let $V_t$ and $\Omega_t$ represent their derivative with respect to time $t$, respectively. Then the depth gradient and second derivative at $t = 0$ are

$$
\begin{align*}
\rho_t &= \rho e_3^T \left( \Omega \times \gamma + \frac{1}{\rho} \Gamma^w + \frac{1}{\rho} V \right) \\
\rho_{tt} &= \rho e_3^T \left( \Omega^2 + [\Omega_t]_x \right) \gamma + \frac{1}{\rho} \left( V - V_t \gamma \right) \\
&\quad + 2 e_3^T \Omega \times \Gamma^w_t + e_3^T \Gamma^w_{tt} + e_3^T V_t,
\end{align*}
$$

(5.1)

and the velocity and acceleration of the projected point at $t = 0$ are given by

$$
\begin{align*}
\gamma_t &= \Omega \times \gamma - \left( e_3^T \Omega \times \gamma \right) \gamma \\
&\quad + \frac{1}{\rho} \left( \Gamma^w_t - e_3^T \Gamma^w_{tt} \gamma \right) + \frac{1}{\rho} \left( V - V_t \gamma \right) \\
\gamma_{tt} &= \left( \Omega^2 + [\Omega_t]_x \right) \gamma + \frac{2}{\rho} \Omega \times \Gamma^w_t + \frac{1}{\rho} \Gamma^w_{tt} \\
&\quad + \frac{1}{\rho} V_t - \frac{2}{\rho} \rho_t \gamma_t - \rho_{tt} \gamma,
\end{align*}
$$

(5.2)

which can be simplified as

$$
\begin{align*}
\gamma_t &= \Omega \times \gamma - \left( e_3^T \Omega \times \gamma \right) \gamma \\
&\quad + \frac{1}{\rho} \left( \Gamma^w_t - e_3^T \Gamma^w_{tt} \gamma \right) + \frac{1}{\rho} \left( V - V_t \gamma \right) \\
&\quad - 2 e_3^T \left( \Omega \times \gamma + \frac{V}{\rho} + \frac{\Gamma^w}{\rho} \right) \\
&\quad \cdot \left( V - V_t \gamma + \Omega \times \gamma - \left( e_3^T \Omega \times \gamma \right) \gamma + \frac{1}{\rho} \Gamma^w_t - \frac{e_3^T \Gamma^w_{tt}}{\rho} \gamma \right) \\
&\quad - e_3^T \left( \Omega \times \gamma + \frac{V}{\rho} + 2 \Omega \times \Gamma^w_t + \frac{\Gamma^w}{\rho} \right) \gamma.
\end{align*}
$$

(5.3)

**Proof** The image velocity $\gamma_t$ is dependent on the velocity $\Gamma^w_t$ of the 3D structure in camera coordinates, which arises from both the motion of $\Gamma^w$ and from the moving camera. Differentiating $\Gamma = R \Gamma^w + \mathcal{F}$, we get

$$
\Gamma_t = R_t \Gamma^w + R \Gamma^w_t + \mathcal{F}_t = \Omega \times R \Gamma^w + R \Gamma^w_t + V.
$$

(5.4)

Differentiating $\Gamma = \rho \gamma$ we get

$$
\Gamma_t = \rho \gamma_t + \rho_t \gamma.
$$

(5.5)

Equating these two expressions leads to

$$
\rho \gamma_t + \rho_t \gamma = \Omega \times R \Gamma^w + R \Gamma^w_t + V
$$

for arbitrary $t$. (5.6)

At $t = 0$ we have $\Gamma^w_t = \Gamma = \rho \gamma$, leading to

$$
\rho \gamma_t + \rho_t \gamma = \rho \Omega \times \gamma + \Gamma^w_t + V
$$

for $t = 0$. (5.7)

The depth gradient $\rho_t$ is then isolated by taking the dot product of both sides of Eq. 5.9 with $e_3$, observing that $e_3^T \gamma = 1$ and $e_3^T \gamma_t = 0$, resulting in Eq. 5.1. The expression for $\rho_t$ is then substituted into Eq. 5.9 from which $\gamma_t$ can be isolated in the form of Eq. 5.3.

The second order expressions $\gamma_{tt}$ and $\rho_{tt}$ require another time derivative of Eq. 5.8,

$$
\rho \gamma_{tt} + 2 \rho_t \gamma_t + \rho_{tt} \gamma = \left( \Omega^2 + [\Omega_t]_x \right) \rho \gamma
$$

+ $2 \Omega \times \Gamma^w_t + \Gamma^w_{tt} + V_t.
$$

(5.10)

Setting $t = 0$ we have

$$
\rho \gamma_{tt} + 2 \rho_t \gamma_t + \rho_{tt} \gamma = \left( \Omega^2 + [\Omega_t]_x \right) \rho \gamma
$$

+ $2 \Omega \times \Gamma^w_t + \Gamma^w_{tt} + V_t.
$$

(5.11)

Now the expression for $\rho_{tt}$ in the theorem can be obtained by dotting with $e_3$, giving Eq. 5.2. Isolating $\gamma_{tt}$ we have

$$
\gamma_{tt} = \left( \Omega^2 + [\Omega_t]_x \right) \gamma
$$

+ $\frac{1}{\rho} \left( 2 \Omega \times \Gamma^w_t + \Gamma^w_{tt} + V_t - 2 \rho_t \gamma_t - \rho_{tt} \gamma \right).
$$

(5.12)

Substituting Eqs. 5.2 and 5.3 into the above, we obtain the final expression for $\gamma_{tt}$. □

**The Special Case of Fixed Points.** The question of how the image of a fixed point moves as the camera moves was studied by Longuet-Higgins and Prazdny (Longuet-Higgins and Prazdny 1980) and later by Waxman and Ullman (Waxman and Ullman 1985), giving the velocity $\gamma_t$ for a fixed point. This calculation also leads to the well-known epipolar
constraint, the notion of Essential matrix (Longuet-Higgins 1981), and the continuous epipolar constraint (Zhuang and Haralick 1984; Maybank 1992; Kanatani 1993; Viéville and Faugeras 1995; Tian et al. 1996; Brooks et al. 1997; Åström and Heyden 1998; Ponce and Genc 1998; Yi Ma 1998; Ma et al. 2004; Stewénius et al. 2007; Lin et al. 2009; Valgaerts et al. 2012; Schneeevoigt et al. 2014). Theorem 9 in the special case of a fixed point gives interesting geometric insight into these classical results. Specifically, setting $\Gamma^w_r = 0$ in first-order computations of Eqs. 5.1 and 5.3 results in

$$\begin{align*}
\frac{\rho}{\rho} &= e^i_j (\Omega x y + \frac{V}{\rho}) \\
y_r &= \Omega x y - \left(e^i_j \Omega x y\right) y + \frac{V}{\rho} - \frac{V}{\rho} y, \quad \text{at } t = 0 \\
(5.13)
\end{align*}$$

Essential constraint. To derive the differential epipolar constraint, eliminate $\rho$ from Eq. 5.14 by first writing out the expression in terms of $\xi$, $\eta$, $u$, and $v$, where $y = [\xi, \eta, 1]^T$, and $y_r = [u, v, 0]$, and use $e^i_j \Omega x y = -\Omega y, \xi + \Omega z, \eta$, giving

$$\begin{align*}
u - \Omega y, \xi^2 + \Omega z, \xi \eta + \Omega z, \eta - \Omega y = \frac{1}{\rho} (V_x - V_y) \\
v + \Omega z, \eta^2 - \Omega y, \xi \eta - \Omega z, \xi + \Omega x = \frac{1}{\rho} (V_y - V_z) \eta \\
(5.15)
\end{align*}$$

and then eliminate $\rho$, giving

$$\begin{align*}
u - \Omega y, \xi^2 + \Omega z, \xi \eta + \Omega z, \eta - \Omega y = \frac{1}{\rho} (V_x - V_y) \\
v + \Omega z, \eta^2 - \Omega y, \xi \eta - \Omega z, \xi + \Omega x = \frac{1}{\rho} (V_y - V_z) \eta \\
\end{align*}$$

which is the epipolar constraint for differential motion. A more direct way of deriving the epipolar constraint equation is to eliminate $\rho$ in Eq. 5.9 with $\Gamma^w_r = 0$ by taking the dot-product with the vector $V x y = V x y$, where $V x$ is a skew-symmetric arrangement of $V$, and using $\Omega^T x = -\Omega x$ gives

$$\rho y_1^T V x y = -\rho y_1^T \Omega x V x y$$

resulting in the differential epipolar constraint

$$y_1^T V x y + y^T \Omega x V x y = 0. \quad (5.16)$$

In comparison, the widely-known essential constraint for relating two views is given by

$$y_2^T T x R y_1 = 0, \quad (5.17)$$

where $T x R$, the essential matrix, combines the effects of translation and rotation to relate two points $y_1$ and $y_2$. In the differential case, the two matrices $V x$ and $\Omega x V x$ play a similar role to $T x R$ in the discrete motion case to relate a point and its velocity.

Remark 1 Observe from Eq. 5.14 that $y_r$ can also be written as the sum of two components, one depending on $V$, and the other on $\Omega$, i.e.,

$$y_r = \frac{1}{\rho} A(y) V + B(y) \Omega,$$

where $A(y) = \left[ \begin{array}{ccc} 1 & -\xi & \eta \\ 0 & 1 & -\eta \\ 0 & 0 & 0 \end{array} \right]$ and $B(y) = \left[ \begin{array}{ccc} -\xi \eta & 1 + \xi^2 - \eta \\ \eta & \xi & \xi \\ 0 & 0 & 0 \end{array} \right]. \quad (5.18)$

That $y_r$ depends linearly on $V$ and $\Omega$, and $\rho$ in the equation, is the basis of subspace methods in structure from motion (Heeger and Jepson 1992). Observations of image velocities $y_{1,1}, y_{1,2}, \ldots, y_{1,N}$ at points $y_1, y_2, \ldots, y_N$ provides $2N$ linear equations in $V$ and $\Omega$, given $\rho_1, \ldots, \rho_N$.

5.2 Differential Relations for a Curve

Theorem 10 (Deforming 3D curve) Consider a deforming 3D curve $\Gamma(s, t)$ projecting to a family of 2D curves $y(s, t)$ with depth $\rho(s, t)$, arising from camera motion with differential velocities of translation and rotation $V$ and $\Omega$, respectively, and let $V_i$ and $\Omega_i$ be their respective derivatives in time. Then, the image velocity $y_i$ is determined from

$$y_i = \alpha + \beta n,$$

where $\alpha = -\Omega \cdot y \times (y \times n)$ and $\beta = \Omega \cdot y \times (y \times t)$. \quad (5.19)

Proof From Eq. 2.15 and using $-R \rho_i = V - \Omega x T$ from Eq. 2.12,

$$\Gamma_i = \Omega x \Gamma + V - \Omega x T + R \Gamma^w_r. \quad (5.21)$$

Using $\Gamma = \rho y$ and $\Gamma_i = \rho_i y + \rho y_t$,

$$\rho y + \rho y_t = \rho \Omega x y + V - \Omega x T + R \Gamma^w_r. \quad (5.22)$$
Taking the dot product with \( y \times n \) and \( y \times t \),
\[
\begin{align*}
\rho y_t \cdot (y \times n) &= \rho (\Omega \times y) \cdot (y \times n) + (V - \Omega \times T) \\
&= -\rho e_3^\top y = -\alpha \\
\rho y_t \cdot (y \times t) &= \rho (\Omega \times y) \cdot (y \times t) + (V - \Omega \times T) \\
&= \rho t \cdot (y \times t) + R\Gamma_t^w \cdot (y \times t).
\end{align*}
\]
(5.23)

Now,
\[
\begin{align*}
y_t \cdot (y \times n) &= (at + \beta n) \cdot (y \times n) = at \cdot (y \times n) \\
&= \alpha n \times t \cdot y = -\alpha e_3^\top y = -\alpha \\
y_t \cdot (y \times t) &= (at + \beta n) \cdot (y \times t) \\
&= \beta n \cdot (y \times t) = \beta t \times n \cdot y = \beta e_3^\top y = \beta.
\end{align*}
\]
(5.24)

So that we can write
\[
\begin{align*}
\alpha &= -(\Omega \times y) \cdot (y \times n) - \left( \frac{V}{\rho} - \Omega \times \frac{T}{\rho} + R\Gamma_t^w \right) \cdot (y \times n) \\
\beta &= (\Omega \times y) \cdot (y \times t) + \left( \frac{V}{\rho} - \Omega \times \frac{T}{\rho} + R\Gamma_t^w \right) \cdot (y \times t).
\end{align*}
\]
(5.25)

Since we can switch the cross and dot products in a triple scalar product, \( \Omega \times y \cdot (y \times n) = \Omega \cdot y \times (y \times n) \) and \( \Omega \times y \cdot (y \times t) = \Omega \cdot y \times (y \times t) \), giving the final result.\( \square \)

**Corollary 2** The spatial variation of the velocity vector field \( y_t \) along the curve and in time can be written as
\[
y_{st} = (\mathbf{t} + V^\prime) \frac{\rho s}{\rho^2} \beta_s - \frac{V_s}{\rho} y_s + \Omega \times y_s - \left( e_3^\top \Omega \times y_s \right) y \\
- \left( e_3^\top \Omega \times y \right) y_s + \frac{1}{\rho} \left( \Gamma^w_{st} - e_3^\top \Gamma^w_s \Omega \times y_s \right) y \\
- \frac{1}{\rho^2} \left( \Gamma^w_t - e_3^\top \Gamma^w_t \Omega \times y \right) \beta_s,
\]
(5.26)

and the time acceleration \( y_{tt} \) is defined by
\[
\begin{align*}
t^\top y_{tt} &= t^\top \left( \Omega^2_{s} + [\Omega, \Omega]_{s} \right) y + \frac{2}{\rho} t^\top \Omega \times y_s + \frac{1}{\rho} t^\top \Gamma^w_{tt} \\
&+ \frac{1}{\rho} t^\top V_t - 2e_3^\top \left( \Omega \times y + \frac{V}{\rho} + \frac{\Gamma^w_{st}}{\rho} \right) \alpha \\
- e_3^\top \left( \Omega^2_{s} + [\Omega, \Omega]_{s} \right) y + \frac{V_s}{\rho} + 2\Omega \times \frac{\Gamma^w_{st}}{\rho} + \frac{\Gamma^w_{tt}}{\rho} \right) \Omega \times y, \\
\end{align*}
\]
\[
\begin{align*}
n^\top y_{tt} &= n^\top \left( \Omega^2_{s} + [\Omega, \Omega]_{s} \right) y + \frac{2}{\rho} n^\top \Omega \times y_s + \frac{1}{\rho} n^\top \Gamma^w_{tt} \\
&+ \frac{1}{\rho} n^\top V_t - 2e_3^\top \left( \Omega \times y + \frac{V}{\rho} + \frac{\Gamma^w_{st}}{\rho} \right) \beta \\
- e_3^\top \left( \Omega^2_{s} + [\Omega, \Omega]_{s} \right) y + \frac{V_s}{\rho} + 2\Omega \times \frac{\Gamma^w_{st}}{\rho} + \frac{\Gamma^w_{tt}}{\rho} \right) n \times y.
\end{align*}
\]
(5.27)

**Proof** The \( y_{st} \) expression in (5.26) is derived by differentiating \( y_t \) with respect to \( s \) in Eq. 5.3. Notice that \( y_t \) in the moving case decomposes into the same terms as for the fixed case, Eq. 5.14, plus terms dependent on \( \Gamma^w_t \) given by \( \frac{1}{\rho} \left( \Gamma^w_{st} - e_3^\top \Gamma^w_s \Omega \times y_s \right) \). Differentiating with respect to \( s \) then gives a term equal to \( y_{st} \) for the fixed case plus terms dependent on \( \Gamma^w_t \) and its spatial derivative, the latter being obtained by differentiating the above expression with respect to \( s \).

The expressions of \( y_{tt} \) in the Frenet frame were obtained by taking the dot product of (5.4) with \( t \) and \( n \), noting that \( y_t \cdot t = \alpha \) and \( y_t \cdot n = \beta \). We then plug in expressions (5.1) and (5.2) for \( \rho_t \) and \( \rho_{tt} \), respectively. \( \square \)

**Special Case: Rigid Stationary Curve**

**Corollary 3** (Rigid stationary 3D curve) Let \( \Gamma(\tilde{s}) \) be a 3D curve projecting to a family of 2D curves \( \gamma(\tilde{s}, t) \) with depth \( \rho(\tilde{s}, t) \), arising from camera motion with differential velocity of translation and rotation \( V \) and \( \Omega \), respectively. Let \( t \) denote the unit tangent to the image curve. Then
\[
y_{st} = -\frac{\rho_s}{\rho} \left( \frac{V}{\rho} - \frac{V_s}{\rho} \right) - \frac{\rho_s}{\rho} \Omega \times t \\
- \left( e_3^\top \Omega \times t \right) y - \left( e_3^\top \Omega \times y \right) t.
\]
(5.28)

**Proof** Follows by setting \( \Gamma^w_t = 0 \) in Eq. 5.26 and using the spatial parameter as the arc-length of the image curve. \( \square \)

**Corollary 4** The tangential and normal velocities of a rigid curve induced by a moving camera are derived from \( \{ y \cdot t, n \cdot t, \Omega \cdot \frac{V}{\rho} \} \) for any \( t \) as
\[
\begin{align*}
\alpha &= -\Omega \cdot y \times (y \times n) \\
&- \left( \frac{V}{\rho} - \Omega \times \frac{T}{\rho} \right) \cdot y \times n \\
\beta &= \Omega \cdot y \times (y \times t) \hspace{1cm} \text{for any } t,
\end{align*}
\]
(5.29)
\[
\begin{align*}
\alpha &= -\Omega \cdot y \times (y \times n) - y \times n \cdot \frac{V}{\rho} \\
\beta &= \Omega \cdot y \times (y \times t) + y \times t \cdot \frac{V}{\rho}
\end{align*}
\]
(5.30)

or
\[
\begin{align*}
\alpha &= -\Omega \cdot y \times (y \times n) - y \times n \cdot \frac{V}{\rho} \\
\beta &= \Omega \cdot y \times (y \times t) + y \times t \cdot \frac{V}{\rho}
\end{align*}
\]
(5.31)

**Proof** Follows directly from Theorem 10 and \( \Gamma^w_t = 0 \). \( \square \)

**Corollary 5** The infinitesimal Essential constraint in the Frenet frame of the image of a rigid curve is given by
\[
(y \times t) \cdot V \left[ \alpha + \Omega \cdot y \times (y \times n) \right] \\
+ (y \times n) \cdot V \left[ \beta - \Omega \cdot y \times (y \times t) \right] = 0.
\]
(5.33)
\[
\alpha = -\left[\beta - \Omega \cdot (\gamma \times t)\right] \frac{V \cdot (\gamma \times n)}{V \cdot (\gamma \times t)} - \Omega \cdot (\gamma \times n).
\]

(5.34)

**Proof** Follows by solving (5.33) for \(\alpha\).

**Special Case: Occluding Contours** A remarkable observation derived below is that the first-order deformation of an apparent contour under epipolar parametrization does not depend on the 3D surface geometry, since the curvature-dependent terms cancel out for an occluding contour, cf. Cipolla and Giblin (1999).

**Theorem 11** (Occluding contours) Let \(\Gamma(s, t)\) be the contour generator for apparent contours \(\gamma(s, t)\). Then the image velocity \(\gamma_i\) at \(t = 0\) can be determined from \(\gamma\) by \(\rho\) and the infinitesimal motion parameters using Eq. 5.14, i.e., the same one used for a stationary contour.

**Proof** Recall from Eq. 2.23 that the velocity of an occluding contour under epipolar parametrization satisfies

\[
\Gamma_i^w = \lambda(\Gamma^w - c)
\]

for some \(\lambda\), so that at \(t = 0\),

\[
\Gamma_i^w = \lambda \rho \gamma_i \Rightarrow \mathbf{e}_3^T \Gamma_i^w = \lambda \rho.
\]

(5.35)

so that \(\Gamma_i^w = (\mathbf{e}_3^T \Gamma_i^w) \gamma_i\) and the terms \(\Gamma_i^w - (\mathbf{e}_3^T \Gamma_i^w) \gamma_i\) = 0 so that all appearances of \(\Gamma_i^w\) cancel-out altogether in Eq. 5.3, giving exactly the same formula as for fixed contours, Eq. 5.14, when \(\Gamma_i^w = 0\).

We now show exactly how the velocity of the 3D occluding contour, \(\Gamma_i^w\), depends on the curvature of the occluding surface (Cipolla 1991; Cipolla and Blake 1992).

**Theorem 12** The velocity of a 3D occluding contour under epipolar parametrization and relative to a fixed world coordinate system (camera at \(t = 0\)) is given by

\[
\begin{align*}
\Gamma_i^w &= \frac{-c_i^T N_i^w}{K^i} \cdot \frac{\Gamma^w - c}{\|\Gamma^w - c\|^2}, & \text{for arbitrary } t. \\
\Gamma_i^w &= \frac{-c_i^T N_i^w}{K^i} \cdot \frac{\gamma}{\rho \|\gamma\|^2}, & \text{for } t = 0,
\end{align*}
\]

(5.36)

(5.37)

where \(K^i\) is the normal curvature of the occluding surface along the visual direction.

**Proof** The desired formulae can be consistently derived by adapting variant (Astrom et al. 1999) of the original result by Cipolla and Blake (Cipolla 1991; Cipolla and Blake 1992) to the proposed notation. This must be performed carefully to establish correctness in a solid way. We thus provide an alternative, clearer proof without using unit view spheres.

The normal curvature of the occluding surface along the visual direction is given by classical differential geometry (Cipolla and Giblin 1999) as

\[
K^i = -\frac{\Gamma_i^{wT} N_i^w}{\Gamma_i^{wT} \Gamma_i^w},
\]

(5.39)

using epipolar parametrization. Substituting the epipolar parametrization condition of the second form of (2.23),

\[
K^i = -\frac{\langle \Gamma^w - c \rangle^T N_i^w}{\lambda \|\Gamma^w - c\|^2},
\]

(5.40)

Isolating \(\lambda\) and plugging back into the epipolar parametrization condition,

\[
\Gamma_i^w = -\frac{(\Gamma^w - c)^T N_i^w}{K^i} \frac{\Gamma^w - c}{\|\Gamma^w - c\|^2}.
\]

(5.41)

We now show that \(\langle \Gamma^w - c \rangle^T N_i^w = -c_i^T N_i^w\), thereby arriving at the desired expression for \(\Gamma_i^w\). In fact, differentiating the occluding contour condition in the second form of Eq. 2.22 gives

\[
(\Gamma_i^w - c_i)^T N_i^w + (\Gamma^w - c)^T N_i^w = 0,
\]

(5.42)

\[-c_i^T N_i^w + (\Gamma^w - c)^T N_i^w = 0
\]

(5.43)

which, together with (5.41) produces the desired result

\[
\Gamma_i^w = -\frac{-c_i^T N_i^w}{K^i} \frac{\gamma}{\rho \|\gamma\|^2}, \quad \text{for arbitrary } t.
\]

(5.44)

At \(t = 0\), we have \(N_i^w = N\) and \(\Gamma^w - c = \Gamma = \rho \gamma\) (but note that \(\Gamma_i(0) \neq \Gamma_i^w(0)\)), hence

\[
\Gamma_i^w = -\frac{c_i^T N}{K^i} \frac{\gamma}{\rho \|\gamma\|^2}, \quad \text{for } t = 0.
\]

(5.45)

Using \(V = -c_i\) from Eq. 2.12 and \(N = \frac{\gamma \times t}{\|\gamma \times t\|}\) gives the alternative form of this equation.

We now present a theorem relating observed quantities to camera motion, which is key for calibrating 3D motion models from families of projected deforming contours observed in video sequences with unknown camera.
motion, among other applications. A form of this theorem appears in (Papadopoulo and Faugeras 1996; Papadopoulo 1996), Equation 11, but this is limited to rigid motion. The following theorem generalizes the results to include occluding contours. The fact that Eq. 5.46 in the theorem is also valid for occluding contours is a new result, to the best of our knowledge. The term \( \Gamma_r^w \) is zero for fixed contours, and is dependent on surface curvature in the case of occluding contours. The equation is not valid for arbitrary non-rigid contours because, in order to derive the normal flow equation, we used \( \Gamma_r^w \cdot (\gamma \times t) = 0 \), which is only true for occluding and fixed contours.

**Theorem 13** (A generalized form of the L1 equation of (Papadopoulo and Faugeras 1996; Papadopoulo 1996))

*Given a 3D occluding contour or fixed curve, and the family of projected curves \( \gamma(t) \) observed in a monocular sequence of images from a moving camera, and given \( t, \kappa, n, \beta, \beta_s \) measurements at one point, then the first and second order camera motion, \( \Omega, V, \Omega_t, V_t \) satisfy the polynomial equation*

\[
\begin{align*}
V_\gamma [\beta - \Omega \cdot \gamma \times (\gamma \times t)]^2 &+ V \cdot t \left( \beta_t - \Omega_t \cdot \gamma \times (\gamma \times t) - \Omega \cdot [\gamma \times (\gamma \times t)]_t \right) \\
- \left[ V_t \cdot \gamma + V \cdot (\gamma \times t)_t \right] [\beta - \Omega \cdot \gamma \times (\gamma \times t)] &+ V \cdot t \left( \eta_1 \Omega + \eta_2 V \right) [\beta - \Omega \cdot \gamma \times (\gamma \times t)] \\
+ V \cdot t \left( \eta_1 \Omega + \eta_2 V \right) [\beta - \Omega \cdot \gamma \times (\gamma \times t)] &+ e_3 \cdot \Omega \times \gamma [\beta - \Omega \cdot \gamma \times (\gamma \times t)]^2 \\
+ (\Omega \times V)(\gamma \times t) [\beta - \Omega \cdot \gamma \times (\gamma \times t)] &= 0. \quad (5.46)
\end{align*}
\]

**Proof** The normal velocity \( \beta \) of an image contour follows Equation (5.30), which holds for both stationary curves, Corollary 4, and for occluding contours, Theorem 11. Differentiating it with respect to time,

\[
\rho_t \beta + \beta_t \rho = \rho \Omega \cdot \gamma \times (\gamma \times t) + \rho \Omega_t \cdot \gamma \times (\gamma \times t) \\
+ \rho \Omega_t \cdot \gamma \times (\gamma \times t) + \rho \Omega \cdot [\gamma \times (\gamma \times t)]_t \\
+ (\gamma \times t)(V - \Omega \times T) \\
+ (\gamma \times t)(V - \Omega_t \times T - \Omega \times V) \quad (5.47)
\]

Rearranging the terms,

\[
\begin{align*}
\rho_t [\beta - \Omega \cdot \gamma \times (\gamma \times t)] &+ \rho [\beta_t - \Omega_t \cdot \gamma \times (\gamma \times t) \\
- \Omega \cdot [\gamma \times (\gamma \times t)]_t] = (\gamma \times t)(V - \Omega \times T) \\
+ (\gamma \times t)(V - \Omega_t \times T - \Omega \times V). \quad (5.48)
\end{align*}
\]

Setting \( t = 0 \),

\[
\begin{align*}
\rho_t [\beta - \Omega \cdot \gamma \times (\gamma \times t)] &+ \rho [\beta_t - \Omega_t \cdot \gamma \times (\gamma \times t) - \Omega \cdot [\gamma \times (\gamma \times t)]_t] \\
= (\gamma \times t)_t V + (\gamma \times t)(V - \Omega \times V) \quad (5.49)
\end{align*}
\]

Now, from Eq. 5.1, we can plug-in an expression for \( \rho \) at \( t = 0 \),

\[
\begin{align*}
(\rho e_3^\top \Omega \times \gamma + V_\gamma + e_3^\top \Gamma_r^w) &\left[ \beta - \Omega \cdot \gamma \times (\gamma \times t) \right] \\
+ \rho (\beta_t - \Omega_t \cdot \gamma \times (\gamma \times t) - \Omega \cdot [\gamma \times (\gamma \times t)]_t) \\
= (\gamma \times t)_t V + (\gamma \times t)(V - \Omega \times V), \quad (5.50)
\end{align*}
\]

which is analogous to Equation 7.28 of Papadopoulo (1996, p.167), but this time with occluding contours also being included. Now, eliminating depth \( \rho \) using Eq. 5.32, e.g., by multiplying the above by \( [\beta - \Omega \cdot \gamma \times (\gamma \times t)] \), we obtain

\[
\begin{align*}
[V_\gamma (\beta - \Omega \cdot \gamma \times (\gamma \times t)) + (V \cdot \gamma \times t)e_3 \cdot \Omega \times \gamma \\
+ e_3 \cdot \Gamma_r^w (\beta - \Omega \cdot \gamma \times (\gamma \times t)) [\beta - \Omega \cdot \gamma \times (\gamma \times t)] \\
+ V \cdot \gamma \times t [\beta_t - \Omega_t \cdot \gamma \times (\gamma \times t) - \Omega \cdot [\gamma \times (\gamma \times t)]_t] \\
= (V_t \cdot \gamma + t V (\gamma \times t)_t - (\Omega \times V)(\gamma \times t)) \\
[\beta - \Omega \cdot \gamma \times (\gamma \times t)]. \quad (5.51)
\end{align*}
\]

Rearranging the terms, we obtain the desired equation. □

**Remark 2** Note that previously reported results for the rigid case (Papadopoulo 1996, eq. 7.12) have an apparently missing term corresponding to the last term in our Eq. 5.46,

\[
(\Omega \times V)(\gamma \times t) [\beta - \Omega \cdot \gamma \times (\gamma \times t)].
\]

This is due to the fact that they used slightly different variables for the translational component of the infinitesimal motion equations, but the results are mathematically the same for the rigid case.

**Theorem 14** *The first spatial derivative of image apparent motion of both a fixed curve and an occluding contour under epipolar correspondence is given by*

\[
\gamma_{st} = \left( \frac{-V}{\rho} + \frac{V_\gamma}{\rho} \right) \gamma_s + \frac{\rho_s}{\rho} - \frac{V_\gamma}{\rho} \gamma_s \\
+ \Omega \times \gamma_s - (e_3^\top \Omega \times \gamma) \gamma_s - (e_3^\top \Omega \times \gamma) \gamma_s. \quad (5.52)
\]

Note that the derivative of depth \( \rho_s \) can be expressed in terms of 3D curve geometry as \( \rho_s = e_3^\top \Gamma_s \).

**Proof** Eq. 5.52 follows by differentiating the fixed flow (5.14) with respect to \( s \), observing that only \( \rho \) and \( \gamma \) depend on \( s \). The formula for \( \rho_s \) is obtained from the observation that the dot product of \( \Gamma = \rho \gamma \) with \( e_3 \) gives \( e_3^\top \Gamma = \rho \). Differentiating this with respect to \( s \) gives \( \rho_s = e_3^\top \Gamma_s \).

Theorem 9 gives an expression for the image acceleration of a moving 3D point, which includes points lying on any type of contour (even non-rigid), in terms of the evolution of the 3D curve. Since the latter is expressed in terms of a
fixed world coordinate system, the motion of the object and the motion of the cameras are written down separately, even though they exert joint effects on image velocity.

**Theorem 15** The image acceleration of an occluding contour under epipolar parametrization is given by

\[
\gamma_{tt} = (\Omega^2 + [\Omega_1]_\times) \gamma - \left[ e_3^T (\Omega^2 + [\Omega_1]_\times) \right] \gamma
+ \frac{2 \Omega \times \Gamma_w}{\rho} + \frac{V_t}{\rho} - \frac{2 \rho_f \gamma_t}{\rho} - \frac{e_3^T v_t}{\rho} - \frac{e_3^T \Gamma_w}{\rho} - \frac{2 e_3^T \Omega \times \Gamma_w}{\rho} \gamma - \frac{e_3^T \Omega \times \gamma}{\rho} - \frac{2 e_3^T \Omega \times \Gamma_w}{\rho} \gamma - \frac{e_3^T \Gamma_w}{\rho} \gamma.
\]

where \( \gamma_t \) and \( \rho_t \) are given by Equations 5.14 and 5.13, and \( \Gamma_w \) is dependent on curvature, Eq. 5.38.

**Proof** Substituting Eq. 5.2 into Eq. 5.12, we get

\[
\gamma_{tt} = (\Omega^2 + [\Omega_1]_\times) \gamma + \frac{2 \Omega \times \Gamma_w}{\rho} + \frac{V_t}{\rho} - \frac{2 \rho_f \gamma_t}{\rho} - \frac{e_3^T v_t}{\rho} - \frac{2 e_3^T \Omega \times \Gamma_w}{\rho} \gamma - \frac{e_3^T \Omega \times \gamma}{\rho} - \frac{2 e_3^T \Omega \times \Gamma_w}{\rho} \gamma - \frac{e_3^T \Gamma_w}{\rho} \gamma.
\]

Now, let \( v \) be the viewing direction in world coordinates, so that

\[
\gamma = R v,
\]

and let \( f \) be the normal to the image plane in world coordinates, so that

\[
e_3 = R f.
\]

Thus,

\[
e_3^T \gamma = f^T R^T R v = f^T v = 1.
\]

Note also that at \( t = 0 \) we have \( f = e_3 \) and \( \gamma = v \). Now, the condition for epipolar parametrization of an occluding contour, Eq. 2.23, can be expressed as

\[
\Gamma_w = \lambda v,
\]

for some scalar factor \( \lambda \). Taking the dot product with \( f \) we have

\[
\left\{ \begin{array}{l}
\lambda = f^T \Gamma_w \\
\Gamma_w = f^T \Gamma_w v.
\end{array} \right.
\]

Differentiating (5.58) with respect to time and using (5.59) gives

\[
\Gamma_w = \lambda_t v + \lambda v_t = \lambda_t v + f^T \Gamma_w v_t.
\]

Taking the dot product with \( f \).

\[
f^T \lambda_t v + f^T \left( f^T \Gamma_w \right) v_t = f^T \Gamma_w v_t.
\]

Thus,

\[
\Gamma_w = e_3^T \Gamma_w \gamma - \left( e_3^T \Gamma_w \right) (e_3^T v_t) v + e_3^T \Gamma_w v_t.
\]

In order to get \( v_t(0) \) in terms of \( \gamma \) we write

\[
\gamma_t = R v + R v_t.
\]

Thus

\[
v_t = \gamma_t - \Omega \times \gamma \quad \text{at } t = 0.
\]

Substituting back into (5.65),

\[
\Gamma_w = e_3^T \Gamma_w \gamma + e_3^T \Gamma_w \gamma_t - e_3^T \Gamma_w \Omega \times \gamma + \left( e_3^T \Gamma_w \right) (e_3^T v_t) \gamma + e_3^T \Gamma_w v_t.
\]

Plugging this equation onto (5.54), the \( e_3^T \Gamma_w \gamma / \rho \) terms cancel out, giving the final equation.

\)

### 6 Mathematical Experiment

To illustrate and test the proposed theoretical framework, we have devised an experiment around a synthetic dataset constructed for this research. This dataset has already been used for validating a pose estimation system (Fabbri et al. 2012). The dataset is composed of the following components:

1. A variety of synthetically generated 3D curves (helices, parabolas, ellipses, straight lines, and saddle curves) with well-known parametric equations, as shown in Fig. 5.
2. Ground-truth camera models for a video sequence around the curves.
3. Differential geometry of the space curves analytically computed up to third-order (torsion and curvature derivative), using Maple when necessary. The dataset together with C++ code implementing these expressions from
Maple are listed in the supplementary material Online Resource 1.

4. The 3D curves are densely sampled, each 3D sample having attributed differential geometry from the analytic computation (up to torsion and curvature).

5. A video sequence containing a family of 2D curve samples with attributed differential geometry is rendered by projecting the 3D samples onto a 500 × 400 view using the ground truth cameras. These subpixel edgels with attributed differential geometry simulate ideal aspects of what in practice could be the output of high-quality subpixel edge detection and grouping (Tamrakar and Kimia 2007; Tamrakar 2008; Guo et al. 2014).

6. Correspondence between all samples obtained by keeping track of the underlying 3D points.

7. Specific analytic expressions for 2D differential geometry were derived using Maple since these are often too long due to perspective projection. These expressions are also provided in the C++ code that synthesizes the dataset.

8. C++ code implementing the formulas in this paper is also provided with the dataset, and can be readily used in other projects.

For the present theoretical paper, the experiments consist of checking the proposed expressions against the analytic expressions that are obtained by differentiating each specific parametric equation. After projecting differential geometry using our formulas applied to the 3D samples attributed with differential geometry, we obtain corresponding 2D projected differential geometry at each sample. We compare this to the differential geometry on the curve projections analytically computed from the parametric equations, observing a match. We then reconstruct these correspondences up to third-order differential geometry using the proposed expressions, and observe that they indeed match to the original analytic expressions. We have also performed a similar experiment for the expressions involving occluding contours, using a 3D ellipsoid and sphere.

We have observed a complete agreement between our code and the specific analytic expressions, confirming that the formulas as presented in this manuscript are correct. The source code of this illustrative experiment also serves as an example of how to use the proposed framework in programming practice, how to check for degenerate conditions stated in the theorems, among others.

7 Conclusion

We presented a unified differential-geometric theory of projection and reconstruction of general curves from multiple views. By gathering previously scattered results using a coherent notation and proof methodology that scale to expressing more sophisticated ideas, we were able to prove novel results and to provide a comprehensive study on how the differential geometry of curves behaves under perspective projection, including the effects of intrinsic parameters. For instance, we derived how the tangent, curvature, and curvature derivative of a space curve projects onto an image, and how the motion of the camera and of the curve relates to the projections. This lead to the novel result that torsion – which characterizes the tri-dimensionality of space curves – projects to curvature derivative in an image, and the novel result of how the parametrization of corresponding image curves are linked across different views, up to third order to reflect the underlying torsion. We also proved formulas for reconstructing differential geometry, given differential geometry at corresponding points measured in at least two views. In particular, this gives the novel result of reconstructing space curve torsion, given corresponding points, tangents, curvatures, and curvature derivatives measured in two views. We determined that there are no correspondence constraints in two views – any pair of points with attributed tangents, curvatures, and curvature derivatives are possible matches, as long as the basic point epipolar constraint is satisfied. There is, however, a constraint in three or more views: from two
views one can transfer differential geometry onto other views and enforce measurements to match the reprojections using local curve shape, avoiding clutter. This has been demonstrated in a recent work in curve-based multiview stereo by the authors (Fabbri and Kimia 2010; Usumezbas et al. 2016), namely for matching linked curve fragments from a subpixel edge detector across many views. Experiments clearly show that differential-geometric curve structure is essential for the matching to be immune to edge clutter and linking instability, by enforcing reprojections to match to image data in local shape.

This paper is part of a greater effort of augmenting multiple view geometry to model general curved surfaces (Fabbri 2010). Work on camera pose estimation based on curves using the formulas in this paper has been recently published (Fabbri et al. 2012), and trifocal relative pose estimation using local curve geometry to bootstrap the system is currently under investigation. In a complete system, once a core set of three views are estimated and an initial set of curves are reconstructed, more views can be iteratively registered (Fabbri et al. 2012). These applications of the theory presented in this paper and their ongoing extensions would enable a practical structure from motion pipeline based on curve fragments, complementing interest point technology. We have also been working on the multiview differential geometry of surfaces and their shading.

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