EIGENVALUES OF THE DIRAC OPERATOR
ON MANIFOLDS WITH BOUNDARY

OUSSAMA HIJAZI, SEBASTIÁN MONTIEL, AND XIAO ZHANG

Abstract. Under standard local boundary conditions or certain global APS boundary conditions, we get lower bounds for the eigenvalues of the Dirac operator on compact spin manifolds with boundary. Limiting cases are characterized by the existence of real Killing spinors and the minimality of the boundary.

1. Introduction

It is well known that the spectrum of the Dirac operator on closed spin manifolds detects subtle information on the geometry and the topology of such manifolds (see for example [3, 5]).

In [17], basic properties of the hypersurface Dirac operator are established. This hypersurface Dirac operator appears as the boundary term in the integral Schrödinger-Lichnerowicz formula (2.3) for compact spin manifolds with compact boundary. In fact, the hypersurface Dirac operator is, up to a zero order operator, the intrinsic Dirac operator of the boundary.

In this paper, we examine the classical local boundary conditions and certain Atiyah-Patodi-Singer boundary conditions for the Dirac operator. Here, the spectral resolution of the intrinsic Dirac operator of the boundary is used to define the APS boundary conditions. We first prove self-adjointness and ellipticity of such conditions.

Then, systematic use of the modified Levi-Civita connections, introduced in [6, 15, 22, 4, 17], is made. Under appropriate curvature
assumptions, these modified connections combined with formula (2.3), yield to the corresponding estimates for compact spin manifolds with boundary. The limiting cases are then studied.

Such estimates are obtained in Sections 3 and 4. In Section 3 we consider both the local and the above mentioned APS boundary conditions. We first introduce the modified connection (3.1) which allows to establish a Friedrich’s type inequality, in case the mean curvature of the boundary is nonnegative. Under the local boundary conditions, the limiting case is then characterized by the existence of a Killing spinor on the compact manifold with minimal boundary (see (3.5)). Then the energy-momentum tensor is used to define the modified connection (3.7), from which one can deduce inequality (3.9).

Finally, in section 4, under the local boundary conditions, the conformal aspect is examined. For example, generalizations of the conformal lower bounds in [13, 15] are obtained (see Remark 9).

2. The elliptic boundary conditions

Let $M$ be an $n$-dimensional Riemannian spin manifold with boundary $\partial M$ endowed with its induced Riemannian and spin structures. Denote by $\mathcal{S}$ the spinor bundle of $M$. Let $\nabla$ (resp. $\nabla^{\partial M}$) be the Levi-Cività connection of $M$ (resp. $\partial M$) and denote by the same symbol their corresponding lift to the spinor bundle $\mathcal{S}$. Consider the Dirac operator $D$ of $M$ defined by $\nabla$ on $\mathcal{S}$. It is known [19] that there exists a positive definite Hermitian metric on $\mathcal{S}$ which satisfies, for any covector field $X^* \in \Gamma(T^*M)$, and any spinor fields $\varphi, \psi \in \Gamma(\mathcal{S})$, the relation

$$ (X^* \cdot \varphi, X^* \cdot \psi) = |X^*|^2(\varphi, \psi), $$

(2.1)

where “$\cdot$” denotes Clifford multiplication. The connection $\nabla$ is compatible with the metric $(\ , \ )$. Fix a point $p \in \partial M$ and an orthonormal basis $\{e_a\}$ of $T_pM$ with $e_0$ the outward normal to $\partial M$ and $e_i$ tangent to $\partial M$ such that for $1 \leq i, j \leq n$

$$ (\nabla^{\partial M}_i e_j)_p = (\nabla_0 e_j)_p = 0. $$

Let $\{e^\alpha\}$ be the dual co-frame. Then, for $1 \leq i, j \leq n$

$$ (\nabla_i e^j)_p = -h_{ij} e^0, $$

$$ (\nabla_i e^0)_p = h_{ij} e^j, $$

where $h_{ij} = (\nabla_i e_0, e_j)$ are the components of the second fundamental form at $p$, and we have

$$ \nabla_i = \nabla^{\partial M}_i + \frac{1}{2} h_{ij} e^0 \cdot e^j. $$

(2.2)
Let $H = \sum h_{ii}$ be the mean curvature of $M$. In the above notation, the standard sphere $S^n_r = \partial B^{n+1}_r$ has positive mean curvature $H = \frac{n}{r}$.

By (2.1), $(\epsilon^0 \cdot \epsilon^j \cdot \varphi, \psi) = (\varphi, \epsilon^j \cdot \epsilon^0 \cdot \psi)$. Therefore (2.2) implies

\[
d(\varphi, \psi) * e_i = \left( (\nabla_i \varphi, \psi) + (\varphi, \nabla_i \psi) \right) * 1.
\]

Hence the connection $\nabla_{\partial M}$ is also compatible with the metric ( , ).

Denote by $D_{\partial M}$ the Dirac operator of $\partial M$. In the above orthonormal coframe $\{e^i\}$ of $M$, $D_{\partial M} = e^i \cdot \nabla_{\partial M}$. Thus $D_{\partial M}$ is self-adjoint with respect to the metric ( , ). The relation (2.2) implies that

\[
\nabla_{\partial M}^i (\epsilon^0 \cdot \varphi) = \epsilon^0 \cdot \nabla_{\partial M}^i \varphi.
\]

Hence

\[
D_{\partial M}^i (\epsilon^0 \cdot \varphi) = -\epsilon^0 \cdot D_{\partial M} \varphi.
\]

Consider the integral form of the Schrödinger–Lichnerowicz formula for a compact manifold with compact boundary

\[
\int_{\partial M} (\varphi, \epsilon^0 \cdot D_{\partial M} \varphi) - \frac{1}{2} \int_{\partial M} H |\varphi|^2 = \int_M |\nabla \varphi|^2 + \frac{R}{4} |\varphi|^2 - |D \varphi|^2.
\]

(2.3)

It is well-known that there are basically two types of elliptic boundary conditions for the Dirac operator: The local boundary condition and the (global) Atiyah-Patodi-Singer (APS) boundary condition. Such boundary conditions are used in the positive mass theorem for black holes, Penrose conjecture in general relativity and the index theory in topology \cite{[3], [11], [12], [23]}. The APS boundary condition exists on any spin manifold with boundary, while the local boundary condition requires certain additional structures on manifolds such as the existence of a Lorentzian structure or a chirality operator, etc \cite{[8], [9]}. Now we shall show that the local boundary condition exists on certain spin manifolds with a “boundary chirality operator”.

An operator $\Gamma$ defined on $C^\infty(\partial M, S|_{\partial M})$ is said to be a boundary chirality operator if it satisfies the following conditions:

\[
\Gamma^2 = Id,
\]

(2.4)

\[
\nabla_{\partial M}^i \Gamma = 0,
\]

(2.5)

\[
e^0 \cdot \Gamma = -\Gamma \cdot e^0,
\]

(2.6)

\[
e^i \cdot \Gamma = \Gamma \cdot e^i,
\]

(2.7)

\[
(\Gamma \cdot \varphi, \Gamma \cdot \psi) = (\varphi, \psi).
\]

(2.8)
If $M$ is a spacelike hypersurface of a spacetime manifold with timelike covector $T$, then we can let $\Gamma = T \cdot e^0$, where $e^0$ is the normal covector on $\partial M$.

Recall that (see [8] for example), an operator $F$ defined on $C^\infty(M, S)$ is called a chirality operator on $M$ if for all $X^* \in \Gamma(T^*M)$, and any spinor fields $\varphi, \psi \in \Gamma(S)$, one has

$$F^2 = Id, \quad \nabla_X F = 0, \quad X^* \cdot F = -F \cdot X^*, \quad (F \cdot \varphi, F \cdot \psi) = (\varphi, \psi).$$

Note that, such an operator exists if the spin manifold $M$ is even dimensional. It is easy to see that if $M$ has a chirality operator $F$, then $\Gamma = F|_{\partial M} \cdot e^0$ is a boundary chirality operator. But, in general, it is not known whether the local boundary condition exists. In this paper, we consider the following boundary conditions:

- **The local boundary condition:**
  As the eigenvalues of the chirality operator $\Gamma$ are $\pm 1$, the corresponding eigenspaces
  $$\Gamma_{loc}^+ = \left\{ \varphi \in C^\infty(\partial M, S|_{\partial M}), \quad \Gamma \cdot \varphi = \varphi \right\},$$
  $$\Gamma_{loc}^- = \left\{ \varphi \in C^\infty(\partial M, S|_{\partial M}), \quad \Gamma \cdot \varphi = -\varphi \right\}$$
  provide local boundary conditions.

- **The APS type boundary condition:**
  The operator $e^0 \cdot D^\partial M$ is self-adjoint with respect to the induced metric $(\ , \ )$ on $\partial M$. Therefore it has a discrete (real) spectrum. Let $(\varphi_k)_{k \in \mathbb{N}}$ be the spectral resolution of $e^0 \cdot D^\partial M$, i.e., $e^0 \cdot D^\partial M \varphi_k = \lambda_k \varphi_k$, and consider the corresponding $L^2$-orthogonal subspaces $\Gamma_{APS}^\pm$ spanned by the positive and negative eigenspaces of $e^0 \cdot D^\partial M$, i.e.,
  $$\Gamma_{APS}^+ = \left\{ \varphi \in C^\infty(\partial M, S|_{\partial M}), \quad \varphi = \sum_{\lambda_k > 0} c_k \varphi_k \right\},$$
  $$\Gamma_{APS}^- = \left\{ \varphi \in C^\infty(\partial M, S|_{\partial M}), \quad \varphi = \sum_{\lambda_k < 0} c_k \varphi_k \right\}.$$

We will consider the APS type boundary conditions corresponding to the projections onto these subspaces. Recall that the original Atiyah-Patodi-Singer (APS) boundary condition refers to the
spectral resolution of \( e^0 \cdot D^\partial M - H/2 \) instead of \( e^0 \cdot D^\partial M \) (see for example [8]).

Note that if \( \varphi, \psi \in \Gamma^\text{loc}_\pm \), then
\[
(e^0 \cdot \varphi, \psi) = (e^0 \cdot \Gamma \cdot \varphi, \Gamma \cdot \psi) = -(\Gamma \cdot e^0 \cdot \varphi, \Gamma \cdot \psi) = -(e^0 \cdot \varphi, \psi),
\]
hence \((e^0 \cdot \varphi, \psi) = 0\). On the other hand, if \( \varphi, \psi \in \Gamma^\text{APS}_\pm \), then \( e^0 \cdot \varphi \in \Gamma^\text{APS}_\mp \), therefore
\[
\int_{\partial M} (e^0 \cdot \varphi, \psi) = 0.
\]

These facts imply that the Dirac operator \( D \) is self-adjoint under either the local boundary conditions or the APS boundary negative and positive conditions. Moreover, it has real eigenvalues. Now we define the \( H^k \) Sobolev norm by
\[
\| \varphi \|^2_{H^k} = \sum_{|\alpha|=k} \| \nabla^\alpha \varphi \|^2_{L^2} + \| \varphi \|^2_{H^{k-1}},
\]
where \( \alpha \) is a multi-index.

**Proposition 1.** Under the local boundary condition \( \varphi \in \Gamma^\text{loc}_\pm \) or the APS boundary condition \( \varphi \in \Gamma^\text{APS}_\pm \), the Dirac operator \( D \) satisfies elliptic estimates: For any \( k \geq 1 \), \( \delta > 0 \), there exists \( C_{k,\delta} \) such that
\[
(2.9) \quad \| \varphi \|^2_{H^k} \leq (1 + \delta) \| \varphi \|^2_{L^2} + C_{k,\delta} \| \varphi \|^2_{H^{k-1}}.
\]

**Proof:** Note that for any \( \varphi \in \Gamma^\text{loc}_+ \) or \( \varphi \in \Gamma^\text{loc}_- \), \( D^\partial M (\Gamma \cdot \varphi) = \Gamma \cdot D^\partial M \varphi \), thus
\[
(e^0 \cdot D^\partial M \varphi) = (\Gamma \cdot e^0 \cdot D^\partial M (\Gamma \cdot \varphi)) = (\Gamma \cdot e^0 \cdot \Gamma \cdot D^\partial M \varphi) = -(e^0 \cdot \varphi, D^\partial M \varphi).
\]
Therefore \((e^0 \cdot D^\partial M \varphi) = 0\). If \( \varphi \in \Gamma^\text{APS}_- \), then
\[
\int_{\partial M} (e^0 \cdot D^\partial M \varphi) = \sum_{\lambda_k < 0} |c_k|^2 \lambda_k \leq 0.
\]
By the Ehrling-Gagliardo-Nirenberg inequality [1], for each \( \varepsilon > 0 \), there exists a constant \( C_\varepsilon > 0 \) such that
\[
\| \varphi \|^2_{L^2(\partial M)} \leq \varepsilon \| \varphi \|^2_{H^1} + C_\varepsilon \| \varphi \|^2_{L^2},
\]
thus \((2.3)\) implies
\[
(2.10) \quad \| \varphi \|^2_{H^1} \leq (1 + \delta) \| D^\partial M \varphi \|^2_{L^2} + C_\delta \| \varphi \|^2_{L^2}.
\]
Then a standard argument gives (2.8). Q.E.D.

The following corollary is a direct consequence of the Sobolev embedding theorem

$$\| \varphi \|_{C^k} \leq C \| \varphi \|_{H^{k+\frac{n}{2}}}$$

**Corollary 2.** Any eigenspinor of the Dirac operator which satisfies either the local boundary condition $\varphi \in \Gamma^{loc}_{\pm}$ or the (negative) APS boundary condition $\varphi \in \Gamma^{APS}_{-}$ is smooth.

### 3. Lower Bounds for the Eigenvalues

In this section, we adapt the arguments used in [17] to the case of spin compact manifolds with boundary. In particular, we get generalizations of basic inequalities on the eigenvalues of the Dirac operator $D$ under the local boundary conditions $\Gamma^{loc}_{\pm}$ or the negative APS boundary condition $\Gamma^{APS}_{-}$. For this, we use the integral identity (2.3) together with an appropriate modification of the Levi-Civita connection.

Let $D\varphi = \lambda \varphi$, where $\lambda$ is a real constant or a real function. For any real functions $a$ and $u$, we define

$$\nabla^{a,u}_{i} = \nabla_{i} + a \nabla_{i} u + \frac{a}{n} \nabla_{j} u e^{i} \cdot e^{j} + \frac{\lambda}{n} e^{i} \cdot.$$ (3.1)

Then

$$\left| \nabla^{a,u}_{i} \varphi \right|^2 = \left| \nabla \varphi \right|^2 + \frac{\lambda^2}{n} \left| \varphi \right|^2 + a^2 (1 - \frac{1}{n}) \left| du \right|^2 \left| \varphi \right|^2$$

$$+ 2a \nabla_{i} u \Re(\nabla_{i} \varphi, \varphi) + \frac{2\lambda}{n} \Re(\nabla_{i} \varphi, e^{i} \cdot \varphi)$$

$$= \left| \nabla \varphi \right|^2 - \frac{\lambda^2}{n} \left| \varphi \right|^2 + a^2 (1 - \frac{1}{n}) \left| du \right|^2 \left| \varphi \right|^2 + a \nabla_{i} u \nabla_{i} \left| \varphi \right|^2.$$

Define the functions $R_{a,u}$ by

$$R_{a,u} = R - 4a \Delta u + 4 \nabla a \nabla u - 4(1 - \frac{1}{n})a^2 \left| du \right|^2$$

where $\Delta$ is the positive scalar Laplacian. Then we have

$$\int_{M} \left| \nabla^{a,u}_{i} \varphi \right|^2 = \int_{M} \left| \nabla \varphi \right|^2 - \frac{\lambda^2}{n} \left| \varphi \right|^2 - \left( \frac{R_{a,u}}{4} - \frac{R}{4} \right) \left| \varphi \right|^2$$

$$+ \int_{\partial M} a \left| du( e_{0} \right) \left| \varphi \right|^2.$$
Therefore (2.3) yields
\[ \int_M |\nabla a_u \varphi|^2 = \int_M \left[ (1 - \frac{1}{n}) \lambda^2 - \frac{R_{a,u}}{4} \right] |\varphi|^2 \]
\[ + \int_{\partial M} (\varphi, e^0 \cdot D_{\partial M} \varphi) + \left[ a \, du(e_0) - \frac{H}{2} \right] |\varphi|^2. \tag{3.3} \]

Now we generalize Lemma 2.3 in [7] to the case where \( a \) is a real function.

**Lemma 3.** Suppose there exist a spinor field \( \varphi \in \Gamma(S) \), a real number \( \lambda \) and a real functions \( a \) and \( u \) on \( M \) such that for all \( i, \ 1 \leq i \leq n \),
\[ \nabla_i \varphi = -\frac{\lambda}{n} e^i \cdot \varphi - a \nabla_i u \varphi - \frac{a}{n} \nabla_k u e^i \cdot e^j \cdot \varphi. \tag{3.4} \]
Then \( \varphi \) is a real Killing spinor, i.e., either \( a = 0 \) or \( du = 0 \). In particular, the manifold is Einstein.

**Proof:** First, observe that (3.4) implies \( D \varphi = \lambda \varphi \). By the Ricci identity (see [7]), we have
\[
\frac{1}{2} R_{ij} e^i \cdot e^j \cdot \varphi = e^i \cdot D(\nabla_i \varphi) - D^2 \varphi
\]
\[ = e^i \cdot e^j \cdot \nabla_j \left( -\frac{\lambda}{n} e^i \cdot \varphi - a \nabla_i u \varphi - \frac{a}{n} \nabla_k u e^i \cdot e^k \cdot \varphi \right) - \lambda^2 \varphi
\]
\[ = \frac{\lambda}{n} e^i \cdot (e^j \cdot e^i \cdot 2\delta_{ij}) \nabla_j \varphi
\]
\[ - \lambda^2 \varphi - du \cdot da \cdot \varphi + a \Delta u \varphi - a \lambda du \cdot \varphi
\]
\[ + \frac{1}{n} e^i \cdot (e^j \cdot e^i \cdot 2\delta_{ij}) e^k \cdot \left( \nabla_j a \nabla_k u \varphi + a \nabla_j u \nabla_k u \varphi \right)
\]
\[ = \left( \frac{2(1-n)}{n} \lambda^2 + \frac{2a}{n} \Delta u - \frac{2(2-n)}{n} \nabla a \nabla u \right) \varphi
\]
\[ - \frac{2}{n} du \cdot da \cdot \varphi + \frac{4a \lambda}{n^2} du \cdot \varphi. \]

This implies either \( a = 0 \) or \( du = 0 \). Q.E.D.

By (3.3) and Lemma 3, we obtain

**Theorem 4.** Let \( M^n \) be a compact Riemannian spin manifold of dimension \( n \geq 2 \), with boundary \( \partial M \), and let \( \lambda \) be any eigenvalue of \( D \) under either the local boundary condition \( \Gamma^{loc} \) or the (negative) APS boundary condition \( \Gamma^{APS} \). If there exist real functions \( a, u \) on \( M \) such that
\[ H \geq 2a \, du(e_0) \]
on $\partial M$, where $H$ is the mean curvature of $\partial M$, then
\begin{equation}
\lambda^2 \geq \frac{n}{4(n-1)} \sup_{a,u} \inf_M R_{a,u},
\end{equation}
where $R_{a,u}$ is given in (3.2). In the limiting case with the local boundary conditions, the associated eigenspinor is a real Killing spinor and $\partial M$ is minimal.

Note that by [16], under the APS boundary conditions equality in (3.5) could not hold.

Now we make use of the energy-momentum tensor (see [15]) to get lower bounds for the eigenvalues of $D$. For any spinor field $\varphi$, we define the associated energy momentum 2-tensor $Q_\varphi$ on the complement of its zero set by,
\begin{equation}
Q_{\varphi,ij} = \frac{1}{2} \Re (e^i \cdot \nabla_j \varphi + e^j \cdot \nabla_i \varphi , \varphi/|\varphi|^2).
\end{equation}

If $\varphi$ is an eigenspinor of $D$, the tensor $Q_\varphi$ is well-defined in the sense of distribution. Let
\begin{equation}
\nabla^{Q,a,u}_i = \nabla_i + a \nabla_i u + \frac{a}{n} \nabla_j u e^i \cdot e^j + Q_{\varphi,ij} e^j.
\end{equation}

It is easy to prove that (see [17])
\begin{equation}
|\nabla^{Q,a,u}_i \varphi|^2 = |\nabla \varphi|^2 - |Q_\varphi|^2 |\varphi|^2 + a^2 (1 - \frac{1}{n}) |du|^2 |\varphi|^2 + a \nabla_i u \nabla_i |\varphi|^2.
\end{equation}

Therefore
\begin{equation}
\int_M |\nabla^{Q,a,u}_i \varphi|^2 = \int_M \left[ \lambda^2 - \left( \frac{R_{a,u}}{4} + |Q_\varphi|^2 \right) \right] |\varphi|^2
+ \int_{\partial M} (\varphi, e^0 \cdot D^{\partial M} \varphi) + \left[ a \ du(e_0) - \frac{H}{2} \right] |\varphi|^2.
\end{equation}

Thus we have

**Theorem 5.** Let $M^n$ be a compact Riemannian spin manifold of dimension $n \geq 2$, with boundary $\partial M$, and let $\lambda$ be any eigenvalue of $D$ under either the local boundary condition $\Gamma^{\text{loc}}_{\pm}$ or the (negative) APS boundary condition $\Gamma_{APS}$. If there exist real functions $a$, $u$ on $M$ such that
\begin{equation}
H \geq 2a \ du(e_0)
\end{equation}
on $\partial M$, where $H$ is the mean curvature of $\partial M$, then
\begin{equation}
\lambda^2 \geq \sup_{a,u} \inf_M \left( \frac{R_{a,u}}{4} + |Q_\varphi|^2 \right).
\end{equation}

In the limiting case, one has $H = 2adu(e_0)$ on $\partial M$. 
Remark 6. Under either the local boundary condition $\Gamma_{\pm}^{\text{loc}}$ or the (negative) APS boundary condition $\Gamma_{\text{APS}}^{-}$, assume that $H \geq 0$. Take $a = 0$ or $u$ constant in (3.3) and (3.9), then one gets Friedrich’s inequality \[\lambda^2 \geq \frac{n}{4(n-1)} \inf_M R\] (3.10) and the following inequality \[\lambda^2 \geq \inf_M \left( \frac{R}{4} + |Q_\varphi|^2 \right).\] (3.11)

4. Conformal Lower Bounds

As in the previous section and under the local boundary conditions $\Gamma_{\pm}^{\text{loc}}$, we show that the conformal arguments used in [17] combined with the integral formula (2.3) yield to generalizations of all known lower bounds for the eigenvalues of the Dirac operator.

Let $g$ be the metric of $M$. For any real function $u$ on $M$, consider a conformal metric $\tilde{g} = e^{2u} g$. Denote by $\tilde{\nabla}$ the Dirac operator with respect to this conformal metric. If $D\varphi = \lambda \varphi$, then $\tilde{D} \tilde{\psi} = \lambda e^{-u} \tilde{\psi}$ where $\tilde{\psi} = e^{-\frac{n-1}{2}u} \varphi$. Note that

\[\nabla_{\tilde{e}_i} = \nabla_{e_i} + a e^{-u} \nabla_i u + \frac{a}{n} e^{-u} \nabla_j u \overleftarrow{e^j} = \overleftarrow{e^j} + \frac{\lambda}{n} e^{-u} \overleftarrow{e}^j,\]
\[\Delta u = -\sum_i e^{-u} (\nabla_i (e^{-u} \nabla_i u)) = e^{-2u}(\Delta u + |du|^2),\]
\[\tilde{R} e^{2u} = R + 2(n-1)\Delta u - (n-1)(n-2)|du|^2,\]
also, on $\partial M$,

\[\tilde{D}\delta M \left( e^{-\frac{(n-2)}{2}u} \varphi \right) = e^{-\frac{n}{2}u} \tilde{D}\delta M \varphi,\]
\[\tilde{H} = e^{-u} \left( H + (n-1) \, du(e_0) \right).\]

Define the function $\hat{R}_{\alpha, u}$ by

\[\hat{R}_{\alpha, u} = R + 4\left(\frac{n-1}{2} - a\right)\Delta u + 4\nabla a \nabla u\]
\[-\left((n-1)(n-2) + 4(2-n)a + 4(1 - \frac{1}{n})a^2 \right)|du|^2\] (4.1)
where $\Delta$ is the positive scalar Laplacian. Then apply (3.3) to the conformal metric $\bar{g}$, to get

$$
\int_M |\overline{\nabla}_{a,u} \psi|^2_{\bar{g}} \, \bar{v}_g = \int_M e^{-u} \left[ (1 - \frac{1}{n}) \lambda^2 - \frac{\hat{R}_{a,u}}{4} \right] |\psi|^2_v \, g
$$

$$+ \int_{\partial M} \left\{ \left(\psi, e^0 \cdot D_{\partial M} \psi\right)_{\bar{g}} + \left[ a \, du(e_0) - \frac{H}{2} \right] |\psi|^2_{\bar{g}} \right\} \, v_g
$$

$$= \int_M e^{-u} \left[ (1 - \frac{1}{n}) \lambda^2 - \frac{\hat{R}_{a,u}}{4} \right] |\psi|^2_v \, g
$$

$$+ \int_{\partial M} e^{-u} \left\{ (\varphi, e^0 \cdot D_{\partial M} \varphi) + \left[ (a - \frac{n-1}{2}) \, du(e_0) - \frac{H}{2} \right] |\varphi|^2 \right\} \, v_g.
$$

(4.2)

Note that $\overline{\nabla}_{a,u} \psi = 0$ implies

$$\nabla_i \varphi = -\frac{\lambda}{n} e^i \cdot \varphi - (a - \frac{n}{2}) \nabla_i u \varphi - \frac{1}{n} (a - \frac{n}{2}) \nabla_j u e^i \cdot e^j \cdot \varphi$$

(see [3]), we thus have either $a = \frac{n}{2}$ or $du = 0$ by Lemma [3]. Thus we obtain:

**Theorem 7.** Let $M^n$ be a compact Riemannian spin manifold of dimension $n \geq 2$, with boundary $\partial M$, and let $\lambda$ be any eigenvalue of $D$ under the local boundary condition $\Gamma^{loc}$. If there exist real functions $a$, $u$ on $M$ such that

$$H \geq (2a - n + 1) \, du(e_0)$$

on $\partial M$, where $H$ is the mean curvature of $\partial M$, then

$$\lambda^2 \geq \frac{n}{4(n-1) \, \sup \inf_{a,u} \hat{R}_{a,u}},$$

(4.3)

where the function $\hat{R}_{a,u}$ is given in (4.1). In the limiting case, the associated eigenspinor is a real Killing spinor and either $H = du(e_0)$ or $H = 0$ on $\partial M$. 

Since \( \overline{Q}_{\varphi,ij} = e^{-u} Q_{\varphi,ij} \) under the conformal transformation \( \overline{g} = e^{2u} g \), we apply (3.8) to the conformal metric \( \overline{g} \), to get

\[
\int_M |\overline{\nabla}_{Q,a,u} \psi|^2_{\overline{g}} = e^{-u} \left[ \lambda^2 - \left( \frac{\overline{R}_{a,u}}{4} + |Q_{\varphi}|^2 \right) \right] |\varphi|^2 v_g \\
+ \int_{\partial M} e^{-u} \left\{ (\varphi, e^0 \cdot D_{\partial M} \varphi) \\
+ \left[ (a - \frac{n-1}{2}) du(e_0) - \frac{H}{2} \right] |\varphi|^2 \right\} v_g.
\]

(4.4)

Thus we have

**Theorem 8.** Let \( M^n \) be a compact Riemannian spin manifold of dimension \( n \geq 2 \), with boundary \( \partial M \), and let \( \lambda \) be any eigenvalue of \( D \) under the local boundary condition \( \Gamma_{\pm}^{loc} \). If there exist real functions \( a, u \) on \( M \) such that

\[
H \geq (2a - n + 1) \ du(e_0)
\]

on \( \partial M \), where \( H \) is the mean curvature of \( \partial M \), then

\[
\lambda^2 \geq \sup_{a,u} \inf_M \left( \frac{\overline{R}_{a,u}}{4} + |Q_{\varphi}|^2 \right).
\]

(4.5)

In the limiting case one has \( H = (2a - n + 1) \ du(e_0) \) on \( \partial M \).

**Remark 9.** If \( n \geq 3 \), take \( a = 0 \) and \( u = \frac{2}{n-2} \log h \) in (4.3) and (4.4), where \( h \) is a positive eigenfunction of the first eigenvalue \( \mu_1 \) of the conformal Laplacian

\[
L := 4 \frac{n-1}{n-2} \Delta + R
\]

under the boundary condition

\[
\overline{dh}(e_0) - \frac{(n-2)H}{2(n-1)} h = 0.
\]

Then, one gets the lower bounds \([13],[15]\)

\[
\lambda^2 \geq \frac{n}{4(n-1)} \mu_1,
\]

(4.6)

and

\[
\lambda^2 \geq \inf_M \left( \frac{\mu_1}{4} + |Q_{\varphi}|^2 \right)
\]

(4.7)

under the local boundary condition \( \Gamma_{\pm}^{loc} \). In the limiting case of (4.4), the associated eigenspinor is a real Killing spinor and \( \partial M \) is minimal.
References

[1] R.A. Adams, *Sobolev spaces*, Academic Press, New York 1978.
[2] C. Bär, *Lower eigenvalue estimates for Dirac operators*, Math. Ann. **293** (1992), 39–46.
[3] H. Baum, T. Friedrich, R. Grunewald, I. Kath, Twistor and Killing Spinors on Riemannian Manifolds, Seminarbericht **108**, Humboldt-Universität zu Berlin, 1990.
[4] J.P. Bourguignon, P. Gauduchon, *Spinors, Opérateurs de Dirac et Variations de Métriques*, Commun. Math. Phys. **144** (1992), 581–599.
[5] J.P. Bourguignon, O. Hijazi, J.-L. Milhorat, A. Moroianu, *A Spinorial Approach to Riemannian and Conformal Geometry*, Monograph (In preparation).
[6] T. Friedrich, *Der erste eigenwert des Dirac-operators einer kompakten, Riemannschen Mannigfaltigkeit nicht negativer skalarkrümmung*, Math. Nachr. **97**(1980), 117-146.
[7] Th. Friedrich, E.-C. Kim, *Some remarks on the Hijazi inequality and generalizations of the Killing equation for spinors*, to appear in J. Geom. Phys.
[8] S. Farinelli, G. Schwarz, *On the spectrum of the Dirac operator under boundary conditions*, J. Geom. Phys. **28** (1998), 67-84.
[9] G. Gibbons, S. Hawking, G. Horowitz, M. Perry, *Positive mass theorems for black holes*, Commun. Math. Phys. **88** (1983), 295–308.
[10] P.B. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, 2nd ed., CRC Press, Boca Raton, 1995.
[11] M. Herzlich, *A Penrose-like inequality for the mass of Riemannian asymptotically flat manifolds*, Commun. Math. Phys. **188** (1998), 121–133.
[12] M. Herzlich, *The positive mass theorem for black holes revisited*, J. Geom. Phys. **26** (1998), 97–111.
[13] O. Hijazi, *A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors*, Commun. Math. Phys. **104** (1986), 151–162.
[14] O. Hijazi, *Première valeur propre de l’opérateur de Dirac et nombre de Yamabe*, C. R. Acad. Sci. Paris, **313** (1991), 865–868.
[15] O. Hijazi, *Lower bounds for the eigenvalues of the Dirac operator*, J. Geom. Phys. **16** (1995), 27-38.
[16] O. Hijazi, S. Montiel, X. Zhang, *Dirac operator on embedded hypersurfaces, to appear in Math. Res. lett.*
[17] O. Hijazi, X. Zhang, *Lower bounds for the Eigenvalues of the Dirac Operator, Part I. The hypersurface Dirac Operator, to appear in Ann. Glob. Anal. Geom.*
[18] O. Hijazi, X. Zhang, *Lower bounds for the Eigenvalues of the Dirac Operator, Part II. The Submanifold Dirac Operator*, preprint.
[19] H. Lawson, M. Michelsohn, *Spin geometry*, Princeton Univ. Press, 1989.
[20] B. Morel, *Eigenvalue Estimates for the Dirac-Schrödinger Operators*, Preprint IÉCN, Nancy (2000).
[21] E. Witten, *A new proof of the positive energy theorem*, Commun. Math. Phys. **80** (1981), 381–402.
[22] X. Zhang, *Lower bounds for eigenvalues of hypersurface Dirac operators*, Math. Res. Lett. 5 (1998), 199–210; *A remark on: Lower bounds for eigenvalues of hypersurface Dirac operators*, Math. Res. Lett. 6 (1999), 465-466.

[23] X. Zhang, *Angular momentum and positive mass theorem*, Commun. Math. Phys. 206 (1999), 137–155.

(Hijazi) Institut Élie Cartan, Université Henri Poincaré, Nancy I, B.P. 239, 54506 Vandœuvre-Lès-Nancy Cedex, France

E-mail address: hijazi@iecn.u-nancy.fr

(Montiel) Departamento de Geometría y Topología, Universidad de Granada, 18071 Granada, Spain

E-mail address: smontiel@goliat.ugr.es

(Zhang) Institute of Mathematics, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100080, P.R. China

E-mail address: xzhang@math08.math.ac.cn