From Kontsevich Graphs to Feynman graphs, a Viewpoint from the Star Products of Scalar Fields

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Abstract

In the present paper a new approach to construct the star products concerning the scalar fields is provided. Due to the structure of the classical Hamiltonians, the star products will be developed at three levels concerning functions, densities (fields) and functionals respectively. The star product at level of functions is the starting point for our setting which includes almost all information of the star products concerning the scalar fields and functionals. The point of the star product at level of functions is that it only concerns the finite dimensional issue, which is a Moyal-like star product on $\mathbb{C}^\infty(\mathbb{R}^d)$ generated by a bi-vector field with abstract coefficients. Thus the Kontsevich graphs will play some roles naturally. Actually we prove that there is an ono-one correspondence between a class of Kontsevich graphs and the Feynman graphs. Additionally the Wick theorem, Wick power and the expectation of Wick-monomial are discussed in terms of the star product at level of functions. Our construction can be considered as the generalisation of the star products in perturbative algebraic quantum fields theory and twist product introduced in [1],[2].

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1 Introduction

The deformation quantization of the fields is an infinite dimensional issue essentially. Up to now, there are a lot of works about the deformation quantization in the infinite dimensional case (for example, see [5], [6], [7], [8], [9], [10], where the list of references is not complete). In some sense the star products in the infinite dimensional space were usually constructed as the copies of the classical Moyal product, a typical point is that the partial derivatives in the classical Moyal bi-vector field are replaced by variational derivatives, for example, Frechet derivative or others. If we focus on the deformation quantization of the fields, for example, the case of scalar fields, we need to pay attention to the following two facts. The first fact is the commutative relation (or Poisson bracket):

\[ \{ \varphi(x), \varphi(y) \} = K(x, y),\]

where \( K(x, y) \) is some propagator and \( \varphi(x) \) is a scalar field in some physical theory. Above commutative relation suggests the possibility of the Moyal-like product. Another one is the variational calculation for a specific functional, for example

\[ F(\varphi) = \int f(\varphi(x_1), \cdots, \varphi(x_d)) \, dx_1 \cdots dx_d, \]

the variation of \( F(\varphi) \) can be calculated in terms of the partial derivatives of function \( f(y_1, \cdots, y_d) \). This fact provides a possibility such that the infinite dimensional calculations can be reduced to the finite dimensional situation. Our setup is motivated by the facts mentioned above.

In the present article we will discuss a new approach to the deformation quantization of scalar fields in the covariant case, i.e. the Moyal-like star product. An approach of the quantization of the scalar fields will pass to Feynman amplitudes or Feynman diagrams with much possibility. On the other hand, the Moyal-like star product will closely connect with Kontsevich’s graph, where Poisson bi-vector field will be replaced by a bi-vector field with abstract coefficients. Thus, our construction about the star product will result in the connection between the Kontsevich graphs (see [13]) and the Feynman graphs. Other applications are discussed also. Now we make some explanation about function \( f(y_1, \cdots, y_d) \) at level of terminology. In general \( f(\varphi(x_1), \cdots, \varphi(x_d)) \) (or more precisely \( f(\varphi(x_1), \cdots, \varphi(x_d)) \, dV \) where \( dV \) is volume form) is called density from the viewpoint of variational theory in classical fields theory. Here we need to distinguish between the \( f(y_1, \cdots, y_d) \) and \( f(\varphi(x_1), \cdots, \varphi(x_d)) \), so we call the function \( f(y_1, \cdots, y_d) \) the density function, or function for short.

Our approach to the star products is divided into three steps. The first step is to construct the star products at level of functions. As the second step, the star product of fields, or densities in the sense mentioned above, can be constructed from the first step simply. Finally, the star product at level of functionals can be constructed based
on the second step. We show our idea in the following table. Our construction can be considered as a generalisation of covariant deformation quantisation of the fields in perturbative algebraic quantum fields theory (see [7], [8], [9]) and twisted product introduced in [1], [2]. Somehow the main outline of our construction is along the idea in our earlier work (see [13]).

The basic starting point of our discussion is the construction of the star product at level of functions. The deformation quantisation in the case of the fields is the infinite dimensional issue basically. But in our approach, as a key point, the construction of the star product at level of functions involves the finite dimensional issue only. Our discussion below will show that the star product of functions contains all algebraic and combinatorial information of deformation quantisation of the fields and functionals almost. Actually, it will be showed that everything can be explicitly calculated based on the calculations at level of functions almost.

Here the star product of functions is a Moyal-like one in $\mathbb{R}^d$. The bi-vector field in the Moyal product is replaced by a bi-vector field with abstract coefficients, this bi-vector field generates the star product of the functions in our setting. With the help of the Moyal-like product in finite dimensional space our discussion goes into the framework of Kontsevich naturally. We prove that for a special class of the Kontsevich graphs, here we call that the graphs of Bernoulli type, there is an one-one correspondence between the graphs of Bernoulli type and the Feynman graphs. In this paper we consider only the Feynman graphs without self-lines. It is well known that the Moyal product is the simplest example in the theory of deformation quantisation on the Poisson manifolds, thus the Kontsevich graphs involving the Moyal product should be the simplest case. Our setting is completely parallel to the Moyal product from the viewpoint of the Kontsevich graphs. Roughly speaking the set of the graphs of Bernoulli type is generated by a special Bernoulli graph (see [11], [12]) which may be the simplest, but non-trivial, graph even in Bernoulli graphs. We will see that the forms of the graphs of Bernoulli type under the structure of product of admissible graphs (see [11], [12]) look like the Feynman amplitudes very much. This similarity results in the existence of one-one correspondence mentioned above.

Moreover, as another application of our construction we discuss the various forms of Wick theorem, Wick power and expectation of Wick-monomial in terms of the coordinates in $\mathbb{R}^d$ from the viewpoint of the star product of functions. In the sense of the star

\[
\begin{align*}
  f(x_i) \ast g(y_j) \quad &\text{functions} \\
  \downarrow \\
  f(\varphi) \ast g(\varphi) \quad &\text{fields or densities} \\
  = f(x_i) \ast g(y_j)|_{x_i=\varphi(\cdot),y_j=\varphi(\cdot)} \\
  \downarrow \\
  \int f(\varphi) \ast g(\varphi) \quad &\text{functionals}
\end{align*}
\]

Table 1: default
product the Wick theorem, Wick power and expectation of Wick-monomial for the case of scalar fields can be obtained from their various forms mentioned above. Observing the procedure to calculate the star product we find the Feynman amplitudes arise from the bi-vector field, that explains also why the Kontsevich graphs are relevant to the Feynman graphs.

This paper is organised as the following. In section 2 we discuss the star products at level of functions. The definitions of star product and Poisson bracket are presented and some properties are discussed. In section 3 we recall some contents of the Kontsevich graphs including admissible graphs and their product, Bernoulli graphs, et cetera (see [11], [12], [13]). A combinatorial notation, adjacency matrix, is introduced. In the end of this section we prove the existence of one-one correspondence between the graphs of Bernoulli type and the Feynman graphs. In section 4 we discuss the Wick theorem, Wick power and notion of expectation of Wick-monomial from the viewpoint of star product of functions on \( \mathbb{R}^d \). Here everything is expressed in terms of functions, or special, coordinates on \( \mathbb{R}^d \). In section 5 we discuss the star products of the fields and functionals based on the star product of functions on \( \mathbb{R}^d \).

2 The star products at level of functions

In this section we discuss the star products of functions which plays the role of underlying structure about the star products of scalar fields and functionals, moreover, includes all of combinatorial and algebraic information concerning the star products of scalar fields almost.

2.1 The star product with tensor form

At first we introduce some notations. Let \( \mathcal{A} \) be a commutative algebra over \( \mathbb{R} \) (or \( \mathbb{C} \)) with finite or countable generators, we consider a free \( C^\infty(\mathbb{R}^d) \) module on \( \mathcal{A} \) denoted by \( C^\infty_{\mathcal{A}}(\mathbb{R}^d) \),

\[
C^\infty_{\mathcal{A}}(\mathbb{R}^d) = \bigoplus_{i \in \mathcal{A}} C^\infty(\mathbb{R}^d).
\]

The elements in \( C^\infty_{\mathcal{A}} \) are the linear combinations of the elements in \( \mathcal{A} \) with coefficients in \( C^\infty(\mathbb{R}^d) \). The partial derivations on \( C^\infty(\mathbb{R}^d) \) can be extended to \( C^\infty_{\mathcal{A}} \) where the elements in \( \mathcal{A} \) are viewed as constants, for example, we have \( \partial_i(\lambda f(\mathbf{x})) = \lambda \partial_i f(\mathbf{x}) \), \( \lambda \in \mathcal{A}, f(\mathbf{x}) \in C^\infty(\mathbb{R}^d) \), here we have used the short symbols, \( \mathbf{x} = (x_1, \ldots, x_d) \), \( \partial_i = \partial_{x_i} \).

In the present article we focus on the situation of real scalar fields, the case of complex ones are similar, thus we discuss the problems over real number field \( \mathbb{R} \) below.

Now we consider the derivations of the tensor of functions. Let \( f_i(\mathbf{x}) \in C^\infty(\mathbb{R}^d), i = 1, \cdots, m \). We define the partial derivations for tensor of the functions \( f_1(\mathbf{x}_1) \otimes \cdots \otimes f_m(\mathbf{x}_m) \) as following:

\[
\partial_i^{(j)}(\bigotimes_{j=1}^m f_j(\mathbf{x}_j)) = f_1(\mathbf{x}_1) \otimes \cdots \otimes \partial_i f_j(\mathbf{x}_j) \otimes \cdots \otimes f_m(\mathbf{x}_m),
\]

(2.1)
where the variables of \( f_j(x_j) \) \( (j = 1, \cdots, m) \) are denoted by \( x_j = (x_j^{(1)}, \cdots, x_j^{(d)}) \). \( \partial^{(j)}_i \) acts on \( j \)th factor in above tensor and it is the partial derivation with respect to the \( i \)th component of \( x_j \).

In this paper the star product at level of functions what we want to construct is Moyal-like one. For simplicity, in the discussions below, we will restrict our considerations in the situations of the functions with one variable, i.e. each factor in the tensor is a function with one variable. Then, the formulas (2.1) will be of the following forms:

\[
\partial_j \left( \bigotimes_{j=1}^{m} f_j(x_j) \right) = f_1(x_1) \otimes \cdots \otimes f'_j(x_j) \otimes \cdots \otimes f_m(x_m),
\]

(2.2)

Let \( K = \{ K_{i,j} | K_{i,j} \in \mathcal{A}, i, j \in \mathbb{Z}^+ \} \), then we have the definition of the star product as follows:

**Definition 2.1.** Let \( f_i(x_i), g_j(y_j) \in \mathcal{C}^\infty(\mathbb{R}), i = 1, \cdots, m, j = 1, \cdots, n \), their star product with tensor form is defined by the following formula:

\[
[ (f_1(x_1) \otimes \cdots \otimes f_m(x_m)) \star_K (g_1(y_1) \otimes \cdots \otimes g_n(y_n)) ] \otimes \exp\{\hbar K_{x,y}\}(f_1(x_1) \otimes \cdots \otimes f_m(x_m)) \otimes (g_1(y_1) \otimes \cdots \otimes g_n(y_n)),
\]

(2.3)

where

\[
K_{x,y} = \sum_{ij} K_{i,m+j} \partial_{x_i} \otimes \partial_{y_j}.
\]

**Remark 2.1.**

- In definition 2.1 the subscripts \( i \) and \( j \) indicate the positions of the factors in the tensor, for example, index \( j \) indicates \( g_j(y_j) \) which is the \( (m+j) \)th factor in \( f_1(x_1) \otimes \cdots \otimes f_m(x_m) \otimes g_1(y_1) \otimes \cdots \otimes g_n(y_n) \). Therefore \( K_{i,m+j} \) concerns \( i \)th and \( (m+j) \)th factors.
- Let

\[
\mathcal{C}_{\mathcal{A}, h}^\infty(\mathbb{R}) = \{ \sum_{n \geq 0} \hbar^n f_n(x) | f_n(x) \in \mathcal{C}_{\mathcal{A}}^\infty(\mathbb{R}) \}.
\]

Then the star product in definition 2.1 defines a map

\[
*_{K} : \left( \mathcal{C}^\infty(\mathbb{R}) \otimes \cdots \otimes \mathcal{C}^\infty(\mathbb{R}) \right)_{m-\text{times}} \times \left( \mathcal{C}^\infty(\mathbb{R}) \otimes \cdots \otimes \mathcal{C}^\infty(\mathbb{R}) \right)_{n-\text{times}} \rightarrow \mathcal{C}_{\mathcal{A}, h}^\infty(\mathbb{R}) \otimes \cdots \otimes \mathcal{C}_{\mathcal{A}, h}^\infty(\mathbb{R})_{(m+n)-\text{times}}.
\]

There is a natural and obvious way to extend the star product as a map.
\[ *_{K} : \mathcal{C}_{\mathcal{A}, \mathcal{B}}(\mathbb{R}) \otimes \cdots \otimes \mathcal{C}_{\mathcal{A}, \mathcal{B}}(\mathbb{R}) \rightarrow \mathcal{C}_{\mathcal{A}, \mathcal{B}}(\mathbb{R}) \otimes \cdots \otimes \mathcal{C}_{\mathcal{A}, \mathcal{B}}(\mathbb{R}) \cdot \]

To simplify the formula (2.3) we introduce short notations in the following way. Let \( F_m(x) = \bigotimes_i f_i(x_i) \), \( G_n(y) = \bigotimes_j g_j(y_j) \). Then the formula (2.3) can be denoted in a short way as following

\[
(F_m(x) *_{K} G_n(y))_\otimes = \exp\{hK_{xy}\}(F_m(x) \otimes G_n(y)).
\]  

(2.4)

With the help of the formula (2.3) or (2.4), by a straightforward computation we know that the associativity is valid, i.e. we have

\[
[(F_m(x) *_{K} G_n(y))_\otimes *_{K} H_p(z)]_\otimes = [F_m(x) *_{K} (G_n(y) *_{K} H_p(z))]_\otimes,
\]

(2.5)

where \( H_p(z) = \bigotimes_{k=1}^p h_k(z_k) \). Actually,

\[
[(F_m(x) *_{K} G_n(y))_\otimes *_{K} H_p(z)]_\otimes = \exp\{hK_{xy} + hK_{yz}\} \exp\{hK_{xy}\}(F_m(x) \otimes G_n(y) \otimes H_p(z))
\]

\[
= \exp\{h(K_{xy} + K_{xz} + K_{yz})\}(F_m(x) \otimes G_n(y) \otimes H_p(z)).
\]

On the other hand,

\[
[F_m(x) *_{K} (G_n(y) *_{K} H_p(z))]_\otimes = \exp\{h(K_{xy} + K_{xz} + K_{yz})\}(F_m(x) \otimes G_n(y) \otimes H_p(z)).
\]

Thus, the both sides of the formula (2.5) are of the following form

\[
\exp\{h(K_{xy} + K_{xz} + K_{yz})\}(F_m(x) \otimes G_n(y) \otimes H_p(z)),
\]

(2.6)

Explicitly,

\[
K_{xy} = \sum_{ijk} K_{x, y, z} \partial_{x_i} \otimes \partial_{y_j} \otimes \partial_{z_k}, \\
K_{xz} = \sum_{i} K_{i, x, z} \partial_{x_i} \otimes \partial_{z_k}, \\
K_{yz} = \sum_{j} K_{y, z, y} \partial_{y_j} \otimes \partial_{z_k}.
\]

Similar to the discussion in remark 2.1, the subscripts \( x, y \) and \( z \) indicate the positions of the factors in the tensor.

**Remark 2.2.** If each factor of the tensor is a function in \( \mathcal{C}^\infty(\mathbb{R}^d) \), in addition one needs to consider the positions of the components of the variables. The star product can be generalized to this situation in a natural way.
2.2 The ordinary star products

Paralleling to definition 2.1 we can define the star product in ordinary sense.

**Definition 2.2.** Let \( f(x_1, \cdots, x_m) \in \mathcal{C}^\infty(\mathbb{R}^m) \), \( g(y_1, \cdots, y_n) \in \mathcal{C}^\infty(\mathbb{R}^n) \), then

\[
\begin{align*}
    f(x_1, \cdots, x_m) \star_K g(y_1, \cdots, y_n) &= \exp\{K_{x,y}\}(f(x_1, \cdots, x_m)g(y_1, \cdots, y_n)), \\
    (2.7)
\end{align*}
\]

where

\[
K_{x,y} = \sum_{i,j} K_{i,m+j} \partial x_i \partial y_j.
\]

It is easy to check that

\[
(m \circ F_m(x)) \star_K (m \circ G_n(y)) = m \circ (F_m(x) \star_K G_n(y)) \odot, \quad (2.8)
\]

where \( m \) denotes the point-wise multiplication of the functions,

\[
m : f_1(x_1) \otimes \cdots \otimes f_m(x_m) \mapsto f_1(x_1) \cdots f_m(x_m).
\]

Actually, by a straightforward calculation one can check that

\[
(m \circ F_m(x)) \star_K (m \circ G_n(y)) = \sum_{k \geq 0} \frac{\hbar^k}{k!} \left( \sum_{1 \leq i \leq m, 1 \leq j \leq n} K_{i,m+j} \partial x_i \partial y_j \right)^k m \circ F_m(x) m \circ G_n(y).
\]

From the formula (2.8), it is easy for us to check that

\[
\begin{align*}
    [(m \circ F_m(x)) \star_K (m \circ G_n(y))] \star_K (m \circ H_p(z)) &= m \circ (F_m(x) \star_K G_n(y)) \odot \star_K (m \circ H_p(z)) \\
    &= m \circ (m \circ F_m(x) \star_K G_n(y) \odot \star_K H_p(z)),
\end{align*}
\]

where \( H_p(z) \) is as above. Therefore, the associativity of the star product defined in definition 2.2 is valid. Furthermore we have

**Definition 2.3.** Let \( F_m(x), G_m(x) \) be as above, we define

\[
(m \circ F_m(x)) \star_K (m \circ G_m(x)) = (m \circ F_m(x)) \star_K (m \circ G_n(y)) |_{x=y}. \quad (2.9)
\]

**Remark 2.3.**

- **Comparing with the Moyal product**

\[
f(x) \star g(x) = f(x)g(x) + \hbar \sum_{i,j} \alpha^{ij} \partial_i f(x) \partial_j g(x) + O(\hbar^2),
\]

in the present paper the constant coefficients \( \alpha^{ij} \) in Moyal star product are replaced by abstract elements \( K_{ij} \) in \( \mathcal{A} \). However, similar to the case of the Moyal product, \( K \) plays the role of bi-vector field with coefficients in \( \mathcal{A} \).
• It is obvious that the star product defined by the formula (2.9) can be extended to the case of $C^\infty_{\mathcal{A},\hbar}$, where

$$C^\infty_{\mathcal{A},\hbar} = \{ \sum_{m \geq 0} \hbar^m F_m | F_m \in C^\infty_{\mathcal{A}}, m \geq 0 \},$$

such that the star product $\star_K$ is a map from $C^\infty_{\mathcal{A},\hbar} \times C^\infty_{\mathcal{A},\hbar}$ to $C^\infty_{\mathcal{A},\hbar}$, or from $C^\infty_{\mathcal{A},\hbar} \otimes C^\infty_{\mathcal{A},\hbar}$ into itself.

• We can extend the star product defined as above to the case where functions depending on some parameters. For example,

$$f(t, x) \star_K g(t, y) = \exp\{\hbar K\}(f(t, x)g(t, y)),$$

where $t = (t_1, \cdots, t_k)$.

More precisely, we have

$$m \circ F_m(x) \star_K m \circ G_n(y)$$

$$= m \circ F_m(x) m \circ G_n(y) + \hbar \sum_{i,j} K_{i,m+j} \partial_i (m \circ F_m(x)) \partial_j (m \circ G_n(y))$$

$$+ \hbar^2 \sum_{i_1,i_2,j_1,j_2} K_{i_1,m+j_1} K_{i_2,m+j_2} \partial_{i_1} \partial_{i_2} (m \circ F_m(x)) \partial_{j_1} \partial_{j_2} (m \circ G_n(y)) + \cdots,$$

thus we have

$$m \circ F_m(x) \star_K m \circ G_n(y) - m \circ G_n(y) \star_K m \circ F_m(x)$$

$$= \hbar \sum_{i \neq j} (K_{i,m+j} - K_{j,m+i}) \partial_i (m \circ F_m(x)) \partial_j (m \circ G_n(y)) + O(\hbar^2).$$

If

$$m \circ F_m(x) \star_K m \circ G_m(y) = m \circ G_m(y) \star_K m \circ F_m(x)$$

we say the star product $\star_K$ is commutative. It is obvious that we have

**Proposition 2.1.** The star products (2.9) are commutative iff the propagator matrix $K_{i,m+j} = K_{j,m+i}$.

For non-commutative case we define the Poisson bracket as following:

$$\{ m \circ F_m(x), m \circ G_m(x) \}_K = \sum_{i \neq j} (K_{i,m+j} - K_{j,m+i}) \partial_i (m \circ F_m(x)) \partial_j (m \circ G_m(x)). \quad (2.10)$$

The Poisson bracket can be extended to $C^\infty_{\mathcal{A}}$ also. It is obvious that the Poisson brackets (2.10) is bi-linear and anti-symmetric, additionally, are derivations for both of $m \circ F_m$ and $m \circ G_m$. The Jacobi identity is valid for the Poisson brackets, i.e. we have:

$$\{ \{ m \circ F_m(x), m \circ G_m(x) \}, m \circ H_m(x) \} + \text{cycles} = 0.$$

Now we extend the star product to the situation with several factors. We have
Proposition 2.2. Let $f_i(x_i) \in C^\infty(\mathbb{R})$, $i = 1, \cdots, m$, we have

$$f_1(x_1) \star_K \cdots \star_K f_m(x_m) = m \circ (f_1(x_1) \star_K \cdots \star_K f_m(x_m)) \otimes (2.11)$$

where

$$(f_1(x_1) \star_K \cdots \star_K f_m(x_m)) \otimes = \exp\{\hbar \sum_{i<j} K_{ij}\}(f_1(x_1) \otimes \cdots \otimes f_m(x_m)),$$

$K_{ij} = K_{ij} \partial_{x_i} \otimes \partial_{x_j}, \partial_{x_k} \text{ acts on } f_k(x_k) \text{ as ordinary partial derivation, } 1 \leq k \leq m.$

Observing the definition 2.1 we know that the different choice of propagator matrices determines the different star products. Now we discuss the connection between the star products depending on the different propagator matrices. We have

Theorem 2.1. For different propagator matrices $K, K'$ we have

$$(m \circ F_m(x) \star_K m \circ G_n(y)) \otimes = \exp\{\hbar(K - K')\}(m \circ F_m(x) \star_{K'} m \circ G_n(y)) \otimes. \quad (2.12)$$

Proof. Actually we have

$$\exp\{\hbar K\} = \exp\{\hbar(K - K')\}\exp\{K'\}. \quad \square$$

Remark 2.4. It is obvious that all formulas in above discussion are valid for elements in $C^\infty_A, \hbar$. Thus we will only discuss the issues concerning the star products for smooth functions below.

3 Kontsevich graphs and Feynman graphs

In this section we will explain how the star products result in the Feynman amplitudes, at same time, Kontsevich graphs result in the Feynman graphs naturally.

3.1 Notations

At beginning as preparations we recall some contents concerning the Kontsevich graphs simply, for more details we direct readers to [13].

Definition 3.1. An oriented graph $\Gamma$ is a pair $(V_\Gamma, E_\Gamma)$ of two finite sets such that $E_\Gamma$ is a subset of $V_\Gamma \times V_\Gamma$. Elements of $V_\Gamma$ are vertices of $\Gamma$, elements of $E_\Gamma$ are edges of $\Gamma$. If $e = (v_1, v_2) \in E_\Gamma \subseteq V_\Gamma \times V_\Gamma$ is an edge of $\Gamma$ then we say that $e$ stars at $v_1$ and ends at $v_2$.

Definition 3.2. (Admissible graphs, [13], p.22) Admissible graph $G_{n,m}$ is an oriented graph with labels such that
• The set of vertices $V_\Gamma$ is $\{v^{(1)}_1, \ldots , v^{(1)}_n\} \cup \{v^{(2)}_1, \ldots , v^{(2)}_m\}$ where $n, m \in \mathbb{Z}_{\geq 0}$, $2m + n - 2 \geq 0$; vertices from $\{v^{(1)}_1, \ldots , v^{(1)}_n\}$ are called vertices of the first type, vertices from $\{v^{(2)}_1, \ldots , v^{(2)}_m\}$ are called vertices of the second type.

• Every edge $e = (v_1, v_2) \in E_\Gamma$ stars at a vertex of the first type, $v_1 \in \{v^{(1)}_1, \ldots , v^{(1)}_n\}$.

• There are no loops, i.e. no edges of the type $(v, v)$.

• For every vertex $k \in \{1, \ldots , n\}$ of the first type, the set of edges $\text{Star}(k) = \{(v_1, v_2) \in E_\Gamma | v_1 = k\}$ starting from $k$, is labeled by symbols $\{e^k_1, \ldots , e^k_{\# \text{Star}(k)}\}$.

In other articles the vertices of the first type are also called internal vertices and vertices of the second type are called boundary vertices. About the operation of graphs we have

**Definition 3.3.** (see [14], p.7 and [12], p.22) If $\Gamma_1 \in G_{n,m}$, $\Gamma_2 \in G_{n',m'}$, we define the product $\Gamma_1 \Gamma_2 \in G_{n+n',m+m'}$ as the graph obtained from disjoint union of two graphs by identification of the vertices of the second type. We call this product Kathotia product. We define $\Gamma \emptyset = \Gamma$.

It is easy to see that the product of admissible graphs defined above is commutative. For convenience we define the embedding of the admissible graphs $G_{n,m} \rightarrow G_{n,m'}$, $m' \geq m$, or, extend a graph in $G_{n,m}$ to a graph in $G_{n,m'}$.

**Definition 3.4.** Let $\Gamma \in G_{n,m}$, the all vertices of the second type $\Gamma$ labeled by $v^{(2)}_1, \ldots , v^{(2)}_m$, for a subset of $I = \{i_1, \ldots , i_m\} \subseteq \{1, \ldots , m'\}$, $1 \leq \bar{i}_1 < \cdots < \bar{i}_m \leq m'$, we extend $\Gamma$ in the following way:

• identifying vertex $v^{(2)}_{i_j}$ with vertex $v^{(2)}_{i_j}$, $j = 1, \ldots , m$.

Above procedure define an embedding $\iota_I : G_{n,m} \rightarrow G_{n,m'}$, and we denote new graph by $\Gamma_I$. We call $I$ the position of $\iota_I$ or $\iota_I(\Gamma)$.

**Remark 3.1.** Combining the definitions 3.3 and 3.4 we can consider the product of general admissible graphs. For instance, let $\Gamma \in G_{n,m}$, $\Gamma' \in G_{n',m'}$, we take $m_1 = \max\{m, m'\}$ and choose two positions $(i_1, \ldots , i_m)$, $(\bar{i}'_1, \ldots , \bar{i}'_{m'})$, then the product

$$\Gamma_{(i_1, \ldots , i_m)}\Gamma'_{(\bar{i}'_1, \ldots , \bar{i}'_{m'})} \in G_{n+n',m_1}$$

makes sense.
3.2 Kontsevich’s rule for star product with tensor form

Bernoulli graphs  Now we embed the star product defined in definition 2.1 into the Kontsevich’s framework. The Moyal product is one generated by constant Poisson bi-vector field, it may be trivial case from the viewpoint of Kontsevich theory. Similar to the case of the Moyal product, the basic graph is Bernoulli graph (see [11], [12])

\[ b \in G_{1,2} \]

which endows one vertex of the first type, two vertices of the second type named by left and right ones respectively, and two edges starting at the unique vertex of the first type, ending at left (or right) vertex of the second type denoted by \( e^L \) and \( e^R \) respectively.

We call \( b \) the basic Bernoulli graph. We consider

\[ b^n = \underbrace{b \cdots b}_{n - \text{times}} \]

where the product of the graphs is Kathotia product in definition 3.3, thus we know that \( b^n \in G_{n,2} \). \( b^n \) is simple graph even in \( G_{n,2} \), actually, there are no edges connecting the different vertices of the first type in \( b^n \). Here we do not distinguish any two vertices of the first type in \( b^n \). In fact, if the vertices of the first type in \( b^n \) are labeled by \( \{1, \ldots, n\} \), and we make a permutation of indices, the new graph is isomorphic to \( b^n \). Each vertex of the first type of \( b^n \) assigns two edges starting at this vertex and ending at different vertices of the second type. The edges of \( b^n \) are labeled by symbols \( e^L_1, e^R_1, \ldots, e^L_n, e^R_n \), where \( e^L_k \) (or \( e^R_k \)) denote the edge starting at \( k \)-th vertex of the first type and ending at left (or right) vertex of the second type. It is obvious that each vertex of the first type corresponds to a pair \((e^L_k, e^R_k)\).

We now turn to a general situation \( G_{n,m} \), which means the Kontsevich’s graphs with \( m \) vertices of the second type. A Bernoulli graph with \( m \) vertices of the second type has two features. There are no edges connecting two vertices of the first type. For each vertex of the first type, there just be two edges starting at this vertex and ending at different vertices of the second type. We indicate the \( m \) vertices of the second type by \( \{v_1, \ldots, v_m\} \), or, the set of the vertices of the second type is denoted by \( \{v_1, \ldots, v_m\} \).

Then, with the help of Kathotia product, any Bernoulli graph can be generated by the Bernoulli graph \( b_{ij} \) \((1 \leq i < j \leq m)\) which is endowed with an unique vertex of the first type and two vertices of the second type, there are two edges of \( b_{ij} \) starting at the unique vertex of the first type and ending at ith and jth vertices of the second type respectively. Actually, \( b_{ij} \) can be regarded as the embedding of \( b \) with position \( \{i, j\} \), that is \( b_{ij} = \iota_{\{i,j\}}b \). Let \( \Gamma \in G_{n,m} \) be a Bernoulli graph, it is easy to check that

\[
\Gamma = \prod_{1 \leq i < j \leq m} b_{ij}^{m_{ij}}.
\]  

(3.1)

In the above formula, the product is Kathotia product, \( m_{ij} \in \mathbb{N} \) \((1 \leq i < j \leq m)\) with \( \sum_{i,j} m_{ij} = n \), when \( m_{ij} = 0 \) \( b_{ij}^{m_{ij}} = \emptyset \). Let \( \sigma \in \mathbb{P}_m \) be permutation on \( \{1, \ldots, m\} \). We define

\[
\sigma(\Gamma) = \prod_{1 \leq i < j \leq m} b_{\sigma(i)\sigma(j)}^{m_{\sigma(i)\sigma(j)}}.
\]
Similar to the discussion in the situation of $b^n$, for fixed $i$ and $j$ ($1 \le i, j \le m$), we do not distinguish any two vertices of the first type which connect to both $i$th and $j$th vertices of the second type. Under this assumption, it is obvious that for two Bernoulli graphs $\Gamma_1, \Gamma_2 \in G_{n,m}$, $\Gamma_1$ is isomorphic to $\Gamma_2$ denoted by $\Gamma_1 \sim \Gamma_2$ if and only if there is a $\sigma \in P_m$ such that $\Gamma_1 = \sigma(\Gamma_2)$. We know $\sim$ is an equivalent relation.

Let

$$B_{n,m} = \{ \Gamma \in G_{n,m} | \Gamma \text{ is a Bernoulli graph} \}, \quad B_m = (\bigcup_n B_{n,m}) \cup \{ \emptyset \}.$$  

Then, we have

**Proposition 3.1.** Under Kathotia product $B_m$ is a monoid with generators $\{b_{ij}\}_{1 \le i < j \le m}$.

It is obvious that $B_m/\sim$ is a monoid also.

**Corollary 3.1.** $\text{Span}_\mathbb{R}(B_m)$ (or $\text{Span}_\mathbb{C}(B_m)$) is an algebra over $\mathbb{R}$ (or $\mathbb{C}$) with generators $\{b_{ij}\}_{1 \le i < j \le m}$.

### Adjacency matrices

We now turn to the adjacency matrices.

**Definition 3.5.** An adjacency matrix is a symmetric matrix with non-negative integer entries. Here we make an additional restriction such that the entries on main diagonal are zeros, i.e. for an adjacency matrix $M = (m_{ij})$ of order $m$, we have $m_{ij} = m_{ji}$, $m_{ii} = 0$ ($1 \le i, j \le m$). We call $\frac{1}{2}\sum_{i,j} m_{ij}$ the degree of $M$ denoted by $\text{deg}M$. Let $M_{\text{adj}}(m, \mathbb{N})$ denote the set of all adjacency matrices of order $m$.

Let $M(i,j) = (m_{kl})_{m \times m}$ satisfying $m_{kl} = \delta_{ik}\delta_{jl}$, where $i < j$, $k < l$. Then $M(i,j)$ is a permutation matrix which is in $M_{\text{adj}}(m, \mathbb{N})$ obviously.

**Proposition 3.2.** $M_{\text{adj}}(m, \mathbb{N})$ is a monoid with generators $\{M(i,j)\}$, $1 \le i < j \le m$.

**Proof.** By definition 3.5 we know $0 \in M_{\text{adj}}(m, \mathbb{N})$. It is obvious that for $M_1, M_2 \in M_{\text{adj}}(m, \mathbb{N})$, $M_1 + M_2 \in M_{\text{adj}}(m, \mathbb{N})$. On the other hand, for each $M \in M_{\text{adj}}(m, \mathbb{N})$, we have $M = \sum_{i,j} m_{ij}M(i,j)$. Because $m_{ij} \in \mathbb{N}$, we know that

$$m_{ij}M(i,j) = M(i,j) + \ldots + M(i,j).$$

Up to now we have proved the conclusion. \qed

Let $M_1, M_2 \in M_{\text{adj}}(m, \mathbb{N})$, we say $M_1 \sim M_2$, if there exists a permutation matrix $P$ of order $m$ such that $M_1 = PM_2P$. $\sim$ is an equivalent relation obviously. It is easy to check that $M_{\text{adj}}(m, \mathbb{N})/\sim$ is a monoid also.

Starting from a given adjacency matrix $M = (m_{ij}) \in M_{\text{adj}}(m, \mathbb{N})$ with $\text{deg}M = n$, one can construct a Bernoulli graph $\Gamma_M = \prod_{1 \le i < j \le m} b_{ij}^{m_{ij}}$ with $n$ vertices of the first type and $m$ vertices of the second type. The previous construction results in a map
\[ \mathcal{A}_B : \text{Adj}(m, \mathbb{N}) \longrightarrow B_m; \quad \mathcal{A}_B(M) = \Gamma_M, \quad M \in \text{Adj}(m, \mathbb{N}). \]  
(3.2)

\( \mathcal{A}_B \) is an one-one correspondence between the adjacency matrices of order \( m \) and Bernoulli graphs with \( m \) vertices of the second type.

It is easy to check that

\[ \Gamma_{M_1 + M_2} = \Gamma_{M_1} \Gamma_{M_2}, \quad M_1, M_2 \in \text{Adj}(m, \mathbb{N}). \]  
(3.3)

In summary, we reach the following conclusion:

**Proposition 3.3.** The following maps

- \( \mathcal{A}_B : \text{Adj}(m, \mathbb{N}) \longrightarrow B_m \),
- \( \mathcal{A}_B : \text{Span}_K(\text{Adj}(m, \mathbb{N})) \longrightarrow \text{Span}_K(B_m) \), \( K = \mathbb{R} \) or \( \mathbb{C} \),

are homomorphisms.

**Remark 3.2.** \( \mathcal{A}_B \) induces a homomorphism \( \mathcal{A}_B : \text{Adj}(m, \mathbb{N})/\sim \longrightarrow B_m/\sim \) also.

**Kontsevich’s rule** We now begin to discuss the Kontsevich’s rule under our consideration. Here the star product works on the tensor of the functions space as the following

\[ \mathcal{C}_h^\infty(\mathbb{R}) \otimes \cdots \otimes \mathcal{C}_h^\infty(\mathbb{R}), \]  

\( m \times \) times

If we indicate the factors in above tensor by \( \{1, \cdots, m\} \) according to the order from left to right, then the \( i \)th factor corresponds to the \( i \)th vertex of the second type of the Bernoulli graphs in \( B_m \). Let \( \mathcal{K} = \{K_{ij} | K_{ij} \in \mathcal{A}, 1 \leq i < j \leq m\} \); \( \partial_i \) denote the derivation acting on \( i \)th factor in the above tensor.

Under the setting of this paper, it is enough to consider Bernoulli graphs when we discuss Kontsevich’s rule. Recalling the previous discussions concerning Bernoulli graphs, \( B_m \) is a monoid with generators \( \{b_{ij}\} \), it is sufficient for us to define Kontsevich’s rule on \( b_{ij} (1 \leq i < j \leq m) \), that is

\[ \mathcal{U}(b_{ij}, \mathcal{K}) = K_{ij}, \quad 1 \leq i < j \leq m, \]  
(3.4)

where \( K_{ij} = K_{ij} \partial_i \otimes \partial_j \) which is same as one in proposition 2.2. For the general Bernoulli graphs, for example, \( \Gamma = \sum_{1 \leq i < j \leq m} b_{ij}^{m_{ij}} \in B_m \), Kontsevich’s rule can be extended in the following way,

\[ \mathcal{U}(\Gamma, \mathcal{K}) = \prod_{1 \leq i < j \leq m} \mathcal{U}(b_{ij}, \mathcal{K})^{m_{ij}} = \prod_{1 \leq i < j \leq m} K_{ij}^{m_{ij}}. \]  
(3.5)
More precisely,
\[
\prod_{1 \leq i < j \leq m} K_{ij}^{m_{ij}} = \prod_{1 \leq i < j \leq m} K_{ij}^{m_{ij}} \partial_1^{\alpha_1} \otimes \cdots \otimes \partial_m^{\alpha_m},
\]
where \(\alpha_1 = \sum_{j > 1} m_{1j}, \alpha_m = \sum_{i < m} m_{im}, \alpha_i = \sum_{j < i} m_{ji} + \sum_{j > i} m_{ij} (1 < i < m)\). The previous discussions result in the following conclusion:

**Proposition 3.4.** Let \(\Gamma_1, \Gamma_2 \in B_m\), we have
\[
\mathcal{U}(\Gamma_1 \Gamma_2, K) = \mathcal{U}(\Gamma_1, K) \mathcal{U}(\Gamma_2, K).
\] (3.6)

Let
\[
D_{A,m} = \left\{ \sum a_\alpha \partial_1^{\alpha_1} \otimes \cdots \otimes \partial_m^{\alpha_m} | a_\alpha \in A, k \in \mathbb{N} \right\},
\]
where \(\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}^m\) are multiple indices, \(|\alpha| = \alpha_1 + \cdots + \alpha_m\). Then the formula (3.6) means that Kontsevich’s rule induces a homomorphism
\[
\mathcal{U}(\cdot, K) : \text{Span}_\mathbb{R}(B_m) \to D_{A,m}.
\] (3.7)
Furthermore
\[
\mathcal{U}(\cdot, K) \circ A_B : M \mapsto K_M.
\] (3.8)

Now we introduce the generating function of the Bernoulli graphs in \(B_m\) with form as the following:

\[
g_{B_m}(t) = \exp\{t \sum_{1 \leq i < j \leq m} b_{ij}\}.
\]

By a straightforward calculation we get

**Lemma 3.1.**
\[
\exp\{t \sum_{1 \leq i < j \leq m} b_{ij}\} = \sum_{M \in M_{ad}(m,\mathbb{N})} \frac{t^{\deg M}}{M!} \Gamma_M,
\] (3.9)
where \(M! = \prod_{1 \leq i < j \leq m} m_{ij}!\).

**Proof.** Noting that
\[
\exp\{t \sum_{1 \leq i < j \leq m} b_{ij}\} = \sum_{k \geq 0} \frac{t^k}{k!} \left( \sum_{1 \leq i < j \leq m} b_{ij} \right)^k,
\]
and
\[
(\sum_{1 \leq i < j \leq m} b_{ij})^k = \sum_{m_{ij} = k, i < j} \frac{k!}{\prod_{i < j} m_{ij}!} \prod_{1 \leq i < j \leq m} b_{ij}^m,
\]
we have
\[
\exp\{t \sum_{1 \leq i < j \leq m} b_{ij}\} = \sum_{k \geq 0} t^k \sum_{M, \text{deg} M = k} \frac{1}{M!} \Gamma_M = \sum_{M \in M_{adj}(m, N)} \frac{\text{deg} M}{M!} \Gamma_M.
\]

Recalling Kontsevich’s rule \( U \) is a homomorphism, we have
\[
U(\exp\{\hbar \sum_{1 \leq i < j \leq m} b_{ij}\}, \mathcal{K}) = \sum_{M \in M_{adj}(m, N)} \frac{\hbar^{\text{deg} M}}{M!} U(\Gamma_M, \mathcal{K}).
\]
Furthermore, combining with the formula (3.4) we have
\[
U(\exp\{\hbar \sum_{1 \leq i < j \leq m} b_{ij}\}, \mathcal{K}) = \sum_{M \in M_{adj}(m, N)} \frac{\hbar^{\text{deg} M}}{M!} K_M
\]
\[
= \exp\{\hbar \sum_{1 \leq i < j \leq m} K_{ij}\}.
\]

In summary, with the help of Kontsevich’s rule, the star product in proposition 2.2 can be expressed in the following way.

**Theorem 3.1.**

\[
U(\exp\{\hbar \sum_{1 \leq i < j \leq m} b_{ij}\}, \mathcal{K}) = \exp\{\hbar \sum_{i < j} K_{ij}\} = \sum_{M \in M_{adj}(m, N)} \frac{\hbar^{\text{deg} M}}{M!} K_M.
\]  

(3.10)

**From Kontsevich graphs to Feynman graphs**  Now we turn to the Feynman amplitudes and Feynman graphs. Firstly, we consider the multiple star product. Recalling proposition 2.2 we know that
\[
(f_1(x_1) \star \cdots \star f_m(x_m)) \otimes = \exp\{\hbar \sum_{i < j} K_{ij}\}(f_1(x_1) \otimes \cdots \otimes f_m(x_m))
\]  

(3.11)

where \( K_{ij} \) are bi-vector fields acting on \( f_i(x_i) \) and \( f_j(x_j), \; i < j \), that is
\[
K_{ij}(f_1(x_1) \otimes \cdots \otimes f_m(x_m)) = K_{i,j}(f_1(x_1) \otimes \cdots \otimes \partial_i f_i(x_i) \otimes \cdots \otimes \partial_j f_j(x_j) \otimes \cdots \otimes f_m(x_m)).
\]
In this sense \( K_{ij} \) denotes a poly-differential operator.
Recalling the discussions about the generating function of the Bernoulli graphs, we have

\[
\exp\{\hbar \sum_{i<j} K_{ij}\} = \sum_{M \in \mathcal{M}_{\text{adj}}(m,N)} \frac{P_{\text{deg}M}}{M!} K_M,
\]  

(3.12)

Here \( K_M \) are poly-differential operators with coefficients in \( A \). But the form of \( K_M = \prod_{i<j} K^{m_{ij}}_{ij} \) and form of the Feynman amplitudes are very much alike. It is seems that here \( K_{ij} \) play the role of the propagators. In fact for the special choice of \( f_i(x) \), \( K_{ij} \) contributes the propagator indeed, we will discuss that in section 4 and section 5 furthermore. Therefore we call \( \prod_{i<j} K^{m_{ij}}_{ij} \) the generalised Feynman amplitude. It is worth to point out that the generalised Feynman amplitudes appearing in the coefficients of the star product. Kontsevich’s rule as mentioned above suggests that the Kontsevich graphs should result in the Feynman graphs.

Recalling discussion about the connection between the adjacency matrices and the Feynman amplitudes, it is obvious that there is a way to build the connection between the Bernoulli graphs and Feynman graphs. Now we get one of our main consequence immediately.

**Theorem 3.2.** There is a one-one correspondence between the Bernoulli graphs and Feynman graphs without loops.

**Proof.** With the help of Kontsevich’s rule

\[
\mathcal{U}(\cdot, \Gamma_M) = K_M, \ M \in \mathcal{M}_{\text{adj}}(m,N), \ \text{deg}M = k,
\]

for a given Bernoulli graph \( \Gamma_M = \prod_{i<j} b^{m_{ij}}_{ij} \in B_{k,m} \), it reduces a Feynman graph by the following rule:

- every vertex of the second type is assigned to a vertex of Feynman graph,
- every \( b_{ij} \) is assigned to a edge of Feynman graph connecting \( i \)-th and \( j \)-th vertices of Feynman graph.

Consequently, according to above rule, from the graph \( \prod_{i<j} b^{m_{ij}}_{ij} \in B_{k,m} \) we get a Feynman graph with \( m \) vertices, \( k \) edges, here there are \( m_{ij} \) edges between the \( i \)-th and \( j \)-th vertices; \( \sum_{i<j} m_{ij} = k \).

Conversely, from a given Feynman graph we can get a graph of Bernoulli type in an obvious way.

The graph \( b_M \) have same form as Feynman amplitudes. Due to the correspondence between the Bernoulli graph \( b_{ij} \) and bi-vector field \( K_{ij} \), the graph \( \prod_{i<j} b^{m_{ij}}_{ij} \) can be viewed as graphical version of the Feynman amplitudes.
4 Wick theorem and Wick power

In this section we want to discuss the Wick theorem and Wick power in terms of the star product at level of functions in $C^\infty(\mathbb{R})$. Therefore we focus on the situation of star product with special form as $f_1(x_1) \star_K \cdots \star_K f_d(x_d)$, where $f_i(\cdot) \in C^\infty(\mathbb{R})$, $i = 1, \cdots, d$. Recalling proposition 2.2 we know that

$$f_1(x_1) \star_K \cdots \star_K f_d(x_d) = \exp\{\hbar \sum_{i<j} K_{ij} \partial_i \partial_j\} f_1(x_1) \cdots f_d(x_d). \quad (4.1)$$

By the notation $K_{ij} = K_{ij} \partial_i \otimes \partial_j$ and discussions concerning the generating function of Bernoulli graphs we have

$$\exp\{\hbar \sum_{i<j} K_{ij}\} = \sum_{M \in \text{M}_{\text{adj}}(d,N)} \frac{\hbar^\deg M}{M!} K_M.$$

More precisely we have

**Proposition 4.1.**

$$f_1(x_1) \star_K \cdots \star_K f_d(x_d) = \sum_{M \in \text{M}_{\text{adj}}(d,N)} \frac{\hbar^\deg M}{M!} K_M \partial^\alpha (f_1(x_1) \cdots f_d(x_d)), \quad (4.2)$$

where $\alpha_i = \sum_j m_{ij}$ ($i = 1, \cdots, d$), and $K_M = \prod_{i<j} K_{ij}^{m_{ij}}$.

In the formula (4.2) the factors $\prod_{i<j} K_{ij}^{m_{ij}}$ reduced from $\prod_{i<j} K_{ij}^{m_{ij}}$ are very close to the original Feynman amplitudes in the standard quantum fields theory.

Now we turn to the discussion of Wick power. In this case, it is necessary for us to consider the following star product:

$$\underbrace{f(x) \star_K \cdots \star_K f(x)}_{l\text{-times}}$$

where $f(\cdot) \in C^\infty(\mathbb{R})$ and $i = 1, \cdots, d$. By definition 2.3 we know

$$f(x) \star_K \cdots \star_K f(x) = (f(x_1) \star_K \cdots \star_K f(x_l))|_{x_1=\cdots=x_l=x}.$$ 

Similar to the formula (4.2) we have:

**Proposition 4.2.**

$$\underbrace{f(x) \star_K \cdots \star_K f(x)}_{l\text{-times}} = \sum_{k \geq 0} \frac{\hbar^k}{k!} \sum_{\deg M = 2k} \left( m_{12}, \cdots , m_{l-1,l} \right) K_M f^{(\alpha_1)}(x) \cdots f^{(\alpha_l)}(x). \quad (4.3)$$
Particularly, if we take $K_{ij} = \Lambda$ for all $1 \leq i < j \leq m$, where $\Lambda \in A$, then we have

$$ f(x) \ast_K \cdots \ast_K f(x), $$

$$ = \sum_{k \geq 0} \frac{(\hbar \Lambda)^k}{k!} \sum_{\text{deg} M = k} \left( m_{12}, \cdots, m_{l-1,l} \right) f^{(\alpha_1)}(x) \cdots f^{(\alpha_l)}(x). $$

Furthermore, if we take $f(x_i) = x_i$ we have

Corollary 4.1.

\[
\frac{x_i \ast_K \cdots \ast_K x_i}{l \text{-times}} = \sum_{k=0}^l \frac{(\hbar \Lambda)^k}{k!} h^K x_i^{l-2k}.
\] (4.4)

Proof. It is enough for us to count the number of terms with form $h^K x_i^{l-2k}$. We consider the star product $x_1 \ast_K \cdots \ast_K x_l$. From the formula (4.2) we have

\[
x_1 \ast_K \cdots \ast_K x_l = \sum_{k=0}^l \frac{(\hbar \Lambda)^k}{k!} (\sum_{i<j} \frac{\partial_i \partial_j}{k!})^k (x_1 \cdots x_l).
\]

It is obvious that

\[
(\sum_{i<j} \frac{\partial_i \partial_j}{k!})^k (x_1 \cdots x_l) = \sum_{i_1, j_1, \ldots, i_k, j_k} \partial_{i_1} \partial_{j_1} \cdots \partial_{i_k} \partial_{j_k} (x_1 \cdots x_l),
\]

where $i_\alpha < j_\alpha$ ($\alpha = 1, \cdots, k$), $\{i_\alpha, j_\alpha\} \cap \{i_\beta, j_\beta\} = 0$ ($\alpha \neq \beta$). The number of the terms in above sum should be

\[
\left( \begin{array}{c} l \\ 2 \end{array} \right) \left( \begin{array}{c} l - 2 \\ 2 \end{array} \right) \cdots \left( \begin{array}{c} l - 2k + 2 \\ 2 \end{array} \right) = \frac{l!}{2^k (l - 2k)!}.
\]

Finally, let $x_1 = \cdots = x_l = x$, we get the formula (4.4).

\[ \square \]

Definition 4.1. The Wick power in the sense of the star product $\ast_K$ of $x_i$ is defined to be

\[
: x_i^l :_{K} = \frac{x_i \ast_K \cdots \ast_K x_i}{l \text{-times}},
\] (4.5)

where $1 \leq i \leq d$.

Remark 4.1. By definition as above we know that the Wick power belongs to $C^\infty_A$. Due to the formula (4.4) the Wick power is expressed by means of Hermite polynomials.

Similar to the theorem 2.2, about the case of $f_1(x_1) \ast_K \cdots \ast_K f_d(x_d)$ we have more precise formula which is a generalisation of the classical Wick theorem.
Theorem 4.1. (Wick theorem) For different propagator matrices $K = (K_{ij})_{d \times d}$ and $K' = (K'_{ij})_{d \times d}$ we have

$$f_1(x_1) \star \cdots \star K f_d(x_d) = \sum_{M \in \text{Adj}(d, N)} \frac{(-\hbar \deg M)}{M!} (K - K')_M f_1^{(\alpha_1)} \star \cdots \star K' f_d^{(\alpha_d)}.$$  \hspace{1cm} (4.6)

Where $(K - K')_M = \sum_{i < j} (K_{ij} - K'_{ij}) m_{ij}$ and

$$\alpha_i = \sum_j m_{ij}, \ i = 1, \ldots, d.$$

Proof. Similar to the proof of theorem 2.2 we have

$$\exp \left\{ \hbar \sum_{i < j} K_{ij} \partial_i \partial_j \right\} = \exp \left\{ \hbar \sum_{i < j} (K_{ij} - K'_{ij}) \partial_i \partial_j \right\} \exp \left\{ \hbar \sum_{i < j} K'_{ij} \partial_i \partial_j \right\}.$$

Therefore

$$f_1(x_1) \star \cdots \star K f_d(x_d) = \exp \left\{ \hbar \sum_{i < j} (K_{ij} - K'_{ij}) \partial_i \partial_j \right\} f_1(x_1) \star \cdots \star K' f_d(x_d).$$

Above formula implies (4.6). \hfill \square

Remark 4.2.

- If we take $K' = 0$ in the formula (4.6), we come back to the formula (4.2).
- If we take $K = 0$, we have

$$f_1(x_1) \star \cdots \star K f_d(x_d) = \sum_{M \in \text{Adj}(d, N)} \frac{(-\hbar \deg M)}{M!} (K' - K')_M f_1^{(\alpha_1)} \star \cdots \star K' f_d^{(\alpha_d)}.$$

Specially, we can get inversion of the formula (4.4):

$$x^l = \sum_{k=0}^{\left\lfloor \frac{l}{2} \right\rfloor} \frac{l!}{2k!(l - 2k)!} (-\hbar)^k \Lambda^k : x^{l-2k} : K.$$

If we make a special choice of $f_i(x_i)$ in the formula (4.6), i.e. $: x_i^{n_i} : K \in C_A^3$, $n_i \in \mathbb{N}$, $i = 1, \ldots, d$, we can get the Wick theorem with expression very similar to the classical Wick theorem in the standard quantum fields theory.

Corollary 4.2.

$$= \sum_{M \in \text{Adj}(d, N)} \frac{(-\hbar \deg M)}{M!} (K^{(1)} - K^{(2)})_M \left( n_1 \at \alpha_1 \right) \cdots \left( n_d \at \alpha_d \right),$$

\hspace{1cm} (4.7)
Where

$$\alpha_i = \sum_j m_{ij}, \alpha_i \leq n_i, i = 1, \cdots, d.$$  

**Remark 4.3.** Observing the formula (4.7), we find that the form of Feynman amplitudes does not depend on the choices of propagator matrices $K^{(1)}$ and $K^{(2)}$. Thus, when we focus on the issues of the Feynman amplitudes, without loss of generality we can choose the the star product $*_{K^{(1)}}$ in Wick-monomial : $x_1^{n_1} : K *_{K^{(1)}} \cdots *_{K^{(1)}} : x_d^{n_d} : K$ to be commutative always.

In the traditional sense the Feynman amplitudes arise from the expectation of Green functions. In the present paper Feynman amplitudes arise from the bi-vector fields or coefficients of the star product. Actually, in perturbative algebraic quantum fields theory the expectation of the Wick-monomial can be defined as the coefficient of the term with the highest power of $\hbar$ in the star product. Now we define the expectation of Wick-monomial : $x_1^{n_1} : K *_{K^{(1)}} \cdots *_{K^{(1)}} : x_d^{n_d} : K$, denoted by

$$\langle : x_1^{n_1} : K *_{K^{(1)}} \cdots *_{K^{(1)}} : x_d^{n_d} : K \rangle,$$

as following:

**Definition 4.2.** If we write the Wick-monomial as a polynomial of $\hbar$

$$: x_1^{n_1} : K *_{K^{(1)}} \cdots *_{K^{(1)}} : x_d^{n_d} : K = \sum_{k=1}^{m} c_k \hbar^k,$$

we define the expectation of above Wick-monomial as following:

- When $n_1 + \cdots + n_d = 2m$,

  $$\langle : x_1^{n_1} : K *_{K^{(1)}} \cdots *_{K^{(1)}} : x_d^{n_d} : K \rangle = c_m, \quad (4.8)$$

  where

  $$c_m = \sum_{M \in M_{\text{adj}}(d, N), \deg M = m} \frac{K^{(1)}_M}{M!}.$$

  above sum is over all adjacency matrices $M = (m_{ij})_{d \times d}$, $\deg M = 2m$, such that

  $$n_i = \sum_j m_{ij}, i = 1, \cdots, d.$$

- When $n_1 + \cdots + n_d > 2m$,

  $$\langle : x_1^{n_1} : K *_{K^{(1)}} \cdots *_{K^{(1)}} : x_d^{n_d} : K \rangle = 0.$$
Definition 4.3.
When the integer sequence \((n_1, \cdots, n_d)\) satisfies the following conditions:

- There is an adjacency matrix \(M = (m_{ij})_{d \times d}\), \(\text{deg} M = 2m\), such that
  \[
  2m = n_1 + \cdots + n_d, \\
  n_i = \sum_j m_{ij}, \quad i = 1, \cdots, d,
  \]
  \[(4.10)\]

we call this integer sequence \((n_1, \cdots, n_d)\) admissible.

Combining the definition 4.2 and 4.3 we have the following conclusion immediately.

Proposition 4.3. The Wick-monomial:
\[
x_1^{n_1} \ast_{K(1)} \cdots \ast_{K(1)} x_d^{n_d} \ast_{K}\endows non-zero expectation
\]
iff \((n_1, \cdots, n_d)\) is admissible.

We have a simpler description of the admissible integer sequence. Here we assume the star product is commutative and \(n_i > 0\), \(i = 1, \cdots, d\).

Theorem 4.2. A integer sequence \((n_1, \cdots, n_d)\) is admissible iff \(n_1 + \cdots + n_d = 2m\) and \(\max_{1 \leq i \leq d} \{n_i\} \leq m\), where \(m\) is a positive integer.

Proof. If the integer sequence \((n_1, \cdots, n_d)\) is admissible, i.e. there is an adjacency matrix \(M = (m_{ij})_{d \times d}\), such that \(n_i = \sum_j m_{ij}, \quad i = 1, \cdots, d\). Then we have

\[
2n_i = \sum_j m_{ij} + \sum_j m_{ji} \leq \sum_{i,j} m_{ij} = 2 \sum_{i<j} m_{ij} = \sum_i n_i.
\]

Conversely we need to prove that for an integer sequence \((n_1, \cdots, n_d)\) satisfying \(n_1 + \cdots + n_d = 2m\), \(m \in \mathbb{N}\) and \(2n_i \leq n_1 + \cdots + n_d, \quad i = 1, \cdots, d\) there is an adjacency matrix \(M = (m_{ij})_{d \times d}\) such that \(n_i = \sum_j m_{ij}, \quad i = 1, \cdots, d\). Now we prove the existence of adjacency matrix by induction for \(d\). Without loss of generality we assume \(n_1 \geq \cdots \geq n_d\).

When \(d = 2\), then \(n_1 = n_2\) at this time. In this case there is an unique suitable adjacency matrix

\[
M = \begin{pmatrix} 0 & n_1 \\ n_2 & 0 \end{pmatrix}
\]

satisfying the conditions what we need.

Suppose the conclusion is valid for \(d\), now we consider the case of \(d + 1\). The case will be divided into a few parts.

Case of \(n_1 = \cdots = n_{d+1}\):

- \(d + 1\) is an even integer: Let \(n_1 = \cdots = n_{d+1} = p\), we can take the entries of \(M = (m_{ij})_{(d+1) \times (d+1)}\) to be
  \[
m_{ij} = p, \quad i + j = d + 2; \quad m_{ij} = 0, \quad \text{for others}.
\]
• \( d + 1 \) is an odd integer: Because \((d + 1)p = 2m, \) \( p \) is an even integer, let \( p = 2q \), the entries of \( M = (m_{ij})_{(d+1)\times(d+1)} \) can be taken to be

\[
m_{i,i+1} = m_{i+1,i} = q, \quad i = 1, \ldots, d, m_{1,d+1} = m_{d+1,1} = q; m_{ij} = 0, \text{ for others.}
\]

**Case of** \( n_1 > n_{d+1} \):

Let \( n'_1 = n_1 - n_{d+1} \), then we know that

\[
n'_1 + n_2 + \cdots + n_d = \sum_{i=1}^{d+1} n_i - 2n_{d+1}
\]

is an even integer and \((n'_1, n_2, \ldots, n_d)\) satisfies the condition (4.11). According to the hypothesis of induction we know that there is an adjacency matrix \( M = (m_{ij})_{d\times d} \) such that

\[
n'_1 = \sum_j m_{1j}, \quad n_i = \sum_j m_{ij}, \quad i = 2, \ldots, d.
\]

Now we take a \((d + 1) \times (d + 1)\) adjacency matrix as following:

\[
M_1 = \begin{pmatrix}
M & n_{d+1} \\
0 & \vdots \\
n_{d+1} & 0 \cdots & 0
\end{pmatrix}.
\]

The matrix \( M_1 \) satisfies all conditions what we need. \( \square \)

We want to talk about theorem 4.2 more from the combinatorial viewpoint. Under the assumption being \( n_i > 0, \quad i = 1, \ldots, d \) the adjacency matrixes need additional restriction which is that there is at least one positive entry at every row or column. Recalling RSK algorithm, there are one-one correspondences between the following objects: (see \([3], [4], [14]\))

- the set of permutations which are involutions without fixed points,
- the set of adjacency matrixes with zeros along the main diagonal,
- the set of semi-standard Young tableaus(SSYT) without odd columns.

From the combinatorial viewpoint the integer sequence \((n_1, \ldots, n_d)\) arising from the monomial: \( x_1^{n_1} \cdot_K \star_K \cdots \star_K \cdot_K \cdots \cdot_K \star_K \cdots \cdot_K \cdot_K \) plays the role of content for some semi-standard Young tableau denoted by \( 1^{n_1} 2^{n_2} \cdots d^{n_d} \cdot_K \). Here we assume \( n_1 \geq \cdots \geq n_d \). Starting from an admissible integer sequence \((n_1, \ldots, n_d)\), now we begin to construct a special SSYT such that under the one-one correspondence mentioned above this Young tableau results in an adjacency matrix which satisfies the conditions in definition 4.2. Firstly we construct an Young diagram with 2 rows and there are \( m \) columns in each row,
where \( m = \frac{1}{2} \sum_i n_i \). Secondly we fill the numbers in above Young diagram as follows. At beginning we put the numbers in the first row. Starting from the up-left corner of Young diagram we put number ”1” with \( n_1 \) times, and then we put all of ”2” and so on until the first row is full. Continuously we put numbers in the second row in the same way. Consequently we get a SSYT as table 3, where \( 2 \leq r_1 \leq r_2 \leq r_3 \leq r_4 \leq d \). Above discussion gives a combinatorial proof of theorem 4.2.

5 The star products at levels of fields and functionals

In this section we construct the star products at levels of fields and functionals base on the star product of functions discussed in previous sections. We will discuss the problems on the general smooth manifold which contains Lorentzian manifold as a special case.

The star product of the scalar fields Let \( X \) be a \( n \)-dimensional real smooth manifold, \( K(x, y) \in \mathcal{D}'(X \times X) \). For simplicity we assume \( K(x, y) \in C^\infty(X \times X) \) which can be considered as regulation of general distribution on \( X \times X \). In this section the bold letters \( x \) or \( y \) will denote the points in \( X \). At beginning of this subsection we discuss the star product of fields. Here we discuss the star product of functions being of form

\[
\varphi(x) \in C^\infty(X),
\]

\[
f(y_1, \cdots, y_d) \in C^\infty(\mathbb{R}^d),
\]

where \( \varphi(x) \in C^\infty(X) \) plays the role of real scalar field on \( X \).

Definition 5.1. Let \( K(x, y) \in C^\infty(X \times X) \), \( f(y_1, \cdots, y_d), g(y_1, \cdots, y_d) \in C^\infty(\mathbb{R}^d) \), \( \varphi(x) \in C^\infty(X) \), we define the star product as following:

\[
f(\varphi(x_1), \cdots, \varphi(x_d)) \star_K g(\varphi(y_1), \cdots, \varphi(y_d)) = m \circ \left[ \exp\{\hbar K\} f(x_1, \cdots, x_d) \otimes g(y_1, \cdots, y_d) \right]_{x_i = \varphi(x_i), y_i = \varphi(y_i)}
\]

(5.1)

Where \( K = \sum_{i,j} K_{ij} \partial_{x_i} \otimes \partial_{y_j}, K_{ij} = K(x_i, y_j) \).

When \( K(x, y) = P^* K(x, y) \), where \( P : X \times X \rightarrow X \times X ; P(x, y) = (y, x) \), is permutation map, we know that the star product \( \star_K \) is commutative. For non-commutative case we have:

Definition 5.2. The Poisson bracket is defined to be

| 1 | \cdots | | 1 | \cdots | | 2 | \cdots | | r_1 |
|---|---|---|---|---|---|---|---|
| r_2 | \cdots | | r_3 | \cdots | | r_4 | \cdots | | d |

Table 2: default
\[ \{ f(\varphi(x_1), \cdots, \varphi(x_d)), g(\varphi(y_1), \cdots, \varphi(y_d)) \}_K = m \circ \left[ \sum_{i,j}(K_{ij} - K_{ji}) \partial_i f(x_1, \cdots, x_d) \otimes \partial_j g(y_1, \cdots, y_d) \big|_{x_i = \varphi(x_i), y_i = \varphi(y_i)} \right]. \quad (5.2) \]

**Remark 5.1.**

- The star product defined in definition 5.1 and Poisson bracket defined in definition 5.2 are well defined and rely on the issues at level functions. Actually as a special case we have

\[ f(\varphi(x_1), \cdots, \varphi(x_d)) *_K g(\varphi(x_1), \cdots, \varphi(x_d)) = f(\varphi(x_1), \cdots, \varphi(x_d)) *_K g(\varphi(y_1), \cdots, \varphi(y_d)) \big|_{x_i = y_i}. \quad (5.3) \]

and

\[ \{ f(\varphi(x_1), \cdots, \varphi(x_d)), g(\varphi(x_1), \cdots, \varphi(x_d)) \}_K = \{ f(y_1, \cdots, y_d), g(y_1, \cdots, y_d) \} \big|_{y_i = \varphi(x_i)}. \quad (5.4) \]

Where \( K_{ij} = K(x_i, x_j) \).

- For all of conclusions in section 2 there are parallel ones in the case of scalar fields.

In the case of the scalar fields in the sense of our setting we have Wick theorem and Wick power similar to the situation of the standard quantum fields theory. Recalling the discussion in section 4, if we take \( x_i = \varphi(x_i), i = 1, \cdots, d \), in each formula in section 4, we can get a corresponding formula in the case of fields.

**Theorem 5.1.** Let \( K(x, y), K'(x, y) \) be smooth functions on \( X \times X \), \( f_1(\cdot), \cdots, f_d(\cdot) \in C^\infty(\mathbb{R}), \varphi(x) \in C^\infty(X) \), we have

\[ f_1(\varphi(x_1)) *_K \cdots *_K f_d(\varphi(x_d)) = \sum_{M \in M_{adj}(d, M)} \frac{\deg M}{M!} (K - K')_M f_1^{(\alpha_1)}(\varphi(x_1)) *_{K'} \cdots *_{K'} f_d^{(\alpha_d)}(\varphi(x_d)), \quad (5.5) \]

where

\[ \alpha_i = \sum_j m_{ij}, i = 1, \cdots, d, \]

\[ K_{ij} = K(x_i, x_j), K'_{ij} = K'(x_i, x_j). \]

Moreover we define the Wick power in the case of fields as follows:

\[ :\varphi^l(x):_K = \varphi(x) *_K \cdots *_K \varphi(x). \quad (5.6) \]
Precisely we have an expression of Wick power in terms of Hermite polynomials

\[ \varphi^l(x_i) : K = \sum_{k=0}^{\lfloor l/2 \rfloor} \frac{l!}{2^k (l - 2k)! k!} h^k K_{ii}^k (\varphi(x_i))^{l-2k}, \quad 1 \leq i \leq d, \quad (5.7) \]

where \( K_{ii} = K(x_i, x_i) \), \( 1 \leq i \leq d \).

The following Wick theorem expressed by means of Wick power is more closed to classical Wick theorem.

**Corollary 5.1.** Let \( K(x, y), K^{(1)}(x, y), K^{(2)}(x, y) \in C^\infty(X \times X) \), then

\[ : \varphi^{n_1}(x_1) : K * K^{(1)} * \cdots * K^{(1)} : \varphi^{n_d}(x_d) : K = \sum_{M \in M_{ad}(d, N)} \frac{h^{\text{deg} M}}{M!} (K^{(1)} - K^{(2)})_M \left( \begin{array}{c} n_1 \\ \alpha_1 \\ \vdots \\ \alpha_d \end{array} \right), \quad (5.8) \]

where

\[ \alpha_i = \sum_j m_{ij}, \quad \alpha_i \leq n_i, \quad i = 1, \cdots, d, \]
\[ K^{(1)}_{ij} = K^{(1)}(x_i, x_j), \quad K^{(2)}_{ij} = K^{(2)}(x_i, x_j). \]

Now we define the expectation of monomial \( : \varphi^{n_1}(x_1) : K * K' * \cdots * K' : \varphi^{n_d}(x_d) : K, \) where \( K(x, y), K'(x, y) \) are smooth functions on \( X \times X \). For convenience we assume the star product \( *_{K'} \) is commutative.

**Definition 5.3.**

- When the integer sequence \( (n_1, \cdots, n_d) \) is admissible, we define

\[ \langle : \varphi^{n_1}(x_1) : K * K' * \cdots * K' : \varphi^{n_d}(x_d) : K \rangle = \sum_{M \in M_{ad}(d, N), \text{deg} M = m} K_M^{K'}, \quad (5.9) \]

where \( 2m = \sum_i n_i \), the sum in (5.9) is over all adjacency matrices \( M = (m_{ij})_{d \times d} \) with \( \text{deg} M = 2m \) satisfying

\[ n_i = \sum_j m_{ij}, \quad i = 1, \cdots, d. \]

- for others

\[ \langle : \varphi^{n_1}(x_1) : K * K' * \cdots * K' : \varphi^{n_d}(x_d) : K \rangle = 0. \]

**Remark 5.2.** For general distribution \( K(x, y) \in \mathcal{D}'(X \times X) \), the power and restriction on diagonal of \( X \times X \) make non-sense generally. In this case only the star product with form \( \varphi(x_1) * K * \cdots * K \varphi(x_d) \) may be well defined, but some analytic conditions, for example, concerning wave front set \( \text{WF}(K) \), may be needed.
The star product of the functionals

Now we turn to situation of the functionals. We consider the functionals with form

\[ F(\varphi) = \int_{X^d} f(x_1, \ldots, x_d, \varphi(x_1), \ldots, \varphi(x_d)) dV_d, \quad (5.10) \]

where \( f \in C^\infty(X^d \times \mathbb{R}^d) \),

\[ X^d = X \times \cdots \times X, \quad d \text{-times} \]

and \( dV_d \) is volume form on \( X^d \).

In the below discussion we assume the integrals make sense always. We state the definitions of star product and Poisson bracket of functionals as following.

**Definition 5.4.** Let \( F(\varphi), G(\varphi) \) be functionals as in (5.10).

- We define their star product to be

\[ F(\varphi) \star_K G(\varphi) = \int_{X^d} \int_{X^d} f(\cdot) \star_K g(\cdot) dV_d dV_d, \quad (5.11) \]

where

\[ f(x_1, \ldots, x_d, \varphi(x_1), \ldots, \varphi(x_d)) \star_K g(y_1, \ldots, y_d, \varphi(y_1), \ldots, \varphi(y_d)) = m \circ \left[ \exp \{ \hbar K \} f(x_1, \ldots, x_1, \ldots, x_d) g(y_1, \ldots, y_1, \ldots, y_d) \right]_{x_i = \varphi(x_i), y_i = \varphi(y_i)}, \]

where \( K = \sum_{i,j} K_{ij} \partial_{x_i} \otimes \partial_{y_j}, \quad K_{ij} = K(x_i, y_j). \)

- For the star product between the functionals and fields we have

\[ F(\varphi) \star_K g(y_1, \ldots, y_d, \varphi(y_1), \ldots, \varphi(y_d)) = \int_{X^d} f(\cdot) \star_K g(\cdot) dV_d, \quad (5.12) \]

in (5.12) the integral concerns the variables of \( f(\cdot) \).

- We define the Poisson bracket of \( F(\varphi) \) and \( G(\varphi) \) to be

\[ \{ F(\varphi), G(\varphi) \}_K = \int_{X^d} \int_{X^d} \{ f(\cdot), g(\cdot) \}_K dV_d dV_d, \quad (5.13) \]

where

\[ \{ f(x_1, \ldots, x_d, \varphi(x_1), \ldots, \varphi(x_d)), g(y_1, \ldots, y_d, \varphi(y_1), \ldots, \varphi(y_d)) \}_K = \sum_{i,j} (K_{ij} - K_{ji}) \partial_{x_i} f(x_1, \ldots, x_1, \ldots, x_d) \partial_{y_j} g(y_1, \ldots, y_1, \ldots, y_d) |_{x_i = \varphi(x_i), y_i = \varphi(y_i)}. \]

The star product and Poisson bracket defined in definition 5.3 are well defined and satisfy all conditions which are needed.
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