RELATIONS BETWEEN BOHL AND GENERAL EXPONENTS

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Abstract. In the paper we study the problem of the influence of the parametric uncertainties on the Bohl exponents of discrete time-varying linear system. We obtain formulas for the computation of the exact boundaries of lower and upper mobility for the supremum and infimum of the Bohl exponents under arbitrary small perturbations of system coefficients matrices on the basis of the transition matrix.

1. Introduction. Consider a discrete time-varying linear system

\[ x(n+1) = A(n)x(n), \quad x(n) \in \mathbb{R}^s, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \]

where \( A = (A(n))_{n \in \mathbb{N}_0} \) is a bounded sequence of invertible s-by-s real matrices such that \( (A^{-1}(n))_{n \in \mathbb{N}_0} \) is bounded. The solution of the system (1) with initial condition \( x(0) = x_0 \) we will denote by \( (x(n,x_0))_{n \in \mathbb{N}_0} \) and by \( (x(n,k,x_k))_{n \in \mathbb{N}_0} \) we will denote the solution satisfying equality \( x(k,k,x_k) = x_k \). The spectral norm will be denoted by \( \|\cdot\| \). Denote

\[ a = \max \left\{ \sup_{n \in \mathbb{N}_0} \|A(n)\|, \sup_{n \in \mathbb{N}_0} \|A^{-1}(n)\| \right\} \]

and let \( \Phi_A(\cdot,\cdot) \) denote the transition matrix of the system (1) which is defined as

\[ \Phi_A(n,k) = A(n-1)\ldots A(k), \quad \text{if} \quad n > k, \]

\[ \Phi_A(n,n) = I_s, \]

and

\[ \Phi_A(n,m) = \Phi_A^{-1}(m,n) \quad \text{if} \quad m > n, \]

where \( I_s \) is the identity matrix of order \( s \). Moreover by \( \mathbb{R}^s_* \) we will denote the set \( \mathbb{R}^s \setminus \{0\} \).

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The lower (upper) Bohl exponent of the nonzero solution \(x(n, x_0)\) of the system (1) is defined as follows

\[
\beta_A(x_0) = \liminf_{n \to m, m \to \infty} \left( \frac{\|x(n, x_0)\|}{\|x(m, x_0)\|} \right)^{1/(n-m)}, \quad x_0 \in \mathbb{R}_+^n \tag{2}
\]

\[
\overline{\beta}_A(x_0) = \limsup_{n \to m, m \to \infty} \left( \frac{\|x(n, x_0)\|}{\|x(m, x_0)\|} \right)^{1/(n-m)}, \quad x_0 \in \mathbb{R}_+^n \tag{3}
\]

Also, by \(\beta[x]\) and \(\overline{\beta}[x]\) we denote lower (upper) Bohl exponents of the sequence \((x(n))_{n \in \mathbb{N}_0}\), respectively. These numbers appear in a natural way in the description of the uniform exponential stability of the solutions of the system (1). Some versions of these exponents for continuous-time systems were called by different authors as the lower (upper) uniform exponent [6], the lower (upper) minimum (maximum) exponent [12], the lower (upper) general exponent [10]. The names Bohl exponents of the lower (upper) uniform exponent [6], the lower (upper) minimum (maximum) exponent [12], the lower (upper) general exponent [10]. The names Bohl exponents for the quantities (2) and (3) were proposed in [16] what is valid from the historical point of view, although Bohl in the work [5] considered a slightly different characteristics described for discrete systems as below and called today [3]-[8] senior upper and junior lower general exponents of the system (1).

\[
\Omega^0(A) = \limsup_{m, n \to \infty} \left( \|\Phi_A(n, m)\| \right)^{1/(n-m)}, \tag{4}
\]

\[
\omega_0(A) = \liminf_{m, n \to \infty} \left( \|\Phi_A^{-1}(n, m)\| \right)^{-1/(n-m)}. \tag{5}
\]

In can be shown (see [8]) that the condition \(m \to \infty\), in the above definitions, may be omitted. Counterpart of the exponent (3) for the continuous-time systems was introduced by Bohl in the work [5] and independently by Persidski in [15]. The concept of general exponents for continuous-time system in Banach spaces was introduced in [10].

To describe relations between the set of the Bohl exponents of solutions of the system (1) and the general exponents let us introduce the following notations

\[
i\beta(A) = \inf_{x_0 \in \mathbb{R}_+^n} \beta_A(x_0), \quad s\beta(A) = \sup_{x_0 \in \mathbb{R}_+^n} \beta_A(x_0),
\]

\[
i\overline{\beta}(A) = \inf_{x_0 \in \mathbb{R}_+^n} \overline{\beta}_A(x_0), \quad s\overline{\beta}(A) = \sup_{x_0 \in \mathbb{R}_+^n} \overline{\beta}_A(x_0).
\]

These numbers will be called junior lower, junior upper, senior lower and senior upper Bohl exponent of the system (1), respectively. Directly from the definition of senior upper and junior lower general exponents the following inequalities follow

\[
\omega_0(A) \leq i\beta(A) \quad \text{and} \quad s\overline{\beta}(A) \leq \Omega^0(A).
\]

It turns out that the general exponents also describe the influence of parametric uncertainties on the Bohl exponents. Together with system (1) consider a system

\[
z(n + 1) = (A(n) + Q(n)) z(n), \tag{6}
\]

where \(Q = (Q(n))_{n \in \mathbb{N}_0}\) is a sequence of \(s\)-by-\(s\) real matrices which represents parametric uncertainties. In the papers [8] and [9] it has been shown that

\[
\omega_0(A) \leq \lim_{\varepsilon \to 0^+} \inf_{\|Q\|_\infty < \varepsilon} i\beta(A + Q) \tag{7}
\]

and

\[
\Omega^0(A) \geq \lim_{\varepsilon \to 0^+} \sup_{\|Q\|_\infty < \varepsilon} s\overline{\beta}(A + Q), \tag{8}
\]
where

\[ \|Q\|_\infty = \sup_{n \in \mathbb{N}_0} \|Q(n)\|. \]

The first main result of the paper is to show that inequalities in the formulas (7) and (8) can be replaced by equalities. In other words, the senior upper (junior lower) general exponents of the system (1) are the exact boundaries of mobility up (down) of the upper (lower) Bohl exponents of the solutions of perturbed systems under arbitrarily small perturbations of the coefficients matrix of the system (1). However, the open question is how to estimate the exact boundaries of mobility up (down) of the lower (upper) Bohl exponents of the perturbed systems, i.e., the following quantities

\[ \lim_{\varepsilon \to 0^+} \sup_{\|Q\|_\infty < \varepsilon} s \beta(A + Q) \]

and

\[ \lim_{\varepsilon \to 0^+} \inf_{\|Q\|_\infty < \varepsilon} i \beta(A + Q). \]

The answer to this question is the second main objective of this paper. The analogical result for the continuous-time system has been published in [4] (in Russian).

2. Method of rotations. To prove the main result of the paper we will use the so-called Millionschikov’s method of rotations ([13], [14]). This method was constructed for continuous-time systems. Below we present the discrete-time counterpart of this method. Firstly, we will introduce the following definition.

**Definition 2.1.** Let us fix \( \varepsilon > 0 \). The solution \( x = (x(n, x_0))_{n \in \mathbb{N}_0} \) of the system (1) will be called \( \varepsilon \)-fast on the interval \([k, m]\) if the following inequality

\[ \|x(m, x_0)\| \geq \frac{\sin \varepsilon}{2} \|\Phi_A(m, k)\| \|x(k, x_0)\| \]

is satisfied. Otherwise, the solution \( x \) will be called \( \varepsilon \)-slow on the interval \([k, m]\).

Observe that there always exists a maximal solution of the system (1) on each interval \([k, m]\). In fact, if \( x_k \in \mathbb{R}^s \) is such that \( \|x_k\| = 1 \) and

\[ \|\Phi_A(m, k)\| = \|\Phi_A(m, k) x_k\|, \]

then for the solution \( (x(n, k, x_k))_{n \in \mathbb{N}_0}\) of the system (1) we have

\[ \|x(m, k, x_k)\| = \|\Phi_A(m, k) x_k\| = \|\Phi_A(m, k)\| \|x(k, k, x_k)\|. \]

Millionschikov’s method of rotations is based on the following two lemmas and a corollary, which are an analogue for difference equations of the theorem proved by Millionschikov in [13] (see also [14]).

Let us introduce the following notation

\[ \text{Con}(y, \tan \varepsilon) = \{x \in \mathbb{R}^s : \angle(x, y) \leq \varepsilon\}, \]

where \( \angle(x, y) \) denotes the angle between \( x \) and \( y \).

**Lemma 2.2.** For all \( \varepsilon \in \mathbb{R}, \varepsilon > 0, \) \( k, m \in \mathbb{N}_0, k < m \) and each \( \varepsilon \)-slow on the interval \([k, m]\) solution \( x = (x(n, x_0))_{n \in \mathbb{N}_0}\) of the system (1) there exists \( x_k \in \text{Con}(x(k), \tan \varepsilon) \) such that the solution \( x = (x(n, x_k))_{n \in \mathbb{N}_0}\) of the system (1) is \( \varepsilon \)-fast on the interval \([k, m]\).
Proof. Let \((x(n, v_0))_{n \in \mathbb{N}_0}\) be the maximal solution of the system \((1)\) on the interval \([k, m]\) and
\[
\|x(k, v_0)\| = \|x(k, x_0)\|.
\]
Since in the case \(v_0 \in \text{Con}(x(k, x_0), \tan \varepsilon)\) the statement of the lemma is obvious, then without loss of generality we may assume that
\[
\varepsilon < \gamma = \angle(x(k, x_0), x(k, v_0)) \leq \frac{\pi}{2}.
\]
Let \(x_{\varepsilon, k}\) denote a vector with the norm
\[
\|x_{\varepsilon, k}\| = \|x(k, x_0)\|,
\]
belonging to the plane of vectors \(x(k, v_0)\) and \(x(k, x_0)\), located between them and forming with the vector \(x(k, x_0)\) the angle \(\varepsilon\). It can be represented in the form of a sum
\[
x_{\varepsilon, k} = \alpha x(k, x_0) + \beta x(k, v_0)
\]
with constants \(\alpha, \beta > 0\). According to the law of sines we obtain firstly equalities
\[
\frac{\beta}{\sin \varepsilon} = \frac{1}{\sin (\pi - \gamma)},
\]
\[
\frac{\alpha}{\sin (\gamma - \varepsilon)} = \frac{\beta}{\sin \varepsilon}
\]
and secondly the inequalities
\[
\beta \geq \sin \varepsilon,
\]
\[
\frac{\alpha}{\beta} \leq \frac{1}{\sin \varepsilon}.
\]
From the equality \((9)\) we also obtain the inequality
\[
\|x(m, k, x_{\varepsilon, k})\| \geq \beta \|x(m, v_0)\| \left(1 - \frac{\alpha}{\beta} \|x(m, v_0)\|\right)
\]
which transforms into the required inequality
\[
\|x(m, k, x_{\varepsilon, k})\| \geq \frac{\sin \varepsilon}{2} \|\Phi_A(m, k)\| \|x_{\varepsilon, k}\|.
\]

Lemma 2.3. For any solution \((x(n, x_0))_{n \in \mathbb{N}_0}\) with \(x_0 \in \mathbb{R}^s\) of the system \((1)\), any \(k \in \mathbb{N}_0\) and any \(z_k \in \mathbb{R}^s\) such that
\[
\|z_k\| = \|x(k, x_0)\|
\]
and
\[
\varepsilon = \angle(x(k, x_0), z_k) < \frac{\pi}{2}
\]
there exists an orthogonal transformation \(U_{\varepsilon}(k)\) which is a transform of rotation in the plane
\[
L = \text{span}\{x(k, x_0), z_k\}
\]
and identity on the orthogonal addition, such that
\[
\|U_{\varepsilon}(k) - I_s\| \leq \varepsilon
\]
and the vectors
\[
z(k) = U_{\varepsilon}(k)x(k, x_0) = z_k \quad \text{and} \quad z(k - 1) = x(k - 1)
\]
Corollary 1. Suppose that satisfy
\[ z(k) = A(k-1)z(k-1) + Q(k-1)z(k-1) \]
with a matrix \( Q(k) \) satisfying
\[ \|Q(k-1)\| \leq \varepsilon a. \]

Proof. Let \( U = [e_1, \ldots, e_n] \) be an orthogonal matrix, with column vectors
\[ e_1 = x(k, x_0)/\|x(k, x_0)\| \]
and such \( e_2 \in L \) that
\[ z_k = (e_1 \cos \varepsilon + e_2 \sin \varepsilon) \|z_k\|. \]
Moreover, let us define
\[ U_\varepsilon(k) = \text{diag}[U_2 \ I_{s-2}] \] \( U^T \)
and
\[ U_2 = \begin{bmatrix} \cos \varepsilon & -\sin \varepsilon \\ \sin \varepsilon & \cos \varepsilon \end{bmatrix}. \]

We will show that the matrix \( U_\varepsilon(k) \) satisfies the requirements of the lemma. It is clear that \( U_\varepsilon(k) \) is orthogonal. Moreover, we have
\[ \|U_\varepsilon(k) - I_s\| = \|\text{diag}[U_2 - I_2 \ 0] \| \leq \|U_2 - I_2\| = 2 \sin \frac{\varepsilon}{2} \leq \varepsilon \] \( \|10\) \( \| \)
and
\[ z(k) = U_\varepsilon(k)x(k, x_0) = U_\varepsilon(k)A(k-1) x(k-1, x_0) = U_\varepsilon(k)A(k-1) z(k-1) = A(k-1) z(k-1) + (U_\varepsilon(k) - I_s) A(k-1) z(k-1). \]
Denoting
\[ Q(k-1) = (U_\varepsilon(k) - I_s) A(k-1) \]
we obtain by inequality \( \|10\) \( \| \)
that
\[ \|Q(k-1)\| \leq \|U_\varepsilon(k) - I_s\| \leq \varepsilon a. \]

From this two lemmas we obtain the following important result.

Corollary 1. Suppose that \( x(n, x_0) \) is an \( \varepsilon \)-slow on the interval \([k, m]\) solution of the system \([7]\). There exists an uncertainties sequence \( Q = (Q(n))_{n \in \mathbb{N}_0} \), which is different from zero only at the point \( k-1 \), such that \( \|Q(k-1)\| \leq \varepsilon a \), and the perturbed system \([8]\) has \( \varepsilon \)-fast on the interval \([k, m]\) solution \( (z(n, z_0))_{n \in \mathbb{N}_0} \) satisfying conditions \( z(k-1, z_0) = x(k-1, x_0) \) and \( \|z(k, z_0)\| = \|x(k, x_0)\| \).

3. Main results. Let us introduce the following notation
\[ \Omega_0(A) = \lim_{T \to \infty} \liminf_{k \to m} \left( \prod_{i=m+1}^k \|\Phi_A(iT, (i-1)T)\| \right)^{\frac{1}{(k-m)^2}}, \] \( [11] \)
\[ \omega^0(A) = \lim_{T \to \infty} \limsup_{k \to m} \left( \prod_{i=m+1}^k \|\Phi_A^{-1}(iT, (i-1)T)\| \right)^{\frac{1}{(k-m)^2}}. \] \( [12] \)
In the proof of Theorem \([3.6]\) we will show that this definition is correct, i.e. that the limits defining \( \Omega_0(A) \) and \( \omega^0(A) \) exist.

The next theorem contains the first main result of our paper.
Theorem 3.1. The following relations hold

\[ \Omega^0(A) = \lim_{\varepsilon \to 0^+} \sup_{\|Q\|_\infty < \varepsilon} s\beta(A + Q), \]  
(13)

\[ \omega_0(A) = \lim_{\varepsilon \to 0^+} \inf_{\|Q\|_\infty < \varepsilon} i\beta(A + Q). \]  
(14)

Proof. As it was mentioned above, the inequalities (11) and (12) hold, so to show the equalities (13) and (14) it is sufficient to prove inequalities

\[ \omega_0(A) \geq \lim_{\varepsilon \to 0^+} \inf_{\|Q\|_\infty < \varepsilon} i\beta(A + Q), \]  
(15)

\[ \Omega^0(A) \leq \lim_{\varepsilon \to 0^+} \sup_{\|Q\|_\infty < \varepsilon} s\beta(A + Q). \]  
(16)

To show inequalities (15) and (16) we will construct uncertainties matrix \( Q_\varepsilon \) for every \( \varepsilon > 0 \) such that \( \|Q_\varepsilon\|_\infty < \varepsilon \), and for some solution \( z = (z(n, z_0))_{n \in \mathbb{N}_0} \) of the perturbed system

\[ z(n + 1) = (A(n) + Q_\varepsilon(n))z(n), \]  
(17)

the inequality \( \beta[z] \leq \omega_0(A) (\beta[z] \geq \Omega^0(A) \), respectively) holds. Such uncertainties matrix \( Q_\varepsilon \) will be constructed by applying the Millionschikov’s method of rotations. Let \( \delta = \varepsilon / a \). Firstly, let us prove the inequality (16). Let \( (z(n, n_i))_{i \in \mathbb{N}} \) be a double sequence of positive integers such that \( m_i \uparrow \infty \), \( n_i - m_i \uparrow \infty \) and

\[ \Omega^0(A) = \lim_{i \to \infty} \|\Phi_A(n_i, m_i)\|^{1/(n_i - m_i)}. \]  
(18)

Without loss of generality we assume that \( n_1 > 2 \) and \( n_i < m_{i+1}, i \in \mathbb{N} \).

Construction of the uncertainties matrix \( Q_\varepsilon \) and the solution \( z \) will be performed by induction on \( i \in \mathbb{N} \). In the \( i \)-step they will be defined on the interval \([m_i, m_{i+1}]\), but in the \((i + 1)\)-step the sequence \( Q_\varepsilon \) may be changed at the point \( m_{i+1} - 1 \), and solution \( z \) at the point \( m_{i+1} \) but during the following steps they will not be changed on this interval. For the convenience of the construction let us make the zero step, i.e. for \( n \in [0, m_1] \) put \( Q_\varepsilon(n) = O_s \) (\( O_s \) is a zero \( s \times s \) matrix) and \( z(n) = x(n) \), where \( x(n) \) is any non-zero solution of the system (1). Suppose that we have done \( i - 1 \) steps, i.e. the sequence \( Q_\varepsilon(n) \) and the solution \( z(n) \) are constructed for all \( n \in [0, m_i] \). If the solution

\[ x(n, m_i - 1, z(m_i - 1)) \]

of the unperturbed system (1) is \( \delta \)-fast on the interval \([m_i, n_i] \), then for \( n \in [m_i, m_{i+1}] \) we define \( Q_\varepsilon(n) = O_s \) and

\[ z(n) = x(n, m_i - 1, z(m_i - 1)). \]

In the opposite case, i.e. if the solution

\[ x(n, m_i - 1, z(m_i - 1)) \]

of the unperturbed system (1) is \( \delta \)-slow on the interval \([m_i, n_i] \), then by Corollary (1) there exists an uncertainties matrix \( Q_\varepsilon(m_i - 1) \) such that

\[ \|Q_\varepsilon(m_i - 1)\| \leq a\delta = \varepsilon \]

and the solution

\[ z(n, z_0) = z(n, m_i - 1, z(m_i - 1)) \]

of perturbed system is \( \delta \)-fast on the interval \([m_i, n_i] \). Despite the fact that the solution \( z(n, z_0) \) was changed at the point \( m_i \) during the \( i \)-step, according to the
assumption, it remains \( \delta \)-fast on the interval \([m_i-1, n_i-1]\). The desired system \([17]\) and its solution \( z \) are constructed.

According to the construction on each interval \([m_i, n_i]\) the solution \((z(n, z_0))_{n \in \mathbb{N}_0}\) of the perturbed system \([17]\) satisfies the following inequality
\[
\|z(m_i, z_0)\| \geq \frac{\sin \delta}{2} \| \Phi_A(n_i, m_i) \| \|z(n_i, z_0)\|,
\]
from which we get the desired estimate of the upper Bohl exponent of \( z \):
\[
\beta[z] \geq \limsup_{i \to \infty} \left( \frac{\|z(m_i, z_0)\|}{\|z(n_i, z_0)\|} \right)^{1/(n_i-1)} \geq \lim_{i \to \infty} \left( \frac{\sin \delta}{2} \right)^{1/(n_i-1)} \times \lim_{i \to \infty} \left( \| \Phi_A(n_i, m_i) \| \right)^{1/(n_i-1)} = \Omega^0(A).
\]
The inequality \([16]\) is proved.

Now let us prove the inequality \([15]\). We will construct uncertainties matrix \( Q_\varepsilon \) for every \( \varepsilon > 0 \) such that \( \|Q_\varepsilon\|_{\infty} < \varepsilon \), and for some solution \( z = (z(n, z_0))_{n \in \mathbb{N}_0} \) of the unperturbed system \([17]\) the inequality \( \beta[z] \leq \omega_0(A) \) holds. Let \((m_i, n_i)_{i \in \mathbb{N}}\) be a double sequence of positive integers such that \( m_i \uparrow \infty, n_i - m_i \uparrow \infty \) and
\[
\omega_0(A) = \lim_{i \to \infty} \| \Phi_A^{-1}(n_i, m_i) \|^{-1/(n_i-1)}.
\]
Without loss of generality we assume that \( m_i - n_{i-1} > 1, i \in \mathbb{N} \).

Let \( \delta = \varepsilon/(2a) \). First, for each \( i \in \mathbb{N} \) we will construct on the interval \([0, n_i]\) the perturbed system
\[
z(n + 1) = (A(n) + Q_i(n))z(n), \quad z(n) \in \mathbb{R}^x, \quad n \in [0, n_i],
\]
where the uncertainties matrix \( Q_i \) satisfies the inequality \( \|Q_i(n)\| < \varepsilon \) for \( n \in [0, n_i] \), and there exists a solution \((z_i(n))_{n \in \mathbb{N}_0}\) satisfying the inequality
\[
\|z_i(m_q)\| \geq \frac{1}{2} \sin \delta \| \Phi(m_q, n_q) \| \|z(n_q)\|, \quad q = 1, \ldots, i.
\]
The perturbed system \([20]\) will be constructed by applying the Millionschikov’s method of rotations for index from \( n_i \) to 0. Indeed, for \( n \in [n_{i-1} + 2, n_i] \) we define \( Q_i(n) = O_i \), and for solution \( z(n) \) we take any \( \delta \)-fast solution on the interval \([n_i, m_i]\) of the unperturbed system \([1]\), i.e. the solution for which the inequality \([21]\) holds for \( q = i \) (we emphasize that \( n_i \) is a start point and \( m_i \) is a finish point of the \( \delta \)-fast solution on the interval \([n_i, m_i]\)). Further, according to Corollary \([1]\) perturbing (if necessary) the system \([1]\) at the point \( n_{i-1} + 1 \) by matrix \( Q_i^2(n_{i-1} + 1) \) such that
\[
\|Q_i^2(n_{i-1} + 1)\| < a\delta = \varepsilon/2,
\]
we construct a desired system \([20]\) and the solution \( z_i \) for which the inequality \([21]\) holds.

Let us now consider the set \( \{z_i : i \in \mathbb{N}\} \) of constructed solutions and the set \( \{Q_i^2 : i \in \mathbb{N}\} \) of constructed uncertainties matrices. Let us choose a subsequence \((i_k^1)_{k \in \mathbb{N}}\) such that the sequence \((z_{i_k^1}(n_{i_k^1} + 1))_{k \in \mathbb{N}}\) of vectors is convergent. From the sequence \((i_k^1)_{k \in \mathbb{N}}\) we choose a subsequence \((i_k^2)_{k \in \mathbb{N}}\) such that the sequence
\[
\left( Q_{i_k^2}^2(n_{i_k^2} + 1) \right)_{k \in \mathbb{N}}
\]
of matrices is convergent. From the sequence \((i_k^2)_{k \in \mathbb{N}}\) we choose a subsequence \((i_k^3)_{k \in \mathbb{N}}\) such that the sequence
\[
(y_{i_k^3}(n_{i_k^3} + 1))_{k \in \mathbb{N}}
\]
of vectors is convergent, and from the sequence \((i_k^2)_{k \in \mathbb{N}}\) we choose a subsequence \((i_k)_{k \in \mathbb{N}}\) such that the sequence
\[
\left( Q_{\epsilon}^2(n_2 + 1) \right)_{k \in \mathbb{N}}
\]
of matrices is convergent, and so on for all \(n_i + 1, i \in \mathbb{N}\). Let \(z\) and \(Q_{\epsilon}\) be a limit solution and a limit uncertainties matrix, respectively. So, we construct the system (17), the uncertainties matrix \(Q_{\epsilon}\), satisfying the inequality
\[
Q_{\epsilon}(n) \leq \epsilon/2 < \epsilon
\]
for any \(n \in \mathbb{N}\), and the solution \(z\), satisfying the inequality
\[
\|z(m_i)\| \geq \frac{1}{2} \sin \delta \|\Phi(m_i, n_i)\| \|z(n_i)\|, \quad i \in \mathbb{N}.
\] (22)

From (22) it follows that
\[
\frac{\|z(n_i)\|}{\|z(m_i)\|} \leq \left( \frac{\sin \delta}{2} \right)^{-1} \|\Phi(m_i, n_i)\|^{-1}.
\]
Therefore
\[
\frac{\beta[z]}{2} \leq \lim_{i \to \infty} \left( \frac{\|z(n_i)\|}{\|z(m_i)\|} \right)^{1/(n_i - m_i)} \leq \lim_{i \to \infty} \|\Phi(m_i, n_i)\|^{-1/(n_i - m_i)} = \lim_{i \to \infty} \|\Phi^{-1}(n_i, m_i)\|^{-1/(n_i - m_i)} = \omega_0(A).
\]
The inequality (15) is proved.

The next lemma shows that in the definition of the Bohl exponents can be assumed that the numbers \(n\) and \(m\) are the elements of any arithmetic sequence.

**Lemma 3.2.** For any \(T \in \mathbb{N}, T \geq 1\) and \(x_0 \in \mathbb{R}^\ast\) we have
\[
\underline{\beta}_A(x_0) = \liminf_{n-m \to \infty} \left( \frac{\|x(Tn, x_0)\|}{\|x(Tm, x_0)\|} \right)^{1/(T(n-m))},
\] (23)
and
\[
\overline{\beta}_A(x_0) = \limsup_{n-m \to \infty} \left( \frac{\|x(Tn, x_0)\|}{\|x(Tm, x_0)\|} \right)^{1/(T(n-m))}.\] (24)

**Proof.** We will show only the equality (23). The proof of (24) is analogical. From the definition of lower limit it follows that
\[
\underline{\beta}_A(x_0) \leq \liminf_{n-m \to \infty} \left( \frac{\|x(Tn, x_0)\|}{\|x(Tm, x_0)\|} \right)^{1/(T(n-m))}.
\] (25)
Let us fix \(T \in \mathbb{N}\) and denote by \((n_k)_{k \in \mathbb{N}_0}\) and \((m_k)_{k \in \mathbb{N}_0}\) such sequences of natural numbers that
\[
\underline{\beta}_A(x_0) = \lim_{k \to \infty} \left( \frac{\|x(n_k, x_0)\|}{\|x(m_k, x_0)\|} \right)^{1/(n_k - m_k)}
\]
and
\[
\lim_{k \to \infty} (n_k - m_k) = \infty.
\]
For \(k \in \mathbb{N}_0\) denote by \(p_k\) and \(q_k\) such natural numbers that
\[
Tp_k \leq n_k < T(p_k + 1)
\]
and

\[ T q_k \leq m_k < T (q_k + 1). \]

We have

\[
\beta_{-A}(x_0) = \lim_{k \to \infty} \left( \frac{\|x(n_k, x_0)\|}{\|x(m_k, x_0)\|} \right)^{1/(n_k - m_k)} = \lim_{k \to \infty} \left( \frac{\Phi_A(n_k, T p_k) x(T p_k, x_0)}{\Phi_A(m_k, T q_k) x(T q_k, x_0)} \right)^{1/(n_k - m_k)} \geq \liminf_{k \to \infty} \left( \frac{\Phi_A(n_k, T p_k) x(T p_k, x_0)}{\Phi_A(m_k, T q_k) x(T q_k, x_0)} \right)^{1/(n_k - m_k)} \]

\[
\liminf_{k \to \infty} \left( \frac{\|\Phi_A^{-1}(n_k, T p_k)\| \|\Phi_A(n_k, T p_k) x(T p_k, x_0)\|}{\|\Phi_A^{-1}(n_k, T p_k)\| \|\Phi_A(n_k, T p_k) x(T p_k, x_0)\|} \right)^{1/(n_k - m_k)} \geq \liminf_{k \to \infty} \left( \frac{\|\Phi_A^{-1}(n_k, T p_k)\| \|\Phi_A(n_k, T p_k) x(T p_k, x_0)\|}{\|\Phi_A^{-1}(n_k, T p_k)\| \|\Phi_A(n_k, T p_k) x(T p_k, x_0)\|} \right)^{1/(n_k - m_k)} = \]

\[
\liminf_{k \to \infty} \left( \frac{\|x(T p_k, x_0)\|}{\|\Phi_A^{-1}(n_k, T p_k)\| \|\Phi_A(n_k, T p_k) x(T p_k, x_0)\|} \right)^{1/(n_k - m_k)} \geq \liminf_{k \to \infty} \left( \frac{\|x(T p_k, x_0)\|}{\|\Phi_A^{-1}(n_k, T p_k)\| \|\Phi_A(n_k, T p_k) x(T p_k, x_0)\|} \right)^{1/(n_k - m_k)} \geq \]

\[
\liminf_{k \to \infty} \left( \frac{1}{\|\Phi_A^{-1}(n_k, T p_k)\| \|\Phi_A(n_k, T p_k) x(T p_k, x_0)\|} \right)^{1/(n_k - m_k)} \times \]

\[
\left( \frac{\|x(T p_k, x_0)\|}{\|x(T p_k, x_0)\|} \right)^{1/(T p_k - T q_k)} \left( \frac{\|x(T p_k, x_0)\|}{\|x(T p_k, x_0)\|} \right)^{(T p_k - T q_k)/(n_k - m_k)} \right) . \]  

\[
(26) \]

From the assumptions about boundedness of \((A(n))_{n \in \mathbb{N}_0}\) and \((A^{-1}(n))_{n \in \mathbb{N}_0}\) it follows that

\[
\lim_{k \to \infty} \left( \frac{1}{\|\Phi_A^{-1}(n_k, T p_k)\| \|\Phi_A(n_k, T q_k)\|} \right)^{1/(n_k - m_k)} = 1,
\]

and by the definition of \((p_k)_{k \in \mathbb{N}_0}\) and \((q_k)_{k \in \mathbb{N}_0}\) we have that

\[
\lim_{k \to \infty} \frac{T p_k - T q_k}{n_k - m_k} = 1.
\]

The last two equalities and (26) together with the definition of lower limit imply the following inequalities

\[
\beta_{-A}(x_0) \geq \liminf_{k \to \infty} \left( \frac{\|x(T p_k, x_0)\|}{\|x(T q_k, x_0)\|} \right)^{1/(T p_k - T q_k)} \geq \liminf_{n \to \infty} \liminf_{m \to \infty} \left( \frac{\|x(T n, x_0)\|}{\|x(T m, x_0)\|} \right)^{1/(T(n - m))} .
\]

Now the inequality (23) follows from (25) and (27).

In the proof of our further theorem we will use the following discrete version of Gronwall’s inequality [1].
Lemma 3.3. Suppose that for two sequences \( u(n) \) and \( f(n) \), \( n = m, m + 1, \ldots \) of nonnegative numbers the following inequality
\[
u(n) \leq p + q \sum_{i=m}^{n-1} u(i)f(i)
\]
holds for certain \( p, q \in \mathbb{R} \) and all \( n = m, m + 1, \ldots \), then
\[
u(n) \leq p \prod_{i=m}^{n-1} (1 + qf(i))
\]
for all \( n = m, m + 1, \ldots \).

Let us introduce the following concepts.

Definition 3.4. Bounded sequences \( R = (R(n))_{n \in \mathbb{N}_0} \) and \( r = (r(n))_{n \in \mathbb{N}_0} \) are called, respectively, the upper and the lower sequence of the system \( \square \) if and only if there exist constants \( C_R \) and \( c_r \) such that
\[
\|\Phi_A(n,k)\| \leq C_R \prod_{i=k}^{n-1} R(i),
\]
\[
\|\Phi_A^{-1}(n,k)\| \leq c_r \left( \prod_{i=k}^{n-1} r(i) \right)^{-1}
\]
for all \( k, n \in \mathbb{N}_0, n > k \).

The next lemma provides an example of upper and lower sequences.

Lemma 3.5. Let us fix \( T \in \mathbb{N} \) and denote
\[
R(n) = \|\Phi_A((i + 1)T, iT)\|^{1/T} \quad \text{for} \quad n \in [iT, (i + 1)T)
\]
and
\[
r(n) = \|\Phi_A^{-1}((i + 1)T, iT)\|^{1/T} \quad \text{for} \quad n \in [iT, (i + 1)T).
\]
The sequence \( R = (R(n))_{n \in \mathbb{N}_0} \) is an upper and \( r = (r(n))_{n \in \mathbb{N}_0} \) is a lower sequence for the system \( \square \).

Proof. Consider \( k, n \in \mathbb{N}_0, k \leq n \) and suppose that \( k \in [iT, (i + 1)T), n \in [jT, (j + 1)T), i, j \in \mathbb{N}_0, k = iT + r_i \) and \( n = jT + r_j \). We have
\[
\|\Phi_A(n,k)\| = \|A(jT + r_j) \ldots A((j + 1)T - 1) A((j + 1)T - 1) \ldots A(jT + r_j) \times A(jT + r_j - 1) \ldots A(iT + r_i) A(iT + r_i - 1) \ldots A(iT) \times A^{-1}(iT) \ldots A^{-1}(iT + r_i - 1)\| \leq
\]
\[
\|A^{-1}(jT + r_j) \ldots A^{-1}((j + 1)T - 1)\| \|A((j + 1)T - 1) \ldots A(jT)\| \times
\]
\[
\|A((j - 1)T) \ldots A((i + 1)T - 1) A(iT)\| \times
\]
\[
\|A^{-1}(iT) \ldots A^{-1}(iT + r_i - 1)\| \leq
\]
\[
a^{T-r_j} \|A((j + 1)T - 1) \ldots A(jT)\| \times
\]
\[
\|A((jT - 1) \ldots A((j - 1)T)\| \|A(iT - 1) \ldots A((i + 1)T)\| \times
\]
\[
\|A((i + 1)T - 1) \ldots A(iT)\| a^{r_i} =
\]
\[
a^{T-r_j + r_i} \|A((j + 1)T - 1) \ldots A(jT)\|^{(T-1)/T} \|A((j + 1)T - 1) \ldots A(jT)\|^{1/T} \times
\]
\[
\|A((jT - 1) \ldots A((j - 1)T)\| \|A(iT - 1) \ldots A((i + 1)T)\| \times
\]
\[
\|A((i + 1)T - 1) \ldots A(iT)\| a^{r_i} =
\]
\[ \|A((i + 1) T - 1) ... A(iT)\|^{1/T} \|A((i + 1) T - 1) ... A(iT)\|^{(T-1)/T} = a^{T-r_j+r_i} \|A((j + 1) T - 1) ... A(jT)\|^{(T-1)/T} \times \]
\[ \prod_{i=k}^{n-1} R(i) \|A((i + 1) T - 1) ... A(iT)\|^{(T-1)/T} \leq \]
\[ a^{T-r_j+r_i} a^{T-1} \prod_{i=k}^{n-1} R(i) a^{T-1} = a^{3T-r_j+r_i-2} \prod_{i=k}^{n-1} R(i) \leq a^{4T-3} \prod_{i=k}^{n-1} R(i). \]

It means that \( R \) is an upper sequence. The proof of the fact that \( r \) is a lower sequence for the system \( [1] \) is analogical.

The next theorem contains the second main result of our paper.

**Theorem 3.6.** The following relations hold

\[ \Omega_0(A) = \lim_{\varepsilon \to 0^+} \sup_{\|Q\|_\infty < \varepsilon} \frac{s\beta(A + Q)}{\|Q\|_\infty}, \quad \text{(30)} \]
\[ \omega^0(A) = \lim_{\varepsilon \to 0^+} \inf_{\|Q\|_\infty < \varepsilon} \frac{i\beta(A + Q)}{\|Q\|_\infty}. \quad \text{(31)} \]

**Proof.** Let us denote

\[ \Omega_0(A) = \lim_{T \to \infty} \sup_{k-m \to \infty} \frac{1}{T} \inf_{T} \left( \prod_{i=m}^{k-1} \|\Phi_A((i + 1) T, i\ell T)\| \right)^{\frac{1}{T}}, \]
\[ \Omega_0(A) = \lim_{T \to \infty} \inf_{k-m \to \infty} \frac{1}{T} \sup_{T} \left( \prod_{i=m}^{k-1} \|\Phi_A((i + 1) T, i\ell T)\| \right)^{\frac{1}{T}}, \]
\[ \omega^0(A) = \lim_{T \to \infty} \sup_{k-m \to \infty} \frac{1}{T} \inf_{T} \left( \prod_{i=m}^{k-1} \|\Phi_A^{-1}((i + 1) T, i\ell T)\| \right)^{\frac{1}{T}}. \]

and
\[ \varpi^0(A) = \lim_{T \to \infty} \inf_{k-m \to \infty} \frac{1}{T} \sup_{T} \left( \prod_{i=m}^{k-1} \|\Phi_A^{-1}((i + 1) T, i\ell T)\| \right)^{\frac{1}{T}}. \]

First, we will prove that

\[ \lim_{\varepsilon \to 0^+} \sup_{\|Q\|_\infty < \varepsilon} s\beta(A + Q) \leq \Omega_0(A) \quad \text{(32)} \]

and

\[ \lim_{\varepsilon \to 0^+} \inf_{\|Q\|_\infty < \varepsilon} i\beta(A + Q) \geq \varpi^0(A). \quad \text{(33)} \]

For the proof of inequality \( [2] \) let us consider an upper sequence \( R = (R(n))_{n \in \mathbb{N}_0} \).

According to the Cauchy formula any solution \( (z(n, z_0))_{n \in \mathbb{N}_0} \) of the system \( [6] \) satisfies the following equality

\[ z(n, z_0) = \Phi_A(n, k)z(k, z_0) + \sum_{i=k}^{n-1} \Phi_A(n, i + 1)Q(i)z(i, z_0) \]

for all \( n, k \in \mathbb{N}_0, n > k \). Taking the norm and assuming that \( \|Q\|_\infty < \varepsilon \) we get

\[ \|z(n, z_0)\| \leq C_R \|z(k, z_0)\| \prod_{i=k}^{n-1} R(i) + C_R \varepsilon \sum_{i=k}^{n-1} \|z(i, z_0)\| \prod_{j=i+1}^{n-1} R(j) \]
and
\[
\|z(n, z_0)\| \left( \prod_{i=0}^{n-1} R(i) \right)^{-1} \leq 
\]
\[
C_R \|z(k, z_0)\| \left( \prod_{i=0}^{k-1} R(i) \right)^{-1} + C_R \varepsilon \sum_{i=k}^{n-1} \|z(i, z_0)\| \left( \prod_{j=0}^{i-1} R(j) \right)^{-1} R^{-1}(i) = 
\]
\[
C_R \|z(k, z_0)\| \left( \prod_{i=0}^{k-1} R(i) \right)^{-1} + C_R \varepsilon \sum_{i=k}^{n-1} \|z(i, z_0)\| \left( \prod_{j=0}^{i-1} R(j) \right)^{-1} R^{-1}(i) \tag{34}
\]
for all \(n, k \in \mathbb{N}_0\), \(n > k\). By the definition of \(a\) we have that
\[
\frac{1}{a} \leq \|A(n)\|
\]
for all \(n \in \mathbb{N}_0\), and by the definition of upper sequence we know that
\[
\|A(n)\| \leq C_R R(n).
\]
Therefore
\[
R^{-1}(n) \leq aC_R.
\]
Taking into account the last inequality and denoting
\[
u(i) = \|z(i, z_0)\| \left( \prod_{j=0}^{i-1} R(j) \right)^{-1}
\]
we may rewrite the inequality (34) as follows
\[
u(n) \leq C_R \nu(k) + aC_R^2 \varepsilon \sum_{i=k}^{n-1} u(i).
\]
Applying Gronwall’s inequality we get the following bound
\[
u(n) \leq C_R \nu(k) \left(1 + aC_R^2 \varepsilon \right)^{n-k}
\]
and
\[
\frac{\|z(n, z_0)\|}{\|z(k, z_0)\|} \leq C_R \left(1 + aC_R^2 \varepsilon \right)^{n-k} \prod_{i=k}^{n-1} R(i).
\]
Taking the upper limit when \(n-k \to \infty\) we obtain that any solution of the system (6) with uncertainties \(Q\) such that \(\|Q\|_\infty \leq \varepsilon\) satisfies the following inequality
\[
\beta_A(z_0) \leq \left(1 + C_R \varepsilon\right) \liminf_{k-m \to \infty} \left( \prod_{i=m}^{n-1} R(i) \right)^{\frac{1}{n-k}}
\]
for any upper sequence \(R = (R(n))_{n \in \mathbb{N}_0}\). Taking in the last inequality the upper sequence from Lemma 3.5 we obtain that
\[
\sup_{\|Q\|_\infty \leq \varepsilon} s\beta(A + Q) \leq (1 + C_R \varepsilon) \liminf_{k-m \to \infty} \left( \prod_{i=m}^{k-1} \|\Phi_A((i + 1) T, iT)\| \right)^{\frac{1}{k-m}}
\]
and
\[
\lim_{\varepsilon \to 0^+} \sup_{\|Q\|_\infty \leq \varepsilon} s\beta(A + Q) \leq \liminf_{k-m \to \infty} \left( \prod_{i=m}^{k-1} \|\Phi_A((i + 1) T, iT)\| \right)^{\frac{1}{k-m}}
\].
Taking the upper limit when $T \to \infty$ we get the inequality (32).

Consider now a lower sequence $r = (r(n))_{n \in \mathbb{N}_0}$. According to the Cauchy formula any solution $(z(n, z_0))_{n \in \mathbb{N}_0}$ of the system (6) satisfies the following equality

$$z(n, z_0) = \Phi_A(n, m)z(m, z_0) - \sum_{i=m-1}^{n} \Phi_A(n, i + 1)Q(i)z(i, z_0)$$

for all $n, m \in \mathbb{N}_0 , n < m$ and therefore

$$\|z(n, z_0)\| \leq \|\Phi_A(n, m)\|\|z(m, z_0)\| + \sum_{i=m-1}^{n} \|\Phi_A(n, i + 1)\|\|Q(i)\|\|z(i, z_0)\| =$$

$$\|\Phi^{-1}_A(m, n)\|\|z(m, z_0)\| + \sum_{i=m-1}^{n} \|\Phi^{-1}_A(i + 1, n)\|\|Q(i)\|\|z(i, z_0)\| =$$

for all $n, m \in \mathbb{N}_0 , n < m$. Suppose that $\|Q\|_{\infty} < \varepsilon$, then by the definition of lower sequence we have that

$$\|z(n, z_0)\| \leq c_r \|z(m, z_0)\| \left( \prod_{i=n}^{m-1} r(i) \right)^{-1} + c_r \varepsilon \sum_{i=n}^{m-1} \|z(i, z_0)\| \left( \prod_{j=n}^{i} r(j) \right)^{-1}$$

and

$$\|z(n, z_0)\| \left( \prod_{i=0}^{n-1} r(i) \right)^{-1} \leq$$

$$c_r \|z(m, z_0)\| \left( \prod_{i=0}^{m-1} r(i) \right)^{-1} + c_r \varepsilon \sum_{i=n}^{m-1} \|z(i, z_0)\| \left( \prod_{j=0}^{i-1} r(j) \right)^{-1} \left( r^{-1}(i) \right).$$

Denoting

$$u(m) = \|z(m, z_0)\| \left( \prod_{i=0}^{m-1} r(i) \right)^{-1}$$

and taking into account that $r^{-1}(i) \leq c_r a$ we obtain the following inequality

$$u(n) \leq c_r u(m) + c_r^2 \varepsilon a \sum_{i=n}^{m-1} u(i)$$

for all $n, m \in \mathbb{N}_0 , n < m$. Applying the same way of reasoning as in the proof of the Gronwall’s inequality $\mathbb{II}$ we get

$$u(n) \leq C_R u(m) \left( 1 + a C_R^2 \varepsilon \right)^{n-m}$$

for all $n, m \in \mathbb{N}_0 , n < m$. Repeating the arguments from the first part of the proof we obtain that

$$\lim_{\varepsilon \to 0^+} \inf_{\|Q\|_{\infty} < \varepsilon} s \beta(A + Q) \geq \limsup_{k \to -\infty} \left( \prod_{i=m}^{k-1} \Phi^{-1}_A(i + 1, T, iT) \right)^{\frac{-1}{|i - m|^r}}.$$
In the next part of the proof we will show that
\[\lim_{\varepsilon \to 0^+} \sup_{\|Q\| < \varepsilon} s\beta(A + Q) \geq \overline{\Omega}_0(A)\] (35)
and
\[\lim_{\varepsilon \to 0^+} \inf_{\|Q\| < \varepsilon} i\beta(A + Q) \leq \omega^0(A)\]. (36)

Let us fix \(\varepsilon > 0\), \(\delta \leq \varepsilon/a\) and consider \(T \in \mathbb{N}\) such that
\[
\lim \inf_{k-m \to \infty} \left( \prod_{i=m}^{k-1} \|\Phi_A((i + 1) T, iT)\| \right)^{1/(k-m)} + \frac{\varepsilon}{2} \geq \overline{\Omega}_0(A) \qquad (37)
\]
and
\[
\left( \frac{\sin \delta}{2} \right)^{1/T} \left( \overline{\Omega}_0(A) - \frac{\varepsilon}{2} \right) \geq \overline{\Omega}_0(A) - \varepsilon. \qquad (38)
\]

Applying the Millionschikov’s method of rotations we will construct uncertainties \(Q\) such that
\[
\|Q\|_\infty < \varepsilon \quad (39)
\]
and such that there exists a solution \(z = (z(n, z_0))_{n \in \mathbb{N}_0}\) of the uncertain system \([6]\)
satisfying the following inequality
\[
\|z((i + 1) T, z_0)\| \geq \frac{\sin \delta}{2} \|\Phi_A((i + 1) T, iT)\| \|z((i T, z_0)\| \quad (40)
\]
for all \(i \in \mathbb{N}_0\). Construction of the uncertainties matrix \(Q\) and the solution \(z\) will be performed by induction on \(i \in \mathbb{N}_0\). In the \(i\)-step they will be defined on the interval \([iT, (i + 1) T]\), but in the \((i + 1)\)-step the sequence \(Q\) may be changed at the point \((i + 1) T - 1\), and solution \(z\) at the point \((i + 1) T\) but during the following steps they will not be changed.

Set \(Q(0) = O_s\) and \(z(0) = z_0\) is any nonzero vector from \(\mathbb{R}^n\). Suppose that we have done the \((i - 1)\)-step. It means that we have defined \(Q(n)\) and \(z(n, z_0)\) for \(n = 0, 1, ..., i N\). Now we define the \(i\)-step. If the solution \(x = (x(n, x_0))_{n \in \mathbb{N}_0}\) of the unperturbed system \([1]\), satisfying the condition \(x(i T, x_0) = z(i T, x_0)\), is \(\delta\)-fast on the interval \([iT, (i + 1) T]\), then we define \(Q(n) = O_s\) and \(z(n, z_0) = x(n, x_0)\) for all \(n = iT + 1, ..., (i + 1) T\). In the opposite case, i.e. if the solution \(x\) is \(\delta\)-slow on the interval \([iT, (i + 1) T]\), then by Corollary \([4]\) there exists an uncertainties matrix \(Q((i + 1) T - 1)\) such that
\[
\|Q((i + 1) T - 1)\| \leq a \delta \leq \varepsilon
\]
and the solution \((z(n, z_0))_{n \in \mathbb{N}_0}\) of the perturbed system continued on the interval \([iT, (i + 1) T]\) is \(\delta\)-fast on the interval \([iT, (i + 1) T]\). Moreover, according to Corollary \([3]\) the solution \((z(n, z_0))_{n \in \mathbb{N}_0}\) defined in the \((i - 1)\)-step and changed in the \(i\)-step remains \(\delta\)-fast on the interval \([i T, (i + 1) T]\). Therefore the constructed uncertainties sequence \(Q\) and the solution \(z\) satisfy the inequality \((39)\) and \((40)\), respectively.

From the inequality \((40)\) we get
\[
\frac{\|z(kT, z_0)\|}{\|z(mT, z_0)\|} \geq \left( \frac{\sin \delta}{2} \right)^{k-m} \prod_{i=m}^{k-1} \|\Phi_A((i + 1) T, iT)\|
\]
and therefore by Lemma 3.2 and the inequality (37)

\[ \beta(z_0) = \liminf_{n-m \to \infty} \left( \frac{\|z(Tn, z_0)\|}{\|z(Tm, z_0)\|} \right)^{1/T(n-m)} \geq \]

\[ \left( \frac{\sin \delta}{2} \right)^{1/T} \liminf_{n-m \to \infty} \left( \prod_{i=m}^{k-1} \|\Phi_A((i+1)T, iT)\| \right)^{1/T(n-m)} \geq \]

\[ \left( \frac{\sin \delta}{2} \right)^{1/T} \left( \Omega_0(A) - \frac{\varepsilon}{2} \right) \geq \Omega_0(A) - \varepsilon. \]

From the last inequality and choice of \( \varepsilon > 0 \) the inequality (35) follows. From the inequalities (32) and (35) we have that

\[ \Omega_0(A) \geq \Omega_0(A) \]

The last inequality implies that there exists the limit defining \( \Omega_0(A) \) in (11) and it ends the proof of equality (30).

Now we will show the inequality (36). Let us fix \( \varepsilon > 0, \delta \leq \varepsilon/a \) and consider \( T \in \mathbb{N} \) such that

\[ \limsup_{k-m \to \infty} \left( \prod_{i=m}^{k-1} \|\Phi_A^{-1}((i+1)T, iT)\| \right)^{-1/(m+1)} \leq \frac{\varepsilon}{2} \leq \omega_0(A) \] (41)

and

\[ \left( \frac{\sin \delta}{2} \right)^{-1/T} \leq \frac{\omega_0(A) + \varepsilon}{\omega_0(A) + \frac{\varepsilon}{2}}. \] (42)

Applying the Millionschikov’s method of rotations we may construct, as above, a sequence \( Q \) such that

\[ \|Q\|_\infty < \varepsilon \] (43)

and a solution \( z = (z(n, z_0))_{n \in \mathbb{N}_0} \) of the uncertain system such that

\[ \|z((i+1)T, z_0)\| \geq \frac{\sin \delta}{2} \||\Phi_A^{-1}((i+1)T, iT)|| \|z((i+1)T, z_0)\| \] (44)

for all \( i \in \mathbb{N}_0 \). Indeed, let us fix \( k \geq 1 \). Repeating the arguments from Section 2 for indexes from \((k+1)T\) to 0 we can construct a finite sequence \( Q_k(i), 0 \leq i < (k+1)T \), and a solution \( z_k(1 \leq \|z_k(0)\| \leq 2) \) of the system

\[ z(i+1) = (A(i) + Q_k(i))z(i), \quad 0 \leq i < (k+1)T, \]

such that \( \|Q_k(i)\| \leq \varepsilon/2 \) and

\[ \|z_k((i+1)T)\| \geq \frac{\sin \delta}{2} \||\Phi_A^{-1}((i+1)T, iT)|| \|z_k((i+1)T)\|, \quad i = k, \ldots, 1. \]

Hence, there exist sequences \( (Q(n))_{n \in \mathbb{N}_0}, \|Q\|_\infty \leq \varepsilon/2 < \varepsilon \), and \( (k_j)_{j \in \mathbb{N}_0} \uparrow \infty \), and a nonzero solution \( z \) of the system

\[ z(n+1) = (A(n) + Q(n))z(n), \quad n \in \mathbb{N}_0, \] (45)

such that \( Q_{k_j} \to Q(n) \) and \( z_{k_j} \to z(n), n \in \mathbb{N}_0 \). Therefore, for solution \( z \) of the system (45) the inequality (44) holds.
Fixing $m < k$, $m, k \in \mathbb{N}_0$ and applying the last inequality for $i = m, \ldots, k - 1$ we obtain that
\[
\| z(kT, z_0) \| \leq \left( \frac{\sin \delta}{2} \right)^{k-m} \prod_{i=m}^{k-1} \| \Phi^{-1}_A((i+1)T, iT) \|^i.
\]
By the inequality (24) and from Lemma [3.2] we get
\[
\bar{\beta}(z_0) = \limsup_{n-m \to \infty} \left( \frac{\| z(Tn, z_0) \|}{\| z(Tm, z_0) \|} \right)^{1/(T(n-m))} \leq \left( \frac{\sin \delta}{2} \right)^{-1/T} \left( \prod_{i=m}^{k-1} \| \Phi^{-1}_A((i+1)T, iT) \| \right)^{-1/(T(n-m))} \leq \omega^0(A) + \varepsilon.
\]
From the last inequality and the choice of $\varepsilon > 0$ the inequality (36) follows. From the inequalities (33) and (36) we get
\[
\omega^0(A) \geq \omega^0(A).
\]
The last inequality implies that there exists the limit $\omega^0(A)$ in (12) defining $\Omega_0(A)$ and it ends the proof of equality (31). The proof of the theorem is completed.

4. Conclusions. In the paper it has been studied the influence of parametric uncertainties on Bohl exponents of the discrete linear time-varying systems. The main result of the paper is to give the exact values of the upper and lower limits of the movability of the supremum of the lower and upper Bohl exponents and upper and lower limits of the infimum of the upper and lower Bohl exponents under arbitrary small uncertainties of the system coefficients. These values are expressed using the transition matrix of the system. The results were obtained by modifying the Millionschikov’s method of rotations proposed originally for the continuous-time systems.

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