On the Stanley depth of edge ideals of k–partite clutters

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Abstract
We give upper bounds for the Stanley depth of edge ideals of certain k–partite clutters. In particular, we generalize a result of Ishaq about the Stanley depth of the edge ideal of a complete bipartite graph. A result of Pournaki, Seyed Fakhari and Yassemi implies that the Stanley’s conjecture holds for d-uniform complete d-partite clutters. Here we give a shorter and different proof of this fact.

1 Introduction
Let \( R = K[x_1, \ldots, x_n] \) be a polynomial ring over a field \( K \). A clutter \( C \), with finite vertex set \( V = \{x_1, \ldots, x_n\} \) is a family of subsets of \( V \), called edges, none of which is included in another. The set of vertices and edges of \( C \) are denoted by \( V(C) \) and \( E(C) \) respectively. For example, a simple graph (no multiple edges or loops) is a clutter. The edge ideal of \( C \), denoted by \( I(C) \), is the ideal of \( R \) generated by all monomials \( x_e = \prod_{x_i \in e} x_i \) such that \( e \in E(C) \). The map

\[ C \mapsto I(C) \]

gives a one to one correspondence between the family of clutters and the family of squarefree monomial ideals. Edge ideals of graphs were introduced and studied in [18, 21]. Edge ideals of clutters correspond to simplicial complexes via the Stanley-Reisner correspondence [20] and to facet ideals [8, 23].

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A \( k \)-partite clutter is a clutter \( C \) where the vertices are partitioned into \( k \) subsets \( V(C) = V_1 \cup V_2 \cup \cdots \cup V_k \) with the following conditions:

1. No two vertices in the same subset are adjacent, i.e., \( |V_i \cap E| \leq 1 \) for all \( 1 \leq i \leq k \) and \( E \in E(C) \).
2. There is no partition of the vertices with fewer than \( k \) subsets where condition (1) holds.

A clutter is called \( d \)-uniform or uniform if all its edges have exactly \( d \) vertices.

Along the paper we introduce most of the notions that are relevant for our purposes. Our main references for combinatorial optimization and commutative algebra are [5, 6, 22].

Let \( M \) be a finitely generated \( \mathbb{Z}^n \)-graded \( R \)-module, \( R = K[x_1, \ldots, x_n] \). If \( u \in M \) is a homogeneous element in \( M \) and \( Z \subseteq \{x_1, \ldots, x_n\} \) then let \( uK[Z] \subset M \) denote the linear \( K \)-subspace of \( M \) of all elements \( uf, f \in K[Z] \). This space is called a Stanley space of dimension \( |Z| \) if \( uK[Z] \) is a free \( K[Z] \)-module. A presentation of \( M \) as a finite direct sum of Stanley spaces

\[
\mathcal{D} : M = \bigoplus_{i=1}^{r} u_iK[Z_i] 
\]

is called a Stanley decomposition of \( M \). The number

\[
s\text{depth}(\mathcal{D}) = \min\{|Z_i| : i = 1, \ldots, r\}
\]

is called the Stanley depth of decomposition \( \mathcal{D} \) and the number

\[
s\text{depth}(M) := \max\{s\text{depth}(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } M\} \leq n.
\]

is called the Stanley depth of \( M \). This is a combinatorial invariant which does not depend on the characteristic of \( K \).

In 1982, [19], Stanley introduced the idea of what is now called the Stanley depth of a \( \mathbb{Z}^n \)-graded module over a commutative ring and conjectured that \( s\text{depth}(M) \geq \text{depth}(M) \). While some special cases of the conjecture have been resolved, it still remains largely open, (for example see [12, 2, 11, 10, 14, 12, 13, 17]). Shen’s proof (see from [17, Lema 2.3, Theorem 2.4]) relies on a theorem of Cimpoeaş, [4, Theorem 2.1], which states that the Stanley depth of a complete intersection monomial ideal is equal to that of its radical, which allows for a focus on squarefree ideals. In [10, Theorem 2.8] Ishaq showed that the Stanley depth of the edge ideal of a complete bipartite graph over \( n \) vertices with \( n \geq 4 \) is less than or equal to \( \frac{n+2}{2} \). In [11] Ishaq and Qureshi, provide an upper bound for the Stanley depth of an edge ideal of a \( k \)-uniform complete bipartite hypergraph which is a kind of generalization to the complete bipartite graph.

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The aim of this paper is to bound the Stanley depth of the edge ideal of a $d$–uniform complete $k$–partite clutter [Theorems 3.3, 3.5, 3.9]. The proofs use the correspondence between a Stanley decomposition of a monomial ideal and a partition of a particular poset into intervals established by Herzog, Vladoiu, and Zheng. In [15, Corollary 2.9] Pournaki, Seyed Fakhari and Yassemi show that the Stanley’s conjecture holds for finite products of monomial prime ideals. This fact implies that the conjecture holds for $d$-uniform complete $d$-partite clutters. Here we give a shorter and different proof of this result [Theorem 3.3]. Finally, we show that the result of Ishaq [10, Theorem 2.8] follows from the Theorem 3.9.

2 Algebraic and combinatorial Stanley depth

For a positive integer $n$, let $[n] = \{1, ..., n\}$ and let $2^n$ denote the Boolean algebra consisting of all subsets of $[n]$. For $x \leq y$ in a poset $P$, we let $[x, y] = \{z : x \leq z \leq y\}$ and call $[x, y]$ an interval in $P$. If $P$ is a poset and $x \in P$, we let $U[x] = \{y \in P : y \geq x\}$ and call this the up-set of $x$. In [9], Herzog et al. introduced a powerful connection between the Stanley depth of a monomial ideal and a combinatorial partitioning problem for partially ordered sets. For $c \in \mathbb{N}^n$, let $x^c := x_1^{c(1)} x_2^{c(2)} \cdots x_n^{c(n)}$. Let $I = (x^{v_1}, \ldots, x^{v_q}) \subset R$ be a monomial ideal. Let $h \in \mathbb{N}^n$ be such that $h \geq v_i$ for all $i$. The characteristic poset of $I$ with respect to $h$, denoted $P^h_I$ is defined as the induced subposet of $\mathbb{N}^n$ with ground set

$$\{c \in \mathbb{N}^n : c \leq h \text{ and there is } i \text{ such that } c \geq v_i\}.$$

Let $D$ be a partition of $P^h_I$ into intervals. For $J = [x, y] \in D$, define

$$Z_J = \{i \in [n] : y(i) = h(i)\}.$$

Define the Stanley depth of a partition $D$ to be

$$sdepth(D) = \min_{J \in D} |Z_J|$$

and the Stanley depth of the poset $P^h_I$ to be

$$sdepth(P^h_I) = \max_D sdepth(D),$$

where the maximum is taken over all partitions $D$ of $P^h_I$ into intervals. Herzog et al. showed in [9] that

$$sdepth(I) = sdepth(P^h_I).$$

If $I$ is a squarefree monomial ideal, then we may take $h = (1, 1, \ldots, 1)$ and work inside $\{0, 1\}^n$, which is isomorphic to $2^n$. A monomial $m$ in $R$ then can be identified with the subset of $[n]$ whose elements correspond to the subscripts of
the variables appearing in $m$. Let $G(I) = \{x^{v_1}, x^{v_2}, \ldots, x^{v_q}\}$ be the set of minimal monomial generators of $I$ and $A_i \subseteq [n]$ corresponds to $v_i$. The characteristic poset of $I$ with respect to $h = (1, 1, \ldots, 1)$, denoted by $P^h_I$ is in fact the set

$$P^h_I = \{C \subset [n] : C \text{ contains the } supp(v_i) \text{ for some } i\}$$

where $supp(v_i) = \{j : x_j \mid v_i\}$. Then the definition of $P^h_I$ clearly simplifies to

$$P^h_I = \bigcup_{i=1}^{\varphi} U[A_i]$$

as a subposet of $2^n$. For an interval $J = [X, Y]$, we then have that $|Z_J|$ corresponds to $|Y|$.

Let $\mathcal{P} : P^h_I = \bigcup_{i=1}^{\varphi} [C_i, D_i]$ be a partition of $P^h_I$, and for each $i$, let $c_i \in \{0, 1\}^n$ be the $n$–tuple such that $supp(x^{c_i}) = C_i$. Then there is a Stanley decomposition $\mathcal{D}(\mathcal{P})$ of $I$

$$\mathcal{D}(\mathcal{P}) : I = \bigoplus_{i=1}^{\varphi} x^{c_i} K[\{x_k : k \in D_i\}].$$

The above description of $sdepth(I) = sdepth(P^h_I)$ shows that

**Lemma 2.1** If $I$ is a squarefree monomial ideal and $G(I)$ is the minimal monomial generating set of $I$, then $\min\{\deg(v) : v \in G(I)\} \leq sdepth(I) \leq n$.

By the previous lemma, if $\mathcal{C}$ is a $d$–uniform clutter, then $d \leq sdepth(I(\mathcal{C})) \leq n$.

### 3 Stanley depth of edge ideals

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field $K$ and let $v_1, \ldots, v_q$ be the column vectors of a matrix $A = (a_{ij})$ whose entries are non-negative integers. For technical reasons, we shall always assume that the rows and columns of the matrix $A$ are different from zero. As usual we use the notation $x^a := x_1^{a_1} \cdots x_n^{a_n}$, where $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$.

Consider the monomial ideal:

$$I = (x^{v_1}, \ldots, x^{v_q}) \subset R$$

generated by $F = \{x^{v_1}, \ldots, x^{v_q}\}$.

Let $A$ be the incidence matrix of $\mathcal{C}$ whose column vectors are $v_1, \ldots, v_q$. The set covering polyhedron of $\mathcal{C}$ is given by:

$$Q(A) = \{x \in \mathbb{R}^n | x \geq 0; xA \geq 1\},$$
A subset \( C \subset V(C) \) is called a minimal vertex cover of \( C \) if: (i) every edge of \( C \) contains at least one vertex of \( C \), and (ii) there is no proper subset of \( C \) with the first property. The map \( C \mapsto \sum_{x_i \in C} e_i \) gives a bijection between the minimal vertex covers of \( C \) and the integral vectors of \( Q(A) \)\(^6\). A polyhedron is called an integral polyhedron if it has only integral vertices.

**Definition 3.1** A \( d \)-uniform clutter \( C(V, E) \) with vertex set \( V \) and edge set \( E \) is called \( k \)-partite if the vertex set \( V \) is partitioned into \( k \) disjoint subset \( V_1, V_2, \ldots, V_k \) and \( |e \cap V_i| \leq 1 \) for all \( e \in E \) and \( 1 \leq i \leq k \).

**Definition 3.2** A \( d \)-uniform clutter \( C(V, E) \) with vertex set \( V \) and edge set \( E \) is called complete \( k \)-partite \((d \leq k \leq n)\) if the vertex set \( V \) is partitioned into \( k \) disjoint subset \( V_1, V_2, \ldots, V_k \) and \( E = \{\{x_{j_1}, \ldots, x_{j_d}\} : |x_{j_i} \cap V_i| \leq 1\} \), in that case we say that \( V = V_1 \cup V_2 \cup \cdots \cup V_k \) is a complete partition. Note that if \( d = k \), then \( V_1, V_2, \ldots, V_k \) are the minimal vertex covers of \( C \).

Let \( I \subset R \) be the edge ideal of a complete bipartite graph over \( n \) vertices with \( n \geq 4 \). In \(^{10}\) Ishaq showed that

\[
\text{sdepth}(I) \leq \frac{n + 2}{2}.
\]

Now let \( C \) be a complete \( k \)-partite \( d \)-uniform clutter with vertex set \( V(C) \) partitioned into \( k \) disjoint subset \( V_1, V_2, \ldots, V_k \); \( V(C) = V_1 \cup \cdots \cup V_k \), with \( |V_i| = r_i \), where \( r_i \in \mathbb{N} \) and \( 2 \leq r_1 \leq \cdots \leq r_k \). Let \( r_1 + \cdots + r_k = n \) and \( V_1 = \{x_1, \ldots, x_{r_1}\} \), \( V_2 = \{x_{r_1+1}, \ldots, x_{r_1+r_2}\} \), \ldots, \( V_k = \{x_{r_1+\cdots+r_{k-1}+1}, \ldots, x_{r_1+\cdots+r_k}\} \). Let \( I_1 = (V_1) \), \( I_2 = (V_2) \), \ldots, \( I_k = (V_k) \) be the monomial ideals in \( R \). Note that

\[
|E(C)| = \sum_{1 \leq j_1 < j_2 < \cdots < j_d \leq k} r_{j_1} r_{j_2} \cdots r_{j_d}.
\]

Then the edge ideal of \( C \) is of the form

\[
I = \sum_{1 \leq j_1 < j_2 < \cdots < j_d \leq k} I_{j_1} \cap I_{j_2} \cap \cdots \cap I_{j_d}.
\]

The next result follows from the fact that the Stanley’s conjecture holds for finite products of monomial prime ideals (see from \(^{15}\) Corollary 2.9); for convenience we include a short proof.

**Theorem 3.3** Let \( I \) be the edge ideal of \( d \)-uniform complete \( d \)-partite clutter. Then Stanley’s Conjecture holds for \( I \).
Proof. We continue to use the notation used in the above description of $I = I(C) = (V_1) \cap \cdots \cap (V_d)$, with $V(C) = V_1 \cup \cdots \cup V_d$. In our situation $V_1, \ldots, V_d$ are the minimal vertex covers of $C$. Therefore

$$I = (V_1) \cap \cdots \cap (V_d)$$

is a reduced intersection of monomial prime ideals of $R$, where $(V_i) \not\subseteq \sum_{j=1, j \neq i}^d (V_j)$ for all $1 \leq i \leq d$. Then by [14, Theorem 3.3],

$$\text{depth}(I) = d \leq \text{sdepth}(I).$$

\[ \square \]

Theorem 3.4 Let $C$ be a $d$–uniform complete $k$–partite clutter. Then

$$d \leq \text{sdepth}(I(C)) \leq d + \frac{1}{|E(C)|} \left( \sum_{1 \leq j_1 < j_2 < \cdots < j_d \leq k} \left( \sum_{i=1}^d \left( \frac{r_{j_i}}{2} \right) \frac{r_{j_1} \cdots r_{j_d}}{r_{j_i}} \right) \right).$$

Proof. Note that $I = I(C)$ is a squarefree monomial ideal generated by monomials of degree $d$. Let $\rho = \text{sdepth}(I)$ and $P : P^h_k = \cup_{i=1}^q [C_i, D_i]$ be a partition of $P^h_k$ satisfying $\text{sdepth}(D(P)) = \rho$, where $D(P)$ is the Stanley decomposition of $I$ with respect to the partition $P$. We may choose $P'$ such that $|D| = \rho$ whenever $C \neq D$ in the interval $[C, D]$, considering these intervals of $P$ with $|D| = \rho$ and 1-dimensional spaces. Now we see that for each interval $[C, D]$ in $P'$ with $|C| = d$ we have $\rho - d$ subsets of cardinality $d + 1$ in this interval. The total number of these kind of intervals is $|E(C)| = \sum r_{j_1}r_{j_2} \cdots r_{j_d}$, where the sum runs over all $1 \leq j_1 < j_2 < \cdots < j_d \leq k$. So we have

$$(\rho - d) \left( \sum r_{j_1}r_{j_2} \cdots r_{j_d} \right)$$

subsets of cardinality $d + 1$. This number is less than or equal to the total number of monomials $m \in I$ with $\deg(m) = d + 1$ and $|\text{supp}(m)| = d + 1$. Furthermore,

$$\{ m : m \in I; \deg(m) = d + 1; |\text{supp}(m)| = d + 1 \} = \{ x^e x_i : e \in E(C); i \notin e \},$$

with cardinality $\sum_{1 \leq j_1 < j_2 < \cdots < j_d \leq k} \left( \sum_{i=1}^d \left( \frac{r_{j_i}}{2} \right) \frac{r_{j_1} \cdots r_{j_d}}{r_{j_i}} \right)$. Hence

$$(\rho - d)|E(C)| \leq \sum_{1 \leq j_1 < j_2 < \cdots < j_d \leq k} \left( \sum_{i=1}^d \left( \frac{r_{j_i}}{2} \right) \frac{r_{j_1} \cdots r_{j_d}}{r_{j_i}} \right).$$

Therefore we obtain the required inequality. \[ \square \]
Theorem 3.5 Let $C$ be a $d$–uniform complete $d$–partite clutter. Then

$$d \leq s\text{depth}(I(C)) \leq d + \sum_{i=1}^{d} \frac{r_i - 1}{2}. $$

Proof. The proof is analogous to the proof of Theorem 3.4, but with $|E(C)| = r_1 r_2 \cdots r_d$ and

$$\{m : m \in I; \deg(m) = d + 1; \text{supp}(m) = d + 1\} = \{x^e x_i : e \in E(C); i \notin e\},$$

has cardinality

$$\sum_{i=1}^{d} \binom{r_i}{2} \frac{r_1 r_2 \cdots r_d}{r_i} = (r_1 r_2 \cdots r_d) \sum_{i=1}^{d} \frac{r_i - 1}{2}. $$

Hence

$$(\rho - d) (r_1 r_2 \cdots r_d) \leq (r_1 r_2 \cdots r_d) \sum_{i=1}^{d} \frac{r_i - 1}{2}. $$

Therefore we obtain the required inequality. \hfill $\Box$

Definition 3.6 A clutter $C(V, E)$, whose set covering polyhedron $Q(A)$ is integral, is called integral.

Lemma 3.7 (See [6]) If $C$ is an integral $d$–uniform clutter, then there exists a minimal vertex cover of $C$ intersecting every edge of $C$ in exactly one vertex.

Proof. Let $B$ be the integral matrix whose columns are the vertices of $Q(A)$. It is not hard to show that a vector $\alpha \in \mathbb{R}^n$ is an integral vertex of $Q(A)$ if and only if $\alpha = \sum_{x \in C} e_i$ for some minimal vertex cover $C$ of $C$. Thus the columns of $B$ are the characteristic vectors of the minimal vertex covers of $C$. Using [5] Theorem 1.17 we get that

$$Q(B) = \{x|x \geq 0; xB \geq 1\}$$

is an integral polyhedron whose vertices are the columns of $A$, where $1 = (1, 1, \ldots, 1)$. Therefore we have the equality

$$Q(B) = \mathbb{R}^n_+ + \text{conv}(v_1, \ldots, v_q).$$

(2)

We proceed by contradiction. Assume that for each column $u_k$ of $B$ there exists a vector $v_{ik}$ in $\{v_1, \ldots, v_q\}$ such that $\langle v_{ik}, u_k \rangle \geq 2$. Here $\langle \cdot, \cdot \rangle$ is the standard inner product. Then

$$v_{ik}B \geq 1 + e_k,$$
where $e_i$ is the $i$-th unit vector.
Consider the vector $\alpha = v_{i_1} + \cdots + v_{i_s}$, where $s$ is the number of columns of $B$.
From the inequality
\[ \alpha B \geq (1 + e_1) + \cdots + (1 + e_s) = (s + 1, \ldots, s + 1) \]
we obtain that $\alpha / (s + 1) \in Q(B)$. Thus, using Eq. (2), we can write
\[ \alpha / (s + 1) = \mu_1 e_1 + \cdots + \mu_n e_n + \lambda_1 v_1 + \cdots + \lambda_q v_q \quad (\mu_i, \lambda_j \geq 0; \sum \lambda_i = 1). \quad (3) \]
Therefore taking inner products with $1$ in Eq. (3) and using the fact that $C$ is $d$-uniform we get that $|\alpha| \geq (s + 1)d$. Then using the equality $\alpha = v_{i_1} + \cdots + v_{i_s}$ we conclude
\[ sd = |v_{i_1}| + \cdots + |v_{i_s}| = |\alpha| \geq (s + 1)d, \]
a contradiction because $d \geq 1$.

Let $C$ be a clutter and let $I = I(C)$ be its edge ideal. Recall that a deletion of $I$ is any ideal $I'$ obtained from $I$ by making a variable equal to 0. A deletion of $C$ is a clutter $C'$ that corresponds to a deletion $I'$ of $I$. Notice that $C'$ is obtained from $I'$ by considering the unique set of square-free monomials that minimally generate $I'$. A contraction of $I$ is any ideal $I'$ obtained from $I$ by making a variable equal to 1. A contraction of $C$ is a clutter $C'$ that corresponds to a contraction $I'$ of $I$. This terminology is consistent with that of [5, p. 23].
A clutter obtained from $C$ by a sequence of deletions and contractions of vertices is called a minor of $C$. The clutter $C$ is considered itself a minor.

The notion of a minor of a clutter is not a generalization of the notion of a minor of a graph in the sense of graph theory [16, p. 25]. For instance if $G$ is a cycle of length four and we contract an edge we obtain that a triangle is a minor of $G$, but a triangle cannot be a minor of $G$ in our sense.

The notion of a minor plays a prominent role in combinatorial optimization [5]. As an application of the power of using minors, this allows us to get a nice decomposition of an integral uniform clutter.

**Proposition 3.8** (See [6]) If $C(V, E)$ be an integral $d$-uniform clutter, then there are $V_1, \ldots, V_d$ mutually disjoint minimal vertex covers of $C$ such that $V = \bigcup_{i=1}^d V_i$. In particular $|\text{supp}(x^e) \cap V_i| = 1$ for all $e \in E; 1 \leq i \leq d$.

**Proof.** By induction on $d$. If $d = 1$, then $E(C) = \{\{x_1\}, \ldots, \{x_n\}\}$ and $V$ is a minimal vertex cover of $C$. In this case we set $V_1 = V$. Assume $d \geq 2$. By Lemma 3.7 there is a minimal vertex cover $V_1$ of $C$ such that $|\text{supp}(x^{v_1}) \cap V_1| = 1$
for all $i$. Consider the ideal $I'$ obtained from $I$ by making $x_i = 1$ for $x_i \in V_1$.
Let $C'$ be the clutter corresponding to $I'$ and let $A'$ be the incidence matrix of $C'$. The ideal $I'$ (resp. the clutter $C'$) is a minor of $I$ (resp. $C$). Recall that
the integrality of $Q(A)$ is preserved under taking minors [16, Theorem 78.2], so $Q(A')$ is integral. Then $C'$ is a $(d - 1)$-uniform clutter whose set covering polyhedron $Q(A')$ is integral. Note that $V(C') = V \setminus V_1$. Therefore by induction hypothesis there are $V_2, \ldots, V_d$ pairwise disjoint minimal vertex covers of $C'$ such that $V \setminus V_1 = V_2 \cup \cdots \cup V_d$. To complete the proof observe that $V_2, \ldots, V_d$ are minimal vertex covers of $C$. Indeed if $e$ is an edge of $C$ and $2 \leq k \leq d$, then $e \cap V_1 = \{x_i\}$ for some $i$. Since $e \setminus \{x_i\}$ is an edge of $C'$, we get $(e \setminus \{x_i\}) \cap V_k \neq \emptyset$. Hence $V_k$ is a vertex cover of $C$. Furthermore if $x \in V_k$, then by the minimality of $V_k$ relative to $C'$ there is an edge $e'$ of $C'$ disjoint from $V_k \setminus \{x\}$. Since $e = e' \cup \{y\}$ is an edge of $C$ for some $y \in V_1$, we obtain that $e$ is an edge of $C$ disjoint from $V_k \setminus \{x\}$. Therefore $V_k$ is a minimal vertex cover of $C$, as required.

**Theorem 3.9** Let $C(V, E)$ be an integral $d$–uniform clutter. Then $C$ is a $d$–partite clutter, with

$$d \leq sdepth(I(C)) \leq d + \frac{r_1 \cdots r_d}{|E(C)|} \sum_{i=1}^{d} \frac{r_i - 1}{2}$$

**Proof.** By Proposition 3.8 we have that $C$ is a $d$–partite clutter. The proof is analogous to the proof of Theorem 3.4 but with $k = d$. Note that $I = I(C)$ is a squarefree monomial ideal generated by monomials of degree $d$. Let $\rho = sdepth(I)$ and $P : P^h = \cup_{i=1}^{\beta} [C_i, D_i]$ be a partition of $P^h$ satisfying $sdepth(D(P)) = \rho$, where $D(P)$ is the Stanley decomposition of $I$ with respect to the partition $P$.
We may choose $P$ such that $|D| = r$ whenever $C \neq D$ in the interval $[C, D]$. Now we see that for each interval $[C, D]$ in $P$ with $|C| = d$ we have $r - d$ subsets of cardinality $d + 1$ in this interval. The total number of these kind of intervals is $|E(C)|$. So we have

$$(\rho - d)|E(C)|$$

subsets of cardinality $d + 1$. This number is less than or equal to the total number of monomials $m \in I$ with $\deg(m) = d + 1$ and $|\text{supp}(m)| = d + 1$. Furthermore,

$$\{m : m \in I; \deg(m) = d + 1; \text{supp}(m) = d + 1\} = \{x^e : e \in E(C); i \notin e\},$$

with cardinality less than or equal $\sum_{i=1}^{d} \binom{r_i}{2} \frac{r_1 \cdots r_d}{r_i} = \sum_{i=1}^{d} \frac{r_i - 1}{2} r_1 \cdots r_d$. Therefore we obtain

$$sdepth(I(C)) \leq d + \frac{1}{|E(C)|} \left( \sum_{i=1}^{d} \left( \frac{r_i - 1}{2} \right) r_1 \cdots r_d \right).$$

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Hence
\[
sdepth(I(C)) \leq d + \frac{r_1 \cdots r_d}{|E(C)|} \sum_{i=1}^{d} \frac{r_i - 1}{2}.
\]
\[\square\]

**Corollary 3.10** Let \( C(V, E) \) be an integral \( d \)-uniform clutter, such that its decomposition \( d \)-partite \( V = V_1 \cup V_2 \cup \cdots \cup V_k \) is complete. Then
\[
d \leq sdepth(I(C)) \leq d + \sum_{i=1}^{d} \frac{r_i - 1}{2}.
\]

**Proof.** It follows from Theorem 3.5 or Theorem 3.9. \[\square\]

**Corollary 3.11** ([10, Theorem 2.8]) The Stanley depth of the edge ideal of a complete bipartite graph over \( n \) vertices with \( n \geq 4 \) is less than or equal to \( \frac{n+2}{2} \).

**Proof.** This follows from the fact that complete bipartite graphs are integral clutters. \[\square\]

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