Analysis of Position and Velocity Variations for Hyperbolic Orbits and Application to Flyby Anomaly*

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Flyby anomaly indicates the existence of an unknown perturbation (i.e., anomalous acceleration) that affects hyperbolic trajectories. Based on the analytical position and velocity variations, this paper investigates the general kinematics of perturbed hyperbolic orbits. As a result, post-interaction approximation formulas are derived. Based on these results, the observation data of the Galileo and NEAR Earth flybys are analyzed. The analysis results derive new constraints for flyby kinematics. The authors of this paper selected a few of the hypothetical acceleration models and analyzed their kinematical properties as representative examples. The simulation results show that the acceleration models fail to reproduce the characteristics of the range and Doppler observation data. This means that, in modeling the flyby anomaly, not only energy variation, but also kinematical constraints must be considered to reproduce the observation data.

Key Words: Orbital Kinematics, Perturbation Analysis, Flyby Anomaly, Analytical State Vector Variation

1. Introduction

In recent decades, planetary flybys (e.g., gravity assist, swing-by) have been widely used in interplanetary missions in order to save propellant, time, and expense. The energy transfer process of a planetary flyby can be explained applying the concept of patched conic approximation. When within the sphere of influence of a planet, a spacecraft’s trajectory can be considered as an approximate hyperbolic trajectory.\(^1\) In practice, all of the well-known perturbations (e.g., air, drag, solar radiation pressure, third-body gravity, relativistic effect) are considered for precise orbit determination and spacecraft navigation.

However, unexpected orbital energy changes during spacecraft’s Earth flybys, known as the “flyby anomaly,” have been detected since 1990. The first anomalous orbital energy change was detected during the first Earth gravity assist (EGA) for Galileo on 8 December 1990.\(^2\) Detailed tracking data analysis revealed unexpected shifts in Doppler and range residuals that indicated a velocity increase at perigee. Galileo’s second and other EGAs, including those of NEAR, Cassini, Rosetta and MESSENGER, also showed anomalous orbital energy changes.\(^3\) There have been many efforts to explain the flyby anomaly using conventional physics\(^2^4^7\) and unconventional physics\(^8^9\) as well. However, none have succeeded in solving the phenomenon completely. The dynamics of the flyby anomaly remains unexplained, and studying the phenomenon has been challenged because only 12 orbital energy variation data points exist. To explain the flyby anomaly, hypothetical acceleration models have been proposed by several authors.\(^10^12\) Such models are usually parameterized to reproduce the energy variations. However, such parameterization may lead to false positive results due to the use of a large number of parameters relative to the smaller number of observation data points. Thus, two things are necessary: first, to reproduce the orbital energy variation, and second, to check the kinematical characteristics of the hypothetical acceleration models (e.g., whether the Doppler residuals show an overshoot or undershoot) for the appropriateness to reproduce the flyby anomaly.

To examine the kinematical characteristics of the flybys, either analytical or numerical approaches can be adopted. Usually, numerical approaches are a convenient option that are well suited to quantitative analysis. On the other hand, analytical approaches can be a bit complicated, but are often better suited to qualitative analysis. If an acceleration model for the flyby anomaly is well-developed and definitive, solely using the numerical approach is sufficient to verify the kinematical behavior of the trajectory with quantitative analysis. However, for an acceleration model of the flyby anomaly that is currently being developed, an analytical approach is more effective to improve the model by incorporating the kinematical behaviors into the model using qualitative analysis.

The main goals of the current study are to derive new analytical formulas to investigate and understand the kinematics of perturbed hyperbolic orbits, and to apply them to few hypothetical acceleration models proposed for the flyby anomaly as representative examples. There have been analytical formulas developed for, and extended to, the study of hyperbolic orbits.\(^13^15\) However, these formulas are only applicable to weak interactions since they use straight-line trajectory approximations\(^13\) or evaluate only conservative perturbations.\(^14\) The new formulas, called post-interaction approximation formulas, are based on the analytical state vector variation developed by the authors in their previous research.\(^16\)
post-interaction approximation formulas aid in analyzing the qualitative behaviors of the perturbed hyperbolic orbit. Especially, these formulas are useful to analytically approximate the radial position and velocity variations, which can be obtained by range and Doppler observations.

As representative examples, the novel formulas of the current study are applied to the acceleration models suggested by Gerrard-Sumner, Hassel et al., and Lewis, and their kinematical properties of flybys are checked against the Doppler tracking data observed in Galileo’s first EGA and NEAR’s EGA.

The linearized Gauss variational equations (GVEs) for hyperbolic orbits are presented in Section 2. The post-interaction approximation formulas are derived in Section 3. The kinematical properties of the observation data for Galileo’s first EGA and NEAR’s EGA are analyzed in Section 4. The hypothetical acceleration models for the flyby anomaly and corresponding kinematics analyses are briefly introduced in Section 5. Conclusions are presented in Section 6.

2. Linearized GVEs

This section introduces linearized GVEs for analyzing hyperbolic orbits. A main distinction between elliptical and hyperbolic orbits is the number of interactions with the central body. For hyperbolic orbits, there is only one interaction between the spacecraft and the central body. Compared to elliptical orbits, this fact reduces the difficulty of addressing perturbed kinematics because there is no reason to address mean elements or separate the perturbation effects into secular, long-term, and short-term components. Therefore, unperturbed orbital elements of the Keplerian motion are defined, including the semi-major axis ($a$), eccentricity ($e$), inclination ($i$), longitude of ascending node ($\Omega$), argument of perigee ($\omega$). The unperturbed mean anomaly ($M$) and true anomaly ($f$) are not constants, but are determined by the hyperbolic Kepler equation. On the other hand, oscillating orbital elements are defined as parameters of an osculating orbit (i.e., equivalently, perturbed orbit) including semi-major axis ($\bar{a}$), eccentricity ($\bar{e}$), inclination ($\bar{i}$), longitude of ascending node ($\bar{\Omega}$), argument of perigee ($\bar{\omega}$), mean anomaly ($\bar{M}$), and true anomaly ($\bar{f}$).

Consider only the point-mass gravity of the Earth without any perturbative acceleration $\mathbf{F}$ on a spacecraft; then the spacecraft’s equation of motion is as follows:

$$\ddot{r} = -\frac{\mu}{r^3} r,$$

where $\mu$ is the standard gravitational constant of the Earth, and $\dot{r}$ and $\dot{v}$ are the unperturbed position and velocity, respectively. In reality, the perturbative acceleration $\mathbf{F}$ must be included:

$$\ddot{r} = -\frac{\mu}{r^3} r + F(t; \bar{r}, \bar{v}),$$

where $\bar{r}$ and $\bar{v}$ are the perturbed position and velocity, respectively. A non-Keplerian orbital element variation, $\Delta c_i$, is defined as:

$$\Delta c_i \equiv \bar{c}_i - c_i,$$

where $c_i$ is an unperturbed orbital element $c_i \in \{a, e, i, \Omega, \omega, M, f\}$, and $\bar{c}_i$ is the corresponding osculating orbital element $\bar{c}_i \in \{\bar{a}, \bar{e}, \bar{i}, \bar{\Omega}, \bar{\omega}, \bar{M}, \bar{f}\}$. Throughout the paper, it is assumed that the perturbations are sufficiently small, such that the first-order approximation holds, as follows:

$$F(t; \bar{r}, \bar{v}) \equiv F(t; r, v).$$

This paper defines two coordinate systems. The $IJK$ coordinate system, which is a typical Earth-centered inertial coordinate system (e.g., Earth MJ2000 equatorial coordinate system), is defined as shown in Fig. 1. The $RSW$ coordinate system, which is a rotating coordinate system, is defined as shown in Fig. 1, where $R$ is parallel to the spacecraft’s unperturbed position vector $r$, $\bar{W}$ is parallel to the spacecraft’s unperturbed specific angular momentum vector $\mathbf{h}(= r \times v)$, and $\bar{S} \equiv \bar{W} \times \bar{R}$. The $RSW$ coordinate system rotates around $\bar{W}$ with angular velocity $h/r^2$. Hereafter, the subscript $RSW$ indicates that a vector is expressed in the rotating $RSW$ coordinate system. For an arbitrary three-dimensional vector $x(\cdot)$, the rotational transformations between the $IJK$ coordinate system and $RSW$ coordinate system are given as:

$$x_{IJK} = T(-\Omega, -i, -u)x_{RSW},$$

$$x_{RSW} = T^{-1}(-\Omega, -i, -u)x_{IJK},$$

where $u(= \omega + f)$ is the unperturbed argument of latitude, and the transformation matrices, $T(-\Omega, -i, -u)$ and $T^{-1}(-\Omega, -i, -u)$, are given in Vallado.

With the approximated perturbative acceleration $F_{RSW}(r; r, v)$, non-Keplerian orbital element variation rates can be derived for hyperbolic orbits as follows:

$$\frac{d}{dt} \Delta a = \frac{2e \sin(f)}{n \sqrt{e^2 - 1}} R_{F} - \frac{2a\sqrt{e^2 - 1}}{nr} F_{S},$$
\[ \frac{d}{dt} \Delta e = -\frac{\sqrt{e^2 - 1}}{an} \sin(f) F_R \]
\[ -\frac{\sqrt{e^2 - 1}}{an} \left( \cos(f) + \frac{e + \cos(f)}{1 + e \cos(f)} \right) F_S, \]
\[ \frac{d}{dt} \Delta i = \frac{r F_W \cos(\omega + f)}{a^2 \sqrt{e^2 - 1}}, \]
\[ \frac{d}{dt} \Delta \Omega = \frac{r F_W \sin(\omega + f)}{a^2 \sqrt{e^2 - 1} - 1}, \]
\[ \frac{d}{dt} \Delta \omega = \frac{\sqrt{e^2 - 1}}{ane} \left[ \cos(f) F_R - \frac{\sin(f)(2 + e \cos(f)) F_S}{1 + e \cos(f)} \right] - \cos(f) \frac{d}{dt} \Delta \Omega, \]
\[ \frac{d}{dt} \Delta M = \frac{1}{2a ne} \left[ (2er - p \cos(f)) F_R + (p + r) \sin(f) F_S \right] - \frac{3n}{2a} \Delta a, \]

where the subscripts \( R, S, \) and \( W \) represent sub-components of a vector, respectively, and the unperturbed mean motion \( n = \sqrt{-\mu/a^3} \), semi-latus rectum \( p = a \cdot (1 - e^2) \), and radial distance \( r = |r| = p(1 + e \cos(f))^{-1} \) are defined as usual. The derivation process of Eqs. (5a)–(5f) is parallel to that of elliptical orbits.\(^{10} \) Note that to avoid recursively updating osculating elements via numerical integration, the GVEs are linearized with respect to the unperturbed orbital elements following the Keplerian motion. In this case, the secular drift term in the perturbed mean motion due to the semi-major axis variation, \( \Delta \dot{n} \equiv \ddot{n} - n \equiv -3n \Delta a/2a \), must be taken into account for the mean motion variation rate, as in Eq. (5f).

The first-order variation of orbital elements \( \Delta c_I \) can be calculated as follows:

\[ \Delta c_I(t) = \int_{t_0}^{t} \left( \frac{d}{dt} \Delta c_I \right) dt, \]

where the subscript \( init \) indicates a parameter is estimated at the initial time. The variation of the true anomaly \( \Delta f \) can be calculated using the partial differential relationships between \( f, M, e \), and the hyperbolic anomaly \((H)\):

\[ \Delta f = \frac{a^2 \sqrt{e^2 - 1}}{r^2} \Delta M + \frac{4 \sin(f) + e \sin(2f)}{2(1 - e^2)} \Delta e. \]

3. Perturbed Hyperbolic Orbit

This section presents the linearized position and velocity variations due to perturbations, and derives post-interaction approximation formulas. The formulas are independent from whether \( \Delta c_I \) is obtained via GVEs or Lagrange Variational Equations (LVEs). If \( \Delta c_I \) is known, the approximate position and velocity vectors can be directly obtained.

For Keplerian motion, the spacecraft’s position and velocity can be represented with its orbital elements\(^{17} \):

\[ \mathbf{r}_{RWS} = [r \ 0 \ 0]^T, \]
\[ \mathbf{v}_{RWS} = \left[ \frac{\mu}{p} \sin(f) \right. \left. \frac{\mu}{p} (1 + e \cos(f)) \ 0 \right]^T. \]

3.1. Position and velocity variation

The complete first-order formulas of the position and velocity variations for hyperbolic orbits are presented in\(^{16} \):

\[ \begin{bmatrix} \frac{1}{a} \Delta a - \frac{a \cos(f)}{r} \Delta e - \frac{ae \sin(f)}{r \sqrt{e^2 - 1}} \Delta M \\ \frac{4 \sin(f) + e \sin(2f)}{(2 - e^2)} \Delta e + \Delta \omega \\ \frac{a^2 \sqrt{e^2 - 1}}{r^2} \Delta M \\ \frac{\sin(u) \Delta i - \cos(u) \sin(i) \Delta \Omega}{r} \end{bmatrix} \]
\[ \mathbf{r}_{WS} = \frac{\mu}{p} \begin{bmatrix} -e \sin(f) \Delta a \\ -\frac{(a \sin(f) \Delta e + p \Delta \omega)}{2a} \\ -\frac{a^2 \sqrt{e^2 - 1}}{r^2} \Delta M \\ -\frac{(1 - e^2)}{2r} \Delta a + \frac{e + \cos(f)}{1 - e^2} \Delta e + e \sin(f) \Delta \omega \end{bmatrix}, \]

where \( \Delta \omega = \Delta \omega + \cos(i) \Delta \Omega, \Delta a = \cos(u) + e \cos(\omega), \) and \( \xi_1 = \sin(u) + e \sin(\omega). \)

3.2. Post-interaction approximation

This subsection introduces post-interaction approximation formulas addressing the orbital kinematics evolution after the closest encounter. The purpose of these formulas is to provide a qualitative description of perturbed hyperbolic orbit by making a few approximations for simplification. Such formulas will be useful for approximating the range and Doppler observables, and for analyzing the observation data of Earth flybys because the radial position and velocity variations are interpreted as range and Doppler observables, respectively.

Assume that the source of the perturbation is Earth-centered; then the acceleration due to the perturbation effect is strongest at the closest approach and quickly weakens thereafter. Consequently, the variations of orbital elements will converge to a specific value. Thus, there is a need to roughly determine when the interaction between the central body and spacecraft ends. To address this, we define the epoch time as the perigee passage, and the dynamical time scale of hyperbolic orbit is defined as follows:

\[ t_p = 0, \]
\[ t^* = e \pi n^{-1}. \]

Note that the form of Eq. (10b) is similar to the regular expression of the period in an elliptical orbit. The eccentricity
term $e$, introduced in Eq. (10b), is closely related to the linearization of a hyperbolic orbit. In the derivation of the post-interaction approximation, the following approximation needs to be made:

$$
\epsilon(t) \approx M e^{-1} \sqrt{1 + (Me^{-1})^2} \approx 1
$$

where $e/M < 1$. Note that $\epsilon(t) \approx 0.95$, and also as $t \to \infty$, $\epsilon(t) \to 1$. Therefore, the dynamical time scale is defined in a way that the approximation $\epsilon(t \geq t^*) \approx 1$ holds with an error of less than 5%.

For cases in which the source of the perturbation is Earth-centered and has greater radial dependence than $\sim 1/r^2$, one can assume the variation of orbital elements is almost constant when $t \gg t^*$. The convergence of orbital element variation is denoted as:

$$
\Delta c_t^\infty \equiv \Delta c(t \gg t^*), \tag{11a}
$$

where $C \in \{a, e, i, \Omega, \omega, M_0\}$

$$
\Delta M^\infty(t) \equiv \frac{3n}{2a} \Delta a^\infty t + \Delta M_0^\infty. \tag{11b}
$$

The next step is to express the time $t$ in terms of the radial distance and unperturbed orbital elements:

$$
 t \cong -\frac{r - a}{an} - \frac{1}{n} \ln \left( \frac{2(a - r)}{ae} \right), \tag{12}
$$

when $t \gg t^*$. The derivation of Eq. (12) is based on Lagrange’s expansion theorem,\(^{18}\) which is given as:

$$
\sin(H) \cong Me^{-1} + e^{-1} \ln \left( Me^{-1} + \sqrt{1 + (Me^{-1})^2} \right).
$$

Although Eq. (12) is not applicable in the non-linear region $-t^* \lesssim t \lesssim t^*$, it is accurate enough to describe the linear region $t \gg t^*$. Now, Eq. (11b) can be rewritten as:

$$
\Delta M^\infty(r) \cong \frac{3n}{2a} \Delta a^\infty \left( \frac{r - a}{an} + \frac{1}{n} \ln \left( \frac{2(a - r)}{ae} \right) \right) + \Delta M_0^\infty. \tag{13}
$$

By ignoring the decaying terms that are proportional to or smaller than $1/r$, the position variation can be approximated as follows when $t \gg t^*$:

$$
\Delta r_{R^2S^2W}^\infty = \begin{cases} 
\left[ 2 - \frac{3}{2} \frac{1}{n} \ln \left( \frac{r - a}{an} \right) \right] \Delta a^\infty - \frac{r \Delta a^\infty}{2a} + \frac{a \Delta e^\infty}{e} \\
-\frac{a \Delta M_0^\infty}{e} \frac{(r - a e^2)}{\sqrt{e^2 - 1}} \frac{\Delta e^\infty}{e} + \frac{3}{2} \sqrt{e^2 - 1} \Delta a^\infty + r \Delta \Omega^\infty \\
\left( \chi_1 \frac{h \gamma_2}{na} \right) \Delta \gamma^\infty + \left( \chi_2 r + \frac{h \chi_1}{na} \right) \sin(i) \Delta \Omega^\infty
\end{cases} \tag{14}
$$

where

$$
\chi_1 = \frac{\cos(\omega) \sqrt{e^2 - 1} - \sin(\omega)}{e},
$$

$$
\chi_2 = \frac{\cos(\omega) + \sin(\omega) \sqrt{e^2 - 1}}{e},
$$

$$
\sin(f) \approx e^{-1} \sqrt{e^2 - 1} (1 - ar^{-1}),
$$

$$
h = na^2 \sqrt{e^2 - 1},
$$

and

$$
1.5(\ln 2 - 2) = 1.96 \approx 2.
$$

The next step is to define the slope coefficients of the position variations as follows:

$$
Z_R \equiv -\frac{1}{2} \frac{\Delta a^\infty}{a}, \tag{15a}
$$

$$
Z_\omega = -\frac{1}{\sqrt{e^2 - 1}} \frac{\Delta \omega^\infty}{e} + \Delta \Omega^\infty + \cos(i) \Delta \Omega^\infty, \tag{15b}
$$

$$
Z_W \equiv \frac{\sin(i) \Delta \Omega^\infty}{e} \left( \cos(\omega) + \sin(\omega) \sqrt{e^2 - 1} \right)
$$

$$
+ \frac{\Delta \Omega^\infty}{e} \left( \cos(\omega) \sqrt{e^2 - 1} - \sin(\omega) \right). \tag{15c}
$$

A position variation $\Delta r^\infty_{R^2S^2W}$ can be characterized as a function of $r$ that is approximately linear with respect to the slope coefficients.

The slope coefficient $Z_\omega$ describes how fast the position variation increases ($Z_\omega > 0$) or decreases ($Z_\omega < 0$). To obtain $\Delta r^\infty_{R^2S^2W}$, apply the known approximation formulas to $\sin(f^\infty)$, $\cos(f^\infty)$, and $\Delta M^\infty$ when $t \gg t^*$:

$$
\sin(f^\infty) \approx e^{-1} \sqrt{e^2 - 1} (1 - ar^{-1}), \tag{16a}
$$

$$
\cos(f^\infty) \approx -e^{-1} (1 - pr^{-1}), \tag{16b}
$$

$$
\frac{a^2 \Delta M^\infty}{r^2} = -\frac{3}{2} \frac{\Delta a^\infty}{a} \frac{t}{r^2} + \frac{a^2 \Delta M_0^\infty}{r^2} \cong 3 \Delta a^\infty \frac{t}{r}, \tag{16c}
$$

where the approximation of the time variable $t$ is given in Eq. (12), and the terms smaller than $1/r$ are ignored, including $\ln(r - a)/r^2$. By inserting the results of Eqs. (16a)–(16c) into Eq. (9b), the following equation is obtained.

$$
\Delta v^\infty_{R^2S^2W} = \begin{bmatrix}
-\frac{naZ_R}{r} - \frac{h}{r} (Z_\omega + Z_\gamma (e^2 - 1)^{-\frac{1}{2}}) \\
-\frac{naZ_\gamma}{r} + \frac{h}{r} (Z_R + Z_\gamma (e^2 - 1)^{-\frac{1}{2}}) \\
-\frac{na}{r} Z_W
\end{bmatrix} \tag{17}
$$

Although Eqs. (14) and (17) are less accurate than Eqs. (9a) and (9b), when $t \gg t^*$, these equations clarify how the position and velocity of a perturbed hyperbolic orbit evolve over time. The general kinematic properties of a perturbed hyperbolic orbit are simpler than those of an elliptical orbit, primarily because the hyperbolic orbit has only single interaction with the perturbation source. Accordingly, position variations are decoupled from each other and grow linearly with $r$. Velocity variations have both constant and decay

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parts, and the decay component is proportional to \(1/r\). Velocity variations in the radial and along-track directions are coupled, whereas the cross-track velocity variation is completely decoupled from the in-plane variations. Thus, along-track position variations \(Z_S\) can be inferred from radial velocity variation \(\Delta v_R\) due to the coupling of in-plane velocity variations.

4. Analysis of Observation Data

This section addresses the observational characteristics of the flyby anomaly. In past decades, anomalous orbital energy changes have been observed in unexpected residuals in the range and Doppler data of spacecraft during Earth flybys. Such range and Doppler tracking data has been produced by the ground-based tracking of the Deep Space Network (DSN), which is operated by NASA’s Jet Propulsion Laboratory.

The flyby anomaly shows anomalous orbital energy variations during Earth flybys. There have been efforts to analyze anomalous orbital energy variations.\(^{3,11}\) This section briefly reviews the analyses of anomalous orbital energy variations. On the other hand, after converting Doppler residuals to radial velocity variation, the post-interaction approximation, which is derived in Section 3, is applied to the observation of Galileo’s first EGA and NEAR’s EGA.

4.1. Anomalous orbital energy variation

Anomalous orbital energy variation has been considered the primary modeling parameter in many analyses.\(^7\)\(^-\)\(^9\)\(^,\)\(^11\)\(^,\)\(^12\)

In Anderson et al.,\(^3\) an empirical equation was proposed in order to reproduce the relative excess velocity variation data \(\Delta v_\infty /v_\infty\) from the six Earth flybys (Galileo’s first and second EGAs, and EGAs of NEAR, Cassini, Rosetta and MESSENGER):

\[
\frac{\Delta v_\infty}{v_\infty} = \frac{1}{2} \frac{\Delta E}{E} = K(\cos(\delta_i) - \cos(\delta_o)),
\]

where \(\Delta v_\infty = \Delta v_R(t = \infty) = -naZ_R\) is the hyperbolic excess velocity variation, \(K = 2\omega_E R_E/c = 3.099 \times 10^{-6}\), and \(\delta_i\) and \(\delta_o\) are the declinations of incoming and outgoing osculating asymptotic velocity vectors, respectively. The empirical equation implies a possible relationship between the flyby anomaly and the rotation of the Earth. For instance, \(K\) can be obtained with the Earth’s angular velocity of rotation \(\omega_E\), the Earth’s mean radius \(R_E\), and the speed of light \(c\). Additionally, \(\delta_i\) and \(\delta_o\) are measured with respect to the equator of the date.\(^10\)

Table 1 presents the orbital energy variations observed and their sigma values for the six flybys.

In Jouannic et al.,\(^7\) a correlation analysis was conducted on \(|\Delta v_\infty|\) with respect to 27 flyby parameters, and the highest correlation coefficient, 0.72, was obtained for time spent below an altitude of 2,000 km. This suggests that the flyby anomaly may originate from an Earth-centered perturbation.

4.2. Analysis of orbital kinematics

Figure 2 presents the radial velocity variation residuals of Galileo’s first EGA and NEAR’s EGA that were observed. Note that the radial velocity variations have been converted from Doppler residuals.\(^19\) The observational data show three characteristics. First, the radial velocity residual shows diurnal variations after the closest approach. This characteristic is caused by the inability of the pre-encounter data to predict the direction of the post-encounter velocity vector with sufficient accuracy.\(^3\) To remove these diurnal signals, three-axis instantaneous impulsive adjustments were needed at the perigee to achieve an adequate fit through the encounter.\(^2\) This implies that the epoch state was changed after the encounter.

Second, the radial distance residuals have an approximately constant rate after the closest approach, and \(\Delta R_\infty\) agrees with \(\Delta v_R\). This characteristic can be analyzed using the results from Section 3.2. Using the same process used for \(\Delta v_\infty\) (i.e., terms smaller than \(1/r\) are ignored including \(\ln(r/a)^2\)), differentiating Eq. (9a) with respect to time leads to the following:

\[
\Delta R_\infty = \frac{\Delta a a_\infty}{a} + \frac{h}{r} \left( a \sin(f) \Delta e_\infty - \frac{ae \cos(f) \Delta M_\infty}{r \sqrt{e^2 - 1}} \right) + \frac{3ne \sin(f) \Delta \Delta e_\infty}{2 \sqrt{e^2 - 1}}
\]

Table 1. Earth flyby parameters for Galileo and NEAR
(data provided by Anderson et al.\(^3\)).

| Parameters | Galileo 1st | Galileo 2nd | NEAR | Cassini | Rosetta | M’Ger |
|------------|------------|------------|------|---------|---------|-------|
| Date       | 12/8/90    | 12/8/92    | 1/23/98 | 8/18/99 | 3/4/05  | 8/2/05 |
| \(\Delta v_\infty\) (mm/s) | 3.92      | -4.60     | 13.46  | -2.00   | 1.80    | 0.02  |
| \(\sigma_{v_\infty}\) (mm/s) | 0.3       | 1.00      | 0.01   | 1       | 0.03    | 0.01  |
| Eq. (18) (mm/s) | 4.12     | -4.67     | 13.28  | -1.07   | 2.07    | 0.06  |

Fig. 2. Radial velocity residuals of (a) Galileo’s first EGA and (b) NEAR’s EGA (observation data provided by Anderson and Nieto\(^10\)).
after about 2.30 h. This characteristic depends on the behavior of various kinds of least-squares fitting curves, but the approximate value given in Table 2 is enough to make a point here. From Eqs. (17) and (19), the following can be obtained:

$$\Delta v_{R}^\infty = -na_{R}Z_{R} - \frac{2}{\sqrt{e^{2} - 1}} Z_{S}.$$  

(20)

Table 2. Post-interaction fitting parameters for Galileo and NEAR.

| Parameters | $\Delta v_{R}^{\text{Obs}}$ | $2Z_{S}^{\text{Obs}}$ | $Z_{S}^{\text{Obs}}$ |
|------------|-----------------------------|----------------------|---------------------|
| (t = 3t*) (mm/s) | $\sqrt{e^{2} - 1}$ | $10^{6}$ | $10^{6}$ |
| Galileo 1st | $-0.41$ | $0.386$ | $-0.612$ |
| NEAR | $-0.226$ | $2.581$ | $-0.245$ |

Note that Eq. (20) indicates that $\Delta v_{R}^\infty$ and $\Delta v_{S}^\infty$ are typically different.

However, no significant traces of $Z_{S}$ predicted by Eq. (20) are observed in Galileo’s first EGA or NEAR’s EGA. It is supported by the observation of $\Delta v_{R}(t = 3t^*)$. There can be more than one estimate of $\Delta v_{R}(t = 3t^*)$ based on approaches. For instance, $\Delta v_{R}(t = 3t^*)$ can be computed using various kinds of least-squares fitting curves, but the approximate value given in Table 2 is enough to make a point here. From Eq. (17), the following holds:

$$Z_{S}^{\text{Obs}} \simeq -\frac{h}{r}(na_{R}Z_{R}^{\text{Obs}} + \Delta v_{R}(t = 3t^*)) - \frac{2Z_{S}^{\text{Obs}}}{\sqrt{e^{2} - 1}},$$  

(21)

where $Z_{S}^{\text{Obs}} = -(na)^{-1}\Delta v_{S}^\infty$. Table 2 presents the values computed for $Z_{S}^{\text{Obs}}$ of Galileo’s first EGA and NEAR’s EGA based on the approximate values of $\Delta v_{R}(t = 3t^*)$. Figure 2 presents the post-interaction fitting curves for Galileo’s first EGA and NEAR’s EGA, which were computed using Eqs. (17) and (21) with Table 2. It can be observed that the post-interaction fitting curves well explain the behaviors of the radial velocity variations in the linear region $t > t^*$. Thus, Eq. (21) and Table 2 show that the effect of the along-track position variation is similar to or less than that of the radial position variation in Galileo’s first EGA and NEAR’s EGA, respectively.

Third, the radial velocity residual observed in Galileo’s first EGA shows a clear overshoot (Fig. 2(a)) and converges after about 2.30h. This characteristic depends on the behavior of the perturbation. In the derivation of Eq. (17), the higher-order terms smaller than $1/r$ are ignored:

$$\Delta v_{R} = \Delta v_{R}^{\infty} + \epsilon_{3}r^{-2} + \epsilon_{4}r^{-3} + \cdots,$$

$$\Delta v_{R} \approx \Delta v_{R}^{\infty} = -na_{R}Z_{R} - hr^{-1} \left( Z_{S} + 2Z_{R}(e^{2} - 1)^{-1} \right).$$

Note that as time passes, $\Delta v_{R}$ converges to the constant $\Delta v_{R}^{\infty} = -na_{R}Z_{R}$ because $r \rightarrow \infty$ as $t \rightarrow \infty$. Additionally, the higher-order terms smaller than $1/r$ decay faster than $1/r$ terms. As a result, the structure of $\Delta v_{R}^{\infty}$ determines how the Doppler residual converges to a specific value. There are two possibilities: an overshoot satisfying $\Delta v_{R}(t) > \Delta v_{R}^{\infty}$ and undershoot satisfying $\Delta v_{R}(t) < \Delta v_{R}^{\infty}$, where $t^* \ll t < \infty$. From Eq. (17), the conditions of overshoot and undershoot are given as follows:

$$\begin{align*}
(Z_{S} + 2Z_{R}(e^{2} - 1)^{-1}) & < 0, \quad \text{overshoot} \quad (22a) \\
(Z_{S} + 2Z_{R}(e^{2} - 1)^{-1}) & > 0, \quad \text{undershoot}. \quad (22b)
\end{align*}$$

Consequently, it must be satisfied that

$$2Z_{R}(e^{2} - 1)^{-1} < -Z_{S},$$  

(23)

for Galileo’s first EGA.

On the other hand, Eqs. (22a) and (22b) cannot give additional information on the NEAR observation data because no Doppler data was acquired during the closest approach (i.e., lasting approximately 3 h) because of the lack of appropriately located DSN tracking stations (Fig. 2(b)).

In brief, the post-interaction formulas derived in Section 3 provide additional constraints, which are Eqs. (21) and (23), on the along-track position variation. Although no direct observation data is available for the along-track position variation, an analysis of the orbital kinematics provides insight. The acceleration model must satisfy not only $\Delta v_{R}^{\infty}$ (or, $Z_{S}^{\text{Obs}}$) provided in Table 1, but also $Z_{S}^{\text{Obs}}$ provided in Table 2 and Eq. (23) to reproduce the radial position and velocity variations observed.

5. Hypothetical Acceleration Models

This section provides orbital kinematics analysis of hypothetical acceleration models suggested by various authors. It will be shown that the constraints on the orbital kinematics are more difficult to satisfy than orbital energy variation $\Delta E$ (i.e., equivalently, hyperbolic excess velocity variation $\Delta v_{R}^{\infty}$). Additionally, the validity of the post-interaction approximation formulas is confirmed by comparing them to numerical analysis results.

This paper tested three hypothetical acceleration models for the flyby anomaly. First, Gerrard-Sumner10) asserts that flyby anomaly can be explained by assuming the existence of an additional large-scale space dimension. An additional large-scale dimension, $r$, that is orthogonal to the three space dimensions and the time dimension is introduced and identified as the radius of curvature of four-dimensional space-time in a closed isotropic and expanding universe. In this space-time geometry, an additional acceleration, which is similar to the Coriolis effect, occurs along with gravity:
\[ \eta = \sin \left( \cos(\beta) \frac{v_K}{|w_{JK}|} - \sin(\beta) \frac{v_J}{|w_{JK}|} \right) \]

This acceleration occurs only during a heliocentric energy transfer such as a flyby.

Hasse et al.\(^{11}\) derived a classical field from the empirical energy change equation proposed by Anderson et al.\(^3\). There, it was shown that only the velocity-dependent field can reproduce the empirical energy change equation. The acceleration equation derived is given as:

\[
F = -K \left( 2(j - 1) \frac{H}{r^2} \cos(\delta) \ddot{R} - \frac{v^2}{r^2} - 2j \frac{H}{r^2} \sin(\delta) \ddot{W} \right),
\]

where \(\delta\) is declination, \(K\) is from Eq. (19), and \(j\) is a free parameter that is introduced to eliminate divergence of the velocity-dependent part at the poles. Note that, in this paper, \(j = 0.5\) and 7.9 are adopted, where \(j = 0.5\) shows an overshoot in Galileo’s first EGA and \(j = 7.9\) shows the best match between computed and observed values of \(\Delta v_p(t = 3\tau)\). This field has a significant feature that energy variation \(\Delta E\) is not accumulated for any bound orbit. Nevertheless, there must be temporal \(\Delta E\) for a bound orbit.

Lewis\(^{12}\) suggested kinetic energy coupling of a scalar field based on the fact that the equatorial/polar asymmetric pattern of \(\Delta E\) resembles the pattern of a quadrupole field:

\[
F = -\frac{1}{2} \frac{v^2}{r^2} \nabla \left( A(3z^2 - r^2) \right),
\]

where \(z\) is displacement along the Earth’s rotation axis and the constant \(A = 1.08 \times 10^{-15} \text{ m}^3\) is chosen to match the data of NEAR’s Earth flyby.

5.1. Simulation results

Using the hypothetical acceleration models introduced in Section 5, earth flyby simulations were conducted for Galileo’s first EGA and NEAR’s EGA. From the flyby data provided in Anderson et al.,\(^3\) the position and velocity vectors at the perigee can be constructed as follows\(^{12}\):

\[
\begin{align*}
\mathbf{r}_p & = (H + R_e) (\mathbf{v}_i - \mathbf{v}_o) / |\mathbf{v}_i - \mathbf{v}_o|, \quad (27a) \\
\mathbf{v}_p & = v_o (\mathbf{v}_i + \mathbf{v}_o) / |\mathbf{v}_i + \mathbf{v}_o|, \quad (27b)
\end{align*}
\]

where \(H\) is the perigee altitude, \(R_e\) is the equatorial radius of the Earth, and \(\mathbf{v}_i\) and \(\mathbf{v}_o\) are the incoming and outgoing asymptote velocities, respectively. Table 3 presents the orbital elements obtained using Eqs. (27a) and (27b).

The simulations were conducted using two dynamics models and a numerical integrator based on the Runge-Kutta-Fehlberg 7/8th algorithm. The default dynamics model includes the JGM3 70 \(\times\) 70 gravity model and third-body gravity by Sun and Luna; the custom dynamics model includes both a hypothetical acceleration model and all the terms in the default dynamics model. To compute the accelerations using the JGM3 70 \(\times\) 70 gravity model and third-body gravity by Sun and Luna, the Orbit Determination Tool Box (ODTBX), a space navigation tool developed by NASA/GSFC, is used for the simulations. Note that although Sections 2–4 assumed the point mass gravity, the method of perturbations guarantees the validity of the post-interaction approximation due to the linearity of the perturbations in the first-order analysis.\(^{17}\)

In order to apply the post-interaction approximation formulas, reference and perturbed trajectories need to be computed for the Galileo first and NEAR flybys. The reference trajectories are computed in two steps with the default dynamics model. First, the initial orbital elements of the reference orbits are computed by backward propagating the reference orbital elements at the perigee, provided in Table 3, from the epoch \(t = 0\) to the initial time \(t = -30\) h. Then, the reference trajectories are computed by forward propagating the initial orbital elements of the reference orbits from \(t_{init}\) to the final time \(t = 30\) h. On the other hand, the perturbed trajectories are computed using the custom dynamics models. Each custom dynamics model includes a hypothetical acceleration model described by one of Eqs. (24)–(26). After that, by comparing two trajectories, the numerical position and velocity variations are obtained. Alternatively, to obtain the post-interaction approximation results, the orbital element variations are calculated using linearized GVEs with Eqs. (5)–(7). Based on these equations, the post-interaction formulas are estimated using Eqs. (14) and (17).

As representative examples, selected numerical and analytical results, which are based on post-interaction formulas, are represented in Fig. 3. In the case of the post-interaction approximation method introduced in Section 3.2, the analytical results are guaranteed to converge to the numerical results; however, the speed of convergence is not guaranteed. Thus, the post-interaction approximation can qualitatively describe perturbed hyperbolic orbits, but only with restricted accuracy.

We will discuss two aspects of the acceleration models: i) the radial position variation, and ii) the radial velocity variation. The simulation results of kinematic parameters are summarized in Table 4 and Table 5 for Galileo’s first EGA and NEAR’s EGA, respectively. First, regarding radial position variation, it can be extrapolated in the nonlinear-region where \(t = 0\) as follows:

\[
\Delta r_p^\infty(t = 0) = (e + 3) \frac{\Delta a^\infty}{2} + \frac{a}{e} \Delta e^\infty - a \Delta M^\infty. \tag{28}
\]

As shown in Fig. 3(a), (c), (e), (g), Eq. (28) can describe the \(t\)-intercept of the post-interaction radial position variation curve. Note that the \(t\)-intercept indicates whether or not there exists a rapid increase or decrease in radial position variation around the periapsis passage. On the other hand, there was no significant effect of \(\Delta r_p^\infty(t = 0)\) in either observation of Galileo’s first EGA or NEAR’s EGA.\(^{21}\) Therefore, the acceleration models of Gerrard-Sumner and Hasse et al. cannot
Fig. 3. Radial position and velocity variations for Galileo’s first EGA.
(a) Position and (b) velocity variations of Gerrard-Sumner model. (c) Position and (d) velocity variations of Hasse’s model with $j = 0.5$. (e) Position and (f) velocity variations of Hasse’s model with $j = 7.9$. (g) Position and (h) velocity variations of Lewis’s model.
explain the range residuals observed in Galileo’s first EGA and NEAR's EGA.

Second, the hyperbolic excess velocity variation $\Delta v_{\text{inf}}$ (equivalently, $\Delta v_{\text{inf}}^R(t = \infty)$) has been used as a design parameter for acceleration models of the flyby anomaly. As a result, the values of $\Delta v_{\text{inf}}$ for each model are similar, but not identical, to the values listed in Table 1 ($\Delta v_{\text{inf}} = 3.92$, 13.46 mm/s for Galileo’s first EGA and NEAR’s EGA, respectively), with the exception of Gerrard-Summer model for NEAR’s EGA. In this case, the predicted value of $\Delta v_{\text{inf}} = 74.2$ mm/s, presented in Table 5, is much larger than the value observed.

In the case of Galileo’s first EGA, it is reported that $\Delta v_R$ converges after approximately 2.30 h (Fig. 2(a)) where its dynamical time scale is 1.20 h (Table 3). Based on these facts, $t = 3t^*$ is long enough to support the comparison of $\Delta v_R(t = 3t^*)$ with the $\Delta v_{\text{inf}}$ observed. Although the values of $\Delta v_{\text{inf}}$ are different for each model in Table 1 ($\Delta v_{\text{inf}} = 3.92$, 13.46 mm/s for Galileo’s first EGA and NEAR’s EGA, respectively), with the exception of Gerrard-Summer model for NEAR’s EGA. In this case, the predicted value of $\Delta v_{\text{inf}} = 74.2$ mm/s, presented in Table 5, is much larger than the value observed.

Table 4. Summary of Galileo’s first EGA simulation results.

| EGA  | Galileo 1st |
|------|-------------|
| Model | Gerrard-Summer | Hasse ($j = 0.5$) | Hasse ($j = 7.9$) | Lewis |
| $\frac{2Z_R}{\sqrt{(e^2 - 1)}} \times 10^5$ | 0.384 | 0.401 | 0.365 | 0.341 |
| $Z_3 \times 10^6$ | 10.9 | -29.2 | 10.7 | 3.19 |
| $\left( Z_5 + \frac{2Z_R}{\sqrt{(e^2 - 1)}} \right) \times 10^6$ | 11.3 | -28.8 | 11.0 | 3.53 |
| $\Delta v_R(t = 3t^*)$ (mm/s) | 1.39 | 27.7 | 3.81 | 0.72 |
| $\Delta v_{\text{inf}}$ (mm/s) | 3.90 | 4.07 | 3.71 | 3.47 |

Table 5. Summary of NEAR EGA simulation results.

| EGA  | NEAR |
|------|------|
| Model | Gerrard-Summer | Hasse ($j = 0.5$) | Hasse ($j = 7.9$) | Lewis |
| $\frac{2Z_R}{\sqrt{(e^2 - 1)}} \times 10^5$ | 14.2 | 2.55 | 2.12 | 2.50 |
| $Z_3 \times 10^6$ | 4.40 | -67.4 | 3.45 | -4.96 |
| $\left( Z_5 + \frac{2Z_R}{\sqrt{(e^2 - 1)}} \right) \times 10^6$ | 18.6 | -64.9 | 5.57 | -2.46 |
| $\Delta v_R(t = 3t^*)$ (mm/s) | 66.9 | 50.2 | 10.3 | 14.6 |
| $\Delta v_{\text{inf}}$ (mm/s) | 74.2 | 13.3 | 11.0 | 13.0 |

the other hand, in the case of NEAR’s EGA, it is difficult to evaluate the convergence of $\Delta v_R$ and whether or not it has overshoot or undershoot due to the lack of data during the closest approach, as shown in Fig. 2(b).

Obviously, none of the models can fully explain the kinematic properties of the flyby anomaly. Accordingly, it can be concluded that none of the acceleration models, which are tested in this study, can explain the flyby anomaly. Therefore, not only energy variation, but also kinematics should be considered in modeling flyby anomaly.

6. Conclusion

Based on the formulas of position and velocity variations, the general kinematics of perturbed hyperbolic orbits is investigated in this paper. As a result, post-interaction approximation formulas are derived and analyzed. According to the post-interaction approximation analysis, the position variations grow linearly after the closest approach as a function of radial distance from the Earth to the spacecraft. Each of the position variation components is decoupled from one another. Due to their linearity, the position variations can be characterized by their slope coefficients. On the other hand, the velocity variations have both constant and decay parts, and the decay part is inversely proportional to the radial distance. Interestingly, it is found that not only the radial slope coefficient, but also the along-track slope coefficient affects radial velocity variation. Radial velocity variation is important because it can be obtained through Doppler observations. Furthermore, the overall shape of radial velocity variation is determined by both the radial and along-track slope coefficients.

As representative examples, the observation data of Galileo and NEAR Earth flybys are analyzed. The radial velocity variations observed in Galileo’s first EGA and NEAR’s EGA show that the effect of the along-track position variations, which is represented as $Z_S$, must be insignificant in those EGAs. Additionally, the radial velocity variation of Galileo’s first EGA shows a clear overshoot. Such analysis results of the flyby kinematics lead to new constraints, which are Eqs. (21) and (23). Furthermore, Eq. (21) is applied to the observation data to compute the approximate values of $Z_S$. Therefore, newly added kinematical constraints can help when modeling the flyby anomaly because one challenge in solving it is that there is a limited number of EGAs.

So far, anomalous orbital energy variation has been used as the primary modeling parameter in many previous investigations of the flyby anomaly. This paper selected a few of those hypothetical acceleration models and analyzed their kinematical properties as representative examples. Such analysis of the kinematics shows that the acceleration models failed to reproduce the properties of the radial velocity variations presented in the observation data. This means that, in the modeling of flyby anomaly, not only the energy variation, but also the kinematic properties must be considered to reproduce the observation data.
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