Differential Weil descent

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ABSTRACT
In this note a differential version of the classical Weil descent is established in all characteristics. This yields a ready-to-deploy tool of differential restriction of scalars for differential varieties over finite differential field extensions.

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1. Introduction

In 1959, André Weil introduced his method of restricting scalars for a finite separable field extension $K \subseteq L$, cf. [13, §1.3]. It says that scalar extension, seen as a functor from $K$-algebras to $L$-algebras, has a left adjoint, which sends an $L$-algebra $D$ to a $K$-algebra $W(D)$, the Weil descent (aka Weil restriction) of $D$ from $L$ to $K$. The construction has been vastly generalized by Grothendieck [3], and used in numerous occasions in number theory [9] and algebraic geometry [7].

We establish a similar descent for differential algebras with respect to a given extension of differential rings $A \subseteq B$, where $B$ is finitely generated and free as an $A$-module. Here a differential ring $A$ is a commutative unital ring equipped with a distinguished set of derivations $A \xrightarrow{d} A$. If $D$ is a differential $B$-algebra with commuting derivations, its descent $W^{\text{diff}}(D)$ is a differential $A$-algebra in commuting derivations, see Theorem 3.4. This is deduced from our main result, which concerns rings and algebras with a single derivation:

Main Theorem (see Theorems 3.2 and 3.3)
Let $d : A \rightarrow A$ be a derivation of a ring $A$ and let $(B, \delta)$ be a differential $(A, d)$-algebra. Assume that $B$ is finitely generated and free as an $A$-module.

i. Let $(D, \partial)$ be a differential $(B, \delta)$-algebra. Then there is a unique derivation $\partial^W$ on the classical Weil descent $W(D)$ such that $(W(D), \partial^W)$ is a differential $(A, d)$-algebra and the unit of the adjunction at $D$ (given by the classical Weil descent), namely the map $W_D : D \rightarrow W(D) \otimes B$, is a differential $(B, \delta)$-algebra homomorphism $(D, \partial) \rightarrow (W(D) \otimes B, \partial^W \otimes \delta)$.

ii. If $B$ is a subring of $D$ and the inclusion is the structure morphism of $D$ as a $B$-algebra, then the assignment $\partial \mapsto \partial^W$ is a Lie-ring and an $A$-module homomorphism.

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In the proof of the main theorem, we give explicit formulas of the involved differential rings and morphisms, rather than only showing that they exist.

A very particular case of this theorem appears in [11, §5], when $A$ and $B$ are differential fields (of characteristic 0), but only under the assumption that $B$ has an $A$-basis consisting of constants. More precisely, assume $K$ is a field of characteristic zero equipped with a finite collection of commuting derivations $\{\delta_1, \ldots, \delta_m\}$ and let $L/K$ be a finite separable field extension. Recall that the derivations extend uniquely to $L$. Suppose $D$ is the differential coordinate ring of an affine differential variety $V$ over $L$; namely, $D = L\{V\} = L\{x_1, \ldots, x_n\}/I$ where $I$ is a radical differential ideal.

In the case when $L$ has a $K$-basis $b_1, \ldots, b_\ell$ of constants (meaning that $\delta_i(b_j) = 0$ for all $i, j$), then the construction of the differential Weil descent $W_{\text{diff}}(L\{V\})$ that appears in [11, §5] is as follows: Let $x = (x_1, \ldots, x_n)$ and $\bar{x} = (x = x^{(1)}, x^{(2)}, \ldots x^{(\ell)})$. For every differential polynomial $f \in I$, let $f^{(1)}, \ldots, f^{(\ell)} \in K\{\bar{x}\}$ be such that

$$f \left( \sum_{i=1}^\ell x^{(i)}b_i \right) = \sum_{i=1}^\ell f^{(i)}(\bar{x})b_i, \quad (1.1)$$

in the differential polynomial ring $L\{\bar{x}\}$. Then, $W_{\text{diff}}(L\{V\})$ is defined as the differential coordinate ring over $K$ of the differential variety whose defining ideal is generated by $f^{(1)}, \ldots, f^{(\ell)}$ as $f$ varies in $I$. We note that when computing the $f^{(i)}$'s using (1.1) the fact that the $b_i$'s are constants of $L$ allows us to treat them as constants in $L\{\bar{x}\}$, making the computation relatively simple. However, a basis of constants does not always exist, as we point out in the following example.

### 1.1. Example

We work in the ordinary case $\Delta = \{\delta\}$. Let $K = \mathbb{Q}(t)$ with $\delta t = 1$ and consider the finite extension $L = K(b)$ where $b^2 = t$. Then the (unique) induced derivation on $L$ is given by $\delta b = \frac{b}{2t} = \frac{b}{2\ell}$. Fix the basis $(1, b)$ of $L$ as a $K$-module. Consider the differential variety $V$ given by $\delta x = 0$ (i.e., $V$ is simply the constants of $L$) viewed as a differential variety over $L$. The differential Weil descent $W_{\text{diff}}(V)$ is obtained as follows; write $x$ as $x_1 + x_2 b$ and compute

$$\delta(x_1 + bx_2) = \delta x_1 + (\delta b)x_2 + b\delta x_2 = \delta x_1 + \frac{b}{2t}x_2 + b\delta x_2 = \delta x_1 + \left( \frac{x_2}{2t} + \delta x_2 \right)b.$$

Thus, $W_{\text{diff}}(V)$ is the affine differential variety over $K$ given by the equations

$$\delta x_1 = 0 \quad \text{and} \quad \delta x_2 + \frac{x_2}{2t} = 0.$$

Note that this is not contained in a product of the constants, as one might expect. Of course, if $\delta(b)$ were zero we would instead obtain the equations $\delta x_1 = 0$ and $\delta x_2 = 0$ (which would occur if $\delta$ were trivial on $K$, for instance).

In a forthcoming paper by the authors several applications of the differential Weil descent are exposed. The main one addresses a method to produce differential fields, in finitely many commuting derivations and of characteristic 0, which possess a minimal differential closure (or in Kolchin’s terminology, constraint closure). First examples of such differential fields were given by Singer in [12], where he showed that, for every closed ordered differential field $K$ in one derivation, the algebraic closure $K[i]$ is differentially closed. In the coming paper, we introduce the notion of a differentially large field (in analogy to the notion of a large field in classical field theory) and use the differential Weil descent to generalize Singer’s result; namely, we will show that algebraic extensions (equipped with the unique induced derivations) of differentially large fields are again differentially large. This is in analogy to the algebraic case where the classical Weil descent is used to show that algebraic extensions of large fields are again large, see [10, Proposition 1.2].
We expect many more applications of the differential Weil descent. For instance, we expect that our results will be a valuable tool in differential Galois cohomology and the parameterized Picard–Vessiot theory for linear differential equations. This will potentially be in the form of finiteness results for cohomology groups of linear differential algebraic groups.

2. Classical Weil descent for algebras

In this section we review the classical construction of Weil descent of scalars for algebras, see for example \[1, \S 7.6\], \[8, \S 2\] and \[3\]. For our purposes we need certain explicit formulas, so we give details.

Convention. Throughout, we assume our rings and algebras to be commutative and unital; ring and algebra homomorphisms are meant to be unital as well.

Let \( A \) be a ring and let \( B \) be an \( A \)-algebra. For each \( A \)-algebra \( C \), the scalar extension by \( B \) is the \( B \)-algebra \( C\otimes_A B \) with structure map \( b \mapsto 1 \otimes b \). This assignment has a natural extension to a covariant functor \( F : A/\text{Alg} \to B/\text{Alg} \).

The functor \( F \) has a right adjoint \( B/\text{Alg} \to A/\text{Alg} \) given by restricting scalars. If \( B \) is finitely generated and free as an \( A \)-module, then \( F \) also has a left adjoint \( W \), called Weil descent, or Weil restriction. We start with a reminder on left adjoints in general, readymade for use later on.

2.1. Fact

\[5, \text{Thm 2, p. 83, Cor. 1,2, p. 84}\]

Let \( F : C \to D \) be a covariant functor between categories \( C \) and \( D \).

i. The following are equivalent.

a. \( F \) has a left adjoint \( W \), i.e., \( W : D \to C \) is a covariant functor such that for all \( D \in D \) the functor \( \text{Hom}_D(D,F(\_)) : C \to \text{Sets} \) is represented by \( W(D) \), meaning that the functors \( \text{Hom}_C(W(D),\_ \) and \( \text{Hom}_D(D,F(\_)) \) are isomorphic.\(^2\)

b. For each \( D \in D \) there are \( W(D) \in C \) and a \( D \)-morphism \( W_D : D \to F(W(D)) \) such that the following condition holds:

\[ (\dagger) : \text{For every } C \in C \text{ and each morphism } f : D \to F(C), \text{ there is a unique } C \text{-morphism } g : W(D) \to C \text{ such that the following diagram commutes} \]

\[
\begin{array}{ccc}
F(W(D)) & \xrightarrow{F(g)} & F(C) \\
\downarrow W_D & & \downarrow \quad \quad f \\
D & & \\
\end{array}
\]

In other words, \( W_D \) gives rise to a bijection \( \tau(D,C) : \text{Hom}_C(W(D),C) \to \text{Hom}_D(D,F(C)), g \mapsto F(g) \circ W_D \).

ii. If (i)a holds, then for every such functor \( W \), all \( D \in D \) and each isomorphism \( \tau(D,\_ : \text{Hom}_C(W(D),\_) \to \text{Hom}_D(D,F(\_)) \) as in (i)a, the choice \( W(D) \) and \( W_D = \tau(D,W(D))(\text{id}_{W(D)}) \) satisfy property \((\dagger)\) of (i)b.

The assignment \( D \mapsto W_D \) is a natural transformation \( \text{id}_D \to F \circ W \) and is called the unit of the adjunction; \( W_D \) is called the component at \( D \) of that unit.

Similarly, for each \( C \in C \) the morphism \( F_C : W(F(C)) \to C \) that is sent to \( \text{id}_{F(C)} \) via \( \tau(F(C),C) \) gives rise to a natural transformation \( W \circ F \to \text{id}_C \), called the counit of the adjunction; \( F_C \) is called the component at \( C \) of that counit.

\(^1\)As a general reference for tensor products, specifically in the category of algebras we refer to \[6, \text{Appendix A}\]

\(^2\)Recall that two functors are isomorphic if there is an invertible natural transformation between them.
iii. If (i)b holds, then for any choice of objects $W(D)$ and morphisms $W_D$ as in (i)b, for $D \in \mathcal{D}$, the assignment $D \mapsto W(D)$ can be extended to a functor $W: \mathcal{D} \to \mathcal{C}$ satisfying (i)a as follows: Take a morphism $f_0: D \to D'$ and set $f := W_{D'} \circ f_0$ and $C := W(D')$. Then define $W(f_0): W(D) \to W(D')$ as the unique $\mathcal{C}$-morphism $W(D) \to W(D')$ such that the diagram

\[
\begin{array}{ccc}
F(W(D)) & \xrightarrow{F(W(f_0))} & F(C) = F(W(D')) \\
\downarrow W_D & & \downarrow W_{D'} \\
D & \xrightarrow{f} & D'
\end{array}
\]

commutes, according to (†).

iv. Any two functors that are left adjoint to $F$ are isomorphic.

v. If $W$ is left adjoint to $F$, then $W$ preserves all co-limits, cf. [5, p. 119, last paragraph]. For example $W$ preserves direct limits and fiber sums (aka pushouts).

### 2.2. Notation and setup

We return to our setup of a ring $A$ and an $A$-algebra $B$. Let

$$F: A \text{- Alg} \to B \text{- Alg}$$

be the functor defined by $F(C) = C \otimes B$ and for $\phi: C \to C'$, $F(\phi) = \phi \otimes \text{id}_B$. Here and below, tensor products are taken over $A$, unless stated otherwise.

We will from now on assume that $B$ is free and finitely generated as an $A$-module of dimension $\ell$ over $A$. We also fix generators $b_1, \ldots, b_\ell$ of the $A$-module $B$.

For $i \in \{1, \ldots, \ell\}$ let

$$\lambda_i: B \to A, \lambda_i \left( \sum_{j=1}^{\ell} a_j b_j \right) = a_i$$

be the $A$-module homomorphism dual to $A \to B, a \mapsto a \cdot b_i$. If $C$ is an $A$-algebra we write $\lambda_i^C = \text{id}_C \otimes \lambda_i: C \otimes B \to C \otimes A = C$ for the base change of $\lambda_i$ to $C$. Since $b = \sum_i \lambda_i(b) b_i$ for $b \in B$ we obtain

$$c \otimes b = c \otimes \sum_i \lambda_i(b) b_i = \sum_i (\lambda_i^C(b) c) \otimes b_i,$$

for $c \in C$. Hence $\lambda_i^C(c \otimes b) = \lambda_i(b) \cdot c$ is the coefficient of $c \otimes b$ at $1 \otimes b_i$ when it is written in the basis $1 \otimes b_1, \ldots, 1 \otimes b_\ell$ of the free $C$-module $C \otimes B$.

This extends to all $f \in C \otimes B$, thus

\[
(*) \quad f = \sum_{i=1}^{\ell} (\lambda_i^C(f) \otimes b_i).
\]

### 2.3. Definition

Let $T$ be a set of indeterminates for $A$ and $B$. We define an $A$-algebra $W(B[T]) = A[T]^{\otimes \ell} (= A[T] \otimes \ldots \otimes A[T])$. For $i \in \{1, \ldots, \ell\}$ and $t \in T$ we write
Let \( W_{B[T]} \) be the unique \( B \)-algebra homomorphism
\[
W_{B[T]} : B[T] \to F(W(B[T])) = A[T]^{\otimes \ell} \otimes B \quad \text{with}
\]
\[
W_{B[T]}(t) = \sum_{i=1}^{\ell} (t(i) \otimes b_i) \quad (t \in T).
\]

Further, let \( F_{A[T]} \) be the unique \( A \)-algebra homomorphism
\[
F_{A[T]} : W(F(A[T])) = A[T]^{\otimes \ell} \to A[T]
\]
with the property \( F_{A[T]}(t(i)) = \lambda_i(1) \cdot t \) for \( t \in T, i \in \{1, \ldots, \ell\} \).

### 2.4. Explicit description of the Weil descent of polynomial algebras

The \( A \)-algebra \( W(B[T]) \) and the morphism \( W_{B[T]} \) described above satisfy condition (\dag) of 2.1(i)b. Hence by 2.1(iii) we may choose \( W(B[T]) \) as the Weil descent of \( B[T] \), and \( W_{B[T]} \) as the unit of the adjunction at \( B[T] \); these choices are then independent of the basis \( b_1, \ldots, b_\ell \) up to a natural \( A \)-algebra isomorphism (see 2.1(iv)).

Explicitly, for every \( C \in \mathcal{A} \), the map
\[
\tau = \tau(B[T], C) : \text{Hom}_{A-\mathcal{A}}(A[T]^{\otimes \ell}, C) \to \text{Hom}_{B-\mathcal{A}}(B[T], C \otimes B)
\]
\[
\phi \mapsto F(\phi) \circ W_{B[T]} = (\phi \otimes \text{id}_B) \circ W_{B[T]}
\]
is bijective, where \( \phi \otimes \text{id}_B = F(\phi) : F(W(B[T])) = A[T]^{\otimes \ell} \otimes B \to C \otimes B \) is the base change of \( \phi \).

For \( t \in T \) we have
\[
\tau(\phi)(t) = \sum_{i=1}^{\ell} (\phi(t(i)) \otimes b_i).
\]

The compositional inverse of \( \tau = \tau(B[T], C) \) is defined as follows. Let \( \psi : B[T] \to C \otimes B \) be a \( B \)-algebra homomorphism. We define an \( A \)-algebra homomorphism \( \phi : A[T]^{\otimes \ell} \to C \) by
\[
\phi(t(i)) := \lambda_i^C(\psi(t))(t \in T, i = 1, \ldots, \ell).
\]

Since \( A[T] \otimes B \cong B[T] \), \( \psi \) is uniquely determined by \( \{\psi(t)|t \in T\} \) and we see that \( \phi \) is the unique preimage of \( \psi \) under \( \tau \).

Further, one checks easily that \( F_{A[T]} \) is the component of the counit of the adjunction at \( A[T] \).

### 2.5. Explicit description of the Weil descent of \( B \)-algebras

Now let \( D \) be a \( B \)-algebra. Take a surjective \( B \)-algebra homomorphism \( \pi_D : B[T] \to D \) for some set \( T \) of indeterminates. Let \( I_D \) be the ideal generated in \( W(B[T]) = A[T]^{\otimes \ell} \) by all the \( \lambda_i^{W(B[T])}(W_{B[T]}(f)) \), with \( i \in \{1, \ldots, \ell\} \) and \( f \in \ker(\pi_D) \). We define
\[
W(D) := W(B[T])/I_D
\]
and write
\[
W(\pi_D) : W(B[T]) \to W(D)
\]
for the residue map. Then the bijection \( \tau(B[T], C) \) from 2.4 induces a bijection
\[
\tau(D, C) : \text{Hom}_{A-\mathcal{A}}(W(D), C) \to \text{Hom}_{B-\mathcal{A}}(D, F(C))
\]
Finally, we display the map together with – straightforward computation. after checking $s^{2.1(iii)}$ gives the definition of $\text{ward}$. For a ring $\Lambda$ we let $\text{Der}_\Lambda(\Lambda)$ denote the family of derivations on $\Lambda$ for all $\Lambda$. Let $\varphi \in \text{Hom}_\Lambda(\Lambda)$. Using $2.1(iii),(iv)$ we have justified our choice of $\tau(\Lambda)$ in 2.1(ii). Take $t \in T$. Then by $(\ast)$ with $C = \Lambda$, $\phi = \text{id}_\Lambda$ we see that

$$\tau(D,C)(\phi) \circ \pi_D = \tau(B[T],C)(\phi \circ \pi_D)$$

$$= ((\phi \circ \pi_D) \otimes \text{id}_B) \circ W_{B[T]}.$$

Finally, we display the map $W_D := \tau(D,W(\Lambda))(\text{id}_{W(\Lambda)}) : D \to F(W(\Lambda))$ explicitly and show that – together with $W(D)$ – it satisfies the mapping property of $(\ast)$ in 2.1(iii). Take $t \in T$. Then by $(\ast)$ with $C = \Lambda$, $\phi = \text{id}_{W(\Lambda)}$ we see that

$$W_D(\pi_D(t)) = \sum_{i=1}^\ell W(\pi_D)(t(i)) \otimes b_i = \sum_{i=1}^\ell (t(i) \mod I_D) \otimes b_i.$$

Pick an $\Lambda$-algebra $C$. Since $\tau(D,C)$ is bijective, the mapping property of $(\ast)$ in 2.1(iii) follows after checking $\tau(D,C)(\phi) = F(\phi) \circ W_D$ for all $\phi \in \text{Hom}_\Lambda(\Lambda)$. Using $(\ast)$ this is a straightforward computation.

Using 2.1(iii),(iv) we have justified our choice of $W(D)$ and $W_D$ for the Weil descent. Finally, 2.1(iii) gives the definition of $W$ on morphisms.

3. Differential Weil descent

In this section we present a construction of a Weil descent functor in the category of differential algebras in arbitrary characteristic. We first recall some basic facts about differential algebras and their tensor products. We continue to assume that our rings and algebras are unital and commutative.

3.1. Generalities about differential algebra

The following are well-known generalities on differential algebras whose proofs are straightforward. For a ring $A$ we let $\text{Der}(A)$ denote the family of derivations on $A$.

i. Let $A$ be a ring and let $T$ be a not necessarily finite set of indeterminates over $A$. For each $t \in T$ let $f_t \in A[T]$. Let $d \in \text{Der}(A)$. Then there is a unique derivation $\delta$ of $A[T]$ extending $d$ with $\delta(t) = f_t$ for all $t \in T$. For $d, \delta \in \text{Der}(A)$ we write $[d, \delta] : A \to A$ for the Lie-bracket of $d$ and $\delta$, defined by $[d, \delta](a) = d\delta(a) - \delta d(a)$. Notice that $[d, \delta]$ is again a derivation of $A$.

ii. Let $A$ be a ring and let $S \subseteq A$ be a set of generators of the ring $A$.

a. Let $d$ and $(\delta_i)_{i \in I}$ be derivations on $A$ and suppose there are $a_i \in A$, all but finitely many zero, with $d(s) = \sum_{i \in I} a_i \delta_i(s)$ for all $s \in S$. Then $d = \sum_{i \in I} a_i \delta_i$.

b. Let $\phi : A \to B$ be a ring homomorphism and let $d : A \to A, \delta : B \to B$ be derivations. If $\phi(ds) = \delta(\phi(s))$ for all $s \in S$, then $\phi$ is a differential homomorphism $(A,d) \to (B,\delta)$. 
iii. Let \( d \in \text{Der}(A) \) and let \((B, \delta), (C, \partial)\) be differential \((A, d)\)-algebras. Then there is a unique derivation \( \delta \otimes \partial \) on \( B \otimes_A C \) such that the natural maps \( B \to B \otimes_A C, C \to B \otimes_A C \) are differential maps, cf. [2, Chapter 2 (1.1), p. 21].

iv. Now let \( d_1, d_2 \in \text{Der}(A), \delta_1, \delta_2 \in \text{Der}(B) \) and \( \partial_1, \partial_2 \in \text{Der}(C) \) such that \((B, \delta_1), (C, \partial_1)\) are differential \((A, d_1)\)-algebras. Then, for \( a_1, a_2 \in A \), straightforward checking shows that

\[
(a_1 \delta_1 + a_2 \delta_2) \otimes (a_1 \partial_1 + a_2 \partial_2) = a_1(\delta_1 \otimes \partial_1) + a_2(\delta_2 \otimes \partial_2).
\]

As in Section 2 we work with a ring \( A \) and an \( A \)-algebra \( B \) that is free and finitely generated by \( b_1, \ldots, b_t \) as an \( A \)-module. We fix a derivation \( d \) on \( A \) and a derivation \( \delta \) on \( B \) such that \((B, \delta)\) is a differential \((A, d)\)-algebra (meaning that the structure map \( A \to B \) is differential).

By 3.1(iii), for any differential \((A, d)\)-algebra \((C, \partial_C)\), there is a unique derivation \( \partial_C \otimes \delta \) on \( F(C) = C \otimes B \) such that the natural map \( C \to F(C) \) is a differential \((B, \delta)\)-algebra morphism.

**3.2. Theorem**

Let \((D, \partial_D)\) be a differential \((B, \delta)\)-algebra. Then there is a unique derivation \( \partial_D^W \) on \( W(D) \) such that \((W(D), \partial_D^W)\) is a differential \((A, d)\)-algebra and

\[ W_D : (D, \partial_D) \to (F(W(D)), \partial_D^W \otimes \delta) \]

is a differential \((B, \delta)\)-algebra homomorphism, i.e., \( W_D \circ \partial_D = (\partial_D^W \otimes \delta) \circ W_D \).

Furthermore, \( \partial_D^W \) only depends on \( \partial_D \) and not on \( \delta \).

**Proof.** Take any set \( T \) of differential indeterminates and a surjective \((B, \delta)\)-algebra homomorphism \( \pi_D : (B \{ T \}, \partial) \to (D, \partial_D) \). Here, the differential polynomial ring \( B \{ T \} \) is considered just as polynomial ring over \( B \) in the algebraic indeterminates \( t_i, \) \( t \in T \) and \( \theta \in \Theta := \{ \partial^i : i \geq 0 \} \). Further, \( \partial = \partial_{B[T]} : B \{ T \} \to B \{ T \} \) is the natural derivation, thus \( \partial t_i = t_{i0} \).

We choose \( W_{B[T]} : B \{ T \} \to F(W(B \{ T \})) \) according to 2.3 for the set of indeterminates \( \{ t_{0i} t \in T, \theta \in \Theta \} \) and \( W_D : D \to F(W(D)) \) according to 2.5. Also recall (x) in 2.5, which says that \( W_D(\pi_D(t_{0i})) = \sum_{i=1}^t W(\pi_D(t_{0i})) \otimes b_i \).

Claim 1. If \( \varepsilon : W(D) \to W(D) \) is a derivation such that \((W(D), \varepsilon)\) is a differential \( A \)-algebra, then for all \( t \in T \) and any \( \theta \in \Theta \) we have

\[
((\varepsilon \otimes \delta) \circ W_D)(\pi_D(t_{0\theta})) = \sum_{i=1}^t \left( \varepsilon(W(\pi_D(t_{0\theta}(i)))) + \sum_{j=1}^t \lambda_i(\delta b_j) \cdot W(\pi_D(t_{0\theta}(j))) \right) \otimes b_i.
\]

See 3.1(iii) for the definition of \( \varepsilon \otimes \delta \).

**Proof.** This is a straightforward calculation using \( \delta b_i = \sum_{j=1}^t \lambda_i(\delta b_j) b_j \).

Claim 2. If \( \varepsilon : W(D) \to W(D) \) is a derivation such that \((W(D), \varepsilon)\) is a differential \( A \)-algebra, then \( W_D \circ \partial = (\varepsilon \otimes \delta) \circ W_D \) if and only if for all \( t_{0\theta}(i) \) we have

\[
(*) \quad \varepsilon(W(\pi_D(t_{0\theta}(i)))) = W(\pi_D(t_{0\theta}(i))) - \sum_{j=1}^t \lambda_i(\delta b_j) \cdot W(\pi_D(t_{0\theta}(j))).
\]

**Proof.** By 3.1(ii)(b), \( W_D \circ \partial = (\varepsilon \otimes \delta) \circ W_D \) if and only if \( ((\varepsilon \otimes \delta) \circ W_D)(\pi_D(t_{0\theta})) = (W_D \circ \partial)(\pi_D(t_{0\theta})) \) for all \( t_{0\theta} \). By Claim 1 this is equivalent to
\[
\sum_{i=1}^{\ell} \left( \varepsilon(W(\pi_D)(t_0(i))) + \sum_{j=1}^{\ell} \lambda_i(\delta b_j) \cdot W(\pi_D)(t_0(j)) \right) \otimes b_i \\
= W_D(\partial(\pi_D(t_0))) \\
= W_D(\pi_D(\partial t_0)), \text{ since } \pi_D \text{ is a differential map} \\
= W_D(\pi_D(t_{0\ell})) \\
= \sum_{i=1}^{\ell} W(\pi_D)(t_{0\ell}(i)) \otimes b_i, \text{ by (2) in 2.5.}
\]

Since \(1 \otimes b_1, ... , 1 \otimes b_\ell\) is a basis of \(F(W(D))\) over \(W(D)\), the identity is equivalent to \((*)\) being true for all \(i \in \{1, ..., \ell\}\).

Claim 2 implies the uniqueness statement of the theorem, because the set of all the \(W(\pi_D)(t_0(i))\) generates \(W(D)\). For existence, we first deal with \(B\{T\}\) instead of \(D\). In that case, Claim 2 says that we only need to find a derivation \(\partial_{B(T)}^W\) on \(W(B\{T\})\) such that \((W(B\{T\}), \partial_{B(T)}^W)\) is a differential \((A, d)\)-algebra with the property

\[
\partial_{B(T)}^W(t_0(i)) = t_{0\ell}(i) - \sum_{j=1}^{\ell} \lambda_i(\delta(b_j)) \cdot t_0(j).
\]

By 3.1(i) applied to the polynomial ring \(W(B\{T\})\) over \(A\), such a derivation indeed exists.\(^3\) It remains to prove that there is a derivation \(\partial_{B(T)}^W\) of \(W(D)\) as required.

Claim 3. The ideal \(I_D\) of \(W(B\{T\})\) (see 2.5) is a differential ideal for \(\partial_{B(T)}^W\).

**Proof.** Let \(f \in \ker(\pi_D)\). Then \(W_{B(T)}(f) = \sum_{i=1}^{\ell} g_i \otimes b_i\), where \(g_i = \lambda_i^{W(B(T))}(W_{B(T)}(f))\). By definition of \(I_D\) it suffices to show that \(\partial_{B(T)}^W(g_i) \in I_D\).\(^4\) Now one checks that

\[
W_{B(T)}(\partial_{B(T)}(f)) = \sum_{i=1}^{\ell} \left( \partial_{B(T)}^W(g_i) + \sum_{j=1}^{\ell} \lambda_i(b_j)g_j \right) \otimes b_i
\]

Since \(1 \otimes b_1, ..., 1 \otimes b_\ell\) is a basis of \(F(W(B\{T\}))\) over \(W(B\{T\})\) we see that

\[
\lambda_i^{W(B(T))}(W_{B(T)}(\partial_{B(T)}(f))) = \partial_{B(T)}^W(g_i) + \sum_{j=1}^{\ell} \lambda_i(b_j)g_j,
\]

The left hand side here is in \(I_D\) by definition of \(I_D\) and because \(\ker(\pi)\) is differential for \(\partial_{B(T)}\). As all \(g_i \in I_D\) this entails \(\partial_{B(T)}^W(g_i) \in I_D\).

By Claim 3, the derivation \(\partial_{B(T)}\) induces a derivation \(\partial_D^W\) of \(W(D) = W(B\{T\})/I_D\) such that \((W(D), \partial_D^W)\) is a differential \((A, d)\)-algebra. It remains to show that \(W_D\) is a differential \((B, \delta)\)-algebra homomorphism, i.e., \(W_D \circ \partial_D = (\partial_D^W \otimes \delta) \circ W_D\). This can be seen by a diagram chase as follows. Consider the diagram of maps

\(^3\)Notice that \(W(B\{T\})\) naturally is a differential polynomial ring over \(A\), but \(\partial_{B(T)}^W\) is in general not the natural derivation of \(W(B\{T\})\).

\(^4\)Notice that the module homomorphism \(\lambda_i^{W(B(T))}: F(W(B\{T\})) \rightarrow W(B\{T\})\) does not in general commute with the derivations.
The claim is that the back side of this cube is commutative. Now, all other sides of the cube are commutative squares, because

- Bottom and top of the cube are identical and commute as a property of the classical Weil descent.
- The front of the cube commutes as we know the theorem already for \( B\{T\}, \partial_{B{T}} \).
- The square on the left hand side commutes by choice of \( B\{T\}, \partial_{B{T}} \).
- The square on the right hand side commutes by applying base change to \( B \) to the definition of \( \partial_W \).

Since \( \pi \) is surjective, we see that the back of the cube also commutes. This finishes the proof of existence and uniqueness of \( \partial_W \). From Claim 2 we see that the definition of \( \partial_{B{T}} \) only depends on \( \partial_{B{T}} \) and not on \( \delta \), because the structure map \( B \to D \) is differential. But then by construction of \( \partial_W \) after Claim 3, \( \partial_W \) only depends on \( \partial_D \) and not on \( \delta \).

\[ \square \]

### 3.3. Theorem

Let again \( B \) be an \( A \)-algebra that is finitely generated and free as an \( A \)-module and let \( D \) be a \( B \)-algebra.

Let \( \text{Der}_B(D) \) be the set of all \( \partial \in \text{Der}(D) \) for which there are derivations \( d \) of \( A \) and \( \delta \) of \( B \) such that the structure maps of \( B \) and \( D \) are differential maps \( (A,d) \to (B,\delta) \) and \( (B,\delta) \to (D,\partial) \), respectively.\(^5\)

Then \( \text{Der}_B(D) \) is an \( A \)-submodule and a Lie subring of \( \text{Der}(D) \) and the map \( \text{Der}_B(D) \to \text{Der}(W(D)) \) that sends \( \partial \) to the derivation \( \partial_W \) defined in 3.2, is an \( A \)-module and a Lie ring homomorphism. Explicitly, given \( \partial_1, \partial_2 \in \text{Der}_B(D) \) we have

i. \( (a_1\partial_1 + a_2\partial_2)_W = a_1\partial_1^W + a_2\partial_2^W \) for all \( a_1, a_2 \in A \).

ii. \( [\partial_1, \partial_2]^W = [\partial_1^W, \partial_2^W] \). In particular, \( \partial_1^W, \partial_2^W \) commute if \( \partial_1, \partial_2 \) commute.

\( ^5 \) If the structure morphism of \( D \) as a \( B \)-algebra is injective and we think of the structure maps \( A \to B \) and \( B \to D \) as inclusions, then \( \text{Der}_B(D) \) is the set of all derivations of \( D \) that restrict to derivations on \( A \) and \( B \).
Proof. In each case, the derivation of $W(D)$ on the right hand side turns $W(D)$ into a differential $A$-algebra, when $A$ is furnished with the derivation $a_1 d_1 + a_2 d_2$ and $[d_1, d_2]$ respectively. By uniqueness in 3.2 we thus only need to verify the defining equation of the left hand side for the right hand side.

i. Using 3.1(iv)(a) we get $$((a_1 \partial_1^W + a_2 \partial_2^W) \otimes (a_1 \delta_1 + a_2 \delta_2)) \circ W_D = a_1 (\partial_1^W \otimes \delta_1) \circ W_D + a_2 (\partial_2^W \otimes \delta_2) \circ W_D = a_1 W_D \circ \partial_1 + a_2 W_D \circ \partial_2 = W_D \circ (a_1 \partial_1 + a_2 \partial_2),$$ since $W_D$ is an $A$-algebra homomorphism.

ii. Using 3.1(iv)(b) we get $$((\partial_1^W, \partial_2^W) \otimes [\delta_1, \delta_2]) \circ W_D = [\partial_1^W \otimes \delta_1, \partial_2^W \otimes \delta_2] \circ W_D = (\partial_1^W \otimes \delta_1) \circ (\partial_2^W \otimes \delta_2) \circ W_D - (\partial_2^W \otimes \delta_2) \circ (\partial_1^W \otimes \delta_1) \circ W_D = W_D \circ [\partial_1 \circ \partial_2 - W_D \circ \partial_2 \circ \partial_1 = W_D \circ [\partial_1, \partial_2].$$

Theorems 3.2 and 3.3 establish

3.4. The differential Weil descent

Let $A$ be a ring and let $d = (d_i)_{i \in I}$ be a family of derivations of $A$. A differential $(A, d)$-algebra is an $A$-algebra $C$ together with derivations $(\eta_i)_{i \in I}$ of $C$ such that the structure map $A \to C$ is a differential morphism $(A, d_i) \to (C, \eta_i)$ for all $i \in I$. Let $(A, d) - \text{Alg}$ be the category of differential $(A, d)$-algebras whose morphisms are ring homomorphisms preserving the appropriate derivations.

We fix a differential $(A, d)$-algebra $(B, \delta)$, with $\delta = (\delta_i)_{i \in I}$, such that $B$ is finitely generated and free as an $A$-module. Then

i. The functor $F^{\text{diff}} : (A, d) - \text{Alg} \to (B, \delta) - \text{Alg}$ that sends $(C, \eta)$ to $(C \otimes B, (\eta_i \otimes \delta_i)_{i \in I})$ has a left adjoint $W^{\text{diff}} : (B, \delta) - \text{Alg} \to (A, d) - \text{Alg}$, which we call the differential Weil descent (or differential Weil restriction) from $(B, \delta)$ to $(A, d)$. It sends $(D, \partial)$ to $(W(D), \partial^W)$ where $\partial^W = (\partial_i^W)_{i \in I}$ with $\partial_i^W$ as defined in 3.2, and a morphism $f$ to $W(f)$.

ii. Let $(C, \eta) \in (A, d) - \text{Alg}$ and let $(D, \partial) \in (B, \delta) - \text{Alg}$. Then the bijection $\tau(D, C) : \text{Hom}_{(A, d) - \text{Alg}}(W(D), C) \to \text{Hom}_{(B, \delta) - \text{Alg}}(D, F(C)), \phi \mapsto F(\phi) \circ W_D$ from the classical Weil descent 2.5 restricts to a bijection

$$\text{Hom}_{(A, d) - \text{Alg}}(W^{\text{diff}}(D, \partial), (C, \eta)) \to \text{Hom}_{(B, \delta) - \text{Alg}}((D, \partial), F^{\text{diff}}(C, \eta)).$$

iii. If $(D, \partial) \in (B, \delta) - \text{Alg}$ and the derivations $\partial = (\partial_i)_{i \in I}$ are Lie commuting with structure coefficients $a_{ij}^k \in A$ $(i, j, k \in I)$, i.e., for fixed $i, j \in I$ only finitely many of the $a_{ij}^k$’s are non-zero and

$$[\partial_i, \partial_j] = \sum_{k \in I} a_{ij}^k \partial_k \quad (i, j \in I),$$

then also the derivations $(\partial_i^W)_{i \in I}$ of $W(D)$ are Lie commuting with structure coefficients $a_{ij}^k$.

Proof. By Theorem 3.2, the map $W_D : D \to F(W(D))$ is differential. Hence, if a morphism $\phi : W(D) \to C$ is differential, so is $F(\phi) \circ W_D$. Thus the map $\tau(D, C)$ restricts to differential morphisms as claimed in (ii). Now recall from 2.5 (and 2.1(iii)) that $W(f)$ is the unique map that corresponds to the morphism $W_D \circ f : D \to F(W(D))$ under the bijection $\tau(D, W(D))$. As the latter morphism is differential, $W(f)$ must be differential. This entails (i), see 2.1(i)a. Item (iii) follows immediately from 3.3. $\square$
Working over differential fields, Theorem 3.4 has the following consequence at the level of rational points. This gives a geometric interpretation (in the sense of Kolchin’s differential algebraic geometry [4]) of the differential Weil descent.

### 3.5. Corollary

Let $K$ be a field equipped with derivations $\delta = (\delta_i)_{i \in I}$. Suppose $L/K$ is a finite separable field extension. Recall that the derivations $(\delta_i)_{i \in I}$ extend uniquely from $K$ to $L$. Then, given a differential $L$-algebra $D$, by 3.4, there is a natural one-to-one correspondence between the differential $L$-points of $D$ and the differential $K$-points of $W^{\text{diff}}(D)$.

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