A CRITERION FOR HOMOGENEOUS PRINCIPAL BUNDLES

INDRANIL BISWAS AND GÜNTER TRAUTMANN

Abstract. We consider principal bundles over $G/P$, where $P$ is a parabolic subgroup of a semisimple and simply connected linear algebraic group $G$ defined over $\mathbb{C}$. We prove that a holomorphic principal $H$–bundle $E_H \rightarrow G/P$, where $H$ is a complex reductive group, is homogeneous if the adjoint vector bundle $\text{ad}(E_H)$ is homogeneous. Fix a faithful $H$–module $V$. We also show that $E_H$ is homogeneous if the vector bundle $E_H \times^H V$ associated to it for the $H$–module $V$ is homogeneous.

1. Introduction

Let $G$ be a semisimple and simply connected linear algebraic group defined over $\mathbb{C}$ and $P \subset G$ a parabolic subgroup. So the quotient $G/P$ is a rational complete homogeneous variety. Let $H$ be any complex algebraic group. A holomorphic principal $H$–bundle $E_H \rightarrow G/P$ is called homogeneous if the left–translation action of $G$ on $G/P$ lifts to an action of $G$ on $E_H$ that commutes with the right action of $H$. The category of holomorphic homogeneous principal $H$–bundles coincides with the category of algebraic homogeneous principal $H$–bundles (see Lemma 2.1).

Now assume that $H$ is reductive. Fix a finite dimensional faithful $H$–module $V$. The following is the main result proved here (see Theorem 2.2).

**Theorem 1.1.** A principal $H$–bundle $E_H \rightarrow G/P$ is homogeneous if and only if the associated vector bundle $E_H \times^H V$ is homogeneous.

The method of proof of Theorem 1.1 yields the following (see Lemma 4.1):

**Lemma 1.2.** A principal $H$–bundle $E_H \rightarrow G/P$ is homogeneous if and only if its adjoint vector bundle $\text{ad}(E_H)$ is homogeneous.

**Application 1.3.** In [Sa2], Sato proved that any infinitely extendable vector bundle on a nested sequence of homogeneous spaces is homogeneous (see [Sa2] p. 171, Main Theorem I] and [Sa2] p. 171, Main Theorem II] for the details). In view of Theorem 2.2 we conclude that the results of [Sa2] extend to principal bundles. In particular, any principal $H$–bundle on an infinite Grassmannian is homogeneous (see also [Sa1] and [DP]). Similarly, the results of Penkov and Tikhomirov (see [PT1], [PT2]) extend to principal bundles.

2000 Mathematics Subject Classification. 14L30, 14F05, 14M17.

Key words and phrases. Homogeneous bundle, principal bundle, homogeneous space.
2. Homogeneous principal bundles and homogeneous vector bundles

Let $G$ be a semisimple and simply connected linear algebraic group defined over $\mathbb{C}$, and let $P \subset G$ be a proper parabolic subgroup. So

$$M := G/P$$

is a rational complete homogeneous variety. In the following, all morphisms, as well as all bundles on $M$, are supposed to be in the holomorphic category; they eventually may also be algebraic.

The left translation action of $G$ on itself defines an action of $G$ on the quotient space $M$. For any $g \in G$, let

$$f_g : M \rightarrow M$$

be the holomorphic automorphism given by the action of $g$. 

Let $H$ be a linear algebraic group defined over $\mathbb{C}$. A principal $H$–bundle $E_H$ on $M$ is called homogeneous if the action of $G$ on $M$ lifts to a holomorphic action of $G$ on $E_H$ that commutes with the right action of $H$. Equivalently, $E_H$ is the extension of structure group of the principal $P$–bundle $G \rightarrow G/P$ by a homomorphism $P \rightarrow H$.

A vector bundle $F$ on $M$ is called homogeneous if the action of $G$ on $M$ lifts to an action of $G$ on the total space of $F$ which is linear on the fibers. Thus a vector bundle of rank $n$ over $M$ is homogeneous if and only if the corresponding principal $\text{GL}(n, \mathbb{C})$–bundle is homogeneous.

The following lemma shows that homogeneous principal $H$–bundles are algebraically homogeneous.

**Lemma 2.1.** Let $E_H \rightarrow M$ be a homogeneous principal $H$–bundle. Then the action of $G$ on $M$ lifts to an algebraic action of $G$ on $E_H$ satisfying the condition that it commutes with the right action of $H$.

**Proof.** Since $E_H$ is homogeneous, for each $g \in G$, the pulled back principal $H$–bundle $f_g^* E_H$ is holomorphically isomorphic to $E_H$, where $f_g$ is constructed in (2.2). Therefore, $f_g^* E_H$ is algebraically isomorphic to $E_H$ for all $g \in G$. Now from Proposition 3.1 of [Bi] we conclude that the action of $G$ on $M$ lifts to an algebraic action of $G$ on $E_H$ satisfying the condition that it commutes with the right action of $H$. 

Let $H$ be a connected reductive linear algebraic group defined over $\mathbb{C}$. Fix a finite dimensional complex representation

$$\rho_0 : H \rightarrow \text{GL}(V)$$

such that $\text{kernel}(\rho_0)$ is a finite group. So the homomorphism of Lie algebras induced by $\rho$

$$\mathfrak{h} := \text{Lie}(H) \rightarrow \text{End}(V)$$

is injective.

Let $E_H \rightarrow M$ be a principal $H$–bundle. Let

$$E_V := E_H \times^H V \rightarrow M$$
be the vector bundle associated to $E_H$ for the $H$–module $V$ in \([2.3]\).

**Theorem 2.2.** Assume that the associated vector bundle $E_V$ is homogeneous. Then the principal $H$–bundle $E_H$ is homogeneous.

### 3. Proof of Theorem 2.2

Let $E_{GL(V)} \rightarrow M$ be the principal $GL(V)$–bundle corresponding to $E_V$. By assumption, we have an action $G \times E_{GL(V)} \rightarrow E_{GL(V)}$ of $G$ on $E_{GL(V)}$ that commutes with the action of $GL(V)$ making $E_{GL(V)}$ a homogeneous principal $GL(V)$–bundle. Our aim is to prove that $E_H$ is homogeneous.

Let $At(E_{GL(V)})$ (respectively, $At(E_H)$) be the Atiyah bundle over $M$ for the principal $GL(V)$–bundle $E_{GL(V)}$ (respectively, principal $H$–bundle $E_H$). We recall that $At(E_{GL(V)})$ (respectively, $H$–invariant vector fields on $E_{GL(V)}$) is the holomorphic vector bundle defined by the sheaf of $GL(V)$–invariant vector fields on $E_{GL(V)}$ (respectively, $H$–invariant vector fields on $E_H$); see [At].

Using the properties of the Atiyah bundle and the injectivity of the homomorphism in \((2.3)\) we have the following diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \text{ad}(E_H) & \rightarrow & \text{At}(E_H) & \rightarrow & TM & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \text{ad}(E_{GL(V)}) & \rightarrow & \text{At}(E_{GL(V)}) & \rightarrow & TM & \rightarrow & 0
\end{array}
\]

(3.1)

Since $H$ is reductive, the homomorphism of $H$–modules in \((2.3)\) splits. In other words, there is a submodule $S$ of the $H$–module $\text{End}(V)$ such that the natural homomorphism

\[
\mathfrak{h} \oplus S \rightarrow \text{End}(V)
\]

(3.2)

is an isomorphism of $H$–modules. Fix such a direct summand $S$, and let

\[E_S := E_H \times^H S\]

be the vector bundle over $M$ associated to the principal $H$–bundle $E_H$ for the $H$–module $S$. From \((3.2)\) we obtain an isomorphism

\[
\text{ad}(E_H) \oplus E_S \cong \text{ad}(E_{GL(V)})
\]

of associated vector bundles. Hence from \((3.1)\) we have the commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \text{ad}(E_H) \oplus E_S & \rightarrow & \text{At}(E_H) \oplus E_S & \rightarrow & TM & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \text{ad}(E_{GL(V)}) & \rightarrow & \text{At}(E_{GL(V)}) & \rightarrow & TM & \rightarrow & 0
\end{array}
\]

(3.3)

where $\iota : \text{ad}(E_H) \rightarrow \text{At}(E_H)$ is the inclusion in \((3.1)\). Let

\[
\sigma : H^0(M, \text{At}(E_{GL(V)})) \rightarrow H^0(M, \text{At}(E_H))
\]

(3.4)

be the surjective homomorphism induced by the projection $\text{At}(E_{GL(V)}) \rightarrow \text{At}(E_H)$ constructed from \((3.3)\).
Let $\mathfrak{g}$ denote the Lie algebra of $G$. Let 
\[ \varphi: \mathfrak{g} \to H^0(M, \text{At}(E_{GL(V)})) \quad \text{and} \quad \varphi_H: \mathfrak{g} \to H^0(M, \text{At}(E_H)) \]
be the homomorphisms of Lie algebras given by the actions of $G$ on $E_{GL(V)}$ and $E_H$ respectively. Note that 
\[ \varphi_H = \sigma \circ \varphi, \]
where $\sigma$ is constructed in (3.4). We have the following commutative diagram of Lie algebras 
\[ \begin{array}{ccc} \mathfrak{g} & \to & H^0(M, \text{At}(E_{GL(V)})) \\
\varphi \downarrow & & \downarrow \varphi_H \\
H^0(M, TM) & \to & H^0(M, \text{At}(E_H)) \\
\beta \downarrow & & \downarrow \alpha \\
H^0(M, TM) & \to & H^0(M, \text{Ad}(E_H)) \\
\end{array} \]
where $\beta$ is the injective Lie algebra homomorphism induced by the natural action of $G$ on $G/P$, and $\varphi$, $\sigma$ are defined above; the remaining two homomorphisms are obtained from (3.1). Consequently, $\beta(\mathfrak{g}) \subset \alpha(H^0(M, \text{At}(E_H)))$. Defining 
\[ H^0(M, \text{At}(E_H)) := \alpha^{-1}(\beta(\mathfrak{g})), \]
from (3.5) and (3.1), we have a short exact sequence of Lie algebras 
\[ 0 \to H^0(M, \text{Ad}(E_H)) \to H^0(M, \text{At}(E_H)) \to \mathfrak{g} \to 0. \]

Since $\mathfrak{g}$ is semisimple, there is a homomorphism of Lie algebras 
\[ \hat{\alpha}: \mathfrak{g} \to H^0(M, \text{At}(E_H)) \]
such that $\alpha \circ \hat{\alpha} = \text{Id}_\mathfrak{g}$; see [Bo, p. 91, Corollaire 3]. Fix such a splitting $\hat{\alpha}$.

Let $\mathcal{G}(E_H)$ denote the group of all biholomorphisms of $E_H$ that commute with the right action of $H$. It is a complex Lie group, and its Lie algebra coincides with $H^0(M, \text{At}(E_H))$ (see [Br] for a proof). The group $G$ being simply connected, the homomorphism of Lie algebras $\hat{\alpha}$ in (3.7) lifts to a homomorphism 
\[ \rho: G \to \mathcal{G}(E_H). \]
Since $\alpha \circ \hat{\alpha} = \text{Id}_\mathfrak{g}$, it follows that the action of $G$ on $E_H$ defined by $\rho$ makes $E_H$ a homogeneous principal $H$–bundle. This completes the proof of Theorem 2.2.

4. ADJOINT BUNDLE CRITERION

The above proof of Theorem 2.2 also gives the following lemma.

Lemma 4.1. Assume that the adjoint vector bundle $\text{ad}(E_H)$ is homogeneous. Then $E_H$ is homogeneous.

Proof. Define $Z := H/[H, H]$. Any holomorphic principal $Z$–bundle on $G/P$ is homogeneous because $Z$ is a product of copies of $\mathbb{C}^*$ and $G$ is simply connected (so any line bundle on $G/P$ is homogeneous). The $H$–module $\mathfrak{h}$ decomposes as 
\[ \mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] \oplus z(\mathfrak{h}), \]
where \( z(\mathfrak{h}) \) is the Lie algebra of \( Z \). The adjoint homomorphism \( \mathfrak{h} \rightarrow \text{End}_C(\mathfrak{h}) \) is injective. So, the homomorphism of Lie algebras corresponding to the homomorphism

\[
H \rightarrow \text{GL}(\mathfrak{h}) \times Z =: \tilde{H}
\]

is injective. Let \( E_{\tilde{H}} \) be the principal \( \tilde{H} \)-bundle on \( M \) obtained by extending the structure group of \( E_H \) using the above homomorphism. Since \( \text{ad}(E_H) \) and all holomorphic line bundles on \( G/P \) are homogeneous, and \( Z \) is a product of copies of \( \mathbb{C}^* \), it follows that \( E_{\tilde{H}} \) is homogeneous. After replacing the principal bundle \( E_{\text{GL}(V)} \) by \( E_{\tilde{H}} \), it is straightforward to check that the proof of Theorem 2.2 also gives a proof of the lemma.

**Remark 4.2.** Let \( E_H \) be a principal \( H \)-bundle on \( M \), and let \( H \rightarrow \text{GL}(V) \) be a faithful representation as in (2.2). If the associated vector bundle \( E_V \) is trivial, then it can be shown that \( E_H \) itself is trivial. To prove this, consider the induced morphism \( E_H \rightarrow E_{\text{GL}(V)} \). The principal \( \text{GL}(V) \)-bundle \( E_{\text{GL}(V)} \) is trivial because \( E_V \) is trivial. Since \( M \) is complete and \( \text{GL}(V)/H \) affine, there are no non-constant maps from \( M \) to \( \text{GL}(V)/H \). This implies that there are trivializing sections of \( E_{\text{GL}(V)} \) which factor through \( E_H \). By this remark, the result of [PT1] on the triviality of vector bundles on twisted ind–Grassmannians can be extended to principal bundles. (See [BCT] for principal bundles on projective spaces.)

**References**

[At] M. F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* 85 (1957) 181–207.

[Bi] I. Biswas, Homogeneous principal bundles and stability, *Forum Math.* 22 (2010) 603–617.

[BCT] I. Biswas, I. Coandă and G. Trautmann, A Babylonian tower theorem for principal bundles over projective spaces, *Jour. Math. Kyoto Univ.* 49 (2009) 69–82.

[Bo] N. Bourbaki, *Éléments de mathématique. XXVI. Groupes et algèbres de Lie. Chapitre 1: Algèbres de Lie*, Actualités Sci. Ind. No. 1285, Hermann, Paris, 1960.

[DP] J. Donin and I. Penkov, Finite rank vector bundles on inductive limits of Grassmannians, *Int. Math. Res. Not.* (2003) 1871–1887.

[PT1] I. Penkov and A. S. Tikhomirov, Triviality of vector bundles on sufficiently twisted ind–Grassmannians, *arXiv:0706.3912*.

[PT2] I. Penkov and A. S. Tikhomirov, Rank 2 vector bundles on ind–Grassmannians, *arXiv:0710.0905*.

[Sa1] E.-i. Sato, On the decomposability of infinitely extendable vector bundles on projective spaces and Grassmann varieties, *Jour. Math. Kyoto Univ.* 17 (1977) 127–150.

[Sa2] E.-i. Sato, On infinitely extendable vector bundles on \( G/P \), *Jour. Math. Kyoto Univ.* 19 (1979) 171–189.

**School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India**

*E-mail address: indranil@math.tifr.res.in*

**FB Mathematik, Universität Kaiserslautern, Postfach 3049, D-67653 Kaiserslautern, Germany**

*E-mail address: trm@mathematik.uni-kl.de*