ON A RESULT BY Y. GROMAN AND J. P. SOLOMON

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ABSTRACT. We give a short proof of a reverse isoperimetric inequality due to Y. Groman and J. P. Solomon.

1. Introduction.

Let $(X, J)$ be a compact almost complex manifold equipped with a hermitian metric and $S \subset X$ a compact totally real submanifold of maximal dimension. Then $J$-holomorphic curves with boundary in $S$ satisfy a reverse isoperimetric inequality. Namely there exists a constant $A > 0$ such that for any compact $J$-holomorphic curve $(C, \partial C) \subset (X, S)$ then

$$\text{long}(\partial C) \leq A \text{area}(C).$$

This statement is due to Y. Groman and J. P. Solomon [2] (with an extra term involving the genus of $C$ on the right hand side). We refer also to [2] for motivation and applications. The proof of [2] is geometric and combinatorial in nature. We propose here an analytic approach of this inequality based on a monotonicity principle, which gives in fact a stronger semi-local statement.

**Theorem.** There exists $A > 0$ such that for any compact $J$-holomorphic curve $(C, \partial C) \subset (X, S)$ then $\text{long}(\partial C) \leq \frac{A}{r} \text{area}(C \cap U_r)$ where $U_r$ is the $r$-neighborhood of $S$, for $r > 0$ small enough.

Our approach is reminiscent of Lelong method [3] for proving the following inequality. Let $C$ be a holomorphic curve in $\mathbb{C}^n$, $m$ its multiplicity at $0$ and $B_r$ the ball of radius $r$ centered at $0$, then

$$m \leq \frac{\text{area}(C \cap B_r)}{\pi r^2}.$$
It is based on the plurisubharmonicity of $\log |z|^2$. It implies that the function $a(r) = \frac{4}{r^2} \text{area}(C \cap B_r) = \frac{1}{r} \int_{C \cap B_r} dd^c |z|^2$ is increasing. Indeed, by Stokes theorem, $a(r) = \int_{C \cap \partial B_r} d^c \log |z|^2$, so $a(r) - a(s) = \int_{C \cap (B_r \setminus B_s)} dd^c \log |z|^2 \geq 0$ where $r > s > 0$. As $\lim_{s \to 0} a = 4\pi m$ the inequality follows.

In our situation the local model is $\mathbf{R}^n \subset \mathbb{C}^n$ where $\mathbf{R}^n = (y = 0)$ with the usual coordinates $z = x + iy$ in $\mathbb{C}^n$. Now $|y|^2$ is strictly plurisubharmonic, so $dd^c |y|^2$ restricts to an area form on holomorphic curves. It turns out that $|y|$ is still plurisubharmonic and will play the role of $\log |z|^2$ above. This can be globalized. Near $S$ there exists a function $\rho$ looking like the square of the distance to $S$; strictly $J$-plurisubharmonic and such that $\sqrt{\rho}$ is still $J$-plurisubharmonic. Let us enter the details.

2. Proof of the theorem.

All objects are supposed smooth except otherwise mentioned. Recall that $S$ is a compact totally real submanifold in $X$ of maximal dimension (say $n$). This means that $TS \oplus JT S = TX|_S$. The point is the following

**Lemma.** Near $S$ there exists a strictly $J$-plurisubharmonic function $\rho \geq 0$ of class $C^2$, vanishing exactly on $S$, such that $\sqrt{\rho}$ is $J$-plurisubharmonic outside $S$.

This means that $dd^J \rho > 0$ and $dd^J \sqrt{\rho} \geq 0$ where $d^J g$ stands for $-dg \circ J$. Recall that a 2-form $\theta$ is non negative (resp. strictly positive) if $\theta(v, Jv) \geq 0$ (resp. $> 0$) for any tangent vector $v \neq 0$. Assuming this lemma for a while let us prove the theorem.

We may take $dd^J \rho$ as the area form of our hermitian metric near $S$. As $\rho$ is comparable to the square of the distance to $S$ we may also take $U_r = (\rho \leq r^2)$. This will only change the constant $A$ in the end.

Take $C$ a compact $J$-holomorphic curve of $X$ with boundary in $S$. Precisely it is the image of a map $f : (\Sigma, i) \to (X, J)$ where $\Sigma$ is a compact Riemann surface with boundary, such that $df \circ i = J \circ df$ and $f(\partial \Sigma) \subset S$. All the integrals below should be meant parametrized by $f$, though we write them on $C$ for simplicity.

As above $a(r) = \frac{1}{r} \text{area}(C \cap U_r) = \frac{1}{r} \int_{C \cap U_r} d^J \rho$ is increasing. Indeed, by Stokes theorem, $a(r) = 2 \int_{C \cap \partial U_r} d^J \sqrt{\rho}$, so $a(r) - a(s) = 2 \int_{C \cap (U_r \setminus U_s)} d^J \sqrt{\rho} \geq 0$ where $r > s > 0$. Hence $\lim_{s \to 0} a \leq a(r)$.

On the other hand, as $\rho$ has a minimum along $S$, there exists $A > 0$ such that $|\nabla \rho| \leq As$ in $U_s$. So $A a(s) \geq \frac{1}{s^2} \int_{C \cap U_s} |\nabla (\rho|_C)| d^J \rho = \frac{1}{s^2} \int_0^{s^2} \text{long}(C \cap (\rho = t)) \, dt$ by the coarea formula (see for instance [1]). Hence $A \lim_{s \to 0} a \geq \text{long}(\partial C)$.

We conclude that $\text{long}(\partial C) \leq A \, a(r).$
Proof of the lemma. Take any function $\rho \geq 0$ near $S$, vanishing on $S$ and non degenerate transversally to $S$. It is known that $\rho$ is strictly $J$-plurisubharmonic (see below). Now $\sqrt{\rho}$ is not necessarily $J$-plurisubharmonic outside $S$ but we will find $B > 0$ such that $\sqrt{\rho} + B\rho$ is. This will do for our lemma replacing $\rho$ by $(\sqrt{\rho} + B\rho)^2$.

So we need only check that $dd^c \sqrt{\rho} \geq O(1)$. We verify it locally.

Parametrize a piece of $S$ by a piece of $\mathbb{R}^n$ via $\phi$. Extend $\phi$ to a local diffeomorphism from $\mathbb{C}^n$ to $X$ such that $d\phi \circ i = J \circ d\phi$ on $\mathbb{R}^n$. This amounts to prescribing the normal derivative of $\phi$ along $\mathbb{R}^n$. Transport the situation via $\phi$ in $\mathbb{C}^n$. Locally we get a function $\rho \geq 0$, vanishing on $\mathbb{R}^n$ and non degenerate transversally, and an almost structure $J$ coinciding with $i$ on $\mathbb{R}^n$. Take the usual coordinates $z = x + iy$ in $\mathbb{C}^n$ such that $\mathbb{R}^n = (y = 0)$.

As $J - i = O(|y|)$ and $\rho = O(|y|^2)$ we infer that $dd^c \sqrt{\rho} = dd^c \sqrt{\rho} + O(1)$. Note that $d^c g$ is nothing but the more familiar $d^c g$. So it is enough to check that $dd^c \sqrt{\rho} \geq O(1)$ where the positivity is meant with respect to $i$. Now by assumption $\rho = q + O(|y|^3)$ where $q = \sum a_{kl}(x) y_k y_l$ with $(a_{kl})$ symmetric positive definite. We may then replace $\rho$ by $q$ in our inequality. Actually we don’t even have to differentiate the coefficients $a_{kl}$ of $q$ as we work modulo $O(1)$.

So everything boils down to proving that $dd^c \sqrt{q} \geq 0$ where $q$ is now a constant positive definite quadratic form in $y$. By a linear change of coordinates this reduces further to the model case $q = |y|^2$. Computing we get $4|y|^3 dd^c |y| = 2|y|^2 dd^c |y|^2 - d|y|^2 \wedge d^c |y|^2 = 2 \sum_{kl} (y_k dy_l - y_l dy_k) \wedge (y_k d^c y_l - y_l d^c y_k) \geq 0$.

In the same way the strict $J$-plurisubharmonicity of $\rho$ near $S$ reduces to the strict plurisubharmonicity of $|y|^2$ which is clear.

References

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3. P. Lelong, Propriétés métriques des variétés analytiques définies par une équation, Ann. Sci. Ecole Norm. Sup. 67 (1950), 393-419.