ON THE TIME OF FIRST LEVEL CROSSING AND INVERSE GAUSSIAN DISTRIBUTION

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Abstract. We propose a new approximation for the distribution of the time of the first level crossing by the random process $V_s - cs$, where $V_s$, $s > 0$, is compound renewal process and $c > 0$. It is competitive with respect to existing approximations, particularly in the region around the critical point $c = c^*$ which separates processes with positive and negative drifts. This approximation is tightly related to inverse Gaussian distributions.

1. Introduction and main result

The inverse Gaussian distribution has probability density function (p.d.f.)

$$f(x; \mu, \lambda, -\frac{1}{2}) = \frac{\lambda^{1/2}}{\sqrt{2\pi x}} x^{-3/2} \exp \left\{ -\frac{\lambda(x-\mu)^2}{2\mu^2 x} \right\},$$

where $x$, $\mu$, and $\lambda$ are positive. Parameter $\lambda$ is called shape parameter, and $\mu$ is called mean parameter.

In the study of this distribution, paramount is finding explicit expression

$$F(x; \mu, \lambda, -\frac{1}{2}) = \int_0^x f(z; \mu, \lambda, -\frac{1}{2}) dz$$

$$= \Phi(0,1) \left( \sqrt{\frac{\lambda}{x}} \left( \frac{x}{\mu} - 1 \right) \right) + \exp \left\{ \frac{2\lambda}{\mu} \right\} \Phi(0,1) \left( -\sqrt{\frac{\lambda}{x}} \left( \frac{x}{\mu} + 1 \right) \right)$$

for cumulative distribution function (c.d.f.) corresponding to p.d.f. $f(x; \mu, \lambda, -\frac{1}{2})$.\footnote{Key words and phrases. Time of first level crossing, Compound renewal processes, Inverse Gaussian distributions.} by $\Phi(0,1)(x)$ and $\varphi(0,1)(x)$ we denote c.d.f. and p.d.f. of a standard normal distribution.

By $f_{T_1}(t)$ and $f_T(t)$ we denote p.d.f. of positive random variable $T_1$, and of positive random variables $T_i \overset{d}{=} T$, $i = 2, 3, \ldots$, all distributed identically. The random variable $T_1$ is the time between starting time zero and time of the first renewal, and the random variables $T_i$ are inter-renewal times. By $f_Y(t)$ we denote p.d.f. of positive random variables $Y_i \overset{d}{=} Y$, $i = 1, 2, \ldots$, all distributed identically. The random variables $Y_i$ are called jump sizes, and the jumps occur only in the moments of renewals. Throughout the entire presentation, p.d.f. $f_T(y)$ and $f_Y(y)$ are assumed bounded from above by a finite constant. Having assumed that $T_1$, i.i.d. $T_i \overset{d}{=} T$, $i = 2, 3, \ldots$, i.i.d. $Y_i \overset{d}{=} Y$, $i = 1, 2, \ldots$, are all mutually independent, we are within renewal model, where the distribution of the first interval $T_1$ may be different from the distribution of the other interclaim intervals, i.e., from the distribution of $T$.\footnote{Key words and phrases. Time of first level crossing, Compound renewal processes, Inverse Gaussian distributions.}
Compound renewal process with time $s \geq 0$ is

$$V_s = \sum_{i=1}^{N_s} Y_i,$$

or $0$, if $N_s = 0$ (or $T_1 > s$), where $N_s = \max \{ n > 0 : \sum_{i=1}^{n} T_i \leq s \}$, or $0$, if $T_1 > s$. The random variable

$$\Upsilon = \inf \{ s > 0 : V_s - cs > u \},$$

or $+\infty$, as $V_s - cs \leq u$ for all $s > 0$, is the time of the first level $u$ crossing by the process $V_s - cs$.

It is easily seen that for $t > 0$

$$P(\Upsilon \leq t) = \int_0^t P\{u + cv - Y_1 < 0\} f_T(u) dv + \int_0^t P\{v < \Upsilon \leq t \ | \ T_1 = v\} f_T(v) dv. \quad (1.4)$$

This distribution of $\Upsilon$ appears in many branches of applied probability, including risk and queuing theories, and was considered by many authors. For it, there are many closed-form formulas and approximations, derived by different techniques. The goal of this paper is to get the approximation that involves inverse Gaussian distribution, and seems new. Remarkable is that it is derived under a set of conditions similar to those usually imposed in the common local central limit theorem.

Set $M = ET/EY$, $D^2 = (ET)^2DY + (EY)^2DT)/(EY)^3$, and introduce

$$\mathcal{M}_d(u, c, v) = \int_0^{(c+1)\frac{v}{cD^2}} \frac{1}{1 + x} \varphi \left( \frac{cM(1+x)}{c^2D^2(1+\lambda)} \right) (x) dx.$$

For $c^* = \frac{1}{M}$, $\lambda = \frac{u + cv}{c^2D^2} > 0$, $\mu = \frac{1}{1-cM}$, and $\hat{\mu} = -\mu = \frac{1}{cM-1}$, using (1.1) and (1.2), we have

$$\mathcal{M}_d(u, c, v) = \begin{cases} 
F(x; \mu, \lambda, -\frac{1}{2}) \left[ \frac{cM(1+x)}{c^2D^2} \right]_{x=1}^{\frac{cM(1+x)}{c^2D^2}+1}, & 0 < c \leq c^*, \\
\exp \left\{ -2 \frac{\hat{\mu}}{\mu} F(x; \hat{\mu}, \lambda, -\frac{1}{2}) \left[ \frac{cM(1+x)}{c^2D^2} \right]_{x=1}^{\frac{cM(1+x)}{c^2D^2}+1}, & c \geq c^* \\
\varphi \left( \frac{u+cv}{cD\sqrt{x}} \right) \left( x(1-cM) - 1 \right) + \exp \left\{ 2 \frac{\mu + cv}{c^2D^2} (1-cM) \varphi \left( \frac{u+cv}{cD\sqrt{x}} \right) \right\} \left[ \frac{cM(1+x)}{c^2D^2} \right]_{x=1}^{\frac{cM(1+x)}{c^2D^2}+1} 
\end{cases}.$$
THEOREM 1.1. In the above model, let p.d.f. \( f_T(y) \) and \( f_Y(y) \) be bounded from above by a finite constant, \( D^2 > 0, E(T^3) < \infty, E(Y^3) < \infty \). Then for \( c > 0 \), for fixed \( 0 < v < t \) we have

\[
\sup_{t>v} \left| \mathbb{P}\{v < Y \leq t \mid T_1 = v\} - \mathcal{M}_1(u,c,v) \right| = O\left( \frac{\ln(u + cv)}{u + cv} \right),
\]

as \( u + cv \to \infty \).

As soon as the distribution of \( T_1 \) is specified, the similar results for \( \mathbb{P}\{Y \leq t\} \) are straightforward from Theorem 1.1. In particular, for \( T_1 \) exponential with parameter \( \beta \)

\[
\mathbb{P}\{Y \leq t\} = \beta \int_0^t \mathbb{P}\{u + cv - Y_1 < 0\} e^{-\beta v} dv + \beta \int_0^t \mathbb{P}\{v < Y \leq t \mid T_1 = v\} e^{-\beta v} dv.
\]

If \( Y \) is exponential with parameter \( \alpha \), we have

\[
\mathbb{P}\{Y \leq t\} = \mathbb{P}\{v < Y \leq t \mid T_1 = v\} e^{-\beta v} dv = \frac{\beta e^{-\alpha u}}{\beta + c\alpha} (1 - e^{-(\beta + c\alpha)t}) + \beta \int_0^t \mathbb{P}\{v < Y \leq t \mid T_1 = v\} e^{-\beta v} dv.
\]

2. Closed-form expression using convolutions

The following result is a modification of a result in Borovkov and Dickson (2008).

THEOREM 2.1. For \( M(s) = \inf\{k \geq 1 : \sum_{i=1}^k Y_i > s\} - 1 \), we have

\[
\mathbb{P}\{v < Y \leq t \mid T_1 = v\} = \int_v^t \frac{u + cv}{u + cz} \sum_{n=1}^{\infty} \mathbb{P}\{M(u + cz) = n\} f_T^n(z - v) dz,
\]

and

\[
\mathbb{P}\{Y \leq t\} = \int_0^t \mathbb{P}\{u + cv - Y_1 < 0\} f_T(v) dv + \int_0^t \left[ \int_v^t \frac{cv + u}{u + cs} \sum_{n=1}^{\infty} \mathbb{P}\{M(u + cs) = n\} f_T^n(s - v) \right] f_T(v) dv.
\]

COROLLARY 2.1 (Theorem 1 in Borovkov and Dickson (2008)). For \( Y \) exponential with parameter \( \alpha \), we have

\[
\mathbb{P}\{Y \leq t\} = \int_0^t e^{-\alpha(s+cz)} \left[ f_T(s) + \frac{1}{u + cs} \sum_{n=1}^{\infty} \frac{(\alpha(u + cs))^n}{n!} \right] \times \int_0^s (u + cv) f_T^n(s - v) f_T(v) dv ds.
\]

PROOF OF THEOREM 2.1. The main idea of this proof is to change jumps direction from “toward the boundary” to “away from the boundary” and then use Kendall’s identity. We set \( T_1 = v \) and write

\[
Y = (\sigma - u)/c,
\]

where \( \sigma \) is the crossing time of the lower level \(-(v + u/c)\) by the process \( Z(s) = \sum_{k \leq M(s)} T_k - s/c \) (here \( M(s) = \inf\{k \geq 1 : \sum_{i=1}^k Y_i > s\} - 1 \)), which is a skip-free in the negative direction \( \text{L} \) Lévy process.

The Kendall’s identity writes as (see Borovkov and Dickson (2008))

\[
p_0(s) = \frac{v + u/c}{s} p_Z(s)(-(v + u/c)),
\]

\(^1\)Recall that this means that this process has no negative jumps and its increments are stationary independent.
where
\[ p_Z(s)(-\frac{v}{u/c}) = \sum_{n=1}^{\infty} P\{M(s) = n\} f_T^n(-\frac{v}{u/c} + s/c). \]

According to (2.1), we have \( p_T(t \mid v) = \alpha \). We observe that \( Y \geq T_1 \) holds always and write
\[ P\{Y \leq t\} = P\{T_1 = Y \leq t\} + P\{T_1 < Y \leq t\} \]
\[ = \int_0^t P\{u + cv - Y_1 < 0\} f_{T_1}(v)dv + \int_0^t P\{v < Y < t \mid T_1 = v\} f_{T_1}(v)dv \]
\[ = \int_0^t P\{u + cv - Y_1 < 0\} f_{T_1}(v)dv + \int_0^t \left[ \int_v^{\infty} P_T(z \mid v)dz \right] f_{T_1}(v)dv \]
\[ = \int_0^t P\{u + cv - Y_1 < 0\} f_{T_1}(v)dv + \int_0^t \left[ \int_v^{\infty} \alpha (u + cs)ds \right] f_{T_1}(v)dv \]
with
\[ \alpha \equiv \frac{cv + u}{u + cs} \sum_{n=1}^{\infty} P\{M(u + cs) = n\} f_T^n(s - v), \]
which is required.

**Proof of Corollary 2.1.** For \( Y_i \equiv Y, i = 1, 2, \ldots \), exponential with parameter \( \alpha \), we have
\[ P\{M(u + cs) = n\} = e^{-\alpha (u + cs)} \frac{(\alpha (u + cs))^n}{n!}, \quad n = 1, 2, \ldots, \]
\[ P\{u + cv - Y_1 < 0\} = e^{-\alpha (u + cv)}, \] and equation (2.2) turns into (2.3), as required.

**3. Proof of Theorem 1.1**

In the sequel, let \( K, K_1, K_2, \ldots \), be “sufficiently large” positive constants, and \( \varepsilon, \varepsilon_1, \varepsilon_2, \) etc., be “sufficiently small” positive constants. All of them do not depend on summation and integration variables, such as \( n, y, z, \) etc., and possibly are different in different equations.

We put \( y = z - v \) in (2.1) and rewrite it as
\[ P\{v < Y \leq t \mid T_1 = v\} = \int_0^{t-v} \frac{u + cv}{u + cv + cy} P\{\sum_{i=2}^{M(u+cv+cy)+1} Y_i\} (y)dy \]
\[ = \sum_{n=1}^{\infty} \int_0^{t-v} \frac{u + cv}{u + cv + cy} \int_0^{\infty} P\{Y_i > z\} f_{T}^n(u + cv + cy - z) dydz \]

(3.1)

Bearing in mind that \( T_i, i = 1, 2, \ldots \) and \( Y_i, i = 1, 2, \ldots \) are mutually independent, the second equality in (3.1) holds true since
\[ P\{M(u + cv + cy) = n\} = P\left\{ \sum_{i=1}^{n} Y_i \leq u + cv + cy < \sum_{i=1}^{n+1} Y_i \right\} \]
\[ = \int_{u+cv+cy}^{\infty} f_{T}^n(u + cv + cy - z) P\{Y_{n+1} > z\} dz. \]

The proof of Theorem 1.1 consists of several steps. The first and the third steps are elimination of the terms that have little impact in (3.1); it may be called preparation for further analysis. The former is elimination of those terms that correspond to small \( n \), i.e., to such \( n \) that the event \( \{M(u + cv + cy) = n\} \) has a small probability, provided that \( u + cv + cy \) is large. The latter is elimination of the terms containing \( z \),
i.e., defect of the random walk $\sum_{i=1}^n Y_i$, $n = 1, 2, \ldots$, as it nearly crosses the high level $u + cv + cy$. In the first step, we use the bounds for large deviations of sums of i.i.d. random variables, like in [Nagaev (1965)]. In the third step, we apply the Taylor formula to the normal p.d.f.

The second step yields the main term of approximation and the corresponding remainder term in a raw form. That is made by means of applying standard non-uniform Berry-Esseen bounds in local CLT formulated in Section 4.4 to the product of $f_{Y_n}^n$ and $f_Y^* n$ in (3.1). The fourth step consists in investigation of the asymptotic behavior of core components in the remainder terms which appear all over the proof. The fifth step is the simplification of the main term of approximation, up to the terms of required order of magnitude. It relies on a standard estimation technique developed on the fourth step.

**3.1. Step 1: reducing of the area of summation.** Let us rewrite (3.1) as

$$P \{ v < Y \leq t \mid T_1 = v \} = \int_0^{t-v} \frac{u + cv}{u + cv + cy} P \{ \sum_{i=2}^{M(u+cv+cy)+1} \} (y)dy$$

$$= \sum_{n=1}^{N_c} \int_0^{t-v} \frac{u + cv}{u + cv + cy} P \{ M(u + cv + cy) = n \} P \{ \sum_{i=2}^{n+1} \} (y)dy,$$

select $N_c = \epsilon (u + cv)$, where $0 < \epsilon < 1$, split the sum $\sum_1^\infty = \sum_1^{N_c} + \sum_{N_c}^\infty$ and show that the first summand may be omitted within the required accuracy of approximation.

Note that $(u + cv) / (u + cv + cy) < 1$ for $y > 0$. Since

$$P \{ M(u + cv + cy) = n \} = P \{ \sum_{i=1}^n Y_i \leq u + cv + cy < \sum_{i=1}^{n+1} Y_i \} \leq P \{ \sum_{i=1}^{n+1} Y_i > u + cv + cy \},$$

we have

$$\sum_{n=1}^{N_c} \int_0^{t-v} P \{ M(u + cv + cy) = n \} P \{ \sum_{i=2}^{n+1} \} (y)dy$$

$$\leq \sum_{n=1}^{N_c} \int_0^{t-v} P \{ \sum_{i=1}^{n+1} Y_i > u + cv + cy \} P \{ \sum_{i=2}^{n+1} \} (y)dy$$

$$\leq N_c \sum_{n=1}^{N_c} P \{ Y_1 + \sum_{i=2}^{n+1} X_i > u + cv \}.$$

For standardized i.i.d. random variables $\hat{X}_i = (X_i - EX)/\sqrt{DX}$, $i = 1, 2, \ldots$, we have

$$P \{ \sum_{i=2}^{n+1} X_i > u + cv \} = P \{ \sum_{i=2}^{n+1} \hat{X}_i > \frac{u + cv - nEX}{\sqrt{DX}} \}.$$ 

It is easily seen that the inequality

$$\frac{u + cv - nEX}{\sqrt{DX}} > K \sqrt{n \ln(n)}$$

holds true for all $n \leq N_c$, and by Lemma 14 we have

$$P \{ \sum_{i=2}^{n+1} \hat{X}_i > \frac{u + cv - nEX}{\sqrt{DX}} \} \leq K \frac{n}{(u + cv - n)^3}.$$
Using simple estimates\(^2\), we have

\[
\sum_{n=1}^{N_r} P \left\{ Y_1 + \sum_{i=2}^{n+1} X_i > u + cv \right\} \leq K \sum_{n=1}^{N_r} \frac{n}{(u+cv-n)^3} = \frac{K}{(u+cv)^3} \sum_{n=1}^{N_r} \left( 1 - \frac{n}{u+cv} \right)^3
\]

\[
\leq \frac{K_1}{(u+cv)^3} \sum_{n=1}^{N_r} n = \frac{K_1}{(u+cv)^3} \frac{N_r(N_r+1)}{2} = O((u+cv)^{-1}),
\]
as \(u + cv \to \infty\), as required.

### 3.2. Step 2: application of CLT.

For i.i.d. random vectors \(\xi_i = (\tilde{Y}_i, \tilde{T}_i) \in \mathbb{R}^2\) with standardized independent components \(\tilde{Y}_i = (Y_i - EY)/\sqrt{D^2Y}\) and \(\tilde{T}_i = (T_i - ET)/\sqrt{DT}\), let us apply Theorem 4.2. Bearing in mind that

\[
f_T^n(x) = \frac{1}{\sqrt{nDT}} p_{n-1/2} \sum_{i=1}^{n} \tilde{Y}_i \left( \frac{x - nET}{\sqrt{nDT}} \right),
\]

we have from Theorem 4.2

\[
P \{ v < Y < t \mid T_1 = v \} - A_t(u, c \mid T_1 = v) \leq R_t(u, c \mid T_1 = v), \tag{3.2}
\]

where \(c, u > 0\), \(0 < v < t\) and

\[
A_t(u, c \mid T_1 = v) = \sum_{n=N_r}^{\infty} \int_0^{t-v} u + cv \frac{u + cv + cy}{u + cv + cy} \int_0^{u + cv + cy} P \{ Y_{n+1} > z \}
\]

\[
\times \frac{1}{\sqrt{nDY}} \varphi(0,1) \left( \frac{u + cv + cy - z - nEY}{\sqrt{nDY}} \right) \frac{1}{\sqrt{nDT}} \varphi(0,1) \left( \frac{u - nET}{\sqrt{nDT}} \right) \, dydz,
\]

\[
R_t(u, c \mid T_1 = v) = K \sum_{n=N_r}^{\infty} \int_0^{t-v} u + cv \frac{u + cv + cy}{u + cv + cy} \int_0^{u + cv + cy} P \{ Y_{n+1} > z \}
\]

\[
\times \frac{1}{n^{3/2}} \left( 1 + \left[ \left( \frac{u + cv + cy - z - nEY}{\sqrt{nDY}} \right)^2 + \left( \frac{u + cv + cy - z - nEY}{\sqrt{nDY}} \right)^2 \right]^{1/2} \right)^{-3} \, dydz.
\]

REMARK 3.1. To get the approximation \(^3\), to the product \(f_T^n(u + cv + cy - z) f_T^n(y)\) we applied Theorem 4.2 which is the Berry-Esseen bounds in two-dimensional local CLT. Instead, we could apply the Berry-Esseen bounds in one-dimensional local CLT to each of these factors, separately. We preferred to use Theorem 4.2 to get the remainder term \(R_t(u, c \mid T_1 = v)\) in a form better suited for the further analysis.

### 3.3. Step 3: bringing the approximation to a convenient form.

We do this in several steps. Major objective is simplification of the main term \(A_t(u, c \mid T_1 = v)\) and verification that the remainder term \(R_t(u, c \mid T_1 = v)\) is of required order of smallness.

#### Change of variables.

Put \(x = cy/(u + cv)\), \(dx = c dy/(u + cv)\). For

\[
Y_{n,z}(u + cv, x) = \frac{(u + cv)(1 + x) - z - nEY}{\sqrt{nDY}}, \quad T_{n}(u + cv, x) = \frac{(u + cv)x/c - nET}{\sqrt{nDT}}
\]

we have\(^4\)

\[
A_t(u, c \mid T_1 = v) = \frac{u + cv}{c \sqrt{D^2T}} \int_0^{t-v} \frac{1}{1 + x} \int_0^{(u + cv)(1 + x)} P \{ Y > z \} \varphi(0,1) \left( Y_{n,z}(u + cv, x) \right) \, dx \, dz \tag{3.3}
\]

\(^2\)Use, e.g., the inequality \(P\{\xi_1 + \xi_2 > x\} \leq P\{\xi_1 > x/2\} + P\{\xi_2 > x/2\}\).

\(^3\)We bear in mind that \(Y_{n+1} \equiv Y\) and that \(cy = (u + cv)x\), \(c dy = (u + cv)dx\).

\(^4\)To get the approximation \(^3\), to the product \(f_T^n(u + cv + cy - z) f_T^n(y)\) we applied Theorem 4.2 which is the Berry-Esseen bounds in two-dimensional local CLT. Instead, we could apply the Berry-Esseen bounds in one-dimensional local CLT to each of these factors, separately. We preferred to use Theorem 4.2 to get the remainder term \(R_t(u, c \mid T_1 = v)\) in a form better suited for the further analysis.
and

\[ R_t(u, c \mid T_1 = v) = K(u + cv) \sum_{n=N_t}^{\infty} n^{-3/2} \int_0^{e^{(1-u)z}} \frac{1}{1+x} \int_0^{(u+cv)(1+x)} P \{ Y > z \} \]

\[ \times (1 + [\gamma_{n,z}^2(u + cv, x) + T_n^2(1+\frac{1}{2})]^{-3}) \] \(dx\) \(dz\). (3.4)

**Use of fundamental identities of Section 4.3.** Denoting

\[ B_1 = (ET)^2DY + (EY)^2DT, \quad B_2 = EYDT, \quad B_3 = ETDY, \quad B_4 = DYDT, \]

we set

\[ \Lambda_{n,z}(u + cv, x) = \frac{(u + cv)(x/c)EY - [(u + cv)(1 + x) - z]ET}{\sqrt{B_1n}}, \]

\[ \gamma_{n,z}(u + cv, x) = B_1n - (B_2[(u + cv)(1 + x) - z] + B_3(u + cv)x/c). \] (3.5)

By Lemma 4.2 we have the identity

\[ \Lambda_{n,z}^2(u + cv, x) + T_n^2(u + cv, x) = \Lambda_{n,z}^2(u + cv, x) + \Delta_{n,z}^2(u + cv, x), \]

and equation (3.3) rewrites as

\[ A_t(u, c \mid T_1 = v) = \frac{u + cv}{2\pi c\sqrt{DYD}} \sum_{n=N_t}^{\infty} n^{-1} \int_0^{e^{(1-u)z}} \frac{1}{1+x} \int_0^{(u+cv)(1+x)} P \{ Y > z \} \]

\[ \times \exp \{ -\frac{1}{2}[\Lambda_{n,z}^2(u + cv, x) + \Delta_{n,z}^2(u + cv, x)] \} \] \(dx\) \(dz\). (3.6)

Equation (3.4) rewrites as

\[ R_t(u, c \mid T_1 = v) = K(u + cv) \sum_{n=N_t}^{\infty} n^{-3/2} \int_0^{e^{(1-u)z}} \frac{1}{1+x} \int_0^{(u+cv)(1+x)} P \{ Y > z \} \]

\[ \times (1 + [\Lambda_{n,z}^2(u + cv, x) + \Delta_{n,z}^2(u + cv, x)]^{-1/2})^{-3} \] \(dx\) \(dz\). \(\Delta_{n,z}^2(u + cv, x) + \Delta_{n,z}^2(u + cv, x)\). (3.7)

**Elimination of terms with \(z\) in (3.6).** Written in terms of elementary functions and considered as functions of \(z\), the expressions \(3.6\) and \(3.7\) are liable to such standard calculus manipulations as, e.g., use of Taylor’s formula.

Let us write

\[ A_{t1}^{(1)}(u, c \mid T_1 = v) = \frac{EY(u + cv)}{2\pi c\sqrt{DYD}} \sum_{n=N_t}^{\infty} n^{-1} \int_0^{e^{(1-u)z}} \frac{1}{1+x} \]

\[ \times \exp \{ -\frac{1}{2}[\Lambda_{n,z}^2(u + cv, x) + \Delta_{n}^2(u + cv, x)] \} \] \(dx\), (3.8)

where (see \(3.5\))

\[ \Lambda_{n}(u + cv, x) = \Lambda_{n,0}(u + cv, x), \quad \Delta_{n}(u + cv, x) = \Delta_{n,0}(u + cv, x). \] (3.9)

We need to show that

\[ \sup_{t>0} \left| A_t(u, c \mid T_1 = v) - A_{t1}^{(1)}(u, c \mid T_1 = v) \right| = O \left( \frac{\ln(u + cv)}{u + cv} \right), \]

as \(u + cv \to \infty\). Writing \(f(z) = \exp \{ -\frac{1}{2}[\Delta_{n,z}(u + cv, x) + \Lambda_{n,z}^2(u + cv, x)] \} \) \(> 0\) and bearing in mind that \(\int_0^{\infty} P \{ Y > z \} \) \(dz = EY\), we divide the region of integration with respect to \(z\) in (3.6) in two parts, \([0, U_{c,x}]\) and \([U_{c,x}, (u + cv)(1 + x)]\), where \(U_{c,x} = c(u + cv)(1 + x), 0 < c < 1\).
Bearing in mind that for $z \in [U_{r,x}, (u + cv)(1 + x)]$

$$\sup_{z > 0} \exp \left\{ - \frac{1}{2} \left[ \Lambda_{n,z}^2(u + cv, x) + \Delta_{n,z}^2(u + cv, x) \right] \right\} = \exp \left\{ - \frac{1}{2} T_n^2(u + cv, x) \right\}$$

and using Chebyshev’s inequality $P \{ Y > z \} \leq EY^3/z^3$ which yields

$$\int_{U_{r,x}}^{(u+cv)(1+x)} P \{ Y > z \} dz \leq EY^3 \int_{U_{r,x}}^{(u+cv)(1+x)} \frac{dz}{z^3} \leq \frac{K}{(u + cv)^2(1 + x)^2},$$

we have

$$\frac{u + cv}{2 \pi \sqrt{D'TY}} \sum_{n=N_0}^{\infty} n^{-1} \int_0^{(1-x)/y} \frac{1}{1 + x} \int_{U_{r,x}}^{(u+cv)(1+x)} P \{ Y > z \}$$

$$\times \exp \left\{ - \frac{1}{2} \left[ \Lambda_{n,z}^2(u + cv, x) + \Delta_{n,z}^2(u + cv, x) \right] \right\} dx dz$$

$$\leq \frac{K}{u + cv} \int_0^{(1-x)/y} \frac{1}{1 + x} \sum_{n=N_0}^{\infty} n^{-1} \exp \left\{ - \frac{1}{2} T_n^2(u + cv, x) \right\} dx = O((u + cv)^{-1}),$$

as $u + cv \to \infty$, which is checked by easy calculus.

For $z \in [0, U_{r,x}]$, bearing in mind that $\int_0^{U_{r,x}} z P \{ Y > z \} dz \leq EY^2/2 < \infty$ and using Taylor’s formula

$$|f(z) - f(0)| \leq z \sup_{\xi \in [0, U_{r,x}]} |f'(\xi)|,$$

where

$$f'(z) = f(z) \left( \frac{D'Y B_1 n}{\sqrt{D'Y}} \right)^{-1/2} \left( ET \sqrt{D'Y} \Delta_{n,z} + EY \sqrt{D'T} \Lambda_{n,z} \right),$$

we have to check that

$$\frac{u + cv}{2 \pi \sqrt{D'TY}} \sum_{n=N_0}^{\infty} n^{-1} \int_0^{(1-x)/y} \frac{1}{1 + x} \int_0^{U_{r,x}} z P \{ Y > z \} dz$$

$$\times \sup_{\xi \in [0, U_{r,x}]} \left( f(\xi) \left( ET \sqrt{D'Y} \Delta_{n,\xi} + EY \sqrt{D'T} \Lambda_{n,\xi} \right) \right) dx = O \left( \frac{\ln(u + cv)}{u + cv} \right),$$

as $u + cv \to \infty$. It follows from

$$(u + cv) \sum_{n=N_0}^{\infty} n^{-3/2} \int_0^{(1-x)/y} \frac{1}{1 + x} \sup_{\xi \in [0, U_{r,x}]} \left( f(\xi) \left| \Lambda_{n,\xi} \right| \right) dx \leq K(u + cv) \sum_{n=N_0}^{\infty} n^{-3/2}$$

$$\times \int_0^{(1-x)/y} \frac{1}{1 + x} \left| \Lambda_n \right| \exp \left\{ - \frac{1}{2} \left[ \Lambda_n^2 + \Delta_n^2 \right] \right\} dx = O \left( \frac{\ln(u + cv)}{u + cv} \right) \quad (3.10)$$

and

$$(u + cv) \sum_{n=N_0}^{\infty} n^{-3/2} \int_0^{(1-x)/y} \frac{1}{1 + x} \sup_{\xi \in [0, U_{r,x}]} \left( f(\xi) \left| \Delta_{n,\xi} \right| \right) dx \leq K(u + cv) \sum_{n=N_0}^{\infty} n^{-3/2}$$

$$\times \int_0^{(1-x)/y} \frac{1}{1 + x} \left| \Delta_n \right| \exp \left\{ - \frac{1}{2} \left[ \Lambda_n^2 + \Delta_n^2 \right] \right\} dx = O \left( \frac{\ln(u + cv)}{u + cv} \right), \quad (3.11)$$

as $u + cv \to \infty$. The proof of (3.10) and (3.11) by means of core asymptotic analysis of the expressions of the second kind is deferred until Step 4.

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4For brevity, here and in the sequel, we omit the arguments of $\Delta_{n,z}(u + cv, x)$ and $\Lambda_{n,z}(u + cv, x)$ whenever it does not create confusion.
Elimination of terms with $z$ in (3.4). Similarly to what just has been done, for

$$\mathcal{R}_t^{(1)}(u, c \mid T_1 = v) = K(u + c v) \sum_{n=N_z}^{\infty} n^{-3/2} \int_0^{\epsilon(t-v)} \frac{1}{1 + x} \times \left(1 + [\Lambda_n^2(u + c v, x) + \Delta_n^2(u + c v, x)]^{1/2}\right)^{-3} dx \quad (3.12)$$

we need to show that

$$\sup_{t > 0} |\mathcal{R}_t(u, c \mid T_1 = v) - \mathcal{R}_t^{(1)}(u, c \mid T_1 = v)| = \mathcal{O}\left(\frac{\ln(u + c v)}{u + c v}\right), \quad (3.13)$$

as $u + c v \to \infty$.

We divide as above the region of integration with respect to $z$ in (3.7) in two parts, $[0, U_{\epsilon, z}]$ and $[U_{\epsilon, z}, (u + c v)(1 + x)]$, where $U_{\epsilon, z} = \epsilon(u + c v)(1 + x)$, $0 < \epsilon < 1$. On the latter, we use Chebyshev’s inequality $P\{Y > z\} \leq \text{EY}^3/z^3$. On the former, bearing in mind that $\int_{U_{\epsilon, z}} z P\{Y > z\} \, dz \leq \text{EY}^2/2$, we put $g(z) = (1 + [\Delta_{n,z}^2(u + c v, x) + \Lambda_{n,z}^2(u + c v, x)]^{1/2})^{-3}$ and use Taylor’s formula

$$|g(z) - g(0)| \leq z \sup_{\xi \in [0, U_{\epsilon, z}]} |g'(\xi)|,$$

where $g'(z) = -\frac{3}{2\sqrt{D_{B_1}^2}}(ET\Delta_{n,z} + \frac{\partial}{\partial z} \Lambda_{n,z})(\Delta_{n,z}^2 + \Lambda_{n,z}^2)^{-1/2}g^{4/3}(z)$. The proof reduces to checking that for all $t > 0$

$$(u + c v) \sum_{n=N_z}^{\infty} n^{-3/2} \int_0^{\epsilon(t-v)} \frac{1}{1 + x} \int_{U_{\epsilon, z}}^{(u + c v)(1 + x)} z^{-3} \, dz \, (1 + [\Delta_{n,z}^2(u + c v, x)]^{1/2})^{-3} \, dx = \mathcal{O}(u + c v)^{-1}) \quad (3.14)$$

and

$$(u + c v) \sum_{n=N_z}^{\infty} n^{-2} \int_0^{\epsilon(t-v)} \frac{1}{1 + x} (K_1|\Lambda_n(u + c v, x)| + K_2|\Lambda_n(u + c v, x)|) \
\times (1 + [\Lambda_n^2(u + c v, x) + \Delta_n^2(u + c v, x)]^{1/2})^{-3} \, dx = \mathcal{O}\left(\frac{\ln(u + c v)}{u + c v}\right), \quad (3.15)$$

as $u + c v \to \infty$. The proof of (3.14) and (3.15) by means of core asymptotic analysis of the expressions of the first kind, is deferred until Step 4.

Estimation of (3.12). We need to show that

$$\sup_{t > 0} \mathcal{R}_t^{(1)}(u, c \mid T_1 = v) = \mathcal{O}\left(\frac{\ln(u + c v)}{u + c v}\right), \quad (3.16)$$

as $u + c v \to \infty$. The proof of (3.16) with $\mathcal{R}_t^{(1)}(u, c \mid T_1 = v)$ written down in (3.12), carried out by means of core asymptotic analysis of the expressions of the first kind, is deferred until Step 4.

3.4. Step 4: core asymptotic analysis. Before we formulate and prove the main results of this section, we examine in more detail $\Lambda_n(u + c v, x)$ and $\Delta_n(u + c v, x)$ defined
in (3.11). From the definition, it follows straightforwardly that
\[ \Lambda_n(u + cv, x) = \frac{B_1 n - (B_2(u + cv)(1 + x) + B_3(u + cv)x/c)}{\sqrt{B_1 B_4 n}}, \]
and
\[ \Delta_n(u + cv, x) = \frac{\((u + cv)x/c\)EY - (u + cv)(1 + x)ET}{\sqrt{B_1 n}}, \]
and that for \(5 \leq n \) identities for \( \Delta_n \) hold true:
\[ \frac{(u + cv)(1 + x)(EY/c - ET) - (u + cv)EY/c}{\sqrt{B_1 n}}, \]
and that for \( 0 < x < \infty \)
\[ -\infty < \Lambda_n(u + cv, x) \leq \frac{B_1 n - (u + cv)B_2}{\sqrt{B_1 B_4 n}} = \frac{\sqrt{B_1}}{\sqrt{B_4}} \left( \frac{\sqrt{n}}{B_1} - \frac{B_2 (u + cv)}{\sqrt{n}} \right). \]

In Lemmas 3.1–3.3 we proved a number of identities for \( \Delta_n(X, V), \Lambda_n(X, V) \) defined in (3.12). In particular, they hold for \( \Delta_n(u + cv, x) = \Delta_n(X, V) \mid_{X=(u+cv)(1+x), V=(u+cv)x/c} \) and \( \Lambda_n(u + cv, x) = \Lambda_n(X, V) \mid_{X=(u+cv)(1+x), V=(u+cv)x/c}. \) Let us establish two more identities for \( \Delta_n(u + cv, x) \) and \( \Lambda_n(u + cv, x) \).

**Lemma 3.1.** The following identities hold true:
\[ \Delta_n(u + cv, x) = \frac{\sqrt{B_4} (EY/c - ET)}{(B_2 + B_3/c)} \left[ -\Lambda_n(u + cv, x) \right. \]
\[ + \frac{\sqrt{B_1}}{\sqrt{B_4}} \left( \frac{\sqrt{n}}{B_1} + \frac{B_3 (u + cv)}{B_1 c \sqrt{n}} \right) \left. \right] - \frac{EY}{c\sqrt{B_1}} \frac{(u + cv)}{\sqrt{n}}, \]
and
\[ 1 + x = \frac{EY \sqrt{B_1 B_4}}{(B_1 + B_3(EY/c - ET))(u + cv)} \left[ -\Lambda_n(u + cv, x) \right. \]
\[ + \frac{\sqrt{B_1}}{\sqrt{B_4}} \left( \frac{\sqrt{n}}{B_1} + \frac{B_3 (u + cv)}{B_1 c \sqrt{n}} \right) \left. \right]. \]

**Proof.** From (3.18) and (3.17), we have two expressions for \((1 + x)\):
\[ 1 + x = \frac{\sqrt{B_1 n} \Delta_n(u + cv, x) + (u + cv)EY/c}{(u + cv)(EY/c - ET)}, \]
\[ 1 + x = \frac{B_1 n + B_3(u + cv)/c - \sqrt{B_1 B_4 n} \Lambda_n(u + cv, x)}{(u + cv)(B_2 + B_3/c)}. \]
To get (3.19), we equate the right-hand sides of both equations (3.21) and do straightforward algebraic transformations. To have (3.20), we transform the right-hand side of the second equation (3.21), bearing in mind that \( B_1 = EY B_2 + ET B_3 \) and consequently that \( B_2 + B_3/c = (B_1 + B_3(EY/c - ET))/EY \).\( \Box \)

**Asymptotic analysis of the expressions of the first kind.** By the expressions of the first kind we call those arising in the analysis of the remainder term in the approximation (3.12). Their integrands contain rational functions modified by a square root. The first expression of this type (cf. (3.12) and (3.13)) is
\[ S = (u + cv) \sum_{n=N_0}^{\infty} n^{-3/2} \int_{0}^{\infty} \frac{1}{1 + x} \left( 1 + \left[ \Delta_2^2(u + cv, x) + \Delta_n^2(u + cv, x) \right]^{1/2} \right)^{-3} dx. \]

\[ \text{Since we are concerned with uniform bounds, we are ready to stretch out the range } 0 < x < \frac{c(u+cv)}{u+cv} \text{ in (3.10) and (3.11) up to } 0 < x < \infty. \]
Other expressions of this type are (cf. (3.15))

\[
S[1] = (u + cv) \sum_{n=N_v}^{\infty} n^{-2} \int_0^\infty \frac{1}{1+x} |\Lambda_n(u + cv, x)|(1 + [\Lambda_n^2(u + cv, x) + \Delta_n^2(u + cv, x)]^{1/2})^{-3} dx
\]

and

\[
S[2] = (u + cv) \sum_{n=N_v}^{\infty} n^{-2} \int_0^\infty \frac{1}{1+x} |\Lambda_n(u + cv, x)|(1 + [\Lambda_n^2(u + cv, x) + \Delta_n^2(u + cv, x)]^{1/2})^{-3} dx.
\]

**Processing of** $S$. Applying both identities of Lemma [3.1] we rewrite it as

\[
S = \frac{(B_1 + B_3(EY/c - ET))}{EY\sqrt{B_1B_4}}(u + cv)^2 \sum_{n=N_v}^{\infty} n^{-1} \int_0^\infty \left\{-\Lambda_n(u + cv, x)ight. \\
+ \sqrt{\frac{B_1}{B_4}}\left(\sqrt{n} + \frac{B_3}{B_1c} \frac{(u + cv)}{\sqrt{n}}\right)\right\}^{-1}
\left(1 + \left[\Lambda_n^2(u + cv, x) + \left\{\sqrt{B_4(EY/c - ET)} \frac{EY}{B_2 + B_3/c} \left(\sqrt{n} + \frac{B_3}{B_1c} \frac{(u + cv)}{\sqrt{n}}\right)\right\}^{1/2}\right)^{-3} dx.
\]

Making the change of variables $\xi = -\Lambda_n(u + cv, x)$ in the integral with respect to $x$ and bearing in mind that

\[
dx = \frac{\sqrt{B_1B_4n}}{(u + cv)(B_2 + B_3/c)} d\xi,
\]

we get

\[
S = (u + cv) \sum_{n=N_v}^{\infty} n^{-3/2} \int_{\xi + R_{u+cv,n}}^\infty (\xi + R_{u+cv,n})^{-1}
\times (1 + [\xi^2 + (K_e (\xi + R_{u+cv,n}) - M_{u+cv,n})^{1/2}]^{-3} d\xi,
\]

where

\[
K_e = \frac{EY\sqrt{B_4}[EY - cET]}{cB_1 + B_3[EY - cET]} \left\{\begin{array}{ll}
> 0, & c < \frac{EY}{cB_1} \\
= 0, & c = \frac{EY}{cB_1} \\
< 0, & c > \frac{EY}{cB_1}
\end{array}\right.,
\]

\[
M_{u+cv,n} = \frac{EY}{c\sqrt{B_1}} \frac{u + cv}{\sqrt{n}} > 0,
\]

\[
L_{u+cv,n} = \frac{\sqrt{B_1}}{\sqrt{B_4}} \left(\frac{B_2 u + cv}{B_1c \sqrt{n}} - \frac{\sqrt{n}}{\sqrt{B_1c}}\right),
\]

\[
R_{u+cv,n} = \frac{\sqrt{B_1}}{\sqrt{B_4}} \left(\frac{B_3 u + cv}{B_1c \sqrt{n}} + \frac{\sqrt{n}}{\sqrt{B_1c}}\right) > 0,
\]

and

\[
M_{u+cv,n} - K_e R_{u+cv,n} = \frac{EY\sqrt{B_1}}{cB_1 + B_3[EY - cET]} \left(\frac{u + cv}{\sqrt{n}} - [EY - cET] \sqrt{n}\right).
\]

In the sequel, put for brevity $c^* = EY/ET$.

**Lemma 3.2.** We have $S = O\left(\frac{\ln(u + cv)}{u + cv}\right)$, as $u + cv \to \infty$.

First, we prove Lemma [3.2] for $c = c^*$, bearing in mind that $K_{c^*} = 0$. Then we prove it for $c \neq c^*$.

\[\text{Bear in mind that } cB_1 + B_3[EY - cET] = EY(cEYDT + ETDY) > 0.\]
Proof of Lemma 3.2 for \(c = c^*\). Let us put for brevity \(U = u + c^* v\) and \(\hat{L} = L_{u+c^*v,n}, \hat{R} = R_{u+c^*v,n}, \hat{M} = M_{u+c^*v,n}\), i.e., for \(A = \frac{ET}{\sqrt{B_1}} \frac{E\sqrt{D_1}}{E\sqrt{DT}} > 0, B = \frac{\sqrt{B_1}}{\sqrt{DT}} > 0, C = \frac{\sqrt{B_1}}{\sqrt{DT}} > 0\), we have

\[
\hat{L} = \frac{C^2}{A \sqrt{n}} U - B \sqrt{n} \left\{ \begin{array}{ll}
> 0, & n < \frac{B}{\sqrt{B_1}} U, \\
< 0, & n > \frac{B}{\sqrt{B_1}} U.
\end{array} \right.
\]

\[
0 < \hat{M} = C \frac{U}{\sqrt{n}} \left\{ \begin{array}{ll}
> 1, & n < \frac{(ET)^2}{B_1} U^2, \\
< 1, & n > \frac{(ET)^2}{B_1} U^2,
\end{array} \right.
\]

and \(\hat{S} = 8|_{c=c^*}\), i.e., \(\hat{S} = U \sum_{n > N} n^{-3/2} \int_L^\infty (\xi + \hat{R})^{-1} (1 + (\xi^2 + \hat{M}^2)^{1/2})^{-3} d\xi\).

It is easily seen that

\[
\hat{S} \leq K_1 I_1 + K_2 I_2, \tag{3.23}
\]

where

\[
I_1 = U \sum_{n > N} n^{-3/2} \int_L^\infty (\xi + \hat{R})^{-1} (\xi^2 + \hat{M}^2)^{-3/2} d\xi,
\]

\[
I_2 = U \sum_{n > \frac{(ET)^2}{B_1} U^2} n^{-3/2} \int_L^\infty (\xi + \hat{R})^{-1} (1 + (2\hat{M}|\xi|^{1/2})^{-3} d\xi.
\]

The essence of (3.23) is the following. For \(n\) such that \(\hat{M} > 1\), we simplify the denominator \((1 + (\xi^2 + \hat{M}^2)^{1/2})^3\) by switching to \((\xi^2 + \hat{M}^2)^{3/2}\). The latter has no singularity since \(\hat{M} > 1\). For \(n\) such that \(\hat{M} < 1\), we keep 1 in the denominator and use the inequality between the arithmetic mean and the geometric mean. Both these estimates are such that the integrals in \(I_1\) and \(I_2\) may be evaluated explicitly.

Examining \(I_1\), the explicit expression for the integral \(\int_L^\infty (\xi + \hat{R})^{-1} (\xi^2 + \hat{M}^2)^{-3/2} d\xi\) is found in Lemma 4.5. Using it, the asymptotic behavior of \(I_1\), as \(U \to \infty\), is checked as required in Section 4.6.

Examining \(I_2\) and bearing in mind that \(\hat{L} < 0\) for \(n > \frac{(ET)^2}{B_1} U^2\), we split the integrand and make the change of variables as follows:

\[
\int_L^\infty (\xi + \hat{R})^{-1} (1 + (2\hat{M}|\xi|^{1/2})^{-3} d\xi
\]

\[
= \int_0^\infty (\xi + \hat{R})^{-1} (1 + (2\hat{M}|\xi|^{1/2})^{-3} d\xi + \int_0^{-\hat{L}} (\hat{R} - \xi)^{-1} (1 + (2\hat{M}|\xi|^{1/2})^{-3} d\xi
\]

\[
= \int_0^\infty (2\hat{M}\hat{R} + \xi)^{-1} (1 + \sqrt{\xi})^{-3} d\xi + \int_0^{-\hat{L}} (2\hat{M}\hat{R} - \xi)^{-1} (1 + \sqrt{\xi})^{-3} d\xi.
\]

The explicit expressions for two latter integrals are found in Lemma 4.7. Using this result, the asymptotic behavior of \(I_2\), as \(U \to \infty\), is checked as required in Section 4.7. The proof is complete.

Proof of Lemma 3.2 for \(c \neq c^*\). Let us put for brevity \(U = u + cv\) and \(\bar{L} = L_{u+cv,n}, \bar{R} = R_{u+cv,n}, \bar{M} = M_{u+cv,n}\), i.e., for \(A = \frac{ET}{\sqrt{B_1}} \frac{E\sqrt{D_1}}{E\sqrt{DT}} > 0, B = \frac{\sqrt{B_1}}{\sqrt{DT}} > 0, C = \frac{\sqrt{B_1}}{\sqrt{DT}} > 0\),

\[
\frac{\sqrt{B_1}}{\sqrt{DT}} > 0,
\]

If we put \(c = c^*\) in these expressions, they will be equal to \(A, B, C\) introduced in the proof of Lemma 3.2 for \(c = c^*\).
\( C = \frac{EY}{c\sqrt{B_1}} > 0 \), we have

\[
\begin{align*}
\tilde{L} &= \frac{cETC^2}{EY\sqrt{n}} - B\sqrt{n} \\
    &> 0, \ n < \frac{B_2U}{B_1}, \\
    &< 0, \ n > \frac{B_2U}{B_1}
\end{align*}
\]

\( \tilde{R} = \frac{AU}{\sqrt{n}} + B\sqrt{n} > 0 \),

\[
0 < \tilde{M} = \frac{C}{\sqrt{n}} \begin{cases} 
1, & n < \frac{(EY)^2}{c^2B_1}U^2, \\
< 1, & n > \frac{(EY)^2}{c^2B_1}U^2,
\end{cases}
\]

\[
\tilde{L} + \tilde{R} = \frac{cETC^2 + EYA^2}{EY\sqrt{n}} U > 0.
\]

Bearing in mind that \( cB_1 + B_3[EY - cET] = EY(cEYDT + ETDY) > 0 \), we have

\[
\tilde{M} - K_c\tilde{R} = \frac{EY\sqrt{B_1}}{cB_1 + B_3[EY - cET]} \left( \frac{U}{\sqrt{n}} - [EY - cET]\sqrt{n} \right)
\]

\[
= (C - K_cA) \frac{U}{\sqrt{n}} - K_cB\sqrt{n}.
\]

Let us rewrite (3.22) as

\[
S = U \sum_{n=N_s}^{\infty} n^{-3/2} \int_{L}^{\infty} (\xi + \tilde{R})^{-1} \left( 1 + [(1 + K_c^2)\xi^2 - 2K_c(\tilde{M} - K_c\tilde{R})\xi + (\tilde{M} - K_c\tilde{R})^2]^{1/2} \right)^{-3} d\xi
\]

and, completing the square and making the change of variables, rewrite it as

\[
S = \frac{U}{(1 + K_c^2)^{3/2}} \sum_{n=N_s}^{\infty} n^{-3/2} \int_{L}^{\infty} \frac{M - K_c\tilde{R}}{1 + K_c^2} \left( \xi + \frac{R + K_c\tilde{M}}{1 + K_c^2} \right)^{-1}
\]

\[
\times \left( \frac{1}{1 + K_c^2} \right)^{1/2} + \left( \frac{1}{1 + K_c^2} \right)^{1/2} \left( \frac{M - K_c\tilde{R}}{1 + K_c^2} \right)^{2/3} d\xi.
\]

**Case c > c*. In this case, \( K_c < 0 \). The second summand in brackets in the integrand in (3.26) is positive since in this case \( \tilde{M} - K_c\tilde{R} > 0 \); it is easily seen from the second equality (3.24). Moreover, for \( K_c < 0 \) the difference \( \tilde{M} - K_c\tilde{R} \) increases, as \( n \) increases, and exceeds \( K_c\sqrt{U} \) for \( n > N_c \). The integrand in (3.26) has no singularities in the region of integration since

\[
\tilde{L} - K_c\frac{M - K_c\tilde{R}}{1 + K_c^2} + \frac{R + K_c\tilde{M}}{1 + K_c^2} = \tilde{L} + \tilde{R} > 0.
\]

We use the estimate \( S \leq K_3\mathcal{I}_3 \), where

\[
\mathcal{I}_3 = U \sum_{n=N_s}^{\infty} n^{-3/2} \int_{L}^{\infty} \left( \frac{1}{1 + K_c^2} \right)^{1/2} + \left( \frac{M - K_c\tilde{R}}{1 + K_c^2} \right)^{2/3} d\xi.
\]

The explicit expression for the integral in \( \mathcal{I}_3 \) is found in Lemma 4.7. Using it, the asymptotic behavior of \( \mathcal{I}_3 \), as \( U \to \infty \), is checked as required in Section 4.8. The proof is complete.

**Case c < c*. In this case, \( K_c > 0 \). We have (see (3.25))

\[
\tilde{M} - K_c\tilde{R} \begin{cases} 
> 0, & n < \frac{U}{cE(c^* - c)}, \\
< 0, & n > \frac{U}{cE(c^* - c)}.
\end{cases}
\]
It is easily seen that $\mathcal{S} \leq K_4 \mathcal{I}_4 + K_5 \mathcal{I}_5 + K_6 \mathcal{I}_6$, where

$$
\mathcal{I}_4 = U \sum_{n < \frac{c}{EY - cET} + \frac{K}{KcB}U} n^{-3/2} \int_{-\infty}^{\infty} \left( \zeta + \frac{\hat{R} + K_c M}{1 + K_c^2} \right)^{-1} \times \left( \zeta^2 + \left( \frac{\hat{M} - K_c \hat{R}}{1 + K_c^2} \right)^2 \right)^{-3/2} d\zeta,
$$

$$
\mathcal{I}_5 = U \sum_{n < \frac{c}{EY - cET} - \frac{K}{KcB}U} n^{-3/2} \int_{-\infty}^{\infty} \left( \zeta + \frac{\hat{R} + K_c M}{1 + K_c^2} \right)^{-1} \times \left( 1 + \left( \frac{\hat{M} - K_c \hat{R}}{1 + K_c^2} \right) |\zeta| \right)^{1/2} \right)^{-3} d\zeta,
$$

$$
\mathcal{I}_6 = U \sum_{n > \frac{c}{EY - cET} + \frac{K}{KcB}U} n^{-3/2} \int_{-\infty}^{\infty} \left( \zeta + \frac{\hat{R} + K_c M}{1 + K_c^2} \right)^{-1} \times \left( \zeta^2 + \left( \frac{\hat{M} - K_c \hat{R}}{1 + K_c^2} \right)^2 \right)^{-3/2} d\zeta.
$$

It is easily seen that since

$$
\frac{C - K_c A}{K_c B} = \frac{1}{EY - cET} = \frac{c^*}{EY(c^* - c)},
$$

(3.27)

which can be verified by direct calculations, the range of summation $n < \frac{c}{EY - cET} - \frac{K}{KcB}U$ in $\mathcal{I}_4$ may be written as $n < \frac{c - Kc A}{KcB} U - \frac{K}{KcB} U$, the range of summation $n < \frac{c}{EY - cET} - \frac{K}{KcB} U < n < \frac{c - Kc A}{KcB} U + \frac{K}{KcB} U$, and the range of summation $n > \frac{c}{EY - cET} + \frac{K}{KcB} U$ in $\mathcal{I}_6$ may be written as $n > \frac{c - Kc A}{KcB} U + \frac{K}{KcB} U$.

The explicit expressions for the integrals in $\mathcal{I}_4$ and $\mathcal{I}_6$ are similar to that one for the integral in $\mathcal{I}_5$. Using it, the asymptotic behavior of $\mathcal{I}_4$ and $\mathcal{I}_6$, as $U \to \infty$, is checked as required in Section 4.4.

The explicit expression for the integral in $\mathcal{I}_5$ is similar to that one for the integral in $\mathcal{I}_2$. Using it, the asymptotic behavior of $\mathcal{I}_5$, as $U \to \infty$, is checked as required in Section 4.10. The proof is complete. $\square$

**Processing of $S^{[1]}$.** The same way as for $S$, rewrite $S^{[1]}$ as

$$
S^{[1]} = (u + cv) \sum_{n = N_c}^{\infty} n^{-2} \int_{L(u + cv, n)}^{\infty} \left( (\xi + R_{u + cv, n})^{-1} |\xi| \right) \times \left( 1 + \left( \xi^2 + (K_c (\xi + R_{u + cv, n}) - M_{u + cv, n}) \right) \right)^{1/2} d\xi.
$$

**Lemma 3.3.** We have $S^{[1]} = O \left( \frac{\ln(u + cv)}{u + cv} \right)$, as $u + cv \to \infty$.

**Proof of Lemma 3.3 for $c = c^*$.** Retaining notation used in Lemma 3.2, consider

$$
\hat{S}^{[1]} = S^{[1]} \bigg|_{c = c^*}, \text{ i.e.,}
$$

$$
\hat{S}^{[1]} = U \sum_{n > N_c} n^{-2} \int_{L}^{\infty} |\xi| (\xi + \hat{R})^{-1} (1 + [\xi^2 + \hat{M}^2]^{1/2})^{-3} d\xi \leq K_1 T_1 + K_2 T_2,
$$
where

\[ T_1 = U \sum_{N_n < n < \frac{bU}{H}} n^{-2} \int_{L}^{\infty} |\xi| (\xi + \hat{R})^{-1} (\xi^2 + \hat{M}^2)^{-3/2} d\xi \]
\[ + U \sum_{n > \frac{(bU)^2}{H}} n^{-2} \int_{L}^{\infty} |\xi| (\xi + \hat{R})^{-1} (\xi^2 + \hat{M}^2)^{-3/2} d\xi, \]
\[ T_2 = U \sum_{n > \frac{(bU)^2}{H}} n^{-2} \int_{L}^{\infty} |\xi| (\xi + \hat{R})^{-1} (1 + (2\hat{M}|\xi|)^{1/2})^{-3} d\xi. \]

The asymptotic behavior of \( T_1 \), as \( U \to \infty \), is checked as required in Section 4.11. The asymptotic behavior of \( T_2 \), as \( U \to \infty \), is checked as required in Section 4.12.

**Proof of Lemma 3.3** for \( c \neq c^* \). This proof is a modification of the proof of Lemma 3.3 for \( c = c^* \), alike the proof of Lemma 3.2 for \( c \neq c^* \) was a modification of that proof for \( c = c^* \). It uses essentially the same techniques and is left to the reader.

**Processing of \( S[2] \).** Just as we did in the analysis of \( S \), rewrite \( S[2] \) as

\[ S[2] = (u + cv) \sum_{n = N_n}^{\infty} n^{-2} \int_{L}^{\infty} (\xi + R_{u+c,v,n})^{-1} \left| K_c \left( \xi + R_{u+c,v,n} \right) - M_{u+c,v,n} \right| \]
\[ \times (1 + [\xi^2 + (K_c \left( \xi + R_{u+c,v,n} \right) - M_{u+c,v,n})^2]^{1/2})^{-3} d\xi. \]

**Lemma 3.4.** We have \( S[2] = O\left( \frac{\ln(u+cv)}{u+cv} \right) \), as \( u + cv \to \infty \).

**Proof.** This proof goes along the same lines as the proof of Lemma 3.3 and is left to the reader.

**Asymptotic analysis of the expressions of the second kind.** By the expressions of the second kind we call those arising when we simplify the main term of approximation \( (8.2) \). Their integrands contain exponential, inherited from CLT, and rational functions. The first expression of this type (cf. \( (8.3) \)) is

\[ g = (u + cv) \sum_{n = N_n}^{\infty} n^{-1} \int_{0}^{\infty} \frac{1}{1 + x} \exp \left\{ - \frac{1}{2} \left[ \Lambda_n^2(u + cv, x) + \Delta_n^2(u + cv, x) \right] \right\} dx. \]

Other expressions of this type are (cf. \( (8.10) \) and \( (8.11) \))

\[ g[1] = (u + cv) \sum_{n = N_n}^{\infty} n^{-3/2} \int_{0}^{\infty} \frac{\left| \Lambda_n(u + cv, x) \right|}{1 + x} \exp \left\{ - \frac{1}{2} \left[ \Lambda_n^2(u + cv, x) \right] \right\} dx, \]
\[ g[2] = (u + cv) \sum_{n = N_n}^{\infty} n^{-3/2} \int_{0}^{\infty} \frac{\left| \Delta_n(u + cv, x) \right|}{1 + x} \exp \left\{ - \frac{1}{2} \left[ \Lambda_n^2(u + cv, x) \right] \right\} dx \]

\[ \text{where}^8 \text{in } T_1, \text{the first integral is with } \hat{L} > 0 \text{ and the second with } \hat{L} < 0. \text{ In } T_2, \text{ the integral is with } \hat{L} < 0 \text{ and } 0 < \hat{M} < 1. \]
and (see Section 3.5 below)

$$G^{[1]} = (u + cv)^{1/2} \sum_{n = N}^{\infty} n^{-1} \int_{0}^{\infty} \frac{|\Delta_n(u + cv, x)|}{(1 + x)^{3/2}} \exp \left\{ -\frac{1}{2} \left[ \Delta_n^2(u + cv, x) + \Delta_0^2(u + cv, x) \right] \right\} dx,$$

$$G^{[4]} = (u + cv)^{1/2} \sum_{n = N}^{\infty} n^{-1} \int_{0}^{\infty} \frac{|\Delta_n(u + cv, x)|}{(1 + x)^{3/2}} \exp \left\{ -\frac{1}{2} \left[ \Delta_n^2(u + cv, x) + \Delta_0^2(u + cv, x) \right] \right\} dx.$$

**Proof of Lemma 3.5**. Applying the identities of Lemma 3.1 and making the change of variables \( \xi = -\Lambda_n(u + cv, x) \) in the integral with respect to \( x \), we rewrite it as

$$G = (u + cv)^{\frac{-3}{2}} \sum_{n = N}^{\infty} n^{-3/2} \int_{\Lambda_n + cv, n}^{\infty} (\xi + R_{u + cv, n})^{-1} \times \exp \left\{ -\frac{1}{2} \left[ \xi^2 + \left\{ K_c(\xi + R_{u + cv, n}) - M_{u + cv, n} \right\}^2 \right] \right\} d\xi. \quad (3.28)$$

**Lemma 3.5.** We have \( G = O\left( \frac{\ln(u + cv)}{u + cv} \right) \), as \( u + cv \to \infty \).

As before, we prove first this lemma in the case \( c = c^* \) and then in the case \( c \neq c^* \). In both cases, we use notation set in respective parts of the proof of Lemma 3.2

**Proof of Lemma 3.5 for \( c = c^* \).** Recall (see (3.24)) that \( \hat{L} > 0 \) for \( n < \frac{B_1}{B_2} U \), \( \hat{L} < 0 \) for \( n > \frac{B_2}{B_1} U \), that \( \hat{L} + \hat{R} > 0 \) for all \( n \), and \( M > 1 \) for \( n < \frac{(E_T)^2}{B_1^2} U^2 \), \( M < 1 \) for \( n > \frac{(E_T)^2}{B_1^2} U^2 \), and consider \( \tilde{G} = G \big|_{c = c^*} \), i.e.,

$$\tilde{G} = U \sum_{n > N} n^{-3/2} \exp \left\{ -\frac{1}{2} M^2 \right\} \int_{\hat{L}}^{\infty} (\xi + \hat{R})^{-1} \exp \left\{ -\frac{1}{2} \xi^2 \right\} d\xi. \quad (3.29)$$

It is easily seen that

$$\tilde{G} \leq K_1 J_1 + K_2 J_2, \quad (3.30)$$

where

$$J_1 = U \sum_{n < \frac{(E_T)^2}{B_1^2} U^2} n^{-3/2} \exp \left\{ -\frac{1}{2} M^2 \right\} \int_{\hat{L}}^{\infty} (\xi + \hat{R})^{-1} \exp \left\{ -\frac{1}{2} \xi^2 \right\} d\xi,$$

$$J_2 = U \sum_{n > \frac{(E_T)^2}{B_1^2} U^2} n^{-3/2} \int_{\hat{L}}^{\infty} (\xi + \hat{R})^{-1} \exp \left\{ -\frac{1}{2} \xi^2 \right\} d\xi.$$

The asymptotic behavior of \( J_1 \), as \( U \to \infty \), is checked as required in Section 4.14. The asymptotic behavior of \( J_2 \), as \( U \to \infty \), is checked as required in Section 4.15. The proof is complete.

**Proof of Lemma 3.5 for \( c \neq c^* \).** As before (see (3.24)), we put \( U = u + cv \) and \( \hat{L} = L_{u + cv, n}, \hat{R} = R_{u + cv, n}, \hat{M} = M_{u + cv, n} \). Rewrite (3.28) as

$$G = U \sum_{n = N}^{\infty} n^{-3/2} \int_{\hat{L}}^{\infty} (\xi + \hat{R})^{-1} \exp \left\{ -\frac{1}{2} \left( (1 + K_c^2) \xi^2 - 2K_c(\hat{M} - K_c \hat{R})\xi + (\hat{M} - K_c \hat{R})^2 \right) \right\} d\xi$$

\( ^{9} \)Therefore, the integrand in (3.29) does not contain singularities within the range of integration.

\( ^{10} \)The unique point of singularity of the first factor lies to the left of \( \hat{L} \) since \( -\hat{R} < \hat{L} \). The second factor is positive everywhere.

\( ^{11} \)While using (3.23) was essential, using of (3.30) is largely for convenience: it emphasizes that \( \hat{M} \) is small for \( n > \frac{(E_T)^2}{B_1^2} U^2 \), and the factor \( \exp \left\{ -\frac{1}{2} M^2 \right\} \) is unessential.
and, completing the square and making the change of variables, as
\[
S = U \sum_{n=N_a}^{\infty} n^{-3/2} \exp \left\{ - \frac{1}{2} \frac{(\bar{M} - K_c \bar{R})^2}{1 + K_c^2} \right\} \int_{L-K_c \bar{M} - K_c \bar{R}}^{\infty} \left( \zeta + \frac{\bar{R} + K_c \bar{M}}{1 + K_c^2} \right)^{-1} \exp \left\{ - \frac{\sqrt{1 + K_c^2}}{2} \zeta^2 \right\} d\zeta.
\]

Since the exponential factor is easier to work, this expression is suitable for its asymptotic analysis without its simplifying.

Case \(c > c^*\). In this case, \(K_c < 0\). Recall that it yields \(\bar{M} - K_c \bar{R} > 0\) and use the arguments outlined in the respective part of the proof of Lemma 3.2 for \(c > c^*\). The asymptotic behavior of the integral
\[
\int_{L-K_c \bar{M} - K_c \bar{R}}^{\infty} \left( \zeta + \frac{\bar{R} + K_c \bar{M}}{1 + K_c^2} \right)^{-1} \exp \left\{ - \frac{\sqrt{1 + K_c^2}}{2} \zeta^2 \right\} d\zeta
\]
is examined by means of a direct extension, as it was done in Section 4.8 of Lemma 4.11. Using it, the asymptotic behavior of \(S\), as \(U \to \infty\), is easily checked as required along the lines traced in Sections 4.8, 4.14, and 4.15. The proof is complete.

Case \(c < c^*\). In this case, when \(K_c > 0\), used should be the arguments outlined in the respective part of the proof of Lemma 3.2, for \(c < c^*\), with the difference that integrals are analyzed along the lines traced in Sections 4.8, 4.14, and 4.15. The proof is complete. □

3.5. Step 5: further simplification of the main term of approximation. In Step 3 of the proof, the main term of approximation \(A_1(u, c \mid T_1 = v)\) (see (3.8)) was simplified up to \(A_1^{(1)}(u, c \mid T_1 = v)\) (see (3.8)). Let us further simplify \(A_1^{(1)}(u, c \mid T_1 = v)\) up to the terms of allowed order of smallness. We use for it core asymptotic analysis developed in Step 4. It is noteworthy that in the rest of the proof this analysis is applied only to the expressions of the second kind.

First step in processing (3.8). Rewrite (3.8) as
\[
A_1^{(1)}(u, c \mid T_1 = v) = \frac{(EY)^{3/2} (u + cv)^{1/2}}{2\pi c \sqrt{DTD}Y} \int_0^{\frac{c(u - v)}{u + cv}} \frac{1}{(1 + x)^{3/2}} \times \sum_{n=N_a}^{\infty} n^{-1/2} \sqrt{\frac{(u + cv)(1 + x)}{nEY}} \exp \left\{ - \frac{1}{2} [A_n^2(u + cv, x) + \Delta_n^2(u + cv, x)] \right\} dx
\]
and introduce
\[
A_1^{(2)}(u, c \mid T_1 = v) = \frac{(EY)^{3/2} (u + cv)^{1/2}}{2\pi c \sqrt{DTD}Y} \int_0^{\frac{c(u - v)}{u + cv}} \frac{1}{(1 + x)^{3/2}} \times \sum_{n=N_a}^{\infty} n^{-1/2} \exp \left\{ - \frac{1}{2} [A_n^2(u + cv, x) + \Delta_n^2(u + cv, x)] \right\} dx.
\]
Using Lemma 1.1 which yields the identity
\[
1 - \sqrt{\frac{(u + cv)(1 + x)}{nEY}} = \left\{ \frac{\sqrt{B_4}}{\sqrt{B_1 n}} A_n(u + cv, x) + \frac{\sqrt{B_3}}{EY \sqrt{B_1 n}} \Delta_n(u + cv, x) \right\} \left( 1 + \sqrt{\frac{(u + cv)(1 + x)}{nEY}} \right)^{-1},
\]

11Recall that dealing with the analogue formula for \(S\) (see (3.26)), due to technical complexities, we had to switch to certain upper bounds for \(S\).
we have to prove that

\[
\sup_{t>0} \left| A_t^{(1)}(u, c \mid T_1 = v) - A_t^{(2)}(u, c \mid T_1 = v) \right| = O\left( \frac{\ln(u + cv)}{u + cv} \right),
\]

as \( u + cv \to \infty \). It is done by means of core asymptotic analysis of the expressions of the second kind described in Step 4. In particular, for this purpose we have to prove that

\[
(u + cv)^{1/2} \int_0^{c(x)} \frac{1}{(1 + x)^{3/2}} \sum_{n=N_x}^{\infty} n^{-1} \left( |\Lambda_n(u + cv, x)| + |\Delta_n(u + cv, x)| \right)
\]

\[
\times \exp \left\{ -\frac{1}{2} \left[ \Lambda_n^2(u + cv, x) + \Delta_n^2(u + cv, x) \right] \right\} dx = O\left( \frac{\ln(u + cv)}{u + cv} \right),
\]

as \( u + cv \to \infty \). This standard check is left to the reader.

**Second step in processing (3.8).** We write

\[
A_t^{(3)}(u, c \mid T_1 = v) = \frac{(EY)^{3/2}(u + cv)^{1/2}}{2\pi \sqrt{D_0}} \int_0^{c(x)} \frac{1}{(1 + x)^{3/2}} \sum_{n=N_x}^{\infty} n^{-1/2} \exp \left\{ -\frac{1}{2} \Lambda_n^2(u + cv, x) \right\} dx.
\]

We have to prove that

\[
\sup_{t>0} \left| A_t^{(2)}(u, c \mid T_1 = v) - A_t^{(3)}(u, c \mid T_1 = v) \right| = O\left( \frac{\ln(u + cv)}{u + cv} \right),
\]

as \( u + cv \to \infty \). It is done by means of core asymptotic analysis of the expressions of the second kind described in Step 4. This standard check is left to the reader.

**Third step in processing (2.8).** Bearing in mind the identity\(^{12}\) (see Lemma 1.3)

\[
\Lambda_{n+1}(u + cv, x) - \Lambda_n(u + cv, x) = \left( \frac{B_1}{B_{3n}} \right)^{1/2} + \Lambda_{n+1}(u + cv, x) \left( 1 - \sqrt{1 + 1/n} \right),
\]

we prove by means of core asymptotic analysis of the expressions of the second kind described in Step 4 that

\[
\sup_{t>0} \left| A_t^{(3)}(u, c \mid T_1 = v) - A_t^{(4)}(u, c \mid T_1 = v) \right| = O\left( \frac{\ln(u + cv)}{u + cv} \right),
\]

as \( u + cv \to \infty \), where

\[
A_t^{(4)}(u, c \mid T_1 = v) = \frac{(u + cv)^{1/2}(EY)^{3/2}}{\sqrt{2\pi} \sqrt{B_1}} \int_0^{c(x)} \frac{1}{(1 + x)^{3/2}} \sum_{n=N_x}^{\infty} \left( \Lambda_{n+1}(u + cv, x) \right.
\]

\[
- \Lambda_n(u + cv, x) \exp \left\{ -\frac{1}{2} \Lambda_n^2(u + cv, x) \right\} dx.
\]

This standard check is left to the reader.

\(^{12}\)Note that \( 1 - \sqrt{1 + x} = -x/2 + x^2/8 - x^3/16 + \ldots \).
Fourth step in processing \((3.8)\). We finally note that \(D^2 = B_1/(EY)^3\) and that
\[
\Delta^2_{[x+c]/(1+x)}(u + cv, x) = \frac{(x - (1 + x)(c/c^*))^2}{c^2 D^2 (1 + x)} = (u + cv) \frac{(x[1/c - 1/c^*] - 1/c^*)^2}{D^2(1 + x)}.
\]
By means of standard core asymptotic analysis of the expressions of the second kind described in Step 4, we prove that
\[
\sup_{t > 0} \left| A_t^{(d)}(u, c | T_1 = v) - A_t^{(b)}(u, c | T_1 = v) \right| = O\left( \frac{\ln(u + cv)}{u + cv} \right),
\]
as \(u + cv \to \infty\), where
\[
A_t^{(b)}(u, c | T_1 = v) = \frac{(u + cv)^{1/2}}{\sqrt{2\pi c^2 D^2}} \int_0^{(x-c)/c} \frac{1}{(1 + x)^{3/2}} \exp \left\{ \frac{-1}{2} \frac{(x - (1 + x)(c/c^*))^2}{c^2 D^2 (1 + x)} \right\} dx,
\]
which yields the required approximation. The proof of Theorem 11 is complete.

4. Main technicalities and auxiliary results

In this section, we gather main auxiliary results used in Section 3.

4.1. Non-uniform Berry-Esseen bounds in local CLT. Let the random vectors \(\xi_i, i = 1, 2, \ldots\), assuming values in \(\mathbb{R}^m\) be i.i.d. with c.d.f. \(P\), with zero mean and with identity covariance matrix \(I\). Put \(S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i, P_n(A) = P\{S_n \in A\}, A \subset \mathbb{R}^m, p_n(x) = \frac{\partial^m}{\partial x_1 \ldots \partial x_m} P\{S_n \leq x\}, x = (x_1, \ldots, x_m) \in \mathbb{R}^m\).

The Berry-Esseen bounds in one-dimensional, as \(m = 1\), central limit theorem (CLT) are well known. The following theorem follows from Theorem 11 in Ch. 7, §2 of Petrov (1975) proved for non-identically distributed random variables \(\xi_i, i = 1, 2, \ldots\).

**Theorem 4.1 (Petrov (1975)).** Let \(E\xi_1^2 > 0, E|\xi_1|^3 < \infty\), and \(\int_{|t| > \epsilon} E|e^{it\xi_1}|^n dt = O(n^{-1})\) for any fixed \(\epsilon > 0\). Then for all sufficiently large \(n\) a bounded p.d.f. \(p_n(x)\) exists and
\[
\sup_{x \in \mathbb{R}} |p_n(x) - \varphi_{(0,1)}(x)| = O(n^{-1/2}), \quad n \to \infty.
\]

The non-uniform Berry-Esseen bounds in integral rather than local one-dimensional CLT may be found in Petrov (1995) (see, e.g., Theorems 15 and 14 in Ch. 5, §6 in Petrov (1995)).

A detailed study of normal approximations and asymptotic expansions in the CLT in \(\mathbb{R}^m\), as \(m > 1\), is conducted in Bhattacharya and Ranga Rao (1976) (see particularly Theorem 19.2 in Bhattacharya and Ranga Rao (1976). The non-uniform Berry-Esseen bounds in \(\mathbb{R}^m\), \(m > 1\), that is used in Section 3 as auxiliary result, is Theorem 4 in §3 of Dubinskai (1982) with \(k = m\) and \(s = 2\). We first formulate the following conditions.

**Condition (Pn):** there exists \(N \geq 1\) such that \(\sup_{x \in \mathbb{R}^m} p_N(x) \leq C < \infty\) and
\[
\int_{\|x\| > \sqrt{n}} \|x\|^2 P(dx) + \frac{1}{n} \int_{\|x\| \leq \sqrt{n}} \|x\|^4 P(dx) + \frac{1}{\sqrt{n}} \sup_{\|x\| = 1} \left| \int_{\|x\| \leq \sqrt{n}} (x, e)^3 P(dx) \right| = O(\epsilon_n),
\]
\(n \to \infty\), where \(\epsilon_n\) is a sequence of positive numbers such that \(\epsilon_n \to 0, as n \to \infty\), and \(\epsilon_n \geq 1/\sqrt{n}\).

**Condition (A2):** \(\beta_2 = E\|\xi_1\|^2 < \infty, \alpha_1(t) = E \xi_1(t) < \infty\).

**Theorem 4.2 (Dubinskai (1982)).** To have
\[
(1 + \|x\|)^3 |p_n(x) - \varphi_{(0,1)}(x)| = O(n^{-1/2}), \quad n \to \infty,
\]
(4.1)
it is necessary and sufficient that conditions \((P_m), (A_2)\), and
\[ z \int_{\|x\|>z} \|x\|^2 P(dx) + \sup_{\|x\|=1} \left| \int_{\|x\| \leq z} (x, e)^3 P(dx) \right| = O(1), \quad z \to \infty, \]
be satisfied.

Remark 4.1. It is known that the estimate \(4.1\) is optimal in terms of dependence on \(\|x\|\), i.e., the power 3 in \(4.1\) can not be replaced by a greater one.

4.2. Large deviations for sums of i.i.d. r.v. The following theorem is Corollary 2 in Nagaev (1965) (see also Nagaev (1979)).

Lemma 4.1. Let \(\xi_i, i=1,2,\ldots\), be i.i.d. random variables such that \(E\xi_1 = 0\) and \(D\xi = 1\). If \(c_m = E|\xi_i|^m < \infty\) with \(m > 2\), then for \(x > 4\sqrt{n} \max \left( \frac{n^{m/2-1}}{K_m c_m}, 0 \right)\)
\[ P \left\{ \sum_{i=1}^{n} \xi_i > x \right\} < \frac{B_m c_m n}{x^m}, \]
where \(K_m = 1 + (m+1)^{m+2} e^{-m}\), and \(B_m\) is an absolute constant depending only on \(m\).

4.3. Fundamental identities. For \(B_1 = (E)2DY + (E)^2 DT, B_2 = EYDT, B_3 = ETDY\), and \(B_4 = DT^2\), we use notation
\[ Y_n(X) = \frac{X - nE Y}{\sqrt{nDY}}, \quad T_n(V) = \frac{V - nE T}{\sqrt{nDT}}, \]
\[ \Lambda_n(X, V) = \frac{\sqrt{EY - XET}}{\sqrt{B_1 n}}, \quad \Lambda_n(X, V) = \frac{B_1 n - (B_2 X + B_3 V)}{\sqrt{B_1 B_4 n}}. \]

Lemma 4.2. We have the identity
\[ Y_n^2(X) + T_n^2(V) = \Lambda_n^2(X, V) + \Lambda_n^2(X, V). \]

Proof. Getting of this identity is based on algebraic manipulations with the left-hand side, aimed at completing the square. Its proof may be done as well by means of a straightforward check.

Lemma 4.3. We have the identity
\[ \Lambda_{n+1}(X, V) - \Lambda_n(X, V) = \left( \frac{B_1}{B_4 n} \right)^{1/2} + \Lambda_{n+1}(X, V)(1 - \sqrt{1 + 1/n}). \]

Proof. We have
\[ (B_1 B_4 n)^{1/2} [\Lambda_{n+1}(X, V) - \Lambda_n(X, V)] \]
\[ = \left\{ (B_1 B_4 n + 1)\right\}^{1/2} \Lambda_{n+1}(X, V) - (B_1 B_4 n)^{1/2} \Lambda_n(X, V) \}
\[ + \left\{ (B_1 B_4 n)^{1/2} \Lambda_{n+1}(X, V) - (B_1 B_4 n + 1)^{1/2} \Lambda_{n+1}(X, V) \right\} \]
\[ = B_1 + \Lambda_{n+1}(X, V)(B_1 B_4 n)^{1/2}(1 - \sqrt{1 + 1/n}). \]

Indeed, since
\[ (B_1 B_4 (n + 1))^{1/2} \Lambda_{n+1}(X, V) = B_1 (n + 1) - (B_2 X + B_3 V), \]
\[ (B_1 B_4 n)^{1/2} \Lambda_n(X, V) = B_1 n - (B_2 X + B_3 V), \]
the first summand is
\[ (B_1 B_4 (n + 1))^{1/2} \Lambda_{n+1}(X, V) - (B_1 B_4 n)^{1/2} \Lambda_n(X, V) \]
\[ = B_1 (n + 1) - (B_2 X + B_3 V) - B_1 n + (B_2 X + B_3 V) = B_1. \]
The second summand is
\[
(B_1B_4n)^{1/2}\Lambda_{n+1}(\mathcal{X}, \mathcal{V}) - (B_1B_4(n+1))^{1/2}\Lambda_{n+1}(\mathcal{X}, \mathcal{V})
\]
\[
= (B_1B_4n)^{1/2}\Lambda_{n+1}(\mathcal{X}, \mathcal{V})\left\{1 - \frac{(B_1B_4(n+1))^{1/2}}{(B_1B_4n)^{1/2}}\right\}
\]
\[
= (B_1B_4n)^{1/2}\Lambda_{n+1}(\mathcal{X}, \mathcal{V})\left\{1 - \sqrt{\frac{n+1}{n}}\right\}.
\]
The proof is complete. □

**Lemma 4.4.** We have the identities
\[
1 - \frac{\mathcal{X}}{n\mathcal{Y}} = \frac{\sqrt{B_4}}{\sqrt{B_1n}}\Lambda_n(\mathcal{X}, \mathcal{V}) + \frac{B_3}{\mathcal{E}_n\sqrt{B_1n}}\Delta_n(\mathcal{X}, \mathcal{V})
\]
and
\[
1 - \sqrt{\frac{\mathcal{X}}{n\mathcal{Y}}} = \left\{\frac{\sqrt{B_4}}{\sqrt{B_1n}}\Lambda_n(\mathcal{X}, \mathcal{V}) + \frac{B_3}{\mathcal{E}_n\sqrt{B_1n}}\Delta_n(\mathcal{X}, \mathcal{V})\right\}\left(1 + \sqrt{\frac{\mathcal{X}}{n\mathcal{Y}}}\right)^{-1}.
\]

**Proof.** Bearing in mind that \(B_1/\mathcal{E}_n - B_2 = (\mathcal{E}T)^2\mathcal{D}/\mathcal{E}_n\), we have
\[
\Lambda_n(\mathcal{X}, \mathcal{V}) = \left(1 - \frac{\mathcal{X}}{\mathcal{E}_n}\right)\left(\frac{B_1n}{B_4}\right)^{1/2} + \frac{\mathcal{X}}{\mathcal{E}_n}\frac{B_3/\mathcal{E}_n - B_2}{\sqrt{B_1B_4n}} - \frac{B_3}{\mathcal{E}_n\sqrt{B_1n}}\mathcal{V}
\]
\[
= \left(1 - \frac{\mathcal{X}}{\mathcal{E}_n}\right)\left(\frac{B_1n}{B_4}\right)^{1/2} - \frac{\mathcal{E}_n\mathcal{D}Y}{\mathcal{E}_n\sqrt{B_4}}\left(\frac{\mathcal{V}}{\mathcal{E}_n} - \frac{\mathcal{X}\mathcal{E}}{\sqrt{B_1n}}\right)
\]
\[
= \left(1 - \frac{\mathcal{X}}{\mathcal{E}_n}\right)\left(\frac{B_1n}{B_4}\right)^{1/2} - \frac{\mathcal{E}_n\mathcal{D}Y}{\mathcal{E}_n\sqrt{B_4}}\Delta_n(\mathcal{X}, \mathcal{V}).
\]
Rewrite it
\[
1 - \frac{\mathcal{X}}{\mathcal{E}_n\mathcal{Y}} = \left(\frac{B_4}{B_1n}\right)^{1/2}\left(\Lambda_n(\mathcal{X}, \mathcal{V}) + \frac{\mathcal{E}_n\mathcal{D}Y}{\mathcal{E}_n\sqrt{B_4}}\Delta_n(\mathcal{X}, \mathcal{V})\right)
\]
\[
= \left(\frac{1}{B_1n}\right)^{1/2}\left(\frac{B_4}{B_1n}\right)^{1/2}\Lambda_n(\mathcal{X}, \mathcal{V}) + \frac{B_3}{\mathcal{E}_n\Delta_n(\mathcal{X}, \mathcal{V})}\right)\left(1 + \sqrt{\frac{\mathcal{X}}{n\mathcal{Y}}}\right)^{-1},
\]
as required. The proof is complete. □

**Remark 4.2.** The identities of Lemmas 4.2, 4.3, 4.4 in a more general form were proved and used first in [Malinovskii (1993)].

**4.4. Sums related to zeta-functions and polygamma functions.** For real \(s > 1\) and for integer \(N > 0\), fairly easy is the upper bound \(\sum_{n>N}^{\infty} \frac{1}{n^s} \leq \frac{1}{N^s} + \int_N^{\infty} \frac{du}{u^s} = \frac{1}{N^s} + \frac{1}{s-1}N^{1-s} \leq \frac{s}{s-1}N^{1-s}\). Much more accurate are the following equalities well known in the theory of Riemann zeta-function and its generalizations.

**Sums related to Riemann zeta-function.** By Riemann zeta-function with \(s > 1\), we call
\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.
\]
It is known that
\[
\sum_{n=N+1}^{\infty} \frac{1}{n^s} = \frac{N^{1-s}}{s-1} - \frac{\rho(s)}{2}N^{-s} + s \int_N^{\infty} \frac{\rho(u)du}{u^{s+1}},
\]
where \( \rho(x) = \frac{1}{2} - \{x\} \), and for \( M > N \)

\[
\sum_{N + \frac{1}{2} < n \leq M + \frac{1}{2}} \frac{1}{n^s} = \int_{N + \frac{1}{2}}^{M + \frac{1}{2}} \frac{du}{(u + 1)^s} + s \int_{N + \frac{1}{2}}^{M + \frac{1}{2}} \frac{\rho(u)du}{(u + 1)^{s+1}} = \frac{(M + \frac{1}{2})^{1-s} - (N + \frac{1}{2})^{1-s}}{1-s} + s \int_{N + \frac{1}{2}}^{M + \frac{1}{2}} \frac{\rho(u)du}{u^{s+1}}.
\]

The former equality is explicit as Corollary 2 in Ch. 1, § 4 of [Karatsuba and Voronin (1992)](#), the latter is shown in the proof of Lemma 3 in Ch. 1, § 4 of [Karatsuba and Voronin (1992)](#).

**Sums related to Hurwitz zeta-function.** By Hurwitz zeta-function with \( s > 1 \) and \( x > 0 \), we call

\[
\zeta(s, x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^s} = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}.
\]

For \( x > 0 \) and for any \( s \neq 1 \), a convergent Newton series representation is known:

\[
\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x+k)^{1-s} \sim \frac{x^{1-s}}{s-1}.
\]

It is easily seen that \( \frac{\partial}{\partial x} \zeta(s, x) = -\sum_{n=0}^{\infty} \frac{\ln(n+x)}{(n+x)^s} = \sum_{n=x}^{\infty} \frac{\ln(n)}{n^s} \), and we have

\[
\sum_{n=x}^{\infty} \frac{\ln(n)}{n^s} \sim -\left( \ln(x) + \frac{1}{s-1} \right) \zeta(s, x) + \frac{1}{2(s-1)x^s} + \ldots
\]

\[
= -\left( \ln(x) + \frac{1}{s-1} \right) \frac{x^{1-s}}{s-1} + \ldots, \quad x \to \infty. \quad (4.3)
\]

**Sums related to polygamma functions.** By polygamma function, we call \( \psi(x) = \Gamma'(x)/\Gamma(x) \sim \ln(x) \). By polygamma function of order \( m \geq 1 \), we call \( \psi^{(m)}(x) = \frac{d^m}{dx^m} \psi(x) \). It is known that for \( x \to +\infty \)

\[
\psi^{(m)}(x) \sim (-1)^{m+1} \sum_{k=0}^{\infty} \frac{(k+m-1)!}{k!} B_k \frac{1}{x^{k+m}},
\]

where \( B_k \) are Stirling’s numbers with \( B_1 = 1/2 \). It is known that \( \sum_{n=M+1}^{\infty} \frac{1}{n} = \psi(1+M) - \psi(1+N) \) for \( N < M \). Consequently, we have, e.g.,

\[
\sum_{n=U}^{U^2} \frac{1}{n} \sim \ln(U) + \frac{1}{2U} + \frac{7}{12U^2} + \ldots, \quad U \to \infty.
\]

Using polygamma functions, we can get the explicit expressions and exact asymptotics of a number of series of this type. In particular, for \( N < M \) we have \( \sum_{n=N}^{M} \frac{1}{n(n+N)} = \frac{1}{N} (\psi(1+M) - \psi(N) + \psi(2N) - \psi(1+M+N)) \) and

\[
\sum_{n=U}^{U^2} \frac{1}{n(n+U)} = \sum_{n=U}^{KU^2} \frac{1}{n^2 (1+\frac{1}{n})} \sim \frac{\ln(2)}{U} - \frac{3}{4U^2} + \frac{9}{16U^3} + \ldots, \quad U \to \infty.
\]

**4.5. Integrals of rational functions.** The following integrals of rational functions modified by a square root (cf. 3.158 in [Gradshteyn and Ryzhik (1980)](#)) can be found in explicit form. We leave to the reader the details of these calculations.
Lemma 4.5. For \( L + R > 0, M > 0 \), we have

\[
\int_L^\infty (y + R)^{-1}(y^2 + M^2)^{-3/2} \, dy = \frac{R}{M^2(R^2 + M^2)} - \frac{M^2 + LR}{M^2\sqrt{L^2 + M^2}(R^2 + M^2)} + \frac{1}{(R^2 + M^2)^{3/2}} \ln \left( \frac{M^2 - LR + \sqrt{L^2 + M^2}\sqrt{R^2 + M^2}}{(L + R)(\sqrt{R^2 + M^2} - R)} \right).
\]

Lemma 4.6. For \( L \geq 0, L + R > 0, M > 0 \), we have

\[
\int_L^\infty \sqrt{y} \, (y + R)^{-1}(y^2 + M^2)^{-3/2} \, dy = \frac{1}{R^2 + M^2} + \frac{R - L}{(R^2 + M^2)^{3/2}} \ln \left( \frac{(R + L)\sqrt{R^2 + M^2} - R}{M^2 - RL + \sqrt{R^2 + M^2}\sqrt{M^2 + L^2}} \right),
\]

and for \( L \leq 0, L + R > 0, M > 0 \), we have

\[
\int_L^\infty |y| (y + R)^{-1}(y^2 + M^2)^{-3/2} \, dy = \frac{2R + M}{M(R^2 + M^2)} - \frac{R - L}{(R^2 + M^2)^{3/2}} \ln \left( \frac{R^2(\sqrt{R^2 + M^2} - R)(M^2 - RL + \sqrt{R^2 + M^2}\sqrt{M^2 + L^2})}{(R + L)(\sqrt{R^2 + M^2} - R)(M^2 + \sqrt{R^2 + M^2}L^2)(R + L)} \right).
\]

To shorten notation in the following two lemmas, we put \( P = -\frac{1}{2M}, K = 2MR \).

Lemma 4.7. For \( K > 0 \), we have

\[
\int_0^\infty (K + y)^{-1}(1 + \sqrt{y})^{-3} \, dy = \frac{K - 3}{(1 + K)^2} - \frac{\pi \sqrt{K}(K - 3)}{(1 + K)^3} + \frac{3K - 1}{(1 + K)^3} \ln(K),
\]

and for \( 0 < P < K \), we have

\[
\int_0^P (K - y)^{-1}(1 + \sqrt{y})^{-3} \, dy = \frac{4\sqrt{P} + (3 + K)P}{(K - 1)^2(1 + \sqrt{P})^2} + \frac{\sqrt{K}(3 + K)}{(K - 1)^3} \ln \left( \frac{\sqrt{K} + \sqrt{P}}{\sqrt{K} - \sqrt{P}} \right) + \frac{3K + 1}{(K - 1)^3} \ln \left( \frac{K - P}{K(1 + \sqrt{P})^2} \right).
\]

Lemma 4.8. For \( K > 0, P < 0 \), we have

\[
\int_P^\infty |y| (K + y)^{-1}(1 + \sqrt{|y|})^{-3} \, dy = \frac{5K + 1}{(1 + K)^2} + \frac{\pi K^{3/2}(K - 3)}{(1 + K)^3} + \frac{5K - 1}{(K - 1)^2}
\]

\[
+ \frac{2\sqrt{P}(1 - 3K) - (5K - 1)}{(K - 1)^2(1 + \sqrt{P})^2} - \frac{K(3K - 1)}{(1 + K)^3} \ln(K)
\]

\[
+ \frac{K^{3/2}(3 + K)}{(K - 1)^3} \ln \left( \frac{\sqrt{P} + \sqrt{K}}{\sqrt{K} - \sqrt{P}} \right) + \frac{K(3K + 1)}{(K - 1)^3} \ln \left( \frac{K - P}{K(1 + \sqrt{P})^2} \right).
\]

4.6. The asymptotic behavior of \( I_1 \). Let us verify that for \( R = A\frac{U}{\sqrt{n}} + B\sqrt{n} \), \( M = C\frac{U}{\sqrt{n}} \), \( L = C^2 \frac{U}{A\sqrt{n}} - B\sqrt{n} \) with \( A, B, C > 0 \)

\[
I_1 = U \sum_{N_x < n < \frac{(R + L)^2}{2U^2}} n^{-3/2} \int_L^\infty (y + R)^{-1}(y^2 + M^2)^{-3/2} \, dy = O \left( \frac{\ln(U)}{U} \right), \quad U \to \infty.
\]
To investigate the asymptotic behavior of $I_1$, $I_2$, $I_3$, as $U \to \infty$, note that the fractions under the summation sign, as well as the argument of the logarithmic function in $I_3$, are rational functions of $n$ modified by a square root. Extracting the highest power of $n$ from both nominators and denominators of these fractions, we have

$$I_{1,1} = \sum_{N_x < n < \frac{E_0^2}{n_1}} \frac{1}{C^2 U} \frac{Bn + AU}{(Bn + AU)^2 + C^2 U^2},$$

$$I_{1,2} = \sum_{N_x < n < \frac{E_0^2}{n_1}} \frac{1}{C^2 U} \frac{AC^2 U^2 - (Bn + AU)(ABn - C^2 U)}{(Bn + AU)^2 + C^2 U^2 + AC^2 U^2},$$

$$I_{1,3} = U \sum_{N_x < n < \frac{E_0^2}{n_1}} \frac{\ln(\vartheta_1/\vartheta)}{(Bn + AU)^2 + C^2 U^2)^{3/2}}.$$

Since for $n > N_x$ the ratio $U/n$ is bounded by a constant, and even monotone decreases to zero, as $n$ growth to infinity, the expressions underlined by a brace and $(\vartheta_1/n^2)/(\vartheta_1/n)$ do not exceed a constant for all $n > N_x$, as $U$ growth to infinity. The proof is completed by summation, as it was done in Section 4.3. $I_{1,1} \sim \ln(U)U^{-1}$, $I_{1,2} \sim \ln(U)U^{-1}$, and $I_{1,3} \sim \ln(U)U^{-1}$, as $U \to \infty$. 

Put the above $R$, $M$, and $L$ in the integral evaluated in Lemma [4.3] It is checked by direct calculations that

$$\frac{U n^{-3/2} R}{M^2 (R^2 + M^2)} = \frac{1}{C^2 U} \frac{Bn + AU}{(Bn + AU)^2 + C^2 U^2},$$

$$\frac{U n^{-3/2} (M^2 + LR)}{M^2 \sqrt{L^2 + M^2 (R^2 + M^2)}} = \frac{1}{C^2 U} \frac{AC^2 U^2 - (Bn + AU)(ABn - C^2 U)}{(Bn + AU)^2 + C^2 U^2 + AC^2 U^2},$$

$$\frac{U n^{-3/2}}{(R^2 + M^2)^{3/2}} \ln \left( \frac{M^2 - LR + \sqrt{L^2 + M^2 \sqrt{R^2 + M^2}}}{(L + R) \left( \sqrt{R^2 + M^2 - R} \right)} \right) = \frac{U \ln(\vartheta_1/\vartheta)}{(Bn + AU)^2 + C^2 U^2)^{3/2}},$$

where

$$\vartheta_1 = B(C^2 - A^2)nU - AB^2 n^2 - ((Bn + AU)^2 + C^2 U^2)^{1/2}((ABn - C^2 U)^2 + A^2 C^2 U^2)^{1/2},$$

$$\vartheta = (A^2 + C^2)U(Bn + AU - ((Bn + AU)^2 + C^2 U^2)^{1/2}).$$

We have $I_1 = I_{1,1} + I_{1,2} + I_{1,3}$, where
4.7. The asymptotic behavior of $I_2$. Let us verify that for $R = A U / \sqrt{n} + B \sqrt{n}$, $M = C U / \sqrt{n}$, $L = \frac{C^2}{A} U / \sqrt{n} - B \sqrt{n}$ with $A, B, C > 0$

$I_2 = U \sum_{n > \frac{(EX)^2}{U^2}} n^{-3/2} \left( \int_0^\infty (2MR + y)^{-1} (1 + \sqrt{y})^{-3} dy \right.

+ \int_0^{\frac{2\pi}{3\sqrt{y}} (2MR - y)^{-1} (1 + \sqrt{y})^{-3} dy) = O \left( \frac{\ln(U)}{U} \right), \quad U \to \infty.$

Put the above $R$, $M$, and $L$ in the integrals evaluated in Lemma 4.7. For the first of them, it is checked by direct calculations that

$$\frac{U}{n^{3/2}} \left. \frac{K - 3}{(1 + K)^2} \right|_{K=2MR} = U \frac{2CU (AU + Bn) - 3n}{\sqrt{n}(2CU (AU + Bn) + n)^2},$$

$$\frac{U}{n^{3/2}} \left. \frac{\pi \sqrt{K(K - 3)}}{(1 + K)^3} \right|_{K=2MR} = U^{3/2} \frac{\pi \sqrt{2C(AU + Bn)(2CU (AU + Bn) - 3n)}}{(2CU (AU + Bn) + n)^3},$$

$$\frac{U}{n^{3/2}} \left. \frac{3K - 1}{(1 + K)^3 \ln(K)} \right|_{K=2MR} = U \frac{\sqrt{n}(6CU (AU + Bn) - n)}{(2CU (AU + Bn) + n)^3} \ln \left( \frac{2CU (AU + Bn)}{n} \right).$$

For the second of them, it is checked by direct calculations that

$$\frac{U}{n^{3/2}} \left. \frac{4\sqrt{P} + (3 + K)P}{(1 + K)^2(1 + \sqrt{P})^2} \right|_{K=2MR} = \frac{U (ABn - C^2U)(2CU (AU + Bn) + 3n) + 4n \sqrt{2ACU(ABn - C^2U)}}{\sqrt{n}(\sqrt{2ACU} + \sqrt{ABn - C^2U})^2 (2CU (AU + Bn) + n)^2},$$

$$\frac{U}{n^{3/2}} \left. \frac{\sqrt{K(3 + K)}}{(1 + K)^3 \ln \left( \frac{\sqrt{K + \sqrt{P}}}{\sqrt{K - \sqrt{P}}} \right)} \right|_{K=2MR} = U \frac{\sqrt{2C\sqrt{Bn} + AU (2CU (AU + Bn) + 3n})}{(2CU (AU + Bn) + n)^3} \ln \left( \frac{\vartheta_2}{\vartheta_3} \right),$$

$$\frac{U}{n^{3/2}} \left. \frac{3K + 1}{(1 + K)^3 \ln \left( \frac{K - P}{(1 + \sqrt{P})^2} \right)} \right|_{K=2MR} = U \frac{\sqrt{n}(n + 6CU (Bn + AU))}{(2CU (AU + Bn) + n)^3} \ln \left( \frac{\vartheta_2}{\vartheta_3} \right),$$

where

$$\vartheta_2 = (ABn - C^2U)n^{1/2} + 2(AC^2U^2 (AU + Bn))^{1/2},$$
$$\vartheta_2 = ((ABn - C^2U)n^{1/2} - 2(AC^2U^2 (AU + Bn))^{1/2},$$

and

$$\vartheta_3 = (C^2U - ABn)n + 4AC^2U^2 (AU + Bn),$$
$$\vartheta_3 = (Bn + AU)CU (2\sqrt{ACU} + (2ABn - C^2U)^{1/2}).$$

Using the standard technique of investigating the asymptotic behavior of the summands in $I_2 = I_{2.1} + I_{2.2} + I_{2.3} + I_{2.4} + I_{2.5} = I_{2.6}$ described in Section 4.6 we have

$$I_{2.1} = U \sum_{n > \frac{(EX)^2}{U^2}} \frac{2CU (AU + Bn) - 3n}{\sqrt{n}(2CU (AU + Bn) + n)^2} = U \sum_{n > \frac{(EX)^2}{U^2}} \frac{1}{n^{3/2}} \frac{2CU (AU / U + B) - 3}{(2CU (AU / U + B) + 1)^2}.$$

Note that in sums with $n > \frac{(EX)^2}{U^2}$ the ratio $U^2/n$ tends to zero, as $U \to \infty$. 

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and similarly
\[ I_{2,2} = U^{3/2} \sum_{n > \frac{EY}{A}} \frac{\pi \sqrt{2}(AU + Bn)(2CU(AU + Bn) - 3n)}{(2CU(AU + Bn) + n)^3} \sim U^{-3/2}, \]
\[ I_{2,3} = U \sum_{n > \frac{EY}{A}} \frac{\sqrt{n}(6CU(AU + Bn) - n^2)}{(2CU(AU + Bn) + n)^3} \ln \left( \frac{2CU(AU + Bn)}{n} \right) \sim \ln(U)U^{-2}, \]
\[ I_{2,4} = U \sum_{n > \frac{EY}{A}} \frac{(ABn - C^2U)(2CU(AU + Bn) + 3n) + 4n\sqrt{2ACU(ABn - C^2U)}}{\sqrt{n}(\sqrt{2ACU} + \sqrt{2ABn - C^2U})^2 (2CU(AU + Bn) - n)^2} \sim U^{-1}, \]
\[ I_{2,5} = U^{3/2} \sum_{n > \frac{EY}{A}} \frac{\sqrt{2C}\sqrt{Bn + AU}}{(2CU(AU + Bn) - n)^3} \ln \left( \frac{\theta_2}{\theta_3} \right) \sim U^{-3/2}, \]
\[ I_{2,6} = U \sum_{n > \frac{EY}{A}} \frac{\sqrt{n}(n + 6CU(Bn + AU))}{(2CU(AU + Bn) - n)^3} \ln \left( \frac{\theta_3}{\theta_3} \right) \sim \ln(U)U^{-2}, \]
as \( U \to \infty \). The proof is complete.

4.8. The asymptotic behavior of \( I_3 \). Let us verify that
\[ I_3 = U \sum_{n>N} \int_l^\infty (y + R)^{-1} y^{-3/2} dy = O \left( \frac{\ln(U)}{U} \right), \quad U \to \infty, \]
where
\[ R = \left( \frac{A + CK_c}{1 + K_c} \right) U \sqrt{n} + \frac{B}{1 + K_c} \sqrt{n}, \quad M = \left( \frac{C - AK_c}{1 + K_c} \right) U \sqrt{n} - \frac{BK_c}{1 + K_c} \sqrt{n}, \]
\[ L = \left( \frac{e^{ETC^2}}{EY} A - \frac{K_c(C - AK_c)}{1 + K_c} \right) \frac{U}{\sqrt{n}} - \frac{B}{1 + K_c} \sqrt{n}, \]
i.e., for \( R = \frac{\tilde{R} + K_c \tilde{M}}{1 + K_c}, \quad M = \frac{\tilde{M} - K_c \tilde{R}}{1 + K_c}, \quad L = \tilde{L} - K_c \frac{\tilde{M} - K_c \tilde{R}}{1 + K_c} \) with \( \tilde{M}, \tilde{L}, \tilde{R} \) defined in (4.21).

Put these \( R, M, L \) into the integral evaluated in Lemma 4.7. It is checked by direct calculations that
\[ \frac{\ln^{-3/2} R}{M^2(R^2 + M^2)} = \frac{U(1 + K_c^2)^2(Bn + (A + K_c)U)}{(CU - K_c(Bn + AU))^2((Bn + AU)^2 + C^2U^2)^3/2}, \]
\[ \frac{\ln^{-3/2}(M^2 + LR)}{M^2\sqrt{L^2 + M^2} (R^2 + M^2)} = \frac{(1 + K_c^2)^2U}{(CU - K_c(Bn + AU))^2((Bn + AU)^2 + C^2U^2)^3/2} \]
\[ \frac{\ln^{-3/2}}{(R^2 + M^2)^{3/2}} \ln \left( \frac{M^2 - LR + \sqrt{L^2 + M^2} \sqrt{R^2 + M^2}}{(L + R) \sqrt{R^2 + M^2}} \right) = \frac{U(1 + K_c^2)^2 \ln (\theta_5/\theta_3)}{(Bn + AU)^2((Bn + AU)^2 + C^2U^2)^3/2}, \]
where
\[ \theta_3 = AC^2U^2 - (Bn + AU)(ABn - e^{ETC^2}CU) + K_c(-4ABn - 3A^2U + e^{ETC^2}CU) \]
\[ + K_c^2(A^2n^2 + 3A^2BnU + Bc^{ETC^2}nCU + 2A^3U^2 + AC^{ETC^2}CU^2 - AC^2U^2) \]
\[ + K_c^3(A^2 + e^{ETC^2}CU^2), \]
\[ \theta_4 = \left( (A(Bn + K_cCU) - A^2K_c^2U - e^{ETC^2}(1 + K_c^2)CU)^2 + A^2(CU - K_c(Bn + AU))^2 \right)^{1/2} \]
14It is noteworthy that \( CU - K_c(Bn + AU) = (C - K_cA)U - K_cBn > 0 \) since \( K_c < 0 \) (see (4.20)).
Lemma 4.5, and the rest of the proof consists in examining the asymptotic behavior, as

\[ \varphi_5 = B(c \frac{\vartheta}{EY} C^2 - A^2) nU + AC^2(c \frac{\vartheta}{EY} - 1) U^2 - AB^2 n^2 + K_c (A^2 + c \frac{\vartheta}{EY} C^2) U^2 
- (1 + K_c^2)^{-1/2} ((Bn + AU)^2 + C^2 U^2)^{1/2} \left( A(Bn + K_c U) - A^2 K_c U \right) 
- c \frac{\vartheta}{EY} (1 + K_c^2) C^2 U^2 + A^2 (K_c (Bn + AU) - CU)^{1/2}, \]

\[ \varphi_5 = (A^2 + c \frac{\vartheta}{EY} C^2) U (Bn + AU - (1 + K_c^2)^{-1/2} ((Bn + AU)^2 + C^2 U^2)^{1/2}) 
+ K_c (A^2 + c \frac{\vartheta}{EY} C^2) U^2 - K_c^2 (1 + K_c^2)^{-1/2} (A^2 + c \frac{\vartheta}{EY} C^2) ((Bn + AU)^2 + C^2 U^2)^{1/2} U. \]

We have \( I = I_{3,1} + I_{3,2} + I_{3,3}, \) where

\[ I_{3,1} = U \sum_{n > N_c} \frac{Bn + (A + K_c) U}{(CU - K_c (Bn + AU))^2 ((Bn + AU)^2 + C^2 U^2)} \sim U^{-1}, \]

\[ I_{3,2} = U \sum_{n > N_c} \frac{1}{(CU - K_c (Bn + AU))^2 ((Bn + AU)^2 + C^2 U^2)} \left( \frac{\varphi_4}{\varphi_4} \right) \sim U^{-1}, \]

and, using \ref{4.15},

\[ I_{3,1} = U \sum_{n > N_c} \frac{1}{((Bn + AU)^2 + C^2 U^2)^{3/2}} \ln \left( \frac{\varphi_2}{\varphi_5} \right) \sim \ln(U) U^{-1}. \]

The proof is complete.

4.9. The asymptotic behavior of \( I_4 \) and \( I_6. \) Recall (see \ref{3.27}) that \( \frac{1}{EY - cU} = \frac{c - K_c A}{K_c B}. \) The difference between \( I_4, I_6 \) and \( I_3 \) lies only in the range of summation: for \( I_4 \) it is \( N_c < n < U \frac{EY - cU}{EY - cU} - K_c B U, \) and for \( I_6 \) it is \( n > U \frac{EY - cU}{EY - cU} + K_c B U. \) The same way as in analyzing \( I_3, \) for \( R, M, L \) set in \ref{4.10}, we turn to the integral evaluated in Lemma \ref{4.8} and the rest of the proof consist in examining the asymptotic behavior, as \( U \to \infty, \) of the sums similar to those in Section \ref{4.8} e.g., of

\[ U \frac{EY - cU}{EY - cU} \sum_{n < N_c} \frac{Bn + (A + K_c) U}{(CU - K_c (Bn + AU))^2 ((Bn + AU)^2 + C^2 U^2)} \sim U^{-1}, \]

\[ U \frac{EY - cU}{EY - cU} \sum_{n > N_c} \frac{Bn + (A + K_c) U}{(CU - K_c (Bn + AU))^2 ((Bn + AU)^2 + C^2 U^2)} \sim U^{-1}. \]

Leaving this checking to the reader, we point the main difference: in this case \( K_c > 0 \) and \( (CU - K_c (Bn + AU))^2 = ((C - K_c A) U - K_c B n)^2 \) vanishes for \( n = \frac{c - K_c A}{K_c B} U. \) But both cases, for \( I_4, \) since \( N_c < n < \frac{c - K_c A}{K_c B} U - \frac{K_c B U}{EY - cU}, \) and for \( I_6, \) since \( n > \frac{c - K_c A}{K_c B} U + \frac{K_c B U}{EY - cU}, \) we have

\[ (CU - K_c (Bn + AU))^2 = ((C - K_c A) U - K_c B n)^2 > K_c^2 U^2. \]

4.10. The asymptotic behavior of \( I_5. \) The investigation of the asymptotic behavior of

\[ I_5 = U \sum_{\frac{c - K_c A}{K_c B} U - \frac{K_c B U}{EY - cU} < n < \frac{c - K_c A}{K_c B} U + \frac{K_c B U}{EY - cU}} n^{-3/2} \left( \int_0^\infty (2MR + y)^{-1} (1 + \sqrt{y})^{-3} dy 
+ \int_0^{- \frac{1}{2M}} (2MR - y)^{-1} (1 + \sqrt{y})^{-3} dy \right), \]
where \( M, R, \) and \( L \) are defined in (4.6), is quite analogous to investigation of the asymptotic behavior of \( T_2 \). We leave it to the reader.

### 4.11. The asymptotic behavior of \( T_1 \).

We have to verify that

\[
T_1 = U \sum_{n < n < \frac{2n}{U}} n^{-2} \int_{U}^{\infty} |y|(y + R)^{-1}(y^2 + M^2)^{-3/2} dy
\]

\[
+ U \sum_{\frac{2n}{U} < n < \frac{2n}{U} + 1/2} n^{-2} \int_{U}^{\infty} |y|(y + R)^{-1}(y^2 + M^2)^{-3/2} dy = O\left(\frac{\ln(U)}{U}\right), \quad U \to \infty,
\]

for \( R = \frac{A}{\sqrt{n}} + B\sqrt{n}, \) \( M = C\frac{V}{\sqrt{n}}, \) \( L = \frac{C^2}{A} \frac{V}{\sqrt{n}} - B\sqrt{n} \) with \( A, B, C > 0 \).

Put the above \( R, M, \) and \( L \) in the first integral (wherein \( L \geq 0 \)) evaluated in Lemma 4.6. It is checked by direct calculations that

\[
\frac{U n^{-2}}{R^2 + M^2} = \frac{U}{n((Bn + AU)^2 + C^2 U^2)},
\]

\[
\frac{U n^{-2} R}{(R^2 + M^2)^{3/2}} \ln \left( \frac{(R + L)(\sqrt{R^2 + M^2} - R)}{M^2 - RL + \sqrt{R^2 + M^2} \sqrt{M^2 + L^2}} \right) = \frac{U (Bn + AU) \ln(\vartheta_0/\vartheta_6)}{n((Bn + AU)^2 + C^2 U^2)^{3/2}},
\]

where

\[
\vartheta_0 = (A^2 + C^2)U((Bn + AU)^2 + C^2 U^2)^{1/2} - (Bn + AU),
\]

\[
\vartheta_6 = B(ABn - C^2U)n + A^2BnU + ((Bn + AU)^2 + C^2 U^2)^{1/2}((ABn - C^2U)^2 + C^2A^2 U^2)^{1/2}.
\]

Put the above \( R, M, \) and \( L \) in the second integral (wherein \( L \leq 0 \)) evaluated in Lemma 4.6. It is checked by direct calculations that

\[
\frac{U n^{-2} (2R + M)}{M(R^2 + M^2)} = \frac{2Bn + (2A + C)U}{nC(Bn^2 + 2ABnU + (A^2 + C^2)U^2)},
\]

\[
\frac{U n^{-2} R}{(R^2 + M^2)^{3/2}} = \frac{U (2ABn + (A^2 - C^2)U)(2\sqrt{R^2 + M^2} - R)(M^2 - RL + \sqrt{R^2 + M^2} \sqrt{M^2 + L^2})}{M^2(M + \sqrt{R^2 + M^2})(R + L)}
\]

\[
\frac{U n^{-2} R}{n((Bn + AU)^2 + C^2 U^2)^{3/2}},
\]

where

\[
\vartheta_7 = (Bn + AU)(Bn + AU - ((Bn + AU)^2 + C^2 U^2)^{1/2})(AB^2n^2 + A^2BnU - BC^2nU
\]

\[
+ ((Bn + AU)^2 + C^2 U^2)^{1/2}((ABn - C^2U)^2 + A^2C^2 U^2)^{1/2},
\]

\[
\vartheta_7 = C^2(A^2 + C^2)U^3(CU + ((Bn + AU)^2 + C^2 U^2)^{1/2})^2.
\]
We have $T_3 = T_{1,1} + T_{1,2} + T_{1,3} + T_{1,4} + T_{1,5} + T_{1,6}$, where

$$T_{1,1} = U \sum_{N_s < n < \frac{U}{U_1} n^2} \frac{1}{n((Bn + AU)^2 + C^2 U^2)} \sim U^{-1},$$

$$T_{1,2} = U \sum_{N_s < n < \frac{U}{U_1} n^2} \frac{2ABn + (A^2 - C^2)U}{n((Bn + AU)^2 + C^2 U^2) \sqrt{(ABn - C^2U)^2 + A^2 C^2 U^2}} \sim U^{-1},$$

$$T_{1,3} = U \sum_{N_s < n < \frac{U}{U_1} n^2} \frac{(Bn + AU) \ln(h_0 / \sqrt{U})}{n((Bn + AU)^2 + C^2 U^2)^{3/2}} \sim \ln(U) U^{-1},$$

and

$$T_{1,4} = \sum_{\frac{U}{U_1} U < n < \frac{U}{U_1} n^2} \frac{2Bn + (2A + C)U}{nC(B^2 n^2 + 2ABnU + (A^2 + C^2) U^2)} \sim U^{-1},$$

$$T_{1,5} = U \sum_{\frac{U}{U_1} U < n < \frac{U}{U_1} n^2} \frac{2ABn + (A^2 - C^2)U}{n((Bn + AU)^2 + C^2 U^2) \sqrt{(ABn - C^2U)^2 + A^2 C^2 U^2}} \sim U^{-1},$$

$$T_{1,6} = U \sum_{\frac{U}{U_1} U < n < \frac{U}{U_1} n^2} \frac{(Bn + AU) \ln(h_0 / \sqrt{U})}{n((Bn + AU)^2 + C^2 U^2)^{3/2}} \sim \ln(U) U^{-1}.$$

The proof is complete.

### 4.12. The asymptotic behavior of $T_2$

We have to verify that

$$T_2 = U \sum_{n > \frac{(ET)^2}{U_1} U^2} n^{-2} \int_L^\infty \frac{|y|(y + R)^{-1} (1 + (2M |y|)^{3/2})^{-3} dy}{\sqrt{U}} \sim U \ln(U), \quad U \to \infty,$$

for $R = A \frac{U}{\sqrt{n}}, \; M = C \frac{U}{\sqrt{n}}, \; L = C^2 \frac{U^2}{A \sqrt{n}} - B \sqrt{n}$ with $A, B, C > 0$.

By making the change of variables, rewrite the integral in $T_2$ as

$$\frac{1}{2M} \int_{\frac{R}{M}}^\infty |y|(y + 2MR)^{-1} (1 + \sqrt{|y|})^{-3} dy$$

and recall that it is evaluated in explicit form in Lemma 4.8. Put the above $R, M,$ and $L$ into thus modified $T_2$. It is checked by direct calculations that

$$\frac{U}{2M n^2} \left| \begin{array}{c} \frac{5K + 1}{(1 + K)^2} \\ \frac{5K - 1}{(K - 1)^2} \\ \frac{\pi K^{3/2} (K - 3)}{(1 + K)^3} \end{array} \right| _{K = 2MR} = \frac{n + 10BCnU + 10ACU^2}{2\sqrt{nC}(n + 2BCnU + 2ACU^2)^2},$$

$$\frac{U}{2M n^2} \left| \begin{array}{c} \pi K^{3/2} (K - 3) \\ \pi K - 1 \\ \pi K - 1 \end{array} \right| _{K = 2MR} = \frac{n\sqrt{2}(CU(Bn + AU))^3/2(2ACU^2 + n(2BCU - 3))}{nC(n + 2BCnU + 2ACU^2)^3},$$

$$\frac{U}{2M n^2} \left| \begin{array}{c} 5K + 1 \\ 5K - 1 \end{array} \right| _{K = 2MR} = \frac{10ACU^2 + n(10BCU - 1)}{2\sqrt{nC}(2ACU^2 + n(2BCU - 1))^2}.$$

\footnote{In the sum with $n > \frac{(ET)^2}{U_1} U^2$ we have $L < 0$.}
so that following integrals of rational and exponential functions can be evaluated in explicit form

\[ \frac{U}{2Mn^2} \frac{2\sqrt{P}(1-3K)-(5K-1)}{(K-1)^2(1+\sqrt{P})^2} \bigg|_{p=-\frac{1}{K}}^{K=2M_R} \]

\[ = \frac{\frac{\sqrt{2\sqrt{AU}}\sqrt{ABn-C^2U}(6CU(AU+Bn)-n)}{nC(2ACU+\sqrt{ABn-C^2U})^2(2CU(AU+Bn)-n)^2}}{AU(n-10CU(Bn+AU))} + \frac{\sqrt{2\sqrt{ACU}}\sqrt{ABn-C^2U})^2(2CU(AU+Bn)-n)^2}{\sqrt{nC(2ACU+\sqrt{ABn-C^2U})^2(2CU(AU+Bn)-n)^2}} \]

and (cf. (4.4) and (4.5))

\[ \text{Lemma 4.9} \]

We have

\[ \frac{U}{2Mn^2} \frac{K(3K-1)}{(1+K)^3} \ln(K) \bigg|_{K=2M_R} = \frac{UBn+AU}{\sqrt{n}} \frac{(6CU(AU+Bn)-n)}{(2CU(AU+Bn)+n)^3} \ln \left( \frac{2CU(AU+Bn)}{n} \right), \]

\[ \frac{U}{2Mn^2} \frac{K^{3/2}(3+K)}{(K-1)^3} \ln \left( \frac{\sqrt{P}+\sqrt{K}}{\sqrt{K}-\sqrt{P}} \right) \bigg|_{p=-\frac{1}{K}}^{K=2M_R} = \frac{UB}{\sqrt{n}} \frac{\sqrt{2C\sqrt{2n+ABU}(2CU(AU+Bn)+3n)}}{(2CU(AU+Bn)-n)^3} \ln \left( -\frac{\varrho_2}{\vartheta_2} \right), \]

\[ \frac{U}{2Mn^2} \frac{K(3K+1)}{(K-1)^3} \ln \left( \frac{K-P}{K(1+\sqrt{P})^2} \right) \bigg|_{p=-\frac{1}{K}}^{K=2M_R} = \frac{UB}{\sqrt{n}} \frac{(n+6CU(Bn+AU))}{(2CU(AU+Bn)-n)^3} \ln \left( \frac{\varrho_3}{\vartheta_3} \right). \]

We have \( T_2 = T_{2,1} + T_{2,2} + T_{2,3} + T_{2,4} + T_{2,5} + T_{2,6} + T_{2,7} \), where, e.g.,

\[ T_{2,1} = \sum_{n>\frac{U}{\sqrt{2}}/U^2} n + 10BCnU + 10ACU^2 \]

\[ \sim U^{-1}, \]

\[ T_{2,2} = \pi\sqrt{2} \sum_{n>\frac{U}{\sqrt{2}}/U^2} \frac{(CU(Bn+AU))^{3/2}(2ACU^2+n(2BCU-3))}{nC(n+2BCnU+2ACU^2)} \sim U^{-1}, \]

and so on, so that \( T_2 \sim \ln(U)U^{-1}, U \to \infty \), as required. The proof is complete.

4.13. Asymptotics for integrals of rational and exponential functions. The following integrals of rational and exponential functions can be evaluated in explicit form in terms of \( \Gamma(0,-1/2x^2) = \int_{-\infty}^x e^{-t^2} dt \) and \( \text{Erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} D(x) \), where \( D(x) = e^{-x^2} \int_0^x e^{t^2} dt \) is Dawson function. Recall that for \( x > 0 \)

\[ D(x) = \frac{1}{2x} + \frac{1}{4x^3} + \frac{3}{8x^5} + \ldots, \quad x \to +\infty, \quad (4.7) \]

and

\[ \Gamma(0,-1/2x^2) = (-1/2x^2)^{-1/2} \left[ 1 + \frac{2}{x^2} + \frac{8}{x^4} + \ldots \right], \quad x \to +\infty, \quad (4.8) \]

so that \( 1/2 e^{-1/2x^2} \Gamma(0,-1/2x^2) \sim -x^{-2} \), as \( x \to +\infty \).

**Lemma 4.9.** For \( R > 0 \), we have

\[ \int_0^\infty (y+R)^{-1} \exp \left\{ -\frac{1}{2} y^2 \right\} dy = \mathcal{O}(R^{-1}), \quad R \to \infty. \]
ON THE TIME OF FIRST LEVEL CROSSING

PROOF. Bearing in mind (4.7) and (4.8), for \( R \to \infty \) we have\(^ {17} \)

\[
\int_0^\infty (y + R)^{-1} \exp \left\{ - \frac{1}{2} y^2 \right\} dy = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} R^2} \left[ \pi \text{Erfi} \left( \frac{1}{\sqrt{2}} R \right) + \Gamma(0, -\frac{1}{2} R^2) - \ln(R) \right]
\]

\[
= \sqrt{\pi}D \left( \frac{1}{\sqrt{2}} R \right) + \frac{1}{\sqrt{2}} e^{-\frac{1}{2} R^2} \left[ \Gamma(0, -\frac{1}{2} R^2) - \ln(R) \right] \sim \frac{\sqrt{\pi}}{2} R^{-1},
\]

which completes the proof.

\( \square \)

**Lemma 4.10.** For \( R > 0 \), we have

\[
\int_0^\infty y (y + R)^{-1} \exp \left\{ - \frac{1}{2} y^2 \right\} dy = \mathcal{Q}(R^{-1}), \quad R \to \infty.
\]

PROOF. Bearing in mind (4.7) and (4.8), for \( R \to \infty \) we have

\[
\int_0^\infty y (y + R)^{-1} \exp \left\{ - \frac{1}{2} y^2 \right\} dy = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} R^2} \left[ \frac{1}{\sqrt{2}} \sqrt{2\pi} - \pi R \text{Erfi} \left( \frac{1}{\sqrt{2}} R \right) - R \Gamma(0, -\frac{1}{2} R^2) \right.
\]

\[+ 2R \ln(R) \right] = \sqrt{\frac{\pi}{2}} \left[ 1 - \sqrt{2} R D \left( \frac{1}{\sqrt{2}} R \right) \right] - \frac{\sqrt{\pi}}{2} e^{-\frac{1}{2} R^2} \left[ \Gamma(0, -\frac{1}{2} R^2) - 2 \ln(R) \right] \sim R^{-1},
\]

which completes the proof.

\( \square \)

Both Lemmas 4.9 and 4.10 are proved by means of very precise calculations which yield exactly the main terms of approximations, rather than investigate the asymptotic behavior. Investigation of mere the asymptotic behavior can be done much simpler (see, e.g., [De Bruijn (1958)]). Indeed, the function \( \exp\{-\frac{1}{2} y^2\} \) is concentrated in a narrow region around the origin. For the remaining factor in the integrand, note that in this region \( 0 < K_1 R^{-1} \leq (y + R)^{-1} \leq K_2 R^{-1} \). Routine estimation completes the proof. Using these considerations, it is easy to check the following lemma.

**Lemma 4.11.** For \( L + R > 0 \), we have

\[
\int_L^\infty (y + R)^{-1} \exp \left\{ - \frac{1}{2} y^2 \right\} dy = \mathcal{Q}(R^{-1}), \quad R \to \infty,
\]

\[
\int_L^\infty |y| (y + R)^{-1} \exp \left\{ - \frac{1}{2} y^2 \right\} dy = \mathcal{Q}(R^{-1}), \quad R \to \infty.
\]

We merely note that the integrand in Lemma 4.11 is positive and has no singularities within the range of integration \([L, \infty)\). Indeed, the point of singularity \( y = -R \) of \((y + R)^{-1}\) lies outside the range of integration since \(-R < L\).

**4.14. The asymptotic behavior of \( J_1 \).** Let us verify that

\[
J_1 = U \sum_{N \leq n < \frac{\delta R^2}{U}} n^{-3/2} \exp \left\{ - \frac{1}{2} M^2 \right\} \int_L^\infty (y + R)^{-1} \exp \left\{ - \frac{1}{2} y^2 \right\} dy = \mathcal{Q}(U^{-1}), \quad U \to \infty,
\]

for \( R = A \frac{U}{\sqrt{n}} + B \sqrt{n}, M = C \frac{U}{\sqrt{n}}, L = \frac{C^2}{A} \frac{U}{\sqrt{n}} - B \sqrt{n} \) with \( A, B, C > 0 \). We use Lemma 4.11 note that

\[
\frac{U n^{-3/2}}{R} \exp \left\{ - \frac{1}{2} M^2 \right\} = \frac{U}{n(\sqrt{A} + \sqrt{B} n)} \exp \left\{ - \frac{C^2 U^2}{2 n} \right\}
\]

and that

\[
U \sum_{N \leq n < \frac{\delta R^2}{U^2}} \frac{1}{n(\sqrt{A} + \sqrt{B} n)} \exp \left\{ - \frac{C^2 U^2}{2 n} \right\} = \frac{1}{U} \sum_{N \leq n < \frac{\delta R^2}{U^2}} \frac{U^2}{A \sqrt{n} + B} \exp \left\{ - \frac{C^2 U^2}{2 n} \right\} = \mathcal{Q}(U^{-1}),
\]

\(^{17}\)Note that \( \text{Erfi}(R/\sqrt{2}) = \frac{2}{\sqrt{\pi}} e^{\frac{1}{2} R^2} D(R/\sqrt{2}) \).
as $U \to \infty$, as required.

4.15. **The asymptotic behavior of $J_2$.** We have to check that

$$J_2 = U \sum_{n > (ET^2/U^2)} n^{-3/2} \int_0^\infty (y + R)^{-1} \exp \left\{ -\frac{1}{2} y^2 \right\} dy = O(U^{-1}), \quad U \to \infty,$$

for $R = A \frac{U^2}{\sqrt{n}} + B \sqrt{n}$ with $A, B > 0$. The same way as in Section 4.14 we have

$$U \sum_{n > (ET^2/U^2)} n^{-3/2} \left( A \frac{U^2}{\sqrt{n}} + B \sqrt{n} \right)^{-1} \exp \left\{ -C^2 \frac{U^2}{n} \right\} \leq U \sum_{n > (ET^2/U^2)} \frac{1}{n^2} \frac{1}{1 + \frac{U}{n}} \leq KU \sum_{n > (ET^2/U^2)} \frac{1}{n^2} = O(U^{-1}),$$

as $U \to \infty$, as required.

**References**

Bhattacharya, R.N., and Ranga Rao, R. (1976) *Normal Approximation and Asymptotic Expansions*. Wiley & Sons, New York, etc.

Borovkov, K., and Dickson, D.C.M. (2008) On the ruin time distribution for a Sparre Andersen process with exponential claim sizes, *Insurance: Mathematics and Economics*, Vol. 42, 1104–1108.

De Bruijn, N.G. (1958) *Asymptotic methods in analysis*. North-Holland.

Dubinskaitė, J. (1982) Limit theorems in $R^k$. I, *Lith. Math. J.*, Vol. 22, No. 2, 129–140, doi:10.1007/BF00969611.

Gradshteyn, I.S., and Ryzhik, I.M. (1980) *Table of Integrals, Series, and Products*. Academic Press, New York.

Kendall, D.G. (1957) Some problems in the theory of dams, *Journal of the Royal Statist. Soc., Ser. B*, Vol. 19, 207–212.

Malinovskii, V.K. (1993) Limit theorems for stopped random sequences. I: rates of convergence and asymptotic expansions, *Theory Probab. Appl.*, Vol. 38, 673–693.

Nagaev, S.V. (1965) Some limit theorems for large deviations, *Theory Probab. Appl.*, Vol. 10, 214–235.

Nagaev, S.V. (1979) Large deviations of sums of independent random variables, *Annals of Probability*, Vol. 7, 754–789.

Petrov, V.V. (1975) *Sums of Independent Random Variables*. Springer, Berlin, etc.

Petrov, V.V. (1995) *Limit Theorems of Probability Theory. Sequences of Independent Random Variables*. Clarendon Press, Oxford Studies in Probability.

Karatsuba, A.A., and Voronin, S.M. (1992) *The Riemann Zeta-Function*. Walter de Gruyter.