ON THE RIEMANN-HILBERT PROBLEM FOR THE BELTRAMI EQUATIONS

Vladimir Ryazanov and Artyem Yefimushkin

December 4, 2021

Abstract

It is developed the theory of the Dirichlet problem for harmonic functions. On this basis, for the nondegenerate Beltrami equations in the quasidisks and, in particular, in smooth Jordan domains, it is proved the existence of regular solutions of the Riemann-Hilbert problem with coefficients of bounded variation and boundary data that are measurable with respect to the absolute harmonic measure (logarithmic capacity). Moreover, it was shown that the dimension of the spaces of the given solutions is infinite.

2010 Mathematics Subject Classification: Primary 31A05, 31A20, 31A25, 31B25, 35Q15; Secondary 30E25, 31C05, 34M50, 35F45

1 Introduction

Let $D$ be a domain in the complex plane $\mathbb{C}$ and let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. The equation of the form

$$f \bar{z} = \mu(z) \cdot f_z$$  \hspace{1cm} (1.1)

where $f \bar{z} = \bar{\partial}f = (f_x + if_y)/2$, $f_z = \partial f = (f_x - if_y)/2$, $z = x + iy$, $f_x$ and $f_y$ are partial derivatives of the function $f$ in $x$ and $y$, respectively, is said to be a Beltrami equation. The Beltrami equation (1.1) is said to be nondegenerate if $||\mu||_\infty < 1$.

Note that there were recently established a great number of new theorems on the existence and on the boundary behavior of homeomorphic solutions and, on this basis, on the Dirichlet problem for the Beltrami equations with essentially
unbounded distortion quotients $K_\mu(z) = (1 + |\mu(z)|)/(1 - |\mu(z)|)$, see, e.g., the monographs [1]–[3] and the papers [4]–[7], and many references therein. However, under the study of the Riemann-Hilbert problem for (1.1) we restrict ourselves here with the nondegenerate case because this investigation leads to a very delicate Lusin’s problem on interconnections of the boundary data of conjugate harmonic functions and with the difficult problem on the distortion of boundary measures under more general mappings.

Boundary value problems for analytic functions are due to the well-known Riemann dissertation (1851), and also to works of Hilbert (1904, 1912, 1924), and Poincaré (1910), see the monograph [8] for details and also for the case of generalized analytic functions.

The first concrete problem of such a type has been proposed by Hilbert (1904) and called at present by the Hilbert problem or the Riemann-Hilbert problem. That consists in finding analytic functions $f$ in a domain bounded by a rectifiable Jordan curve $C$ with the linear boundary condition

$$\text{Re } \overline{\lambda(\zeta)} \cdot f(\zeta) = \varphi(\zeta) \quad \forall \zeta \in C \quad (1.2)$$

where it was assumed by him that the functions $\lambda$ and $\varphi$ are continuously differentiable with respect to the natural parameter $s$ on $C$ and, moreover, $|\lambda| \neq 0$ everywhere on $C$. Hence without loss of generality one may assume that $|\lambda| \equiv 1$ on $C$.

The first way for solving this problem based on the theory of singular integral equations was given by Hilbert, see [9]. This attempt was not quite successful because of the theory of singular integral equations has been not yet enough developed at that time. However, just that way became the main approach in this research direction, see e.g. [8], [10] and [11]. In particular, the existence of solutions to this problem was in that way proved for Hölder continuous $\lambda$ and $\varphi$, see e.g. [10].

Another way for solving this problem based on a reduction to the corresponding two Dirichlet problems was also proposed by Hilbert, see e.g. [12]. A very general solution of the Riemann-Hilbert problem by this way was recently given in [13] for the arbitrary Jordan domains with functions $\varphi$ and $\lambda$ that are only measurable with respect to the harmonic measure.
We follow the second scheme of Hilbert under the study of the generalized Riemann-Hilbert problem for the Beltrami equations. However, as it follows from the known Ahlfors–Beurling–Bishop examples, see e.g. [14], the harmonic measure zero is not invariant under quasiconformal mappings. Hence here we apply the so-called absolute harmonic measure (logarithmic capacity).

Recall that homeomorphic solutions with distributional derivatives of the nondegenerate Beltrami equations (1.1) are called quasiconformal mappings, see e.g. [15] and [16]. The images of the unit disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$ under the quasiconformal mappings $\mathbb{C}$ onto itself are called quasidisks and their boundaries are called quasicircles or quasiconformal curves. Recall that a Jordan curve is a continuous one-to-one image of the unit circle in $\mathbb{C}$. It is known that every smooth (or Lipschitz) Jordan curve is a quasiconformal curve and, at the same time, quasiconformal curves can be nonrectifiable as it follows from the known examples, see e.g. the point II.8.10 in [16].

Note that a Jordan curve generally speaking has no tangents. Hence we need a replacement for the notion of a nontangential limit usually applied. In this connection, recall the Bagemihl theorem in [17], see also Theorem III.1.8 in [18], stated that, for any function $\Omega : D \to \overline{\mathbb{C}}$, except at most countable set of points $\zeta \in \partial D$, for all pairs of arcs $\gamma_1$ and $\gamma_2$ in $D$ terminating at $\zeta \in \partial D$,

$$C(\Omega, \gamma_1) \cap C(\Omega, \gamma_2) \neq \emptyset,$$

where $C(\Omega, \gamma)$ denotes the cluster set of $\Omega$ at $\zeta$ along $\gamma$, i.e.,

$$C(\Omega, \gamma) = \{ w \in \overline{\mathbb{C}} : \Omega(z_n) \to w, \ z_n \to \zeta, \ z_n \in \gamma \}.$$

Immediately by the theorems of Riemann and Caratheodory, this result is extended to an arbitrary Jordan domain $D$ in $\mathbb{C}$. Given a function $\Omega : D \to \overline{\mathbb{C}}$ and $\zeta \in \partial D$, denote by $P(\Omega, \zeta)$ the intersection of all cluster sets $C(\Omega, \gamma)$ for arcs $\gamma$ in $D$ terminating at $\zeta$. Later on, we call the points of the set $P(\Omega, \zeta)$ principal asymptotic values of $\Omega$ at $\zeta$. Note that if $\Omega$ has a limit along at least one arc in $D$ terminating at a point $\zeta \in \partial D$ with the property (1.3), then the principal asymptotic value is unique.

Recall also that a mapping $f : D \to \mathbb{C}$ is called discrete if the pre-image $f^{-1}(y)$ consists of isolated points for every $y \in \mathbb{C}$, and open if $f$ maps every
open set $U \subseteq D$ onto an open set in $\mathbb{C}$.

The regular solution of a Beltrami equation (1.1) is a continuous, discrete and open mapping $f : D \rightarrow \mathbb{C}$ with distributional derivatives satisfying (1.1) a.e. Note that, in the case of nondegenerate Beltrami equations (1.1), a regular solution $f$ belongs to class $W^{1,p}_{\text{loc}}$ for some $p > 2$ and, moreover, its Jacobian $J_f(z) \neq 0$ for almost all $z \in D$, and it is called a quasiconformal function, see e.g. Chapter VI in [16].

A regular solution of the Riemann-Hilbert problem (1.2) for the Beltrami equation (1.1) is a regular solution of (1.1) satisfying the boundary condition (1.2) in the sense of unique principal asymptotic value for all $\zeta \in \partial D$ except a set of logarithmic capacity zero.

2 On the logarithmic capacity

The most important notion for our research is the notion of logarithmic capacity, see e.g. [18], [21] and [22]. First of all, given a bounded Borel set $E$ in the plane $\mathbb{C}$, a mass distribution on $E$ is a nonnegative completely additive function of a set $\nu$ defined on its Borel subsets with $\nu(E) = 1$. The function

$$U^\nu(z) := \int_E \log \left| \frac{1}{z - \zeta} \right| \, d\nu(\zeta)$$

is called a logarithmic potential of the mass distribution $\nu$ at a point $z \in \mathbb{C}$. A logarithmic capacity $C(E)$ of the Borel set $E$ is the quantity

$$C(E) = e^{-V}, \quad V = \inf_\nu V_\nu(E), \quad V_\nu(E) = \sup_z U^\nu(z).$$

Note that it is sufficient to take the supremum in (2.2) over the set $E$ only. If $V = \infty$, then $C(E) = 0$. It is known that $0 \leq C(E) < \infty$, $C(E_1) \leq C(E_2)$ if $E_1 \subseteq E_2$, $C(E) = 0$ if $E = \bigcup_{n=1}^\infty E_n$, with $C(E_n) = 0$, $n = 1, 2, \ldots$, see e.g. Lemma III.4 in [21].

It is also well-known the following geometric characterization of the logarithmic capacity, see e.g. the point 110 in [22]:

$$C(E) = \tau(E) := \lim_{n \to \infty} V_n^{\frac{2}{n(n-1)}}$$

(2.3)
where $V_n$ denotes the supremum (really, maximum) of the product

$$V(z_1, \ldots, z_n) = \prod_{k<l} |z_k - z_l|$$

(2.4)

taken over all collections of points $z_1, \ldots, z_n$ in the set $E$. Following Fekete, see [23], the quantity $\tau(E)$ is called the transfinite diameter of the set $E$. By the geometric interpretation of the logarithmic capacity as the transfinite diameter we immediately see that if $C(E) = 0$, then $C(f(E)) = 0$ for an arbitrary mapping $f$ that is continuous by H"older and, in particular, for conformal and quasiconformal mappings on the compact sets, see e.g. Theorem II.4.3 in [16].

In order to introduce sets that are measurable with respect to logarithmic capacity, we define, following [21], inner $C_*$ and outer $C^*$ capacities:

$$C_*(E) := \sup_{F \subseteq E} C(E)$$

(2.5)

where supremum is taken over all compact sets $F \subseteq \mathbb{C}$, and

$$C^*(E) := \inf_{E \subseteq O} C(O)$$

(2.6)

where infimum is taken over all open sets $O \subseteq \mathbb{C}$. Further, a bounded set $E \subseteq \mathbb{C}$ is called measurable with respect to the logarithmic capacity if

$$C^*(E) = C_*(E),$$

(2.7)

and the common value of $C_*(E)$ and $C^*(E)$ is still denoted by $C(E)$. Note, see e.g. Lemma III.5 in [21], that the outer capacity is semiadditive, i.e.,

$$C^* \left( \bigcup_{n=1}^{\infty} E_n \right) \leq \sum_{n=1}^{\infty} C^*(E_n).$$

(2.8)

A function $\varphi : E \to \mathbb{C}$ defined on a bounded set $E \subseteq \mathbb{C}$ is called measurable with respect to logarithmic capacity if, for all open sets $O \subseteq \mathbb{C}$, the sets

$$\Omega = \{ z \in E : \varphi(z) \in O \}$$

(2.9)

are measurable with respect to logarithmic capacity. It is clear from the definition that the set $E$ is itself measurable with respect to logarithmic capacity.
Note also that sets of logarithmic capacity zero coincide with sets of the so-called **absolute harmonic measure** zero introduced by Nevanlinna, see Chapter V in [22]. Hence a set $E$ is of (Hausdorff) length zero if $C(E) = 0$, see Theorem V.6.2 in [22]. However, there exist sets of length zero having a positive logarithmic capacity, see e.g. Theorem IV.5 in [21].

**Remark 2.1.** It is known that Borel sets and, in particular, compact and open sets are measurable with respect to logarithmic capacity, see e.g. Lemma I.1 and Theorem III.7 in [21]. Moreover, as it follows from the definition, any set $E \subset \mathbb{C}$ of finite logarithmic capacity can be represented as a union of the sigma-compactum (union of countable collection of compact sets) and the set of logarithmic capacity zero. It is also known that the Borel sets and, in particular, compact sets are measurable with respect to all Hausdorff’s measures and, in particular, with respect to measure of length, see e.g. theorem II(7.4) in [24]. Consequently, any set $E \subset \mathbb{C}$ of finite logarithmic capacity is measurable with respect to measure of length. Thus, on such a set any function $\varphi : E \to \mathbb{C}$ being measurable with respect to logarithmic capacity is also measurable with respect to measure of length on $E$. However, there exist functions that are measurable with respect to measure of length but not measurable with respect to logarithmic capacity, see e.g. Theorem IV.5 in [21].

We are especially interested by functions $\varphi : \partial \mathbb{D} \to \mathbb{C}$ defined on the unit circle $\partial \mathbb{D} = \{ z \in \mathbb{C} : |z| = 1 \}$. However, in view of (2.3), it suffices to examine the corresponding problems on the segments of the real axis because any closed arc on $\partial \mathbb{D}$ admits a bi-Lipschitz (even infinitely smooth, the so-called stereographic) mapping $g$ onto such a segment.

In this connection, recall that a mapping $g : X \to X'$ between metric spaces $(X, d)$ and $(X', d')$ is said to be **Lipschitz** if $d'(g(x_1), g(x_2)) \leq C \cdot d(x_1, x_2)$ for any $x_1, x_2 \in X$ and for a finite constant $C$. If, in addition, $d(x_1, x_2) \leq c \cdot d'(g(x_1), g(x_2))$ for any $x_1, x_2 \in X$ and for a finite constant $c$, then mapping $g$ is called **bi-Lipschitz**.

Recall also, see e.g. Subsection IV.10 in [24], that a point $x_0 \in \mathbb{R}$ is called a **density point** for a measurable (with respect to the length, i.e., with respect
to the Lebesgue measure) set $E \subset \mathbb{R}$ if $x_0 \in E$ and
\[
\lim_{\varepsilon \to 0} \frac{|(x_0 - \varepsilon, x_0 + \varepsilon) \setminus E|}{2\varepsilon} = 0 . \tag{2.10}
\]

Similarly, we say that a point $x_0 \in \mathbb{R}$ is the **density point with respect to logarithmic capacity** for a measurable (with respect to $C$) set $E \subset \mathbb{R}$ if $x_0 \in E$ and
\[
\lim_{\varepsilon \to 0} \frac{C([x_0 - \varepsilon, x_0 + \varepsilon] \setminus E)}{C([x_0 - \varepsilon, x_0 + \varepsilon])} = 0 . \tag{2.11}
\]

Note that the logarithmic capacity of a segment of length $l$ is equal to $l/4$, see e.g. [25], p. 25 and 45. Hence $C([x_0 - \varepsilon, x_0 + \varepsilon]) = \varepsilon/2$. Note simultaneously from the same place that logarithmic capacity of a circle (and a disk) of a radius $r$ is equal to $r$ and, in particular, logarithmic capacity of the unit circle (and the unit disk) is equal to 1.

Finally, recall that a function $\varphi : [a, b] \to \mathbb{C}$ is **approximately continuous (with respect to logarithmic capacity)** at a point $x_0 \in (a, b)$ if it is continuous on a set $E \subseteq [a, b]$ for which $x_0$ is a density point (with respect to logarithmic capacity), see e.g. Subsection IV.10 in [24], and the Subsection 2.9.12 in [26], correspondingly.

Further, it is important that the following analog of the Denjoy theorem holds, see e.g. Theorem 2.9.13 in [26], cf. Theorem IV(10.6) in [24].

**Proposition 2.1.** A function $\varphi : [a, b] \to \mathbb{C}$ is measurable with respect to logarithmic capacity if and only if it is approximately continuous for a.e. $x \in (a, b)$ with respect to logarithmic capacity.

**Remark 2.2.** As known, $C([a; b]) = (b-a)/4$ and, moreover, $C(E) \geq |E|/4$ where $|E|$ – длина $E$, see e.g. II.4.17 in [36]. Thus, if $x_0$ is a density point for a set $E$ with respect to logarithmic capacity, then $x_0$ is also a density point for the set $E$ with respect to measure of length. Consequently, each point of approximate continuity of a function $\varphi : [a, b] \to \mathbb{R}$ with respect to logarithmic capacity is also the point of approximate continuity of the function $\varphi$ with respect to the Lebesgue measure on the real axis.

Hence, in particular, we obtain the following useful lemma.
Lemma 2.1. Let a function $\varphi : [a, b] \to \mathbb{R}$ be bounded and measurable with respect to logarithmic capacity and let $\Phi(x) = \int_a^x \varphi(t) \, dt$ be its indefinite Lebesgue integral. Then $\Phi'(x) = \varphi(x)$ a.e. on $(a, b)$ with respect to logarithmic capacity.

Proof. Indeed, let $x_0 \in (a, b)$ be a point of approximate continuity for a function $\varphi$. Then there is a set $E \subseteq [a, b]$ for which point $x_0$ is a density point and $\varphi$ is continuous on this set. Since $|\varphi(x)| \leq C < \infty$ for all $x \in [a, b]$, we obtain that at small $h$

$$\left| \frac{\Phi(x_0 + h) - \Phi(x_0)}{h} - \varphi(x_0) \right| \leq \max_{x \in E \cap [x_0, x_0 + h]} |\varphi(x) - \varphi(x_0)| + 2C \frac{|(x_0, x_0 + h) \setminus E|}{|h|},$$

i.e., $\Phi'(x_0) = \varphi(x_0)$. Thus, Lemma 2.1 follows from Proposition 2.1, see also Remarks 2.1 and 2.2.

3 One analog of the Lusin theorem

The following remarkable theorem of Lusin says that, for any measurable finite a.e. (with respect to the Lebesgue measure) function $\varphi$ on the segment $[a, b]$, there is a continuous function $\Phi$ such that $\Phi'(x) = \varphi(x)$ a.e. on $[a, b]$, see e.g. Theorem VII(2.3) in [24]. This statement was well-known long ago for integrable functions $\varphi$ with respect to its indefinite integral $\Phi$, see e.g. Theorem IV(6.3) in [24]. However, this result is completely nontrivial for nonintegrable functions $\varphi$.

In the proof of one analog of the Lusin theorem in terms of logarithmic capacity, the following lemma on singular functions of the Cantor type will take the key part.

Lemma 3.1. There is a continuous nondecreasing function $\Psi : [0, 1] \to [0, 1]$ such that $\Psi(0) = 0$, $\Psi(1) = 1$ and $\Psi'(t) = 0$ a.e. with respect to logarithmic capacity.

Proof. To prove this fact we use the construction of sets of Cantor’s type of logarithmic capacity zero due to Nevanlinna. Namely, let us consider a sequence
of numbers $p_k > 1$, $k = 1, 2, \ldots$, and define the corresponding sequence of the sets $E(p_1, \ldots, p_n)$, $n = 1, 2, \ldots$, by the induction in the following way. Let $E(p_1)$ be the set consisting of two equal-length segments obtained from the unit segment $[0, 1]$ by removing the central interval of length $1 - 1/p_1$; $E(p_1, p_2)$ be the set consisting of $2^2 = 4$ equal-length segments obtained by removing from each segment of the previous set $E(p_1)$ the central interval with $1 - 1/p_2$ fraction of its length and so on. Denote by $E(p_1, p_2, \ldots)$ the intersection of all the sets $E(p_1, \ldots, p_n)$, $n = 1, 2, \ldots$. By Theorem V.6.3 in [22] the set $E(p_1, p_2, \ldots)$ has logarithmic capacity zero if and only if the series $\sum 2^{-k} \log p_k$ is divergent. This condition holds, for example, if $p_k = e^{2^k}$.

It is known that all sets of Cantor’s type are homeomorphic each to other. In particular, there is a homeomorphism $h : [0, 1] \to [0, 1]$, $h(0) = 0$ and $h(1) = 1$, under which $E(p_1, p_2, \ldots)$ is transformed into the classical Cantor set, see e.g. 8.23 in [29]. Thus, if $\kappa$ is the classical Cantor function, see e.g. 8.15 in [29], then $\Psi = \kappa \circ h$ is the desired function. ✷

Lemma 3.2. Let a function $g : [a, b] \to \mathbb{R}$ be bounded and measurable with respect to logarithmic capacity. Then, for every $\varepsilon > 0$, there is a continuous function $G : [a, b] \to \mathbb{R}$ such that $|G(x)| \leq \varepsilon$ for all $x \in [a, b]$, $G(a) = G(b) = 0$, and $G'(x) = g(x)$ a.e. on $[a, b]$ with respect to logarithmic capacity.

Proof. Let $H(x) = \int_a^x g(t) \, dt$ be the indefinite Lebesgue integral of the function $g$. Choose on $[a, b]$ a finite collection of points $a = a_0 < a_1 < \ldots < a_n = b$ such that the oscillation of $H$ on each segment $[a_k, a_{k+1}]$, $k = 0, 1, \ldots, n - 1$ is less than $\varepsilon/2$. Applying linear transformations of independent and dependent variables to the function $\Psi : [0, 1] \to [0, 1]$ from Lemma 3.1 we obtain the function $F_k$ on each segment $[a_k, a_{k+1}]$, $k = 0, 1, \ldots, n - 1$, that coincides with the function $H$ at its endpoints and whose derivative is equal to zero a.e. with respect to logarithmic capacity. Let $F$ be the function on $[a, b]$ glued of the functions $F_k$. Then $G = H - F$ gives us the desired function by Lemmas 2.1 and 3.1. ✷

Lemma 3.3. Let a function $g : [a, b] \to \mathbb{R}$ be bounded and measurable with respect to logarithmic capacity and let $P$ be a closed subset of the segment
such that $|G(x+h)| \leq \varepsilon|h|$ for all $x \in P$ and all $h \in \mathbb{R}$ such that $x + h \in [a, b]$, $G(x) = G'(x) = 0$ for all $x \in P$ and $G'(x) = g(x)$ a.e. on $[a, b] \setminus P$ with respect to logarithmic capacity.

Proof. Let $I = (a, b)$. Then the set $I \setminus P$ is open and can be represented as the union of a countable collection of mutually disjoint intervals $I_k = (a_k, b_k)$. Choose in each interval $I_k$ an increasing sequence of numbers $c_k^{(j)}$, $j = 0, \pm 1, \pm 2, \ldots$ such that $c_k^{(j)} \to a_k$ as $j \to -\infty$ and $c_k^{(j)} \to b_k$ as $j \to +\infty$. Denote by $\varepsilon_k^{(j)}$ the minimal of the two numbers $\varepsilon(c_k^{(j)} - a_k)/(k + |j|)$ and $\varepsilon(b_k - c_k^{(j)})/(k + |j|)$. Then by Lemma 3.2 in each interval $I_k$ there is a continuous function $G_k$ such that $|G(x)| \leq \varepsilon_k^{(j)}$ for all $x \in [c_k^{(j)}, c_k^{(j+1)}]$, $G(c_k^{(j)}) = 0$ for all $j = 0, \pm 1, \pm 2, \ldots$, and $G'(x) = g(x)$ a.e. on $I_k$ with respect to logarithmic capacity. Thus, setting $G(x) = G_k(x)$ on each interval $I_k$ and $G(x) = 0$ on the set $P$, we obtain the desired function. □

Finally, we prove the following analog of the Lusin theorem mentioned above.

**Theorem 3.1.** Let $\varphi : [a, b] \to \mathbb{R}$ be a measurable function with respect to logarithmic capacity. Then there is a continuous function $\Phi : [a, b] \to \mathbb{R}$ such that $\Phi'(x) = \varphi(x)$ a.e. on $(a, b)$ with respect to logarithmic capacity. Furthermore, the function $\Phi$ can be chosen such that $\Phi(a) = \Phi(b) = 0$ and $|\Phi(x)| \leq \varepsilon$ for a prescribed $\varepsilon > 0$ and all $x \in [a, b]$.

Proof. First we define by induction a sequence of closed sets $P_n \subseteq [a, b]$ and a sequence of continuous functions $G_n : [a, b] \to \mathbb{R}$, $n = 0, 1, \ldots$, whose derivatives exist a.e. and are measurable with respect to logarithmic capacity such that, under the notations $Q_n = \bigcup_{k=0}^{n} P_k$ and $\Phi_n = \sum_{k=0}^{n} G_k$, the following conditions hold: (a) $\Phi_n(x) = \varphi(x)$ for $x \in Q_n$, (b) $G_n(x) = 0$ for $x \in Q_{n-1}$, (c) $|G_n(x+h)| \leq |h|/2^n$ for all $x \in Q_{n-1}$ and all $h$ such that $x + h \in [a, b]$, (d) $C(I \setminus Q_n) < 1/n$ where $I = [a, b]$.

So, let $G_0 \equiv 0$ and $P_0 = \emptyset$ and let $G_n$ and $P_n$ be already constructed with the given conditions for all $n = 1, 2, \ldots, m$. Then there is a compact set $E_m \subseteq I \setminus Q_m$ such that

$$C(I \setminus (Q_m \cup E_m)) < 1/(m+1) \quad (3.1)$$

[1] Theorem 3.1 is a special case of a more general theorem stated by A. I. Markushevich in his book "Theory of Functions of a Complex Variable". The theorem states that for any measurable function $\varphi : [a, b] \to \mathbb{R}$ with respect to logarithmic capacity, there exists a continuous function $\Phi : [a, b] \to \mathbb{R}$ such that $\Phi'(x) = \varphi(x)$ a.e. on $(a, b)$ with respect to logarithmic capacity. Moreover, $\Phi$ can be chosen such that $\Phi(a) = \Phi(b) = 0$ and $|\Phi(x)| \leq \varepsilon$ for a prescribed $\varepsilon > 0$ and all $x \in [a, b]$.

[2] The proof of Theorem 3.1 involves constructing a sequence of functions $G_n$ and sets $P_n$ by induction such that $G_n$ is continuous, $G_n(x) = \varphi(x)$ a.e. on $Q_n$, and $G_n(x) = 0$ for $x \in Q_{n-1}$. The sequence $P_n$ is constructed to ensure that the capacity of $I \setminus Q_n$ is less than $1/n$. The existence of such a sequence of functions and sets is guaranteed by the properties of logarithmic capacity.

[3] The notation $\Phi_n = \sum_{k=0}^{n} G_k$ implies that $\Phi_n$ is the sum of the functions $G_k$ for $k = 0, 1, \ldots, n$. The continuity of $\Phi_n$ follows from the continuity of $G_k$, and the fact that $\Phi_n(x) = \varphi(x)$ for all $x \in Q_n$ ensures that $\Phi_n$ satisfies the condition $\Phi'(x) = \varphi(x)$ a.e. on $(a, b)$ with respect to logarithmic capacity.
and the functions \( \Phi'_m \) and \( \varphi \) are continuous on \( E_m \), see e.g. Theorem 2.3.5 in [26].

By Lemma 3.3 with the set \( P = Q_m \) and the function \( g : I \to \mathbb{R} \) that is equal to \( \varphi(x) - \Phi'_m(x) \) on \( E_m \) and zero on \( I \setminus E_m \), there is a continuous function \( G_{m+1} : I \to \mathbb{R} \) such that (i) \( G'_{m+1}(x) = \varphi(x) - \Phi'_m(x) \) a.e. on \( I \setminus Q_m \) with respect to logarithmic capacity, (ii) \( G_{m+1}(x) = G'_{m+1}(x) = 0 \) for all \( x \in Q_m \), and (iii) \( |G_{m+1}(x+h)| \leq |h|/2^{m+1} \) for all \( x \in Q_m \) and all \( h \) such that \( x+h \in I \).

By the definition of logarithmic capacity, conditions (i) and (3.4), there is a compact set \( P_{m+1} \subseteq E_m \) such that
\[
C(I \setminus (Q_m \cup P_{m+1})) < 1/(m + 1) ,
\]
\[
G'_{m+1}(x) = \varphi(x) - \Phi'_m(x) \quad \forall x \in P_{m+1} .
\]
By (3.2), (3.3), (ii), (iii) it is easy to see that conditions (a), (b), (c) and (d) hold for \( n = m + 1 \), too.

Set now on the basis of the above construction of the sequences \( G_n \) and \( P_n \): \[
\Phi(x) = \lim_{k \to \infty} \Phi_k(x) = \sum_{k=1}^{\infty} G_k(x) , \quad Q = \lim_{k \to \infty} Q_k = \bigcup_{k=1}^{\infty} P_k .
\]
Note that \( \Phi_k \to \Phi \) uniformly on the segment \( I \) because of the condition (c) and hence the function \( \Phi \) is continuous. By the construction, for each \( x_0 \in Q \) we have that \( x_0 \in Q_n \) for large enough \( n \) and, since
\[
\frac{\Phi(x_0 + h) - \Phi(x_0)}{h} = \frac{\Phi_n(x_0 + h) - \Phi_n(x_0)}{h} + \sum_{k=n+1}^{\infty} \frac{G_k(x_0 + h) - G_k(x_0)}{h} ,
\]
we obtain from the conditions (a), (b) and (c) that
\[
\limsup_{h \to 0} \left| \frac{\Phi(x_0 + h) - \Phi(x_0)}{h} - \varphi(x_0) \right| < \frac{1}{2^n} ,
\]
i.e., \( \Phi'(x_0) = \varphi(x_0) \). Moreover, by condition (d) we see that \( C(I \setminus Q) = 0 \). Thus, \( \Phi'(x) = \varphi(x) \) a.e. on \([a,b]\) with respect to logarithmic capacity.

Finally, applying the construction of the proof of Lemma 3.2 to the function \( \Phi \) instead of the indefinite integral, we find a new function \( \Phi_* \) such that \( \Phi'_*(x) = \varphi(x) \) a.e. on \([a,b]\) with respect to logarithmic capacity with \( \Phi_*(a) = \Phi_*(b) = 0 \) and \( |\Phi_*(x)| \leq \varepsilon \) for a prescribed \( \varepsilon > 0 \) and all \( x \in [a,b] \). \( \square \)
4 On the Dirichlet problem for harmonic functions in the unit circle

Gehring in [30] has established the following brilliant result: if $\varphi : \mathbb{R} \to \mathbb{R}$ is $2\pi$-periodic, measurable and finite a.e. with respect to the Lebesgue measure, then there is a harmonic function in $|z| < 1$ such that $u(z) \to \varphi(\vartheta)$ for a.e. $\vartheta$ as $z \to e^{i\vartheta}$ along any nontangential path.

It will be useful the following analog of the Gehring theorem.

**Theorem 4.1.** Let $\varphi : \mathbb{R} \to \mathbb{R}$ be $2\pi$-periodic, measurable and finite a.e. with respect to logarithmic capacity. Then there is a harmonic function $u(z)$, $z \in \mathbb{D}$, such that $u(z) \to \varphi(\vartheta)$ as $z \to e^{i\vartheta}$ along any nontangential path for all $\vartheta \in \mathbb{R}$ except a set of logarithmic capacity zero.

**Proof.** By Theorem 3.1 we are able to find a continuous $2\pi$-periodic function $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\Phi'(\vartheta) = \varphi(\vartheta)$ for a.e. $\vartheta$ with respect to logarithmic capacity. Set

$$U(re^{i\vartheta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\vartheta - t) + r^2} \Phi(t) \, dt$$  \hspace{1cm} (4.1)

for $r < 1$. Next, by the well-known result due to Fatou, see e.g. 3.441 in [31], p. 53, see also Theorem IX.1.2 in [19], $\frac{\partial}{\partial \vartheta} U(z) \to \Phi'(\vartheta)$ as $z \to e^{i\vartheta}$ along any nontangential path whenever $\Phi'(\vartheta)$ exists. Thus, the conclusion follows for the function $u(z) = \frac{\partial}{\partial \vartheta} U(z)$. \qed

**Remark 4.1.** Note that the given function $u$ is harmonic in the punctured unit disk $\mathbb{D} \setminus \{0\}$ because the function $U$ is harmonic in $\mathbb{D}$ and the differential operator $\frac{\partial}{\partial \vartheta}$ is commutative with the Laplace operator $\Delta$. Setting $u(0) = 0$, we see that

$$u(re^{i\vartheta}) = -\frac{r}{\pi} \int_0^{2\pi} \frac{(1 - r^2) \sin(\vartheta - t)}{(1 - 2r \cos(\vartheta - t) + r^2)^2} \Phi(t) \, dt \to 0 \quad \text{as } r \to 0 ,$$

i.e. $u(z) \to u(0)$ as $z \to 0$, and, moreover, the integral of $u$ over each circle $|z| = r$, $0 < r < 1$, is equal to zero. Thus, by the criterion for a harmonic
function on the averages over circles we have that \( u \) is harmonic in \( \mathbb{D} \). The alternative argument for the latter is the removability of isolated singularities for harmonic functions, see e.g. [22].

It is known that every harmonic function \( u(z) \) in \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) has a conjugate function \( v(z) \) such that \( f(z) = u(z) + iv(z) \) is an analytic function in \( \mathbb{D} \). Hence we have the following corollary:

**Corollary 4.1.** Under the conditions of Theorem 4.1, there is an analytic function \( f \) in \( \mathbb{D} \) such that \( \Re f(z) \to \varphi(\vartheta) \) as \( z \to e^{i\vartheta} \) along any nontangential path for a.e. \( \vartheta \) with respect to logarithmic capacity.

Note that the boundary values of the conjugate function \( v \) cannot be prescribed arbitrarily and simultaneously with the boundary values of \( u \) because \( v \) is uniquely determined by \( u \) up to an additive constant, see e.g. I.A in [20].

Denote by \( h^p, p \in (0, \infty) \), the class of all harmonic functions \( u \) in \( \mathbb{D} \) with

\[
\sup_{r \in (0,1)} \left\{ \int_0^{2\pi} |u(re^{i\vartheta})|^p \, d\vartheta \right\}^{1/p} < \infty.
\]

**Remark 4.2.** It is clear that \( h^p \subseteq h^{p'} \) for all \( p > p' \) and, in particular, \( h^p \subseteq h^1 \) for all \( p > 1 \). It is important that every function in the class \( h^1 \) has a.e. nontangential boundary limits, see e.g. Corollary IX.2.2 in [19].

It is also known that a harmonic function \( U \) in \( \mathbb{D} \) can be represented as the Poisson integral (4.1) with a function \( \Phi \in L^p(-\pi, \pi), p > 1 \), if and only if \( U \in h^p \), see e.g. Theorem IX.2.3 in [19]. Thus, \( U(z) \to \Phi(\vartheta) \) as \( z \to e^{i\vartheta} \) along any nontangential path for a.e. \( \vartheta \), see e.g. Corollary IX.1.1 in [19]. Moreover, \( U(z) \to \Phi(\vartheta_0) \) as \( z \to e^{i\vartheta_0} \) at points \( \vartheta_0 \) of continuity of the function \( \Phi \), see e.g. Theorem IX.1.1 in [19].

Note also that \( v \in h^p \) whenever \( u \in h^p \) for all \( p > 1 \) by the M. Riesz theorem, see [32]. Generally speaking, this fact is not trivial but it follows immediately for \( p = 2 \) from the Parseval equality, see e.g. the proof of Theorem IX.2.4 in [19]. The latter will be sufficient for our goals.
5 Correlations of boundary data of conjugate functions

It is known the very delicate fact due to Lusin that harmonic functions in the unit circle with continuous (even absolutely continuous!) boundary data can have conjugate harmonic functions whose boundary data are not continuous functions, furthermore, they can be even not essentially bounded in neighborhoods of each point of the unit circle, see e.g. Theorem VIII.13.1 in [35]. Thus, a correlation between boundary data of conjugate harmonic functions is not a simple matter, see also I.E in [20].

We call $\lambda : \partial \mathbb{D} \rightarrow \mathbb{C}$ a function of bounded variation, write $\lambda \in BV(\partial \mathbb{D})$, if

$$V_\lambda(\partial \mathbb{D}) : = \sup \sum_{j=1}^{j=k} |\lambda(\zeta_{j+1}) - \lambda(\zeta_j)| < \infty$$  \hspace{1cm} (5.1)

where the supremum is taken over all finite collections of points $\zeta_j \in \partial \mathbb{D}$, $j = 1, \ldots, k$, with the cyclic order meaning that $\zeta_j$ lies between $\zeta_{j+1}$ and $\zeta_{j-1}$ for every $j = 1, \ldots, k$. Here we assume that $\zeta_{k+1} = \zeta_1 = \zeta_0$. The quantity $V_\lambda(\partial \mathbb{D})$ is called the variation of the function $\lambda$.

**Remark 5.1.** It is clear by the triangle inequality that if we add new intermediate points in the collection $\zeta_j$, $j = 1, \ldots, k$, then the sum in (5.1) does not decrease. Thus, the given supremum is attained as $\delta = \sup_{j=1,\ldots,k} |\zeta_{j+1} - \zeta_j| \rightarrow 0$. Note also that by the definition $V_\lambda(\partial \mathbb{D}) = V_{\lambda \circ h}(\partial \mathbb{D})$, i.e., the variation is invariant under every homeomorphism $h : \partial \mathbb{D} \rightarrow \partial \mathbb{D}$ and, thus, the definition can be extended in a natural way to an arbitrary Jordan curve in $\mathbb{C}$.

**Lemma 5.1.** Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$, $\alpha : \partial \mathbb{D} \rightarrow \mathbb{R}$ be a function of bounded variation, $u : \mathbb{D} \rightarrow \mathbb{R}$ be a bounded harmonic function such that

$$\lim_{z \rightarrow \zeta} u(z) = \alpha(\zeta)$$  \hspace{1cm} (5.2)

at every point of continuity of $\alpha$ and let $v$ be its conjugate harmonic function.

Then for a.e. $\zeta \in \partial \mathbb{D}$ with respect to logarithmic capacity

$$\lim_{z \rightarrow \zeta} v(z) = \beta(\zeta)$$  \hspace{1cm} (5.3)
along any nontangential path in $\mathbb{D}$ terminating at $\zeta$ where $\beta : \partial \mathbb{D} \to \mathbb{R}$ is a function that is measurable with respect to logarithmic capacity.

Proof. Indeed, $\alpha$ as a function of bounded variation has at most a countable set $S$ of points of discontinuity and, consequently, $C(S) = 0$. Hence by the generalized maximum principle, see e.g. the point 115 in [22], such a function $u$ is unique. Moreover, $\alpha \in BV(\partial \mathbb{D})$ is bounded and by the Denjoy theorem, see e.g. Theorem IV(10.6) in [24], cf. Proposition 2.1 above, it is measurable with respect to the length measure (as well as with respect to logarithmic capacity), i.e., $\alpha \in L^\infty(\partial \mathbb{D})$, and, consequently, $u$ can be represented as the Poisson integral of the function $\alpha$, see e.g. Theorem I.D.2.2 in [20],

$$u(re^{i\vartheta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r \cos(\vartheta - t) + r^2} \alpha(e^{it}) \, dt.$$ (5.4)

Here the Poisson kernel is a real part of the analytic function $(\zeta + z)/(\zeta - z)$, $\zeta = e^{it}$, $z = re^{i\vartheta}$, and by the Weierstrass theorem, see e.g. Theorem 1.1.1 in [19], the Schwartz integral

$$f(z) := \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \alpha(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta}$$ (5.5)

gives the analytic function $f = u + iv$ in $\mathbb{D}$ with $u = \text{Re} \, f$, $v = \text{Im} \, f$, and

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \alpha(e^{it}) \frac{e^{it} + z}{e^{it} - z} \, dt = C + \frac{z}{\pi} \int_{-\pi}^{\pi} \frac{F(t)}{1 - e^{-it}z} \, dt$$ (5.6)

where $F(t) = e^{-it} \alpha(e^{it})$ and $C = \frac{1}{2\pi} \int_{-\pi}^{\pi} \alpha(e^{it}) \, dt$. By Theorem 2(c) in [28] the function $f(z)$ has angular limits $f(\zeta)$ as $z \to \zeta$ for a.e. $\zeta \in \partial \mathbb{D}$ with respect to logarithmic capacity because the function $F$ is of bounded variation. It remains to note that $f(\zeta) = \lim_{n \to \infty} f_n(\zeta)$, where $f_n(\zeta) = f(r_n \zeta)$, for an arbitrary sequence $r_n \to 1 - 0$ as $n \to \infty$ for a.e. $\zeta \in \partial \mathbb{D}$ with respect to logarithmic capacity and, thus, $f(\zeta)$ is measurable with respect to logarithmic capacity because the functions $f_n(\zeta)$ are so as continuous functions on $\partial \mathbb{D}$, see e.g. 2.3.10 in [26].

Remark 5.2. One can show that the statement is valid for analytic functions in quasidisks $D$ and, in particular, in Jordan domains with smooth and
Lipschitz boundaries. By the given proof, see especially Theorem 2(c) in [28], it
is also clear that $f = u + iv$ admits limits along wide classes of tangential paths to a.e. boundary point of $\partial \mathbb{D}$ with respect to logarithmic capacity. However, it is not interesting for us in the context of our research.

We also prove the following statement that will be useful later on.

**Proposition 5.1.** For every function $\lambda : \partial \mathbb{D} \to \partial \mathbb{D}$ of the class $\mathcal{BV}(\partial \mathbb{D})$ there is a function $\alpha_\lambda : \partial \mathbb{D} \to \mathbb{R}$ of the class $\mathcal{BV}(\partial \mathbb{D})$ with $V_{\alpha_\lambda} \leq V_\lambda \cdot 3\pi/2$ such that $\lambda(\zeta) = \exp\{i\alpha_\lambda(\zeta)\}, \zeta \in \partial \mathbb{D}$.

We will call the function $\alpha_\lambda$ a function of argument of $\lambda$.

**Proof.** Let us consider the function $\Lambda(\varphi) = \lambda(e^{i\varphi}), \varphi \in [0, 2\pi]$. It is clear that $V_\Lambda = V_\lambda$ and, thus, $\Lambda$ has not more than a countable collection of jumps $j_n$ where the series $\sum j_n$ is absolutely convergent, $\sum |j_n| \leq V_\lambda$, and $\Lambda(\varphi) = J(\varphi) + C(\varphi)$ where $C(\varphi)$ is a continuous function and $J(\varphi)$ is the function of jumps of $\Lambda$ that is equal to the sum of all its jumps in $[0, \varphi]$, see e.g. Corollary VIII.3.2 and Theorem VIII.3.7 in [33]. We have that $V_J \leq V_\lambda$ and $V_C \leq 2V_\lambda$, see e.g. Theorem 6.4 in [34]. Let us associate with the complex quantity $j_n$ the real quantity

$$\alpha_n = -2 \arctg \frac{\text{Re } j_n}{\text{Im } j_n} \in [-\pi, \pi].$$

By the geometric interpretation of these quantities ($|j_n|$ is equal to the length of the chord for an arc of the unit circle of the length $|\alpha_n|$) and elementary calculations, we have that $|j_n| \leq |\alpha_n| \leq |j_n| \cdot \pi/2$.

The first inequality is evident because the length $|\alpha|$ of an arc $\alpha$ of a circle is always greater than the length $|h|$ of a chord $h$ connecting its ends. For the proof of the second inequality, note that $|h| = 2|\sin \frac{\alpha}{2}|$. The necessary condition for extremum of the functional $f(\beta) := \frac{\sin^2 \beta}{\beta^2}, \beta \in [0, \frac{\pi}{2}]$ in the points of its differentiability is the equality $f'(\beta_0) = 0 = \frac{2\sin \beta_0}{\beta_0^3} (\beta_0 \cos \beta_0 - \sin \beta_0)$, i.e. $\tan \beta_0 = \beta_0$. However, the functional $g(\beta) := \tan \beta - \beta, \beta \in [0, \frac{\pi}{2}]$, is strictly increasing because $g'(\beta) = \tan^2 \beta > 0$ for $\beta \in (0, \frac{\pi}{2})$, $g(0) = 0$, and hence the equality $\tan \beta_0 = \beta_0$ cannot hold for $\beta \in (0, \frac{\pi}{2})$. Thus, the extreme points of the functional $f$ are the ends of the interval.

Now, let us associate with the function $J(\varphi)$ the function $j(\varphi)$ that is equal
to the sum of all $\alpha_n$ corresponding to jumps of $\Lambda$ in $[0, \vartheta]$, $V_j \leq V_j \cdot \pi/2$. Next, let us associate with the complex-valued function $C(\vartheta)$ a real-valued function $c(\vartheta)$ in the following way. As $C(\vartheta)$ is uniformly continuous on the segment $[0, 2\pi]$, the latter can be split to the segments $S_k = [\theta_{k-1}, \theta_k]$, $\theta_k = 2\pi k/m$, $k = 1, \ldots, m$, with a large enough $m \in \mathbb{N}$ such that $|\Lambda(\vartheta) - \Lambda(\vartheta')| < 2$ for all $\vartheta$ and $\vartheta' \in S_k$. Set by induction

$$c(\vartheta) = c(\theta_{k-1}) - 2 \arctg \frac{\text{Re}[C(\vartheta) - C(\theta_{k-1})]}{\text{Im}[C(\vartheta) - C(\theta_{k-1})]} \quad \forall \vartheta \in S_k, k = 1, \ldots, m,$$

where

$$c(0) := \arctg \frac{\text{Re}[C(0) - 1]}{\text{Im}[C(0) - 1]}.$$

Moreover, let $\gamma_\lambda(\vartheta) = j(\vartheta) + c(\vartheta)$, $\vartheta \in [0, 2\pi]$. By the construction $\Lambda(\vartheta) = e^{i\gamma_\lambda(\vartheta)}$, $\vartheta \in [0, 2\pi]$, $V_{\gamma_\lambda} \leq V_\lambda \cdot 3\pi/2$. Finally, setting $\alpha_\lambda(\zeta) = \gamma_\lambda(\vartheta)$ for $\zeta = e^{iz}$, $\vartheta \in [0, 2\pi)$, we obtain the desired function $\alpha_\lambda$ of the class $\mathcal{BV}(\partial \mathbb{D})$.

\section{The Riemann-Hilbert problem for analytic functions}

\textbf{Theorem 6.1.} Let $\lambda : \partial \mathbb{D} \to \partial \mathbb{D}$ be of bounded variation and $\varphi : \partial \mathbb{D} \to \mathbb{R}$ be measurable with respect to logarithmic capacity. Then there is an analytic function $f : \mathbb{D} \to \mathbb{C}$ such that along any nontangential path

$$\lim_{z \to \zeta} \text{Re} \left\{ \Lambda(\zeta) \cdot f(z) \right\} = \varphi(\zeta) \quad \text{for a.e. } \zeta \in \partial \mathbb{D} \quad (6.1)$$

with respect to logarithmic capacity.

\textbf{Proof.} By Proposition 5.1 the function of argument $\alpha_\lambda \in \mathcal{BV}(\partial \mathbb{D})$. Therefore

$$g(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \alpha(\zeta) \frac{z + \zeta}{z - \zeta} \frac{d\zeta}{\zeta}, \quad z \in \mathbb{D},$$

is an analytic function with $u(z) = \text{Re} \ g(z) \to \alpha(\zeta)$ as $z \to \zeta$ for every $\zeta \in \partial \mathbb{D}$ except a countable collection of points of discontinuity of $\alpha_\lambda$, see e.g. Corollary IX.1.1 in \cite{19} and Theorem I.D.2.2 in \cite{20}. Note that $A(z) = \exp\{ig(z)\}$ is also an analytic function.

By Lemma 5.1 there is a function $\beta : \partial \mathbb{D} \to \mathbb{R}$ that is finite a.e. and measurable with respect to logarithmic capacity such that $v(z) = \text{Im} \ g(z) \to \beta(\zeta)$ as $z \to \zeta$ for every $\zeta \in \partial \mathbb{D}$ except a countable collection of points of discontinuity of $\beta$, see e.g. Corollary IX.1.1 in \cite{19} and Theorem I.D.2.2 in \cite{20}. Note that $B(z) = \exp\{-ig(z)\}$ is also an analytic function.
\( \beta(\zeta) \) as \( z \to \zeta \) for a.e. \( \zeta \in \partial \mathbb{D} \) with respect to logarithmic capacity along any nontangential path. Thus, by Corollary \ref{cor4.1} there is an analytic function \( B : \mathbb{D} \to \mathbb{C} \) such that \( U(z) = \Re B(z) \to \varphi(\zeta) \cdot \exp\{\beta(\zeta)\} \) as \( z \to \zeta \) along any nontangential path for a.e. \( \zeta \in \partial \mathbb{D} \). Finally, elementary calculations show that the desired function \( f = A \cdot B \). \( \square \)

By the Bagemihl theorem, see Introduction, we obtain directly from Theorem \ref{thm6.1} the following result.

**Theorem 6.2.** Let \( D \) be a Jordan domain in \( \mathbb{C} \), \( \lambda : \partial D \to \partial \mathbb{D} \) be a function of bounded variation and \( \varphi : \partial D \to \mathbb{R} \) be a measurable function with respect to logarithmic capacity. Then there is an analytic function \( f : D \to \mathbb{C} \) such that

\[
\lim_{z \to \zeta} \Re \{ \lambda(\zeta) \cdot f(z) \} = \varphi(\zeta) \quad \text{for a.e.} \quad \zeta \in \partial D \quad (6.2)
\]

with respect to logarithmic capacity in the sense of the unique principal asymptotic value.

In particular, choosing \( \lambda \equiv 1 \) in (6.2), we obtain the following consequence.

**Proposition 6.1.** Let \( D \) be a Jordan domain and let \( \varphi : \partial D \to \mathbb{R} \) be a measurable function with respect to logarithmic capacity. Then there is an analytic function \( f : D \to \mathbb{C} \) such that

\[
\lim_{z \to \zeta} \Re f(z) = \varphi(\zeta) \quad \text{for a.e.} \quad \zeta \in \partial D \quad (6.3)
\]

with respect to logarithmic capacity in the sense of the unique principal asymptotic value.

**Corollary 6.1.** Under the conditions of proposition \ref{prop6.1}, there is a harmonic function \( u \) in \( D \) such that in the same sense

\[
\lim_{z \to \zeta} u(z) = \varphi(\zeta) \quad \text{for a.e.} \quad \zeta \in \partial D \quad . \quad (6.4)
\]

**Remark 6.1.** It is easy to see that here in comparison with the paper \cite{13}, we strengthen the conditions as well as the conclusions of these theorems, see Remark \ref{rem2.1}.
The Riemann-Hilbert problem in quasidisks

Theorem 7.1. Let $D$ be a Jordan domain in $\mathbb{C}$ bounded by a quasiconformal curve, $\mu : D \to \mathbb{C}$ be a measurable (by Lebesgue) function with $||\mu||_{\infty} < 1$, $\lambda : \partial D \to \mathbb{C}, |\lambda(\zeta)| \equiv 1$, be a function of bounded variation and let $\varphi : \partial D \to \mathbb{R}$ be a measurable function with respect to logarithmic capacity. Then the Beltrami equation (1.1) has a regular solution of the Riemann-Hilbert problem (1.2). If in addition $\partial D$ is rectifiable, then the limit in (1.2) holds a.e. with respect to the natural parameter along any nontangential path.

In particular, the latter conclusion of Theorem 7.1 holds in the case of smooth boundaries.

Proof. Without loss of generality we may assume that $0 \in D$ and $1 \in \partial D$. Extending $\mu$ by zero everywhere outside of $D$, we obtain the existence of a quasiconformal mapping $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$ satisfying the Beltrami equation (1.1) with the given $\mu$, see e.g. Theorem V.B.3 in [15]. By the theorems of Riemann and Caratheodory, the Jordan domain $f(D)$ can be mapped by a conformal mapping $g$ with the normalization $g(0) = 0$ and $g(1) = 1$ onto the unit disk $\mathbb{D}$. It is clear that $h := g \circ f$ is a quasiconformal homeomorphism with normalization $h(0) = 0$ and $h(1) = 1$ satisfying the same Beltrami equation.

By the reflection principle for quasiconformal mappings, using the conformal reflection (inversion) with respect to the unit circle in the image and quasiconformal reflection with respect to $\partial D$ in the preimage, we can extend $h$ to a quasiconformal mapping $H : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with the normalization $H(0) = 0$, $H(1) = 1$ and $H(\infty) = \infty$, see e.g. I.8.4, II.8.2 and II.8.3 in [16]. Note that $\Lambda = \lambda \circ H^{-1}$ is a function of bounded variation, $V_\Lambda(\partial \mathbb{D}) = V_\Lambda(\partial D)$.

The mappings $H$ and $H^{-1}$ transform sets of logarithmic capacity zero on $\partial D$ into sets of logarithmic capacity zero on $\partial \mathbb{D}$ and vice versa because quasiconformal mappings are continuous by Hölder on $\partial D$ and $\partial \mathbb{D}$ correspondingly, see e.g. Theorem II.4.3 in [16].

Further, the function $\Phi = \varphi \circ H^{-1}$ is measurable with respect to logarithmic capacity. Indeed, under this mapping measurable sets with respect to logarith-
mic capacity are transformed into measurable sets with respect to logarithmic capacity because such a set can be represented as the union of a sigma-compactum and a set of logarithmic capacity zero and compacta under continuous mappings are transformed into compacta and compacta are measurable sets with respect to logarithmic capacity.

Thus, the original Riemann-Hilbert problem for the Beltrami equation \((1.1)\) is reduced to the Riemann-Hilbert problem for analytic functions \(F\) in the unit circle:

\[
\lim_{z \to \zeta} \overline{\Lambda(\zeta)} \cdot F(z) = \Phi(\zeta) \tag{7.1}
\]

and by Theorem 6.1 there is an analytic function \(F : \mathbb{D} \to \mathbb{C}\) for which this boundary condition holds for a.e. \(\zeta \in \partial \mathbb{D}\) with respect to logarithmic capacity along any nontangential path.

So, the desired solution of the original Riemann-Hilbert problem \((1.2)\) for the Beltrami equation \((1.1)\) exists and can be represented as \(f = F \circ H\).

Finally, since the distortion of angles under the quasiconformal mapping is bounded, see e.g. [37]–[39], then in the case of a rectifiable boundary of \(D\) condition \((1.2)\) can be understood along any nontangential path a.e. with respect to the natural parameter. \(\square\)

8 On the dimension of spaces of solutions

By the known Lindelöf maximum principle, see e.g. Lemma 1.1 in [40], it follows the uniqueness theorem for the Dirichlet problem in the class of bounded harmonic functions on the unit disk \(\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}\). In general there is no uniqueness theorem in the Dirichlet problem for the Laplace equation. Furthermore, it was proved in [13] that the space of all harmonic functions in \(\mathbb{D}\) with nontangential limit 0 at a.e. point of \(\partial \mathbb{D}\) has the infinite dimension.

Let us show that in view of Lemma 3.1 one can similarly prove the more refined results on harmonic functions with respect to logarithmic capacity instead of the measure of the length on \(\partial \mathbb{D}\).
Theorem 8.1. The space of all harmonic functions $u : \mathbb{D} \to \mathbb{R}$ such that $\lim_{z \to \zeta} u(z) = 0$ along any nontangential path for a.e. $\zeta \in \partial \mathbb{D}$ with respect to logarithmic capacity has the infinite dimension.

Proof. Indeed, let $\Phi : [0, 2\pi] \to \mathbb{R}$ be integrable, differentiable and $\Phi'(t) = 0$ a.e. on $\partial \mathbb{D}$ with respect to logarithmic capacity. Then the function

$$U(z) : = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\vartheta - t) + r^2} \Phi(t) \, dt, \quad z = re^{i\vartheta}, \ r < 1,$$

is harmonic on $\mathbb{D}$ with $U(z) \to \Phi(\Theta)$ as $z \to e^{i\Theta}$, see e.g. Theorem 1.3 in [40] or Theorem IX.1.1 in [19], and $\frac{\partial}{\partial \vartheta} U(z) \to \Phi'(\Theta)$ as $z \to e^{i\Theta}$ along any nontangential path whenever $\Phi'(\Theta)$ exists, see e.g. 3.441 in [31], p. 53, or Theorem IX.1.2 in [19]. Thus, the harmonic function $u(z) = \frac{\partial}{\partial \vartheta} U(z)$ has nontangential limit 0 at a.e. point of $\partial \mathbb{D}$ with respect to logarithmic capacity.

Let us give a subspace of such functions $u$ with an infinite basis. Namely, let $\varphi : [0, 1] \to [0, 1]$ be a function of the Cantor type, see Lemma 3.1, and let $\varphi_n : [0, 2\pi] \to [0, 1]$ be equal to $\varphi((t - a_{n-1})/(a_n - a_{n-1}))$ on $[a_{n-1}, a_n)$ where $a_0 = 0$ and $a_n = 2\pi(2^{-1} + \ldots + 2^{-n}), \ n = 1, 2, \ldots$ and 0 outside of $[a_{n-1}, a_n)$. Denote by $U_n$ and $u_n$ the harmonic functions corresponding to $\varphi_n$ as in the first item.

By the construction the supports of the functions $\varphi_n$ are mutually disjoint and, thus, the series $\sum_{n=1}^{\infty} \gamma_n \varphi_n$ is well defined for every sequence $\gamma_n \in \mathbb{R}$, $n = 1, 2, \ldots$. If in addition we restrict ourselves to the sequences $\gamma = \{\gamma_n\}$ in the space $l$ with the norm $\|\gamma\| = \sum_{n=1}^{\infty} |\gamma_n| < \infty$, then the series is a suitable function $\Phi$ for the first item.

Denote by $U$ and $u$ the harmonic functions corresponding to the function $\Phi$ as in the first item and by $\mathcal{H}_0$ the class of all such $u$. Note that $u_n, \ n = 1, 2, \ldots$, form a basis in the space $\mathcal{H}_0$ with the locally uniform convergence in $\mathbb{D}$ which is metrizable.

Firstly, $\sum_{n=1}^{\infty} \gamma_n u_n \neq 0$ if $\gamma \neq 0$. Really, let us assume that $\gamma_n \neq 0$ for some $n = 1, 2, \ldots$. Then $u \neq 0$ because the limits $\lim_{z \to \zeta} U(z)$ exist for all $\zeta = e^{i\vartheta}$ with $\vartheta \in (a_{n-1}, a_n)$ and can be arbitrarily close to 0 as well as to $\gamma_n$.
Secondly, \( u_m^* = \sum_{n=1}^{m} \gamma_n u_n \to u \) locally uniformly in \( \mathbb{D} \) as \( m \to \infty \). Indeed, elementary calculations give the following estimate of the remainder term

\[
|u(z) - u_m^*(z)| \leq \frac{2r(1+r)}{(1-r)^3} \cdot \sum_{n=m+1}^{\infty} |\gamma_n| \to 0 \quad \text{as} \quad m \to \infty
\]
in every disk \( \mathbb{D}(r) = \{ z \in \mathbb{C} : |z| \leq r \} \), \( r < 1 \).

**Remark 8.1.** Note that harmonic functions \( u \) found by us in Theorem 8.1 themselves cannot be represented in the form of the Poisson integral with any function \( \Phi \in L_p([0, 2\pi]), p > 1 \), because this integral would have nontangential limits \( \Phi \) a.e., see e.g. Corollary IX.9.1 in [19]. Thus, \( u \) do not belong to the classes \( h_p \) for any \( p > 1 \), see e.g. Theorem IX.2.3 in [19].

**Corollary 8.1.** Given a measurable function \( \varphi : \partial \mathbb{D} \to \mathbb{R} \), the space of all harmonic functions \( u : \mathbb{D} \to \mathbb{R} \) with the limits \( \lim_{z \to \zeta} u(z) = \varphi(\zeta) \) for a.e. \( \zeta \in \partial \mathbb{D} \) with respect to logarithmic capacity along nontangential paths has the infinite dimension.

Indeed, we have at least one such harmonic function \( u \) by Theorem 4.1 and, combining this fact with Theorem 8.1, we obtain the conclusion of Corollary 8.1.

The statements on the infinite dimension of the space of solutions can be extended to the Riemann-Hilbert problem because we have reduced this problem in Theorem 6.1 to the corresponding two Dirichlet problems.

**Theorem 8.2.** Let \( \lambda : \partial \mathbb{D} \to \partial \mathbb{D} \) be of bounded variation and \( \varphi : \partial \mathbb{D} \to \mathbb{R} \) be measurable with respect to logarithmic capacity. Then the space of all analytic functions \( f : \mathbb{D} \to \mathbb{C} \) such that along any nontangential path

\[
\lim_{z \to \zeta} \text{Re} \left\{ \lambda(\zeta) \cdot f(z) \right\} = \varphi(\zeta) \quad \text{for a.e.} \quad \zeta \in \partial \mathbb{D} \quad (8.1)
\]

with respect to logarithmic capacity has the infinite dimension.

**Proof.** Let \( u : \mathbb{D} \to \mathbb{R} \) be a harmonic function with nontangential limit 0 at a.e. point of \( \partial \mathbb{D} \) with respect to logarithmic capacity from Theorem 8.1. Then there is the unique harmonic function \( v : \mathbb{D} \to \mathbb{R} \) with \( v(0) = 0 \) such
that $C = u + iv$ is an analytic function. Thus, setting in the proof of Theorem 6.1 $g = A(B + C)$ instead of $f = A \cdot B$, we obtain by Theorem 8.1 the space of solutions of the Riemann-Hilbert problem (8.1) for analytic functions of the infinite dimension.

**Remark 8.2.** The dimension of the spaces of solutions of the Riemann-Hilbert problem for the Beltrami equation in Theorem 7.1 is also infinite because this case is reduced to the case of Theorem 8.2 as in the proof of Theorem 7.1.

### 9 Extension of results to countably bounded variation

We call $\lambda : \partial \mathbb{D} \rightarrow \mathbb{C}$ a function of **countably bounded variation**, write $\lambda \in CBV(\partial \mathbb{D})$, if there is a countable collection of mutually disjoint arcs $\gamma_n$, $n = 1, 2, \ldots$ on each of which the restriction of $\lambda$ is of bounded variation $V_n$, 

$$
\sum_{n=1}^{\infty} V_n \cdot |\gamma_n| < \infty,
$$

and the set $\partial \mathbb{D} \setminus \bigcup_{n=1}^{\infty} \gamma_n$ is countable. The definition is also extended in a natural way to an arbitrary Jordan curve $\Gamma$ in $\mathbb{C}$. All the above results on the Riemann-Hilbert problem have been extended to the case of $\lambda \in CBV$ in the paper [41]. The latter was based on the following analogs of Proposition 5.1 and Lemma 5.1.

**Proposition 9.1.** For every function $\lambda : \partial \mathbb{D} \rightarrow \partial \mathbb{D}$ of the class $CBV(\partial \mathbb{D})$ there is a function $\alpha_\lambda : \partial \mathbb{D} \rightarrow \mathbb{R}$ of the class $L_1(\partial \mathbb{D}) \cap CBV(\partial \mathbb{D})$ such that $\lambda(\zeta) = \exp\{i\alpha_\lambda(\zeta)\}$, $\zeta \in \partial \mathbb{D}$.

**Proof.** Denote by $\lambda_n$ the complex valued function on $\partial \mathbb{D}$ that is equal to $\lambda$ on $\gamma_n$ and to 1 outside of $\gamma_n$. Let $\alpha_n$ correspond to $\lambda_n$ by Proposition 5.1. Then its variation $V_n^* \leq V_n \cdot 3\pi/2$. With no loss of generality we may assume that $\alpha_n \equiv 0$ outside of $\gamma_n$. Set $\alpha = \sum_{n=1}^{\infty} \alpha_n$. Then $\alpha \in CBV(\partial \mathbb{D})$ and $\lambda(\zeta) = \exp\{i\alpha(\zeta)\}$, $\zeta \in \partial \mathbb{D}$. Applying the corresponding shifts (divisible $2\pi$) we may change $\alpha_n$ on $\gamma_n$ through $\alpha_n^*$ with $|\alpha_n^*| \leq \pi$ at the middle point of $\gamma_n$. Then it is clear that the new function $\alpha^* \in CBV(\partial \mathbb{D})$ and $\lambda(\zeta) = \exp\{i\alpha^*(\zeta)\}$, $\zeta \in \partial \mathbb{D}$, and, moreover, $|\alpha^*| \leq \pi + V_n \cdot 3\pi/2$ on every $\gamma_n$ and hence $\|\alpha^*\|_1 \leq 2\pi^2 + \frac{3\pi}{2} \sum_{n=1}^{\infty} V_n < \infty$, i.e. $\alpha^* \in L_1(\partial \mathbb{D})$. □
We prove the following statement similarly to Lemma 5.1, however, without the reference to the paper [28] because in the case the function $\alpha$, generally speaking, will be not of bounded variation but we are able instead it to apply Lemma 5.1 itself.

Lemma 9.1. Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$, $\alpha : \partial \mathbb{D} \to \mathbb{R}$ be a bounded function of the class $CBV(\partial \mathbb{D})$, $u : \mathbb{D} \to \mathbb{R}$ be a bounded harmonic function such that

$$
\lim_{z \to \zeta} u(z) = \alpha(\zeta) \quad (9.1)
$$

at every point of continuity of $\alpha$ and let $v$ be its conjugate harmonic function. Then for a.e. $\zeta \in \partial \mathbb{D}$ with respect to logarithmic capacity

$$
\lim_{z \to \zeta} v(z) = \beta(\zeta) \quad (9.2)
$$

along any nontangential path in $\mathbb{D}$ terminating at $\zeta$ where $\beta : \partial \mathbb{D} \to \mathbb{R}$ is a function that is measurable with respect to logarithmic capacity.

Proof. Indeed, the function $\alpha \in CBV(\partial \mathbb{D})$ has at most a countable set $S$ of points of discontinuity. Hence by the generalized maximum principle such a function $u$ is unique. Moreover, $\alpha \in L_1(\partial \mathbb{D})$ and, consequently, $u$ can be represented as the Poisson integral of the function $\alpha$

$$
u(re^{i\vartheta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r \cos (\vartheta - t) + r^2} \alpha(e^{it}) \, dt \quad (9.3)$$

The Schwartz integral

$$
f(z) := \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \alpha(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} \quad (9.4)
$$

gives the analytic function $f = u + iv$ in $\mathbb{D}$, where

$$
v(re^{i\vartheta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2r \sin (\vartheta - t)}{1 - 2r \cos (\vartheta - t) + r^2} \alpha(e^{it}) \, dt \quad (9.5)
$$

Let us apply the linearity of the integral operator (9.5). Namely, denote by $\chi$ the characteristic function of an arc $\gamma_*$ of $\partial \mathbb{D}$ where $\alpha$ is of bounded variation.
from the definition of $\mathcal{CBV}$. Setting $\alpha_* = \alpha \cdot \chi$ and $\alpha_0 = \alpha - \alpha_*$, we have that $\alpha = \alpha_* + \alpha_0$. Then $v = v_* + v_0$ where $v_*$ and $v_0$ correspond to $\alpha_*$ and $\alpha_0$ by formula \((9.5)\). By Lemma 5.1 for a.e. $\zeta \in \partial \mathbb{D}$ with respect to logarithmic capacity
\[
\lim_{z \to \zeta} v_*(z) = \beta_*(\zeta) \tag{9.6}
\]
along any nontangential path in $\mathbb{D}$ terminating at $\zeta$ where $\beta_* : \partial \mathbb{D} \to \mathbb{R}$ is a function that is measurable with respect to logarithmic capacity. Moreover, it is evident from formula \((9.5)\) that $v_0(z) \to \beta_0(\zeta)$ as $z \to \zeta$ for all $\zeta \in \gamma_*$ where $\beta_0 : \gamma_* \to \mathbb{R}$ is even continuous on $\gamma_*$. Thus, setting $\beta = \beta_* + \beta_0$ on $\gamma_*$, we obtain the conclusion of Lemma 9.1 by countability of the collection of such arcs $\gamma_*$ and by countability of the completion of this collection on $\partial \mathbb{D}$. □

References

[1] Astala K., Iwaniec T., Martin G.J. Elliptic differential equations and quasiconformal mappings in the plane. – Princeton Math. Ser., 48. – Princeton: Princeton Univ. Press, 2009.

[2] Gutlyanskii V., Ryazanov V., Srebro U., Yakubov E. The Beltrami Equation: A Geometric Approach. – Developments in Mathematics, 26. - New York etc.: Springer, 2012.

[3] Martio O., Ryazanov V., Srebro U., Yakubov E. Moduli in Modern Mapping Theory. – Springer Monographs in Mathematics. – New York etc.: Springer, 2009.

[4] Kovtonyuk D., Petkov I., Ryazanov V. On the boundary behaviour of solutions to the Beltrami equations // Complex Var. Elliptic Eqns. - 2013. - 58, no. 5. - P. 647-663.

[5] Kovtonyuk D., Petkov I., Ryazanov V. On the Dirichlet problem for the Beltrami equations in finitely connected domains // Ukr. Mat. Zh. – 2012. – 64, no. 7. – P. 932–944 [in Russian]; transl. in Ukrainian Math. J. – 2012. – 64, no. 7. – P. 1064 – 1077.
[6] Kovtonyuk D., Petkov I., Ryazanov V., Salimov R. The boundary behaviour and the Dirichlet problem for the Beltrami equations // Algebra and Analysis. – 2013. – 25, no. 4. – P. 102-125 [in Russian]; transl in St. Petersburg Math. J. - 2014. - 25. - P. 587- 603.

[7] Ryazanov V., Salimov R., Srebro U., Yakubov E. On Boundary Value Problems for the Beltrami Equations // Contemporary Math. – 2013. – 591. – P. 211 – 242.

[8] Vekua I.N. Generalized analytic functions. – London etc.: Pergamon Press, 1962.

[9] Hilbert D. Über eine Anwendung der Integralgleichungen auf eine Problem der Funktionentheorie. – Verhandl. des III Int. Math. Kongr., Heidelberg, 1904.

[10] Gakhov F.D. Boundary value problems. – New York: Dover Publications. Inc., 1990.

[11] Muskhelishvili N.I. Singular integral equations. Boundary problems of function theory and their application to mathematical physics. – New York: Dover Publications. Inc., 1992.

[12] Hilbert D. Grundzüge einer allgemeinen Theorie der Integralgleichungen. – Leipzig, Berlin, 1912.

[13] Ryazanov V.I. On the Riemann-Hilbert problem without Index // Ann. Univ. Bucharest, Ser. Math. - 2014. - 5 (LXIII), no. 1. - P. 169-178.

[14] Ahlfors L., Beurling A. The boundary correspondence under quasiconformal mappings // Acta Math. – 1956. – 96. – P. 125–142.

[15] Ahlfors L. Lectures on Quasiconformal Mappings. – New York: Van Nostrand, 1966.

[16] Lehto O., Virtanen K.J. Quasiconformal mappings in the plane. – Berlin, Heidelberg: Springer-Verlag, 1973.
[17] Bagemihl F. Curvilinear cluster sets of arbitrary functions // Proc. Nat. Acad. Sci. U.S.A. – 1955. – 41. – P. 379–382.

[18] Noshiro K. Cluster sets. – Berlin etc.: Springer-Verlag, 1960.

[19] Goluzin G. M. Geometric theory of functions of a complex variable. Transl. of Math. Monographs, 26. – Providence, R.I.: American Mathematical Society, 1969.

[20] Koosis P. Introduction to $H_p$ spaces. – Cambridge Tracts in Mathematics, 115. – Cambridge: Cambridge Univ. Press, 1998.

[21] Carleson L. Selected Problems on Exceptional Sets. – Princeton etc.: Van Nostrand Co., Inc., 1971.

[22] Nevanlinna R. Eindeutige analytische Funktionen. – Michigan: Ann Arbor, 1944.

[23] Fekete M. Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten // Math. Z. – 1923. – 17. – P. 228-249.

[24] Saks S. Theory of the integral. – New York: Dover Publications Inc., 1964.

[25] Saff E.B., Totik V. Logarithmic potentials with external fields. – Grundlehren der Mathematischen Wissenschaften, 316. – Berlin: Springer-Verlag, 1997.

[26] Federer H. Geometric Measure Theory. – Berlin: Springer-Verlag, 1969.

[27] Adams D.R., Hedberg L.I. Function spaces and potential theory. – Springer-Verlag, Berlin, 1996.

[28] Twomey J.B. Tangential boundary behaviour of the Cauchy integral // J. London Math. Soc. (2). – 1988. – 37, no. 3. – P. 447–454.

[29] Gelbaum B.R., Olmsted J.M.H. Counterexamples in Analysis. – San Francisco etc.: Holden-Day, 1964.
[30] Gehring F.W. On the Dirichlet problem // Michigan Math. J. – 1955–1956. – 3. – P. 201.

[31] Zygmund A. Trigonometric series. – Wilno, 1935.

[32] Riesz M. Sur les fonctions conjuguées // Math. Z. – 1927. – 27. – P. 218-244.

[33] Natanson I.P. The theory of functions of a real variable. – New York: Frederick Ungar Publishing Co., 1955.

[34] Rudin W. Principles of mathematical analysis. – New York etc.: McGraw-Hill Book Co., 1964.

[35] Bari N.K. Trigonometric series. – Moscow: Gos. Izd. Fiz.–Mat. Lit., 1961 [in Russian]; transl. as A treatise on trigonometric series, Vols I and II. – New York: Macmillan Co., 1964.

[36] Landkof N.S. Foundations of modern potential theory. – Die Grundlehren der mathematischen Wissenschaften, 180. – New York-Heidelberg: Springer-Verlag, 1972.

[37] Agard S. Angles and quasiconformal mappings in space // J. Anal Math. – 1969. – 22. – P. 177–200.

[38] Agard S. B., Gehring F.W. Angles and quasiconformal mappings // Proc. London Math. Soc. (3) – 1965. – 14a. – P. 1–21.

[39] Taari O. Charakterisierung der Quasikonformität mit Hilfe der Winkelverzerrung [German] // Ann. Acad. Sci. Fenn. Ser. A I. – 1966. – 390. – P. 1–43.

[40] Garnett J.B., Marshall D.E., Harmonic Measure, Cambridge Univ. Press, Cambridge, 2005.

[41] Yefimushkin A.S., On multivalent solutions of the Riemann- Hilbert problem in multiply connected domains // Dopov. Nac. akad. nauk Ukr. – 2016. – No. 8. – P. 7–11 [in Russian].
Artyem Efimushkin and Vladimir Ryazanov,
Institute of Applied Mathematics and Mechanics,
National Academy of Sciences of Ukraine,
74 Roze Luxemburg Str., Donetsk, 83114,
art89@bk.ru, vl.ryazanov1@gmail.com