Uniqueness and Stability of Optimizers for a Membrane Problem

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Abstract

We investigate a PDE-constrained optimization problem, with an intuitive interpretation in terms of the design of robust membranes made out of an arbitrary number of different materials. We prove existence and uniqueness of solutions for general smooth bounded domains, and derive a symmetry result for radial ones. We strengthen our analysis by proving that, for this particular problem, there are no non-global local optima. When the membrane is made out of two materials, the problem reduces to a shape optimization problem. We lay the preliminary foundation for computable analysis of this type of problem by proving stability of solutions with respect to some of the parameters involved.

Key Words: Optimization, Stability, Radial symmetry, Boundary value problem, Rearrangements of functions.

Mathematics Subject Classification: 65K10, 74H55, 35J20, 35J25.

1 Introduction

Consider the boundary value problem:

\[
\begin{cases}
-\Delta u + g(x)u = f(x), & \text{in } D, \\
u = 0, & \text{on } \partial D,
\end{cases}
\]

in which \( D \subseteq \mathbb{R}^N \) is a smooth domain, \( N \in \{2, 3\} \), \( g \) is a non-negative function in \( \mathcal{L}^\infty(D) \), and \( f \) is a non-negative function in \( \mathcal{L}^2(D) \).

When the range of the function \( g \) is a finite set, say, \( \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \), this equation may be interpreted in a very intuitive way. Indeed, the boundary value problem (1.1) models an elastic membrane, constructed out of \( n \) different materials, fixed around the boundary, and subject to a vertical force \( f(x) \) at each point \( x \). The solution \( u \) denotes the displacement of the membrane from the rest position.

Let us assume that we have been given the \( n \) constituent materials, together with the force \( f \) and the geometry of the domain, and our task is to construct a robust membrane out of the given materials. In this paper, we demonstrate how this may be achieved. More formally, we associate the following energy functional with the boundary value problem (1.1):

\[
\Phi(g) := \int_D f u_g \, dx = \int_D |\nabla u_g|^2 \, dx + \int_D g u_g^2 \, dx,
\]

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in which, \textit{u}_g is the unique solution of (1.1). At an intuitive level, this energy functional is mean to measure the vulnerability of the membrane. It should be straightforward to verify that the following identity follows from the variational formulation of \textit{u}_g:

\[
\Phi(g) = \sup_{v \in H_0^1(D)} \left\{ 2 \int_D f v \, dx - \int_D (|\nabla v|^2 + g v^2) \, dx \right\}.
\]  

(1.3)

We assume that the information about the constituent materials is provided in a given function \textit{g}_0 which satisfies \(0 \leq g_0 \leq 1\), and which is is not identically zero. We let \(R = R(g_0)\) denote the rearrangement class generated by \textit{g}_0 (Definition 2.3 on the following page). To obtain a robust membrane, we need to obtain the arrangement of \textit{g}_0 with the least vulnerability, i.e., we need to solve the following minimization problem:

\[
\inf_{g \in R} \Phi(g).
\]  

(1.4)

\textbf{Remark 1.1.} The minimization problem (1.4) is of interest from a pure mathematical perspective as well. Indeed, the maximum principle ensures that \textit{u}_g, the solution of (1.1), is positive. Hence, the integral \(\int_D f u_g \, dx\) is the \(L^1(\mu)\)-norm of \textit{u}_g, with \(d\mu\) being the measure which is absolutely continuous with respect to the Lebesgue measure \(dx\), having \(f\) as its Radon-Nikodym derivative with respect to \(dx\). Minimization of various norms of solutions of partial differential equations is a classical topic of interest among mathematicians.

\section{1.1 Approach and contributions}

Our approach towards proving the solvability of (1.4) is based on the well-developed theory of rearrangements of functions \cite{24}. Specifically, we use the theory developed by G. R. Burton \cite{2,3} for optimization over rearrangement classes. To this end, we first relax the minimization problem (1.4) by extending the admissible set \(R\) to its weak closure \(\overline{R}\) with respect to \(L^2\)-topology. Once the relaxed problem is shown to be solvable, we will demonstrate how the appropriate restrictions on the force function \(f\) imply that solutions of the relaxed problem are indeed solutions of the original problem (1.4).

We strengthen our results by proving that the optimization problem (1.4) has no non-global local optima, and by showing that, when \(D\) is a ball and \(f\) is radial, then the solution of (1.4) is radial and non-increasing.

\textbf{Remark 1.2.} An appealing aspect of our method is that it can also be used when the function \(g\) belongs to the larger class \(L^p(D)\), for \(1 < p < \infty\), in which case, only minor modifications will be required. We prefer, however, to focus on the case \(g \in L^\infty(D)\), in order to minimize technicalities, and keep the model more realistic.

In the second part of the paper, we discuss some stability results. These results are of utmost importance in setting up a framework for computable analysis of problems such as our main problem (1.4).

\section{1.2 Related work}

For any given set \(E \subseteq D\), by \(\chi_E\) we denote the characteristic function of \(E\), i.e., \(\chi_E(x) = 1\) if \(x \in E\), and \(\chi_E(x) = 0\) if \(x \notin E\). Henrot and Maillot \cite{15} have investigated the special case of the minimization problem (1.4), in which \(g_0 = \chi_{E_0}\), for some \(E_0 \subseteq D\) with \(|E_0| = \alpha\). Under this assumption, one would get \(R = \{\chi_E : E \subseteq D \land |E| = \alpha\}\). In simple terms, the rearrangement class generated by \(g_0\) would be exactly the set of all characteristic functions of those measurable subsets of \(D\) that have the same Lebesgue measure as \(E_0\).

Henrot and Maillot \cite{15} prove the solvability for this special case, and state the minimality condition in terms of tangent cones. Since the underlying function space is \(L^\infty(D)\), they are able to derive a convenient formulation of the tangent cone of an appropriate convex set.

The method employed in \cite{15} is inadequate for addressing the optimization problem (1.4) for general generators \(g_0\). The theory that we shall introduce in this paper, however, not only furnishes an answer to the aforementioned question, but also can be used for a broader range that includes other design problems.

The second part of the current paper addresses some further issues, including stability properties of the solutions. This is part of a broader programme of laying the foundations for robust computable analysis of
rearrangement optimization problems in particular, and shape optimization problems in general. In this regard, we have carried out some general stability analyses pertaining to rearrangement optimization classes, which may be found in [19].

Remark 1.3. Parts of an earlier draft of this article have appeared in the PhD dissertation of one of the co-authors [18, Sec. 3.3].

1.3 Structure of the paper

The remainder of the paper is structured as follows:

- Section 2 contains preliminary material from the theory of rearrangements of functions.
- In Section 3 we prove existence and uniqueness of optimal solutions, and provide a radial symmetry result as well. For the minimization problem, we will show that there are no non-global local optima. Finally, we provide some remarks on the corresponding maximization problem.
- In Section 4 we discuss the shape optimization variant of the main problem. Specifically, we will discuss monotonicity and stability results related to the case where the generator is two-valued.
- In Section 5, we provide some remarks on the numerical simulation of the optimization problem.
- In order to avoid breaking the flow of the paper, the lengthy proof of Lemma 3.3 (from Section 3) is moved to Section 6.
- In Section 7, we finish the paper with some concluding remarks.

2 Preliminaries

In this section, we recall some well-known results from the theory of rearrangements of functions. Henceforth, we denote the $N$-dimensional Lebesgue measure of a measurable set $E$ by $|E|$. Moreover, for a Lebesgue measurable function $h : D \to [0, \infty)$ and $\alpha \geq 0$, we let:

$$\lambda_h(\alpha) = |\{x \in D : h(x) \geq \alpha\}|.$$

Definition 2.1. Let $g, g_0 : D \to [0, \infty)$ be Lebesgue measurable. We say that $g$ is a rearrangement of $g_0$ if and only if $\forall \alpha \geq 0 : \lambda_{g_0}(\alpha) = \lambda_g(\alpha)$.

Definition 2.2. For a Lebesgue measurable $g : D \to [0, \infty)$, the essentially unique decreasing rearrangement $g^\Delta$ is defined on $(0, |D|)$ by $g^\Delta(s) := \max\{\alpha : \lambda_g(\alpha) \geq s\}$. The essentially unique increasing rearrangement $g^\Delta$ of $g$ is defined by $g^\Delta(s) := g^\Delta(|D| - s)$.

Definition 2.3. The set $\mathcal{R} \equiv \mathcal{R}(g_0)$, called the rearrangement class generated by $g_0$, is defined as follows

$$\mathcal{R}(g_0) := \{g : D \to [0, \infty) : g \text{ is a rearrangement of } g_0\}.$$

Definition 2.4. For a function $f : D \to [0, \infty)$, we say that the graph of $f$ has no significant flat sections on $D$ if $\forall c \geq 0 : ||x \in D : f(x) = c|| = 0$.

Henceforth, the support of $g$ will be denoted by $S(g) \equiv \{x \in D : g(x) > 0\}$, and the reader should distinguish this definition of support from the usual topological definition. We use $\overline{\mathcal{R}}$ to denote the weak closure of $\mathcal{R}$ in $L^2(D)$. It is well-known that $\overline{\mathcal{R}}$ is convex, and weakly compact in $L^2(D)$.

Lemma 2.1. Let $\overline{\mathcal{R}}$ be the weak closure of $\mathcal{R}$ in $L^2(D)$. Then, $\overline{\mathcal{R}} \subseteq L^\infty(D)$, and $\forall g \in \overline{\mathcal{R}} : ||g||_{\infty} \leq ||g_0||_{\infty}$. 

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Proof. In order to derive a contradiction, we suppose \( g \not\in L^\infty(D) \). Hence, for every positive \( M \):

| \{ x \in D : g(x) > M \} | > 0.

Let us choose \( M = \|g_0\|_\infty \), and set \( E := \{ x \in D : g(x) > \|g_0\|_\infty \} \). Since \( g \in \overline{R} \), there exists \( \{g_n\} \subseteq \mathcal{R} \) such that \( g_n \rightharpoonup g \) in \( L^2(D) \). Then, we have:

\[
\int_E g_n \, dx = \int_D g_n \chi_E \, dx \to \int_D g \chi_E \, dx = \int_E g \, dx.
\] (2.1)

From the definition of \( E \) and the fact that \( \int_E g_n \, dx \leq \|g_0\|_\infty |E| \), in conjunction with (2.1), we deduce:

\[
\|g_0\|_\infty |E| < \int_E g \, dx = \lim_{n \to \infty} \int_E g_n \, dx \leq \|g_0\|_\infty |E|.
\] (2.2)

Obviously, (2.2) is a contradiction. The above argument implies that the measure of \( E \) is zero. Hence, \( \|g\|_\infty \leq \|g_0\|_\infty \). This completes the proof of the lemma. \( \square \)

Lemma 2.2. Suppose \( \{g_n\} \subseteq L^2_\infty(D) \), and \( g \in L^2(D) \). Suppose \( g_n \rightharpoonup g \) in \( L^2(D) \). Then, \( g \) is non-negative a.e. in \( D \).

Proof. This is an immediate consequence of Mazur’s Lemma. Indeed, by Mazur’s Lemma, there exists a sequence \( \{v_n\} \) in the convex hull of the set \( \{g_n : n \in \mathbb{N}\} \) such that \( v_n \rightharpoonup g \) in \( L^2(D) \). Therefore, \( v_n \rightharpoonup g \) in measure. Whence, there exists a subsequence of \( \{v_n\} \) which converges to \( g \) a.e. in \( D \). This completes the proof. \( \square \)

The next lemma is easy to prove:

Lemma 2.3. Suppose that \( f : D \to [0, \infty) \) is measurable. Then, for every measurable subset \( E \subseteq D \):

\[
\int_E f \, dx \geq \int_0^{\|E\|} f_\Delta(s) \, ds.
\]

Lemma 2.4. For every \( g \) in \( \overline{R} \) we have \( |S(g_0)| \leq |S(g)| \).

Proof. In order to derive a contradiction, let us assume that \( |S(g)| < |S(g_0)| \). Hence, \( \alpha = \int_0^{\|S(g)\|} g_0 \, dx \) is positive. Since \( g \in \overline{R} \), there exists \( \{g_n\} \subseteq \mathcal{R} \) such that \( g_n \rightharpoonup g \) in \( L^2(D) \). Then, we have:

\[
\alpha = \int_0^{\|S(g)\|} g_0 \, dx = \int_0^{\|S(g)\|} g_{n_0} \, dx \leq \int_{S(g)} g_{n_0} \, dx = \int_D g_{n_0} \chi_{S(g)} \, dx \to \int_D g \chi_{S(g)} \, dx = \int_{S(g)} g \, dx = 0, \quad (2.3)
\]

which is a contradiction. The inequality in (2.3) is a consequence of Lemma 2.3. \( \square \)

We make use of the following lemmata from [2] and [3].

Lemma 2.5. The following characterization for the weak closure of \( \mathcal{R} \) holds:

\[
\overline{R} = \left\{ g \in L^1(D) : \int_D g \, dx = \int_D g_0 \, dx \quad \text{and} \quad \forall s \in (0,|D|) : \int_0^s g^\Delta \, dt \leq \int_0^s g_0^\Delta \, dt \right\}.
\]

Proof. See Lemma 2.3 in [3]. \( \square \)

In line with the established convention of [2, 3], in what follows we often write ‘increasing’ instead of non-decreasing and ‘decreasing’ instead of non-increasing.

Lemma 2.6. Suppose that \( f : D \to [0, \infty) \) is measurable and has no significant flat sections on \( D \). Then, there exists an increasing function \( \psi \) such that \( \psi(f) \) is a rearrangement of \( g_0 \). Moreover, there is a decreasing function \( \tilde{\psi} \) such that \( \tilde{\psi}(f) \) is a rearrangement of \( g_0 \).
Proof. See Lemma 2.9 in [3]. □

Lemma 2.7. Let \( f \in L^2(D) \) be a non-negative and non-trivial function (i.e., it is not identically zero), and assume that there is an increasing function \( \psi \) such that \( \psi(f) \in \mathbb{R} \). Then \( \psi(f) \) is the unique maximizer of the linear functional \( L(h) := \int_D fh \, dx \) relative to \( h \in \mathbb{R} \).

Proof. See Lemma 2.4 in [3]. □

We will also need the following rearrangement result for the Dirichlet integral (see, e.g., [1]). Note that here \( v^* \) denotes the Schwarz symmetrization of \( v \) (see, e.g., [17]):

Lemma 2.8.

(i) If \( v \in H^1_0(\mathbb{R}^N) \) is non-negative, then, \( v^* \in H^1_0(\mathbb{R}^N) \), and the following inequality holds:

\[
\int_{\mathbb{R}^N} |\nabla v^*|^2 \, dx \leq \int_{\mathbb{R}^N} |\nabla v|^2 \, dx.
\] (2.4)

(ii) If \( v \in H^1_0(\mathbb{R}^N) \) is non-negative and equality holds in (2.4), then, for every \( 0 \leq \alpha < M := \text{ess sup} \, v \), \( v^{-1}(\alpha, \infty) \) is a translate of the disk \( v^{-1}(\alpha, \infty) \), almost everywhere. If, in addition, \( \{x \in \mathbb{R}^N : \nabla v = 0, \, 0 < v(x) < M\} \) has zero measure, then \( v \) is a translate of \( v^* \).

3 Optimal solutions

We need to make certain assumptions on the force function \( f \) in order to be able to obtain our main results. Henceforth, \( v_f \in H^1_0(D) \) will denote the unique solution of the Poisson boundary value problem:

\[
\begin{aligned}
-\Delta v_f &= f \quad \text{in } D, \\
v_f &= 0 \quad \text{on } \partial D.
\end{aligned}
\] (3.1)

Here is the main assumption on which our results will hinge:

\begin{center}
A1: \( v_f \leq f \), in \( D \).
\end{center}

A minor problem with this assumption is that its statement involves the solution to the Poisson boundary value problem (3.1). It turns out that we can also work with the following assumption, whose statement involves just the function \( f \) and its Laplacian:

\begin{center}
A2: \( f \leq -\Delta f \), in \( D \).
\end{center}

Proposition 3.1. A2 implies A1.

Proof. Notice that we have:

\[
\begin{aligned}
-\Delta(v_f - f) &= f + \Delta f \quad \text{in } D, \\
v_f - f &\leq 0 \quad \text{on } \partial D.
\end{aligned}
\] (3.2)

Since \( f + \Delta f \) is non-positive, we can apply the maximum principle to (3.2) to deduce \( v_f \leq f \). □

As a consequence, all of the results that will be proved based on A1 will also hold for assumption A2.
Remark 3.1. Our assumptions are valid, in the sense that there are non-negative functions satisfying $A_2$, and by implication $A_1$. Indeed, consider the boundary value problem

$$\begin{cases}
-\Delta u - u = N & \text{in } D, \\
u = 0 & \text{on } \partial D,
\end{cases} \tag{3.3}$$

in which $N \in [0, \infty)$. The energy functional associated with (3.3) is:

$$I(u) = \frac{1}{2} \int_D |\nabla u|^2 \, dx - \frac{1}{2} \int_D u^2 \, dx - \int_D Nu \, dx.$$  

It is clear from the Poincaré inequality that, if $D$ is thin, then $I(u)$ will be coercive. So, by an application of the direct method of calculus of variations to the functional $I(u)$, we infer the existence of a critical point which is a solution of (3.3). In order to show that (3.3) has a non-negative solution, it suffices to point out that $I(|u|) \leq I(u)$.

3.1 Existence, uniqueness, and optimality condition

Our assumptions guarantee that the solution $u_g$ of the boundary value problems (1.1) has no significant flat sections on $S(g)$, a fact which will be used in the proof of our main result:

**Lemma 3.2.** Suppose that $f$ satisfies assumption $A_1$, and $g$ is a measurable function such that $0 \leq g \leq 1$. Then, $u_g$ has no significant flat sections on $S(g)$.

**Proof.** From the boundary value problems (1.1) and (3.1), we deduce:

$$\begin{cases}
-\Delta(u_g - v_f) + g(u_g - v_f) = -gv_f, & \text{in } D, \\
u_g - v_f = 0, & \text{on } \partial D.
\end{cases}$$

Since $g$ and $v_f$ are non-negative, $u_g < v_f$ in $D$ by the strong maximum principle.

In order to derive a contradiction, we assume that there exists an $L \subseteq S(g)$ such that the measure of $L$ is positive, and $u_g$ is constant on $L$. By applying Lemma 7.7 in [14], we infer $f = gu_g$ in $L$. Hence:

$$f = gu_g < gv_f \leq v_f \leq f, \text{ in } L,$$

which is a contradiction. □

Next, we turn to the energy functional. In order to prove the existence and uniqueness of solutions of the minimization problem (1.4), we need the following basic result regarding the energy functional $\Phi$:

**Lemma 3.3.** The energy functional $\Phi$ satisfies the following:

(i) $\Phi$ is weakly continuous on $\overline{\mathcal{R}}$ with respect to the $L^2$–topology.

(ii) $\Phi$ is strictly convex on $\overline{\mathcal{R}}$.

(iii) Given $g$ and $h$ in $\overline{\mathcal{R}}$, the following formula holds:

$$\lim_{t \to 0^+} \frac{\Phi(\xi_t) - \Phi(g)}{t} = -\int_D (h-g)u^2 \, dx, \quad 0 < t < 1, \tag{3.4}$$

in which $\xi_t = g + t(h-g)$, and $u = u_g$.

**Proof.** The proof of this lemma is quite long and involved. In order not to break the flow of the discussion, the proof is placed in a separate section altogether. Please see Sect. 6. □

The main result of the paper is the following:
Theorem 3.4. Suppose that $f$ satisfies assumption A1. Then the minimization problem (1.4) has a unique solution $\hat{g} \in \mathcal{R}$. Moreover, there exists an increasing function $\psi$ such that:

$$\hat{g} = \psi(\hat{u}) \quad a.e. \text{ in } D,$$

where $\hat{u} = u_{\tilde{R}}$.

Proof. We relax the minimization problem (1.4) first by extending the admissible set $\mathcal{R}$ to $\tilde{\mathcal{R}}$. Thus, we consider:

$$\inf_{g \in \tilde{\mathcal{R}}} \Phi(g)$$

By Lemma 3.3 (i), $\Phi$ is weakly continuous on $\tilde{\mathcal{R}}$ with respect to the $L^2$-topology. Hence, the minimization problem (3.6) is solvable. Furthermore, thanks to the strict convexity of $\Phi$ (Lemma 3.3 (ii)) the solution to (3.6) is unique. Let us denote this solution by $\hat{g}$.

Fix $g \in \tilde{\mathcal{R}}$, and set $g_t = \hat{g} + t(g - \hat{g})$, for $t \in (0, 1)$. Due to the convexity of $\tilde{\mathcal{R}}$, $g_t \in \tilde{\mathcal{R}}$. From Lemma 3.3 (iii) we can derive $\int_D (g - \hat{g})\hat{u}^2 dx \leq 0$. Whence, $\hat{g}$ maximizes the linear functional $L(h) := \int_D h\hat{u}^2 dx$, relative to $h \in \tilde{\mathcal{R}}$. From Lemma 2.1 and Lemma 3.2, it follows that the graph of $\hat{u}$, the restriction of $\hat{u}$ to the set $S(\hat{g})$, has no significant flat sections on $S(\hat{g})$. From Lemma 2.4, we know that there exists a $g_1 \in \mathcal{R}$ such that $S(g_1) \subseteq S(\hat{g})$. Therefore, if we denote by $\mathcal{R}_S$ the functions which are rearrangements of $g_1$ on $S(\hat{g})$, then by Lemma 3.3 (iii) we infer the existence of an increasing function $\psi_S$ such that $\psi_S((\hat{u})^2) \in \mathcal{R}_S$. We now proceed to extending $\psi_S$ to an increasing function $\psi$ in such a way that $\psi(\hat{u}) \in \mathcal{R}(g_1) = \mathcal{R}$. Let us assume for the moment that this task has been accomplished. Then, from Lemma 2.7, it follows that $\psi(\hat{u})$ is the unique maximizer of the functional $L$, whence we must have $\hat{g} = \psi(\hat{u})$, which is the desired result.

We now come to the issue of extending $\psi_S$. This is done in two steps. The first step is to show that $\hat{u}$ attains its largest values on $S(\hat{g})$. To this end, it suffices to prove the following inequality:

$$\alpha = \operatorname{ess inf} \hat{u} \geq \operatorname{ess sup} \hat{u} = \beta,$$

where $S(\hat{g})^c$ denotes the complement of $S(\hat{g})$. We prove (3.7) by contradiction. So, let us suppose that $\alpha < \beta$. Hence, there exist constants $\gamma, \delta$, and sets $A \subseteq S(\hat{g}), B \subseteq S(\hat{g})^c$, such that $\beta > \gamma > \delta > \alpha$, and:

$$\begin{cases} 
\hat{u} \leq \delta & \text{on } A, \\
\hat{u} \geq \gamma & \text{on } B.
\end{cases}$$

We may assume that $|A| = |B|$, otherwise we consider subsets of $A$ and $B$. Let $\eta : A \to B$ be a measure preserving bijection. Next, we define a new function $\tilde{g}$ as follows:

$$\tilde{g}(x) = \begin{cases} 
\hat{g}(x) & x \in (A \cup B)^c, \\
\hat{g}(\eta(x)) & x \in A, \\
\hat{g}(\eta^{-1}(x)) & x \in B.
\end{cases}$$

Clearly $\tilde{g}$ is a rearrangement of $\hat{g}$. Since $\hat{g} \in \mathcal{R}$, it follows from Lemma 2.5 that $\tilde{g} \in \mathcal{R}$. Thus:

$$\int_D \tilde{g}\hat{u}^2 dx - \int_D \hat{g}\hat{u}^2 dx = \int_{A \cup B} \tilde{g}\hat{u}^2 dx - \int_{A \cup B} \hat{g}\hat{u}^2 dx = \int_B \tilde{g}\hat{u}^2 dx - \int_A \hat{g}\hat{u}^2 dx$$

$$= \int_B \hat{g}(\eta^{-1}(x))\hat{u}^2 dx - \int_A \hat{g}\hat{u}^2 dx = \int_A \hat{g}(x)\hat{u}^2(\eta(x)) dx - \int_A \hat{g}\hat{u}^2 dx \geq (\gamma^2 - \delta^2) \int_A \hat{g} dx > 0,$$

which contradicts the maximality of $\hat{g}$.

\footnote{Such a map exists. See, e.g., [21].}
In the second step, we give an explicit formula for the extended function as follows:

\[
\psi(t) = \begin{cases} 
\psi_0(t) & t > \alpha^2, \\
0 & t \leq \alpha^2,
\end{cases}
\]

where \( \alpha \) is defined in (3.7). Clearly, \( \psi \) is increasing and \( \psi(\hat{t}^2) \in \mathcal{R}(g_1) = \mathcal{R} \). Hence, by setting \( \psi(t) := \psi(t^2) \) we derive (3.5). The proof of the theorem is completed.

\[\Box\]

Remark 3.2. As mentioned earlier, in the special case of \( g_0 = \chi_{E_0} \) with \( |E_0| = \alpha \), the minimization problem (1.4) reduces to the one considered in [15]. So, \( \hat{g} = \chi_{\hat{E}} \) with \( |\hat{E}| = \alpha \). Hence, from (3.5) we deduce that \( \hat{E} = \{ \hat{t} > \gamma \} \), for some \( \gamma > 0 \). Whence, we derive the following boundary value problem:

\[
\begin{cases}
-\Delta \hat{u} + \hat{u} \chi_{(\hat{t} > \gamma)} = f(x) & \text{in } D, \\
\hat{u} = 0 & \text{on } \partial D.
\end{cases}
\]

By setting \( U = \hat{u} - \gamma \), the differential equation in (3.8) becomes:

\[
\Delta U = (U + \gamma - f)\chi_{(U > 0)} - f \chi_{(U \leq 0)}.
\]

So, (3.9) is an obstacle problem of type:

\[
\Delta U = G(x)\chi_{(U > 0)} - H(x)\chi_{(U \leq 0)},
\]

where \( G \leq 0 \), because \( \hat{u} \leq f \), and \( H(x) \geq 0 \). Since \( G(x) + H(x) \geq 0 \), we can apply the result of [23] to deduce that the free boundary has \( C^{1,1} \) regularity.

### 3.2 Local minimizers

Even though \( \hat{g} \) in Theorem 3.4 is a global minimizer, is it possible for \( \Phi \) to have non-global local minimizers over \( \mathcal{R} \)? The answer to this question is negative. To prove this, we need a less restrictive version of Theorem 3.3 (iii) in [3], stated as follows:

**Lemma 3.5.** Let \( N : L'(D) \to \mathbb{R} \) be weakly sequentially continuous, and let \( \mathcal{R} = \mathcal{R}(h_0) \) denote the rearrangement class generated by some \( h_0 \in L'(D) \). Assume that for every pair \( (h_1, h_2) \in \mathcal{R} \times \mathcal{R} \) the following relation holds:

\[
\lim_{t \to 0^+} \frac{N(th_2 + (1-t)h_1) - N(h_1)}{t} = \int_D (h_2 - h_1) G \, dx,
\]

for some \( G \in L'(D) \). Suppose \( \mathcal{U} \) is a strong neighborhood (relative to \( \mathcal{R} \)) of \( \hat{h} \in \mathcal{R} \), for which we have:

\[\forall h \in \mathcal{U} : N(\hat{h}) \leq N(h).\]

Then, \( \hat{h} \) minimizes the linear functional \( \mathcal{L}(h) := \int_D hG \, dx \), relative to \( h \in \mathcal{R} \).

Now we state our result concerning local minimizers.

**Theorem 3.6.** Let the hypotheses of Theorem 3.4 hold. If \( g_1 \) and \( g_2 \) are two local minimizers of \( \Phi(g) \) relative to \( g \in \mathcal{R} \), then \( g_1 = g_2 \).

**Proof.** For simplicity we set \( u_1 := u_{g_1} \) and \( u_2 := u_{g_2} \). Lemma 3.5 in conjunction with Lemma 3.3 (iii), implies that \( g_1 \) and \( g_2 \) are maximizers of the linear functionals:

\[
\mathcal{L}_1(g) := \int_D gu_1^2 \, dx,
\]

\[
\mathcal{L}_2(g) := \int_D gu_2^2 \, dx.
\]
and
\[ L_2(g) := \int_D gu^2 \, dx, \]
relative to \( g \in \mathcal{R} \), respectively. In particular, we infer:
\[
\int_D g_2 u_1^2 \, dx \leq \int_D g_1 u_1^2 \, dx \quad \text{and} \quad \int_D g_1 u_2^2 \, dx \leq \int_D g_2 u_2^2 \, dx.
\]
(3.11)
Thus, we obtain:
\[
2 \int_D f u_1 \, dx - \int_D (|\nabla u_1|^2 + g_1 u_1^2) \, dx \leq 2 \int_D f u_1 \, dx - \int_D (|\nabla u_2|^2 + g_2 u_2^2) \, dx
\]
\[
\leq 2 \int_D f u_2 \, dx - \int_D (|\nabla u_2|^2 + g_1 u_2^2) \, dx
\]
\[
\leq 2 \int_D f u_1 \, dx - \int_D (|\nabla u_1|^2 + g_1 u_1^2) \, dx
\]
(3.12)
where the first and third inequalities are consequences of (3.11), whereas the second and the fourth inequalities follow from (1.3). From (3.12) we see that all inequalities must in fact be equalities. This, in turn, implies that \( u_1 = u_2 \), due to the uniqueness. Whence, we deduce \( g_1 = g_2 \) as desired. \( \square \)

### 3.3 Radial domain

Here we present our result regarding radial symmetry of the optimizers. Note how, compared with similar results in the literature, in our approach, such result may be obtained with minimal technicalities:

**Theorem 3.7.** Suppose that \( f \) is radial and satisfies assumption \( A1 \). Then the solution of (1.4) is radial and non-increasing.

**Proof.** Let \( g \) denote the solution of (1.4) and let \( R \) be a rotational map about the origin. Since \( f \) is radial, we infer \( u_g \circ R = u_{g,R} \). Thus, \( \Phi(g \circ R) = \Phi(g) \), and \( g \circ R \) is also a solution of (1.4). By uniqueness, we deduce \( g \circ R = g \), for every rotational map \( R \). Whence, \( g \) is radial, as desired. To prove that \( g \) is non-increasing, we observe that, since \( u = u_g \) is radial, we can write the equation in (1.1) as:
\[
-(r^{N-1}u')' = r^{N-1}(f - gu).
\]
Since \( f \geq v_f \) by \( A1 \), and \( g \leq 1 \) by assumption, we have \( f - gu \geq v_f - u \). Furthermore, \( v_f - u > 0 \) by the proof of Lemma [3.2]. Hence,
\[
-(r^{N-1}u')' > 0, \quad -r^{N-1}u' > 0, \quad u' < 0.
\]
By Theorem [3.4] \( g = \psi(u) \), for some non-decreasing \( \psi \). As a result, \( g \) is non-increasing, as desired. \( \square \)

### 3.4 Some remarks on maximization

In addition to the minimization problem (1.4), one can also consider the maximization problem:
\[
\sup_{g \in \mathcal{R}} \Phi(g). \quad (3.13)
\]
Since \( \Phi \) is weakly continuous and convex, \( \Phi \) reaches its maximum value at the extremal points of the convex set \( \mathcal{R} \) (i.e., the elements of \( \mathcal{R} \)). Hence, problem (3.13) is solvable (see Theorem 7 of [2] or Remark 3.1 of [15]). Moreover, if the assumption \( A1 \) holds, along the same lines as in the proof of Theorem [3.4] it can be shown that, if \( \tilde{g} \) is a maximizer, then:
\[
\tilde{g} = \tilde{\psi} (\tilde{u}), \quad (3.14)
\]
almost everywhere in $D$, for some decreasing function $\tilde{\psi}$. Here $\tilde{u} = u\tilde{g}$, the solution of (1.1) with $g = \tilde{g}$.

Note that, for maximizers we do not have uniqueness in general. However, we are going to prove that, in case $D$ is a ball and $f$ is radially symmetric and non-increasing, any maximizer is radially symmetric and non-decreasing, hence unique. Indeed, let $v = u\tilde{g}$, where $\tilde{g}$, is the increasing Schwarz symmetrization of $\tilde{g}$ (see [17]). For simplicity, we write $u$ instead of $u\tilde{g}$. By Lemma 2.8(1):

$$-\frac{1}{2} \Phi(\tilde{g}) = \frac{1}{2} \int_D |\nabla u|^2 \, dx + \frac{1}{2} \int_D \tilde{g} u^2 \, dx - \int_D f u \, dx$$

(3.15)

Now, by applying the Hardy-Littlewood inequality to the last two integrals in (3.15), keeping in mind that $f = f^*$, we obtain:

$$-\frac{1}{2} \Phi(\tilde{g}) \geq \frac{1}{2} \int_D |\nabla u|^2 \, dx + \frac{1}{2} \int_D \tilde{g} u^2 \, dx - \int_D f u^\ast \, dx$$

(3.16)

Recalling that $v$ minimizes the functional

$$I(w) = \frac{1}{2} \int_D |\nabla w|^2 \, dx + \frac{1}{2} \int_D \tilde{g} w^2 \, dx - \int_D f w \, dx$$

relative to $w \in H_0^1(D)$, we infer from (3.16) that:

$$-\frac{1}{2} \Phi(\tilde{g}) \geq \frac{1}{2} \int_D |\nabla v|^2 \, dx + \frac{1}{2} \int_D \tilde{g} v^2 \, dx - \int_D f v \, dx = -\frac{1}{2} \Phi(\tilde{g}^\ast).$$

(3.17)

As $\tilde{g}$ is maximal for $\Phi$, then $\Phi(\tilde{g}) \leq \Phi(\tilde{g})$, which together with (3.15), (3.16) and (3.17) yield:

$$\int_D |\nabla u|^2 \, dx = \int_D |\nabla u|^2 \, dx.$$

Thus, from Lemma 2.8(1), we see that $u^{-1}(\alpha, \infty)$ is a ball for every $0 \leq \alpha < M = \text{ess sup} u$. We now proceed to show that $u = u^\ast$. Recalling Lemma 2.8(1), it suffices to verify that the set $\{x \in D : \nabla u = 0, \ 0 < u(x) < M\}$ is measure zero. To this end, consider $x_0 \in D$, and set $S := \{u \geq u(x_0)\}$. We know that $S$ is a disk (ball), and by continuity of $u$, $x_0 \in \partial S \subseteq \{u = u(x_0)\}$. So we can apply the Hopf lemma (see, e.g., [13]), and deduce that $\frac{\partial u}{\partial \nu}(x_0) < 0$, where $\nu$ denotes the unit outward normal vector to $\partial S$ at $x_0$. Whence, in particular, $\nabla u(x_0) \neq 0$. Thus, in fact, $\{x \in D : \nabla u = 0, \ 0 < u(x) < M\}$ is empty, so its measure is zero, as desired. This implies $u = u^\ast$, and by (3.14), $\tilde{g} = \tilde{\psi}(u^\ast)$ almost everywhere in $D$. Since $\tilde{\psi}$ is decreasing, $\tilde{g}$ is radial and non-decreasing, as claimed.

Remark 3.3. A consequence of (3.5) is that the larger values of $\hat{u}$ are attained where $\tilde{g}$ is large. Whence, in case the set $\{\tilde{g} = 0\}$ has positive measure, it will contain a layer around the boundary $\partial D$, since $\tilde{u}$ is continuous, and vanishes on $\partial D$. Physically, this means that in the construction of a robust membrane one should use the material with least density near the boundary. The dual conclusion can be drawn similarly regarding the maximization problem (3.13).

Remark 3.4. Note that Theorem 3.7 can be improved. Indeed, if $D$ is Steiner symmetric with respect to a hyperplane $l$ (see, e.g., [17]), then $\tilde{g}$ (the solution of (1.4)) will also be Steiner symmetric with respect to $l$. Of course, in this case, one needs to use the inequality:

$$\int_D |\nabla u|^2 \, dx \geq \int_D |\nabla u^\ast|^2 \, dx,$$

instead of (2.4), in which $u^\ast$ stands for the Steiner symmetrization of $u$. A similar result can be obtained for the maximization problem (3.13). Of course, for the maximization problem we do not necessarily have uniqueness of optimal solutions.
4 Shape optimization

In this section, we focus on the shape optimization variant of our main problem, i.e., the case where the generator \( g_0 \) is two-valued. Thus, we consider the following boundary value problem:

\[
\begin{aligned}
-\Delta u + (\alpha \chi_{E} + \beta \chi_{E^c}) u &= f, \quad \text{in } D, \\
u &= 0, \quad \text{on } \partial D,
\end{aligned}
\]

in which, \( D \) is a smooth bounded domain in \( \mathbb{R}^N, N \in \{2, 3\} \), \( f \in L^2(D) \) is a given non-negative function, \( 1 \geq \alpha > \beta \geq 0, E \) is a measurable subset of \( D \), and \( E^c \) is the complement of \( E \) in \( D^c \).

Denoting the unique solution of (4.1) by \( u_E \), we are interested in the following minimization problem:

\[
\inf_{|E| = \gamma} \int_D f u_E \, dx, \tag{4.2}
\]

where \( 0 < \gamma < |D| \). By Theorem 3.4, we know that, if \( f \) satisfies A1, then (4.2) has a unique solution \( \hat{D} \subset D \), with \( |\hat{D}| = \gamma \). Also, we have \( \hat{D} = \{ x \in D : u_{\hat{D}}(x) > c \} \), for some positive \( c \), which, in turn, implies:

\[ u_{\hat{D}}(x) = c, \quad \text{on } \partial \hat{D}. \]

Our aim is to analyze monotonicity and stability of solutions with respect to the parameters \( \alpha \) and \( \gamma \). Analyses of this kind are crucial for laying the foundation for computable analysis of shape optimization problems such as (4.1).

**Remark 4.1.** In what follows, we keep the presentation succinct, and as such, many of the claims will be listed with the proofs omitted. The interested reader may refer to Sect. 3.3 of [18] for the details of the omitted proofs. Nonetheless, we present the proofs of a few of the more interesting cases.

4.1 Monotonicity and stability results with respect to \( \gamma \)

We know that, for each \( 0 < \gamma < |D| \), the minimization problem (4.2) has a unique solution. Now, consider \( 0 < \gamma_1 < \gamma_2 < |D| \), and their corresponding unique solutions:

\[
\hat{D}_{\gamma_1} = \{ x \in D : u_{\gamma_1}(x) > c_{\gamma_1} \} \quad \text{and} \quad \hat{D}_{\gamma_2} = \{ x \in D : u_{\gamma_2}(x) > c_{\gamma_2} \},
\]

for some positive \( c_{\gamma_1} \) and \( c_{\gamma_2} \), where \( u_{\gamma_1} \) and \( u_{\gamma_2} \) satisfy:

\[
\begin{aligned}
-\Delta u_{\gamma_1} + (\alpha \chi_{\hat{D}_{\gamma_1}} + \beta \chi_{\hat{D}_{\gamma_1}^c}) u_{\gamma_1} &= f, \quad \text{in } D, \\
u_{\gamma_1} &= 0, \quad \text{on } \partial D,
\end{aligned}
\]

and

\[
\begin{aligned}
-\Delta u_{\gamma_2} + (\alpha \chi_{\hat{D}_{\gamma_2}} + \beta \chi_{\hat{D}_{\gamma_2}^c}) u_{\gamma_2} &= f, \quad \text{in } D, \\
u_{\gamma_2} &= 0, \quad \text{on } \partial D.
\end{aligned}
\]

We also restate the minimization problem (4.2), with \( \gamma \) as an input parameter:

\[
\Psi(\gamma) := \inf_{|E| = \gamma} \int_D f u_E \, dx. \tag{4.6}
\]

**Proposition 4.1.** If \( 0 < \gamma_1 < \gamma_2 < |D| \), then

(i) \( c_{\gamma_1} \geq c_{\gamma_2} \).

(ii) \( \hat{D}_{\gamma_1} \subseteq \hat{D}_{\gamma_2} \).

\[ \text{To see why the assumption } 1 \geq \alpha \text{ is imposed, see Lemma 3.2 on page 6.} \]
Theorem 4.3. If $\gamma_1$ tends to $\gamma_2$ in $(0, |D|)$, then $u_{\gamma_1}$ converges to $u_{\gamma_2}$ in $C(\bar{D})$. Moreover, $c_{\gamma_1}$ converges to $c_{\gamma_2}$, where $c_{\gamma_1} = u_{\gamma_1}(\partial \bar{D}_{\gamma_1})$ and $c_{\gamma_2} = u_{\gamma_2}(\partial \bar{D}_{\gamma_2})$.

Corollary 4.2. $\Psi(\gamma)$ is a decreasing function on $(0, |D|)$.

Corollary 4.4. $\Psi(\gamma)$ is continuous on $(0, |D|)$.

From Corollary 4.2 we infer that $\Psi(\gamma)$ is differentiable almost everywhere. However, the following theorem shows that it is actually continuously differentiable on $(0, |D|)$.

Theorem 4.5. $\Psi(\gamma)$ is continuously differentiable on $(0, |D|)$. Moreover:

$$\Psi'(\gamma) = -(\alpha - \beta)c_{\gamma}^2,$$

where $c_{\gamma} = u_{\gamma}(\partial \bar{D}_{\gamma})$.

Proof. Fix $0 < \gamma_2 < |D|$, and let $\gamma_1$ increase to $\gamma_2$. We claim that $\frac{\Psi(\gamma_1) - \Psi(\gamma_2)}{\gamma_1 - \gamma_2}$ converges to $-(\alpha - \beta)c_{\gamma_2}^2$. From (4.4) and (4.5), we deduce

$$\begin{cases}
-\Delta(u_{\gamma_1} - u_{\gamma_2}) + (a\chi_{D_{\gamma_1}} + \beta\chi_{\bar{D}_{\gamma_1}})(u_{\gamma_1} - u_{\gamma_2}) = -(\alpha - \beta)u_{\gamma_2}(\chi_{D_{\gamma_1}} - \chi_{\bar{D}_{\gamma_2}}) & \text{in } D \\
u_{\gamma_1} - u_{\gamma_2} = 0 & \text{on } \partial D.
\end{cases}$$

(4.7)

Multiplying the differential equation in (4.7) by $u_{\gamma_1} + u_{\gamma_2}$, integrating the result over $D$, followed by an application of divergence theorem, in conjunction with $\bar{D}_{\gamma_1} \subseteq \bar{D}_{\gamma_2}$ (Proposition 4.1 (iii)) yields:

$$\int_D (|\nabla u_{\gamma_1}|^2 - |\nabla u_{\gamma_2}|^2) \, dx + \int_D (a\chi_{D_{\gamma_1}} + \beta\chi_{\bar{D}_{\gamma_1}})(u_{\gamma_1}^2 - u_{\gamma_2}^2) \, dx$$

$$= -(\alpha - \beta)\int_D u_{\gamma_2}(u_{\gamma_1} + u_{\gamma_2})(\chi_{D_{\gamma_1}} - \chi_{\bar{D}_{\gamma_2}}) \, dx$$

$$= (\alpha - \beta)\int_{D_{\gamma_2}\setminus\bar{D}_{\gamma_1}} u_{\gamma_2}(u_{\gamma_1} + u_{\gamma_2}) \, dx. \quad (4.8)$$

Furthermore, from (4.4), (4.5), and (4.8), we deduce:

$$\Psi(\gamma_1) - \Psi(\gamma_2) = \int_D u_{\gamma_1}f \, dx - \int_D u_{\gamma_2}f \, dx$$

$$= \left[\int_D |\nabla u_{\gamma_1}|^2 \, dx + \int_D (a\chi_{D_{\gamma_1}} + \beta\chi_{\bar{D}_{\gamma_1}})u_{\gamma_1}^2 \, dx\right] - \left[\int_D |\nabla u_{\gamma_2}|^2 \, dx + \int_D (a\chi_{D_{\gamma_2}} + \beta\chi_{\bar{D}_{\gamma_2}})u_{\gamma_2}^2 \, dx\right]$$

$$= \int_D (|\nabla u_{\gamma_1}|^2 - |\nabla u_{\gamma_2}|^2) \, dx + \int_D (a\chi_{D_{\gamma_1}} + \beta\chi_{\bar{D}_{\gamma_1}})(u_{\gamma_1}^2 - u_{\gamma_2}^2) \, dx + \int_{D_{\gamma_1}\setminus\bar{D}_{\gamma_2}} (\beta u_{\gamma_1}^2 - a u_{\gamma_2}^2) \, dx$$

$$= (\alpha - \beta)\int_{D_{\gamma_2}\setminus\bar{D}_{\gamma_1}} u_{\gamma_2}(u_{\gamma_1} + u_{\gamma_2}) \, dx - \beta \int_{D_{\gamma_2}\setminus\bar{D}_{\gamma_1}} (u_{\gamma_1}^2 - u_{\gamma_2}^2) \, dx + \int_{D_{\gamma_1}\setminus\bar{D}_{\gamma_2}} (\beta u_{\gamma_1}^2 - a u_{\gamma_2}^2) \, dx$$

$$= (\alpha - \beta)\int_{D_{\gamma_2}\setminus\bar{D}_{\gamma_1}} u_{\gamma_2}u_{\gamma_1} \, dx. \quad (4.9)$$

(iii) $u_{\gamma_1} > u_{\gamma_2}$ in $D$.

Since $f$ is non-negative and non-trivial, the following is an easy consequence of Proposition 4.1 (iii).
where we have used the fact that \( \bar{D}_{\gamma_1} \subseteq \bar{D}_{\gamma_2} \) in the third and fourth equality, and also applied (4.8) in the fourth equality. By using (4.9) and the fact that \(|\bar{D}_{\gamma_2} \setminus \bar{D}_{\gamma_1}| = \gamma_2 - \gamma_1\), we calculate:

\[
\frac{|\Psi(\gamma_1) - \Psi(\gamma_2)|}{\gamma_1 - \gamma_2} = \frac{\alpha - \beta}{\gamma_2 - \gamma_1} \left| \int_{\bar{D}_{\gamma_2} \setminus \bar{D}_{\gamma_1}} (u_{y_2} u_{y_1} - c_2^2) \, dx \right| \\
\leq (\alpha - \beta) \|u_{y_2} u_{y_1} - c_2^2\|_{L^\infty(\bar{D}_{\gamma_2} \setminus \bar{D}_{\gamma_1})}.
\]

By (4.3) and Proposition 4.1 (iii), in \( \hat{\bar{D}}_{\gamma_2} \setminus \bar{D}_{\gamma_1} \) we have \( c_2 < u_{y_2} < u_{y_1} \leq c_1 \). So, by applying Theorem 4.3 we infer:

\[
\|u_{y_2} u_{y_1} - c_2^2\|_{L^\infty(\bar{D}_{\gamma_2} \setminus \bar{D}_{\gamma_1})} \leq |c_1^2 - c_2^2|,
\]

which converges to zero. From (4.10), we obtain the desired result.

Similarly, when \( \gamma_1 \) decreases to \( \gamma_2 \), the ratio \( \frac{\Psi(\gamma_1)}{\Psi(\gamma_2)} \) converges to \( -(\alpha - \beta)c_1^2 \). By Theorem 4.3 we know that \( c_1 \) is continuous with respect to \( \gamma \). Hence, we infer that \( \Psi(\gamma) \) is continuously differentiable with \( \Psi'(\gamma) = -(\alpha - \beta)c_1^2 \) on \((0, |D|)\). \(\Box\)

### 4.2 Monotonicity and stability results with respect to \( \alpha \)

Assume that \( 0 \leq \beta < \alpha_1, \alpha_2 \leq 1 \). For each of \( \alpha_1 \) and \( \alpha_2 \), the minimization problem (4.2) has a unique solution, which we denote by \( \hat{\bar{D}}_{\alpha_1} \) and \( \hat{\bar{D}}_{\alpha_2} \), respectively. We know that \(|\bar{D}_{\alpha_1}| = |\bar{D}_{\alpha_2}| = \gamma \), and:

\[
\hat{\bar{D}}_{\alpha_1} = \{ x \in D : u_{\alpha_1}(x) > c_{\alpha_1} \} \quad \text{and} \quad \hat{\bar{D}}_{\alpha_2} = \{ x \in D : u_{\alpha_2}(x) > c_{\alpha_2} \},
\]

for some positive \( c_{\alpha_1} \) and \( c_{\alpha_2} \), where \( u_{\alpha_1} \) and \( u_{\alpha_2} \) satisfy:

\[
\begin{aligned}
-\Delta u_{\alpha_1} + (\alpha_1 \chi_{D_{\alpha_1}} + \beta \chi_{D_{\alpha_2}}) u_{\alpha_1} &= f, \quad \text{in } D, \\
u_{\alpha_1} &= 0, \quad \text{on } \partial D,
\end{aligned}
\]

and

\[
\begin{aligned}
-\Delta u_{\alpha_2} + (\alpha_2 \chi_{D_{\alpha_2}} + \beta \chi_{D_{\alpha_2}}) u_{\alpha_2} &= f, \quad \text{in } D, \\
u_{\alpha_2} &= 0, \quad \text{on } \partial D.
\end{aligned}
\]

This time, we restate the minimization problem (4.2), with \( \alpha \) as an input parameter:

\[
\Psi(\alpha) := \inf_{|\bar{D}| = \gamma} \int_D f u_{E,\alpha} \, dx = \int_D f u_{\alpha} \, dx.
\]

**Proposition 4.6.** If \( 0 < \alpha_1 < \alpha_2 \leq 1 \), then:

(i) \( c_{\alpha_1} > c_{\alpha_2} \).

(ii) \( u_{\alpha_1} > u_{\alpha_2} \) in \( D \).

(iii) \( \hat{\bar{D}}_{\alpha_1} \cap \hat{\bar{D}}_{\alpha_2} \neq \emptyset \).

**Theorem 4.7.** If \( \beta > 0 \) and \( \alpha_1 \) converges to \( \alpha_2 \) in \((\beta, 1]\), then \(|\hat{\bar{D}}_{\alpha_1} \cup \hat{\bar{D}}_{\alpha_2}| \) converges to zero.

**Proof.** Fix \( \beta < \alpha_2 \leq 1 \) and let \( \alpha_1 \) increase to \( \alpha_2 \). We claim that \( |\hat{\bar{D}}_{\alpha_1} \cup \hat{\bar{D}}_{\alpha_2}| \) converges to zero. First, let us introduce the following auxiliary boundary value problem

\[
\begin{aligned}
-\Delta \hat{u}_{\alpha_1} + (\alpha_1 \chi_{D_{\alpha_2}} + \beta \chi_{D_{\alpha_2}}) \hat{u}_{\alpha_1} &= f, \quad \text{in } D, \\
\hat{u}_{\alpha_1} &= 0, \quad \text{on } \partial D.
\end{aligned}
\]

(4.15)
From (4.13) and (4.15), we deduce:
\[
\begin{cases}
-\Delta \tilde{u}_{a_1} - u_{a_2} + (\alpha_1 \chi_{D_{a_1}} + \beta \chi_{D_{a_2}})(\tilde{u}_{a_1} - u_{a_2}) = (\alpha_2 - \alpha_1)u_{a_2}\chi_{D_{a_2}} & \text{in } D, \\
\tilde{u}_{a_1} - u_{a_2} = 0 & \text{on } \partial D.
\end{cases}
\]  
(4.16)

Since \(\alpha_2 > \alpha_1\), we infer that \((\alpha_2 - \alpha_1)u_{a_2}\chi_{D_{a_2}}\) is non-negative. So, by applying the strong maximum principle to (4.16), we obtain \(\tilde{u}_{a_1} > u_{a_2}\) in \(D\). Furthermore, by (4.11), we have:
\[
\hat{D}_{a_1} = \{ x \in D : \tilde{u}_{a_1}(x) > c_{a_2} \} \supseteq \{ x \in D : u_{a_2}(x) > c_{a_2} \} = \hat{D}_{a_2}.
\]  
(4.17)

Multiplying the differential equation in (4.16) by \(\tilde{u}_{a_1} - u_{a_2}\), integrating the result over \(D\), followed by an application of divergence theorem yields:
\[
\int_D |\nabla (\tilde{u}_{a_1} - u_{a_2})|^2 dx + \int_D (\alpha_1 \chi_{D_{a_1}} + \beta \chi_{D_{a_2}})(\tilde{u}_{a_1} - u_{a_2})^2 dx
\]
\[
= (\alpha_2 - \alpha_1) \int_D u_{a_2}(\tilde{u}_{a_1} - u_{a_2})\chi_{D_{a_2}} dx
\]
\[
\leq (\alpha_2 - \alpha_1) \|u_{a_2}\|_4 \|\tilde{u}_{a_1} - u_{a_2}\|_4 \|\hat{D}_{a_2}\|_4^{\frac{1}{2}}
\]
\[
\leq C(\alpha_2 - \alpha_1) \|u_{a_2}\|_{H^1_0(D)} \|\tilde{u}_{a_1} - u_{a_2}\|_{H^1_0(D)} \|D_{a_2}\|^{\frac{1}{2}},
\]  
(4.18)

where we have used general Hölder’s inequality in the first inequality, and Sobolev embedding theorem in the second inequality. Since the second term of the first line of (4.18) is non-negative, we obtain:
\[
\|\tilde{u}_{a_1} - u_{a_2}\|_{H^1_0(D)} \leq C(\alpha_2 - \alpha_1) \|u_{a_2}\|_{H^1_0(D)} \|D_{a_2}\|^{\frac{1}{2}}.
\]  
(4.19)

Noting that \(\alpha_1\) increases to \(\alpha_2\), we infer \(\tilde{u}_{a_1}\) converges to \(u_{a_2}\) in \(H^1_0(D)\). By using elliptic regularity theory and Sobolev embedding theorem, we infer \(\tilde{u}_{a_1}\) converges to \(u_{a_2}\) in \(C(D)\). So, from (4.17) and the fact that \(|\hat{D}_{a_2}| = \gamma\), in conjunction with Lemma 3.2, we deduce that \(|\hat{D}_{a_1} \setminus \hat{D}_{a_2}|\) decreases to zero, and
\[
|\hat{D}_{a_1}| \rightarrow \gamma^+.
\]  
(4.20)

On the other hand, from (4.12) and (4.15), we have:
\[
-\Delta (u_{a_1} - \tilde{u}_{a_1}) + (\alpha_1 \chi_{D_{a_1}} + \beta \chi_{D_{a_2}})(u_{a_1} - \tilde{u}_{a_1}) = (\alpha_1 - \beta)\tilde{u}_{a_1}(\chi_{D_{a_2}} \setminus \hat{D}_{a_2} - \chi_{D_{a_1}} \setminus \hat{D}_{a_1})
\]  
in \(D\),
\[
(4.21)
\]
with \(u_{a_1} - \tilde{u}_{a_1} = 0\) on \(\partial D\). Now, let us introduce the following subsets of \(D\):
\[
\hat{E} := \{ x \in D : u_{a_1}(x) - \tilde{u}_{a_1}(x) \leq c_{a_1} - c_{a_2} \},
\]
\[
\hat{F} := \{ x \in D : u_{a_1}(x) - \tilde{u}_{a_1}(x) > c_{a_1} - c_{a_2} \}.
\]

Using (4.11) and (4.17), we infer \(\hat{D}_{a_1} \setminus \hat{D}_{a_2} \subseteq \hat{F}\) and \(\hat{D}_{a_1} \setminus \hat{D}_{a_2} \subseteq \hat{E}\). Moreover, by (4.17), we have \(\hat{F} = (\hat{E})^c \subseteq (\hat{D}_{a_1} \setminus \hat{D}_{a_1})^c \subseteq (\hat{D}_{a_1} \setminus \hat{D}_{a_1})\). So, (4.21) leads to:
\[
-\Delta (u_{a_1} - \tilde{u}_{a_1}) + (\alpha_1 \chi_{D_{a_1}} + \beta \chi_{D_{a_2}})(u_{a_1} - \tilde{u}_{a_1}) = -(\alpha_1 - \beta)\tilde{u}_{a_1}(\chi_{D_{a_2}} \setminus \hat{D}_{a_2} + \chi_{D_{a_1}} \setminus \hat{D}_{a_1})
\]  
in \(\hat{F} \subseteq (\hat{D}_{a_2} \setminus \hat{D}_{a_1})\).  
(4.22)

Since \(c_{a_1} > c_{a_2}\) (by Proposition 4.6[4]), we have \(u_{a_1} - \tilde{u}_{a_1} = c_{a_1} - c_{a_2} > 0\) on \(\partial \hat{F}\). By applying the maximum principle to (4.22), we deduce \(u_{a_1} - \tilde{u}_{a_1} \leq c_{a_1} - c_{a_2}\) in \(\hat{F}\). Recalling the definition of \(\hat{F}\), we have \(\hat{F} = \emptyset\). Since \(\hat{D}_{a_1} \setminus \hat{D}_{a_2} \subseteq \hat{F}\), we infer \(\hat{D}_{a_1} \setminus \hat{D}_{a_2} = \emptyset\), i.e. \(\hat{D}_{a_1} \subseteq \hat{D}_{a_2}\). So, from (4.20) and the fact that \(|\hat{D}_{a_1}| = \gamma\), we deduce \(|\hat{D}_{a_1} \setminus \hat{D}_{a_2}|\) decreases to zero. Furthermore, recalling that \(|\hat{D}_{a_1} \setminus \hat{D}_{a_2}|\) decreases to zero, from (4.17) we have:
\[
|\hat{D}_{a_1} \Delta \hat{D}_{a_2}| = |(\hat{D}_{a_1} \setminus \hat{D}_{a_2}) \cup (\hat{D}_{a_2} \setminus \hat{D}_{a_1})| \leq |\hat{D}_{a_1} \setminus \hat{D}_{a_2}| + |\hat{D}_{a_1} \setminus \hat{D}_{a_1}| \rightarrow 0^+,
\]
when \(\alpha_1\) increases to \(\alpha_2\) as desired. Similarly, when \(\alpha_1\) decreases to \(\alpha_2\), with \(\beta < \alpha < 1\), we will have \(|\hat{D}_{a_1} \Delta \hat{D}_{a_2}|\) converging to zero. This completes the proof. □

**Theorem 4.8.** If \(\beta > 0\) and \(\alpha_1\) converges to \(\alpha_2\) in \((\beta, 1]\), then \(u_{a_1}\) converges to \(u_{a_2}\) in \(C(\hat{D})\).

**Corollary 4.9.** If \(\beta > 0\) and \(\alpha_1\) converges to \(\alpha_2\) in \((\beta, 1]\), then \(\Psi(\alpha_1)\) converges to \(\Psi(\alpha_2)\).
5 Numerical simulation

Numerical algorithms for solving rearrangement optimization problems have appeared in the literature (see, e.g., [5, 11, 10]). As there are no non-global local minima for problem (1.4), a simple gradient descent algorithm suffices. Thus, we do not discuss the details of the algorithm here.

Nonetheless, we highlight a few issues regarding numerical simulation of the problem (1.4). It is clear from the variational formulation (1.3) that the optimization problem (1.4) is a minmax one. Speeding up algorithms for rearrangement problems of this kind requires dealing with certain heuristics, which are discussed in detail by Kao and Su [16].

The optimization problem (1.4) of the current paper should be contrasted with (say) the optimal harvesting problem of [10], or the steady vortex problem considered in [3, 4]. Here are two major differences:

(1) Whereas the steady vortex and optimal harvesting problems can have uncountably many local optima and saddle points—which may only be partially overcome through the use of randomized algorithms [10]—problem (1.4) has no non-global local minima.

(2) On the other hand, the maxmax nature of the steady vortex problem and the minmin nature of the optimal harvesting problem provide for highly efficient algorithms that generate optimizing sequences. For problem (1.4), however, careful use of heuristics is needed.

Using an approach similar to that of [16], we have implemented an algorithm for the shape optimization problem (4.2). Figure 1 (generated by MATLAB®) illustrates one of our monotonicity results, as stated in Proposition 4.1(ii).

6 Proof of Lemma 3.3

(i) We follow the ideas in [7] (also, see [20]). Let \( \{g_n\} \subseteq \overline{R} \) and \( g \in \overline{R} \), such that \( g_n \rightharpoonup g \) in \( L^2(D) \). For simplicity, let us set \( u_n := u_{g_n} \) and \( u := u_g \). We have:

\[
\begin{aligned}
-\Delta u_n + g_n u_n &= f & \text{in } D, \\
 u_n &= 0 & \text{on } \partial D.
\end{aligned}
\]  

(6.1)

Multiplying the differential equation in (6.1) by \( u_n \), and integrating the result over \( D \), yields

\[ \int_D |\nabla u_n|^2 \, dx + \int_D g_n u_n^2 \, dx = \int_D f u_n \, dx. \]  

(6.2)

From Lemma 2.2, we know that \( g_n \) are non-negative. Therefore (6.2) implies

\[ \int_D |\nabla u_n|^2 \, dx \leq \int_D f u_n \, dx. \]  

(6.3)
By applying H"older’s inequality and the Poincaré inequality to the right hand side of (6.3) we obtain
\[ \int_D |\nabla u_n|^2 \, dx \leq C \|f\|_2 \|u_n\|_{H^1_0(D)}, \tag{6.4} \]
in which $C$ is a positive constant. Whence, $\{u_n\}$ is a bounded sequence in $H^1_0(D)$. This in turn implies existence of a subsequence of $\{u_n\}$, still denoted $\{u_n\}$, and $w \in H^1_0(D)$, such that:
\[ u_n \rightharpoonup w \quad \text{in } H^1_0(D) \quad \text{and} \quad u_n \to w \quad \text{in } L^2(D). \]
Let us prove that $w = u$, where $u$ is the solution of
\[ \begin{cases} -\Delta u + gu = f & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases} \tag{6.5} \]
Indeed, by (6.1) we have
\[ \int_D \nabla u_n \cdot \nabla \phi \, dx + \int_D g_n u_n \phi \, dx = \int_D f \phi \, dx, \quad \forall \phi \in C_0^\infty(D). \]
Since $u_n \to w$ in $H^1_0(D)$, $g_n \to g$ in $L^2(D)$, and $u_n \to w$ strongly in $L^2(D)$, from the latter equation we find
\[ \int_D \nabla w \cdot \nabla \phi \, dx + \int_D gw \phi \, dx = \int_D f \phi \, dx, \quad \forall \phi \in C_0^\infty(D). \]
This means that $w$ is a solution of (6.5), and by uniqueness, we must have $w = u$. To prove (i), we observe that
\[ |\Phi(g_n) - \Phi(g)| = \left| \int f(u_n - u) \, dx \right| \leq \|f\|_2 \|u_n - u\|_2 \]
which together with the fact that $\lim_{n \to \infty} \|u_n - u\|_2 = 0$ implies (i).

(ii) Let $h, g \in \overline{\mathbb{R}}$, $0 < t < 1$, and $\xi_t = th + (1 - t)g$. For $v \in H^1_0(D)$, we have
\[ 2 \int_D fv \, dx - \int_D |\nabla v|^2 \, dx - \int_D \xi_t v^2 \, dx = \]
\[ t \left( 2 \int_D f v \, dx - \int_D |\nabla v|^2 \, dx - \int_D hv^2 \, dx \right) \]
\[ + (1 - t) \left( 2 \int_D f v \, dx - \int_D |\nabla v|^2 \, dx - \int_D gv^2 \, dx \right) \tag{6.6} \]
By taking the supremum of (6.6) with respect to $v \in H^1_0(D)$, we obtain
\[ \Phi(th + (1 - t)g) \leq t\Phi(h) + (1 - t)\Phi(g). \tag{6.7} \]
This proves the convexity of $\Phi$. We now show, by contradiction, that $\Phi$ is in fact strictly convex. To this end, we assume that there exists $t \in (0, 1)$ such that $\Phi(th + (1 - t)g) = t\Phi(h) + (1 - t)\Phi(g)$. For simplicity, we use $u_t$ in place of $u_{th+(1-t)g}$. So, we have:
\[ 2 \int_D fu \, dx - \int_D |\nabla u|^2 \, dx - \int_D \xi_t u_t^2 \, dx = \]
\[ t \left( 2 \int_D fu \, dx - \int_D |\nabla u|^2 \, dx - \int_D hu_t^2 \, dx \right) \]
\[ + (1 - t) \left( 2 \int_D fu \, dx - \int_D |\nabla u|^2 \, dx - \int_D gu_t^2 \, dx \right) \tag{6.8} \]
From (6.8), we deduce the following equations:

\[ 2 \int_D fu_h \, dx - \int_D |\nabla u_h|^2 \, dx - \int_D hu_h^2 \, dx = 2 \int_D fu_t \, dx - \int_D |\nabla u_t|^2 \, dx - \int_D hu_t^2 \, dx, \quad (6.9) \]

and

\[ 2 \int_D fu_g \, dx - \int_D |\nabla u_g|^2 \, dx - \int_D gu_g^2 \, dx = 2 \int_D fu_t \, dx - \int_D |\nabla u_t|^2 \, dx - \int_D gu_t^2 \, dx. \quad (6.10) \]

From the maximality of \( u_h \) coupled with (6.9), we infer \( u_h = u_t \). Similarly, from the maximality of \( u_g \) and (6.10), we find \( u_g = u_t \). Hence, \( u_t = u_h = u_g \). On the other hand, from the differential equations

\[-\Delta u_h + hu_h = f, \quad \text{a.e. in } D,\]

and

\[-\Delta u_g + gu_g = f, \quad \text{a.e. in } D,\]

we infer \((h - g)u_h = 0\), almost everywhere in \( D \). Since \( u_h \) is positive by the strong maximum principle, we must have \( h = g \) almost everywhere in \( D \). Therefore, the strict convexity is proved.

(iii) For simplicity, we set \( u_t := u_{\xi_t} \). We know that:

\[ \begin{cases} -\Delta u_t + \xi_t u_t = f & \text{in } D, \\ u_t = 0 & \text{on } \partial D, \end{cases} \quad (6.11) \]

and

\[ \begin{cases} -\Delta u + gu = f & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases} \quad (6.12) \]

From (6.11) and (6.12), we obtain:

\[-\Delta(u_t - u) + g(u_t - u) = gu_t - \xi_t u_t = (g - \xi_t)u_t. \quad (6.13)\]

Multiplying (6.13) by \( u_t + u \), and integrating the result over \( D \), we get:

\[ \int_D |\nabla u_t|^2 \, dx - \int_D |u|^2 \, dx + \int_D gu_t^2 \, dx - \int_D gu^2 \, dx = \int_D (g - \xi_t)u_t(u_t + u) \, dx = -t \int_D (h - g)u_t(u_t + u) \, dx. \quad (6.14) \]

From (6.14), we derive \( \Phi(\xi_t) - \Phi(g) = -t \int_D (h - g)u_t u \, dx \), which in turn implies:

\[ \Phi(\xi_t) - \Phi(g) + t \int_D (h - g)u_t^2 \, dx = -t \int_D (h - g)u_t u \, dx + t \int_D (h - g)u^2 \, dx = -t \int_D (h - g)(u_t - u) u \, dx. \quad (6.15) \]

By applying Hölder’s inequality to the right hand side of (6.15), we find

\[ \left| \Phi(\xi_t) - \Phi(g) + t \int_D (h - g)u_t^2 \, dx \right| \leq t \||h - g||_\infty \|u_t - u\|_2 \|u\|_2. \quad (6.16) \]

Since \( \xi_t \to g \) weakly in \( L^2(D) \) (and even strongly), by the proof of part (i), we have \( \|u_t - u\|_2 \to 0 \) as \( t \to 0 \). Hence, dividing by \( t \) in (6.16) and letting \( t \to 0 \) we get the desired result. \( \Box \)
7 Concluding remarks

In the main result of the current paper, i.e., Theorem 3.4, we proved existence and uniqueness of solutions for an optimization problem arising in construction of robust membranes, with no restriction on the number of materials used. This is yet another witness to the power and elegance of the theory behind optimization of convex functionals over rearrangement classes, as laid out by Burton [2]. Although the theory was originally devised for studying vortex rings, i.e., in the context of fluid dynamics, ever since its introduction, there has been a steady flow of contribution to the theory and its applications, in fluid mechanics [6][5], finance [12][22], free boundary problems [8], population biology [10], and eigenvalue problems [9], to name a few.

For the particular problem considered in the current paper, we showed that there cannot be any non-global local optima (Theorem 3.6). This has to be contrasted with other rearrangement optimization problems where local optima and saddle points abound [3][4][11][10]. Furthermore, we managed to deepen our understanding of the problem through some stability results, which are, very difficult to prove, or even formulate, in the presence of symmetry breaking, such as those occurring in [11][10].

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