An Analytical Approach to Computing Blocking Probabilities in Flexi-Grid Optical Networks

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Abstract—Due to the analytical complexity of routing and spectrum allocation policies – random-fit and first-fit, in flexi-grid optical networks, it is necessary to analyze the approximate blocking probabilities. To this end, we present a novel model for reduced link state occupancy, whereby each state of a Markov chain represents the total occupied spectrum slices in a flexi-grid optical link. Using the independence link model, we further develop a method to calculate state-dependent per-class connection setup rates in multi-class flexi-grid optical networks. To alleviate the spectrum fragmentation problem, we extend the analysis with a defragmentation process as a reactive connection reconfiguration scheme, taking into account the probability of observing a link being in a fragmentation state, and show that a reactive defragmentation process can indeed reduce blocking considerably. We show that the defragmentation process is most effective under the medium load condition as analysed in a large scale flexi-grid optical networks. Through various numerical and simulation results we show that our model, with or without defragmentation, can be efficiently used to calculate accurate blocking probabilities in multi-class flexi-grid optical networks, as the approximate results match closely with the exact and simulation results under different scenarios, including varying link capacities, classes of demands, and traffic loads.

Index Terms—Flexi-grid optical networks, spectrum allocation, fragmentation, defragmentation, blocking probability, performance evaluation, approximation.

I. INTRODUCTION

Flexi-grid optical networks (FGONs) divide optical spectrum into flexible grids (slices), thus offering flexible and just-enough spectrum to variable bandwidth demands [1]. Flexible spectrum allocation generally reduces blocking and can increase spectrum utilization. At the same time, due to the flexible spectrum allocation, and in absence of the optimal spectrum allocation, connection requests in FGONs are blocked due to the fragmentation of spectrum occupancy, i.e., even though sufficient, but scattered, spectrum is available in the network, a connection request is blocked if there is no required number of continuous and contiguous slices available on the connection’s route to satisfy its bandwidth demand. The latter constraints, i.e., spectrum continuity and spectrum contiguity, are fundamental constraints for routing and spectrum allocation in FGONs. The spectrum continuity constraint is analogous to wavelength continuity constraint in wavelength routed wavelength division multiplexing (WDM) networks, which requires an incoming connection (optical channel) request to be provisioned over the same set of subcarrier slices in all links it traverses. The constraint called spectrum contiguity constraint means that a connection request demanding multiple subcarriers needs to be allocated over adjacent slices. These two constrains along with the resulting spectrum fragmentation have been subject to much research in optimal routing and spectrum allocation (RSA) schemes.

Since the spectrum continuity and contiguity constraints make the RSA in FGON an NP hard problem [2], the exact blocking analysis remains intractable [3]. To be precise, the availability of a free continuous and contiguous spectrum path in FGONs depends not only on the total occupancies on each link of the traversed route, but also on the exact locations of occupied slices. On the other hand, to overcome the intractability of blocking analysis in FGONs, it is important to depart from the exact spectrum representation model to the reduced state model describing total occupancy per link. It is analytically hard however to taking into account the effect of fragmentation while deriving the connection setup rate, i.e., the effective arrival rate at which a link allows connections to be setup in a given spectrum occupancy state. The reason is that a given number of occupied slices could be represented by fragmented as well as non-fragmented spectrum patterns, and the fragmented states could not accept an incoming connection request. The estimation of the connection setup rates taking into account the fraction of time a link stays in non-blocking states can be obtained by monitoring the link state occupancy over a long period of time, as shown by simulations by Reyes et al. [4]. In absence of such monitoring information, a tractable and relatively accurate approximate blocking analysis for the FGONs would be useful to calculate blocking probabilities and has not been addressed to date.

In this paper, we first develop an approach that allows us to depart from an exact link state occupancy model, i.e., where the location of spectrum occupancy is taken into account, to a simple link state occupancy model considering only the total number of occupied spectrum slices in a link. More importantly, we derive approximate state-dependent connection setup rates in a flexi-grid optical link (FGOL). Furthermore, we assume the link independency and compute the connection setup rates on all links and apply a recursive algorithm to compute approximate blocking in an FGON. Notably, we analyze blocking under two different spectrum allocation schemes: random fit (RF), which randomly allocates spectrum to connection requests; and first-fit (FF), which allocates first available slices to an incoming request. In addition to these
approach. Zhang et al. [4] proposed a method to depart from the exact state description to a reduced state (denoted by vertices. Recently, Reyes et al. [4] proposed a method to depart from the exact state description to a reduced state (denoted by connection setup rates, and a procedure to calculate blocking probability in optical networks. The blocking models for a link and a network without defragmentation (DF) are presented in Section VI. In Section VII we extend this model with the DF process and compute the blocking in a link and networks with DF. We evaluate the performance in Section VII and conclude the paper in Section VIII.

II. RELATED WORK AND OUR CONTRIBUTION

The analytical performance evaluation of connection blocking in FGONs has started only recently [3], [5], showing that RSA problem in FGONs is a very complex process to model analytically. This is due to spectrum continuity and contiguity constraints that also result in load correlation among spectrum slices and links as well as spectrum fragmentation. Moreover, the blocking is affected by the type of the RSA scheme used. In [5], an exact blocking probability of a link was evaluated for the first time by modeling the bandwidth occupancy as continuous-time Markov chain (CTMC) under the random-fit (RF) and first-fit (FF) spectrum allocation methods. The first step in calculating blocking probabilities in FGONs was given by Beyranvand et al. [3], which presented an approximation based on a Kaufman’s one-dimensional recursion formula [6] to calculate the link occupancy distribution recursively, without consideration of the RSA constraints. Authors used another approach to consider both RSA constraints based on the binomial distribution presented in [7], where it was shown that the accuracy of approximate blocking results deteriorates significantly with the increase in link capacity. Recently, [9] proposed an iterative procedure—a parallel birth-death process having different initial states, to approximate path request (connection) blocking in FGONs enabled with spectrum converters. Recently, Reyes et al. [4] proposed a method to depart from the exact state description to a reduced state (denoted by connections per class) model, by estimating the connection setup rates in non-blocking states using the link-simulation approach. Zhang et al. in [9] proposed both proactive and reactive, sequential and parallel defragmentation schemes based on integer linear programming in FGONs. However, an analytical model for computing approximate blocking with defragmentation schemes is still missing in the literature.

Our contribution in this paper is two fold. First, we provide an approximation to translating exact Markov model to a Microstate model that only keeps the information of total occupied slices per link, which is essential in reducing the complexity of the Markov chain. It should be noted that the accuracy of the connection setup and departure rates dictates the accuracy of approximate blocking probability in the Microstate model. Second, we consider the spectrum fragmentation factor in the performance evaluations, by calculating the effective arrival (connection setup) rate per state. We propose an approximation taking into account the fragmentation factor to calculate the connection setup rates without observing the link states through any real simulation. As spectrum fragmentation plays an important role in the blocking performance of FGONs, in our prior works [10], [11] we evaluated a spectrum reconfiguration strategy, called spectrum defragmentation (DF), to reallocate spectrum to connections. In [10], we modelled a reactive DF in a link for the first time, and showed that the overall blocking can be reduced if reconfiguration time is much lower than the connections’ inter-arrival times and service times. A framework to model the DF process in a generalized form that allows both proactive and reactive triggering mechanism for reconfiguration was considered in [11] for a single hop network using the multiclass CTMC model. However, we assumed that all arrivals are blocked during the connection reconfiguration process, and also services of active connections are interrupted. Although these two assumptions made the DF model simpler, they resulted in reducing the gain obtained by defragmenting the spectrum. In contrast, the present paper relaxes these assumptions to evaluate the blocking performance in a DF-enabled network.

It should also be noted that we adopt a multirate loss framework developed by Birman [12], and later generalized for the multi-class scenario in WDM networks by Kuppuswamy et al. [13]. The multirate loss model for WDM mainly uses an independence link assumption that spectrum occupancies of links are statistically independent, and a reduced-load approximation approach, which calculates the effective load of the combined streams (connections) on a link. We use these two approaches to compute per-class approximate blocking probabilities in a multi-class FGON. However, it is important to note that the WDM approaches cannot be applied directly to FGONs, since the spectrum fragmentation resulted by the RSA constraints require the connection setup rates to be calculated differently taking into account by fragmented and non-fragmented states, which enable us to model a reactive DF process to reconfigure existing connections and accept incoming requests even in fragmented states, thus leading to lower blocking in a DF-enabled FGONs. We propose an approximation to calculate the probability of acceptance of a new connection request in a given occupancy state, which is later used to compute connection setup rates. As the approximation is based on the uniform distribution assumption, which distributes the steady state link occupancy probability equally among all possible corresponding exact states, it does not handle fragmented and non-fragmented exact states differently. This is the main drawback of our model, and we also analyze its effect on the approximate
possible spectrum occupancies (states) of a link with capacity $C = 7$ slices, demands $d_k = \{3, 4\}$ slices under RF policy.

| Exact State Description | Macrostate | Microstate |
|-------------------------|------------|------------|
| $s = (s_1, s_2, s_3, s_4, s_5, s_6, s_7)$ | $n = (n_1, n_2)$ | $X = x = d \cdot n^T$ |
| $(0, 0, 0, 0, 0, 0, 0)$ | $(0, 0)$ | $0$ |
| $(1, \infty, \infty, 0, 0, 0, 0)$ | $(1, 0)$ | $3$ |
| $(0, 1, \infty, \infty, 0, 0, 0)$ | $(0, 1)$ | $4$ |
| $(0, 0, 1, \infty, \infty, 0)$ | $(2, 0)$ | $6$ |
| $(0, 0, 0, 1, \infty, \infty)$ | $(1, 1)$ | $7$ |


table I

blocking probability, especially for FF spectrum allocation policy. This paper also presents the analytical models for a link and a network separately, mainly because the link model requires only the contiguity constraint to be fulfilled, which enables us to derive a more accurate probability of acceptance considering the contiguity constraint and fragmentation factor. In the network case, on the other hand, deriving the probability of acceptance is a challenge, as one needs to consider both RSA constraints and the fragmentation factor that depends on the classes of demands and link occupancy. Thus, in order to calculate an approximate probability of acceptance of a connection path request, we assume that the occupancies of links are statistically independent, and they are uniformly distributed among slices on each link. While these assumptions are common, they are strong and critical in any network blocking analysis, as they ignore the load correlation factor among spectrum slices and links. At the same time, nonetheless, the approximate blocking results we obtain in this paper are promising, as they match closely to the exact, or simulation results for most of the traffic loads and classes of demands. More precisely, these assumptions work well for the RF spectrum allocation policy, even though not for the FF policy. This is because that under the FF policy, the spectrum allocation over slices is not uniform, and thus the variance of exact states’ probabilities representing an occupancy state is quite significant, as we will show in the paper.

III. REDUCED STATE DESCRIPTION AND APPROXIMATION

In this Section, we first present the reduced link state representation in order to tackle the intractability of exact blocking analysis, and identify blocking and non-blocking (NB) exact states for a given link occupancy. Then, using the definition of NB states we derive an approximate probability of acceptance of a connection request in a link, which is used later to derive the connection setup rates for a link. We also test the accuracy of our approximation by comparing it with exact probability of acceptance under the RF and FF spectrum allocation policies. The notations and definitions of some of the parameters used in the model are listed in Table II.

A. Complexity of the Exact Solution

Let us first highlight the complexity involved in the exact solution by an example, and how a reduced state model could simplify it. Let us consider an example of 7-slice fiber link, and two classes of demands $d_k = \{3, 4\}$ consecutive slices. We adapt the link state occupancy representation proposed in [5], where 0 represents a free slice, and 1 followed by infinities ($\infty$) represents the first occupied slice and the remaining ones as required by a class $k$ connection bandwidth. For example, an empty state of a 7-slice fiber link without any connection is represented by $(0, 0, 0, 0, 0, 0, 0)$, and when a new connection request with bandwidth demand $d_1 = 3$ consecutive slices arrives, then under the RF policy spectrum can be allocated in one of the five different ways, as shown in the first column by the rows corresponding to total occupied slices (macrostates) $x = 3$ in Table I. On the other hand, under the FF spectrum allocation policy, a new arrival ($d_3 = 3$ slices) in an empty state will trigger the system transition to only one exact state $s = (1, \infty, \infty, 0, 0, 0, 0)$, where first 3 slices are allocated. Note that the exact state space $\Omega_S$ for a given classes of demands and link capacity vary based on the spectrum allocation policy. Table II lists all possible exact states of a 7-slice fiber and demands $d_k = \{3, 4\}$ slices under the RF spectrum allocation policy. In this simple example, there are 15 exact states, and as the link capacity increases, it will be impossible to compute the blocking probability using the exact state representation due to computational complexity. However, the number of states can be reduced considerably if we represent the link occupancy by the number of connections per class $k$ ($n_k$) that it serves, which we call a macrostate $n = n(s) = (n_1, n_2, \cdots, n_K)$ as a $K$-dimensional vector, where $K$ is the total number of connection classes. Note that the number of macrostates in Table II is only 5. Generally, the number of states could be further reduced by associating a link state with the number of spectrum slices it occupies, which we call a microstate $X = x = \sum_{k=1}^{K} n_k d_k$.

In this illustrative example, the number of macrostates and microstates are same, as shown in the second and third columns of Table II. However, generally adopting microstate representation of the link state occupancy further reduces the number of states, thus the complexity of blocking analysis. For example, even in a small-scale link with 20 slices and bandwidth demands $d_k = \{3, 4, 5\}$, under the RF policy the number of exact states is 5885, which could be reduced to 51 and 19 with macrostates and microstates representations, respectively. In this paper, we use Exact, Macrostate, and Microstate models to obtain blocking probabilities, where link occupancies are represented by exact states, macrostates and microstates, respectively. Furthermore, the term “state” is also used in the context of the models, i.e., a state in the Microstate model has the same meaning as a microstate.

Departing from the exact state representation to a macro/micro state representation causes some inaccuracy in finding the connection setup rates in reduced link state models.
TABLE II
Notations and the parameters used in the models

| Notation | Description |
|-----------|-------------|
| $C$       | Total number of spectrum slices (or capacity units) per link |
| $K$       | Number of connection classes. Note that classes $k = 1, 2, \ldots, K$ |
| $X_k^c$   | Arrival rate of class $k$ connections on an OD pair $o \in \mathcal{O}$ |
| $\mu_k$   | Service rate of a class $k$ connection; mean service time $t_z = 1/\mu_k$ |
| $\mu_d$   | Mean defragmentation rate, $t_F = 1/\mu_d$; mean reconfiguration time |
| $d$       | $\equiv \{d_1, d_2, \ldots, d_K\}$, where $d_k$: # class $k$ bandwidth (in slices) |
| $n$       | $\equiv \{n_1, n_2, \ldots, n_K\}$, where $n_k$: number of class $k$ connections |
| $n(x)$    | $\equiv \{n_1(x), \ldots, n_K(x)\}$, realization of $n$ with occupancy $x$ |
| $n_k(n)$  | Number of class $k$ connections in $n$ |
| $o_j^l(x_j)$ | Class $k$ connection setup rate on link $j$ with $x_j$ occupied slices |
| $r(o)$    | Route of an OD pair $o \in \mathcal{O}$ consisting of some links $j \in J$ |
| $\Omega_S(x)$ | Set of exact states $s$ representing total occupancy of $x$ slices |
| $E(n)$    | Number of free slices in $n = C - d \cdot [n]^2$; $E(x) = C - x$ |
| $f_m(s)$  | Size of the largest block of consecutive free slices in state $s$ |
| $\text{NB}(x,k)$ | Set of NB exact states with occupancy of $x$ slices for class $k$ requests |
| $\text{FB}(x,k)$ | Set of FB exact states with occupancy of $x$ slices for class $k$ requests |

The reason is that a macro-/micro- state could be represented by different class-dependent blocking and non-blocking (NB) exact states. For example, a macrostate with occupancy $x = 3$ slices is represented by five different exact states out of which one state $(0, 0, 1, \infty, 0, 0)$ is a blocking state for both classes of demands (3 and 4 slices), and $(0, 1, \infty, 0, 0, 0)$ and $(0, 0, 1, \infty, 0, 0)$ are additional blocking states for a 4-slice demand. Let us first define a blocking state for an incoming class $k$ request in Eq. (1), which can not admit a demand $d_k$ due to the fact that the size of the largest consecutive free slices ($f_m(s)$) is not sufficient.

$$\mathcal{B}(x, k) = \{s|d_k > f_m(s), s \in \Omega_S(x)\}$$

Therefore, the exact NB states can be given by

$$\text{NB}(x, k) = \Omega_S(x) \setminus \mathcal{B}(x, k).$$

The blocking in a link happens either due to insufficient free spectrum, called as resource blocking (RB) or due the fragmentation of free spectrum resources, called as fragmentation blocking (FB). Fragmentation blocking states ($\text{FB}(x,k)$) do have enough free slices, but they are scattered and the largest block of consecutive free slices ($f_m(s)$) cannot satisfy demand $d_k$. Thus, a class-dependent set of FB states corresponding to a macrostate $X = x$ is given by

$$\text{FB}(x,k) = \{s|f_m(s) < d_k \leq C - x, s \in \Omega_S(x)\}$$

The set of resource blocking states $\mathcal{R}\mathcal{B}(x,k) \subseteq \mathcal{B}(x,k)$ is $\mathcal{B}(x,k) \setminus \text{FB}(x,k)$. Note that when all classes are taken into account, then using the union operation, for example, we have

$$\mathcal{F}(x,k) = \bigcup_k \mathcal{F}(x,k).$$

B. Probability of Acceptance of a Connection in a Link

Let us now use the above definitions to reduce an exact link state model into a reduced macrostate model. In the reduced model the transition rate from a macrostate $X = x$ to another macrostate due to an arrival of a class $k$ request (i.e., connection setup rate) depends on the connection arrival rate and the probability of its acceptance. Noting that only NB exact states corresponding to the macrostate $X = x$ will accept the incoming request, in a single link system (route $r = j$) the probability of acceptance of a class $k$ connection request with bandwidth $d_k$ in a given occupancy (microstate) $X = x$, i.e., $p_k(x)$ is obtained by

$$p_k(x) = Pr[|Z_r| \geq d_k|X = x] = \sum_{s \in \Omega_S(x)} \text{Pr}[f_m(s) \geq d_k|s, X = x] \times \text{Pr}[s|X = x]$$

where the event $\{Z_r \geq d_k\}$ represents that the route $r$ (here a link $j$) must have equal or more than $d_k$ consecutive free slices to accept a class $k$ request. In a given microstate $X = x$, only a subset of exact states representing a microstate $x$ that have sufficient consecutive free slices would accept the class $k$ request $(\forall s \in \Omega_S(x))$. The first multiplication term in (4) is a probability function resulting in a value 1 if a state $s$ is a NB state, 0 otherwise. The second term is the probability of observing the link in an exact state $s$ among the set of exact states representing occupancy of $x$ slices, i.e., $\Omega_S(x)$. However, the calculation of exact state probabilities (for the second term) in a large link is analytically intractable. We need, therefore, some kind of approximation to calculate the class- and state-dependent probability of acceptance and connection setup rates. Assuming that all exact states corresponding to a given microstate have uniform state probability distribution, i.e., $Pr[s|X = x] = 1/|\Omega_S(x)|$, $\forall s \in \Omega_S(x)$, where $|\Omega_S(x)|$ is the number of exact states representing the microstate $x$. As only NB exact states would allow a connection to be accepted, therefore, the first multiplication term in (4) would add up to the total number of exact NB states representing a occupancy $x$. Thus, the probability of acceptance in (4) can be approximated as

$$p_k(x) = \frac{|\text{NB}(x,k)|}{|\Omega_S(x)|}$$

where $|\text{NB}(x,k)|$ is the number of exact NB states for class $k$ in the microstate $X = x$. The state-dependent per-class connection setup rate in a link is given as the class $k$ arrival rate $\lambda_k$ multiplied by the probability of acceptability of an incoming demand $d_k$ in a microstate $x$, i.e., $\alpha_k(x) = \lambda_k \times p_k(x)$.

To illustrate the transitions and connection setup rates, we consider an example in Fig. 1a where the state transition diagram of a 7-slice link occupancy is shown with two classes of demands $d_k = \{3, 4\}$ slices using an exact state representation in Fig. 1a and a macrostate representation in Fig. 1b under the RF policy. As can be seen in Fig. 1a, a class-1 (class-2) request arrival in an empty state $(0, 0, 0, 0, 0, 0, 0)$, i.e., $n = (0, 0)$ or $x = 0$ will lead the system to one of the 5 (4) possible states with rate $\lambda_1/5$ ($\lambda_2/4$) corresponding to $n = (1, 0)$ ($n = (0, 1)$). Since the empty state is an NB state for both connection classes, the overall connection setup rate in the empty state is $\lambda_1$ and $\lambda_2$ for class-1 (3-slice) and class-2 (4-slice) connection request, respectively, as seen in Fig. 1b. However, in the microstate $x = 3$, which represents 5 different exact states of $n = (1, 0)$, four (two) states are NB for class
$k = 1$($k=2$). Using the uniform steady state distribution assumption, the connection setup rate for class-1 (class-2) in the microstate $x = 3$ is $\frac{1}{2}\lambda_1$ ($\frac{2}{3}\lambda_2$), as shown in Fig. 1(b). It should be noted that we use this assumption for modeling a link under RF and FF spectrum allocation policies. Moreover, this assumption will be used when we calculate departure rate in Sec. IV-B. The connection setup and departure rates and the probability of acceptance of a connection path request in a multi-hop scenario have been discussed in the subsequent sections.

The spectrum allocation in a 7-slice fiber link under the FF policy is depicted in Fig. 1(c). The main difference between the states generated under both policies is that in contrast to RF, FF does not have all possible spectrum occupancy patterns, and in fact has a fewer number of exact states. In this example, there is only a single exact state $s = (0, 0, 0, 1, \infty, \infty, 0)$ that blocks a class-2 demand, and taking this into the account and using (5) the class-2 connection setup rate in the microstate $x = 3$ is $\alpha_2(x = 3) = \frac{2}{3}\lambda_2$ as shown in Fig. 1(d), since $x = 3$ is represented by three exact states, and two of them are NB for class-2 demand. To test the uniform state probability distribution assumption under both spectrum allocation policies, in Fig. 2 we plot the average probability of acceptance $p(x) = \sum_{x \in \mathcal{E}} p_k(x)$ considering all classes: using an exact method in Eq. (4) which requires the knowledge of exact state probabilities; and using our approximation in Eq. (5), which only requires the number of NB and blocking states per microstate per class. As can be seen, the approximate $p(x)$ follows its exact value in the RF policy. Under the FF policy, on the other hand, the deviation between the two is significant for some states ($10 \leq x \leq 17$). Thus, the model accuracy could be compromised under the FF policy.

IV. COMPUTING APPROXIMATE BLOCKING PROBABILITIES

In this Section, we present the analysis, as adopted for FGONs from the known models for circuit-switched optical networks [12], [13], to compute approximate blocking probabilities in FGONs. We list all assumptions, calculate the connection setup and departure rates, and present a well-known fixed-point iterative procedure to compute blocking.

A. Assumptions

To compute the blocking in an arbitrary FGON topology with $N$ nodes, $J$ unidirectional fiber links (belongs to set $\mathcal{J}$), and $C$ spectrum slices per link (link capacity), below we list...
all assumptions related to traffic and routing policies in our analytical model.

- Arrivals of class \( k \in \{1, 2, \cdots, K\} \) connection requests on an origin-destination (OD) node-pair \( o \in O \) follow Poisson process with arrival rate \( \lambda_k^o \), and connection holding (service) time is exponentially distributed with mean \( 1/\mu_k \). We assume that the arrivals and departures are statistically independent.

- Each OD pair connection request is routed on a predetermined shortest path, and we apply one of the two spectrum allocation policies: RF and FF. While the RF policy is easy to model, FF is very complex to model analytically with accuracy. Therefore, we mention FF explicitly whenever a different approach is required.

- An OD connection request \( o \) with bandwidth demand of \( d_k \) slices is accepted in an FGON iff there are sufficient (\( \geq d_k \)) contiguous and continuous free slices on its predetermined route \( r(o) \).

- Spectrum occupancy in a link \( j \) is independent from other links \( i \neq j; i, j \in J \), which is called the independence link assumption.

B. Calculating State Probabilities, Connection Setup and Departure Rates in FGONs

Let \( X_j \) be the random variable representing the number of occupied slices (\( x_j \)) on link \( j \). We define the probability that the link \( j \) is in state \( x_j \) as

\[
\pi_j(x_j) \equiv Pr[X_j = x_j].
\]

(6)

Following the link independence assumption, the random variables \( X_j \)'s are independent, i.e., \( Pr[X_j = x_j | X_i = x_i] = Pr[X_j = x_j], i \neq j \). Let \( Z_r \) be the random variable representing the size of the largest continuous and contiguous free slices on route \( r \). The connection setup rate, which is a function of state, class, and spectrum allocation policy, can be given by (7), with the assumption that the time until the next connection is setup on a link \( j \) with \( x_j \) occupied slices is exponentially distributed with parameter \( \alpha_k^x(x_j) \) as in [13], [14].

\[
\alpha_k^x(x_j) = \sum_{x \in \{0, \cdots, d_k \}} \lambda_k^o Pr(Z_r \geq d_k | X_j = x_j)
\]

(7)

In (7), the summation takes into account the effective arrival rates of all OD pairs \( o \in O \) whose routes \( r(o) \) pass through link \( j \). It should be noted that the effective (reduced) load contribution of an OD pair \( o \) on the link \( j \) depends on the continuous and contiguous free slices available on its route \( r(o), j \in r(o) \). Additionally, as we will see in Sec. V-B the term \( Pr(Z_r \geq d_k | X_j = x_j) \) is a function of link state probabilities \( \pi_j(x_j) \) for a given number of occupied slices \( x_j \), thus the connection setup rate \( \alpha_k^x(x_j) \) is a function of \( \pi_j(x_j) \), and finding the probability that a link \( j \) is in the state \( x_j \) (i.e., \( \pi_j(x_j) \)) requires the information of \( \alpha_k^x(x_j) \). Thus, these coupled equations are generally solved iteratively [12], [13], as we also adopt the same, to obtain blocking probabilities which depend on the carried traffic in FGONs. However, as a special case, due to our assumption of uniform exact state probabilities corresponding to an occupancy state \( x_j \), in a single-hop network with route \( r = \{j\} \), we can write \( \lambda_k^o = \lambda_k, x_j = x \) and \( Pr(Z_r \geq d_k | X_j = x_j) = p_k(x) \) using (6) and (5), the connection setup rates in (7) become independent of the link state probabilities, thus making them uncoupled. The expected departure rate of a class \( k \) connection in state \( x_j \) is obtained by

\[
\gamma_k^x(x_j) = \mu_k \times E[\eta_k | X_j = x_j]
\]

\[
= \mu_k \times \frac{1}{|n(x)|} \sum_{n \in n | x_j} \eta_k(n)
\]

(8)

where \( E[\eta_k | X_j = x_j] \) is the expected number of class \( k \) connections in the microstate \( x_j \), which is given by assuming all \( n \) that results into the same \( x_j \) (i.e., \( n(x) \)) have uniform distribution \( \frac{1}{|n(x)|} \), and \( \eta_k(n) \) is the number of class \( k \) connections in \( n \). In Fig. 1(b) the transition rate from the state occupancy \( x \) to \( x = 6 \) to \( x = 3 \) is \( 2\mu_1 \), since the transition occurs due to the departure of a class-1 connection (3 slice bandwidth), and the expected number of class-1 connections in \( x = 6 \) is 2, because \( x = 6 \) is represented by only one \( n = (2, 0) \) as shown in Table I.

C. Algorithm for Computing Blocking Probabilities in FGONs

The calculation of blocking probability per class per OD pair (\( BP_k^o \)) requires the information of steady state link occupancy probabilities (\( \pi_j(x_j) \)) of each traversed link of route \( r(o) \). In a single-hop network (link) the approximate link occupancy distribution can be obtained in a single step: using our uniform exact state distribution assumption which makes \( \alpha_k^x(x_j) \) independent of \( \pi_j(x_j) \); and solving (10) (or (17) and (18) for DF) to obtain \( \pi_j(x_j) \). In a multi-hop FGON, on the other hand, the probability that a class \( k \) connection on route \( r(o) \) is blocked can be calculated by finding \( \pi_j(x_j) \) through solving the nonlinear coupled equations (10) (or (17) and (18) for DF), as described in Sec. VI. However, these nonlinear coupled equations, which are a function of \( \alpha \) and \( \pi \), could be made linear by repeated substitution or iterative procedure as follows [13].

1) For all classes \( k \in \{1, 2, \cdots, K\} \) and OD pairs \( o \in O \), initialize blocking probabilities \( BP_k^o = 0 \), and set \( \alpha_k^x() \) for each link \( j \in J \) as \( \sum_{x \in \{0, \cdots, d_k \}} \lambda_k^o p_k(x) \) as described in Sec. IV-B.

2) Determine the link state occupancy distribution \( \pi_j = (\pi_j(x_0), \cdots, \pi_j(x_C)) \) for each link \( j \in J \) by solving \( \pi_j \cdot Q_j = 0 \) and \( \sum_{x=0}^C \pi_j(x_j) = 1 \) using LSQR method [15] or successive over-relaxation [16]. Here, \( Q_j \) is the transition rate matrix formed by the connection setup rates \( \alpha_k^x() \) and the expected departure rates \( \gamma_k^x() \). When the DF is considered, form \( \pi_j \) and \( Q_j \) using all states, including the DF states \( x_h \), and solve \( \pi_j \cdot Q_j = 0 \) and \( \sum_{x=0}^C \pi_j(x_j) = 1 + \sum_{x=0}^C \pi_j(x_h) \cdot \pi_j(x_h) = 1 \).

3) Calculate state-dependent per-class connection setup rate \( \alpha_k^x() \pi_j \forall j \in J, \forall k \in \{1, 2, \cdots, K\} \) using (7). The term \( Pr(Z_r \geq d_k | X_j = x_j) \) in (7) is calculated separately for the case of link-wise modeling, network-wise modeling, with and without defragmentation in Sec. V and Sec. VI respectively.
4) Calculate $BP_k^o$ for all OD pairs $o$ and classes $k$ in a link using (11), and using (14) in an FGON. In a DF-enabled link and network, use (19) and (21) respectively.

5) If $\max_{x,k} |BP_k^o - BP_k^r| < \epsilon$ then terminate. Else, let $BP_k = BP_k^o$ and go to step (2).

V. Modeling Blocking in FGONs

This section first presents the blocking analysis in a single-hop network (link), and then the blocking analysis in FGONs is given.

A. Link Model

Here, we present the technique for calculating approximate blocking in a single (unidirectional) fiber link $j$ considering a single OD pair $o$ with route $r(o) = \{j\}$ and class $k$ arrival rate $\lambda_k^o = \lambda_k$. Using the uniform state probability distribution assumption in (5), the connection setup rate is given as follows (omitting the subscript $j$ from all notations for a link).

$$\alpha_k(x) = \lambda_k \times \frac{\#NB(x, k)}{\#TS(x)}.$$  (9)

In Appendix A using the inclusion-exclusion principle we derive the number of exact NB states ($\#NB(x, k)$) and the number of all possible exact states ($\#TS(x)$) corresponding to a microstate $X = x$ under the RF spectrum allocation policy, and we also provide an Algorithm [1] to compute them for the FF or other spectrum allocation policies.

Before we calculate blocking probability (BP), we need to find out the steady state link occupancy distribution $\pi(x), 0 \leq x \leq C$, which can be obtained by solving a set of global balance equations (GBEs) with a normalizing condition $\sum_{x=0}^{C} \pi(x) = 1$. The GBE of a microstate $X = x$ is obtained by (10).

$$\sum_{k=1}^{K} (\alpha_k(x) + \gamma_k(x)) \pi(x) =$$

$$\sum_{k=1,d_k \leq x \leq C} \alpha_k(x - d_k) + \sum_{k=1,0 \leq x \leq C-d_k} \gamma_k(x + d_k)$$  (10)

In (10), left hand side (LHS) represents the output flow rate from the microstate $X = x$ taking into account the connection set up rates $\alpha_k(x)$ and departure of connection(s) with expected rate $\gamma_k(x)$, while the right hand side (RHS) represents input flow rate into the microstate $x$ from other states $x - d_k$ $(x + d_k)$ due to an arrival (departure) of a class $k$ connection demand of $d_k$ slices. Thus, for example, using (10) the GBE of a microstate $x = 3$ in Fig. 1(b) can be written as

$$(\frac{4}{2} \lambda_1 + \frac{2}{2} \lambda_3 + \mu_1) \pi(x_3) = \lambda_1 \pi(x_0) + 2 \mu_1 \pi(x_6) + \mu_2 \pi(x_7),$$

where suffix in $x$ i.e., $x_s$ represents the number of occupied slices. Here, the LHS of the GBE of the state $x = 3$, i.e., $x_3$ takes into the account of a 3-slice (4-slice) demand arrival in $x_3$ with effective connection setup rate $4 \lambda_1 / 5$ $(2 \lambda_3 / 5)$, and the RHS terms are due to an arrival in $x_0$, and departures in states $x_6$ and $x_7$. As described in Sec. IV-C, the LSQR method [15] or successive over-relaxation [16] methods can be used to solve the above linear equations (10) for all microstates $0 \leq x \leq C$, and the steady state link occupancy distribution $\pi(x)$ can be obtained for all microstates in a single step, since $\alpha_k(x)$ does not depend on the $\pi(x)$ in the reduced link model.

The class $k$ BP in a single link system can be obtained by (11).

$$BP_k = Pr[Z_r < d_k] = 1 - Pr[Z_r \geq d_k]$$

$$= 1 - \sum_{0 \leq x \leq C-d_k} \pi(x) Pr[Z_r \geq d_k | X = x]$$

$$= 1 - \sum_{0 \leq x \leq C-d_k} \pi(x) \frac{\#NB(x, k)}{\#TS(x)}$$  (11)

B. Network Model

Unlike the single-hop network, where only spectrum contiguity constraint needs to be taken care of in the blocking analysis, the spectrum must be allocated on a multi-hop path in an FGON (assuming that there is no spectrum converters, and traffic splitting is not allowed) with both spectrum contiguity and continuity constraints being satisfied. In this section we model the spectrum occupancy of links $j \in J$ belonging to a given route $r$ in order to calculate the blocking probability of connection requests $o$ opting for that route ($r(o)$). The very first step in calculating blocking probability in FGONs is to find per-class state-dependent connection setup rate in (7), which requires the term $Pr[Z_r \geq d_k | X_j = x_j]$ to be calculated. Let us consider a 2-hop route $r = \{j_1 = j, j_2\}$. Then, this term is given as follows.

$$Pr[Z_r \geq d_k | X_j = x_j]$$

$$= \sum_{x_{j_2}=0}^{C} Pr[Z_r \geq d_k | X_j = x_j, X_{j_2} = x_{j_2}]$$

$$= \sum_{x_{j_2}=0}^{C-d_k} \pi_{j_2}(x_{j_2}) \times Pr[Z_r \geq d_k | X_j = x_j, X_{j_2} = x_{j_2}]$$  (12)

Note that the random variables $X_j$ and $X_{j_2}$ are independent so $Pr[X_{j_2} = x_{j_2} | X_j = x_j] = Pr[X_{j_2} = x_{j_2}] = \pi_{j_2}(x_{j_2})$. In general, the above term can be calculated as in (12), (13), for an OD pair traversing route $r = \{j_1 = j, j_2, j_3, \ldots, j_l\}$ with $l$ hops, using (13).

$$Pr[Z_r \geq d_k | X_j = x_j] = \sum_{x_{j_l}=0}^{C-d_k} \cdots \sum_{x_{j_1}=0}^{C-d_k} \pi_{j_l}(x_{j_l}) \cdots \pi_{j_1}(x_{j_1})$$  (13)

In (13) the term after multiplication, which we call the probability of acceptance of a connection path request with demand $d_k$ as $p_k(x_r) = Pr[Z_r \geq d_k | X_j = x_j, X_{j_2} = x_{j_2}, \ldots, X_{j_l} = x_{j_l})$ is given by (28) in Appendix A. Note that $x_r = (x_j, x_{j_2}, \ldots, x_{j_l})$.

Using (13), BP of class $k$ bandwidth requests on an OD pair...
With equal probability, thus omitting the need of some of the unnecessary DF states as shown by green dotted arrows in Fig. 3(a). Fig. 3(b) shows a reduced DF link model, wherein three DF occupancy states and related transition are included in addition to regular occupancy microstates (see also Fig. 1(b)). Similar to the exact DF model, a DF microstate is represented by \((x_s, x_w)\), where \(x_s\) and \(x_w\) are the serving bandwidth (in slices) and additional bandwidth required by the waiting connection requests, respectively. As described in the previous section, in the approximate model, the most important thing is to derive the connection setup rate per class per microstate. Unlike the network case (explained in Section VI-B), a link ensures that a demand which could not be accepted due to fragmentation of spectrum gets accepted after the DF operation. Therefore, in our reduced DF link model, the probability of acceptance of a class \(k\) request in a microstate \(x\) is given by

\[
p_k(x) = Pr[Z_r \geq d_k | X = x]
= \frac{[NB(x,k)] + [FB(x,k)]}{\Omega_S(x)}. \tag{15}
\]

Note that we again assume that all exact states corresponding to a microstate have uniform distribution. Using (15), the connection setup rate in a microstate \(x\) is

\[
\alpha_k(x) = \lambda_k \times p_k(x).
\]

The arrival of a class \(k\) request in a microstate \(x\) triggers the system to transit to a DF state \((x_s, x_w)\) with rate \(\lambda_k \times [FB(x,k)] / \Omega_S(x)\). The connection As described before, a new arrival, departure, or completion of a DF process would transit the system to the respective microstates with rates \(\lambda_k, \mu_k\) and \(\mu_d\), respectively. For example, in Fig. 3(b) a class-2 arrival in a microstate \(x = 3\) would lead the system to a regular occupancy state with probability 2/5 and to a DF state \((x_s = 3, x_w = 4)\) with probability 3/5. Notably, this process eliminates fragmentation blocking in a single hop network (a link), and blocking is only due to resource unavailability. However, this gain is obtained at the cost of delaying the admission of some of the connections. Let us denote a DF microstate \(x_d = (x_s, x_w)\), and define the following transition function to help map a DF microstate \(x_d\) to a target microstate \(x\).

\[
\beta(x_d, x) = \begin{cases} 
\mu_d & \text{if } x = x_s + x_w \\
\mu_k & \text{if } x_s = d_k, x_w = x \\
0 & \text{otherwise}
\end{cases} \tag{16}
\]

In (16), the function \(\beta(x_d, x)\) selects the correct DF transition rate from a DF to a regular microstate based on the completion of a DF process (with rate \(\mu_d\)) or a departure (with rate \(\mu_k\)) resulting in no servicing connections. It should be noted that \(\beta(x_d, x) = 0\) means that the microstate \(x\) is not a target state of the DF microstate \(x_d\).

Since the Kaufmann one-dimensional recursion cannot be applied here (due to the asymmetrical traffic characteristics), we need to solve the global balance equations (GBEs) in \(17\) and \(18\) for each regular \((x)\) and DF \((x_d)\) microstate in the reduced link model to obtain the steady state link occupancy distribution \(\pi(x)\) and \(\pi(x_d)\), which can be obtained easily by
Different DF microstates could generate the same target state and input flow rates from and into the microstate $x$ in microstates: due to an arrival of bandwidth request of $d$ in Eq. (17) represent the transitions from other regular microstates, respectively. More precisely, the first two terms of the RHS described in Sec. IV-C.

$$ \sum_{k=1}^{K} \alpha_k(x) + \gamma_k(x) \pi(x) = \sum_{k=1}^{K} \alpha_k(x - d_k) \pi(x - d_k) + \sum_{k=1}^{K} \gamma_k(x + d_k) \pi(x + d_k) + \sum_{k \neq K} \beta(x, d_k) \pi(x + d_k) \pi(x) (17) $$

Similar to (10), in (17), LHS and RHS represent the output and input flow rates from and into the microstate $X = x$, respectively. More precisely, the first two terms of the RHS in Eq. (17) represent the transitions from other regular microstates: due to an arrival of bandwidth request of $d_k$ slices in $x - d_k$; and due to a departure of bandwidth of $d_k$ slices from the microstate $x + d_k$, while the third term adds the rates from all DF microstates to the microstate $x$, which means that different DF microstates could generate the same target state, which is identified by the function $\beta(x, d_k, x)$.

The GBE of a DF microstate $x_d = (x_s, x_w)$ is given as

$$ \left( \sum_{k \neq K} \lambda_k \pi(x_d) + \sum_{k \neq K} \gamma_k(x_s, x_d) + \mu_d \right) \pi(x_s, x_w) = \sum_{k \neq K} \lambda_k \pi(x_d) \Omega S(x_s, x_w) + \sum_{k \neq K} \gamma_k(x_s, x_w - d_k) + \sum_{k \neq K} \gamma_k(x_s + d_k) \mu_d \pi(x_s, x_w) (18) $$

where the LHS includes outgoing transition rates from $x_d$ due to arrivals of new requests with rate $\lambda_k$, $1 \leq k \leq K$, due to departures of serving class $k$ connections with expected rate $\gamma_k(x_s)$ during the reconfiguration phase, and also due to the completion of the DF process with rate $\mu_d$. Note that the departure of a connection could lead to another DF state, which is captured by the condition $x_s - d_k > 0$, and also to a microstate $x = x_s$ if $x_s - d_k = 0$, which also marks the completion of the current DF period. The RHS includes input flow rates possibly due to an arrival of a DF triggering request in a regular microstate $x_s$ taking into account the fragmentation factor, due to arrivals of different classes of requests in other DF microstates $(x_s, x_w - d_k)$, and also due to the departure of different class $k$ connections in other DF states $(x_s + d_k, x_w)$.

The class $k$ blocking probability in a DF-enabled link system is given as follows.

$$ BP_k = Pr[Z_r < d_k] = 1 - Pr[Z_r \geq d_k] = 1 - \sum_{C-d_k \leq x \leq C} \pi(x, x_w) - \sum_{C-x_w+d_k > C} \pi(x, x_w) $$

In (19) blocking in such microstates (including DF) happen which do not have enough free slices i.e., $x + d_k > C$ in regular microstates or $x_s + x_w + d_k > C$ in DF microstates.

It should be noted that the waiting list of connection requests vary per DF microstate, since we allow arrivals and departures during the DF process. Moreover, the creation of extra DF microstates increases the total number of states thus the model complexity in a reduced DF link model. To simplify this, we could model a large scale link with DF by assuming mean reconfiguration time $t_{DF} = 0$, meaning that the requests are immediately accepted even though they arrive in FB states, which reduces the above detailed DF model into a reduced link model as described in Sec. IV-A, where the connection setup rate is $\lambda_k \times p_k(x)$. Here, the probability of acceptance $p_k(x)$ is given by (15).

Fig. 3. Defragmentation process and related transitions in a 7-slice fiber link with two classes of demands $d_k = \{3, 4\}$ slices under RF spectrum allocation policy. (a) Partial state transitions due to arrivals in an exact state $0, 0, 1, \infty, \infty, 0, 0$. (b) Complete defragmentation model with microstates representation.

Fig. 4. An example of fragmentation in a 4 node ring network with 4 unidirectional links $1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 4$ and $4 \rightarrow 1$; and 2 wavelengths (slices) per link, where a DF process could not eliminate fragmentation in order to accept a request $4 \rightarrow 1 \rightarrow 2$.

B. Modeling FGONs with Defragmentation

Defragmenting spectrum in a single-hop network is simpler, modeling is however still complex, due to the fact that
only contiguity constraint needs to be satisfied. Spectrum defragmentation in FGONs, on the other hand, is an NP hard problem [9], and there are multiple solutions which can be said to be sub-optimal as a result of a defragmentation process. To simplify the DF model in FGONs, we assume that the mean reconfiguration time $t_{DF} = 0$, i.e., when a request arrives and could not be accepted due to fragmentation of spectrum, then we try to reconfigure some of the existing connections to set free the required number of continuous and contiguous slices on its route, and this reconfiguration time is assumed to be zero, which enable us to immediately accept such requests.

However, it should also be noted that, unlike a link, an FGON does not guarantee the acceptance of an incoming request even though the sufficient free spectrum on a route is fragmented. For example, when a new OD pair connection $o$ arrives on a route $4 \rightarrow 1 \rightarrow 2$ in Fig. 4, where two connections, each with 1-slice bandwidth, are allocated over wavelengths (or frequencies) $f_1$ and $f_2$, then the connection request $o$ is blocked even though there is a free wavelength (inner or outer) on both unidirectional links $4 \rightarrow 1$ and $1 \rightarrow 2$. In the current scenario, any DF processes would not be able to reconfigure existing connection(s) in order to accept the new request on the route $4 \rightarrow 1 \rightarrow 2$, assuming that there is no wavelength converters in the 4-node unidirectional ring network. Noting that blocking also occurs due to scattered free spectrum in a single link as well as their misalignment over the constituent links of a route, only a fraction of arrivals into fragmented states could be accepted in FGONs. However, for simplicity, we again assume the statistical link independency, and the term $Pr[Z_r \geq d_k \mid X_j = x_j]$ in the DF network model needs to take into account the probability of acceptance term $p_k(x_r)$ and the probability of each constituting link of a route in a fragmentation state, thus using (13) it can be given as follows.

$$Pr(Z_r \geq d_k \mid X_j = x_j) = \sum_{x_{j_2}=0}^{C-d_k} \cdots \sum_{x_{j_i}=0}^{C-d_k} \pi_{j_2}(x_{j_2}) \cdots \pi_{j_i}(x_{j_i}) \times p_k(x_r) + \prod_{x_{j_i} \in x_r} \frac{F^{RB}(x_{j_i}, k)}{F^{SB}(x_{j_i}, k)}$$ \hspace{1cm} (20)

Note that in (20) the probability of acceptance term in an FGON without DF $p_k(x_r)$, where $x_r = (x_{j_1}, x_{j_2}, \ldots, x_{j_i})$ and $x_{j_1} = x_j$, on a route $r(o)$ is calculated in Appendix using (28), and the probability of observing a route, with link occupancies $x_{j_1}, x_{j_2}, \ldots, x_{j_i} \in r(o)$, in a fragmentation state is given by the second product term, assuming the links occupancies on route $r(o)$ are independent. Additionally, the blocking probability in FGONs with DF can be calculated by (21).

$$BP_k^o = Pr[Z_r < d_k] = 1 - Pr[Z_r \geq d_k] = 1 - \sum_{x_{j_1}=0}^{C-d_k} \sum_{x_{j_2}=0}^{C-d_k} \cdots \sum_{x_{j_i}=0}^{C-d_k} \pi_{j_1}(x_{j_1}) \pi_{j_2}(x_{j_2}) \cdots \pi_{j_i}(x_{j_i}) \times p_k(x_r) + \prod_{x_{j_i} \in x_r} \frac{F^{RB}(x_{j_i}, k)}{F^{SB}(x_{j_i}, k)}$$ \hspace{1cm} (21)

VII. NUMERICAL AND SIMULATION RESULTS

In this section, we investigate the accuracy of the approximate BP by comparing them with the discrete event simulation results obtained in a unidirectional fiber link, a 2-hop network and a 14-node (42 links) NSF network. Additionally, for a small capacity link we compare our approximate BP with the exact blocking results [11] under two different spectrum allocation policies, RF and FF. Furthermore, we compare blocking in a regular FGONs, i.e., without defragmentation (denoted as w/o DF), to a network enabled with defragmentation (DF). Although approximate BP results are presented using the Microstate model for both link and network examples due to its better scalability, we additionally show in Sec. VII-C the accuracy of Macrostate model in finding BP results. Blocking results are depicted versus offered load, which is defined as $\sum o \sum_k \sum_k x_k^o$, where $o \in O, k = 1, 2, \ldots, K$. We assume that the service (holding) times of connection requests between an OD pair are exponentially distributed with mean $1/\mu_k = 1$ unit [3], [5], and per-class, per OD pair connection requests arrive according to a Poisson process with uniformly distributed rate $\lambda_k = offered \ load/(|O| \times K)$. We present the average BP considering all OD pairs $o \in O$ and classes $k = 1, 2, \ldots, K$. All results presented here consider both spectrum contiguity and spectrum continuity constraints. We generated $10^6$ and $10^7$ connection requests to simulate small, and large scale link as well as FGONs, respectively.

A. Link Model

Note that for a unidirectional link only one OD pair exists, thus $\lambda_k = \lambda_k^o = offered \ load/K$. Table [11] presents exact (denoted by Exact), verifying simulation (Sim.) and approximate (Approx.) BP in a link with different link capacities and set of demands for various offered loads. In a small scale scenario, $C = 10, d_k = \{3, 4\}$, slices, it can be seen that verifying simulation results are very close to the exact results, thus in a large scale case where the exact solution is intractable it can be used to verify the approximate solutions. Additionally, Approx. solutions are also very close to the exact solutions under both RF and FF spectrum allocation policies. As expected, BP under the FF policy is lower than that of RF due to the lower spectrum fragmentation [10], [17]. Interestingly, with the increase in the link capacity $C = 20$ and $C = 100$ the Approx. results are still very close to the exact or Sim. results under the RF policy for various loads. Note that the probability of acceptance term $p_k(x)$ is calculated using (5) for both policies, however we derive the number of NB states in (26) and total states corresponding to a microstate $x$ in (23) for the RF policy, and use Algorithm 1 to find the ratio of these numbers, i.e., $p_k(x)$ in Appendix A for a link scenario. However, the Approx. results under the FF policy deviate with the increase in load, as can be seen in the scenario $C = 20, d_k = \{3, 4, 5\}$. The reason is that although Approx. BP in the FF is obtained by considering the exact probability of acceptance $p_k(\cdot)$ using the Algorithm 1, the uniform state probability distribution assumptions is very critical in the FF case. In other words, the probabilities of being in the blocking and NB exact states having same occupancy are not close.
enough at larger loads, which makes the connection setup rates and expected departure rates in a microstate model less accurate. This can be seen in Fig. 5 where the variance of exact state probabilities representing a microstate, i.e., total occupied slices, is much higher for a few states (3 ≤ x ≤ 5) in the case of FF policy even at lower loads, and it increases for both RF and FF policies when the offered load is higher.

In a DF-enabled scenario, on the other hand, BP considerably reduces as compared to w/o DF case with increasing link capacities, C = 10 to C = 100, as can be verified by the Exact/Sim. and the matching Approx. results. Moreover, the blocking gain or the reduction obtained due to the DF process is higher in the RF policy as compared to the FF policy, since the RF policy exhibits higher blocking in w/o DF case. It should be noted that we present the BP results in a DF-enabled small scale link (C = 10, 20) in Table III with mean reconfiguration time 1/μ_d = 1 unit time, same as the mean service time of connections. On the other hand, in a large scale scenario C = 100, we omitted DF states by assuming zero reconfiguration time to keep the Approx. DF model tractable. To show the effect of mean reconfiguration time t_DF = 1/μ_D, we plot BP in Fig. 6 in a relatively larger scenario C = 20, d_k = {3, 4, 5} for various fraction of reconfiguration time as compared to mean connection service time (t_c = 1/μ_c), i.e., 1/μ_D = t_DF/ t_c. As can be seen, with increasing DF time t_DF, BP also increases, and it is minimum when t_DF = 0, which gives the lower bound on the BP that can be achieved with DF. Thus, in a practical system, the DF process must be performed with a very low reconfiguration time t_DF → 0 to increase the connection acceptance probability. The authors in [18], [19] have shown that reconfiguration can be performed in much lower time (~ microseconds) as compared to the connection holding time (~ seconds). Furthermore, like [11], where the effect of t_DF on the blocking reduction was shown (although limited by the assumption that all arrivals are blocked during the DF process), our DF model in this paper proves that even though we accept requests during the DF operation, if capacity is sufficient, a relatively larger t_DF has adverse effect on the blocking probability.

![Fig. 5. Variance of steady state probabilities of a set of exact states representing a link occupancy (microstate) for C = 20, d_k = {3, 4, 5}.](image)

![Fig. 6. Effect of reconfiguration time on the blocking probability for C = 20, d_k = {3, 4, 5}.](image)
It should be noted that some of the arriving connection demands need to be buffered during the DF process. The average queue length (AQL) required to buffer the bandwidth of waiting requests ($x_w$) can be given by expectation of $x_w$ over all DF microstates ($x_s, x_w$) as follows.

$$AQL = \sum_{x_s, x_w} x_w \frac{\pi(x_s, x_w)}{\sum_{x_s, x_w} \pi(x_s, x_w)}$$ \hspace{1cm} (22)

Fig. 7 depicts the AQL required to buffer the waiting demands. As expected, the AQL increases with the $t_{DF}$ and load, however limited by the capacity of the link. Interestingly, for a much lower $t_{DF}$ with respect to the mean holding time ($t_c = 1/\mu_k$) and the mean interarrival time ($\sim 1/\lambda_k$), the AQL size does not vary much, and therefore the system operator does not need to buffer more requests even at higher loads.

In Table IV we present computational run time of the exact and approximate solutions, which are obtained on a PC with Intel 6-core i7 3.20 GHz processor with 32 GB RAM. We observe that the run times for the exact and approximate solutions, with or without DF, are nearly same for both RF and FF spectrum allocation policies in a link with smaller capacity ($C = 10$). However, when capacity of the link increases to $C = 20$, finding exact solutions becomes extremely time consuming and increases exponentially with respect to capacity $C$. On the other hand, the approx solution under RF policy takes only a fraction of seconds for $C = 20$, and a few seconds for $C = 100$. Interestingly, the approx solution for FF policy is possible to obtain using the Algorithm 1 for capacity $C = 20$, which is time consuming.

### Table IV

| C=10 | C=20 | C=100 |
|------|------|-------|
| $d_k = \{3, 4\}$ | $d_k = \{3, 4, 5\}$ | $d_k = \{1, 2, 8\}$ |
| **Exact** | **Approx.** | **Exact** | **Approx.** | **Approx.** |
| RF | 0.044 | 0.060 | 10207.2 | 0.104 | 112.3 |
| RF-DF | 0.063 | 0.065 | 10899.9 | 0.106 | 112.5 |
| FF | 0.054 | 0.064 | 2280.8 | 2213.9 | – |
| FF-DF | 0.062 | 0.067 | 2519.5 | 2218.2 | – |

### B. Network Model

Firstly, we consider a 2-hop ($1 \rightarrow 2$ and $2 \rightarrow 3$) FGON with 3 OD pair routes $1 \rightarrow 2$, $2 \rightarrow 3$ and $1 \rightarrow 2 \rightarrow 3$ over which connection requests arrive according to a Poisson process. In addition to a small scale scenario $C = 10$, $d_k = \{3, 4\}$, and bandwidth demands $d_k = \{1, 2, 8\}$ for a practical scenario which can be seen as supporting high-speed connection with demands $\{40, 100, 400\}$ Gb/s using a modulation format $m = 4$, such as DP-QPSK, where each optical subcarrier (12.5 GHz slice bandwidth) supports at max 50 Gb/s. We use the Algorithm 1 to obtain the probability of acceptance $p_k(\cdot)$ for the small-scale scenario under the RF and FF policies. On the other hand, an approximate $p_k(\cdot)$ is derived in Appendix using (28) to obtain the $p_k(\cdot)$ for the large-scale RF case. As we discussed in Sec. VI, the DF operation is very complex to perform in multi-hop FGONs, since the spectrum continuity and contiguity constraints must be satisfied during the connection reconfiguration process. Thus, for the discrete event simulation, we adopt a strategy [20], [21] to construct an auxiliary graph, which transforms the DF problem into a matter of finding non-overlapping set of free spectrum paths for the connections to be reconfigured in order to create a free spectrum block for an incoming demand which could not be accepted w/o DF. Thus, like our analytical model, in our simulation the DF is triggered reactively, and arrivals during the DF operations are immediately accepted if the reconfiguration time ($t_{DF}$) is zero.

Table IV presents the Approx. BP results and verifying simulations in for a 2-hop FGON under various scenarios. As can be seen, under both spectrum allocation policies the Approx. BP matches closely to the Sim. results in a small scale $C = 10$ case, however, they are slightly higher than the Sim. results in a large scale FGON ($C = 100$), possibly due to the fact that at lower load the probability of being in NB exact states would be higher than that of observing a link in exact blocking states in a given occupancy state. On a contrary, our uniform state probability distribution assumption does not consider this fact. When the network is enabled with the DF process, the BP considerably reduces in a 2-hop FGON, and in NSFnet in Table VI at a medium load due to the fact that fragmentation blocking due to the non-alignment of free spectrum also happens in addition to the scattered free spectrum within a link, and the blocking reduction increases with the capacity, i.e., with the increasing scope of connections to get reconfigured in the network. Table VI indicates the similar trend in the NSFnet topology, where all possible OD pairs ($|O| = 182$) are considered for calculating average BP in the NSF network. It should be noted that the computational run time of the Approx. BP in an FGON increases with the number of links, OD pairs, and the link capacity.

As we see in Table VII that the Approx. BP deviates from Sim. results when link capacity increases, i.e., with the number of microstates. The obvious reason is that the variance of exact link state probability distributions, which has not been taken into the account for calculating $\alpha_k(\cdot) \text{ and } \gamma_k(\cdot)$, increases with the capacity and the network load. Importantly, type of demands is also a factor, since smaller bandwidth demands
lead to more number of exact states than a set of larger demands for a fixed link capacity. To cope up with this issue, one can also use a relatively less scalable Macrostate model.

**C. Comparison of Macrostate and Microstate models**

Unlike the Microstate model, a state in the Macrostate model represents connections per class \( n \) served by it, therefore the class \( k \) departure rate in a macrostate \( n \) can be given accurately by the number of connections served times per-class departure rate i.e., \( \gamma_k(n) = n_k(n) \mu_k \). Although throughout the paper we used the scalable Microstate model to derive BP, the BP results in the Macrostate model can be obtained in a similar way as in the case of the Microstate model with minimal changes to the formulas presented in this paper (see Appendix A). Tables VII and VIII present the Approx. BP results under both Macrostate and Microstate models and compare them to the Sim. results for a link and for the NSFNet, respectively. The Approx. results prove our hypothesis: smaller the number of exact states associated with a reduced state (as in the case of the Macrostate model), better is the approximation. The BP results in the case of Macrostate model is very close to the Sim. results under both RF and FF policies and for different loads as well as with DF. However, again, the accuracy comes at the cost of scalability issue in the Macrostate model.

**D. Model Limitations and Scope of Improvements**

Although this paper shows promising approximation results, there are some limitations which if addressed could further improve the accuracy of the approximate blocking results in FGONs. First, this paper assumes the uniform exact state probabilities, which is the case in the large-scale states representing the same link occupancy generally have different state probabilities, which is the case in the large-scale link scenario, and for the FF scenario. One could consider a different distribution to cope up with this issue. Second, we do not provide an exact probability of acceptance \( p_k(x_c) \) of a connection path requests in a large network scenario. We consider an approximate \( p_k(x_c) \), which assumes the uniform distribution of link load (occupancies) among slices. The main drawback is that it ignores the spectrum occupancy patterns created by a set of given classes of demands with continuity and contiguity constraints. Due to this reason, this paper does not provide an approximate blocking results for a large-scale link and network under the FF spectrum allocation policy. Although deriving an exact \( p_k(x_c) \) is an unsolved challenge for a network, it should be noted that it has huge influence on the approximate blocking probability. Third, in a network defragmentation scenario, the independence link
assumption is necessary for the analysis but rather strong, because reconfiguration of a connection depends not only on its route occupancies, but also on their relative distribution on all constituent links as well as other routes which partially overlap with the connection’s route. Again, this is a grand challenge that needs to be worked on in the future. Nevertheless, the accuracy of our blocking model could be further improved by considering the link correlation model, which is relatively complex compared to the link independence model but essential to analyze the effect of correlation of loads among links under different spectrum allocation policies in a network.

VIII. CONCLUSIONS

In this paper, we proposed a method to reduce the exact link state occupancy model into a reduced occupancy model, and presented an analytical methodology to calculate state-dependent connection setup rates to computing approximate blocking in multi-class FGONs. Notably, we considered both RSA constraints: spectrum continuity and contiguity, in the approximate BP models in a regular FGON without DF and in a DF-enabled FGONs under two different spectrum allocation policies, namely random-fit and first-fit. We used the principle of inclusion-exclusion to calculate the probability of a link being in a non-blocking or fragmentation states, which is essential in finding the class and state-dependent connection setup rates. Our numerical results show that the approximate solutions can be very useful in a broad range of scenarios, as they are close to the exact or simulation results in both with and w/o DF scenarios. At the same time, the accuracy of the methods proposed depend on the various factors, such as the spectrum allocation policies, link capacity, and traffic loads. The next steps in this analysis include grand challenges that have not been solved yet and are analytically rather complex, including the demand-dependent spectrum patterns, impact of fragmentation factor, and the interdependency of links and slices in a defragmentation-enabled flexible grid optical network.

APPENDIX A

DERIVATION OF THE PROBABILITY OF ACCEPTANCE

In this appendix, we first derive the probability of acceptance $p_k(x)$ of a class $k$ connection in a single-hop path (Eq. 4) for the case of random fit (RF) spectrum allocation policy using the inclusion-exclusion principle. Following which, we present an exact procedure to compute it for any spectrum allocation policies in a multi-hop path. As the exact calculation of the probability of acceptance is not possible for a large-scale network, we provide an approximation method for its computation.

For a given microstate $x, 0 \leq x \leq C$, where the number of empty slices $E(x) = C - x$, we can find connection patterns or macrostates (i.e., connections per class) $n$ that satisfies $n \cdot \mathbf{d}^T = x$ as shown in Table II. Now, for each connection pattern $n = (n_1, n_2, \ldots, n_K)$, the $E(x)$ empty slices can be distributed at $N(n) + 1$ places (including the start, end, and in between each two connections), where the total number of connections $N(n) = \sum_{k=1}^{K} n_k(n)$. Noting that there are $N(n)!$ distinct permutations of connections per class in $n$, and there are $(E(x) + N(n))!$ different ways to distribute $E(x)$ empty slices at $N(n) + 1$ places in each unique permutation of $n$, the number of exact states with all connection patterns representing a microstate occupancy $x$ is, thus, given by

$$|\Omega_S(x)| = \sum_{n \in \Omega_S(x)} \frac{N(n)!}{\prod_{k=1}^{K} n_k(n)!} \times \binom{E(x) + N(n)}{N(n)}.$$  

(23)

For example, see Table II where the number of exact states corresponding to $x = 3$, which is represented by a unique $n = (1, 0)$, is $\frac{1}{2} \times 1 \times 1 = 5$. It should be noted that the number of exact states for a microstate as given in (23) is only valid under RF spectrum allocation policy, the number of such states under FF policy is generally much lower.

Importantly, only some of the exact states ($s \in \Omega_S(x)$) are NB states, as defined in (2). To compute the number of NB states, let us solve the following equation for each permutation of the connection pattern $\forall n \in \Omega_S(x)$:

\begin{table}
\centering
\caption{The BP in a Link using the Macrostate and Microstate Models.}
\begin{tabular}{llllllll}
\hline
Scenarios & \multicolumn{2}{c}{\text{offered load=0.1}} & \multicolumn{2}{c}{\text{offered load=1.2}} \\
\hline
& Sim. & Approx. (Macro.) & Sim. & Approx. (Micro.) & Sim. & Approx. (Macro.) & Sim. & Approx. (Micro.) \\
RF w/o DF & $8.2 \times 10^{-5}$ & $7.5 \times 10^{-5}$ & $1.1 \times 10^{-4}$ & $3.9 \times 10^{-2}$ & $4.0 \times 10^{-2}$ & $4.2 \times 10^{-2}$ \\
RF w/o DF & $1.4 \times 10^{-6}$ & $1.5 \times 10^{-5}$ & $1.2 \times 10^{-6}$ & $1.4 \times 10^{-2}$ & $1.4 \times 10^{-2}$ & $1.2 \times 10^{-2}$ \\
FF w/o DF & $3.1 \times 10^{-6}$ & $7.8 \times 10^{-5}$ & $1.1 \times 10^{-4}$ & $2.1 \times 10^{-2}$ & $3.7 \times 10^{-2}$ & $8.5 \times 10^{-2}$ \\
FF w/o DF & $1.4 \times 10^{-6}$ & $1.5 \times 10^{-5}$ & $1.2 \times 10^{-6}$ & $1.4 \times 10^{-2}$ & $1.4 \times 10^{-2}$ & $1.2 \times 10^{-2}$ \\
\hline
\end{tabular}
\end{table}

\begin{table}
\centering
\caption{The BP in an NSFNet using the Macrostate and Microstate Models.}
\begin{tabular}{llllllll}
\hline
Scenarios & \multicolumn{2}{c}{\text{offered load=5}} & \multicolumn{2}{c}{\text{offered load=10}} \\
\hline
& Sim. & Approx. (Macro.) & Sim. & Approx. (Micro.) & Sim. & Approx. (Macro.) & Sim. & Approx. (Micro.) \\
RF w/o DF & $4.1 \times 10^{-3}$ & $1.3 \times 10^{-3}$ & $7.7 \times 10^{-4}$ & $2.1 \times 10^{-2}$ & $1.5 \times 10^{-2}$ & $7.9 \times 10^{-3}$ \\
RF w/o DF & $1.4 \times 10^{-3}$ & $1.1 \times 10^{-3}$ & $5.0 \times 10^{-4}$ & $9.3 \times 10^{-3}$ & $1.3 \times 10^{-2}$ & $6.5 \times 10^{-3}$ \\
\hline
\end{tabular}
\end{table}
\( a_1 + a_2 + \cdots + a_{N(n)+1} = E(x), \text{s.t., } \exists i: a_i \geq d_k \) (24)

Using the inclusion-exclusion principle (hint: consider the event \( A_i = \{ a_i \geq d_k \}, 1 \leq i \leq N(n)+1 \) and find \( |\cup_i A_i| \)), the number of NB exact states corresponding to each permutation of connections in \( n \in \Omega_S(x) \) is given by

\[
W(n) = \sum_{i=1}^{N(n)+1} (-1)^{i+1} \binom{N(n)+1}{i} \left( E(x) + N(n) - id_k \right)^{-1} \binom{N(n)}{n} \tag{25}
\]

Now, considering all permutations of \( n \), and adding all NB states corresponding to each \( n \) belonging to the microstate \( x \) would result in the number of class \( k \) NB exact states for a given microstate \( x \), given by \( \{ \mathbb{NB}(x, k) \} \)

\[
|\mathbb{NB}(x, k)| = \sum_{n \in \Omega_S(x)} W(n) \cdot \frac{N(n)!}{\prod_{i=1}^{N(n)+1} n_k(n)!} \tag{26}
\]

Thus, using equations (26) and (23), the ratio of NB exact states and total exact states gives the probability of acceptance \( p_k(x) \) in (5) for regular link without DF process. For example, the number of NB exact states for a class 2 demand \( (d_k = 4 \text{ slices}) \) in a microstate \( x = 3 \) i.e., \( n = (1, 0) \) and \( E(x = 3) = 4 \text{ slices} \), is \((1+1) \times (1+1+1) = 2\), which can be seen in Table [I]. Thus, the class 2 connection setup rate in the microstate \( x = 3 \) in Fig. [I] is \( \frac{1}{2} \lambda_2 \). Note that when we use the Macrostate model, where a state represents \( n \), the probability of acceptance \( p_k(n) \) can be given by the ratio of \( |\mathbb{NB}(n, k)| \) and \( \Omega_S(n) \), where these two terms can be calculated by omitting the summation over all \( n \) in \( \{ \mathbb{NB}(x, k) \} \) and \( \Omega_S(n) \), respectively.

The probability of acceptance of a class \( k \) request in (13) also needs the number of FB states to be calculated, which is easy to give. Since the set of microstates \( x \), \( 0 < x \leq C - d_k \) has sufficient free slices to accept an incoming demand \( d_k, C > d_k \), the blocking in their corresponding exact states could happen due to the fragmentation. Thus, the number of FB exact states for a class \( k \) request in a microstate \( x, 0 < x \leq C - d_k \) is \( |\mathbb{FB}(x, k)| = |\Omega_S(x)| - |\mathbb{NB}(x, k)| \). On the other hand, the exact states representing a microstate \( x, C - d_k < x \leq C \) are RB states for class \( k \) request. For \( C - d_k < x \leq C \), the number of FB states is zero, and the number of RB states \( |\mathbb{FB}(x, k)| = |\Omega_S(x)| - |\mathbb{NB}(x, k)| \).

Noting that the above derivations can only be used for RF spectrum allocation policy in a link, we now present a simple procedure for computing the probability of acceptance of a connection request in a multi-hop path in small-scale FGONs that can be used for any spectrum allocation policies. In Algorithm [I] all possible combinations of exact states satisfying the occupancy on each link of the route \( r(o) \) is generated first (Step 3). Then for each occupancy vector \( \mathbf{v} \) we check using a union operation if the vector \( \mathbf{v} \) has sufficient consecutive slices on its route \( r(o) \) (in Step 5). Note that the union operation takes care of the continuity constraint. For example, the union of occupancies on link 1, i.e., \( s_1 = (1, \infty, \infty, 0, 0, 0, 0) \) and on link 2, \( s_2 = (0, 1, \infty, \infty, 0, 0, 0) \), means that this 2-link path can accept an incoming demand \( d_k \leq 3 \). The normalized number of all favorable occupancy vectors gives the probability of acceptance in Step 7. Notably, Algorithm [I] can be used for deriving the probability of acceptance in a single-hop (i.e., \( p_k(x) \)) case as well, wherein \( l = 1 \) and \( x_j = x \), and step 7 is gives the ratio of the number of NB states to the total states representing a microstate \( x \). In the link scenario, each occupancy vector \( \mathbf{v} \) contains a unique exact state \( s_i \) which represents the microstate \( x \). Thus, we can also use Algorithm [I] to find \( p_k(x) \) for the FF policy in the link scenario.

Algorithm 1 Find \( p_k(x_j, x_{j2}, x_{j3}, \ldots, x_{j_l}) \)

1. Given: \( \Omega_S(x), X_j = x_j, d_k, r(o) = \{j, j_2, \ldots, j_l\} \)
2. if \( d_k \leq C - x_{j}, \forall j \in r(o), j_1 = j \)
   then
3. Generate all combinations (\( \Omega_V \)) of a \( l \) elements vector \( \mathbf{v} = (s_1, \ldots, s_l) \) of exact states s. t. \( s_i \in \Omega_S(x_{j_i}), 1 \leq i \leq l \);
4. for each \( \mathbf{v} : \Omega_V \) do
   \( t++ \), if \( f_m(\cup_{i=1}^{l} s_i) \geq d_k, s_i \in \mathbf{v} \)
   end for
5. \( p_k(x_j, x_{j2}, x_{j3}, \ldots, x_{j_l}) = t/|\mathbf{v}| \).
6. else
7. \( p_k(x_j, x_{j2}, x_{j3}, \ldots, x_{j_l}) = 0 \).
8. end if
9. end if

It should be noted that the Algorithm [I] requires all exact states to be generated for a given spectrum allocation policy, thus it is not suitable for a large-scale FGONs. Therefore, we provide an approximate probability of acceptance \( p_k(x) \equiv p_k(x_j, x_{j2}, x_{j3}, \ldots, x_{j_l}) = Pr[Z_r \geq d_k|X_j = x_j, \ldots, X_{j_l} = x_{j_l}] \) for the general purpose FGONs. Considering the independence link assumption, we further assume that the occupancy of slices are independent and identically distributed in each link. This means that total occupied slots can be represented in all possible ways, without considering the contiguous allocation of slots and are not limited to the spectrum patterns generated by a given classes of demands. Now, for a given occupancy of links of a route \( r \), the probability that there are \( n \) continuous slices free on its route (but not necessarily contiguous) is obtained by the following recursive relationship [12], [13]:

\[
g_n(x_j, x_{j2}, \ldots, x_{j_l}) = Pr[Z_r \geq n|X_j = x_j, \ldots, X_{j_l} = x_{j_l}] = \sum_{i=n}^{\infty} g_n(C - i, x_{j_i}) g_i(x_{j_1}, x_{j_2}, \ldots, x_{j_{l-1}}) \tag{27}
\]

where \( i^* = \min(C - x_{j1}, C - x_{j2}, \ldots, C - x_{j_{l-1}}) \) and \( g_n(x, y) = \binom{C-x}{n} / \binom{x+y}{C-y} \). Now, the approximate probability of acceptance \( p_k(x) \) is given by \( \{28\} \).

\[
p_k(x) = \sum_{n=0}^{d_k-1} g_n(x_j, x_{j2}, \ldots, x_{j_l}) \tag{28}
\]

Although Eq. (28) provides a good approximation for \( p_k(x) \), and for that reason it is used in the performance evaluation, we could also find the probability that the route \( r \) has equal or more than \( d_k \) free contiguous slices across links on its route \( \{Z_r \geq d_k\} \), given the link occupancy vector \( x_r \) and also there are exactly \( n \) continuous free slices on the route \( \{X_r = n\} \), i.e., \( p_k(x, n) = Pr[Z_r \geq d_k|X_{j_1} = x_{j_1}, \ldots, X_{j_l} = x_{j_l}, X_r = n] \). Using the inclusion-exclusion
principle, it can be given by
\[ p_k(x_r, n) = \frac{\sum_{i=1}^{C-n} (-1)^{i+1} \binom{C-n+1}{i} \binom{C-d_k}{C-n}}{\binom{C}{r}} \] (29)

The above equation seems to be independent of \( x_r \), but actually a factor which is a function of \( x_r \), is multiplied in both numerator and denominator, thus cancels the effect of \( x_r \). Thus, \( p_k(x_r) = P_{dr}[Z_r \geq d_k | X_{ji} = x_{ji}, \ldots, X_{ji} = x_{ji}] \) can be given as follows:
\[ p_k(x_r) = \sum_{n=d_k}^{\min(C-x_r)} p_k(x_r, n) g_n(x_r) \] (30)

It is important to note that we provide (30) for the sake of completeness and could be beneficial for the readers to know that it has not been used in the performance evaluation mainly because the use of (30) reduces \( p_k(x_r) \) further (as compare to (28)) and thus the approximate blocking probability is much higher than the exact or the simulation results.

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**REFERENCES**

[1] M. Jinno, H. Takara, B. Kozicki, Y. Tsukishima, Y. Sone, and S. Matsumoto, “Spectrum-efficient and scalable elastic optical path network: architecture, benefits, and enabling technologies,” *Communications Magazine, IEEE*, vol. 47, no. 11, pp. 66–73, 2009.

[2] Y. Wang, X. Cao, and Y. Pan, “A study of the routing and spectrum allocation in spectrum-sliced elastic optical path networks,” in *INFOCOM, 2011 Proceedings IEEE*. IEEE, 2011, pp. 1503–1511.

[3] H. Beyranvand, M. Maier, and J. Salehi, “An analytical framework for the performance evaluation of node-and network-wise operation scenarios in elastic optical networks,” *Communications, IEEE Transactions on*, vol. 62, no. 5, pp. 1621–1633, 2014.

[4] R. R. Reyes and T. Bauschert, “Reward-based online routing and spectrum assignment in flex-grid optical networks,” in *Telecommunications Network Strategy and Planning Symposium (Networks), 2016 17th International*. IEEE, 2016, pp. 101–108.

[5] Y. Yu, J. Zhang, Y. Zhao, X. Cao, X. Lin, and W. Gu, “The first single-link exact model for performance analysis of flexible grid WDM networks,” in *National Fiber Optic Engineers Conference*. Optical Society of America, 2013, pp. JW2A–68.

[6] J. Kaufman, “Blocking in a shared resource environment,” *IEEE Transactions on communications*, vol. 29, no. 10, pp. 1474–1481, 1981.

[7] L. Peng, C.-H. Youn, and C. Qiao, “Theoretical analyses of lightpath blocking performance in co-ODM optical networks with/without spectrum conversion,” *IEEE Communications Letters*, vol. 17, no. 4, pp. 789–792, 2013.

[8] F. Ge and L. Tan, “Blocking performance approximation in flexi-grid networks,” *Optical Fiber Technology*, vol. 32, pp. 58–65, 2016.

[9] M. Zhang, C. You, and Z. Zhu, “On the parallelization of spectrum defragmentation reconfigurations in elastic optical networks,” *Networking, IEEE/ACM Transactions on*, vol. PP, no. 99, pp. 1–1, 2015.

[10] S. K. Singh, W. Bziuk, and A. Jukan, “Defragmentation-as-a-service (Daas): How beneficial is it?” in *Optical Fiber Communication Conference*. Optical Society of America, 2016, pp. P2A–35.

[11] ——, “Analytical performance modeling of spectrum defragmentation in elastic optical link networks,” *Optical Switching and Networking*, vol. 24, pp. 25–38, 2017.

[12] A. Birman, “Computing approximate blocking probabilities for a class of all-optical networks,” *IEEE Journal on Selected Areas in Communications*, vol. 14, no. 5, pp. 852–857, 1996.

[13] K. Kupraswamy and D. C. Lee, “An analytic approach to efficiently computing call blocking probabilities for multi-class WDM networks,” *IEEE/ACM Transactions on Networking (TON)*, vol. 17, no. 2, pp. 658–670, 2009.