Mining $\mathcal{EL}^\perp$ Bases with Adaptable Role Depth

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Abstract. In Formal Concept Analysis, a base for a finite structure is a set of implications that characterizes all valid implications of the structure. This notion can be adapted to the context of Description Logic, where the base consists of a set of concept inclusions instead of implications. In this setting, concept expressions can be arbitrarily large. Thus, it is not clear whether a finite base exists and, if so, how large concept expressions may need to be. We first revisit results in the literature for mining $\mathcal{EL}^\perp$ bases from finite interpretations. Those mainly focus on finding a finite base or on fixing the role depth but potentially losing some of the valid concept inclusions with higher role depth. We then present a new strategy for mining $\mathcal{EL}^\perp$ bases which is adaptable in the sense that it can bound the role depth of concepts depending on the local structure of the interpretation. Our strategy guarantees to capture all $\mathcal{EL}^\perp$ concept inclusions holding in the interpretation, not only the ones up to a fixed role depth.

1 Introduction

Among its many applications in artificial intelligence, logic is used to formally represent knowledge. Such knowledge, often in the form of facts and rules, enables machines to process complex relational data, deduce new knowledge from it, and extract hidden relationships in a specific domain. A well-studied formalism for knowledge representation is given by a family of logics known as description logics (DLs) \cite{BA03}. DL is the logical formalism behind the design of many knowledge-based applications. However, it is often difficult and time-consuming to manually model in a formal language rules and constraints that hold in a domain of knowledge.

In this work, we consider an automatic method to extract rules (concept inclusions (CIs)) formulated in DL from data. This data can be, for instance, a collection of facts in a database or a knowledge graph. For instance, in the DBpedia knowledge graph \cite{DBPedia}, one can represent the relationship between a city ‘a’ and the region ‘b’ it belongs to with the facts city(a), region(b), partof(a, b), and capital(b, a). From this data, one can mine a CI expressing that a capital is a city that is part of a region.

To mine CIs that hold in a dataset, we combine notions of Formal Concept Analysis (FCA) \cite{Ganter07} and DLs. FCA is a subfield of lattice theory that provides
methods for analysing datasets and identifying the dependencies in them. In FCA a dataset, also called a formal context, is a table showing which objects have which attributes. Given a formal context, FCA methods are used to extract the dependencies between the attributes, also called implications (Figure 1). A base is a set of implications that entails every valid implication of the dataset and only those (soundness and completeness). It can be used for detecting erroneous or missing items in the dataset [5]. In the DL setting, a base is a set of CIs (an ontology) which can serve as a starting point for ontology engineers to build an ontology in a domain of interest.

However, for some DLs and datasets, it may happen that no finite base exists. Cyclic relationships are common in knowledge graphs and they are the main challenge for finding a finite DL base. With only one cyclic relationship, we already have that infinitely many concepts hold in the dataset. Strategies for limiting the size of concepts in the presence of cyclic dependencies have already been investigated in the literature. Baader and Distel (2008) and Distel (2011) propose a way of mining DL finite bases expressible in the DL $\mathcal{EL}^\perp_{gfp}$ which is the addition of greatest fix-point semantics to the DL language $\mathcal{EL}^\perp$ [4, 11]. The semantics offered by $\mathcal{EL}^\perp_{gfp}$ elegantly solves the difficulty of mining CIs from cyclic relationships in the data. However, this semantics comes with two drawbacks. Firstly, $\mathcal{EL}^\perp_{gfp}$ concepts may be difficult to understand, and learned CIs may be too complex to validate by domain experts. Secondly, there is no efficient implementation of a reasoner for $\mathcal{EL}^\perp_{gfp}$, even though the reasoning complexity is tractable, like for $\mathcal{EL}^\perp$. The authors also show how to transform an $\mathcal{EL}^\perp_{gfp}$ base into an $\mathcal{EL}$ base. However, it is far from being trivial to avoid the step of creating an $\mathcal{EL}^\perp_{gfp}$ base in their approach.

A simplification of the mentioned work has been proposed by Borchmann, Distel, and Kriegel (2016) where they show how to mine $\mathcal{EL}^\perp$ finite bases with a predefined and fixed role depth for concept expressions [8]. As a consequence, the base is sound and complete only w.r.t. CIs containing concepts with bounded role depth. Their approach avoids the step of creating an $\mathcal{EL}^\perp_{gfp}$ base but also avoids the main challenge in creating a finite base for $\mathcal{EL}^\perp$, which is the fact that the role depth of concepts can be arbitrarily large.

Our work brings together the best of the approaches by Distel (2011) and Borchmann, Distel, and Kriegel (2016): we directly compute a finite $\mathcal{EL}^\perp$ base that captures the whole language (not only up to a certain role depth). In particular, we present a new approach for computing the role depth of concepts which adapts depending on the objects considered during the computation of CIs.

**Related work.** Several authors have worked on combining FCA and DLs or on applying methods from one field to the other [21]. Baader uses FCA to compute the subsumption hierarchy of the conjunction of predefined concepts [4], uses FCA to compute the subsumption hierarchy of the conjunction of predefined concepts. Rudolph uses the DL $\mathcal{FLE}$ for the definition of FCA attributes and FCA techniques for generating a knowledge base [22, 23]. Baader et al. uses FCA for completing missing knowledge in a DL knowledge base [3]. Baader et al. proposes a method for building DL ontologies through the interaction of
domain experts [7]. Sertkaya presents a survey on applications of FCA methods in DLs [24]. Borchmann and Distel provide a practical application of the theory developed by Distel on knowledge graphs [10]. Borchmann shows how a base of confident $\mathcal{EL}^\bot$ concept inclusions can be extracted from a DL interpretation [9]. Monnin et al. compare, using FCA techniques, data present in DBpedia with the constraints of a given ontology to check if the data is compliant with it [20]. Kriegel [15] among other contributions employs FCA notions to build ontologies in logics more expressive than $\mathcal{EL}^\bot$, building upon the framework already established for $\mathcal{EL}^\bot$ [8] and $\mathcal{EL}^\bot_{gfp}$ [11]. He also investigates the same problem for probabilistic DLs [16].

In the next section, we present basic definitions and notation. In Section 3 we present the problem of mining $\mathcal{EL}^\bot$ CIs and establish lower bounds for this problem. In Section 4 we present our main result for mining $\mathcal{EL}^\bot$ bases with adaptable role depth. Our result uses a notion that relates each vertex in a graph to a set of vertices, called maximum vertices from (MVF). In Section 5 we show that the MVF of a vertex in a graph can be computed in linear time in the size of the graph. Missing proofs can be found in the long version [13].

2 Preliminaries

We introduce the syntax and semantics of $\mathcal{EL}^\bot$ and basic definitions related to description graphs used in the paper.

The Description Logic $\mathcal{EL}^\bot$

$\mathcal{EL}^\bot$ [3] is a lightweight DL, which only allows for expressing conjunctions and existential restrictions. Despite this rather low expressive power, slight extensions of it have turned out to be highly successful in practical applications, especially in the medical domain [25].

We use two finite and disjoint sets, $\mathbb{N}_C$ and $\mathbb{N}_R$, of concept and role names to define the syntax and semantics of $\mathcal{EL}^\bot$. $\mathcal{EL}^\bot$ concept expressions are built according to the grammar rule $C, D ::= A | \top | \bot | C \sqcap D | \exists r.C$ with $A \in \mathbb{N}_C$ and $r \in \mathbb{N}_R$. We write $\exists r^{n+1}.C$ as a shorthand for $\exists r.(\exists r^n.C)$, where $\exists r^n.C := \exists r.C$.

An $\mathcal{EL}^\bot$ TBox is a finite set of concept inclusions (CIs) $C \sqsubseteq D$, where $C, D$ are $\mathcal{EL}^\bot$ concept expressions. We may omit `\mathcal{EL}^\bot` when we speak of concept expressions, CIs, and TBoxes, if this is clear from the context. We may write $C \equiv D$ (an equivalence) as a short hand for when we have both $C \sqsubseteq D$ and...
$D \subseteq C$. The \textit{signature} of a concept expression, a CI, or a TBox is the set of concept and role names occurring in it.

The semantics of $\mathcal{EL} \perp$ is based on \textit{interpretations}. An interpretation $\mathcal{I}$ is a pair $(\Delta^\mathcal{I}, \mathcal{J})$ where $\Delta^\mathcal{I}$ is a non-empty set, called the \textit{domain} of $\mathcal{I}$, and $\mathcal{J}$ is a function mapping each $A \in \mathbb{N}_C$ to a subset $A^\mathcal{I}$ of $\Delta^\mathcal{I}$ and each $r \in \mathbb{N}_R$ to a subset $r^\mathcal{I}$ of $\Delta^\mathcal{I} \times \Delta^\mathcal{I}$. The function $\mathcal{J}$ extends to arbitrary $\mathcal{EL} \perp$ concept expressions as usual:

\[
\begin{align*}
(C \cap D)^\mathcal{I} & := C^\mathcal{I} \cap D^\mathcal{I} & (\top)^\mathcal{I} & := \Delta^\mathcal{I} & (\bot)^\mathcal{I} & := \emptyset \\
(\exists r.C)^\mathcal{I} & := \{ x \in \Delta^\mathcal{I} \mid (x, y) \in r^\mathcal{I} \text{ and } y \in C^\mathcal{I} \}
\end{align*}
\]

An interpretation $\mathcal{I}$ \textit{satisfies} a CI $C \subseteq D$, in symbols $\mathcal{I} \models C \subseteq D$, iff $C^\mathcal{I} \subseteq D^\mathcal{I}$. It satisfies a TBox $\mathcal{T}$ if it satisfies all CIs in $\mathcal{T}$. A TBox $\mathcal{T}$ \textit{entails} a CI $C \subseteq D$, written $\mathcal{T} \models C \subseteq D$, iff all interpretations satisfying $\mathcal{T}$ also satisfy $C \subseteq D$. We write $\Sigma_\mathcal{I}$ for the set of concept or role names $X$ such that $X^\mathcal{I} \neq \emptyset$. A \textit{finite interpretation} is an interpretation with a finite domain.

\textbf{Description Graphs, Products, and Unravellings}

We also use the notion of description graphs $\mathcal{G}$. The \textit{description graph} $\mathcal{G}(\mathcal{I}) = (V_I, E_I, L_I)$ of an interpretation $\mathcal{I}$ is defined as (e.g. Figure 1):

1. $V_I = \Delta^\mathcal{I}$;
2. $E_I = \{(x, r, y) \mid r \in \mathbb{N}_R \text{ and } (x, y) \in r^\mathcal{I}\}$;
3. $L_I(x) = \{A \in \mathbb{N}_C \mid x \in A^\mathcal{I}\}$.

The \textit{description tree} of an $\mathcal{EL} \perp$ concept expression $C$ over the signature $\Sigma$ is the finite directed tree $\mathcal{G}(C) = (V_C, E_C, L_C)$ where $V_C$ is the set of nodes, $E_C \subseteq V_C \times \mathbb{N}_R \times V_C$ is the set of edges, and $L_C : V \to 2^{\mathbb{N}_C}$ is the labelling function. $\mathcal{G}(C)$ is defined inductively:

1. for $C = \top$, $V_C = \{\rho_C\}$ and $L_C(\rho_C) = \emptyset$ where $\rho_C$ is the root node of the tree;
2. for $C = A \in \mathbb{N}_C$, $V_C = \{\rho_C\}$ and $L_C(\rho_C) = A$;
3. for $C = D_1 \cap D_2$, $\mathcal{G}(C)$ is obtained by merging the roots $\rho_{D_1}, \rho_{D_2}$ in one $\rho_C$ with $L_C(\rho_C) = L_{D_1}(\rho_{D_1}) \cup L_{D_2}(\rho_{D_2})$;
4. for $C = \exists r.D$, $\mathcal{G}(C)$ is built from $\mathcal{G}(D)$ by adding a new node (root) $\rho_C$ to $V_D$ and an edge $(\rho_C, r, \rho_D)$ to $E_D$.

The \textit{concept expression} (unique up to logical equivalence) $\mathbf{C}(\mathcal{G}_v)$ of a tree shaped graph $\mathcal{G}_v = (V, E, L)$ rooted in $v$ is

\[
\prod_{i=1}^{k} P_i \cap \prod_{j=1}^{l} \exists r_j. C(\mathcal{G}_{w_j}),
\]

where $L(v) = \{P_i \mid 1 \leq i \leq k\}$, $(v, r_j, w_j) \in E$ (and there are $l$ such edges) and $\mathbf{C}(\mathcal{G}_{w_j})$ is inductively defined, with $\mathcal{G}_{w_j}$ being the subgraph of $\mathcal{G}$ rooted in $w_j$. 
A walk in a description graph $G = (V, E, L)$ between two nodes $u, v \in V$ is a word $w = v_0 r_0 v_1 r_1 \ldots r_{n-1} v_n$ where $v_0 = u$, $v_n = v$, $v_i \in V$, $r_i \in \mathbb{N}_R$ and $(v_i, r_i, v_{i+1}) \in E$ for all $i \in \{0, \ldots, n - 1\}$. The length of $w$ in this case is $n$, in symbols, $|w| = n$. Walks with length $n = 0$ are possible, it means that the walk has just one vertex (no edges). Vertices and edges may occur multiple times in a walk. Let $G = (V, E, L)$ be an $\mathcal{EL}^+$ description graph with $x \in V$ and $d \in \mathbb{N}$. Denote by $\delta(w)$ the last vertex in the walk $w$. The unravelling of $G$ up to depth $d$ is the description graph $G^x_d = (V_d, E_d, L_d)$ starting at node $x$ defined as follows:

1. $V_d$ is the set of all directed walks in $G$ that start at $x$ and have length at most $d$;
2. $E_d = \{(w, r, wrv) \mid v \in V, r \in \mathbb{N}_R, w, wrv \in V_d\}$;
3. $L_d(w) = L(\delta(w))$.

A path is a walk where vertices do not repeat.

Let $G_1, \ldots, G_n$ be $n$ description graphs such that $G_i = (V_i, E_i, L_i)$. Then the product of $G_1, \ldots, G_n$ is the description graph $(V, E, L)$ defined as:

1. $V = \times_{i=1}^n V_i$;
2. $E = \{(v_1, \ldots, v_n), r, (w_1, \ldots, w_n) \mid r \in \mathbb{N}_R, (v_i, r, w_i) \in E_i, \text{ for all } 1 \leq i \leq n\}$;
3. $L(v_1, \ldots, v_n) = \bigcap_{i=1}^n L_i(v_i)$.

If each $G_i$ is a tree with root $v_i$ then we denote by $\prod_{i=1}^n G_i$ the tree rooted in $(v_1, \ldots, v_n)$ contained in the product graph of $G_1, \ldots, G_n$. 
3 Mining $\mathcal{EL}^\perp$ Bases

The set of all $\mathcal{EL}^\perp$ CIs that are satisfied by an interpretation $\mathcal{I}$ is in general infinite because whenever $\mathcal{I} \models C \sqsubseteq D$, $\mathcal{I} \models \exists r.C \sqsubseteq \exists r.D$ as well. Therefore one is interested in a finite and small set of CIs that entails the whole set of valid CIs. For mining such a set of CIs from a given interpretation we employ ideas from FCA and recall literature results.

Definition 1. A TBox $\mathcal{T}$ is a base for a finite interpretation $\mathcal{I}$ and a DL language $L$, if for every CI $C \sqsubseteq D$, formulated within $L$ and $\Sigma_I$: $\mathcal{I} \models C \sqsubseteq D$ iff $\mathcal{T} \models C \sqsubseteq D$.

We say that a DL has the finite base property (FBP) if, for all finite interpretations $\mathcal{I}$, there is a finite base with CIs formulated within the DL language and $\Sigma_I$. Not all DLs have the finite base property. Consider for instance the fragments $\mathcal{EL}^\perp_{\text{rhs}}$ (and $\mathcal{EL}^\perp_{\text{lhs}}$) of $\mathcal{EL}^\perp$ that allows only concept names on the left-hand (right-hand) side but complex $\mathcal{EL}^\perp$ concept expressions on the right-hand (left-hand) side of CIs.

Proposition 1. $\mathcal{EL}^\perp_{\text{rhs}}$ and $\mathcal{EL}^\perp_{\text{lhs}}$ do not have the FBP.

Proof.(Sketch) No finite base $\mathcal{EL}^\perp_{\text{rhs}}$ exists for the interpretation in Figure 5 (i). For every $n \geq 1$, the $\mathcal{EL}^\perp_{\text{rhs}}$ base should entail the CI $A \sqsubseteq \exists r^n.\top$. Similarly, no finite $\mathcal{EL}^\perp_{\text{lhs}}$ base exists for the interpretation in Figure 5 (ii). For every $n \geq 1$, the $\mathcal{EL}^\perp_{\text{lhs}}$ base should entail the CI $\exists s.\exists n.B \sqsubseteq A$. 

The main difficulty in creating an $\mathcal{EL}^\perp$ base is knowing how to define the role depth of concept expressions in the base. In a finite interpretation, an arbitrarily
large role depth means the presence of a cyclic structure in the interpretation. However, $\mathcal{EL}^\bot$ concept expressions cannot express cycles. The difficulty can be overcome by extending $\mathcal{EL}^\bot$ with greatest fix-point semantics. It is known that the resulting DL, called $\mathcal{EL}^\bot_{gfp}$, has the FBP \cite{4, 11}. The authors then show how to transform an $\mathcal{EL}^\bot_{gfp}$ base into an $\mathcal{EL}^\bot$ base, thus, establishing that $\mathcal{EL}^\bot$ also enjoys the FBP.

In the following, we show that, although finite, the role depth of a base for $\mathcal{EL}^\bot$ and a (finite) interpretation $I$ can be exponential in the size of $I$.

**Example 2.** Consider $I$ represented in the shaded area in Figure 6. For $p_1 = 2, p_2 = 3, p_3 = 5$ and for all $k \in \mathbb{N}^+$, we have that $x_i \in (\exists r^{k \cdot p_i - 1}.A)^I$, where $1 \leq i \leq 3$. We know that $30 = \min(\bigcap_{i=1}^3 \{k \cdot p_i \mid k \in \mathbb{N}^+\}) = \prod_{i=1}^3 p_i$ (which is the least common multiple). We also know that for any $n, p \in \mathbb{N}^+$, $n + 1$ is a multiple of $p$ iff $n$ is a multiple of $p$ minus 1. Therefore, the number

$$d = \min\left(\bigcap_{i=1}^3 \{k \cdot p_i - 1 \mid k \in \mathbb{N}^+\}\right),$$

such that $\{x_1, x_2, x_3\} = B^X = (\exists r^d.A)^I$, is $\prod_{i=1}^3 p_i - 1 = 29$. A base for $I$ should have the CI with role depth at least $d$ because it has to entail the CI $B \sqsubseteq \exists r^d.A$.

**Theorem 3.** There is a finite interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ such that any $\mathcal{EL}^\bot$ base for $\mathcal{I}$ has a concept expression with role depth exponential in the size of $\mathcal{I}$.

**Proof.** (Sketch) We can generalise Example 2 to the case where we have an interpretation $\mathcal{J}$ that for an arbitrary $n > 1$, and for every $i \in \{1, \cdots, n\}$ and $k \in \mathbb{N}^+$, there is an $x \in \Delta^\mathcal{J}$ that satisfies $x \in (\exists r^{k \cdot p_i - 1}.A)^\mathcal{J}$ where $p_i$ is the $i$-th prime number. In this case, the minimal role depth of concepts in any base for $\mathcal{J}$ must be $d \geq \prod_{i=1}^n p_i - 1 \geq 2^n$. \[\Box\]

**Fig. 6.** Description graph of an interpretation $\mathcal{I}$. Let $X = \{x_1, x_2, x_3\}$. For all $d < 29$ we have $x_4 \in C((\prod_{x \in X} \mathcal{G}(\mathcal{J}^d))^X = (B \cap \exists r^d.\top)^I$. However, for all $k \geq 29$, $x_4 \not\in C((\prod_{x \in X} \mathcal{G}(\mathcal{J}^d))^X$ since $x_4 \not\in (\exists r^{29}.A)^I$.

In addition to the role depth of the concept expressions in the base, the size of the base itself can also be exponential in the size of the data given as
input, which is a well-known result in classical FCA \cite{17}. The DL setting is more challenging than classical FCA, and so, this lower bound also holds in the problem we consider. In Section 4, we present our definition of an $\mathcal{EL}^\perp$ base for a finite interpretation $\mathcal{I}$ and highlight cases in which the role depth is polynomial in the size of $\mathcal{I}$.

4 Adaptable Role Depth

We present in this section our main result which is our strategy to construct $\mathcal{EL}^\perp$ bases with adaptable role depth. To define an $\mathcal{EL}^\perp$ base, we use the notion of a model-based most specific concept (MMSC) up to a certain role depth. The MMSC plays a key role in the computation of a base from a given finite interpretation.

**Definition 2.** An $\mathcal{EL}^\perp$ concept expression $C$ is a model-based most specific concept of $X \subseteq \Delta^I$ with role depth $d \geq 0$ iff (1) $X \subseteq C^I$, (2) $C$ has role depth at most $d$, and (3) for all $\mathcal{EL}^\perp$ concept expressions $D$ with role depth at most $d$, if $X \subseteq D^I$ then $\emptyset \models C \sqsubseteq D$.

For a given $X \subseteq C^I$ and a role depth $d$ there may be multiple MMSCs (always at least one \cite{8}) but they are logically equivalent. So we write ‘the’ MMSC of $X$ with role depth $d$ (in symbols $\text{mmsc}(X, I, d)$), meaning a representative of such class of concepts. As a consequence of Definition 2, if $X = \emptyset$ then $\text{mmsc}(X, I, d) \equiv \bot$ for any interpretation $I$ and $d \in \mathbb{N}$.

**Example 4.** Consider the interpretation $\mathcal{I}$ in Figure 4 and let $X = \{x_1, x_2\}$. We have that $\text{mmsc}(X, I, 1) \equiv \text{City} \sqcap \exists \text{government.Party} \sqcap \exists \text{partof.Region}$. With an increasing $k$, the concept expression $\text{mmsc}(X, I, k)$ can become more and more specific. Indeed, $\text{mmsc}(X, I, 2) \equiv \text{mmsc}(X, I, 1) \sqcap \exists \text{partof.(Region} \sqcap \exists \text{capital.\top})$ which is more specific than $\text{mmsc}(X, I, 1)$. However, for any $k \geq 2$, we have that $\text{mmsc}(X, I, 2) \equiv \text{mmsc}(X, I, k)$.

A straightforward (and inefficient) way of computing $\text{mmsc}\{X, I, d\}$, for a fixed $d$, would be conjoining every $\mathcal{EL}^\perp$ concept expression $C$ (over $\mathbb{N}_C \cup \mathbb{N}_R$) such that $X \subseteq C^I$ and the depth of $C$ is bounded by $d$. A more elegant method for computing MMSCs is based on the product of description graphs and unravelling cyclic concept expressions up to a sufficient role depth.

The MMSC can be written as the concept expression obtained from the product of description graphs of an interpretation \cite{9}. Formally, if $\mathcal{I} = (\Delta^I, \cdot^I)$ is a finite interpretation, $X = \{x_1, \ldots, x_n\} \subseteq \Delta^I$ and $a d \geq 0$, then $\text{mmsc}\{X, I, d\} \equiv C([\prod_{i=1}^n \Theta(I)^d])$.

The interesting challenge is how to identify the smallest $d$ that satisfies the property: if $x \in \text{mmsc}(X, I, d)^I$, then $x \in \text{mmsc}(X, I, k)^I$ for every $k > d$. In the following, we develop a method for computing MMSCs with a role depth that is suitable for building an $\mathcal{EL}^\perp$ base of the given interpretation. This method is based on the already mentioned MVF notion, defined as follows.
Definition 3. Given a description graph $\mathcal{G} = (V, E)$ with $u \in V$, we define the maximum vertices from (or MVF) $u$ in $\mathcal{G}$, denoted $\text{mvf}(\mathcal{G}, u)$, as:

$$\max\{v_{\text{num}}(w) \mid w \text{ is a walk in } \mathcal{G} \text{ starting at } u\}$$

where $v_{\text{num}}(w)$ is the number of distinct vertices occurring in $w$. Additionally, we define the function $\text{mmvf}$ as follows:

$$\text{mmvf}(\mathcal{G}) := \max_{u \in V} \text{mvf}(\mathcal{G}, u).$$

In other words, MVF measures the maximum number of distinct vertices that a walk with a fixed starting point can visit in the graph.

Example 5. Consider the interpretation $\mathcal{I}$ in Figure 4. Any walk in the description graph of $\mathcal{I}$ starting at $x_1$ will visit at most three distinct vertices (including $x_1$). Although there are four vertices reachable from $x_1$, we have that $\text{mvf}(\mathcal{G}(\mathcal{I}), x_1) = 3$. For the vertex $x_2$, there are walks of any finite length, but we visit at most three distinct vertices, namely, $x_2, x_4, x_7$, and $\text{mvf}(\mathcal{G}(\mathcal{I}), x_2) = 3$.

For computing the MMSC up to a sufficient role depth based on MVF we use the following notion of simulation.

Definition 4. Let $\mathcal{G}_1 = (V_1, E_1, L_1)$, $\mathcal{G}_2 = (V_2, E_2, L_2)$ be $\mathcal{EL}^\perp$ description graphs and $(v_1, v_2) \in V_1 \times V_2$. A relation $Z \subseteq V_1 \times V_2$ is a simulation from $(\mathcal{G}_1, v_1)$ to $(\mathcal{G}_2, v_2)$, if (1) $(v_1, v_2) \in Z$, (2) $(w_1, w_2) \in Z$ implies $L_1(w_1) \subseteq L_2(w_2)$, and (3) $(w_1, w_2) \in Z$ and $(w_1, r, w'_2) \in E_1$ imply there is $w'_2 \in V_2$ such that $(w_2, r, w'_2) \in E_2$ and $(w'_1, w'_2) \in Z$.

Simulations can be used to decide whether an individual from an interpretation domain belongs to the extension of a given concept expression.

Lemma 6 ([8]). Let $\mathcal{I}$ be an interpretation, let $C$ be an $\mathcal{EL}^\perp$ concept expression, and let $\mathcal{G}(C) = (V_C, E_C, L_C)$ be the $\mathcal{EL}^\perp$ description graph of $C$ with root $\rho_C$. For every $x \in \Delta^{\mathcal{I}}$, there is a simulation from $(\mathcal{G}(C), \rho_C)$ to $(\mathcal{G}(\mathcal{I}), x)$ iff $x \in C^{\mathcal{I}}$.

Lemma 6 together with other previous results is used below to prove Lemma 7 which is crucial for defining the adaptable role depth. It shows the upper bound on the required role depth of the MMSC.

Lemma 7. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \mathcal{I})$ be a finite interpretation and take an arbitrary $X = \{x_1, \ldots, x_n\} \subseteq \Delta^{\mathcal{I}}$, $x' \in \Delta^{\mathcal{I}}$, and $k \in \mathbb{N}$. Let

$$d = \text{mvf} \left( \prod_{i=1}^{n} \mathcal{G}(\mathcal{I}), (x_1, \ldots, x_n) \right) - \text{mvf}(\mathcal{G}(\mathcal{I}), x').$$

If $x' \in C \left( \prod_{i=1}^{n} \mathcal{G}(\mathcal{I})_{d}^{x_i} \right)^{\mathcal{I}}$ then $x' \in C \left( \prod_{i=1}^{n} \mathcal{G}(\mathcal{I})_{d}^{x_i} \right)^{\mathcal{I}}$.

Proof. (Sketch) We show in the long version [13] the following claim.
Claim. For all description graphs $G = (V, E, L)$ and $G' = (V', E', L')$, all vertices $v \in V$ and $v' \in V'$, and

$$d = \text{mvf}(G, v) \cdot \text{mvf}(G', v')$$

if there is a simulation $Z_d : (G_d^v, v) \mapsto (G', v')$, then there is a simulation $Z_k : (G_k^v, v) \mapsto (G', v')$ for all $k \in \mathbb{N}$.

If $k \leq d$, one can restrict $Z_d$ to the vertices of $G_k^v$, which would be a subgraph of $G_d^v$. Otherwise, the intuition behind this claim is that the pairs in $Z_d$ define a walk in $G'$ for each walk in $G$ that has length at most $d - 1$. And if a walk in $G$ has length at least $d$, then there is a vertex $w$ that this walk visits twice while the image of this walk in $G'$ also repeats a vertex at the same time. This paired repetition can be used to find a matching vertex in $V'$ for each vertex of $G_k^v$ by recursively shortening the walk that this vertex corresponds to if it has length $d$ or larger.

Lemma 6 and $x' \in C [(\prod_{i=1}^n G(I)_{d_i}^x)]^T$ imply that there is a simulation $Z_d$ from $(\prod_{i=1}^n G(I)_{d_i}^x, (x_1, \ldots, x_n))$ to $(G(I), x')$. Then, by Claim 4 there is a simulation $Z_k : (\prod_{i=1}^n G(I)_{d_i}^x, (x_1, \ldots, x_n)) \mapsto (G(I), x')$ (we just need to take $G = \prod_{i=1}^n G(I), G' = G(I), v = (x_1, \ldots, x_n)$ and $v' = x'$). Therefore, Lemma 6 implies that $x' \in C [(\prod_{i=1}^n G(I)_{d_i}^x)]^T$.

Lemma 7 shows that even for vertices that are parts of cycles, there is a certain depth of unravellings, which we call a fixpoint, that is guaranteed to be an upper bound.

Proposition 8 gives an intuition about how large the MVF of a vertex in a product graph can be when compared to the MVF of the corresponding vertices in the product's factors.

**Proposition 8.** Let $\{G_i \mid 1 \leq i \leq n\}$ be $n$ description graphs such that $G_i = (V_i, E_i, L_i)$. Also let $v_i \in V_i$. Then:

$$\text{mvf}\left(\prod_{i=1}^n G_i, (v_1, \ldots, v_n)\right) \leq \prod_{i=1}^n \text{mvf}(G_i, v_i).$$

**Proof.** Let $w$ be an arbitrary walk in $\prod_{i=1}^n G_i, (v_1, \ldots, v_n)$ that starts in $(v_1, \ldots, v_n)$ and let $(w_1, \ldots, w_n)$ be a vertex in this walk. It follows from the definition of product that each $w_i$ belongs to a walk in $G_i$ that begins in $v_i$. Therefore, there are only $\text{mvf}(G_i, v_i)$ options for each $w_i$. Hence, there are at most $\prod_{i=1}^n \text{mvf}(G_i, v_i)$ possible options for $(w_1, \ldots, w_n)$. In other words, $\nu_{\text{num}}(w) \leq \prod_{i=1}^n \text{mvf}(G_i, v_i)$. Since $w$ is arbitrary, we can conclude that

$$\text{mvf}\left(\prod_{i=1}^n G_i, (v_1, \ldots, v_n)\right) \leq \prod_{i=1}^n \text{mvf}(G_i, v_i).$$

$\Box$
Although the MVF of a product can be exponential in $|\Delta^2|$, there are many cases in which it is linear in $|\Delta^2|$. Example 9 illustrates one such case.

**Example 9.** Consider the interpretation of Figure 4. The elements $x_1, x_3, x_4, x_5$ and $x_6$ never reach cycles, therefore, each of them can only have walks up to a finite length. Take $X = \{x_1, x_2\}$. Since every walk in $G(I)$ starting from $x_1$ has length at most 2, the longest walk possible in $\prod_{i=1}^{2} G(I)$ starting at node $(x_1, x_2)$ is: $(x_1, x_2), \text{partof}, (x_5, x_7), \text{capital}, (x_6, x_2)$. Thus

$$\text{mvf} \left( \prod_{i=1}^{2} G(I), (x_1, x_2) \right) = 2.$$ 

Take $X = \{x_1, x_7\}$. Since $x_1$ and $x_7$ do not share labels in their outgoing edges

$$\text{mvf} \left( \prod_{i=1}^{1} G(I), (x_1, x_7) \right) = 1.$$ 

The observations about the MVF in Example 9 are generalised in Lemma 10 which shows a sufficient condition for polynomial (linear) role depth.

**Lemma 10.** Let $\mathcal{I} = (\Delta^2, \mathcal{I})$ be a finite interpretation and $X = \{x_1, \ldots, x_n\} \subseteq \Delta^2$. If for some $1 \leq i \leq n$ it holds that every walk in $G(I)$ starting at $x_i$ has length at most $m$ for some $m \in \mathbb{N}$, then $\text{mvf} \left( \prod_{i=1}^{n} G(I), (x_1, \ldots, x_n) \right) \leq \text{mvf} \left( G(\mathcal{I}), x_i \right)$.

**Proof.** (Sketch) As it happens in Example 9, it can be proven that whenever there is a vertex $x_i$ for which every walk starting at it has length at most $m$, then $m$ also bounds the lengths of the walks starting at $(x_1, \ldots, x_n)$ in $\prod_{i=1}^{n} G(I)$. \hfill \square

Combining the bounds for the fixpoint and MVF given by Lemmas 7 and 10, we can define a function that returns an upper approximation of the fixpoint, for any subset of the domain of an interpretation, as follows.

**Definition 5.** Let $\mathcal{I} = (\Delta^2, \mathcal{I})$ be a finite interpretation and $X = \{x_1, \ldots, x_n\} \subseteq \Delta^2$. Also let

$$X_{lim} = \{x \in X \mid \exists m \in \mathbb{N} : \text{every walk starting at } x \text{ in } G(\mathcal{I}) \text{ has length } \leq m\}.$$ 

The function $d_{\mathcal{I}} : \mathcal{P}(\Delta^2) \mapsto \mathbb{N}$ is defined as follows:

$$d_{\mathcal{I}}(X) = \begin{cases} 
  d - 1 & \text{if } X_{lim} \neq \emptyset \\
  d \cdot \text{mmvf}(G(\mathcal{I})) & \text{otherwise,}
\end{cases}$$

where $d = \text{mvf} \left( \prod_{i=1}^{n} G(\mathcal{I}), (x_1, \ldots, x_n) \right)$. 
Next, we prove that function $d_I$ is indeed an upper bound for the fixpoint of an MMSC. The idea sustaining Lemma 11 is that if $x \in X \subseteq \Delta^I$ and every walk in $G(I)$ starting at $x$ has length at most $m$, then $m$ can be used as a fixpoint depth for the MMSC of $X$ in $I$. Lemma 7 covers the cases where vertices are the starting point of walks of any length.

**Lemma 11.** Let $I = (\Delta^I, \cdot)$ be a finite interpretation and $X \subseteq \Delta^I$. Then, for any $k \in \mathbb{N}$, it holds that:

$$\text{mmsc} (X, I, d_I(X))^I \subseteq \text{mmsc} (X, I, k)^I.$$  

**Proof.** (Sketch) Let $X = \{x_1, \ldots, x_n\} \subseteq \Delta^I$. If $k \leq d_I(X)$, the lemma holds trivially. For $k > d_I(X)$ we divide the proof in two cases. First, if there is a $x_i \in X$ such that every walk in $G(I)$ starting at $x_i$ has length at most $m$ for some $m \in \mathbb{N}$, then as stated in Lemma 10, every walk in $\prod_{i=1}^n G(I)$ starting at $(x_1, \ldots, x_n)$ has length at most $\text{mvf} (\prod_{i=1}^n G(I), (x_1, \ldots, x_n)) - 1$.

In other words, even when $k > d_I(X)$, we have:

$$\prod_{i=1}^n G(I)_{k} = \prod_{i=1}^n G(I)_{d_I(X)}$$

and therefore, we can apply Lemma 9 to conclude that: $\text{mmsc} (X, I, d_I(X))^I \subseteq \text{mmsc} (X, I, k)^I$. Otherwise, if $X_{lim} \neq \emptyset$, the lemma is a direct consequence of Definition 5 and Lemma 7. □

In the remaining of this paper, we write $\text{mmsc} (X, I)$ as a shorthand for $\text{mmsc} (X, I, d_I(X))^I$. An important consequence of Lemma 11 and the definition of MMSC is that, for any $\mathcal{EL}^\perp$ concept expression $C$ and finite interpretation $I$, it holds that $C^I = \text{mmsc} (C^I, I)^I$.

**Lemma 12.** Let $I = (\Delta^I, \cdot)$ be a finite interpretation. Then, for all $\mathcal{EL}^\perp$ concept expression $C$ it holds that: $\text{mmsc} (C^I, I)^I = C^I$.

**Proof.** Direct consequence of Lemma 4.4 (vi) of 5 and Lemma 11. □

We use this result below to define a finite set of concept expressions $M_I$ for building a base of the CIs valid in $I$.

**Definition 6.** Let $I = (\Delta^I, \cdot)$ be a finite interpretation. The set $M_I$ is the union of $\{\bot\} \cup \mathbb{N}_C$ and

$$\{\exists r. \text{mmsc} (X, I) | r \in \mathbb{N}_R \text{ and } X \subseteq \Delta^I, X \neq \emptyset\}$$

We also define $A_I = \{\bigcap U | U \subseteq M_I\}$.

Building the base mostly relies on the fact that, given a finite interpretation $I$, for any $\mathcal{EL}^\perp$ concept expression $C$, there is a concept expression $D \in A_I$ such that $C^I = D^I$. 

Theorem 13. Let $\mathcal{I}$ be a finite interpretation and let $\Lambda_\mathcal{I}$ be defined as above. Then,

$$
\mathcal{B}(\mathcal{I}) = \{ C \equiv \text{mmsc}(C^\mathcal{I}, \mathcal{I}) \mid C \in \Lambda_\mathcal{I} \} \cup \{ C \sqsubseteq D \mid C, D \in \Lambda_\mathcal{I} \text{ and } \mathcal{I} \models C \sqsubseteq D \}
$$

is a finite $\mathcal{EL}^\perp$ base for $\mathcal{I}$.

Proof. (Sketch) As $\Lambda_\mathcal{I}$ is finite, so is $\mathcal{B}(\mathcal{I})$. The CIs are clearly sound and the soundness of the equivalences is due to Lemma 12. For completeness, assume that $\mathcal{I} \models C \sqsubseteq D$. Using an adaptation of Lemma 5.8 from [11] and Lemma 12 above, we can prove, by induction on the structure of the concept expressions $C$ and $D$, that there are concept expressions $E, F \in \Lambda_\mathcal{I}$ such that $\mathcal{B}(\mathcal{I}) \models E \equiv \text{mmsc}(C^\mathcal{I}, \mathcal{I})$, $\mathcal{B}(\mathcal{I}) \models F \equiv \text{mmsc}(D^\mathcal{I}, \mathcal{I})$, $\mathcal{B}(\mathcal{I}) \models C \equiv \text{mmsc}(C^\mathcal{I}, \mathcal{I})$, and $\mathcal{B}(\mathcal{I}) \models D \equiv \text{mmsc}(D^\mathcal{I}, \mathcal{I})$. By construction, as $E \sqsubseteq F \in \mathcal{B}(\mathcal{I})$, we can prove that whenever $\mathcal{I} \models C \sqsubseteq D$, so does $\mathcal{B}(\mathcal{I})$. □

Recall the interpretation $\mathcal{I}$ in Figure 6. In order to compute a base for $\mathcal{I}$, we should compute an MMSC with role depth at least 29. An important benefit of our approach is that the role depth of the other MMSCs, which are part of the mined CIs in the base may be smaller. For instance, the role depth of $\text{mmsc}(\{x_1\}, \mathcal{I})$ is 10. In the next section, we show that one can compute the MVF of a vertex in a graph in linear time in the size of the graph.

5 Computing the MVF

As discussed in Section 4, the MVF is the key to provide an upper bound for the fixpoint for each MMSC. An easy way to estimate the MVF would consist in computing the number of vertices reachable from $v$ in the description graph $\mathcal{G}$. Let $\text{reach}(\mathcal{G}, v)$ be such a function. By definition it holds that $\text{mvf}(\mathcal{G}, v) \leq \text{reach}(\mathcal{G}, v)$. Although $\text{reach}(\mathcal{G}, v)$ can be computed in polynomial time, the difference between these two metrics can be quite large. For instance, consider that $v$ is the root of a description graph $\mathcal{G}$ that is a binary tree with $2^n$ nodes. Then $\text{mvf}(\mathcal{G}, v) = n$, while $\text{reach}(\mathcal{G}, v) = 2^n$.

In this section, we present an algorithm to compute $\text{mvf}(\mathcal{G}, v)$ that takes linear time in the size of $\mathcal{G}$, but first we need to recall some fundamental concepts from Graph Theory, one of them is the notion of strongly connected components (Definition 7).

Definition 7. Let $\mathcal{G} = (V, E, L)$ be a description graph. The strongly connected components (SCCs) of $\mathcal{G}$, in symbols $\text{SCC}(\mathcal{G})$, are the partitions $V_1, \ldots, V_n$ of $V$ such that for all $1 \leq i \leq n$: if $u, v \in V_i$ then there is a path from $u$ to $v$ and a path from $v$ to $u$ in $\mathcal{G}$. Additionally, we define a function $\text{scc}(\mathcal{G}, v)$, which returns the SCC of $\mathcal{G}$ that contains $v$.

A compact way of representing a description graph $\mathcal{G}$ consists in regarding each SCC in $\mathcal{G}$ as a single vertex. This compact graph is a directed acyclic graph (DAG), also called condensation of $\mathcal{G}$ [14], and it is formalised in Definition 8.
Definition 8. Let $\mathcal{G} = (V, E, L)$ be a description graph. The condensation of $\mathcal{G}$ is the directed acyclic graph $\mathcal{G}^* = (V^*, E^*)$ where $V^* = \{\text{scc}(\mathcal{G}, u) \mid u \in V\}$ and $E^* = \{(\text{scc}(\mathcal{G}, u), \text{scc}(\mathcal{G}, v)) \mid (u, r, v) \in E \text{ and scc}(\mathcal{G}, u) \neq \text{scc}(\mathcal{G}, v)\}$. Also, if $w^*$ is path in $\mathcal{G}^*$, the weight of $w^*$, in symbols $\text{weight}(\mathcal{G}^*)$, is the sum of the sizes of the SCCs that appear as vertices of $w^*$.

We use these notions to link the MVF (Definition 3) to the paths in the condensation graph in Lemma 14.

Lemma 14. Let $\mathcal{G} = (V, E, L)$ be a description graph, let $\mathcal{G}^* = (V^*, E^*)$ be the condensation of $\mathcal{G}$, and $v \in V$. Then:

$$\text{mvf}(\mathcal{G}, v) = \max \{\text{weight}(w^*) \mid w^* \text{ is a path in } \mathcal{G}^* \text{ starting at } \text{scc}(\mathcal{G}, v)\}.$$ 

Proof. (Sketch) First we prove that every path $w^* = V_1, \ldots, V_m$ in $\mathcal{G}^*$ starting at $\text{scc}(\mathcal{G}, v)$ induces a walk $w$ in $\mathcal{G}$ starting at $v$ with $v_{\text{num}}(w) = \text{weight}(w^*)$. Then, we show that if $w^*$ has maximal weight, then no walk in $\mathcal{G}$ starting at $v$ can visit more than $\text{weight}(w^*)$ vertices. 

By Lemma 14 we only need to compute the maximum weight of a path in $\mathcal{G}^*$ that starts at $\text{scc}(\mathcal{G}^*, v)$ to obtain the MVF of a vertex $v$ in a description graph $\mathcal{G}$. Algorithm 1 relies on this result and proceeds as follows: first, it computes the SCCs of the description graph and the condensation graph. Then, the algorithm transverses the condensation graph, using an adaptation of depth-first search to determine the maximum path weight for the initial SCC.

Algorithm 1 assumes that the SCCs and condensation are computed correctly. Besides keeping the computed values, the array $\text{ugt}$ prevents recursive calls on SCCs that have already been processed. According to Lemma 14 to prove that Algorithm 1 is correct we just need to prove that the function $\text{maxWeight}$ in fact returns the maximum weight of a path in the condensation given a starting vertex (which corresponds to an SCC in the original graph).

Lemma 15. Given $\mathcal{G} = (V, E, L)$ and $v \in V$ as input, Algorithm 1 returns the maximum weight of a path in the condensation of $\mathcal{G}$ starting at $\text{scc}(\mathcal{G}, v)$.

Proof. (Sketch) Let $\mathcal{G}^* = (V^*, E^*)$ be the condensation of $\mathcal{G}$. If $\text{scc}(\mathcal{G}, v)$ has no successor in $\mathcal{G}^*$, then the output of $\text{maxWeight}$ is correct. If $\text{scc}(\mathcal{G}, v)$ has successors, then the maximum weight of a path staring at $\text{scc}(\mathcal{G}, v)$ in $\mathcal{G}^*$

Fig. 7. Condensation of the description graph in Figure 4. Every vertex is an SCC of the original graph and the edges indicate accessibility between the SCCs. Also, the condensation has no labels.
Algorithm 1: Computing MVF via Lemma 14

**Input:** A description graph $G = (V, E, L)$ and a vertex $v \in V$

**Output:** The MVF of $v$ in $G$, i.e., $\text{mvf}(G, v)$

1. $V^* \leftarrow \text{SCC}(G)
2. E^* \leftarrow \text{condense}(G, V^*)
3. $G^* \leftarrow (V^*, E^*)$
4. for $V' \in V^*$ do
5. \hspace{1em} $wgt[V'] \leftarrow \text{null}$
6. \hspace{1em} return $\text{maxWeight}(G^*, \text{scc}(G, v), wgt)$

// Auxiliary function

7. Function $\text{maxWeight}(G^*, V', wgt)$:
8. \hspace{1em} current $\leftarrow 0$
9. \hspace{1em} for $W' \in \{U' \in V^* \mid (V', U') \in E^*\}$ do
10. \hspace{2em} if $wgt[W'] = \text{null}$ then
11. \hspace{2em} \hspace{1em} current $\leftarrow \max(\text{current}, \text{maxWeight}(G^*, W', wgt))$
12. \hspace{2em} else
13. \hspace{2em} \hspace{1em} current $\leftarrow wgt[W']$
14. \hspace{2em} \hspace{1em} $wgt[V'] \leftarrow current + |V'|$
15. \hspace{1em} return $wgt[V']$

is given by $|\text{scc}(G, v)|$ plus the maximum value computed among its successors. This equation holds because $G^*$ is a DAG.

Lemmas 14 and 15 imply that Algorithm 1 computes the MVF of $v$ in $G$ correctly. Moreover, the computation of SCCs can be done in time $O(|V| + |E|)$ \[26\], the condensation in time $O(|E|)$ \[19\] and the depth-first transversal via $\text{maxWeight}$ in time $O(|V| + |E|)$. Hence, it is possible to compute the MVF of a vertex in a graph in linear time in the size of the description graph even if it consists solely of cycles. Yet, given an interpretation $\mathcal{I} = (\Delta^2, \cdot^\mathcal{I})$ the graph given as input to Algorithm 1 might be a product graph with an exponential number of vertices in $|\Delta^2|$. Also, Algorithm 1 can be modified to compute the MVF for all vertices by starting the function $\text{maxWeight}$ from an unvisited SCC until all vertices are visited in polynomial time in the size of the graph.

### 6 Conclusion

In this work, we introduce a way of computing $\mathcal{EL}^\bot$ bases from finite interpretations that adapts the role depth of concepts according to the the structure of interpretations. Our definition relies on a notion that relates vertices in a graph to sets of vertices, called MVF. We have also shown that the MVF computation can be performed in polynomial time in the size of the underlying graph structure. Our $\mathcal{EL}^\bot$ base, however, is not minimal. As future work, we plan to build on previous results combining FCA and DLs to define a base with minimal cardinality. We will also investigate the problem of mining CIs in the presence of noise in the dataset. We plan to use the support and confidence measures from
association rule mining to deal with noisy data and implement our approach using knowledge graphs as datasets.

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A Proofs for Section 3

We prove that $\mathcal{EL}_{\text{rhs}}$ and $\mathcal{EL}_{\text{lhs}}$ do not have the finite base property (Propositions 16 and 17).

**Proposition 16.** $\mathcal{EL}_{\text{rhs}}$ does not have the finite base property.

**Proof.** Consider the interpretation $I = (\{x_1, x_2\}, \mathcal{T})$ where $\{x_1, x_2\} = r^T$, $\{x_1\} = A^T$ and every other concept and role name is mapped by $\mathcal{T}$ to $\emptyset$ (Figure 5 (i)). In $I$, $A^T = \{x_1\}$ and for all $n \in \mathbb{N}^+$, $x_1 \in (\exists r^n \top)^I$. Assume that $B$ is a base for $I$ and $\mathcal{EL}_{\text{rhs}}$. As $B$ is a (finite) TBox formulated in $\mathcal{EL}_{\text{rhs}}$ with symbols from $\Sigma_I$, it can only have CIs of the form $A \sqsubseteq C$. Since $I \models A \sqsubseteq \exists r^n \top$, for all $n \in \mathbb{N}^+$, it follows that $B$ is infinite, which is a contradiction. □

**Proposition 17.** $\mathcal{EL}_{\text{lhs}}$ does not have the finite base property.

**Proof.** In this proof, assume that CIs are formulated in $\mathcal{EL}_{\text{lhs}}$. Consider the interpretation

$I = (\{x_1, x_2, x_3, x_4\}, \mathcal{T})$

with

$r^T = \{(x_2, x_2), (x_1, x_4), (x_3, x_2)\}$

$s^T = \{(x_1, x_2), (x_3, x_4)\}$

$A^T = \{x_1\}$

$B^T = \{x_2\}$

and every other concept and role name is mapped by $\mathcal{T}$ to $\emptyset$ (see Figure 5 (ii)).

By definition of $I$, for all $n \in \mathbb{N}^+$, we have that $I \models \exists s.\exists r^n.B \sqsubseteq A$. So if $B$ is a base for $\mathcal{EL}_{\text{lhs}}$ and $I$ then $B \models \exists s.\exists r^n.B \sqsubseteq A$ for all $n \in \mathbb{N}^+$. Now, observe that there is no $D$ such that

1. $\emptyset \models \exists s.\exists s.\exists r^n.B \sqsubseteq D$,
2. $\emptyset \not\models D \sqsubseteq \exists s.\exists s.\exists r^n.B$ (where $\not\models$ means ‘does not entail’),
3. and $I \models D \sqsubseteq A$.

The reason for the above is because $x_3 \in D^T$ for all $D$ satisfying (1) and (2) but $x_3 \not\in A^T$. Moreover, there is no $k \in \mathbb{N}^+$ such that $I \models \exists s.\exists r^k.B \sqsubseteq B$ or $I \models \exists s.\exists r^k.B \sqsubseteq A$ (because $x_3 \not\in B^T$ and $x_2, x_3 \not\in A^T$ but $x_2, x_3 \in (\exists r^k.B)^I$). So, $B$ can only entail $\exists s.\exists r^n.B \sqsubseteq A$ if there is a CI in $B$ with a concept equivalent to $\exists s.\exists r^n.B$. This concept needs to have role depth $n$. Since $B \models \exists s.\exists r^n.B \sqsubseteq A$ for all $n \in \mathbb{N}^+$, there are CIs with role depth $n$ for all $n \in \mathbb{N}^+$. This means that $B$ cannot be finite. □
Next, we prove the result which shows that the depth of roles in a base has an exponential lower bound.

**Theorem 3.** There is a finite interpretation $\mathcal{I} = (\Delta^Z, \mathcal{I})$ such that any $\mathcal{EL}^\perp$ base for $\mathcal{I}$ has a concept expression with role depth exponential in the size of $\mathcal{I}$.

**Proof.** For any $n \geq 1$, we consider the interpretation $\mathcal{I}$ where for every $i \in \{1, \cdots, n\}$ and $k \geq 1$, there is $x_i \in \Delta^Z$ that satisfies $x_i \in (\exists^{k \cdot p_i - 1} A)^\mathcal{I}$, $x_i \in B^\mathcal{I}$, and $x_i \notin (\exists r^d A)^\mathcal{I}$ for $l \notin \{k \cdot p_i - 1 | k \geq 1\}$ where $p_i$ is the $i$-th prime number.

We know that $\min(\bigcap_{i=1}^n \{k \cdot p_i | k \geq 1\}) = \prod_{i=1}^n p_i$ (which is the least common multiple). We also know that for any $n, p \in \mathbb{N}^+$, $n + 1$ is a multiple of $p$ iff $n$ is a multiple of $p$ minus 1. Therefore, $d = \min(\bigcap_{i=1}^n \{k \cdot p_i - 1 | k \geq 1\})$, is the minimal number such that $B^\mathcal{I} = (\exists r^d A)^\mathcal{I}$. Since $d = \prod_{i=1}^n p_i - 1 \geq 2^n$, the statement holds because a base for $\mathcal{I}$ should entail the CI $B \subseteq \exists r^d A$. For this to happen, it should have a CI with role depth at least $d$. \hfill \Box

### B Proofs for Section 4

Now we will prove Claim 18 which is part of Lemma 1 which underlies our approach. Before that, we need an additional result regarding simulations, which allows us to view them as functions.

**Lemma 18.** Let $G_1 = (V_1, E_1, L_1)$ and $G_2 = (V_2, E_2, L_2)$ be two $\mathcal{EL}^\perp$ description graphs. Let $Z : (G_1, v_1) \mapsto (G_2, v_2)$ be a simulation. Then, there exists a simulation $Z' : (G_1, v_1) \mapsto (G_2, v_2)$ such that for every $v \in V_1$, there is at most one $w \in V_2$ such that $(v, w) \in Z'$.

**Proof.** Assume that $Z$ is a simulation from $(G_1, v_1)$ to $(G_2, v_2)$. If it satisfies the property that for every $v \in V_1$, there is at most one $w \in V_2$ such that $(v, w) \in Z$, we can take $Z = Z'$.

Let $Z'$ be such that $\{(v_1, v_2)\} \subseteq Z' \subseteq Z$. Also, suppose that it satisfies the following properties.

- If $(v, w) \in Z$, then exists $w' \in V_2$ such that $(v, w') \in Z'$.
- If $(v, w) \in Z'$ and $(v, w') \in Z'$ then $w = w'$.

A subset $Z' \subseteq Z$ satisfying these two properties always exists: we just leave in $Z'$ one pair $(v, w)$ for each $v \in V_1$ such that $(v, w') \in Z$ for some $w' \in V_2$.

We will show that if $(v_1, v_2) \in Z'$, then $Z'$ is a simulation from $(G_1, v_1)$ to $(G_2, v_2)$.

1. $(v_1, v_2) \in Z'$ is assumed.
2. If $(v, w) \in Z'$, then $(v, w) \in Z$, therefore $L_1(v) \subseteq L_2(w)$ since $Z$ is a simulation.
3. If \((v, w) \in Z'\) and \((v, r, w') \in E_1\), then we know that \((w, r, w') \in E_2\) and that \((w', w') \in Z\) for some \(w' \in V_2\). If \((w', w') \in Z'\) then (3) holds for \(Z'\).

Otherwise, by construction of \(Z'\), there is a \(w_2' \in V_2\) such that \((w', r, w_2') \in E_2\) and \((v', w_2') \in Z'\), which proves (3) for \(Z'\) in this case.

Therefore, \(Z': (G_1, v_1) \rightarrow (G_2, v_2)\) is a simulation such that for every \(v \in V_1\), there is at most one \(w \in V_2\) such that \((v, w) \in Z'\).

Now we proceed to the claim’s actual proof.

**Claim.** For all description graphs \(G = (V, E, L)\) and \(G' = (V', E', L')\), all vertices \(v \in V\) and \(v' \in V'\), and

\[
d = \text{mf}(G, v) \cdot \text{mf}(G', v')
\]

if there is a simulation \(Z_d : (G_d^v, v) \rightarrow (G', v')\), then there is a simulation \(Z_k : (G_k^v, v) \rightarrow (G', v')\) for all \(k \in \mathbb{N}\).

**Proof.** Let \(G, G', v\) and \(v'\) as stated earlier and consider the unravellings

\[
G_d^v = (V_d, E_d, L_d) \text{ and } G_k^v = (V_k, E_k, L_k)
\]

of \(G\). Now, assume that there is a simulation \(Z_d : (G_d^v, v) \rightarrow (G', v')\). By Lemma 18, we can assume w.l.o.g. that for each \(w \in V_d\) there exists at most one \(u \in V'\) such that \((w, u) \in Z_d\). Therefore, we can define a function \(z\) such that \(z(w)\) is the only vertex in \(V'\) such that \((w, z(w)) \in Z_d\).

If \(k \leq d\) then, as \(G_k^v\) is a subtree of \(G_d^v\) and thus, \(V_k \subseteq V_d\), one can just take \(Z_k = \{(w, z(w)) \mid w \in V_k\}\) as simulation. We now argue about the case where \(k > d\). Recall that the function \(\delta\) returns the vertex of a graph that occurs at end of a path. We show that in any path of length \(d\) in \(G_d^v\), there are two vertices \(w_1\) and \(w_2\) such that \(\delta(w_1) = \delta(w_2)\) and \(z(w_1) = z(w_2)\).

In what follows, we use the fact that unravellings are trees, and thus, for each vertex in an unravelling, there is exactly one path starting from the root to it. So we can refer to this path without ambiguity. Moreover, if \(w\) is a vertex in an unravelling with root \(v\), then the path distance of \(w\) is the length of the path from \(v\) to \(w\).

Now, let \(w = w_0r_0\ldots r_{n-1}w_n\) be a vertex in \(V_d\) and let \(w_i = w_0r_0\ldots r_{i-1}w_i\) for \(0 \leq i \leq n\) be the vertices in the path from \(v\) to \(w\) \((w_0 = v\) and \(w_n = w\)). The path from \(v\) to \(w\) in \(G_d^v\) determines a walk \(w^*\) in \(G\) starting at \(v\) as follows:

\[
w^* = \delta(w_0)r_0\ldots r_{n-1}\delta(w_n).
\]

Due to the definition of \(\text{mf}\) there can be at most \(\text{mf}(G, v)\) distinct values of \(\delta\) for all vertices in the path from \(v\) to \(w\), that is, \(|\{\delta(w_i) \mid 0 \leq i \leq n\}| \leq \text{mf}(G, v)\).

As \(Z_d\) is a simulation, the path from \(v\) to \(w\) also determines a walk in \(G'\) starting at \(v'\):

\[
w' = z(w_0)r_0\ldots r_{n-1}z(w_n).
\]

Again, due to the definition of \(\text{mf}\) there can be at most \(\text{mf}(G', v')\) distinct values of \(z\) for all vertices in the path from \(v\) to \(w\), that is, \(|\{z(w_i) \mid 0 \leq i \leq n\}| \leq \text{mf}(G', v')\).
the pigeonhole principle implies that there will be two vertices
from \(v\) by the function \(f\), where \(w'\) is a vertex in the path
from \(v\) to \(w\), i.e.,

\[
|\{(\delta(w_i), z(w_i)) \mid 0 \leq i \leq n\}| \leq d.
\]

If a vertex \(w\) has path distance \(d\) from \(v\) in \(G_d^n\), then there are \(d + 1\) vertices in the path from \(v\) to \(w\). As there are at most \(d\) distinct pairs \((\delta(w'), z(w'))\), where \(w'\) is a vertex in this path, and \(d + 1\) vertices in the path from \(v\) to \(w\), the pigeonhole principle implies that there will be two vertices \(w_1, w_2 \in V_d\) in the path from \(v\) to \(w\) such that both \(z(w_1) = z(w_2)\) and \(\delta(w_1) = \delta(w_2)\).

Let \(\Gamma \subseteq V_d\) be the set of all vertices such that there are no two distinct vertices \(w_1\) and \(w_2\) on the path from \(v\) to \(w\) with \(\delta(w_1) = \delta(w_2)\) and \(z(w_1) = z(w_2)\). Because of the previous argument, \(\Gamma\) contains only vertices whose path distance from \(v\) is strictly less than \(d\).

Since \(G_d^n\) is a description tree with root \(v\), there is exactly one directed path from \(v\) to any given vertex \(w \in V_d\). Hence, if \(w \in \Gamma\) then every vertex \(w'\) on the path from \(v\) to \(w\) in \(G_d^n\) is also in \(\Gamma\). In other words, \(\Gamma\) spans a subtree of \(G_d^n\).

Now, let us consider the set \(\Gamma^n\) composed by the direct successors of the leaves of the subtree determined by \(\Gamma\), that is, \(\Gamma^n = \{w_0r_0 \ldots r_{n-1}w_n \in V_d \setminus \Gamma \mid w_0r_0 \ldots r_{n-2}w_{n-1} \in \Gamma\}\). Since each vertex in \(\Gamma\) has path distance at most \(d - 1\) from \(v\), each vertex \(\Gamma^n\) has path distance at most \(d\) from \(v\). Together with the fact that \(\Gamma\) spans a subtree of \(G_d^n\), for each vertex \(w \in V_d\) with path distance \(d\) from \(v\), there is exactly one vertex \(w' \in \Gamma^n\) in the path from \(v\) to \(w\) (including the extremities).

As we assume \(k > d\), we know that \(G_d^n\) is a subtree of \(G_k^n\), hence \(\Gamma\) also spans a subtree of \(G_k^n\). Therefore \(\Gamma \cup \Gamma^n \in V_k\) and for every vertex \(w \in V_k\) there is exactly one vertex \(w'\) in \(\Gamma^n\) in the path from \(v\) to \(w\) in \(G_k^n\). For each vertex \(w \in V_k\), such \(w'\) can be used to build a simulation from \(Z_d\) that includes \(w\), as we will show next.

For each vertex \(w \in \Gamma \cup \Gamma^n\), there is exactly one vertex \(w'\) in \(\Gamma\) in the path from \(v\) to \(w\) such that \(\delta(w) = \delta(w')\) and \(z(w) = z(w')\). Therefore, we can define a function \(s : \Gamma \cup \Gamma^n \mapsto \Gamma\) which retrieves such vertex for every \(w \in \Gamma \cup \Gamma^n\).

Now, we can use this function \(s\) to find an alternative path in \(V_k\) for each vertex in \(V_k\) when extending \(Z_d\) to the vertices in \(V_k \setminus V_d\). This notion is formalised by the function \(f : V_k \mapsto \Gamma\) defined next, where \(w = w_0r_0 \ldots w_{|w|-1}r_{|w|-1}w_{|w|}\).

\[
f(w) = \begin{cases} 
  s(w) & \text{if } w \in \Gamma \cup \Gamma^n \\
  f(f(w_0r_0 \ldots w_{|w|-1}r_{|w|-1}w_{|w|})) & \text{otherwise}.
\end{cases}
\]

To clarify the rôle of \(f\) in this proof, consider a vertex \(w = w_0r_0 \ldots r_{n-1}w_n \in V_k\) with \(n > d\). As before, let \(w_i = w_0r_0 \ldots r_{i-1}w_i\) for \(0 \leq i \leq n\) be the vertices in the path from \(v\) to \(w\). Since the path distance from \(v\) to \(w\) is more than \(d\), we know that there is one \(1 \leq m \leq n\) such that \(w_m \in \Gamma^n\). We also know that there
is one $0 \leq j < m$ such that $s(w_j) = w_m$. When applying $f$ to $w$, we obtain the following:

$$f(w) = f(\ldots f(f(w_m)r_{m+1}) \ldots)r_{n-1}w_n).$$

Since $w_m \in \overrightarrow{V}$, we have that $f(w_m) = w_j$, which is closer to $v$ than $w_m$. As a consequence of $s(w_j) = w_m$ and the definitions of unravelling and simulation, we know that $(\delta(w_j), r_m, \delta(w_{m+1})) \in E$ and $(z(w_j), r_m, z(w_{m+1})) \in E'$. Because the relation between vertices in $\overrightarrow{V}$ and their image via the function $s$ holds in each step of the recursion, we can add $(w, z(f(w)))$ to $Z_d$ for every vertex in $V_k$ creating a new simulation.

We use this observation to define the relation $Z_k$ as:

$$Z_k = \{(w, z(f(w)) \mid w \in V_k\}.$$

Now we show that $Z_k$ is a simulation from $(G_k^v, v)$ to $(G', v')$.

1. Since $v \in \overrightarrow{V}$ and $Z_d$ is a simulation satisfying the property of Lemma 18 $(v, z(f(v))) = (v, z(v)) = (v, v')$.
2. Since $Z_d$ is a simulation and $f(w) \in V_d$:

$$L_k(w) = L(\delta(w)) = L(\delta(f(w)))$$

$$= L_d(f(w)) \subseteq L'(z(f(w))).$$

3. Let $w \in V_k$ and assume that $(w, r, wry) \in E_k$.

If $wry \in \overrightarrow{V} \cup \overrightarrow{V}$, then $w \in \overrightarrow{V}$. Therefore, $(w, r, wry) \in E_d$. We also have:

$$(z(f(w)), r, z(f(wry))) = (z(s(w)), r, z(s(wry)))$$

$$= (z(w), r, z(wry)).$$

Moreover, $(z(w), r, z(wry)) \in E'$ because $Z_d$ is a simulation. Finally, by construction, $(wry, z(wry)) \in Z_k$.

Otherwise, if $wry \not\in \overrightarrow{V} \cup \overrightarrow{V}$, we have that $f(wry) = f(f(w)ry)$. Since $f(w) \in \overrightarrow{V}$, $f(w)ry \in \overrightarrow{V} \cup \overrightarrow{V}$ and consequently $f(f(w)ry) = s(f(w)ry)$.

By the definition of $s$: $z(s(f(w)ry)) = z(f(w)ry) = z(f(wry))$. Since $Z_d$ is a simulation and $f(w) \in V_d$, it holds that $(z(f(w)), r, z(f(w)ry)) = (z(f(w)), r, z(f(wry))) \in E'$. Thus, $(wry, z(f(w)ry)) = (wry, z(f(wry))) \in Z_k$ which concludes the proof of (S3) for $Z_k$.

Therefore, $Z_k$ is a simulation from $(G_k^v, v)$ to $(G', v')$, which proves the claim. \[\Box\]
Lemma 10 refers to walks in a product graph. To simplify its proof we highlight a relationship between walks in the product graph and walks in their factors via Proposition 19.

**Proposition 19.** Let $G_1, \ldots, G_n$ be $n$ description graphs, with $G_i = (V_i, E_i, L_i)$ for $1 \leq i \leq n$. It holds that, for each walk $w$ in $\prod_{i=1}^{n} G_i$ starting at $(v_1, \ldots, v_n)$, there is a walk in $G_i$ starting at $v_i$ with the same length, for all $1 \leq i \leq n$.

**Proof.** Let $w$ be a walk in $\prod_{i=1}^{n} G_i$ starting in $(v_1, \ldots, v_n)$ with length $m$ as follows:

$$w = (w_{1,0}, \ldots, w_{n,0})r_0 \ldots r_{m-1}(w_{1,m-1}, \ldots, w_{n,m}).$$

The walk $w_i = w_{i,0}r_0 \ldots r_{m-1}w_{i,m}$ is a walk in $G_i$ because $w_{i,j} \in V_i$ for $0 \leq j < m$ and $(w_{i,j}, r_j, w_{i,j+1}) \in E$ due to the definition of product. As $w_{i,0} = v_i$, by construction of $w$, $w_i$ starts at $v_i$. Additionally, by construction, $w_i$ has length $m$, which concludes the proof.

We use Proposition 19 in Lemma 10 below.

**Lemma 10.** Let $\mathcal{I} = (\Delta^\mathcal{I}, \mathcal{I})$ be a finite interpretation and $X = \{x_1, \ldots, x_n\} \subseteq \Delta^\mathcal{I}$. If for some $1 \leq i \leq n$ it holds that every walk in $G(\mathcal{I})$ starting at $x_i$ has length at most $m$ for some $m \in \mathbb{N}$, then $\operatorname{mvf}(\prod_{i=1}^{n} G(\mathcal{I}), (x_1, \ldots, x_n)) \leq \operatorname{mvf}(G(\mathcal{I}), x_i)$.

**Proof.** Let $X = \{x_1, \ldots, x_n\} \subseteq \Delta^\mathcal{I}$ and let

$$X_{lim} = \{x \in X \mid \exists m \in \mathbb{N} : \text{every walk starting from } x \text{ in } G(\mathcal{I}) \text{ has length } \leq m\}.$$

Assume $X_{lim} \neq \emptyset$ and let $x' \in X_{lim}$ be such that

$$\operatorname{mvf}(G(\mathcal{I}), x') = \min_{x \in X_{lim}} \operatorname{mvf}(G(\mathcal{I}), x).$$

Since $x' \in X_{lim}$, every walk in $G(\mathcal{I})$ starting at $x'$ has length bounded by $\operatorname{mvf}(G(\mathcal{I}), x') - 1$. Due to the definition of product of description graphs (recall how the edges are built), this limitation extends to every walk in $\prod_{i=1}^{n} G(\mathcal{I})$ starting at $(x_1, \ldots, x_n)$: they have length at most $\min_{x \in X_{lim}} \operatorname{mvf}(G(\mathcal{I}), x) - 1$. If there was a longer walk, there would be also a walk in in $G(\mathcal{I})$ starting at $x'$ with the same length due to Proposition 19.

In the following, we prove that our adaptable role depth yields an upper bound of the actual fixpoint for an MMSC.

**Lemma 11.** Let $\mathcal{I} = (\Delta^\mathcal{I}, \mathcal{I})$ be a finite interpretation and $X \subseteq \Delta^\mathcal{I}$. Then, for any $k \in \mathbb{N}$, it holds that:

$$\operatorname{mmsc}(X, \mathcal{I}, d_\mathcal{I}(X))^\mathcal{I} \subseteq \operatorname{mmsc}(X, \mathcal{I}, k)^\mathcal{I}.$$
Proof.
Let \( X = \{ x_1, \ldots, x_n \} \subseteq \Delta^I \) and
\[
X_{\text{lim}} = \{ x \in X \mid \exists m \in \mathbb{N} : \text{every walk starting from } x \text{ in } G(I) \text{ has length } \leq m \}.
\]

If \( k \leq d_I(X) \) the lemma holds trivially. For \( k > d_I(X) \) we divide the proof into two cases. First, if \( X_{\text{lim}} \neq \emptyset \) then as stated in Lemma 10, every walk in \( \prod_{i=1}^{n} G(I) \) starting at \( (x_1, \ldots, x_n) \) has length at most \( \text{mvf}(\prod_{i=1}^{n} G(I), (x_1, \ldots, x_n)) - 1 = d_I(X) \).

In other words, even when \( k > d_I(X) \), we have:
\[
\prod_{i=1}^{n} G(I)_{\text{lim}} = \prod_{i=1}^{n} G(I)_{d_I(X)},
\]
and therefore, we can apply Lemma 6 to conclude that:
\[
\text{mmsc}(X, I, d_I(X))^T \subseteq \text{mmsc}(X, I, k)^T.
\]

Otherwise, if \( X_{\text{lim}} = \emptyset \), we can use the fact that \( \text{mmvf}(I) \geq \text{mvf}(I, x') \forall x' \in \Delta_I \) to obtain:
\[
d_I(X) \geq \text{mvf}\left(\prod_{i=1}^{n} G(I), (x_1, \ldots, x_n)\right) \cdot \text{mvf}(G(I), x').
\]

Hence, if \( X_{\text{lim}} = \emptyset \), the lemma is a direct consequence of Definition 5 and Lemma 7. \( \square \)

To prove that \( B(I) \) defined in Theorem 13 is a base, we first recall a result related to the notion of MMSC.

Lemma 20. [8] Let \( I = (\Delta^I, \cdot^I) \) be a finite \( \mathcal{EL}^\bot \) interpretation. For all \( X \subseteq \Delta^I \) and \( k \in \mathbb{N} \), it holds that \( \emptyset \models \text{mmsc}(\text{mmsc}(X, I, k)^I, I, k) = \text{mmsc}(X, I, k) \).

We will also need a property regarding the construction of concept expressions with MMSCs.

Lemma 21 (Adaptation of Proposition A.1 from [8]). For all \( \mathcal{EL}^\bot \) concept expressions \( C, D \) over \( \mathbb{N}_C \cup \mathbb{N}_R \) and all \( r \in \mathbb{N}_R \) it holds that:
\[
(\text{mmsc}(C^T, I) \cap D)^T = (C \cap D)^T,
\]
\[
(\exists r. (\text{mmsc}(C^T, I))^T = (\exists r. C)^T.
\]

Then, we define for each concept expression \( C \) and interpretation \( I \) a specific concept in \( A_I \) which is called the lower approximation of \( C \) in \( I \). We recall that, for \( X \subseteq \Delta^I \), we write \( \text{mmsc}(X, I) \) as a shorthand for \( \text{mmsc}(X, I, d_I(X)) \).

Definition 9 (Lower Approximation (adapted from Definition 5.4 in [11])). Let \( C \) be an \( \mathcal{EL}^\bot \) concept expression and \( I = (\Delta^I, \cdot^I) \) a model. Also let \( \mathbb{N}_C \cup \mathbb{N}_R \) be a finite signature and \( \mathcal{EL}^\bot(\mathbb{N}_C, \mathbb{N}_R) \) the set of all \( \mathcal{EL}^\bot \) concept
expressions over $\mathbb{N}_C \cup \mathbb{N}_R$. Then, there are concept names $U \subseteq \mathbb{N}_C$ and pairs $\Pi \subseteq \mathbb{N}_R \times \mathbb{EL}^+(\mathbb{N}_C, \mathbb{N}_R)$ such that:

$$C = \bigcap U \cap \bigcap_{(r,E) \in \Pi} \exists r.E$$

We define the lower approximation of $C$ in $\mathcal{I}$ as:

$$\text{approx}(C, \mathcal{I}) = \begin{cases} \bigcap U \cap \bigcap_{(r,E) \in \Pi} \exists r.\text{msc}(E^\mathcal{I}, \mathcal{I}) & \text{if } C \neq \bot, \\ \bot & \text{otherwise.} \end{cases}$$

Concept expressions built according to Definition 9 are always elements of $A_\mathcal{I}$ because they are a conjunction of elements in $M_\mathcal{I}$ (Definition 10). Next, with a straightforward, but nevertheless important, adaptation of the Lemma 5.8 from [11] we prove that the lower approximation of a concept and the concept itself have the same extension.

**Lemma 22.** Let $C$ be an $\mathbb{EL}^+$ concept expression and $\mathcal{I} = (\Delta^\mathcal{I}, \mathcal{I})$ a model. It holds that

$$\text{msc}(C^\mathcal{I}, \mathcal{I})^\mathcal{I} = \text{approx}(C, \mathcal{I})^\mathcal{I} = C^\mathcal{I}.$$ 

**Proof.** If $C = \bot$ then $\text{msc}(C^\mathcal{I}, \mathcal{I})^\mathcal{I} = \text{approx}(C, \mathcal{I})^\mathcal{I} = \emptyset$. Otherwise, there are concept names $U \subseteq \mathbb{N}_C$ and pairs $\Pi \in \mathbb{N}_R \times \mathbb{EL}^+(\mathbb{N}_C, \mathbb{N}_R)$ such that

$$C = \bigcap U \cap \bigcap_{(r,E) \in \Pi} \exists r.E$$

Using Lemma 21 we obtain:

$$C^\mathcal{I} = (\bigcap U \cap \bigcap_{(r,E) \in \Pi} \exists r.E)^\mathcal{I}$$

$$= (\bigcap U \cap \bigcap_{(r,E) \in \Pi} \exists r.\text{msc}(E^\mathcal{I}, \mathcal{I}))^\mathcal{I}$$

$$= (\text{approx}(C, \mathcal{I}))^\mathcal{I}$$

Finally, we can apply Lemma 12 obtaining $\text{msc}(C^\mathcal{I}, \mathcal{I})^\mathcal{I} = \text{approx}(C, \mathcal{I})^\mathcal{I}$.

Using these results, we can conclude that for each MMSC there is a concept expression in $A_\mathcal{I}$ with the same extension in $\mathcal{I}$. With this observation we can we proceed to Theorem 13’s proof.

**Theorem 13.** Let $\mathcal{I}$ be a finite interpretation and let $A_\mathcal{I}$ be defined as above. Then,

$$\mathcal{B}(\mathcal{I}) = \{ C \equiv \text{msc}(C^\mathcal{I}, \mathcal{I}) \mid C \in A_\mathcal{I} \} \cup \{ C \sqsubseteq D \mid C, D \in A_\mathcal{I} \text{ and } \mathcal{I} \models C \sqsubseteq D \}.$$
is a finite $\mathcal{EL}^\bot$ base for $I$.

**Proof.** As $A_I$ is finite, so it is $B(I)$. The concept inclusions are clearly sound and the soundness of the equivalences is due to Lemma [12].

Let $J = (\Delta^J, \cdot^J)$ be an arbitrary interpretation such that $J \models B(I)$. For completeness, we prove that for any $\mathcal{EL}^\bot$ concept expression $C$, $J \models C \equiv \text{mmsc} (C^J, I)$. We prove this claim by induction of the structure of $C$.

**Base case:** If $C = \bot$ or $C = A$ where $A \in \mathcal{NC}$, then $C \in A_I$, by definition of $A_I$. Then, by definition of $B(I)$, we have that $C \equiv \text{mmsc} (C^J, I) \in B(I)$.

**Step case $(\cap)$:** Suppose $C = E \cap F$ and the claim holds for $E$ and $F$. By the inductive hypothesis, $B(I) \models E \equiv \text{mmsc} (E^J, I)$ and $B(I) \models F \equiv \text{mmsc} (F^J, I)$. Hence, for all interpretations $J$ such that $J \models B(I)$, we have that $E^J = \text{mmsc} (E^J, I)^J$ and $F^J = \text{mmsc} (F^J, I)^J$. By Lemma [22] there are $E, F \in A_I$ such that $\text{mmsc} (E^J, I)^J = E^I$ and $\text{mmsc} (F^J, I)^J = F^I$. Moreover, by Lemma [12] $\text{mmsc} (E^J, I)^J = E^J$ and $\text{mmsc} (F^J, I)^J = F^J$. Therefore $(E \cap F)^J = E^J \cap F^J = E^J \cap F^J = (E \cap F)^J$.

As $E \cap F \in A_I$ (up to logical equivalence), $E \cap F \equiv \text{mmsc} ((E \cap F)^J, I) \in B(I)$ (again up to logical equivalence). Since $J$ is a model of $B(I)$, by Lemma [21]

$$(E \cap F)^J = \text{mmsc} ((E \cap F)^J, I)^J = \text{mmsc} ((E \cap F)^J, I)^J = \text{mmsc} (C^J, I)^J.$$

To prove that $C^J = \text{mmsc} (C^J, I)^J$, in the following, we write $C$ as a shorthand for $E \cap F$ and show that $C = C^J$. Since $E \in A_I$, we have that $B(I) \models E \equiv \text{mmsc} (E^J, I)$. Moreover, as $\text{mmsc} (E^J, I)^J = E^J$, we have that

$$B(I) \models E \equiv \text{mmsc} \left( \text{mmsc} (E^J, I)^J, I \right).$$

By Lemma [20] it follows that

$$\emptyset \models \text{mmsc} \left( \text{mmsc} (E^J, I)^J, I \right) \equiv \text{mmsc} (E^J, I).$$

Therefore, $B(I) \models E \equiv \text{mmsc} (E^J, I)$ and as $B(I) \models E \equiv \text{mmsc} (E^J, I)$, then $B(I) \models E \equiv E$. Similarly we obtain $B(I) \models F \equiv F$ and that $B(I) \models C \equiv C$. As $J$ was an arbitrarily chosen model of $B(I)$, we conclude that $B(I) \models C \equiv \text{mmsc} (C^J, I)$ and $B(I) \models C \equiv C$. 
**Step case (3):** In this case, \( C = \exists r.E \) for some \( r \in N_R \) and \( \mathcal{EL}^+ \) concept expression \( E \). Let \( J \) be an interpretation such that \( J \models B(I) \). We know that:

\[
x \in C^J \iff x \in (\exists r.E)^J \iff \exists y \in E^J : (x, y) \in r^J.
\]

By our induction hypothesis, \( B(I) \models E \equiv \text{mmsc} \left( E^I, I \right) \), hence:

\[
x \in C^J \iff \exists y \in \text{mmsc} \left( E^I, I \right)^J : (x, y) \in r^J \iff x \in (\exists r. \text{mmsc} \left( E^I, I \right))^J.
\]

In short, we proved that \( C^J = (\exists r. \text{mmsc} \left( E^I, I \right))^J \). Next, as \( \exists r. \text{mmsc} \left( E^I, I \right) \in M_I \), we know that:

\[
\exists r. \text{mmsc} \left( E^I, I \right) \equiv \text{mmsc} \left( \exists r. \text{mmsc} \left( E^I, I \right)^I, I \right) \in B(I)
\]

With Lemma 21 we obtain:

\[
(\exists r. \text{mmsc} \left( E^I, I \right))^J = (\text{mmsc} \left( \exists r. \text{mmsc} \left( E^I, I \right)^I, I \right))^J
\]

\[
= (\text{mmsc} \left( (\exists r. E)^I, I \right))^J
\]

\[
= (\text{mmsc} \left( C^I, I \right))^J.
\]

Thus, \( C^J = (\text{mmsc} \left( C^I, I \right))^J \). Since \( J \) was chosen arbitrarily, we can conclude that \( B(I) \models C \equiv \text{mmsc} \left( C^I, I \right) \).

Now, we prove that if \( I \models C \subseteq D \), then \( B(I) \models \text{mmsc} \left( C^I, I \right) \subseteq \text{mmsc} \left( D^I, I \right) \). Let \( J \) be a model of \( B(I) \). We know from Lemmas 12 and 22 that there are \( U, V \subseteq M_I \) such that \( C^I = (\bigcap U)^I \) and \( D^I = (\bigcap V)^I \). From the definition of \( B(I) \), we obtain \( \text{mmsc} \left( (\bigcap U)^I, I \right) \subseteq \text{mmsc} \left( (\bigcap V)^I, I \right) \in B(I) \). Therefore:

\[
J \models \text{mmsc} \left( (\bigcap U)^I, I \right) \subseteq \text{mmsc} \left( (\bigcap V)^I, I \right)
\]

Replacing \((\bigcap U)^I\) with \( C^I \) and \((\bigcap V)^I\) with \( D^I \) yields:

\[
J \models \text{mmsc} \left( C^I, I \right) \subseteq \text{mmsc} \left( D^I, I \right).
\]

Therefore, using the fact that \( J \models C \equiv \text{mmsc} \left( C^I, I \right) \) for every \( \mathcal{EL}^+ \) concept expression \( C \) (proved earlier) we can conclude that \( J \models C \subseteq D \).

Since all the required concept inclusions hold in an arbitrary model of \( B(I) \), whenever they hold in \( I \) we have that \( B(I) \) is also complete for the \( \mathcal{EL}^+ \) CIs. \( \square \)
C Proofs for Section 5

In the following we present the proofs related to the computation of the MVF function. In particular, we provide proofs for the relationship between the condensation graph and the MVF function (Lemma 14), and the correctness of Algorithm 1 (Lemma 15).

Lemma 14. Let $G = (V, E, L)$ be a description graph, let $G^* = (V^*, E^*)$ be the condensation of $G$, and $v \in V$. Then:

$$\text{mvf}(G, v) = \max \{ \text{weight}(w^*) \mid w^* \text{ is a path in } G^* \text{ starting at } \text{scc}(G, v) \}.$$

Proof. First we prove that every path $w^* = v_1, \ldots, v_m$ in $G^*$ starting at $\text{scc}(G, v)$ induces a walk in $G$ starting at $v$ with $v_{\text{num}}(w) = \text{weight}(w^*)$. Let $v_1 = v$. For each $1 \leq i < m$, the induced walk must: visit $v_i$, then pass through all vertices in $V_i$ (repeating vertices whenever needed), then visit a vertex $u_i \in V_i$ such that there is an edge $(u_i, v_{i+1}) \in E$ with $v_{i+1} \in V_{i+1}$ (this is possible due to the definitions of SCCs and condensation). When the walk reaches a vertex $u_{m-1}$, it must visit $v_m$ and pass through every vertex in $V_m$ before stopping. Such walk visits every vertex in $\bigcup_{i=1}^{m} V_i$, thus $v_{\text{num}}(w) = \text{weight}(w^*)$.

Now let $w$ be a walk in $G$ starting at $v$ which is induced (as explained earlier) by a path $w^*$ in $G^*$ starting at $\text{scc}(G, v)$ with maximum weight. Assume that there is a walk $\overline{w}$ in $G$ starting at $v$ such that $v_{\text{num}}(\overline{w}) > v_{\text{num}}(w)$. Due to the definitions of SCC and condensation we know that there is a path $\overline{w}$ in $G^*$ starting at $\text{scc}(G, v)$ such that $v_{\text{num}}(\overline{w}) \leq \text{weight}(w^*)$. However, this would imply that: $\text{weight}(w^*) = v_{\text{num}}(w) < v_{\text{num}}(\overline{w}) \leq \text{weight}(w^*)$, which is a contradiction since we assume that $w^*$ has maximal weight. Therefore, no walk in $G$ starting at $v$ can visit more vertices than $\text{weight}(w^*)$.

Since we have shown that for every path $w^*$ in $G^*$ starting at $\text{scc}(G, v)$, there is a walk $w$ in $G$ starting at $v$, with $v_{\text{num}}(w) = \text{weight}(w^*)$, we can conclude that the statement of this lemma holds.

Lemma 15. Given $G = (V, E, L)$ and $v \in V$ as input, Algorithm 1 returns the maximum weight of a path in the condensation of $G$ starting at $\text{scc}(G, v)$.

Proof. Let $G^* = (V^*, E^*)$ be the condensation of $G$. If $W^* \in V^*$ is unreachable from $\text{scc}(G, v)$ then $\text{wgt}[W^*]$ will remain null as it will never be visited. Otherwise, $W^*$ will be visited in some call of $\text{maxWeight}$. If it has no successors, the loop in Line 9 will not do anything, and thus $\text{wgt}[W^*] = |W^*|$ as expected. Instead, if $\text{scc}(G, v)$ has successors, then the maximum weight of a path starting at $\text{scc}(G, v)$ in $G^*$ is given by $|\text{scc}(G, v)|$ plus the maximum value computed among its successors. This equation holds because $G^*$ is a DAG. Since, the loop in Line 9 forces the maximum weights of the successors of $W^*$ to be calculated first, the value returned in Line 13 is correct.