Unbroken Discrete Supersymmetry

Gerhart Seidl
Institut für Physik und Astrophysik, Universität Würzburg, Am Hubland, D 97074 Würzburg, Germany

We consider an interacting theory with unbroken supersymmetry. Different from usual supersymmetry, our Fermion-Bose symmetry is discrete. As a consequence, this allows to put all exotic superpartners into a hidden sector without breaking the symmetry. In the hidden sector, the new symmetry predicts Lorentz-violating interactions that may be probed gravitationally for a low Planck scale. Apart from its possible relevance for the problem of vacuum energies in the universe, the model provides dark matter candidates in the hidden sector.

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INTRODUCTION

Supersymmetry (SUSY) \[ \mathbb{R}^\mathbb{R} \] is widely considered the chief candidate for physics beyond the Standard Model (SM) (see, e.g., \[ \mathbb{1}, \mathbb{2}, \mathbb{3} \] ). Actually, the unified description of fermions and bosons achieved in SUSY harbors several remarkable and unique features both on the theoretical as well as on the phenomenological side, such as an improved ultraviolet behavior. In its local form, supergravity, offers a very promising possibility for unifying gravity with all other forces of Nature. In fact, SUSY is the only way to combine non-trivially external with internal symmetries \[ \mathbb{4}, \mathbb{5} \]. When broken at the TeV scale, it provides a solution to the Higgs mass problem and leads to the attractive prospect of being testable at colliders such as the LHC. Moreover, by offering a dark matter candidate, SUSY yields an answer to one of the most pressing questions about the matter composition of the universe (see, e.g., \[ \mathbb{6} \] and references therein). However, despite these highly attractive properties, the only striking evidence for SUSY so far is the enhanced unification of the SM gauge couplings. And unfortunately, as long as the mechanism of SUSY breaking \[ \mathbb{9} \] is unknown, we are confronted at low energies with a plethora of extra parameters that seem to limit, in practice, its notion as a really predictive theory.

It appears to be an obvious fact that SUSY must be broken. Unbroken SUSY would imply that the SM particles and their predicted superpartners be degenerate in mass and should, in contrast with observation, have already been detected. This holds at least for usual continuous SUSY that describes infinitesimal transformations mixing fermions with bosons. In this paper, we consider, instead, a simple model with an unbroken global Fermion-Boson symmetry that is discrete. We will call this symmetry “discrete SUSY”. The involved discrete symmetry transformations share some similarities with the infinitesimal variations of conventional \( N = 1 \) SUSY but there are also important differences. In discrete SUSY, e.g., we do not necessarily seek a unification with the Poincaré group or try to give an answer to the Higgs mass problem. Instead, we are mainly interested in its implications for cosmology related to the energy and matter composition of the universe.

The discrete nature of our Fermion-Bose symmetry allows to hide all exotic superpartner fields in a hidden sector, implying that discrete SUSY can be left unbroken, thereby protecting a mass degeneracy between the matter particles and their superpartners. To account for an interacting theory, we discuss the simple case of a Yukawa interaction. The fermions plus the bosons mediating the Yukawa interaction define the visible sector communicating with the hidden sector only gravitationally. This structure roughly resembles the idea of gauge mirror models \( \mathbb{10}, \mathbb{11} \), but differs in essential ways. In the hidden sector, the discrete SUSY predicts new Lorentz-violating interactions that are transmitted to the visible sector only gravitationally at the loop level and can, thus, lead only to small observable effects. Just as continuous SUSY, our discrete symmetry predicts stable dark matter candidates residing in the hidden sector.

SCALAR-FERMION SECTOR

We will be concerned here first with the free theory of scalars and fermions in four dimensions that have the Lagrangians

\[
L_s = -\frac{1}{2}(\partial\mu A\partial\mu A + \partial\mu B\partial\mu B) + \frac{m^2}{2}(A^2 + B^2), \quad (1a)
\]

and

\[
L_f = \frac{1}{2}\overline{\psi}(i\slashed{\partial} - m)\psi, \quad (1b)
\]

where \( A \) and \( B \) are real scalar fields, \( \Psi \) is a Majorana fermion field, and \( m \) is the mass. In \( \mathbb{14} \), we have, as usual, \( \mu = 0, 1, 2, 3 \), \( \slashed{\partial} = \gamma^\mu\partial_\mu \), \( \slashed{\Psi} = \Psi\gamma_0 \), and we adopt the chiral representation for the gamma matrices that are given by

\[
\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad (2)
\]

where \( \sigma_i \) (\( i = 1, 2, 3 \)) are the Pauli matrices. The Majorana spinor in \( \mathbb{14} \) reads

\[
\Psi = (\psi_1, \psi_2, \psi_2^*, -\psi_1^*)^T, \quad (3)
\]

\[
\hat{\gamma}_\mu = \gamma_\mu, \quad \hat{\gamma}_5 = \gamma_5, \quad (4)
\]

where \( \hat{\gamma}_\mu \) and \( \hat{\gamma}_5 \) are the Dirac gamma matrices.
where $\psi_1$ and $\psi_2$ and their complex conjugates are Grassmann numbers. The scalar sector contains two degrees of freedom on- and off-shell, whereas the fermion sector has two degrees of freedom on-shell and four degrees of freedom off-shell. Fermi-Bose balance is therefore maintained for the total system on-shell. We will later extend the free theory described here to an interacting theory by including a Yukawa interaction.

Usual SUSY transformations are continuous and described by infinitesimal variations of the fields. As a consequence, they mix fermionic and bosonic degrees of freedom. In a Lagrangian with unbroken continuous SUSY, invariance under the SUSY transformations occurs through a cancellation of the terms becoming proportional to the infinitesimal SUSY parameter describing the transformation. For example, the action for the sum $L_0 = L_s + L_t$ of the Lagrangians in (1) and (3) by

$$\gamma^\mu \partial_\mu \bar{\psi} \gamma^\mu + m^2 \bar{\psi}$$

will nevertheless have to expect significant departures from the structure of the field vari-

\[\begin{align*}
A &\rightarrow A + \delta A, \\
B &\rightarrow B + \delta B, \\
\Psi &\rightarrow \Psi + \delta \Psi,
\end{align*}\]

with the infinitesimal variations

$$\delta A = \bar{\zeta} \Psi, \quad \delta B = i\bar{\zeta} \gamma_5 \Psi, \quad \delta \Psi = -(i\bar{\theta} + m) (A - \gamma_5 B) \zeta,$$

\[\begin{align*}
\text{where} & \quad \zeta = i \gamma^\mu \gamma_1 \gamma^\mu \gamma_3 \zeta \quad \text{and} \quad \zeta \text{ is a Grassmann-valued Majorana spinor with mass dimension } -1/2 \text{ that parameterizes the infinitesimal transformation. The Lagrangian } L_0 \\
\text{along with } & \quad \Psi \quad \text{is, of course, just the on-shell formulation of the Wess-Zumino model.}
\end{align*}\]

In the following, we wish to introduce a finite SUSY transformation that does not mix fermions with bosons but completely interchanges the fermionic degrees of freedom with the bosonic ones and vice versa. Different from the continuous case, the SUSY parameter describing such a discrete finite symmetry will appear explicitly in the Lagrangians so that we choose this parameter to be $c$-number-valued instead of being Grassmann-number-valued as in (4b). In formulating our discrete symmetry, we will try to stick as closely as possible to the familiar continuous case, but will nevertheless have to expect significant departures from the structure of the field variations in (4b).

**DISCRETE SUSY TRANSFORMATIONS**

To implement the discrete SUSY, we will represent the scalar and fermionic fields introduced in (1) and (3) by the block-diagonal $4 \times 4$ matrices

\[\begin{align*}
(A_{ij}) &= \text{diag} \left( \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}, \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \right), \\
(B_{ij}) &= \text{diag} \left( \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}, \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix} \right), \\
(\Psi_{ij}) &= \text{diag} \left( \begin{pmatrix} \psi_1 & \psi_2 \\ \psi_2 & -\psi_1 \end{pmatrix}, \begin{pmatrix} \psi^*_1 & \psi^*_2 \\ \psi^*_2 & -\psi^*_1 \end{pmatrix} \right)
\end{align*}\]

where $i,j = 1,2,3,4$. The components of the matrix-valued fields are then $A_{11} = 0, A_{12} = A, A_{21} = -A$, etc. (correspondingly for $B_{ij}$) and $\Psi_{11} = \psi_1, \Psi_{12} = \psi_2, \Psi_{22} = -\psi_1, etc.$ The Lagrangians in (1) can be written in terms of these components as

$$\mathcal{L}_s = -\frac{1}{4} \xi^T \gamma_0 (\bar{\psi} \gamma^\mu \partial_\mu \bar{\psi} + m^2) \xi_2, \quad \mathcal{L}_t = \frac{1}{2} \bar{\Psi} (i \bar{\theta} - m) \bar{\Psi},$$

where $\xi_1$ and $\xi_2$ denote the four-component objects

$$\xi_1 = (A_{34}, B_{34}, A_{12}, B_{12})^T, \quad \xi_2 = (A_{21}, B_{21}, A_{43}, B_{43})^T,$$

which are $c$-number-valued, whereas

$$\Psi_1^* = (-\Psi_{33}, \Psi_{43}, \Psi_{12}, \Psi_{22})^T, \quad \Psi_2 = (\Psi_{11}, \Psi_{21}, -\Psi_{34}, \Psi_{44})^T,$$

are Grassmann-number-valued Majorana spinors. Inserting the components in (5) explicitly, $\xi_1$ and $\xi_2$ become $\xi_{1,2} \rightarrow \pm (A, B, A, B)^T$, while $\Psi_1$ and $\Psi_2$ take both the form $\Psi_{1,2} \rightarrow \Psi$. We, thus, immediately see that $\xi_1$ and $\xi_2$ in (6) reproduce the expressions for the Lagrangians in (1). Unless otherwise stated, we will from now on refer to $\xi_1$ and $\xi_2$ as defined in (6).

Let us now arrange the quantities in (7) into two multiplets as

$$\varphi_1 = \left( \begin{array}{c} \xi_1 \\ \frac{1}{\sqrt{m}} \Psi_1^* \end{array} \right), \quad \varphi_2 = \left( \begin{array}{c} \xi_2 \\ \frac{1}{\sqrt{m}} \Psi_2 \end{array} \right),$$

which have both mass-dimension one. We then define a discrete SUSY transformation acting on the $\varphi_{1,2}$ by

$$\varphi_1 \rightarrow Q_1 \varphi_1, \quad \varphi_2 \rightarrow Q_2 \varphi_2,$$

where the symmetry operators $Q_1$ and $Q_2$ are given by the matrices

$$Q_1 = i \left( \begin{array}{cc} 0 & c_1 \cdot 1_4 \\ -c_2 \frac{1}{m} (i \bar{\theta} + m) \cdot & 0 \end{array} \right), \quad Q_2 = i \left( \begin{array}{cc} 0 & c_1 \cdot 1_4 \\ -c_2 \frac{1}{m} (i \bar{\theta} + m) \cdot & 0 \end{array} \right),$$

and where $c_1, c_2$ are some ordinary real numbers. In the free theory, we set $c_1 = c_2 = 1$ but will allow in general also other values, e.g., when considering the interacting theory. Note that the discrete transformations in (9) exhibit some similarities with the infinitesimal variations of the fields (4b) for the continuous case. The discrete SUSY transformations partly resemble also features of SUSY quantum mechanics [12].
DISCRETE SUSY INVARIANCES

Under the discrete transformations in (9), the scalar Lagrangian in (5), (6) gets transformed as
\[ L_s = \frac{1}{4} \xi_1^T \gamma_0 (\xi_2 \partial \mu \partial^\nu + m^2) \xi_2 \]
\[ + \frac{1}{2m} \Psi_1 (\xi_2 \partial \mu \partial^\nu + m^2) \Psi_2 = \frac{1}{2} \Psi_1 (i \partial \mu - m) \Psi_2 = L_t, \]
where we have set \( c_1 = 1 \) and used, after partial integration, the equation of motion for \( \Psi_1 \). The fermionic Lagrangian in (6) gets mapped as
\[ L_t = \frac{1}{2} \Psi_1 (i \partial \mu - m) \Psi_2 \]
\[ + \frac{1}{4m} \xi_1^T \gamma_0 (\xi_2 \partial \mu \partial^\nu + m^2) \xi_2 \]
\[ = \frac{1}{4} \xi_1^T \gamma_0 (\xi_2 \partial \mu \partial^\nu + m^2) \xi_2 \]
\[ = \frac{1}{4} \xi_1^T \gamma_0 (\xi_2 \partial \mu \partial^\nu + m^2) \xi_2 = L_s, \]
where we have set \( c_2 = 1 \) and used, after partial integration, the equations of motion for the scalar fields to let the square bracket vanish. In total, we see that the discrete SUSY transformations in (9) map in (6) the scalar Lagrangian and vice versa, when using the equations of motion. The transformations in (9) therefore describe an on-shell discrete symmetry.

We see from the relations in (10) and (11) that the total free Lagrangian \( L = L_s + L_t \) can, using the equations of motion, also be put for the multiplets \( \phi_1 \) and \( \phi_2 \) into the alternative forms
\[ L_0 = \frac{1}{4} \phi_1^T \gamma_0 (\phi_2 \partial \mu \partial^\nu + m^2) \phi_2, \]
\[ L_0 = \frac{m}{4} \phi_1^T \gamma_0 (i \phi - m) \phi_2, \]
where we have introduced the matrix \( \gamma_0 = \text{diag} (\gamma_0, \gamma_0) \). Note that the two expressions resemble respectively the Lagrangians for Klein-Gordon fields and a Majorana fermion.

Let us now summarize the result of successive discrete transformations of the scalar and fermionic fields. Under two successive discrete SUSY transformations, the scalar object \( \xi_1 \) is mapped as follows
\[ \xi_1 \rightarrow \delta \xi_1 = i \sqrt{2} c_1 \Psi_1^1, \]
\[ \delta \xi_1 \rightarrow \frac{c_1 c_2}{m} (i \phi + m) \xi_1, \]
and correspondingly for \( \xi_2 \), with \( \Psi_1^1 \) replaced by \( \Psi_2^1 \) and \( (i \phi + m)^* \) replaced by \( (i \phi + m) \). On the other hand, \( \Psi_1^1 \) is mapped under the same transformations as
\[ \Psi_1^1 \rightarrow \delta \Psi_1^1 = - \frac{i c_2}{\sqrt{2m}} (i \phi + m)^* \xi_1, \]
\[ \delta \Psi_1^1 \rightarrow \frac{c_1 c_2}{m} (i \phi + m)^* \Psi_1^1 = 2 c_1 c_2 \Psi_1^1, \]
where we have used in the last step the equations of motion for \( \Psi_1^1 \), and correspondingly for \( \Psi_2^1 \), with \( \xi_1 \) replaced by \( \xi_2 \) and \( (i \phi + m)^* \) replaced by \( (i \phi + m) \).

MODEL FOR YUKAWA INTERACTION

So far, we have been studying on-shell discrete SUSY for the free theory of matter fields only. Let us now consider an interacting theory by including a Yukawa interaction for \( \Psi \). In doing so, we will assume that the discrete SUSY transformations (9) are formulated in the interaction (Dirac) picture, where the fields are subject to the free field equations of motion even in presence of the interaction.

To implement the Yukawa interaction, we introduce a copy of the matter sector discussed earlier, which is described by representations and Lagrangians \( \mathcal{L}_s' \) and \( \mathcal{L}_t' \) for the scalars and fermions that are identical to those given in (10) and (11), but where all fields are now replaced by primed fields \( A', B', \Psi' \), etc. The primed fields have a mass \( m' \) that needs not be equal to \( m \). This new sector is then invariant under discrete SUSY transformations analogous to (9), (13), and (14). We let the new sector couple to the matter sector discussed previously via the Yukawa interaction
\[ L_Y = - \frac{1}{4m} \eta^T (\xi_1^1 \gamma_0 + \eta_2^1 \gamma_0) \Psi_2, \]
where \( \eta = (Y_A, Y_B, Y_A, Y_B)^T \) and \( Y_{A,B} \) are real Yukawa couplings. In (15), \( \xi_1^1 \) and \( \xi_2^2 \) are defined for the sector of primed fields in complete analogy with (7). Inserting the components of \( \xi_1^1 \) and \( \xi_2^2 \) explicitly, one finds that the kinetic term in (15) actually drops out such that the Yukawa interaction reduces to the more familiar form
\[ L_Y = - (Y_A A' + Y_B B') \Psi, \]
The expression in (15) manifestly establishes the invariance of the Yukawa interaction after two discrete SUSY transformations. From (13) and (14), we obtain that (15) becomes after the first and the second discrete SUSY transformation
\[ L_Y \rightarrow L_Y' = - i c_1 c_2 \eta^T (\Psi_1^1 - \Psi_2^1) (\xi_1^1, \gamma_0 \xi_2^2), \]
\[ L_Y' \rightarrow \frac{1}{m'} c_1 c_2 \eta^T (\Psi_1^1 - \Psi_2^1) (\xi_1^1, \gamma_0 \xi_2^2), \]
where we have restored again the parameters \( c_1 \) and \( c_2 \) from (9) and \( c_1' \) and \( c_2' \) are the corresponding real parameters for the analogous discrete transformations of the primed fields. In the first transformation in (17), we have used the equations of motion of \( \xi_1^1 \) and \( \xi_2^2 \). To reproduce the original interaction \( L_Y \) after two SUSY transformations, i.e. for \( L_Y \rightarrow L_Y' \), we thus require \( c_1 c_2 (c_1 c_2)' = \frac{1}{2} \). Note that even without having \( c_1, c_2, c_1', \) and \( c_2' \), all equal to one, we can always use the equations of motion to rescale the free parts of the Lagrangians after application of the discrete SUSY transformations and thus establish

invariance of the total Lagrangian under these operations on-shell.

The total Lagrangian of the interacting model, invariant under the discrete SUSY transformations, is then given by

$$\mathcal{L}_{\text{total}} = \mathcal{L}_\text{f} + \mathcal{L}_\text{s} + \mathcal{L}_\gamma + \mathcal{L}_s + \mathcal{L}'_\text{f} + \mathcal{L}'_\text{s} \quad (18)$$

where $\mathcal{L}_\text{f}$, $\mathcal{L}_\text{s}$, and $\mathcal{L}_\gamma$ define a visible and $\mathcal{L}_s$, $\mathcal{L}'_\text{f}$, and $\mathcal{L}'_\text{s}$, a hidden sector. While $\mathcal{L}_\text{f}$ represents our usual matter fields, their scalar superpartners are mass degenerate but cannot be directly observed because they are confined to the hidden sector. This allows discrete SUSY to be left unbroken without running into immediate conflict with observation. The visible and the hidden sector can communicate only gravitationally.

The interaction $\mathcal{L}_\gamma$ violates Lorentz invariance (see, e.g., [15] and references therein) but the Lorentz-violating effects can be transmitted to the visible sector gravitationally only at the loop-level and will therefore be suppressed by at least four powers of the Planck scale $M_{\text{Pl}} \simeq 10^{18}$ GeV, which should be sufficiently small to be in agreement with current bounds. Apart from $\mathcal{L}_\gamma$, there may be additional interactions invariant under the discrete SUSY, but their coupling strengths can be dialed to be small enough to be in agreement with current bounds on Lorentz violation and the non-observation of exotic superpartners, as well.

This model may be relevant for cosmology in at least two ways. First, the hidden sector provides stable scalar or fermionic dark matter candidates with masses $m$ or $m'$. Second, the improved UV behavior of unbroken discrete SUSY may be related to the problem of vacuum energies. To understand the actually observed small but non-zero vacuum energy density in the universe $\rho \simeq 10^{-47}$ (GeV)$^4$ requires, however, an additional explanation.

## SUMMARY AND CONCLUSIONS

In this paper, we have presented a model for discrete unbroken on-shell SUSY. The model consists of a visible and a hidden sector. In the visible sector, we have fermions plus scalars that mediate a Yukawa interaction. The hidden sector is the SUSY copy of the visible sector and contains all exotic superpartners. As a result, discrete SUSY can be left unbroken with exotic superpartners that have the same mass as the visible sector particles. The discrete symmetry predicts from the Yukawa interaction in the visible sector Lorentz-violating interactions in the hidden sector that can be transmitted to the visible sector only gravitationally at the loop level.

Similar to conventional SUSY, our discrete Fermion-Boson symmetry predicts dark matter candidates in the hidden sector that may be bosons or fermions. One possibility to observe the Lorentz-violating interactions may be in scenarios with a lowered Planck scale [16, 17]. Apart from the dark matter candidates provided by the model, the mass degeneracy between particles and their superpartners may be relevant for the problem of vacuum energies in the universe. An explanation of the non-zero value of the observed vacuum energy density, however, would still call for other ideas [14]. The discrete SUSY model is mainly targeting questions relevant for cosmology and exhibits clear differences to conventional SUSY. In its present form, it does not provide a unification with the Poincaré group or offers a solution to the Higgs mass problem which would then require alternative mechanisms for electroweak symmetry breaking [15]. Also, it is not yet clear how the improved gauge coupling unification of continuous SUSY could be achieved for discrete SUSY, too.

As a next step, it would be interesting to extend this model to include also gauge interactions and give a formulation for the SM. Moreover, one should study the role of condensates in the hidden sector that may form due to confinement in QCD. Finally, it would be exciting to attempt an off-shell formulation of the discrete symmetry and try to embed it into a continuous local form to establish also a possible connection with gravity.

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[1] Yu. A. Golfand and E. P. Likhtman, JETP Lett. 13, 323 (1971) [Pisma Zh. Eksp. Teor. Fiz. 13, 452 (1971)].
[2] D. V. Volkov and V. P. Akulov, JETP Lett. 16, 438 (1972) [Pisma Zh. Eksp. Teor. Fiz. 16, 621 (1972)].
[3] J. Wess and B. Zumino, Nucl. Phys. B 188, 39 (1974).
[4] H. P. Nilles, Phys. Rept. 110, 1 (1984).
[5] S. P. Martin, arXiv:hep-ph/9709356.
[6] S. R. Coleman and J. Mandula, Phys. Rev. 159, 1251 (1967).
[7] R. Haag, J. T. Lopuszanski and M. Sohnius, Nucl. Phys. B 88, 257 (1975).
[8] K. A. Olive, arXiv:1001.5014 [hep-ph].
[9] M. A. Luty, arXiv:hep-th/0509029.
[10] R. Foot and R.R. Volkas, Phys. Rev. D 52 (1995) 6595; Z.G. Berezhiani and R.N. Mohapatra, Phys. Rev. D 52 (1995) 6607.
[11] K. S. Babu and G. Seidl, Phys. Lett. B 591, 127 (2004); Phys. Rev. D 70, 113014 (2004).
[12] E. Witten, Nucl. Phys. B 188, 513 (1981).
[13] S. Weinberg, Rev. Mod. Phys. 61, 1 (1989); arXiv:astro-ph/0005265.
[14] M. S. Carroll, Living Rev. Rel. 4, 1 (2001).
[14] C. Wetterich, Nucl. Phys. B 302, 668 (1988); B. Ratra and P. J. E. Peebles, Phys. Rev. D 37, 3406 (1988).

[15] T. Jacobson, S. Liberati and D. Mattingly, Annals Phys. 321, 150 (2006).

[16] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B 429, 263 (1998); Phys. Rev. D 59, 086004 (1999); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B 436, 257 (1998).

[17] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999); Phys. Rev. Lett. 83, 3370 (1999).

[18] S. Weinberg, Phys. Rev. D 13, 974 (1976); L. Susskind, Phys. Rev. D 20, 2619 (1979); W. A. Bardeen, C. T. Hill and M. Lindner, Phys. Rev. D 41, 1647 (1990); C. Csaki, C. Grojean, H. Murayama, L. Pilo and J. Terning, Phys. Rev. D 69, 055006 (2004).

[19] The representations in (5) are anti-symmetric (scalars) and symmetric-traceless (fermion) matrices that are invariant under a transformation acting on the matrices from the left and right by $\pm \text{diag}(1_2, -1_2)$. 