A manifold of pure Gibbs states of the Ising model on the Lobachevsky plane

Daniel Gandolfo, Jean Ruiz, and Senya Shlosman

October 23, 2013

Abstract

In this paper we construct many ‘new’ Gibbs states of the Ising model on the Lobachevsky plane, the millefeuilles. Unlike the usual states on the integer lattices, our foliated states have infinitely many interfaces. The interfaces are rigid and fill the Lobachevsky plane with positive density.

Keywords: geodesical family, Ising model, cross-ratio, interface, rigidity.

1 Introduction

There is a common belief in the field of statistical mechanics that the qualitative properties of the systems living on Cayley trees $\mathcal{T}_n$ and on the (tesselations $\mathcal{L}_{p,q}$ of the) Lobachevsky plane $\mathcal{L}$ should be the same. Indeed, the behavior of the ratio $\frac{|\partial D_r|}{|D_r|}$, where $|D_r|$ is the volume of the ball $D_r$ of radius $r$, and $|\partial D_r|$ is the volume of the sphere $\partial D_r$ of the same radius, is the same for the two families of graphs, as $r \to \infty$. Yet the models on the Cayley trees are studied in much more details, due to the fact that there one can use the recurrent relations, which were used in many papers, like [BRZ] or [E]. On the Lobachevsky plane no such relations exists, since the graphs $\mathcal{L}_{p,q}$ have cycles. In fact, the Cayley trees $\mathcal{T}_n$ are ‘limits’ of the tesselations $\mathcal{L}_{n+1,q}$ when $q \to \infty$, see Fig.1. Nevertheless, some results for the Cayley trees were also obtained for the Lobachevsky plane as well – see [E], for example. One such result is about the non translation invariant states of the Ising model on the Cayley trees and Lobachevsky plane. For the Cayley trees the analogs
of Dobrushin non translation invariant states $\langle \cdot \rangle^\pm$, [D], where an interface separates (+)-phase from the (−)-phase, were constructed by Blekher and Ganikhodzhaev in [BG]. For the Lobachevsky plane these were obtained by Series and Sinai, [SS].

Later it was discovered that there are much more non translation invariant states on Cayley trees $T_n$ than there are those for the case of the lattice $\mathbb{Z}^\nu$. Namely, in [RR] the authors constructed Gibbs states of the ferromagnetic Ising model on $T_n$, $n \geq 4$, which they called ‘weakly periodic’. For such a state $\langle \cdot \rangle$ the expectation $\langle \sigma_t \rangle$ of the spin $\sigma_t$ at $t \in T_n$ is a ‘periodic’ function on $T_n$, taking finitely many different values. Thus, it is different both from the (+)-state $\langle \cdot \rangle^+$ and the (−)-state $\langle \cdot \rangle^-$, as well as from the Dobrushin $\langle \cdot \rangle^\pm$ states. Indeed, for every $t$ we have $\langle \sigma_t \rangle^+ \equiv m^*$, $\langle \sigma_t \rangle^- \equiv -m^*$, while the function $\langle \sigma_t \rangle^\pm$ takes infinitely many values – provided the temperature $\beta^{-1}$ is low enough. Here $m^* \equiv m^* (\beta)$ is the spontaneous magnetization. Then in [GRS] we have given a general construction of a grand family of pure states on Cayley trees $T_n$, again for $n \geq 4$. One can think of these states as having infinitely many rigid ±-interfaces, and moreover the density of those interfaces is positive. (The reader should not think that the trees $T_n$ with $n \geq 4$ have properties which the lighter trees $T_2$ and $T_3$ do not enjoy. Indeed, in the Section 4 of the present paper we explain how to modify the
constructions of [GRS] in order to include all the Cayley trees $T_n$, $n \geq 2$. So the trees $T_n$ with $n \geq 4$ are not special in that respect.)

The purpose of the present paper is to construct similar family of states on the tessellations $L_{p,q}$ of the Lobachevsky plane $L$. Here $L_{p,q}$ is the planar graph, such that each face has $p$ edges, every vertex is incident to $q$ edges, and $1/q + 1/p < 1/2$. Every such graph can be isometrically embedded into $L$. The main result of the present paper is the construction of Gibbs states of the Ising model on $L_{p,q}$, which have infinitely many Dobrushin $\pm$-interfaces. In our case the interfaces are 1D random curves. They are rigid, as is the interface of the Dobrushin state $\langle \cdot \rangle^\pm$ of the Ising model on $\mathbb{Z}^3$. But unlike the 3D lattice, when there can be at most one such interface, on the Lobachevsky plane $L$ one can have infinitely many of them. Moreover, they can have positive density. We call these states the Foliated States, or just the millefeuilles.

In the next section we introduce special families of geodesics, which we call geodesical families. We provide examples of such families and establish their properties. The geodesical families define foliated ground state configurations. We prove in the Section 3 that these ground state configurations are stable: the thermal fluctuations do not destroy them. In other words, there are low-temperature Gibbs states, which are small perturbations of the foliated ground states. In the Section 4 we study the foliated states on the Cayley trees.

2 Geodesics and geodesical families

Let $M$ be a locally compact Riemannian manifold. A (finite or countable) family $\Gamma$ of smooth curves $\gamma_i \in M$, either closed or coming from and going to infinity, will be called a geodesical family, if no bounded surgery can decrease its length. By this we mean the following: let $s = \{s_i = [\alpha_{2i-1}, \alpha_{2i}] \subset \gamma_i, i = 1, ..., k\}$ be a collection of segments of the curves $\gamma_i$, $i = 1, ..., k$. We want to remove the segments $s_i$ and to interconnect the remaining family of curves, $\{\gamma_i \setminus s_i, i = 1, ..., k\}$ in a different way. To do this, let us consider a permutation $\pi$ of the set $\{\alpha_1, ..., \alpha_{2k}\}$, and let $\bar{s} = \{\bar{s}_i = [\pi (\alpha_{2i-1}), \pi (\alpha_{2i})]\}$ be some collection of continuous curves, con-
necting the points $\pi(\alpha_{2i-1})$ and $\pi(\alpha_{2i})$, $i = 1, \ldots, k$. The length increment

$$\Delta(s, \bar{s}) = \sum_{i=1}^{k} |\bar{s}_i| - \sum_{i=1}^{k} |s_i|$$

is well-defined. The family \(\{\gamma_i\}\) is called geodesical, if for any \(k \geq 1\), any collection \(s\), and permutation \(\pi\) and any collection \(\bar{s} \neq s\) the increment \(\Delta(s, \bar{s})\) is strictly positive. In particular, each curve \(\gamma_i\) from a geodesical family has to be a geodesic.

(A reader with background from stat. mechanics recognizes immediately that our definition is inspired by that of a ground state configuration. The generalization to the case of minimal surfaces is immediate, but we will not need it this paper.)

For example, if \(\gamma_1\) and \(\gamma_2\) are two geodesics, and \(\gamma_1 \cap \gamma_2 \neq \emptyset\), then the pair \(\gamma_1, \gamma_2\) is never a geodesical family. If \(M\) is \(\mathbb{R}^2\) or \(\mathbb{R}^n\), \(n > 2\), then any geodesical family contains at most one geodesic, which is a straight line.

The situation is different when \(M\) is the Lobachevsky plane, \(\mathcal{L}\). Let \(\gamma_1, \gamma_2\) be two geodesics, and \(x_i', x_i''\) be their end-points at the absolute, \(i = 1, 2\). The cross-ratio of the two pairs of points \((x_1', x_1'')\) and \((x_2', x_2'')\) is defined by

$$R(x_1', x_1'', x_2', x_2'') = \frac{(x_2' - x_1')(x_2'' - x_1'')}{(x_2' - x_1')(x_2'' - x_1')}.$$

For us it will be more convenient to use another version of it, which is given by

$$R(x_1', x_1'', x_2', x_2'') = -\frac{(x_1'' - x_1')(x_2'' - x_2')}{(x_1'' - x_2')(x_1'' - x_1')} = R - 1.$$

Then the pair \(\gamma_1, \gamma_2\) of non-crossing geodesics is a geodesical family, iff the cross-ratio \(R(x_1', x_1'', x_2', x_2'')\) of the quadruple \(x_1', x_1'', x_2', x_2''\) is less than 1.

In what follows we will use for the quantity \(R(x_1', x_1'', x_2', x_2'')\) the notation \(R(\gamma_1, \gamma_2)\). For later use we need to choose a scale length on \(\mathcal{L}\). We do it by imposing the condition that for every pair \(\gamma_1, \gamma_2\) of non-intersecting geodesics with the cross-ratio of the quadruple \(x_1', x_1'', x_2', x_2''\) equal to 1, we have \(\text{dist} (\gamma_1, \gamma_2) = 1\). When \(R(\gamma_1, \gamma_2) \to 0\), we have that \(\text{dist} (\gamma_1, \gamma_2) \to \infty\).

We are going to present a countable geodesical family \(\Gamma\), having positive density. That means that there exists a value \(R > 0\), such that for every point \(x \in \mathcal{L}\) the disc \(D_x(R)\), centered at \(x\) and having radius \(R\) intersects some of the curves from \(\Gamma\). The construction is the following:
Construction 1. Let $0 \in \mathcal{L}$ be an arbitrary point, which we will fix and will call an origin. Our construction will be inductive, and will start ‘near’ $0$ and will proceed away from $0$ to infinity.

Figure 2: Construction 1, the first three steps.

Let us fix a small $\alpha < 1$. Let $\gamma_1, \gamma_2$ be two geodesics, with $\gamma_1$ being
centrally symmetric to $\gamma_2$ with respect to 0. Additionally we suppose that for their endpoints $x'_1, x''_1, x'_2, x''_2$ at the absolute we have $R(x'_1, x''_1, x'_2, x''_2) = \alpha$.

Let $\gamma_3$ be an axial reflection of $\gamma_1$ in $\gamma_2$, $\gamma_4$ - a reflection of $\gamma_2$ in $\gamma_1$. Let $\gamma_5, \gamma_6, \gamma_7$ be reflections of $\gamma_1, \gamma_2, \gamma_4$ in $\gamma_3$, and so on. Proceeding in this way, we obtain an infinite sequence of geodesics, which we denote by $\Gamma_1$.

Note that if the two geodesics $\gamma, \gamma'$ intersect, then the cross-ratio $R(\gamma, \gamma') = \alpha$. For every geodesic $\gamma_i$ in $\Gamma_1$ denote by $c(\gamma_i)$ the point on $\gamma_i$ which is the closest to 0. Clearly, there exists a geodesic $\delta_1$, passing through 0 and containing all the points $c(\gamma_i)$. Let $\delta_2$ be a geodesic passing through 0 and orthogonal to $\delta_1$. Clearly, $\gamma_2$ is a reflection of $\gamma_1$ in $\delta_2$.

Let us now fill in the spaces between every two consecutive geodesics $\gamma, \gamma' \in \Gamma_1$. Because of the above symmetries it is sufficient to fill in the strip $S(\gamma_1, \gamma_2)$ between $\gamma_1$ and $\gamma_2$; reflecting this ‘filling’ in curves of $\Gamma_1$ we get it for all neighboring curves in $\Gamma_1$. We do it as follows. Let $\chi_1$ and $\chi_2$ be two non-intersecting geodesics in $S(\gamma_1, \gamma_2)$, having the property: $R(\chi_i, \chi_j) = \alpha$.

The existence and uniqueness of this pair is straightforward. Note that $\chi_2$ is a reflection of $\chi_1$ in $\delta_1$, and that the centers $c(\chi_j)$ belong to $\delta_2$. Let us fill in the ‘inside’ of $\chi_1$ by the sequence of geodesics $\chi_i, i = 1, 2, ..., \chi_0 \equiv \chi_1$, which are defined by the properties: $R(\chi_i, \chi_{i+1}) = \alpha$, $c(\chi_i) \in \delta_2$, and $\text{dist}(c(\chi_i), 0) \nearrow \infty$, $i = 0, 1, 2, ...$. We do the same for $\chi_2$. Applying the reflection symmetries we fill in all the strips between the neighboring curves in $\Gamma_1$. The obtained family, together with curves in $\Gamma_1$, is denoted by $\Gamma_2$, see Fig.2(b).

Note that for every two neighboring curves $\gamma, \gamma' \in \Gamma_2$ we have $R(\gamma, \gamma') = \alpha$. The notion ‘neighboring’ means that one can choose ends $x, x'$ for $\gamma, \gamma'$ in such a way that between them there are no ends of other members of $\Gamma_2$. For each such pair of ends $x, x'$ of the curves $\gamma, \gamma'$ we reproduce the above construction, obtaining the sequence $\chi_i(x, x'), i = 0, 1, 2, ...$ of geodesics with ends between $x$ and $x'$. In particular, $R(\chi_i(x, x'), \chi_{i+1}(x, x')) = \alpha = R(\chi_0(x, x'), \gamma) = R(\chi_0(x, x'), \gamma'')$. The resulting family is denoted by $\Gamma_3$, see Fig.2(c).

The last step can be iterated, so we inductively can define the families $\Gamma_1 \subset \Gamma_2 \subset ...$; the final result $\Gamma_\infty = \cup_k \Gamma_k$ is the desired collection. $\blacksquare$

The Millefeuille Construction 1 above results in a ‘ground state’ with ‘zero mean magnetization’. We present now a generalization, which will have a non-zero magnetization.

The Construction 2 will be defined by the two parameters: $\alpha$ and $\eta$. 
The construction of the family $\Gamma_\infty = \Gamma_\infty (\alpha, \eta)$ is inductive. The data after the completion of the $k$-th step consists from

- the family of $k$ non-intersecting geodesics $\gamma_i$, $i = 1, \ldots, k$,
- a choice of one arc among the $2k$ non-intersecting arcs on the absolute, which are defined by the $2k$ end-points of the curves $\gamma_i$,
- the chess-board assignment of the $+$ or $-$ signs ($\equiv$ phases) to each of the $k + 1$ regions of the plane $L$.

Let $(x, y)$ be the selected arc, and $(y, z)$ be the next one (clockwise). Then after the $k + 1$-th step one extra geodesic $\gamma_{k+1}$ is added, with endpoints inside the arc $(x, y)$, while the arc $(y, z)$ is declared as chosen. Now we have to describe the rule of constructing the curve $\gamma_{k+1}$ given all the previous curves $\gamma_i$, the new assignment of signs, plus we need to specify the first step of induction.

The sign assignment is simple. Let $L_0, L_1, \ldots, L_k$ are connected components of $L$, which are defined by the lines $\gamma_1, \ldots, \gamma_k$. Each of them already has its sign. The geodesics $\gamma_{k+1}$ belongs to one of this components, say $L_j$, and it splits $L_j$ into two connected components, one of which borders some of the previous curves $\gamma_i$. Let us call this component exterior to $\gamma_{k+1}$, while other one will be called interior, and will be denoted by $L_{k+1}$. Let the ‘new’ component $L_j \equiv L_j^{\text{new}} = L_j^{\text{old}} \setminus L_{k+1}$. Then the ‘new’ component $L_j$ retains the sign of the ‘old’ one, while the component $L_{k+1}$ gets the opposite one. All other components retain their signs.

In the beginning we pick a point 0 on the plane, which will be called the origin. We take for $\gamma_1$ an arbitrary (maximal) geodesic, passing through 0, and we choose arbitrarily one of the two arcs (semicircles) on the absolute, which are defined by the endpoints of $\gamma_1$. The resulting two components $L_0$ and $L_1$ of the plane $L$ get opposite signs $+$ and $-$. The construction of the curve $\gamma_{k+1}$ is defined by one of our extra real parameters $\alpha, \eta > 0$; we use $\alpha$ if the curve $\gamma_{k+1}$ traverses the ‘$(+)$-phase’, and $\eta$ if it goes through ‘$(-)$-phase’. The construction is the same in both cases, so we consider only the case of the ‘$(+)$-phase’, where we use the parameter $\alpha$. So let $(x, y)$ is our chosen arc on the absolute, and we want to pick two points $(u, v)$ on it, which uniquely define the geodesic $\gamma_{k+1} = \gamma (u, v)$, joining them, see Fig.[3] Suppose first that the two points $x, y$ are endpoints of the
same geodesic $\gamma_i$, constructed earlier. Then the pair $u, v$ is uniquely defined by the two properties:

- the cross-ratio $(u, v; y, x) = \alpha,$
- the geodesic, which passes through 0 and is perpendicular to $\gamma_i$, is also perpendicular to $\gamma(u, v)$.

In the remaining case the point $x$ is an end-point of a geodesic $\gamma_i = \gamma(t, x)$, while $y$ is an end-point of a geodesic $\gamma_j = \gamma(y, z)$. Then the pair $u, v$ is uniquely defined by the two properties:

- the cross-ratio $(u, v; y, z) = \alpha,$
- the cross-ratio $(u, v; t, x) = \alpha.$

If $\alpha = \eta$, the Construction 2 gives the same geodesical family as Construction 1.

Let us check that indeed the above families of geodesics are geodesical families. We will represent the Lobachevsky plane $\mathcal{L}$ as a disc $D_1$ of unit radius in $\mathbb{C}^1$, centered at the origin 0; then the absolute $\mathcal{L}^\infty$ will be represented
by the unit circle $C$. Let $z_1, z_2, z_3, z_4 \in \mathcal{L}^\infty$ be four points, going clockwise. Note that when the points $z_1, z_2, z_3, z_4$ are made by an isometry of $\mathcal{L}$ into a rectangle, and the cross-ratio $R(z_1, z_2; z_3, z_4)$ is small, the side $[z_1, z_2]$ of this rectangle is much shorter than $[z_2, z_3]$.

Let $\gamma_1, \gamma_2, \ldots, \gamma_k \subset \mathcal{L}$ be a family of non-intersecting doubly-infinite geodesics on Lobachevsky plane $\mathcal{L}$. For two points $x', x''$ on the absolute $\mathcal{L}^\infty$ we denote by $\gamma(x', x'')$ the geodesic connecting them. Let $\gamma_i = \gamma(x'_i, x''_i), i = 1, 2, \ldots, k$. The following lemma claims that if all the geodesics $\gamma_1, \gamma_2, \ldots, \gamma_k$ are far away from each other, then they form a geodesical family.

**Lemma 1** Suppose that for some $\alpha$ small enough the geodesics $\gamma_i = \gamma(x'_i, x''_i), i = 1, \ldots, k$ have the property that for any $i \neq j$

$$R(x'_i, x''_i; x'_j, x''_j) < \alpha.$$  

Then the family $\gamma_1, \gamma_2, \ldots, \gamma_k$ is geodesical.

**Proof.** Note that the condition $R(x'_i, x''_i; x'_j, x''_j) < \alpha$ implies that the Lobachevsky distance $\lambda$ satisfies

$$\lambda(\gamma_i, \gamma_j) = \inf_{y_i \in \gamma_i, y_j \in \gamma_j} \lambda(y_i, y_j) > L(\alpha),$$

with $L(\alpha) \to \infty$ as $\alpha \to 0$. Let $\delta_{ij}$ be the common perpendicular to $\gamma_i$ and $\gamma_j$, and $\Delta_{ij} \in \gamma_i, \Delta_{ji} \in \gamma_j$ be the feet of this perpendicular. We have that the length $|\delta_{ij}| = \lambda(\Delta_{ij}, \Delta_{ji}) \equiv \lambda(\gamma_i, \gamma_j) > L(\alpha)$. Define $L = \max_{ij} |\delta_{ij}|$.

For the future use we will compute the quantity $L(\alpha)$. Let $x > 0$ be large, and $\gamma_1(-x - 1, -x + 1), \gamma_2(x - 1, x + 1)$ be two unit semicircles in the upper half-plane, centered at $-x$ and $x$, i.e. geodesics in the upper half-plane model. Then the cross-ratio $R(-x - 1, -x + 1; x - 1, x + 1) = \frac{1}{x^2 - 1}$, while the distance $\text{dist}(\gamma_1, \gamma_2) \approx \text{dist}((-x, 1), (x, 1)) = \cosh^{-1}(1 + 2x^2) \approx \ln x^2$, so

$$L(\alpha) \approx \ln \alpha^{-1}. \quad (1)$$

Let us start with the case of two geodesics, $\gamma_1 = \gamma_1(x_1, x_2)$ and $\gamma_2 = \gamma_2(x_3, x_4)$. We can suppose that $\gamma_2$ is symmetric to $\gamma_1$ with respect to the origin $0 \in \mathcal{L}$. The common perpendicular $\delta_{12}$ passes through 0, and $\lambda(0, \Delta_{12}) = \lambda(0, \Delta_{21})$. Consider now the surgery, which is almost the same as the pair of geodesics $\mathcal{x}_1 = \mathcal{x}_1(x_1, x_4)$ and $\mathcal{x}_2 = \mathcal{x}_2(x_2, x_3)$. We want to show that $|\mathcal{x}_1| + |\mathcal{x}_2| \gg |\gamma_1| + |\gamma_2|$ (renormalized in the obvious way), provided $L(\alpha)$ is large.
For that let us consider the triangle $0, \Delta_{12}, A$, where $A \in \gamma_1$ with $\lambda(A, 0) > L(\alpha)$, and $A$ is on the same side from $0$ as $x_1$. Note that both the distances $\lambda(0, \gamma_1)$ and $\lambda(A, \gamma_1)$ become small for small $\alpha$. Now we will use the following

**Estimate** 1$_H$ from [SS], p. 69: Let $XYZ$ be a triangle on $\mathcal{L}$, made by three geodesics, with the angle $\angle XYZ = \frac{\pi}{2}$. Then

$$|XZ| \geq |XY| + |YZ| - c,$$

(2)

for some universal constant $c$, independent of $X, Y$ and $Z$.

It follows from (2) that $|0A| \geq \frac{1}{2} \delta_{12} + |\Delta_{12}A| - c$, which implies that $|\gamma_1| + |\gamma_2| \geq |\gamma_1| + |\gamma_2| + 2\delta_{12} - 4c - \varepsilon(\alpha)$, with $\varepsilon(\alpha) \to 0$ as $\alpha \to 0$. Since $\delta_{12} \to \infty$ as $\alpha \to 0$, that proves our claim.

Consider now the general case, when we have $2k$ points at the absolute, $x'_1, x''_1, x'_2, \ldots, x''_k$, going clockwise. Note that the only permutation $\bar{\pi}$ we have to consider is given by

$$\bar{\pi}(x'_1, x''_1, x'_2, \ldots, x''_{k-1}, x'_k, x''_k) = x'_1, x''_k, x'_k, x''_{k-1}, \ldots, x'_2, x''_1.$$

First of all, we do not have to consider permutations, for which the geodesics $\gamma(\pi(x'_i), \pi(x''_i))$ and $\gamma(\pi(x'_j), \pi(x''_j))$ intersect for some $i \neq j$. Indeed, let $\tau$ be a transposition, exchanging the points $\pi(x''_i)$ and $\pi(x''_j)$; it is easy to see that

$$|\gamma(\pi(x'_i), \pi(x''_i))| + |\gamma(\pi(x'_j), \pi(x''_j))| >$$

$$|\gamma(\tau \pi(x'_i), \tau \pi(x''_i))| + |\gamma(\tau \pi(x'_j), \tau \pi(x''_j))|.$$ 

Next, for every remaining permutation $\pi$ consider the union of all the curves $\gamma(x'_i, x''_i)$ and all the curves $\gamma(\pi(x'_i), \pi(x''_i))$. This union consists of several - say, $l$ - connected components, $l \leq k$. If $l > 1$, then we can consider each component separately, thus reducing the number $2k$ of points $x'_1, x''_1, x'_2, \ldots, x''_k$ to smaller values and use induction on $k$.

We will compare the permutation $\bar{\pi}$ with another one, $\bar{\pi}'$, given by

$$\bar{\pi}'(x'_1, x''_1, x'_2, x''_2, \ldots, x''_{k-1}, x'_k, x''_k) = x''_1, x'_1, x''_2, x'_2, x''_k, x'_k, x''_{k-1}, \ldots, x''_k.$$ 

This one has two components: the first one contains the two points $x'_1, x''_1$, while the second one incorporates all the rest, see Fig[3]. We will show that by doing surgery $\bar{\pi} \sim \bar{\pi}'$ just described – which makes two cycles from one – we diminish the total length. The argument is the same for all $k$, so we will consider the case
Figure 4: The length decrease through surgeries.

\( k = 3 \). The computations are simpler in the half–plane model, in which case the absolute is just the real line \( \mathbb{R}^1 \).

Without loss of generality we can take two pairs of points to be \(-x - 1, -x + 1; x - 1, x + 1\), see Fig. 5. Their cross-ratio is

\[
R (-x - 1, -x + 1; x - 1, x + 1) = \frac{1}{x^2 - 1},
\]

and our assumption that the two geodesics \( \gamma (-x - 1, -x + 1), \gamma (x - 1, x + 1) \) are far away is satisfied once \( x \) is large enough. Let there be another pair, \( y - z, y + z \), lying in between, \(-x + 1 < y - z, y + z < x - 1\), and we want the two cross-ratios to be as small:

\[
R (-x - 1, -x + 1; y - z, y + z) = \frac{4z}{(x + y)^2 - (z + 1)^2} \leq \frac{1}{x^2 - 1}.
\]

\[
R (y - z, y + z; x - 1, x + 1) = \frac{4z}{[x - 1 - y - z][x + 1 - y + z]} \\
\equiv \frac{4z}{(x - y)^2 - (z + 1)^2} \leq \frac{1}{x^2 - 1}.
\]

Our goal is to show that under our assumptions we have

\[
|\gamma (-x + 1, y - z)| + |\gamma (y + z, x - 1)| \gg |\gamma (y - z, y + z)| + |\gamma (-x + 1, x - 1)|.
\]
Figure 5: The first surgery step in the half-plane model.

As we know already, for this it is enough to check that the cross-ratio
\( R(y - z, y + z; x - 1, -x + 1) \) is small. Indeed, as we will check now,

\[
R(y - z, y + z; x - 1, -x + 1) < [R(-x - 1, -x + 1; x - 1, x + 1)]^{1/2} \sim \frac{1}{x},
\]

so we will be done.

Let \( y > 0 \), then we will use only the relation
\[
\frac{4z}{(x - y)^2 - (z + 1)^2} \leq \frac{1}{x^2 - 1}.
\]

The worst case is when
\[
\frac{4z}{(x - y)^2 - (z + 1)^2} = \frac{1}{x^2 - 1}.
\]

It implies that

\[
y = x - \sqrt{4z(x^2 - 1) + (z + 1)^2},\tag{3}
\]

\[
y^2 = x^2 + 4z(x^2 - 1) + (z + 1)^2 - 2x\sqrt{4z(x^2 - 1) + (z + 1)^2}
\]

We want to estimate the cross-ratio

\[
R(y - z, y + z; x - 1, -x + 1)
= \frac{2z(2x - 2)}{[x - z - 1 + y][x - z - 1 - y]} = \frac{2z(2x - 2)}{(x - z - 1)^2 - y^2}
\]
By (3),
\[ R(y-z, y+z; x-1, -x+1) = \frac{2z(2x-2)}{-2x(z+1) + 4z(x^2 - 1) + 2x\sqrt{4z(x^2 - 1) + (z+1)^2}}. \]
Since \( x \) is large, and \( z < 1 \) we have
\[ \frac{2z(2x-2)}{-2x(z+1) + 4z(x^2 - 1) + 2x\sqrt{4z(x^2 - 1) + (z+1)^2}} \sim \frac{4z}{-2(z+1) + 4zx + 2\sqrt{4zx^2 + 1}} < \frac{4z}{4z(x-1)} = \frac{1}{x-1}, \]
which proves our claim. ■

3  Foliated states

3.1 Rigidity of a single interface

The Ising model on \( \mathcal{L}_{p,q} \) is defined by the formal Hamiltonian
\[ H(\sigma) = -\sum_{u \sim v} \sigma(u) \sigma(v), \]
where \( u, v \in \mathcal{L}_{p,q} \) are vertices of the graph \( \mathcal{L}_{p,q} \), the function \( \sigma(\cdot) \) takes values \( \pm 1 \), and the summation goes over the nearest neighbours.

Let \( \Gamma \) be a geodesical family with phase assignment, see Construction 2 above. For every tessellation \( \mathcal{L}_{p,q} \) this family defines a spin configuration \( \sigma^\pm_\Gamma \) on \( \mathcal{L}_{p,q} \) in an evident way.

The rigidity of the interface \( \Sigma_\Gamma \) of the low temperature Ising model on \( \mathcal{L}_{p,q} \), corresponding to the boundary condition \( \sigma^\pm_\Gamma \), in the case when \( \Gamma \) consists of a single geodesic \( \gamma \), is the main result of Series and Sinai, where the following statement is proven:

**Theorem 2** (see [SS]) Let \( \gamma \) be a geodesics, \( z \in \gamma \) be an arbitrary point, and \( \Sigma_{\gamma V} \) be the interface in the box \( V \), corresponding to the boundary condition \( \sigma^\pm_\gamma \). Define the neighborhood \( C_m(\gamma, z) \) of the curve \( \gamma \) by
\[ C_m(\gamma, z) = \cup_{y \in \gamma} B(y, r_m(y)), \quad (4) \]
where \( B(y,r) \subset \mathcal{L} \) is a ball of radius \( r \), centered at \( y \), and the radius \( r_m(y) \) is given by

\[
    r_m(y) = \max \{ m, \text{dist}(y, z) \}.
\]

then

\[
    \mathbb{P}_{V, \beta, \sigma^\pm} \{ \Sigma_{\gamma} V \nsubseteq C_m(\gamma, z) \} \leq \exp\{-\beta m\}
\]

uniformly in \( V \).

Here \( \mathbb{P}_{V, \beta, \sigma^\pm} \) is the Ising model Gibbs state in the finite box \( V \subset \mathcal{L}_{p,q} \), corresponding to the inverse temperature \( \beta \) and boundary conditions \( \sigma^\pm \).

### 3.2 Rigidity of finitely many interfaces

When \( \Gamma \) consists of finitely many geodesics \( \gamma_1, \ldots, \gamma_k \), a similar result holds, and the proof needs only one extra element, as compared with the theorem \[2\] which element was already used for the lemma \[1\]. We start with some definitions.

Let our geodesical family \( \Gamma \) be defined by the sequence \( X_{2k} \) of points on the absolute, \( \{ x'_1, x''_1, \ldots, x'_k, x''_k \} \), going clockwise. In other words, \( \Gamma = \{ \gamma_1(x'_1, x''_1), \ldots, \gamma_k(x'_k, x''_k) \} \). The family \( \Gamma \) defines the partition \( \Pi_0 \) of \( X_{2k} \) into \( k \) pairs: \( (x'_1, x''_1), \ldots, (x'_k, x''_k) \). Every spin configuration \( \sigma \) has its interface collection \( \Sigma_{\Gamma}(\sigma) \), and so defines a partition \( \Pi(\sigma) \) of \( X_{2k} \) into pairs. The partition \( \Pi_0 \) should be called the ground state partition.

(Of course, the observable \( \Pi(\sigma) \), as defined, is not local. One has to talk about finite boxes \( V \) with boundary condition \( \sigma^\pm \), and then the corresponding observable \( \Pi_V(\sigma) \) is local, evidently. However, our estimates will be uniform in \( V \), so we will talk about the observable \( \Pi(\sigma) \), omitting the index \( V \).)

**Theorem 3.1.** Suppose the geodesical family \( (\Gamma, \pm) = \{ \gamma_1, \ldots, \gamma_k \} \) satisfies the conditions of Lemma \[4\] with \( \alpha \) small. Then for every \( V \) finite the probability of the event \( \Pi(\sigma) \neq \Pi_0 \) satisfies

\[
    \mathbb{P}_{V, \beta, \sigma^\pm} \{ \Pi(\sigma) \neq \Pi_0 \} \leq C(k) \exp\{-\beta C(\alpha)\},
\]

where \( C(k) \sim k^2 \), while \( C(\alpha) \sim \ln \alpha^{-1} \to \infty \) as \( \alpha \to 0 \). In words, the typical collection \( \Sigma_{\Gamma} \) of \( k \) interfaces pairs the points \( x'_1, x''_1, \ldots, x'_k, x''_k \) in the ‘correct’ way: \( (x'_1, x''_1), \ldots, (x'_k, x''_k) \).
2. Let \( z_i \in \gamma_i, i = 1, \ldots, k \) be an arbitrary collection of points on geodesics \( \gamma_i \), and the sets \( C_m (\gamma_i, z_i) \) are defined by (4.5). Then under condition that \( \Pi (\sigma) = \Pi_0 \) we have

\[
P_{\nu, \beta, \sigma_1^*} \{ \Sigma (\sigma) \not\subset \bigcup_{i=1}^{k} (C_m (\gamma_i, z_i)) \mid \Pi (\sigma) = \Pi_0 \} \leq k \exp \{ -\beta m \},
\]

uniformly in \( V \) and the set \( \{ z_i, i = 1, \ldots, k \} \). In particular, for the value \( m (\alpha) = \frac{1}{2} \log \alpha^{-1} \) we have

\[
P_{\nu, \beta, \sigma_1^*} \{ \Sigma (\sigma) \not\subset \bigcup_{i=1}^{k} (C_m (\gamma_i, z_i)) \} \leq C (k) \exp \{ -\beta m (\alpha) \}.
\]

**Proof.** The relation (8) follows in a straightforward way from the Proposition 4.1 of [SS], even after replacing the partition \( \Pi_0 \) by any other allowed partition.

To see (7), let us start with the case \( k = 2 \). The event we are interested in is that the partition \( \Pi (\sigma) \) is ‘wrong’: the two interfaces from \( \Sigma (\sigma) \) connect \( x'_1 \) to \( x''_2 \), and \( x''_1 \) to \( x'_2 \). Consider the ‘wrong’ geodesics \( \tilde{\gamma}_1 (x'_1, x''_2) \) and \( \tilde{\gamma}_2 (x'_2, x''_1) \). As we know already, the surplus – or the difference \( \mid \tilde{\gamma}_1 + \tilde{\gamma}_2 \mid - \mid \gamma_1 \mid - \mid \gamma_2 \mid \sim 2L (\alpha) = 2 \log \alpha^{-1} \) is the minimal extra length the interface \( \Sigma (\sigma) \) has to pay for the wrong connection, and this is the reason for (7) to hold. The argument goes as follows.

Let \( d (\sigma) \) be the distance between the two interfaces \( \eta_1, \eta_2 \), making \( \Sigma (\sigma) \). Suppose first that \( d (\sigma) \leq c_1 L (\alpha) \), for some suitable constant \( c_1 \) to be specified later. Let \( \delta (\sigma) \) be the corresponding path, connecting \( \eta_1 \) to \( \eta_2 \), \( \mid \delta \mid \leq c_1 L (\alpha) \). Denote by \( D (\sigma) = \max (\text{dist}(\delta, \gamma_1), \text{dist}(\delta, \gamma_2)) \). Note that

\[
D (\sigma) \geq \frac{1}{2} (1 - c_1) L (\alpha).
\]

Consider the tubular neighborhood \( \Delta (\sigma) \) of \( \delta (\sigma) \) of width \( C_3 \), and let us perform a Peierls transformation of \( \sigma \to \sigma' \), flipping \( \sigma \) inside the contour \( \partial = \partial \Delta (\sigma) \). The result on \( \eta_1, \eta_2 \) of this surgery is a new pair \( \eta'_1, \eta'_2 \), connecting now \( x'_1 \) to \( x''_1 \), and \( x''_2 \) to \( x'_2 \), so \( \Pi (\sigma') = \Pi_0 \). Note that the Hausdorff distance \( \text{dist}_H \) satisfies

\[
\max_{i=1,2} \text{dist}_H (\eta'_i, \gamma_i) \geq D (\sigma) - C_3,
\]

thus the event \( \Sigma (\sigma') \not\subset \bigcup_{i=1}^{k} (C_m (\gamma_i, z_i)) \) happens, with \( m = D (\sigma) - C_3 \). Here the points \( z_i \in \gamma_i \) are chosen in such a way that \( \text{dist} (z_1, z_2) = \text{dist} (\gamma_1, \gamma_2) \). On the other hand, \( H (\sigma') - H (\sigma) \leq C (m, q) \mid \delta (\sigma) \mid \leq C (m, q) c_1 L (\alpha) \) for
some constant $C(m, q)$, which arises due to the difference of the metrics on $L$ and $L_{p,q}$. The usual Peierls argument computations together with (8) imply that

$$
P_{V, \beta, \sigma}^\pm \{ \sigma : \Pi(\sigma) \neq \Pi_0, d(\sigma) \leq c_1 L(\alpha) \} \leq C_4 \exp \{ \beta C(p, q) c_1 L(\alpha) \} \exp \{ -\beta (D(\sigma) - C_3) \}$$

$$= C_4 \exp \left\{ -\beta \left( \frac{1}{2} (1 - c_1) L(\alpha) - C_3 - C(p, q) c_1 L(\alpha) \right) \right\}. \quad (9)$$

In the opposite case, when $d(\sigma) > c_1 L(\alpha)$, we note that necessarily $\Sigma_{\Gamma}(\sigma) \not\subset \bigcup_{i=1}^2 (C_m(\gamma_i, \bar{z}_i))$ for $\bar{z}_1, \bar{z}_2$ defined by dist $(\bar{z}_1, \bar{z}_2) = \text{dist}(\gamma_1, \gamma_2)$, and $m = \frac{1}{2} c_1 L(\alpha)$. (Here we use the obvious fact that dist $(\gamma_1, \gamma_2)$ goes to 0 as $\alpha \to 0$.) Therefore, again by (8),

$$P_{V, \beta, \sigma}^\pm \{ \sigma : \Pi(\sigma) \neq \Pi_0, d(\sigma) \leq c_1 L(\alpha) \} \leq 2 \exp \left\{ -\frac{1}{2} \beta c_1 L(\alpha) \right\}. \quad (10)$$

Comparing (9) with (11), we see that the optimal choice of $c_1$ is given by

$$c_1 = \frac{1}{2 + C(p, q)},$$

so our claim follows. (It is easy to see, by the way, that $C(p, q)$ is bounded by a universal constant.)

The case of $k > 2$ is treated in the same way as above, using the computations made in the proof of the Lemma.

3.3 Rigidity of the Foliated State interfaces

As we saw in the previous subsection, in the low-temperature state defined by the boundary condition $\sigma_{\Gamma}^\pm$, corresponding to some finite geodesical family $(\Gamma, \pm) = \{ \gamma_1, ..., \gamma_k \}$, we have $\Pi(\sigma) = \Pi_0$ for a typical configuration $\sigma$. However, the temperature for which this claim is true, goes to 0 as $k \to \infty$. Moreover, in the case of infinitely many interfaces, corresponding to the millefeuille family $\Gamma_\infty(\alpha, \eta)$ and the boundary condition $\sigma_{\Gamma_\infty}^\pm$, the probability $P_{V, \beta, \sigma_{\Gamma_\infty}^\pm} \{ \Pi(\sigma) = \Pi_0 \}$ goes to zero as $V \to \infty$ for every finite temperature. Indeed, for every geodesic $\gamma \in \Gamma_\infty$ there are infinitely many curves $\bar{\gamma}$ in $\Gamma_\infty$, for which dist $(\gamma, \bar{\gamma}) \leq \max \{ L(\alpha), L(\eta) \}$. Therefore, the probability in the state $P_{\beta, \sigma_{\Gamma_\infty}^\pm}$ that somewhere along $\gamma$ the surgery between it and $\bar{\gamma}$ will
happen, equals 1 for any positive temperature $\beta^{-1}$. Nevertheless, the density of these surgeries, for $\alpha$ small, goes to zero as $\beta \to \infty$, which means that in the vicinity of any fixed point the probability that such a surgery happens there is small (but it does depend on the size of the neighborhood -- as is also the case for the rigidity property of the Dobrushin interface in $\mathbb{Z}^3$).

Now we will formulate one version of the theorem which makes this claim rigorous. Let $\gamma_1, \gamma_2 \in \Gamma_\infty (\alpha, \eta)$ be the first two geodesics of the family, say, and let the point $Z \in \mathcal{L}$ be defined by $\operatorname{dist} (Z, \gamma_1) = \operatorname{dist} (Z, \gamma_2) = \frac{1}{2} \operatorname{dist} (\gamma_1, \gamma_2)$. In other words, the point $Z$ is the symmetry center of the pair $\gamma_1, \gamma_2$. As before, we denote by $\sigma_{\Gamma_\infty}^\pm$ some ground state configuration, corresponding to the geodesical family $\Gamma_\infty (\alpha, \eta)$. Without loss of generality we can assume that $\sigma_{\Gamma_\infty}^\pm (t) = +1$ for sites $t \in U_r (Z) \subset \mathcal{L}_{p,q}$ which are at distance $\leq r$ from the point $Z \in \mathcal{L}$; here $r$ is some finite radius, much smaller than our scale $\ln \alpha^{-1}$. Let $V \subset \mathcal{L}_{p,q}$ be a finite box with boundary condition $\sigma_{\Gamma_\infty}^\pm |_{V^c}$ defined by $\sigma_{\Gamma_\infty}^\pm$, and $\sigma_V \in \Omega_V$ be a spin configuration in $V$. Let $\Sigma (\sigma_{\Gamma_\infty}^\pm)$ be the (countable) collection of all (infinite) contours of $\sigma_{\Gamma_\infty}^\pm$, while $\Sigma (\sigma_{\Gamma_\infty}^\pm |_{V^c} \cup \sigma_V)$ -- the collection of all infinite contours of $\sigma_{\Gamma_\infty}^\pm |_{V^c} \cup \sigma_V$.

Denote by $\Lambda (\sigma_V)$ the symmetric difference $\Sigma (\sigma_{\Gamma_\infty}^\pm) \Delta \Sigma (\sigma_{\Gamma_\infty}^\pm |_{V^c} \cup \sigma_V)$; it consists of finitely many closed contours.

**Theorem 4** Let the parameters $\alpha$ and $\eta$ are small enough, and the temperature $\beta^{-1}$ is low. Then typically the interfaces $\Sigma (\sigma_{\Gamma_\infty}^\pm |_{V^c} \cup \sigma_V)$ are far away from the point $Z$, and the phase $\langle \cdot \rangle_{\beta, \sigma_{\Gamma_\infty}^\pm}$ is $e^{-\beta \ln \alpha^{-1}}$-close to the phase $\langle \cdot \rangle_+$, when restricted to the box $U_r (Z)$. Explicitly, for some $C$ and $C (r)$

$$
\mathbb{P}_{V, \beta, \sigma_{\Gamma_\infty}^\pm} (\Lambda (\sigma_V) \cap U_r (Z) \neq \emptyset) \leq C (r) \exp \{-C \beta \ln \alpha^{-1}\}
$$

uniformly in $V$. In words, the interfaces typically are far away from $U_r (Z)$.

**Proof.** Let the event $\Lambda (\sigma_V) \cap U_r (Z) \neq \emptyset$ does happen, and $\theta \equiv \theta (\sigma_V, Z) \in \Lambda (\sigma_V)$ be a contour which contributes to the event $\Lambda (\sigma_V) \cap U_r (Z) \neq \emptyset$. Then the loop $\theta$ is made by fragments of several interfaces from the family $\Sigma (\sigma_{\Gamma_\infty}^\pm)$ of contours of $\sigma_{\Gamma_\infty}^\pm$, and by the same number of interfaces from $\Sigma (\sigma_{\Gamma_\infty}^\pm |_{V^c} \cup \sigma_V)$. Denote this observable by $k (\theta (\sigma_V, Z))$, and extend it to all configurations $\sigma_V$ by defining $k (\theta (\sigma_V, Z)) = 0$ if $\Lambda (\sigma_V) \cap U_r (Z) = \emptyset$.

We will prove our theorem by induction on $k$, estimating the probabilities of the events $\mathbb{P}_{V, \beta, \sigma_{\Gamma_\infty}^\pm} (k (\theta (\sigma_V, Z)) = k)$ for each $k \geq 1$. 17
Case $k = 1$. That means that for one of the geodesics $\gamma_i \in \Gamma_\infty$ the event $\Sigma_{\gamma_i \cup C} \notin \mathcal{C}_{m_i} (\gamma_i, z_i)$ happens, where $z_i \in \gamma_i$ is chosen to be the point on $\gamma_i$ closest to $Z$, and $m_i = \text{dist} (Z, \gamma_i)$. We know already, that

$$
P_{V,\beta,\sigma_\infty} \left\{ \Sigma_{\gamma_i \cup C} \notin \mathcal{C}_{m_i} (\gamma_i, z_i) \right\} \leq \exp \left\{ -\beta m_i \right\}.
$$

(12)

So we need to know how fast the sequence $m_i$ grows.

The computations are easier in the upper half–plane model. Without loss of generality we can assume that the first two geodesics $\gamma_1, \gamma_2$ are the two semicircles, $O_0^-, O_0^+$ in the upper half–plane, centered at points $(-X, 0)$ and $(X, 0)$, with radius 1. $X$ is related to $\alpha$ by $X \sim \ln \alpha^{-1}$. Let us take for the point $Z$ the point $(0, X)$. It is sufficient to consider only those geodesics $\gamma_i$ – i.e. semicircles $O_i$ – which are located between the $y$-axis and the semicircle $O_0^+$. This is only ‘a quarter’ of all geodesics from $\Gamma_\infty$, but it is sufficient for our question. So let $X > x_1 > x_2 > ... > x_n > 0$, $r_1, ..., r_n > 0$ be a sequence of centers and a sequence of radii of non-intersecting semicircles $O_i$, see Fig.6.

It turns out that the divergence we need does not require the precise information about the structure of the family $\{O_i\}$; in particular, we will not use in the essential way the fact that the cross-ratios for certain pairs $O_i, O_j$ are our small numbers $\alpha$ and $\eta$. The only thing needed is that the semicircles $O_i$ do not intersect, and that all the radii $r_i \leq \frac{1}{2}$. We want to estimate the distances $\rho_i = \text{dist} (Z, O_i)$; we need them to diverge to infinity.
fast enough. It is sufficient to estimate the distances \( g_i = \text{dist}(O_0, O_i) \), since \( \text{dist}(o, O_0) \) is just a constant. The latter is simpler, since the distances \( g_i \) can be expressed via the cross-ratios:

\[
g_i \approx -\ln \frac{(2)(2r_i)}{(X + 1 + x_i + r_i)(X - 1 + x_i - r_i)} \gtrsim -\ln \frac{r_i}{X^2}.
\]

According to (12), the probability in question is bounded by

\[
\sum_{i=1}^{n} \exp \{-\beta g_i\} \lesssim \sum_{i=1}^{n} \exp \left\{ \beta \ln \frac{r_i}{X^2} \right\} = \sum_{i=1}^{n} \left( \frac{r_i}{X^2} \right)^\beta = X^{-2\beta} \sum_{i=1}^{n} (r_i)^\beta.
\]

Note that \( 2(r_1 + \ldots + r_n) \leq X \), and that \( r^\beta + r'^\beta < (r + r')^\beta \) for \( \beta > 1 \). Therefore, for any \( n \) we have

\[
\sum_{i=1}^{n} (r_i)^\beta \leq \sum_{i=1}^{[X]} 1 = X,
\]

so

\[
\sum_{i=1}^{n} \exp \{-\beta g_i\} \leq X^{-2\beta + 1}.
\]

**Case** \( k = 2 \). The events \( \mathbb{P}_{V,\beta,\sigma_V}^\pm(k(\theta(\sigma_V, Z)) = 2) \) means that there are two geodesics \( \gamma'(x_1, x_2), \gamma''(x_3, x_4) \in \Gamma_\infty \), such that the configuration \( \sigma_\Gamma^\pm|_{V \cup \sigma_V} \) has, among other, two interfaces \( \Sigma(x_1, x_4) \) and \( \Sigma(x_2, x_3) \), so that the "contour \( \Sigma(x_1, x_4) \cup \Sigma(x_2, x_3) \Delta[\gamma'(x_1, x_2) \cup \gamma''(x_3, x_4)]\)" surrounds the point \( Z \). As is established during the proof of the Theorem 3, the probability of such an event is of the order of \( \exp \{-\beta (\text{dist}(Z, \gamma') + \text{dist}(Z, \gamma''))\} \). In the previous paragraph we have shown that

\[
\sum_{\gamma', \gamma'' \in \Gamma_\infty} \exp \{-\beta (\text{dist}(Z, \gamma') + \text{dist}(Z, \gamma''))\} \leq \left( \ln \alpha^{-1} \right)^{-2\beta + 1}^2,
\]

and we are done.

The rest of the proof goes by induction on \( k \). ■

**4 Periodic Gibbs states on Cayley trees \( T_n \) for \( n = 2, 3 \)**

Motivated by the paper [RR], we have constructed in [GRS] a huge manifold of extremal Gibbs states on Cayley trees \( T_n \). However, our construction, as
well as that in [RR], was restricted to the case $n \geq 4$; we were using the fact that each vertex of our tree has at least 5 neighbors. However, that, happily, does not mean that the trees $T_2$ and $T_3$ are that different from the rest. The difference between $T_2, T_3$ and $T_{n \geq 4}$ lies only in the fact that the former trees do not carry states with period two. Thus, the equations of [RR] do not have solutions for $n = 2, 3$, since they are period two functions, while the equations for functions with bigger periods are too complicated to be analyzed. On the other hand, our method to construct ‘dimer’ states and their analogs, needs just a tiny modification to be applicable to all $n \geq 2$. This modification is explained below. To simplify the exposition we will consider only the case of $T_2$. We start with a definition.

**Definition 5** A $k$-chain $C \subset E$ is a sequence $x_0, x_1, ..., x_k \in V$ of distinct n.n. vertices of $T_2 \equiv (V, E)$. (For example, a dimer is a 1-chain.) The vertices $x_0$ and $x_k$ are called the ends of the chain.

The set of all $k$-chains will be denoted by $C_k$.

A collection $R_k \subset C_k$ of $k$-chains will be called a covering of $T_2$, if every vertex $x$ of $T_2$ belongs to precisely one $k$-chain from $R_k$. The existence of coverings is evident.

Let $R_k$ be such a covering, and suppose that $k$ is odd, $k = 2m + 1$. Then every $k$-chain has a middle dimer. The collection $D = \{d_i = (y_i, y'_i) \in E\}$ of these dimers is called a $k$-covering, associated with $R_k$, $D = D(R_k)$. (Of course, a $k$-covering is not a covering for $k > 1$.)

Let a $k$-covering $D$ be fixed. Consider the Ising spin configuration $\sigma_D$ on $T_2$, defined by the property: for every two n.n. sites $z, z' \in V$

$$\sigma_D(z) \sigma_D(z') = \begin{cases} -1 & \text{for } (z, z') \in D \\ +1 & \text{for } (z, z') \notin D \end{cases}.$$ 

In fact, there are exactly two such configurations, which differ by a global spin-flip. We choose one, see Fig.7. Our main observation is that

**Claim 6** For $k \geq 5$ the configuration $\sigma_D$ is a ground state configuration. Moreover, it is a stable ground state, which means that there exist a family of low temperature Ising model Gibbs states $\langle \cdot \rangle_\beta$ on $T_2$, which converges to $\sigma_D$ weakly, as $\beta \to \infty$. In fact, for trees $T_{n \geq 4}$ this is true even for $k = 1$, see [GRS], but for $T_2$ this is not the case.
Figure 7: Covering $R_3$ of $T_2$, 3–covering $D(R_3)$ (black bonds) and spin configuration $\sigma_D$. Blue and red sites have opposite signs. The first six chains are marked.

To see this, we put $T_2$ on $\mathbb{R}^2$, since all Cayley trees are planar, and we will talk about contours, which are closed loops on $\mathbb{R}^2$, which intersect the bonds of our tree, but which do not pass through the vertices.

For every loop $\gamma$, which is a Peierls-like contour, we define the length $|\gamma|$ of $\gamma$ to be the number of bonds of $T_2$ that $\gamma$ traverses, and let $|\gamma|_D \leq |\gamma|$ be the number of bonds among them which are from $D$.

Consider the ratio $\frac{|\gamma|_D}{|\gamma|}$, and let

$$\varphi(D) = \sup_{\gamma} \frac{|\gamma|_D}{|\gamma|}.$$
where the sup is taken over all finite contours. Our claim follows immediately from the following Peierls stability property:

**Lemma 7**

\[ \varphi(D) \leq \frac{1}{m+1}. \]  

(13)

Note that for paths \( \gamma \) which are not loops this is not true. Moreover, one can find paths \( \gamma - \) even long paths - for which \( |\gamma|_D = |\gamma| \).

We will have Peierls condition satisfied, if for some (small) \( c > 0 \) and any \( \gamma \) we have \( |\gamma| - 2 |\gamma|_D > c |\gamma| \), which means that we need \( |\gamma|_D < \frac{1-c}{2} |\gamma| \).

According to (13), the desired relation holds once \( \frac{1}{m+1} < \frac{1}{2} \), i.e. when \( m \geq 2 \) and \( k \geq 5 \).

**Proof.** Let \( \mathcal{T}_2 \) be embedded into upper half-plane \( \mathbb{R}^{2+} \). Choose an arbitrary vertex \( 0 \in \mathcal{T}_2 \), which will be called the root of the tree. Let it be the only vertex with \( y \)-coordinate equals to zero. We suppose that all the bonds are of the length one. First we will construct a concrete covering \( R \) of \( \mathcal{T}_2 \) by \( k \)-chains. The construction is by induction. The first chain starts from \( 0 \in \mathcal{T}_2 \), and each of its bonds is the left one among the two (or three in the first step) possible options. Suppose the \( k \)-chains \( C_i \) are already defined, \( i \leq n - 1 \). Let the vertex \( x \in \mathcal{T}_2 \) be the closest to \( 0 \) among those not yet covered by all the \( C_i \). If there are several such vertices, we take the leftmost one. The chain \( C_n \) starts at \( x \) and then always uses the ‘left’ bonds.

Let \( \gamma \subset \mathbb{R}^2 \) be a loop, surrounding \( 0 \), and such that \( \gamma \cap V = \emptyset \). We will prove the inequality

\[ \frac{|\gamma|_D}{|\gamma|} \leq \frac{1}{m+1} \]

by induction in the number \( |\text{Int}(\gamma) \cap V| \) of the vertices of \( \mathcal{T}_2 \), surrounded by \( \gamma \). Note that if \( |\text{Int}(\gamma) \cap V| \leq m \), then \( |\gamma|_D = 0 \), so the initial step of induction is done.

For any \( y \neq 0 \in V \) denote by \( l_y \) and \( r_y \) the left and the right bonds, starting at \( y \) and going away from \( 0 \), and by \( b_y \) the bond going ‘back’ to \( 0 \).

Suppose the lemma is true for all \( \gamma \) with \( |\text{Int}(\gamma) \cap V| \leq N - 1 \). Take some loop \( \gamma \) with \( |\text{Int}(\gamma) \cap V| = N \). Let \( x \in \text{Int}(\gamma) \) be the point in \( \text{Int}(\gamma) \) with maximal distance from \( 0 \). Because of maximality of \( x \), \( \gamma \) intersects both \( l_x \) and \( r_x \).

Suppose first that \( l_x \notin D \). (By construction, \( r_x \notin D \).) Let us deform \( \gamma \) in the vicinity of \( x \) into \( \gamma' \), moving \( \gamma \) in such a way that instead of intersecting \( l_x \)

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and \( r_x \) it intersects just \( b_x \). The situation for all other bonds stays the same. In other words, \( \text{Int} (\gamma') = \text{Int} (\gamma) \setminus x \). Then \( |\gamma'| = |\gamma| - 1 \), while \( |\gamma'|_D = |\gamma|_D \), so \( |\gamma'|_D < |\gamma|_D \). By induction, \( |\gamma'|_D < \frac{1}{m+1} \), and we are done.

Now consider the case when \( l_x \in D \). Let \( x_0, x_1, \ldots, x_m = x \) be the ‘lower’ half of the \((2m+1)\)-chain \( C^x \), to which \( x \) belongs. Note, that, by construction, none of the bonds \( r_{x_i}, i = 0,1,\ldots, m \) belong to the \((2m+1)\)-chains from our covering \( R \). Let \( T_0, T_1, \ldots, T_m \subset T_2 \) be the subtrees, defined as follows: \( T_i \) is the maximal connected component of the complement \( T_2 \setminus C^x \), containing the bond \( r_{x_i} \). Since \( 0 \in \text{Int} (\gamma) \), the loop \( \gamma \) has to intersect every tree \( T_i \) at least once. However, due to the maximality property of \( x \), the intersection \( [\gamma \cap (T_0 \cup T_1 \cup \ldots \cup T_m)] \) is disjoint with \( D \); in other words, \( [\gamma \cap (T_0 \cup T_1 \cup \ldots \cup T_m)] \setminus D = \emptyset \). On the other hand, this intersection \( [\gamma \cap (T_0 \cup T_1 \cup \ldots \cup T_m)] \) involves at least \( m+1 \) bonds – at least one bond per any subtree \( T_i \). Let \( T_{x_m} \subset T_2 \) be the subtree, rooted at \( x_m \) and consisting of all its descendants. Let us deform \( \gamma \) into \( \gamma'' \), where the latter loop is defined by two conditions:

1. \( \text{Int} (\gamma'') \cap T_{x_m} = \emptyset \);
2. \( \text{Int} (\gamma'') \cap [T_2 \setminus T_{x_m}] = \text{Int} (\gamma) \cap [T_2 \setminus T_{x_m}] \).

One sees immediately that the bonds that contribute to \( |\gamma''| \) are these bonds in \( T_2 \setminus T_{x_m} \), which contribute to \( |\gamma| \), plus the bond \( r_{x_m} \). So \( |\gamma''| \leq |\gamma| - (m + 1) \). By construction, \( |\gamma''|_D = |\gamma|_D - 1 \). By induction, we have \( \frac{|\gamma|_D - 1}{|\gamma| - (m + 1)} \leq \frac{1}{m+1} \), which implies that \( \frac{|\gamma|_D}{|\gamma|} \leq \frac{1}{m+1} \).

### 4.1 Periodic states with non-zero magnetization

The previous construction results in a ground state with zero mean magnetization. We present now a generalization, which will have a non-zero magnetization. It is analogous to our construction of such states on the Lobachevsky plane.

Let \( R \) be the covering by \( k \)-chains, constructed in the proof above, \( D \) is the associated \( k \)-covering by the dimers, and \( \sigma_D \) is the corresponding ground state. For every \( k \)-chain \( C = \{x_0, x_1, \ldots, x_k\} \in R \) we call the site \( x_0 \) a \((+)-end\) iff \( \sigma_D (x_0) = +1 \). We denote it by \( e_+ (C) \). Then the opposite end-point \( x_k \) is, naturally, called a \((-)-end\), and denoted by \( e_- (C) \). Let \( k = l + n - 1 \), \( l > n > 0 \). Define the collection of dimers \( D_{l,n} \) as follows: on every chain \( C \in R \) let us take the bond \( d(C) \), which is at distance \( l \) from \( e_+ (C) \) (and so at distance \( n \) from \( e_- (C) \)).
The collection $D_{t,n}$ satisfies
\[ \varphi(D_{t,n}) \leq \frac{1}{n+1}, \]
as the Lemma above shows, so the configuration $\sigma_{D_{t,n}}$ is a stable ground state, once $n$ is big enough. The nice property of it is that, unlike the collection $D \equiv D_{(m+1):(m+1)}$, constructed earlier, its mean magnetization $m(\sigma_{D_{t,n}})$ is non-zero.

Acknowledgement. We would like to thank M. Aizenman, P. Bleher, A. van Enter and S. Pirogov for valuable discussions and comments.

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Daniel Gandolfo  
Aix Marseille Université, CNRS, CPT, UMR 7332, 13288 Marseille, France.  
Université de Toulon, CNRS, CPT, UMR 7332, 83957 La Garde, France.  
gandolfo@cpt.univ-mrs.fr, gandolfo@univ-tln.fr

Jean Ruiz  
Aix Marseille Université, CNRS, CPT, UMR 7332, 13288 Marseille, France.  
Université de Toulon, CNRS, CPT, UMR 7332, 83957 La Garde, France.  
ruiz@cpt.univ-mrs.fr

Senya Shlosman  
Aix Marseille Université, CNRS, CPT, UMR 7332, 13288 Marseille, France.  
Université de Toulon, CNRS, CPT, UMR 7332, 83957 La Garde, France.  
shlosman@cpt.univ-mrs.fr, shlosman@univ-amu.fr  
Inst. of the Information Transmission Problems, RAS, Moscow, Russia.  
shlos@iitp.ru