A family of sequences with large size and good correlation property arising from $M$-ary Sidelnikov sequences of period $q^d - 1$

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Abstract—Let $q$ be any prime power and let $d$ be a positive integer greater than 1. In this paper, we construct a family of $M$-ary sequences of period $q-1$ from a given $M$-ary, with $M|q-1$, Sidelnikov sequence of period $q^d - 1$. Under mild restrictions on $d$, we show that the maximum correlation magnitude of the family is upper bounded by $(2d - 1:\sqrt{q} + 1$ and the asymptotic size, as $q \to \infty$, of that is $\frac{q^d - 1}{d}$. This extends the pioneering work of Yu and Gong for $d = 2$ case in [9] and [10].

II. PRELIMINARIES

We will use the following notations throughout this paper.

- $p$ a prime number,
- $n$ a positive integer,
- $q = p^n$,
- $\mathbb{F}_q$ the finite field with $q$ elements,
- $\mathbb{F}_{q^d}$ the finite field with $q^d$ elements, with $d \geq 2$,
- $M$ a positive divisor of $q - 1$, with $M \geq 2$,
- $w_M = \exp\left(\frac{2\pi i}{M}\right)$,
- $\alpha$ a fixed primitive element of $\mathbb{F}_{q^d}$,
- $\beta = \alpha^d = N(\alpha)$ a primitive element of $\mathbb{F}_q$,
- $N$ the norm map from $\mathbb{F}_{q^d} \to \mathbb{F}_q$, given by $N(x) = x^{q^d - 1}$,
- $Tr$ the trace map from $\mathbb{F}_{q^d} \to \mathbb{F}_q$, given by $Tr(x) = \sum_{j=0}^{d-1} x^j$,
- $\psi$ the multiplicative character of $\mathbb{F}_q$ of order $M$, defined by $\psi(x) = \exp\left(\frac{2\pi i \log_q x}{M}\right) = w_M^\log_q x$.

Here we recall that, for any fixed primitive element $\beta$ of $\mathbb{F}_q$, a logarithm over $\mathbb{F}_q$ is defined by

$$\log_{\beta} x = \begin{cases} t, & \text{if } x = \beta^t \ (0 \leq t \leq q - 2), \\ 0, & \text{if } x = 0. \end{cases}$$

so that, in particular, $\psi(0) = 1$. This convention is not the usual one requiring $\psi(0) = 0$. However, this agreement turns out to be very convenient, as this has been fruitfully demonstrated in the papers [8]-[11].

In this paper, we consider $M$-ary Sidelnikov sequences, with $M|q-1$, of period $q^d - 1$ (with $q = p^n$ a prime power) and study the $(q - 1) 	imes \left(\frac{q^d - 1}{q - 1}\right)$ array structure of such sequences. Then we construct a family of $M$-ary sequences with period $q - 1$, with large size and good correlation property. It is shown that the constant multiples of those column sequences corresponding to a set of $q$-cyclotomic coset representatives mod $\left(\frac{q^d - 1}{q - 1}\right)$. Under the mild restrictions on $d$ (cf. (9)), it is shown that the maximum correlation magnitude of the family is upper bounded by $(2d - 1:\sqrt{q} + 1$, and the asymptotic size, as $q \to \infty$, of that is $\frac{q^d - 1}{d}$. Also, we derive an exact but less explicit expression of the size of the family of sequences by using a result of Yucas [12]. One refers to the tables either in [2] or [9] to compare our result with the known ones. This generalizes the pioneering work of Yu and Gong for $d = 2$ case in [9] and [10].
generalized his estimate to the case of multiple multiplicative character sums. On the other hand, Yu and Gong(cf. [3]-[11]) introduced a refined version of Wan’s bound that works under the assumption that the value of the multiplicative characters at 0 are equal to 1 rather than the traditional 0. Here we state only a special case that is just suitable for our purpose.

**Theorem 1 ([7], [9]):** Let \( f_1(x), \ldots, f_m(x) \) be monic distinct irreducible polynomials over \( \mathbb{F}_q \) with degrees \( d_1, \ldots, d_m \), with \( e_j \) the number of distinct roots in \( \mathbb{F}_q \) of \( f_j(x) \), \( j = 1, \ldots, m \). Let \( \psi_1, \ldots, \psi_m \) be nontrivial multiplicative characters of \( \mathbb{F}_q \), with \( \psi_j(0) = 1 \), \( j = 1, \ldots, m \). Then, for \( a_1, \ldots, a_m \in \mathbb{F}_q^x \), we have the estimate

\[
\left| \sum_{x \in \mathbb{F}_q} \psi_1(a_1 f_1(x)) \cdots \psi_m(a_m f_m(x)) \right| \leq \left( \sum_{j=1}^m d_j - 1 \right) \sqrt{q} + \sum_{j=1}^m e_j.
\]

(3)

**III. ARRAY STRUCTURE OF THE M-ARY SIDelnikov SEQUENCES OF PERIOD \( q^d - 1 \)**

Here we investigate the \( (q - 1) \times (\frac{q^d - 1}{q - 1}) \) array structure of M-ary Sidelnikov sequences of period \( q^d - 1 \), with \( M|q - 1 \). This is a generalization of the \( d = 2 \) case in [9] and [10] that has its origin in the paper [11].

**Theorem 2:** Let \( \{ s(t) \} \) be an M-ary Sidelnikov sequences of period \( q^d - 1 \), with \( M|q - 1 \). Then

\[
s(t) \equiv \log_\alpha(N(\alpha^t + 1)) \mod M,
\]

(4)

where \( 0 \leq t \leq q^d - 2 \).

In other words,

\[
s(t) = \begin{cases} 0, & \text{if } N(\alpha^t + 1) = 0, \\ k, & \text{if } N(\alpha^t + 1) \in S_k, \end{cases}
\]

where \( S_k = \{ \beta^{Mj+k} | 0 \leq j < \frac{q-1}{M} \} \), for \( 0 \leq k \leq M - 1 \).

**Remark 1:** Note here that the sets \( S_k \) are different from those \( D_k \) in [11].

**Proof:** By definition of Sidelnikov sequence,

\[
s(t) \equiv y(t) \mod M, \text{ with } y(t) = \log_\alpha(\alpha^t + 1).
\]

To prove the statement, we may assume that \( N(\alpha^t + 1) \neq 0 \). Then, with \( N(\alpha^t + 1) = \beta^z(t) \),

\[
\frac{q^d - 1}{q - 1} y(t) \equiv \log_\alpha(\alpha^t + 1) \frac{q^d - 1}{q - 1} \equiv \log_\alpha N(\alpha^t + 1) \equiv \log_\alpha N(\alpha^t + 1) \equiv (\log_\alpha(\alpha^t + 1))(t) \equiv \frac{q^d - 1}{q - 1} x(t) \mod q^d - 1.
\]

This implies that

\[
x(t) \equiv y(t) \mod q - 1,
\]

and hence that, as \( M|q - 1 \),

\[
x(t) \equiv y(t) \mod M,
\]

Thus

\[
s(t) \equiv y(t) \equiv x(t) \equiv \log_\beta N(\alpha^t + 1) \mod M.
\]

We list the sequence \( \{ s(t) \} \) \( \{ 0 \leq t \leq q^d - 2 \} \) as an \( (q - 1) \times (\frac{q^d - 1}{q - 1}) \) array so that the \( l \)-th column \( v_l(t) \), \( 0 \leq t \leq q^d - 2 \), of the array is given by:

\[
v_l(t) = s \left( \left( \frac{q^d - 1}{q - 1} \right) t + l \right), \quad (0 \leq l \leq \frac{q^d - 1}{q - 1} - 1).
\]

Then

\[
v_l(t) \equiv \log_\beta(\log_\alpha(\alpha^t + 1)) \mod M.
\]

Let \( f_1(x) \) be the polynomial of degree \( d \) over \( \mathbb{F}_q \) given by:

\[
f_1(x) = N(\alpha x^l + 1) = (\alpha x^l + 1)(\alpha x^{l-1} + 1) \cdots (\alpha x + 1) = \beta^l x^d + \cdots + \text{Tr}(\alpha^l)x + 1.
\]

Then

\[
v_l(t) \equiv \log_\beta(f_1(t^l)) \mod M.
\]

(5)

For each \( l(0 \leq l \leq \frac{q^d - 1}{q - 1} - 1) \),

\[
f_l(x) = \beta^l N(x + \alpha^{-l}) = \beta^l(x + \alpha^{-l})(x + \alpha^{-l+1}) \cdots (x + \alpha^{-lq^{d-1}}) \equiv \beta^l p_l(x)^M,
\]

where \( p_l(x) \) is the irreducible polynomial over \( \mathbb{F}_q \) of \(-\alpha^l \) of degree \( d_l \). Note here that \( d_l|d \).

**Remark 2:** Note that the \( q \)-cyclotomic coset containing \( l(0 \leq l \leq q^d - 2) \) modulo \( q^d - 1 \) is

\[
C_l = \{ l, ql, \cdots, q^{d-1}l \},
\]

where each \( q^l l \) is reduced modulo \( q^d - 1 \), \( d_l \) is the smallest positive integer satisfying \( q^d l \equiv 1 \) modulo \( q^d - 1 \), and

\[
p_l(x) = \prod_{j \in C_l} (x + \alpha^{-j}).
\]

Here \( l \) is taken as the smallest positive integer in \( C_l \) modulo \( q^d - 1 \), as usual.

**Proposition 1:**

1) \( v_l(t) = v_{q^l}(t) \).
2) \( p_l(x) \) has no roots in \( \mathbb{F}_q \), for \( l \), with \( 1 \leq l \leq \frac{q^d - 1}{q - 1} - 1 \).
3) For nonnegative integers \( l_1, l_2 \), with \( l_1 \equiv l_2 \mod \frac{q^d - 1}{q - 1} \), \( v_{l_1}(t) \) and \( v_{l_2}(t) \) are cyclically equivalent.
4) \( v_{q^l - 1} \equiv v_{q^l - 1} \mod M \), so that \( v_{q^l - 1} \equiv v_{q^l - 1} \mod M \), and \( v_l(t) \) are cyclically equivalent for each \( l \) \( 1 \leq l \leq q \).

**Proof:**

1) \( f_1(x) = f_{l_1}(x) \), so that \( v_l(t) = v_{l_1}(t) \), by (6).
2) This follows from the observation that $d_l = 1$ iff $\alpha \in \mathbb{F}_q$ 
iff $\frac{d - 1}{q - 1}l$.
3) is easy to see.
4) This is a generalization of the result for $d = 2$ discovered by Yu and Gong in [9] and [10]: for each $l(1 \leq l \leq q)$,
$$v_{\frac{d - 1}{q - 1}l}(t) = \log_\beta(N(\alpha^{d - 1}|\beta^{l+1} + 1))$$
$$= \log_\beta(N(\alpha^{d - 1}|\beta^{l+1} + 1))$$
$$= \log_\beta(N(\alpha^{d - 1}|\beta^{l+1} + 1))$$
$$= \log_\beta(N(\alpha^{d - 1}|\beta^{l+1} + 1))$$
$$= v_l(t - l + 1) \mod M.$$

Also, this follows from 1) and 3), since
$$\left(\frac{d - 1}{q - 1} - \frac{d - 1}{q - 1}l\right) \equiv l \mod \frac{d - 1}{q - 1}.$$ 

**Remark 3:** Because of 1) and 3) of Proposition 1, we are led to consider the $q$-cyclotomic cosets $\mod \frac{d - 1}{q - 1}$. Recall that the $q$-cyclotomic coset containing $l(0 \leq l \leq \frac{d - 1}{q - 1} - 1)$ \mod $\frac{d - 1}{q - 1}$ is
$$\hat{C}_l = \{l, ql, \ldots, q^{m_l - 1}l\},$$
where each $q^l$ is reduced modulo $\frac{d - 1}{q - 1}$, $m_l$ is the smallest positive integer satisfying $q^{m_l}l \equiv l \mod \frac{d - 1}{q - 1}$. Again, here $l$ is taken as the smallest positive integer in $\hat{C}_l$ such that $\mod \frac{d - 1}{q - 1}$, as usual. Here $m_l|d_l$. So if $q(x) = \prod_{l \in C_l}(x + \alpha^{-j})$, then
$$p_l(x) = q_l(x)q_l(x)^{\alpha^{m_l}}q_l(x)^{\alpha^{2m_l}}\ldots q_l(x)^{\alpha^{d_l - m_l}}.$$ 

Here $\sigma$ is the Frobenius automorphism of $\mathbb{F}_q$ over $\mathbb{F}_q$, given by $\sigma(\alpha) = \alpha^q$, so that
$$q_l(x)^{\sigma^{m_l}} = \alpha^{m_l}q_l(x)^{\sigma^{m_l}}(0 \leq i \leq \frac{d_l}{m_l} - 1).$$

**IV. Construction of a Family of Sequences**

Here we construct a family $\Sigma$ of $M$-ary sequences with period $q - 1$, consisting of the constant multiples of those column sequences $v_l(t)$ corresponding to a set of $q$-cyclotomic coset representatives $\mod \frac{d - 1}{q - 1}$, for the set consisting of $l(1 \leq l \leq \frac{d - 1}{q - 1})$. Then it is shown that, under mild restrictions on $d$ (cf. [9], it has a large family size and good correlation property. Actually, we show that the maximum correlation magnitude of the family is upper bounded by $(2d - 1)/\sqrt{q + 1}$, and the asymptotic size, as $q \to \infty$, of that is $(\frac{M - 1}{d - 1})^{d - 1}$. Also, we derive an exact but less explicit expression of the size of the family of sequences by using a result of Yucas(cf. Theorem 5). This generalizes the pioneering work of Yu and Gong for $d = 2$ case in [9] and [10].

**Definition 1:** Let $\Lambda$ be the set of all integers $l(0 \leq l \leq \frac{d - 1}{q - 1} - 1)$ consisting of the smallest $q$-cyclotomic coset representative from each $q$-cyclotomic coset $\mod \frac{d - 1}{q - 1}$. 

**Proposition 2:** 1) $|\Lambda| = \frac{d - 1}{q - 1} = \frac{d - 1}{q - 1}$ is the number of monic irreducible factors of $x^{\frac{d - 1}{q - 1}} - 1$.
2) Let $p(x) = x^c + \cdots + (-1)^c b$ be a monic irreducible factor of $x^{\frac{d - 1}{q - 1}} - 1$. Then $e|d$ and $b^e = 1$.

**Proof:** 1) The first equality is just Definition [1]. Let $\gamma = \alpha^{q - 1}$ be a primitive $(\frac{d - 1}{q - 1})$-th root of unity in $\mathbb{F}_q$. Then, with $M(0)(x) = \prod_{l \in C_l}(x - \gamma^l)$ denoting the irreducible polynomial of $\gamma^l$ over $\mathbb{F}_q$, we have
$$x^{\frac{d - 1}{q - 1}} - 1 = \prod_{l \in \Lambda} M(0)(x).$$

Thus we have the desired equality. 

2) Clearly, $e|d$. For a root $\alpha$ of $p(x)$ in $\mathbb{F}_q$, $N(\alpha) = 1$, and $((-1)^c b)^{\frac{d}{e}} = ((-1)^d b^e)$ is the constant term of $p(x)^{\frac{d}{e}} = x^d + \cdots + ((-1)^d N(\alpha))$.

Assume from now on that
$$(d, q - 1) = 1, \quad d < \frac{\sqrt{q - 2} + 1}{2}.$$ 

**Proposition 3:** Let $l_1, l_2$ be elements in $\Lambda \setminus \{0\}$, and let $\tau(0 \leq \tau \leq q - 2)$ be an integer. Then $p_{l_1}(x)$ and $\beta^{-\tau d_{l_2}}p_{l_2}(\beta^\tau x)$ are distinct irreducible polynomials over $\mathbb{F}_q$, unless $l_1 = l_2$ and $\tau = 0$. Here
$$\beta^{-\tau d_{l_2}}p_{l_2}(\beta^\tau x) = (x + \alpha^{-l_2} \beta^{-\tau})(x + \alpha^{-l_2 q \beta^{-\tau}})\ldots (x + \alpha^{-l_2 q^{d - 1}} \beta^{-\tau}).$$ 

**Proof:** We know that $p_{l_1}(x)$ and $\beta^{-\tau d_{l_2}}p_{l_2}(\beta^\tau x)$ are irreducible polynomials over $\mathbb{F}_q$. Assume that they are the same. Then $\alpha^{-l_1} = \alpha^{-l_2 q \beta^{-\tau}}$, for some nonnegative integer $s(0 \leq s \leq d_{l_2} - 1)$, and hence $l_1 \equiv l_2 q^s + \tau(\frac{d - 1}{q - 1}) \mod d - 1$. So $l_1 \equiv l_2 q^s \mod \frac{d - 1}{q - 1}$ and thus $l_1$ and $l_2$ are in the same $q$-cyclotomic coset $\mod \frac{d - 1}{q - 1}$. This implies $l_1 \equiv l_2$. Now, $l_1 \equiv l_1 q^s \mod \frac{d - 1}{q - 1}$, and hence $l_1(q^s - 1) = \tau(\frac{d - 1}{q - 1})$.

Observe that we have $\frac{d - 1}{q - 1} = f(q)(q - 1) + d$, for $f(q) = \sum_{j=1}^{d - 1} j q^{-j - 1}$, and hence that $(q - 1, \frac{d - 1}{q - 1}) = (q - 1, d) = 1$. Hence $l_1(q^s - 1) \equiv 0 \mod d - 1$, and so $d_{l_1}|s$. As $0 \leq s \leq d_{l_1} - 1 = d_{l_2} - 1$, we have $s = 0$. In all, $l_1 \equiv l_2 + \tau(\frac{d - 1}{q - 1}) \mod d - 1$, which implies $q - 1|\tau$, and therefore $\tau = 0$.

**Definition 2:** Let $\Sigma$ be the family consisting of $M$-ary sequences of period $q - 1$, given by
$$\Sigma = \{vw_l(t)|1 \leq c \leq M - 1, l \in \Lambda \setminus \{0\}\}.$$

**Remark 4:** When $d = 2, \Lambda \setminus \{0\} = \{1, \ldots, \frac{d - 1}{q - 1}\}$. This follows from the simple observation that the $q$-cyclotomic coset containing $l \mod q + 1$ is $\hat{C}_l = \{l, ql\}$, and $q \equiv q - 1 \mod q + 1$. So the family $S_\beta$ considered in [9] and [10] is identical to our $\Sigma$, for $q = p^h$ even and contains $M - 1$ less sequences, namely $\text{cv}_{\frac{d - 1}{q - 1}}(t)(1 \leq c \leq M - 1)$, for $q$ odd.
Recall that the maximum correlation of $\Sigma$, $\delta_{\text{max}} = \delta_{\text{max}}(\Sigma)$, is defined as the maximum absolute value of all nontrivial auto- and cross-correlations of the sequences in $\Sigma$.

**Theorem 3:** For the family $\Sigma = \{cv_i(t)|1 \leq c \leq M-1,l \in \Lambda \setminus \{0\}\}$ of $M$-ary sequences of period $q-1$, we have

$$\delta_{\text{max}}(\Sigma) \leq (2d-1)\sqrt{q} + 1.$$  

**Proof:** Assume that $l_1 \neq l_2(l_1,l_2 \in \Lambda \setminus \{0\})$ or $\tau$ is in the range $1 \leq \tau \leq q-2$. Then $p_{l_1}(x)$ and $\beta^{-\tau}p_{l_2}(\beta^\tau x)$ are distinct irreducible polynomials over $\mathbb{F}_q$, by Proposition 3. The cross-correlation function $R(\tau) = R_{c_1l_1,c_2l_2}(\tau)$ between the sequence $c_1v_{l_1}(t)$ and $c_2v_{l_2}(t)$ in $\Sigma$ is given by

$$R(\tau) = \sum_{t=0}^{q-2} w_{c_1l_1} c_1v_{l_1}(t) - c_2v_{l_2}(t + \tau)$$

$$= \sum_{t=0}^{q-2} \psi^t(f_1(\beta^t)) \psi^{M-c_2} f_2(\beta^{t+1})$$

$$= \sum_{x \in \mathbb{F}_q} \psi_1(p_{l_1}(x)) \psi_2(\beta^{-\tau}p_{l_2}(\beta^\tau x)) - 1,$$

where $\psi_1 = \psi^{c_1 \frac{q-\tau}{d_1}}$ and $\psi_2 = \psi^{c_2 \frac{q-\tau}{d_2}}$. Observe that both $c_1 \frac{d_1}{d}$ and $c_2 \frac{d_2}{d}$ are not divisible by $M$ and hence $\psi_1$ and $\psi_2$ are both nontrivial, since $(d, q-1) = 1$. In view of (3), the sum in (11) in absolute value is

$$\sum_{x \in \mathbb{F}_q} |\psi_1(p_{l_1}(x)) \psi_2(\beta^{-\tau}p_{l_2}(\beta^\tau x))|$$

$$\leq (d_{l_1} + d_{l_2} - 1) \sqrt{q}$$

$$\leq (2d-1)\sqrt{q}.$$  

So we get the desired result in this case. Note here that $p_{l_1}(x)$ and $\beta^{-\tau}p_{l_2}(\beta^\tau x)$ have no roots in $\mathbb{F}_q$, by Proposition [2], and (10). Then we consider the case that $c_1 \neq c_2$, but $l_1 = l_2$ and $\tau = 0$. In this case,

$$R(\tau) = \sum_{t=0}^{q-2} w_{c_1l_1} c_1v_{l_1}(t)$$

$$= \sum_{x \in \mathbb{F}_q} \psi^{c_1}(p_{l_1}(x)) - 1,$$

where $\psi^{c_1} = \psi^{(c_1-c_2) \frac{q-\tau}{d_1}}$ is nontrivial, as $(c_1 - c_2) \frac{d}{d_1}$ is not divisible by $M$. So, by the classical Weil’s theorem (the $m = 1$ case of Theorem [1]),

$$|R(\tau)| \leq (d_{l_1} - 1)\sqrt{q} + 1$$

$$\leq (d-1)\sqrt{q} + 1.$$  

Note that these take care of the cases that $(c_1, l_1) \neq (c_2, l_2)$ and $(c_1, l_1) = (c_2, l_2)$, but with $\tau \neq 0$.

**Theorem 4:** The sequences in the family $\Sigma = \{cv_i(t)|1 \leq c \leq M-1,l \in \Lambda \setminus \{0\}\}$ are cyclically inequivalent.

**Proof:** If $c_1v_{l_1}(t)$ and $c_2v_{l_2}(t)$ are cyclically equivalent, then, for some $\tau(0 \leq \tau \leq q-2), c_1v_{l_1}(t) = c_2v_{l_2}(t + \tau)$ and hence

$$q-1 = \sum_{t=0}^{q-2} w_{c_1l_1} c_1v_{l_1}(t) - c_2v_{l_2}(t + \tau)$$

$$= |\sum_{t=0}^{q-2} w_{c_1l_1} c_1v_{l_1}(t) - c_2v_{l_2}(t + \tau)|$$

$$\leq \sum_{x \in \mathbb{F}_q} |\psi_1(p_{l_1}(x))| |\psi_2(\beta^{-\tau}p_{l_2}(\beta^\tau x))| + 1$$

$$\leq (2d-1)\sqrt{q} + 1,$$

if $(c_1, l_1) \neq (c_2, l_2)$. Here $\psi_1 = \psi^{c_1 \frac{q-\tau}{d_1}}$ and $\psi_2 = \psi^{c_2 \frac{q-\tau}{d_2}}$. This is impossible in view of our assumption in (9). Thus $c_1v_{l_1}(t)$ and $c_2v_{l_2}(t)$ are the same.

**Remark 5:** Under the mild restrictions in (9), we proved Proposition [3] and Theorems [3] and [4] Assume that $d = 2$. The second condition in (9) needed in proving Theorem [4] misses only a few values of $q$. Namely, $q = 2, 4, 8, 3, 9, 5, 7,$ and $11$. Note that $(2, q-1) = 1$ for $q$ even and $(2, q-1) = 2$ for $q$ odd. Suppose we are in the latter case. Then the first condition in (9) is not necessary in showing Theorems [3] and [4] since $\frac{d}{d_1} = d_2 = 1$, and so the $\psi_1$ and $\psi_2$ are nontrivial. In addition, if we replace $\Lambda \setminus \{0\}$ by $\Lambda \setminus \{0, \frac{q-1}{2}\} = \{1, \ldots, \frac{q-1}{2}\}$, then one easily checks that the statement of Proposition [3] holds true.

**Theorem 5 (12, Theorem 3.5):** Let $A_f = \{r | r\phi(r)\neq 1 \text{ but } r \text{ does not divide } q^g - 1 \text{ for } 1 \leq g < f\}$, for each positive integer $f$, and, for $r \in A_f$, write $r = d_r m_r$, with $d_r = (r, \frac{q-1}{2})$.

Assume $b \in \mathbb{F}_q^*$ has order $m$, and let $N(f, b, q)$ denote the number of monic irreducible polynomials over $\mathbb{F}_q$ of degree $f$ with constant term $(-1)^f b$. Then

$$N(f, b, q) = \frac{1}{f \phi(m)} \sum_{r \in A_f} \phi(r).$$  

**Theorem 6:** The size of the family $\Sigma = \{cv_i(t)|1 \leq c \leq M-1,l \in \Lambda \setminus \{0\}\}$, with the notations in the above, can be expressed as:

$$|\Sigma| = (M - 1)(|\Lambda| - 1),$$  

where the number of monic irreducible factors $|\Lambda|$ of $x^{d_r - 1} - 1$ is given by

$$\sum_{c|d} \sum_{e|d_r} \sum_{m_r | m} \phi(r).$$  

**Proof:** Clearly, we have (13). By Proposition [2], the size of $\Sigma$ is also given by

$$|\Sigma| = (M - 1) \times ((\text{the number of monic irreducible factors of } x^{d - 1} - 1) - 1).$$  

Thus we only need to verify that the number of irreducible factors $|\Lambda|$ of $x^{\frac{d-1}{d_1}} - 1$ is given by the expression in (14). In
view of Proposition 2, that number is equal to
\[
\sum_{d|n} \sum_{b^d=1} (\text{# of monic irreducible factors over } \mathbb{F}_q \text{ of } x^{q^d} - 1, \text{ with degree } e \text{ and the constant term equal to } (-1)^eb)
\]

\[
= \sum_{d|m} \sum_{e|d} \sum_{a(b)=m} (\text{# of monic irreducible polynomials over } \mathbb{F}_q \text{ with degree } e \text{ and the constant term equal to } (-1)^eb)
\]

\[
= \sum_{e|m} \sum_{a(b)=m} N(e, b, q).
\]

(15)

The desired result now follows from (12).

\[\text{Remark 6: Let's consider the case of } d = 2. \text{ In that case,}
\]

\[
|A| = \sum_{r \in A_1} \phi(r) + \sum_{r \in A_2} \phi(r) + \frac{1}{2} \sum_{r \in A_2} \phi(r)
\]

\[= 1 + \frac{1}{2}\sum_{r \neq 1, 2} \phi(r)
\]

and hence

\[|A| - 1 = \left[\frac{q + 1}{2}\right] = \left\{\begin{array}{ll}
\frac{q+1}{2}, & \text{if } q \text{ odd}, \\
\frac{q}{2}, & \text{if } q \text{ even}.
\end{array}\right.\]

(16)

This is what is expected (cf. Remark 4).

The next theorem follows from [7, Theorem 5.1] by taking \(f(T) = T\). It gives an estimate for \(N(f, b, q)\) in (12).

**Theorem 7 ([7]):** Let \(N(f, b, q)\) denote the number of monic irreducible polynomials over \(\mathbb{F}_q\) of degree \(f\) with constant term \((-1)^fb\). Then

\[N(f, b, q) - \frac{q^f}{f(q - 1)} \leq \frac{2}{f^2} q^f.\]  

(17)

**Theorem 8:** The asymptotic size of \(\Sigma = \{cv(t)|1 \leq c \leq M - 1, l \in \Lambda \setminus \{0\}\}\), as \(q \to \infty\), is given by:

\[|\Sigma| \sim \frac{(M - 1)q^{d-1}}{d}, \text{ as } q \to \infty.\]

**Proof:** Assume first that \(d > 2\). From (15) and (17),

\[|A| - d \sum_{c|d} \frac{q^c}{d^2(q - 1)} \leq 2d \sum_{c|d} \frac{q^c/2}{c^2}.
\]

This implies that

\[|A| \sim \frac{q^{d-1}}{d}, \text{ as } q \to \infty,
\]

and hence

\[|\Sigma| \sim \frac{(M - 1)q^{d-1}}{d}, \text{ as } q \to \infty.\]

(18)

Even for \(d = 2\), we get the same result as in (18). Indeed, from (16), we have

\[|\Sigma| = (M - 1)\left[\frac{q + 1}{2}\right] \sim (M - 1)q/2, \text{ as } q \to \infty.\]

V. Conclusion

In this paper, starting with \(M\)-ary Sidelnikov sequences, with \(M|q - 1\), of period \(q^d - 1(q = p^n \text{ a prime power})\) and considering the \((q - 1) \times (\frac{q^n - 1}{q - 1})\) array structure of such sequences, we constructed a family of \(M\)-ary sequences with period \(q - 1\), with large size and good correlation property. It is formed as the constant multiples of those column sequences corresponding to a set of \(q\)-cyclotomic coset representatives \(\mathcal{D}_{q-1} \). Then, under the mild restrictions on \(d\) (cf. (9)), it is shown that the maximum correlation magnitude of the family is upper bounded by \((2d - 1)^{\frac{1}{2}}\), and the asymptotic size, as \(q \to \infty\), of that is \(\frac{(M - 1)q}{d}\). Also, we derived an exact but less explicit expression of the size of the family of sequences by using a result of Yucas [12]. This generalizes the pioneering work of Yu and Gong for \(d = 2\) case in [9] and [10].

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