Idempotents and one-sided units II.
Lattice invariants and a semigroup of functors on the category of monoids

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Abstract
For a monoid $M$, we denote by $G(M)$ the group of units, $E(M)$ the submonoid generated by the idempotents, and $G_L(M)$ and $G_R(M)$ the submonoids consisting of all left or right units. Writing $\mathcal{M}$ for the (monoidal) category of monoids, $G$, $E$, $G_L$ and $G_R$ are all (monoidal) functors $\mathcal{M} \to \mathcal{M}$. There are other natural functors associated to submonoids generated by combinations of idempotents and one- or two-sided units. The above functors generate a monoid with composition as its operation. We show that this monoid has size 15, and describe its algebraic structure. We also show how to associate certain lattice invariants to a monoid, and classify the lattices that arise in this fashion. A number of examples are discussed throughout, some of which are essential for the proofs of the main theoretical results.

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1 Introduction

Idempotent-generated semigroups arise naturally in many settings, and include semigroups of singular transformations, matrices, partitions, and endomorphisms of various structures [1, 3–12, 18, 22–26, 36, 42–44, 72, 76, 73]. Free idempotent-generated semigroups associated to abstract bidergated sets have long been a crucial tool in the structure theory of (regular) semigroups [13, 14, 16, 37, 38, 61]. Many well-known monoids are generated by their idempotents and units, and several studies have calculated the submonoids generated by all idempotents and units of important monoids [6, 17, 19, 22, 23, 27–31, 35, 42, 47, 49, 55].

One-sided units have also played an important role in many of the above studies (and others), sometimes implicitly. As an example, consider the full transformation monoid over a set $X$; this monoid is denoted $T_X$, and consists of all mappings $X \rightarrow X$ under composition. If functions are composed right-to-left, then the left and right units of $T_X$ are precisely the injective and surjective mappings, respectively, while the two-sided units are the bijections, which together form the symmetric group $S_X$. It was shown in [47] that finite $T_X$ is generated by $S_X$ and a single idempotent, and that infinite $T_X$ is generated by $S_X$ together with two additional mappings, one a left unit and the other a right unit (both of a certain special form). It follows that infinite $T_X$ is generated by its one-sided units, a property that holds for a number of other important monoids [1, 20–22, 41, 42]. The submonoids of $T_X$ consisting of all left units or all right units (i.e., all injective or all surjective mappings $X \rightarrow X$) have of course been studied in a number of settings as well [15, 21, 42, 57, 59, 66].

To the author’s knowledge, the article [22] was the first to systematically study submonoids generated by all combinations of idempotents and one- or two-sided units of a monoid, though the article [42] is a forerunner, as it considered set products of subsets consisting of such elements (in the monoid of partial transformations of an infinite set). The principal object of study in [22] was the so-called partial Brauer monoid $PB_X$ [54], which consists of certain graphs under a natural diagrammatic multiplication. The main results in [22] were descriptions of the various submonoids of $PB_X$, as well as the relationships between them. These relationships were described using notions such as relative rank [42, 47], Sierpiński rank [2, 60, 65] and the Bergman property [3, 53]. The article [22] also contained the beginnings of a general theory of submonoids generated by idempotents and one- or two-sided units of arbitrary monoids. The current article develops this general theory further, as we now describe.

In Section 2 we define the submonoids we will be concerned with, and show that the set of all such submonoids of a given monoid $M$ forms a lattice $L(M)$; this lattice is a natural invariant of the isomorphism class of $M$, and its generic shape is shown in Figure 1. We also show that each such submonoid arises from a functor on the category of monoids, and end the section with a number of examples. Section 3 contains preliminary results, mostly concerning intersections of the submonoids, and collapse within the lattice $L(M)$; we will also describe some connections with Green’s relations and stability of the identity element. In Section 4 we classify the lattice invariants $L(M)$, and show that their structure is completely determined by a certain binary quadruple $T(M)$, which we call the type of $M$. The main results of Section 4 are summarised in Theorem 4.4, and the possible shapes of $L(M)$ are shown in Figures 4, 6 and 7. In Section 5 we study the monoid $F^+$ generated by all of the above-mentioned functors. This involves calculating all compositions of the functors, and introducing four new ones; in the end we are able to calculate the size of $F^+$ and describe its algebraic structure (see Table 3 and Figure 9), using GAP [58] for some computations. Finally, in Section 6 we show that the monoid $F^+$ may be used to associate a (sometimes) larger lattice $L^+(M)$ to an arbitrary monoid $M$; we classify these lattices as well (see Figures 11 and 12), and show that they provide essentially the same information as the original invariant $L(M)$.

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2 Definitions and basic examples

In this section we introduce the submonoids (Section 2.1), functors (Section 2.2) and lattices (Section 2.3) that will be at the heart of our investigations, and consider some fundamental examples (Section 2.4).
2.1 Submonoids

A monoid is a set $M$ with an associative binary operation, and an identity element $1_M$; the latter will usually be abbreviated to 1, and the product represented as juxtaposition. Note that the identity is part of the signature of a monoid, so submonoids must contain the identity, and monoid homomorphisms must map the identity to the identity.

Following the terminology of [7, Section 1.7], an element $x$ of a monoid $M$ is:

- an idempotent if $x = x^2$,
- a left unit if $ax = 1$ for some $a \in M$; the element $a$ is a left inverse of $x$,
- a right unit if $xa = 1$ for some $a \in M$; the element $a$ is a right inverse of $x$,
- a (two-sided) unit if it is a left and right unit.

In general, $x$ could have multiple left or right inverses; however, if it has at least one of each, then it has a unique left unit and a unique right unit, which must be equal, and which we denote by $x^{-1}$. We write

$$E(M), \quad G_L(M), \quad G_R(M), \quad G(M) = G_L(M) \cap G_R(M)$$

for the sets of all idempotents, left units, right units and (two-sided) units of $M$, respectively. Note that $G_L(M)$, $G_R(M)$ and $G(M)$ are all submonoids of $M$, with $G(M)$ a group. We also denote by

$$E(M) = \langle E(M) \rangle$$

the submonoid of $M$ generated by all idempotents, and further define

$$F(M) = \langle E(M) \cup G(M) \rangle, \quad G_{LR}(M) = \langle G_L(M) \cup G_R(M) \rangle, \quad F_L(M) = \langle E(M) \cup G_L(M) \rangle, \quad F_{LR}(M) = \langle E(M) \cup G_{LR}(M) \rangle,$$

$$F_R(M) = \langle E(M) \cup G_R(M) \rangle, \quad F_{LR}(M) = \langle E(M) \cup G_{LR}(M) \rangle.$$

It will also be convenient to write

$$I(M) = M \quad \text{and} \quad \mathbb{O}(M) = \{1_M\}.$$

The relative containments of the submonoids defined above are shown in Figure 1. Note that Figure 1 pictures the generic case, but that these submonoids need not be distinct in general; cf. Figures 4, 6 and 7.

2.2 Functors

We write $\mathcal{M}$ for the (locally small) category of all monoids. The hom-set $\mathcal{M}(M, N)$ consists of all monoid homomorphisms $M \to N$ (each of which, recall, maps $1_M$ to $1_N$).

Now suppose $\mathcal{X}$ is one of $E$, $G$, $G_L$, $G_R$, $G_{LR}$, $F$, $F_L$, $F_R$, $F_{LR}$, $I$ or $\mathbb{O}$. For any monoid $M$, $\mathcal{X}(M)$ is a submonoid of $M$, so it follows that $\mathcal{X}$ is an operator $\mathcal{M} \to \mathcal{M}$. In fact, since any monoid homomorphism $f : M \to N$ maps idempotents (respectively, left units, right units, or units) of $M$ to idempotents (respectively, left units, right units, or units) of $N$, it is clear that $f$ maps $\mathcal{X}(M)$ into $\mathcal{X}(N)$. Thus, we may define $\mathcal{X}(f) : \mathcal{X}(M) \to \mathcal{X}(N)$ to be the restriction of $f$ to $\mathcal{X}(M)$. It then quickly follows that $\mathcal{X}$ is a functor $\mathcal{M} \to \mathcal{M}$. We will write

$$\mathcal{F} = \{\mathbb{O}, E, G, G_L, G_R, G_{LR}, F, F_L, F_R, F_{LR}, I\}$$

for the set of all these functors.

The direct product operation gives $\mathcal{M}$ the structure of a (symmetric) monoidal category; see [51, Chapters VII and XI] and [48]. The next lemma says that the functors from $\mathcal{F}$ are monoidal.

Lemma 2.1. For any $\mathcal{X} \in \mathcal{F}$, and for any two monoids $M$ and $N$, we have

$$\mathcal{X}(M \times N) = \mathcal{X}(M) \times \mathcal{X}(N).$$

Proof. This is clear if $\mathcal{X}$ is $\mathbb{O}$ or $I$. For the other functors, it follows quickly from the fact that $(x, y)$ is an idempotent (or a left, right or two-sided unit) of $M \times N$ if and only if $x$ and $y$ are idempotents (or left, right or two-sided units) of $M$ and $N$, respectively. $\square$
Figure 1: The generic shape of the lattice $\mathcal{L}(M)$. In general these submonoids need not be distinct.

It will also be convenient to record the following obvious fact. For a monoid $M$, we write $M^0$ for the monoid obtained by adjoining a new zero element 0 to $M$.

**Lemma 2.2.** For any monoid $M$ we have

$$X(M^0) = \begin{cases} X(M) & \text{if } X \text{ is one of } \mathcal{O}, \mathcal{G}, \mathcal{G}_L, \mathcal{G}_R \text{ or } \mathcal{G}_{LR} \\ X(M) \cup \{0\} & \text{if } X \text{ is one of } \mathcal{I}, \mathcal{E}, \mathcal{F}, \mathcal{F}_L, \mathcal{F}_R \text{ or } \mathcal{F}_{LR}. \end{cases}$$

**2.3 Lattices**

For a monoid $M$, we write

$$\mathcal{L}(M) = \{X(M) : X \in \mathcal{F}\}$$

for the set of all submonoids of $M$ defined in Section 2.1. The set $\mathcal{L}(M)$ is partially ordered by inclusion; its Hasse diagram in the generic case is shown in Figure 1.

We denote by $\text{Sub}(M)$ the set of all submonoids of $M$, and we note that $\text{Sub}(M)$ is a lattice with meet and join operations defined by

$$S \land T = S \cap T \quad \text{and} \quad S \lor T = \langle S \cup T \rangle$$

for submonoids $S$ and $T$ of $M$.

Throughout this article, the $\lor$ symbol will be used exclusively for the join operation in $\text{Sub}(M)$.

**Proposition 2.3.** For any monoid $M$, the set $\mathcal{L}(M)$ is a finite $\lor$-subsemilattice of $\text{Sub}(M)$, with top element $\mathcal{I}(M) = M$ and bottom element $\mathcal{O}(M) = \{1\}$. Consequently, $\mathcal{L}(M)$ is a lattice.

**Proof.** It is clear that $M$ and $\{1\}$ are the top and bottom elements of $\mathcal{L}(M)$. Since a finite $\lor$-semilattice with a bottom element is a lattice (with the meet of two elements equal to the join of all common lower bounds), it suffices to show that $\mathcal{L}(M)$ is closed under $\lor$. This is easily checked, using the definitions of the submonoids. For example:

$$F_L(M) \lor G_R(M) = \langle E(M) \cup G_L(M) \rangle \lor G_R(M) = \langle E(M) \cup G_L(M) \cup G_R(M) \rangle = F_{LR}(M).$$

**Remark 2.4.** The previous result did not say that $\mathcal{L}(M)$ is a sublattice of $\text{Sub}(M)$ because this is not the case in general. Specifically, $\mathcal{L}(M)$ is not always a $\land$-subsemilattice of $\text{Sub}(M)$, meaning that the intersection of two submonoids from $\mathcal{L}(M)$ might not belong to $\mathcal{L}(M)$; cf. Remark 2.6.
### 2.4 Examples

Before we move on, we pause to consider some basic examples. These should serve to illustrate the above ideas, but will also be useful later for proving some of our main results.

First, if $G$ is a group, then clearly every element is a (two-sided) unit, and the only idempotent is the identity element. It quickly follows that the submonoids $\mathbb{X}(G)$, $X \in \mathcal{F}$, are as listed in the first column of Table 1.

Next, suppose $E$ is an idempotent-generated monoid. Clearly $E(E) = E$. It follows from [22, Lemma 2.1] that $G_L(E) = G_R(E) = G(E) = \{1\}$. Thus, the submonoids $\mathbb{X}(G)$, $X \in \mathcal{F}$, are as listed in the second column of Table 1.

Next, we denote by $P = \{1, 2, 3, \ldots\}$ the multiplicative monoid of positive integers. This time, 1 is the unique unit, and also the unique idempotent. The submonoids $\mathbb{X}(P)$, $X \in \mathcal{F}$, are listed in the third column of Table 1.

The **bicyclic monoid** $B$ is defined by the monoid presentation $B = \langle a, b : ba = 1 \rangle$. Because of the relation $ba = 1$, we may think of the elements of $B$ as words of the form $a^mb^n$, where $m, n \geq 0$. Two such words $a^mb^n$ and $a^kb^l$ represent the same element of $B$ if and only if $m = k$ and $n = l$, and the product in $B$ is given by

$$a^mb^n \cdot a^kb^l = a^{m+n}b^{l+k} \quad \text{where } \mu = \max(n, k). \quad (2.5)$$

Any monoid generated by two elements $x, y$ for which $yx = 1 \neq xy$ is isomorphic to $B$; see [45, pp. 31–32] for more details. Idempotents of $B$ are words of the form $a^mb^n$ ($m \geq 0$), and it is easily checked that idempotents commute, so that $E(B) = E(B)$. Using (2.5), it is easy to see that

$$a^mb^n \cdot a^kb^l = 1 \iff m = l = 0 \text{ and } n = k,$$

so that

$$G_L(B) = \langle a \rangle = \{a, a^2, \ldots\} \quad \text{and} \quad G_R(B) = \langle b \rangle = \{1, b, b^2, \ldots\}.$$

The fourth column of Table 1 lists the submonoids $\mathbb{X}(B)$, $X \in \mathcal{F}$; verification for the submonoids not discussed so far is an exercise. The fifth column of Table 1 lists the corresponding submonoids of $B^0$ (the bicyclic monoid with a zero adjoined); cf. Lemma 2.2. The lattices $\mathcal{L}(B)$ and $\mathcal{L}(B^0)$ are pictured in Figure 2.

| $X$ | $\mathbb{X}(G)$ | $\mathbb{X}(E)$ | $\mathbb{X}(P)$ | $\mathbb{X}(B)$ | $\mathbb{X}(B^0)$ |
|-----|----------------|----------------|----------------|----------------|----------------|
| $\mathbb{O}$ | $\{1\}$ | $\{1\}$ | $\{1\}$ | $\{1\}$ | $\{1\}$ |
| $\mathbb{E}$ | $\{1\}$ | $E$ | $\{1\}$ | $\{a^mb^n : m \geq 0\}$ | $\{a^mb^n : m \geq 0\} \cup \{0\}$ |
| $\mathbb{G}$ | $G$ | $\{1\}$ | $\{1\}$ | $\langle a \rangle$ | $\{a\}$ |
| $\mathbb{G}_L$ | $G$ | $\{1\}$ | $\{1\}$ | $\langle a \rangle$ | $\{a\}$ |
| $\mathbb{G}_R$ | $G$ | $\{1\}$ | $\{1\}$ | $\langle b \rangle$ | $\{b\}$ |
| $\mathbb{G}_{LR}$ | $G$ | $\{1\}$ | $\{1\}$ | $B$ | $B$ |
| $\mathbb{F}$ | $G$ | $E$ | $\{1\}$ | $\{a^mb^n : m \geq 0\}$ | $\{a^mb^n : m \geq 0\} \cup \{0\}$ |
| $\mathbb{F}_L$ | $G$ | $E$ | $\{1\}$ | $\{a^mb^n : m \geq n\}$ | $\{a^mb^n : m \geq n\} \cup \{0\}$ |
| $\mathbb{F}_R$ | $G$ | $E$ | $\{1\}$ | $\{a^mb^n : m \leq n\}$ | $\{a^mb^n : m \leq n\} \cup \{0\}$ |
| $\mathbb{F}_{LR}$ | $G$ | $E$ | $\{1\}$ | $B$ | $B^0$ |
| $\mathbb{I}$ | $G$ | $E$ | $\{1\}$ | $B$ | $B^0$ |

Table 1: The submonoids $\mathbb{X}(M)$, $X \in \mathcal{F}$, for $M = G$ (a group), $M = E$ (an idempotent-generated monoid), $M = P$ (the positive integers under multiplication), $M = B$ (the bicyclic monoid) and $M = B^0$ (the bicyclic monoid with a zero adjoined).

**Remark 2.6.** Proposition 2.3 showed that the lattice $\mathcal{L}(M)$ is a $\vee$-subsemilattice of $\text{Sub}(M)$, and we claimed in Remark 2.4 that $\mathcal{L}(M)$ is not always a $\wedge$-subsemilattice. We can use the above example of $M = B^0$ to verify this. Indeed, using Table 1 we see that the meet in $\text{Sub}(B^0)$ of the submonoids $\mathbb{F}_L(B^0)$ and $\mathbb{G}_{LR}(B^0)$ is

$$\mathbb{F}_L(B^0) \cap \mathbb{G}_{LR}(B^0) = \{a^mb^n : m \geq n\},$$

and the submonoid $\mathbb{X}(B)$, $X \in \mathcal{F}$, is not listed in the fourth column of Table 1.
which does not belong to \( \mathcal{L}(B^0) \). Of course, the submonoids \( F_L(B^0) \) and \( G_{LR}(B^0) \) do have a meet in \( \mathcal{L}(B^0) \) itself, as the latter is a lattice, but this meet in \( \mathcal{L}(B^0) \) is \( G_L(B^0) = \{ a \} \); cf. Figure 2.

On the other hand, the lattice \( \mathcal{L}(B) \) is a sublattice of \( \text{Sub}(B) \), as may be easily verified using Table 1.

\section{Preliminary results}

We now gather a number of technical results that will be useful in subsequent sections. Section 3.1 concerns intersections of various submonoids from the lattice \( \mathcal{L}(M) \), and Section 3.2 concerns equalities between such submonoids. Section 3.3 establishes connections with Green’s relations, in particular with stability (or otherwise) of the identity element.

Throughout this section, unless otherwise stated, \( M \) will denote an arbitrary monoid. It will also be convenient to abbreviate the submonoids \( \mathcal{X}(M) \), \( \mathcal{X} \in \mathcal{F} \), in obvious ways. Specifically, we will often write

\[
G = G(M), \quad G_L = G_L(M), \quad G_R = G_R(M), \quad G_{LR} = G_{LR}(M),
\]

\[
E = E(M), \quad F = F(M), \quad F_L = F_L(M), \quad F_R = F_R(M), \quad F_{LR} = F_{LR}(M).
\]

A further piece of notation will also be convenient. Often we will wish to give a statement or argument that holds regardless of subscripts, so will sometimes write \( G_\circ(M) \) or \( G_\triangle(M) \) to stand for any of \( G(M) \), \( G_L(M) \), \( G_R(M) \) or \( G_{LR}(M) \). Similarly, we will at times write \( G_\triangledown \) or \( F_\triangle \), etc.

\subsection{Intersections}

The next two results concern intersections of various submonoids of \( M \). We will sometimes make use of them without explicit reference. The first concerns intersections with \( E = E(M) \).

\begin{lemma}
For any monoid \( M \) we have
\begin{enumerate}
    \item \( E \cap G = E \cap G_L = E \cap G_R = \{ 1 \} \),
    \item \( E \cap F = E \cap F_L = E \cap F_R = E \cap F_{LR} = E \).
\end{enumerate}
\end{lemma}

\begin{proof}
Part (i) is part of [22, Lemma 2.1]. Part (ii) is clear, since \( E \) is contained in each of \( F, F_L, F_R, F_{LR} \).
\end{proof}

\begin{remark}
The previous result did not say anything about \( E \cap G_{LR} \). Certainly \( \{ 1 \} \subseteq E \cap G_{LR} \subseteq E \), but we cannot say any more than this in general, since any of the following situations are possible (cf. Table 1):
\begin{itemize}
    \item \( \{ 1 \} = E \cap G_{LR} = E \): e.g., if \( M \) is a group,
    \item \( \{ 1 \} = E \cap G_{LR} \nsubseteq E \): e.g., if \( M \) is a group with a zero adjoined,
    \item \( \{ 1 \} \nsubseteq E \cap G_{LR} = E \): e.g., if \( M \) is the bicyclic monoid,
    \item \( \{ 1 \} \nsubseteq E \cap G_{LR} \nsubseteq E \): e.g., if \( M \) is the bicyclic monoid with a zero adjoined.
\end{itemize}
\end{remark}
The next result concerns intersections with $G_L = \mathbb{G}_L(M)$. There is an obvious dual result concerning intersections with $G_R = \mathbb{G}_R(M)$, but we will not state it.

**Lemma 3.5.** For any monoid $M$ we have

(i) $G_L \cap G_L = G_L \cap G_{LR} = G_L \cap F_L = G_L \cap F_{LR} = G_L,$  
(ii) $G_L \cap G = G_L \cap G_R = G_L \cap F = G_L \cap F_R = G.$

**Proof.** (i). This is clear, since $G_L$ is contained in each of $G_L, G_{LR}, F_L, F_{LR}$.

(ii). Since each of the stated intersections contains $G$, and since each of $G, G_R, F$ are contained in $F_R$, it suffices to show that $G_L \cap F_R \subseteq G$. To do so, suppose $x \in G_L \cap F_R$. Since $x \in G_L$ we have $1 = ax$ for some $a \in M$. Since $F_R = G_R E$ by [22, Lemma 2.5], we also have $x = ge_1 \cdots e_k$ for some $k \geq 0$, and some $g \in G_R$ and $e_1, \ldots, e_k \in E(M)$. We may assume that $k$ is minimal among all such expressions; in particular, $e_i \neq 1$ for all $1 \leq i \leq k$. If $k \geq 1$, then $e_k \neq 1$ and $x = x e_k$, which gives $e_k = 1 e_k = ax \neq 1$, a contradiction. Thus, $k = 0$, so that $x = g \in G_R$. It follows that $x \in G_L \cap G_R = G$, as required. \[\Box\]

### 3.2 Collapse

We have already observed that the submonoids of $M$ defined in Section 2.1 are not always distinct. Roughly speaking, this means that certain “collapse” can occur in the lattice $\mathcal{L}(M)$. The next two results show that such collapse happens in a somewhat controlled manner, in the sense that equalities between certain submonoids imply other such equalities.

**Lemma 3.6.** For a monoid $M$, the following are equivalent:

(i) $G, G_L, G_R$ and $G_{LR}$ are not all equal,

(ii) $G, G_L, G_R$ and $G_{LR}$ are pairwise distinct,

(iii) $F, F_L, F_R$ and $F_{LR}$ are not all equal,

(iv) $F, F_L, F_R$ and $F_{LR}$ are pairwise distinct,

(v) $G_{LR}$ contains infinitely many idempotents,

(vi) $G_{LR}$ contains a nontrivial idempotent,

(vii) $E \cap G_{LR} \neq \{1\}.$

**Proof.** We begin by establishing the equivalence of items involving submonoids of the form $G_{\emptyset}$.

(i) $\Rightarrow$ (ii). We prove the contrapositive: i.e., that if any two of the stated submonoids are equal, then all four are equal.

- If $G = G_L$, then also $G = G_R$ (cf. [22, Lemma 2.3]), and $G_{LR} = G_L \vee G_R = G \vee G = G$.
  
  - The $G = G_R$ case is dual.

- If $G = G_{LR}$, then $G = G_L \cap G = G_L \cap G_{LR} = G_L$, reducing to the first case.

- If $G_L = G_R$, then $G = G_L \cap G_R = G_L \cap G_L = G_L$.

- If $G_L = G_{LR}$, then $G = G_L \cap G_R = G_{LR} \cap G_R = G_R$.

  - The $G_R = G_{LR}$ case is again dual.

(ii) $\Rightarrow$ (v). Suppose $G \neq G_L$, and let $x \in G_L \setminus G$ be arbitrary. Then $1 = ax$ for some $a \in M$, and we note that $a \in G_R$. Since $x \notin G$ we have $xa \neq 1$. It follows that $\langle a, x \rangle$ is bicyclic, and hence contains infinitely many idempotents (of the form $x^m a^m$ for each $m \geq 0$). Since $x \in G_L$ and $a \in G_R$, it follows that $\langle a, x \rangle \subseteq G_{LR}$.

(v) $\Rightarrow$ (vi) and (vi) $\Rightarrow$ (vii). These are clear.

(vii) $\Rightarrow$ (i). If $E \cap G_{LR} \neq \{1\}$, then $G_{LR} \neq G$ because $E \cap G = \{1\}$. 

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Now that we know (i), (ii), (v)–(vii) are equivalent, it is time to tie these in with (iii) and (iv).

(ii) ⇒ (iv). Suppose (ii) holds. From Lemma 3.5, we have

\[ G_L \cap F_L = G_L \cap F_{LR} = G_L \neq G = G_L \cap F = G_L \cap F_R, \]

and it follows that \( \{F_L, F_{LR}\} \cap \{F, F_R\} = \emptyset \). We similarly obtain \( \{F_R, F_{LR}\} \cap \{F, F_L\} = \emptyset \) from the dual of Lemma 3.5.

(iv) ⇒ (iii). This is clear.

(iii) ⇒ (i). Aiming to prove the contrapositive, suppose \( G_\Diamond = G_\heartsuit \) for distinct subscripts \( \Diamond, \heartsuit \). Then \( F_\Diamond = E \vee G_\Diamond = E \vee G_\heartsuit = F_\heartsuit \).

The previous lemma concerned collapse in \( \mathcal{L}(M) \) within the two “diamonds” \( \{G, G_L, G_R, G_{LR}\} \) and \( \{F, F_L, F_R, F_{LR}\} \). The next concerns collapse at the very bottom of the lattice, namely between \( \{1\} \) and \( G \) or \( E \). In particular, it shows that \( E = \{1\} \) has the significant consequence of collapsing the whole “cube” section of the lattice: i.e., the interval from \( G \) to \( F_{LR} \).

**Lemma 3.7.** For any monoid \( M \) we have

(i) \( G = \{1\} \iff F = E \),

(ii) \( E = \{1\} \iff F_{LR} = G \iff \{G, G_L, G_R\} \cap \{F, F_L, F_R, F_{LR}\} \neq \emptyset \).

**Proof.** (i). If \( G = \{1\} \) then \( F = E \vee G = E \vee \{1\} = E \). Conversely, if \( F = E \), then since \( G \subseteq F \), we have \( G = F \cap G = E \cap G = \{1\} \).

(ii). If \( E = \{1\} \), then \( G_{LR} \) contains no nontrivial idempotents, so by Lemma 3.6 we have \( G = G_{LR} \); but then \( F_{LR} = E \vee G_{LR} = \{1\} \vee G = G \).

If \( F_{LR} = G \), then obviously \( \{G, G_L, G_R\} \cap \{F, F_L, F_R, F_{LR}\} \neq \emptyset \).

Finally, suppose the two stated sets of submonoids have nonempty intersection, say \( G_\Diamond = F_\heartsuit \), noting that \( \Diamond \neq LR \). Then Lemma 3.3 gives \( E = E \cap F_\Diamond = E \cap G_\Diamond = \{1\} \).

**Remark 3.8.** The submonoid \( G_{LR} = G_{LR}(M) \) was not mentioned in Lemma 3.7(ii), since it is possible to have \( G_{LR} = F_{LR} \) but \( E \neq \{1\} \). For example, this happens when \( M \) is the bicyclic monoid; cf. Table 1 and Figure 2.

### 3.3 Green’s relations and stability

Recall that for elements \( x \) and \( y \) of a monoid \( M \), we write

\[ x \mathcal{L} y \iff Mx = My, \quad x \mathcal{R} y \iff xM = yM, \quad x \mathcal{J} y \iff MxM = MyM. \]

We also set \( \mathcal{H} = \mathcal{L} \cap \mathcal{R} \) and \( \mathcal{D} = \mathcal{L} \vee \mathcal{R} \) (the join in the lattice of equivalences). These five equivalences, \( \mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H} \) and \( \mathcal{D} \), are called Green’s relations [39], and are essential tools in semigroup theory. Equivalent formulations in terms of divisibility may also be given; for example, \( x \mathcal{L} y \) if and only if \( x = ay \) and \( y = bx \) for some \( a, b \in M \). See [7, Chapter 2] or [45, Chapter 2] for more background on Green’s relations.

If \( \mathcal{H} \) is one of Green’s relations, we denote by \( K_x = \{y \in M : x \mathcal{H} y\} \) the \( \mathcal{H} \)-class of \( x \in M \). One may easily check that the submonoids consisting of one- or two-sided units are certain Green’s classes containing the identity:

\[ G_L = \mathbb{G}_L(M) = L_1, \quad G_R = \mathbb{G}_R(M) = R_1, \quad G = \mathbb{G}(M) = H_1. \]

An element \( x \) of a monoid \( M \) is stable the following implications hold for all \( a \in M \):

\[ ax \mathcal{J} x \Rightarrow ax \mathcal{L} x \quad \text{and} \quad xa \mathcal{J} x \Rightarrow xa \mathcal{R} x. \quad (3.9) \]

If \( x \) is not stable, we will call it unstable. For more on stability, see [50, Section 2.3], [64, Section A.2] or [25].

Taking \( x = 1 \) to be the identity element in (3.9), and keeping in mind that \( \mathcal{H} = \mathcal{L} \cap \mathcal{R} \), we see that 1 is stable if and only if \( a \mathcal{J} 1 \Rightarrow a \mathcal{H} 1 \) for all \( a \in M \). This implication is equivalent to \( J_1 \subseteq H_1 \). Since \( H_x \subseteq J_x \) for any \( x \), it follows that 1 is stable if and only if \( J_1 = H_1 \); i.e., \( J_1 = G \).
Lemma 3.10. For a monoid \( M \), the following are equivalent:

(i) \( G = G_L \),
(ii) \( G = G_R \),
(iii) \( G = J_1 \),
(iv) \( H_1 = L_1 = R_1 = D_1 = J_1 \),
(v) \( M \) has no bicyclic submonoid,
(vi) the identity element 1 is stable.

Proof. (i) \( \iff \) (ii) \( \iff \) (v). These are part of [22, Lemma 2.3]; cf. Lemma 3.6.

(iii) \( \iff \) (iv). This follows from \( G = H_1 \subseteq L_1, R_1 \subseteq D_1 \subseteq J_1 \), which itself follows from \( \mathcal{H} \subseteq \mathcal{L}, \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J} \).

(iii) \( \iff \) (vi). This was discussed before the statement of the lemma.

(i) \( \Rightarrow \) (iii). If \( G = G_1 \) holds, then so too does \( G = G_R \) (as (i) \( \iff \) (ii)). Since \( G = H_1 \subseteq J_1 \), it is enough to show that \( J_1 \subseteq G \). To do so, let \( x \in J_1 \). Then \( 1 = axb \) for some \( a, b \in M \). Since \( 1 = a(xb) \) we have \( a \in G_R = G \), and similarly \( b \in G \). But then \( x = a^{-1}(axb)b^{-1} = a^{-1}b^{-1} \in G \).

(iv) \( \Rightarrow \) (i). If (iv) holds, then \( G = H_1 = L_1 = G_L \). \( \square \)

Remark 3.11. The second condition of Lemma 3.6 and the first condition of Lemma 3.10 are clearly mutually exclusive. It follows that a monoid either satisfies all of the conditions of Lemma 3.6 and none of the conditions of Lemma 3.10, or vice versa. This yields a dichotomy that will allow a convenient split in the argument of the next section.

4 Classification of lattice invariants

In this section we classify the lattices \( \mathcal{L}(M) \), for monoids \( M \). To do so, we first define in Section 4.1 the type of a monoid, as a certain binary tuple of length 4; we show in Proposition 4.2 that all sixteen such tuples occur as the type of a monoid. We then show in Sections 4.2 and 4.3 that the type of \( M \) uniquely determines the structure of \( \mathcal{L}(M) \); these sections concern the cases in which the identity of \( M \) is stable or unstable, respectively. In Section 4.4 we summarise the results of Sections 4.1–4.3 in Theorem 4.4; cf. Figures 3–7.

Throughout this section, unless otherwise specified, \( M \) denotes an arbitrary monoid, and we continue to use the abbreviations (3.1) and (3.2).

4.1 The type of a monoid

Consider the following questions concerning a monoid \( M \):

(T1) Does \( G = G_L \) hold?
(T2) Does \( F_{LR} = M \) hold?
(T3) Does \( F_{LR} = G_{LR} \) hold?
(T4) Does \( G = \{1\} \) hold?

We denote the Yes (=1) or No (=0) answers to these questions by \( T_1(M), T_2(M), T_3(M) \) and \( T_4(M) \), respectively. We also define the binary quadruple

\[ T(M) = (T_1(M), T_2(M), T_3(M), T_4(M)), \]

and call this the type of \( M \). There are sixteen quadruples over \( \{0, 1\} \), and Proposition 4.2 below shows that each such quadruple is the type of some monoid.

By Lemma 2.1, if \( X, Y \in \mathcal{F} \) then for any monoids \( M \) and \( N \), we have

\[ X(M \times N) = Y(M \times N) \iff X(M) = Y(M) \text{ and } X(N) = Y(N). \]
It follows that the integers $T_i(M)$ are multiplicative, in the sense that for monoids $M$ and $N$, we have $T_i(M \times N) = T_i(M) \times T_i(N)$; here the first $\times$ is monoid direct product, and the second is ordinary integer multiplication in $\{0,1\}$. It follows that types are multiplicative as well:

$$T(M \times N) = T(M) \times T(N) \quad \text{for monoids } M \text{ and } N. \quad (4.1)$$

In the second expression, we mean the coordinate-wise product of tuples.

**Proposition 4.2.** For any $i, j, k, l \in \{0,1\}$, there exists a monoid $M$ with type $T(M) = (i, j, k, l)$.

**Proof.** Consulting Table 1, we see that

- $T(G) = (1,1,1,0)$ for a nontrivial group $G$,
- $T(E) = (1,1,0,1)$ for a nontrivial idempotent-generated monoid $E$,
- $T(P) = (1,0,1,1)$ for the multiplicative monoid of positive integers $P$,
- $T(B) = (0,1,1,1)$ for the bicyclic monoid $B$.

Thus, in light of (4.1), we can obtain a monoid with any type by taking a suitable direct product of some (possibly empty) collection of $G$, $E$, $P$, $B$, as above.

The rest of Section 4 is devoted to showing that the type of the monoid $M$ determines the entire structure of the lattice $\mathcal{L}(M)$. In Sections 4.2 and 4.3, we consider separate cases according to whether the identity of $M$ is stable or unstable, respectively.

### 4.2 Stable identity

We first consider the case in which the identity of $M$ is stable. By Lemma 3.10, this is equivalent to having $G = G_L$; i.e., to having $T_1(M) = 1$. In this case, the conditions in Lemma 3.10 all hold, but the conditions in Lemma 3.6 do not (cf. Remark 3.11). In particular, we have $G = G_L = G_R = G_{LR}$ and $F = F_L = F_R = F_{LR}$. Thus, the lattice $\mathcal{L}(M) = \{\{1\}, E, G, F, M\}$ simplifies substantially, and has the generic shape pictured in Figure 3. In this diagram and others to follow, the trivial submonoid $\{1\}$ is abbreviated to 1.

![Figure 3: The generic shape of the lattice $\mathcal{L}(M)$ when $M$ has a stable identity.](image)

In general, some of the submonoids pictured in Figure 3 could be equal, but by Lemma 3.7 (and the fact that $F = F_{LR}$) we have

$$G = \{1\} \iff F = E \quad \text{and} \quad E = \{1\} \iff F = G.$$ 

Also note that since $F = F_{LR}$ and $G = G_{LR}$, questions $(T_2)$ and $(T_3)$ are equivalent (in the case of $M$ having a stable identity) to:

- $(T_2')$ Does $F = M$ hold?
- $(T_3')$ Does $F = G$ (equivalently, $E = \{1\}$) hold?

Figure 4 shows the lattice $\mathcal{L}(M)$ for monoids of type $(1, i, j, k)$. The values of $i = T_2(M)$, $j = T_3(M)$ and $k = T_4(M)$ determine which edges (if any) in Figure 3 are contracted.
Figure 4: The lattice $\mathcal{L}(M)$ when $M$ has a stable identity, according to the type $T(M) = (1, i, j, k)$. In each case, the nodes represent distinct submonoids of $M$.

4.3 Unstable identity

We now consider the case in which the identity of $M$ is unstable, which is equivalent to having $G \neq G_L$; i.e., to having $T_1(M) = 0$. In this case, the conditions in Lemma 3.6 all hold, but the conditions in Lemma 3.10 do not. In particular, $G, G_L, G_R$ and $G_{LR}$ are four distinct submonoids; so too are $F, F_L, F_R$ and $F_{LR}$. Moreover, $G_{LR}$ (and hence $M$) contains infinitely many idempotents (cf. Lemma 3.6), so certainly $E \neq 1$; it follows from Lemma 3.7(ii) that $\{G, G_L, G_R\} \cap \{F, F_L, F_R, F_{LR}\} \neq \emptyset$. All of the above shows that the following seven submonoids of $M$ are distinct:

$$G, G_L, G_R, F, F_L, F_R \text{ and } F_{LR}. \quad (4.3)$$

These submonoids are shaded red in Figure 5, which gives the generic shape of $\mathcal{L}(M)$ in the unstable case. Again we note that $G = \{1\} \iff F = E$; cf. Lemma 3.7(i).

Figure 5: The generic shape of the lattice $\mathcal{L}(M)$ when $M$ has an unstable identity. The submonoids shaded red are distinct, and thick lines indicate proper containment.

Figure 6 shows the shapes the lattice $\mathcal{L}(M)$ takes for monoids of each type $(0, i, j, k)$, and again the values of $i, j, k$ determine which thin edges (if any) in Figure 5 are contracted.
4.4 The classification

The results of Sections 4.1–4.3 may be summarised as follows:

Theorem 4.4. (i) If a monoid $M$ has a stable identity, then the lattice $\mathcal{L}(M)$ is as shown in Figure 4, according to its type $T(M) = (1, i, j, k)$, as defined in Section 4.1.

(ii) If a monoid $M$ has an unstable identity, then the lattice $\mathcal{L}(M)$ is as shown in Figure 6, according to its type $T(M) = (0, i, j, k)$, as defined in Section 4.1.

(iii) Each of the lattices pictured in Figures 4 and 6 arises as $\mathcal{L}(M)$ for some monoid $M$.

(iv) Up to isomorphism, the lattice $\mathcal{L}(M)$ associated to a monoid $M$ has one of the forms shown in Figure 7. \hfill \Box

5 A semigroup of functors

In this section we study the semigroup of functors $\mathcal{M} \to \mathcal{M}$ generated (via composition) by the functors considered so far:

$$\mathcal{F} = \{\mathcal{O}, \mathcal{E}, \mathcal{G}, \mathcal{G}_L, \mathcal{G}_R, \mathcal{G}_{LR}, \mathcal{F}, \mathcal{F}_L, \mathcal{F}_R, \mathcal{F}_{LR}, \mathcal{I}\}.$$ 

We begin in Section 5.1 by calculating compositions of the functors from $\mathcal{F}$, and observe that four such compositions do not seem to belong to $\mathcal{F}$. In Section 5.2 we define a suitably enlarged set $\mathcal{F}^+$ of functors, and associate an enhanced lattice $\mathcal{L}^+(M)$ to each monoid $M$. In Section 5.3 we show that $\mathcal{F}^+$ is a semigroup, indeed a monoid; we calculate its size in Section 5.4, and describe its algebraic structure in Section 5.5. In Section 5.6 we calculate the lattice $\mathcal{L}(\mathcal{F}^+)$. Throughout this section, unless otherwise specified, $M$ denotes an arbitrary monoid, and we continue to use the abbreviations (3.1) and (3.2).
5.1 Compositions

Since each functor from $\mathcal{F}$ maps $\mathcal{M} \to \mathcal{M}$, these functors may be composed. For example, we may consider the functor $E \circ G : \mathcal{M} \to \mathcal{M}$. Since groups have only one idempotent, we have $E(G(M)) = \{1\} = O(M)$ for any monoid $M$, and this means that $E \circ G = O$. On the other hand, we have $E \circ E = E$. We also clearly have $O \circ X = X \circ O = X$ and $\mathbb{I} \circ X = X \circ \mathbb{I} = X$ for any $X \in \mathcal{F}$.

Various results from [22, Section 2] may be interpreted as further such compositional equations. For example, [22, Lemmas 2.1 and 2.9] say that if $X$ is one of $G$, $G_L$ or $G_R$, then

$$E \circ X = X \circ E = O$$ and $$G \circ X = G_L \circ X = G_R \circ X = F \circ X = F_L \circ X = F_R \circ X = G.$$ Similarly, [22, Lemma 2.8] says that if $X$ is any of $F$, $F_L$ or $F_R$, then

$$E \circ X = E, \quad G \circ X = G_L \circ X = G_R \circ X = G, \quad F \circ X = F_L \circ X = F_R \circ X = F.$$ If $\heartsuit$ represents any subscript other than $LR$, then since $G_{\heartsuit} \circ E = O$ (noted above), we have

$$F_{\heartsuit} \circ E(M) = F_{\heartsuit}(E) = E(E) \lor G_{\heartsuit}(E) = E(M) \lor \{1\} = E(M),$$

so that $F_{\heartsuit} \circ E = E$. The above composition rules are recorded as the black entries in Table 2.

Table 2 contains a number of other entries in blue (and some missing entries, which we will discuss in Section 5.2). The blue entries follow from Lemmas 5.2 and 5.3 below. The proofs of these will use the following simple fact.

**Lemma 5.1.** If $N$ is a submonoid of $M$, and if $G_L(M), G_R(M) \subseteq N$, then

$$X(N) = X(M)$$ for $X = G, G_L, G_R, G_{LR}$.

**Proof.** We first prove the claim for $X = G_L$. Since $N \subseteq M$, we clearly have $G_L(N) \subseteq G_L(M)$. Conversely, suppose $x \in G_L(M)$. So $x \in N$ by assumption. We also have $1 = ax$ for some $a \in M$. But this implies that $a \in G_R(M) \subseteq N$, so in fact $x \in G_L(N)$ as required.

The claim for $X = G_R$ is dual, and the others follow since

$$G(N) = G_L(N) \cap G_R(N) = G_L(M) \cap G_R(M) = G(M),$$

with a similar calculation for $G_{LR}(N) = G_L(N) \lor G_R(N)$.

![Figure 7: The possible lattices $\mathcal{L}(M)$ for a monoid $M$, up to lattice isomorphism.](image)
The next statement concerns compositions with \( G_{LR} \), but we note that it says nothing about \( X \circ G_{LR} \) for \( X = E, F, F_L, F_R \).

**Lemma 5.2.** For \( X \in \mathcal{F} \) we have

(i) \( X \circ G_{LR} = \begin{cases} X & \text{if } X = \emptyset, G, G_L, G_R, G_{LR} \\ G_{LR} & \text{if } X = F_{LR}, \mathbb{I}, \end{cases} \)

(ii) \( G_{LR} \circ X = \begin{cases} \emptyset & \text{if } X = \emptyset, E \\ G & \text{if } X = G, G_L, G_R, F, F_L, F_R \\ G_{LR} & \text{if } X = G_{LR}, F_{LR}, \mathbb{I}. \end{cases} \)

**Proof.** (i). This is clear for \( X = \emptyset \) or \( \mathbb{I} \). For \( X = G, G_L, G_R, G_{LR} \) we apply Lemma 5.1 with \( N = G_{LR} \):
\[
X \circ G_{LR}(M) = X(G_{LR}) = X(N) = X(M).
\]

For \( X = F_{LR} \) we have
\[
G_{LR} \supseteq F_{LR}(G_{LR}) = E(G_{LR}) \lor G_{LR}(G_{LR}) = E(G_{LR}) \lor G_{LR} = G_{LR},
\]
where we again used Lemma 5.1 in the third step. Thus, \( F_{LR}(G_{LR}) = G_{LR} \): i.e., \( F_{LR} \circ G_{LR}(M) = G_{LR}(M) \).

(ii). This is again clear for \( X = \emptyset \) or \( \mathbb{I} \), and follows from Lemma 5.1 for \( X = G_{LR}, F_{LR} \). For the other choices of \( X \), and writing \( X = X(M) \), the claim follows from previously calculated compositions, in light of \( G_{LR}(X) = G_L(X) \lor G_R(X) \).

Now we treat compositions with \( F_{LR} \).

**Lemma 5.3.** For \( X \in \mathcal{F} \) we have

(i) \( X \circ F_{LR} = \begin{cases} F_{LR} & \text{if } X = \mathbb{I} \\ X & \text{otherwise,} \end{cases} \)

(ii) \( F_{LR} \circ X = \begin{cases} X & \text{if } X = \emptyset, E, G_{LR}, F_{LR} \\ G & \text{if } X = G, G_L, G_R \\ F & \text{if } X = F, F_L, F_R \end{cases} \)

**Proof.** (i). The \( X = \emptyset, \mathbb{I} \) cases are clear, and Lemma 5.1 (with \( N = F_{LR} \)) again gives the \( X = G, G_L, G_R, G_{LR} \) cases. The \( X = E \) case is clear since \( E(M) \subseteq F_{LR} \). The \( X = F \circ \varnothing \) case follows from the others since
\[
F \circ F_{LR}(M) = F \circ (F_{LR}) = E(F_{LR}) \lor G \circ (F_{LR}) = E(M) \lor G \varnothing(M) = F \varnothing(M).
\]

(ii). The \( X = \emptyset, \mathbb{I} \) cases are clear, and the \( X = G_{LR} \) case is part of Lemma 5.2(i). The others follow from previously calculated compositions, in light of \( F_{LR}(X) = E(X) \lor G_{LR}(X) \), where \( X = X(M) \).
5.2 More functors

We have already noted that Table 2 has four missing entries. At this stage it is conceivable that these missing compositions could be among the functors considered so far, but we will see in Section 5.4 that they are indeed four new functors. For now, we simply deal with the missing entries in Table 2 by defining the functors

\[ Q = E \circ G_{LR}, \quad P = F \circ G_{LR}, \quad P_L = F_L \circ G_{LR}, \quad P_R = F_R \circ G_{LR}. \]  

(5.4)

We also define the enlarged set of functors

\[ \mathcal{F}^+ = \mathcal{F} \cup \{Q, P, P_L, P_R\} = \{O, E, G, G_L, G_R, G_{LR}, F, F_L, F_R, F_{LR}, Q, P, P_L, P_R, I\}. \]

For a monoid \( M \), the functors in (5.4) yield (at most) four additional submonoids:

\[ Q(M) = E(G_{LR}(M)), \quad P(M) = F(G_{LR}(M)), \quad P_L(M) = F_L(G_{LR}(M)), \quad P_R(M) = F_R(G_{LR}(M)). \]

Accordingly, we also define

\[ \mathcal{L}^+(M) = \{X(M) : X \in \mathcal{F}^+\}. \]

We will show in Section 6 (see Proposition 6.1) that \( \mathcal{L}^+(M) \) is a lattice. Figure 8 displays the generic shape of \( \mathcal{L}^+(M) \), with the new submonoids shown in red; cf. Figure 1. The inclusion relations claimed in Figure 8 are all easily verified. For example,

\[ O(M) \subseteq Q(M) = E(G_{LR}(M)) \subseteq \begin{cases} E(M) \\ F(G_{LR}(M)) = P(M), \end{cases} \]

and for \( \odot \neq LR \),

\[ G_\odot(M) = G_\odot(G_{LR}(M)) \subseteq F_\odot(G_{LR}(M)) \subseteq F_\odot(M) \Rightarrow G_\odot(M) \subseteq P_\odot(M) \subseteq F_\odot(M). \]

Figure 8: The generic shape of the lattice \( \mathcal{L}^+(M) \). In general these submonoids need not be distinct.
5.3 More compositions

Now that we have enlarged our list of functors to \( \mathcal{F}^+ \), we have a number of further compositions to calculate, namely those of the form \( X \circ Y \) and \( Y \circ X \) for \( X, Y \in \mathcal{F}^+ \). These compositions are shown in blue in Table 3. All of these entries can be readily verified using Table 2, associativity of functor composition, and the definition of the new functors. For example,

\[
E \circ Q = E \circ E \circ G_{LR} = E \circ G_{LR} = Q \quad \text{and} \quad Q \circ E = E \circ G_{LR} \circ E = E \circ O = O.
\]

As before, some calculations can be performed simultaneously; for example,

\[
F \circ P = F \circ F \circ G_{LR} = F \circ G_{LR} = P \quad \text{and} \quad P \circ F = F \circ G_{LR} \circ F \circ G_{LR} = F \circ G \circ G_{LR} = G \circ G_{LR} = G.
\]

Since \( \mathcal{F}^+ \) is closed under composition (cf. Table 3), it is therefore a semigroup, indeed a monoid with identity \( I \). Note that \( \mathcal{F}^+ \setminus \{I\} \) is also a semigroup, although it is not a monoid; however, \( F_{LR} \) is a right (but not left) identity element of this subsemigroup. We will say more about the size and structure of the monoid \( \mathcal{F}^+ \) in Sections 5.4 and 5.5.

| o | O | E | G | G_L | G_R | G_{LR} | F | F_L | F_R | F_{LR} | Q | P | P_L | P_R | I |
|---|---|---|---|----|----|-------|---|-----|-----|-------|---|---|-----|-----|---|
| O | O | O | O | O  | O  | O     | O | O   | O   | O     | O | O | O   | O   | O |
| E | O | E | O | O  | O  | Q     | E | E   | E   | Q    | Q | Q | Q   | Q   | E |
| G | O | G | G | G  | G  | G     | G | G   | G   | G    | G | G | G   | G   | G |
| G_L| O | G | G | G  | G  | G     | G | G   | G   | G    | G | G | G   | G   | G |
| G_R| O | G | G | G  | G  | G     | G | G   | G   | G    | G | G | G   | G   | G |
| G_{LR}| O | G | G | G  | G  | G     | G | G   | G   | G    | G | G | G   | G   | G |
| F | O | E | G | G  | P  | F     | F | F   | F   | Q    | P | P | P   | P   | F |
| F_L| O | E | G | G  | P  | F     | F | F   | F   | F_L | Q | P | P   | P   | F_L |
| F_R| O | E | G | G  | P  | F_R   | F | F   | F_R | Q_P | P | P | P   | F_R |
| F_{LR}| O | E | G | G  | P  | F_{LR} | F | F   | F_{LR} | Q_P | P | P | P   | F_{LR} |
| Q | O | O | O | O  | O  | Q     | O | O   | O   | Q    | O | O | O   | O   | Q |
| P | O | O | G | G  | G  | G     | G | G   | G   | G    | G | G | G   | G   | G |
| P_L| O | O | G | G  | G  | G     | G | G   | G   | G    | G | G | G   | G   | G |
| P_R| O | O | G | G  | G  | G_P   | G | G   | G   | G    | G | G | G   | G   | G |
| I | O | E | G | G_L| G_R| G_{LR}| F | F_L| F_R| F_{LR}| Q | P | P_L| P_R| I |

Table 3: Composition of the functors from \( \mathcal{F}^+ \).

5.4 Size

We now know that the set \( \mathcal{F}^+ \) is a monoid under composition, and that its size is at most 15. We also know that \( |\mathcal{F}^+| \geq 11 \), since \( |\mathcal{L}^+(M)| \geq |\mathcal{L}(M)| = 11 \) for \( M \) of type \((0,0,0,0)\); cf. Figure 6. To show that the size of \( \mathcal{F}^+ \) is in fact 15, as we will in Proposition 5.6 below, we will construct a monoid \( M \) such that \( \mathcal{L}^+(M) \) has size 15. We begin by showing that the functors from \( \mathcal{F}^+ \) respect the direct product operation:

**Lemma 5.5.** For any \( X \in \mathcal{F}^+ \), and for any two monoids \( M \) and \( N \), we have

\[
X(M \times N) = X(M) \times X(N).
\]

**Proof.** In light of Lemma 2.1, it suffices to demonstrate this for any \( X \in \mathcal{F}^+ \setminus \mathcal{F} \). For any such \( X \), we have \( X = Y \circ Z \) for some \( Y, Z \in \mathcal{F} \). Two applications of Lemma 2.1 then give

\[
X(M \times N) = Y(Z(M \times N)) = Y(Z(M) \times Z(N)) = Y(Z(M)) \times Y(Z(N)) = X(M) \times X(N).
\]

Table 1 listed the submonoids \( X(M), X \in \mathcal{F} \), for various monoids \( M \) defined in Section 2.4. Table 4 gives the submonoids \( X(M) \) for the additional functors \( X \in \mathcal{F}^+ \setminus \mathcal{F} \). The entries for \( M = G, E \) and \( P \) are clear, while those for \( M = B \) and \( B^0 \) follow quickly from Table 1 and the fact that \( G_{LR}(B^0) = G_{LR}(B) = B \).
and is also the so-called idempotent-generated submonoid, $M = P$ (the positive integers under multiplication), $M = B$ (the bicyclic monoid) and $M = B^0$ (the bicyclic monoid with a zero adjoined); cf. Table 1.

| $X$   | $X(G)$ | $X(E)$ | $X(P)$ | $X(B)$ | $X(B^0)$ |
|-------|--------|--------|--------|--------|----------|
| $Q$   | {1}    | {1}    | {1}    | $\{a^mb^m : m \geq 0\}$ | $\{a^mb^m : m \geq 0\}$ |
| $P$   | $G$    | {1}    | {1}    | $\{a^mb^m : m \geq 0\}$ | $\{a^mb^m : m \geq 0\}$ |
| $P_L$ | $G$    | {1}    | {1}    | $\{a^mb^m : m \geq n\}$ | $\{a^mb^m : m \geq n\}$ |
| $P_R$ | $G$    | {1}    | {1}    | $\{a^mb^m : m \leq n\}$ | $\{a^mb^m : m \leq n\}$ |

Table 4: The submonoids $X(M)$, $X \in \mathcal{F}^+ \setminus \mathcal{F}$, for $M = G$ (a group), $M = E$ (an idempotent-generated monoid), $M = P$ (the positive integers under multiplication), $M = B$ (the bicyclic monoid) and $M = B^0$ (the bicyclic monoid with a zero adjoined); cf. Table 1.

Proposition 5.6. The monoid $\mathcal{F}^+$ has size 15.

Proof. Consider the monoid $M = G \times E \times P \times B$, where $G$ is a nontrivial group, $E$ a nontrivial idempotent-generated monoid, $P$ the positive integers under multiplication, and $B$ the bicyclic monoid. By consulting Tables 1 and 4, and keeping in mind that $X(M) = X(G) \times X(E) \times X(P) \times X(B)$ for all $X \in \mathcal{F}^+$ (cf. Lemma 5.5), one may easily check that $\mathcal{L}^+(M)$ has size 15.

5.5 Structure

Now that we know the size of the monoid $\mathcal{F}^+$, it is natural to seek more information about its algebraic structure. The most common way to structurally decompose a semigroup is by using Green’s relations. These were defined in Section 3.3, but we recall a number of additional definitions here.

For a monoid $M$, Green’s $J$-preorder is the relation $\leq_J$ defined by $x \leq_J y$ if and only if $MxM \subseteq MyM$. Again, this may be reformulated in terms of divisibility: $x \leq_J y$ if and only if $x = ayb$ for some $a, b \in M$. Green’s $J$ relation (as defined in Section 3.3) is then given by $J = \leq_J \cap \geq_J$. Note that $\leq_J$ is a partial order if and only if $M$ is $J$-trivial.

Using the composition table (cf. Table 3), the computational algebra system GAP [58] can perform many calculations in the monoid $\mathcal{F}^+$. For example, GAP verifies that the monoid $\mathcal{F}^+$ is in fact $J$-trivial, and hence $L$, $R$, $H$- and $D$-trivial as well. GAP was also used to produce Figure 9, which displays the $\leq_J$ order in $\mathcal{F}^+$. In fact, since $\mathcal{F}^+$ is $J$-trivial, Figure 9 is also the so-called eggbox diagram of $\mathcal{F}^+$, as defined for example in [40, Section 1.2].

As is customary, the idempotents of $\mathcal{F}^+$ (i.e., the functors $X = X \circ X$) are coloured grey in Figure 9. The idempotent-generated submonoid

$$E(\mathcal{F}^+) = \{0, E, G, G_{L,R}, F, F_{L,R}, Q, P, I\}$$

(5.7)

is also pictured in Figure 9, along with its $\leq_J$ ordering, again with assistance from GAP.

GAP also shows that $\mathcal{F}^+$ has 2904 subsemigroups (exactly half of which are submonoids), and 1613 congruences, of which 76 are principal. (A congruence on a semigroup is an equivalence relation compatible with the product; these are used to form quotient semigroups; see [45, Section 1.5] for more details.)

It is interesting to compare the two posets ($\mathcal{L}^+(M), \subseteq$) and ($\mathcal{F}^+, \leq_J$), which are pictured in Figures 8 and 9, respectively (the former in the generic case). Although there are certainly some superficial similarities between them, the two posets are not (quite) isomorphic. For example, we have $G_L(M) \subseteq P_L(M)$ for any monoid $M$ (cf. Figure 8), while $G_L \not\leq_J P_L$ in $\mathcal{F}^+$ (cf. Figure 9). We can also see that $G_L \not\leq_J P_L$ directly; using Table 3, it is easy to verify that for any $X, Y \in \mathcal{F}^+$, we have $X \circ P_L \circ Y \in \{0, G, Q, P, P_L\}$, which means that $G_L \not\in \mathcal{F}^+ \circ P_L \circ \mathcal{F}^+$.

5.6 The lattice of the monoid of functionors

Since $\mathcal{F}^+$ is a monoid, it is natural to calculate its associated lattice $\mathcal{L}(\mathcal{F}^+)$. Consulting Table 3, we see that the only solution in $\mathcal{F}^+$ to $X \circ Y = I$ is $X = Y = I$, so it follows that $G(\mathcal{F}^+) = G_L(\mathcal{F}^+) = \{1\}$: i.e., that $T_1(\mathcal{F}^+) = T_4(\mathcal{F}^+) = 1$. From (5.7) we have $\{1\} \subseteq E(\mathcal{F}^+) = F(\mathcal{F}^+) \subseteq \mathcal{F}^+$; note that $E(\mathcal{F}^+) = F(\mathcal{F}^+)$ because $G(\mathcal{F}^+) = \{1\}$; cf. Lemma 3.7(i). It follows that $T_2(\mathcal{F}^+) = T_3(\mathcal{F}^+) = 0$. All of the above shows that the monoid $\mathcal{F}^+$ has type $T(\mathcal{F}^+) = (1, 0, 0, 1)$, and so

$$\mathcal{L}(\mathcal{F}^+) = \{\{1\}, E(\mathcal{F}^+), \mathcal{F}^+\}$$

is the three-element chain displayed in the second diagram on the top row of Figure 4.
6 An enhanced lattice invariant?

Section 4 concerned the lattice \( \mathcal{L}(M) = \{X(M) : X \in \mathcal{F}\} \) consisting of the submonoids of a monoid \( M \) arising from the functors from \( \mathcal{F} \). Section 5 concerned the monoid \( \mathcal{F}^+ \) of functors generated by \( \mathcal{F} \), and we defined \( \mathcal{L}^+(M) = \{X(M) : X \in \mathcal{F}^+\} \). As promised earlier, we now show that \( \mathcal{L}^+(M) \) is a lattice.

**Proposition 6.1.** For any monoid \( M \), the set \( \mathcal{L}^+(M) \) is a finite \( \lor \)-subsemilattice of \( \text{Sub}(M) \), with top element \( I(M) = M \) and bottom element \( O(M) = \{1\} \). Consequently, \( \mathcal{L}^+(M) \) is a lattice.

**Proof.** As in the proof of Proposition 2.3, it suffices to show that \( \mathcal{L}^+(M) \) is closed under \( \lor \). To do so, let \( X, Y \in \mathcal{F}^+ \); we must show that

\[ X(M) \lor Y(M) = Z(M) \quad \text{for some } Z \in \mathcal{F}^+. \quad (6.2) \]

In light of Proposition 2.3, and by commutativity of \( \lor \), we may assume that \( X \in \mathcal{F}^+ \setminus \mathcal{F} = \{Q, P, P_L, P_R\} \).

If \( Y = I \) or \( O \), then (6.2) is clear; we take \( Z = I \) or \( X \), respectively.

Next suppose \( Y \) is one of \( G \lor, Q \) or \( P \). Then, consulting Table 3, we see that \( X = U \circ G_{LR} \) and \( Y = V \circ G_{LR} \) for some \( U, V \in \mathcal{F} \). But then

\[ X(M) \lor Y(M) = U(G_{LR}) \lor V(G_{LR}) = W(G_{LR}) = W \circ G_{LR}(M) \quad \text{for some } W \in \mathcal{F}, \]

using Proposition 2.3 (applied in the monoid \( G_{LR} \)) in the second step. We then take \( Z = W \circ G_{LR} \in \mathcal{F}^+ \).

Next suppose \( Y = E \). If \( X = Q \), then (6.2) is clear since \( Q \subseteq E \). Now suppose \( X = P \). Then

\[ P \lor E = F \lor G_{LR} \lor E = E(G_{LR}) \lor G \lor G_{LR} \lor E = G \lor E \lor G = E \lor G = E \lor G_{LR} = E \lor F \],

where we used \( E(G_{LR}) \subseteq E \) and \( G \lor G_{LR} = G \) in the third step. Thus, we may take \( Z = F \lor G \) in this case.

Finally, suppose \( Y = F \). Again (6.2) is clear for \( X = Q \), as \( Q \subseteq F \), so suppose \( X = P \). Then writing \( G \lor G = G \), we have

\[ P \lor F \lor E = P \lor F \lor G_{LR} \lor E = E(G_{LR}) \lor G \lor G_{LR} \lor E \lor G = G \lor E \lor G = E \lor E \lor G = E \lor F \],

so we may take \( Z = F \) in this case. \( \square \)
We now have two lattice invariants $\mathcal{L}(M)$ and $\mathcal{L}^+(M)$, associated to a monoid $M$. Given that $\mathcal{L}^+(M)$ is defined in terms of a larger set of functors, one might hope that it allows us to distinguish monoids not distinguished by $\mathcal{L}(M)$. However, it follows from the results of this section that this is not the case. In Section 6.1 we prove some preliminary results about collapse in the enhanced lattice $\mathcal{L}^+(M)$, and then we classify the lattices $\mathcal{L}^+(M)$ in Section 6.2.

### 6.1 More collapse

We begin with some results analogous to Lemmas 3.6 and 3.7, but involving the functors from $\mathcal{F}^+ \setminus \mathcal{F}$. For a monoid $M$, we continue to use the abbreviations (3.1) and (3.2), as well as

$$Q = \mathbb{Q}(M), \quad P = \mathbb{P}(M), \quad P_L = \mathbb{P}_L(M), \quad P_R = \mathbb{P}_R(M).$$

**Lemma 6.3.** For a monoid $M$, conditions (i)–(vii) of Lemma 3.6 are also equivalent to each of the following:

(viii) $P$, $P_L$, $P_R$ and $G_{LR}$ are not all equal,

(ix) $P$, $P_L$, $P_R$ and $G_{LR}$ are pairwise distinct.

**Proof.** Writing $N = \mathbb{G}_{LR}(M)$, note that

$$P = \mathbb{F}(N), \quad P_L = \mathbb{F}_L(N), \quad P_R = \mathbb{F}_R(N), \quad G_{LR} = \mathbb{F}_{LR}(N).$$

Thus, the equivalence of (viii) and (ix) follows from the equivalence of (iii) and (iv) in the monoid $N$.

(ii) \(\Rightarrow\) (viii). Aiming to prove the contrapositive, suppose (viii) does not hold. In particular, we have $P = G_{LR}$: i.e., $\mathbb{P}(M) = \mathbb{G}_{LR}(M)$. But then

$$G = \mathbb{G}(M) = \mathbb{G}_{LR}(\mathbb{P}(M)) = \mathbb{G}_{LR}(\mathbb{G}_{LR}(M)) = \mathbb{G}_{LR}(M) = G_{LR},$$

so that (ii) does not hold.

(ix) \(\Rightarrow\) (i). Again we prove the contrapositive. If (i) does not hold, then $G = G_{LR}$, from which it follows that

$$P = \mathbb{P}(M) = \mathbb{F}(\mathbb{G}_{LR}(M)) = \mathbb{F}(\mathbb{G}(M)) = \mathbb{G}(M) = G = G_{LR},$$

so that (ix) does not hold. \[\square\]

**Lemma 6.4.** For any monoid $M$ we have

(i) $G = \{1\} \iff P = Q \iff F = E$,

(ii) $E = Q \iff F = P \iff F_L = P_L \iff F_R = P_R \iff F_{LR} = G_{LR}$.

**Proof.** (i). In light of Lemma 3.7(i), it is enough to show that $G = \{1\} \iff P = Q$. If $G = \{1\}$, then

$$P = \mathbb{F}(G_{LR}) = \mathbb{E}(G_{LR}) \lor \mathbb{G}(G_{LR}) = Q \lor G = Q \lor \{1\} = Q.$$  

Conversely, if $P = Q$, then

$$G = \mathbb{G}(M) = \mathbb{G} \circ \mathbb{P}(M) = \mathbb{G} \circ \mathbb{Q}(M) = \mathbb{O}(M) = \{1\}.$$

(ii). For convenience during this part of the proof, we will write $\mathbb{P}_{LR} = \mathbb{G}_{LR}$ and $P_{LR} = G_{LR}$. So we wish to show that $E = Q \iff F_\triangledown = P_\triangledown$ for any subscript $\triangledown$. First, if $E = Q$, then

$$F_\triangledown = E \lor G_\triangledown = Q \lor G_\triangledown = \mathbb{E}(G_{LR}) \lor \mathbb{G}_\triangledown(G_{LR}) = \mathbb{F}_\triangledown(G_{LR}) = P_\triangledown.$$

(Note that the last step holds by definition apart from the $\triangledown = LR$ case, when it follows instead from Lemma 5.2(i) and the $P_{LR} = G_{LR}$ convention.) Conversely, if $F_\triangledown = P_\triangledown$ for some choice of $\triangledown$, then

$$E = \mathbb{E}(M) = \mathbb{E} \circ \mathbb{F}_\triangledown(M) = \mathbb{E} \circ \mathbb{P}_\triangledown(M) = \mathbb{Q}(M) = Q.$$ \[\square\]
6.2 Classification of enhanced lattice invariants

We now wish to classify the enhanced lattice invariants $\mathcal{L}^+(M)$, for monoids $M$. To do so, we will again use the type of $M$, as defined in Section 4.1.

First note that if $T_1(M) = 1$, then $G = G_L$ and so $G_{LR} = G$ (cf. Lemma 3.6), so it follows that

$$Q = E(G_{LR}) = E(G) = \{1\} \quad \text{and} \quad P_\varnothing = F_\varnothing(G_{LR}) = F_\varnothing(G) = G,$$

which means that $\mathcal{L}^+(M) = \mathcal{L}(M)$ in this case.

If $T_3(M) = 1$, then $G_{LR} = F_{LR}$, and this time

$$Q = E(G_{LR}) = E(F_{LR}) = E \quad \text{and} \quad P_\varnothing = F_\varnothing(G_{LR}) = F_\varnothing(F_{LR}) = F_\varnothing,$$

so that $\mathcal{L}^+(M) = \mathcal{L}(M)$ in this case as well.

This leaves us to consider monoids $M$ of type $T(M) = (0, i, 0, j)$. As explained at the beginning of Section 4.3, the seven submonoids of $M$ listed in (4.3) are distinct (as $T_1(M) = 0$). By Lemma 6.3, the four submonoids $P, P_L, P_R, G_{LR}$ are distinct as well. Because also $F_{LR} \neq G_{LR}$ (as $T_3(M) = 0$), it follows from Lemma 6.4(ii) that the following containments are strict:

$$Q \subsetneq E, \quad P \subsetneq F, \quad P_L \subsetneq F_L, \quad P_R \subsetneq F_R, \quad G_{LR} \subsetneq F_{LR}.$$

We claim that the following containments are also strict:

$$\{1\} \subsetneq Q, \quad G \subsetneq P, \quad G_L \subsetneq P_L, \quad G_R \subsetneq P_R.$$

Indeed, to see this, note first that $\{1\}$, $G$, $G_L$ and $G_R$ have only one idempotent (cf. Lemma 3.3(i)). On the other hand, $Q = E(G_{LR}(M))$ contains infinitely many idempotents (cf. Lemma 3.6, and note that $G \neq G_L$ since $T_1(M) = 0$), and so too do each of $P, P_L$ and $P_R$, since all three of these contain $Q$. This completes the proof of the claim. All of the above shows that the following eleven submonoids of $M$ are distinct:

$$G, \quad G_L, \quad G_R, \quad G_{LR}, \quad P, \quad P_L, \quad P_R, \quad F, \quad F_L, \quad F_R \text{ and } F_{LR}.$$

These submonoids are shaded red in Figure 10, which gives the generic shape of $\mathcal{L}^+(M)$ in the case that $T(M) = (0, i, 0, j)$.

![Figure 10: The generic shape of the lattice $\mathcal{L}^+(M)$ when $M$ has type $T(M) = (0, i, 0, j)$. The submonoids shaded red are distinct, and thick lines indicate proper containment.](image)

The exact shape of $\mathcal{L}^+(M)$ depends on the values of $i = T_2(M)$ and $j = T_4(M)$, and these determine which thin edges (if any) of Figure 10 to contract (but keep Lemma 6.4(i) in mind). The possible shapes are shown in Figure 11. Figure 12 shows the possible lattices $\mathcal{L}^+(M)$, for an arbitrary monoid $M$, up to lattice isomorphism (cf. Figure 7).
Figure 11: The lattice $\mathcal{L}^+(M)$ when $M$ has type $T(M) = (0, i, 0, j)$. In each case, the nodes represent distinct submonoids of $M$. For other types we have $\mathcal{L}^+(M) = \mathcal{L}(M)$; cf. Figure 4 and 6.

Remark 6.5. Recall that $B^0$ is the bicyclic monoid with a zero adjoined. We noted in Remark 2.6 that $\mathcal{L}(B^0)$ is not a sublattice of Sub($B^0$), citing the fact that $F_L \cap G_{LR} \notin \mathcal{L}(B^0)$, using the usual abbreviations. However, consulting Tables 1 and 4, we see that $F_L \cap G_{LR} = P_L \in \mathcal{L}^+(B^0)$. In fact, $\mathcal{L}^+(B^0)$ is closed under arbitrary intersections, as one may easily check using the aforementioned tables, which means that $\mathcal{L}^+(B^0)$ is a sublattice of Sub($B^0$). The author does not currently know if $\mathcal{L}^+(M)$ is a sublattice of Sub($M$) for an arbitrary monoid $M$.

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Figure 12: The possible lattices \( \mathcal{L}^+(M) \) for a monoid \( M \), up to lattice isomorphism.
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