On b-Chromatic Number for a Certain Triangular Line graphs

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Abstract

The b-Chromatic number of a graph is the maximum number of colors that can be used to color the vertices of G. Let us consider the color classes for different line graphs, where every line graph with i has a representative (the color dominating vertex) that is able to communicate with all the other line graphs. Further we investigate the b-chromatic number for certain triangular graphs such as triangular snake graphs, triangular fan graphs, triangular cylindrical graphs and we also generalize some of the results based on these graphs.

Key words : b-chromatic, triangular line graph, cycles, coloring,

Introduction

A b-coloring by k colors is a proper coloring of the vertices of G such that in each color classes there exists a vertex has neighbors in all the other k-1 color classes. In other words each color class contains a vertex which has at least one neighbor in all the other color cases. Such vertex is called color dominating vertex. The b-chromatic number \( \varphi(G) \) is the largest integer k such that G admits a b-coloring with k-colors. The basic concept of b-chromatic number was introduced Irving and Manlove by considering proper coloring that are minimal with respect to a partial order defined on the set of all partition of V(G). Naturally, a proper coloring of G with \( \chi(G) \) colors is a b-coloring of G since it cannot be improved so that \( \chi(G) \leq b(G) \). The NP completeness results have incited researchers to establish bounds on the b-chromatic number in general or to find its exact values for classes of graphs.

The b-chromatic number of certain product of some families of graphs is examined by Balakrishnan et al. and b-coloring for square of Cartesian product of two cycles is discovered by Chandra Kumar and Nicholas. Some results on b-coloring and b-chromatic are introduced by Alkhteeb. The b-chromatic of some cycle related graph has examined by Vaidhya and Shukla and also they established the b-chromatic number of certain wheel graphs.

Definition 1.1 : A line graph \( L(G) \) of a simple graph G is obtained by associating a vertex with each edge of the graph and the connecting two vertices with an edge if the corresponding edges of G have a common vertex.

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Definition 1.2: The triangle graph, \( T(G) \) of a graph \( G \) has as its vertices the triangles of \( G \) and two vertices of \( T(G) \) are adjacent if the corresponding triangles in \( G \) have a common edge.

Definition 1.3: A triangular snake is a triangular cactus whose block cut point graph path \( P = a_1a_2\ldots a_{n+1} \) by joining \( a_i \) and \( a_{i+1} \) to a new vertex \( b_i \) for \( i = 1, 2, 3, \ldots, n \). We denote this snake graph as \( S(n) \).

Line graphs of triangular snakes:
The line graph \( L(G) \) of an undirected graph \( G \) is another graph \( L(G) \) that represents the adjacencies between edges of \( G \). In other words, given graph \( G \), its line graph \( L(G) \) is a graph such that (1) each vertex of \( L(G) \) represents an edge of \( G \) and (2) two vertices of \( L(G) \) are adjacent if their corresponding edges are adjacent in \( G \).

Theorem 1: Let \( L[S(n)] \) be a line graph of a triangular snake graph \( S(n) \), \( n \geq 9 \). Then \( b(L[S(n)]) = \delta(L[S(n)]) + 1 \).

Proof: Let us consider that \( V_n \) be the set of vertices of \( S(n) \). We also assign the colors as \( v_j = 1 \).

Net us assign the coloring \( C \) to the vertices as follows:

\[
\begin{align*}
    c(v_j) &= \begin{cases} 
        (i+2) \mod 7 & \text{if } 1 \leq i \leq n \\
        (i+2) \mod 7 + 7 & \text{otherwise}
    \end{cases} \\
    c(v_{2n+i}) &= \begin{cases} 
        (i+6) \mod 7 & \text{if } 1 \leq i \leq n \\
        (i+3) \mod 7 + 7 & \text{otherwise}
    \end{cases}
\end{align*}
\]

Fig 1.1
That clearly shows that $C$ is a 7-coloring.

**Case 1:** $C$ is a $b$-coloring

It is sufficient to consider the vertices $v_{2n+1}, v_{2n+2}, \ldots, v_{n}$ of $L[S(n)]$. It is clear that all the vertices except the odd degree vertices have maximum degree 6. As we specified that for 7 coloring of any vertex of maximum degree is adjacent to all the vertices.

**Case 2:** The $b$-chromatic number of $L[S(n)]$ is 7.

If $C(v_{3n}) = 8$ then $V_{3n-1}$ are not has adjacent to 3 different colors. Thus the vertices $L(G)$ maximum degree 6 minimum degree 2.

**Definition 1.3:** A fan graph $F_{m,n}$ is defined as the graph joining $K_{m} + P_{n}$, Where $K_{m}$ is the empty graph on $m$ nodes and $P_{n}$ is the path graph on $n$ nodes. The case of $m=1$ corresponds to the usual fan graph. The graph in fig 1.3 shows a triangular fan graph and its line graph.

**Definition 1.4:** A fan graph or a friendship graph $F(nC3)$ is defined as the following connected graph containing $n$ copies of circuits of each length 3.

**Theorem 1.1:** If $F_n$ be the line graph of the fan graph, the $\phi(L(F_n)) = (n-1)$

**Proof:** Let us consider the $F_n$ be the fan graph with $n$-vertices with set of $V(F_n) = \{f_1, f_2, f_3, \ldots, f_n\}$ with $L(F_n)$ as a complete graph $K_{n-1}$ with $V(K_{n-1}) = \{u_1, u_2, \ldots, u_{n-1}\}$ and a cycle $C$ with $V(C) = \{v_1, v_2, \ldots, v_{n-1}\}$.

Now to assign the proper coloring and the $b$-chromatic for the set of cycles $C = \{1, 2, 3, \ldots, n\}$

And define the coloring for as function $f : V \rightarrow C$. Assign the color $C$ to the vertex set $V(K_{n-1}) = u_i$ for $i = 1, 2, 3, \ldots, (n-1)$

Now we will color the vertices of a cycle. If one more color is introduced, we say $C_n$ to be colored for any one of the vertex of the cycle $C$. In the outer layer of the cycle $C$ are adjacent with $v_2, v_{n-1}, u_1, u_2, v_{n-1}$ is adjacent with $v_1, v_{n-2}, u_{n-1}, u_1$.

In general each vertex $v_i$ is adjacent to $v_{i+1}, v_{i-1}$ for $i = 1, 2, 3, \ldots, (n-1)$. Hence we can assign the (n-1) colors which are assigned to the complete graph. Now the vertices in the cycle $v_i$ for $i = 3, 4, \ldots, (n-1)$ should be the color $C_i$ for $i = 1, 2, 3, \ldots, n-3$ and the vertex $v_1$ should be assigned by the color $C_{n-2}$ by $v_2$ by $C_{n-1}$.

Hence $\phi(L(F_n)) = n$.

**Definition 1.3:** A Cylindrical graph $C_n$ is a graph with $n$ vertices $(n \geq 6)$, formed by connecting with a multi vertex to all the vertices of an $(n+1)$ cycle.

Line graph of a Cylindrical graph: In the following figure 1.4 illustrates a Cylindrical graph and Line graph of cylindrical graph.

**Theorem 1.3:** If is a complete cylindrical graph, then $\chi(C_n) = n$, for any value of $n$.

**Proof:** Let us consider $C_n$ be the cylindrical graph with $n$ vertices. Let $V(C_n) = \{v_1, v_2, v_3, \ldots, v_n\}$
be the set of vertex with cylindrical graph $C_n$. Then $|V(G)| = n$ and $|E(G)| = n(n-1)/2$.

To determine the proper coloring we consider the following condition.

**Case 1:** $C_n$ is a proper coloring with $n=4$

$V(C_4) = \{v_1, v_2, v_3, v_4\}$, $V(G) = 4$, $E(G) = 4$. $\chi(C_n) = n$

In this case, $G$ has a three vertices of degree 3. Maximum degree 3. Using one the preposition we have $\chi(C_4) \leq 4$.

To assign a proper coloring and b-coloring consider the color of set $C = \{1, 2, 3, 4\}$ and define the color function $f : V \rightarrow C$ so that $f(V_1) = 1, f(V_2) = 2, f(V_3) = 3, f(V_4) = 4$. Therefore $\chi(C_4) = 4$.

**Case 2:** $C_n$ is a b-coloring

It is observed in the above case that $G$ has four vertices of degree three. Maximum degree is 3. In the graph $G$ must have three vertices of a degree 3 which is possible. To assign the proper coloring and for b-coloring consider the color set $C = \{1, 2, 3, 4\}$ and define the function as we above and color the graph. Hence we get the required result as $\chi(C_4) = 4$.

From these two conditions, we say that the b-chromatic number of the given cylindrical triangular line graph with $n$ vertices is $n$.

Hence $\chi(C_n) = n$.

**Conclusion**

Graph theory has many applications in biochemistry, electrical networks engineering computer science and operational research. Graph coloring is also many practical applications in bandwidth allocation and matching of such patterns. Here we have obtained some the b-chromatic number of line graphs and certain triangular line graphs such as snake triangular graphs, triangular fan graphs, cylindrical triangular pattern line graphs.

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