Quantum contextuality implies a logic that does not obey the principle of bivalence

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Abstract

In the paper, a value assignment for projection operators relating to a quantum system is equated with assignment of truth-values to the propositions associated with these operators. In consequence, the Kochen-Specker theorem (its localized variant, to be exact) can be treated as the statement that a logic of those projection operators does not obey the principle of bivalence. This implies that such a logic has a gappy (partial) semantics or many-valued semantics.

Keywords: Quantum mechanics; Kochen-Specker theorem; Contextuality; Truth values; Partial semantics; Many-valued semantics.

1 Introduction

Consider the triple (H, |Ψ⟩, O) in which H is the Hilbert space of a quantum system, |Ψ⟩ ∈ H is the normalized vector describing the state of this system, and O = {P} is a finite set of projection operators P on H.

Define an assignment function h as a function from the set O to the set of numerical values {0, 1}, namely,

\[ h : O \rightarrow \{0, 1\} \] ,

such that

\[ h(\hat{0}) = 0 \] , \hspace{1cm} (2)

\[ h(\hat{1}) = 1 \] , \hspace{1cm} (3)

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where \( \hat{0} \) and \( \hat{1} \) are the zero-projection and identity-projection operators, respectively. According to [1], the assignment function \( h \) expresses the notion of a hidden variable, namely, \( h(\hat{P}) \) specifies in advance the result obtained from the measurement of an observable corresponding to the particular projection operator \( \hat{P} \).

Also, define a subset \( C \subset O \) as a context if \( |C| \geq 2 \) and any two projection operators from \( C \), say, \( \hat{P}_i \) and \( \hat{P}_j \), are orthogonal

\[
\hat{P}_i, \hat{P}_j \in C, i \neq j \implies \hat{P}_i \hat{P}_j = \hat{P}_j \hat{P}_i = \hat{0} .
\]  

The context \( C \) is maximal (or complete) if the projection operators from \( C \) resolve to the identity-projection operator:

\[
\sum_{\hat{P}_i \in C} \hat{P}_i = \hat{1} .
\]  

Now consider the following sentences:

(a) The set \( O \) is value definite under \( h \); in other words, \( h \) is a total function.

(b) The value \( h(\hat{P}_i) \) depends only on \( \hat{P}_i \) and not the context \( C \) containing \( \hat{P}_i \).

(c) For a maximal \( C \), the values of its projection operators \( h(\hat{P}_i) \) add up to 1; otherwise stated, the next entailment holds:

\[
h \left( \sum_{\hat{P}_i \in C} \hat{P}_i \right) = 1 \implies \sum_{\hat{P}_i \in C} h(\hat{P}_i) = 1 .
\]  

Provided that the sentences (b) and (c) are true, the sentence (a) must be denied in accordance with the Kochen-Specker theorem [2, 3]. That is, the assignment function \( h \) cannot be total and, hence, at least one projection operator from \( O \) must be value indefinite under \( h \) (i.e., must have the value neither 0 nor 1). What is more, according to the variant of the Kochen-Specker theorem localizing value indefiniteness [4], there is a set \( O \) containing projection operators \( \hat{P}_i \) and \( \hat{P}_j \) such that if the system is prepared in the pure state \( |\Psi_i\rangle \) in which \( h(\hat{P}_i) = 1 \), then both \( h(\hat{P}_j) = 1 \) and \( h(\hat{P}_j) = 0 \) lead to contradictions.

On the other hand, consider a truth-value assignment function \( v_C \) that denotes a truth valuation in a circumstance \( C \), that is, a mapping from some subset of propositions \( P \subseteq \{\diamond\} \) related to the quantum system (where the symbol \( \diamond \) stands for any proposition, compound or simple) to the set of truth-values \( \{0, 1\} \) (where the value 0 represents “false” and the value 1 represents “true”) relative to a circumstance of valuation indicated by \( C \) (such a circumstance can be, for example, the state \( |\Psi\rangle \) in which the system is prepared or found):

\[
v_C : P \rightarrow \{0, 1\} .
\]
Commonly, it is written using the double-bracket notation, namely, \( v_C(\diamond) = [\diamond]_C \). The truth-value assignment function \( v_C \) expresses the notion of not-yet-verified truth values: It specifies in advance the truth-value obtained from the verification of the proposition \( \diamond \).

Let the following valuational axiom hold true

\[
v_C(\hat{P}_\diamond) = [\diamond]_C ,
\]

(8)

where \( \hat{P}_\diamond \) is the projection operator uniquely (i.e., one-to-one) associated with the proposition \( \diamond \in \{\diamond\} \).

Assume that the function \( h \) coincides with the function \( v_C \). Then, the localized variant of the Kochen-Specker theorem is equivalent to the statement that a logic defined as the relations between projection operators \( \hat{P}_\diamond \) on \( \mathcal{H} \) does not obey the principle of bivalence (according to which a proposition must be either true or false [5]). In other words, a logic of the projection operators \( \hat{P}_\diamond \) has a non-bivalent semantics, e.g., a gappy one (in which the function \( v_C \) is partial and thus some propositions may have absolutely no truth-value) or a many-valued one (in which there are more than two truth-values).

Let us demonstrate this equivalence in the presented paper.

2 Truth-value assignment for projection operators

Consider the lattice \( \mathcal{L}(\mathcal{C}) \) formed by the column spaces (a.k.a. ranges) of the projection operators \( \hat{P}_i \in \mathcal{C} \), the closed subspaces of the Hilbert space \( \mathcal{H} \). Let the lattice operation meet \( \wedge \) correspond to the intersection of the column spaces, while the lattice operation join \( \vee \) correspond to the smallest closed subspace of \( \mathcal{H} \) containing their union. Let the lattice \( \mathcal{L}(\mathcal{C}) \) be bounded, i.e., let it have the greatest element \( \text{ran}(\hat{1}) = \mathcal{H} \) and the least element \( \text{ran}(\hat{0}) = \{0\} \).

One can define the lattice operations on \( \mathcal{L}(\mathcal{C}) \) as follows:

\[
\text{ran}(\hat{A}) \wedge \text{ran}(\hat{B}) = \text{ran}(\hat{A}) \cap \text{ran}(\hat{B}) = \text{ran}(\hat{A} \hat{B}) ,
\]

(9)

\[
\text{ran}(\hat{A}) \vee \text{ran}(\hat{B}) = \left( \left( \text{ran}(\hat{A}) \right)^\perp \cap \left( \text{ran}(\hat{B}) \right)^\perp \right)^\perp ,
\]

(10)

where \( \text{ran}(\hat{A}), \text{ran}(\hat{B}) \in \mathcal{L}(\mathcal{C}) \) and \( (\cdot)^\perp \) stands for the orthogonal complement of \( (\cdot) \). Given that the orthogonal complement of the column space is the null space (a.k.a. kernel), that is,

\[
\left( \text{ran}(\hat{A}) \right)^\perp = \ker(\hat{A}) = \text{ran}(\neg \hat{A}) ,
\]

(11)
where

\[ \neg \hat{A} = \hat{1} - \hat{A} \]  

(12)

is understood as negation of \( \hat{A} \) such that

\[ \text{ran}(\hat{A}) + \text{ran}(\neg \hat{A}) = \text{ran}(\hat{1}) = \text{ran}(\hat{A} + \neg \hat{A}) \]  

(13)

it holds that

\[ \text{ran}(\hat{A}) \lor \text{ran}(\hat{B}) = \left(\text{ran}(\neg \hat{A} \neg \hat{B})\right)^\perp = \text{ran}(\hat{A} + \hat{B}) \]  

(14)

As the closed subspaces \( \text{ran}(\hat{P}_i) \) and \( \text{ran}(\hat{P}_j) \) where \( \hat{P}_i \perp \hat{P}_j \), i.e., \( \hat{P}_i \hat{P}_j = 0 \), are orthogonal to each other, one finds

\[ \text{ran}(\hat{P}_i) \land \text{ran}(\hat{P}_j) = \text{ran}(\hat{P}_i) \cap \text{ran}(\hat{P}_j) = \{0\} \]  

(15)

Next, let us consider the truth-value assignments of the projection operators from the lattice \( \mathcal{L}(\mathcal{C}) \).

Given that \( \text{ran}(\hat{1}) = \mathcal{H} \), any arbitrary state of the system \( |\Psi\rangle \in \mathcal{H} \) resides in the column space of the identity-projection operator, i.e., \( |\Psi\rangle \in \text{ran}(\hat{1}) \). But then, being in \( \text{ran}(\hat{P}_i) \) means \( \hat{P}_i |\Psi\rangle = |\Psi\rangle \); so, in agreement with the eigenstate assumption \( \mathbb{I} \), one can presume that the function \( v_{|\Psi\rangle} \) assigns the truth value 1 to the projection operator \( \hat{1} \) in any admissible state of the system \( |\Psi\rangle \in \mathcal{H} \).

At the same time, any admissible state of the system \( |\Psi\rangle \in \mathcal{H} \) also resides in the null space of the null-projection operator, i.e., \( |\Psi\rangle \in \text{ran}(\hat{0}) \). It gives \( \hat{0} |\Psi\rangle = 0 \cdot |\Psi\rangle \), consequently, one can presume that the function \( v_{|\Psi\rangle} \) assigns the truth value 0 to \( \hat{0} \) in any admissible state of the system \( |\Psi\rangle \in \mathcal{H} \).

This can be written down as

\[ |\Psi\rangle \in \left\{ \begin{array}{l} \text{ran}(\hat{1}) = \mathcal{H} \\
\text{ran}(\hat{0}) = \mathcal{H} \end{array} \right. \iff \left\{ \begin{array}{l} v_{|\Psi\rangle}(\hat{1}) = 1 \\
 v_{|\Psi\rangle}(\hat{0}) = 0 \end{array} \right. \]  

(16)

Let the system be prepared in a pure state \( |\Psi_i\rangle \) lying in the column space of the projection operator \( \hat{P}_i \in \mathcal{C} \). Since \( \hat{P}_i |\Psi_i\rangle = |\Psi_i\rangle \), one can assume that the function \( v_{|\Psi_i\rangle} \) assigns the truth value 1 to the projection operator \( \hat{P}_i \) in the state \( |\Psi_i\rangle \). Contrariwise, if the truth value of the projection operator \( \hat{P}_i \) is 1 in the state \( |\Psi_i\rangle \), one can deduce that the state \( |\Psi_i\rangle \) is in the column space of the projection operator \( \hat{P}_i \). These two suppositions can be recorded together as the following logical biconditional:

\[ |\Psi_i\rangle \in \text{ran}(\hat{P}_i) \iff v_{|\Psi_i\rangle}(\hat{P}_i) = 1 \]  

(17)
In view of (15), the vector $|\Psi_i\rangle$ must also reside in the null space of any other projection operator $\hat{P}_j$ in the context $C$, and therefore all other truth values $v_{|\Psi_i\rangle}(\hat{P}_j)$ relating to the context $C$ must be zero:

$$|\Psi_i\rangle \in \text{ran}(-\hat{P}_j) \iff v_{|\Psi_i\rangle}(\hat{P}_j) = 0$$

(18)

This obviously gives

$$\sum_{j \neq i} v_{|\Psi_i\rangle}(\hat{P}_j) = 1$$

(19)

By contrast, consider the state $|\Phi_i\rangle$ where $v_{|\Phi_i\rangle}(\hat{P}_i) = 0$ implying $|\Phi_i\rangle \in \text{ran}(\hat{P}_i)$. According to (9), in the maximal context $C$ there is another $\hat{P}_k$, $k \neq i$, such that

$$\text{ran}(-\hat{P}_i) \cap \text{ran}(\hat{P}_k) = \text{ran}(\hat{P}_k - \hat{P}_i) = \text{ran}(\hat{P}_k)$$

(20)

and so

$$|\Phi_i\rangle \in \text{ran}(-\hat{P}_i) \subseteq \text{ran}(\hat{P}_k) \iff v_{|\Phi_i\rangle}(\hat{P}_i) = v_{|\Phi_i\rangle}(\hat{P}_k) = 1$$

(21)

$$\sum_{j \neq i, k} v_{|\Phi_i\rangle}(\hat{P}_j) = 1$$

(22)

Subsequently, if the system is prepared (found) in the state lying in the column or null space of any projection operator from the maximal context $C$, then among all the propositions $\mathcal{P}_C = \{\diamond\}_C$ associated with $C$ exactly one would be true while the others would be false.

Assume that there is a different context $C' \subset \mathcal{O}$, where some members $\hat{P}_l' \in \mathcal{C}'$ do not commute with $\hat{P}_i \in \mathcal{C}$. Suppose that the state $|\Omega\rangle$ is arranged in the subspace from the lattice $\mathcal{L}(C')$, e.g., $|\Omega\rangle \in \text{ran}(\hat{P}_l')$, entailing $v_{|\Omega\rangle}(\hat{P}_l') = 1$ and $v_{|\Omega\rangle}(-\hat{P}_l') = 0$.

Let us show that the vector $|\Omega\rangle$ resides in neither the column space nor the null space of at least one projection operator, say, $\hat{P}_l$, from the other lattice $\mathcal{L}(C)$ and, as a result, the truth-value function $v_{|\Omega\rangle}(\hat{P}_l)$ must assign neither 1 nor 0 to this operator under the valuations (17) and (18), that is,

$$|\Omega\rangle \notin \{ \text{ran}(\hat{P}_l), \text{ran}(-\hat{P}_l) \iff v_{|\Omega\rangle}(\hat{P}_l) \notin \{0, 1\} \}$$

(23)
3 Cabello’s set of 4×4 matrices

Consider the projection operators \( \hat{P}_i^{(1)} \in \mathcal{C}^{(1)} \) and \( \hat{P}_i^{(6)} \in \mathcal{C}^{(6)} \) on the Hilbert space \( \mathcal{H} = \mathbb{C}^4 \) from the set \( O = (\mathcal{C}^{(Q)})_{Q=1}^9 \) of 18 four-dimensional matrices used in the paper [6] by Cabello et al. to prove the Bell-Kochen-Specker theorem:

\[
\begin{align*}
\hat{P}_1^{(1)} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, & \hat{P}_2^{(1)} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, & \hat{P}_3^{(1)} &= \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, & \hat{P}_4^{(1)} &= \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\hat{P}_1^{(6)} &= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, & \hat{P}_2^{(6)} &= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, & \hat{P}_3^{(6)} &= \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, & \hat{P}_4^{(6)} &= \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\end{align*}
\]

(24)

where \( \bar{1} \) stands for \( -1 \).

As it can be readily seen, all \( \hat{P}_i^{(1)} \) are orthogonal to each other and \( \sum_{i=1}^{4} \hat{P}_i^{(1)} = \hat{1} \). The same is true for \( \hat{P}_i^{(6)} \), which means that \( \mathcal{C}^{(1)} \) and \( \mathcal{C}^{(6)} \) are the maximal contexts. Their column and null spaces are:

\[
\begin{align*}
\text{ran}(\hat{P}_1^{(1)}) &= \left\{ \begin{bmatrix} 0 \\ 0 \\ a \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\}, & \text{ran}(\neg\hat{P}_1^{(1)}) &= \left\{ \begin{bmatrix} b \\ c \\ d \\ 0 \end{bmatrix} : b, c, d \in \mathbb{R} \right\}, \\
\text{ran}(\hat{P}_2^{(1)}) &= \left\{ \begin{bmatrix} 0 \\ a \\ 0 \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\}, & \text{ran}(\neg\hat{P}_2^{(1)}) &= \left\{ \begin{bmatrix} b \\ 0 \\ c \\ d \end{bmatrix} : b, c, d \in \mathbb{R} \right\}, \\
\text{ran}(\hat{P}_3^{(1)}) &= \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a \end{bmatrix} : a \in \mathbb{R} \right\}, & \text{ran}(\neg\hat{P}_3^{(1)}) &= \left\{ \begin{bmatrix} -c \\ b \\ c \\ d \end{bmatrix} : b, c, d \in \mathbb{R} \right\}, \\
\text{ran}(\hat{P}_4^{(1)}) &= \left\{ \begin{bmatrix} a \\ 0 \\ 0 \\ -a \end{bmatrix} : a \in \mathbb{R} \right\}, & \text{ran}(\neg\hat{P}_4^{(1)}) &= \left\{ \begin{bmatrix} c \\ b \\ c \\ d \end{bmatrix} : b, c, d \in \mathbb{R} \right\}; \\
\text{ran}(\hat{P}_1^{(6)}) &= \left\{ \begin{bmatrix} a \\ -a \\ -a \\ a \end{bmatrix} : a \in \mathbb{R} \right\}, & \text{ran}(\neg\hat{P}_1^{(6)}) &= \left\{ \begin{bmatrix} b + c - d \\ b \\ c \\ d \end{bmatrix} : b, c, d \in \mathbb{R} \right\},
\end{align*}
\]

(26)

(27)

(28)

(29)

(30)
Consider the intersections

In consequence,

accordance with (17) and (18), implies

and so in the null spaces of the rest of the projections from the context $C^{(1)}$, which, in accordance with (17) and (18), implies

$$|1^{(1)}⟩ ∈ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a \end{bmatrix} \implies v_{|1^{(1)}⟩} (\hat{P}_1^{(1)}) = 1 \quad ,$$

$$|1^{(1)}⟩ ∈ \begin{cases} \begin{bmatrix} b \\ c \\ 0 \\ -a \end{bmatrix} \implies v_{|1^{(1)}⟩} (\hat{P}_2^{(1)}) = 0 \\ \begin{bmatrix} b \\ c \\ d \\ -c \end{bmatrix} \implies v_{|1^{(1)}⟩} (\hat{P}_3^{(1)}) = 0 \\ \begin{bmatrix} b \\ c \\ d \\ d \end{bmatrix} \implies v_{|1^{(1)}⟩} (\hat{P}_4^{(1)}) = 0 \end{cases} .$$

In consequence, $\sum_{i=1}^4 v_{|1^{(1)}⟩} (\hat{P}_i^{(1)}) = 1.$

Consider the intersections

$$\text{ran}(\hat{P}_1^{(1)}) \cap \text{ran}(\hat{P}_1^{(6)}) = \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \cap \begin{bmatrix} 0 \\ -a \\ -a \\ a \end{bmatrix} = \{0\} \quad .$$
Another contradiction, namely, \( \forall \)

From these intersections it follows that

Because every one of these intersections is the zero subspace, \( \text{ran}(\hat{P}_1^{(1)}) \) is orthogonal to every \( \text{ran}(\hat{P}_i^{(6)}) \) and, hence, \( v_{1(1)}(\hat{P}_1^{(1)}) = v_{1(1)}(\hat{P}_i^{(6)}) = 0 \). However, this leads to a contradiction, namely, \( v_{1(1)}(\sum_{i=1}^{4} \hat{P}_i^{(6)}) = 1 = \sum_{i=1}^{4} v_{1(1)}(\hat{P}_i^{(6)}) = 0 \).

Now, consider additional intersections:

\[
\text{ran}(\hat{P}_1^{(1)}) \cap \text{ran}(\hat{P}_2^{(6)}) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a \end{bmatrix} \right\} \cap \left\{ \begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix} \right\} = \{0\} ,
\]

\[
\text{ran}(\hat{P}_1^{(1)}) \cap \text{ran}(\hat{P}_3^{(6)}) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a \end{bmatrix} \right\} \cap \left\{ \begin{bmatrix} a \\ 0 \\ 0 \\ -a \end{bmatrix} \right\} = \{0\} ,
\]

\[
\text{ran}(\hat{P}_1^{(1)}) \cap \text{ran}(\hat{P}_4^{(6)}) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a \end{bmatrix} \right\} \cap \left\{ \begin{bmatrix} 0 \\ -a \\ a \\ 0 \end{bmatrix} \right\} = \{0\} .
\]

From these intersections it follows that \( v_{1(1)}(\hat{P}_i^{(6)}) = 1 \) and \( v_{1(1)}(\hat{P}_4^{(6)}) = 0 \). But this leads to another contradiction, namely, \( v_{1(1)}(\sum_{i=1}^{4} \hat{P}_i^{(6)}) = 1 = \sum_{i=1}^{4} v_{1(1)}(\hat{P}_i^{(6)}) = 3 \).
So, if the system is prepared in the pure state $|1^{(1)}\rangle$ in which $v_{|1^{(1)}\rangle}(\hat{P}_1^{(1)}) = 1$, both $v_{|1^{(1)}\rangle}(\hat{P}_{i\neq 4}^{(6)}) = 1$ and $v_{|1^{(1)}\rangle}(\hat{P}_{i\neq 4}^{(6)}) = 0$ lead to contradictions. Hence, $\hat{P}_{i\neq 4}^{(6)}$ must be value indefinite under $v_{|1^{(1)}\rangle}$, that is,

$$
|1^{(1)}\rangle \notin \{ \text{ran}(\hat{P}_{i\neq 4}^{(6)}) \text{ \text{ran}(\neg \hat{P}_{i\neq 4}^{(6)})} \iff v_{|1^{(1)}\rangle}(\hat{P}_{i\neq 4}^{(6)}) \notin \{0, 1\} \).
$$

(44)

4 Interpretation of the Kochen-Specker theorem

The failure of being a total function for the evaluation relation $v_{|1^{(1)}\rangle} : \{\hat{P}_i^{(6)}\} \to \{0, 1\}$ can be described by way of the truth-value gaps, namely,

$$
|\Omega\rangle \notin \{ \text{ran}(\hat{P}_\diamond) \text{ \text{ran}(\neg \hat{P}_\diamond)} \iff \{v_{|\Omega\rangle}(\hat{P}_\diamond)\} = \emptyset \).
$$

(45)

This expression means that in a state $|\Omega\rangle$ not residing in the column or null space of a projection operator $\hat{P}_\diamond$, a proposition $\diamond$ associated with $\hat{P}_\diamond$ has no truth-value at all, i.e., $\{\{\diamond\}|\Omega\rangle\} = \emptyset$. A semantics defined by set of these truth-value gaps in conjunction with the valuations (17) and (18) is gappy and yet two-valued. Accordingly, it can be called a supervaluationist semantics (for details of such semantics see [7, 8] and also [9, 10]).

This semantics is, in general, not truth-functional: Thus, according to (17), (18) and (45), in any admissible state of the system, the truth-value assignment function assigns the value of the truth to the sum of the projection operators $\hat{P}_\diamond$ in a maximal context $C$, even though there is a state $|\Omega\rangle$ where at least one of these projection operators has no truth-value:

$$
v_{|\Omega\rangle}(\hat{1}) = 1 \iff v_{|\Omega\rangle}(\sum_{\hat{P}_\diamond \in C} \hat{P}_\diamond) = 1 \iff \sum_{\hat{P}_\diamond \in C} v_{|\Omega\rangle}(\hat{P}_\diamond) = \emptyset \). \) \)

(46)

For example, the values $v_{|1^{(1)}\rangle}(\hat{P}_{i\neq 4}^{(6)})$ are nonexistent within the supervaluationist semantics, and so $\{\sum_{i=1}^{4} v_{|1^{(1)}\rangle}(\hat{P}_i^{(6)})\} = \emptyset$.

Alternatively, the failure of the principle of bivalence can be described using a multivalued semantics in which projection operators $\hat{P}$ may have more than two values, specifically,

$$
v_{C} : \{\hat{P}\} \to V_N \), \)

(47)

where $V_N$ denotes a set of truth-values whose cardinality is $N > 2$ and whose upper and lower bounds are 1 (that represents “true” or “absolutely true”) and 0 (that represents “false” or “absolutely false”), respectively.
To accomplish that, instead of the truth-value gaps (45) one can introduce the following valuation

$$\Omega \notin \{ \frac{\text{ran}(\hat{P} \cdot \Omega)}{\text{ran}(\neg \hat{P} \cdot \Omega)} \Leftrightarrow v|_{\Omega}(\hat{P}) = \langle \Omega | \hat{P}_{\cdot} \Omega \rangle \in \{ x \in \mathbb{R} | 0 < x < 1 \} \}, \quad (48)$$

where the function $v|_{\Omega}$ is determined by the probability $P[[\cdot]_{\cdot} = 1] = \langle \Omega | \hat{P}_{\cdot} \Omega \rangle$. As it is said in [11, 12] the value $v|_{\Omega}(\hat{P})$ represents the degree to which the projection operator $\hat{P} \in \mathcal{C}$ has the value 1 in the state $|\Omega\rangle$. Because $\langle \Omega | \hat{P}_{\cdot} \Omega \rangle \in [0, 1]$, a semantics defined by the set of the valuations (17), (18) and (48) is infinite-valued.

For example, in the infinite-valued semantics, the values $v|_{1(\cdot)}(\hat{P}_{\cdot}^{(6)})$ are defined by $v|_{1(\cdot)}(\hat{P}_{\cdot}^{(6)}) = v|_{1(\cdot)}(\hat{P}_{\cdot}^{(6)}) = \frac{1}{4}$ and $v|_{1(\cdot)}(\hat{P}_{\cdot}^{(6)}) = \frac{1}{2}$ inferring $\sum_{i=1}^{4} v|_{1(\cdot)}(\hat{P}_{\cdot}^{(6)}) = 1$. This shows that despite its value indefiniteness under the bivaluation, the context $\mathcal{C}^{(6)}$ is value definite under the infinite-valued assignment. (48).

Hence, only within the supervaluationist semantics, the negation of the sentence (a) (i.e., the principle of bivalence) can be interpreted as a sign that the verification of the “gappy” (i.e., having no truth-values) propositions must result in the ex nihilo creation of the bivalent values of these propositions.

While on the contrary, in the many-valued semantics, the failure of bivalence implies that non-classical (i.e., different from 1 and 0) truth-values must exist before the verification and they become bivalent as a result of the verification (thus, a bivalent semantic merely emerges at the end of the verification process). In such a sense, one may say that the measurements (verifications) produce the output that yield pre-existing elements of physical reality.

For that reason, the Kochen-Specker theorem (along with its localized variant) cannot justify the belief that quantum mechanics is indeterministic, that is, that there are no hidden variables (or not-yet-verified truth values of the propositions) determining somehow the outcome of a measurement (verification) in advance. This theorem only shows that if those hidden variables were to exist, they would have to comply with a logic which does not obey the principle of bivalence.

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