Comment on the Riemann Hypothesis

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Abstract

The Riemann hypothesis is identified with zeros of $\mathcal{N} = 4$ supersymmetric gauge theory four-point amplitude. The zeros of the $\zeta(s)$ function are identified with the complex dimension of the spacetime, or the dimension of the toroidal compactification. A sequence of dimensions are identified in order to map the zeros of the amplitude to the Riemann hypothesis.
1 Introduction

The century old Riemann hypothesis \cite{1} states that the only nontrivial zeros of the zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod (1 - p^{-s})^{-1}, \quad (1.1)$$

are on the set of points $s = \frac{1}{2} + it$. Tremendous numerical computations support this conjecture. The purpose of this article is to identify that under certain conditions imposed on the $\mathcal{N} = 4$ amplitude, the zeros of the Riemann zeta function are found in a formal sense with the zeros of these amplitudes. In precise terms, after an identification of the real parts of a sequence of derived dimensions, all gauge theory amplitudes vanish when the zeta function has zeros on the real axis $s = 1/2 + it$. (The Riemann zeta function on this axis has some similarities with the vanishing of the partition function of certain condensed matter theories as a function of couplings, i.e. Lee-Yang zeros.)

2 Review of the S-duality derivative expansion

The $\mathcal{N} = 4$ spontaneously broken theory is examined in this work. The Lagrangian is,

$$S = \frac{1}{\kappa^2} \text{Tr} \int d^4x [F^2 + \phi \Box \phi + \psi / D\psi + [\phi, \phi^2]]. \quad (2.1)$$

The quantum theory is believed to have a full S-duality, which means that the gauge amplitudes are invariant: under $A \rightarrow A_D$ and $\tau \rightarrow (a\tau + b)/(c\tau + d)$ the functional form of the amplitude is invariant. The series supports a tower of dyonic multiplets satisfying the mass formula $m^2 = 2|n^i a_i + m^i a_{d,i}|^2$ with $a_i$ and $a_{d,i}$ the vacuum values of the scalars and their duals; $a_{d,i} = \tau a_i$. The two couplings parameterizing the simplest SU(2)$\rightarrow$U(1) theory is,

$$\frac{\theta}{2\pi} + \frac{4\pi i}{g^2} = \tau = \tau_1 + i\tau_2, \quad (2.2)$$

taking values in the Teichmuller space of the keyhole region in the upper half plane, i.e. $|\tau| \geq 1/2$ and $|\tau_1| \leq 1/2$. The S-duality invariant scattering within the derivative
expansion is constructed in [2]. Derivative expansions in general are examined in [2]-[12].

The full amplitudes of $\mathcal{N} = 4$ theory may be constructed either in a gauge coupling perturbative series, i.e. the usual diagrammatic expansion formulated via unitarity methods, or as an expansion in derivatives, with the latter approach being nonperturbative in coupling. Both expansions are equivalent, found from a diagram by diagram basis.

The full set of operators to create a spontaneously broken $\mathcal{N} = 4$ gauge theory amplitude is found from

$$\mathcal{O} = \prod_{j=1}^{\mathcal{O}} \text{Tr} F_j^k ,$$

with possible $\ln^{m_1}(\Box) \ldots \ln^{m_n}$ (from the massless sector) and combinations with the covariant derivative; the derivatives are gauge covariantized and the tensor contractions are implied. The dimensionality of the operator is compensated by a factor of the vacuum expectation value, $\langle \phi^2 \rangle^m$. The generic tensor has been suppressed in the combination, and we did not include the fermions of scalars as in [2] because the gauge vertices are only required (the coefficients of course are found via the sewing, involving the integrations [2], [4]-[7]).

The generating function of the gauge theory $\mathcal{N} = 4$ four-point amplitude is given

$$S_4 = \sum \int d^4 x \ h_n(\tau, \bar{\tau}) \mathcal{O}_n ,$$

with the ring of functions spanning $h_n(\tau, \bar{\tau})$ consisting of the elements,

$$\prod E_{s_j}^{(q_j, -q_j)}(\tau, \bar{\tau}) ,$$

and their weights

$$\sum_j s_j = n/2 , \quad \sum_j q_j = 0 ,$$

with $s = m/2 + 1$, and $n$ the number of gauge bosons. The general covariant term in the effective theory has terms,
\[ \prod_{i=1}^{n_\partial} \nabla_{\mu_{s(i)}} \prod_{i=1}^{m_\phi^A} A_{\mu_{s(i)},a_{s(i)}} \prod_{j=1}^{n_\phi'} \phi_{a_{s(j)}} \prod_{j=1}^{m_\psi} \psi_{a_{s(j)}} , \]  

(2.7)

with the derivatives placed in various orderings (multiplying fields and products of combinations of fields; this is described in momentum space in [2]). The multiplying Eisenstein series possessing weights,

\[ s = n_A + n_\phi + n_\psi/2 + n_\partial/2 + 2 \quad q = n_\psi/2 . \]  

(2.8)

These terms span the general operator \( \mathcal{O} \) in the generating functional. The non-holomorphic weight \( q \) is correlated with the R-symmetry.

The perturbative coupling structure, for the gauge bosons as an example, has the form,

\[ g^{n-1}(g^2)^{n_{\text{max}}/2} \left[ \left( \frac{1}{g^2} \right)^{n_{\text{max}}/2}, \ldots, \left( \frac{1}{g^2} \right)^{-n_{\text{max}}/2+1} \right] . \]  

(2.9)

The factor in brackets agrees with the modular expansion of the Eisenstein series pertinent to the scattering amplitudes, and the prefactor may be absorbed by a field redefinition,

\[ A \to g^{-2}A \quad x \to gx , \]  

(2.10)

which maps the gauge field part of the Lagrangian into

\[ \int d^4x \frac{1}{g^2} \text{Tr} \left( \partial A + \frac{1}{g} A^2 \right)^2 . \]  

(2.11)

This field redefinition, together with the supersymmetric completion, agrees with the \( \mathcal{N} = 4 \) S-duality self-mapping in a manifest way (the factor in front may be removed by a Weyl rescaling).

Fermionic (and mixed) amplitudes would have a non-vanishing \( q_j \) sum. The Eisenstein functions have the representation

\[ E_{s_j}^{(q_j,-q_j)}(\tau, \bar{\tau}) = \sum_{(p,q) \neq (0,0)} \frac{\tau_2^{s-q}}{(p+q\tau)^{s-q}(p+q\bar{\tau})^{s+q}} , \]  

(2.12)
with an expansion containing two monomial terms and an infinite number of exponential (representing instanton) terms,

$$E_s(\tau, \bar{\tau}) = 2\zeta(2s)\tau_2^s + \sqrt{\pi} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \zeta(2s - 1)\tau_2^{1-2s} + O(e^{-2\pi\tau}) \ldots$$ \hspace{1cm} (2.13)

with a modification in the non-holomorphic counterpart, \(E_s^{(q,-q)}\), but with the same zeta function factors. The latter terms correspond to gauge theory instanton contributions to the amplitude; via S-duality all of the instantonic terms are available from the perturbative sector. (At \(s = 0\) or \(s = \frac{1}{2}\) the expansion is finite: \(\zeta(0) = -1\) and both \(\zeta(2s - 1)|_{s=0}\) have simple poles.) The \(n\)-point amplitudes, with the previously discussed modular weight, are

$$\langle A(k_1) \ldots A(k_n) \rangle = \sum_q h_q^{(n)}(\tau, \bar{\tau}) f_q(k_1, \ldots, k_n),$$ \hspace{1cm} (2.14)

where the modular factor is \(h\) (with the weights \(n_A/2+2\)) and the kinematic structure of the higher derivative term \(f_q\). The \(n_{\text{max}}\) follows from the modular expansion \(n_A/2 + n_\partial/2 + 2\), and corresponds to a maximum loop contribution of \(n_A + n_\partial + 1\).

We shall not review in detail the sewing relations that allow for a determination of the coefficients of the modular functions at the various derivative orders. This is discussed in detail in [4]-[7].

3 Rescaling of coupling

A rescaling of the coupling constant via \(g \rightarrow g^{1+\epsilon}\) changes the expansion in (2.9) to,

$$\langle A(k_1) \ldots A(k_n) \rangle = \sum_q h_q^{(n)}(\tau, \bar{\tau}) f_q(k_1, \ldots, k_n),$$ \hspace{1cm} (2.14)

where the modular factor is \(h\) (with the weights \(n_A/2+2\)) and the kinematic structure of the higher derivative term \(f_q\). The \(n_{\text{max}}\) follows from the modular expansion \(n_A/2 + n_\partial/2 + 2\), and corresponds to a maximum loop contribution of \(n_A + n_\partial + 1\).

The rescaling of the couplings into the metric and the gauge fields would naively generate a derivative expansion with modular functions labeled by \(E_s(1+\epsilon)\), and hence different coefficients for the expansion. These terms can always be supersymmetrized to obtain the remaining couplings involving the fermions and scalars. Within the loop expansion the zeta function takes values in accord with the dimension of the loop integrals, which suggests that the theory is in a different dimension from 4 to 4(1 + \(\epsilon/2\)); comparison with the loop expansion is required to determine this (note...
that the tree-level terms found from the first term in (2.9) are invariant after including the gauge field rescaling; this is true for the scattering after changing dimension).

Note that for $\epsilon = -1$ the entire scattering has no coupling dependence; gauge theory in $d = 2$ is topological, and the gauge field and coupling may be gauged away in a background without topology. The self-consistency via the sewing knocks out the coefficients of the covariant gauge field operators and one is left with the scalar interactions; the fermionic terms vanish as they only couple to the gauge field. The dimension changes as $4(1 + \epsilon/2)$, or rather to a dimension of $4(1 + (d - 4)/2) = 4(-1 + d/2) = -4 + 2d$.

In the altered theory the ring of functions consists of

$$\prod_{E_{s_i(1+\epsilon)}} E_{s_i(1+\epsilon)}(\tau, \bar{\tau}) \sum s_i = s,$$

with $s = n/2 + 1$, and $n$ being the number of external gauge bosons. The expansion at $\epsilon = -1$ has finite coefficients.

### 4 Amplitudes and zeros of the Riemann function

The arguments of the Riemann zeta function for a given derivative term of the gauge theory scattering amplitude are $2s$ and $2s - 1$. In terms of $s = (n+2)/2$ the arguments of the zeta function are

$$2(-2 + d)(n + 2) \quad \text{and} \quad 2(-2 + d)(n + 2) - 1.$$  (4.1)

If all of the real parts of the dimensions

$$d_R = \frac{1}{4(n + 2)} + 2, \quad d_R = \frac{3}{4(n + 2)} + 2$$  (4.2)

are identified then the arguments of the zeta functions are on the real $s = 1/2$ axis. These series have $d = 2$ as a limit point, with a maximum dimension of $2+1/8 = 2.125$. The gauge sector vanishes for $d = 2$, i.e. at the limit point.

If the amplitudes vanished via the identification on the $s = 1/m$ axis, then the real part of the dimension would be
\[ d_R = \frac{1}{2m(n + 2)} + 2, \quad d_R = \frac{3}{2m(n + 2)} + 2. \]  \hspace{1cm} (4.3)

Example dimensions pertaining to the Riemann hypothesis, \( m = 2 \) in (4.3), are

\[ d_R = 2 + \frac{1}{12} = \frac{25}{12}, \quad 2 + 1/16 = \frac{33}{16}, \quad 2 + 1/20 = \frac{41}{20}, \]  \hspace{1cm} (4.4)

\[ d_R = 2 + \frac{1}{4} = \frac{9}{4}, \quad 2 + \frac{3}{16} = \frac{35}{16}, 2 + \frac{3}{20} = \frac{43}{20}. \]  \hspace{1cm} (4.5)

The identification can be thought of as toroidal compactification with the dimensions identified, or as a series of identified four-manifolds.

## 5 Discussion

\( \mathcal{N} = 4 \) supersymmetric gauge theory amplitudes, including the nonperturbative corrections, are examined as a function of complex dimension. The zeros of the Riemann zeta function enforce the vanishing of the four-point gauge theory amplitudes. More precisely, the Riemann hypothesis is equivalent to the vanishing of the amplitudes of \( \mathcal{N} = 4 \) four-point functions when the theory is dimensionally reduced on identified tori of dimension \( d \), with \( d = id_I + d_R \),

\[ d_R = \frac{1}{2m(n + 2)} + 2 \quad \text{and} \quad d_R = \frac{3}{2m(n + 2)} + 2. \]  \hspace{1cm} (5.1)

The real parts of these dimensions range from 2 to 2.125, with \( d = 2 \) (\( d_I = 0 \)) special from the point of the triviality of the gauge field (pure gauge).
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