ON GENERALIZED SPHERICAL SURFACES IN EUCLIDEAN SPACES

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Abstract

In the present study we consider the generalized rotational surfaces in Euclidean spaces. Firstly, we consider generalized spherical curves in Euclidean $(n+1)$-space $\mathbb{E}^{n+1}$. Further, we introduce some kind of generalized spherical surfaces in Euclidean spaces $\mathbb{E}^3$ and $\mathbb{E}^4$ respectively. We have shown that the generalized spherical surfaces of first kind in $\mathbb{E}^4$ are known as rotational surfaces, and the second kind generalized spherical surfaces are known as meridian surfaces in $\mathbb{E}^4$. We have also calculated the Gaussian, normal and mean curvatures of these kind of surfaces. Finally, we give some examples.

1 Introduction

The Gaussian curvature and mean curvature of the surfaces in Euclidean spaces play an important role in differential geometry. Especially, surfaces with constant Gaussian curvature [18], and constant mean curvature conform nice classes of surfaces which are important for surface modelling [3]. Surfaces with constant negative curvature are known as pseudo-spherical surfaces [14].

Rotational surfaces in Euclidean spaces are also important subject of differential geometry. The rotational surfaces in $\mathbb{E}^3$ are called surface of revolution. Recently V. Velickovic classified all rotational surfaces in $\mathbb{E}^3$ with constant Gaussian curvature [17]. Rotational surfaces in $\mathbb{E}^4$ was first introduced by C. Moore in 1919. In the recent years some mathematicians have taken an interest in the rotational surfaces in $\mathbb{E}^4$, for example G. Ganchev and V. Milousheva [12], U. Dursun and N. C. Turgay [11], K. Arslan, at all. [1] and D.W.Yoon [19]. In [12], the authors applied invariance theory of surfaces in the four dimensional Euclidean space to the class of general rotational surfaces whose meridians lie in two-dimensional planes in order to find all minimal super-conformal surfaces. These surfaces were further studied in [11], which found all minimal surfaces by solving the differential equation that characterizes minimal surfaces. They then
determined all pseudo-umbilical general rotational surfaces in $\mathbb{E}^4$. K. Arslan et.al in [1] gave the necessary and sufficient conditions for generalized rotation surfaces to become pseudo-umbilical, they also shown that each general rotational surface is a Chen surface in $\mathbb{E}^4$ and gave some special classes of generalized rotational surfaces as examples. See also [9] and [2] rotational surfaces with Constant Gaussian Curvature in Four-Space. For higher dimensional case N.H. Kuiper defined rotational embedded submanifolds in Euclidean spaces [15].

The meridian surfaces in $\mathbb{E}^4$ was first introduced by G. Ganchev and V. Milousheva (See, [13] and [5]) which are the special kind of rotational surfaces. Basic source of examples of surfaces in 4-dimensional Euclidean or pseudo-Euclidean space are the standard rotational surfaces and the general rotational surfaces. Further, Ganchev and Milousheva defined another class of surfaces of rotational type which are one-parameter system of meridians of a rotational hypersurface. They constructed a family of surfaces with at normal connection lying on a standard rotational hypersurface in $\mathbb{E}^4$ as a meridian surfaces. The geometric construction of the meridian surfaces is different from the construction of the standard rotational surfaces with two dimensional axis in $\mathbb{E}^4$.

This paper is organized as follows: Section 2 gives some basic concepts of the surfaces in $\mathbb{E}^n$. Section 3 explains some geometric properties of spherical curves $\mathbb{E}^{n+1}$. Section 4 tells about the generalized spherical surfaces in $\mathbb{E}^{n+m}$. Further, this section provides some basic properties of generalized spherical surfaces in $\mathbb{E}^4$ and the structure of their curvatures. We also shown that every generalized spherical surfaces in $\mathbb{E}^4$ have constant Gaussian curvature $K = 1/c^2$. Finally, we present some examples of generalized spherical surfaces in $\mathbb{E}^4$.

2 Basic Concepts

Let $M$ be a smooth surface in $\mathbb{E}^n$ given with the patch $X(u, v) : (u, v) \in D \subset \mathbb{E}^2$. The tangent space to $M$ at an arbitrary point $p = X(u, v)$ of $M$ span \{X$_u$, X$_v$\}. In the chart $(u, v)$ the coefficients of the first fundamental form of $M$ are given by

$$g_{11} = \langle X_u, X_u \rangle, \ g_{12} = \langle X_u, X_v \rangle, \ g_{22} = \langle X_v, X_v \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. We assume that $W^2 = g_{11}g_{22} - g_{12}^2 \neq 0$, i.e. the surface patch $X(u, v)$ is regular. For each $p \in M$, consider the decomposition $T_p \mathbb{E}^n = T_p M \oplus T^\perp_p M$ where $T^\perp_p M$ is the orthogonal component of $T_p M$ in $\mathbb{E}^n$.

Let $\chi(M)$ and $\chi^\perp(M)$ be the space of the smooth vector fields tangent to $M$ and the space of the smooth vector fields normal to $M$, respectively. Given any local vector fields $X_1, X_2$ tangent to $M$, consider the second fundamental map $h : \chi(M) \times \chi(M) \to \chi^\perp(M)$;

$$h(X_i, X_j) = \bar{\nabla}X_i X_j - \nabla X_i X_j \quad 1 \leq i, j \leq 2$$

(2)

where $\nabla$ and $\bar{\nabla}$ are the induced connection of $M$ and the Riemannian connection of $\mathbb{E}^n$, respectively. This map is well-defined, symmetric and bilinear [6].
For any arbitrary orthonormal frame field \( \{N_1, N_2, ..., N_{n-2}\} \) of \( M \), recall the shape operator \( A : \chi^\perp(M) \times \chi(M) \to \chi(M) \):

\[
A_{N_k}X_j = -\langle \nabla_{X_j}N_k \rangle T, \quad X_j \in \chi(M).
\]  

(3)

This operator is bilinear, self-adjoint and satisfies the following equation:

\[
\langle A_{N_k}X_j, X_i \rangle = \langle h(X_i, X_j), N_k \rangle = L_{kij}, \quad 1 \leq i, j \leq 2, \quad 1 \leq k \leq n-2
\]  

(4)

where \( L_{kij} \) are the coefficients of the second fundamental form. The equation (2) is called Gaussian formula, and

\[
h(X_i, X_j) = \sum_{k=1}^{n-2} L_{ij}^k N_k, \quad 1 \leq i, j \leq 2
\]  

(5)

holds. Then the Gauss curvature \( K \) of a regular patch \( X(u,v) \) is given by

\[
K = \frac{1}{W^2} \sum_{k=1}^{n-2} (L_{11}^k L_{22}^k - (L_{12}^k)^2).
\]  

(6)

Further, the mean curvature vector of a regular patch \( X(u,v) \) is given by

\[
\vec{H} = \frac{1}{2W^2} \sum_{k=1}^{n-2} (L_{11}^k g_{22} + L_{22}^k g_{11} - 2L_{12}^k g_{12}) N_k.
\]  

(7)

We call the functions

\[
H_k = \frac{(L_{11}^k g_{22} + L_{22}^k g_{11} - 2L_{12}^k g_{12})}{2W^2},
\]  

(8)

the \( k \).th mean curvature functions of the given surface. The norm of the mean curvature vector \( H = \|\vec{H}\| \) is called the mean curvature of \( M \). Recall that a surface \( M \) is said to be \textit{flat} (resp. \textit{minimal}) if its Gauss curvature (resp. mean curvature vector) vanishes identically [7], [8].

The normal curvature \( K_N \) of \( M \) is defined by (see [10])

\[
K_N = \left\{ \frac{1}{n-2} \sum_{1=\alpha<\beta}^n \left\langle R^\perp(X_1, X_2)N_\alpha, N_\beta \right\rangle \right\}^{1/2}.
\]  

(9)

where

\[
R^\perp(X_i, X_j)N_\alpha = h(X_i, A_{N_\alpha} X_j) - h(X_j, A_{N_\alpha} X_i),
\]  

(10)

and

\[
\left\langle R^\perp(X_i, X_j)N_\alpha, N_\beta \right\rangle = \left\langle [A_{N_\alpha}, A_{N_\beta}]X_i, X_j \right\rangle,
\]  

(11)

is called the \textit{equation of Ricci}. We observe that the normal connection \( D \) of \( M \) is flat if and only if \( K_N = 0 \), and by a result of Cartan, this equivalent to the diagonalisability of all shape operators \( A_{N_\alpha} \) [6].
3 Generalized Spherical Curves

Let $\gamma$ be a regular oriented curve in $\mathbb{E}^{n+1}$ that does not lie in any subspace of $\mathbb{E}^{n+1}$. From each point of the curve $\gamma$ one can draw a segment of unit length along the normal line corresponding to the chosen orientation. The ends of these segments describe a new curve $\beta$. The curve $\gamma \in \mathbb{E}^{n+1}$ is called a generalized spherical curve if the curve $\beta$ lies in a certain subspace $\mathbb{E}^n$ of $\mathbb{E}^{n+1}$. The curve $\beta$ is called the trace of $\gamma$.

Let $\gamma(u) = (f_1(u), ..., f_{n+1}(u))$, be the radius vector of the curve $\gamma$ given with arclength parametrization $u$, i.e., $\|\gamma'(u)\| = 1$. The curve $\beta$ is defined by the radius vector

$$\beta(u) = (\gamma + c^2 \gamma''(u)) = ((f_1 + c^2 f_1''(u)), ..., (f_{n+1} + c^2 f_{n+1}'')(u)),$$

where $c$ is a real constant. If $\gamma$ is a generalized spherical curve of $\mathbb{E}^{n+1}$ then by definition the curve $\beta$ lies in the hyperplane $\mathbb{E}^n$ if and only if $f_{n+1} + c^2 f_{n+1}'' = 0$. Consequently, this equation has a non-trivial solution

$$f_{n+1}(u) = \lambda \cos \left( \frac{u}{c} + c_0 \right),$$

with some constants $\lambda$ and $c_0$. By a suitable choose of arclength we may assume that

$$f_{n+1}(u) = \lambda \cos \left( \frac{u}{c} \right),$$

with $\lambda > 0$. Thus, the radius vector of the generalized spherical curve $\gamma$ takes the form

$$\gamma(u) = (f_1(u), ..., f_n(u), \lambda \cos \left( \frac{u}{c} \right)).$$

Moreover, the condition for the arclength parameter $u$ implies that

$$(f_1')^2 + ... + (f_n')^2 = 1 - \frac{\lambda^2}{c^2} \sin^2 \left( \frac{u}{c} \right).$$

For convenience, we introduce a vector function

$$\phi(u) = (f_1(u), ..., f_n(u); 0).$$

Then the radius vector can be represented in the form

$$\gamma(u) = \phi(u) + \lambda \cos \left( \frac{u}{c} \right) e_{n+1},$$

where $e_{n+1} = (0, 0, ..., 0, 1)$. Consequently, the condition gives

$$\|\phi'(u)\|^2 = 1 - \frac{\lambda^2}{c^2} \sin^2 \left( \frac{u}{c} \right).$$

Hence, the radius vector of the trace curve $\beta$ becomes

$$\beta(u) = \phi(u) + c^2 \phi''(u).$$
Consider an arbitrary unit vector function
\[ a(u) = (a_1(u), ..., a_n(u); 0), \] (20)
in \( \mathbb{E}^{n+1} \) and use this function to construct a new vector function
\[ \phi(u) = \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \left( \frac{u}{c} \right)} a(u) du, \] (21)
whose last coordinate is equal to zero. Consequently, the vector function \( \phi(u) \) satisfies the condition (18) and generates a generalized spherical curve with radius vector (17).

**Example 1** The ordinary circular curve in \( \mathbb{E}^2 \) is given with the radius vector
\[ \gamma(u) = \left( \left( \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \left( \frac{u}{c} \right)} du \right), \lambda \cos \left( \frac{u}{c} \right) \right). \] (22)

**Example 2** Consider the unit vector \( a(u) = (\cos \alpha(u), \sin \alpha(u); 0) \) in \( \mathbb{E}^2 \). Then using (21), the corresponding generalized spherical curve in \( \mathbb{E}^3 \) is defined by the radius vector
\[ f_1(u) = \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \left( \frac{u}{c} \right)} \cos \alpha(u) du, \]
\[ f_2(u) = \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \left( \frac{u}{c} \right)} \sin \alpha(u) du, \]
\[ f_3(u) = \lambda \cos \left( \frac{u}{c} \right). \] (23)

**Example 3** Consider the unit vector
\[ a(u) = (\cos \alpha(u), \cos \alpha(u) \sin \alpha(u), \sin^2 \alpha(u); 0) \] in \( \mathbb{E}^3 \). Then using (21), the corresponding generalized spherical curve in \( \mathbb{E}^4 \) is defined by the radius vector
\[ f_1(u) = \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \left( \frac{u}{c} \right)} \cos \alpha(u) du, \]
\[ f_2(u) = \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \left( \frac{u}{c} \right)} \cos \alpha(u) \sin \alpha(u) du, \]
\[ f_3(u) = \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \left( \frac{u}{c} \right)} \sin^2 \alpha(u) du; \]
\[ f_4(u) = \lambda \cos \left( \frac{u}{c} \right). \] (24)
4 Generalized Spherical Surfaces

Consider the space $\mathbb{E}^{n+1} = \mathbb{E}^n \oplus \mathbb{E}^1$ as a subspace of $\mathbb{E}^{n+m} = \mathbb{E}^n \oplus \mathbb{E}^m$, $m \geq 2$ and Cartesian coordinates $x_1, x_2, \ldots, x_{n+m}$ and orthonormal basis $e_1, \ldots, e_{n+m}$ in $\mathbb{E}^{n+m}$. Let $M^2$ be a local surface given with the regular patch (radius vector) $X(u, v) = \phi(u) + \lambda \cos \left( \frac{u}{c} \right) \rho(v)$, \hspace{1cm} (25)

where the vector function $\phi(u) = (f_1(u), \ldots, f_n(u), 0, \ldots, 0)$, satisfies (18) and generates a generalized spherical curve with radius vector $\gamma(u) = \phi(u) + \lambda \cos \left( \frac{u}{c} \right) e_{n+1}$, \hspace{1cm} (26)

and the vector function $\rho(v) = (0, \ldots, 0, g_1(v), \ldots, g_m(v))$, satisfying the conditions $\| \rho(v) \| = 1$, $\| \rho'(v) \| = 1$, and specifies a curve $\rho = \rho(v)$ parametrized by a natural parameter on the unit sphere $S^{m-1} \subset \mathbb{E}^m$. Consequently, the surface $M^2$ is obtained as a result of the rotation of the generalized spherical curve $\gamma$ along the spherical curve $\rho$, which is called generalized Spherical surface in $\mathbb{E}^{n+m}$.

In the sequel, we will consider some type of generalized spherical surface;

**CASE I.** For $n = 1$ and $m = 2$, the radius vector \hspace{1cm} (25) satisfying the indicated properties describes the spherical surface in $\mathbb{E}^3$ with the radius vector $X(u, v) = (\phi(u), \lambda \cos \left( \frac{u}{c} \right) \cos v, \lambda \cos \left( \frac{u}{c} \right) \sin v)$, \hspace{1cm} (27)

where the function $\phi(u)$ is found from the relation $|\phi'(u)| = \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \left( \frac{u}{c} \right)}$. The surface given with the parametrization (27) is a kind of surface of revolution which is called ordinary sphere.

The tangent space is spanned by the vector fields

$X_u(u, v) = (\phi'(u), -\frac{\lambda}{c} \sin \left( \frac{u}{c} \right) \cos v, -\frac{\lambda}{c} \sin \left( \frac{u}{c} \right) \sin v)$,

$X_v(u, v) = (0, -\lambda \cos \left( \frac{u}{c} \right) \sin v, \lambda \cos \left( \frac{u}{c} \right) \cos(v))$.

Hence, the coefficients of the first fundamental form of the surface are

$g_{11} = \langle X_u(u, v), X_u(u, v) \rangle = 1$

$g_{12} = \langle X_u(u, v), X_v(u, v) \rangle = 0$

$g_{22} = \langle X_v(u, v), X_v(u, v) \rangle = \lambda^2 \cos^2 \left( \frac{u}{c} \right)$,

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{E}^3$. 
For a regular patch $X(u,v)$ the unit normal vector field or surface normal $N$ is defined by

$$N(u,v) = \frac{X_u \times X_v}{\|X_u \times X_v\|(u,v)} = \left( -\frac{\lambda}{c} \sin \left( \frac{u}{c} \right), -\phi'(u) \cos v, -\phi'(u) \sin v \right),$$

where

$$\|X_u \times X_v\| = \sqrt{g_{11}g_{22} - g_{12}^2} = \lambda \cos \left( \frac{u}{c} \right) \neq 0.$$

The second partial derivatives of $X(u,v)$ are expressed as follows

$$X_{uu}(u,v) = (\phi''(u), -\frac{\lambda}{c^2} \cos \left( \frac{u}{c} \right) \cos v, -\frac{\lambda}{c^2} \cos \left( \frac{u}{c} \right) \sin v),$$

$$X_{uv}(u,v) = (0, \frac{\lambda}{c} \sin \left( \frac{u}{c} \right) \sin v, -\frac{\lambda}{c} \sin \left( \frac{u}{c} \right) \cos v),$$

$$X_{vv}(u,v) = (0, -\lambda \cos \left( \frac{u}{c} \right) \cos v, -\lambda \cos \left( \frac{u}{c} \right) \sin v).$$

Similarly, the coefficients of the second fundamental form of the surface are

$$L_{11} = \langle X_{uu}(u,v), N(u,v) \rangle = -\kappa_1(u),$$

$$L_{12} = \langle X_{uv}(u,v), N(u,v) \rangle = 0,$$

$$L_{22} = \langle X_{vv}(u,v), N(u,v) \rangle = \phi'(u) \lambda \cos \left( \frac{u}{c} \right),$$

where

$$\kappa_1(u) = -\frac{\lambda}{c^2} \phi'(u) \cos \left( \frac{u}{c} \right) + \frac{\lambda}{c} \phi''(u) \sin \left( \frac{u}{c} \right),$$

is the differentiable function. Furthermore, substituting (28) into (6)-(7) we obtain the following result.

**Proposition 4** Let $M$ be a spherical surface in $\mathbb{E}^3$ given with the parametrization (27). Then the Gaussian and mean curvature of $M$ become

$$K = \frac{1}{c^2},$$

and

$$H = \frac{2\lambda^2 \cos^2 \left( \frac{u}{c} \right) - \frac{\lambda^2}{c^2} + 1}{2\lambda \cos \left( \frac{u}{c} \right) \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \left( \frac{u}{c} \right)}},$$

respectively.

**Corollary 5** Let $M$ be a spherical surface in $\mathbb{E}^3$ given with the parametrization (27). Then we have the following assertions

i) If $\lambda = c$ then the corresponding surface is a sphere with radius $c$ and centered at the origin,

ii) If $\lambda > c$ then the corresponding surface is a hyperbolic spherical surface,

iii) If $\lambda < c$ then the corresponding surface is an elliptic spherical surface.
are differentiable functions.

where these surfaces are the special type of rotational surfaces [12], see also [2].

where $N$ radius vector indicated properties describes the

where

The normal space is spanned by the vector fields

$$X(u, v) = (f_1(u), f_2(u), \lambda \cos \left(\frac{u}{c}\right) \cos v, \lambda \cos \left(\frac{u}{c}\right) \sin v), \quad (30)$$

where

$$f_1(u) = \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \left(\frac{u}{c}\right)} \cos \alpha(u) du,$$

$$f_2(u) = \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \left(\frac{u}{c}\right)} \sin \alpha(u) du. \quad (31)$$

are differentiable functions.

We call this surface the generalized spherical surface of first kind. Actually, these surfaces are the special type of rotational surfaces [12], see also [2].

The tangent space is spanned by the vector fields

$$X_u(u, v) = (f_1'(u), f_2'(u), -\frac{\lambda}{c} \sin \left(\frac{u}{c}\right) \cos v, -\frac{\lambda}{c} \sin \left(\frac{u}{c}\right) \sin v),$$

$$X_v(u, v) = (0, 0, -\lambda \cos \left(\frac{u}{c}\right) \sin v, \lambda \cos \left(\frac{u}{c}\right) \cos (v)).$$

Hence, the coefficients of the first fundamental form of the surface are

$$g_{11} = \langle X_u(u, v), X_u(u, v) \rangle = 1$$

$$g_{12} = \langle X_u(u, v), X_v(u, v) \rangle = 0$$

$$g_{22} = \langle X_v(u, v), X_v(u, v) \rangle = \lambda^2 \cos^2 \left(\frac{u}{c}\right),$$

where $\langle , \rangle$ is the standard scalar product in $\mathbb{E}^4$.

The second partial derivatives of $X(u, v)$ are expressed as follows

$$X_{uu}(u, v) = (f_1''(u), f_2''(u), -\frac{\lambda}{c^2} \cos \left(\frac{u}{c}\right) \cos v, -\frac{\lambda}{c^2} \cos \left(\frac{u}{c}\right) \sin v),$$

$$X_{uv}(u, v) = (0, 0, \frac{\lambda}{c} \sin \left(\frac{u}{c}\right) \sin v, -\frac{\lambda}{c} \sin \left(\frac{u}{c}\right) \cos (v)),$$

$$X_{vv}(u, v) = (0, 0, -\lambda \cos \left(\frac{u}{c}\right) \cos v, -\lambda \cos \left(\frac{u}{c}\right) \sin (v)).$$

The normal space is spanned by the vector fields

$$N_1 = \frac{1}{\kappa} (f_1''(u), f_2''(u), -\frac{\lambda}{c^2} \cos \left(\frac{u}{c}\right) \cos v, -\frac{\lambda}{c^2} \cos \left(\frac{u}{c}\right) \sin v)$$

$$N_2 = \frac{1}{\kappa} \left(\frac{-\lambda f_2''(u)}{c^2} \cos \left(\frac{u}{c}\right) \cos v, \frac{\lambda f_2''(u)}{c} \sin \left(\frac{u}{c}\right), \frac{\lambda \cos \left(\frac{u}{c}\right)}{c} \sin \left(\frac{u}{c}\right) + \frac{\lambda f_1'(u)}{c^2} \cos \left(\frac{u}{c}\right)ight),$$

where

$$(f_1'(u)f_2''(u) - f_1''(u)f_2'(u)) \cos v, (f_1'(u)f_2''(u) - f_1''(u)f_2'(u)) \sin v)$$

$$\kappa = \sqrt{(f_1'')^2 + (f_2'')^2 + \frac{\lambda^2}{c^2} \cos \left(\frac{u}{c}\right)}), \quad (32)$$

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is the curvature of the profile curve $\gamma$. Hence, the coefficients of the second fundamental form of the surface are

\[
L_{11}^1 = <X_{uu}(u,v), N_1(u,v)> = \kappa(u),
\]

\[
L_{12}^1 = <X_{uv}(u,v), N_1(u,v)> = 0,
\]

\[
L_{22}^1 = <X_{vv}(u,v), N_1(u,v)> = \frac{\lambda^2 \cos^2 \left( \frac{u}{c} \right)}{\kappa^2(u)},
\]

(33)

\[
L_{11}^2 = <X_u(u,v), N_2(u,v)> = 0,
\]

\[
L_{12}^2 = <X_{uv}(u,v), N_2(u,v)> = 0,
\]

\[
L_{22}^2 = <X_{vv}(u,v), N_2(u,v)> = -\frac{\lambda \cos \left( \frac{u}{c} \right) \kappa_1(u)}{\kappa(u)}.
\]

where

\[
\kappa_1(u) = f_1'(u)f_2''(u) - f_1''(u)f_2'(u),
\]

(34)

is the differentiable function.

Furthermore, by the use of (33) with (6)-(7) we obtain the following results.

**Proposition 6** The generalized spherical surface of first kind has constant Gaussian curvature $K = 1/c^2$.

**Proposition 7** Let $M$ be a generalized spherical surface of first kind given with the surface patch (30). Then the mean curvature vector of $M$ becomes

\[
\bar{H} = \frac{1}{2} \left\{ \left( \frac{\kappa^2c^2 + 1}{c^2K} \right) N_1 - \frac{\kappa_1}{\kappa \lambda \cos \left( \frac{u}{c} \right)} N_2 \right\}.
\]

(35)

where

\[
\kappa = \sqrt{(\varphi')^2 + \varphi^2 \left( (\alpha')^2 + \frac{1}{c^2} \right) + \frac{\lambda^2}{c^2} \left( 1 - \frac{c^2}{\lambda^2} \right)}, \quad \kappa_1 = \varphi^2 \alpha',
\]

(36)

and

\[
\varphi = \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \left( \frac{u}{c} \right)}.
\]

(37)

**Corollary 8** Let $M$ be a generalized spherical surface of first kind given with the surface patch (30). If the second mean curvature $H_2$ vanishes identically then the angle function $\alpha(u)$ is a real constant.

For any local surface $M \subset \mathbb{E}^4$ given with the regular surface patch $X(u,v)$ the normal curvature $K_N$ is given with the following result.

**Proposition 9** Let $M \subset \mathbb{E}^4$ be a local surface given with a regular patch $X(u,v)$ then the normal curvature $K_N$ of the surface becomes

\[
K_N = \frac{g_{11}(L_{12}^2L_{22}^2 - L_{12}^2L_{22}^1) - g_{12}(L_{11}^2L_{22}^2 - L_{12}^1L_{22}^1) + g_{22}(L_{11}^1L_{22}^2 - L_{12}^1L_{22}^1)}{W^3}.
\]

(38)
As a consequence of (33) with (35) we get the following result.

**Corollary 10** Any generalized spherical surface of first kind has flat normal connection, i.e., \( K_N = 0 \).

**Example 11** In 1966, T. Otsuki considered the following special cases

\[
\begin{align*}
a) \quad f_1(u) &= \frac{4}{3} \cos^3\left(\frac{u}{2}\right), \quad f_2(u) = \frac{4}{3} \sin^3\left(\frac{u}{2}\right), \quad f_3(u) = \sin u, \\
b) \quad f_1(u) &= \frac{1}{2} \sin^2 u \cos(2u), \quad f_2(u) = \frac{1}{2} \sin^2 u \sin(2u), \quad f_3(u) = \sin u.
\end{align*}
\]

For the case a) the surface is called Otsuki (non-round) sphere in \( \mathbb{E}^4 \) which does not lie in a 3-dimensional subspace of \( \mathbb{E}^4 \). It has been shown that these surfaces have constant Gaussian curvature \([16]\).

**CASE III.** For \( n = 1 \) and \( m = 3 \), the radius vector (25) satisfying the indicated properties describes the generalized spherical surface given with the radius vector

\[
X(u, v) = \phi(u)e_1^\perp + \lambda \cos\left(\frac{u}{c}\right)\rho(v),
\]

where

\[
\phi(u) = \int \sqrt{1 - \lambda^2 \frac{c^2}{c^2} \sin^2\left(\frac{u}{c}\right)} \, du,
\]

and \( \rho = \rho(v) \) parametrized by

\[
\rho(v) = (g_1(v), g_2(v), g_3(v)),
\]

\[
\|\rho(v)\| = 1, \|\rho'(v)\| = 1,
\]

which lies on the unit sphere \( S^2 \subset \mathbb{E}^4 \). The spherical curve \( \rho \) has the following Frenet Frames;

\[
\begin{align*}
\rho'(v) &= T(v) \\
T'(v) &= \kappa_\rho(v)N(v) - \rho(v) \\
N'(v) &= -\kappa_\rho(v)T(v).
\end{align*}
\]

We call this surface a generalized spherical surface of second kind. Actually, these surfaces are the special type of meridian surface defined in \([13]\), see also \([5]\).

**Proposition 12** Let \( M \) be a meridian surface in \( \mathbb{E}^4 \) given with the parametrization (39). Then \( M \) has the Gaussian curvature

\[
K = -\frac{\kappa_\gamma \phi'(u)}{\lambda \cos\left(\frac{u}{c}\right)},
\]

where

\[
\kappa_\gamma(u) = -\frac{\lambda}{c^2} \phi'(u) \cos\left(\frac{u}{c}\right) + \frac{\lambda}{c} \phi''(u) \sin\left(\frac{u}{c}\right).
\]

10
Proof 13 Let $M$ be a meridian surface in $\mathbb{E}^4$ defined by (39). Differentiating (39) with respect to $u$ and $v$ and we obtain

\begin{align*}
X_u &= \phi'(u) \vec{e}_1 - \frac{\lambda}{c} \sin \left( \frac{u}{c} \right) \rho(v), \\
X_v &= \lambda \cos \left( \frac{u}{c} \right) \rho'(v), \\
X_{uu} &= \phi''(u) \vec{e}_1 - \frac{\lambda}{c^2} \cos \left( \frac{u}{c} \right) \rho(v), \\
X_{uv} &= -\frac{\lambda}{c} \sin \left( \frac{u}{c} \right) \rho'(v), \\
X_{vv} &= \lambda \cos \left( \frac{u}{c} \right) \rho''(v).
\end{align*}

The normal space of $M$ is spanned by

\begin{align*}
N_1 &= N(v), \\
N_2 &= -\frac{\lambda}{c} \sin \left( \frac{u}{c} \right) \vec{e}_1 - \phi'(u) \rho(v),
\end{align*}

where $N(v)$ is the normal vector of the spherical curve $\rho$.

Hence, the coefficients of first and second fundamental forms are becomes

\begin{align*}
g_{11} &= \langle X_u(u, u), X_u(u, u) \rangle = 1, \\
g_{12} &= \langle X_u(u, v), X_v(u, v) \rangle = 0, \\
g_{22} &= \langle X_v(v, v), X_v(v, v) \rangle = \lambda^2 \cos^2 \left( \frac{u}{c} \right),
\end{align*}

and

\begin{align*}
L^1_{11} &= L^1_{12} = L^2_{12} = 0, \\
L^1_{22} &= \kappa_\rho(v) \lambda \cos \left( \frac{u}{c} \right), \\
L^2_{11} &= -\kappa_\gamma(u), \\
L^2_{11} &= \phi'(u) \lambda \cos \left( \frac{u}{c} \right).
\end{align*}

respectively, where

\[
\kappa_\gamma(u) = f'_1(u) f''_2(u) - f'_2(u) f''_1(u) = -\frac{\lambda}{c} \phi'(u) \cos \left( \frac{u}{c} \right) + \frac{\lambda}{c} \phi''(u) \sin \left( \frac{u}{c} \right).
\]

Consequently, substituting (44)-(45) into (6) we obtain the result.

As a consequence of (45) with (38) we get the following result.
Proposition 14 Any generalized spherical surface of second kind has flat normal connection, i.e., $K_N = 0$.

Corollary 15 Every generalized spherical surface of second kind is a meridian surface given with the parametrization

$$f_1(u) = \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \left( \frac{u}{c} \right)} \, du,$$

$$f_2(u) = \lambda \cos \left( \frac{u}{c} \right).$$

(46)

By the use of (40)-(41) with (46) we get the following result.

Corollary 16 The generalized spherical surface of second kind has constant Gaussian curvature $K = 1/c^2$.

As consequence of (7) we obtain the following result.

Proposition 17 Let $M$ be a generalized spherical surface of second kind given with the parametrization (39). Then the mean curvature vector of $M$ becomes

$$\vec{H} = \frac{1}{2f_2(u)} \left\{ \kappa_\rho(v) N_1 + (-\kappa_\gamma f_2(u) + f'_1(u)) N_2 \right\}.$$

(47)

where

$$\kappa_\rho(v) = \sqrt{g''_1(v)^2 + g''_2(v)^2 + g''_3(v)^2}.$$

Corollary 18 Let $M$ be a generalized spherical surface of second kind given with the parametrization (39). If

$$\kappa_\gamma(u) = \frac{f_1'(u)}{f_2(u)},$$

(48)

then $M$ has vanishing second mean curvature, i.e., $H_2 = 0$.

Example 19 Consider the curve $\rho(v) = (\cos v, \cos v \sin v, \sin^2 v)$ in $S^2 \subset \mathbb{E}^3$. The corresponding generalized spherical surface

$$x_1(u, v) = \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \left( \frac{u}{c} \right)} \, du$$

$$x_2(u, v) = \lambda \cos \left( \frac{u}{c} \right) \cos v$$

$$x_3(u, v) = \lambda \cos \left( \frac{u}{c} \right) \cos v \sin v$$

$$x_4(u, v) = \lambda \cos \left( \frac{u}{c} \right) \sin^2 v.$$

(49)

is of second kind.
References

[1] K. Arslan, B. Bayram, B. Bulca and G. Öztürk, *General rotation surfaces in $E^4$*, Results. Math., 2012, DOI 10.1007/s00025-011-0103-3.

[2] B. Bulca, K. Arslan, B.K. Bayram and G. Öztürk, *Spherical product surfaces in $E^4$*, An. St. Univ. Ovidius Constanta, 20(2012), 41-54.

[3] B. Bulca, K. Arslan, B.K. Bayram, G. Öztürk and H. Ugail, *Spherical product surfaces in $E^3$*, IEEE Computer Society, Int. Conference on CYBERWORLDS, 2009.

[4] B. Bulca, *$E^4$ deki Yüzeylerin Bir Karakterizasyonu*, PhD.Thesis, Bursa, 2012.

[5] K. Arslan, B. Bulca, and V. Milousheva, *Meridian Surfaces in $E^4$ with Pointwise 1-type Gauss map*. Bull. Korean Math. Soc., 51(2014), 911-922.

[6] B.Y., Chen, *Geometry of Submanifolds*, Dekker, New York, 1973.

[7] B. Y. Chen, *Pseudo-umbilical surfaces with constant Gauss curvature*, Proceedings of the Edinburgh Mathematical Society (Series 2), 18(2) (1972), 143-148.

[8] B.-Y. Chen, *Geometry of Submanifolds and its Applications*, Science University of Tokyo, 1981.

[9] D.V. Cuong, *Surfaces of Revolution with Constant Gaussian Curvature in Four-Space*, arXiv:1205.2143v3.

[10] DeSmet, P.J., Dillen, F., Verstrelen, L., Vrancken, L., *A pointwise inequality in submanifold theory*. Arch. Math.(Brno) 35(1999), 115-128.

[11] Dursun, U. and Turgay, N.C. *General rotational surfaces in Euclidean space $E^4$ with pointwise 1-type Gauss map*. Math. Commun., 17(2012), 71-81.

[12] G. Ganchev and V. Milousheva, *On the Theory of Surfaces in the Four-dimensional Euclidean Space*. Kodai Math. J. 31 (2008), 183-198.

[13] G. Ganchev and V. Milousheva, *Invariants and Bonnet-type theorem for surfaces in $R^4$*, Cent. Eur. J. Math., 8 (2010), no. 6, 993-1008.

[14] V. A. Gor’kavyi and E. N. Nevmerzhitskaya, *Two-dimensional Pseudo-spherical surfaces with degenerate bianchi transformation*, Ukrainian Mathematical Journal, Vol. 63(2012), No. 11, Translated from Ukrains’kyi Matematychnyi Zhurnal, Vol. 63(2011), No. 11, pp. 1460–1468.

[15] N.H. Kuiper, *Minimal Total Absolute Curvature for Immersions*. Invent. Math., 10(1970), 209-238.

[16] T. Otsuki, *Surfaces in the 4-dimensional Euclidean Space Isometric to a Sphere*, Kodai Math. Sem. Rep. 18(1966), 101-115.
[17] V. Velickovic, *On Surface of Rotation of a given Constant Gaussian Curvature and Their Visualization*, Preprint.

[18] Y.C. Wong, *Contributions to the theory of surfaces in 4-space of constant curvature*, Trans. Amer. Math. Soc, 59 (1946), 467-507.

[19] D.W. Yoon, *Some Properties of the Clifford Torus as Rotation Surfaces*, Indian J. Pure Appl. Math. 34(2003), 907-915.

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