Burchnall-Chaundy annihilating polynomials for commuting elements in Ore extension rings

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Abstract. In this article further progress is made in extending the Burchnall-Chaundy type determinant construction of annihilating polynomial for commuting elements to broader classes of rings and algebras by deducing an explicit general formula for the coefficients of the annihilating polynomial obtained by the Burchnall-Chaundy type determinant construction in Ore extension rings. It is also demonstrated how this formula can be used to compute the annihilating polynomials in several examples of commuting elements in Ore extensions. Also it is demonstrated that additional properties which may be possessed by the endomorphism, such as for example injectivity, may influence strongly the annihilating polynomial.

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1. Introduction

It is a classical result, going back to [1, 2, 3], that all pairs of commuting elements in the Heisenberg (Weyl) algebra are algebraically dependent over \(\mathbb{C}\). This result was later rediscovered and applied to the study of non-linear partial differential equations [9, 10, 14].

In 1994, one of the authors of the present paper, S. Silvestrov, based on consideration of the previous literature and a series of trial computations, made the following three part conjecture.

- The first part of the conjecture stated that the Burchnall–Chaundy type result on algebraic dependence of commuting elements can be proved in greater generality, that is for much more general classes of non-commutative algebras and rings than the Heisenberg algebra and related algebras of differential operators treated by Burchnall and Chaundy and in subsequent literature [1, 2, 3, 5, 9, 10, 14].
- The second part stated that the Burchnall–Chaundy eliminant construction of annihilating polynomials formulated in determinant (resultant) form works after some appropriate modifications for most or possibly all classes of algebras where the Burchnall–Chaundy type result on algebraic dependence of commuting elements can be proved.
- Finally, the third part of the conjecture stated that the proof of the vanishing of the corresponding determinant polynomial on the commuting elements can be performed in a purely algebraic way for all classes of algebras or rings where this fact is true, that is...
using only the internal structure and calculations with the elements in the corresponding algebras or rings and the algebraic combinatorial expansion formulas for the corresponding determinants, that is, without any need of passing to operator representations and use of analytic methods as in the Burchnall–Chaundy type proofs.

In the first and the second part of the conjecture more progress has been made. In [6], the key Burchnall–Chaundy type theorem on algebraic dependence of commuting elements in $q$-deformed Heisenberg algebras (and thus as a corollary for $q$-difference operators as operators representing $q$-deformed Heisenberg algebras) was obtained. The result and the methods have been extended to more general algebras and rings generalizing $q$-deformed Heisenberg algebras (generalized Weyl structures and graded rings) in [7]. The proof in [6] is totally different from the Burchnall–Chaundy type proof. It is an argument based only on the intrinsic properties of the elements and internal structure of $q$-deformed Heisenberg algebras, thus supporting the first part of the conjecture. It can be used successfully for an algorithmic implementation for computing the corresponding annihilating polynomial for given commuting elements. However, it does not give any specific information on the structure or properties of such polynomials or any general formulae. It is thus important to have a way of describing such annihilating polynomials by some explicit formulae, as for example those obtained using the Burchnall–Chaundy eliminant construction for the $q = 1$ case, i.e., for the classical Heisenberg algebra. In [11], a step in that direction was taken by offering a number of examples, all supporting the claim that the eliminant determinant method should work in the general case. However, no general proof for this was provided. The complete proof following the Burchnall–Chaundy approach in the case of $q$ not a root of unity has been recently obtained [8], by showing that the determinant eliminant construction, properly adjusted for the $q$-deformed Heisenberg algebras, gives annihilating polynomials for commuting elements in the $q$-deformed Heisenberg algebra when $q$ is not a root of unity, thus confirming the second part of the conjecture for these algebras. That proof was obtained by adapting the Burchnall–Chaundy eliminant determinant method of the case $q = 1$ of differential operators to the $q$-deformed case, after passing to a specific faithful representation of the $q$-deformed Heisenberg algebra on Laurent series and then performing a detailed analysis of the kernels of arbitrary operators in the image of this representation. While exploring the determinant eliminant construction of the annihilating polynomials, we also obtained some further information on such polynomials and some other results on dimensions and bases in the eigenspaces of the $q$-difference operators in the image of the chosen representation of the $q$-deformed Heisenberg algebra. In the case of $q$ being a root of unity the algebraic dependence of commuting elements holds only over the center of the $q$-deformed Heisenberg algebra [6], and it is still unknown how to modify the eliminant determinant construction to yield annihilating polynomials for this case.

This third part of the conjecture remains widely open for most classes of non-commutative algebras and rings, even in the case of the usual Heisenberg algebra and differential operators, and with only a series of examples calculated for the Heisenberg algebra, $q$-Heisenberg algebra and some more general algebras, all supporting the conjecture. An interesting partial progress in this direction has been made in the recent work by Daniel Larsson on extension of Burchnall–Chaundy theory to Ore extensions [12]. An algebraic proof has been given in this article in case of Ore extension algebras. However it uses some general properties of resultants in Ore extension rings in order to deduce that Burchnall–Chaundy type determinant polynomial for commuting elements is annihilating. This proof does not involve explicit combinatorics of computations in Ore extensions based on their defining commutation relations, and thus still does not reveal explicit formulas for annihilating polynomials and does not add to understanding of why it again works for such broader class of algebras defined by this more general class of commutation relations.

In this article we make further progress in this third part of the conjecture by deducing an
explicit general formula for the coefficients of annihilating polynomial obtained by the Burchnall–Chaundy type determinant construction in Ore extensions, and we also demonstrate how this formula can be used to compute the annihilating polynomials in several examples of commuting elements.

2. Extension of Burchnall–Chaundy theory to Ore extensions

In this section we describe an extension of Burchnall–Chaundy theory to Ore extensions developed by Daniel Larsson in [12].

Given a commutative ring $\mathbf{R}$ with an endomorphism $\sigma$ and another additive map $\delta$ that satisfies the $\sigma$-Leibniz’ rule:

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b,$$

we define a multiplication on $\mathbf{R}[x]$ by

$$xa = \sigma(a)x + \delta(a).$$

As usual we write multiplication as juxtaposition. This is known as an Ore polynomial ring or an Ore extension of $\mathbf{R}$. The multiplication in an Ore polynomial ring is associative and distributive, but typically not commutative. There is a normal form in $\mathbf{R}[x]$ where we can write all elements uniquely as $\sum_{j \in \mathbb{N}} a_j x^j$, where $\mathbb{N} = \{n \in \mathbb{Z} : n \geq 0\}$. We can also write explicitly how the multiplication operation acts in this normal form. For this purpose we introduce, following [12], the functions $\pi^n_i$ defined as the sum of all possible compositions of $i$ copies of $\sigma$ and $n-i$ copies of $\delta$. Thus for example $\pi^1_0(a) = \sigma(a)$ and $\pi^2_1 = \sigma(\delta(a)) + \delta(\sigma(a))$. Also it is convenient to define that $\pi^n_i(a) = 0$ if $i < 0$ or $i > n$ or $x = 0$. Using these functions we get the following expression for the multiplication

$$\left(\sum_{i=0}^{n} a_ix^i\right) \cdot \left(\sum_{j=0}^{m} b_j x^j\right) = \sum_{z=0}^{n+m} \sum_{i=0}^{n} \sum_{j=0}^{m} a_i \pi^n_{z-j}(b_j)x^{z}. $$

2.1. Determinant polynomial

Let $M$ be a $r \times c$ matrix with entries in $\mathbf{R}$ where $r \leq c$. Then we define the determinant polynomial of $M$, denoted by $|M|$, as follows

$$|M| = \sum_{i=0}^{c-r} \det(M_i)x^i$$

where $M_i$ is the $r \times r$ matrix whose first $r-1$ columns coincide with $M$ and whose last column equals the $(c-i)th$ column of $M$.

Larsson and Li show how to rewrite the determinant polynomial as a determinant of a $r \times r$ matrix, where the entries in the last column are elements of $\mathbf{R}[x]$ and all other entries lie in $\mathbf{R}$. The expansion of such a determinant is defined by the convention that the element of $\mathbf{R}[x]$ is placed on the right-hand side of the entries in $\mathbf{R}$, in the usual expansion that defines the determinant.

The following two useful Propositions easily follow from basic properties of the determinant.

**Proposition 1.** Let $M$ be a $r \times c$ matrix, $r \leq c$, with determinant polynomial $|M|$. Let $H_i$ denote the polynomial

$$m_{i,1}x^{c-1} + \ldots + m_{i,r}x^{c-r} + \ldots + \ldots + m_{i,c}.$$
Then

$$\left| M \right| = \det \begin{pmatrix} m_{1,1} & \ldots & m_{1,r-1} & H_1 \\ m_{2,1} & \ldots & m_{2,r-1} & H_2 \\ \vdots & \ddots & \ddots & \vdots \\ m_{r,1} & \ldots & m_{r,r-1} & H_r \end{pmatrix}$$

Now assume we have a sequence $A_1, A_2, \ldots, A_r$ of polynomials in $\mathbb{R}[x]$ and let $d$ be the maximum degree of the polynomials. We assume that $d \geq r$. We form a $r \times (d + 1)$ matrix, denoted $\text{mat}(A)$, whose entry in the $i$th row and $j$th column is the coefficient of $x^{d+1-j}$ in $A_i$. The determinant polynomial of $A$ is defined as $\left| \text{mat}(A) \right|$ and denoted $\left| A \right|$.

**Proposition 2.** Let $A_1, A_2, \ldots, A_r$ be a sequence of elements of $\mathbb{R}[x]$ of maximum degree $d$. Then

$$\left| A \right| = \det \begin{pmatrix} a_{1,d} & \ldots & a_{1,d-r+1} & A_1 \\ a_{2,d} & \ldots & a_{2,d-r+1} & A_2 \\ \vdots & \ddots & \ddots & \vdots \\ a_{r,d} & \ldots & a_{r,d-r+1} & A_r \end{pmatrix}$$

where $a_{i,j}$ is the coefficient of $x^j$ in $A_i$.

### 2.2. The resultant

Let $P$ and $Q$ be two elements of $\mathbb{R}[x]$, of degree $m$ and $n$ respectively. We define their resultant, $\text{Res}(P,Q)$, as the determinant polynomial of the sequence

$$P, xP, \ldots, x^{n-1}P, Q, xQ, \ldots, x^{m-1}Q.$$ 

It is easy to see that this will be a determinant of size $m+n$.

**Proposition 3.** For all $P$ and $Q$ in $\mathbb{R}[x]$ there exists element $S, T$ in $\mathbb{R}[x]$ such that

$$\text{Res}(P,Q) = SP + TQ.$$ 

**Proof.** By the definition and the previous proposition we know that for some $m_{i,j} \in \mathbb{R}$

$$\text{Res}(P,Q) = \det \begin{pmatrix} m_{n,1} & \ldots & m_{n,n+m-1} \\ m_{2,1} & \ldots & m_{2,n+m-1} \\ m_{1,1} & \ldots & m_{1,n+m-1} \\ m_{n+m} & \ldots & m_{n+m,n+m-1} \\ \vdots & \ddots & \ddots \\ m_{n+1} & \ldots & m_{n+1,n+m-1} \end{pmatrix} \begin{pmatrix} P \\ x^{n-2}P \\ x^{n-1}P \\ x^{n-1}Q \end{pmatrix}.$$ 

If we expand this using the definition of the determinant we see that the theorem is true. \[ \square \]

Now let $P$ and $Q$ be commuting elements, and let $s$ and $t$ be variables that commute with each other and everything in $\mathbb{R}[x]$. (So they can be seen as elements in the larger algebra $\mathbb{R}[x][s,t]$. Then $\text{Res}(P - s, Q - t) = S(P - s) + T(Q - t)$ for some $S, T$ in $\mathbb{R}[x][s,t]$. We note that $\text{Res}(P - s, Q - t)$ is a element of $\mathbb{R}[s,t]$, a fact that follows from the definition. It is also true that $\text{Res}(P - s, Q - t)$ will depend polynomially on $s$ and $t$, once again by the definition of the resultant. Finally if we formally replace $s$ by $P$ and $t$ by $Q$ the resultant becomes zero. Putting it all together we have proven

**Theorem 1.** If $P$ and $Q$ are commuting elements of $\mathbb{R}[x]$ then

$$f(s, t) = \text{Res}(P - s, Q - t)$$

is a polynomial in two commuting variables such that $f(P, Q) = 0$. 

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3. Recursive construction of the matrix of the resultant
We can also construct the matrix used for computing the resultant in a recursive way. For the first row we simply take the coefficients of \( P - s \). Let \( d_{i,j} \) denote the element in row \( i \) and column \( j \) of the resultant. If \( i \) or \( j \) is non-positive we agree to set \( d_{i,j} = 0 \). Further we set \( \sigma(s) = s \) and \( \delta(s) = 0 \). This means that \( xs = \sigma(s)x + \delta(s) \) in accordance with the \( \sigma \)-Leibniz rule. For \( 1 < i \leq n \) and \( n + 1 < i \leq n + m \) we then have the recursive formula

\[
d_{i,j} = \delta(d_{i-1,j}) + \sigma(d_{i-1,j-1}).
\]

This formula simply expresses the definition of the resultant and the \( \sigma \)-Leibniz rule.

3.1. The Heisenberg algebra case
We illustrate the result in the classical Heisenberg (Weyl) algebra case.
Assume that \( P \) and \( Q \) are commuting elements in the Heisenberg algebra, of degree \( m \) and \( n \) respectively. We change notation and write \( P = \sum a_j D^j \) and \( Q = \sum b_i D^i \), where the \( a_i \) and \( b_i \) are polynomials over \( \mathbb{C} \) in one variable, which we denote by \( y \).

We want to give a more explicit expression for the resultant of \( P \) and \( Q \). To this end we use Leibniz’ rule. In the context of Heisenberg algebras this takes the form

\[
D^n p = \sum_{k=0}^{n} \binom{n}{k} p^{(k)} D^{n-k}
\]

for any polynomial \( p \) in \( y \).

Thus

\[
D^e P = D^e \sum_{j=0}^{m} a_j D^j = \sum_{j=0}^{m} \left( \sum_{k=0}^{n} \binom{e}{k} a_j^{(k)} D^{e-k} \right) D^j = \sum_{j=0}^{m} \left( \sum_{l=0}^{n+m} \binom{e}{n+j-l} a_j^{(n+j-l)} D^l \right) = \sum_{j=0}^{m} \sum_{l=0}^{n} \binom{e}{n+j-l} a_j^{(n+j-l)} D^l.
\]

We can now write down the expression for \( \text{Res}(P,Q) \). At place \((e,f)\), when \( e \leq n \), we get

\[
\sum_{j=0}^{m} \left( e - 1 + j - n - m + f \right) a_j^{(e-1+j-n-m+f)}
\]

and similarly for the last \( m \) rows.

The expression for the resultant becomes

\[
\text{Res}(P,Q) = \begin{vmatrix}
a_{n+m} & \ldots & a_1 & P \\
a_{n+m} + a_{n+m-1} & \ldots & a_1 + a_0 & DP \\
\vdots & \ddots & \vdots & \vdots \\
a_{m} & \ldots & \sum_{j=0}^{n-1} a_j^{(n-j-2)} & D^{n-1}P \\
b_{n+m} & \ldots & b_1 & Q \\
b_{n+m} + b_{n+m-1} & \ldots & b_1 + b_0 & DQ \\
\vdots & \ddots & \vdots & \vdots \\
b_{n} & \ldots & \sum_{i=0}^{m-1} b_i^{(m+i-2)} & D^{m-1}Q
\end{vmatrix}
\]
As an example take $P = yD$ and $Q = y^2D^2$. Then
\[ \text{Res}(P - s, Q - t) = \begin{vmatrix} 0 & y & yD - s \\ y & (1 - s)D & yD^2 + (1 - s)D \\ y^2 & 0 & y^2D^2 - t \end{vmatrix}. \]

Expanding this we get
\[ \text{Res}(P - s, Q - t) = y^3(yD^2 + (1 - s)D) - y^2(y^2D^2 - t) - y^2(1 - s)(yD - s) = y^4D^2 + y^3D - sy^3D - y^4D^2 + y^2t + sy^2 - y^3D - y^2s^2 + sy^3D = (t + s - s^2)y^2. \]

We note that $Q + P - P^2 = 0$.

4. Necessity of injectivity

We give an example to show what can happen if we do not require $\sigma$ to be injective. Let $k$ be a field and set $R = k[y]$, the polynomials over $k$ in the variable $y$.

Set $\delta(f) = 0$ and $\sigma(f) = f(0)$, for all polynomials $f$. Then $\sigma$ is an endomorphism and $\delta$ is a $\sigma$-derivation. So $R[x]$ is well-defined.

Now set $P = yx^2$ and $Q = P$. Then $P$ and $Q$ commute. But the resultant we get is
\[ \text{Res}(P - s, Q - t) = \begin{vmatrix} 0 & y & -s \\ 0 & y & 0 \\ 0 & 0 & -t \end{vmatrix}. \]

This resultant is zero. We note that as long as the leading coefficient of both $P$ and $Q$ belong to the kernel of $\sigma$ the resultant will be zero.

**Theorem 2.** Let $P, Q$ be commuting elements in some Ore extension $R[x; \sigma, \delta]$ of degrees $m$ and $n$ respectively in $x$. Suppose the highest coefficients $a_m$ and $b_n$ both belong to the kernel of $\sigma$. Then $\text{Res}(P - s, Q - t) = 0$.

**Proof.** Form the matrix of the resultant according to the definition. Consider the first column of the matrix. The only potentially non-zero elements in it are the elements in row $n$ and $n + m$ where the elements $\sigma^{n-1}(a_m)$ respectively $\sigma^{m-1}(b_n)$ appear. But since $a_m$ and $b_n$ belong to the kernel of $\sigma$ these elements must also be zero and thus the determinant is zero. $\square$

5. The case $\delta = 0$

When we set $\delta(r) = 0$, for all $r \in R$ the formulae for the elements in the determinant simplify. We transpose the columns of the resultant to simplify the formulae further.

Let $D_{i,j}$ denote the element in place $i, j$ of $\text{Res}(P - s, Q - t)$, where $P = \sum_{i=0}^{m} a_i x^i$ has degree $m$ and $Q = \sum_{j=0}^{n} b_j x^j$ has degree $n$. Multiplying by $x$ repeatedly and using the commutation rule $xa = \sigma(a)x$ we find that if $i \leq n$ then
\[ D_{i,j} = \sigma^{i-1}(a_{j-i}) - \Delta_{i,j}s \]

where $\Delta$ is the Kronecker delta-function. If $i > n$ then
\[ D_{i,j} = \sigma^{i-n-1}(b_{j-i-n}) - \Delta_{i-n,j}t. \]

The formula is in some sense simpler than it looks. The first row simply has the same coefficients as $P - s$. Then you apply $\sigma$ to the coefficients and shift them one step to the left.
to get the second row. (Where we use the rule \( \sigma(s) = s \).) This gives us the first \( n \) rows. For the last \( m \) rows we do the same with \( Q - t \). We note that it is a generalization of the classical resultant for polynomials which we recover if we set \( \sigma \equiv 1 \).

We can expand the determinant along the last column to get

\[
\text{Res}(P - s, Q - t) = (-1)^{m+n+1} \left[ \begin{array}{cccc}
\sigma(a_{n+m-1}) & \ldots & \sigma(a_1) & \sigma(a_0) - s \\
\sigma^2(a_{n+m-2}) & \ldots & \sigma^2(a_0) - s & 0 \\
\vdots & \ddots & \vdots & \vdots \\
\sigma^{m-1}(b_n) & \ldots & \sigma^{m-1}(b_{n-1}) & 0 \\
\end{array} \right] (a_0 - s) + \\
+ (-1)^{m+2n+1} \left[ \begin{array}{cccc}
a_{n+m} & \ldots & a_2 & a_1 \\
\sigma(a_{n+m-1}) & \ldots & \sigma(a_1) & \sigma(a_0) - s \\
\vdots & \ddots & \vdots & \vdots \\
\sigma(b_{n+m-1}) & \ldots & \sigma(b_1) & \sigma(b_0) - t \\
\sigma^{m-1}(b_n) & \ldots & \sigma^{m-1}(b_{n-1}) & 0 \\
\end{array} \right] (b_0 - t).
\]

Now in every row there is only one element that contains \( s \) or \( t \). This helps us to determine the coefficient of \( s^n \) and \( t^m \) which are the highest possible powers of \( s \) and \( t \). Start with \( s \). In the preceding expression for \( \text{Res}(P - s, Q - t) \) it is easy to see that we need only expand the first determinant. Further, when we have expanded this determinant we see that there is only one relevant subdeterminant and so on. It help with the presentation of the result to make a new definition. Set \( p_k \) to be the coefficient of \( x^k \) in \( P - s \). (So \( p_k \) equals \( b_k \) plus possibly a \((-s)\)-term.) We set \( \sigma(b_k - s) = \sigma(b_k) - s \).

Doing this we get the coefficient of \( s^n \) as

\[
(-1)^{m+n+1} \left| \begin{array}{c}
0 \\
\vdots \\
\sigma^{m-1}(p_n) \\
\end{array} \right| p_n \\
\vdots \\
\sigma^{m-1}(p_{n-m}) \\
\right|.
\]

Similarly we get (with \( q_k \) the coefficient of \( x^k \) in \( Q - t \)) that the coefficient of \( t^m \) is

\[
(-1)^{m+2n+1} \left| \begin{array}{c}
0 \\
\vdots \\
\sigma^{n-1}(q_m) \\
\end{array} \right| q_m \\
\vdots \\
\sigma^{n-1}(q_{m-n}) \\
\right|.
\]

These matrices are of rather special form ("triangular with respect to the wrong diagonal") so we can compute them explicitly. The coefficient of \( s^n \) becomes \((-1)^{n+1} b_n \sigma(b_n) \cdot \ldots \cdot \sigma^{m-1}(b_n) \) and the coefficient of \( t^m \) becomes \((-1)^{m+n+1} a_m \sigma(a_m) \cdot \ldots \cdot \sigma^{n-1}(a_m) \).

5.1. The leading coefficients in general
We can extend our computation of the leading coefficients to the general case. So let \( P \) and \( Q \) be two Ore-polynomials of degree \( m \) and \( n \) respectively. If we form the determinant \( \text{Res}(P - s, Q - t) \) we notice that all elements containing \( s \) lie on the anti-diagonal. We also note that there are \( n \) such elements so the highest possible power of \( s \) that can occur is \( s^n \).

We have the following expression for the resultant

\[
\text{Res}(P - s, Q - t) = \sum_{\alpha} p(\alpha) \prod_{i=1}^{m+n} d_{i,\alpha(i)}
\]
where α runs over all the permutations of m + n elements, p(α) denotes the sign of the permutation and d_{i,j} denotes the element in place (i, j) of the resultant.

Now if i > n and i < m + n − j then d_{i,j} = 0. This helps us compute the coefficient of $s^n$. The only element of the sum given earlier that contains $s^n$ is the one that comes from multiplication of all the elements on the anti-diagonal. Thus up to some sign we get that $s^n$ has the coefficient $\prod_{k=0}^{m-1} \sigma^k(b_n)$, where the exponent denotes functional iteration. Similarly we get that the coefficient of $t^m$ is $\prod_{k=0}^{n-1} \sigma^k(a_m)$. As a corollary we obtain

**Theorem 3.** $\text{Res}(P - s, Q - t)$ is non-zero if $R$ is an integral domain and $\sigma$ is injective.

**Proof.** We have just shown that $s^n$ has the coefficient $\prod_{k=0}^{m-1} \sigma^k(b_n)$. $b_n$ is non-zero and if $\sigma$ is injective so must all the $\sigma^k(b_n)$ be. Finally, if $R$ is an integral domain the product of non-zero elements is non-zero.

The reason that the formulae for the leading coefficients remain the same in the general case as when $\delta = 0$ is that the highest term when you multiply a Ore-polynomial by $x$ from the left is determined by $\sigma$ alone.

6. The lower-order coefficients

One of our main goals in this article is to obtain formula for calculating the annihilating polynomial. In other words we want to derive a formula not only for the highest coefficient but also for the lower-order coefficients of $\text{Res}(P - s, Q - t)$. In other words we want to derive a formula not only for the highest coefficient but also for the lower-order coefficients of $\text{Res}(P - s, Q - t)$. To do this we start with formulating a formula for the k-derivative of a determinant in our general context of Ore extension rings. This formula has been obtained in the special case of determinants of ordinary enough times differentiable real valued functions in [4].

**Theorem 4.** Let $A$ be a square matrix of size m with entries in $R[t]$ for some commutative ring $R$. We denote the determinant of $A$ by $|A|$. Then

$$\frac{\partial^k |A|}{\partial t^k} = \sum_{k_1+k_2+\ldots+k_m=k} \frac{k!}{k_1! k_2! \ldots k_m!} \begin{vmatrix} a_{11}^{(k_1)} & a_{12}^{(k_1)} & \ldots & a_{1n}^{(k_1)} \\ a_{21}^{(k_2)} & a_{22}^{(k_2)} & \ldots & a_{2n}^{(k_2)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^{(k_m)} & a_{m2}^{(k_m)} & \ldots & a_{mn}^{(k_m)} \end{vmatrix}$$

with the sum taken over all combinations of non-negative integers $k_1, k_2 \ldots k_m$ that sum to $k$. Multiplication by a positive integer here denotes the obvious and usual way that $\mathbb{N}$ acts on $R[t]$.

**Proof.** We give a proof by induction on both $k$ and $m$. The formula is clearly true when $k = 0$ for any $m$. It is also easy to check that it is true when $k = 1$ and $m = 2$. So we assume it is true for $k = 1$ and $m$ and try to prove it is true for $m + 1$ and $k = 1$. Let $A$ be any $m + 1$ sized square matrix. Then

$$\frac{\partial |A|}{\partial t} = \frac{\partial}{\partial t} \left( \sum_{i=1}^{m+1} (-1)^{i+1} a_{i1} |A_{1i}| \right) = \sum_{i=1}^{m+1} a_{i1}' |A_{1i}| + \sum_{i=1}^{m+1} a_{i1} |A_{1i}'|$$

$$= \sum_{i=1}^{m+1} \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1,m+1} \\ a_{21} & a_{22} & \ldots & a_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{m,m+1} \end{vmatrix}$$
This proves that the claim is valid for \( k = 1 \) and \( m \) arbitrary. So we now assume the theorem is true for \( k \) and try to prove it for \( k + 1 \). If \( A \) is a square matrix of size \( m \), then

\[
\frac{\partial^{k+1}|A|}{\partial t^{k+1}} = \frac{\partial}{\partial t} \left( \frac{\partial^k|A|}{\partial t^k} \right) = \frac{\partial}{\partial t} \sum_{k_1+k_2+\ldots+k_m=k} \frac{k!}{k_1!k_2!\ldots k_m!} \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,m} \end{vmatrix}
\]

\[
= \sum_{k_1+k_2+\ldots+k_m=k} \frac{(k+1)!}{k_1!k_2!\ldots k_m!} \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,m} \end{vmatrix}
\]

where we have used a well-known property of multinomial coefficients.

We now apply Proposition 4 to get a formula for the lower order coefficients of \( \text{Res}(P-s, Q-t) \). We note that any time we differentiate a row of the determinant twice we end up with zero and that only the last \( m \) rows contains any \( t \)-terms. Set \( A = \text{Res}(P-s, Q-t) \) and let \( a_i \) denote the vector consisting of the elements in row \( i \) of \( A \).

We find that

\[
\frac{\partial^k|A|}{\partial t^k} = \sum_{k_1+k_2+\ldots+k_m=k, 0 \leq k_i \leq 1} k! \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,m} \end{vmatrix}
\]

We differentiate this expression with respect to \( s \), and get for non-negative integers \( k, q \),

\[
\frac{\partial^{k+q}|A|}{\partial s^q \partial t^k} = \sum_{k_1+k_2+\ldots+k_m=k, q_1+q_2+\ldots+q_m=q, 0 \leq k_i \leq 1, 0 \leq q_i \leq 1} k!q! \begin{vmatrix} \frac{\partial^{k+1}a_{1,1}}{\partial s} & \frac{\partial^{k+1}a_{1,2}}{\partial s} & \cdots & \frac{\partial^{k+1}a_{1,m}}{\partial s} \\ \frac{\partial^{k+2}a_{2,1}}{\partial s} & \frac{\partial^{k+2}a_{2,2}}{\partial s} & \cdots & \frac{\partial^{k+2}a_{2,m}}{\partial s} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^{k+m}a_{m,1}}{\partial s} & \frac{\partial^{k+m}a_{m,2}}{\partial s} & \cdots & \frac{\partial^{k+m}a_{m,m}}{\partial s} \end{vmatrix}
\]

\[\square\]
This gives almost immediately an expression for the coefficient of \( s^q t^k \). (At least if \( \mathbb{R} \) torsion-free, seen as a module over \( \mathbb{Z} \).)

The coefficient in front of \( s^q t^k \) is the same as the derivative of \( \text{Res}(P - s, Q - t) \) with respect to \( s \) and \( t \) taken respectively \( q \) and \( k \) times and then evaluated at \( t = 0 \) and \( s = 0 \). Thus we get the following theorem describing the formula for all coefficients of \( \text{Res}(P - s, Q - t) \).

**Theorem 5.** Let \( \mathbb{R} \) be torsion-free as a module over \( \mathbb{Z} \) with the natural action of \( \mathbb{Z} \). Let \( P \) and \( Q \) be two commuting elements of \( \mathbb{R}[x] \). Denote the coefficient of \( s^q t^k \) in \( \text{Res}(P - s, Q - t) \) by \( c_{q,k} \). Denote row number \( i \) in the resultant matrix by \( a_i \).

Then

\[
c_{q,k} = \sum_{k_1+k_2+\ldots+k_m = k \atop q_1+q_2+\ldots+q_m = q \atop 0 \leq k_1 \leq 1 \atop 0 \leq q_1 \leq 1} \left| \begin{array}{ccc} \frac{\partial^q a_1}{\partial s} |_{s=0,t=0} & \cdots & \frac{\partial^q a_m}{\partial s} |_{s=0,t=0} \\ \vdots & \ddots & \vdots \\ \frac{\partial^{q+1} a_1}{\partial s} |_{s=0,t=0} & \cdots & \frac{\partial^{q+1} a_m}{\partial s} |_{s=0,t=0} \\ \frac{\partial^{q+2} a_1}{\partial t} |_{s=0,t=0} & \cdots & \frac{\partial^{q+2} a_m}{\partial t} |_{s=0,t=0} \\ \vdots & \ddots & \vdots \\ \frac{\partial^{q+m} a_1}{\partial t} |_{s=0,t=0} & \cdots & \frac{\partial^{q+m} a_m}{\partial t} |_{s=0,t=0} \\ \end{array} \right|.
\]

We will illustrate now the preceding theorem with several examples.

**Example 1.** We choose the Heisenberg algebra setting we considered earlier for our example. So we set \( \mathbb{R} = \mathbb{C}[y] \) and denote the Ore extension by \( \mathbb{R}[D] \).

Set \( P = yD \) and \( Q = y^2 D^2 \). We have already computed that \( \text{Res}(P - s, Q - t) = (t + s - s^2)y^2 \).

We want to check that we get the same result using the theorem. We compute

\[
c_{0,0} = \begin{vmatrix} 0 & y & 0 \\ y & 1 & 0 \\ y^2 & 0 & 0 \end{vmatrix} = 0, \quad c_{0,1} = \begin{vmatrix} 0 & y & 0 \\ y & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = y^2,
\]

\[
c_{1,0} = \begin{vmatrix} 0 & 0 & -1 \\ y & 1 & 0 \\ y^2 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & y & 0 \\ y^2 & 0 & 0 \end{vmatrix} = y^2,
\]

\[
c_{1,1} = \begin{vmatrix} 0 & 0 & -1 \\ y & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} + \begin{vmatrix} 0 & y & 0 \\ 0 & 0 & -1 \end{vmatrix} = 0,
\]

\[
c_{2,0} = \begin{vmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ y^2 & 0 & 0 \end{vmatrix} = -y^2.
\]

Thus we get the same result as before.

**Example 2.** We now give an example using the quantum plane. This can be defined by setting \( \mathbb{R} = \mathbb{C}[y] \), setting \( \sigma(y) = qy \) for some constant \( q \) and \( \sigma(a) = a \) for all \( a \in \mathbb{C} \) and finally defining \( \delta \equiv 0 \). This defines an Ore extension that is known as the quantum plane.

In this algebra \( P = xy \) and \( Q = (yx)^2 = qy^2x^2 \) commute. We compute \( \text{Res}(P - s, Q - t) \) in two ways. Using the definition we get

\[
\text{Res}(P - s, Q - t) = \begin{vmatrix} 0 & y & -s \\ qy & -s & 0 \\ qy^2 & 0 & -t \end{vmatrix} = (t - s^2)qy^2.
\]
We now compute the same thing using Theorem 5:

\[
\begin{align*}
&c_{0,0} = \begin{vmatrix} 0 & y & 0 \\ qy & 0 & 0 \\ qy^2 & 0 & 0 \end{vmatrix} = 0, \\
&c_{0,1} = \begin{vmatrix} 0 & y & 0 \\ qy & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} = qy^2, \\
&c_{1,0} = \begin{vmatrix} 0 & 0 & -1 \\ qy & 0 & 0 \\ qy^2 & 0 & 0 \end{vmatrix} = 0, \\
&c_{1,1} = \begin{vmatrix} 0 & 0 & -1 \\ qy & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 0, \\
&c_{2,0} = \begin{vmatrix} 0 & 0 & -1 \\ qy^2 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = -qy^2, \\
&c_{2,1} = \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \end{vmatrix} = 0.
\end{align*}
\]

**Example 3.** We now describe a larger example, using the \(q\)-Heisenberg algebra. We generate this as an Ore extension by setting \(R = \mathbb{C}[y]\), \(\sigma(y) = qy\) and \(\delta(y) = 1\) for some \(q \in \mathbb{C}\).

Set \(P = Q = (yx)^2 = qy^2x^2 + xy\). Then

\[
\text{Res}(P - s, Q - t) = \begin{vmatrix} 0 & qy^2 & y & -s \\ q^3y^2 & (2q + q^2)y & 1 & 0 \\ q^3y^2 & y & 0 & 0 \\ q^3y^2 & (2q + q^2)y & 1 & 0 \end{vmatrix}.
\]

On computing the determinant we find that

\[\text{Res}(P - s, Q - t) = q^4y^4t^2 - 2q^4y^4st + q^4y^4s^2.\]

We now wish to compute the resultant using our formula. With the same notation as in the previous examples we get

\[
\begin{align*}
&c_{0,0} = \begin{vmatrix} 0 & qy^2 & y & 0 \\ q^3y^2 & (2q + q^2)y & 1 & 0 \\ q^3y^2 & qy^2 & 0 & 0 \\ q^3y^2 & (2q + q^2)y & 1 & 0 \end{vmatrix} = 0, \\
&c_{0,1} = \begin{vmatrix} 0 & qy^2 & y & 0 \\ q^3y^2 & (2q + q^2)y & 1 & 0 \\ q^3y^2 & 0 & 0 & -1 \\ q^3y^2 & (2q + q^2)y & 1 & 0 \end{vmatrix} = 0 + 0 = 0, \\
&c_{1,0} = \begin{vmatrix} 0 & 0 & -1 \\ q^3y^2 & (2q + q^2)y & 1 & 0 \\ 0 & qy^2 & 0 & 0 \\ 0 & q^3y^2 & (2q + q^2)y & 1 \end{vmatrix} = 0 + 0 = 0, \\
&c_{1,1} = \begin{vmatrix} 0 & 0 & -1 \\ q^3y^2 & (2q + q^2)y & 1 & 0 \\ 0 & 0 & 0 & -1 \\ q^3y^2 & (2q + q^2)y & 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & -1 \\ q^3y^2 & (2q + q^2)y & 1 & 0 \\ 0 & 0 & 0 & -1 \\ q^3y^2 & (2q + q^2)y & 1 & 0 \end{vmatrix} = 0 - q^4y^4 - q^4y^4 + 0 = -2q^4y^4.
\end{align*}
\]
Example 4. We give a second example in the $q$-Heisenberg algebra. Take $P = (y^2 x)^2 + y^2 x$ and $Q = y^2 x$. Then $P = q^2 y^4 x^2 + (y^3 + q y^3 + y^2) x$ and

$$\text{Res}(P - s, Q - t) = \begin{vmatrix} y^4 & q y^3 + y^3 + y^2 & -s \\ 0 & q y^3 + y^3 + y^2 & -t \\ q^2 y^2 & q y + y - t & 0 \end{vmatrix} = -q^2 y^4 t^2 - q^2 y^4 t + q^2 y^4 s.$$

As expected we get the same result using both calculations.
7. Annihilating polynomials for elements in a specific commutative subalgebra

Suppose that \( R = S[y] \) for some commutative ring \( S \), by which we mean simply the ordinary polynomial ring over \( S \). In this case it is natural to suppose that \( \sigma \) and \( \delta \) are \( S \)-linear functions. The important case of the Heisenberg algebra is one such example as are the more general \( q \)-deformed Heisenberg algebras. Under the preceding assumptions we can specify \( \sigma \) and \( \delta \) completely by just giving \( \sigma(y) \), respectively \( \delta(y) \).

The elements of the form \((yx)^n\), for natural numbers \( n \), generate a subalgebra of \( S[y][x; \sigma, \delta] \). It will be a commutative sub-algebra, as is easy to see. If we assume that \( P \) and \( Q \) come from this subalgebra we can give a formula for the elements in the determinant, using the function \( \pi_i^m \) introduced in section 2. We first have that

\[
x^n(yx)^m = \sum_{i_1=1}^{n+m} \sum_{i_2=1}^{n+m-1} \cdots \sum_{i_m=1}^{n+1} \pi^{n}_{i_m-1}(y) \pi^{i_m}_{i_{m-1}-1}(y) \cdots \pi^{i_2}_{i_1}(y) x^1
\]

which is proven by induction. If \( P = \sum_{i=0}^{m} a_i(yx)^m \) and \( Q = \sum_{j=0}^{n} b_j(yx)^n \) we can from this get an expression for the elements of the matrix whose determinat is \( \text{Res}(P - s, Q - t) \). Setting \( e_{i,j} \) to be the element of row \( i \) and column \( n + m - j \) and letting \( d_{i,j} \) denote the Kronecker delta-function we have that

\[
d_{i,j} = e_{i,j} + e_{i,j} = \sum_{k=0}^{m} a_k \sum_{i_2=1}^{i_1-2} \cdots \sum_{i_k=1}^{i_{k-1}-1} \pi^{i_1-1}_{i_{k-1}-1}(y) \cdots \pi^{i_2}_{i_1}(y)
\]

if \( j \leq n \) and similarly if \( j > n \)

\[
d_{i,j+n} = e_{i,j} + e_{i,j} = \sum_{k=0}^{n} b_k \sum_{i_2=1}^{i_1-2} \cdots \sum_{i_k=1}^{i_{k-1}-1} \pi^{i_1-1}_{i_{k-1}-1}(y) \cdots \pi^{i_2}_{i_1}(y)
\]

Assume now that \( R = k[y] \) for some field \( k \), and that \( \delta \equiv 0 \).

Set \( P = \sum_{i=0}^{m} a_i(yx)^m \) and \( Q = \sum_{j=0}^{n} b_j(yx)^n \). In this special case we can prove the following

**Theorem 6.** Let \( R = k[y] \) for some field \( k \). Let \( \sigma \) be any endomorphism and assume \( \delta \) is identically zero. If \( P \sum_{i=0}^{m} a_i(yx)^m \) and \( Q = \sum_{j=0}^{n} b_j(yx)^n \) then \( P \) and \( Q \) commute and

\[
\text{Res}(P - s, Q - t) = G(s, t) \prod_{j=0}^{n-1} \prod_{i=0}^{m-1} \sigma^j(y) \text{ where } G(s, t) \text{ does not contain any non-zero power of } y.
\]

**Proof.** Denote the element in row \( i \) and column \( n + m - j + 1 \) by \( r_{i,j} \). Then it can be seen that

\[
r_{i,j} = C_{i,j} \prod_{k=1}^{j-1} (r_{i+k-2}(y))
\]

if \( i \leq n \), where \( C_{i,j} \) does not depend on \( y \) and we interpret the empty product as 1. If \( i > n \) we get that

\[
r_{i,j} = C_{i,j} \prod_{k=1}^{j+n-1} \sigma^{i+k-2-n}(y).
\]

If \( i \leq n \) and \( i > j \) we must have that \( C_{i,j} = 0 \). Similarly if \( i > n \) and \( i - n > j \) we have that \( C_{i,j} = 0 \).

We know that

\[
\text{Res}(P - s, Q - t) = \sum_{\alpha \in S_{m+n}} \text{sign}(\alpha) \prod_{i=1}^{m+n} r_{i,\alpha(n+m-i+1)}
\]
where the sum is over all possible permutations of \(n + m\) elements. We let \(\gamma\) denote the permutation that maps \(n + m - j + 1\) to \(j\) for all \(1 \leq j \leq n + m\). Then we can write

\[
\text{Res}(P - s, Q - t) = \sum_{\alpha \in S_{n+m}} \text{sign}(\alpha \circ \gamma) \prod_{i=1}^{m+n} r_{i,\alpha(i)}(\gamma(n+m-i+1)) = \sum_{\alpha \in S_{n+m}} \text{sign}(\alpha \circ \gamma) \prod_{i=1}^{m+n} r_{i,\alpha(i)}.
\]

Now, we compute one term of this expansion, ignoring the sign:

\[
\prod_{i=1}^{n+m} r_{i,\alpha(i)} = \prod_{i=1}^{n} C_{i,\alpha(i)} \left( \prod_{k=1}^{\alpha(i)-i} \sigma^{k+i-2}(y) \right)^{-1} \prod_{i=n+1}^{n+m} C_{i,\alpha(i)} \left( \prod_{k=1}^{\alpha(i)+n-i} \sigma^{i+k-2-n} \right) = \prod_{i=1}^{n+m} C_{i,\alpha(i)} \left( \prod_{i=n+1}^{n+m} \sigma^{i+k-2-n} \right).
\]

We count the number of times a factor \(\sigma^p\) appears in this expression. It can be seen that for any natural number \(p\), \(\sigma^p(y)\) occurs \(S = |A_1 \cap C_1| + |A_2 \cap C_1|\) number of times in the factorization, where

\[
A_1 = \{i\mid 1 \leq i \leq n \text{ and } i - 1 \leq p\},
\]

\[
A_2 = \{i\mid n + 1 \leq i \leq n + m \text{ and } i \leq p + n + 1\},
\]

\[
C_1 = \{i\mid \alpha(i) \geq p + 2\}
\]

and \(|F|\) denotes the number of elements of set \(F\). We are not interested in those \(\alpha\) that make any \(C_{i,\alpha(i)}\) zero since the claim is trivially true for them. We will use this fact. So we assume that \(\alpha(i) \geq i\) if \(i \leq n\) and \(\alpha(i) \geq i - n\) otherwise.

Define

\[
B_1 = \{i\mid 1 \leq i \leq n \text{ and } p \leq i - 2\},
\]

\[
B_2 = \{i\mid n + 1 \leq i \leq n + m \text{ and } p + n + 2 \leq i\},
\]

\[
C_2 = \{i\mid \alpha(i) \leq p + 1\}
\]

We can see that the \(A_i\) and \(B_i\) form one partition of the set \(\{1, \ldots, n + m\}\) and the \(C_i\) another. We further note that if \(i \in C_2 \cap B_1\) we must have that \(i > \alpha(i)\) and \(i \leq n\) which we have assumed can not happen. Similarly \(C_2 \cap B_2\) is empty. We use this to rewrite \(S\) as

\[
S = |A_1 \cap C_1| + |A_2 \cap C_1| - |B_1 \cap C_2| - |B_2 \cap C_2|.
\]

Setting \(A = A_1 \cup A_2\) and \(B = B_1 \cup B_2\) we then compute \(S\) as follows

\[
S = |A \cap C_1| - |B \cap C_2| = |A \cap C_1| - |A^c \cap C_1^c| = |A \cap C_1| - |(A \cup C_1)^c| = |A \cap C_1| - (m + n - |A \cup C_1|) = |A \cap C_1| + |A \cup C_1| - m - n = |A \cap C_1| + |A| + |C_1| - |A \cap C_1| - m - n = |A| + |C_1| - m - n
\]

where we have used the principle of inclusion-exclusion in the last step. This result is independent of the permutation \(\alpha\) which almost proves the proposition. Simply see that if we multiply all the elements along the anti-diagonal we get the claimed formula to finish the proof.
As a special case of the theorem we get the following

**Corollary 1.** Assume that $R = k[y]$. If $P, Q$ are linear combinations of elements of the form $(yx)^i$, where $P$ is of degree $m$ and $Q$ of degree $n$ in $x$, $\delta$ is identically zero and $\sigma(y) = qy^p$, where $q$ is an element of the field $k$ and $p$ is a positive integer, then $P$ and $Q$ commute and

$$\text{Res}(P - s, Q - t) = G(s, t) \cdot y^{(\sum_{i=0}^{n-1} p^i)(\sum_{i=0}^{m-1} p^i)}.$$ 

For general commuting $P$ and $Q$ we can write the resultant as

$$\text{Res}(P - s, Q - t) = \sum G_i(s, t)y^i$$

where the $G_i$ do not contain any power of $y$. It would be interesting to find conditions guaranteeing that $G_i(P, Q) = 0$ for all $i$. This will not be true in general but if $R[x] = S[y][x; \sigma, 0]$ where $S$ is an integral domain and $\sigma$ is injective and $P$ and $Q$ are of the form considered in this section then $G_i(P, Q) = 0$ for all $i$, since all the $G_i$ will in fact be equal.

**Example 5.** We include an example to illustrate Corollary 1. Set $P = (yx)^2$ and $Q = yx$. Then $P = qy^{k+1}x^2$ and $xQ = qy^{k}x^2$ so

$$\text{Res}(P - s, Q - t) = \begin{vmatrix} qy^{k+1} & 0 & -s \\ 0 & y & -t \\ qy^k & -t & 0 \end{vmatrix} = q(s^2 - t) \cdot y^{k+1}.$$ 

With the notation of the theorem $G(s, t) = q(s - t^2)$ in this case.

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