Low-Energy Effective Action
in N=4 Super Yang-Mills Theory

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Abstract: We consider $N = 4$ supersymmetric Yang-Mills theory formulated in terms of $N = 2$ superfields in harmonic superspace. Using the background field method we define manifestly gauge invariant and $N = 2$ supersymmetric effective action depending on $N = 2$ strength superfields and develop a general procedure for its calculation in one-loop approximation. Explicit form for this effective action is found for the case of $SU(2)$ gauge group broken down to $U(1)$.

Maximally extended $N = 4$ supersymmetric Yang-Mills (SYM) theory attracts much attention due to remarkable properties on quantum level and profound links with modern string/brane activity. It is natural to expect that an effective action describing many aspects of quantum theory, such as symmetry breaking, low- and high-energy behaviour, anomalies and so on, also must possess the many remarkable features in $N = 4$ SYM. Unfortunately, the effective action of $N = 4$ SYM was studied insufficiently so far and ones know little both about its general structure and about its form in different approximations.

One of the main obstacles to investigate the effective action in $N = 4$ SYM is absence of adequate quantum formulation such a theory. In our opinion, an efficient technique of carrying out the calculations of effective action in $N = 4$ SYM preserving $N = 4$ supersymmetry manifestly is still undeveloped. The best we have at present is treatment of $N = 4$ SYM as $N = 2$ SYM coupled to a specific (hypermultiplet) matter and use the methods of $N = 2$ supersymmetric quantum field theory.

Recently Dine and Seiberg [1] has shown that part of low-energy effective action of $N = 4$ SYM depending only on $N = 2$ superfield strengths $W$ and $\bar{W}$ can be found exactly up to numerical factor on the base of restrictions imposed by $N = 4$ supersymmetry. To be more precise, it was shown that low-energy effective action $\Gamma[W, \bar{W}]$ for $N = 4 SU(2)$ theory is expressed in terms of non-holomorphic effective potential $\mathcal{H}(W, \bar{W})$ as follows

$$\Gamma[W, \bar{W}] = \int d^4x d^8\theta \mathcal{H}(W, \bar{W})$$

where
\( \mathcal{H}(W, \bar{W}) = c \log \frac{W^2}{\Lambda^2} \log \frac{\bar{W}^2}{\Lambda^2} \) 

with some numerical coefficient \( c \) and some scale \( \Lambda \). The effective action (1) with \( \mathcal{H}(W, \bar{W}) \) (2) is \( \Lambda \)-independent. The explicit calculation of the coefficient \( c \) have been given in Refs [7, 8]. The generalizations for arbitrary \( SU(n) \) groups were considered in Refs [9, 10]. The present paper is a brief overview of the approach to calculation of low-energy effective action in \( N = 4 \) SYM developed in Refs [8, 9].

We consider \( N = 4 \) SYM formulated in terms of \( N = 2 \) superfields in harmonic superspace [2]. Such an approach is still the only one operating with unconstrained \( N = 2 \) superfields and explicitly realizing \( SU(2)_R \)-symmetry of \( N = 2 \) Poincare superalgebra. Manifestly \( N = 2 \) supersymmetric calculation of low-energy (holomorphic) effective action in harmonic superspace was given in Refs [3, 4, 6].

From point of view of \( N = 2 \) supersymmetry, the \( N = 4 \) SYM theory describes coupling of \( N = 2 \) vector multiplet to the hypermultiplet in the adjoint representation. In harmonic superspace approach, the vector multiplet is realized by an unconstrained analytic gauge superfield \( V^{++} \) and hypermultiplet can be realized either by real unconstrained analytic superfield \( \omega \) (\( \omega \)-hypermultiplet) or by a complex unconstrained analytic superfield \( q^+ \) (\( q \)-hypermultiplet) [2]. In the \( \omega \)-hypermultiplet realization, the action of \( N = 4 \) SYM theory reads

\[
S[V^{++}, \omega] = \frac{1}{2g^2} \int d^4x d^8\theta \text{tr} W^2 - \frac{1}{2g^2} \int d\zeta^{(-4)} \text{tr} \nabla^{++} \omega \nabla^{++} \omega
\]

In the \( q \)-hypermultiplet representation, the \( N = 4 \) SYM theory is given by the action

\[
S[V^{++}, q^+, \bar{q}^+] = \frac{1}{2g^2} \int d^4x d^4\theta \text{tr} W^2 - \frac{1}{2g^2} \int d\zeta^{(-4)} \text{tr} q^+ i \nabla^{++} q_i
\]

where

\[
q_i^+ = (q^+, \bar{q}^+), \quad q^{+i} = \varepsilon^{ij} q^+_j = (q^+, -q^+)
\]

Both models (3,4) are manifestly \( N = 2 \) supersymmetric. However, they possess the two extra hidden supersymmetries [2] and, as a result, one gets \( N = 4 \) supersymmetric theories with \( N = 4 \) SYM content. Of course, the theories (3,4) are classically equivalent. Our purpose is to describe a calculation of low-energy effective action for the above theories.

The first step of calculation is background field quantization of the model under consideration allowing to preserve manifest gauge invariance and \( N = 2 \) supersymmetry. Background field formulation for \( N = 2 \) supersymmetric field theories in harmonic superspace was developed in Ref [3] (see also [2, 4, 5]). Within this formulation, the one-loop effective action \( \Gamma^{(1)}[W, \bar{W}] \) for the theories (3,4) looks like

\[
\Gamma^{(1)}[W, \bar{W}] = i \frac{2}{2} \text{Tr}_{(2,2)} \log \Box - i \frac{2}{2} \text{Tr}_{(4,0)} \log \Box
\]

where \( \Box \) is the analytic d’Alambertian introduced in Ref [4] and the formal definition of the \( \text{Tr}_{(2,2)} \log \Box \) and \( \text{Tr}_{(4,0)} \log \Box \) are given in Ref [3].

The second step is convenient path integral representation of the effective action \( \Gamma^{(1)}[W, \bar{W}] \) for special background. The low-energy effective action (5) depends only on \( W \) \( \bar{W} \), therefore it is sufficient to choose the background gauge superfield on shell

\[
\mathcal{D}^a(i \mathcal{D}^a)_0 = 0
\]
where $\mathcal{D}_\alpha^i$ are the standard $N = 2$ supercovariant derivatives. In this case the effective action (5) can be represented in the form

$$\exp(i\Gamma^{(1)}[W, \bar{W}]) = \frac{\int \mathcal{D}G^{++} \exp\left\{ -\frac{i}{2} \text{tr} \int d\zeta (-4) G^{++} \bar{\Box} G^{++} \right\}}{\int \mathcal{D}G^{++} \exp\left\{ -\frac{i}{2} \text{tr} \int d\zeta (-4) G^{++} G^{++} \right\}}$$

(7)

where analytic integration superfield $G^{++}$ is constrained by

$$\nabla^{++} G^{++} = 0$$

(8)

The result (7) is quite general and preserves manifestly all symmetries of the theory.

The third step is transformation of the path integral (7) to ones over unconstrained $N = 1$ superfields [8]. This transformation can be treated as some replacement of integration variables under the path integral. We introduce also the $N = 1$ projections of $W$: $\phi = W|$, $2i\bar{W}_\alpha = \mathcal{D}_\alpha W$. The details of such a projection are given in Ref [8]. Here $\mathcal{D}_\alpha^2 \equiv \mathcal{D}_\alpha^i |_{i=2}$.

As a result, ones obtain

$$\exp(i\Gamma^{(1)}) = \int \mathcal{D}\bar{\phi} \mathcal{D}\phi \exp\left\{ \frac{i}{2} \int d^8 z \bar{\phi} \Delta V \right\}$$

(9)

$$\Delta = \mathcal{D}^\alpha \mathcal{D}_\alpha - eW^\alpha \mathcal{D}_\alpha + e\bar{W}_\alpha \bar{\mathcal{D}}^\alpha + e^2|\phi|^2$$

Here $\mathcal{D}_\alpha, \mathcal{D}_\alpha^\dagger, \bar{\mathcal{D}}_\alpha$ are the $N = 1$ supercovariant derivatives and $\Gamma^{(1)}$ depends on $\phi, W_\alpha, \bar{\phi}, \bar{W}_\alpha$, $V$ and $\bar{V}$ are unconstrained $N = 1$ complex scalar superfields.

The final step is calculation of $\Gamma^{(1)}$ in low-energy limit. We consider the background gauge superfield corresponding to the unbroken $U(1)$ subgroup of $SU(2)$ group in the Coulomb branch. In this case the effective action $\Gamma^{(1)}$ depending on $\phi, \bar{\phi}, W_\alpha, \bar{W}_\alpha$ can be written as follows

$$\Gamma^{(1)} = \int d^8 z W^\alpha W_\alpha \bar{W}_\alpha \bar{W}_\alpha \frac{\partial^4 \mathcal{H}(\phi, \bar{\phi})}{\partial \bar{\phi}^2 \partial \bar{\phi}^2} + \ldots$$

(10)

To calculate $\partial^4 \mathcal{H}(\phi, \bar{\phi})/\partial \phi^2 \partial \bar{\phi}^2$ we use $N = 1$ superfield proper-time technique introducing Schwinger kernel for the operator $\Delta$ [8] (see the details of superfield proper-time technique in Ref [11]).

$$\Gamma^{(1)} = -i \int_0^\infty \frac{ds}{s} e^{-i(\phi^2 - \bar{\phi}^2)s} \int d^8 z \mathcal{U}(z, z|s)$$

(11)

where for constant $W$ and $\bar{W}$ ones obtain

$$\mathcal{U}(z, z'|s) = \frac{i}{(4\pi is)^2} e^{i(s(eW^\alpha \mathcal{D}_\alpha - e\bar{W}_\alpha \mathcal{D}_\alpha^\dagger)\delta^4(\theta - \theta') - \frac{(z - z')^2}{4is})}$$

(12)

It leads to

$$\partial^4 \mathcal{H}(\phi, \bar{\phi})/\partial \phi^2 \partial \bar{\phi}^2 = (4\pi \delta(\phi))^2$$

(13)

One can easily find a general solution of this equation and restore the function $\mathcal{H}(W, \bar{W})$. We finally get

$$\mathcal{H}(W, \bar{W}) = \frac{1}{4(4\pi)^2} \log \frac{W^2}{\Lambda^2} \log \frac{\bar{W}^2}{\Lambda^2}$$

(14)

Thus the coefficient $c$ in (2) is equal to $1/(4(4\pi)^2)$. All details of the calculations can be found in Ref [8].
To conclude, we have formulated a general procedure of calculating the low-energy effective action depending on $N = 2$ strength superfields for $N = 4$ SYM theories.

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References

[1] M. Dine, N. Seiberg, Phys. Lett. B409, 239, 1997.

[2] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetcky, E. Sokatchev, Class. Quant. Grav. 1, 469, 1984; A. Galperin, E. Ivanov, V. Ogievetcky, E. Sokatchev, Class. Quant. Grav. 2, 601; 617, 1985.

[3] I.L. Buchbinder, E.I. Buchbinder, E.A. Ivanov, S.M. Kuzenko, B.A. Ovrut, Phys. Lett. B412, 309, 1997; E.I. Buchbinder, I.L. Buchbinder, E.A. Ivanov, S.M. Kuzenko, Mod. Phys. Lett. A13, 1071, 1998.

[4] I.L. Buchbinder, E.I. Buchbinder, S.M. Kuzenko, B.A. Ovrut, Phys. Lett. B417, 61, 1998.

[5] I.L. Buchbinder, S.M. Kuzenko, B.A. Ovrut, Phys. Lett. B433, 335, 1998.

[6] I.L. Buchbinder, Effective action of N=2 supersymmetric field theories in harmonic superspace approach, Proc. Second Int. Conf. “Quantum Field Theory and Gravity”, Tomsk, 1998, pp. 41-52, [hep-th/9802153]. I.L. Buchbinder, B.A. Ovrut, Background field method and structure of effective action in N=2 super Yang-Mills theories, in “Theory of Elementary Particles”, Proc. 31st Int. Symposium Ahrenshoop, 1997, Buckow, Wiley-Vch, 1998, pp. 33-39, [hep-th/9802156]. I.L. Buchbinder, S.M. Kuzenko, B.A. Ovrut, Covariant harmonic supergraphity for N=2 super Yang-Mills theories, [hep-th/9810040].

[7] V. Periwal, R. von Unge, Phys. Lett. B430, 71, 1998; F. Gonzalez-Rey, M. Roček, Phys. Lett. B434, 303, 1998.

[8] I.L. Buchbinder, S.M. Kuzenko, Mod. Phys. Lett. A13, 1623, 1998.

[9] E.I. Buchbinder, I.L. Buchbinder, S.M. Kuzenko, Phys. Lett. B446, 216, 1999.

[10] F. Gonzalez-Rey, B. Kulik, I.Y. Park, M. Roček, Self-dual effective action of N=4 super Yang-Mills, [hep-th/9810152]. D.A. Love, R. von Unge, Constraints on higher derivative operators in maximally supersymmetric gauge theory, [hep-th/9811017].

[11] I.L. Buchbinder, S.M. Kuzenko, Ideas and methods of supersymmetry and supergravity, IOP Publ., Bristol and Philadelphia, 1995; Revised edition 1998.