Polynomial estimates for the method of cyclic projections in Hilbert spaces

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Abstract

We study the method of cyclic projections when applied to closed and linear subspaces $M_i, i = 1, \ldots, m,$ of a real Hilbert space $\mathcal{H}$. We show that the average distance to individual sets enjoys a polynomial behavior $o(k^{-1/2})$ along the trajectory of the generated iterates. Surprisingly, when the starting points are chosen from the subspace $\sum_{i=1}^m M_i^\perp$, our result yields a polynomial rate of convergence $O(k^{-1/2})$ for the method of cyclic projections itself. Moreover, if $\sum_{i=1}^m M_i^\perp$ is not closed, then both of the aforementioned rates are best possible in the sense that the corresponding polynomial $k^{1/2}$ cannot be replaced by $k^{1/2+\varepsilon}$ for any $\varepsilon > 0$.

Keywords Product space · Rates of asymptotic regularity · Rates of convergence

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1 Introduction

Let $\mathcal{H}$ be a real Hilbert space with its inner product denoted by $\langle \cdot, \cdot \rangle$ and its induced norm denoted by $\| \cdot \|$. Throughout this paper we assume that for each $i = 1, \ldots, m$, the set $M_i$ is a closed and linear subspace of $\mathcal{H}$ and we put $M := \bigcap_{i=1}^m M_i$. We denote by $P_M$ and $P_{M_i}$ the orthogonal projections onto $M$ and $M_i$, respectively, $i = 1, \ldots, m$. The method of cyclic projections for the subspaces $M_i$ is defined by

$$y_0 \in \mathcal{H}, \quad y_k := (P_{M_m} \ldots P_{M_1})^k(y_0), \quad k = 1, 2, \ldots$$ (1.1)
where in order to shorten the notation, we put

$$T := P_{M_m} \cdots P_{M_1}. \quad (1.2)$$

Thanks to von Neumann [27] ($m = 2$) and Halperin [24] ($m \geq 2$), we know that:

**Theorem 1.1** For each $y_0 \in \mathcal{H}$, we have $\|y_k - P_M(y_0)\| \to 0$ as $k \to \infty$.

Since then, the method of cyclic projections has been extensively studied in the literature; see, for example, [9, 14, 17, 23, 29]. In this paper we study the asymptotic properties of the error term

$$\|y_k - P_M(y_0)\|, \quad (1.3)$$

the average distance to the individual sets

$$\sqrt{\frac{1}{m} \sum_{i=1}^{m} d^2(y_k, M_i)}, \quad (1.4)$$

and the increment

$$\|y_k - y_{k-1}\| \quad (1.5)$$

along the trajectory $(y_k)_{k=0}^{\infty}$, assuming that the subspace $\sum_{i=1}^{m} M_i^\perp$ is not closed. Below we present a very brief overview of the relevant literature.

We begin with a result by Bauschke, Borwein and Lewis [10, Theorem 3.7.4] according to which:

**Theorem 1.2** The subspace $\sum_{i=1}^{m} M_i^\perp$ is closed if and only if $\|T - P_M\| < 1$.

For historical developments concerning Theorem 1.2 we refer the interested reader to [17, p. 235]. We note here briefly that for $m = 2$, the subspace $M_1^\perp + M_2^\perp$ is closed $\iff \cos(M_1, M_2) < 1$, where $\cos(M_1, M_2) := \sup\{\langle x_1, x_2 \rangle : x_i \in M_i \cap (M_1 \cap M_2)^\perp \text{ and } \|x_i\| \leq 1, i = 1, 2\}$ is the cosine of the Friedrichs angle. In fact, for $m = 2$ we have $\|T^k - P_M\| = \cos^{2k-1}(M_1, M_2)$; see [2, 25]. Interestingly enough, analogous formulas involving $\cos(M_1, M_2)$ have been established for other projection methods; see, for example, [7, 8, 30] and [1, Table 1].

Because of the inequality $\|T^k - P_M\| \leq \|T - P_M\|^k$, which holds for $m \geq 2$ (see [25, Corollary 1]), the closedness of $\sum_{i=1}^{m} M_i^\perp$ implies linear rate of convergence for the error term (1.3), that is,

$$\|y_k - P_M(y_0)\| = O(q^k) \quad (1.6)$$

for some $q \in (0, 1)$. Moreover, the same linear rate of convergence $O(q^k)$ holds for the average distance (1.4) and the increment (1.5) as both of them can be bounded from above by $2\|y_k - 1 - P_M(y_0)\|$.

The question what happens with the error term (1.3) when the subspace $\sum_{i=1}^{m} M_i^\perp$ is not closed was answered much later by Bauschke, Deutsch and Hundal in [11, Theorem 1.4] for $m = 2$ and in [18, Theorem 6.4] for $m \geq 2$.

**Theorem 1.3** Assume that $\sum_{i=1}^{r} M_i^\perp$ is not closed. Then for each $y_0 \in \mathcal{H}$, the sequence $(y_k)_{k=1}^{\infty}$ converges in norm to $P_M(y_0)$, but the convergence is arbitrarily slow, that
is, for any sequence \((a_k)_{k=0}^\infty\) of positive numbers converging to zero, there is \(y_0 \in \mathcal{H}\) such that
\[
\|y_k - P_M(y_0)\| \geq a_k, \quad k = 1, 2, \ldots \tag{1.7}
\]

The first example of two subspaces with the arbitrarily slow convergence phenomena was presented by Franchetti and Light in [22]. The arbitrary slow convergence for the method of cyclic projections is also discussed in [3, 19, 20]. Interestingly, analogous results hold for other projection methods; see, for example, [5, 11, 12, 30, 31]. The alternative between linear and arbitrarily slow convergence is known as the dichotomy theorem.

Theorem 1.3 implies that if the subspace \(\sum_{i=1}^r M_i^\perp\) is not closed, then there cannot be a polynomial upper bound \(O(k^{-p})\) for (1.3) that holds for some \(p > 0\) and all \(y_0 \in \mathcal{H}\). This, however, does not rule out the existence of such upper bounds for the increment (1.5). In fact:

**Theorem 1.4** For each \(y_0 \in \mathcal{H}\), we have \(\|y_k - y_{k-1}\| = o(k^{-1})\).

This result has been established by Badea and Seifert in [4, Theorem 2.1 and Remark 4.2(b)] for the product of orthogonal projections in a complex Hilbert space. Since Theorem 1.4 plays an important role in our analysis, we elaborate more on its proof for a real Hilbert space in the Appendix. We also present an alternative proof by using [16, Lemma 5.2]. A similar result can be found in [15, Proposition 2.2] for the product of conditional expectations.

It turns out that in contrast to the arbitrarily slow convergence, we can still expect polynomial behavior for the error term (1.3) if the starting points \(y_0\) belong to a certain subspace of \(\mathcal{H}\). For example, in view of Theorem 1.4, one of the candidates is the subspace \(M \oplus (I - T)(\mathcal{H})\), which is dense in \(\mathcal{H}\) and on which (1.3) converges with the rate \(o(k^{-1})\). A more general result of Badea and Seifert [4, Theorem 4.3] (see the Appendix) asserts that:

**Theorem 1.5** If \(\sum_{i=1}^r M_i^\perp\) is not closed, then for each \(y_0 \in X_p := M \oplus (I - T)^p(\mathcal{H})\) (which is a dense linear subspace of \(\mathcal{H}\)), the convergence is polynomial and we have
\[
\|y_k - P_M(y_0)\| = o(k^{-p}), \tag{1.8}
\]
where \(p = 1, 2, \ldots\). Moreover, for each \(y_0 \in X := \bigcap_{p=1}^\infty X_p\) (which is also a dense linear subspace of \(\mathcal{H}\)), the convergence is super-polynomially fast as (1.8) holds for all \(p > 0\).

The authors commented in [4, Remark 2.5 (c)] that the rate in (1.8) is optimal in the sense that it cannot be improved for all \(y_0 \in X_p\). In particular, the inclusions \(X_{p+1} \subset X_p\) are strict for all \(p = 1, 2, \ldots\). We elaborate further on this below. It is also worth mentioning that [4, Theorem 4.3] allows real values of \(p > 0\) in (1.8) for which the corresponding subspaces \(X_p\) are defined by using the so-called fractional powers of operators.

Subsequently, Borodin and Kopecká [12, Theorems 3 and 4] managed to show two polynomial error bounds when the set of starting points is restricted to the subspace \(\sum_{i=1}^m M_i^\perp\) and when \(M = \{0\}\). We slightly rephrase their result allowing \(M \neq \{0\}\).
Theorem 1.6  For each $y_0 \in Y := M \oplus \sum_{i=1}^{m} M_i^\perp$, we have
\[ \|y_k - P_M(y_0)\| = \mathcal{O}(k^{-1/(4m\sqrt{m} + 2)}). \] (1.9)

Moreover, if the number of subspaces $m = 2$, then for each $y_0 \in Y = (M_1 \cap M_2) \oplus M_1^\perp + M_2^\perp$, we have
\[ \|y_k - P_M(y_0)\| = \mathcal{O}(k^{-1/2}). \] (1.10)

Furthermore, the rate in (1.10) is best possible as the corresponding polynomial $k^{1/2}$ cannot be replaced by $k^{1/2+\varepsilon}$ for any $\varepsilon > 0$.

Finding the best possible power $p > 0$ for the upper bound $\mathcal{O}(k^{-p})$ in (1.9) was left as an open problem when $m \geq 3$; see [12, Problem 3]. It is worth emphasizing that the optimality of (1.10) is shown in [12] by using an example of two subspaces of a separable Hilbert space $\mathcal{H}$ for which $M_1^\perp + M_2^\perp$ is not closed. In that particular example, for each $\varepsilon > 0$, the authors define an $\varepsilon$-dependent starting point $y_0 \in M_1^\perp + M_2^\perp$ for which the sequence $\{y_k\}_{k=0}^\infty$ satisfies the lower bound
\[ \|y_k\| \geq Ck^{-1/2-\varepsilon}, \quad k = 1, 2, \ldots, \] (1.11)
where $C = C(y_0) > 0$. The example extends to the case where $m \geq 3$ by simply putting $M_i := M_2$ for $i \geq 3$. In particular, the power $p > 0$ in (1.9) cannot be larger than $1/2$. We return to [12, Problem 3] and the lower bound property (1.11) below.

Similarly to the subspaces $X_p$ considered in Theorem 1.5, the subspace $Y$ defined in Theorem 1.6 is dense in $\mathcal{H}$. This follows, for example, from the identity $\sum_{i=1}^{m} M_i^\perp = M^\perp$; see [17, Theorem 4.6]. Moreover, it was suggested in [4, Remark 4.4(b)] that when $\sum_{i=1}^{m} M_i^\perp$ is not closed, then, in general, the inclusion $X_1 \subset Y$ is strict.

After this short literature overview, we may now present the contributions of our paper, which are as follows:

(C1) We show that for all $y_0 \in \mathcal{H}$, the average distance (1.4) exhibits a polynomial rate $o(k^{-1/2})$.

(C2) Moreover, we show that for all $y_0 \in Y$, the error term (1.3), the average distance (1.4) and the increment (1.5) satisfy polynomial upper bounds $\mathcal{O}(k^{-1/2}), \mathcal{O}(k^{-1})$ and $\mathcal{O}(k^{-3/2})$, respectively.

(C3) Furthermore, we prove that if $\sum_{i=1}^{m} M_i^\perp$ is not closed, then all of the above-mentioned rates, including $o(k^{-1})$ in Theorem 1.4, cannot be improved.

(C4) In addition, we verify that if $\sum_{i=1}^{m} M_i^\perp$ is not closed, then the inclusions $X_{p+1} \subset X_p \subset Y$ are indeed strict for $p = 1, 2, \ldots$.

(C5) Finally, we demonstrate that if $\sum_{i=1}^{m} M_i^\perp$ is not closed, then there is a dense subset of starting points $V \subset Y$ on which
\[ \limsup_{k \to \infty} k^{1/2+\varepsilon} \|y_k - P_M(y_0)\| = \infty \] (1.12)
for all $\varepsilon > 0$. An analogous property holds for the average distance (1.4) and for the increment (1.5).
The first three statements (C1)–(C3) can be found in Theorem 4.3. In particular, we fully solve [12, Problem 3] so that the upper bound in (1.9) is $O(k^{-1/2})$. It is worth pointing out that our “optimality argument” for (C3) significantly differs from the one used in [12, Theorem 3] and is based on Lemma 4.2. In particular, this approach not only allowed us to verify (C4), but also led us to (C5); see Corollary 4.4 and Theorem 4.5. Note here that when $M = \{0\}$, then (1.12) implies (1.11) on $V$ for infinitely many $k$’s; see Remark 4.7.

Our paper is organized as follows. In Section 2 we recall basic properties of the product space formulation of Pierra, on which we rely heavily throughout the paper. In Section 3 we develop basic inequalities that connect (1.4) and (1.5). Note that the results of Section 3, in particular Lemma 3.1, hold for all closed and convex sets and not only for closed and linear subspaces. Section 4 is where we present our main results. In the Appendix we present the proofs of Theorems 1.4 and 1.5.

2 Product space formulation of Pierra

We consider the product space

$$H := \mathcal{H} \times \ldots \times \mathcal{H}$$

(2.1)

equipped with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$ given by

$$\langle x, y \rangle := \frac{1}{m} \sum_{i=1}^{m} \langle x_i, y_i \rangle \quad \text{and} \quad \|x\| := \sqrt{\frac{1}{m} \sum_{i=1}^{m} \|x_i\|^2},$$

(2.2)

where $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m) \in H$. In order to distinguish subsets and operators defined in $\mathcal{H}$ from those defined in $H$, we use bold font in the latter case. Following Pierra [28], let

$$C := M_1 \times \cdots \times M_m \quad \text{and} \quad D := \{x = (x, \ldots, x) : x \in \mathcal{H}\},$$

(2.3)

The subspace $D$ is called the diagonal of $H$. In addition, we define

$$M := M \times \cdots \times M \quad \text{and} \quad M_i := M_i \times \cdots \times M_i,$$

(2.4)

$i = 1, \ldots, m$. It is not difficult to see that $H$ is a Hilbert space while all of the above-mentioned sets are closed and linear subspaces of $H$. Moreover, one can verify that

$$C^\perp = M_1^\perp \times \cdots \times M_m^\perp,$$

(2.5)

where “$\perp$” stands for the orthogonal complement in both $\mathcal{H}$ and $H$; see, for example, [17, Theorem 4.6]. Furthermore, we have the following theorem:
Theorem 2.1 Let \( x = (x_1, \ldots, x_m) \) and let \( s = \frac{1}{m} \sum_{i=1}^{m} x_i \). Then,
\[
PC(x) = (P_{M_i}(x_i))_{i=1}^{m}, \quad PD(x) = (s_{i=1}^{m}) \quad \text{and} \quad P_{C \cap D}(x) = (P_{M}(s))_{i=1}^{m}. \tag{2.6}
\]

Proof See, for example, [28, Lemma 1.1] or [14, Section 4.4.1]. \( \square \)

Analogously to the projection onto \( C \), one can obtain coordinate-wise formulas for the orthogonal projections \( P_{C \perp} \) and \( PM_j, j = 1, \ldots, m \), that is,
\[
P_{C \perp}(x) = (PM_{\perp i}(x_i))_{i=1}^{m} \quad \text{and} \quad PM_j(x) = (PM_{i}(x_i))_{i=1}^{m}. \tag{2.7}
\]

In particular, the coordinate-wise formulas apply to the product of orthogonal projections in \( H \) defined by
\[
T(x) := PM_m \cdots PM_1(x) = (T(x_i))_{i=1}^{m}. \tag{2.8}
\]

Note, however, that unlike the projection \( P_C \), the projections \( PM_i \) do commute with \( PD \).

Proposition 2.2 For each \( i = 1, \ldots, m \), we have
\[
PM_i PD = PD PM_i. \tag{2.9}
\]

In particular,
\[
TPD = PDPD. \tag{2.10}
\]

Proof Let \( x = (x_1, \ldots, x_m) \). Then, by (2.6) and (2.7), we have
\[
PD PM_i(x) = \left(\frac{1}{m} \sum_{j=1}^{m} PM_i(x_j)\right)_{i=1}^{m} = \left(PM_i \left(\frac{1}{m} \sum_{j=1}^{m} x_j\right)\right)_{i=1}^{m} = PM_i PD(x). \tag{2.11}
\]
Equation (2.10) follows from the definition of \( T \). \( \square \)

We finish this section with a few simple equalities and inequalities, which are used in the sequel.

Lemma 2.3 For each \( k = 1, 2, \ldots \), we have
\[
\|T^k - T^{k-1}\| = \|T^k - T^{k-1}\|, \tag{2.12}
\]
\[
\|P_C \perp T^k PD\| \leq \max_{i=1, \ldots, m} \|P_{M_i} \perp T^k\| \leq \sqrt{m} \|P_C \perp T^k PD\|, \tag{2.13}
\]
\[
\|T^k PD P_C \perp\| \leq \max_{i=1, \ldots, m} \|T^k P_{M_i} \perp\| \leq \sqrt{m} \|T^k P_D P_C \perp\|, \tag{2.14}
\]
\[
\|T^k - T^{k-1}\| P_D P_C \perp \leq \max_{i=1, \ldots, m} \|T^k - T^{k-1}\| P_{M_i} \perp \leq \sqrt{m} \|T^k - T^{k-1}\| P_D P_C \perp, \tag{2.15}
\]
\[
\|P_C \perp T^k PD P_C \perp\| \leq \max_{i,j=1, \ldots, m} \|P_{M_j} \perp T^k P_{M_i} \perp\| \leq m \|P_C \perp T^k PD P_C \perp\|. \tag{2.16}
\]
Proof Equality (2.12) can be easily obtained by a direct calculation of the corresponding norms. Indeed, we have

\[ \| (T^k - T^{k-1}) P_D \| = \sup \{ \| (T^k - T^{k-1}) P_D (x) \| : x = (x, \ldots, x) \in D, \| x \| \leq 1 \} = \sup \{ \| (T^k - T^{k-1}) (x) \| : x \in \mathcal{H}, \| x \| \leq 1 \}. \]  

(2.17)

In order to show (2.13), let \( x = (x_1, \ldots, x_m) \in H \). Then, by using the convexity of \( \| \cdot \| \), we have

\[ \| P_{C^k} T^k P_D (x) \|^2 = \frac{1}{m} \sum_{i=1}^{m} \| P_{M_i^+} T^k \left( \frac{1}{m} \sum_{j=1}^{m} x_j \right) \|^2 \leq \max_{i=1,\ldots,m} \| P_{M_i^+} T^k \|^2 \cdot \| x \|^2. \]  

(2.18)

On the other hand, for \( x \in \mathcal{H} \) and \( x := (x, \ldots, x) \) (so that \( \| x \| = \| x \| \)), we have

\[ \| P_{M_i^+} T^k (x) \|^2 \leq \sum_{i=1}^{m} \| P_{M_i^+} T^k (x) \|^2 = m \| P_{C^k} T^k P_D (x) \|^2 \leq m \| P_{C^k} T^k P_D \|^2 \cdot \| x \|^2. \]  

(2.19)

It now suffices to take the supremum over \( \| x \| = 1 \) in (2.18) and over \( \| x \| = 1 \) in (2.19). Inequalities (2.14) follow from (2.13). Indeed, if we change the order of projections in (2.13), for example, by using a permutation \( \sigma = (\sigma(1), \ldots, \sigma(m)) \), then, the corresponding operators \( T_\sigma := P_{M_\sigma(1)} \ldots P_{M_\sigma(m)} \) and \( T_\sigma := P_{M_\sigma(1)} \ldots P_{M_\sigma(1)} \) satisfy

\[ \| P_{C^k} T_\sigma^k P_D \| \leq \max_{i=1,\ldots,m} \| P_{M_i^+} T_\sigma^k \| \leq \sqrt{m} \| P_{C^k} T_\sigma^k P_D \|. \]  

(2.20)

In particular, for the adjoints \( T^* = P_{M_1} \ldots P_{M_m} \) and \( T^* = P_{M_1} \ldots P_{M_m} \), we get

\[ \| P_{C^k} (T^*)^k P_D \| \leq \max_{i=1,\ldots,m} \| P_{M_i^+} (T^*)^k \| \leq \sqrt{m} \| P_{C^k} (T^*)^k P_D \|. \]  

(2.21)

Using the equality between the norms of a bounded linear operator and its adjoint, and by Proposition 2.2, we get

\[ \| P_{C^k} (T^*)^k P_D \| = \| (P_{C^k} (T^*)^k P_D)^* \| = \| T^k P_D P_{C^k} \|. \]  

(2.22)

and

\[ \| P_{M_i^+} (T^*)^k \| = \| (P_{M_i^+} (T^*)^k)^* \| = \| T^k P_{M_i^+} \|. \]  

(2.23)

which, when combined with (2.21) proves (2.14).

We now proceed to showing (2.15). The proof is a combination of arguments used for (2.13) and (2.14) with \( T^k \) replaced by \( T^k - T^{k-1} \) and with \( T^k \) replaced by \( T^k - T^{k-1} \).

Indeed, observe that by repeating the calculation from (2.18) and (2.19), we get

\[ \| P_{C^k} (T^k - T^{k-1}) P_D \| \leq \max_{i=1,\ldots,m} \| P_{M_i^+} (T^k - T^{k-1}) \| \leq \sqrt{m} \| P_{C^k} (T^k - T^{k-1}) P_D \|. \]  

(2.24)
Obviously, inequalities (2.24) hold true if we change the order of projections by using the operators $T_\sigma$ and $T_\sigma^\ast$; compare with (2.20). In particular, (2.24) holds for the adjoints $T_\ast$ and $T_\ast^\ast$. Knowing that
\[
\| P_{C_\perp} ((T\ast)^k - (T\ast)^{k-1}) P D \| = \| (T^k - T^{k-1}) P D P_{C_\perp} \| \quad (2.25)
\]
and
\[
\| P_{M_i^\perp} ((T\ast)^k - (T\ast)^{k-1}) \| = \| (T^k - T^{k-1}) P_{M_i^\perp} \|, \quad (2.26)
\]
we arrive at (2.15), as claimed.

Finally, we proceed to showing inequalities (2.16). On the one hand, using the convexity of $\| \cdot \|_2$, for $x = (x_1, \ldots, x_m) \in H$, we get
\[
\| P_{C_\perp} T^k P D P_{C_\perp} (x) \|^2 = \frac{1}{m} \sum_{i=1}^m \left\| P_{M_i^\perp} T^k \left( \frac{1}{m} \sum_{j=1}^m P_{M_j^\perp} (x_j) \right) \right\|^2 
\leq \frac{1}{m^2} \sum_{i=1}^m \sum_{j=1}^m \left\| P_{M_i^\perp} T^k P_{M_j^\perp} (x_j) \right\|^2 
\leq \max_{i,j=1,\ldots,m} \| P_{M_i^\perp} T^k P_{M_j^\perp} \|^2 \cdot \| x \|^2. \quad (2.27)
\]
On the other hand, for each $x \in \mathcal{H}$ and for $x_j := (0, \ldots, mx, \ldots, 0) \in H$, we have
\[
P_{C_\perp} T^k P D P_{C_\perp} (x_j) = (P_{M_1^\perp} T^k P_{M_1^\perp} (x), \ldots, P_{M_m^\perp} T^k P_{M_m^\perp} (x)) \quad (2.28)
\]
and
\[
\| P_{M_i^\perp} T^k P_{M_j^\perp} (x) \|^2 \leq \sum_{i=1}^m \| P_{M_i^\perp} T^k P_{M_j^\perp} (x) \|^2 
= m \| P_{C_\perp} T^k P D P_{C_\perp} (x_j) \|^2 
\leq m^2 \| P_{C_\perp} T^k P D P_{C_\perp} \|^2 \cdot \| x \|^2. \quad (2.29)
\]
as $\| x_j \| = \sqrt{m} \| x \|$. Thus, by taking the supremum over $\| x \| = 1$ in (2.27) and over $\| x \| = 1$ in (2.29), we arrive at (2.16).

\[\square\]

3 Closed and convex subsets

Throughout this section we assume that for each $i = 1, \ldots, m$, the set $C_i$ is a closed and convex subset of $\mathcal{H}$, and we put $C := \bigcap_{i=1}^m C_i$. The following lemma corresponds to [6, Lemma 8 (iii)].
Lemma 3.1 Let the sequence \( \{y_k\}_{k=0}^{\infty} \) be defined by the method of cyclic projections using the subsets \( C_i \), that is,

\[
y_0 \in \mathcal{H}, \quad y_k := (P_{C_m} \ldots P_{C_1})^k(y_0), \quad k = 1, 2, \ldots
\]  

(3.1)

Assume that the intersection \( C \neq \emptyset \). Then, for each \( k = 1, 2, \ldots \), we have

\[
\frac{1}{m} \sum_{i=1}^{m} d(y_k, C_i)^2 \leq \frac{m}{2} \| y_k - y_{k-1} \| \cdot d(y_{k-1}, C)
\]  

(3.2)

while

\[
\max_{i=1, \ldots, m} d(y_k, C_i)^2 \leq m \| y_k - y_{k-1} \| \cdot d(y_{k-1}, C).
\]  

(3.3)

Proof We follow the argument from [6, Lemma 8 (iii)] which we adjust for a simple product of the nearest point projections. Put \( Q_0 := I \) (the identity operator) and \( Q_i := P_{C_i} \ldots P_{C_1}, i = 1, \ldots, m \). By using the properties of the projections \( P_{C_i} \) (see [14, Corollaries 2.2.24 and 4.5.2]), for each \( z \in C \), we have

\[
\sum_{i=1}^{m} \| Q_i(y_k) - Q_{i-1}(y_k) \|^2 \leq \| y_k - z \|^2 - \| y_{k+1} - z \|^2.
\]  

(3.4)

Moreover, by the Cauchy-Schwarz inequality, we get

\[
\| y_{k+1} - z \|^2 = \| y_{k+1} - y_k \|^2 + \| y_k - z \|^2 + 2(y_{k+1} - y_k, y_k - z) \\
\geq \| y_{k+1} - y_k \|^2 + \| y_k - z \|^2 - 2\| y_{k+1} - y_k \| \cdot \| y_k - z \|.
\]  

(3.5)

In particular, by combining (3.4) and (3.5), we obtain

\[
\sum_{i=1}^{m} \| Q_i(y_k) - Q_{i-1}(y_k) \|^2 \leq 2\| y_{k+1} - y_k \| \cdot \| y_k - z \|.
\]  

(3.6)

Furthermore, since the product of projections \( Q_m \) is nonexpansive and \( \text{Fix} Q_m = C \), for all \( k = 1, 2, \ldots \), we have

\[
\| y_{k+1} - y_k \| \leq \| y_k - y_{k-1} \| \quad \text{and} \quad \| y_k - z \| \leq \| y_{k-1} - z \|.
\]  

(3.7)

Let \( j \in \{1, \ldots, \lfloor \frac{m}{2} \rfloor \} \). Then, by the definition of the metric projection, by using the triangle and the Cauchy-Schwarz inequalities, and by combining this with (3.6)–(3.7), we obtain

\[
d(y_k, C_j)^2 = \| y_k - P_{C_j}(y_k) \|^2 \leq \| y_k - Q_j(y_k) \|^2
\]
\[ \leq \left( \sum_{i=1}^{j} \| Q_i(y_k) - Q_{i-1}(y_k) \| \right)^2 \]
\[ \leq j \sum_{i=1}^{j} \| Q_i(y_k) - Q_{i-1}(y_k) \|^2 \]
\[ \leq 2j \cdot \| y_k - y_{k-1} \| \cdot \| y_{k-1} - z \|. \quad (3.8) \]

Let now \( j \in \{ \lfloor \frac{m}{2} \rfloor + 1, \ldots, m - 1 \} \). By using similar arguments, we obtain

\[ d(y_k, C_j)^2 = \| Q_m(y_{k-1}) - P_{C_j}(Q_m(y_{k-1})) \|^2 \leq \| Q_m(y_{k-1}) - Q_j(y_{k-1}) \|^2 \]
\[ \leq \left( \sum_{i=j+1}^{m} \| Q_i(y_{k-1}) - Q_{i-1}(y_{k-1}) \| \right)^2 \]
\[ \leq (m - j) \sum_{i=j+1}^{m} \| Q_i(y_{k-1}) - Q_{i-1}(y_{k-1}) \|^2 \]
\[ \leq 2(m - j) \cdot \| y_k - y_{k-1} \| \cdot \| y_{k-1} - z \|. \quad (3.9) \]

By combining (3.8) and (3.9), we arrive at

\[ \frac{1}{m} \sum_{i=1}^{m} d(y_k, C_i)^2 \leq s_m \| y_k - y_{k-1} \| \cdot \| y_{k-1} - z \|, \quad (3.10) \]

where

\[ s_m = \frac{2}{m} \left( \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} i + \sum_{i=\lfloor \frac{m}{2} \rfloor + 1}^{m-1} (m - i) \right) = \begin{cases} m/2, & \text{if } m \text{ is even} \\ m/2 - 1/(2m), & \text{if } m \text{ is odd} \end{cases} \quad (3.11) \]

This shows inequality (3.2). Inequality (3.3) follows directly from (3.8) and (3.9). \( \square \)

**Theorem 3.2** Let the sequence \( \{ y_k \}_{k=0}^{\infty} \) be defined as in Lemma 3.1 and assume that \( C \neq \emptyset \). Then, we have

\[ \| y_k - y_{k-1} \| = o(k^{-1/2}) \quad (3.12) \]

and

\[ \sqrt{\frac{1}{m} \sum_{i=1}^{m} d^2(y_k, C_i)} = o(k^{-1/4}). \quad (3.13) \]

**Proof** Let \( z \in C \). Knowing that \( P_{C_m} \ldots P_{C_1} \) is \((1/m)\)-strongly quasi-nonexpansive, we have

\[ \| y_{k+1} - y_k \|^2 \leq m(\| y_k - z \|^2 - \| y_{k+1} - z \|^2). \quad (3.14) \]
See, for example, [14, Corollary 4.5.3]. Consequently,

\[ \sum_{k=1}^{\infty} \| y_k - y_{k-1} \|^2 \leq m \| y_0 - z \|^2 < \infty. \]  

(3.15)

In particular, \( \sum_{n=k}^{\infty} \| y_n - y_{n-1} \|^2 \to 0 \) as \( k \to \infty \). By (3.7), we have

\[ \frac{k}{2} \| y_k - y_{k-1} \|^2 \leq \left( \frac{k}{2} \right) \| y_k - y_{k-1} \|^2 \leq \sum_{n=[k/2]+1}^{k} \| y_n - y_{n-1} \|^2 \to 0 \]  

(3.16)

as \( k \to \infty \). This proves (3.12). The rate of (3.13) follows immediately from Lemma 3.1. \( \square \)

4 Closed and linear subspaces

In this section we oftentimes use the product space notation introduced in Section 2. The following result is a direct consequence of Theorem 1.4, Lemmas 2.3 and 3.1.

**Lemma 4.1** For the operators \( T \) defined in (1.2) and \( T \) defined in (2.8), we have:

(i) \( \| T^k - T^{k-1} \| = O(k^{-1}) \);

(ii) \( \max_{i=1, \ldots, m} \| P_{M_i} T^k \| = O(k^{-1/2}) \);

(iii) \( \max_{i=1, \ldots, m} \| T^k P_{M_i} \| = O(k^{-1/2}) \);

(iv) \( \max_{i=1, \ldots, m} \| (T^k - T^{k-1}) P_{M_i} \| = O(k^{-3/2}) \);

(v) \( \max_{i,j=1, \ldots, m} \| P_{M_i} T^k P_{M_j} \| = O(k^{-1}) \);

(vi) \( \| (T^k - T^{k-1}) P_D \| = O(k^{-1}) \);

(vii) \( \| P_{C}^\perp T^k P_D \| = O(k^{-1/2}) \);

(viii) \( \| T^k P_D P_{C^\perp} \| = O(k^{-1/2}) \);

(ix) \( \| (T^k - T^{k-1}) P_D P_{C^\perp} \| = O(k^{-3/2}) \);

(x) \( \| P_{C}^\perp T^k P_D P_{C^\perp} \| = O(k^{-1}) \).

**Proof** Statement (i) follows from Theorem 1.4 and the uniform boundedness principle [13, Theorem 2.2]. In view of Lemma 2.3, it suffices to show statements (vii)–(x).

(vii). We show that

\[ \| P_{C}^\perp T^k P_D \| \leq \sqrt{\frac{m}{2} \| (T^k - T^{k-1}) P_D \|}. \]  

(4.1)
Let \( x = (x_1, \ldots, x_m) \in H \). Moreover, let \( \{y_k\}_{k=0}^{\infty} \) be defined by the method of cyclic projections (1.1) with \( y_0 := \frac{1}{m} \sum_{i=1}^{m} x_i \). Then, using the coordinate-wise projection formulas from Section 2 and Lemma 3.1 (see (3.2)), we obtain

\[
\| P_{C^\perp} T^k P_D(x) \|_2^2 = \frac{1}{m} \sum_{i=1}^{m} \| P_{M^\perp} (y_k) \|_2^2 = \frac{1}{m} \sum_{i=1}^{m} d^2(y_k, M_i) \\
\leq \frac{m}{2} \| y_k - y_{k-1} \| \cdot \| y_{k-1} \| \\
= \frac{m}{2} \| (T^k - T^{k-1}) P_D(x) \| \cdot \| T^{k-1} P_D(x) \| \\
\leq \frac{m}{2} \| (T^k - T^{k-1}) P_D \| \cdot \| x \|^2. \tag{4.2}
\]

\((viii)\). We show that

\[
\| T^k P_D P_{C^\perp} \| \leq \sqrt{\frac{m}{2} \| (T^k - T^{k-1}) P_D \|}, \tag{4.3}
\]

where we use an argument similar to the one used in the proof of (2.14). Indeed, observe that if we change the order of projections in (4.1) by using a permutation \( \sigma = (\sigma(1), \ldots, \sigma(m)) \), then the operator \( T_\sigma := P_{M_{\sigma(m)}} \ldots P_{M_{\sigma(1)}} \) satisfies

\[
\| P_{C^\perp} T_\sigma^k P_D \| \leq \sqrt{\frac{m}{2} \| (T_\sigma^k - T_\sigma^{k-1}) P_D \|}. \tag{4.4}
\]

In particular, for the adjoint \( T^* \), we get

\[
\| P_{C^\perp} (T^*)^k P_D \| \leq \sqrt{\frac{m}{2} \| (T^*)^k - (T^*)^{k-1} P_D \|}. \tag{4.5}
\]

Note that in view of Proposition 2.2, the projection \( P_D \) commutes with the operator \( T \). Moreover, using the equality between the norms of a bounded linear operator and its adjoint, we get

\[
\| T^k P_D P_{C^\perp} \| = \| (T^k P_D P_{C^\perp})^* \| = \| P_{C^\perp} (T^*)^k P_D \| \tag{4.6}
\]

and

\[
\| (T^*)^k - (T^*)^{k-1} P_D \| = \| (T^k - T^{k-1}) P_D \|. \tag{4.7}
\]

This proves \((viii)\).

\((ix)\). Since \( P_D \) is idempotent and commutes with \( T \), we have

\[
\| (T^k - T^{k-1}) P_D P_{C^\perp} \| = \| ((T^{[k/2]} - T^{[k/2]-1}) P_D) (T^{[k/2]-1} P_D P_{C^\perp}) \| \\
\leq \| (T^{[k/2]} - T^{[k/2]-1}) P_D \| \cdot \| (T^{[k/2]-1} P_D P_{C^\perp}) \|. \tag{4.8}
\]
By combining this with \((vi)\) and \((viii)\) we arrive at \((ix)\). \((x)\). We show that

\[
\| P_{C^\perp} T^k P_D P_{C^\perp} \| \leq \sqrt{\frac{m}{2}} \| (T^k - T^{k-1}) P_D P_{C^\perp} \| \cdot \| T^k P_D P_{C^\perp} \|. \tag{4.9}
\]

We slightly adjust the argument from the proof of case \((vii)\). Indeed, let \(x = (x_1, \ldots, x_m) \in H\). Moreover, let \(\{y_k\}_{k=0}^\infty\) be defined by the method of cyclic projections \((1.1)\), but this time with \((vii)\). In our next result we show that the thresholds established in Lemma 4.1 are critical as they distinguish polynomial from linear rates of convergence.

**Lemma 4.2** Let \(\varepsilon > 0\) and assume that for the operators \(T\) defined in \((1.2)\) and \(T\) defined in \((2.8)\), one of the following conditions holds:

\(\begin{align*}
(i) & \quad \| T^k - T^{k-1} \| = \mathcal{O}(k^{-1-\varepsilon}); \\
(ii) & \quad \max_{i=1, \ldots, m} \| P_{M^\perp_i} T^k \| = \mathcal{O}(k^{-1/2-\varepsilon}); \\
(iii) & \quad \max_{i=1, \ldots, m} \| T^k P_{M^\perp_i} \| = \mathcal{O}(k^{-1/2-\varepsilon}); \\
(iv) & \quad \max_{i=1, \ldots, m} \| (T^k - T^{k-1}) P_{M^\perp_i} \| = \mathcal{O}(k^{-3/2-\varepsilon}); \\
v) & \quad \max_{i, j=1, \ldots, m} \| P_{M^\perp_i} T^k P_{M^\perp_j} \| = \mathcal{O}(k^{-1-\varepsilon}); \\
(vi) & \quad \| (T^k - T^{k-1}) P_D \| = \mathcal{O}(k^{-1-\varepsilon}); \\
(vii) & \quad \| P_{C^\perp} T^k P_D \| = \mathcal{O}(k^{-1/2-\varepsilon}); \\
(viii) & \quad \| T^k P_D P_{C^\perp} \| = \mathcal{O}(k^{-1/2-\varepsilon}); \\
(ix) & \quad \| (T^k - T^{k-1}) P_D P_{C^\perp} \| = \mathcal{O}(k^{-3/2-\varepsilon}); \\
(x) & \quad \| P_{C^\perp} T^k P_D P_{C^\perp} \| = \mathcal{O}(k^{-1-\varepsilon}).
\end{align*}\)

Then \(\sum_{i=1}^m M^\perp_i\) is closed (equivalently, \(\| T - P_M \| < 1\)). In particular, all of the above-mentioned rates are linear and take the form \(\mathcal{O}(q^k)\) for some \(q \in (0, 1)\).

**Proof** The road map of the proof is to show the following implications:

\(\begin{align*}
(i) & \quad \implies \sum_{i=1}^m M^\perp_i\) is closed, \(\{(ii), (iii), (v)\} \implies (i)\) and \((iv) \implies (v)\). \tag{4.11}
\end{align*}\)
The equivalences $(i) \iff (vi)$, $(ii) \iff (vii)$, $(iii) \iff (viii)$, $(iv) \iff (ix)$ and $(v) \iff (x)$ follow from Lemma 2.3.

“(ii) $\implies \sum_{i=1}^m M_i^\perp$ is closed.” Assume that $\|T^k - T^{k-1}\| = O(k^{1-\varepsilon})$ for some $\varepsilon > 0$. Then there are $N \geq 1$ and $C > 0$, such that

$$q := \sum_{n=N}^{\infty} \|T^n - T^{n-1}\| \leq C \sum_{n=N}^{\infty} \frac{1}{n^{1+\varepsilon}} < 1.$$  \hspace{1cm} (4.12)

In particular, by using the triangle inequality, for each $x \in H$, $\|x\| = 1$, and for all $k = 1, 2, \ldots$, we have

$$\|T^N(x) - T^{N+k}(x)\| \leq \|T^N - T^{N+k}\| \leq \sum_{n=N+1}^{N+k} \|T^n - T^{n-1}\| \leq q.$$  \hspace{1cm} (4.13)

Using Theorem 1.1, we see that $\lim_{k \to \infty} T^{N+k}(x) = P_M(T^N(x))$. On the other hand, recall that $P_M P_{M_i} = P_M$ for all $i = 1, \ldots, m$ (see [17, Lemma 9.2]). Thus $P_M(T^N(x)) = P_M(x)$. Therefore, by passing to the limit as $k \to \infty$ and then, by taking the supremum over $\|x\| = 1$ on the left-hand side of (4.13), we arrive at

$$\|T^N - P_M\| \leq q < 1.$$  \hspace{1cm} (4.14)

By applying Theorem 1.2 to $T^N$ (seen as the product of $m \cdot N$ projections) and $M$ (seen as the intersection of $m \cdot N$ subspaces), we get

$$\sum_{i=1}^m M_i^\perp = \sum_{i=1}^m M_i^\perp + \cdots + \sum_{i=1}^m M_i^\perp \text{ is closed,}$$  \hspace{1cm} (4.15)

which completes the proof of the implication.

“(iii), (iiii), (v) $\implies (i)$”. We begin by showing that

$$\|T^k - T^{k-1}\|^2 \leq \frac{C}{n} \cdot \max_{i,j=1, \ldots, m} \|P_{M_i} T^{2n} P_{M_j}\|$$  \hspace{1cm} (4.16)

for some $C > 0$, where $k \geq 4$ and where $n := \lfloor (k - 1)/3 \rfloor$ (so that $3n \leq k - 1$). Indeed, let $x \in H$ be such that $\|x\| = 1$. Observe that $I - T$ commutes with $T$ and that

$$I - T = \sum_{i=1}^m (Q_{i-1} - Q_i) = \sum_{i=1}^m P_{M_i} Q_{i-1}.$$  \hspace{1cm} (4.17)
where \(Q_0 := I\) and \(Q_i := P_{M_i} \ldots P_{M_1}, i = 1, \ldots, m\). Using the fact that the orthogonal projection is idempotent and self-adjoint, and that \(\|T\| \leq 1\), we get

\[
\|T^k(x) - T^{k-1}(x)\|^2 \leq \|(I - T)T^{3n}(x)\|^2 = \langle (I - T)T^{3n}(x), T^{2n}(I - T)T^n(x) \rangle
\]

\[
= \left( \sum_{i=1}^m P_{M_i} Q_1 T^{3n}(x), T^{2n} \left( \sum_{j=1}^m P_{M_j} Q_{j-1} \right) T^n(x) \right)
\]

\[
= \sum_{i=1}^m \sum_{j=1}^m \left( P_{M_i} Q_1 T^{3n}(x), (P_{M_i} T^{2n} P_{M_j}) P_{M_j}^{-1} Q_{j-1} T^n(x) \right)
\]

\[
\leq m^2 \max_{i=1,...,m} \|P_{M_i} Q_1 T^n\|^2 \cdot \max_{i,j=1,...,m} \|P_{M_i} T^{2n} P_{M_j}^{-1}\|. \quad (4.18)
\]

On the other hand, by Lemma 4.1 (ii) applied to different orders of projections, we have

\[
\max_{i=1,...,m} \|P_{M_i} Q_1 T^n\|^2 \leq \max_{i=1,...,m} \|P_{M_i} (Q_1 P_{M_m} \ldots P_{M_1})^n\|^2 \leq \frac{C'}{n}, \quad (4.19)
\]

for some \(C' > 0\), which shows (4.16).

Assume now that condition (v) holds, that is,

\[
\max_{i,j=1,...,m} \|P_{M_i} T^k P_{M_j}^{-1}\| \leq \frac{C'}{k^{1+\varepsilon}} \quad (4.20)
\]

for some \(\varepsilon > 0\) and some \(C' > 0\). Then, using (4.16) and knowing that \(n \geq k/4\) (so that \(1/n \leq 4/k\)), we get

\[
\|T^k - T^{k-1}\|^2 \leq \frac{C}{n} \cdot \frac{C'}{(2n)^{1+\varepsilon}} \leq \frac{4^{2+\varepsilon} CC'}{k^{2+\varepsilon}}. \quad (4.21)
\]

Thus we have arrived at condition (i).

Assume now that condition (ii) holds, that is,

\[
\max_{i=1,...,m} \|P_{M_i}^{-1} T^k\| \leq \frac{C'}{k^{1/2+\varepsilon}} \quad (4.22)
\]

for some \(\varepsilon > 0\) and some \(C' > 0\). Then,

\[
\max_{i,j=1,...,m} \|P_{M_i} T^{2n} P_{M_j}^{-1}\| \leq \max_{i=1,...,m} \|P_{M_i} T^n\| \cdot \max_{j=1,...,m} \|T^n P_{M_j}^{-1}\|
\]

\[
\leq \frac{C'}{n^{1/2+\varepsilon}} \max_{j=1,...,m} \|T^n P_{M_j}^{-1}\|. \quad (4.23)
\]
By Lemma 4.1 (iii) we know that
\[
\max_{j=1,\ldots,m} \| T^n P_{M_j^\perp} \| \leq \frac{C''}{\sqrt{n}}
\] (4.24)
for some $C'' > 0$. This, when combined with (4.16), and with the inequality $n \geq k/4$, leads to
\[
\| T^k - T^{k-1} \| \leq C = \frac{C'}{n^{1/2+\epsilon}} \cdot \frac{C''}{n^{1/2}} \leq \frac{4^{2+\epsilon} C' C''}{k^{2+\epsilon}}.
\] (4.25)
We have again arrived at condition $(i)$. An analogous argument can be used if we assume condition $(iii)$, that is, when
\[
\max_{j=1,\ldots,m} \| T^k P_{M_j^\perp} \| \leq \frac{C'}{k^{1/2+\epsilon}}
\] (4.26)
for some $\epsilon > 0$ and some $C' > 0$. Then, instead of (4.23), we use
\[
\max_{i,j=1,\ldots,m} \| P_{M_i^\perp} T^{2n} P_{M_j^\perp} \| \leq \max_{i=1,\ldots,m} \| P_{M_i^\perp} T^n \| \cdot \max_{j=1,\ldots,m} \| T^n P_{M_j^\perp} \|
\] \[
\leq \frac{C'}{n^{1/2+\epsilon}} \max_{i=1,\ldots,m} \| P_{M_i^\perp} T^n \|
\] (4.27)
combined with (4.16) and Lemma 4.1 (ii).

“(i\,v) \Rightarrow (v).” Assume that (i\,v) holds. Then, in view of Lemma 2.3, we also obtain condition (i\,x). However, inequality (4.9) together with Lemma 4.1 (viii) lead us to condition (x) with $\epsilon' := \epsilon/2 > 0$. Again, thanks to Lemma 2.3 we obtain condition (v) with $\epsilon' > 0$, which completes the proof. \hfill \square

We now arrive at the main result of our paper.

**Theorem 4.3** For each $y_0 \in \mathcal{H}$, the sequence $\{y_k\}_{k=0}^\infty$ defined by (1.1) satisfies
\[
\| y_k - y_{k-1} \| = o(k^{-1})
\] (4.28)
and
\[
\sqrt{\frac{1}{m} \sum_{i=1}^m d^2(y_k, M_i)} = o(k^{-1/2}).
\] (4.29)
Moreover, for each $y_0 \in Y := M \oplus \sum_{i=1}^m M_i^\perp$, the sequence $\{y_k\}_{k=0}^\infty$ defined by (1.1) satisfies
\[
\| y_k - P_M(y_0) \| = O(k^{-1/2}),
\] (4.30)
\[
\| y_k - y_{k-1} \| = O(k^{-3/2})
\] (4.31)
and
\[
\sqrt{\frac{1}{m} \sum_{i=1}^m d^2(y_k, M_i)} = O(k^{-1}).
\] (4.32)
Furthermore, if \( \sum_{i=1}^{m} M_{i}^{\perp} \) is not closed, then all of the above-mentioned rates (4.28)–(4.32) are best possible as the corresponding polynomials \( k^{1/2} \), \( k \) and \( k^{3/2} \) cannot be replaced by \( k^{1/2+\varepsilon} \), \( k^{1+\varepsilon} \) and \( k^{3/2+\varepsilon} \), respectively, for any \( \varepsilon > 0 \).

**Proof**  Let \( y_{0} \in \mathcal{H} \). Statement (4.28) is a repetition of Theorem 1.4 while (4.29) follows from Lemma 3.1 (see (3.3)).

To this end, assume that \( y_{0} \in Y \), say \( y_{0} = x_{0} + \frac{1}{m} \sum_{i=1}^{m} x_{i} \), where \( x_{0} \in M \) and \( x_{i} \in M_{i}^{\perp} \). Moreover, let \( x := (x_{1}, \ldots, x_{m}) \in \mathbf{H} \) and let \( T \) be defined by (2.8). Recall that \( P_{M} P_{M} = P_{M} P_{M} = P_{M} \mathcal{P} \mathcal{P}_{M} = 0 \) (use, for example, [17, Lemma 9.2]). In particular, using the identities \( x_{0} = P_{M}(x_{0}) \) and \( x_{i} = P_{M_{i}}^{\perp}(x_{i}) \), we have \( T(x_{0}) = x_{0} \) and \( P_{M}(x_{i}) = 0 \). Then, by (2.7) and by Lemma 4.1, we get

\[
\|y_{k} - P_{M}(y_{0})\| = \left\| T^{k} \left( \frac{1}{m} \sum_{i=1}^{m} P_{M_{i}}^{\perp}(x_{i}) \right) \right\| = \|T^{k} P_{D} P_{C_{\mathcal{P}}}(x)\| = \mathcal{O}(k^{-1/2}),
\]

(4.33)

\[
\|y_{k} - y_{k-1}\| = \left\| \left( T^{k} - T^{k-1} \right) \left( \frac{1}{m} \sum_{i=1}^{m} P_{M_{i}}^{\perp}(x_{i}) \right) \right\| = \|\mathcal{T}^{k} - T^{k-1}\| P_{D} P_{C_{\mathcal{P}}}(x)\| = \mathcal{O}(k^{-3/2})
\]

(4.34)

and, since \( d(y_{k}, M_{i}) = \| P_{M_{i}}^{\perp}(y_{k}) \| \), we also get

\[
\sqrt{\frac{1}{m} \sum_{i=1}^{m} d^{2}(y_{k}, M_{i})} = \sqrt{\frac{1}{m} \sum_{i=1}^{m} \left\| P_{M_{i}}^{\perp} T^{k} \left( \frac{1}{m} \sum_{i=1}^{m} P_{M_{i}}^{\perp}(x_{i}) \right) \right\|^{2}} = \left\| P_{C_{\mathcal{P}}} T^{k} P_{D} P_{C_{\mathcal{P}}}(x)\right\| = \mathcal{O}(k^{-1}).
\]

(4.35)

The fact that (4.28)–(4.32) cannot be improved follows directly from Lemma 4.2 and the uniform boundedness principle [13, Theorem 2.2]. For the convenience of the reader we sketch the proof for the average distance in (4.29) and in (4.32).

To this end, let \( \varepsilon > 0 \) and suppose to the contrary that

\[
\sqrt{\frac{1}{m} \sum_{i=1}^{m} d^{2}(y_{k}, M_{i})} = \mathcal{O}(k^{-1/2-\varepsilon})
\]

(4.36)

for all \( y_{0} \in \mathcal{H} \). Then, for each \( x = (x_{1}, \ldots, x_{m}) \in \mathbf{H} \) and for \( y_{0} := \frac{1}{m} \sum_{i=1}^{m} x_{i} \), we get

\[
\sup_{k=1,2,\ldots} k^{1/2+\varepsilon} \| P_{C_{\mathcal{P}}} T^{k} P_{D}(x)\| = \sup_{k=1,2,\ldots} k^{1/2+\varepsilon} \sqrt{\frac{1}{m} \sum_{i=1}^{m} d^{2}(y_{k}, M_{i})} < \infty.
\]

(4.37)
By the uniform boundedness principle [13, Theorem 2.2] applied to the family of operators \( \{k^{1/2+\varepsilon} P_{C^+} T^k P_D: k = 1, 2, \ldots \} \), we obtain
\[
\sup_{k=1,2,\ldots} k^{1/2+\varepsilon} \|P_{C^+} T^k P_D\| < \infty, \tag{4.38}
\]
which corresponds to condition (vii) in Lemma 4.2. This implies that \( \sum_{i=1}^{m} M_{i}^{\perp} \) is closed, which is to contradiction with our assumption.

A similar argument can be used when we assume that
\[
\sqrt{\frac{1}{m} \sum_{i=1}^{m} d^2(y_k, M_i)} = O(k^{-1-\varepsilon}) \tag{4.39}
\]
for all \( y_0 \in Y \). Indeed, for each \( x = (x_1, \ldots, x_m) \in H \) and since \( y_0 := \frac{1}{m} \sum_{i=1}^{m} P_{M_i^\perp}(x_i) \in Y \), we get
\[
\sup_{k=1,2,\ldots} k^{1+\varepsilon} \|P_{C^+} T^k P_D P_{C^\perp}(x)\| = \sup_{k=1,2,\ldots} k^{1+\varepsilon} \sqrt{\frac{1}{m} \sum_{i=1}^{m} d^2(y_k, M_i)} < \infty. \tag{4.40}
\]
Again, by using the uniform boundedness principle [13, Theorem 2.2], but this time applied to the family of operators \( \{k^{1+\varepsilon} P_{C^+} T^k P_D P_{C^\perp}: k = 1, 2, \ldots \} \), we obtain
\[
\sup_{k=1,2,\ldots} k^{1+\varepsilon} \|P_{C^+} T^k P_D P_{C^\perp}\| < \infty, \tag{4.41}
\]
which corresponds to condition (x) in Lemma 4.2. This again leads to a contradiction with our assumption.

Analogously, we can show that if
\[
\|y_k - P_M(y_0)\| = o(k^{-1-\varepsilon}) \tag{4.42}
\]
holds for all \( y_0 \in H \), then we arrive at condition (i) (or (vi)) of Lemma 4.2. Furthermore, if any of the conditions
\[
\|y_k - P_M(y_0)\| = O(k^{-1/2-\varepsilon}) \quad \text{or} \quad \|y_k - y_{k-1}\| = O(k^{-3/2-\varepsilon}) \tag{4.43}
\]
holds for all \( y_0 \in Y \), then we obtain conditions (viii) or (ix) from Lemma 4.2, respectively.

**Corollary 4.4** Assume that \( \sum_{i=1}^{m} M_{i}^{\perp} \) is not closed. Let \( X_p \) be defined as in Theorem 1.5, \( p = 1, 2, \ldots \), and let \( Y \) be defined as in Theorem 4.3. Then the polynomial \( k^p \) cannot be replaced by \( k^{p+\varepsilon} \) in (1.8) for any \( \varepsilon > 0 \). In particular, the inclusions \( X_{p+1} \subset X_p \subset Y \) are strict.

**Proof** In order to show that the rate in (1.8) cannot be improved we use an induction argument with respect to \( p \).

Suppose first that
\[
\|y_k - P_M(y_0)\| = o(k^{-1-\varepsilon}) \tag{4.44}
\]
holds for all $y_0 \in X_1$ and some $\varepsilon > 0$. Then, for all $y'_0 \in \mathcal{H}$ with $y'_k := T^k(y'_0)$, we obtain
\[
\|y'_k - y'_{k+1}\| = \|T^k(y'_0 - y'_1)\| = o(k^{-1-\varepsilon'})
\]  
(4.45)
as $y'_0 - y'_1 \in X_1$ and $P_M(y'_0 - y'_1) = 0$. This, however, contradicts Theorem 4.3 in view of which the rate in (4.28) cannot be improved.

Suppose now that
\[
\|y_k - P_M(y_0)\| = o(k^{-p-\varepsilon})
\]  
(4.46)
holds for all $y_0 \in X_p$ and some $\varepsilon > 0$, where $p \geq 2$. We show that an analogous relation holds for $p - 1$ with $\varepsilon/2$. Indeed, let $y_0 \in X_{p-1}$, say $y_0 = x + (I - T)^{p-1}(y)$, where $x \in M$ and $y \in \mathcal{H}$. Then, for each $n > k$, we get
\[
\|y_k - P_M(y_0)\| \leq \sum_{i=k}^{n} \|y_i - y_{i+1}\| + \|y_{n+1} - P_M(y_0)\|.
\]  
(4.47)
Note that
\[
y_i - y_{i+1} = T^i(I - T)(y_0) = T^i(I - T)^p(y),
\]  
(4.48)
where $(I - T)^p(y) \in X_p$. Moreover, because of our assumption (see (4.46)) combined with the uniform boundedness principle [13, Theorem 2.2],
\[
C := \sup_{k=1,2,...} k^{p+\varepsilon} \|T^k(I - T)^p\| < \infty.
\]  
(4.49)
Thus, by letting $n \to \infty$ in (4.47), we obtain
\[
\|y_k - P_M(y_0)\| \leq \sum_{i=k}^{\infty} \|T^i(I - T)^p(y)\| \leq \sum_{i=k}^{\infty} \frac{C}{i^{p+\varepsilon}} 
\leq \int_{i=k-1}^{\infty} \frac{C}{x^{p+\varepsilon}}dx = \frac{C(p - 1 + \varepsilon)^{-1}}{(k - 1)^{p-1+\varepsilon}}
\]  
(4.50)
In particular, for all $y_0 \in X_{p-1}$, we get
\[
\|y_k - P_M(y_0)\| = o(k^{-(p-1)-\varepsilon/2}),
\]  
(4.51)
as claimed.

By repeating the above-mentioned argument, we arrive at (4.44) with some $\varepsilon' > 0$. Consequently, we have shown that the rate in (1.8) cannot be improved.

Observe that the latter statement implies that the subspaces $X_p$ are distinct for different values of $p = 1, 2, \ldots$. Indeed, if we suppose otherwise, that $X_p = X_{p+1}$ for some $p \geq 1$, then this would imply (4.46) with $\varepsilon = 1$. However, as we have shown above, this situation cannot happen.

Similarly, if $X_p = Y$ for some $p \geq 1$, then this would imply that $\|y_k - P_M(y_0)\| = o(k^{-1/2-\varepsilon})$ for all $y_0 \in Y$, where $\varepsilon = p - 1/2$. This however would contradict Theorem 4.3 in view of which the rate in (4.30) cannot be improved. We note here that the inclusion $X_1 \subset Y$ can be easily deduced from (4.17).
The following result provides an alternative explanation for the fact that the rates of (4.28)–(4.32) cannot be improved.

**Theorem 4.5** Assume that \( \sum_{i=1}^{m} M_i^\perp \) is not closed. Then there is a dense subset \( U \) of \( \mathcal{H} \) such that for each \( y_0 \in U \) the sequence \( \{y_k\}_{k=0}^\infty \) defined in (1.1) satisfies

\[
\limsup_{k \to \infty} k^{1+\varepsilon} \|y_k - y_{k-1}\| = \infty
\]

and

\[
\limsup_{k \to \infty} k^{1/2+\varepsilon} \sqrt{\frac{1}{m} \sum_{i=1}^{m} d^2(y_k, M_i)} = \infty
\]

for all \( \varepsilon > 0 \). Moreover, there is a dense subset \( V \) of \( Y = M \oplus \sum_{i=1}^{m} M_i^\perp \) such that for each \( y_0 \in V \) the sequence \( \{y_k\}_{k=0}^\infty \) defined in (1.1) satisfies

\[
\limsup_{k \to \infty} k^{1/2+\varepsilon} \|y_k - PM(y_0)\| = \infty,
\]

\[
\limsup_{k \to \infty} k^{3/2+\varepsilon} \|y_k - y_{k-1}\| = \infty
\]

and

\[
\limsup_{k \to \infty} k^{1+\varepsilon} \sqrt{\frac{1}{m} \sum_{i=1}^{m} d^2(y_k, M_i)} = \infty
\]

for all \( \varepsilon > 0 \).

**Proof** We first define the subset \( V \) and then show equalities (4.54)–(4.56). To this end, let \( \{\varepsilon_n\}_{n=0}^\infty \subset (0, \infty) \) be such that \( \varepsilon_n \downarrow 0 \). By Lemma 4.2, for each \( n = 1, 2, \ldots, \) we have

\[
\sup_{k=1,2,\ldots} k^{1+\varepsilon_n} \|P_{C^\perp} T^k P_D P_{C^\perp}\| = \infty.
\]

Consequently, by applying the strong contrapositive of the uniform boundedness principle [32, Theorem 5.4.10] (and the successive Remark on p. 399 in [32]) to the family of operators \( \{k^{1/2+\varepsilon_n} P_{C^\perp} T^k P_D P_{C^\perp}: k = 1, 2, \ldots\} \), we see that

\[
V_n := \left\{ v: \sup_{k=1,2,\ldots} k^{1+\varepsilon_n} \|P_{C^\perp} T^k P_D P_{C^\perp}(v)\| = \infty \right\}
\]

is a dense \( G_\delta \) subset of \( \mathcal{H} \). By the Baire category theorem [32, Theorem 5.4.1], the subset \( V := \bigcap_{n=0}^\infty V_n \) is also a dense \( G_\delta \) subset of \( \mathcal{H} \). In fact, we have

\[
V = \left\{ v: \limsup_{k \to \infty} k^{1+\varepsilon} \|P_{C^\perp} T^k P_D P_{C^\perp}(v)\| = \infty \text{ for all } \varepsilon > 0 \right\}.
\]
We may now define the aforementioned subset $V$ in $H$ by

$$V := \left\{ v = v_0 + \frac{1}{m} \sum_{i=1}^{m} P_{M_i}^\perp (v_i) : v_0 \in M \text{ and } v = (v_1, \ldots, v_m) \in V \right\} \subset Y. \quad (4.60)$$

We show that $V$ is dense in $Y$, that is, for each $y \in Y$, there is a sequence $\{v_k\}_{k=0}^{\infty} \subset V$ such that $v_k \to y$. To this end, suppose that $y = x_0 + \frac{1}{m} \sum_{i=1}^{m} x_i$, where $x_0 \in M$ and where $x_i \in M_i^\perp$. Moreover, let $x := (x_1, \ldots, x_m)$. Since $V$ is a dense subset of $H$, there is a sequence $\{v_k\}_{k=0}^{\infty} \subset V$, with $v_k = (v_k, 1, \ldots, v_k, m)$, satisfying $v_k \to x$ as $k \to \infty$. Equivalently, $v_{k,i} \to x_i$ as $k \to \infty$ for all $i = 1, \ldots, m$. In particular, for $v_k := x_0 + \frac{1}{m} \sum_{i=1}^{m} P_{M_i}^\perp (v_{k,i}) \in V$, we get

$$\|v_k - y\| = \left\| \frac{1}{m} \sum_{i=1}^{m} P_{M_i}^\perp (v_{k,i} - x_i) \right\| \leq \frac{1}{m} \sum_{i=1}^{m} \|v_{k,i} - x_i\| \to 0 \quad (4.61)$$

as $k \to \infty$. Since $Y$ is a dense subset of $H$, we have also established that $V$ is dense in $H$.

We may now turn our attention to equalities (4.54)–(4.56). Note that for each $y_0 \in V$, say $y_0 = v_0 + \frac{1}{m} \sum_{i=1}^{m} P_{M_i}^\perp (v_i)$, we have

$$\sqrt{\frac{1}{m} \sum_{i=1}^{m} d^2(y_k, M_i)} = \| P_{C_\perp} T^k P_D P_{C_\perp} (v) \|, \quad (4.62)$$

where $v = (v_1, \ldots, v_m) \in V$. This, when combined with (4.59), shows (4.56). On the other hand, by Theorem 4.3, we see that

$$\|y_k - P_M(y_0)\| \leq \frac{C}{k^{1/2}} \quad \text{and} \quad \|y_k - y_{k-1}\| \leq \frac{C}{k^{3/2}} \quad (4.63)$$

for some $C > 0$, where $k = 1, 2, \ldots$. Thus, by Lemma 3.1 (see (3.2)), we arrive at

$$k^{2+\epsilon} \frac{1}{m} \sum_{i=1}^{m} d^2(y_k, M_i) \leq k^{2+\epsilon} \frac{m}{2} \|y_k - y_{k-1}\| \|y_k - P_M(y_0)\| \leq \begin{cases} \frac{Cm}{2} k^{3/2+\epsilon} \|y_k - y_{k-1}\| \\ \frac{Cm}{2} k^{1/2+\epsilon} \|y_k - P_M(y_0)\| \end{cases} \quad (4.64)$$

After taking the lim sup as $k \to \infty$ in (4.64), and using (4.56), we arrive at (4.54) and (4.55).
A similar argument can be used in order to define a dense subset $U$ on which equalities (4.52) and (4.53) hold. Indeed, by Lemma 4.2, we have
\[
\sup_{k=1,2,\ldots} k^{1/2+\varepsilon_n} \| P_{C^\bot} T^k P_D \| = \infty,
\] (4.65)
where $\{\varepsilon_n\}_{n=0}^\infty$ is as above. By using the above-mentioned strong contrapositive of the uniform boundedness principle, but this time applied to the family of operators $\{k^{1/2+\varepsilon_n} P_{C^\bot} T^k P_D : k = 1, 2, \ldots\}$, we obtain that
\[
U_n := \left\{ u : \sup_{k=1,2,\ldots} k^{1/2+\varepsilon_n} \| P_{C^\bot} T^k P_D(u) \| = \infty \right\}
\] (4.66)
is a dense $G_\delta$ subset of $H$. By again invoking the Baire category theorem, the set $U := \bigcap_{n=0}^\infty U_n$ is a dense $G_\delta$ subset of $H$. Since $U$ satisfies
\[
U = \left\{ u : \limsup_{k \to \infty} k^{1/2+\varepsilon} \| P_{C^\bot} T^k P_D(u) \| = \infty \text{ for all } \varepsilon > 0 \right\},
\] (4.67)
it suffices to put
\[
U := \left\{ u = u_0 + \frac{1}{m} \sum_{i=1}^m u_i : u_0 \in M \text{ and } u = (u_1, \ldots, u_m) \in U \right\}.
\] (4.68)
It is not difficult to see that $U$ is a dense subset of $\mathcal{H}$ (because $U$ is dense in $H$). Moreover, for each $y_0 = u_0 + \frac{1}{m} \sum_{i=1}^m u_i \in U$, we have
\[
k^{1+\varepsilon} \| P_{C^\bot} T^k P_D(u) \|^2 = k^{1+\varepsilon} \frac{1}{m} \sum_{i=1}^m d^2(y_k, M_i) \leq k^{1+\varepsilon} \frac{m}{2} \| y_k - y_{k-1} \| \| y_0 \|, \] (4.69)
where $u = (u_1, \ldots, u_m) \in U$. By taking the lim sup as $k \to \infty$ we arrive at (4.52) and (4.53).

**Remark 4.6** Thanks to the inequalities
\[
\sqrt{\frac{1}{m} \sum_{i=1}^m d^2(y_k, C_i)} \leq \max_{i=1,\ldots,m} d(y_k, M_i) \leq \sqrt{\sum_{i=1}^m d^2(y_k, C_i)},
\] (4.70)
which hold for all $y_k \in \mathcal{H}$, we may equivalently replace the average distance by the maximum distance in Theorems 4.3 and 4.5.

**Remark 4.7** (Lower Bounds) Theorem 4.5 implies that for each $y_0 \in U$, $\varepsilon > 0$ and $C > 0$, the lower bounds
\[
\| y_k - y_{k-1} \| \geq \frac{C}{k^{1+\varepsilon}} \quad \text{and} \quad \sqrt{\frac{1}{m} \sum_{i=1}^m d^2(y_k, C_i)} \geq \frac{C}{k^{1/2+\varepsilon}} \] (4.71)
hold for infinitely many $k$’s. Similarly, for each $y_0 \in V$, $y_0 \in U$, $\varepsilon > 0$ and $C > 0$, the lower bounds
\begin{align}
\|y_k - P_M(y_0)\| &\geq \frac{C}{k^{1/2+\varepsilon}}, \\
\|y_k - y_{k-1}\| &\geq \frac{C}{k^{3/2+\varepsilon}}
\end{align}
(4.72)
and
\begin{align}
\sqrt{\frac{1}{m} \sum_{i=1}^{m} d^2(y_k, C_i)} &\geq \frac{C}{k^{1+\varepsilon}}
\end{align}
(4.73)
hold for infinitely many $k$’s. This corresponds to (1.11). We do not know whether it is possible to show that the above-mentioned lower bounds hold for all sufficiently large $k$’s. Equivalently, we do not know if lim sup of Theorem 4.5 can be replaced by lim inf. We leave this as an open problem.

**Remark 4.8** We have become aware of a paper by Evron et al. [21] in which the method of cyclic projections is studied in the context of machine learning. Although formulated in a different setting and employing different assumptions in its analysis (in particular, $\mathcal{H} = \mathbb{R}^d$), this paper is related to the results presented here.

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**Data availability** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

**Declarations**

**Conflict of interest** The authors declare no competing interests.

**Appendix**

In this section we sketch how to derive Theorems 1.4 and 1.5 in a real Hilbert space, having in mind that the corresponding results of [4] were established in a complex Hilbert space. We also present an alternative proof of Theorem 1.4 by using [16, Lemma 5.2]. For this purpose, we use a complexification argument. For more details concerning the complexification, we refer the reader to [26].

To this end, let $H_C := \mathcal{H} + i\mathcal{H}$ be the (external) complexification of $\mathcal{H}$ with scalar multiplication given by
\begin{align}
(\alpha + i\beta)(x + iy) &:= \alpha x - \beta y + i(\alpha y + \beta x)
\end{align}
(4.74)
and inner product $\langle \cdot, \cdot \rangle_C$ defined by
\begin{align}
\langle x + iy, x' + iy' \rangle_C &:= \langle x, x' \rangle + \langle y, y' \rangle + i(\langle x', y \rangle - \langle x, y' \rangle),
\end{align}
(4.75)
where $\alpha, \beta \in \mathbb{R}$ and $x, y, x', y' \in \mathcal{H}$. Thus, the induced norm on $\mathbb{H}$, denoted by $\| \cdot \|_C$, satisfies
\[
\| x + iy \|_C^2 = \| x \|^2 + \| y \|_C^2
\] (4.76)
for all $x + iy \in \mathbb{H}$. It is not difficult to see that $(\mathbb{H}, \langle \cdot, \cdot \rangle_C)$ is indeed a complex Hilbert space.

**Proof of Theorem 1.4** For each $j = 1, \ldots, m$, let $M_j := M_j + iM_j$. Observe that $M_j$ is a closed linear subspace of $\mathbb{H}$. Denote by $P_{M_j}$ the orthogonal projection onto $M_j$. Then, for each $z = x + iy \in \mathbb{H}$, we have $P_{M_j}(z) = P_{M_j}(x) + iP_{M_j}(y)$. This implies that the product $T := P_{M_m} \ldots P_{M_1}$ satisfies $T(z) = T(x) + iT(y)$, where $T$ is defined as in (1.2). Using induction, we get
\[
T^k(z) = T^k(x) + iT^k(y).
\] (4.77)
By [4, Remark 4.2(b) and Theorem 2.1], we have
\[
k\| T^k(z) - T^{k-1}(z) \|_C \to 0 \quad \text{as} \quad k \to \infty
\] (4.78)
for all $z \in \mathbb{H}$. In particular, by taking $z := y_0 + i0$, we obtain
\[
\| y_k - y_{k-1} \| = \| T^k(z) - T^{k-1}(z) \|_C.
\] (4.79)
This implies Theorem 1.4. \hfill \Box

**Alternative proof of Theorem 1.4** By [16, Lemma 5.2] applied to the operator $T$, we have
\[
\sum_{k=1}^{\infty} k\| T^k(z) - T^{k-1}(z) \|_C^2 < \infty
\] (4.80)
for all $z \in \mathbb{H}$. In particular, by taking $z := y_0 + i0$, and knowing that the sequence $\{\| y_k - y_{k-1} \|\}_{k=1}^{\infty}$ is decreasing, we have
\[
k^2\| y_k - y_{k-1} \|^2 \leq 2k[k/2]\| y_k - y_{k-1} \|^2 \leq 4 \sum_{n=[k/2]+1}^{k} n\| y_n - y_{n-1} \|^2
\]
\[
\leq 4 \sum_{n=\lceil k/2 \rceil+1}^{k} n\| y_n - y_{n-1} \|^2 = 4 \sum_{n=\lceil k/2 \rceil+1}^{k} n\| T^n(z) - T^{n-1}(z) \|_C^2 \to 0
\] (4.81)
as $k \to \infty$. Thus we have shown that $\| y_k - y_{k-1} \| = o(k^{-1})$ for all $y_0 \in \mathcal{H}$. \hfill \Box

**Proof of Theorem 1.5** It is not difficult to see that $M_j^\perp = M_j + iM_j$ for all $j = 1, \ldots, m$; compare with (2.5). Consequently,
\[
\sum_{j=1}^{m} M_j^\perp = \sum_{j=1}^{m} M_j^\perp + i \sum_{j=1}^{m} M_j^\perp
\] (4.82)
\[\Box\] Springer
and thus
\[ \sum_{j=1}^{m} M_j \| is (not) closed \iff \sum_{j=1}^{m} M_j \| is (not) closed. \] (4.83)

Consider now the subspaces of \( H_C \) given by \( X_p := M \oplus (I - T)p(H_C) \) and \( X := \bigcap_{p=1}^{\infty} X_p \), where \( M := M + iM \) and \( I \) is the identity operator on \( H_C \), \( p = 1, 2, \ldots \). Obviously, \( X_p \) and \( X \) are the analogues of \( X_p \) and \( X \) considered in Theorem 1.5. In fact, we have
\[ X_p = X_p + iX_p \quad \text{and} \quad X = X + iX. \] (4.84)

Consequently, we obtain
\[ X_p(X) \text{ is dense in } H \iff X_p(X) \text{ is dense in } H_C. \] (4.85)

By [4, Theorem 4.3], for each \( z \in X_p \), we get
\[ \| T^k(z) - P_M(z) \| = o(k^{-p}) \] (4.86)
where \( p = 1, 2, \ldots \). Thus, for each \( y_0 \in X_p \) it suffices to take \( z := y_0 + i0 \in X_p \), to see that
\[ \| T^k(y_0) - P_M(y_0) \| = \| T^k(z) - P_M(z) \| = o(k^{-p}), \] (4.87)
which shows (1.8). Similarly, for each \( y_0 \in X \), it suffices to take \( z := y_0 + i0 \in X \) to see that (4.87) holds for all \( p > 0 \). Moreover, by [4, Theorem 4.3], we know that \( X_p \) and \( X \) are dense in \( H_C \).

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