Anomalous Quantum Hall Bilayer Effect

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Quantum Hall bilayers (QHB) are two-component fractional quantum Hall (FQH) systems that provide a unique platform to realize exotic Abelian/non-Abelian FQH and excitonic condensate states, but in the presence of a high magnetic field. Here, we show that these states can be realized in an analogue of bilayer Kagome lattice with flat bands (FBs) in each layer, leading to anomalous Quantum Hall bilayer effect, without magnetic field. Using exact diagonalization of finite size bilayer Kagome lattices, we demonstrate the stabilization of excitonic condensate Halperin’s (1, 1, 1) and (3, 3, 3) state at the total filling \( v_F = 1 \) and \( v_F = 1/3 \) of the two FBs, respectively. We further show that by tuning the interlayer tunneling at \( v_F = 2/3 \), one can expect a phase transition from Halperin’s (3, 3, 1) state to particle-hole conjugate of Laughlin’s 1/3 state, as previously observed in QHB systems. Our work significantly enriches the field of FB physics by demonstrating bilayer FB systems as an attractive avenue for realizing exotic QHB states including non-Abelian anyons without magnetic field.

When two-dimensional (2D) free electron gas is subject to a magnetic field, electron-electron interactions in the Landau levels of electrons lead to remarkable fractional quantum Hall (FQH) effect with fractionally quantized Hall conductance \( 1/2 \). A peculiar property of emergent excitations of FQH state is that they carry fractional charges \( \frac{1}{3} \) following anyonic statistics in 2D \( 1/3 \). Possibility of non-Abelian fractional statistics \( 6 \) obeyed by FQH states are particularly interesting owing to their potential in topological quantum computation \( 7 \) and are expected to arise in a single-component FQH system due to electron-electron interactions in the excited Landau levels of free electron gas \( 8–10 \).

Beyond single-component FQH, multi-component FQH systems offer a richer and more exotic quantum phase diagram with extra degrees of freedom allowing for additional tuning parameters \( 11–14 \). Quantum Hall Bilayers (QHB), such as GaAs quantum wells (QWs) \( 15–18 \), are spectacular examples of multi-component FQH systems. In such systems either a wide QW or a double QW can be mapped to two layers of free electron gas separated by a finite distance forming the two components that lead to exotic multi-component quantum Hall plateaus \( 15–18 \). One of the most notable plateaus manifests at \( v_F = 1 \) (Halperin’s (1, 1, 1) state) where interlayer excitonic coherence is observed by counter-flow measurements \( 19–21 \). In addition, QHBs have generated a lot of theoretical \( 22–24 \) interest, and are known to host a plethora of Abelian and non-Abelian FQH phase transitions owing to exceptional tunability of parameters; for example, tuning the interlayer tunneling can induce transitions from Abelian Halperin’s (3, 3, 0) to particle-hole conjugate of 1/3 Laughlin’s state at \( v_F = 2/3 \) \( 25 \), as well as from Abelian Halperin’s (3, 3, 1) to non-Abelian Moore-Read Pfaffian state at \( v_F = 1/2 \) \( 25 \).

Interestingly, in the past decade 2D lattices with topologically non-trivial Chern FB \( 29–30 \) have been shown to stabilize fractional quantum Hall effect (FQH) without magnetic field, giving rise to the so-called fractional Chern insulator (FCI) \( 31–34 \). In this sense, non-trivial FBs can be considered as solid-state analogue of dispersion-less Landau levels of the free electron gas. At fractional filling of FBs, ground state of the system mimics the Laughlin’s FQH state with features like ground-state degeneracy on a torus \( 31–33 \) spectral flow under flux insertion \( 35–37 \), quasihole statistics \( 33 \), and entanglement spectra \( 37 \). On the other hand, however, the realization of non-Abelian anyons in FCI is not straightforward since in a single-component FQH they arise at partial filling of excited Landau levels \( (v_F > 1) \) \( 8–10 \) while topological FB mimics only the lowest Landau level of free electron gas. A largely unexplored research direction, which is actually a natural extension to FCI, is the realization of exotic multi-component quantum Hall bilayer states without magnetic field in topological FBs. This is especially appealing since QHBs can host non-Abelian anyons at \( v_F < 1 \) \( 25–28 \). A recent work \( 35 \) discussed the possibility of stabilizing multi-component Halperin’s \( (m, m, m) \) state in a single layer topological FB partially filled by spin-up and spin-down electrons, but this system lacks the inter-component tunability present in QHBs.

In this Letter, we provide evidence for a direct analogue of QHB states in bilayer Kagome lattices without magnetic field. Kagome lattice is the most widely studied 2D lattice and material system hosting topological FBs \( 39–47 \). Recently, it has generated a lot of experimental interest with the realization of strongly correlated Kagome metals \( 48–52 \). Importantly, bilayer Kagome lattice materials have also been experimentally synthesized and known to be a host of quantum spin liquids \( 53–55 \). In addition, a variety of topological FBs are proposed in multi-layer heterostructures \( 56–57 \), as well as in metal organic frameworks \( 58–61 \). The feasibility in synthesizing multi-layer FB lattices with FBs belonging to individual layers, which can be treated similar to Landau levels in QHB systems, motivates the investigation.
into its potential in realizing QHB states without magnetic field. Here, we use exact diagonalization (ED) of finite-size bilayer Kagome lattice and study characteristic features of QHB states. Specifically, we show topological degeneracy of ground-state manifold on a torus, spectral flow under flux insertion and excitonic off-diagonal long-range order, indicating the stabilization of excitonic condensate states at total filling $v_T = 1$ and $v_T = 1/3$ of FBs, and further illustrate the potential of this lattice in stabilizing exotic tunable QHB states at $v_T = 2/3$, which potentially allows for the realization of non-Abelian anyons without magnetic field.

We use a typical nearest-neighbor (NN) tight-binding model of bilayer Kagome lattice (Fig. 1(a)), and consider the extreme limit of totally spin polarized electrons in partially filed bands with spin-orbital coupling (SOC) that conserves the out-of-plane spin-component [10], leading to kinetic part of total Hamiltonian,

$$H_{\text{kin}} = \sum_{\sigma} \left[ -t \sum_{<i,j>\alpha} c_{i\alpha\sigma}^\dagger c_{j\alpha\sigma} + i\lambda \sum_{<i,j>\alpha\beta} \frac{2}{\sqrt{3}} (r_{ij}^1 \times r_{ij}^2).\sigma_{\alpha\beta} c_{i\alpha\sigma}^\dagger c_{j\beta\sigma} \right]$$

(1)

where $t$ is the NN hopping integral, $\lambda$ represents SOC strength, $r_{ij}^{1,2}$ are the two NN unit vectors, $\alpha(\beta)$ are spin indices, and $\sigma(\sigma')$ represents each layer of the bilayer lattice, while $\sigma_{\alpha\beta}$ is the Pauli matrix. Interactions between electrons are described by Hubbard model which includes NN intra-layer ($V$), and direct inter-layer ($V_{\perp}$) interactions. $V_{\perp}$ depends on the distance $d$ between the layers as shown in Fig. 1(a) which can be independently varied, allowing for the tunability of inter-component interactions. We also include tunneling between the layers ($t_{\perp}$) in the interaction part of the Hamiltonian. This is closely related to the experimental observation of excitonic condensate in bilayer graphene where a tunneling barrier is introduced between the two layers to extend the lifetime of electron-holes pairs [62,63]. In essence, one can tune tunneling between the layers independent of the layer distance. Here, we imagine a similar tunneling barrier whose strength can be varied depending on the tunneling material being used between the layers. It also implies that this tunneling interaction makes the layer index a good quantum number although the number of electrons in each layer individually is not conserved, in contrast to introducing tunneling in $H_{\text{kin}}$ that would mix the layers while still conserving the number of electrons in each band. Hence, the interaction part of Hamiltonian in our model is given as,

$$H_{\text{int}} = \sum_{\sigma} \left[ V \sum_{<r,r'>\text{ intra}} n_{r\sigma} n_{r'\sigma} + V_{\perp} \sum_{r} n_{r\sigma} n_{r\sigma'} \right] + t_{\perp} \sum_{r} \left[ c_{r\sigma}^\dagger c_{r\sigma'} \right]$$

(2)

where $n_{r\sigma} = c_{r\sigma}^\dagger c_{r\sigma}$ is the electron density operator. The spin indices are suppressed. Throughout this work we set $V = 2t$, which is smaller than the single-particle gap separating the FBs (Fig. 1(b)), while the distance dependence of $V_{\perp}$ is described as: $V_{\perp} = V$ if $d \to 0$; $V_{\perp} = t$ if $d$ is finite.

Single-particle band structure is shown in Fig. 1(b). In the absence of tunneling, the two FBs, both with non-zero and same Chern number [30], belong to individual layers of the bilayer Kagome lattice, exactly matching the two Landau levels in each layer of QHB. In our model, we use a slightly high SOC ($\lambda = 0.3t$) to isolate the bottom FBs from other bands. This procedure is widely used in realizing FCIIs where a high flatness ratio of FBs is desirable. Interestingly, such conditional parameters can be realized using Floquet band engineering in real materials [47-67]. To study the effect of interactions, we exactly diagonalize the full Hamiltonian ($H = H_{\text{kin}} + H_{\text{int}}$) in reciprocal space with interactions projected to the two FBs of finite lattice containing total number of sites $N_s = 6 \times N_x \times N_y$. Total filling factor is given by $\nu_T = N_e/(N_x \times N_y)$, where $N_e$ is the number of electrons in the system. Under periodic boundary conditions (PBC), we implement translational symmetries and diagonalize the Hamiltonian in each momentum sector $q = (2\pi k_x/N_x, 2\pi k_y/N_y)$ with $k_x$ and $k_y$ being the integers, labelled as $(k_x, k_y)$. One can also assign a pseudospin notation to each FB and define $S_z = (N_r - N_\perp)/2$, where $N_r$ is the number of electrons in FB belonging to upper (lower) layer. Complete methodology of ED for multiple bands is presented in our previous work [6].

First, we study the $\nu_T = 1$ case. In QHBs with negligible $t_\perp$, $\nu_T = 1$ plateau manifests excitonic condensate with the wavefunction described by Halperin’s (1,1,1) state featuring a non-degenerate ground state on two tori [38]. In Fig. 2(a) we show the energy spectra of $H$ for $\nu_T = 1$ with negligible $d$ and $t_\perp$ at multiple $S_z$. The degeneracy of states with varying $S_z$ is a signature of spontaneous symmetry breaking, also known as the “which layer” uncertainty for QHBs [20]. At a finite distance, the $S_z = 0$ system or the case of balanced layers with $N_r = N_{\perp}$, becomes energetically stable (Fig. 2(b) in [68]). In Fig. 2(b) we plot the energy spectra for $S_z = 0$ case at a finite $d = 0.3a$ and negligible $t_\perp$. There is a clear presence
of a non-degenerate ground state separated from excited states for two finite system sizes, which also illustrates the convergence of our results. For all our subsequent calculations we use the system size $4 \times 3$. Halperin’s (1,1,1) state on two tori with PBC should belong to the total momentum sector $(\sum_{i=1}^{N} k_{x}^{i}, \sum_{i=1}^{N} k_{y}^{i})$, where $N = N_{x} \times N_{y}$, implying all the reciprocal points are occupied; for a $4 \times 3$ system size, it is the $(2,0)$ sector, as also shown in Fig. 2(b), implying a non-trivial topological characteristic of the ground state.

In addition to the case of $v_{T} = 1$, it is expected that similar excitonic condensate state could be present at $v_{T} = 1/3$ described by Halperin’s (3,3,3) wavefunction [38]. But so far, no experimental evidence for such a state has been reported. In the case of bilayer Kagome, we next study the characteristics of system at $v_{T} = 1/3$ with negligible $d$ and $t_{1}$. In Fig. 2(a), one clearly sees the signature of “which layer” uncertainty with degenerate ground state manifold at multiple $S_{z}$. Different form $v_{T} = 1$, the ground state manifold in this case has a three-fold degeneracy slightly lifted due to finite size effects. Moreover, as can be seen from Fig. 2(b), the three states belong to the momentum sectors $(2,0)$, $(2,1)$, and $(2,2)$ respectively, as expected from generalized Pauli’s principle [33], implying fractional statistics and non-trivial topology. In Fig. 2(c) we plot the spectral flow of these states under flux insertion in $y$-direction. The three states warp around each other and go back to their original configuration after insertion of three flux quanta while the gap never closes. This correlation indicates a fractional Hall conductance [35, 36]. We also check for the excitonic off-diagonal long-range order by plotting eigenvalues of two-body reduced density matrix in Fig. 2(d) and find the existence of one large eigenvalue although not as prominent as for the case of $v_{T} = 1$. Our results here suggest that Halperin’s (3,3,3) state featuring excitonic condensate with fractional Hall conductance can in principle be realized in bilayer Kagome system by only controlling the filling factor of electrons in the system in contrast to tuning the magnetic field in QHBs.

Next, we move on to illustrating the tunability of QHB states as realized in this lattice system. Similar to QHBs, tunneling strength between the two layers of Kagome lattice can be tuned by varying the tunneling material used between the layers. At $v_{T} = 2/3$ and finite $d = 0.3a$, we show a tunneling driven transition from Halperin’s (3,3,0) state to the particle-hole conjugate of Laughlin’s 1/3 state for $S_{z} = 0$, as depicted in Fig. 2(a). Topological degeneracy of ground-state manifold on a torus is a powerful tool that can distinguish the two phases. This was shown earlier by Haldane [70] for single component FQH and later expanded by McDonald [71] for multi-component FQH on a torus. Inter-component correlations in Halperin’s $(m, n, n)$ state can be studied using K-matrix, $K = \begin{pmatrix} m & n \\ n & m \end{pmatrix}$, as formulated within the field theory framework [72, 73]. Topological degeneracy of ground state for a system with $v_{T} = p/q$ on a torus is given by the determinant of K-matrix, $\text{det} K = qN'$, where $N'$ is an integer describing the different translations of the center-of-mass of different components, while $q$ determines the overall center-of-mass degeneracy. Moreover, $N'$ can be expressed as the sum of $N_{3}$, the number of states at $(0,0)$, and $N_{B}$, number of states
at zone boundary momentum sectors $(0, \overline{N}/2)$, $(\overline{N}/2, 0)$, and $(\overline{N}/2, \overline{N}/2)$, where $\overline{N} = N_e/p$. For homogenous liquids, $N' = N_0 + 3N_B$ with two possibilities: if $N_e/(N_0p)$ is even, $N_0 = N_B = N'/4$, otherwise, $N_0 = N'$ and $N_B = 0$, where $N_i$ is the number of electrons with component $i$.

In the case of Halperin’s $(3, 3, 0)$ state at $v_T = 2/3$, $\det K = 9$, $p = 2$, and $q = 3$, implying $N' = 3$. For a finite system size $4 \times 3$, $N_e = 8$, and since we are working with balanced layers ($S_z = 0$), $N_i = 4$. Based on the above analysis, the 3 degenerate states corresponding to $N' = 3$ should lie at $\{0, 0\}$. Taking center-of-mass degeneracy into account there should be additional 3-fold degeneracy for each of these states. The center-of-mass degenerate points are given by Pauli’s generalized principle [33] - if one state in the ground-state manifold lies in the momentum sector $(k_1, k_2)$, the next state can be always found at $(k_1 + N_e, k_2 + N_e)$ [modulo $(N_x, N_y)$], i.e., for $(N_x = 4, N_y = 3)$ if there’re three degenerate states at $(0, 0)$, there should be three more states at $(0, 1)$ and $(0, 2)$ each. In Fig. 4(a), one can see at small tunneling, the ground state manifold is composed of nine states, exactly as predicted by the above analysis, confirming it is the Halperin’s $(3, 3, 0)$ state. At strong tunneling, the bilayer effectively behaves as a single layer due to strong correlations and stabilizes a particle-hole conjugate of Laughlin’s $1/3$ state $(1/3)$ with only 3-fold center-of-mass degeneracy at $(0, 0)$, $(0, 1)$, and $(0, 2)$, which is also seen in Fig. 4(a), similar to QHB system [25].

We finally do a similar analysis for the case of $v_T = 1/2$ at a finite $d = 0.3\alpha$, negligible $t_\perp$, and $S_z = 0$. The ground state in this case is described by Halperin’s $(3, 3, 3)$ wavefunction [25]. Here with $q = 2$, and $N' = 4$, the K-matrix analysis requires an 8-fold ground-state degeneracy. The four states corresponding to $N' = 4$ belong to sectors $(0, 0)$, $(0, \overline{N}/2 = 3)$, $(\overline{N}/2 = 3, \overline{N}/2 = 3)$, and $(\overline{N}/2 = 3, 0)$. Since $N_x = 4$, and $N_y = 3$, this corresponds to two states at $(0, 0)$ and $(3, 0)$ each, in addition to $q = 2$ center-of-mass degeneracy. The 8-fold degenerate ground states, hence, should belong to the $(0, 0)$, $(1, 0)$, $(0, 2)$, and $(0, 3)$ momentum sectors, each two-fold degenerate. In Fig. 4(b), we plot the energy spectrum for this lattice with 8-fold degenerate ground-state manifold exactly matching the K-matrix analysis. Importantly, in this system a tunneling driven transition to non-Abelian Moore-Read state is proposed whose investigation, however, requires numerical calculations with much larger system sizes as shown earlier for QHBs [25], which has been left for future work.

In conclusion, we have demonstrated the potential of bilayer Kagome lattice, as a prototypical example of experimentally feasible multi-layer FB lattices, to realize anomalous quantum Hall bilayer effect. In addition to the stabilization of excitonic condensate Halperin’s $(1, 1, 1)$ and $(3, 3, 3)$ states, we demonstrate striking tunability of QHB states with interlayer tunneling strength which could lead to the realization of non-Abelian anyonic states without magnetic field. Interestingly, a non-Abelian interlayer Pfaffian state with 9-fold ground degeneracy is proposed for $v_T = 2/3$ QHB when interlayer interaction becomes slightly attractive [25]. We see a similar 9-fold degeneracy of ground state manifold at negative interlayer interaction strength in our calculations (Fig. S2 [28]) but more concrete evidence requires the analysis of orbital-cut entanglement spectra demanding further investigation in this direction. With our work, we hence provide a new platform to study anyonic statistics and further elaborate the role of unique topological prop-

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**FIG. 3.** ED calculation results at $v_T = 1/3$. (a) Energy spectra of $H$ at $d = 4$ with varying $S_z$ for $N_e = 6 \times 4 \times 3$ system size. Red diamonds, blue and green triangles denote the ground state manifold while yellow denotes the excited states. (b) Momentum-resolved spectra of $H$ at $d = 0$, $t_\perp = 0$, and $S_z = 0$. The three quasi-degenerate ground states are denoted by red, blue and green colors respectively. (c) Spectral flow of the system considered in (b) under flux insertion in y-direction. (d) Eigenvalues of excitonic reduced two-body density matrix for (b) plotted in descending order.

**FIG. 4.** (a) Tunneling-driven phase transition at $v_T = 2/3$, $d = 0.3\alpha$, $S_z = 0$ for $N_e = 6 \times 4 \times 3$ system size. (b) Energy spectra of $H$ at $v_T = 1/2$, $d = 0.3\alpha$, $t_\perp = 0$, and $S_z = 0$ for $N_e = 6 \times 4 \times 3$ system size. Red diamonds denote the 8-fold degenerate ground states while yellow denotes the excited states.
properties of FBs in realizing QHB states without magnetic field.
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SUPPLEMENTARY MATERIAL

FIG. S1. Energy spectra of $H$ at $v_T = 1$, $d = 0.3a$, and $t_\perp = 0$ with varying $S_z$ for $N_s = 6 \times 4 \times 3$ system size. Red diamonds denote the ground states while yellow denotes the excited states.

[1] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Two-dimensional magnetotransport in the extreme quantum limit, Physical Review Letters 48, 1559 (1982).
[2] R. B. Laughlin, Anomalous quantum hall effect: an incompressible quantum fluid with fractionally charged excitations, Physical Review Letters 50, 1395 (1983).
[3] F. D. M. Haldane, Fractional quantization of the hall effect: a hierarchy of incompressible quantum fluid states, Physical Review Letters 51, 605 (1983).
[4] B. I. Halperin, Statistics of quasiparticles and the hierarchy of fractional quantized hall states, Physical Review Letters 52, 1583 (1984).
[5] D. Arovas, J. R. Schrieffer, and F. Wilczek, Fractional statistics and the quantum hall effect, Physical review letters 53, 722 (1984).
[6] G. Moore and N. Read, Non-Abelions in the fractional quantum hall effect, Nuclear Physics B 360, 362 (1991).
[7] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. D. Sarma, Non-Abelian anyons and topological quantum computation, Reviews of Modern Physics 80, 1083 (2008).
[8] M. Greiter, X.-G. Wen, and F. Wilczek, Paired hall state at half filling, Physical review letters 66, 3205 (1991).
[9] N. Read and E. Rezayi, Beyond paired quantum hall states: Parafermions and incompressible states in the first excited landau level, Physical Review B 59, 8084 (1999).
[10] E. Rezayi and N. Read, Non-Abelian quantized hall states of electrons at filling factors 12/5 and 13/5 in the first excited landau level, Physical Review B 79, 075306 (2009).
[11] B. I. Halperin, Theory of the quantized hall conductance, Helvetica Physica Acta 56, 75 (1983).
[12] F. Haldane and E. Rezayi, Spin-singlet wave function for the half-integral quantum hall effect, Physical review letters 60, 956 (1988).
[13] X.-G. Wen, Continuous topological phase transitions between clean quantum hall states, Physical review letters 84, 3950 (2000).
[14] M. Barkeshli and X.-G. Wen, Bilayer quantum hall phase transitions and the orbifold non-abelian fractional quantum hall states, Physical Review B 84, 115121 (2011).
[15] Y. Suen, L. Engel, M. Santos, M. Shayegan, and D. Tsui, Observation of a $\nu = 1/2$ fractional quantum hall state in a double-layer electron system, Physical review letters 68, 1379 (1992).
[16] J. Eisenstein, G. Boebinger, L. Pfeiffer, K. West, and S. He, New fractional quantum hall state in double-layer two-dimensional electron systems, Physical review letters 68, 1383 (1992).
[17] Y. Suen, H. Manoharan, X. Ying, M. Santos, and M. Shayegan, Origin of the $\nu = 1/2$ fractional quantum hall state in wide single quantum wells, Physical review letters 72, 3405 (1994).
[18] J. Shabani, Y. Liu, M. Shayegan, L. Pfeiffer, K. West, and K. Baldwin, Phase diagrams for the stability of the $\nu = 1/2$ fractional quantum hall effect in electron systems confined to symmetric, wide GaAs quantum wells, Physical Review B 88, 245413 (2013).

[19] J. Eisenstein and A. MacDonald, Bose-Einstein condensation of excitons in bilayer electron systems, Nature 432, 691 (2004).

[20] J. Eisenstein, Exciton condensation in bilayer quantum hall systems, Annu. Rev. Condens. Matter Phys. 5, 159 (2014).

[21] J. Eisenstein, L. Pfeiffer, and K. West, Precursors to exciton condensation in quantum hall bilayers, Physical Review Letters 123, 066802 (2019).

[22] D. Sheng, L. Balents, and Z. Wang, Phase diagram for quantum hall bilayers at $\nu = 1$, Physical review letters 91, 116802 (2003).

[23] S. H. Simon, E. Rezayi, and M. V. Milovanovic, Coexistence of composite bosons and composite fermions in $\nu = 1/2 + 1/2$ quantum hall bilayers, Physical review letters 91, 046803 (2003).

[24] G. M"oller, S. H. Simon, and E. H. Rezayi, Paired composite fermion phase of quantum hall bilayers at $\nu = 1/2 + 1/2$, Physical review letters 101, 176803 (2008).

[25] W. Zhu, Z. Liu, F. Haldane, and D. Sheng, Fractional quantum hall bilayers at half filling: Tunneling-driven non-Abelian phase, Physical Review B 94, 245147 (2016).

[26] I. McDonald and F. Haldane, Topological phase transition in the $\nu = 2/3$ quantum hall effect, Physical Review B 53, 15845 (1996).

[27] A. Vaezi and M. Barkeshli, Fibonacci anyons from abelian bilayer quantum hall states, Physical review letters 113, 236804 (2014).

[28] S. Geraedts, M. P. Zaletel, Z. Papić, and R. S. Mong, Competing abelian and non-abelian topological orders in $\nu = 1/3 + 1/3$ quantum hall bilayers, Physical Review B 91, 205139 (2015).

[29] E. J. Bergholtz and Z. Liu, Topological flat band models and fractional Chern insulators, International Journal of Modern Physics B 27, 1330017 (2013).

[30] Z. Liu, F. Liu, and Y.-S. Wu, Exotic electronic states in the world of flat bands: From theory to material, Chinese Physics B 23, 077308 (2014).

[31] W. Li, Z. Liu, Y.-S. Wu, and Y. Chen, Exotic fractional topological states in a two-dimensional organometallic material, Physical Review B 89, 125411 (2014).

[32] D. Sheng, Z.-C. Gu, K. Sun, and L. Sheng, Fractional quantum hall effect in the absence of Landau levels, Nature communications 2, 1 (2011).

[33] N. Regnault and B. A. Bernevig, Fractional Chern insulator, Physical Review X 1, 021014 (2011).

[34] Y.-L. Wu, B. A. Bernevig, and N. Regnault, Zeology of fractional Chern insulators, Physical Review B 85, 075116 (2012).

[35] R. B. Laughlin, Quantized hall conductivity in two dimensions, Physical Review B 23, 5632 (1981).

[36] Q. Niu, D. J. Thouless, and Y.-S. Wu, Quantized hall conductance as a topological invariant, Physical Review B 31, 3372 (1985).

[37] A. Sterdyniak, N. Regnault, and B. A. Bernevig, Extracting excitations from model state entanglement, Physical review letters 106, 100405 (2011).

[38] T.-S. Zeng, D. Sheng, and W. Zhu, Quantum hall effects of exciton condensate in topological flat bands, Physical Review B 101, 195310 (2020).

[39] A. Mielke, Exact ground states for the hubbard model on the kagome lattice, Journal of Physics A: Mathematical and General 25, 4335 (1992).

[40] E. Tang, J.-W. Mei, and X.-G. Wen, High-temperature fractional quantum hall states, Physical review letters 106, 236802 (2011).

[41] G. Liu, P. Zhang, Z. Wang, and S.-S. Li, Spin hall effect on the kagome lattice with Rashba spin-orbit interaction, Physical Review B 79, 035323 (2009).

[42] H.-M. Guo and M. Franz, Topological insulator on the kagome lattice, Physical Review B 80, 113102 (2009).

[43] S. Yan, D. A. Huse, and S. R. White, Spin-liquid ground state of the $S=1/2$ kagome Heisenberg antiferromagnet, Science 332, 1173 (2011).

[44] Z. Wang and P. Zhang, Quantum spin hall effect and spin-charge separation in a kagomé lattice, New Journal of Physics 12, 043055 (2010).

[45] Y. Zhou, G. Sethi, H. Liu, Z. Wang, and F. Liu, Excited quantum anomalous and spin Hall effect: dissociation of flat-bands-enabled excitonic insulator state, Nanotechnology 33, 415001 (2022).

[46] G. Sethi, Y. Zhou, L. Zhu, L. Yang, and F. Liu, Flat-band-enabled triplet excitonic insulator in a diatomic kagome lattice, Physical Review Letters 126, 190403 (2021).

[47] H. Liu, G. Sethi, D. Sheng, Y. Zhou, J.-T. Sun, S. Meng, and F. Liu, High-temperature fractional quantum hall state in the flouket kagome flat band, Physical Review B 105, L161108 (2022).

[48] M. Kang, L. Ye, S. Fang, J.-S. You, A. Levitan, M. Han, J. I. FACIO, C. JOZWIAK, A. BOSTWICK, E. ROTENBERG, ET AL., Dirac fermions and flat bands in the ideal kagome metal FeSn, Nature materials 19, 163 (2020).

[49] L. Ye, M. Kang, J. Liu, F. Von Cube, C. R. Wicker, T. Suzuki, C. Jozwiak, A. Bostwick, E. Rotenberg, D. C. Bell, ET AL., Massive dirac fermions in a ferromagnetic kagome metal, Nature 555, 638 (2018).

[50] H. Tan, Y. Liu, Z. Wang, and B. Yan, Charge density waves and electronic properties of superconducting kagome metals, Physical review letters 127, 046401 (2021).

[51] G. Liu, X. Ma, K. He, Q. Li, H. Tan, Y. Liu, J. Xu, W. Tang, K. Watanabe, T. Taniguchi, ET AL., Observation of anomalous amplitude modes in the kagome metal CsV$_3$Sb$_5$, Nature communications 13, 1 (2022).

[52] H. Zhao, H. Li, B. R. Ortiz, S. M. Teicher, T. Park, M. Ye, Z. Wang, L. Balents, S. D. Wilson, and I. Zeljkovic, Cascade of correlated electron states in the kagome superconductor CsV$_3$Sb$_5$, Nature materials 599, 216 (2021).

[53] J. Ni, Q. Liu, Y. Yu, E. Cheng, Y. Huang, Z. Liu, X. Wang, Y. Sui, and S. Li, Ultralow-temperature heat transport in the quantum spin liquid candidate Ca$_2$Cr$_7$O$_{28}$ with a bilayer kagome lattice, Physical Review B 97, 104413 (2018).

[54] C. Balz, B. Lake, J. Reuther, H. Luetkens, R. Schönenmann, T. Herrmannsdörfer, Y. Singh, A. Niazmul Islam, E. M. Wheeler, J. A. Rodriguez-Rivera, ET AL., Physical realization of a quantum spin liquid based on a complex frustration mechanism, Nature Physics 12, 942 (2016).

[55] A. Balodhi and Y. Singh, Synthesis and pressure and field-dependent magnetic properties of the kagome-
bilayer spin liquid Ca$_{10}$Cr$_7$O$_{28}$, Physical Review Materials \textbf{1}, 024407 (2017).

[56] L. Balents, C. R. Dean, D. K. Efetov, and A. F. Young, Superconductivity and strong correlations in moiré flat bands, Nature Physics \textbf{16}, 725 (2020).

[57] Y.-H. Zhang, D. Mao, Y. Cao, P. Jarillo-Herrero, and T. Senthil, Nearly flat chern bands in moiré superlattices, Physical Review B \textbf{99}, 075127 (2019).

[58] T. Kambe, R. Sakamoto, K. Hoshiko, K. Takada, M. Miyachi, J.-H. Ryu, S. Sasaki, J. Kim, K. Nakazato, M. Takata, \textit{et al.}, $\pi$-Conjugated nickel bis (dithiolene) complex nanosheet, Journal of the American Chemical Society \textbf{135}, 2462 (2013).

[59] W. Jiang, X. Ni, and F. Liu, Exotic topological bands and quantum states in metal-organic and covalent-organic frameworks, Accounts of Chemical Research \textbf{54}, 416 (2021).

[60] X. Ni, H. Li, F. Liu, and J.-L. Brédas, Engineering of flat bands and dirac bands in two-dimensional covalent organic frameworks (COFs): relationships among molecular orbital symmetry, lattice symmetry, and electronic-structure characteristics, Materials Horizons \textbf{9}, 88 (2022).

[61] X. Zhang, Z. Wang, M. Zhao, and F. Liu, Tunable topological states in electron-doped HTT-Pt, Physical Review B \textbf{93}, 165401 (2016).

[62] M. Y. Kharitonov and K. B. Efetov, Electron screening and excitonic condensation in double-layer graphene systems, Physical Review B \textbf{78}, 241401 (2008).

[63] J. Li, T. Taniguchi, K. Watanabe, J. Hone, and C. Dean, Excitonic superfluid phase in double bilayer graphene, Nature Physics \textbf{13}, 751 (2017).

[64] X. Liu, K. Watanabe, T. Taniguchi, B. I. Halperin, and P. Kim, Quantum hall drag of exciton condensate in graphene, Nature Physics \textbf{13}, 746 (2017).

[65] M. S. Fuhrer and A. R. Hamilton, Chasing the exciton condensate, Physics \textbf{9}, 80 (2016).

[66] R.-C. Ge and M. Kolodrubetz, Floquet engineering flat bands for bosonic fractional quantum hall with superconducting circuits, Physical Review B \textbf{104}, 035427 (2021).

[67] A. G. Grushin, Á. Gómez-León, and T. Neupert, Floquet fractional chern insulators, Physical review letters \textbf{112}, 156801 (2014).

[68] See supplementary material at for ED results on finite distance bilayer Kagome at $\nu_T=1$ with varying $S_z$, and possible non-Abelian phase $\nu_T=2/3$.

[69] G. Sethi, M. Cuma, and F. Liu, Excitonic condensate in flat valence and conduction bands of opposite chirality, arXiv preprint arXiv:2210.03252 (2022).

[70] F. Haldane, Many-particle translational symmetries of two-dimensional electrons at rational landau-level filling, Physical review letters \textbf{55}, 2095 (1985).

[71] I. McDonald, Degeneracy of multicomponent quantum hall states satisfying periodic boundary conditions, Physical Review B \textbf{51}, 10851 (1995).

[72] X.-G. Wen and A. Zee, Classification of Abelian quantum hall states and matrix formulation of topological fluids, Physical Review B \textbf{46}, 2290 (1992).

[73] B. Blok and X.-G. Wen, Effective theories of the fractional quantum hall effect at generic filling fractions, Physical Review B \textbf{42}, 8133 (1990).