A STOCHASTIC DELAY MODEL FOR PRICING DEBT AND LOAN GUARANTEES:
Theoretical results

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May 3, 2014

Abstract

We consider that the price of a firm follows a non linear stochastic delay differential equation. We also assume that any claim value whose value depends on firm value and time follows a non linear stochastic delay differential equation. Using self-financed strategy and replication we are able to derive a random partial differential equation (RPDE) satisfied by any corporate claim whose value is a function of firm value and time. Under specific final and boundary conditions, we solve the RPDE for the debt value and loan guarantees within a single period and homogeneous class of debt.

Keywords: Corporate claim, Levered firm, Debt security, Loan guarantees.

1 INTRODUCTION

The valuation of corporate claims has always been an important topic for finance researchers. On one hand, bond issuers would like to know what factors affect prices and yields, as yields represent their cost of capital. On the other hand, prospective bond buyers are interested in knowing how sensitive yields and yield spreads are to various relevant factors (e.g. leverage) as they develop investment strategies. Due to the significant growth of the credit derivatives market, the interest in corporate claims values models and risk structure has recently increased. This growth is explained by the need of better prediction models to fit the real market data.

Corporate bankruptcy is central to the theory of the firm. A firm is generally considered bankrupt when it cannot meet a current payment on a debt obligation. In this case the equityholders lose all claims on the firm, and the remaining loss which is the difference between the face value of the fixed claims and the market value of the firm, is supported by the debtholders. This is the

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definition of bankruptcy that we adopt in this paper. Loan guarantees have been proposed by several authors as a way to encourage new investments for companies when they become insolvent.

The risk structure of interest rates on bonds with the same maturity is degree of the likelihood of default on the payment of interest and the principal. Returns are measured by yields to maturity of each bond. The difference between the yields of bonds with default risk and default free bonds, the yield spreads, is defined as the spread between their interest rates. This yield spread is sometimes called risk premium since it is supposed to measure the additional yield that risky bond pay in order to motivate investors to buy risky bonds instead of the less risky ones. It does not seem to be a consensus among the researchers what the determinants of the risk structure are. Different variables have been considered to represent a valid measure of risk depending on whether the same maturity or different maturities, see (5).

The current model in corporate finance was developed by Merton [5, 6]. This model is closed to Black and Scholes model for stock price. As the Black and Scholes model [12, 11], the fitness of Merton model can be questioned since the model assume that the volatility is constant and empirical evidence shows that volatility actually depends on time in a way that is not predictable. Beside, the need for better ways of understanding the behavior of many natural processes has motivated the development of dynamic models of these processes that take into consideration the influence of past events on the current and future states of the system [2, 3, 9, 10]. Following a delayed Black and Scholes formula proposed in [1] we developed the corresponding delayed model in corporate finance, which has not yet been introduced. Because of the isomorphic relationship between levered equity and a European call option (see Merton [7]) on one hand, and the isomorphic correspondence between loan guarantees and common stock put options (see Merton [6]), we can claim and prove that results obtained in the theory of option pricing are feasible in corporate liabilities pricing.

The paper is organized as follows. In section 2 we present the stochastic delay model for corporate claims. In this section we provide keys definitions in corporate finance used in this work and develop the Random Partial Differential Equation for claims. In section 3, we evaluate the debt in a levered firm. The evaluation of the loan guarantees is provided in section 4. We end by studying the impact of an additional debt on the firm’s risk structure in section 5.

2 STOCHASTIC DELAY MODEL FOR CORPORATE CLAIMS

Let us start by providing some keys definitions in finance which will be extensively used in this work.

2.1 Keys Definitions

Definition 2.1. [Firm value or Company value]

The firm value or Company value is the market value of the company’s machines and commercial activities. This value is equal to the market value of the equityholders plus the market value of the net financial debt.
Definition 2.2. [Equity Value]
The equity is the total dollar market value of all of a company’s outstanding shares. Market value of equity is calculated by multiplying the company’s current stock price by its number of outstanding shares. It’s the total value of the business after taking out the amount owed to debtholders.

Definition 2.3. [Corporate claim or corporate liability]
A corporate claim or corporate liability is an official request for money usually in the form of compensation, from a corporation.

Definition 2.4. [Debt security]
A Debt security is a security issued by a company or government which represents money borrowed from the security’s purchaser and which must be repaid at a specified maturity date, usually at a specified interest rate.

Definition 2.5. [Loan Guarantees]
Loan on which a promise is made by a third party or guarantor that he or she will be liable if the creditor fails to fulfill their contractual obligations.

2.2 Stochastic Model and Random Partial Differential Equation for claims
In order to obtain a better prediction of the company value, we need to include its history. Let us assume that the price of the firm \( V(t) \) at time \( t \in [0, T] \) follows a non linear Stochastic Delay Differential Equation (SDDE) of the form

\[
\begin{align*}
\frac{dV(t)}{dt} &= (\alpha V(t) V(t-L_1) - C) dt + g(V(t-L_2)) V(t) dW(t) \\
V(t) &= \varphi(t), \quad t \in [-L, 0]
\end{align*}
\]

(1)
on a probability space \((\Omega, \mathcal{F}, P)\) with a filtration \( (\mathcal{F}_t)_{0 \leq t \leq T} \) satisfying the usual conditions. The constants \( L_1 \) and \( L_2 \) are positive, \( \alpha \) is the riskless interest rate of return on the firm per unit time, \( C \) is the total amount payout by the firm per unit time to either the shareholders or claims-holders (e.g., dividends or interest payments) if positive, and it is the net amount received by the firm from new financing if negative. The constant \( L = \max(L_1, L_2) \) represents the past length while \( T \) is the maturity date. The function \( g : \mathbb{R} \to \mathbb{R} \) is a continuous representing the volatility function on the firm value per unit time. The initial process \( \varphi : \Omega \to C([-L, 0], \mathbb{R}) \) is \( \mathcal{F}_0 \)-measurable with respect to the Borel \( \sigma \)-algebra of \( C([-L, 0], \mathbb{R}) \), actually \( \varphi \) is the past price of the firm. The process \( W \) is a one dimensional standard Brownian motion adapted to the filtration \( (\mathcal{F}_t)_{0 \leq t \leq T} \).

The following theorem ensure that the price model (1) is feasible in the sense that it admit pathwise unique solution such that \( V(t) > 0 \) almost surely for all \( t \geq 0 \) whenever the initial path \( \varphi(t) > 0 \) for all \( t \in [0, t] \).

**Theorem 2.6.**
The firm price model (1) has a unique solution. Furthermore, if \( C = 0 \) the
solution is represented by the formula

\[ V(t) = \varphi(0) \exp \left( \int_0^t \alpha(s)V(s-L_1)ds - \frac{1}{2} \int_0^t (g(V(s-L_2)))^2 ds \right) \]

(2)

\[ + \int_0^t g(V(s-L_2))dW(s) \]

(3)

Proof. Proof can be found in [1], where the authors deal with stock price. □

Following the work in [5], in order to derive a random partial differential equation which must be satisfied by any security whose value can be written as a function of the value of the firm and time. We assume that any claim with market value \( Y(t) \) (which can be replicated using self-financed strategy) at time \( t \) with \( Y(t) = F(V(t), t) \) follows a non-linear stochastic delay differential equation

\[
\begin{cases}
  dY(t) = (\alpha_y Y(t) - C_y)dt + g_y(Y(t-L_2))Y(t)dW_y(t), \ t \in [0, T] \\
  Y(t) = \varphi_y(t), \ t \in [-L, 0],
\end{cases}
\]

(4)

on a probability space \( (\Omega, \mathcal{F}, P) \). The constant \( \alpha_y \) is the riskless interest rate of return per unit time on this claim; \( C_y \) is the amount payout per unit time to this claim; \( g_y: \mathbb{R} \to \mathbb{R} \) is a continuous function representing the volatility function of the return on this claim per unit time; the initial process \( \varphi_y: \Omega \to C([-L, 0], \mathbb{R}) \) is \( \mathcal{F}_0 \)-measurable with respect to the Borel \( \sigma \)-algebra of \( C([-L, 0], \mathbb{R}) \). The process \( W_y \) is a one dimensional standard Brownian motion adapted to the filtration \( (\mathcal{F}_t)_{0 \leq t \leq T} \).

Assumption 2.7. The value of the company is unaffected by how it is financed (the capital structure irrelevance principle).

**Theorem 2.8.** Assume that the value of the firm \( V(t), t \in [0, T] \) follows the SDDE [1]. Furthermore, suppose that Assumption 2.7 is satisfied and that the debt value accumulates interest compounded continuously at a rate of \( r \), that is \( B(t) = B(0)e^{rt} \). For any claim whose value is a function of the firm value and time i.e. \( Y(t) = F(V(t), t) \), where \( F \) is twice continuously differentiable with respect to \( V \) and once differentiable with respect to \( t \), the following RPDE should be satisfied:

\[
\frac{1}{2}g^2(V(t-L_2))v^2F_{vv} + (rv - C)F_v + F_t - rF + C_y = 0, \ (t,v) \in (0, T) \times \mathbb{R}^+ \quad (5)
\]

with

\[
F_t(v,t) = \frac{\partial F(v,t)}{\partial t}, \quad F_v(v,t) = \frac{\partial F(v,t)}{\partial v}, \quad F_{vv}(v,t) = \frac{\partial^2 F(v,t)}{\partial v^2}.
\]

Proof. The proof is closed to one in [5] for non delayed model. Note that, for a given \( Y(t) = F(V(t), t) \), there are similarities between the \( \alpha_y, g_y, dW_y \) and the corresponding \( \alpha, g, dW \) in SDDE [1]. Knowing that \( V(t) \) is an Ito process and since we assumed that \( F \) is twice continuously differentiable with respect
to $v$ and once differentiable with respect to $t$, Ito formula, allows us to write the following for $Y(t) = F(V(t), t)$

\[ df(V(t), t) = F_t(V(t), t)dt + F_v(V(t, L_1) - C)dt + g(V(t, L_2)V(t)dw(t)) + \frac{1}{2}F_{vv}[g^2(V(t, L_2)V^2(t)](dw(t))^2. \]

Hence the dynamic equation for $Y(t)$ is

\[ \begin{align*}
    dY(t) &= \left[ \frac{1}{2}F_{vv}g^2(V(t, L_2)V^2(t) + F_v\alpha V(t)\alpha V(t, L_1) - C) + F_t(V(t), t) \right] dt \\
    &+ F_vg(V(t, L_2)V(t)dw(t)
\end{align*} \]

From the uniqueness of solution to stochastic delay differential equation in [1], we have the equality almost surely of the coefficients of the corresponding terms $dt$ and $dW(t)$ in [4] and [5] as follow

\[ \begin{align*}
    \alpha g(Y(t) - C_y = \alpha gF(Y(t), t) - C_y \\
    \equiv \frac{1}{2}g^2(V(t, L_2)V^2(t)F_v(V(t, t) + \alpha V(t)V(t, L) - C)F_v + F_t
\end{align*} \]

\[ \begin{align*}
    g_y(Y(t, L_2))Y(t) &= g_y(F(V(t, L_2), t))F(V(t), t) \\
    &\equiv g(V(t, L_2))V(t)F_v(V(t), t) \\
    dW_y(t) &= dW(t).
\end{align*} \]

Following the self-financing and replication strategy ([1]), let $z_1$ be the instantaneous number corresponding to the amount invested in the firm, $z_2$ be the instantaneous number corresponding to the amount invested in the security and $z_3$ be the instantaneous number corresponding to the amount invested in riskless debt. Consider $dx$ the instantaneous return to the portfolio and assume the total investment in the portfolio is zero, we may write $z_1 + z_2 + z_3 = 0$ and then

\[ \begin{align*}
    dx &= \frac{z_1 dV(t) + Cdt}{V(t)} + \frac{z_2 dY(t) + C_y dt}{Y(t)} + z_3 dt \\
    &= \frac{z_1 [(\alpha V(t)V(t, L_1) - C)dt + g(V(t, L_2)V(t)dw(t)] + C_2 dt}{V(t)} \\
    &+ \frac{z_2 [(\alpha_y Y(t) - C_y)dt + g_y(Y(t, L_2)Y(t)dw_y(t)] + C_v z_2 dt}{Y(t)} + z_3 r dt \\
    &= z_1 \alpha V(t, L_1) dt + z_1 g(V(t, L_2)dw(t) + z_2 \alpha_y dt + z_2 g_y(Y(t, L_2)dw_y(t) \\
    &- (z_1 + z_2)rdt.
\end{align*} \]

Hence from the equivalence (4), we have

\[ dx = [z_1 \alpha V(t, L_1) - r + z_2 (\alpha_y - r)] dt + [z_1 g(V(t, L_2)) + z_2 g_y(Y(t, L_2))] dw(t). \]

Since the return on the portfolio is non stochastic and there is no arbitrage condition we have: $z_1 g(V(t, L_2)) + z_2 g_y(Y(t, L_2)) = 0$ and $z_1 (\alpha V(t, L_1) - r) + z_2 (\alpha_y - r) = 0$ leading to the following system:

\[ \begin{align*}
    z_1 g(V(t, L_2)) + z_2 g_y(Y(t, L_2)) &= 0 \\
    z_1 (\alpha V(t, L_1) - r) + z_2 (\alpha_y - r) &= 0.
\end{align*} \]
A non trivial solution \((z_i \neq 0)\) to this system exists if and only if
\[
\begin{pmatrix}
\alpha V(t - L_1) - r \\
g(V(t - L_2))
\end{pmatrix} = \begin{pmatrix}
\alpha_y - r \\
g_y(F(V(t - L_2), t))
\end{pmatrix}.
\] (11)

But from (7) and (8) substituting for \(\alpha\) and \(g_y(F(V(t - L_2), t))\), we get
\[
\alpha_y = \frac{1}{2} g^2(V(t - L_2)) V^2(t) F_{vv} + (\alpha V(t)V(t - L_1) - C) F_v + F_t + C_y
\]
and
\[
g_y(F(V(t - L_2), t)) = \frac{g(V(t - L_2)) V(t) F_v}{F(V(t), t)}
\]
Replacing \(\alpha_y\) and \(g_y(F(V(t - L_2), t))\) in (11), we obtain
\[
\frac{\alpha V(t - L_1) - r}{g(V(t - L_2))} = \frac{1}{2} g^2(V(t - L_2)) V^2(t) F_{vv} + (\alpha V(t) V(t - L_1) - C) F_v + F_t + C_y - r F(V(t), t)
\]
By rearranging terms and simplifying, we get
\[
a V(t)\frac{V(t - L_1) F_v - r V(t) F_v}{F(V(t - L_2))} = \frac{1}{2} g^2(V(t - L_2)) V^2(t) F_{vv} + (\alpha V(t) V(t - L_1) - C) F_v + F_t + C_y - r F(V(t), t).
\] (12)

Therefore, we can rewrite equation (12) as the following random parabolic partial differential equation for \(F\)
\[
\frac{1}{2} g^2(V(t - L_2)) V^2 F_{vv} + (r v - C) F_v + F_t + C_y - r F = 0.
\]

For any claim whose value depends on the value of the firm and time, the equation (5) must be satisfied under some specific boundary conditions and initial condition. From these boundary conditions, we will be able to distinguish the debt of a firm from its equity. By definition the value \(V\) of the company can be written as
\[
V(t) = F(V(t), t) + f(V(t), t),
\] (13)
where \(f(V(t), t)\) is the value of the equity, \(F(V(t), t)\) the value of debt a any time \(t\) before the maturity. Because both \(F\) and \(f\) can only take on non-negative values, we have that for initial condition
\[
F(0, t) = f(0, t) = 0.
\] (14)
Further \(F(V(t), t) \leq V(t)\) which implies the regular condition
\[
\frac{F(V(t), t)}{V(t)} \leq 1.
\] (15)
Using relation (13), for \(F\) satisfying the RPDE (5), the equity \(f\) therefore satisfy the following equation
\[
\frac{1}{2} g^2(V(t - L)) V^2 f_{vv} + (r v - C) f_v + f_t + C - C_y - r f = 0.
\] (16)
Remark 2.9. Notice that $C$ and $\alpha$ ($C_y$ and $\alpha_y$) in the SDDE (3) and (4) respectively can be time dependent functions, in which case they will be measurable and integrable in the interval $[0, T]$.

In the accompanied paper [4] and in [3], efficient numerical methods to solve (5) and (16) subject to final and boundary conditions (14), (17) and (18) are presented.

In the sequel we will assume that $C = C_y = 0$ and provide the representations of the exact solutions.

3 EVALUATION OF DEBT IN A LEVERED FIRM

In this section, we consider a claim market value as the simplest case of corporate debt and therefore use the following assumption.

Assumption 3.1. We assume that:

(a) The company is financed by:

1. A single class of debt
2. The equity.

(b) The following restrictions and provisions are stipulated in the contract according to the bond issue

1. The firm must pay an amount $B(T)$ to the debtholders at the maturity date $T$;
2. In case the firm cannot make the payment, the debtholders take over the company and the equityholders lose their investment;
3. The firm is not allow neither to issue a new senior claim on the firm nor to pay cash dividend during the option life. In other words, there is no coupon payment nor dividends prior to the maturity of the debt (i.e. $C = C_y = 0$).

From Assumption 3.1 we have the following final conditions

$$F(V, T) = \min[V, B(T)], \quad f(V, T) = \max(V - B(T), 0).$$

(17)

Toward infty, as for option prices, we also have the following boundary conditions

$$F(V, t) \sim v - B(T)e^{-r(T-t)}, \quad f(V, t) \sim V - B(T)e^{-r(T-t)}, \quad \text{as } v \to \infty.$$  

(18)

Since there are no coupon payments, the values of $C_y$ and $C$ in equation 4 are zero. Equations (4) and (18) coupled with final and boundary conditions are given respectively by

$$\begin{align*}
\frac{1}{2}g^2(V(t-L))v^2f_{vv} + rvf_v + f_t - rf &= 0, \quad 0 < t < T \\
f(0, T) &= \max[v - B(T), 0], \quad v > 0 \\
f(0, t) &= 0, \quad f(v, t) \sim v - B(T)e^{-r(T-t)}, \quad \text{as } v \to \infty,
\end{align*}$$

(19)
and

\[
\begin{cases}
\frac{1}{2} g^2(V(t - L)) v^2 F_{vv} + rv F_v + F_t - r F = 0, \quad 0 < t < T \\
\bar{F}(v, T) = \min[V, B(T)], \quad v > 0 \\
F(0, t) = 0, \quad F(v, t) \sim B(T)e^{-r(T-t)}, \text{ as } v \to \infty,
\end{cases}
\]  

(20)

We shall solve the above parabolic partial differential equation (19) directly using the standard method based on Fourier transforms.

**Lemma 3.2.** The backward parabolic RPDE of the form (17) (where \( C = C_y = 0 \)) with final solution \( f(v, T) = \phi(v) \) can be transformed to the well known heat equation

\[ h_\tau = h_{xx}, \quad h = h(x, \tau), \quad \tau = \tau(t), x = x(v). \]  

(21)

**Proof.** Let us make the following change of variables

\[
\begin{align*}
\tau & = \frac{1}{2} \int_t^T g^2(V(s - L_2)) ds. \\

\end{align*}
\]  

(22)

The corresponding partial derivatives are given as

\[ h_\tau = \frac{e^{-rt}}{B(T)} \frac{\partial}{\partial x} f_v = \frac{e^{-rt}}{B(T)} v f_v, \]

\[
\begin{aligned}
f_t & = B(T)e^{rt} h \left( x - \frac{1}{2} \int_t^T g^2(V(s - L_2)) ds - r(T - t), \frac{1}{2} \int_t^T g^2(V(s - L_2)) ds \right) \\
& \quad + B(T)e^{rt} \left[ \left( \frac{1}{2} \int_t^T g^2(V(t - L_2)) \right) h_x - \frac{1}{2} g^2(V(t - L_2)h_\tau) \right].
\end{aligned}
\]

Then

\[
h_\tau = \frac{-f_t + B(T)e^{rt} h \left( x - \frac{1}{2} \int_t^T g^2(V(s - L_2)) ds - r(T - t), \frac{1}{2} \int_t^T g^2(V(s - L_2)) ds \right)}{\frac{1}{2} B(T)e^{rt} g^2(V(t - L_2)) + \frac{1}{2} B(T)e^{rt} g^2(V(t - L_2)) h_x - r B(T)e^{rt} h_\tau}.
\]

Applying the change of variables, we get

\[
\begin{align*}
h_\tau & = \frac{-f_t + rf(v, t) + \frac{1}{2} g^2(V(t - L_2)) v f_v - r v f_v}{\frac{1}{2} B(T)e^{rt} g^2(V(t - L_2))}, \quad (23) \\
h_{xx} & = \frac{e^{-rt}}{B(T)} \frac{\partial}{\partial v} \frac{\partial}{\partial x} \phi = \frac{e^{-rt}}{B(T)} v (f_v + v f_{vv}). \quad (24)
\end{align*}
\]
Plugging (23) and (24) into (21), we get

\[
\frac{1}{2}g^2(V(t-L_2))v^2fv + rvf_v + f_t - rf = 0, 
\]  

(25)

We can observe from (22) that the final condition in (19) correspond to the initial condition in (21).

Theorem 3.3. Assume that the value of the firm \( V(t), t \in [0,T] \) follows the SDDE (7). Furthermore, suppose that Assumption 2.7 and 3.1 are satisfied. The equity function \( f \), solution of (19) and the debt function \( F \), solution of (20) are given respectively by

\[
f(V(t), t) = V(t)\Phi(x_1) - Be^{-r(T-t)}\Phi(x_2), \quad (26)
\]

\[
F(V(t), t) = Be^{-r(T-t)} \left[ \Phi(x_2) + \frac{1}{d} \Phi(-x_1) \right], \quad (27)
\]

where

\[
\begin{align*}
\Phi(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy, \\
x_1 &= \log \frac{V(t)}{B} + r(T-t) + \frac{1}{2} \int_t^T g^2(V(s-L_2))ds \quad \sqrt{\int_t^T g^2(V(s-L_2))ds}, \\
x_2 &= x_1 - \sqrt{\int_t^T g^2(V(s-L_2))ds} \\
d &= \frac{Be^{-r(T-t)}}{V(t)}
\end{align*}
\]

(28)

Proof. From Lemma 3.2 we have

\[
f(V, t) = B(T)e^{rt}h(x, \tau),
\]

\[
= B(T)e^{rt}h(x - \frac{1}{2} \int_t^T g^2(V(s-L_2))ds + r(T-t), \frac{1}{2} \int_t^T g^2(V(s-L_2))ds).
\]

But, the fundamental solution to the diffusion equation (21) is given by the Green’s function

\[
G(x, \tau) = \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{x^2}{4\tau}}.
\]

Furthermore, the general solution \( h \) with initial condition \( h(x, 0) = \phi(x) \) is given by the convolution

\[
h(x, \tau) = h(x, 0) * G(x, \tau)
\]

\[
= \int_{-\infty}^{\infty} G(x - \eta, \tau)\phi(\eta)d\eta
\]

\[
= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\tau}} \phi(\eta) \exp \left[ -\frac{(x - \eta)^2}{4\tau} \right] d\eta.
\]
Now,

\[ h(x, \tau) = e^{-rT} \int_{-\infty}^{\infty} \max[e^{\eta} - 1, 0] \exp \left[ -\frac{(x - \tau - rt + rT - \eta)^2}{4\tau} \right] d\eta \]

\[ = e^{-rT} \int_{0}^{\infty} \exp \left[ \frac{4\eta^2 - (x - \tau - rt + rT - \eta)^2}{4\tau} \right] d\eta \]

\[ = e^{-rT} \int_{0}^{\infty} \exp \left[ -\frac{(x - \tau - rt + rT - \eta)^2}{4\tau} \right] d\eta \]

\[ = I_1 - I_2, \]  

(30)

where

\[ I_1 = e^{-rT} \int_{0}^{\infty} \exp \left[ \frac{4\eta^2 - (x - \tau - rt + rT - \eta)^2}{4\tau} \right] d\eta \]

and

\[ I_2 = e^{-rT} \int_{0}^{\infty} \exp \left[ -\frac{(x - \tau - rt + rT - \eta)^2}{4\tau} \right] d\eta. \]

We first solve \( I_1 \). We make the following change of variable

\[ z = \frac{4\eta^2 - (x - \tau - rt + rT - \eta)^2}{\sqrt{2\tau}} \rightarrow d\eta = \sqrt{2\tau}dz. \]

Completing the perfect square in the exponential of the integrand, we have

\[ 4\eta^2 - (x - \tau - rt + rT - \eta)^2 \]

\[ = -((x + \tau - rt + rT) - \eta)^2 + (x + \tau - rt + rT)^2 - (x - \tau - rt + rT)^2 \]

\[ = -((x + \tau - rt + rT) - \eta)^2 + 4\tau(x - rt + rT). \]

Moreover, we define the lower limit of the integration as

\[ x_1 = \frac{(x + \tau - rt + rT)}{\sqrt{2\tau}}. \]

Hence we can write

\[ I_1 = \exp \left[ -rT + \frac{4\tau(x - rt + rT)}{4\tau} \right] \int_{-x_1}^{\infty} \exp \left[ -\frac{z}{\sqrt{2\pi}} \right] dz \]

\[ = e^{x - rt} \Phi(x_1). \]

We will compute \( I_2 \) in a similar way. Let us make the following change of variable

\[ y = \frac{-(x + \tau + rt - rT + \eta)}{\sqrt{2\tau}} \rightarrow \eta = \sqrt{2\tau}dy. \]

As before we define the lower limit of the integration as

\[ x_2 = \frac{(x - \tau - rt + rT)}{\sqrt{2\tau}}. \]

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\[ I_2 = \frac{e^{-rT}}{\sqrt{4\pi \tau}} \int_0^\infty \exp \left[ -\frac{(x - \tau - rt + rT - \eta)^2}{4\tau} \right] d\eta \]

\[ = e^{-rT} \int_{-x_2}^\infty \exp \left[ -\frac{\eta^2}{2\tau} \right] \frac{1}{\sqrt{2\pi}} = e^{-rT} \Phi(x_2). \]

Using the results above, we obtain the solution of the heat equation

\[ h(x, \tau) = e^{x-rT} \Phi(x_1) - e^{-rT} \Phi(x_2). \quad (31) \]

Now, we want to proceed backward using the relation between \( h, x \) and \( \tau \), and \( f, v \) and \( t \), respectively to get the solution for the RPDE. From relation (29), we can write the solution of the RPDE (19) as follow

\[ f(v, t) = v \Phi(x_1) - B(T)e^{-(T-t)} \Phi(x_2). \]

Finally, if we replace \( v \) by the value of the company \( V(t) \) in the above equation, we obtain the formula for the equity value

\[ f(V(t), t) = V(t) \Phi(x_1) - B(T)e^{-(T-t)} \Phi(x_2), \quad (32) \]

The debt function \( F \) is obtained using the relation \( F(V(t), t) = V(t) - f(V(t), t) \). More details can be found in [3].

Notice that we can consider \( F \) as a function of \( V \) and \( \tau_1 = T-t \) (see [5] when \( g \) is constant). Let \( R(\tau_1) \) the yield to the maturity on the risky debt provided that the firm does not default defined by

\[ e^{-R(\tau_1)T} = \frac{F(V(t), \tau_1)}{B}. \quad (33) \]

We therefore have

\[ R(\tau_1) - r = -\frac{1}{\tau_1} \log \left\{ \Phi[x_2] + \frac{1}{d} \Phi[-x_1] \right\}. \quad (34) \]

It seems reasonable to call \( R(\tau_1) - r \) a risk premium in which case equation (34) defines a risk structure of interest rates. As in Merton ([5]), the risk premium is a function of the volatility function \( g \) and \( d \).

### 4 EVALUATION OF LOAN GUARANTEES

Now let us examine the impact of a guarantor, that is a government or an institution insuring payment to the bondholders in any case. Now we make the following assumption.

**Assumption 4.1.** We assume that

(a) The company is financed by:

1. A single class of debt.
2. The equity.
3. The guarantee on the debt.
(b) The following restrictions and provisions are stipulated in the contract according to the loan guarantees issue. The contract stipulate that

1. In case the management on the maturity date is unable to make the payment promised, the government will meet these payments with no uncertainty;

2. The firm is expected to pay an amount at least equal to its actuarial cost for the guarantee, so that in case this happens, the firm is required to default all its assets to the guarantor;

3. The firm is not allow neither to issue a new senior claim on the firm nor to pay cash dividend during the option life i.e $C = C_y = 0$.

Notice that the presence of a guarantor transforms the debt which was a risky asset to a riskless asset. If the firm value is less than the promised payment, then the debtholders receive $B$, the equity holders receive nothing. Therefore, the guarantor lose the amount $B - V(T)$. However, if the firm value is greater than the promised payment, then the debtholders receive $B$ and the equity holders receive $V(T) - B$ as without the guarantee. In other words, the guarantor has no impact on the equity value (max[$V(T) - B, 0]$) at the maturity date, but the debt value is riskless and always known as $B$. However, the value of the guarantor is the non positive value min[$V(T) - B, 0$]. In effect, the result of the guarantee is to create an additional cash inflow to the firm of the amount $-\min[V(T) - B, 0]$. But, $-\min[V(T) - B, 0] = \max[B - V(T), 0]$. Therefore, if $G(T)$ is the cost we are looking for, where the length of time until the maturity date of the bond is $T$, we can write

$$G(T) = G(V(T), T) = \max[B - V(T), 0],$$

with

$$\frac{1}{2}g^2(V(t - L))v^2G_{vv} + rvG_v + G_t - rG = 0, \quad 0 < t < T$$

$$G(0, t) = 0,$$

where $B$ can be taken as the strike price and $V(T)$ as the stock price $S(T)$. These similarities between the evaluation of $G(T)$ and the evaluation of an European put option allow us to say that loan guarantees works as an European put option on the firm value giving to the management the right but not an obligation to sell the amount $B$ to a guarantor.

**Theorem 4.2.** Assume that the value of the firm $V(t)$, $t \in [0, T]$ follows the SDDE (2). Furthermore, suppose that Assumption 2.7 and 4.1 are satisfied. Then loan guarantees (a fair premium equal to the present value of the cash flows from the option) is given by

$$G(V(t), t) = Be^{-r(T-t)}\Phi [x_1] - V(t)\Phi [x_2],$$

with (28).

**Proof.** The proof is similar to the proof of Theorem 3.3 using Lemma 3.2 and can be found in [3].
Remark 4.3. Notice that the probabilistic methods can be used to derive the
the equity $f$, the debt $F$ and the loan guarantee $G$. The technique is the same
as in [2] for options price. More details can be found in [3, 4]. The analysis of
the risk structure for an homogeneous class of debt is done in the same way as
in Merton model in [5](see [3]).

Remark 4.4. Notice that when the volatility function $g$ is constant, all results
in this paper are the same as Merton’s results in [5].

5 IMPACT OF AN ADDITIONAL DEBT ON
THE FIRM’S RISK STRUCTURE

Let us verify the impact of the guarantee on the company. Assume a levered
company financed by equity and debt. Assume the face value of the debt is
$B$. Let us compute the probability of default of this company given by $P(V(T) < B)$. From the work in [1, 2], by setting $V(t) = e^{-rt}V(t)$ we have

$$
\tilde{V}(t) = \varphi(0) \exp \left( \int_0^t g(V(s - L_2))dW^*(s) - \frac{1}{2} \int_0^t g^2(V(s - L_2))ds \right),
$$

(38)

where

$$
W^*(t) := W(t) + \int_0^t \frac{\alpha V(s - L_1) - r}{g(V(s - L_2))} ds, \quad t \in [0, T],
$$

Indeed,

$$
P(V(T) < B)
= P \left( \tilde{V}(T)e^{-rT} < B \right)
= P \left( \tilde{V}(t) \exp \left( -\frac{1}{2} \int_t^T g^2(V(s - L_2))ds + \int_t^T g(V(s - L_2))dW^*(s) \right) < Be^{-rT} \right)
= P \left( \exp \left( -\frac{1}{2} \int_t^T g^2(V(s - L_2))ds + \int_t^T g(V(s - L_2))dW^*(s) \right) < \frac{Be^{-r(T-t)}}{V(t)} \right)
= P \left( \int_t^T g(V(s - L_2))dW^*(s) < \log d + \frac{1}{2} \int_t^T g^2(V(s - L_2))ds \right)
= P \left( \frac{\int_t^T g(V(s - L_2))dW^*(s)}{\sqrt{\int_t^T g^2(V(s - L_2))ds}} < \frac{\log d + \frac{1}{2} \int_t^T g^2(V(s - L_2))ds}{\sqrt{\int_t^T g^2(V(s - L_2))ds}} \right)
= \Phi \left( \frac{\log d + \frac{1}{2} \int_t^T g^2(V(s - L_2))ds}{\sqrt{\int_t^T g^2(V(s - L_2))ds}} \right)
= \Phi \left( x_1 \right)

$$

Notice that we have used the fact that

$$
\frac{\int_t^T g(V(s - L))dW^*(s)}{\sqrt{\int_t^T g^2(V(s - L))ds}}
$$

is normally distributed with mean 0 variance 1 so that $\int_t^T g(V(s - L))dW^*(s)$ is normally distributed with mean 0 and variance $\int_t^T g^2(V(s - L))ds$. 

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Now let us consider that a different debt is added to value of the company $V(t)$ and compare the probability of default with the previous. An additional debt of face value $B'$ will increase the total face value which becomes $B + B'$. If $V(t) > B$, that is

\[
\frac{V(t)}{B} > \frac{V(t) + B'}{B + B'} = \frac{V'(t)}{B + B'}
\]

where $V'(t) = V(t) + B'$. Since logarithm is an increasing function, we can write

\[
\log \left( \frac{B + B'}{V(t)} \right) < \log \left( \frac{B + B'}{V'(t)} \right).
\]

So that, $x_1 < x'_1$ where

\[
x'_1 = \frac{\log \left( \frac{B + B'}{V'(t)} \right) - r(T - t) + \frac{1}{2} \int_t^T g^2(V(s - L))ds}{\sqrt{\int_t^T g^2(V(s - L))ds}}
\]

and therefore $\Phi(x_1) < \Phi(x'_1)$. From the previous analysis, we can say that loan guarantees do not prevent bankruptcy. They mainly care about debt holders investments.

Now, suppose $V(t) < B$ then $\Phi(x_1) > \Phi(x'_1)$. This means, if the firm is already less than the face value of the debt, they may be a chance that an additional debt may decrease the probability of default. But the question is will this additional debt able to avoid firm to a new bankruptcy situation? For this reason, we need to compute what is the profitability index for a new project that we want to invest in and decide.

**ACKNOWLEDGEMENTS**

Antoine Tambue is supported by the Research Council of Norway under grant number 190761/S60.

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