ASPECTS ON WEAK, $s$-CS AND ALMOST INJECTIVE RINGS

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Abstract. It is not known whether right $CF$-rings ($FGF$-rings) are right artinian (quasi-Frobenius). This paper gives a positive answer of this question in the case of weak $CS$ ($s$-CS) and $GC^2$ rings. Also we get some new results on almost injective rings.

1. Introduction

A module $M$ is said to satisfy $C1$-condition or called $CS$-module if every submodule of $M$ is essential in a direct summand of $M$. Patrick F. Smith [20] introduced weak $CS$ modules. A right $R$-module $M$ is called weak $CS$ if every semisimple submodule of $M$ is essential in a summand of $M$. I. Amin, M. Younis and N. Zeyada [11] introduced soc-injective and strongly soc-injective modules, for any two modules $M$ and $N$, $M$ is soc-$N$-injective if any $R$-homomorphism $f : soc(N) \to M$ extends to $N$. $R$ is called right (self-) soc-injective, if the right $R$-module $R_R$ is soc-injective. $M$ is strongly soc-injective if $M$ is soc-$N$-injective for any module $N$. They proved that every strongly soc-injective module is weak $CS$.

N. Zeyada [24] introduced the notion of $s$-CS, for any right $R$-module $M$, $M$ is called $s$-CS if every semisimple submodule of $M$ is essential in a summand of $M$.

A ring $R$ is called a right $CF$ ring if every cyclic right $R$-module can be embedded in a free module. A ring $R$ is called a right $FGF$ ring if every finitely generated right $R$-module can be embedded in a free right $R$-module. In section 2, we show that the right $CF$, weak $CS$ ($s$-CS) and $GC^2$ rings are artinian.

Zeyada, Hussein and Amin introduced the notions of almost and rad-injectivity [23]. In the third section we make a correction to the result [23, Theorem 2.12] and we get a new results using these notions.

Throughout this paper $R$ is an associative ring with identity and all modules are unitary $R$-modules. For a right $R$-module $M$, we denote the socle of $M$ by $soc(M)$. $S_r$ and $S_l$ are used to indicate the right socle and the left socle of $R$, respectively. For a submodule $N$ of $M$, the notations $N \subseteq^{\text{ess}} M$ and $N \subseteq^{\oplus} M$ mean that $N$ is essential and direct summand, respectively. We refer to [2], [5], [7], [12] and [14] for all undefined notions in this paper.

2. Generalizations of $CS$-modules and rings

Lemma 1. For a right $R$-module $M$, the following statements are equivalent:

1. $M$ is weak $CS$.
2. $M = E \oplus T$ where $E$ is $CS$ with $soc(M) \subseteq^{\text{ess}} E$.

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(3) For every semisimple submodule \( A \) of \( M \), there is a decomposition \( M = M_1 \oplus M_2 \) such that \( A \subseteq M_1 \) and \( M_2 \) is a complement of \( A \) in \( M \).

Proof. (1) \( \implies \) (2). Let \( M \) be a weak \( CS \). Then \( \text{soc}(M) \) is essential in a summand, so \( M = E \oplus T \) with \( \text{soc}(M) \subseteq_{\text{ess}} E \). Now if \( K \) is a submodule of \( E \), then \( \text{soc}(K) \subseteq_{\text{ess}} L \) where \( L \) is a summand of \( M \) and \( L \subseteq_{\text{ess}} (K + L) \). But \( L \) is closed, so \( K \subseteq L \). Since \( E \subseteq_{\text{ess}} (L + E) \) and \( E \) is closed in \( M \), so \( L \subseteq E \) and \( E \) is \( CS \).

(2) \( \implies \) (1). If \( E \) is \( CS \) and a summand of \( M \) with \( \text{soc}(M) \subseteq_{\text{ess}} E \), then every submodule of \( \text{soc}(M) \) is a summand of \( E \) and a summand of \( M \).

(1 \( \implies \) 3). Let \( A \) be a submodule of \( \text{soc}(M) \). By (1), there exists \( M_1 \subseteq M \) such that \( A \subseteq_{\text{ess}} M_1 \). Write \( M = M_1 \oplus M_2 \) for some \( M_2 \subseteq M \). Since \( M_2 \) is a complement of \( M_1 \) in \( M \) and \( A \) is essential in \( M_1 \), then \( M_2 \) is a complement of \( A \) in \( M \).

(3 \( \implies \) 1). Let \( A \) be a submodule of \( \text{soc}(M) \). By (2), there exists a decomposition \( M = M_1 \oplus M_2 \) such that \( A \subseteq M_1 \) and \( M_2 \) is a complement of \( A \) in \( M \). Then \( (A \oplus M_2) \subseteq_{\text{ess}} M = M_1 \oplus M_2 \) and \( A \subseteq M_1 \) then \( A \subseteq_{\text{ess}} M_1 \). Hence \( M \) is weak \( CS \) module. \( \square \)

Recall that, a right \( R \)-module \( M \) is \( s-CS \) if every singular submodule of \( M \) is essential in a summand [24].

**Proposition 1.** If \( M \) is a right \( R \)-module, then the following statements are equivalent:

(1) \( M \) is \( s-CS \).

(2) The second singular submodule \( Z_2(M) \) is \( CS \) and a summand of \( M \).

(3) For every singular submodule \( A \) of \( M \), there is a decomposition \( M = M_1 \oplus M_2 \) such that \( A \subseteq M_1 \) and \( M_2 \) is a complement of \( A \) in \( M \).

Proof. (1) \( \iff \) (2). [24] Proposition 14].

(1) \( \iff \) (3). Similar argument of the proof of the above Lemma. \( \square \)

Given a right \( R \)-module \( M \) we will denote by \( \Omega(M) \) [respectively \( C(M) \)] a set of representatives of the isomorphism classes of the simple quotient modules (respectively simple submodules) of \( M \). In particular, when \( M = R_R \), then \( \Omega(R) \) is a set of representatives of the isomorphism classes of simple right \( R \)-modules.

**Lemma 2.** Let \( R \) be a ring, and let \( P_R \) be a finitely generated quasi-projective \( CS \)-module, such that \( |\Omega(P)| \leq |C(P)| \). Then \( |\Omega(P)| = |C(P)| \), and \( P_R \) has finitely generated essential socle.

Proof. See [17] Lemma 7.28]. \( \square \)

**Proposition 2.** Let \( R \) be a ring. Then \( R \) is a right \( PF \)-ring if and only if \( R_R \) is a cogenerator and \( R \) is weak \( CS \).

Proof. Every right \( PF \)-ring is right self-injective and is a right cogenerator by [17] Theorem 1.56]. Conversely, if \( R \) is weak \( CS \) and \( R \) is cogenerator then \( R = E \oplus T \) where \( E \) is \( CS \) with \( S_r \subseteq_{\text{ess}} E \). By the above Lemma, \( E \) has a finitely generated, essential right socle. Since \( E \) is right finite dimensional and \( R_R \) is a cogenerator, let \( S_r = S_1 \oplus S_2 \oplus \ldots \oplus S_m \) and \( I_i = I(S_i) \) be the injective hull of \( S_i \), then there exists an embedding \( \sigma : I_i \rightarrow R^I \) for some set \( I \). Then \( \pi \circ \sigma \neq 0 \) for some projection \( \pi : R^I \rightarrow R \), so \( (\pi \circ \sigma)|S_i \neq 0 \) and hence is monic. Thus \( \pi \circ \sigma : I_i \rightarrow R \) is monic, and so \( R = E_1 \oplus \ldots \oplus E_m \oplus T \) where \( \text{soc}(T) = 0 \). So \( R \) is a right \( PF \)-ring. \( \square \)
Proposition 3. [24] Proposition 16] Let $R$ be a ring. Then $R$ is a right PF-ring if and only if $R_R$ is a cogenerator and $(Z^2_R)_R$ is CS.

Proposition 4. The following statements are equivalent:

1. Every right $R$-module is weak CS.
2. Every right $R$-module with essential socle is CS.
3. For every right $R$-module $M$, $M = E \oplus K$ where $E$ is CS with $\text{soc}(M) \subseteq \text{ess}\text{-soc} E$.

Proposition 5. The following statements are equivalent:

1. Every right $R$-module is s-CS.
2. Every Goldie torsion right $R$-module is CS.
3. For every right $R$-module $M$, $M = Z_2(M) \oplus K$ where $Z_2(M)$ is CS.

Dinh Van Huynh, S. K. Jain and S. R. López-Permouth [?] proved that if $R$ is simple such that every cyclic singular right $R$-module is CS, then $R$ is right noetherian.

Corollary 1. If $R$ is simple such that every cyclic right $R$-module is s-CS, then $R$ is right noetherian.

Proposition 6. If $R$ is a weak CS and GC2, and right Kasch, then $R$ is semiperfect.

Proof. Since $R$ is a weak CS, so $E$ is CS by Lemma [1] and $R = E \oplus K$ for some right ideal $K$ of $R$ and so $E$ is a finitely generated projective module. By Lemma [2] $E$ has a finitely generated essential socle. Then, by hypothesis, there exist simple submodules $S_1, \cdots, S_n$ of $E$ such that $\{S_1, \cdots, S_n\}$ is a complete set of representatives of the isomorphism classes of simple right $R$-modules. Since $E$ is CS, there exist submodules $Q_1, \cdots, Q_n$ of $E$ such that $Q_1, \cdots, Q_n$ is an direct summands of $E$ and $(S_i)_R \subseteq \text{ess}\text{-soc} (Q_i)_R$ for $i = 1, \cdots, n$. Since $Q_i$ is an indecomposable projective and GC2 $R$-module, it has a local endomorphism ring; and since $Q_i$ is projective, $J(Q_i)$ is maximal and small in $Q_i$. Then $Q_i$ is a projective cover of the simple module $Q_i/J(Q_i)$. Note that $Q_i \cong Q_j$ clearly implies $Q_i/J(Q_i) \cong Q_j/J(Q_j)$; and the converse also holds because every module has at most one projective cover up to isomorphism. It is clear that $Q_i \cong Q_j$ if and only if $S_i \cong S_j$ if and only if $i = j$. Thus, $\{Q_1/J(Q_1), \cdots, Q_n/J(Q_n)\}$ is a complete set of representatives of the isomorphism classes of simple right $R$-modules. Hence every simple right $R$-module has a projective cover. Therefore $R$ is semiperfect. \qed

The following example show that the proof of [24] Proposition 13 is not true, since the endomorphism ring of an indecomposable projective module which is an essential extension of a simple module may be not a local ring. So we add an extra condition that $R$ is right GC2 to prove the Proposition.

**Example 1.** Let $R$ be the ring of triangular matrices, $R = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a \in \mathbb{Z}, b, c \in Q \right\}$.

Take $P_1 = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a \in \mathbb{Z}, b \in Q \right\}$ and $P_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} : c \in Q \right\}$, we see that $P_1$ is indecomposable projective module with simple essential socle and $P_2$ is projective simple module. The socle of $P_1$ is isomorphic to $P_2$ and its endomorphism ring is isomorphic to $Z$ which is not local.
Proposition 7. If $R$ is right $s$-$CS$ and $GC2$, and right Kasch, then $R$ is semiperfect.

Proof. Since $R$ is a weak $CS$, so $E$ is $CS$ by Lemma 1 and $R = E \oplus K$ for some right ideal $K$ of $R$ and so $E$ is a finitely generated projective module. By Lemma 2, $E$ has a finitely generated essential socle. Then, by hypothesis, there exist simple submodules $S_1, \ldots, S_n$ of $E$ such that $\{S_1, \ldots, S_n\}$ is a complete set of representatives of the isomorphism classes of simple right $R$-modules. Since $E$ is $CS$, there exist submodules $Q_1, \ldots, Q_n$ of $E$ such that $Q_1, \ldots, Q_n$ is an direct summands of $E$ and $(S_i)_R \subseteq_{\text{ess}} (Q_i)_R$ for $i = 1, \ldots, n$. Since $Q_i$ is an indecomposable projective and $GC2$ $R$-module, it has a local endomorphism ring; and since $Q_i$ is projective, $J(Q_i)$ is maximal and small in $Q_i$. Then, $Q_i$ is a projective cover of the simple module $Q_i/J(Q_i)$. Note that $Q_i \cong Q_j$ clearly implies $Q_i/J(Q_i) \cong Q_j/J(Q_j)$; and the converse also holds because every module has at most one projective cover up to isomorphism. It is clear that $Q_i \cong Q_j$ if and only if $Q_i \cong Q_j$ if and only if $i = j$. Thus, $\{Q_1/J(Q_1), \ldots, Q_n/J(Q_n)\}$ is a complete set of representatives of the isomorphism classes of simple right $R$-modules. Hence every simple right $R$-module has a projective cover. Therefore $R$ is semiperfect. □

Lemma 3. Let $R$ be a semiperfect, left Kasch, left min-$CS$ ring. Then the following hold:

(1) $S_i \subseteq_{\text{ess}} R$ and $soc(Re)$ is simple and essential in $Re$ for all local idempotents $e \in R$.

(2) $R$ is right Kasch if and only if $S_i \subseteq S_r$.

(3) If $\{e_1, \ldots, e_n\}$ are basic local idempotents in $R$ then $\{soc(Re_1), \ldots, soc(Re_n)\}$ is a complete set of distinct representatives of the simple left $R$-modules.

Proof. See [17, Lemma 4.5]. □

Recall that a ring $R$ is right minfull if it is semiperfect, right mininjective, and $soc(eR) \neq 0$ for each local idempotent $e \in R$.

Corollary 2. If $R$ is commutative $s$-$CS$ (weak $CS$) and Kasch, then $R$ is minfull.

Proof. Since every Kasch ring is $C2$, so $R$ is semiperfect by Proposition 6 (Proposition 7). Thus using the above Lemma and [17, Proposition 4.3] $R$ is minfull. □

Theorem 1. If $R$ is right weak CS (s-CS), GC2 and every cyclic right $R$-module can be embedded in a free module (right $CF$ ring ) then $R$ is right artinian.

Proof. If $R$ is right weak CS (s-CS) right $CF$, then by Lemma 1 (Lemma 1) $R = E \oplus K$ where $E$ is $CS$ and $soc(K) = 0$ ($Z(K) = 0$). Thus by Proposition 6 (Proposition 7), $R$ is semiperfect. The above Lemma gives $S_i \subseteq_{\text{ess}} R_R$, so $K = 0$. Hence $R$ is $CS$ and $R$ is right artinian by [9, Corollary 2.9]. □

Proposition 8. Let $R$ be a right $FGF$, right weak CS (s-CS) and right $GC2$ ring. Then $R$ is QF.

Proof. It clear by Proposition 6 (Proposition 7), and [8, Theorem 3.7]. □
3. Almost injective Modules

Definition 1. A right $R$-module $M$ is called almost injective, if $M = E \oplus K$ where $E$ is injective and $K$ has zero radical. A ring $R$ is called right almost injective, if $RR$ is almost injective.

In [23], the statement of Theorem 2.12 is not true, so the proof of (3) $\implies$ (1). The following Proposition is the true version of [23, Theorem 2.12]. Then we rewrite the related results.

Proposition 9. For a ring $R$ the following are true:

1. $R$ is semisimple if and only if every almost-injective right $R$-module is injective.
2. If $R$ is semilocal, then every rad-injective right $R$-module is injective.

Proof. (1). Assume that every almost-injective right $R$-module is injective, then every right $R$-module with zero radical is injective. Thus every semisimple right $R$-module is injective and $R$ is right $V$-ring. Hence, every right $R$-module has a zero radical. Therefore, every right $R$-module is injective and $R$ is semisimple. The converse is clear.

(2). Let $R$ be a semilocal ring and $M$ be a rad-injective right $R$-module. Consider a homomorphism $f : K \rightarrow M$ where $K$ is a right ideal of $R$. Since $R$ is semilocal, there exists a right ideal $L$ of $R$ such that $K + L = R$ and $K \cap L \subseteq J$ [13]. Then there exists a $R$-homomorphism $g : R \rightarrow M$ such that $g(x) = f(x)$ for every $x \in K \cap L$. Define $F : R \rightarrow M$ by $F(x) = f(k) + g(l)$ for any $x = k + l$ where $k \in K$ and $l \in L$. It is clear that $F$ is a well-defined $R$-homomorphism such that $F|_K = f$, i.e. $F$ extends $f$. Therefore $M$ is injective. □

A ring $R$ is called quasi-Frobenius ($QF$) if $R$ is right (or left) artinian and right (or left) self-injective. Also, $R$ is $QF$ if and only if every injective right $R$-module is projective.

Theorem 2. $R$ is a quasi-Frobenius ring if and only if every rad-injective right $R$-module is projective.

Proof. If $R$ is quasi-Frobenius, then $R$ is right artinian, and by Proposition 9(2), every rad-injective right $R$-module is injective. Hence, every rad-injective right $R$-module is projective. Conversely, if every rad-injective right $R$-module is projective, then every injective right $R$-module is projective. Thus, $R$ is quasi-Frobenius. □

Recall that a ring $R$ is called a right pseudo-Frobenius ring (right $PF$-ring) if the right $R$-module $RR$ is an injective cogenerator.

Proposition 10. The following are equivalent:

1. $R$ is a right $PF$-ring.
2. $R$ is a semiperfect right self-injective ring with essential right socle.
3. $R$ is a right finitely cogenerated right self-injective ring.
4. $R$ is a right Kasch right self-injective ring.

Theorem 3. If $R$ is right Kasch right almost-injective, then $R$ is semiperfect.

Proof. Let $R$ be right Kasch and $RR = E \oplus T$, where $E$ is injective and $T$ has zero radical. If $J = 0$, then every simple right ideal of $R$ is projective and $R$ is semiperfect (for $R$ is right Kasch). Now suppose that $J \neq 0$. Clearly, every
Proposition 11. The following are equivalent:

1. $R$ is a right PF-ring.
2. $R$ is a semiperfect right rad-injective ring with $\text{soc}(eR) \neq 0$ for each local idempotent $e$ of $R$.
3. $R$ is a right finitely cogenerated right rad-injective ring.
4. $R$ is a right Kasch right rad-injective ring.
5. $R$ is a right rad-injective ring and the dual of every simple left $R$-module is simple.

Proof. (1) $\iff$ (2) By Proposition 9 (2).
(1) $\Rightarrow$ (3) Clear.
(3) $\Rightarrow$ (1) Since $R$ is a right rad-injective ring, it follows from [23] Proposition 2.5 that $R = E \oplus K$, where $E$ is injective and $K$ has zero radical. Since $R$ is a right finitely cogenerated ring, $K$ is a finitely cogenerated right $R$-module with zero radical. Hence, $K$ is semisimple. Therefore, by [22] Corollary 8, $R$ is a right PF-ring.
(1) $\Rightarrow$ (4) Clear.
(4) $\Rightarrow$ (1) If $R$ is right Kasch right rad-injective, then $R$ is right almost-injective ([23] Proposition 2.5). Thus $R$ is semiperfect (9). Hence $R$ is injective by Proposition 9 (2). Therefore, $R$ is right PF.
(1) $\Rightarrow$ (5) Since every right PF-ring is left Kasch and left mininjective, the dual of every simple left $R$-module is simple by [16] Proposition 2.2.
(5) $\Rightarrow$ (1) By [23] Proposition 2.10, $R$ is a right min-\textit{CS} ring (i.e., every minimal right ideal of $R$ is essential in a summand). Thus, by [10] Theorem 2.1, $R$ is semiperfect with essential right socle. Proposition 9 (2) entails that $R$ is right self-injective, and hence right PF by Proposition 11.
(1) $\iff$ (4) and (1) $\iff$ (5) are direct consequences of [23] Proposition 2.5. \qed

A result of Osowsky [13] Proposition 2.2] asserts that a ring $R$ is QF if and only if $R$ is a left perfect, left and right self-injective ring. This result remains true for rad-injective rings.
Proposition 12. The following are equivalent:

(1) $R$ is a quasi-Frobenius ring.
(2) $R$ is a left perfect, left and right rad-injective ring.

Proof. (1) $\Rightarrow$ (2) It is well known.
(2) $\Rightarrow$ (1) By hypothesis, $R$ is a semiperfect right and left rad-injective ring. By Proposition 9, $R$ is quasi-Frobenius.

Note that the ring of integers $\mathbb{Z}$ is an example of a commutative noetherian almost-injective ring which is not quasi-Frobenius.

Definition 2. A ring $R$ is called right $CF$-ring ($FGF$-ring) if every cyclic (finitely generated) right $R$-module embeds in a free module. It is not known whether right $CF$-rings ($FGF$-rings) are right artinian (quasi-Frobenius rings). In the next result, a positive answer is given if we assume in addition that the ring $R$ is right rad-injective.

Proposition 13. The following are equivalent:

(1) $R$ is quasi-Frobenius.
(2) $R$ is right $CF$ and right rad-injective.

Proof. (1) $\Rightarrow$ (2) It is well known.
(2) $\Rightarrow$ (1) Since every simple right $R$-module embeds in $R$, $R$ is a right Kasch ring. By Proposition 11, $R$ is right self-injective with finitely generated essential right socle. Thus, every cyclic right $R$-module has a finitely generated essential socle, and by [21, Proposition 2.2], $R$ is right artinian, hence quasi-Frobenius.

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