LOOP DYNAMICS OF A FULLY DISCRETE SHORT PULSE EQUATION

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ABSTRACT
In this article, a fully discrete short pulse (SP) equation is presented as an integrability condition of a linear system of difference equations (also known as discrete Lax pair). Additionally, two semi-discrete versions of the SP equation have also been obtained from fully discrete SP equation under continuum limits. Darboux transformation is employed to compute multi-soliton solutions of fully discrete and semi-discrete SP equations. Explicit expressions of first and second nontrivial soliton solutions are computed. We also derived explicit expression of breather solution for fully discrete SP equation. The dynamics of single loop soliton and interaction mechanism of loop-loop and loop-antiloop solutions has been explored and illustrated.

Keywords Short pulse (SP) equation · Darboux transformation · Loop soliton · Multi-soliton solutions · breather solutions

1 Introduction
Nonlinear Schrödinger equation can best describe the propagation of slowly varying envelopes in nonlinear dispersive media. However, when the width of the optical pulse further reduces to the order of femtoseconds, then NLS equation cannot be derived for such ultra-short optical pulses [1]. So in order to describe the propagation of ultra-short optical signals, slowly varying optical models needs to be modified. Shäfer and Wayne introduced the short pulse (SP) equation [2]

\[ q_{xt} = q + \frac{1}{6}(q^3)_{xx}, \]

(1)
to describe the processes including ultra-short optical pulses. Here real-valued function \( q = q(x,t) \) represents the magnitude of electric field and the subscripts \( x, t \) indicate usual space and time derivatives. Short pulse equation first emerged during the study of pseudospherical surfaces [3]. SP equation is completely integrable equation and its integrability has been proven by many means, for example, existence of associated isospectral problem also known as Wadati-Konno-Ichikawa (WKI) scheme [4], bi-Hamiltonian structure [5], infinite set of conservation laws [6], soliton solutions [7, 8, 9, 10, 11, 12]. By using a suitable hodograph transformation, the SP equation can be transformed into some well-known integrable equations, such as sine-Gordon equation [8, 7].

The general integrable short pulse (SP) equation as presented in [13] is given by

\[ q_{xt} = q + \frac{1}{2}(qrq_x)_x, \]

\[ r_{xt} = r + \frac{1}{2}(rqr_x)_x, \]

(2)
where \( q(x,t) \) and \( r(x,t) \) are the dynamical variables that represent the magnitude of the electric field. The coupled system \( \{q, r\} \) can be written as the consistency condition of the following linear eigenvalue problem of Wadati-Konno-

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Additionally, the results obtained here can be reducible to already known versions of SP equation under continuum limits.

Hodograph transformation is exclusively used to solve a nonlinear systems with a switched role of dependent and independent variables. This transformation is more or less similar to reciprocal transformation except for a difference that is the reciprocal transformation can put a system into conservative form. The reciprocal transformation was presented by Kingston and Roger in 1982 [14]. Lets define the hodograph transformation \((x, t) \rightarrow (y, \tau)\) by the means of following transformations relating the old variables \((x, t)\) to the new ones \((y, \tau)\) as

\[
dy = \omega dx + \frac{1}{2} qr \omega dt, \quad d\tau = dt,
\]

where \(\omega = \sqrt{1 + q_x r_x}\), also

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial t} = \frac{1}{2} q r \omega \frac{\partial}{\partial y} + \frac{\partial}{\partial \tau}.
\]

Now under the transformation (4)-(5) the equations (2) reduce to following system of equations

\[
x_{y\tau} = -\frac{1}{2} (qr)_y, \quad q_{y\tau} = qx_y, \quad r_{y\tau} = rx_y.
\]

Under the reduction \(r = -q\) system of equations (6) becomes

\[
x_{y\tau} = \frac{1}{2} (q^2)_y, \quad q_{y\tau} = qx_y.
\]

The integrable system of equations (7) can be written as the consistency condition of the following WKI scheme

\[
\Psi_y = \lambda \left( \begin{array}{c} x_y \\ r_y \\ -x_y \\ -r_y \end{array} \right) \Psi, \quad \Psi_x = \frac{1}{2} \left( \begin{array}{c} \frac{1}{2x} \\ -\frac{q}{x} \\ -\frac{1}{2x} \end{array} \right) \Psi.
\]

During last two decades, discretizations of linear and nonlinear differential equations (ordinary and partial) have been studied extensively. Generally speaking, many physical phenomenon are modeled as differential-difference or difference-difference equations. Discretization of nonlinear integrable models plays an important role in various fields of science, e.g., in biological sciences, nonlinear optical communications, other nonlinear systems, optical fiber communication, quantum mechanics, field theories, etc [15, 16, 17, 18]. Such systems comprise of inherent nonlocality.

The construction of discrete analogue of any particular integrable system has a remarkable history. First of all, Ablowitz and Ladik presented a linear system of difference-differential equations also known as Ablowitz-Ladik scheme [19, 20]. Later on, Hirota studied discrete integrable analogues of some famous PDEs through Hirota bilinearization method [21, 22, 23, 24, 25]. Integrable discretization of SP equation and its multi-component generalizations have been presented in the articles such as [26, 27, 28, 29] by using Hirota’s direct method and self-adaptive moving mesh schemes.

In recent times, different semi-discrete versions of SP equations have been extensively investigated. However, fully discrete SP equation has never been given much attention. In this paper, we would present fully discrete SP equation by considering both variables (time and space) as discrete integral variables. In section 2, a generalized version of fully discrete SP equation will be presented as consistency condition of a discrete linear system. Moreover, under various continuum limits, fully discrete system will be deduced to two semi-discrete versions of SP equation (discrete in time and space separately). Section 3 will be devoted to a brief review of discrete Darboux transformation to construct discrete multi-soliton solutions through the covariance of discrete Lax pair. In section 4, explicit expressions of first two nontrivial solutions will be computed. Fascinating interactions of loop-loop and loop-antiloop will be depicted. Furthermore, discrete breather for a particular choice of spectral parameters will be obtained too. The article will end up with a concise summary.

## 2 Discretization

In this section, we will present the fully discrete, time discretized and space discretized versions of SP equation. Additionally, the results obtained here can be reducible to already known versions of SP equation under continuum limits.
2.1 Fully discrete SP equation

A generic fully discrete SP equation is given by

\[
\Delta_n \Delta_m x_{n,m} = \frac{b}{4} (\Delta_n x_{n,m+1} \Delta_m x_{n,m} - \Delta_m x_{n+1,m} \Delta_n x_{n,m}) + \frac{b}{4}
\]

\[
(\nabla_m \Delta_n q_{n,m} - \nabla_m \Delta_n r_{n,m} - \nabla_m \Delta_n q_{n,m} + \Delta_n r_{n,m} \nabla_m \Delta_n q_{n,m} + \Delta_n q_{n,m} \nabla_m \Delta_n r_{n,m}) = 0.
\]

(9)

\[
\Delta_n \Delta_m q_{n,m} - \frac{b}{4} (\Delta_n \Delta_m x_{n,m} \nabla_m q_{n,m} + \Delta_n \Delta_m x_{n,m} \Delta_n q_{n,m} + \Delta_n x_{n,m} \nabla_m q_{n,m} - \Delta_n x_{n,m} \Delta_n q_{n,m}) = 0.
\]

(10)

\[
\Delta_n \Delta_m r_{n,m} - \frac{b}{4} (\Delta_n \Delta_m x_{n,m} \nabla_m r_{n,m} + \Delta_n \Delta_m x_{n,m} \Delta_n r_{n,m} + \Delta_n x_{n,m} \nabla_m r_{n,m} - \Delta_n x_{n,m} \Delta_n r_{n,m}) = 0.
\]

(11)

Here difference operator \((\Delta_n)\) and forward shift operator \((\nabla_n)\) are defined on an arbitrary discrete-valued function \(l_{n,m}\) as

\[
\Delta_n l_{n,m} = l_{n+1,m} - l_{n,m}, \quad \Delta_m l_{n,m} = l_{n,m+1} - l_{n,m}, \quad \nabla_n l_{n,m} = l_{n+1,m} + l_{n,m}, \quad \nabla_m l_{n,m} = l_{n,m+1} + l_{n,m}.
\]

The generalized fully discrete SP equation \((9)-(11)\) appears as compatibility condition of the following linear system of difference equations

\[
L_n \Psi_{n,m} \equiv \Psi_{n+1,m} = (I_2 + \lambda Q_{n,m}) \Psi_{n,m} = \left( \begin{array}{cc} 1 + \lambda \Delta_n x_{n,m} & \lambda \Delta_n q_{n,m} \\ \lambda \Delta_n r_{n,m} & 1 - \lambda \Delta_n x_{n,m} \end{array} \right) \Psi_{n,m},
\]

\[
L_m \Psi_{n,m} \equiv \Psi_{n,m+1} = \left( I_2 + \frac{b}{4\lambda} \sigma_3 + \frac{b}{4} R_{n,m} \right) \Psi_{n,m} = \left( \begin{array}{cc} 1 + \frac{b}{4\lambda} + \frac{b}{4} \Delta_m x_{n,m} & -\frac{b}{4} \nabla_m q_{n,m} - \frac{b}{4} \Delta_m x_{n,m} \\ \frac{b}{4} \nabla_m r_{n,m} & 1 - \frac{b}{4\lambda} + \frac{b}{4} \Delta_m x_{n,m} \end{array} \right) \Psi_{n,m}.
\]

(12)

where

\[
Q_{n,m} = \left( \begin{array}{cc} \Delta_n x_{n,m} & \Delta_n q_{n,m} \\ \Delta_n r_{n,m} & -\Delta_n x_{n,m} \end{array} \right), \quad R_{n,m} = \left( \begin{array}{cc} \Delta_m x_{n,m} & -\nabla_m q_{n,m} \\ \nabla_m r_{n,m} & \Delta_m x_{n,m} \end{array} \right).
\]

(13)

Here \(n\) and \(m\) are discrete variables and \(\lambda\) is a real or complex-valued spectral parameter. The integrability condition of \((12)\) i.e., \(L_n \Psi_{n,m} = L_m \Psi_{n,m}\) will equivalently yield discrete SP equation \((9)-(11)\). Therefore, the linear system of difference-difference equations \((12)\) is also named as fully discrete Lax pair. Under the continuum limit \(a \to 0\) and \(b \to 0\) discrete SP equation \((9)-(11)\) and its associated Lax pair \((12)\) will reduce into continuous SP equation and its associated linear system \((6)\) and \((8)\) respectively. Under the reduction \(r_{n,m} = -q_{n,m}\), system \((9)-(11)\) reduces to

\[
\Delta_n \Delta_m x_{n,m} + \frac{b}{4} (\Delta_n x_{n,m+1} \Delta_m x_{n,m} - \Delta_m x_{n+1,m} \Delta_n x_{n,m}) - \frac{b}{4} (\nabla_m \Delta_n q_{n,m} \nabla_n q_{n,m} + \Delta_m \nabla_n q_{n,m} \nabla_m q_{n,m} + 2 \Delta_n q_{n,m} \nabla_m \Delta_n q_{n,m}) = 0,
\]

(14)

\[
\Delta_n \Delta_m q_{n,m} - \frac{b}{4} (\Delta_n \Delta_m x_{n,m} \nabla_m q_{n,m} + \Delta_n \Delta_m x_{n,m} \nabla_m q_{n,m} - \Delta_n x_{n,m} \Delta_n q_{n,m}) = 0.
\]

(15)

Equations \((14)-(15)\) represent a fully discrete parametric (transformed) version of SP equation \((7)\).

2.2 Semi-discrete SP equation (discrete in time)

Under the continuum limit \(a \to 0\) and \(n \to \infty\), the discrete linear system \((12)\) will reduce to following pair of differential-difference equations:

\[
\frac{d}{dy} \Psi_m = \left( \begin{array}{cc} \lambda \frac{d}{dy} x_m & -\lambda \frac{d}{dy} q_m \\ \lambda \frac{d}{dy} r_m & -\lambda \frac{d}{dy} x_m \end{array} \right) \Psi_m,
\]

\[
L_m \Psi_m \equiv \Psi_{m+1} = \left( 1 + \frac{b}{4\lambda} + \frac{b}{4} \Delta_m x_m & -\frac{b}{4} \nabla_m q_m - \frac{b}{4} \Delta_m x_m \\ \frac{b}{4} \nabla_m r_m & 1 - \frac{b}{4\lambda} + \frac{b}{4} \Delta_m x_m \end{array} \right) \Psi_m.
\]

(16)
The consistency condition of linear system (16), that is, \( L_m \left( \frac{d}{dy} \Psi_m \right) = \frac{d}{dy} (L_m \Psi_m) \), yield semi-discrete SP equation (discrete in time) given by

\[
\left( \frac{1}{b} + \frac{1}{4} \Delta_m x_m \right) \frac{d}{dy} \Delta_m x_m + \frac{1}{8} \left( \frac{d}{dy} \nabla_m q_m \nabla_m r_m + \nabla_m q_m \frac{d}{dy} \nabla_m r_m \right) = 0,
\]

\[
\left( \frac{1}{b} + \frac{1}{4} \Delta_m x_m \right) \frac{d}{dy} \Delta_m q_m - \frac{1}{4} \nabla_m q_m \frac{d}{dy} \nabla_m x_m = 0,
\]

\[
\left( \frac{1}{b} + \frac{1}{4} \Delta_m x_m \right) \frac{d}{dy} \Delta_m r_m - \frac{1}{4} \nabla_m r_m \frac{d}{dy} \nabla_m x_m = 0.
\]

System of equation (17) and its associated linear system (16) under the continuum limit will reduce to respective continuous counterparts (6) and (8), respectively. This time discretized version of SP equation is a new addition in literature.

2.3 Space discretized version of SP equation

Similarly another semi-discrete version of SP equation can be obtained from (9)-(11) under the continuum limit \( b \to 0 \) and \( m \to \infty \) as

\[
\frac{d}{d\tau} (x_{n+1} - x_n) + \frac{1}{2} (q_{n+1} r_{n+1} - q_n r_n) = 0,
\]

\[
\frac{d}{d\tau} (q_{n+1} - q_n) - \frac{1}{2} (x_{n+1} - x_n) (q_{n+1} + q_n) = 0,
\]

\[
\frac{d}{d\tau} (r_{n+1} - r_n) - \frac{1}{2} (x_{n+1} - x_n) (r_{n+1} + r_n) = 0.
\]

(18)

Above system of equations arises as consistency condition of the following pair of difference-differential equations (also known as semi-discrete Lax pair)

\[
\Psi_{n+1} = (I_2 + \lambda Q_n) \Psi_n,
\]

\[
\frac{d}{d\tau} \Psi_n = \left( \frac{1}{4\lambda} \sigma_3 + \frac{1}{2} R_n \right) \Psi_n,
\]

(19)

where \( 2 \times 2 \) matrices \( Q_n \) and \( R_n \) are defined below

\[
Q_n = \left( \begin{array}{cc} \Delta_n x_n & \Delta_n q_n \\ \Delta_n r_n & -\Delta_n x_n \end{array} \right), \quad R_n = \left( \begin{array}{cc} 0 & -q_n \\ r_n & 0 \end{array} \right).
\]

(20)

This was introduced by Feng and coworkers in their work [27]. It can easily be verified that semi discrete SP equation (18) and its associated linear system (19) can be obtained from fully discrete SP equation (9)-(11) and its discrete linear system (12) under the limit \( b \to 0 \). Also under \( n \to \infty \), above defined equation and linear system will reduce to their respective continuous counterparts. We have already investigated this system (18) under nonlocal symmetry reduction in our work [30].

3 Discrete Darboux transformation

Darboux transformation (DT) is a well-established mathematical technique which has been used to compute multisoliton solutions of many integrable systems [31]. It is more or less like a gauge transformation that relates two matrix-valued solutions of the associated linear system via a Darboux matrix whose application keeps the linear system and associated equations covariant. The covariance of the system is then used to compute higher-order nontrivial solutions under successive iterations of DT. One-fold DT on the matrix-valued solution \( \Psi_{n,m} \) of the discrete Lax pair (12) is defined as:

\[
\Psi_{n,m}^{[1]} = T_{n,m}^{[1]} \Psi_{n,m},
\]

(21)

where \( T_{n,m}^{[1]} \) is the Darboux matrix and is given by

\[
T_{n,m}^{[1]} = \left( \begin{array}{cc} \lambda^{-1} + \alpha_n^{[0]} & \beta_n^{[0]} \\ \gamma_n^{[0]} & \lambda^{-1} + \delta_n^{[0]} \end{array} \right),
\]

(22)
We have to compute four unknown discrete-valued functions. After rearranging we have,

\[ \alpha = \text{Employing renowned technique of Cramer's rule, we eventually arrive at} \]

\[ \text{and} \]

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\[ \text{After rearranging we have,} \]

\[ \text{By employing renowned technique of Cramer's rule, we eventually arrive at} \]

\[ \text{Covariance of discrete linear system (12) under the action of one-fold DT (21), allows us to write} \]

\[ \Psi_{n+1,m}[1] = (I_2 + \lambda Q_{n,m}[1]) \Psi_{n,m}[1], \quad \Psi_{n,m+1}[1] = \left( f + \frac{b}{4\lambda} \sigma_3 + \frac{b}{4} R_{n,m}[1] \right) \Psi_{n,m}[1], \] (24)

that implies

\[ x_{n,m}[1] = x_{n,m} + \frac{1}{2} \left( \alpha_{n,m} - \delta_{n,m} \right), \]

\[ q_{n,m}[1] = q_{n,m} + \beta_{n,m}, \]

\[ r_{n,m}[1] = r_{n,m} + \gamma_{n,m}. \] (25)

The result (25) enables us to explicitly calculate one-loop soliton solution of discrete SP equation (14). Similarly, the second iteration of DT on the matrix-valued discrete function is defined as

\[ \Psi_{n,m}[2] = T_{n,m}^{[2]} \Psi_{n,m} = \left( \begin{array}{ccc} \lambda^{-2} + \lambda^{-1} \alpha_{n,m} & \alpha_{n,m} & \lambda^{-1} \beta_{n,m} + \beta_{n,m} \\ \alpha_{n,m} & \lambda^{-1} \gamma_{n,m} + \gamma_{n,m} & \lambda^{-1} \delta_{n,m} + \delta_{n,m} \\ \lambda^{-2} + \lambda^{-1} \gamma_{n,m} + \gamma_{n,m} & \lambda^{-1} \delta_{n,m} + \delta_{n,m} & \end{array} \right) \Psi_{n,m}. \] (26)

Covariance of discrete Lax pair (12) under DT (26) demands

\[ x_{n,m}[2] = x_{n,m} + \frac{1}{2} \left( \alpha_{n,m} - \delta_{n,m} \right), \]

\[ q_{n,m}[2] = q_{n,m} + \beta_{n,m}, \]

\[ r_{n,m}[2] = r_{n,m} + \gamma_{n,m}. \] (27)
Above systems can be easily solved for unknown discrete functions by using a trivial technique of Cramer’s method. Eventually, we obtain

\[ \Psi_{n,m} \mapsto \lambda = \begin{bmatrix} f_{n,m} \\ g_{n,m} \end{bmatrix} \]

expressed in matrix form,

\[ \begin{vmatrix} f_{n,m} \\ g_{n,m} \end{vmatrix} = \begin{vmatrix} \alpha_{n,m} \\ \beta_{n,m} \end{vmatrix} \]

where \( i = 1, 2, 3, 4 \). Under this condition, we will arrive at following systems of equation expressed in matrix form,

\[ \begin{vmatrix} f_{n,m} \\ g_{n,m} \end{vmatrix} = \begin{vmatrix} \alpha_{n,m} \\ \beta_{n,m} \end{vmatrix} \]

\[ \begin{vmatrix} f_{n,m} \\ g_{n,m} \end{vmatrix} = \begin{vmatrix} \gamma_{n,m} \\ \delta_{n,m} \end{vmatrix} \]

Above systems can be easily solved for unknown discrete functions by using a trivial technique of Cramer’s method. Eventually, we obtain

\[ \alpha_{n,m}^{[1]} = - \frac{\det \begin{vmatrix} f_{n,m} & g_{n,m} \\ \lambda_1 f_{n,m} & \lambda_2 f_{n,m} \\ \lambda_3 f_{n,m} & \lambda_4 f_{n,m} \end{vmatrix}}{\det \begin{vmatrix} f_{n,m} & g_{n,m} \\ \lambda_1 f_{n,m} & \lambda_2 f_{n,m} \\ \lambda_3 f_{n,m} & \lambda_4 f_{n,m} \end{vmatrix}} \]

\[ \beta_{n,m}^{[1]} = - \frac{\det \begin{vmatrix} f_{n,m} & g_{n,m} \\ \lambda_1 f_{n,m} & \lambda_2 f_{n,m} \\ \lambda_3 f_{n,m} & \lambda_4 f_{n,m} \end{vmatrix}}{\det \begin{vmatrix} f_{n,m} & g_{n,m} \\ \lambda_1 f_{n,m} & \lambda_2 f_{n,m} \\ \lambda_3 f_{n,m} & \lambda_4 f_{n,m} \end{vmatrix}} \]

\[ \gamma_{n,m}^{[1]} = - \frac{\det \begin{vmatrix} f_{n,m} & g_{n,m} \\ \lambda_1 f_{n,m} & \lambda_2 f_{n,m} \\ \lambda_3 f_{n,m} & \lambda_4 f_{n,m} \end{vmatrix}}{\det \begin{vmatrix} f_{n,m} & g_{n,m} \\ \lambda_1 f_{n,m} & \lambda_2 f_{n,m} \\ \lambda_3 f_{n,m} & \lambda_4 f_{n,m} \end{vmatrix}} \]

\[ \delta_{n,m}^{[1]} = - \frac{\det \begin{vmatrix} f_{n,m} & g_{n,m} \\ \lambda_1 f_{n,m} & \lambda_2 f_{n,m} \\ \lambda_3 f_{n,m} & \lambda_4 f_{n,m} \end{vmatrix}}{\det \begin{vmatrix} f_{n,m} & g_{n,m} \\ \lambda_1 f_{n,m} & \lambda_2 f_{n,m} \\ \lambda_3 f_{n,m} & \lambda_4 f_{n,m} \end{vmatrix}} \]

6
Again, the expression (27) gives the interactions of loop-loop, loop-antiloop soliton solutions and breather solution of discrete SP equation 14-15.

The N-fold DT can be generalized as

\[
\Psi_{n,m}[N] = T_{n,m}^{[N]} \Psi_{n,m},
\]  

(28)

with

\[
T_{n,m}^{[N]} = \left( \lambda^{-N} + \sum_{k=0}^{N-1} \alpha_{n,m}^{[k]} \lambda^{-k} \right) \left( \lambda^{-N} + \sum_{k=0}^{N-1} \beta_{n,m}^{[k]} \lambda^{-k} \right). 
\]  

(29)

Under the action of DT (28) the discrete dynamical variables transform as

\[
\begin{align*}
x_{n,m}[N] &= x_{n,m} + \frac{1}{2} \left( \alpha_{n,m}^{[N-1]} - \delta_{n,m}^{[N-1]} \right), \\
g_{n,m}[N] &= g_{n,m} + \beta_{n,m}^{[N-1]}, \\
r_{n,m}[N] &= r_{n,m} + \gamma_{n,m}^{[N-1]}.
\end{align*}
\]  

(30)

The unknowns \( \alpha_{n,m}^{[k]}, \beta_{n,m}^{[k]}, \gamma_{n,m}^{[k]} \) and \( \delta_{n,m}^{[k]} \) (0 \( \leq k \) \( \leq N-1 \)) can be uniquely determined by requiring

\[
T_{n,m}^{[N]} \Psi_{n,m}|_{\lambda \rightarrow \lambda_i} = [0], \quad (i = 1, 2, \ldots, 2N)
\]  

(31)

which yields system of equations

\[
\begin{align*}
\sum_{k=0}^{N-1} \alpha_{n,m}^{[k]} \lambda_i^{-k} f_{n,m}^{(i)} + \sum_{k=0}^{N-1} \beta_{n,m}^{[k]} \lambda_i^{-k} g_{n,m}^{(i)} &= -\lambda_i^{-N} f_{n,m}^{(i)}, \\
\sum_{k=0}^{N-1} \gamma_{n,m}^{[k]} \lambda_i^{-k} f_{n,m}^{(i)} + \sum_{k=0}^{N-1} \delta_{n,m}^{[k]} \lambda_i^{-k} g_{n,m}^{(i)} &= -\lambda_i^{-N} g_{n,m}^{(i)}.
\end{align*}
\]  

(32)

where \( \lambda_i (1 \leq i \leq 2N) \) are spectral parameters and \( \Psi_{n,m}^{(i)} (\lambda) = \left( f_{n,m}^{(i)} \, g_{n,m}^{(i)} \right)^T \) are the vector solutions of system (12). The linear system (32) can be solved by using Cramer’s rule, the coefficient matrices provide the following results

\[
\begin{align*}
\alpha_{n,m}^{[N-1]} &= \frac{\Delta_{\alpha_{n,m}}^{[N-1]}}{\Delta_{\alpha_{n,m}}^{[N-1]}}, \\
\beta_{n,m}^{[N-1]} &= \frac{\Delta_{\beta_{n,m}}^{[N-1]}}{\Delta_{\beta_{n,m}}^{[N-1]}}, \\
\gamma_{n,m}^{[N-1]} &= \frac{\Delta_{\gamma_{n,m}}^{[N-1]}}{\Delta_{\gamma_{n,m}}^{[N-1]}}, \\
\delta_{n,m}^{[N-1]} &= \frac{\Delta_{\delta_{n,m}}^{[N-1]}}{\Delta_{\delta_{n,m}}^{[N-1]}},
\end{align*}
\]  

(33)

with

\[
\begin{align*}
\Delta_{\alpha_{n,m}}^{[N-1]} &= \det \begin{pmatrix} f_{n,m}^{(i)} & g_{n,m}^{(i)} & \cdots & \lambda_i^{-N+2} f_{n,m}^{(i)} & \lambda_i^{-N+2} g_{n,m}^{(i)} & \lambda_i^{-N+1} f_{n,m}^{(i)} & \lambda_i^{-N+1} g_{n,m}^{(i)} \end{pmatrix}, \\
\Delta_{\beta_{n,m}}^{[N-1]} &= \det \begin{pmatrix} f_{n,m}^{(i)} & g_{n,m}^{(i)} & \cdots & \lambda_i^{-N+2} f_{n,m}^{(i)} & \lambda_i^{-N+2} g_{n,m}^{(i)} & \lambda_i^{-N+1} f_{n,m}^{(i)} & \lambda_i^{-N+1} g_{n,m}^{(i)} \end{pmatrix}, \\
\Delta_{\gamma_{n,m}}^{[N-1]} &= \det \begin{pmatrix} f_{n,m}^{(i)} & g_{n,m}^{(i)} & \cdots & \lambda_i^{-N+2} f_{n,m}^{(i)} & \lambda_i^{-N+2} g_{n,m}^{(i)} & \lambda_i^{-N+1} f_{n,m}^{(i)} & \lambda_i^{-N+1} g_{n,m}^{(i)} \end{pmatrix}, \\
\Delta_{\delta_{n,m}}^{[N-1]} &= \det \begin{pmatrix} f_{n,m}^{(i)} & g_{n,m}^{(i)} & \cdots & \lambda_i^{-N+2} f_{n,m}^{(i)} & \lambda_i^{-N+2} g_{n,m}^{(i)} & \lambda_i^{-N+1} f_{n,m}^{(i)} & \lambda_i^{-N+1} g_{n,m}^{(i)} \end{pmatrix}.
\end{align*}
\]  

(34)

The higher-order loop-soliton solutions of discrete SP equation 14-15 can be computed through the results (30) and (33). In what follows next, we will compute first two nontrivial solution of the fully discrete SP equation 14-15. We also apply continuum limits to relate our results with already known solutions of semi-discrete and continuous SP equations.
4 Explicit solutions

In this section explicit expressions of first two nontrivial solutions of discrete SP equation (14)-(15) will be computed. Consider a seed solution \( x_{n,m} = na \) and \( q_{n,m} = r_{n,m} = 0 \), the associated Lax pair (12) will become

\[
\Psi_{n+1,m} = \begin{pmatrix} 1 + a\lambda & 0 \\ 0 & 1 - a\lambda \end{pmatrix} \Psi_{n,m}, \quad \Psi_{n,m+1} = \begin{pmatrix} 1 + \frac{b}{4\lambda} & 0 \\ 0 & 1 - \frac{b}{4\lambda} \end{pmatrix} \Psi_{n,m},
\]

(34)

where \( \Psi_{n,m} = (f_{n,m}, g_{n,m})^T \). The solution of linear system of difference-difference equations (34) becomes

\[
f_{n,m} = A (1 + a\lambda)^n \left( 1 + \frac{b}{4\lambda} \right)^m, \quad g_{n,m} = B (1 - a\lambda)^n \left( 1 - \frac{b}{4\lambda} \right)^m,
\]

(35)

where \( A \) and \( B \) are the constants. The particular column solutions at \( \lambda = \lambda_k \) are defined as

\[
f^{(i)}_{n,m} = A_i (1 + a\lambda_i)^n \left( 1 + \frac{b}{4\lambda_i} \right)^m, \quad g^{(i)}_{n,m} = B_i (1 - a\lambda_i)^n \left( 1 - \frac{b}{4\lambda_i} \right)^m.
\]

(36)

The reduction requirement \( q_{n,m} = -r_{n,m} \) will be realized when

\[
\lambda_{2l} = -\lambda_{2l-1}, \quad f^{(2l)}_{n,m} = \pm g^{(2l-1)}_{n,m}, \quad g^{(2l)}_{n,m} = \mp f^{(2l-1)}_{n,m}.
\]

(37)

4.1 First-order nontrivial loop soliton solution

The expression (35) gives us explicit expression for one-loop soliton solutions as

\[
x_{n,m}[1] = an - \frac{1}{2} \begin{pmatrix} \lambda_1^{-1} f^{(1)}_{n,m} & \lambda_2^{-1} f^{(1)}_{n,m} \\ \lambda_1^{-1} f^{(2)}_{n,m} & \lambda_2^{-1} f^{(2)}_{n,m} \end{pmatrix},
\]

\[
q_{n,m}[1] = -\begin{pmatrix} \lambda_1^{-1} f^{(1)}_{n,m} \\ \lambda_2^{-1} f^{(2)}_{n,m} \end{pmatrix}.
\]

(38)

Substituting \( f^{(i)} \) and \( g^{(i)} \) from (35) and also applying reduction constraints (37) in above expressions we get,

\[
x_{n,m}[1] = an - \frac{1}{\lambda_i} \left( 1 - A_2B_1 - A_1B_2 (4\lambda - b) - 2m (4\lambda - b)^2 (1 + a\lambda_i)^2(1 + a\lambda_i)^2 \right),
\]

\[
q_{n,m}[1] = \frac{2A_2B_1}{\lambda_i} \left( (1 - \frac{b^2}{4\lambda_i^2})^m (1 + a\lambda_i)^2 - 2m (1 + a\lambda_i)^2 \right).
\]

(39)

Equation (39) corresponds to one-soliton solution and the parametric correlation leads us to the loop solutions of fully discrete SP equation (14)-(15) for \( A_{2l} = A_{2l-1} = 1 \) and \( B_{2l-1} = -B_{2l} = 1 \). Profile of loop soliton solution for \( \lambda_1 = -1 \) and \( a = b = 0.5 \) is shown in Fig. (1). Under the continuum limit as \( a \to 0 \), \( n \to \infty \) and \( y = na \), expression (39) gives the solution of time-discrete SP equation (17)

\[
x_m(y)[1] = y - \frac{1}{\lambda_1} \left( 1 - \frac{2}{1 + 4\lambda_1^2} \right),
\]

\[
q_m(y)[1] = -\frac{2e^{2\lambda_1 y} \left( 1 - \frac{b^2}{4\lambda_1^2} \right)^m}{\lambda_1 \left( 1 + \frac{b^2}{4\lambda_1^2} \right)^2 + e^{4\lambda_1 y} \left( 1 + \frac{b}{4\lambda_1^2} \right)^2}.
\]

(40)

Likewise, the continuum limit as \( b \to 0 \), \( m \to \infty \) and \( \tau = mb \), expression (39) yields the solution of space-discrete SP equation (27)-(30)

\[
x_n(\tau)[1] = an - \frac{1}{\lambda_1} \left( 1 - \frac{2}{1 + e^{\frac{\tau}{\lambda_1}} \left( (1 + a\lambda_1)^2 \right)^{2n}} \right).
\]

(41)
which represents the loop soliton solution for the continuous SP equation (7). If we apply the continuum limit

\[
q_n(\tau)[1] = -\frac{2e^{\frac{\pi}{2\sqrt{t}}}(1-a^2\lambda_1^2)^n}{\lambda_1 \left(1 - a\lambda_1\right)^{2n} + e^{\frac{\pi}{2\sqrt{t}}} \left(1 + a\lambda_1\right)^{2n}}.
\]

while, again under the limit as \(a \to 0\), \(n \to \infty\) and \(y = na\), the above expression (41) reduces to

\[
x(y, \tau)[1] = x - \frac{1}{\lambda_1} \tanh \left(\frac{2y\lambda_1 + \frac{\tau}{2\lambda_1}}{2}\right),
\]

\[
q(y, t)[1] = -\frac{1}{\lambda_1} \sec \left(\frac{2y\lambda_1 + \frac{\tau}{2\lambda_1}}{2}\right),
\]

which represents the loop soliton solution for the continuous SP equation (7). If we apply the continuum limit \(b \to 0\), \(m \to \infty\) and \(\tau = mb\), expression (40) will reduce to (42).

### 4.2 Second-order nontrivial solutions

Equation (27) gives the explicit form of two-loop soliton solutions for \(A_{2l} = A_{2l-1} = 1\) and \(B_{2l-1} = -B_{2l} = 1\), \(\lambda_2 = -\lambda_1\) and \(\lambda_4 = -\lambda_3\) as

\[
x_{n,m}[2] = \frac{\lambda_1^2 - \lambda_3^2}{\lambda_1 \lambda_3} \left(-8\lambda_1 \lambda_3 \Gamma^{(a)}_{n,m} + \left(\lambda_2^2 - \lambda_2^2\right) \Gamma^{(s)}_{n,m} + \left(\lambda_2^2 + \lambda_2^2\right) \Gamma^{(d)}_{n,m}\right),
\]

\[
q_{n,m}[2] = -\frac{2\lambda_1^2 - \lambda_3^2}{\lambda_1 \lambda_3} \left(-8\lambda_1 \lambda_3 \Gamma^{(a)}_{n,m} + \left(\lambda_2^2 - \lambda_2^2\right) \Gamma^{(s)}_{n,m} + \left(\lambda_2^2 + \lambda_2^2\right) \Gamma^{(d)}_{n,m}\right),
\]

with

\[
\Gamma^{(\alpha_1)}_{n,m} = (1 + a\lambda_1)^2 \left(1 + \frac{b}{4\lambda_1}\right)^m \left(1 - a\lambda_1\right)^{2m},
\]

\[
\Gamma^{(\alpha_2)}_{n,m} = (1 + a\lambda_1)^2 \left(1 + \frac{b}{4\lambda_1}\right)^m \left(1 - a\lambda_1\right)^{2m},
\]

\[
\Gamma^{(\beta_1)}_{n,m} = (1 + a\lambda_3)^2 \left(1 + \frac{b}{4\lambda_3}\right)^m \left(1 - a\lambda_3\right)^{2m},
\]

\[
\Gamma^{(\beta_2)}_{n,m} = (1 + a\lambda_3)^2 \left(1 + \frac{b}{4\lambda_3}\right)^m \left(1 - a\lambda_3\right)^{2m},
\]

\[
\Gamma^{(\gamma)}_{n,m} = (1 + a\lambda_3)^m \left(1 + \frac{b}{4\lambda_3}\right)^m \left(1 - a\lambda_3\right)^{2m},
\]

\[
\Gamma^{(\delta)}_{n,m} = (1 + a\lambda_3)^m \left(1 + \frac{b}{4\lambda_3}\right)^m \left(1 - a\lambda_3\right)^{2m},
\]

\[
\Gamma^{(\delta)}_{n,m} = (1 + a\lambda_3)^m \left(1 + \frac{b}{4\lambda_3}\right)^m \left(1 - a\lambda_3\right)^{2m}.
\]
There exist two types of loop soliton interactions. The first one is a loop-antiloop interface that is an attractive process in which a loop and an antiloop while crossing form a spiral formation and split in their respective course as shown in the Fig. (2). This profile is obtained for \(\lambda_1 = -1, \lambda_3 = -0.7\), and \(a = b = 0.5\). The second one is a loop-loop interface that is a repulsive progression in such collisions as two loops approach they repel each other and also exchange their energies and diverged as shown in the Fig. (3). This profile is obtained for \(\lambda_1 = 1, \lambda_3 = -0.7\) and \(a = b = 0.5\).

### 4.3 Breather solutions

Breather solutions are bound state solutions of soliton and anti-soliton under the specific parametric domain. In such a case, attractive loop-antiloop pair form a bound structure and oscillate about each other irrespective of their original shapes. For breather solution we take \(A_{2l} = A_{2l-1} = 1\) and \(B_{2l-1} = -B_{2l} = 1\) and \(\lambda_1 = -\lambda_2, \lambda_3 = -\lambda_4 = \lambda_3^*,\) thus equation (27) become

\[x_{n,m}[2] = an + \frac{\xi^{(a)}_{n,m}}{\xi^{(c)}_{n,m}}, \quad q_{n,m}[2] = -\frac{\xi^{(b)}_{n,m}}{\xi^{(c)}_{n,m}}.\]
A generalized fully discrete SP equation is explored as an integrability condition of a linear pair of difference equations. Analytical solutions of the systems can also be considered. The discretization pairs also fused into one another to form breather solutions that oscillate in time irrespective of their original shape. Discrete loop solutions are obtained and the dynamics of repulsive as well as attractive interactions between loop-loop and loop-antiloop solutions are discussed. These discrete solutions under covariance of the discrete linear system under Darboux transformation, nontrivial multi-soliton solutions are calculated in terms of ratios of determinants. Darboux transformation is applied to the contemporary discrete linear system. Furthermore, the by considering two continuum limits, generic case has been shown to reduce to space semi-discrete and time discretized SP equations. By taking two continuum limits, generic case has been shown to reduce to space semi-discrete and time discretized SP equations.

First-order breather solution is shown in Fig. 4 for $a = b = c = d = 0.5$.

**5 Concluding remarks**

A generalized fully discrete SP equation is explored as an integrability condition of a linear pair of difference equations. By considering two continuum limits, generic case has been shown to reduce to space semi-discrete and time discretized SP equations. Darboux transformation is applied to the contemporary discrete linear system. Furthermore, the covariance of the discrete linear system under Darboux transformation, nontrivial multi-soliton solutions are calculated in terms of ratios of determinants. Discrete loop solutions are obtained and the dynamics of repulsive as well as attractive interactions between loop-loop and loop-antiloop solutions are discussed. These discrete solutions under continuum limits were reduced to their respective continuous counterpart results that existed in literature. Loop-antiloop pairs also fused into one another to form breather solutions that oscillate in time irrespective of their original shape. Moreover, solutions established in this article may have applications in physics and engineering as the SP equation is regarded as the physical model for the ultra-short pulse propagations in a nonlinear medium. The discretization procedure used in this article can also be extended to have discrete versions of other nonlinear integrable systems and analytical solutions of the systems can also be considered.
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