Emergence of N-body tunable interactions in universal few-atom systems

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Abstract

A three-atom molecule AAB, formed by two identical bosons A and a distinct one B, is studied by considering coupled channels close to a Feshbach resonance. It is assumed that the subsystems AB and AA have, respectively, one and two channels, where, in this case, AA has open and closed channels separated by an energy gap. The induced three-body interaction appearing in the single channel description is derived using the Feshbach projection operators for the open and closed channels. An effective three-body interaction is revealed in the limit where the trap setup is tuned to vanishing scattering lengths. The corresponding homogeneous coupled Faddeev integral equations are derived in the unitarity limit. The s-wave transition matrix for the AA subsystem is obtained with a zero-range potential by a subtractive renormalization scheme with the introduction of two finite parameters, besides the energy gap. The effect of the coupling between the channels in the coupled equations is identified with the energy gap, which essentially provides an ultraviolet scale that competes with the van der Waals radius - this sets the short-range physics of the system in the open channel. The competition occurring at short distances exemplifies the violation of the “van der Waals universality” for narrow Feshbach resonances in cold atomic setups. In this sense, the active role of the energy gap drives the short-range three-body physics.

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I. INTRODUCTION - AN OVERVIEW OF RECENT PROBLEMS INVOLVING UNIVERSEALITY IN FEW-BODY SYSTEMS

The existence of several nuclear potentials with many free parameters, usually set to reproduce scattering observables, raises the question about the possibility to make something relevant with a zero-range potential. From the absolute point-of-view it is not possible to describe the energy spectrum or the nuclei structure with a such quasi-no-parameter potential. However, the description of physical observables for specific nuclear systems, after identifying the relevant scales, is completely suitable for a Dirac-delta potential: this is in the core of the universality concept [1–3].

When applied to few-body systems, universality means, *grosso modo*, independence on details of the short-range part of the interactions which are describing such systems. These systems appear in several few-body areas, being characterized for having their typical sizes, represented by the absolute value of the two-body scattering length $a_0$ and potential range $r_0$, such that the first is much greater that the second, $|a| \gg r_0$ - this property defines a weakly-bound system. There are many molecules in the atomic context, which satisfy the relation $|a|/r_0 \gg 1$ [4]. In the nuclear context, these weakly-bound structures are well represented by halo nuclei, in which we have one or more halo nucleons weakly-bound to a core nucleus. [5].

Universality in few-body physics provided explanation to many interesting phenomena, which have been extensively studied in recent years. The remarkable one is the Efimov effect [6], which was established by studying bosonic-like three-body system with at least two subsystems having infinite scattering length. This effect became a paradigm when mentioning universal aspects of few-body systems - nowadays, the study of the relevant physical scales close to the unitary limit is known as “Efimov physics”.

A natural three-body system with two-body subsystems exactly at zero energy does not exist. One of the systems found in nature that approaches the ideal situation is the helium trimer. It was suggested that the Efimov effect could be verified in this molecule [7]. The lack of any other natural three-body system close to the Efimov limit was the main reason why the original article from Efimov was considered for thirty years uniquely as a theoretical allegory. However, with the recent modern techniques involving ultracold traps, it was possible to measure the ground and first excited states of the helium trimer [8], confirming
the original suggestion and many other theoretical studies from different groups [14–25].

The confirmation of Efimov states is also possible by producing them artificially. Some experiments with cold-atom systems were able to observe signals of Efimov resonant levels by manipulating the two-body interactions. The possibility to freely tune the two-body interaction in ultracold atomic traps brought unprecedented possibilities to artificially create the ideal condition for Efimov states [9]. The use of Feshbach resonances technique [10] to alter the two-body energies [11–13] in atomic traps produced a rich playground with plenty of possibilities to investigate the correlations between the physical scales. Experimentalists are now able to continuously vary $a_0$ from large positive to large negative values, across the threshold at which a bound state turns into a resonance or a virtual state (see e.g. Ref. [5]).

The continuous change of spatial dimension is also another interesting aspect [26] that can now be explored in actual experimental realizations [27]. The study of one, two or three-dimensional solitons has already started long ago (See [28, 29] and references therein). However, it still missing systematic experimental studies about the effect of the dimensional change in few-body observables, e.g., the Efimov effect is drastically affected by the spatial dimensional reduction. The disappearance of Efimov effect in fractional dimensions between three and two dimensions, including another few-body aspects involving a continuous change of spatial dimensions, has recently being investigated by some groups [26, 30–36].

All the experimental possibilities regarding the change of dimensionality and atomic interactions are not possible in the nuclear context where the nucleon-nucleon interaction is rigid. A conjecture was raised in Ref. [37] that a resonant Hoyle state in $^{12}$C could emerge from an Efimov bound state. In order to investigate this conjecture, a recent work studied the screening of the Coulomb potential in three alphas [38].

The strong repulsion at short distances coming from the Coulomb potential is what prevents the appearance of Efimov effect in three alphas. However, in hot dense plasmas the electrons can, in principle, shield the repulsion coming from the protons favoring the formation of the Hoyle resonance. A continuous increase of the electronic density can also produce a similar effect, occurring in atomic traps, that is the change of the effective two-body interaction of the system. However, recent calculations made in Ref. [38] exclude the possibility of a Hoyle state emerging from an Efimov bound state.

Another interesting aspect in the context of few-atom systems, which is the central point of this article, is that, at least theoretically, it is possible that induced few-atom (beyond
two) forces appear [39] when the cold atomic trap set-up is tuned near a narrow Feshbach resonance, where the closed channel is strongly coupled to the open channel [40]. If the control of only two-body interactions [13] revealed a wide horizon to study correlations among their observables, due to the change of few-body scales [41], the possibility to control three and more atomic interaction would give an unprecedented freedom to few-body community.

Formally, the origin of the induced few-particle effective interactions in a single channel representation of few-atom system goes back to the Feshbach decomposition of the Hilbert space in open channels ($P$-space) and closed channels ($Q$-space), given that $P + Q = 1$ (details on the coupled-channel formalism applied to potential scattering can be found in the textbook by Canto and Hussein in Ref. [42]). The $Q$-space represents the state where the two atoms interacts in a region of the potential where the scattering states are closed. The $P$-space corresponds to potential well lower in energy where the scattering states are open. Actually, $P$ and $Q$ are associated with different spin states of the low partial waves of the atom-atom system (see e.g. [40]). In such framework, attractive effective few-body forces arise from connected diagrams with the intermediate virtual propagation of the system in the $Q$-space as illustrated in [39]. The strength is enhanced for narrow resonances, as the coupling between open and closed channels is larger in this case [40].

The experimental evidences suggesting the possibility of few-atom forces come from the so called “van der Waals universality” [43–48] of Efimov states across broad and narrow Feshbach resonances [49, 50] with Lithium-Caesium ($^6\text{Li} - ^{133}\text{Cs}$) mixtures. The “van der Waals universality” associates the position ($a_0 < 0$) of the resonant three-atom recombination peak, originated when an Efimov state dives into the continuum, with van der Waals radius ($\ell_{vdW}$), such that the ratio $a_0/\ell_{vdW} \sim -9$ is verified for broad resonances [43, 50]. However, the experimental results from Ref. [50] indicated a dependence of the position of the Efimov resonance on the Feshbach resonance strength, deviating from the prediction of the single channel “van der Waals” universality. Such observation suggests that close to narrow Feshbach resonances a single channel description may be poor, and beyond the expected large variations of the atom-atom scattering length, the three-body scale can also change, as also supported by the observation of resonant recombinations in the $^6\text{Li} - ^{133}\text{Cs}_2$ experiments [50].

Note that the length scale associated with the position of the triatomic recombination resonance is observed to be larger than the van der Waals length for narrow resonances [50].
This was interpreted in Ref. [41] as a manifestation of the attraction due to the induced three-atom interaction, which dislocates the effective repulsive barrier [51], where the triatomic continuum resonance [52] is formed, to distances larger than $\ell_{\text{vdW}}$. Therefore, the position of the narrow recombination resonance appears dislocated towards larger absolute values of the scattering length with respect to broad Feshbach resonances, as experimentally observed [50].

One should expect that, in few-atom systems driven by the Feshbach resonance mechanism, which can induce from three up to N-body interactions, systems beyond the atom-atom scattering length can be manipulated by tuning also the short-range scales associated with three, four and more particles [39, 41, 53, 54]. Such exciting possibilities motivated us to explore the ideas we have sketched in Ref. [39]. We write here the formalism of the three-body Faddeev equations for the bound state in the presence of open and closed channels. Following that, a practical framework can be formulated by identifying the relevant parameters of the Feshbach resonance, which control the induced three-body interaction and the associated short-range scale.

In this contribution, the AAB atomic bound state problem with coupled open-closed channels close to a Feshbach resonance is formulated through the Faddeev equations for the wave function. The effective three-body force appearing in the single channel description is detailed by using the Feshbach projection operators in the open and closed channels. We discuss the interesting case where the scattering lengths vanish, such that in the open channel the direct interaction vanishes, while we show formally that the Faddeev components of the wave function in the open channel survives, allowing the system to effectively interact in a single channel description through the coupling of the open channel with the closed one. Furthermore, the Faddeev formalism is derived explicitly for a zero-range model in one particular example of three-particle AAB system, where the AA subsystem has open and closed channels, by formulating in practice the ideas proposed in Ref. [39]. The dependence of the short-range physics on the parameters of the Feshbach resonance in the AA subsystem is obtained with the corresponding meaning being explored qualitatively.

This work is organized as follows. In the next Sect. II, the Faddeev equation for the bound state wave function for a general three-body model with open-closed channels is formulated. It is also discussed the single channel reduction and induced three-body interaction in the open channel. In addition, the effective three-body interaction appearing when the system
has vanishing scattering length is derived within the closed channel description of the three-body bound state. In Sect. III, the AAB Faddeev equations for the bound state within a zero-range model with open and closed channels is derived in the case that the AA bosonic subsystem has one open and one closed channels, and by considering the AB interaction acting only in a single channel. Once obtained the two-body T-matrix for the AA subsystem within the zero-range interaction, the integral equations for the AAB coupled channel model in the unitarity limit ($a \to \pm \infty$) are derived, by considering the dependence on the Feshbach resonance parameters. In Sect. IV, we present our final considerations.

II. OPEN-CLOSED CHANNELS THREE-BODY MODEL

The open-closed channels model for three-atom interactions has altogether eight channels, as each pair can interact in an open or closed channel. The notation has to reflect such different physical situations, with an index $\alpha$ or $\beta$ for the corresponding channel wave function, in which the atom pairs can be in open or closed channels, being indicated by $\alpha_{ij}$, which runs over the two-body channels. Then, the possibilities are

$$\alpha = (\alpha_{ij}, \alpha_{jk}, \alpha_{ki}),$$

and in a situation where only two of them are possible, namely the open and closed one, it allows altogether eight three-body channels. Assuming, two-atom interactions, the potential is an operator that allows transitions between the open and closed channels: $V_{ij}^{\alpha, \beta}$, which for the moment we are not specifying.

After setting the structure of the Hilbert space where the wave function is defined, we write the three-body Hamiltonian as a matrix, with operators as matrix elements in the open-closed channel two-atom states, as

$$H_{\alpha\beta} = (H_0 + \Delta_\alpha)\delta_{\alpha\beta} + \sum_{i>j} V_{ij}^{\alpha_{ij}, \beta_{ij}} \delta_{\alpha_{jk}, \beta_{jk}} \delta_{\alpha_{ki}, \beta_{ki}},$$

where, to simplify the presentation, we introduce the following notation

$$V_{ij}^{\alpha, \beta} := V_{ij}^{\alpha_{ij}, \beta_{ij}} \delta_{\alpha_{jk}, \beta_{jk}} \delta_{\alpha_{ki}, \beta_{ki}},$$

for the potential. The kinetic energy operator is $\hat{H}_0$ and $\Delta_\alpha$ is the energy level of channel
The energy of the channels are given by
\[ \Delta_\alpha = \sum_{i>j} e(\alpha_{ij}), \] (4)
where \( e(\alpha_{ij}) \) is the \( \alpha_{ij} \) channel energy. For the open channel we set it as \( e = 0 \), therefore the three-body open channel continuum has \( \Delta = 0 \). Furthermore, the standard translational invariance applies to \( H_{\alpha\beta} \) as long as the two-atom potential depends only on relative coordinates.

The eigenvalue equation for the energy states of the system reads:
\[ \sum_\beta H_{\alpha\beta} |\Psi^\beta\rangle = \sum_\beta \left( (H_0 + \Delta_\alpha) \delta_{\alpha\beta} + \sum_{i>j} V_{\alpha\beta}^{ij} \right) |\Psi^\beta\rangle = E |\Psi^\alpha\rangle. \] (5)

We write the Faddeev components of the particular case of the bound-state wave function as [55],
\[ |\Psi^\alpha\rangle = \frac{1}{E - H_0 - \Delta_\alpha + i\varepsilon} \sum_\beta \sum_{i>j} V_{\alpha\beta}^{ij} |\Psi^\beta\rangle, \] (6)
where
\[ G_0^\alpha(E) \equiv \frac{1}{E - H_0 - \Delta_\alpha + i\varepsilon} \] (7)
is the resolvent. The corresponding Faddeev equations can be written as
\[ |\Psi_i^\alpha\rangle = G_0^\alpha(E)t_{jk}^{\alpha\beta}(E)(|\Psi_j^\beta\rangle + |\Psi_k^\beta\rangle), \] (8)
with the two-body T-matrix within the three-body system being a solution of
\[ t_{jk}^{\alpha\beta}(E) = V_{jk}^{\alpha\beta} + V_{jk}^{\alpha\gamma} G_0^\gamma(E)t_{jk}^{\gamma\beta}(E). \] (9)

For the sake of clarity, when written solely for the two-body subsystem, it reads:
\[ t_{jk}^{ab}(E) = V_{jk}^{ab} + V_{jk}^{ac} \frac{1}{E - H_0^{(2)} - e(c) + i\varepsilon} t_{jk}^{cb}(E), \] (10)
where \( H_0^{(2)} \) is the two-body kinetic energy operator.

Let us consider the most simple AAB case, when only the AA subsystem has open and closed channels, while AB does not interact in a closed channel. In such simplified situation the three-body system has only two channels. Then, close to a s-wave Feshbach resonance, the two-body scattering amplitude (evidently associated with the open channel \( o \) and \( a = b = o \)) is customarily approximated by the corresponding effective range expansion:
\[ \langle \vec{k}' | t_{jk}^{oo}(E) | \vec{k} \rangle = -\frac{1}{(2\pi)^2 \mu_{ij}} \left[ -a^{-1} + \frac{1}{2} r_0 k^2 + \cdots - ik \right]^{-1}, \] (11)
where \( k = \sqrt{2\mu_{ij} E} \), \( a_0 \) is the scattering length and \( r_0 \) the effective range.
A. Single channel reduction and induced three-atom interaction

It is possible to reduce the Hamiltonian eigenvalue equation to the open channel, at the expenses of introducing an effective Hamiltonian, which can be derived by using the Feshbach projection operators to the open channels \((P\text{-space})\) and closed channels \((Q\text{-space})\), respectively, such that \(P + Q = 1\). In our notation,

\[
P_{\alpha\beta} = \delta_{\alpha\alpha}\delta_{\alpha\beta} \quad \text{and} \quad Q_{\alpha\beta} = \delta_{\alpha\beta} - P_{\alpha\beta},
\]

where \(o\) indicates that the three pair of particles are in two-body open channels. However, if we do the common procedure, we loose the re-sum of two-body intermediate state propagation in the closed channels which builds the two-body T-matrix in the open channel.

This motivates us to use the Feshbach projection \(P\) and \(Q\) operators directly in the Faddeev equations. We can exemplify the procedure by considering the bound-state in (8), such that

\[
|\Psi_{i}^{o}\rangle = G_{0}^{o}(E)t_{jk}^{\alpha\beta}(E)(P_{\beta\gamma} + Q_{\beta\gamma})(|\Psi_{j}^{o}\rangle + |\Psi_{k}^{o}\rangle)
\]

\[
= G_{0}^{o}(E)t_{jk}^{\alpha\beta}(E)(|\Psi_{j}^{o}\rangle + |\Psi_{k}^{o}\rangle) + G_{0}^{o}(E)t_{jk}^{\alpha\beta}(E)(Q_{\beta\gamma}|\Psi_{j}^{o}\rangle + Q_{\beta\gamma}|\Psi_{k}^{o}\rangle).
\]

For the open channel we have that

\[
|\Psi_{i}^{o}\rangle = G_{0}^{o}(E)t_{jk}^{\alpha\beta}(E)(|\Psi_{j}^{o}\rangle + |\Psi_{k}^{o}\rangle) + G_{0}^{o}(E)t_{jk}^{\alpha\beta}(E)(Q_{\beta\gamma}|\Psi_{j}^{o}\rangle + Q_{\beta\gamma}|\Psi_{k}^{o}\rangle).
\]

For the closed channels, the Faddeev equations can be written as an inhomogeneous integral equations, given by

\[
Q_{\alpha\beta}|\Psi_{i}^{o}\rangle = Q_{\alpha\beta}G_{0}^{\beta}(E)t_{jk}^{\alpha\beta}(E)(|\Psi_{j}^{o}\rangle + |\Psi_{k}^{o}\rangle) + Q_{\alpha\beta}G_{0}^{\beta}(E)t_{jk}^{\alpha\beta}(E)(Q_{\beta\gamma}|\Psi_{j}^{o}\rangle + Q_{\beta\gamma}|\Psi_{k}^{o}\rangle).
\]

Due to the elimination of the closed channels to describe the dynamics of the system only using the open channel, the interactions in the Faddeev equations for the open channel gain new terms coming from the virtual propagation of the three atoms in a closed channel.

Such interactions are given automatically by three-particle connected operators, that means an induced three-atom interaction. To illustrate this, we introduce the iterative solution corresponding to Eq. (15), given by

\[
Q_{\alpha\beta}|\Psi_{i}^{o}\rangle = Q_{\alpha\beta}G_{0}^{\beta}(E)t_{jk}^{\alpha\beta}(E)(|\Psi_{j}^{o}\rangle + |\Psi_{k}^{o}\rangle) + \cdots,
\]

in Eq. (14), which results in the single channel Faddeev equations:

\[
|\Psi_{i}^{o}\rangle = G_{0}^{o}(E)t_{jk}^{\alpha\beta}(E)(|\Psi_{j}^{o}\rangle + |\Psi_{k}^{o}\rangle) + G_{0}^{o}(E)t_{jk}^{\alpha\beta}(E)Q_{\beta\gamma}G_{0}^{\beta}(E)t_{ki}^{\alpha\beta}(E)(|\Psi_{j}^{o}\rangle + |\Psi_{k}^{o}\rangle)
\]

\[
+ G_{0}^{o}(E)t_{jk}^{\alpha\beta}(E)Q_{\beta\gamma}G_{0}^{\beta}(E)t_{ij}^{\gamma\alpha}(E)(|\Psi_{i}^{o}\rangle + |\Psi_{j}^{o}\rangle)) + \cdots.
\]
The new connected operators, contributing to the kernel of the first terms of Eq. (17) are
\[ t_{jk}^{\beta}(E)Q_{\beta\gamma}G_{0}(E)t_{ij}^{\alpha}(E) \quad \text{and} \quad t_{jk}^{\beta}(E)Q_{\beta\gamma}G_{0}(E)t_{ij}^{\alpha}(E), \] (18)
these operators clearly indicate the propagation in closed channels, which characterizes the nature of a three-body force as sketched in Ref. [39].

It is not a complex exercise to extrapolate the above result to the full series of connected kernel, considering any number of intermediate propagation in closed channels. Therefore, it is clear that the Feshbach resonance also drives a triatomic interaction, besides the two-atom scattering length, and potentially could be controlled, at least from the theoretical point-of-view. The framework developed so far can also be generalized to four atoms with the corresponding Faddeev-Yakubovski equations [56].

B. Effective three-body interaction and vanishing scattering length

In atomic traps, the scattering length can be tuned by Feshbach resonance techniques, as shown in Ref. [11], such that one can reach a particular situation is which the two-body scattering length is exactly zero. In this limit, the dynamics of the Bose-Einstein condensate, as described by the Gross-Pitaevskii equation, says that the interaction between identical bosonic atoms ceases to exist. However, as discussed in Sect. II A, in the single channel Faddeev equation we have the contribution of an effective three-body force when the closed channels are eliminated in favor of the open channel.

By assuming that the s-wave two-atom T-matrix in the open channel is obtained for a short-range interaction, close to a zero-range form, its matrix element will not depend on the relative momentum, except for the dependence on the energy of the system. In this case, the matrix elements will resemble the amplitude written in Eq. (11), such that in the \( a_0 \to 0 \) limit, the two-body T-matrix in the open channel vanishes, i.e. \( t_{ij}^{\alpha}(E) = 0 \), and for the identical bosons system, the coupled open-closed channels Faddeev equation (14) will simplify and given by

\[ |\Psi_{0}^{\alpha}(E) = G_{0}(E)t_{jk}^{\beta}(E)\left(Q_{\beta\gamma}|\Psi_{j}^{\gamma}\rangle + Q_{\beta\gamma}|\Psi_{k}^{\gamma}\rangle\right), \] (19)
where the term carrying the open-channel two-body T-matrix disappear, allowing to determine the open channel Faddeev components of the wave function from the ones in the closed
channel.

By eliminating the open channel components in the closed-channel Faddeev equations, (13), by using Eq. (19) written above, one gets the following set of coupled equations in the closed channels:

\[
Q_{\alpha\beta}|\Psi^\beta_i\rangle = Q_{\alpha\beta}G^\beta_0(E)t^\beta_0(E)G^\gamma_0(E)t^\gamma_0(E)Q_{\gamma\delta}|\Psi^\delta_i\rangle + Q_{\gamma\delta}|\Psi^\delta_i\rangle
\]  
\[+ Q_{\alpha\beta}G^\beta_0(E)t^\beta_0(E)G^\gamma_0(E)t^\gamma_0(E)Q_{\gamma\delta}|\Psi^\delta_i\rangle + Q_{\gamma\delta}|\Psi^\delta_i\rangle
\]  
\[+ Q_{\alpha\delta}G^\delta_0(E)t^\delta_0(E)(Q_{\beta\gamma}|\Psi^\gamma_j\rangle + Q_{\beta\gamma}|\Psi^\gamma_j\rangle).
\]  

The connected operators in the first two terms of the equation, which can be interpreted as an effective three-body interaction, come from the coupling of the closed and open channels with a virtual propagation of the system to the open channel coming back to the closed one. This is can be seen, for example, in the first term of the kernel

\[
Q_{\alpha\beta}G^\beta_0(E)t^\beta_0(E)G^\gamma_0(E)t^\gamma_0(E)Q_{\gamma\delta}.
\]  

The indices \(\beta\) and \(\gamma\) in \(t^\beta_0(E) G^\gamma_0(E)t^\gamma_0(E)\) are referring to the closed channels and \(G^\gamma_0(E)\) corresponds to the virtual propagation of the three-body system in the open channel.

This extreme situation can be studied in schematic models in the limit of zero-range interactions, for example. In what follows we will exemplify another case modeled with a two-channel s-wave zero-range interaction model, close to the Feshbach resonance. This setup allows the derivation of an analytically two-body T-matrix and the built of the bound-state Faddeev equations.

### III. AAB System with Open and Closed Channels: Zero-Range Model

We substantiate the model proposed in Ref. [39] by assuming a Feshbach resonance in only one pair. The two identical bosons A interact with a third particle B, and all s-wave interactions have zero-range. First, we derive the two-body T-matrix for the AA system, and then the bound-state Faddeev equations for the open-closed channel AAB model are derived in the unitarity limit, \(a_0 \rightarrow \pm \infty\). The parametric dependencies on the Feshbach resonance are pointed out.
A. Two-channel zero range interaction model

The T-matrix of a two channel zero-range interaction in s-wave can be analytically derived, having a simple form when the renormalized interaction strength in the open channel vanishes. Such assumption, taken for simplicity, allows us to study the three-body dynamics close to the Feshbach resonance, where the coupling with the closed channel puts the system in the unitarity limit.

This model can be renormalized by subtracting the resolvent at some given scale (see e.g. [5]), which for convenience we choose for zero energy. The two-body T-matrix elements of the Lippmann-Schwinger equation can be written, in this model, in an operatorial-matrix notation as

\[
T(E) = \begin{bmatrix} 0 & \eta \\ \eta & \lambda \end{bmatrix} + \int d^3p \begin{bmatrix} 0 & \eta g_0(p^2, E-e; 0) \\ \eta g_0(p^2, E; 0) & \lambda g_0(p^2, E-e; 0) \end{bmatrix} T(E),
\]

where the renormalized strengths \( \lambda \) and \( \eta \) refer to the interaction in the closed channel and the coupling between the open and close channels, respectively. The resolvent, with outgoing boundary condition, is given by

\[
g_0(p^2, E; \mu^2) = \frac{E + \mu^2}{(\mu^2 + \frac{p^2}{2m_r} - i\varepsilon)(E - \frac{p^2}{2m_r} + i\varepsilon)},
\]

where \( m_r \) is the reduced mass of the two atoms. Note that, the resolvent in the closed channel carries the energy \( e \), which allows it to open only when \( E > e \) and when the scattered particles can transitioned from one channel to the other one. In what follows, we will study the situation where \( E < e \). The operator \( T(E) \), in momentum space, has matrix elements that do not depend on any momenta, only depending on \( E \) - this is a consequence of the zero-range interaction, used in this example.

The T-matrix equation (22) allows analytical solution in the following form, with the channel terms given by

\[
t^{oo}(E) = \left[ \frac{1 - \lambda B(E-e)}{\eta^2 B(E-e)} - B(E) \right]^{-1},
\]

\[
t^{oc}(E) = t^{co}(E) = \eta \left[ 1 - B(E-e)(\eta^2 B(E) + \lambda) \right]^{-1},
\]

\[
t^{cc}(E) = \left[ \frac{1}{\eta^2 B(E) + \lambda} - B(E-e) \right]^{-1},
\]

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where

$$B(E) \equiv \int d^3p \frac{E}{(\frac{p^2}{2m_r} - i\varepsilon)(E - \frac{p^2}{2m_r} + i\varepsilon)} = -i(2\pi)^2 \sqrt{2m_r^3E},$$

(27)

where we work with the hypothesis of a closed channel, \(E < e\), such that \(B(E - e) = (2\pi)^2 \sqrt{2m_r^3(e - E)}\). Therefore,

$$t^{co}(E) = \left\{ 4\pi^2m_r \left[ \frac{1}{a_0(E)} + i\sqrt{2m_rE} \right] \right\}^{-1},$$

(28)

$$t^{oc}(E) = t^{co}(E) = \eta \left[ 1 - \zeta(E)\sqrt{2m_r(e - E)} \right]^{-1},$$

(29)

$$t^{cc}(E) = \left\{ 4\pi^2m_r \left[ \frac{1}{\zeta(E)} - \sqrt{2m_r(e - E)} \right] \right\}^{-1},$$

(30)

where

$$a_0(E) \equiv \frac{(2\pi)^4 m_r^2 \eta^2 \sqrt{2m_r(e - E)}}{1 - (2\pi)^2 m_r \lambda \sqrt{2m_r(e - E)}},$$

(31)

$$\zeta(E) \equiv 4\pi^2m_r \left[ \lambda - iiv^2(2\pi)^2m_r\sqrt{2m_rE} \right].$$

(32)

This example shows that by varying \(e\) one can tune \(a_0(0)\), namely the scattering length. The model could be enriched by also considering the interaction strength in the open channel. This is left for another work.

One interesting situation that emerges in this model corresponds to the unitarity limit, \(a_0(0) \to \pm \infty\). In this situation \(\lambda^{-1} = (2\pi)^2 m_r \sqrt{2m_r e}\), giving

$$t^{oo}(E) = \left\{ 4\pi^2m_r \left[ \frac{1}{a_U(E)} + i\sqrt{2m_rE} \right] \right\}^{-1},$$

(33)

$$t^{co}(E) = t^{oo}(E) = -i \left[ 2(2\pi)^4 m_r^3 \eta \sqrt{E(e - E)} \right]^{-1},$$

(34)

$$t^{cc}(E) = \frac{1}{(2\pi)^2 m_r} \left[ 1 - iiv^2(2\pi)^4 m_r^3(e - E) \sqrt{2m_rE} \right],$$

(35)

where

$$a_U(E) \equiv \frac{(2\pi)^4 m_r^2 \eta^2 \sqrt{2m_r e - E}}{\sqrt{e - \sqrt{2m_rE}}}.$$  

(36)

For \(E < 0\), which corresponds to the three-body bound state,

$$t^{cc}(E) = -\frac{1}{(2\pi)^2 m_r} \left[ 1 + \eta^2(2\pi)^4 m_r^3 \sqrt{E|e - E|} \right],$$

(37)

leading to a strong attraction in the closed channel within the kernel of the Faddeev equations. The other term \(t^{oc}(E)\), from the coupling between the open and closed channels, has an attractive effect in three-atom system. This will be discussed in the next section.
B. AAB coupled channel model in the unitarity limit

Our starting point is Eq. (8) and the two-body T-matrix (33) for \( a_0 = \pm \infty \). We are assuming that the subsystem AA has two channels and AB has only one open channel. The first observation is that the matrix elements of the T-matrix for the AA system, derived in Sect. III A in momentum space, depends only on the energy, as a consequence of the zero-range interaction s-wave. This is also the case for the single channel AB T-matrix. For this particular AAB system, in units of \( m_A = m_B = 1 \) and \( \hbar = 1 \), it follows that

\[
\langle \vec{q}_A(B)\vec{p}_A(B) | \Psi_{A(B)}^\alpha \rangle = f_\alpha \left( \vec{q}_A(B) \right) = f_\alpha \left( \vec{q}_A \right), \tag{38}
\]

where \( \vec{q}_A(B) \) is the relative momentum of particle \( A(B) \) with respect to the center of mass of the pair \( A'B(AA) \), with \( \vec{p}_A(B) \) the relative momentum of the particles \( A' \) and \( B \) \((A \text{ and } A')\).

Considering that the bosonic wave function is symmetric by the exchange of the identical bosons \( A \) and \( A' \), it follows that the spectator functions \( f_i^\alpha \) should be:

\[
f_\alpha \left( \vec{q}_A \right) = f_\alpha \left( \vec{q}_{A'} \right) \equiv f_\alpha \left( \vec{q} \right), \tag{39}
\]

for \( \vec{q}_A = \vec{q}_{A'} \equiv \vec{q} \). For the AAB system only two channels are present: AA standing in the open or closed channels. The correspondent spectator functions are \( f^\alpha \), with \( \alpha = 0, 1 \), respectively, for the open, \((o)\), and closed, \((c)\), channels, indicating the pair AA propagating in the open or closed channels, and correspondingly \( \Delta_0 = \alpha e \). Altogether for this model, the spectator functions are four: \( f_0^A(\vec{q}) \), \( f_1^A(\vec{q}) \), \( f_0^B(\vec{q}) \) and \( f_1^B(\vec{q}) \).

In the unitary limit, i.e., by assuming \( a_{AB} = a_{AA} \to \pm \infty \) for both interactions, the set of four coupled Faddeev equations are derived from Eq. (8) by using Eq. (38):
\[ f^0_A(\vec{q}) = \frac{1}{2\pi^2 \sqrt{|E| + \frac{3}{4}q^2}} \int d^3k \mathcal{K}^0_E(\vec{q}, \vec{k}) \left( f^0_A(\vec{k}) + f^0_B(\vec{k}) \right), \]

\[ f^0_B(\vec{q}) = \frac{1/(2\pi^4 \eta)}{\sqrt{(|E| + e + \frac{3}{4}q^2)(|E| + \frac{3}{4}q^2)}} \int d^3k \mathcal{K}^0_E(\vec{q}, \vec{k}) f^0_A(\vec{k}) \]

\[ + \frac{1/(2\pi^2)}{\alpha_{\nu} (-|E| - \frac{3}{4}q^2) + \sqrt{|E| + \frac{3}{4}q^2}} \int d^3k \mathcal{K}^0_E(\vec{q}, \vec{k}) f^0_A(\vec{k}), \]

\[ f^1_A(\vec{q}) = \frac{1/(2\pi^2)}{\sqrt{|E| + e + \frac{3}{4}q^2}} \int d^3k \mathcal{K}^1_E(\vec{q}, \vec{k}) \left( f^1_A(\vec{k}) + f^1_B(\vec{k}) \right), \]

\[ f^1_B(\vec{q}) = \frac{1/(2\pi^4 \eta)^2}{\pi^2(|E| + e + \frac{3}{4}q^2) \sqrt{|E| + \frac{3}{4}q^2}} \int d^3k \mathcal{K}^1_E(\vec{q}, \vec{k}) f^1_A(\vec{k}), \]

where the three-body resolvent in terms of the spectator momenta is given by:

\[ \mathcal{K}^\alpha_E(\vec{q}, \vec{k}) = \frac{1}{|E| + \alpha e + q^2 + k^2 + \vec{q}.\vec{k}^{-1}}, \]

which takes into account the energy gap between the two channels.

The coupled set of equations (40) and (41) for the spectator functions \( f^0(\vec{q}) \) and \( f^0_B(\vec{q}) \) reduces to the Skorniakov and Ter-Martirosian equations \([57]\) for an AAB system, which for three atoms in s-wave present both to the Efimov effect and to the Thomas collapse. The Efimov effect and Thomas collapse are related to the breaking of continuous scale invariance to a discrete one. This demands the necessity of an ultraviolet (UV) scale associated to the three-body one, which carries all correlations of physical observables of the s-wave three-particle system. These correlations can easily be represented by scaling functions (see \([5]\)).

We have discussed in Sect. II A the appearance of an effective three-body interaction, when the Faddeev equations in the open and closed channels are reduced to a single one. The set of coupled integral equations for the spectator functions of the AAB system, (40)-(43), illustrates such dynamical mechanism. The coupling with \( f^1_A \) in the open channel equation (41) could be translated to an effective three-body interaction acting in the open channel equations for \( f^0_A(\vec{q}) \) and \( f^0_B(\vec{q}) \).

The coupled set of equations (40)-(43) needs to be regularized at the UV region to avoid the collapse of the AAB system for vanishing total angular momentum. The regularization
can be done by resorting to a subtraction in the kernel. Then instead of (44) we can use the subtracted resolvent [58]:

\[
K_{E}^{\beta}(\vec{q}, \vec{k}) = \frac{1}{|E| + \alpha e + q^2 + k^2 + \vec{q}.\vec{k}} - \frac{1}{\mu^2 + \alpha e + q^2 + k^2 + \vec{q}.\vec{k}}
\]

\[
= \frac{\mu^2 - E}{\left(|E| + \alpha e + q^2 + k^2 + \vec{q}.\vec{k}\right) \left(\mu^2 + \alpha e + q^2 + k^2 + \vec{q}.\vec{k}\right)},
\]

such that \(\mu^2 >> e\). It will imply that, even with \(\mu^2\) fixed, the effect of the coupling between the open and closed channels, even at the unitarity limit, allows to drive the Efimov states by changing \(\eta\) and/or \(e\). In this model, the relevant dimensionless quantities for driving the Efimov states at the unitarity are \(e/\mu\) and \(\eta \mu\). The energy of the AAB system is given in units of \(\mu\).

The short-range scale is related to the van der Waals interaction as \(\mu \sim 1/\ell_{vdW}\). At unitarity, the single channel description of the AAB system makes that all s-wave three-body observables be scaled with powers of \(\mu\). This is a direct consequence of the possibility to find the particles simultaneously at the same position, i.e., depending on the configuration of the constituents of the system, this collapse can exist for bosonic or fermionic systems. For example, in the single channel framework given by Eqs. (40) and (41), with the subtracted kernel (45), once the coupling with the closed channel is disregarded, one can easily verify that at the unitarity the three-body binding energy is proportional to \(\mu^2\), as no other scales are present in the integral equations in limit of a zero-range interaction. The separation and coupling between the open and closed channel should represent other length scales, namely \(\eta\) and \(1/\sqrt{e}\), which controls the scattering length for \(E = p^2 < e\)

\[
p \cot \delta_{0}^{U} = -\frac{\sqrt{e} - \sqrt{e - p^2}}{4\pi^4 \eta^2 \sqrt{e} \sqrt{e - p^2}} \bigg|_{p^2 \to 0} = -\frac{1}{2} R^* p^2 \text{ with } R^* = \frac{1}{4\pi^4 \eta^2 e \sqrt{e}},
\]

where the parameter \(R^*\) is controlled by the length scale \(\gamma = \eta^{-2} e^{-\frac{7}{2}}\), with \(R^*\) related to the width of the Feshbach resonance [59].

The separation energy \(e\) between the two channels gives the momentum scale \(\sqrt{e}\), which competes in the UV region with the subtraction scale, as it is seen in the coupling term between the open and closed channels in Eq. (41). Note that for momentum \(k \sim \sqrt{e}\), the virtual propagation of the system brings to the bound state a new scale in the UV region.
competing with $\mu^2$. The van der Waals radius is associate to an equivalent energy scale of $\hbar^2/(m\ell_{vdW}^2)$. For example, in the $^{133}$Cs$_2$ system, where $\ell_{vdW} = 101 a_0$ [4] ($a_0$ the Bohr radius) the energy $\mu^2 \sim 127 \mu$K sets the UV scale for the $^{133}$Cs$_3$ system. The energy gap between the open and closed channels will compete, in the UV region, with the subtraction scale that can drive the triatomic system, beyond the scattering and effective range, by moving the Feshbach resonance parameters.

IV. FINAL CONSIDERATIONS

In summary, the three-atom bound state problem with coupled open-closed channels close to a Feshbach resonance is formulated through the Faddeev equations for the wave function. The effective three-body force appearing in the single channel description was derived using the Feshbach projection operators in the open and closed channels. Following this formal presentation, we investigated the interesting case for which the trap setup is tuned to vanishing scattering lengths, such that the direct open channel interaction disappears. In this case, we demonstrated that an effective interaction in the open channel appears due to an effective three-body interaction built from the coupling between the open and closed channels.

We have provided one explicit example of a three-body system composed by two identical bosons and a third different particle (AAB system) with open and closed channels, which realizes in practice the schematic discussion we have performed in Ref. [39]. In this illustrative case, the subsystem AA has an open and a closed channel separated by an energy gap, and interacting through a zero range s-wave potential. The transition matrix for the AA subsystem is derived resorting to a subtractive renormalization scheme, at the expense of introducing two finite parameters, besides the energy gap. The subsystem AA and AB are tuned to the unitary limit, and the Faddeev equations for the bound AAB system was derived explicitly, to expose the dependences of the kernel with the two body parameters and energy gap. The set of coupled equations allowed to identify the effect of the coupling between the Faddeev components in the open channel with the closed channel ones. The energy gap between the channels essentially provides an UV scale probed by the virtual propagation of the three-body system in the closed channel. This short-range scale competes with the van der Waals radius that sets the short-range physics of the system in the
open channel. Therefore, we have clearly illustrated how the three-atom system can be tuned by the Feshbach resonance beyond the scattering length and van der Waals radius.

As a final comment, considering the different possibilities to study few-body physics in atomic traps, we should mention that few-atom system can also be driven by the change in the effective dimensions, by squeezing the trap, while tuning both the two and few-body parameters controlling the Feshbach resonance. Controlled cold chemical reactions [60–62] with forces driven by external fields is also a field of intense activity (see e.g. [63–70]).

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