Characterization of the spatial complex behavior and transition to chaos in flow systems

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(November 8, 2018)

Abstract

We introduce a “spatial” Lyapunov exponent to characterize the complex behavior of non chaotic but convectively unstable flow systems. This complexity is of spatial type and is due to sensitivity to the boundary conditions. We show that there exists a relation between the spatial-complexity index we define and the comoving Lyapunov exponents. In these systems the transition to chaos, i.e. the appearing of a positive Lyapunov exponent, can take place in two different ways. In the first one (from neither chaotic nor spatially complex behavior to chaos) one has the typical scenario; that is, as the system size grows up the spectrum of the Lyapunov exponents gives rise to a density. In the second one (when the chaos develops from a convectively unstable situation) one observes only a finite number of positive Lyapunov exponents.

PACS: 05.45.+b
I. INTRODUCTION

The dynamical chaos is considered to be one of the main sources of complex behavior in a dynamical system. One of the main properties of dynamical chaos is the sensitive dependence of the evolution on the initial conditions, i.e.: a small error on the initial state grows exponentially in time [1]. This behavior is usually assumed as the characterizing property of chaos, and it is quantified by a positive value of the maximal Lyapunov exponent $\lambda_1$.

However, highly nontrivial behaviors can appear also in systems which are not chaotic (i.e., $\lambda_1 \leq 0$). Let us mention the systems with asymptotically stable fixed points, but with fractal boundaries of the attraction basins [2], and the chaotic scattering phenomenon [3], where the “chaos” is just transient.

An interesting situation can occur in high dimensional systems, like the following chain of maps with unidirectional coupling:

$$x_n(t + 1) = (1 - c)f_a(x_n(t)) + cf_a(x_{n-1}(t)),$$

where $t$ is the discrete time, $n = 1, 2, 3, \ldots, N$ is a spatial index and $x_0(t)$ is a given boundary condition. These models are quite natural candidates for the description of flow systems, that are systems with a privileged direction, e.g.: boundary layer, thermal convection and wind-induced water waves [4].

After the seminal papers of Deissler and Kaneko [5] it is now well known that nontrivial phenomena can take place in systems with asymmetric couplings, even in the absence of chaos ($\lambda_1 \leq 0$). In particular, if the system is convectively unstable the spatial structure can be very complex and the external noise can have an important role in the formation and the maintenance of the structure [3][4]. In spite of the clear evidence of a spatial “complexity” in these non chaotic systems, up to now, as far as we know, there is not a simple and systematic quantitative characterization of this phenomenon. To answer this purpose, we define a quantity that measures the degree of sensitivity of the system to the boundary conditions, and we study its possible quantitative relation with the comoving Lyapunov exponents – the quantities by means of which one can define the convective instabilities. The definition of the comoving Lyapunov exponents $\lambda(v)$, for these extended systems, may be given as follows [3]. If $\delta x_0(0)$ is a perturbation on the boundary at the time $t = 0$, in a frame of reference that moves along the system with velocity $v > 0$, at large $t$, this perturbation is $O(\delta x_0(0) \exp[\lambda(v)t])$. If $\lambda(v) < 0$ for all $v > 0$ the system is said to be absolutely stable; if there exists a range of velocity for which $\lambda(v)$ is positive, then the system is said to be convectively unstable. The interesting situation, in a convectively unstable system, arises when the usual Lyapunov exponent, $\lambda_1 = \lambda(v = 0)$, is negative.

In sect. II we discuss some results about the flow system model (1) with $f_a(x) = a x (1 - x)$, that is the local map we use for all the computations. In particular we report on the qualitative spatio-temporal behaviors at varying the control parameters $c$ and $a$.

In sect. III we introduce an index – we call it “spatial Lyapunov exponent”– that supplies us with a quantitative characterization of the spatial complexity, in terms of the spatial sensitivity to the boundary conditions. We show that there exists a strong relation between the “spatial” Lyapunov exponent and the comoving Lyapunov exponents.

In sect. IV the reader can find a study on the transition to chaos for the flow systems. The transition to a positive value of $\lambda_1$ can take place in two different settings:
I) from an absolutely stable state (i.e., a state for which the comoving Lyapunov exponents are negative for all velocities), that is not spatially complex;

II) from a state that is convectively unstable (i.e., a state for which the comoving Lyapunov exponents are positive for some velocities), that has a certain degree of spatial complexity.

The case I) is a rather standard transition, by which we mean that, for large $N$, there exists a finite density of positive Lyapunov exponents. On the other hand, in the case II), at varying $N$, one obtains only a finite number, not a finite fraction, of positive Lyapunov exponents. The latter behavior is a clear indication that the transition is dominated by a sort of boundary effect.

Sect. V is devoted to conclusions and discussion.

II. QUALITATIVE BEHAVIOR OF THE MODEL

In this section we present some dynamical features of the unidirectionally coupled map lattice (1), with $f_a(x) = a x (1 - x)$. In many papers the boundary condition is kept fixed, i.e. $x_0(t) = x^*$, and often $x^*$ is an unstable fixed point of the single map $x(t+1) = f_a(x(t))$. Here, following Deissler [5] and Pikovsky [8], we adopt a more general time dependent boundary condition: $x_0(t) = y(t)$ with $y(t)$ a known function, that may be periodic, quasi-periodic or obtained by a chaotic system.

At varying the control parameters, $c$ and $a$, one observes a plethora of different spatio-temporal behaviors. In Fig. 1 we show the results of a numerical exploration of the phase space of the system, with a quasi-periodic boundary condition $x_0(t) = 0.5 + 0.4 \sin(\omega t)$, with $\omega = \pi(\sqrt{5} - 1)$. These results can be summarized by saying that, basically, there exist four qualitatively different spatio-temporal behaviors.

A) Non chaotic and convectively stable. The comoving Lyapunov exponents are negative for all values of $v$ ($\lambda(v) < 0$, $\forall v$) and $x_n \to 0$ for $n \to \infty$. The region of the parameter space corresponding to this behavior (absolute stability) is identified by the ‘□’ symbols in Fig. 1. We call it “region A”. One can say that the quasi-periodic (or chaotic) boundary condition $x_0(t)$ is not able to excite the bulk of the system (see Fig. 2).

B) Non chaotic and marginally convectively stable. The comoving Lyapunov exponents have a maximum value equal to zero for a $v^* \neq 0$: $\lambda_{\max}(v) = \lambda(v^*) = 0$ (the region with ‘+’ symbols, in Fig. 1, that we call “region B”). The quasi-periodic boundary conditions produce spatio-temporal “strips patterns” (see Fig. 3).

C) Non chaotic but convectively unstable. The maximal Lyapunov exponent, $\lambda_1 = \lambda(0)$, is negative, but the comoving Lyapunov exponent spectrum is positive in a certain interval of $v > 0$ (the region with ‘∗’ symbols, in Fig. 1, that we call “region C”). The spatio-temporal behavior appears irregular (see Fig. 4).

D) Standard chaotic (the region with ‘■’, in Fig. 1, that we call “region D”). In this case one has $\lambda(0) > 0$, and the spatio-temporal behavior is irregular, and is similar to the one of Fig. 4.
We remark that the results discussed above remain valid, with slight changes, for chaotic boundary conditions; e.g. with $x_0(t)$ given by the $y$ variable of the Hénon map: $y(t + 1) = -\alpha y(t)^2 + \beta y(t - 1) + 1$, with typical values of the parameters $\alpha = 1.4$ and $\beta = 0.3$.

### III. Quantitative Characterization of the Spatial Behavior

We discuss here how to characterize the convectively unstable region $C$ of Fig. 1. Part of the results in this section have been briefly discussed in ref. [9]. Some authors, e.g. Pikovsky [8] and Kozlov et al. [10], stressed the fact that the “irregularity” of these systems seems to increase with $n$. An analysis of $x_n$ as a function of $t$ (by means of some standard methods for the characterization of dynamical systems, like, for instance, the power spectrum, the return map, the Grassberger-Procaccia correlation dimension [11]) typically shows that $x_1$ is more irregular than $x_0$, $x_2$ more irregular than $x_1$, and so on. Fig. 5 shows $x_n(t+1)$ vs $x_n(t)$ for different $n$: it is evident an increasing of the irregularity as $n$ increases. A simple way to characterize quantitatively the spatial complexity is by studying the spatial correlation functions:

$$C(n, m) = \frac{\langle x_n x_m \rangle - \langle x_n \rangle \langle x_m \rangle}{\langle x_n^2 \rangle - \langle x_n \rangle^2},$$

(2)

where the average is with respect to the time. $C(n, m)$ vs $m$, computed in the convectively unstable region, is shown in Fig. 6, for different $n$: two facts are evident:

- the shape of $C(n, m)$, at least for $n \gg 1$, does not depend on $n$, but only on $|n - m|$, thus revealing that we are observing a bulk property of the system;
- the correlation decays exponentially, $C(n, m) \sim \exp(-|n - m|/\xi)$.

It is natural to wonder how an uncertainty $\delta x_0(t) = O(\epsilon)$ – with $\epsilon \ll 1$ – on the knowledge of the boundary conditions will affect the system. In this paper we consider only the case of infinitesimal perturbations, so that we may safely assume that $\delta x_n$ evolves according to the tangent vector equations of the system (1):

$$\delta x_n(t + 1) = (1 - c)f'_a(x_n(t))\delta x_n(t) + cf'_a(x_{n-1}(t))\delta x_{n-1}(t).$$

(3)

For the moment we do not consider, for sake of simplicity, intermittency effects, that is, we neglect finite time fluctuations of the comoving Lyapunov exponents. The uncertainty $\delta x_n(t)$, on the determination of the variable at the site $n$, is given by the superposition of the evolved $\delta x_0(t - \tau)$ with $\tau = n/v$:

$$\delta x_n(t) \sim \int \delta x_0(t - \tau)e^{\lambda(v)\tau}dv = \epsilon \int e^{[\lambda(v)/v]n}dv.$$  

(4)

Since we are interested in the asymptotic spatial behavior, i.e. the large $n$ one, we can write:

$$\delta x_n(t) \sim \epsilon e^{\gamma n},$$

(5)

where, in the particular case of a non intermittent system, one has:
\[ \gamma = \max_v \frac{\lambda(v)}{v}. \] (6)

Equation (6) is a link between the comoving Lyapunov exponent and the “spatial” Lyapunov exponent \( \gamma \), a more precise and operative definition of which is given by:

\[ \gamma = \lim_{n \to \infty} \frac{1}{n} \ln \left( \frac{\delta x_n}{\epsilon} \right), \] (7)

where the brackets mean a time average.

So equation (6) establishes a relation between the convective instability of a system and its sensitivity to the boundary conditions, which can be considered a sort of spatial complexity.

We remark again that equation (6) holds exactly only in the absence of intermittency; in this case it can be shown from (6) that our spatial index can be written in a simple way in terms of another kind of spatial Lyapunov exponents, \( \mu(\Lambda) \), introduced in ref. [12]:

\[ \gamma = \max_v \frac{\lambda(v)}{v} = \mu(\Lambda = 0). \] (8)

In the general case the relation is rather more complicated. We introduce the effective comoving Lyapunov exponent, \( \tilde{\lambda}_t(v) \), that gives the exponential changing rate of a perturbation, in the frame of reference moving with velocity \( v \), on a finite time interval \( t \).

Then, instead of (4) we obtain

\[ \delta x_n(t) \sim \epsilon \int e^{\tilde{\lambda}_t(v)/v} dv, \] (9)

and therefore:

\[ \gamma = \lim_{n \to \infty} \frac{1}{n} \ln \left( \frac{\delta x_n}{\epsilon} \right) = \lim_{n \to \infty} \frac{1}{n} \ln \left( \frac{\delta x_{n \text{typical}}}{\epsilon} \right) = \left\langle \max_v \frac{\tilde{\lambda}_t(v)}{v} \right\rangle. \] (10)

In a generic case, because of the fluctuations, it is not possible to write \( \gamma \) in terms of \( \lambda(v) \). Nevertheless it is possible to state a lower bound:

\[ \gamma \geq \max_v \frac{\tilde{\lambda}_{t}(v)}{v} = \max_v \frac{\lambda(v)}{v} \equiv \gamma^*. \] (11)

The evaluation of the function \( \lambda(v) \) needs a heavy computational effort, however one can find good approximations of the quantity \( \gamma^* \). A first simple approximation for it, actually a lower bound, is given by

\[ \gamma_1 = \frac{\lambda(v^*)}{v^*}, \] (12)

where \( v^* \) is the velocity at which \( \lambda \) attains its maximum value. The analysis of the long time behavior of many impulsive perturbations makes it possible to obtain \( v^* \) and \( \lambda(v^*) \) without the knowledge of \( \lambda(v) \) as a function of \( v \). An improvement of this approximation can be performed in the following way. Beside \( \lambda(v^*) \), one computes the usual Lyapunov
exponent $\lambda_1 = \lambda(0)$, then one estimates the function $\lambda(v)$, by assuming it is the parabola $\lambda_p(v)$ passing in the point $(0, \lambda_1)$ with maximum $\lambda(v^*)$ for $v = v^*$ and, finally, one determines $\gamma_p = \max_v[\lambda_p(v)/v]$. Typically $\gamma_p$ is very close (within a few percent) to $\gamma^*$.

In Fig. 7 we show $\gamma$, $\gamma^*$, $\gamma_1$ and $\gamma_p$ versus $a$ at a fixed value of $c$ ($c = 0.7$), again for the logistic map with the quasi-periodic boundary condition $x_0(t) = 0.5 + 0.4 \sin((\sqrt{5} - 1)\pi t)$. There is a large range of values of the parameter $a$ for which $\gamma$ is rather far from $\gamma^*$; for instance, at $a = 3.74$ we have $\gamma = 0.28$ and $\gamma^* = 0.26$.

The difference is an effect of the intermittency; this may be pointed out by looking at what happens with the map $f_a(x) = ax \mod 1$: in this case we find that, all over the explored range of variation of $a$, $\gamma$ and $\gamma^*$, from a numerical point of view, are indistinguishable (their relative difference is smaller than $10^{-6}$).

We may obtain a further indication of the fact that the non negligible fluctuations of the comoving Lyapunov exponents are at the origin of the marked difference of $\gamma$ from its lower bound, by introducing, following ref. [13], the generalized spatial Lyapunov exponents, $L_s(q)$. These quantities allow us to characterize the fluctuations in the growth of the perturbations along the chain:

$$L_s(q) = \lim_{n \to \infty} \frac{1}{n} \ln \left( \left\langle \frac{\delta x_n}{\epsilon} \right\rangle^q \right). \tag{13}$$

By means of standard arguments of probability theory, one has that:

- $L_s(q)/q$ is a monotonic non decreasing function of $q$;
- $dL_s(q)/dq|_{q=0} = \gamma$;
- $L_s(q) = \gamma q + \frac{1}{2} \sigma^2 q^2$, for small $q$, where $\sigma^2 = \lim_{n \to \infty} \langle (\ln|\delta x_n|/\epsilon) - \gamma n \rangle^2/n$.

The shape of $L_s(q)/q$ depends on the details of the dynamics, however $L_s(q)$ is fairly determined – at least for small values of $q$ – by the two parameters $\gamma$ and $\sigma^2$. The reason for having introduced this function is that one expects some relation between the fluctuations of the spatial-complexity index $\gamma$, and the fluctuations of the effective comoving Lyapunov exponents, and it is much more easy to compute the former than the latter. Fig. 8, shows that, in the case of the logistic map, as we expected, the parameter $\sigma^2$ (that is related to the variance of the spatial fluctuations) is small (large) in the region where $\gamma^*$ is a good (bad) approximation of $\gamma$.

We stress that the definition (7) and the bound (11) have a general validity. It is not difficult to understand that they are valid not only for 1-D flow systems, such as model (1), but also for continuous-time systems (as in the case of the asymmetric Ginsburg-Landau equation). As a matter of fact, all the arguments discussed above hold unaltered whenever one can introduce the comoving Lyapunov exponents.

At the end of this section we want to note that there is not a simple relation between the correlation length $\xi$ and the exponent $\gamma$, such as, for instance, $\xi \propto \gamma^{-1}$. Indeed – in analogy with the case for the corresponding quantities (characteristic correlation time and maximal Lyapunov exponent) used to characterize the temporal behavior of the dynamical systems with few degrees of freedom – we do not expect a simple relation between $\xi$ and $\gamma$. 
IV. TRANSITION TO CHAOS

From Fig. 1 one can see that, for the system under investigation, there exist two routes for reaching the chaos:

I) the way from non chaotic and convectively stable behavior, i.e. from region $A$ to region $D$;

II) the way from non chaotic, but convectively unstable behavior, i.e. from region $C$ to region $D$.

In this section we study with a certain detail this twofold way to chaos, by looking at the features of the Lyapunov exponents of the system.

It is easy to understand that, for the system (1) the computation of all the Lyapunov exponents is much easier than in the generic case. The origin of this lucky fact is in the triangular structure of the Jacobian matrix $M[x(t)]$ that rules the evolution of the tangent vector

$$
\delta x(t + 1) = M[x(t)] \delta x(t),
$$

where $\delta x = (\delta x_1, \delta x_2, \ldots, \delta x_N)$, $x = (x_1, x_2, \ldots, x_N)$ and

$$
M[x] = \begin{pmatrix}
(1 - c)f'_a(x_1) & 0 & 0 & 0 & \cdots \\
0 & (1 - c)f'_a(x_2) & 0 & 0 & \cdots \\
0 & 0 & (1 - c)f'_a(x_3) & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

(15)

Since the product of triangular matrices is again a triangular matrix, all the Lyapunov exponents $\lambda_1, \lambda_2, \ldots, \lambda_N$ (ordered, as usual, according to the decreasing values) can be computed in a simple way – without using the standard orthonormalization method of Benettin et al. [14] – from the quantities

$$
\Lambda_i = \ln(1 - c) + \lim_{t \to \infty} \frac{1}{t} \sum_{n=1}^{t} \ln |f'_a(x_i(n))|, \quad i = 1, 2, \ldots, N.
$$

(16)

The Lyapunov exponents $\{\lambda_j\}$ then are nothing but the $\{\Lambda_i\}$ after a reordering by decreasing values. We stress that, for the system under investigation, the computer time $T_N$ for the computation of all the Lyapunov exponents is proportional to the system size $N$: $T_N \sim N$; while in a generic map, with local coupling, one has $T_N \sim N^2$.

The transition of type I) – that takes place in the zone of the region $D$ close to the boundary with the region $A$ – shows the behavior already observed in the maps with a generic symmetric local coupling [17]. By looking at Fig. 4 one clearly sees that there exists a very well established thermodynamic limit for the Lyapunov spectrum. When $N \to \infty$ there exists a limiting function $G(x)$ such that

$$
\lambda_i \simeq G(i/N).
$$

(17)

The existence of this limit entails various consequences. From [17] one can conclude that the number of the non negative Lyapunov exponents $N_0$ is proportional to $N$: 

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So, by the Pesin formula \( [1] \), one obtains a finite Kolmogorov-Sinai entropy per degree of freedom, \( h \):

\[
h = \lim_{N \to \infty} \frac{H}{N} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \lambda_i \theta(\lambda_i) = \int_{0}^{1} G(x) \theta(G(x)) dx,
\]

where \( H \) is the Kolmogorov-Sinai entropy and \( \theta \) is the step function. In addition, from eq. (17) and the Kaplan-Yorke conjecture \([1]\) one infers that the information dimension \( d_I \) of the attractor is proportional to \( N \):

\[d_I \sim N.\]

For the transition of type \( \text{II} \) – taking place in the part of the region \( D \) close to the boundary with the region \( C \) – the features of the Lyapunov exponents are very different from those described above. Fig. 10 shows that the behavior of the \( \{\lambda_i\} \) do not follow eq. (17); on the contrary one has

\[
\lambda_i \approx F(i).
\]

Therefore, in this case, the Kolmogorov-Sinai entropy and the information dimension of the attractor are not extensive quantities: \( H = O(1) \) and \( d_I = O(1) \) \( \forall N \). Loosely speaking we can say that this transition can be described in terms of a finite layer.

At the end of this section we want to stress the following point. In Fig. 10, as a matter of fact, we show \( \Lambda_i \) vs \( i \) since, in the case of the transition of type \( \text{II} \), the ordering of \( \Lambda_i \) according to their decreasing values coincides with their ordering according to the label of the lattice site on which they are computed (see eq. (17)): \( \Lambda_n \geq \Lambda_{n+1} \) (with the exception of very few sites close to the boundary); this means that the horizontal coordinate in Fig. 10 is just the site label of the chain. Therefore one realizes that, in this case, the positive Lyapunov exponents are in correspondence with the sites close to the boundary: this gives further support to the idea that we are observing a kind of finite layer phenomenon. On the other hand, for the transition of type \( \text{I} \), where one has a good thermodynamic limit, there is no correspondence between the sites on the lattice and the Lyapunov exponents. This is well evident from Fig. 11.

V. CONCLUSIONS AND DISCUSSION

In this paper we characterized, in a quantitative way, the spatial complex behavior and the transition to chaos of flow systems. We have shown that in a non chaotic, but convectively unstable flow – where the convective instability induces a spatial sensitivity to the boundary conditions – it is possible to introduce an index (a sort of “spatial” Lyapunov exponent) for the quantitative characterization of this “spatial complexity”. Moreover, there exists a relation (a bound) between this spatial complexity and the comoving Lyapunov exponents.

The transition to chaos can take place in two possible scenarios: either from a state that is both non chaotic and spatially non complex, or from a state that is non chaotic
but convectively unstable. In the first case one has a standard thermodynamic limit: the
Kolmogorov-Sinai entropy and the information dimension of the attractor are proportional
to the system size. In the second case one has a sort of layer phenomenon, where only a
finite number of Lyapunov exponents – related to the sites near the boundary – are positive.

We conclude noting that all the results above do not depend too much on the details
of the used boundary conditions $x_0(t)$. Indeed we have that if $x_0(t)$ has a chaotic behavior
(like, for instance, that of a variable of the Hénon map) $\gamma$ and $\sigma^2$, as functions of $a$ are not
very different from the case with $x_0(t)$ a quasi-periodic function. The same is true for the
properties of the two kind of transitions to chaos.

ACKNOWLEDGMENTS

We thank K. Kaneko and A. Pikovsky for useful discussions and correspondences. We
thank A. Politi, S. Ruffo and A. Torcini for having pointed out to us the existence of eq.
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FIGURES

FIG. 1. The behavior of the system in the space of the control parameters and : 'square' = absolute stability; '+' = marginal convective stability; '*' = convective instability; '=' = absolute instability.

FIG. 2. Evolution of a state of the system, with and ; the boundary condition is quasi-periodic: , with .

FIG. 3. Evolution of a state of the system, with and ; the boundary condition is quasi-periodic, as in Fig. For a better graphical effect only the configurations at even times have been reported.

FIG. 4. Evolution of a state of the system, with and ; the boundary condition is quasi-periodic, as in Fig.

FIG. 5. Graph of at different ; the values of the parameters are and ; the boundary condition is quasi-periodic, as in Fig.

FIG. 6. Spatial correlation function as a function of , for with parameters and ; the boundary condition is quasi-periodic, as in Fig.

FIG. 7. vs at fixed ; the boundary condition is quasi-periodic, as in Fig.

FIG. 8. vs at ; the boundary condition is quasi-periodic, as in Fig.

FIG. 9. vs for the system , with and , in the cases (dotted line), (dashed line) and (full line); the boundary condition is quasi-periodic, as in Fig.

FIG. 10. vs for the system , with and , in the cases (full line) and (dotted line); the boundary condition is quasi-periodic, as in Fig.

FIG. 11. Non ordered Lyapunov spectra vs for the system , with and , obtained with two different initial conditions (identified with 'o' or 'x').
