A Distributed Privacy-preserving Incremental Update Algorithm for Probability Distribution of Wind Power Forecast Error

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Abstract—Establishing the conditional probability distribution (PD) of wind power forecast error (WFE) is a prerequisite for many stochastic analysis considering wind power integration. However, with the increasingly emergence of new data, the update burden of the conditional PD is getting heavier as the size of training data set grows rapidly. Meanwhile, the centralized training manner of the conditional PD may reveals the data privacy of wind farms (WFs) belonging to different stakeholders. To solve these problems, we propose a distributed privacy-preserving (DPP) incremental update algorithm (DPP-IUA) for updating each WF’s conditional PD of WFE considering their correlation. This algorithm consists of two original methods: (1) a DPP incremental Gaussian-mixture-model algorithm (DPP-IGA) for updating the joint PD of the correlated WFs; and (2) a DPP mechanism for deriving each WF’s conditional PD of WFE under a given forecast vector. The DPP-IUA keeps each WF’s conditional PD of WFE up to date with extremely low update burden. Meanwhile, this algorithm is also fully distributed and strictly protects the data privacy of different WFs. The effectiveness, correctness and efficiency of the proposed DPP-IUA is empirically verified using historical data.

Index Terms—Wind power forecast error, Probability distribution, Gaussian mixture model, Incremental update, Distributed, Privacy-preserving

I. INTRODUCTION

WIND energy continues rapid growth [1]. But wind power forecasting methods still could not generate a perfect forecast under current technologies. Wind power forecast error (WFE) significantly affects the decision-making in economic dispatch [2], [3], reserve schedule [4] as well as bidding [5] and punishing [6] in electricity market. Among them, establishing the conditional probability distribution (PD) of WFE under a given forecast value serves as a foundation for the stochastic analysis in [2]–[6].

Many efforts have been made on probability modeling of WFE. Representing WFE as normally distributed random variable with forecast value as expected value is a common way [7], [8]. Beta distribution [5] and Cauchy distribution [9] are another two common models to characterize the uncertainty of WFE. Yet neither Gaussian, Beta or Cauchy distribution is universal especially when time scales change [10]. Other modified models, e.g., Levy alpha-stable distribution [4], mixed Beta distribution [6], Gamma-like distribution [11], T-location distribution [12] are proposed as well to better fit the features of WFE. Subsequently, to improve the generality of the probability distribution (PD) model, researchers tend to utilize or develop models with more adjustable parameters. Zhang et al. presents a versatile distribution with three adjustable shape parameters estimated by nonlinear least square [2], [13], which not only can better represent WFE than Gaussian and Beta [2], but also has analytical forms of CDF and its inverse function. However, the unimodal feature of the versatile distribution limits its application [14]. To solve this problem, an improved versatile distribution is first proposed with more accurate representation. Then a versatile mixture distribution is developed by the convex combination of several improved versatile distributions [14]. Recently, Gaussian mixture model (GMM) is applied to stochastic analysis of power system as it can characterize multidimensional random variable subjecting to arbitrary distribution with high precision [15], [16], and has been utilized to represent the conditional PD of WFE in [3], [17], [18] under the consideration of multiple correlated wind farms (WFs). The modeling process of the GMM-based conditional PD is divided into two parts: the first part is establishing the joint PD of wind power and wind power forecast for correlated WFs by expectation–maximization (EM) algorithm [3], [15]–[18], which need to collect the historical raw wind power and wind power forecast data of correlated WFs to a data center for training. The second part is deriving the conditional PD from the joint PD and substituting the latest multidimensional forecast value of correlated WFs into it, where each WF needs to exchange its forecast value with every other WFs or send it to the same center [3], [17], [18].

However, three problems of establishing the GMM-based conditional PD still remain: (1) the EM algorithm is a static training algorithm requiring a complete historical data set for each training. As new data with the latest information continues to emerge, the size of historical data set is growing, directly leading to the increasing computation cost; (2) the EM algorithm has a centralized structure, which may suffer from some issues inherent in centralized methods, like single point failure or limitation of scalability; (3) the complete modeling process for the GMM-based conditional PD of WFE may reveal the data privacy of WFs for two reasons. First, as the bidding of WF in electrical market is dependent on its predicted value, exchanging it with WFs belonging to different stakeholders will leak its commercial secrets. Second, even though the data center is considered trustworthy, these raw data may still be accessible by third parties due to the susceptibility of the communicating infrastructure to attacks. In fact, WFs attach great importance to its data protection in China, even
if researchers want to do wind power data research, they also have to sign a confidentiality contract with the WF for keeping its data secret and safe.

Solving the above three problems is essentially finding the answers to the following two questions: first, how to efficiently update the GMM-based joint PD in a distributed manner with the protection of data privacy; second, how to derive the latest conditional PD from the updated joint PD by privacy-preserving distributed way. To the best of our knowledge, there are few studies that can answer the above two questions. Therefore, this paper aims to answer them. The original contributions of this paper is as follows:

- To efficiently update the GMM-based joint PD of correlated WFs, we first introduce the incremental GMM algorithm (IGA) to realize incremental update. This algorithm is an unsupervised incremental learning algorithm which is already applied in video recognition [19]–[21] and mobile robotics [22], [23]. Because IGA only requires new data for the update calculation, the repeated training of old data is avoided and the efficiency is improved.
- To achieve a distributed privacy-preserving (DPP) feature when updating the joint PD, we then propose a DPP-IGA based on the IGA, the average consensus algorithm and a private-mean design. The DPP-IGA is fully distributed that only local communication between neighboring WFs is required. Meanwhile this algorithm strictly protect the data privacy of WFs during communications between them. Moreover, since only new data is required for update calculation, the DPP-IGA has extremely low cost for update, making it an online algorithm.
- To achieve a DPP feature when deriving the latest conditional PD, we design a DPP mechanism based on the conditional probability invariance of GMM, the average consensus algorithm and the private-mean design. Eventually, we combine the DPP-IGA and the DPP mechanism to present a DPP incremental updating algorithm (DPP-IUA) for updating each WF’s conditional PD of WFE. By this algorithm, each WF can only update its own conditional PD of WFE in an incremental manner, yet the correlation information of all WFs is contained in the conditional PD through the DPP communication.

The rest of the paper is organized as follows. In Section II, the GMM-based conditional PD of WFE is demonstrated. The IGA is introduced in Section III and investigated in Section IV. The DPP-IGA is proposed in Section V and the DPP-IUA is developed in Section VI. Case studies are described in Section VII. Finally Section VIII concludes this paper.

II. GMM-BASED CONDITIONAL PD OF WFE

In this section, the foundation for the GMM-based conditional PD of WFE, i.e., the joint PD of wind power and wind power forecast of correlated WFs, is first demonstrated. Then the derivation for each WF’s conditional PD of WFE from the joint PD is detailed. Finally, the key to update the conditional PD is summarized.

A. The joint Probability Distribution

Denote the wind power, wind power forecast and WFE of M WFs by random vectors \( X, Y, Z \in \mathbb{R}^M \) respectively, where \( Z = X - Y \). The \( m \)-th elements of \( X, Y \) and \( Z \), i.e., \( x_m, y_m \) and \( z_m \), represent the wind power, wind power forecast and WFE of the \( m \)-th WF. The GMM-based joint PD of \( X \) and \( Y \) with \( J \) Gaussian components and corresponding weighted coefficient \( w_j \) is given in (4). The conditional PD of \( WFE \), i.e., the joint PD of wind power and wind power forecast observations of correlated WFs \( X, Y \) with \( J \) Gaussian components and corresponding weighted coefficient \( w_j \) is shown in (8), where its weighted coefficient \( \alpha_j \) is given in \( \alpha_j \), mean vector \( \mu_j \) is given in \( \mu_j \) and its covariance matrix \( \Sigma_j \) is given in \( \Sigma_j \). Because the conditional PD in this paper always refer to the conditional PD of WFE, we omit

\[
P(X, Y) = \sum_{j=1}^{J} w_j N_j(X, Y) = \sum_{j=1}^{J} w_j N_j(U; \mu_j, \Sigma_j) \tag{1}
\]

\[
N_j(U; \mu_j, \Sigma_j) = \exp \left[ -\frac{1}{2} (U - \mu_j)^T \Sigma_j^{-1} (U - \mu_j)^T \right] / \sqrt{(2\pi)^{2M} \det(\Sigma_j)} \tag{2}
\]

where \( N_j(\cdot) \) is the \( j \)-th \( 2M \)-dimensional Gaussian distribution in (2) with mean vector \( \mu_j \in \mathbb{R}^M \) and covariance matrix \( \Sigma_j \in \mathbb{R}^{2M \times 2M} \). \( U \) is the compact form of \([X, Y]\). The parameter set of the joint PD is defined as \( \theta = \{ w_j, \mu_j, \Sigma_j | j = 1, 2, \ldots, J \} \), where the details of \( \mu_j \) and \( \Sigma_j \) is in \( \mu_j \) and \( \Sigma_j \).

\[
\mu_j = [\mu_{j,x} \; \mu_{j,y} \; \cdots \; \mu_{j,x_M} \; \mu_{j,y_M}] \tag{3}
\]

\[
\mu_{j,x} = \mu_{j,x_1} \quad \mu_{j,x_2} \quad \cdots \quad \mu_{j,x_M} \tag{4}
\]

\[
\mu_{j,y} = \mu_{j,y_1} \quad \mu_{j,y_2} \quad \cdots \quad \mu_{j,y_M} \tag{5}
\]

\[
\sigma_{j,x_1,x_1} \quad \sigma_{j,x_1,x_2} \quad \cdots \quad \sigma_{j,x_1,x_M} \tag{6}
\]

\[
\sigma_{j,x_M,x_M} \quad \sigma_{j,x_{M+1},x_{M+1}} \quad \cdots \quad \sigma_{j,x_{M+1},x_{M+1}} \tag{7}
\]

The EM algorithm is the common way to estimate \( \theta \) on the basis of a complete training data set consisting of wind power and wind power forecast observations of correlated WFs [3], [15]–[18]. Many commercial software tools, e.g., Matlab and Python, provide reliable off-the-shelf solvers of the EM algorithm.

B. The Derivation of The conditional PD

Once the joint PD in (1) is obtained, each WF’s conditional PD of WFE under a given wind power forecast \( y^0 \in \mathbb{R}^M \) can be derived from the joint PD by the conditional probability invariance of GMM [24]. Details of the \( m \)-th WF’s conditional PD is shown in \( \beta \), where its weighted coefficient \( \alpha_j \) is given in \( \alpha_j \), mean vector \( \lambda_j \) is given in \( \lambda_j \) and its covariance matrix \( \Delta_j \) is given in \( \Delta_j \). Because the conditional PD in this paper always refer to the conditional PD of WFE, we omit
‘WFE’ for the sake of simplicity.

\[
P(z_m|y^0) = \sum_{j=1}^{J} x_j N_j(z_m + y^0|y^0; \lambda_j, \Delta_j)\]  

(8)

\[
\alpha_j = \frac{w_j N_j(y^0; \mu_{j,y}, C_j)}{\sum_{j=1}^{J} w_j N_j(y^0; \mu_{j,y}, C_j)}\]  

(9)

\[
\lambda_j = \mu_{j,x_m} + b_{j,m} C_j^{-1}(y^0 - \mu_{j,y})\]  

(10)

\[
\Delta_j = \sigma_{j,x_m,x_m} - b_{j,m} C_j^{-1} b_{j,m}^T\]  

(11)

To update the conditional PD of each WF, we first has to initialize the joint PD, then, update the parameters set \(\theta\) and \(\Sigma\) of the \(j\)-th WF, represented by \(w_j\) and \(\mu_j\) respectively. The IGA can continually adjust a probabilistic model consistent to all sequentially presented data after each data point presentation \([22], [25]–[29]\). It consists of 3 steps in this paper: (1) judging step; (2) updating step; (3) creating step. Details are as follows.

### III. THE INCREMENTAL GMM ALGORITHM

In this section, we introduce the IGA for efficiently update the joint PD of wind power and wind power forecast of correlated WFs. The IGA can continually adjust a probabilistic model consistent to all sequentially presented data after each data point presentation \([22], [25]–[29]\). It consists of 3 steps in this paper: (1) judging step; (2) updating step; (3) creating step. Details are as follows.

#### A. Judging Step

The judging step is to decide whether a piece of new data belongs to the existing old GMM. Given a piece of new data \(u = [x, y] \in \mathbb{R}^{2M}\), the \(m\)-th and the \((M+m)\)-th elements of \(u\) are new wind power and wind power forecast of the \(m\)-th WF, represented by \(u_{xm}\) and \(u_{ym}\), respectively. The judging step is provided in (12) based on the squared Mahalanobis distance \(d_j^2(u, j)\) between the \(j\)-th component and \(u\) \([25]–[29]\). \(\chi^2_{D,1-\beta}\) is the \(1 - \beta\) percentile of a chi-squared distribution with \(D\) degrees-of-freedom as given in (13).

\[
d_j^2(u, j) = (u - \mu_j^{old})\Sigma_j^{old, -1}(u - \mu_j^{old})^T \leq \chi^2_{D,1-\beta} \]  

(12)

\[
1 - \frac{1}{\Gamma(D/2)^\gamma} \left( \frac{D}{2} \right)^{D/2} \chi^2_{D,1-\beta} = 1 - \beta \]  

(13)

If (12) holds for any \(d_j^2(u, j), j = 1, ..., J\), then \(u\) is considered to belong to the old GMM, resulting in subsequent updates to the parameters of the old GMM, known as the updating step in III-B. If not, a new component will be created on the basis of \(u\), known as the creating step in III-C.

#### B. Updating Step

If (12) holds, \(\theta\) will be updated based on \(u\) in the updating step. As the key part of the IGA, the updating step follows the Robbins-Monro stochastic approximation \([30]\) to derive the incremental recursive equations for maximizing the likelihood of \(u\). Derivation and analysis refer to \([23], [31]\).

The incremental recursive equations for updating \(\mu_j, \Sigma_j\) and \(w_j\) of the \(j\)-th component are given in (14)-(16), where the superscript ‘old’ and ‘new’ represent the parameters before and after the update, respectively.

\[
\mu_j^{new} = \mu_j^{old} + \Delta \mu_j \]  

(14)

\[
\Sigma_j^{new} = (1 - r_j) \Sigma_j^{old} + r_j e_j^T e_j - \Delta \mu_j^T \Delta \mu_j \]  

(15)

\[
w_j^{new} = h_j^{new} / \sum_{j=1}^{J} h_j^{new} \]  

(16)

Besides, \(r_j, \Delta \mu_j\) and \(e_j\) are auxiliary parameters as given in (17)-(19). Details of those 3 parameters are given in (17)-(19), where \(p(j|u)\) is the posterior probability for the \(j\)-th component as given in (20) and \(h_j\) is the accumulator of the posterior probability as given in (21).

\[
r_j = p(j|u)/h_j^{new} \]  

(17)

\[
\Delta \mu_j = r_j (u - \mu_j^{old}) \]  

(18)

\[
e_j = u - \mu_j^{new} \]  

(19)

\[
p(j|u) = \frac{w_j^{old} N_j(u; \mu_j^{old}; \Sigma_j^{old})}{\sum_{j=1}^{J} w_j^{old} N_j(u; \mu_j^{old}; \Sigma_j^{old})} \]  

(20)

\[
h_j^{new} = h_j^{old} + p(j|u) \]  

(21)

#### C. Creating Step

If (12) does not hold, the new data \(u\) must carry new information that has never been learned before. Thus a new component should be created to accommodate this information. The parameters \(\mu_j, \Sigma_j\) and \(w_j\) of this new component is initialized in (22)-(24). Note that, because the summation of \(h_j^{new}\) is changed due to the newly created component, the weighted coefficients of the old components should be updated accordingly by (16) as well.

\[
\mu_j^{new} = u; \quad \Sigma_j^{new} = \delta_{ini} \]  

(22)

\[
w_j^{new} = h_j^{new} / \sum_{j=1}^{J} h_j^{new} \]  

(23)

\[
h_j^{new} = 1; \quad J^{new} = J^{old} + 1 \]  

(24)

Review the three steps above and one can find that, \(u\) is involved in the whole process of the IGA. The traditional way to realize the IGA is to collect or exchange \(u_{xm}\) and \(u_{ym}\) of all WF to form \(u\) and update the parameters in a centralized manner, leading to the inherent problems of the centralized method as well as privacy leakage.

### IV. THE INVESTIGATION OF THE IGA

In this section, the essence of the IGA is investigated, laying the foundation for proposing the following DPP-IGA in next section.
A. The Essence of the Judging Step

Details of (12) is investigated and given in (25), where \( \tau_{j,x} \) and \( \tau_{j,y} \) are given in (26) and (27), respectively. Calculating (26) or (27) is essentially computing the summation of \( \varphi_{2,m} \) or \( \varphi_{3,m} \) among WFs. The calculation of (25) is essentially computing the summation of \( \varphi_{1,m} \) when \( \tau_{j,x} \) and \( \tau_{j,y} \) are obtained. Because the summation of \( \varphi_{1,m} \), \( \varphi_{2,m} \) or \( \varphi_{3,m} \) are similar, we omit subscript 1, 2 and 3 in the later derivations. Therefore, the essence of the judging step lies in the summation of \( \varphi_m \) among WFs.

\[
d^2_M(u,j) = \sum_{m=1}^{M} \tau_{j,x,m}(u_{x,m}^j - \mu_{j,x,m}^{old}) + \tau_{j,y,m}(u_{y,m}^j - \mu_{j,y,m}^{old})
\]

\[
\tau_{j,x} = \sum_{m=1}^{M} \rho_{j,x,m}^{old}(u_{x,m}^j - \mu_{j,x,m}^{old}) + \rho_{j,y,m}^{old}(u_{y,m}^j - \mu_{j,y,m}^{old})
\]

\[
\tau_{j,y} = \sum_{m=1}^{M} \rho_{j,x,m}^{old}(u_{x,m}^j - \mu_{j,x,m}^{old}) + \rho_{j,y,m}^{old}(u_{y,m}^j - \mu_{j,y,m}^{old})
\]

\[
\Sigma_j^{old} = \begin{bmatrix}
\rho_{j,x,x}^{old} & \cdots & \rho_{j,x,y}^{old}
\vdots & \ddots & \vdots
\rho_{j,y,x}^{old} & \cdots & \rho_{j,y,y}^{old}
\end{bmatrix}
\]

B. The Essence of the Updating Step

For the update of \( w_j \), further detail is given in (28). Once the posterior probability \( p(j|u) \) in (20) is obtained, \( h_j^{new} \) in (21) will be available. Then \( w_j \) can be directly updated based on the result of \( h_j^{new} \). Therefore, the calculation of \( p(j|u) \) is the essence for updating \( w_j \).

\[
w_j^{new} = \frac{h_j^{old} + p(j|u)}{\sum_{j'=1}^{M} h_j^{old} + p(j|u)}
\]

For the update of \( \mu_j \), its \( m \)-th and \( (M+m) \)-th elements are defined by \( \mu_{j,x,m} \) and \( \mu_{j,y,m} \), and their updates are given in (31)-(34). Obviously, once \( r_j \) is obtained, \( \mu_{j,x,m} \) and \( \mu_{j,y,m} \) can be directly updated by the \( m \)-th WF who owns \( u_{x,m} \) and \( u_{y,m} \). Since the calculation of \( p(j|u) \) is the basis for that of \( r_j \), the calculation of \( p(j|u) \) is also the essence for updating \( \mu_j \).

\[
\mu_j^{new} = \mu_j^{old} + r_j(u_{x,m}^j - \mu_{j,x,m}^{old})
\]

\[
\mu_j^{new} = \mu_j^{old} + r_j(u_{y,m}^j - \mu_{j,y,m}^{old})
\]

\[
\Sigma_j^{new} = \begin{bmatrix}
\rho_{j,x,x}^{old} & \cdots & \rho_{j,x,y}^{old}
\vdots & \ddots & \vdots
\rho_{j,y,x}^{old} & \cdots & \rho_{j,y,y}^{old}
\end{bmatrix}
\]

C. The Essence of the Creating Step

For the creation of \( \mu_j \), its elements can be directly created via \( \mu_{j,x,m} = u_{x,m} \) and \( \mu_{j,y,m} = u_{y,m} \) by the \( m \)-th WF who owns \( u_{x,m} \) and \( u_{y,m} \). For the creation of \( \Sigma_j \), since \( \delta_{mi} \) in (27) is a preset parameter as public knowledge, this creation is also straightforward.

For the creation of \( w_j \), the summation of \( h_j^{new} \) is required. As mentioned above, once \( p(j|u) \) is calculated, \( h_j^{new} \) can be obtained by (21). Therefore, the essence for creating \( w_j \) is still the calculation of \( p(j|u) \). Meanwhile, this essence is also the essence of the whole creating step.

V. The DPP Incremental GMM Algorithm

In this section, a DPP-IGA is proposed based on the above essence analysis. Its distributed feature is developed by the average consensus algorithm (ACA). And its privacy-preserving feature is achieved by a private-mean design. The motivation and advantages of the private-mean design is first discussed. Then the key part of the DPP-IGA is demonstrated. Finally, a detailed DPP-IGA is given at the end of this section.

A. The Private-mean Design

Before the demonstrations of the DPP-IGA, some considerations should be briefly discussed: the updated parameters of the joint PD is required by all WFs, because they need to eventually derive their latest conditional PD by those updated parameters. However, if each WF obtains the complete \( \mu_j^{new} \), since \( r_j \) must have been calculated during the update process, each WF can deduce \( u_{x,m} \) from (29) or \( u_{y,m} \) from (30) for \( \forall m \in \{1, 2, ..., M\} \). To avoid this situation, we propose a private-mean design, i.e., each WF only updates or creates its \( \mu_{j,x,m} \) and \( \mu_{j,y,m} \) by (29) or (30) or by \( \mu_{j,x,m} = u_{x,m} \) and \( \mu_{j,y,m} = u_{y,m} \) from the very beginning without sharing it with others. Meanwhile, each WF will not know the private means of others as well. Therefore, instead of updating the complete \( \theta \), the proposed DPP-IGA actually aims to enable each WF only to update \( \mu_{j,x,m}, \mu_{j,y,m}, w_j \) and \( \Sigma_j \) in a DPP manner.

Advantages of this private-mean design are as follows: (1) this design avoids the disclosure of data privacy as there is no chance for other WFs to deduce \( u_{x,m} \) from (29) or \( u_{y,m} \) from (30); (2) this design will not affect the derivation of conditional PD later, because knowing \( \mu_{j,x,m}, \mu_{j,y,m} \) is enough for the \( m \)-th WF to derive its conditional PD; (3) this design reduces unnecessary calculation and communication; (4) this design of both \( \epsilon_m \) and \( \varepsilon \) require \( r_j \), thus the calculation of \( p(j|u) \) is one essence for updating \( \mu_j \). Besides, another essence for updating \( \mu_j \) lies in computing \( \xi_m \) for \( m,j \in \{1, 2, ..., 2M\} \).

\[
\sigma_j^{new} = \sigma_j^{old} \varepsilon + \epsilon(u_{x,m} - \mu_{j,x,m}^{old})(u_{x,m} - \mu_{j,x,m}^{old})
\]

\[
\epsilon_m = (1 - r_j)\sigma_j^{old}
\]

\[
\varepsilon = r_j(1 + r_j^2 - 3r_j)
\]

In summary, the essence of the updating step lie in the calculation of \( p(j|u) \) and \( \xi_m \) for \( m,j \in \{1, 2, ..., 2M\} \).
helps to achieve privacy protection during the communication between WFs; (5) this design makes each WF unable to obtain others’ conditional PD. Details of the advantages will be discussed below.

B. The Key Parts of The DPP-IGA

Based on the above analysis, there are three essences to perform the IGA: (1) calculating the summation of $\varphi_m$ among WFs; (2) calculating $p(j|u)$; and (3) calculating $\zeta_m\xi_i$ for $m, i \in \{1, 2, ..., 2M\}$. In fact, $p(j|u)$ is a function of $d^2_M(u, j)$ as given in (34). Once the $d^2_M(u, j)$ is obtained, the calculation of $p(j|u)$ is available. Thus, the second essence is equivalent to the first one, resulting in the final two critical essences of the IGA: calculating the summation of $\varphi_m$ in (35) and calculating $\zeta_m\xi_i$ for $m, i \in \{1, 2, ..., 2M\}$ in (36). If these two calculations can be achieved in a DPP manner, the DPP-IGA will be obtained. To achieve this goal, except for the private-mean design, we also require the help of the ACA.

\[
p(j|u) = f(d^2_M(u, j)) = \frac{c_j w_j^{old}\exp[-\frac{1}{2}d^2_M(u, j)]}{\sum_{j=1}^{J} c_j w_j^{old}\exp[-\frac{1}{2}d^2_M(u, j)]} (34)
\]

\[
c_j = \sqrt{(2\pi)^2M} \det(\Sigma_j^{old})
\]

\[
g = \sum_{m=1}^{M} \varphi_m (35)
\]

\[
l = \zeta_m\xi_i, \ \forall m, i \in \{1, 2, ..., 2M\} (36)
\]

Consider $M$ WFs as $M$ nodes in a connected communication network, where each WF can only communicate with its neighbors. The neighbor set of the $m$-th WF is defined as $\Phi_m$. Based on the local information exchange between WFs in each iteration of the ACA, every WF can finally achieve the average value of their initial input after the convergence of the ACA. Derivation and discussion refer to [32].

To deal with the first critical essence of IGA, each WF can calculate (35) in a distributed manner by the ACA in (37) with their initial input in (38), where $t$ represents the iteration number. Adjacency coefficient $\zeta_m\xi_i$ is defined by (39). Once the convergence of the ACA is achieved, each WF can obtain the average value as given in (40). Since the value of $M$ is public knowledge, each WF can eventually obtain the result of (35) from the result of (40). Note that, WFs need to exchange its $g^t_m$ with its neighbors during each iteration. For the first iteration, the value of $\varphi_m$, i.e., the initial value $g^0_m$, is directly shared. However, due to the private-mean design, the value of $\varphi_m$ is privacy-preserving as no WF can deduce other WFs’ raw data from it. Therefore, combined with the private-mean design, the ACA becomes a natural DPP algorithm without the need for modification with cryptography techniques as in [33], [34]. This is also the fourth advantage of the private-mean design. Thus, the goal of calculating (35) in a DPP manner is achieved.

\[
g^{t+1}_m = g^t_m + \sum_{i \in \Phi_m} \zeta_{m,i} \left[ g^t_i - g^t_m \right] (37)
\]

\[
g^0_m = \varphi_m (38)
\]

\[
\zeta_{m,i} = \frac{2}{D_m + D_i + 1}, \ \forall i \in \Phi_m (39)
\]

\[
\lim_{t \to \infty} g^t_m = \frac{1}{M} \sum_{m=1}^{M} \varphi_m (40)
\]

To deal with the second critical essence of IGA, each WF can also calculate (36) in a distributed manner by the ACA through certain design of its initial input. The initial input $l^0_m \in \mathbb{R}_M^M$ of the $m$-th WF is given in (41), which is a sparse vector with only value on the $m$-th element, i.e., $\xi_m$, while other elements are 0. Based on this design for the initial value, each WF can perform the ACA by (42). The convergent result of the $m$-th WF is defined as $l_m$ and provided in (43). Note that each WF can only obtain the result of $\zeta_m$ for $m = 1, ..., M$ from (43). But the consensus calculation of $\zeta_m$ for $m = 1, ..., M$ is the same as computing $\xi_m$ for $m = 1, ..., M$. Finally, each WF can obtain the result of $\zeta_m$ for $m = 1, ..., 2M$ by the ACA algorithm and the design for the initial value. Then the calculation of (36) is straightforward for each WF. Similar to $\varphi_m$, the private-mean design also makes $\xi_m$ into a privacy-preserving value, turning the above consensus calculation process into a DPP one.

\[
l^0_m = [0,...,0,\xi_m,0,...,0] (41)
\]

\[
l^{t+1}_m = l^t_m + \sum_{i \in \Phi_m} \zeta_{m,i} \left[ l^t_i - l^t_m \right] (42)
\]

\[
l_m = \frac{1}{M} \left[ \xi_1, \xi_2, ..., \xi_m, ..., \xi_M \right] (43)
\]

C. The Detailed DPP-IGA

Based on the above derivations and discussions of the first and the second critical essences of the IGA, a DPP-IGA is proposed. Details of the proposed algorithm are given in Algorithm 1. By this algorithm, each WF can update $\mu_{j,x_m}$, $\mu_{j,y_m}$, $w_j$ and $\Sigma_j$ in a DPP manner. Those updated parameters are the foundation for deriving the latest conditional PD in the next section.

VI. THE DPP-IUA FOR THE CONDITIONAL PD

Although the parameters of the joint PD can be updated by each WF via the DPP-IGA, to derive each WF’s conditional PD under a given wind power forecast, a DPP mechanism is still required as directly collecting or exchanging the wind power forecast data of WFs will reveal privacy as well. In this section, we first propose a DPP mechanism for each WF to derive its latest conditional PD by the updated parameters in a DPP manner. Then we combine the DPP-IGA and the DPP mechanism to propose a DPP-IUA, which is a complete solution to the DPP of each WF’s conditional PD.
Each WF calculates (34) by (39); each WF updates $\alpha$, as a function of $d$.

A. The DPP Mechanism for Deriving The conditional PD

To derive each WF’s latest conditional PD in (8), the critical part is calculating the summation of $\vartheta_{1,m}$ and $\vartheta_{2,m}$ among WFs in a distributed way without privacy leakage. Similar to DPP calculation of (43), the sum of $\vartheta_{1,m}$ and $\vartheta_{2,m}$ can be calculated by the ACA in (47) in a distributed manner. The only difference is the initial value becomes $y_{m}^{0}$ or $g_{m}^{0}$.

B. The DPP-IUA for The conditional PD

Combine the DPP-IGA and the DPP mechanism for deriving each WF’s conditional PD, we finally obtain the DPP-IUA as a complete solution for the DPP update of each WF’s conditional PD:

1. Each WF updates the parameters of the conditional PD in (8), the critical part is calculating the summation of $\vartheta_{1,m}$ and $\vartheta_{2,m}$ among WFs in a distributed way without privacy leakage. Similar to DPP calculation of (43), the sum of $\vartheta_{1,m}$ and $\vartheta_{2,m}$ can be calculated by the ACA in (47) in a distributed manner. The only difference is the initial value becomes $y_{m}^{0}$ or $g_{m}^{0}$.

2. Meanwhile, due to the private-mean design, $\mu_{j,y}$ is a natural DPP algorithm. Based on the above discussions, the DPP mechanism for deriving each WF’s conditional PD is given in Algorithm 2.
Algorithm 2: The DPP Mechanism for Deriving The conditional PD

| Input: WFs with their $y_m^n$ |
|-----------------------------|
| Output: WFs with their $\Sigma_j^new$, $\mu_j^new$ and $\mu_j^new$ |

for $j = 1$ to $J$ do
  for $t = 2$ to $T$ do
    if convergence criterion is not met then
      Each WF calculates $\beta_j$ by (47) ;
      $t = t + 1$ ;
    end
    Each WF obtains $\Sigma_m^j$ by (40) ;
    Each WF obtains $\Delta_j^new$ by (11) ;
  end
end

presented data and keep it up to date with high accuracy. Regardless of whether the refined classification of wind power data is required in the early stage, and whether the update time window is required to set, this algorithm is suitable for all situations as long as there is a need for updating the conditional PD.

- **Highly efficient.** This algorithm enables each WF to update its conditional PD in an incremental manner, which only need to perform update calculation of a piece of new data. Compared with the traditional EM algorithm that needs to consistently reconstruct train the whole historical data set, this algorithm is much more efficient with extremely low update cost.

- **Fully distributed.** By this algorithm, each WF only needs to communicate with its neighbors to realize the whole update process without any center or coordinator. Meanwhile the correlation between those correlated WFs is involved into the update results, leading to a very concentrated conditional PD of each WF and making the characterization of uncertainty more precise.

- **Privacy-preserving.** This algorithm protects each WF’s privacy during local communication through the private-mean design, which turns the ACA into a natural DPP algorithm without the need for modification by cryptography techniques. Meanwhile, although the correlation is considered, each WF can only derive its own conditional PD without knowing others’. Since decision-making of WFs will depend on their conditional PD, turning each WF’s conditional PD into a secret information to others is also necessary and practical.

VII. Case Study

For case study, we use the eastern wind integration data set published by the National Renewable Energy Laboratory (NREL) [29]. Nine correlated WFs in Maryland is chosen with a preset communication network topology in Fig.1 where each WF can only communicate with the WFs that connect to it. Besides, we select 40 days of hourly wind power and wind power forecast data as the historical data set of the 9 WFs to build their old PD, while we choose the following 40 days of hourly wind power and wind power forecast data as the new data set waiting for update. Note that, for situations where data classification or time window settings are considered, the proposed algorithms are also suitable and the verifications are straightforward.

For parameters settings, the Bayesian information criterion (BIC) is utilized to set the number of Gaussian components based on the selected data set. The lowest BIC value is achieved when $J = 4$, thus we set $J$ as 4. For setting parameter of IGA, i.e., $\beta$ in (12), through a certain step adjustment, we have tried several different values within a certain range, and finally set $\beta = 0.01$ due to its best performance. Besides, the maximum iteration times of the ACA is set as 50.

Moreover, all experiments are conducted on a laptop with a dual-core Core i5-7267U processor running at 3.1GHz and 8GB of RAM. Meanwhile, the benchmark algorithm, i.e., the centralized EM algorithm, is performed by the off-the-shelf solver named gmdistribution.fit in MATLAB.

Note that, since the privacy-preserving features of the proposed DPP-IGA and DPP-IUA have already been discussed above, the following cases only aim to verify the effectiveness, correctness and efficiency of the proposed algorithms.

A. Verification of The DPP-IGA

The DPP-IGA enables each WF to update the parameters of the joint PD, leading to 9 groups of update results. To save space, we only choose the 1st group, i.e., the update result of the 1st WF for illustrations. Thus we need first to verify the rationality of this choice, i.e., to verify the consensus effect of the DPP-IGA.

From a virtual global perspective, we combine each WF’s updated private mean value to form a complete updated mean vector, and then to form 9 complete updated joint PDs. Thereafter, we derive a virtual shared PD, e.g., the updated marginal PD of the 1st dimension from each WF’s joint PD to illustrate the differences in Fig.1. From this figure, we can see that the differences between the updated results of different WFs are very small as the PDFs built by them are basically the same. In order to quantify these differences, we calculate...
the Jensen–Shannon divergence (JSD) between the complete updated joint PD built by 1st WF and by other WFs. JSD is a common measure to quantify similarity between two PDs with a range from 0 to 1. Details are provided in Table I. Because all the JSDs are lower than $9.83 \times 10^{-12}$, which is an extremely low value compared to 1, the differences between the updated results of each WF are indeed small. Therefore, the consensus effect of the DPP-IGA is verified. Meanwhile, the updated results of each WF are indeed small. Therefore, the correctness of the proposed DPP-IGA is verified as well.

![Fig. 2. The marginal PDF of the first dimension built by different WFs](image)

**TABLE I**

| Wind Farm | 1   | 2   | 3   | 4   | 5   |
|-----------|-----|-----|-----|-----|-----|
| JSD ($\times 10^{-12}$) | 0.00 | 1.28 | 9.83 | 2.00 | 4.76 |

To verify the effectiveness and correctness of the DPP-IGA, we utilize the whole data set, i.e., the combined historical and the new data set, as the training set, and use the centralized EM algorithm for training. This training result is considered as the final benchmark for the incremental update. Then we use the proposed DPP-IGA to update the old PD built only by the historical data set. Because the new data set has 960 pieces of new data, the DPP-IGA needs to be performed 960 times for the final complete update. For clear illustration, we derive the marginal PDF of the 1st dimension from the update results, and the marginal PD functions (PDFs) built by the two algorithms are shown in Fig. 3.

In Fig. 3, the empirical distribution is formed by the whole data set, while the benchmark is the result of the centralized EM algorithm with the whole data set. 'D-U' represents the result of the DPP-IGA after 960 updates, and 'D-N' represents the result of the old PD built by the historical data set. Legend 'D-U9' denotes the update result of the DPP-IGA after $n \times 96$ updates. From this illustration, we can draw two conclusions: (1) the effectiveness of the DPP-IGA is verified, because as the number of updates increases, the updated PDF curves built by the DPP-IGA is gradually moving away from the 'D-N' curve and approaching the benchmark in the direction of the black arrow. (2) The correctness of the DPP-IGA is verified, because the curve 'D-U' and the benchmark are ideally coincident. Meanwhile, the curve 'D-U' also matches the empirical distribution well.

For further verification of the DPP-IGA's correctness, we also use the centralized IGA to update the old PD for 960 times. Thereafter, we derive the 2-dimensional PDF consisting of the 2nd and the 3rd dimension from the update results of the centralized IGA and the proposed DPP-IGA. Details are provided in Fig. 4. It can be observed that the 2-dimensional PDF built by the DPP-IGA also perfectly matches the result of the centralized IGA. Therefore, the correctness of the proposed DPP-IGA is verified as well.

![Fig. 3. The marginal PDF of the first dimension](image)

![Fig. 4. The joint PDF of the 2nd and 3rd dimension](image)

For the verification of efficiency of the DPP-IGA, we focus on the comparison of the computing time of the centralized EM algorithm and the proposed DPP-IGA for each update. Note that, the centralized EM algorithm has to reconstruct the training data set whenever new data is generated and train this set to obtain the update PD result. Although there are 960 updates, for clear demonstration, we only illustrate the computing time for the 6th, the 12th, the 18th, ....... and the 960th updates. Details are in Fig. 5. From this figure we can see that the computing time of the DPP-IGA is much lower than the computing time of the centralized EM algorithm. Further comparisons of the computing time for all 960 updates are given in Table II. Results show that the proposed DPP-IGA is much more efficient than the centralized EM algorithm. Due to the low computational cost of the DPP-IGA, continuous updates are acceptable for each WF without the need for considering the trade-off between the update effect and the update cost.
B. Verification of The DPP-IUA

To verify the effectiveness of the DPP-IUA, we first use the whole data set to build the empirical distributions under a given wind power forecast data of correlated WFs. Then we use the EM algorithm to build the latest marginal PD of each WF’s WFE based on the whole data set without considering the correlation between WFs. Finally, we use the proposed DPP-IUA to update each WF’s conditional PD until the conditional PDs are the latest. The conditional PDs are shown in Fig. 7 in the form of PDF. Only the conditional PDFs of the first 4 WFs are shown. Due to the limited space, only the conditional PD of the first 4 WFs are shown.

It can be seen that the conditional PDFs built by the DPP-IUA is more concentrated than the marginal ones, making the characterization of uncertainty more precise. Meanwhile, the conditional PDFs also match the empirical distributions well.

It should be noted that, if the decision makers use the marginal PDF for decision, even if this distribution is up to date, huge unnecessary cost will be brought because much reserves are scheduled to deal with the situations that actually will never happen. On the contrary, the proposed DPP-IUA will greatly help the decision makers to reduce cost as the characterization of uncertainty is more concentrated and precise, avoiding unnecessary waste of reverse. Therefore, the proposed DPP-IUA has highly practical value.

To verify the correctness of the DPP-IUA, we build each WF’s conditional PD by different ways. First, we use centralized EM algorithm to build the joint PD by the whole data set, and then to derive each WF’s conditional PD by collecting their wind power forecast data. These results are also considered as the benchmark and represented by the legend ‘C-S’. Second, we use the centralized EM algorithm to build the old PD by the historical data set and then to derive each WF’s conditional PD in a centralized way. These results are represented by the legend ‘C-N’. Finally, we use the proposed DPP-IUA to update each WF’s conditional PD until the conditional PDs are the latest. And we use legend ‘D-U’ to represent these results. The conditional PDs in the form of cumulative distribution function (CDF) built by the 3 ways are demonstrated in Fig. 7 while due to the limited space, only the conditional PD of the first 4 WFs are shown.

It can be seen that the curve ‘D-U’ is coincident with benchmark curve ‘C-S’, while curve ‘C-N’ greatly differs from the benchmark. The differences between the benchmark with the curve ‘C-N’ and the curve ‘D-U’ are measured by relative standard error (RSE) and provided in Table III. Based on these illustrations, we know that the conditional PD updated by the DPP-IUA is correct as the RSE value between curve ‘D-U’ and the benchmark is lower than 4 × 10⁻⁴, and the curve ‘D-U’ fits the benchmark well. On the contrary, the RSE value between curve ‘C-N’ and the benchmark is much higher than that of curve ‘D-U’.

Fig. 5. The computing time for each update

|                  | Centralized EM | DPP-IGA     | Reduction |
|------------------|----------------|-------------|-----------|
| Maximum          | 1344 × 10⁻³s   | 11 × 10⁻³s  | 12218%    |
| Minimum          | 69 × 10⁻³s     | 3 × 10⁻³s   | 2300%     |
| Mean             | 195 × 10⁻³s    | 6 × 10⁻³s   | 3250%     |
| Median           | 165 × 10⁻³s    | 6 × 10⁻³s   | 2750%     |

Fig. 6. The PDFs comparison

Fig. 7. The CDF of the first 4 WFs’ WFE
In this paper, we propose a DPP-IUA for incremental update of each WF’s conditional PD in distributed and privacy-preserving manner under the consideration of WFs’ correlation. To achieve this algorithm, a DPP-IGA for updating the parameters of the joint PD and a DPP mechanism for deriving conditional PDs are proposed as well. The DPP-IUA can makes each WF’s conditional PD stay up to date with extremely low update cost. Meanwhile, this algorithm is fully distributed without any center or coordinator. Furthermore, the data privacy of WFs belonging to different stakeholders is strictly protected by this algorithm during local communications.

The proposed DPP-IUA is highly efficient, thus there is no need to consider the trade-off between the update effect and the update cost. Although data classification or update time window are required in some situations when updating PDs, this algorithm is also suitable and those situations can still benefit from the high efficiency of this algorithm. Besides, the updated results, i.e., the conditional PDs of WFs, are much more concentrated than the PDs that have not considered the correlation of WFs, thus the characterization of uncertainty by the proposed algorithm is more precise and the unnecessary waste of reverse can be avoided.

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