Tremendous and continuous efforts have been devoted to testing the equality of the inertial and gravitational masses, with no difference found so far \cite{1}. Since relativity requires that gravity comes from not only the mass, but also the momentum and the flow of energy and momentum, all of which round together to make the energy-momentum tensor, the test of equivalence between gravity and inertia should naturally be extended to the whole energy-momentum tensor. In this paper, we show that the gravitational and inertial energy-momentum tensors may not coincide: Einstein’s general relativity requires the gravitational energy-momentum tensor to be symmetric, but we show that a symmetric inertial energy-momentum tensor would ruin the concordance between conservations of quantized energy and charge. The nonsymmetric feature of the inertial energy-momentum tensor can be verified unambiguously by measuring the transverse flux of a collimated spin-polarized electron beam, and leads to a serious implication that the equivalence principle and Einstein’s gravitational theory cannot be both exact.

The gravitational and inertial energy-momentum tensors ($T^\mu_\nu_g$ and $T^\mu_\nu_i$) are relativistic extensions of the gravitational and inertial masses ($m_g$ and $m_i$):

\begin{align}
F & = \frac{m_g M_g}{r^2} \Rightarrow \mathcal{T}_\mu^\rho - \frac{1}{2} g_{\mu\rho} R = -8\pi T^\mu_\nu_g, \\
\tilde{f} & = m_i \tilde{a} \Rightarrow \begin{cases} f^\mu = \frac{\partial}{\partial t} \mathcal{L}^\mu_i & \text{(for a particle)} \\
f^\mu = \partial_\nu T^\mu_\nu_i & \text{(for a field)} \end{cases} \tag{2}
\end{align}

(In this paper we take the units with $c = \hbar = G = 1$.)

As for $m_g$ and $m_i$, the relation between $T^\mu_\nu_g$ and $T^\mu_\nu_i$ is not known \textit{a priori}, and has to be answered by experiment. In Einstein’s theory, $T^\mu_\nu_g$ is well restricted, though it is very hard to measure the gravitational effect of the components other than $T^0_0$ (the energy). In comparison, $T^\mu_\nu_i$ has much better experimental accessibility, and receives strong theoretical constrain as well. The concrete knowledge of $T^\mu_\nu_i$ can have much to say about the validity of the equivalence principle and Einstein’s equation.

It is not unreasonable to suspect that $T^\mu_\nu_i$ and $T^\mu_\nu_g$ might differ. After all, in field theory the conservation law alone does not give a unique expression for the energy-momentum tensor. From Noether’s theorem, one can derive the canonical expression from any covariant Lagrangian density $\mathcal{L}(\psi^\alpha, \partial_\mu \psi^\alpha)$:

\begin{equation}
\mathcal{T}^\mu_\nu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi^\alpha)} \partial^\nu \psi^\alpha - g^{\mu\nu} \mathcal{L}, \quad \text{with } \partial_\mu \mathcal{T}^\mu_\nu = 0. \tag{3}
\end{equation}

But one can always add to $\mathcal{T}^\mu_\nu$ a total-divergence term

\begin{equation}
\mathcal{J}^\mu_\nu = \partial_\rho B^{(\rho\nu)}, \tag{4}
\end{equation}

with $B^{(\rho\nu)}$ antisymmetric in $[\rho\nu]$. It gives $\partial_\nu \mathcal{J}^\mu_\nu = 0$, and, for a finite system, \int \mathcal{J}^0_0 dV = 0. Therefore, $\mathcal{J}^\mu_\nu$ changes neither the conservation law nor the conserved four-momentum. By choosing a suitable $\mathcal{J}^\mu_\nu$, one can obtain a symmetric tensor $\Theta^\mu_\nu = \Theta^{\mu\nu}$. Such tensor is what can fit into the Einstein’s equation, the left-hand side of which requires $T^\mu_\nu_g$ to be symmetric and conserved in a flat space-time (namely, when gravity is neglected):

\begin{equation}
T^\mu_\nu_g = T^\nu_\mu_g, \quad \partial_\mu T^\mu_\nu_g = 0. \tag{5}
\end{equation}

We are going to show the key point that any conserved and symmetric energy-momentum tensor $\Theta^\mu_\nu$ leads to a concrete and testable consequence, which is unacceptable for the inertial energy-momentum tensor $T^\mu_\nu_i$ when spin plays a role.

As is well-known, the symmetric and conserved $\Theta^\mu_\nu$ can be used to construct a conserved angular momentum tensor:

\begin{equation}
\hbar^\lambda_\mu_\nu = x^\mu \Theta^{\lambda_\nu} - x^\nu \Theta^{\lambda_\mu}, \quad \text{with } \partial_\lambda \hbar^\lambda_\mu_\nu = 0. \tag{6}
\end{equation}

A remarkable feature of such a construction is that the total (spin+orbital) angular momentum density,

\begin{equation}
\mathcal{J}^k = \frac{1}{2} \epsilon_{ijk} \hbar^{0ij}, \tag{7}
\end{equation}

is given by an orbital-like expression:

\begin{equation}
\mathcal{J}^k = \vec{z} \times \vec{\mathcal{P}} = \vec{z} \times \vec{\mathcal{K}}, \tag{8}
\end{equation}
where \( \mathcal{J}^\mu = \Theta^{0\mu} \) and \( \mathcal{K}^i = \Theta^{i0} \) are the momentum density and energy-flow density defined by the symmetric energy-momentum tensor, which gives \( \mathcal{J} = \mathcal{K} \). (In general the two quantities can disagree.)

Simple as it looks, Eq. (5) (particularly the latter expression) predicts a strong sum rule for experimental test. Consider a collimated, polarized beam of spin-\( s \) particles moving in the \( z \) direction (cf. Fig. 1 below). We have

\[
\int (\vec{x} \times \vec{\mathcal{K}})^2 dV = \int J^2 dV = N s, \quad (9)
\]

with \( N \) the number of particles in the integrated volume. This says that if the transverse flow of energy is measured locally, one can integrate the result to tell whether or not the energy-momentum tensor can be symmetric.

As we mentioned, it is certainly too hard to employ gravitational measurement to test Eq. (9), and thus to tell whether the gravitational energy-momentum tensor \( T_{\mu \nu}^g \) can indeed be symmetric. On the other hand, the inertial energy flow is in principle measurable, therefore the symmetry of the inertial energy-momentum tensor \( T_{\mu \nu}^i \) can be put into concrete test by experiment.

One may think that such an experiment is not at all urgent or even necessary, since \( T_{\mu \nu}^i \) must be symmetric should one believe in both Einstein's equation and an exact equivalence between gravity and inertia. However, we reveal an alerting fact that \( T_{\mu \nu}^i \) receives other theoretical constraint which contradicts Eq. (9).

Consider the spin-polarized beam of free electrons moving in the \( z \) direction, with the same energy \( E \) for each electron. Besides its energy and momentum, the electron has another conserved quantity, the electric charge. Since the electrons are quantized in actual detection, the flow of charge must be proportional to the flow of energy:

\[
\frac{\vec{K}^i}{\mathcal{E}} = \frac{j}{e} = \vec{n}. \quad (10)
\]

Here \( K^i = T_{i0} \) is the density of inertial energy flow, \( j \) is the electric current and \( n \) is the flux density of electron number.

Noticing that \( \frac{1}{2} \vec{x} \times \vec{j} = \vec{\mu} \) is the density of electron magnetic moment, we have

\[
\int (\vec{x} \times \vec{K}^i)^2 dV = \frac{\mathcal{E}}{e} \int (\vec{x} \times \vec{j})^2 dV = \frac{\mathcal{E}}{e} \int 2\mu^2 dV
\]

\[
= \frac{\mathcal{E}}{e} (N \cdot 2 \frac{e}{\mathcal{E}} \cdot \frac{1}{2}) = N \cdot 1. \quad (11)
\]

However, should \( T_{\mu \nu}^i \) be symmetric, Eq. (9) gives

\[
\int (\vec{x} \times \vec{K}^i)^2 dV = N \cdot \frac{1}{2}, \quad (12)
\]

which differs from Eq. (11) by a factor of two!

It is interesting and important to note that the canonical expression can satisfy Eq. (10) and hence reconcile conservations of quantized energy and charge. For the electron, the canonical energy-momentum tensor is easily derived to be,

\[
\mathcal{T}^{\mu \nu} = \frac{i}{2} \bar{\psi} \gamma^\mu \partial^\nu \psi + h.c., \quad (13)
\]

where \( h.c. \) stands for hermitian conjugate. This gives the canonical momentum and energy-flow densities

\[
\mathcal{P}^i = \mathcal{T}^{0i} = \frac{i}{2} \bar{\psi} \gamma^0 \partial^i \psi + h.c., \quad (14)
\]

\[
\mathcal{K}^i = \mathcal{T}^{i0} = \frac{i}{2} \bar{\psi} \gamma^i \partial^0 \psi + h.c. \rightarrow \mathcal{E} \bar{\psi} \gamma^i \psi. \quad (15)
\]

Here the last expression applies to the electron state with fixed energy \( \mathcal{E} \), in which case we find \( \mathcal{K}^i = \frac{\mathcal{E}}{e} j^i \) by comparing to the electric current expression \( j^\mu = e \bar{\psi} \gamma^\mu \psi \).

It is worthwhile to remind that in the canonical expression \( \vec{x} \times \vec{\mathcal{P}} = \vec{\mathcal{L}} \) is merely the orbital angular momentum. For the spin-polarized electron beam with no orbital angular momentum, one finds

\[
\int (\vec{x} \times \vec{\mathcal{P}})^2 dV = \int d^3 x \mathcal{L}^2 = 0, \quad (16)
\]

\[
\int (\vec{x} \times \vec{\mathcal{K}})^2 dV = \frac{\mathcal{E}}{e} \int (\vec{x} \times \vec{j})^2 dV = N \cdot 1. \quad (17)
\]

In comparison, the symmetric energy-momentum tensor of the electron is

\[
\Theta^{\mu \nu} = \frac{i}{4} \bar{\psi} (\gamma^\mu \partial^\nu + \gamma^\nu \partial^\mu) \psi + h.c. \quad (18)
\]

The corresponding momentum and energy-flow densities are

\[
\Theta^i = \Theta^{0i} = \Theta^{i0} = \mathcal{J}^i = \frac{i}{4} \bar{\psi} (\gamma^0 \partial^i + \gamma^i \partial^0) \psi + h.c.
\]

\[
= \frac{1}{2} \mathcal{P}^i + \frac{1}{2} \mathcal{K}^i. \quad (19)
\]

This shows clearly that if there is no orbital angular momentum one finds

\[
\int \vec{x} \times \vec{\mathcal{J}} dV = \int \vec{x} \times \vec{\mathcal{P}} dV = \frac{1}{2} \int \vec{x} \times \vec{\mathcal{L}} dV. \quad (20)
\]

Another interesting and important fact to note is that if the beam has only orbital angular momentum \( l \) and no spin polarization, we have

\[
\int (\vec{x} \times \vec{\mathcal{P}})^2 dV = \int \mathcal{L}^2 dV = N \cdot l, \quad (21)
\]

\[
\int (\vec{x} \times \vec{\mathcal{K}})^2 dV = \frac{\mathcal{E}}{e} \int (\vec{x} \times \vec{j})^2 dV = N \cdot l, \quad (22)
\]

\[
\int (\vec{x} \times \vec{\mathcal{J}})^2 dV = \int (\vec{x} \times \vec{\mathcal{P}})^2 dV = N \cdot l. \quad (23)
\]

Namely, orbital angular momentum is unable to discriminate the symmetric energy-momentum tensor from the canonical one. A key difference between Eqs. (11) and
The detector window is placed in the plane containing the $z$ axis. Since the detector can only count the incoming electrons, the fluxes in both $\phi$ and $-\phi$ directions need to be measured and subtracted to get the net flux $K_\phi$. If the detector has good enough spatial resolution (say, 1/10 of the beam radius), then the detector can be held fixed to record directly the local flux density (see left panel of Fig. 1). Otherwise, one has to gradually move the detector into and out of the beam (see right panel of Fig. 1), measuring the total flux at each step, and finally differentiating with respect to the detector position to get the local flux.

To test the symmetry of the inertial energy-momentum tensor, the integrated result in Eq. (24) is to be compared with $N/2$ as predicted by Eqs. (11) or (12). In actual practice, $N$ is the normalized number of electrons in the integrated region, given by

$$N = I \frac{L}{v} \cdot \xi \eta,$$

where $I$ is the beam intensity, $L$ is the detecting length along the beam direction, $v$ is the electron velocity, $\xi$ is the counting efficiency, and $\eta$ is the polarization rate.

Before closing this paper, we comment on the energy-momentum tensor for other particles, particularly photons, composite and higher-spin particles. For the photon, the symmetric and canonical energy-momentum tensors are easily derived to be

$$\Theta^{\mu\nu} = F^{\mu\rho} F_{\rho}{}^\nu + \frac{1}{4} g^{\mu\nu} F^2,$$

$$\mathcal{T}^{\mu\nu} = -F^{\mu\rho} \partial_\rho A_\nu + \frac{1}{4} g^{\mu\nu} F^2.$$

The canonical expression is gauge-dependent and hence often abandoned. But recent intensive studies show that $\mathcal{T}^{\mu\nu}$ can be revised gauge-invariantly. (The issue is, though, highly tricky and controversial. We refer interested readers to Refs. [2, 4] and references therein.)

The familiar Poynting vector $\vec{E} \times \vec{B}$ is the momentum and energy-flow density of $\Theta^{\mu\nu}$. Interestingly, for a free photon, $\vec{E} \times \vec{B}$ is also the energy-flow density of $\mathcal{T}^{\mu\nu}$ in the radiation gauge. Therefore, the type of measurement discussed above is unable to distinguish the symmetric and canonical energy-momentum tensors for the photon. To explore the symmetry of the photon energy-momentum tensor, one has to measure the momentum-flow tensor, which is different for $\Theta^{\mu\nu}$ and $\mathcal{T}^{\mu\nu}$. (The momentum density itself is also different for $\Theta^{\mu\nu}$ and $\mathcal{T}^{\mu\nu}$, but it is hard to conceive how to measure the momentum density.) Knowledge of the photon energy-momentum tensor would shed invaluable light on the gauge-dependence problem in canonical expressions, and offer a deeper or even new understanding of the gauge symmetry [2, 4].

(25)

FIG. 1: A sketch of transverse flux measurement to test the symmetry of the energy-momentum tensor. A cylindrically symmetric beam goes along the $z$ axis. The thick solid line is the symmetric axis of the beam, dashed lines mark the beam width. The shaded box represents the detector window.

(24) is that the gyromagnetic ratio is 1 for orbital magnetic moment but 2 for spin magnetic moment.

It is in order here to comment on the implication of our above observations. First of all, as we showed vividly, the energy-momentum tensor does not have the long-assumed arbitrariness; especially, its symmetry can be tested unambiguously by experiment. On the theoretical side, Einstein's equation dictates a symmetric gravitational energy-momentum tensor, while in a quantum measurement the concordance between conservations of quantized energy and electric charge refutes a symmetric inertial energy-momentum tensor and favors the nonsymmetric canonical expression. Apparently, one has to give up at least one long-cherished holy grail: either the exact equivalence between gravity and inertia, or the beautiful Einstein's theory, or the conservation law in quantum measurement.

We believe that it would not take too long before the experiment sheds some light: The measurable difference we predict for the symmetric and canonical energy-momentum tensors is surprisingly large. Should it be confirmed that the inertial energy-momentum tensor is indeed nonsymmetric (and thus in no conflict with the hardly questionable quantum conservation law), then one has to infer that, either (i) the equivalence between gravity and inertia is exact, thus $T^{\mu\nu}_{\text{grav}}$ must take the same nonsymmetric expression of $T^{\mu\nu}_{\text{can}}$, then the theory cannot be that of Einstein for spin-polarized objects; or (ii) Einstein's equation is correct and thus $T^{\mu\nu}_{\text{grav}}$ is symmetric, then the equivalence principle must be violated when spin plays a role.

We discuss in some detail our proposed measurement as sketched in Fig. 1. For convenience, it is better to prepare a cylindrically symmetric beam, then the integral in Eqs. (11) or (12) simplifies to

$$\int (\vec{E} \times \vec{K})^2 dV = \int \rho K_\phi dV = 2\pi L \int \rho^2 K_\phi d\rho,$$
Studying the energy-momentum tensors of composite and higher-spin particles is also very interesting in its own right. If a beam of such particles is used, then, unlike for the fundamental electrons or photons, we have basically no prediction for the possible outcome of measuring the transverse flux as in Fig. 1. It does not help to express the energy-momentum tensor of a composite particle in terms of its fundamental constituents, since it is the composite particle as a whole that is quantized in a measurement. Actually performing the type of measurement as in Fig. 1 for composite and high-spin particles would help to study how their effective degrees of freedom can be described by a quantum wavefunction.

This work is supported by the China NSF via Grants No. 11275077 and No. 11035003, and by the NCET Program of the China Ministry of Education.

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