Long-range correlations and nonstationarity in the Brazilian stock market

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Abstract

We report an empirical study of the Ibovespa index of the São Paulo Stock Exchange in which we detect the existence of long-range correlations. To analyze our data we introduce a rescaled variant of the usual Detrended Fluctuation Analysis that allows us to obtain the Hurst exponent through a one-parameter fitting. We also compute a time-dependent Hurst exponent $H(t)$ using three-year moving time windows. In particular, we find that before the launch of the Collor Plan in 1990 the curve $H(t)$ remains, in general, well above 1/2, while afterwards it stays close to 1/2. We thus argue that the structural reforms set off by the Collor Plan has lead to a more efficient stock market in Brazil. We also suggest that the time dependence of the Ibovespa Hurst exponent could be described in terms of a multifractional Brownian motion.

Key words: Long memory processes, Detrended fluctuation analysis, Hurst exponent, Econophysics, Multifractional Brownian motion

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1 Introduction

In recent years, it has been realized that many problems from Economics can be studied with the standard “toolkit” of statistical physics [1,2]. For instance, asset prices in financial markets are commonly described in terms of a geometric Brownian motion, an assumption that is at the heart of the so-called Efficient Market Hypothesis (EMH). Thus, in the EMH scenario the returns

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on a given stock follow an uncorrelated Gaussian process (white noise). In more common parlance, the EMH says that all information available about a given financial asset is already reflected on its current price, so that knowing the asset past history does not in any way help us to predict future prices. Although the EMH stands as a cornerstone of modern Finance [3], deviations from efficiency have been recently observed in many different financial markets [1,2]. In such ‘inefficient’ markets the empirical data violate either the independence or the Gaussian assumptions of the EMH. The former instance of ‘inefficiency’ is of particular interest to us here because it implies the existence of long-range correlations that are not accounted for in the standard EMH model.

A time series can be tested for correlation in many different ways [4]. One general methodology consists in estimating how a certain fluctuation measure, to be denoted here generically by $F$, scales with the size $\tau$ of the time window considered. Specific methods, such as the Hurst rescaled range analysis [5] or the Detrended Fluctuation Analysis [6,7], differ basically on the choice of the fluctuation measure; see below for more details about these two methods. If the time series is uncorrelated one expects that $F \sim \tau^{1/2}$, as is the case for the standard Brownian motion. On the other hand, if $F \sim \tau^H$ with $H \neq 1/2$ one then says that the time series has long-term memory, with $H > 1/2$ ($H < 1/2$) meaning persistence (antipersistence). The exponent $H$ is generally referred to as the Hurst exponent.

In the present paper we analyze the behavior of the São Paulo Stock Exchange Ibovespa index, which is the main index of the Brazilian stock market. We have carried out a Detrended Fluctuation Analysis (DFA) of the Ibovespa covering over 30 years of data, since its inception in 1968 until the year 2001. Here we introduce however a rescaled variant of the original DFA [6] that has the advantage of allowing us to determine the Hurst exponent $H$ with a one-parameter fitting. For the complete Ibovespa time series we find $H = 0.6$, indicating that the Ibovespa index exhibits persistence. We have also computed a ‘local’ Hurst exponent $H(t)$ in moving three-year time windows and found that $H(t)$ varies considerably over time. In particular, we find that the curve $H(t)$ undergoes a distinct change in character around the year 1990. More specifically, we observe that before 1990 the exponent $H(t)$ is always greater than 1/2. Then around the year 1990 $H(t)$ drops quite rapidly towards 1/2 and stays (within some fluctuation) around this value afterwards. We identify this drastic change in $H(t)$ as a consequence of the economic plan adopted in March 1990, the so-called Collor Plan, that marked the beginning of structural reforms in the Brazilian economy. This interpretation is confirmed by separate analyses of the Ibovespa returns prior and after the Collor Plan, where in the former case we find $H = 0.63$ while for the latter period we have $H \approx 0.5$. We thus argue that the process of opening and modernization of the economy started by the Collor Plan has led to a more efficient stock market.
in Brazil, in the sense that the Hurst exponent $H$ for this period is close to the value predicted by the EMH.

The fact that its Hurst exponent changes over time indicates that the Ibovespa follows a multifractal (rather than monofractal) process. We thus suggest that the Ibovespa basic dynamics could perhaps be captured by the so-called multifractional Brownian motion (MFBM) \cite{8}, which is a generalized fractional Brownian motion with a time-dependent Hurst exponent $H(t)$. One interesting property of the MFBM is that it is a Gaussian additive process, while the multifractal processes often considered in physics \cite{9,10} and economics \cite{11} are in general multiplicative.

The paper is organized as follows. In Sec. 2 we describe our data. In Sec. 3 we present some background material about fractional Brownian motion and give a brief description of the Detrended Fluctuation Analysis. In particular, we discuss a rescaled version of the DFA that allows us to determine the Hurst exponent $H$ through a one-parameter fitting. The results of this rescaled DFA for the Ibovespa time series are shown in Sec. 4. In Sec. 5 we offer a succinct discussion of the multifractional Brownian motion and its possible relevance to financial time series. In Sec. 6 we review our main results and conclusions.

For completeness, we present in Appendix A the results of the Hurst analysis for our Ibovespa time series. It is shown there that in this case the Hurst method is slightly less reliable than the DFA in detecting long-correlation, thus justifying our choice of restricting ourselves to the DFA in the main portion of the present paper.

## 2 The Data

The Ibovespa index represents the present value of a self-financing hypothetical portfolio made up of the most traded stocks on the São Paulo Stock Exchange. In this paper we have analyzed the daily closing values of the Ibovespa since its inception (January 02, 1968) until recent times (May 31, 2001), amounting to 8209 trading days. The nominal figures of the daily Ibovespa were deflated by Brazil’s general price index (IGP-DI, in its Portuguese acronym) and converted to Brazilian Reais (BRL) in values of August 1994. The corresponding time series for the deflated Ibovespa is shown in Fig. 1, while the IGP-DI price index is plotted in Fig. 2 in monthly rates.

In the period comprised in our study many economic events took place in Brazil and abroad that had a direct impact on the Brazilian stock market. For the benefit of the reader less familiar with Brazil’s recent economic history, it was thought desirable to recall here some of the main events whose effects can be clearly distinguished in the data shown in Figs. 1 and 2. For instance, the
first large peak in the Ibovespa in the early 1970’s corresponds to the so-called ‘Brazilian economic miracle’ (during the military regime that took power in 1964 and ended in 1985) when Brazil’s GDP grew over 10% a year. After the first oil crises in 1973 the Brazilian economy entered a long period of slower growth and steadily increasing inflation, as reflected in the consistent decline of the Ibovespa from 1973 to 1984 (Fig. 1) and a corresponding surge of the inflation rate (Fig. 2). From 1986 to 1991 six largely unsuccessful economic plans were adopted by the succeeding governments in attempts to control inflation. The effects of these economic plans are very clearly distinguished in

Fig. 1. Daily closing values of the deflated Ibovespa index in the period January 1968–May 2001.

Fig. 2. Brazilian monthly inflation rate in the period January 1968–May 2001. The arrows indicate the time of launch of the following economic plans: Cruzado Plan (A), Collor Plan (B), and the Real Plan (C); see text.
Fig. 3. Daily returns of the Ibovespa in the period January 02, 1968 ($t = 0$) through May 30, 2001 ($t = 8207$).

Fig. 2 as they correspond to sudden drops in the inflation rate. For example, the first of such plans, the so-called Cruzado Plan, was launched in February 1986 and led not only to a sharp decline in the inflation rate (arrow A in Fig. 2) but also to a strong peak in the Ibovespa index (see Fig. 1). Shortly afterwards, however, inflation picks up again and the Ibovespa recedes to pre-Cruzado levels. Although it also failed to rein in inflation, a plan of more lasting impact on the Brazilian economy was the Collor Plan (arrow B in Fig. 2), as we shall see later. It should also be recalled that the Real Plan, launched in July 1994 (arrow C in Fig. 2), has so far succeeded in stabilizing the economy and keeping inflation at bay.

In what follows we shall carry out a detailed analysis of the Ibovespa daily returns, which we define in the usual way. Let $y_t$ denote the closing value of the Ibovespa at time $t$, where $t$ is measured in trading days and $t = 0$ corresponds to the date January 02, 1968, when the Ibovespa was created. The return $r_t$ at time $t$ is given by the logarithmic difference between consecutive daily values:

$$ r_t \equiv \ln y_{t+1} - \ln y_t = \ln \left( \frac{y_{t+1}}{y_t} \right). \tag{1} $$

In Fig. 3 we plot the daily returns $r_t$ for the Ibovespa index. In what follows we wish to investigate the existence of long-term dependence in the Ibovespa daily returns. Before going into this analysis, however, we shall first present some background material about correlated time series and describe briefly the method we will use to detect correlation in our data.
3 Correlated Time Series

One of the simplest and most important stochastic models that display long-term dependence is the fractional Brownian motion (FBM), whose main properties we will briefly review below. Following this, we shall then describe the Detrended Fluctuation Analysis (DFA) which, as already noted, is the main method we will use to estimate the Hurst exponent $H$ for our empirical data. (Several other estimators for $H$ have been discussed in the literature; see, e.g., Ref. [4] for a comparison among some of them.)

3.1 Fractional Brownian motion

We recall here that the fractional Brownian motion $\{B_H(t), t > 0\}$ is a Gaussian process with zero mean and stationary increments whose variance and covariance are given by

$$E[B_H^2(t)] = \sigma^2 t^{2H}, \quad (2)$$

$$E[B_H(s)B_H(t)] = \frac{1}{2} \sigma^2 \left(s^{2H} + t^{2H} - |t - s|^{2H}\right), \quad (3)$$

where $0 < H < 1$, $\sigma > 0$, and $E[.]$ denotes expected value. The process $B_H(t)$ is statistically self-similar (or more exactly self-affine) in the sense of finite-dimensional distributions:

$$B_H(at) \overset{d}{=} a^H B_H(t), \quad (4)$$

for all $a > 0$, where $\overset{d}{=} \text{ means equality in distribution.}$

The parameter $H$ is called the self-similarity exponent or the Hurst exponent. For $H = 1/2$ the process $B_H(t)$ corresponds to the standard Brownian motion, in which case the increments $X_t = B_H(t+1) - B_H(t)$ are statistically independent and represent the usual white noise. On the other hand, for $H \neq 1/2$ the increments $X_t$, known as fractional white noise, display long-range correlation in the sense that

$$E[X_{t+h}X_t] \approx \sigma^2 2H(2H - 1)h^{2H-2} \quad \text{for} \quad h \to \infty, \quad (5)$$

as one can easily verify from (2) and (3). Thus, if $1/2 < H < 1$ the increments of the FBM are positively correlated and we say that the process $B_H(t)$ exhibits persistence. Likewise, for $0 < H < 1/2$ the increments are negatively correlated and the FBM is said to show antipersistence.
3.2 Detrended fluctuation analysis

The Detrended Fluctuation Analysis (DFA), which is a modification of the usual variance analysis, was proposed independently albeit with different names in Refs. [6] and [7]. (In Ref. [7] the analogous of the fluctuation function \( F(\tau) \) defined below was termed Straight Line Roughness.) The DFA has the advantage over the standard variance analysis of being able to detect long-term dependence in nonstationary time series [6]. The idea of the method is to subtract possible deterministic trends from the original time series and then analyze the fluctuation of the detrended data, as explained below.

To implement the DFA, we first integrate the original time series \( \{r_t\}_{t=1,...,T} \) to obtain the cumulative time series \( X(t) \):

\[
X(t) = \sum_{t'=1}^{t} (r_{t'} - \bar{r}), \quad t = 1, ..., T,
\]

where

\[
\bar{r} = \frac{1}{T} \sum_{t'=1}^{T} r_{t'}.
\]

Next we break up \( \{X(t)\} \) into \( N \) non-overlapping time intervals, \( I_n \), of equal size \( \tau \), where \( n = 0, 1, ..., N - 1 \) and \( N \) corresponds to the integer part of \( T/\tau \). We then introduce the local trend function \( Y_\tau(t) \) defined by

\[
Y_\tau(t) = a_n + b_n t \quad \text{for} \quad t \in I_n,
\]

where the coefficients \( a_n \) and \( b_n \) represent the least-square linear fit of \( X(t) \) in the interval \( I_n \). We now compute the fluctuation function \( F(\tau) \) defined as the root mean deviation of \( X(t) \) with respect to the trend function \( Y_\tau(t) \):

\[
F(\tau) = \sqrt{\frac{1}{T} \sum_{t=1}^{T} [X(t) - Y_\tau(t)]^2}.
\]

In the original DFA [6] the Hurst exponent \( H \) is obtained directly from the scaling behavior of the above fluctuation function: \( F(\tau) \sim \tau^H \). For reasons that will become apparent shortly, we find it convenient to introduce a rescaled (dimensionless) fluctuation function \( F_S(\tau) \),

\[
F_S(\tau) \equiv \frac{F(\tau)}{S},
\]
where $S$ is the data standard deviation

$$S = \sqrt{\frac{1}{T} \sum_{i=1}^{T} (r_t - \bar{r})^2}. \quad (11)$$

The rescaled fluctuation function $F_S(\tau)$ obviously has the same functional form as the original $F(\tau)$ and so we write

$$F_S(\tau) = C \tau^H, \quad (12)$$

where $C$ is a constant independent of the time lag $\tau$. From here on, we shall drop the subscript from $F_S(\tau)$, with the understanding that we will be working solely with the rescaled fluctuation function.

In a double-logarithmic plot the relationship (12) yields a straight line whose slope is precisely the exponent $H$, and so the empirical value for $H$ can be easily obtained from a linear regression (in log-log scale) of the corresponding data for $F(\tau)$. One practical problem with this method, however, is that one needs to choose an appropriate interval within which to perform the linear fit. For instance, the first few points at the low end of the graph of $F(\tau)$ should be disregarded because in this region the detrending procedure removes too much of the fluctuation—this effect accounts for the usual ‘bending down’ of $F(\tau)$ for small $\tau$. For large $\tau$, on the other hand, there are few boxes $I_n$ for a proper averaging to be made and hence the values of $F(\tau)$ are not statistically reliable in this case. Furthermore, it has been found [12] that the values obtained for $H$ using this procedure are somewhat dependent on the choice of the fitting interval. In order to avoid some of these difficulties we shall use here a slightly different procedure to estimate the exponent $H$ from the graph of $F(\tau)$.

To do this, we shall rely on the fact that for the fractional Brownian motion, i.e., for $X(t) = B_H(t)$, the (rescaled) fluctuation function $F(\tau)$ can be computed exactly [4]:

$$F_H(\tau) = C_H \tau^H, \quad (13)$$

where

$$C_H = \frac{2}{2H + 1} + \frac{1}{H + 2} - \frac{2}{H + 1}. \quad (14)$$

In (13) we have added a subscript $H$ to the fluctuation function $F(\tau)$ to denote explicitly that it refers to $B_H(t)$. Now, to estimate $H$ for our time series we shall simply adjust the parameter $H$ so as to obtain the best agreement between the theoretical curve predicted by $F_H(\tau)$ and our empirical data for
In this way, we can estimate $H$ with a one-parameter fitting! In principle, one could carry out a formal nonlinear regression using (13) and (14) to determine $H$, but in practice it suffices to vary $H$ incrementally and decide by visual inspection when the theoretical curve best matches the empirical data. We shall now apply this methodology to estimate the Hurst exponent for the Ibovespa returns.

### 4 Fluctuation analysis of the Ibovespa

In this section we discuss the results of the DFA applied to our Ibovespa time series, which we recall spans the period from January 1968 through May 2001. In Fig. 4 we plot in double-logarithmic scale the corresponding fluctuation function $F(\tau)$ against the window size $\tau$. Using the procedure outlined at the end of the previous section, we obtain the following estimate for the Hurst exponent: $H = 0.60 \pm 0.01$. In Fig. 4 we have also plotted (straight line) the corresponding theoretical curve $F_H(\tau)$ given by (13) and (14), which as one can see is in excellent agreement with the empirical data for intermediate values of $\tau$. Since $H > 1/2$ we conclude that the Ibovespa returns show persistence. We also note that for $\tau > 130$ (indicated by the arrow in Fig. 4) the empirical data deviate somewhat from the initial scaling behavior and appear to cross over to a regime with a slope closer to $1/2$. This indicates that the Ibovespa tends to lose its 'memory' after a period of about 6 months. It is perhaps worth pointing out that similar behavior was seen, for example, in several equity indices of the London Stock Exchange [13]. There, the break from the scaling regime typically occurred around 160 trading days [13], which is not very different from what we found for the Ibovespa, especially considering that the location of the crossover point is difficult to determine precisely.

In order to check the validity of our analysis above, we have also computed the Hurst exponent for a shuffled version of our data in which we have randomly mixed the time series of the Ibovespa returns. Because the process of shuffling tends to destroy any previously existing correlation, we would now expect a Hurst exponent equal (or very close) to $1/2$. This is indeed the case, as shown in Fig. 5, where we plot $F(\tau)$ for the shuffled data. In this figure one sees that the data points are extremely well described by the theoretical curve (straight line) given by $F_H(\tau)$ with $H = 0.5$. This thus confirms the fact that the long-term dependence seen in the original time series was not an spurious effect.

We now wish to investigate whether the Hurst exponent for the Ibovespa varies in time, which would indicate the existence of nonstationary fluctuations in the Brazilian stock market. In fact, a simple visual inspection of the distribution of the Ibovespa returns shown in Fig. 3 already reveals evidences of nonstationary
behavior, which we would like to quantify better. Furthermore, in the recent past not only the Brazilian economy was plagued by runaway inflation and but it also endured several ‘shock-therapy’ economic plans, as we have already mentioned. It is thus natural to ask how such stressful events may have affected the dynamics of the Ibovespa index, particularly in regard to its degree of correlation. We now turn our attention to this point.

In order to estimate a time-varying Hurst exponent using a fluctuation analysis (or any method for that matter) one is presented with a difficult challenge, for one has to try to satisfy competing requirements. Indeed, to determine a ‘local’ exponent $H$ at a particular time $t$ one has to consider a time interval

Fig. 4. Fluctuation function $F(\tau)$ as a function of window size $\tau$ for the returns of the Ibovespa index in the period 1968–2001. The straight line gives the theoretical curve $F_H(\tau)$ for $H = 0.6$; see text.

Fig. 5. Fluctuation function $F(\tau)$ for the shuffled Ibovespa returns. The straight line represents the theoretical curve $F_H(\tau)$ for $H = 0.5$.
Fig. 6. The Hurst exponent $H$ for the Ibovespa as a function of time. Here $H$ was computed in three-year periods and the variable $t$ represents the origin of each such interval.

around $t$ that is considerably smaller than the total time spanned by the data but still sufficiently large to contain enough points for a meaningful statistics. In the case of the Ibovespa we have found that a three-year period (736 trading days) is an acceptable compromise between these two opposing demands. (For smaller periods the fluctuations in the values of $H$ are too large to be trusted.) We have accordingly applied the DFA to the Ibovespa returns in moving three-year intervals. More specifically, our methodology is as follows: starting at the beginning of our time series we compute the exponent $H$ considering only the data points within three years from the initial point, then we advance our three-year time window by 20 days and compute $H$ for this new period, and so on. Proceeding in this way we obtain the curve for $H(t)$ shown in Fig. 6, where $t$ denotes the origin of each three-year time interval.

The most striking feature seen in Fig. 6 is perhaps the fact that the Ibovespa shows persistence ($H > 1/2$) all the way up to the early 1990’s, after which it ‘switches’ to a regime with alternating persistent and antipersistent behavior but where $H$ remains somewhat close to 1/2. Notice in particular that during the 1970’s and 1980’s the curve $H(t)$ stays well above 1/2, the only exception to this trend occurring around the year 1986 when $H$ dips momentarily towards 1/2—we associate this effect with the launch of the Cruzado Plan in February 1986; see Sec. 2. Shortly after the Cruzado Plan, however, inflation picks up again (Fig. 2), the Ibovespa declines (Fig. 1), and the exponent $H$ returns to pre-Cruzado levels (Fig. 6). Then in the early 1990’s, at the time of the Collor Plan (see below), we observe a dramatic decline in the curve $H(t)$ towards 1/2, with $H$ remaining (within some fluctuation) around this value.
afterwards. Note also that around the time of the launch of the Real Plan in July 1994 the curve $H(t)$ reaches its lowest values. We have thus seen that the Ibovespa Hurst exponent declines following the adoption of a major economic plan (such as the Cruzado, Collor and Real Plans), thus confirming the fact [15] that a firm intervention on the market tends to reduce $H$ momentarily.

As we already mentioned, the drop in $H$ that occurs in the early 1990’s can be unambiguously traced back to the launch, in March 1990, of the Collor Plan which marked the beginning of structural reforms in Brazil [14]. In this context, it is important to realize that this decline in $H$ takes place well before the adoption of the Real Plan. It is indeed quite surprising that the Hurst exponents found in the early 1990’s, in a macroeconomic environment dominated by very high inflation, are on about the same level as those found after the Real Plan when the economy had been stabilized. This fact in itself is a clear evidence of the profound impact that the structural reforms set off by the Collor Plan had on the Brazilian economy. In order to study this effect in more detail, we have applied the DFA separately to the Ibovespa returns prior and after the Collor Plan. Our results are shown in Fig. 7. In this figure one sees that before the Collor Plan the Brazilian stock market shows a significant amount of persistence ($H = 0.63$), while after the Collor Plan the Ibovespa displays essentially no long-term dependence ($H = 0.5$).

In light of the preceding results, it thus seems legitimate to conclude that the opening and consequent modernization [14] of the Brazilian economy that begun in the early 1990’s has led to a more efficient stock market, in the sense that $H = 0.5$ for the Ibovespa after 1990. By the same token, we associate the period of higher $H$ before 1990 with a less efficient market. In those years,
the Brazilian economy was considerably closed to foreign competition and its financial markets were not readily accessible to international investors. In such closed economic environment the Brazilian stock market was conceivably more prone to ‘correlated fluctuations’ (and perhaps also more susceptible to being pushed around by aggressive investors), which may explain in part a Hurst exponent greater than 0.5 for that period.

5 Multifractional Brownian motion

We have seen above that the Hurst exponent $H$ for the Ibovespa stock index changes appreciably over time. In this regard, the fractional Brownian motion appears to be a somewhat restrictive model, unable to capture more fully the complex dynamics of the Brazilian stock market. Similar time-dependent Hurst exponents have also been observed in other financial markets [15,16,17] and in physical processes such as traffic flow in the Internet [18]. To circumvent this shortcoming of the FBM, Peltier and Levy-Vehel [8] have proposed a multifractional Brownian motion (MFBM) in which the scaling exponent $H$ is allowed to vary in time. For a rigorous exposition of the definition and mathematical properties of MFBM the interested reader is referred to Ref. [8]. Here we shall present only the main properties of the MFBM that are of interest to us. Before we proceed, it should however be emphasized that our aim in this section is not to develop a specific model for the Ibovespa but rather to present a generic theoretical framework based on the MFBM in which the time-dependence of the Ibovespa Hurst exponent could, in principle, be understood.

Let $H(t)$ be a Hölder-continuous function in the interval $t \in [0, 1]$ with Hölder exponent $\beta > 0$, such that for any $t > 0$ we have $0 < H(t) < \min(1, \beta)$ [8]. (For ease of notation, we shall sometimes indicate the $t$-dependence as a subscript and write $H_t$.) The multifractional Brownian motion $\{W_{H_t}(t), t > 0\}$ is the Gaussian process defined by

$$W_{H_t}(t) = \frac{1}{\Gamma(H_t + \frac{1}{2})} \int_{-\infty}^{t} \left[ (t - s)^{H_t - 1/2} - (-s)^{H_t - 1/2} \right] dB(t), \quad (15)$$

where $\Gamma(x)$ is the gamma function, $(x)_+ = x$ if $x > 0$ and zero otherwise, and $B(t)$ denotes the usual Brownian motion. One important aspect of the process defined above is that its increments are no longer stationary since it can be shown [8] that

$$E[\{W_{H_{t+h}}(t+h) - W_{H_t}(t)\}^2] \approx h^{2H(t)} \quad \text{as} \quad h \to 0. \quad (16)$$
Because of its non-stationarity the MFBM is no longer a self-similar process either. However, it is possible to define the concept of locally asymptotically similarity at the point $t_0 > 0$ in the following way

$$W_{H_{t_0+at}}(t_0 + at) - W_{H_{t_0}}(t_0) \overset{d}{=} W_{H_{t_0}}(t) \quad \text{as} \quad a \to 0. \quad (17)$$

[Compare this with the global self-similar property of the FBM given in (4).] In this sense, $W_{H_{t_0}}(t)$ can be thought of as a process that locally at time $t$ ‘resembles’ a fractional Brownian motion $B_{H_{t_0}}(t)$. In fact, a practical way to generate a multifractional Brownian path consists in generating for each time $t \in [0, 1]$ a fractional Brownian motion $B_{H_{t}}(s)$, $s \in [0, 1]$, and then setting $W_{H_{t}}(t) = B_{H_{t}}(t)$. More details about this method together with an algorithm for its implementation can be found in Ref. [8]. (We remark parenthetically, however, that the algorithm presented there for the computation of $H(t)$ is not appropriate for empirical time series, as inadvertently used in Ref. [17], because such algorithm requires previous knowledge of the variance of the FBM used to generate the MFBM, which of course is not known for empirical data.)

In Fig. 8 we show an example of a path of multifractional Brownian motion characterized by the Hurst function

$$H(t) = 0.63 - 0.076 \arctan(30t - 24). \quad (18)$$

Although somewhat arbitrary, this example was chosen to mimic the generic trend seen in the Ibovespa where $H(t)$ can be viewed as having (on average) two distinct ‘plateaus,’ one before and the other after 1990, with a swift crossover between them. For comparison purpose, we have generated a MFBM path with approximately the same number of data points ($N = 2^{13} = 8192$) as in the original Ibovespa time series ($N = 8209$). We have then applied the detrended fluctuation analysis to our MFBM path, with the result being shown in Fig. 9. Since a MFBM path is not characterized by a constant Hurst exponent, the functional dependence of the DFA fluctuation function $F(\tau)$ will not, in general, be a power law. Nevertheless, we see from Fig. 9 that $F(\tau)$ exhibits an approximate scaling regime, $F(\tau) \sim \tau^{\bar{H}}$ with $\bar{H} = 0.6$, up to about $\tau = 160$. For comparison, we also show in Fig. 9 the corresponding relationship (straight line) predicted by $F_H(\tau)$ given in (13), which as one sees is in good agreement with the data in the scaling region.

We thus interpret the scaling exponent $\bar{H}$ as a kind of average Hurst exponent for our MFBM time series. Note, however, that although the region of small $H$ represents only about 20% of the total time series it has nonetheless a significant weight on the exponent $\bar{H}$, since its value is considerably smaller than what one would obtain by simply averaging the curve $H(t)$ [in this case
We have carried out a Detrended Fluctuation Analysis of the daily returns of the São Paulo Stock Exchange Index (Ibovespa), spanning over 30 years of data from January 1968 up to May 2001. For this time series we have obtained a Hurst exponent greater than 1/2, indicating that the Ibovespa has long-term dependence (persistence). This memory effect seems to last for up to about 6 months (130 trading days), after which time the fluctuation function $F(\tau)$ deviates from the initial scaling behavior and crosses over to a regime with a
slope closer to 1/2. Similar behavior has been observed in other stock indices [13], and it thus appears that stock markets tend to lose their ‘memory’ typically in about half a year.

We have also performed a more localized (in time) analysis of the Ibovespa returns, where we have calculated the local Hurst exponent $H(t)$ in three-year moving time windows. Here the most striking feature is the fact that after 1990 the Hurst exponent stays close to 1/2 (within some fluctuation), whereas in the preceding decades it was, in general, considerably larger than 1/2. We have thus argued that the liberalizing measures (e.g., substantial reduction of import tariffs, cutting down the ‘red tape’ for foreign investments, and privatisation of state companies [14]) brought about by the Collor Plan in the early 1990’s and continued with the Real Plan in 1994 resulted in a more efficient stock market in the sense that $H$ remained close to 1/2 after 1990.

The fact that its Hurst exponent changes over time time indicates that the Ibovespa time series cannot be satisfactorily modeled in terms of a simple fractional Brownian motion. In other words, the Ibovespa follows a multifractal process. We have thus suggested that the multifractal Brownian motion, in which the scaling exponent $H$ is allowed to vary in time, could perhaps be a more appropriate description for the Brazilian stock market dynamics. It thus remains an interesting question to investigate to what extent the MFBM is a valid model for other financial time series.

Fig. 9. Detrended fluctuation analysis of the multifractional Brownian motion shown in Fig. 8(b).
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A Appendix: Hurst rescaled range analysis

The Hurst rescaled range \((R/S)\) analysis is a technique devised by the hydrologist Henry Hurst in 1951 [5] to test for the presence of correlations in empirical time series. A detailed description of the Hurst analysis (including its historical origin) can now be found in textbooks [9,10], so that here we shall give only a brief summary of the method and then proceed to apply it to the Ibovespa returns.

In the Hurst analysis, we start by dividing our time series \(\{r_t\}_{t=1}^T\) in \(N\) non-overlapping time intervals \(I_n\) of equal size \(\tau\). For each interval \(I_n\) we compute the range

\[
R_n = \max_{1 \leq k \leq \tau} \left[ \sum_{t=1}^{k} (r_{n\tau+t} - \tau_n) \right] - \min_{1 \leq k \leq \tau} \left[ \sum_{t=1}^{k} (r_{n\tau+t} - \tau_n) \right],
\]

(A.1)

where

\[
\tau_n = \frac{1}{\tau} \sum_{t=1}^{\tau} r_{n\tau+t},
\]

(A.2)

and the data standard deviation

\[
S_n = \sqrt{\frac{1}{\tau} \sum_{t=1}^{\tau} (r_{n\tau+t} - \tau_n)^2}.
\]

(A.3)

The rescaled range \((R/S)\) is then defined as the average of the ratio \(R_n/S_n\) over all intervals \(I_n\)

\[
(R/S)_{\tau} \equiv \left\langle \frac{R_n}{S_n} \right\rangle = \frac{1}{N} \sum_{n=0}^{N-1} \frac{R_n}{S_n}.
\]

(A.4)
Fig. A.1. Rescaled range $R/S$ versus the time lag $\tau$ for the Ibovespa returns in the period 1968—2001 and for the corresponding shuffled data. Also shown for comparison is a straight line with slope equal to $1/2$.

The Hurst exponent $H$ is obtained from the scaling behavior of $(R/S)_\tau$:

$$(R/S)_\tau \sim \tau^H.$$  \hspace{1cm} (A.5)

A plot of the rescaled range $R/S$ as a function of $\tau$ for the Ibovespa returns over the entire period (1968–2001) analyzed in this paper is shown in the upper curve of Fig. A.1. The data in this case show a scaling regime that goes from $\tau = 10$ up to about $\tau = 160$. A linear regression in this region yields the value $H = 0.65$, which is somewhat larger than the exponent ($H = 0.6$) obtained via the DFA for the same time series. In order to understand the origin of such discrepancy, we have also applied the Hurst method to the shuffled Ibovespa data (lower graph in Fig. A.1) and found $H = 0.57$ in this case. We thus see that the Hurst analysis of the shuffled data predicts an unexpected residual correlation. (Recall that applying the DFA to the shuffled data yields $H = 0.5$, as expected.) This ‘excess of correlation’ found in the $R/S$ analysis of the shuffled Ibovespa is perhaps not entirely surprising, given that it is known that the Hurst method tends to overestimate the Hurst exponent for time series of small sizes [9]. This effect may perhaps explain why the exponent $H$ obtained via the Hurst method is usually larger than that found via the DFA [19]. We thus conclude that the DFA appears to be a more reliable method for detecting true correlation in financial time series. It is worth therefore emphasizing that the numerical values obtained for the exponent $H$ via the Hurst method should be interpreted with some caution, specially when the time series is not sufficiently large, as is commonly the case in practice.
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