ON THE (NON)EXISTENCE OF STATES ON ORTHOGONALLY CLOSED SUBSPACES IN AN INNER PRODUCT SPACE

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Abstract. Suppose that \( S \) is an incomplete inner product space. In [2] A. Dvurečenskij shows that there are no finitely additive states on orthogonally closed subspaces, \( F(S) \), of \( S \) that are regular with respect to finitely dimensional spaces. In this note we show that the most important special case of the former result—the case of the evaluations given by vectors in the “Gleason manner”—allows for a relatively simple proof. This result further reinforces the conjecture that there are no finitely additive states on \( F(S) \) at all.

1. Introduction

Let \( S \) be a real or complex separable inner product space and let \( \langle \cdot, \cdot \rangle \) denote the inner product of \( S \). Let us denote by \( F(S) \) the set of all orthogonally closed subspaces of \( S \). A subspace \( M \) of \( S \) is in \( F(S) \) if \( M = M^\perp \), where \( M^\perp = \{ x \in S : \langle x, y \rangle = 0 \text{ for all } y \in M \} \). It turns out that if we understand \( F(S) \) with the ordering given by the inclusion relation and with orthocomplementation relation \( M \rightarrow M^\perp \) as defined above, then \( F(S) \) becomes a complete orthocomplemented lattice. However, \( F(S) \) does not have to be orthomodular. In fact, Amemiya and Araki [1] proved the following algebraic criterion for the (topological) completeness of an inner product space \( S \): an inner product space \( S \) is complete if and only if \( F(S) \) is orthomodular.
Let us now turn to measure-theoretic criteria for the completeness of $S$. The following result by Hamhalter and Pták \cite{Hamhalter-Ptak} initiated a series of interesting measure theoretic characterizations for the completeness of an inner product space $S$ \cite{Hamhalter-Ptak}.

**Theorem 1.1.** An inner product space $S$ is complete if and only if $F(S)$ possesses a $\sigma$-additive state.

In 1988 \cite{Ptak}, Pták asked whether $S$ has to be complete if $F(S)$ possesses a finitely additive state. Recently, Dvurečenskij and Pták \cite{Dvurecenskij-Ptak} proved that if $S$ is an incomplete inner product space, then the assumption that there is a finitely additive state on $F(S)$ implies that the range of this state has to be the entire interval $[0, 1]$. In this note we show that an inner product space $S$ is complete if, and only if, there exists $u \in \overline{S}$ such that $s_u$ defines a state on $F(S)$, where by $\overline{S}$ is denoted the completion of $S$. Here, for any vector $u \in S$ with $\|u\| = 1$, by $s_u$ is meant the “Gleason” assignment defined by

$$s_u : F(S) \to [0, 1]$$

$$M \mapsto \langle P_M u, u \rangle.$$

Before we launch on the proof proper, let us summarize the “state of the art” of the state problem for $F(S)$. If there are states on $F(S)$ then there are pure states on $F(S)$ (Krein–Milman). But in view of the previous two facts these pure states must be rather bizarre. Thus, a conjecture remains that for an incomplete space $S$ the lattice $F(S)$ is stateless.

2. Results

Let $S$ be a separable inner product space and let $\overline{S}$ be its completion. In this section we mainly prove the result formulated in the introduction.

**Theorem 2.1.** A separable inner product space $S$ is complete if, and only if, there exists $u \in \overline{S}$ such that

$$s_u : M \mapsto \langle P_M u, u \rangle$$

defines a state on $F(S)$.

**Proof.** If $S$ is complete then, obviously, for every $u \in S = \overline{S}$, $s_u$ is a ($\sigma$–additive) state on $F(S)$ ($F(S) = L(S)$ and this follows from Gleason’s theorem).
For the second implication, suppose that there exists a vector \( u \in S \) such that \( s_u \) is a state on \( F(S) \). We divide the proof into auxiliary results. We believe that they could be of certain importance in their own right.

**Claim 1.** Suppose that there exists \( u \in S \) such that

\[
 s_u : M \mapsto \langle P_M u, u \rangle
\]

defines a state on \( F(s) \). Then for every unit vector \( v \in S \), \( s_v \) defines a state on \( F(S) \).

**Proof.** Let \( \tilde{S} \) be a subspace of \( S \) generated by \( s \) and \( u \). Let \( v(\neq u) \) be a unit vector in \( S \) and put \( P = [u] + [v] \). Then \( \tilde{S} = P \oplus P^\perp \). Set \( \tilde{w} = v - \langle v, u \rangle u \), and let \( w = \frac{\tilde{w}}{\|\tilde{w}\|} \).

Similarly, let \( \tilde{z} = u - \langle v, u \rangle v \), and put \( z = \frac{\tilde{z}}{\|\tilde{z}\|} \). Then \( P = [v] \oplus [z] = [u] \oplus [w] \).

Define the map

\[
 T : \tilde{S} \to \tilde{S}
\]

\[
 P \oplus P^\perp \to \tilde{S}
\]

\[
 p + p' = \alpha v + \beta z + p' \mapsto \alpha u + \beta w + p'.
\]

\( T \) is a unitary operator on \( \tilde{S} \), that is \( T \) is a bijective linear operator satisfying

\[
 \langle x, y \rangle = \langle Tx, Ty \rangle
\]

for all \( x, y \in \tilde{S} \).

By the continuity of \( T \) we can extend it over \( \tilde{S} \). With a harmless abuse of notation let us denote the extension again by \( T \). We now show that if \( A \) is a subspace of \( S \), then \( TA \subset TA \). Since \( T \) is continuous it follows immediately that \( TA \subset TA \). Let \( x \in TA \). Then \( x = \lim_{i \to \infty} x_i \) where \( x_i \in TA \) for all \( i \in \mathbb{N} \). Let \( y_i \in A \) be such that \( Ty_i = x_i \). Then we have

\[
 \|x_i - x_j\|^2 = \langle Ty_i - Ty_j, Ty_i - Ty_j \rangle
\]

\[
 = \langle T(y_i - y_j), T(y_i - y_j) \rangle
\]

\[
 = \langle y_i - y_j, y_i - y_j \rangle
\]

\[
 = \|y_i - y_j\|^2.
\]

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This implies that \( \{y_i\} \) is Cauchy and therefore it converges to some \( y \in \overline{A} \). That \( Ty = x \) follows again by the continuity of \( T \).

We now show that for any \( A \in F(S) \), we have

\[
\| P_A^v \|^2 = \| P_{TA}^u \|^2.
\]

Let \( \{a_i\} \subset A \) be an ONB of \( \overline{A} \). Then \( \{Ta_i\} \) is an ONB of \( T\overline{A} (= \overline{TA}) \) in \( TA \). We then have

\[
a_i = \alpha_i v + \beta_i z + p_i \quad \text{and therefore} \quad Ta_i = \alpha_i u + \beta_i w + p_i.
\]

This implies that

\[
\| P_A^v \|^2 = \sum |(a_i, v)|^2 = \sum |\alpha_i|^2 = \sum |(Ta_i, u)|^2 = \| P_{TA}^u \|^2.
\]

Thus, for any \( A \in F(S) \), \( s_v(A) = \| P_A^v \|^2 = \| P_{TA}^u \|^2 = s_u(TA) \), and therefore \( s_v \) does indeed define a state on \( F(S) \). \( \square \)

Claim 2. Suppose that, for each \( u \in S \), \( s_u \) defines a state on \( F(S) \). Then for every unit vector \( v \in \overline{S} \), \( s_v \) defines a state on \( F(S) \).

Proof. Let \( v \in \overline{S} \setminus S \). There exists a sequence \( \{v_i\} \subset S \) such that \( v = \lim_{i \to \infty} v_i \).

For any \( A \in F(S) \),

\[
P_A^v = P_A \lim_{i \to \infty} v_i = \lim_{i \to \infty} P_A^{v_i}
\]

and therefore

\[
s_v(A) = \lim_{i \to \infty} s_{v_i}(A).
\]

It is then not difficult to check that \( s_v \) defines a state on \( F(S) \) (pointwise limits of finitely additive states are finitely additive states). \( \square \)
Claim 3. Let for any $v \in S$ $s_v$ defines a state on $F(S)$. Let $M$ be a closed subspace of $S$. Then

$$M \in F(S) \text{ if, and only if, } \overline{M^\perp} = \overline{M^\perp S}.$$  

Proof. Let $M \in F(S)$. We need to show that $\overline{M^\perp} = \overline{M^\perp S}$. It is sufficient to prove that $\overline{M^\perp} \supset \overline{M^\perp S}$. Let $\{n_i : i \in I_M\}$ be an orthonormal basis (ONB) in $M^\perp$ of $\overline{M^\perp}$, and let $\tilde{x} \in \overline{M^\perp S}$ ($\tilde{x} \neq 0$) be arbitrary. Put $x = \frac{\tilde{x}}{\|\tilde{x}\|}$. Consider the state $s_x$ on $F(S)$.

$$1 = s_x(S) = s_x(M \vee M^\perp S)$$

$$= s_x(M) + s_x(M^\perp S)$$

$$= s_x(M^\perp S) \quad \text{since } x \perp M$$

$$= \sum_{i \in I_M} |\langle x, n_i \rangle|^2.$$  

This implies that for all $\tilde{x} \in \overline{M^\perp S}$,

$$\|\tilde{x}\|^2 = \sum_{i \in I_M} |\langle \tilde{x}, n_i \rangle|^2.$$  

Therefore it follows, by Parseval’s identity, that $\{n_i : i \in I_M\}$ is an ONB of $\overline{M^\perp S}$ and hence $M^\perp = \overline{M^\perp S}$.

Now we prove the converse. Suppose that $\overline{M^\perp} = \overline{M^\perp S}$. To reach a contradiction, assume that $M \notin F(S)$. There exists $v \in M^\perp S \setminus M$ such that $v \perp M^\perp$ and $v \notin M$. This implies that $v \perp M^\perp S$ and hence $v \in \overline{M^\perp S} = \overline{M}$. But this would imply that $v \in \overline{M} \cap S = M$, since $M$ is closed in $S$. This is the required contradiction. □

Claim 4. Suppose that for every $u \in S$ the mapping

$$M \mapsto \langle P_M^u, u \rangle$$

defines a state on $F(S)$. Let $M \in F(S)$ and let $\{x_i\}$ be any maximal orthonormal system (MONS) in $M$. Then $M = \{x_i\}^\perp S$. 

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Proof. Let \( \{m_i\} \subset M \) be an ONB of \( \overline{M} \) and \( \{n_i\} \subset M^{\perp_S} \) be an ONB of \( M^{\perp_S} = \overline{M}^{\perp_S} \). Then \( \{x_i\} \cup \{n_i\} \) is a MONS of \( S \). This implies that
\[
M \lor M^{\perp_S} = \lor [m_i] \lor [n_i] = S = \lor [x_i] \lor [n_i] = \{x_i\}^{\perp_S} \lor M^{\perp_S}.
\]
Certainly, we have \( \{x_i\}^{\perp_S} \subset M \). Take any unit vector \( y \in \overline{M} \) and consider the state \( s_y \). We have
\[
1 = s_y(S) = s_y(\{x_i\}^{\perp_S} \lor M^{\perp_S}) = s_y(\{x_i\}^{\perp_S}) = \|P_{\{x_i\}^{\perp_S}} y \|.
\]
This implies that \( y \in \{x_i\}^{\perp_S} \) and therefore
\[
M = \{x_i\}^{\perp_S}
\]
which yields
\[
M = \{x_i\}^{\perp_S}.
\]

Claim 5. \( F(S) \) is orthomodular.

Proof. Let \( A \subset B \) be in \( F(S) \). Let \( \{a_i\} \subset A \) be an ONB of \( \overline{A} \). Extend \( \{a_i\} \) to a MONS \( \{a_i\} \cup \{b_i\} \) of \( B \). It is not difficult to see that \( \{b_i\} \) is a MONS in \( A^{\perp_S} \land B \) and that therefore
\[
A \lor (A^{\perp_S} \land B) = \{a_i\}^{\perp_S} \lor \{b_i\}^{\perp_S} = \lor [a_i] \lor [b_i] = B.
\]
This completes the proof of Theorem 2.1.
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