IDEAL RELATED $K$-THEORY WITH COEFFICIENTS

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Abstract. In this paper, we define an invariant, which we believe should be the substitute for total $K$-theory in the case when there is one distinguished ideal. Moreover, some diagrams relating the new groups to the ordinary $K$-groups with coefficients are constructed. These diagrams will in most cases help to determine the new groups, and will in a companion paper be used to prove a universal multi-coefficient theorem for the one distinguished ideal case for a large class of algebras.

1. Introduction

To characterize the automorphism groups of purely infinite $C^*$-algebras up to, say, approximate unitary equivalence, one naturally looks at the work of Dadarlat and Loring, which gave such a characterization of the automorphism groups of certain stably finite $C^*$-algebras of real rank zero as a corollary to their Universal Multi-Coefficient Theorem (UMCT), cf. [DL96]. But even for nuclear, separable, purely infinite $C^*$-algebras with real rank zero, finitely generated $K$-theory and only one non-trivial ideal, there are substantial problems in doing so. The work of Rørdam (cf. [Rør97]) clearly indicates that the right invariant contains the associated six term exact sequence in $K$-theory, and the work of Dadarlat and Loring indicates that one should consider $K$-theory with coefficients in a similar way.

In the paper [ERR11], the authors gave a series of examples showing that the naïve approach — of combining the six term exact sequence with total $K$-theory — does not work. There are several obstructions given in the paper, and they can even be obtained using Cuntz-Krieger algebras of type (II) with exactly one non-trivial ideal.

With this as motivation, a new invariant — ideal-related $K$-theory with coefficients — is defined, and we will argue that it should be thought of as the substitute for total $K$-theory, when working with $C^*$-algebras with one specified ideal. It is easy to show that all the obstructions from the paper [ERR11] vanish when using this invariant. Furthermore, a lot of diagrams, which are part of the new invariant, are exhibited. These diagrams can — in many cases — be very useful when computing the new groups which go into the invariant. Also these diagrams are used in another paper (ERR) by the three authors, where they show a UMCT for $KK_\varepsilon$ for a class of $C^*$-algebras.
including all Cuntz-Krieger algebras of type (II) with one specified ideal — in this case, the invariant can actually be reduced quite a lot.

The paper is organized as follows. The first section contains an introduction. The second section sets up notation and conventions, and proves some results related to suspensions, cones, and mapping cone sequences. The third section recalls definitions and results about homology and cohomology theories for C∗-algebras. The fourth section exhibits relations between homology (and cohomology) theories and mapping cone sequences. The fifth section gives some concrete examples of the results developed in Section 4. In the sixth section, Ideal-related K-theory with coefficients is defined. In the seventh section, some important diagrams involved in the invariant are constructed. The eighth section contains the proof of Theorem 7.14 from Section 7. The ninth section contains some examples of the theory developed in this paper.

Parts of this paper have appeared in the second named author’s PhD-thesis [Res08].

2. Suspensions, cones, and mapping cones

This section is devoted to setting up notation, recalling basic concepts, and proving some results that will be needed later. Throughout the paper, $\mathbb{N}_0$, $\mathbb{N}$, and $\mathbb{N}_\geq 2$ will denote the set of non-negative integers, the set of positive integers, and the set of integers greater or equal to 2, respectively. Moreover, $M_n$ will denote the algebra of $n \times n$ matrices with complex entries.

**Definition 2.1.** Let $\mathcal{A}$ be a C∗-algebra. Define the suspension and the cone of $\mathcal{A}$ as

$$S\mathcal{A} = \{ f \in C([0,1], \mathcal{A}) \mid f(0) = 0, f(1) = 0 \},$$

$$C\mathcal{A} = \{ f \in C([0,1], \mathcal{A}) \mid f(0) = 0 \},$$

respectively.

**Remark 2.2.** For each C∗-algebra $\mathcal{A}$, there is a canonical short exact sequence:

$$S\mathcal{A} \hookrightarrow C\mathcal{A} \rightarrow \mathcal{A}. $$

It is well-known, that $S$ and $C$ are exact functors.

**Notation 2.3.** Whenever convenient, $C^*\mathcal{A}$, $SC\mathcal{A}$, $CS\mathcal{A}$, and $SS\mathcal{A}$ will be identified with subalgebras of $C([0,1]^2, \mathcal{A})$ by writing $f(x,y)$ for $(f(x))(y)$. In this way $ev_1(f)$ will become $f(1,-)$ while $(S ev_1)(f)$ or $(C ev_1)(f)$ will be $f(-,1)$.

The operation on $C([0,1]^2, \mathcal{A})$ that flips a function on $[0,1]^2$ along the diagonal will be denoted by flip, i.e., $\text{flip}(f)(x,y) = f(y,x)$.

**Definition 2.4.** Let $\mathcal{A}$ and $\mathcal{B}$ be C∗-algebras, and let $\phi: \mathcal{A} \rightarrow \mathcal{B}$ be a ∗-homomorphism. The mapping cone of $\phi$, $C_\phi$, is the pullback of the maps $\mathcal{A} \xrightarrow{\phi} \mathcal{B}$ and $C\mathcal{B} \xrightarrow{ev_1} \mathcal{B}$. As usual, the pullback can be realized as the restricted direct sum:

$$C_\phi = \mathcal{A} \oplus_{\phi, ev_1} C\mathcal{B} = \{ (x,y) \in \mathcal{A} \oplus C\mathcal{B} \mid \phi(x) = ev_1(y) = y(1) \}. $$

**Remark 2.5.** Let $\phi: \mathcal{A} \rightarrow \mathcal{B}$ be a ∗-homomorphism between C∗-algebras. Then there is a canonical short exact sequence

$$S\mathcal{B} \hookrightarrow C_\phi \rightarrow \mathcal{A}$$

called the mapping cone sequence. This sequence is natural in $\mathcal{A}$ and $\mathcal{B}$, i.e., if there exists a commuting diagram:

$$\begin{align*}
\mathcal{A}_1 & \xrightarrow{\phi_1} \mathcal{B}_1 \\
\mathcal{A}_2 & \xrightarrow{\phi_2} \mathcal{B}_2 \\
\mathcal{A} & \xrightarrow{f} \mathcal{A}_1 \\
& \mathcal{A} \xrightarrow{g} \mathcal{B}_2
\end{align*}$$

1Note that some authors place the algebra at 0 rather than 1 — e.g. Blackadar in [Bla08]
then there is a (canonical) $*$-homomorphism $\omega: C_{\phi_1} \to C_{\phi_2}$ making the diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{\omega} & S\mathcal{A}_1 \\
\downarrow & & \downarrow \omega \\
0 & \xrightarrow{\omega} & S\mathcal{A}_2 \\
\end{array}
\]

commutative (cf. [Bla98 Section 19.4]). Actually, a concrete description of $\omega$ is as follows: $\omega(a, h) = (f(a), g \circ h)$ for all $(a, h) \in \mathcal{A}_1 \oplus_{\phi_1, ev_1} C\mathcal{B}_1 = C_{\phi_1}$.

**Remark 2.6.** The mapping cone sequence of the identity homomorphism $id_\mathcal{A}$ is the canonical sequence $S\mathcal{A} \to C\mathcal{A} \to \mathcal{A}$. For each $*$-homomorphism $\phi: \mathcal{A} \to \mathcal{B}$ between $C^*$-algebras, there exists a canonical $*$-isomorphisms $S$-flip from $SC_\phi$ to $C_{S\phi}$, and $C$-flip from $CC_\phi$ to $C_{CC\phi}$, given by

\[
SC_\phi = S(\mathcal{A} \oplus_{\phi, ev_1} C\mathcal{B}) \ni (x, y) \mapsto (x, flip(y)) \in S\mathcal{A} \oplus_{S\phi, ev_1} CS\mathcal{B} = CS_\phi,
\]

\[
CC_\phi = C(\mathcal{A} \oplus_{\phi, ev_1} C\mathcal{B}) \ni (x, y) \mapsto (x, flip(y)) \in C\mathcal{A} \oplus_{C\phi, ev_1} CCC\mathcal{B} = CC_{\phi},
\]

respectively. See Definition 2.10 and Lemma 2.11 for more on these isomorphisms.

**Definition 2.7.** Define functors $mc$, $S$ and $C$ on the category of all extensions of $C^*$-algebras (with the morphisms being triples of $*$-homomorphisms making the obvious diagram commutative) as follows. For an extension $e: \mathcal{A}_0 \subseteq \mathcal{A}_1 \to \mathcal{A}_2$ set

\[
mc(e): S\mathcal{A}_2 \xrightarrow{mc(e)} C\mathcal{A}_1 \xrightarrow{\pi} \mathcal{A}_1,
\]

\[
S(e) = S: \mathcal{A}_0 \xrightarrow{\iota} \mathcal{A}_1 \xrightarrow{\pi} \mathcal{A}_2.
\]

\[
C(e) = C: \mathcal{A}_0 \xrightarrow{\iota} \mathcal{A}_1 \xrightarrow{\pi} \mathcal{A}_2.
\]

For a morphism $\phi = (\phi_0, \phi_1, \phi_2)$ from $e$ to $e'$, let $mc(\phi)$ be the morphism $(S\phi_2, \omega, \phi_1)$ defined using the naturality of the mapping cone construction (see above), let $S(\phi) = S\phi$ be the morphism $(S\phi_0, S\phi_1, S\phi_2)$, and we let $C(\phi) = C\phi$ be the morphism $(C\phi_0, C\phi_1, C\phi_2)$.

It is easy to verify that these are functors. Moreover, one easily verifies, that they are exact (i.e., they send short exact sequences of extensions to short exact sequences of extensions).

**Definition 2.8.** Let $e: \mathcal{A}_0 \subseteq \mathcal{A}_1 \xrightarrow{\pi} \mathcal{A}_2$ be an extension of $C^*$-algebras. Then construct two new extensions, $i(e)$ and $q(e)$, from $e$ as follows. Let $i(e)$ denote the extension $\mathcal{A}_0 = \mathcal{A}_0 \to 0$, and let $q(e)$ denote the extension $0 \to \mathcal{A}_2 = \mathcal{A}_2$. Then there exists a canonical short exact sequence of extensions:

\[
i(e) \xrightarrow{i} e \xrightarrow{q} q(e).
\]

**Remark 2.9.** Note that if $e: \mathcal{A}_0 \subseteq \mathcal{A}_1 \xrightarrow{\pi} \mathcal{A}_2$ is an extension of $C^*$-algebras, then there exists a commuting diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{i} & S\mathcal{A}_2 \xrightarrow{\iota} C\mathcal{A}_2 \\
\downarrow & & \downarrow \pi \\
0 & \xrightarrow{i} & \mathcal{A}_0 \xrightarrow{\iota} \mathcal{A}_1 \xrightarrow{\pi} \mathcal{A}_2 \\
\end{array}
\]

with short exact rows and columns. The map $f_x: \mathcal{A}_0 \to C_x$ induces isomorphism on the level of $K$-theory (actually, this holds more generally for additive, homotopy-invariant, half-exact functors, cf. [Bla98 Proposition 21.4.1]). Actually, this diagram is nothing but the short exact sequence $mc(i(e)) \xrightarrow{mc(i_e)} mc(e) \xrightarrow{mc(q_e)} mc(q(e))$ induced by applying the functor $mc$ to the short exact sequence $i(e) \xrightarrow{i} e \xrightarrow{q} q(e)$. 
Definition 2.10. Let there be given an extension $e: A_0 \to A_1 \to A_2$ of $C^*$-algebras. Form the extensions $S mc(e)$, $mc(S(e))$, $C mc(e)$, and $mc(C(e))$ as above. Then define morphisms $\theta_e$ from $S mc(e)$ to $mc(Se)$ and $\eta_e$ from $C mc(e)$ to $mc(Ke)$ as follows:

$$
\begin{align*}
S mc(e): & \quad 0 \to S A_2 \to S A_1 \to 0 \\
mc(Se): & \quad 0 \to S A_2 \to S A_1 \to 0,
\end{align*}

where the $*$-homomorphisms $SC_\pi \to C S_\pi$ and $CC_\pi \to C C_\pi$ are the canonical isomorphisms from Remark 2.6.

Lemma 2.11. The above morphisms, $\theta_e$ and $\eta_e$, are functorial, i.e., they implement isomorphisms from the functor $S \circ mc$ to the functor $mc \circ S$ and from the functor $C \circ mc$ to the functor $mc \circ C$, respectively.

Proof. This is a long, straightforward verification.

Lemma 2.12. Let $e$ be an extension of $C^*$-algebras. Then there exists an isomorphism of short exact sequences of extensions as follows:

$$
\begin{array}{ccc}
0 & \to & S mc(e) \\
\downarrow \theta_e & & \downarrow \eta_e \\
0 & \to & mc(Se)
\end{array}

\begin{array}{ccc}
0 & \to & C mc(e) \\
\downarrow \phi_e & & \downarrow \psi_e \\
0 & \to & mc(Ke)
\end{array}

\begin{array}{ccc}
0 & \to & mc(e)
\end{array}

Proof. This is a straightforward verification.

Lemma 2.13. Let there be given a commutative diagram

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\phi_1} & \mathcal{Y}_1 \\
\phi_2 & & \psi_1 \\
\mathcal{Y}_2 & \xleftarrow{\psi_2} & \mathcal{Y}
\end{array}
$$

of $C^*$-algebras and $*$-homomorphisms. Then there are canonically induced $*$-homomorphisms $C_{\phi_1} \to C_{\phi_2}$ and $C_{\phi_2} \to C_{\psi_1}$. The mapping cones $C_{\phi_1} \to C_{\phi_2}$ and $C_{\phi_2} \to C_{\psi_1}$ are canonically isomorphic to

$$
\begin{cases}
(x, f_1, f_2, h) \in \mathcal{X} \oplus \mathcal{C} \mathcal{Y}_1 \oplus \mathcal{C} \mathcal{Y}_2 \oplus \mathcal{C} \mathcal{Y} \\
\phi_1(x) = f_1(1), \quad \psi_1 \circ f_1(-) = h(1, -), \\
\phi_2(x) = f_2(1), \quad \psi_2 \circ f_2(-) = h(-, 1)
\end{cases}

\begin{cases}
(x, f_2, f_1, h) \in \mathcal{X} \oplus \mathcal{C} \mathcal{Y}_2 \oplus \mathcal{C} \mathcal{Y}_1 \oplus \mathcal{C} \mathcal{Y} \\
\phi_1(x) = f_1(1), \quad \psi_1 \circ f_1(-) = h(-, 1), \\
\phi_2(x) = f_2(1), \quad \psi_2 \circ f_2(-) = h(1, -)
\end{cases}
$$

respectively. So $(x, f_1, f_2, h) \mapsto (x, f_2, f_1, \text{flip}(h))$ is an isomorphism from $C_{\phi_1} \to C_{\phi_2}$ to $C_{\phi_2} \to C_{\psi_1}$.

Proof. This is straightforward to check by writing out the mapping cones as restricted direct sums. Note that we only need to check the first statement, since the second follows by symmetry (by interchanging 1 and 2).

For a morphism $\phi = (\phi_0, \phi_1, \phi_2)$ between extensions of $C^*$-algebras, let $C_\phi$ denote the object $C_{\phi_0} \to C_{\phi_1} \to C_{\phi_2}$ (cf. also [Bon02 Definition 3.4.1]).
Lemma 2.14. Let there be given a commuting diagram

with the rows and columns being short exact sequences of C*-algebras — we will write this short as $e_\alpha \xrightarrow{x} e_\beta \xrightarrow{y} e_\gamma$. Then there exists an isomorphism $\xi_y$ from $C_{mc(y)}$ to $mc(C_y)$ given as follows:

$$\begin{align*}
C_{mc(y)}: & 0 \longrightarrow C_{S_{y_2}} \quad \xi_y \quad \cong \quad \xi_y \quad \cong \quad mc(C_y): 0 \longrightarrow S_{y_2} \longrightarrow C_{C_{y_1} \rightarrow C_{y_2}} \longrightarrow C_{y_1} \longrightarrow 0
\end{align*}$$

where the isomorphism from $C_{C_{y_1} \rightarrow C_{y_2}}$ to $C_{C_{y_1} \rightarrow C_{y_2}}$ is given as in the above lemma. Moreover, the map given by the matrix

$$\begin{pmatrix}
0 & \theta_{\epsilon_\alpha} & \theta_{\epsilon_\gamma} \\
\text{id} & \xi_y & \eta_{\epsilon_\gamma}
\end{pmatrix}$$

between the standard diagrams

makes everything commutative.

Proof. Using the above, we have that $C_{C_{y_1} \rightarrow C_{y_2}}$ is isomorphic to

$$\{(x, f_2, f_1, h) \in B_1 \oplus C B_2 \oplus C C_1 \oplus C C_2 \mid \begin{align*}
y_1(x) &= f_1(1), & \pi_\epsilon \circ f_1(-) &= h(-, 1), \\
\pi_\Theta(x) &= f_2(1), & \pi_\Theta(x) &= f_2(-) &= h(1, -)
\end{align*}\}$$

and $C_{C_{y_1} \rightarrow C_{y_2}}$ is isomorphic to

$$\{(x, f_1, f_2, h) \in B_1 \oplus C C_1 \oplus C B_2 \oplus C C_2 \mid \begin{align*}
y_1(x) &= f_1(1), & \pi_\epsilon \circ f_1(-) &= h(1, -), \\
\pi_\Theta(x) &= f_2(1), & \pi_\Theta(x) &= f_2(-) &= h(-, 1)
\end{align*}\}$$

and, moreover,

$$\begin{align*}
C_{S_{y_2}} &= \{(f_2, h) \in S B_2 \oplus C C_2 \mid y_2 \circ f_2(-) = h(1, -)\}, \\
SC_{y_2} &= \{(f_2, h) \in S B_2 \oplus S C C_2 \mid y_2 \circ f_2(-) = h(-, 1)\}, \\
C_{y_1} &= \{(x, f_1) \in B_1 \oplus C C_1 \mid y_1(x) = f_1(1)\}.
\end{align*}$$

Using these identifications, we compute the extensions:

$$\begin{align*}
C_{mc(y)}: & 0 \longrightarrow C_{S_{y_2}} \quad (f_2, h) \rightarrow (0, f_2, 0, h) \quad \cong \quad C_{C_{y_1} \rightarrow C_{y_2}} \quad (x, f_2, f_1, h) \rightarrow (x, f_1) \quad \cong \quad C_{y_1} \longrightarrow 0, \\
mc(C_y): & 0 \longrightarrow SC_{y_2} \quad (f_2, h) \rightarrow (0, f_2, 0, h) \quad \cong \quad C_{C_{y_1} \rightarrow C_{y_2}} \quad (x, f_2, f_1, h) \rightarrow (x, f_1) \quad \cong \quad C_{y_1} \longrightarrow 0.
\end{align*}$$

Now it is routine to check that the given diagram commutes.
Second part: The above results show that every square which does not involve $C_{mc(\gamma)}$ and $mc(C_f)$ commutes. The long and straightforward proof of the commutativity of the remaining four squares of morphisms of extension is left to the reader.

3. HOMOLOGY AND COHOMOLOGY THEORIES FOR $C^*$-ALGEBRAS

We recall some of the definitions and results about homology and cohomology theories on $C^*$-algebras. The reader is referred to [Bla98] Chapters 21 and 22 (these two chapters are primarily due to Cuntz, Higson, Rosenberg, and Schochet — see the monograph for further references).

**Definition 3.1.** Let $S$ be a subcategory of the category of all $C^*$-algebras, which is closed under quotients, extensions, and closed under suspension in the sense that if $A$ is an object of $S$, then the suspension $SA$ of $A$ is also an object of $S$, $S\phi$ is a morphism in $S$ whenever $\phi$ is, $SC$ is an object of $S$ and every $*$-homomorphism from $SC$ to every object of $S$ is a morphism in $S$.

Let $Ab$ denote the category of abelian groups. We will consider functors $F$ from $S$ to $Ab$. Such functors may or may not satisfy each of the axioms Homotopy-invariance (H), Stability (S), $\sigma$-additivity ($A$), completely additive, additive, Half-exactness (HX) defined in [Bla98] Chapters 21 and 22.

- **Theorem 3.3.** Let $F$ be an additive functor from $S$ to $Ab$ satisfying (H) and (HX). If $F$ is covariant, then $(F_n)_{n=0}^\infty$ is a homology theory. If $F$ is contravariant, then $(F^n)_{n=0}^\infty$ is a cohomology theory.

**Corollary 3.4.** If $F$ is an additive functor from $S$ to $Ab$ satisfying (H) and (HX), then $F$ is split-exact, i.e., $F$ sends split-exact sequences of $S$ to split-exact sequences of abelian groups.

**Proof.** Let $A_0 \rightarrow A_1 \rightarrow A_2$ be an arbitrary $S$-sequence. Then $F(A_0) \rightarrow F(A_1) \rightarrow F(A_2)$ is an exact sequence of $F(A_1)$-algebras, and assume that $F$ is covariant. It is clear that $F(\pi)$ and $F(S\pi)$ are surjective (since $F$ and $F \circ S$ are functors). From preceding theorem it follows that $\partial_1 = 0$, so $F\pi$ is injective. The proof in the contravariant case is dual.

**Theorem 3.5.** Let $F$ be an additive functor from $S$ to $Ab$ satisfying (H), (S), and (HX). Then $F$ is naturally isomorphic to $F \circ S^2$.

**Definition 3.6.** Let $F$ be an additive functor from $S$ to $Ab$ satisfying (H), (S), and (HX), and let $\beta_2: F(A) \rightarrow F(S^2A)$ denote the natural isomorphism. Then for each short exact sequence of $C^*$-algebras we make the following definition. If $F$ is covariant, then define $\partial_0: F(A_2) \rightarrow F(SA_0)$ as the composition of the homomorphisms

- $F(A_2) \xrightarrow{\beta_2} F(S^2A_2) \xrightarrow{\partial_1} F(SA_0)$.

If $F$ is contravariant, then define $\widetilde{\partial}_1: F(SA_0) \rightarrow F(A_2)$ as the composition of the homomorphisms

- $F(SA_0) \xrightarrow{\partial_1} F(S^2A_2) \xrightarrow{\beta_2^{-1}} F(A_2)$.

Note that $S^n$ denotes the composition of $S$ with itself $n$ times, while the superscript in $F^n$ indicates that this is some kind of $n$th cohomology.
So with each such short exact sequence we have associated a cyclic six term exact sequence

\[
\begin{align*}
\text{F}(\mathcal{A}_0) & \xrightarrow{F(\iota)} \text{F}(\mathcal{A}_1) \xrightarrow{F(\pi)} \text{F}(\mathcal{A}_2) \\
\partial_0 & \downarrow \\
\text{F}(\mathcal{S}\mathcal{A}_2) & \xrightarrow{F(\hat{\pi})} \text{F}(\mathcal{S}\mathcal{A}_1) \xrightarrow{F(\hat{\iota})} \text{F}(\mathcal{S}\mathcal{A}_0) \\
\partial_1 & \uparrow \\
\text{F}(\mathcal{S}\mathcal{A}_0) & \xrightarrow{F(S\pi)} \text{F}(\mathcal{S}\mathcal{A}_1) \xrightarrow{F(S\iota)} \text{F}(\mathcal{S}\mathcal{A}_2)
\end{align*}
\]

which is natural with respect to morphisms of short exact sequences of C*-algebras. We will occasionally misuse the notation and write \(\partial^1\) instead of \(\hat{\partial}^1\) (which should not cause any confusions).

**Remark 3.7.** While it is obvious how to generalize homotopy-invariance, stability, additivity, and split-exactness for a functor from \(S\) to an additive category \(A\), it is not obvious how to generalize half-exactness.

In Section 21.4 in [Bla98], Blackadar defines half-exactness for such functors (i.e., \(\text{Hom}_{A}(X, F(-))\) and \(\text{Hom}_{A}(F(-), X)\) should be half-exact for all objects \(X\)). It is natural to ask whether this extends the original definition, and the answer is no. This is seen by applying \(\text{Hom}_{Z}(\mathbb{Z}_3, K_1(-))\) to the short exact sequence \(SM_3 \hookrightarrow I_3 \rightarrow C\) (cf. Definition [BL]). On the other hand, for the category of modules over a unital ring, \(\text{Hom}_{R}(R, M)\) is naturally isomorphic to \(M\) — so this property is stronger than the ordinary half-exactness. To avoid confusions, we will not use this terminology.

### 4. (Co-)Homology theories and mapping cone sequences

In this section we show exactly how the cyclic six term exact sequence of the mapping cone sequence for an extension of C*-algebras is related to the cyclic six term exact sequence of the original extension. First we will need the following lemma, which Bonkat uses a version of in the proof of [Bon02, Lemma 7.3.1]. The proof given here is much more elementary.

**Lemma 4.1.** Let \(F_0\) and \(F_1\) be covariant additive functors from the category \(S\) to the category \(\text{Ab}\), which have the properties \((H), (S), \text{and (HX)}\). Assume that \(\partial_0^-\) and \(\partial_1^-\) are boundary maps making \((F_1, \partial_1^-)_{i=0}^1\) into a cyclic homology theory on \(S\). Let there also be given a commuting diagram

\[
\begin{array}{ccc}
\mathcal{A}_0 & \xrightarrow{\iota} & \mathcal{A}_1 \\
\downarrow & & \downarrow \\
\mathcal{B}_0 & \xrightarrow{\iota} & \mathcal{B}_1 \\
\downarrow & & \downarrow \\
\mathcal{C}_0 & \xrightarrow{\iota} & \mathcal{C}_1 \\
\end{array}
\]

with the rows and columns being short exact sequences of C*-algebras. Let \(e_0, e_1, e_2\) denote the three horizontal extensions, while \(e_0, e_1\) and \(e_2\) denote the three vertical extensions. Then there exists
a diagram

\[
\begin{array}{cccccccc}
\partial_1^0 & \partial_0^1 & \partial_1^2 & \partial_0^3 & \partial_0^4 & \partial_1^5 & \partial_0^6 & \partial_1^7 \\
F_0(\mathcal{A}_1) & F_0(\mathcal{A}_2) & F_1(\mathcal{A}_0) & F_1(\mathcal{A}_1) & F_1(\mathcal{A}_2) & F_2(\mathcal{A}_0) & F_2(\mathcal{A}_1) & F_2(\mathcal{A}_2) \\
\partial_1^m & \partial_0^m & \partial_1^m & \partial_0^m & \partial_0^m & \partial_1^m & \partial_0^m & \partial_1^m \\
F_0(\mathcal{B}_1) & F_0(\mathcal{B}_2) & F_1(\mathcal{B}_0) & F_1(\mathcal{B}_1) & F_1(\mathcal{B}_2) & F_2(\mathcal{B}_0) & F_2(\mathcal{B}_1) & F_2(\mathcal{B}_2) \\
\partial_1^\varepsilon & \partial_0^\varepsilon & \partial_1^\varepsilon & \partial_0^\varepsilon & \partial_0^\varepsilon & \partial_1^\varepsilon & \partial_0^\varepsilon & \partial_1^\varepsilon \\
F_0(\mathcal{C}_0) & F_0(\mathcal{C}_1) & F_1(\mathcal{C}_0) & F_1(\mathcal{C}_1) & F_1(\mathcal{C}_2) & F_2(\mathcal{C}_0) & F_2(\mathcal{C}_1) & F_2(\mathcal{C}_2) \\
\partial_1^\alpha & \partial_0^\alpha & \partial_1^\alpha & \partial_0^\alpha & \partial_0^\alpha & \partial_1^\alpha & \partial_0^\alpha & \partial_1^\alpha \\
F_0(\mathcal{A}_0) & F_0(\mathcal{B}_0) & F_1(\mathcal{A}_1) & F_1(\mathcal{B}_1) & F_1(\mathcal{A}_2) & F_2(\mathcal{A}_0) & F_2(\mathcal{B}_1) & F_2(\mathcal{A}_2) \\
\partial_1^\varepsilon & \partial_0^\varepsilon & \partial_1^\varepsilon & \partial_0^\varepsilon & \partial_0^\varepsilon & \partial_1^\varepsilon & \partial_0^\varepsilon & \partial_1^\varepsilon \\
F_1(\mathcal{C}_0) & F_1(\mathcal{C}_1) & F_2(\mathcal{C}_0) & F_2(\mathcal{C}_1) & F_2(\mathcal{C}_2) & F_3(\mathcal{C}_0) & F_3(\mathcal{C}_1) & F_3(\mathcal{C}_2) \\
\end{array}
\]

with the cyclic six term exact sequence both horizontally and vertically. The two squares

\[
\begin{array}{ccc}
F_0(\mathcal{E}_2) & F_1(\mathcal{E}_0) & F_0(\mathcal{E}_0) \\
\partial_0^\varepsilon & \partial_1^\alpha & \partial_0^\alpha \\
F_1(\mathcal{A}_2) & F_0(\mathcal{A}_0) & F_1(\mathcal{A}_0) \\
\end{array}
\]

antcommute, while all the other squares (in the big diagram) commute.

If \( F \) is contravariant instead, the dual statement holds.

**Proof.** That all the other squares commute, is evident (using that \( F_0 \) and \( F_1 \) are functors and that the maps \( \partial_0 \) and \( \partial_1 \) are natural). Let \( D \) denote the pullback of \( \mathcal{E}_2 \) along \( \mathcal{B}_2 \to \mathcal{E}_2 \) and \( \mathcal{A}_1 \to \mathcal{E}_2 \). Then there exist short exact sequences

\[
\begin{align*}
\varepsilon_{\text{sum}} : & \quad \mathcal{A}_0 \rightarrowtail \mathcal{A}_1 + \mathcal{B}_0 \twoheadrightarrow \mathcal{A}_2 \oplus \mathcal{E}_0 \\
\varepsilon_{\text{pullback}} : & \quad \mathcal{A}_2 \oplus \mathcal{E}_0 \twoheadrightarrow \mathbf{D} \rightarrowtail \mathcal{E}_2,
\end{align*}
\]

where we identify \( \mathcal{A}_1 \) and \( \mathcal{B}_0 \) with their images inside \( \mathcal{B}_1 \). Split-exactness of \( F_0 \) and \( F_1 \), cf. Corollary 3.4 and naturality of \( \partial_0 \) and \( \partial_1 \) together with the morphisms of extensions

\[
\begin{array}{ccc}
\mathcal{A}_0 & \mathcal{B}_0 & \mathcal{E}_0 \\
\mathbf{D} & \mathcal{E}_2 \\
\mathcal{A}_2 \oplus \mathcal{E}_0 & \mathbf{D} & \mathcal{E}_2 \\
\end{array}
\]

give that the map \( \partial_1^\text{sum} \partial_0^\text{pullback} : F_j(\mathcal{E}_2) \to F_j(\mathcal{A}_0) \) is exactly \( \partial_1^\varepsilon - \partial_0^\alpha + \partial_0^\alpha \partial_1^\varepsilon \), for \( j = 0, 1 \). But it turns out that \( \partial_1^\varepsilon - \partial_0^\alpha \partial_1^\varepsilon = 0 \) proving anticommutativity. For we have the following commuting
diagram with short exact rows and columns

\[
\begin{array}{c}
\mathcal{A}_0 \to \mathcal{A}_1 + \mathcal{B}_0 \to \mathcal{A}_2 \oplus \mathcal{C}_0 \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{A}_1 \to \mathcal{B}_1 \to \mathcal{D} \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{C}_2 \to \mathcal{E}_2,
\end{array}
\]

so the map \( \partial_j \) factors through \( F(1_j(\mathcal{A}_1 + \mathcal{B}_0)) \to F(1_j(\mathcal{A}_2 \oplus \mathcal{C}_0)) \).

The proof in the case that \( F \) is contravariant is dual. \( \square \)

**Lemma 4.2.** Let \( F \) be an additive functor from the category \( S \) to the category \( \text{Ab} \), which has the properties \((H), (S), \) and \((HX)\). Let \( \mathfrak{A} \) be an arbitrary \( C^\ast \)-algebra. The standard cyclic six term exact sequence\(^3\) associated with \( \mathcal{S} \mathfrak{A} \hookrightarrow \mathcal{C} \mathfrak{A} \to \mathfrak{A} \) is the sequence

\[
\begin{array}{ccc}
F(\mathcal{S} \mathfrak{A}) & \longrightarrow & 0 \\
\downarrow \text{id} & \cong & \downarrow \beta \mathfrak{A} \\
F(\mathcal{S} \mathfrak{A}) & \longrightarrow & 0 \\
\end{array}
\]

in the covariant case, and the sequence

\[
\begin{array}{ccc}
F(\mathfrak{A}) & \longrightarrow & 0 \\
\cong & \downarrow \beta^{-1} \mathfrak{A} \\
F(\mathcal{S} \mathfrak{A}) & \longrightarrow & 0 \\
\end{array}
\]

in the contravariant case.

**Proof.** Assume that \( F \) is covariant. Since the cone, \( \mathcal{C} \mathfrak{A} \), of \( \mathfrak{A} \) is homotopy equivalent to the zero \( C^\ast \)-algebra, \( F(\mathcal{C} \mathfrak{A}) \cong F(\mathcal{S} \mathfrak{A}) \cong 0 \) (cf. [RLL, Example 4.1.5]). We have the commutative diagram

\[
\begin{array}{c}
\mathcal{S} \mathfrak{A} \quad \quad \mathcal{S} \mathfrak{A} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\mathcal{C} \mathfrak{A} \quad \quad \mathcal{C} \mathfrak{A} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\mathfrak{A} \quad \quad \mathfrak{A}
\end{array}
\]

with short exact rows and columns. Note that \( \mathcal{C} \mathfrak{A} \) is realized as \( \{(x, y) \in \mathcal{C} \mathfrak{A} \oplus \mathcal{C} \mathfrak{A} \mid x(1) = y(1)\} \). Using this picture there exists a \(*\)-homomorphism \( \varphi : \mathcal{C} \mathfrak{A} \ni x \mapsto (x, x) \in \mathcal{C} \pi \). Note that the composed \(*\)-homomorphism \( \varphi \circ \iota \) is just \( f + \tau_{mc} \). Since \( F(\mathcal{C} \mathfrak{A}) = 0 \), we must have \( F(\varphi \circ \iota) = 0 \). Using the split-exactness of \( F \) (cf. Corollary \[3\]), we get a canonical identification of \( F(\mathcal{S} \mathfrak{A} \oplus \mathcal{S} \mathfrak{A}) \) with \( F(\mathcal{S} \mathfrak{A}) \oplus F(\mathcal{S} \mathfrak{A}) \). Under this identification, we get

\[
\begin{array}{ccc}
F(\mathcal{S} \mathfrak{A}) & \longrightarrow & F(\mathcal{S} \mathfrak{A} \oplus \mathcal{S} \mathfrak{A}) \\
\downarrow \quad \downarrow \quad \downarrow \\
F(\mathcal{C} \pi) & \longrightarrow & F(\mathcal{S} \mathfrak{A}) \oplus F(\mathcal{S} \mathfrak{A})
\end{array}
\]

Consequently,

\[
F(f) + F(\tau_{mc}) = F(f + \tau_{mc}) = F(\varphi \circ \iota) = 0,
\]

and hence \( F(f) = -F(\tau_{mc}) \). Therefore, we have \( \partial_1 = F(f)^{-1} \circ F(\tau_{mc}) = -\text{id} \).

\(^3\)as defined in Definitions \[3\] and \[\[6\].
The map \( \partial_0 : F(\mathfrak{A}) \to F(S^2 \mathfrak{A}) \) is the composition of the maps
\[
F(\mathfrak{A}) \xrightarrow{\partial_0} F(S^2 \mathfrak{A}) \xrightarrow{\beta_3} F(S^2 \mathfrak{A}),
\]
where \( \partial_2 = F(Sf)^{-1} \circ F(\varepsilon_{mc}) \). It is easy to see that the matrix
\[
\begin{pmatrix}
0 & \text{flip} & \text{flip} \\
\text{flip} & (\text{flip}, \text{flip}) & \text{flip} \\
\text{flip} & \text{flip} & \text{id}
\end{pmatrix}
\]
implements a map between the diagrams

![Diagram](https://via.placeholder.com/150)

such that everything commutes. So by the above, we have \( \partial_0 = -\beta_3 \).

The proof when \( F \) is contravariant is dual. \( \blacksquare \)

**Proposition 4.3.** Let \( F \) be an additive functor from \( S \) to the category \( \text{Ab} \), which has the properties (H), (S), and (HX). Let there be given an extension
\[
e : \mathfrak{A}_0 \xrightarrow{\iota} \mathfrak{A}_1 \xrightarrow{\pi} \mathfrak{A}_2.
\]
Then there exist isomorphisms of cyclic six term exact sequences as follows:

![Diagram](https://via.placeholder.com/150)

in the covariant case, and

![Diagram](https://via.placeholder.com/150)

in the contravariant case.

**Proof.** Assume that \( F \) is covariant. The diagram
\[
\begin{array}{ccc}
\mathfrak{A}_1 & \xrightarrow{\pi} & \mathfrak{A}_2 \\
\downarrow & & \downarrow \\
\mathfrak{A}_2 & \xrightarrow{\pi} & \mathfrak{A}_2
\end{array}
\]
induces the morphism of extensions
\[
\begin{array}{ccccccc}
0 & \xrightarrow{\cdot} & S\mathfrak{A}_2 & \xrightarrow{C_\pi} & \mathfrak{A}_1 & \xrightarrow{\iota} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{\cdot} & S\mathfrak{A}_2 & \xrightarrow{C_{id_{\mathfrak{A}_2}}} & \mathfrak{A}_2 & \xrightarrow{\pi} & 0.
\end{array}
\]
Note that $C_{\tilde{\alpha}_2}$ is canonically isomorphic to $C\mathfrak{a}_2$. According to Lemma 4.2, this induces a morphism between cyclic six term exact sequences:

$$\begin{array}{cccccccc}
\partial^{mc}_0 & F(S\mathfrak{a}_2) & F(\pi_{s_0}) & F(\mathfrak{a}_2) & F(\pi_{s_1}) & F(\mathfrak{a}_1) & F(S\mathfrak{a}_1) & F(\pi) \\
\cong & \text{id} & 0 & F(\pi) & 0 & F(\pi) & \text{id} & 0 \\
\end{array}$$

This takes care of the commutativity of two of the six squares.

Commutativity of

$$\begin{array}{cccc}
F(\mathfrak{a}_0) & F(\pi_{s_0}) & F(\mathfrak{a}_1) & F(\pi_{s_1}) \\
\cong & F(\pi_0) & F(\pi_1) & F(\pi_2) \\
\end{array} \quad \begin{array}{cccc}
F(S\mathfrak{a}_0) & F(\pi_{s_0}) & F(S\mathfrak{a}_1) & F(\pi_{s_1}) \\
\cong & F(S\pi_0) & F(S\pi_1) & F(S\pi_2) \\
\end{array}$$

follows directly from the $3 \times 3$-diagram above. Now we only need to check commutativity of

$$\begin{array}{cccc}
F(\mathfrak{a}_0) & \partial^{mc}_0 & F(\mathfrak{a}_1) & \partial^{mc}_0 \\
\cong & \beta_{\alpha_2} & \cong & \beta_{\alpha_2} \\
\end{array} \quad \begin{array}{cccc}
F(S\mathfrak{a}_0) & \partial^{mc}_0 & F(S\mathfrak{a}_1) & \partial^{mc}_0 \\
\cong & \beta_{\alpha_2} & \cong & \beta_{\alpha_2} \\
\end{array}$$

Since $C_\pi$ is the pullback, we get a canonical map $C\mathfrak{a}_1 \rightarrow C_\pi$ and commuting diagrams

with exact rows and columns. Using Lemma 4.1 and Lemma 4.2, these diagrams give rise to the following commutative diagrams

$$\begin{array}{cccc}
F(\mathfrak{a}_0) & \partial^{mc}_0 & F(\mathfrak{a}_1) & \partial^{mc}_0 \\
\cong & \beta_{\alpha_2} & \cong & \beta_{\alpha_2} \\
\end{array} \quad \begin{array}{cccc}
F(S\mathfrak{a}_0) & \partial^{mc}_0 & F(S\mathfrak{a}_1) & \partial^{mc}_0 \\
\cong & \beta_{\alpha_2} & \cong & \beta_{\alpha_2} \\
\end{array}$$

where $e'$ denotes the extension $S\mathfrak{a}_0 \hookrightarrow C\mathfrak{a}_1 \rightarrow C_\pi$. Consequently,

$$F(t_{mc}) = (\partial^{mc}_0)^{-1} \circ \partial^{mc}_0 = -F(f_e) \circ (\beta_{\alpha_2}^{-1}) \circ \partial^{mc}_0$$

$$= F(f_e) \circ \beta_{\alpha_2}^{-1} \circ \beta_{\alpha_0} \circ \partial^{mc}_0 = F(f_e) \circ \partial^{mc}_0,$$

$$F(S_{mc}) = (\partial^{mc}_0)^{-1} \circ \partial^{mc}_0 = -F(Sf_e) \circ \partial^{mc}_0$$

$$= F(Sf_e) \circ \partial^{mc}_0 \circ \beta_{\alpha_2}^{-1}.$$

The proof in the contravariant case is dual. \[ \blacksquare \]
Corollary 4.4. Let $F$ be an additive functor from $S$ to the category $\text{Ab}$, which has the properties (H), (S), and (HX). Let there be given a $\ast$-homomorphism

$$\phi : A \to B$$

from a $C^\ast$-algebra $A$ to a $C^\ast$-algebra $B$, and let

$$e : S \overrightarrow{\pi_\ast} C \overrightarrow{\pi_\ast} A$$

denote the corresponding mapping cone sequence. Then there exist isomorphisms of cyclic six term exact sequences as follows:

\[
\begin{array}{cccccccccc}
& & & & F(\phi) & & & & F(\pi_\ast) & & \\
& & & & F(S\phi) & & & & F(C) & & \\
& & & & \partial_0' & & \phi & & \eta_0' & & \\
& & & & F(C) & & F(\pi_\ast) & & F(S\phi) & & \\
& & & & F(S\phi) & & F(C) & & F(\pi_\ast) & & \\
& & & & \partial_0' & & \phi & & \eta_0' & & \\
& & & & F(\phi) & & F(\pi_\ast) & & F(S\phi) & & \\
& & & & F(F(S\phi)\circ\beta_\ast) & & F(F(C)\circ\beta_\ast) & & F(F(S\phi)\circ\beta_\ast) & & F(F(\phi)) \\
& & & & F(S\phi) & & F(C) & & F(\pi_\ast) & & \\
& & & & \partial_0' & & \phi & & \eta_0' & & \\
& & & & F(\phi) & & F(\pi_\ast) & & F(S\phi) & & \\
& & & & \partial_0' & & \phi & & \eta_0' & & \\
& \cong & & \cong & & \cong & & \cong & & \cong & \\
& & & & F(\phi) & & F(\pi_\ast) & & F(S\phi) & & \\
& & & & F(S\phi) & & F(C) & & F(\pi_\ast) & & \\
& & & & \partial_0' & & \phi & & \eta_0' & & \\
& & & & F(\phi) & & F(\pi_\ast) & & F(S\phi) & & \\
& & & & \partial_0' & & \phi & & \eta_0' & & \\
\end{array}
\]

in the covariant case, and

\[
\begin{array}{cccccccccc}
& & & & F(\phi) & & & & F(\pi_\ast) & & \\
& & & & F(S\phi) & & & & F(C) & & \\
& & & & \partial_1 & & \phi & & \eta_1 & & \\
& & & & F(C) & & F(\pi_\ast) & & F(S\phi) & & \\
& & & & F(S\phi) & & F(C) & & F(\pi_\ast) & & \\
& & & & \partial_1 & & \phi & & \eta_1 & & \\
& & & & F(\phi) & & F(\pi_\ast) & & F(S\phi) & & \\
& & & & \partial_1 & & \phi & & \eta_1 & & \\
& \cong & & \cong & & \cong & & \cong & & \cong & \\
& & & & F(\phi) & & F(\pi_\ast) & & F(S\phi) & & \\
& & & & F(S\phi) & & F(C) & & F(\pi_\ast) & & \\
& & & & \partial_1 & & \phi & & \eta_1 & & \\
& & & & F(\phi) & & F(\pi_\ast) & & F(S\phi) & & \\
& & & & \partial_1 & & \phi & & \eta_1 & & \\
\end{array}
\]

in the contravariant case.

Proof. This follows from the first part of the proof of the previous proposition. \hfill \blacksquare

5. Examples of concrete homology and cohomology theories

Example 5.1. Let $S$ be the full subcategory of the category of all $C^\ast$-algebras, consisting of separable, nuclear algebras. For each separable $C^\ast$-algebra $A$, both $KK(-, A)$ and $KK(A, -)$ are additive functors from $S$ to $\text{Ab}$, which have the properties (H), (S), and (HX). The first one is contravariant while the second is covariant. So the above theory applies to these, and identifies the cyclic six term exact sequences associated with extensions in these two cases (as defined in [Blu98]).

Example 5.2. The functors $K_0$ and $K_1$ are additive, covariant functors from the category of all separable $C^\ast$-algebras to the category $\text{Ab}$, which have the properties (H), (S), and (HX). So the above theory applies to these two functors.

We have also a standard cyclic six term exact sequence in $K$-theory (as defined in [RLL00]). To avoid confusions, we write $\delta_0$ and $\delta_1$ for the exponential map and the index maps, respectively. We will recall the definition. We have an isomorphism $\theta_\ast$ of functors from $K_1(-)$ to $K_0(S(-))$, i.e., for each $C^\ast$-algebra $A$ we have an isomorphism $\theta_A : K_1(A) \to K_0(SA)$ and, moreover, for all $C^\ast$-algebras $A$ and $B$ and all $\ast$-homomorphisms $\varphi : A \to B$, the diagram

\[
\begin{array}{ccc}
K_1(A) & \xrightarrow{K_1(\varphi)} & K_1(B) \\
\theta_A & \Downarrow & \theta_B \\
K_0(SA) & \xrightarrow{K_0(\varphi)} & K_0(SB) \\
\end{array}
\]

commutes (cf. [RLL00] Theorem 10.1.3).

The exponential map $\delta_0 : K_0(A_2) \to K_1(A_0)$ associated with a short exact sequence $A_0 \hookrightarrow A_1 \to A_2$ is defined as the composition of the maps

\[
K_0(A_2) \xrightarrow{\beta_{A_2}} K_1(SA_2) \xrightarrow{\varphi_1} K_0(SA_0) \xrightarrow{\theta_{A_0}^{-1}} K_1(A_0),
\]

where $\varphi_1$ is the index map associated with the short exact sequence

\[
S\mathcal{A}_0 \hookrightarrow S\mathcal{A}_1 \to S\mathcal{A}_2.
\]
**Lemma 5.3.** Let \( \mathfrak{A} \) be a \( C^* \)-algebra. The standard cyclic six term exact sequence in \( K \)-theory associated with \( S\mathfrak{A} \hookrightarrow C\mathfrak{A} \twoheadrightarrow \mathfrak{A} \) (as in \([RLL00]\)) is the sequence

\[
\begin{array}{ccccccccc}
K_0(S\mathfrak{A}) & \longrightarrow & 0 & \longrightarrow & K_0(\mathfrak{A}) \\
\delta_\alpha & \cong & \cong & \cong & -\beta_\alpha \\
K_1(\mathfrak{A}) & \longrightarrow & 0 & \longrightarrow & K_1(S\mathfrak{A}).
\end{array}
\]

**Proof.** Since the cone, \( C\mathfrak{A} \), of \( \mathfrak{A} \) is homotopy equivalent to the zero \( C^* \)-algebra, \( K_0(C\mathfrak{A}) \cong K_1(C\mathfrak{A}) \cong 0 \) (cf. \([RLL00]\) Example 4.1.5)). That the index map is \( \theta_\mathfrak{A} \) follows directly from the definition of \( \theta_\mathfrak{A} \) (cf. \([RLL00]\) Proof of Theorem 10.1.3)). The exponential map \( \delta_0 : K_0(\mathfrak{A}) \to K_1(S\mathfrak{A}) \) is defined as the composition of the maps

\[
\begin{array}{ccc}
K_0(\mathfrak{A}) & \xrightarrow{\beta_\mathfrak{A}} & K_1(S\mathfrak{A}) \\
& \pi_1 & \longleftarrow K_0(S\mathfrak{A}) \\
& \xrightarrow{\delta_{S\mathfrak{A}}} & K_1(S\mathfrak{A}),
\end{array}
\]

where \( \delta_1 \) is the index map associated with the short exact sequence

\[
S\mathfrak{A} \hookrightarrow S(C\mathfrak{A}) \twoheadrightarrow \mathfrak{A}.
\]

We have a commuting diagram

\[
\begin{array}{cccccc}
S(S\mathfrak{A}^0) & \xrightarrow{\cong} & S(C\mathfrak{A}^0) & \xrightarrow{\cong} & S(\mathfrak{A}^0) \\
& \downarrow & \downarrow & \downarrow & \downarrow \\
C(S\mathfrak{A}^0) & \xrightarrow{\cong} & C(C\mathfrak{A}^0) & \xrightarrow{\cong} & C(\mathfrak{A}^0) \\
& \downarrow & \downarrow & \downarrow & \downarrow \\
S\mathfrak{A}^0 & \xrightarrow{\cong} & C\mathfrak{A}^0 & \xrightarrow{\cong} & \mathfrak{A}^0
\end{array}
\]

with exact rows and columns. This gives — by Lemma 4.11 and the above (applied to \( S\mathfrak{A} \) instead of \( \mathfrak{A} \) — rise to an anticommuting square

\[
\begin{array}{ccc}
K_0(\mathfrak{A}) & \xrightarrow{\delta_0} & K_1(S\mathfrak{A}) \\
\cong & \cong & \cong \\
K_1(S(\mathfrak{A})) & \xrightarrow{\delta_1} & K_0(S(S\mathfrak{A}))
\end{array}
\]

Consequently, \( \delta_1 = -\theta_{S\mathfrak{A}} \). Now it follows that \( \delta_0 = -\beta_{S\mathfrak{A}} \). \( \blacksquare \)

Since the index and exponential maps are unique up to signs (cf. \([WO93]\) Exercise 9.F)), we have that the standard cyclic six term exact sequence in \( K \)-theory as defined here differs from the cyclic six term exact sequence defined as above by change of sign of the index map (under the identification \( \theta_\mathfrak{A} \) of \( K_1 \) with \( K_0 \circ S \)).

Thus we get the following corollaries:

**Corollary 5.4.** Let there be given an extension

\[
e : \mathfrak{A}_0 \xrightarrow{i} \mathfrak{A}_1 \xrightarrow{\pi} \mathfrak{A}_2.
\]

Then there exists an isomorphism of cyclic six term exact sequences as follows:

\[
\begin{array}{cccccccc}
K_1(\mathfrak{A}_2) & \xrightarrow{-\delta^e_{\mathfrak{A}_2}} & K_0(\mathfrak{A}_2) & \xrightarrow{\theta_{\mathfrak{A}_2}} & \xrightarrow{-\delta^e_{\mathfrak{A}_2}} & K_0(\mathfrak{A}_2) & \xrightarrow{\theta_{\mathfrak{A}_2}} & K_1(\mathfrak{A}_2) \\
K_0(\mathfrak{A}_1) & \xrightarrow{-\delta^e_{\mathfrak{A}_1}} & K_0(\mathfrak{A}_1) & \xrightarrow{\theta_{\mathfrak{A}_1}} & K_1(\mathfrak{A}_1) & \xrightarrow{\theta_{\mathfrak{A}_1}} & K_1(\mathfrak{A}_1) \\
K_0(\mathfrak{A}_0) & \xrightarrow{-\delta^e_{\mathfrak{A}_0}} & K_0(\mathfrak{A}_0) & \xrightarrow{\theta_{\mathfrak{A}_0}} & K_1(\mathfrak{A}_0) & \xrightarrow{\theta_{\mathfrak{A}_0}} & K_1(\mathfrak{A}_0) \\
\cong & \cong & \cong & \cong & \cong & \cong & \cong \\
K_0(S\mathfrak{A}_2) & \xrightarrow{\cong} & K_0(C\mathfrak{A}_2) & \xrightarrow{\cong} & K_0(\mathfrak{A}_1) & \xrightarrow{\cong} & K_1(S\mathfrak{A}_2) & \xrightarrow{\cong} & K_1(C\mathfrak{A}_2) & \xrightarrow{\cong} & K_1(\mathfrak{A}_0) & \xrightarrow{\cong} & K_1(S\mathfrak{A}_0)
\end{array}
\]

where the second sequence is the standard cyclic six term exact sequence in \( K \)-theory associated with \( mc(e) \).
Corollary 5.5. Let there be given a \(*\)-homomorphism
\[ \phi : \mathfrak{A} \to \mathfrak{B} \]
from a \(C^*\)-algebra \(\mathfrak{A}\) to a \(C^*\)-algebra \(\mathfrak{B}\), and let
\[ e : S\mathfrak{B} \overset{i_\phi}{\to} C_\phi \overset{\pi_\phi}{\to} \mathfrak{A} \]
denote the mapping cone sequence. Then there exists an isomorphism of exact sequences as follows:
\[
\begin{array}{ccccccc}
K_1(\mathfrak{B}) & \overset{K_0(\phi)}{\to} & K_1(\mathfrak{B}) & \overset{K_0(\phi)}{\to} & K_1(\mathfrak{B}) & \overset{K_0(\phi)}{\to} & K_1(\mathfrak{B}) \\
\cong & \beta_m & \cong & \beta_m & \cong & \beta_m & \cong \\
K_0(5\mathfrak{B}) & \to & K_0(\mathfrak{A}) & \to & K_0(\mathfrak{A}) & \to & K_0(\mathfrak{A}) \\
\end{array}
\]
where the second sequence is the standard cyclic six term exact sequence in \(K\)-theory associated with \(e\).

Remark 5.6. Note that the way Bonkat associates cyclic six term exact sequences in ideal-related \(KK\)-theory with short exact sequences with completely positive contractive coherent splittings is completely analogous to the definitions of Section 3 (cf. [Bon02, Section 3.4]).

Example 5.7. An instructive example to get a better understanding of Lemma 4.1 is to look at
\[
\begin{array}{cccccccc}
S \otimes S & \to & S \otimes C & \to & S \otimes C \\
\downarrow & & \downarrow & & \downarrow \\
C \otimes S & \to & C \otimes C & \to & C \otimes C \\
\downarrow & & \downarrow & & \downarrow \\
C \otimes S & \to & C \otimes C & \to & C \otimes C \\
\end{array}
\]
where \(S = SC\) and \(C = CC\). It is tempting to guess that the maps
\[
K_0(C \otimes C) \to K_1(S \otimes C) \to K_0(S \otimes S)
\]
\[
K_0(C \otimes C) \to K_1(C \otimes S) \to K_0(S \otimes S)
\]
are equal (after all, \(S \otimes C\) is canonically isomorphic to \(C \otimes S\)) — but this is not the case. One map gives the Bott map while the other gives the anti-Bott map. After some thought this seems reasonable after all, since the map \(S \otimes S \ni x \otimes y \mapsto y \otimes x \in S \otimes S\) corresponds to the flip along the diagonal in \(C_0((0,1) \times (0,1))\), which induces the automorphism \(-id\) on \(K_0\).

6. Ideal-related \(K\)-theory with coefficients

In this section, we introduce a new invariant, Ideal-related \(K\)-theory with coefficients.

Definition 6.1. Let \(n \in \mathbb{N}_{\geq 2}\). Denote the (non-unital) dimension-drop interval by \(I_{n,0}\), i.e., \(I_{n,0}\) is the mapping cone of the unital \(*\)-homomorphism from \(C\) to \(M_n\).

Definition 6.2. Let \(n \in \mathbb{N}_{\geq 2}\) and let \(\epsilon_{n,0}\) denote the mapping cone sequence
\[ \epsilon_{n,0} : SM_n \hookrightarrow I_{n,0} \to C \]
corresponding to the unital \(*\)-homomorphism from \(C\) to \(M_n\). Moreover, let \(\epsilon_{n,i} = mc^i(\epsilon_{n,0})\), for all \(i \in \mathbb{N}\) and write
\[ \epsilon_{n,1} : SC \hookrightarrow I_{n,1} \to I_{n,0}, \]
\[ \epsilon_{n,i} : I_{n,i-2} \hookrightarrow I_{n,i} \to I_{n,i-1}, \text{ for } i \geq 2. \]
Similarly, set \(f_{n,0} : C \overset{id}{\to} \overset{id}{\to} 0\) and \(f_{n,0} : I_{n,0} \overset{id}{\to} I_{n,0} \overset{id}{\to} 0\), for all \(n \in \mathbb{N}_{\geq 2}\). Moreover, set \(f_{n,i} = mc^i(f_{n,0})\) for all \(n \in \mathbb{N}\) and all \(i \in \mathbb{N}\).
Definition 6.3. Let $K_{\text{six}}$ denote the functor, which to each extension of $C^*$-algebras associates the corresponding standard cyclic six term exact sequence (as defined in [RLL00]) — cf. Example 5.2. Let $\text{Hom}_{\text{six}}(K_{\text{six}}(e_1), K_{\text{six}}(e_2))$ denote the group of cyclic six term chain homomorphisms.

As in [DL98], set $K_i(-; \mathbb{Z}_n) = KK^i(\mathbb{Z}_n, -)$. Moreover, let $\underline{K}$ denote total $K$-theory as defined in [DL98].

Remark 6.4. As is easily seen, the above cyclic six term exact sequence in $K$-theory differs from that defined by Bonkat in [Bon02] $\S 7.3$ by the index and exponential maps having the opposite signs. This makes no difference for the arguments and results in [Bon02] (the important thing here is that we change the sign of either the index map or the exponential map compared with the definition of the connecting homomorphisms in $KK$-theory).

By applying Lemma 4.1 and Lemma 5.3 to

there exists a commuting diagram

$$
\begin{array}{ccccccc}
\delta^e_5 & K_0(\mathbb{A}_0) & K_0(\mathbb{A}_1) & K_0(\mathbb{A}_2) & K_1(\mathbb{A}_0) & K_1(\mathbb{A}_1) & K_1(\mathbb{A}_2) & \delta^e_5 \\
\cong & \beta_{a_0} & \cong & \beta_{a_1} & \cong & \beta_{a_2} & \cong & \beta_{a_2} \\
\delta^{\phi}_5 & K_1(\mathbb{S}_0) & K_1(\mathbb{S}_1) & K_1(\mathbb{S}_2) & K_0(\mathbb{S}_0) & K_0(\mathbb{S}_1) & K_0(\mathbb{S}_2) & \delta^{\phi}_5 \\
\end{array}
$$

Consequently, the definition of "$(K_{n+1}; A_j)$" in [Bon02] is just $K_{\text{six}}(S\ell)$ (up to canonical identification with our terminology). The same argument works if we choose to work with the slightly different cyclic six term exact sequence defined in [Bon02]. Note also that this is not true if we define the cyclic six term sequence using the abstract machinery of Section 3.

Definition 6.5. For each extension $e$ of separable $C^*$-algebras, define the ideal-related $K$-theory with coefficients, $\underline{K}_{\mathcal{E}}(e)$, of $e$ to be the (graded) group

$$
\underline{K}_{\mathcal{E}}(e) = \bigoplus_{n=1}^5 \bigoplus_{n,i} \left( KK_{\mathcal{E}}(f_{n,i}, e) \oplus \bigoplus_{n=2} KK_{\mathcal{E}}(e_{n,i}, e) \right).
$$

A homomorphism $\phi$ from $\underline{K}_{\mathcal{E}}(e_1)$ to $\underline{K}_{\mathcal{E}}(e_2)$ is a group homomorphism respecting the direct sum decomposition and the natural homomorphisms induced by the elements of $KK_{\mathcal{E}}(e, e')$, for $j = 0, 1$, where $e$ and $e'$ are in \{ $e_{n,i}, f_{n,i}, \tilde{f}_{n,i}$ | $n \in \mathbb{N}_{\geq 2}$, $i = 0, 1, 2$ \}. The set of homomorphisms from $\underline{K}_{\mathcal{E}}(e_1)$ to $\underline{K}_{\mathcal{E}}(e_2)$ will be denoted by $\text{Hom}_{\mathcal{E}}(\underline{K}_{\mathcal{E}}(e_1), \underline{K}_{\mathcal{E}}(e_2))$.

Let $x \in KK_{\mathcal{E}}(e_1, e_2)$. Then $x$ induces an element of $\text{Hom}_{\mathcal{E}}(\underline{K}_{\mathcal{E}}(e_1), \underline{K}_{\mathcal{E}}(e_2))$ by

$$
y \in KK_{\mathcal{E}}(f_{n,i}, e_1) \mapsto y \times x \in KK_{\mathcal{E}}(f_{n,i}, e_2), \quad n \in \mathbb{N},
$$

$$
y \in KK_{\mathcal{E}}(e_{n,i}, e_1) \mapsto y \times x \in KK_{\mathcal{E}}(e_{n,i}, e_2), \quad n \in \mathbb{N}_{\geq 2}.
$$

Hence, if $\phi: e_1 \to e_2$ is a homomorphism, then $\phi$ induces an element $\underline{K}_{\mathcal{E}}(\phi) \in \text{Hom}_{\mathcal{E}}(\underline{K}_{\mathcal{E}}(e_1), \underline{K}_{\mathcal{E}}(e_2))$. In this way $\underline{K}_{\mathcal{E}}$ becomes a functor on the category of extensions.

Remark 6.6. For extensions $e_1: \mathbb{A}_0 \to \mathbb{A}_1 \to \mathbb{A}_2$ and $e_2: \mathbb{B}_0 \to \mathbb{B}_1 \to \mathbb{B}_2$ of separable $C^*$-algebras, there are natural homomorphisms $G_i: KK_{\mathcal{E}}(e_1, e_2) \to KK_{\mathcal{E}}(\mathbb{A}_i, \mathbb{B}_i)$, for $i = 0, 1, 2$. As in the proof of [Bon02] Satz 7.5.6, the obvious diagram

$$
\begin{array}{ccccccc}
\text{Ext}_{\text{six}}(K_{\text{six}}(e_1), K_{\text{six}}(e_2)) & \to & KK_{\mathcal{E}}(e_1, e_2) & \to & \text{Hom}_{\text{six}}(K_{\text{six}}(e_1), K_{\text{six}}(e_2)) \\
\text{Ext}(K_{\text{six}}(\mathbb{A}_1), K_{\text{six}}(\mathbb{B}_1)) & \to & KK_{\mathcal{E}}(\mathbb{A}_1, \mathbb{B}_1) & \to & \text{Hom}(K_{\text{six}}(\mathbb{A}_1), K_{\text{six}}(\mathbb{B}_1)) \\
\end{array}
$$

transforms into

$$
\begin{array}{ccccccc}
\text{Ext}_{\text{six}}(K_{\text{six}}(e_1), K_{\text{six}}(e_2)) & \to & KK_{\mathcal{E}}(e_1, e_2) & \to & \text{Hom}_{\text{six}}(K_{\text{six}}(e_1), K_{\text{six}}(e_2)) \\
\text{Ext}(K_{\text{six}}(\mathbb{A}_1), K_{\text{six}}(\mathbb{B}_1)) & \to & KK_{\mathcal{E}}(\mathbb{A}_1, \mathbb{B}_1) & \to & \text{Hom}(K_{\text{six}}(\mathbb{A}_1), K_{\text{six}}(\mathbb{B}_1)) \\
\end{array}
$$
commutes and is natural in $e_2$, for $i = 0, 1, 2$ — provided that $e_1$ belongs to the UCT class considered by Bonkat.

Let $e : \mathfrak{A}_0 \xleftarrow{\pi} \mathfrak{A}_1 \xrightarrow{\pi} \mathfrak{A}_2$ and $e' : \mathfrak{B}_0 \xleftarrow{\pi'} \mathfrak{B}_1 \xrightarrow{\pi'} \mathfrak{B}_2$ be two given extensions. Then define

$$\Lambda_{e, e'} : \text{Hom}_{\text{six}}(K_{\text{six}}(e), K_{\text{six}}(e')) \rightarrow \text{Hom}_{\text{six}}(K_{\text{six}}(mc(e)), K_{\text{six}}(mc(e'))$$

as follows: Let $(\alpha_i)_{i=0}^3$ be natural in $K_{\text{six}}$ be given. Then by Corollary 6.4, the diagram

$$\begin{array}{ccccccc}
\delta_{\pi}(\mathfrak{A}_2) & K_0(\mathfrak{A}_2) & K_0(\mathfrak{A}_1) & K_0(\mathfrak{A}_0) & K_0(\mathfrak{A}_0) & K_0(\mathfrak{A}_0) & K_0(\mathfrak{A}_0) \\
\delta_{\pi}(\mathfrak{A}_3) & K_1(\mathfrak{A}_3) & K_1(\mathfrak{A}_2) & K_1(\mathfrak{A}_1) & K_1(\mathfrak{A}_1) & K_1(\mathfrak{A}_1) & K_1(\mathfrak{A}_1) \\
& \delta_{\pi}(\mathfrak{A}_4) & K_2(\mathfrak{A}_4) & K_2(\mathfrak{A}_3) & K_2(\mathfrak{A}_2) & K_2(\mathfrak{A}_2) & K_2(\mathfrak{A}_2) \\
\end{array}$$

commutes. Let $\Lambda_{e, e'}((\alpha_i)_{i=0}^3)$ denote the composition. Clearly, $\Lambda_{e, e'}$ is an isomorphism. A computation shows that $\Lambda$ from $\text{Hom}_{\text{six}}(K_{\text{six}}(e), K_{\text{six}}(-))$ to $\text{Hom}_{\text{six}}(K_{\text{six}}(mc(e)), K_{\text{six}}(mc(-)))$ defined by $\Lambda(e') = \Lambda_{e, e'}$ is a natural transformation such that $\Lambda_{e, e'}(K_{\text{six}}(id_e)) = K_{\text{six}}(id_{mc(e)})$.

Let $\mathcal{SE}$ be the subcategory of $\mathcal{E}$ consisting only of extensions of separable $\ast$-algebras and morphism being triples of $\ast$-homomorphisms such that the obvious diagram commutes. Consider the category $\mathcal{KK}_e$ whose objects are the objects of $\mathcal{SE}$ and the group of morphisms is $\mathcal{KK}_e(e_1, e_2)$. Consider the the composed functor $\mathcal{KK}_e \circ mc$ from $\mathcal{SE}$ to $\mathcal{KK}_e$, which sends an object $e$ of $\mathcal{SE}$ to $mc(e)$, and sends a morphism $(\phi_0, \phi_1, \phi_2)$ of $\mathcal{SE}$ to $\mathcal{KK}_e(mc((\phi_0, \phi_1, \phi_2)))$. This is a stable, homotopy invariant, split exact functor, so by [Bonk22 Satz 3.5.10 und Satz 6.2.4], there exists a unique functor $\mathcal{KK}_e \circ mc$ from $\mathcal{KK}_e$ to $\mathcal{KK}_e$ such that the diagram

$$\begin{array}{ccc}
\mathcal{SE} & \xrightarrow{mc} & \mathcal{SE} \\
\downarrow \mathcal{KK}_e & & \downarrow \mathcal{KK}_e \\
\mathcal{KK}_e \circ mc & \xrightarrow{mc} & \mathcal{KK}_e \circ mc \\
\end{array}$$

commutes. By the universal property, the diagram

$$\begin{array}{ccc}
\mathcal{KK}_e(e, e') & \rightarrow & \text{Hom}_{\text{six}}(K_{\text{six}}(e), K_{\text{six}}(e')) \\
\downarrow \mathcal{KK}_e(e, e') & & \downarrow \mathcal{KK}_e(e, e') \\
\mathcal{KK}_e(mc(e), mc(e')) & \rightarrow & \text{Hom}_{\text{six}}(K_{\text{six}}(mc(e)), K_{\text{six}}(mc(e'))) \\
\end{array}$$

commutes, where the horizontal arrows are the natural maps in the UCT.

**Lemma 6.7.** Let $e$ and $e'$ be extensions of separable, nuclear $\ast$-algebras in the bootstrap category $\mathcal{N}$. Then $mc$ induces an isomorphism from $\mathcal{KK}_e(e, e')$ to $\mathcal{KK}_e(mc(e), mc(e'))$, which is natural in both variables.

**Proof.** Let $\alpha_{e, e'}$ denote the map from $\mathcal{KK}_e(e, e')$ to $\mathcal{KK}_e(mc(e), mc(e'))$ induced by the functor $mc$. Since $mc$ is a functor, clearly the map is going to be natural (in both variables). From Proposition 3.5.6 in [Bonk22] (cf. also [Hig87 Lemma 3.2]), it follows that $mc$ is a group homomorphism. Since $\Lambda_{e, e'}$ is an isomorphism, from the above diagram and the UCT of Bonkat [Bonk22], we have that $\alpha_{e, e'}$ is an isomorphism whenever $K_{\text{six}}(e')$ is injective.

When $e'$ is an arbitrary extension, then by [Bonk22 Proposition 7.4.3], there exist an injective geometric resolution $e_1 \hookrightarrow e_2 \rightarrow \mathcal{S} e'$ of $e'$, i.e., there exists a short exact sequence $e_1 \hookrightarrow e_2 \rightarrow \mathcal{S} e'$ of extensions from $\mathcal{SE}$, with a completely positive contractive coherent splitting, such that the induced
six term exact $K_{six}$-sequence degenerates to a short exact sequence $K_{six}(SSe') \rightarrow K_{six}(e_1) \rightarrow K_{six}(e_2)$, which is an injective resolution of $K_{six}(SSe')$.

The cyclic six term exact sequences in $KK_\mathcal{E}$-theory give a commuting diagram

$$
\begin{array}{c}
KK_\mathcal{E}(e,S_{e_1}) \xrightarrow{\alpha_{e,S_{e_1}}} KK_\mathcal{E}(mc(e),mc(S_{e_1})) \xrightarrow{\alpha_{e,S_{e_1}}} KK_\mathcal{E}(mc(e),mc(S_{e_1})) \\
KK_\mathcal{E}(e,S_{e_2}) \xrightarrow{\alpha_{e,S_{e_2}}} KK_\mathcal{E}(mc(e),mc(S_{e_2})) \xrightarrow{\alpha_{e,S_{e_2}}} KK_\mathcal{E}(mc(e),mc(S_{e_2})) \\
KK_\mathcal{E}(e,Se') \xrightarrow{\alpha_{e,Se'}} KK_\mathcal{E}(mc(e),mc(Se')) \xrightarrow{\alpha_{e,Se'}} KK_\mathcal{E}(mc(e),mc(Se')) \\
KK_\mathcal{E}(e,e_1) \xrightarrow{\alpha_{e,e_1}} KK_\mathcal{E}(mc(e),mc(e_1)) \xrightarrow{\alpha_{e,e_1}} KK_\mathcal{E}(mc(e),mc(e_1)) \\
KK_\mathcal{E}(e,e_2) \xrightarrow{\alpha_{e,e_2}} KK_\mathcal{E}(mc(e),mc(e_2)) \xrightarrow{\alpha_{e,e_2}} KK_\mathcal{E}(mc(e),mc(e_2))
\end{array}
$$

with exact columns. Naturality of $\alpha_{e,-}$ gives us commutativity of the squares on the left hand side, while naturality of the isomorphism from the functor $mcS$ to the functor $Smc$ gives us commutativity of the squares on the right hand side (cf. Lemma 2.11). The remaining rectangle is seen to commute by using the definition of the connecting homomorphisms and Lemma 2.14. By the Five Lemma, we have that $\alpha_{e,Se'}$ is an isomorphism. Therefore also $\alpha_{e,e'}$.

Remark 6.8. Similarly, there exists a unique functor $\hat{S}$ from $KK_\mathcal{E}$ to $KK_\mathcal{E}$ such that the diagram

$$
\begin{array}{ccc}
S\mathcal{E} & \xrightarrow{\hat{S}} & S\mathcal{E} \\
KK_\mathcal{E} & \xrightarrow{\hat{S}} & KK_\mathcal{E}
\end{array}
$$

commutes.

7. Some diagrams

In this section we construct 19 diagrams involving the groups of the new invariant. These diagrams can in many cases be used to determine the new groups introduced in the invariant (see Section [5] for some examples). They are also used in the paper [ERR], where the three authors prove a Universal Multi-Coefficient Theorem (UMCT) for a certain class of $C^*$-algebras with one specified ideal, which includes all the Cuntz-Krieger algebras of type (II) with one specified ideal. The long proof of these diagrams is outlined in the next section.

Assumption 7.1. Throughout this section, $e: \mathfrak{A}_0 \hookrightarrow \mathfrak{A}_1 \xrightarrow{\pi} \mathfrak{A}_2$ is a (fixed) extension of separable $C^*$-algebras.

Definition 7.2. Set $F_{1,1} = KK_\mathcal{E}(1,1,e)$, $F_{n,1} = KK_\mathcal{E}(1_n,1,e)$, and $H_{n,1} = KK_\mathcal{E}(e_{n,1},1)$, for all $n \in \mathbb{N}_{\geq 2}$ and all $i = 0, 1, 2, 3, 4, 5$. For convenience, identify indices modulo 6, i.e., we write $F_{n,6} = F_{n,0}$, $F_{n,7} = F_{n,1}$ etc.

Remark 7.3. Let $e: \mathfrak{A}_0 \hookrightarrow \mathfrak{A}_1 \xrightarrow{\pi} \mathfrak{A}_2$ be a given extension of $C^*$-algebras. Then consider the extensions

$$
mc^3(e): \mathcal{S} \mathcal{C}_\pi \xrightarrow{((\pi_{me})_{me})_{me}} \mathcal{C}_{(\pi_{me})_{me}} \xrightarrow{(\pi_{me})_{me}} \mathcal{C}_{\pi_{me}}
$$

and

$$
S(e): \mathcal{S} \mathcal{A}_0 \xrightarrow{\pi} \mathcal{S} \mathcal{A}_1 \xrightarrow{\pi} \mathcal{S} \mathcal{A}_2.
$$

There are canonical $\ast$-homomorphisms $\mathcal{S} \mathcal{A}_0 \rightarrow \mathcal{S} \mathcal{C}_\pi$, $\mathcal{S} \mathcal{A}_1 \rightarrow \mathcal{C}_{(\pi_{me})_{me}}$, and $\mathcal{S} \mathcal{A}_2 \rightarrow \mathcal{C}_{\pi_{me}}$, which all induce isomorphisms on the level of $K$-theory. But these do not, in general, induce a morphism of
extensions — in fact not even of the corresponding cyclic six term exact sequences. Using Corollary 5.4, we easily see, that the diagram

\[
\begin{align*}
\delta_1^e &= \text{K}_0(\mathcal{A}_0) \to \text{K}_0(\mathcal{A}_1) \\
\delta_1^{mc} &= \text{K}_0(\mathcal{B}_0) \to \text{K}_0(\mathcal{B}_1) \\
\delta_1^\pi &= \text{K}_0(\mathcal{C}_0) \to \text{K}_0(\mathcal{C}_1) \\
\delta_1 &= \text{K}_1(\mathcal{A}_0) \to \text{K}_1(\mathcal{A}_1) \\
\delta_1^{mc} &= \text{K}_1(\mathcal{B}_0) \to \text{K}_1(\mathcal{B}_1) \\
\delta_1^\pi &= \text{K}_1(\mathcal{C}_0) \to \text{K}_1(\mathcal{C}_1) \\
\delta_1 &= \text{K}_2(\mathcal{A}_0) \to \text{K}_2(\mathcal{A}_1)
\end{align*}
\]

commutes, where \(\alpha_i\) are the induced maps as mentioned above, and \(\psi'\) and \(\pi'\) denote the maps \(((\pi mc) mc) mc\) and \(((\pi mc) mc) mc\), respectively. We expect that it is possible to find a functorial way to implement the \(KK_\mathcal{E}\)-equivalences between \(mc^3(e)\) and \(Se\), but can not see how to do this — not even how to make a canonical choice of \(KK_\mathcal{E}\)-equivalences.

**Definition 7.4.** The previous remark showed that \(mc^3(e)\) and \(Se\) are \(KK_\mathcal{E}\)-equivalent (assuming the UCT), but the remark did not give a canonical way to choose a specific \(KK_\mathcal{E}\)-equivalence (so we get a functorial identification of the two functors).

For our purposes, it is enough to have the following lemma. Let \(e: \mathcal{A}_0 \to \mathcal{A}_1 \to \mathcal{A}_2\) be a given extension of separable, nuclear \(C^\ast\)-algebras in the bootstrap category \(\mathcal{N}\). Assume, moreover, that \(Ext_{six}(K_{six}(e), K_{six}(Se))\) is the trivial group. For each such extension \(e\), define

\[
x_e \in KK_\mathcal{E}(Se, mc^3(e))
\]
to be the unique element inducing \((\alpha_0, -\alpha_1, \alpha_2, -\alpha_3, -\alpha_4, \alpha_5)\) in \(Hom_{six}(K_{six}(Se), K_{six}(mc^3(e)))\) (as defined in the preceding remark).

**Lemma 7.5.** Let \(e\) and \(e'\) be two given extensions of separable, nuclear \(C^\ast\)-algebras in the bootstrap category \(\mathcal{N}\). Assume, moreover, that \(Ext_{six}(K_{six}(e), K_{six}(Se)), Ext_{six}(K_{six}(e'), K_{six}(Se'))\), and \(Ext_{six}(K_{six}(e), K_{six}(Se'))\) are trivial groups. Let \(\phi\) be a morphism from \(e\) to \(e'\), and set \(x = KK_\mathcal{E}(\phi)\) in \(KK_\mathcal{E}(e, e')\). Then

\[
KK_\mathcal{E}(S\phi) \times x_{e'} = \hat{S}x \times x_{e'} = x_e \times \hat{mc}^3(x) = x_e \times KK_\mathcal{E}(mc^3(\phi)).
\]

**Proof.** From the assumptions and the UCT of Bonkat, we see that the canonical homomorphisms

\[
\begin{align*}
KK_\mathcal{E}(e, e')&\to Hom_{six}(K_{six}(e), K_{six}(e')), \\
KK_\mathcal{E}(Se, Se')&\to Hom_{six}(K_{six}(Se), K_{six}(Se')), \\
KK_\mathcal{E}(mc^3(e), mc^3(e'))&\to Hom_{six}(K_{six}(mc^3(e)), K_{six}(mc^3(e'))), \\
KK_\mathcal{E}(Se, mc^3(e'))&\to Hom_{six}(K_{six}(Se), K_{six}(mc^3(e')))
\end{align*}
\]

are functorial isomorphisms. Consequently, it is enough to prove that the result holds for the induced maps in \(K\)-theory, i.e.,

\[
K_{six}(x_e) \circ K_{six}(\hat{S}x) = K_{six}(\hat{mc}^3(x)) \circ K_{six}(x_e).
\]

Again to prove this, it is enough to show that

\[
\psi'_i \circ S\phi_i = (mc^3(\phi))_i \circ \psi_i,
\]

for \(i = 0, 1, 2\), where \(\psi_0, \psi_1\), and \(\psi_2\), (respectively) is the canonical \(*\)-homomorphisms from the ideal, the extension, and the quotient, respectively) of \(Se\) to the ideal (the extension, and the quotient, respectively) of \(mc^3(e)\) — and correspondingly for \(\psi'_i\). This equation is straightforward to check.

**Remark 7.6.** Let \(e\) be an extension of separable, nuclear \(C^\ast\)-algebras in the bootstrap category \(\mathcal{N}\), and assume that \(Ext_{six}(K_{six}(e), K_{six}(Se))\) is the trivial group. Then there exists a \(KK_\mathcal{E}\)-equivalence \(\hat{S} x_e \times mc^3(e)\) from \(SSe\) to \(mc^3(e)\). Composed with the standard \(KK_\mathcal{E}\)-equivalence from \(e\) to \(SSe\), this gives a canonical \(KK_\mathcal{E}\)-equivalence from \(e\) to \(mc^3(e)\).

It is also easy to show that

\[
x_{mc^3(e)} = -KK_\mathcal{E}(\theta_e) \times \hat{mc}(x_e).
\]

\(^4\)Here we also use that the canonical identifications \(K_i(\mathcal{A}_1) \to K_{i-1}(SSe)\) give an isomorphism of the corresponding cyclic six term exact sequences.
Definition 7.7. For an extension $e$, let $b_e$ denote the element of $KK\varepsilon(e, SSe)$ induced by the Bott element — this is a $KK\varepsilon$-equivalence. Moreover, let $z_n$ denote the $KK\varepsilon$-equivalence in $KK\varepsilon(Sf_{1,0}, i(\varepsilon_{n,0}))$ induced by the canonical embedding $\mathbb{C} \to M_n$. Let $w_n$ denote the $KK\varepsilon$-equivalence from $0 \to SM_n \to SM_n$ to $q(\varepsilon_{n,2})$ induced by the canonical embedding $SM_n \to \mathbb{I}_{n,1}$.

For each $n \in \mathbb{N}_{\geq 2}$, we will, during the following three definitions, define $36$ homomorphisms,

$$
\begin{align*}
F_{1,i+1} &\xrightarrow{h_{n,i}^{1,\text{in}}} H_{n,i} &\xrightarrow{h_{n,i}^{1,\text{out}}} F_{1,i+3} \\
F_{n,i} &\xrightarrow{h_{n,i}^{1,\text{in}}} H_{n,i} &\xrightarrow{h_{n,i}^{1,\text{out}}} F_{1,i+2} \\
F_{1,i+2} &\xrightarrow{h_{n,i}^{1,\text{in}}} H_{n,i} &\xrightarrow{h_{n,i}^{1,\text{out}}} F_{n,i+1},
\end{align*}
$$

where we identify indices modulo $6$ (so we write e.g. $h_{n,0}^{1,*,*} = h_{n,0}^{1,*,*}$).

Definition 7.8. For each $n \in \mathbb{N}_{\geq 2}$, there exists a short exact sequence $i(\varepsilon_{n,0}) \xrightarrow{i_{n,0}} \varepsilon_{n,0} \xrightarrow{q_{n,0}} q(\varepsilon_{n,0})$ of extensions. We define $h_{n,0}^{1,\text{in}}$ and $h_{n,0}^{1,\text{out}}$ by

$$
\begin{align*}
F_{1,1} &\xrightarrow{h_{n,0}^{1,\text{in}}} H_{n,0} &\xrightarrow{h_{n,0}^{1,\text{out}}} F_{1,3} \\
KK\varepsilon(q(\varepsilon_{n,0}), e) &\xrightarrow{KK\varepsilon(q(\varepsilon_{n,0}), e) \times -} KK\varepsilon(q(\varepsilon_{n,0}), e) &\xrightarrow{KK\varepsilon(q(\varepsilon_{n,0}), e) \times -} KK\varepsilon(\varepsilon_{n,0}, e) \\
\end{align*}
$$

By applying the functor $\hat{m_1}$, we define $h_{n,i}^{1,\text{in}}$ and $h_{n,i}^{1,\text{out}}$, for $i = 1, 2, 3, 4, 5$, i.e.,

$$
\begin{align*}
h_{n,i}^{1,\text{in}} &= KK\varepsilon(\hat{m_1}(KK\varepsilon(q(\varepsilon_{n,0}))), e), \\
h_{n,i}^{1,\text{out}} &= KK\varepsilon(\hat{m_1}(x_{1,0}^{-1} \times z_n \times KK\varepsilon(\varepsilon_{n,0}))), e),
\end{align*}
$$

for all $i = 0, 1, 2, 3, 4, 5$ (of course we use the canonical $KK\varepsilon$-equivalences from Remark 7.6 to identify $KK\varepsilon(f_{1,j+6}, e)$ with $KK\varepsilon(f_{1,j}, e)$).

Definition 7.9. For each $n \in \mathbb{N}_{\geq 2}$, there exists a short exact sequence $i(\varepsilon_{n,1}) \xrightarrow{i_{n,1}} \varepsilon_{n,1} \xrightarrow{q_{n,1}} q(\varepsilon_{n,1})$ of extensions. Define $h_{n,1}^{1,\text{in}}$ and $h_{n,1}^{1,\text{out}}$ by

$$
\begin{align*}
F_{n,1} &\xrightarrow{h_{n,1}^{1,\text{in}}} H_{n,1} &\xrightarrow{h_{n,1}^{1,\text{out}}} F_{1,3} \\
KK\varepsilon(q(\varepsilon_{n,1}), e) &\xrightarrow{KK\varepsilon(q(\varepsilon_{n,1}), e) \times -} KK\varepsilon(q(\varepsilon_{n,1}), e) &\xrightarrow{KK\varepsilon(q(\varepsilon_{n,1}), e) \times -} KK\varepsilon(\varepsilon_{n,1}, e) \\
\end{align*}
$$

By applying the functor $\hat{m_1}$, we define $h_{n,i}^{1,\text{in}}$ and $h_{n,i}^{1,\text{out}}$, for $i = 0, 2, 3, 4, 5$, i.e.,

$$
\begin{align*}
h_{n,i}^{1,\text{in}} &= KK\varepsilon(\hat{m_1}^{-1}(KK\varepsilon(q(\varepsilon_{n,1}))), e), \\
h_{n,i}^{1,\text{out}} &= KK\varepsilon(\hat{m_1}^{-1}(x_{1,0}^{-1} \times KK\varepsilon(\varepsilon_{n,1}))), e),
\end{align*}
$$

for all $i = 1, 2, 3, 4, 5, 6$. 


Definition 7.11. For each \( n \in \mathbb{N}_{>2} \), there exists a short exact sequence \( i(\varepsilon_{n,2}) \xrightarrow{i_{n,2}} \varepsilon_{n,2} \xrightarrow{q_{n,2}} q(\varepsilon_{n,2}) \) of extensions. Define \( h_{n,2}^{1,n_{in}} \) and \( h_{n,2}^{1,n_{out}} \) by

\[
\begin{array}{ccc}
F_{1,4} & \xrightarrow{h_{n,2}^{1,n_{in}}} & H_{n,2} \\
\downarrow & & \downarrow \\
KK\varepsilon(h_{1,4,e}) & \xrightarrow{KK\varepsilon(q(\varepsilon_{n,2})))xw_{n-1}xmc(z_{n-1}^{-1}xh_{1,1}x)} & KK\varepsilon(h_{n,2,e}) \\
\downarrow & & \downarrow \\
H_{n,2} & \xrightarrow{h_{n,2}^{1,n_{out}}} & F_{n,3}
\end{array}
\]

By applying the functor \( \overline{mc} \), we define \( h_{n,1,i}^{1,n_{in}} \) and \( h_{n,1,i}^{1,n_{out}} \), for \( i = 0, 1, 3, 4, 5 \), i.e.,

\[
\begin{align*}
& h_{n,1,i}^{1,n_{in}} = KK\varepsilon(\overline{mc}i^{-2}(KK\varepsilon(q(\varepsilon_{n,2})))xw_{n-1}xmc(z_{n-1}^{-1}xh_{1,1}), e), \\
& h_{n,1,i}^{1,n_{out}} = KK\varepsilon(\overline{mc}i^{-2}(x_{n,0}^{-1}xKK\varepsilon(i(\varepsilon_{n,2})), e),
\end{align*}
\]

for all \( i = 2, 3, 4, 5, 6, 7 \).

**Definition 7.11.** Now, we define homomorphisms \( f_{n,i} \) from \( F_{n,i} \) to \( F_{n,i+1} \), for all \( n \in \mathbb{N} \) and \( i = 0, 1, 2, 3, 4, 5 \). We set

\[
\begin{array}{ccc}
KK\varepsilon(\overline{S}f_{n,0,e}) & \xrightarrow{b_{n,0}^{-1}x-} & KK\varepsilon(f_{n,1,e}) \\
\downarrow & & \downarrow \\
F_{n,0} & \xrightarrow{f_{n,0}} & F_{n,1} \\
\downarrow & & \downarrow \\
F_{n,5} & \xrightarrow{f_{n,5}} & F_{n,6} \\
\downarrow & & \downarrow \\
KK\varepsilon(\overline{S}f_{n,2,e}) & \xrightarrow{KK\varepsilon(S_{f_{n,1}}e)} & KK\varepsilon(f_{n,1,e}) \\
\end{array}
\]

where the outer sequence is the cyclic six term exact sequence in \( KK\varepsilon \)-theory induced by the short exact sequence \( i(\varepsilon_{n,2}) \xrightarrow{i_{n,2}} \varepsilon_{n,2} \xrightarrow{q_{n,2}} q(\varepsilon_{n,2}) \) (which is exactly \( \overline{S}f_{n,0} \xrightarrow{\overline{S}f_{n,2}} \overline{S}f_{n,1} \)).

**Definition 7.12.** Now, we will define the Bockstein operations,

\[
\begin{array}{ccc}
F_{1,i} & \xrightarrow{\rho_{n,i}} & F_{n,i} \\
\downarrow & & \downarrow \\
F_{1,i+3} & \xrightarrow{\beta_{n,i}} & F_{1,i+3}
\end{array}
\]

for all \( n \in \mathbb{N}_{>2} \) and \( i = 0, 1, 2, 3, 4, 5 \).

The extension \( \varepsilon_{n,0}: SM_{n} \xrightarrow{\varepsilon_{n,0}} \mathbb{C} \) induces a short exact sequence \( i(\varepsilon_{n,0}) \xrightarrow{x} f_{n,0} \xrightarrow{y} f_{1,0} \). We set

\[
\begin{array}{ccc}
F_{1,0} & \xrightarrow{\rho_{n,0}} & F_{n,0} \\
\downarrow & & \downarrow \\
F_{1,3} & \xrightarrow{\beta_{n,0}} & F_{1,3}
\end{array}
\]

By applying the functor \( mc \), we define \( \rho_{n,i} \) and \( \beta_{n,i} \), for \( i = 1, 2, 3, 4, 5 \), i.e.,

\[
\begin{align*}
& \rho_{n,i} = KK\varepsilon(mc(KK\varepsilon(y)), e), \\
& \beta_{n,i} = KK\varepsilon(mc(x_{n,0}^{-1}xzKK\varepsilon(x), e),
\end{align*}
\]

for all \( i = 0, 1, 2, 3, 4, 5 \) (of course we use the canonical \( KK\varepsilon \)-equivalences from Remark 7.6 to make identifications modulo 6).

**Definition 7.13.** For each \( n \in \mathbb{N} \), set \( \tilde{f}_{n,i} = f_{n,i} \) for \( i = 1, 2, 4, 5 \) and \( \tilde{f}_{n,i} = -f_{n,i} \) for \( i = 0, 3 \).
Theorem 7.14. For all $n \in \mathbb{N}$ and all $i = 0, 1, 2, 3, 4, 5$,

$$F_{n,i-1} \xrightarrow{f_{n,i-1}} F_{n,i} \xrightarrow{f_{n,i}} F_{n,i+1}$$

is exact. For all $n \in \mathbb{N}_{\geq 2}$ and all $i = 0, 1, 2, 3, 4, 5$,

$$F_{1,i+1} \xrightarrow{h_{n,i}^{1,1,in}} H_{n,i} \xrightarrow{h_{n,i}^{1,1,out}} F_{1,i+3} \quad F_{n,i} \xrightarrow{h_{n,i}^{n,1,in}} H_{n,i} \xrightarrow{h_{n,i}^{n,1,out}} F_{n,i+2} \quad F_{1,i} \xrightarrow{h_{n,i}^{1,1,in}} H_{n,i} \xrightarrow{h_{n,i}^{1,1,out}} F_{1,i+2}$$

$$F_{1,i+2} \xrightarrow{h_{n,i}^{1,1,in}} H_{n,i} \xrightarrow{h_{n,i}^{1,1,out}} F_{n,i+1} \quad F_{n,i+4} \xrightarrow{h_{n,i}^{n,1,in}} H_{n,i} \xrightarrow{h_{n,i}^{n,1,out}} F_{n,i+5}$$

are exact, and, moreover, all the three diagrams

(1) $\quad F_{1,i} \xrightarrow{f_{1,i}} F_{1,i+1}$

(2) $\quad F_{1,i+1} \xrightarrow{f_{1,i+1}} F_{1,i+3}$

(3) $\quad F_{n,i+5} \xrightarrow{\beta_{n,i+5}} F_{n,i}$

commute.

Proof. See next section.  ■
Corollary 7.15. For each \( n \in \mathbb{N}_{\geq 2} \) and \( i = 0, 1, 2, 3, 4, 5 \), the two squares

\[
\begin{array}{c}
F_{1,i} \xrightarrow{f_{1,i}} F_{1,i+1} \\
\rho_{n,i} \quad \rho_{n,i+1} \quad \beta_{n,i} \quad \beta_{n,i+1}
\end{array}
\]

\[
\begin{array}{c}
F_{n,i} \xrightarrow{f_{n,i}} F_{n,i+1} \\
\end{array}
\]

and

\[
\begin{array}{c}
F_{1,i+1} \xrightarrow{f_{1,i+3}} F_{1,i+4} \\
\end{array}
\]

commute.

Proof. This follows directly from the previous theorem:

\[
\tilde{f}_{n,i} \circ \rho_{n,i} = \tilde{f}_{n,i} \circ h_{n,i}^{1,n,\text{out}} \circ h_{n,i}^{1,1,\text{in}} \quad \text{by } (2)
\]

\[
= \rho_{n,i+1} \circ h_{n,i-1}^{1,n,\text{out}} \circ h_{n,i-1}^{1,1,\text{in}} \quad \text{by } (3)
\]

\[
= \rho_{n,i+1} \circ \tilde{f}_{1,i} \quad \text{by } (1)
\]

\[
\beta_{n,i+1} \circ \tilde{f}_{n,i} = \beta_{n,i+1} \circ h_{n,i}^{1,n,\text{out}} \circ h_{n,i}^{n,1,\text{in}} \quad \text{by } (3)
\]

\[
= -\tilde{f}_{1,i+3} \circ h_{n,i}^{1,1,\text{out}} \circ h_{n,i}^{n,1,\text{in}} \quad \text{by } (2)
\]

\[
= -\tilde{f}_{1,i+3} \circ \beta_{n,i} \quad \text{by } (1)
\]

Remark 7.16. From the preceding theorem and corollary, it follows that, for each \( n \in \mathbb{N}_{\geq 2} \) and \( i = 0, 1, 2 \), we have the following — both horizontally and vertically six term cyclic — commuting diagrams with exact rows and columns:

\[
\begin{array}{cccccccccccc}
\tilde{f}_{1,0} & \times n & \tilde{f}_{1,0} & \times n & \tilde{f}_{1,1} & \times n & \tilde{f}_{1,2} & \times n & \tilde{f}_{1,3} & \times n & \tilde{f}_{1,4} & \times n & \tilde{f}_{1,5} \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
\tilde{f}_{n,0} & \times n & \tilde{f}_{n,0} & \times n & \tilde{f}_{n,1} & \times n & \tilde{f}_{n,2} & \times n & \tilde{f}_{n,3} & \times n & \tilde{f}_{n,4} & \times n & \tilde{f}_{n,5} \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
\tilde{f}_{1,0} & \times n & \tilde{f}_{1,0} & \times n & \tilde{f}_{1,1} & \times n & \tilde{f}_{1,2} & \times n & \tilde{f}_{1,3} & \times n & \tilde{f}_{1,4} & \times n & \tilde{f}_{1,5} \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
\tilde{f}_{n,0} & \times n & \tilde{f}_{n,0} & \times n & \tilde{f}_{n,1} & \times n & \tilde{f}_{n,2} & \times n & \tilde{f}_{n,3} & \times n & \tilde{f}_{n,4} & \times n & \tilde{f}_{n,5} \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
\tilde{f}_{1,0} & \times n & \tilde{f}_{1,0} & \times n & \tilde{f}_{1,1} & \times n & \tilde{f}_{1,2} & \times n & \tilde{f}_{1,3} & \times n & \tilde{f}_{1,4} & \times n & \tilde{f}_{1,5} \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
\tilde{f}_{1,0} & \times n & \tilde{f}_{1,0} & \times n & \tilde{f}_{1,1} & \times n & \tilde{f}_{1,2} & \times n & \tilde{f}_{1,3} & \times n & \tilde{f}_{1,4} & \times n & \tilde{f}_{1,5} \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
\tilde{f}_{1,0} & \times n & \tilde{f}_{1,0} & \times n & \tilde{f}_{1,1} & \times n & \tilde{f}_{1,2} & \times n & \tilde{f}_{1,3} & \times n & \tilde{f}_{1,4} & \times n & \tilde{f}_{1,5} \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
\tilde{f}_{1,0} & \times n & \tilde{f}_{1,0} & \times n & \tilde{f}_{1,1} & \times n & \tilde{f}_{1,2} & \times n & \tilde{f}_{1,3} & \times n & \tilde{f}_{1,4} & \times n & \tilde{f}_{1,5} \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
\tilde{f}_{1,0} & \times n & \tilde{f}_{1,0} & \times n & \tilde{f}_{1,1} & \times n & \tilde{f}_{1,2} & \times n & \tilde{f}_{1,3} & \times n & \tilde{f}_{1,4} & \times n & \tilde{f}_{1,5} \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
\tilde{f}_{1,0} & \times n & \tilde{f}_{1,0} & \times n & \tilde{f}_{1,1} & \times n & \tilde{f}_{1,2} & \times n & \tilde{f}_{1,3} & \times n & \tilde{f}_{1,4} & \times n & \tilde{f}_{1,5} \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
\tilde{f}_{1,0} & \times n & \tilde{f}_{1,0} & \times n & \tilde{f}_{1,1} & \times n & \tilde{f}_{1,2} & \times n & \tilde{f}_{1,3} & \times n & \tilde{f}_{1,4} & \times n & \tilde{f}_{1,5} \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
\tilde{f}_{1,0} & \times n & \tilde{f}_{1,0} & \times n & \tilde{f}_{1,1} & \times n & \tilde{f}_{1,2} & \times n & \tilde{f}_{1,3} & \times n & \tilde{f}_{1,4} & \times n & \tilde{f}_{1,5} \\
\end{array}
\]
The purpose of this section is to prove Theorem 7.14. First we need some results, which will be useful in the proof.

**Remark 7.17.** Note that Diagrams $(D_i)$ and $(D'_i)$ with two extra conditions each are equivalent, for $i = 1, 2, 3$. For instance, Diagram $(D_1)$ with the extra condition

$$nh_{n,i}^{1,1,\text{out}} = f_{1,i+2} \circ h_{n,i}^{1,1,\text{out}}, \quad nh_{n,i+3}^{1,1,\text{out}} = f_{1,i+5} \circ h_{n,i+3}^{1,1,\text{out}}$$

is equivalent to Diagram $(D'_1)$ with the extra condition

$$h_{n,i}^{1,1,\text{in}} \circ \rho_{n,i} = h_{n,i}^{1,1,\text{in}} \circ f_{1,i}, \quad h_{n,i+3}^{1,1,\text{in}} \circ \rho_{n,i+3} = h_{n,i+3}^{1,1,\text{in}} \circ f_{1,i+3}.$$ 

8. **Proof of Theorem 7.14**

The purpose of this section is to prove Theorem 7.14. First we need some results, which will be useful in the proof.

**Remark 8.1.** Let $\mathfrak{A}$ be a separable, nuclear $C^*$-algebra in the bootstrap category $\mathcal{N}$. Set $e_0 : \mathfrak{A} \xrightarrow{id} \mathfrak{A} \twoheadrightarrow 0$, and set $e_1 = mc(e_0)$. As earlier we know that

$$e_0 : \mathfrak{A} \xrightarrow{id} \mathfrak{A} \twoheadrightarrow 0,$$

$$e_1 : 0 \xrightarrow{\epsilon} \mathfrak{A} \xrightarrow{id} \mathfrak{A},$$

$$e_2 : \mathfrak{S}\mathfrak{A} \xrightarrow{\mathfrak{ev}} \mathfrak{C}\mathfrak{A},$$

$$e_3 : \mathfrak{S}\mathfrak{A} \xrightarrow{(0,1)} \mathfrak{C}\mathfrak{A} \oplus \mathfrak{ev}_{1,\epsilon} \mathfrak{C}\mathfrak{A} \xrightarrow{\pi_1} \mathfrak{C}\mathfrak{A},$$

where $\pi_1$ is the projection onto the first coordinate.

Note that there exists a canonical morphism, $\phi = (\mathfrak{id}, (0, \epsilon), 0)$, from $S e_0$ to $e_3$, which induces a $KK_{\mathcal{E}}$-equivalence. It is evident that $KK_{\mathcal{E}}(\phi)$ is exactly $x_{e_0}$ in the case that $\text{Ext}_{\mathcal{E}}^1(\mathfrak{K}_i(\mathfrak{A}), K_{1-i}(\mathfrak{A})) = 0$, for $i = 0, 1$. Also note that in this case, $KK_{\mathcal{E}}(mc(\phi)) = -x_{e_1}$ (according to Remark 7.16).
Note that \( i(e_2) = S e_0, q(e_2) = e_1, \) and \( \text{mc}(i(e_2)) = S e_1. \) So if applying \( \text{mc}^0, \text{mc}^1, \) and \( \text{mc}^2 \) to the short exact sequence \( i(e_2) \xrightarrow{i_{e_2}} e_2 \xrightarrow{q_{e_2}} q(e_2), \) we get just \( S e_0 \xrightarrow{i_{e_2}} e_2 \xrightarrow{q(e_2)} e_1, S e_1 \xrightarrow{mc(i_{e_2})} e_1 \rightarrow e_3 \rightarrow e_2, \) and \( \text{mc}S e_1 \xrightarrow{mc(i_{e_2})} e_4 \rightarrow e_3, \) respectively.

**Proposition 8.2.** Let \( \mathfrak{A} \) be a separable, nuclear \( C^* \)-algebra in the bootstrap category \( \mathcal{N} \) satisfying \( \text{Ext}^1(K_i(\mathfrak{A}), K_{i-1}(\mathfrak{A})) = 0, \) for \( i = 0, 1, \) and let \( e \) be an extension of separable \( C^* \)-algebras. Set \( e_0 : \mathfrak{A} \xrightarrow{id} \mathfrak{A} \rightarrow 0, \) and set \( e_i = \text{mc}^i(e_0). \) Then we have

\[
\begin{array}{c}
\text{KK}_e(Se_1, e) \\
\downarrow \text{Commutes} \\
\text{KK}_e(e_1, e) \\
\downarrow \text{Commutes} \\
\text{KK}_e(Se_0, e) \\
\downarrow \text{Commutes} \\
\text{KK}_e(e_0, e) \\
\downarrow \text{Commutes} \\
\text{KK}_e(e_1, e)
\end{array}
\]

where the inner and outer sequences are the cyclic six term exact sequences in \( \text{KK}_e \)-theory induced by \( i(e_2) \xrightarrow{i_{e_2}} e_2 \xrightarrow{q(e_2)} q(e_2) \) and \( \text{mc}(i(e_2)) \xrightarrow{mc(i_{e_2})} \text{mc}(e_2) \xrightarrow{mc(q_{e_2})} \text{mc}(q(e_2)), \) respectively. Moreover, we have that

\[
\begin{array}{c}
\text{KK}_e(Se_1, e) \\
\downarrow \text{Commutes} \\
\text{KK}_e(e_2, e) \\
\downarrow \text{Commutes} \\
\text{KK}_e(Se_3, e) \\
\downarrow \text{Commutes} \\
\text{KK}_e(e_3, e) \\
\downarrow \text{Commutes} \\
\text{KK}_e(e_4, e)
\end{array}
\]

where the inner and outer sequences are the cyclic six term exact sequences in \( \text{KK}_e \)-theory induced by \( \text{mc}(i(e_2)) \xrightarrow{mc(i_{e_2})} \text{mc}(e_2) \xrightarrow{mc(q_{e_2})} \text{mc}(q(e_2)), \) and \( \text{mc}^2(i(e_2)) \xrightarrow{mc^2(i_{e_2})} \text{mc}^2(e_2) \xrightarrow{mc^2(q_{e_2})} \text{mc}^2(q(e_2)), \) respectively.

**Proof.** First, writing out the short exact sequences \( S e_0 \xrightarrow{i_{e_2}} e_2 \xrightarrow{q(e_2)} e_1, S e_1 \xrightarrow{mc(i_{e_2})} e_1 \rightarrow e_3 \rightarrow e_2, \) and \( \text{mc}S e_1 \xrightarrow{mc(i_{e_2})} e_4 \rightarrow e_3 \) we get:
Now, we write out the cyclic six term exact sequences of cyclic six term exact sequences corresponding to these three short exact sequences—where we horizontally use the $KK$-boundary maps and vertically use the $K_{six}$-boundary maps. For convenience we will identify $K_1$ with $K_0 \circ S$. Moreover, we let

\[ \tilde{A} = C\mathfrak{A} \oplus_{\pi, \pi} C\mathfrak{A} \]

\[ \tilde{A} = \tilde{A} \oplus_{\pi_1, ev_1} CC\mathfrak{A}. \]

The diagrams are:
Note that $x_{e_0}$, $-x_{e_1}$, and $KK_E(\theta_{e_1})$ are induced by the morphisms

![Diagram]

Using all these diagrams, a long, tedious, straightforward verification shows the proposition. ■

**Remark 8.3.** What we actually showed in the proof of the preceding proposition, is that the corresponding diagrams of morphisms in the category $KK_E$ (i.e., before we apply $KK_E(-, e)$) commute respectively anti-commute. This observation will be useful in the sequel.

**Proof of the first part of Theorem 7.14** By definition, $F_{n,i-1} \xrightarrow{f_{n,i-1}} F_{n,i} \xrightarrow{f_{n,i}} F_{n,i+1}$ is exact for all $n \in \mathbb{N}$ and all $i = 0, 1, 2, 3, 4, 5$.

Note that there exists a commuting square

![Square]

where the maps $C \to M_n$ are the unital $\ast$-homomorphisms. By naturality of the mapping cone construction, this induces a morphism $\phi = (\phi_0, \phi_1, \phi_2)$ from $f_{1,2}$ to $e_{n,0}$. This gives a commuting diagram

![Diagram]

of short exact sequences. If we apply $KK_E(-, e)$ to this diagram we will get a morphism between two cyclic six term exact sequences in $KK_E$-theory. Using the standard equivalences introduced so far, we
arrive at the commuting diagrams

\[
\begin{array}{cccccc}
\cong & s_{1,0} & & & & \\
\cong & s_{1,1} \times - & & & & \\
\cong & s_{1,2} & & & & \\
\cong & s_{1,3} & & & & \\
\cong & s_{1,0} \times - & & & & \\
\cong & s_{1,1} \times - & & & & \\
\cong & s_{1,2} \times - & & & & \\
\cong & s_{1,3} \times - & & & & \\
\cong & s_{1,0} \times - & & & & \\
\cong & s_{1,1} \times - & & & & \\
\cong & s_{1,2} \times - & & & & \\
\cong & s_{1,3} \times - & & & & \\
\cong & s_{1,0} \times - & & & & \\
\cong & s_{1,1} \times - & & & & \\
\cong & s_{1,2} \times - & & & & \\
\cong & s_{1,3} \times - & & & & \\
\cong & s_{1,0} \times - & & & & \\
\cong & s_{1,1} \times - & & & & \\
\cong & s_{1,2} \times - & & & & \\
\cong & s_{1,3} \times - & & & & \\
\cong & s_{1,0} \times - & & & & \\
\cong & s_{1,1} \times - & & & & \\
\cong & s_{1,2} \times - & & & & \\
\cong & s_{1,3} \times - & & & & \\
\cong & s_{1,0} \times - & & & & \\
\end{array}
\]

with exact rows. We use Lemma 5.3 for commutativity of the two squares between row three and four in the lower part of the diagram — and we use that

\[
\text{Ext}^1_{\text{six}}(K_{\text{six}}(\epsilon_{n,0}), K_{\text{six}}(S_{\mathfrak{f}1,1})) = 0,
\]

\[
\text{Ext}^1_{\text{six}}(K_{\text{six}}(S\mathfrak{f}1,3), K_{\text{six}}(S\epsilon_{n,3})) = 0.
\]

This is easily verified using projective resolutions.

It is easy to verify that, up to a sign, we have

\[
KK_{\mathcal{E}}(\mathbb{Z}_n^{-1} \times KK_{\mathcal{E}}((\phi_0, \phi_0, 0)), e) = \text{nid} \quad \text{and} \quad KK_{\mathcal{E}}(\mathbb{Z}_n^{-1} \times \tilde{S}KK_{\mathcal{E}}((\phi_0, \phi_0, 0)), e) = \text{nid}.
\]

Consequently, \(n f_{1,0}\) and \(n f_{1,3}\) are exactly the connecting homomorphisms of the cyclic six term exact sequence in the bottom.

This proves exactness of the first of the four cyclic sequences in the theorem, for \(i = 0, 3\).

This same result also works for \(i = 1, 2, 4, 5\), by invoking Proposition 8.2 (recall that we do not care about the signs, because that does not change exactness).

Note that there exists a commuting square

\[
\begin{array}{ccc}
\mathbb{Z}_{n,0} & \cong & \mathbb{Z}_{n,0} \\
\downarrow & & \downarrow \\
\mathbb{Z}_{n,0} & \cong & \mathbb{C},
\end{array}
\]

where the maps \(\mathbb{Z}_{n,0} \to \mathbb{C}\) are the canonical surjective \(\ast\)-homomorphisms. By naturality of the mapping cone construction, this induces a morphism \(\phi = (\phi_0, \phi_1, \phi_2)\) from \(f_{n,2}\) to \(\epsilon_{n,1}\). This gives a commuting
of short exact sequences. If we apply $KK^E(-, e)$ to this diagram we will get a morphism between two cyclic six term exact sequences in $KK^E$-theory. Using the standard equivalences introduced so far, we arrive at the commuting diagram

\[
\begin{array}{cccccc}
& f_{n,1} & \to & f_{n,2} & \to & f_{n,3} \\
\downarrow (\phi_0, \phi_0, 0) & \phi & \phi & \phi \\
& c_{n,1} & \to & c_{n,2} & \to & c_{n,3} \\
\end{array}
\]

with exact rows. We use Lemma 7.3 for commutativity of the two squares between row three and four in the lower part of the diagram — and we use that

\[
\begin{align*}
\text{Ext}^1_{\text{six}}(K_{\text{six}}(S\ell_{n,1}), K_{\text{six}}(S\ell_{n,4})) &= 0, \\
\text{Ext}^1_{\text{six}}(K_{\text{six}}(S\ell_{1,3}), K_{\text{six}}(S\ell_{4,4})) &= 0.
\end{align*}
\]

This is easily verified using projective resolutions.

Using naturality of $b_-$ and Lemma 7.4 it is easy to see that

\[
KK^E(x_{l_{1,n}}^{-1} \times KK^E((\phi_0, \phi_0, 0)) \times x_{f_{1,n}}, e) = \rho_{n,3}, \quad \text{and}
\]

\[
KK^E(b_{l_{1,n}} \times KK^E((\phi_0, \phi_0, 0)) \times b_{f_{1,n}}^{-1}, e) = \rho_{n,0}.
\]

Consequently, $f_{n,3} \circ \rho_{n,3}$ and $f_{n,0} \circ \rho_{n,0}$ are exactly the connecting homomorphisms of the cyclic six term exact sequence in the bottom.

This proves exactness of the second of the four cyclic sequences in the theorem, for $i = 1, 4$. This same result also works for $i = 0, 2, 3, 5$, by invoking Proposition 5.2.
Note that there exists a commuting square

\[
\begin{array}{ccc}
\mathbb{I}_{n,1} & \to & \mathbb{I}_{n,0} \\
\downarrow & & \downarrow \\
\mathbb{I}_{n,0} & \to & \mathbb{n}_{n,0},
\end{array}
\]

where the maps \( \mathbb{I}_{n,1} \to \mathbb{I}_{n,0} \) are the canonical surjective \(*\)-homomorphisms. By naturality of the mapping cone construction, this induces a morphism \( \phi = (\phi_0, \phi_1, \phi_2) \) from \( \epsilon_{n,2} \) to \( f_{n,2} \). This gives a commuting diagram

\[
\begin{array}{c}
i(\epsilon_{n,2}) \ar@{->}[r] & \epsilon_{n,2} \ar[r] & q(\epsilon_{n,2}) \\
\phi \ar@{|->}[rr] & & (0,\phi_2,\phi_2)
\end{array}
\]

\[
\begin{array}{c}
i(f_{n,2}) \ar@{->}[r] & f_{n,2} \ar[r] & q(f_{n,2}) \\
\phi \ar@{|->}[rr] & & (0,\phi_2,\phi_2)
\end{array}
\]

of short exact sequences. If we apply \( KK_E(\cdot, e) \) to this diagram we will get a morphism between two cyclic six term exact sequences in \( KK_E \)-theory. Using the standard equivalences introduced so far, we arrive at the commuting diagram

\[
\begin{array}{cccc}
F_1 & \to & H_{n,2} & \to & F_{n,3} \\
B_{n,2}^{1, n, \text{in}} & \to & H_{n,2} & \to & B_{n,2}^{1, n, \text{out}} \\
KK_E(f_{n,1}, e) & \to & KK_E(\epsilon_{n,2}, e) & \to & KK_E(f_{n,3}, e) \\
S(w_n^{-1} \times \tilde{\omega}(z_n^{-1})) \times b_{n,1}^{-1} \times - & \to & KK_E(q(\epsilon_{n,2}), e) & \to & KK_E(Sf_{n,0}, e) \\
KK_E(Sf_{n,3}, e) & \to & KK_E(S\epsilon_{n,2}, e) & \to & KK_E(Sf_{n,0}, e) \\
F_{n,0} & \to & F_{n,1} & \to & H_{n,5} & \to & F_{n,0} \\
B_{n,5}^{1, n, \text{in}} & \to & H_{n,5} & \to & B_{n,5}^{1, n, \text{out}} \\
KK_E(f_{n,1}, e) & \to & KK_E(\epsilon_{n,5}, e) & \to & KK_E(f_{n,6}, e) \\
S(w_n^{-1} \times \tilde{\omega}(z_n^{-1})) \times b_{n,1}^{-1} \times - & \to & KK_E(\epsilon_{n,2}, e) & \to & KK_E(\epsilon_{n,5}, e) \\
KK_E(Sf_{n,3}, e) & \to & KK_E(S\epsilon_{n,2}, e) & \to & KK_E(Sf_{n,0}, e) \\
F_{n,3} & \to & F_{n,4} & \to & H_{n,5} & \to & F_{n,0} \\
B_{n,5}^{1, n, \text{in}} & \to & H_{n,5} & \to & B_{n,5}^{1, n, \text{out}} \\
KK_E(f_{n,1}, e) & \to & KK_E(\epsilon_{n,5}, e) & \to & KK_E(f_{n,6}, e) \\
S(w_n^{-1} \times \tilde{\omega}(z_n^{-1})) \times b_{n,1}^{-1} \times - & \to & KK_E(\epsilon_{n,2}, e) & \to & KK_E(\epsilon_{n,5}, e)
\end{array}
\]

with exact rows. We use Lemma 7.5 for commutativity the two squares on the right hand side between row two and three — and we use that

\[
\begin{align*}
\text{Ext}^1_{\text{six}}(K_{\text{six}}(S\epsilon_{n,2}), K_{\text{six}}(Sf_{1,1})) &= 0, \\
\text{Ext}^1_{\text{six}}(K_{\text{six}}(f_{n,0}), K_{\text{six}}(SS\epsilon_{n,2})) &= 0.
\end{align*}
\]

This is easily verified using projective resolutions.
Using naturality of $b_-$ and Lemma 7.3 it is easy to see that

$$KK_\mathcal{E}(x_{f,1} \times \tilde{m}(z_n) \times w_n \times KK_\mathcal{E}((0, \phi_2, \phi_2)), e) = -\beta_{n,1},$$

and

$$KK_\mathcal{E}(b_{h_1,1} \times \tilde{m}(z_n) \times \tilde{w}_n \times KK_\mathcal{E}(S(0, \phi_2, \phi_2)), e) = -\beta_{n,3}.$$

Consequently, $\beta_{n,4} \circ f_{n,3}$ and $\beta_{n,1} \circ f_{n,0}$ are exactly the connecting homomorphisms of the cyclic six term exact sequence in the top (up to a sign, of course).

This proves exactness of the third of the four cyclic sequences in the theorem, for $i = 2, 5$.

This same result also works for $i = 0, 1, 3, 4$, by invoking Proposition 8.2.

That the last one of the sequences is exact for all $i = 0, 1, 2$ is straightforward to check. $\blacksquare$

**Proof of the second part of Theorem 7.14.** Diagram (1). First we prove it for $i = 1$. We have a commuting diagram

$$
\begin{array}{ccc}
F_{1,1} & \xrightarrow{f_{1,1}} & F_{1,2} \\
\rho_{n,1} \downarrow & & \downarrow h_{n,1}^{1,1} \\
F_{n,1} & \xrightarrow{h_{n,1}^{n,1}} & H_{n,1} \\
\beta_{n,1} \downarrow & & \downarrow \beta_{n,1}^{1,1} \\
F_{1,4} & \xrightarrow{\tilde{f}_{1,4}} & F_{1,3}
\end{array}
$$

of objects from $\mathcal{C}$ with short exact rows and short exact columns. Note that the short exact sequences $i(\epsilon_{n,1}) \hookrightarrow mc(i(\epsilon_{n,0})) \rightarrow f_{n,1}$ and $mc(i(\epsilon_{n,0})) \twoheadrightarrow q(\epsilon_{n,1}) \twoheadrightarrow f_{n,1}$ are exactly the short exact sequences $i(\phi_{1,2}) \hookrightarrow \phi_{1,2} \twoheadrightarrow q(\phi_{1,2})$ and $mc$ applied to the sequence $i(\epsilon_{n,0}) \xrightarrow{\epsilon_{n,0}} f_{n,0} \xrightarrow{q} f_{n,1}$ from Definition 7.12 respectively. Now apply $KK_\mathcal{E}(\cdot, e)$, then one easily shows the commutativity of the diagram (using the definitions of the different maps)

$$
\begin{array}{ccc}
F_{1,1} & \xrightarrow{f_{1,1}} & F_{1,2} \\
\rho_{n,1} \downarrow & & \downarrow h_{n,1}^{1,1} \\
F_{n,1} & \xrightarrow{h_{n,1}^{n,1}} & H_{n,1} \\
\beta_{n,1} \downarrow & & \downarrow \beta_{n,1}^{1,1} \\
F_{1,4} & \xrightarrow{\tilde{f}_{1,4}} & F_{1,3}
\end{array}
$$

If we apply $mc^k$ to the diagram, for $k = 1, 2, 3, 4, 5$, we obtain commutativity of the corresponding part of Diagram (1), for $i = 2, 3, 4, 5, 0$, respectively — this is, indeed, a very long and tedious verification using the identifications and results above.

Now we prove commutativity of the remaining square in Diagram (1) for $i = 1$. Note that there exists a commuting diagram

$$
\begin{array}{ccc}
Sf_{1,1} & \xrightarrow{Sf_{1,0}} & Sf_{1,3} \\
\downarrow \phi & & \downarrow \phi \\
q(\epsilon_{n,2}) & \xrightarrow{\phi} & \epsilon_{n,1},
\end{array}
$$

where the horizontal morphisms are the unique morphism which are the identity on the extension algebra, and the vertical morphism from $Sf_{1,1}$ to $q(\epsilon_{n,2})$ is the morphism induced by the $*$-homomorphism $SC \rightarrow \mathcal{I}_{n,1}$ in the extension $\epsilon_{n,1}$. It is easy to see that $mc(i_{\epsilon_{n,0}})$ is exactly $\phi \circ w_n$, where $w_n$ is
the morphism inducing \( w_n \). Now we get commutativity of

\[
\begin{array}{ccc}
H_{n,1}^3 & \to & F_{1,3} \\
\downarrow_{h_{n,1}^{1,0}} & & \downarrow_{\beta_{n,3}} \\
F_{1,4} & \to & F_{1,0},
\end{array}
\]

by applying \( KK_E(-,e) \) to the above diagram. If we first apply \( mc^k \) to the diagram, for \( k = 1, 2, 3, 4, 5 \), we obtain commutativity of the corresponding diagram of Diagram (1), for \( i = 2, 3, 4, 5, 0 \), respectively.

Diagram (2). We first prove it for \( i = 2 \). We have a commuting diagram

\[
\begin{array}{ccc}
i(mc^2(Sf_{1,0})) & \to & q(mc^2(Sf_{1,0})) \\
h_{n,2}^{1,0} & \to & h_{n,2}^{1,0} \\
\downarrow & \ & \downarrow \\
\xi_{n,2} & \to & q(\epsilon_{n,2}) \\
Sf_{1,0} & \to & F_{1,3}
\end{array}
\]

where \( \xi_{1,0}, x_n, z_n \) and \( w_n \) denote the morphisms inducing \( \xi_{1,0}, x_n, \) and \( w_n \), respectively, and the first column is the suspension of the short exact sequence introduced in Definition 7.12 (note that we do not claim the columns and rows to be exact).

A computation shows that this gives rise to a commutative diagram (by applying \( KK_E(-,e) \))

\[
\begin{array}{ccc}
F_{1,3} & \to & F_{1,5} \\
\downarrow_{\rho_{n,3}} & & \downarrow_{f_{1,3}} \\
H_{n,2} & \to & F_{1,0} \\
\downarrow_{h_{n,2}} & & \downarrow_{f_{1,0}} \\
F_{1,4} & \to & F_{1,0} \\
\downarrow_{h_{n,2}^{1,0}} & & \downarrow_{f_{1,0}} \\
F_{1,2} & \to & F_{1,0} \\
\end{array}
\]

If we apply \( mc^k \) to the diagram, for \( k = 1, 2, 3, 4, 5 \), we obtain commutativity of the corresponding square of Diagram (3), for \( i = 3, 4, 5, 0, 1 \), respectively — this is, indeed, a very long and tedious verification using the identifications and results above.

Now we prove commutativity of the remaining square in Diagram (2) for \( i = 2 \). Note that there exists a commuting diagram

\[
\begin{array}{ccc}
\epsilon_{n,2} & \to & q(\epsilon_{n,2}) \\
\downarrow_{(\star,\text{id},0)} & & \downarrow_{(0,\text{id},0)} \\
[I_{n,2} \leftrightarrow \tilde{I}_{n,2} \to 0] & & [I_{n,1} \leftrightarrow \tilde{I}_{n,1} \to 0],
\end{array}
\]

where the bottom horizontal morphism is the morphism induced by the \( \star \)-homomorphism from \( l_{n,2} \) to \( \tilde{l}_{n,1} \) in the extension \( \epsilon_{n,2} \). It is easy to see that there exists a commuting square

\[
\begin{array}{ccc}
Sf_{1,1} & \to & F_{1,4} \\
\downarrow_{KKE(\xi(0,\text{id},0))} & & \downarrow_{KKE(\xi(0,\text{id},0))} \\
Sf_{1,0} & \to & F_{1,3}
\end{array}
\]
If we first apply $\text{Ext}^{1}_{\text{six}}(\varepsilon_{n,2},\sigma_{1,0})$ to the diagram, for $k = 1, 2, 3, 4, 5$, we obtain commutativity of the corresponding square of Diagram (2), for $i = 3, 4, 5, 0, 1$, respectively.

Diagram (3). First we prove it for $i = 2$. We have a commuting diagram

$$
\begin{array}{ccc}
F_{1,3} & \xrightarrow{\times n} & F_{1,3} \\
\downarrow & & \downarrow \\
F_{1,4} & \xrightarrow{\beta_{n,2}} & H_{n,2}.
\end{array}
$$

with short exact rows and columns. The short exact sequence $i(\varepsilon_{n,2}) \hookrightarrow \text{mc}(q(\varepsilon_{n,1})) \hookrightarrow f_{n,1}$ is exactly the short exact sequence $i(\varepsilon_{n,2}) \hookrightarrow \text{mc}(q(\varepsilon_{n,1})) \hookrightarrow f_{n,1}$ is exactly the short exact sequence $\text{mc}(i(\varepsilon_{n,1})) \hookrightarrow q(\varepsilon_{n,2}) \hookrightarrow f_{n,1}$ induced by the extension $\varepsilon_{n,1}: SC \hookrightarrow I_{n,1} \hookrightarrow I_{n,0}$, where $e$ is $0 \hookrightarrow I_{n,1} \hookrightarrow I_{n,0}$.

A computation shows that this gives rise to a commutative diagram (by applying $\text{KK}_F(-,e)$)

$$
\begin{array}{ccc}
F_{1,4} & \xrightarrow{\times n} & F_{1,4} \\
\downarrow & & \downarrow \\
F_{1,3} & \xrightarrow{\beta_{n,1}} & F_{1,3}.
\end{array}
$$

If we apply $\text{mc}^k$ to the diagram, for $k = 1, 2, 3, 4, 5$, we obtain commutativity of the corresponding part of Diagram (3), for $i = 3, 4, 5, 0, 1$, respectively — this is, indeed, a very long and tedious verification using the identifications and results above.

Now we prove commutativity of the remaining square in Diagram (3) for $i = 2$. We have a commuting diagram

$$
\begin{array}{ccc}
Sf_{n,1} & \xrightarrow{e} & Sf_{n,1} \\
\downarrow & & \downarrow \\
i(\varepsilon_{n,2}) & \xrightarrow{i_{n,2}} & \varepsilon_{n,2}.
\end{array}
$$
where \( e \) is the extension \( 0 \hookrightarrow \mathbb{I}_{n,2} \xrightarrow{id} \mathbb{I}_{n,2} \), the map from \( \mathcal{S}\mathfrak{f}_{n,1} \) to \( e \) is the one induced by the map \( \mathcal{S}\mathfrak{f}_{n,0} \to \mathbb{I}_{n,2} \) in \( \mathfrak{e}_{n,2} \), the map from \( \mathcal{S}\mathfrak{f}_{n,1} \) to \( \mathcal{S}\mathfrak{f}_{n,0} = \mathfrak{i}(\mathfrak{e}_{n,2}) \) is the unique morphism which is the identity on the extension algebra, and the morphism from \( e \) to \( \mathfrak{e}_{n,2} \) is the unique morphism which is the identity on the extension algebra. It is elementary to see that if we compose the morphism from \( e \) to \( \mathfrak{e}_{n,2} \) with the canonical identification of \( e \) with \( \mathcal{S}\mathfrak{f}_{1} \), we get exactly the morphism \( \text{mc}^3(\mathfrak{e}) \). If \( \phi \) denotes the obvious morphism from \( \mathfrak{f}_{n,1} \) to \( \mathfrak{f}_{n,0} \), it is elementary to show that \( \text{mc}^3(\phi) \) is \( \text{mc}^2(\mathfrak{f}_{n,2}) \).

Using all this, we see that this gives rise to a commuting square

\[
\begin{array}{ccc}
H_{n,2} & \xrightarrow{h_{n,2}} & F_{n,3} \\
\downarrow & & \downarrow \\
F_{1,4} & \xrightarrow{p_{1,4}} & F_{n,4} \\
\end{array}
\]

If we first apply \( \text{mc}^k \) to the diagrams, for \( k = 1, 2, 3, 4, 5 \), we obtain commutativity of the corresponding square of Diagram (3), for \( i = 3, 4, 5, 0, 1 \), respectively.

9. Examples

In the article [ERR11], the authors examine the invariant \( K_{\text{six}} \) of extensions. For an extension \( \mathfrak{A}_0 \hookrightarrow \mathfrak{A}_1 \to \mathfrak{A}_2 \), the invariant consists of the six term exact sequences:

\[
\begin{array}{ccc}
K_0(\mathfrak{A}_0) & \to & K_0(\mathfrak{A}_1) \\
\downarrow & & \downarrow \\
K_0(\mathfrak{A}_1) & \to & K_0(\mathfrak{A}_2) \\
\end{array}
\]

\[
\begin{array}{ccc}
K_0(\mathfrak{A}_0; \mathbb{Z}_n) & \to & K_0(\mathfrak{A}_1; \mathbb{Z}_n) \\
\downarrow & & \downarrow \\
K_0(\mathfrak{A}_1; \mathbb{Z}_n) & \to & K_0(\mathfrak{A}_2; \mathbb{Z}_n) \\
\end{array}
\]

together with all the Bockstein operations. A homomorphism between the invariants is a family of group homomorphisms respecting all the above maps as well as all the individual Bockstein operations. We let \( \text{Hom}_\Gamma(\tilde{K}_{\text{six}}(\mathfrak{e}), \tilde{K}_{\text{six}}(\mathfrak{e}')) \) denote the group of such homomorphisms.

In the article [ERR11], the authors prove that there is a natural homomorphism from \( KK\ell(e, e') \) to \( \text{Hom}_\Gamma(\tilde{K}_{\text{mix}}(e), \tilde{K}_{\text{mix}}(e')) \). Moreover, the authors prove through a series of examples that this homomorphism is neither surjective nor injective. We take a closer look at this series of examples here.

Example 9.1. We will compute the invariant \( \tilde{K}_{\ell}(\mathfrak{e}_{n,i}) \) for \( n \in \mathbb{N}_{\geq 2} \) and \( i = 0, 1, \ldots, 5 \). The groups are as in the table below.

\[
\begin{array}{ccc}
K_0(\mathfrak{A}_0; \mathbb{Z}_n) & \otimes & K_0(\mathfrak{A}_1; \mathbb{Z}_n) \\
\downarrow & & \downarrow \\
K_0(\mathfrak{A}_1; \mathbb{Z}_n) & \otimes & K_0(\mathfrak{A}_2; \mathbb{Z}_n) \\
\end{array}
\]
First case ($\epsilon_{n,0}$): Using the first sequence from Theorem 7.14 with $i = 0$, we get that $H_{k,0} = \mathbb{Z}$. Using the first sequence from Theorem 7.14 with $i = 1$, we get that $H_{k,4} = \mathbb{Z}$. Using the first sequence from Theorem 7.14 with $i = 2$, we get that $H_{k,2} = \mathbb{Z}_{m,k}$ and that $H_{k,5} = 0$. Using the third sequence from Theorem 7.14 with $i = 0$, we get that $H_{k,3} = \mathbb{Z}_{n,k}$. If we put $i = 1$, we see from the second sequence that $H_{k,1}$ fits into a short exact sequence

$$0 \longrightarrow \mathbb{Z}_{n,k} \longrightarrow H_{k,1} \longrightarrow \mathbb{Z} \longrightarrow 0.$$ 

Since $\mathbb{Z}$ is projective, $H_{k,1} = \mathbb{Z} \oplus \mathbb{Z}_{n,k}$.

The other cases follow by symmetry.

The $H_{k,i,j}$’s above tell us what $KK_\mathbb{E}(\epsilon_{k,i}, \epsilon_{n,j})$ is for all $k, n \in \mathbb{N}_{>2}$ and $i, j = 0, 1, 2, 3, 4, 5$. A lengthy, quite straightforward computation shows that the group of homomorphisms from $KK_\mathbb{E}(\epsilon_{k,i})$ to $KK_\mathbb{E}(\epsilon_{n,j})$ agrees with these groups for all $k, n \in \mathbb{N}_{>2}$ and $i, j = 0, 1, 2, 3, 4, 5$, and thus all of the counterexamples from [ERR11] become examples when we augment the invariant this way.

A more systematic proof of the above claims (and some generalizations) are given in the paper [ERR].

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