A method for minimum risk portfolio optimization under hybrid uncertainty

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Abstract. In this paper, we investigate a minimum risk portfolio model under hybrid uncertainty when the profitability of financial assets is described by fuzzy random variables. According to Feng, the variance of a portfolio is defined as a crisp value. To aggregate fuzzy information the weakest (drastic) t-norm is used. We construct an equivalent stochastic problem of the minimum risk portfolio model and specify the stochastic penalty method for solving it.

1. Introduction
Nowadays, various methods for solving portfolio optimization problems under hybrid uncertainty are being developed [1-3]. In the case of hybrid uncertainty of possibilistic-probabilistic type, portfolio optimization problems can be solved by techniques of possibilistic-probabilistic programming.

In this work we investigate a model of minimum risk portfolio where the risk of a portfolio is modeled by the variance. The variance of a portfolio can be defined in two different ways: as a crisp value or as a fuzzy variable. We use the former definition [4]. To aggregate fuzzy information the weakest (drastic) t-norm is used, that is, we consider that all fuzzy parameters are mutually $T_w$-related.

This mathematical model of a minimum risk portfolio has been studied in [3] where equivalent crisp problems have been constructed. However, obtaining equivalent crisp problems requires the calculation of the precise values of the expected value and the variance of a random function. In the case of the weakest t-norm, the expected value and the variance of a portfolio are of such a complex nature that the calculation of their precise values is almost impossible. Moreover, the results of this calculation are nonlinear non-convex functions, which lead to a non-convex optimization problem.

To avoid difficult calculations the methods of stochastic programming, such as the stochastic quasigradient methods, can be used [5]. In [6, 7] the stochastic quasigradient method is used to solve possibilistic-probabilistic optimization problems in the case of the weakest t-norm.

In this work we construct an equivalent stochastic problem of the minimum risk portfolio optimization problem. To solve it we specify the stochastic penalty method, which is a modification of the stochastic quasigradient method. Using of the stochastic penalty method is demonstrated on numerical examples and the solutions are compared with the results obtained in [3].
2. Materials and methods

2.1. Necessary concepts and definitions

We introduce a number of definitions and concepts from the possibility theory following [8-10]. Let \((\Gamma, \mathcal{F}(\Gamma), \tau)\) and \((\Omega, \mathcal{B}, \mathcal{P})\) be possibilities and probabilities spaces where \(\Omega\) is a sample space with possible outcomes \(\omega \in \Omega\), \(\Gamma\) is a pattern space with elements \(\gamma \in \Gamma\), \(\mathcal{B}\) is an \(\sigma\)-algebra of events, \(\mathcal{F}(\Gamma)\) is the discrete topology on \(\Gamma\), \(\tau \in \{\pi, \nu\}\), \(\pi\) and \(\nu\) are measures of possibility and necessity respectively, \(\mathcal{P}\) is a probability measure and \(E^1\) is the real line.

**Definition 1.** Fuzzy random variable \(Y\) is a real function \(Y : \Omega \times \Gamma \rightarrow E^1\), which is \(\sigma\)-measurable for each fixed \(\gamma\) and

\[
\mu_\gamma(\omega, t) = \pi \{\gamma \in \Gamma : Y(\omega, \gamma) = t\}
\]

is called its distribution function.

It follows from Definition 1 that the distribution function of a fuzzy random variable depends on a random parameter, that is, it is a random function.

**Definition 2.** Let \(Y(\omega, \gamma)\) be a fuzzy random variable. Its expected value \(E[Y]\) is a fuzzy variable with the possibility distribution function

\[
\mu_{E[Y]}(t) = \pi \{\gamma \in \Gamma : Y(\omega, \gamma) = t\},
\]

where \(E\) is the mathematical expectation operator

\[
E[Y(\omega, \gamma)] = \int Y(\omega, \gamma) \mathcal{P}(d\omega).
\]

In this case, the distribution function of the expected value of a fuzzy random variable is no longer dependent on a random parameter and is therefore deterministic. We define second central moment following [4]. Let \(X\) and \(Y\) be fuzzy random variables.

**Definition 3.** A covariance of fuzzy random variables \(X\) and \(Y\) is defined as:

\[
\text{cov}(X, Y) = \frac{1}{2} \int \left( \text{cov}(X_\omega(\alpha), Y_\omega(\alpha)) + \text{cov}(X_\omega^+(\alpha), Y_\omega^+(\alpha)) \right) d\alpha,
\]

where \(X_\omega(\alpha), Y_\omega(\alpha), X_\omega^+(\alpha), Y_\omega^+(\alpha)\) are left and right boundaries of \(\alpha\)-level sets of fuzzy variables \(X_\omega\) and \(Y_\omega\).

**Definition 4.** A variance of a fuzzy random variable \(Y\) is

\[
\text{D}[Y] = \text{cov}(Y, Y).
\]

The expected value, variance and covariance of fuzzy random variables determined in such a way inherit basic properties of the corresponding characteristics of real-valued random variables.

The most interesting representation of a fuzzy random variable is a shift-scale representation [10]:

\[
Y(\omega, \gamma) = a(\omega) + \sigma(\omega) Z(\gamma),
\]

where \(a(\omega), \sigma(\omega)\) are random variables defined on the probability space \((\Omega, \mathcal{B}, \mathcal{P})\) and \(Z(\gamma)\) is a fuzzy variable defined on the possibility space \((\Gamma, \mathcal{F}(\Gamma), \tau)\). Random components \(a(\omega)\) and \(\sigma(\omega)\) are called the shift and the scale respectively.

LR-type distributions are often used to model fuzzy numbers [11].
Definition 5. $Z(\gamma)$ is called an LR-type fuzzy variable if its distribution function has the form

$$
\mu_Z(t) = \begin{cases} 
L \left( \frac{m-t}{d} \right), & t < m, \\
1, & m \leq t \leq \bar{m}, \\
R \left( \frac{t-\bar{m}}{d} \right), & t > \bar{m},
\end{cases}
$$

where $L(t), R(t)$ are shape functions [11].

In this case $Z(\gamma)$ is written in the form $Z = [m, \bar{m}, d, \bar{d}]_{LR}$, where $m \leq \bar{m}$ are left and right limits of tolerance (modal) interval, $d > 0, \bar{d} > 0$ are coefficients of fuzziness.

We use triangular norms and conorms (t-norms and t-conorms) as an instrument for aggregation of fuzzy information. They extend min and max operations on which actions on fuzzy sets and fuzzy variables are based [12].

In particular, in this work we consider two extreme t-norms

$$
T_m(x, y) = \min\{x, y\} \quad \text{and} \quad T_w(x, y) = \begin{cases} 
\min\{x, y\}, & \text{if } \max\{x, y\} = 1, \\
0, & \text{otherwise.}
\end{cases}
$$

$T_m$ and $T_w$ are called the strongest and the weakest t-norm respectively, since for any arbitrary t-norm $T$ and $\forall x, y \in [0, 1]$ the following inequality holds [12]:

$$
T_w(x, y) \leq T(x, y) \leq T_m(x, y).
$$

One of the main properties of t-norms is their ability to control uncertainty (“fuzziness”) growth. The growth of fuzziness can appear, for example, when performing arithmetic operations on fuzzy numbers: if two fuzzy numbers of LR-type are summed using the strongest t-norm $T_m$, corresponding coefficients of fuzziness are also summed, therefore uncertainty is growing. With the help of t-norms other than $T_m$ we can slow the growth of fuzziness. The extreme triangular norms, which are considered in this work, give us boundaries for control of fuzziness in our minimum risk portfolio model.

Following [13] we introduce the notion of mutual t-relatedness of fuzzy sets and fuzzy variables. It is used as an instrument for constructing joint possibility distribution functions.

Definition 6. Fuzzy sets $A_1, \ldots, A_n \in \mathcal{P}(\Gamma)$ are called mutually T-related, if for any index set $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}, \ k = 1, n$, we have

$$
\pi(A_{i_1} \cap \ldots \cap A_{i_k}) = T(\pi(A_{i_1}), \ldots, \pi(A_{i_k})),
$$

where

$$
T(\pi(A_{i_1}), \ldots, \pi(A_{i_k})) = T \left( \ldots T \left( T \left( \pi(A_{i_1}, \pi(A_{i_2})), \pi(A_{i_3}) \right), \ldots, \pi(A_{i_k}) \right) \right).
$$

We can extend the notion of mutual T-relatedness of fuzzy sets to T-relatedness of fuzzy variables. Let $Z_i(\gamma), \ldots, Z_n(\gamma)$ be fuzzy variables defined on the possibility space $(\Gamma, \mathcal{P}(\Gamma), \pi)$. 
Definition 7. Fuzzy variables $Z_i(\gamma), \ldots, Z_n(\gamma)$ are called mutually $T$-related if for any index set, 
$\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, $k = 1, n$, we have
\[
\mu_{Z_{i_1}, \ldots, Z_{i_k}}(t_{i_1}, \ldots, t_{i_k}) = \pi \left\{ \gamma \in \Gamma : Z_{i_1}(\gamma) = t_{i_1}, \ldots, Z_{i_k}(\gamma) = t_{i_k} \right\}
\]
\[
= \pi \left[ Z_{i_1}^{-1}[t_{i_1}] \cap \ldots \cap Z_{i_k}^{-1}[t_{i_k}] \right] = T \left( \pi \left( Z_{i_1}^{-1}[t_{i_1}] \right), \ldots, \pi \left( Z_{i_k}^{-1}[t_{i_k}] \right) \right), 

\]

2.2. Minimum risk portfolio under conditions of hybrid uncertainty
In conditions of hybrid uncertainty of the possibilistic-probabilistic type the return of an investment portfolio can be represented as a fuzzy random function
\[
R_p(x, \omega, \gamma) = \sum_{i=1}^{n} R_i(\omega, \gamma) x_i,
\]
where fuzzy random variables $R_i(\omega, \gamma)$ model the profitability of financial assets and have a shift-scale representation [10]:
\[
R_i(\omega, \gamma) = a_i(\omega) + \sigma_i(\omega) Z_i(\gamma).
\]
Further in this work we assume that in this representation fuzzy variables $Z_i(\gamma) = [m_i, \bar{m}_i, d_i, \bar{d}_i]_{\mu_i}$ are mutually $T$-related and random parameters $a_i(\omega)$ and $\sigma_i(\omega)$ are independent.

For a better intuitive understanding of such a representation of a fuzzy random variable imagine a situation where some financial expert is asked to estimate a return of a certain financial asset. Both the return and its estimation by the expert are uncertain quantities. We assume that the uncertainty determined by market conditions has a probabilistic nature. On the other hand, uncertainty of the estimation by the expert is described by some possibility distribution. This model seems to be quite plausible if we assume that the degree of fuzziness of the expert depends mainly on the scale of variation of the estimated variable and not on its true value [10].

In accordance with the classical Markowitz approach [14], and with the help of [4], we can construct a risk function for the portfolio and include its expected return in the system of constraints. Since the expected portfolio return in the case of fuzzy random data is fuzzy the uncertainty of possibilistic type can be removed by imposing requirements on the possibility/necessity of fulfilling an investor’s constraints on an acceptable level of the expected portfolio return. In this case the model of feasible portfolios by Markowitz can be represented in the following form
\[
D[R_p(x, \omega, \gamma)] \rightarrow \min_{x \in E^n},
\]
\[
G_p = \begin{cases} 
\tau \{ E[R_p(x, \omega, \gamma)] \mathcal{R} m_d \} \geq \alpha, \\
\sum_{i=1}^{n} x_i = 1, \\
x \in E^n.
\end{cases}
\]

Here $D[R_p(x, \omega, \gamma)]$ is the variance and $E[R_p(x, \omega, \gamma)]$ is the expected return of a portfolio; $E^n = \{ x \in E^n : x \geq 0 \}$; $\tau$ is the measure of possibility $\pi$ or necessity $\nu$; $\mathcal{R}$ is a crisp relation, $\mathcal{R} \in \{ \geq, = \}$; $\alpha \in (0, 1]$ and $m_d$ is an acceptable level of the expected return of a portfolio.
3. Results and discussion

3.1. Equivalent stochastic problem of the model of minimum risk portfolio

Let $a_i(\omega)$ and $\sigma_i(\omega)$ be independent random variables and fuzzy components $Z_i(\gamma)$ be represented by symmetrical fuzzy numbers of LR-type, i.e. $Z_i(\gamma) = [m_i, m_i, d_i, d_i]_{LR}$, $i = 1, n$ and $L(t) = R(t)$, $\forall t \geq 0$.

Then for each $x$ the expected return of a portfolio is a fuzzy variable with the following possibility distribution [10]

$$E_p(x, \gamma) = [m_{ER_p}(x), m_{ER_p}(x), d_{ER_p}(x), d_{ER_p}(x)]_{LR},$$

and the risk of the portfolio takes the form [3]

$$D_p(x) = m_{DR_p}(x) + \eta d_{DR_p}(x).$$

Here

$$m_{ER_p}(x) = E[m_{R_p}(x, \omega)], \quad d_{ER_p}(x) = E[d_{R_p}(x, \omega)],$$

$$m_{DR_p}(x) = D[m_{R_p}(x, \omega)], \quad d_{DR_p}(x) = D[d_{R_p}(x, \omega)],$$

$$\eta = \frac{1}{0} \left((L(\alpha))^2\right) d\alpha,$$

and

$$m_{R_p}(x, \omega) = \sum_{i=1}^{n} \left(a_i(\omega) + \sigma_i(\omega)m_i\right)x_i, \quad d_{R_p}(x, \omega) = \max_{i=1,2} \left\{\sigma_i(\omega)d_{x_i}\right\}.$$

The constraint on the expected return can be replaced by [7]

$$E_p(x) = \lambda m_{DR_p}(x) + \beta d_{DR_p}(x) \leq \lambda m_d,$$

where $\lambda = \pm 1$, $\beta = \pm L^{-1}(\alpha)$ or $\pm L^{-1}(1-\alpha)$ depending on $\mathcal{R}$ and $\tau$.

Let us write the constraints on assets of a portfolio in the form

$$X = \begin{cases} \sum_{i=1}^{n} x_i = 1, \\
\quad x \in E^p. \end{cases}$$

Thus, the model of a minimum risk portfolio under conditions of hybrid uncertainty of possibilistic-probabilistic type can be replaced by the following stochastic program

$$D_p(x) \rightarrow \min_{x \in E_+},$$

$$F_p = \begin{cases} E_p(x) \leq \lambda m_d, \\
\quad x \in X. \end{cases}$$

Note that in general the model (1)-(2) is not convex. However, in [15] Nurminski studied one class of non-convex non-differentiable functions which he named weakly convex. For this class methods similar to the stochastic quasigradient methods [5, 15] can be specified.
Definition 8. A function $F(x)$ is called weakly convex if for any $x$ there exists a non-empty set of vectors $G(x)$ that for all $z$ and $\hat{F}(x) \in G(x)$ the following inequality holds
\[ F(z) - F(x) \geq \langle \hat{F}(x), z - x \rangle + r(z, x), \tag{3} \]
where $r(x, y) \| x - y \|^{-1} \to 0$ for $y \to x$ uniformly in $X$. A vector $\hat{F}(x)$ that satisfies (3) is called quasi-gradient of a weakly convex function $F(x)$.

The term “weakly convex” has been suggested by analogy to “strongly convex” functions [15] that satisfy the inequality (3) with additional constraints on $r(x, y) = \delta(\|x - y\|) \leq 0$ and $\delta(t)/t \to 0$ for $t \to +0$.

While the class of strongly convex functions are a subset of the class of convex functions, the class of weakly convex functions includes convex functions as well as non-convex differentiable. For non-convex differentiable functions the inequality (3) transforms into
\[ F(z) - F(x) = \langle F_i(x), z - x \rangle + r(z, x) \]
where $F_i(x)$ is the gradient of $F(x)$.

Theorem 1. If the function $d_{\text{E}P}(x)$ is differentiable then $D_{\rho}(x)$ and $E_{\rho}(x)$ are weakly convex.

Proof. Since $m_{\text{E}P}(x)$ is differentiable, then $E_{\rho}(x)$ is also differentiable as well as weakly convex. On the other hand the function
\[ d_{\text{E}P}(x) = \mathbb{E} \left[ \max_{i=1, \ldots, n} \left( \sigma_{\rho} \left| d_{ri} \right| \right) \right]^2 - d_{\text{E}P}^2(x) \]
is weakly convex because it is the sum of a convex function and a differentiable function. Hence, $D_{\rho}(x)$ is weakly convex and this completes the proof.

3.2. The stochastic penalty method

The model (1)-(2) can be solved by deterministic techniques of mathematical programming. However, these techniques require the calculation of the precise values of the expected value and the variance of a random function. An alternative approach is to use the stochastic quasigradient (SQG) methods [5, 15], for example the stochastic penalty methods.

The SQG methods allow us to solve optimization problems with objective functions and constraints of such a complex nature that the calculation of the precise values of these functions (all the more of their derivatives) is impossible. The main idea of these methods consists of using statistical estimates for the values of the functions and of their derivatives instead of their precise values. The SQG methods generalize the well-known stochastic approximation methods for unconstrained optimization of the expectation of random functions.

Instead of the model (1)-(2) we consider the following minimization problem
\[ F(x) = D_{\rho}(x) + c \left( E_{\rho}(x) \right) E_{\rho}(x) \to \min_{x \in X}, \tag{4} \]
where $c(\alpha)$ is a penalty function
\[ c(\alpha) = \begin{cases} C, & \alpha > 0, \\ 0, & \alpha \leq 0, \end{cases} \]
where small $s$ satisfy the requirements of convergence of the function $F(x)$ we can specify the stochastic quasigradient method with the averaging operation [5].

Let $Y$ and $Z$ be convex compact sets such that $d_{E_p}(x) \in Y$ and $E_p(x) \in Z, \forall x \in X$. We will define a sequence of points $(x^s, y^s, z^s)$ by the following relations

$$ x^{s+1} = \begin{cases} x^s - \rho^s \xi^s, & \|x^s\| \leq 1, \\ b^{s+1} \in B, & \|x^s\| > 1, \end{cases} \quad y^{s+1} = \Pi_Y \left( y^s - \rho^s \left( y^s - \xi^s \right) \right), \quad z^{s+1} = \Pi_Z \left( z^s - \rho^s \left( z^s - \chi^s \right) \right), $$

where $(x^0, y^0, z^0)$ is an initial point, $s = 0, 1, \ldots$, $B \in A = \{ x \mid x \parallel x \parallel \leq 1 \}$; the step sizes $\rho^j$, $j = 1, 2, 3$, are non-negative and measurable with respect to a $\sigma$-algebra $\mathcal{B}_s$ generated by the sequence of points $(x^s, y^s, z^s)$. $\Pi_Y$ and $\Pi_Z$ are projections on the sets $Y$ and $Z$ respectively; $\xi^s$, $\chi^s$ are random variables such that

$$ \xi^s = \delta_{E_p} + \eta \theta^s - 2 \eta \gamma \theta^s + c(z^s) \left( \lambda \delta_{E_p} + \beta \theta^s \right), \quad \chi^s = \delta_{E_p} \left( x^s, \omega^s \right), \quad \gamma \chi^s = \lambda \delta_{E_p} \left( x^s, \omega^s \right) + \beta d_{E_p} \left( x^s, \omega^s \right). $$

If we consider $\hat{a}_i = D[a_i(\omega)], \hat{\sigma}_i = D[\sigma_i(\omega)]$ and $\bar{a}_i = E[a_i(\omega)], \bar{\sigma}_i = E[\sigma_i(\omega)]$ then

$$ \delta_{E_p} = \left( \bar{a}_i + \sigma_i m_1, \ldots, \bar{a}_i + \sigma_i m_n \right), \quad \theta^s = (0, \ldots, \left| \sigma_i(\omega^s) \right| d^s, \ldots, 0), $$

$$ \delta_{E_p} = \left( 2(\hat{a}_i + \hat{\sigma}_i m_1^2) x_i, \ldots, 2(\hat{a}_i + \hat{\sigma}_i m_n^2) x_n \right), \quad \theta^s = (0, \ldots, 2 \gamma^s(\omega^s) d^s x_i, \ldots, 0). $$

Here $x^s$ is the value of $x$ at the $s$th iteration, $\sigma_i(\omega^s)$ are the realization of a random variable $\sigma_i(\omega)$ at the $s$th iteration, $i^*$ is such an index that for any $i = 1, n$

$$ \left| \sigma_i(\omega) \right| d^s x_i \geq \left| \sigma_i(\omega) \right| d^s x_i. $$

We will show that under the assumptions mentioned above the vector $\xi^s$ and the variables $\xi^s$, $\chi^s$ satisfy the requirements of convergence of the stochastic penal method [5].

**Theorem 2.** If $\left| y^s - d_{E_p}(x^s) \right| \rightarrow 0$ and $\left| z^s - E_p(x^s) \right| \rightarrow 0$ with probability 1 for $s \rightarrow \infty$, then for small $\varepsilon$ and $\|x - x^s\| \leq \varepsilon$ the following inequality holds

$$ \left| E \left[ \frac{\xi^s}{(x^s, y^s, z^s)} - \hat{F}(x^s) \right] \right| 0, $$

where $\hat{F}(x)$ is a quasi-gradient of the weakly convex function $F(x)$.

**Proof.** To prove this theorem it is sufficient to show that the following statements hold:

1. Vectors $E[\delta_{E_p} / B_i]$ and $E[\delta_{E_p} / B_i]$ are quasi-gradients of the functions $m_{E_p}(x)$ and $m_{E_p}(x)$ respectively.
2. Vectors $E[\delta_{E_p} / B_i]$ and $E[\delta_{E_p} / B_i]$ are quasi-gradients of the functions $d_{E_p}(x)$ and $E[d_{E_p}(x)]$ respectively.
3. If $\left| y^s - d_{E_p}(x^s) \right| \rightarrow 0$ for $s \rightarrow \infty$ then a vector $E[-2y^s \delta_{E_p} / B_i]$ converges to a quasi-gradient of the function $d^2_{E_p}(x)$.
4. If $|z^s - E_p(x^s)| \to 0$ for $s \to \infty$ then a vector $E \left[ c(z^s)(\lambda \delta_{E_{r_p}} + \beta \theta_i) / B_i \right]$ converges to a quasi-gradient of the function $E_p(x)$.

It is obvious that the statement 1 holds. The proof that the vector $E(\theta_i / B_i)$ is a quasi-gradient of the function $d_{E_{r_p}}(x)$ is presented in [6]. Since $|\sigma_i(\omega)|d_{x_i} \geq 0 \ \forall i = 1, n$, then

$$d_{E_{r_p}}(x) = E \left[ \max_{i=1,n} \{ |\sigma_i(\omega)|d_{x_i} \} \right]^2 = E \left[ \max_{i=1,n} \{ \sigma_i(\omega)d_i^2 x_i^2 \} \right].$$

Here $(0, \ldots, 2\sigma_i^2(\omega)d_i^2 x_i, \ldots, 0)$ is a quasi-gradient of a function $\sigma_i^2(\omega)d_i^2 x_i$. The following proof is similar to the proof for the vector $E[\theta_i / B_i]$.

Since the vector $E[\theta_i / B_i]$ is the quasi-gradient of $d_{E_{r_p}}(x)$ then the vector

$$E[-2y^s \theta_i / B_i] = -2y^s \cdot E[\theta_i / B_i]$$

for $y^s \to d_{E_{r_p}}(x^s)$ converges to a quasi-gradient of $d_{E_{r_p}}(x)$.

Considering that $E[\lambda \delta_{E_{r_p}} + \beta \theta_i / B_i]$ is the quasi-gradient of $E_p(x)$ [6] and the form of the penalty function $c(\alpha)$, the vector $E \left[ c(z^s)(\lambda \delta_{E_{r_p}} + \beta \theta_i) / B_i \right]$ for $z^s \to E_p(x^s)$ converges to the quasi-gradient of the function $c(E_p(x))E_p(x)$ and this completes the proof.

**Theorem 3.** Assume that the condition (5) is fulfilled, the random variables $a_i(\omega)$ and $\sigma_i(\omega)$ are bounded and their expectation and variance are finite; the function $d_{E_{r_p}}(x)$, $E_p(x)$ are continuous at $x$; the step sizes satisfy with probability 1 the conditions

$$\rho_j^s \geq 0, \quad \frac{\rho_j^{s+1}}{\rho_j^s} \to 1, \quad \sum_{s=0}^{\infty} \rho_j^s = \infty,$$

$$\sum_{s=0}^{\infty} E \left[ (\rho_j^s)^2 \right] < \infty, \quad j = 1, 2, 3.$$

Also assume that for $s \to \infty$

$$\frac{\rho_{1}^s}{\rho_{2}^s} \to 0 \quad \text{and} \quad \frac{\rho_{1}^s}{\rho_{3}^s} \to 0,$$

the functions $d_{E_{r_p}}(x)$ and $E_p(x)$ satisfy a local Lipschitz condition and

$$\max_{x \in B} F(x) < \inf_{x \in B} F(x).$$

Then $\lim_{x \to \infty} F(x^s) = F(x^s), \ x^s \in X^s$ with probability 1.

**Proof.** Let us show that there exists a constant $C$ that

$$\|x^s\| + |x^s| + |x'| + \left\| \tilde{F}(x', y', z') \right\| \leq C.$$

It is easy to see that there exist such constants $C_x$ and $C_z$ that

$$|x^s| = \left| d_{E_p}(x^s, \omega^s) \right| = \left| \sigma_i(\omega^s)d_{x_i}^s, x_i^s \right| < C_x$$

and

$$\|x^s\| + |x^s| + |x'| + \left\| \tilde{F}(x', y', z') \right\| \leq C.$$
\[ \chi' = \left| \lambda m'_{\xi'} (x', \omega') + \beta d_{\xi'} (x', \omega') \right| \leq \left| m'_{\xi'} (x', \omega') \right| + \left| \beta \sigma_{\xi'} (\omega') d_{\xi'} \right| < C_{\chi}. \]

The components of the vector \( \xi' \) can be written as
\[
\xi'_j = 2(\hat{a}_j + \hat{\sigma}_j m_i^2) x_j + 2 \eta \sigma_{\omega'} d_{\xi'} x_j
- 2 \eta y' \sigma_{\omega'} d_{\xi'} + c(z') \cdot \left[ \lambda \left( \hat{a}_j + \hat{\sigma}_j m_i \right) + \beta \sigma_{\omega'} x_j \right].
\]
\[ \xi'_j = 2(\hat{a}_j + \hat{\sigma}_j m_i^2) x_j + c(z') \cdot \lambda \left( \hat{a}_j + \hat{\sigma}_j m_i \right), \quad j = 1, n, \ j \neq i'. \]

It is easy to see that \( \left\|\xi'_j\right\| \) and \( \left\|\xi'_{j'}\right\| \) are bounded. So there exists \( C_{\xi} \) that \( \left\|\xi'\right\| < C_{\xi} \). On the other hand
\[ \left\|\hat{F}(x', y', z')\right\| = \left\|E \xi'\right\|, \]
which means that \( \left\|\hat{F}(x', y', z')\right\| < C_{\xi} \).

The following steps are similar to the proof presented in [5].

### 3.3. Numerical examples

In this section, we will give numerical examples for \( n = 2 \) and compare the results of using the stochastic penalty method with the results that was obtained in [3]. We consider that the shift and scale coefficients \( a_i(\omega) \) and \( \sigma_i(\omega) \) are uniformly distributed on \([0,1]\), all fuzzy variables have symmetrical triangular forms with \( \mathcal{R}(t) = L(t) = \max\{0, 1 - t\}, \ t \geq 0 \).

Let \( Z_1 = [0.3, 0.3, 3.5, 3.5]_x \) and \( Z_2 = [2.8, 2.8, 1.5, 1.5]_x \), \( \alpha = 0.65 \), the relation \( \mathcal{R} \) is “greater than or equal” \((\geq)\). Then the equivalent stochastic problem of the minimum risk portfolio (1)-(2) takes the following forms in possibility context
\[
 \frac{1}{12} \left( 1.09 x_1^2 + 8.84 x_2^2 \right) + d_{\text{DR}_p} (x) \rightarrow \min, \quad \text{sec}_{\text{FP}}, \quad (6)
\]
\[
 F_p = \begin{cases} 
 -0.65 x_1 - 1.9 x_2 - 0.35 \cdot d_{\text{DR}_p} (x) \leq -m_d, \\
 x \in X,
 \end{cases} \quad (7)
\]
and in necessity context
\[
 \frac{1}{12} \left( 1.09 x_1^2 + 8.84 x_2^2 \right) + d_{\text{DR}_p} (x) \rightarrow \min, \quad \text{sec}_{\text{FP}}, \quad (8)
\]
\[
 F_p = \begin{cases} 
 -0.65 x_1 - 1.9 x_2 + 0.65 \cdot d_{\text{DR}_p} (x) \leq -m_d, \\
 x \in X,
 \end{cases} \quad (9)
\]
where
\[
 d_{\text{DR}_p} (x) = E[d_{\text{DR}_p} (x, \omega)], \quad d_{\text{DR}_p} (x) = D[d_{\text{DR}_p} (x, \omega)],
\]
and
\[
 d_{\text{DR}_p} (x, \omega) = \max \left\{ 3.5 \cdot \sigma_1(\omega) x_1, \ 1.5 \cdot \sigma_2(\omega) x_2 \right\}. 
\]
In [16] it is shown that in the case of uniformly distributed random components the function $d_{EP}(x)$ is differentiable. Moreover, it is easy to show that all requirements of Theorem 3 are fulfilled, so we can solve these examples using the stochastic penalty method.

The figure 1 shows solutions of the problems (6)-(7) and (8)-(9) depending on the acceptable level of the expected portfolio return $m_d$. Curves (a) and (b) represent the dependence of the minimum risk of the portfolio on its expected return in necessity context (8)-(9) and in possibility context (6)-(7) respectively.

![Figure 1](image)

**Figure 1.** Dependence of the minimum risk of the portfolio on its expected return in necessity context (a) and possibility context (b).

The numerical examples confirm the theoretical results presented in this paper. Their solutions are similar to the results obtained in [3] where the problems (6)-(7) and (8)-(9) were replaced with their equivalent crisp problems and a genetic algorithm was used to solve them.

4. Conclusion
In the present paper, we have contributed to the research area of the portfolio optimization in the following aspects: 1) a mathematical model of the minimum risk portfolio has been described under the conditions of a hybrid uncertainty of possibilistic-probabilistic type for the case of $T_w$-related fuzzy components; 2) the stochastic equivalent problem for the minimum risk portfolio model has been obtained; 3) the stochastic penalty method has been specified for solving the equivalent stochastic program. The approach is demonstrated on numerical examples when its random parameters are uniformly distributed on [0,1] and fuzzy factors are symmetrical fuzzy numbers.

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