When are Vector Fields Hamiltonian?

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There is reason to believe that at small scales and low temperatures, quantum mechanical effects will play a role in dissipative systems which arise in solid state physics. References to and a brief discussion of various approaches to the quantisation of the damped harmonic oscillator, can be found in [1]. Although there exist successful and physically intuitive ways to deal with the quantisation of the damped harmonic oscillator, for reasons outlined in [1] we consider a more direct approach to the problem, one which requires an understanding of the Hamiltonian structure of the classical flow. This is why we ask the question from which the article gets its title.

Locally it is possible to consider any flow to be Hamiltonian in a sense which we will later make precise. This is true even if the vector field is not conservative. Having understood the Hamiltonian structure of the classical flow, it is possible in principle to *-quantise the system by constructing an associated Moyal-type algebra. This step is based on an insight of Bayen, Flato, Fronsdal Lichnerowicz and Sternheimer [2], who pointed out that the Moyal algebra based on the product

\[ A(q, p) * B(q, p) = \exp \left( \frac{i}{\hbar} \Lambda^{ij} \partial_i^{(1)} \partial_j^{(2)} \right) A^{(1)}(q, p) B^{(2)}(q, p) \]

where \( \Lambda^{ij} \) is the standard symplectic matrix, is an associative deformation of the classical commutative associative algebra of observables. Although the Moyal algebra is widely used in the context of the Wigner formalism, the general problem of constructing Moyal-like algebras associated with a given Hamiltonian structure is not understood, and there remain many problems to solve before such a program for quantisation can be completed.

The purpose of this talk is to highlight the fact that it is possible to consider dissipative flows as autonomous Hamiltonian systems, to clarify what this means, and to outline further problems that this path to quantisation presents. Our starting point is the following theorem [3].

**Theorem 1** In a finite but sufficiently small neighbourhood \( U \) of a non-critical point of a smooth vector field \( v = v^i \partial_i \) on \( \mathbb{R}^{n+1} \), there exist independent functions \( \Phi^{(k)} \in C^2(U) \), a scalar \( \rho \in C^1(\mathbb{R}^{n+1}) \) and smooth independent rank 2 Poisson tensors \( \Lambda^{(k)} \), \( k = 1, \ldots, n \), such that

\[ v^i = \Lambda^{ij} \partial_j \Phi^{(k)} \]

where

\[ \Lambda^{ij} = \frac{1}{2} \left( \delta^{ij} + \frac{\partial^k \partial_k}{\partial_i \partial_j} \right) \]
The $\Phi^{(k)}$ can be thought of as local Hamiltonians of the flow. The Poisson tensors $\Lambda^{ij}_{(k)}$ are so-called because each operation defined by $\{A, B\}_{(k)} = \Lambda^{ij}_{(k)} \partial_i A \partial_j B$, has all the algebraic properties of the familiar Poisson bracket. In addition a local measure constructed from $\rho$, and given by $d\mu = \rho dx^0 \wedge \ldots \wedge dx^n$, is invariant under the flow.

1 The Kepler system

First we consider a system which is Hamiltonian by all of the usual criteria, the Kepler system on $(\mathbb{R}^3 - \{0\}) \times \mathbb{R}^3$. This system is exceptional even among fully integrable systems on a 2n-dimensional phase space, since it has a maximum number of global conservation laws $2n - 1$, instead of $n$. These can be chosen from the components of the angular momentum vector

$$L_1 = yp_z - zp_y$$
$$L_2 = zp_x - xp_z$$
$$L_3 = xp_y - yp_x$$

and those of the Runge-Lenz vector

$$R_1 = p_y L_3 - p_z L_2 - mx/r$$
$$R_2 = p_z L_1 - p_x L_3 - my/r$$
$$R_3 = p_x L_2 - p_y L_1 - mz/r$$

where $r^2 = x^2 + y^2 + z^2$, and $m$ is the mass. Putting $\rho = -L_3(L_1^2 + L_2^2 + L_3^2)$, $\Phi^{(2)} = L_2$, $\Phi^{(3)} = L_3$, $\Phi^{(4)} = R_1$, and $\Phi^{(5)} = R_2$, we can verify that

$$\Lambda^{ij}_{(1)} = \rho^{-1} \epsilon^{ij} s_3 s_4 s_5 \partial_{s_3} \Phi^{(2)} \partial_{s_4} \Phi^{(3)} \partial_{s_4} \Phi^{(4)} \partial_{s_5} \Phi^{(5)}$$

has all the properties required of a Poisson bracket. Taking $\Phi^{(1)} = L_1$, to be the Hamiltonian, and computing $\dot{x}^i = \Lambda^{ij}_{(1)} \partial_j \Phi^{(1)}$ we recover the familiar Kepler flow on $\mathbb{R}^6$, and we can check that $\rho dx^1 \wedge \ldots \wedge dx^6$ does indeed provide an invariant measure. This is not the only way in which to construct the Kepler system. In fact it is easy to show that this can be done in an infinite number of different ways.

2 The simple predator-prey equation

For flows on $\mathbb{R}^2$, the picture is a little simpler. The Poisson tensors all have the form $\Lambda^{ij} = \rho^{-1} \epsilon^{ij}$, where $\rho \in C^1(\mathbb{R}^2)$. Hamiltonian vector fields with respect to any such Poisson structure, have the form $\dot{x}^i = \Lambda^{ij} \partial_j \Phi$. The corresponding symplectic two-forms are given by $\rho dx \wedge dy$ and these co-incide with the invariant measures mentioned above.

We will illustrate this with a system which is not usually treated within the Hamiltonian framework, the simple predator-prey equation. This system is not Hamiltonian with respect to the canonical Hamiltonian structure on $\mathbb{R}^2$, and the flow does not conserve the standard volume element. On the other hand it is not dissipative, and it can be considered to be globally Hamiltonian with respect to an infinite number of non-standard Hamiltonian structures. It is defined by
\[ \dot{x} = x(b - y) \]
\[ \dot{y} = y(x - a) \]

where \( a, b \in \mathbb{R} \geq 0 \). Taking

\[
\Phi = \exp(x + y) x^{-a} y^{-b} \\
\rho = -\exp(x + y) x^{-(a+1)} y^{-(b+1)}
\]

it is easy to check that the simple predator-prey system is Hamiltonian with Hamiltonian \( \Phi \), with respect to the Poisson structure \( \Lambda^{ij} = \rho^{-1} \epsilon^{ij} \). To see that it has an infinite number of other Hamiltonian structures, it suffices to note that replacing \( \Phi \) with any non-constant \( F(\Phi) \), and \( \rho \) with \( \rho F'(\Phi) \) we get a different Hamiltonian function and a different Poisson structure, for the same system.

\( \Phi \) is well behaved except when \( x = 0 \) or \( y = 0 \), and \( \Lambda \) becomes singular along these lines. We know that the flow never crosses these axes, so it is globally Hamiltonian on \((0, \infty) \times (0, \infty)\).

### 3 The damped harmonic oscillator

Now we come to one of the simplest dissipative systems, the damped harmonic oscillator. This is given by

\[
\dot{x} = y \\
\dot{y} = -x - \kappa y
\]

where \( \kappa \) is the damping constant. The qualitatively different phases - undamped (H), damped (D), critically damped (C), and over-damped (O), correspond to \( \kappa = 0 \), \( 0 < \kappa < 2 \), \( \kappa = 2 \) and \( \kappa > 2 \) respectively. For each phase it is possible to determine an infinite number of Hamiltonian structures. However the following choice has the property that as \( \kappa \) descends from high values down through the value 2 to \( \kappa = 0 \), the Hamiltonians vary continuously from \( \Phi^O \) to \( \Phi^C \), from \( \Phi^C \) to \( \Phi^D \) eventually to \( \Phi^H \) at \( \kappa = 0 \) where we recover the familiar case of the harmonic oscillator.

\[
\begin{align*}
\Phi^H &= \frac{1}{2}(x^2 + y^2) \\
\Phi^D &= \frac{1}{2}(x^2 + y^2 + \kappa xy) \exp\left(2 \frac{\kappa}{\Delta} \arctan\left(\frac{\Delta x}{\kappa x + 2y}\right)\right) \\
\Phi^C &= \frac{1}{2}(x + y)^2 \exp\left(\frac{2x}{x + y}\right) \\
\Phi^O &= \frac{1}{2}(x^2 + y^2 + \kappa xy) \left|\frac{\kappa x + 2y + \Delta x}{\kappa x + 2y - \Delta x}\right|^{\frac{\kappa}{2}}
\end{align*}
\]

For damped motion \( \Delta = +(4 - \kappa^2)^{1/2} \) and for overdamped motion \( \Delta = +(\kappa^2 - 4)^{1/2} \). On \( \mathbb{R}^2 \) the invariant volume element provided by \( \rho \, dx \wedge du \) is also the symplectic structure corresponding to

\[ \dot{x} = y \]
\[ \dot{y} = -x - \kappa y \]
\[ \rho^H = 1 \]
\[ \rho^D = \exp \left( \frac{2\kappa}{\Delta} \arctan \left( \frac{\Delta x}{\kappa x + 2y} \right) \right) \]
\[ \rho^C = \exp \left( \frac{2x}{x+y} \right) \]
\[ \rho^O = \left| \frac{\kappa x + 2y + \Delta x}{\kappa x + 2y - \Delta x} \right|^{\frac{\pi}{2}} \]

In this way we can consider the Hamiltonians and Poisson structures of the damped harmonic oscillator as one-parameter deformations of the simple harmonic oscillator. In [1] we deal with this in more detail and relate the different phases of the flow to the nature of the singularity in the various structures.

4 Discussion

We have shown by examples how many different kinds of vector fields can be considered to be Hamiltonian. These three examples present different kinds of problems for the *-quantisation program. The Kepler system is well understood within the standard framework of Hamiltonian dynamics on \( \mathbb{R}^3 \times \mathbb{R}^3 \), however the existence of many different Hamiltonian structures raises the question of whether these will give rise to many different quantisations. The predator-prey system was shown to be globally Hamiltonian on the upper right quadrant of the plane. In this case the usual division of phase space into position and momentum does not readily make sense. Both co-ordinates have the same physical interpretation as representing the size of the corresponding population. From this point of view an explicit phase-space quantisation seems more natural. However none of its Hamiltonian structures coincide with the standard one on \( \mathbb{R}^2 \), and a-priori we have no way of choosing which one is correct. The Poisson structures which arise in the case of the damped harmonic oscillator are either multi-valued or singular. This leaves the problem of boundary conditions for the wave-function along the singularity. We think that this is a physically reasonable consideration as singularities must also occur in conserved measures in the classical case. If eventually, we succeed in consistently applying the *-quantisation approach to the damped harmonic oscillator, it will be interesting to see if there occur qualitatively different phases, which correspond with those of the classical flow.

As an approach to the quantisation of general dynamical systems, this program raises more questions than it answers. However we think that these are questions which should be asked anyway of standard quantum mechanics, and that the answers will be of mathematical and physical interest in their own right.

References

[1] P. Crehan: Preprint Kyoto-Math 94-04: On the Hamiltonian structure of non-conservative flows on \( \mathbb{R}^2 \).

[2] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz & D. Sternheimer: Ann. Physics (NY) vol.111 1978 p.61, p.111.

[3] P. Crehan: Preprint Kyoto-Math 94-03: On the local Hamiltonian structure of vector fields.