MACROSCOPIC QUANTUM GAME

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Abstract
The game in which acts of participants don’t have an adequate description in terms of Boolean logic and classical theory of probabilities is considered. The model of the game interaction is constructed on the basis of a non-distributive orthocomplemented lattice. Mixed strategies of the participants are calculated by the use of probability amplitudes according to the rules of quantum mechanics. A scheme of quantization of the payoff function is proposed and an algorithm for the search of Nash equilibrium is given. It is shown that differently from the classical case in the quantum situation a discrete set of equilibria is possible.

It often occurs that mathematical structures discovered when solving some class of problems find their natural application in totally different areas. The mathematical formalism of quantum mechanics operating with such notions as "observable", "state", "probability amplitude" is not an exception to this rule. The goal of the present paper is to show that the language of quantum mechanics, initially applied to the description of the microworld, is adequate for the description of some macroscopic systems and situations where Planck’s constant plays no role. It is natural to look for applications of the formalism of quantum mechanics in those situations when one has interactions with the element of indeterminacy. In [1] as well as more recently [2] it was shown that the quantum mechanical formalism can be applied to description of macroscopic systems when the distributive property for random events is broken. In the physics of the microworld non-distributivity has an objective status and must be present in principle. For macroscopic systems the non-distributivity of random events expresses some specific case of the observer’s "ignorance".

In the present paper a quantum mechanical formalism is applied to the analysis of a conflict interaction, the mathematical model for which is an antagonistic game of two persons. The game is based on a generalization of examples of the

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macroscopical automata simulating the behaviour of some quantum systems considered earlier in [3, 4]. A special feature of the game considered is that the players acts go in contradiction with the usual logic. The consequence is breaking of the classical probability interpretation of the mixed strategy: the sum of the probabilities for alternate outcomes may be larger than one. The cause of breaking of the basic property of the probability is in the non-distributivity of the logic. The partners relations are such that the disjunction "or", conjunction "and" and the operation of negation do not form a Boolean algebra but an orthocomplemented non-distributive lattice. However this ortholattice happens to be just that which describes some properties of a quantum system with spin one half. This leads to new "quantum" rules for the calculations of the average profit and new representation of the mixed strategy, the role of which is played by the "wave function" – the normalized vector in a finite dimensional Hilbert space. Calculations of probabilities are made according to the standard rules of quantum mechanics. Differently from the examples of quantum games considered in [5, 6, 7] where the "quantum" nature of the game was conditioned by the microparticles or quantum computers based on them, in our case we deal with a macroscopic game, the quantum nature of which has nothing to do with microparticles. This gives the hope that our example is one of many analogous situations in biology, economics etc where the formalism of quantum mechanics can be used.

The game "Wise Alice" formulated in our paper is a modification of the well known game when each of the participants names one of some previously considered objects. In the case if the results differ, one of the players wins from the other some agreed sum of money. The participants of our game A and B, call them Alice and Bob have a quadratic box in which a ball is located. Bob puts his ball in one of the corners of the box but doesn’t tell his partner which corner.

![Figure 1: Bob’s ball moves into the place asked by Alice](image)

Alice must guess in which corner Bob has put his ball. The rules of the game are such that Alice can ask Bob questions supposing the two-valued answer: "yes" or "no". It is supposed that Bob is honest and always tells the truth. In the case
of a "yes" answer Alice is satisfied, in the opposite case she asks Bob to pay her some compensation. However, differently from other such games [8, 9], the rules of this game (see Figure 1) have one specific feature: Bob has the possibility to move the ball to any of the adjacent vertices of the square after Alice asks her question. This additional condition decisively changes the behavior of Bob, making him become active under the influence of Alice’s questions. Due to the fact that negative answers are not profitable for him, in all possible cases, moves his ball to the convenient adjacent vertex.

So being in vertices 2 or 4 and getting from Alice the question "Are you in the vertex 1?" Bob quickly puts his ball in the asked vertex and honestly answers "yes". However, if the Bob’s ball was initially in the vertex 3 he cannot escape the negative answer notwithstanding to what vertex he moves his ball and he fails. One must pay attention that in this case Alice not only gets the profit but also obtains the exact information on the initial position of the ball: Bob’s honest answer immediately reveals his initial position. The interaction of our players can be described by a four on four matrix ($h_{jk}$) representing payoffs of Alice in each of the 16 possible game situations where $a, b, c, d > 0$ are her payoffs in those situations when Bob cannot answer her questions affirmatively. Our game is an antagonistic game, so the payoff matrix of Bob is the opposite to that of Alice: ($-h_{jk}$). The main problem of game theory is to find so-called points of equilibrium or saddle points – game situations, optimal for all players at once. It is easy to see that the classical game with our payoff matrix does not have such equilibrium points. Nonexistence of the saddle point follows from the strict inequality valid for our game

$$\max_j \min_k h_{jk} < \min_k \max_j h_{jk}$$

So there are no stable strategies to follow for Bob and Alice in each separate turn of the game. In spite of the absence of a rational choice at each turn of the game, when the game is repeated many times some optimal lines of behaviour can be found. To find them in the theory of classical games one must, following von Neumann [10], look for the so-called mixed generalization of the game. The optimal mixed strategies for Alice and Bob are defined as such probability distributions on the sets of pure strategies $x^0 = (x^0_1, x^0_2, x^0_3, x^0_4)$ and $y^0 = (y^0_1, y^0_2, y^0_3, y^0_4)$ that for all distributions of $x, y$ the von Neumann-Nash

|     | 1   | 2   | 3   | 4   |
|-----|-----|-----|-----|-----|
| 1   | 0   | 0   | a   | 0   |
| 2   | 0   | 0   | 0   | b   |
| 3   | c   | 0   | 0   | 0   |
| 4   | 0   | d   | 0   | 0   |

Table 1: The Payoff-matrix of Alice
inequalities are valid:

\[ H_A(x^0, y^0) \geq H_A(x, y^0), \quad H_B(x^0, y^0) \geq H_B(x^0, y), \]  

(1)

where \( H_A, H_B \) – payoff functions of Alice and Bob are the expectation values of their wins

\[ H_A(x, y) = \sum_{j,k=1}^{4} h_{jk} x_j y_k, \quad H_B(x, y) = -\sum_{j,k=1}^{4} h_{jk} x_j y_k \]

The combination of strategies, satisfying the von Neumann-Nash inequalities, is called the situation of equilibrium in Nash’s sense. However in the case of our game the logic of behaviour of the players is such that usual classical theory does not work. To see this consider Hasse diagram (Fig. 2) where to atoms correspond different possibilities for Bob from the point of view of Alice when she pays attention only to his negative answers. One has the special structure of disjunction so that for example 1 or 2 is true when 1 true or 2 true but not always true. If one considers all outcomes equally possible, then the probability of the always true event, i.e. disjunction of any of two events occurs to be one half! The distributivity property is broken. So a classical probabilistic description of the behaviour of the players in the repeated game is impossible in principle.

The solution for the situation arising is given by the ideas of quantum mechanics. Following A.A.Grib and R.R.Zapatrin we pay attention to the fact that the ortholattice of the logic of interaction of partners of the ”Wise Alice” is isomorphic to the ortholattice of invariant subspaces of the Hilbert space of the quantum system with spin \( \frac{1}{2} \) and observables of the type of \( S_x, S_\theta \). As it is well known one can represent this lattice by considering on the plane two pairs of mutually orthogonal direct lines \( \{a_1; a_3\}, \{a_2; a_4\} \). One of these pairs makes diagonal the operator \( S_x \), the other \( S_\theta \). If one takes as representations of logical conjunction and disjunction their intersection and linear envelope and if negation corresponds to the orthogonal complement one obtains the ortholattice isomorphic to the logic of our players. We saw that in one ”experiment” neither

![Figure 2: Lattice of Alice’s questions and Bob’s answers](image-url)
Alice nor Bob have a stable strategy. However if the game is repeated many times one can ask about optimal frequencies of the corresponding pure strategies. Due to the non-distributivity of the logic it is impossible to define on the sets $S_A$ and $S_B$ of pure strategies a probabilistic measure. The main problem is calculation of an adequate procedure of averaging. Following well known constructions of quantum mechanics we take instead of the sets of pure strategies of Alice and Bob $S_A, S_B$ the pair of two-dimensional Hilbert spaces $H_A, H_B$. So pure strategies are represented by one-dimensional subspaces or normalized vectors of Hilbert space (wave functions). Use of Hilbert space permits us without any difficulties to realize the non-distributive logic of our players. So the average payoff for the given types of behaviour of the players:

$$E_{\psi \otimes \psi} \hat{H}_A = \sum_{j,k=1}^{4} h_{jk} \langle \alpha_j \varphi, \varphi \rangle \cdot \langle \beta_k \psi, \psi \rangle$$

Putting into this formula the elements of our payoff matrix and using the notations $p_j = \langle \alpha_j \varphi, \varphi \rangle$, $q_k = \langle \beta_k \psi, \psi \rangle$ one obtains

$$E_{\psi \otimes \psi} \hat{H}_A = a p_1 q_3 + cp_3 q_1 + bp_2 q_4 + dp_4 q_2$$

(2)

The definition of the Nash equilibrium for the quantum case is not much different from the classical case (1) and can be written as

$$E \hat{H}_A(\varphi^0, \psi^0) \geq E \hat{H}_A(\varphi, \psi^0), \quad E \hat{H}_B(\varphi^0, \psi^0) \geq E \hat{H}_B(\varphi^0, \psi)$$

It is convenient to find the equilibrium points in the coordinate form. To do this let us fix in the space of strategies of Alice $H_A$ eigenbasis $\{\xi_1^+, \xi_1^-, \xi_2^+, \xi_2^- \}$ corresponding to two projectors $\hat{\alpha}_1, \hat{\alpha}_2$ and let us do the same for Bob, taking bases $\{\eta_1^+, \eta_1^-, \eta_2^+, \eta_2^- \}$. The angles between the largest eigenvectors denote as $\theta_A$ and $\theta_B$. Then one can write in the quantum payoff function

$$E \hat{H}_A(\varphi, \psi) = a p_1 q_3 + cp_3 q_1 + bp_2 q_4 + dp_4 q_2$$

the squares of moduli of the amplitudes $p_j, p_k$ as

$$p_1 = \cos^2 \alpha, \quad p_3 = \sin^2 \alpha, \quad p_2 = \cos^2 (\alpha - \theta_A), \quad p_4 = \sin^2 (\alpha - \theta_A),$$

$$q_1 = \cos^2 \beta, \quad q_3 = \sin^2 \beta, \quad q_2 = \cos^2 (\beta - \theta_B), \quad q_4 = \sin^2 (\beta - \theta_B),$$

where $\alpha, \beta$ are the angles of vectors $\varphi, \psi$ to the corresponding axises. For values of angles one can take the interval $[0^0; 180^0]$. In the result the problem of search of the equilibrium points of the quantum game became the problem of finding a minimax of the function of two angle variables

$$F(\alpha, \beta) = a \cos^2 \alpha \sin^2 \beta + c \sin^2 \alpha \cos^2 \beta +$$

$$+ b \cos^2 (\alpha - \theta_A) \sin^2 (\beta - \theta_B) + d \sin^2 (\alpha - \theta_A) \cos^2 (\beta - \theta_B)$$
on the square $[0^0; 180^0] \times [0^0; 180^0]$. Differently from the geometrical saddle points the conditions of the Nash equilibrium are not just putting to zero values of the corresponding partial derivatives. So in the situation of absence of simple analytical solutions one must look for numerical methods. To do calculations we use an algorithm based on the construction of "curves of reaction" or "curves of the best answers" of the participants of the game. The definition of curves of reaction is based on the following consideration. If Alice knew what decision Bob will take she could make an optimal choice. But the essence of the game situation is that she doesn’t know it. She must take into account his different strategies and on each possible act of the partner she must find the optimal way to act. Her considerations look like considerations of the player, expressed by the formula: "if he does this, then I shall do that". Bob thinks the same way. So one must consider two functions,

$$\alpha = R_A(\beta) \quad \text{and} \quad \beta = R_B(\alpha)$$

the plots of which are called the curves of reactions of Alice and Bob. Due to the definition of these functions

$$\max_{\lambda} F(\lambda, \beta) = F(R_A(\beta), \beta), \quad \min_{\mu} F(\alpha, \mu) = F(\alpha, R_B(\alpha))$$

It is easy to see that intersections of curves of reaction give points of Nash equilibrium. Numerical experiments show that dependent on the values of the parameters $a, b, c, d$ of the payoff function and the angles characterizing the type of player one has qualitatively different pictures. Intersections can be absent, there can be one intersection and lastly there can be the case with two equilibrium points with different values of the payoff of the game, which is absent in the case of the classical matrix game.

1. Two equilibrium points arise in the case of the payoff matrix and an operator representation of the ortholattice corresponding to angles $\theta_A = 10^0$, $\theta_B = 70^0$. One of the equilibrium points is inside the square, the other one is on its boundary (see Fig. 3). The curves of reaction in this case happen to be discontinuous. For convenience the discontinuities are shown by thin lines. The discontinuous character of the curve of reaction of Alice made it impossible for one more equilibrium point to occur. One of the equilibrium takes place for $\alpha = 145,5^0, \beta = 149,5^0$ and gives the following values for the squares of moduli of amplitudes:

for Alice $p_1 = 0,679; p_2 = 0,509; p_3 = 0,321; p_4 = 0,491$;
for Bob $q_1 = 0,258; q_2 = 0,967; q_3 = 0,742; q_4 = 0,033$. 

| $A \setminus B$ | 1 | 2 | 3 | 4 |
|-----------------|---|---|---|---|
| 1               | 0 | 3 | 0 |   |
| 2               | 0 | 0 | 3 |   |
| 3               | 5 | 0 | 0 |   |
| 4               | 0 | 1 | 0 |   |
The price of the quantum game, i.e. the equilibrium value of the profit for Alice in this case is equal to $E\hat{H}_A = 2.452$. The second equilibrium point corresponds to angles $\alpha = 180^0$, $\beta = 123.5^0$ and the squares of the amplitude moduli

for Alice $p_1 = 1.000; p_2 = 0.967; p_3 = 0.000; p_4 = 0.033$;
for Bob $q_1 = 0.695; q_2 = 0.646; q_3 = 0.305; q_4 = 0.354$.

The price of the game in the second equilibrium point is equal to $E\hat{H}_A = 1.926$.

2. A unique equilibrium is observed for example in the case when all nonzero payoffs are equal and are equal to one and for equal angles $\theta_A = 45^0$, $\theta_B = 45^0$. The equilibrium point is located in the upper right vertex of the square (see Fig. 4): The curve of Bob’s reaction is shown on the Fig. 4 as

Figure 3: Two points of Nash equilibrium

Figure 4: The unique Nash equilibrium
continuous while the analogous curve of Alice is discontinuous when Bob is using the strategy corresponding to the angle $\beta = 90^0$. To make it more explicit the discontinuity is shown by drawing the thin line. In reality both lines are discontinuous. This becomes evident if one prolongs both functions on the whole real axis taking into account the periodicity: the plots of one of them is obtained by the shift of the other one on the halfperiod – $90^0$. The squares of the amplitude moduli in this case have the following values

for Alice: $p_1 = 1$; $p_2 = 0.5$; $p_3 = 0$; $p_4 = 0.5$;
for Bob: $q_1 = 1$; $q_2 = 0.5$; $q_3 = 0$; $q_4 = 0.5$.

The payoff of the "wise" Alice in this case is $\hat{E}H_A = 0.5$. The unique equilibrium located inside the square takes place for the initial payoff matrix $a = 3$, $b = 3$, $c = 5$, $d = 1$ and angles $\theta_A = 15^0$, $\theta_B = 35^0$ (see Fig. 5).

The other example of the unique Nash equilibrium

3. Absence of equilibrium is perhaps one of the most interesting phenomena, because as it is known for classical matrix games, equilibrium in mixed strategies always exist. One can obtain absence of equilibrium by taking the same payoff matrix for which one as well as two points of equilibrium were found. For this it is sufficient to take the operator representation of the ortholattice with typical angles: $\theta_A = 30^0$, $\theta_B = 0^0$. Absence of equilibrium in this case as it is seen from the Fig. 6 is due to the discontinuity of the functions of reaction which is impossible in the classical case. We met this phenomenon in the first example when two equilibrium points were obtained. This last example shows the importance of the realization of a nondistributive lattice. In the language of the game theory one can understand it as follows: having the same interests the players can form their behaviour qualitatively in different ways. So the mathematician can give to the client, for example to Alice, strategic recommendations: how she can organize the style of her behaviour to make the profit larger for the same payoff conditions. For this, however, he must know the choice of the representation of Bob’s logic.
Acknowledgements

One of the authors (A.A.G.) is indebted to the Foundation of the Ministry of Education of Russia, grant E0-00-14 for the financial support of this work.

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