FEDOSOV CONNECTIONS ON JET BUNDLES AND DEFORMATION QUANTIZATION

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Abstract. We review our construction of star-products on Poisson manifolds and discuss some examples. In particular, we work out the relation with Fedosov’s original construction in the symplectic case.

1. Introduction

In [8] Kontsevich solved the problem of describing associative deformations of the algebra of functions on a manifold. In particular, he gave an explicit formula for the associative deformations (“star-products”) of the product of functions on an open subset of \( \mathbb{R}^d \), with bidifferential operators \( B_j \) and so that \( f \star 1 = 1 \star f = f \). Here \( \epsilon \) is a formal parameter (\( \epsilon = \hbar / 2 \) in the notation of physics). He also described a (non-explicit) way to obtain star-products on arbitrary manifolds, as a corollary of his formality theorem, see [8] and, for more details, Appendix A.3 of [9].

Here we review a more explicit version of the construction of star-products on arbitrary manifolds. For a more physics-oriented review and the relation to quantum field theory, see [4]. Our approach is based on the notion of Fedosov connections on jet bundles. A special case of this notion was introduced by Fedosov [7] to construct star-products such that the Poisson bracket defined by

\[
f \star g - g \star f = 2 \epsilon \{ f, g \} + O(\epsilon^2)
\]

comes from a symplectic structure.

In general, one is given a Poisson manifold \((M, \alpha)\), which is a manifold \( M \) with a bivector field \( \alpha \in \Gamma(M, \wedge^2 TM) \) so that the bracket \( \{ f, g \} = \alpha(df \otimes dg) \) obeys the Jacobi identity and asks for star-products as above so that

\[
\frac{1}{2}(B_1(f, g) - B_1(g, f)) = \{ f, g \}.
\]

To construct such products using Kontsevich’s local formula, we first describe the local data given by the Poisson structure. Let \( E_0 \) be the bundle of (infinite) jets of smooth functions on \( M \). The fiber over \( x \in M \) consists of jets of functions at \( x \), namely equivalence classes of smooth functions defined on some open neighborhood of \( x \), where two functions are considered to be equivalent if they have the same Taylor expansion at \( x \) (with respect to any choice of local coordinates around \( x \)). Similarly, one considers jets of multivector fields and multidifferential operators. To any function \( f \in C^\infty(M) \) there corresponds a section of \( E_0 \), the jet of \( f \); its value at \( x \) is the jet of \( f \) at \( x \). The bundle \( E_0 \) comes with a canonical flat connection \( D_0 : \Gamma(M, E_0) \to \Omega^1(M, E_0) \), which is the unique connection so that its horizontal sections are precisely the jets of globally defined smooth functions. Moreover the

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jets of \( \alpha \) define a Poisson bracket on each fiber and we have the Leibniz rules

\[
D_0(fg) = D_0(f)g + fD_0(g), \quad D_0\{f, g\} = \{D_0(f), g\} + \{f, D_0(g)\},
\]

for any sections \( f, g \in \Gamma(M, E_0) \). Thus \( E_0 \) is a bundle of Poisson algebras with flat connection. The Leibniz rules imply that we have an isomorphism of Poisson algebras

\[
\iota : C^\infty(M) \to H^0(E_0, D_0) = \text{Ker}(D_0),
\]

from the algebra of smooth functions onto the algebra of horizontal sections.

The idea is now to “quantize” \((E_0, D_0)\): we construct a bundle \( E = E_0 \otimes \mathbb{R}[[\epsilon]] \) of associative \( \mathbb{R}[[\epsilon]] \)-algebras using Kontsevich’s local formula in each fiber. Then we deform the connection \( D_0 \) to a flat connection \( \bar{D} = D_0 + O(\epsilon) \) on \( E \) obeying the Leibniz rule

\[
\bar{D}(f \ast g) = \bar{D}(f) \ast g + f \ast \bar{D}(g), \quad f, g \in \Gamma(M, E).
\]

Thus the product induces a product on the space of horizontal sections \( H^0(E, \bar{D}) \). Finally we construct a quantization map, an isomorphism of \( \mathbb{R}[[\epsilon]] \)-modules \( \rho : H^0(E_0, D_0)[[\epsilon]] \to H^0(E, \bar{D}) \) given by differential operators and such that \( \rho(1) = 1 \). Then \( f \ast_M g = \rho^{-1}(\rho(f) \ast \rho(g)) \) is a star-product on \( M \).

2. Formality theorem for \( \mathbb{R}^d \)

Let \( \mathcal{A}_d = \mathbb{R}[[y^1, \ldots, y^d]] \) be the algebra of formal power series in \( d \) indeterminates. A (formal) Poisson bracket on \( \mathcal{A}_d \) is a Lie algebra structure on \( \mathcal{A}_d \) of the form

\[
\{f, g\} = \sum_{i,j=1}^d \alpha_{ij} \frac{\partial f}{\partial y^i} \frac{\partial g}{\partial y^j}, \quad f, g \in \mathcal{A}_d,
\]

for some \( \alpha_{ij} \in \mathbb{R}[[y^1, \ldots, y^d]] \). M. Kontsevich constructed in \( \mathbb{R} \) an \( \mathbb{R}[[\epsilon]] \)-bilinear associative product \( \ast_\alpha \) on \( \mathcal{A}_d[[\epsilon]] \) which is a deformation of the product on \( \mathcal{A}_d \): for \( f, g \in \mathcal{A}_d \), the product has the form

\[
f \ast_\alpha g = fg + \sum_{n=1}^\infty \frac{\epsilon^n}{n!} U_n(\alpha, \ldots, \alpha)(f \otimes g),
\]

where \( U_n(\alpha, \ldots, \alpha) \in \text{Hom}_\mathbb{R}(\mathcal{A}_d \otimes \mathcal{A}_d, \mathcal{A}_d) \) is a certain bidifferential operator whose coefficients are homogeneous polynomials of degree \( n \) in the partial derivatives of the coordinates \( \alpha_{ij} \) of the Poisson bivector field \( \alpha \). For example, \( U_1(\alpha)(f \otimes g) = \sum_{i,j=1}^d \alpha_{ij} \partial_i f \partial_j g = \{f, g\} \).

More generally, \( U_n(\alpha_1, \ldots, \alpha_n) \) is defined not only for Poisson bivector fields but also for multivector fields (skew-symmetric contravariant tensor fields) \( \alpha_i \). If \( \alpha_i \) has rank \( m_i, i = 1, \ldots, n \), then \( U_n(\alpha_1, \ldots, \alpha_n) \in \text{Hom}_\mathbb{R}(\mathcal{A}_d^{\otimes m}, \mathcal{A}_d) \), where \( m = \sum m_i - 2n + 2 \), is a multidifferential operator. The coefficients of these multidifferential operators are polynomials in the partial derivatives of the coordinates of \( \alpha_1, \ldots, \alpha_n \), linear in each \( \alpha_i \). They are given in terms of integrals of differential forms over configuration spaces of \( n \)-points in the upper half plane, see \( \mathbb{R} \), and can be interpreted as Feynman amplitudes for a topological string theory.

The associativity of the product is a special case of Kontsevich’s formality theorem, which is a sequence of quadratic relations for the operators \( U_n \). For our purpose we need a further set of special cases, namely all cases where \( \alpha_i \) is either a given Poisson bivector field, a vector field or a function. Leaving functions apart for
the time being, the relations may be expressed as follows: introduce the generating series

\[(1) \quad P(\alpha) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} U_n(\alpha, \ldots, \alpha) \in \text{Hom}_R(\mathcal{A}_d \otimes \mathcal{A}_d, \mathcal{A}_d),\]

\[(2) \quad A(\xi, \alpha) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} U_{n+1}(\xi, \alpha, \ldots, \alpha) \in \text{Hom}_R(\mathcal{A}_d, \mathcal{A}_d),\]

\[(3) \quad F(\eta, \alpha) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} U_{n+2}(\xi, \eta, \alpha, \ldots, \alpha) \in \mathcal{A}_d.\]

Let \(W_d = \oplus_1^d \mathcal{A}_d \partial / \partial y^i\) be the Lie algebra of formal vector fields on \(\mathbb{R}^d\). This Lie algebra acts on \(\mathcal{A}_d\) and thus on the Hochschild cochain complex \(\text{Hoch}_1(\mathcal{A}_d, \mathcal{A}_d) = \oplus_{n=0}^{\infty} \text{Hom}_R(\mathcal{A}_d^\otimes n, \mathcal{A}_d)\). Let

\[C^*_\text{Lie}(W_d, \text{Hoch}^1(\mathcal{A}_d, \mathcal{A}_d)) = \text{Hom}_R(\Lambda^\cdot W_d, \text{Hoch}^1(\mathcal{A}_d, \mathcal{A}_d))\]

be the corresponding Lie algebra complex, with differential \(\delta\). The maps \(\xi_1 \wedge \cdots \wedge \xi_k \mapsto U_{n+k}(\xi_1, \ldots, \xi_k, \alpha, \ldots, \alpha)\), for \(k = 0, 1, 2\) can be then considered as elements of this complex. The formality theorem for these objects may be written as

(i) \(P(\alpha) \circ (P(\alpha) \otimes \text{Id}) = P(\alpha) \circ (\text{Id} \otimes P(\alpha))\).
(ii) \(P(\alpha) \circ (A(\xi, \alpha) \otimes \text{Id} + \text{Id} \otimes A(\xi, \alpha)) - A(\xi, \alpha) \circ P(\alpha) = \delta P(\xi, \alpha)\).
(iii) \(P(\alpha) \circ (F(\eta, \xi, \alpha) \otimes \text{Id} - \text{Id} \otimes F(\xi, \eta, \alpha)) - A(\xi, \alpha) \circ A(\eta, \alpha) + A(\eta, \alpha) \circ A(\xi, \alpha) = \delta A(\xi, \eta, \alpha)\).
(iv) \(-A(\xi, \alpha) \circ F(\eta, \xi, \alpha) - A(\eta, \alpha) \circ F(\xi, \eta, \alpha) - A(\xi, \alpha) \circ F(\xi, \eta, \alpha) = \delta F(\xi, \eta, \xi, \alpha)\).

The first equation is the associativity of the product. The second equation says that changing coordinates leads to an equivalent product. Indeed \(\delta P(\xi, \alpha) = \frac{\partial}{\partial t} P(\phi_t \alpha)|_{t=0}\) is the infinitesimal variation of the product under a coordinate transformation given by the flow \(\phi_t\) of \(\xi\). Before explaining the meaning of the remaining equations, which will appear in Prop. 5.1 below, we need to discuss the action of the Lie algebra \(\text{gl}_d \subset W_d\) of linear vector fields and the lowest order terms of \(P, A, F\). The action of \(\text{gl}_d\) on \(P, A, F\) is described by two properties:

A. The operators \(U_n(\alpha_1, \ldots, \alpha_n)\) are \(\text{gl}_d\) equivariant, in the sense that

\[g^* \circ U_n(g \cdot \alpha_1, \ldots, g \cdot \alpha_n) = U_n(\alpha_1, \ldots, \alpha_n) \circ (g^*)^\otimes m,\]

where \(g^* f(y) = f(gy)\) and \(g \cdot \alpha_i(y) = (g \otimes \cdots \otimes g) \alpha_i(g^{-1} y)\), if \(g \in \text{GL}(d, \mathbb{R})\), \(f \in \mathcal{A}_d\), and \(\alpha_i\) is a formal multivector field. Therefore \(P, A, F\) are \(\text{gl}_d\)-equivariant.

B. For any \(\xi \in \text{gl}_d, \eta \in W_d, A(\xi, \alpha) = \xi\), viewed as a first order differential operator and \(F(\xi, \eta, \alpha) = 0\). This property follows from the vanishing of certain integrals, see property P5 in Sect. 7 of [8].

The lowest order terms in \(\epsilon\) are also the result of explicit calculation of simplest Feynman integrals:

C. \(P(\alpha)(f \otimes g) = fg + \epsilon \alpha(df \otimes fg) + O(\epsilon^2)\).
D. \(A(\xi, \alpha) = \xi + O(\epsilon)\).
E. \(F(\xi, \eta) = O(\epsilon)\).

It is worth noticing that the equations (i)–(iv) may be written succinctly as a Maurer–Cartan equation \(\delta S + \frac{1}{2}[S, S]_G = 0\) for \(S = P + A + F\), where \([\cdot, \cdot]_G\) is the Gerstenhaber bracket on the Hochschild complex.
3. Formal geometry

In order to apply Kontsevich’s formula to the jet bundle $E_0$ we need to introduce coordinates. This is done using ideas of formal geometry, see [3]: let $M^{\text{coor}}$ be the manifold of jets of coordinates systems on $M$. A point in $M^{\text{coor}}$ is an infinite jet at zero of local diffeomorphisms $U \subset \mathbb{R}^d \rightarrow M$ defined on some open neighborhood $U$ of $0 \in \mathbb{R}^d$. Two such maps define the same infinite jet if and only if their Taylor expansions at zero (for any choice of local coordinates on $M$) coincide. We have a projection $\pi : M^{\text{coor}} \rightarrow M$ sending $\varphi$ to $\varphi(0)$.

The group $\text{GL}(d, \mathbb{R})$ of linear diffeomorphisms acts on $M^{\text{coor}}$ and we set $M^{\text{aff}} = M^{\text{coor}}/\text{GL}(d, \mathbb{R})$. Define $\tilde{E}_0 := M^{\text{coor}} \times_{\text{GL}(d, \mathbb{R})} \mathbb{R}[[y^1, \ldots, y^d]]$. Moreover the Lie algebra $W_d$ of formal vector fields acts on $M^{\text{coor}}$ by infinitesimal coordinate transformations. The action is given by an isomorphism from $W_d$ to the tangent space at any point of $M^{\text{coor}}$. The inverse map defines the Maurer–Cartan form $\omega_{MC}$, a $W_d$-valued one-form on $M^{\text{coor}}$. The fact that this one-form comes from a Lie algebra action implies that it obeys the Maurer–Cartan equation $d\omega_{MC} + \frac{1}{2} [\omega_{MC}, \omega_{MC}] = 0$.

As a consequence, $\tilde{D}_0 = d + \omega_{MC}$ is a flat connection on the trivial bundle $M^{\text{coor}} \times A_d$ over $M^{\text{coor}}$ and it is easy to check that it descends to $\tilde{E}_0$.

We will need the fact that the fibers of the bundle $M^{\text{aff}} \rightarrow M$ are contractible so that there exist sections $\varphi^{\text{aff}} : M \rightarrow M^{\text{aff}}$. For example, the exponential map of a torsion free connection on the tangent bundle defines such a section.

**Lemma 3.1.** Let $\varphi^{\text{aff}} : x \mapsto [\varphi_x]$ be a section of $M^{\text{aff}}$. The map sending a point $(x, f)$ of the jet bundle $E_0$ ($f$ is the jet of a function at $x$) to the class of $(\varphi_x, \text{Taylor expansion of } f \circ \varphi_x)$ is an isomorphism of vector bundles with flat connection from $(E_0, D_0)$ to $(\varphi^{\text{aff}} \ast \tilde{E}_0, \varphi^{\text{aff}} \ast \tilde{D}_0)$.

From now on, we fix a section $\varphi^{\text{aff}}$ and identify $(E_0, D_0)$ with $(\varphi^{\text{aff}} \ast \tilde{E}_0, \varphi^{\text{aff}} \ast \tilde{D}_0)$. Here is an explicit local description of $E_0$ and $D_0$. On any open contractible subset $U \subset M$ we may choose a section $\varphi$ of $M^{\text{coor}} \rightarrow M$ such that $\pi \circ \varphi = \varphi^{\text{aff}}$ where $\pi : M^{\text{coor}} \rightarrow M^{\text{aff}}$ is the canonical projection. We call such a section a local lift of $\varphi^{\text{aff}}$. Local lifts on $U$ differ by a $\text{GL}(d, \mathbb{R})$-gauge transformation $\varphi \mapsto \varphi \circ g$, $g : U \rightarrow \text{GL}(d, \mathbb{R})$. A local lift induces a local trivialization of $E_0|_U \simeq U \times A_d$, so that the isomorphism $C^\infty(M) \rightarrow \mathcal{H}^0(E_0, D_0)$ sends $f$ to the Taylor expansion at zero of $f \circ \varphi$. One way to choose a local lift is induced by a choice of local coordinates: let $x^1, \ldots, x^d : U \rightarrow \mathbb{R}$ be coordinates on $U \subset M$. Then there is a unique local lift $x \mapsto \varphi_x$ so that $\varphi_x^* = x^1 \circ \varphi_x$ has the form

$$\varphi_x^*(y) = x^1 + y^j + \text{higher order terms in } y.$$ 

The flat connection $D_0$ on a local section $x \mapsto f_x(y) \in \mathbb{R}[[y^1, \ldots, y^d]]$ is then

$$D_0 f_x(y) = \sum_{j=1}^d dx^j \left( \frac{\partial f_x(y)}{\partial x^j} - \sum_{k,l=1}^d T^j_k(x,y) \frac{\partial \varphi_x^k(y)}{\partial x^j} \frac{\partial f_x(y)}{\partial y^k} \right),$$

where $T(x,y)$ is the matrix inverse to $(\partial \varphi_x^k(y)/\partial y^k)$. Indeed, it follows from the chain rule that Taylor expansions of globally defined functions are $D_0$-closed sections. Conversely, observe that $\sum_{k,l=1}^d T^j_k(x,y) \frac{\partial \varphi_x^k(y)}{\partial x^j}$ is a formal power series in $y$ beginning with $\delta_k^j$ and whose coefficients are smooth in $x$. By this property it follows immediately that the coefficients of a section $\sigma$ of $E_0$ satisfying $D_0 \sigma = 0$ are determined by the zeroth coefficient $\sigma^0(x)$. If we set $\tilde{\sigma} = \sigma^0 \circ \varphi$, we have $D_0(\sigma - \tilde{\sigma}) = 0$.
and \((\sigma - \tilde{\sigma})|_{y=0} = 0\); but this implies \(\sigma = \tilde{\sigma}\). This shows that a section of \(E_0\) is the Taylor expansion of a globally defined function if and only if it is \(D_0\)-closed.

4. A deformation of the canonical connection

Let \(E = E_0[[\epsilon]]\) be the bundle of jets of \(\mathbb{R}[[\epsilon]]\)-valued functions. By definition, sections of \(E\) are formal power series in \(\epsilon\) whose coefficients are sections of \(E_0\). We fix as above a section \(\phi^\text{aff}\) of \(M^\text{aff}\). Then \(E\), with fiberwise Kontsevich product associated to the Taylor expansion of \(\alpha\), is a bundle of \(\mathbb{R}[[\epsilon]]\)-algebras: choose a local lift \(x \mapsto \phi_x\) of \(\phi^\text{aff}\) to a section of \(M^\text{coor}\). This induces a local trivialization of \(E_0\) so that local sections are given by \(A_x[[\epsilon]]\)-valued functions \(x \mapsto f_x(y)\). The product of local sections \(f, g\) is then the section

\[
x \mapsto (f \ast g)_x = P(\alpha_x)(f_x \otimes g_x), \quad \alpha_x = \text{Taylor expansion of } (\varphi^{-1}_x)_x \alpha.
\]

It does not depend on the choice of local lift since \(P\) is \(\text{GL}(d, \mathbb{R})\)-equivariant, see Property A in Sect. 3.

We now use \(A\) of equation (2) to construct a deformation \(D = D_0 + \epsilon D_1 + \cdots\) of the canonical connection \(D_0\). Let \(x \in M\) and \(\xi \in T_x M\). Again we choose a local lift \(\varphi\) of \(\varphi^\text{aff}\) and set

\[
\hat{\xi}_x = \varphi_x^* \omega_\text{MC}(\xi) \in W_d.
\]

The connection is then defined as

\[
Df_x = df_x + A_x^M f_x,
\]

where \(A_x^M(\xi) = A(\hat{\xi}_x, \alpha_x), \xi \in T_x M\). By properties A and B, this formula defines a connection on \(E\) which is independent of the choice of local lift. By property D, it is a deformation of \(D_0\).

Equation (ii) in the formality theorem now implies the crucial statement:

**Proposition 4.1.** For any \(f, g \in \Gamma(M, E)\), we have the Leibniz rule

\[
D(f \ast g) = Df \ast g + f \ast Dg.
\]

However \(D\) is not flat, so that the algebra of horizontal sections with respect to \(D\) will not be isomorphic to \(C^\infty(M)[[\epsilon]]\) as an \(\mathbb{R}[[\epsilon]]\)-module. We are going to discuss this difficulty in the next section.

5. Flattening the connection \(D\)

The connection \(D\) can be extended to a (graded) derivation

\[
D : \Omega^\cdot(M, E) \to \Omega^{\cdot+1}(M, E)
\]

of the algebra of differential forms on \(M\) with values in \(E\) (the product on this algebra is defined by the star-product in the fibers and the wedge product on differential forms). Its curvature \(D^2\) is then an \(\text{End}(E)\) valued two-form. Let \(F^M \in \Omega^2(M, E)\) be the two-form \(x \mapsto F^M_x\) with

\[
F^M_x(\xi, \eta) = F(\hat{\xi}_x, \hat{\eta}_x, \alpha_x), \quad \xi, \eta \in T_x M.
\]

By properties A and B, \(F^M\) is independent of the local lift \(\varphi_x\) needed to define \(\hat{\xi}_x, \hat{\eta}_x\) and \(\alpha_x\).

Then the formality identities (iii), (iv) translate into the following statements.
Proposition 5.1. For any \( f \in \Gamma(M, E) \),
\[
(4) \quad D^2 f = F^M \star f - f \star F^M,
\]
\[
(5) \quad D F^M = 0.
\]

The two-form \( F^M \) is called the Weyl curvature of \( D \). In general, a connection on a bundle of associative algebras with the above properties, i.e. to be a derivation whose curvature is an inner derivation such that its Weyl curvature satisfies the Bianchi identity \((\ref{eq:Bianchi})\), is called a Fedosov connection.

We want now to modify \( D \) so that it becomes flat still remaining a derivation. The first observation is that
\[
\bar{D} := D + [\gamma, \cdot],
\]
is still a derivation for any \( \gamma \in \Omega^1(M, E) \). The star-commutator is defined by
\[
[a, b]_\star = a \star b - b \star a.
\]
Moreover, \( \bar{D} \) turns out to be again a Fedosov connection with Weyl curvature \( \bar{F}^M = F^M + D \gamma + \gamma \star \gamma \). If we are able to find \( \gamma \) so that \( \bar{F}^M = 0 \),
\[
(6) \quad F^M + D \gamma + \gamma \star \gamma = 0,
\]
then \( \bar{D} \)-closed sections will form a nontrivial subalgebra of \( \Gamma(M, E) \).

The one-form \( \gamma \) can be found as follows. Since \( F^M \) starts at order \( \epsilon \), see property \( E \), we may write \( F = \epsilon F_1 + \epsilon^2 F_2 + \cdots \). The correction \( \gamma \) does not need to have a term of order zero, so we write \( \gamma = \epsilon \gamma_1 + \epsilon^2 \gamma_2 + \cdots \). The equation \( \bar{F} = 0 \) at order \( \epsilon \) reads
\[
F_1 + D_0 \gamma_1 = 0,
\]
while the Bianchi identity imply at this order \( D_0 F_1 = 0 \). Hence, \( D_0 \gamma_1 \) is equal to a \( D_0 \)-closed expression. But the \( D_0 \)-cohomology is trivial in degree 2, so it is possible to find a \( \gamma_1 \) that solves the equation. At higher order in \( \epsilon \), one proves by induction that one always has an equation of the form \( D_0 \gamma_k \) equal to a known \( D_0 \)-closed form depending on the lower order coefficients of \( \gamma \) and \( F^M \), so a solution exists by the same argument.

Thus we have a flat connection \( \bar{D} \) on the bundle of algebras \( E \), obeying the Leibniz rule. Let \( \mathcal{H}^0(E, \bar{D}) \) be the \( \mathbb{R}[[\epsilon]] \)-algebra of horizontal sections. We are left to show the existence of a quantization map, an \( \mathbb{R} \)-linear homomorphism
\[
\rho : C^\infty(M) \simeq \mathcal{H}^0(E_0, D_0) \to \mathcal{H}^0(E, \bar{D}),
\]
inducing an isomorphism \( C^\infty(M)[[\epsilon]] \to \mathcal{H}^0(E, \bar{D}) \) of \( \mathbb{R}[[\epsilon]] \)-modules. Then the pull-back of the product defines a star-product on \( C^\infty(M)[[\epsilon]] \). To construct \( \rho \) one looks for a bundle map, also denoted by \( \rho \), from \( E_0 \) to \( E \) of the form \( \rho = \text{Id} + \epsilon \rho_1 + \cdots \) so that
\[
(7) \quad \rho \circ D_0 = \bar{D} \circ \rho,
\]
and that \( \rho_i \) are given by differential operators. Again, there are no cohomological obstructions to find such a \( \rho \): the \( \rho_i \) can be found recursively by solving equations of the form \( D_0(\rho_i) := \rho_i \circ D_0 - D_0 \circ \rho_i \) known. The known term on the right is \( D_0 \)-closed by the induction hypothesis, and \( \rho_i \) can be found as a differential operator, as the \( D_0 \)-cohomology of the subbundle of \( \text{End}(E_0) \) given by differential operators is trivial in degree 1. Furthermore \( \rho \) is unique if we require that \( \rho(f) \epsilon(y = 0) = f_\epsilon(y = 0) \), for any section \( f \). The result is then:
Theorem 5.2. Let \((M, \alpha)\) be a Poisson manifold. Fix a section \(\varphi^{aff} : M \to M^{aff}\). Let \(\star, D, F^M\) be the corresponding product on \(\Gamma(M, E)\), the connection on \(E\) and its curvature, respectively. Let \(\gamma \in \Omega^2(M, E)\), and \(\rho \in \Gamma(M, \text{Hom}(E_0, E))\) be solutions of (6), (7), respectively. Then \(\bar{D} = D + [\gamma, \cdot , \cdot] \star\) is a flat connection on \(E\) obeying the Leibniz rule \(\bar{D}(f \star g) = \bar{D}(f) \star g + f \star \bar{D}(g)\) and \(\rho\) induces an isomorphism of \(\mathbb{R}[\epsilon]\)-modules \(C^\infty(M)[[\epsilon]] \to H^0(E, \bar{D})\). The pull-back of the product to \(C^\infty(M)[[\epsilon]]\) is a star-product with first order term \(\alpha\).

Thus the construction of a star-product for Poisson manifolds requires solving an equation for \(\gamma\) and an equation for \(\rho\). We want to show that these equations can be solved in an explicit way. The equation for \(\gamma = \sum \epsilon^j \gamma_j\) can be solved recursively: at the \(j\)th step of the recursion, one has an equation of the form

\[D_0 \gamma_j = \beta_j,\]

where \(\beta_j \in \Omega^2(M, E_0)\) obeys \(D_0 \beta_j = 0\) and is known. The point is that \(\gamma_j\) may be constructed by a purely local calculation: namely, let us introduce local coordinates and a local trivialization of \(E_0\) as in Sect. 3.

It is useful to define the total degree of a form on \(M\) taking values in sections of \(E_0\) as the sum of the form degree and the degree in \(y\). Then we write

\[D_0 = -\delta + D'_0,\]

where

\[\delta := \sum_{i=1}^d dx^i \frac{\partial}{\partial y^i}\]

is the zero-degree part and \(D'_0\) has positive degree. It follows immediately that \(\delta^2 = 0\). We can define a dual operator to \(\delta\) on \(E_0\)-valued differential forms:

\[\delta^* := \sum_{i=1}^d y^i \iota \frac{\partial}{\partial x^i},\]

where \(\iota\) denotes inner multiplication. It is easy to verify that \((\delta \delta^* + \delta^* \delta) \rho = k \rho\) for every form \(\rho\) of total degree \(k\). Thus, if we restrict to \(\delta\)-closed forms of positive total degree \(k\), we may then invert \(\delta\) by \(\delta^{-1} \rho = \frac{1}{k} \delta^* \rho\). This inverse yields the unique form \(\sigma\) such that \(\delta \sigma = \rho\) and \(\delta^* \sigma = 0\). This proves that the \(\delta\)-cohomology is concentrated in degree zero, i.e., functions on \(M\) (independent of \(y\)).

Then the solution of eq. (8) obeying \(\delta^* \gamma_i = 0\) is

\[\gamma_i = -\sum_{n=0}^\infty (\delta^{-1} D'_0)^n \delta^{-1} \beta_i.\]

The infinite sum converges in the sense of formal power series since the \(n\)-th term is of degree at least \(n\) in \(y\).

The equation for \(\rho\) can be solved similarly.

6. Casimir and central functions

We describe variants of our constructions which are relevant for the treatment of the center and for the comparison with Fedosov’s construction in the symplectic case.
6.1. Central closed two-forms. Let $\omega = \omega_0 + \epsilon \omega_1 + \cdots \in \Omega^2(M, E)$ be such that $D\omega = 0$ and $[\omega, f] = 0 \forall f \in \Gamma(M, E)$. Then we may construct a more general flat connection $\tilde{D}$ by replacing (9) by

$$F^M + D\gamma + \gamma \ast \gamma = \omega.$$  

Indeed, the Bianchi identity holds also for $F^M - \omega$ and $\tilde{D}^2 = 0$ since $\omega$ is central. We get thus a family of products parametrized by $D$-closed central two forms $\omega$.

6.2. The second construction of a quantization map. The second variation concerns the quantization map. It is possible to make it compatible with the center in the following sense: let $Z_0(M) = \{ f \in C^\infty(M) \mid \{ f, \cdot \} = 0 \}$ be the algebra of Casimir functions. There exists a quantization map $\rho : H^0(E, D_0) \to H_0(E, \tilde{D})$ whose restriction to $Z_0(M)$ induces an algebra isomorphism from $Z_0(M)[[\epsilon]]$ onto the center of $H^0(E, \tilde{D})$. It is constructed using the two remaining special cases of the formality theorem involving the Poisson vector field, vector fields and functions, see [5].

This quantization map can be used to construct a linear map from the space of $D_0$-closed Casimir two-forms $B = \{ \omega \in \Omega^2(M, E_0) \mid D_0\omega = 0, \{ \omega, \cdot \} = 0 \}$ onto the space of $\tilde{D}$-closed central two-forms. Thus we may parametrize our products by two-forms in $B[[\epsilon]]$.

6.3. Dependence on choices. If we choose the homotopy $\delta^{-1}$ to determine canonical choices of $\gamma$ and $\rho$, our construction of star-products depends on a Poisson bivector field $\alpha$, a section $\varphi^\text{aff}$ and a central closed 2-form in $B[[\epsilon]]$. It is possible to show that different choices of $\varphi^\text{aff}$ lead to equivalent products (two star-products $\ast, \ast'$ are called equivalent if there is a series $\psi = \text{Id} + \epsilon \psi_1 + \epsilon^2 \psi_2 + \cdots$ so that $\psi(f \ast g) = \psi(f) \ast' \psi(g), \forall f, g \in C^\infty(M)$). Also, if we replace $\omega \in B[[\epsilon]]$ by $\omega + D_0\beta$ for a Casimir one-form $\beta \in \Omega^1(M, E_0)$, we obtain an equivalent star-product. Still in general we do not get all star-products up to equivalence in this way. To obtain all equivalence classes of star-products, we have to take $\alpha$ to be a power series in $\epsilon$. Then the equivalence classes of star-products are in one-to-one correspondence with $\mathbb{R}[[\epsilon]]$-valued Poisson vector fields $\alpha$ modulo formal paths in the group of diffeomorphisms generated by flows of vector fields $\epsilon \xi_1 + \epsilon^2 \xi_2 + \cdots$. This correspondence is realized by taking $\omega = 0$ in our construction. If we take another $\omega$, we get thus a product corresponding to $\omega = 0$ for a different $\alpha$. This defines a map from $B[[\epsilon]]$ modulo $D_0$-exact central two-forms to the equivalence classes of $\mathbb{R}[[\epsilon]]$-valued Poisson bivector fields. It would be interesting to describe this map. For infinitesimal $\omega$ and to leading order in $\epsilon$ this map has the following description. Let $C^\cdot: = \Omega(M, \wedge^\cdot \text{Det}(E_0))$ be the double complex of differential forms with values in the jets of formal multivector fields. The differentials are $D_0$ and the differential $\delta_\alpha$ of Poisson cohomology (the Schouten–Nijenhuis bracket with $\alpha$) on the fibers. Then $\omega$ defines a class in $H^2(H^0(C^\cdot, \delta_\alpha), D_0)$. As the $D_0$-cohomology is trivial except in degree zero, we have a map

$$j : H^2(H^0(C^\cdot, \delta_\alpha), D_0) \to H^0(H^2(C^\cdot, \delta_\alpha), D_0) = H_\alpha^2(M),$$

given by $\omega \mapsto \delta_\alpha D_0^{-1} \delta_\alpha D_0^{-1} \omega$. Thus $j$ sends the class of $\omega$ to an element of the second Poisson cohomology group, which consists of equivalence classes of infinitesimal variations of the Poisson bivector field $\alpha$. 

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7. Examples, related results

7.1. Open subsets of $\mathbb{R}^d$. Let us check that our construction gives back the original Kontsevich formula for open subsets of $\mathbb{R}^d$. In this case we may take $\varphi^{aff}$ to be given by

$$\varphi_x^j(y) = x^j + y^j.$$ 

Then $D_0 = \sum_{j=1}^d dx^j (\frac{\partial}{\partial y^j} - \frac{\partial}{\partial x^j}).$ Since $A(\xi, \alpha) = \xi$ for constant $\xi$, we have $D = D_0$ and $F = 0$. We may then choose $\gamma = \rho = \text{Id}$ and the result is Kontsevich’s formula $f \ast g = \sum_{n=0}^\infty \frac{\alpha^n}{n!} U_n(\alpha, \ldots, \alpha)(f \otimes g)$.

7.2. Leaves of Poisson foliations. If $N$ is a foliated Poisson manifold so that the Poisson bivector field is tangent to the leaves, then each leaf is a Poisson submanifold. An adapted chart with domain $V \subset M$ is a chart $\psi : V \rightarrow \mathbb{R}^d$ so that the leaves are given by equations $\psi_j(x) = \text{const}$, $j = m+1, \ldots, d$. A star-product is called tangential if on the domain $V$ of any adapted chart and for any $f \in C^\infty(M)$ so that $f|_V$ is constant on each leaf, one has $f \ast g|_V = fg|_V$, $\forall g \in C^\infty(M)$. In this case it is shown that if $\varphi^{aff}$ is chosen to be adapted to the foliation, in the sense that for all $x \in M$, there is a representative $\varphi_x$ of $\varphi^{aff}(x)$ so that $\varphi_x^{-1}$ is an adapted chart, then $\ast$, with $\rho$ from the “second construction”, is tangential. Using this property, one can restrict star-products to the leaves of a foliated Poisson manifold.

7.3. The case of a symplectic manifold. Suppose $(M, \alpha)$ is a Poisson manifold whose Poisson bracket comes from a symplectic form $\Omega$.

A Darboux section of $M^{aff}$ is a section $x \mapsto [\varphi_x]$ of $M^{aff}$ such that, for all $x \in M$, $\varphi_x^\ast \Omega$ is a constant two-form. Symplectic manifolds always have Darboux sections. For example, a torsion free symplectic connection gives rise to a Darboux section (but not through the exponential map), see [1], Section 2.5.

In Darboux coordinates the Kontsevich product reduces to the Moyal product, and also the other objects $A, F$ may be described explicitly. As a result, we have explicit formulae in terms of a local lift of $\varphi^{aff}$ with the property that $\alpha_x = (\varphi_x^{-1})_\ast \alpha$ is a constant bivector field $\alpha_0$. In this case, $\varphi_x^\ast \omega_{MC}$ is a 1-form on $M$ with values in the Hamiltonian formal vector fields. The corresponding hamiltonian functions $h_x$, normalized by $h_x(y = 0) = 0$, define an $E_0$-valued one-form $x \mapsto h_x$. Then we have

\begin{enumerate}
  \item $(f \ast g)_x(y) = f_x \ast g_x(y) = \exp \left( \epsilon \sum \alpha_0 \frac{\partial}{\partial y_1} \frac{\partial}{\partial y_2} \right) f_x(y_1) g_x(y_2)|_{y_1 = y_2 = y}$.
  \item $Df_x = d_x f_x + \frac{1}{2}\{[h_x, f_x]\}$.
  \item $F^x: (\xi, \eta) = \frac{1}{2}(\{h_x, \xi\}, \{h_x, \eta\}) - 2\epsilon(\{h_x, \xi\}, \{h_x, \eta\})$.
\end{enumerate}

Moreover, in the symplectic case, the space $B$ of closed Casimir two-forms consists of closed two-forms in $\Omega^2(M)[\epsilon]) \subset \Omega^2(M, E)$. Using these formulae the comparison with Fedosov’s original construction becomes clear. The latter starts with the definition of the Weyl bundle: let $F(M)$ be the principal $\text{GL}(d, \mathbb{R})$-bundle of frames (bases of tangent spaces). The Weyl bundle is the associated vector bundle $W = F(M) \times \text{GL}(d, \mathbb{R}) \mathbb{R}[y^1, \ldots, y^d][[\epsilon]]$. The fiber over $x$ is the space of (formal) functions on the tangent space at $x$, a symplectic vector space. So there is a Moyal product in each fiber, and $W$ is a bundle of algebras. A symplectic connection $\nabla$ on the tangent bundle induces a connection $D_E$ on $W$ obeying the Leibniz rule on sections. Its curvature is $D^2_E = (2\epsilon)^{-1}[R, -]$, where the $W$-valued 2-form $R = -\frac{1}{2} \omega(\nabla^2 y, y)$ is the quadratic form on $TM$ associated with the curvature of $\nabla$. For any closed
two-form of the form $\Omega(\epsilon) = -(2\epsilon)^{-1}\Omega + \Omega_0 + \epsilon\Omega_1 + \cdots \in \Omega^2(M)[[\epsilon]]$, Fedosov shows that the equation $DF \psi F + \psi F \ast DF + R = \Omega(\epsilon)$ has a solution $\psi F$ of the form $(2\epsilon)^{-1}\Omega_{ij}y^idx^j + \psi F_1 + \epsilon\psi F_2 + \cdots$, obeying the normalization condition $\psi F|_{y=0} = 0$. It follows that connection $DF = DF + [\psi F, \cdot]$ is flat.

Let $\nabla$ be the symplectic connection on $TM$ whose Christoffel symbols are obtained from the 2-jet of the Darboux section via

$$\varphi^j_x(y) = x^j + y^j - \frac{1}{2} \sum_{k,l} \Gamma^j_{kl}(x)y^k y^l + \cdots.$$  

We have a bundle isomorphism $E \to W$ sending the jet at $x$ of a function $f$ to the class of $(e, f \circ \varphi^j_x)$, where $\varphi^j_x$ is the unique representative of $\varphi^{\text{aff}}_x$ that maps the standard frame of $\mathbb{R}^d$ to the frame $e$. By (i) this is an isomorphism of algebra bundles. The connection $DF$ is then of the form $D = D + [\gamma, \cdot]$ with $\gamma = (2\epsilon)^{-1}h_x + \gamma$, where $\gamma$ is a solution of (10) with $\omega = \Omega_0 + \epsilon\Omega_1 + \cdots$.

The result is that our star-product constructed using a closed two-form $\omega \in \Omega^2(M)[[\epsilon]]$ is equivalent to Fedosov’s star-product associated to the class of $\Omega(\epsilon) = -(2\epsilon)^{-1}\Omega + \omega$. Details will be presented elsewhere [6].

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