Dilatonic Brane-World Black Holes, Gravity Localization and Newton Constant

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ABSTRACT

The family of brane-world solutions of d+1-dimensional dilatonic gravity is presented. It includes flat brane with small cosmological constant and (anti) de Sitter brane, dilatonic brane-world black holes (Schwarzschild-(anti-) de Sitter, Kerr, etc). Gravitational and dilatonic perturbations around such branes are found. It is shown that near dilatonic brane-world black hole the gravity may be localized in a standard form. The brane corrections to Newton law are estimated. The proposal to take into account the dilaton coupled brane matter quantum effects is made. The corresponding effective action changes the structure of 4d de Sitter wall. RG flow of four-dimensional Newton constant in IR and UV is briefly discussed.

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# 1 Introduction

According to idea clearly expressed in ref. [1] our observed world could be a brane embedded in a higher-dimensional space. Moreover, in the scenario [1] four-dimensional gravity with acceptable properties may be recovered. This fact initiated enormous activity in the study of different aspects of brane worlds.

In some versions of brane-world scenarios one can construct the black holes on the branes [2, 3, 4]. The investigation of their properties is extremely interesting because this may help to understand better the fundamental problems of quantum gravity. For example, one can try to describe from brane point of view such phenomena as Hawking radiation, Black Hole entropy origin, etc. It is most likely that brane-world scenario should be realized within the context of AdS/CFT correspondence (say, in its simplest form as 5d gauged SG/4d CFT). If it is so one should start from the scalar-tensor gravity as a bulk theory. Then the role of scalars (dilaton if only single scalar presents) should be carefully addressed. In the present paper working in this direction we construct the family of dilatonic brane-world black holes (including regular cases, like de Sitter space) and carefully investigate their properties, the problem of localization of 4d gravity in such spaces and brane corrections to Newton constant.

In the next section we start from d+1-dimensional dilatonic gravity with d-dimensional brane vacuum energy and formulate the corresponding equations of motion. A flat brane solution with very small cosmological constant is possible. A family of brane-world black holes solutions (curved branes) is presented. They correspond to (anti) de Sitter space, Nariai space, Kerr, and Schwarzschild-de Sitter black holes, etc. Some their properties are investigated. In section 3 we look to gravity perturbations around such backgrounds. Not only graviton but also dilaton perturbations are found. It is explicitly shown that in some cases 4d gravity may be localized in the same fashion as in ref. [1]. Section four is devoted to the study of corrections to Newton constant. The Newton potential is calculated and it is shown that corrections to Newton law near branes are very small. In section 5 we analyse the regular solution where de Sitter Universe is realised on the brane. Brane vacuum energy is found. Simple analysis indicates that there may be problems with gravity localization. The role of quantum effects of brane matter is investigated in section 6. We suggest to add the conformal anomaly
induced effective action of boundary, dilaton coupled matter to the complete action. In such a way, Randall-Sundrum compactification may be naturally fitted with AdS/CFT correspondence. It is shown that with account of such effective action there still exists 4d de Sitter wall (our Universe) living in 5d dilatonic spacetime which is asymptotically AdS. Quantum corrections change the brane vacuum energies, they become explicitly time-dependent. RG flow of Newton constant in IR and UV is briefly discussed. Some resume is given in last section.

2 Dilatonic black hole solutions in the brane world

In analogy with Randall-Sundrum model [1], we start with the following action of the gravity coupled with dilaton $\phi$:

$$S = \frac{1}{16\pi G_{d+1}} \left[ \int_M d^{d+1}x \sqrt{-g} \left( R - \frac{1}{2} \partial_{\mu}\phi \partial^{\mu}\phi - V(\phi) \right) - \sum_{i=\text{hid},\text{vis}} \int_{B_i} d^{d}x \sqrt{-\gamma} U_i(\phi) \right]. \quad (1)$$

Here $M$ is the bulk manifold which usually corresponds to AdS and $B_{\text{hid}}$ and $B_{\text{vis}}$ are branes corresponding to hidden and visible sectors respectively. $\gamma$ is the metric on the brane induced by the metric $g$ in the bulk. Here $U_i(\phi)$ corresponds to the vacuum energies on the branes in $\{i\}$. One assumes $U(\phi)$ is dilaton dependent and its form is explicitly given later from the consistency of the equations of motion. Some important examples of the dilaton potential are presented in [3], where $V(\phi)$ is given in terms of the superpotential $W(\phi)$:

$$V = \left( \frac{\partial W}{\partial \phi} \right)^2 - \frac{D - 1}{2(D - 2)} W^2, \quad (2)$$

and $W$ has the following form:

$$W = \sqrt{\frac{\mathcal{N}}{2}} g \left( \frac{1}{a_1} e^{\frac{a_1 \phi}{2}} \pm \frac{1}{a_2} e^{\frac{a_2 \phi}{2}} \right). \quad (3)$$

Here $D = d + 1$, $\mathcal{N}$ is the number of the supercharges and the parameter $g, a_1$ and $a_2$ depend on the model features but $a_1 > a_2 > 0$ in general. As
in \cite{3}, we only consider the case of $\pm$ sign of $\pm_3$ in \cite{3}. Note that potentials of above type appear as a result of sphere reduction in M-theory or string theory \cite{4}.

We now assume the metric has the following form:

$$ds^2 = dz^2 + e^{2A(z)}\eta_{ij}dx^i dx^j,$$

and $\phi$ only depends on $z$. We also suppose the hidden and visible branes sit on $z = z_{\text{hid}}$ and $z = z_{\text{vis}}$, respectively. Then the equations of motion are given by

$$\phi'' + (D - 1)A' \phi' = \frac{\partial V}{\partial \phi} + \sum_{i=\text{hid,vis}} \frac{\partial U_i(\phi)}{\partial \phi} \delta(z - z_i),$$

$$(D - 1)A'' + (D - 1)(A')^2 + \frac{1}{2}(\phi')^2$$

$$= -\frac{V}{D - 2} - \frac{D - 1}{2(D - 2)} \sum_{i=\text{hid,vis}} U_i(\phi) \delta(z - z_i),$$

$$(D - 1)(A')^2 = -\frac{1}{D - 2} V - \frac{1}{2(D - 2)} \sum_{i=\text{hid,vis}} U_i(\phi) \delta(z - z_i).$$

Here $' \equiv \frac{d}{dz}$. For purely bulk sector ($z_{\text{hid}} < z < z_{\text{vis}}$, as $z_{\text{hid}} < z_{\text{vis}}$), the explicit solutions are given in \cite{3}. Eqs. (3-4) have the following first integrals (in the bulk sector):

$$\phi' = \sqrt{2} \frac{\partial W}{\partial \phi}, \quad A' = -\frac{1}{\sqrt{2(D - 2)}} W.$$

Near the branes, Eqs. (3-7) have the following form:

$$\phi'' \sim \frac{\partial U_i(\phi)}{\partial \phi} \delta(z - z_i), \quad A'' \sim -\frac{U_i(\phi)}{2(D - 2)} \delta(z - z_i),$$

or

$$2\phi' \sim \frac{\partial U_i(\phi)}{\partial \phi}, \quad 2A' \sim -\frac{U_i(\phi)}{2(D - 2)},$$

at $z = z_i$. Comparing (10) with (8), we find

$$U_{\text{hid}}(\phi) = 2\sqrt{2} W(\phi), \quad U_{\text{vis}}(\phi) = -2\sqrt{2} W(\phi).$$
For simplicity, let $z_{\text{hid}} = 0$, where $\phi = 0$ in the solution in [4] and we only consider this solution in the following. Then at $z = z_{\text{vis}}$, $\phi$ is negative. Since the superpotential is given by the exponential of $\phi$ with positive sign, the vacuum energy $U_{\text{vis}}$, which can be identified with the cosmological constant, can be small even if $|\phi|$ is not so large. The result might explain why the cosmological constant is small. We should note that the visible brane corresponds to the 4d universe where we live. Then the visible brane part in the action (11) mainly describes the dynamics of the universe and $U_{\text{vis}}$ corresponds to the vacuum energy or cosmological constant in our universe.

As an extension, one can consider the case that the brane is curved. Instead of (11), we take the following metric:

$$ds^2 = dz^2 + e^{2A(z)} \tilde{g}_{ij} dx^i dx^j , \quad (12)$$

Here $\tilde{g}_{ij}$ is the metric of the Einstein manifold, which is defined by

$$\tilde{R}_{ij} = k \tilde{g}_{ij} , \quad (13)$$

where $\tilde{R}_{ij}$ is the Ricci tensor given by $\tilde{g}_{ij}$ and $k$ is a constant. Then Eqs.(3) and (5) do no change but one obtains the following equation instead of (7):

$$A'' + (D - 1)(A')^2 = ke^{2A} - \frac{1}{D - 2} V - \frac{1}{2(D - 2)} \sum_{i=\text{hid,vis}} U_i(\phi) \delta(z - z_i) . \quad (14)$$

 Especially when $k = 0$, we get the previous solution for $\phi$, $A$ and $U_i$. We should note, however, that $k = 0$ does not always mean the brane is flat. As well-known, the Einstein equations are given by,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{1}{2} \Lambda g_{\mu\nu} = T^\text{matter}_{\mu\nu} . \quad (15)$$

Here $T^\text{matter}_{\mu\nu}$ is the energy-momentum tensor of the matter fields. If we consider the vacuum solution where $T^\text{matter}_{\mu\nu} = 0$, Eq.(15) can be rewritten when $d = 4$ as (see also [3])

$$R_{\mu\nu} = \frac{\Lambda}{2} g_{\mu\nu} . \quad (16)$$

If we put $\Lambda = 2k$, Eq.(16) is nothing but the equation for the Einstein manifold. The Einstein manifolds are not always homogeneous manifolds like flat Minkowski, (anti-)de Sitter space

$$ds^2_4 = -V(r) dt^2 + V^{-1}(r) dr^2 + r^2 d\Omega^2 , \quad V(r) = 1 - \frac{\Lambda}{6} r^2, \quad (17)$$
or Nariai space
\[ ds^2_3 = \frac{1}{\Lambda} \left( \sin^2 \chi d\psi^2 - d\chi^2 - d\Omega^2 \right). \] (18)

but they can be some black hole solutions like Schwarzschild-(anti-)de Sitter black hole
\[ ds^2_4 = -V(r)dt^2 + V^{-1}(r)dr^2 + r^2d\Omega^2, \quad V(r) = 1 - \frac{\tilde{G}_4 M}{r} - \frac{\Lambda}{6} r^2. \] (19)

As a special case, one can also consider \( k = 0 \) solution like Schwarzschild black hole,
\[ ds^2_4 \equiv \tilde{g}_{ij}dx^i dx^j = - \left( 1 - \frac{\tilde{G}_4 M}{r} \right) dt^2 + \frac{dr^2}{\left( 1 - \frac{\tilde{G}_4 M}{r} \right)} + r^2d\Omega^2, \] (20)
or Kerr one
\[ ds^2_4 = \Delta \tilde{A}dt^2 - \frac{\Sigma^2}{\Delta} dr^2 - \Sigma^2d\theta^2 - \frac{\sin^2 \theta}{\tilde{A}} (d\varphi - \Omega dt) \]
\[ \tilde{A} \equiv \frac{\Sigma^2 \left( \Delta - a^2 \sin^2 \theta \right)}{\Sigma^4 \Delta - 4a^2 \tilde{G}_4^2 M^2 r^2 \sin^2 \theta}, \quad \Omega \equiv \frac{2a\tilde{G}_4 M r \tilde{A}}{\Sigma^2} \]
\[ \Delta \equiv r^2 - 2\tilde{G}_4 M r + a^2, \quad \Sigma^2 \equiv r^2 + a^2 \cos^2 \theta. \] (21)

In (19), (20) and (21), \( M \) is the mass of the black hole on the brane and the effective gravitational constant \( \tilde{G}_4 \) on the 3-brane (here \( d = 4 \)) is given by
\[ \frac{1}{\tilde{G}_4} = \frac{1}{G_5} \int_{z_{\text{hid}}}^{z_{\text{vis}}} dz e^{(d-2)A}. \] (22)

In (21), the parameter \( a \) is related with the angular momentum \( J \) of the black hole on the brane by
\[ J = Ma. \] (23)

In these solutions, the curvature singularity at \( r = 0 \) has a form of line penetrating the bulk 5d universe and the horizon makes a tube surrounding the singularity. The singularity and the horizon connect the hidden and visible branes. Thus we presented a big family of dilatonic solutions in the brane world.
It would be interesting to see how the black hole looks like in the bulk (see corresponding discussion in [3]). We now consider the case \(|z_{vis}| \text{ is large but } z_{vis} < 0 \text{ or } z_{vis} > 0\). When \(z_{vis}\) is negative and its absolute value \(|z_{vis}|\) is large, the asymptotic behaviour of \(A\) is given by

\[
A \sim \gamma z , \quad \gamma \equiv \frac{g}{a_1 d \sqrt{\mathcal{N}}} .
\]  

Then since the metric has the form in (12), the typical proper size \(l_{BH}^-\) of the black hole (or the typical proper distance to the horizon) transverse to the brane would be given by

\[
l_{BH}^- = \frac{1}{\gamma} \ln (\gamma r_0) = \frac{1}{\gamma} \ln \left( \gamma \tilde{G}_4 M \right) .
\]  

Then the transverse size \(l_{BH}^-\) grows like \(\ln M\) when the black hole mass \(M\) increases although the (horizon) size \(r_0\) along the brane is linear to \(M\). This might tell that matter can pass around the black hole through the fifth dimension similar to the effect suggested in [3].

On the other hand, we can consider the case that \(z_{vis}\) is positive and large. The asymptotic behaviour of \(A\) when \(z\) is large, is given by

\[
e^{2A} \sim \alpha^2 z^{2\beta} , \quad \alpha \equiv \left\{ -c \sqrt{\frac{\mathcal{N}}{8} g \left( \frac{1}{4} a_2 \sqrt{\mathcal{N}} g \right)^{\frac{a_1}{a_2}}} \right\}^{\frac{1}{3}} , \quad \beta \equiv \frac{a_1}{a_2 d} .
\]  

Then if one defines a new coordinate \(w\) by

\[
y = \frac{z^{1-\beta}}{\alpha (1-\beta)} ,
\]  

the metric in (12) can be rewritten in the following form:

\[
ds^2 \sim \alpha^2 \{ (1-\beta) y \}^{\frac{a_1}{a_2}} \left( dw^2 + \tilde{g}_{ij} dx^i dx^j \right) .
\]  

Then we can expect the horizon in the bulk is typically given by \(y \sim r_0\) and the typical proper size \(l_{BH}^+\) of the black hole transverse to the brane would be given by

\[
l_{BH}^+ = \{ (1-\beta) r_0 \}^{\frac{1}{1-\beta}} = \{ (1-\beta) \tilde{G}_4 M \}^{\frac{1}{1-\beta}} .
\]
Since $a_1 = 2\sqrt{\frac{2}{3}}$ and $a_2 = \frac{4}{\sqrt{15}}$ for $d = D - 1 = 4$ and $\mathcal{N} = 1$ model in [3], we have $\frac{1}{l_{\text{BH}}} = \frac{2}{3}$ and the transverse size $l_{\text{BH}}^+$ grows like $M^\frac{2}{3}$ when the black hole mass $M$ increases. This would tell that matter cannot pass through a black hole, which is different from the case when $z_{\text{vis}} \to -\infty$. Of course, this is a preliminary qualitative discussion of brane-world black holes properties.

3 Gravity perturbations

Our next problem will be the description of the gravity on the brane. For that purpose one can consider the perturbation from the above obtained solution by assuming the metric in the following form:

$$ds^2 = dz^2 + e^{2A(z)}\tilde{g}_{ij}dx^i dx^j + h_{ij}dx^i dx^j, \quad |h_{ij}| \ll 1.$$  \hspace{1cm} (30)

Here we choose the gauge where $g_{zz} = 1$, $g_{zi} = g_{iz} = 0$, $D^i h_{ij} = 0$ and $h^i_\ i = 0$. We should note that the metric $\tilde{g}_{ij}$ on the brane is not necessary to be flat but can be any Einstein manifold (13). We put $k = 0$ for simplicity. Then $A(z)$ is given by solving (8) [5]. From the Einstein equation, we obtain the following linearized equation:

$$0 = e^{-2A}\Box h_{ij} + \partial^2 h_{ij} + (d - 4)\partial z A \partial z h_{ij} - 4 \left\{ (d - 1)\partial z A^2 + \partial^2 A \right\} h_{ij}. \hspace{1cm} (31)$$

Here $\Box$ is the d’Alembertian on the brane given by $\tilde{g}_{ij}$. Changing the coordinate $z$ to $\zeta$ by

$$d\zeta = -e^{-A}dz,$$  \hspace{1cm} (32)

one rewrites (31) in the following form:

$$0 = \Box h_{ij} + \partial^2 \zeta h_{ij} + (d - 5)\partial \zeta A \partial \zeta h_{ij} - 4 \left\{ (d - 2)\partial \zeta A^2 + \partial^2 \zeta A \right\} h_{ij}. \hspace{1cm} (33)$$

In order to solve Eq.(33), we assume the following form for $h_{ij}$:

$$h_{ij} = \psi(\zeta)\hat{h}_{ij}(x)$$  \hspace{1cm} (34)

and assume $\hat{h}_{ij}(x)$ satisfies

$$\Box \hat{h}_{ij}(x) = m^2 \hat{h}_{ij}(x).$$  \hspace{1cm} (35)
Here \( m \) corresponds to the mass. Then one has

\[
0 = m^2 \psi + \partial_z^2 \psi + (d - 5) \partial_z A \partial_z \psi - 4 \left\{ (d - 2) \partial_z A^2 + \partial_z^2 A \right\} \psi .
\] (36)

As we have a dilaton field \( \phi \), we should consider the perturbation of \( \phi \):

\[
\phi = \phi^{(0)} + \varphi , \quad |\varphi| \ll |\phi^{(0)}| .
\] (37)

Here \( \phi^{(0)} \) is given by solving (3). Linearizing the equation of motion given by the variation of \( \phi \),

\[
\Box \varphi = \frac{\partial V(\phi)}{\partial \phi} + \sum_i \frac{\partial U_i(\phi)}{\partial \phi^2} \delta(z - z_i) ,
\] (38)

one gets

\[
e^{-2A} \Box^{(0)} \varphi + \partial_z^2 \varphi + d \partial_z A \partial_z \varphi = \left\{ \frac{\partial^2 V(\phi)}{\partial \phi^2} + \sum_i \frac{\partial^2 U_i(\phi)}{\partial \phi^2} \delta(x - x_i) \right\} \varphi .
\] (39)

Using the coordinate \( \zeta \) defined by (32) and assuming \( \phi \) in the following form:

\[
\varphi(\zeta, x) = \theta(\zeta) \hat{\varphi}(x) , \quad \Box^{(0)} \hat{\varphi}(x) = m^2 \hat{\varphi}(x) ,
\] (40)

we can rewrite Eq.(33) as:

\[
m^2 \theta + \partial_z^2 \theta + (d - 1) \partial_z A \partial_z \theta = e^{2A} \left\{ \frac{\partial^2 V(\phi)}{\partial \phi^2} + \sum_i \frac{\partial^2 U_i(\phi)}{\partial \phi^2} \delta(z - z_i) \right\} \theta .
\] (41)

In the solutions found in [5], there appear two branches. In the first branch, \( z \) runs from \(-\infty\) to 0, \( \phi \) from 0 to \(+\infty\) and there appears a curvature singularity at \( z = 0 \). In the second one \( z \) runs from \(-\infty\) to \(+\infty\), \( \phi \) from 0 to \(-\infty\) and there does not appear any curvature singularity. In both of the branches, the spacetime approaches to AdS.

We now consider the perturbation when \( z_{\text{vis}} \) is negative and its absolute value \( |z_{\text{vis}}| \) is large and the region \( z \leq z_{\text{vis}} \). Then we find the following asymptotic behaviour of \( A \) and \( \phi \):

\[
A \sim \gamma z , \quad \phi \sim e^{d\gamma z} \rightarrow 0 , \quad \gamma \equiv \frac{g}{a_1 d \sqrt{N}} .
\] (42)
Then $\zeta$ in (32) is given by
\[ \zeta = \frac{e^{-\gamma z}}{\gamma} \] (43)
and Eq.(36) has the following form
\[ 0 = m^2 \psi + \partial_z^2 \psi + \frac{5 - d}{\zeta} \partial_z \psi - \frac{4(d - 1)}{\zeta^2} \psi . \] (44)

When $m > 0$, the corresponding solution is called Kaluza-Klein (KK) mode and the solution of $m = 0$ corresponds to the graviton on the brane. When $m^2 = 0$, the solution of (44) is given by the power of $\zeta$:
\[ \psi = \zeta^a . \] (45)
The exponent $a$ can be found by solving the following algebraic equation:
\[ 0 = a^2 + (4 - d)a - 4(d - 1) . \] (46)
Especially when $d = 4$, one gets
\[ a = \pm 2\sqrt{3} . \] (47)

The existence of the normalizable solution tells that the gravity is localized near the brane. This situation does not change even if the brane is flat or a (4d) black hole spacetime.

When $m^2 > 0$, the solution of (44) is given by Bessel functions $J_\nu$ and $N_\nu$ ($= Y_\nu$):
\[ \psi(\zeta) = \zeta^b (c_1 J_\nu(m\zeta) + c_2 N_\nu(m\zeta)) \]
\[ 1 - 2b = 5 - d , \quad b^2 - \nu^2 = -4(d - 1) . \] (48)

Here $c_1$ and $c_2$ are constants of integration, which should be determined by the boundary condition at $z = z_{\text{vis}}$:
\[ \partial_z \psi(z = z_{\text{vis}}) = -e^{-A(\zeta = \zeta_{\text{vis}})} \partial_\zeta \psi(\zeta = \zeta_{\text{vis}}) \]
\[ = -\frac{2\sqrt{2}}{d - 3} W(\phi(z = z_{\text{vis}})) \psi(z = z_{\text{vis}}) . \] (49)
Here $\zeta_{\text{vis}}$ is the value of $\zeta$ corresponding to $z = z_{\text{vis}}$. The boundary condition comes from the $\delta$-function behaviour of $\partial^2_z A$ at $z = z_{\text{vis}}$. Especially for $d = 4$, we find

$$b = 0, \quad \nu = 2\sqrt{3}. \quad (50)$$

If there is a solution with $m^2 < 0$, the system becomes unstable. When $m^2 < 0$, the solution of (44) is given by modified Bessel functions $I_\nu$ and $K_\nu$:

$$\psi(\zeta) = \zeta^b (c_1 I_\nu(\mu \zeta) + c_2 K_\nu(\mu \zeta)) \quad 1 - 2b = 5 - d, \quad b^2 - \nu^2 = -4(d - 1), \quad \mu^2 = -m^2. \quad (51)$$

Since $I_\mu$ increases exponentially for large $\zeta$, $c_1$ must vanish. On the other hand, $K_\nu$ behaves as $K_\mu(\mu \zeta) \sim e^{-\mu \zeta}$ for large $\zeta$. When $\zeta$ is large, $z \to -\infty$, $\phi \to 0$ and $W \to \sqrt{\frac{N}{2}} g \left( \frac{1}{a_1} - \frac{1}{a_2} \right) < 0$. Therefore there is an unstable solution which satisfies the boundary condition (49) if one chooses

$$\mu \sim \frac{2\sqrt{2}}{d-3} e^{A(z = z_{\text{vis}})} W(\phi(z = z_{\text{vis}})) \to -\frac{2\sqrt{2}}{d-3} e^{\gamma z_{\text{vis}}} \sqrt{\frac{N}{2}} g \left( \frac{1}{a_1} - \frac{1}{a_2} \right). \quad (52)$$

As $\mu$ vanishes in the limit of $z \to -\infty$, the brane will be driven to $z \to -\infty$.

We now consider the perturbation by the dilaton $\phi$. When $\zeta$ is large, Eq.(44) can be rewritten as

$$m^2 \theta + \partial^2_\zeta \theta - \frac{d-1}{\zeta} \partial_\zeta \theta - \frac{V''}{\gamma^2 \zeta^2} \theta = e^{2A} \sum_i \frac{\partial^2 U_i(\phi)}{\partial \phi^2} \delta(z - z_i) \theta. \quad (53)$$

Here

$$V''(\phi) \bigg|_{\phi=0} = \frac{1}{g^2} \left( 1 - \frac{a_2}{a_1} \right) > 0. \quad (54)$$

Then one finds $\theta$ is given by the power of $\zeta$ when $m^2 = 0$ or Bessel functions when $m^2 > 0$. From the $\delta$-function, we find that $\theta$ should satisfy the following boundary condition $\zeta = \zeta_{\text{vis}}$:

$$\partial_\zeta \theta = -\frac{\sqrt{2}}{\gamma \zeta_{\text{vis}}} \frac{\partial^2 W}{\partial \phi^2} \bigg|_{\phi=0} \theta. \quad (55)$$

Since

$$\frac{\partial^2 W}{\partial \phi^2} \bigg|_{\phi=0} = \sqrt{\frac{N}{2}} g(a_1 - a_2) > 0, \quad (56)$$
the coefficient \( -\frac{\sqrt{\gamma}}{\gamma_{\text{vis}}} \frac{\partial^2 W}{\partial \phi^2} \bigg|_{\phi=0} \) is negative. Then the boundary condition is consistent with the modified Bessel function \( K_\nu \) and the condition could be satisfied by properly choosing \( m^2 \). This tells that there exists an unstable mode corresponding to \( m^2 < 0 \).

In the second branch in the solution in [5], we can consider the case that \( z_{\text{vis}} \) is positive and large. We also consider the region \( z \leq z_{\text{vis}} \). Since the asymptotic behaviour of \( A \) when \( z \) is large is given by (26), one gets

\[
- \zeta \sim \frac{z^{1-\beta}}{1-\beta}
\]

and Eq.(33) has the following form

\[
0 = \Box^{(0)} h_{ij} + \partial_\zeta^2 h_{ij} + \frac{(d-5)\tilde{\beta}}{\zeta} \partial_\zeta h_{ij} - 4 \left\{ (d-2)\tilde{\beta}^2 - \tilde{\beta} \right\} h_{ij} .
\]

Here

\[
\tilde{\beta} \equiv \frac{\beta}{1-\beta} .
\]

and we obtain

\[
0 = m^2 \psi + \partial_\zeta^2 \psi + \frac{(d-5)\tilde{\beta}}{\zeta} \partial_\zeta \psi - 4 \left\{ (d-2)\tilde{\beta}^2 - \tilde{\beta} \right\} \psi .
\]

When \( m = 0 \), the solution is given by

\[
\psi = (-\zeta)^a .
\]

Here \( a \) is the solution of the following algebraic equation:

\[
0 = a^2 + \left\{ (d-5)\tilde{\beta} - 1 \right\} a - 4 \left\{ (d-2)\tilde{\beta}^2 - \tilde{\beta} \right\} .
\]

For \( d = D - 1 = 4 \) and \( \mathcal{N} = 1 \) model in [3], where \( a_1 = 2\sqrt{\frac{5}{3}} \) and \( a_2 = \frac{4}{\sqrt{15}} \), we have

\[
a = \frac{4 \pm 2\sqrt{39}}{3} = 5.49 \cdots, -2.83 \cdots
\]

Since we are considering the case that \( z_{\text{vis}} \) is positive and \( z \leq z_{\text{vis}} \), the first positive \( a \) would correspond to a normalizable solution. The existence of the normalizable solution tells that the gravity near the brane is localized.
When \( m > 0 \), the solution of (60) is given by Bessel functions \( J_\nu \) and \( N_\nu \) (\( = Y_\nu \)):

\[
\psi(\zeta) = z^b (c_1 J_\nu(-m\zeta) + c_2 N_\nu(-m\zeta)) \quad 1 - 2b = (d - 5)\tilde{\beta} \quad b^2 - \nu^2 = -4 \left\{ (d - 2)\tilde{\beta}^2 - \tilde{\beta} \right\} .
\]

(64)

Here \( c_1 \) and \( c_2 \) are constants of integration, which should be again determined by the boundary condition (49) at \( z = z_{\text{vis}} \). The boundary condition comes from the \( \delta \)-function behaviour of \( \partial_z^2 \hat{A} \) at \( z = z_{\text{vis}} \). When \( \zeta(z) \) is large, the Bessel functions \( J_\nu(-m\zeta) \) and \( N_\nu(-m\zeta) \) behave as

\[
J_\nu(-m\zeta) \sim \sqrt{-\frac{2}{\pi m\zeta}} \cos \left( -m\zeta - \frac{(2\nu + 1)\pi}{4} \right) ,
\]

\[
N_\nu(-m\zeta) \sim \sqrt{-\frac{2}{\pi m\zeta}} \sin \left( -m\zeta - \frac{(2\nu + 1)\pi}{4} \right) .
\]

(65)

For \( d = D - 1 = 4 \) and \( \mathcal{N} = 1 \) model in [5], we have

\[
b = \frac{4}{3} \quad \nu^2 = \frac{4 \cdot 39}{9} = (4.16 \ldots)^2 .
\]

(66)

The existence of the unstable mode corresponding to modified Bessel function \( K_\mu(\mu\zeta) \) (\( \mu^2 = -m^2 \)) depends on the asymptotic behaviour of \( W \) as in \( z_{\text{vis}} \to -\infty \) case. From (5) (\( - \) sign is chosen), we find \( W < 0 \) when \( \phi \to -\infty \) and \( e^{A(\phi(z))} W(\phi(z)) \) vanishes when \( z(\zeta) \to +\infty \). Therefore an unstable solution given by \( K_\mu(\mu\zeta) \) exists and the brane moves to \( +\infty \), where the vacuum energy of the brane vanishes since \( \mu \to 0 \) when \( z_{\text{vis}} \to \infty \).

We also consider the perturbation by the dilaton \( \phi \). When \( \zeta \) is large, Eq.(41) can be rewritten as

\[
m^2 \theta + \partial_\zeta^2 \theta + \frac{\tilde{\beta}(d - 1)}{\zeta} \partial_\zeta \theta + \frac{\tilde{V}_0}{\gamma^2 \zeta^2} = e^{2A} \sum_i \frac{\partial^2 U_i(\phi)}{\partial \phi^2} \delta(z - z_i) \theta .
\]

(67)

Here

\[
\tilde{V}_0 \equiv \frac{1}{2} g^2 \left( 1 - \frac{a_1}{d a_2} \right)^{-2} \left( \frac{a_2}{a_1} \right)^2 \left( \frac{1}{4} a_2 \sqrt{\mathcal{N} g} \right)^{-2} .
\]

(68)

Then we find \( \theta \) is given by the power of \( \zeta \) when \( m^2 = 0 \) or Bessel functions when \( m^2 > 0 \). The existence of the unstable mode corresponding the modified
Bessel function depends on the sign of $\frac{\partial^2 W}{\partial \phi^2}$ at $z = z_{\text{vis}}$. Since

$$\left. \frac{\partial^2 W}{\partial \phi^2} \right|_{z=z_{\text{vis}}} \rightarrow - \sqrt{\frac{N}{2}} \frac{a_2}{4} e^{\frac{a_2 \phi}{2}} < 0,$$

for large positive $z_{\text{vis}}$, the sign is negative, which is different from the case $z_{\text{vis}} \rightarrow -\infty$. Therefore, remarkably there does not exist an unstable mode corresponding to $m^2 < 0$ when we choose $z_{\text{vis}} \rightarrow +\infty$, where the cosmological constant becomes very small. To conclude, the above analysis shows the possibility to get the localized gravity near the brane corresponding to our black hole solution. Note that excellent introduction to universal aspects of gravity localization in brane world can be found in ref. [18].

4 Newton law correction

It is interesting to discuss now the correction to the Newton law coming from KK mode.

First we consider $z_{\text{vis}} \rightarrow -\infty$ case in (48). The constants of integration should be determined by the boundary condition at $z = z_{\text{vis}}$ in (49). As the overall factor $\sqrt{\frac{2}{\pi mc}}$ should be absorbed into the measure for the normalization, we replace it with unity and we impose a constraint as follows:

$$c_1^2 + c_2^2 = 1.$$  \hfill (70)

Then when $d = 4$, and $m\zeta$ is large, $\psi(\zeta)$ behaves as

$$\psi(\zeta) \sim c_1 \cos \left( m\zeta - \frac{(4\sqrt{3} + 1)\pi}{4} \right) + c_2 \sin \left( m\zeta - \frac{(4\sqrt{3} + 1)\pi}{4} \right),$$

so then $\partial_\zeta \psi(\zeta)$ is given by

$$\partial_\zeta \psi(\zeta) \sim -mc_1 \sin \left( m\zeta - \frac{(4\sqrt{3} + 1)\pi}{4} \right) + mc_2 \cos \left( m\zeta - \frac{(4\sqrt{3} + 1)\pi}{4} \right).$$

We should note that there might be some ambiguities in the limiting procedure. Since we consider $\zeta_{\text{vis}} \rightarrow +\infty$, we first put $m\zeta$ to be large and after that we choose the mass $m$ in the KK mode to be small, which is relevant.
to the long range force. Then by using the boundary condition (49) and the constraint (70), we get $c_1, c_2$ in following forms:

\begin{align}
    c_1 &= \pm \kappa \sqrt{\frac{1}{\kappa^2 + 1}} \\
    c_2 &= \pm \sqrt{\frac{1}{\kappa^2 + 1}}
\end{align}

(73)

Here $\kappa$ is defined by

\begin{align}
    \kappa &\equiv -\cos \zeta' + Z(\zeta = \zeta_{\text{vis}}) \sin \zeta' \\
    Z(\zeta = \zeta_{\text{vis}}) &\equiv \frac{2\sqrt{2}e^{A(\zeta = \zeta_{\text{vis}})}W(\phi(z = z_{\text{vis}}))}{m} \\
    \zeta' &\equiv m\zeta_{\text{vis}} - \frac{(4\sqrt{3} + 1)\pi}{4}
\end{align}

(74)

One can find the value of $\psi$ (71) at $\zeta = \zeta_{\text{vis}}$ using (73), (74):

\begin{align}
    \psi(\zeta_{\text{vis}}) &\sim \sqrt{\frac{1}{\kappa^2 + 1}} (\kappa \cos \zeta' + \sin \zeta') \\
    &= \sqrt{\frac{1}{\kappa^2 + 1}} \left( \sin \zeta' + Z(\zeta = \zeta_{\text{vis}}) \cos \zeta' \right) \\
    &= \sin \zeta' + Z(\zeta = \zeta_{\text{vis}}) \cos \zeta' \left( \frac{1}{\sin \zeta' + Z(\zeta = \zeta_{\text{vis}}) \cos \zeta'} \right) \\
    &= \frac{1}{\sqrt{1 + Z(\zeta = \zeta_{\text{vis}})^2}}
\end{align}

(75)

If we take $m$ is small (or $|Z|$ is large), then $\psi(\zeta_{\text{vis}})$ is given by

\begin{align}
    \psi &\sim \frac{1}{Z(\zeta = \zeta_{\text{vis}})} = \frac{m}{2\sqrt{2}e^{A(\zeta = \zeta_{\text{vis}})}W(\phi(z = z_{\text{vis}}))}
\end{align}

(76)

Then the correction to Newton’s Law is

\begin{align}
    V(r) &\sim \tilde{G}_4 \frac{m_1 m_2}{r} + \int_0^\infty dmG_5 \frac{m_1 m_2 e^{-mr}}{r} \psi(\zeta = \zeta_{\text{vis}})^2
\end{align}
\[
\begin{align*}
G_4 \frac{m_1 m_2}{r} + \int_0^\infty dm G_5 \frac{m_1 m_2 e^{-mr}}{r} \frac{m^2}{8 e^{2A(\zeta = \zeta_{\text{vis}})} W(\phi(z = z_{\text{vis}}))^2} \\
= G_4 \frac{m_1 m_2}{r} \left( 1 + \frac{G_5}{G_4 r^3 e^{2A(\zeta = \zeta_{\text{vis}})} W(\phi(z = z_{\text{vis}}))^2} \right) 
\end{align*}
\] (77)

We should note that the correction is not given by \( \frac{1}{r^3} \) as in [1] but \( \frac{1}{r^4} \). This is mainly due to the limiting procedure where we first have put \( m\zeta \) to be large and after that we have chosen the mass \( m \) in the KK mode to be small.

Next we consider the case \( z_{\text{vis}} \to \infty \) given in (64). When \( \zeta(z) \) is large, the Bessel functions \( J_\nu(-m\zeta) \) and \( N_\nu(-m\zeta) \) behave as

\[
\begin{align*}
J_\nu(-m\zeta) &\sim \sqrt{-\frac{2}{\pi m\zeta}} \cos \left( -m\zeta - \frac{(2\nu + 1)\pi}{4} \right), \\
N_\nu(-m\zeta) &\sim \sqrt{-\frac{2}{\pi m\zeta}} \sin \left( -m\zeta - \frac{(2\nu + 1)\pi}{4} \right).
\end{align*}
\] (78)

For \( d = D - 1 = 4 \) and \( \mathcal{N} = 1 \) model in [3], we have the parameters \( b, \nu \) given in (60). Then \( c_1 \) and \( c_2 \) in (64) should be determined by the boundary condition at \( z = z_{\text{vis}} \) in (39) and constraint (70) again. Since

\[
\partial_\zeta \psi(\zeta) \sim mc_1 \sin \left( -m\zeta - \frac{(2\nu + 1)\pi}{4} \right) - mc_2 \cos \left( -m\zeta - \frac{(2\nu + 1)\pi}{4} \right),
\] (79)

we can get \( c_1, c_2 \) in the same way as (73).

\[
\begin{align*}
c_1 &= \pm \kappa \sqrt{\frac{1}{\kappa^2 + 1}} \\
c_2 &= \pm \sqrt{\frac{1}{\kappa^2 + 1}}
\end{align*}
\] (80)

Here \( \kappa \) is defined by

\[
\begin{align*}
\kappa &\equiv \frac{-\cos \zeta' + Z(\zeta = \zeta_{\text{vis}}) \sin \zeta'}{-\sin \zeta' - Z(\zeta = \zeta_{\text{vis}}) \cos \zeta'} \\
Z(\zeta = \zeta_{\text{vis}}) &\equiv \frac{2\sqrt{2} e^{A(\zeta = \zeta_{\text{vis}})} W(\phi(z = z_{\text{vis}}))}{-m} \\
\zeta' &\equiv -m\zeta - \frac{(2\nu + 1)\pi}{4}
\end{align*}
\] (81)
And $\psi$ (64) is written by using (80), (81).

$$\psi(\zeta) \sim \sqrt{\frac{1}{\kappa^2 + 1}} (\kappa \cos \zeta' + \sin \zeta')$$

$$= \frac{1}{\sqrt{1 + Z(\zeta = \zeta_{\text{vis}})^2}}$$  \hfill (82)

Then the correction to Newton’s law in the limit that $m$ is small (or $|Z|$ is large) is

$$V(r) \sim \tilde{G}_4 \frac{m_1 m_2}{r} + \int_0^\infty dm G_5 \frac{m_1 m_2 e^{-mr}}{r} \psi^2$$

$$= \tilde{G}_N \frac{m_1 m_2}{r} + \int_0^\infty dm G_5 \frac{m_1 m_2 e^{-mr}}{r} \frac{m^2}{8e^{2A(\zeta = \zeta_{\text{vis}})}W(\phi(z = z_{\text{vis}}))^2}$$

$$= \tilde{G}_4 \frac{m_1 m_2}{r} \left( 1 + \frac{G_5}{4e^{2A(\zeta = \zeta_{\text{vis}})}W(\phi(z = z_{\text{vis}}))^2} \right).$$  \hfill (83)

This is qualitatively the same type of correction as in ref.[1]. Thus, the observer living on the brane Universe does not see drastic changes in the Newton law.

## 5 Dilatonic de Sitter brane Universe

When $V$ is constant, instead of (2), the solution is given in [3, 4], as follows

$$ds^2 = f(y)dy^2 + y \sum_{i,j=0}^{d-1} \tilde{g}_{ij}(x^k)dx^i dx^j$$

$$f = \frac{d(d-1)}{4y^2 \lambda^2 \left( 1 + \frac{c^2}{2\lambda^2 y^2} + \frac{kd}{\lambda^2 y} \right)}$$

$$\phi = c \int dy \left[ \frac{d(d-1)}{4y^{d+2} \lambda^2 \left( 1 + \frac{c^2}{2\lambda^2 y^2} + \frac{kd}{\lambda^2 y} \right)} \right].$$  \hfill (84)

Here we define $V = -\lambda^2$ and $g_{ij}$ is the metric of the Einstein manifold, which is defined by $r_{ij} = k\tilde{g}_{ij}$, where $r_{ij}$ is the Ricci tensor constructed with $\tilde{g}_{ij}$ and
$k$ is a constant. Especially when $k$ is given by $k = d\tilde{c}^2 > 0$, the metric on the brane can correspond to the cosmological solution

$$\sum_{i,j=0}^{d-1} \tilde{g}_{ij}(x^k) dx^i dx^j = \frac{1}{c^2 t^2} \left( -dt^2 + \sum_{i=1}^{d-1} (dx^i)^2 \right). \quad (85)$$

This solution describes the wall expanding and travelling in 5d universe. It may be presented as regular solution in terms of radial coordinate similarly to black hole.

If one defines a new coordinate $z$ by

$$z = \int dy \sqrt{\frac{d(d-1)}{4y^2 \lambda^2 \left( 1 + \frac{c^2}{2\lambda^2 y^2} + \frac{k d}{\lambda^2 y} \right)}} \quad \text{ (86)}$$

and solves $y$ with respect to $z$, we obtain the warp factor $e^{2A} = y(z)$. We should note that there is a curvature singularity at $y = 0$ \cite{7, 8}. Therefore we cannot put only one brane but two branes and consider the region sandwiched by them to avoid the singularity.

Using (10), one finds the vacuum energy on the brane as follows

$$U_{\text{hid}} = -4\lambda \sqrt{\frac{d(d-1)}{d} \left( 1 + \frac{c^2}{2\lambda^2 y_{\text{hid}}^2} + \frac{k d}{\lambda^2 y_{\text{hid}}} \right)} \quad \text{ (87)}$$

Here we assume $z_{\text{vis}} > z_{\text{hid}}$ and consider the region $z_{\text{vis}} \geq z \geq z_{\text{hid}}$. $y_{\text{hid,vis}}$ is the value of $y$ corresponding to $z = z_{\text{hid,vis}}$. From (10), one gets

$$\frac{\partial U_{\text{hid}}}{\partial \phi} = \frac{2c}{y_{\text{hid}}^2}, \quad \frac{\partial U_{\text{vis}}}{\partial \phi} = -\frac{2c}{y_{\text{vis}}^2}, \quad (88)$$

which tells that the vacuum energies $U_{\text{vis}}$ on the brane depend on the dilaton field.

One can consider the perturbation around the solution (84), (85). As it is difficult to consider the general case, we first investigate the region $y < y_{\text{vis}}$ and $y_{\text{vis}} \to 0$, that is, the brane is near the singularity. We should
note, however, it is sufficient to consider the asymptotic region to check the existence of the normalized zero mode and continuous KK modes. Using (86), when $y \to 0$, one gets
\[ \zeta \sim \frac{1}{c} \sqrt{\frac{d}{d-1}} y^{\frac{d-1}{2}}. \]  
\[ (89) \]
This tells that we now consider the region where $\zeta$ is small. Since $A = \frac{1}{2} \ln y$, Eq.(36) has the following form:
\[ 0 = m^2 \psi + \partial_\xi^2 \psi + \frac{d-5}{d-1} \partial_\xi \psi + \frac{4}{(d-1)^2} \left( \zeta (\partial_\xi \psi) \right). \]  
\[ (90) \]
When $m^2 = 0$, the solution is given by
\[ \psi_0 = \zeta^2 \left( c_1 + c_2 \ln \zeta \right). \]  
\[ (91) \]
On the other hand, when $m^2 > 0$, we find
\[ \psi_m = \zeta^2 \left( c_1 J_0(m\zeta) + c_2 N_0(m\zeta) \right). \]  
\[ (92) \]
The mode corresponding to $m = 0$ does not seem to be normalizable as in flat dilatonic brane [10]. Then in order to localize the gravity, we need to put a brane corresponding to the hidden sector.

6 Quantum effective action for dilatonic brane and RG flow of Newton constant

Our starting point is again the following action of 5d dilatonic gravity (gauged supergravity):
\[ S = \frac{1}{16\pi} \left[ \int_M d^{d+1}x \sqrt{-g} \left( R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right) \right]. \]  
\[ (93) \]
The dilatonic potential is not specified for the moment. As it was discussed in section 2, the very common choice for $V$ is the exponential of dilaton: It corresponds to the effective action for the breathing-mode scalar and gravity which follows from KK sphere reduction from M-theory or strings [10].
the exponential potentials there are singular domain wall solutions in above theory \[5\]. It is interesting that there are usually problems with localization of 4d gravity when using only the action (93). As a result one should consider the inclusion of four dimensional action (for walls they correspond to wall source terms). Then

\[
S_{\text{source}} = \sum_{i=\text{hid,vis}} \int d^4x \sqrt{-\gamma} \left\{ L_{\text{QFT}} e^{\alpha_i \phi} + U_i(\phi) \right\} .
\]

(94)

Here \(U_i(\phi)\) are vacuum energies on branes, normally \(U_i(\phi)\) are dictated by the form of dilatonic potential. For example, if \(V \sim e^{\kappa \phi}\) then \(U_i\) has also the exponential form. \(L_{\text{QFT}}\) is an arbitrary Lagrangian corresponding to massless QFT (say, QED, QCD, SM, GUT) which is classically conformally invariant in the background \(\tilde{g}_{\mu\nu}\). There is dilaton coupling on the brane which is typical for Brans-Dicke gravity. Usually it is assumed to be invisible in 4d world.

We are going to search for the solutions of the sort

\[
ds^2 = dz^2 + e^{2A(z)} \tilde{g}_{ij} dx^i dx^j
\]

(95)

where 4 dimensional \(\tilde{g}_{ij} = a^2(\eta) \eta_{ij}\). It is assumed that branes sit on \(z = z_{\text{hid}}\) and \(z = z_{\text{vis}}\).

We integrate over quantum fields in theory (94). Supposing that there is only gravitational background, the interaction of dilaton coupled quantum fields leads to effective action induced by conformal anomaly [14]:

\[
\Gamma_{\text{source}} = \sum_{i=\text{hid,vis}} V_3 \int d\eta \left\{ 2 b_1 \sigma_1 \sigma_1''' - 2(b_1 + b) (\sigma_1'' - \sigma_1')^2 \right\}
\]

(96)

where \(\sigma = \ln a(\eta)\), \(\sigma_1 = A + \sigma + \frac{\alpha_1 \phi}{3}\). For the sake of simplicity, we adopt the large \(N\)-expansion (that justifies the neglect of proper quantum gravity contribution to (96)). If spinors give the leading contribution then \(b = \frac{3N}{60(4\pi)^2}\), \(b_1 = -\frac{11N}{360(4\pi)^2}\). One can take the contribution above as corresponding to maximally SUSY Yang-Mills theory (which only changes the coefficients of above effective action). That corresponds to implementing above compactification to AdS/CFT scheme. Note that the suggestion to take into account the boundary matter quantum effects (via conformal anomaly induced effective action for SUSY Yang-Mills theory) in brane-world scenario has appeared
in ref. [15]. It was shown the possibility of creation of de Sitter or Anti-de Sitter 4d Universe in 5d AdS space. In ref. [16] the same idea on application of conformal anomaly has been expressed and effective brane tension due to such boundary quantum contribution for 4d de Sitter world has been found.

Thus our complete action will be given by sum of three terms:

\[ S_{\text{complete}} = S + \Gamma_{\text{source}} + \sum_{i=\text{hid,vis}} V_3 \int d\eta e^{4A} a^4(\eta) U_i(\phi). \quad (97) \]

One can now consider the solution of the equations of motion given from the action (97). In the bulk 5d universe, the action is identical with the previous one (1), then the solutions in the bulk are also given by the previous ones. Especially when \( V(\phi) \) is a constant, we obtain the solution in (84).

Near the brane, however, one obtains the following equations for \( d = D - 1 = 4 \) instead of (9):

\[ \phi'' \sim \left[ \frac{\partial U_i(\phi)}{\partial \phi} + \frac{\alpha_1}{3} e^{4A} \left\{ 4b_1 \sigma_1''' - 4(b + b_1)(\sigma_1''' - 6\sigma_1'^2 \sigma_1'') \right\} \right] \delta(z - z_i), \]
\[ A'' \sim -\frac{1}{6} \left\{ U_i(\phi) 4b_1 \sigma_1''' - 4(b + b_1)(\sigma_1''' - 6\sigma_1'^2 \sigma_1'') \right\} \delta(z - z_i). \quad (98) \]

Then by substituting the solution in (84), (86), we find \( U_i \) and \( \frac{\partial U_i}{\partial \phi} \) become time dependent:

\[ U_{\text{hid}} = -4\lambda \sqrt{\frac{3}{4} \left( 1 + \frac{c^2}{2\lambda^2 y_{\text{hid}}^4} + \frac{4k}{\lambda^2 y_{\text{hid}}} \right) - \frac{6b_1 y_{\text{hid}}^2}{t^4}} \]
\[ U_{\text{vis}} = 4\lambda \sqrt{\frac{3}{4} \left( 1 + \frac{c^2}{2\lambda^2 y_{\text{vis}}^4} + \frac{4k}{\lambda^2 y_{\text{vis}}} \right) - \frac{6b_1 y_{\text{vis}}^2}{t^4}} \]
\[ \frac{\partial U_{\text{hid}}}{\partial \phi} = \frac{2c}{y_{\text{hid}}} - \frac{8\alpha_1 b_1 y_{\text{hid}}^2}{t^4} \]
\[ \frac{\partial U_{\text{vis}}}{\partial \phi} = -\frac{2c}{y_{\text{vis}}} - \frac{8\alpha_1 b_1 y_{\text{vis}}^2}{t^4}. \quad (99) \]

Hence, the price one pays for keeping the same de Sitter brane-world solution is in change of brane vacuum energies. Quantum corrections explicitly give contribution to vacuum energies which become time-dependent (or
dependent from the radius of de Sitter space in radial coordinates. That indicates that effective brane tension will be changed.

Let us discuss now RG flow of 4d Newton constant (for a recent review of holographic RG, see \[17\]). Using (22) for the solution in (84) for \(d = 4\), we find

\[
\frac{1}{G_4} = \frac{1}{G_5} \int_{y_{\text{hid}}}^{y_{\text{vis}}} dy \frac{\sqrt{3}}{\lambda \sqrt{1 + \frac{c^2}{2\lambda^2 y^4} + \frac{k_d}{\lambda^2 y}}}.
\]

(100)

If we define \(U\) by \(y_{\text{vis}} = U^2\), we can identify \(U\) with the energy scale on the visible brane from AdS/CFT correspondence \[11, 12, 13\]. Therefore Eq.(100) expresses the scale dependence of the gravitational coupling. When \(U (y_{\text{vis}})\) is small, we find

\[
\frac{1}{G_4} \sim \frac{1}{G_5} \frac{(y_{\text{vis}}^3 - y_{\text{hid}}^3)}{c\sqrt{3}} = \frac{1}{G_5} \frac{(U^6 - y_{\text{hid}}^3)}{c\sqrt{3}}.
\]

(101)

Therefore the gravitational coupling \(G_4\) becomes large in the IR region, which would be due to the curvature singularity at \(y = 0\). On the other hand, when \(U (y_{\text{vis}})\) is large, we find

\[
\frac{1}{G_4} \sim \frac{1}{G_5} \frac{\sqrt{3}}{\lambda} (y_{\text{vis}} + Y(y_{\text{hid}})) = \frac{1}{G_5} \frac{\sqrt{3}}{\lambda} (U^2 + Y(y_{\text{hid}}))
\]

(102)

Here \(Y\) depends on \(y_{\text{hid}}\) but does not on \(y_{\text{vis}}\). Eq.(102) seems to tell that the gravitational coupling \(G_4\) becomes small in the UV region.

We can also consider the scale dependence of the Newton constant for the black hole type solution in Section 2. If the scale \(U = e^{A(z_{\text{vis}})}\) is small (IR region), \(z_{\text{vis}}\) becomes negative and large. Then from (24) and (22), we find

\[
U \sim e^{\gamma z_{\text{vis}}},
\]

\[
\frac{1}{G_4} = \frac{1}{G_5} \int_{z_{\text{hid}}}^{z_{\text{vis}}} dz e^{2A} \sim \frac{1}{G_5} \int_{z_{\text{vis}}}^{z_{\text{hid}}} dz e^{2\gamma z} = \frac{1}{G_5} \frac{e^{2\gamma z_{\text{vis}}}}{2\gamma} = \frac{1}{G_5} \frac{U^2}{2\gamma}.
\]

(103)

Here we assumed \(z_{\text{vis}} > z_{\text{hid}} \rightarrow -\infty\) and put the constant of the integration to vanish. Therefore \(G_4\) becomes large in the IR region. On the other hand, when the scale \(U = e^{A(z_{\text{vis}})}\) is large (UV region), \(z_{\text{vis}}\) becomes positive and large. By using (24), we obtain

\[
U \sim \alpha z_{\text{vis}}^\beta,
\]
Here $\tilde{Y}$ depends on $y_{\text{hid}}$ but does not on $y_{\text{vis}}$. For $d = D - 1 = 4$ and $\mathcal{N} = 1$ model in [5], we have $2 + \frac{1}{\beta} = \frac{18}{5}$. Therefore the gravitational coupling $\tilde{G}_4$ becomes small in the UV region again. We should note that the parameters specifying the black hole on the brane do not enter in the above expressions (103) and (104), which is significantly different from the case that there is a black hole in the bulk [19].

7 Discussion

In summary, we presented the family of brane-world solutions of 5d dilatonic gravity. This family includes flat brane with effectively small cosmological constant, (anti) de Sitter and Nariai spaces, and brane-world dilatonic black holes. The study of gravity perturbations around such black holes shows that 4d gravity may be trapped. Corrections to Newton law near branes are calculated. The proposal to take into account brane matter quantum effects is made. (Actually, such proposal presented earlier in refs. [15, 16] helps to formulate the problem in terms of AdS/CFT.) The corresponding anomaly induced effective action is used to estimate the role of quantum effects in realization of de Sitter branes in asymptotically AdS dilatonic space. It is demonstrated that quantum corrections change the brane vacuum energies. RG flow of Newton constant in IR and UV is discussed.

There are many related problems which are left for future study. In particular, it looks that the picture under discussion should be realized within AdS/CFT correspondence. As it has been partially demonstrated it is possible to do (at least in case of de Sitter brane). However, the details of such quantum corrected brane-world Universe should be investigated deeply. The research of the role of brane matter quantum effects to black holes is also extremely interesting topic. Another open problem is the correct interpretation of dilatonic brane-world black holes as holographic RG flows. The presented example of RG flow of 4d Newton constant for dilatonic black hole represents the modest step in this direction.
8 Acknowledgements.

We thank M.Ryan for the interest in this work. This research has been supported in part by CONACyT grant 2845E.

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