Orbital angular momentum in a nonchiral topological superconductor

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We investigate the bulk orbital angular momentum in a two-dimensional time-reversal broken topological superconductor with the Rashba spin-orbit interaction, the Zeeman interaction, and the s-wave pairing potential. Prior to the topological phase transition, we find the crossover from s-wave to p-wave. For the large spin-orbit interaction, even in the topological phase, \( \Delta_s \) does not reach \(-1/2\), which is the universal value in chiral p-wave superconductors. Here \( \Delta_s \) and \( N \) are the bulk orbital angular momentum and the total number of electrons at zero temperature, respectively. Finally, we discuss the effects of nonmagnetic impurities.

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I. INTRODUCTION

Topological superconductors (SCs) have been intensively studied due to their fundamental interest in the physical realization of Majorana fermions and their future application to topological quantum computation. A typical design principle for two-dimensional time-reversal broken topological SCs is to realize a spinless chiral p-wave SC\(^2\). Possible candidates are the surface of a topological insulator\(^2\) and a quantum anomalous Hall insulator in proximity to s-wave SC\(^3\). More promising candidates are conventional spin-orbit coupled systems with the Zeeman interaction and the s-wave pairing potential\(^4\). In fact, their one-dimensional analog was theoretically proposed\(^5,6\) and later experimentally realized in semiconductor nanowires contacted with s-wave SC\(^7,8\).

In a chiral p-wave SC, each Cooper pair carries the orbital angular momentum (AM) \( \ell = 1 \), which leads to the bulk orbital AM \( L_z/N = 1/2 \) at zero temperature\(^9,10\). Here \( N \) is the total number of electrons, and we set \( \hbar = 1 \). In addition, a chiral p-wave SC has the midgap bound states induced by nonmagnetic impurities\(^11\) and hence is fragile, i.e., the critical temperature \( T_c \) is rapidly suppressed as the impurity density increases. This is in sharp contrast to an s-wave SC, which is robust according to Anderson’s theorem\(^12\). Thus, it is natural to relate the bulk orbital AM to the robustness against nonmagnetic impurities.

Impurity effects in topological SCs have also been studied. Although topological invariants at zero temperature are robust against symmetry-preserving perturbations, topological SCs are not always robust against nonmagnetic impurities\(^13,14\). Especially, in the specific model with the Rashba spin-orbit interaction (SOI) \( \alpha \), the Zeeman interaction \( h \), and the s-wave pairing potential \( \Delta \), which is dubbed the Rashba+Zeeman+s-wave model below, the appearance of the midgap bound states and the reduction of \( T_c \) by nonmagnetic impurities become more relevant as \( h \) increases\(^16,17\). This is consistent with the chiral p-wave behavior for large \( h \). The similar tendency was observed in a three-dimensional time-reversal symmetric topological SC by changing the chemical potential\(^18,19\).

There is another evidence that the bulk orbital AM is related to the robustness against nonmagnetic impurities. The impurity effects on the bound state in a single vortex core were investigated in the Rashba+Zeeman+s-wave mode\(^20\). There are two types of vortices; parallel (antiparallel) vortex when the signs of the vorticity and the Zeeman interaction are opposite (same). Note that our sign convention of the Zeeman interaction is opposite to that in Ref.\(^23\). The authors found that the low-energy scattering rate is suppressed in the case of an antiparallel vortex compared to the case of a parallel vortex. This result alone might suggest that the impurity effects are characterized by the Chern number because the sign of the Chern number is determined by the sign of the Zeeman interaction. However, they also showed that the robustness of the bound state against nonmagnetic impurities depends on the Rashba SOI. Thus, the impurity effects may be characterized by the bulk orbital AM rather than the Chern number.

In this paper, we investigate the bulk orbital AM in a two-dimensional time-reversal broken topological SC. We focus on the Rashba+Zeeman+s-wave model\(^20\). We calculate the bulk orbital AM both by the Berry-phase formula\(^21\) and in the circular disk. One advantage of the Berry-phase formula is that the wavefunctions in the reciprocal space are simpler compared to those in the real space. Note that we do not take into account any impurities. Nonetheless, we qualitatively explain the effects of nonmagnetic impurities in the absence and presence of a vortex.

II. MODEL AND METHOD

The Rashba+Zeeman+s-wave model\(^20\) is represented by

\[
H = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \Psi_\uparrow^\dagger \begin{bmatrix} k_x \tau_z + \alpha (k_x \sigma_y - k_y \sigma_x) \tau_z - h \sigma_z + \Delta \tau_x \end{bmatrix} \Psi_\uparrow,
\]

where \( \Psi_\uparrow = [c_{\uparrow \uparrow}, c_{\downarrow \downarrow}, c_{\downarrow \uparrow}, c_{\uparrow \downarrow}]^T \) is the Nambu spinor, and \( \sigma^z \) and \( \tau^z \) are the Pauli matrices for the spin and Nambu spaces, respectively. \( \xi_k = k^2/2m - \mu \) is the kinetic term in reference to the chemical potential \( \mu \). There are four dispersions, \( \pm E_{k \pm} = \pm \sqrt{\xi^2 + \alpha^2 k^2 + h^2 + \Delta^2 + 2g_{\Delta}^2} \) with \( g_{\Delta}^2 = \sqrt{\xi^2(\alpha k)^2 + \xi^2 k^2 + \Delta^2 - \xi^2} \). Therefore, the gap between \( \pm E_{k-} \) closes at \( k = 0 \) for \( \hbar c = \sqrt{\mu^2 + \Delta^2} \), above which this model has the nontrivial Chern number \( C = -1 \).
First, we calculate the bulk orbital AM in the reciprocal space. The wavefunctions with the dispersions \(+E_{k-}, +E_+\), \(-E_{k+}\), and \(-E_{k-}\) are simply given by

\[
N_{k-} = \begin{pmatrix}
(h^2 - g_{k}^2 - hE_{k-}) (\xi_{k}^2 - g_{k}^2 + \xi_{k}E_{k-}) & -i\alpha k e^{-i\phi} (h^2 - g_{k}^2 + \xi_{k}E_{k-}) \\
-i\alpha k e^{-i\phi} (h - \xi_{k}) (\xi_{k}^2 - g_{k}^2 + \xi_{k}E_{k-}) & \Delta(h - \xi_{k})(h^2 - g_{k}^2 - hE_{k-})
\end{pmatrix}
\]

\[
N_{k+} = \begin{pmatrix}
-i\alpha k e^{-i\phi} (h + \xi_{k}) (\xi_{k}^2 + g_{k}^2 + \xi_{k}E_{k+}) \\
(h^2 + g_{k}^2 + hE_{k+}) (\xi_{k}^2 + g_{k}^2 + \xi_{k}E_{k+}) & -i\alpha k e^{-i\phi} (h^2 - g_{k}^2 + \xi_{k}E_{k+})
\end{pmatrix}
\]

\[
N_{k-} = \begin{pmatrix}
-i\alpha k e^{-i\phi} (h - \xi_{k}) (\xi_{k}^2 - g_{k}^2 + \xi_{k}E_{k-}) \\
(h^2 - g_{k}^2 - hE_{k-}) (\xi_{k}^2 - g_{k}^2 - \xi_{k}E_{k-})
\end{pmatrix}
\]

integrand for the bulk orbital AM is given by \((\hat{A}_{kn}' \times \hat{k})_z = -i(u_{kn}' | \partial_r u_{kn}'| - | \partial_r u_{kn}'| \partial_r u_{kn}') / k\), where \(A_{kn}' = i(\partial_{kn} | \partial_{kn} u_{kn}') - (\partial_{kn} u_{kn}' | \partial_{kn} u_{kn}')\) is the Berry connection. Since the Berry connection can be interpreted as the expectation value of the position operator \(\hat{x}\) in the reciprocal space, the integrand is interpreted as that of \(\hat{x} \times \hat{p}\). The Chern number \(C\), the total number of electrons \(N\), the total spin \(S_z\), and the bulk orbital AM \(L_z\) are calculated by

\[
C = \sum_{ij} (i - j)[u_{kij}^2 - v_{kij}^2] \cdot 0,
\]

\[
N = \int d^2k (2\pi)^2 \sum_{ij} 2v_{kij}^2,
\]

\[
S_z = \int d^2k (2\pi)^2 \sum_{ij} (v_{kij}^2 - v_{kij}^2),
\]

\[
L_z = -\int d^2k (2\pi)^2 \sum_{ij} (i - j)(u_{kij}^2 - v_{kij}^2),
\]

respectively. The bulk orbital AM can be calculated not only in the reciprocal space but in the real space. We consider the circular disk with the radius \(r_c\). The Bogoliubov-de Gennes (BdG) equations in the real space are given as

\[
[H_0(-i\nabla)_{z} + \Delta \tau_z] \begin{pmatrix} u(r) \\ v(r) \end{pmatrix} = E \begin{pmatrix} u(r) \\ v(r) \end{pmatrix},
\]

where

\[
H_0 = \begin{pmatrix} \xi(r, \theta) - h & -\alpha L_-(r, \theta) \\ \alpha L_+(r, \theta) & \xi(r, \theta) + h \end{pmatrix},
\]

with

\[
\xi(r, \theta) = -(\partial^2/\partial r^2 + (1/r)\partial / \partial r + (1/r^2)(\partial^2 / \partial \theta^2)/2m - \mu)\]

and \(L_{\pm} = e^{\pm i\phi}(\partial / \partial r \mp (i/r)\partial / \partial \theta)\).

With the use of the rotational symmetry along the z axis in the circular disk, the solutions can be expressed as

\[
\begin{pmatrix} u_n(r) \\ v_n(r) \end{pmatrix} = e^{in\theta} \begin{pmatrix} u_1(r) \\ v_1(r) \end{pmatrix}.
\]

Here \(n\) is the quantum number. When \([u_n(r)^T, v_n(r)^T]\) is a solution with the energy \(E_n\), \([u_{n-1}(r)^T, v_{n-1}(r)^T]\) is \([-i\sigma_y v_n(r)^T, i\sigma_y u_n(r)^T]\) is the solution with the energy \(-E_n\). We use the normalization condition,

\[
\sum_{\sigma} \int_0^{r_c} \int_0^{2\pi} rdrd\theta (|u_{\sigma}(r)|^2 + |v_{\sigma}(r)|^2) = 1,
\]

and the boundary conditions,

\[
\left. \frac{\partial}{\partial r} \begin{pmatrix} u_1(r) \\ v_1(r) \end{pmatrix} \right|_{r=0} = 0,
\]

\[
\left. \begin{pmatrix} u_1(r_c) \\ v_1(r_c) \end{pmatrix} \right| = \begin{pmatrix} u_1(r_c) \\ v_1(r_c) \end{pmatrix} = 0.
\]
The bulk orbital AM in the real space is expressed as

$$L_z=2\pi \sum_{l} \sum_{n=0}^{N_c} \int_0^{r_c} r dr \left[ (n+1)|u_n^l(r)|^2 f(E_n^l) - n|u_n^l(r)|^2 f(-E_n^l) \right],$$

with the l-th eigenvalue $E_n^l$ with fixed $n$ and the Fermi-Dirac distribution function $f(\omega) \equiv 1/(e^{\omega/(T+1)} + 1)$. Note that we focus on $T=0$. In order to solve the radial BdG equations, the second order finite difference method with $N$ real-space grid points are used. Thus, the BdG equations become the $2N \times 2N$ matrix eigenvalue equations. We set the number of the grid points $N=1000$, the cutoff $n_c=512-1$, and the disk radius $r_c=100$. The unit of the real space is defined by the lattice spacing of the tight-binding model in Ref.30.

### III. RESULTS

Figures 2 and 3 show the $h$- and $\alpha$-dependences of $C$, $S_z/N$, and $L_z/N$. $L_z/N$ obtained by the Berry-phase formula completely coincides with that in the circular disk. For small $\alpha$, we obtain $L_z/N \approx 0$ for $h<\Delta$ and $L_z/N \approx -1/2$ for $h>h_c$ as shown in Fig.2(a). These are expected from the $s$- and $(p-i\nu)$-wave behaviors, respectively. Remarkably, in the intermediate region $\Delta < h < h_c$, we obtain the nonzero $L_z/N$. In contrast to $C$, $L_z/N$ does not jump but shows the crossover. As $\alpha$ increases, the region with $L_z/N \approx 0$ gets narrower. For large $\alpha$, as shown in Figs.2(c) and 2(d), $L_z/N$ never goes to $-1/2$ even in the topological phase. In Fig.3, we find that $\alpha$ tends to suppress $L_z/N$. Although the Rashba-Zeeman++-wave model is mapped onto a chiral $p$-wave SC, these two are different in terms of the bulk orbital AM. These results are summarized in Tab.1.

![Fig. 2.](image1.png)  ![Fig. 3.](image2.png)

**FIG. 2.** $h$-dependence of $C$ (blue broken line), $S_z/N$ (red dashed line), and $L_z/N$ (black solid line). Black filled square indicates $L_z/N$ obtained by numerical calculations in the circular disk. We set (a) $\alpha=0.1$, (b) $\alpha=0.25$, (c) $\alpha=0.5$, and (d) $\alpha=1$.

**FIG. 3.** $\alpha$-dependence of $C$, $S_z/N$, and $L_z/N$. The legends are the same as in Fig.2. We set (a) $h=0.33<\Delta$, (b) $\Delta < h < 0.5 < h_c$, (c) $h=0.67 > h_c$, and (d) $h=1 > h_c$.

| Small $\alpha$ | Large $\alpha$ | Chern number $C$ |
|----------------|-----------------|------------------|
| $h < h^*$      | 0               | Finite           |
| $h^* < h < h_c$| Finite          | 0                |
| $h_c < h$      | $-1/2$          | Finite ($\neq -1/2$) |

Table 1. Summary of the $h$- and $\alpha$-dependences of $L_z/N$. For small $\alpha$, $h^* = \Delta$.

In the case of small $\alpha$, we obtain $L_z/N \approx 0$ for $h<\Delta$. To see this, we evaluate the triplet pairing amplitudes

$$\langle c_{k^+l^1} c_{k^1l^1} \rangle = -ie^{-i\theta}(u_{k11}v_{k12} + u_{k12}v_{k11}),$$

$$\langle c_{k^z l^1} c_{k^1 l^z} \rangle = -ie^{i\theta}(u_{k21}v_{k22} + u_{k22}v_{k21}),$$

whose pairing symmetries are $(p+i\nu)$-wave. Figure 4 shows the $h$-dependence of the integrated pairing amplitudes

$$\mathcal{F}(p-i\nu) = \int \frac{d^2k}{(2\pi)^2} (u_{k11}v_{k12} + u_{k12}v_{k11}),$$

$$\mathcal{F}(p+i\nu) = \int \frac{d^2k}{(2\pi)^2} (u_{k21}v_{k22} + u_{k22}v_{k21}),$$

divided by the bulk gap $E_g = \min_k E_{k-}$. For $h < h^*$, both $\mathcal{F}(p+i\nu)/E_g$ are suppressed.

On the other hand, for $h^* < h < h_c$, $s$-wave is no longer dominant although it is topologically trivial. In Fig.4 both $\mathcal{F}(p+i\nu)/E_g$ increase besides their divergence at $h = h_c$ because of $E_g \to 0$. Owing to the intraband $(p+i\nu)$-wave pairing potentials, the lower and upper bands acquire the orbital AM $\pm \langle \ell_z \rangle$ per electron, respectively. When the total numbers of electrons in the lower and upper bands are denoted by $N_\uparrow$, respectively, the bulk orbital AM is represented by $L_z = \langle \ell_z \rangle (N_- - N_+)$, and hence $L_z/N = \langle \ell_z \rangle (N_- - N_+)/ (N_- + N_+)$ negatively increases as a function of $h$. Note that $N_+ > N_-$ and $\langle \ell_z \rangle < 0$. Especially, in the case of small $\alpha$ and $\Delta$, we obtain $L_z/N = \langle \ell_z \rangle h/\mu$ because of $N_\uparrow = m(\mu \pm h)\theta(\mu \pm h)/2\pi$.

For $h > h_c$, since the upper band goes away from the chemical potential, we obtain $N_+ = 0$ and $L_z/N = \langle \ell_z \rangle$. However,
between the pairing amplitudes is almost suppressed in Figs. 4(a–c). As
\[ \langle \alpha \rangle \]
tilted, and corresponding, the whose eigenvalues are \( \pm 1 \)
and \( \pm 2 \). Therefore, the bulk total AM \( J_z = L_z + S_z \) always vanishes as seen in Figs. 2 and 3. For small \( \alpha \), spins are polarized in the \( z \) direction owing to the Zeeman interaction, which leads to \( \langle \ell_z \rangle \simeq -1/2 \). Correspondingly, the \((p - ip)\)-wave pairing amplitude \( F(p+ip)/E_g \) is almost suppressed in Figs. 4(a–c). As \( \alpha \) increases, spins are tilted, and \( \langle \ell_z \rangle \) goes to zero. Although \( F(p+ip)/E_g \) itself is no longer suppressed in Fig. 4(d), we find that the difference between the pairing amplitudes \( (F(p-ip) - F(p+ip))/E_g \) decreases as a function of \( \alpha \).

IV. DISCUSSION

Here we discuss the effects of nonmagnetic impurities in the light of the bulk orbital AM. In the topological phase, a strong nonmagnetic impurity behaves like a vortex core, which gives rise to the midgap bound state. Since a vortex carries the orbital AM, such behavior seems plausible in the topological phase with the bulk orbital AM. The ratio \( E_b/E_g \) monotonically decreases as \( h \) increases, where \( E_b \) represents the energy level of the midgap bound state. Correspondingly, the reduction of \( T_c \) becomes relevant. This crossover of the robustness against nonmagnetic impurities is consistent with the crossover of the bulk orbital AM we have found. Moreover, we expect that the Rashba SOI makes the topological SC robust because it suppresses the bulk orbital AM, which was already observed in a three-dimensional time-reversal symmetric topological SC.

We can also explain the vorticity-dependent impurity effects on the vortex bound state. As already mentioned in Sec. I, the bound state in an antiparallel vortex is more robust against nonmagnetic impurities than that in a parallel vortex. Furthermore, the low-energy scattering rate in the case of an antiparallel vortex is suppressed for small \( \alpha \) as in chiral \( p \)-wave SCs, while it is similar to that in the case of a parallel vortex for large \( \alpha \) as in \( s \)-wave SCs. The orbital AM carried by the antiparallel vortex is almost canceled by the intrinsic orbital AM for small \( \alpha \), but not for large \( \alpha \). For sufficiently large \( \alpha \), the intrinsic orbital AM is totally suppressed, and two types of vortices cannot be distinguished. Therefore, the impurity effects are characterized not by the integer Chern number but by the nonuniversal bulk orbital AM.

Although it may be difficult to measure the bulk orbital AM directly, there are two alternatives. One is the Hall viscosity, which is equal to half the bulk orbital AM in two-dimensional gapped systems at zero temperature. The other is the spin magnetization, whose absolute value is equal to that of the bulk orbital AM but sign is opposite as discussed above. Especially, in Fig. 2, the slope of \( S_z/N \) at \( h \to 0 \) is nothing but the spin susceptibility \( \chi_{zz} = \partial S_z/N/\partial h \). In the presence of the SOI, \( \chi_{zz} \) is nonzero due to the interband Van Vleck spin susceptibility. This is an indirect but simple way to measure the bulk orbital AM.

V. SUMMARY

To summarize, we have calculated the bulk orbital AM in the two-dimensional Rashba+Zeeman+\( s \)-wave topological SC. The Berry-phase formula coincides with numerical calculations in the circular disk. Prior to the topological phase transition at \( h = h_c \), we have found the crossover from \( s \)-wave to \( p \)-wave at \( h = h^* \), where \( L_z/N \) starts to increase negatively. We have also found that the Rashba SOI tends to suppress \( S_z/N \) and \( L_z/N \) because the bulk total AM always vanishes. For large \( \alpha \), even in the topological phase, \( L_z/N \) does not reach \(-1/2\), which is the universal value in chiral \( p \)-wave SCs. We have explained the impurity effects on the topological SC; the appearance of the midgap bound states, the reduction of the critical temperature, and the vorticity-dependence of the vortex bound state.

Finally, let us comment on one possible future direction. In the Rashba+Zeeman+\( s \)-wave model, the bulk orbital AM obtained by the Berry-phase formula coincides with that in the circular disk. Recently, it was found that the bulk orbital AM in the circular disk is totally suppressed in higher-order chiral SC. Therefore, these two formalisms give the different results. To examine this issue in more detail, other two-dimensional time-reversal broken topological SCs with \( C = 1 \) and \( C = 3 \) are interesting.

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