Continuity of the attractors in time-dependent spaces and applications

Yanan Li¹, Zhijian Yang²,*

¹ College of Mathematical Sciences, Harbin Engineering University, 150001, China
² School of Mathematics and Statistics, Zhengzhou University, 450001, China

Abstract

In this paper, we investigate the continuity of the attractors in time-dependent phase spaces. (i) We establish two abstract criteria on the upper semicontinuity and the residual continuity of the pullback ℰ-attractor with respect to the perturbations, and an equivalence criterion between their continuity and the pullback equi-attraction, which generalize the continuity theory of attractors developed recently in [27, 28] to that in time-dependent spaces. (ii) We propose the notion of pullback ℰ-exponential attractor, which includes the notion of time-dependent exponential attractor [33] as its spacial case, and establish its existence and Hölder continuity criterion via quasi-stability method introduced originally by Chueshov and Lasiecka [12, 13]. (iii) We apply above-mentioned criteria to the semilinear damped wave equations with perturbed time-dependent speed of propagation: \( \epsilon \rho(t) u_{tt} + \alpha u_t - \Delta u + f(u) = g \), with perturbation parameter \( \epsilon \in (0, 1] \), to realize above mentioned continuity of pullback ℰ and ℰ-exponential attractors in time-dependent phase spaces, and the method developed here allows to overcome the difficulty of the hyperbolicity of the model. These results deepen and extend recent theory of attractors in time-dependent spaces in literatures [15, 20, 29].

Keywords: Time-dependent phase space, pullback ℰ-attractor, pullback ℰ-exponential attractor, continuity of attractors, semilinear damped wave equation.

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*Corresponding author: Zhijian Yang, e-mail: liyn@hrbeu.edu.cn, yzjzzut@tom.com. The first author is supported by the National Natural Science Foundation for Young Scientists of China (No.12101155) and the Natural Science Foundation of Heilongjiang Province of China (No. LH2021A001). The second author is supported by the National Natural Science Foundation of China (No.12171438).
1 Introduction

In the last decade, there have been more and more concern on the theory of attractors in time-dependent spaces, which can be seen as a development of the theory of nonautonomous infinite dimensional dynamical system. Some key notations such as pullback $D$-attractors, time-dependent global and exponential attractors in time-dependent spaces have been defined and the related theory including existence theorems has been established, and there have been many applications in various mathematical physical models (cf. [15, 16, 18–20, 29, 33]). In this paper, we shall investigate the theory on the continuity of the pullback $D$-attractors, the existence and the continuity of the pullback $D$-exponential attractors and give their application to the semilinear damped wave equation with perturbed time-dependent speed of propagation

$$\rho_\varepsilon(t)u_{tt} + \alpha u_t - \Delta u + f(u) = g,$$

where $$\rho_\varepsilon(t) = \frac{1}{\varepsilon \rho(t)}$$ is called perturbed time-dependent speed of propagation [15] and the perturbation parameter $\varepsilon \in (0, 1]$.

In 2008, Kloeden, Marín-Rubio and Real [29] proposed the concept of pullback $D$-attractors for the process acting on time-dependent phase spaces, established an abstract criterion on the existence of the pullback $D$-attractors and gave its application to the semilinear heat equation in a non-cylindrical domain. For narrative convenience, we first state a few related definitions.

**Definition 1.1.** A process acting on time-dependent metric spaces $(X_t, d_t)$ is a two-parametrical family of operators $\{U(t, \tau) : X_\tau \to X_t \mid \tau \leq t \in \mathbb{R}\}$ satisfying that

(i) $U(\tau, \tau)$ is the identity mapping on $X_\tau$, $\forall \tau \in \mathbb{R}$;

(ii) $U(t, s)U(s, \tau) = U(t, \tau)$, $-\infty < \tau \leq s \leq t < \infty$.

Define the universe

$$D = \{D = \{D(t)\}_{t \in \mathbb{R}} \mid \emptyset \neq D(t) \subset X_t, t \in \mathbb{R}, \text{ and } D \text{ is of some properties}\}.$$  

**Definition 1.2.** A family $A = \{A(t)\}_{t \in \mathbb{R}}$ is called a pullback $D$-attractor of the process $U(t, \tau)$ acting on time-dependent metric spaces $(X_t, d_t)$, if

(i) $A(t)$ is a compact subset of $X_t$ for each $t \in \mathbb{R}$;

(ii) $A$ is invariant, that is,

$$U(t, \tau)A(\tau) = A(t), \quad -\infty < \tau \leq t < +\infty;$$

(iii) $A$ is a pullback $D$-attracting family, that is, for any $D \in D$,

$$\lim_{\tau \to -\infty} \text{dist}_{X_t}(U(t, \tau)D(\tau), A(t)) = 0, \quad \forall t \in \mathbb{R},$$

hereafter, $\text{dist}_{X_t}(B, C) = \sup_{x \in B} \inf_{y \in C} d_t(x, y)$ denotes the Hausdorff semidistance between two nonempty subsets $B, C$ of $X_t$.

In addition, the pullback $D$-attractor $A$ is said to be minimal if $A(t) \subset C(t)$ for all $t \in \mathbb{R}$ whenever $C = \{C(t)\}_{t \in \mathbb{R}}$ is a pullback $D$-attracting family of non-empty closed sets.
Remark 1.3. (i) Let \{X_t\}_{t \in \mathbb{R}} be a family of Banach spaces, \(U(t, \tau) \in C(X_\tau, X_t)\), and for every \(\mathcal{D} \in \mathcal{D}\), \(\mathcal{D}\) be backward bounded, i.e., \(\sup_{s \in (-\infty, t]} \|D(s)\|_{X_s} < +\infty\). Then the related pullback \(\mathcal{D}\)-attractor \(\mathcal{A}\) becomes the time-dependent global attractor proposed by Di Plinio et al. \[20\].

(ii) Let \{X_t\}_{t \in \mathbb{R}} be a family of normed linear spaces, for every \(\mathcal{D} \in \mathcal{D}\), \(\mathcal{D}\) be uniformly bounded, i.e., \(\sup_{s \in \mathbb{R}} \|D(s)\|_{X_s} < \infty\), \(A \in \mathcal{D}\) and replace the invariance (ii) by the minimality in Definition 1.2. Then \(A\) becomes the time-dependent global attractor proposed by Conti et al. \[15\], where the minimality ensures the uniqueness of \(A\), and such an attractor is invariant whenever the process \(U(t, \tau)\) is \(T\)-closed for some \(T > 0\), i.e., \(U(t, t-T)\) is closed for all \(t \in \mathbb{R}\).

Conti et al. \[15\] developed the theory of time-dependent global attractor initialized by Di Plinio et al. \[20\] and gave its application to the nonautonomous semilinear damped wave equations with time-dependent speed of propagation

\[\rho(t)u_{tt} + \alpha u_t - \Delta u + f(u) = g, \quad (1.3)\]

where \(\frac{1}{\rho(t)}\) stands for the time-dependent speed of propagation. They exploited new framework to prove that the related process acting on time-dependent phase spaces \{\mathcal{H}_t\}_{t \in \mathbb{R}} has an invariant time-dependent global attractor \(\mathcal{A} = \{A(t)\}_{t \in \mathbb{R}}\). After that, Conti and Pata \[16\] supplemented the general theory with two new results: the first gives the structure of the time-dependent attractor \(\mathcal{A} = \{A(t)\}_{t \in \mathbb{R}}\) in terms of complete bounded trajectories of the system; the second provides sufficient conditions in order for the section \(A(t)\) to be close to (in terms of Hausdorff semidistance) the global attractor \(A_\infty\) of the limiting equation:

\[\alpha u_t - \Delta u + f(u) = g\]

as \(t \to +\infty\) provided that \(\lim_{t \to +\infty} \rho(t) = 0\), precisely,

\[\lim_{t \to +\infty} \text{dist}_{H^1}(\Pi_t A(t), A_\infty) = 0\]

where \(\Pi_t : \mathcal{H}_t \to H^1\) is the projection on the first component of \(\mathcal{H}_t\), \(\Pi_t A(t) = \{\xi \in H^1 | (\xi, \eta) \in A(t)\}\).

Replacing the weak damping \(\alpha u_t\) in model (1.3) by more complex nonlinear one \([1 + \rho(t)f'(u)] u_t\), Conti and Pata \[17\] studied the same issue for the corresponding one dimensional heat conduction model of Cattaneo type.

These pioneering works promote the development of the attractor theory in time-dependent phase space. Since then, there are many researches on the existence criteria of the pullback \(\mathcal{D}\)-attractor or time-dependent global attractor, as well as their applications in different mathematical physical models (cf. \[30, 35, 36, 39, 41, 43, 45\] and references therein).

More recently, Conti et al. \[18, 19\] established the well-posedness of solutions, the existence and the regularity of the invariant time-dependent global attractor \(\mathcal{A} = \{A(t)\}_{t \in \mathbb{R}}\) for the following viscoelastic wave model with time-dependent memory kernel \(k(t, s)\) in \(\Omega \subset \mathbb{R}^3\):

\[u_{tt}(t) - [1 + k(t, 0)] \Delta u(t) - \int_0^\infty \partial_s k(t, s) \Delta u(t-s) ds + f(u(t)) = g, \quad (1.4)\]

and showed that the section \(A(t)\) is close to (in the sense of the Hausdorff semidistance \(\text{dist}_{H^2 \times H^1}(\cdot, \cdot)\)) the global attractor \(\hat{A}\) of the following Kelvin-Voigt type model

\[u_{tt}(t) - \Delta u(t) - m \Delta u(t) + f(u(t)) = g\]

as \(t \to \infty\) provided that \(\lim_{t \to \infty} k(t, \cdot) = m \delta_0(\cdot)\) (in the distributional sense), where \(\delta_0(\cdot)\) denotes the Dirac mass.
Based on the works of [18, 19], Li and Yang [33] further presented a notion of time-dependent exponential attractor, provided an abstract existence criterion and gave its application to the model (1.4) to establish the existence and the regularity of the related time-dependent exponential attractors.

For comparison, we give a short survey for the continuity theory of the global and exponential attractors for the dynamical system on fixed phase space, i.e., $X_t \equiv X, \forall t \in \mathbb{R}$. One of the most common ways to describe the stability of global attractors is the upper semicontinuity in the sense of Hausdorff semidistance, there have been some abstract criteria and many of their applications to a variety of model equations with various perturbations (see e.g. [4, 6–8, 10, 11, 23, 42] and references therein). While the lower semicontinuity and therefore the continuity of the attractors is very difficult for it requires strict conditions on the structure of the unperturbed attractor, which are rarely satisfied even for a slightly simpler global attractor of complicated systems (see Hale et al [25, 26], Stuart and Humphries [40]).

Recently, some advances on this issue have been archived. Exploiting the Baire category theorem (cf. [37]) instead of discussing the structure of the attractor, Babin and Pilyugin [5] showed a residual continuity criterion, that is, the global attractors are continuous with respect to (w.r.t.) perturbation parameter $\lambda$ in a residual set of parameter space $\Lambda$. Under more relaxed conditions, Hoang, Olson and Robinson [27, 28] simplified the proof of [5] and established a few residual continuity criteria of global, pullback and uniform attractors, respectively. Moreover, based on the works in [31, 32], the authors [27, 28] also proved a continuity criterion of the above-mentioned attractors on $\Lambda$ (rather than a residual subset of $\Lambda$), and showed the equivalence between the continuity and the equi-attraction of the attractors. For the related research on this topic, one can see [2, 5, 9, 34].

However, as it says in [21, 22], the global attractor may have two essential drawbacks: (i) the concept lacks the description on its fractal dimension; (ii) its attracting rare for the bounded subset in phase space may be arbitrarily slow, which leads to that it is difficult (if not impossible) to estimate in terms of the physical parameters of the system and even makes it unobservable. Moreover, the global attractor is of, in general, upper semi-continuity and residual continuity w.r.t. perturbations, which cause that it may change drastically in the complementary set of the residual set under very small perturbations.

The exponential attractor suggested originally in [21] overcomes these drawbacks of global attractor in attracting rate, finiteness of fractal dimension and stability. But due to the non-uniqueness of the exponential attractors, the “optimal” choice of an exponential attractor which is stable at every point in perturbation parameter space is very important.

Comparing the attractor theory in time-dependent phase spaces with that in fixed phase space, one sees that the former is more complex and far from complete. For example, what about the upper and lower semicontinuity of the pullback $\mathcal{D}$-attractors w.r.t. perturbations? Can we give a proper notion of the pullback $\mathcal{D}$-exponential attractor and an existence criterion? What about the continuity of the pullback $\mathcal{D}$-exponential attractor w.r.t. perturbations? What about the applications of these abstract criteria to the mathematical physical models? All these questions are unsolved.

The purpose of this paper is to solve these questions. The main contributions are as follows:

(i) We establish two abstract criteria on the upper semicontinuity and residual continuity of the pullback $\mathcal{D}$-attractor in the time-dependent phase spaces, respectively. (see Theorem 2.1 and Theorem 2.4)

(ii) We show the equivalence between the continuity of the pullback $\mathcal{D}$-attractor w.r.t. the perturbation parameter and its equi-attraction. (see Theorem 2.8)

(iii) We present the notion of the pullback $\mathcal{D}$-exponential attractor in time-dependent phase spaces, and provide an abstract criterion on its existence and Hölder continuity via quasi-stability method introduced originally by Chueshov and Lasiecka [12, 13], which is a continuation of the researches on exponential attractors in time-dependent phase spaces in recent literature [33]. (see Theorem 3.2, Corollary 3.4 and Corollary 3.5)
(iv) Applying above-mentioned criteria to the perturbed semilinear damped wave equation (1.1) we show that under the same assumptions as in [15],

(a) the related evolution process \( U_\epsilon(t, \tau) \) has a pullback \( \mathcal{D} \)-attractor \( A_\epsilon = \{ A_\epsilon(t) \} _{t \in \mathbb{R}} \) for each \( \epsilon \in (0, 1] \), which is upper semicontinuous and residual continuous w.r.t. the perturbation parameter \( \epsilon \), respectively; (see Theorem 4.1)

(b) for every \( \epsilon_0 \in (0, 1] \), there exists a family of pullback \( \mathcal{D} \)-exponential attractors \( E_\epsilon = \{ E_\epsilon(t) \} _{t \in \mathbb{R}} \) depending on \( \epsilon_0 \), which is Hölder continuous at the point \( \epsilon_0 \). (see Theorem 4.2)

The features of these results are that:

(i) In the upper semicontinuity and the residual continuity criteria of the pullback \( \mathcal{D} \)-attractor, we replace the uniform requirement for all \( t \in \mathbb{R} \) appearing in the corresponding criteria in fixed phase space with the relaxed backward uniform requirement for \( t \leq t_0 \), and remove the requirement for the uniform compactness of the sections of attractors as in literatures [28], which made them being more applicable. In addition, we show the equivalence between the continuity and the equi-attraction of the pullback \( \mathcal{D} \)-attractor w.r.t. perturbation parameter.

(ii) In the existence and continuity criterion of the pullback \( \mathcal{D} \)-exponential attractors in time-dependent phase spaces, we replace the Banach spaces with more general normed linear spaces, and also replace the uniform requirement for all \( t \in \mathbb{R} \) with the relaxed \( t \leq t_0 \) because the former is rarely satisfied for the the nonlinear hyperbolic models in time-dependent phase spaces.

(iii) The results of application can be seen as an extension of those in [15] because under the same assumptions as in [15], we show not only the upper semicontinuity and residual continuity of the pullback \( \mathcal{D} \)-attractor of model (1.1) w.r.t. perturbation parameter \( \epsilon \), but also the existence and the regularity of the pullback \( \mathcal{D} \)-exponential attractors and their Hölder continuity w.r.t. perturbation parameter \( \epsilon \in (0, 1] \), which implies that the fractal dimension of the sections of the invariant time-dependent global attractor as shown in [15] are uniformly bounded. The method developed here allows to overcome the difficulty of the hyperbolicity of the model.

The paper is organized as follows. In Section 2, we establish two abstract criteria on the upper semicontinuity and the residual continuity of the pullback \( \mathcal{D} \)-attractor, respectively. In Section 3, we give the definition of the pullback \( \mathcal{D} \)-exponential attractors and discuss their existence and Hölder continuity criterion at an abstract level. In Section 4, we apply the above mentioned criteria to model (1.1) to show the continuity of the related pullback \( \mathcal{D} \)-attractors, and the existence and the Hölder continuity of the related pullback \( \mathcal{D} \)-exponential attractors w.r.t. the perturbation parameter \( \epsilon \in (0, 1] \).

2 Continuity of pullback \( \mathcal{D} \)-attractor

In this section, we discuss the continuity of pullback \( \mathcal{D} \)-attractor \( A_\lambda = \{ A_\lambda(t) \} _{t \in \mathbb{R}} \) w.r.t. parameter \( \lambda \in \Lambda \). For clarity, we first quote some notations, which will be used in the following sections.

Let \( \{ (X_t, d_t) \} _{t \in \mathbb{R}} \) be a family of metric spaces,

\[
B_t(x_0; R) := \{ x \in X_t \mid d_t(x, x_0) \leq R \}
\]
denote the \( R \)-ball of \( X_t \) centered at \( x_0 \in X_t \), especially, \( B_t(0) = B_t(0; R) \), and

\[
\mathcal{O}_t^\epsilon(B) := \bigcup_{x \in B} \mathcal{O}_t^\epsilon(x) := \bigcup_{x \in B} \{ y \in X_t \mid d_t(y, x) < \epsilon \}
\]
denote the \( \epsilon \)-neighborhood of a set \( B \subset X_t \), \( [B]_X \), denote the closure of \( B \) in \( X_t \) and the letters \( \mathcal{A}, B, \cdots \mathcal{E} \) denote the time-dependent families, respectively, for example

\[
\mathcal{D} = \{ D(t) \} _{t \in \mathbb{R}} \text{ with } D(t) \subset X_t, \forall t \in \mathbb{R}.
\]
We denote the symmetric Hausdorff distance of two nonempty sets \(B, C \subset X_t\) by
\[
\text{dist}_{X_t}^{symm}(B, C) := \max\{\text{dist}_{X_t}(B, C), \text{dist}_{X_t}(C, B)\}.
\]
Then \((CB(X_t), \text{dist}_{X_t}^{symm})\) is a complete metric space, where \(CB(X_t)\) is the collection of all nonempty closed and bounded subsets of \(X_t\).

**Assumption 2.1.** Let \((\Lambda, \rho(\cdot, \cdot))\) be a complete metric space, and \(\{U_{\lambda}(t, \tau)\}_{\lambda \in \Lambda}\) be a family of parameterized processes acting on time-dependent metric spaces \((X_t, d_t)\). Assume that

\((L_1)\) \(U_{\lambda}(t, \tau)\) has a pullback \(\mathcal{D}\)-attractor \(A_{\lambda} = \{A_{\lambda}(t)\}_{t \in \mathbb{R}}\) for each \(\lambda \in \Lambda\);

\((L_2)\) there exist a family \(B = \{B(t)\}_{t \in \mathbb{R}}\), with \(\emptyset \neq B(t) \subset X_t\), and a \(t_0 \in \mathbb{R}\) such that \(C(t) := \bigcup_{\lambda \in \Lambda} A_{\lambda}(t) \subset B(t), \forall t \leq t_0\), \(2.1\)

and \(A_{\lambda}\) pullback attracts \(B\), that is,
\[
\lim_{\tau \to -\infty} \text{dist}_{X_t}(U_{\lambda}(t, \tau)B(\tau), A_{\lambda}(t)) = 0, \forall t \in \mathbb{R}, \lambda \in \Lambda;
\]

\((L_3)\) for every \(t \in \mathbb{R}\) and \(s \leq \min\{t, t_0\},
\[
\lim_{\lambda \to \lambda_0} \max_{x \in B(s)} d_t(U_{\lambda}(t, s)x, U_{\lambda_0}(t, s)x) = 0, \forall \lambda_0 \in \Lambda.
\]

### 2.1 Upper semicontinuity of pullback \(\mathcal{D}\)-attractor

We show a criterion on the upper semicontinuity of pullback \(\mathcal{D}\)-attractor in time-dependent phase space in this subsection. For the related criterion on that in fixed metric space, i.e., \((X_t, d_t) \equiv (X, d), \forall t \in \mathbb{R}\), one can see [11].

**Theorem 2.1.** Let Assumption 2.1 hold. Then the family of pullback \(\mathcal{D}\)-attractors \(A_{\lambda} = \{A_{\lambda}(t)\}_{t \in \mathbb{R}}\) is upper semicontinuous at each point \(\lambda_0 \in \Lambda\), that is,
\[
\lim_{\lambda \to \lambda_0} \text{dist}_{X_t}(A_{\lambda}(t), A_{\lambda_0}(t)) = 0, \forall t \in \mathbb{R}.
\]

Moreover, if the metric space \(\Lambda\) is compact, then the family \(C = \{C(t)\}_{t \in \mathbb{R}}\) given by (2.1) possesses the following properties:

\((i)\) \(C(t)\) is compact in \(X_t\) for each \(t \in \mathbb{R}\);

\((ii)\) \(A_{\lambda}\) pullback attracts the family \(C\) for each \(\lambda \in \Lambda\), that is,
\[
\lim_{\tau \to -\infty} \text{dist}_{X_t}(U_{\lambda}(t, \tau)C(\tau), A_{\lambda}(t)) = 0, \forall t \in \mathbb{R}.
\]

**Proof.** It follows from condition \((L_2)\) that for any \(\lambda_0 \in \Lambda, t \in \mathbb{R}\) and \(\epsilon > 0\), there exists a \(T > 0\) such that \(t - T \leq t_0\) and
\[
\text{dist}_{X_t}(U_{\lambda_0}(t, t - T)B(t - T), A_{\lambda_0}(t)) < \frac{\epsilon}{2},
\]
which means
\[
U_{\lambda_0}(t, t - T)B(t - T) \subset \mathcal{O}_{t/2}^f(A_{\lambda_0}(t)).
\]

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By condition \((L_2)\) and the invariance of the pullback \(\mathcal{D}\)-attractor,

\[
A_\lambda(t) = U_\lambda(t, t - T) A_\lambda(t - T) \subset U_\lambda(t, t - T) B(t - T), \quad \forall \lambda \in \Lambda. \tag{2.4}
\]

By condition \((L_3)\),

\[
\lim_{\lambda \to \lambda_0} \sup_{x \in B(t - T)} d_t (U_\lambda(t, t - T) x, U_{\lambda_0}(t, t - T) x) = 0,
\]

which means that for the given \(\varepsilon > 0\), there exists a \(\delta = \delta(\varepsilon) > 0\) such that

\[
U_\lambda(t, t - T) B(t - T) \subset O(t) (U_{\lambda_0}(t, t - T) B(t - T)) \quad \text{as} \quad \rho(\lambda, \lambda_0) < \delta. \tag{2.5}
\]

The combination of \((2.3)-(2.5)\) shows that

\[
A_\lambda(t) \subset O(t) (U_{\lambda_0}(t, t - T) B(t - T)) \subset O(t) (A_{\lambda_0}(t)) \quad \text{as} \quad \rho(\lambda, \lambda_0) < \delta.
\]

That is, \(\text{dist}_{X_t} (A_\lambda(t), A_{\lambda_0}(t)) < \varepsilon\). Hence, the upper semicontinuity \((2.2)\) holds.

Moreover, for any sequence \(\{x_n\} \subset C(t) = \cup_{\lambda \in \Lambda} A_\lambda(t) \subset B(t)\), there exists a sequence \(\{\lambda_n\} \subset \Lambda\) such that \(x_n \in A_{\lambda_n}(t)\). Taking into account the compactness of \(\Lambda\), we have (subsequence if necessary) \(\lambda_n \to \lambda_0\) in \(\Lambda\). By the compactness of \(A_{\lambda_0}(t)\) in \(X_t\) and formula \((2.2)\), there exists a sequence \(\{y_n\} \subset A_{\lambda_0}(t)\) such that

\[
\lim_{n \to \infty} d_t (x_n, y_n) = \lim_{n \to \infty} \text{dist}_{X_t} (x_n, A_{\lambda_0}(t)) \leq \lim_{n \to \infty} \text{dist}_{X_t} (A_{\lambda_n}(t), A_{\lambda_0}(t)) = 0,
\]

and there exists a subsequence \(\{y_{n_k}\} \subset \{y_n\}\) such that

\[
y_{n_k} \to x_0 \in A_{\lambda_0}(t) \quad \text{in} \quad X_t.
\]

Therefore,

\[
\lim_{k \to \infty} d_t (x_{n_k}, x_0) \leq \lim_{k \to \infty} d_t (x_{n_k}, y_{n_k}) + \lim_{k \to \infty} d_t (y_{n_k}, x_0) = 0,
\]

which means that the set \(C(t)\) is compact in \(X_t\) for each \(t \in \mathbb{R}\).

Taking into account the fact that \(C(t) \subset B(t)\) for all \(t \leq t_0\) and that \(A_\lambda\) pullback attracts the family \(\mathcal{B}\), we have that \(A_\lambda\) pullback attracts the family \(\mathcal{C}\) for each \(\lambda \in \Lambda\).

\(\square\)

**Remark 2.2.** Theorem 2.1 implies that when \(\Lambda\) is a compact metric space, we can take the family \(\mathcal{B} = \{B(t)\}_{t \in \mathbb{R}} = \mathcal{C} = \{C(t)\}_{t \in \mathbb{R}}\) in conditions \((L_2) - (L_3)\), in other words, conditions \((L_2) - (L_3)\) hold on \(\mathcal{C}\).

### 2.2 Residual continuity of pullback \(\mathcal{D}\)-attractor

In this subsection, we show a residual continuity criterion on the pullback \(\mathcal{D}\)-attractor in time-dependent phase spaces. For the related criteria on the residual continuity of global, uniform and pullback attractors in fixed metric space, one can see [5, 27, 28]. For clarity, we first quote a definition of the residual subset.

**Definition 2.3.** A set is said to be nowhere dense if its closure contains no nonempty open sets. A set is said to be a residual set if its complement is a countable union of nowhere dense sets.

Any residual subset of the complete metric space \(\Lambda\) is dense in \(\Lambda\).
Theorem 2.4. Let Assumption 2.1 be valid and \(\{U_\lambda(t, \tau)\}_{\lambda \in \Lambda}\) be a family of continuous processes, i.e., \(U_\lambda(t, \tau) : X_\tau \to X_t\) is a continuous operator for each \(t \geq \tau\) and \(\lambda \in \Lambda\). Assume that either (i) the set \(B(t)\) is compact in \(X_t\) for all \(t \leq t_0\) or (ii) the metric space \(\Lambda\) is compact. Then there exists a residual subset \(\Lambda^*\) of \(\Lambda\) such that the pullback \(\mathscr{D}\)-attractor \(A_\lambda = \{A_\lambda(t)\}_{t \in \mathbb{R}}\) is continuous at each point \(\lambda_0 \in \Lambda^*\), i.e.,

\[
\lim_{\lambda \to \lambda_0} \text{dist}_{X_t}^{\text{symm}}(A_\lambda(t), A_{\lambda_0}(t)) = 0, \quad \forall t \in \mathbb{R}.
\] (2.6)

By Remark 2.2, when \(\Lambda\) is a compact metric space, the conditions \((L_2) \sim (L_3)\) of Assumption 2.1 hold on the family \(C = \{C(t)\}_{t \in \mathbb{R}}\) as shown in (2.1), and Theorem 2.1 shows that each section \(C(t)\) is compact in \(X_t\). Thus, it is enough to prove Theorem 2.4 in case (i) for one only needs to replace \(B\) there by \(C\) in case (ii). In order to prove Theorem 2.4, we need the following lemmas.

Lemma 2.5. [27] Let \(f_n : X \to Y\) be a continuous map for each \(n \in \mathbb{N}\), where \(X\) is a complete metric space and \(Y\) is a metric space. Assume that \(f\) is the pointwise limit of \(f_n\), i.e.,

\[
f(x) = \lim_{n \to \infty} f_n(x), \quad \forall x \in X.
\]

Then, there exists a residual subset \(\mathcal{R}\) of \(X\) such that \(f\) is continuous at every point \(x \in \mathcal{R}\).

Lemma 2.6. Let the assumptions of Theorem 2.4 be valid. Then the mapping \(\lambda \mapsto [U_\lambda(t, s)B(s)]_{X_t}\) is continuous from \(\Lambda\) into \(CB(X_t)\) for all \(t \in \mathbb{R}\) and \(s \leq \min\{t, t_0\}\).

Proof. For any \(\lambda \in \Lambda\), \(t \in \mathbb{R}\) and \(s \leq \min\{t, t_0\}\), it follows from the continuity of operator \(U_\lambda(t, s) : X_s \to X_t\) and the compactness of \(B(s)\) in \(X_s\) that

\[
U_\lambda(t, s)B(s) = [U_\lambda(t, s)B(s)]_{X_t} \in CB(X_t).
\]

Thus, the mapping \(\lambda \mapsto [U_\lambda(t, s)B(s)]_{X_t}\) is from \(\Lambda\) into \(CB(X_t)\). Repeating the similar argument as Lemma 3.1 in [27, 28], one easily obtains the continuity of this mapping. We omit the details here.

\(\square\)

Lemma 2.7. Let the assumptions of Theorem 2.4 be valid. Then the mapping \((\lambda, B) \mapsto U_\lambda(t, s)B\) is continuous at every point \((\lambda, B) \in \Lambda \times CB(B(s))\) for all \(t \in \mathbb{R}\) and \(s \leq \min\{t, t_0\}\).

Proof. It follows from condition \((L_3)\) that for any \((\lambda_0, B_0) \in \Lambda \times CB(B(s))\) and \(\varepsilon > 0\), there exists a \(\delta = \delta(\varepsilon) > 0\) such that

\[
\sup_{x \in B(s)} d_t(U_\lambda(t, s)x, U_{\lambda_0}(t, s)x) < \frac{\varepsilon}{2} \quad \text{as} \quad \rho(\lambda, \lambda_0) < \delta.
\] (2.7)

Since \(B(s)\) is compact in \(X_s\) and the operator \(U_\lambda(t, s) : X_s \to X_t\) is continuous, \(U_\lambda(t, s)\) is uniformly continuous on \(B(s)\) and \(U_\lambda(t, s)B\) is compact in \(X_t\) for all \((\lambda, B) \in \Lambda \times CB(B(s))\). Thus there exists a constant \(\delta_1 : 0 < \delta_1 < \delta\) such that for every \(x, y \in B(s)\) with \(d_s(x, y) < \delta_1\),

\[
d_t(U_{\lambda_0}(t, s)x, U_{\lambda_0}(t, s)y) < \frac{\varepsilon}{2}.
\] (2.8)

For any point \((\lambda, B) \in \Lambda \times CB(B(s))\) with

\[
\rho(\lambda, \lambda_0) < \delta_1 \quad \text{and} \quad \text{dist}_{X_s}^{\text{symm}}(B, B_0) < \delta_1,
\] (2.9)

it follows from the compactness of \(B_0\) in \(X_s\) that for every \(b \in B\), there exists the best approximating element \(b_0 \in B_0\) such that

\[
d_s(b, b_0) = \text{dist}_{X_s}(b, B_0) \leq \text{dist}_{X_s}^{\text{symm}}(B, B_0) < \delta_1.
\] (2.10)
The combination of (2.7)-(2.10) turns out
\[
\text{dist}_{X_0} (U_{\lambda}(t, s)b, U_{\lambda_0}(t, s)B_0) \\
\leq d_t (U_{\lambda}(t, s)b, U_{\lambda_0}(t, s)b) \\
\leq d_t (U_{\lambda}(t, s)b, U_{\lambda_0}(t, s)b) + d_t (U_{\lambda_0}(t, s)b, U_{\lambda_0}(t, s)b) \\
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
By the arbitrariness of \(b \in B\),
\[
\text{dist}_{X_0} (U_{\lambda}(t, s)B, U_{\lambda_0}(t, s)B_0) \leq \epsilon.
\]
On the other hand, by the compactness of \(B\) in \(X_0\) and the symmetry of \(\text{dist}^{\text{symm}}_{X_0} (B, B_0) < \delta_1\), we also have
\[
\text{dist}_{X_0} (U_{\lambda}(t, s)B_0, U_{\lambda_0}(t, s)B) \leq \epsilon.
\]
Therefore, for any \((\lambda, B) \in \Lambda \times CB(B(s))\), we arrive at
\[
\text{dist}^{\text{symm}}_{X_0} (U_{\lambda}(t, s)B, U_{\lambda_0}(t, s)B_0) \leq \epsilon \text{ as (2.9) holds.}
\]
By the arbitrariness of \((\lambda_0, B_0) \in \Lambda \times CB(B(s))\), we obtain the desired conclusion. \(\quad \Box\)

**Proof of Theorem 2.4.** It follows from condition (L2) and the invariance of pullback \(\mathcal{D}\)-attractor that
\[
A_{\lambda}(n) = U_{\lambda}(n, s)A_{\lambda}(s) \subset U_{\lambda}(n, s)B(s), \quad \forall \lambda \in \Lambda, \quad n \in \mathbb{Z}, \quad s \leq \min\{n, t_0\},
\]
which implies
\[
\text{dist}_{X_0} (A_{\lambda}(n), U_{\lambda}(n, s)B(s)) = 0, \quad \forall \lambda \in \Lambda, \quad n \in \mathbb{Z}, \quad s \leq \min\{n, t_0\}. \quad (2.11)
\]
While condition (L2) shows that
\[
\lim_{s \to -\infty} \text{dist}_{X_0} (U_{\lambda}(n, s)B(s), A_{\lambda}(n)) = 0, \quad \forall \lambda \in \Lambda, \quad n \in \mathbb{Z}. \quad (2.12)
\]
The combination of (2.11)-(2.12) yields
\[
\lim_{s \to -\infty} \text{dist}^{\text{symm}}_{X_0} (U_{\lambda}(n, s)B(s), A_{\lambda}(n)) = \lim_{s \to -\infty} \text{dist}_{X_0} (U_{\lambda}(n, s)B(s), A_{\lambda}(n)) = 0,
\]
that is,
\[
A_{\lambda}(n) = \lim_{s \to -\infty} U_{\lambda}(n, s)B(s), \quad \forall \lambda \in \Lambda, \quad n \in \mathbb{Z}. \quad (2.13)
\]
For every \(n \in \mathbb{Z}\) and \(s \leq \min\{n, t_0\}\), we define the mappings \(f^n_s, f^n : \Lambda \to CB(X_n)\),
\[
f^n_s (\lambda) = U_{\lambda}(n, s)B(s), \quad f^n (\lambda) = A_{\lambda}(n), \quad \forall \lambda \in \Lambda.
\]
Then formula (2.13) reads \(f^n (\lambda) = \lim_{s \to -\infty} f^n_s (\lambda)\). Lemma 2.6 shows that \(f^n (\lambda)\) is continuous from \(\Lambda\) into \(CB(X_n)\) for each \(n \in \mathbb{Z}\) and \(s \leq \min\{n, t_0\}\). Therefore, by Lemma 2.5, there exists a residual subset \(\Lambda_n \subset \Lambda\) for each \(n \in \mathbb{Z}\) such that the limiting function \(f^n\) is continuous on \(\Lambda_n\), that is,
\[
\lim_{\lambda \to \lambda_0} \text{dist}^{\text{symm}}_{X_n} (f^n (\lambda), f^n (\lambda_0)) = 0, \quad \forall \lambda_0 \in \Lambda_n, \quad n \in \mathbb{Z}. \quad (2.14)
\]
Let
\[
\Lambda^* = \bigcap_{n \in \mathbb{Z}} \Lambda_n.
\]
Obviously, $\Lambda^*$ is a residual subset of $\Lambda$ for the countable intersection of residual subsets is also a residual subset, and by (2.14)
\[
\lim_{\lambda \to \lambda_0} \operatorname{dist}^{symm}_{X_t} (A_\lambda(n), A_{\lambda_0}(n)) = 0, \quad \forall \lambda_0 \in \Lambda^*, \quad \forall n \in \mathbb{Z}.
\]
(2.15)

Take $n \leq \min\{t, t_0\}$ for any given $t \in \mathbb{R}$. By the invariance of $A_\lambda$, we have
\[
A_\lambda(t) = U_\lambda(t, n)A_\lambda(n), \quad \forall \lambda \in \Lambda.
\]
(2.16)

Taking into account $A_{\lambda_0}(n) \in CB(B(n)), \forall \lambda_0 \in \Lambda^*$, we infer from Lemma 2.7 that for any $\epsilon > 0$, there exists a $\delta > 0$ such that
\[
\operatorname{dist}^{symm}_{X_t} (U_\lambda(t, n)B, U_{\lambda_0}(t, n)A_{\lambda_0}(n)) < \epsilon
\]
whenever $(\lambda, B) \in \Lambda \times CB(B(n))$ and $\rho(\lambda, \lambda_0) + \operatorname{dist}^{symm}_{X_n}(B, A_{\lambda_0}(n)) < \delta$. Formula (2.15) means that there exists a constant $\delta_1 : 0 < \delta_1 < \delta/2$ such that
\[
\operatorname{dist}^{symm}_{X_n} (A_\lambda(n), A_{\lambda_0}(n)) < \delta/2 \quad \text{as} \quad \rho(\lambda, \lambda_0) < \delta_1.
\]
(2.17)

The combination of (2.16)-(2.18) arrives at
\[
\operatorname{dist}^{symm}_{X_t} (A_\lambda(t), A_{\lambda_0}(t)) = \operatorname{dist}^{symm}_{X_t} (U_\lambda(t, n)A_\lambda(n), U_{\lambda_0}(t, n)A_{\lambda_0}(n)) < \epsilon
\]
whenever $\lambda \in \Lambda$ and $\rho(\lambda, \lambda_0) < \delta_1$ for $A_\lambda(n) \in CB(B(n))$. That is, the mapping $\lambda \mapsto A_\lambda(t)$ is continuous on $\Lambda^*$ for all $t \in \mathbb{R}$.

\[\square\]

2.3 The equivalence between the continuity and the equi-attraction

In this section, motivated by the idea in [27, 28], we show that the continuity of pullback $\mathcal{D}$-attractor w.r.t. the parameter $\lambda \in \Lambda$ (not only the residual subset $\Lambda^*$) is equivalent to the pullback equi-attraction under some assumptions. For the related equivalence criterion on that of global, pullback and uniform attractors, one can see [27, 28], and for a comprehensive summary on this topic, one can see [31, 32].

Theorem 2.8. Let the assumptions of Theorem 2.4 be valid.

(i) If the family $\mathcal{B} = \{B(t)\}_{t \in \mathbb{R}}$ as shown in condition (L2) satisfies the pullback equi-attraction at time $t \in \mathbb{R}$, that is,
\[
\lim_{s \to -\infty} \sup_{\lambda \in \Lambda} \operatorname{dist}_{X_t} (U_\lambda(t, s)B(s), A_\lambda(t)) = 0,
\]
(2.19)
then the mapping $\lambda \mapsto A_\lambda(t)$ is continuous on $\Lambda$, that is,
\[
\lim_{\lambda \to \lambda_0} \operatorname{dist}^{symm}_{X_t} (A_\lambda(t), A_{\lambda_0}(t)) = 0, \quad \forall \lambda_0 \in \Lambda.
\]

(ii) Assume that $\Lambda$ is compact and there is a function $\gamma(t)$, with $\gamma(t) \leq \min\{t, t_0\}$, such that
\[
\bigcup_{\lambda \in \Lambda} U_\lambda(t, s)B(s) \subset B(t), \quad \forall s \leq \gamma(t).
\]
(2.20)

If the mapping $\lambda \mapsto A_\lambda(t)$ is continuous on $\Lambda$, then the pullback equi-attraction (2.19) holds.

In order to prove Theorem 2.8, we need the following Dini’s theorem.

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Lemma 2.9. (Theorem 4.1 in [27]) Let $K$ be a compact metric space, $Y$ be a metric space, and $f_n : K \rightarrow Y$ be a continuous mapping for each $n \in \mathbb{N}$. Assume that $f_n$ converges to a continuous function $f : K \rightarrow Y$ as $n \rightarrow \infty$ in the following monotonic way

$$
d_Y (f_{n+1}(x), f(x)) \leq d_Y (f_n(x), f(x)), \ \forall x \in K, \ \forall n \geq 1.
$$

Then $f_n$ converges to $f$ uniformly on $K$ as $n \rightarrow \infty$.

Proof of Theorem 2.8. (i) By condition $(L_2)$ and the invariance of pullback $\mathcal{D}$-attractor,

$$A_\lambda(t) = U_\lambda(t, s)A_\lambda(s) \subset U_\lambda(t, s)B(s), \ \forall s \leq \min\{t, t_0\}, t \in \mathbb{R}, \ \lambda \in \Lambda,$$

which implies

$$\sup_{\lambda \in \Lambda} \text{dist}_{X_t} (A_\lambda(t), U_\lambda(t, s)B(s)) = 0, \ \forall s \leq \min\{t, t_0\}. \hspace{1cm} (2.21)$$

The combination of (2.19) and (2.21) turns out

$$\lim_{s \rightarrow -\infty} \sup_{\lambda \in \Lambda} \text{dist}_{X_t}^{symm} (A_\lambda(t), U_\lambda(t, s)B(s)) = \lim_{s \rightarrow -\infty} \sup_{\lambda \in \Lambda} \text{dist}_{X_t} (U_\lambda(t, s)B(s), A_\lambda(t)) = 0,$$

which means that for any $\epsilon > 0$, there exists a $s_0 \leq \min\{t, t_0\}$ such that

$$\sup_{\lambda \in \Lambda} \text{dist}_{X_t}^{symm} (A_\lambda(t), U_\lambda(t, s_0)B(s_0)) < \frac{\epsilon}{3}. \hspace{1cm} (2.22)$$

Lemma 2.6 shows that for any $\lambda_0 \in \Lambda$, there exists a $\delta > 0$ such that

$$\text{dist}_{X_t}^{symm} (U_\lambda(t, s_0)B(s_0), U_{\lambda_0}(t, s_0)B(s_0)) < \frac{\epsilon}{3} \text{ as } \rho(\lambda, \lambda_0) < \delta. \hspace{1cm} (2.23)$$

The combination of (2.22)-(2.23) yields

$$\text{dist}_{X_t}^{symm} (A_\lambda(t), A_{\lambda_0}(t)) \leq \text{dist}_{X_t}^{symm} (A_\lambda(t), U_\lambda(t, s_0)B(s_0))$$

$$+ \text{dist}_{X_t}^{symm} (U_\lambda(t, s_0)B(s_0), U_{\lambda_0}(t, s_0)B(s_0))$$

$$+ \text{dist}_{X_t}^{symm} (U_{\lambda_0}(t, s_0)B(s_0), A_{\lambda_0}(t))$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \text{ as } \rho(\lambda, \lambda_0) < \delta,$$

that is, the mapping $\lambda \mapsto A_\lambda(t)$ is continuous at $\lambda_0$. By the arbitrariness of $\lambda_0 \in \Lambda$ we obtain that the mapping $\lambda \mapsto A_\lambda(t)$ is continuous on $\Lambda$.

(ii) If formula (2.20) holds, then for any given $t \in \mathbb{R}$, we take $s_0 = \gamma(t) - 1 \leq \min\{t, t_0\} - 1$ and

$$s_n = \gamma(s_{n-1}) - 1 \leq \min\{s_{n-1}, t_0\} - 1, \ \forall n \geq 1.$$ 

Obviously, the sequence $\{s_n\}_{n=0}^{\infty}$ is strictly decreasing and $\lim_{n \rightarrow \infty} s_n = -\infty$. By formula (2.20),

$$U_\lambda(t, s_{n+1})B(s_{n+1}) = U_\lambda(t, s_n)U_\lambda(s_n, s_{n+1})B(s_{n+1}) \subset U_\lambda(t, s_n)B(s_n),$$

and hence,

$$\text{dist}_{X_t} (U_\lambda(t, s_{n+1})B(s_{n+1}), A_\lambda(t)) \leq \text{dist}_{X_t} (U_\lambda(t, s_n)B(s_n), A_\lambda(t)), \ \forall \lambda \in \Lambda, \ n \geq 1. \hspace{1cm} (2.24)$$
The combination of (2.21) and (2.24) shows that
\[ \text{dist}_{X_t}^{symm} (U_\lambda(t, s_{n+1})B(s_{n+1}), A_\lambda(t)) \leq \text{dist}_{X_t}^{symm} (U_\lambda(t, s_n)B(s_n), A_\lambda(t)), \quad \forall \lambda \in \Lambda, \quad n \geq 1. \tag{2.25} \]

Let the mappings \( f^t_n, f^t : \Lambda \to CB(X_t) \),
\[ f^t_n(\lambda) = U_\lambda(t, s_n)B(s_n), \quad f^t(\lambda) = A_\lambda(t), \quad \forall \lambda \in \Lambda, \quad n \geq 1. \]

Lemma 2.6 shows that \( f^t_n : \Lambda \to CB(X_t) \) is continuous for each \( n \geq 1 \), and formula (2.25) reads
\[ \text{dist}_{X_t}^{symm} \left( f^t_{n+1}(\lambda), f^t(\lambda) \right) \leq \text{dist}_{X_t}^{symm} \left( f^t_n(\lambda), f^t(\lambda) \right), \quad \forall \lambda \in \Lambda, \quad n \geq 1. \]

By the continuity of the mapping \( f^t : \Lambda \to CB(X_t) \) and Lemma 2.9 we know that \( f^t_n \) converges to \( f^t \) uniformly on \( \Lambda \) as \( n \to \infty \), that is,
\[ \lim_{n \to \infty} \sup_{\lambda \in \Lambda} \text{dist}_{X_t}^{symm} (U_\lambda(t, s_n)B(s_n), A_\lambda(t)) = 0. \tag{2.26} \]

For any \( s \in (s_{n+2}, s_{n+1}) \), we have that \( s < s_{n+1} \leq \gamma(s_n) \leq \min\{s_n, t_0\} \). Thus it follows from formula (2.20) that
\[ U_\lambda(t, s)B(s) = U_\lambda(t, s_n)U_\lambda(s_n, s)B(s) \subset U_\lambda(t, s_n)B(s_n), \quad \forall \lambda \in \Lambda, \]
and hence,
\[ \text{dist}_{X_t} (U_\lambda(t, s)B(s), A_\lambda(t)) \leq \text{dist}_{X_t} (U_\lambda(t, s_n)B(s_n), A_\lambda(t)) \]
\[ \leq \text{dist}_{X_t}^{symm} (U_\lambda(t, s_n)B(s_n), A_\lambda(t)), \quad \forall \lambda \in \Lambda. \]

Therefore, we infer from formula (2.26) that
\[ \lim_{s \to -\infty} \sup_{\lambda \in \Lambda} \text{dist}_{X_t} (U_\lambda(t, s)B(s), A_\lambda(t)) \leq \lim_{n \to \infty} \sup_{\lambda \in \Lambda} \text{dist}_{X_t}^{symm} (U_\lambda(t, s_n)B(s_n), A_\lambda(t)) = 0. \]

This completes the proof. \( \square \)

3 Continuity of the pullback \( \mathcal{D} \)-exponential attractors

The purpose of this section is to show an existence and continuity criterion on the pullback \( \mathcal{D} \)-exponential attractors, which can be seen as an extension of the related criterion on the pullback exponential attractors in fixed metric space (cf. [44]). We first give a proper notion of the pullback \( \mathcal{D} \)-exponential attractor.

**Definition 3.1.** A family \( \mathcal{E} = \{E(t)\}_{t \in \mathbb{R}} \) is called a pullback \( \mathcal{D} \)-exponential attractor of the process \( U(t, \tau) : X_{\tau} \to X_t \), if

(i) each section \( E(t) \) is compact in \( X_t \) and the fractal dimension of \( E(t) \) in \( X_t \) is uniformly bounded, that is,
\[ \sup_{t \in \mathbb{R}} \text{dim}_f (E(t), X_t) < +\infty; \]

(ii) \( \mathcal{E} \) is semi-invariant, that is, \( U(t, \tau)E(\tau) \subset E(t) \) for all \( t \geq \tau; \)

(iii) there exists a positive constant \( \beta \) such that
\[ \text{dist}_{X_t} (U(t, t - \tau)D(t - \tau), E(t)) \leq C(\mathcal{D}, t) \exp^{-\beta \tau}, \quad \forall \tau \geq \tau_0(\mathcal{D}, t), \]
for all \( t \in \mathbb{R} \) and \( D \in \mathcal{D} \), where \( C(\mathcal{D}, t) \) and \( \tau_0(\mathcal{D}, t) \) are positive constants depending only on \( \mathcal{D} \) and \( t \).
**Assumption 3.1.** Let \((\Lambda, \rho(\cdot, \cdot))\) be a complete metric space and \(\{U_\lambda(t, \tau)\}_{\lambda \in \Lambda}\) be a parameterized family of processes acting on time-dependent normed linear spaces \(\{X_t\}_{t \in \mathbb{R}}\). Suppose that

(H1) There exist a time-dependent family \(\mathcal{B} = \{B(t)\}_{t \in \mathbb{R}}\) and positive constants \(T, R_0\) such that

\[
B(t) \text{ is closed in } X_t \text{ and } B(t) \subseteq B_t(R_0), \quad \forall t \in \mathbb{R},
\]

\[
\cup_{\lambda \in \Lambda} U_\lambda(t, t - \tau)B(t - \tau) \subseteq B(t), \quad \forall \tau \geq T, t \in \mathbb{R},
\]

hereafter, \(B_t(R_0) = B_t(0; R_0)\) is the \(R_0\)-ball of \(X_t\) centered at 0.

(H2) (Quasi-stability) There exist a Banach space \(Z\) with the compact seminorm \(n_Z(\cdot)\) and a \(t_0 \in \mathbb{R}\) such that

\[
\|U_\lambda(t, t - T)x - U_\lambda(t, t - T)y\|_{X_t} \leq \eta\|x - y\|_{X_{t - T}} + n_Z(K^\lambda_t x - K^\lambda_t y)
\]

(3.2)

for all \(x, y \in B(t - T), \lambda \in \Lambda\) and \(t \leq t_0\), where \(\eta \in (0, 1/2)\) and the mapping \(K^\lambda_t : B(t - T) \to Z\) is uniformly Lipschitz continuous, that is,

\[
\sup_{\lambda \in \Lambda} \|K^\lambda_t x - K^\lambda_t y\|_Z \leq L\|x - y\|_{X_{t - T}}, \quad \forall x, y \in B(t - T), t \leq t_0,
\]

for some constant \(L > 0\).

(H3) (Lipschitz continuity) There exists a uniform Lipschitz constant \(L_1 \geq 2\) such that

\[
\sup_{\lambda \in \Lambda} \|U_\lambda(t, t - \tau)x - U_\lambda(t, t - \tau)y\|_{X_t} \leq L_1\|x - y\|_{X_{t - \tau}},
\]

(3.4)

for all \(x, y \in B(t - \tau), \tau \in [0, T]\) and \(t \in \mathbb{R}\).

For any given \(\lambda_0 \in \Lambda\), we define the function

\[
\Gamma(\lambda, \lambda_0) := \sup_{t \leq t_0} \sup_{s \in [0, T]} \sup_{x \in B(t - s)} \|U_\lambda(t, t - s)x - U_{\lambda_0}(t, t - s)x\|_{X_t}, \quad \forall \lambda \in \Lambda,
\]

(3.5)

and the set

\[
\Lambda_0 := \{\lambda \in \Lambda \mid 0 < \Gamma(\lambda, \lambda_0) < 1\}.
\]

**Theorem 3.2.** Let Assumption 3.1 be valid. Then for any given \(\lambda_0 \in \Lambda\), there is a semi-invariant family \(\mathcal{E}_\lambda = \{E_\lambda(t)\}_{t \in \mathbb{R}}\), with \(\lambda \in \Lambda\), possessing the following properties:

(i) each section \(E_\lambda(t) \subset B(t)\) is compact in \(X_t\) and its fractal dimension in \(X_t\) is uniformly bounded, that is,

\[
\sup_{\lambda \in \Lambda} \sup_{t \in \mathbb{R}} \dim_f(E_\lambda(t), X_t) \leq \left[\ln\left(\frac{1}{2\eta}\right)\right]^{-1} \ln m_Z\left(\frac{2L}{\eta}\right) < \infty,
\]

where \(m_Z(R)\) is the maximal number of elements \(z_i\) in the ball \(\{z \in Z \mid \|z\|_Z \leq R\}\) such that \(n_Z(z_i - z_j) > 1, i \neq j\);

(ii) \(\mathcal{E}_\lambda\) pullback attracts the family \(\mathcal{B}\) at an exponential rate, that is,

\[
dist_{X_t}(U_\lambda(t, t - \tau)B(t - \tau), E_\lambda(t)) \leq C(t)e^{-\beta t}, \quad \forall \tau \geq \tau_t, \ t \in \mathbb{R},
\]

(3.7)

where \(\beta, C(t)\) and \(\tau_t\) are positive constants independent of \(\lambda \in \Lambda\);
(iii) \( \mathcal{E}_\lambda \) is continuous at the point \( \lambda_0 \) in the following sense,

\[
\text{dist}_{X_t}^{\text{sym}} (E_\lambda(t), E_{\lambda_0}(t)) \leq C_1(t) \Gamma_t(\lambda, \lambda_0), \quad \forall t \in \mathbb{R}, \lambda \in \Lambda_0,
\]

where \( C_1(t) \) is a positive constant depending only on \( t \) and

\[
\Gamma_t(\lambda, \lambda_0) = \begin{cases} 
[\Gamma(\lambda, \lambda_0)]^\gamma, & t \leq t_0, \\
[\Gamma(\lambda, \lambda_0)]^\gamma + \sup_{x \in B(n_\ast T)} \|U_\lambda(t, n_\ast T)x - U_{\lambda_0}(t, n_\ast T)x\|_{X_t}, & t > t_0,
\end{cases}
\]

with \( \gamma \in (0, 1) \), \( n_\ast \in \mathbb{Z} \) such that \( t_0 - n_\ast T \in [0, T) \).

However, we can not ensure that the family \( \mathcal{E}_\lambda = \{ E_\lambda(t) \}_{t \in \mathbb{R}} \) as shown in Theorem 3.2 is the pullback \( \mathcal{D} \)-exponential attractor of the process \( U_\lambda(t, \tau) \) as it possesses the semi-invariance, the compactness and the boundedness of the fractal dimension because it only pullback attracts the family \( B \) (rather than every family \( \mathcal{D} \in \mathcal{D} \)) at an exponential rate. In order to guarantee that the family \( \mathcal{E}_\lambda \) is exactly the desired pullback \( \mathcal{D} \)-exponential attractor, we give the following two corollaries with additional assumptions. We first introduce the definition of pullback \( \mathcal{D} \)-absorbing family which will be used here.

**Definition 3.3. (Pullback \( \mathcal{D} \)-absorbing family)** A family of nonempty sets \( B = \{ B(t) \}_{t \in \mathbb{R}} \) is called a pullback \( \mathcal{D} \)-absorbing family of the process \( U(t, \tau) : X_\tau \to X_t \), if for any \( t \in \mathbb{R} \) and any \( \mathcal{D} \in \mathcal{D} \), there exists a \( \tau_0(t, \mathcal{D}) \leq t \) such that

\[
U(t, \tau)D(\tau) \subset B(t) \quad \text{for} \quad \tau \leq \tau_0(t, \mathcal{D}).
\]

In particular, if for any \( \mathcal{D} \in \mathcal{D} \), there exists a constant \( e(\mathcal{D}) > 0 \) such that

\[
U(t, \tau)D(\tau) \subset B(t), \quad \forall \tau \leq t - e(\mathcal{D}), \quad t \in \mathbb{R},
\]

then \( B \) is called a uniformly pullback \( \mathcal{D} \)-absorbing family.

**Corollary 3.4.** Let Assumption 3.1 be valid, and the family \( B = \{ B(t) \}_{t \in \mathbb{R}} \) as shown in (H1) is a uniformly pullback \( \mathcal{D} \)-absorbing family of the process \( U_\lambda(t, \tau) \). Then the family \( \mathcal{E}_\lambda = \{ E_\lambda(t) \}_{t \in \mathbb{R}} \) given by Theorem 3.2 is a pullback \( \mathcal{D} \)-exponential attractor of the process \( U_\lambda(t, \tau) \).

**Proof.** By Theorem 3.2, it is enough to prove that \( \mathcal{E}_\lambda \) pullback attracts every family \( \mathcal{D} \subset \mathcal{D} \) at an exponential rate. For any \( \mathcal{D} \in \mathcal{D} \), there exists a constant \( e(\mathcal{D}) > 0 \) such that

\[
U(t, t - \tau)D(t - \tau) \subset B(t), \quad \forall \tau \geq e(\mathcal{D}), \quad \forall t \in \mathbb{R}.
\]

Then it follows from (3.7) that

\[
\text{dist}_{X_t}(U_\lambda(t, t - \tau)D(t - \tau), E_\lambda(t)) \\
\leq \text{dist}_{X_t}(U_\lambda(t, t - \tau + e(\mathcal{D}))U_\lambda(t - \tau + e(\mathcal{D}), t - \tau)D(t - \tau), E_\lambda(t)) \\
\leq \text{dist}_{X_t}(U_\lambda(t, t - \tau + e(\mathcal{D})))B(t - \tau + e(\mathcal{D})), E_\lambda(t)) \\
\leq C(t)e^{\beta e(\mathcal{D})}e^{-\beta \tau}, \quad \forall t \in \mathbb{R}, \quad \tau \geq e(\mathcal{D}) + \tau_t.
\]

This completes the proof.

**Corollary 3.5.** Let Assumption 3.1 be valid, and the process \( U_\lambda(t, \tau) \) also possess a uniformly pullback \( \mathcal{D} \)-absorbing family \( \mathcal{D}_0 = \{ D_0(t) \}_{t \in \mathbb{R}} \in \mathcal{D} \) satisfying the following conditions:
(i) there are positive constants \( \kappa, \tau_1 \) and \( C_0 \) such that
\[
\operatorname{dist}_{X_t}(U_\lambda(t, t - \tau)D_0(t - \tau), B(t)) \leq C_0 e^{-\kappa \tau}, \quad \forall t \in \mathbb{R}, \ \tau \geq \tau_1; \tag{3.8}
\]

(ii) there is a positive constant \( \mathcal{R} > R_0 \) such that \( D_0(t) \subset \mathbb{B}_t(\mathcal{R}) \) for all \( t \in \mathbb{R} \) and
\[
\|U_\lambda(t, t - \tau)x - U_\lambda(t, t - \tau)y\|_{X_t} \leq C_1 e^{\gamma \tau} \|x - y\|_{X_{t-\tau}}, \tag{3.9}
\]
for all \( x, y \in \mathbb{B}_{t-\tau}(\mathcal{R}), \ \tau \geq 0 \) and \( t \in \mathbb{R} \), where \( C_1 \) and \( \gamma \) are positive constants depending only on \( \mathcal{R} \).

Then the family \( \mathcal{E}_\lambda = \{E_\lambda(t)\}_{t \in \mathbb{R}} \) given by Theorem 3.2 is a pullback \( \mathcal{D} \)-exponential attractor of the process \( U_\lambda(t, \tau) \).

**Proof.** For any given \( t \in \mathbb{R} \), we take
\[
\theta = \frac{\kappa}{2(\gamma + \kappa)} \quad \text{and} \quad \beta' = \min \left\{ \frac{\kappa}{2}, \frac{\kappa \beta}{2(\gamma + \kappa)} \right\}.
\]
A simple calculation shows that
\[
\theta \in (0, 1), \quad \beta' > 0, \quad \gamma \theta + \kappa \theta - \kappa = -\frac{\kappa}{2}.
\]

Due to \( D_0 \in \mathcal{D} \), by Definition 3.3, there exists a positive constant \( \epsilon_1 > \tau_1 \) such that
\[
U_\lambda(t - \tau \theta, t - \tau)D_0(t - \tau) \subset D_0(t - \tau) \subset \mathbb{B}_{t-\tau}(\mathcal{R})
\]
for all \( \tau \geq (1 - \theta)^{-1} \epsilon_1 \) and \( t \in \mathbb{R} \). Thus it follows from formulas (3.7)-(3.9) and the fact (see \( (H_1) \))
\[
B(t - \tau) \subset \mathbb{B}_{t-\tau}(R_0) \subset \mathbb{B}_{t-\tau}(\mathcal{R})
\]
that
\[
\operatorname{dist}_{X_t}(U_\lambda(t, t - \tau)D_0(t - \tau), E_\lambda(t)) \\
\leq \operatorname{dist}_{X_t}(U_\lambda(t, t - \tau \theta)U_\lambda(t - \tau \theta, t - \tau)D_0(t - \tau), U_\lambda(t, t - \tau \theta)B(t - \tau \theta)) \\
+ \operatorname{dist}_{X_t}(U_\lambda(t, t - \tau \theta)B(t - \tau \theta), E_\lambda(t)) \\
\leq C_1 e^{\gamma \tau \theta} \operatorname{dist}_{X_{t-\tau \theta}}(U_\lambda(t - \tau \theta, t - \tau)D_0(t - \tau), B(t - \tau \theta)) + C(1) e^{-\beta' \tau \theta} \\
\leq C_1 C_0 e^{(\gamma \theta + \kappa \theta - \kappa) \tau} + C(1) e^{-\beta' \tau \theta} \\
\leq C(1) e^{-\beta' \tau \theta}, \quad \forall \tau \geq \max\{\theta^{-1} \tau_1, (1 - \theta)^{-1} \epsilon_1\}, \ t \in \mathbb{R}.
\]

That is, \( \mathcal{E}_\lambda \) pullback attracts \( D_0 \) at an exponential rate. Then repeating the same proof as Corollary 3.4 (replacing \( B(t) \) there by \( D_0(t) \)), we obtain that \( \mathcal{E}_\lambda \) is a pullback \( \mathcal{D} \)-exponential attractor of the process \( U_\lambda(t, \tau) \). \( \square \)

**Proof of Theorem 3.2.** For clarity, we divide the proof into four steps.

**Step 1.** (Construction of the family \( \{E_\lambda(n)\}_{n \in \mathbb{N}} \), with \( \lambda \in \Lambda \setminus \Lambda_0 \)) For simplicity and without loss of generality, we take \( T = 1 \). It follows from Assumption 3.1 that for each \( \lambda \in \Lambda \), the discrete process \( U_\lambda(m, n) : X_n \rightarrow X_m \) possesses the following properties:
\[
\bigcup_{\lambda \in \Lambda} U_\lambda(m, n)B(n) \subset B(m), \quad \forall m \geq n \in \mathbb{Z}, \tag{3.11}
\]
\[
\sup_{\lambda \in \Lambda} \|U_\lambda(n, n - 1)x - U_\lambda(n, n - 1)y\|_{X_n} \leq L_1 \|x - y\|_{X_{n-1}}, \quad \forall x, y \in B(n - 1), \ n \in \mathbb{Z}, \tag{3.12}
\]
\[ \| U_\lambda(n, n-1)x - U_\lambda(n, n-1)y \|_{X_n} \leq \eta \| x - y \|_{X_{n-1}} + n Z_k \left( K^k_n x - K^k_n y \right), \quad (3.13) \]

\[ \sup_{\lambda \in \Lambda} \| K^k_n x - K^k_n y \| \leq L \| x - y \|_{X_{n-1}}, \quad \forall x, y \in B(n-1), n \leq n^*, \quad (3.14) \]

where \( n^* \in \mathbb{Z} \) satisfies \( t_0 - n^* \in [0, 1) \).

Repeating the similar arguments as Theorem 2.5 in [33] (or Theorem 2.3 in [44]), one easily infers from formulas (3.11)-(3.14) that

\[ N_n(k) := N_n \left( U_\lambda(n, n-k)B(n-k), (2\eta)^k R_0 \right) \leq \left[ m_Z \left( \frac{2L}{\eta} \right) \right]^k, \quad \forall n \leq n^*, k \geq 1, \lambda \in \Lambda, \quad (3.15) \]

where \( N_n(B, \epsilon) \) denotes the cardinality of minimal covering of the set \( B \subset X_n \) by the closed subsets of \( X_n \) with diameter \( \leq 2\epsilon \).

It follows from (3.15) and (3.11) that there exists a finite subset \( V_k^\lambda(n) \) of \( X_n \) for each \( \lambda \in \Lambda, n \leq n^* \) and \( k \geq 1 \), which possesses the following properties:

\[ \text{Card} \left( V_k^\lambda(n) \right) \leq \left[ m_Z \left( \frac{2L}{\eta} \right) \right]^k, \quad (3.16) \]

\[ V_k^\lambda(n) \subset U_\lambda(n, n-k)B(n-k) \subset B(n) \subset X_n, \quad (3.17) \]

\[ U_\lambda(n, n-k)B(n-k) \subset \bigcup_{h \in V_k^\lambda(n)} B_n(h; 2R_0(2\eta)^k). \quad (3.18) \]

Given \( \lambda_0 \in \Lambda \), for every \( \lambda \in \Lambda \setminus \Lambda_0 \) and \( n \leq n^* \), we define by induction the sets

\[
\begin{cases}
E_1^\lambda(n) = V_1^\lambda(n), \\
E_k^\lambda(n) = V_k^\lambda(n) \cup U_\lambda(n, n-k)E_{k-1}^\lambda(n-1), \quad k \geq 2, \\
E_\lambda(n) = \left[ \bigcup_{k \geq 1} E_k^\lambda(n) \right]_{\mathbb{X}_n}.
\end{cases}
\]

It follows from (3.11) and (3.17)-(3.18) that

\[ E_k^\lambda(n) = \bigcup_{l=0}^{k-1} U_\lambda(n, n-l)V_{k-1}^\lambda(n-l) \subset U_\lambda(n, n-k)B(n-k), \quad (3.20) \]

\[ U_\lambda(n+1, n)E_k^\lambda(n) \subset E_{k+1}^\lambda(n+1), \quad (3.21) \]

\[ E_\lambda(n) \subset U_\lambda(n, n-1)B(n-1) \subset B(n), \quad (3.22) \]

for all \( \lambda \in \Lambda \setminus \Lambda_0, n \leq n^*, k \geq 1 \). Moreover, we infer from (3.16) and (3.20) that

\[ \text{Card} \left( E_k^\lambda(n) \right) \leq \sum_{l=0}^{k-1} \text{Card} \left( V_{k-1}^\lambda(n-l) \right) \leq \left[ m_Z \left( \frac{2L}{\eta} \right) \right]^{k+1}, \quad \forall \lambda \in \Lambda \setminus \Lambda_0, n \leq n^*, k \geq 1. \quad (3.23) \]

**Step 2.** (The properties of the family \( \{ E_\lambda(n) \}_{n \leq n^*} \), with \( \lambda \in \Lambda \setminus \Lambda_0 \)) We show that, for every \( \lambda \in \Lambda \setminus \Lambda_0 \), the family \( \{ E_\lambda(n) \}_{n \leq n^*} \) is of the following properties:
(i) (Semi-invariance) It follows from formulas (3.12), (3.19) and (3.21) that
\[ U_\lambda(n, l)E_\lambda(l) \subset \left[ \bigcup_{k \geq 1} U_\lambda(n, l)E_k^\lambda(l) \right] \subset \left[ \bigcup_{k \geq 1} E_k^\lambda_{n-l}(n) \right] \subset E_\lambda(n), \quad \forall l \leq n \leq n_*. \quad (3.24) \]

(ii) (Pullback exponential attractiveness) Formula (3.19) shows \( V_k^\lambda(n) \subset E_\lambda(n) \) for all \( n \leq n_* \) and \( k \geq 1 \). Thus by (3.18), we have
\[ \text{dist}_{X_n} (U_\lambda(n, n - k)B(n - k), E_\lambda(n)) \leq \text{dist}_{X_n} \left( U_\lambda(n, n - k)B(n - k), V_k^\lambda(n) \right) \leq 2(2\eta)^k R_0, \quad \forall k \geq 1, \ n \leq n_. \quad (3.25) \]

(iii) (The compactness and the boundedness of the fractal dimension) For any \( \epsilon \in (0, 1) \), there exists unique \( k_\epsilon \in \mathbb{N} \) such that
\[ 2(2\eta)^{k_\epsilon} R_0 < \epsilon \leq 2(2\eta)^{k_\epsilon-1} R_0. \quad (3.26) \]
Obviously, \( \lim_{\epsilon \to 0^+} k_\epsilon = +\infty \). It follows from (3.11) and (3.20) that
\[ E_k^\lambda(n) \subset U_\lambda(n, n - k)B(n - k) \subset U_\lambda(n, n - k_\epsilon)B(n - k_\epsilon), \quad \forall k \geq k_\epsilon, \ n \leq n_, \]
which implies
\[ E_\lambda(n) \subset \left( \bigcup_{k < k_\epsilon} E_k^\lambda(n) \right) \bigcup \left[ U_\lambda(n, n - k_\epsilon)B(n - k_\epsilon) \right]_{X_n}, \quad \forall n \leq n_. \]
Thus by (3.15), (3.23) and (3.26), we obtain
\[ N_n (E_\lambda(n), \epsilon) \leq N_n \left( E_\lambda(n), 2(2\eta)^{k_\epsilon} R_0 \right) \leq \sum_{k < k_\epsilon} \text{Card} \left( E_k^\lambda(n) \right) + N_n \left( U_\lambda(n, n - k_\epsilon)B(n - k_\epsilon), 2(2\eta)^{k_\epsilon} R_0 \right) \leq \sum_{k < k_\epsilon} \left[ mZ \left( \frac{2L}{\eta} \right) \right]^{k+1} + N_n(k_\epsilon) \leq 2 \left[ mZ \left( \frac{2L}{\eta} \right) \right]^{k_\epsilon+1} < +\infty. \quad (3.27) \]
Formula (3.27) shows that \( E_\lambda(n) \) is a compact subset of \( X_n \) for the arbitrariness of \( \epsilon \in (0, 1) \). Moreover, by virtue of estimates (3.26)-(3.27) and a simple calculation, we have
\[ \frac{\ln N_n (E_\lambda(n), \epsilon)}{\ln (1/\epsilon)} \leq \frac{(k_\epsilon + 1) \ln \left[ mZ \left( \frac{2L}{\eta} \right) \right] + \ln 2}{(k_\epsilon - 1) \ln (1/2\eta) - \ln (2R_0)}, \quad \forall \epsilon \in (0, 1), \]
which implies
\[ \dim_f (E_\lambda(n); X_n) = \limsup_{\epsilon \to 0^+} \frac{\ln N_n (E_\lambda(n), \epsilon)}{\ln (1/\epsilon)} \leq \left[ \ln \left( \frac{1}{2\eta} \right) \right]^{-1} \ln \left[ mZ \left( \frac{2L}{\eta} \right) \right], \quad \forall n \leq n_. \quad (3.28) \]

Step 3. (The Hölder continuity of the family \( \{ E_\lambda(n) \}_{n \leq n_*} \) at the point \( \lambda_0 \)) Taking into account \( \lambda_0 \in \Lambda \setminus \Lambda_0 \), by (3.20) we have \( E_{k_0}^\lambda(n) \subset U_{\lambda_0}(n, n - k)B(n - k) \) for all \( k \geq 1 \) and \( n \leq n_* \), so there must be a subset \( \tilde{E}_k(n - k) \subset B(n - k) \) satisfying
\[ U_{\lambda_0}(n, n - k)\tilde{E}_k(n - k) = E_{k_0}^\lambda(n), \quad \text{Card} \left( \tilde{E}_k(n - k) \right) = \text{Card} \left( E_{k_0}^\lambda(n) \right). \quad (3.29) \]
By (3.21) and (3.29),
\[ U_{\lambda_0}(n + 1, n - k) \tilde{E}_k(n - k) = U_{\lambda_0}(n + 1, n) U_{\lambda_0}(n, n - k) \tilde{E}_k(n - k) \]
\[ = U_{\lambda_0}(n + 1, n) E_{k_0}(n) \subset E_{k_0}^{n+1}(n + 1) = U_{\lambda_0}(n + 1, n - k) \tilde{E}_{k+1}(n - k), \]
which implies
\[ \tilde{E}_k(n - k) \subset \tilde{E}_{k+1}(n - k), \quad \forall k \geq 1, n \leq n_*. \]

For each \( \lambda \in \Lambda_0, n \leq n_* \) and \( k \geq 1 \), we define the set
\[ \tilde{E}_{k}^\lambda(n) := U_{\lambda}(n, n - k) \tilde{E}_k(n - k). \]

Obviously, \( \tilde{E}_{k}^\lambda(n) \subset U_{\lambda}(n, n - k) B(n - k) \), and it follows from formula (3.29) and estimate (3.32) that
\[ \text{Card} \left( \tilde{E}_{k}^\lambda(n) \right) \leq \text{Card} \left( \tilde{E}_k(n - k) \right) \leq \left[ m_Z \left( \frac{2l}{\eta} \right) \right]^{k+1}, \quad \forall k \geq 1, n \leq n_. \]

Therefore, by the definition of \( \Gamma(\lambda, \lambda_0) \) (see (3.5)) and Assumption 3.1,
\[ ||U_{\lambda}(n, n - k)x - U_{\lambda_0}(n, n - k)x||_{X_n} \]
\[ \leq ||U_{\lambda}(n, n - 1)U_{\lambda}(n - 1, n - k)x - U_{\lambda}(n, n - 1)U_{\lambda_0}(n - 1, n - k)x||_{X_n} \]
\[ + ||U_{\lambda}(n, n - 1)U_{\lambda_0}(n - 1, n - k)x - U_{\lambda_0}(n - 1, n - k)x||_{X_n} \]
\[ \leq L_1 ||U_{\lambda}(n, n - 1)x - U_{\lambda_0}(n, n - 1)x||_{X_{n-1}} + \Gamma(\lambda, \lambda_0) \]
\[ \leq \cdots \leq \Gamma(\lambda, \lambda_0) \left( L_1^{k-1} + \cdots + 1 \right) \]
\[ \leq \Gamma(\lambda, \lambda_0) L_1^k, \quad \forall x \in B(n - k), \ n \leq n_*, k \geq 1, \]
that is,
\[ \sup_{x \in B(n - k)} \sup_{n \leq n_*} ||U_{\lambda}(n, n - k)x - U_{\lambda_0}(n, n - k)x||_{X_n} \leq \Gamma(\lambda, \lambda_0) L_1^k, \quad \forall k \geq 1. \]

When \( \lambda \in \Lambda_0 \), which means \( 0 < \Gamma(\lambda, \lambda_0) < 1 \), we let
\[ k_T := \frac{-\log L_{1}(\Gamma(\lambda, \lambda_0))}{1 - \log L_{1}(2\eta)} \quad \text{and} \quad \gamma := \frac{-\log L_{1}(2\eta)}{1 - \log L_{1}(2\eta)}. \]

A simple calculation shows that
\[ \Gamma(\lambda, \lambda_0) L_1^{k_T} = (2\eta)^{k_T} = [\Gamma(\lambda, \lambda_0)]^{\gamma}. \]

Hence, it follows from formulas (3.29), (3.32), (3.34) and \( \tilde{E}_k(n - k) \subset B(n - k) \) that
\[ \sup_{n \leq n_*} \text{dist}^{\text{symm}}_{X_n} \left( \tilde{E}_{k}^\lambda(n), E_{k}^{\lambda_0}(n) \right) \]
\[ = \sup_{n \leq n_*} \text{dist}^{\text{symm}}_{X_n} \left( U_{\lambda}(n, n - k) \tilde{E}_k(n - k), U_{\lambda_0}(n, n - k) \tilde{E}_k(n - k) \right) \]
\[ \leq \Gamma(\lambda, \lambda_0) L_1^k \leq [\Gamma(\lambda, \lambda_0)]^{\gamma}, \quad 1 \leq k \leq k_T. \]
For each $b \in B(n - k)$, by the finiteness of the subset $E_k^\lambda (n)$, there must be a $b_0 \in E_k^\lambda (n)$ such that

$$
\|U(\lambda, n, n - k)b - b_0\|_{X_n} = \text{dist}_{X_n} \left( U(\lambda, n, n - k)b, E_k^\lambda (n) \right) \leq \text{dist}_{X_n} \left( U(\lambda, n, n - k)B(n - k), V_k^\lambda (n) \right) \leq 2R_0(2\eta)^k, \ \forall k \geq 1, \ n \leq n_*,
$$

where we have used formula (3.18). The combination of (3.34)-(3.36) yields, for any $b \in B(n - k),

$$
\text{dist}_{X_n} \left( U(\lambda, n, n - k)b, \tilde{E}_k^\lambda (n) \right) \leq ||U(\lambda, n, n - k)b - U(\lambda, n, n - k)b||_{X_n} + ||U(\lambda, n, n - k)b - b_0||_{X_n} + \text{dist}_{X_n} \left( b_0, \tilde{E}_k^\lambda (n) \right) \leq \Gamma(\lambda, \lambda_0) L_1^k + 2R_0(2\eta)^k + [\Gamma(\lambda, \lambda_0)]^\gamma \leq (2R_0 + 2)(2\eta)^k, \ 1 \leq k \leq k_\Gamma.
$$

By the arbitrariness of $b \in B(n - k),

$$
\text{dist}_{X_n} \left( U(\lambda, n, n - k)B(n - k), \tilde{E}_k^\lambda (n) \right) \leq (2R_0 + 2)(2\eta)^k, \ \forall 1 \leq k \leq k_\Gamma, \ n \leq n_*, \ \lambda \in \Lambda_0.
$$

For each $\lambda \in \Lambda_0$ and $n \leq n_*$, we define by induction the sets

$$
E_k^\lambda (n) = \left\{ \begin{array}{ll}
\tilde{E}_k^\lambda (n), & 1 \leq k \leq k_\Gamma, \\
V_k^\lambda (n) \cup \text{U}(\lambda, n, n - 1)E_{k-1}^\lambda (n - 1), & k > k_\Gamma,
\end{array} \right.
$$

and

$$
E_\lambda (n) = \left[ \bigcup_{k \geq 1} E_k^\lambda (n) \right]_{X_n}.
$$

It follows from formulas (3.16)-(3.18) and (3.33) that

$$
E_k^\lambda (n) \subset U(\lambda, n, n - k)B(n - k), \quad (3.39)
$$

$$
U(\lambda, n + 1, n)E_k^\lambda (n) \subset E_{k+1}^\lambda (n + 1), \quad (3.40)
$$

$$
E_\lambda (n) \subset U(\lambda, n, n - 1)B(n - 1) \subset B(n), \quad (3.41)
$$

$$
\text{Card} \left( E_k^\lambda (n) \right) \leq \left[ mZ \left( \frac{2L}{\eta} \right) \right]^{k+1} \quad (3.42)
$$

for all $\lambda \in \Lambda, \ n \leq n_*$ and $k \geq 1$.

For every $\lambda \in \Lambda_0$, repeating the same arguments as in Step 2, we obtain that the family $\{E_\lambda (n)\}_{n \leq n_*}$ also possesses the properties (i)-(iii) as shown in Step 2. Furthermore, it follows from formulas (3.18) and (3.37) that

$$
\text{dist}_{X_n} \left( U(\lambda, n, n - k)B(n - k), E_k^\lambda (n) \right) \leq \left\{ \begin{array}{ll}
\text{dist}_{X_n} \left( U(\lambda, n, n - k)B(n - k), \tilde{E}_k^\lambda (n) \right), & 1 \leq k \leq k_\Gamma, \\
\text{dist}_{X_n} \left( U(\lambda, n, n - k)B(n - k), V_k^\lambda (n) \right), & k > k_\Gamma,
\end{array} \right. \leq (2R_0 + 2)(2\eta)^k, \ \forall n \leq n_*, k \geq 1.
$$
Now, we show the Hölder continuity of the family \( \{ E_\lambda(n) \}_{n \leq n_*} \) at the point \( \lambda_0 \).

For any \( \lambda \in \Lambda_0, n \leq n_* \) and any \( a \in \cup_{k \geq 1} E_k^\lambda(n) \), there must be \( a \in E_k^\lambda(n) \) for some \( k \geq 1 \).

When \( 1 \leq k \leq k, E_k^\lambda(n) = \bar{E}_k^\lambda(n) \), by (3.35) we obtain

\[
\text{dist}_{X_n} (a, E_{\lambda_0}(n)) \leq \text{dist}_{X_n} \left( a, E_{\lambda_0}^0(n) \right) \leq \text{dist}_{X_n} \left( \bar{E}_k^\lambda(n), E_{\lambda_0}^0(n) \right) \leq [\Gamma(\lambda, \lambda_0)]^\gamma.
\]

When \( k > k \), by (3.11) and (3.39),

\[
a \in E_k^\lambda(n) \subset U_\lambda(n, n - k)B(n - k) = U_\lambda(n, n - [k])U_\lambda(n - [k], n - k)B(n - k),
\]

then there exists an element \( b \in U_\lambda(n - [k], n - k)B(n - k) \) such that \( a = U_\lambda(n, n - [k])b \), and by (3.34) and (3.25),

\[
\text{dist}_{X_n} (a, E_{\lambda_0}(n)) \\
\leq \text{dist}_{X_n} (U_\lambda(n, n - [k])b, E_{\lambda_0}(n)) \\
\leq ||U_\lambda(n, n - [k])b - U_\lambda(n, n - [k])b||_{X_n} + \text{dist}_{X_n} (U_{\lambda_0}(n, n - [k])b, E_{\lambda_0}(n)) \\
\leq \Gamma(\lambda, \lambda_0)L_1^{[k]} + (2R_0 + 2)(2n)^[k] \leq C [\Gamma(\lambda, \lambda_0)]^\gamma,
\]

where \( C = 1 + \frac{R_0 + 1}{\eta} \) and \([k] \) denotes the integer part of \( k \). By the arbitrariness of \( a \in \cup_{k \geq 1} E_k^\lambda(n) \), we obtain

\[
\text{dist}_{X_n} (E_\lambda(n), E_{\lambda_0}(n)) = \text{dist}_{X_n} \left( \cup_{k \geq 1} E_k^\lambda(n), E_{\lambda_0}(n) \right) \leq C [\Gamma(\lambda, \lambda_0)]^\gamma. \tag{3.44}
\]

Repeating the proof of (3.44) (changing the position of \( \lambda \) and \( \lambda_0 \)) and making use of (3.35) and (3.43), we have

\[
\text{dist}_{X_n} (E_{\lambda_0}(n), E_\lambda(n)) \leq C [\Gamma(\lambda, \lambda_0)]^\gamma. \tag{3.45}
\]

The combination of (3.44) and (3.45) gives

\[
\sup_{n \leq n_*} \text{dist}_{X_n}^{\text{symm}} (E_\lambda(n), E_{\lambda_0}(n)) \leq C [\Gamma(\lambda, \lambda_0)]^\gamma, \forall \lambda \in \Lambda_0. \tag{3.46}
\]

**Step 4.** (The structure of the family \( \mathcal{E}_\Lambda = \{ E_\lambda(t) \}_{t \in \mathbb{R}} \), with \( \lambda \in \Lambda \), and its properties) For any \( t \in \mathbb{R} \), when \( t \leq t_0 \), there exists unique integer \( n_t = [t] \leq n_* \) such that \( t = n_t + s_t \) with \( s_t \in [0, 1) \); when \( t > t_0 \), we take \( n_t \equiv n_* \) and \( t = n_* + s_t \). Let

\[
E_\lambda(t) = U_\lambda(t, n_t)E_\lambda(n_t), \forall t \in \mathbb{R}, \lambda \in \Lambda. \tag{3.47}
\]

By (3.22), (3.41) and condition (3.1) we have \( E_\lambda(t) \subset B(t) \) for all \( t \in \mathbb{R} \) and \( \lambda \in \Lambda \). We show that for every \( \lambda \in \Lambda, \mathcal{E}_\lambda = \{ E_\lambda(t) \}_{t \in \mathbb{R}} \) is the desired family of Theorem 3.2.

(i) (Semi-invariance) For every \( t \leq r \in \mathbb{R} \), we have \( n_t \leq n_r \). When \( t \leq t_0 \), we have \( n_t \leq n_* \), and it follows from the semi-invariance (3.24) of \( \{ E_\lambda(n) \}_{n \leq n_*} \) and formula (3.47) that

\[
U_\lambda(r, t)E_\lambda(t) = U_\lambda(r, t)U_\lambda(t, n_t)E_\lambda(n_t) \\
= U_\lambda(r, n_t)U_\lambda(n_t, n_t)E_\lambda(n_t) \\
\subset U_\lambda(r, n_t)E_\lambda(n_t) = E_\lambda(r), \forall \lambda \in \Lambda;
\]

When \( t > t_0 \), \( n_t \equiv n_* \) and we also have

\[
U_\lambda(r, t)E_\lambda(t) = U_\lambda(r, t)U_\lambda(t, n_*)E_\lambda(n_*) = U_\lambda(r, n_*)E_\lambda(n_*) = E_\lambda(r), \forall \lambda \in \Lambda.
\]
(ii) (The compactness and the boundedness of the fractal dimension) For each \( t \in \mathbb{R} \), let

\[
L_t = \begin{cases} 
L_1, & t \leq t_0, \\
L_1 + t - t_0, & t > t_0.
\end{cases}
\]

We infer from condition \((H_3)\) that

\[
\sup_{\lambda \in \Lambda} \|U_\lambda(t, n_t)x - U_\lambda(t, n_t)y\|_{X_t} \leq L_t \|x - y\|_{X_{n_t}}, \quad \forall x, y \in B(n_t),
\]

that is, the mapping \( U_\lambda(t, n_t) : B(n_t) \to X_t \) is Lipschitz continuous. Therefore, the set \( E_\lambda(t) \) is compact in \( X_t \) for each \( \lambda \in \Lambda \), and by \((3.28)\),

\[
\dim_f (E_\lambda(t); X_t) \leq \dim_f (E_\lambda(n_t); X_{n_t}) \leq \left[ \ln \left( \frac{1}{2\eta} \right) \right]^{-1} \ln \left[ m_Z \left( \frac{2L}{\eta} \right) \right].
\]

(iii) (Pullback exponential attractiveness) For each \( t \in \mathbb{R} \), let

\[
\tau_t = \begin{cases} 
3, & t \leq t_0, \\
3 + t - t_0, & t > t_0.
\end{cases}
\]

For every \( \tau \geq \tau_t \), there exists unique integer \( k_\tau \in \mathbb{N} \) such that \( \tau \in [k_\tau + 2, k_\tau + 3) \), which implies

\[
n_t - k_\tau - (t - \tau) > 1 \quad \text{and} \quad -k_\tau \leq -\tau + t - n_t + 2 \leq -\tau + \tau_t.
\]

Then it follows from \((3.1)\) and \((3.49)\)

\[
U_\lambda(t, t - \tau)B(t - \tau) = U_\lambda(t, n_t)U_\lambda(n_t, n_t - k_\tau)U_\lambda(n_t - k_\tau, t - \tau) B(t - \tau)
\]

\[
\subset U_\lambda(t, n_t)U_\lambda(n_t, n_t - k_\tau)B(n_t - k_\tau),
\]

which combining with formulas \((3.48), (3.25)\) and \((3.47)\) yields

\[
\text{dist}_{X_t}(U_\lambda(t, t - \tau)B(t - \tau), E_\lambda(t)) \leq \text{dist}_{X_t}(U_\lambda(t, n_t)U_\lambda(n_t, n_t - k_\tau)B(n_t - k_\tau), U_\lambda(t, n_t)E_\lambda(n_t))
\]

\[
\leq L_t \text{dist}_{X_{n_t}}(U_\lambda(n_t, n_t - k_\tau)B(n_t - k_\tau), E_\lambda(n_t))
\]

\[
\leq 2L_t R_0(2\eta)^{k_\tau} = 2L_t R_0 e^{-\beta k_\tau} \leq 2L_t R_0 e^{\beta \tau} e^{-\beta \tau} = C(t)e^{-\beta \tau},
\]

with \( \beta = \ln \frac{1}{2\eta} \) and \( C(t) = 2L_t R_0 e^{\beta \tau} \).

(iv) (The continuity of the family \( E_\lambda = \{E_\lambda(t)\}_{t \in \mathbb{R}} \) at the point \( \lambda_0 \)) For each \( \lambda \in \Lambda_0 \), it follows from \((3.46)-(3.48), (3.41)\) and the definition of \( \Gamma_t(\lambda, \lambda_0) \) that

\[
\text{dist}_{X_t}^{\text{symm}}(E_\lambda(t), E_{\lambda_0}(t)) \leq \text{dist}_{X_t}^{\text{symm}}(U_\lambda(t, n_t)E_\lambda(n_t), U_{\lambda_0}(t, n_t)E_{\lambda_0}(n_t))
\]

\[
+ \text{dist}_{X_t}^{\text{symm}}(U_{\lambda_0}(t, n_t)E_{\lambda_0}(n_t), U_{\lambda_0}(t, n_t)E_{\lambda_0}(n_t))
\]

\[
\leq \Gamma(\lambda, \lambda_0) + L_t \text{dist}_{X_{n_t}}^{\text{symm}}(E_{\lambda}(n_t), E_{\lambda_0}(n_t)), \quad t \leq t_0,
\]

\[
\leq \sup_{x \in B(n_t)} \|U_\lambda(t, n_t)x - U_{\lambda_0}(t, n_t)x\|_{X_t} + L_t \text{dist}_{X_{n_t}}^{\text{symm}}(E_{\lambda}(n_t), E_{\lambda_0}(n_t)), \quad t > t_0
\]

\[
\leq C_1(t)\Gamma_t(\lambda, \lambda_0),
\]

with the constant \( C_1(t) = 1 + L_t \) depending only on \( t \). This completes the proof. 
\qed

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4 Application to the semilinear damped wave equation

In this section, we give an application of the above mentioned criteria to the following semilinear damped wave equations with perturbed time-dependent speed of propagation

\[ \rho(t)u_{tt} + \alpha u_t - \Delta u + f(u) = g(x), \quad x \in \Omega, t > \tau, \] (4.1)

\[ u|_{\partial\Omega} = 0, \] (4.2)

\[ u(x, \tau) = u_0(x), \quad u_t(x, \tau) = u_1(x), \] (4.3)

where \( \Omega \subset \mathbb{R}^3 \) is a bounded domain with the smooth boundary \( \partial\Omega \), \( \alpha > 0 \) is the damping coefficient, \( g \in L^2(\Omega) \), the function \( \rho(t) = \epsilon \rho(t) \) with perturbation parameter \( \epsilon \in (0, 1] \), and the assumptions on \( \rho(t) \) and the nonlinearity \( f(u) \) are the same as in [15]. For clarity, we quote them as follows.

**Assumption 4.1.** (i) \( \rho \in C^1(\mathbb{R}) \) is a decreasing bounded function satisfying

\[ \lim_{t \to +\infty} \rho(t) = 0 \quad \text{and} \quad \sup_{t \in \mathbb{R}} \left[ |\rho(t)| + |\rho'(t)| \right] \leq L, \] (4.4)

for some constant \( L > 0 \).

(ii) \( f \in C^2(\mathbb{R}) \) with \( f(0) = 0 \) satisfies

\[ |f''(s)| \leq c(1 + |s|), \quad \forall s \in \mathbb{R}, \] (4.5)

for some \( c \geq 0 \), and the dissipation conditions

\[ \liminf_{|s| \to \infty} \frac{f(s)}{s} > -\lambda_1, \quad 2f(s)s \geq F(s) - \nu s^2 - c_1, \quad \forall s \in \mathbb{R}, \]

where \( F(s) = \int_0^s f(r)dr, \) \( 0 < \nu < \lambda_1, \) \( c_1 \geq 0 \) and \( \lambda_1 > 0 \) is the first eigenvalue of the operator \( A = -\Delta \) with the Dirichlet boundary condition.

4.1 Notations and main results

Let \( V_0 = L^2(\Omega) \), with the inner product \((\cdot, \cdot)\) and norm \( \| \cdot \| \). For any \( \sigma > 0 \), let the spaces

\[ V_\sigma = D(A^{\sigma/2}) \quad \text{and} \quad V_{-\sigma} = \text{the dual space of } V_\sigma \]

equipped with the inner product and the norm

\[ (w, v)_\sigma = (A^{\sigma/2}w, A^{\sigma/2}v), \quad \|w\|_\sigma = \|A^{\sigma/2}w\|, \quad \forall \sigma \in \mathbb{R}. \]

Then \( V_\sigma \) are the Hilbert spaces and \( V_{\sigma_1} \hookrightarrow V_{\sigma_2} \) if \( \sigma_1 > \sigma_2 \). In particular,

\[ V_2 = H^2(\Omega) \cap H^1_0(\Omega), \quad V_1 = H^1_0(\Omega), \quad \|w\|_1 = \|A^{1/2}w\| = \|\nabla w\|. \]

More precisely, one can infer from [1] that

\[ V_\sigma \hookrightarrow H^\sigma(\Omega), \quad \sigma \geq 0, \]

\[ V_\sigma \hookrightarrow L^r(\Omega), \quad 1 \leq r \leq \frac{6}{3 - 2\sigma}, \quad L^q(\Omega) \hookrightarrow V_{-\sigma}, \quad q \geq \frac{6}{3 + 2\sigma}, \quad 0 \leq \sigma < \frac{3}{2}. \]
For each $t \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and $\epsilon \in (0, 1]$, let the time-dependent phase spaces
\[
\mathcal{H}_{t, \sigma}^\epsilon = V_{\sigma+1} \times V_{\sigma} \quad \text{and} \quad \mathcal{H}_{t}^\epsilon = \mathcal{H}_{t,0}^\epsilon = V_1 \times V_0
\]
equipped with the time-dependent product norms
\[
\| (u, v) \|_{\mathcal{H}_{t, \sigma}^\epsilon}^2 = \| u \|_{\sigma+1}^2 + \rho_\epsilon(t) \| v \|_{\sigma}^2 \quad \text{and} \quad \| (u, v) \|_{\mathcal{H}_{t}^\epsilon}^2 = \| u \|_1^2 + \rho_\epsilon(t) \| v \|_2^2,
\]
respectively. In particular, when $\epsilon = 1$, we omit the superscript $\epsilon$ and let
\[
\mathcal{H}_{t, \sigma} = \mathcal{H}_{t, \sigma}^1 \quad \text{and} \quad \mathcal{H}_t = \mathcal{H}_t^1,
\]
respectively. A simple calculation shows that
\[
\epsilon \| (u, v) \|_{\mathcal{H}_{t, \sigma}}^2 \leq \| (u, v) \|_{\mathcal{H}_{t, \sigma}^\epsilon}^2 \leq \| (u, v) \|_{\mathcal{H}_{t, \sigma}^\epsilon}^2, \quad \forall \epsilon \in (0, 1], \tag{4.6}
\]
and
\[
a \| (u, v) \|_{\mathcal{H}_{t, \sigma}^\epsilon}^2 \leq \| (u, v) \|_{\mathcal{H}_{t, \sigma}^\epsilon}^2 \leq \| (u, v) \|_{\mathcal{H}_{t, \sigma}^\epsilon}^2, \quad \forall \epsilon \in [a, 1] \subset (0, 1]. \tag{4.7}
\]
Obviously, all the spaces $\mathcal{H}_t^\epsilon$ are same as the linear space, while formula (4.6) shows $\| \cdot \|_{\mathcal{H}_t^\epsilon} \sim \| \cdot \|_{\mathcal{H}_t}$ for each $\epsilon \in (0, 1]$.

We rewrite problem (4.1)-(4.3) at an abstract level
\[
\rho_\epsilon(t) u_{tt} + \alpha u_t + A u + f(u) = g, \quad t > \tau, \tag{4.8}
\]
\[
u(\tau) = u_0, \quad u_\tau(\tau) = u_1. \tag{4.9}
\]

Under Assumption 4.1, repeating the similar argument as Theorem 9.1 in [15], one easily obtains the existence of the continuous process $U_\epsilon(t, \tau) : \mathcal{H}_\tau \to \mathcal{H}_t$ corresponding to problem (4.8)-(4.9), that is
\[
U_\epsilon(t, \tau) z_\tau = z^+(t), \quad \forall z_\tau = (u_0, u_1) \in \mathcal{H}_\tau, \quad t \geq \tau,
\]
where $z_\epsilon(t) = (u_\epsilon(t), u_\epsilon(t)) \in C(\mathbb{R}^+; V_1 \times V_0)$ is the weak solution of problem (4.8)-(4.9) corresponding to the initial data $z_\tau \in \mathcal{H}_\tau$, with $\epsilon \in (0, 1]$. Besides, for any $z_\tau \in \mathcal{H}_\tau$ with $\| z_\tau \|_{\mathcal{H}_\tau} \leq R$, $i = 1, 2$, and any interval $[a, 1] \subset (0, 1]$,
\[
\sup_{\epsilon \in [a, 1]} \| U_\epsilon(t, \tau) z_{1\tau} - U_\epsilon(t, \tau) z_{2\tau} \|_{\mathcal{H}_t} \leq Q_\alpha(R) e^{Q_\alpha(R)(t-\tau)} \| z_{1\tau} - z_{2\tau} \|_{\mathcal{H}_\tau}, \quad t \geq \tau, \tag{4.10}
\]
hereafter $Q_\alpha(\cdot) = a^{-1} Q(\cdot)$ and $Q(\cdot)$ is an increasing positive function independent of $t$.

Let the universe of problem (4.8)-(4.9) be
\[
\mathcal{D} := \{ D = \{ D(t) \}_{t \in \mathbb{R}} | \emptyset \neq D(t) \subset \mathcal{H}_t, \forall t \in \mathbb{R} \ \text{and} \ \sup_{t \in \mathbb{R}} \| D(t) \|_{\mathcal{H}_t} < +\infty \}.
\]

**Theorem 4.1.** Let Assumption 4.1 be valid. Then the process $U_\epsilon(t, \tau) : \mathcal{H}_\tau \to \mathcal{H}_t$ has a minimal pullback $\mathcal{D}$-attractor $A_\epsilon = \{ A_\epsilon(t) \}_{t \in \mathbb{R}}$ for each $\epsilon \in (0, 1]$, and $A_\epsilon$ possesses the following properties:

(i) the sections $A_\epsilon(t)$ are uniformly bounded in $\mathcal{H}_{t,1}$, and their fractal dimensions in $\mathcal{H}_t$ are uniformly bounded, that is,
\[
\sup_{t \in \mathbb{R}} \| A_\epsilon(t) \|_{\mathcal{H}_{t,1}} < +\infty \ \text{and} \ \sup_{t \in \mathbb{R}} \dim_f(A_\epsilon(t), \mathcal{H}_t) < +\infty, \quad \forall \epsilon \in (0, 1];
\]

(ii) the sections $A_\epsilon(t)$ are uniformly bounded in $\mathcal{H}_t$, and their fractal dimensions in $\mathcal{H}_t$ are uniformly bounded, that is,
\[
\sup_{t \in \mathbb{R}} \| A_\epsilon(t) \|_{\mathcal{H}_t} < +\infty \ \text{and} \ \sup_{t \in \mathbb{R}} \dim_f(A_\epsilon(t), \mathcal{H}_t) < +\infty, \quad \forall \epsilon \in (0, 1].
\]
(ii) (Upper semicontinuity) for any point $\epsilon_0 \in (0, 1]$,
\[ \lim_{\epsilon \to \epsilon_0} \text{dist}_{\mathcal{H}_t} (A_\epsilon(t), A_{\epsilon_0}(t)) = 0, \quad \forall t \in \mathbb{R}; \]

(iii) (Residual continuity) there exists a residual subset $\Lambda^*$ of $(0, 1]$ such that
\[ \lim_{\epsilon \to \epsilon_0} \text{dist}_{\mathcal{H}_t}^{\text{symm}} (A_\epsilon(t), A_{\epsilon_0}(t)) = 0, \quad \forall \epsilon_0 \in \Lambda^*, \ t \in \mathbb{R}. \]

**Theorem 4.2.** Let Assumption 4.1 be valid. Then the process $U_\epsilon(t, \tau) : \mathcal{H}_\tau \to \mathcal{H}_t$ has a pullback $\mathcal{D}$-exponential attractor $\mathcal{E}_\epsilon = \{ E_\epsilon(t) \}_{t \in \mathbb{R}}$ for each $\epsilon \in (0, 1]$, and $\mathcal{E}_\epsilon$ possesses the following properties:

(i) (Regularity) the sections $E_\epsilon(t)$ are uniformly bounded in $\mathcal{H}_{t, 1}$, that is,
\[ \sup_{t \in \mathbb{R}} \| E_\epsilon(t) \|_{\mathcal{H}_{t, 1}} < \infty, \quad \forall \epsilon \in (0, 1]; \]

(ii) (Hölder continuity) for any $\epsilon_0 \in (0, 1]$, there exists a $\delta = \delta(\epsilon_0) \in (0, 1)$ such that
\[ \text{dist}_{\mathcal{H}_t}^{\text{symm}} (E_\epsilon(t), E_{\epsilon_0}(t)) \leq C(t)|\epsilon - \epsilon_0|^{\gamma}, \quad \forall t \in \mathbb{R}, \]
whenever $|\epsilon - \epsilon_0| < \delta$, where $C(t)$ and $\gamma : 0 < \gamma < 1$ are some positive constants.

## 4.2 Some essential estimates

In order to prove Theorem 4.1 and Theorem 4.2, we need the following lemmas which will play key role later. In this subsection, we always assume the interval $[a, 1] \subset (0, 1]$.

**Lemma 4.3.** Let Assumption 4.1 be valid, and $z_\tau \in \mathcal{H}_\tau$ with $\| z_\tau \|_{\mathcal{H}_\tau} \leq R$. Then there exist positive constants $\kappa = \kappa(a)$ and $R_a$ independent of $R$ and $\epsilon$ such that
\[ \sup_{\epsilon \in [a, 1]} \| U_\epsilon(t, \tau)z_\tau \|_{\mathcal{H}_t} \leq Q_a(R)e^{-\kappa(t-\tau)} + R_a, \quad \forall t \geq \tau, \]
and
\[ \sup_{\epsilon \in [a, 1]} \int_\tau^\infty \| u_{\epsilon t}(t) \|^2 dt \leq Q_a(R), \quad \forall t \geq \tau, \quad (4.11) \]
hereafter $(u_\epsilon(t), u_{\epsilon t}(t)) = U_\epsilon(t, \tau)z_\tau$, $Q_a(\cdot)$ and $Q(\cdot)$ are as shown in (4.10), and $R_a$ satisfies $\lim_{a \to 0^+} R_a = +\infty$.

**Proof.** Repeating the similar argument as Lemma 10.3 in [15], we obtain
\[ a\| U_\epsilon(t, \tau)z_\tau \|_{\mathcal{H}_t} \leq \| U_\epsilon(t, \tau)z_\tau \|_{\mathcal{H}_t^\prime} \leq Q(\| z_\tau \|_{\mathcal{H}_t^\prime})e^{-\kappa(t-\tau)} + R_0, \quad \forall \epsilon \in [a, 1], \ t \geq \tau, \quad (4.12) \]
and formula (4.11), that is, the conclusions of Lemma 4.3 hold.

**Remark 4.4.** Lemma 4.3 shows that the family $\{ B_t(R_1) \}_{t \in \mathbb{R}} \in \mathcal{D}$, with $R_1 > R_a$, is a uniformly (w.r.t. $\epsilon \in [a, 1]$) pullback $\mathcal{D}$-absorbing family of the processes $U_\epsilon(t, \tau) : \mathcal{H}_\tau \to \mathcal{H}_t, \ \epsilon \in [a, 1]$. 

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Lemma 4.5. [3, 24] Let Assumption 4.1 be valid. Then $f$ can be split into the sum $f = f_0 + f_1$, where $f_0, f_1 \in C^2(\mathbb{R})$ satisfying

$$f_0(0) = f'_0(0) = 0, \quad f_0(s) \geq 0, \quad |f''_0(s)| \leq C(1 + |s|), \quad \forall s \in \mathbb{R},$$

and

$$|f'_1(s)| \leq C, \quad \forall s \in \mathbb{R}. \quad (4.14)$$

By virtue of Lemma 4.5, we split the solution $(u_\epsilon(t), u_{\epsilon t}(t)) = U_\epsilon(t, \tau)z_{\tau}$ into the sum

$$U_\epsilon(t, \tau)z_{\tau} = U_{\epsilon,0}(t, \tau)z_{\tau} + U_{\epsilon,1}(t, \tau)z_{\tau},$$

where $U_{\epsilon,0}(t, \tau)z_{\tau} = (v_\epsilon(t), v_{\epsilon t}(t))$ solves

$$\begin{cases} \rho_\epsilon(t)v_{\epsilon tt} + \alpha v_{\epsilon t} + Av_\epsilon + f_0(v_\epsilon) = 0, & t > \tau, \\ U_{\epsilon,0}(\tau, \tau)z_{\tau} = z_{\tau}, \end{cases}$$

and $U_{\epsilon,1}(t, \tau)z_{\tau} = (w_\epsilon(t), w_{\epsilon t}(t))$ solves

$$\begin{cases} \rho_\epsilon(t)w_{\epsilon tt} + \alpha w_{\epsilon t} + Aw_\epsilon + f(u_\epsilon) - f_0(v_\epsilon) = g, & t > \tau, \\ U_{\epsilon,1}(\tau, \tau)z_{\tau} = 0. \end{cases}$$

Lemma 4.6. Let Assumptions 4.1 be valid, and $z_{\tau} \in \mathcal{H}_{\tau}$ with $\|z_{\tau}\|_{\mathcal{H}_{\tau}} \leq R$. Then

$$\sup_{\epsilon \in [a, 1]} \|(v_\epsilon(t), v_{\epsilon t}(t))\|_{\mathcal{H}_t} \leq Q_a(R)e^{-\kappa(t-\tau)}, \quad (4.15)$$

$$\sup_{\epsilon \in [a, 1]} \|(w_\epsilon(t), w_{\epsilon t}(t))\|_{\mathcal{H}_{t,1/3}} \leq Q_a(R), \quad \forall t \geq \tau, \quad (4.16)$$

where $\kappa > 0$ is as shown in Lemma 4.3.

Proof. Repeating the same proof as Lemma 11.2 and Lemma 11.3 in [15], we obtain

$$a\|U_{\epsilon,0}(t, \tau)z_{\tau}\|_{\mathcal{H}_{t}} \leq \|U_{\epsilon,0}(t, \tau)z_{\tau}\|_{\mathcal{H}_{t}} \leq Q(R)e^{-\kappa(t-\tau)},$$

$$a\|U_{\epsilon,1}(t, \tau)z_{\tau}\|_{\mathcal{H}_{t,1/3}} \leq \|U_{\epsilon,1}(t, \tau)z_{\tau}\|_{\mathcal{H}_{t,1/3}} \leq Q(R), \quad \forall \epsilon \in [a, 1], \quad t \geq \tau,$$

which imply (4.15) and (4.16). \qed

On the basis of Lemma 4.6, we further give a delicate estimate.

Lemma 4.7. Let Assumption 4.1 be valid, and $z_{\tau} \in \mathcal{H}_{\tau,1/3}$ with $\|z_{\tau}\|_{\mathcal{H}_{\tau}} \leq R$. Then

$$\sup_{\epsilon \in [a, 1]} \|U_{\epsilon}(t, \tau)z_{\tau}\|_{\mathcal{H}_{t,1/3}} \leq Q_a(\|z_{\tau}\|_{\mathcal{H}_{t,1/3}} + R)e^{-\kappa(t-\tau)} + Q_a(R), \quad \forall t \geq \tau.$$

Proof. For simplicity, we omit the subscript $\epsilon$ and let $u = u_\epsilon$. For any $\epsilon \in [a, 1]$ and $\delta \in (0, 1)$, we define the functional along the solution $(u(t), u_t(t)) = U_\epsilon(t, \tau)z_{\tau}$ as follows

$$\Phi_\epsilon(t) = \rho_\epsilon(t)\|u_t\|_{L^1/3}^2 + \|u\|_{L^1/3}^2 + 2\langle f(u) - g, A_{1/3}u \rangle + \delta \left[2\rho_\epsilon(t)\langle u_t, A_{1/3}u \rangle + \alpha\|u\|_{L^1/3}^2 \right], \quad t \geq \tau. \quad (4.17)$$

It follows from Lemma 4.3 and condition (4.5) that

$$\|f(u)\| \leq C\|1 + |u|^3\| \leq C \left(1 + \|u\|_{L^0(\Omega)}^3 \right) \leq C \left(1 + \|u\|_{L^0(\Omega)}^3 \right) \leq Q_a(R), \quad t \geq \tau,$$
which implies that
\[2|\langle f(u) - g, A^\frac{1}{2} u \rangle| \leq 2 \|f(u)\| + \|g\| \|u\|_{2/3} \leq \frac{1}{4} \|u\|^2_{1/3} + Q_a(R), \quad t \geq \tau. \quad (4.18)\]

By condition (4.4),
\[2\rho_\epsilon(t)\|u_t, A^\frac{1}{2} u\| \leq 2\rho_\epsilon(t)\|u_t\|_{1/3} \|u\|_{1/3} \leq \alpha\|u\|^2_{1/3} + \frac{L}{\alpha}\rho_\epsilon(t)\|u_t\|^2_{1/3}. \quad (4.19)\]

The combination of (4.7) and (4.17)-(4.19) yields
\[\frac{a}{2} \|u(t), u_t(t)\|_{\mathcal{H}_{t,1/3}}^2 - Q_a(R) \leq \Phi_\epsilon(t) \leq 2\|u(t), u_t(t)\|_{\mathcal{H}_{t,1/3}}^2 + Q_a(R) \quad (4.20)\]
for \(\delta > 0\) suitably small.

Taking the multiplier \(2A^\frac{1}{2} u_t + 2\delta A^\frac{1}{2} u\) in Eq. (4.8) yields
\[\frac{d}{dt}\Phi_\epsilon(t) + \delta\Phi_\epsilon(t) + (2\alpha - \rho_\epsilon'(t) - 3\delta\rho_\epsilon(t)) \|u_t\|_{1/3}^2 + \delta\|u\|_{1/3}^2 - \delta^2\alpha\|u\|_{1/3}^2 = (2\delta^2\rho_\epsilon(t) + 2\delta\rho_\epsilon'(t)) \langle u_t, A^\frac{1}{2} u \rangle + I_1 + I_2 + I_3, \quad (4.21)\]
where
\[I_1 = 2\langle [f'_0(u) - f'_0(v)] u_t, A^\frac{1}{2} u \rangle, \quad I_2 = 2\langle f'_0(v) u_t, A^\frac{1}{2} u \rangle, \quad I_3 = 2\langle f'_1(u) u_t, A^\frac{1}{2} u \rangle. \]

A simple calculation shows that
\[(2\alpha - \rho_\epsilon'(t) - 3\delta\rho_\epsilon(t)) \|u_t\|_{1/3}^2 + \delta\|u\|_{1/3}^2 - \delta^2\alpha\|u\|_{1/3}^2 \geq \frac{3}{2}\alpha\|u_t\|_{1/3}^2 + \frac{\delta}{2}\|u\|_{1/3}^2, \]
\[(2\delta^2\rho_\epsilon(t) + 2\delta\rho_\epsilon'(t)) \langle u_t, A^\frac{1}{2} u \rangle \leq 4L\|u_t\|\|A^\frac{1}{2} u\| \leq \frac{\delta}{4}\|u\|_{1/3}^2 + Q_a(R) \]
for \(\delta > 0\) suitably small. Taking into account the Sobolev embedding:
\[V_1 \hookrightarrow L^6(\Omega), \quad V_{4/3} \hookrightarrow L^{18}(\Omega), \quad V_{2/3} \hookrightarrow L^{18}(\Omega), \quad V_{1/3} \hookrightarrow L^{18}(\Omega), \quad (4.22)\]
and making use of Lemma 4.3, formulas (4.14)-(4.16), we have
\[I_1 \leq C (1 + \|u\|_{L^6(\Omega)} + \|v\|_{L^6(\Omega)}\|w\|_{L^{18}(\Omega)} \|u_t\|\|A^\frac{1}{2} u\|_{L^{18}(\Omega)} \]
\[\leq C (1 + \|u\|_1 + \|v\|_1) \|w\|_{1/3} \|u_t\|\|u\|_{4/3} \]
\[\leq \frac{\delta}{8}\|u\|^2_{4/3} + Q_a(R), \]
\[I_2 \leq C\|v\|_{L^6(\Omega)} \|u_t\|_{L^{18}(\Omega)} \|A^\frac{1}{2} u\|_{L^{18}(\Omega)} \]
\[\leq C\|u_t\|^2_{1/3} + Q_a(R)\|v\|^2_{1/3} \|u\|^2_{4/3}, \]
and
\[I_3 \leq C\|u_t\|\|A^\frac{1}{2} u\| \leq \frac{\delta}{8}\|u\|^2_{4/3} + Q_a(R). \]

Inserting above estimates into (4.21) and making use of estimates (4.19)-(4.20) receive
\[\frac{d}{dt}\Phi_\epsilon(t) + \delta\Phi_\epsilon(t) \leq q(t)\Phi_\epsilon(t) + Q_a(R), \quad t > \tau, \quad (4.23)\]
where \( q(t) = Q_a(R)\|v(t)\|_1^2 \) satisfies (see estimate (4.15))
\[
\int_\tau^\infty q(s)ds \leq Q_a(R) \int_\tau^\infty \|v(s)\|_1^2ds \leq Q_a(R).
\]

Applying the Gronwall-type lemma (cf. [14]) to (4.23) and making use of estimate (4.20) turn out the conclusions of Lemma 4.7. 

For any fixed \( \epsilon \in (0, 1] \), \( \tau \in \mathbb{R} \) and \( z_\tau \in \mathcal{H}_{r,1/3} \), to avoid using too many symbols we still write
\[
U_\epsilon(t, \tau)z_\tau = U_{\epsilon,0}(t, \tau)z_\tau + U_{\epsilon,1}(t, \tau)z_\tau,
\]
where \( U_{\epsilon,0}(t, \tau)z_\tau = (v_\epsilon(t), v_{\epsilon t}(t)) \) solves
\[
\begin{cases}
\rho_\epsilon(t)v_{\epsilon tt} + \alpha v_{\epsilon t} + A v_\epsilon = 0, & t > \tau, \\
U_{\epsilon,0}(\tau, \tau)z_\tau = z_\tau,
\end{cases}
\]
and \( U_{\epsilon,1}(t, \tau)z_\tau = (w_\epsilon(t), w_{\epsilon t}(t)) \) solves
\[
\begin{cases}
\rho_\epsilon(t)w_{\epsilon tt} + \alpha w_{\epsilon t} + A w_\epsilon + f(u_\epsilon) = g, & t > \tau, \\
U_{\epsilon,1}(\tau, \tau)z_\tau = 0.
\end{cases}
\]
For any \( z_\tau \in \mathcal{H}_{r,1/3} \) with \( \|z_\tau\|_{\mathcal{H}_{r,1/3}} \leq R \), by Lemma 4.7 and estimate (4.7), we have
\[
\sup_{\epsilon \in [a,1]} \|U_\epsilon(t, \tau)z_\tau\|_{\mathcal{H}_{r,1/3}} \leq Q_a(R), \quad t \geq \tau.
\] (4.24)

Then repeating the same argument as Lemma 11.6 in [15], we have

**Lemma 4.8.** Let Assumption 4.1 be valid, and \( z_\tau \in \mathcal{H}_{r,1/3} \) with \( \|z_\tau\|_{\mathcal{H}_{r,1/3}} \leq R \). Then
\[
\sup_{\epsilon \in [a,1]} \|U_{\epsilon,0}(t, \tau)z_\tau\|_{\mathcal{H}_r} \leq Q_a(R) e^{-\kappa(t-\tau)},
\] (4.25)
\[
\sup_{\epsilon \in [a,1]} \|U_{\epsilon,1}(t, \tau)z_\tau\|_{\mathcal{H}_{r,1}} \leq Q_a(R), \quad t \geq \tau.
\] (4.26)

Based on Lemma 4.8, we further give the desired estimate.

**Lemma 4.9.** Let Assumption 4.1 be valid, and \( z_\tau \in \mathcal{H}_{r,1} \) with \( \|z_\tau\|_{\mathcal{H}_{r,1/3}} \leq R \). Then
\[
\sup_{\epsilon \in [a,1]} \|U_\epsilon(t, \tau)z_\tau\|_{\mathcal{H}_{r,1}} \leq Q_a(\|z_\tau\|_{\mathcal{H}_{r,1}}) e^{-\kappa(t-\tau)} + Q_a(R), \quad t \geq \tau.
\]

**Proof.** For simplicity, we omit the subscript \( \epsilon \) and let \( u = u_\epsilon \). For any \( \epsilon \in [a, 1] \) and \( \delta \in (0, 1) \), we define the functional along the solution \( (u(t), u_t(t)) = U_\epsilon(t, \tau)z_\tau \):
\[
L_\epsilon(t) = \|u\|_2^2 + \rho_\epsilon \|u_t\|_1^2 - 2\langle g, Au \rangle + \delta \left[ 2\rho_\epsilon \langle u_t, Au \rangle + \alpha \|u\|_1^2 \right].
\]

It follows from condition (4.4) that
\[
2|\langle g, Au \rangle| \leq 2\|g\| \|u\|_2 \leq \frac{1}{4}\|u\|_2^2 + 4\|g\|^2,
\]
\[
2\rho_\epsilon |\langle u_t, Au \rangle| \leq 2\rho_\epsilon \|u_t\|_1 \|u\|_1 \leq \alpha \|u\|_1^2 + \frac{L}{\alpha} \rho_\epsilon \|u_t\|_1^2.
\] (4.27)
Inserting (4.27) into $\mathcal{L}_\epsilon(t)$ receives

$$\frac{d}{dt}(u(t), u_t(t))\|^{2}_{\mathcal{H}_{t,1}} - 4\|g\|^2 \leq \mathcal{L}_\epsilon(t) \leq 2\|u(t), u_t(t))\|^{2}_{\mathcal{H}_{t,1}} + 4\|g\|^2$$  \hspace{1cm} (4.28)

for $\delta > 0$ suitably small.

Taking the multiplier $2Au_t + 2\delta Au$ in Eq. (4.8) gives

$$\frac{d}{dt}\mathcal{L}_\epsilon(t) + \delta\mathcal{L}_\epsilon(t) + \left(2\alpha - \rho'_e - 3\delta \rho_e\right)\|u_t\|^2 + \delta\|u\|^2 - \delta^2 \alpha\|u\|^2_1$$

$$= \left(2\delta^2 \rho_e + 2\delta \rho'_e\right)(u_t, Au) - \langle f(u), 2Au_t + 2\delta Au, t > \tau.$$  \hspace{1cm} (4.29)

Exploiting the Sobolev embedding (4.22), condition (4.5) and estimate (4.24), we have

$$\|f(u)\|_1^2 = \|f'(u)A^\frac{1}{2}u\|^2$$

$$\leq C\int_\Omega (1 + |u|^4) |A^\frac{1}{2}u|^2 \, dx$$

$$\leq C\left(1 + \|u\|_{L^4(\Omega)}^4\right) \|A^\frac{1}{2}u\|^2_{L^4(\Omega)}$$

$$\leq C\left(1 + \|u\|_{L^4/3}^4\right) \|u\|^2_{L^4/3} \leq Q_\alpha(R),$$

and hence,

$$-\langle f(u), 2Au_t + 2\delta Au\rangle \leq 2\|f(u)\|_1 \|u_t\|_1 + \|u\|_1 \leq \alpha\|u_t\|^2 + \frac{\delta}{4}\|u\|^2_2 + Q_\alpha(R).$$

Due to

$$\left(2\alpha - \rho'_e - 3\delta \rho_e\right)\|u_t\|^2 + \delta\|u\|^2 - \delta^2 \alpha\|u\|^2_1 \geq \frac{3}{2}\alpha\|u_t\|^2_1 + \frac{\delta}{2}\|u\|^2_2,$$

$$\left(2\delta^2 \rho_e + 2\delta \rho'_e\right)(u_t, Au) \leq 4\delta L\|u_t\|\|u\|_2 \leq \frac{\delta}{4}\|u\|^2_2 + Q_\alpha(R)$$

for $\delta > 0$ suitably small, inserting above estimates into (4.29), we have

$$\frac{d}{dt}\mathcal{L}_\epsilon(t) + \delta\mathcal{L}_\epsilon(t) \leq Q_\alpha(R), \ t > \tau.$$  \hspace{1cm} (4.30)

Applying the Gronwall inequality to (4.30) and exploiting estimate (4.28) turn out the conclusion of Lemma 4.9.

\begin{lemma}
Let Assumption 4.1 be valid, and $z_{i\tau} \in \mathcal{H}_{\tau,1}$ with $\|z_{i\tau}\|_{\mathcal{H}_{\tau,1}} \leq R, i = 1, 2$. Then, for all $\epsilon \in [a, 1]$, we have

$$\|U_\epsilon(t, \tau)z_{1\tau} - U_\epsilon(t, \tau)z_{2\tau}\|_{\mathcal{H}_t} \leq Q_\alpha(R)e^{-\kappa(t-\tau)}\|z_{1\tau} - z_{2\tau}\|_{\mathcal{H}_\tau} + Q_\alpha(R) \sup_{s \in [\tau, t]} \|\bar{u}_\epsilon(s)\|, \ t \geq \tau,$$

where

$$(\bar{u}_\epsilon(t), \bar{u}_{\epsilon t}(t)) = (u_{\epsilon 1}(t), u_{\epsilon 1t}(t)) - (u_{\epsilon 2}(t), u_{\epsilon 2t}(t)) = U_\epsilon(t, \tau)z_{1\tau} - U_\epsilon(t, \tau)z_{2\tau}.$$  \hspace{1cm} (4.31)
\end{lemma}
Proof. It follows from Lemma 4.9 that
\[
\sup_{\epsilon \in [a, 1]} \| U_\epsilon(t, \tau) z_{1\tau} \|_{H_{t,1}} + \| U_\epsilon(t, \tau) z_{2\tau} \|_{H_{t,1}} \leq Q_a(R), \ \forall t \geq \tau. \tag{4.31}
\]

For simplicity, we omit the subscript \( \epsilon \) and let \( u_i = u_{i\epsilon}, i = 1, 2 \). Obviously, the difference \( \tilde{u} \) solves
\[
\begin{cases}
\rho_i(t) \tilde{u}_t + \alpha \tilde{u}_t + A \tilde{u} + f(u_1) - f(u_2) = 0, t > \tau, \\
(\tilde{u}(\tau), \tilde{u}_t(\tau)) = z_{1\tau} - z_{2\tau}.
\end{cases} \tag{4.32}
\]

Taking the multiplier \( 2 \tilde{u}_t + 2\delta \tilde{u} \) in Eq. (4.32) gives
\[
\begin{aligned}
\frac{d}{dt} \Psi_\epsilon(t) + 2\delta \| \tilde{u} \|^2 + (2\alpha - \rho_\epsilon' - 2\delta \rho_\epsilon) \| \tilde{u}_t \|^2 & = 2\delta \rho_\epsilon' \langle \tilde{u}_t, \tilde{u} \rangle - (f(u_1) - f(u_2), 2 \tilde{u}_t + 2\delta \tilde{u}), \ t > \tau, \\
\end{aligned} \tag{4.33}
\]
where
\[
\begin{aligned}
\Psi_\epsilon(t) & = \rho_\epsilon \| \tilde{u} \|^2 + \| \tilde{u}_t \|^2 + \delta \left[ 2 \rho_\epsilon \langle \tilde{u}_t, \tilde{u} \rangle + \alpha \| \tilde{u} \|^2 \right] \sim \| (\tilde{u}, \tilde{u}_t) \|^2_{H_t}, \\
(2\alpha - \rho_\epsilon' - 2\delta \rho_\epsilon) \| \tilde{u}_t \|^2 & \geq \alpha \| \tilde{u}_t \|^2, \\
2\delta \rho_\epsilon' \langle \tilde{u}_t, \tilde{u} \rangle & \leq \frac{\alpha}{4} \| \tilde{u}_t \|^2 + Q_a(R) \| \tilde{u} \|^2.
\end{aligned} \tag{4.34}
\]

for \( \delta > 0 \) suitably small, and where we have used condition (4.4) and formula (4.7). By the Sobolev embedding \( V_2 \hookrightarrow L^\infty(\Omega) \), condition (4.5) and estimate (4.31), we have
\[
- \langle f(u_1) - f(u_2), 2 \tilde{u}_t + 2\delta \tilde{u} \rangle \\
\leq C \int_{\Omega} (1 + |u_1|^3 + |u_2|^3) |\tilde{u}| |\tilde{u}_t| + |\tilde{u}| \, dx \\
\leq C \left( 1 + \| u_1 \|^2_{L^\infty(\Omega)} + \| u_2 \|^2_{L^\infty(\Omega)} \right) \left[ \| \tilde{u} \|^2 + \| \tilde{u}_t \|^2 \right] \\
\leq Q_a(R) \| \tilde{u} \|^2 + \frac{\alpha}{4} \| \tilde{u}_t \|^2.
\]

Inserting above estimates into (4.33) and making use of (4.34) yield
\[
\frac{d}{dt} \Psi_\epsilon(t) + \delta \Psi_\epsilon(t) \leq Q_a(R) \| \tilde{u} \|^2, \ t > \tau. \tag{4.35}
\]

Applying the Gronwall inequality to (4.35) and making use of formula (4.34) receive
\[
\| (\tilde{u}(t), \tilde{u}_t(t)) \|_{H_t} \leq Q_a(R) e^{-\kappa(t-\tau)} \| z_{1\tau} - z_{2\tau} \|_{H_t} + Q_a(R) \sup_{s \in [\tau, t]} \| \tilde{u}(s) \|, \ t \geq \tau.
\]

This completes the proof. \( \square \)

Lemma 4.11. Let Assumption 4.1 be valid, and \( z_\tau \in H_{t,1} \) with \( \| z_\tau \|_{H_{t,1}} \leq R \). Then for any \( \epsilon_1, \epsilon_2 \in [a, 1] \),
\[
\| U_{\epsilon_1}(t, \tau) z_\tau - U_{\epsilon_2}(t, \tau) z_\tau \|_{H_t} \leq Q_a(R) e^{Q_a(R) (t-\tau)} | \epsilon_1 - \epsilon_2 |, \ \forall t \geq \tau. \tag{4.36}
\]

Proof. It follows from Lemma 4.9 that
\[
\| U_{\epsilon_1}(t, \tau) z_\tau \|_{H_{t,1}} + \| U_{\epsilon_2}(t, \tau) z_\tau \|_{H_{t,1}} \leq Q_a(R), \ \forall t \geq \tau. \tag{4.37}
\]
Let
\[(u_i(t), u_{it}(t)) = U_{e_i}(t, \tau)z_{\tau}, \quad t \geq \tau, \quad i = 1, 2.\]

Obviously, the difference \(\tilde{u} = u_1 - u_2\) solves
\[
\begin{align*}
\rho_{\epsilon_2} \tilde{u}_{tt} + (\rho_{\epsilon_1} - \rho_{\epsilon_2}) u_{1tt} + \alpha \tilde{u}_t + A \tilde{u} + f(u_1) - f(u_2) &= 0, \quad t > \tau, \\
(\tilde{u}(\tau), \tilde{u}_t(\tau)) &= 0.
\end{align*}
\]

(4.38)

Taking the multiplier \(2\tilde{u}_t\) in Eq. (4.38) gives
\[
\frac{d}{dt}\|\tilde{u}_t\|_{H^2_t}^2 + (2\alpha - \rho'_{\epsilon_2}) \|\tilde{u}_t\|^2 = - (\rho_{\epsilon_1} - \rho_{\epsilon_2}) \langle u_{1tt}, 2\tilde{u}_t \rangle - \langle f(u_1) - f(u_2), 2\tilde{u}_t \rangle.
\]

Taking into account \(\epsilon_1, \epsilon_2 \in [a, 1]\), by Eq. (4.8) and formula (4.37) we have
\[
\rho(t)\|u_{1tt}\| \leq \frac{1}{a} [\alpha \|u_{1tt}\| + \|Au_1\| + \|f(u_1)\| + \|g\|] \leq Q_1(R), \quad \forall t \geq \tau,
\]

where we have used the fact \(\|f(u_1)\| \leq Q_1(R)\) for \(V_2 \leftrightarrow L^\infty(\Omega)\). It follows from estimates (4.37) and (4.40) that
\[
- (\rho_{\epsilon_1} - \rho_{\epsilon_2}) \langle u_{1tt}, 2\tilde{u}_t \rangle \leq 2|\epsilon_1 - \epsilon_2|\rho(t)\|u_{1tt}\||\tilde{u}_t|| \leq \frac{\alpha}{2} \|\tilde{u}_t\|^2 + Q_1(R)|\epsilon_1 - \epsilon_2|^2,
\]
\[
- \langle f(u_1) - f(u_2), 2\tilde{u}_t \rangle \leq Q_1(R)\|\tilde{u}_t\|\|\tilde{u}_t\| \leq \frac{\alpha}{2} \|\tilde{u}_t\|^2 + Q_1(R)\|\tilde{u}_t\|^2.
\]

Inserting above estimates into formula (4.39) and making use of the fact \(\rho'_{\epsilon} < 0\), we obtain
\[
\frac{d}{dt}\|\tilde{u}_t\|_{H^2_t}^2 \leq Q_1(R)\|\tilde{u}_t\|_{H^2_t}^2 + Q_1(R)|\epsilon_1 - \epsilon_2|^2, \quad t > \tau.
\]

Applying the Gronwall inequality and estimate (4.7) give formula (4.36). \(\square\)

### 4.3 Proof of the main results

Based on the technical preparation in previous subsection, we prove Theorem 4.1 and Theorem 4.2 by applying the abstract criteria obtained in Section 3 and Section 4, respectively. These arguments are challenging because of the hyperbolicity of model (4.8). The method developed here allows to overcome this difficulty. We first construct a desired family \(B\) belonging to universe \(\mathcal{D}\).

**Lemma 4.12.** Let Assumption 4.1 be valid, and the interval \([a, 1] \subset (0, 1]\). Then there exists a family \(B = \{B(t)\}_{t \in \mathbb{R}} \in \mathcal{D}\) possessing the following properties:

(i) each section \(B(t)\) is closed in \(\mathcal{H}_t\) and
\[
B(t) \subset \mathcal{B}_t(R_0) \cap \mathcal{B}_t^1(R), \quad \forall t \in \mathbb{R}
\]

for some constants \(R = R(a) > 0\) and \(R_0 > R_1\), where \(R_1\) is as shown in Remark 4.4, \(\mathcal{B}_t^1(R)\) is the \(R\)-ball in \(\mathcal{H}_{t,1}\) centered at 0;

(ii) there exist positive constants \(\kappa_1\) and \(\tau_1\) such that
\[
\sup_{t \geq \tau} \text{dist}_{\mathcal{H}_t} (U_\epsilon(t, \tau)\mathcal{B}_t(R_1), B(t)) \leq Q_1(R_1)e^{-\kappa_1(t-\tau)}, \quad \forall t \geq \tau + \tau_1,
\]

where the family \(\{\mathcal{B}_t(R_1)\}_{t \in \mathbb{R}} \in \mathcal{D}\) is as shown in Remark 4.4;
(iii) there exists a positive constant $T_1$ such that

$$\bigcup_{t \geq \tau + T_1} U_\epsilon(t, \tau) B(\tau) \subset B(t), \quad \forall t \geq \tau + T_1. \quad (4.43)$$

**Proof.** For any $z_\tau \in \mathbb{B}_\tau(R_1)$, it follows from Lemma 4.6 that

$$\sup_{\epsilon \in [a, 1]} \|U_{\epsilon,0}(t, \tau) z_\tau\|^2_{\mathcal{H}_t} \leq Q_a(R_1) e^{-\kappa(t-\tau)} \quad \text{and} \quad \sup_{\epsilon \in [a, 1]} \|U_{\epsilon,1}(t, \tau) z_\tau\|^2_{\mathcal{H}_t,1} \leq Q_a(R_1), \quad \forall t \geq \tau,$$

which imply that there exists a positive constant $R_1 = R_1(R_1)$ such that

$$\sup_{\epsilon \in [a, 1]} \text{dist}_{\mathcal{H}_t} \left( U_\epsilon(t, \tau) \mathbb{B}_\tau(R_1), \mathbb{B}_t^{1/3}(R_1) \right) \leq Q_a(R_1) e^{-\kappa(t-\tau)}, \quad \forall t \geq \tau. \quad (4.44)$$

Similarly, for any $z_\tau \in \mathbb{B}_\tau^{1/3}(R_1)$, we infer from Lemma 4.8 that

$$\sup_{\epsilon \in [a, 1]} \|U_{\epsilon,0}(t, \tau) z_\tau\|^2_{\mathcal{H}_t} \leq Q_a(R_1) e^{-\kappa(t-\tau)} \quad \text{and} \quad \sup_{\epsilon \in [a, 1]} \|U_{\epsilon,1}(t, \tau) z_\tau\|^2_{\mathcal{H}_t,1} \leq Q_a(R_1), \quad t \geq \tau,$$

which imply that there exists a positive constant $R_2 = R_2(R_1)$ such that

$$\sup_{\epsilon \in [a, 1]} \text{dist}_{\mathcal{H}_t} \left( U_\epsilon(t, \tau) \mathbb{B}_\tau^{1/3}(R_1), \mathbb{B}_t^{1/3}(R_1) \right) \leq Q_a(R_1) e^{-\kappa(t-\tau)}, \quad \forall t \geq \tau. \quad (4.45)$$

Remark 4.4 shows that the family \{\mathbb{B}_\tau(R_1)\}_{t \in \mathbb{R} \in \mathcal{D}} is a uniformly (w.r.t. $\epsilon \in [a, 1]$) pullback $\mathcal{D}$-absorbing family of the processes $U_\epsilon(t, \tau), \epsilon \in [a, 1]$, that is, there exists a positive constant $e(R_1)$ such that

$$\bigcup_{t \geq \tau + e(R_1)} U_\epsilon(t, \tau) \mathbb{B}_\tau(R_1) \subset \mathbb{B}_t(R_1), \quad \forall t \geq \tau + e(R_1). \quad (4.46)$$

Let $\theta = \frac{\kappa}{Q_a(R_1) + 2 \kappa}$. Obviously,

$$\theta \in (0, 1) \quad \text{and} \quad -\kappa \theta = -\kappa + (Q_a(R_1) + \kappa) \theta.$$

We infer from formula (4.46) that there exists a constant $e_1 = \frac{e(R_1)}{1-\theta} > 0$ such that

$$\bigcup_{t \geq \tau + e_1} U_\epsilon((1 - \theta)t + \theta \tau, \tau) \mathbb{B}_\tau(R_1) \subset \mathbb{B}_{(1-\theta)t+\theta\tau}(R_1), \quad \forall t \geq \tau + e_1. \quad (4.47)$$

For any $t \geq \tau$, let $t_1 = (1 - \theta)t + \theta \tau$. It follows from Lemma 4.10 and formulas (4.44)-(4.47) that

$$\sup_{\epsilon \in [a, 1]} \text{dist}_{\mathcal{H}_t} \left( U_\epsilon(t, \tau) \mathbb{B}_\tau(R_1), \mathbb{B}_t^{1/3}(R_2) \right) \leq \sup_{\epsilon \in [a, 1]} \text{dist}_{\mathcal{H}_t} \left( U_\epsilon(t, t_1) U_\epsilon(t_1, \tau) \mathbb{B}_\tau(R_1), U_\epsilon(t, t_1) \mathbb{B}_t^{1/3}(R_1) \right)$$

$$+ \sup_{\epsilon \in [a, 1]} \text{dist}_{\mathcal{H}_t} \left( U_\epsilon(t, t_1) \mathbb{B}_t^{1/3}(R_1), \mathbb{B}_t^{1/3}(R_2) \right), \quad (4.48)$$

$$\leq Q_a(R_1) \exp \{Q_a(R_1)(t - t_1)\} \sup_{\epsilon \in [a, 1]} \text{dist}_{\mathcal{H}_{t_1}} \left( U_\epsilon(t_1, \tau) \mathbb{B}_\tau(R_1), \mathbb{B}_t^{1/3}(R_1) \right)$$

$$+ Q_a(R_1) e^{-\kappa(t-t_1)} \leq Q_a(R_1) \exp \{-\kappa + (Q_a(R_1) + \kappa) \theta \} (t - \tau) + Q_a(R_1) e^{-\kappa \theta(t-\tau)}$$

$$\leq Q_a(R_1) e^{-\kappa \theta(t-\tau)}, \quad \forall t \geq \tau + e_1.$$
For every $\xi \in \mathcal{B}_t^1(\mathcal{R}_2)$,
\[\|\xi\|_{\mathcal{H}_t} \leq \lambda_1^{-1/2}\|\xi\|_{\mathcal{H}_{t,1}} \leq \lambda_1^{-1/2}\mathcal{R}_2, \ \forall t \in \mathbb{R},\]
which implies
\[\mathcal{B}_t^1(\mathcal{R}_2) \subset \mathcal{B}_t^1(\lambda_1^{-1/2}\mathcal{R}_2) \subset \mathcal{B}_t(\mathcal{R}_3) \text{ and } \mathcal{B}_t(R_1) \subset \mathcal{B}_t(\mathcal{R}_3), \ \forall t \in \mathbb{R}, \tag{4.49}\]
with $\mathcal{R}_3 = R_1 + \lambda_1^{-1/2}\mathcal{R}_2(R_1) = \mathcal{R}_3(R_1)$. By Remark 4.4 and formula (4.49), there exists a constant $e_2 = e_2(R_1) > 0$ such that
\[\bigcup_{e \in [a,1]} U_e(t,\tau)\mathcal{B}_t(\mathcal{R}_3) \subset \mathcal{B}_t(R_1) \subset \mathcal{B}_t(\mathcal{R}_3), \ \forall t \geq \tau + e_2. \tag{4.50}\]
It follows from Lemma 4.7 that for any $z_\tau \in \mathcal{B}_t(\mathcal{R}_3) \cap \mathcal{H}_{\tau,1/3}$,
\[\sup_{e \in [a,1]} \|U_e(t,\tau)z_\tau\|_{\mathcal{H}_{t,1/3}}^2 \leq Q_a \left(\mathcal{R}_3 + \|z_\tau\|_{\mathcal{H}_{\tau,1/3}}\right) e^{-\kappa(t-\tau)} + \mathcal{R}_4, \ \forall t \geq \tau, \tag{4.51}\]
with the positive constant $\mathcal{R}_4 = Q_a(\mathcal{R}_3(R_1)) = \mathcal{R}_4(R_1)$.
Similarly, for every $\xi \in \mathcal{B}_t^1(\mathcal{R}_2)$, we have
\[\|\xi\|_{\mathcal{H}_{t,1/3}} \leq \lambda_1^{-1/3}\|\xi\|_{\mathcal{H}_{t,1}} \leq \lambda_1^{-1/3}\mathcal{R}_2, \ \forall t \in \mathbb{R},\]
which means
\[\mathcal{B}_t^1(\mathcal{R}_2) \subset \mathcal{B}_t^{1/3}(\lambda_1^{-1/3}\mathcal{R}_2) \subset \mathcal{B}_t^{1/3}(\mathcal{R}_5), \ \forall t \in \mathbb{R}, \tag{4.52}\]
where $\mathcal{R}_5 = \mathcal{R}_4 + \lambda_1^{-1/3}\mathcal{R}_2 = \mathcal{R}_5(R_1)$. It follows from formula (4.51) that there exists a positive constant $e_3 = e_3(R_1)$ such that
\[\bigcup_{e \in [a,1]} U_e(t,\tau) \left[\mathcal{B}_t(\mathcal{R}_3) \cap \mathcal{B}_t^{1/3}(\mathcal{R}_5)\right] \subset \mathcal{B}_t^{1/3}(\mathcal{R}_5), \ \forall t \geq \tau + e_3. \tag{4.53}\]
Lemma 4.9 shows that for any $z_\tau \in \mathcal{B}_t^{1/3}(\mathcal{R}_5) \cap \mathcal{H}_{\tau,1}$,
\[\sup_{e \in [a,1]} \|U_e(t,\tau)z_\tau\|_{\mathcal{H}_{t,1}}^2 \leq Q_a \left(\mathcal{R}_5 + \|z_\tau\|_{\mathcal{H}_{\tau,1}}\right) e^{-\kappa(t-\tau)} + \mathcal{R}_6, \ \forall t \geq \tau, \tag{4.54}\]
where the positive constant $\mathcal{R}_6 = Q_a(\mathcal{R}_5) = \mathcal{R}_6(R_1)$. Obviously,
\[\mathcal{B}_t^1(\mathcal{R}_2) \subset \mathcal{B}_t^1(\mathcal{R}_7) \text{ with } \mathcal{R}_7 = \mathcal{R}_2 + \mathcal{R}_6 = \mathcal{R}_7(R_1), \ \forall t \in \mathbb{R}. \tag{4.55}\]
Formula (4.54) implies that there exists a positive constant $e_4 = e_4(R_1)$ such that
\[\bigcup_{e \in [a,1]} U_e(t,\tau) \left[\mathcal{B}_t^{1/3}(\mathcal{R}_5) \cap \mathcal{B}_t^1(\mathcal{R}_7)\right] \subset \mathcal{B}_t^1(\mathcal{R}_7), \ \forall t \geq \tau + e_4. \tag{4.56}\]
Let
\[B(t) = \mathcal{B}_t(\mathcal{R}_3) \cap \mathcal{B}_t^{1/3}(\mathcal{R}_5) \cap \mathcal{B}_t^1(\mathcal{R}_7), \ \forall t \in \mathbb{R}.\]
We show that \(\{B(t)\}_{t \in \mathbb{R}} \in \mathcal{D}\) is the desired family.
(i) Obviously, for every $t \in \mathbb{R}$, $B(t)$ is closed in $\mathcal{H}_t$ and

$$B(t) \subset \mathbb{B}_t(\mathcal{R}_3) \cap \mathbb{B}_t^1(\mathcal{R}_7), \quad \forall t \in \mathbb{R},$$

that is, formula (4.41) holds, with $\mathcal{R}_0 = \mathcal{R}_3 > R_1$ and $\mathcal{R} = \mathcal{R}_7 = \mathcal{R}(a)$.

(ii) It follows from formulas (4.49), (4.52) and (4.55) that $\mathbb{B}_t^1(\mathcal{R}_2) \subset B(t)$ for all $t \in \mathbb{R}$. Then we infer from estimate (4.48) that

$$\sup_{\epsilon \in [a,1]} \text{dist}_{\mathcal{H}_t} \left( U_\epsilon(t, \tau) \mathbb{B}_\tau(R_1), B(t) \right) \leq \sup_{\epsilon \in [a,1]} \text{dist}_{\mathcal{H}_t} \left( U_\epsilon(t, \tau) \mathbb{B}_\tau(R_1), \mathbb{B}_t^1(\mathcal{R}_2) \right)$$

$$\leq Q_a(R_1)e^{-\kappa\theta(t-\tau)}, \quad \forall t \geq \tau + e_1,$$

that is, formula (4.42) holds, with $\kappa_1 = \kappa\theta$ and $\tau_1 = e_1$.

(iii) Taking $T_1 = \max\{e_2, e_3, e_4\}$ and making use of formulas (4.50), (4.53) and (4.56) yield

$$\bigcup_{\epsilon \in [a,1]} U_\epsilon(t, \tau)B(\tau) \subset \left\{ \begin{array}{ll}
\bigcup_{\epsilon \in [a,1]} U_\epsilon(t, \tau) \mathbb{B}_\tau(\mathcal{R}_3) \subset \mathbb{B}_t(\mathcal{R}_3), \\
\bigcup_{\epsilon \in [a,1]} U_\epsilon(t, \tau) \left[ \mathbb{B}_\tau(\mathcal{R}_3) \cap \mathbb{B}_t^{1/3}(\mathcal{R}_5) \right] \subset \mathbb{B}_t^{1/3}(\mathcal{R}_5), \\
\bigcup_{\epsilon \in [a,1]} U_\epsilon(t, \tau) \left[ \mathbb{B}_\tau^{1/3}(\mathcal{R}_5) \cap \mathbb{B}_t^1(\mathcal{R}_7) \right] \subset \mathbb{B}_t^1(\mathcal{R}_7),
\end{array} \right.$$ 

for all $t \geq \tau + T_1$. Therefore,

$$\bigcup_{\epsilon \in [a,1]} U_\epsilon(t, \tau)B(\tau) \subset \mathbb{B}_t(\mathcal{R}_3) \cap \mathbb{B}_t^{1/3}(\mathcal{R}_5) \cap \mathbb{B}_t^1(\mathcal{R}_7) = B(t), \quad \forall t \geq \tau + T_1.$$

This completes the proof. \qed

**Proof of Theorem 4.2.** For any $\epsilon_0 \in (0,1]$, there must be an interval $[a,1] \subset (0,1]$ such that $\epsilon_0 \in [a,1]$. Lemma 4.12 shows that there exists a family $\mathcal{B} = \{B(t)\}_{t \in \mathcal{S}}$ with the properties (i)-(iii) there.

Take $T > T_1$ satisfying $\eta := Q_a(\mathcal{R})e^{-\kappa T} < 1/4$, formulas (4.41) and (4.43) mean that condition $(H_1)$ of Assumption 3.1 holds, with $X_t = \mathcal{H}_t$ and $\Lambda = [a,1]$ there.

Let the space

$$Z = \{ \phi \in C([0,T]; V_1) \mid \phi_t \in C([0,T]; L^2) \}$$

be equipped with the norm

$$\| \phi \|_Z = \sup_{s \in [0,T]} \| (\phi(s), \phi_t(s)) \|_{V_1 \times L^2}.$$ 

Obviously, $Z$ is a Banach space, and

$$n_Z(\phi) = Q_a(\mathcal{R}) \sup_{s \in [0,T]} \| \phi(s) \|, \quad \forall \phi \in Z$$

is a compact seminorm on $Z$ (cf. [38]). Taking $t_0 = 0$, we define the mapping

$$K_t^\epsilon : B(t-T) \subset \mathcal{H}_{t-T} \rightarrow Z, \quad K_t^\epsilon \xi = u_\epsilon(\cdot + t - T), \quad \forall \xi \in B(t-T), \quad \epsilon \in [a,1], \quad t \leq t_0,$$

where $u_\epsilon(\cdot + t - T)$ means $u_\epsilon(s + t - T), s \in [0,T]$ and

$$(u_\epsilon(s + t - T), u_{\epsilon t}(s + t - T)) = U_\epsilon(s + t - T, t - T)\xi.$$ 

Formula (4.10) shows that for all $\xi_1, \xi_2 \in B(t-T)(\subset \mathbb{B}_{t-T}(\mathcal{R}_0) \cap \mathbb{B}_{t-T}^1(\mathcal{R})), \tau \in [0,T]$,

$$\sup_{\epsilon \in [a,1]} \| U_\epsilon(t, t - \tau)\xi_1 - U_\epsilon(t, t - \tau)\xi_2 \|_{\mathcal{H}_t} \leq L_1 \| \xi_1 - \xi_2 \|_{\mathcal{H}_{t-T}}, \quad \forall t \in \mathbb{R},$$

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with \( L_1 = Q_\alpha(R_0)e^{Q_\alpha(R_0)T} \), and Lemma 4.10 shows that when \( t \leq t_0 = 0 \),
\[
\|U_\epsilon(t, t-T)\xi_1 - U_\epsilon(t, t-T)\xi_2\|_{\mathcal{H}_t} \leq \eta\|\xi_1 - \xi_2\|_{\mathcal{H}_{t-T}} + n_Z (K_1^t \xi_1 - K_1^t \xi_2),
\]
where
\[
\sup_{\epsilon \in [a,1]} \|K_1^t \xi_1 - K_1^t \xi_2\|_Z = \sup_{\epsilon \in [a,1]} \sup_{s \in [0,T]} \|U_\epsilon(s + t - T, t-T)\xi_1 - U_\epsilon(s + t - T, t-T)\xi_2\|_{V_1 \times L^2} 
\leq \sup_{s \in [0,T]} \left[ 1 + \left( \rho(s + t - T) \right)^{-\frac{1}{2}} \right] \sup_{\epsilon \in [a,1]} \|U_\epsilon(s + t - T, t-T)\xi_1 - U_\epsilon(s + t - T, t-T)\xi_2\|_{H_{s+t-T}} 
\leq \left[ 1 + \left( \rho(0) \right)^{-\frac{1}{2}} \right] Q_\alpha(R)e^{Q_\alpha(R)T} \|\xi_1 - \xi_2\|_{\mathcal{H}_{t-T}},
\]
where we have used the fact that \( \rho \) is a decreasing function. That is, the conditions \((H_2)-(H_3)\) of Assumption 3.1 hold.

Moreover, Remark 4.4 shows that the process \( U_\lambda(t, \tau) \) has a uniformly pullback \( \mathcal{D} \)-absorbing family \( D_0 = \{ \mathbb{B}_t(R_1) \}_{t \in \mathbb{R}} \in \mathcal{D} \) possessing the properties: (see (4.42), (4.41) and (4.10))
\[
(i) \quad \sup_{\epsilon \in [a,1]} \text{dist}_{X_1}(U_\epsilon(t, t-\tau)\mathbb{B}_{t-\tau}(R_1), B(t)) \leq Q_\alpha(R_1)e^{-\kappa_1 \tau}, \ \forall t \in \mathbb{R}, \ \tau \geq \tau_1; \quad (4.57)
\]
\[
(ii) \quad \mathbb{B}_t(R_1) \subset \mathbb{B}_t(R_0), \ \forall t \in \mathbb{R} \text{ for } \mathcal{R}_0 > R_1, \text{ and}
\sup_{\epsilon \in [a,1]} \|U_\epsilon(t, t-\tau)\xi_1 - U_\epsilon(t, t-\tau)\xi_2\|_{\mathcal{H}_t} \leq Q_\alpha(R_0)e^{Q_\alpha(R_0)\tau} \|\xi_1 - \xi_2\|_{\mathcal{H}_{t-\tau}}, \quad (4.58)
\]
for all \( x, y \in \mathbb{B}_{t-\tau}(R_0), \ \tau \geq 0 \text{ and } t \in \mathbb{R} \).

That is, the conditions of Corollary 3.5 hold. Therefore, by Corollary 3.5, the process \( U_\epsilon(t, \tau) \) has a pullback \( \mathcal{D} \)-exponential attractor \( E_{\epsilon} = \{ E_{\epsilon}(t) \}_{t \in \mathbb{R}} \) for each \( \epsilon \in [a,1] \), and \( E_{\epsilon}(t) \subset B(t) \subset \mathbb{B}_1^1(\mathcal{R}) \) for all \( t \in \mathbb{R} \). By the arbitrariness \( a \in (0,1) \) we have
\[
\sup_{t \in \mathbb{R}} \|E_{\epsilon}(t)\|_{\mathcal{H}_{t,1}} < +\infty, \ \forall \epsilon \in (0,1].
\]

Lemma 4.11 shows that
\[
\Gamma(\epsilon, \epsilon_0) := \sup_{t \leq t_0} \sup_{\xi \in B(t-T)} \sup_{\epsilon \in [0,1]} \|U_\epsilon(t, t-s)\xi - U_{\epsilon_0}(t, t-s)\xi\|_{\mathcal{H}_t} \leq Q_\alpha(R)e^{Q_\alpha(R)T}|\epsilon - \epsilon_0|.
\]
Taking \( \delta = \delta(\epsilon_0) = \left( Q_\alpha(R)e^{Q_\alpha(R)T} \right)^{-1} \), we obtain that \( \Gamma(\epsilon, \epsilon_0) < 1 \) whenever \( |\epsilon - \epsilon_0| < \delta \). Then by Theorem 3.2,
\[
\text{dist}_{\mathcal{H}_t}^{\text{symm}}(E_{\epsilon}(t), E_{\epsilon_0}(t)) \leq C(t)|\epsilon - \epsilon_0|^\gamma \text{ as } |\epsilon - \epsilon_0| < \delta, \ \forall t \in \mathbb{R}.
\]
where \( \gamma \in (0,1) \) is a positive constant. By the arbitrariness of \( \epsilon_0 \in (0,1] \), we complete the proof.

In order to prove Theorem 4.1, we first quote a few notations and lemmas on the pullback \( \mathcal{D} \)-attractor.

**Definition 4.13.** (Pullback \( \mathcal{D} \)-asymptotically compact) A process \( U(t, \tau) : X_\tau \to X_t \) is said to be pullback \( \mathcal{D} \)-asymptotically compact if for any \( \{ \tau_n \} \subset (-\infty, t] \) with \( \tau_n \to -\infty \), and \( y_n \in D(\tau_n) \subset \mathcal{D} \in \mathcal{D} \), the sequence \( \{ U(t, \tau_n)y_n \} \) is relatively compact in \( X_t \).
Lemma 4.14. [29] If the process \( U(t, \tau) : X_\tau \rightarrow X_t \) has a compact pullback \( \mathcal{D} \)-attracting family, then it is pullback \( \mathcal{D} \)-asymptotically compact.

Lemma 4.15. [29] Assume that the process \( U(t, \tau) : X_\tau \rightarrow X_t \) is continuous, and (i) it has a pullback \( \mathcal{D} \)-absorbing family \( B_0 = \{ B_0(t) \}_{t \in \mathbb{R}} \in \mathcal{D} \); (ii) it is pullback \( \mathcal{D} \)-asymptotically compact. Then the family \( A = \{ A(t) \}_{t \in \mathbb{R}} \), with

\[
A(t) := \bigcap_{s \leq t} \left[ \bigcup_{\tau \leq s} U(t, \tau)B_0(\tau) \right]_{X_t}, \quad t \in \mathbb{R},
\]

is the minimal pullback \( \mathcal{D} \)-attractor of \( U(t, \tau) \).

Proof of Theorem 4.1. For any \( n \in \mathbb{N}^+ = \{1, 2, \cdots \} \), let

\[
[a_n, 1] = [1/(n + 1), 1].
\]

Obviously, \( \cup_{n \in \mathbb{N}^+} [a_n, 1] = (0, 1] \).

(i) Remark 4.4 shows that the process \( U_\epsilon(t, \tau) \) has a uniformly pullback \( \mathcal{D} \)-absorbing family \( \{ B(t) \}_{t \in \mathbb{R}} \), that is, for any \( D \in \mathcal{D} \), there exists a constant \( e(D) > 0 \) such that

\[
\bigcup_{\epsilon \in [a_n, 1]} U_\epsilon(t, t - \tau)D(t - \tau) \subset B_t(R_1), \quad \forall \tau \geq e(D), \quad t \in \mathbb{R}.
\]

(4.60)

Lemma 4.12 shows that there exists a family \( B = \{ B(t) \}_{t \in \mathbb{R}} \in \mathcal{D} \) possessing the properties (i)-(iii) there. Formula (4.41) implies that the section \( B(t) \) is compact in \( \mathcal{H}_t \) for each \( t \in \mathbb{R} \) because of \( \mathcal{H}_{t,1} \hookrightarrow \hookrightarrow \mathcal{H}_t \). The combination of (4.42) and (4.60) receives

\[
\sup_{\epsilon \in [a_n, 1]} \text{dist}_{\mathcal{H}_t} (U_\epsilon(t, t - \tau)D(t - \tau), B(t)) \leq \sup_{\epsilon \in [a_n, 1]} \text{dist}_{\mathcal{H}_t} (U_\epsilon(t, t - \tau + e(D))U_\epsilon(t - \tau + e(D), t - \tau)D(t - \tau), B(t)) \leq \sup_{\epsilon \in [a_n, 1]} \text{dist}_{\mathcal{H}_t} (U_\epsilon(t, t - \tau + e(D))B_{t - \tau + e(D)}(R_1), B(t)) \leq Q_{a_n}(R_1)e^{\epsilon 1 e(D)}e^{-\epsilon \tau}, \quad \forall \tau \geq e(D) + \tau_1, \quad t \in \mathbb{R},
\]

that is, \( B = \{ B(t) \}_{t \in \mathbb{R}} \) is a compact pullback \( \mathcal{D} \)-attracting family of the process \( U_\epsilon(t, \tau) \). Therefore, by Lemma 4.14 and Lemma 4.15, the family \( A_\epsilon = \{ A_\epsilon(t) \}_{t \in \mathbb{R}} \), with

\[
A_\epsilon(t) := \bigcap_{s \leq t} \left[ \bigcup_{\tau \leq s} U_\epsilon(t, \tau)B(t) \right]_{\mathcal{H}_t}, \quad t \in \mathbb{R}
\]

is the minimal pullback \( \mathcal{D} \)-attractor of the process \( U_\epsilon(t, \tau) \), and (see (4.41))

\[
\bigcup_{\epsilon \in [a_n, 1]} A_\epsilon(t) \subset B(t) \subset B^1_t(\mathcal{R}), \quad \forall t \in \mathbb{R},
\]

(4.61)

where the constant \( \mathcal{R} = \mathcal{R}(a_n) \). Therefore, by the arbitrariness of \( n \in \mathbb{N}^+ \),

\[
\sup_{t \in \mathbb{R}} \| A_\epsilon(t) \|_{\mathcal{H}_{t,1}} < \infty, \quad \forall \epsilon \in (0, 1],
\]

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We know from the minimality of $\mathcal{A}_\epsilon$ that $A_\epsilon(t) \subset E_\epsilon(t)$ for all $t \in \mathbb{R}$, where $\mathcal{E}_\epsilon = \{E_\epsilon(t)\}_{t \in \mathbb{R}}$ is a pullback $\mathcal{D}$-exponential attractor of the process $U_\epsilon(t, \tau)$ as shown in Theorem 4.2. Therefore,

$$\sup_{t \in \mathbb{R}} \dim_f (A_\epsilon(t), \mathcal{H}_t) \leq \sup_{t \in \mathbb{R}} \dim_f (E_\epsilon(t), \mathcal{H}_t) < +\infty, \quad \forall \epsilon \in (0, 1].$$

(ii) (Upper semicontinuity) For any $\epsilon_0 \in (0, 1]$, there must be $\epsilon_0 \in \Lambda_n := [a_n, 1]$ for some $n \in \mathbb{N}^+$. For any $\xi \in B(t)(\subset \mathbb{B}_1^H(\mathcal{R}))$, by Lemma 4.11 we have

$$\sup_{\xi \in B(\tau)} \left\| U_\epsilon(t, \tau)\xi - U_{\epsilon_0}(t, \tau)\xi \right\|_{\mathcal{H}_t} \leq Q_{a_n}(\mathcal{R})e^{Q_{a_n}(\mathcal{R})(t-\tau)}|\epsilon - \epsilon_0| \rightarrow 0 \text{ as } \epsilon \rightarrow \epsilon_0.$$

And $\mathcal{A}_\epsilon$ pullback attracts $B = \{B(t)\}_{t \in \mathbb{R}}$ for all $\epsilon \in \Lambda_n$ for $B \in \mathcal{D}$, that is,

$$\lim_{\tau \rightarrow -\infty} \text{dist}_{\mathcal{H}_t} (U_\epsilon(t, \tau)B(\tau), A_\epsilon(t)) = 0, \quad \forall t \in \mathbb{R}, \epsilon \in \Lambda_n. \quad (4.62)$$

Formulas (4.61)-(4.62) mean that the conditions $(L_1)$-$(L_3)$ of Assumption 2.1 hold for $\epsilon \in \Lambda_n$. By Theorem 2.1, the pullback $\mathcal{D}$-attractor $\mathcal{A}_\epsilon$ is upper semicontinuous at $\epsilon_0$. By the arbitrariness of $\epsilon_0 \in (0, 1]$, $\mathcal{A}_\epsilon$ is upper semicontinuous on $(0, 1]$.

(iii) (Residual continuity) Since $\Lambda_n = [a_n, 1]$ is a compact metric space for each $n \in \mathbb{N}^+$, we infer from Theorem 2.4 that there exists a residual subset $\Lambda^*_n \subset \Lambda_n$ such that $\mathcal{A}_\epsilon$ is continuous on $\Lambda^*_n$. Let

$$\Lambda^* = \bigcup_{n \in \mathbb{N}^+} \Lambda^*_n.$$

Then $\Lambda^*$ is still a residual subset of $(0, 1]$ and $\mathcal{A}_\epsilon$ is continuous on $\Lambda^*$ for the arbitrariness of $n \in \mathbb{N}^+$. \hfill \Box

References

[1] H. Amann, Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problem, in: Schmeisser/Triebel: Function Spaces, Differential Operators and Nonlinear Analysis, Teubner Texte zur Mathematik, vol. 133, Teubner, 1993, pp. 9-126.

[2] G. S. Aragão, F. D. M. Bezerra, R. N. Figueroa-López, M. J. D. Nascimento, Continuity of pullback attractors for evolution processes associated with semilinear damped wave equations with time-dependent coefficients, J. Differential Equations 298 (2021) 30-67.

[3] J. Arrieta, A. N. Carvalho, J. K. Hale, A damped hyperbolic equation with critical exponent, Comm. Partial Differential Equations 17 (1992) 841-866.

[4] J. M. Arrieta, A. N. Carvalho, Spectral convergence and nonlinear dynamics of reactiondiffusion equations under perturbations of the domain, J. Differential Equations 199 (2004) 143-178.

[5] A. V. Babin, S. Yu Pilyugin, Continuous dependence of attractors on the shape of domain, J. Math. Sci. 87 (1997) 3304-3310.

[6] F. D. M. Bezerra, A. N. Carvalho, J. W. Cholewa, M. J. D. Nascimento, Parabolic approximation of damped wave equations via fractional powers: fast growing nonlinearities and continuity of dynamics, J. Math. Anal. Appl. 450 (2017) 377-405.

[7] M. C. Bortolan, A. N. Carvalho, J. A. Langa, Attractors under autonomous and non-autonomous perturbations, Mathematical Surveys and Monographs, 246. Amer. Math. Soc., Providence, RI, 2020.

[8] S. M. Bruschi, A. N. Carvalho, Upper semicontinuity of attractors for the discretization of strongly damped wave equation, Mat. Contemp. 32 (2007) 39-62.

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[9] T. Caraballo, A. N. Carvalho, H. B. Costa, J. A. Langa, Equi-attraction and continuity of attractors for skew-product semiflows, Discrete Contin. Dyn. Syst. Ser. B 21 (2016) 2949-2967.

[10] A. N. Carvalho, T. Dłotko, H. M. Rodrigues, Upper semicontinuity of attractors and synchronization, J. Math. Anal. Appl. 220 (1998) 13-41.

[11] A. N. Carvalho, J. A. Lange, J. C. Robinson, On the continuity of pullback attractors for evolution processes, Nonlinear Anal. 71 (2009) 1812-1824.

[12] I. Chueshov, I. Lasiecka, Long-time behavior of second order evolution equations with nonlinear damping. Amer. Math. Soc., Providence, RI, 2008.

[13] I. Chueshov, Dynamics of Quasi-Stable Dissipative Systems, Springer, New York, 2015.

[14] M. Conti, V. Pata, Weakly dissipative semilinear equations of viscoelasticity, Commun. Pure Appl. Anal. 4 (2005) 705-720.

[15] M. Conti, V. Pata, R. Temam, Attractors for the processes on time-dependent spaces. Application to wave equations, J. Differential Equations 255 (2013) 1254-1277.

[16] M. Conti, V. Pata, Asymptotic structure of the attractor for processes on time-dependent spaces, Nonlinear Anal., Real World Appl. 19 (2014) 1-10.

[17] M. Conti, V. Pata, On the time-dependent Cattaneo law in space dimension one, Appl. Math. Comput. 259 (2015) 32-44.

[18] M. Conti, V. Danese, C. Giorgi, V. Pata, A model of viscoelasticity with time-dependent memory kernels, Amer. J. Math. 140 (2018) 349-389.

[19] M. Conti, V. Danese, V. Pata, Viscoelasticity with time-dependent memory kernels.II: asymptotical behavior of solutions, Amer. J. Math. 140 (2018) 1687-1729.

[20] F. Di Plinio, G. S. Duane, R. Temam, Time-dependent attractor for the oscillon equation, Discrete Contin. Dyn. Syst. 29 (2011) 141-167.

[21] A. Eden, C. Foias, B. Nicolaenko, R. Temam, Exponential attractors for dissipative evolution equations, John-Wiley, New York, 1994.

[22] M. Efendiev, A. Miranville, S. Zelik, Exponential attractors and finite-dimensional reduction for non-autonomous dynamical systems, Proc. Roy. Soc. Edinburgh Sect. A 135 (2005) 703-730.

[23] M. M. Freitas, P. Kalita, J. A. Langa, Continuity of non-autonomous attractors for hyperbolic perturbation of parabolic equations, J. Differential Equations 264 (2018) 1886-1945.

[24] M. Grasselli, V. Pata, Asymptotic behavior of a parabolic-hyperbolic system, Commun. Pure Appl. Anal. 3 (2004) 849-881.

[25] J. K. Hale, R. Geneviève, Lower semicontinuity of attractors of gradient systems and applications, Ann. Mat. Pura Appl. 154(1989) 281-326.

[26] J. K. Hale, G. Raugel, Lower semicontinuity of the attractor for a singularly perturbed hyperbolic equation, J. Dyn. Differ. Equ. 2 (1990) 19-67.

[27] L. T. Hoang, E. J. Olason, J. C. Robinson, On the continuity of global attractors, Proc. Amer. Math. Sc. 143 (10) (2015) 4389-4395.

[28] L. T. Hoang, E. J. Olason, J. C. Robinson, Continuity of pullback and uniform attractors, J. Differential Equations 264 (2018) 4067-4093.

[29] P. E. Kloeden, P. María-Rubio, J. Real, Pullback attractors for a semilinear heat equation in a non-cylindrical domain, J. Differential Equations 244 (2008) 2062-2090.

[30] P. E. Kloeden, J. Real, C. Y. Sun, Pullback attractors for a semilinear heat equation on time-varying domains, J. Differential Equations, 246 (2009) 4702-4730.
[31] D. S. Li, P. E. Kloeden, Equi-attraction and the continuous dependence of attractors on parameters, Glasgow Math. J. 46 (2004) 131-142.

[32] D. S. Li, P. E. Kloeden, Equi-attraction and the continuous dependence of pullback attractors on parameters, Stoch. Dyn. 4 (2004) 373-384.

[33] Y. N. Li, Z. J. Yang, Exponential attractor for the viscoelastic wave model with time-dependent memory kernels, J. Dyn. Differ. Equ. 2021, https://doi.org/10.1007/s10884-021-10035-z.

[34] Y. R. Li, S. Yang, Hausdorff sub-norm spaces and continuity of random attractors for bi-stochastic g-Navier-Stokes Equations with respect to tempered forces, J. Dyn. Differ. Equ. 2021, https://doi.org/10.1007/s10884-021-10026-0.

[35] T. F. Ma, P. Mário-Rubio, C. M. Surco Cháno, Dynamics of wave equations with moving boundary, J. Differential Equations 262 (2017) 3317-3343.

[36] F. J. Meng, M. H. Yang, C. K. Zhong, Attractors for wave equations with nonlinear damping on time-dependent space, Discrete Contin. Dyn. Syst. B 21 (2016) 205-225.

[37] J. C. Oxtoby, Measure and Category, 2nd ed, Springer-Verlag, New York, 1980.

[38] J. Simon, Compact sets in the space $L^p(0, T; B)$, Ann. Mat. Pura Appl. 146 (1986) 65-96.

[39] X. Y. Song, C. Y. Sun, L. Yang, Pullback attractors for 2D Navier-Stokes equations on time-varying domains, Nonlinear Anal., Real World Appl. 45 (2019) 437-460.

[40] A. M. Stuart, A. R. Humphries, Dynamical Systems and Numerical Analysis, Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, Cambridge, 1996.

[41] C. Y. Sun, Y. B. Yuan, $L^p$-type pullback attractors for a semilinear heat equation on time-varying domains, Proc. Roy. Soc. Edinburgh Sect. A 145 (2015) 1029-1052.

[42] Y. H. Wang, C. K. Zhong, Upper semicontinuity of pullback attractors for nonautonomous Kirchhoff wave models, Discrete Contin. Dyn. Syst. 33 (2013) 3189-3209.

[43] Y. P. Xiao, C. Y. Sun, Higher-order asymptotic attraction of pullback attractors for a reaction-diffusion equation in non-cylindrical domains, Nonlinear Anal. 113 (2015) 309-322.

[44] Z. J. Yang, Y. N. Li, Criteria on the existence and stability of pullback exponential attractors and their application to non-autonomous Kirchhoff wave models, Discrete Contin. Dyn. Syst. 38 (2018) 2629-2653.

[45] F. Zhou, C. Y. Sun, J. Q. Cheng, Dynamics for the complex Ginzburg-Landau equation on non-cylindrical domains II: The monotone case, J. Math. Phys. 59 (2018) 022703.