Collinear and triangular solutions to the three-body problem in the parameterized post-Newtonian formalism

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This paper investigates the three-body problem in the parameterized post-Newtonian (PPN) formalism, for which we focus on a coplanar case in a class of fully conservative theories characterized by the Eddington-Robertson parameters $\beta$ and $\gamma$. It is shown that there can still exist a collinear equilibrium configuration and a triangular one, each of which is a generalization of the post-Newtonian equilibrium configuration in general relativity. The collinear configuration can exist for arbitrary mass ratio, $\beta$ and $\gamma$. On the other hand, the PPN triangular configuration depends on the nonlinearity parameter $\beta$ but not on $\gamma$. For any value of $\beta$, the equilateral configuration is possible, if and only if three finite masses are equal or two test masses orbit around one finite mass. For general mass cases, the PPN triangle is not equilateral as in the post-Newtonian case. It is shown also that the PPN displacements from the standard Lagrange points $L_1$, $L_2$ and $L_3$ depend on $\beta$ and $\gamma$, whereas those to $L_4$ and $L_5$ rely only on $\beta$.

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I. INTRODUCTION

The three-body problem is among the classical ones in physics. It led to the notion of the chaos [1]. On the other hand, particular solutions such as Euler’s collinear solution and Lagrange’s equilateral one [2, 3] express regular orbits and they have still attracted interest e.g. [4-8]. The collinear solution and triangular one correspond to Lagrange points $L_1$, $L_2$, $L_3$, $L_4$ and $L_5$ as particular solutions for the coplanar restricted three-body problem.

In his pioneering work [9], Nordtvedt found that the position of the triangular points is very sensitive to the ratio of the gravitational mass to the inertial mass in gravitational experimental tests, where the post-Newtonian (PN) terms are not fully taken into account. Kretz [10] and Maindl [11] studied the restricted three-body problem in the PN approximation and found the PN triangular configuration for general masses. These investigations were extended to the PN three-body problem for general masses [12-17], and the PN counterparts for Euler’s collinear [12, 13] and Lagrange’s equilateral solutions [14, 15] were obtained. It should be noted that the PN triangular solutions are not necessarily equilateral for general mass ratios and they are equilateral only for either the equal mass case or two test masses. The stability of the PN solution and the radiation reaction at 2.5PN order were also examined [16, 17].

In a scalar-tensor theory of gravity, a collinear configuration for three-body problem was discussed [18]. In addition to such fully classical treatments, a possible quantum gravity correction to the Lagrange points was proposed [19, 20].

Moreover, the recent discovery of a relativistic hierarchical triple system including a neutron star [21] has generated renewed interest in the relativistic three-body problem and the related gravitational experiments [22-24].

The main purpose of the present paper is to reexamine the three-body problem especially in the PPN formalism. One may ask if collinear and triangular configurations are still solutions for the coplanar three-body problem in the PPN gravity. If so, how large are the PPN effects of the three-body configuration? We focus on the Eddington-Robertson parameters $\beta$ and $\gamma$, because the two parameters are the most important ones; $\beta$ measures how much nonlinearity there is in the superposition law for gravity and $\gamma$ measures how much space curvature is produced by unit rest mass [23, 25]. Hence, preferred locations, preferred frames or a violation of conservation of total momentum will not be considered in this paper. We confine ourselves to a class of fully conservative theories. See e.g. [27] for the celestial mechanics in this class of PPN theories.

This paper is organized as follows. In Section II, collinear configurations are discussed in the PPN formalism. Section III investigates PPN triangular configurations. In Section IV, the PPN corrections to the standard Lagrange points are examined. Section V summarizes this paper. Throughout this paper, $G = c = 1$. $A(= 1, 2, 3)$ denotes a mass, and B, C denote the other masses ($B \neq A, C \neq A$ and $C \neq B$).

II. COLLINEAR CONFIGURATION IN PPN GRAVITY

A. Euler’s collinear solution in Newton gravity

Let us begin with briefly mentioning the Euler’s collinear solution for the circular three-body problem in Newton gravity [2, 3], for which each mass $M_A$ ($A = 1, 2, 3$) at $x_A$ is orbiting around the common center of
Eq. (4) is solved for \( z \) for which there exists the only positive root \([2, 3]\). Once \( \omega \) is chosen, we define the relative vector between masses as \( \mathbf{v}_{AB} \equiv \mathbf{v}_A - \mathbf{v}_B \), for which the relative length is \( R_{AB} = \| \mathbf{v}_{AB} \| \). See Figure 1 for a configuration of the Euler’s collinear solution.

The coordinate origin \( x = 0 \) is chosen between \( M_1 \) and \( M_3 \), such that \( R_1 > R_2 > R_3 \), \( R_1 > 0 \) and \( R_3 < 0 \). By taking account of this sign convention, the equation of motion becomes

\[
R_1 \omega^2 = \frac{M_2}{R_{12}^2} + \frac{M_3}{R_{13}^2},
\]

\[
R_2 \omega^2 = -\frac{M_1}{R_{12}^2} + \frac{M_3}{R_{23}^2},
\]

\[
R_3 \omega^2 = -\frac{M_1}{R_{13}^2} + \frac{M_2}{R_{23}^2}.
\]

We define the distance ratio as \( z \equiv R_{23}/R_{12} \), which plays a key role in the following calculations. Note that \( z > 0 \) by definition. We subtract Eq. (2) from Eq. (1) and Eq. (3) from Eq. (2). By combining the results including the same angular velocity \( \omega \), we obtain a fifth-order equation for \( z \) as

\[
(M_1 + M_2)z^5 + (3M_1 + 2M_2)z^4 + (3M_1 + M_2)z^3 - (M_2 + 3M_3)z^2 - (2M_2 + 3M_3)z - (M_2 + M_3) = 0,
\]

for which there exists the only positive root \([2, 3]\). Once Eq. (4) is solved for \( z \), we can obtain \( \omega \) by substituting \( z \) into any of Eqs. (1-3).

B. PPN collinear configuration

In a class of fully conservative theories including only the Eddington-Robertson parameters \( \beta \) and \( \gamma \), the equation of motion is [25][26]

\[
\mathbf{a}_A = -\sum_{B \neq A} \frac{M_B}{R_{AB}^2} \mathbf{n}_{AB} \cdot \mathbf{v}_B - \frac{\gamma}{2} \mathbf{n}_{AB} - \frac{3}{2} \mathbf{n}_{AB} \cdot \mathbf{v}_B^2 - \left( 2\gamma + 2\beta + 1 \right) \frac{M_A}{R_{AB}}
\]

\[
-2(\gamma + \beta) \frac{M_B}{R_{AB}} \mathbf{n}_{AB}
\]

\[
+ \sum_{B \neq A} \frac{M_B}{R_{AB}^2} \left( \mathbf{n}_{AB} \cdot (2(\gamma + 1)\mathbf{v}_A - (2\gamma + 1)\mathbf{v}_B) \right)(\mathbf{v}_A - \mathbf{v}_B)
\]

\[
+ \sum_{B \neq A} \sum_{C \neq A,B} \frac{M_B M_C}{R_{AB} R_{AC} R_{BC}} \left[ 2(\gamma + \beta) \frac{M_A}{R_{AB}} - 2(\gamma + \beta) \frac{M_B}{R_{AC}} + 2(\gamma + \beta) - 1 \frac{M_C}{R_{BC}} \right] \mathbf{n}_{BC}
\]

\[
- \frac{1}{2} (4\gamma + 3) \sum_{B \neq A} \sum_{C \neq A,B} \frac{M_B M_C}{R_{AB} R_{BC}^2} \mathbf{n}_{BC} + O(c^{-4}),
\]

(5)

where

\[
\mathbf{n}_{AB} \equiv \frac{R_{AB}}{R_{AB}^2}
\]

(6)

For three aligned masses, Eq. (5) becomes the force-balance equation as

\[
\ell \omega^2 = F_N + F_M + F_V \omega^2,
\]

(7)

where we define the mass ratio \( \nu_A \equiv M_A/M \) for \( M = \sum_A M_A \), and

\[
F_N = \frac{M}{\ell z^2} \left[ 1 - \nu_1 - \nu_3 + 2(1 - \nu_1 - \nu_3)z + (2 - \nu_1 - \nu_3)z^2 \right] + 2(1 - \nu_1 - \nu_3)z^3 + (1 - \nu_1 - \nu_3)z^4,
\]

(8)
By rearranging Eq. (5) for the collinear configuration and

\[ F_M = -\frac{M^2}{\ell^2 z^3} \{ (2(\beta + \gamma)\nu_2 + (1 + 2\beta + 2\gamma)\nu_3)\nu_2 \]
\[ + \{(1 - 4\beta + 2\gamma)\nu_1 + 6(\beta + \gamma)\nu_2 + 3(1 + 2\beta + 2\gamma)\nu_3\nu_2z \]
\[ + \{(1 + 10\beta + 4\gamma)\nu_1 + 6(\beta + \gamma)\nu_2 + (12\beta + 4\gamma - 5)\nu_3\nu_2z \]
\[ + \{3(1 + 2\beta + 2\gamma)\nu_1 + 6(\beta + \gamma)\nu_2 + (1 + 4\beta + 2\gamma)\nu_3\nu_2z \]
\[ + \{(1 + 2\beta + 2\gamma)\nu_1 + 2(\beta + \gamma)\nu_2\nu_2z^6\}, \]

(9)

and

\[ F_V = \frac{M}{(1 + z)^2 z^2} \times \left[ -\nu_2^2 \nu_2 - 2\nu_1\nu_2(2\nu_1 + \nu_2)z \right.
\[ + \{\nu_1^3 + ((-2 + 4\gamma)\nu_1 + 3(1 + \gamma)\nu_3)\nu_2^2 \]
\[ + (2\nu_2 + \nu_3)(\nu_2^2 + (1 + 2\gamma)\nu_2\nu_3 + \gamma\nu_3^2) \]
\[ + ((-1 + 5\gamma)\nu_2^2 + 8(1 + \gamma)\nu_2\nu_3 + 3(1 + \gamma)\nu_3^2)\nu_1\nu_2^2 \]
\[ + 2(\nu_1 + 2\nu_2 + \nu_3)\{\nu_1^2 + \gamma\nu_2^2 + (1 + 2\gamma)\nu_2\nu_3 + \gamma\nu_3^2\} \]
\[ + (1 + 4\gamma)\nu_2 + (3 + 2\gamma)\nu_3\nu_2^2 \]
\[ + \{\nu_1^3 + 2\nu_2^2 + (1 - 5\gamma)\nu_2\nu_3 - 2(1 - 2\gamma)\nu_2\nu_3^2 \]
\[ + \nu_2^2 + 3(1 + 4\gamma)\nu_2 + 3(1 + 4\gamma)\nu_2 \]
\[ + \{(2 + 5\gamma)\nu_2^2 + 8(1 + \gamma)\nu_2\nu_3 + 3(1 + \gamma)\nu_3^2\}\nu_1\nu_2^2 \]
\[ - 2\nu_2\nu_3(\nu_2^2 + \nu_3^2)z^5 - \nu_2^2\nu_3^6 \}. \]

(10)

By rearranging Eq. (5) for the collinear configuration by the same way as in subsection II.A, we find a seventh-order equation for \( z \) as

\[ \sum_{k=0}^{7} A_k z^k = 0, \]

(11)

where the coefficients are

\[ A_7 = \frac{M}{\ell} \left[ -2(\beta + \gamma) - 2\nu_1 + 4(\beta + \gamma)\nu_3 + 2\nu_1^2 + 4\nu_1\nu_3 \right. \]
\[ - 2(\beta + \gamma)\nu_3^2 - 2\nu_1^2\nu_3 - 2\nu_1\nu_3^2 \].

(12)

\[ A_6 = -\nu_3 \]
\[ + \frac{M}{\ell} \left[ -6(\beta + \gamma) - 6(2\beta + 2\gamma)\nu_1 \right. \]
\[ - (2 - 8\beta - 11\gamma)\nu_1 + 4\nu_1^2 + (12 + 2\beta + 2\gamma)\nu_1\nu_3 \]
\[ + (4 - 2\beta - 4\gamma)\nu_3^2 + 2\nu_1^2 - 6\nu_1\nu_3^2 - 2\nu_3^2 \].

(13)

\[ A_5 = 2 + \nu_1 - 2\nu_3 \]
\[ + \frac{M}{\ell} \left[ -3(2\beta + 3\gamma) - 3(2 + 2\beta + 2\gamma)\nu_1 - (6 - 11\gamma)\nu_1 \right. \]
\[ + (12 + 6\beta + 2\gamma)\nu_1\nu_3 + (12 + 6\beta - 2\gamma)\nu_3^2 \]
\[ + 6\nu_1^2 - 6\nu_1\nu_3^2 - 6\nu_3^2 \].

(14)

\[ A_4 = 1 + 2\nu_1 - \nu_3 \]
\[ + \frac{M}{\ell} \left[ -2\beta - 4\gamma - (2\beta + 8\gamma)\nu_1 - (6 + 6\beta - 8\gamma)\nu_3 \right. \]
\[ - (6 + 4\beta - 2\gamma)\nu_1^2 + (4 + 2\beta - 2\gamma)\nu_1\nu_3 \]
\[ + (12 + 8\beta - 4\gamma)\nu_3^2 + 6\nu_1^2 + 4\nu_1\nu_3^2 - 4\nu_1\nu_3^2 - 6\nu_3^2 \].

(15)

\[ A_3 = -1 + \nu_1 - 2\nu_3 \]
\[ + \frac{M}{\ell} \left[ 2\beta + 4\gamma + (6 + 6\beta - 8\gamma)\nu_1 + (2\beta + 8\gamma)\nu_3 \right. \]
\[ - (12 + 8\beta - 4\gamma)\nu_1^2 - (4 + 2\beta - 2\gamma)\nu_1\nu_3 \]
\[ + (6 + 4\beta - 2\gamma)\nu_3^2 + 6\nu_1^2 + 4\nu_1\nu_3^2 - 2\nu_1\nu_3^2 - 6\nu_3^2 \].

(16)

\[ A_2 = -2 + 2\nu_1 - \nu_3 \]
\[ + \frac{M}{\ell} \left[ 6\beta + 9\gamma + (6 + 11\gamma)\nu_1 + (6 + 6\beta + 6\gamma)\nu_3 \right. \]
\[ - (12 + 6\beta - 2\gamma)\nu_1^2 - (12 + 6\beta + 2\gamma)\nu_1\nu_3 \]
\[ + 6\nu_1^3 + 4\nu_1^2\nu_3 - 6\nu_3^3 \].

(17)

\[ A_1 = 1 + \nu_1 \]
\[ + \frac{M}{\ell} \left[ 6\beta + 7\gamma + (2 - 8\beta - 11\gamma)\nu_1 + (6 + 2\beta + 2\gamma)\nu_3 \right. \]
\[ - (4 - 2\beta - 4\gamma)\nu_1^2 - (12 + 2\beta + 2\gamma)\nu_1\nu_3 \]
\[ - 4\nu_3^2 + 2\nu_1^2 + 4\nu_1\nu_3^2 + 6\nu_1^2\nu_3 - 2\nu_3^2 \].

(18)

\[ A_0 = \frac{M}{\ell} \left[ 2\beta + 2\gamma - 4(\beta + \gamma)\nu_1 + 2\nu_3 + 2(\beta + \gamma)\nu_1^2 \right. \]
\[ - 4\nu_1\nu_3 - 2\nu_1^2 + 2\nu_1\nu_3^2 + 2\nu_1\nu_3^2 \].

(19)
It follows that Eq. (11) recovers the PN collinear configuration by Eq. (13) of Reference [13] if and only if \( \beta = \gamma = 1 \). The uniqueness is because the number of the parameters \( \beta, \gamma \) is two for eight coefficients \( A_0, \ldots, A_7 \).

From Eq. (7) for \( z \) obtained above, the angular velocity \( \omega_{PPN} \) of the PPN collinear configuration is obtained as

\[
\omega_{PPN} = \omega_N \left( 1 + \frac{F_M}{2F_N} + \frac{F_V}{2F} \right),
\]

where \( \omega_N = (F_N/\ell) \) is the Newtonian angular velocity. The subscript \( N \) denotes the Newtonian case.

**III. TRIANGULAR CONFIGURATION IN PPN GRAVITY**

**A. Lagrange’s equilateral solution in Newtonian gravity**

In this subsection, we suppose that the three masses are in circular motion with keeping the same separation between the masses, namely \( R_{AB} = a \) for a constant \( a \).

The coordinate origin is chosen as the COM

\[
\sum A M_A x_A = 0.
\]

The equation of motion becomes

\[
d^2 x_A \over dt^2 = -\frac{M}{a^3} x_A.
\]

A triangular configuration is a solution, if the Newtonian angular velocity \( \omega_N \) satisfies

\[
(\omega_N)^2 = \frac{M}{a^3}.
\]

The orbital radius \( \ell_A \) of each mass around the COM is [2]

\[
\ell_1 = a \sqrt{\nu_1^2 + \nu_2 \nu_3}, \quad \ell_2 = a \sqrt{\nu_2^2 + \nu_1 \nu_3}, \quad \ell_3 = a \sqrt{\nu_3^2 + \nu_1 \nu_2}.
\]

**B. PPN orbital radius**

We suppose again that three masses in circular motion are in a triangular configuration with a constant angular velocity \( \omega \). By noting that a vector in the orbital plane can be expressed as a linear combination of \( x_1 \) and \( v_1 \), Eq. (5) becomes

\[
-g_1 = M/a \left( \frac{2\beta + \gamma + (\nu_2 + \nu_3)(\nu_2 + \nu_3 - 1) - \frac{7}{16} \nu_2 \nu_3}{\nu_2^2 + \nu_2 \nu_3 + \nu_3^2} \right) \omega_N v_1,
\]

where Eq. (23) is used and

\[
g_1 = \frac{M}{a} \left[ \left( 2\beta + \gamma + (\nu_2 + \nu_3)(\nu_2 + \nu_3 - 1) - \frac{7}{16} \nu_2 \nu_3 \right) + \frac{3}{16} \nu_2 \nu_3 \{9\nu_2 \nu_3 + 2(\nu_2 + \nu_3)(8\beta - 5)\} \right].
\]

By a cyclic permutation, we obtain the similar equations for \( M_2 \) and \( M_3 \).

The second and third terms in the right-hand side of Eq. (27) are the PPN forces. The second term is parallel to \( x_1 \), whereas the third term is parallel to \( v_1 \). Note that \( v_1 \parallel x_1 \) in circular motion.

The location of the COM in the fully conservative theories of PPN [28, 29] remains the same as that in the PN approximation of general relativity [30, 31]

\[
G_{PN} = \frac{1}{E} \sum A M_A x_A \left[ 1 + \frac{1}{2} \left( v_A^2 - \sum_{B \neq A} M_B \frac{M_B}{R_{AB}} \right) \right].
\]

where \( E \) is defined as

\[
E = \sum A M_A \left[ 1 + \frac{1}{2} \left( v_A^2 - \sum_{B \neq A} M_B \frac{M_B}{R_{AB}} \right) \right].
\]

This coincidence allows us to obtain the PPN orbital radius \( \ell_A^{PPN} \) around the COM by straightforward calculations. The orbital radius of \( M_1 \) is formally obtained as

\[
(\ell_1^{PPN})^2 = (\ell_1)^2 + \frac{a M}{2} \left( 1 - \frac{a^3 \omega_N^2}{M} \right) \times (-2\nu_1^2 \nu_2 - 2\nu_1^2 \nu_3 - 2\nu_2^2 \nu_1
\]

\[
+ 2\nu_1 \nu_2^3 + \nu_2 \nu_3^3 + \nu_3 \nu_1^3 + 2\nu_1^3 \nu_1
\]

\[
- 2\nu_1^2 \nu_2 \nu_3 + \nu_1 \nu_2 \nu_3^2 + \nu_1 \nu_2 \nu_3^2),
\]

and the similar expressions of \( \ell_2^{PPN} \) and \( \ell_3^{PPN} \) for the orbital radius of \( M_2 \) and \( M_3 \) are obtained. The second term of the right-hand side in Eq. (31) vanishes because \( \omega_N \) satisfies Eq. (23). As a result, the PPN orbital radius remains the same as the Newtonian one. Namely, \( \ell_A^{PPN} = \ell_A \).

**C. Equilateral condition**

First, we discuss a condition for an equilateral configuration.

For Eq. (27) to hold, the coefficient of the velocity vector \( v_1 \) must vanish, because there are no other terms including \( v_1 \). The coefficient is proportional to \( \nu_2 \nu_3 (\nu_2 - \nu_3) \). The same thing is true also of \( M_2 \) and \( M_3 \). For any value of \( \beta \), therefore, the equilateral configuration in the PPN gravity can be present if and only if three
finite masses are equal or two test masses orbit around one finite mass.

Note that one can find a very particular value of \( \beta \) satisfying

\[
16 \beta - 1 - 9 \nu_1 = 0, \tag{32}
\]

which leads to the vanishing coefficient of the velocity vector \( \nu_1 \). However, this choice is very unlikely, because the particular value of \( \beta \) is dependent on the mass ratio \( \nu_1 \) and is not universal. Hence, this case will be ignored.

\section{PPN triangular configuration for general masses}

Next, let us consider a PPN triangle configuration for general masses. For this purpose, we introduce a nondimensional parameter \( \varepsilon_{AB} \) at the PPN order, such that each side length of the PPN triangle can be expressed as

\[
R_{AB} = a(1 + \varepsilon_{AB}). \tag{33}
\]

The equilateral case is achieved by assuming \( \varepsilon_{AB} = 0 \) for every masses. See Figure 2 for the PPN triangular configuration.

In order to fix the degree of freedom corresponding to a scale transformation, we follow Reference [15] to suppose that the arithmetic mean of the three side lengths is unchanged as

\[
\frac{R_{12} + R_{23} + R_{31}}{3} = a \left[ 1 + \frac{1}{3} (\varepsilon_{12} + \varepsilon_{23} + \varepsilon_{31}) \right]. \tag{34}
\]

The left-hand side of Eq. (34) is a in the Newtonian case, which leads to

\[
\varepsilon_{12} + \varepsilon_{23} + \varepsilon_{31} = 0. \tag{35}
\]

This is a gauge fixing in \( \varepsilon_{AB} \).

In terms of \( \varepsilon_{AB} \), Eq. (27) is rearranged as

\[
-\omega^2 x_1 = -(\omega_N)^2 x_1
- \frac{3}{2} \frac{(\omega_N)^2}{\nu_1^2 + \nu_2^2 + \nu_3^2}
\times \left[ \nu_2(\nu_1 - \nu_2 - 1)\varepsilon_{12} + \nu_3(\nu_1 - \nu_3 - 1)\varepsilon_{31} \right] x_1
+ \sqrt{3} \nu_2 \nu_3 (\varepsilon_{12} - \varepsilon_{31}) \frac{v_1}{\omega_N} + \delta_1, \tag{36}
\]

where

\[
\delta_1 = g_1(\omega_N)^2 x_1 + \frac{\sqrt{3} \nu_2 \nu_3 (\nu_2 - \nu_3)(16 \beta - 1 - 9 \nu_1)}{16a(\nu_1^2 + \nu_2^2 + \nu_3^2)} \omega_N v_1. \tag{37}
\]

By a cyclic permutation, the equations for \( M_2 \) and \( M_3 \) can be obtained.

A triangular equilibrium configuration can exist if and only if the two conditions (A) and (B) are simultaneously satisfied; (A) Each mass satisfies Eq. (36), and (B) the configuration is unchanged in time. Condition (B) means \( \omega_1 = \omega_2 = \omega_3 \).

Condition (A) is equivalent to Condition (A2); The coefficient of \( v_A \) in the equation of motion vanishes as

\[
\varepsilon_{12} - \varepsilon_{31} - \frac{M}{24a} (\nu_2 - \nu_3)(16 \beta - 1 - 9 \nu_1) = 0, \tag{38}
\]

\[
\varepsilon_{23} - \varepsilon_{21} - \frac{M}{24a} (\nu_3 - \nu_1)(16 \beta - 1 - 9 \nu_2) = 0, \tag{39}
\]

\[
\varepsilon_{31} - \varepsilon_{23} - \frac{M}{24a} (\nu_1 - \nu_2)(16 \beta - 1 - 9 \nu_3) = 0. \tag{40}
\]

From Eqs. (38)-(40) and the gauge fixing as \( \varepsilon_{12} + \varepsilon_{23} + \varepsilon_{31} = 0 \), we obtain

\[
\varepsilon_{12} = \frac{M}{72a} \left[ (\nu_2 - \nu_3)(16 \beta - 1 - 9 \nu_1) \right. \\
- (\nu_3 - \nu_1)(16 \beta - 1 - 9 \nu_2), \tag{41}
\]

\[
\varepsilon_{23} = \frac{M}{72a} \left[ (\nu_3 - \nu_1)(16 \beta - 1 - 9 \nu_2) \right. \\
- (\nu_1 - \nu_2)(16 \beta - 1 - 9 \nu_3), \tag{42}
\]

and

\[
\varepsilon_{31} = \frac{M}{72a} \left[ (\nu_1 - \nu_2)(16 \beta - 1 - 9 \nu_3) \right. \\
- (\nu_2 - \nu_3)(16 \beta - 1 - 9 \nu_1). \tag{43}
\]

Therefore, the PPN triangle is inequilateral depending on \( \beta \) via \( \varepsilon_{AB} \) but not on \( \gamma \). This suggests that also the
PPN Lagrange points corresponding to $L_4$ and $L_5$ are sensitive to $\beta$ but are free from $\gamma$, as shown in Section IV.

It follows that Eqs. (41)-(43) recover the PN counterpart of Eq. (26)-(28) of Reference [15], if and only if $\beta = 1$. The uniqueness is because the PPN parameter is only $\beta$ for three equations as Eqs. (41), (42), (43).

By substituting Eqs. (41) and (43) into Eq. (36), the angular velocity $\omega_{PPN}$ of the PPN configuration is obtained as

$$\omega_{PPN} = \omega_N (1 + \delta_\omega),$$

where the PPN correction $\delta_\omega$ is

$$\delta_\omega = \frac{3}{4} \frac{\nu_2 (\nu_1 - \nu_2 - 1) \varepsilon_{12} + \nu_3 (\nu_1 - \nu_3 - 1) \varepsilon_{31}}{\nu_2^2 + \nu_2 \nu_3 + \nu_3^2} - \frac{1}{2} g_1.$$

IV. PPN CORRECTIONS TO THE STANDARD LAGRANGE POINTS

A. PPN Lagrange points $L_1$, $L_2$ and $L_3$

In this section, we discuss PPN modifications of the standard Lagrange points that are originally defined in the restricted three-body problem in Newtonian gravity. We choose $\nu_A = 1 - \nu$, $\nu_B = \nu$ and $\nu_C = 0$, where $\nu$ is the mass ratio of the secondary object (a planet).

First, we seek PPN corrections to $L_1$, $L_2$ and $L_3$. There are three choices of how to correspond $M_1$, $M_2$ and $M_3$ to the Sun, a planet and a test mass in the collinear configuration. Indeed the thee choices lead to the Lagrange points $L_1$, $L_2$ and $L_3$.

We consider the collinear solution by Eq. (11). We denote the physical root for Eq. (11) as $z = z_N (1 + \varepsilon)$ for the Newtonian root $z_N$ with using a small parameter $\varepsilon (|\varepsilon| \ll 1)$ at the PPN order. We substitute $z$ into Eq. (11) and rearrange it to obtain $\varepsilon$ as

$$\varepsilon = - \sum_{k=0}^{7} A_k^{PPN} (z_N)^k \sum_{k=1}^{6} k A_k^N (z_N)^k,$$

where $O(\varepsilon^2)$ is discarded because of being at the 2PN order, and $A_k^N$ and $A_k^{PPN}$ denote the Newtonian and PPN parts of $A_k$, respectively, as $A_k = A_k^N + \varepsilon A_k^{PPN}$ ($A_0^N = 0$ and $A_2^N = 0$ because there are no counterparts in the Newtonian case).

Eq. (46) is used for calculating the PPN corrections to $L_1$, $L_2$ and $L_3$. The PPN displacement from the standard Lagrange point $L_1$ is thus obtained as

$$\delta_{PPN} R_{31} \equiv R_{31} - (R_{31})_N$$

where $M_1$, $M_2$ and $M_3$ are chosen as a planet, a test mass and the Sun, respectively.

Similarly, the PPN displacement from the standard $L_2$ becomes

$$\delta_{PPN} R_{23} \equiv R_{23} - (R_{23})_N$$

where $M_1$, $M_2$ and $M_3$ are chosen as the Sun, a planet and a test mass, respectively. The PPN displacement from the standard $L_3$ is

$$\delta_{PPN} R_{31} \equiv R_{31} - (R_{31})_N$$

where $M_1$, $M_2$ and $M_3$ are chosen as a planet, the Sun and a test mass, respectively. Here, a value of $z_N$ depends on $L_1$, $L_2$ or $L_3$, which is given by Eq. (11).

B. PPN Lagrange points $L_4$ and $L_5$

Next, we discuss PPN corrections to the standard Lagrange points $L_4$ and $L_5$, for which we consider the PPN triangular solution. Let $a$ denote the orbital separation between the primary object and the secondary one, which equals to $R_{12} = \ell (1 + \varepsilon_{12})$. Therefore, $\ell = a (1 - \varepsilon_{12}) + O(a \varepsilon^2)$, where $\varepsilon^2$ denotes the second order in $\varepsilon_{AB}$. By using this for $R_{23}$ and $R_{31}$, we obtain $R_{23} = a (1 + \varepsilon_{23} - \varepsilon_{12}) + O(a \varepsilon^2)$ and $R_{31} = a (1 + \varepsilon_{31} - \varepsilon_{12}) + O(a \varepsilon^2)$.

The PPN displacement from the standard Lagrange point $L_4$ (and $L_5$) with respect to the Sun is obtained as

$$\delta_{PPN} R_{31} \equiv R_{31} - a$$

$$= a (\varepsilon_{31} - \varepsilon_{12}) + O(a \varepsilon^2)$$

$$= - \frac{16 \beta - 10 + 9 \nu}{24} M$$

$$+ O \left( \frac{M^2}{a} \right),$$

where $\nu_1 = 1 - \nu$, $\nu_2 = \nu$ and $\nu_3 = 0$ are used in the last line.

In the similar manner, the PPN displacement from the standard Lagrange point $L_4$ (and $L_5$) with respect to the planet

$$\delta_{PPN} R_{23} \equiv R_{23} - a$$

$$= a (\varepsilon_{23} - \varepsilon_{12}) + O(a \varepsilon^2)$$

$$= - \frac{(1 - \nu)(16 \beta - 10 + 9 \nu)}{24} M$$

$$+ O \left( \frac{M^2}{a} \right).$$
TABLE I. The PPN displacement from the standard Lagrange points of the Sun-Jupiter system. The PPN corrections to $L_1$, $L_2$, $L_3$ and $L_4$ are listed in this table, where the sign convention for $L_1$, $L_2$, $L_3$ is chosen along the direction from the Sun to the Jupiter, and the correction to $L_5$ is identical to that to $L_4$. The PPN displacement from the standard $L_4$ is two-dimensional and hence they are indicated by the deviations from the Sun and from the Jupiter.

| Lagrange points | PPN displacement [m] |
|-----------------|----------------------|
| $L_1$           | $-0.000051 + 40.000\beta - 9.905\gamma$ |
| $L_2$           | $0.000040 - 50.27\beta + 12.40\gamma$ |
| $L_3$           | $0.000122 + 1.424\beta + 0.01882\gamma$ |
| $L_4(L_5)$-Sun  | $-0.05875 \times (-9.991 + 16\beta)$ |
| $L_4(L_5)$-Jupiter | $61.47 \times (-1 + 16\beta)$ |

C. Example: the Sun-Jupiter case

The PPN corrections to the standard $L_1$, $L_2$ and $L_3$ can be expressed as a linear function in $\beta$ and $\gamma$. The PPN corrections to $L_4$ and $L_5$ are in a linear function only of $\beta$. The results for the Sun-Jupiter system are summarized in Table I, where the sign convention is chosen along the direction from the Sun to a planet.

Before closing this section, we mention gravitational experiments. The lunar laser ranging experiment put a constraint on $\eta = 4\beta - \gamma - 3$ as $|\eta| < O(10^{-4})$ \cite{8, 9}. If one wish to constrain $1 - \beta$ at the level of $O(10^{-4})$ by using the location of the Lagrange points, the Lagrange point accuracy of about a few millimeters (e.g. for $L_4$) is needed in the solar system, though this is very unlikely in the near future.

On the other hand, possible PPN corrections in a three-body system may be relevant with relativistic astrophysics in e.g. a relativistic hierarchical triple system and a supermassive black hole with a compact binary \cite{23, 24}. This subject is beyond the scope of the present paper.

VI. CONCLUSION

The coplanar three-body problem was investigated for a class of fully conservative theories in the PPN formalism, characterized by the Eddington-Robertson parameters $\beta$ and $\gamma$.

The collinear configuration can exist for arbitrary mass ratio, $\beta$ and $\gamma$. On the other hand, the PPN triangular configuration depends on the nonlinearity parameter $\beta$ but not on $\gamma$. This is far from trivial, because the parameter $\beta$ is not separable from $\gamma$ apparently at the level of Eq. (1). For any value of $\beta$, the equilateral configuration in the PPN gravity is possible, if and only if three finite masses are equal or two test masses orbit around one finite mass. For general mass cases, the PPN triangle is not equilateral.

We showed also that the PPN displacements from the standard Lagrange points $L_1$, $L_2$ and $L_3$ depend on both $\beta$ and $\gamma$, while those to $L_4$ and $L_5$ rely only upon $\beta$. It is left for future to study the stability of the PPN configurations.

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