Classical Poisson structures and $r$-matrices from constrained flows

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Abstract

We construct the classical Poisson structure and $r$-matrix for some finite dimensional integrable Hamiltonian systems obtained by constraining the flows of soliton equations in a certain way. This approach allows one to produce new kinds of classical, dynamical Yang-Baxter structures. To illustrate the method we present the $r$-matrices associated with the constrained flows of the Kaup-Newell, KdV, AKNS, WKI and TG hierarchies, all generated by a 2-dimensional eigenvalue problem. Some of the obtained $r$-matrices depend only on the spectral parameters, but others depend also on the dynamical variables. For consistency they have to obey a classical Yang-Baxter-type equations, possibly with dynamical extra terms.

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1 Introduction

Integrable finite dimensional systems that admit a classical $r$-matrix depending only on the spectral parameters has been studied extensively \[1\]. Recently it has been found that for many integrable systems the $r$-matrix depends also on the dynamical variables \[2, 3, 4, 5, 6, 7\]. For example, the celebrated Calogero-Moser system has been shown to possess a dynamical $r$-matrix \[6, 7\]. In contrast with the well-studied case of $r$-matrices depending only on spectral parameters, the general theory of dynamical $r$-matrices has not yet been established. New examples of dynamical $r$-matrices are therefore needed for the search for the underlying structure, and the method presented below seems to be quite useful for this purpose.

In recent years, many types constrained flows of soliton hierarchies have been discussed in the literature. For one such class the Lax representation can be deduced from the adjoint representation of the auxiliary linear problems of the soliton equation \[8, 9\], or derived by using the Gelfand-Dikii approach \[10\]. By means of the Lax representation one can then construct the classical Poisson structure and $r$-matrix for the constrained flows. For some constrained flows the $r$-matrix depends only on the spectral parameters, but for others it also depends on the dynamical variables.

For consistency the Poisson bracket has to obey the Jacobi identity and this implies an equation for the $r$-matrix. In some cases this equation is just the classical Yang-Baxter equation, but in other cases there will be dynamical extra terms.

In present paper, to illustrate the above, we describe the classical Poisson structure and the related classical Yang-Baxter equations associated with the constrained flows for the Kaup-Newell hierarchy, the AKNS hierarchy and the KdV hierarchy. We present also two examples of dynamical $r$-matrix associated with the G. Tu (TG) hierarchy and the Wadati-Konno-Ichikawa (WKI) hierarchy and discuss the related “classical, dynamical” Yang-Baxter equation. At same time some new solutions of the dynamical Yang-Baxter equations are obtained.

2 Integrable constrained flows

To make the paper self contained we first briefly describe how finite dimensional integrable systems and their Lax representation can be constructed from constrained flows of soliton equations. We will use the Kaup-Newell
(KN) hierarchy as an illustration, for further details, see [9].

2.1 The hierarchy of Hamiltonian flows

Let us start by considering the the Kaup-Newell eigenvalue problem [11]

\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}_x = U(u, \lambda) \begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}, \quad U(u, \lambda) = \begin{pmatrix}
-\lambda^2 & \lambda q \\
\lambda r & \lambda^2
\end{pmatrix}.
\] (1)

[Here and in the following we denote \(u^t = (q, r)\).] First, we solve the adjoint representation of (1) [12, 13]

\[ V_x = [U, V] \equiv UV - VU, \] (2)

where \(V\) has a Laurent series expansion

\[ V(u, \lambda) = \sum_{m=0}^{\infty} \begin{pmatrix}
a_m(u) & b_m(u) \\
c_m(u) & -a_m(u)
\end{pmatrix} \lambda^{-m}. \] (3)

Eqs (2) and (3) lead to the recursion relations

\[
\begin{align*}
b_{m+2} &= -qa_{m+1} - \frac{1}{2} b_{m,x}, \\
c_{m+2} &= -ra_{m+1} + \frac{1}{2} c_{m,x}, \\
a_m &= \frac{1}{2} \partial_x^{-1} (qc_{m-1,x} + rb_{m-1,x}),
\end{align*}
\] (4)

and to the parity constraints \(a_{2m+1} = b_{2m} = c_{2m} = 0\). The first few terms are as follows:

\[
\begin{align*}
a_0 &= 1, & a_2 &= -\frac{1}{2} q r, & a_4 &= \frac{3}{8} q^2 r^2 + \frac{1}{4} (r q_x - q r_x), \\
b_1 &= -q, & b_3 &= \frac{1}{2} (q^2 r + q_x), \\
c_1 &= -r, & c_3 &= \frac{1}{2} (r^2 q - r_x),
\end{align*}
\] (5)

The recursion relation (3) can also be expressed as

\[
\begin{pmatrix}
c_{2m+1} \\
b_{2m+1}
\end{pmatrix} = L \begin{pmatrix}
c_{2m-1} \\
b_{2m-1}
\end{pmatrix}, \quad L = \frac{1}{2} \begin{pmatrix}
\partial_x - r \partial_x^{-1} q \partial_x & -r \partial_x^{-1} r \partial_x \\
-q \partial_x^{-1} q \partial_x & -\partial_x - q \partial_x^{-1} r \partial_x
\end{pmatrix}.
\] (6)

Next let us consider a “truncation” of the expression (3)

\[ V^{(n)}(u, \lambda) \equiv (\lambda^{2n} V)_+ \equiv \sum_{m=0}^{n-1} \begin{pmatrix}
a_{2m} \lambda^{2n-2m} & b_{2m+1} \lambda^{2n-2m-1} \\
c_{2m+1} \lambda^{2n-2m-1} & -a_{2m} \lambda^{2n-2m}
\end{pmatrix}, \] (7)
and using it let us define the \( n \)'th flow of the eigenfunction by
\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
_{t_n} = V^{(n)}(u, \lambda)
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}.
\]
Then the compatibility condition of (1) and (8) gives rise to a zero-curvature representation
\[
U_{t_n} - V^{(n)}_x + [U, V^{(n)}] = 0, \quad n = 1, 2, \cdots.
\]
Due to the construction of \( V^{(n)} \) in (7) only terms lowest order in \( \lambda \) contribute, yielding the KN hierarchy
\[
\begin{pmatrix}
q \\
r
\end{pmatrix}
_{t_n} = J
\begin{pmatrix}
c_{2n-1} \\
b_{2n-1}
\end{pmatrix} = J \frac{\delta H_{2n-2}}{\delta u}, \quad J = \begin{pmatrix}
0 & \partial_x \\
\partial_x & 0
\end{pmatrix},
\]
where
\[
H_{2m} = \frac{1}{2m}(4a_{2m+2} - rb_{2m+1} - qc_{2m+1}), \quad H_0 = -qr.
\]
In the above construction all other steps are straightforward, except the fact that the flow (10) can be written in terms of a Hamiltonian \( H_{2n-2} \), and that the Hamiltonians so obtained are in involution with respect to the ordinary infinite-dimensional Poisson bracket [13]. [Also in some cases one has to add a lowest order in \( \lambda \) correction term to \( V^{(n)} \)]. One elegant method to derive this is by using certain trace identities [12].

### 2.2 The constrained flow

In order to construct a finite dimensional integrable system we will take \( N \) copies of (1) with distinct \( \lambda_j \)'s
\[
\begin{pmatrix}
\psi_{1j} \\
\psi_{2j}
\end{pmatrix}_x = U(u, \lambda_j)
\begin{pmatrix}
\psi_{1j} \\
\psi_{2j}
\end{pmatrix}, \quad j = 1, \ldots, N,
\]
and these \( \psi \)'s will be the new dynamical variables (although sometimes there will be others as well). The additional ingredient we need is a constraint that relates \( u \) to the \( \psi \)'s. Furthermore this constraint must be such that it preserves the integrability of the original system, i.e., it must be invariant under the flows (10).

One suitable constraint is obtained as follows [14]. It is known [15] that for systems (12) with \( Tr(U) = 0 \) we have (up to a constant factor)
\[
\frac{\delta \lambda}{\delta u_i} = \frac{1}{2} Tr \left[ \begin{pmatrix}
\psi_1 \psi_2 & -\psi_2^2 \\
\psi_2 & -\psi_1 \psi_2
\end{pmatrix} \frac{\partial U(u, \lambda)}{\partial u_i} \right],
\]

(13)
which in the present case implies
\[
\frac{\delta \lambda}{\delta u} = \frac{1}{2} \begin{pmatrix}
\lambda \psi_2^2 \\
-\lambda \psi_1^2
\end{pmatrix}.
\] (14)

It is easy to verify that
\[
L \begin{pmatrix}
\lambda \psi_2^2 \\
-\lambda \psi_1^2
\end{pmatrix} = \lambda^2 \begin{pmatrix}
\lambda \psi_2^2 \\
-\lambda \psi_1^2
\end{pmatrix},
\] (15)

We then take as our constraint the restriction of the variational derivatives of conserved quantities \( H_{2k_0} \) (for any fixed \( k_0 \)) and \( \lambda_j \) [9, 14]:
\[
\frac{\delta H_{2k_0}}{\delta u} - \beta \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u} = 0,
\] (16)

which in the present case implies
\[
\begin{pmatrix}
c_{2k_0+1} \\
b_{2k_0+1}
\end{pmatrix} - \frac{1}{2} \beta \begin{pmatrix}
< \Lambda \Psi_2, \Psi_2 > \\
-< \Lambda \Psi_1, \Psi_1 >
\end{pmatrix} = 0.
\] (17)

[The constant \( \beta \) has been introduced for later convenience.] Hereafter we denote the inner product in \( \mathbb{R}^N \) by \( \langle ., . \rangle \) and
\[
\Psi_1 = (\psi_{11}, \cdots, \psi_{1N})^T, \quad \Psi_2 = (\psi_{21}, \cdots, \psi_{2N})^T, \quad \Lambda = diag(\lambda_1, \cdots, \lambda_N).
\] (18)

It is shown in [9] that (14) is invariant under all flows of (14). The system consisting of (12) and (16) is called a constrained flow and can be transformed into a finite-dimensional integrable Hamiltonian system (FDIHS) by introducing the so-called Jacobi-Ostrogradsky coordinates.

To deduce the Lax representation for the system (12) and (17) from the adjoint representation (2), we have to find the expressions of \( a_m, b_m, c_m \) under (12) and (17). Due to (3), (13) and (17), we may define the higher order terms [9] by
\[
\begin{pmatrix}
\tilde{c}_{2m+1} \\
\tilde{b}_{2m+1}
\end{pmatrix} = \frac{1}{2} \beta \begin{pmatrix}
< \Lambda^{2m-2k_0+1} \Psi_2, \Psi_2 > \\
-< \Lambda^{2m-2k_0+1} \Psi_1, \Psi_1 >
\end{pmatrix}, \quad m \geq k_0,
\] (19)

and according to (3) and (12)
\[
\tilde{a}_m = -\frac{1}{q} (\tilde{b}_{2m+1} + \frac{1}{2} \tilde{b}_{2m-1,x}) = \frac{1}{2} \beta < \Lambda^{2m-2k_0} \Psi_1, \Psi_2 >, \quad m > k_0.
\] (20)
By using (4), (3), (12) and (17), a direct calculation gives then expressions for the lower order terms \(a_{2m}\) for \(m \leq k_0\) and \(b_{2m+1}, c_{2m+1}\) for \(m < k_0\), which are denoted also by \(\tilde{a}_{2m}, \tilde{b}_{2m+1}, \tilde{c}_{2m+1}\), respectively.

The construction of \(\tilde{a}_m, \tilde{b}_m, \tilde{c}_m\) ensures that under (12) and (17)

\[
\tilde{V} = \sum_{m=0}^{\infty} \begin{pmatrix} \tilde{a}_m & \tilde{b}_m \\ \tilde{c}_m & -\tilde{a}_m \end{pmatrix} \lambda^{-m},
\]

(21)
satisfies (2) as well. By a direct calculation we find

\[
M^{(k_0)} = \begin{pmatrix} A^{(k_0)} & B^{(k_0)} \\ C^{(k_0)} & -A^{(k_0)} \end{pmatrix} \equiv \lambda^{2k_0} \tilde{V},
\]

\[
A^{(k_0)} = \sum_{m=0}^{k_0} \tilde{a}_{2m} \lambda^{2k_0-2m} + \frac{1}{2} \beta \sum_{j=1}^{N} \frac{\lambda_j^2 \psi_{1j} \psi_{2j}}{\lambda^2 - \lambda_j^2},
\]

\[
B^{(k_0)} = \sum_{m=0}^{k_0-1} \tilde{b}_{2m+1} \lambda^{2k_0-2m-1} - \frac{1}{2} \beta \sum_{j=1}^{N} \frac{\lambda_j \lambda_j^2 \psi_{2j}^2}{\lambda^2 - \lambda_j^2},
\]

\[
C^{(k_0)} = \sum_{m=0}^{k_0-1} \tilde{c}_{2m+1} \lambda^{2k_0-2m-1} + \frac{1}{2} \beta \sum_{j=1}^{N} \frac{\lambda_j \lambda_j^2 \psi_{2j}^2}{\lambda^2 - \lambda_j^2}.
\]

Since \(\tilde{V}\) under (12) and (17) satisfies (2), the \(M^{(k_0)}\) under (12) and (17) satisfies (2), too, namely

\[
M_x^{(k_0)} = [U, M^{(k_0)}].
\]

Conversely, the construction of \(M^{(k_0)}\) guarantees that (23) is just the Lax representation for the system (12) and (17). This can also be verified by a direct calculation.

We present first three systems of (12) and (17) below.

(a) When \(k_0 = 0, \beta = 1\), (17) becomes

\[
q = \frac{1}{2} < \Lambda \Psi_1, \Psi_1 >, \quad r = -\frac{1}{2} < \Lambda \Psi_2, \Psi_2 >,
\]

(24)
and then (12) can be written in canonical Hamiltonian form

\[
\Psi_{1x} = -\Lambda^2 \Psi_1 + \frac{1}{2} < \Lambda \Psi_1, \Psi_1 > \Lambda \Psi_2 = \frac{\partial \tilde{H}_0}{\partial \Psi_2},
\]

\[
\Psi_{2x} = -\frac{1}{2} < \Lambda \Psi_2, \Psi_2 > \Lambda \Psi_1 + \Lambda^2 \Psi_2 = -\frac{\partial \tilde{H}_0}{\partial \Psi_1},
\]

(25)

\[
\tilde{H}_0 = - < \Lambda^2 \Psi_1, \Psi_2 > + \frac{1}{4} < \Lambda \Psi_1, \Psi_1 > < \Lambda \Psi_2, \Psi_2 > .
\]

(26)
The \(A^{(0)}, B^{(0)}, C^{(0)}\) in (22) read

\[
A^{(0)}(\lambda) = 1 + \frac{1}{2} \sum_{j=1}^{N} \frac{\lambda_j^2 \psi_{1j} \psi_{2j}}{\lambda^2 - \lambda_j^2},
\]

\[
B^{(0)}(\lambda) = -\frac{1}{2} \lambda \sum_{j=1}^{N} \frac{\lambda_j \psi_{2j}^2}{\lambda^2 - \lambda_j^2},
\]

\[
C^{(0)}(\lambda) = \frac{1}{2} \lambda \sum_{j=1}^{N} \frac{\lambda_j \psi_{2j}^2}{\lambda^2 - \lambda_j^2}.
\]

(27)
(b) When \( k_0 = 1, \beta = \frac{1}{2} \), then (17) gives rise to the constraint
\[
q_x = -q^2 r - \frac{1}{2} < \Lambda \Psi_1, \Psi_1 >, \quad r_x = r^2 q - \frac{1}{2} < \Lambda \Psi_2, \Psi_2 >, \tag{28}
\]
and by introducing
\[
q_1 = q, \quad p_1 = r, \tag{29}
\]
the system (12) and (28) can be written in canonical Hamiltonian form
\[
\Psi_{1x} = \frac{\partial \tilde{H}_1}{\partial \Psi_2}, \quad q_{1x} = \frac{\partial \tilde{H}_1}{\partial p_1}, \quad \Psi_{2x} = -\frac{\partial \tilde{H}_1}{\partial \Psi_1}, \quad p_{1x} = -\frac{\partial \tilde{H}_1}{\partial q_1}, \tag{30}
\]
\[
\tilde{H}_1 = -\frac{1}{2} q_1^2 p_1^2 - < \Lambda^2 \Psi_1, \Psi_2 > + \frac{1}{2} q_1 < \Lambda \Psi_2, \Psi_2 > - \frac{1}{2} p_1 < \Lambda \Psi_1, \Psi_1 >. \tag{31}
\]
The \( A^{(1)}, B^{(1)}, C^{(1)} \) for \( M^{(1)} \) are of the form
\[
A^{(1)}(\lambda) = \lambda^2 - \frac{1}{2} q_1 p_1 + \frac{1}{4} \sum_{j=1}^{N} \frac{\lambda_j^2 \psi_{1j} \psi_{2j}}{\lambda^2 - \lambda_j^2},
B^{(1)}(\lambda) = -q_1 \lambda - \frac{1}{4} \lambda \sum_{j=1}^{N} \frac{\lambda_j \psi_{1j}^2}{\lambda^2 - \lambda_j^2},
C^{(1)}(\lambda) = -p_1 \lambda + \frac{1}{4} \lambda \sum_{j=1}^{N} \frac{\lambda_j \psi_{2j}^2}{\lambda^2 - \lambda_j^2}. \tag{32}
\]
(c) When \( k_0 = 2, \beta = 1 \), (17) leads to the constraint
\[
\begin{align*}
\frac{1}{2} r_{xx} - \frac{3}{2} qrr_x + \frac{3}{4} q^2 r^3 &= -< \Lambda \Psi_2, \Psi_2 >, \\
\frac{1}{2} q_{xx} + \frac{3}{2} qrr_q + \frac{3}{4} q^3 r^2 &= < \Lambda \Psi_1, \Psi_1 >,
\end{align*} \tag{33}
\]
and by introducing the following Jacobi-Ostrogradsky coordinates:
\[
q_1 = q, \quad q_2 = r, \quad p_1 = -\frac{3}{16} r^2 q + \frac{1}{4} r_x, \quad p_2 = \frac{3}{16} q^2 r + \frac{1}{4} q_x, \tag{34}
\]
the system (12) and (33) can be written in canonical Hamiltonian form
\[
\Psi_{1x} = \frac{\partial \tilde{H}_2}{\partial \Psi_2}, \quad q_{1x} = \frac{\partial \tilde{H}_2}{\partial p_1}, \quad \Psi_{2x} = -\frac{\partial \tilde{H}_2}{\partial \Psi_1}, \quad p_{1x} = -\frac{\partial \tilde{H}_2}{\partial q_1}, \tag{35}
\]
where
\[
\tilde{H}_2 = 4p_1 p_2 - \frac{3}{2} q_1^2 q_2 p_1 + \frac{3}{2} q_2^2 q_1 p_2 - \frac{1}{2} q_1^3 q_2^2 - < \Lambda^2 \Psi_1, \Psi_2 > + \frac{1}{2} q_1 < \Lambda \Psi_2, \Psi_2 > - \frac{1}{2} q_2 < \Lambda \Psi_1, \Psi_1 >, \tag{36}
\]
and the \( A^{(2)}, B^{(2)}, C^{(2)} \) for \( M^{(2)} \) are of the form
\[
A^{(2)}(\lambda) = \lambda^4 - \frac{1}{2} q_1 q_2 \lambda^2 + q_2 p_2 - q_1 p_1 + \frac{1}{2} \sum_{j=1}^{N} \frac{\lambda_j^2 \psi_{1j} \psi_{2j}}{\lambda^2 - \lambda_j^2},
B^{(2)}(\lambda) = -q_1 \lambda^3 + (\frac{3}{8} q_2 q_1^2 + 2p_2) \lambda - \frac{1}{2} \lambda \sum_{j=1}^{N} \frac{\lambda_j \psi_{1j}^2}{\lambda^2 - \lambda_j^2},
C^{(2)}(\lambda) = -q_2 \lambda^3 + (\frac{3}{8} q_1 q_2^2 - 2p_1) \lambda + \frac{1}{2} \lambda \sum_{j=1}^{N} \frac{\lambda_j \psi_{2j}^2}{\lambda^2 - \lambda_j^2}. \tag{37}
\]
3 The main results

3.1 The classical Poisson structure

We now present the classical Poisson structure associated with the Lax representation for (23), (30) and (35). With respect to the standard Poisson bracket, it is found by a direct calculation that both \( A^{(0)}, B^{(0)}, C^{(0)} \) and \( A^{(1)}, B^{(1)}, C^{(1)} \) as well as \( A^{(2)}, B^{(2)}, C^{(2)} \) satisfy the following relations

\[
\begin{align*}
\{A(\lambda), A(\mu)\} &= \{B(\lambda), B(\mu)\} = \{C(\lambda), C(\mu)\} = 0, \\
\{A(\lambda), B(\mu)\} &= \frac{\beta \mu}{\mu^2 - \lambda^2}(\mu B(\mu) - \lambda B(\lambda)), \\
\{A(\lambda), C(\mu)\} &= \frac{\beta \mu}{\mu^2 - \lambda^2}(\lambda C(\lambda) - \mu C(\mu)), \\
\{B(\lambda), C(\mu)\} &= \frac{2 \beta \mu}{\mu^2 - \lambda^2}(A(\mu) - A(\lambda)).
\end{align*}
\]

(38)

In [3] it was pointed out, that the integrability of a system (along with many other useful properties) can be shown straightforwardly (see Sec. 3.2), if the Poisson structure can be written in the form (we follow the notation of [3])

\[
\{M^{(1)}(\alpha_1), M^{(2)}(\alpha_2)\} = [r^{(12)}(\alpha_1, \alpha_2), M^{(1)}(\alpha_1)] - [r^{(21)}(\alpha_2, \alpha_1), M^{(2)}(\alpha_2)].
\]

(39)

Here the superscripts refer to the vector space on which the matrices act non-trivially, and \( \alpha_i, \alpha_j \) are the corresponding spectral parameters. The equation itself is defined on \( V_1 \otimes V_2 \), where \( V_i \) are identical 2-dimensional vector spaces, so all matrices are \( 2^2 \times 2^2 \)-dimensional, for example \( M^{(1)}(\alpha_1) = M(\alpha_1) \otimes 1 \) and \( M^{(2)}(\alpha_2) = 1 \otimes M(\alpha_2) \). From (39) we can see that the spectral parameters \( \alpha_i \) are associated with the vector spaces, so it is not necessary to write them explicitly. Note that the usual permutation matrix \( P \) permutes only the vector spaces:

\[
r^{(21)}(\alpha_1, \alpha_2) = P^{(12)}r^{(12)}(\alpha_1, \alpha_2)P^{(12)}.
\]

For the Poisson brackets (38) one finds that (39) holds with

\[
r^{(ij)} = \frac{\beta \alpha_i \alpha_j}{\alpha_j^2 - \alpha_i^2} p^{(ij)} - \frac{\beta \alpha_i}{\alpha_j + \alpha_i} s^{(ij)}, \quad s^{(ij)} = \frac{1}{2}(\sigma_0^{(i)} \otimes \sigma_0^{(j)} + \sigma_3^{(i)} \otimes \sigma_3^{(j)}).
\]

(40)

(Here the \( \sigma_i \)'s are the standard Pauli matrices, and the permutation matrix \( P \) is given by \( P^{(ij)} = \frac{1}{2} \sum_{n=0}^3 \sigma_n^{(i)} \otimes \sigma_n^{(j)} \). In fact (39) hold for all FDIHS obtained from the constrained flows (12) and (17). The classical Poisson structure (34, 40) contains all necessary information of the present system, and is more rich than the Lax representation (1).)

It is well known that any Poisson bracket must satisfy the Jacobi identity

\[
\{M^{(1)}, \{M^{(2)}, M^{(3)}\}\} + \{M^{(2)}, \{M^{(3)}, M^{(1)}\}\} + \{M^{(3)}, \{M^{(1)}, M^{(2)}\}\} = 0.
\]

(41)
This equation is defined on \( V_1 \otimes V_2 \otimes V_3 \) so, e.g., \( M^{(2)} = 1 \otimes M(\alpha_2) \otimes 1 \). If we allow for the possibility that the \( r \)'s depend on the dynamical variables, then a direct application of (39) to (41) leads to the requirement

\[
[R^{(123)}, M^{(1)}] + [R^{(231)}, M^{(2)}] + [R^{(312)}, M^{(3)}] = 0,
\]

where

\[
R^{(ijk)} := r^{(ijk)} + \{M^{(j)}, r^{(ik)}\} - \{M^{(k)}, r^{(ij)}\},
\]

\[
r^{(ijk)} := [r^{(ij)}, r^{(ik)}] + [r^{(ij)}, r^{(jk)}] + [r^{(kj)}, r^{(ik)}].
\]

If the \( r \)'s do not depend on dynamical variables, then the Jacobi identity (42) should be satisfied by \( r^{(ijk)} = 0 \). This equation is almost the classical Yang-Baxter equation, which would be obtained if we had \( r^{(kj)} = -r^{(jk)} \) (in which case the last term in (44) could be written as \( [r^{(ik)}, r^{(jk)}] \)). It turns out, however, that most of the examples presented here, e.g., (40), do not have such a antisymmetry, so the index order in (44) is crucial.

For the dynamical \( r \)-matrices presented in the next section one finds that \( R^{(ijk)} \neq 0 \). In [6] Sklyanin observed that in such a case the Jacobi identity (42) can nevertheless be satisfied, if

\[
R^{(ijk)} = [X^{(ijk)}, M^{(j)}] - [X^{(kij)}, M^{(k)}],
\]

for some matrix \( X \). We call this equation the dynamical, classical Yang-Baxter equation. [The special case \( X^{(ijk)} = X^{(kij)} \) was used before in [5].] For the examples presented in Sec. 4 the Jacobi identity is indeed satisfied due to (43), where \( X^{(ijk)} \neq X^{(kij)} \).

### 3.2 Integrability

An immediate consequence of (39) is that

\[
\{M_1^2(\lambda), M_2^2(\mu)\} = [\tau_{12}(\lambda, \mu), M_1(\lambda)] - [\tau_{21}(\mu, \lambda), M_2(\mu)],
\]

where [4]

\[
\tau_{ij}(\lambda, \mu) = \sum_{k=0}^{1} \sum_{l=0}^{1} M_{1}^{1-k}(\lambda) M_{2}^{1-l}(\mu) r^{(ij)}(\lambda, \mu) M_{1}^{k}(\lambda) M_{2}^{l}(\mu).
\]

Then it follows from (44) immediately that

\[
4\{Tr M^2(\lambda), Tr M^2(\mu)\} = Tr \{M_1^2(\lambda), M_2^2(\mu)\} = 0,
\]
which ensures the involution property of the integrals of motion obtained from expanding \( M^2 \) in powers of \( \lambda \).

For system (22) one obtains

\[
TrM^2(\lambda) = (A^{(0)}(\lambda))^2 + B^{(0)}(\lambda)C^{(0)}(\lambda) = 1 + \sum_{j=1}^{N} \frac{F_0^{(j)}}{\lambda^2 - \lambda_j^2}, \quad (49)
\]

where

\[
F_0^{(j)} = \lambda_j^2 \psi_{1j} \psi_{2j} - \frac{1}{4} < \lambda \Psi_1, \Psi_1 > \lambda_j \psi_{2j}^2 + \frac{1}{4} \sum_{k \neq j} \frac{\lambda_j \lambda_k}{\lambda_k^2 - \lambda_j^2} (\lambda_j \psi_{1j} \psi_{2k} - \lambda_k \psi_{1k} \psi_{2j})^2, \quad j = 1, \ldots, N. \quad (50)
\]

and we have \( \tilde{H}_0 = - \sum_{j=1}^{N} F_0^{(j)} \).

For system (23) we find

\[
TrM^2(\lambda) = (A^{(1)}(\lambda))^2 + B^{(1)}(\lambda)C^{(1)}(\lambda) = \lambda^4 - 2\tilde{H}_1 + \frac{1}{2} \sum_{j=1}^{N} \frac{F_1^{(j)}}{\lambda^2 - \lambda_j^2}, \quad (51)
\]

where

\[
F_1^{(j)} = \lambda_j^3 \psi_{1j} \psi_{2j} - \frac{1}{2} q_1 \lambda_j^3 \psi_{2j}^2 + \frac{1}{2} p_1 \lambda_j^3 \psi_{1j}^2 - \frac{1}{8} < \lambda \Psi_1, \Psi_1 > \lambda_j \psi_{2j}^2 + \frac{1}{8} \sum_{k \neq j} \frac{\lambda_j \lambda_k}{\lambda_k^2 - \lambda_j^2} (\lambda_j \psi_{1j} \psi_{2k} - \lambda_k \psi_{1k} \psi_{2j})^2, \quad j = 1, \ldots, N. \quad (52)
\]

For system (33) one gets

\[
TrM^2(\lambda) = (A^{(2)}(\lambda))^2 + B^{(2)}(\lambda)C^{(2)}(\lambda) = \lambda^8 - \tilde{H}_2 \lambda^2 + F_2^{(0)} + \sum_{j=1}^{N} \frac{F_2^{(j)}}{\lambda^2 - \lambda_j^2}, \quad (53)
\]

where

\[
F_2^{(0)} = \lambda^4 \Psi_1, \Psi_2 > - \frac{1}{2} q_1 q_2 \lambda^2 \Psi_1, \Psi_2 > - \frac{1}{2} q_1 \lambda^3 \Psi_2, \Psi_2 > + \frac{1}{2} q_2 \lambda^2 \Psi_1, \Psi_1 > + (p_2 q_2 - q_1 p_1)^2 + (p_2 + \frac{1}{16} q_1^2 q_2) < \lambda \Psi_2, \Psi_2 > + (p_1 \frac{1}{16} q_2^2 q_1) < \lambda \Psi_1, \Psi_1 >,
\]

\[
F_2^{(j)} = \lambda_j^4 \frac{1}{2} q_1 q_2 \lambda_j^2 + p_2 q_2 - q_1 p_1) \lambda_j^3 \psi_{1j} \psi_{2j} + (p_2 - \frac{1}{2} q_1 \lambda_j^2 + \frac{1}{16} q_1^2 q_2) \lambda_j^3 \psi_{2j}^2 + (p_1 + \frac{1}{2} q_2 \lambda_j^2 - \frac{1}{16} q_2^2 q_1) \lambda_j^3 \psi_{1j}^2 - \frac{1}{4} < \lambda \Psi_1, \Psi_1 > \lambda_j \psi_{2j}^2 + \frac{1}{4} \sum_{k \neq j} \frac{\lambda_j \lambda_k}{\lambda_k^2 - \lambda_j^2} (\lambda_j \psi_{1j} \psi_{2k} - \lambda_k \psi_{1k} \psi_{2j})^2, \quad j = 1, \ldots, N. \quad (54)
\]
Then equation (48) and, for example, (53) guarantees that the functionally independent integrals of motion \( \tilde{H}_2 \) and \( F^{(j)} \), \( j = 0, 1, \ldots, N \), are in involution. This shows the integrability of (25), (30) and (35) in the sense of Liouville \([16]\).

### 3.3 Two further examples of classical \( r \)-matrix

#### 3.3.1 The KdV hierarchy

For the KdV hierarchy \([17]\), the eigenvalue problem is of the form

\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}_x = U(q, \lambda)
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}, \quad U(q, \lambda) = \begin{pmatrix} 0 & 1 \\ -\lambda - q & 0 \end{pmatrix}. \tag{55}
\]

the second constrained flow with constraint \( q = \frac{1}{8} < \Psi_1, \Psi_1 > \) reads \([14]\)

\[
\Psi_{1x} = \Psi_2 = \frac{\partial \tilde{H}}{\partial \Psi_2}, \quad \Psi_{2x} = -\frac{1}{8} < \Psi_1, \Psi_1 > - \Lambda \Psi_1 = -\frac{\partial \tilde{H}}{\partial \Psi_1}, \tag{56}
\]

with the Hamiltonian

\[
\tilde{H} = \frac{1}{2} < \Lambda \Psi_1, \Psi_1 > + \frac{1}{2} < \Psi_2, \Psi_2 > + \frac{1}{32} < \Psi_1, \Psi_1 >^2. \tag{57}
\]

The Lax representation for \((56)\) is given by \((23)\)

\[
M(\lambda) \equiv \begin{pmatrix}
A(\lambda) & B(\lambda) \\
C(\lambda) & -A(\lambda)
\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\lambda - \frac{1}{16} < \Psi_1, \Psi_1 > & 0 \end{pmatrix} + \frac{1}{16} \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} \psi_{1j} \psi_{2j} & -\psi_{1j}^2 \\ \psi_{2j}^2 & -\psi_{1j} \psi_{2j} \end{pmatrix}, \tag{58}
\]

and we have

\[
\{A(\lambda), A(\mu)\} = \{B(\lambda), B(\mu)\} = 0,
\]

\[
\{C(\lambda), C(\mu)\} = \frac{1}{4}(A(\lambda) - A(\mu)),
\]

\[
\{A(\lambda), B(\mu)\} = \frac{1}{8(\mu - \lambda)}(B(\mu) - B(\lambda)),
\]

\[
\{A(\lambda), C(\mu)\} = \frac{1}{8(\mu - \lambda)}(C(\lambda) - C(\mu)) - \frac{1}{8} B(\lambda),
\]

\[
\{B(\lambda), C(\mu)\} = \frac{1}{4(\mu - \lambda)}(A(\mu) - A(\lambda)). \tag{59}
\]

Then \((58)\) gives rise to the classical Poisson structure \((39)\) for the system \((56)\) (in fact for all constrained flows of KdV hierarchy) with the \( r \)-matrix given by

\[
r^{(ij)}(\alpha_i, \alpha_j) = \frac{1}{8(\alpha_j - \alpha_i)} P^{(ij)} + \frac{1}{8} S^{(ij)} = \sigma^{(i)}_+ \otimes \sigma^{(j)}_. \tag{60}
\]

and this \( r \) satisfies the classical Yang-Baxter equations of the form \( r^{(ijk)} = 0 \) \([14]\). [In this case \( r^{(ij)} \neq -r^{(ji)} \) and the index order in \((14)\) is important.]
3.3.2 The AKNS hierarchy

For the AKNS eigenvalue problem [17]
\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}_x = U(u, \lambda)
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix},
\quad U(u, \lambda) = \begin{pmatrix}
-\lambda & q \\
r & \lambda
\end{pmatrix}.
\]
(61)

the first constraint is 
\[
r = \frac{1}{2} < \Psi_2, \Psi_2 >, q = -\frac{1}{2} < \Psi_1, \Psi_1 >
\]
and the corresponding flow reads
\[
\begin{align*}
\Psi_{1x} &= -\Lambda \Psi_1 - \frac{1}{2} < \Psi_1, \Psi_1 > \Psi_2 = \frac{\partial \tilde{H}}{\partial \Psi_2}, \\
\Psi_{2x} &= \frac{1}{2} < \Psi_2, \Psi_2 > \Psi_1 + \Lambda \Psi_2 = -\frac{\partial \tilde{H}}{\partial \Psi_1},
\end{align*}
\]
(62)

where
\[
\tilde{H} = -< \Lambda \Psi_1, \Psi_2 > - \frac{1}{4} < \Psi_2, \Psi_2 > < \Psi_1, \Psi_1 >.
\]
(63)

The Lax representation for (62) is given by (23) with [18]
\[
\begin{pmatrix}
A(\lambda) & B(\lambda) \\
C(\lambda) & -A(\lambda)
\end{pmatrix} = \begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix} + \frac{1}{2} \sum_{j=1}^N \frac{1}{\lambda - \lambda_j}
\begin{pmatrix}
\psi_{1j} \psi_{2j} & -\psi_{1j}^2 \\
\psi_{2j}^2 & -\psi_{1j} \psi_{2j}
\end{pmatrix},
\]
(64)

and one gets
\[
\begin{align*}
\{ A(\lambda), A(\mu) \} &= \{ B(\lambda), B(\mu) \} = \{ C(\lambda), C(\mu) \} = 0, \\
\{ A(\lambda), B(\mu) \} &= \frac{1}{\mu - \lambda} (B(\mu) - B(\lambda)), \\
\{ A(\lambda), C(\mu) \} &= \frac{1}{\mu - \lambda} (C(\lambda) - C(\mu)), \\
\{ B(\lambda), C(\mu) \} &= \frac{2}{\mu - \lambda} (A(\mu) - A(\lambda)).
\end{align*}
\]
(65)

Then (65) gives rise to the classical Poisson structure (39) for the system (62) (in fact for all constrained flows of AKNS hierarchy) with the r-matrix given by
\[
r^{(ij)}(\alpha_i, \alpha_j) = \frac{1}{\alpha_j - \alpha_i} P^{(ij)}.
\]
(66)

4 Two examples of dynamical r-matrix

The above examples had an r-matrix that depended only on the spectral parameters. We will now present two restricted flows that lead to a dynamical r-matrix.
4.1 The TG hierarchy

Let us first consider the TG hierarchy associated with the following eigenvalue problem \[19\]

\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}_x = U(u, \lambda) \begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}, \quad U(u, \lambda) = \begin{pmatrix}
-\lambda + \frac{1}{2} q & r \\
r & \lambda - \frac{1}{2} q
\end{pmatrix}.
\]

(67)

The first constrained flow, with constraint \( q = (\langle \Psi_2, \Psi_2 \rangle - \langle \Psi_1, \Psi_1 \rangle) / G, \ r = 2G \) reads

\[
\begin{align*}
\Psi_{1x} &= -\Lambda \Psi_1 + \frac{1}{2G} (\langle \Psi_2, \Psi_2 \rangle - \langle \Psi_1, \Psi_1 \rangle) \Psi_1 + 2G \Psi_2 = \frac{\partial \tilde{H}}{\partial \Psi_1}, \\
\Psi_{2x} &= 2G \Psi_1 + \Lambda \Psi_2 - \frac{1}{2G} (\langle \Psi_2, \Psi_2 \rangle - \langle \Psi_1, \Psi_1 \rangle) \Psi_2 = -\frac{\partial \tilde{H}}{\partial \Psi_2},
\end{align*}
\]

(68)

where

\[
\tilde{H} = - < \Lambda \Psi_1, \Psi_2 > + G (< \Psi_2, \Psi_2 > - < \Psi_1, \Psi_1 >), \quad G = \sqrt{< \Psi_1, \Psi_2 >}.
\]

(69)

Using the method in \[8\], we obtain the Lax representation for (68) given by \[23\] with

\[
\begin{pmatrix}
A(\lambda) & B(\lambda) \\
C(\lambda) & -A(\lambda)
\end{pmatrix} = \begin{pmatrix}
-\frac{1}{2} \lambda & G \\
G & \frac{1}{2} \lambda
\end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix}
\psi_{1j} \psi_{2j} & -\psi_{1j}^2 \\
\psi_{2j}^2 & -\psi_{1j} \psi_{2j}
\end{pmatrix}.
\]

(70)

One gets

\[
\{ A(\lambda), A(\mu) \} = 0, \quad \{ B(\lambda), B(\mu) \} = \frac{1}{G}(B(\lambda) - B(\mu)), \\
\{ C(\lambda), C(\mu) \} = -\frac{1}{G}[C(\lambda) - C(\mu)], \quad \{ A(\lambda), B(\mu) \} = \frac{2}{\mu - \lambda}(B(\mu) - B(\lambda)), \\
\{ A(\lambda), C(\mu) \} = \frac{2}{\mu - \lambda}(C(\lambda) - C(\mu)), \\
\{ B(\lambda), C(\mu) \} = \frac{4}{\mu - \lambda}(A(\mu) - A(\lambda)) + \frac{1}{G}(B(\lambda) + C(\mu))
\]

(71)

which gives rise to the classical Poisson structure \[39\] for the system (68) with the dynamical \( r \)-matrix given by

\[
r^{(ij)}(\alpha_i, \alpha_j) = \frac{2}{\alpha_j - \alpha_i} P^{(ij)} + \frac{1}{2G} S^{(ij)}, \quad S^{(ij)} = \sigma_3^{(i)} \otimes \sigma_1^{(j)}.
\]

(72)

This satisfies the classical, dynamical Yang-Baxter equations \[43, 45\] with

\[
X^{(ijk)} = -\frac{1}{2G} \sigma_3^{(i)} \otimes \sigma_3^{(j)} \otimes \sigma_1^{(k)}.
\]

(73)
4.2 The Wadati-Konno-Ichikawa hierarchy

Finally consider the Wadati-Konno-Ichikawa (WKI) hierarchy associated with the following eigenvalue problem \[20\]

\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}_x = U(u, \lambda) \begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}, \quad U(u, \lambda) = \begin{pmatrix}
\lambda & \lambda q \\
\lambda r & -\lambda
\end{pmatrix}.
\]

(74)

By using the method in \[8, 9, 14\], we obtain the first constrained flow, with the constraint \(q = -<\Lambda \Psi_1, \Psi_1 > /G, r = <\Lambda \Psi_2, \Psi_2 > /G\) as follows

\[
\Psi_{1x} = \Lambda \Psi_1 - \frac{1}{G} <\Lambda \Psi_1, \Psi_1 > \Lambda \Psi_2 = \frac{\partial \tilde{H}}{\partial \psi_2},
\]

\[
\Psi_{2x} = \frac{1}{G} <\Lambda \Psi_2, \Psi_2 > \Lambda \Psi_1 - \Lambda \Psi_2 = -\frac{\partial \tilde{H}}{\partial \psi_1},
\]

(75)

where

\[
\tilde{H} = <\Lambda \Psi_1, \Psi_2 > -G, \quad G = \sqrt{1 + <\Lambda \Psi_1, \Psi_1 > <\Lambda \Psi_2, \Psi_2 >}.
\]

(76)

The Lax representation for (75) is given by (23) with

\[
\begin{pmatrix}
A(\lambda) & B(\lambda) \\
C(\lambda) & -A(\lambda)
\end{pmatrix} = \frac{1}{2} \begin{pmatrix} G & 0 \\
0 & -G \end{pmatrix} + \frac{1}{2} \sum_{j=1}^{N} \frac{\lambda_j}{\lambda - \lambda_j} \begin{pmatrix}
\lambda_j \psi_j \psi_{2j} & -\lambda \psi_j^2 \\
\lambda \psi_{2j}^2 & -\lambda_j \psi_j \psi_{2j}
\end{pmatrix}.
\]

(77)

and we have

\[
\begin{align*}
\{B(\lambda), B(\mu)\} &= \{C(\lambda), C(\mu)\} = 0, \\
\{A(\lambda), A(\mu)\} &= \frac{1}{2G} <\Lambda \Psi_2, \Psi_2 > (\lambda B(\lambda) - \mu B(\mu)) \\
&+ \frac{1}{2G} <\Lambda \Psi_1, \Psi_1 > (\lambda C(\lambda) - \mu C(\mu)), \\
\{A(\lambda), B(\mu)\} &= \frac{\lambda \mu}{\lambda - \mu} B(\lambda) - \frac{\mu^2}{\lambda - \mu} B(\mu) + \frac{1}{G} <\Lambda \Psi_1, \Psi_1 > \mu A(\mu), \\
\{A(\lambda), C(\mu)\} &= \frac{\lambda \mu}{\lambda - \mu} C(\lambda) - \frac{\mu^2}{\lambda - \mu} C(\mu) + \frac{1}{G} <\Lambda \Psi_2, \Psi_2 > \mu A(\mu), \\
\{B(\lambda), C(\mu)\} &= \frac{2\lambda \mu}{\lambda - \mu} (A(\lambda) - A(\mu)),
\end{align*}
\]

(78)

This leads to the classical Poisson structure \[33\] for (75) with the \(r\)-matrix given by

\[
\begin{align*}
r^{(ij)}(\alpha_i, \alpha_j) &= \frac{\alpha_i \alpha_j}{\alpha_i - \alpha_j} P^{(ij)} - \alpha_i S^{(ij)} + \frac{\alpha_i}{G} E^{(ij)}, \\
S^{(ij)} &= \frac{1}{2} (\sigma_0^{(i)} \otimes \sigma_0^{(j)} + \sigma_3^{(i)} \otimes \sigma_3^{(j)}), \\
E^{(ij)} &= F^{(i)} \otimes \sigma_3^{(j)}, \\
F^{(i)} &= <\Lambda \Psi_1, \Psi_1 > \sigma_+^{(i)} - <\Lambda \Psi_2, \Psi_2 > \sigma_-^{(i)}.
\end{align*}
\]

(79)

This \(r\)-matrix satisfies the dynamical classical Yang-Baxter equation \[18, 19\], with

\[
X^{(ijk)} = -\frac{\alpha_i \alpha_j}{2G^3} [F^{(i)} \otimes F^{(j)} \otimes \sigma_3^{(k)} + 2\sigma_+^{(i)} \otimes \sigma_-^{(j)} \otimes \sigma_3^{(k)} + 2\sigma_-^{(i)} \otimes \sigma_+^{(j)} \otimes \sigma_3^{(k)}].
\]

(80)
5 Conclusions

In this paper we have discussed the classical Poisson structure and the related (dynamical) \( r \)-matrix for some finite dimensional integrable Hamiltonian systems. These integrable systems were derived by constraining the integrable flow of an evolution equation in a particular way \([8, 9, 14, 18]\).

The possibility of a dynamical \( r \)-matrices has been known for some time now, but no general theory for such systems has been developed so far. It is therefore important to derive examples using different methods, in order to find what the essential features are. For example, one may ask in what way the Jacobi identities are satisfied. The examples presented in Sec. 4 belong to the class that satisfy them only through the most general form proposed so far, Eq. (45). The method presented in this paper can probably be used to generate still other types of interesting examples.

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