Integrability in string/field theories is known to emerge when considering dynamics in the moduli space of physical theories. This implies that one has to look at the dynamics with respect to unusual time variables like coupling constants or other quantities parameterizing configuration space of physical theories. The dynamics given by variations of coupling constants can be considered as a canonical transformation or, infinitesimally, a Hamiltonian flow in the space of physical systems. We briefly consider here an example of mechanical integrable systems. Then, any function \( T(\vec{p}, \vec{q}) \) generates a one-parametric family of integrable systems in vicinity of a single system. For integrable system with several coupling constants the corresponding “Hamiltonians” \( T_i(\vec{p}, \vec{q}) \) satisfy Whitham equations and after quantization (of the original system) become operators satisfying the zero-curvature condition in the space of coupling constants:

\[
\left[ \frac{\partial}{\partial g_a} - \hat{T}_a(\hat{\vec{p}}, \hat{\vec{q}}), \frac{\partial}{\partial g_b} - \hat{T}_b(\hat{\vec{p}}, \hat{\vec{q}}) \right] = 0
\]

1 Introduction

One of the main lessons we learnt from string theory is that, in contrast to the usual field theory approach, one should not study the theory at fixed values of parameters, but instead needs to vary as many parameters as possible in order to reveal some new structures behind the theory. In fact, most of the structures realized during last years just can not be discovered at fixed values of parameters. Moreover, if the theory is restricted by some additional requirements (which is often the case in the standard field theory, where one typically asks for renormalizability, unitarity etc.) the simple structures are typically absent, or become realized in a very non-linear complicated way. The situation is much similar to the integrable system which is often can be described by as a simple free system on a manifold but gets a non-trivial dynamics after the Hamiltonian reduction on the phase space.

Among new interesting structures realized in such a way is integrability of complete effective actions in quantum field theory [1]. By complete effective action we understand the generating function of all correlation functions. It turns out that the same effective action also enjoys other interesting features, typically of topological or similar nature.

In this short review, in ss.2 and 3 we are going to list several typical examples of this kind of structures. However, one should remark that the set of examples is far not exhausted by the effective actions. One sometimes needs to deal with some more refined quantities like metrics of the \( \sigma \)-model describing the low-energy effective action. This is the case in Seiberg-Witten theory discussed in ss.4

* mironov@lpi.ac.ru, mironov@itep.ru
and 5. In ss.6 and 7 we develop, following [2], a formalism that could explain all observed structures of the effective theories, in particular, we propose some explanation of integrability of the effective actions. At last, we consider several manifest examples in ss.8 and 9.

2 Matrix model as a toy example

To be more specific, let us consider a toy example of the matrix integral which imitates path integral for a field theory. This example has come from the theory of non-critical strings, [3]. Namely, we consider the integral over Hermitian $N \times N$ matrix with the corresponding Haar measure. The partition function of this theory is given in the simplest case by the integral

$$Z_N(g) = \int dMe^{TrM^2 + gTrM^3}$$

(1)

The first term in the exponential is a counterpart of the kinetic term, while the second one generates the cubic interaction. Therefore, one can deal with this integral using the perturbation theory in the parameter $g$. We restrict ourselves here with maximum the cubic interaction in order to reproduce situation of the field theory with typically finite number of different (polynomial) interactions.

Now, if one wants to learn something about this integral, one should deform the theory to include more interactions and allow constants in integral (1) to vary. Say, let the size of matrix $N$ be a variable, while another variable $t$ be the coefficient in front of linear term that is put zero in (1). Then, one can easily show [4] that the partition function as a function of these two parameters satisfies the difference-differential equation

$$\frac{\partial^2 \phi_N}{\partial t^2} = e^{\phi_{N+1}-\phi_N} - e^{\phi_{N-1}} - e^{\phi_N} - 1,$$

$$e^{\phi_N} \equiv \frac{Z_{N+1}}{Z_N}$$

(2)

One recognizes in this equation the Toda chain equation, $Z_N$ being its $\tau$-function [5]. The Toda chain is an integrable system, therefore, there are (infinitely) many Poisson-commuting conserved quantities in the system, each of them can be taken as a Hamiltonian, i.e. each of them gives rise to a time flow. Since all these Hamiltonians are Poisson-commuting, we have (infinitely) many commuting time flows which lead to a whole hierarchy of equations of motion. This hierarchy is called Toda chain hierarchy. Note that all the equations of the hierarchy are described by a single function of (infinitely) many time variables called $\tau$-function. We expect this $\tau$-function to be the partition function of our system.

Now, what are these other time variables? In fact, they have very simple meaning: these are nothing but other couplings, the couplings with $TrM^k$ [4]. Thus, in order to obtain complete integrable system, one needs to consider all single trace interactions in (1), the corresponding coupling constants being time variables in the Toda chain hierarchy and the partition function being the Toda chain $\tau$-function. Note that this partition function is the generating function for all correlators in the theory.

In fact, introducing infinitely many interactions allows one to construct some other underlying structures in the theory apart from integrability. Say, if one makes polynomial changes of variable in (1) with all single trace interactions included, it leads [3] to a set of constraints (Ward identities) satisfied by the partition function:

$$L_n \left( \int dMe^{\sum_k t_k TrM^k} \right) = 0, \quad n \geq -1, \quad L_n \equiv \sum_k k t_k \frac{\partial}{\partial t_{k+n}} + \sum_{k=0}^n \frac{\partial^2}{\partial t_{n-k} \partial t_k}$$

(3)

These constraints form the Borel subalgebra of the Virasoro algebra. Note that they essentially involve the whole set of times and, therefore, can not be observed for a truncated “action” like (1) at all (in
contrast to integrability which we partially observed already with two parameters included). Indeed, the Ward identities contain more subtle information about the theory, basically related to the exact (Polchinski) renormalization group \([7]\). This is why their solution typically unambiguously gives the partition function \([8]\).

3 Integrability in string/field theories

Thus, the main lesson we should learn from our simple example is that one has to switch on and vary as many couplings as possible. Then, for the complete effective action (generating function of all correlators) one realizes a series of different underlying structures like integrability, complete set of Ward identities etc.

Note that the very phenomenon of classical integrability in different quantum systems was observed many times. The examples were mostly done in quantum integrable systems, just as only these systems could be exactly solved and the integrability could be \textit{a posteriori} realized. However, the examples constructed come from different fields (of two-dimensional quantum field theories, statistical lattice partition functions, etc) \([5, 11]\) which implies the phenomenon is very general. Moreover, it was also observed in matrix models of absolutely different type (so called continuous matrix models \([12]\), in two-dimensional gauge theories \([13]\) etc. These examples are all two-dimensional, however, we mention below higher-dimensional examples too. Moreover, within the string theory approach it is also expected that the sum of the whole perturbative series for the amplitudes (and the partition function) can be described by a quantum integrable system \([14, 1]\). Thus, the very fact that the effective action for the \textit{quantum} system is a \(\tau\)-function of some \textit{classical} integrable system\(^1\) is already known for many years, although has never been explained. In section 6, we propose a way for an explanation of this fact, following \([2]\).

However, effective actions depends not only on coupling constants. Other parameters essentially determining the theory are boundary conditions \([16]\) or, which is basically the same, vacuum expectation values of different fields. It turns out that involving these parameters makes general structures behind the theory much richer. Strikingly amazing, in this case integrability is also presented although realized in a different way.

4 Integrability in Seiberg-Witten theory

A basic example of the effective action depending on vacuum expectation values is the low-energy effective action in \(N = 2\) supersymmetric gauge theories in the dimensions 4, 5 and 6 with various matter contents. The exact solution for such an action was constructed by N.Seiberg and E.Witten \([17]\), while its integrable properties were revealed in \([18]\) (one may find a lot of discussions of Seiberg-Witten solutions and their integrable structures in the book \([19]\)). More concretely, one studies the

\(^1\)It is interesting that all the examples can be parted into two large classes of systems naturally depending on times (coupling constants) and on the so called Miwa variables, these latter ones being just eigenvalues of an (infinite) external matrix. An instance of the first class system where the matrix model \([1]\) also lies is the quantum non-linear Schrödinger model. The generating functional of the correlators in this model is a \(\tau\)-function of some classical integrable non-linear equation of the same (non-linear Schrödinger) kind \([4]\). A typical example of the other type is given by the partition function in the six-vertex model with non-trivial boundary conditions. This partition function turns out to be the \(\tau\)-function of the two-dimensional Toda system expressed in Miwa variables \([1]\). Another typical representative of the same class of theories is the continuous matrix models \([2, 13]\). In particular, such matrix models have much to do with the very low-energy limit of QCD \([13]\).
theory with the gauge group $G$ such that the symmetry breaks down to $U(1)^n$ with $n$ scalar fields getting non-zero vacuum expectation values, $<\phi_i> = a_i$ and $n$ photons remaining massless (Coulomb branch). One then effectively obtains at low energies $n \, N = 2$ supersymmetric photodynamics. The scalar part of the action can be described by a sigma-model with the metric being the second derivative of a single holomorphic function, $T_{ij}(\phi) = \partial^2 F_j \phi(\phi)$ called prepotential. Singularities of this function are known from physical arguments (of duality and compatibility with the renormalization group flows) and fix $F(\phi)$. The prepotential completely fixes the exact low energy amplitudes in the theory. It can be constructed as follows.

First of all, one has to find out proper variables whose modular properties fit the field theory interpretation. These variables are the integrals of a meromorphic 1-form $dS$ over the cycles on a two-dimensional Riemann surface, $a_i$ and $a^D_i$

$$a_i = \oint_{A_i} dS, \quad a^D_i = \oint_{B_i} dS,$$

(4)

(where $i, j = 1, ..., N_c - 1$ for the gauge group $SU(N_c)$).

These integrals play the two-fold role in the Seiberg-Witten approach. First of all, one may calculate the prepotential $F$ and, therefore, the low energy effective action through the identification of $a^D_i$ and $\partial F / \partial a_i$ with $a$ defined as a function of moduli (values of condensate) by formula (4). Then, using the property of the differential $dS$ that its variations w.r.t. moduli are holomorphic one may also calculate the matrix of coupling constants in the gauge theory

$$T_{ij}(u) = \frac{\partial^2 F}{\partial a_i \partial a_j},$$

(5)

The second role of formula (4) is that, as was shown these integrals define the spectrum of the stable states in the theory which saturate the Bogomolny-Prasad-Sommerfeld (BPS) limit. For instance, the formula for the BPS spectrum in the $SU(2)$ theory reads as

$$M_{n,m} = |na(u) + ma^D(u)|,$$

(6)

where the quantum numbers $n, m$ correspond to the “electric” and “magnetic” states.

It was realized in that each Seiberg-Witten solution can be associated with an integrable system with finite number degrees of freedom, and the prepotential can be immediately constructed from integrable data.

Indeed, the structures underlying Seiberg-Witten theory are the following set of data:

- Riemann surface $\mathcal{C}$
- moduli space $\mathcal{M}$ (of the curves $\mathcal{C}$), the moduli space of vacua of the gauge theory
- meromorphic 1-form $dS$ on $\mathcal{C}$

Exactly this input can be naturally described within the framework of some underlying integrable system.

To this end, first, we introduce bare spectral curve $E$ that is torus $y^2 = x^3 + g_2 x^2 + g_3$ for the UV-finite gauge theories with the associated holomorphic 1-form $d\omega = dx/y$. This bare spectral curve degenerates into the double-punctured sphere (annulus) for the asymptotically free theories (where dimensional transmutation occurs): $x \rightarrow w + 1/w$, $y \rightarrow w - 1/w$, $d\omega = dw/w$. On this bare curve,
there are given a matrix-valued Lax operator $L(x, y)$. The corresponding dressed spectral curve $C$ is defined from the formula $\text{det}(L - \lambda) = 0$.

This spectral curve is a ramified covering of $E$ given by the equation

$$P(\lambda; x, y) = 0 \quad (7)$$

In the case of the gauge group $G = SU(N_c)$, the function $P$ is a polynomial of degree $N_c$ in $\lambda$.

Thus, we have the spectral curve $C$, the moduli space $M$ of the spectral curve being given just by coefficients of $P$. The third important ingredient of the construction is the generating 1-form $dS \sim = \lambda d\omega$ meromorphic on $C$ ("\sim" denotes the equality modulo total derivatives). From the point of view of the integrable system, it is just the shortened action "$pdq$" along the non-contractible contours on the Hamiltonian tori. This means that the variables $a_i$ in (4) are nothing but the action variables in the integrable system. The defining property of $dS$ is that its derivatives with respect to the moduli (ramification points) are holomorphic differentials on the spectral curve. This, in particular, means that

$$\frac{\partial dS}{\partial a_i} = d\omega_i \quad (8)$$

where $d\omega_i$ are the canonical holomorphic differentials\(^2\). Integrating this formula over $B$-cycles and using that $a_D = \partial F/\partial a$, one immediately obtains (5).

So far we reckoned without matter hypermultiplets. In order to include them, one just needs to consider the surface $C$ with punctures. Then, the hypermultiplet masses are proportional to the residues of $dS$ at the punctures, and the moduli space has to be extended to include these mass moduli. All other formulas remain in essence the same.

The prepotential $F$ and other "physical" quantities are defined in terms of the cohomology class of $dS$, formula (4). Note that formula (5) allows one to identify the prepotential with logarithm of the $\tau$-function of the Whitham hierarchy [21]: $F = \log \tau$. This illustrates how another, Whitham integrability also emerges in Seiberg-Witten theories [22]. In fact, this phenomenon is quite general: in matrix models one observes the Whitham integrability as well [23]. However, any clear general reason for this is still missed.

Note that Seiberg-Witten theory celebrates even more interesting properties. One of them is that the prepotential $F$ satisfies a set of highly non-linear equations called associativity (or WDVV) equations [24].

5 Duality in Seiberg-Witten theory

In order to understand another interesting feature of Seiberg-Witten theory, one needs to look at the whole web of $N = 2$ SUSY gauge theories, in different space-time dimensions and with different matter contents [19]. Restricting oneself to the theories with one adjoint matter hypermultiplet and varying the dimensions, one rolls among the members of the dual Calogero-Ruijsenaars integrable family [25]. For instance, the perturbative limit of the four-dimensional theory is described by the trigonometric Calogero system which is dual to a degenerated perturbative limit of the five-dimensional theory described by the rational Ruijsenaars system. At the same time, the perturbative limit of the five-dimensional theory is described by the self-dual trigonometric Ruijsenaars system. Meanwhile, dealing

\(^2\)I.e. satisfying the conditions

$$\oint_{A_i} d\omega_j = \delta_{ij}, \quad \oint_{B_i} d\omega_j = T_{ij}$$
with complete non-perturbative contributions requires elliptic models\(^4\) and ultimately knowledge of the most general (self-dual) member of the family, the double-elliptic system \([21, 22]\). This system describes the (non-perturbative) six-dimensional gauge theory compactified onto two-dimensional torus and is constructed so far only basing on (self-)duality arguments.

In order to understand what this duality means we develop below the scheme \([2]\) which simultaneously allows one to understand a reason for quantum systems to reveal an integrability in coupling constant flows we discussed in sections 2 and 3. Meanwhile, we realize that the notion of (self-)duality emerged is not very restrictive for the theory, moreover, there are a lot of self-dual systems. Self-duality of the Calogero-Ruijsenaars family turns out to have rather more to do with the group theory interpretation of this family but does not pick up it among other integrable systems.

In order to make the main idea of \([2]\) clear, we start with a simple mechanical integrable system\(^4\). Integrable system with \(N\) coordinates \(q_i\) and \(N\) momenta \(p_i\) is characterized by existence of \(N\) Poisson-commuting Hamiltonians \(H_i(\vec{p}, \vec{q})\), \(\{H_i, H_j\} = \frac{\partial H_i}{\partial \vec{p}} \frac{\partial H_j}{\partial \vec{q}} - \frac{\partial H_i}{\partial \vec{q}} \frac{\partial H_j}{\partial \vec{p}} = 0\). For such a system one can consider a canonical transformation, treating these Hamiltonians as new momenta-like or coordinate-like variables. From now on, this transformation (of which the infinitesimal version is a certain Hamiltonian flow) will be the main subject of the paper.

To make the problem precise, let us consider a one-parametric family of integrable models, parameterized by a single coupling constant \(g\) such that the model is **free** when \(g = 0\). This means that at \(g = 0\) the Hamiltonians \(H_i^{(0)}(\vec{p}) = H_i(\vec{p}, \vec{q}|g = 0)\) are functions only of momenta \(\vec{p}\), though, for conventional choices of Hamiltonians in particular applications, these functions can be non-trivial. The typical examples are: \(H_k^{(0)}(\vec{p}) = \sum_{i=0}^{N} p_i^k\) and \(H_k^{(0)}(\vec{p}) = \sum_{|I|=k} \prod_{i \in I} e^{ip_i}\) for \(p\)-rational and \(p\)-trigonometric models respectively and some elliptic functions of \(\vec{p}\) for their elliptic generalizations.

The adequate definition of the new canonical variables \(\vec{P}_g = \vec{P}(\vec{p}, \vec{q}|g)\) and \(\vec{Q}_g = \vec{Q}(\vec{p}, \vec{q}|g)\)\(^5\) is \([2]\)

\[
\begin{align*}
H_i^{(0)}(\vec{P}) &= H_i(\vec{p}, \vec{q}|g) \\
\vec{H}_i^{(0)}(\vec{Q}) &= \vec{H}_i(\vec{Q}, \vec{P}|g)
\end{align*}
\]

where \(\vec{H}_i(\vec{p}, \vec{q}|g)\) define the Hamiltonians of the dual integrable system. This is the exact definition of what is the duality in integrable mechanical system, in particular, in Seiberg-Witten theory. It should not be mixed with “physical” electro-magnetic duality of this theory \([17]\) which is the duality between “electric” and “magnetic” states in \([9]\).

Note that in \([3]\) the Hamiltonians depend on the “dressed” variables \(\vec{P}\) and \(\vec{Q}\) and, moreover, \(\vec{P}\) and \(\vec{Q}\) are interchanged. The shape of the dual Hamiltonians is dictated by the requirement that the new variables \(\vec{P}\) and \(\vec{Q}\) are canonical, i.e.

\[
\sum_i dP_i \wedge dQ_i = \sum_i dp_i \wedge dq_i
\]

\(^4\)Note that the system dual to the complete non-perturbative four-dimensional theory is the perturbative six-dimensional theory compactified onto two-dimensional torus. Thus, taking into account non-perturbative (instanton) corrections is effectively equivalent to summing up Kaluza-Klein modes when compactifying onto two-dimensional torus.

\(^5\)We do not discuss here to what extent we need here the integrability of the system on the whole phase space. Indeed, since our consideration is local, one may expect that we need only \(N\) Poisson-commuting Hamiltonians given locally, at a patch of the phase space. For the sake of simplicity, we ignore this point here, merely assuming our system is completely integrable.

\(^5\)In what follows we often omit the label \(g\), implying that the capital letters \(P\) and \(Q\) always denote the “dressed” momenta and coordinates \(P_g\) and \(Q_g\).
and the Poisson brackets are
\[ \{ , \} = \sum_i \left( \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial q_i} - \frac{\partial}{\partial q_i} \otimes \frac{\partial}{\partial p_i} \right) = \sum_i \left( \frac{\partial}{\partial P_i} \otimes \frac{\partial}{\partial Q_i} - \frac{\partial}{\partial Q_i} \otimes \frac{\partial}{\partial P_i} \right) \] (11)

In examples of section 9 we shall see that this definition of the dual Hamiltonians, indeed, makes the trigonometric Calogero system dual to the rational Ruijsenaars one, trigonometric Ruijsenaars system self-dual etc. However, this same definition implies that any integrable system has its dual counterpart, i.e. the dualities between different members of the Calogero-Ruijsenaars family says nothing special about these specific integrable systems. This is only their interpretation in group theory terms that makes the duality of special interest in this case. We, however, do not address more to this point here, dealing instead with generic properties of integrable systems.

6 Integrability and flows in coupling constants: General theory

Relations (9) define \( \vec{P}_g \) as functions of \( \vec{p}, \vec{q} \) and the coupling constant \( g \). Of special interest and importance is the infinitesimal version of this canonical transformation, considered as a Hamiltonian flow along the \( g \) direction in the space of coupling constants. Such transformation is generated by a new Hamiltonian \( T(\vec{p}, \vec{q}; g) \), according to the rule:
\[
\frac{\partial \vec{P}_g}{\partial g} = \left\{ T(\vec{P}_g, \vec{Q}_g|g), \vec{P}_g \right\} = -\frac{\partial T}{\partial \vec{Q}_g},
\]
\[
\frac{\partial \vec{Q}_g}{\partial g} = \left\{ T(\vec{P}_g, \vec{Q}_g|g), \vec{Q}_g \right\} = \frac{\partial T}{\partial \vec{P}_g},
\] (12)

This new Hamiltonian does not commute with the old ones, but it converts them into a new set of commuting Hamiltonians.

Moreover, instead of considering a pre-given family of integrable systems, one can use any function \( T(\vec{p}, \vec{q}|g) \) to define a whole one-parametric family of integrable systems, though explicit construction of the corresponding Poisson-commuting Hamiltonians is rarely possible. Further, a multi-parametric family of integrable systems is similarly generated by any collection of functions \( T_a(\vec{p}, \vec{q}|g) \) satisfying the compatibility (Whitham) equations:
\[
\frac{\partial T_b}{\partial g_a} - \frac{\partial T_a}{\partial g_b} + \left\{ T_a, T_b \right\} = 0
\] (13)

Note that after quantization of the original system these \( T_a(\vec{p}, \vec{q}|g_b) \) become a family of operators \( \hat{T}_a(\vec{p}, \vec{q}|g_b) \), satisfying the zero-curvature condition in the space of coupling constants:
\[
\left[ \frac{\partial}{\partial g_a} - \hat{T}_a(\vec{p}, \vec{q}), \frac{\partial}{\partial g_b} - \hat{T}_b(\vec{p}, \vec{q}) \right] = 0
\]

This can serve as an explanation of emergency of classical integrability (=the zero-curvature equations) in the study of quantum integrable systems we discussed in the previous sections. In particular, it would be interesting to deal with the cases described in sections 2 and 3 and associated to integrable mechanical systems with infinitely many degrees of freedom along the line of this paper.

The coupling constant variation for the dual integrable system is governed by the dual Hamiltonian \( \tilde{T}(\vec{p}, \vec{q}|g) \): for
\[
\tilde{H}_i^{(0)}(\vec{P}) = \tilde{H}_i(\vec{p}, \vec{q}|g)
\] (14)
(and $H_i^{(0)}(\mathcal{q}) = H_i(\mathcal{Q}, \mathcal{P} | g)$) we have
\begin{equation}
\frac{\partial \tilde{P}_g}{\partial g} = \{ \tilde{T}(\tilde{P}_g, \tilde{Q}_g | g), \tilde{P}_g \} = - \frac{\partial \tilde{T}}{\partial \tilde{Q}_g},
\end{equation}
\begin{equation}
\frac{\partial \tilde{Q}_g}{\partial g} = \{ \tilde{T}(\tilde{P}_g, \tilde{Q}_g | g), \tilde{Q}_g \} = \frac{\partial \tilde{T}}{\partial \tilde{P}_g}.
\end{equation}

Relation between $\tilde{T}(\mathcal{p}, \mathcal{q} | g)$ and $T(\mathcal{p}, \mathcal{q} | g)$ becomes especially simple at any self-dual point (where $\tilde{H}_i(\mathcal{p}, \mathcal{q} | g_{SD}) = H_i(\mathcal{p}, \mathcal{q} | g_{SD})$):
\begin{equation}
\tilde{T}(\mathcal{p}, \mathcal{q} | g_{SD}) = T(\mathcal{q}, \mathcal{p} | g_{SD})
\end{equation}
As immediate consequence, \textit{any} symmetric function $T(\mathcal{p}, \mathcal{q} | g) = T(\mathcal{q}, \mathcal{p} | g)$ defines a one-parametric family of self-dual integrable systems. Therefore, not just the duality but even the property of self-duality of models from the Calogero-Ruijsenaars family (rational Calogero, trigonometric Ruijsenaars and double elliptic systems) is in no way specific. We again should claim that the (self-)duality of this family is interesting only in the group theory context.

Note that it is not that immediate calculation to restore the Hamiltonian starting from arbitrarily given $T(\mathcal{p}, \mathcal{q} | g)$. Say, one can build a perturbative series in the coupling constant $g$. At the self-dual point for the system with one degree of freedom one takes the expansion $T(p, q | g) = \sum_{i=1}^{\infty} T_i(p, q) g^{i-1}$, $P = H(p, q) = p + \sum_{i=1}^{\infty} \Phi_i(p, q) g^i$ etc and obtains that $\Phi_1(p, q) = \partial_q T_1(p, q)$, $\Phi_2(p, q) = \partial_q T_2(p, q) + ...$. The ultimate result is that the symmetric part of $\partial_q \Phi_i(p, q) = \partial^2_{q,p} T_i(p, q)$ is given independently at any order, while the antisymmetric parts are fixed by $\Phi_i$’s at lower orders:
\begin{align*}
\Phi_1'(p, q) - \Phi_1'(q, p) &= 0, \\
\Phi_2'(p, q) - \Phi_2'(q, p) &= \Phi''_1(p, q) \Phi_1(p, q) - (p \leftrightarrow q), \\
\Phi_3'(p, q) - \Phi_3'(q, p) &= \Phi''_1(p, q) \Phi_2(p, q) + \Phi''_2(p, q) \Phi_1(p, q) - \frac{1}{2} \Phi'''_1(p, q) \Phi^2_1(p, q) - \\
&- \Phi''_1(p, q) \Phi_1(p, q) \Phi_1(p, q) - (p \leftrightarrow q), \\
&\ddots
\end{align*}
where all the derivatives are taken w.r.t. to the first variable. A trivial solution to these equations is $\Phi_i(p, q) = p^{n_i+1} q^{n_i}$ with arbitrary $\{n_i\}$. This implies that the Hamiltonian $H(p, q) = pf(pq)$ is self-dual with arbitrary function $f$. Indeed, one easily checks this is the case (note that for such a Hamiltonian $pq = PQ$). In particular, the rational Calogero Hamiltonian gets to this class of Hamiltonians.

### 7 Generating functions and quantization

Along with the Hamiltonians $T(\mathcal{P}, \mathcal{Q} | g)$ one can also consider the generating functions of canonical transformations in question, like $S(\mathcal{Q}, \mathcal{q} | g)$ or its Legendre transform $F(\mathcal{P}, \mathcal{q} | g) = \mathcal{P} \mathcal{Q} - S(\mathcal{Q}, \mathcal{q} | g)$, such that
\begin{equation}
-\mathcal{P} = \frac{\partial S}{\partial \mathcal{Q}}, \quad \mathcal{P} = \frac{\partial S}{\partial \mathcal{Q}}
\end{equation}
and
\begin{equation}
T = \frac{\partial S}{\partial g}
\end{equation}
Of course, for canonical transformation

$$\frac{\partial P_i}{\partial q_j} + \frac{\partial p_j}{\partial Q_i} = 0,$$

(20)
as implied by (18), but one should be more careful when drawing similar conclusion from (19): the second g derivatives satisfy eq.(13), because g-derivative is taken at constant $\vec{P}_g$ and $\vec{Q}_g$, which themselves depend on g.

Similarly,

$$-\vec{Q} = \frac{\partial F}{\partial \vec{P}}, \quad \vec{P} = \frac{\partial F}{\partial \vec{q}}$$

(21)

At self-dual points $F(\vec{P}, \vec{q} | g_{SD}) = F(\vec{q}, \vec{P} | g_{SD})$ is a symmetric function.

Note that

$$\frac{\partial F}{\partial g} = T$$

(22)
(with p and Q constant), i.e. T is, in a sense, a more invariant quantity than S and F, which does not depend on the choice of independent variables.

Another way to see it is to consider a flow from an integrable system with coupling constant $g_1$ to the same system with coupling constant $g_2$. Then, T depends only on $g_2$, but not on $g_1$, while the generating functions depend on both $g_1$ and $g_2$.

Basically, if reexpressed in terms of $\vec{Q}$ and $\vec{q}$ (instead of $\vec{P}$ and $\vec{Q}$) the Hamiltonian $T = \partial S / \partial g$. Therefore, $e^{iS}$ can be considered as a kind of an evolution operator (kernel) in the space of coupling constants, which performs a canonical transformation from the free system to the integrable one. After quantization it can be symbolically represented as

$$e^{iS(\vec{Q}, \vec{q} | g)} = \oplus \lambda |\psi_\lambda^{(0)}(\vec{Q})\rangle c_\lambda \langle \psi_\lambda(\vec{q})|$$

(23)

where $|\psi_\lambda\rangle$ and $|\psi_\lambda^{(0)}\rangle$ are eigenfunctions of the system and $c_\lambda$ are some coefficients, depending on the spectral parameter $\lambda$. The evolution operator satisfies the quantum version of eq.(1),

$$H^{(0)}(\partial / \partial \vec{Q}) = e^{-iS(\vec{Q}, \vec{q} | g)} H(\partial / \partial \vec{q}, \vec{q} | g) e^{iS(\vec{Q}, \vec{q} | g)}$$

(24)

Since eigenfunctions $|\psi_\lambda^{(0)}(Q)\rangle$ of a free system are just exponents of the spectral parameter $\lambda$, the dressed eigenfunctions $|\psi_\lambda(q)\rangle$ are basically Fourier transforms of the evolution operator $e^{iS(Q,q)}$:

$$\psi_\lambda(q) \sim \int e^{iS(Q,q)} e^{i\lambda Q} dQ$$

(25)

Similarly,

$$\psi_\lambda(q) \sim e^{i\tilde{F}(P,q)} \delta(\lambda - P) dP \sim e^{i\tilde{F}(\lambda,q)}$$

(26)

Note that there is a freedom in solutions of eq.(2) to shift $\{Q_i\}$ by an arbitrary function of $\{P_i\}$. This shift is quite complicated in terms of the generating function S, but in F it is just an addition of the term depending only on $\{P_i\}$. In the quantum case, this ambiguity in the definition of F is just a matter of normalization of the eigenfunction.

One can also consider the quantum counterpart of $T(p, q | g)$ which is a Hamiltonian that gives rise to a Schrödinger equation w.r.t. the coupling constant

$$\frac{\partial \psi}{\partial g} = i\tilde{T} \psi$$

(27)
Therefore, the wavefunction \( \psi \) can be also realized as a path integral over the phase space variables \( P(g), Q(g) \).

The generating functions \( S(Q, q), F(P, q) \) and \( T(p, q) \) satisfy the quasiclassical versions of these relations (when multiple derivatives of \( S, F \) and \( T \) are neglected). Exact definition of quantum evolution operators, including effects of discrete spectra and identification of the spectra at different values of coupling constants, i.e. precise definition of the spectral parameter \( \lambda \) in such a way that eq.(23) is diagonal in \( \lambda \), will be considered elsewhere.

8 Particular examples

In order to illustrate our consideration of the previous sections, we briefly discuss here several manifest examples (further details can be found in \(^3\)) restricting ourselves to the systems with one degree of freedom only. We start with the simplest example of harmonic oscillator. Then, \( H(p, q|\omega) = \frac{1}{2}(p^2 + \omega^2 q^2) \). Let the frequency \( \omega \) play the role of the coupling constant so that \( H(0)(p) = \frac{1}{2}p^2 \). Then

\[
P = \sqrt{p^2 + \omega^2 q^2},
Q = \sqrt{p^2 + \omega^2 q^2} \frac{\omega q}{p} \text{ arctan} \frac{\omega q}{p}
\]

The inverse transformation looks like

\[
p = P \cos \frac{\omega Q}{P},
q = \frac{P \sin \frac{\omega Q}{P}}{\omega}
\]

i.e. the Hamiltonian of the dual flow is

\[
\tilde{H}(\tilde{p}, \tilde{q}|\omega) = \frac{\tilde{q}}{\omega} \sin \frac{\omega \tilde{p}}{\tilde{q}}
\]

(Note that with this definition \( \tilde{H}(0)(p) = p \) and also note that \( p, q \)-duality does not respect conventional dimensions of \( p \) and \( q \) so that it should not be a surprise that \( \omega q/p \) in \( H \) got substituted by \( \omega \tilde{p}/\tilde{q} \) in \( \tilde{H} \). Of course, \( \tilde{p} \) and \( \tilde{q} \) are nothing but \( Q_\omega \) and \( P_\omega \).)

The generator of \( \omega \)-evolution is

\[
T = \frac{P^2}{4\omega^2} \sin \frac{2\omega Q}{P} - \frac{PQ}{2\omega} = \frac{pq - PQ}{2\omega}
\]

For the dual system we similarly have

\[
\tilde{T} = -\omega \tilde{P} \tilde{Q} + (\tilde{Q}^2 + \omega^2 \tilde{P}^2) \text{ arctan} \frac{\omega \tilde{p}}{\tilde{q}} = \frac{\tilde{pq} - \tilde{P} \tilde{Q}}{2\omega} = \frac{PQ - pq}{2\omega} = -T
\]

The generating function

\[
F(P, q|\omega) = \int pdq = \int \sqrt{P^2 - \omega^2 q^2} dq = \frac{1}{2}q \sqrt{P^2 - \omega^2 q^2} + \frac{P^2}{2\omega} \log \left( \frac{i \omega q}{P} + \sqrt{1 - \omega^2 q^2} \right)
\]
so that the relation \( e^{iF(\lambda, q)} \) in quasiclassical approximation is

\[
e^{iF(\lambda, q)} = \left( \frac{i\omega q}{\lambda} - \sqrt{1 - \frac{\omega^2 q^2}{\lambda^2}} \right)^{\lambda^2/2\omega} \sim e^{\frac{i\omega}{2} \sqrt{\lambda^2 - \omega^2 q^2}} \sim e^{-\omega q^2/2} H_{\nu}(\sqrt{\omega q})
\]  

(34)

where \( \frac{\lambda^2}{2\omega} = \nu + \frac{1}{2} \) and the Hermite polynomials, satisfying

\[
(-\partial_x^2 + x^2) e^{-x^2/2} H_\nu(x) = (2\nu + 1) e^{-x^2/2} H_\nu(x)
\]  

(35)

are given by inverse Laplace transform of the generating function \( e^{-x^2/4 + tx} \)

\[
H_{\nu}(x) \sim \int \frac{dt}{\nu+1} e^{-t^2/4 + tx}
\]  

(36)

Taking this integral in the saddle point approximation, one gets

\[
H_\nu(x) \sim \left( x - \sqrt{x^2 - 2(\nu + 1)} \right)^{\nu+1} e^{x^2/2 + x^2/\sqrt{x^2 - 2(\nu+1)}}
\]  

(37)

which, at large \( \nu \), is in accordance with eq. (34).

For a single particle (one degree of freedom) any dynamics is integrable, and any Hamiltonian is canonically equivalent to the free one. Therefore, it may make sense to look at the general Hamiltonian. For \( H = \sqrt{p^2 + g^2 V(q)} \) the dual Hamiltonian is \( q = \tilde{H}(P, Q) \). The canonicity condition then gives

\[
\frac{\partial \tilde{H}(P, Q)}{\partial Q} = \frac{p(P, Q)}{P}
\]  

(38)

Therefore, the dual Hamiltonian can be obtained via solving the equation

\[
Q = \int q \frac{d\xi}{\sqrt{1 - g^2 V(\xi)/P^2}}
\]  

(39)

w.r.t. \( q \). Now taking the derivative of \( P = \sqrt{p^2 + g^2 V(q)} \) w.r.t. \( g \), one obtains

\[
- \frac{\partial T}{\partial Q} = \frac{g}{P} V(q(P, Q))
\]  

(40)

where \( q(P, Q) \) is given by (39).

9 Calogero-Ruijsenaars family

To complete this our review, we also list some results for the rational and trigonometric members of the Calogero-Ruijsenaars family (which describe different limits of Seiberg-Witten theory with adjoint matter hypermultiplet in different space-time dimensions):

- The rational-rational case (rational Calogero model)
In this case the single Hamiltonian is \( H = \frac{1}{2} \left( p^2 + \frac{g^2}{q^2} \right) \), thus \( H^0(p) = \frac{1}{2} p^2 \), and

\[
P^2 = p^2 - \frac{g^2}{q^2},
\]

\[
q^2 = Q^2 - \frac{g^2}{P^2}
\]

The rational Calogero model is self-dual. The flow is generated by the Hamiltonian

\[
T(p, q; g) = \frac{1}{2} \log \frac{pq - g}{pq + g} = \frac{1}{2} \log \frac{PQ - g}{PQ + g}
\]

(42)

This is obviously a symmetric function of \( p \) and \( q \), thus, \( T_D(p, q) = T(p, q) \) as it should be for the self-dual system. The generating function is

\[
S = g \arccosh \frac{Q}{q} = g \log \left( \frac{Q - \sqrt{Q^2 - q^2}}{q} \right) = \frac{g}{2} \log \frac{Q - \sqrt{Q^2 - q^2}}{Q + \sqrt{Q^2 - q^2}}
\]

(43)

In this case, \( S \) is a simple linear function of \( g \), and \( \frac{\partial S}{\partial g} = S/g \). Similarly,

\[
F(P, q) = \sqrt{P^2 q^2 + g^2} + \frac{g}{2} \log \frac{\sqrt{P^2 q^2 + g^2} - g}{\sqrt{P^2 q^2 + g^2} + g}
\]

\[
\frac{\partial F(P, q|g)}{\partial g} = T(p(P, q), q|g)
\]

(44)

In accordance with (25)

\[
\psi_{\lambda}(q) \sim \int e^{iS(q, Q|g)} e^{iQ\lambda} dQ \sim q J_{i\lambda}(i\lambda q)
\]

(45)

where \( J_{i\lambda}(x) \) is the Bessel function. The exact quantum wavefunction, solving the equation \((-\partial_x^2 + \frac{g^2}{x^2})\psi_{\lambda}(x) = \lambda^2 \psi_{\lambda}(x)\), is \( \sqrt{\pi} J_{\nu}(i\lambda x) \), \( \nu^2 = -g^2 + 1/4 \). It coincides with (45) in quasiclassical approximation.

- The rational-trigonometric case (trigonometric Calogero model)

\[
P^2 = 2H = p^2 - \frac{g^2}{\sinh^2 q},
\]

\[
\cosh^2 q = \bar{H}^2 = \cosh^2 Q \left( 1 - \frac{g^2}{P^2} \right),
\]

(46)

\[
T = \frac{1}{2} \log \frac{p \tanh Q - g}{p \tanh Q + g} = \frac{1}{2} \log \frac{\tanh q - g}{\tanh q + g}
\]

In this case \( S \) is still a simple linear function of \( g \)

\[
S = g \arcsinh \left( \frac{\sinh Q}{\sinh q} \right)
\]

(47)
The trigonometric-rational case (rational Ruijsenaars model)

$$\cosh^2 P = H^2 = \cosh^2 p \left(1 - \frac{\sinh^2 \epsilon}{q^2}\right),$$

$$q^2 = 2 \tilde{H} = Q^2 - \frac{\sinh^2 \epsilon}{\sinh^2 P},$$

$$T = \frac{1}{2} \log \frac{Q \tanh P - \tanh \epsilon}{Q \tanh P + \sinh \epsilon} = \frac{1}{2} \log \frac{q \tanh p - \tanh \epsilon}{q \tanh p + \sinh \epsilon}.$$  \hfill (48)

In this case, $S$ is no longer a simple linear function of the coupling constant $\epsilon$.

The trigonometric-trigonometric case (trigonometric Ruijsenaars model)

$$\cosh^2 P = H^2 = \cosh^2 p \left(1 - \frac{\sinh^2 \epsilon}{\sinh^2 q}\right),$$

$$\cosh^2 q = \tilde{H}^2 = \cosh^2 Q \left(1 - \frac{\sinh^2 \epsilon}{\sinh^2 P}\right),$$

$$T = \frac{1}{2} \log \frac{\tanh P \tanh Q - \tanh \epsilon}{\tanh P \tanh Q + \tanh \epsilon} = \frac{1}{2} \log \frac{\tanh p \tanh q - \tanh \epsilon}{\tanh p \tanh q + \tanh \epsilon}.$$  \hfill (49)

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