ANOSOV STRUCTURE ON MARGULIS SPACE TIME

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Abstract. In this paper we describe the stable and unstable leaves for the geodesic flow on the space of non-wandering spacelike geodesics of a Margulis Space Time and prove contraction properties of the leaves under the flow. We also show that monodromy of Margulis Space Times are “Anosov representations in non semi-simple Lie groups”.

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1. Introduction

A Margulis Space Time $M$ is a quotient manifold of the three dimensional affine space by a free, non-abelian group acting as affine transformations with discrete linear part. It owes its name to Grigory Margulis, who was the first to use these spaces, in [2] and [4], as examples to answer Milnor’s following question in the negative.

**Question 1.** Is the fundamental group of a complete, flat, affine manifold virtually polycyclic? [1]

Observe that if $M$ is a Margulis Space Time then the fundamental group $\pi_1(M)$ does not contain any translation. By combining results of Fried, Goldman and Mess from [3], [7], a complete flat affine manifold either has a polycyclic fundamental group or is a Margulis Space Time. In this paper we will only consider Margulis Space Times whose linear part contains no parabolic, although by Drumm there exists Margulis Space Time whose linear part contains parabolics. Fried and Goldman showed in [3] that a conjugate of the linear part of the affine action of the fundamental group forms a subgroup of $SO(2,1)$ in $GL(3,\mathbb{R})$. Therefore, a Margulis Space Time comes with a parallel Lorentz metric.

The parallelism classes of timelike geodesics of $M$ can be parametrized by a non-compact complete hyperbolic surface $\Sigma$. Recent work by Danciger, Gueritaud and Kassel in [12] have shown that $M$ is a $\mathbb{R}$-bundle over $\Sigma$ and the fibers are time like geodesics. Previous works of Jones, Charette, Goldman, Labourie and Margulis in [5], [8] and [9] showed that the dynamics of $M$ is closely related to that of $\Sigma$. Jones, Charette and Goldman showed in [5] that bispiralling geodesics in $M$ exists and they correspond to bispiralling geodesics in $\Sigma$. Goldman and Labourie showed in [9] that non-wandering spacelike geodesics in $M$ correspond to non-wandering geodesics in $\Sigma$.

In this paper, we first chalk out some preliminary notions, inorder to prepare the grounds to explicitly describe the stable and unstable laminations of $T^+_\text{rec}M$, the space of non-wandering spacelike geodesics in $M$, under the geodesic flow. We carry on to show that the stable lamination contracts under the forward flow and the unstable lamination contracts under the backward flow. More precisely, we prove the following,

**Theorem 1.0.1.** Let $\mathcal{L}^+$ and $\mathcal{L}^-$ be two laminations of the metric space $T^+_\text{rec}M$ as defined in definition [3.2.3]. The geodesic flow on the space of non-wandering spacelike geodesics in $M$ contracts $\mathcal{L}^+$ exponentially in the forward direction of the flow and contracts $\mathcal{L}^-$ exponentially in the backward direction of the flow.

Moreover, in the last section using a natural extension of the definition of Anosov representation given in section 2.0.7 of [6] we define the notion of an Anosov representation in our context replacing manifolds by metric spaces. Using this definition we can restate our theorem by the following theorem:

**Theorem 1.0.2.** Let $L$ be a subgroup of the Lie group $SO^0(2,1) \ltimes \mathbb{R}^3$ as defined in [10.14]. There exist a pair of foliations on the homogeneous space $N := SO^0(2,1) \ltimes \mathbb{R}^3/L$ so that $T^+_\text{rec}M$ admits a $(N,SO^0(2,1) \ltimes \mathbb{R}^3)$ Anosov
structure whose monodromy is the fundamental group of the Margulis Space Time $\mathcal{M}$.

In other words, monodromy of Margulis Space Times are “Anosov representations in non semi-simple Lie groups”.

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2. Background

2.1. Affine Geometry. An affine space is a set $\mathbb{E}$ together with a vector space $\mathbb{V}$ and a faithful and transitive group action of $\mathbb{V}$ on $\mathbb{E}$. We call $\mathbb{V}$ the underlying vector space of $\mathbb{E}$ and refer to its elements as translations. An affine transformation $F$ between two affine spaces $\mathbb{E}_1$ and $\mathbb{E}_2$, is a map such that for all $x$ in $\mathbb{E}_1$ and for all $v$ in $\mathbb{V}_1$, $F$ satisfies the following property:

\begin{equation}
F(x + v) = F(x) + L(F).v
\end{equation}

for some linear transformation $L(F)$ between $\mathbb{V}_1$ and $\mathbb{V}_2$. Therefore, by fixing an origin $O$ in $\mathbb{E}$, one can represent an affine transformation $F$, from $\mathbb{E}$ to itself as a combination of a linear transformation and a translation. More precisely,

\begin{equation}
F(O + v) = O + L(F).v + (F(O) - O).
\end{equation}

We denote $(F(O) - O)$ by $u(F)$. The space of affine automorphisms of $\mathbb{E}$ onto itself form a group and we denote this group by $\text{Aff}(\mathbb{E})$. Using equation 2.1.2 we get an isomorphism between $\text{Aff}(\mathbb{E})$ and $\text{GL}(\mathbb{V}) \ltimes \mathbb{V}$ which sends $F$ in $\text{Aff}(\mathbb{E})$ to $(L(F), u(F))$ in $\text{GL}(\mathbb{V}) \ltimes \mathbb{V}$. Let us denote the tangent bundle of $\mathbb{E}$ by $T\mathbb{E}$. The tangent bundle $T\mathbb{E}$ of an affine space $\mathbb{E}$ is a trivial bundle and is canonically isomorphic to $\mathbb{E} \times \mathbb{V}$ as a bundle. The geodesic flow on $T\mathbb{E}$ is as follows,

\begin{equation}
\tilde{\Phi}_t: T\mathbb{E} \rightarrow T\mathbb{E}
(p, v) \mapsto (p + tv, v).
\end{equation}

2.2. Hyperboloid Model of Hyperbolic Geometry. Consider the three dimensional vector space $\mathbb{V}$ over $\mathbb{R}$ with a quadratic form, called the Lorentz form, given by the following matrix:

$$Q := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$ 

We denote $v^t.Q.v$ by $\langle v, v \rangle$ where $v$ is a vector in $\mathbb{V}$. The Lorentz cross product $\times$ associated with the Lorentz form is defined as follows:

\begin{equation}
u \times v := (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_2v_1 - u_1v_2)^t
\end{equation}
where \( u, v \) is denoted by \( (u_1, u_2, u_3)^t \) and \( (v_1, v_2, v_3)^t \) respectively. The Lorentz cross product satisfies the following properties for all \( u, v \) in \( \mathbb{V} \),

\begin{align}
(2.2.2) & \quad 1. \langle u, v \boxtimes w \rangle = \det[u, v, w], \\
& \quad 2. \langle u, u \boxtimes v \rangle = \langle v, u \boxtimes v \rangle = 0, \\
& \quad 3. \langle u \boxtimes v, u \boxtimes v \rangle = \langle u, v \rangle^2 - \langle u, u \rangle \langle v, v \rangle, \\
& \quad 4. u \boxtimes v = -v \boxtimes u.
\end{align}

Now for all real number \( k \) we define,

\begin{equation}
(2.2.3) \quad S^k := \{ v \in \mathbb{V} \mid \langle v, v \rangle = k \}.
\end{equation}

We note that \( S^{-1} \) has two components. We denote the component which contains the vector \((0, 0, 1)^t\) as \( \mathbb{H} \). The quadratic form gives rise to a Riemannian metric of constant negative curvature on the subspace \( \mathbb{H} \) of \( \mathbb{V} \). The space \( \mathbb{H} \) is called the hyperboloid model of hyperbolic geometry. Geodesics of this Riemannian metric in \( \mathbb{H} \) are the intersection curves between \( \mathbb{H} \) and two dimensional sub vector spaces of \( \mathbb{V} \) provided they intersect. The unit tangent vectors at a point \( h \) in \( \mathbb{H} \) are the vectors lying in the intersection of \( W \) and \( S^1 \) where \( W \) is a vector subspace of dimension two of \( \mathbb{V} \) such that \( q + W \) is the space of tangent vectors of \( \mathbb{H} \) at \( q \). The space of unit tangent vectors of \( \mathbb{H} \) forms a bundle, \( U\mathbb{H} \) over \( \mathbb{H} \), the bundle \( U\mathbb{H} \) is called the \textit{unit tangent bundle} of \( \mathbb{H} \). In particular, the point \(( (0, 0, 1)^t, (0, 1, 0)^t )\) belongs to \( U\mathbb{H} \).

Let \( SO^0(2, 1) \) denote the connected component containing the identity of \( SO(2, 1) \). We notice that the space \( \mathbb{H} \) is invariant under the action of \( SO^0(2, 1) \) on \( \mathbb{V} \). Furthermore, the map \( \Theta \) defined by,

\begin{equation}
(2.2.4) \quad \Theta : SO^0(2, 1) \longrightarrow U\mathbb{H} \\
\quad \quad \quad g \mapsto (g.(0, 0, 1)^t, g.(0, 1, 0)^t),
\end{equation}

is simply transitive, and hence, defines an identification between \( SO^0(2, 1) \) and \( U\mathbb{H} \). Under the identification \( \Theta \), the geodesic passing through \( g \) is given by \( g.a(t) \), where \( a(t) \) is the following matrix:

\begin{equation}
(2.2.5) \quad a(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(t) & \sinh(t) \\ 0 & \sinh(t) & \cosh(t) \end{pmatrix}.
\end{equation}

We denote the geodesic flow on \( U\mathbb{H} = SO^0(2, 1) \) by \( \tilde{\phi}_t \). Note that \( \tilde{\phi}_t(g) \) is equal to \( g.a(t) \).

There is a canonical metric \( d_{U\mathbb{H}} \) on the unit tangent bundle \( U\mathbb{H} \) whose restriction on \( \mathbb{H} \) is the hyperbolic metric. The metric \( d_{U\mathbb{H}} \) is unique upto the action of the maximal compact subgroup of \( SO^0(2, 1) \). Let \( g \) be an element of \( U\mathbb{H} \). We denote the collection of all \( h \) in \( U\mathbb{H} \) with \( d_{U\mathbb{H}}(\tilde{\phi}_t g, \tilde{\phi}_t h) \) going to zero as \( t \) goes to infinity, by \( \mathcal{H}^+_g \) and the collection of all \( h \) in \( U\mathbb{H} \) with \( d_{U\mathbb{H}}(\tilde{\phi}_t g, \tilde{\phi}_t h) \) going to zero as \( -t \) goes to infinity, by \( \mathcal{H}^-_g \). The collections \( \mathcal{H}^{\pm}_g \) are called the \textit{horocycles} of the geodesic flow. Under the identification
\( \Theta \), the horocycle \( H^\pm_g \) passing through \( g \) is given by \( g. u^\pm(t) \), where \( u^\pm(t) \) is given as follows:

\[
(2.2.6) \quad u^+(t) := \begin{pmatrix} 1 & -2t & 2t \\ 2t & 1 - 2t^2 & 2t^2 \\ 2t & -2t^2 & 1 + 2t^2 \end{pmatrix}
\]

\[
(2.2.7) \quad u^-(t) := \begin{pmatrix} 1 & 2t & 2t \\ -2t & 1 - 2t^2 & -2t^2 \\ 2t & 2t^2 & 1 + 2t^2 \end{pmatrix}
\]

Let \( \Gamma \) be a free, nonabelian subgroup with finitely many generators. We consider the left action of \( \Gamma \) on \( U \mathbb{H} \). We notice that the action of \( \Gamma \) being from the left and the action of \( a(t) \) being from the right, the two actions commute. Furthermore, given a free and proper action of \( \Gamma \) on \( U \mathbb{H} \), one gets an isomorphism between \( \Gamma \backslash U \mathbb{H} \) and \( U(\Gamma \backslash \mathbb{H}) \), where \( U(\Gamma \backslash \mathbb{H}) \) is the unit tangent bundle of the surface \( \Gamma \backslash \mathbb{H} \).

Let \( x_0 \) be a point in \( \mathbb{H} \). Let \( \Gamma x_0 \) denote the orbit of \( x_0 \) under the action of \( \Gamma \). We denote the closure of \( \Gamma x_0 \) inside the closure of \( \mathbb{H} \) by \( \overline{\Gamma x_0} \). We define the boundary of the group \( \Gamma \) to be the collection \( \overline{\Gamma x_0} \setminus \Gamma x_0 \) and denote it by \( \Lambda_\infty \Gamma \). We note that the collection \( \overline{\Gamma x_0} \setminus \Gamma x_0 \) is independent of the particular choice of \( x_0 \). We also know that \( \Lambda_\infty \Gamma \) is compact.

A non-wandering geodesic on the surface \( \Gamma \backslash \mathbb{H} \) is a geodesic whose lift in \( \mathbb{H} \) has end points lying in the set \( \Lambda_\infty \Gamma \) inside the boundary of \( \mathbb{H} \). The space of all non-wandering orbits of the geodesic flow on \( \Gamma \backslash \mathbb{H} \) can be parametrized by a compact subspace of \( U(\Gamma \backslash \mathbb{H}) \) and is denoted by \( U_{\text{rec}}(\Gamma \backslash \mathbb{H}) \). We denote the preimage of \( U_{\text{rec}}(\Gamma \backslash \mathbb{H}) \) in \( U \mathbb{H} \) by \( U_{\text{rec}}(\Gamma \backslash \mathbb{H}) \). The subspace \( U_{\text{rec}}(\Gamma \backslash \mathbb{H}) \) can be described as follows

\[
U_{\text{rec}}(\Gamma \backslash \mathbb{H}) = \left\{(x, v) \in U \mathbb{H} \mid \lim_{t \to \pm \infty} \tilde{\phi}_t x = \Lambda_\infty \Gamma \right\}
\]

where \( \tilde{\phi}_t(x, v) = (\tilde{\phi}_t^1 x, \tilde{\phi}_t^2 v) \).

2.3. Metric Anosov Property. The definitions in this section, which can also be found in section 6.1 of [11], has been included here for the sake of completeness.

Let \((X, d)\) be a metric space. A lamination \( \mathcal{L} \) of \( X \) is an equivalence relation on \( X \) such that for all \( x \) in \( X \) there exist an open neighborhood \( U_x \) of \( x \) in \( X \), two topological spaces \( U_1 \) and \( U_2 \) and a homeomorphism \( f_x \) from \( U_1 \times U_2 \) onto \( U_x \) satisfying the following properties,

1. for all \( w, z \) in \( U_x \cap U_y \) we have \( p_2 (f_x^{-1}(w)) = p_2 (f_y^{-1}(z)) \) if and only if \( p_2 (f_y^{-1}(w)) = p_2 (f_x^{-1}(z)) \) where \( p_2 \) is the projection from \( U_1 \times U_2 \) onto \( U_2 \).

2. for all \( w, z \) in \( X \) we have \( w \mathcal{L} z \) if and only if there exists a finite sequence of points \( w_1, w_2, ..., w_n \) in \( X \) with \( w_1 = w \) and \( w_n = z \), such that \( w_{i+1} \) is in \( U_{w_i} \), where \( U_{w_i} \) is a neighborhood of \( w_i \) and \( p_2 (f_{w_i}^{-1}(w_i)) = p_2 (f_{w_i}^{-1}(w_{i+1})) \) for all \( i \) in \( \{1, 2, ..., n-1\} \).
The homeomorphism $f_x$ is called a chart and the equivalence classes are called the leaves.

A plaque open set in the chart corresponding to $f_x$ is a set of the form $f_x(V_1 \times \{x_2\})$ where $x = f_x(x_1, x_2)$ and $V_1$ is an open set in $U_1$. The plaque topology on $L_x$ is the topology generated by the plaque open sets. A plaque neighborhood of $x$ is a neighborhood for the plaque topology on $L_x$.

A local product structure on $X$ is a pair of two laminations $L_1, L_2$ satisfying the following property: for all $x$ in $X$ there exist two plaque neighborhoods $U_1, U_2$ of $x$, respectively in $L_1, L_2$ and a homeomorphism $f_x$ from $U_1 \times U_2$ onto a neighborhood $W_x$ of $x$, such that $f_x$ defines a chart for both the laminations $L_1$ and $L_2$.

Now, let us assume that $\psi_t$ be a flow on $X$. A lamination $L$ invariant under the flow $\psi_t$ is called transverse to the flow, if for all $x$ in $X$ there exists a plaque neighborhood $U_x$ of $x$ in $L_x$, a topological space $V$, a positive $\epsilon$ and a homeomorphism $f_x$ from $U_x \times V \times (\epsilon, \epsilon)$ onto an open neighborhood $W_x$ of $x$ in $X$ satisfying the following condition:

\[ d(\psi_t y, \psi_t z) < \frac{1}{2} d(y, z) \]

for all $t$ bigger than $T_0$.

Remark 2.3.1. We note that a lamination ‘contracts under a flow’ if and only if the lamination contracts exponentially under the flow.

Definition 2.3.2. A flow $\psi_t$ on a compact metric space is called Metric Anosov, if there exist two laminations $L^+, L^-$ of $X$ such that the following conditions hold:

1. $(L^+, L^{-, 0})$ defines a local product structure on $X$,
2. $(L^-, L^{+, 0})$ defines a local product structure on $X$,
3. the leaves of $L^+$ are contracted by the flow,
4. the leaves of $L^-$ are contracted by the inverse flow.

In such a case we call $L^+, L^-, L^{+, 0}$ and $L^{-, 0}$ respectively the stable, unstable, central stable and central unstable laminations.
2.4. Margulis Space Times and Surfaces. Let $\mathbb{E}$ be the affine 3-space, $\mathbb{V}$ be the underlying vector space and $M$ be a Margulis Space Time. We know that $M$ can be identified with the space $\Gamma \backslash \mathbb{E}$ where $\Gamma$ is a free, discrete subgroup of $\text{Aff}(\mathbb{E})$. Using a result proved by Fried and Goldman in [3], one can think of $\Gamma$ as a subgroup of $SO^0(2, 1) \ltimes \mathbb{V}$ inside $GL(\mathbb{V}) \ltimes \mathbb{V}$. Hence, every Margulis Space Time come with a parallel Lorentz metric, denoted by $b$. Furthermore, the group $\Gamma$ is isomorphic to the fundamental group $\pi_1(M)$ of $M$. Now, we define a subbundle $T^{+1}M$ of the tangent bundle $TM$ over $M$ as follows:

\[(2.4.1) \quad T^{+1}M := \{ (x, v) \in TM \mid b_x(v, v) = 1 \} \]

Under our identification, we get a bundle isomorphism between $T^{+1}M$ and $\Gamma \backslash (\mathbb{E} \times S^{+1})$. If $\Sigma$ is the surface parametrizing the parallelism classes of time-like geodesics in $M$, then $\Sigma$ can be identified with $\Gamma \backslash H$.

The tangent bundle $TSO^0(2, 1)$ of $SO^0(2, 1)$ can be identified with $\mathbb{V} \ltimes SO^0(2, 1)$, the semi-direct product of $\mathbb{V}$ and $SO^0(2, 1)$, where the multiplication in $\mathbb{V} \ltimes SO^0(2, 1)$ is given by:

\[(2.4.2) \quad (v_1, g_1)(v_2, g_2) = (v_1 + g_1.v_2, g_1.g_2) \]

We note that the Killing form on the Lie algebra of $SO^0(2, 1)$ defines a Lorentz metric on $\mathbb{V}$. Now we are ready to define a few important sections of the bundle $\mathbb{V} \ltimes SO^0(2, 1)$ over $SO^0(2, 1)$ as follows,

\[(2.4.3) \quad \tilde{\nu} : SO^0(2, 1) \rightarrow \mathbb{V} \quad g \mapsto g.(1, 0, 0)^t, \]

\[(2.4.4) \quad \tilde{\nu}^\pm : SO^0(2, 1) \rightarrow \mathbb{V} \quad g \mapsto g.\left(0, \pm \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^t. \]

The section $\tilde{\nu}$ is called the neutral section. We list a few properties of the sections $\tilde{\nu}$ and $\tilde{\nu}^\pm$ below: for all real number $t$ and for all $g, h$ in $SO^0(2, 1)$ we have

\[(2.4.5) \quad 1. \quad \tilde{\nu}(\tilde{\phi}_t(g)) = \tilde{\nu}(g), \]

\[2. \quad \tilde{\nu}(h.g) = h.\tilde{\nu}(g), \]

\[3. \quad \tilde{\nu}^\pm(\tilde{\phi}_t(g)) = e^{\pm t}\tilde{\nu}^\pm(g), \]

\[4. \quad \tilde{\nu}^\pm(h.g) = h.\tilde{\nu}^\pm(g), \]

\[5. \quad \tilde{\nu}^+(g.u^+)(t) = \tilde{\nu}^+(g), \]

\[6. \quad \tilde{\nu}^-(g.u^-)(t) = \tilde{\nu}^-(g). \]

We also note that the image of $SO^0(2, 1)$ under the map $\tilde{\nu}$ is a subset of $S^{+1}$ and the image of $SO^0(2, 1)$ under $\tilde{\nu}^\pm$ is a subset of the light cone $S^0$.

We notice that the left action of $\mathbb{V} \ltimes SO^0(2, 1)$ on itself behave like an affine transformation on its first coordinate $\mathbb{V}$. We recall that $\mathbb{E}$ is the affine space whose underlying vector space is $\mathbb{V}$. Hence, the action of the discrete subgroup $\Gamma$ on $\mathbb{V} \ltimes SO^0(2, 1)$ by left multiplication, gives an affine action of
Γ on $E$. We also recall that $\widetilde{U}_{rec}\Sigma$ is a subspace of $U_H$ and using the map $\Theta$ we can think of $\widetilde{U}_{rec}\Sigma$ as a subspace of $SO^0(2,1)$. Recent works of Margulis, Goldman and Labourie in [8] have shown the existence of a $\Gamma$-equivariant continuous section, $\tilde{N}$, of $E$ over $\widetilde{U}_{rec}\Sigma$ and a $\Gamma$-invariant positive continuous function $\tilde{\nu}$ on $\widetilde{U}_{rec}\Sigma$ satisfying the following property,

$\tilde{N}(\phi_t(g)) = \tilde{N}(g) + \left(\int_0^t \tilde{f}(\phi_s(g))ds\right)\tilde{\nu}(g)$. \hfill (2.4.6)

The section $\tilde{N}$ is called the neutralised section. Furthermore, Goldman and Labourie have shown in [9] that there exists an injective map $\hat{N}$ such that the following diagram commutes,

\[
\begin{array}{ccc}
\widetilde{U}_{rec}\Sigma & \xrightarrow{(\tilde{N},\tilde{\nu})} & E \times S^+ \\
\pi \downarrow & & \downarrow \pi \\
U_{rec}\Sigma & \xrightarrow{\tilde{N}} & T_{rec}^1M
\end{array}
\] \hfill (2.4.7)

and moreover $\tilde{N}$ is an orbit equivalence sending the set of non-wandering orbits of $\Sigma$ onto the set of spacelike non-wandering orbits of $M$ i.e. $\tilde{N}(U_{rec}\Sigma) = T_{rec1}^1M$. We denote $(\tilde{N},\tilde{\nu})(U_{rec}\Sigma)$ by $(E \times S^+)^{rec}$.

3. Metric Anosov structure on Margulis Space Time

In this section, first we define a distance function $d$ on $T_{rec1}^1M$ such that $(T_{rec1}^1M, d)$ is a metric space. Later, we define two laminations $L^\pm$ on $(T_{rec1}^1M, d)$ which are invariant under the geodesic flow $\Phi_t$ of the affine structure on $T_{rec1}^1M$. Finally, we show that the lamination $L^+$ is a stable lamination and the lamination $L^-$ is an unstable lamination for the flow $\Phi_t$ on $(T_{rec1}^1M, d)$. We note that the method used in this paper to construct the distance function $d$ and to prove contraction properties of the lamination is inspired by [11].

3.1. Metric space structure. The restriction of any euclidean metric on $E \times V$ to the subspace $(E \times S^+)^{rec}$, defines a distance on $(E \times S^+)^{rec}$. We call this distance the euclidean distance on $(E \times S^+)^{rec}$. In this section we will define a distance on the space $(E \times S^+)^{rec}$ such that the distance is locally bilipschitz equivalent to any euclidean distance on $(E \times S^+)^{rec}$ and also is $\Gamma$-invariant, so as to get a distance on the quotient space $T_{rec1}^1M$.

We note that any two euclidean metric on $E \times V$ are bilipschitz equivalent with each other and hence any two euclidean distances on $(E \times S^+)^{rec}$ are also bilipschitz equivalent with each other. Fix an euclidean distance $d$ on $(E \times S^+)^{rec}$. The action of $\Gamma$ on the space $E \times V$ gives rise to a collection of distances related to $d$ defined as follows: for any $\gamma$ in $\Gamma$ define,

\[
d_\gamma : (E \times S^+)^{rec} \times (E \times S^+)^{rec} \rightarrow \mathbb{R} \\
(x, y) \mapsto d(\gamma^{-1}x, \gamma^{-1}y) \hfill (3.1.1)
\]
Since each element of \( \Gamma \) acts as a bilipschitz automorphism with respect to any euclidean distance, any two distances in the family \( \{d_{\gamma}\}_{\gamma \in \Gamma} \) are bilipschitz equivalent with each other.

Compactness of \( U_{rec} \Sigma \) implies that \( T_{rec}^1 M \) is compact and hence we can choose a pre-compact fundamental domain \( D \) of \( T_{rec}^1 M \) inside \( (E \times S^1)_{rec} \) with an open interior. We can also choose a suitable precompact open set \( U \) which contains the closure of \( D \). We note that properness of the action of \( \Gamma \) on \( (E \times S^1)_{rec} \) implies that the cover of \( (E \times S^1)_{rec} \) by the open sets \( \{\gamma.U\}_{\gamma \in \Gamma} \), is locally finite.

A path joining two points \( x \) and \( y \) in \( (E \times S^1)_{rec} \) is a pair of tuples,

\[
P = ((z_0, z_1, ..., z_n), (\gamma_1, \gamma_2, ..., \gamma_n))
\]

where \( z_i \in (E \times S^1)_{rec} \) and \( \gamma_i \in \Gamma \) such that the following two conditions hold,

1. \( x = z_0 \in \gamma_1.U \) and \( y = z_n \in \gamma_n.U \).
2. for all \( n > i > 0 \), \( z_i \in \gamma_i.U \cap \gamma_{i+1}.U \).

**Definition 3.1.1.** The length of a path is defined by,

\[
l(P) := \sum_{i=0}^{n-1} d_{\gamma_{i+1}}(z_i, z_{i+1})
\]

**Definition 3.1.2.** We then define,

\[
\tilde{d}(x, y) := \inf \{ l(P) \mid P \text{ joins } x \text{ and } y \}
\]

**Lemma 3.1.3.** \( \tilde{d} \) is a \( \Gamma \)-invariant pseudo-metric.

**Proof.** If \( P = ((z_0, z_1, ..., z_n), (\gamma_1, \gamma_2, ..., \gamma_n)) \) is a path joining \( \gamma x \) and \( \gamma y \), then the path,

\[
\gamma^{-1} P := ((\gamma^{-1} z_0, \gamma^{-1} z_1, ..., \gamma^{-1} z_n), (\gamma^{-1} \gamma_1, \gamma^{-1} \gamma_2, ..., \gamma^{-1} \gamma_n))
\]

is a path joining \( x \) and \( y \). Moreover,

\[
l(P) = \sum_{i=0}^{n-1} d_{\gamma_{i+1}}(z_i, z_{i+1}) = \sum_{i=0}^{n-1} d\left((\gamma^{-1} \gamma_i z_i, \gamma^{-1} \gamma_{i+1} z_{i+1})\right)
\]

\[
= \sum_{i=0}^{n-1} d\left((\gamma^{-1} \gamma_{i+1})^{-1} \gamma^{-1} z_i, (\gamma^{-1} \gamma_{i+1})^{-1} \gamma^{-1} z_{i+1}\right)
\]

\[
= \sum_{i=0}^{n-1} d_{\gamma^{-1} \gamma_{i+1}}(\gamma^{-1} z_i, \gamma^{-1} z_{i+1})
\]

\[
= l\left(\gamma^{-1} P\right).
\]

Hence, using the definition of \( \tilde{d} \) we get \( \tilde{d}(\gamma x, \gamma y) \) is equal to \( \tilde{d}(x, y) \).

We also notice that \( l(P) \) is a sum of distances. So \( l(P) \) is non-negative and hence \( \tilde{d} \) is non-negative.

It remains to show that \( \tilde{d} \) is a metric and \( \tilde{d} \) is locally bilipschitz equivalent to any euclidean distance. As all euclidean distances are bilipschitz equivalent with each other, it suffices to show that \( \tilde{d} \) is locally bilipschitz equivalent with \( d \).
Lemma 3.1.4. \( \tilde{d} \) is a metric and \( \tilde{d} \) is locally bilipschitz equivalent to \( d \).

Proof. Let \( z \) be a point in \((E \times \mathbb{S}^{+1})_{rec}\). There exists a neighbourhood \( V \) of \( z \) in \((E \times \mathbb{S}^{+1})_{rec}\) such that
\[
A := \{ \gamma \mid \gamma.U \cap V \neq \emptyset \}
\]
is a finite set. We fix \( V \) and choose a positive real number \( \alpha \) so that
\[
\bigcup_{\gamma \in A} \{ x \mid d_\gamma(z, x) \leq \alpha \} \subset V.
\]
We have seen that any two distances in the family \( \{d_\gamma\}_{\gamma \in \Gamma} \) are bilipschitz equivalent with each other. Hence \( A \) being a subset of \( \Gamma \), any two distances in \( A \) are bilipschitz equivalent with each other. Now finiteness of \( A \) implies that we can choose a constant \( K \) such that for all \( \beta_1, \beta_2 \) in \( A \) we have that
\[
d_{\beta_1} \text{ and } d_{\beta_2} \text{ are } K\text{-bilipschitz equivalent with each other.}
\]
We set,
\[
W := \bigcap_{\gamma \in A} \{ x \mid d_\gamma(z, x) \leq \frac{\alpha}{10K} \}.
\]
We note that \( W \) is a subset of \( V \) because \( K \) is bigger than 1.

By construction, if \( x, y \) is in \( W \) then for all \( \gamma \in A \) we have,
\[
(3.1.2) \quad d_\gamma(x, y) \leq d_\gamma(x, z) + d_\gamma(z, y) \leq \frac{\alpha}{5K}.
\]
Now let \( x \) be any point in \( W \), \( y \) be any general point and
\[
P = ((z_0, z_1, \ldots, z_n), (\gamma_1, \gamma_2, \ldots, \gamma_n))
\]
be a path joining \( x \) and \( y \).

We notice that \( x = z_0 \) is in \( \gamma_1U \). On the other hand \( x \) is also an element of \( W \), which is a subset of \( V \). Therefore,
\[
\gamma_1U \cap V \neq \emptyset
\]
Hence \( \gamma_1 \) is in \( A \). If there exists \( k \) such that \( \gamma_k \) is not in \( A \) then we choose \( j \) to be the smallest \( k \) such that \( \gamma_k \) is not in \( A \).
\[
(3.1.3) \quad l(P) = \sum_{i=0}^{n-1} d_{\gamma_{i+1}}(z_i, z_{i+1}) \geq \sum_{i=0}^{j-1} d_{\gamma_{i+1}}(z_i, z_{i+1}).
\]
Now using the fact that \( d_{\gamma_{j-1}} \) is K-bilipschitz equivalent with \( d_{\gamma_i} \) for any \( \gamma_i \) in \( A \) we get,
\[
(3.1.4) \quad \sum_{i=0}^{j-1} d_{\gamma_{i+1}}(z_i, z_{i+1}) \geq \frac{1}{K} \sum_{i=0}^{j-1} d_{\gamma_{j-1}}(z_i, z_{i+1}).
\]
Now from the triangle inequality it follows that
\[
(3.1.5) \quad \frac{1}{K} \sum_{i=0}^{j-1} d_{\gamma_{j-1}}(z_i, z_{i+1}) \geq \frac{1}{K} d_{\gamma_{j-1}}(z_0, z_j)
\]
\[
\geq \frac{1}{K} (d_{\gamma_{j-1}}(z, z_j) - d_{\gamma_{j-1}}(z, z_0)).
\]
The point $z_0 = x$, belongs to $W$ and $\gamma_{j-1}$ belongs to $A$. Therefore, by the definition of $W$ we get that
\begin{equation}
(3.1.6) \quad d_{\gamma_{j-1}}(z, z_0) \leq \frac{\alpha}{10K}.
\end{equation}
We also know that $\gamma_j$ is not in $A$. Hence $\gamma_jU$ does not intersect with $V$.
The point $z_j$ by definition belongs to $\gamma_jU$ and so $z_j$ is not in $V$. Therefore by the choice of $\alpha$ it follows that
\begin{equation}
(3.1.7) \quad d_{\gamma_{j-1}}(z, z_j) > \alpha.
\end{equation}
Using the inequalities $3.1.3$ and $3.1.6$ we get that
\begin{equation}
(3.1.8) \quad \frac{1}{K} (d_{\gamma_{j-1}}(z, z_j) - d_{\gamma_{j-1}}(z, z_0)) > \frac{1}{K} \left( \alpha - \frac{\alpha}{10K} \right).
\end{equation}
Now as $K$ is bigger than 1 we have,
\begin{equation}
(3.1.9) \quad \frac{1}{K} \left( \alpha - \frac{\alpha}{10K} \right) > \frac{1}{K} \left( \alpha - \frac{\alpha}{10} \right) > \frac{\alpha}{5K}.
\end{equation}
Finally, using the inequalities from $3.1.3$ to $3.1.9$ we get that if there exists $k$ such that $\gamma_k$ is not in $A$ then,
\begin{equation}
(3.1.10) \quad l(\mathcal{P}) > \frac{\alpha}{5K}.
\end{equation}
On the other hand, if for all $k$ we have $\gamma_k$ in $A$, then for all $\gamma \in A$ we have,
\begin{equation}
(3.1.11) \quad l(\mathcal{P}) = \sum_{i=0}^{n-1} d_{\gamma_{i+1}}(z_i, z_{i+1}) \geq \frac{1}{K} \sum_{i=0}^{n-1} d_{\gamma}(z_i, z_{i+1}).
\end{equation}
And using triangle inequality it follows that
\begin{equation}
(3.1.12) \quad \frac{1}{K} \sum_{i=0}^{n-1} d_{\gamma}(z_i, z_{i+1}) \geq \frac{1}{K} d_{\gamma}(x, y).
\end{equation}
Therefore, in the case when for all $k$, $\gamma_k$ is in $A$, we have for all $\gamma$ in $A$,
\begin{equation}
(3.1.13) \quad l(\mathcal{P}) \geq \frac{1}{K} d_{\gamma}(x, y).
\end{equation}
Combining the inequalities $3.1.10$ and $3.1.13$ and using the definition of $\tilde{d}$ we have that for any point $x$ in $W$, any general point $y$ and for all $\gamma$ in $A$,
\begin{equation}
(3.1.14) \quad \tilde{d}(x, y) \geq \frac{1}{K} \inf \left( \frac{\alpha}{5}, d_{\gamma}(x, y) \right).
\end{equation}
Therefore for any point $y$ distinct from $z$ we have,
\begin{equation}
(3.1.15) \quad \tilde{d}(z, y) > 0.
\end{equation}
The above is true for any arbitrary choice of $z$ and hence it follows that $\tilde{d}$ is a metric.

Moreover, if $x, y$ are points in $W$ and $\gamma$ is in $A$ then from the inequality $3.1.2$ we get,
\begin{equation}
(3.1.16) \quad d_{\gamma}(x, y) \leq \frac{\alpha}{5K} \leq \frac{\alpha}{5}.
\end{equation}
and hence for all \( x, y \) in \( W \) and \( \gamma \) in \( A \),
\[
\inf \left( \frac{\alpha}{5}, d_\gamma(x, y) \right) = d_\gamma(x, y).
\]
(3.1.16)

Therefore, from the inequalities 3.1.14 and 3.1.16 it follows that for \( x, y \) in \( W \) and for any \( \gamma \) in \( A \),
\[
\tilde{d}(x, y) \geq \frac{1}{K} d_\gamma(x, y).
\]
(3.1.17)

We know that there exists \( \gamma_a \) such that the point \( z \) is inside the open set \( \gamma_a U \). We note that the above defined \( \gamma_a \) is also an element of \( A \). Finally, we set \( W_a \) to be the intersection of of the set \( W \) with the set \( \gamma_a U \). Let \( x, y \) be any two points in \( W_a \). We choose the path \( \mathcal{P}_0 = ((x, y), (\gamma_a, \gamma_a)) \) and get that
\[
\tilde{d}(x, y) = \inf \{ l(\mathcal{P}) \mid \mathcal{P} \text{ joins } x \text{ and } y \} \leq l(\mathcal{P}_0) = d_{\gamma_a}(x, y).
\]
Hence, \( \tilde{d} \) is bilipschitz equivalent to \( d_{\gamma_a} \) on \( W_a \) and the distance \( d \) is bilipschitz equivalent to \( d_{\gamma_a} \). Therefore, \( d \) is bilipschitz to \( \tilde{d} \) on \( W_a \). Since \( z \) was arbitrarily chosen it follows that \( d \) is locally bilipschitz equivalent to \( \tilde{d} \).

\[\square\]

3.2. The lamination and its lift. In this section, we explicitly describe two laminations of \((E \times S^1)_{rec}\) for the geodesic flow on \((E \times S^1)_{rec}\) and show that the laminations are equivariant under the action of the geodesic flow and the action of \( \Gamma \). We also define the notion of a leaf lift.

Let \( Z \) be a point in \((E \times S^1)_{rec}\). We know from the commutative diagram 2.4.4 that for all \( Z \) in \((E \times S^1)_{rec}\) there exists an unique \( g \) in \( U_{rec} \Sigma \) such that \( Z = (\tilde{N}(g), \tilde{\nu}(g)) \).

Definition 3.2.1. The positive and central positive partition of \((E \times S^1)_{rec}\) are respectively given by,
\[
\mathcal{L}^+_{(\tilde{N}(g), \tilde{\nu}(g))} := \tilde{\mathcal{L}}^+_{(\tilde{N}(g), \tilde{\nu}(g))} \cap (E \times S^1)_{rec}
\]
\[
\mathcal{L}^{+0}_{(\tilde{N}(g), \tilde{\nu}(g))} := \tilde{\mathcal{L}}^{+0}_{(\tilde{N}(g), \tilde{\nu}(g))} \cap (E \times S^1)_{rec}
\]
where
\[
\tilde{\mathcal{L}}^+_{(\tilde{N}(g), \tilde{\nu}(g))} := \left\{ \left( (\tilde{N}(g) + s^+_1 \tilde{\nu}^+(g), \tilde{\nu}(g) + s^+_2 \tilde{\nu}^+(g)), t \tilde{\nu}(g), \tilde{\nu}(g) \right) \right\}
\]
\[
\tilde{\mathcal{L}}^{+0}_{(\tilde{N}(g), \tilde{\nu}(g))} := \left\{ \left( (\tilde{N}(g) + s^+_1 \tilde{\nu}^+(g), t \tilde{\nu}(g), \tilde{\nu}(g) \right) \right\}
\]
\[
\left. \right| s^+_1, s^+_2 \in \mathbb{R} \}
\]

Definition 3.2.2. The negative and central negative partition of \((E \times S^1)_{rec}\) are respectively given by,
\[
\mathcal{L}^-_{(\tilde{N}(g), \tilde{\nu}(g))} := \tilde{\mathcal{L}}^-_{(\tilde{N}(g), \tilde{\nu}(g))} \cap (E \times S^1)_{rec}
\]
\[
\mathcal{L}^{-0}_{(\tilde{N}(g), \tilde{\nu}(g))} := \tilde{\mathcal{L}}^{-0}_{(\tilde{N}(g), \tilde{\nu}(g))} \cap (E \times S^1)_{rec}
\]
where
\[
\tilde{\mathcal{L}}^-_{(\tilde{N}(g), \tilde{\nu}(g))} := \left\{ \left( (\tilde{N}(g) + s^-_1 \tilde{\nu}^+(g), \tilde{\nu}(g) + s^-_2 \tilde{\nu}^-(g)), s^-_1, s^-_2 \in \mathbb{R} \right) \right\}
\]
\[
\tilde{\mathcal{L}}^{-0}_{(\tilde{N}(g), \tilde{\nu}(g))} := \left\{ \left( (\tilde{N}(g) + s^-_1 \tilde{\nu}^-(g), t \tilde{\nu}(g), \tilde{\nu}(g) \right) \right\}
\]
\[
\left. \right| s^-_1, s^-_2 \in \mathbb{R} \}
\]
such that the above equation is equivalent to the following equation, the form $a^t. h = a(t_1)$. 

**Definition 3.2.3.** We denote the projection of $L^\pm, L^{\pm, 0}$ on the space $T_{\text{rec}}^1 M$ by $L^\pm, L^{\pm, 0}$ respectively.

**Lemma 3.2.4.** Let $g, h$ be two points in $U \mathbb{H}$ then

$h$ is in $\bigcup_{t \in \mathbb{R}} \hat{L}^+_{\phi g}$ if and only if $\tilde{\nu}(h) = \tilde{\nu}(g) + \frac{\langle \tilde{\nu}(h), \tilde{\nu}^-(g) \rangle}{\langle \tilde{\nu}^+(g), \tilde{\nu}^-(g) \rangle} \tilde{\nu}^+(g)$.

**Proof.** Let $h$ be a point of $\bigcup_{t \in \mathbb{R}} \hat{L}^+_{\phi g}$. Hence there exist real numbers $t_1, t_2$ such that $h = \tilde{\nu}(g a(t_1) u^+(t_2))$. Therefore, we have

$$
\begin{align*}
\tilde{\nu}(h) &= \tilde{\nu}(g a(t_1) u^+(t_2)) = ga(t_1) u^+(t_2) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = ga(t_1) \begin{pmatrix} 1 \\ 2t_2 \\ 2t_2 \end{pmatrix} \\
&= ga(t_1) \left( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2t_2 \\ 2t_2 \end{pmatrix} \right) = \tilde{\nu}(g) + 2t_2 ga(t_1) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\
&= \tilde{\nu}(g) + 2t_2 (\cosh t_1 + \sinh t_1) g \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\
&= \tilde{\nu}(g) + 2\sqrt{2} t_2 (\cosh t_1 + \sinh t_1) \tilde{\nu}^+(g).
\end{align*}
$$

Now we notice that

$$
\langle \tilde{\nu}(h), \tilde{\nu}^-(g) \rangle = \langle \tilde{\nu}(g) + 2\sqrt{2} t_2 (\cosh t_1 + \sinh t_1) \tilde{\nu}^+(g), \tilde{\nu}^-(g) \rangle \\
= 2\sqrt{2} t_2 (\cosh t_1 + \sinh t_1) \langle \tilde{\nu}^+(g), \tilde{\nu}^-(g) \rangle.
$$

Combining the above two calculations we get

\[
\tilde{\nu}(h) = \tilde{\nu}(g) + \frac{\langle \tilde{\nu}(h), \tilde{\nu}^-(g) \rangle}{\langle \tilde{\nu}^+(g), \tilde{\nu}^-(g) \rangle} \tilde{\nu}^+(g).
\]

Now, let $g, h$ be two points in $U \mathbb{H}$ satisfying,

$$
\tilde{\nu}(h) = \tilde{\nu}(g) + a_1 \tilde{\nu}^+(g)
$$

for some real number $a_1$. Using the definition of $\tilde{\nu}$ and $\tilde{\nu}^+$ we observe that the above equation is equivalent to the following equation,

\[
\left( g.u^+ \left( \frac{a_1}{2\sqrt{2}} \right) \right)^{-1} . h \left( \begin{array}{c} 0 \\ 1 \end{array} \right) = \left( u^+ \left( \frac{a_1}{2\sqrt{2}} \right) \right)^{-1} \left( \begin{array}{c} 1 \\ a_1 / \sqrt{2} \\ a_1 / \sqrt{2} \end{array} \right) = \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right).
\]

We know that the only elements of $SO^0(2, 1)$ fixing the vector $\left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right)$ are of the form $a(t)$ for some real number $t$. Hence there exist a real number $t_1$ such that

\[
\left( g.u^+ \left( \frac{a_1}{2\sqrt{2}} \right) \right)^{-1} . h = a(t_1).
\]
Therefore,
\[ h = g. a^+ \left( \frac{a_1}{2\sqrt{2}} \right) . a(t_1) = g. a(t_1). a^+ \left( \frac{a_1 \exp(-t_1)}{2\sqrt{2}} \right) \]
and the result follows.

**Corollary 3.2.5.** Let \( g, h \) be two points in \( U^\perp \) and \( h \) is in \( \bigcup_{t \in \mathbb{R}} \bar{H}_{\phi_{tg}}^+ \) then
\[
\frac{\langle \tilde{\nu}(g), \tilde{\nu}^-(h) \rangle}{\langle \tilde{\nu}^+(h), \tilde{\nu}^-(h) \rangle} \tilde{\nu}^+(h) = -\frac{\langle \tilde{\nu}(h), \tilde{\nu}^-(g) \rangle}{\langle \tilde{\nu}^+(g), \tilde{\nu}^-(g) \rangle} \tilde{\nu}^+(g).
\]

**Proof.** We know that if \( h \) is in \( \bigcup_{t \in \mathbb{R}} \bar{H}_{\phi_{tg}}^+ \) then \( g \) is in \( \bigcup_{t \in \mathbb{R}} \bar{H}_{\phi_{th}}^+ \). Therefore using lemma 3.2.4 we get
\[
\tilde{\nu}(h) = \tilde{\nu}(g) + \frac{\langle \tilde{\nu}(h), \tilde{\nu}^-(g) \rangle}{\langle \tilde{\nu}^+(g), \tilde{\nu}^-(g) \rangle} \tilde{\nu}^+(g)
\]
and
\[
\tilde{\nu}(g) = \tilde{\nu}(h) + \frac{\langle \tilde{\nu}(g), \tilde{\nu}^-(h) \rangle}{\langle \tilde{\nu}^+(h), \tilde{\nu}^-(h) \rangle} \tilde{\nu}^+(h).
\]
Hence
\[
\frac{\langle \tilde{\nu}(g), \tilde{\nu}^-(h) \rangle}{\langle \tilde{\nu}^+(h), \tilde{\nu}^-(h) \rangle} \tilde{\nu}^+(h) = -\frac{\langle \tilde{\nu}(h), \tilde{\nu}^-(g) \rangle}{\langle \tilde{\nu}^+(g), \tilde{\nu}^-(g) \rangle} \tilde{\nu}^+(g).
\]

**Definition 3.2.6.** For all \( g \) in \( \widetilde{U_{\text{rec}}} \Sigma \) we define,
\[
\bar{H}_g^+ := \bar{H}_g^+ \cap \widetilde{U_{\text{rec}}} \Sigma.
\]

**Proposition 3.2.7.** The following equations are true for all \( g \) in \( \widetilde{U_{\text{rec}}} \Sigma \),
\[
1. \quad \mathcal{L}_{(N(g),\tilde{\nu}(g))}^{+0} = \left\{ (\tilde{N}(h), \tilde{\nu}(h)) \mid h \in \bigcup_{t \in \mathbb{R}} \bar{H}_{\phi_{tg}}^+ \right\}
\]
\[
2. \quad \mathcal{L}_{(N(g),\tilde{\nu}(g))}^{-0} = \left\{ (\tilde{N}(h), \tilde{\nu}(h)) \mid h \in \bigcup_{t \in \mathbb{R}} \bar{H}_{\phi_{tg}}^- \right\}
\]

**Proof.** We start with defining a function,
\[
(3.2.1) \quad F : \widetilde{U_{\text{rec}}} \Sigma \times \widetilde{U_{\text{rec}}} \Sigma \rightarrow \mathbb{R}
\]
\[
(g, h) \mapsto \det[\tilde{N}(g) - \tilde{N}(h), \tilde{\nu}(g), \tilde{\nu}(h)].
\]
Using equations 2.2.2, 2.4.5 and 2.4.6 we get that
\[
(3.2.2) \quad F(\tilde{\phi}_{tg}, \tilde{\phi}_{th}) = F(g, h).
\]
As the neutralised section and the neutral section are equivariant under the action of \( \Gamma \), for all \( \gamma \) in \( \Gamma \) we have,
\[
(3.2.3) \quad F(\gamma g, \gamma h) = \det[\tilde{N}(\gamma g) - \tilde{N}(\gamma h), \tilde{\nu}(\gamma g), \tilde{\nu}(\gamma h)]
\]
\[
= \det[\gamma (\tilde{N}(g) - \tilde{N}(h)), \gamma \tilde{\nu}(g), \gamma \tilde{\nu}(h)]
\]
\[
= \det[\gamma \det[\tilde{N}(g) - \tilde{N}(h)), \tilde{\nu}(g), \tilde{\nu}(h)]
\]
\[
= \det[\tilde{N}(g) - \tilde{N}(h)), \tilde{\nu}(g), \tilde{\nu}(h)]
\]
Using lemma 3.2.4 we get that

\[ h = F(g, h). \]

Now for a fixed real number \( c_0 \) we consider the space,

\[ \mathcal{R} := \{(g_1, g_2) \mid d_{\text{HH}}(g_1, g_2) \leq c_0\} \subset \overline{U_{\text{rec}} \Sigma} \times \overline{U_{\text{rec}} \Sigma}. \]

Compactness of \( U_{\text{rec}} \Sigma \) implies that \( \mathcal{R}_\Gamma \), the projection of \( \mathcal{R} \) in \( \Gamma \backslash (\overline{U_{\text{rec}} \Sigma} \times \overline{U_{\text{rec}} \Sigma}) \), is compact. Now continuity of \( F \) implies that \( F \) is uniformly continuous on \( \mathcal{R}_\Gamma \).

Let \( g \) and \( h \) be two points in \( \overline{U_{\text{rec}} \Sigma} \) such that \( h \) is in \( \mathcal{H}_g^+ \). Given any such choice of \( g \) and \( h \) we can choose a sufficiently large \( t_0 \) such that \( d_{U\mathbb{E}}(\hat{\phi}_t g, \hat{\phi}_t h) \) is arbitrarily close to zero, hence we have \( F(\hat{\phi}_t g, \hat{\phi}_t h) \) arbitrarily close to zero. Therefore by using equation 3.2.2 it follows that \( F(g, h) \) is zero for all \( h \) in \( \mathcal{H}_g^+ \).

Now, using equations 3.2.2, 2.4.5 and lemma 3.2.4 we have,

\[ 0 = F(\hat{\phi}_t g, \hat{\phi}_t h) = \det[(\tilde{N}(\hat{\phi}_t g) - \tilde{N}(\hat{\phi}_t h)), \tilde{\nu}(\hat{\phi}_t g), \tilde{\nu}(\hat{\phi}_t h)] = \det[(\tilde{N}(\hat{\phi}_t g) - \tilde{N}(\hat{\phi}_t h)), \tilde{\nu}(g), \tilde{\nu}(h)] = \det[(\tilde{N}(\hat{\phi}_t g) - \tilde{N}(\hat{\phi}_t h)), \tilde{\nu}(g), \tilde{\nu}(g) + \frac{\tilde{\nu}(h)}{\tilde{\nu}(g)}, \tilde{\nu}(g)] = \langle \tilde{\nu}(g), \tilde{\nu}^-(g) \rangle \det[(\tilde{N}(\hat{\phi}_t g) - \tilde{N}(\hat{\phi}_t h)), \tilde{\nu}(g), \tilde{\nu}^+(g)]. \]

Therefore for all \( h \) in \( \mathcal{H}_g^+ \) and for all real number \( t \) we have

\[ \det[(\tilde{N}(\hat{\phi}_t g) - \tilde{N}(\hat{\phi}_t h)), \tilde{\nu}(g), \tilde{\nu}^+(g)] = 0. \]

Hence there exist real numbers \( a_1, b_1 \) such that

\[ (3.2.4) \quad \tilde{N}(\hat{\phi}_t h) = \tilde{N}(\hat{\phi}_t g) + a_1 \tilde{\nu}(g) + b_1 \tilde{\nu}^+(g) = \tilde{N}(g) + \left( a_1 + \int_0^t \hat{f}(\hat{\phi}_s(g)) ds \right) \tilde{\nu}(g) + b_1 \tilde{\nu}^+(g). \]

Combining lemma 3.2.4 and equation 3.2.4 we get that

\[ \mathcal{L}^{+,0}_{(\tilde{N}(g), \tilde{\nu}(g))} \supseteq \left\{ (\tilde{N}(h), \tilde{\nu}(h)) \mid h \in \bigcup_{t \in \mathbb{R}} \mathcal{H}_g^+ \right\} \]

Now, let \( W \) be in \( \mathcal{L}^{+,0}_{(\tilde{N}(g), \tilde{\nu}(g))} \). By the commutative diagram 2.4.7 we know that there exist \( h \) in \( U_{\text{rec}} \Sigma \) such that \( W = (\tilde{N}(h), \tilde{\nu}(h)) \). Now the choice of \( W \) implies that there exist some real number \( a_2 \) such that

\[ \tilde{\nu}(h) = \tilde{\nu}(g) + a_2 \tilde{\nu}^+(g). \]

Using lemma 3.2.4 we get that \( h \) belongs to \( \bigcup_{t \in \mathbb{R}} \mathcal{H}_g^+ \). Therefore \( h \) is in

\[ \bigcup_{t \in \mathbb{R}} \mathcal{H}_g^+ = \left( \overline{U_{\text{rec}} \Sigma} \cap \bigcup_{t \in \mathbb{R}} \mathcal{H}_g^+ \right). \]
and we have
\[ \mathcal{L}^+_{(\tilde{N}(g),\tilde{\nu}(g))} \subseteq \left\{ (\tilde{N}(h),\tilde{\nu}(h)) \mid h \in \bigcup_{t \in \mathbb{R}} \mathcal{H}^+_{\phi_t g} \right\}. \]

Similarly the other equality follows. \( \square \)

**Theorem 3.2.8.** The laminations \((\mathcal{L}^+,\mathcal{L}^-)\) and \((\mathcal{L}^-,\mathcal{L}^+)\) define a local product structure on \((\mathbb{E} \times S^{1})_{\text{rec}}\).

**Proof.** Let \( g \) be a point in \( U_{\text{rec}} \Sigma \). We denote the limit of the projection of \( \tilde{\phi}_t g \) onto \( \mathbb{H} \) as the real number \( t \) goes to infinity by \( g^+ \) and the limit of the projection of \( \tilde{\phi}_t g \) onto \( \mathbb{H} \) as the real number \( t \) goes to negative infinity by \( g^- \). We note that for \( g \) in \( U_{\text{rec}} \Sigma \) the points \( g^\pm \) lies in \( \Lambda_\infty \Gamma \).

We observe that \( \partial \mathbb{H} \setminus \{ g^+ \} \) is homeomorphic to \( \mathbb{R} \). Given any \( g \), let \( \mathcal{V}_g^- \) denote a connected bounded open neighborhood of \( g^- \) in \( \partial \mathbb{H} \setminus \{ g^+ \} \) and \( \mathcal{V}_g^+ \) be a connected open neighborhood of \( g^+ \) in \( \partial \mathbb{H} \setminus \{ g^- \} \) such that \( \mathcal{V}_g^- \cap \mathcal{V}_g^+ \) is empty and \( \mathcal{V}_g^- \times \mathcal{V}_g^+ \) is a subset of \( \partial \mathbb{H} \times \partial \mathbb{H} \setminus \Delta \). We define \( U_{g^\pm} := \mathcal{V}_{g^\pm} \cap \Lambda_\infty \Gamma \). Let \( U_g \) be the open subset of \( U_{\text{rec}} \Sigma \) corresponding to the open set \( U_{g^-} \times U_{g^+} \times \mathbb{R} \). We consider the following map,

\[ \tilde{\mathcal{H}}_g : U_g \rightarrow \mathbb{E} \]

\[ h \mapsto \tilde{N}(h) - \left( \tilde{N}(h) - \tilde{N}(g), \frac{\tilde{\nu}^-(g) \otimes \tilde{\nu}^+(h)}{\langle \tilde{\nu}^-(g), \tilde{\nu}^+(h) \rangle} \right) \tilde{\nu}(h) \]

We notice that \( \tilde{\mathcal{H}}_g(\tilde{\phi}_t h) = \tilde{\mathcal{H}}_g(h) \) for all real number \( t \). Now we define the continuous map \( \Pi_g \) as follows:

\[(3.2.5) \quad \Pi_g : U_{g^-} \times U_{g^+} \times \mathbb{R} \rightarrow (\mathbb{E} \times S^{1})_{\text{rec}} \]

\[ (h^-, h^+, t) \mapsto \left( \tilde{\mathcal{H}}_g + t\tilde{\nu}, \tilde{\nu} \right) (h^-, h^+, t). \]

We also define another continuous map \( \Pi_{(\tilde{N}(g),\tilde{\nu}(g))} \) as follows:

\[(3.2.6) \quad \Pi_{(\tilde{N}(g),\tilde{\nu}(g))} : U_{(\tilde{N}(g),\tilde{\nu}(g))} \rightarrow (\Lambda_\infty \Gamma \times \Lambda_\infty \Gamma \setminus \Delta) \times \mathbb{R} \]

\[ (\tilde{N}(h), \tilde{\nu}(h)) \mapsto \left( h^-, h^+, \left( \tilde{N}(h) - \tilde{N}(g), \frac{\tilde{\nu}^-(g) \otimes \tilde{\nu}^+(h)}{\langle \tilde{\nu}^-(g), \tilde{\nu}^+(h) \rangle} \right) \right) \]

where \( U_{(\tilde{N}(g),\tilde{\nu}(g))} := (\tilde{N}, \tilde{\nu})(U_g) \).

We notice that \( \Pi_{(\tilde{N}(g),\tilde{\nu}(g))} \circ \Pi_g = \text{Id} \). Indeed, for \( h = (h^-, h^+, t) \) we have,

\[ \Pi_{(\tilde{N}(g),\tilde{\nu}(g))} \circ \Pi_g(h) = \Pi_{(\tilde{N}(g),\tilde{\nu}(g))} \left( \tilde{\mathcal{H}}_g(h) + t\tilde{\nu}(h), \tilde{\nu}(h) \right) \]

\[ = \left( h^-, h^+, \left( \tilde{\mathcal{H}}_g(h) + t\tilde{\nu}(h) - \tilde{N}(g), \frac{\tilde{\nu}^-(g) \otimes \tilde{\nu}^+(h)}{\langle \tilde{\nu}^-(g), \tilde{\nu}^+(h) \rangle} \right) \right) \]

\[ = \left( h^-, h^+, t + \left( \tilde{\mathcal{H}}_g(h) - \tilde{N}(g), \frac{\tilde{\nu}^-(g) \otimes \tilde{\nu}^+(h)}{\langle \tilde{\nu}^-(g), \tilde{\nu}^+(h) \rangle} \right) \right) \]
and
\[
\left\langle \tilde{N}(h), \frac{\tilde{v}^-(g) \boxplus \hat{v}^+(h)}{\tilde{v}^-(g), \tilde{v}^+(h)} \right\rangle = -\left\langle \tilde{v}(h), \frac{\tilde{v}^-(g) \boxplus \hat{v}^+(h)}{\tilde{v}^-(g), \tilde{v}^+(h)} \right\rangle \left\langle \tilde{N}(h) - \tilde{N}(g), \frac{\tilde{v}^-(g) \boxplus \hat{v}^+(h)}{\tilde{v}^-(g), \tilde{v}^+(h)} \right\rangle \\
+ \left\langle \tilde{N}(h) - \tilde{N}(g), \frac{\tilde{v}^-(g) \boxplus \hat{v}^+(h)}{\tilde{v}^-(g), \tilde{v}^+(h)} \right\rangle.
\]

= 0.

We also notice that \( \Pi_g \circ \Pi_\nu(N(h), \tilde{v}(h)) = \text{Id} \). Indeed, for \((\tilde{N}(h), \tilde{v}(h))\) in \(\mathcal{U}(N(g), \tilde{v}(g))\) we have,

\[
\Pi_g \circ \Pi_\nu(N(h), \tilde{v}(h)) = \Pi_g \left( h^-, \tilde{h}^+, \left\langle \tilde{N}(h) - \tilde{N}(g), \frac{\tilde{v}^-(g) \boxplus \hat{v}^+(h)}{\tilde{v}^-(g), \tilde{v}^+(h)} \right\rangle \right).
\]

= \left( \tilde{N}_g + t_0 \tilde{v}, \tilde{v} \right) \left( h^-, \tilde{h}^+, t_0 \right)

where

\[
t_0 = \left\langle \tilde{N}(h) - \tilde{N}(g), \frac{\tilde{v}^-(g) \boxplus \hat{v}^+(h)}{\tilde{v}^-(g), \tilde{v}^+(h)} \right\rangle.
\]

Now as both \( \tilde{N}_g \) and \( \tilde{v} \) are invariant under the geodesic flow. We have

\[
\tilde{N}_g \left( h^-, \tilde{h}^+, t_0 \right) = \tilde{N}_g(h)
\]

and

\[
\tilde{v} \left( h^-, \tilde{h}^+, t_0 \right) = \tilde{v}(h).
\]

Hence

\[
\left( \tilde{N}_g + t_0 \tilde{v}, \tilde{v} \right) \left( h^-, \tilde{h}^+, t_0 \right)
= \left( \tilde{N}_g(h) + \left\langle \tilde{N}(h) - \tilde{N}(g), \frac{\tilde{v}^-(g) \boxplus \hat{v}^+(h)}{\tilde{v}^-(g), \tilde{v}^+(h)} \right\rangle \tilde{v}(h) \right)
= (\tilde{N}(h), \tilde{v}(h)).
\]

Therefore \( \Pi_g \) is a local homeomorphism.

Now we show that \( L^+ \) defines a lamination. Let \( g_1, g_2 \) be two points in \( \mathcal{U}_{\text{rec}} \sum, h_1, h_2 \) be two points in the intersection \( \mathcal{U}_{g_1} \cap \mathcal{U}_{g_2} \) and \( p^{+,0} \) be the projection from \( \mathcal{U}_{g^-} \times \mathcal{U}_{g^+} \times \mathbb{R} \) onto \( \mathcal{U}_{g^+} \times \mathbb{R} \).

We notice that if

\[
p^{+,0} \left( \Pi_\nu(N(g_1), \tilde{v}(g_1)) \right) = p^{+,0} \left( \Pi_\nu(N(g_2), \tilde{v}(g_2)) \right)
\]

then \( h_1^+ = h_2^+ \) and

\[
\left\langle \tilde{N}(h_1) - \tilde{N}(g_1), \frac{\tilde{v}^-(g_1) \boxplus \hat{v}^+(h_1)}{\tilde{v}^-(g_1), \tilde{v}^+(h_1)} \right\rangle = \left\langle \tilde{N}(h_2) - \tilde{N}(g_1), \frac{\tilde{v}^-(g_1) \boxplus \hat{v}^+(h_2)}{\tilde{v}^-(g_1), \tilde{v}^+(h_2)} \right\rangle.
\]
Now using proposition \[3.2.7\] corollary \[3.2.5\] and \(h_1^+ = h_2^+\) we get that
\[
\tilde{N}(h_2) = \tilde{N}(h_1) + s\tilde{\nu}^+(h_1) + t\tilde{\nu}(h_1)
\]
and
\[
\tilde{\nu}^+(h_2) = c\tilde{\nu}^+(h_1)
\]
for some real numbers \(c, s, t\). Hence for \(i \in \{1, 2\}\)
\[
\frac{\tilde{\nu}^-(g_i) \boxtimes \tilde{\nu}^+(h_2)}{\langle \tilde{\nu}^-(g_i), \tilde{\nu}^+(h_2) \rangle} = \frac{\tilde{\nu}^-(g_i) \boxtimes \tilde{\nu}^+(h_1)}{\langle \tilde{\nu}^-(g_i), \tilde{\nu}^+(h_1) \rangle}.
\]
Finally we have
\[
\left\langle \tilde{N}(h_2) - \tilde{N}(h_1), \tilde{\nu}^-(g_1) \boxtimes \tilde{\nu}^+(h_1) \right\rangle = 0
\]
and which gives us that \(t = 0\). Therefore
\[
\left\langle \tilde{N}(h_2) - \tilde{N}(h_1), \frac{\tilde{\nu}^-(g_2) \boxtimes \tilde{\nu}^+(h_1)}{\langle \tilde{\nu}^-(g_2), \tilde{\nu}^+(h_1) \rangle} \right\rangle = s\tilde{\nu}^+(h_1), \tilde{\nu}^+(h_1) = 0.
\]
Hence
\[
\left\langle \tilde{N}(h_1) - \tilde{N}(g_2), \frac{\tilde{\nu}^-(g_2) \boxtimes \tilde{\nu}^+(h_1)}{\langle \tilde{\nu}^-(g_2), \tilde{\nu}^+(h_1) \rangle} \right\rangle = 0.
\]
and it follows that
\[
p^{+,0} \left( \Pi_{(\tilde{N}(g_2), \tilde{\nu}(g_2))} (\tilde{N}(h_1), \tilde{\nu}(h_1)) \right) = p^{+,0} \left( \Pi_{(\tilde{N}(g_2), \tilde{\nu}(g_2))} (\tilde{N}(h_2), \tilde{\nu}(h_2)) \right).
\]
similarly one can show that if
\[
p^{+,0} \left( \Pi_{(\tilde{N}(g_2), \tilde{\nu}(g_2))} (\tilde{N}(h_1), \tilde{\nu}(h_1)) \right) = p^{+,0} \left( \Pi_{(\tilde{N}(g_2), \tilde{\nu}(g_2))} (\tilde{N}(h_2), \tilde{\nu}(h_2)) \right)
\]
then
\[
p^{+,0} \left( \Pi_{(\tilde{N}(g_1), \tilde{\nu}(g_1))} (\tilde{N}(h_1), \tilde{\nu}(h_1)) \right) = p^{+,0} \left( \Pi_{(\tilde{N}(g_1), \tilde{\nu}(g_1))} (\tilde{N}(h_2), \tilde{\nu}(h_2)) \right).
\]

Let \((\tilde{N}(h_i), \tilde{\nu}(h_i))\) be a sequence of points for \(i \in \{1, 2, ..., n\}\) such that \((\tilde{N}(h_{i+1}), \tilde{\nu}(h_{i+1}))\) is in \(\mathcal{U}(\tilde{N}(h_i), \tilde{\nu}(h_i))\) and
\[
p^{+,0} \left( \Pi_{(\tilde{N}(h_i), \tilde{\nu}(h_i))} (\tilde{N}(h_i), \tilde{\nu}(h_i)) \right) = p^{+,0} \left( \Pi_{(\tilde{N}(h_i), \tilde{\nu}(h_i))} (\tilde{N}(h_{i+1}), \tilde{\nu}(h_{i+1})) \right)
\]
for all \(i \in \{1, 2, ..., n - 1\}\). Hence we have \(h_1^+ = h_{i+1}^+\) and
\[
0 = \left\langle \tilde{N}(h_i) - \tilde{N}(h_i), \frac{\tilde{\nu}^-(h_i) \boxtimes \tilde{\nu}^+(h_i)}{\langle \tilde{\nu}^-(h_i), \tilde{\nu}^+(h_i) \rangle} \right\rangle = \left\langle \tilde{N}(h_{i+1}) - \tilde{N}(h_i), \frac{\tilde{\nu}^-(h_i) \boxtimes \tilde{\nu}^+(h_{i+1})}{\langle \tilde{\nu}^-(h_i), \tilde{\nu}^+(h_{i+1}) \rangle} \right\rangle.
\]
Now using proposition \[3.2.7\] corollary \[3.2.5\] and \(h_i^+ = h_{i+1}^+\) we get that
\[
\tilde{N}(h_{i+1}) = \tilde{N}(h_i) + s_i\tilde{\nu}^+(h_i) + t_i\tilde{\nu}(h_i)
\]
and
\[
\tilde{\nu}^+(h_{i+1}) = c_i\tilde{\nu}^+(h_i).
\]
for some real numbers $c_{i}, s_{i}$ and $t_{i}$. Hence

$$\frac{\tilde{\nu}^{-}(h_{i}) \boxtimes \tilde{\nu}^{+}(h_{i})}{\langle \tilde{\nu}^{-}(h_{i}), \tilde{\nu}^{+}(h_{i}) \rangle} = \frac{\tilde{\nu}^{-}(h_{i}) \boxtimes \tilde{\nu}^{+}(h_{i+1})}{\langle \tilde{\nu}^{-}(h_{i}), \tilde{\nu}^{+}(h_{i+1}) \rangle}.$$ 

Now

$$\left\langle \tilde{N}(h_{i+1}) - \tilde{N}(h_{i}), \frac{\tilde{\nu}^{-}(h_{i}) \boxtimes \tilde{\nu}^{+}(h_{i+1})}{\langle \tilde{\nu}^{-}(h_{i}), \tilde{\nu}^{+}(h_{i+1}) \rangle} \right\rangle = 0$$

gives us that $t = 0$. Hence we have

$$\mathcal{L}^{+}_{(\tilde{N}(h_{i}), \tilde{\nu}(h_{i}))} = \mathcal{L}^{+}_{(\tilde{N}(h_{i+1}), \tilde{\nu}(h_{i+1}))}.$$ 

Therefore we conclude that

$$\mathcal{L}^{+}_{(\tilde{N}(h_{i}), \tilde{\nu}(h_{i}))} = \mathcal{L}^{+}_{(\tilde{N}(h_{i+1}), \tilde{\nu}(h_{i+1}))}.$$ 

Now let $h$ be a point in $\bigcup_{i \in \Sigma} \tilde{g}$ such that $(\tilde{N}(h), \tilde{\nu}(h))$ is in $\mathcal{L}^{+}_{(\tilde{N}(g), \tilde{\nu}(g))}$. Using proposition 3.2.7 we get that $h^{+} = g^{+}$. Let $\mathcal{V}_{g}^{-}$ be a connected bounded open neighborhood of $g^{-}$ in $\partial_{\infty} \mathbb{H} \setminus \{g^{+}\}$ containing the point $h^{-}$ and let $\mathcal{V}_{g}^{+}$ be a connected open neighborhood of $g^{+}$ in $\partial_{\infty} \mathbb{H} \setminus \{g^{-}\}$ such that the intersection $\mathcal{V}_{g}^{+} \cap \mathcal{V}_{g}^{-}$ is empty. We denote the sets $\mathcal{V}_{g} \cap \Lambda_{\infty} \Gamma$ respectively by $\mathcal{U}_{g}^{\pm}$, the open subset of $\bigcup_{i \in \Sigma} \tilde{g}$ corresponding to the open set $\mathcal{U}_{g}^{-} \times \mathcal{U}_{g}^{+} \times \mathbb{R}$ by $\mathcal{U}_{g}$ and the open set $\mathcal{U}(\tilde{N}, \tilde{\nu})$ around $(\tilde{N}(g), \tilde{\nu}(g))$ by $\mathcal{U}(\tilde{N}(g), \tilde{\nu}(g))$. Now we consider the chart $\left( \Pi(\tilde{N}(g), \tilde{\nu}(g)), \Pi(\tilde{N}(g), \tilde{\nu}(g)) \right)$ and notice that

$$p^{+0} \left( \Pi(\tilde{N}(g), \tilde{\nu}(g)) (\tilde{N}(g), \tilde{\nu}(g)) \right) = (g^{+}, 0).$$

Since $(\tilde{N}(h), \tilde{\nu}(h))$ is in $\mathcal{L}^{+}_{(\tilde{N}(g), \tilde{\nu}(g))}$, using the definition of $\mathcal{L}^{+}_{(\tilde{N}(g), \tilde{\nu}(g))}$ we get

$$\left\langle \tilde{N}(h) - \tilde{N}(g), \frac{\tilde{\nu}^{-}(g) \boxtimes \tilde{\nu}^{+}(g)}{\langle \tilde{\nu}^{-}(g), \tilde{\nu}^{+}(g) \rangle} \right\rangle = 0.$$ 

Now using corollary 3.2.5 and the fact that $h^{+} = g^{+}$ we obtain

$$\frac{\tilde{\nu}^{-}(g) \boxtimes \tilde{\nu}^{+}(g)}{\langle \tilde{\nu}^{-}(g), \tilde{\nu}^{+}(g) \rangle} = \frac{\tilde{\nu}^{-}(g) \boxtimes \tilde{\nu}^{+}(h)}{\langle \tilde{\nu}^{-}(g), \tilde{\nu}^{+}(h) \rangle}.$$ 

Hence

$$\left\langle \tilde{N}(h) - \tilde{N}(g), \frac{\tilde{\nu}^{-}(g) \boxtimes \tilde{\nu}^{+}(h)}{\langle \tilde{\nu}^{-}(g), \tilde{\nu}^{+}(h) \rangle} \right\rangle = 0$$

and we finally have

$$p^{+0} \left( \Pi(\tilde{N}(g), \tilde{\nu}(g)) (\tilde{N}(g), \tilde{\nu}(g)) \right) = p^{+0} \left( \Pi(\tilde{N}(g), \tilde{\nu}(g)) (\tilde{N}(h), \tilde{\nu}(h)) \right).$$ 

Therefore we conclude that $\mathcal{L}^{+}$ defines a laminated with plaque neighborhoods given by the image of the open sets $\mathcal{U}_{g}^{-}$ for $g^{-}$ in $\Lambda_{\infty} \Gamma \setminus \{g^{+}\}$.

Now we show that $\mathcal{L}^{-0}$ defines a laminated. Let $g_{1}, g_{2}$ be two points in $\bigcup_{i \in \Sigma} \tilde{g}$, $h_{1}, h_{2}$ be two points in the intersection $\mathcal{U}_{g_{1}} \cap \mathcal{U}_{g_{2}}$, and $p^{+0}$ be the projection from $\mathcal{U}_{g_{1}} \times \mathcal{U}_{g_{2}} \times \mathbb{R}$ onto $\mathcal{U}_{g_{1}} \times \mathcal{U}_{g_{2}} \times \mathbb{R}$. We see that

$$p^{-} \left( \Pi(\tilde{N}(g_{1}), \tilde{\nu}(g_{1})) (\tilde{N}(h_{1}), \tilde{\nu}(h_{1})) \right) = p^{-} \left( \Pi(\tilde{N}(g_{2}), \tilde{\nu}(g_{1})) (\tilde{N}(h_{2}), \tilde{\nu}(h_{2})) \right).$$
Now let $h$ be a point in $\mathring{U}_{\text{rec}}\Sigma$ such that $(\tilde{N}(h), \tilde{\nu}(h))$ is in $\mathcal{L}^{-,0}_{(\tilde{N}(h), \tilde{\nu}(h))}$. Using proposition 3.2.7 we get that $h^- = g^-$. Let $\Sigma_{g^+}$ be a connected bounded open neighborhood of $g^+$ in $\partial_{\infty}\mathbb{H} \setminus \{g^+\}$ containing the point $h^+$ and let $\Sigma_{g^-}$ be a connected open neighborhood of $g^-$ in $\partial_{\infty}\mathbb{H} \setminus \{g^-\}$ such that $\Sigma_{g^+} \cap \Sigma_{g^-}$ is empty. We denote the sets $\Sigma_{g^\pm} \cap \Lambda_{\infty}\Gamma$ respectively by $\Sigma_{g^\pm}$, the open subset of $\mathring{U}_{\text{rec}}\Sigma$ corresponding to the open set $\mathcal{U}_{g^-} \times \mathcal{U}_{g^+} \times \mathbb{R}$ by $\mathcal{U}_g$ and the open set $(\tilde{N}, \tilde{\nu})(\mathcal{U}_g)$ around $(\tilde{N}(g), \tilde{\nu}(g))$ by $\mathcal{U}_{(\tilde{N}(g), \tilde{\nu}(g))}$. Now we consider the chart $\left(\mathcal{U}_{(\tilde{N}(g), \tilde{\nu}(g))}, \mathcal{U}_{(\tilde{N}(g), \tilde{\nu}(g))}\right)$ and notice that

$$p^- \left(\Pi_{(\tilde{N}(g), \tilde{\nu}(g))}(\tilde{N}(g), \tilde{\nu}(g))\right) = g^- = \left(\Pi_{(\tilde{N}(g), \tilde{\nu}(g))}(\tilde{N}(h), \tilde{\nu}(h))\right).$$

Therefore we conclude that $\mathcal{L}^{-,0}$ defines a lamination with plaque neighborhoods given by the image of the open sets $\mathcal{U}_{g^+} \times \mathcal{U}_{g^-} \times \mathbb{R}$ for $g^+ \in \Lambda_{\infty}\Gamma \setminus \{g^+\}$. Hence $(\mathcal{L}^+, \mathcal{L}^{-,0})$ defines a local product structure. In a similar way one can show that $(\mathcal{L}^+, \mathcal{L}^{+,0})$ also defines a local product structure. \(\square\)

**Proposition 3.2.9.** The laminations are equivariant under the action of $\Gamma$.  

Proposition 3.2.10. The laminations are equivariant under the geodesic flow.

Proof. Let $Z$ be in $(\mathbb{E} \times S^1)_{rec}$ such that $Z = (\tilde{N}(g), \tilde{v}(g))$ for some $g$ in $U_{\text{rec}} \Sigma$ and $W$ be in $\mathcal{L}_Z^+$. Therefore, there exist real numbers $s_1$, $s_2$ such that $W = (\tilde{N}(g) + s_1 \tilde{v}^+(g), \tilde{v}(g) + s_2 \tilde{v}^+(g))$. Now for all $\gamma$ in $\Gamma$ we get,
\[
\gamma.Z = \gamma.(\tilde{N}(g), \tilde{v}(g)) = (\tilde{N}(\gamma.g), \tilde{v}(\gamma.g))
\]
and
\[
\gamma.W = \gamma.(\tilde{N}(g) + s_1 \tilde{v}^+(g), \tilde{v}(g) + s_2 \tilde{v}^+(g)) = (\tilde{N}(\gamma.g) + s_1 \gamma.\tilde{v}^+(g), \gamma.\tilde{v}(g) + s_2 \gamma.\tilde{v}^+(g)) = (\tilde{N}(\gamma.g) + s_1 \tilde{v}^+(\gamma.g), \tilde{v}(\gamma.g) + s_2 \tilde{v}^+(\gamma.g)).
\]
Therefore $\gamma.W$ lies in $\mathcal{L}_{\gamma.Z}^+$ and $(\mathbb{E} \times S^1)_{rec}$ is invariant under the action of $\Gamma$ implies that $\gamma.W$ lies in $\mathcal{L}_{\gamma.Z}^+$. Hence we get that for all $\gamma$ in $\Gamma$,
\[
\mathcal{L}_{\gamma.Z}^+ = \gamma.\mathcal{L}_Z^+.
\]
Similarly one can show that for all $\gamma$ in $\Gamma$,
\[
\mathcal{L}_{\gamma.Z}^- = \gamma.\mathcal{L}_Z^-.
\]
\[\square\]

Now, we define the notion of a leaf lift. The leaf lift is a map from the leaves of the lamination through a point, to the tangent space of $(\mathbb{E} \times S^1)$ at that point. We will use this leaf lift to compare distance between the metric $\tilde{d}$ and the norm on the tangent space on any point of the leaves. We
define the leaf lift as follows:

The positive leaf lift is the map,

\[
i^+_{\tilde{N}(g),\tilde{\nu}(g)} : \tilde{L}^+_{\tilde{N}(g),\tilde{\nu}(g)} \to T_{\tilde{N}(g),\tilde{\nu}(g)}(E \times S^+) = (\tilde{N}(g) + s_1\tilde{\nu}^+(g), \tilde{\nu}(g) + s_2\tilde{\nu}^+(g))
\]

and the negative leaf lift is the map,

\[
i^-_{\tilde{N}(g),\tilde{\nu}(g)} : \tilde{L}^-_{\tilde{N}(g),\tilde{\nu}(g)} \to T_{\tilde{N}(g),\tilde{\nu}(g)}(E \times S^+) = (\tilde{N}(g) + s_1\tilde{\nu}^-(g), \tilde{\nu}(g) + s_2\tilde{\nu}^-(g)).
\]

where we identify \( T_{\tilde{N}(g),\tilde{\nu}(g)}(E \times S^+) \) with \( T_{\tilde{N}(g),\tilde{\nu}(g)}(E) \times T_{\tilde{\nu}(g)}S^+ \).

3.3. Contraction Properties. In this subsection we will prove that the leaves denoted by \( \tilde{L}^+ \) contracts in the forward direction of the geodesic flow and the leaves denoted by \( \tilde{L}^- \) contracts in the backward direction of the geodesic flow. We will prove it only for the forward direction of the flow. The other case will follow similarly. We start with the following construction whose raison d’être would be apparent in proposition 3.3.2.

**Proposition 3.3.1.** There exists a \( \Gamma \)-invariant map from \( (E \times S^+)_{\text{rec}} \) into the space of euclidean metrics on \( V \times V \) sending \( Z \) to \( \| \cdot \|_Z \) such that for all positive integer \( n \), there exists a positive real number \( t_n \) such that if \( t \) is bigger than \( t_n \), \( Z \) is in \( (E \times S^+)_{\text{rec}} \) and \( W \) is in \( L^+_Z \) then,

\[
\| i^+_{\tilde{\Phi}_tZ,\tilde{\Phi}_tZ} - i^+_{\tilde{\Phi}_tZ,\tilde{\Phi}_tZ} \|_{\tilde{\Phi}_tZ} \leq \frac{1}{2^n} \| i^+_{\tilde{\Phi}_tZ} - i^+_{\tilde{\Phi}_tZ} \|_Z.
\]

**Proof.** Let \( \langle \cdot, \cdot \rangle_{\tilde{N}(g),\tilde{\nu}(g)} \) be a positive definite bilinear form on the tangent space \( T_{\tilde{N}(g),\tilde{\nu}(g)}(E \times V) \) satisfying the following properties,

1. \( \langle \tilde{\nu}^\alpha(g), 0 \rangle_{\tilde{N}(g),\tilde{\nu}(g)} = \langle \tilde{\nu}^\alpha(g), 0 \rangle_{\tilde{N}(g),\tilde{\nu}(g)} = \delta_{\alpha,\beta} \)

2. \( \langle \tilde{\nu}^\alpha(g), 0 \rangle_{\tilde{N}(g),\tilde{\nu}(g)} = \langle \tilde{\nu}^\alpha(g), 0 \rangle_{\tilde{N}(g),\tilde{\nu}(g)} = 0 \)

where \( \delta_{\alpha,\beta} \) is the dirac delta function with \( \alpha, \beta \) in \( \{+, +, -\} \). We define the map \( \| \cdot \| \) as follows,

\[
\| X \|_{\tilde{N}(g),\tilde{\nu}(g)} := \sqrt{\langle X, X \rangle_{\tilde{N}(g),\tilde{\nu}(g)}},
\]

where \( X \) is in \( T_{\tilde{N}(g),\tilde{\nu}(g)}(E \times V) \). Now, from the equations \( 2.4.5 \) and \( 2.4.6 \) we get that \( \| \cdot \| \) is \( \Gamma \)-invariant, that is,

\[
\| \gamma X \|_{\gamma\tilde{N}(g),\gamma\tilde{\nu}(g)} = \| X \|_{\tilde{N}(g),\tilde{\nu}(g)}.
\]

Let \( Z = (\tilde{N}(g),\tilde{\nu}(g)) \) be in \( (E \times S^+)_{\text{rec}} \) and \( W \) be in \( L^+_Z \). Therefore there exists real numbers \( s_1 \) and \( s_2 \) such that \( W = (\tilde{N}(g) + s_1\tilde{\nu}^+(g), \tilde{\nu}(g) + s_2\tilde{\nu}^+(g)). \)

Hence the norm is,

\[
\| i^+_Z(W) - i^+_Z(Z) \|_Z = \| (s_1\tilde{\nu}^+(g), s_2\tilde{\nu}^+(g)) \|_Z = \sqrt{s_1^2 + s_2^2}.
\]

We note that \( \tilde{\Phi}_tZ = (\tilde{N}(g) + t\tilde{\nu}(g),\tilde{\nu}(g)) \) and using equation \( 2.4.6 \) we get that there exists a positive real number \( t_1 \) such that \( (\tilde{N}(g) + t\tilde{\nu}(g)) \) is equal
to $\tilde{N}(\tilde{\phi}, g)$. Moreover $t$ and $t_1$ are related by the following formula,

$$t = \int_0^{t_1} \tilde{f}(\tilde{\phi}, g) ds.$$

Therefore we have,

$$\Vert i_{\tilde{\phi}_1}^+(\tilde{\Phi}_1 W) - i_{\tilde{\phi}_1}^+ (\tilde{\Phi}_1 Z) \Vert_{\Phi_1 Z} = \Vert \tilde{N}(\tilde{\phi}_1 g, \tilde{\phi}_1 g) \Vert_{\tilde{\Phi}_1 Z}$$

$$= \sqrt{(s_1 + ts_2 + s_2^2) \cdot e^{-t_1} (\tilde{\phi}_1(Z), 0)} \Vert \tilde{N}(\tilde{\phi}_1 g, \tilde{\phi}_1 g) \Vert_{\tilde{\Phi}_1 Z}$$

Hence the norm is,

$$\text{(3.3.2)}$$

We also know that $U = \Sigma$ is compact. Hence, $\tilde{f}$ is bounded on $U$. Therefore, there exists a constant $c_1$ such that

$$t = \int_0^{t_1} \tilde{f}(\tilde{\phi}, g) ds \leq \int_0^{t_1} c_1 ds = c_1 t_1.$$

We choose a constant $c$ bigger than $\max\{1, 2c_1\}$ and get that

$$\text{(3.3.3)}$$

Now, by combining equation (3.3.1) inequalities (3.3.2) and (3.3.3) we get that

$$\Vert i_{\tilde{\phi}_1}^+(\tilde{\Phi}_1 W) - i_{\tilde{\phi}_1}^+ (\tilde{\Phi}_1 Z) \Vert_{\tilde{\Phi}_1 Z} \leq \sqrt{2} \sqrt{s_1^2 + s_2^2} (1 + t) e^{-t_1}.$$

Hence, for all positive integer $n$, there exists a positive real number $t_n$ depending on $n$ such that if $t$ is bigger than $t_n$, $Z$ in $(E \times S^1)_{rec}$ and $W$ in $L_{\tilde{\Phi}_1}^+$ then,

$$\Vert i_{\tilde{\phi}_1}^+(\tilde{\Phi}_1 W) - i_{\tilde{\phi}_1}^+ (\tilde{\Phi}_1 Z) \Vert_{\tilde{\Phi}_1 Z} \leq \frac{1}{2^n} \Vert i_{\tilde{\phi}_1}^+(W) - i_{\tilde{\phi}_1}^+ (Z) \Vert_{\tilde{\Phi}_1 Z}.$$

Proposition 3.3.2. Let $d$ be a $\Gamma$-invariant distance on $(E \times S^1)_{rec}$ which is locally bilipschitz equivalent to an euclidean distance and let $\|\|$ be the $\Gamma$-invariant map from $(E \times S^1)_{rec}$ to the space of euclidean metrics on $V \times V$ as constructed in the proof of proposition 3.3.1. There exist positive constants $K$ and $\alpha$ such that for any $Z$ in $(E \times S^1)_{rec}$ and for any $W$ in $L_{\tilde{\Phi}_1}^+$, the following statements are true,

1. If $d(W, Z) \leq \alpha$, then $\Vert i_{\tilde{\phi}_1}^+(Z) - i_{\tilde{\phi}_1}^+(W) \Vert \leq K d(W, Z)$,

2. If $\Vert i_{\tilde{\phi}_1}^+(Z) - i_{\tilde{\phi}_1}^+(W) \Vert \leq \alpha$, then $d(W, Z) \leq K \Vert i_{\tilde{\phi}_1}^+(Z) - i_{\tilde{\phi}_1}^+(W) \Vert$.
Proof. Since $\Gamma$ acts cocompactly on $(E \times S^1)_{rec}$ and both $d$ and $\|\cdot\|$ are $\Gamma$-invariant, it suffices to prove the above assertion for $Z$ in a compact subset $D$ of $(E \times S^1)_{rec}$, where $D$ is the closure of a suitably chosen fundamental domain.

We can define an euclidean distance $d_Z$ on $(E \times S^1)_{rec}$, uniquely using the euclidean metric $\|\cdot\|_Z$ on $V \times V$, by taking the embedding of $(E \times S^1)_{rec}$ in $E \times V$. We notice that for any $Z$ in $(E \times S^1)_{rec}$ and for any $W$ in $L_Z^+$, $d_Z(W, Z)$ is equal to $\|i^+_Z(W) - i^+_Z(Z)\|_Z$. Now, any two euclidean distances are bilipschitz equivalent with each other and by our hypothesis, $d$ is locally bilipschitz equivalent to an euclidean distance. Therefore, in particular, $d$ is locally bilipschitz equivalent with $d_Z$ for $Z$ in $D$, that is, there exist constants $K_Z$ depending on $Z$, and open sets $U_Z$ around $Z$, such that the distance $d_Z$ and $d$ are $K_Z$ bilipschitz equivalent with each other on $U_Z$.

Let $C_{(X, Y)}$ for any $X$ and $Y$ in $D$, be a constant such that the distance $d_X$ and $d_Y$ are $C_{(X, Y)}$ bilipschitz equivalent with each other. It follows from the construction of the norm $\|\cdot\|$, as done in proposition 3.3.1 that we can choose the constants $C_{(X, Y)}$ in such a way that $C_{(X, Y)}$ vary continuously on $(X, Y)$. As $D$ is compact it follows that $C_{(X, Y)}$ is bounded above by some constant $C$. Hence, for all $X$ and $Y$ in $D$, $d_X$ and $d_Y$ are $C$ bilipschitz equivalent with each other.

Now, we consider the open cover of $D$ by the open sets $U_Z$. As $D$ is compact, there exist points $Z_1, Z_2, ..., Z_n$ in $D$ such that $U_{Z_1}, U_{Z_2}, ..., U_{Z_n}$ covers $D$. Let $\beta$ be the Lebesgue number of this cover for the distance $d$ and $K_0$ be the maximum of $K_{Z_1}, K_{Z_2}, ..., K_{Z_n}$. Therefore, for any $Z$ in $D$, the open ball of radius $\beta$ around $Z$ for the metric $d$, denoted by $B_d(Z, \beta)$, lies inside $U_{Z_j}$ for some $j$ in $\{1, 2, ..., n\}$. Hence, $d$ and $d_{Z_j}$ are $K_0$ bilipschitz equivalent with each other on $B_d(Z, \beta)$. As $d_Z$ and $d_{Z_j}$ are $C$ bilipschitz equivalent with each other, it follows that $d$ and $d_Z$ are $CK_0$ bilipschitz equivalent with each other on $B_d(Z, \beta)$. Moreover, we note that the constants $\beta, C, K_0$ and hence also $CK_0$, does not depend on $Z$. Therefore, $d$ and $d_Z$ are $CK_0$ bilipschitz equivalent with each other on $B_d(Z, \beta)$, for all $Z$ in $D$.

As any two distances $d_X$ and $d_Y$, for all $X, Y$ in $D$ are $C$ bilipschitz equivalent with each other. Without loss of generality we can choose a point $X$ in $D$ and consider the distance $d_X$. The note that the set $\{B_d(Z, \beta) : Z \in D\}$ is an open cover of $D$. Let $\beta_1$ be a Lebesgue number for this cover for the metric space $(D, d_X)$. Therefore, the open ball $B_{d_X}(Y_1, \beta_1)$ for any $Y_1$ in $D$, lies inside an open ball $B_d(Y_2, \beta)$ for some point $Y_2$ in $D$. Now, as $d$ and $d_Z$ are $CK_0$ bilipschitz equivalent with each other on the ball $B_d(Z, \beta)$ for all $Z$ in $D$, it follows that $d$ and $d_X$ are $CK_0$ bilipschitz equivalent with each other on the ball $B_{d_X}(Y_2, \beta_1)$. As $Y_2$ was chosen arbitrarily we have that $d$ and $d_X$ are $CK_0$ bilipschitz equivalent with each other on the ball $B_{d_X}(Y, \beta_1)$, for all $Y$ in $D$.

Now, we know that $d_X$ and $d_Z$ are $C$ bilipschitz equivalent with each other. Therefore we get that $d$ and $d_Z$ are $CK_0$ bilipschitz equivalent with each other on the ball $B_{d_Z}(Y, \frac{\beta_1}{C})$, for all $Y$ in $D$. In particular one has, $d$ and $d_Z$ are $CK_0$ bilipschitz equivalent with each other on the ball $B_{d_Z}(Z, \frac{\beta_1}{C})$.

Finally, set $\alpha$ to be $\min\{\frac{\beta_1}{C}, \beta\}$ and $K$ to be $CK_0$ to get that for any $Z$ in
ANOSOV STRUCTURE ON MARGULIS SPACE TIME

\((\mathbb{E} \times S^1)_{\text{rec}}\) and \(W\) in \(L^+_{\text{Z}}\) we have,

1. If \(d(W, Z) \leq \alpha\), then \(\|i^+_Z(Z) - i^+_Z(W)\|_Z \leq K d(W, Z)\),
2. If \(\|i^+_Z(Z) - i^+_Z(W)\|_Z \leq \alpha\), then \(d(W, Z) \leq K\|i^+_Z(Z) - i^+_Z(W)\|_Z\).

\[\square\]

**Theorem 3.3.3.** Let \(L^\pm\) be two laminations on \((\mathbb{E} \times S^1)_{\text{rec}}\) as defined in definitions 3.2.1 and 3.2.2 and let \(\tilde{d}\) be the \(\Gamma\) invariant metric, as defined in definition 3.1.2. Under these assumptions, for the metric \(\tilde{d}\) on \((\mathbb{E} \times S^1)_{\text{rec}}\) we have that

1. \(L^+\) is contracted in the forward direction of the geodesic flow, and,
2. \(L^-\) is contracted in the backward direction of the geodesic flow.

**Proof.** Let \(\|\cdot\|\) be the \(\Gamma\)-invariant map from \((\mathbb{E} \times S^1)_{\text{rec}}\) to the space of euclidean metrics on \(V \times V\) as constructed in the proof of proposition 3.3.1 and let \(K\) and \(\alpha\) be as in the proposition 3.3.2 for the distance \(\tilde{d}\). We choose a positive integer \(n\) such that

\[
\frac{K}{2^n} < 1, \quad \frac{K^2}{2^n} \leq \frac{1}{2}.
\]

Let \(t_n\) be the constant as in proposition 3.3.1 for our chosen \(n\). Also let \(Z\) be in \((\mathbb{E} \times S^1)_{\text{rec}}\) and \(W\) be in \(L^+_Z\), so that \(\tilde{d}(W, Z) \leq \alpha\). Then using proposition 3.3.2 we get,

\[
\|i^+_Z(W) - i^+_Z(Z)\|_Z \leq K \tilde{d}(W, Z).
\]

Furthermore, using proposition 3.3.1 we get that for all \(t\) bigger than \(t_n\),

\[
\|i^+_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t W) - i^+_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t Z)\|_{\tilde{\Phi}_t Z} \leq \frac{1}{2^n} \|i^+_Z(W) - i^+_Z(Z)\|_Z.
\]

It follows that

\[
\|i^\pm_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t W) - i^\pm_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t Z)\|_{\tilde{\Phi}_t Z} \leq \frac{K\alpha}{2^n} \leq \alpha.
\]

Hence, again using proposition 3.3.2 we have,

\[
\tilde{d}(\tilde{\Phi}_t W, \tilde{\Phi}_t Z) \leq K \|i^\pm_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t W) - i^\pm_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t Z)\|_{\tilde{\Phi}_t Z}.
\]

Combining the above inequalities, for all \(t\) bigger than \(t_n\) we get,

\[
\tilde{d}(\tilde{\Phi}_t W, \tilde{\Phi}_t Z) \leq \frac{K^2}{2^n} \tilde{d}(W, Z) \leq \frac{1}{2} \tilde{d}(W, Z).
\]

Hence, \(L^+\) is contracted in the forward direction of the geodesic flow. The proof of the contraction of \(L^-\) follows similarly.

\[\square\]
3.4. Anosov structure on the quotient. Let us now consider what happens in the quotient, that is, $T^+_\text{rec}M$. Let $Z$ be in $(\mathbb{E} \times S^+)_{\text{rec}}$ and $\epsilon$ be a positive real number, we define,

$$\mathcal{L}^+_{\epsilon}(Z) := \mathcal{L}^+_{\frac{Z}{Z}} \cap B_d(Z, \epsilon),$$

and

$$\mathcal{K}_\epsilon(Z) := \Pi_Z (\mathcal{L}^+_{\epsilon}(Z) \times \mathcal{L}^-_{\epsilon}(Z) \times (-\epsilon, \epsilon)) \subset (\mathbb{E} \times S^+)_{\text{rec}}$$

where $\Pi_Z$ is the local product structure at $Z$ defined by the stable and unstable leaves.

We know that there exists a positive real number $\epsilon_0$ such that for any non identity element $\gamma$ of $\Gamma$ and for $Z$ in $(\mathbb{E} \times S^+)_{\text{rec}}$ we have,

$$\gamma(\mathcal{K}_{\epsilon_0}(Z)) \cap \mathcal{K}_{\epsilon_0}(Z) = \emptyset.$$  

Proof of Theorem 1.0.1. Let us fix $\alpha$ as in proposition 3.3.2 and let $\epsilon_1$ be from the open interval $(0, \min \{\alpha, \frac{\alpha}{2}\})$. Now, let $z$ be any point of $T^+\text{rec}M$ and let $Z$ be a point in $(\mathbb{E} \times S^+)_{\text{rec}}$ in the preimage of $z$. Our choice of $\epsilon_1$ gives us that the inequality\footnote{Definition 3.3.4} holds for the geodesic flow on $(\mathbb{E} \times S^+)_{\text{rec}}$ for the points in the chart $\mathcal{K}_{\epsilon_1}(Z)$. Hence, the inequality\footnote{Definition 3.3.4} also holds for the geodesic flow on $T^+\text{rec}M$ for points in the chart which is in the projection of $\mathcal{K}_{\epsilon_1}(Z)$. Therefore, $\mathcal{L}^+$, the projection of $\mathcal{L}^+$ in $T^+\text{rec}M$, is contracted in the forward direction of the geodesic flow on $T^+\text{rec}M$. A similar proof holds for $\mathcal{L}^-$, the projection of $\mathcal{L}^-$ in $T^+\text{rec}M$, too. \qed

4. Anosov representations

In this section we define the notion of an Anosov representation and restate our main theorem using this definition. We start by defining the following three subgroups of the Lie group $G := SO(2,1) \ltimes \mathbb{R}^3$:

Definition 4.0.1. Let $a(t)$ be as in equation\footnote{Equation 2.2.4} and $u^\pm(t)$ be as in equation\footnote{Equation 2.2.6} for real number $t$. We denote the vectors $(1,0,0)^t$, $\left(0, \pm \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^t$ respectively by $v_0, v_0^+, v_0^-$ and define

$$P^+ := \{(a(t_1)u^+(t_2), t_3v_0 + t_4v_0^+) \mid t_1, t_2, t_3, t_4 \in \mathbb{R}\},$$

$$P^- := \{(a(t_1)u^-(t_2), t_3v_0 + t_4v_0^-) \mid t_1, t_2, t_3, t_4 \in \mathbb{R}\},$$

$$L := \{(a(t_1), t_2v_0) \mid t_1, t_2 \in \mathbb{R}\} = P^+ \cap P^-.$$

Consider the space

$$\tilde{\mathcal{F}}^{+0}_{O,v_0} := \{(O + t_0v_0 + t^+v_0^+, v_0 + s^+v_0^+) \mid t_0, t^+, s^+ \in \mathbb{R}\}$$

$$\tilde{\mathcal{F}}^{-0}_{O,v_0} := \{(O + t_0v_0 + t^-v_0^-, v_0 + s^-v_0^-) \mid t_0, t^-, s^- \in \mathbb{R}\}$$

$$G.\tilde{\mathcal{F}}^{\pm0}_{O,v_0} := \{(g, u)\tilde{\mathcal{F}}^{\pm0}_{O,v_0} \mid (g, u) \in G\}.$$

We consider the left action of $G$ on $G.\tilde{\mathcal{F}}^{\pm0}_{O,v_0}$ and observe that

$$P^+ \cdot \tilde{\mathcal{F}}^{\pm0}_{O,v_0} = \tilde{\mathcal{F}}^{\pm0}_{O,v_0}.$$
Hence $G_0 \tilde{F}^\pm_{\mathcal{O},v_0}$ can be identified with $G/P^\pm$ respectively. Now we define

$$\mathcal{J} := \{(F_1, F_2) \in G_0 \tilde{F}^+_{\mathcal{O},v_0} \times G_0 \tilde{F}^-_{\mathcal{O},v_0} \mid F_1 \text{ is transverse to } F_2$$

in $E \times S^1\}.$

As transversality is an open condition we get that $\mathcal{J}$ is open.

**Proposition 4.0.2.** If we consider the diagonal action of $G$ on the space $G_0 \tilde{F}^+_{\mathcal{O},v_0} \times G_0 \tilde{F}^-_{\mathcal{O},v_0}$ then

$$G_0 \tilde{F}^+_{\mathcal{O},v_0}, \tilde{F}^-_{\mathcal{O},v_0} = \mathcal{J}.$$  

**Proof.** We notice that $\tilde{F}^+_{\mathcal{O},v_0}$ is transversal to $\tilde{F}^-_{\mathcal{O},v_0}$ in $E \times S^1$ and diagonal action of $G$ respects transversality. Hence we have

$$G_0 \tilde{F}^+_{\mathcal{O},v_0}, \tilde{F}^-_{\mathcal{O},v_0} \subseteq \mathcal{J}.$$  

Now let $(F_1, F_2)$ be in $\mathcal{J}$. As $F_1$ is transversal to $F_2$ in $E \times S^1$, there exists a point $(X, v)$ in $F_1 \cap F_2$. Also there exist a $g$ in $SO^0(2, 1)$ such that $gv_0 = v$. We notice that

$$(g, X - O) \tilde{F}^+_{\mathcal{O},v_0} = F_1$$

$$(g, X - O) \tilde{F}^-_{\mathcal{O},v_0} = F_2.$$  

Hence we have

$$\mathcal{J} \subseteq G_0 \tilde{F}^+_{\mathcal{O},v_0}, \tilde{F}^-_{\mathcal{O},v_0}$$  

and our result follows. \hfill $\square$

We notice that the stabilizer of $(\tilde{F}^+_{\mathcal{O},v_0}, \tilde{F}^-_{\mathcal{O},v_0})$ for the diagonal action of $G$ on $G_0 \tilde{F}^+_{\mathcal{O},v_0} \times G_0 \tilde{F}^-_{\mathcal{O},v_0}$ is $L$. Hence $\mathcal{J}$ can be identified with $G/L$. Now we define a continuous $\Gamma$ equivariant map from $(E \times S^1)_{rec}$ into $G/L$ as follows:

$$(X, v) \mapsto \tilde{N}(X, v) \in G_0 \tilde{F}^+_{\mathcal{O},v_0} \times G_0 \tilde{F}^-_{\mathcal{O},v_0}.$$ 

$$G_0 \tilde{F}^+_{\mathcal{O},v_0}, \tilde{F}^-_{\mathcal{O},v_0} \subseteq \mathcal{J}.$$  

We note that the map $\tilde{N}$ is invariant under the geodesic flow $\tilde{\Phi}$ on $(E \times S^1)_{rec}$. Moreover, we observe that

$$(g, \tilde{N}(g) - O) \tilde{F}^+_{\mathcal{O},v_0} = \tilde{L}^+_{(\tilde{N}(g), \tilde{\nu}(g))}$$

$$(g, \tilde{N}(g) - O) \tilde{F}^-_{\mathcal{O},v_0} = \tilde{L}^-_{(\tilde{N}(g), \tilde{\nu}(g))}.$$  

Let $\pi^\pm$ denote the projection from $G/P^+ \times G/P^-$ onto $G/P^\pm$ respectively. We observe that the image of $(\tilde{N}(h_1), \tilde{\nu}(h_1))$ under $\pi^+ \circ \tilde{N}$ is $\tilde{L}^+_{(\tilde{N}(g), \tilde{\nu}(g))}$, where $(\tilde{N}(h_1), \tilde{\nu}(h_1))$ is an element of $L^+_{(\tilde{N}(g), \tilde{\nu}(g))}$ and similarly the image of $(\tilde{N}(h_2), \tilde{\nu}(h_2))$ under $\pi^- \circ \tilde{N}$ is $\tilde{L}^-_{(\tilde{N}(g), \tilde{\nu}(g))}$, where $(\tilde{N}(h_2), \tilde{\nu}(h_2))$ is an element of $L^-_{(\tilde{N}(g), \tilde{\nu}(g))}$.

The pair $G/P^\pm$ gives a continuous set of foliations on the space $\mathcal{N} := G/L$ whose tangential distributions $E^\pm$ satisfy

$$\text{TN} = E^+ \oplus E^-.$$
Definition 4.0.3. We say that a vector bundle $E$ over a compact topological space whose total space is equipped with a flow $\{\psi_t\}_{t \in \mathbb{R}}$ of bundle automorphisms is contracted by the flow if for any metric $\| \cdot \|$ on $E$, there exists positive constants $A$ and $c$ such that

$$\|\psi_t(v)\| \leq A e^{-ct}\|v\|$$

for any $v$ in $E$.

Definition 4.0.4. Let $\mathbb{N}^\pm := \mathbb{N}\mathbb{E}^\pm$ and let $\mathcal{F}$ denote the orbit foliation of $T^{++}_{rec}M$ under the geodesic flow. By the flow invariance, these bundles are equipped with a parallel transport along the orbit of $\tilde{\Phi}$. We say that $(T^{++}_{rec}M, \mathcal{F})$ admits a $(\mathbb{N}, G)$ Anosov structure if $\mathbb{N}^+$ gets contracted by the lift of the flow $\Phi_{-t}$ and $\mathbb{N}^-$ gets contracted by the lift of the flow $\Phi_1$.

Theorem 4.0.5. The space $(T^{++}_{rec}M, \mathcal{F})$ admits a $(\mathbb{N}, G)$ Anosov structure.

Proof. Let $H$ be a neighborhood around $(g, \tilde{N}(g) - O)$ in $G$ and $U$ be a neighborhood around $(\tilde{N}(g), \tilde{\nu}(g))$ in $\tilde{L}^-_{(\tilde{N}(g), \tilde{\nu}(g))}$. As $(g, \tilde{N}(g) - O)\tilde{F}^{+,0}_{\nu_0} = \tilde{L}^{+,0}_{(\tilde{N}(g), \tilde{\nu}(g))}$ is transverse to $\tilde{L}^-_{(\tilde{N}(g), \tilde{\nu}(g))}$ and $\tilde{L}^{+,0}_{(\tilde{N}(g), \tilde{\nu}(g))} \cap \tilde{L}^-_{(\tilde{N}(g), \tilde{\nu}(g))}$ is a singleton we can choose $H$ in such a way that for any $(g, u)$ in $H$ the intersection $(g, u)\tilde{F}^{+,0}_{\nu_0} \cap \tilde{L}^-_{(\tilde{N}(g), \tilde{\nu}(g))}$ is also a singleton. We denote the element of $(g, u)\tilde{F}^{+,0}_{\nu_0} \cap \tilde{L}^-_{(\tilde{N}(g), \tilde{\nu}(g))}$ by $\Xi_1((g, u)\tilde{F}^{+,0}_{\nu_0})$ and define

$$\Xi_1 : H\tilde{F}^{+,0}_{\nu_0} \rightarrow \tilde{L}^-_{(\tilde{N}(g), \tilde{\nu}(g))}$$

$$(g, u)\tilde{F}^{+,0}_{\nu_0} \mapsto \Xi_1((g, u)\tilde{F}^{+,0}_{\nu_0}).$$

We note that the map $\Xi_1$ is smooth. We define another smooth map as follows:

$$\Theta_1 : U \rightarrow G\tilde{F}^{+,0}_{\nu_0}$$

$$(X, v) \mapsto (g, X - O)\tilde{F}^{+,0}_{\nu_0}$$

where $g$ is such that $gv_0 = v$. We observe that $\Xi_1 \circ \Theta_1 = \text{Id}$ and $\Theta_1 \circ \Xi_1 = \text{Id}$. Hence $\Xi_1$ is a diffeomorphism. Similarly we can construct a local diffeomorphism $\Xi_2$ between $G\tilde{F}^{-,0}_{\nu_0}$ and $\tilde{L}^+_{(\tilde{N}(g), \tilde{\nu}(g))}$. Now using these diffeomorphisms we consider the pullback of the metric defined in the proof of proposition 4.0.1 and notice that $\mathbb{N}^+$ gets contracted by the lift of the flow $\Phi_{-t}$ and $\mathbb{N}^-$ gets contracted by the lift of the flow $\Phi_t$. Moreover, as $T^{++}_{rec}M$ is compact we have that the convergence is independent of the choice of the metric. \hfill \Box

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