The onset of zonal modes in two-dimensional Rayleigh–Bénard convection

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We study the stability of steady convection rolls in two-dimensional Rayleigh–Bénard convection with free-slip boundaries and horizontal periodicity over 12 orders of magnitude in the Prandtl number ($10^{-6} \leq Pr \leq 10^{6}$) and 6 orders of magnitude in the Rayleigh number ($8\pi^4 < Ra \leq 10^8$). The analysis is facilitated by partitioning our modal expansion into so-called even and odd modes. With aspect ratio $\Gamma = 2$, we observe that zonal modes (with horizontal wavenumber equal to zero) can emerge only once the steady convection roll state consisting of even modes only becomes unstable to odd perturbations. We determine the stability boundary in the $(Pr, Ra)$ plane and observe remarkably intricate features corresponding to qualitative changes in the solution, as well as three regions where the steady convection rolls lose and subsequently regain stability as the Rayleigh number is increased. We study the asymptotic limit $Pr \to 0$ and find that the steady convection rolls become unstable almost instantaneously, eventually leading to nonlinear relaxation oscillations and bursts, which we can explain with a weakly nonlinear analysis. In the complementary large-$Pr$ limit, we observe that the zonal modes at the instability switch off abruptly at a large, but finite, Prandtl number.

Key words: Bénard convection, bifurcation, low-dimensional models

1. Introduction
Rayleigh–Bénard convection typically begins with a steady cellular pattern (for example, rolls, squares or hexagons), and the stability of these cellular patterns has been studied for decades (Busse 1967, 1983; Busse & Bolton 1984; Bolton & Busse 1985; Rucklidge & Matthews 1996; Paul et al. 2012). In a two-dimensional periodic box, the cellular pattern takes the form of convection rolls which are invariant under reflections both in the vertical

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plane that separates them and in the horizontal mid-plane. The vertical mirror symmetry can be broken in a pitchfork bifurcation, generating a net zonal flow in which any motion in one direction at the top will be balanced by an equal and opposite motion at the bottom (Rucklidge & Matthews 1996). The physical mechanism behind this instability is well understood (Thompson 1970; Busse 1983; Howard & Krishnamurti 1986). Suppose a pair of initially symmetric rolls tilts over, say to the right. The rising plume will now transport rightward momentum to the top of the layer, while the descending plume will transport leftward momentum to the bottom of the layer. The resulting horizontal streaming motion causes a net zonal flow across the layer, which may be enough to sustain the original tilt of the rolls. The vertical mirror symmetry may also be broken in a Hopf bifurcation, leading to oscillations in which the direction of the zonal flow alternates (Landsberg & Knobloch 1991; Proctor & Weiss 1993; Rucklidge & Matthews 1996).

Large-scale zonal flow in buoyancy-driven convection has been found in the atmosphere of Jupiter (Heimpel, Aurnou & Wicht 2005; Kaspi et al. 2018; Kong, Zhang & Schubert 2018), the Earth’s oceans (Maximenko, Bang & Sasaki 2005; Nadiga 2006; Richards et al. 2006), nuclear fusion devices (Diamond et al. 2005; Fujisawa 2008) and laboratory experiments (Krishnamurti & Howard 1981; Read et al. 2015; Zhang & van Gils 2020). The shearing instability which generates this net zonal flow has been studied both in the full Boussinesq equations (Rucklidge & Matthews 1996; Paul et al. 2012) and in several modal truncations (Howard & Krishnamurti 1986; Hermiz, Guzdar & Finn 1995; Horton, Hu & Laval 1996; Rucklidge & Matthews 1996; Aoyagi, Yagi & Itoh 1997; Berning & Spatschek 2000). More recently, prominent zonal flow has been found to greatly suppress convective heat transfer (Goluskin et al. 2014) and to depend strongly on the geometry of the studied domain (Wang et al. 2020; Fuentes & Cumming 2021).

Winchester, Dallas & Howell (2021) showed how zonal flow can emerge at Prandtl number $Pr = 30$ in two-dimensional Rayleigh–Bénard convection through a sequence of bifurcations as Rayleigh number $Ra$ increases. First the system undergoes a Hopf bifurcation, resulting in an oscillating zonal flow, which then grows in amplitude until the system becomes attracted to one of the two symmetric metastable shearing states. At intermediate Rayleigh numbers, the system performs apparently random-in-time transitions between these states, resulting in abrupt zonal flow reversals. As $Ra$ increases further, the system ultimately converges to one of the shearing states, resulting in a persistent zonal flow.

In this article, we study in further detail the initial emergence of zonal modes, with horizontal wavenumber equal to zero. We analyse the stability of steady two-dimensional convective rolls over 12 orders of magnitude in the Prandtl number $(10^{-6} \leq Pr \leq 10^6)$ and 6 orders of magnitude in the Rayleigh number $(8\pi^4 < Ra \leq 10^8)$. We focus on the case of free-slip boundary conditions on the horizontal boundaries and periodic boundary conditions on the vertical boundaries in a rectangular domain of width-to-height aspect ratio $\Gamma = 2$.

The equations governing Rayleigh–Bénard convection and our modal decomposition are outlined in § 2. In § 3, we describe the numerical scheme used to calculate steady states and their stability. We determine the stability boundary in the $(Pr, Ra)$ plane and demonstrate how remarkably intricate features on the boundary correspond to qualitative changes in the solution. In § 4, the regime of small Prandtl number is examined in more detail, by using formal asymptotic analysis to construct a system of two amplitude equations in the limit $Pr \to 0$. The complementary limit of $Pr \to \infty$ is studied in § 5. Concluding remarks on the article’s findings appear in § 6.
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2. Problem formulation

We consider two-dimensional Rayleigh–Bénard convection governed by the dimensionless equations

$$\nabla^2 \psi_t + \{\psi, \nabla^2 \psi\} = RaPr \theta_x + Pr \nabla^4 \psi,$$  \(2.1a\)

$$\theta_t + \{\psi, \theta\} = \psi_x + \nabla^2 \theta,$$  \(2.1b\)

where $\psi(x, y, t)$ is the streamfunction and $\theta(x, y, t)$ is the field of temperature fluctuations from the heat-conducting temperature profile $T = 1 - y + \theta$. Here, subscripts denote partial derivatives, and $\{f, g\} = fxgy - gxfy$ is the usual Poisson bracket. Equations (2.1) have been non-dimensionalised using $d$, $d^2/\kappa$ and $\Delta T$ as the relevant scales for length, time and temperature, respectively, where $d$ is the height of the fluid layer, $\kappa$ is the thermal diffusivity and $\Delta T$ is the temperature difference imposed across the layer. The two dimensionless parameters in the system (2.1) are the Prandtl and Rayleigh numbers:

$$Pr = \frac{\nu}{\kappa}, \quad Ra = \frac{\alpha \Delta T gd^3}{\nu \kappa},$$  \((2.2a,b)\)

where $\nu$ is the kinematic viscosity, $\alpha$ is the thermal expansion coefficient and $g$ is the gravitational acceleration.

The dimensionless spatial domain of our problem is $(x, y) \in [0, \Gamma] \times [0, 1]$, where $\Gamma$ is the width-to-height aspect ratio. The aspect ratio in this study is fixed at $\Gamma = 2$. At the lower ($y = 0$) and upper ($y = 1$) boundaries the temperature satisfies isothermal conditions while the velocity field satisfies no-penetration and stress-free boundary conditions, i.e.

$$\theta|_{y=0,1} = 0, \quad \psi|_{y=0,1} = \psi_{yy}|_{y=0,1} = 0,$$  \((2.3a,b)\)

and periodic boundary conditions are imposed in the $x$ direction.

Given the above boundary conditions, it is convenient to decompose the streamfunction into basis functions with Fourier modes in the $x$ direction and sine modes in the $y$ direction:

$$\psi(x, y, t) = \sum_{k_x \in \mathbb{Z}} \sum_{k_y \in \mathbb{Z}_{>0}} \hat{\psi}_{k_x, k_y}(t) e^{2\pi i k_x x / \Gamma} \sin (\pi k_y y)$$

$$= \sum_{k_x, k_y} \hat{\psi}_{k_x, k_y}(t) \phi_{k_x, k_y}(x, y),$$  \(2.4\)

where $\hat{\psi}_{k_x, k_y}$ is the amplitude of the $(k_x, k_y)$ mode of $\psi$, and we have the constraint $\hat{\psi}_{k_x, k_y} = \hat{\psi}^*_{-k_x, k_y}$ (with the asterisk denoting complex conjugation). We decompose $\theta$ in the same way.

Rayleigh (1916) has shown that the static conduction state ($\psi = \theta = 0$) bifurcates supercritically to a steady pattern of counter-rotating convection rolls vertically spanning the layer ($k_y = 1$) when $Ra > Ra_c$ with

$$Ra_c = \min_{k_x} \left( \frac{\pi^4}{4 \Gamma^4} \frac{(4k_X^2 + \Gamma^2)^3}{k_X^2} \right).$$  \(2.5\)

The minimum is taken over integer wavenumbers $k_x$, and occurs at $k_x = 1$ provided that $\Gamma < 2^{4/3} \sqrt{1 + 2^{2/3}} \approx 4.05$. Consequently, for our set-up with $\Gamma = 2$, the modes $\hat{\psi}_{1,1}$ and $\hat{\theta}_{1,1}$ from (2.4) become excited through a supercritical pitchfork bifurcation at $Ra_c = 8\pi^4$. In the remainder of the article, we refer to the resulting steady pattern as the steady
convection roll state (SCRS). An instance of the SCRS with the parameters \((Pr, Ra) = (1, 10^5)\) is shown in Figure 1. A further decomposition which will aid our analysis is to partition the modes into odd modes indicated by \(O\) with \(k_x + k_y \in 2\mathbb{Z} + 1\), and even modes indicated by \(E\) with \(k_x + k_y \in 2\mathbb{Z}\) similarly to previous studies (Chandra & Verma 2011; Verma, Ambhire & Pandey 2015). Consistent with this notion of odd and even modes, we decompose \(\psi\) as

\[
\psi^O(x, y, t) = \frac{1}{2}[\psi(x, y, t) + \psi(x + \Gamma/2, 1 - y, t)],
\]

\[
\psi^E(x, y, t) = \frac{1}{2}[\psi(x, y, t) - \psi(x + \Gamma/2, 1 - y, t)],
\]

and similarly for \(\theta^O\) and \(\theta^E\). Thus, \(\psi^O\) and \(\theta^O\) consist only of odd modes as defined above, while \(\psi^E\) and \(\theta^E\) consist only of even modes. Applying the decomposition (2.6) to (2.1), we find that

\[
\nabla^2 \psi^O_t + \{\psi^E, \nabla^2 \psi^O\} + \{\psi^O, \nabla^2 \psi^E\} = RaPr\theta^O_x + Pr\nabla^4 \psi^O, \tag{2.7a}
\]

\[
\theta^O_t + \{\psi^E, \theta^O\} + \{\psi^O, \theta^E\} = \psi^O_x + \nabla^2 \theta^O \tag{2.7b}
\]

and

\[
\nabla^2 \psi^E_t + \{\psi^E, \nabla^2 \psi^E\} - RaPr\theta^E_x - Pr\nabla^4 \psi^E = -\{\psi^O, \nabla^2 \psi^O\}, \tag{2.8a}
\]

\[
\theta^E_t + \{\psi^E, \theta^E\} - \psi^E_x - \nabla^2 \theta^E = -\{\psi^O, \theta^O\}. \tag{2.8b}
\]

We observe that \((\psi^O, \theta^O)\) satisfy the linear partial differential equations (2.7), with coefficients that depend on \((\psi^E, \theta^E)\), while \((\psi^E, \theta^E)\) satisfy the autonomous nonlinear partial differential equations (2.8), with forcing terms that depend on \((\psi^O, \theta^O)\).

At the onset of steady convection, the excited modes \(\psi^\dagger_{1,1}\) and \(\theta^\dagger_{1,1}\) are even modes, and indeed the SCRS has the property that \(\psi^O \equiv \theta^O \equiv 0\), which remains true as we increase \(Ra\). In this case, the system of (2.7) and (2.8) reduces to

\[
\nabla^2 \psi^E_t + \{\psi^E, \nabla^2 \psi^E\} = RaPr\theta^E_x + Pr\nabla^4 \psi^E, \tag{2.9a}
\]

\[
\theta^E_t + \{\psi^E, \theta^E\} = \psi^E_x + \nabla^2 \theta^E, \tag{2.9b}
\]

where the even modes are decoupled from the odd modes. The SCRS has a reflection symmetry about the vertical line which separates the counter-rotating convection rolls. Without loss of generality, we choose this vertical line to be at \(x = 0\). Thus, the steady-state
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solution \( \psi^E, \theta^E \) of the system (2.9) is invariant under the symmetry

\[
\psi^E(x, y) \mapsto -\psi^E(-x, y), \quad \theta^E(x, y) \mapsto \theta^E(-x, y).
\] (2.10a,b)

Since the zonal modes \( \hat{\psi}_{0,k_y} \) are \( x \)-independent, (2.10a,b) implies that \( \hat{\psi}_{0,k_y}^E = 0 \). In other words, the SCRS has no zonal flow, and zonal modes can emerge only once the SCRS has become unstable.

3. Linear stability analysis

In this section, we present the linear stability analysis of the SCRS. For given values of \( Pr \) and \( Ra \), we first solve numerically the steady-state problem of (2.9), following the procedure described in Appendix A. We then exploit the partial decoupling of odd and even modes to consider odd and even perturbations separately. We analyse odd perturbations by setting

\[
\psi(x, y, t) = \psi^E(x, y) + \delta e^{\sigma ot} \phi^o(x, y),
\]

\[
\theta(x, y, t) = \theta^E(x, y) + \delta e^{\sigma ot} \phi^o(x, y),
\]

where \( \psi^E, \theta^E \) is the SCRS, \( \delta \ll 1 \) and \( \phi^o, \theta^o \) consist only of odd modes. By substituting (3.1) into (2.1) and linearising with respect to \( \delta \), we obtain an eigenvalue problem for the odd growth rates \( \sigma^o \) (see Appendix B for details). Then, we solve this eigenvalue problem numerically and determine that the SCRS is unstable with respect to odd perturbations if \( \max [\mathcal{R}(\sigma^o)] > 0 \). An analogous approach is used to determine the corresponding even growth rates \( \sigma^e \) and thus the stability of the SCRS with respect to even perturbations.

Figure 2 displays the stability of the SCRS in the \((Pr, Ra)\) plane, highlighting the stable (S) regime in white and the unstable (US) regime in blue. The pink region denotes \( Ra < Ra_c \) where the static conduction state is stable. Although the SCRS does become unstable to even perturbations within the region US, we always observe that \( \max [\mathcal{R}(\sigma^o)] > \max [\mathcal{R}(\sigma^e)] \) and the dominant unstable perturbation therefore consists of odd modes. Moreover, for \( Pr \lesssim 7 \times 10^5 \) the most unstable eigenfunction has the property that \( \hat{\psi}_{0,k_y}^o \) and \( i\hat{\theta}_{0,k_y}^o \) are purely real, so that \( \psi^o \) and \( \theta^o \) break the reflection symmetry (2.10a,b). Zonal modes \( \hat{\psi}_{0,k_y} \) with \( k_y \in 2\mathbb{Z} + 1 \) are present in the most unstable perturbation until we reach very large Prandtl numbers (see § 5). In particular, we find that the largest scale zonal mode \( \hat{\psi}_{0,1} \) is present in the most unstable eigenfunction at the stability boundary for \( Pr \lesssim 7 \times 10^5 \), and is the dominant mode for \( Pr \lesssim 4.79 \) and \( 8.58 \lesssim Pr \lesssim 40 \).

The points \( p_i \) annotated in figure 2 highlight turning points in the stability boundary. The behaviour of \( \sigma^o \) near each such point is plotted in figure 3. Figure 3(c) shows that, close to \( Pr = 0.2 \) and with increasing Rayleigh number, a real eigenvalue crosses the origin twice, and the SCRS first loses and then regains stability. The steady state unexpectedly regains stability as the Rayleigh number increases at point \( p_1 \) with \( Pr \approx 0.055 \). Similar behaviour has been observed by Paul et al. (2012), with \( Pr = 6.8 \) and \( \Gamma = 2\sqrt{2} \). At point \( p_2 \), the
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Figure 2. The $(Pr, Ra)$ plane highlighting when the SCRS is linearly stable (white, $S$) and unstable (blue, $US$). Only the static conduction state is stable in the red shaded region below $Ra = Ra_c$ (dashed line). When $Pr \ll 1$, the SCRS becomes unstable at $0 < Ra - Ra_c \ll 1$. The highlighted points in red are at $p_1 = (5.48 \times 10^{-2}, 1.16 \times 10^3)$, $p_2 = (1.81 \times 10^{-2}, 4.12 \times 10^3)$, $p_3 = (0.2175, 2.59 \times 10^3)$, $q_1 = (0.2, 3.32 \times 10^5)$, $q_2 = (4.97, 3.949 \times 10^3)$, $q_3 = (8.58, 1.27 \times 10^6)$, $p_4 = (6.16, 5.43 \times 10^4)$, $p_5 = (9.53, 3.24 \times 10^5)$ and $q_4 = (7 \times 10^5, 2.54 \times 10^4)$. Visualisations of the SCRS at all the points $p_i$ and $q_i$ are given as supplementary figures available at https://doi.org/10.1017/jfm.2022.185.

Figure 3. The largest odd growth rate $\sigma_o$, plotted with one dimensionless parameter varied and the other fixed. The plots show the behaviour close to each of the points $p_i$ highlighted in figure 2. In (a–c) $\sigma_o$ is real, while in (d,e) the motion of $\sigma_o$ in the complex plane is tracked, with the arrows indicating the direction of increasing $Ra$ or $Pr$.

opposite occurs with the SRCS first regaining and then losing stability as $Ra$ increases. At points $p_4$ and $p_5$ we instead find complex eigenvalues crossing the imaginary axis twice as $Ra$ increases with fixed $Pr$ or vice versa.

At each of the points labelled $q_i$, the stability boundary is not smooth, and there is a qualitative change in the eigenfunction to which the SCRS becomes unstable. Figure 4(a) shows how, with $Pr = 0.2$, the SCRS regains stability with a real eigenvalue crossing zero at $Ra \approx 3.32859 \times 10^5$. With further increase in $Ra$, the eigenvalue splits into a complex conjugate pair, which then recrosses the imaginary axis as the SCRS loses stability in a Hopf bifurcation. The cusp at $q_1$ occurs when these three events happen simultaneously. These observations have been confirmed with direct numerical simulation (DNS) of the system (2.7) and (2.8), using the pseudospectral scheme described by...
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Figure 4. The largest odd growth rate(s) $\sigma^o$, plotted with one dimensionless parameter varied and the other fixed. The plots show the behaviour close to each of the points $q_i$ highlighted in figure 2. The motion of $\sigma^o$ in the complex plane is tracked, with the arrows indicating the direction of increasing $Ra$ or $Pr$.

Figure 5. Time series of the $\hat{\psi}_{0,1}$ mode from the linearised DNS with the right-hand side of (2.8) set to zero. Here we fix $Pr = 0.2$ ($q_1$) and have $Ra = 3.32 \times 10^5$ (a), $Ra = 3.32859 \times 10^5$ (b), $Ra = 3.33 \times 10^5$ (c) and $Ra = 3.35 \times 10^5$ (d). The initial conditions are the SCRS with a small perturbation.

Winchester et al. (2021). Figure 5 shows numerical results for the largest scale odd mode $\hat{\psi}_{0,1}$ obtained with the right-hand side of (2.8) set to 0, so that we have persistent exponential growth or decay in the odd modes. We observe that, with increasing $Ra$, the odd perturbations regain stability, then become oscillatory and finally become unstable again.

Figure 4(c) shows that, with $Pr = 8.58$, the SCRS regains stability through a Hopf bifurcation at $Ra = 1.28 \times 10^6$ but loses stability shortly after at $Ra = 1.29 \times 10^6$ with much larger oscillation frequency. The corner $q_3$ occurs when these two pairs of eigenvalues cross the imaginary axis simultaneously. Again, this observation has been verified with DNS close to $q_3$. In figure 6 we fix $Pr = 8.57$ and see that the odd mode $\hat{\psi}_{0,1}$ grows at $Ra = 1.2 \times 10^6$, decays at $Ra = 1.27 \times 10^6$ and grows again at $Ra = 1.3 \times 10^6$, but now with a much greater oscillation frequency. As can been seen in figure 4(b), similar behaviour occurs at corner $q_2$. The final corner point $q_4$, which lies in the large-$Pr$ region, is considered in more detail in § 5. How the dominant eigenfunction changes at each of the points $q_i$ is shown in the supplementary material.

Figure 2 hints that the Rayleigh number at which the SCRS loses stability approaches $Ra_c$ as $Pr \to 0$. In figure 7 we plot

$$\delta Ra := Ra - Ra_c$$ (3.3)

versus $Pr$ and indeed see that the stability boundary follows a clear power law with $\delta Ra \propto Pr^2$. When this boundary is crossed, at small $Pr$ the SCRS becomes unstable through a pitchfork bifurcation as a new steady state containing both even and odd modes emerges.
Figure 6. Time series of the $\hat{\psi}_{0,1}$ mode close to $q_3$ from the linearised DNS with the right-hand side of (2.8) set to zero. We fix $Pr = 8.57$ and the initial conditions are the SCRS with a small perturbation. Longer time series clearly showing the exponential growth and decay are in the supplementary material.

Figure 7. The $(Pr, \delta Ra)$ plane for small $Pr$ highlighting when the SCRS is linearly stable (white, $S$) and unstable (blue, $US$). On the vertical axis we plot $\delta Ra = Ra - Ra_c$, where $Ra_c = 8\pi^4$. The red dashed line highlights the power law $\delta Ra \propto Pr^2$, where the prefactor has been calculated by fitting.

Figure 8. (a) Bifurcation diagram plotting $|\hat{\psi}_{1,1}|$ (blue) and $|\hat{\psi}_{0,1}|$ (red) against $\delta Ra$ with $Pr = 10^{-2}$. Corresponding time series of $|\hat{\psi}_{1,1}|$ and $|\hat{\psi}_{0,1}|$ when (b) $\delta Ra = 10.7$ and (c) $\delta Ra = 2.22 \times 10^3$.

This prediction has been verified with fully nonlinear DNS solving (2.1) using the pseudospectral scheme described in Winchester et al. (2021). In figure 8(a) we plot a bifurcation diagram showing the amplitudes of the even mode $\hat{\psi}_{1,1}$ and the odd mode $\hat{\psi}_{0,1}$ versus $\delta Ra$ with $Pr = 0.01$. The SCRS emerges at $\delta Ra = 0$ and becomes unstable at $\delta Ra = \delta Ra_c \approx 6.135$ as the new steady state emerges. In figure 8(b) we plot the time series of $|\hat{\psi}_{1,1}|$ and $|\hat{\psi}_{0,1}|$ at $Pr = 0.01$ and $\delta Ra = 10.7$. At this point the mixed steady state has become unstable and the mode $|\hat{\psi}_{1,1}|$ undergoes nonlinear relaxation oscillations whilst $|\hat{\psi}_{0,1}|$ exhibits bursts that quickly decay away. At $\delta Ra = 2.22 \times 10^3$, the relaxation oscillations and the bursts persist, but are separated by a much slower buildup phase, as shown in figure 8(c).
In summary, the dynamics at low Prandtl number with $\delta Ra > \delta Ra_c$ can be partitioned into three phases, as follows.

(i) In the first phase we are close to the static conduction state with $|\hat{\psi}_{1,1}|, |\hat{\psi}_{0,1}| \approx 0$. However, when $\delta Ra > 0$, the static conduction state is unstable, so that $\hat{\psi}_{1,1}$ and other even modes grow exponentially.

(ii) The even modes grow until we are in the SCRS. Since $\delta Ra > \delta Ra_c$, the SCRS is unstable to odd perturbations, so that $\hat{\psi}_{0,1}$ and other odd modes grow exponentially.

(iii) Once $\hat{\psi}_{1,1}$ and $\hat{\psi}_{0,1}$ are of comparable amplitude, the system quickly collapses back to the static conduction state and the cycle continues.

This dynamics differs markedly from the chaotic behaviour found close to $Ra_c$ in small-$Pr$ and zero-$Pr$ convection in three dimensions (Thual 1992; Pal et al. 2009). In the following section, we carry out a weakly nonlinear analysis to help us explain the observed power law (see figure 7) as well as the dynamics of the nonlinear relaxation oscillations and of the bursts (see figure 8).

4. Weakly nonlinear analysis at small Prandtl number

We perform a weakly nonlinear analysis with the Prandtl number being our small parameter, $\epsilon = Pr \ll 1$. The analysis is carried out for arbitrary aspect ratio $\Gamma < 2^{4/3} \sqrt{1 + 2^{2/3}}$, and the other parameters and variables are scaled as follows:

\[ Ra = Ra_c + \epsilon^2 = \pi^4 \left( \frac{4 + \Gamma^2}{4\Gamma^4} \right) + \epsilon^2, \quad (4.1a) \]

\[ t = \frac{\tau_o}{\epsilon} = \frac{\tau_e}{\epsilon^3}, \quad (4.1b) \]

\[ \psi^E = \epsilon \psi_1^E(\tau_E, x, y) + \epsilon^2 \psi_2^E(\tau_E, x, y) + O(\epsilon^3), \quad (4.1c) \]

\[ \theta^E = \epsilon \theta_1^E(\tau_E, x, y) + \epsilon^2 \theta^E(\tau_E, x, y) + O(\epsilon^3), \quad (4.1d) \]

\[ \psi^O = \epsilon^2 \psi_1^O(\tau_O, x, y) + \epsilon^3 \psi_2^O(\tau_O, x, y) + O(\epsilon^4), \quad (4.1e) \]

\[ \theta^O = \epsilon^2 \theta_1^O(\tau_O, x, y) + \epsilon^3 \theta_2^O(\tau_O, x, y) + O(\epsilon^4). \quad (4.1f) \]

We have introduced two time scales $\tau_o$ and $\tau_e$, due to the distinct time-scale separation between the odd and even modes, observed for example in figure 8(c). Equation (4.1a) is inspired by the power law $\delta Ra \propto Pr^2$. The scalings of the dependent variables are constructed such that weak nonlinearity and coupling between even and odd modes enter at the same order, as we will see below.

First we examine the evolution of the even modes. At $O(\epsilon)$ and $O(\epsilon^2)$ we find

\[ \psi_1^E = \frac{4 + \Gamma^2}{\Gamma} \pi A \sin \left( \frac{2\pi x}{\Gamma} \right) \sin \left( \frac{\pi y}{\Gamma} \right), \quad (4.2a) \]

\[ \theta_1^E = 2A \cos \left( \frac{2\pi x}{\Gamma} \right) \sin \left( \frac{\pi y}{\Gamma} \right), \quad (4.2b) \]

and

\[ \psi_2^E = 0, \quad (4.3a) \]

\[ \theta_2^E = -\pi A^2 \frac{4 + \Gamma^2}{2\Gamma^2} \sin \left( \frac{2\pi y}{\Gamma} \right), \quad (4.3b) \]

which describe the asymptotic behaviour of the SCRS as $\delta Ra \rightarrow 0$. 

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At $O(\epsilon^3)$, we have
\begin{equation}
\nabla^2 \psi^E_{1E} + \{\psi^O_1, \nabla^2 \psi^O_1\} = Ra_c \theta^E_{3x} + \nabla^4 \psi^E_{1x} + r \theta^E_{1x}, \tag{4.4a}
\end{equation}
\begin{equation}
\{\psi^E_1, \theta^E_2\} = \psi^E_{3x} + \nabla^2 \theta^E_3. \tag{4.4b}
\end{equation}
the solvability condition for which gives us
\begin{equation}
\frac{(4 + \Gamma^2)^2}{2 \Gamma^3} \frac{dA}{d\tau_E} - \frac{2 \pi}{\Gamma} A + \frac{(4 + \Gamma^2)^4}{4 \Gamma^7} \pi^3 A^3 = \frac{2i}{\Gamma} \langle \{\psi^O_1, \nabla^2 \psi^O_1\}, \phi_{1,1} \rangle. \tag{4.5}
\end{equation}
The right-hand side of (4.5) is the inner product of the nonlinear forcing term with the $(1, 1)$ basis function, and captures the net effect of the odd modes on the amplitude $A$ of the dominant even mode. If this term is set to zero, then (4.5) reduces to the standard Landau equation governing the amplitude $A$ of weakly nonlinear perturbations to the static conduction state (Fowler 1997).

To evaluate the right-hand side of (4.5), we now turn to the odd modes. At lowest order we obtain the problem
\begin{equation}
\nabla^2 \psi^O_{1_{\tau O}} + \{\psi^O_1, \nabla^2 \psi^O_1\} + \{\psi^E_1, \nabla^2 \psi^O_1\} = Ra_c \theta^O_{1x} + \nabla^4 \psi^O_1, \tag{4.6a}
\end{equation}
\begin{equation}
0 = \psi^O_{1x} + \nabla^2 \theta^O_{1}, \tag{4.6b}
\end{equation}
which can be reduced to
\begin{equation}
\nabla^2 \psi^O_{1_{\tau O}} + \frac{\pi (4 + \Gamma^2)}{\Gamma} A \left\{ \sin \left( \frac{2 \pi x}{\Gamma} \right) \sin(\pi y), \nabla^2 \psi^O_1 + \frac{\pi^2 (4 + \Gamma^2)}{\Gamma^2} \psi^O_1 \right\}
\end{equation}
\begin{equation}
= \nabla^4 \psi^O_1 - Ra_c \nabla^{-2} \psi^O_{1xx}. \tag{4.7}
\end{equation}
Using the Fourier decomposition (2.4), i.e.
\begin{equation}
\psi^O_1 = \sum_{k_x + k_y \in 2\mathbb{Z} + 1} \hat{\psi}_{k_x,k_y} \phi_{k_x,k_y}, \tag{4.8}
\end{equation}
we can express (4.7) as a linear dynamical system of the form
\begin{equation}
\frac{d\psi^O_{1_{\tau O}}}{d\tau_O} = (AM + D)\psi^O_{1}. \tag{4.9}
\end{equation}
Here $M$ and $D$ are known constant matrices, calculated as described in Appendix C, and $\psi^O_1 = (\hat{\psi}_{k_x,k_y})_{k_x + k_y \in 2\mathbb{Z} + 1}$ is the vector of odd modes.

We recall that $A$ evolves on the much slower time scale $\tau_E = \epsilon^2 \tau_O$. As justified in Appendix C, to leading order in $\epsilon$, it follows that $\psi^O_1$ is proportional to the most unstable eigenvector of the problem (4.9). We can therefore write
\begin{equation}
\psi^O_{1_{\tau O}; A} \sim B(\tau_O)v(A), \tag{4.10}
\end{equation}
where
\begin{equation}
(AM + D)v(A) = \sigma(A)v(A), \tag{4.11}
\end{equation}
with $\sigma(A)$ the eigenvalue of (4.11) with the largest real part, and $v$ normalised such that $\|v\| = 1$. At leading order in $\epsilon$, the net amplitude $B$ of the odd modes thus evolves
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![Graph showing \( \sigma(A) \) and \( \lambda(A) \) as functions of \( A \).

Figure 9. The functions \((a)\) \( \sigma(A) \) and \((b)\) \( \lambda(A) \) as they appear in \((4.16)\) with \( \Gamma = 2 \). Highlighted is the point \( A_* \) at which \( \sigma(A_*) = 0 \).

According to

\[
\frac{dB}{d\tau_O} = \sigma(A)B. \tag{4.12}
\]

With \( \psi_1^O \) decomposed as in \((4.10)\), we can express the right-hand side of \((4.5)\) in the form

\[
\frac{2i}{\Gamma} \{\{\psi_1^O, \nabla^2 \psi_1^O\}, \phi_{1,1}\} = -B^2\lambda(A), \tag{4.13}
\]

where

\[
\lambda(A) = -\frac{2i}{\Gamma} \{\{v(A), \nabla^2 v(A)\}, \phi_{1,1}\}. \tag{4.14}
\]

For a given value of \( \Gamma \), the functions \( \sigma(A) \) and \( \lambda(A) \) can both be computed once and for all from \((4.11)\) and \((4.14)\). Equations \((4.5)\) and \((4.12)\) then provide a closed two-dimensional autonomous system for the amplitudes \( A \) and \( B \) of the even and odd modes, respectively.

By using the scalings

\[
A = \left(\frac{4\Gamma^7}{(4 + \Gamma^2)^4\pi^7}\right)^{1/3} \hat{\lambda} = a\hat{A}, \quad B = a(4 + \Gamma^2)^2\pi^3\hat{B} = b\hat{B}, \quad r = \frac{\Gamma}{2\pi a}\hat{r}, \quad \tau_E = b\hat{\tau}_E, \quad \lambda = \frac{\hat{\lambda}}{b^2}, \quad \sigma = \frac{\hat{\sigma}}{b},
\]

we normalise the system \((4.5)\) and \((4.12)\) and are left with the two ordinary differential equations (after dropping hats)

\[
\dot{A} = rA - A^3 - B^2\lambda(A), \tag{4.16a}
\]

\[
\epsilon^2\dot{B} = \sigma(A)B, \tag{4.16b}
\]

where the time derivative is taken with respect to the slow time scale \( \tau_E \). In figure 9 we show the functions \( \sigma(A) \) and \( \lambda(A) \) (normalised according to \((4.15)\)) in the instance when \( \Gamma = 2 \). The highlighted point \( A_* \), where \( \sigma(A_*) = 0 \), takes the value \( A_* \approx 83.1 \).
10\((\text{Time series of Point})\) the predator–prey population itemised above in practice, we solve (4.16) numerically. Non-Point fol
Point plane (Goluskin 2003; Malkov, Diamond & Rosenbluth 2001; Garcia et al. 2003; Decristoforo, Theodorsen & Garcia 2020). In this analogy, the place of the predator population is taken by a quantity undergoing relaxation oscillations, and the place of the prey population is taken by a bursting quantity. In contrast, our model (4.16) has been derived systematically from the governing equations without the need for any \(ad hoc\) closure assumptions.

Since \(\sigma(A)\) is an even function and \(\lambda(A)\) is an odd function, we need only consider solutions of the phase plane problem (4.16) in the quadrant \(A, B \geq 0\). The critical points are the following.

(i) Point \((A, B) = (0, 0)\) corresponds to the pure conduction state, which loses stability through the primary pitchfork bifurcation as \(r\) increases through zero. This bifurcation excites the even modes in the system.

(ii) Point \((\sqrt{r}, 0)\) exists for \(r > 0\) and represents the SCRS, with no odd modes. This state loses stability through a secondary pitchfork bifurcation at \(r = r_c = A^2_\ast \approx 6.91 \times 10^3\) (when \(\Gamma = 2\)), at which point the SCRS becomes unstable to odd perturbations.

(iii) Point \((A_\ast, \sqrt{(rA_\ast - A^3_\ast)/\lambda(A_\ast)})\) exists for \(r > r_c\), when a mixed steady state including odd and even modes emerges. Finally, this steady state loses stability in a Hopf bifurcation at \(r = r_\ast = A^2_\ast + (2A^2_\ast\lambda(A_\ast))/\lambda(A_\ast) - A_\ast \lambda'(A_\ast))\). When \(\Gamma = 2\), we calculate \(r_\ast \approx 1.53 \times 10^4\).

We can use the value of \(r_c\) from (ii) to infer the prefactor in the power law found in § 4 for the stability boundary at small \(Pr\). Rescaling back to our original variables using (4.15), we find that \(r_c = \delta Ra_c/Pr^2 \approx 6.35 \times 10^4\) which indeed gives excellent agreement with the fit obtained in figure 7.

To see the bifurcations itemised above in practice, we solve (4.16) numerically. Figure 10(a) is a bifurcation diagram showing how the non-trivial steady states identified in (ii) and (iii) above emerge at \(r = 0\) and \(r = r_c = A^2_\ast\), respectively. The Hopf bifurcation occurs at \(r = r_\ast\), beyond which point the system undergoes the nonlinear relaxation oscillations and bursts displayed in figures 10(b) and 10(c) for \(r = 2 \times 10^4\) and \(r = 3 \times 10^4\), respectively.

There is great qualitative agreement between the behaviours of our vastly simplified two-dimensional system and of the full system, displayed in figures 10 and 8, respectively. Although the system (4.16) was formally derived in the asymptotic limit as \(Pr \rightarrow 0\), similar relaxation oscillations are found in many parts of the \((Pr, Ra)\) plane (Goluskin et al. 2014). The resemblance of these oscillations to the behaviour of simple predator–prey population models and the Lotka–Volterra equations has often been studied (Leboeuf, Charlton & Carreras 1993; Malkov, Diamond & Rosenbluth 2001; García et al. 2003; Decristoforo, Theodorsen & García 2020). In this analogy, the place of the predator population is taken by a quantity undergoing relaxation oscillations, and the place of the prey population is taken by a bursting quantity. In contrast, our model (4.16) has been derived systematically from the governing equations without the need for any \(ad hoc\) closure assumptions.
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Figure 11. The \((Pr, Ra)\) parameter space for large \(Pr\) highlighting when the SCRS is linearly stable (white, \(S\)) and unstable (blue, \(US\)). The point \(q_4\) is where the stability boundaries corresponding to the symmetry breaking (red, \(ESB\)) and period doubling (green, \(EPD\)) eigenfunctions cross. The insets show the odd eigenfunctions, \(\psi^O\), associated with \(ESB\) and \(EPD\) where red, green and blue indicate \(\psi^O > 0\), \(\psi^O \approx 0\) and \(\psi^O < 0\), respectively. The parameters are \((Ra, Pr) = (1.56 \times 10^7, 10^5)\) and \((Ra, Pr) = (2.54 \times 10^7, 7 \times 10^5)\).

5. Stability at large Prandtl number

Figure 11 displays the \((Pr, Ra)\) plane for \(10^5 < Pr < 10^6\), highlighting where the SCRS is stable (white, \(S\)) or unstable (blue, \(US\)). At \(q_4 = (7 \times 10^5, 2.54 \times 10^7)\) there is a corner in the stability boundary and two complex conjugate pairs of eigenvalues cross the imaginary axis simultaneously, as was discussed in §3.

The stability boundaries for the two most unstable eigenfunctions therefore cross at \(q_4\), as indicated by the dashed curves in the inset in figure 11. As noted in §3, the symmetry breaking eigenfunction (labelled \(ESB\)) that is excited as we cross the red dashed curve has the property that \(\hat{\psi}^O_{k_x, k_y}\) and \(i\hat{\theta}^O_{k_x, k_y}\) are purely real and therefore breaks the reflection symmetry between counter-rotating convection rolls in the SCRS. It follows that the horizontal velocity is an even function of \(x\) which allows for instantaneous zonal flow in the solution.

The period doubling eigenfunction (labelled \(EPD\)) corresponding to the green dashed curve has the complementary symmetry that \(\hat{\psi}^O_{k_x, k_y}\) and \(i\hat{\theta}^O_{k_x, k_y}\) are purely imaginary, and the zonal modes are therefore all zero: \(\hat{\psi}^O_{0, k_y} = 0\) for all \(k_y\). The horizontal velocity is now an odd function of \(x\) and, although the \(EPD\) eigenfunction introduces odd modes to the solution, it does not break the reflection symmetry (2.10a,b) in the SCRS, but instead causes a period doubling in \(x\). Consequently, zonal modes at the instability switch off as we pass the point \(q_4\). The qualitative change in the structure of the dominant eigenfunction is demonstrated by the insets in figure 11, and how these eigenfunctions evolve in time is shown in the supplementary material. To measure the proportion of energy held in zonal modes in the eigenfunction \(ESB\), we define

\[
R_{\text{zonal}} = \sum_{k_x} |K_{k_x,0} \hat{\psi}_{k_x,0}|^2 / \sum_{k_x,k_y} |K_{k_x,k_y} \hat{\psi}_{k_x,k_y}|^2 ,
\]

where \(K_{k_x,k_y} = \sqrt{(2\pi k_x/\Gamma)^2 + (\pi k_y)^2}\) is the non-dimensional wavenumber. As shown in figure 12, \(R_{\text{zonal}}\) decreases as we increase \(Pr\) along the stability boundary. Nevertheless,
Figure 12. The proportion of energy held in zonal modes, $R_{\text{zonal}}$ defined in (5.1), in the eigenfunction $E_{SB}$ at the stability threshold plotted versus $Pr$. The red dashed line highlights the point $q_4$ beyond which the eigenfunction $E_{PD}$, with no zonal modes, becomes the most unstable.

$E_{SB}$ still has a small but non-zero zonal component at the point $q_4$ where the eigenfunction $E_{PD}$ takes over and zonal modes disappear completely. The instance of $E_{SB}$ in figure 11 is at the large Prandtl number $Pr = 10^5$, and hence the zonal modes do not carry much energy. Supplementary movie 6 shows $E_{SB}$ with $Pr = 100$, in which case the presence of zonal modes is clear.

As we increase the Prandtl number past $q_4$, the green dashed line in figure 11 appears to approach a limiting value $Ra = 2.54 \times 10^7$, at which the SCRS is unstable for all $Pr$. To examine this statement, we consider the $Pr \to \infty$ limit in which the governing equations (2.1) reduce to

$$\nabla^4 \psi = -Ra \theta_x, \quad (5.2a)$$

$$\theta_t + \{\psi, \theta\} = \psi_x + \nabla^2 \theta. \quad (5.2b)$$

We analyse the linear stability of the SCRS for the reduced system (5.2) as in § 3, and we find that it becomes unstable at $Ra = 2.54 \times 10^7$, consistent with the results presented in figure 11. It is clear from (5.2a) that $\hat{\psi}_{0,k_y} = 0$ in the $Pr \to \infty$ limit, again in agreement with the period doubling eigenfunction identified above. We conclude that $Pr > 7 \times 10^5$ is a large enough Prandtl number such that the system exhibits the asymptotic behaviour of identically zero zonal modes, at least linearly at the SCRS instability.

We note that the disparity between the thermal and viscous time scales makes it very difficult to reproduce the behaviour close to $q_4$ using DNS. The linear stability analysis predicts that $E_{PD}$ oscillates with a frequency of order 10 with respect to the thermal time scale, and it is therefore necessary to integrate the underlying equations over at least $Pr/10 \approx 7 \times 10^4$ viscous time units to observe just one complete oscillation. At the required spatial resolution, it proved unfeasible to compute enough cycles to reliably observe the exponential growth or decay associated with the Hopf bifurcation.

6. Conclusions

This article concerns two-dimensional Rayleigh–Bénard convection with free-slip boundaries and horizontal periodicity. The horizontal periodicity permits so-called zonal flow, with horizontal wavenumber equal to zero. When present, the zonal flow can dominate the flow, significantly suppress convective heat transfer (Goluskin et al. 2014) and undergo random-in-time reversals (Winchester et al. 2021). The onset of zonal flow is therefore of great importance.
With aspect ratio $\Gamma = 2$ and increasing Rayleigh number, the first state to be excited as the pure conducting state loses stability consists of steady convection rolls, referred to in the text as the SCRS. The SCRS consists of only so-called even modes and becomes unstable to odd modes as the Rayleigh number is increased. We determine the stability boundary in the $(Pr, Ra)$ plane and identify corners and cusps in the boundary where qualitative changes occur to the most unstable eigenfunction. We observe three regions where the SCRS loses and subsequently regains stability as the Rayleigh number is increased. Such behaviour seems to defy our intuition that increasing the importance of buoyancy relative to viscous and thermal dissipation should make the system less stable, although we note that reorganisation with increasing $Ra$ has been observed experimentally by Fauve et al. (1984) in convective flow of mercury at small Prandtl number.

The excitation of odd modes is necessary, but not sufficient, for net zonal flow to exist. The possible qualitative behaviours in our system are delineated by the points $q_1$ and $q_4$ identified in the parameter space shown in figure 2. Below and to the left of point $q_1$, the SCRS loses stability through a pitchfork bifurcation, and the resulting steady shear flow then itself becomes unstable and undergoes nonlinear oscillations and bursts. In either case the system produces a net zonal flow. On the other hand, to the right of point $q_4$, the dominant unstable odd eigenfunction preserves the reflection symmetry in the SCRS, and the zonal flow is identically zero.

Between points $q_1$ and $q_4$, the initial instability occurs through a Hopf bifurcation leading to oscillations in which the direction of the shear flow alternates (Rucklidge & Matthews 1996). In this case, although the solution exhibits zonal flow instantaneously, the net (time-averaged) zonal flow is zero. This range includes the case $Pr = 30$ for which Winchester et al. (2021) showed that the initial Hopf bifurcation is the first stage of a process that ultimately results in either persistent or intermittent zonal flow in the turbulent regime. We find that zonal modes become less prominent in the most unstable odd eigenfunction as the Prandtl number increases, but are still present up to $Pr = 7 \times 10^5$, when they abruptly switch off at point $q_4$.

As $Pr \to 0$, the SCRS becomes unstable almost immediately as a new steady state, consisting of both even and odd modes, emerges in a pitchfork bifurcation. We observe a power law $\delta Ra \propto Pr^2$ along the stability boundary. With a weakly nonlinear analysis, we derive a two-dimensional approximate model that successfully predicts the power law in the stability boundary, including its prefactor, as well as the nonlinear relaxation oscillations and bursts observed in DNS at low $Pr$.

At large Prandtl numbers, we observe a corner in the stability boundary at $q_4 = (7 \times 10^5, 2.54 \times 10^7)$ where the zonal modes in the most unstable eigenfunction switch off. As the Prandtl number increases beyond $q_4$, the stability boundary converges to the asymptotic value $Ra = 2.54 \times 10^7$. This observation is consistent with formally taking the $Pr \to \infty$ limit, which completely removes zonal modes. Our conclusion is that no zonal modes are excited at the SCRS instability boundary for Prandtl numbers greater than $7 \times 10^5$. Instead, an unsteady state is produced consisting of both even and odd modes. It remains open to determine when this state in turn becomes susceptible to growing zonal perturbations.

We focus here on the case of aspect ratio $\Gamma = 2$. Further analysis of the weakly nonlinear model (4.16) suggests that the type and sequence of bifurcations that occur in the small-$Pr$ limit may be completely different for different values of $\Gamma$. At large $Pr$, both the asymptotic value of the Rayleigh number and the Prandtl number at which zonal modes switch off at onset can depend on the geometry and the boundary conditions.

For general Prandtl number, we can anticipate that the structure of the stability boundary and the properties of the excited solutions also depend significantly on the value of $\Gamma$. 

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In particular, if \( \Gamma \geq 2^{4/3} \sqrt{1 + 2^{2/3}} \) then both odd and even modes become excited as \( Ra \) increases past \( Ra_c \), and it is no longer clear that the odd/even mode decomposition which proves so convenient in our set-up is still able to give some insight.

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**Appendix A. Computation of the SCRS**

We wish to find a solution \((\psi^E, \theta^E)\) of the steady ‘even’ problem (2.9). We expand both \(\psi^E\) and \(\theta^E\) as in (2.4), i.e. in modes of the form

\[
\phi_{k_x, k_y}(x, y) = e^{2\pi i k_x x / \Gamma} \sin(\pi k_y y), \quad k_x + k_y \in 2\mathbb{Z},
\]

and define the inner product

\[
\langle \phi_{k_{x_1}, k_{y_1}}, \phi_{k_{x_2}, k_{y_2}} \rangle = \int_0^1 \int_0^\Gamma \phi_{k_{x_1}, k_{y_1}} \phi_{k_{x_2}, k_{y_2}}^* \, dx \, dy = \frac{\Gamma}{2} \delta_{k_{x_1}, k_{x_2}} \delta_{k_{y_1}, k_{y_2}}.
\]

Acting with this inner product on (2.9) we find, for each \((k_x, k_y)\),

\[
\begin{align*}
\frac{i}{2} \sum_{k_{x_1} + k_{x_1} = k_x} \frac{k_{y_1} = k_y}{|k_{y_1} \pm k_{y_2}| = k_y} & \hat{\psi}_{k_{x_1}, k_{y_1}} \hat{\psi}_{k_{x_2}, k_{y_2}} K_{k_{x_1}, k_{x_2}, k_{y_2}}^2 G(k_{x_1}, k_{x_2}, k_{y_1}, k_{y_2}) \\
& + i k_x Ra Pr \hat{\theta}_{k_x, k_y} + Pr K_{k_x, k_y}^4 \hat{\psi}_{k_x, k_y} = 0, \quad (A3a) \\
\frac{i}{2} \sum_{k_{x_1} + k_{x_1} = k_x} \frac{k_{y_1} = k_y}{|k_{y_1} \pm k_{y_2}| = k_y} & \hat{\psi}_{k_{x_1}, k_{y_1}} \hat{\theta}_{k_{x_2}, k_{y_2}} G(k_{x_1}, k_{x_2}, k_{y_1}, k_{y_2}) \\
& - i k_x \hat{\psi}_{k_x, k_y} + K_{k_x, k_y}^2 \hat{\theta}_{k_x, k_y} = 0, \quad (A3b)
\end{align*}
\]

where we have introduced the wavenumber and the functions

\[
K_{k_x, k_y} = \sqrt{\left( \frac{2\pi k_x}{\Gamma} \right)^2 + (\pi k_y)^2}, \quad (A4a)
\]

\[
G(k_{x_1}, k_{y_1}, k_{x_2}, k_{y_2}, k_y) = \frac{2\pi^2}{\Gamma} (k_{x_1} k_{x_2} h(k_{y_2}, k_{y_1}, k_y) - k_{y_1} k_{x_2} h(k_{y_1}, k_{y_2}, k_y)), \quad (A4b)
\]

\[
h(x, y, z) = \begin{cases} 
-1 & \text{if } x = y + z, \\
1 & \text{otherwise}. 
\end{cases} \quad (A4c)
\]
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Table 1. Resolution (N) used in all computations in the DNS, Newton’s method to find the SCRS and the linear stability analysis, for \( Ra \in (8\pi^4, 10^8) \).

| \( N \)  | \( Ra \)                  |
|---------|---------------------------|
| 64      | \( (8\pi^4, 3 \times 10^3] \) |
| 152     | \( (3 \times 10^3, 6 \times 10^5] \) |
| 256     | \( (6 \times 10^5, 4 \times 10^6] \) |
| 400     | \( (4 \times 10^6, 10^8] \) |

The quadratic system of algebraic equations (A3) is solved using Newton’s method repeated as

\[ \mathbf{v}_{n+1} = \mathbf{v}_n - \mathbf{J}^{-1}(\mathbf{v}_n) \mathbf{f}(\mathbf{v}_n), \]  

where \( \mathbf{v}_n \) is a vector of the \( \psi \) and \( \theta \) modes, \( \mathbf{f}(\mathbf{v}_n) \) and \( \mathbf{J}(\mathbf{v}_n) \) are the left-hand side of (A3) and its Jacobian evaluated at \( \mathbf{v}_n \), respectively, and \( \mathbf{v}_{n+1} \) is a vector containing the subsequent values of the modes following one iteration. We utilise the reflection symmetry (2.10a,b) of the SCRS to reduce the number of unknowns.

For all computations in this study, we use an equal number of Fourier modes in both \( x \) and \( y \). So that our results are consistent with those from the DNS, we employ the two-thirds dealiasing rule, meaning that the largest retained wavenumbers in \( x \) and \( y \) are \( N/3 \) and \( 2N/3 \), respectively. The values of \( N \) used for different values of \( Ra \) are listed in Table 1.

To validate our computations of the SCRS, in the supplementary material we list the Nusselt number obtained for parameter values in the range \((Pr, Ra) \in [10^{-6}, 10^6] \times (8\pi^4, 10^8] \), demonstrating excellent agreement with results presented previously (Wen et al. 2020).

Appendix B. Linear stability analysis

Here, we provide further details of the linear stability analysis discussed in § 3. For given \( Ra \) and \( Pr \), we construct a steady solution \((\psi^E, \theta^E)\) to the even system (2.9) as described in Appendix A. Odd perturbations to this steady state satisfy the ‘odd’ part of the Boussinesq equations, namely the linear, time-autonomous system of partial differential equations (2.7). We make the ansatz (3.1) and act on (2.7) with the inner product (A2) to obtain an eigenvalue problem for the odd growth rate \( \sigma^o \), namely

\[
\begin{align*}
\frac{i}{2} \sum_{k_{x0} + k_{xE} = k_x} \hat{\psi}^o_{k_{x0}, k_y} \hat{\psi}^E_{k_{xE}, k_{yE}} G(k_{xE}, k_{yE}, k_{x0}, k_{y0}, k_y) [K^2_{k_{x0}, k_{y0}} - K^2_{k_{xE}, k_{yE}}] \\
- ik_x Ra Pr \hat{\theta}^o_{k_x, k_y} - Pr K^4_{k_x, k_y} \hat{\psi}^o_{k_x, k_y} &= \sigma^o K^2_{k_x, k_y} \hat{\psi}^o_{k_x, k_y}, \tag{B1a}
\end{align*}
\]

\[
\begin{align*}
\frac{i}{2} \sum_{k_{x0} + k_{xE} = k_x} [\hat{\theta}^o_{k_{x0}, k_y} \hat{\psi}^E_{k_{xE}, k_{yE}} - \hat{\psi}^o_{k_{x0}, k_y} \hat{\theta}^E_{k_{xE}, k_{yE}}] G(k_{xE}, k_{yE}, k_{x0}, k_{y0}, k_y) \\
+ ik_x \hat{\psi}^o_{k_x, k_y} - K^2_{k_x, k_y} \hat{\theta}^o_{k_x, k_y} &= \sigma^o \hat{\theta}^o_{k_x, k_y}. \tag{B1b}
\end{align*}
\]

where \( K^2, G \) and \( h \) are as in (A4).
Similarly, for even perturbations, we let \( \psi = \psi^E + \delta \psi^E e^{\sigma t} \) and \( \theta = \theta^E + \delta \theta^E e^{\sigma t} \), and linearise (2.8) with respect to \( \delta \) to obtain the same system as in (B1), but with \( o \mapsto e \).

Appendix C. Weakly nonlinear analysis

In this appendix, we provide some details of the construction of the eigenvalue problem (4.9) and the derivation of the amplitude equation (4.12) for the odd modes.

First, substitution of the decomposition (4.8) into the first-order odd equation (4.7) results in the linear system

\[
\frac{d\hat{\psi}_{kx,ky}}{d\tau_O} = A(\tau_E) \sum_{(n,m) \in S} \frac{\pi \Gamma}{2} \left( K_{n,m}^2 - \frac{K_{1,1}^4}{K_{n,m}^2} \right) (nh(n, k_x, 1) - mh(m, k_y, 1))\hat{\psi}_{n,m} \\
+ \frac{\Gamma}{2} \left( 4 \pi^2 R \frac{k_x^2}{\Gamma^2 K_{k_x,k_y}^4} - K_{k_x,k_y}^4 \right) \hat{\psi}_{kx,ky},
\]

(C1)

where \( S \) denotes the set

\[
S = \{(k_x + 1, k_y + 1), (k_x - 1, k_y + 1), (k_x + 1, k_y - 1), (k_x - 1, k_y - 1)\},
\]

(C2)

and \( K_{k_x,k_y}^2 \) and \( h \) are as in (A4). The first and second terms on the right-hand side of (C1) respectively define the elements of the matrix \( M \) and of the diagonal matrix \( D \) in (4.9), which takes the form

\[
\epsilon^2 \frac{d\psi^O_1}{d\tau_E} = (A(\tau_E)M + D)\psi^O_1
\]

(C3)

on the slow time scale \( \tau_E \).

We seek the solution for \( \psi^O_1 \) as a Wentzel–Kramers–Brillouin (WKB) asymptotic expansion of the form

\[
\psi^O_1(\tau_E) \sim e^{\phi(\tau_E)/\epsilon^2} C(\tau_E; \epsilon) \sim e^{\phi(\tau_E)/\epsilon^2} \sum_n C_n(\tau_E) \epsilon^{2n},
\]

(C4)

following which (C3) becomes

\[
(\dot{\phi}I - A(\tau_E)M - D)C + \epsilon^2 \dot{C} = 0.
\]

(C5)

To leading order in \( \epsilon \), we find that \( \dot{\phi} = \sigma(A) \) and \( C_0 = b(\tau_E)\psi(A) \), where \( \sigma \) and \( \psi \) satisfy the eigenvalue problem

\[
(\sigma I - AM - D)\psi = 0.
\]

(C6)

For a modal truncation such that our system is of size \( N \) we (in general) have \( N \) eigenvalues \( \sigma_j(A) \), ordered such that \( \mathcal{R}(\sigma_j(A)) \geq \mathcal{R}(\sigma_{j+1}(A)) \), and corresponding eigenvectors \( \psi_j(A) \), assumed to be normalised such that \( \| \psi_j(A) \|_2 = 1 \). The general leading-order solution for \( \psi^O_1 \) is then

\[
\psi^O_1(\tau_E) \sim \sum_{j=1}^N b_j(\tau_E) e^{\phi_j(\tau_E)/\epsilon^2} \psi_j(A(\tau_E)),
\]

(C7)

with \( \dot{\phi}_j(\tau_E) = \sigma_j(A(\tau_E)) \).
Onset of zonal modes in 2-D Rayleigh–Bénard convection

Since $\mathcal{R}(\sigma_j) \geq \mathcal{R}(\sigma_{j+1})$, the solution (C7) is dominated by the first term in the sum. In fact, we observe that the dominant eigenvalue $\sigma_1$ is real and that $\mathcal{R}(\sigma_2) < 0$, meaning that all terms except the first are exponentially small after a short transient. We thus have

$$\psi^0_1(\tau_E) \sim B(\tau_E)\psi_1(\tau_E),$$  \hspace{1cm} (C8)

where the leading-order amplitude is given by $B(\tau_E) = b_1(\tau_E) e^{\psi_1(\tau_E)/\epsilon^2}$. Taking derivatives with respect to $\tau_E$ leaves us with

$$\epsilon^2 \dot{B}(\tau_E) \sim \sigma_1(\psi_1(\tau_E)) B + O(\epsilon^2),$$  \hspace{1cm} (C9)

which is equivalent to (4.12). An equation for $b_1$ can be obtained in principle from the solvability condition for $C_1$ in (C5), but does not affect the leading-order evolution of $B$.

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