Properly stratified algebras and tilting

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Abstract

We study the properties of tilting modules in the context of properly stratified algebras. In particular, we answer the question when the Ringel dual of a properly stratified algebra is properly stratified itself, and show that the class of properly stratified algebras for which the characteristic tilting and cotilting modules coincide is closed under taking the Ringel dual. Studying stratified algebras, whose Ringel dual is properly stratified, we discover a new Ringel-type duality for such algebras, which we call the two-step duality. This duality arises from the existence of a new (generalized) tilting module for stratified algebras with properly stratified Ringel dual. We show that this new tilting module has a lot of interesting properties, for instance, its projective dimension equals the projectively defined finitistic dimension of the original algebra, it guarantees that the category of modules of finite projective dimension is contravariantly finite, and, finally, it allows one to compute the finitistic dimension of the original algebra in terms of the projective dimension of the characteristic tilting module.

1 Introduction

The concept of Ringel duality for quasi-hereditary algebras, introduced in [Ri], has been shown to be a very important and useful tool for the study of different situations in which quasi-hereditary algebras arise in a natural way, for example, for the study of the category $\mathcal{O}$, Schur algebras and algebraic groups. The list of natural generalizations of quasi-hereditary algebras starts with the so-called standardly stratified algebras, introduced in [CPS2], and the so-called properly stratified algebras, introduced in [Dl]. The study of the Ringel duality for stratified algebras was originated in [AHLU1], where the results, analogous to those of Ringel, were obtained. Alternative approach to the study of tilting modules for stratified algebras was developed in [PR, Xi].

The class of properly stratified algebras is a proper subclass of the class of standardly stratified algebras. In particular, the results of [AHLU1] perfectly apply to properly stratified algebras. However, the very definition of the properly stratified algebras suggests that such algebras must possess much more symmetric properties, for example, because of the left-right symmetry of the class of such algebras. The principle motivation for the present paper was the question whether one can obtain any nice new properties for the Ringel duals of properly stratified algebras.
Our first (relatively unexpected) observation is that the Ringel dual of a properly stratified algebra does not have to be properly stratified, see the example in Subsection 9.2. This leads to the following natural question: when the Ringel dual of a stratified algebra is properly stratified? We answer this question in Section 3 in terms of the existence of special filtrations on tilting modules. In Section 4 we show that a natural class of properly stratified algebras is closed with respect to taking of the Ringel dual. This is the class of properly stratified algebras for which the characteristic tilting and cotilting modules coincide. This class appears naturally in [MO] during the study of the finitistic dimension of stratified algebras and contains, in particular, all quasi-hereditary algebras.

Assume now that $A$ is a stratified algebra, whose Ringel dual $R$ is properly stratified. As a properly stratified algebra, the algebra $R$ has both tilting and cotilting modules. The classical Ringel duality identifies the characteristic cotilting $R$-module with the injective cogenerator of $A$-mod, whereas the characteristic tilting $R$-module is identified with a possibly different $A$-module, which we call $H$. It happens that this module $H$ carries a lot of very important information about the algebra $A$ and the major part of our paper is devoted to the study of the properties of $H$. In particular, in Sections 5 and 6 we obtain the following:

(i) the module $H$ is a (generalized) tilting module;

(ii) the module $H$ is relatively injective with respect to the subcategory of $A$-modules of finite projective dimension;

(iii) the projective dimension of $H$ equals the finitistic dimension of $A$;

(iv) existence of $H$ guarantees that the category of $A$-modules of finite projective dimension is contravariantly finite in $A$-mod;

(v) if the algebra $A$ has a simple preserving duality, then the existence of $H$ guarantees that the finitistic dimension of $A$ equals twice the projective dimension of the characteristic tilting $R$-module.

The classical tilting theory (see for example [Ha]) motivates the study of the endomorphism algebra of a (generalized) tilting module, whenever one has such a module. Hence in Section 8 we consider the opposite of the endomorphism algebra $B$ of the module $H$. We show that this algebra is standardly stratified and that its Ringel dual is properly stratified. This allows us to consider the corresponding module $H^{(B)}$. It happens that $H^{(B)}$ can be naturally identified with the injective cogenerator for $A^{opp}$-mod, which leads to a new covariant Ringel-type duality for stratified algebras having properly stratified Ringel dual.

We show that this duality induces an equivalence between the category of all $A$-modules of finite projective dimension and the category of all $B$-modules of finite injective dimension. It is not surprising that this duality was not discovered in the theory of quasi-hereditary algebras for in that case, and, more generally, in the case of properly stratified algebras for which the characteristic tilting and cotilting modules coincide, it degenerates to the identity functor. In particular, in such case we always have $B = A$. 

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We finish the paper with various examples illustrating the new duality, which include, in particular, certain categories of Harish-Chandra bimodules over complex semi-simple Lie algebras, and tensor products of quasi-hereditary and local algebras.

2 Notation

2.1 General conventions

Let $A$ be a finite dimensional associative algebra with identity over a field $k$. For simplicity we assume that $k$ is algebraically closed. However, slightly modifying the dimension arguments, one easily extends all the results to the case of an arbitrary field. Denote by $\Lambda$ an index set for the isomorphism classes of simple $A$-modules, which we denote by $L(\lambda)$, $\lambda \in \Lambda$. We write $P(\lambda)$ for the projective cover and $I(\lambda)$ for the injective hull of $L(\lambda)$.

Throughout the paper a module means a left module. Our main object will be some associative algebra $A$ (satisfying certain conditions, which we will specify later). Since we will consider more than one algebra, to avoid confusion we adopt the following convention: $A$-modules will be written without any additional notation, for example $M$; modules over any other algebra, $B$ say, will be written as $M(B)$ or $BM$. By $A$-$\text{mod}$ we denote the category of all finite-dimensional left $A$-modules.

Let $C$ be a subclass of objects from $A$-$\text{mod}$. Define $F(C)$ as the full subcategory of $A$-$\text{mod}$, which consists of all modules $M$ having a filtration, whose subquotients are isomorphic to modules from $C$. Given an $A$-module, $M$, we define $\text{add}(M)$ to be the full subcategory of $A$-$\text{mod}$ containing all modules, which are isomorphic to direct summands of $M^k$ for some $k \geq 0$. Let $M$ and $N$ be two $A$-modules. We define the trace $\text{Tr}_M(N)$ of $M$ in $N$ as the sum of images of all $A$-homomorphisms from $M$ to $N$. We denote by $D$ the usual duality functor $D(\_)=\text{Hom}_k(\_,k)$. Recall also that an algebra, $A$, has a simple preserving duality if there exists an exact contravariant and involutive equivalence $\circ : A$-$\text{mod} \to A$-$\text{mod}$, which preserves the isomorphism classes of simple $A$-modules.

For an algebra, $A$, we denote by $\mathcal{P}(A)^{<\infty}$ the full subcategory of $A$-$\text{mod}$, consisting of all modules of finite projective dimension; and by $\mathcal{I}(A)^{<\infty}$ the full subcategory of $A$-$\text{mod}$, consisting of all $A$-modules of finite injective dimension. We also denote by $\mathcal{D}(A)$ the derived category of $A$-$\text{mod}$, by $\mathcal{D}^b(A)$ its full subcategory consisting of all bounded complexes, and by $\mathcal{D}^-(A)$ its full subcategory consisting of all right bounded complexes. By $\mathcal{K}(A$-$\text{mod})$ we denote the homotopy category of the category of all complexes of $A$-modules, by $\mathcal{K}^b(A$-$\text{mod})$ its full subcategory consisting of all bounded complexes, and by $\mathcal{K}^-(A$-$\text{mod})$ its full subcategory consisting of all right bounded complexes.

Let $\mathcal{X}^\bullet$ be a complex in $\mathcal{K}(A$-$\text{mod})$ and $j \in \mathbb{Z}$. We define the truncated complex $t_j \mathcal{X}^\bullet$ to be the complex

$$t_j \mathcal{X}^\bullet : \cdots \to X_{j-2} \xrightarrow{d_{j-2}} X_{j-1} \xrightarrow{d_{j-1}} X_j \to 0 \to 0 \ldots,$$

where we keep the differentials $d_i$, $i < j$. A complex, $\mathcal{X}^\bullet = \{X_i : i \in \mathbb{Z}\}$, is called positive.
or *negative* provided that $X_i = 0$ for all $i < 0$ or $i > 0$ respectively. For an $A$-module, $M$, we denote by $M^\bullet$ the corresponding complex in $\mathcal{K}^b(A\text{-mod})$, concentrated in degree zero.

### 2.2 Stratified algebras and quasi-hereditary algebras

Let $\leq$ be a linear order on $\Lambda$. For $\lambda, \mu \in \Lambda$ we will write $\lambda < \mu$ provided that $\lambda \leq \mu$ and $\lambda \neq \mu$. For $\lambda \in \Lambda$ set $P^{>\lambda} = \oplus_{\mu>\lambda}P(\mu)$ and $I^{>\lambda} = \oplus_{\mu>\lambda}I(\mu)$. For each $\lambda \in \Lambda$ we define the *standard module*

$$\Delta(\lambda) = P(\lambda)/\text{Tr}_{P^{>\lambda}}(P(\lambda)),$$

and the *proper standard module*

$$\overline{\Delta}(\lambda) = \Delta(\lambda)/\sum f \text{ im } f,$$

where the sum is taken over all $f \in \text{rad End}_A(\Delta(\lambda))$. Dually, we define the *costandard module*

$$\nabla(\lambda) = \bigcap_{f : I(\lambda) \to I^{>\lambda}} \text{Ker } f,$$

and the *proper costandard module*

$$\overline{\nabla}(\lambda) = \bigcap_f \text{Ker } f,$$

where the intersection is taken over all homomorphisms $f \in \text{rad End}_A(\nabla(\lambda))$.

We can now define different classes of algebras which we will consider in this paper. The pair $(A, \leq)$ is called a *strongly standardly stratified algebra* or simply an *SSS-algebra* if

(SS) the kernel of the canonical epimorphism $P(\lambda) \twoheadrightarrow \Delta(\lambda)$ has a filtration, whose subquotients are $\Delta(\mu)$ with $\lambda < \mu$.

In several papers, see for example [AHLU1, ADL2], the authors used the name *standardly stratified algebras* for the algebras defined above. This gives rise to a confusion with a more general original definition from [CPS2], where standardly stratified algebras were defined with respect to a *partial pre-order* on $\Lambda$. Since in the present paper we will work with linear orders, we decided to use a different name.

The SSS-algebra $(A, \leq)$ is said to be *properly stratified* (see [Dl]) if the following condition is satisfied:

(PS) for each $\lambda \in \Lambda$ the module $\Delta(\lambda)$ has a filtration, whose subquotients are isomorphic to $\overline{\Delta}(\lambda)$.

Since the order $\leq$ will be fixed throughout the paper, we will usually omit it in the notation. It is easy to see (consult [Dl]) that an SSS-algebra, $A$, is properly stratified if and only if $A^{\text{opp}}$ is an SSS-algebra as well (with respect to the same order $\leq$). In particular, the algebra $A$ is properly stratified if and only if $A^{\text{opp}}$ is.
The smallest class of algebras we will treat is the class of quasi-hereditary algebras, defined in the following way: the SSS-algebra \((A, \leq)\) is called quasi-hereditary (see [CPS1]) if the following condition is satisfied:

\[(QH) \text{ for each } \lambda \in \Lambda \text{ we have } \Delta(\lambda) = \overline{\Delta}(\lambda).\]

If \(A\) is standardly (properly) stratified then we denote by \(F(\Delta)\) the category \(F(\mathcal{C})\), where \(\mathcal{C} = \{\Delta(\lambda) | \lambda \in \Lambda\}\), and define \(F(\overline{\Delta}), F(\nabla)\) and \(F(\overline{\nabla})\) similarly.

Let \(A\) be an SSS-algebra. It was shown in [AHLU1] that the category \(F(\Delta) \cap F(\nabla)\) is closed under taking direct summands, and that the indecomposable modules in this category are indexed by \(\lambda \in \Lambda\) in a natural way. The objects of \(F(\Delta) \cap F(\nabla)\) are called tilting modules. For \(\lambda \in \Lambda\) we denote by \(T(\lambda)\) the (unique up to isomorphism) indecomposable object in \(F(\Delta) \cap F(\nabla)\) for which there exists an exact sequence,

\[0 \to \Delta(\lambda) \to T(\lambda) \to \text{Coker} \to 0,\]

such that \(\text{Coker} \in F(\Delta)\) (see [AHLU1, Lemma 2.5]). For \(T = \oplus_{\lambda \in \Lambda} T(\lambda)\) we have \(F(\Delta) \cap F(\nabla) = \text{add}(T)\). The module \(T\) is usually called the characteristic tilting module.

It also follows from [AHLU1] that, in the case when \(A\) is properly stratified, the category \(F(\overline{\Delta}) \cap F(\nabla)\) is closed under taking direct summands, and that the indecomposable modules in this category are indexed by \(\lambda \in \Lambda\) in a natural way. The objects of \(F(\overline{\Delta}) \cap F(\nabla)\) are called cotilting modules. For \(\lambda \in \Lambda\) we denote by \(C(\lambda)\) the (unique up to isomorphism) indecomposable object in \(F(\overline{\Delta}) \cap F(\nabla)\) for which there exists an exact sequence,

\[0 \to \text{Ker} \to C(\lambda) \to \nabla(\lambda) \to 0,\]

such that \(\text{Ker} \in F(\nabla)\). For \(C = \oplus_{\lambda \in \Lambda} C(\lambda)\) we have \(F(\overline{\Delta}) \cap F(\nabla) = \text{add}(C)\). The module \(C\) is usually called the characteristic cotilting module.

We would like to point out that the name \((co)\text{tilting module}\) introduced above is slightly confusing. In the terminology of the classical tilting theory the \((co)\text{tilting modules}\) as defined above are only partial \((co)\text{tilting modules}\). On the other hand, \(A\text{–mod}\) usually contains many other modules, which are \((co)\text{tilting in the classical sense but are not related to the stratified structure and thus to the (co)tilting modules defined above. However, we will keep the above name since it is now standard and commonly accepted in the theory of quasi-hereditary and stratified algebras. Later on, in Subsection 6.1 we will recall the definition of a (generalized) tilting modules from the classical tilting theory. A tilting module as defined above is a (generalized) tilting module in the classical sense if and only if it contains a direct summand, which is isomorphic to the characteristic tilting module.}

We set \(L = \oplus_{\lambda \in \Lambda} L(\lambda), \Delta = \oplus_{\lambda \in \Lambda} \Delta(\lambda), \nabla = \oplus_{\lambda \in \Lambda} \nabla(\lambda), \overline{\Delta} = \oplus_{\lambda \in \Lambda} \overline{\Delta}(\lambda), \overline{\nabla} = \oplus_{\lambda \in \Lambda} \overline{\nabla}(\lambda), I = \oplus_{\lambda \in \Lambda} I(\lambda), \text{and } P = \oplus_{\lambda \in \Lambda} P(\lambda)\).

Throughout the paper we will multiply the maps from the left to the right, for instance the composition \(g \circ f\) of the map \(f : X \to Y\) and the map \(g : Y \to Z\) is denoted by \(fg\).
2.3 Ringel dual for standardly stratified algebras

Let \((A, \leq)\) be an SSS-algebra. Following [AHLU1] we define the Ringel dual \(R = R(A)\) of \(A\) as the algebra \(R = \text{End}_A(T)\). This algebra comes together with the Ringel duality functor

\[
F = F_A : A\text{-mod} \to R\text{-mod}, \quad \text{defined by} \quad F(-) = \text{Hom}_A(T, -). \quad (1)
\]

Due to [AHLU1, Theorem 2.6] the algebra \(R^{-}\) is an SSS-algebra with the same indexing set \(\Lambda\), but with respect to the order \(\leq^{-}\), which is opposite to the original order \(\leq\). The Ringel duality asserts that the algebra \(A^{-}\) is Morita equivalent to the Ringel dual of the algebra \(R^{-}\). Moreover, the functor \(F\) sends \(\nabla\) to \(\Delta^{(R)}\) (the latter being defined with respect to \(\leq^{-}\)) and induces an equivalence between \(F(\nabla)\) and \(F(\Delta^{(R)})\).

3 When is the Ringel dual properly stratified?

Let \((A, \leq)\) be an SSS-algebra. First of all we would like to determine when the Ringel dual \(R\) of \(A\) is properly stratified.

For \(\lambda \in \Lambda\) set \(T^{<\lambda} = \bigoplus_{\mu < \lambda} T(\mu)\). Define further \(S(\lambda) = \text{Tr}_{T^{<\lambda}}(T(\lambda))\) and put \(N(\lambda) = T(\lambda)/S(\lambda)\) giving the following short exact sequence:

\[
0 \to S(\lambda) \to T(\lambda) \to N(\lambda) \to 0. \quad (2)
\]

Finally, set \(N = \bigoplus_{\lambda \in \Lambda} N(\lambda)\) and \(F(N) = F(C)\), where \(C = \{N(\lambda)|\lambda \in \Lambda\}\). We start with another description of \(S(\lambda)\).

**Lemma 1.** For every \(\lambda \in \Lambda\) the module \(S(\lambda)\) is the unique submodule \(M\) of \(T(\lambda)\) which is characterized by the following properties:

(a) \(M \in F(\{\nabla(\mu)|\mu < \lambda\})\).

(b) \(T(\lambda)/M \in F(\{\nabla(\lambda)\})\).

**Proof.** Let

\[
0 = T_0 \subset T_1 \subset \cdots \subset T_k = T(\lambda)
\]

be a proper costandard filtration of \(T(\lambda)\). Using

\[
\text{Ext}^1_A(\nabla(\mu_1), \nabla(\mu_2)) = 0 \quad \text{for all } \mu_1 < \mu_2,
\]

we can assume that there exists \(0 \leq l < k\) such that \(T_i/T_{i-1} \cong \nabla(\mu)\) with \(\mu < \lambda\) for all \(i \leq l\), and \(T_i/T_{i-1} \cong \nabla(\lambda)\) for all \(i > l\). For \(\mu < \lambda\) we apply the functor \(\text{Hom}_A(T(\mu), -)\) to the short exact sequence

\[
0 \to T_l \to T(\lambda) \to \text{Coker} \to 0,
\]
and obtain
\[ 0 \to \text{Hom}_A(T(\mu), T) \to \text{Hom}_A(T(\mu), T(\lambda)) \to \text{Hom}_A(T(\mu), \text{Coker}). \]
As Coker $\in \mathcal{F}(\{\nabla(\lambda)\})$, $T(\mu) \in \mathcal{F}(\Delta)$ and $[T(\mu) : \Delta(\lambda)] = 0$ we get $\text{Hom}_A(T(\mu), \text{Coker}) = 0$. This implies that $\text{Hom}_A(T(\mu), T_1) = \text{Hom}_A(T(\mu), T(\lambda))$ and thus the image of any map $f : T(\mu) \to T(\lambda)$ belongs to $T_1$. In particular, $S(\lambda) \subset T_1$.

To prove that $S(\lambda) = T_1$ we show by induction on $i$ that for every $0 \leq i \leq l$ the module $T_i$ is a quotient of some $T^{(i)} \in \text{add}(\oplus_{\nu<\lambda} T(\nu))$. The statement is obvious for $i = 0$, and thus we have to prove only the induction step $i \implies i + 1$. Consider the short exact sequence
\[ 0 \to T_i \to T_{i+1} \to \nabla(\nu) \to 0, \]
where $\nu < \lambda$. Then we have the map $T^{(i)} \to T_i \to T_{i+1}$, and an epimorphism, $T(\nu) \to \nabla(\nu)$. Since $T_i, T_{i+1}, \nabla(\nu) \in \mathcal{F}(\nabla)$, and $T(\nu) \in \mathcal{F}(\Delta)$, the epimorphism $T(\nu) \to \nabla(\nu)$ can be lifted to a map, $T(\nu) \to T_{i+1}$, giving an epimorphism, $T^{(i)} \oplus T(\nu) \to T_{i+1}$. Therefore we finally obtain $S(\lambda) = T_1$. In particular, we see that $S(\lambda)$ satisfies both (a) and (b), moreover, it is unique by construction. This completes the proof.

From the proof of Lemma 1 we also obtain the following information:

**Corollary 2.** (1) For each $\lambda \in \Lambda$ the module $N(\lambda)$ has a filtration, whose subquotients are isomorphic to $\nabla(\lambda)$.

(2) If $\mu < \lambda$ then $\text{Hom}_A(T(\mu), N(\lambda)) = 0$.

Now we are ready to formulate and prove a characterization of those SSS-algebras, whose Ringel dual is properly stratified.

**Theorem 3.** Let $(A, \leq)$ be an SSS-algebra. Then the following assertions are equivalent:

(I) The Ringel dual $(R, \leq_R)$ is properly stratified.

(II) For each $\lambda \in \Lambda$ we have $S(\lambda) \in \mathcal{F}(N)$.

(III) For each $\lambda \in \Lambda$ we have $T(\lambda) \in \mathcal{F}(N)$.

**Proof.** The equivalence of the conditions (II) and (III) follows from (2).

Since the functor $F$ induces an exact equivalence from $F(\nabla)$ to $F(\nabla^{(R)})$, the short exact sequence (2) gives the short exact sequence
\[ 0 \to F(S(\lambda)) \to F(T(\lambda)) \to F(N(\lambda)) \to 0. \] (3)
Moreover, $F(T(\lambda)) = P^{(R)}(\lambda)$ by definition.

(II) $\implies$ (I). The algebra $R^{opp}$ is an SSS-algebra by [AHLU1], hence it is enough to prove that so is $R$. We start with the following useful statement, which we will also use later on.
Lemma 4. Let \((A, \leq)\) be an SSS-algebra. Then \(F(N(\lambda)) \cong \Delta^{(R)}(\lambda)\).

Proof. Using Lemma 1, the exactness of \(F\) on \(\mathcal{F}(\nabla)\), and \(F(\nabla(\nu)) = \Sigma^{(R)}(\nu)\) for all \(\nu \in \Lambda\) we obtain that \(F(S(\lambda))\) has a filtration with subquotients \(\Sigma^{(R)}(\mu)\), \(\mu < \lambda\). This implies that \(F(S(\lambda))\) includes into the trace of \(\oplus_{\nu \in R^\lambda} F(T(\nu))\) in \(F(T(\lambda))\). On the other hand, from Corollary 2(2) it follows that the latter inclusion is in fact the equality. Now the statement follows from the definition of a standard module.

Exactness of \(F\) on \(\mathcal{F}(\nabla)\) and Lemma 4 imply that \(P^{(R)}(\lambda) \in \mathcal{F}(\{F(N(\lambda))|\lambda \in \Lambda\})\). This means that \(R\) is an SSS-algebra, and hence is properly stratified, completing the proof of the implication (II) \(\implies\) (I).

(I) \(\implies\) (II). Assume that \(R\) is properly stratified, in particular is an SSS-algebra. For every \(\lambda \in \Lambda\) consider the short exact sequence

\[0 \to K^{(R)}(\lambda) \to P^{(R)}(\lambda) \to \Delta^{(R)}(\lambda) \to 0.\]

Conditions (SS) and (PS) ensure that \(P^{(R)}(\lambda) \in \mathcal{F}(\Sigma^{(R)}), \Delta^{(R)}(\lambda) \in \mathcal{F}(\{\Sigma^{(R)}(\lambda)\}), K^{(R)}(\lambda) \in \mathcal{F}(\{\Sigma^{(R)}(\mu)|\mu > R \lambda\})\). Now we can apply \(F^{-1}\), which is exact on \(\mathcal{F}(\Sigma^{(R)})\), and obtain the exact sequence

\[0 \to F^{-1}(K^{(R)}(\lambda)) \to T(\lambda) \to F^{-1}(\Delta^{(R)}(\lambda)) \to 0,\]

moreover, \(F^{-1}(\Delta^{(R)}(\lambda)) \in \mathcal{F}(\{\overline{\nabla}(\lambda)\})\), and \(F^{-1}(K^{(R)}(\lambda)) \in \mathcal{F}(\{\overline{\nabla}(\mu)|\mu < \lambda\})\). From Lemma 1 we obtain that \(S(\lambda) = F^{-1}(K^{(R)}(\lambda))\) and \(N(\lambda) = F^{-1}(\Delta^{(R)}(\lambda))\). Using \(K^{(R)}(\lambda) \in \mathcal{F}(\Delta^{(R)})\) we get \(S(\lambda) \in F^{-1}(\mathcal{F}(\Delta^{(R)})) = \mathcal{F}(N)\). This proves the implication (I) \(\implies\) (II) and thus the proof is complete. \(\square\)

4 The Ringel dual of a properly stratified algebras

In general the Ringel dual of a properly stratified algebra does not need to be properly stratified, see the example in Subsection 9.2. In this section we show that for one natural class of properly stratified algebras this can always be guaranteed (in fact, we prove even more, namely, that this class is closed under taking the Ringel dual). The class consists of all properly stratified algebras for which the characteristic tilting and cotilting modules coincide. This class appeared in [MO] where it was shown that the finitistic dimension of an algebra with duality from this class is twice the projective dimension of the characteristic tilting module (we will extend this result in Subsection 6.4). This class contains, in particular, all quasi-hereditary algebras.

Theorem 5. Assume that \((A, \leq)\) is a properly stratified algebra for which the characteristic tilting and cotilting modules coincide. Then the Ringel dual \((R, \leq_R)\) is properly stratified, moreover, the characteristic tilting and cotilting \(R\)-modules coincide as well.
Proof. Let \((A, \leq)\) be a properly stratified algebra. By Theorem 3, to show that \((R, \leq_R)\) is properly stratified it is enough to show that \(S(\lambda) \in \mathcal{F}(N)\) for each \(\lambda \in \Lambda\). Recall that \(T(\lambda) = C(\lambda)\) by our assumption. This gives us the following short exact sequence:

\[
0 \to \text{Ker} \to T(\lambda) \to \nabla(\lambda) \to 0,
\]

where \(\text{Ker}\) has a filtration by \(\nabla(\mu), \mu < \lambda\). Since \(A\) is properly stratified, filtrations by costandard modules extend to filtrations by the corresponding proper costandard modules. Hence we can use Lemma 1 to obtain \(S(\lambda) = \text{Ker}\) and \(N(\lambda) = \nabla(\lambda)\). This implies that \((R, \leq_R)\) is properly stratified.

It is left to show that \(T(R) = C(R)\). By [AHLU1, Theorem 2.6(vi)], \(F(I)\) is the characteristic cotilting \(R\)-module, and, since \(R\) is properly stratified, we need only to show that \(F(I) \in \mathcal{F}(\Delta(R))\). However, since \(A\) is properly stratified, we have \(I \in \mathcal{F}(\nabla) = \mathcal{F}(N)\), \(F\) is exact on \(\mathcal{F}(\nabla)\), and \(F(N) = \Delta(R)\) by Lemma 4. This implies that \(F(I) \in \mathcal{F}(\Delta(R))\) and completes the proof. \(\square\)

5 Module \(H\): definition and basic properties

Now we can make the principal assumption until the end of the paper: \(A, \leq\) is an SSS-algebra such that the Ringel dual \((R, \leq_R)\) is properly stratified.

There are two possibilities. The first one is the case when \(T(R) = C(R)\). In this case Theorem 5 says that the algebra \(A\) itself is properly stratified and, moreover, that \(T = C\). Hence the behavior of the Ringel duality in this case is completely similar to the classical case of quasi-hereditary algebras (which is in fact a special case of this situation).

The second case is when \(T(R) \not\cong C(R)\). In this case it is possible that the algebras \(\text{End}_R(T(R))\) and \(\text{End}_R(C(R))\) are quite different (see the example in Subsection 9.2). Later on in the paper we will show that this situation leads to a new Ringel type duality, which we call the two-step duality, on a subclass of SSS-algebras. In the special case of quasi-hereditary algebras, and even in the more general situation of properly stratified algebras for which the characteristic tilting and cotilting modules coincide, the two-step duality degenerates to the identity functor.

Since the Ringel duality functor \(F\) induces an equivalence between \(\mathcal{F}(\nabla)\) and \(\mathcal{F}(\Delta(R))\), we can take the module \(T(R)(\lambda) \in \mathcal{F}(\Delta(R))\) and define \(H(\lambda) = F^{-1}(T(R)(\lambda))\) for all \(\lambda \in \Lambda\). Set \(H = \bigoplus_{\lambda \in \Lambda} H(\lambda)\). The module \(H\) will be the main object of our interest in the sequel. We start with some basic properties of the modules \(N\) and \(H\).

**Proposition 6.** (i) All tilting modules belong to \(\mathcal{F}(N)\) and are exactly the relatively \((\text{ext})\)-projective modules in \(\mathcal{F}(N)\), that is for \(M \in \mathcal{F}(N)\) the module \(M\) is tilting if and only if \(\text{Ext}^i_{\lambda}(M, N) = 0\) for all \(i > 0\).

(ii) \(\text{add}(H) \subset \mathcal{F}(N)\) and the modules from \(\text{add}(H)\) are exactly the relatively \((\text{ext})\)-injective modules in \(\mathcal{F}(N)\), that is for \(M \in \mathcal{F}(N)\) we have \(M \in \text{add}(H)\) if and only if \(\text{Ext}^i_{\lambda}(N, M) = 0\) for all \(i > 0\).
Proof. We start with (i). That all tilting modules belong to \( \mathcal{F}(N) \) follows from Theorem 3(III). Assume that \( M \) is a tilting module. Then \( \text{Ext}_A^i(M, N) = 0 \) for all \( i > 0 \) by [AHLU1, Lemma 1.2] since \( M \in \mathcal{F}(\Delta) \) and \( N \in \mathcal{F}(\nabla) \). To prove the opposite statement, assume that \( M \in \mathcal{F}(N) \) and \( \text{Ext}_A^i(M, N) = 0 \), for all \( i > 0 \). Since \( M \in \mathcal{F}(N) \subset \mathcal{F}(\nabla) \), we only need to prove that \( M \in \mathcal{F}(\Delta) \).

Let \( \lambda \in \Lambda \) and \( N \) be a submodule of \( N(\lambda) \) such that we have the following short exact sequence:

\[
0 \to X \to N(\lambda) \to \nabla(\lambda) \to 0,
\]

where \( X \) has a filtration with subquotients \( \nabla(\lambda) \). Applying \( \text{Hom}_A(M, -) \) to this short exact sequence we get the following fragment in the long exact sequence

\[
\cdots \to \text{Ext}_A^i(M, N(\lambda)) \to \text{Ext}_A^i(M, \nabla(\lambda)) \to \text{Ext}_A^{i+1}(M, X) \to \text{Ext}_A^{i+1}(M, N(\lambda)) \cdots.
\]

The condition \( \text{Ext}_A^i(M, N(\lambda)) = 0, i > 0 \), gives us a dimension shift between the spaces \( \text{Ext}_A^i(M, \nabla(\lambda)) \) and \( \text{Ext}_A^{i+1}(M, X) \).

To proceed we need the following statement:

Lemma 7. The module \( N \) has finite projective dimension.

Proof. We prove that \( \text{p.d.}(N(\lambda)) < \infty \) by induction on \( \lambda \in \Lambda \). Suppose \( \lambda \) is minimal. Then \( N(\lambda) \cong T(\lambda) \) and hence \( \text{p.d.}(N(\lambda)) < \infty \). Now assume by induction that for all \( \mu < \lambda \) we have \( \text{p.d.}(N(\mu)) < \infty \). Since \( S(\lambda) \) is filtered by \( N(\mu) \), \( \mu < \lambda \), it follows that \( \text{p.d.}(S(\lambda)) < \infty \). The exact sequence (2) and \( \text{p.d.}(T(\lambda)) < \infty \) now implies \( \text{p.d.}(N(\lambda)) < \infty \) and completes the proof.

Since \( M \in \mathcal{F}(N) \), Lemma 7 implies that \( \text{p.d.}(M) < \infty \). This forces the equality \( \text{Ext}_A^i(M, \nabla(\lambda)) = 0 \) for all \( i \) big enough. Thus, the dimension shift and the fact that \( X \) is filtered by \( \nabla(\lambda) \), guarantees that \( \text{Ext}_A^i(M, \nabla(\lambda)) = 0 \) for all \( i > 0 \). From [AHLU1, Theorem 1.6] we now obtain \( M \in \mathcal{F}(\Delta) \), which completes the proof of (i).

Now let us prove (ii). Since \( F : \mathcal{F}(\nabla) \to \mathcal{F}(\Delta^R) \) is an equivalence with \( \mathcal{F}(\nabla) = \{ X \in A\text{-mod} | \text{Ext}_A^i(T, M) = 0 \} \) and \( \mathcal{F}(N) \subset \mathcal{F}(\nabla) \), we obtain that for all \( i \geq 0 \) and for all \( M_1, M_2 \in \mathcal{F}(N) \) we have

\[
\text{Ext}_A^i(M_1, M_2) \cong \text{Ext}_R^i\left(F(M_1), F(M_2)\right).
\]

Put \( M_1 = N \) and \( M_2 = H \). By Lemma 4 we have \( F(N) = \Delta^R \) and by the definition we have \( F(H) = T^R \). Therefore \( H \in \mathcal{F}(N) \) and (4) guarantees

\[
\text{Ext}_A^i(N, H) \cong \text{Ext}_R^i(\Delta^R, T^R) = 0,
\]

as \( T^R \in \mathcal{F}(\nabla^R) \). This proves that \( H \) is relatively injective in \( \mathcal{F}(N) \).

To prove the opposite statement, we assume that \( M \in \mathcal{F}(N) \) and \( \text{Ext}_A^i(N, M) = 0 \) for all \( i > 0 \). Using (4) we get that \( \text{Ext}_R^i(\Delta^R, F(M)) = 0 \), which implies that \( F(M) \in \mathcal{F}(\nabla^R) \) by [AHLU1, Theorem 1.6]. Moreover, \( F(M) \in \mathcal{F}(\Delta^R) \) since \( M \in \mathcal{F}(N) \), \( F(N) = \Delta^R \), and \( F \) is exact on \( \mathcal{F}(N) \subset \mathcal{F}(\nabla) \). Hence \( F(M) \) is a tilting \( R \)-module and thus \( M \in \text{add}(H) \) by the definition of \( H \). This completes the proof.
Proposition 8. Let $\lambda \in \Lambda$. Then there exist the following exact sequences:

$$0 \to \nabla(\lambda) \to H(\lambda) \to \text{Coker}_1 \to 0, \quad (5)$$
$$0 \to N(\lambda) \to H(\lambda) \to \text{Coker}_2 \to 0, \quad (6)$$

where $\text{Coker}_1$ has a filtration with subquotients $\nabla(\mu), \mu \geq \lambda$ and $\text{Coker}_2$ has a filtration with subquotients $N(\mu), \mu > \lambda$.

Proof. Applying $F^{-1}$ to the exact sequence

$$0 \to \Delta^{(R)}(\lambda) \to T^{(R)}(\lambda) \to \widetilde{\text{Coker}}_{(R)}^{(R)} \to 0, \quad (7)$$

where $\widetilde{\text{Coker}}_{(R)} \in \mathcal{F}(\Delta^{(R)})$ (see [AHLU1, Lemma 2.5(iv)]), gives (6).

Since $R$ is properly stratified, all standard modules have proper standard filtrations and hence (7) gives rise to the exact sequence

$$0 \to \Delta^{(R)}(\lambda) \to T^{(R)}(\lambda) \to \widetilde{\text{Coker}}_{1}^{(R)} \to 0, \quad (8)$$

where $\widetilde{\text{Coker}}_{1}^{(R)} \in \mathcal{F}(\Delta^{(R)})$. Applying $F^{-1}$ to (8) gives (5). This completes the proof. \qed

6 Module $H$: advanced properties

6.1 $H$ is a (generalized) tilting module

Recall that a module, $M$, over an associative algebra $A$ is called a (generalized) tilting module (see for example [Ha, Chapter III]) if the following three conditions are satisfied:

(i) $\text{Ext}^i_A(M, M) = 0$, $i > 0$;

(ii) $\text{p.d.}(M) < \infty$;

(iii) there is an exact sequence $0 \to AA \to M_0 \to M_1 \to \cdots \to M_k \to 0$, where $k \geq 0$ and $M_i \in \text{add}(M)$.

We would like to emphasize once more that a tilting module, $M$, over an SSS-algebra is a (generalized) tilting module in the above sense if and only if $M$ contains a direct summand, which is isomorphic to the characteristic tilting module, see Subsection 2.2. Now we can state the following:

Proposition 9. The module $H$ is a (generalized) tilting module.

Proof. We split the proof into a sequence of lemmas.

Lemma 10. $\text{Ext}^i(H, H) = 0$ for all $i > 0$. 

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Proof. This follows directly from the second statement of Proposition 6.

Lemma 11. The module $H$ has finite projective dimension.

Proof. Since $H \in \mathcal{F}(N)$, the statement follows from Lemma 7.

**Lemma 12.** For every $X \in \text{add}(T)$ there exists a minimal coresolution,

$$0 \to X \to H_0 \to H_1 \to \cdots \to H_k \to 0,$$

where $H_i \in \text{add}(H)$ for all $i$ and $0 \leq k \leq \text{p.d.}(T^{(R)})$.

Note that the length of the coresolution in Lemma 12 is estimated in terms of the projective dimension of the tilting module over the Ringel dual $R$.

Proof. The module $F(X)$ is a projective $R$-module and hence, by [Ha, Section III.2.2], there exists a finite coresolution,

$$0 \to F(X) \to T_0^{(R)} \to T_1^{(R)} \to \cdots \to T_k^{(R)} \to 0,$$

where $T_i^{(R)} \in \text{add}(T^{(R)})$ for all $i$, of length $k \leq \text{p.d.}(T^{(R)})$. Applying $F^{-1}$ to (9) yields the exact sequence

$$0 \to X \to F^{-1}(T_0^{(R)}) \to F^{-1}(T_1^{(R)}) \to \cdots \to F^{-1}(T_k^{(R)}) \to 0,$$

where $F^{-1}(T_i^{(R)}) \in \text{add}(H)$ for all $i$. This completes the proof.

**Lemma 13.** There is an exact sequence

$$0 \to A^0 \to H_0 \to H_1 \to \cdots \to H_m \to 0,$$

where $H_i \in \text{add}(H)$.

Proof. Choose a minimal tilting coresolution,

$$0 \to A^0 \to T_0 \to T_1 \to \cdots \to T_k \to 0,$$

where $T_i \in \text{add}(T)$ and $k = \text{p.d.}(T)$, and consider the corresponding positive complex

$$\cdots \to 0 \to T_0 \to T_1 \to \cdots \to T_k \to 0 \to \cdots,$$

in $K^b(\text{add}(T))$, whose only non-zero homology is in degree zero and equals $A^0$. Applying Lemma 39 to $A = A$, $X^{(A)} = T$, $Y^{(A)} = H$ and the complex (10) we obtain a complex,

$$\cdots \to 0 \to H_0 \to H_1 \to \cdots \to H_{k'} \to 0 \to \cdots,$$

in $K^b(\text{add}(H))$, which is quasi-isomorphic to (10). This completes the proof.

The proof of Proposition 9 is now also complete.

We remark that, since $H$ is a (generalized) tilting module, the minimal length of the coresolution, given by Lemma 13, is equal to $\text{p.d.}(H)$, see for example [Ha, Chapter III].
6.2 \(H\) and \(\text{fin.dim.}(A)\)

Recall that the (projectively defined) finitistic dimension of \(A\) is defined as follows:

\[
\text{fin.dim}(A) = \sup\{\text{p.d.}(M)|M \in A\text{-mod}, \text{p.d.}(M) < \infty\}.
\]

In [AHU2] it was shown that the projectively defined finitistic dimension of a properly stratified (and even SSS-) algebra is finite (and even that the injectively defined finitistic dimension of such algebra is finite as well). In this subsection we show that the module \(H\) can be used to effectively compute \(\text{fin.dim}(A)\).

**Proposition 14.** Let \(M \in A\text{-mod}\) and \(\text{p.d.}(M) < \infty\). Then there exists a finite coreolution,

\[
0 \to M \to H_0 \to \cdots \to H_k \to 0,
\]

where \(H_i \in \text{add}(H)\) and \(k \geq 0\).

**Proof.** First we choose a minimal projective resolution,

\[
0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0,
\]

of \(M\), and obtain the complex

\[
\mathcal{P}^\bullet: \quad \cdots \to 0 \to P_n \to \cdots \to P_1 \to P_0 \to 0 \to \ldots,
\]

where \(P_0\) stays in degree zero. The complex \(\mathcal{P}^\bullet\) has the only non-zero homology in degree zero, which equals \(M\). Applying Lemma 39 to \(A = A, X^{(k)} = P, Y^{(k)} = H\) and the complex \(\mathcal{P}^\bullet[-n]\) we obtain a complex, shifting which by \(n\) in the derived category gives the complex

\[
\mathcal{H}^\bullet: \quad \cdots \to 0 \to H_{-n} \to \cdots \to H_{-1} \to H_0 \to H_1 \to \cdots \to H_k \to 0 \to \ldots,
\]

in \(K^b(\text{add}(H))\), where \(k \geq 0\), which is quasi-isomorphic to \(\mathcal{P}^\bullet\). This implies that the only non-zero homology of \(\mathcal{H}^\bullet\) is in degree zero and equals \(M\). Let us show that \(\mathcal{H}^\bullet\) is quasi-isomorphic to a positive complex from \(K^b(\text{add}(H))\). In fact we will show by a downward induction on \(l\) that for every \(0 \leq l \leq n\) there exists a complex,

\[
\mathcal{H}(l)^\bullet: \quad \cdots \to 0 \to H(l)_{-l} \to \cdots \to H(l)_{-1} \to H(l)_0 \to H(l)_1 \to \cdots \to H(l)_k \to 0 \to \ldots,
\]

in \(K^b(\text{add}(H))\), which is quasi-isomorphic to \(\mathcal{H}^\bullet\).

For \(l = n\) the statement is obvious with \(\mathcal{H}(n)^\bullet = \mathcal{H}^\bullet\). Assume that the complex \(\mathcal{H}(l)^\bullet, 0 < l \leq n\), is constructed and let us construct the complex \(\mathcal{H}(l - 1)^\bullet\).

From \(\mathcal{H}(l)^\bullet\) we obtain the short exact sequence

\[
0 \to H(l)_{-l} \to H(l)_{-l+1} \to H(l)_{-l+1}/H(l)_{-l} \to 0.
\] (11)

We are going to show that \(H(l)_{-l+1}/H(l)_{-l} \in \text{add}(H)\). Since \(H(l)_{-l}, H(l)_{-l+1} \in \mathcal{F}(\nabla)\), and \(\mathcal{F}(\nabla)\) is closed with respect to taking cokernels of monomorphisms, we obtain that
$H(l)_{-t+1}/H(l)_{-t} \in \mathcal{F}(\nabla)$ as well. Hence we can apply the Ringel duality functor $F$ to (11) and obtain the exact sequence

$$0 \to F(H(l)_{-t}) \to F(H(l)_{-t+1}) \to F(H(l)_{-t+1}/H(l)_{-t}) \to 0 \quad (12)$$

in $R$–mod. The modules $F(H(l)_{-t})$ and $F(H(l)_{-t+1})$ are tilting $R$–modules and therefore are contained in $\mathcal{F}(\nabla(R))$. Thus $F(H(l)_{-t}/H(l)_{-t+1})$ is contained in $\mathcal{F}(\nabla(R))$ as well by the same arguments as above. Moreover, the modules $F(H(l)_{-i})$, $F(H(l)_{-i+1})$, and $F(H(l)_{-i}/H(l)_{-i+1})$ are contained in $\mathcal{F}(\Delta(R))$ by the Ringel duality.

Both $F(H(l)_{-i})$ and $F(H(l)_{-i+1})$ have finite projective dimension and therefore the projective dimension of $F(H(l)_{-i+1}/H(l)_{-i})$ is also finite. To show that $F(H(l)_{-i+1}/H(l)_{-i}) \in \mathcal{F}(\Delta(R))$, we use the following lemma:

**Lemma 15.** Let $A$ be a properly stratified algebra. Then

(i) $\mathcal{F}(\Delta(A)) = \{M^{(A)} \in \mathcal{F}(\Delta(A)) \mid \text{p.d.}(M^{(A)}) < \infty\}$, and

(ii) $\mathcal{F}(\nabla(A)) = \{M^{(A)} \in \mathcal{F}(\nabla(A)) \mid \text{i.d.}(M^{(A)}) < \infty\}.$

**Proof.** We prove (i). The statement (ii) is proved by similar arguments.

The inclusion $\mathcal{F}(\Delta(A)) \subset \{M^{(A)} \in \mathcal{F}(\Delta(A)) \mid \text{p.d.}(M^{(A)}) < \infty\}$ can be found for example in [PR, Proposition 1.3] or in [AHLU1, Proposition 1.8].

Let us prove the inverse inclusion. Let $M^{(A)} \in \mathcal{F}(\nabla(A))$ and $\lambda \in \Lambda$. Consider a short exact sequence,

$$0 \to X^{(A)} \to \nabla^{(A)}(\lambda) \to \nabla^{(A)}(\lambda) \to 0,$$

where $X^{(A)}$ has a filtration with subquotients $\nabla^{(A)}(\lambda)$. Applying $\text{Hom}_A(M^{(A)}, \_)$ to (13) we get the long exact sequence

$$\ldots \to \text{Ext}_A^i(M^{(A)}, X^{(A)}) \to \text{Ext}_A^i(M^{(A)}, \nabla^{(A)}(\lambda)) \to \text{Ext}_A^i(M^{(A)}, \nabla^{(A)}(\lambda)) \to$$

$$\to \text{Ext}_A^{i+1}(M^{(A)}, X^{(A)}) \to \text{Ext}_A^{i+1}(M^{(A)}, \nabla^{(A)}(\lambda)) \to \ldots. \quad (14)$$

But $\text{Ext}_A^i(M^{(A)}, \nabla^{(A)}(\lambda)) = 0$ for $i > 0$, which gives us a dimension shift between the spaces $\text{Ext}_A^i(M^{(A)}, \nabla^{(A)}(\lambda))$ and $\text{Ext}_A^{i+1}(M^{(A)}, X^{(A)})$. Since p.d.($M^{(A)}$) is finite, we derive that $\text{Ext}_A^i(M^{(A)}, \nabla^{(A)}(\lambda)) = 0$ for all $i$ big enough. But the dimension shift and the fact that $X^{(A)}$ has a filtration with subquotients $\nabla^{(A)}(\lambda)$ imply that $\text{Ext}_A^i(M^{(A)}, \nabla^{(A)}(\lambda)) = 0$ for all $i$. Hence $M^{(A)} \in \mathcal{F}(\Delta(A))$ by [AHLU1, Theorem 1.6(iii)]. This completes the proof of Lemma 15.

From Lemma 15 it follows that $F(H(l)_{-i+1}/H(l)_{-i})$ has a standard filtration and thus is a tilting module. Applying $F^{-1}$ we obtain $H(l)_{-i+1}/H(l)_{-i} \in \text{add}(H)$. From Lemma 10 it now follows that the short exact sequence (11) splits. This means that $H(l)_{-t}$ is a direct summand of $H(l)_{-t+1}$ and by deleting it we construct the complex $\mathcal{H}(l-1)$. This proves the existence of the complex $\mathcal{H}(0)$, which happens to be a positive complex in $\mathcal{K}^b(\text{add}(H))$ quasi-isomorphic to $\mathcal{P}$. This completes the proof.
Corollary 16. Let $M \in \mathcal{P}(A)^{<\infty}$. Then the module $M$ belongs to $\text{add}(H)$ if and only if $\text{Ext}_A^i(X,M) = 0$ for all $i > 0$ and all $X \in \mathcal{P}(A)^{<\infty}$.

Proof. Let $M \in \text{add}(H)$. That $\text{Ext}_A^i(X,M) = 0$ for all $i > 0$ and all $X \in \mathcal{P}(A)^{<\infty}$ follows easily from $\text{Ext}_A^i(H,H) = 0$, $i > 0$, and Proposition 14 by induction on the length of the $\text{add}(H)$-coresolution of $M$.

Let $M \in \mathcal{P}(A)^{<\infty}$ be such that $\text{Ext}_A^i(X,M) = 0$ for all $i > 0$ and all $X \in \mathcal{P}(A)^{<\infty}$, and let $H_0 \in \text{add}(H)$ be such that $M \hookrightarrow H_0$ (such $H_0$ exists by Proposition 14). Then the cokernel of the latter embedding belongs to $\mathcal{P}(A)^{<\infty}$ and hence the above condition on $M$ implies that $M$ is in fact a direct summand of $H_0$, that is $M \in \text{add}(H)$. $\square$

Theorem 17. Let $A$ be an SSS-algebra such that the Ringel dual $R$ of $A$ is properly stratified. Then

$$\text{fin.dim}(A) = \text{p.d.}(H).$$

Proof. If $0 \to X \to Y \to Z \to 0$ is an exact sequence then the long exact sequence implies that

$$\text{p.d.}(X) \leq \max\{\text{p.d.}(Y), \text{p.d.}(Z)\}.\tag{15}$$

Let $M \in \mathcal{P}(A)^{<\infty}$. Applying (15) inductively to the $\text{add}(H)$-coresolution of $M$, constructed in Proposition 14, one obtains $\text{p.d.}(M) \leq \text{p.d.}(H)$. This proves the statement of the theorem. $\square$

Note that Theorem 17 implies, in particular, that $\text{fin.dim}(A)$ is finite (we have never used the corresponding result from [AHLU2]).

6.3 Existence of $H$ guarantees that the category of modules of finite projective dimension is contravariantly finite

For a full subcategory, $\mathcal{C}$, of $A$-mod, denote by $\hat{\mathcal{C}}$ the full subcategory of $A$-mod, which contains all modules $M$ for which there is a finite exact sequence,

$$0 \to M \to C_0 \to C_1 \to \cdots \to C_k \to 0,$$

with $C_i \in \mathcal{C}$. The category $\hat{\mathcal{C}}$ is defined dually.

Recall that a full subcategory, $\mathcal{C}$, of $A$-mod is called contravariantly finite provided that it is closed under direct summands and isomorphisms, and for each $A$-module $X$ there exists a homomorphism $f : C_X \to X$, where $C_X \in \mathcal{C}$, such that for any homomorphism $g : C \to X$ with $C \in \mathcal{C}$ there is a homomorphism $h : C \to C_X$ such that $f \circ h = g$.\footnote{Note that in general $\mathcal{P}(A)^{<\infty}$ is not contravariantly finite, see [IST].}

Recall also that a subcategory, $\mathcal{B}$, of $A$-mod is called resolving if it contains all projective modules and is closed under extensions and kernels of epimorphisms. Obviously, $\mathcal{P}(A)^{<\infty}$ is a resolving category. However, $\mathcal{P}^{<\infty}$ is not contravariantly finite in general, see [IST].

Our main result in this subsection is the following:

Theorem 18. Let $A$ be an SSS-algebra, whose Ringel dual is properly stratified. Then $\mathcal{P}^{<\infty}$ is contravariantly finite in $A$-mod.
Proof. Since \( H \) is a (generalized) tilting module, the subcategory \( \text{add}(H) \) is contravariantly finite and resolving by [AR, Section 5]. From Proposition 14 we see that \( \mathcal{P}^{< \infty} \subset \text{add}(H) \). On the other hand, \( \text{p.d.}(H) < \infty \) implies \( \mathcal{P}^{< \infty} \supset \text{add}(H) \). This completes the proof. \( \square \)

6.4 \( H \) and \( \text{fin.dim.}(A) \) for algebras with duality

In this section we calculate the finitistic dimension of a properly stratified algebra \( A \) having a simple preserving duality in terms of the projective dimension of the characteristic tilting module. This generalizes the main result in [MO], where analogous result is obtained under the assumption that the characteristic tilting and cotilting \( A \)-modules coincide. The main result of the section is:

**Theorem 19.** Let \( A \) be a properly stratified algebra having a simple preserving duality, whose Ringel dual \( R \) is also properly stratified. Then

\[
\text{fin.dim}(A) = 2 \text{p.d.}(T(R)).
\]

To prove the statement we will need several lemmas.

**Lemma 20.** Let \( A \) be as in Theorem 19. Then

\[
\text{p.d.}(T(R)) \leq \text{p.d.}(T).
\]

**Proof.** Since \( T \) is a tilting module, we can choose a minimal tilting coresolution,

\[
0 \rightarrow P \rightarrow T_0 \rightarrow T_1 \rightarrow \cdots \rightarrow T_a \rightarrow 0,
\]

of \( P \), where \( a = \text{p.d.}(T) \). From this coresolution we obtain the complex

\[
\cdots \rightarrow 0 \rightarrow T_0 \rightarrow T_1 \rightarrow \cdots \rightarrow T_a \rightarrow 0 \rightarrow \ldots,
\]

in which the only non-zero homology is in degree zero and equals \( P \).

On the other hand, the module \( F(H) \) is tilting over \( R \), in particular, it has finite projective dimension. Hence we can choose a minimal projective resolution,

\[
0 \rightarrow P_0^{(R)} \rightarrow \cdots \rightarrow P_1^{(R)} \rightarrow P_0^{(R)} \rightarrow F(H) \rightarrow 0,
\]

of \( F(H) \), where \( b = \text{p.d.}(T(R)) \). Applying \( F^{-1} \) we get a minimal tilting resolution,

\[
0 \rightarrow T_{(0)} = F^{-1}(P_0^{(R)}) \rightarrow \cdots \rightarrow T_{(1)} = F^{-1}(P_1^{(R)}) \rightarrow T_{(0)} = F^{-1}(P_0^{(R)}) \rightarrow H \rightarrow 0,
\]

(16)

of \( H \). Hence we also obtain the complex

\[
J^\bullet : \quad \cdots \rightarrow 0 \rightarrow T_{(b)} \rightarrow \cdots \rightarrow T_{(1)} \rightarrow T_{(0)} \rightarrow 0 \rightarrow \ldots,
\]

in which the only non-zero homology is in degree zero and equals \( H \).
If \(0 \to X \to Y \to Z \to 0\) is an exact sequence, then \(p.d.(Z) \leq \max\{p.d.(X), p.d.(Y)\} + 1\). Applying this inequality inductively to (16) we get
\[
p.d.(H) \leq a + b. \tag{17}
\]

Applying \(^\circ\) to \(\mathcal{J}^\bullet\) gives the complex
\[
\mathcal{C}^\bullet : \quad \cdots \to 0 \to T^0_0 \to T^0_1 \to \cdots \to T^0_b \to 0 \to \cdots,
\]
where the only non-zero homology is in degree zero and equals \(H^0\). Remark that the complex \(\mathcal{C}^\bullet\) consists of cotilting \(A\)-modules. We would like to substitute \(\mathcal{C}^\bullet[b]\) by a negative complex from \(\mathcal{K}(\text{add}(T))\). To be able to do this we need the following lemma:

**Lemma 21.** Let \(A\) be a properly stratified algebra. Then for each \(\lambda \in \Lambda\) there exists a (possibly infinite) minimal tilting resolution of \(C(\lambda)\), which has the following form:
\[
\cdots \to T_1 \to T(\lambda) \oplus T_0 \to C(\lambda) \to 0.
\]

**Proof.** Let \(\lambda \in \Lambda\). Since \(C(\lambda)\) has a proper costandard filtration, we can apply the Ringel duality functor \(F\) and get the \(R\)-module \(F(C(\lambda))\). Choose a minimal projective resolution,
\[
\cdots \to P^1(R) \to P^0(R) \to F(C(\lambda)) \to 0, \tag{18}
\]
of \(F(C(\lambda))\). Since \(C(\lambda)\) surjects onto \(\nabla(\lambda)\) and \(\nabla(\lambda)\) has a filtration with subquotients \(\nabla(\lambda)\), applying \(F\) it follows that there is an epimorphism from \(F(C(\lambda))\) to \(\Delta(R)\)(\(\lambda\)). Hence we conclude that the head of \(F(C(\lambda))\) contains \(L(R)\)(\(\lambda\)), and therefore \(P^0(R) = P(R)(\lambda) \oplus \hat{P}(R)\), where \(\hat{P}(R)\) is some projective \(R\)-module. Since \(F(C(\lambda))\) \(\in \mathcal{F}(\Delta(R))\) and (18) is exact, it follows that the kernels of all morphisms in (18) belong to \(\mathcal{F}(\Delta(R))\) as well. The statement of the lemma now follows by applying \(F^{-1}\) to (18) and taking \(P^0(R) = P(R)(\lambda) \oplus \hat{P}(R)\) into account. \(\square\)

From Lemma 21 it follows that we can apply Lemma 40 to \(A = A, X(\lambda) = C, Y(\lambda) = T\) and the complex \(\mathcal{C}^\bullet[b]\), and obtain a negative complex in \(\mathcal{K}(\text{add}(T))\), which is quasi-isomorphic to \(\mathcal{C}^\bullet[b]\). Shifting the latter one by \(-b\) in the derived category gives a complex,
\[
\mathcal{T}^\bullet : \quad \cdots \to T(0) \to \cdots \to T(b - 1) \to T(b) \to 0 \to \cdots,
\]
in \(\mathcal{K}^-(\text{add}(T))\) with the only non-zero homology in degree zero, which equals \(H^0\). We assume that \(\mathcal{T}^\bullet\) is minimal that does not contain trivial direct summands. Lemma 21 implies that \(T(b) = T(b) \oplus \hat{T}\) for some tilting module \(\hat{T}\).

We have
\[
\text{Ext}^2_A(H, H^\circ) = \text{Hom}_{\mathcal{D}^-(A)}(H^\bullet[-b], (H^\circ)^\bullet[b]) = \text{Hom}_{\mathcal{D}^-(A)}(\mathcal{J}^\bullet[-b], \mathcal{T}^\bullet[b]).
\]

From [Ha, Chap. III, Lemma 2.1] we have
\[
\text{Hom}_{\mathcal{D}^-(A)}(\mathcal{J}^\bullet[-b], \mathcal{T}^\bullet[b]) = \text{Hom}_{\mathcal{K}^-(A)}(\mathcal{J}^\bullet[-b], \mathcal{T}^\bullet[b]).
\]

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Let $f : T(b) \hookrightarrow T(b)$ be the inclusion, defined via the isomorphism of $T(b)$ with the first direct summand of $T(b)$. Denote by $g : T(b) \to T(b-1)$ and $h : T(b-1) \to T(b)$ the differentials in the complexes $\mathcal{J}^\bullet$ and $\mathcal{T}^\bullet$ respectively and consider the following diagram:

$$
\begin{array}{ccc}
F(T(b)) & \overset{F(g)}{\longrightarrow} & F(T(b-1)) \\
\downarrow \alpha & & \downarrow \beta \\
F(T(b-1)) & \overset{F(h)}{\longrightarrow} & F(T(b))
\end{array}
$$

The minimality of $\mathcal{J}^\bullet[-b]$ and $\mathcal{T}^\bullet[b]$ implies that the images of the morphisms $F(g)$ and $F(h)$ belong to the radicals of the corresponding modules. Hence for every $\alpha$ and $\beta$ as depicted on the diagram the image of $F(h) \circ \alpha + \beta \circ F(g)$ belongs to the radical of $F(T(b))$. However the image of $F(f)$ does not belong to the radical of $F(T(b))$. This means that the morphism $f$ induces a non-zero homomorphism from $\mathcal{J}^\bullet[-b]$ to $\mathcal{T}^\bullet[b]$ in $K^-(A)$. From this we conclude that $\text{Ext}_A^{2b}(H, H^\circ) \neq 0$ and hence

$$2b \leq \text{p.d.}(H). \quad (19)$$

Combining (17) and (19) we obtain $b \leq a$, which completes the proof of Lemma 20. \hfill \Box

The arguments above immediately imply:

**Corollary 22.** $\text{Ext}_A^i(H, H^\circ) = 0$ for all $i > 2 \cdot \text{p.d.}(T^R)$.

Further, we can derive the following inequality (compare with [EP, Theorem 2.2.1]):

**Lemma 23.** Let $A$ be as in Theorem 19 and $k = \text{p.d.}(H)$. Then $\text{Ext}_A^k(H, H^\circ) \neq 0$.

**Proof.** Choose a minimal projective resolution,

$$0 \to P_k \to \cdots \to P_1 \to P_0 \to H \to 0,$$

of $H$ and let

$$\mathcal{P}^\bullet : \quad \cdots \to 0 \to P_k \to \cdots \to P_1 \to P_0 \to 0 \to \cdots$$

be the corresponding complex in $\mathcal{K}^b(\text{add}(P))$. Choose also a minimal (possibly infinite) projective resolution,

$$\cdots \to Q_1 \to Q_0 \to H^\circ \to 0,$$

of $H^\circ$, and construct the corresponding (possibly infinite) complex

$$\mathcal{Q}^\bullet : \quad \cdots \to Q_1 \to Q_0 \to 0 \to \cdots.$$

Applying $^\circ$ to the short exact sequence

$$0 \to \nabla \to H \to \text{Coker} \to 0$$

Applying $^\circ$ to the short exact sequence

$$0 \to \nabla \to H \to \text{Coker} \to 0$$

applying $^\circ$ to the short exact sequence

$$0 \to \nabla \to H \to \text{Coker} \to 0$$

applying $^\circ$ to the short exact sequence

$$0 \to \nabla \to H \to \text{Coker} \to 0$$

given by Proposition 8, we obtain the short exact sequence

\[ 0 \to \text{Coker}^\circ \to H^0 \to \Delta \to 0. \]

It follows that the head of \( H^0 \) contains the head of \( \Delta \), which coincides with \( L \). Hence \( Q_0 = P \oplus Q \), where \( Q \) is some projective module. Using the same arguments as in the proof of Lemma 20 we obtain \( \text{Hom}_{D^-(A)}(P^\bullet, Q^\bullet[k]) \neq 0 \) and hence \( \text{Ext}_A^k(H, H^0) \neq 0 \). This completes the proof.

Now we are ready to prove Theorem 19.

**Proof of Theorem 19.** Let \( k = \text{p.d.}(H) \). Using Theorem 17 we have \( k = \text{fin.dim}(A) \). From Lemma 23 we obtain that \( \text{Ext}_A^k(H, H^0) \neq 0 \). Certainly, \( \text{Ext}_A^i(H, H^0) = 0 \) for all \( i > k \). But from the proof of Lemma 20 and from Corollary 22 we conclude \( k = 2b \). This completes the proof.

To relate the finitistic dimension of \( A \) to \( \text{p.d.}(T) \) we will need a stronger assumption.

**Proposition 24.** Let \( A \) be as in Theorem 19 and assume that \( R \) also has a simple preserving duality. Then

\[ \text{fin.dim}(A) = 2p.d.(T). \]

**Proof.** From Lemma 20 we have \( \text{p.d.}(T^{(R)}) \leq \text{p.d.}(T) \). The existence of the duality for both \( A \) and \( R \) implies that \( \text{p.d.}(T^{(R)}) = \text{i.d.}(C^{(R)}) \) and \( \text{p.d.}(T) = \text{i.d.}(C) \). Applying Lemma 20 to \( A^{\text{opp}} \) and \( R^{\text{opp}} \) and using the usual duality we obtain \( \text{i.d.}(C) \leq \text{i.d.}(C^{(R)}) \). Altogether we deduce \( \text{p.d.}(T^{(R)}) = \text{p.d.}(T) \) and the statement follows from Theorem 19.

We remark that if \( A \) is as in Theorem 19, then \( R \) does not necessarily have a simple preserving duality, see the example in Subsection 9.3.

### 7 Two-step duality for standardly stratified algebras

Since \( H \) is a (generalized) tilting module, the classical tilting theory suggests to study the algebra \( B(A) = \text{End}_A(H) \). Consider the functor \( G : A\text{-mod} \to B(A)\text{-mod} \) defined via

\[ G(-) = \mathbb{D} \circ \text{Hom}_A(-, H). \]

From the definition it follows that \( G(H) \) is an injective cogenerator of \( B(A) \).

Consider also the functor \( G' : B(A)\text{-mod} \to A\text{-mod} \) defined via

\[ G'(-) = \text{Hom}_{B(A)}(G(A), -). \]

We start with establishing some necessary properties of the functor \( G \).

**Lemma 25.** 1. \( G \) is exact on \( \mathcal{P}(A)<\infty \), and maps \( \mathcal{P}(A)<\infty \) to \( \mathcal{I}(B(A))<\infty \).

2. \( G \) is full and faithful on \( \mathcal{P}(A)<\infty \).
Proof. That $G$ is exact on $\mathcal{P}(A)^{<\infty}$ follows from Corollary 16. Let $M \in A\text{-mod}$ be such that \( \text{p.d.}(M) < \infty \). By Proposition 14 there exists a coresolution

$$0 \to M \to H_0 \to \cdots \to H_k \to 0, \tag{20}$$

where $H_i \in \text{add}(H)$ and $k \geq 0$. Applying the exact functor $G$ gives the exact sequence

$$0 \to G(M) \to G(H_0) \to \cdots \to G(H_k) \to 0,$$

in $B(A)\text{-mod}$. Since $G(H_j)$ is $B(A)$-injective for all $j$, we obtain that the injective dimension of $G(M)$ is finite. This proves the first statement.

Let us now prove that $G$ is full and faithful on $\mathcal{P}(A)^{<\infty}$. We start with showing that $G$ is full and faithful on $\text{add}(H)$. For this we calculate the following:

$$G' \circ G(AH_{B(A)}) =$$

$$= \text{Hom}_{B(A)}((\mathbb{D} \circ \text{Hom}_{A}(AA, AH_{B(A)})), \mathbb{D} \circ \text{Hom}_{A}(AH_{B(A)}, AH_{B(A)})) =$$

$$= \text{Hom}_{B(A)}(\text{Hom}_{A}(AH_{B(A)}, AH_{B(A)}), \text{Hom}_{A}(AA, AH_{B(A)})) =$$

$$= \text{Hom}_{B(A)}(B(A), AH_{B(A)}) = AH_{B(A)}, \tag{21}$$

and

$$G \circ G'(B(A)\mathbb{D}(B(A)^{\text{opp}})) =$$

$$= \mathbb{D} \circ \text{Hom}_{A}(\text{Hom}_{B(A)}((\mathbb{D} \circ \text{Hom}_{A}(AA, AH_{B(A)}), B(A)\mathbb{D}(B(A)^{\text{opp}})), AH_{B(A)})) =$$

$$= \mathbb{D} \circ \text{Hom}_{A}(\text{Hom}_{B(A)}(B(A)B(A), \text{Hom}_{A}(AA, AH_{B(A)})), AH_{B(A)})) =$$

$$= \mathbb{D} \circ \text{Hom}_{A}(B(A)B(A), AH_{B(A)})), AH_{B(A)}) =$$

$$= \mathbb{D} \circ \text{Hom}_{A}(AH_{B(A)}, AH_{B(A)}) = B(A)\mathbb{D}(B(A)^{\text{opp}}), \tag{22}$$

which implies that $G$ is full and faithful on $\text{add}(H)$.

Now the fact that $G$ is full and faithful on $\mathcal{P}(A)^{<\infty}$ follows from the existence of (20) for all $M \in \mathcal{P}(A)^{<\infty}$ by induction on the length of the coresolution (20). This completes the proof. \hfill \square

**Lemma 26.**

1. The functor $G$ maps $N(\lambda)$ to $\nabla^{(B(A))}(\lambda)$ for all $\lambda \in \Lambda$.

2. The algebra $(B(A)^{\text{opp}}, \leq)$ is an SSS-algebra.

**Proof.** For $\lambda \in \Lambda$ consider the short exact sequence

$$0 \to N(\lambda) \to H(\lambda) \to Y(\lambda) \to 0,$$

where $Y(\lambda)$ has a filtration with subquotients $N(\mu), \mu > \lambda$ (see Proposition 8). Because of the exactness of $G$ on $\mathcal{P}(A)^{<\infty}$, see Lemma 25, and the fact that $\text{p.d.}(N) < \infty$, the sequence above yields the short exact sequence

$$0 \to G(N(\lambda)) \to I^{(B(A))}(\lambda) \to G(Y(\lambda)) \to 0,$$
where \( G(Y(\lambda)) \) has a filtration with subquotients \( G(N(\mu)), \mu > \lambda \). Further, using the classical Ringel duality, for \( \lambda < \mu \) we have
\[
\text{Hom}_A(N(\lambda), H(\mu)) = \text{Hom}_R(\Delta^{(R)}(\lambda), T^{(R)}(\mu)) = 0
\]
(we recall that \( \Delta^{(R)}(\lambda) \) are defined with respect to \( \leq_R \), which is opposite to the original order \( \leq \)). Using the fact that \( G \) is full and faithful on \( \mathcal{F}(N) \subset \mathcal{P}(A)^{<\infty} \) we obtain
\[
\left[ G(N(\lambda)) : L^{(B(A))}(\mu) \right] = \dim_k \text{Hom}_{B(A)}(G(N(\lambda)), I^{(B(A))}(\mu)) = \dim_k \text{Hom}_A(N(\lambda), H(\mu)) = 0.
\]

So the family \( \{ G(N(\lambda)) | \lambda \in \Lambda \} \subset B(A)-\text{mod} \) satisfies all the conditions, characterizing the costandard modules for SSS-algebras (see for example [AHLU1, Lemma 1.5]). This implies \( G(N(\lambda)) = \nabla^{(B(A))}(\lambda) \). Exactness of \( G \) on \( \mathcal{F}(N) \subset \mathcal{P}(A)^{<\infty} \) guarantees that the injective cogenerator \( G(H) \) of \( B(A) \) is filtered by costandard modules. This completes the proof of both statements of the lemma. \( \square \)

**Lemma 27.**

1. \( G(A) \) is a (generalized) cotilting module for \( B(A) \).

2. \( G(A) \in \mathcal{F}\left(\nabla^{(B(A))}\right) \).

**Proof.** Using Lemma 25 one easily obtains
\[
\text{Ext}^i_{B(A)}(G(A), G(A)) = \text{Ext}^i_A(A, A) = 0
\]
for all \( i > 0 \). Since \( \text{p.d.}(A) = 0 < \infty \), from Lemma 25 it also follows that \( \text{i.d.}(G(A)) < \infty \). Since \( H \) has finite projective dimension, we can take a minimal projective resolution,
\[
0 \to P_k \to \cdots \to P_1 \to P_0 \to H \to 0,
\]
of \( H \). Since all modules in this resolution have finite projective dimension we can apply \( G \) and use Lemma 25 to obtain a resolution of the injective cogenerator \( G(H) \) over \( B(A) \) by modules \( G(P_i) \) from \( \text{add}(G(A)) \). Hence \( G(A) \) is a (generalized) cotilting module for \( B(A) \).

Using Lemma 25 and Lemma 26 one gets
\[
\text{Ext}^i_{B(A)}(G(A), \nabla^{(B(A))}) = \text{Ext}^i_{B(A)}(G(A), G(N)) = \text{Ext}^i_A(A, N) = 0,
\]
implies \( G(A) \in \mathcal{F}\left(\nabla^{(B(A))}\right) \). This completes the proof. \( \square \)

**Lemma 28.** The functor \( G \) induces an equivalence between \( \mathcal{F}(N) \) and \( \mathcal{F}\left(\nabla^{(B(A))}\right) \) with the inverse functor \( G' \).

**Proof.** By Lemma 25 for \( M \in \mathcal{P}(A)^{<\infty} \) we have
\[
G' \circ G(M) = \text{Hom}_{B(A)}(G(A), G(M)) = \text{Hom}_A(A, M) = M,
\]
(23)
which implies that \( G' \circ G \) is isomorphic to the identity functor on \( \mathcal{F}(N) \).

After Lemma 25 we have that \( G \) is full and faithful on \( \mathcal{F}(N) \), and thus we have only to prove that it is dense.

From the second statement of Lemma 27 we obtain that \( G' \) is exact on \( \mathcal{F}(\nabla(B(A))) \). (21) and (22) imply that \( G' \) is full and faithful on \( \text{add}(\mathbb{D}(B(A)_{\text{opp}})) \) that is on all \( B(A) \)-injective modules. Since every module with a costandard filtration has a finite coresolution by injective modules we obtain that \( G' \) is full and faithful on \( \mathcal{F}(\nabla(B(A))) \). But (23) implies that \( G' : \mathcal{F}(\nabla(B(A))) \to \mathcal{F}(N) \) and is dense. Thus \( G' \) is an equivalence and (23) implies that \( G \) is inverse to \( G' \), hence is dense as well. This completes the proof. \( \square \)

Now we can formulate the main result of this section.

**Theorem 29.** Let \((A, \leq)\) be an SSS-algebra, whose Ringel dual \((R, \leq_R)\) is properly stratified. Then

(i) The algebra \( B(A)^{\text{opp}} \) is an SSS-algebra and is isomorphic to the opposite algebra of the second Ringel dual \( \text{End}_R(T(R)) \).

(ii) \( B(A)^{\text{opp}} \) has the Ringel dual \((R^{\text{opp}}, \leq_R)\), which is properly stratified, and the algebra \( B(B(A)^{\text{opp}})^{\text{opp}} \) is Morita equivalent to \( A \).

**Proof.** That \((B(A)^{\text{opp}}, \leq)\) is an SSS-algebra was proved in Lemma 26. The statement about the second Ringel dual follows from the usual Ringel duality ([AHLU1, Theorem 2.6]) by

\[
\text{End}_A(H) = \text{End}_R(F(H)) = \text{End}_R(T(R)),
\]

since \( F(H) = T(R) \). This proves (i).

Now we prove (ii). We start with calculation of the indecomposable tilting modules in \( B(A)^{\text{opp}} \). Composing the functor \( G \) with the duality \( \mathbb{D} : B(A)\text{-mod} \to B(A)^{\text{opp}}\text{-mod} \), we get the contravariant functor \( \mathbb{D} \circ G : A\text{-mod} \to B(A)^{\text{opp}}\text{-mod} \). Applying this functor to the short exact sequence (2) and using the exactness of \( G \) on \( \mathcal{F}(N) \), we get the exact sequence

\[
0 \to \mathbb{D} \circ G(N(\lambda)) \to \mathbb{D} \circ G(T(\lambda)) \to \mathbb{D} \circ G(S(\lambda)) \to 0,
\]

where \( \mathbb{D} \circ G(N(\lambda)) = \mathbb{D} \nabla^i(B(A))_{\lambda} = \Delta^i(B(A)^{\text{opp}})_{\lambda} \), and \( \mathbb{D} \circ G(S(\lambda)) \) has a filtration with subquotients \( \Delta^i(B(A)^{\text{opp}})(\mu), \mu < \lambda \). Moreover, using Lemma 28, we have

\[
0 = \text{Ext}^i_A(T(\lambda), N(\mu)) \cong \text{Ext}^i_{B(A)^{\text{opp}}}(\Delta^i(B(A)^{\text{opp}})(\mu), \mathbb{D} \circ G(T(\lambda)))
\]

for all \( \lambda, \mu \) and \( i > 0 \). Hence \( \mathbb{D} \circ G(T(\lambda)) \in \mathcal{F}(\nabla^i(B(A)^{\text{opp}})) \) and we can conclude that \( \mathbb{D} \circ G(T(\lambda)) \) is an indecomposable tilting module, and, moreover, that \( \mathbb{D} \circ G(T(\lambda)) = T(B(A)^{\text{opp}})(\lambda) \). The Ringel dual to \( B(A)^{\text{opp}} \) is now computed by

\[
\text{End}_{B(A)^{\text{opp}}}(T(B(A)^{\text{opp}})) = \text{End}_A(T)^{\text{opp}} = R^{\text{opp}},
\]

and thus is properly stratified.
Hence we have the corresponding functor $F^{B(A)^{\text{opp}}} : B(A)^{\text{opp}}\text{-mod} \to R^{\text{opp}}\text{-mod}$. Since $R^{\text{opp}}$ is also properly stratified, we can construct $N^{(B(A)^{\text{opp}})}(\lambda) = (F^{B(A)^{\text{opp}}})^{-1} (\Delta^{(R^{\text{opp}})}(\lambda))$ and $H^{(B(A)^{\text{opp}})}(\lambda) = (F^{B(A)^{\text{opp}}})^{-1} T^{(R^{\text{opp}})}(\lambda)$. Then $B(B(A)^{\text{opp}}) = \text{End}_{B(A)^{\text{opp}}}(H^{B(A)^{\text{opp}}})$ and we compute

$$B(B(A)^{\text{opp}})^{\text{opp}} = \left(\text{End}_{B(A)^{\text{opp}}}(H^{B(A)^{\text{opp}}})\right)^{\text{opp}} = \left(\text{End}_{R^{\text{opp}}}(T^{(R^{\text{opp}})})\right)^{\text{opp}} = \text{End}_R(C(R)) = \text{End}_A(I) \simeq A,$$

where $\simeq$ denotes Morita equivalence. This completes the proof.

Lemma 25 and Lemma 28 admit the following extension:

**Proposition 30.** The functor $G$ induces an equivalence between $\mathcal{P}(A)^{<\infty}$ and $\mathcal{I}(B(A))^{<\infty}$ with the inverse functor $G'$.

To prove this we will need the following lemma:

**Lemma 31.** Let $M^{(B(A))} \in \mathcal{I}(B(A))^{<\infty}$. Then there exists a resolution,

$$0 \to Y_k^{(B(A))} \to \cdots \to Y_1^{(B(A))} \to Y_0^{(B(A))} \to M^{(B(A))} \to 0,$$

where $Y_i^{(B(A))} \in \text{add}(G(A))$ for all $i$.

**Proof.** Consider an injective coresolution,

$$0 \to M^{(B(A))} \to I_0^{(B(A))} \to I_1^{(B(A))} \to \cdots \to I_m^{(B(A))} \to 0,$$

of $M^{(B(A))}$ and let $\mathcal{M}^\bullet$ be the corresponding complex in $\mathcal{K}(I^{(B(A))})$. Applying Lemma 39 to $A = B(A)^{\text{opp}}$, $X^\bullet = \mathbb{D}(I^{(B(A))})$, $Y^\bullet = \mathbb{D}(G(A))$ and the complex $\mathbb{D}(\mathcal{M}^\bullet)$ gives a finite complex, the dual of which belongs to $\mathcal{K}(\text{add}(G(A)))$ and is quasi-isomorphic to $\mathcal{M}^\bullet$. The necessary resolution is now obtained using the projectivity of $A A$, properties of $G$ given by Lemma 25, and arguments dual to those used in Proposition 14. We omit the details.

We are now ready to prove Proposition 30.

**Proof of Proposition 30.** By Lemma 25 for $M \in \mathcal{P}(A)^{<\infty}$ we have

$$G' \circ G(M) = \text{Hom}_{B(A)}(G(A), G(M)) = \text{Hom}_A(A, M) = M,$$

which implies that $G' \circ G$ is isomorphic to the identity functor on $\mathcal{P}(A)^{<\infty}$. From Lemma 25 we have that $G$ is full and faithful on $\mathcal{P}(A)^{<\infty}$ and thus we are only left to prove that it is dense. From Lemma 31, using induction on the length of the $\text{add}(G(A))$-resolution, we obtain that $\text{Ext}_i^{B(A)}(G(A), X) = 0$ for all $i > 0$ and $X \in \mathcal{I}(B(A))^{<\infty}$, Thus $G'$ is exact on $\mathcal{I}(B(A))^{<\infty}$. One now completes the proof using the same arguments as in Lemma 28.

After Theorem 29 we can define $N^* = \mathbb{D}(N^{(B(A)^{\text{opp}})})$ and $H^* = \mathbb{D}(H^{(B(A)^{\text{opp}})})$. With this notation we have the following images of the two-step duality functor $G$. 

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Proposition 32. For every $\lambda \in \Lambda$ we have
\[ G(H(\lambda)) = I^{(B(A))}(\lambda), \quad G(N(\lambda)) = \nabla^{(B(A))}(\lambda), \quad G(T(\lambda)) = C^{(B(A))}(\lambda) \]
\[ G(\Delta(\lambda)) = (N^*)(^{(B(A))})(\lambda), \quad G(P(\lambda)) = (H^*)(^{(B(A))})(\lambda). \]

Proof. The first three equalities were proved during the proof of Theorem 29. The fourth equality follows from the third one and the fact that $G$ is exact on $\mathcal{P}(A)^{<\infty}$ by induction on $\lambda$, which starts from that $\lambda$ for which $T(\lambda) = \Delta(\lambda)$. The fifth equality follows from the fourth one and the fact that $G$ is exact on $\mathcal{P}(A)^{<\infty}$ by induction on $\lambda$, which starts from that $\lambda$ for which $P(\lambda) = \Delta(\lambda)$. \qed

We would like to end this section with the following two remarks: firstly, the Ringel and the two-step dualities give rise to the following schematic picture of functors on module categories of SSS-algebras:

\[ \begin{array}{ccc}
R-\text{mod} & & A-\text{mod} \\
\downarrow F & & \downarrow \mathcal{D} \circ F \circ R \\
B(A)^{\text{opp}}-\text{mod} & & \mathcal{D} \circ G \\
& \mathcal{D} \circ F^{\text{opp}} \circ \mathcal{R} & \downarrow F^{B(A)^{\text{opp}}} \\
R^{\text{opp}}-\text{mod} & & \\
\end{array} \]

Note that the picture above is not a commutative diagram. In particular, the two-step duality functor $G$ is not the composition of the Ringel dualities $F$ and $F^R$. Each functor on this picture induces an equivalence of certain subcategories. However, these subcategories are not well-coordinated with each other. A deeper understanding of the picture above might be an interesting problem.

Secondly, in Section 3 we have shown that the information about the proper stratification of the Ringel dual of an SSS-algebra, $A$, can be derived directly from $A-\text{mod}$. An interesting problem seems to be whether $A-\text{mod}$ contains enough information to answer directly the question about the proper stratification of the the two-step dual.

8 $\mathcal{F}(N)$-filtration dimension

Recall (see for example [MO, Section 4.2]) that for an algebra, $A$, a family, $\mathcal{M}$, of $A$-modules, and an $A$-module, $M$, one says that the $\mathcal{M}$-filtration codimension $\text{codim}_\mathcal{M}(M)$ of $M$ equals $n$ provided that there exists an exact sequence,
\[ 0 \to M \to M_0 \to M_1 \to \cdots \to M_n \to 0, \]
where $M_i \in \mathcal{M}$, and $n$ is minimal with this property. In this section we study the $\mathcal{F}(N)$-filtration codimension for $A$-modules. We start with determining the modules for which the notion of $\mathcal{F}(N)$-filtration codimension makes sense.
Lemma 33. Let $M \in A\text{-mod}$. Then codim$_{\mathcal{F}(N)}(M)$ is defined and finite if and only if p.d.$(M) < \infty$.

Proof. The “if” part follows from Proposition 14 and the fact that $H \in \mathcal{F}(N)$.

To prove the “only if” part, we take a finite $\mathcal{F}(N)$-coresolution,

$$0 \to M \to X_0 \to X_1 \to \cdots \to X_l \to 0,$$  \hspace{1cm} (25)

of $M$. From Lemma 7 it follows that all $X_i$ have finite projective dimension, which implies that $M$ has finite projective dimension as well. $\square$

Theorem 34. Let $M$ be an $A$-module of finite projective dimension. Then

$$\text{codim}_{\mathcal{F}(N)}(M) = \text{codim}_{\mathcal{F}^{\dagger}(\nabla)}(M).$$

Proof. Since $\mathcal{F}(N) \subset \mathcal{F}^{\dagger}(\nabla)$ it follows directly that codim$_{\mathcal{F}(N)}(M) \leq$ codim$_{\mathcal{F}^{\dagger}(\nabla)}(M)$. To prove that codim$_{\mathcal{F}(N)}(M) \leq$ codim$_{\mathcal{F}^{\dagger}(\nabla)}(M)$, we let (25) to be an $\mathcal{F}(N)$-coresolution of $M$ of minimal length and $\mathcal{N}^\bullet_M$ be the corresponding complex, whose only non-zero homology is in degree zero and equals $M$. Applying Lemma 39 to $A = A^{\text{opp}}$, $X^{(k)} = \mathbb{D}(\oplus_{i=0}^l X_i)$, $Y^{(k)} = \mathbb{D}(T)$ and the complex $\mathbb{D}(\mathcal{N}^\bullet_M[l])$ gives a complex, the dual $\mathcal{Z}^\bullet_M$ of which belongs to $K^b(\text{add}(T))$ and is quasi-isomorphic to $\mathcal{N}^\bullet_M[l]$.

From a tilting coresolution of $\Delta$ we get the complex

$$J^\bullet_\Delta : \cdots \to 0 \to Q_0 \to Q_1 \to \cdots \to Q_r \to 0 \ldots$$

in $K^b(\text{add}(T))$, in which $Q_0$ is isomorphic to the characteristic tilting module $T$, and whose only non-zero homology is in degree zero and equals $\Delta$.

Let $T(\lambda)$ be a direct summand of $\mathcal{Z}^0_M \neq 0$ and $f : T \to T(\lambda) \hookrightarrow \mathcal{Z}^0_M$ be the projection on this direct summand. By [MO, Lemma 1] the homomorphism $f$ gives rise to a non-zero homomorphism in $\mathcal{D}^b(A)$ from $J^\bullet_\Delta$ to $\mathcal{Z}^\bullet_M$. The last implies that Ext$_A^{l}(\Delta, M) \neq 0$ and from [MP, Lemma 1] we obtain that codim$_{\mathcal{F}^{\dagger}(\nabla)}(M) \geq l$. This completes the proof. $\square$

9 Examples

9.1 Quasi-hereditary algebras

Assume that $A$ is quasi-hereditary (or, more generally, a properly stratified algebra, for which the characteristic tilting and cotilting modules coincide). Then the tilting $A$-module $T$ is also cotilting and hence there is an epimorphism, $T \to \nabla$, whose kernel is filtered by costandard modules. In this situation Lemma 1 implies that $N = \nabla$ and hence $H = I$. Thus, in this case we get that $B(A)$ is Morita equivalent to $A$. Moreover, the two-step duality functor $G$ is isomorphic to the identity functor.
9.2  An algebra with a non-trivial two-step duality

This example presents an SSS-algebra, $(A, \leq)$, such that the Ringel dual $(R, \leq_R)$ is a properly stratified algebra with $T(R) \neq C(R)$. Let $A$ be the quotient of the path algebra of the following quiver

modulo the relations $\gamma \beta = \alpha \gamma \alpha = 0$. We set $\Lambda = \{1 < 2\}$.

The radical filtrations of the projective module $P(\lambda)$, the standard module $\Delta(\lambda)$ and the proper standard module $\overline{\Delta}(\lambda)$, $\lambda = 1, 2$, look as follows:

and $\Delta(2) = P(2)$, $\Delta(1) = \overline{\Delta}(1) = L(1)$. Here we see that $A$ is an SSS-algebra, but not properly stratified.

The injective module $I(\lambda)$, the costandard module $\nabla(\lambda)$ and the proper costandard module $\overline{\nabla}(\lambda)$, $\lambda = 1, 2$, have the following socle filtrations:

and $\nabla(2) = I(2)$, $\nabla(1) = \overline{\nabla}(1) = L(1)$.
The modules $T(\lambda)$, $N(\lambda)$, $S(\lambda)$ and $H(\lambda)$ have the following radical filtrations:

$$
\begin{array}{c|c}
T(2) & H(1) \\
\hline
\begin{array}{c}
1 \\
\gamma \\
\alpha \\
2 \\
\beta \\
1 \\
\gamma \\
2 \\
\beta \\
1 \\
\end{array} & \begin{array}{c}
1 \\
\gamma \\
\alpha \\
1 \\
\gamma \\
2 \\
\beta \\
2 \\
\beta \\
1 \\
\end{array}
\end{array}
$$

and $T(1) = N(1) = L(1)$, $N(2) = \nabla(2)$, $S(1) = 0$, $S(2) = L(1) \oplus L(1)$, $H(2) = I(2)$.

Since $S(1)$ and $S(2)$ are in $\mathcal{F}(N)$, we can conclude, by Theorem 3, that the Ringel dual $(R, \leq_R)$ is properly stratified. Also $T(R) \neq C(R)$, because $H(1) \neq I(1)$.

By a straightforward calculation one gets that the Ringel dual $R$ is the quotient of the path algebra of the following quiver

modulo the relations $\gamma \beta = \gamma^2 = \alpha \beta = 0$. The projective modules over this algebra have the following radical filtrations:

$$
\begin{array}{c|c}
P^{(R)}(1) & P^{(R)}(2) \\
\hline
\begin{array}{c}
1 \\
\beta \\
2 \\
\gamma \\
2 \\
\alpha \\
1 \\
\beta \\
2 \\
\end{array} & \begin{array}{c}
2 \\
\alpha \\
1 \\
\beta \\
2 \\
\end{array}
\end{array}
$$

This algebra has dimension 8 and is properly stratified with respect to the opposite order $2 <_R 1$ on $\Lambda$.

By a straightforward calculation one gets that the two-step dual algebra $B(A)$ is the quotient of the path algebra of the following quiver

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modulo the relations $\gamma^2 = \gamma \beta = \beta \alpha = 0$. The projective modules over this algebra have the following radical filtrations:

$$P^{(B(A))}(1) \quad P^{(B(A))}(2)$$

$$\begin{array}{c}
1 \\
\downarrow \beta \\
2 \\
\downarrow \alpha \\
1
\end{array} \quad \begin{array}{c}
1 \\
\downarrow \beta \\
2 \\
\downarrow \alpha \\
1
\end{array}$$

The dimension of $B(A)$ is 7, while the dimension of $A$ is 11. Since both algebras are basic, it follows that $A$ and $B(A)^{\text{opp}}$ are neither isomorphic nor Morita equivalent.

### 9.3 A properly stratified algebra with duality whose two-step dual is not properly stratified

This example presents a properly stratified algebra $(A, \leq)$ having a simple preserving duality, $\circ$, whose Ringel dual $(R, \leq_R)$ is properly stratified, does not have any simple preserving duality and such that the two-step dual algebra $B(A)$ is not properly stratified.

Let $A$ be the quotient of the path algebra of the following quiver

modulo the relations $\gamma^2 = \delta^2 = \gamma \delta = \delta \gamma = \alpha \beta = 0$. We set $\Lambda = \{1 < 2\}$. The algebra $A$ is isomorphic to the opposite algebra via the antiautomorphism $\iota : A \to A$, defined by $\iota(e_i) = e_i$, $i = 1, 2$, $\iota(\alpha) = \beta$, $\iota(\beta) = \alpha$, $\iota(\gamma) = \delta$, $\iota(\delta) = \gamma$. Since $\iota$ stabilizes the primitive idempotents, it induces a simple preserving duality for $A$.

The radical filtrations of the projective module $P(\lambda)$, the standard module $\Delta(\lambda)$ and the proper standard module $\overline{\Delta}(\lambda)$, $\lambda = 1, 2$, look as follows:

$$P(1) \quad P(2) \quad \overline{\Delta}(2)$$

$$\begin{array}{c}
1 \\
\downarrow \alpha \\
2 \\
\downarrow \beta \\
\downarrow \delta \\
2
\end{array} \quad \begin{array}{c}
1 \\
\downarrow \beta \\
1
\end{array} \quad \begin{array}{c}
1 \\
\downarrow \beta \\
1
\end{array}$$

and $\Delta(2) = P(2)$, $\Delta(1) = \overline{\Delta}(1) = L(1)$. It follows that $A$ is properly stratified.
The injective module $I(\lambda)$, the costandard module $\nabla(\lambda)$ and the proper costandard module $\nabla(\lambda)$, $\lambda = 1, 2$, have the following socle filtrations (dual to the corresponding radical filtrations above):

$I(1)$  
\[
\begin{array}{c}
1 \\
\alpha \\
\gamma \\
\beta \\
1 \\
2 \\
\end{array}
\]

$I(2)$  
\[
\begin{array}{c}
1 \\
\alpha \\
\gamma \\
\beta \\
1 \\
2 \\
\end{array}
\]

$\nabla(2)$  
\[
\begin{array}{c}
1 \\
\alpha \\
\gamma \\
\beta \\
1 \\
2 \\
\end{array}
\]

and $\nabla(2) = I(2)$, $\nabla(1) = \nabla(1) = L(1)$.

The tilting module $T(2)$ has the following radical filtration:

$T(2)$  
\[
\begin{array}{c}
1 \\
\alpha \\
\gamma \\
\beta \\
1 \\
2 \\
\end{array}
\]

and $T(1) = L(1)$.

By a straightforward calculation one gets that the Ringel dual $R$ is the quotient of the path algebra of the following quiver

modulo the relations $d^2 = c^2 = de = ed = ea = db = ca = cb = cda = ceb = 0$ and $da = eb$.

The projective modules over this algebra have the following radical filtrations:

$P^{(R)}(1)$  
\[
\begin{array}{c}
a \\
1 \\
b \\
2 \\
d \\
e \\
1 \\
\end{array}
\]

$P^{(R)}(2)$  
\[
\begin{array}{c}
da \\
2 \\
1 \\
c \\
2 \\
e \\
1 \\
\end{array}
\]
This algebra is properly stratified with respect to the opposite order $2 <_R 1$ on $\Lambda$. Further it is easy to get the following equalities: \( \dim_k \text{Ext}^1_R(L^{(R)}(1), L^{(R)}(2)) = 2 \) and \( \dim_k \text{Ext}^1_R(L^{(R)}(2), L^{(R)}(1)) = 1 \) and hence $R$ does not have any simple preserving duality.

By a straightforward calculation one also gets that the two-step dual algebra $B(A)$ is the quotient of the path algebra of the following quiver

mod

modulo the relations $\beta_1 \gamma = \beta_2 \gamma = \beta_3 \gamma = \beta_4 \gamma = \gamma \alpha_1 = \gamma \alpha_2 = 0$, $\beta_3 \alpha_1 \beta_1 = \beta_4 \alpha_2 \beta_1 = 0$, $\beta_3 \alpha_1 \beta_1 = \beta_4 \alpha_2 \beta_2$, $\beta_3 \alpha_1 \beta_4 = \beta_4 \alpha_2 \beta_4 = \beta_3 \alpha_1 \beta_3 = \beta_4 \alpha_2 \beta_3 = 0$, $\beta_1 \alpha_1 = \beta_2 \alpha_1 = \beta_1 \alpha_2 = \beta_2 \alpha_2 = \beta_3 \alpha_2 = \beta_4 \alpha_1 = 0$ and $\gamma^2 = 0$. The projective modules over this algebra have the following radical filtrations:

$P^{(B(A))}(1)$

$P^{(B(A))}(2)$

One immediately sees that $B(A)^{\text{opp}}$ is an SSS-algebra, while $B(A)$ is not (since the trace of $P^{(B(A))}(2)$ in $P^{(B(A))}(1)$ is not a direct sum of several copies of $P^{(B(A))}(2)$). In particular, $B(A)$ is neither properly stratified nor has a simple preserving duality.
Let \( g \) be a semi-simple finite dimensional complex Lie algebra, \( U(g) \) be its universal enveloping algebra and \( Z(g) \) be the center of \( U(g) \). Denote by \( \mathcal{H} \) the category of all Harish-Chandra \( g \)-bimodules, that is finitely generated \( g \)-bimodules, which are direct sums of finite dimensional \( g \)-modules under the adjoint action of \( g \). Fix a positive integer \( n \) and two maximal ideals \( \chi \) and \( \xi \) of \( Z(g) \). Suppose for simplicity that \( \chi \) and \( \xi \) correspond to regular central characters and denote by \( \chi \mathcal{H}^n_\xi \) the full subcategory in \( \mathcal{H} \) which consists of all bimodules \( M \) satisfying \( M\xi^n = 0 \) and \( \chi^m M = 0 \), \( m \gg 0 \). We refer the reader to [Ja, Kapitel 6] and [So] for details. It is well-known, see for example [So, Section 5], that \( \chi \mathcal{H}^n_\xi \) is equivalent to the module category of some finite dimensional associative algebra \( A \). In [Ma, Section 5.6] it was shown that \( A \) is properly stratified. At the same time, if \( \text{rank}(g) > 1 \) and \( n > 1 \), it is easy to see that the cotilting modules for \( A \) have infinite projective dimension and hence are not tilting modules.

However, using the translation functors one can easily show that the Ringel dual of \( A \) is properly stratified. The simple bimodules in \( \chi \mathcal{H}^n_\xi \) are indexed by the elements of the Weyl group \( W \), see [Ja, Kapitel 6]. Let \( w_0 \) be the longest element in \( W \). Consider the tilting bimodule \( T(w_0) \) corresponding to \( w_0 \). Then all tilting bimodules in \( \chi \mathcal{H}^n_\xi \) are direct summands of \( F \otimes T(w_0) \) for some finite dimensional \( g \)-module \( F \). Let us construct the bimodules \( N(w) \), \( w \in W \), inductively. Set \( N(w_0) = T(w_0) \), and for \( w \in W \) and a simple reflection \( s \) such that the length of \( sw \) is smaller than the length of \( w \), let \( N(sw) \) be the cokernel of the adjunction morphism from \( N(w) \) to \( \theta_s(N(w)) \), where \( \theta_s \) denotes the translation functor through the \( s \)-wall. It is easy to see that this adjunction morphism is always injective in the situation above. Using the standard properties of the translation functors and the fact that all tilting bimodules are obtained by translating \( T(w_0) \), we obtain that all tilting bimodules are filtered by \( N(w) \), \( w \in W \). This implies that the Ringel dual \( R \) of \( A \) is properly stratified.

In fact, using Arkhipov’s functor, [Ar], one can show that \( R \) is isomorphic to \( A \). Iterating this we also obtain \( A \cong B(A) \). However, in this case the two-step duality functor is quite far from being trivial, because the injective \( A \)-modules have infinite projective dimension in general. In fact, the two-step duality functor does something remarkable, namely, it defines a covariant equivalence between the categories of \( \mathcal{P}(A) \) and \( \mathcal{I}(A) \). That these two categories are contravariantly equivalent follows from the existence of a simple preserving duality for \( A \) (the last comes from the equivalence of \( \chi \mathcal{H}^n_\xi \) with a block of the thick category \( \mathcal{O} \), [So, Theorem 3]). In particular, the category \( \mathcal{P}(A) \) happens to be equivalent to \( (\mathcal{P}(A) \otimes \mathcal{I}(A))^{opp} \). Obviously, this gives us a contravariant equivalence from \( \mathcal{P}(A) \) to itself, which sends \( P \) to \( H \). In particular, from Proposition 24 it follows that the finitistic dimension of \( \chi \mathcal{H}^n_\xi \) equals twice the projective dimension of the characteristic tilting module in \( \chi \mathcal{H}^n_\xi \).
9.5 A tensor construction for properly stratified algebras

In this section we present one general construction of properly stratified algebras using quasi-hereditary and local algebras. In this way one obtains a huge family of stratified algebras, moreover, there is an easy criterion when all tilting modules over such algebras are cotilting. In fact, this happens to be the case if and only if the local algebra we start from is self-injective. Thus this gives us a possibility of constructing series of properly stratified algebras for which tilting and cotilting modules do not coincide, hence showing that the two-step duality we worked out is not that rare.

We remark that any local algebra is properly stratified with simple proper standard and costandard modules, projective standard modules and injective costandard modules.

Throughout this subsection we fix a quasi-hereditary algebra, \((A, \leq)\), with \(A\) being an index set for the isomorphism classes of simple \(A\)-modules. Let further \(B\) be a fixed local algebra and consider the algebra \(D = A \otimes_k B\). Since \(B\) is local, \(A\) also indexes the isomorphism classes of simple \(D\)-modules in a natural way. Note that we have the following:

\[
\Delta^{(A)}(\lambda) = \Delta^{(A)}(\lambda), \quad \nabla^{(A)}(\lambda) = \nabla^{(A)}(\lambda) \quad \text{for all } \lambda.
\]

Moreover, \(P^{(B)} = \Delta^{(B)}, \quad I^{(B)} = \nabla^{(B)}, \quad \Delta^{(B)} = \nabla^{(B)} = L^{(B)}\).

**Proposition 35.** The algebra \((D, \leq)\) is properly stratified. Moreover, for each \(\lambda \in \Lambda\) we have

\[
X^{(D)}(\lambda) = X^{(A)}(\lambda) \otimes_k X^{(B)}, \quad \text{where } X \in \{\Delta, \Delta, \nabla, \nabla\}.
\]

**Proof.** Let \(\lambda \in \Lambda\). Then we obtain

\[
P^{(D)}(\lambda) = D(e_\lambda \otimes_k 1_B) = Ae_\lambda \otimes_k B1_B = P(\lambda) \otimes_k P^{(B)},
\]

where the idempotent \(e_\lambda \in A\) is chosen such that \(P^{(A)}(\lambda) = Ae_\lambda\). The functor \(\_ \otimes_k P^{(B)} : A\text{-mod} \rightarrow D\text{-mod}\) is exact (as the tensor product over a field) and if we apply \(\_ \otimes_k P^{(B)}\) to the short exact sequence

\[
0 \rightarrow K^{(A)} \rightarrow P^{(A)}(\lambda) \rightarrow \Delta^{(A)}(\lambda) \rightarrow 0,
\]

where \(K^{(A)}\) has a filtration with subquotients \(\Delta^{(A)}(\mu), \mu > \lambda\), we obtain the short exact sequence

\[
0 \rightarrow K^{(A)} \otimes_k P^{(B)} \rightarrow P^{(D)}(\lambda) \rightarrow \Delta^{(A)}(\lambda) \otimes_k P^{(B)} \rightarrow 0,
\]

where, using the exactness of \(\_ \otimes_k P^{(B)}\), \(K^{(A)} \otimes_k P^{(B)}\) has a filtration with subquotients \(\Delta^{(A)}(\mu) \otimes_k P^{(B)}, \mu > \lambda\). Exactness of \(\_ \otimes_k P^{(B)}\), \(L^{(A)}(\nu) \otimes_k \), \(\nu \in \Lambda\), and the equality

\[
L^{(D)}(\mu) = L^{(A)}(\mu) \otimes_k L^{(B)}\]

also implies that \([\Delta^{(A)}(\lambda) \otimes_k P^{(B)} : L^{(D)}(\mu)] = 0\) for \(\mu > \lambda\). Therefore we conclude that the property (SS) holds for \(A \otimes_k B\).

To show that (PS) holds, apply the exact functor \(\Delta^{(A)}(\lambda) \otimes_k - : B\text{-mod} \rightarrow D\text{-mod}\) to the short exact sequence

\[
0 \rightarrow K^{(B)} \rightarrow P^{(B)} \rightarrow L^{(B)} \rightarrow 0,
\]

where \(K^{(B)}\) has a filtration with simple subquotients \(L^{(B)}\), and obtain

\[
0 \rightarrow \Delta^{(A)}(\lambda) \otimes_k K^{(B)} \rightarrow \Delta^{(A)}(\lambda) \otimes_k P^{(B)} \rightarrow \Delta^{(A)}(\lambda) \otimes_k L^{(B)} \rightarrow 0,
\]

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where \( \Delta^A(\lambda) \otimes K^B \) has a filtration with subquotients \( \Delta^A(\lambda) \otimes_k L^B \). We conclude that property (PS) holds. By dual arguments it follows that \( I(D)(\lambda) = I(A)(\lambda) \otimes_k f(B) \) has a filtration with subquotients \( \nabla(D) = \nabla(A) \otimes_k \nabla(B) \), and \( \nabla(D) \) has a filtration with subquotients \( \nabla(D) = \nabla(A) \otimes_k L^B \). Hence the lemma is proved.

As an immediate corollary we obtain:

**Corollary 36.** The module \( \Delta(D) \) is projective and the module \( \nabla(D) \) is injective if viewed as \( B \)-module.

**Corollary 37.** We have:

1. \( T(D)(\lambda) = T(A)(\lambda) \otimes_k T(B) = T(A)(\lambda) \otimes_k P(B) \);
2. \( C(D)(\lambda) = C(A)(\lambda) \otimes_k C(B) = T(A)(\lambda) \otimes_k f(B) \).

**Proof.** Follows from Proposition 35 and exactness of the functors \( \cdot \otimes_k P(B) \) and \( \cdot \otimes_k f(B) \). \( \square \)

And, finally, we can formulate our main result in this section:

**Corollary 38.** The tilting modules for \( D \) are cotilting if and only if \( B \) is self-injective.

**Proof.** Since \( D \) is properly stratified, we have that \( T(D) = C(D) \) is equivalent to \( C(D) \in \mathcal{F}(\Delta(D)) \). By Lemma 15 the last is equivalent to \( \text{p.d.}(C(D)) < \infty \). From Corollary 36 and the fact that \( D \) is properly stratified we get that all projective \( D \)-modules are also projective as \( B \)-modules.

Thus \( \text{p.d.}(C(D)) < \infty \) implies that \( C(D) \) is a \( B \)-module of finite projective dimension, hence projective as \( B \) is local. But from Corollary 36 it also follows that \( C(D) \) is \( B \) injective and hence \( B \) is self-injective.

On the other hand, Corollary 37 implies that, when \( B \) is self-injective, then

\[
T(D) = T(A) \otimes_k T(B) = T(A) \otimes_k P(B) = C(A) \otimes_k I(B) = C(A) \otimes_k C(B) = C(D).
\]

This completes the proof. \( \square \)

## 10 Appendix: Two technical lemmas

In this appendix we prove two auxiliary technical lemmas similar to [MO, Lemma 4], which were used in the paper.

**Lemma 39.** Let \( \mathbf{A} \) be a finite-dimensional associative \( k \)-algebra, \( X^A \) be a \( \mathbf{A} \)-module, and \( Y^A \) be a (generalized) (co)tilting \( \mathbf{A} \)-module. Assume that \( X^A \) has a finite coresolution by modules from \( \text{add}(Y^A) \). Let \( X^* \) be a positive complex in \( \mathcal{K}(\text{add}(X^A)) \). Then there is a positive complex, \( Y^* \), in \( \mathcal{K}(\text{add}(Y^A)) \) such that \( X^* \) is quasi-isomorphic to \( Y^* \). Moreover, if \( \lambda^* \in \mathcal{K}^b(\text{add}(X^A)) \), then \( \gamma^* \in \mathcal{K}^b(\text{add}(Y^A)) \).

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Proof. Let $X^\bullet : \cdots \to 0 \to X^{(a)}_0 \to X^{(a)}_1 \to \cdots$ be a positive complex in $\mathcal{K}(\text{add}(X^{(a)}))$ and put $X^\bullet_j = t_j X^\bullet$. We will show by induction that for each $j \geq 0$ there exist a complex, $Y^\bullet_j \in \mathcal{K}(\text{add}(Y^{(a)}))$, and a quasi-isomorphism, $\Phi_j : X^\bullet_j \to Y^\bullet_j$. Moreover, we will choose the family $\{Y^\bullet_j\}_{j \geq 0}$ such that for all $k < j$ we have $t_k Y^\bullet_j = t_k Y^\bullet_k$.

Since $X^{(a)}_i \in \text{add}(X^{(a)})$ for all $i$ we can choose by our assumptions a coresolution, $Z^\bullet_i \in \mathcal{K}(\text{add}(Y^{(a)}))$, of $X^{(a)}_i$, and a quasi-isomorphism, $\varphi_i : (X^{(a)}_i)^\bullet \to Y^\bullet_i$. In the case $j = 0$ we put $Y^\bullet_0 = Z^\bullet_0$ and we are done.

Now suppose by induction that there exists a quasi-isomorphism, $\Phi_{j-1} : X^\bullet_{j-1} \to Y^\bullet_{j-1}$. The map $d_{j-1} : X^{(a)}_{j-1} \to X^{(a)}_j$ induces the distinguished triangle

$$X^\bullet_{j-1} \overset{d^\bullet_{j-1}}{\longrightarrow} (X^{(a)}_j)^\bullet[-j + 1] \to \text{Cone}(d^\bullet_{j-1}) \to X^\bullet_{j-1}[1],$$

and we have $\text{Cone}(d^\bullet_{j-1}) = X^\bullet_j[1]$.

Using $\text{Ext}_4(Y^{(a)}, Y^{(a)}) = 0$ and [Ha, Chap.III, Lemma 2.1] we can choose a representative, $\psi_j : Y^\bullet_{j-1} \to Z^\bullet_j[-j + 1]$, in $\mathcal{K}(\text{add}(Y^{(a)}))$ of the composition $\varphi_j[-j + 1] \circ d^\bullet_{j-1} \circ \Phi_{j-1}^{-1}$ (in $\mathcal{D}^b(\mathfrak{A} \text{-mod})$). This gives us a diagram in $\mathcal{D}^b(\mathfrak{A} \text{-mod})$, which can be completed to the following commutative diagram:

$$\begin{array}{ccc}
X^\bullet_{j-1} & \overset{d^\bullet_{j-1}}{\longrightarrow} & (X^{(a)}_j)^\bullet[-j + 1] \\
\downarrow \Phi_{j-1} & & \downarrow \varphi_j[-j + 1] \\
Y^\bullet_{j-1} & \overset{\psi_j}{\longrightarrow} & Z^\bullet_j[-j + 1] \\
\downarrow \Phi_{j-1} & & \downarrow \Phi_{j-1}[1] \\
Y^\bullet_{j-1} & \longrightarrow & \text{Cone}(\psi_j) \longrightarrow Y^\bullet_{j-1}[1].
\end{array}$$

Since both $\Phi_{j-1}$ and $\varphi_j[-j + 1]$ are isomorphisms the morphism $\Phi_j$ is an isomorphism too. Hence we get the quasi-isomorphism $\Phi_j : X^\bullet_j \to \text{Cone}(\psi_j)[-1]$, where $\text{Cone}(\psi_j)[-1]$ is a positive complex in $\mathcal{K}^b(\text{add}(Y^{(a)}))$ with the property $t_{j-1} \text{Cone}(\psi_j)[-1] = t_{j-1} Y^\bullet_{j-1}$. Set $Y^\bullet_j = \text{Cone}(\psi_j)[-1]$ and the induction follows.

Define the limit complex $Y^\bullet \in \mathcal{K}(\text{add}(X^{(a)}))$ by $t_j Y^\bullet = t_j Y^\bullet_j$ for all $j \geq 0$ (that for all $k < j$ we have $t_k Y^\bullet_j = t_k Y^\bullet_k$, guarantees that $Y^\bullet$ is well-defined). Moreover, we have a quasi-isomorphism $\Phi : Y^\bullet \to X^\bullet$. Hence the general statement is true and we see by the construction that $Y^\bullet$ is bounded, whenever $X^\bullet$ is. \hfill \Box

**Lemma 40.** Let $\mathfrak{A}$ be a finite-dimensional associative algebra, $X^{(a)}$ be a $\mathfrak{A}$-module and $Y^{(a)}$ be a (generalized) (co)tilting $\mathfrak{A}$-module. Assume that $X^{(a)}$ admits a (possibly infinite) resolution by modules from $\text{add}(Y^{(a)})$. Then for every negative complex $X^\bullet \in \mathcal{K}^b(\text{add}(X^{(a)}))$ there exists a (possibly infinite) negative complex, $Y^\bullet \in \mathcal{K}(\text{add}(Y^{(a)}))$, which is quasi-isomorphic to $X^\bullet$.

**Proof.** The statement is proved by induction analogous to that used in the previous lemma. Moreover, since we start with a finite complex from the very beginning, no truncation is needed. We leave the details out. \hfill \Box
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