A Note On $G$-normal Distributions

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Abstract

As is known, the convolution $\mu * \nu$ of two $G$-normal distributions $\mu, \nu$ with different intervals of variances may not be $G$-normal. We shows that $\mu * \nu$ is a $G$-normal distribution if and only if $\frac{\sigma_\mu}{\sigma_\nu} = \frac{\sigma_\nu}{\sigma_\mu}$.

Key words: $G$-normal distribution, Cramer’s decomposition

MSC-classification: 60E10

1 Introduction

Peng (2007) introduced the notion of $G$-normal distribution via the viscosity solutions of the $G$-heat equation below

$$\partial_t u - G(\partial^2_x u) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R},$$

$$u(0, x) = \varphi(x),$$

where $G(a) = \frac{1}{2}(\sigma^2 x^+ - \sigma^2 x^-)$, $a \in \mathbb{R}$ with $0 \leq \underline{\sigma} \leq \overline{\sigma} < \infty$, and $\varphi \in C_{b, Lip}(\mathbb{R})$, the collection of bounded Lipschitz functions on $\mathbb{R}$.

Then the one-dimensional $G$-normal distribution is defined by

$$N_G[\varphi] = u^\varphi(1, 0),$$

where $u^\varphi$ is the viscosity solution to the $G$-heat equation with the initial value $\varphi$. We denote by $\mathcal{G}^0$ the collection of functions $G$ defined above. For $G_1, G_2 \in \mathcal{G}^0$, the convolution between $N_{G_1}, N_{G_2}$ is defined as $N_{G_1} * N_{G_2} [\varphi] := N_{G_1}[\phi]$ with $\phi(x) := N_{G_2}[\varphi(x + \cdot)]$.

As is well-known, the convolution of two normal distributions is also normal. How about $G$-normal distributions? We state the question in the PDE language: For $G_1, G_2 \in \mathcal{G}^0$, set $G(t, a) = G_2(a)$, $t \in (0, \frac{1}{2}]$ and $G(t, a) = G_1(a)$, $t \in (\frac{1}{2}, 1]$. Let $v^\varphi$ be the viscosity solution to the following PDE

$$\partial_t v - G(t, \partial^2_x v) = 0,$$

$$v(0, x) = \varphi(x).$$

Can we find a function $G \in \mathcal{G}^0$ such that

$$v^\varphi(1, 0) = u^\varphi(1, 0), \quad \text{for all } \varphi \in C_{b, Lip}(\mathbb{R}).$$
Here $u^\varphi$ is the viscosity solution to the $G$-heat equation. However, [2] gave a counterexample to show that generally $N_{G_1} * N_{G_2}$ is not $G$-normal any more.

We denote by $\mathcal{G}$ the subset of $\mathcal{G}^0$ consisting of the elements with $\sigma > 0$. $N_G$ is called non-degenerate if $G$ belongs to $\mathcal{G}$. For $G \in \mathcal{G}$, set

$$\beta_G := \frac{\sigma}{\sigma} \text{ and } \sigma_G = \frac{\sigma + \sigma}{2}. $$

For abbreviation, we write $\beta, \sigma$ instead of $\beta_G, \sigma_G$ when no confusion can arise.

We shall prove that $N_{G_1} * N_{G_2}$ is a $G$-normal distribution if and only if $\beta_{G_1} = \beta_{G_2}$. Besides, the convolution between $G$-normal distributions is not commutative. We also prove that $N_{G_1} * N_{G_2} = N_{G_2} * N_{G_1}$ if and only if $\beta_{G_1} = \beta_{G_2}$.

Therefore, in order to emphasize the importance of the ratio $\beta$, we denote the $G$-normal distribution $N_G$ by $N_{\beta}(0, \sigma^2)$.

## 2 Characteristic Functions

First we shall consider the solutions of special forms to the $G$-heat equation

$$\partial_t u - G(\partial^2_x u) = 0. \tag{2.1}$$

Assume that $u(t, x) = a(t)\phi(x)$ with $a(t) \geq 0$ is a solution to the $G$-heat equation (2.1). Then we conclude that

$$\frac{a'(t)}{a(t)} = \frac{G(\phi''(x))}{\phi(x)}$$

is a constant. Assuming that they are equal to $-\rho^2$, we have $a(t) = e^{\frac{-\rho^2}{2}t}$ and

$$G(\phi''(x)) = -\frac{\rho^2}{2} \phi(x). \tag{2.2}$$

If $\phi''(x)$ is positive, the equation (2.2) reduces to

$$\phi''(x) = -\frac{\rho^2}{\sigma^2} \phi(x). \tag{2.3}$$

and the solution is

$$\phi(x) = \lambda \cos(\frac{\rho}{\sigma} x + \varphi).$$

On the other hand, if $\phi''(x)$ is negative, the equation (2.2) reduces to

$$\phi''(x) = -\frac{\rho^2}{\sigma^2} \phi(x). \tag{2.4}$$

and the solution is

$$\phi(x) = \lambda \cos(\frac{\rho}{\sigma} x + \varphi).$$

Motivated by the arguments above, we shall construct the solutions to the equation (2.2) in the following way. Denoting by $\beta = \frac{\rho}{\sigma}$, set

$$\phi_\beta(x) = \begin{cases} \frac{2}{1+\beta} \cos(\frac{1+\beta}{2\beta} x) & \text{for } x \in [-\frac{\pi}{1+\beta}, \frac{\pi}{1+\beta}); \\
\frac{2\beta}{1+\beta} \cos(\frac{1+\beta}{2\beta} x + \frac{\beta-1}{2\beta} \pi) & \text{for } x \in [\frac{\pi}{1+\beta}, -\frac{(2\beta+1)\pi}{1+\beta}). \end{cases} \tag{2.5}$$
This is a variant of the trigonometric function \( \cos x \) (see Figure 1).

Then extend the definition of \( \phi_\beta \) to the whole real line by the property \( \phi(x + 2k\pi) = \phi(x), k \in \mathbb{Z} \). Clearly, \( \phi_1(x) = \cos x \) and \( \phi_\beta \) belongs to \( C^{2,1}(\mathbb{R}) \), the space of bounded functions on \( \mathbb{R} \) with uniformly Lipschitz continuous second-order derivatives.

**Proposition 2.1** \( \phi_\beta \) is a solution to equation (2.2) with \( \rho = \sigma^2 : = \sigma \).

**Proof.** The proof follows from simple calculations. For \( x \in (-\frac{\pi}{1+\beta}, \frac{\pi}{1+\beta}) \),

\[
\phi''_\beta(x) = -\frac{1 + \beta}{2} \cos\left(\frac{1 + \beta}{2} x\right) = -\frac{(1 + \beta)^2}{4} \phi_\beta(x) \leq 0.
\]

So

\[
G(\phi''_\beta(x)) = -\frac{(1 + \beta)^2}{8} \phi_\beta(x) = -\frac{\sigma^2}{2} \phi_\beta(x).
\]

For \( x \in [-\frac{\pi}{1+\beta}, \frac{(2\beta+1)\pi}{1+\beta}) \),

\[
\phi''_\beta(x) = -\frac{1 + \beta}{2\beta} \cos\left(\frac{1 + \beta}{2\beta} x + \frac{\beta - 1}{2\beta} \pi\right) = -\frac{(1 + \beta)^2}{4\beta^2} \phi_\beta(x) \geq 0.
\]

So

\[
G(\phi''_\beta(x)) = -\frac{(1 + \beta)^2}{8\beta^2} \phi_\beta(x) = -\frac{\sigma^2}{2} \phi_\beta(x).
\]

\[\square\]

**Corollary 2.2** Let \( \sigma := \frac{\sigma + \pi}{2} \) and \( \beta := \frac{\pi}{2} \). \( e^{-\frac{\sigma^2}{2} t} \phi_\beta(x) \) is the solution to equation (2.1) with the initial value \( \phi_\beta(x) \). In other words, \( N_G[\phi_\beta(x + \sqrt{t})] = e^{-\frac{\sigma^2}{2} t} \phi_\beta(x) \).

For \( \lambda > 0 \) and \( c, \theta \in \mathbb{R} \), set \( \phi_\beta^{\lambda,c,\theta}(x) := \lambda \phi_\beta(cx + \theta) \). It’s easy to check that

\[
G((\phi_\beta^{\lambda,c,\theta})'') = -\frac{c^2 \sigma^2}{2} \phi_\beta^{\lambda,c,\theta}.
\]

So \( e^{-\frac{c^2 \sigma^2}{2} t} \phi_\beta^{\lambda,c,\theta}(x) \) is the solution to equation (2.1) with the initial value \( \phi_\beta^{\lambda,c,\theta}(x) \). For any \( \beta > 1 \), we call

\[
(\phi_\beta^{\lambda,c,\theta}(x))_{\lambda,c,\theta}
\]

the characteristic functions of \( G \)-normal distributions \( N_\beta(0, \sigma^2) \).
3 Cramer’s Type Decomposition for $G$-normal Distributions

Let’s introduce more properties on the characteristic functions $\{\phi_\beta(x)\}_{\beta \geq 1}$. For $1 \leq \alpha < \beta < \infty$, we have

$$\phi_\alpha(x) - \phi_\beta(x) \geq \frac{2(\beta - \alpha)}{(1 + \alpha)(1 + \beta)} =: e(\alpha, \beta) > 0.$$  

**Theorem 3.1** For $G_1, G_2 \in \mathcal{G}$, the convolution $N_{G_1} * N_{G_2}$ is a $G$-normal distribution if and only if $\beta_{G_1} := \frac{\sigma_{G_1}}{\sigma_{G_1}} = \frac{\sigma_{G_2}}{\sigma_{G_2}} =: \beta_{G_2}$.

**Proof.** The sufficiency is obvious, and we only prove the necessity. Assume $N := N_{G_1} * N_{G_2}$ is a $G$-normal distribution. Then $\sigma_N^2 = \sigma_{G_1}^2 + \sigma_{G_2}^2$ and $\sigma_N^2 = \sigma_{G_1}^2 + \sigma_{G_2}^2$. If $\beta_{G_1} \neq \beta_{G_2}$, we may assume that $\beta_{G_1} < \beta_{G_2}$, and the other case can be proved similarly. Then we have $\beta_{G_1} < \beta_N < \beta_{G_2}$. On one hand we have

$$N[\phi_{\beta_N}(\sqrt{t} \cdot)] = e^{-\frac{\sigma_N^2}{2} t} \phi_{\beta_N}(0) = \frac{2}{1 + \beta_N} e^{-\frac{\sigma_N^2}{2} t}.$$  

On the other hand we have

$$N_{G_2}[\phi_{\beta_{G_2}}(\sqrt{t} x + \cdot)] 
\geq N_{G_2}[\phi_{\beta_{G_2}}(\sqrt{t} x + \cdot)] + e(\beta_N, \beta_{G_2})
= e^{-\frac{\sigma_{G_2}^2}{2} t} \phi_{\beta_{G_2}}(\sqrt{t} x) + e(\beta_N, \beta_{G_2})
\geq -\frac{2\beta_{G_2}}{1 + \beta_{G_2}} e^{-\frac{\sigma_{G_2}^2}{2} t} + e(\beta_N, \beta_{G_2}).$$  

Consequently,

$$N[\phi_{\beta_N}(\sqrt{t} \cdot)] = N_{G_1} * N_{G_2}[\phi_{\beta_N}(\sqrt{t} \cdot)] \geq -\frac{2\beta_{G_2}}{1 + \beta_{G_2}} e^{-\frac{\sigma_{G_2}^2}{2} t} + e(\beta_N, \beta_{G_2}).$$  

Combining the above arguments we have

$$\frac{2}{1 + \beta_N} e^{-\frac{\sigma_N^2}{2} t} \geq -\frac{2\beta_{G_2}}{1 + \beta_{G_2}} e^{-\frac{\sigma_{G_2}^2}{2} t} + e(\beta_N, \beta_{G_2}), \text{ for any } t > 0,$$

which is a contradiction noting that $e(\beta_N, \beta_{G_2}) > 0$. $\square$
Theorem 3.2 For $G_1, G_2 \in \mathcal{G}$, $N_{G_1} \ast N_{G_2} = N_{G_2} \ast N_{G_1}$ if and only if $\beta_{G_1} = \beta_{G_2}$.

Proof. We shall only prove the necessity. Assume that $\beta_{G_1} < \beta_{G_2}$. Then

$$N_{G_1}[\phi_{\beta_{G_1}}(\sqrt{t} \cdot)] = e^{-\frac{\sigma_{G_1}^2}{2}t} \phi_{\beta_{G_1}}(\sqrt{t} \cdot) \leq \frac{2}{1 + \beta_{G_1}} e^{-\frac{\sigma_{G_1}^2}{2}t}.$$

So

$$N_{G_2} \ast N_{G_1}[\phi_{\beta_{G_1}}(\sqrt{t} \cdot)] \leq \frac{2}{1 + \beta_{G_1}} e^{-\frac{\sigma_{G_1}^2}{2}t}.$$

On the other hand

$$N_{G_2}[\phi_{\beta_{G_1}}(\sqrt{t} \cdot)] \geq N_{G_2}[\phi_{\beta_{G_2}}(\sqrt{t} \cdot)] + e(\beta_{G_1}, \beta_{G_2})$$

$$= e^{-\frac{\sigma_{G_2}^2}{2}t} \phi_{\beta_{G_2}}(\sqrt{t} \cdot) + e(\beta_{G_1}, \beta_{G_2})$$

$$\geq -\frac{2\beta_{G_2}}{1 + \beta_{G_2}} e^{-\frac{\sigma_{G_2}^2}{2}t} + e(\beta_{G_1}, \beta_{G_2}),$$

which implies that

$$N_{G_1} \ast N_{G_2}[\phi_{\beta_{G_1}}(\sqrt{t} \cdot)] \geq -\frac{2\beta_{G_2}}{1 + \beta_{G_2}} e^{-\frac{\sigma_{G_2}^2}{2}t} + e(\beta_{G_1}, \beta_{G_2}).$$

Noting that $e(\beta_{G_1}, \beta_{G_2}) > 0$, we have

$$N_{G_1} \ast N_{G_2}[\phi_{\beta_{G_1}}(\sqrt{t} \cdot)] > N_{G_2} \ast N_{G_1}[\phi_{\beta_{G_1}}(\sqrt{t} \cdot)] \text{ for } t \text{ large enough.}$$

□

References

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[2] Hu, M. (2010) *Nonlinear Expectations and Related Topics*, Doctoral thesis.