On the order derivatives of Bessel functions

T. M. Dunster*

August 5, 2016

Abstract

The derivatives with respect to order for the Bessel functions $J_\nu(x)$ and $Y_\nu(x)$, where $\nu > 0$ and $x \neq 0$ (real or complex), are studied. Representations are derived in terms of integrals that involve the products of Bessel functions, and in turn series expansions are obtained for these integrals. From the new integral representations for $\partial J_\nu(x)/\partial \nu$ and $\partial Y_\nu(x)/\partial \nu$, asymptotic approximations involving Airy functions are constructed for the case $\nu$ large, which are uniformly valid for $0 < x < \infty$.

1 Introduction

Due to their importance in mathematics and physics, Bessel functions have been extensively studied in the literature. Their main properties can be found by perusing various classic textbooks on the subject ([1], [7], [11], [13], [16]), and, most recently, [14, Chap. 10]. The purpose of this paper is to study the order derivatives of Bessel functions. The literature is relatively sparse on the properties of these functions, with none currently existing on their uniform asymptotic properties. We remark that the order derivatives are used in the study of the monotonicity with respect to order of Bessel functions, which in turn has applications in quantum mechanics [9].

For some integral representations see [3] and [12] (the latter for the special case $\nu = \pm \frac{1}{2}$), and for certain series representations see [4], [8, §8.486(1)], [10, §9.4.4] and [15]. We also mention that in [5] modified Bessel functions play a role in the study of the order derivatives of Legendre functions.

In this paper we derive new integral representations for the derivatives with respect to order of Bessel and Hankel functions (§2), new series expansions for the integrals appearing in these representations (§3), and uniform asymptotic approximations for the order derivatives of $J_\nu(x)$ and $Y_\nu(x)$ (§§4 and 5).

*Department of Mathematics and Statistics, San Diego State University, San Diego, CA 92182-7720, U.S.A. email: mdunster@mail.sdsu.edu
2 Integral representations for order derivatives of Bessel functions

Our first result is as follows.

Proposition 1 For \( \nu > 0 \), \(|\arg(z)| \leq \pi\), and \( z \neq 0 \)

\[
\frac{\partial J_\nu(z)}{\partial \nu} = \nu \pi Y_\nu(z) \int_0^z \frac{J_\nu^2(t)}{t} \, dt + \nu \pi J_\nu(z) \int_z^\infty \frac{J_\nu(t) Y_\nu(t)}{t} \, dt.
\]

(1)

If \( z \) is complex the paths of integration must lie in \( C \setminus (-\infty, 0] \) (except at the end point for the first integral), and with the end point at infinity in the second integral on the positive real axis.

Proof. For convenience, here and throughout we denote

\[
\hat{J}_\nu(z) = \frac{\partial J_\nu(z)}{\partial \nu}, \quad \hat{Y}_\nu(z) = \frac{\partial Y_\nu(z)}{\partial \nu}.
\]

Then, from the \( \nu \) derivative of Bessel’s equation, we have

\[
z^2 \frac{d^2 \hat{J}_\nu(z)}{dz^2} + z \frac{d \hat{J}_\nu(z)}{dz} + (z^2 - \nu^2) \hat{J}_\nu(z) = 2 \nu J_\nu(z).
\]

(2)

We then apply the method of variation of parameters, to obtain

\[
\hat{J}_\nu(z) = \nu \pi Y_\nu(z) \int_0^z t^{-1} J_\nu^2(t) \, dt + \nu \pi J_\nu(z) \int_z^\infty t^{-1} J_\nu(t) Y_\nu(t) \, dt \\
+ c_\nu J_\nu(z) + d_\nu Y_\nu(z),
\]

(3)

for certain constants \( c_\nu \) and \( d_\nu \). Recalling the behavior of Bessel functions as \( z \to 0^+ \) (see for example, \([14, \text{Eqs. (10.7.3, (10.7.4) and (10.15.1))}]\)), we observe that the LHS is bounded, whereas the RHS is bounded if and only if \( d_\nu = 0 \), which we conclude must be the case.

To determine \( c_\nu \) we compare both sides as \( z \to \infty \). To do so, we use the well-known approximations \([14, \text{\S 10.17(i)}]\)

\[
\left( \frac{\pi z}{2} \right)^{1/2} J_\nu(z) = \cos \left( z - \frac{1}{2} \nu \pi - \frac{1}{4} \pi \right) \left\{ 1 - \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{128z^2} \right\} \]

\[- \sin \left( z - \frac{1}{2} \nu \pi - \frac{1}{4} \pi \right) \frac{4\nu^2 - 1}{8z} + O \left( \frac{1}{z^3} \right),
\]

(4)

and

\[
\left( \frac{\pi z}{2} \right)^{1/2} Y_\nu(z) = \sin \left( z - \frac{1}{2} \nu \pi - \frac{1}{4} \pi \right) \left\{ 1 - \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{128z^2} \right\} \]

\[+ \cos \left( z - \frac{1}{2} \nu \pi - \frac{1}{4} \pi \right) \frac{4\nu^2 - 1}{8z} + O \left( \frac{1}{z^3} \right).
\]

(5)
Taking the \(\nu\) derivative of (4), we have (formally) for the LHS of (3)

\[
\hat{J}_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} \sin \left(z - \frac{1}{2} \nu \pi - \frac{1}{4} \pi\right) \quad (z \to \infty).
\]

This can be verified, for example, by a straightforward modification of the saddle point method found in [13, Chap. 4, §9.1] to the integral representation

\[
\hat{J}_\nu(z) = -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{t}{z} \exp \left\{-z \sinh (t) + \nu t\right\} dt \quad (|\arg z| < \frac{1}{2} \pi).
\]

On the other hand, from (4) and (5), we find the RHS of (3) takes the form

\[
\left(\frac{\pi}{2z}\right)^{1/2} \sin \left(z - \frac{1}{2} \nu \pi - \frac{1}{4} \pi\right) + c_\nu \left(\frac{2}{\pi z}\right)^{1/2} \cos \left(z - \frac{1}{2} \nu \pi - \frac{1}{4} \pi\right) + O\left(\frac{1}{z^{3/2}}\right).
\]

In arriving at this we have used [16, p. 405]

\[
\int_{0}^{\infty} \frac{J^2_\nu(t)}{t} dt = \frac{1}{2\nu}.
\]

Comparing (6) with (7) we deduce that \(c_\nu = 0\), and the result (1) follows.

**Proposition 2** Under the same conditions as Proposition 1.1

\[
\frac{\partial Y_\nu(z)}{\partial \nu} = \nu \pi J_\nu(z) \int_{z}^{\infty} \frac{Y^2_\nu(t)}{t} dt - \nu \pi Y_\nu(z) \int_{z}^{\infty} \frac{J_\nu(t) Y_\nu(t)}{t} dt - \frac{1}{2} \pi J_\nu(z).
\]

**Proof.** From [14, Eq. (10.22.6)] we have, for \(\mu \neq \pm \nu\),

\[
\int C_\mu(z) D_\nu(z) \frac{dz}{z} = z \left\{C_\mu(z) D_{\nu+1}(z) - C_{\nu+1}(z) D_\nu(z)\right\} + \frac{C_\mu(z) D_\nu(z)}{\mu + \nu} + C,
\]

where \(C\) is some constant. We first choose \(C = J\) and \(D = Y\), and we shall select \(C\) so that the RHS is finite as \(\mu \to \nu\). To do so, note from [14, Eq. (10.5.2)]

\[
J_{\nu+1}(z) Y_\nu(z) - J_\nu(z) Y_{\nu+1}(z) = \frac{2}{\pi z}.
\]

Thus the choice

\[
C = \frac{2}{\pi (\mu^2 - \nu^2)} + C_0 = \frac{z \left\{J_{\nu+1}(z) Y_\nu(z) - J_\nu(z) Y_{\nu+1}(z)\right\}}{\mu^2 - \nu^2} + C_0,
\]

ensures that the RHS of (10) is indeed finite as \(\mu \to \nu\), for any bounded \(C_0\). Thus we can assert that

\[
\int_{z}^{\infty} \frac{J_\mu(t) Y_\nu(t)}{t} dt = -\frac{z Y_{\nu+1}(z)}{\mu + \nu} \left\{J_\mu(z) - J_\nu(z)\right\} + \frac{z Y_\nu(z)}{\mu + \nu} \left\{J_{\mu+1}(z) - J_{\nu+1}(z)\right\} - \frac{1}{\mu + \nu} + C_0.
\]
where \( C_0 \) is now a specific constant which must be determined. To do so, we let \( \mu \to \nu \), and (12) then becomes

\[
\int_\infty^\infty \frac{J_\nu(t)Y_\nu(t)}{t} dt = \frac{z}{2\nu} \left[ \frac{\partial J_{\nu+1}(z)}{\partial \nu} Y_\nu(z) - \frac{\partial J_\nu(z)}{\partial \nu} Y_{\nu+1}(z) \right] - \frac{J_\nu(z)Y_\nu(z)}{2\nu} + C_0. \tag{13}
\]

Letting \( z \to \infty \) with \( \nu \) fixed, we find with the aid of (4) and (5) that \( C_0 = 0 \).

Moreover, it is seen that

\[
\int_\infty^\infty \frac{J_\nu(t)Y_\nu(t)}{t} dt = \frac{\sin(2z - \nu\pi)}{2\pi z^2} + O\left(\frac{1}{z^3}\right) \quad (z \to \infty), \tag{14}
\]

for each fixed \( \nu \).

We next interchange \( J \) and \( Y \) in (13), to obtain

\[
\int_\infty^\infty \frac{J_\nu(t)Y_\nu(t)}{t} dt = \frac{z}{2\nu} \left[ \frac{\partial Y_{\nu+1}(z)}{\partial \nu} J_\nu(z) - \frac{\partial Y_\nu(z)}{\partial \nu} J_{\nu+1}(z) \right] - \frac{J_\nu(z)Y_\nu(z)}{2\nu}. \tag{15}
\]

We can obtain a similar expression by replacing \( J \) by \( Y \) in (13), noting that the value of \( C_0 \) may differ. In fact, from comparing both sides of (13) as \( z \to \infty \) and referring to (4) and (5) we find this time that \( C_0 = 1/(2\nu) \). Consequently, we arrive at

\[
\int_\infty^\infty \frac{Y_{\nu+1}(t)}{t} dt = \frac{z}{2\nu} \left[ \frac{\partial Y_{\nu+1}(z)}{\partial \nu} Y_\nu(z) - \frac{\partial Y_\nu(z)}{\partial \nu} Y_{\nu+1}(z) \right] - \frac{Y_{\nu+1}(z)}{2\nu}. \tag{16}
\]

Finally, eliminating \( \partial Y_{\nu+1}(x)/\partial \nu \) from (15) and (16), and using (11), yields (9).

Using the behavior of Hankel functions at infinity, and in a similar manner to the proof of (1), one can show that the following result holds. Details of the proof are left to the reader.

**Lemma 3** For \( \nu > 0 \) and \( z \neq 0 \)

\[
\frac{\partial H^{(1)}_\nu(z)}{\partial \nu} = \frac{1}{2}\pi i \left[ \nu H^{(1)}_\nu(z) \int_\infty^\infty t^{-1} H^{(1)}_\nu(t) H^{(2)}_\nu(t) dt - \nu H^{(2)}_\nu(z) \int_\infty^\infty t^{-1} \left( H^{(1)}_\nu(t) \right)^2 dt - H^{(1)}_\nu(z) \right]. \tag{17}
\]

Likewise, for \( \partial H^{(2)}_\nu(z)/\partial \nu \), in (17) interchange the superscripts (1) and (2) and replace \( i \) by \( -i \) throughout.

## 3 Series expansions for integrals involving products of Bessel functions

Here we obtain a number of series expansions for the integrals appearing in §2. While power series expansions exist for \( J_\nu(z) \) and \( Y_\nu(z) \) themselves (see
[14, §10.15], these are only useful for small \( z \), or for special values of \( \nu \). Also, expansions of the integrals themselves is of importance in scattering theory, as noted below.

Our results appear to be new, and are complementary to certain results appearing in the literature. In particular, from [16, p. 152], we have the following expansion.

**Lemma 4** For positive \( x \) and \( \nu \)

\[
\int_0^x \frac{J_\nu^2(t)}{t} \, dt = \frac{1}{2\nu} \sum_{n=0}^\infty \varepsilon_n J_{\nu + n}^2(x),
\]

(18)

where \( \varepsilon_0 = 1 \) and \( \varepsilon_n = 2 \) otherwise.

This result was generalized in [6]; in that paper it is shown how series of this type allows one to write the spherical harmonic component of the Coulomb kernel \(|r - r'|^{-1}\) as a numerically stable series.

The integral in (18) diverges when \( \nu = 0 \). However, from (8), we see that (18) can be re-expressed in the form

\[
\int_0^\infty \frac{J_\nu^2(t)}{t} \, dt = \frac{1}{2\nu} - \frac{1}{2\nu} \sum_{n=0}^\infty \varepsilon_n J_{\nu + n}^2(x).
\]

The integral here now converges when \( \nu = 0 \) for each positive \( x \), and limiting value of the RHS can therefore be found as \( \nu \to 0 \). In particular, using L’Hopital’s rule, we find that

\[
\int_0^\infty \frac{J_0^2(t)}{t} \, dt = - \sum_{n=0}^\infty \varepsilon_n J_n(x) \hat{J}_n(x).
\]

(19)

To obtain a similar result to (18) for the integrals appearing in (1) and (9), we shall require the following.

**Lemma 5** For each positive fixed \( x \)

\[
\int_x^\infty \frac{J_\nu(t)}{t} Y_\nu(t) \, dt = \frac{1}{\nu \pi} \ln \left( \frac{x}{2\nu} \right) + O \left( \frac{1}{\nu^3} \right),
\]

(20)

as \( \nu \to \infty \).

**Proof.** Using

\[
J_\nu(x) = \left( \frac{x}{2} \right)^\nu \sum_{k=0}^\infty (-1)^k \frac{\left( \frac{1}{2} \right)^k}{k! \Gamma(\nu + k + 1)},
\]

(21)

and, assuming temporarily that \( \nu \) is not an integer,

\[
Y_\nu(x) = \frac{J_\nu(x) \cos(\nu \pi) - J_{-\nu}(x)}{\sin(\nu \pi)}.
\]

(22)

we find from (13) that (20) holds. By continuity the condition \( \nu \) not an integer can now be relaxed.
Proposition 6 For positive $x$, and $\nu \neq 0, -1, -2, -3, \cdots$

\[
\int_{x}^{\infty} \frac{J_{\nu} (t) Y_{\nu} (t)}{t} \, dt = \frac{\ln \left( \frac{1}{\pi} x \right) - \psi (\nu)}{\nu \pi} - \frac{1}{2\nu} \sum_{n=0}^{\infty} \left\{ \varepsilon_{n} J_{\nu+n} (x) Y_{\nu+n} (x) + \frac{2}{\pi (\nu + n)} \right\},
\]

where $\psi (\nu)$ is the Psi function [14, §5.2(i)]. For $\nu = -1, -2, -3, \cdots$ (23) holds with $\nu$ replaced by $|\nu|$ on the RHS.

Proof. Firstly we assume $\nu \neq 0, -1, -2, -3, \cdots$. Then using

\[
C_{p}' (t) = \frac{1}{2} C_{p-1} (t) - \frac{1}{2} C_{p+1} (t),
\]

we have

\[
\{J_{\nu+n} (t) Y_{\nu+n} (t)\}' = \frac{1}{2} \{J_{\nu+n} (t) Y_{\nu+n-1} (t) - J_{\nu+n+1} (t) Y_{\nu+n} (t)\}
\]

\[+ \frac{1}{2} \{J_{\nu+n-1} (t) Y_{\nu+n} (t) - J_{\nu+n} (t) Y_{\nu+n+1} (t)\}.\]

On summing both sides from $n = 0$ to $n = N$ yields, via telescoping,

\[
\sum_{n=0}^{N} \{J_{\nu+n} (t) Y_{\nu+n} (t)\}' = \frac{1}{2} J_{\nu} (t) Y_{\nu-1} (t) + \frac{1}{2} J_{\nu-1} (t) Y_{\nu} (t)
\]

\[+ \frac{1}{2} J_{\nu+N+1} (t) Y_{\nu+N} (t) - \frac{1}{2} J_{\nu+N} (t) Y_{\nu+N+1} (t) .\]

We plan to let $N \to \infty$. Next we use

\[
C_{p-1} (t) = (p/t) C_{p} (t) + C_{p}' (t),
\]

to obtain from (24)

\[
\sum_{n=0}^{N} \{J_{\nu+n} (t) Y_{\nu+n} (t)\}' = (\nu/t) J_{\nu} (t) Y_{\nu} (t) + \frac{1}{2} \{J_{\nu} (t) Y_{\nu} (t)\}'
\]

\[- (\nu/N + t) J_{\nu+N} (t) Y_{\nu+N} (t) - \frac{1}{2} \{J_{\nu+N} (t) Y_{\nu+N} (t)\}'.\]

Integrating this from $t = x$ to $t = \infty$, and using (4) and (5), then yields

\[
\int_{x}^{\infty} t^{-1} J_{\nu} (t) Y_{\nu} (t) \, dt - (\nu + N) \int_{x}^{\infty} t^{-1} J_{\nu+N} (t) Y_{\nu+N} (t) \, dt.
\]

But, for fixed $x > 0$, we have from (21) and (22)

\[
\int_{x}^{\infty} \frac{J_{\nu+n} (x) Y_{\nu+n} (x)}{t} \, dt = - \frac{1}{\pi (\nu + n)} - \frac{x^2}{2\pi (\nu + n)} + O \left( \frac{1}{n^3} \right),
\]

as $n \to \infty$. Thus, in order to ensure that the series in (25) converges as $N \to \infty$, we rewrite the equation as

\[
\int_{x}^{\infty} \frac{J_{\nu} (x) Y_{\nu} (x)}{t} \, dt - \frac{1}{\pi} \sum_{n=0}^{N} \frac{1}{\nu+n} - (\nu + N) \int_{x}^{\infty} \frac{J_{\nu+N} (t) Y_{\nu+N} (t)}{t} \, dt.
\]
Moreover, from (20),
\[ (\nu + N) \int_0^\infty \frac{J_{\nu+N}(t)}{t} J_{\nu+N}(t) \, dt = \frac{1}{\pi} \ln \left( \frac{x}{2N} \right) + O \left( \frac{1}{N} \right). \]  
(28)

Using (27) and (28), and then letting \( N \to \infty \) in (26), yields (29).

Finally, for \( \nu = -1, -2, -3, \ldots \), we observe that \( J_\nu(t) Y_\nu(t) = J_{-\nu}(t) Y_{-\nu}(t) \) ([14, §10.4]), and hence (28) holds with \( \nu \) replaced by \( |\nu| \) on the RHS. \( \blacksquare \)

In the case \( \nu = 0 \) we take limits for the RHS of (23), and we obtain
\[ \int_0^\infty \frac{J_0(t) Y_0(t)}{t} \, dt = -\frac{1}{2} \sum_{n=0}^{\infty} \varepsilon_n \left. \frac{\partial}{\partial \nu} \{ J_{\nu+n}(x) Y_{\nu+n}(x) \} \right|_{\nu=0}. \]  
(29)

In deriving this we used L’Hopital’s rule, along with the relation
\[ \frac{1}{\nu^2} - \psi'(\nu) = -\frac{\pi^2}{6} + O(\nu) \quad (\nu \to 0). \]

We note in passing that if we multiply both sides of (23) by \( \nu \), and then set \( \nu = 0 \), we obtain the following new result.

**Corollary 7** For positive \( x \)
\[ \sum_{n=1}^{\infty} \left\{ J_n(x) Y_n(x) + \frac{1}{\pi n} \right\} = \frac{\ln \left( \frac{x}{2} \right) + \gamma}{\pi} - \frac{1}{2} J_0(x) Y_0(x). \]  
(30)

Finally, in order to obtain a complementary representation for \( \int_0^\infty t^{-1} Y_\nu^2(t) \, dt \), one can use the identity
\[ J_{-\nu}(t) Y_{-\nu}(t) = \cos(2\nu \pi) J_\nu(t) Y_\nu(t) + \frac{1}{2} \sin(2\nu \pi) \left\{ J_\nu^2(t) - Y_\nu^2(t) \right\}. \]  
(31)

(see [14, (10.4.5)]). Thus, on integrating from \( t = x \) to \( t = \infty \), we obtain
\[ \int_0^\infty \frac{Y_\nu^2(t)}{t} \, dt = \int_0^\infty \frac{J_\nu^2(t)}{t} \, dt + 2 \cot(2\nu \pi) \int_0^\infty \frac{J_\nu(t) Y_\nu(t)}{t} \, dt \\
-2 \csc(2\nu \pi) \int_0^\infty \frac{J_{-\nu}(t) Y_{-\nu}(t)}{t} \, dt, \]  
(32)

provided \( 2\nu \) is not an integer. Then one can substitute (31), (18), and (23) (with \( \nu \) replaced by \( -\nu \) for the third integral in (32)) to obtain the desired result.

If \( 2\nu = p \) is an integer we can take limits in (32), and in particular use
\[ \lim_{2\nu \to p} \cos(2\nu \pi) J_\nu(t) Y_\nu(t) - J_{-\nu}(t) Y_{-\nu}(t) \]
\[ = \frac{1}{2\pi} \left. \frac{\partial}{\partial \nu} \{ J_\nu(t) Y_\nu(t) \} \right|_{\nu=p/2} + \frac{(-1)^p}{2\pi} \left. \frac{\partial}{\partial \nu} \{ J_\nu(t) Y_\nu(t) \} \right|_{\nu=-p/2}. \]
As a result, we have
\[
\int_x^\infty \frac{Y_{\nu/2}^2(t)}{t} dt = \int_x^\infty \frac{J_{\nu/2}^2(t)}{t} dt + \frac{1}{\pi} \frac{\partial}{\partial \nu} \int_x^\infty J_\nu(t) Y_\nu(t) dt \bigg|_{\nu=p/2} + \frac{(-1)^{\nu}}{\pi} \frac{\partial}{\partial \nu} \int_x^\infty J_\nu(t) Y_\nu(t) dt \bigg|_{\nu=-p/2}.
\]

(33)

4 Uniform asymptotic approximations for \( J_\nu (\nu x) \) and \( Y_\nu (\nu x) \)

We now obtain uniform asymptotic approximations for the order derivatives of \( J_\nu (x) \) and \( Y_\nu (x) \) for large \( \nu \), in terms of the Airy functions \( \text{Ai} (x) \) and \( \text{Bi} (x) \). For brevity we restrict our consideration to real argument and leading order approximations, although extensions to complex argument and expansions are feasible from our methods. Explicit error bounds are also available from the corresponding ones for the Bessel functions themselves, but again for brevity we only include order estimates.

We note in passing the well-known behavior of Airy functions, given by
\[
\text{Ai} (x) \sim \exp \left(-\frac{2}{3} x^{3/2} \right) \ , \quad \text{Bi} (x) \sim \exp \left(\frac{2}{3} x^{3/2} \right) \sqrt{\pi x^{1/4}} \quad (x \to \infty),
\]

(34)

and
\[
\text{Ai} (x) \sim \cos \left(\frac{2}{3} |x|^{3/2} - \frac{1}{4} \pi \right) \frac{1}{\sqrt{\pi |x|^{1/4}}} , \quad \text{Bi} (x) \sim -\sin \left(\frac{2}{3} |x|^{3/2} - \frac{1}{4} \pi \right) \frac{1}{\sqrt{\pi |x|^{1/4}}} \quad (x \to -\infty).
\]

(35)

Moreover, \( \text{Ai} (x) \) and \( \text{Bi} (x) \) have no zeros for \( 0 \leq x < \infty \).

We shall employ the corresponding uniform approximations for the Bessel functions themselves, a brief description of which is given as follows. Let \( c = -0.36604 \cdots \) be the largest real root of the equation \( \text{Ai} (x) = \text{Bi} (x) \), and define the weight function of \( \text{Ai} (x) \) and \( \text{Bi} (x) \) by
\[
E (x) = 1 \quad (\infty < x \leq c)
E (x) = \{ \text{Bi} (x) / \text{Ai} (x) \}^{1/2} \quad (c \leq x < \infty),
\]

and the modulus function by
\[
M (x) = \{ \text{Ai}^2 (x) + \text{Bi}^2 (x) \}^{1/2} \quad (\infty < x \leq c),
M (x) = \{ 2 \text{Ai} (x) \text{Bi} (x) \}^{1/2} \quad (c \leq x < \infty).
\]

Next, introduce a new variable \( \zeta \) by
\[
\frac{2}{3} \zeta^{3/2} = \ln \left\{ \frac{1 + (1 - x^2)^{1/2}}{x} \right\} - (1 - x^2)^{1/2} \quad (0 < x \leq 1),
\]

(36)
\[
\frac{2}{3} (-\zeta)^{3/2} = (x^2 - 1)^{1/2} - \sec^{-1}(x) \quad (1 \leq x < \infty),
\]

all functions taking their principal values, with \( \zeta = \infty, 0, -\infty \), corresponding to \( x = 0, 1, \infty \), respectively. Note \( \zeta \to -\infty \) as \( x \to \infty \) such that

\[
x = \frac{2}{3} (-\zeta)^{3/2} + \frac{1}{2} \pi + O \left( |\zeta|^{-3/2} \right),
\]

and \( \zeta \to \infty \) as \( x \to 0^+ \) such that

\[
x = 2 \exp \left( -\frac{2}{3} \zeta^{3/2} - 1 \right) + O \left\{ \exp \left( -\frac{4}{3} \zeta^{3/2} \right) \right\}.
\]

For complex \( x \) and \( \zeta \), the transformation is given by (36), where the branches take their principal values when \( x \in (0, 1) \) and \( \zeta \in (0, \infty) \), and are continuous elsewhere.

From [13, Chap. 11, §10] we then have

\[
Y_\nu (\nu x) = -\frac{1}{\nu^{1/3}} \left( \frac{4\zeta}{1 - x^2} \right)^{1/4} \left\{ \text{Bi} \left( \nu^{2/3} \zeta \right) + \varepsilon_1 (\nu, \zeta) \right\},
\]

and

\[
J_\nu (\nu x) = \frac{1}{\nu^{1/3}} \left( \frac{4\zeta}{1 - x^2} \right)^{1/4} \left\{ \text{Ai} \left( \nu^{2/3} \zeta \right) + \varepsilon_2 (\nu, \zeta) \right\},
\]

where

\[
\varepsilon_1 (\nu, \zeta) = O \left( \nu^{-1} \right) E \left( \nu^{2/3} \zeta \right) M \left( \nu^{2/3} \zeta \right),
\]

and

\[
\varepsilon_2 (\nu, \zeta) = O \left( \nu^{-1} \right) M \left( \nu^{2/3} \zeta \right) / E \left( \nu^{2/3} \zeta \right),
\]

uniformly for \( 0 < x < \infty \) (i.e. \( -\infty < \zeta < \infty \)).

We shall also utilize the uniform approximation

\[
H_\nu^{(1)} (\nu x) = \frac{2e^{-\pi i/3}}{\nu^{1/3}} \left( \frac{4\zeta}{1 - x^2} \right)^{1/4} \text{Ai} \left( e^{2\pi i/3} \nu^{2/3} \zeta \right) \left\{ 1 + O \left( \frac{1}{\nu} \right) \right\},
\]

where (for our purposes) \( x \) and \( \zeta \) are complex, with \( -\pi \leq \arg (\zeta) < -\frac{1}{3} \pi \), and correspondingly \( x \) lying in a certain unbounded subset of the upper half plane (see [13, Chap. 11, §10]). A fundamental property of \( \text{Ai} \left( e^{2\pi i/3} \nu^{2/3} \zeta \right) \) is that for large \( \nu \) it is exponentially small in the sector \( -\pi < \arg (\zeta) < -\frac{1}{3} \pi \) (and likewise \( H_\nu^{(1)} (\nu x) \) for \( x \) in the upper half plane and lying outside a certain domain which contains the origin: see Figs. 10.1 and 10.2 of [13, Chap. 11]).

Now, from (1) and (3) we have

\[
\hat{J}_\nu (\nu x) = \nu \pi J_\nu (\nu x) I_1 (\nu, \nu x) + \nu \pi Y_\nu (\nu x) I_2 (\nu, \nu x),
\]

and

\[
\hat{Y}_\nu (\nu x) = \nu \pi J_\nu (\nu x) I_3 (\nu, \nu x) - \nu \pi Y_\nu (\nu x) I_1 (\nu, \nu x) - \frac{1}{2} \pi J_\nu (\nu x),
\]
where
\[ I_1 (\nu, t) = \int_t^\infty \frac{J_\nu (s) Y_\nu (s)}{s} ds, \]  
\[ I_2 (\nu, t) = \int_0^t \frac{J_\nu^2 (s)}{s} ds, \]  
\[ I_3 (\nu, t) = \int_t^\infty \frac{Y_\nu^2 (s)}{s} ds. \]

The Bessel functions in \((45)\) and \((46)\) can immediately be replaced by their Airy function approximations, given above. Regarding the integrals \(I_j (\nu, t)\) \((j = 1, 2, 3)\) we note, from \((40)\) - \((43)\) and \((35)\), that the integrands in \((47)\) and \((49)\) are highly oscillatory in the interval \(\nu < s < \infty\), and likewise for \((48)\) if \(t > \nu\). This is detrimental in numerical evaluations, so our goal is to approximate these integrals by readily computable functions and/or integrals with monotonic or slowly varying integrands.

Our results are summarized in Theorem 4.2 below. In this section and the next, \(\zeta (t)\) appearing in an integral denotes the function given by \((36)\) and \((37)\) with \(x\) replaced by \(t\), and similarly \(x (\eta)\) is given by \((36)\) and \((37)\) with \(\zeta\) replaced by \(\eta\).

Before stating our main theorem, we shall require the following result.

**Lemma 8** Let
\[ L (\nu, t) = \int_t^\infty \left\{ \frac{H_\nu^{(1)} (s)}{2s} \right\}^2 ds. \]  
Then as \(\nu \to \infty\)
\[ L (\nu, \nu x) = \frac{4e^{-2\pi i/3}}{\nu^{2/3}} \int_x^{i\infty} \left( \frac{\zeta (t)}{1 - t^2} \right)^{1/2} \frac{\text{Ai}^2 \left( e^{2\pi i/3} \nu^{2/3} \zeta (t) \right)}{t} dt \left( 1 + O \left( \frac{1}{\nu} \right) \right), \]  
uniformly for \(1 \leq x < \infty\). Here the path of integration lies in the first quadrant and is bounded away from the boundary EPB in Fig. 10.1 of [13, Chap. 11, §10], except for the endpoint when it is located at \(x = 1\).

**Proof.** We make a simple change of variable \(s = \nu t\) to obtain from \((50)\)
\[ L (\nu, \nu x) = \int_x^{i\infty} \left( \frac{H_\nu^{(1)} (\nu t)}{2t} \right)^2 dt. \]  
We then use \((44)\) in the integrand, and deform the path of integration from the positive \(x\)-axis to the stated path. The deformation is justifiable since the integrand is holomorphic in the first quadrant, and exponentially small at \(x = i\infty\). Note that \(e^{2\pi i/3} \zeta \to +\infty\) as \(x \to i\infty\), and hence the Airy function in the integrand of \((51)\) is non-zero and (in absolute value) rapidly decreasing along the path of integration.
Theorem 9 For large $\nu$, uniformly for $0 < x \leq 1$ ($0 \leq \zeta < \infty$),

\begin{align*}
I_1 (\nu, \nu x) &= \text{Im} \, L (\nu, \nu) \\
&- \frac{2}{\nu^{2/3}} \int_x^1 \left( \frac{\zeta (t)}{1 - t^2} \right)^{1/2} \frac{\text{Ai} (\nu^{2/3} \zeta (t)) \, \text{Bi} (\nu^{2/3} \zeta (t))}{t} \, dt \left\{ 1 + O \left( \frac{1}{\nu} \right) \right\}, \\
I_2 (\nu, \nu x) &= \frac{2}{\nu^{2/3}} \int_0^x \left( \frac{\zeta (t)}{1 - t^2} \right)^{1/2} \frac{\text{Ai}^2 (\nu^{2/3} \zeta (t))}{t} \, dt \left\{ 1 + O \left( \frac{1}{\nu} \right) \right\}.
\end{align*}

For large $\nu$, uniformly for $1 \leq x < \infty$ ($-\infty < \zeta \leq 0$)

\begin{align*}
I_1 (\nu, \nu x) &= \text{Im} \, L (\nu, \nu x), \\
I_2 (\nu, \nu x) &= \frac{1}{2 \nu} - \text{Re} \, L (\nu, \nu x) \\
&- \frac{1}{\nu^{2/3}} \int_x^\infty \left( \frac{\zeta (t)}{1 - t^2} \right)^{1/2} \frac{\text{Ai}^2 (\nu^{2/3} \zeta (t)) + \text{Bi}^2 (\nu^{2/3} \zeta (t))}{t} \, dt \left\{ 1 + O \left( \frac{1}{\nu} \right) \right\}, \\
I_3 (\nu, \nu x) &= - \text{Re} \, L (\nu, \nu x) \\
&+ \frac{1}{\nu^{2/3}} \int_x^\infty \left( \frac{\zeta (t)}{1 - t^2} \right)^{1/2} \frac{\text{Ai}^2 (\nu^{2/3} \zeta (t)) + \text{Bi}^2 (\nu^{2/3} \zeta (t))}{t} \, dt \left\{ 1 + O \left( \frac{1}{\nu} \right) \right\}.
\end{align*}

In these, $L (\nu, \nu x)$ is approximated by (51).

Remark. The Airy functions in the integrals appearing in (53), (54), and the first integral on the RHS of (55) are all non-oscillatory, since their arguments are nonnegative. From (55) we also note that

\[ \text{Ai}^2 \left( \nu^{2/3} \zeta \right) + \text{Bi}^2 \left( \nu^{2/3} \zeta \right) \sim 1 \left( \frac{\pi \nu^{1/3} |\zeta|^{1/2}}{\nu^{2/3} \zeta \to -\infty} \right), \]

and indeed, as seen (for example) from [14, Eq. (9.11.4)], this function is monotonic. Therefore the second integral on the RHS of (55), as well as integrals appearing in (57) and (58) are slowly varying, and consequently all the integrals in question are numerically stable.

Proof. Making the change of variable $s = \nu t$ in (47) we have

\[ I_1 (\nu, \nu x) = \int_x^\infty J_\nu (\nu t) Y_\nu (\nu t) \frac{dt}{t}, \]
(with similar representations used for $I_2(\nu, \nu x)$ and $I_3(\nu, \nu x)$). We next use $H^{(1)}_{\nu}(t) = J_{\nu}(t) + iY_{\nu}(t)$ to give the relation

$$J_{\nu}(t) Y_{\nu}(t) = \frac{1}{2} \text{Im} \left\{ \left( H^{(1)}_{\nu}(t) \right)^2 \right\},$$

for $t$ positive. Then, using this in (59) and referring to (52), we see that (56) follows.

For $0 < x \leq 1$ we express $I_1(\nu, \nu x)$ in the form

$$I_1(\nu, \nu x) = \int_x^1 \frac{J_{\nu}(\nu t) Y_{\nu}(\nu t)}{t} dt + \int_1^\infty \frac{J_{\nu}(\nu t) Y_{\nu}(\nu t)}{t} dt.$$  \hspace{1cm} (60)

The first integral here has a desired monotonic integrand, and using (40) and (41) it can be approximated by

$$\int_x^1 \frac{J_{\nu}(\nu t) Y_{\nu}(\nu t)}{t} dt = -\frac{2}{\nu^{2/3}} \int_x^1 \left( \frac{\zeta(t)}{1 - t^2} \right)^{1/2} \frac{\text{Ai} \left( \nu^{2/3} \zeta(t) \right)}{t} \frac{\text{Bi} \left( \nu^{2/3} \zeta(t) \right)}{t} dt \left\{ 1 + O \left( \frac{1}{\nu} \right) \right\}.$$  \hspace{1cm} (61)

For the second integral, we have from (56),

$$\int_1^\infty \frac{J_{\nu}(\nu t) Y_{\nu}(\nu t)}{t} dt = I_1(\nu, \nu) = \text{Im} L(\nu, \nu).$$  \hspace{1cm} (62)

Thus (60), (61) and (62) establishes (53).

Similarly to (61), we use (41) in (48) to obtain (54) for the case $0 < x \leq 1$.

For $1 \leq x < \infty$ we use (8) to rewrite

$$I_2(\nu, \nu x) = \frac{1}{2\nu} - \int_x^\infty \frac{J^2_{\nu}(\nu t)}{t} dt.$$  \hspace{1cm} (63)

Next, in this we use

$$J^2_{\nu}(t) = \frac{1}{2} \text{Re} \left\{ \left( H^{(1)}_{\nu}(t) \right)^2 \right\} + \frac{1}{2} H^{(1)}_{\nu}(t) H^{(2)}_{\nu}(t),$$

to arrive at

$$\int_x^\infty \frac{J^2_{\nu}(\nu t)}{t} dt = \text{Re} \int_x^\infty \frac{H^{(1)}_{\nu}(\nu t)}{2t} dt + \int_x^\infty \frac{J^2_{\nu}(\nu t) + Y^2_{\nu}(\nu t)}{2t} dt.$$  \hspace{1cm} (64)

Thus (57) follows from (40), (41), (50), (63) and (64).

Similarly to (64), we use (49) along with

$$Y^2_{\nu}(t) = \frac{1}{2} H^{(1)}_{\nu}(t) H^{(2)}_{\nu}(t) - \frac{1}{2} \text{Re} \left\{ \left( H^{(1)}_{\nu}(t) \right)^2 \right\},$$

12
to arrive at (58).

Finally, for $0 < x < 1$, we write

$$I_3 (\nu, \nu x) = \int_x^1 \frac{Y_{\nu}^2 (\nu t)}{t} \, dt + \int_1^\infty \frac{Y_{\nu}^2 (\nu t)}{t} \, dt.$$  

The approximation [55] then follows from using (40) in the first of these integrals, and using (58) (with $x = 1$) for the second integral.

As an illustration, let $\nu = 100$ and $x = 0.5$. The exact values that follow are obtained using the $\texttt{fdiff}$ routine in MAPLE with sufficient precision. From (45) we obtain the approximation $\hat{J}_{100} (50) \approx -1.47735 \times 10^{-21}$, compared to the exact value $\hat{J}_{100} (50) = -1.47702 \ldots \times 10^{-21}$. Similarly, from (46), we obtain the approximation $\hat{Y}_{100} (50) \approx -4.31473 \times 10^{18}$, compared to the value $\hat{Y}_{100} (50) = -4.31569 \times 10^{18}$.

Likewise, if we choose $\nu = 100$ and $x = 5$, we obtain the approximations $\hat{J}_{100} (500) \approx 0.0150731$ and $\hat{Y}_{100} (500) \approx -0.0470087$. In comparison, the exact values are $\hat{J}_{100} (500) = 0.0150695 \ldots$ and $\hat{Y}_{100} (500) = -0.0470099 \ldots$.

5 Asymptotic approximations of the integrals appearing in Theorem 4.2

The integrals in Theorem 4.2 involve Airy functions, and as such it is desirable to have explicit asymptotic approximations for them. These will involve the following integrals

$$G_{i,j}^k (\nu, \zeta) = \int \zeta^k y_i (\nu^{2/3} \zeta) y_j (\nu^{2/3} \zeta) \, d\zeta, \quad (65)$$

for $i = 1, 2, j = 1, 2, k = 0, 1, 2$; here $y_1 (t) = \text{Ai} (t)$ and $y_2 (t) = \text{Bi} (t)$. The key is that these integrals can be explicitly evaluated, and in particular from [2] we have (with $i$ and $j$ not necessarily distinct)

$$G_{1,j}^0 (\nu, \zeta) = \zeta y_j (\nu^{2/3} \zeta) \left( \nu^{2/3} \zeta \right) - \nu^{-2/3} y_j' (\nu^{2/3} \zeta) y_j (\nu^{2/3} \zeta), \quad (66)$$

and

$$G_{1,j}^1 (\nu, \zeta) = \frac{1}{6} \nu^{-4/3} \left\{ y_j (\nu^{2/3} \zeta) y_j' (\nu^{2/3} \zeta) + y_j' (\nu^{2/3} \zeta) y_j (\nu^{2/3} \zeta) \right\}. \quad (66)$$

Next let

$$a_0 = 2^{-2/3}, \quad a_1 = \frac{2}{5}, \quad a_2 = \frac{3}{30} 2^{2/3}, \quad (69)$$

13
Theorem 10  For large $\nu$, uniformly for $0 < x \leq 1$ ($0 \leq \zeta < \infty$)

\[
\int_{x}^{\infty} \left( \frac{\zeta(t)}{1 - t^2} \right)^{1/2} \frac{\text{Ai} \left( \nu^{2/3} \zeta(t) \right) \text{Bi} \left( \nu^{2/3} \zeta(t) \right)}{t} \, dt \\
= \left\{ 1 + O \left( \frac{1}{\nu} \right) \right\} \sum_{k=0}^{\infty} a_k \left[ G_{1,2}^k (\nu, \min \{ \zeta, \zeta_0 \}) - G_{1,2}^k (\nu, 0) \right] \\
+ \frac{1}{2 \pi \nu^{1/3}} \left\{ 1 + O \left( \frac{1}{\nu} \right) \right\} \max \left\{ \cosh^{-1} \left( \frac{1}{x} \right) - \cosh^{-1} \left( \frac{1}{x_0} \right), 0 \right\}.
\]

(71)

For the integrals (72), (73), and (74) that have rapidly changing in ter-

\[
\int_{0}^{x} \left( \frac{\zeta(t)}{1 - t^2} \right)^{1/2} \frac{\text{Ai} \left( \nu^{2/3} \zeta(t) \right) \text{Bi} \left( \nu^{2/3} \zeta(t) \right)}{t} \, dt \\
= - \left( \frac{\zeta}{1 - x^2} \right) \left[ G_{1,1}^0 (\nu, \zeta) \left\{ 1 + O \left( \frac{1}{\nu} \right) \right\} \right]
\]

(72)

and

\[
\int_{x}^{1} \left( \frac{\zeta(t)}{1 - t^2} \right)^{1/2} \frac{\text{Bi} \left( \nu^{2/3} \zeta(t) \right)}{t} \, dt \\
= \left[ \left( \frac{\zeta}{1 - x^2} \right) G_{2,2}^0 (\nu, \zeta) - a_0 G_{2,2}^0 (\nu, 0) \right] \left\{ 1 + O \left( \frac{1}{\nu} \right) \right\}
\]

(73)

In (73) the nonnegative branch of the inverse hyperbolic cosine is taken.

For large $\nu$, uniformly for $1 \leq x < \infty$ ($-\infty < \zeta \leq 0$)

\[
L (\nu, \nu x) = 4 e^{-2 \pi i / 3} \left( \nu^{2/3} \right) \left( \frac{\zeta}{1 - x^2} \right) G_{1,1}^0 \left( e^{\pi i / 3} \nu, \zeta \right) \left\{ 1 + O \left( \frac{1}{\nu} \right) \right\}
\]

(74)

and

\[
\int_{x}^{\infty} \left( \frac{\zeta(t)}{1 - t^2} \right)^{1/2} \frac{\text{Ai} \left( \nu^{2/3} \zeta(t) \right) + \text{Bi} \left( \nu^{2/3} \zeta(t) \right)}{t} \, dt \\
= \frac{1}{\pi \nu^{1/3}} \left\{ 1 + O \left( \frac{1}{\nu} \right) \right\} \left[ \sin^{-1} \left( \frac{1}{\max \{ x, x_1 \} } \right) \right]
\]

+ \max \left\{ 2 \sum_{k=0}^{\infty} a_k \left[ G_{1,1}^k (\nu, \zeta) + G_{2,2}^k (\nu, \zeta) - G_{1,1}^k (\nu, \zeta_1) - G_{2,2}^k (\nu, \zeta_1) \right], 0 \right\}
\]

(75)

with the principal inverse sine taken in (73).

Proof.  For the integrals (72), (73), and (74) that have rapidly changing inte-
grands (i.e. exponentially decreasing or increasing), we use integration by parts. We consider the latter, with other two results being similarly proven. Applying integration by parts to the integral appearing in (51) we obtain

\[ \int_{\infty}^{\zeta} \left( \frac{\eta}{1 - x(\eta)^2} \right) \exp \left( \frac{4}{3} \nu \zeta^{3/2} \right) d\eta = \int_{\infty}^{\zeta} \left( \frac{\zeta}{1 - x(\eta)^2} \right) G^0_{1,1} \left( e^{\pi i/3} \nu, \zeta \right) + O(\zeta), \]

(76)

Here the prime represents differentiation with respect to \( \eta \), and from (66) we have

\[ G^0_{1,1} \left( e^{\pi i/3} \nu, \zeta \right) = \zeta \exp \left( \frac{4}{3} \nu \zeta^{3/2} \right) + O(\zeta) \]

(77)

Now the first of (63) holds for complex \( x \) with \( |\arg(x)| \leq \pi - \delta \) (\( \delta > 0 \)) and the principal value of \( x^{1/2} \) taken. Then, from that equation, along with (69) and (77), we can readily show that \( G^0_{1,1} \left( e^{\pi i/3} \nu, \zeta \right) \) is non-vanishing for nonnegative \( \zeta \), with

\[ G^0_{1,1} \left( e^{\pi i/3} \nu, \zeta \right) = -\frac{e^{-2\pi i/3} \nu^{1/3} \nu^{2/3} \left( \frac{3}{2} \right)}{4 \pi^2 \nu^{2/3}} + O(\zeta), \]

as \( \zeta \to 0 \), and

\[ G^0_{1,1} \left( e^{\pi i/3} \nu, \zeta \right) = -\frac{e^{-4\pi i/3} \exp \left( \frac{4}{3} \nu \zeta^{3/2} \right)}{8 \pi \nu^{1/3} \zeta^{1/3}} \left( 1 + O \left( \frac{1}{\nu^{1/3} \zeta^{1/3}} \right) \right), \]

as \( \nu \zeta^{3/2} \to \infty \) such that \( |\arg(e^{2\pi i/3} \zeta)| \leq \pi - \delta \). Hence we can assert that

\[ K_1 \frac{|\exp \left( \frac{4}{3} \nu \zeta^{3/2} \right)|}{\nu^{1/3} |\zeta| + \nu^{2/3}} \leq \left| G^0_{1,1} \left( e^{\pi i/3} \nu, \zeta \right) \right| \leq K_2 \frac{|\exp \left( \frac{4}{3} \nu \zeta^{3/2} \right)|}{\nu^{1/3} |\zeta| + \nu^{2/3}}, \]

(78)

for some positive constants \( K_1 \) and \( K_2 \). Next, from the complex variable extension of (36) it is straightforward to show that

\[ \left( \frac{\zeta}{1 - x^2} \right) = O \left( \frac{1}{\zeta^2} \right), \quad \frac{d}{d\zeta} \left( \frac{\zeta}{1 - x^2} \right) = O \left( \frac{1}{\zeta^3} \right), \]

(79)

as \( \zeta \to \infty \) in the sector \( |\arg(-\zeta)| < 2\pi/3 \). Therefore from this (with \( x, \zeta \) replaced by \( x(\eta), \eta \), respectively), and the upper bound in (78), we have for large \( \nu \)

\[ \int_{\infty}^{\zeta} \left( \frac{\eta}{1 - x(\eta)^2} \right) G^0_{1,1} \left( e^{\pi i/3} \nu, \eta \right) d\eta = O(1) \int_{\infty}^{\zeta} \left| \exp \left( \frac{4}{3} \nu \eta^{3/2} \right) \right| \left( \frac{1}{\nu^{1/3} |\eta| + \nu^{2/3}} \right) \left| \frac{d|\eta|}{d\eta} \right| d|\eta|, \]

15
uniformly for \(-\infty < \zeta \leq 0\). Now express \(\eta = re^{-i\theta(r)}\) on the contour, where \(r = |\eta|\), and \(\theta(r)\) decreases continuously from \(\pi\) to \(2\pi/3\) as \(r\) increases from \(|\zeta|\) to \(\infty\). From Laplace’s method ([13, Chap. 3, §7]), we arrive at

\[
\int_{\infty}^{\zeta} e^{-2\pi i/3 (\eta_1 - x(\eta_2))} G_{1,1}^0 \left( e^{\pi i/3 \nu, \eta} \right) d\eta
= O(1) \int_{|\zeta|}^{\infty} \frac{1}{\nu^{2/3} (\nu^{2/3} |\zeta| + 1) (|\zeta| + 1)^3} \, dr.
\]

(80)

The result (74) follows from (51), (76), (78), (79) and (80). The proofs of (72) and (73) are similar.

Consider next the integrals with slowly varying (non-exponential) integrands. The method of integration by parts is not effective in approximating these, and instead we resort to a combination of a Maclaurin series expansion for small values of the integration variable, coupled with asymptotic approximations of Airy functions for the other unbounded values. To this end, on using

\[
d\zeta = -\frac{1}{x} \left( \frac{1 - x^2}{\zeta} \right)^{1/2} \, dx,
\]

we express (71) in the form

\[
\int_x^1 \left( \frac{\zeta(t)}{1 - t^2} \right)^{1/2} \frac{\Ai\left( \nu^{2/3} \zeta(t) \right) \Bi\left( \nu^{2/3} \zeta(t) \right)}{t} \, dt
= \int_0^\zeta \frac{\eta}{1 - x(\eta)^2} \Ai\left( \nu^{2/3} \eta \right) \Bi\left( \nu^{2/3} \eta \right) d\eta
+ \int_x^{x_0} \left( \frac{\zeta(t)}{1 - t^2} \right)^{1/2} \frac{\Ai\left( \nu^{2/3} \zeta(t) \right) \Bi\left( \nu^{2/3} \zeta(t) \right)}{t} \, dt.
\]

(81)

If \(0 \leq \zeta \leq \zeta_0\) \((x_0 \leq x \leq 1)\) the upper limit in the first integral on the RHS is replaced by \(\zeta\) and the second integral is null.

For the first integral on the RHS of (81) we find from (70), and recalling \(\zeta_0 = (2/\nu)^{1/3} + O(\nu^{-2/3})\), that

\[
\int_0^\zeta \frac{\eta}{1 - x(\eta)^2} \Ai\left( \nu^{2/3} \eta \right) \Bi\left( \nu^{2/3} \eta \right) d\eta
= \left\{ 1 + O\left( \frac{1}{\nu} \right) \right\} \int_0^\zeta \left( a_0 + a_1 \eta + a_2 \eta^2 \right) \Ai\left( \nu^{2/3} \eta \right) \Bi\left( \nu^{2/3} \eta \right) d\eta.
\]

(82)

Explicit integration and referring to (65) yields the first term on the RHS of (71). Clearly this also holds if \(\zeta_0\) is replaced by \(\zeta\) when \(0 \leq \zeta \leq \zeta_0\).
Next, from \[ \text{Ai} \left( \nu^{2/3} \zeta \right) \text{Bi} \left( \nu^{2/3} \zeta \right) = \frac{1}{2 \pi \nu^{1/3} \zeta^{1/2}} \left\{ 1 + O \left( \frac{1}{\nu^{2} \zeta^{3}} \right) \right\}, \]
as \( \nu^{2/3} \zeta \to \infty \). Hence, assuming \( 0 < x < x_{0} \) \((\zeta_{0} < \zeta < \infty)\), we have \( \zeta^{-1} = O \left( \nu^{1/3} \right) \) in the second integral on the RHS of (81), and so it follows that

\[
\int_{x_{0}}^{x} \left( \frac{\zeta(t)}{1-t^{2}} \right)^{1/2} \frac{\text{Ai} \left( \nu^{2/3} \zeta(t) \right) \text{Bi} \left( \nu^{2/3} \zeta(t) \right)}{t} dt \]

\[
= \frac{1}{2 \pi \nu^{1/3}} \left\{ 1 + O \left( \frac{1}{\nu} \right) \right\} \int_{x_{0}}^{x} \left( \frac{1}{1-t^{2}} \right)^{1/2} \frac{1}{t} dt. \]

Now for \( 0 < t \leq 1 \)

\[
\int \left( \frac{1}{1-t^{2}} \right)^{1/2} \frac{1}{t} dt = - \cosh^{-1} \left( \frac{1}{t} \right). \]

On combining (83) and (84) we then arrive at the second term on the RHS of (71).

The proof of (75) is similar. For \( 1 \leq x < x_{1} \) \((\zeta_{1} < \zeta \leq 0)\) we write this as

\[
\int_{x}^{\infty} \left( \frac{\zeta(t)}{1-t^{2}} \right)^{1/2} \frac{\text{Ai}^{2} \left( \nu^{2/3} \zeta(t) \right) + \text{Bi}^{2} \left( \nu^{2/3} \zeta(t) \right)}{t} dt \]

\[
= \int_{x_{1}}^{\infty} \left( \frac{\zeta(t)}{1-t^{2}} \right)^{1/2} \frac{\text{Ai}^{2} \left( \nu^{2/3} \zeta(t) \right) + \text{Bi}^{2} \left( \nu^{2/3} \zeta(t) \right)}{t} dt \]

\[
+ \int_{\zeta_{1}}^{\zeta} \left( \frac{\eta}{1-x(\eta)^{2}} \right) \left\{ \text{Ai}^{2} \left( \nu^{2/3} \eta \right) + \text{Bi}^{2} \left( \nu^{2/3} \eta \right) \right\} d\eta, \]

with no such splitting being necessary if \(-\infty < \zeta \leq \zeta_{1} \) \((x_{1} \leq x < \infty)\). Now using

\[
\text{Ai}^{2} \left( \nu^{2/3} \zeta \right) + \text{Bi}^{2} \left( \nu^{2/3} \zeta \right) = \frac{1}{\pi \nu^{1/3} (-\zeta)^{1/2}} \left\{ 1 + O \left( \frac{1}{\nu^{2} \zeta^{3}} \right) \right\} \quad \left( \nu^{2/3} \zeta \to -\infty \right), \]

we find for the first integral on the RHS of (85) that

\[
\int_{x_{1}}^{\infty} \left( \frac{\zeta(t)}{1-t^{2}} \right)^{1/2} \frac{\text{Ai}^{2} \left( \nu^{2/3} \zeta(t) \right) + \text{Bi}^{2} \left( \nu^{2/3} \zeta(t) \right)}{t} dt \]

\[
= \frac{1}{\pi \nu^{1/3}} \left\{ 1 + O \left( \frac{1}{\nu} \right) \right\} \int_{x_{1}}^{\infty} \left( \frac{1}{t^{2} - 1} \right)^{1/2} \frac{1}{t} dt. \]

Then with

\[
\int_{x_{1}}^{\infty} \left( \frac{1}{t^{2} - 1} \right)^{1/2} \frac{1}{t} dt = \sin^{-1} \left( \frac{1}{x_{1}} \right), \]
we arrive at the first term on the RHS of (75).

In the case when splitting is necessary (i.e. $1 \leq x < x_1$), we follow \[m2\] and employ the Maclaurin expansion (70) for the second integral on the RHS of (85), and the second term on the RHS of (75) is obtained. \[m2\]

Table 1 illustrates the results \[m1\], \[m2\], \[m3\] and \[m4\] numerically for the case $\nu = 50$ and various values of $x$. In this, we denote the relative errors $\eta_l(\nu, x)$ ($l = 1, 2, 3, 4$) by

$$f_l(\nu, x) = g_l(\nu, x) \{1 + \eta_l(\nu, x)\},$$

where

$$f_1(\nu, x) = \int_x^1 \left(\frac{\zeta(t)}{1 - t^2}\right)^{1/2} \frac{\text{Ai}(\nu^{2/3} \zeta(t)) \text{Bi}(\nu^{2/3} \zeta(t))}{t} \, dt,$$

$$g_1(\nu, x) = \sum_{k=0}^2 a_k \left[ G_{1,2}^k(\nu, \min\{\zeta, \zeta_0\}) - G_{1,2}^k(\nu, 0) \right] + \frac{1}{2\pi \nu^{1/3}} \max \left\{ \cosh^{-1} \left( \frac{1}{\nu} \right) - \cosh^{-1} \left( \frac{1}{\nu x_0} \right), 0 \right\},$$

$$f_2(\nu, x) = \int_0^x \left(\frac{\zeta(t)}{1 - t^2}\right)^{1/2} \frac{\text{Ai}(\nu^{2/3} \zeta(t))}{t} \, dt,$$

$$g_2(\nu, x) = -\left( \frac{\zeta}{1 - x^2} \right) G_{1,1}^0(\nu, \zeta),$$

$$f_3(\nu, x) = \int_x^1 \left(\frac{\zeta(t)}{1 - t^2}\right)^{1/2} \frac{\text{Bi}^2(\nu^{2/3} \zeta(t))}{t} \, dt,$$

$$g_3(\nu, x) = \left[ \left( \frac{\zeta}{1 - x^2} \right) G_{2,2}^0(\nu, \zeta) - a_0 G_{2,2}^0(\nu, 0) \right],$$

$$f_4(\nu, x) = \int_x^\infty \left(\frac{\zeta(t)}{1 - t^2}\right)^{1/2} \frac{\text{Ai}^2(\nu^{2/3} \zeta(t)) + \text{Bi}^2(\nu^{2/3} \zeta(t))}{t} \, dt,$$

and

$$g_4(\nu, x) = \frac{1}{\pi \nu^{1/3}} \sin^{-1} \left( \frac{1}{\max\{x, x_1\}} \right) + \max \left\{ \sum_{k=0}^2 a_k \left[ G_{1,1}^k(\nu, \zeta) + G_{2,2}^k(\nu, \zeta) - G_{1,1}^k(\nu, \zeta_1) - G_{2,2}^k(\nu, \zeta_1) \right], 0 \right\}.$$

All calculations were performed with MAPLE, with the integrals $f_l(\nu, x)$ being evaluated numerically using Simpson’s method.

| $x$  | $|\eta_1(50, x)|$ | $|\eta_2(50, x)|$ | $|\eta_3(50, x)|$ | $|\eta_4(50, x)|$ |
|------|-----------------|-----------------|-----------------|-----------------|
| 0.1  | 1.6240E-04      | 3.1202E-03      | 3.1466E-03      | -               |
| 0.5  | 3.4440E-04      | 6.9778E-03      | 7.2109E-03      | -               |
| 0.75 | 2.1710E-04      | 1.0326E-02      | 1.1859E-02      | -               |
| 0.99 | 1.7825E-08      | 2.0756E-02      | 1.8467E-02      | -               |
| 1    | -               | -               | -               | 2.6086E-04      |
| 5    | -               | -               | -               | 6.2709E-07      |
| 10   | -               | -               | -               | 1.1871E-07      |
Table 1.

Acknowledgement. I thank the referees for a number of helpful suggestions and comments.

References

[1] M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. National Bureau of Standards Applied Mathematics Series, U.S. Government Printing Office, Washington, D.C., 1964.

[2] J. R. Albright, Integrals of products of Airy functions. J. Phys. A 10 (4) (1977), 485-490.

[3] A. Apelblat and N. Kravitsky, Integral Representations of Derivatives and Integrals with Respect to the Order of the Bessel Functions $J_\nu(t)$, $I_\nu(t)$, the Anger Function $J_{\nu}\prime(t)$, and the Integral Bessel Function. $J_{\nu}\prime(t)$. IMA J. Appl. Math. 34 (2) (1985), 187-210.

[4] Yu. A. Brychkov and K. O. Geddes, On the derivatives of the Bessel and Struve functions with respect to the order. Integral Transforms Spec. Funct. 16 (2005), 187-198.

[5] H. S. Cohl, Derivatives with respect to the degree and order of associated Legendre functions for $|z|>1$ using modified Bessel functions. Integral Transforms Spec. Funct. 21 (2010), no. 7-8, 581-588

[6] D. E. Dominici, and P. M. W. Gill, A remarkable identity involving Bessel functions. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 468 (2012), 2667-2681.

[7] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*. Vol. II. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953.

[8] I. S. Gradshteyn and I. M. Ryzik. *Table of integrals, series and products*. Academic Press, 2006.

[9] L. J. Landau, Bessel functions: monotonicity and bounds. J. London Math. Soc. (2) 61 (2000), 197-215.

[10] Y. D. Luke, *The special functions and their approximations*, vol. II. Academic Press, 1969.

[11] W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*. 3rd edition, Springer-Verlag, New York-Berlin, 1966.
[12] F. Oberhettinger, On the derivative of Bessel functions with respect to the order. J. Math. and Phys. 37 (1958), 75-78.

[13] F. W. J. Olver, *Asymptotics and special functions*. Reprint of the 1974 original [Academic Press, New York]. AKP Classics. A K Peters, Ltd., Wellesley, MA, 1997.

[14] F. W. J. Olver, D. W. Lozier, R. Boisvert, and C. W. Clark, eds., *NIST Handbook of Mathematical Functions*, Cambridge Univ. Press, Cambridge, 2010. Available at [http://dlmf.nist.gov/](http://dlmf.nist.gov/)

[15] J. Sesma, Derivatives with respect to the order of the Bessel function of the first kind. (2014), arXiv:1401.4850 [math.CA]

[16] G. N. Watson, *A Treatise on the Theory of Bessel Functions*. 2nd edition, Cambridge University Press, Cambridge, England, 1944.