Finite-dimensional output stabilization for a class of linear distributed parameter systems — a small-gain approach

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Abstract

A small-gain approach is proposed to analyze closed-loop stability of linear diffusion-reaction systems under finite-dimensional observer-based state feedback control. For this, the decomposition of the infinite-dimensional system into a finite-dimensional slow subsystem used for design and an infinite-dimensional residual fast subsystem is considered. The effect of observer spillover in terms of a particular (dynamic) interconnection of the subsystems is thoroughly analyzed for in-domain and boundary control as well as sensing. This leads to the application of a small-gain theorem for interconnected systems based on input-to-output stability and unbounded observability properties. Moreover, an approach is presented for the computation of the required dimension of the slow subsystem used for controller design. Simulation scenarios for both scalar and coupled linear diffusion-reaction systems are used to underline the theoretical assessment and to give insight into the resulting properties of the interconnected systems.

Keywords: Output stabilization, small-gain theory, diffusion-reaction systems, spillover, observer design, input-to-output stability, distributed parameter systems, partial differential equations, modal approximation.

1. Introduction

Spillover is an inherent performance and stability issue when addressing the control of distributed parameter systems based on finite-dimensional approximations. The term spillover was characterized in, e.g., \cite{1, 4, 20} and refers to deterioration of the control performance due to the infinite-dimensional residual dynamics that is neglected during control design when taking into account approximation schemes such as modal, Galerkin or weighted residuals methods \cite{6, 13, 14, 16}. In particular the so-called observation spillover might be a source of instability of the closed-loop control system. Observation spillover can arise when applying the combination of state feedback controller and state observer — both designed based on the finite-dimensional approximation — to the original distributed parameter system due to the additional feedback loop generated by the injection of the contribution of the residual dynamics to the system output into the observer.

Finite-dimensional compensator design for distributed parameter systems (DPSs) has a long history with contributions from different authors, e.g., \cite{18, 2, 4, 13, 16, 15, 34}. The particular combined controller and observer structure used in this paper seems to be used first in \cite{2} and later in, e.g., \cite{33}. Explicit formulas to determine the effect of a finite-dimensional modal controller on the original infinite-dimensional system are derived in \cite{14}. For this degenerate operator perturbations are studied but without providing a criterion concerning the order of the (modal) subsystem to design the finite-dimensional compensator and observer. Related results are provided in \cite{12} based on Hankel-norm approximation. The connection between spillover and robustness is analyzed in \cite{8}. A Lyapunov-based stability analysis of the closed-loop control system with finite-dimensional modal controller is presented in, e.g., \cite{22}. To reduce spillover effect when controlling distributed parameter systems different measured have been suggested. These include residual mode filters \cite{6}, augmented observers \cite{10} or so-called cascaded output observers \cite{23}. The latter reference also considers the a priori determination of the order of the stabilizing compensator for systems with bounded input and output operator while in general the necessary order of the reduced system is not specified explicitly but should be chosen sufficiently large without providing a computational criteria.

Feedback stabilization based on reduced order models for large scale (converged) approximations of linear and nonlinear distributed parameter systems are suggested in, e.g., \cite{35, 9}. In \cite{9} numerical tools are used to determine a lower bound on the order of the reduced system so that the stabilization of a steady state is ensured. Lyapunov theory and modal decomposition are applied, e.g., in \cite{11} for a
semilinear 1D heat equation or in \cite{31} for a 1D linear heat with input delay. Herein a separation between the finite-dimensional and the infinite-dimensional residual dynamics is considered for the stability analysis by assuming direct availability of the modal states without amending the control loop by an observer. These results are extended in \cite{27} by developing a finite-dimensional observer-based control for a 1D heat equation which relies on Lyapunov’s stability theory and linear matrix inequalities to formulate conditions for the determination of the dimension of the reduced order system. Delayed input and output are addressed in \cite{28} for a scalar diffusion-reaction equation. Related results are proposed in \cite{29} for observer-based PI-control and in \cite{30} taking into account saturated control.

Differing from previous work this contribution makes use of a small-gain theorem to assess closed-loop stability of the interconnection between a finite-dimensional state feedback control with observer using modal approximation and the infinite-dimensional residual system. This enables us to verify that the stabilization of a suitable low-order subsystem ensures stability of the infinite-dimensional system under this feedback control and to compute a lower bound on the order of this subsystem. Here, the classical decomposition into slow and fast dynamics is exploited and an observer-based state feedback control is designed for the slow subsystem. Observer spillover arises as the sensor signal contains information of both slow and fast dynamics, which induces additional feedback loops that are not considered during the design. Based on the eigenvalue distribution of the system operator and certain characteristic features of the input and output operators a sequence of estimates for the fast (residual) dynamics under observer-based state feedback control is determined addressing both in-domain and boundary actuation and sensing. The preliminary results lead to the application of a small-gain theorem for interconnected systems based on input-to-output stability and unbounded observability properties. To address the dimension of the slow subsystem used for controller design, a numerical approach is presented and illustrated in simulation scenarios for both scalar and coupled linear diffusion-reaction systems.

The paper is organized as follows. A prototype example is introduced in Section 2 to motivate the formulation of an abstract model in Section 3 and the decomposition into slow and fast dynamics as well as observer-based state feedback control design. Based on this, auxiliary estimates and results are provided in Section 4 to prepare the main small-gain result in Section 5 to confirm closed-loop stability. Section 6 summarizes a computational approach to determine the minimal order of the slow subsystem used for control design. Simulation results for scalar and coupled linear diffusion-reaction systems in Section 7 are presented to confirm the theoretical assessment. Some final remarks conclude the paper.

Notation

Given vectors $x_s(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$, $e_s(t) = [e_1(t), \ldots, e_n(t)]^T \in \mathbb{R}^n$, $x_f(t) = [x_{n+1}(t), x_{n+2}(t), \ldots]^T \in \mathbb{R}^{\infty}$ we use the following norms:

\[
\|x_s\|_2 := \sum_{k=1}^{n} x_k^2, \quad \|e_s\|_2 := \sum_{k=1}^{n} e_k^2,
\]

\[
\|x_s\|_{2,\infty} := \sup_{t \geq 0} \|x_s(t)\|_2, \quad \|e_s\|_{2,\infty} := \sup_{t \geq 0} \|e_s(t)\|_2,
\]

\[
\|x_s\|_{[t_1,t_2]} := \sup_{\tau \in [t_1,t_2]} \|x_s(\tau)\|_p, \quad p \in \{1,2\},
\]

\[
\|x_f\|_1 := \sum_{k \geq n+1} |x_k|, \quad \|x_f\|_2 := \sqrt{\sum_{k \geq n+1} x_k^2},
\]

\[
\|x_f\|_{p,\infty} := \sup_{t \geq 0} \|x_f(t)\|_p, \quad p \in \{1,2\}.
\]

Moreover, for matrices and linear operators we use the usual induced operator norms.

2. A prototype system

We motivate the study in this paper by considering the (unstable) linear diffusion–reaction system

\[
\begin{align*}
\partial_t x &= \partial_x^2 x + rx + bu_1, \quad z \in (0,1), \ t > 0 \quad (1a) \\
\partial_t x|_{z=0} &= 0, \quad x|_{z=1} = u_2, \quad t > 0 \quad (1b) \\
x|_{t=0} &= x_0, \quad z \in [0,1]. \quad (1c)
\end{align*}
\]

Let $X = L^2(0,1)$ denote the state space and introduce the self-adjoint operator $Ax = \partial_x^2 x$ with domain $D(A) = \{x \in H^2(0,1) | \partial_x x(0) = x(1) = 0\}$. The eigenproblem for $A$ reads $A\phi = \mu \phi$, $\phi \in D(A)$. Its solution can be obtained by directly solving the differential equation and taking into account the boundary conditions. In particular it follows that $\phi_k = \sqrt{2} \cos(\omega_k z)$, $\mu_k = -\omega_k^2$, $k \in \mathbb{N}$ for $\omega_k = \frac{2k-1}{2} \pi$. The sequence $\langle \phi_k \rangle_{k \in \mathbb{N}}$ form an orthonormal Riesz basis for $L^2(0,1)$. Taking into account either operator extensions \cite{31} Section 13.7, Green’s theorem or modal transformation the system \cite{4} can be projected onto the basis $\langle \phi_k \rangle_{k \in \mathbb{N}}$ even taking into account the inhomogeneous boundary condition at $z = 1$. Let $\langle \cdot, \cdot \rangle$ denote the inner product in $L^2(0,1)$, then

\[
\langle \partial_t x, \phi_k \rangle = \langle \partial_x^2 x, \phi_k \rangle + r\langle x, \phi_k \rangle + \langle b, \phi_k \rangle u_1.
\]

Interchanging time differentiation and integration and using integration by parts taking into account the boundary conditions provides

\[
\begin{align*}
\partial_t \langle x, \phi_k \rangle &= -\partial_x \langle x, \phi_k \rangle|_{z=1} u_2 + \langle x, \partial_x^2 \phi_k \rangle + r\langle x, \phi_k \rangle + \langle b, \phi_k \rangle u_1 \\
&= -\partial_x \langle x, \phi_k \rangle|_{z=1} u_2 + \mu_k \langle x, \phi_k \rangle + r\langle x, \phi_k \rangle + \langle b, \phi_k \rangle u_1.
\end{align*}
\]
Denoting \( x_k = \langle x, \phi_k \rangle, b_{1,k} = \langle b, \phi_k \rangle \) and \( b_{2,k} = -\partial_x \phi_k |_{x=1} \) the latter equation can be re-written as the infinite-dimensional system of ODEs in diagonal form

\[
\dot{x}_k = (r + \mu_k)x_k + b_{1,k}u_1 + b_{2,k}u_2, \quad k \in \mathbb{N} \tag{2a}
\]
\[
x_k(0) = \langle x_0, \phi_k \rangle = \delta_{k,0}. \tag{2b}
\]

Let subsequently \( \lambda_k := r + \mu_k \). In view of the coefficient \( b_{1,k} \) the prototype problem \([2]\) involves both in-domain \((l = 1)\) and boundary control \((l = 2)\), respectively.

3. An abstract model

In this section we specify an abstract model that captures the properties of the prototype system just discussed. The results in the paper will be formulated for this abstract model. All necessary assumptions will be summarised in the next section.

3.1. Problem setup

We consider systems given by the abstract Cauchy problem

\[
\begin{align*}
\dot{x} &= Ax + Bu, \quad t > 0, \quad x(0) = x_0 \in D(A) \tag{3a} \\
y &= Cx, \quad t \geq 0 \tag{3b}
\end{align*}
\]

on the Hilbert space \( X \) equipped with the inner product \( \langle \cdot, \cdot \rangle_X \). The system operator is denoted by \( A \) with domain \( D(A) \subset X \).

Assumption 3.1. The abstract system fullfills the following assumptions:

(A1) The operator \( A \) is a Riesz spectral operator in the sense of \([7, Section 2.3]\), i.e., \( A \) has only discrete eigenvalues \( \lambda_k \) with \( \sup_{k \in \mathbb{N}} \text{Re}\{\lambda_k\} < \infty \) and the eigenvectors \( \phi_k \in D(A) \) and the adjoint eigenvectors \( \psi_k \in D(A^*) \) form orthonormal Riesz bases so that \( \langle \phi_k, \psi_k \rangle_X = \delta_{k,l} \) with \( \delta_{k,l} \) denoting the Kronecker delta.

(A2) The eigenvalues \( \lambda_k \) of \( A \) are real-valued with \( \lambda_k \to -\infty \) as \( k \to \infty \) and are arranged so that \( \lambda_{k+1} \leq \lambda_k \) for all \( k \).

(A3) The possibly unbounded operators \( B \) and \( C \) are assumed to be admissible control and output operators, respectively, in the sense of \( [24] \) or \( [36], Chapter 4 \) with finite dimensional control space \( U = \mathbb{R}^m \) and output space \( Y = \mathbb{R}^q \).

Assumption (A1) implies that the state \( x \in X \) can be represented by the Fourier series \( x = \sum_{k=1}^{\infty} x_k \phi_k \) and the operator \( A : D(A) \to X \) admits the decomposition

\[
Ax = \sum_{k=1}^{\infty} \lambda_k x_k \phi_k \quad \forall x \in D(A) \tag{4}
\]
\[
D(A) = \left\{ x \in X : \sum_{k=1}^{\infty} (1 + \lambda_k^2) |x_k|^2 < \infty \right\}. \tag{5}
\]

where \( x_k := \langle x, \phi_k \rangle_X \) represents the \( k \)-th Fourier or modal coefficient \([19, 17, 38, 21]\). The operator \( A \) is also called diagonalizable \([36, Section 2.6]\). By \((A1)\) it also follows that the adjoint operator is diagonalizable with eigenvalues \( \overline{\lambda_k} = -\lambda_k \) noting Assumption 3.1 \((A2)\). Proceeding as in \([36, Section 2.10]\), let \( X_1 \) denote the space \( D(A) \) equipped with the norm \( ||x|| = \| (\beta I - A)x \|_X \) for some \( \beta \in \rho(A) \neq \emptyset \). Note that the norms generated for different \( \beta \) are equivalent in the graph norm so that \( ||x||_1 \) is independent of the particular choice of \( \beta \). Let \( X_{-1} \) denote the dual of \( X_1 \) with respect to the pivot space \( X, i.e., X_1 \subset X \subset X_{-1} \) with continuous dense injections. Similar to the Fourier representation of \( x \) via the sequence \((x_k)_k\), any linear operator \( z \) can be represented by the sequence \((z_k)_k \) given by \( z_k := z \phi_k \) for \( k \in \mathbb{N} \). With this representation, the space \( X_{-1} \) can be identified with the space of sequences \( z = (z_k)_k \) for which

\[
||z||_{-1}^2 = \sum_{k=1}^{\infty} \frac{|z_k|^2}{1 + |\lambda_k|^2} < \infty. \tag{6}
\]

In a similar fashion, the Riesz basis property and \( A, A^* \) being diagonalizable imply, see, e.g., \([22, 23]\) that any input operator \( B \in \mathcal{L}(U, X_{-1}) \) can be represented by a sequence in \( U \) according to

\[
Bu = \sum_{k=1}^{\infty} (b_k, u)_U \phi_k, \quad b_k = B^* \psi_k. \tag{7}
\]

If we define the sequence \( v = ((b_k, u)_U)_k \), then \( ||v||_{-1} < \infty \). Taking into account \((A2)\) and \((A3)\) it can be shown, see, e.g., \([35]\) that \( m_{\beta} > 0 \) so that \( ||b_k|| \leq m_{\beta}(1 + |\lambda_k|) \) for all \( k \in \mathbb{N} \). We also note that the input operator \( B \) is called admissible, if \([36]\) is considered as an abstract Cauchy problem with values in \( X_{-1} \) has a continuous \( X \)-valued mild solution for any \( u \in L^2([0, \infty); U) \) \([24, 36, Definition 4.1.5]\). Throughout this paper, we consider these mild solutions. We refer to \([24]\) and \([36, Chapter 10]\) for the formulation of boundary control problems in the form \([3] \) with unbounded input operator \( B \) using so-called operator extensions.

Let \( C_j \) denote the \( j \)-th component of the output operator \( C \). Then, using \( c_{j,k} = C_j \phi_k \), the identity

\[
C_j = \sum_{k=1}^{\infty} c_{j,k} x_k \tag{8}
\]

holds provided the infinite sum is absolutely convergent. Due the fact that the admissibility assumption \((A3)\) demands that \( C_j \in \mathcal{L}(X_1, \mathbb{R}) \), by \([3]\) this is in particular the case if \( x \in X_1 \). Theorem 5.3.2 from \([36]\) and the eigenvalue condition in \((A2)\) imply that the admissibility condition in \((A3)\) for \( C \) is equivalent to the existence of a constant \( m_c \geq 0 \) such that

\[
\frac{1}{\eta} \sum_{k \geq -h} |c_{j,k}|^2 \leq m_c. \tag{9}
\]
for all \( h > 0 \). We note that in general this is a more demanding condition than \( [6] \).

**Remark 3.2.** In order to simplify the exposition we restrict ourselves to \( [3] \) in the SISO-case, i.e., \( Bu = bu \), \( y = Cx = y \), and explain the necessary changes for the MIMO-case in Remark \([7, 8]\).  

3.2. System decomposition and finite-dimensional observer-based control design

In view of the orthonormality property of the eigenvectors, the orthogonal projections, and the eigenvalues, \( \Lambda_f \) and \( \Lambda_s \) can be split into a finite-dimensional slow and an infinite-dimensional fast dynamics. In particular, if we identify \( x_s = P_s x \) with \( [x_1, x_2, \ldots, x_n]^T \) and \( y_f = P_f x \) with \( [x_{n+1}, x_{n+2}, \ldots]^T \), then \([3]\) can be written in the form

\[
\dot{x}_s = \Lambda_s x_s + b_s u \\
\dot{x}_f = \Lambda_f x_f + b_f u
\]

with \( \Lambda_s = \text{diag}\{\lambda_1, \ldots, \lambda_s\} \) and \( \Lambda_f = \text{diag}\{\lambda_{s+1}, \lambda_{s+2}, \ldots\} \), \( b_s = [b_1, \ldots, b_s]^T \), and \( b_f = [b_{s+1}, b_{s+2}, \ldots]^T \) where \( b_j = \langle b, \psi_k \rangle \). We assume that \( \lambda_{s+1} < 0 \), which for our model problem can always be achieved if \( n \) is sufficiently large.

The output of the system can be equivalently split into

\[
y = Cx = c_s^T x_s + Cx_f,
\]

where \( c_s^T = [c_1, \ldots, c_s] \) with \( c_k = C\phi_k \). If the infinite sum in \([8]\) is absolutely convergent, then we can write

\[
Cx_f = \sum_{k=n+1}^{\infty} C\phi_k x_k = c_f^T x_f
\]

with \( c_f^T = [c_{n+1}, c_{n+1}, \ldots] \) with \( c_k = C\phi_k \). Whenever we use the representation \([12]\) in this paper, we will check that the absolute convergence property holds (cf. \([24]\) and Lemma \([4, 4]\) as well as \([29]\) and Lemma \([14, 7]\).  

Consider now the finite-dimensional (slow) system

\[
\dot{x}_s = \Lambda_s x_s + b_s u \\
y_s = c_s^T x_s
\]

Assuming stabilizability and detectability, we can design a stabilizing dynamic output feedback law based on a Luenberger observer. The observer is of the form

\[
\dot{\hat{x}}_s = \Lambda_s \hat{x}_s + b_s u + l(y_s - \hat{y}_s) \\
\dot{\hat{y}}_s = c_s^T \hat{x}_s
\]

and the resulting control reads \( u = -k^T \hat{x}_s \). Defining \( e_s := x_s - \hat{x}_s \) we can rewrite this as

\[
\dot{e}_s = (\Lambda_s - le_c^T) e_s \\
u = -k^T (x_s - e_s)
\]

If we entirely neglect the fast, infinite-dimensional subsystem \([10b]\), we end up with the finite-dimensional closed-loop system

\[
\dot{x}_s = \Lambda_s x_s - b_s k^T (x_s - e_s) \\
\dot{e}_s = (\Lambda_s - le_c^T) e_s
\]

which is asymptotically stable if \( k \) and \( l \) are appropriately chosen. We note that while \( k \) and \( l \) depend on \( n \), we impose in Assumption \([4, 1]\) below, that their norm is bounded independent of \( n \). This assumption is satisfied, e.g., if we only shift a finite number of eigenvalues with these feedback laws. We illustrate this for \( k \). Suppose we want to design \( k \) such that \( \Lambda_s - b_s k^T \) has the eigenvalues \( \kappa_1, \kappa_2, \ldots, \kappa_j, \lambda_{j+1}, \lambda_{j+2}, \ldots \), where \( \lambda_i \) are the eigenvalues of \( \Lambda_s \). Recalling that \( \Lambda_s \) is a diagonal matrix, we can then write \( \Lambda_s, b_s \) and \( k \) as

\[
\Lambda_s = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}, \quad b_s = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad \text{and } k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}
\]

with \( \Lambda_1 \in \mathbb{R}^{j \times j}, b_1, k_1 \in \mathbb{R}^j, \Lambda_2 \in \mathbb{R}^{(n-j) \times (n-j)} \) and \( b_2, k_2 \in \mathbb{R}^{n-j} \). Then we choose \( k_1 \) such that the matrix \( \Lambda_1 - b_1 k_1^T \) has the eigenvalues \( \kappa_1, \kappa_2, \ldots, \kappa_j \). Clearly, the entries of \( k_1 \) and thus the norm of this vector do not depend on \( n \). Setting \( k^T : = (k_1^T, 0)^T \) with \( 0 \in \mathbb{R}^{n-j} \), the norm of \( k \) is also independent of \( n \). This feedback law yields the desired eigenvalues, since

\[
\Lambda_s - b_s k^T = \begin{pmatrix} \Lambda_1 & -b_1 k_1^T \\ -b_1 k_1^T & 0 - \Lambda_2 \end{pmatrix}
\]

is a block lower triangular matrix whose eigenvalues coincide with that of \( \Lambda_1 - b_1 k_1^T \) and \( \Lambda_2 \).

In practice, the output \( y_s = c_s^T x_s \) will not be available for implementation. Rather, \( y = Cx = c_s^T x_s + Cx_f \) can be measured. This means that \( y_s \) in \([13a]\) is replaced by \( y_s + y_f \) with \( y_f = Cx_f \). As a consequence, \([15a]\) becomes

\[
\dot{e}_s = (\Lambda_s - lc_c^T) e_s - ly_f = (\Lambda_s - lc_c^T) e_s - LCx_f
\]

and the true closed-loop system is described by

\[
\dot{\hat{x}}_s = \Lambda_s \hat{x}_s - b_s k^T (x_s - e_s) \\
\dot{\hat{y}}_s = c_s^T \hat{x}_s
\]

which needs to be completed with

\[
\dot{x}_f = \Lambda_f x_f - b_f k^T (x_s - e_s)
\]

i.e., with the fast subsystem \([10b]\) for \( u = -k^T (x_s - e_s) \). Note that \([17a]\) and \([17b]\) are finite-dimensional, while \([17a]\) is infinite-dimensional.

If we abbreviate \( \hat{x}_s = [x_s^T, e_s^T]^T \), then we can write \([17a]\), \([17b]\) as

\[
\hat{x}_s = A\hat{x}_s + LCx_f
\]

where \( \hat{A} \) is a Hurwitz matrix and \( \hat{l} = \begin{pmatrix} 0 \\ l \end{pmatrix} \). We pick \( \hat{\lambda} > 0 \) and \( M > 0 \) such that \( \|e^{\hat{A}t}\| \leq Me^{-\hat{\lambda}t} \) for all \( t \geq 0 \).
Proposition 3.3. Let
\[ \sum_{k=n+1}^{\infty} \frac{b_k^2}{\lambda_k^2} < \infty, \] (19)
then the bound \( \bar{M} \) is independent of the dimension \( n \) of slow subsystem \( \{0\} \).

We note that (19) follows from the admissibility of \( B \), using that \( \|v\|_{-1} < \infty \) for \( v \) defined after (7) and that \( \lambda_k \to -\infty \) as \( k \to \infty \) and \( \lambda_k < 0 \) for all \( k \geq n + 1 \).

Proof. Let \( x_{s,j} = [x_1, \ldots, x_j]^T, e_{s,j} = [e_1, \ldots, e_j]^T \) and \( \tilde{x}_{s,j} = \{x_{s,j}^T, e_{s,j}^T\}^T \). Then, for each \( k > j \) with \( j \) from the controller and observer construction, we obtain the two equations
\[
\dot{x}_k = \lambda_k x_k + b_k z_k^T \tilde{x}_{s,j}, \\
\dot{e}_k = \lambda_k e_k + b_k z_k^T \tilde{x}_{s,j},
\]
where \( z_k^T \) and \( z_k^T \) are suitable column vectors independent of \( n \) determined by \( k^T \) and \( l \). By construction and the suitable choice of \( \lambda_j, \tilde{M}_j > 0 \) with \( \lambda_j + \lambda_k < 0 \) for all \( k > j \) so that \( \|\tilde{x}_{s,j}\|_2 \leq \tilde{M}_je^{\lambda_j t}\|\tilde{x}_{s,j}(0)\|_2 \). By the variations of constants formula we obtain
\[
x_k(t) = e^{\lambda_k t}x_k(0) + \int_0^t e^{\lambda_k(t-s)}b_k z_k^T \tilde{x}_{s,j}(s) ds.
\]
Using the bound on \( \tilde{x}_{s,j} \), the Cauchy-Schwarz inequality \( |z_k^T \tilde{x}_{s,j}(s)| \leq \|z_k\|_2 \|\tilde{x}_{s,j}(s)\|_2 \), and noting \( e^{\lambda_k t} \leq e^{-\lambda_j t} \) this implies the sequence of estimates
\[
|x_k(t)| \leq e^{\lambda_k t}|x_k(0)| + \int_0^t e^{\lambda_k(t-s)}|b_k| \|z_k\|_2 \tilde{M}_je^{-\lambda_j s}\|\tilde{x}_{s,j}(0)\|_2 ds \\
= e^{\lambda_k t}|x_k(0)| + |b_k| \|z_k\|_2 \tilde{M}_je^{\lambda_k t} \int_0^t e^{-(\lambda_j + \lambda_k)s} ds \\
\leq e^{\lambda_k t}|x_k(0)| + \|z_k\|_2 \|\tilde{x}_{s,j}(0)\|_2 \tilde{M}_j |b_k| e^{\lambda_k t} - e^{-\lambda_j t} \lambda_k + \lambda_j \\
\leq e^{-\lambda_j t}\left(x_k(0) + \|z_k\|_2 \|\tilde{x}_{s,j}(0)\|_2 \tilde{M}_j \left|\frac{|b_k|}{\lambda_k + \lambda_j}\right|\right).
\]
The analogous inequality holds for the components of \( e_{s,j} \) with \( |x_k(0)| \) replaced by \( |e_k(0)| \) and \( \|z_k\|_2 \) by \( \|z_{2,k}\|_2 \). Since \( (a + b)^2 \leq 2a^2 + 2b^2 \) and \( e^{2\lambda_k t} \leq e^{-2\lambda_j t} \), we obtain for the 2-norm after some intermediate but straightforward computations making use of \( \|z\|_2 = \|z_{2,k}\|_2 + \|z_{1,k}\|_2 \) that
\[
\|\tilde{x}_{s,n}\|_2^2 \leq \|\tilde{x}_{s,j}\|_2^2 + \sum_{k=j+1}^{n} |x_k|^2 + |e_k|^2 \\
\leq \tilde{M}_je^{-2\lambda_j t}\|\tilde{x}_{s,j}(0)\|_2^2 + 2e^{-2\lambda_j t} \sum_{k=j+1}^{n} |x_k(0)|^2 + |e_k(0)|^2 \\
+ 2\tilde{M}_je^{-2\lambda_j t}\|z\|_2^2 \|\tilde{x}_{s,j}(0)\|_2^2 \tilde{M}_j \left(\frac{|b_k|}{\lambda_j + \lambda_k}\right)^2 \\
\leq \max\{2, \tilde{M}_j\} e^{-2\lambda_j t}\|\tilde{x}_{s,n}(0)\|_2^2 + 2\tilde{M}_je^{-2\lambda_j t}\|z\|_2^2 \|\tilde{x}_{s,n}(0)\|_2^2 \sum_{k=j+1}^{n} \left(\frac{|b_k|}{\lambda_j + \lambda_k}\right)^2 \\
\leq \max\{2, \tilde{M}_j\} e^{-2\lambda_j t}\|\tilde{x}_{s,n}(0)\|_2^2 \times \left(1 + \|z\|_2^2 \sum_{k=j+1}^{\infty} \left(\frac{|b_k|}{\lambda_j + \lambda_k}\right)^2 \right).
\]
Assumption (19) implies the convergence of the sum in the latter term which together with \( \|z\|_2^2 \) being independent of \( n \) by construction of \( k^T \) and \( l \) verifies the claim. \( \square \)

4. Assumptions and auxiliary estimates

In this section we formulate the precise assumptions on our abstract model and provide auxiliary estimates, which we will need for the proof of the main theorem in the next section.

Assumption 4.1. We impose the following assumptions on (10) and (17).

(A4) The matrices \( \Lambda_s - lc_l^T \) and \( \Lambda_s - b_kk \) are Hurwitz with eigenvalues whose real parts are smaller than \( \delta \), where \( \delta < 0 \) is independent of \( n \), and \( \|k\|, \|l\| \) and \( M \) are bounded with bounds independent of \( n \).

(A5) The sequence of eigenvalues \( \lambda_k \) satisfies \( \lambda_k < 0 \) for all \( k \in \mathbb{N} \), \( \lambda_k \to -\infty \) for \( k \to \infty \), \( \lambda_{n+1} < 0 \) and
\[
\sum_{k=1}^{\infty} \frac{1}{|\lambda_k|} = M < \infty.
\]

(A6) The following assumptions will be used alternatively.

(i) There is \( \alpha > 1 \) and \( d_1 > 0 \) such that \( |d_1| \leq \frac{\alpha}{\delta_{l}} \), and there is \( c_2 > 0 \) with \( |c_k| \leq c_2 \) for all \( k \geq n + 1 \).

(ii) Any of the conditions is fulfilled:
- The inequality \( \sum_{k=n+1}^{\infty} \frac{d_1}{|\lambda_k|} < \infty \) holds.
- There are constants \( c_1, c_2, c_3 > 0 \) with \( |b_k| \leq c_1k, |c_k| \leq c_2, \lambda_k \leq -c_3k^2 \) for all \( k \geq n \).
- For each \( m \in \mathbb{N} \) there is \( \gamma_m > 0 \) such that \( |\lambda_k^{-1} - \lambda_{k+m}^{-1}| \leq \gamma_m k^{-3} \) for all \( k \geq n \).
• There exist \( c_4, c_5, k_1 > 0 \) and pairwise disjoint sets \( S_j \subset \mathbb{N}, j \in \mathbb{N} \), each with at most \( s \in \mathbb{N} \) elements, \( \mathbb{N} = \bigcup_{j \in \mathbb{N}} S_j \), such that \( \min S_j \geq c_4 j \), \( \max S_j \leq \min S_{j+1} + k_1 \), and \( \sum_{j \in S_j} c_5 b_k / |\lambda_j| \leq c_5 j^{-2} \).

Assumption \((A4)\) imposes bounds on the norms of \( k \) and \( I \), which, as discussed after \((10)\), are satisfied if only finitely many eigenvalues are shifted. Moreover, Assumption \((A5)\) imposes a restriction on the growth of the eigenvalues. In particular this condition is fulfilled, if \( \lambda_k \sim k^2 \) as is typically the case for diffusion-reaction problems. Assumptions \((A6)(i)\) and \((A6)(ii)\) can be applied in case of boundary control. Assumption \((A6)(ii)\) is satisfied for the prototype system from Section 4 in the boundary control case. This can be checked using \( \lambda_k = r - ((2k - 1)\pi / |S|)^2 \) defined after \((3b)\) together with the \( b_{2k} = (-1)^k (2k - 1)\pi / \sqrt{2} \) defined before \((3j)\) and the coefficients \( c_k \) obtained from point measurements at some point \( \xi \in (0, 1) \cup \mathbb{Q} \). We can then define the \( S_j \) to be of the form
\[
S_j = \{ mk_1 + k, (m+1)k_1 - k + 1 \}
\]
with \( m = [2j/k_1] \) and \( j = m - mk_1/2 \). Here \( k_1 \in \mathbb{N} \) chosen such that \( c_{mk_1+k} = c_{(m+1)k_1-k+1} \) for all these \( m \) and \( k \). Such a \( k_1 \) exists in case of a point measurements at some point \( \xi \in (0, 1) \cap \mathbb{Q} \) due to the periodicity of the cosine function and \( [2j/k_1] \) denotes the integer part of \( 2j/k_1 \). For \( \xi \in (0, 1) \setminus \mathbb{Q} \), the existence of \( k_1 \) with \( c_{mk_1+k} = c_{(m+1)k_1-k+1} \), which is crucial for ensuring the last item of Assumption \((A6)(ii)\), cannot be guaranteed. Numerical tests have, however, revealed that one can still find \( k_1 \) such that this identity is satisfied approximately with a small error. We thus expect that Assumption \((A6)(ii)\) is also satisfied for \( \xi \not\in \mathbb{Q} \), although a formal proof of this property is beyond the scope of this paper.

In the remainder of this section we derive auxiliary estimates for the solutions of the closed loop system \((17)\) in case \((A5)\) and \((A6)(i)\) or \((A6)(ii)\) are satisfied.

### 4.1. Estimates under Assumptions \((A5)\) and \((A6)(i)\)

**Lemma 4.3.** Let \((A5)\) and \((A6)(i)\) be satisfied. Then there exists a constant \( C_1' \) such that for all \( k \geq n + 1 \) the inequalities
\[
|\tau_k(t)| \leq e^{\lambda_k t} |\tau_k(0)| + C_1' \frac{\|k\|_2}{\lambda_k} \|\tau_k(t) - \tau_s(s)\|_{2, \infty}
\]
\[
\leq e^{\lambda_{n+1} t} |\tau_k(0)| + C_1' \frac{\|k\|_2}{\lambda_k} \|\tau_k(t) - \tau_s(s)\|_{2, \infty}
\]
hold, where \( \alpha = 1 \) in case \((A6)(ii)\) holds. In particular, this implies the existence of \( C_2 > 0 \) with
\[
\|\tau_k(t)\|_2 \leq e^{\lambda_{n+1} t} \|\tau_k(0)\|_2 + \frac{C_2}{\sqrt{n^2}} \|k\|_2 \|\tau_k(t) - \tau_s(s)\|_{2, \infty}
\]
and in case that \((A6)(i)\) holds, additionally
\[
\|\tau_k(t)\|_1 \leq e^{\lambda_{n+1} t} \|\tau_k(0)\|_1 + \frac{C_1'}{\lambda_k} \|k\|_2 \|\tau_k(t) - \tau_s(s)\|_{2, \infty}.
\]

**Proof.** By the variation of constants formula we obtain the estimate
\[
|\tau_k(t)| \leq e^{\lambda_k t} |\tau_k(0)\|
\]
\[
+ \int_0^t e^{\lambda_k (t - \tau)} \|k\|_2 \|\tau_k(\tau) - \tau_s(s)\|_{2, \infty} \, d\tau
\]
\[
\leq e^{\lambda_k t} |\tau_k(0)| + \frac{1}{\lambda_k} \|k\|_2 \|\tau_k(t) - \tau_s(s)\|_{2, \infty}.
\]

The inequalities in \((20)\) then follow directly from \((A6)(i)\) with \( C_1 = C_{1'} \) or \((A6)(ii)\) with \( C_1 = c_1 / c_5 \) and the fact that \( \lambda_k \leq \lambda_{n+1} \) for \( k \geq n + 1 \) due to \((A5)\).

The additional inequalities \((21)\) and \((22)\) follow by taking the \( \ell_2 \) - or \( \ell_1 \)-norm, respectively, of the expressions on both sides, making use of the triangle inequality. For \((21)\) we additionally use the inequality
\[
\sum_{k=n+1}^{\infty} \frac{1}{k^{2\alpha}} \leq \frac{1}{n^{2\alpha-1}} \sum_{k=n+1}^{\infty} \frac{1}{k^{2\alpha}} \leq \frac{\pi^2}{6} \left( \frac{1}{n^{2\alpha-1}} \right) = \frac{\pi^2}{6} \frac{1}{n^{2\alpha-1}}
\]
as well as \( n^{2\alpha-1} \geq n^\alpha \) and for \((22)\) we use the definition of the Hurwitz zeta function with \( \alpha > 1 \) and \( n \geq 1 \) if \((A6)(i)\) holds.

### 4.2. Estimates under Assumptions \((A5)\) and \((A6)(ii)\)

We now define the quantities
\[
z_k := c_k x_k, \quad k \geq n + 1,
\]
so that
\[
y_f = C x_f = c_f^T x_f = \sum_{k=n+1}^{\infty} z_k
\]
holds provided the infinite sum is absolutely convergent, i.e., \( \|z\|_1 < \infty \). The following input-to-output stability (IOS) estimate for \( \|z\|_1 \) ensures this absolute convergence.

**Lemma 4.4.** Let \((A5)\) and \((A6)(ii)\) hold. Then there are constants \( C^1_a, C^2_a > 0 \) such that with \( \eta_n(t) = \sum_{k=n+1}^{\infty} e^{\lambda_k t} \) for all \( t > 0 \) we obtain
\[
\|z(t)\|_1 \leq C^1_a \eta_n(t) \|x_f(0)\|_2 + \frac{C^2_a}{n^2} \|\tau_k(t) - \tau_s(s)\|_{2, \infty}.
\]
Proof. Taking into account the definition of the Hurwitz zeta function note that
\[ \mathcal{C}_1 \zeta(a, n + 1)\|k\|_2 = \frac{1}{n^{\alpha - 1}} \mathcal{C}_1 \|k\|_2 \sum_{k=1}^{\infty} \left( \frac{n}{n+k} \right)^\alpha. \]

The second summand on the right hand side of the inequality then results by summing up the second terms in (A6)(ii). Since
\[ \sum_{k=1}^{\infty} \left( \frac{n}{n+k} \right)^\alpha = \sum_{k=1}^{\infty} \left( \frac{1}{1+k/n} \right)^\alpha \leq \sum_{k=0}^{\infty} \left( \frac{1}{1+k/n} \right)^\alpha \leq n \sum_{k=0}^{\infty} \left( 1 + k/n \right)^\alpha, \]
defining \( \mathcal{C}_2 := c_2 \mathcal{C}_1 \|k\|_2 \sum_{k=1}^{\infty} (n/(n+k))^\alpha < \infty \) with \( c_2 \) from (A6)(ii) yields a constant that is independent of \( n \).

To derive the first summand, consider (4.3) and note that
\[ \sum_{k=1}^{\infty} e^{\lambda_k t} |x_k(0)| \leq \sum_{k=1}^{\infty} e^{\lambda_k t} \|x_f(0)\|_2 \]
as \( |x_k(0)| \leq \|x_f(0)\|_2 \). The sum \( \sum_{k=1}^{\infty} e^{\lambda_k t} \) with \( \lambda_{k+1} < \lambda_k < 0 \) for \( k \geq n+1 \) converges absolutely to a function \( \eta_k(t) \) fulfilling \( \lim_{t \to 0} \eta_k(t) = \infty \) and \( \lim_{t \to \infty} \eta_k(t) = 0 \) with exponential convergence. Let \( a_k(t) = e^{\lambda_k t} \) for \( k \geq n+1 \), then absolute convergence for \( t > 0 \) becomes apparent as \( e^{\lambda_k t} \leq q/|\lambda_k| \) for suitable \( q > 0 \) and \( 1/|\lambda_k| \) is absolutely convergent by (A5). For \( t = 0 \) we have \( e^{\lambda_k t} = 1 \) for all \( k \) so that the series approaches infinity. This yields the first summand on the right hand side of the estimate with \( \mathcal{C}_1 = \mathcal{C}_2 \) from (A6)(ii).

\[ \blacksquare \]

4.3. Estimates under Assumptions (A5) and (A6)(ii)

In this section we derive a counterpart for Lemma 4.4 in case (A5) and (A6)(ii) are satisfied. The difficulty here is that Lemma 4.2 does not give us an immediate estimate for the \( l_1 \)-norm \( \|x_f(t)\|_1 \) if (A6)(ii) does not hold. In order to circumvent this problem we have to use another definition of \( z_j \). We start with an auxiliary lemma.

Lemma 4.5. Let (A5) and (A6)(ii) hold. Then there is a constant \( \mathcal{C}_3 > 0 \) such that
\[ \int_0^t \left| \sum_{k \in S_j} c_k b_k e^{\lambda_k \tau} \right| d\tau \leq \frac{\mathcal{C}_3}{j^2} \]
for all \( t \geq 0 \) and all \( j \in \mathbb{N} \) with \( j \geq n/c_4 \) with \( n \) and \( c_4 \) from (A6)(ii).

Proof.Abbreviate \( a_k = c_k b_k \) and consider the function
\[ \tau \mapsto h_j(\tau) := \sum_{k \in S_j} a_k e^{\lambda_k \tau}. \]
This function is continuous and, since \( S_j \) has \( s \) elements, according to Descartes’ rule of signs \( 2^s \) Theorem 3.1] it has \( \sigma \leq s \) zeros \( t_1, \ldots, t_\sigma > 0 \), which we number in ascending order. This means that
\[ t \mapsto H_j(t) := \int_0^t h_j(\tau) d\tau \]
has at most \( \sigma \leq s \) local maxima and minima \( H_j(t_l), l = 1, \ldots, \sigma \) and \( h_j \) can change its sign only at the times \( t_l \). Since \( h_j(t) \to 0 \) as \( t \to \infty \) exponentially fast, the limit \( H_j^\infty := \lim_{t \to \infty} H_j(t) \) exists and is finite. Let
\[ H_j^+ := \max \{0, H_j(t_1), \ldots, H_j(t_\sigma), H_j^\infty\} \]
and
\[ H_j^- := \min \{0, H_j(t_1), \ldots, H_j(t_\sigma), H_j^\infty\}. \]
Then
\[ \left| \int_{t_1}^{t_2} h_j(\tau) d\tau \right| = |H(s_2) - H(s_1)| \leq H_j^+ - H_j^- \]
for all \( 0 \leq s_1 \leq s_2 \).

Now we pick an arbitrary \( t > 0 \), let \( \sigma_1 \leq \sigma \) be the largest index with \( t_{\sigma_1} < t \) and set \( \tau_0 := 0, \tau_1 := t \) for \( l = 1, \ldots, \sigma_1 \) and \( \tau_{\sigma_1+1} := t \). Then, since \( h \) can only change sign at the times \( t_l \) we get
\[ \int_0^t |h_j(\tau)| d\tau = \sum_{l=0}^{\sigma_1} \left| \int_{\tau_l}^{\tau_{l+1}} h_j(\tau) d\tau \right| \]
\[ \leq \sum_{l=0}^{\sigma_1} H_j^+ - H_j^- \leq s(H_j^+ - H_j^-). \]
It thus suffices to prove that there is \( c_0 > 0 \) with \( |H_j(t_l)| \leq c_0 j^{-2} \) for all \( l = 1, \ldots, \sigma \) and \( \lim_{t \to \infty} |H_j(t)| \leq c_0 j^{-2} \). Then the assertion follows with \( \mathcal{C}_3 = 2sc_0 \).

To this end, observe that
\[ H_j(t) = \sum_{k \in S_j} \frac{a_k}{p_k} e^{p_k t} - \frac{a_k}{p_k} \]
Hence,
\[ \lim_{t \to \infty} |H_j(t)| = \left| \sum_{k \in S_j} \frac{a_k}{p_k} \right| \leq \frac{\mathcal{C}_3}{j^2}. \]

For the \( t_l \) we obtain
\[ H_j(t_l) = \sum_{k \in S_j} \frac{a_k}{p_k} p_k t_l - \frac{a_k}{p_k} \]
\[ \leq \left| \sum_{k \in S_j} \frac{a_k}{p_k} p_k t_l \right| + \left| \sum_{k \in S_j} \frac{a_k}{p_k} \right| \]
The second term is bounded by \( \frac{\mathcal{C}_3}{j^2} \), as above. For the first term, observe that \( h_j(t_l) = 0 \) implies
\[ a_k e^{p_k \tau} = - \sum_{k \in S_j \setminus \{k\}} a_k e^{p_k \tau}, \]
7
where $\hat{k}$ is an arbitrary element in $S_j$. This yields
\[
\left| \sum_{k \in S_j} \frac{a_k}{p_k} e^{p_k t} \right| = \left| \sum_{k \in S_j \setminus \{\hat{k}\}} \left( \frac{a_k}{p_k} - \frac{a_{\hat{k}}}{p_{\hat{k}}} \right) e^{p_k t} \right| \\
\leq \sum_{k \in S_j \setminus \{\hat{k}\}} \frac{a_k}{p_k} \left| 1 - \frac{1}{p_k} \right| \\
\leq \sum_{k \in S_j \setminus \{\hat{k}\}} c_{\hat{k}} \gamma(k_1) k^{-3} \\
\leq \sum_{k \in S_j \setminus \{\hat{k}\}} c_{\hat{k}} \gamma(k_1) k^{-2} \\
\leq \frac{(s - 1) c_{\hat{k}} \gamma(k_1)}{c_{\hat{k}}} \frac{1}{j^2}.
\]

All in all, we obtain the desired bound on $|H_j(t_j)|$ with $c_6 = \frac{(s - 1) c_{\hat{k}} \gamma(k_1)}{c_{\hat{k}}} + c_5$.

Using the notation of Lemma 4.5, we consider the quantities
\[
z_j := \sum_{k \in S_j} c_k x_k
\]
for $j \in \mathbb{N}$. We note that then for $n = \min S_{j_0}$ we obtain
\[
y_j = C x_j = e_j^T x_j = \sum_{j = j_0}^n z_j
\]
holds, provided the infinite sum is absolutely convergent, i.e., provided $\|z\|_1 < \infty$ for $z = (z_{j_0}, z_{j_0+1}, \ldots)^T$.

**Lemma 4.6.** Let (A5) and (A6)(ii) hold. Then there is a constant $C_j > 0$ independent of $n$, such that for all $j \in \mathbb{N}$ with $n = j_0 e_4 - 1$ satisfying $\lambda_{n+1} < 0$, and all $t \geq 0$ the inequality
\[
|z_j(t)| \leq C_j \sum_{k \in S_j} e^{\lambda_k t} x_k(0) \\
+ \frac{C_j}{j^2} \|k\|_2 \|\| (x_j - e_j)\|_{[0, t], 2, \infty}
\]
holds.

**Proof.** By the variation of constants formula we obtain the estimate
\[
|z_j(t)| = \left| \sum_{k \in S_j} c_k x_k(t) \right| \\
\leq \left| \sum_{k \in S_j} e^{\lambda_k t} x_k(0) \right| \\
+ \left| \int_0^t \sum_{k \in S_j} c_k e^{\lambda_k (t-\tau)} b_k k^T (x_j(\tau) - e_j(\tau)) d\tau \right| \\
\leq \left| \sum_{k \in S_j} e^{\lambda_k t} x_k(0) \right| \\
+ \left| \int_0^t \sum_{k \in S_j} c_k b_k e^{\lambda_k (t-\tau)} d\tau \|k\|_2 \| (x_j - e_j)\|_{[0, t], 2, \infty} \right|.
\]

Observing that
\[
\int_0^t \left| \sum_{k \in S_j} c_k b_k e^{\lambda_k (t-\tau)} \right| d\tau = \int_0^t \left| \sum_{k \in S_j} c_k b_k e^{\lambda_k \tau} \right| d\tau,
\]
the assertion follows from Lemma 4.5 with $C_4 = c_2$.

Based on Lemma 4.6 we can obtain the counterpart of Lemma 4.4.

**Lemma 4.7.** Let (A5) and (A6)(i) hold with $n = j_0 e_4 - 1$, $j_0 \in \mathbb{N}$. Then there are constants $C_4, C_5 > 0$ such that for all $\eta_n(t) = \sum_{j=n+1}^\infty e^{\lambda_i t}$ the inequality
\[
\|z(t)\|_1 \leq C_4 \eta_n(t) \|x_j(0)\|_2 + C_5 \frac{\|x_j - e_j\|_{[0, t], 2, \infty}}{n}
\]
holds for all $t > 0$.

**Proof.** The inequality follows by summing the right hand sides of Lemma 4.6 over $j \geq j_0$. Then the first terms sum up to
\[
\sum_{j = j_0}^\infty C_j \sum_{k \in S_j} e^{\lambda_k t} x_k(0) \leq C_j \sum_{k = n+1}^\infty e^{\lambda_k t} \|x_k(0)\|_2 \leq \eta_n(t) \|x_j(0)\|_2
\]
with $C_4 = C_j$. Using the same inequality as in the proof of Lemma 4.3 we can estimate
\[
\sum_{j = j_0}^\infty \frac{1}{j^2} \leq \frac{\pi^2}{6 j_0} \leq \frac{C_4 \pi^2}{6(n+1)} \leq \frac{C_4 \pi^2}{6n}
\]
and thus the second terms sum up to $\frac{C_4^2 \pi^2}{6} \|x_j - e_j\|_{[0, t], 2, \infty}$ if we set
\[
C_5^2 := \frac{C_4 \pi^2}{6}.
\]

This shows the claim.

**4.4. A further estimate under Assumption (A5).**

In order to apply a small-gain argument, we also need an estimate for the slow subsystem. The following lemma yields this result if (A5) holds.

**Lemma 4.8.** Let (A5) hold and let $\tilde{x}_s = (x_s, e_s)$. There exists $C_4^s, C_5^s > 0$ such that the inequality
\[
\|\tilde{x}_s(t)\| \leq C_4^s e^{-\lambda t} \|\tilde{x}_s(0)\| + C_5^s \int_0^t \|z(\tau)\|_1 d\tau
\]
holds for all $t \geq 0$ with $\lambda > 0$ defined after (18).

**Proof.** The variation of constants formula applied to (18) or (26) and the definition of $z$ provide
\[
\|\tilde{x}_s(t)\| = \left\| e^{\tilde{A} t} \tilde{x}_s(0) + \int_0^t e^{\tilde{A} (t-\tau)} \| I c^T x_f(\tau) d\tau \right\| \\
\leq \|e^{\tilde{A} t} \tilde{x}_s(0)\| + \int_0^t e^{\tilde{A} (t-\tau)} \| I z(\tau)\|_1 d\tau.
\]
Note that the norm of $\tilde{l}$ satisfies the same $n$-independent bound as the norm of $l$. Setting
\[ c_2^2 := \int_0^{\infty} \|e^{\tilde{A}(t-\tau)}\tilde{l}\| d\tau < \infty, \]
the integral term can be estimated via
\[
\left\| \int_0^t e^{\tilde{A}(t-\tau)}\|z(\tau)\| d\tau \right\| \\
\leq \int_0^t \|e^{\tilde{A}(t-\tau)}\|\|z(\tau)\| d\tau \leq c_2^2 \int_0^t \|z(\tau)\| d\tau.
\]
Together with the definition of $\hat{\lambda} > 0$ and $\hat{M} > 0$ after (18) this yields the claim with $c_1^2 = \hat{M}$. □

5. Main result

The proof of our following main stability theorem is inspired by [3, 26], which give stability criteria for interconnected systems based on input-to-output stability (IOS) and unbounded observability (UO) properties. There are, however, two important differences. On the one hand, the proof is significantly simplified here as we can make use of the linearity of the system. On the other hand, the term $\eta_\nu(t)$ on the right-hand sides of the estimates in Lemma 4.3 and 4.7 tends to $\infty$ as $t \to 0$, which requires a slightly more involved treatment of this term.

Theorem 5.1. Consider the closed-loop system (17) satisfying (A5) and (A6)(i) or (A6)(ii). Then, for sufficiently large $n$ there are constants $C > 0$ and $\lambda > 0$ such that the slow subsystem satisfies the estimate
\[ \|\tilde{x}_s(t)\|_2 \leq Ce^{-\lambda t}\|\tilde{x}_s(0)\|_2 + Ce^{-\lambda t}\|x_f(0)\|_2 \]
for all $t \geq 0$ and all initial conditions $\tilde{x}_s(0) \in \mathbb{R}^n$ and $x_f(0) \in \ell^2$. In particular, this implies exponential stability of (17) in $\ell^2$.

Proof. Lemma 4.5 yields
\[ \|\tilde{x}_s(t)\|_2 \leq C_1 e^{-\lambda t}\|\tilde{x}_s(0)\|_2 + C_2 \int_0^t \|z(\tau)\|_1 d\tau. \]

The $z$-dependent term can be suitably bounded. Using Lemma 4.3 in case (A6)(i) holds or Lemma 4.7 in case (A6)(ii) holds we obtain
\[
\int_0^t \|z(\tau)\|_1 d\tau \\
\leq \int_0^t \eta_\nu(\tau) d\tau \|x_f(0)\|_2 + \frac{c_2^2}{\nu^2}\|x_s - e_s\|_{0,t} \|x_f(0)\|_2.
\]
with $* = a$ and $\beta = a - 1 > 0$ if (A6)(i) holds and $* = b$ and $\beta = 1$ if (A6)(ii) holds. The function $\eta_\nu(t) = \sum_{j=n+1}^{\infty} e^{\beta t}$ is obtained as the limit of the sequence $(a_p(t))_p \geq n+1$ with $a_p(t) = \sum_{j=n+1}^{\infty} e^{\beta t}$. Observing that $a_p(t)$ is monotone, positive and absolutely integrable, Beppo Levi’s monotone convergence theorem implies that
\[
\int_0^t \eta_\nu(t) d\tau = \lim_{p \to \infty} \int_0^t a_p(t) d\tau = \lim_{p \to \infty} \int_0^t \sum_{j=n+1}^{p} e^{\beta t} d\tau
\]
\[ = \sum_{j=n+1}^{\infty} \frac{1 - e^{\beta t}}{\beta} \leq \sum_{j=n+1}^{\infty} \frac{1}{\beta} \leq M_\lambda, \]
where the last inequality follows from Assumption (A5).

As a consequence
\[
\int_0^t \|z(\tau)\|_1 d\tau \leq C_1 M_\lambda \|x_f(0)\|_2 + \frac{c_2^2}{\nu^2}\|x_s - e_s\|_{0,t} \|x_f(0)\|_2.
\]

Using $\|x_s - e_s\|_{0,t} \|x_f(0)\|_2 \leq \sqrt{2}\|\tilde{x}_s\|_{0,t} \|x_f(0)\|_2$ we obtain
\[ \|\tilde{x}_s(t)\|_2 \leq C_1 e^{-\lambda t}\|\tilde{x}_s(0)\|_2 + C_5 \|x_f(0)\|_2 + \frac{c_6}{\nu^2}\|\tilde{x}_s\|_{0,t} \|x_f(0)\|_2, \]
with $C_5 = C_1^2 M_\lambda C_4$ and $C_6 = C_2^2 \sqrt{2} C_2^2$. Taking the supremum for $t \in [0, \tau]$ of both sides of this inequality yields
\[ \|\tilde{x}_s\|_{0,t} \|x_f(0)\|_2 \leq C_1^2 \|\tilde{x}_s(0)\|_2 + C_5 \|x_f(0)\|_2 + \frac{c_6}{\nu^2}\|\tilde{x}_s\|_{0,t} \|x_f(0)\|_2, \]
Choosing $n$ so large that $C_6/\nu^2 \leq 1/2$ holds and subtracting the last term on the right-hand side implies
\[ \|\tilde{x}_s\|_{0,t} \|x_f(0)\|_2 \leq 2C_1^2 \|\tilde{x}_s(0)\|_2 + 2C_5 \|x_f(0)\|_2. \]

Moreover, from (21) we obtain
\[ \|x_f(t)\|_2 \leq e^{\lambda n + 1} \|x_f(0)\|_2 + \frac{C_2}{\sqrt{2} n^2} \|k\|_2 \|\tilde{x}_s\|_{0,t} \|x_f(0)\|_2, \]
We increase $n$ further, if necessary, such that $2C_1^2 \|C_5/\nu^2 \leq 1/(8(C_5 + 1))$, $2C_5/\nu^2 \leq 1/(8(C_5 + 1))$, $2C_1 C_5/\nu^2 \leq 1/(8(C_5 + 1))$, and $2C_5/\nu^2 \leq 1/(8(C_5 + 1))$ hold. This implies
\[ \|\tilde{x}_s(t)\|_2 \leq C_1^4 e^{-\lambda t}\|\tilde{x}_s(0)\|_2 + C_5 \|x_f(0)\|_2 \\
+ \frac{1}{8(C_5 + 1)} \|\tilde{x}_s(0)\|_2 + \|x_f(0)\|_2, \]
and
\[ \|x_f(t)\|_2 \leq e^{\lambda n + 1} \|x_f(0)\|_2 \\
+ \frac{1}{8(C_5 + 1)} \|\tilde{x}_s(0)\|_2 + \|x_f(0)\|_2. \]

Choosing $\tau > 0$ such that $C_1^2 e^{-\lambda \tau} \leq 1/(8(C_5 + 1))$ and $e^{\lambda n + \tau} \leq 1/(8(C_5 + 1))$, we obtain
\[ \|\tilde{x}_s(\tau)\|_2 \leq \frac{1}{4(C_5 + 1)} \|\tilde{x}_s(0)\|_2 + (C_5 + 1) \|x_f(0)\|_2 \\
\|x_f(\tau)\|_2 \leq \frac{1}{4(C_5 + 1)} \|x_f(0)\|_2 + \frac{C_5}{4(C_5 + 1)} \|\tilde{x}_s(0)\|_2. \]
Since the system is time-invariant, for \( t_k = k\tau, k \in \mathbb{N}_0 \), by the same reasoning we obtain the inequalities

\[
\|\tilde{x}_s(t_{k+1})\| \leq \frac{1}{4(C_5 + 1)} \|\tilde{x}_s(t_k)\|_2 + (C_5 + 1)\|x_f(t_k)\|_2
\]

\[
\|x_f(t_{k+1})\| \leq \frac{1}{4(C_5 + 1)} \|x_f(t_k)\|_2 + \frac{1}{4(C_6 + 1)} \|x_s(t_k)\|.
\]

Considering equality the two equations can be considered as coupled difference equations that admit a closed-form solution. Based on this the following estimates are obtained for all \( k \geq 1 \)

\[
\|\tilde{x}_s(t_k)\|_2 \leq 2 \left( \frac{3}{4} \right)^k (\|\tilde{x}_s(t_0)\|_2 + (C_5 + 1)\|x_f(t_0)\|_2)
\]

\[
\|x_f(t_k)\|_2 \leq 2 \left( \frac{3}{4} \right)^k (\|x_f(t_0)\|_2 + \|x_s(t_0)\|_2).
\]

Between the times \( t_k \) with the same arguments as those leading to (28) we obtain

\[
\|\tilde{x}_s[t_k, t_{k+1}]\|_2, \infty \leq 2C_5 \|\tilde{x}_s(t_k)\| + 2C_5 \|x_f(t_k)\|_2.
\]

Together this yields the claimed inequality (27) with \( \lambda = \log(4/3) \) and \( C = \max\{C_1, C_5\}(C_5 + 1)6/3 \).

To prove exponential stability in \( \mathcal{L}_2 \) of the overall system, it remains to be shown that also \( \|x_f(t)\|_2 \) has an upper bound that is linear in the norms of the initial conditions \( \|x_f(0)\|_2 \) and \( \|\tilde{x}_s(\cdot)\|_2 \) and decays exponentially. This follows by combining the already proved inequality (27) with estimate (21) from Lemma 3.3 as follows: We again pick \( \tau > 0 \) and consider the times \( t_k = k\tau \). Applying (21) on the interval \( [t_k, t_{k+1}] \) yields

\[
\|x_f(t_{k+1})\|_2 \leq e^{\lambda_0 + \tau} \|x_f(t_k)\|_2 + \frac{C_2}{\sqrt{\eta^3}} \|k\|_2 (\|x_s - e_s\|_{t_k, t_{k+1}}) \|2, \infty \]

Then \( e^{\lambda_0 + \tau} \|x_f(t_k)\|_2 \leq 2 \left( \frac{3}{4} \right)^k (\|x_f(t_0)\|_2 + \|x_s(t_0)\|_2, \infty) \]

\[
\leq 2 \left( \frac{3}{4} \right)^k (\|x_f(t_0)\|_2 + \|x_s(t_0)\|_2, \infty) \].
\]

Thus we have

\[
\|x_f(t_{k+1})\|_2 \leq e^{\lambda_0 + 1 + \tau} \|x_f(t_k)\|_2 + \frac{C_2}{\sqrt{\eta^3}} \|k\|_2 \|x_f(t_k)\|_2 + \|x_s(t_k)\|_2.
\]

with \( \tilde{C} = \frac{2C_2}{\sqrt{\eta^3}} \|k\|_2 \). By induction we obtain

\[
\|x_f(t_{k+1})\|_2 \leq e^{\lambda_0 + 1 + \tau} \|x_f(t_k)\|_2 + \frac{C_2}{\sqrt{\eta^3}} \|k\|_2 \|x_f(t_k)\|_2 + \|x_s(t_k)\|_2.
\]

\[
\leq e^{\lambda_0 + 1 + \tau} \|x_f(t_k)\|_2 + \|x_s(t_k)\|_2.
\]

\[
+ (k + 1)e^{-\lambda_0 t_0} (\|x_s(t_0)\|_2 + \|x_f(t_0)\|_2)
\]

\[
\leq e^{\lambda_0 + \tau} \|x_f(t_0)\|_2 + \|x_f(t_0)\|_2.
\]

\[
+ \tilde{C}e^{-\lambda_0 t_0} (\|x_s(t_0)\|_2 + \|x_f(t_0)\|_2).
\]

Here we used \( \tilde{\lambda} = \min\{\lambda, -\lambda_{n+1}\} \) and the fact that for any \( \lambda \leq \tilde{\lambda} \) there is \( M > 0 \) with \( (k + 1)e^{-\lambda t_0} \leq Me^{-\lambda t_{k+1}} \). Between the times \( t_k \) we can again use the same arguments as those leading to (28), establishing the desired exponential estimate

\[
\|x_f(t)\|_2 \leq C e^{-\lambda t} (\|\tilde{x}_s(0)\|_2 + \|x_f(0)\|_2) \quad (29)
\]

with suitably redefined \( C, \lambda > 0 \) for all \( t \geq 0 \).

**Remark 5.2.** The particular value \( \lambda = \log(4/3) \) in the proof could be changed to any arbitrary positive constant by adjusting the constants, the controller and the value of \( n \) appropriately. This means that the resulting exponential decay rate \( -\lambda = -\min\{\lambda, -\lambda_{n+1}\} \) can be made as negative as desired. Note, however, that other performance measures such as the phase and gain margin of the closed-loop transfer function cannot be directly determined from our approach.

**Corollary 5.3.** Let the assumptions of Theorem 5.1 hold true. Then the closed-loop system composed of (3) with state feedback \( u = -k^T \tilde{x}_s \) evaluated using the observer (14) is exponentially stable in \( X = (L^2(0, 1))^N \).

**Proof.** The result follows by combining the Riesz basis property of the operator \( A \) and the estimates (27) and (29), which imply the existence of constants \( C > 0 \) and \( \lambda > 0 \) so that

\[
\|x(t)\|_X = \left( \sum_{k=1}^{\infty} \|x(t_k)\phi_k\|_X \right)^{\frac{1}{2}} \leq \|x(t_0)\|_X + (\sum_{k=1}^{\infty} (\|x(t_k)\|_X)^2)^{\frac{1}{2}}
\]

\[
\leq \|x(0)\|_X + \|x_f(t_0)\|_X + \tilde{C}e^{-\lambda t_0} (\|x_s(0)\|_X + \|x_f(0)\|_X)
\]

\[
\leq 2Ce^{-\lambda t} (\|x_s(0)\|_X + \|x_f(0)\|_X)
\]

\[
\leq 2Ce^{-\lambda t} (\|x_s(0)\|_X + \|x_f(0)\|_X + \|x_f(0)\|_X)
\]

\[
= 2Ce^{-\lambda t} (\|x(0)\|_X + \|x_f(0)\|_X).
\]

Here in the last step we used the inequalities \( \|x_s(0)\|_X \leq \|x(0)\|_X \) and \( \|x_f(0)\|_X \leq \|x(0)\|_X \). This shows the claim.

**Remark 5.4.** In the MIMO-case, the operators \( b \) and \( e^T \) are replaced by operators \( B \) with \( m \) columns and \( C \) with \( l \) rows, respectively. This implies that the vectors \( k \) and \( l \) become matrices \( K \) and \( L \) of appropriate dimensions. All proofs can be straightforwardly extended to this case if the following modifications are made in Assumption 4.1: the respective lemmas and their proofs.

- The vector norms \( \|k\| \) and \( \|l\| \) are replaced by the corresponding induced matrix norms \( \|K\| \) and \( \|L\| \).
- The fraction \( \|b_k\|/\|k\| \) is replaced by \( \|b_k\|/\|k\| \), where \( b_k \) is now the \( k \)-th row of \( B \).
- The modulus \( |c_k| \) is replaced by the norm \( \|c_k\| \), where \( c_k \) is the \( k \)-th column of \( C \).
- The modulus \( \sum_{k \in S_k} c_k b_k/\|\lambda\| \) is replaced by the induced matrix norm \( \sum_{k \in S_k} c_k b_k^T/\|\lambda\| \). Note that \( c_k b_k^T \) is an \( l \times m \)-matrix in the MIMO-case.
6. Numerical computation of $n$

While our approach in principle allows for computing a bound on $n$ for which the inequalities required in the proof of Theorem 5.1 hold, this bound will be very conservative. We can, however, use a numerical approach that leads to a tighter bound: We fix a second index $m > n$ and split the state of the fast subsystem into

$$x_{f1} = [x_{n+1}, \ldots, x_m]^T \text{ and } x_{f2} = [x_{m+1}, x_{m+2}, \ldots]^T.$$ 

Then the overall closed loop system becomes

$$\dot{x} = \Lambda x - b_1 k^T (x - e_s) \quad \text{(30a)}$$
$$\dot{e}_s = (\Lambda - l c_1^T) e_s - l c_1^T f_{f1} x_{f1} - l c_1^T f_{f2} x_{f2} \quad \text{(30b)}$$
$$\dot{x}_{f1} = \Lambda_{f1} x_{f1} - b_{f1} k^T (x - e_s) \quad \text{(30c)}$$
$$\dot{x}_{f2} = \Lambda_{f2} x_{f2} - b_{f2} k^T (x - e_s). \quad \text{(30d)}$$

If we neglect the infinite-dimensional part $x_{f2}$ of the fast dynamics, then we obtain

$$\dot{x} = \Lambda x - b_1 k^T (x - e_s) \quad \text{(31a)}$$
$$\dot{e}_s = (\Lambda - l c_1^T) e_s - l c_1^T f_{f1} x_{f1} \quad \text{(31b)}$$
$$\dot{x}_{f1} = \Lambda_{f1} x_{f1} - b_{f1} k^T (x - e_s), \quad \text{(31c)}$$

Equation (31) defines a finite-dimensional LTI system, whose stability can be easily checked by analyzing the eigenvalues of the overall system matrix. The question, however, is, whether stability of (31) implies stability of the true closed-loop system (30). In order to see whether this is the case, we define

$$x_{sf} := \begin{pmatrix} x_s \\ x_{f1} \end{pmatrix}, \quad b_{sf} := \begin{pmatrix} b_s \\ b_{f1} \end{pmatrix},$$
$$\tilde{k} := \begin{pmatrix} k \\ 0 \end{pmatrix}, \quad \tilde{c}_s^T := (0^T c_{f1}), \quad c_{sf}^T := (c_s^T c_{f1}),$$
and $\Lambda_{sf} := \text{diag}(\Lambda_s, \Lambda_{f1})$. With these vectors and matrices, (30) can be rewritten as

$$\dot{x}_{sf} = \Lambda_{sf} x_{sf} - b_{sf} \tilde{k}^T (x_{sf} - e_{sf}) \quad \text{(32a)}$$
$$\dot{e}_s = (\Lambda_s - l c_1^T) e_s - l c_1^T x_{sf} - l c_1^T f_{f2} x_{f2} \quad \text{(32b)}$$
$$\dot{x}_{f2} = \Lambda_{f2} x_{f2} - b_{f2} \tilde{k}^T (x_{sf} - e_{sf}). \quad \text{(32c)}$$

while (31) becomes

$$\dot{x}_{sf} = \Lambda_{sf} x_{sf} - b_{sf} \tilde{k}^T (x_{sf} - e_{sf}) \quad \text{(33a)}$$
$$\dot{e}_s = (\Lambda_s - l c_1^T) e_s - l c_1^T x_{sf}. \quad \text{(33b)}$$

Now one sees that (32) have a similar structure as (17). Particularly, if (A5) and [(A6)(i)] or [(A6)(ii)] hold, then the subsystem (32) satisfies all the requirements of the respective lemmata in Sections 1.1 and 1.3 with $m$ in place of $n$. The only assumption on subsystem (32) needed for Theorem 5.1 is imposed in Lemma 4.8 in Section 4.1. There it is assumed that its overall system matrix, i.e., the matrix $\hat{A}$ in (33) is Hurwitz with max $\text{Re}(\lambda_i) < \lambda < 0$. Hence, if this assumption is satisfied, then Theorem 5.1 can be applied to (32a)–(32c) with $m$ in place of $n$. This means that stability of (32) (or, equivalently, of (31)) for sufficiently large $m$ and with $\text{max Re}(\lambda_i) < \lambda < 0$ with $\lambda$ independent of $m$ implies stability of (32) and thus of (17). This leads to the following numerical test in order to check whether a certain number $n$ of modes taken into account in the controller is sufficient for stabilization:

(i) Fix $n$ and compute $\rho_m = \max_i \text{Re}(\lambda_i)$ for the eigenvalues $\lambda_i$ of the matrix governing the LTI system (31) for growing numbers of $m$ in order to find $\lambda \in \mathbb{R}$ and $m_0 \in \mathbb{N}$ such that $\rho_m \leq \lambda$ holds for all $m \geq m_0$.

(ii) If $\lambda < 0$, then system (32) is exponentially stable for the given $n$.

Clearly, by means of numerical computations it is not possible to rigorously ensure $\rho_m \leq \lambda$ for all $m \geq m_0$. However, often—as in the examples in the next section—convergence of $\rho_m$ for $m \to \infty$ can be observed numerically, which provides a strong evidence for the desired inequality since $\rho_m$ hardly changes anymore for large $m$.

7. Simulation results

In the following the previous analysis and main results are evaluated for three simulation scenarios covering both scalar and coupled diffusion-reaction systems.

7.1. Scalar diffusion-reaction problem

Based on the introductory problem formulation in Section 2 boundary and in-domain control as well as sensing, respectively, are considered to numerically evaluate the formulated preliminaries and results.

7.1.1. Boundary control and point sensing

We illustrate the algorithm using the equation

$$\partial_t x = \partial^2 x + \omega x, \quad z \in (0, 1), \quad t > 0$$
$$\partial_x x|_{z=0} = 0, \quad x|_{z=1} = u, \quad t > 0$$
$$x|_{t=0} = x_0, \quad z \in [0, 1].$$

We set $r = 15$, leading to the first three eigenvalues $\lambda_1 \approx 12.5326$, $\lambda_2 \approx -7.2066$, and $\lambda_3 \approx -46.6850$. As output we use a point measurement at $\xi = 1/4$, leading to the components $c_k = \sqrt{2}\cos(\omega_k/4)$ of the operator $C$ and rendering the actuator/sensor configuration non-collocated. The stabilizing feedback $k$ and the observer matrix $\hat{l}$ are designed to shift the open-loop eigenvalues $\lambda_k$ to the desired eigenvalues $\kappa_k = 0$.

1We note that $\lambda$ must be independent of $m$ (resp. $n$) because the constant $\xi_2$ from Lemma 4.8 which determines the size of the “sufficiently large” $n$ in the proof of Theorem 5.1 via the constant $\xi_6$, depends on $\lambda$. 

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with width \( 2\epsilon \) and height \( 1/(2\epsilon) \). Furthermore let the spatial input characteristics be given by \( b = f_{\xi,\epsilon} \) so that \( b(\cdot) \) approaches a Dirac delta function centered at \( z = \zeta \) in the limit as \( \epsilon \to 0 \), which refers to a pointwise in-domain control. The output is taken as a pointwise measurement at position \( \xi \in (0, 1) \) so that \( c_k = \sqrt{2}\cos(\omega_k \xi) \). Making use of the eigenvalue and eigenfunction computations in Section 2, we obtain \( b_{1,k} = \langle b, \phi_k \rangle \) in terms of

\[
 b_{1,k} = 2\sqrt{2}\cos(\frac{\xi}{2}(2k - 1)\zeta) \sin(\frac{\pi}{2}(2k - 1)\epsilon),
\]

which is used to confirm Assumptions [A5] and [A6](i) instead of \( b_k \). Provided that the parameter pair \((\zeta, \epsilon, \xi)\) is such that the slow subsystem (13) is stabilizable and detectable, then [A4] and [A5] are immediately fulfilled as in the example of Section 7.1.1 [A6](i) follows since

\[
 \left| \frac{b_{1,k}}{\lambda_k} \right| \leq \frac{2\sqrt{2}}{(2k - 1)\pi\epsilon} \leq \frac{1}{\sqrt{2\pi}k^2}
\]

for \( k \geq n \) with \( n \) chosen sufficiently large depending on \( r \). With this, proceed as in Section 7.1.1 by determining the stabilizing feedback \( k \) and the observer matrix \( l \) to shift the open-loop eigenvalues \( \lambda_k \) to the desired eigenvalues \( \kappa_1 = -10, \kappa_2 = -11, \kappa_k = \lambda_k \) for \( k \in [3,n] \) for the feedback control and \( \nu_1 = -15, \nu_2 = -16, \nu_k = \lambda_k \) for \( k \in [3,n] \) for the observer error dynamics. By assigning \( \zeta = 0.7, \epsilon = 0.05 \) and \( \xi = 0.4 \), the resulting \( \rho_m \) for \( m \in [3,8] \) and \( n = m + 1, \ldots, 150 \) are shown in Figure 2a and clearly confirms the closed-loop stability assessment. Respective results are provided in Figure 2b for \( \zeta = 0.4 \) and Figure 2c for \( \zeta = 0.1 \). The values of \( \rho_m \) computed for \( m = 150 \) depending on the order \( n \) is depicted in Figure 2d for the three actuator/sensor configurations. While closed-loop stability is achieved in all scenarios only the collocated configuration with \( \zeta = \xi = 0.4 \) shows an (almost) uniform decay to the assigned smallest closed-loop eigenvalue \( \kappa_1 = -10 \).

\[ k \in [3,n] \] for the feedback control and \( \nu_1 = -15, \nu_2 = -16, \) and \( \nu_k = \lambda_k \) for \( k \in [3,n] \) for the observer error dynamics. Note that it is subsequently not aimed at studying closed-loop performance but to illustrate the main stability result. The verification of the remaining conditions formulated in Assumption 4.1 can be found in Remark 4.2.

Assumption 4.1 can be found in Remark 4.2.

\[ \text{Re}(\lambda_{\max}) \]

\[ \text{Im}(\lambda_{\max}) \]

(a) Values of \( \rho_m \) for \( n \in [3,8] \) when varying \( m = n + 1, \ldots, 200 \).

(b) Values of \( \rho_m \) for \( m = 200 \) and variation of the dimension \( n \) of the slow subsystem.

Figure 1: Results for the scalar example of Section 7.1.1.
7.2. Coupled diffusion-reaction problem

As a second problem consider
\[ \partial_t x = D \partial_x^2 x + Rx + B u, \quad z \in (0, 1), \quad t > 0 \] (37a)
\[ G_0 \partial_x x|_{z=0} + F_0 x|_{z=0} = 0, \quad t > 0 \] (37b)
\[ G_1 \partial_x x|_{z=1} + F_1 x|_{z=1} = 0, \quad t > 0 \] (37c)
\[ x|_{t=0} = x_0, \quad z \in [0, 1] \] (37d)

for \( N = 2 \) with the diffusion and reaction matrices
\[ D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad R = \begin{pmatrix} \alpha & r_{12} \\ r_{21} & \alpha \end{pmatrix}, \] (37e)
and
\[ G_0 = F_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad G_1 = F_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \] (37f)

In-domain control is considered by means of
\[ B = \begin{pmatrix} f_{x_1, x_1} \\ f_{x_2, x_2} \end{pmatrix} \] (37g)
with \( f_{x_1, x_1} \) as defined in (40). The output is taken as the pointwise measurement of the state, i.e.,
\[ y_1 = x_1|_{z=\xi_1}, \quad y_2 = x_2|_{z=\xi_2}. \] (37h)

Problem can be recast into the abstract form on the state space \( X = (L_2(0, 1))^2 \) by introducing
\[ Ax = D \partial_x^2 x + Rx \] (38)

with domain
\[ D(A) = \{ x \in X | Ax \in X, \ G_0 \partial_x x|_{z=0} + F_0 x|_{z=0} = 0, \ G_1 \partial_x x|_{z=1} + F_1 x|_{z=1} = 0 \} \] (39)

For the verification of Assumption 4.1 it is necessary to determine the eigenvalue distribution and the respective eigenvectors from \( A \phi_k = \lambda_k \phi_k, \ \phi_k \in D(A) \) with \( A \) and \( D(A) \) defined in 3. Contrary to the scalar example this can no longer be performed analytically. However, it is possible to compute asymptotic results as \( k \to 1 \). In particular, after some tedious but straightforward computations the two asymptotic eigenvalue branches \( \lambda_{j,k} = \lambda_{j,k}^a + O(k^{-2}) \), \( j = 1, 2 \), with
\[ \lambda_{1,k}^a = 3 \mu_k^2 - \sqrt{\mu_k^2 + 4r_{12}r_{21}}, \quad \lambda_{2,k}^a = 3 \mu_k^2 + \sqrt{\mu_k^2 + 4r_{12}r_{21}} \] (40)
and \( \mu_k = (2k - 1)\pi/2 \), can be deduced for \( k \gg 0 \). The corresponding eigenvectors \( \phi_{1,k}, \phi_{2,k} \) of the operator \( A \) and the mutually orthogonal eigenvectors \( \psi_{1,k}, \psi_{2,k} \) of the adjoint operator \( A^* x = D \partial_x^2 x + R^T x, \ D(A^*) = D(A) \) can be determined by direct evaluation and scaled so that \( \langle \psi_{i,k}, \phi_{j,k} \rangle_X = \delta_{i,k} \) for \( i = 1, 2 \). The analysis in Appendix A allows us to conclude that the eigenvectors of \( A \) generate a Riesz basis. As a consequence \( A \) is a Riesz spectral operator [17, Section 2.3] so that using Fourier series expansion

Figure 2: Results for the scalar example of Section 7.1.2 with varying location of in-domain control and sensing for \( n \in \{3, 8\} \).
and projection the system \( \mathbf{Ax} + \mathbf{Bu} = \mathbf{0} \) can be re-written as the infinite-dimensional system of ODEs in diagonal form

\[
\begin{align*}
\dot{x}_k &= \lambda_k x_k + b_k^T u, & k \in \mathbb{N} \\
x_k(0) &= (x_0, \psi_k)x = x_k^0
\end{align*}
\]

with \( x_k = (x, \psi_k)x \) and \( b_k^T u = (Bu, \psi)x \). Herein, \( \lambda_k \) is asymptotically determined by \( \lambda_k \) for \( k \) sufficiently large. In any case for \( k \) the eigenvalues can be approximately computed by a suitable discretization of \( \mathbf{A} \).

Assuming that actuator and sensors determined by the parameter pairs \((c_1, \epsilon_1, \xi_1)\) and \((c_2, \epsilon_2, \xi_2)\) are chosen so that the slow finite-dimensional subsystem \( \mathbf{A} \) is stabilizable and detectable, then Assumption \( \mathbf{A} \) is fulfilled. Taking into account the eigenvalue asymptotics yields that \( \mathbf{A} \) holds true. Assumption \( \mathbf{A} \) follows from the fact that \( \mathbf{B} \) and thus the \( b_k \) are bounded and that \( |\lambda_k| \) grows quadratically.

For numerical evaluation we consider \( \alpha = 10, \tau_2 = 5, \tau_{21} = 10 \), which yields the open-loop eigenvalues \( \lambda_1 = 11.56, \lambda_2 = 2.84, \lambda_3 = -13.09, \lambda_4 = -33.79, \lambda_k < \lambda_4, k \geq 5 \) so that two eigenvalues have to be shifted to the complex left half-plane by the feedback control. In particular, in view of the discussed preliminaries, the stabilizing feedback gain matrix \( \mathbf{K} \) and the observer matrix \( \mathbf{L} \) are determined to place the two eigenvalues \( \lambda_1 \), \( \lambda_2 \) to the desired eigenvalues \( \kappa_1 = -10, \kappa_2 = -11, \kappa_k = \lambda_k \) for \( k \in [3, n] \) for the feedback control and \( \nu_1 = -15, \nu_2 = -16, \nu_3 = \lambda_k \) for \( k \in [3, n] \) for the observer error dynamics. Actuators and sensors are parametrized by \( c_1 = 0.3, \epsilon_1 = 0.05, \xi_1 = 0 \) and \( c_2 = 0.6, \epsilon_2 = 0.05, \xi_2 = 1 \) so that the outputs \( y_1 = x_1|_{z=0} \) and \( y_2 = x_2|_{z=1} \) denote the boundary values of the state variables. The resulting values of \( \rho_m \) for the dimension of the slow subsystem restricted to \( n \in [3, 8] \) and \( m = n + 1, \ldots, 200 \) residual modes is shown in Figure \ref{fig:rho}. The obtained results clearly confirm the closed-loop stability assessment. Similar to the previous examples the values to which the \( \rho_m \) converge vary with \( n \) and may grow as \( n \) is increased. To further study this behavior Figure \ref{fig:rho} shows the \( \rho_m \) for \( m = 200 \) as they change over \( n \). Here after some initial variation the expected behavior becomes visible with the values settling to the assigned smallest closed-loop eigenvalue \( \kappa_1 = -10 \).

Remark 7.1. As indicated before it is noteworthy to mention that the numerical results for all examples show that the desired dominating eigenvalue assigned during the state feedback control and state observer design (here \( \kappa_1 \)) is obtained only for sufficiently large values of the order \( n \) of the slow subsystem used for design. However, the results clearly indicate that closed-loop stability, as assessed in previous sections, is given for much lower values of \( n \). This is an interesting observation that needs further examination.

8. Conclusions

The closed-loop stability of linear diffusion-reaction systems under finite-dimensional observer-based state feedback control, i.e., dynamic output feedback control, is addressed based on the classical decomposition of the considered class of infinite-dimensional diffusion-reaction systems into a finite-dimensional slow subsystem and an infinite-dimensional (residual) fast subsystem. State feedback control and observer design is performed based on the slow subsystem but remains interconnected to the residual system, which leads to control and observation spillover. By thoroughly analyzing the (dynamic) feedback interconnection of the subsystems a small-gain theorem can be applied to verify closed-loop stability of the infinite-dimensional system. For practical purposes an approach for the computation of the required dimension of the slow subsystem used for controller design is presented together with simulation results scalar and coupled linear diffusion-reaction systems that confirm the theoretical assessment.

Appendix A. Riesz basis generation for coupled diffusion-reaction problem

To analyze the Riesz basis property of the set of eigenfunctions of the operator \( A \) for problem \( \mathbf{A} \) it is necessary to take into account the two eigenvalue branches provided in \( \mathbf{A} \) in terms of their asymptotics. These follow from solving \( A\phi = \lambda\phi \) with \( \phi \in D(A) \), which after some tedious computations yields the characteristic equation

\[
\cos(\epsilon_-(\bar{\lambda})) \cos(\epsilon_+(\bar{\lambda})) = -4\tau_{12}\tau_{21} \times \\
(\lambda^2 - \tau_{12}\tau_{21}) + \frac{3}{\sqrt{\lambda}} \sqrt{\lambda^2 - \tau_{12}\tau_{21} \sin(\epsilon_-(\bar{\lambda})) \sin(\epsilon_+(\bar{\lambda}))} \\
(\lambda^2 - \tau_{12}\tau_{21})(\lambda^2 + 4\tau_{12}\tau_{21})
\]

(A.1)

with \( \bar{\lambda} = \lambda - \alpha \) and

\[
\epsilon_\pm(\bar{\lambda}) = \frac{1}{2} \sqrt{-3\lambda \pm \sqrt{\lambda^2 + 8\tau_{12}\tau_{21}}}.
\]

Appendix A.1. Asymptotic analysis of the eigenvalues

To deduce \( \mathbf{A} \) consider \( \epsilon_+(\lambda) = \mu \in \mathbb{R}^+ \) and solve for \( \bar{\lambda} \), i.e.,

\[
\bar{\lambda} = \lambda - \alpha = \frac{-3\mu^2}{2} + \sqrt{\mu^4 + 4\tau_{12}\tau_{21}}.
\]

This admits to conclude the following relationships

(i) \( \sqrt{\lambda^2 + 8\tau_{12}\tau_{21}} = 3\lambda + 4\mu^2 \geq 0 \)

(ii) \( \epsilon_-(\bar{\lambda}) = \frac{1}{2} \sqrt{-6\lambda - 4\mu^2, -6\lambda - 4\mu^2 \geq 0} \)

(iii) \( \epsilon_-(\lambda) = \frac{1}{2} \sqrt{5\mu^2 - 3\mu^4 + 4\tau_{12}\tau_{21}} \geq 0 \) as \( 5\mu^2 - 3\sqrt{\mu^4 + 4\tau_{12}\tau_{21}} \geq 0 \).

Using that for \( \gamma \gg 1 \) and fixed \( c \in \mathbb{R} \) the inequality \( \sqrt{\gamma^2 + c} = \gamma + O(\gamma^{-1}) \) holds, for \( \mu \gg 1 \) we obtain
(iv) \( \bar{\lambda} = -\mu^2 + O(\mu^{-2}) \)
(v) \( 5\mu^2 - 3\sqrt{\mu^4 + 4r_{12}r_{21}} = 2\mu^2 + O(\mu^{-2}) \)
(vi) \( \bar{\lambda}^2 - r_{12}r_{21} = \mu^4 + O(1) \)
(vii) \( \bar{\lambda}^2 + 4r_{12}r_{21} = \mu^4 + O(1) \)

For each fixed (finite) value of \( r_{12}r_{21} \). Property (v) together with (iii) implies \( \epsilon_-(\bar{\lambda}) = \mu + O(\mu^{-3}) \) and \( \epsilon_+(\bar{\lambda}) = \mu/\sqrt{2} + O(\mu^{-3}) \) for \( \mu \gg 1 \). Let \( f(\lambda) = \cos(\epsilon_-(\bar{\lambda})) \cos(\epsilon_+(\lambda)) \), let \( g(\lambda) \) denote the right hand side of (A.1). In view of properties (iv) to (vii) we obtain for \( \mu \gg 1 \)

\[
f(\lambda) = \cos(\mu) \cos \left( \frac{\mu}{\sqrt{2}} \right) + O(\mu^{-3})
g(\lambda) = \mu^4 + O(1) + (\mu^4 + O(1)) \sin(\epsilon_-(\bar{\lambda})) \sin(\epsilon_+(\lambda))
\]

\[
= O \left( \frac{\mu^4 + O(1)}{(\mu^4 + O(1))^2} \right) = O(\mu^{-4}).
\]

Hence \( f(\lambda) = g(\lambda) \) implies

\[
\cos(\mu) \cos \left( \frac{\mu}{\sqrt{2}} \right) = O(\mu^{-3}). \quad (A.3)
\]

A sequence of solutions for this equation is \( \mu = \mu_k + O(k^{-3}) \) with \( \mu_k = (2k - 1)\pi/2, k \in \mathbb{N} \). The corresponding eigenvalue branch follows from the substitution into

\[
\bar{\lambda}_{1,k} = \lambda_{1,k} - \alpha = \frac{3\mu_k^2}{2} + \sqrt{\mu_k^4 + 4r_{12}r_{21}}\]

\[
= \bar{\lambda}_{1,k}^0 + O(k^{-2}) \quad (A.4)
\]

with

\[
\bar{\lambda}_{1,k}^0 = \lambda_{1,k}^0 - \alpha = \frac{3\mu_k^2}{2} + \sqrt{\mu_k^4 + 4r_{12}r_{21}},
\]

where the second equality follows from (iv) and \( \mu = \mu_k + O(k^{-3}) \). The second sequence of asymptotic solutions to (A.3), \( \mu = \bar{\mu}_k + O(k^{-3}) \) with \( \bar{\mu}_k = \sqrt{2}(2k - 1)\pi/2 \), corresponds to the analysis of the second branch determined from \( \epsilon_+(\bar{\lambda}) = \mu \in \mathbb{R} \) so that following a similar argumentation the second eigenvalue branch can be determined in the form

\[
\lambda_{2,k} = \lambda_{2,k}^0 - \alpha = -\frac{3\mu_k^2}{2} - \sqrt{\mu_k^4 + 4r_{12}r_{21}}
\]

\[
= \bar{\lambda}_{2,k}^0 + O(k^{-2}) \quad (A.5)
\]

with

\[
\bar{\lambda}_{2,k}^0 = \lambda_{2,k}^0 - \alpha = -\frac{3\mu_k^2}{2} - \sqrt{\mu_k^4 + 4r_{12}r_{21}}.
\]

### Appendix A.2. Asymptotic analysis of the eigenvectors

Taking into account the two branches (A.3) and (A.5) the solution of the eigenproblem \( A\phi = \lambda \phi \) with \( \phi \in D(A) \) can be asymptotically determined. A closed-form general solution, which is determined up to a normalization constant, can be computed for each of the two eigenvalue branches. The normalization constant is obtained by evaluating \( ||\phi_j,k||^2_X = \langle \phi_j,k, \psi_j,k \rangle_X = 1 \) for the branches \( j \in \{1,2\} \). Here, \( \psi_j,k \) denotes the eigenvector for the adjoint operator \( A^* \), which is given by \( A^*x = D\partial_x^2x + R^T x \) with \( D(A^*) = D(A) \). This implies that \( \psi_{j,k} \) follows from \( \phi_{j,k} \) by mutually interchanging \( r_{12} \) and \( r_{21} \).

The resulting expressions are rather lengthy and are thus subsequently omitted. However, they allow to deduce the following asymptotics

\[
\phi_{j,k} = \phi_{j,k}^0 + O(k^{-1}), \quad j \in \{1,2\} \quad (A.6)
\]

with

\[
\phi_{1,k}^0 = \frac{\sqrt{2}}{\sqrt{1 + \frac{\mu_k^2}{\mu_k^2}}} \begin{bmatrix} 1 \\ \frac{\mu_k}{\mu_k^2} \end{bmatrix} \cos(\mu_k z) \quad (A.7)
\]

for branch 1 with (A.3) and

\[
\phi_{2,k}^0 = \frac{\sqrt{2}}{\sqrt{1 + \frac{\mu_k^2}{\mu_k^2}}} \begin{bmatrix} \frac{\mu_k}{\mu_k^2} \\ 1 \end{bmatrix} \sin(\mu_k z) \quad (A.8)
\]

for branch 2 with (A.5). To illustrate the asymptotic behavior, consider branch 1 with (A.3), (A.8), which yields

\[
A\phi_{1,k}^0 - \lambda_{1,k}^0 \phi_{1,k}^0 = \frac{\sqrt{2}}{\sqrt{1 + \frac{\mu_k^2}{\mu_k^2}}} \begin{bmatrix} \lambda_{1,k}^0 - \alpha \end{bmatrix} \frac{O(k^{-2})}{O(k^{-4})}
\]

\[
= O(k^{-4}).
\]
with boundary conditions
\[
(G_0 \partial_t \phi^0_{1,k} + F_0 \phi^0_{1,k})|_{t=0} = \sqrt{2}r_{21} \begin{bmatrix} \mu_k^2 + r_{12} \mu_k \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} O(k^{-2}) \end{bmatrix}
\]
\[
(G_1 \partial_t \phi^0_{1,k} + F_1 \phi^0_{1,k})|_{t=1} = \sqrt{2}(1-(1)^k \mu_k) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} = O(k^{-1}).
\]

A similar analysis can be performed for branch 2 with \(A_{3} \), \(A_{4} \).

**Appendix A.3. Riesz basis property**

To analyze that \( \{ \phi^0_{1,k} \}_{k \in \mathbb{N}} = \{ \phi^0_{1,k}, \phi^0_{2,k} \}_{k \in \mathbb{N}} \) generates a Riesz basis we make use of Bari’s theorem [38, 21] taking into account \(A_{7} \), \(A_{8} \) to show that \( \{ \phi^0_{k} \}_{k \geq n} = \{ \phi^0_{1,k}, \phi^0_{2,k} \}_{k \geq n} \) for sufficiently large \( n \) is quadratically close to a (known) Riesz basis \( \{ e_k \}_{k \in \mathbb{N}} = \{ e_{1,k}, e_{2,k} \}_{k \in \mathbb{N}} \). For the latter we consider the basis spanned by the eigenvectors of the decoupled problem, i.e., \(E_{12} \), \(E_{21} \) with matrix \( R = 0 \). This implies
\[
e_{1,k} = \sqrt{2} \begin{bmatrix} \cos(\mu_k z) \\ 0 \end{bmatrix}, \quad e_{2,k} = \sqrt{2} \begin{bmatrix} 0 \\ \sin(\mu_k z) \end{bmatrix}
\]
with \( \mu_k = (2k-1)\pi/2, k \in \mathbb{N} \). As each of the sets \( \{ \sqrt{2} \cos(\mu_k z) \}_{k \in \mathbb{N}} \) and \( \{ \sqrt{2} \sin(\mu_k z) \}_{k \in \mathbb{N}} \) generates a Riesz basis for \( L_2(0,1) \) we conclude that \( \{ e_k \}_{k \in \mathbb{N}} \) defines a Riesz basis for \( X = (L_2(0,1))^2 \). We remark that \( e_{1,k} \) and \( e_{2,k} \) refer to the eigenvalues branches 1 and 2 for the decoupled problem as do \( \phi^0_{1,k} \) and \( \phi^0_{2,k} \) for the considered coupled problem.

To verify that \( \{ \phi^0_{k} \}_{k \geq n} \) is quadratically close to the Riesz basis \( \{ e_k \}_{k \in \mathbb{N}} \) it is necessary to show that
\[
\sum_{k \geq n} \| \phi^0_k - e_k \|_X^2 < \infty
\]  
(A.10)
for sufficiently large finite \( n \in \mathbb{N} \). Taking into account \(A_{6} \) provides
\[
\| \phi_k - e_k \|_X^2 = \| \phi^0_{1,k} + O(k^{-1}) - e_{1,k} \|_X^2 + \| \phi^0_{2,k} + O(k^{-1}) - e_{2,k} \|_X^2
\]
with
\[
\| \phi^0_{1,k} + O(k^{-1}) - e_{1,k} \|_X^2 = \| \phi^0_{2,k} - e_{2,k} \|_X^2 + O(k^{-2})
\]
\[
+ 2O(k^{-1}) \int_0^1 \{(\phi^0_{1,k})_1 - (e_{1,k})_1 + (\phi^0_{2,k})_2 - (e_{2,k})_2 \} dz,
\]
where \( (\phi^0_{l,k})_l, (e_{l,k})_l, l \in \{ 1, 2 \} \) refer to the \( l \)-th component of the vectors. Making use of \(A_{7} \), \(A_{8} \) and \(A_{9} \) we obtain
\[
\| \phi^0_{1,k} - e_{1,k} \|_X^2 = \frac{r_{21}(r_{12} + r_{21}) + 2\mu_k^4}{r_{12}r_{21} + \mu_k^2} \left(1 - \sqrt{1 + \frac{2\mu_k^4}{r_{12}r_{21}}} \right) + O(k^{-2})
\]
\[
\| \phi^0_{2,k} - e_{2,k} \|_X^2 = \frac{r_{12}(r_{12} + r_{21}) + 2\mu_k^4}{r_{12}r_{21} + \mu_k^2} \left(1 - \sqrt{1 + \frac{2\mu_k^4}{r_{12}r_{21}}} \right) + O(k^{-2})
\]
and
\[
\int_0^1 \{(\phi^0_{1,k})_1 - (e_{1,k})_1 + (\phi^0_{2,k})_2 - (e_{2,k})_2 \} dz = \frac{r_{21} - r_{12}}{\mu_k^2 + r_{12}r_{21}} O(1) + 2O(k^{-1})\int_0^1 \left( \frac{1}{\mu_k^2 + r_{12}r_{21}} \right) dz = O(1) + O(k^{-1})\int_0^1 \left( \frac{1}{\mu_k^2 + r_{12}r_{21}} \right) dz = O(1).
\]

As \((*) = O(k^{-1})\) and \((***) = o(1)\) the product fullfills \((*) = O(k^{-1})\). Since \((***) = O(k^{-3})\) we obtain
\[
2O(k^{-1})\int_0^1 \{(\phi^0_{1,k})_1 - (e_{1,k})_1 + (\phi^0_{2,k})_2 - (e_{2,k})_2 \} dz = O(k^{-1})(o(k^{-1}) + O(k^{-3})) = O(k^{-2} + O(k^{-4})�)
\]
Moreover observing \(1 - \sqrt{1 + r_{12}r_{21}/\mu_k^2} = O(\mu_k^{-4})\) for \( k \gg 1 \) it follows that
\[
\| \phi^0_{1,k} - e_{1,k} \|_X = \frac{r_{21}(r_{12} + r_{21}) + 2\mu_k^4}{r_{12}r_{21} + \mu_k^2} O(\mu_k^{-4}) = O(k^{-4})
\]
\[
\| \phi^0_{2,k} - e_{2,k} \|_X = \frac{r_{12}(r_{12} + r_{21}) + 2\mu_k^4}{r_{12}r_{21} + \mu_k^2} O(\mu_k^{-4}) = O(k^{-4}).
\]

Thus, \( \| \phi^0_k - e_k \|_X = O(k^{-2}) \) and consequently \( \sum_{k \geq n} \| \phi^0_k - e_k \|_X^2 \) converges and \(A_{10} \) is fulfilled, so that \( \{ \phi^0_k \}_{k \in \mathbb{N}} = \{ \phi^0_{1,k}, \phi^0_{2,k} \}_{k \in \mathbb{N}} \) generates a Riesz basis.

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