DIRECT METHODS ON FRACTIONAL EQUATIONS

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Abstract. In this paper, we summarize some of the recent developments in the area of fractional equations with focus on the ideas and direct methods on fractional non-local operators. These results have more or less appeared in a series of previous literature, in which the ideas were usually submerged in detailed calculations. What we are trying to do here is to single out these ideas and illustrate the inner connections among them, so that the readers can see the whole picture and quickly grasp the essence of these useful methods and apply them to a variety of problems in this area.

1. Introduction. In this paper, we summarize some of the recent developments on the study of qualitative properties of solutions for nonlinear equations involving the fractional Laplacian, the fractional p-Laplacian, and more general nonlinear non-local operators. We will illustrate the major differences between these kinds of non-local operators and the classical local differential operators, and will focus on the ideas and techniques in analyzing qualitative properties of solutions.

The fractional Laplacian is a pseudo-differential operator defined by

\[
(-\Delta)^{\alpha/2} u(x) \equiv C_{n,\alpha} \text{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+\alpha}} dy \\
= C_{n,\alpha} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x-y|^{n+\alpha}} dy,
\]

for any real number \(0 < \alpha < 2\), where PV stands for the Cauchy principal value.
Alternatively, it can be expressed without using the Cauchy principal value:

\[ (-\Delta)^{\alpha/2} u(x) \equiv \frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x + y) - u(x - y)}{|y|^{n+\alpha}} dy. \]

This operator is well defined in the Schwartz space of rapidly decreasing \( C^\infty \) functions in \( \mathbb{R}^n \). In this space, it can also be equivalently defined in terms of the Fourier transform

\[ (-\hat{\Delta})^{\alpha/2} u(\xi) = |\xi|^\alpha \hat{u}(\xi). \]

Let

\[ L_\alpha = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}} \, dx < \infty \right\}. \]

Then it is easy to verify that for \( u \in L_\alpha \cap C^{1,1}_{\text{loc}}, \) the integral on the right hand side of (1) converges, and hence \( (-\Delta)^{\alpha/2} \) is well-defined on such \( u \).

The definition of the fractional Laplacian can be further extended to the distributions \( u \) in the space \( L_\alpha \) by

\[ \langle (-\Delta)^{\alpha/2} u, \phi \rangle = \int_{\mathbb{R}^n} u (-\Delta)^{\alpha/2} \phi \, dx, \quad \text{for all} \quad \phi \in C^\infty_0(\mathbb{R}^n). \]

The non-locality of the fractional Laplacian makes it difficult to investigate. To circumvent this difficulty, Caffarelli and Silvestre [4] introduced the extension method that reduced this nonlocal problem into a local one in higher dimensions. For a function \( u : \mathbb{R}^n \to \mathbb{R}, \) consider its extension \( U : \mathbb{R}^n \times [0, \infty) \to \mathbb{R} \) that satisfies

\[ \begin{cases} \text{div}(y^{1-\alpha} \nabla U) = 0, & (x, y) \in \mathbb{R}^n \times [0, \infty), \\ U(x, 0) = u(x). \end{cases} \]

Then it can be shown that

\[ (-\Delta)^{\alpha/2} u(x) = -C_{n,\alpha} \lim_{y \to 0^+} y^{1-\alpha} \frac{\partial U}{\partial y}, \quad x \in \mathbb{R}^n. \]

Because the extended problem (2) involves only the usual second order elliptic operators, which are of local nature, many known results in elliptic theories can be applied directly to study it. Hence this extension method has been employed successfully to investigate equations involving the fractional Laplacian, and a series of fruitful results have been obtained (see [3] [23] and the references therein).

Another effective method in dealing with fractional equations is the integral equation approach introduced earlier in [18] and [48] (see also [12]), including the method of moving planes in integral forms and the regularity lifting.

However, either by extension or via integral equations, one usually needs to impose extra conditions on the solutions, which would not be necessary if we consider the pseudo differential equations directly. Moreover, for equations involving the uniformly elliptic nonlocal operators

\[ C_{n,\alpha} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{a(x - z)(u(x) - u(z))}{|x - z|^{n+\alpha}} \, dz = f(x, u(x)), \]

where

\[ 0 < c_0 \leq a(y) \leq C_1; \]

and for equations involving fully nonlinear non-local operators, such as

\[ F_\alpha(u) = f(x, u) \]
where
\[ F_\alpha(u(x)) = C_{n,\alpha} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{G(u(x) - u(z))}{|x - z|^{n+\alpha}} \, dz, \] (4)
with \( G(\cdot) \) being a Lipschitz continuous function (see [5]), so far as we know, there has neither been any corresponding extension methods nor equivalent integral equations that one can work at. The fractional \( p \)-Laplacian \( (-\Delta)_p^s \) is a special case of the fully nonlinear operator \( F_\alpha(\cdot) \) with \( G(t) = |t|^{p-2}t \) and \( \alpha = sp \).

This motivated us to develop direct approaches (without going through extension) on these kinds of general non-local operators, which has become one of our main objectives in the last few years. To name a few:

In [15], we introduced a direct method of moving planes on fractional equations and applied it to obtain symmetry and nonexistence of solutions for nonlinear equations involving the fractional Laplacian on various domains. This method has become a powerful tool in investigating qualitative properties of fractional equations and has been applied by many researchers in this field to solve a wide variety of problems (see [24] [13] [14] [6] [43]). We [14] [13] then modified this idea and employed it to study semi-linear equations (4) involving fully nonlinear nonlocal operators and equations involving fractional \( p \)-Laplacians (degenerate nonlinear nonlocal operators), and obtained symmetry, monotonicity, and nonexistence of solutions. Several new ideas were introduced. So far as we know, we were the first ones to apply the method of moving planes on the fractional \( p \)-Laplacian.

In [21], we introduced a direct method of moving spheres on fractional equations and applied it to study qualitative properties of solutions, in particular, we studied prescribing \( Q_\alpha \) curvature equations on Riemannian manifolds (the fractional Nirenberg problem) and obtained a non-existence result, which provides a stronger necessary condition than the Kazdan-Warner type condition.

In [16], we introduced a direct blowing-up and re-scaling method for fractional equations and used it to obtain a prior estimates for positive solutions on a bounded domain, which then enabled us to establish existence of solutions via a topological degree theory.

In this paper, we will summarized these methods as well as other direct approaches, such as super and sub-solutions, Poisson’s representation of \( s \)-harmonic functions, regularity arguments, and integral equations approaches. We will use simple examples to show the applications of these methods, to explain the difference between local and non-local operators. And whenever possible, we will illustrate the ideas in deriving these methods and the inner connections among these ideas.

In Section 2, we introduce preliminaries including the solutions of fractional equations in three senses: distributional, weak, and classical (pointwise); the main differences between local differential operator and nonlocal pseudo differential operators; and sub and supper solutions and their applications.

We construct sub-solutions to derive a Hopf Lemma and to obtain a lower bound on the decay rate near infinity of positive solutions. We also construct a super solution to deduce Holder continuity of solutions up to the boundary. One will see that, in general, it is much more difficult to find a sub or super solution in the nonlocal cases, because they need to satisfy exterior conditions rather than boundary conditions.

In Section 3, we introduce the Poisson representation
\[ u(x) = \int_{|y|>r} P_r(y, x) u(y) \, dy, \forall x \in B_r(0) \]
of \(s\)-harmonic functions in \(\mathbb{R}^n\), where \(P_r(y,x)\) will be defined precisely in that section. We observe that this resembles the spherical average of harmonic functions, and as \(s\to 1\),
\[
\int_{|y|>r} P_r(y,0)u(y)dy \rightarrow \frac{1}{|S_r|} \int_{S_r(0)} u(y)d\sigma_y.
\]
As applications, it is used to establish Liouville type theorems in the whole space and in the half space for \(s\)-harmonic functions.

In Section 4, we present the direct method of moving planes for fractional equations. We illustrate the ideas behind the proofs of the two key ingredients—the narrow region principle and the decay at infinity and how to use them to carry out the method of moving planes to derive the symmetry and monotonicity of solutions for nonlinear equations and systems involving the fractional Laplacian, the fractional p-Laplacian, and more general fully nonlinear nonlocal operators.

In Section 5, we deal with the direct method of moving spheres for fractional equations. The key ingredient is the spherical narrow region principle. As an application, we consider prescribing \(Q_{\alpha}\) curvature on \(n\)-dimensional sphere \(\mathbb{S}^n\) to obtain a stronger Kazdan-Warner type necessary conditions on the existence of solutions.

In Section 6, we review a decade old technique introduced in [18], the method of moving planes in integral forms, which has been widely applied by numerous researchers to solve a series of problems in nonlinear PDEs, in particular, in fractional equations and higher order equations. It is known that to carry out the method of moving planes for PDEs and for fractional equations, one relies heavily on some kinds of maximum principles such as narrow region principles. While for equations of order greater than two, these maximum principles are usually no longer true, and one has to reduce this higher order one into systems of lower order ones. This difficulty can be circumvented by applying the method of moving planes in integral forms which works indiscriminately for equations of any order less than the dimension \(n\). In this approach, the first key step is to show that a given PDE or fractional equation is equivalent to an integral equation. Recently, some new ideas were introduced in [8] and [54] to show such equivalences, such as by using Liouville Theorems for \(s\)-harmonic functions and by applying the method of moving planes, which will be deliberated in the section.

In Section 7, we introduce a direct blowing up and re-scaling argument and use it to derive a priori estimates for positive solutions to a semi-linear fractional equation on a bounded domain with prescribed Dirichlet condition. Then by a topological degree theory, we obtain the existence of solutions.

In Section 8, we cite some current results on the regularity of solutions established in [50], and provide an outline of the proof.

Throughout this paper, we will sometimes write \((-\Delta)^{\alpha/2}\) as \((-\Delta)^s\) with \(0 < s < 1\) whenever it is more convenient.

2. Preliminaries.

2.1. Solutions in 3 senses. Let \(\Omega\) be a domain in \(\mathbb{R}^n\), bounded or unbounded. Consider
\[
\begin{cases}
(-\Delta)^{\alpha/2} u = f(x) & x \in \Omega \\
u(x) = 0 & x \in \Omega^C,
\end{cases}
\] (5)
where \(\Omega^C\) is the complement of \(\Omega\) in \(\mathbb{R}^n\).
(i) In the sense of distributions

We say that \( u \in L_\alpha \) solves the problem (5) in the sense of distribution if and only if

\[
\int_{\mathbb{R}^n} u (-\triangle)^{\alpha/2} \phi \, dx = \int_{\mathbb{R}^n} f(x) \phi(x) \, dx, \quad \text{for all } \phi \in C^\infty_0(\Omega).
\]

One may notice that, for an integer order differential operator, say for \( \triangle \), in order the integral

\[
\int_{\mathbb{R}^n} u (-\triangle) \phi \, dx
\]

to be valid, we only require \( u \) to be locally integrable, i.e. \( u \in L^1_{\text{loc}} \), because \( \triangle \phi \) has a compact support. Then why here do we need \( u \in L_\alpha \)? This actually is the essential difference between a local and nonlocal operator. For the latter, even \( x \) is beyond the support of \( \phi \), \((-\triangle)^{\alpha/2} \phi(x)\) may still not vanish as oppose to the case of \( \triangle \phi(x) \).

However, for \( |x| \) large, \( \phi(x) = 0 \), and we have

\[
|(-\triangle)^{\alpha/2} \phi(x)| = \left| \int_{\mathbb{R}^n} \frac{\phi(x) - \phi(y)}{|x - y|^{n+\alpha}} \, dy \right|
\leq \int_{\text{supp } \phi} \frac{|\phi(y)|}{|x - y|^{n+\alpha}} \, dy
\leq \int_{\text{supp } \phi} \frac{C}{|x - y|^{n+\alpha}} \, dy
\sim \frac{C}{|x|^{n+\alpha}}.
\]

Together with \( u \in L_\alpha \), it implies that

\[
\int_{\mathbb{R}^n} u (-\triangle)^{\alpha/2} \phi \, dx < \infty.
\]

(ii) In the weak sense

Let \( H_0^{\alpha/2}(\mathbb{R}^n) \) be the usual fractional Sobolev space endowed with the norm

\[
\|u\|_{H_0^{\alpha/2}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |\xi|^\alpha |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2},
\]

where \( \hat{u} \) is the Fourier transform of \( u \). Denote \( H_0^{\alpha/2}(\Omega) \) the completion of \( C_0^\infty(\Omega) \) under this norm.

We say that \( u \in H_0^{\alpha/2}(\Omega) \) is a weak solution of (5) if

\[
\int_{\mathbb{R}^n} |\xi|^\alpha \hat{u}(\xi) \overline{\hat{v}(\xi)} \, d\xi = \int_{\mathbb{R}^n} f(x, u(x))v(x) \, dx, \quad \forall v \in H_0^{\alpha/2}(\Omega).
\]

(iii) In the classical (point-wise) sense

We say that \( u \) is a classical solution of (5), if \( u \in C^{1,1}_{\text{loc}}(\Omega) \cap L_\alpha \) and satisfies (5) point-wise in \( \Omega \).
2.2. The non-locality. Let $u$ be a smooth function and let $D$ be an usual differential operator. We say that $D$ is a local operator in a sense that the value of $Du$ at a given point $x$ depends only on the values of $u$ at the nearby points. For instance, assume that $\Omega$ is a bounded domain in $\mathbb{R}^n$, and
\[
\begin{cases}
  u(x) > 0 & x \in \Omega \\
  u(x) = 0 & x \in \Omega^C.
\end{cases}
\]
Then for any point $x$ not in $\bar{\Omega}$, we obviously have $Du(x) = 0$ for any differential operator $D$, and in particular, $\Delta u(x) = 0$. However, this is not the case for the fractional Laplacian, instead we have
\[
(\Delta)^{\alpha/2}u(x) = C_{n,\alpha} \int_{\mathbb{R}^n} \frac{-u(y)}{|x-y|^{n+\alpha}} \, dy
\]
\[
< 0.
\]
From here one can see that the value of $(\Delta)^{\alpha/2}u(x)$ depends not only on the values of $u$ near $x$, but also on its values all over $\mathbb{R}^n$.

It is well-known that the Dirichlet problem
\[
\begin{cases}
  \Delta u(x) = 0 & x \in \Omega \\
  u(x) = 0 & x \in \partial \Omega
\end{cases}
\]
has a unique trivial solution. However, this is no longer true for the fractional Laplacian. To see this, we need the following Poisson representation of the $\alpha$ harmonic functions.

Let
\[
P_r(y, x) = \begin{cases}
  \frac{\Gamma(n/2)}{\pi^{n/2}} \sin \frac{\pi \alpha}{2} \left( \frac{r^2-|x|^2}{|y|^2-|x|^2} \right)^\frac{n}{2} \frac{1}{|y|^n}, & |y| > r, \\
  0, & |y| < r.
\end{cases}
\]
(6)

It is known as the fractional Poisson kernel (see [37]). Given any good function $g(x)$ on the complement of $B_r(0)$, let
\[
u(x) = \left\{ \begin{array}{ll}
  \int_{|y|>r} P_r(y, x)g(y) \, dy & x \in B_r(0) \\
  g(x) & x \in (B_r(0))^C.
\end{array} \right.
\]
(7)

One can verify that (see [37])
\[
(\Delta)^{\alpha/2}u(x) = 0 \quad x \in B_r(0).
\]
From here, we deduce that for any function $g(x)$ that vanishes on the boundary of $B_r(0)$, we all have
\[
\begin{cases}
  (\Delta)^{\alpha/2}u(x) = 0 & x \in B_r(0) \\
  u(x) = 0 & x \in \partial B_r(0).
\end{cases}
\]
(8)

Now, due to (7), Dirichlet problem (8) possesses infinitely many nontrivial solutions! Hence, in order the solution be unique, the Dirichlet data need to be prescribed on the entire complement of $B_r(0)$ instead of just on its boundary.

Due to the non-locality of the fractional Laplacian, a correct version of the maximum principle is
Theorem 2.1. (see [51] or [15]) Assume that $\Omega$ is a bounded domain in $\mathbb{R}^n$, $u \in L_\alpha$ is lower semi-continuous on $\bar{\Omega}$ and satisfies
\[
\begin{cases}
(-\Delta)^{\alpha/2} u(x) \geq 0 & x \in \Omega \\
u(x) \geq 0 & x \in \Omega^C
\end{cases}
\]
in the sense of distribution, then
\[u(x) \geq 0 \quad x \in \Omega.
\]
Based on this maximum principle, to find a super solution $\bar{u}$, it is necessary to require that
\[\bar{u} \geq u \quad (9)
\]
in the whole complement of $\Omega$, while for local operators, such as $-\Delta$, one only need inequality (9) to hold on the boundary of $\Omega$. This makes it much more difficult to construct a super or a sub-solution.

2.3. Boundary regularity. Another remarkable difference between the fractional Laplacian and the regular Laplacian is the regularity of solutions on the boundary.

Let $u$ be a solution of
\[
\begin{cases}
(-\Delta)^s u(x) = f(x) & x \in \Omega \\
u(x) = 0 & x \in \Omega^C.
\end{cases}
\]

The Schauder Interior Estimate of $u$ is similar to that for the Poisson equation (associated with regular Laplacian) when $s = 1$. It states roughly that if $f \in C^\gamma(\Omega)$, then in any proper subset of $\Omega$, the regularity of the solution $u$ can be raised by the order of $2s$, the same order as the operator $(-\Delta)^s$. By introducing proper weighted Hölder norms as in the case of Poisson equations, one will be able to control a weighted $C^{2s+\gamma}$ norm of $u$ in $\Omega$ in terms of another weighted $C^\gamma$ norm of $f$ in $\Omega$.

However, when consider regularity up to the boundary, the situation in the fractional order equation is quite different from that in the integer order equation (when $s = 1$, the Poisson equation). For integer order equations, a typical example of regularity up to the boundary reads

**Proposition 2.1.** (See [29] Theorem 4.13) Let $B$ be a ball in $\mathbb{R}^n$ and $u$ and $f$ be functions on $B$ satisfying
\[u \in C^2(B) \cap C^0(\bar{B}), \ f \in C^\gamma(B), \ \Delta u = f \text{ in } B, \ u = 0 \text{ on } \partial B.
\]

Then
\[u \in C^{2+\gamma}(\bar{B}).
\]

This proposition tells us that if $f$ is Hölder continuous, and if the boundary data is smooth enough, then the regularity up to the boundary of the solution $u$ can still be raised by order 2. This is no longer the case for the fractional equation as illustrated by the following example.

It is well-known that the function
\[
\phi(x) = \begin{cases}
(1 - |x|^2)^s, & |x| < 1 \\
0, & |x| \geq 1
\end{cases}
\]
satisfies
\[
\begin{cases}
(-\Delta)^s \phi(x) = c, & x \in B_1(0), \\
u = 0, & x \in (B_1(0))^C.
\end{cases}
\]
Here $f(x) \equiv c$ is smooth and the Dirichlet data is also smooth, however, one can easily see that it is only $C^s$ ($s$-Hölder continuous) up to the boundary for $0 < s < 1$. 

Another typical example is
\[ \psi(x) = (x_n)_+^s = \begin{cases} x_n^s & \text{if } x_n \geq 0 \\ 0 & \text{if } x_n < 0. \end{cases} \]

It satisfies
\[ \begin{cases} (-\Delta)^s \psi(x) = 0, & x \in \mathbb{R}^n_+ \equiv \{ x \mid x_n > 0 \} \\ u = 0, & x \in (\mathbb{R}^n_+)^C. \end{cases} \]

Again here \( f(x) \equiv 0 \) is smooth and the Dirichlet data is also smooth, however, it is obvious that \( \psi(x) \) is only \( C^s \) up to the boundary.

For interested readers, please see [50] for more details.

### 2.4. Sub-solutions.

#### 2.4.1. Use a sub-solution to prove a Hopf type lemma.

Assume that \( \Omega \) is a domain in \( \mathbb{R}^n \) with smooth boundary and satisfies the “interior ball property”:

For every point \( x \) on \( \partial \Omega \), there exists a ball in \( \Omega \) of radius \( r(x) \) tangent to \( \partial \Omega \) at point \( x \) and \( r(x) \) is bounded away from 0 for all \( x \) on \( \partial \Omega \).

Suppose that \( u \) is a positive solution of
\[ \begin{cases} (-\Delta)^s u(x) \geq 0, & x \in \Omega \\ u(x) = 0, & x \in \Omega^C, \end{cases} \tag{11} \]
where \( 0 < s < 1 \). Let \( \delta(x) = \text{dist}(x, \partial \Omega) \). We will construct a sub-solution to show that

**Theorem 2.2. (A Hopf Type Lemma)** There exists a positive constant \( c_o \), such that for all \( x \) near the boundary of \( \Omega \), we have
\[ u(x) \geq c_o \delta^s(x). \tag{12} \]

That is
\[ \lim_{x \to \partial \Omega} \frac{u(x)}{\delta^s(x)} \geq c_o > 0. \]

This theorem implies that, up to the boundary, \( u \) is at most \( s \)-Hölder continuous. It is known as the fractional version of the Hopf Lemma. When \( s = 1 \), (12) implies that
\[ \frac{\partial u}{\partial \nu}(x) < 0 \quad x \in \partial \Omega, \]
where \( \nu \) is the outward normal.

**Proof.** Let \( x^o \) be a point on \( \partial \Omega \). Without loss of generality, we may assume that there is a ball of radius 1 tangent to \( \partial \Omega \) at point \( x^o \). For simplicity of notation, we assume that the center of the ball is the origin.

Let \( \phi(x) = C(1 - |x|^2)_+^s \), and choose \( C \), such that
\[ (-\Delta)^s \phi(x) = 1, \quad x \in B \equiv B_1(0). \tag{13} \]

Let \( D \) be a compact subset which has a positive distance from \( B \). Set \( v = \chi_D u \), then for any \( x \in B \)
\[ (-\Delta)^s v(x) = C \int_D \frac{-u(y)}{|x-y|^{n+2s}} dy < 0, \]
and hence there exists a positive constant \( a \), such that
\[ (-\Delta)^s v(x) \leq -a \quad \forall x \in B. \tag{14} \]
Let \( w(x) = v(x) + a\phi(x) \), then by (13) and (14), we have
\[
\begin{cases}
(-\triangle)^s w(x) \leq (-\triangle)^s u(x) & x \in B \\
w(x) \leq u(x) & x \in B^C.
\end{cases}
\]
This \( w \) is a sub-solution. Now by the maximum principle (Theorem 2.1), we derive
\[
u(x) \geq w(x) \quad \forall \ x \in B.
\]
Noticing that both \( u \) and \( w \) vanish at \( x_0 \), we arrive at (12).

2.4.2. Use a sub-solution to obtain an asymptotic behavior of positive solutions.

**Lemma 2.1.** Assume that \( u \in L_\alpha \cap C^{1,1}_{loc}(\mathbb{R}^n) \) is a nonnegative solution for
\[
(-\triangle)^{\alpha/2} u(x) = u^\tau(x), \quad x \in \mathbb{R}^n,
\]
with \( \tau > 0 \). Then there exist constants \( R > 0 \) and \( c > 0 \), such that
\[
u(x) \geq \frac{c}{|x|^{n-\alpha}} \quad \text{for} \ |x| \geq R.
\] (15)

**Proof.** Let \( \eta \) be a smooth cutoff function such that \( \eta(x) \in [0,1] \) in \( \mathbb{R}^n \), \( \text{supp} \eta \subset B_2 \) and \( \eta(x) \equiv 1 \) in \( B_1 \). Let
\[
(-\triangle)^{\alpha/2} \varphi(x) = \eta(x) u^\tau(x).
\]
Then
\[
\varphi(x) = c_{n,-\alpha} \int_{\mathbb{R}^n} \frac{\eta(y) u^\tau(y)}{|x-y|^{n-\alpha}} \ dy.
\]
It’s easy to see that for all \( |x| \) sufficiently large, there exists a constant \( c > 0 \) such that
\[
\frac{c}{|x|^{n-\alpha}} \leq \varphi(x) \leq \frac{2c}{|x|^{n-\alpha}}.
\] (16)

Let
\[
g(x) = u(x) - \varphi(x) + \frac{2c}{R^{n-\alpha}}.
\]
Then for \( R > 0 \) large, we have
\[
\begin{cases}
(-\triangle)^{\alpha/2} g(x) \geq 0, & \text{in } B_R, \\
g(x) \geq 0, & \text{in } B_R^c.
\end{cases}
\]
It follows from the maximum principle (see [51]) that
\[
g(x) \geq 0, \quad \text{in } B_R.
\]
Thus
\[
u(x) - \varphi(x) + \frac{2c}{R^{n-\alpha}} \geq 0, \quad \text{in } B_R.
\]
Fix \( x \), as \( R \to \infty \),
\[
u(x) \geq \varphi(x), \quad \text{in } \mathbb{R}^n.
\] (17)
Combining (16) and (17), we arrive at
\[
u(x) \geq \frac{c}{|x|^{n-\alpha}}.
\]
2.5. **Super solutions.** Let $\Omega$ be a domain in $\mathbb{R}^n$, bounded or unbounded, with the exterior ball condition:

For every point $x$ on $\partial \Omega$, there exists a ball in $\Omega^C$ of radius $r(x)$ tangent to $\partial \Omega$ at point $x$ and $r(x)$ is bounded away from 0 for all $x$ on $\partial \Omega$.

Let $u$ be a bounded nonnegative function with $(-\Delta)^{\alpha/2}$ bounded in $\Omega$. We may assume that
\[
\begin{cases}
(-\Delta)^{\alpha/2}u(x) \leq 1, & x \in \Omega \\
u(x) = 0, & x \in \Omega^C.
\end{cases}
\]

(18)

**Theorem 2.3.** Let $u$ be a solution of (18), then it is $C^{\alpha/2}$ up to the boundary.

**Proof.** Without loss of generality, let $B_1(0) \subset \Omega^C$ be the tangent ball at $y \in \partial \Omega$.

We start the construction of the auxiliary function with $\varphi(x) = (1 - |x|^2)^{\alpha/2}$. The Kelvin transform of $\varphi(x)$ is denoted by
\[
\psi(x) = \frac{1}{|x|^{n-\alpha}}\varphi\left(\frac{x}{|x|^2}\right).
\]

Then
\[
(-\Delta)^{\alpha/2}\psi(x) = \frac{c}{|x|^{n+\alpha}}, \quad x \in B_1^C(0).
\]

Let $D = (B_2 \setminus B_1) \cap \Omega$. By choosing sufficiently large $t > 0$, we have
\[
(-\Delta)^{\alpha/2}(t\psi(x) - u(x)) \geq 0, \quad x \in D.
\]

To apply the maximum principle, we need to check the condition on $\mathbb{R}^n \setminus D$. Obviously,
\[
t\psi(x) - u(x) > 0, \quad x \in \Omega^C,
\]

since $u$ vanishes in $\Omega^C$. However, in an unbounded domain $\Omega$, $t\psi(x)$ may not be able to control $u(x)$ when $|x|$ large. To overcome this difficulty, we may add a positive constant, for example 1, to $t\psi(x)$.

It’s easy to see that $t\psi(x) + 1 - u(x)$ satisfies both conditions required in the maximum principle, and consequently
\[
t\psi(x) + 1 \geq u(x), \quad x \in D.
\]

However, $t\psi(x) + 1$ does not vanish at $y \in \partial \Omega$. Only when $u(x)$ is controlled by a Hölder continuous function that vanishes at the point $y$, will we be able to derive Hölder continuity for $u(x)$ itself at $y$. To this end, we modify the auxiliary function by adding a cut-off function to $t\psi(x)$. Let $\xi(x)$ be a smooth cutoff function such that
\[
0 \leq \xi(x) \leq 1, \quad \text{in} \ \mathbb{R}^n
\]

and
\[
\xi(x) = \begin{cases}
0, & x \in B_1(0) \\
1, & x \in B_2^C(0).
\end{cases}
\]

Then it is easy to see that $t\psi(y) + \xi(y) = 0$. We also have
\[
(-\Delta)^{\alpha/2}\xi(x) = \frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n} \frac{2\xi(x) - \xi(x + y) - \xi(x - y)}{|y|^{n+\alpha}} \, dy \\
\geq -C_1.
\]

Let
\[
\varphi_1(x) = t\psi(x) + \xi(x),
\]
then for $t$ sufficiently large and for all $x \in D$,
\[
(-\Delta)^{\alpha/2} \varphi_1(x) = \{ (-\Delta)^{\alpha/2} \psi(x) + (-\Delta)^{\alpha/2} \xi(x)
\geq \frac{tC}{|x|^{n+\alpha}} - C_1
\geq 1.
\]
Thus
\[
\begin{cases}
(\alpha/2)(\varphi_1 - u) \geq 0, & x \in D, \\
\varphi_1(x) - u(x) \geq 0, & x \in D^C.
\end{cases}
\quad (19)
\]
Applying the maximum principle to (19), we arrive at
\[
\varphi_1(x) \geq u(x), \quad x \in D.
\]
Hence for $x \in D$,
\[
0 \leq u(x) - u(y) = u(x) \leq \varphi_1(x) = \varphi_1(x) - \varphi_1(y) \leq C|x-y|^{\alpha/2}.
\]
This shows that $u$ is Hölder continuous near $y$ indeed.

3. S-harmonic functions, Poisson’s representations, and Liouville theorems. We say that a function $u$ is $s$-harmonic, if
\[
(-\Delta)^s u(x) = 0.
\]
In the case $s = 1$, $u$ is a harmonic function, and it can be expressed in terms of its average on the sphere:
\[
u(x) = \frac{1}{|S_r|} \int_{S_r(x)} u(y) d\sigma_y
\]
where $S_r(x)$ is the sphere of radius $r$ centered at $x$. Consequently, one can derive the Harnack inequality and the well-known Liouville theorem:
**If $u$ is bounded from either above or below, then it must be constant.**

Do we have similar properties for $s$-harmonic functions?

The answer is affirmative.

**Proposition 3.1.** Assume that $u$ is $s$-harmonic in $\mathbb{R}^n$, then
\[
u(x) = \int_{|y|>r} P_r(y,x)u(y)dy, \quad \forall x \in B_r(0),
\]
where
\[
P_r(y,x) = \begin{cases}
\frac{\Gamma(n/2)}{\pi^{n/2}s} \sin(\pi s) \left(\frac{x^2-|y|^2}{|y|^2-r^2}\right)^s \frac{1}{|x-y|^n}, & |y| > r, \\
0, & |y| < r,
\end{cases}
\]
is the so-called Poisson kernel and (20) is the Poisson representation of $u(x)$.

The idea of the proof was given in [37], and more details were provided in [7].

Notice that the total integral
\[
\int_{\mathbb{R}^n} P_r(y,x) dy = 1
\]
which is independent of $r$. One may regard this as the distribution of a unit mass (with density $P_r(y,x)$ at point $y$) over the whole region outside of $B_r(0)$ when $s < 1$, and as $s \to 1$, all the mass is concentrating on the boundary $S_r(0)$ due to the term $\frac{1}{(|y|^2-r^2)^s}$ in the kernel $P_r(y,x)$, and the limit becomes a uniform distribution of the unit mass over the sphere. Hence the Poisson representation of $s$-harmonic functions
(20) approaches the spherical average of harmonic functions. More precisely, we have

**Proposition 3.2.** Assume that \( u \in L_{2s} \) is continuous. Then as \( s \to 1 \),

\[
\int_{|y| > r} P_r(y, 0) u(y) dy \to \frac{1}{|S_r|} \int_{S_r(0)} u(y) d\sigma_y.
\]

**Proof.** By virtue of (21), we have

\[
\int_{|y| > r} P_r(y, 0) u(y) dy = \frac{\Gamma(\frac{d}{2})}{\pi^{d/4}} \sin(\pi s) \int_{|y| > r} \left( \frac{r^2}{|y|^2 - r^2} \right)^s u(y) dy
\]

\[
= \frac{\Gamma(\frac{d}{2})}{\pi^{d/4}} \sin(\pi s) \cdot \left(1 - \frac{s}{1 - s}\right) \int_{|y| > r} \left( \frac{r^2}{|y|^2 - r^2} \right)^s u(y) dy \quad (22)
\]

:= J_1(s) \cdot J_2(s).

It then follows from the L'Hospital's rule that

\[
\lim_{s \to 1} J_1(s) = \frac{\Gamma(\frac{d}{2})}{\pi^{d/4}} = \frac{2}{|S_1(0)|}. \quad (23)
\]

For any \( \epsilon > 0 \), \( J_2(s) \) can be rewritten as

\[
J_2(s) = (1 - s) \int_{r < |y| \leq r + \epsilon} \left( \frac{r^2}{|y|^2 - r^2} \right)^s u(y) dy
\]

\[
+ (1 - s) \int_{|y| > r + \epsilon} \left( \frac{r^2}{|y|^2 - r^2} \right)^s u(y) dy \quad (24)
\]

:= J_{2,1}^s(s) + J_{2,2}^s(s).

Noting that

\[
|J_{2,2}^s(s)| \leq (1 - s) \int_{|y| \geq r + \epsilon} \frac{r^{2s}}{\left(1 - \frac{r^2}{|y|^2}\right)^2} \frac{|u(y)|}{|y|^{n+2s}} dy,
\]

which along with \( u \in L_{2s} \) verifies

\[
\lim_{s \to 1} J_{2,2}^s(s) = 0, \quad \text{for any fixed } \epsilon > 0. \quad (25)
\]

Applying the polar coordinate exchange to \( J_{2,1}^s(s) \) yields

\[
J_{2,1}^s(s) = (1 - s) \int_r^{r+\epsilon} \int_{S_r(0)} \frac{r^{2s}}{(\tau^2 - r^2)^s} u(y) \frac{d\sigma_y}{\tau^n} d\tau
\]

\[
= (1 - s) \int_{S_r(0)} u(y) d\sigma_y \cdot \int_r^{r+\epsilon} \frac{1}{(\tau^2 - r^2)^s} \frac{d\tau}{\tau^n} \quad (\delta \in [r, r + \epsilon])
\]

\[
= \int_{S_r(0)} u(y) d\sigma_y \cdot \frac{(1 - s) r^{2s}}{1 - s} \int_r^{r+\epsilon} \frac{1}{(\tau - r)^s} d\tau \quad (\delta \in [r, r + \epsilon])
\]

\[
= \left( \frac{r^{2s}}{(\xi + r)^s \xi^{n+s}} \right) \int_{S_r(0)} u(y) d\sigma_y,
\]

where we have used the mean value theorem in the second and third equalities. Consequently, we obtain

\[
\lim_{s \to 1} J_{2,1}^s(s) = \frac{r^2}{(\xi + r)^s \xi^n} \int_{S_r(0)} u(y) d\sigma_y \quad (\xi, \delta \in [r, r + \epsilon]).
\]
Letting $\epsilon \to 0$, we arrive at
\[
\lim_{\epsilon \to 0} \lim_{s \to 1} J_{2,1}^\epsilon(s) = \frac{1}{2r_n-1} \int_{\partial S_r(0)} u(y) d\sigma_y,
\]
which together with (22), (23), (24) and (25) completes the proof. \qed

This Poisson’s representation can be used to prove the Liouville theorem for $s$-harmonic functions.

**Proposition 3.3.** Assume that $u \in L_{2s} \cap C(R^n)$ is $s$-harmonic in the sense of distribution, and $u$ is either bounded from below or from above, then $u \equiv C$.

The earlier proof was given in [2], and more general cases were considered in [7] and [26]. The idea is, for a fixed $x$, differentiate both sides of (20) with respect to each $x_i$, then let $r \to \infty$ to show that $\partial u/\partial x_i(x) = 0$.

Poisson’s representation can also be used to solve the non-homogeneous Dirichlet problem
\[
\begin{cases}
(- \triangle)^* u(x) = f(x), & x \in B_r(0), \\
u(x) = g(x), & x \in B_{r}^c(0).
\end{cases}
\]

We split the solution into two parts $u = u_1 + u_2$, where $u_1$ is the solution of the homogeneous Dirichlet problem
\[
\begin{cases}
(- \triangle)^* u_1(x) = f(x), & x \in B_r(0), \\
u_1(x) = 0, & x \in B_{r}^c(0).
\end{cases}
\]

It is given by (see Chapter 2 in [17])
\[
u_1(x) = \int_{B_r(0)} G(x,y) f(y) dy,
\]
where $G(x,y)$ is the Green’s function associated with $(- \triangle)^*$ on $B_r(0)$.

While $u_2$ satisfies
\[
\begin{cases}
(- \triangle)^* u_2(x) = 0, & x \in B_r(0), \\
u_2(x) = g(x), & x \in B_{r}^c(0).
\end{cases}
\]

One can show that (see Chapter 4 in [17])
\[
u_2(x) = \int_{|y| > r} P_r(y,x) g(y) dy, \quad \forall x \in B_r(0),
\]

Similar results on the half space
\[R^n_+ = \{ x \in R^n \mid x_n > 0 \}\]
were obtained in [53].

Consider the Dirichlet problem for $s$-harmonic functions
\[
\begin{cases}
(- \triangle)^* u(x) = 0, & x \in R^n_+, \\
u(x) \geq 0, & x \in R^n_+.
\end{cases}
\]

It is well-known that
\[
u(x) = \begin{cases}
C x_n^s, & x \in R^n_+, \\
0, & x \notin R^n_+.
\end{cases}
\]
is a family of solutions for problem (30) with any nonnegative constant $C$.

Then one may naturally ask: *Are these the only solutions?*

The answer is affirmative as given in [53]:
Proposition 3.4. Let \( 0 < s < 1 \), \( u \in L_{2s} \). Assume \( u \) is a solution of (30) in the sense of distribution, then
\[
    u(x) = \begin{cases} 
        Cx^n, & x \in \mathbb{R}_+^n, \\
        0, & x \notin \mathbb{R}_+^n,
    \end{cases}
\]
for some nonnegative constant \( C \).

The proof is mainly based on the Poisson representation of the solutions:
Proposition 3.5. Assume that \( u \in L_{2s} \) and is lower semi-continuous on \( \mathbb{R}_+^n \). If \( u \) is a solution of (30) in the sense of distribution, then for \( |x - x_r| < r \),
\[
    u(x) = \int_{|y - x_r| > r} P_r(x - x_r, y - x_r)u(y)dy, \tag{31}
\]
where \( x_r = (0, \cdots, 0, r) \).

The idea of proof for Proposition 3.4 is slightly different from that in \( \mathbb{R}^n \). For each fixed \( x \in \mathbb{R}_+^n \), we evaluated first derivatives of \( u \) by using (31). Letting \( r \to \infty \), we derive
\[
    \frac{\partial u}{\partial x_i}(x) = 0, \quad i = 1, 2, \cdots, n - 1,
\]
and
\[
    \frac{\partial u}{\partial x_n}(x) = \frac{s}{x_n} u(x).
\]
These yield the desired results. It is quite delicate to derive
\[
    \frac{\partial u}{\partial x_i}(x) = 0, \quad i = 1, 2, \cdots, n - 1,
\]
and some interesting techniques were employed.

4. Direct method of moving planes.

4.1. Single equation with the fractional Laplacian. In [33], Jarohs and Weth introduced antisymmetric maximum principles and applied them to carry on the method of moving planes directly on nonlocal problems to show the symmetry of solutions. The operators they considered are quite general, however, their maximum principles only apply to bounded regions \( \Omega \), and they only considered weak solutions defined by \( H^{\alpha/2}(\Omega) \) inner product.

In [15], the authors developed a systematic approach to directly carry on the method of moving planes for nonlinear equations involving the fractional Laplacian, either on bounded or unbounded domains. To illustrate the major difference between local and nonlocal operators, and the major difficulty encountered by the latter, we consider the following simple example:
\[
    (-\Delta)^{\alpha/2} u(x) = f(u(x)), \quad u(x) > 0, \quad x \in \mathbb{R}^n. \tag{32}
\]

As usual, let
\[
    T_\lambda = \{ x \in \mathbb{R}^n | \ x_1 = \lambda, \ \text{for} \ \lambda \in \mathbb{R} \}
\]
be the moving planes,
\[
    \Sigma_\lambda = \{ x \in \mathbb{R}^n | \ x_1 < \lambda \}
\]
be the region to the left of the plane, and
\[
    x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)
\]
be the reflection of \( x \) about the plane \( T_\lambda \).
Assume that $u$ is a solution of pseudo differential equation (32). To compare the values of $u(x)$ with $u(x^\lambda)$, we denote

$$w_\lambda(x) = u(x^\lambda) - u(x).$$

The first step is to show that for $\lambda$ sufficiently negative, we have

$$w_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda.$$  \hspace{1cm} (33)

This provides a starting point to move the plane. Then in the second step, we move the plane to the right as long as inequality (33) holds to its limiting position to show that $u$ is symmetric about the limiting plane.

To prove (33), usually a maximum principle is employed. One can derive from (32) that $w_\lambda$ satisfies the equation

$$(-\Delta)^{\alpha/2} w_\lambda(x) - f'(\xi_\lambda(x)) w_\lambda(x) = 0, \quad x \in \Sigma_\lambda.$$ \hspace{1cm} (34)

In the above equation, if we replace the fractional Laplacian by a local operator such as $-\Delta$, then in order to apply a maximum principle, we have the needed boundary condition that

$$w_\lambda(x) = 0, \quad x \in \partial \Sigma_\lambda$$

which is satisfied automatically by the definition of $w_\lambda$. However, for nonlocal operators, as we explained in Section 3, the boundary condition is far from enough and one generally requires an exterior condition that

$$w_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda^C$$

be satisfied in the whole complement of $\Sigma_\lambda$! This is almost impossible. To bypass such a condition, we need to explore the anti-symmetry property of $w_\lambda$:

$$w_\lambda(x) = -w_\lambda(x^\lambda).$$

In [15], a simple maximum principle for anti-symmetric functions was proved.

**Proposition 4.1.** Let $\Omega$ be a bounded domain in $\Sigma_\lambda$. Assume that $w_\lambda \in L_\alpha \cap C^{1,1}_{loc}(\Omega)$ and is lower semi-continuous on $\bar{\Omega}$. If

$$\begin{cases}
(-\Delta)^{\alpha/2} w_\lambda(x) \geq 0 & \text{in } \Omega, \\
w_\lambda(x) \geq 0 & \text{in } \Sigma_\lambda \setminus \Omega,
\end{cases}$$

then

$$w_\lambda(x) \geq 0 \text{ in } \Omega.$$

Furthermore, if $w_\lambda = 0$ at some point in $\Omega$, then

$$w_\lambda(x) = 0 \text{ almost everywhere in } \mathbb{R}^n.$$

These conclusions hold for unbounded region $\Omega$ if we further assume that

$$\lim_{|x| \to \infty} w_\lambda(x) \geq 0.$$

The idea of proof is to evaluate $(-\Delta)^{\alpha/2} w_\lambda(\cdot)$ at a negative minimum $x^o$ of $w_\lambda$ in $\Sigma_\lambda$ to show that

$$(-\Delta)^{\alpha/2} w_\lambda(x^o) < 0.$$

This contradicts with the equation

$$(-\Delta)^{\alpha/2} w_\lambda(x) \geq 0.$$

In this process, we need to exploit the anti-symmetry of $w_\lambda$ and monotonicity of the kernel $\frac{1}{|x-y|^{n+\alpha}}$. 

Now, if we impose the following strong conditions on \( f \) and on the solution \( u \):
\[
f'(\cdot) \leq 0 \quad \text{and} \quad \lim_{|x| \to \infty} u(x) = 0;
\]
en then the above Proposition 4.1 implies, without any exterior condition, that
\[
w_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda.
\]

In many cases, \( w_\lambda \) may not satisfy the equation
\[
(-\triangle)^{\alpha/2} w_\lambda \geq 0 \tag{35}
\]
as required in Proposition. For example, if \( f(u) = u^p \), then \( f'(u) > 0 \), and (35) is violated. However, as in (34), one can derive that
\[
(-\triangle)^{\alpha/2} w_\lambda + c_\lambda(x)w_\lambda(x) \geq 0
\]
for some function \( c_\lambda(x) \) depending on \( u \). One can easily see that the above maximum principle is still valid for nonnegative \( c_\lambda \), however this is usually not the case in practice. Fortunately, in the process of moving planes, each time we only need to move \( T_\lambda \) a little bit to the right, hence the increment of \( \Sigma_\lambda \) is a narrow region, and a maximum principle is easier to hold in a narrow region provided \( c(x) \) is not “too negative”, as you will see below.

**Proposition 4.2.** (Narrow Region Principle.) Let \( \Omega \) be a bounded narrow region in \( \Sigma_\lambda \), such that it is contained in
\[
\{ x \mid \lambda - \delta < x_1 < \lambda \}
\]
with small \( \delta \). Suppose that \( w_\lambda \in L_\alpha \cap C^{1,1}_{loc}(\Omega) \) and is lower semi-continuous on \( \overline{\Omega} \). If \( c(x) \) is bounded from below in \( \Omega \) and
\[
\begin{cases}
(-\triangle)^{\alpha/2} w_\lambda(x) + c(x)w_\lambda(x) \geq 0 & \text{in } \Omega, \\
w_\lambda(x) \geq 0 & \text{in } \Sigma_\lambda \setminus \Omega,
\end{cases}
\tag{36}
\]
then for sufficiently small \( \delta \), we have
\[
w_\lambda(x) \geq 0 \text{ in } \Omega.
\]

Furthermore, if \( w_\lambda = 0 \) at some point in \( \Omega \), then
\[
w_\lambda(x) = 0 \text{ almost everywhere in } \mathbb{R}^n.
\]

These conclusions hold for unbounded region \( \Omega \) if we further assume that
\[
\lim_{|x| \to \infty} w_\lambda(x) \geq 0.
\]

The key ingredient is the following estimate at a negative minimum \( x^o \) of \( w_\lambda \) in \( \Sigma_\lambda \):
\[
(-\triangle)^{\alpha/2} w_\lambda(x^o) \leq 2w_\lambda(x^o)C_{n,\alpha} \int_{\Sigma_\lambda} \frac{1}{|x^o - y|^n + \alpha} \, dy. \tag{37}
\]
When \( x^o \) is within \( \delta \) distance from the complement of \( \Sigma_\lambda \), we have
\[
\int_{\Sigma_\lambda} \frac{1}{|x^o - y|^n + \alpha} \, dy \sim \frac{1}{\delta^\alpha}.
\]
This, together with the lower bounded-ness of \( c_\lambda(x) \), leads to
\[
(-\triangle)^{\alpha/2} w_\lambda(x^o) + c_\lambda(x^o)w_\lambda(x^o) < 0,
\]
a contradiction with the inequality in (36).

Since the contradiction arguments are conducted at a negative minimum of \( w_\lambda \), hence when working on an unbounded domain, one needs to rule out the possibility
that such minima would “leak” to infinity. This can be done when \( c(x) \) decays “faster” than \( 1/|x|^\alpha \) near infinity.

**Proposition 4.3. (Decay at Infinity.)** Let \( \Omega \) be an unbounded region in \( \Sigma_\lambda \). Assume \( w_\lambda \in L_\alpha \cap C^{1,1}_{\text{loc}}(\Omega) \) is a solution of

\[
\begin{cases}
-\Delta^{\alpha/2}w_\lambda(x) + c(x)w_\lambda(x) \geq 0 & \text{in } \Omega, \\
w_\lambda(x) \geq 0 & \text{in } \Sigma_\lambda \setminus \Omega,
\end{cases}
\]

with

\[
\lim_{|x| \to \infty} |x|^\alpha c(x) \geq 0,
\]

then there exists a constant \( R_0 > 0 \) (depending on \( c(x) \), but independent of \( w_\lambda \)), such that if

\[w_\lambda(x_o) = \min_{\Omega} w_\lambda(x) < 0,\]

then

\[|x_o| \leq R_0.\]

The key element in the proof is again (37). When a negative minimum \( x_o \) is far away from the origin, one can derive

\[
\int_{\Sigma_\lambda} \frac{1}{|x^\alpha - y^\alpha|^n + \alpha} dy \sim \frac{1}{|x^\alpha|^\alpha}.
\]

This would contradict with the inequality in (36) under the decay assumption on \( c(x) \).

To illustrate the applications of the narrow region principle and decay at infinity, we consider

\[(-\Delta)^{\alpha/2} u = u^p(x), \quad x \in \mathbb{R}^n.\]

**Proposition 4.4.** Assume that \( 0 < \alpha < 2 \) and \( u \in L_\alpha \cap C^{1,1}_{\text{loc}} \) is a nonnegative solution of equation (38). Then

(i) In the critical case \( p = \frac{n+\alpha}{n-\alpha} \), \( u \) is radially symmetric and monotone decreasing about some point.

(ii) In the subcritical case \( 1 < p < \frac{n+\alpha}{n-\alpha} \), \( u \equiv 0 \).

**Outline of Proof.** Without decay assumption on the solution \( u \) near infinity, we need to make a Kelvin transform. Let \( v(x) = \frac{1}{|x|^{n-\alpha}} u\left(\frac{x}{|x|}\right) \). Then

\[(-\Delta)^{\alpha/2} v(x) = \frac{1}{|x|^\gamma} u^p(x)\]

with \( \gamma = n + \alpha - p(n - \alpha) \). Let \( w_\lambda(x) = v(x^\lambda) - v(x) \). Then at points where \( w_\lambda \leq 0 \), one has

\[(-\Delta)^{\alpha/2} w_\lambda + c(x)w_\lambda(x) \geq 0\]

with

\[c(x) = -\frac{p}{|x|^\gamma} u^{p-1}(x),\]

which obviously satisfies the decay condition in Proposition 4.3. Hence when \( \lambda \) is sufficiently negative, there should be no negative minimum of \( w_\lambda \) in \( \Sigma_\lambda \), and hence

\[w_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda.\]

This completes Step 1.
In Step 2, we move the plane $T_\lambda$ to the right as long as (39) holds to its limiting position and show that $v$ is symmetric about the limiting plane. Take the subcritical case for example. Let

$$\lambda_0 = \sup \{ \lambda \leq 0 \mid w_\mu(x) \geq 0, \ x \in \Sigma_\mu \setminus \{0^\mu \}, \mu \leq \lambda \}.$$  

We show that $\lambda_0 = 0$. Otherwise, if $\lambda_0 < 0$, then $w_{\lambda_0} \not\equiv 0$, which can be derived from

$$(-\Delta)^{\alpha/2} w_{\lambda_0}(x) = \frac{1}{|x|^{2\alpha/\gamma}} v_{\lambda_0}^p(x) - \frac{1}{|x|^\gamma} v^p(x).$$

Then by the strong maximum principle, we must have

$$w_{\lambda_0}(x) > 0, \ x \in \Sigma_{\lambda_0} \setminus \{0^\lambda\}.$$  

Notice that $w_{\lambda_0}(x)$ has a possible singularity $0^{\lambda_0}$ in $\Sigma_{\lambda_0}$, and one needs to show that in a neighborhood of this singular point, $w_{\lambda_0}(x)$ is positively bounded away from 0. This can be derived directly from a recent fundamental result of Li, Wu, and Xu [40], in which a Bocher type theorem for fractional equations was established.

Now using both decay at infinity and narrow region principle, we will be able to move the plane to the right a little bit (note this “little bit” is a narrow region), while keeping inequality (39) valid. This would contradict the definition of $\lambda_0$. Therefore, we must have $\lambda_0 = 0$ and hence

$$v_0(x) \geq v(x), \ x \in \Sigma_0.$$  

Since $x_1$ direction can be chosen arbitrarily, $v(x)$ must be radially symmetric about the origin, and so does $u(x)$.

Let $x^o$ be any point in $\mathbb{R}^n$, make a Kelvin transform centered at $x^o$:

$$v_{x^o}(x) = \frac{1}{|x-x^o|^{n-\alpha}} u \left( \frac{x-x^o}{|x-x^o|^2} + x^o \right).$$

Then similarly, by carrying the method of moving planes on $v_{x^o}(x)$, one can show that $v_{x^o}(x)$ must be radially symmetric about $x^o$, and hence $u(x)$ is also radially symmetric about $x^o$. A function that is radially symmetric about any point has to be constant, and due to equation (38), we have $u(x) \equiv 0$.

4.2. For equations when Kelvin transform are not applicable. Consider a fractional equation with indefinite nonlinearities

$$(-\Delta)^{\alpha/2} u = a(x_1) u^p(x)$$  

(40)

for $0 < \alpha < 2$ and $1 < p < \infty$.

Usually, to carry on the method of moving planes, one needs to assume that the solution $u$ decays to zero at some rate near $\infty$. As we have seen in the previous section, for equation (38), without decay assumption on $u$, in the critical and subcritical cases, one can exploit the Kelvin transform $v(x) = \frac{1}{|x|^{n-\alpha}} u \left( \frac{x}{|x|^2} \right)$ to derive

$$(-\Delta)^{\alpha/2} v(x) = \frac{1}{|x|^{\gamma}} v^p(x), \ \text{with} \ v(x) \sim \frac{1}{|x|^{n-\alpha}} \ \text{near} \ \infty.$$  

(41)

Here $v(x)$ has the required decay rate near infinity, and $\gamma \geq 0$ guarantees that the coefficient $\frac{1}{|x|^{\gamma}}$ possesses the needed monotonicity to carry on the method of moving planes on the transformed equation (41).

Now for equation (40), due to the presence of $a(x_1)$, the coefficient of the transformed equation does not have the required monotonicity, and this renders the Kelvin transform useless. Even for equation (38), in the super critical case where
p > \frac{n+\alpha}{n-\alpha}$, after the Kelvin transform, the right hand side of the equation in (41) becomes $|x|^\gamma v^p(x)$ for some $\gamma > 0$, which also does not have the needed monotonicity to carry on the method of moving planes.

To assume $\lim_{|x|\to \infty} u(x) = 0$ is impractical, because in the process of applying this Liouville Theorem (nonexistence of solutions) in the blowing up and re-scaling arguments to establish a priori estimate, the solution of the limiting equation is known to be only bounded. Hence it is reasonable to assume that $u$ is bounded when we consider equation (40). Without the condition $\lim_{|x|\to \infty} u(x) = 0$, in order to use the method of moving planes, in [22], an auxiliary function was introduced, so that the planes can still be moved along $x_1$ direction all the way up to $\infty$ to derive

**Proposition 4.5.** ([22]) Let $0 < \alpha < 2$ and $1 < p < \infty$. Assume that

\[ a(t) \text{ is nondecreasing for } t \in (-\infty, \infty) \text{ and } \limsup_{t \to -\infty} a(t)|t|^{\alpha} \leq 0. \tag{42} \]

Then any positive bounded solution $u$ of equation (40) is monotone increasing in $x_1$ direction.

By comparing the solution $u$ with the first eigenfunction at a unit ball far away from the origin, a contradiction was derived and a non-existence result has been obtained.

**Proposition 4.6.** ([22]) Let $0 < \alpha < 2$ and $1 < p < \infty$. Then equation (40) possesses no positive bounded solutions.

The main ideas in deriving the monotonicity of solutions (Proposition 4.5)

(i) The difficulty for the fractional equation

From the assumption (42), it follows that

\[ (-\Delta)^{\alpha/2}w_\lambda(x) \geq a(x_1) p \xi_\lambda^{-1}(x) w_\lambda(x), \tag{43} \]

where $\xi_\lambda(x)$ is valued between $u(x)$ and $u_\lambda(x)$.

We want to show that

$w_\lambda \geq 0 \ \forall x \in \Sigma_\lambda$ and for all $\lambda \in (-\infty, \infty)$.

To this end, usually a contradiction argument is used. Suppose $w_\lambda$ has a negative minimum in $\Sigma_\lambda$, then one would derive a contradiction with inequality (43). However, here we only assume that $u$ is bounded, which cannot prevent the minimum of $w_\lambda(x)$ from leaking to $\infty$. To overcome this difficulty, for integer order equations (see [42])

\[ -\Delta u = x_1 u^p(x) \in \mathbb{R}^n, \]

an auxiliary function was introduced:

\[ \tilde{w}_\lambda(x) = \frac{w_\lambda(x)}{g(x)} \text{ with } g(x) \to \infty, \text{ as } |x| \to \infty. \]

Now

\[ \lim_{|x| \to \infty} \tilde{w}_\lambda(x) = 0 \]

and hence $\tilde{w}_\lambda$ can attain its negative minimum in the interior of $\Sigma_\lambda$. The corresponding left hand side of (43) becomes

\[ -\Delta w_\lambda = -\Delta \tilde{w}_\lambda \cdot g - 2\nabla \tilde{w}_\lambda \cdot \nabla g - \tilde{w}_\lambda \cdot \Delta g. \tag{44} \]
At a minimum of $\bar{w}_\lambda$, the middle term on the right hand side vanishes since $\nabla \bar{w}_\lambda = 0$. This makes the analysis relatively easier. However, the fractional counterpart of (44) is

\[
(-\Delta)^{\alpha/2} w_\lambda = \nabla \bar{w}_\lambda \cdot g - 2C \int_{\mathbb{R}^n} \frac{(\bar{w}_\lambda(x) - \bar{w}_\lambda(y))(g(x) - g(y))}{|x - y|^{n+\alpha}} \, dy + \bar{w}_\lambda \cdot (-\Delta)^{\alpha/2} g.
\]

At a minimum of $\bar{w}_\lambda$, the middle term on the right hand side (the integral) neither vanishes nor has a definite sign. This is the main difficulty encountered by the fractional nonlocal operator, and to circumvent which, the authors in [22] introduced a different auxiliary function and estimate $(-\Delta)^{\alpha/2} w_\lambda$ in an entirely different approach.

(ii) The key estimate

Different from the logarithmic auxiliary function chosen in [42] and [23], we choose the auxiliary function as

\[
g(x) = |x - (\lambda + 1)e_1|^\sigma, \quad \bar{w}_\lambda(x) = \frac{w_\lambda(x)}{g(x)},
\]

where

\[
e_1 = (1, 0, \ldots, 0),
\]

and $\sigma$ is a small positive number.

Now $\lim_{|x| \to \infty} \bar{w}_\lambda(x) = 0$. If for a given $\lambda$, the inequality

\[
w_\lambda \geq 0 \quad \forall x \in \Sigma_\lambda
\]

is violated, then there exists a point $x^o$ in $\Sigma_\lambda$, such that

\[
\bar{w}_\lambda(x^o) = \min_{x \in \Sigma_\lambda} \bar{w}_\lambda(x).
\]

By directly evaluating the singular integral defining $(-\Delta)^{\alpha/2} w_\lambda(x^o)$ and through a delicate calculation, one arrives at the key estimate

\[
(-\Delta)^{\alpha/2} w_\lambda(x^o) \leq C w_\lambda(x^o) \frac{1}{|x_1^o - \lambda|^\sigma},
\]

where $x_1^o$ is the first component of $x^o$ and $C$ is a positive constant independent of $\lambda$.

Combining this with (43), we arrive at

\[
\frac{C}{|x_1^o - \lambda|^\sigma} \leq a(x_1^o) p_{\xi}^p - 1 (x^o).
\]

Here $\xi_\lambda^{p-1}(x)$ is uniformly bounded due to the bounded-ness assumption on $u(x)$.

In Step 1, for $\lambda$ sufficiently negative, if (45) is violated, then one derives (47), and this contradicts our assumption (42) on $a(\cdot)$.

In Step 2, if $x^o$ falls in a narrow region $\Sigma_\lambda \setminus \Sigma_{\lambda-\delta}$ for sufficiently small $\delta$, then the left hand side of (47) becomes very large while the right hand side is bounded, again a contradiction. This actually implies the narrow region principle. Through Step 1 and Step 2, one is able to move the plane $T_\lambda$ all the way to infinity to derive that $u(x)$ is monotone increasing along $x_1$ direction.
4.3. Systems with the fractional Laplacian.

4.3.1. On bounded domains. To illustrate the idea, we consider a system of two equations involving fractional Laplacians with different powers:

\[
\begin{aligned}
\begin{cases}
(-\Delta)^{\alpha/2}u(x) = f(v(x)) & x \in B_1(0) \\
(-\Delta)^{\beta/2}v(x) = g(u(x)) & x \in B_1(0).
\end{cases}
\end{aligned}
\]

(48)

Let \( U_\lambda(x) = u_\lambda(x) - u(x) \) and \( V_\lambda(x) = v_\lambda(x) - v(x) \), then by (48) and the mean value theorem, we have

\[
\begin{aligned}
\begin{cases}
(-\Delta)^{\alpha/2}U_\lambda(x) - f'(\lambda_\lambda(x))V_\lambda(x) = 0 & x \in \Sigma \cap B_1(0) \\
(-\Delta)^{\beta/2}V_\lambda(x) - g'(\lambda_\lambda(x))U_\lambda(x) = 0 & x \in \Sigma \cap B_1(0).
\end{cases}
\end{aligned}
\]

(49)

Using the method of moving planes, we can prove

**Proposition 4.7.** Assume that \((u, v)\) is a pair of positive solutions for (48). Suppose

\[f'(\cdot), g'(\cdot) > 0\] are bounded from above.

Then \((u, v)\) must be radially symmetric about 0.

For this bounded domain, we only need the following

**Proposition 4.8** (Narrow Region Principle). Let \( \Omega \) be a bounded narrow region in \( \Sigma_\lambda \), such that it is contained in \( \{ x \mid \lambda - \delta < x_1 < \lambda \} \) with small \( \delta \). For \( 0 < \alpha, \beta < 2 \), assume that \( U_\lambda \in L_\alpha(\mathbb{R}^n) \cap C_{loc}^{1,1}(\Omega) \), \( V_\lambda \in L_\beta(\mathbb{R}^n) \cap C_{loc}^{1,1}(\Omega) \), and \( U, V \) are lower semi-continuous on \( \Omega \). If \( c_i(x) < 0 \), \( i = 1, 2 \), are bounded from below in \( \Omega \) and

\[
\begin{aligned}
\begin{cases}
(-\Delta)^{\alpha/2}U_\lambda(x) + c_1(x)V_\lambda(x) \geq 0 & x \in \Omega, \\
(-\Delta)^{\beta/2}V_\lambda(x) + c_2(x)U_\lambda(x) \geq 0 & x \in \Omega, \\
U_\lambda(x), V_\lambda(x) \geq 0 & x \in \Sigma \setminus \Omega
\end{cases}
\end{aligned}
\]

(50)

then for sufficiently small \( \delta \), we have

\[ U_\lambda(x), V_\lambda(x) \geq 0 \text{ in } \Omega. \]

(51)

If \( \Omega \) is unbounded, the conclusion still holds under the condition that

\[ \lim_{|x| \to \infty} U_\lambda(x), V_\lambda(x) \geq 0. \]

Further, if \( U_\lambda(x) \) or \( V_\lambda(x) \) attains \( \theta \) somewhere in \( \Omega \), then

\[ U_\lambda(x) = V_\lambda(x) \equiv \theta, \quad x \in \mathbb{R}^n. \]

(52)

**Idea of the Proof of the Narrow Region Principle**

Similar to the single equation case, we can obtain the following two key estimates at negative minima \( x^o \) and \( x^1 \) of \( U_\lambda \) and \( V_\lambda \) respectively:

\[ (-\Delta)^{\alpha/2}U_\lambda(x^o) \leq U_\lambda(x^o) \frac{A}{\delta^\alpha} \]

(53)

and

\[ (-\Delta)^{\beta/2}V_\lambda(x^1) \leq V_\lambda(x^1) \frac{B}{\delta^\beta} \]

(54)

The condition \( c_i(x) < 0, i = 1, 2 \) is to ensure, via inequalities in (50) that \( U_\lambda \) and \( V_\lambda \) attain their negative minima in \( \Omega \) simultaneously. To derive a contradiction, the key idea is to use the following iteration argument. Starting from the first inequality
in (50), then applying (53), second inequality in (50), and (54) successively, we arrive at
\[
-c_1(x^0)\lambda(x^0) \leq (-\Delta)^{\alpha/2}U_\lambda(x^0) \leq U_\lambda(x^0)\frac{A}{\delta^\alpha} \tag{55}
\]
\[
\leq U_\lambda(x^1)\frac{A}{\delta^\alpha} \leq -\frac{(-\Delta)^{\beta/2}V_\lambda(x^1)}{c_2(x^1)} A \frac{A}{\delta^\alpha}
\]
\[
\leq -\frac{V_\lambda(x^1)}{c_2(x^1)} \frac{AB}{\delta^\alpha + \beta}.
\]
It follows that
\[
c_1(x^0)c_2(x^1) \geq \frac{AB}{\delta^\alpha + \beta}, \tag{56}
\]
because \(c_2(x^1)\) and \(V_\lambda(x^0)\) (due to (55)) are both negative.

Since \(c_1(x)\) and \(c_2(x)\) are assumed both bounded, for sufficiently small \(\delta\), we derive a contradiction with (56).

Similar approaches apply to a system of \(m\) equations.

Applying this narrow regions principle in both step 1 and step 2 in the method of moving planes, one can prove Proposition 4.7.

4.3.2. On unbounded domains. When considering a system on an unbounded domain, for instance
\[
\begin{cases}
(-\Delta)^{\alpha/2}u(x) = v^q(x) & x \in \mathbb{R}^n \\
(-\Delta)^{\beta/2}v(x) = u^p(x) & x \in \mathbb{R}^n,
\end{cases} \tag{57}
\]
to carry on the method of moving planes to derive symmetry of solutions, one also need

**Proposition 4.9** (Decay at infinity). For \(0 < \alpha, \beta < 2\), assume that \(U_\lambda \in L_\alpha(\mathbb{R}^n) \cap C^{1,1}_{loc}(\Omega), V_\lambda \in L_\beta(\mathbb{R}^n) \cap C^{1,1}_{loc}(\Omega)\), and \(U_\lambda, V_\lambda\) are lower semi-continuous on \(\bar{\Omega}\). If
\[
\begin{cases}
(-\Delta)^{\alpha/2}U_\lambda(x) + c_1(x)V_\lambda(x) \geq 0, & x \in \Omega, \\
(-\Delta)^{\beta/2}V_\lambda(x) + c_2(x)U_\lambda(x) \geq 0, & x \in \Omega, \\
U_\lambda(x), V_\lambda(x) \geq 0, & x \in \Sigma \setminus \Omega,
\end{cases} \tag{58}
\]
with
\[
c_1(x) \sim o\left(\frac{1}{|x|^\alpha}\right), \quad c_2(x) \sim o\left(\frac{1}{|x|^\beta}\right), \quad \text{for } |x| \text{ large}, \tag{59}
\]
and
\[
c_i(x) < 0, \quad i = 1, 2.
\]
Then there exists a constant \(R > 0\) depending on \(c_i(x)\), but is independent of \(U_\lambda, V_\lambda\), such that if \(\exists \hat{x}, \check{x} \in \Omega\),
\[
U_\lambda(\hat{x}) = \min_{\Omega} U_\lambda(x) < 0, \quad V_\lambda(\check{x}) = \min_{\bar{\Omega}} V_\lambda(x) < 0,
\]
then at least one of \(\hat{x}\) and \(\check{x}\) satisfies
\[
|x| \leq R.
\]

The proof can be conducted by using the similar iteration technique as for the narrow region principle, interested readers may see [17] for more details.

Now we use system (57) as an example to illustrate the ideas on how to employ the decay at infinity and the narrow region principle to obtain the radial symmetry
of positive solutions. From this process, one can see that there are major differences between a system and a single equation, for the former, one needs to consider more cases. To focus on the main approaches, we simply assume that \( u \) and \( v \) decay at a certain rate near infinity. As we mentioned before, without these decay assumption, one may resort to Kelvin transforms.

**Proposition 4.10.** Assume that \( u \in L^\alpha_\alpha(\mathbb{R}^n) \cap C^{1,1}_\text{loc} \) and \( v \in L^\beta_\beta(\mathbb{R}^n) \cap C^{1,1}_\text{loc} \) are a pair of positive solutions of (57) satisfying

\[
\frac{1}{|x|}\frac{x}{|x|^2} \quad \text{and} \quad \frac{1}{|x|}\frac{x}{|x|^2}
\]

for \(|x|\) sufficiently large.

Then \((u,v)\) must be radially symmetric about some point.

**Outline of Proof.** From system (57), we derive

\[
\begin{cases}
(-\Delta)^{\alpha/2} U_\lambda(x) + a_\lambda(x) V_\lambda(x) = 0 & x \in \Sigma_\lambda \\
(-\Delta)^{\beta/2} V_\lambda(x) + b_\lambda(x) U_\lambda(x) = 0 & x \in \Sigma_\lambda,
\end{cases}
\]

where

\[
a_\lambda(x) = -q \xi^{p-1}_\lambda(x) \quad \text{and} \quad b_\lambda(x) = -p \eta^{p-1}_\lambda(x)
\]

with \( \xi_\lambda(x) \) valued between \( u(x) \) and \( u_\lambda(x) \), and \( \eta_\lambda(x) \) valued between \( v(x) \) and \( v_\lambda(x) \).

**Step 1.** We want to show that, for \( \lambda \) sufficiently negative,

\[
U_\lambda(x), V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda.
\]  

(63)

Suppose (63) is violated, then there are the following 3 possibilities:

(i) both \( U_\lambda \) and \( V_\lambda \) are negative somewhere in \( \Sigma_\lambda \); or

(ii) only \( U_\lambda \) is negative somewhere in \( \Sigma_\lambda \); or

(iii) only \( V_\lambda \) is negative somewhere in \( \Sigma_\lambda \).

This shows the difference between a system and a single equation, since for the latter there is only one possibility. In order to apply Proposition 4.9 (**decay at infinity**), we first need to rule out possibilities (ii) and (iii). This can be achieved by using system (61). In fact, if \( U_\lambda \) is negative somewhere in \( \Sigma_\lambda \), then there exists \( \tilde{x} \in \Sigma_\lambda \), such that

\[
(-\Delta)^{\alpha/2} U_\lambda(\tilde{x}) < 0,
\]

then from the first equation in (61) and (62), we must have \( V_\lambda(\tilde{x}) < 0 \), hence case (ii) cannot happen. Similarly, one can rule out case (iii).

Now by Proposition 4.9, we deduce (63).

**Step 2.** In this step, we move the plane \( T_\lambda \) to the right as long as inequality (63) holds to its limiting position. Let

\[
\lambda_o = \sup \{ \lambda \mid U_\mu(x), V_\mu(x) \geq 0, x \in \Sigma_\mu, \mu \leq \lambda \}.
\]

We want to show that

\[
U_{\lambda_o}(x), V_{\lambda_o}(x) = 0, \quad x \in \Sigma_{\lambda_o}.
\]  

(64)

Otherwise, by the **strong maximum principle**, we have

\[
U_{\lambda_o}(x), V_{\lambda_o}(x) > 0, \quad x \in \Sigma_{\lambda_o}.
\]  

(65)

As in the single equation case, we show that, based on (65), the plane can be moved past \( \lambda_o \) a little bit while keeping inequality (63) valid, which would contradict the
definition of $\lambda_o$. In this process, there are also some essential differences between a system and a single equation as illustrated below.

For $\lambda > \lambda_o$ sufficiently close to $\lambda_o$, if (63) is violated, then due to the argument in Step 1, both $U_\lambda$ and $V_\lambda$ achieve their negative minima in $\Sigma_\lambda$. We will derive contradictions by showing that these minima can fall nowhere in $\Sigma_\lambda$.

Let $R$ (independent of $\lambda$) be the number in Proposition 4.9, such that the negative minima of at least one of $U_\lambda$ and $V_\lambda$ will fall into $B_R(0)$. Without loss of generality, we assume that all negative minima of $U_\lambda$ are in $B_R(0)$. By (65), for any $\delta_1 > 0$, there exist a $c_o > 0$, such that

$$U_\lambda(x), V_\lambda(x) \geq c_o, \ x \in \Sigma_{\lambda_o - \delta_1} \cap B_R(0).$$

Consequently, by the continuity of $U_\lambda$ and $V_\lambda$ with respect to $\lambda$, there is an $\delta_2 > 0$, such that

$$U_\lambda(x), V_\lambda(x) \geq 0, \ x \in \Sigma_{\lambda_o - \delta_1} \cap B_R(0), \ \forall \lambda \in [\lambda_o, \lambda_o + \delta_2]. \quad (66)$$

Choose $\delta_1$ and $\delta_2$ small, so that

$$\Omega \equiv (\Sigma_{\lambda_o + \delta_2} \setminus \Sigma_{\lambda_o - \delta_1}) \cap B_R(0)$$

is the narrow region as described in Proposition 4.8 (narrow region principle). We want to show that for all $\lambda \in [\lambda_o, \lambda_o + \delta_2]$, it holds

$$U_\lambda(x), V_\lambda(x) \geq 0, \ x \in \Sigma_\lambda. \quad (67)$$

Although we are not able to apply this principle directly, we will use the idea in the proof. Due to (66), the negative minima (if any) of $U_\lambda$ in $\Sigma_\lambda$ can only fall into the narrow region $\Omega$. If a negative minimum (if any) of $V_\lambda$ in $\Sigma_\lambda$ is also in $\Omega$, then by (55) and (56), we derive a contradiction.

Now to prove (67), what’s left to be ruled out is the possibility that there is an $\bar{x} \in \Sigma_\lambda \cap (B_R(0))^C$, such that

$$V_\lambda(\bar{x}) = \min_{\Sigma_\lambda} V_\lambda < 0.$$  

Consequently, there is an $A > 0$, such that (see [17], the proof for the decay at infinity)

$$(-\Delta)^{\beta/2} V_\lambda(\bar{x}) \leq \frac{AV_\lambda(\bar{x})}{|\bar{x}|^\beta}. \quad (68)$$

From the second equation in (61), $U_\lambda(\bar{x}) < 0$, and there exists $\tilde{x} \in \Sigma_\lambda$, such that

$$U_\lambda(\tilde{x}) = \min_{\Sigma_\lambda} U_\lambda < 0,$$

and it follows that, there is a $B > 0$, such that

$$(-\Delta)^{\alpha/2} U_\lambda(\tilde{x}) \leq \frac{BU_\lambda(\tilde{x})}{\delta^\alpha}, \quad (69)$$

with $\delta = \delta_1 + \delta_2$ being the width of the narrow region.

At negative points of $U_\lambda$ and $V_\lambda$, system (61) becomes

$$\begin{cases} (-\Delta)^{\alpha/2} U_\lambda(x) + a(x)V_\lambda(x) \geq 0 \quad x \in \Sigma_\lambda \\ (-\Delta)^{\beta/2} V_\lambda(x) + b(x)U_\lambda(x) \geq 0 \quad x \in \Sigma_\lambda, \end{cases} \quad (70)$$

where

$$a(x) = -qv^{q-1}(x) \quad \text{and} \quad b(x) = -pu^{p-1}(x) \quad (71)$$
Combining (68), (69), and (70), we carry on the following iteration
\[-a(\tilde{x})V_{\lambda}(\tilde{x}) \leq (-\triangle)^{\alpha/2}U_{\lambda}(\tilde{x}) \leq U_{\lambda}(\tilde{x}) \frac{B}{\delta^\alpha} \]
\[\leq \frac{V_{\lambda}(\tilde{x})}{b(\tilde{x})} \frac{A}{|\tilde{x}|^\beta \delta^\alpha} \]
\[\leq -\frac{V_{\lambda}(\bar{x})}{b(\bar{x})} AB \frac{|\bar{x}|^\beta}{\delta^\alpha}.\]

It follows that
\[a(\tilde{x})\delta^\alpha b(\bar{x})|\bar{x}|^\beta \geq AB.\]

This is impossible because \(a(\tilde{x})\) is bounded, \(\delta\) is sufficiently small, and
\[b(\bar{x})|\bar{x}|^\beta = o(1)\]
due to decay assumption (60) on \(u\). This verifies (67), which contradicts the definition of \(\lambda_o\). Therefore, (64) must be true, that is, \((u(x), v(x))\) is symmetric about the plane \(T_{\lambda_o}\). Since \(x_1\) direction can be chosen arbitrarily, we conclude that \((u(x), v(x))\) is radially symmetric about some point.

### 4.4. Fully nonlinear nonlocal operators and the fractional p-Laplacian.

Consider nonlinear equations involving fully nonlinear nonlocal operators

\[F_{\alpha}(u) = f(x, u)\]  
(72)

where

\[F_{\alpha}(u(x)) = C_{n,\alpha} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{G(u(x) - u(z))}{|x - z|^{n+\alpha}} dz\]
\[= C_{n,\alpha} \text{PV} \int_{\mathbb{R}^n} \frac{G(u(x) - u(z))}{|x - z|^{n+\alpha}} dz,\]  
(73)

where PV stands for the Cauchy principal value. This kind of operator was introduced by Caffarelli and Silvestre in [5].

In order the integral to make sense, we require that

\[u \in C^{1,1}_{loc} \cap L_{G,\alpha}\]

with

\[L_{G,\alpha} = \{G(u) \in L^1_{loc} \mid \int_{\mathbb{R}^n} \frac{|G(1 + u(x))|}{1 + |x|^{n+\alpha}} dx < \infty\};\]

\(G\) being at least local Lipschitz continuous, and \(G(0) = 0\).

Two special cases are

i) When \(G(\cdot)\) is an identity map, \(F_{\alpha}\) becomes the fractional Laplacian \((-\triangle)^{\alpha/2}\).

ii) When \(G(t) = |t|^{p-2}t\) and \(\alpha = ps\), \(F_{\alpha}\) become the fractional \(p\)-Laplacian defined by

\[(-\triangle)^{s}_p u(x) = C_{n,s} \text{PV} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+ps}} dy.\]

One of the significance in considering such fully nonlinear nonlocal operator is the following.
Lemma 4.1. (See [14]) Assume that $u \in C^2_{\text{loc}} \cap L_{G, \alpha}$, $G(\cdot)$ is second order differentiable, $G'$ is bounded and $G(0) = 0$. Then
\[
\lim_{\alpha \to 2} F_{\alpha}(u(x)) = a(-\triangle u)(x) + b|\nabla u(x)|^2,
\]
where $a$ and $b$ are constant multiples of $G'(0)$ and $G''(0)$ respectively.

In the case of fractional $p$-Laplacian, if $p > 2$ is fixed, then as $s \to 1$, one can show that, for each $x$,
\[
(-\triangle)^p u(x) \to -\text{div}(|\nabla u|^{p-2} \nabla u(x)) \equiv -\triangle^p u(x).
\]

In terms of the method of moving planes, the main feature of $F_{\alpha}$ depends on whether $G'(\cdot)$ is bounded away from 0, which will be illustrated by the simple example:
\[
F_{\alpha}(u(x)) = g(u(x)), \quad x \in \mathbb{R}^n.
\]

As usual, subtracting the two equations of (74) satisfied by $u_{\lambda}$ and $u$ respectively, we derive
\[
F_{\alpha}(u_{\lambda}(x)) - F_{\alpha}(u(x)) = g'(\xi_{\lambda}(x))w_{\lambda}(x).
\]

4.4.1. The case when $G'(\cdot)$ is bounded away from 0. This case is relatively easier, since at a negative minimum $x^0$ of $w_{\lambda}$ in $\Sigma_{\lambda}$, we have the following key estimate
\[
F_{\alpha}(u_{\lambda}(x^0)) - F_{\alpha}(u(x^0)) \leq \int_{\Sigma_{\lambda}} G'(\xi)w(x^0) + G''(\eta)w(x^0) \frac{1}{|x - y|^{n+\alpha}} dy \leq Cw_{\lambda}(x^0) \int_{\Sigma_{\lambda}} \frac{1}{|x - y|^{n+\alpha}} dy,
\]
with positive constant $C$ depending on the lower bound of $G'(\cdot)$.

From this key estimate, one can derive the narrow region principle and the decay at infinity.

Proposition 4.11 (Narrow Region Principle). Let $\Omega$ be a bounded narrow region in $\Sigma_{\lambda}$, such that it is contained in
\[
\{x | \lambda - \delta < x_1 < \lambda\}
\]
with small $\delta$. Suppose that $c(x)$ is bounded from below in $\Omega$ and
\[
\begin{cases}
F_{\alpha}(u_{\lambda}(x)) - F_{\alpha}(u(x)) + c(x)w_{\lambda}(x) \geq 0 & \text{in } \Omega, \\
w_{\lambda}(x) \geq 0 & \text{in } \Sigma_{\lambda}\backslash \Omega,
\end{cases}
\]
then for sufficiently small $\delta$, we have
\[
w_{\lambda}(x) \geq 0 \text{ in } \Omega.
\]

Furthermore, if $w_{\lambda} = 0$ at some point in $\Omega$, then
\[
w_{\lambda}(x) \equiv 0 \text{ in } \mathbb{R}^n.
\]

These conclusions hold for unbounded region $\Omega$ if we further assume that
\[
\lim_{|x| \to \infty} w_{\lambda}(x) \geq 0.
\]
Proposition 4.12 (Decay at Infinity). Let $\Omega$ be an unbounded region in $\Sigma$. Assume that
\[
\begin{cases}
F_\alpha(u_\lambda(x)) - F_\alpha(u(x)) + c(x)w_\lambda(x) \geq 0 & \text{in } \Omega, \\
w_\lambda(x) \geq 0 & \text{in } \Sigma \setminus \Omega,
\end{cases}
\]
with
\[
\lim_{|x| \to \infty} |x|^\gamma c(x) \geq 0,
\]
then there exists a constant $R_0 > 0$ (depending on $c(x)$, but independent of $w_\lambda$), such that if $w_\lambda(x^0) = \min_{\Omega} w_\lambda(x) < 0$, then $|x^0| \leq R_0$.

Using the narrow region principle and decay at infinity, through a standard moving planes approach, we can prove Proposition 4.13. (See [14]) Assume that $u \in C^{1,1}_{loc} \cap L^{G,\alpha}$ is a positive solution of
\[
F_\alpha(u(x)) = g(u(x)), \quad x \in \mathbb{R}^n.
\]
Suppose, for some $\gamma > 0$,
\[
u(x) = o\left(\frac{1}{|x|^\gamma}\right), \quad \text{as } |x| \to \infty,
\]
and
\[
g'(s) \leq s^q, \quad \text{with } q\gamma \geq \alpha.
\]
Then $u$ must be radially symmetric about some point in $\mathbb{R}^n$.

4.4.2. The fractional $p$-Laplacian. In the case where $G'(\cdot)$ is not bounded away from 0, we consider, in particular, the fractional $p$-Laplacian, in which $G'(0) = 0$. Now, we are not able to obtain an estimate as good as (76). We have to exploit the term that we had thrown away to arrive at
\[
(-\triangle)_p^s u_\lambda(x) - (-\triangle)_p^s u(x) \leq
\]
\[
C \int_{\Sigma_\lambda} \left[\frac{1}{|x-y|^{n+sp}} - \frac{1}{|x-y^\gamma|^{n+sp}}\right] [G(u_\lambda(x) - u_\lambda(y)) - G(u(x) - u(y))] \, dy,
\]
where $G(t) = |t|^{p-2}t$.

Based on this estimate, through a more delicate analysis, we obtain

Proposition 4.14. (A Boundary Estimate, see [13])
Assume that $w_{\lambda_0} > 0$, for $x \in \Sigma_{\lambda_0}$. Suppose $\lambda_k \searrow \lambda_0$, and $x^k \in \Sigma_{\lambda_k}$, such that $w_{\lambda_k}(x^k) = \min_{\Sigma_{\lambda_k}} w_{\lambda_k} \leq 0$ and $x^k \to x^o \in \partial \Sigma_{\lambda_o}$.

Let
\[
\delta_k = \text{dist}(x^k, \partial \Sigma_{\lambda_k}) \equiv |\lambda_k - x^k|.
\]
Then
\[
\lim_{\delta_k \to 0} \frac{1}{\delta_k} \left\{(-\triangle)_p^s u_{\lambda_k}(x^k) - (-\triangle)_p^s u(x^k)\right\} < 0.
\]
Remark 4.1. The limit in (80) may be $-\infty$.

This boundary estimate plays a similar role as the narrow region principle does in carrying out the method of moving planes and we can use it to establish the radial symmetry of positive solutions.

Consider
\[-\triangle_{p}u(x) = g(u(x)), \quad x \in \mathbb{R}^{n}.\]  

Proposition 4.15. Let
\[L_{sp} = \{u \in L_{loc}^{p-1} | \int_{\mathbb{R}^{n}} \frac{(1 + |u(x)|)^{p-1}}{1 + |x|^p} dx < \infty\}.\]

Assume that $u \in C^{1,1}_{loc} \cap L_{sp}$ is a positive solution of (81) with
\[\lim_{|x| \to \infty} u(x) = 0.\]

Suppose
\[g'(s) \leq 0 \text{ for } s > 0 \text{ sufficiently small.} \tag{82}\]

Then $u$ must be radially symmetric and monotone decreasing about some point in $\mathbb{R}^{n}$.

Outline of Proof. The condition (82) is used to guarantee that negative minima of $w_{\lambda}$ are confined in $B_{R}(0)$ with $R$ independent of $\lambda$. This completes the Step 1, that is, for $\lambda$ sufficient negative, we have
\[w_{\lambda}(x) \geq 0, \quad x \in \Sigma_{\lambda}. \tag{83}\]

In Step 2, as usual, we let
\[\lambda_{o} = \sup\{\lambda | w_{\mu}(x) \geq 0, \quad x \in \Sigma_{\mu}, \quad \mu \leq \lambda\},\]
we show that $u$ is symmetric about the limiting plane $T_{\lambda_{o}}$, or
\[w_{\lambda_{o}}(x) \equiv 0, \quad x \in \Sigma_{\lambda_{o}}. \tag{84}\]

Suppose (84) is false, then by the strong maximum principle, we have
\[w_{\lambda_{o}}(x) > 0, \quad \forall x \in \Sigma_{\lambda_{o}}.\]

On the other hand, by the definition of $\lambda_{o}$, there exists a sequence
\[\lambda_{k} \searrow \lambda_{o} \quad \text{and} \quad x^{k} \in \Sigma_{\lambda_{k}},\]
such that
\[w_{\lambda_{k}}(x^{k}) = \min_{\Sigma_{\lambda_{k}}} w_{\lambda_{k}} < 0, \quad \text{and} \quad \nabla w_{\lambda_{k}}(x^{k}) = 0. \tag{85}\]

Again the assumption (82) guarantees that $\{x^{k}\}$ is bounded, and hence there is a subsequence of $\{x^{k}\}$ that converges to some point $x^{o}$.

Now from (85), we have
\[w_{\lambda_{o}}(x^{o}) \leq 0, \quad \text{hence} \quad x^{o} \in \partial\Sigma_{\lambda_{o}}; \quad \text{and} \quad \nabla w_{\lambda_{o}}(x^{o}) = 0.\]

It follows that
\[\frac{w_{\lambda_{k}}(x^{k})}{\delta_{k}} \to 0, \quad \text{as} \quad k \to \infty.\]

This contradicts (80) in Proposition 4.14 and the equation
\[-\triangle_{p}u_{\lambda_{k}}(x^{k}) - (\triangle_{p}u(x^{k}) = g'(\xi_{\lambda_{k}}(x^{k}))w_{\lambda_{k}}(x^{k}).\]
5. Direct method of moving spheres. In [21], another direct method—the method of moving spheres on the fractional Laplacian was introduced, which is more convenient than the method of moving planes in applications in some contexts. For instance, in [9] and [10], for the case $\alpha = 2$, the authors applied the method of moving spheres to prove a non-existence result for the prescribing scalar curvature equation, and hence answered an open question posed by Kazdan [36] that

*whether the well-known Kazdan-Warner necessary condition is also sufficient for the existence of a solution in the case the given curvature function is rotationally symmetric.*

Here, we will extend this non-existence result to all real values of $\alpha$ between 0 and 2 by applying the direct method of moving spheres in the fractional setting.

Similar to the method of moving planes, the method of moving spheres is a continuous application of maximum principles, however, in a spherical pattern. Given $0 < \alpha < 2$, $\lambda > 0$ and $x_0 \in \mathbb{R}^n$, we define the Kelvin transform of a function $u$ with respect to the sphere of radius $\lambda$ centered at $x_0$ as follows,

$$
\lambda \left( \frac{x}{|x-x_0|} \right)^{n-\alpha} u \left( \frac{x}{\lambda} \right),
$$

where $x_\lambda \equiv \frac{\lambda^2 (x-x_0)}{|x-x_0|^2} + x_0$ is the inversion of $x$ with respect to the sphere $S_\lambda(x_0) \equiv \{ x \mid |x-x_0| = \lambda \}$.

For a given solution $u(x)$, we compare its value with the value of $u_\lambda(x)$. Again, let $w_\lambda(x) = u_\lambda(x) - u(x)$. Since $w_\lambda$ is not defined at $x_0$, we will consider it in

$$
\tilde{B}_\lambda(x_0) \equiv B_\lambda(x_0) \setminus \{ x_0 \}
$$

with $B_\lambda(x_0) \equiv \{ x \mid |x-x_0| < \lambda \}$.

In Step 1, we show that for certain range of values of $\lambda$, we have

either $w_\lambda(x) \geq 0$ or $w_\lambda(x) \leq 0$ for $x \in \tilde{B}_\lambda(x_0)$. (87)

This provides a starting point to move the sphere—to change the radius continuously while fixing the center.

In Step 2, we either increase or decrease the radius $\lambda$ while holding inequality (87). If one can increase such $\lambda$ to $\infty$, or decrease it to 0, then one derives some kind of monotonicity for $u$, which would usually lead to non-existence of solutions. If the supremum or infimum of such $\lambda$ is finite, one may apply some calculus techniques to classify all the solutions (see [41] and [21]).

The key ingredient in carrying out this method is the (spherically) narrow region principle:

**Proposition 5.1.** Let $\Omega \subset \tilde{B}_\lambda(x_0)$ and $w_\lambda$ be defined as above. Assume $u \in L_\alpha \cap C^{1,1}_{loc}(\Omega)$ is lower semi-continuous on $\Omega$, $c(x)$ is bounded from below in $\Omega$, and

$$
\begin{cases}
(-\Delta)^{\alpha/2} w_\lambda(x) + c(x) w_\lambda(x) \geq 0, & x \in \Omega \\
w_\lambda(x) \geq 0, & x \in \tilde{B}_\lambda(x_0) \setminus \Omega,
\end{cases}
$$

Then there exists some sufficiently small $\delta > 0$ such that if

$$
\Omega \subset \{ x \in \mathbb{R}^n | \lambda - \delta < |x-x_0| < \lambda \}
$$
(a spherically narrow region), then we have
\[ w_\lambda(x) \geq 0, \ \forall x \in \Omega. \]

Furthermore, if \( w_\lambda(x) = 0 \) for some \( x \in \Omega \), then
\[ w_\lambda(x) = 0 \ \text{for almost all} \ x \in \mathbb{R}^n. \]

To illustrate the application of the method of moving spheres, we study prescribing \( Q_\alpha \) curvature equations on Riemannian manifolds and obtain a non-existence of solutions.

**Proposition 5.2.** Let \( (\mathbb{S}^n, g_1) \) be the round sphere with standard metric of dimension \( n \geq 2 \). Assume that \( Q(x) \) is continuous and rotationally symmetric on \( \mathbb{S}^n \), monotone in the region where \( Q > 0 \), and \( Q \neq C \). Then for every \( 0 < \alpha \leq 2 \), the prescribing \( Q_\alpha \)-curvature equation
\[ P_\alpha(u) = Q(x)u^{\frac{n+\alpha}{n-\alpha}}, \ x \in \mathbb{S}^n \] 
(88)
does not admit any positive solution.

For the precise definition of \( P_\alpha \), please see [21]. In particular, it is the conformal Laplacian when \( \alpha = 2 \) and the Penitz operator for \( \alpha = 4 \).

**Remark 5.1.** In [34], Jin, Li, and Xiong obtained a necessary condition—a Kazdan-Warner type identity for (88) to have a positive solution (see Proposition A.1 there). In the case \( Q \) is rotationally symmetric, the condition becomes that, in order (88) to have a positive solution, \( Q \) must not be monotone. Our Proposition 5.2 actually provides a stronger necessary condition that
\[ Q \text{ must not be monotone in the region where it is positive.} \] 
(89)
The significant part of this stronger necessary condition is that it is almost the sufficient condition to guarantee the existence of a solution. Actually, in the case \( \alpha = 2 \), it is proved that, besides (89), if further assume that \( Q \) is non-degenerate, then problem (88) possesses a positive solution for \( n \geq 2 \) (see [52] for \( n = 2 \) and [9] for \( n \geq 3 \)). We believe that, the same existence results can be established for all real values of \( \alpha \) between 0 and 2.

By a stereo-graphic projection, one can reduce (88) into an equation in Euclidean space
\[ (-\triangle)^{\alpha/2}u(x) = Q(x) \cdot u^p(x), \ \forall x \in \mathbb{R}^n, \] 
(90)
with
\[ p \equiv \frac{n + \alpha}{n - \alpha} \text{ and } u(x) \sim \frac{1}{|x|^{n-\alpha}} \text{ near } \infty. \] 
(91)

Without loss of generality, we may assume that \( Q(x) \equiv Q(|x|) \) and
\[
\begin{aligned}
&Q \in C^\infty(\mathbb{R}^n), \\
&Q(r) > 0, \ Q'(r) \leq 0, \ \forall r < 1, \\
&Q(r) \leq 0, \ \forall r \geq 1,
\end{aligned}
\] 
(92)
where \( r \equiv |x| \).
Now, to prove the Proposition, it suffice to show the non-existence of positive \( C^{1,1} \) solution to the equation (90) with (91) and (92).

Outline of the Proof. Take the center \( x_0 = 0 \). Then
\[
x(\lambda) = \frac{\lambda^2 x}{|x|^2}, \quad u_\lambda(x) \equiv \left( \frac{\lambda}{|x|} \right)^{n-\alpha} u(x),
\]
and
\[
w_\lambda(x) \equiv u(x) - u_\lambda(x).
\]
Straightforward computations give that
\[
(-\Delta)^{\alpha/2} u_\lambda(x) = Q \left( \frac{\lambda^2}{r} \right) u^p_\lambda(x),
\]
and then
\[
(-\Delta)^{\alpha/2} w_\lambda(x) = Q(r) u^p(x) - Q \left( \frac{\lambda^2}{r} \right) u^p_\lambda(x) \quad (93)
\]
Since \( Q(r) > 0, Q(1/r) \leq 0 \) for every \( 0 < r < 1 \), it holds that for \( \lambda = 1 \),
\[
(-\Delta)^{\alpha/2} w_1(x) > -Q(1/r) u^p_1(x) \geq 0.
\]
By a simple maximum principle (see [21]), \( w_1(x) > 0 \) for every \( x \in B_1(0) \). This provides a starting point to shrink the sphere. From this point, we will decrease \( \lambda \) all the way to 0 and show that
\[
w_\lambda(x) \geq 0, \quad x \in \tilde{B}_\lambda(0), \quad \text{for all } 0 < \lambda \leq 1. \quad (94)
\]
Define
\[
\lambda_0 \equiv \inf \{ \lambda > 0 \mid w_\mu(x) \geq 0, \quad \forall x \in \tilde{B}_\mu(0), \quad \forall \mu \leq 1 \}.
\]
We will apply the narrow region principle (Proposition 5.1) to show that
\[
\lambda_0 = 0.
\]
Given any \( 0 < \lambda \leq 1 \), equation (93) implies that
\[
(-\Delta)^{\alpha/2} w_\lambda(x) = Q(r) u^p(x) - Q \left( \frac{\lambda^2}{r} \right) u^p_\lambda(x)
\]
where \( \psi_\lambda(x) \) is valued between \( u(x) \) and \( u_\lambda(x) \).

The decay condition (91) implies that \( \psi_\lambda(x) \) is uniformly bounded, and hence,
\[
(-\Delta)^{\alpha/2} w_\lambda(x) + c_\lambda(x) w_\lambda(x) > 0, \quad (95)
\]
with \( c_\lambda(x) \) uniformly bounded.

By the definition of \( \lambda_0 \),
\[
w_{\lambda_0}(x) \geq 0, \quad \forall x \in \tilde{B}_{\lambda_0}(0).
\]
Furthermore, applying the strong maximum principle (see [21]), we have
\[
w_{\lambda_0}(x) > 0, \quad \forall x \in \tilde{B}_{\lambda_0}(0). \quad (96)
\]
Let \( \delta > 0 \) be the width of the narrow region as in Proposition 5.1, then by (96), there is a constant \( c_\delta > 0 \), such that
\[
w_{\lambda_0}(x) \geq c_\delta, \quad \forall x \in \tilde{B}_{\lambda_0-\delta}(0).
By the continuity of \( w_\lambda \) with respect to \( \lambda \), there exists \( 0 < \epsilon \leq \delta \), such that, for all \( \lambda_0 - \epsilon < \lambda < \lambda_0 \)
\[
w_\lambda(x) \geq 0 \quad \forall \ x \in \hat{B}_{\lambda_0 - \delta}(0).
\]
(97)

For such \( \lambda \), let \( \Omega = \hat{B}_\lambda(0) \setminus \hat{B}_{\lambda_0 - \delta}(0) \). Then it is a spherically narrow region with width less than \( \delta \). By Proposition 5.1, we have
\[
w_\lambda(x) \geq 0 \quad \forall \ x \in \Omega.
\]

This, together with (97), implies
\[
w_\lambda(x) \geq 0 \quad \forall \ x \in \hat{B}_\lambda(0).
\]
This contradicts the definition of \( \lambda_0 \), thus we must have \( \lambda_0 = 0 \), and hence arrive at (94), from which there exists a sequence \( \lambda_j \to 0 \) such that
\[
\frac{1}{\lambda_j^{n-\alpha}} \left( \frac{\lambda_j^2}{|x|} \right)^{n-\alpha} u \left( \frac{\lambda_j^2 \cdot x}{|x|^2} \right) < u(x), \ \forall \ x \in \hat{B}_{\lambda_j}(0).
\]

By the decay condition (91),
\[
u(0) > \frac{M_0}{\lambda_j^{n-\alpha}},
\]
and hence,
\[
u(0) > \lim_{j \to \infty} \frac{M_0}{\lambda_j^{n-\alpha}} = +\infty.
\]
A contradiction.

6. Integral equations approaches. Another powerful tool in studying qualitative properties of solutions for fractional equations is the method of moving planes in integral forms. Since its introduction in [18], this method has been widely applied by numerous researchers to solve a series of problems arising from nonlinear PDEs, in particular, from fractional equations to higher order equations (see [19] [20] [25] [27] [31] [30] [38] [39] [44] [45] [46] [47] [48] [49] [54] and the references therein). One major advantage of this method is that it works indiscriminately for equations of any order less than the dimension \( n \).

In order to apply this method, one first needs to show the equivalence between a PDE or a pseudo differential equation and an integral equation. The Liouville theorems introduced in Section 3 play a crucial role in this process.

6.1. Equivalence. We use two examples, a single equation and a system in \( \mathbb{R}^n \), to illustrate how to show the equivalence between these pseudo differential equations and the integral equations.

**Proposition 6.1.** Assume \( u \in L_\alpha \) is lower semi-continuous. Suppose \( u \) is a locally bounded positive solution of
\[
\begin{cases}
(\Delta)^{\alpha/2} u(x) = u^p(x), & x \in \mathbb{R}^n, \\
u(x) > 0, & x \in \mathbb{R}^n,
\end{cases}
\]
in the sense of distribution.

Then it is also a solution of
\[
u(x) = \int_{\mathbb{R}^n} \frac{C_{n,\alpha}}{|x - y|^{n-\alpha}} u^p(y) dy.
\]
The converse is also true.

Outline of Proof. Assume that $u$ is a positive solution of (98). We first show that
\[ \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-\alpha}} dy < \infty. \]

Let
\[ v_R(x) = \int_{B_R(0)} G_R(x,y) u^p(y) dy, \]
where $G_R(x,y)$ is the Green's function associated to $(-\Delta)^{\alpha/2}$ with the Dirichlet condition. Our assumption $u \in L^\infty_{loc}(\mathbb{R}^n)$ guarantees that for each given $R > 0$, $v_R(x)$ is well-defined and continuous. It's easy to see that
\[ v_R(x) = 0, \quad x \notin B_R(0). \]

Let
\[ w_R(x) = u(x) - v_R(x). \]
Then
\[ w_R(x) \geq 0, \quad x \notin B_R(0). \]

By the maximum principle, we have
\[ w_R(x) \geq 0, \quad x \in B_R(0). \]

As $R \to \infty$
\[ v_R(x) \to v(x) := \int_{\mathbb{R}^n} \frac{C_{n,\alpha}}{|x-y|^{n-\alpha}} u^p(y) dy, \]
and
\[ w_R(x) \to w(x) := u(x) - v(x). \]

It's easy to see that
\[ w(x) \equiv c \geq 0, \quad x \in \mathbb{R}^n. \]

By the Liouville’s theorem, we deduce that
\[ u(x) = w(x) + v(x) \geq c. \]

If $c > 0$, then
\[ u(x) \geq v(x) = \int_{\mathbb{R}^n} \frac{C_{n,\alpha}}{|x-y|^{n-\alpha}} u^p(y) dy \geq \int_{\mathbb{R}^n} \frac{C_{n,\alpha}}{|x-y|^{n-\alpha}} c^p dy = \infty. \]
This is a contradiction, hence $c = 0$ and
\[ u(x) = v(x) = \int_{\mathbb{R}^n} \frac{C_{n,\alpha}}{|x-y|^{n-\alpha}} u^p(y) dy. \]

Now the situation is much more complicated in the case of systems. Take the following simple system in $\mathbb{R}^n$ for example:
\[ \begin{cases} (-\Delta)^{\alpha/2} u = u^{p_1} v^{q_1} \\ (-\Delta)^{\alpha/2} v = u^{p_2} v^{q_2}. \end{cases} \]
Using the above maximum principle and Liouville theorem arguments, we arrive quickly at
\[
\begin{align*}
  u(x) &= c_1 + \int_{\mathbb{R}^n} \frac{C_{n, \alpha}}{|x-y|^{n-\alpha}} u^{p_1}(y)v^{q_1}(y)dy \\
  v(x) &= c_2 + \int_{\mathbb{R}^n} \frac{C_{n, \alpha}}{|x-y|^{n-\alpha}} u^{p_2}(y)v^{q_2}(y)dy,
\end{align*}
\]
with \( c_1, c_2 \geq 0 \).

If both \( c_1 \) and \( c_2 \) are positive, then we derive a contradiction easily as we did for the above single equation. However, there are two other possibilities: \( c_1 > 0 \) and \( c_2 = 0 \), or \( c_1 = 0 \) and \( c_2 > 0 \).

In these two cases, the above simple argument for single equation (98) does not give us any contradiction. It is even more complicated for a system of \( m \) equations. To circumvent this complexity, we can apply the method of moving planes in integral forms to each equation to show that \( c_i \) must be zero for each \( i = 1, 2 \). For more details, please see [54] or [17].

6.2. The method of moving planes in integral forms. To illustrate the idea, let’s consider the simple example
\[
u(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} u^p(y)dy, \quad x \in \mathbb{R}^n.
\]
(99)

As in the previous sections, let
\[
T_\lambda = \{ x \in \mathbb{R}^n \mid x_1 = \lambda, \text{ for some } \lambda \in \mathbb{R} \},
\]
\[
\Sigma_\lambda = \{ x \in \mathbb{R}^n \mid x_1 < \lambda \},
\]
\[
x_\lambda = (2\lambda - x_1, x_2, ..., x_n), \quad \text{and}
\]
\[
w_\lambda(x) = u(x_\lambda) - u(x) \equiv u_\lambda(x) - u(x).
\]

First we derive
\[
u(x) - u_\lambda(x) = \int_{\Sigma_\lambda} \left( \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_\lambda-y|^{n-\alpha}} \right) (u^p(y) - u_\lambda^p(y))dy.
\]
(100)

In order to show that
\[
w_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda,
\]
(101)

instead of using maximum principles (decay at infinity and narrow region principle), we show the emptiness of the set
\[
\Sigma_\lambda^- = \{ x \in \Sigma_\lambda \mid w_\lambda(x) < 0 \},
\]
where inequality (101) is violated.

Based on (100) and using Hardy-Littlewood-Sobolev inequality, we obtain the key estimate:
\[
\| w_\lambda \|_{L^q(\Sigma_\lambda^-)} \leq C \| u^{p-1} \|_{L^{\frac{n}{p}}(\Sigma_\lambda^+)} \| w_\lambda \|_{L^q(\Sigma_\lambda^-)}
\]
(102)

for some \( q > 1 \). It plays a crucial role in both steps.

In Step 1, to derive (101) for \( \lambda \) sufficiently negative, we assume that
\[
\| u \|_{L^{(p-1)n/(n-p)}(\mathbb{R}^n)} < \infty.
\]
(103)

Then for \( \lambda \) sufficiently negative, \( \Sigma_\lambda^- \) is contained in a neighborhood of infinity, and thus \( \| u^{p-1} \|_{L^{\frac{n}{p}}(\Sigma_\lambda^+)} \) becomes very small. It follows from (102) that \( \| w_\lambda \|_{L^q(\Sigma_\lambda^-)} = 0, \)
which implies the emptiness of the set $\Sigma_{\lambda}$. Condition (103) is kind of strong, and one can weaken it considerably by applying a Kelvin transform
\[ v(x) = \frac{1}{|x - x^0|^{n-\alpha}} \left( \frac{x - x^0}{|x - x^0|^2} + x^0 \right) \]
centered at any point $x^0 \in \mathbb{R}^n$. It can be verified that $v$ satisfies a similar equation
\[ v(x) = \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-\alpha}|y - x^0|^\gamma} v^p(y) \, dy, \quad x \in \mathbb{R}^n \setminus \{x^0\}, \]
where $\gamma = (n + \alpha) - p(n - \alpha)$. Now $v(x)$ satisfies the desired integrability condition, i.e. its $L^{(p-1)n/\alpha}$ norm is sufficiently small in the neighborhood of infinity, hence we can carry on the method of moving planes on $v(x)$.

In Step 2, as usual, we define
\[ \lambda_o = \sup \{ \lambda \mid w_{\mu}(x) \geq 0, \ x \in \Sigma_{\mu}, \mu \leq \lambda \}. \]
We show that if $\lambda_o$ is finite, then $w_{\lambda_o}(x) \equiv 0, \ x \in \Sigma_{\lambda_o}$.

Otherwise, by (100), we derive that $w_{\lambda_o}(x) > 0, \ x \in \Sigma_{\lambda_o}$. This will enable us, based on (102) and (103), to move the plane a little bit forward to the right while keeping inequality (101) valid, which would contradict the definition of $\lambda_o$. For more details, please see [18] or [17].

One can also apply this method to systems of integral equations (see [20] [54] [17]).

One advantage of the method of moving planes in integral forms is that it works indiscriminately for all real values $\alpha$ between 0 and $n$. Hence it can be conveniently employed to study higher ($\alpha > 2$) order PDEs or pseudo PDEs, in which case, the maximum principles, such as narrow region principles, are no longer valid.

7. Blowing up and re-scaling, a priori estimates, and existence. It is well-known that the a priori estimates play important roles in establishing the existence of solutions. Once there is such an a priori estimate, one can use various approaches, such as continuation methods or topological degrees arguments, to derive the existence of solutions.

Usually, for a nonlocal fractional equation, to obtain a priori estimates for solutions, one would first use the Caffarelli and Silvestre’s extension method to reduced it into a local second order elliptic PDE, then apply the blowing up and re-scaling argument on the extended equation to derive an a priori estimate on the solutions. In this process, many standard elliptic theory can be employed.

In [16] and [1], a direct blowing up and re-scaling method was introduced to obtain the the a priori estimates for solutions of fractional equations.

7.1. A priori estimates. To illustrate the idea, we consider the simple example
\[
\begin{aligned}
&\left\{\begin{array}{ll}
(\Delta)^{\alpha/2} u(x) = u^p(x), & x \in \Omega, \\
u(x) \equiv 0, & x \in \Omega^C.
\end{array}\right.
\end{aligned}
\]  

Proposition 7.1. For $0 < \alpha < 2$, and $1 < p < \frac{n+\alpha}{n-\alpha}$, suppose
\[ u \in L_{\alpha} \cap C^{1,1}_{loc}(\Omega) \] is upper semi-continuous on $\Omega$,
and is a positive solution of (104). Then
\[ \|u\|_{L^\infty(\Omega)} \leq C, \] (105)
for some positive constant \(C\) independent of \(u\).

This method can be applied to obtain a priori estimates for positive solutions for uniformly elliptic nonlocal operators with more general nonlinearities:
\[ \begin{cases} (-\Delta)^{\alpha/2}u(x) = f(x,u), & x \in \Omega, \\ u(x) \equiv 0, & x \in \Omega^C, \end{cases} \] (106)
where
\[ (-\Delta)^{\alpha/2}u(x) = \frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x + y) - u(x - y)}{|y|^{n+\alpha}} a(y) dy, \] (107)
and
\[ 0 < c_0 \leq a(x) \leq C_0, \quad x \in \mathbb{R}^n. \]

Outline of Proof of Proposition 7.1. Suppose (105) does not hold, then there exists a sequence of solutions \(\{u_k\}\) to (104) and a sequence of points \(\{x_k\} \subset \Omega\) such that
\[ u_k(x_k) = \max_{\Omega} u_k := m_k \to \infty. \]

Let
\[ \lambda_k = m_k^{1-p} \quad \text{and} \quad v_k(x) = \frac{1}{m_k} u_k(\lambda_k x + x_k), \] (108)
then we have
\[ (-\Delta)^{\alpha/2} v_k(x) = v_k^p(x), \quad x \in \Omega_k := \{x \in \mathbb{R}^n \mid x = \frac{y - x_k}{\lambda_k}, \ y \in \Omega\}. \] (109)

Let \(d_k = \text{dist}(x_k, \partial \Omega)\). We will carry out the proof using the contradiction argument to exhaust all three possibilities.

Case (i). \(\lim_{k \to \infty} \frac{d_k}{\lambda_k} = \infty\) (\(x_k\) does not approaches the boundary or approach the boundary “slowly”).

In this case,
\[ \Omega_k \to \mathbb{R}^n \quad \text{as} \ k \to \infty, \]
and we show that there exists a function \(v\) such that as \(k \to \infty\),
\[ u_k(x) \to v(x) \quad \text{and} \quad (-\Delta)^{\alpha/2} u_k(x) \to (-\Delta)^{\alpha/2} v(x), \] (110)
thus
\[ (-\Delta)^{\alpha/2} v(x) = v^p(x), \quad x \in \mathbb{R}^n. \] (111)
This will contradict the well-known Liouville theorem.

The key is to prove the second part in (110). To this end, the main difference between the fractional order equation and the integer order equation
\[ -\Delta u = u^p \]
is that for the latter, one uses standard elliptic estimates involving Sobolev embeddings, while for the former, we estimate the singular integral defining the fractional Laplacian:
\[ \begin{align*}
(-\Delta)^{\alpha/2} v_k(x) &= \frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n} \frac{2v_k(x) - v_k(x + y) - v_k(x - y)}{|y|^{n+\alpha}} dy.
\end{align*} \]
Using the dominant function for the sequence of integrands:
\[
\left| \frac{2v_k(x) - v_k(x + y) - v_k(x - y)}{|y|^{n+\alpha}} \right| \leq \frac{C}{1 + |y|^{n+\alpha}} \left( 1 + \frac{1}{|y|^{n+\alpha-2}} \right),
\]
we exchange the limit with the integral and arrive at
\[
(-\triangle)^{\alpha/2}v_k(x) \to (-\triangle)^{\alpha/2}v(x).
\] (112)

**Case (ii).** \( \lim_{k \to \infty} \frac{d_k}{\lambda_k} = C > 0 \) (\( x^k \) approaches the boundary “quickly”).

In this case,
\[
\Omega_k \to \mathbb{R}_+^n := \{ x_n \geq -C \mid x \in \mathbb{R}^n \} \text{ as } k \to \infty.
\]
Similar to Case (i), here we’re able to establish the existence of a function \( v \) and a subsequence of \( \{v_k\} \), such that as \( k \to \infty \),
\[
v_k(x) \to v(x) \text{ and } (-\triangle)^{\alpha/2}v_k(x) \to (-\triangle)^{\alpha/2}v(x),
\] (113)
and thus
\[
(-\triangle)^{\alpha/2}v(x) = v^p(x), \quad x \in \mathbb{R}_+^n C.
\] (114)

This contradicts the non-existence result obtained in [8].

**Case (iii).** \( \lim_{k \to \infty} \frac{d_k}{\lambda_k} = 0 \) (\( x^k \) approaches the boundary “very quickly”).

In this case, there exists a point \( x^o \in \partial \Omega \) and a subsequence of \( \{x^k\} \), still denoted by \( \{x^k\} \), such that
\[
x^k \to x^o, \quad k \to \infty.
\]
Let \( p^k = \frac{x^o - x^k}{\lambda_k} \). Obviously,
\[
v_k(p^k) = 0, \text{ and } v_k(0) - v_k(p^k) = 1.
\] (115)

On the other hand,
\[
p^k \to 0, \quad k \to \infty.
\]
This will contradict the uniform Hölder continuity of \( \{v_k\} \) near \( p^k \):
\[
|v_k(x) - v_k(p^k)| \leq C|x - p^k|^{\alpha/2},
\] (116)
and thus rules out the possibility of Case (iii).

To prove (116), we construct a super solution \( \phi(x) \) in a neighborhood \( D \) of \( p^k \). For an integer order equation, one require
\[
\phi(x) \geq v_k(x)
\]
only on the boundary of \( D \); however, for our fractional equation, we need this inequality to hold in the whole complement of \( D \). Which obviously is a much more difficult task. For more details, please see [16] or [17].

7.2. **Existence of solutions.** To prove the existence of positive solutions, we use the fixed point theorem on a closed positive cone \( P \).

**Proposition 7.2.** Suppose that \( (X, P) \) is an ordered Banach space, \( U \subset P \) is bounded and open and contains 0. Assume that there exists \( \rho > 0 \) such that \( B_{\rho}(0) \cap P \subset U \) and that \( K : U \to P \) is compact and satisfies:

1. For any \( x \in P \) with \( |x| = \rho \), and \( \lambda \in [0, 1) \), \( x \neq \lambda Kx \);
2. There exists some \( y \in P \setminus \{0\} \), such that \( x - Kx \neq ty \) for any \( t \geq 0 \) and \( x \in \partial U \).

Then \( K \) possesses a fixed point on the closure of \( U \setminus B_{\rho}(0) \).
Combining this fixed point theorem and the a priori estimate (Proposition 7.1), we establish the existence of positive solutions.

**Proposition 7.3.** Assume \( 1 < p < \frac{n + \alpha}{n - \alpha} \), then problem (104) possesses a positive solution.

8. Regularity of solutions. Let \( \Omega \) be a bounded domain with smooth boundary in \( \mathbb{R}^n \). Consider

\[
(-\Delta)^su(x) = f(x), \quad x \in \Omega. \tag{117}
\]

For this fractional equation, the *Schauder interior estimate* is similar to that for the corresponding Poisson equation when \( s = 1 \). It states roughly that if \( f \in C^\gamma(D) \) in some open set \( D \subset \Omega \), then in any proper subset of \( D \), the regularity of the solution \( u \) can be lifted by the order of \( 2s \), the same order as the operator \((-\Delta)^s\).

By introducing proper weighted Hölder norms as in the case of Poisson equations, we will be able to control a weighted \( C^{2s+\gamma} \) norm of \( u \) in \( \Omega \) in terms of another weighted \( C^\gamma \) norm of \( f \) in \( \Omega \).

However, when considering regularity up to the boundary, the situation in the fractional order equation is quite different from that in the integer order equation (when \( s = 1 \), the Poisson equation). For integer order equations, if \( f \) is Hölder continuous, and if the boundary data is smooth enough, then the regularity up to the boundary of the solution \( u \) can still be raised by order 2. This is no longer the case for the fractional equation as illustrated in Section 2.3. In most cases, no matter how smooth \( f(x) \) is, the solution of (117) can only go up to \( s \)-Holder continuous as indicated by the example

\[
\phi(x) = \begin{cases} 
(1 - |x|^2)^s, & |x| < 1 \\
0, & |x| \geq 1
\end{cases}
\]

and by Theorem 2.2.

8.1. Interior regularity. To state the *interior regularity* more precisely, we introduce the weighted Hölder norms. Let

\[
d_x = \text{dist}(x, \partial \Omega), \quad d_{x,y} = \min\{d_x, d_y\}.
\]

Write \( 2s + \gamma = k + \beta \), where \( k = 0, 1, 2 \) and \( \beta \in (0, 1) \). Recall that the usual \( C^k \) norm and \( C^{k+\beta} \) semi-norm are defined by

\[
\|u\|_{C^k(\Omega)} = \sum_{|\gamma| \leq k} \sup_{x \in \Omega} |D^\gamma u(x)|
\]

and

\[
[u]_{C^{k+\beta}(\Omega)} = \sum_{|\gamma| = k} \sup_{x \in \Omega} \frac{|D^\gamma u(x) - D^\gamma u(y)|}{|x - y|^{\beta}},
\]

while \( C^{k+\beta} \) norm is the sum of the two:

\[
\|u\|_{C^{k+\beta}(\Omega)} = \|u\|_{C^k(\Omega)} + [u]_{C^{k+\beta}(\Omega)}.
\]

Different from the above norms, the following weighted norms contain scaling factors \( d_x \) and \( d_{x,y} \).

Let

\[
\|u\|^*_{C^k(\Omega)} = \sum_{|\gamma| \leq k} \sup_{x \in \Omega} d_x^{\gamma} |D^\gamma u(x)|
\]
and
\[ [u]_{C^{\alpha + \beta}(\Omega)}^{*} = \sum_{|\gamma| = k} \sup_{x \in \Omega} \frac{d^{\alpha + \beta} D^\gamma u(x) - D^\gamma u(y)}{|x-y|^\beta}, \]

Define
\[ \|u\|_{C^{\alpha + \beta}(\Omega)}^{*} = \|u\|_{C^{\alpha}(\Omega)}^{*} + [u]_{C^{\alpha + \beta}(\Omega)}^{*}. \]

When \( s = 1 \), equation (117) becomes
\[-\Delta u(x) = f(x) \quad x \in \Omega,\]
and the well-known Schauder estimate states that
\[ \|u\|_{C^{2+\gamma}(\Omega)}^{*} \leq C(n, \gamma)(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^{\gamma}(\Omega)}^{(2s)}). \]

Notice that the difference between this and the classical Schauder estimate is that here we have the term \( \|u\|_{L^\infty(\mathbb{R}^n)} \) instead of \( \|u\|_{L^\infty(\Omega)} \), because of the nonlocal nature of the fractional Laplacian:

For any point \( x \) in \( \Omega \), the value of \( (-\Delta)^s u(x) \) depends not only on the values of \( u \) in \( \Omega \), but also on its values in \( \mathbb{R}^n \setminus \Omega \).

\textbf{Outline of the Proof.} (i) First we start with ball regions in \( \mathbb{R}^n \) by extending \( f \) to be 0 outside of \( B_{3}(0) \). Estimate the potential
\[ w(x) = \int_{\mathbb{R}^n} \frac{C_{n,s}}{|x-y|^{n-2s}} f(y) dy, \]
since \( w(x) \) is one of the solutions of fractional equation (117). One derives the interior estimate
\[ \|w\|_{C^{2+\gamma}(B_{1}(0))} \leq C \|f\|_{C^{\gamma}(B_{2}(0))}. \]

(ii) Then the difference between \( w \) and \( u \) is an \( s \)-harmonic function \( h \), which can be expressed in terms of a Poisson kernel:
\[ h(x) = \int_{|y| < 2} P_{2}(y, x) h(y) dy, \quad \forall |x| < 2. \]

Using this integral, one obtain:
\[ \|h\|_{C^{2+\gamma}(B_{1}(0))} \leq C(\|f\|_{C^{\gamma}(B_{2}(0))} + \|u\|_{L^\infty(\mathbb{R}^n)}). \]

It follows from (119) and (120) that
\[ \|u\|_{C^{2+\gamma}(B_{1}(0))} \leq C(\|f\|_{C^{\gamma}(B_{2}(0))} + \|u\|_{L^\infty(\mathbb{R}^n)}). \]
(iii). For any point \( x^o \in \Omega \), let \( R = \frac{d_{x^o}}{3} \), then \( B_{3R}(x^o) \subset \Omega \). Through a rescaling on (121), we deduce
\[
\sum_{i=0}^{k} R^i \sup_{x \in B_R} |D^i u(x)| + R^{2s+\gamma} |D^k u|_{C^\gamma(B_R)} 
\leq C(\|u\|_{L^\infty(B_{3R})} + R^{2s} \sup_{x \in B_{3R}} |f(x)| + R^{2s+\gamma} |f|_{C^\gamma(B_{3R}(0))}). \tag{122}
\]

Based on (122), we finally arrive at (118).

In the above, for simplicity we abuse notation by denoting all the \( k \)-th order partial derivatives of \( u \) by \( D^k u \).

8.2. Holder continuity up to the boundary. From Theorem 2.2, one can see that, in many cases, a solution is only \( s \)-Holder continuous up to the boundary. If the right hand side of (117) is bounded, this is exactly the situation as proved in [50].

**Proposition 8.1.** ([50]) Let \( \Omega \) be a bounded Lipschitz domain satisfying the interior ball condition, \( f \in L^\infty(\Omega) \), and \( u \) be a solution of (117). Then, \( u \in C^s(\mathbb{R}^n) \) and
\[
\|u\|_{C^s(\mathbb{R}^n)} \leq C \|f\|_{L^\infty(\Omega)},
\]
where \( C \) is a constant depending on \( \Omega \) and \( s \).

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