GENERALIZED SCHUR FUNCTION DETERMINANTS USING THE BAZIN IDENTITY

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Abstract. In the literature there are several determinant formulas for Schur functions: the Jacobi–Trudi formula, the dual Jacobi–Trudi formula, the Giambelli formula, the Lascoux–Pragacz formula, and the Hamel–Goulden formula, where the Hamel–Goulden formula implies the others. In this paper we use an identity proved by Bazin in 1851 to derive determinant identities involving Macdonald’s 9th variation of Schur functions. As an application we prove a determinant identity for factorial Schur functions conjectured by Morales, Pak, and Panova. We also obtain a generalization of the Hamel–Goulden formula, which contains a result of Jin, and prove a converse of the Hamel–Goulden theorem and its generalization.

1. Introduction

The Schur functions $s_\lambda$ are an important family of symmetric functions. They form a linear basis of the space of symmetric functions and have connections to different areas of mathematics including combinatorics and representation theory. Schur functions have been extensively studied and there are numerous generalizations and variations of them in the literature. In particular, Macdonald \[10\] introduced nine variations of Schur functions. Macdonald’s 9th variation of Schur functions generalize Schur functions and many of their variations. The main focus of this paper is to find determinant identities for Macdonald’s 9th variation of Schur functions.

We briefly review several known determinant formulas for Schur functions. The classical Jacobi–Trudi formula and its dual formula express a Schur function $s_\lambda$ as a determinant in terms of complete homogeneous symmetric functions $h_k$ and elementary symmetric functions $e_k$:

$$s_\lambda = \det (h_{\lambda_i+j-1})_{i,j=1}^{\ell(\lambda)}, \quad s_\lambda = \det (e_{\lambda'_i+j-1})_{i,j=1}^{\ell(\lambda')},$$

where $\ell(\lambda)$ is the number of parts in the partition $\lambda$ and $\lambda'$ is the transpose of $\lambda$. Observe that $h_k$ and $e_k$ are also Schur functions whose shapes are partitions with one row and one column, respectively. The Giambelli formula \[6\] and the Lascoux–Pragacz formula \[9\] express a Schur function as a determinant of Schur functions whose shapes are hooks and border strips, respectively. Using so-called outside decompositions, Hamel and Goulden \[7\] found a determinant formula for a Schur function, which generalizes all of the aforementioned formulas. Chen, Yan, and Yang \[8\] restated the Hamel–Goulden formula using certain border strips called cutting strips, and showed that all these formulas are equivalent up to simple matrix operations. Recently, Jin \[8\] generalized the Hamel–Goulden formula by considering more general border strips called thickened border strips.

In this paper, inspired by the original proof of the Lascoux–Pragacz formula in \[9\, Section 2\], we find determinant identities for Macdonald’s 9th variation of Schur functions using the Bazin identity \[2\]. Our results restricted to Schur functions are also new and include the Hamel–Goulden formula \[7\] and its generalization due to Jin \[8\]. As an application we prove a determinant identity (after some correction) for factorial Schur functions conjectured by Morales, Pak, and Panova \[12\]. Another special case of our results is the Hamel–Goulden formula for Macdonald’s 9th variation of Schur functions. We note that recently Bachmann and Charlton \[1\], and Foley and King \[5\] also proved this using the Lindström–Gessel–Viennot lemma.

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To illustrate, we compare one of our results with the Lascoux–Pragacz formula [9].

**Theorem 1.1 (Lascoux–Pragacz formula).** Suppose that \( \mu \subseteq \lambda \) and \( \theta = (\theta_1, \ldots, \theta_k) \) is the Lascoux–Pragacz decomposition of \( \lambda/\mu \). Then

\[
s_{\lambda/\mu} = \det \left( s_{\lambda^0[p_j,q_i]} \right)_{i,j=1}^k,
\]

where \( p_i \)'s and \( q_i \)'s are the contents of the starting and ending cells of \( \theta_i \), respectively.

In the theorem above, the shape \( \lambda^0[p_j,q_i] \) in each entry is a border strip. See Sections 2 and 4 for the undefined terms. The following theorem will be proved in Section 3.

**Theorem 1.2.** Let \( \lambda, \mu, \) and \( \nu \) be partitions. Suppose that \( \mu \subseteq \lambda \) and \( \theta = (\theta_1, \ldots, \theta_k) \) is the Lascoux–Pragacz decomposition of \( \lambda/\mu \). Then we have

\[
S_{\lambda/\nu}^{k-1} S_{\mu/\nu} = \det \left( (-1)^{p_j > q_i} s_{\lambda(q_i,p_j-1)/\nu} \right)_{i,j=1}^k,
\]

\[
S_{\nu/\lambda}^{k-1} S_{\nu/\mu} = \det \left( (-1)^{p_j > q_i} s_{\nu/p_j-1}/\lambda(\theta_i-1) \right)_{i,j=1}^k.
\]

Here \( S_{\lambda/\mu} \) is Macdonald’s 9th variation of Schur functions and \( \lambda(q_i, p_j-1) \) is a partition obtained from \( \lambda \) by either adding or deleting a border strip. See Section 2 for their precise definitions. If \( \nu = \lambda \) in (1.2) we obtain the Lascoux–Pragacz formula for \( S_{\lambda/\mu} \), thus Theorem 1.1 follows. If \( \nu = \emptyset \) and Macdonald’s 9th variation is specialized to the factorial Schur functions in (1.1), then we obtain a corrected version of the conjecture of Morales, Pak, and Panova [12].

Now we recall the reformulation of the Hamel–Goulden formula [7] due to Chen, Yan, and Yang [3].

**Theorem 1.3 (Hamel–Goulden formula).** Let \( \lambda \) and \( \mu \) be partitions and let \( \gamma \) be a border strip. Suppose that \( \mu \subseteq \lambda \) and \( \theta = (\theta_1, \ldots, \theta_k) \) is the decomposition of \( \lambda/\mu \) determined by the cutting strip \( \gamma \). Then we have

\[
s_{\lambda/\mu} = \det \left( s_{\gamma[p_j,q_i]} \right)_{i,j=1}^k,
\]

where \( p_i \) and \( q_i \) are the contents of the starting and ending cells of \( \theta_i \), respectively.

In Theorem 1.3, \( \gamma[p_j,q_i] \) is the set of cells in \( \gamma \) whose contents are in the closed interval \([p_j,q_i] \). See Sections 2 and 4 for the undefined terms. Observe that in Theorem 1.3 the \( p_i \)'s and \( q_i \)'s are particular integers determined by \( \lambda, \mu, \) and \( \gamma \). A natural question is whether the determinant in this theorem can represent a skew Schur function for arbitrary \( p_i \)'s and \( q_i \)'s. We show that this is indeed true up to sign. This may be considered as a converse of the Hamel–Goulden theorem. More generally, we can take \( \gamma \) to be any connected skew shape, not necessarily a border strip.

**Theorem 1.4.** Let \( \alpha \) be any connected skew shape. Suppose that \((a_1, \ldots, a_k)\) and \((b_1, \ldots, b_k)\) are sequences of integers such that \( \alpha[a_j,b_i] \) is a skew shape for all \( i \) and \( j \). Then either \( \det \left( s_{\alpha[a_j,b_i]} \right)_{i,j=1}^k = 0 \) or there exists a skew shape \( \rho \) such that

\[
\det \left( s_{\alpha[a_j,b_i]} \right)_{i,j=1}^k = \pm s_{\rho}.
\]

The rest of this paper is organized as follows. In Section 2 we give basic definitions. In Section 3 we derive a determinant identity involving \( S_{\lambda/\mu} \) using the Bazin identity. In Section 4 we restate the result in the previous section using border strip decompositions. As an application we prove a conjecture of Morales, Pak, and Panova [12]. In Section 5 we prove a generalization of the Hamel–Goulden formula. In Section 6 we prove Theorem 1.4.

2. Definitions

In this section we give basic definitions which will be used throughout this paper.

A **partition** is a weakly decreasing sequence \( \lambda = (\lambda_1, \ldots, \lambda_k) \) of positive integers. Each \( \lambda_i > 0 \) is called a part of \( \lambda \). The **length** \( \ell(\lambda) \) of \( \lambda \) is the number of parts in \( \lambda \). For integers \( r > \ell(\lambda) \), we use the convention \( \lambda_r = 0 \). The set of partitions with at most \( n \) parts is denoted by \( \text{Par}_n \). By appending zeros at the end if necessary we will write each element \( \lambda \in \text{Par}_n \) as \( \lambda = (\lambda_1, \ldots, \lambda_n) \).
A pair \((i, j)\) of integers is called a cell. The Young diagram of a partition \(\lambda\) is the set of cells \((i, j)\) with \(1 \leq i \leq \ell(\lambda)\) and \(1 \leq j \leq \lambda_i\). We will often identify a partition \(\lambda\) with its Young diagram. The Young diagram \(\lambda\) is visualized as an array of squares so that there is a square in row \(i\) and column \(j\) for each \((i, j) \in \lambda\), according to the matrix coordinates. See Figure 1.

For two partitions \(\lambda\) and \(\mu\), we write \(\mu \subseteq \lambda\) to mean that the Young diagram of \(\mu\) is contained in that of \(\lambda\). A skew shape, denoted by \(\lambda/\mu\), is a pair \((\lambda, \mu)\) of partitions satisfying \(\mu \subseteq \lambda\). We also consider the skew shape \(\lambda/\mu\) as the set-theoretic difference \(\lambda \setminus \mu\) of their Young diagrams. Note, however, that when we consider a skew shape \(\lambda/\mu\) we have the information on the partitions \(\lambda\) and \(\mu\) as well as the difference \(\lambda - \mu\) of their Young diagrams. For example, the two skew shapes in Figure 1 have the same set of cells but are considered as different skew shapes.

For a cell \(x = (i, j)\), the content \(c(x)\) of \(x\) is defined by \(c(x) = j - i\). For a skew shape \(\alpha\), we define

\[
\text{Cont}(\alpha) = \{c(x) : x \in \alpha\}.
\]

For a skew shape \(\alpha\) we define \(\alpha + (r, s)\) to be the skew shape obtained by shifting \(\alpha\) by \((r, s)\), i.e.,

\[
\alpha + (r, s) = \{x + (r, s) : x \in \alpha\}.
\]

In this paper “connected” means edgewise connected. A connected component of a skew shape \(\alpha\) is a maximal connected subset of \(\alpha\), see Figure 2.

A border strip is a connected skew shape that contains no \(2 \times 2\) block of squares. For a border strip \(\gamma\), the contents of the starting and ending cells of \(\gamma\) are denoted by \(p(\gamma)\) and \(q(\gamma)\), respectively:

\[
p(\gamma) = \min(\text{Cont}(\gamma)), \quad q(\gamma) = \max(\text{Cont}(\gamma)).
\]

For a border strip \(\gamma\) and integers \(a\) and \(b\), we define

\[
\gamma[a, b] = \begin{cases} 
\{x \in \gamma : a \leq c(x) \leq b\} & \text{if } a \leq b, \\
\emptyset & \text{if } a = b + 1, \\
\text{undefined} & \text{if } a > b + 1.
\end{cases}
\]

See Figure 3. If \(\gamma[a, b]\) is undefined, then we define \(S_{\gamma[a, b]}\) to be 0.

Consider a skew shape \(\alpha\) and a border strip \(\gamma\) such that \(\text{Cont}(\alpha) \subseteq \text{Cont}(\gamma)\). Consider the diagonal shifts \(\gamma + (i, i)\) of \(\gamma\), for \(i \in \mathbb{Z}\), that cover \(\alpha\). The intersection of each diagonal shift of \(\gamma\) with \(\alpha\) is a union of border strips. Let \(\theta\) be the collection of the border strips obtained in this way.
\[ \gamma = \begin{array}{cccccc}
-1 & 0 & 1 \\
-3 & -2
\end{array} \]
\[ \gamma[-2,3] = \begin{array}{cccccc}
2 & 3 \\
-1 & 0 & 1 \\
-2
\end{array} \]

Figure 3. The left diagram shows a border strip \( \gamma \) with \( p(\gamma) = -3 \) and \( q(\gamma) = 5 \). The right diagram shows \( \gamma[-2,3] \). In both diagrams the contents of the cells are shown. We have \( \gamma[3,2] = \emptyset \) and \( \gamma[2,-1] \) is undefined.

\[
\begin{array}{ccccccc}
-1 & 0 & 1 & 2 & 3 & 4 & 5 \\
-2 & -1 & 0 & 1 & 2 & 3 & 4 \\
-3 & -2 & -1 & 0 & 1 \\
-4 & -3 & -2 \\
-5 \\
-6
\end{array}
\]

Figure 4. The left diagram shows the diagonal shifts of a border strip \( \gamma \) covering a skew shape \( \alpha \). The right diagram shows the decomposition \( \theta \) with respect to the cutting strip \( \gamma \).

\[
\begin{array}{ccccccc}
-1 & 0 & 1 & 2 & 3 & 4 & 5 \\
-2 & -1 & 0 & 1 & 2 & 3 & 4 \\
-3 & -2 & -1 & 0 & 1 \\
-4 & -3 & -2 \\
-5 \\
-6
\end{array}
\]

Figure 5. The Young diagram of \( \lambda = (6,6,4,2) \), where the contents of the cells in \( \lambda \cup \{(i,0) : 1 \leq i \leq 6\} \) are shown. The circled integers are the elements in \( C_6(\lambda) \).

Then \( \theta \) is a decomposition of \( \alpha \). In this case we say that \( \gamma \) is the **cutting strip** of \( \theta \). See Figure 4 for an example.

For a partition \( \lambda \) with at most \( n \) parts, let

\[ C_n(\lambda) = \{ \lambda_i - i : 1 \leq i \leq n \} \]

Figure 6. Note that for any partition \( \lambda \in \text{Par}_n \), we have \( |C_n(\lambda)| = n \) and \( \min(C_n(\lambda)) \geq -n \). Conversely, one can easily see that for any set \( C \) of integers with \( |C| = n \) and \( \min(C) \geq -n \), there is a unique partition \( \lambda \in \text{Par}_n \) with \( C_n(\lambda) = C \).

**Definition 2.1.** The **outer strip** \( \lambda^0 \) of a partition \( \lambda \) is the set of cells \( x \in \lambda \) such that \( x + (1,1) \notin \lambda \). The **extended outer strip** \( \lambda^+ \) of \( \lambda \) is the (infinite) set of cells \( x \in \mathbb{Z}_{\geq 0}^2 \setminus \lambda \) such that \( x + (-1,-1) \notin \mathbb{Z}_{\geq 0}^2 \setminus \lambda \). See Figure 6.

Note that by definition, for \( p \leq q \), we always have \( \lambda^0[p,q] \subseteq \lambda \), but \( \lambda^+[p,q] \not\subseteq \lambda \).
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Finally, we define Macdonald’s 9th variation of Schur functions. We define Macdonald's 9th variation of Schur functions. See Figure 7. It is easy to check that under the above conditions on $a$ and $b$, the partition $\lambda(a, b)$ is well defined and

$$
\lambda(a, b) = \begin{cases} 
\lambda & \text{if } a = b, \\
\lambda \setminus \lambda^0[b + 1, a] & \text{if } a > b, \\
\lambda \cup \lambda^+(a + 1, b) & \text{if } a < b.
\end{cases}
$$

Remark 2.3. The notations $\lambda^0[p, q]$ and $\lambda(p, q)$ will be used frequently throughout this paper. Note that $\lambda^0[p, q]$ is a border strip and $\lambda(p, q)$ is a partition obtained from $\lambda$ by adding or deleting a (possibly empty) border strip. The notation $\lambda / \mu(a, b)$ always means $\lambda/(\mu(a, b))$ and not $(\lambda/\mu)(a, b)$.

Finally, we define Macdonald’s 9th variation of Schur functions.

Definition 2.4. Let $h_r, s$, for $r, s \in \mathbb{Z}$ with $r \geq 1$, be independent indeterminates. Define $h_{0, s} = 1$ and $h_{r, s} = 0$ for all $r < 0$ and $s \in \mathbb{Z}$. For any partitions $\lambda$ and $\mu$ with at most $n$ parts, the Macdonald’s 9th variation $S_{\lambda/\mu}$ of Schur functions is defined by

$$S_{\lambda/\mu} = \det \left( h_{\lambda_i - \mu_j - i + j, \mu_j - j + 1} \right)_{i, j=1}^n.$$ 

If $h_{r,s}$ is set to be equal to the complete homogeneous symmetric function $h_r$ for all $r$ and $s$, then $S_{\lambda/\mu}$ reduces to the Schur function $S_{\lambda/\mu}$. Note that the Schur functions have the property that $s_\alpha = s_{\alpha+(r,s)}$ for any skew shape $\alpha$ and integers $r, s$ such that $\alpha + (r, s)$ is a skew shape. However, $S_{\alpha} \neq S_{\alpha+(r,s)}$ unless $r = s$.

3. The Bazin identity

In this section we recall the Bazin identity and derive a determinant identity for Macdonald’s 9th variation of Schur functions from it.
Definition 3.1. Let $M = (M_{ij})_{i \leq j \leq n}$ be a matrix whose rows and columns are indexed by $\mathbb{Z}$ and $\{1, 2, \ldots, n\}$ respectively. For any sequence $a = (a_1, \ldots, a_n)$ of $n$ integers (not necessarily distinct), we define

$$M[a] = \det(M_{i,j})_{1 \leq i, j \leq n}.$$ 

Note that if $a$ has repeated elements, then $[a] = 0$, and otherwise $[a]$ is, up to sign, equal to the minor of $M$ obtained by selecting the rows of $M$ indexed by the integers in $a$.

For sequences $a = (a_1, \ldots, a_r)$ and $b = (b_1, \ldots, b_s)$ of integers, let $inv(a)$ be the number of inversions in $a$. The Bazin identity from it.

The key lemma in this paper is the following result proved by Bazin [2] in 1851, see also [9, Lemma 2.1].

Lemma 3.2 (Bazin identity). Let $a = (a_1, \ldots, a_k), b = (b_1, \ldots, b_k)$ and $c = (c_1, \ldots, c_{n-k})$ be any sequences of integers. Then

$$[a \sqcup c]^{k-1} \cdot [b \sqcup c] = (-1)^{\frac{k}{2}} \det([b_j \sqcup (a_i \sqcup c)]_{i,j=1}).$$

Remark 3.3. The original statement of the Bazin identity is

$$[a \sqcup c]^{k-1} [b \sqcup c] = \det([a_1, \ldots, a_{i-1}, b_j, a_{i+1}, \ldots, a_k] \sqcup c)_{i,j=1}.$$ 

Since

$$[[a_1, \ldots, a_{i-1}, b_j, a_{i+1}, \ldots, a_k] \sqcup c] = \chi^{i-1} [b_j \sqcup (a_i \sqcup c)],$$

the above identity is equivalent to the one in Lemma 3.2. The Bazin identity is also attributed to Sylvester (see [9, p. 563]), Reiss, and Picquet (see [4, p. 195]).

We note that Okada [13, Corollary 3.2] found a more general determinant identity and derived the Bazin identity from it.

For a permutation $\pi$ of $\{1, 2, \ldots, n\}$ we denote by $inv(\pi)$ the number of pairs $(i,j)$ of integers $1 \leq i < j \leq n$ satisfying $\pi(i) > \pi(j)$. For two sequences $x = (x_1, \ldots, x_r)$ and $y = (y_1, \ldots, y_s)$ of integers, let $inv(x,y)$ denote the number of pairs $(i,j)$ of integers $1 \leq i \leq r$ and $1 \leq j \leq s$ satisfying $x_i > y_j$.

For a statement $p$ we define $\chi(p) = 1$ if $p$ is true and $\chi(p) = 0$ otherwise.

Now we derive a determinant identity for Macdonald’s 9th variation of Schur functions using the Bazin identity. In later sections we will give some applications of this identity.

Theorem 3.4. Let $\lambda, \mu,$ and $\nu$ be partitions with at most $n$ parts. Suppose that $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_k)$ are the sequences defined by

$$C_n(\lambda) \setminus C_n(\mu) = \{a_1 > a_2 > \cdots > a_k\},$$

$$C_n(\mu) \setminus C_n(\lambda) = \{b_1 > b_2 > \cdots > b_k\}.$$ 

Then we have

$$S_{\lambda/\mu}^{k-1} S_{\mu/\nu} = (-1)^{inv(b,a) + \frac{k}{2}} \det\left((-1)^{\chi(b_j > a_i)} S_{\lambda(a_i,b_j)/\nu}\right)_{i,j=1},$$

and

$$S_{\nu/\lambda}^{k-1} S_{\mu/\nu} = (-1)^{inv(b,a) + \frac{k}{2}} \det\left((-1)^{\chi(b_j > a_i)} S_{\nu(b_i,a_j)/\lambda}\right)_{i,j=1}.$$ 

Proof. Let

$$M = (h_i - h_j, \nu_{1+j} - \nu_{j+1})_{1 \leq i, j \leq n}$$

and $c = (c_1, \ldots, c_{n-k})$, where $c_1 > \cdots > c_{n-k}$ are the elements of $C_n(\lambda) \setminus a = C_n(\mu) \setminus b$ written in decreasing order. Then Lemma 3.2 says that

$$[a \sqcup c]^{k-1} [b \sqcup c] = (-1)^{\frac{k}{2}} \det([b_j \sqcup (a_i \sqcup c)]_{i,j=1}).$$
Since \( a \sqcup c \) is a rearrangement of the decreasing sequence \((\lambda_1 - 1, \lambda_2 - 2, \ldots,\lambda_n - n)\), we have
\[
(a \sqcup c) = (-1)^{\text{inv}(c,a)} \det(h_{\lambda_1-i-j+i,j+j+1})_{i,j=1}^n = (-1)^{\text{inv}(c,a)} S_{\lambda/\mu}.
\]
Similarly, we have
\[
(b \sqcup c) = (-1)^{\text{inv}(c,b)} S_{\mu/\nu}.
\]
Now, we claim that
\[
(b_j \sqcup (a \setminus a_i) \sqcup c) = (-1)^{\text{inv}(c,a) - \text{inv}(c,a_i) + \text{inv}(c,b_j) + \chi(a_i > b_j)} S_{\lambda(a_i, b_j)/\nu}.
\]
If \( b_j \in a \setminus a_i \), both sides of (3.6) are zero. Suppose \( b_j \notin a \setminus a_i \). Then \( b_j \sqcup (a \setminus a_i) \sqcup c \) is a rearrangement of \((\rho_1 - 1, \rho_2 - 2, \ldots, \rho_n - n)\), where \( \rho = (\rho_1, \ldots, \rho_n) \) is the partition \( \lambda(a_i, b_j) \). Thus
\[
(b_j \sqcup (a \setminus a_i) \sqcup c) = (-1)^t S_{\lambda(a_i, b_j)/\nu},
\]
where
\[
t = \text{inv}(a \setminus a_i, b_j) + \text{inv}(c, b_j) + \text{inv}(c, a \setminus a_i)
\]
\[
= \text{inv}(a, b_j) - \text{inv}(a_i, b_j) + \text{inv}(c, b_j) + \text{inv}(c, a) - \text{inv}(c, a_i).
\]
Since \( \text{inv}(a_i, b_j) = \chi(a_i > b_j) \), we obtain (3.6).

By factoring out common factors from each row and each column using (3.6), we get
\[
\det((b_j \sqcup (a \setminus a_i) \sqcup c))_{i,j=1}^k = (-1)^{(k-1)\text{inv}(c,a) + \text{inv}(c,b) + \text{inv}(a,b)} det((-1)^{\chi(a_i > b_j)} S_{\lambda(a_i, b_j)/\nu}).
\]
By substituting (3.4), (3.5) and (3.7) to (3.3), we obtain
\[
S_{\lambda/\mu}^{-1} S_{\mu/\nu} = (-1)^{\text{inv}(a,b)+\frac{k}{2}} \text{det}\left((-1)^\chi(a_i > b_j) S_{\lambda(a_i, b_j)/\nu}\right)_{i,j=1}^k.
\]
Since \( a_i \neq b_j \) for all \( i,j \), we have \( \text{inv}(a,b) = k^2 - \text{inv}(b,a) \) and \( \chi(a_i > b_j) = 1 - \chi(b_j > a_i) \). Thus
\[
(3.8)
\]
is equivalent to (3.1), which completes the proof of the first identity.

The second identity (3.2) is proved by the same arguments except that in this case we use the matrix \( N = (h_{i,j+i,j+1})_{i \in \mathbb{Z}, 1 \leq j \leq n} \) in place of \( M \).

4. Lascoux–Pragacz and Kreiman decompositions

In this section we restate Theorem 3.4 using the Lascoux–Pragacz and Kreiman decompositions for the case \( \mu \subseteq \lambda \). As a corollary we prove (a corrected version of) a conjecture of Morales, Pak, and Panova [12].

Recall that for a border strip \( \gamma \), we denote by \( p(\gamma) \) (resp. \( q(\gamma) \)) the content of the starting (resp. ending) cell of \( \gamma \).

**Definition 4.1.** A (border strip) decomposition of a skew shape \( \alpha \) is a sequence \( \theta = (\theta_1, \ldots, \theta_k) \) of border strips satisfying the following conditions.

- \( \theta_i \cap \theta_j = \emptyset \) for all \( i \neq j \),
- \( \theta_1 \cup \cdots \cup \theta_k = \alpha \).

If there is no possible confusion, we will simply write \( p_i = p(\theta_i) \), \( q_i = q(\theta_i) \), \( p = (p_1, \ldots, p_k) \) and \( q = (q_1, \ldots, q_k) \). By convention we will always label the border strips so that \( q_1 \geq q_2 \geq \cdots \geq q_k \).

Recall that the outer strip of a partition \( \lambda \) is the set of cells \( x \in \lambda \) satisfying \( x + (-1,1) \notin \lambda \). For a skew shape \( \lambda/\mu \), the outer strip of \( \lambda/\mu \) is defined to be the outer strip of \( \lambda \). Let \( \gamma \) be the outer strip of \( \lambda/\mu \) and let \( \rho \) be the set of cells \( x \in \lambda/\mu \) with \( x + (-1,-1) \notin \lambda/\mu \). We define the inner strip of \( \lambda/\mu \) to be the set
\[
\rho \cup \{ x \in \gamma : c(x) \notin \text{Cont}(\rho) \}.
\]
See Figure 8

**Definition 4.2.** Let \( \mu \) and \( \lambda \) be partitions with \( \mu \subseteq \lambda \). The Lascoux–Pragacz decomposition (resp. Kreiman decomposition) of \( \lambda/\mu \) is the decomposition of \( \lambda/\mu \) obtained by using the outer (resp. inner) strip of \( \lambda \) as the cutting strip. See Figure 8.
Combining the above results we obtain the lemma for set of consecutive integers. This shows that $p$ for all $2$ if $\text{Cont}(\theta)$ true for $1$.

**Proof.** We will only consider the case that $\theta$ is the Lascoux–Pragacz decomposition of $\lambda/\mu$ since it can be proved similarly for the case of the Kreiman decomposition. We proceed by induction on $k$. If $k = 0$, then $\lambda = \mu$ and there is nothing to prove. Suppose that $k \geq 1$ and the lemma is true for $k - 1$.

By the definition of the Lascoux–Pragacz decomposition, the border strip $\theta_1$ is a connected component of $(\lambda/\mu) \cap \lambda^0$. This implies that $p_1 - 1, q_1 + 1 \not\in \text{Cont}(\lambda/\mu)$ and the starting and ending cells of $\theta_1$ are the only cells in $\lambda/\mu$ whose contents are $p_1$ and $q_1$, respectively. Therefore, if $\text{Cont}(\theta_i) \cap \text{Cont}(\theta_1) \neq \emptyset$ for some $i \geq 2$, then $\text{Cont}(\theta_i) \subseteq [p_1 + 1, q_1 - 1]$ because $\text{Cont}(\theta_i)$ is a set of consecutive integers. This shows that $p_i \neq p_1, q_i \neq q_1, p_i \neq q_1, p_i - 1 \neq q_1$, and $p_1 - 1 \neq q_i$ for all $2 \leq i \leq k$.

Let $\rho = \lambda \setminus \theta_1$. Then $\mu \subseteq \rho$ and $(\theta_2, \theta_3, \ldots, \theta_k)$ is the Lascoux–Pragacz decomposition of $\rho/\mu$. By the induction hypothesis, we have $p_i \neq p_j, q_i \neq q_j, p_i \neq q_j$, and $p_i - 1 \neq q_j$ for all $2 \leq i \neq j \leq k$, and

$$C_n(\rho) \setminus C_n(\mu) = \{q_2 > q_3 > \cdots > q_k\},$$

$$C_n(\mu) \setminus C_n(\rho) = \{p_2 - 1, p_3 - 1, \ldots, p_k - 1\}.$$

It is straightforward to check that $p_1 - 1 \in C_n(\mu), q_1 \not\in C_n(\mu), p_1 - 1 \not\in C_n(\lambda), q_1 \in C_n(\lambda)$, and

$$C_n(\rho) = (C_n(\lambda) \setminus \{q_1\}) \cup \{p_1 - 1\}.$$

Combining the above results we obtain the lemma for $k$. The induction then completes the proof.
By Lemma 4.3 if $\mu \subseteq \lambda$, then we can restate Theorem 3.4 using the Lascoux–Pragacz or Kreiman decompositions as follows.

**Theorem 4.4 (Theorem 1.2).** Let $\lambda$, $\mu$, and $\nu$ be partitions and suppose that $\mu \subseteq \lambda$ and $\theta = (\theta_1, \ldots, \theta_k)$ is the Lascoux–Pragacz decomposition of $\lambda/\mu$. Then we have

\begin{align}
\mathbf{S}^{k-1}_{\lambda/\mu} \mathbf{S}_{\mu/\nu} &= \det \left( (-1)^{\lambda(p_i > q_i)} \mathbf{S}_{\lambda(p_i, p_j - 1)/\nu} \right)_{i,j=1}^k, \\
\mathbf{S}^{k-1}_{\nu/\lambda} \mathbf{S}_{\nu/\mu} &= \det \left( (-1)^{\lambda(p_i > q_i)} \mathbf{S}_{\nu(q_i, q_j - 1)/\mu} \right)_{i,j=1}^k.
\end{align}

**Proof.** Let $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_k)$ be the sequences defined by

\[ C_n(\lambda) \setminus C_n(\mu) = \{a_1 > a_2 > \cdots > a_k\}, \]
\[ C_n(\mu) \setminus C_n(\lambda) = \{b_1 > b_2 > \cdots > b_k\}. \]

By (3.1), we have

\[ \mathbf{S}^{k-1}_{\lambda/\mu} \mathbf{S}_{\mu/\nu} = (-1)^{\text{inv}(b,a)} \det \left( (-1)^{\chi(b_i > a_i)} \mathbf{S}_{\lambda(a_i,b_j)/\nu} \right)_{i,j=1}^k. \]

By Lemma 4.3 we have $a = b$ and $\{p_1 - 1, p_2 - 1, \ldots, p_k - 1\} = \{b_1, b_2, \ldots, b_k\}$. Let $b' = (b'_1, b'_2, \ldots, b'_k) = (b_k, b_{k-1}, \ldots, b_1)$ be the increasing rearrangement of $b$. Then we can rewrite the above equation as

\[ \mathbf{S}^{k-1}_{\lambda/\mu} \mathbf{S}_{\mu/\nu} = (-1)^{\text{inv}(b,q)} \det \left( (-1)^{\chi(b'_i > q_i)} \mathbf{S}_{\lambda(q_i,b'_j)/\nu} \right)_{i,j=1}^k = (-1)^{\text{inv}(b,q) + \text{inv}(\pi)} \det \left( (-1)^{\chi(p_i - 1 > q_i)} \mathbf{S}_{\lambda(q_i,p_j - 1)/\nu} \right)_{i,j=1}^k, \]

where $\pi$ is the permutation of $\{1, 2, \ldots, k\}$ satisfying

\[ p - 1 := (p_1 - 1, p_2 - 1, \ldots, p_k - 1) = (b'_1, b'_2, \ldots, b'_{\pi(k)}). \]

Note that $\text{inv}(\pi)$ is the number of pairs $(i, j)$ with $i < j$ and $\pi(i) > \pi(j)$, where $\pi(i) > \pi(j)$ is equivalent to $b'_i > b'_j$, which in turn is equivalent to $p_i > p_j$. Thus $\text{inv}(\pi)$ is equal to the number of pairs $(i, j)$ with $i < j$ and $p_i > p_j$. By the construction of the Lascoux–Pragacz decomposition, if $i < j$, then we must have either $p_j < q_j < p_i < q_i$ or $p_i < p_j \leq q_j < q_i$. This shows that $\text{inv}(\pi)$ is equal to the number of pairs $(\theta_i, \theta_j)$ of border strips such that $p(\theta_i) > q(\theta_j)$, which automatically implies $i < j$. Therefore $\text{inv}(\pi) = \text{inv}(p, q)$. On the other hand, by Lemma 4.3 we have $p_i - 1 \neq q_j$ for all $1 \leq i, j \leq n$. Thus $\text{inv}(\pi) = \text{inv}(p, q) = \text{inv}(p - 1, q) = \text{inv}(b, q)$ and $\chi(p_j - 1 > q_i) = \chi(p_j > q_i)$. Therefore the right hand sides of (4.3) and (4.1) are equal, which completes the proof of (4.1).

The second identity (4.2) can be proved similarly. \[\square\]

**Theorem 4.5.** Let $\lambda$, $\mu$, and $\nu$ be partitions. Suppose that $\mu \subseteq \lambda$ and $\theta = (\theta_1, \ldots, \theta_k)$ is the Kreiman decomposition of $\lambda/\mu$. Then we have

\begin{align}
\mathbf{S}^{k-1}_{\mu/\nu} \mathbf{S}_{\lambda/\nu} &= \det \left( (-1)^{\chi(p_i > q_i)} \mathbf{S}_{\mu(p_i - 1, q_j)/\nu} \right)_{i,j=1}^k, \\
\mathbf{S}^{k-1}_{\nu/\mu} \mathbf{S}_{\nu/\lambda} &= \det \left( (-1)^{\chi(p_i > q_i)} \mathbf{S}_{\nu(q_i - 1, p_j)/\mu} \right)_{i,j=1}^k.
\end{align}

**Proof.** Let $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_k)$ be the sequences defined by

\[ C_n(\mu) \setminus C_n(\lambda) = \{a_1 > a_2 > \cdots > a_k\}, \]
\[ C_n(\lambda) \setminus C_n(\mu) = \{b_1 > b_2 > \cdots > b_k\}. \]

Then by (3.1) with the roles of $\lambda$ and $\mu$ switched, we have

\[ \mathbf{S}^{k-1}_{\mu/\nu} \mathbf{S}_{\lambda/\nu} = (-1)^{\text{inv}(b,a)+\binom{k}{2}} \det \left( (-1)^{\chi(b_i > a_i)} \mathbf{S}_{\mu(a_i,b_j)/\nu} \right)_{i,j=1}^k. \]
By Lemma 4.3, we have $S^{k-1}_{\lambda/\nu} = (-1)^{\text{inv}(q,a)} \det \left( (-1)^{\delta(q_j,a_i)} S_{\mu(a_i,q_j)/\nu} \right)_{i,j=1}^k$

(4.6)

$$S^{k-1}_{\lambda/\nu} = (-1)^{\text{inv}(q,a)+\text{inv}(\pi)} \det \left( (-1)^{\delta(q_j,p_i-1)} S_{\mu(p_i-1,q_j)/\nu} \right)_{i,j=1}^k,$$

where $\pi$ is the permutation of $\{1, 2, \ldots, k\}$ satisfying $p-1 := (p_1-1, p_2-1, \ldots, p_k-1) = (a'_{\pi(1)}, a'_{\pi(2)}, \ldots, a'_{\pi(k)})$.

By the same argument as in the proof of Theorem 4.4, we have

$$\text{inv}(\pi) = \text{inv}(p,q) = \text{inv}(p-1,q) = \text{inv}(a,q) = k^2 - \text{inv}(q,a).$$

Thus we can rewrite (4.6) as

$$S^{k-1}_{\lambda/\nu} = \det \left( (-1)^{1-\delta(q_j,p_i-1)} S_{\mu(p_i-1,q_j)/\nu} \right)_{i,j=1}^k.$$

Since $q_j \neq p_i - 1$ for all $1 \leq i,j \leq k$ by Lemma 4.3, we have $1 - \delta(q_j,p_i-1) = \delta(q_j,p_i) = \delta(q_j,p_i), which together with the above equation shows (4.4).

The second identity (4.5) can be proved similarly. 

**Example 4.6.** Let $\lambda = (6, 6, 6, 3, 3)$, $\mu = (4, 3, 2)$ and $\nu = (7, 6, 6, 3, 2)$. The Lascoux–Pragacz decomposition of $\lambda/\mu$ and the skew shape $\nu/\lambda$ are shown in Figure 10. Then (4.2) of Theorem 4.4 implies that $S^2_{\nu/\lambda} S_{\nu/\mu}$ is equal to the determinant shown in Figure 11.

If $\nu = \lambda$, then the $(i,j)$-entry of the matrix in (4.2) is

$$(-1)^{\delta(p_j,q_i)} S_{\lambda/\nu(p_i,q_j-1)} = \begin{cases} S_{\lambda[p_j,q_i]} & \text{if } p_j \leq q_i, \\ S_{\lambda} & \text{if } p_j - 1 = q_i, \\ 0 & \text{if } p_j - 1 \geq q_i, \end{cases}$$

where the right hand side is exactly the same as the definition of $S_{\lambda[p_j,q_i]}$. Therefore we obtain the following Lascoux–Pragacz identity for $S_{\lambda/\mu}$, in which the case $\mu = \emptyset$ is proved by Macdonald [10 (9.9)].

**Corollary 4.7.** Suppose that $\mu \subseteq \lambda$ and $\theta = (\theta_1, \ldots, \theta_k)$ is the Lascoux–Pragacz decomposition of $\lambda/\mu$. Then we have

$$S_{\lambda/\mu} = \det \left( S_{\lambda[p_j,q_i]} \right)_{i,j=1}^k.$$

By setting $\nu = \emptyset$ in (4.1) and (4.4), we obtain the following two corollaries.

**Corollary 4.8.** Suppose that $\mu \subseteq \lambda$ and $\theta = (\theta_1, \ldots, \theta_k)$ is the Lascoux–Pragacz decomposition of $\lambda/\mu$. Then we have

$$S^{k-1}_{\lambda} S_{\mu} = \det \left( (-1)^{\delta(p_j,q_i)} S_{\lambda(p_j,q_i-1)} \right)_{i,j=1}^k.$$
Corollary 4.9. Suppose that $\mu \subseteq \lambda$ and $\theta = (\theta_1, \ldots, \theta_k)$ is the Krewman decomposition of $\lambda/\mu$. Then we have

$$S_{\mu}^{k-1}S_{\lambda} = \det \left( (-1)^{\chi(p_i > q_j)} S_{\mu(p_i, q_j - 1)} \right)_{i,j=1}^k.$$ 

Example 4.10. For $\lambda/\mu = (6, 6, 3, 3)/4, 3, 2)$, the Lakou–Pragacz decomposition of $\lambda/\mu$ is shown in Figure 10. Then Corollary 4.8 implies that $S_{\lambda}^2 S_{\mu}$ is equal to the determinant shown in Figure 12.

Macdonald’s 6th variation $s_\lambda(x|\alpha)$ of Schur functions [10], also known as factorial Schur functions, are defined by

$$s_\lambda(x|\alpha) = \frac{\det \left( (x_i - a_1)(x_i - a_2) \cdots (x_i - a_{\lambda_j} + d - j) \right)^{d}_{i,j=1}}{\prod_{1 \leq i < j \leq d} (x_i - x_j)},$$

where $x = (x_1, \ldots, x_d)$ is a sequence of variables and $\alpha = (a_1, a_2, \ldots)$ is a sequence of parameters. Note that $s_\lambda(x|\alpha)$ is a symmetric polynomial in the variables $x$ with parameters $\alpha$. If $a_i = 0$ for all $i$, then $s_\lambda(x|\alpha)$ becomes the Schur polynomial $s_\lambda(x)$. The factorial Schur functions $s_\lambda(x|\alpha)$ are a special case of Macdonald’s 9th variation $S_\lambda$ of Schur functions.

By specializing the Macdonald’s 9th variation to the factorial Schur functions in Corollary 4.8 we obtain the following result, which is a corrected version of a conjecture proposed by Morales, Pak, and Panova [12].
Corollary 4.11. Let \( \theta = (\theta_1, \ldots, \theta_k) \) be the Lascoux–Pragacz decomposition of a skew shape \( \lambda/\mu \) with \( \ell(\lambda) \leq d \). Then

\[
(4.7) \quad s_\mu(x|a)s_\lambda(x|a)^{k-1} = \det \left( (-1)^{\chi(p_j > q_i)} S_{\lambda(q_i, p_j - 1)} \right)_{i,j=1}^k,
\]

where \( x = (x_1, \ldots, x_d) \) and \( a = (a_1, a_2, \ldots) \).

Remark 4.12. The original conjecture of Morales, Pak, and Panova [12 Conjecture 7.9] states that, under the same assumption in Corollary 4.11

\[
(4.8) \quad s_\mu(x|a)s_\lambda(x|a)^{k-1} = \det \left( s_{\lambda \setminus \lambda^0[p_j, q_i]}(x|a) \right)_{i,j=1}^k.
\]

In fact (4.8) is not true even for the Schur function case. As a counterexample, if \( \lambda = (2,1), \mu = (1), x = (x_1, x_2) \), and \( a = (0,0,\ldots) \), then the Lascoux–Pragacz decomposition \( \theta = (\theta_1, \theta_2) \) has two border strips with \( p_1 = q_1 = 1 \) and \( p_2 = q_2 = -1 \). One can easily check using Pieri’s rule that the left hand side of (4.8) is

\[
(4.9) \quad s_\mu(x)s_\lambda(x) = s_{(3,1)}(x) + s_{(2,2)}(x) + s_{(2,1,1)}(x) = s_{(3,1)}(x) + s_{(2,2)}(x).
\]

However, the right hand side of (4.8) is

\[
\det \left( s_{\lambda \setminus \lambda^0[p_j, q_i]}(x) \right)_{i,j=1}^2 = \det \begin{pmatrix} s_{\lambda \setminus \lambda^0[1,1]}(x) & s_{\lambda \setminus \lambda^0[-1,1]}(x) \\ s_{\lambda \setminus \lambda^0[1,-1]}(x) & s_{\lambda \setminus \lambda^0[-1,-1]}(x) \end{pmatrix} = \det \begin{pmatrix} s_{(1,1)}(x) & s_{(2,1)}(x) \\ 0 & s_{(2)}(x) \end{pmatrix} = s_{(3,1)}(x) + s_{(2,1,1)}(x) = s_{(3,1)}(x).
\]

Note that the right hand side of (4.7) for the running example is

\[
\det \left( (-1)^{\chi(p_j > q_i)} s_{\lambda(q_j, p_j - 1)}(x) \right)_{i,j=1}^2 = \det \begin{pmatrix} s_{(1,1)}(x) & s_{(2)}(x) \\ -s_{(2,2)}(x) & s_{(2)}(x) \end{pmatrix} = s_{(3,1)}(x) + s_{(2,1,1)}(x) + s_{(2,2)}(x) = s_{(3,1)}(x) + s_{(2,2)}(x),
\]

which is equal to \( s_\mu(x)s_\lambda(x) \) as shown in (4.9).
5. A generalized Hamel–Goulden formula

In this section we give a generalization of the Hamel–Goulden formula. Our result involves generalizations of cutting strips as well as Schur functions. As corollaries we obtain Jin’s result \[8\] and a generalized Giambelli formula.

We first introduce some definitions.

**Definition 5.1.** Let \( \lambda \) and \( \nu \) be partitions with \( \lambda \subseteq \nu \). A border strip \( \gamma \) is \( \nu/\lambda \)-compatible if the following condition holds:

- \( \text{Cont}(\nu/\lambda) \subseteq \text{Cont}(\lambda) \), and
- for any connected component \( \alpha \) of \( \nu/\lambda \), we have
  \[
  \lambda^0[a-1,b+1] = \gamma[a-1,b+1],
  \]
  where \( a = \min(\text{Cont}(\alpha)) \) and \( b = \max(\text{Cont}(\alpha)) \).

See Figure 13.

**Definition 5.2.** Let \( \lambda \) and \( \nu \) be partitions with \( \lambda \subseteq \nu \). A partition \( \mu \subseteq \lambda \) is \( \nu/\lambda \)-compatible if the following condition holds:

- \( \text{Cont}(\nu/\lambda) \subseteq \text{Cont}(\lambda) \), and
- for any connected component \( \alpha \) of \( \nu/\lambda \), we have
  \[
  \lambda^0[a,b] = \mu^+[a,b],
  \]
  where \( a = \min(\text{Cont}(\alpha)) \) and \( b = \max(\text{Cont}(\alpha)) \).

See Figure 14.

In Definitions 5.1 and 5.2 one can use \( \lambda^+ \) instead of \( \lambda^0 \); if \( \alpha \) is a connected component of \( \nu/\lambda \), then since \( \text{Cont}(\alpha) \subseteq \text{Cont}(\lambda) \), we have

\[
\lambda^0[a-1,b+1] = \lambda^+[a-1,b+1],
\]
  where \( a = \min(\text{Cont}(\alpha)) \) and \( b = \max(\text{Cont}(\alpha)) \).

Note that if \( \nu = \lambda \), then by definition every border strip \( \gamma \) and every partition \( \mu \subseteq \lambda \) are \( \nu/\lambda \)-compatible.

Recall that we have defined \( \gamma[a,b] = \{ x \in \gamma : c(x) \in [a,b] \} \) for a border strip \( \gamma \). We extend this definition to any connected skew shape \( \alpha \), that is,

\[
\alpha[a,b] = \{ x \in \alpha : c(x) \in [a,b] \}.
\]
Figure 14. A $\nu/\lambda$-compatible partition $\mu$ for the skew shape $\nu/\lambda$ in Figure 13.

Figure 15. The skew shape $\gamma \oplus \nu/\lambda$ for the border strip $\gamma$ and the skew shape $\nu/\lambda$ in Figure 13. The contents of the cells in $\gamma$ are shown and the cells coming from $\nu/\lambda$ are colored red.

Figure 16. The diagrams $(\gamma \oplus \nu/\lambda)[-11, -7]$ (left), $(\gamma \oplus \nu/\lambda)[-6, -1]$ (middle), and $(\gamma \oplus \nu/\lambda)[-7, 7]$ (right), for the $\gamma \oplus \nu/\lambda$ in Figure 15.

Definition 5.3. Suppose that $\gamma$ is a $\nu/\lambda$-compatible border strip. Define $\gamma \oplus \nu/\lambda$ to be the skew shape obtained from $\gamma$ by gluing each connected component $\alpha$ of $\nu/\lambda$ below $\gamma$ after shifting $\alpha$ diagonally so that the southeast boundary of $\gamma$ and the northwest boundary of $\alpha$ have common edges. See Figure 15.

Observe that if $\gamma$ is a $\nu/\lambda$-compatible border strip and

$$(5.1) \quad a, b \in (\text{Cont}(\lambda) \setminus \text{Cont}(\nu/\lambda)) \cup \{\min(\text{Cont}(\lambda)), \max(\text{Cont}(\lambda))\},$$

then $(\gamma \oplus \nu/\lambda)[a, b]$ is a connected skew shape. See Figure 16.

We now state the generalized Hamel–Goulden formula.
**Theorem 5.4.** Let $\mu \subseteq \lambda \subseteq \nu$ be partitions. Suppose that $\mu$ is $\nu/\lambda$-compatible and $\gamma$ is a $\nu/\lambda$-compatible border strip. Let $\theta = (\theta_1, \ldots, \theta_k)$ be the decomposition of $\lambda/\mu$ determined by the cutting strip $\gamma$. If $\alpha_1, \ldots, \alpha_\ell$ are the connected components of $\nu/\lambda$, we have

$$S_{\nu/\mu} \prod_{s=1}^{\ell} S_{\alpha_s}^{r_s-1} = \det \left( S_{\gamma \cup \nu/\lambda} \right)_{i,j=1}^{k},$$

where $r_s$ is the number of strips $\theta_i$ such that $\text{Cont}(\alpha_s) \subseteq \text{Cont}(\theta_i)$.

Before proving Theorem 5.4 we give an example and some of its applications.

**Example 5.5.** Consider the partitions $\nu = (8, 8, 6, 6, 6, 5, 4)$, $\lambda = (8, 8, 6, 6, 6, 3, 3)$, and $\mu = (4, 3)$ as shown in Figure 17. The border strip $\gamma$ in Figure 18 is $\nu/\lambda$-compatible. The decomposition $\theta$ of $\lambda/\mu$ with cutting strip $\gamma$ is also shown in Figure 18. Since $C(\alpha_1) \subseteq C(\theta_i)$ for $i = 1, 2, 4$ and $C(\alpha_2) \subseteq C(\theta_j)$ for $j = 1, 2$, we have $r_1 = 3$ and $r_2 = 2$. Thus by Theorem 5.4, $S_{\nu/\mu} S_{\alpha_1}^2 S_{\alpha_2}$ is equal to the determinant shown in Figure 19.

If $\nu = \lambda$ in Theorem 5.4 then we obtain the following Hamel–Goulden formula for Macdonald’s 9th variation of Schur functions, which was also proved by Bachmann and Charlton [1], and Foley and King [5] using the Lindström–Gessel–Viennot lemma.

**Corollary 5.6.** Let $\lambda$ and $\mu$ be partitions with $\mu \subseteq \lambda$. Suppose that $\gamma$ is a border strip with $\text{Cont}(\lambda) \subseteq \text{Cont}(\gamma)$ and $\theta = (\theta_1, \ldots, \theta_k)$ is the decomposition of $\lambda/\mu$ with cutting strip $\gamma$. Then

$$S_{\lambda/\mu} = \det \left( S_{\gamma \cup \nu/\lambda} \right)_{i,j=1}^{k}.$$

If $\nu/\lambda$ is a disjoint union of single cells and if we restrict Theorem 5.4 to Schur functions, then we obtain Jin’s result [8, Theorem 2] for the cases when the “enriched diagrams” are not necessary.

For sequences $a = (a_1, \ldots, a_r)$ and $b = (b_1, \ldots, b_r)$ with $a_1 > \cdots > a_r \geq 0$ and $b_1 > \cdots > b_r \geq 0$, the *Frobenius notation* $(a|b)$ denotes the partition

$$\{(i, i) : 1 \leq i \leq r\} \cup \{(i, j) : 1 \leq i \leq r, i < j \leq a_i\} \cup \{(i, j) : 1 \leq j \leq r, j < i \leq b_i\}.$$
If we take \( \nu = (a \sqcup c|b \sqcup d), \lambda = (a|b), \mu = (c|d), \) and \( \theta \) to be the Kreiman decomposition of \( \lambda/\mu \) in Theorem 5.4, then we obtain the following generalized Giambelli formula.

**Corollary 5.7.** Let \( a = (a_1, \ldots, a_r), b = (b_1, \ldots, b_r), c = (c_1, \ldots, c_s), \) and \( d = (d_1, \ldots, d_s) \) be sequences of integers such that \( r, s \geq 0 \) and

\[
a_1 > \cdots > a_r > c_1 > \cdots > c_s \geq 0, \quad b_1 > \cdots > b_r > d_1 > \cdots > d_s \geq 0.
\]

Then

\[
S_{(c|d)}^{-1} S_{(a\sqcup c|b\sqcup d)} = \det \left( S_{(a_i \sqcup c_j|b_j \sqcup d_j)} \right)_{1 \leq i, j \leq r}.
\]
If \( c = d = \emptyset \), then Corollary 5.7 reduces to the Giambelli formula. If \( c = d = (0) \) in Corollary 5.7, we obtain
\[
S_{(1)}^{r-1}S_{(\alpha, \emptyset)(\beta, \emptyset)} = \det \left( S_{(a_j, 0)(b_j, 0)} \right)_{1 \leq i, j \leq r},
\]
where each entry has a near hook shape \(((a_i, 0)|(b_j, 0))\).

For the rest of this section we give a proof of Theorem 5.4. We first recall a known property of \( S_{\lambda/\mu} \).

Let \( \alpha \) and \( \beta \) be skew shapes and let \( a \) and \( b \) be, respectively, the top-right corner of \( \alpha \) and the bottom-left corner of \( \beta \). We define \( \alpha \to \beta \) (resp. \( \alpha \uparrow \beta \)) to be the skew shape obtained from \( \alpha \) by attaching \( \beta \) so that \( b \) is to the right of \( \alpha \) (resp. above \( a \)).

**Lemma 5.8.** [11 Chapter I, §5, Example 30 (d)] Let \( \alpha \) and \( \beta \) be skew shapes such that the top-right corner of \( \alpha \) and the bottom-left corner of \( \beta \) satisfy \( c(b) = c(a) + 1 \). Then

\[
S_{\alpha}S_{\beta} = S_{\alpha \to \beta} + S_{\alpha \uparrow \beta}.
\]

Chen, Yan, and Yang [8] showed that the Hamel–Goulden formula can be obtained from the Lascoux–Pragacz formula using simple matrix operations. Their proof uses only the fact that the Schur functions satisfy
\[
s_{\alpha}s_{\beta} = s_{\alpha \to \beta} + s_{\alpha \uparrow \beta}.
\]

Hence, by Lemma 5.8 their proof extends to the identity in Theorem 5.4. This implies that it is sufficient to prove this theorem for the case when \( \theta \) is the Lascoux–Pragacz decomposition of \( \lambda/\mu \).

To this end we need three lemmas.

**Lemma 5.9.** The following properties hold.

1. If \( \mu \not\subseteq \lambda \), then \( S_{\lambda/\mu} = 0 \).
2. If \( \mu \subseteq \lambda \) and \( \alpha_1, \ldots, \alpha_s \) are the connected components of \( \lambda/\mu \), then \( S_{\lambda/\mu} = S_{\alpha_1} \cdots S_{\alpha_s} \).

**Proof.** Both properties can be proved similarly as in the Schur function case [11 Chapter I, (5.7)].

**Lemma 5.10.** Suppose that \( \mu \subseteq \lambda \subseteq \nu \) are partitions such that \( \mu \) is \( \nu/\lambda \)-compatible. Let \( \alpha_1, \ldots, \alpha_\ell \) be the connected components of \( \nu/\lambda \) and let \( \gamma \) be the outer strip of \( \lambda \). Finally, let \( \theta = (\theta_1, \ldots, \theta_k) \) be the Lascoux–Pragacz decomposition of \( \lambda/\mu \). Then

\[
S_{\nu/\lambda(q_i, p_j - 1)} = S_{\gamma/\nu/\lambda}[p_j, q_i] \prod_{s=1}^{\ell} S_{\chi(q_i < \min(\text{Cont}(\alpha_s))) + \chi(p_j > \max(\text{Cont}(\alpha_s)))}.
\]

**Proof.** By Lemma 5.9 we have \( p_j - 1 \neq q_i \) for all \( i, j \). Thus there are two cases \( p_j \leq q_i \) and \( p_j - 1 > q_i \).

**Case 1:** \( p_j \leq q_i \). Since \( \lambda(q_i, p_j - 1) = \lambda - \lambda^{0}_{[p_j, q_i]} \), the connected component \( \alpha_s \) of \( \nu/\lambda \) remains the same in \( \nu/\lambda(q_i, p_j - 1) \) if \( q_i < \min(\text{Cont}(\alpha_s)) \) or \( p_j > \max(\text{Cont}(\alpha_s)) \). Otherwise, \( \text{Cont}(\alpha_s) \subseteq [p_j, q_i] \) and therefore \( \alpha_s \) is contained in \( (\gamma \oplus \nu/\lambda)[p_j, q_i] \), which is a connected component of \( \nu/\lambda(q_i, p_j - 1) \). Thus by Lemma 5.9

\[
S_{\nu/\lambda(q_i, p_j - 1)} = S_{\gamma/\nu/\lambda}[p_j, q_i] \prod_{s=1}^{\ell} S_{\chi(q_i < \min(\text{Cont}(\alpha_s))) + \chi(p_j > \max(\text{Cont}(\alpha_s)))}.
\]

Since \( q_i < \min(\text{Cont}(\alpha_s)) \) and \( p_j > \max(\text{Cont}(\alpha_s)) \) cannot be satisfied at the same time, the above equation is the same as the one in the lemma.

**Case 2:** \( p_j - 1 > q_i \). Since \( S_{\gamma/\nu/\lambda}[p_j, q_i] = 0 \), it suffices to show that \( \nu \not\subseteq \lambda(q_i, p_j - 1) \), which implies \( S_{\nu/\lambda(q_i, p_j - 1)} = 0 \) by Lemma 5.9. Since \( \lambda(q_i, p_j - 1) = \lambda \cup \lambda^+_{[q_i + 1, p_j - 1]} \) has a cell with content \( p_j - 1 \), to show \( \nu \not\subseteq \lambda(q_i, p_j - 1) \) it is enough to show that \( p_j - 1 \not\in \text{Cont}(\nu/\lambda) \).

For a contradiction suppose \( p_j - 1 \in \text{Cont}(\nu/\lambda) \). Then \( p_j - 1 \in \text{Cont}(\alpha_s) \) for some \( s \). Since \( \mu \) is \( \nu/\lambda \)-compatible and \( p_j - 1 \in \text{Cont}(\alpha_s) \), we must have \( \lambda^+_{[p_j - 1, p_j]} = \mu^+_{[p_j - 1, p_j]} \). On the other hand, we have \( p_j - 1 \in \mathcal{C}_n(\mu) \setminus \mathcal{C}_n(\lambda) \) by Lemma 1.3. Since \( p_j - 1 \in \mathcal{C}_n(\mu) \), the border strip \( \mu^+_{[p_j - 1, p_j]} \) must be a vertical domino. However, since \( p_j - 1 \not\in \mathcal{C}_n(\lambda) \), the border strip \( \lambda^+_{[p_j - 1, p_j]} \) must be a horizontal domino. Then \( \lambda^+_{[p_j - 1, p_j]} \neq \mu^+_{[p_j - 1, p_j]} \), which is a contradiction. Therefore we must have \( p_j - 1 \not\in \text{Cont}(\nu/\lambda) \), which completes the proof. \( \square \)
Lemma 5.11. Under the same assumptions in Lemma 5.11, we have

\[ S_{\nu/\mu} \prod_{s=1}^{\ell} S_{\alpha_s}^{k-1-I_s-J_s} = \det \left( S_{(\gamma \oplus \nu/\lambda) \setminus [p_j, q_i]} \right)_{i,j = 1}^{k}, \]

where \( I_s \) is the number of \( 1 \leq i \leq k \) such that \( q_i < \min(\text{Cont}(\alpha_s)) \) and \( J_s \) is the number of \( 1 \leq j \leq k \) such that \( p_j > \max(\text{Cont}(\alpha_s)) \).

Proof. By (5.2), we have

\[ S_{\nu/\mu} S_{\nu/\lambda}^{k-1} = \det \left( (-1)^{\chi(p_j > q_i)} S_{\nu/\lambda(q_i, p_j-1)} \right)_{i,j = 1}^{k}. \]

By Lemmas 5.9 and 5.10, the above equation can be written as

\[ \prod_{s=1}^{\ell} S_{\alpha_s}^{k-1} = \det \left( S_{(\gamma \oplus \nu/\lambda) \setminus [p_j, q_i]} \prod_{s=1}^{\ell} S_{\alpha_s}^{\chi(q_i < \min(\text{Cont}(\alpha_s)))+ \chi(p_j > \max(\text{Cont}(\alpha_s)))} \right)_{i,j = 1}^{k}, \]

where the factor \((-1)^{\chi(p_j > q_i)}\) can be omitted because \( S_{(\gamma \oplus \nu/\lambda) \setminus [p_j, q_i]} = 0 \) if \( p_j > q_i \). By factoring out the factor \( S_{\alpha_s}^{\chi(q_i < \min(\text{Cont}(\alpha_s)))} \) from each row \( i \) and the factor \( \prod_{s=1}^{\ell} S_{\alpha_s}^{\chi(p_j > \max(\text{Cont}(\alpha_s)))} \) from each column \( j \) and dividing both sides of (5.2) by these factors we obtain the desired identity. \( \square \)

Now we are ready to prove Theorem 5.4.

Proof of Theorem 5.4. As we have already discussed, it suffices to show the theorem for the case when \( \theta \) is the Lascoux–Pragacz decomposition of \( \lambda/\mu \). Then, by Lemma 5.11 it suffices to show that \( k - I_s - J_s = r_s \) for all \( s \). By definition, \( k - I_s - J_s \) is the number of border strips \( \theta_i \) such that \( p(\theta_i) \leq \max(\text{Cont}(\alpha_s)) \) and \( \min(\text{Cont}(\alpha_s)) \leq q(\theta_i) \). Since \( \mu = \nu/\lambda \)-compatible and \( \theta \) is the Lascoux–Pragacz decomposition, every border strip \( \theta_i \) with \( \text{Cont}(\alpha_s) \cap \text{Cont}(\theta_i) \neq \emptyset \) must satisfy \( \text{Cont}(\alpha_s) \subseteq \text{Cont}(\theta_i) \). This implies that \( p(\theta_i) \leq \max(\text{Cont}(\alpha_s)) \) and \( \min(\text{Cont}(\alpha_s)) \leq q(\theta_i) \) if and only if \( \text{Cont}(\alpha_s) \subseteq \text{Cont}(\theta_i) \). Therefore the number of such border strips \( \theta_i \) is equal to \( r_s \), which completes the proof. \( \square \)

6. A generalization of a converse of Hamel–Goulden’s theorem

In this section we prove Theorem 1.3, which is a generalization of a converse of Hamel–Goulden’s theorem. We restate Theorem 1.3 as follows.

Theorem 6.1. Let \( \alpha \) be any connected skew shape. Suppose that \((a_1, \ldots, a_k)\) and \((b_1, \ldots, b_k)\) are sequences of integers such that \( a_i \) and \( b_j \) is a skew shape for all \( i,j \). Then either \( \det (s_{a_i, b_j})_{i,j=1}^{k} = 0 \) or there exists a skew shape \( \rho \) such that

\[ \det (s_{a_i, b_j})_{i,j=1}^{k} = \pm s_{\rho}. \]

To prove this theorem we need the following lemma.

Lemma 6.2. Let \( \gamma \) be a border strip and let \((a_1 < \cdots < a_k)\) and \((b_1 < \cdots < b_k)\) be sequences of integers such that \( a_i, b_i \in \text{Cont}(\gamma) \) for all \( i \), and for any integer \( \alpha \),

\[ |\{i: a_i \leq a\}| \geq |\{i: b_i \leq a\}|. \]

Then there is a skew shape \( \tau \) such that if \( \theta = (\theta_1, \ldots, \theta_r) \) is the decomposition of \( \tau \) with cutting strip \( \gamma \), then \( r = k \) and \( \{p(\theta_1), \ldots, p(\theta_k)\} = \{a_1, \ldots, a_k\} \) and \( \{q(\theta_1), \ldots, q(\theta_k)\} = \{b_1, \ldots, b_k\} \).

Proof. We proceed by induction on \( k \). If \( k = 0 \), we can take \( \tau = \emptyset \). Let \( k \geq 1 \) and suppose that the lemma is true for \( k - 1 \).

We consider the following two cases depending on the shape of \( \gamma[b_1, b_1 + 1] \).
the two sequences (a₁, a₂, a₃) and (b₁, b₂, b₃) on the left and the corresponding skew shape and its decomposition on the right.

**Figure 21.** A border strip γ and the two sequences (a₁, a₂, a₃) and (b₁, b₂, b₃) on the left and the corresponding skew shape and its decomposition on the right.

**Figure 22.** The decomposition in Figure 21 is obtained recursively using these three border strips.

**Case 1:** γ[b₁, b₁ + 1] is a horizontal domino. Let θ₀ = γ[a₁, b₁]. It is easy to see that the two sequences (a₂ < ⋯ < aₖ) and (b₂ < ⋯ < bₖ) satisfy (6.1). Thus, by the induction hypothesis, there is a skew shape σ whose decomposition with cutting strip γ is (θ₁, ⋯, θₖ₋₁) such that \{p(θ₁), ⋯, p(θₖ₋₁)\} = {a₂, ⋯, aₖ} and \{q(θ₁), ⋯, q(θₖ₋₁)\} = {b₂, ⋯, bₖ}. The fact that a₁ < a₂ and γ[b₁, b₁ + 1] is a horizontal domino guarantees that τ = σ ∪ θ₀, where θ₀ is attached below σ after an appropriate diagonal shift, is a skew shape satisfying the desired properties.

**Case 2:** γ[b₁, b₁ + 1] is a vertical domino. Let θ₀ = γ[aₘ, b₁], where m is the largest integer such that aₘ ≤ b₁. It is easy to see that the two sequences (a₁ < ⋯ < aₘ₋₁ < aₘ+₁ < ⋯ < aₖ) and (b₂ < ⋯ < bₖ) satisfy (6.1). Thus, by the induction hypothesis, there is a skew shape σ whose decomposition with cutting strip γ is (θ₁, ⋯, θₖ₋₁) such that \{p(θ₁), ⋯, p(θₖ₋₁)\} = \{a₁, ⋯, aₘ₋₁, aₘ+₁, ⋯, aₖ\} and \{q(θ₁), ⋯, q(θₖ₋₁)\} = \{b₂, ⋯, bₖ\}. The fact that aᵢ < aₘ, for all 1 ≤ i < m, and γ[b₁, b₁ + 1] is a vertical domino guarantees that τ = σ ∪ θ₀, where θ₀ is attached above σ after an appropriate diagonal shift, is a skew shape satisfying the desired properties.

The above two cases show that the lemma is true for k and the proof is completed by induction.

For an illustration of the construction in the proof of Lemma 6.2, consider the border strip γ and the two sequences (a₁, a₂, a₃) and (b₁, b₂, b₃) in Figure 21. Since γ[b₁, b₁ + 1] is a horizontal domino, we construct the border strip γ[a₁, b₁] as in Figure 22(a). Since γ[b₂, b₂ + 1] is a vertical domino, we construct the border strip γ[a₂, b₂] as in Figure 22(b). Finally we construct the border strip γ[a₂, b₃] as in Figure 22(c). By combining these border strips after appropriate diagonal shifts we obtain the skew shape shown in Figure 21.

We now prove Theorem 6.1.

**Proof of Theorem 6.1** We may assume det \( s_{a[a_i, b_i]} \) \( i,j=1 \) ≠ 0 because otherwise there is nothing to prove. Since permuting the aᵢ’s or bᵢ’s only changes the sign of the determinant, we may
can be expressed as a single skew Schur function, the theorem follows from (6.2).

\[ \lambda \]\n
obtained from \( \text{Cont}(\tau) \) det \( \alpha \) is diagonal shift, see Figure 24. The assumption that \( \mu \) and \( \gamma \) are \( \nu/\lambda \)-compatible. Therefore, by Theorem 5.4 with \( s_r \) specialized to \( s_r \) for all skew shapes \( \tau \), we have

\[ \det (s_{\gamma \oplus \nu/\lambda}[a_j, b_i])_{i,j=1}^k = \pm s_{\tau \mu} / \mu \prod_{s=1}^{\ell} s_{\alpha_s - 1}^{r_s}, \]

where \( \alpha_1, \ldots, \alpha_\ell \) are the connected components of \( \nu/\lambda \), and \( r_s \) is the number of strips \( \theta_i \) such that \( \text{Cont}(\alpha_s) \subseteq \text{Cont}(\theta_i) \).

By the construction, we have \( \gamma \oplus \nu/\lambda \)[a_j, b_i] = \( \alpha \)[a_j, b_i]. Since a product of skew Schur functions can be expressed as a single skew Schur function, the theorem follows from (6.2). \( \square \)
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