Effects of the Central Mass Concentration on Bar Formation in Disk Galaxies

Daejong Jang1 and Woong-Tae Kim1,2

1 Department of Physics & Astronomy, Seoul National University, Seoul 08826, Republic of Korea; unitree@snu.ac.kr
2 SNU Astronomy Research Center, Seoul National University, Seoul 08826, Republic of Korea

Received 2022 October 5; revised 2022 November 18; accepted 2022 November 29; published 2023 January 18

Abstract

While bars are common in disk galaxies, their formation conditions are not well understood. We use N-body simulations to study the formation and evolution of a bar in isolated galaxies consisting of a stellar disk, a classical bulge, and a dark halo. We consider 24 galaxy models that are similar to the Milky Way but differ in the mass and compactness of the classical bulge and halo concentration. We find that the bar formation requires \( Q_{T,\text{min}}/1.2^2 + (\text{CMC}/0.05)^2 \lesssim 1 \), where \( Q_{T,\text{min}} \) and CMC refer to the minimum value of the Toomre stability parameter and the central mass concentration, respectively. Bars tend to be stronger and longer, and to rotate more slowly, in galaxies with a less massive and less compact bulge and halo. All bars formed in our models correspond to slow bars. A model with the bulge mass of \( \sim 10\% -20\% \) of the disk under a concentrated halo produces a bar similar to that of the Milky Way. We discuss our findings in relation to other bar formation criteria suggested by previous studies.

Unified Astronomy Thesaurus concepts: Disk galaxies (391); Milky Way Galaxy (1054); Galaxy bulges (578); Galaxy disks (589); Barred spiral galaxies (136); Galaxy bars (2364)

1. Introduction

Bars are common in the universe. More than \( \sim 60\% \) of disk galaxies in the local universe in optical and near-infrared images are known to possess a weak or strong bar (de Vaucouleurs 1963; Sellwood & Wilkinson 1993; Knapen et al. 2000; Whyte et al. 2002; Laurikainen et al. 2004; Marinova & Jogee 2007; Menéndez-Delmestre et al. 2007; Aguerrí et al. 2009; Méndez-Abreu et al. 2012; Buta et al. 2015; Díaz-García et al. 2016, 2019). The fraction of barred disk galaxies decreases with redshift (Sheth et al. 2008; Melvin et al. 2014), with a tendency that more massive galaxies are more likely barred. Bar formation appears inhibited in dispersion-dominated galaxies and in halo-dominated galaxies at low redshift (Sheth et al. 2012). These features indicate that the bar formation occurs preferentially in a late secular phase of galaxy formation when the disks become dynamically cold (Kraljic et al. 2012).

Theoretically, bar formation is due to gravitational instability of a rotationally supported stellar disk (Toomre 1964): nonaxisymmetric perturbations grow via swing amplification, and initially circular stellar orbits are deformed to elongated \( x_1 \) orbits that form and support a bar (see, e.g., Sellwood 2014). A number of simulations have shown that the presence of a dark halo affects the formation and evolution of a bar (Ostriker & Peebles 1973; Hohl 1976; Debattista & Sellwood 2000; Valenzuela & Klypin 2003; Holley-Bockelmann et al. 2005; Weinberg & Katz 2007). While the gravity of a halo tends to suppress the bar formation by reducing the relative strength of the disk’s self-gravity in equilibrium (Ostriker & Peebles 1973), angular momentum exchange between a bar and a live halo allows the former to grow longer and stronger (Athanassoula 2002). Also, the halo parameters such as the axial ratio (Athanassoula 2002; Athanassoula et al. 2013) and spin (Collier et al. 2018, 2019; Kataria & Shen 2022) lead to considerable changes in the evolution of the bar.

In addition to a halo, a classical bulge can also strongly affect the formation and evolution of a bar. Classical bulges are produced as a result of major/minor mergers during galaxy formation (Kauffmann et al. 1993; Baugh et al. 1996; Bournaud et al. 2007; Hopkins et al. 2009, 2010; Naab et al. 2014). Unlike halos, classical bulges are highly centrally concentrated and can thus stabilize the inner regions of disks without affecting the outer regions much. Early studies found that a strong bulge suppresses swing amplification by interrupting a feedback loop that transforms propagating trailing waves to leading ones (Sellwood 1980; Toomre 1981; Binney & Tremaine 2008), inhibiting bar formation (e.g., Kataria & Das 2018; Saha & Elmegreen 2018). Also, a live bulge can make a bar longer and stronger by removing angular momentum from the latter, just like a live halo (Sellwood 1980).

A bar that forms can increase a central mass, for example, by driving gas inflows (e.g., Athanassoula 1992; Buta & Combes 1996; Kim et al. 2012), which in turns weakens or destroys the bar by disturbing bar-supporting \( x_1 \) orbits (e.g., Pfenniger & Norman 1990; Hasan et al. 1993; Norman et al. 1996; Shen & Sellwood 2004; Bournaud et al. 2005; Athanassoula et al. 2013).

While many numerical studies mentioned above are useful for understanding the effects of a bulge and a halo on the formation and evolution of a bar, the quantitative conditions for bar-forming instability are still debated. Using numerical models with a fixed halo and a disk with surface density \( \Sigma_d \sim R^{-1} \), Ostriker & Peebles (1973) suggested that bar formation requires

\[
T_{\text{OP}} \equiv T/|W| > 0.14,
\]

where \( T \) and \( W \) stand for the total rotational and gravitational potential energies of a galaxy, respectively. Using two-dimensional (2D) models with a fixed halo and an exponential disk, Efstathiou et al. (1982, ELN) showed that a bar forms in
where $V_{\text{max}}$, $M_b$, and $R_d$ refer to the maximum rotational velocity, mass, and scale radius of the disk, respectively.

It is not until recent years that galaxy models for bar formation have treated all three components (disk, bulge, and halo) as being live (Polyachenko et al. 2016; Salo & Laurikainen 2017; Fujii et al. 2018; Kataria et al. 2018; Saha & Elmegreen 2018; Kataria & Das 2020). In particular, Kataria & Das (2018) used self-consistent $N$-body simulations with differing bulge masses, and showed that bar formation requires that the ratio of the bulge to total radial force initially satisfies

$$f_{\text{KD}} \equiv \frac{GM_b}{R_d V_{\text{tot}}^2} < 0.35,$$

where $M_b$ is the bulge mass and $V_{\text{tot}}$ is the total rotational velocity at $R = R_d$. Using three-component galaxy models with differing disk and bulge densities, Saha & Elmegreen (2018) argued that their models evolve to barred galaxies provided

$$D_{\text{SE}} \equiv \frac{\langle \rho_b \rangle}{\langle \rho_d \rangle} < \frac{1}{10},$$

where $\langle \rho_b \rangle$ and $\langle \rho_d \rangle$ are the mean densities of the bulge and disk, respectively, within the half-mass radius of the bulge.

The several different conditions given above imply that there has not been consensus regarding the quantitative criterion for bar formation. Part of the reason for the discrepancies in the proposed conditions may be that some models considered a fixed (rather than live) halo, and that some authors explored parameter space by fixing either bulge or halo parameters. Also, it is questionable whether the effects of the complicated physical processes (swing amplification and feedback loop) involved in the bar formation can be encapsulated by the single parameters given above. In this paper, we revisit the issue of bar formation by varying both bulge and halo parameters together. Our models will be useful for clarifying what conditions are necessary to produce a bar when the mass and compactness of the bulge and halo vary. We will show that the two key elements that govern the bar formation are the minimum value of the Toomre stability parameter $Q_{\text{T, min}}$ and the central mass concentration (CMC), defined as the total galaxy mass inside the central 0.1 kpc relative to the total disk mass: bars form more easily in galaxies with smaller $Q_{\text{T, min}}$ and CMC. We also measure the strength, length, and pattern speed of the bars that form in our simulations and explore their dependence on the halo and bulge parameters.

This paper is organized as follows. In Section 2, we describe our galaxy models and the numerical methods we employ. In Section 3, we present temporal changes of the bar properties such as bar strength, pattern speed, length, and angular momentum transfer from a disk to halo and bulge. In Section 4, we compare our numerical results with the previous bar formation conditions mentioned above, and propose the new conditions in terms of $Q_{\text{T, min}}$ and CMC. We also use our numerical model to constrain the classical bulge of the Milky Way. Finally, we conclude our work in Section 5.
The gravitational susceptibility of a disk can be measured by the Toomre (1966) stability parameter

$$Q_t = \frac{\kappa \sigma_R}{3.36 G \Sigma_d}.$$  (7)

where $\kappa$ is the epicycle frequency and $\Sigma_d$ is the disk surface density. Figure 2 plots the radial distributions of $Q_t$ for models with $M_b/M_d = 0$ and 0.1. Overall, $Q_t$ is large at both small $R$ (due to increase in $\kappa$) and large $R$ (due to decrease in $\Sigma_d$) and it attains a minimum value $Q_{T, \text{min}}$ at $R \sim 4-6$ kpc. Our galaxy models have $Q_{T, \text{min}}$ in the range between 0.95 and 1.16 (Table 1): $Q_{T, \text{min}}$ tends to be larger for a galaxy with a centrally concentrated halo and/or more massive bulge, while it is almost independent of the bulge compactness.

2.2. Numerical Method

To construct the initial galaxy models, we make use of the GALIC code (Yurin & Springel 2014), which solves the collisionless Boltzmann equations to find a desired equilibrium state by optimizing the velocities of individual particles. We distribute $N_d = 1.0 \times 10^6$, $N_b = 5 \times 10^5-5 \times 10^6$, and $N_h = 2.6 \times 10^7$ particles for the disk, bulge, and halo, respectively. We set the mass of each particle to $m = 5 \times 10^4 M_\odot$, which is the same for all three components.

We evolve our galaxy models by using a public version of the Gadget-4 code (Springel et al. 2021). This version has improved force accuracy, time-stepping, computational efficiency, and parallel scalability from Gadget-3. It offers the fast multipole method, in which the tree is accelerated by multipole expansion not only on the source side but also on the sink side. For our galaxy models, we find the multipole expansion to

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**Table 1**

Model Parameters and Various Dimensionless Quantities of the Initial Galaxy Models

| Model | $a_b$ (kpc) | $M_b/M_d$ | $a_0$ (kpc) | $Q_{T, \text{min}}$ | CMC | $\nu_{\text{SP}}$ | $\nu_{\text{ELN}}$ | $\nu_{\text{FD}}$ | $\nu_{\text{SE}}$ |
|-------|-------------|------------|-------------|---------------------|-----|-----------------|-----------------|-----------------|-----------------|
| C00   | 30          | 0.0        | –           | 1.082               | 0.04 x 10^-2 | 0.449 | 0.89 | 0.10 | 0.00 |
| C05   | 30          | 0.05       | 0.4         | 1.079               | 0.24 x 10^-2 | 0.449 | 0.90 | 0.10 | 0.67 |
| C10   | 30          | 0.1        | 0.4         | 1.085               | 0.44 x 10^-2 | 0.447 | 0.91 | 0.19 | 1.30 |
| C20   | 30          | 0.2        | 0.4         | 1.110               | 0.84 x 10^-2 | 0.446 | 0.92 | 0.33 | 2.60 |
| C30   | 30          | 0.3        | 0.4         | 1.140               | 1.22 x 10^-2 | 0.445 | 0.94 | 0.44 | 3.90 |
| C40   | 30          | 0.4        | 0.4         | 1.164               | 1.62 x 10^-2 | 0.444 | 0.95 | 0.52 | 5.20 |
| L00   | 40          | 0.0        | –           | 0.954               | 0.04 x 10^-2 | 0.442 | 0.80 | 0.12 | 0.64 |
| L05   | 40          | 0.05       | 0.4         | 0.961               | 0.24 x 10^-2 | 0.441 | 0.81 | 0.12 | 0.64 |
| L10   | 40          | 0.1        | 0.4         | 0.975               | 0.44 x 10^-2 | 0.440 | 0.82 | 0.22 | 1.30 |
| L20   | 40          | 0.2        | 0.4         | 1.000               | 0.84 x 10^-2 | 0.439 | 0.84 | 0.37 | 2.61 |
| L30   | 40          | 0.3        | 0.4         | 1.023               | 1.24 x 10^-2 | 0.438 | 0.86 | 0.49 | 3.91 |
| L40   | 40          | 0.4        | 0.4         | 1.046               | 1.64 x 10^-2 | 0.437 | 0.89 | 0.58 | 5.19 |
| L50   | 40          | 0.5        | 0.4         | 1.056               | 2.06 x 10^-2 | 0.436 | 0.90 | 0.64 | 7.05 |
| C05c  | 30          | 0.05       | 0.2         | 1.082               | 0.58 x 10^-2 | 0.447 | 0.90 | 0.10 | 3.01 |
| C10c  | 30          | 0.1        | 0.2         | 1.086               | 1.16 x 10^-2 | 0.446 | 0.91 | 0.18 | 5.76 |
| C20c  | 30          | 0.2        | 0.2         | 1.106               | 2.26 x 10^-2 | 0.442 | 0.92 | 0.32 | 11.30 |
| C30c  | 30          | 0.3        | 0.2         | 1.127               | 3.34 x 10^-2 | 0.439 | 1.07 | 0.42 | 16.80 |
| C40c  | 30          | 0.4        | 0.2         | 1.158               | 4.40 x 10^-2 | 0.437 | 1.23 | 0.50 | 22.21 |
| L05c  | 40          | 0.05       | 0.2         | 0.961               | 0.58 x 10^-2 | 0.439 | 0.81 | 0.12 | 2.89 |
| L10c  | 40          | 0.1        | 0.2         | 0.972               | 1.16 x 10^-2 | 0.438 | 0.82 | 0.21 | 5.72 |
| L20c  | 40          | 0.2        | 0.2         | 0.999               | 2.26 x 10^-2 | 0.435 | 0.88 | 0.36 | 11.28 |
| L30c  | 40          | 0.3        | 0.2         | 1.024               | 3.38 x 10^-2 | 0.432 | 1.07 | 0.46 | 16.84 |
| L40c  | 40          | 0.4        | 0.2         | 1.049               | 4.48 x 10^-2 | 0.430 | 1.23 | 0.54 | 22.06 |
| L50c  | 40          | 0.5        | 0.2         | 1.049               | 5.62 x 10^-2 | 0.428 | 1.38 | 0.59 | 25.53 |

![Figure 1](image.png)

Figure 1. Radial distributions of the total rotational velocity $v_{rot}$ for models C00, L00 (top), C10, L10 (middle), and C10c, L10c (bottom). A more massive and compact bulge increase $v_{rot}$ at small $R$. Models in the C series have higher $v_{rot}$ by $\sim$20 km s$^{-1}$ on average, than those in the L series.

rotational velocity in the inner regions with $R \lesssim a_b$. At $R \lesssim 20$ kpc, models in the C series have larger $v_{rot}$, by $\sim$20 km s$^{-1}$ on average, than their L series counterparts.
and calculate the amplitudes of the $m = 2$ Fourier modes as
\begin{align}
a_2(R) & = \sum_i m_i \cos(2\theta_i), \\
b_2(R) & = \sum_i m_i \sin(2\theta_i),
\end{align}

where $\theta_i$ and $m_i$ are the azimuthal angle and mass of the $i$th disk particle in the annulus, respectively. We then define the bar strength as
\begin{equation}
A_2 = \max \left( \frac{\sqrt{a_2^2 + b_2^2}}{\Sigma_i m_i} \right)
\end{equation}

Note that $A_2/A_0$ measures the strength of $m = 2$ spirals when a bar is absent or weak. For spirals, the position angle
\begin{equation}
\psi(R) \equiv \frac{1}{2} \tan^{-1} \left( \frac{b_2}{a_2} \right)
\end{equation}

varies systematically with $R$, while $\psi(R)$ remains almost constant for a bar.

Following Algory et al. (2017), we regard galaxies with $A_2/A_0 \geq 0.2$ and relatively constant $\psi(R)$ as being barred: features with $A_2/A_0 < 0.2$ are considered as ovals if $\psi(R)$ is constant or spirals if $\psi(R)$ changes with $R$. Figure 5 plots the temporal evolution of $A_2/A_0$ for models with less compact bulges in the C series (upper panel) and for models with $M_b/M_d = 0.1$ (lower panel). The evolution of $A_2/A_0$ is dominated by spirals at early times ($\lesssim 1$–$2$ Gyr) and then by a bar. Although the spirals are strong in the outer regions, they can affect the inner disk where a bar exists before it fully grows (see the $t \lesssim 3$ Gyr panels in Figure 3). The spirals and bar rotate about the galaxy center at different pattern speeds. When the spirals and bar become in phase, $A_2/A_0$ achieves its peak value temporarily (at $t = 2.3$ Gyr for model C10). Subsequently, $A_2/A_0$ decreases as they become out of phase, although it increases again as the bar grows further and dominates the inner disk. The presence of a more massive bulge makes the bar form later and weaker. The bar formation is completely suppressed in models with $M_b/M_d \gtrsim 0.4$ in the C series.

The compactness of halo and bulge has a significant effect on the bar formation. In the L series with a less concentrated halo, disks are unstable to form a bar even when the bulge mass amounts to $\sim$50% of the disk mass. In contrast, disks in the C series with a compact halo do not produce a bar when $M_b/M_d \gtrsim 0.35$. Similarly, a more compact bulge tends to suppress the bar formation. For example, the maximum bulge mass for bar formation is decreased to 20% and 10% in the L and C series, respectively, when the bulge is compact. Our result that a bar does not form in galaxies with a very massive and compact bulge is qualitatively consistent with previous studies (e.g., Kataria & Das 2018; Saha & Elmegreen 2018).

3.2. Buckling Instability

Figure 6 plots the temporal changes in the ratio, $\sigma_z/\sigma_R$, of the vertical to radial velocity dispersion of the disk particles at $R = 1$ kpc for models with $M_b/M_d \lesssim 0.2$ in the C series together with model L00. Since a bar is supported by $x_1$ orbits elongated along its semimajor axis, its growth naturally involves the increase in $\sigma_R$. At the same time, the bar and spirals can excite the vertical motions of star particles, enhancing $\sigma_z$ (e.g., Quillen et al. 2014). When a bar grows
rapidly, as in models C00 and L00, $\sigma_R$ increases faster than $\sigma_z$, resulting in a decrease in the ratio $\sigma_z/\sigma_R$ at early time. When a bar grows slowly, in contrast, $\sigma_z/\sigma_R$ remains more or less constant.

The increase in $\sigma_z$ leads to the disk thickening and the formation of a boxy/peanut (B/P) bulge. All the bars that form in our models evolve to B/P bulges. Figure 7 plots the evolution of the B/P strength, defined as

$$P_z = \max\left(\frac{|z|}{z_0}\right),$$

where the tilde denotes the median and $z_0$ is the initial value (Iannuzzi & Athanassoula 2015; Fragkoudi et al. 2017; Seo et al. 2019), for models with $M_b/M_d \leq 0.2$. In most models, the disk thickening occurs secularly. However, $P_z$ (as well as $\sigma_z/\sigma_R$) in models C00 and L00 increases rapidly at $t \sim 5$ Gyr, which is due to vertical buckling instability.

It is well known that a bar can undergo buckling instability when $\sigma_z/\sigma_R$ is small. Toomre (1966) and Araki (1987) suggested that nonrotating thin disks are unstable to the buckling instability if $\sigma_z/\sigma_R \leq 0.3$. For realistic disks with spatially varying $\sigma_z/\sigma_R$, Raha et al. (1991) found that the buckling instability occurs if $\sigma_z/\sigma_R \lesssim 0.25$–0.55 in the mid-disk regions. By varying the values of $\sigma_z/\sigma_R$ in N-body simulations, Martinez-Valpuesta et al. (2006) suggested that the critical value is at $\sigma_z/\sigma_R \sim 0.6$ (see also Kwak et al. 2017). This is consistent with our numerical results that models L00 and C00 have $\sigma_z/\sigma_R \lesssim 0.6$ before undergoing the buckling instability, as shown in Figure 6.

Figure 8 plots the edge-on views of the projected density distributions of the disks at $t = 6.0$ Gyr for models with $M_b/M_d \leq 0.2$. The $x$- and $z$-directions correspond to the direction parallel to the semimajor axis of the bar and the vertical direction, respectively. The density distributions in models L00 and C00 with no bulge are asymmetric with respect to the $z = 0$ plane, evidencing the operation of buckling instability.
Figure 4. Snapshots of the disk surface density at $t = 8.0$ Gyr in all the models. Each image is rotated such that the semimajor axis of a bar (or an oval) is aligned parallel to the $x$-axis.
We note that the other models with a bulge also possess a $B/P$ bulge, which develops on a timescale longer than in models $L00$ and $C00$. This is consistent with Sellwood & Gerhard (2020), who showed that the presence of a nuclear mass with only a small ($\sim 2.5\%$) fraction of the disk mass tends to suppress the buckling instability.

A rapid increase of $\sigma_z/\sigma_R$ at $t \sim 5$ Gyr in models $C00$ and $L00$ is due to buckling instability. We note that the other models with a bulge also possess a $B/P$ bulge, which develops on a timescale longer than in models $L00$ and $C00$. This is consistent with Sellwood & Gerhard (2020), who showed that the presence of a nuclear mass with only a small ($\sim 2.5\%$) fraction of the disk mass tends to suppress the buckling instability. In models with a bulge, the disks appear to thicken as the bar particles are excited vertically by the passage through the 2:1 vertical resonance (Quillen et al. 2014; Sellwood & Gerhard 2020).
3.3. Angular Momentum and Pattern Speed

We calculate the angular momentum of each component as

\[ L_i = \sum m_i (x_i v_y - y_i v_x). \]

Figure 9 plots temporal changes of \( L_z \) relative to the initial disk angular momentum for a disk (orange), halo (blue), bulge (green), as well as the total (black) in model C10. The disk loses its angular momentum right after the bar formation, while the halo and bulge absorb it. Since the bulge occupies a relatively small volume in space in model C10, the amount of angular momentum it gains is limited to \( \sim 4\% \), while the halo absorbs the remaining \( \sim 96\% \). In model L50 with a large bulge mass, however, the bulge absorbs about \( \sim 26\% \) of the angular momentum lost by the disk. The total angular momentum is conserved within \( \sim 0.1\% \) in all models.

To calculate the bar pattern speed \( \Omega_b \), we use two methods: (1) the cross-correlation of the disk surface density in the annular regions with width \( \Delta R = 0.1 \text{ kpc} \) at \( R = 2 \text{ kpc} \) where most bars attain their maximum strength, and (2) the temporal rate of change in the position angle \( \psi \), i.e., \( \Omega_b = d\psi / dt \mid_{R=2 \text{ kpc}} \). We check that the two methods yield pattern speeds that agree within \( \sim 1\% \). Figure 10 plots the evolution of the bar pattern speeds for selected models. The initial bar pattern speed depends on the bulge mass in such a way that a more massive bulge tends to have larger \( \Omega_b \) (Kataria & Das 2018, 2019). In all models, a bar becomes slower over time due to angular momentum transfer from a bar to both halo and bulge.

Figure 11 plots the temporal changes in the bar slowdown rate, \( -d\Omega_b / dt \), for the models shown in Figure 10(a), showing that there is no systematic relation between the bar slowdown rate and the bulge mass. This result is different from Kataria & Das (2019), who found that the rate is higher for a more massive bulge. The discrepancy may be due to the differences in the bulge (and halo) compactness. The models considered by Kataria & Das (2019) have \( R_b/R_d \leq 0.18 \) with \( R_b \) being the half-mass bulge radius, which is more compact than our models in the C series that have \( R_b/R_d = (1 + \sqrt{2})a_0/R_d = 0.32 \). Figure 12 of Kataria & Das (2018) shows no systematic trend between the bar slowdown rate and the bulge mass for models with \( 0.43 < R_b/R_d < 0.47 \). This suggests the bulge should be sufficiently compact to control the temporal evolution of the bar pattern speed. In our models with less compact bulge than...
in Kataria & Das (2019), angular momentum is predominantly absorbed by the halo (see Figure 9).

### 3.4. Bar Length

One can use the position angle \( \psi(R) \) defined in Equation (10) to measure the bar length (e.g., Athanassoula & Misiriotis 2002; Scannapieco & Athanassoula 2012). Figure 12 plots the radial distribution of the position angle of the \( m = 2 \) mode in the disk of model C10 at \( t = 5.5 \) Gyr. Note that \( \psi(R) \) that remains more or less constant at small \( R \) but changes abruptly at \( R \gtrsim 6.8 \) kpc, indicating that the bar has a semimajor axis \( R_b = 6.8 \) kpc at this time.

Figure 13 plots temporal changes of \( R_b \) for the models shown in Figure 10. First of all, bars are longer in models with a less massive and/or less compact bulge since these allow stronger swing amplifications. Overall, the bar length in our models increases with time, expedited by angular momentum exchange with the halo and bulge (Athanassoula 2003). The increasing rate of the bar length is lower in models with a more massive and compact bulge. We note that the decrease in the bar length at \( t \sim 3.8 \) Gyr in model C00, \( t \sim 5.3 \) Gyr in model C05, and \( t \sim 3.3 \) Gyr in model C20 is caused by the interactions with surrounding spiral arms (or an inner ring), which tend to shorten the bar by perturbing particles on outer \( x_1 \) orbits. In model L10, outer spiral arms are in phase with the bar at \( t \sim 1 \) Gyr, making \( R_b \) longer than the true bar length temporarily.

Figure 14 plots the dependence of the bar pattern speed \( \Omega_b \) and the corotation radius \( R_{CR} \) on the bar length \( R_b \) in all models that form a bar, with the symbol size representing the simulation time. In general, longer bars tend to be slower. The ratio \( \mathcal{R} = R_{CR}/R_b \) is useful to classify slow or fast bars: bars with \( \mathcal{R} > 1.4 \) are considered slow, while those with \( \mathcal{R} < 1.4 \) are termed fast bars. Models with a massive and compact bulge have larger \( \mathcal{R} \) since they have short bars compared to those with a less compact bulge. Note that all bars are slow rotators for almost all times.

### 4. Discussion

In the preceding section, we have shown that models with a massive and compact bulge and a concentrated halo are less likely to form a bar. In this section we compare our numerical results with the previous bar formation conditions mentioned in Section 1. We then propose a new two-parameter condition that is consistent with the theory of bar formation. We also use our numerical results to indirectly measure the mass of the classical bulge in the Milky Way.

#### 4.1. Criteria for Bar Formation

Figure 15 plots the simulation outcomes in the \( t_{OP}-\epsilon_{ELN} \) plane, with the blue and red symbols representing unstable and stable models to bar formation, respectively; the values of \( t_{OP} \) and \( \epsilon_{ELN} \) of each model are listed in Columns (7) and (8) of Table 1. Circles and triangles mark the models in the L and C series, respectively, with the open (filled) symbols corresponding to the compact (less compact) bulges. While all the models have \( t_{OP} > 0.42 \), some of them do not evolve to form a bar, suggesting that \( t_{OP} \) is not a good indicator of the disk stability against bar formation. This is most likely because
Ostriker & Peebles (1973) employed models with a fixed halo, neglecting halo–disk interactions, which are crucial for the bar growth. Saha & Elmegreen (2018) also noted that the initial value of $t_{\text{OP}}$ cannot determine whether a bar forms or not. The abscissa of Figure 15 shows that all the bar-forming models satisfy the ELN criterion (Equation (2)). However, some galaxies with a massive bulge under a concentrated halo remain stable even with $t_{\text{OP}} < 1.1$. The discrepancy between the ELN criterion and our results is because it—based on 2D thin-disk models with a fixed halo—does not capture the disk–halo interactions (e.g., Athanassoula 2008; Fujii et al. 2018).

Analyses of galaxies in recent simulations of cosmological galaxy formation such as EAGLE and IllustrisTNG, etc. have also found that $t_{\text{OP}}$ is incomplete to predict whether galaxies formed are barred or not (Yurin & Springel 2015; Algorry et al. 2017; Izquierdo-Villalba et al. 2022; Marioni et al. 2022). A recent observational test based on statistically unbiased sample of barred/non-barred galaxies shows that the ELN criterion is largely inaccurate (Romeo et al. 2023).

Figure 16 plots our results in the $\mathcal{F}_{\text{KD}}$–$D_{\text{SE}}$ plane, with the blue and red symbols corresponding to the unstable and stable models, respectively: the values of $\mathcal{F}_{\text{KD}}$ and $D_{\text{SE}}$ of each model are given in Columns (9) and (10) of Table 1. The ordinate of Figure 16 shows that Equation (3) is overall consistent with the simulation results for the models in the C series, although it fails for the models in the L series: some models with a massive bulge under a less concentrated halo form a bar even with $\mathcal{F}_{\text{KD}} > 0.35$. In fact, all the models in Kataria & Das (2018) belong to our C series, so that their criterion is unable to predict bar formation in models with less concentrated halos.3

Saha & Elmegreen (2018) found that a compact bulge suppresses feedback loops by making the ILR strong. According to Equation (4) for bar formation, all of our models

3 The halos employed in Kataria & Das (2018) have the scale radius of $a_h = 17.88$ kpc for the MA models and $a_h = 25.54$ kpc for the MB models (S. K. Kataria 2022, private communication), which are smaller than $a_h = 30$ kpc for the models in our C series.
except models C00 and L00 with no bulge should not form a bar. However, the abscissa of Figure 16 shows that most models with $D_{SIE} \lesssim (4–10)$ are unstable to bar formation. It is unclear why our results are so different from Equation (4), but part of the reason may be that compared with our models, their halos are small in mass with $M_h \sim 4M_d$ and their disks are thin with $z_d \sim 0.02R_d$.

As mentioned earlier, bar formation in a disk involves several cycles of swing amplifications and feedback loops. This naturally requires two conditions: (1) the disk should have small $Q_{T,\text{min}}$ to be sufficiently susceptible to self-gravitational instability, and (2) the ILR should be weak enough for incoming waves pass through the center, which is achieved when the CMC is small. Motivated by these physical considerations, Figure 17 plots the simulation outcomes in the $Q_{T,\text{min}}$–CMC plane. Models with a more compact bulge and halo have a higher CMC than their less concentrated counterparts with the same $M_b/M_d$. Models with concentrated halos tend to have higher $Q_{T,\text{min}}$, although $Q_{T,\text{min}}$ is insensitive to the bulge compactness. Note that all the bar-forming models satisfy

$$\left(\frac{Q_{T,\text{min}}}{1.2}\right)^2 + \left(\frac{\text{CMC}}{0.05}\right)^2 < 1,$$  

(13)

marked by the shaded area in Figure 17. In models with $Q_{T,\text{min}}$ or CMC larger than Equation (13) implies, swing amplifications with suppressed feedback loops are not strong enough to promote bar formation: these models end up with only weak spiral arms in outer disks (see Figure 4).

Failure of Equations (3) and (4) as a criterion for bar formation is because they account only for a bulge in setting the ILR. However, our results show that not only the bulge mass but also the halo mass in the galaxy center is important in determining the strength of the ILR.

4.2. Fast or Slow Bars

The fact that all bars in our models are slow is consistent with the results of Roshan et al. (2021), who found that bars formed in cosmological hydrodynamical simulations are preferentially slow, with a mean value of $R \sim 1.9–3.0$. However, Cuomo (2020) showed that most observed bars in 77 nearby galaxies are fast, with a mean value of $R \sim 0.92$. What causes the discrepancy in the bar properties between observations and simulations is a challenging question. Frankel et al. (2022) argued that the discrepancy arises not because the simulated bars are too slow but because they are too short.

There is much room for improvement in both simulations and observations for more reliable comparisons. In simulations, our models of isolated galaxies need to be more realistic by including a gaseous component, star formation, halo spin, etc., which may affect the bar pattern speeds significantly. Cosmological simulations still suffer from issues such as insufficient resolution and calibration of feedback from star formation and active galactic nuclei. In observations, the often-used Tremaine–Weinberg method in measuring the bar pattern speeds depends critically on the assumptions that galaxies are in a steady state and that there is a well-defined pattern (Tremaine & Weinberg 1984), the validity of which is not always guaranteed. In addition, the bar length depends considerably on the measurement methods such as Fourier analysis, force ratio, ellipse fitting, etc. (Lee et al. 2022). Theoretically, it is impossible to have a long-lived, quasi-steady bar with $R < 1$ since the bar-supporting $x_1$ orbits exist only inside the corotation radius (e.g., Contopoulos 1980; Contopoulos & Grosbøl 1989; Binney & Tremaine 2008).

4.3. Classical Bulge of the Milky Way

The Milky Way is a barred galaxy dominated by a B/P bulge (e.g., Dwek et al. 1995; Martinez-Valpuesta & Gerhard 2011; Ness et al. 2013). Some early studies reported that the bar in the Milky Way is fast and short, with $50 < \Omega_b < 60 \text{ km s}^{-1}\text{ kpc}^{-1}$ and $R_b \sim 3 \text{ kpc}$ (Fux 1999; Debattista et al. 2002; Bissantz et al. 2003; Fragkoudi et al. 2019; Dehnen 2000), while recent studies suggested that it is rather slow and long, with $33 < \Omega_b < 45 \text{ km s}^{-1}\text{ kpc}^{-1}$ and $R_b \sim 4.5–5 \text{ kpc}$ (Sormani et al. 2015; Wegg et al. 2015; Bland-Hawthorn & Gerhard 2016; Portail et al. 2017; Clarke & Gerhard 2022). By comparing observed proper motions in the bar and bulge regions with dynamical models, Clarke & Gerhard (2022) most recently reported $\Omega_b = 33.29 \pm 1.81 \text{ km s}^{-1}\text{ kpc}^{-1}$, placing the corotation resonance at $R_{CR} \sim 5–7 \text{ kpc}$.

Using our numerical results, we attempt to constrain the mass of the classical bulge of the Milky Way. As Figures 10 and 13 show, the bar in model C10 has $\Omega_b \sim 30–35 \text{ km s}^{-1}\text{ kpc}^{-1}$ and $R_b \sim 4.5–5 \text{ kpc}$ at $t = 2.5–3.5 \text{ Gyr}$, which are well matched to the observed properties of the Milky Way bar. Model C20 produces a bar with $\Omega_b \sim 36 \text{ km s}^{-1}\text{ kpc}^{-1}$ and $R_b \sim 4.2 \text{ kpc}$ at $t = 8 \text{ Gyr}$. The bars in models C00 and C05 have a length of $R_b \sim 5 \text{ kpc}$ at $t \sim 2.5 \text{ Gyr}$, but their pattern speeds are less than $30 \text{ km s}^{-1}\text{ kpc}^{-1}$. These results suggest that the Milky Way may possess a classical bulge with mass $\sim 10%–20%$ of the disk mass. This is consistent with the claim of Shen et al. (2010) that the classical bulge of the Milky Way should be less than 25% of the disk mass to be fitted well with the observed stellar kinematics (see also Di Matteo et al. 2015). If the age of the Milky Way bar is $\sim 3 \text{ Gyr}$, as proposed by Cole & Weinberg (2002) based on the ages of infrared carbon stars, the bar in model C10 best represents the Milky Way bar. If it is instead $\sim 8 \text{ Gyr}$ old, as proposed by Bovy et al. (2019) based on the kinematic analyses of APOGEE and Gaia data, it would be better described by the bar in model C20.
5. Conclusions

We have presented the results of N-body simulations to study the effects of spherical components including a classical bulge and a dark halo on the formation and evolution of a bar. For this, we have constructed 3D galaxy models with physical conditions similar to the Milky Way and varied the bulge-to-disk mass ratio as well as the compactness of the halo and bulge components, while fixing the disk and halo masses. Our main conclusions are highlighted below.

1. Bar properties. The presence of a massive bulge delays the bar formation. A bar forms later and is weaker in models with a more massive and compact bulge and under a more concentrated halo. Bars are shorter and thus rotate faster in models with more massive and compact bulges. Angular momentum transfer from a bar to both halos makes the bar slower and longer over time, although most of the angular momentum lost by the bar is absorbed by the halo. All the bars in our models are slow rotators with \( R = R_{\text{bar}} / R_0 > 1.4 \).

2. B/P bulge and buckling instability. All the models that form a bar undergo disk thickening and eventually develop a B/P bulge. In all models with a bulge, this proceeds secularly as the bulge tends to suppress the bar formation. However, two models (L00 and C00) without a bulge experience buckling instability at \( t \sim 5 \) Gyr during which the bar thickens rapidly. The buckling instability occurs when \( \sigma_z / \sigma_R \) is kept below \( \sim 0.6 \) and involves asymmetric density distribution of the disk across the \( z = 0 \) plane.

3. Conditions for bar formation. Our numerical results for bar formation are not well explained by the single-parameter criteria proposed by previous studies. We instead find that the bar formation in our galaxy models needs to satisfy Equation (13). In models with larger \( Q_{T, \text{min}} \) or larger CMC, the growth of perturbations via swing amplifications combined with feedback loops is too weak to produce bar-supporting \( x_i \) orbits.

4. Classical bulge of the Milky Way. Among our models, the bar at \( t \sim 2.5 \)–3.5 Gyr in model C10 or at \( t \sim 8 \) Gyr in model C20 is matched well with the observed ranges of the bar length and pattern speed in the Milky Way. This suggests that the Milky Way is most likely to possess a classical bulge with mass \( \sim 10\%–20\% \) of the disk mass.

We are grateful to the referee, Dr. Sandeep Kumar Katari, for an insightful report. This work was supported by the grants of National Research Foundation of Korea (2020R1A4A2002885 and 2022R1A2C1004810). Computational resources for this project were provided by the Supercomputing Center/Korea Institute of Science and Technology Information with supercomputing resources including technical support (KSC-2022-CRE-0017).

ORCID iDs

Dajeong Jang https://orcid.org/0000-0002-7202-4373
Woong-Tae Kim https://orcid.org/0000-0003-4625-229X

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