Trapped modes for periodic structures in waveguides

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Abstract

The Laplace operator is considered for waveguides perturbed by a periodic structure consisting of \( N \) congruent obstacles spanning the waveguide. Neumann boundary conditions are imposed on the periodic structure, and either Neumann or Dirichlet conditions on the guide walls. It is proven that there are at least \( N \) (resp. \( N - 1 \)) trapped modes in the Neumann case (resp. Dirichlet case) under fairly general hypotheses, including the special case where the obstacles consist of line segments placed parallel to the waveguide walls.

1 Introduction

Let \( N \) be a positive integer. Consider the region in \( \mathbb{R}^2 \):

\[
\Omega = \left\{ (x, y) : x \in (-\infty, \infty), \ y \in (0, 2N) \right\} - \bigcup_{m=1}^{N} O_m,
\]

with

\[
O_m = \left\{ (x, y), \ x \in [-a, a], \ y \in [2m - 1 - g(x), 2m + 1 + g(x)] \right\}.
\]

Here \( a > 0 \), and \( g \) is a continuous function with \( g(x) \in [0, 1) \) and \( g(\pm a) = 0 \). Thus \( \Omega \) can be viewed as a waveguide with \( N \) congruent obstacles placed periodically along the cross section. In [6], Linton and McIvor studied the trapped modes in such regions under the hypothesis that \( g(x) \) was not identically zero. Assuming Neumann boundary conditions on the obstacles, and
either Dirichlet or Neumann boundary conditions on the guide walls, they proved the existence of at least $N$ trapped modes in the Neumann case, and at least $N-1$ trapped modes in the Dirichlet case. The existence of these trapped modes was indicated earlier in numerical studies by Utsumomiya and Eatock Taylor [8] and Evans and Porter [5]. These studies were motivated by a variety of possible applications to wave propagation in fluid and vibrating membranes; the reader is referred to [6] for a thorough discussion of these.

Although the methods of Linton-McIvor apply for a wide variety on assumptions of the geometry of the structure, they do not apply to the important special case where the structure consists of $N$ identical line segments placed parallel to the guide walls, i.e. $g \equiv 0$.

Furthermore, the numerical results in [8], [5] also fail to indicate any trapped modes in this setting. The main purpose of this note is prove the existence of at least $N$ (resp. $N-1$) trapped modes in this setting for the Neumann (resp. Dirichlet) case. The methods of this paper also apply to the more general periodic structures described by Eqs. 1, 2, and this work might also be of interest because the upper bounds proven here on the associated frequencies will sometimes be sharper than those found in Linton-McIvor.

In addition, we consider the trapped modes of the region

$$\tilde{\Omega} = \{(x,y) : x \in (-\infty, \infty), y \in (0,2N)\} - \bigcup_{m=1}^{N-1} \mathcal{O}_m,$$

(3)

with

$$\mathcal{O}_m = \{(x,y), x \in [-a,a], y = 2m\}.$$  

(4)

For such regions with Neumann boundary conditions both on the obstacles and the guide walls, we prove the existence of $N-1$ embedded eigenvalues. The methods of this paper seem to fail in this case when one has Dirichlet boundary conditions on the guide walls.

We note that a number of papers have proven the existence of at least one trapped mode for the case of a single line segment is placed parallel to the guide walls: see [1], [4], [3], [2], [7], and the references found therein.

To prove the announced results, we use the symmetry of the problem to decompose the ambient Hilbert space into a direct sum of $N+1$ invariant subspaces, as in [8]. However, we differ in our choice of test functions. In the direction transversal to the guide walls, Linton and McIvor’s test function is essentially sinusoidal. In this paper, we choose a test function which has a
jump discontinuity across the obstacles, and which arises naturally from the Hilbert space decomposition.

2 Statement of results and proofs

We define the Laplace operator as

\[ \Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}. \]

Let \( C_b^\infty(\Omega) \) be the smooth functions of bounded support on \( \Omega \). We shall study the self-adjoint operators \( \Delta_1, \Delta_2 \) on \( L^2(\Omega) \), where \( \Delta_1 \) has operator core

\[ \{ u \in C_b^\infty(\Omega) : \frac{\partial u}{\partial \eta}|_{\partial \Omega} = 0 \}, \]

and \( \Delta_2 \) has operator core

\[ \{ u \in C_b^\infty(\Omega) : \frac{\partial u}{\partial \eta}|_{\partial \Omega_j} = 0, j = 1, \ldots, m, \ u|_{y=0} = u|_{y=2N} = 0 \}. \]

Of course, \( \Delta_1 \) is simply the Neumann Laplacian, while we shall refer to \( \Delta_2 \) as the Dirichlet case.

2.1 Neumann boundary conditions

In this section we prove the following:

**Theorem 1** Suppose \( \Omega \) satisfies Eqs. 1, 2. Then, the operator \( \Delta_1 \) has at least \( N \) linearly independant trapped modes. Among the trapped modes, there exist \( N \) whose associated frequencies, denoted \( \mu_m \), satisfy

\[ \mu_m \leq \left( \frac{m\pi}{2N} \right)^2, \ m = 1, \ldots, N. \]

We begin the proof of the theorem by recalling the decomposition, used in [6], of \( L^2(\Omega) \) into \( \Delta_1 \)-invariant subspaces. Suppose \( f \in L^2[0, 2N] \). We extend \( f \) to \( L^2(-\infty, \infty) \) using the rules

\[ f(y) = f(-y), \ f(2N - y) = f(2N + y). \]
Set $f(y) = \sum_{m=0}^{N} f_{m}(y)$, where
\[ f_{m}(y) = \frac{\gamma_{m}^{N}}{2N} \sum_{n=-N}^{N-1} c_{m,n}^{N} f(y + 2n), \tag{5} \]
with
\[ c_{m,n}^{N} = \cos(mn\pi/N), \tag{6} \]
and $\gamma_{m}^{N} = 2/(1 + \delta_{m0} + \delta_{mN})$. Here $\delta_{ij}$ is the Kronecker delta function. It is shown in [6] that $f(y) = \sum_{m=0}^{N} f_{m}(y)$, and in fact
\[ L^{2}(\Omega) = S_{0} \oplus \ldots S_{N}, \]
with the $S_{m}$ the image of the mapping $f \rightarrow f_{m}$. The subspaces $\{S_{m}\}$ are mutually orthogonal and are invariant under the Laplacian. We label $\Delta_{1}|_{S_{m}} = A_{m}$. Then
\[ \inf \sigma_{ess}(A_{m}) = \frac{m^{2}\pi^{2}}{4N^{2}}. \]
Recall the Rayleigh quotient is given by
\[ Q(\phi) = \frac{\int_{\Omega} |\nabla \phi|^{2}}{\int_{\Omega} |\phi|^{2}}, \phi \neq 0, \tag{7} \]
where $\phi$ is in the quadratic form domain of $A_{m}$. For Neumann boundary conditions, the quadratic form domain of $A_{m}$ is
\[ S_{m} \cap H^{1}(\Omega) = \{ u \in S_{m} : |\nabla u| \in L^{2}(\Omega) \}. \]
To prove the existence of eigenvalues below the essential spectrum of $A_{m}$, (and hence the existence of trapped modes for $\Delta_{1}$), it suffices to construct $\phi$ such that $Q(\phi) < \frac{m^{2}\pi^{2}}{4N^{2}}$.

Fix $x$ and $m$, and for notational simplicity set $c_{m,n}^{N} = c_{n}$. We label the intervals $(0, 1 - g(x))$, $(1 + g(x), 3 - g(x))$, $\ldots$, $(2N - 1 + g(x), 2N)$ as $I_{0}, I_{1}, \ldots, I_{N}$ respectively.

Let $\tilde{v}(y) = 1$ on $I_{0}$ and 0 elsewhere. Let $v$ be the image of $\tilde{v}$ under the mapping $f \rightarrow f_{m}$. Then, by Eq. [5],
\[ v(y) = \frac{\gamma_{m}^{N}}{2N} \cos\left(\frac{m\pi}{N} \cdot j \right) \text{ for } y \in I_{j}. \tag{8} \]
Let \( b \in [0, a) \), let \( \alpha > 0 \), and define a piecewise differentiable functions \( \chi \) and \( \psi_\alpha \) on \( \Omega \) by

\[
\chi(x) = \begin{cases} 
0, & |x| > a, \\
1, & |x| < b, \\
\frac{a-x}{a-b}, & x \in (b, a), \\
\frac{a+x}{a-b}, & x \in (-a, -b); 
\end{cases}
\]

\[
\psi_\alpha(x) = \begin{cases} 
e^{-\alpha(|x|-a)}, & |x| > a, \\
1, & |x| \leq a.
\end{cases}
\]

Our test function for Eq. 7 will be:

\[
\phi(x, y) = \chi(x)v(y) + \lambda \psi_\alpha(x) \cos \left(\frac{m\pi y}{2N}\right),
\]

where \( \lambda > 0 \) is a parameter to be chosen later. Since \( v \) is in the image of the mapping \( f \rightarrow f_m \), it follows that \( \chi v \in S_m \). Also, \( \psi_\alpha(x) \cos \left(\frac{m\pi y}{2N}\right) \in S_m \) by [6], Eqs.2.12, 2.13. Thus \( \phi \) is in the quadratic form domain of \( A_m \). This test function can be compared to the one used in [6], Eq.4.13.

In what follows, it is convenient to set

\[
\|v\|_2^2 \equiv |v_0|^2 + 2|v_2|^2 + 2|v_3|^2 + \ldots + 2|v_{N-1}|^2 + |v_N|^2,
\]

where \( v_j \equiv v|_{I_j} \).

**Proposition 1** Let \( p = \frac{m\pi}{2N} \). Then:

\[
\frac{\int |\nabla \phi|^2}{\int |\phi|^2} - \frac{\lambda^2 \alpha N + \frac{a}{a} \int_{x=-a}^a \left( |v||^2(1-g)((\chi')^2 - p^2 \chi^2) - \frac{\lambda C \alpha N^2}{m\pi} \chi \sin(p(1-g))\right) dx}{\lambda^2 N/\alpha + \frac{a}{a} \int_{x=-a}^a \left( \lambda^2 N(1-g(x)) + \frac{C \alpha N^2}{m\pi} \sin\left(\frac{m\pi}{2N}(1-g)\right)\chi + \|v||^2 \chi^2(1-g(x))\right) dx};
\]

here \( C = 4 \) for \( m = 1, \ldots, N-1 \) and \( C = 8 \) for \( m = N \).

The proof this result appears in the appendix.

We now complete the proof of the theorem. By Eq. [3] and the remarks that follow it, it suffices for the right hand side of the last equation to be negative. Note that the denominator is positive, and the same holds for the term \( \int_{|x|<a} \chi(x) \sin(p(1-g(x))) dx \). Choose \( \lambda \) sufficiently large that

\[
\|v\|_2^2 \int_{|x|<a} (1-g)((\chi')^2 - p^2 \chi^2) - \lambda \frac{C \alpha N^2}{m\pi} \int_{|x|<a} \chi \sin(p(1-g)) < 0.
\]
Fixing this $\lambda$, we then choose $\alpha > 0$ so that
\[
\|v\|_2^2 \int_{|x|<a} (1 - g)(\chi'\chi - p^2\chi^2) - \lambda \frac{CN^2p^2}{m\pi} \int_{|x|<a} \chi \sin(p(1 - g)) + \lambda^2 \alpha N < 0.
\]
The theorem is proven.

### 2.2 Dirichlet case

In this section we prove the following:

**Theorem 2** Suppose $\Omega$ satisfies Eqs. [1, 2]. Then, the operator $\Delta_2$ has at least $N - 1$ linearly independent trapped modes. Among the trapped modes, there exist $N - 1$ whose associated frequencies $\mu_j$ satisfy
\[
\mu_m \leq \left(\frac{m\pi}{2N}\right)^2, \ m = 1, \ldots, N - 1.
\]

For Dirichlet boundary conditions, we extend $f \in L^2(0, 2N)$ to $L^2(-\infty, \infty)$ via the equations
\[
f(-y) = -f(y), \ f(2N + y) = -f(2N - y).
\]

Then, using Eq. 4 as in the Neumann case, we have the decomposition $L^2(\Omega) = \bigoplus_{m=0}^{N} S_m$, where the $S_m$ are mutually orthogonal and $S_m$ are invariant under $\Delta_2$. Setting $A_m \equiv \Delta_2|_{S_m}$, we have
\[
\inf \sigma_{ess}(A_m) = \begin{cases} 
\pi^2, & m = 0 \\
\left(\frac{m\pi}{2N}\right)^2, & m = 1, \ldots, N.
\end{cases}
\]

As in the Neumann case, we construct a test function for the Rayleigh quotient. In this case the quadratic form domain for $A_m$ is the closure in $H^1(\Omega)$-norm of the set
\[
\{u \in S_m \cap C^\infty_b(\Omega) : \text{support}(u) \cap \{y = 0\} = \text{support}(u) \cap \{y = 2N\} = \phi\}.
\]

Let $\{I_j\}$ be as in the Neumann case. Let
\[
\tilde{v}_j(y) = \begin{cases} 1, & y \in I_j, \\
0, & \text{elsewhere}.
\end{cases}
\]

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Denote by $v_j$ the image of $\tilde{v}_j$ under the mapping $f \to f_m$. Then, for $j = 1, \ldots, N - 1$, we have by Eq. 5,

$$v_j(y) = \frac{\gamma_N}{2N} (c_{j-s} - c_{-j-s}), \text{ for } y \in I_s$$

$$= \frac{\gamma_N}{N} \sin\left(\frac{m\pi}{N} \cdot j\right) \sin\left(\frac{m\pi}{N} \cdot s\right) \text{ for } y \in I_s. \quad (9)$$

Here we have used $c_J = c_{-J}$, $c_{J-2N} = c_J$, and the identity \(\cos(A - B) - \cos(A + B) = 2 \sin A \sin B\).

We chose $j$ so that $\sin\left(\frac{m\pi}{N} \cdot j\right) \neq 0$. Thus $v_j \equiv 0$ if and only if $m = 0$ or $m = N$.

The test function is defined as:

$$\phi(x, y) = \chi(x)v_j(y) + \lambda \psi_\alpha(x) \sin\left(\frac{m\pi y}{2N}\right),$$

where $\lambda > 0$ is a parameter to be chosen later, and $\chi, \psi_\alpha$ as in the Neumann case. Note that $\chi v_j \in S_m$, and also that $\chi v_j$ vanishes at $y = 0, y = 2N$ (see Eq. 6). Hence, $\chi v_j$ is in the quadratic form domain of $A_m$. Also, by (4, Eq.2.15), $\psi_\alpha(x) \sin\left(\frac{m\pi y}{2N}\right)$ is in the quadratic form domain of $A_m$, and hence $\phi$ is in the quadratic form domain of $A_m$.

The proof of the theorem now follows from a word to word repetition of the argument used in the Neumann case.

### 2.3 Line segments placed on $y = 2, 4, \ldots, 2N - 2$

In this section, we prove

**Theorem 3** Suppose $\Omega$ satisfies Eqs. 3, 4. Then, the operator $\Delta_1$ has at least $N - 1$ linearly independent trapped modes. Among the trapped modes, there exist $N - 1$ whose associated frequencies $\mu_m$ satisfy

$$\mu_m \leq \left(\frac{m\pi}{2N}\right)^2, \quad m = 1, \ldots, N - 1.$$  

First, we note that the region $\Omega$ under these hypotheses satisfies the conditions necessary for the decomposition

$$L^2(\Omega) = S_0 \oplus \ldots S_N,$$
with $f \rightarrow f_m$ defined exactly as above (see [3], p.3).

We label the intervals $(0,2), (2,4), \ldots, (2N-2,2N)$ as $I_1, I_2, \ldots, I_N$ respectively. Let $\tilde{v} = 1$ on $I_1$, and $\tilde{v} = 0$ elsewhere.

Denote by $v$ the image of $\tilde{v}$ under the mapping $f \rightarrow f_m$. Then we have by Eq. 5,

$$v(y) = \frac{\gamma^N}{2N}(c_{-1-s} + c_{-s}), \text{ for } y \in I_s$$

$$= \frac{\gamma^N}{N} \cos\left(\frac{m\pi}{N} \cdot \frac{1}{2}\right) \cos\left(\frac{m\pi}{N} \cdot \left(\frac{1}{2} + s\right)\right), \text{ for } y \in I_s.$$

Here we have used $\cos(A - B) + \cos(A + B) = 2 \cos A \cos B$.

Note that $v \equiv 0$ if and only if $m = N$.

The test function is defined as:

$$\phi(x,y) = \chi(x)v(y) + \lambda \psi_\alpha(x) \cos\left(\frac{m\pi y}{2N}\right),$$

where $\lambda > 0$ is a parameter to be chosen later, and $\chi, \psi_\alpha$ as in the Neumann case. The proof of the theorem now follows by mimicking the argument used in the Neumann case.

### 3 Appendix

In this section we prove Proposition 1.

We use the notation of Section 2.1. In what follows, we denote by $\{ |x| < a \}$ the set $\{ (x,y) : |x| < a \}$. We have

$$\int_{\{ |x| > a \}} |\phi|^2 = 2\lambda^2 \int_{x=-a}^{x=a} \int_{y=0}^{2N} \cos^2\left(\frac{m\pi y}{2N}\right) e^{-2\alpha(x-a)} \ dy \ dx$$

$$= \lambda^2 N/\alpha.$$

Next, we calculate:

$$\int_{\{ |x| < a \}} |\phi|^2 = \lambda^2 \int_{\{ |x| < a \}} \cos^2\left(\frac{m\pi y}{2N}\right) + \int_{\{ |x| < a \}} v^2 \chi^2 + 2\lambda \int_{\{ |x| < a \}} \chi v \cos\left(\frac{m\pi y}{2N}\right).$$

The first of the integrals on the right hand side of the last equation we compute as follows:

$$\int_{\{ |x| < a \}} \cos^2\left(\frac{m\pi y}{2N}\right) = \int_{-a}^{a} \left( \int_{y=0}^{y=1-g(x)} \cos^2\left(\frac{m\pi y}{2N}\right) \ dy \right)$$
\[
+ \sum_{n=1}^{N-1} \int_{y=2n-1+g(x)}^{2n+1-g(x)} \cos^2 \left( \frac{m\pi y}{2N} \right) dy + \int_{2N-1+g(x)}^{2N} \cos^2 \left( \frac{m\pi y}{2N} \right) dy \right) dx
= N \int_{x=-a}^{a} \left( (1 - g(x)) + \frac{\sin \left( \frac{m\pi (1-g)}{N} \right)}{m\pi} \cdot \left( \sum_{i=0}^{N-1} \cos \left( \frac{m2\pi i}{N} \right) \right) \right) dx
= N \int_{x=-a}^{a} (1 - g(x)) dx.
\] (10)

Here we have used the identity \( \sum_{i=0}^{N-1} \cos \left( \frac{2\pi i}{N} \right) = 0 \), which follows from \( \sum_{j=0}^{N-1} e^{2ij\pi/N} = 0 \).

For the third integral on the right hand side of Eq. (10), a similar computation yields
\[
\int_{\{|x|<a\}} v^2 \chi^2 = \|v\|^2 \int_{-a}^{a} \chi^2 (1 - g(x)) dx.
\]

Thus,
\[
\int_{\Omega} |\phi|^2 = \lambda^2 N/\alpha + \int_{x=-a}^{a} \left\{ \lambda^2 N (1 - g(x)) + \frac{C\lambda N^2}{m\pi} \sin \left( \frac{m\pi}{2N} (1 - g) \right) + \|v\|^2 \chi^2 (1 - g(x)) \right\} dx.
\] (11)

Here, \( C = 8 \) for \( m = 0, N \) and \( C = 4 \) otherwise.

In what follows, it is convenient to set \( \frac{m\pi}{2N} = p \). Then
\[
\int_{\Omega} |\nabla \phi|^2 = \left( \lambda^2 p^2 \sin^2(py) \psi'_{\alpha}^2 + \lambda^2 \cos^2(py) (\psi'_{\alpha})^2 \right) + v^2 (\chi')^2 + 2\lambda v \cos(py) \psi'_{\alpha} \chi'.
\] (12)

The fourth integrand is identically zero because \( \psi'_{\alpha} \chi' = 0 \). Next, we calculate the first three terms in Eq. (12). First,\[
\lambda^2 p^2 \int_{\Omega} \sin^2(py) \psi_{\alpha}^2 = \lambda^2 p^2 \int_{\{|x|<a\}} (1 - \cos^2(py)) + 2\lambda^2 p^2 \int_{x=a}^{\infty} e^{-2\alpha(x-a)} \sin^2(py) \ dy \ dx
\]
\[ \int_{x=a}^{x=-a} (1 - g(x)) \, dx + \lambda^2 p^2 N/\alpha; \]

here we have used Eq. 10. Similarly,

\[ \lambda^2 \int_\Omega \cos^2(py)(\psi'_\alpha)^2 = \lambda^2 N\alpha, \]

and

\[ \int_\Omega v^2(\chi')^2 = \|v\|_2^2 \int_{|x|<a} (\chi')^2(1 - g(x)) \, dx. \]

Thus

\[ \int_\Omega |\nabla \phi|^2 = \lambda^2 p^2 N \int_{|x|<a} (1-g(x)) \, dx + \lambda^2 p^2 N/\alpha + \lambda^2 N\alpha + \|v\|_2^2 \int_{|x|<a} (\chi')^2(1-g(x)) \, dx. \]  

The proposition now follows from Eqs. 13, 11.

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