Momentum Imparted by Gravitational Waves

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Abstract

We calculate momentum imparted by colliding gravitational waves in a closed Friedmann Robertson-Walker background and also by gravitational waves with toroidal wavefronts using an operational procedure. The results obtained for toroidal wavefronts are well behaved and reduce to the spherical wavefronts for a special choice.

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1 Introduction

The localization of energy and momentum has been one of the most interesting and important problems in Einstein’s theory of General Relativity (GR). The different attempts at constructing an energy-momentum density do not provide a generally accepted definition. Consequently, different people have different points of view. In one of his papers Cooperstock [1] argued that in GR, energy and momentum are localized in regions of the non-vanishing energy and momentum tensor. As a result gravitational waves are not carriers of energy and momentum in vacuum. Since the gravitational waves, by definition, have zero stress-energy tensor. Thus the natural question about the existence of these waves was raised. However, the theory of GR indicates the existence of gravitational waves as solutions of Einstein’s field equations [2]. This contrary result arises because energy is not well defined in GR.

Ehlers and Kundt [3], Pirani [4] and Weber and Wheeler [5] resolved this problem for gravitational waves by considering a sphere of test particles in the path of the waves. They showed that when gravitational waves passed through the test particles these acquired a constant momentum from the waves. However, the procedure [6] does not provide a simple prescription that can be used for arbitrary case. Qadir and Sharif [7] used an operational procedure, embodying the same principle and found a closed form formula which can be applied to arbitrary spacetimes. This procedure provided the similar results when we applied to plane and cylindrical gravitational waves [7]. In a recent work [8], this approach has been used to find the momentum imparted by gravitational waves with spherical wavefronts. Interestingly, we obtain physically acceptable results which coincide with results obtained by using Möller’s prescription. This paper focuses the problem of finding the momentum imparted by colliding gravitational waves in a closed Friedmann Robertson-Walker (FRW) background and also by gravitational waves with toroidal wavefronts.

We use an operational procedure for expressing the consequences of Relativity in terms of the Newtonian concept of gravitational force. The pseudo-Newtonian gravitational force is defined as the vector whose intrinsic derivative along the separation vector is the maximum tidal force, which is given by the acceleration vector for preferred class of observers. We will not discuss this formalism here as it is available in detail elsewhere [9]. We shall restrict ourselves to briefly mention the essential points of the formalism. In the free fall rest-frame the extended pseudo-Newtonian four-vector force can be written in the form [7,9]

\[ F_0 = m\{\ln(A/\sqrt{g_{00}})\}_{,0} - g_{ij,0}g^{ij}/4A, \quad F_i = m(\ln\sqrt{g_{00}})_{,i}, \]  \(1\)

where \( A = (\ln \sqrt{\gamma})_{,0}, \quad g = \text{det}(g_{ij}), \quad i, j = 1, 2, 3. \) This force formula is not uniquely fixed rather than it depends on the choice of frame. We define the quantity whose proper time derivative is \( F_a, \ (a = 0, 1, 2, 3) \) as the momentum four-vector of the test particle. Its spatial components give the momentum imparted to test particles in the preferred frame (where \( g_{00} = 0 \)). Thus the
momentum four-vector, denoted by $p_a$, is given as \[ \tag{2} p_a = \int F_a dt, \]

The rest of the paper is organized as follows. In the next section, we shall describe the colliding gravitational waves in a closed FRW background and toroidal waves. In section three, we use the operational procedure to evaluate momentum imparted to test particles by these waves. Finally, section four is devoted to a brief summary of the results and concluding remarks.

2 Gravitational Waves

2.1 Colliding Gravitational Waves

The general line element describing the gravitational waves can be written in the form \[ \tag{3} ds^2 = e^{-M}(dt^2 - dx^2) - e^{-U}(e^{-V}dy^2 + e^V dz^2), \]

where $U$, $V$ and $M$ are functions of $t$ and $x$.

In the presence of a stiff fluid or a scalar field, the field equations imply that $e^{-U}$ satisfies the wave equation

\[ (e^{-U})_{tt} - (e^{-U})_{xx} = 0. \]

It is well known that the fluid can be defined in terms of a potential function $\sigma(t, x)$ which satisfies the wave equation

\[ (e^{-U}\sigma_t)_t - (e^{-U}\sigma_x)_x = 0. \]

In terms of this potential, the fluid density is given by

\[ 32\pi\rho = e^M(\sigma_t^2 - \sigma_x^2) \]

and its 4-velocity is proportional to the gradient of $\sigma$.

The main gravitational field equation takes the form

\[ (e^{-U}V_t)_t - (e^{-U}V_x)_x = 0 \]

and the remaining equations, which are now automatically integrable, become

\[ U_t M_t + U_x M_x + U_{tt} + U_{xx} = \frac{1}{2}(U_t^2 + U_x^2 + V_t^2 + V_x^2 + \sigma_t^2 + \sigma_x^2), \]

\[ U_x M_t + U_t M_x + 2U_{tx} = U_t U_x + V_t V_x + \sigma_t \sigma_x. \]

In principle, these equations can always be integrated for $M$. Techniques for obtaining exact solutions of these equations have been given by Carmeli et al [11].
It may be noted that the solution describing the closed FRW stiff fluid model can be given by

\[ e^{-U} = \sin 2t \sin 2x = \sin^2(t + x) - \sin^2(t - x), \quad V = \ln \tan x, \]

\[ M = -\ln \sin 2t - \ln \gamma, \quad \sigma = \sqrt{3} \ln \tan t, \quad (9) \]

where \(0 < t < \pi/2\) and \(0 < x < \pi/2\). In this spacetime, we can define two null hypersurfaces \(t - x + a = 0\) and \(t + x - b = 0\), where \(0 < a < \pi/4\) and \(\pi/4 < b < \pi/2\). These can be taken to be wavefronts of approaching gravitational waves which then collide when \(t = \frac{1}{2}(b - a)\) and \(x = \frac{1}{2}(a + b)\). The FRW solution (9) is taken only as the background region into which the waves propagate.

### 2.2 Toroidal Gravitational Waves

The general line element of the form given by Eq.(3) can be written in cylindrical coordinates as

\[ ds^2 = e^{-M}(dt^2 - d\rho^2) - e^{-U}(e^{-V}d\phi^2 + e^{V}dz^2), \quad (10) \]

where \(U, V\) and \(M\) are functions of \(t\) and \(\rho\). As a result, the vacuum field equations imply that \(e^{-U}\) satisfies the wave equation

\[ (e^{-U})_{tt} - (e^{-U})_{\rho\rho} = 0. \quad (11) \]

The function \(V\) satisfies the linear equation

\[ V_{tt} - U_t V_t - V_{\rho\rho} + U_{\rho}V_{\rho} = 0. \quad (12) \]

and the remaining equations for \(M\) are

\[ U_{tt} - U_{\rho\rho} = \frac{1}{2}(U_t^2 + U_{\rho}^2 + V_t^2 + V_{\rho}^2) - U_tM_t - U_{\rho}M_{\rho}, \]

\[ 2U_{t\rho} = U_t U_\rho - U_t M_\rho - U_\rho M_t + V_t V_\rho. \quad (13) \]

If Eqs.(11) and (12) are satisfied then Eq.(13) can always be integrated for \(M\).

We can now consider a gravitational wave with toroidal wavefront by putting [12]

\[ U = -\ln t - \ln \rho, \quad V = \ln t - \ln \rho + \tilde{V}(t, \rho), \quad (14) \]

where \(\tilde{V}\) takes the form

\[ \tilde{V}(t, \rho) = \int_{1/2}^{\infty} \phi(k)(t\rho)^k H_k\left(\frac{t^2 + \rho^2 - a^2}{2t\rho}\right) dk \quad (15) \]

with an arbitrary function \(\phi(k)\). Clearly these solutions reduce to the spherical-fronted waves of [13] when \(a = 0\).
3 Momentum Imparted to Test Particles

In this section, we use the procedure outlined in the first section to calculate momentum imparted to test particles by colliding gravitational waves in the closed FRW spacetime background and also the gravitational waves with toroidal wavefronts. Using the four-vector force formula, given by Eq.(1) in Eq.(3), we have

$$F_0 = m\left[\dot{U} + \frac{\dot{M} + 2\dot{U}}{M + 2U} - \frac{3\dot{U}^2 + \dot{V}^2}{M + 2U}\right], \quad F_1 = -\frac{m}{2}M', \quad F_2 = 0 = F_3. \quad (16)$$

The corresponding four-vector momentum takes the form

$$p_0 = m[U + \ln(\dot{M} + 2\dot{U}) - \int \frac{3\dot{U}^2 + \dot{V}^2}{M + 2U}dt] + f_1(x), \quad (17)$$

$$p_1 = -\frac{m}{2} \int M' dt + f_2(x), \quad p_2 = \text{constant} = p_3. \quad (18)$$

where $f_1(x)$ and $f_2(x)$ are arbitrary integration functions of coordinate $x$. Here dot and prime indicate derivatives with respect to time $t$ and coordinate $x$ respectively. It is to be noticed that Eqs.(17) and (18) give the general result of momentum four-vector for colliding and toroidal gravitational waves. Since we are interested to find out the momentum imparted by these gravitational waves, we would use Eq.(18) to calculate the term $p_1$. As, it is obvious from this expression, we need to have the value of $M$.

3.1 Momentum Imparted by Colliding Gravitational Waves

The solution describing the closed FRW stiff fluid model is given by Eq.(9). Substituting these values in Eqs.(17) and (18), we obtain

$$p_0 = -m \ln(-6 \cot 2t \sin 2x) + f_1(x), \quad p_1 = \text{constant}. \quad (19)$$

We see that the momentum imparted to test particles becomes constant which can be made zero for a special choice of constant to be zero.

3.2 Momentum Imparted by Toroidal Gravitational Waves

The solution in the wave region ($t \geq \rho - a$) can be found by solving Eqs.(11) and (12) and is given in the form [12]

$$U = -\ln t - \ln \rho, \quad V = \ln t - \ln \rho + \tilde{V}(t, \rho), \quad (20)$$

Using Eq.(20), we can write the equations for $M$, for $t \geq \rho - a$, in the form

$$M_t + M_\rho = \left(\frac{t-\rho}{t+\rho}\right)(\tilde{V}_t + \tilde{V}_\rho) - \frac{t\rho}{2(t+\rho)}(\tilde{V}_t + \tilde{V}_\rho)^2, \quad (21)$$

$$M_t - M_\rho = \left(\frac{t+\rho}{t-\rho}\right)(\tilde{V}_t - \tilde{V}_\rho) + \frac{t\rho}{2(t-\rho)}(\tilde{V}_t - \tilde{V}_\rho)^2.$$
We now consider the case of a single component for which \(V(t, \rho)\) becomes

\[
\tilde{V}(t, \rho) = a_k(t\rho)^k H_k\left(\frac{t^2 + \rho^2 - a^2}{2t\rho}\right),
\]

where \(k\) is an arbitrary real parameter such that \(k \geq \frac{1}{2}\), \(a_k\) is constant and \(t^2 + \rho^2 - a^2 = 2t\rho\). The parameter \(k\) can also be complex, but in this case one should consider for \(V\) the real or imaginary part of the right hand side of Eq.(22).

By substituting Eq.(22) into the transformed form of Eq.(12), it is found that the functions \(H_k(\frac{t^2 + \rho^2 - a^2}{2t\rho})\) must satisfy the linear ordinary differential equation which can be reduced to a hypergeometric equation. The condition that \(\tilde{V} = 0\) can be expressed by the constraint \(H_k(1) = 0\). It can be seen that the outgoing toroidal wave includes an impulsive component if the lowest term in the expansion for \(V\) has \(k = \frac{1}{2}\). The front of gravitational wave has a step (or shock) if the lowest term has \(k = \frac{1}{2}\). The value of \(M\) for the wave region \(t \geq \rho - a\) turns out to be

\[
M = \frac{1}{2k}a_k(t^2 - \rho^2)(t\rho)^{k-1}H_{k-1} - \frac{1}{2k}(t\rho)^{2k}a_k^2[H_k^2 - \frac{(t^2 - \rho^2)^2}{4t^2\rho^2}H_{k-1}^2].
\]

It is mentioned here that the dimension of \(a_k\) is \(L^{-2k}\). To simplify the problem, we choose a particular value of \(k\), i.e., \(k = 1\) for which the above equation reduces to

\[
M = \frac{1}{2}a_1(t^2 - \rho^2)H_0 - \frac{1}{2}(t\rho)^2a_1^2[H_1^2 - \frac{(t^2 - \rho^2)^2}{4t^2\rho^2}H_0^2],
\]

where

\[
H_0 = \ln\left(\frac{t}{\rho - a}\right), \quad H_1 = \frac{1}{2}\left(\frac{t}{\rho - a} + \frac{\rho - a}{t}\right)\ln\left(\frac{t}{\rho - a}\right) - \left(\frac{t}{\rho - a} - \frac{\rho - a}{t}\right).
\]

Differentiating this value of \(M\) with respect to \(\rho\) and substituting in Eq.(18), then after some algebra, we arrive at the following

\[
p_1 = ma_1\left[\frac{t^3}{4(3(\rho - a))} - \frac{(\rho - a) + 2t\rho}{\rho - a}t + 2t\rho\ln\left(\frac{t}{\rho - a}\right)\right] + ma_1^2\left[\frac{1}{4}\frac{t}{50(\rho - a)}(\rho - a) \right]
\]

\[
- \frac{7t^2}{3(\rho - a)} - 1)t^5 + \frac{21}{9}(\rho - a) - \frac{5}{6})t^3 + \frac{1}{2}(1)(\rho - a) - (\rho - a)^2 - \frac{\rho^2}{\rho - a}
\]

\[
-4\rho^2 t - \left(\frac{11}{5(\rho - a)}\right)\left(\frac{t}{5(\rho - a)}\right) - \frac{7t^2}{10(\rho - a)^2} - \frac{1}{2}t^5 + \frac{2}{3}(\rho - a) + \frac{2}{3})t^3
\]

\[
+ \frac{1}{2}(\rho - a - \frac{\rho^2}{\rho - a} - 4\rho)t\ln\left(\frac{t}{\rho - a}\right) - \left(\frac{\rho}{10(\rho - a)^2}\right)(\rho - a) - 1)t^5 - \frac{2}{3}t^3 \rho
\]

\[
- \frac{1}{2}(\rho - a) + (\rho - a)^2 - 2\rho^3 t)(\ln\left(\frac{t}{\rho - a}\right))^2 + f_2(\rho).
\]

This is the momentum imparted to test particles by gravitational waves with toroidal wavefronts. We see that the momentum term approaches to zero in
the limit $t \to 0$ for the particular choice of $f_2 = 0$. This result reduces to the momentum imparted by gravitational waves with spherical wavefronts for $a = 0$ [8]. The interpretation of the term $p_0$ of the four-vector momentum has been discussed in detail elsewhere [14].

The character of the gravitational wave near the wavefront can be determined by the component of $\tilde{V}$ with the minimum value of $k$. It can be shown that, near the wavefront as $t \to \rho - a$,

$$\tilde{V} \sim a_k(t + \rho)^{-1}(t - \rho + a)^{1+2k}, \quad M \sim ((1 + 2k)/2k)a_k(t - \rho + a)^{2k}. \quad (26)$$

Making use of this value of $M$ in Eq.(18), we obtain

$$p_1 = \frac{m}{4k}(1 + 2k)a_k(t - \rho + a)^{2k} + f_2(\rho). \quad (27)$$

This is the momentum imparted to test particles near the toroidal wavefront.

4 Discussion

We have evaluated momentum imparted to test particles by colliding gravitational waves in a closed FRW background and also for toroidal gravitational waves by using an entirely different approach. It can be seen that the momentum becomes constant for colliding gravitational waves in the closed FRW background. This is what one would expect from this procedure. It also follows from Eq.(25) that the momentum imparted by toroidal gravitational waves is physically well behaved. This turns out to be zero in the limit $t \to 0$ for a particular value of arbitrary function. We also see that this becomes singular at $\rho = a$. If we take $a = 0$, it reduces to the result of spherical wavefronts [8] and the singularity shifts to $\rho = 0$ which also acts as a source of the gravitational waves inside the wave region. It is interesting to note that all these results exactly coincide with the results obtained by using Møller prescription [15] for a particular choice of arbitrary function $f_2$. 


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