Global Stabilization of Controlled Nonlinear System "Inverted Pendulum on a Cart" Using Method of Two Lyapunov Functions

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In this work a mathematical apparatus for method of several global Lyapunov functions is applied to study the stability properties of nonlinear model of concrete singularly perturbed mechanical system "inverted pendulum on a cart" with discontinuous relay-type control. In this analysis, the concept of solution in sense of Gelig et al. [8] is used.

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I. INTRODUCTION

The nonlinear controlled model system "inverted pendulum on a cart" is one of intensively studied last time (see, e.g. [1]). The results of investigation of this system are applicable at analysis of a number of concrete problems of stability, e.g. monorail [2], fault tolerant control [3], inverted pendulum on a rotating arm as well as satellites and underwater vehicle with internal rotors etc. [4].

In the work [5] it was performed a study of stability of nonlinear system "inverted pendulum on a controlled cart" for a number of cases. There the control was continuous and was taken as sum of the controls

\[ u = \sum_i \lambda_i u_i, \]

and it was applied to the cart moving on the plane surface.

In recent work of Bloch et al. [4], it was performed the investigation of stabilization of this mechanical system using the method of controlled Lagrangian. The control law was obtained in the form

\[ u = \frac{k b \sin \beta (a \dot{\beta}^2 + c \cos \beta)}{a - b^2/\bar{m}(1 + k) \cos \beta^2}, \]

where \( b = ml, c = -mgl \). \( m \) is the pendulum mass, \( l \) is the pendulum length, \( g \) is the acceleration due to gravity, \( \bar{m} = M + m \), \( M \) is the cart mass. And equilibrium \( \beta = \dot{\beta} = r = \dot{r} = 0 \) is achieved if dimensionless constant

\[ k > \frac{a \bar{m} - b^2}{b^2} = \frac{M}{m} > 0 \]

The control force acts to a cart while there are no direct action to a pendulum. At analysis it is used a character of the Lie symmetry group for nonlinear system, which in the case of plane system "inverted pendulum on a controlled cart" appears to be that of translation.

Also in that work it is studied an asymptotic stabilization of the system "inverted pendulum on a controlled cart", and control under investigation is brokeed into "conservative" and "dissipative" piece

\[ u = u^{\text{cons}} + u^{\text{dis}}, \]

which are analysed separately.

Introduced in the work Lyapunov function for system with control is taken so that its total time derivative was non-negative elsewhere \( (dV/dt \leq 0) \) and disappeared at the set \( M \), which is determined by equating to zero of dissipative control component \( u^{\text{dis}} = 0 \).

As follows from simulation results for concrete parameters of nonlinear system "inverted pendulum on a controlled cart" (for given stabilizing action with added dissipation), obtained using the MATLAB system, the pendulum begins the motion from almost horizontal position what indicates that attraction region is large enough, even at given positive (down) initial velocity. Initial position of the cart is \( s(0) = 0 \), and initial velocity is \( \dot{s} = -3 \text{ m/sec} \). The cart comes to an equilibrium state in the position which is far enough from the initial one. The control law was taken with an initial peak to provide the initial large acceleration. The Lyapunov function is negative in the beginning and then it strictly increases up to zero at the equilibrium.
In present work, it is performed a detailed analysis of the behavior of the controlled mechanical system "inverted pendulum on a cart" with discontinuous relay-type control on the basis of the method of investigation of the global asymptotic stability of nonlinear dynamical systems with using of two Lyapunov functions. The distinct feature of such analysis from the case of continuous right-hand sides is that total time derivatives of the Lyapunov functions appears to be also discontinuous. Because of this to study the global stability of this system it is used the method of two Lyapunov functions (see, e.g. [6]) which permits to perform the stability analysis for discontinuous systems, in particular, for the case of mechanical systems with dry friction (see, e.g. [8]). At such analysis the solution is considered in the sense of Gelig et al. [8].

II. DYNAMICS OF CONTROL OBJECT IN NEW VARIABLES

It is considered the problem of global stabilization of inverted pendulum on a controlled cart in the presence of unknown disturbance acting at cart. This mechanical dynamic system is described by nonlinear system of differential equations in the form (see, e.g. Brusin et al. [7])

\[
\begin{align*}
M \ddot{r} + L \ddot{\beta} \cos \beta + N \dot{r} - L \dot{\beta}^2 \sin \beta &= Gu(t) + D(t) \\
L \cos \beta \ddot{\beta} + I \ddot{\beta} + c \dot{\beta} + \kappa \cos \beta - L g \sin \beta &= 0,
\end{align*}
\]

(5)

where \(D(t)\) is uniformly bounded \((D(t) \leq \Pi)\) where \(\Pi\) is known value, continuous function of \(t\), describing the external action at cart; \(\bar{m} = M + m, L = ml, c = c_0 + \kappa l, I = J + ml^2\) where \(M > 0, m > 0\) are the cart and the pendulum masses, respectively. \(N > 0\) and \(\kappa > 0\) are the coefficients of friction force for motion of cart and pendulum, respectively; \(c_0\) is the coefficient of elastic force moment for rotation friction of the pendulum; \(g\) is the acceleration due to gravity; \(G > 0\) is the coefficient of amplification of motor; \(J > 0\) is the inertion moment for pendulum relative to the mass center; \(r\) is the coordinate of cart mass center; \(|\beta| < \pi/2\) is the angle between pendulum axis and the vertical, numbered from vertical unstable positon of pendulum; \(u\) is the value of controlled signal of regulator.

The purpose of control is to reduce controlled "cart" in asymptotics to given position and attached "pendulum" in vertical position (at which its gravity center is situated above fixed point) from any initial position (such that \(|\beta| < \pi/2\) of "cart" and "reversed pendulum" in the presence of immeasurable perturbation at cart, i.e. \(\beta \to 0, r \to 0\) as \(t \to \infty\).

Let introduce new variables \((s, \dot{s}, \Omega, \dot{\Omega})\) for state vector of the object of control using relations

\[
s = r + \rho \Omega(\beta), \quad \Omega(\beta) = -\ln |\tan(\frac{\pi}{4} - \frac{\beta}{2})|, \beta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \rho > 0.
\]

(6)

\[
\beta = 2(\pi/4 - \arctg(exp(-\Omega))).
\]

(7)

The variables \((s, \dot{s})\) are connected due to system (5) by the following equation [2]

\[
(p + a)A \psi(\beta)(s + \dot{s}) + bs = \sum_{i=1}^{11} \lambda_i u_i + b(\dot{s} + \alpha s) - u - \frac{D(t)}{G},
\]

(8)

\[
\psi(\beta) = MI - L^2 \cos^2 \beta,
\]

where \(p = d/dt; a > 0, \alpha > 0, b \geq 0\) are some numbers. \(A = |G(I - \rho L)| > 0\); \(u_i\) are the continuous functions of new variables

\[
\begin{align*}
\{ u_1 &= \dot{r}, \quad u_2 = \dot{r} \cos^2 \beta, \quad u_3 = \dot{\beta}^2 \sin \beta, \quad u_4 = \dot{\beta} \cos \beta, \quad u_5 = \sin 2 \beta, \\
\{ u_6 &= \dot{\beta} / \cos \beta = \Omega, \quad u_7 = \dot{t} \dot{\beta}, \quad u_8 = a \dot{\beta} + \rho \dot{\beta}^2 \sin \beta / \cos \beta, \\
\{ u_9 &= (\dot{\beta} \sin 2 \beta - \cos^2 \beta) \dot{s} - \rho \dot{\beta}^2 \sin \beta, \quad u_{10} = \dot{s} + \alpha s, \quad u_{11} = (p + a)(\cos^2 \dot{\beta}) = (\dot{s} + \alpha s) \cos^2 \beta - \rho \sin 2 \beta.
\end{align*}
\]

(9)
Here functions of old variables are expressed via new ones with taking into account the relations directly following from (6)

\[ \dot{r} = \dot{s} - \rho \dot{\Omega}(\beta), \quad \dot{\beta} = \dot{\Omega} \cos\beta, \] (10)

and angle \( \beta \) is expressed from (7).

The constants \( \lambda_i \) is in the form

\[ \lambda_1 = \frac{\rho M \kappa + I N - \rho N L}{A}; \quad \lambda_2 = -\frac{L \kappa}{A}; \quad \lambda_3 = \frac{L}{G}; \]

\[ \lambda_4 = -\frac{L c}{A}; \quad \lambda_5 = -\frac{L^2 g}{2A}; \quad \lambda_6 = -\frac{\rho M c}{A}; \quad \lambda_7 = -\frac{\rho L M g}{A}; \]

\[ \lambda_8 = -\frac{L M I}{A}; \quad \lambda_9 = \frac{\Gamma^2}{A}; \quad \lambda_{10} = -\frac{L I}{A}; \quad \lambda_{11} = \frac{L^2}{A}. \] (11)

Further, if the relations

\[ \ddot{r} = \ddot{s} - \rho \ddot{\Omega}, \quad \ddot{\beta} = \ddot{\Omega} \cos\beta - \dot{\Omega}^2 \sin\beta \cos\beta, \]

following from (10), will be taken into account, then equation for \( \Omega \) can be expressed in new variables as

\[ \ddot{\Omega} + d_1 \dot{\Omega} + d_2 t g \beta + d_3 \dot{\Omega}^2 \sin\beta = \frac{L \ddot{s} + \kappa \dot{s}}{\rho L - I} \] (12)

where new designations

\[ d_1 = \frac{\kappa \rho - c}{\rho L - I}, \quad d_2 = \frac{L g}{\rho L - I}, \quad d_3 = \frac{I}{\rho L - I} \] (13)

are introduced. The obtained equations (8) and (12) describe the behavior of initial system in new variables.

Let take the control law in the form

\[ u(t) = \bar{u}(t) + \Delta u(t), \] (14)

where function

\[ \bar{u}(t) = \sum_{i=1}^{11} \lambda_i u_i(t) \] (15)

is taken so that right-hand side of (8) was equal to zero if \( \Delta u(t) = 0 \). \( \Delta u(t) \) - correcting term to compensate perturbation \( D(t) \), which is a piecewise function with a jump discontinuity along some smooth surface.

Then (8) for closed system at \( b = 0 \) will be wrote in the form

\[ (p + a) \left[ \frac{1}{A} \psi(\beta)(\dot{s} + \alpha s) \right] = -\Delta u(t) - \frac{D(t)}{G}. \] (16)

Let denote

\[ \psi(\beta)(\dot{s} + \alpha s) = \gamma(t), \] (17)

then obtain equation for \( \dot{s} \)

\[ \dot{s} = \frac{\gamma}{\psi(\beta)} - \alpha s. \] (18)

Substitution of (17) in (16) results in

\[ (p + a) \left[ \frac{\gamma}{A} \right] = -\Delta u(t) - \frac{D(t)}{G}, \] (19)
\[
\dot{\gamma} = -a\gamma - A\Delta u(t) - BD(t),
\]  
(20)

\[B = A/G.\]

The equations for \(y = \text{col}(\Omega, \dot{\Omega})\) (see (12) can be obtained, if express \(\dot{s}\) from (18)

\[
\dot{s} = \frac{\ddot{s}\psi - \gamma\dot{\psi}}{\psi^2} - \alpha \dot{s}
\]  
(21)

or, with taking into account (18)

\[
\dot{s} = \frac{\ddot{s}\psi - \gamma\dot{\psi}}{\psi^2} - \alpha \left(\frac{\gamma}{\psi} - \alpha s\right)
\]  
(22)

For numerator of right-hand side of (12) with taking into account (21) one obtains

\[
L\dddot{s} + \kappa \dot{s} = \frac{L}{\psi^2}(\ddot{s}\psi - \gamma\dot{\psi}) + (\kappa - \alpha L)\dot{s}
\]  
(23)

or with taking into account (20)

\[
L\dddot{s} + \kappa \dot{s} = \frac{L}{\psi^2}((-a\gamma - A\Delta u(t) - BD(t))\psi - \gamma\dot{\psi}) + \dot{s}(\kappa - \alpha L)
\]  
(24)

Substituting (24) in (12) one obtains

\[
\ddot{\Omega} = -d_1\dot{\Omega} - d_2tg\beta - d_3\dot{\Omega}^2\sin\beta +
\]

\[
+ \frac{1}{\rho L - I}(L(-a\gamma - A\Delta u(t) - BD(t))\psi - \gamma\dot{\psi}) + \dot{s}(\kappa - \alpha L)
\]  
(25)

\[\dot{\psi} = -L^2\sin2\beta.\]

So, the dynamics of the control object (5) in new variables \(s, \gamma, \Omega, \dot{\Omega}\) will be determined by system of four differential equations of the first order

\[
\begin{cases}
\frac{ds}{dt} = \frac{\gamma}{\psi(\beta)} - \alpha s \\
\frac{d\gamma}{dt} = -a\gamma - A\Delta u(t) - BD(t) \\
\frac{d\Omega}{dt} = \dot{\Omega} \\
\frac{d\dot{\Omega}}{dt} = -d_1\dot{\Omega} - d_2tg\beta - d_3\dot{\Omega}^2\sin\beta + \\
+ \frac{L}{\rho L - I}((-a\gamma - A\Delta u(t) - BD(t))\psi - \gamma\dot{\psi}) + \dot{s}(\kappa - \alpha L)
\end{cases}
\]  
(26)

III. THE CASE OF CONTINUOUSLY ACTING PERTURBATION \(D(t)\)

To stabilize the system it will be used a discontinuous relay-type function

\[
\Delta u(t) = \Pi \text{sign} \gamma, \text{ sign } \gamma = \begin{cases}
1, & \gamma > 0 \\
-1, & \gamma < 0 \\
(-1,1), & \gamma = 0
\end{cases}
\]  
(27)
So, as a control it will be used a function

\[ u = \ddot{u} + \Delta u(t), \]

where \( \ddot{u} \) is taken from (15).

If denote \( x = \text{col}(s, \gamma) \), \( y = \text{col}(\Omega, \dot{\Omega}) \), then system (26) can be attributed to the class of systems considered in [7].

To use theorems 1-3 from [7], there is necessary the verification of all conditions and proposals used.

It can be obtained that conditions 1.2, 1.3 of theorem 1 from [7] are fulfilled if functions \( V, W, \psi, \eta \) are taken in the form

\[
\begin{align*}
V(s, \gamma) &= s^2 + k\gamma^2, \\
W(\Omega, \dot{\Omega}) &= \frac{\Omega^2}{\cos^2\beta(\Omega)} + \frac{r}{\cos^\gamma\beta(\Omega)} - r > 0 \\
\psi &= \varepsilon(|s| + |\gamma|), \\
\eta &= \frac{2a\dot{\Omega}^2}{\cos^2\beta(\Omega)}
\end{align*}
\]

where \( k > 0, \varepsilon > 0 \) are the numbers taken by corresponding manner \( q = 2gL(gL - I)^{-1} > 1, r = 2(q - 1)^{-1}Lg(gL - I)^{-1} > 0 \).

IV. THE CASE OF IMMEASURABLE VELOCITIES: REDUCING TO SINGULARLY PERTURBED SYSTEM

Consider system (5) at \( D(t) \equiv 0 \). Let that in the regulator velocities \( \dot{r}, \dot{\beta} \) are measured inexacty, and their approximate values \( \dot{z}_1, \dot{z}_2 \) are generated with using the system [8, 10].

\[
\begin{align*}
\mu \frac{d\dot{z}_1}{dt} &= \dot{r} - \dot{z}_1, \\
\mu \frac{d\dot{z}_2}{dt} &= \dot{\beta} - \dot{z}_2
\end{align*}
\]

where \( 0 < \mu < 1 \) is the small parameter. The processes \( \dot{z}_{1,2} \) approximate processes \( \dot{r}, \dot{\beta} \). It is essentially that in contrast to later they can be calculated without using of differentiation operation

\[ \dot{z}_1(t) = D^1_\mu(r(t)), \quad \dot{z}_2(t) = D^1_\mu(\beta(t)). \]

In this case instead of globally stabilizing control \( u = \sum_i \lambda_i u_i(t) \) consider control in the form

\[ u = \sum_i \lambda_i \tilde{u}_i(t) \]

where \( \tilde{u}_i \) is determined from continuous functions \( u_i \) by replacement \( \dot{r}, \dot{\beta} \) to \( z_1, z_2 \), respectively.

\[
\begin{align*}
u_1 &= \ddot{z}_1, \\
u_2 &= \dddot{z}_1 s^2 \beta, \\
u_3 &= \dddot{z}_2 \sin\beta, \\
u_4 &= \dddot{z}_2 \cos\beta, \\
u_5 &= u_5, \\
u_6 &= \ddot{z}_2 \cos\beta = \ddot{\Omega}, \\
u_7 &= u_7, \\
u_8 &= a \dddot{z}_3 + q \dddot{z}_2 \sin\beta \cos^2\beta \\
u_9 &= (\dddot{z}_2 \sin^2\beta - a \dddot{z}_3 \cos^2\beta) \dddot{z}_3 - \dddot{z}_2 \dddot{z}_3 \sin\beta; & u_{10} &= \dddot{z}_3 + ar - a\lambda n|tg(\pi/2 - \beta/2) + \dddot{z}_2 \dddot{z}_3 \beta \cos^2\beta.
\end{align*}
\]

Let introduce a vector process \( z(t) = \text{col}(z_1(t), z_2(t)) \)

\[ z_1 = \dot{z}_1 - \dot{r}, \quad z_2 = \dot{z}_2 - \dot{\beta} \]
and also denote

\[ u_i^\mu = \ddot{u}_i - u_i. \quad (34) \]

Then system (26), (29), (31) with taking into account (33), (34) can be presented in the form of six differential equations of the first order (cf. with (5))

\[
\begin{cases}
\frac{ds}{dt} + \alpha s = \gamma / \psi(\beta) \\
\frac{d\gamma}{dt} + a\gamma = -\sum_{i=1}^{n} \lambda_i u_i^\mu \\
\frac{d\Omega}{dt} = \Omega \\
\frac{d\dot{z}_1}{dt} = f_\Omega(s, \gamma, \mu), \\
\mu \frac{dz_1}{dt} = -z_1 - \mu f_r(s, \gamma, \Omega, \dot{z}_1, z_2) \\
\mu \frac{dz_2}{dt} = -z_2 - \mu f_\beta(s, \gamma, \Omega, \dot{z}_1, z_2).
\end{cases} \quad (35)
\]

From above presented it’s seen that \( f_\Omega, f_r, f_\beta \) are the continuous functions. Moreover, at \( \mu = 0 \) system (35) is reduced to system (26) (at \( D(t) = 0 \) and \( \Delta u = 0 \)).

To obtain functions \( f_r \) and \( f_\beta \) in direct form express \( \dot{z}_1, \dot{\dot{z}}_1, \dot{z}_2, \dot{\dot{z}}_2 \) via \( z_1, z_2 \)

\[ \dot{z}_1 = z_1 + \dot{r}, \quad \dot{\dot{z}}_1 = \dot{z}_1 + \ddot{r}, \quad \dot{z}_2 = z_2 + \dot{\beta}, \quad \dot{\dot{z}}_2 = \dot{\dot{z}}_2 + \ddot{\beta}. \quad (36) \]

Substituting these expressions in (4.5.1) one obtains

\[ \mu(\dot{z}_1 + \ddot{r}) = -z_1, \quad \mu(\dot{\dot{z}}_2 + \ddot{\beta}) = -z_2 \quad (37) \]

from where

\[
\begin{cases}
\mu \frac{d\dot{z}_1}{dt} = -z_1 - \mu \ddot{r} \\
\mu \frac{d\dot{z}_2}{dt} = -z_2 - \mu \ddot{\beta}
\end{cases} \quad (38)
\]

So, for vector process (33)

\[ f_r(s, \gamma, \Omega, \dot{z}_1, z_2) = \dot{\beta}, \quad f_\beta(s, \gamma, \Omega, \dot{z}_1, z_2) = \ddot{\beta} \quad (39) \]

Express values \( \dot{\beta} \) and \( \ddot{\beta} \) (due to (5)) as continuous functions of state of the control object and regulator \( \Omega \). Let write the system (5) in the form

\[
\begin{cases}
M \ddot{\beta} + L \dddot{\beta} \cos \beta = Gu(t) + D(t) - N \dddot{r} + L \dddot{r}^2 \sin \beta \\
L \dddot{\beta} \cos \beta + I \dddot{\beta} = L \dddot{\beta} \sin \beta - c \dddot{\beta} - \kappa \dddot{r} \cos \beta.
\end{cases} \quad (40)
\]

The solution of this system of linear algebraic equations relative to variables \( \dot{\beta} \) and \( \dddot{\beta} \) has a form

\[
\begin{cases}
\dot{\beta} = \frac{I(Gu(t) - Br + L \dddot{\beta} \cos^2 \beta + D(t)) - L \dddot{\beta} \sin \beta \dddot{\beta} - c \dddot{\beta} - b \dddot{r} \cos \beta}{MI - L^2 \cos^2 \beta} \\
\dddot{\beta} = \frac{M(L \dddot{\beta} \sin \beta - c \dddot{\beta} - b \dddot{r} \cos \beta) - L \dddot{\beta} \sin \beta \dddot{\beta} - c \dddot{\beta} - b \dddot{r} \cos \beta}{MI - L^2 \cos^2 \beta} \quad (41)
\end{cases}
\]

In new variables the values \( \dot{\beta} \) and \( \dddot{\beta} \) will be expressed with taking into account (10) as

\[ \dot{\beta} = \dddot{\beta} - \rho \Omega, \quad \dddot{\beta} = \dddot{\beta} \cos \beta - \dddot{\beta} \sin \beta \cos \beta. \quad (42) \]
\[
\ddot{s} = \frac{\dot{s} \psi - \gamma \dot{\psi}}{\psi^2} - \alpha \dot{s}
\]  

(43)

In correspondence with (26)

\[
\dot{\gamma} = -a\gamma - A\Delta u(t) - BD(t)
\]

(44)

from where

\[
\ddot{\Omega} = \sum_{i=1}^{11} \lambda_i \dot{u}_i + \frac{\gamma}{\psi(\beta)} + \alpha s \left( aL \right)
\]

(45)

Here \( \psi \) and its derivative are determined with using of relations

\[
\psi = M - L^2 \cos \beta, \quad \dot{\psi} = 2L^2 \cos \beta \sin \beta = L^2 \sin^2 \beta
\]

(46)

and \( \beta \), in its turn is determined by relation (7).

Now, turn out to verification of fulfilment of conditions of theorem 3 from [9]. To present the system considered in the form of [9], replace (38), (39) to system

\[
\begin{aligned}
\frac{d\dot{z}_1}{dt} &= -z_1 - \mu_2 f_r \\
\mu_1 \frac{d\dot{z}_2}{dt} &= -z_2 - \mu_2 f_\beta
\end{aligned}
\]

(47)

So, system (35), (38) with functions

\[
\begin{aligned}
f_r &= \ddot{s}(s,\gamma,\Omega,\dot{\Omega}) - \rho \ddot{\Omega}(s,\gamma,\Omega,\dot{\Omega}), \\
f_\beta &= \dddot{\beta} = \dot{\Omega}(s,\gamma,\Omega,\dot{\Omega}) \cos \beta - \dot{\Omega}^2 \sin \beta \cos \beta
\end{aligned}
\]

(48)

can be reduced to the form [9], if denote \( x = col(s,\gamma,\Omega,\dot{\Omega}) \quad z = col(z_1, z_2) \).

\[
\begin{pmatrix}
\dot{s} \\
\dot{\gamma} \\
\dot{\Omega} \\
\ddot{\Omega}
\end{pmatrix} = \begin{pmatrix}
\gamma/\psi(\beta) - \alpha s \\
-a\gamma - \sum_{i=1}^{11} \lambda_i u_i \\
\dot{\Omega} \\
\ddot{\Omega} \text{ from (45)}
\end{pmatrix}
\]

(49)

\[
\dot{z} = \begin{pmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{pmatrix}, \quad -\Gamma z = \begin{pmatrix}
-z_1 \\
-z_2
\end{pmatrix}, \quad \ddot{b}(x, z) = \begin{pmatrix}
\ddot{s} - \ddot{\Omega} \\
\ddot{\Omega} \cos \beta - \dot{\Omega} \sin \beta \cos \beta
\end{pmatrix}
\]

(50)

At \( \mu_2 = 0, z = 0 \) it obtained the initial unperturbed system studied in [11]. As it was demonstrated in that work this system will have the dissipativity domain in the form of \( D = \{(s, \gamma, \Omega, \dot{\Omega}) : V(s, \gamma, \Omega, \dot{\Omega}) \leq C\} \), where \( V(s, \gamma, \Omega, \dot{\Omega}) \) is the continuous piecewise positive definite in \( \mathbb{R}^m \) (in sense [12]) function, satisfying the condition a) of theorem 3 in [3], i.e.

\[
\left. \frac{dV(s, \gamma, \Omega, \dot{\Omega})}{dt} \right|_{(s, \gamma) \in S(R)} \leq -\varepsilon, \quad V(s, \gamma, \Omega, \dot{\Omega}) \to \infty \quad \text{at} \quad |s, \gamma, \Omega, \dot{\Omega}| \to \infty,
\]

where \( S(R) \) is the bounded closed (hyper)surface:

\[
\{(s, \gamma, \Omega, \dot{\Omega}), \quad V(s, \gamma, \Omega, \dot{\Omega}) = R\},
\]

and \( R, \varepsilon \) are the positive numbers.
Further, for system (38)

\[
\begin{cases}
\mu \frac{dz_1}{dt} = -z_1 - \mu (\dot{s} - \rho \dot{\Omega}), \\
\mu \frac{dz_2}{dt} = -z_2 - \mu (\dot{\Omega} \cos\beta - \dot{\Omega}^2 \sin\beta \cos\beta)
\end{cases}
\]

(51)

where \(\dot{s}\) and \(\dot{\Omega}\) are determined from (21), (25), it will take place a condition b) of the theorem 3 from [2], and as \(W(z_1, z_2)\) it can be taken quadratic form

\[
W = <z, Bz>, \quad \Gamma^T B + B\Gamma = -E
\]

(52)
i.e.

\[
\frac{dW(z_1, z_2)}{dt} \leq -\frac{L_1}{\mu_1} |z|^2, \quad (\exists \mu_1), (\forall \mu_1 \in (0, \bar{\mu}_1]) \quad (W(z) \rightarrow \infty \quad |z| \rightarrow \infty).
\]

(53)

Also from (32), (33) it will follow a fulfilment of condition c) of theorem 3 in [2]. I.e. all conditions of theorem 3 will be hold and thus the system (5), (31)-(33), (38) will have the dissipativity domain \(D = \{y, W_\rho(x, z) \leq C\}\), independen of \(\rho \in (0, \bar{\rho})\) \(\exists \bar{\rho} > 0\), \(W_\rho(y) = V(s, \gamma, \Omega, \dot{\Omega}) + \rho W(z_1, z_2)\) is the global Lyapunov function satisfacting due to the system considered the equality [2]

\[
\frac{dW_\rho(y)}{dt} \biggm|_{\mu_2=0} = -(|x|^2 + \rho |z|^2).
\]

Here \(y = (x, z), |y|^2 = |(x, z)|^2 = |x|^2 + |z|^2, \ C \ \rho\) are some positive numbers.

Then, it can be stated the following theorem.

**Theorem.** Given system has attraction domain of equilibrium state \(x = 0, z = 0\) in the form \(W(x, z) \leq C\) at all \(0 < \mu \leq \bar{\mu}_1\), and \(C \rightarrow \infty \quad \bar{\mu}_1 \rightarrow 0\).

The proof is based on estimation of total time derivative of above introduced function

\[
W_\rho(x, z) = W_\rho(s, \gamma, \Omega, \dot{\Omega}; z_1, z_2) = V_\rho(s, \gamma, \Omega, \dot{\Omega}) + \rho W(z_1, z_2)
\]

with using of inequality \(W_\rho(x, z) \leq C\) as well as following from it inequality \(|s, \gamma, \Omega, \dot{\Omega}| \leq R_{W_\rho}(C)\), where \(R_{W_\rho}(C) = sup\{s, \gamma, \Omega, \dot{\Omega}\}, W_\rho(s, \gamma, \Omega, \dot{\Omega}; z_1, z_2) \leq C\) following from conditions of the theorems estimations of right-hand sides

\[
\bar{f}(s, \gamma, \Omega, \dot{\Omega}; z_1, z_2) \leq const(C)|\{s, \gamma, \Omega, \dot{\Omega}\}|(z_1, z_2)|
\]

\[
\bar{h}(s, \gamma, \Omega, \dot{\Omega}; z_1, z_2) \leq const(C)|\{s, \gamma, \Omega, \dot{\Omega}\}|(z_1, z_2)|
\]

correct in this region.

V. RESULTS OF NUMERICAL SIMULATION

In work [3] it was performed a numerical simulation for the case of closed-loop system with continuous control, described by equation (5). The behavior of mass-center cart coordinate as well as angular coordinate with increase of \(t\) evident about stabilization of the system for a short enough time interval at any values of parameters used. Moreover, with increase of parameter \(a\) oscillatory regime of decrease of angular as well as coordinate amplitude is changed to damping one.

In present work, the numerical simulation of the behavior of the nonlinear system "inverted pendulum on a cart" with relay-type control under acting of continuously acting immeasurable perturbation was performed on the basis of MATLAB system. In particular, it was performed a numerical simulation of nonlinear dynamical system on the basis of the system (26) where as control force it was taken discontinuous relay-type (27) function with following parameter values \(a = 1; \ a = 0.5; \ k = 0.1; \ I = 1; \ L = 1; \ M = 1.5; \ d_1 = d_2 = d_3 = 1; \ \rho_m = 2\). The initial values of variables were the following \(s = -0.7; \ \gamma = 0.7; \ \Omega = 1.0; \ \dot{\Omega} = 0.5\).
For the case of absence of the control and external action \((A = 0, B = 0)\) in the system all time dependencies for four variables \(s, \gamma, \Omega, \dot{\Omega}\) are damping and asymptotically approaching zero but the character of approaching zero appears to be essentially different. The value \(s\) coming from the point with negative coordinate then becomes to be positive and approach zero from the positive value side. In contrast, the variable \(\gamma\) is damping with increasing time up to zero being positive value beginning from \(t = 0\) \((\gamma(0) = 0.7)\). Then, time dependences \(\Omega(t)\) and \(\dot{\Omega}(t)\) are of oscillatory damping character. At \(t > 10\) the system comes to equilibrium state \(s = 0, \gamma = 0, \Omega = 0, \dot{\Omega} = 0\) and rate of approaching the equilibrium state is determined by the friction force given by parameters \(L\) and \(\kappa\) in the nonlinear system (35).

The appearance of discontinuous relay-type control (27) in the system \((A = 0.03)\) in absence of external action \((B = 0)\) essentially decreases time of approach the equilibrium state in the system, moreover in \(\gamma(t)\) dependence there appear horizontal regions \(\gamma = 0\) corresponding to the sliding regime at the breaking surface in the state space. Note also that presence of relay-type control leads to significant decrease of time necessary for coming of the system in the equilibrium state \(\beta = 0, r = 0\).

The numerical simulation in the case of external immeasurable sinusoidal perturbation in the system 
\[
D(t) = \sin(t)
\]
in absence of the control \((A = 0)\) all time dependences don’t approach the stationary solution of the system (4.3.21) \(x = 0\) but are of oscillatory character with amplitude \(\Delta x = 0.1\) and period equal to that of external force \(D(t)\). The presence of discontinuous relay-type control (27) \((A = 0.05)\) permit approach the stationary solution with sufficient accuracy \(\Delta x = 0.03\) already at \(t > 10\). Such stabilization is determined, in essential degree, by appearance at \(\gamma(t)\) dependence of horizontal regions corresponding to sliding regime at the breaking surface \(\gamma = 0\) in the state space of the system (26) what don’t permit to variables reach sufficient oscillation amplitude for time interval between horizontal regions.

The presence of discontinuous relay-type control (27) in the case of action of external sinusoidal perturbation leads to a fast stabilization of the system guiding the trajectory at the beaking surface \(\gamma = 0\) of the system (26) and further it does not permit this trajectory essentially decline from this surface at periodical sign change in external perturbation \(D(t)\). The essential fact here is that at more high-frequency perturbation the stabilization of the system occurs for a shorter time as compared with low-frequency (or even constant) external action.

VI. CONCLUSION

In this work it was performed a study of global asymptotic stability of nonlinear mechanical system "inverted pendulum on a controlled cart" in conditions of continuously acting perturbation with using method of two Lyapunov functions. It was demonstrated that at given choice of Lyapunov functions and comparison functions the conditions of the basic theorem of the method are fulfilled, i.e. system "inverted pendulum on a controlled cart" with discontinuous relay-type control, in conditions of continually acting perturbation is globally asymptotically stable. Here the solution is considered in the sense of Gelig et al. \[8\]. There is established the robustness property of above obtained stabilization algorithms relatively nonideality of measurements of both cart and pendulum, since in practice there are no strict signal differentiators (instead of this the approximative velocity measurements are used). Also, using the method of two Lyapunov functions it is demonstrated that at given algorithm but with replace of ideal derivatives, the stability region persists, and its size at \(\mu \rightarrow 0\) increases unlimitedly and in the limit coincides with the full phase space.

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