COMPLETE NEGATIVELY CURVED IMMERSED ENDS IN $\mathbb{R}^3$

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To my beloved wife, Cristina Marques, whose love and sincerity inspire me

Abstract. This paper extends, in a sharp way, the famous Efimov’s Theorem to immersed ends in $\mathbb{R}^3$. More precisely, let $M$ be a non-compact connected surface with compact boundary. Then there is no complete isometric immersion of $M$ into $\mathbb{R}^3$ satisfying that $\int_M |K| = +\infty$ and $K \leq -\kappa < 0$, where $\kappa$ is a positive constant and $K$ is the Gaussian curvature of $M$. In particular Efimov’s Theorem holds for complete Hadamard immersed surfaces, whose Gaussian curvature $K$ is bounded away from zero outside a compact set.

In 1901 Hilbert ([Hi]) proved that there is no complete immersed hyperbolic plane $\mathbb{H}^2$ in $\mathbb{R}^3$. In 1902 Holmgren ([Ho]) presented a new version with a more rigorous proof. Blaschke ([Bs]) and Bieberbach ([Bi]) presented new versions of the proof. In 1955 Blanuša ([Bl]) presented an example of a smooth complete isometric embedding of $\mathbb{H}^2$ in $\mathbb{R}^4$. In 1960 Rozendorn ([Ro]) obtained a smooth complete isometric immersion of $\mathbb{H}^2$ in $\mathbb{R}^5$.

In 1936 Cohn-Vossen ([C-V]) have conjectured that the hyperbolic plane in Hilbert’s Theorem could be replaced by a complete immersed surface with Gaussian curvature not greater than a negative constant. The solution for this problem came only in 1964 with the work of Efimov ([Ef]):

**Theorem 0.1** (Efimov’s Theorem). There is no complete isometric immersion $\varphi : M \to \mathbb{R}^3$ with Gaussian curvature $K \leq -\kappa < 0$, where $M$ is a connected surface.

An extension of the Efimov’s Theorem to higher dimensions was the work of B. Smith and F. Xavier in [SX], in which it is proved that there exists no codimension one complete isometric immersion $f : M \to \mathbb{R}^n$ with the Ricci curvature $\text{Ric}_M \leq -\kappa < 0$, provided that $n = 3$, or that $n \geq 4$ and the sectional curvatures of $M$ do not assume all values in $\mathbb{R}$.

Tilla Klotz Milnor in [Mi] published a more detailed version of the proof of the Efimov’s Theorem. In her version the immersion $\varphi : M \to \mathbb{R}^3$ is $C^2$ and the induced Riemannian metric on $M$ is just supposed to be $C^1$. With this hypothesis we may not define the Gaussian curvature by the usual intrinsic method. Instead we follow [Mi] and define $K = \frac{e_0^2-f^2}{E_0-FF}$ for the Gaussian curvature in terms of the first and second fundamental forms of the immersion: $I = Edx^2 + 2Fdxdy + Gdy^2$ and $II = e dx^2 + 2f dxdy + g dy^2$.

Our main result is the following

**Theorem A.** Let $M$ be a non-compact connected surface with compact boundary. Then does not exist any $C^2$ immersion $\varphi : M \to \mathbb{R}^3$ inducing a $C^1$ complete
Riemannian metric on $M - \partial M$ with the Gaussian curvature satisfying $\int_M |K| = +\infty$ and $K \leq -\kappa < 0$ for some constant $\kappa$.

Note that if $\partial M = \emptyset$ Theorem A reduces to the Efimov’s Theorem. Indeed, if there exists an immersion $\varphi : M \to \mathbb{R}^3$ as in the statement of the Efimov’s Theorem, we may compose it with the universal covering $P : \tilde{M} \to M$. The induced metric on $\tilde{M}$ will have infinite area (see [Mi]), hence the fact that $K \leq -\kappa < 0$ implies that the total curvature of $\tilde{M}$ is infinite, which contradicts Theorem A.

By applying Theorem A to the complement of an open ball, we obtain the following application to Hadamard surfaces.

**Corollary B.** Let $M$ be an open simply-connected surface without boundary. Then does not exist any $C^2$ immersion $\varphi : M \to \mathbb{R}^3$ inducing a $C^1$ complete metric on $M$ with the Gaussian curvature satisfying $K \leq 0$ on $M$ and $K \leq -\kappa < 0$ on the complement $M - B$, where $B$ is an open ball and $\kappa$ is a positive constant.

**Remark 1.** Each hypothesis in Theorem A is essential. Indeed, consider the set

$$S = \left\{ \left( \cos u \sin t, \sin u \sin t, \cos t + \log \tan \frac{t}{2} \right) \mid u \in [0, 2\pi], t \in \left[ \frac{\pi}{2} + \epsilon, \pi \right) \right\},$$

for some $\epsilon > 0$, which is a smooth surface with boundary, contained in a pseudosphere with Gaussian curvature $-1$ and finite area. Thus the surface $S \subset \mathbb{R}^3$ shows that the condition that the total curvature is infinite may not be dropped. Its universal covering $P : M = \tilde{S} \to S \subset \mathbb{R}^3$ with the induced metric shows that the compactness of the boundary $\partial M$ is also essential, since the other conditions hold. Now consider the incomplete surface $S' = S - \partial S$ and its universal covering $P' : H \to S' \subset \mathbb{R}^3$ with induced metric. It shows that the hypothesis that $\varphi$ is complete may not be removed from Theorem A. The helicoid shows that the condition $K \leq -\kappa < 0$ may not be replaced by the condition $K < 0$. M. Kuiper showed in [Ku] that the condition that $\varphi$ is $C^2$ is essential even in the Hilbert’s theorem.

**Remark 2.** As a byproduct of this paper we provide a more detailed presentation of the proof of the Efimov’s theorem.

To prove Theorem A, the proof of the Efimov’s Theorem will be used as written in [Mi]. The author would like to thank Heudson Mirandola, Cristina Marques and Manolo Heredia for useful discussions during the reading of that paper.

**Remark 3.** It came to our knowledge that the paper [GMT] proves Theorem A independently.

1. **Notations**

Given a Riemannian metric $\omega$ on a manifold $M$, let $A_\omega$, $L_\omega$, $d_\omega$ denote, respectively, the area, length, distance associated to the metric $\omega$. Similarly a $\omega$-geodesic will denote a geodesic with respect to the metric $\omega$. Given a subset $C \subset M$ we set

$$B_\omega(C, r) = \{ x \in M \mid d_\omega(x, C) < r \},$$

$$\bar{B}_\omega(C, r) = \{ x \in M \mid d_\omega(x, C) \leq r \}$$

and

$$S_\omega(C, r) = \{ x \in M \mid d_\omega(x, C) = r \}.$$
2. An idea of the proof of Theorem A

To give an idea of the proof of Theorem A, we consider an immersion \( \varphi : M \to \mathbb{R}^3 \) as in the statement of Theorem A. By passing to the orientable double cover with induced metric if necessary, we still have the same curvature conditions, the completeness of the induced metric \( \alpha \) and the compactness of the boundary. Thus we may assume, without loss of generality, that \( M \) is orientable. As a consequence there exists a \( C^1 \) Gauss map \( N : M \to S^2 \), where \( S^2 \) is the sphere with the standard round metric \( \nu \) of curvature 1. Since \( K < 0 \) the map \( N \) is a local \( C^1 \) diffeomorphism on \( M - \partial M \). Let \( \beta \) be the \( C^0 \) Riemannian metric induced by \( N \) on \( M - \partial M \). We have:

\[
A_\beta(M) = \int_M dA_\beta = \int_M |\det(dN)| \, dA_\alpha = \int_M |K| \, dA_\alpha = +\infty.
\]

In Lemma 4.3 below we will show that \((M, \beta)\) is bounded. To do this we will fix some point \( p \in M \) far from the boundary \( \partial M \). Then we will apply on a large ball centered at \( p \) similar ideas as in [Mi] obtaining a contradiction if the distance from \( \partial M \) is greater than \( 5\pi \). We will present some arguments in a way different from [Mi]. In some points our proof is shorter and in another points we prefer to present more details, in order to make the proof clearer. We also need to be careful to assure that the constructions used never involve points in \( \partial M \). Finally we will show that boundedness in this case implies pre-compactness and finite area of \((M, \beta)\), contradicting equation (1) and proving Theorem A.

3. Some basic facts

The following simple lemma, which will be used in the proof of Lemma 4.2, is possibly known, but we didn’t find it in the literature. For completeness we will present its proof in the Appendix, which is partially inspired in the proof of Lemma 3.1 in [Mi].

**Lemma 3.1.** Let \((S, g)\) be a Riemannian smooth surface. Let \( D \subset S \) be a connected surface with piecewise smooth boundary with internal angles at the vertices different from 0 and \( 2\pi \). For \( p, q \in D \), let \( d_g(p, q) \) denote the distance induced by the Riemannian metric \( g \), and \( d_{int}(p, q) \) the infimum of the \( g \)-lengths of piecewise smooth curves \( \gamma : [0, 1] \to D \) joining \( p \) to \( q \), such that \( \gamma([0, 1]) \subset \text{int}(D) \), where \( \text{int}(D) \) is the set of interior points of \( D \) with respect to the metric \( d_g \). Then \( d_{int} \) is a distance on \( D \), and the distances \( d_g \) and \( d_{int} \) induce the same topology on \( D \).

Take positive numbers \( r, s \) with \( s + 2r < \pi \). Consider a unit speed minimal \( \nu \)-geodesic \( \gamma : \mathbb{R} \to S^2 \). Fix \( z = \gamma(0) \) and \( u = \gamma(s) \). Consider the antipodal points \( p = \gamma \left( \frac{s + \pi}{2} \right), p^* = \gamma \left( \frac{s + \pi}{2} \right) \), which satisfy \( d_\nu(p, z) = d_\nu(p^*, u) = \frac{s + \pi}{2} \). Set

\[
X = \overline{B_\nu(z, r)} \cup \gamma([0, s]) \cup \overline{B_\nu(u, r)}.
\]

We will recall the construction of the convex hull of \( X \), which will be needed in the proof of Lemma 4.4 below. Fixing the antipodal points \( p \) and \( p^* \), we rotate \( \gamma \) in both directions until we obtain exactly two geodesics \( \gamma_1 : \left[ \frac{s - \pi}{2}, \frac{s + \pi}{2} \right] \to S^2 \) and \( \gamma_2 : \left[ \frac{s - \pi}{2}, \frac{s + \pi}{2} \right] \to S^2 \) from \( p \) to \( p^* \), which intersect tangentially both \( S_\nu(z, r) \) and \( S_\nu(u, r) \). This is possible since \( s + 2r < \pi \). For each \( i \in \{1, 2\} \), consider points \( p_i \) and \( q_i \) given by

\[
\{p_i\} = S_\nu(z, r) \cap \gamma_i \left( \left[ \frac{s - \pi}{2}, \frac{s + \pi}{2} \right] \right), \quad \{q_i\} = S_\nu(u, r) \cap \gamma_i \left( \left[ \frac{s - \pi}{2}, \frac{s + \pi}{2} \right] \right).
\]
Let $Y(X)$ be the compact domain containing $X$, whose boundary of $C^1$ class is the image of a curve which follows $\gamma_1$ from $p_1$ to $q_1$, then the arc in $S_x(u, r)$ from $q_1$ to $q_2$ which contains $\gamma(s + r)$, then the image of $\gamma_2$ in the opposite direction from $q_1$ to $p_2$, and then the arc in $S_y(z, r)$ from $p_2$ to $p_1$ which contains $\gamma(-r)$. Since $r < \frac{\pi}{2}$, the balls $B_v(z, r)$ and $B_u(u, r)$ are strongly convex, hence it is easy to prove the following

**Lemma 3.2.** Given positive numbers $r, s$ such that $s + 2r < \pi$ and $X$ defined as in (2), the set $Y = Y(X)$ is the convex hull of $X$, and $Y$ is strongly convex.

Fix a point $p \in S^2$ and $0 < r \leq \frac{s}{2}$. Consider the circle $S = S_v(p, r) = S, (p^*, \pi - r)$. Let $W_{pr}$ be the set of pairs $(x, v)$ in the normal fiber bundle of $S$ satisfying one of the following three conditions: $v = 0$; $0 < |v| < r$ and $v$ points to $B_v(p, r)$; $0 < |v| < \pi - r$ and $v$ points to $B_v(p^*, \pi - r)$. By using spherical coordinates it is easy to show that the normal exponential map $exp^*: W_{pr} \to S^2 - \{p, p^*\}$ is a diffeomorphism and $exp^*(\partial W_{pr}) = \{p, p^*\}$. Thus it is easy to prove the following well known Lemma, which will be used in the proof of Lemma 3.3 below.

**Lemma 3.3.** Fix $0 < r \leq \frac{s}{2}$ and $p \in S^2$. If $x \in B_v(p, r) - \{p\}$ and $\gamma: [0, d] \to S^2$ is a unit speed geodesic from $x$ to $S = S_v(p, r)$ with $0 < d < r$ and $\gamma'(d)$ orthogonal to $S$, then $\gamma$ is the unique unit speed geodesic from $x$ to $S$ such that $d_v(x, S) = L_v(\gamma)$. If $\eta \in B_v(p^*, \pi - r) - \{p^*\}$ and $\sigma: [0, e] \to S^2$ is a unit speed geodesic from $z$ to $S$ with $0 < e < \pi - r$ and $\gamma'(e)$ orthogonal to $S$, then $\sigma$ is the unique unit speed geodesic from $z$ to $S$ such that $d_v(z, S) = L_v(\sigma)$.

**Lemma 3.4.** Consider unit speed $\nu$-geodesics $\gamma: [0, \mu] \to S^2$, with $0 < \mu < \pi$ and $\eta: [0, \pi] \to S^2$ with $\eta(0) = \gamma(\frac{\mu}{2})$ and $\eta'(0)$ orthogonal to $\gamma$. For $0 \leq s < \frac{s}{2}$, set $q_s = \eta(-\frac{s}{2} + s)$ and $q^s = \eta(\frac{s}{2} - s)$. If $z = \gamma(0)$ and $u = \gamma(\mu)$, consider the distance $d_s = d_v(q_s, z) = d_v(q_s, u) = d_v(q^s, z) = d_v(q^s, u)$. Set $D_0 = \gamma([0, \mu])$ and

$$D_s = \bar{B}_v(q_s, d_s) \cap \bar{B}_v(q^s, d_s),$$

if $0 < s \leq \frac{s}{2}$. Then for $0 \leq s < s' < \frac{s}{2}$ it holds that $D_s \subset D_{s'}$.

**Proof.** We first observe that

$$(3) \quad \{z, u\} \subset \partial L_s,$$

for all $0 \leq s < \frac{s}{2}$.

By the spherical law of cosines we have that

$$(4) \quad \cos d_s = \sin s \cos \frac{\mu}{2},$$

for $0 \leq s < \frac{s}{2}$.

Fix $0 < s < \frac{s}{2}$. Equation (4) implies that $0 < d_s < \frac{s}{2}$, hence $D_s$ is strongly convex. In particular we have by (3) that $D_0 \subset D_s$.

Now fix $0 < s < s' < \frac{s}{2}$. To prove that $D_s \subset D_{s'}$ we take $x \in D_s$. Thus we have that $d_v(x, q_s) \leq d_s$. By triangle inequality we have that

$$(5) \quad d_v(x, q_{s'}) \leq d_v(x, q_s) + (s' - s) \leq d_s + s' < \frac{s}{2} + s' < \pi.$$

Fix $x_0 \in D_s$ such that $d_v(x_0, q_{s'})$ is a maximum. By (5) there exists a unique $\nu$-geodesic $\chi: [0, 1] \to S^2$ from $q_{s'}$ to $x_0$. The inequality (5) also implies that $\chi$ may be extended to a minimal geodesic $\tilde{\chi}: [0, 1 + \epsilon] \to S^2$ for some small $\epsilon > 0$. Thus the maximality of $d_v(x_0, q_{s'})$ implies that $x_0 \in \partial D_s$. 


First assume that \( x_0 \) is in the interior of the arc \( D_s \cap S_\nu(q_s,d_s) \). By the first variation formula and the maximality of \( d_\nu(x_0,q_s) \), it follows from (5) that \( \chi'(1) \) is orthogonal to \( D_s \cap S_\nu(q_s,d_s) \). Since \( q_s \) is distinct from \( q_s \) and \( (q_s)^* \), we have from Lemma 3.3 that the distance from \( q_s \) attains a strict minimum at \( x_0 \), which is a contradiction. Thus \( x_0 \) may not belong to the interior of \( D_s \cap S_\nu(q_s,d_s) \). Since \( q_s \) is distinct from \( q_s \) and \( (q_s)^* \), we obtain similarly that \( x_0 \) may not belong to the interior of \( D_s \cap S_\nu(q^*,d_s) \). We conclude that \( x_0 \in \{z,u\} \), hence \( d_\nu(x,q_s) \leq d_\nu(x_0,q_s) = d_\nu(q_s) \). As a consequence we have that \( x \in \bar{B}_\nu(q_s,d_\nu(q_s)) \). Similarly we show that \( x \in \bar{B}_\nu(q^*,d_\nu(q^*)) \), hence \( x \in D_s \). Lemma 3.4 is proved. \( \square \)

4. Proof of Theorem A

If \( (N,\beta) : M \to (S^2,\nu) \) is a \( C^1 \) isometric immersion, by a \( \beta \)-geodesic in \( M \) we will mean a curve that locally minimizes length. In [Mi], a metric ball \( B \) starting at \( v \) is defined on \([0,r]\). The fact that \( N \) preserves length of curves implies easily that \( N \) is injective on a full geodesic disk \( B_\beta(p,r) \) if \( 0 < r < \pi \), as well as on the image of any \( \beta \)-geodesic of length smaller than \( 2\pi \). From now on, for simplicity we will use the expression ‘normal ball’ instead of ‘full geodesic disk’.

If \( U \subset M - \partial M \) is an open set, such that \( N|_U : U \to N(U) \subset S^2 \) is injective and \( U, N(U) \) are strongly convex with respect to \( \beta \), then \( N|_U : (U,d_\beta|_U \times U) \to (N(U),d_\nu|_{N(U) \times N(U)}) \) is an isometry. In particular \( N : (M - \partial M, \beta) \to (S^2,\nu) \) is a local isometry.

From the non-compactness of \( M \) and the Myers Theorem ([My]) it follows that the metric \( \beta \) cannot be complete. Indeed, if \( (M,\beta) \) is complete and non-compact there exist a minimal \( \beta \)-geodesic \( \gamma \) in \( M \) with \( \beta \)-length greater than \( \pi \), which would contradict the Myers Theorem for \( N \circ \gamma \).

If \( (M,\beta) \) is some connected \( C^2 \) surface with compact boundary, let \( \tilde{M} \) be the metric completion of \( (M,\beta) \). We set \( \delta M = \tilde{M} - M \). Assume that there exists some \( C^1 \) local isometry \( N : (M,\beta) \to (S^2,\nu) \). Since \( N \) preserves length of curves, it is easy to see that \( N \) maps Cauchy sequences in \( M \) to Cauchy sequences in \( S^2 \), hence \( N \) has a unique continuous extension \( \bar{N} : \tilde{M} \to S^2 \). For some \( X \subset M \) we denote by \( \bar{X} \) the closure of \( (X,d_\beta|_{X \times X}) \) in \( \tilde{M} \). Unless otherwise stated we will consider on \( X \) the topology induced by \( d_\beta|_{X \times X} \). Similarly, for \( Y \subset S^2 \) we will always consider the topology induced by the inclusion in \( S^2 \).

The next definition is the natural extension of Definition 2 in [Mi] to a manifold with boundary.

**Definition 4.1.** Consider a connected noncompact surface \( M \) and a \( C^1 \) local isometry \( N : (M,\beta) \to (S^2,\nu) \). We will say that \( \tilde{M} \) is concave at some \( q \in \delta M \) if there exist \( p \in S^2 \) and \( \frac{\pi}{2} < r < \pi \) such that:

1. \( \bar{N}(q) \in S_\nu(p,r) \);
2. there exists \( \epsilon > 0 \) such that \( \left(B_\nu(\bar{N}(q),\epsilon) \cap B_\nu(p,r)\right) = N(U) \), where \( U \) is an open set in \( (M - \partial M) \) such that \( q \in \bar{U} \) and \( \bar{N} \) is injective on \( \bar{U} \).

**Lemma 4.1.** Let \( \varphi : M \to \mathbb{R}^3 \) be as in the statement of Theorem A. Then there is no point \( q \in \delta M \) at which \( \tilde{M} \) is concave.
The proof of Lemma 4.1 is exactly the same as in the proof of Lemma A in [Mi], since all arguments there refer to a neighborhood of the end point \( q \) as in Definition 4.1. Lemma 4.1 is the unique result in this paper where the completeness of \( (M, \alpha) \) is needed. We observe that the hypothesis that the total curvature of \( M \) is infinite is not used to prove Lemma 4.1.

The following lemma will be used several times in this paper. Compare condition (3) in Lemma 4.2 with Definition 4.1 above.

**Lemma 4.2.** Consider a connected surface \( M \) and a \( C^1 \) local isometry \( N : (M, \beta) \rightarrow (S^2, \nu) \) such that there exists no point \( q \in \partial M \) at which \( \tilde{M} \) is concave. Fix \( w \in M - \partial M \) and a compact domain \( F \subset M - \partial M \) containing \( w \) such that \( N|_F \) is injective. Assume that there exist compact domains \( \tilde{F}_t \subset S^2 \), for \( s_0 \leq t < T \) satisfying the following conditions:

1. \( \tilde{F}_{s_0} = N(F) \);
2. if \( s_0 \leq t \leq s < T \) then \( \tilde{F}_t \subset \tilde{F}_s \);
3. there exists a continuous deformation \( s_0 \leq t < T \mapsto \sigma_t : S^1 \rightarrow S^2 \) such that the boundary \( \partial \tilde{F}_t = \sigma_t(S^1) \), where \( \sigma_t \) is a piecewise smooth simple closed curve with internal angles with respect to \( \tilde{F}_t \) different from 0 and \( 2\pi \);
4. for any \( s_0 < t < T \) and any \( z \in \partial \tilde{F}_t \), one of the following assumptions hold:
   a. \( z \in \tilde{F}_{s_0} = N(F) \);
   b. there exist \( p \in S^2 \) and \( \frac{\pi}{2} < r < \pi \) such that there exists a nonempty open arc \( C \subset (\partial \tilde{F}_t \cap S_r(p, r)) \) containing \( z \), and there exists \( \epsilon > 0 \) such that \( \left( B_{\nu}(z, \epsilon) \cap \text{int}(\tilde{F}_t) \right) \subset B_{\nu}(p, r) \);
5. in the case that \( \partial M \neq \emptyset \), it holds that, for each \( s_0 < t < T \) and any \( z \in (\partial \tilde{F}_t \cap (S^2 - N(F))) \), there exists a piecewise smooth simple curve \( \gamma_{zt} : [0, 1] \rightarrow \tilde{F}_t \) joining \( N(w) \) to \( z \) with \( L_{\nu}(\gamma_{zt}) < d_{\beta}(w, \partial M) \).

Then there exists a connected set \( U \subset M - \partial M \) containing \( F \) such that \( N|_U : U \rightarrow \bigcup_{s_0 \leq t < T} \tilde{F}_t \) is a bijection.

**Proof.** Let \( Z \) be the set of numbers \( t \in [s_0, T) \) such that there exists a family \( (F_s)_{s_0 \leq s < t} \) of compact sets contained in \( M - \partial M \) such that \( N|_{F_s} : F_s \rightarrow \tilde{F}_s \) is a bijection for any \( s_0 \leq s \leq t \) and \( F = F_{s_0} \subset F_u \subset F_v \), if \( s_0 \leq u < v \leq t \). By condition (3) we see easily that \( s_0 \in Z \). If \( \sup(Z) = T \), the local \( C^1 \) diffeomorphism \( N \) is injective on \( U = \bigcup_{s_0 \leq t < T} F_t \), hence \( N|_U : U \rightarrow N(U) \) is a homeomorphism. Since the set \( N(U) = \bigcup_{s_0 \leq t < T} \tilde{F}_t \) is connected, we conclude that \( U \) is a connected set. Thus, to prove Lemma 4.2 it suffices to show that \( \sup(Z) = T \).

We assume by contradiction that \( t_0 = \sup(Z) < T \). First we prove the following

**Claim 4.1.** \( t_0 \notin Z \) and \( s_0 < t_0 \).

In fact, if \( t_0 \in Z \) then for each \( s_0 \leq t \leq t_0 \) the map \( N|_{F_t} : F_t \rightarrow \tilde{F}_t \) is injective and a local \( C^1 \) diffeomorphism, hence it is a homeomorphism. Furthermore it holds that \( F = F_{s_0} \subset F_u \subset F_v \subset (M - \partial M) \), if \( s_0 \leq u \leq v \leq t_0 \). Since \( F_{t_0} \) is compact and \( N \) is a local diffeomorphism which is injective on \( F_{t_0} \), it is easy to see that there exists an open set \( W \subset (M - \partial M) \) containing \( F_{t_0} \) such that \( N|_W : W \rightarrow S^2 \) is injective. By condition (3) in Lemma 4.2, the map \( t \mapsto \sigma_t \) is continuous, hence the compactness of \( \partial \tilde{F}_{t_0} \) implies that there exists \( t_0 < s < T \) such that for any \( t_0 < t < s \) it holds that \( \tilde{F}_t \) is contained in the open set \( N(W) \), hence we may
set $F_t = (N|_W)^{-1}(\hat{F}_t)$, obtaining that $N|_{F_t}: F_t \to \hat{F}_t$ is a homeomorphism. In particular we have that each $F_t$ is a compact set and by condition (2) in Lemma 4.2 the family $(F_t)_{s_0 \leq t \leq s}$ satisfies that $F = F_{s_0} \subset F_u \subset F_v \subset W \subset (M - \partial M)$, if $s_0 \leq u \leq v \leq t$. We conclude that $s \in Z$, which contradicts the fact that $t_0 = \sup(Z)$. Since we proved that $t_0 \notin Z$ and $s_0 \in Z$ we have that $s_0 < t_0$. Claim 4.1 is proved.

Set $V = \bigcup_{s_0 \leq t < t_0} F_t$. Since $F_u \subset F_v$ if $u \leq v$, it is easy to see that $N|_V : V \to N(V)$ is injective, hence it is a homeomorphism. We have that

$$N(V) = N \left( \bigcup_{s_0 \leq t < t_0} F_t \right) = \bigcup_{s_0 \leq t < t_0} \hat{F}_t \subset \hat{F}_{t_0}.$$ 

Take $x \in \text{int}(\hat{F}_{t_0}) = \hat{F}_{t_0} - \partial \hat{F}_{t_0}$. By compactness we have that $d = d_v(x, \partial \hat{F}_{t_0}) > 0$. By compactness again and using the continuous deformation $t \mapsto \sigma_t$, we obtain that there exists $s_0 < s < t_0$ such that $\hat{F}_{t_0} - \hat{F}_s \subset B_r(\partial \hat{F}_{t_0}, d)$, hence $x \in \hat{F}_s$. Thus we obtain that

$$(6) \quad \text{int}(\hat{F}_{t_0}) \subset N(V) \quad \text{and} \quad \hat{F}_{t_0} - N(V) \subset \hat{F}_{t_0} - \text{int}(\hat{F}_{t_0}) = \partial \hat{F}_{t_0}.$$ 

**Claim 4.2.** $\tilde{N}|_V : \tilde{V} \to S^2$ is injective.

Fix $x, y \in \tilde{V}$ such that $\tilde{N}(x) = \tilde{N}(y)$. Take sequences $x_n, y_n \in V$ such that $x_n \to x$ and $y_n \to y$ in $\tilde{M}$. Since $N(x_n)$ and $N(y_n)$ converge to $\tilde{N}(x) = \tilde{N}(y)$ in $S^2$, we have that $N(x) \in \hat{F}_{t_0}$. Since $N(x_n)$ and $N(y_n)$ belong to $\hat{F}_{t_0}$ and converge to $\tilde{N}(x)$, Lemma 3.1 implies that $d_{\text{int}}(N(x_n), N(y_n)) \to 0$, hence there exists a sequence of piecewise smooth curves $\tilde{\gamma}_n : [0, 1] \to \hat{F}_{t_0}$ joining $N(x_n)$ to $N(y_n)$, with $L_v(\tilde{\gamma}_n) \to 0$ and $\tilde{\gamma}_n((0, 1)) \subset \text{int}(\hat{F}_{t_0}) \subset N(V)$, where we used (4) above. As a consequence we have that $\tilde{\gamma}_n([0, 1]) \subset N(V)$. The curve $\gamma_n = (N|_V)^{-1} \circ \tilde{\gamma}_n : [0, 1] \to V$ joins $x_n$ to $y_n$ and satisfies $L_\beta(\gamma_n) = L_v(\tilde{\gamma}_n) \to 0$. In particular we have that $d_\beta(x_n, y_n) \to 0$, hence $x = y$. Claim 4.2 is proved.

**Claim 4.3.** $\tilde{V} \subset M$.

In fact, if this is not true, there exists $q \in \delta M \cap \tilde{V}$. Since $q \notin V$, Claim 4.2 implies that $\tilde{N}(q) \notin N(V)$. Thus (6) above implies that

$$\tilde{N}(q) \in \left( \hat{F}_{t_0} - N(V) \right) \subset \partial \hat{F}_{t_0}.$$ 

Since $\tilde{N}(q) \notin N(V)$, the point $\tilde{N}(q)$ does not satisfy condition (2)-(a) in Lemma 4.2. By condition (2)-(b) in Lemma 4.2 there exist $p \in S^2$, $\frac{\pi}{2} < r < \pi$ and $\epsilon > 0$ such that

$$\left( B_r(\tilde{N}(q), \epsilon) \cap \text{int}(\hat{F}_{t_0}) \right) \subset B_v(p, r).$$ 

Again by condition (2)-(b) in Lemma 4.2 the boundaries of $B_v(p, r)$ and $\hat{F}_{t_0}$ coincide in a neighborhood of $\tilde{N}(q)$. More precisely, there exists a nonempty open arc $C$ satisfying

$$\tilde{N}(q) \in C \subset \left( \partial \hat{F}_{t_0} \cap S_v(p, r) \right).$$ 

Thus we obtain that for some $\epsilon' > 0$ sufficiently small it holds that

$$\Omega = \left( B(\tilde{N}(q), \epsilon') \cap \text{int}(\hat{F}_{t_0}) \right) = \left( B(\tilde{N}(q), \epsilon') \cap B_v(p, r) \right).$$
By \( \emptyset \) above we have that \( \Omega \subset \text{int}(\tilde{F}_{t_0}) \subset N(V) \). We consider the open set \( U = (N|_V)^{-1}(\Omega) \). Since \( U \subset V \) we obtain from Claim 4.2 that \( \tilde{N} \) is injective on \( U \).

Consider a sequence \( q_n \in V \) converging to \( q \). For sufficiently large \( n \) we have that \( N(q_n) \in \Omega \), hence \( q_n = (N|_V)^{-1}(N(q_n)) \in U \), hence \( q \in U \). We conclude that \( \tilde{M} \) is concave at \( q \), which contradicts our hypotheses. Claim 4.3 is proved.

**Claim 4.4.** \( \tilde{N}|_V : V \to \tilde{F}_{t_0} \) is a bijection.

Indeed, by \( \emptyset \) above we have that \( \text{int}(\tilde{F}_{t_0}) \subset N(V) \). Since Claim 4.2 holds, in order to show Claim 4.4 it suffices to prove that any point in \( \partial \tilde{F}_{t_0} \) is in the image of \( N|_V \). By Condition 3 in the statement of Lemma 4.2, given \( z \in \partial \tilde{F}_{t_0} \) there exists a unit vector \( v \in T_z(S^2) \) pointing to \( \text{int}(\tilde{F}_{t_0}) \). We consider the unit speed \( \nu \)-geodesic \( \tilde{\gamma} : [0, \eta] \to \tilde{F}_{t_0} \) given by \( \tilde{\gamma}(t) = \exp_z(\eta - t)v \), for some small \( \eta > 0 \) such that \( \tilde{\gamma}([0, \eta]) \subset \text{int}(\tilde{F}_{t_0}) \). Set \( \gamma : [0, \eta] \to V \) given by \( \gamma = (N|_V)^{-1} \circ \tilde{\gamma}([0, \eta]) \). Take a sequence \( t_n \in [0, \eta) \) converging to \( \eta \). Since \( \gamma(t_n) \) is a Cauchy sequence in \( M \), it converges to some \( q \in [0, \eta] \) and \( \tilde{N}(q) = \lim \tilde{N}(\gamma(t_n)) = \lim \tilde{\gamma}(t_n) = z \). Claim 4.4 is proved.

**Claim 4.5.** \( \tilde{V} \subset M - \partial M \).

By Claim 4.3 there is nothing to prove if \( \partial M = \emptyset \). Thus we will assume that \( \partial M \neq \emptyset \). Fix \( q \in \tilde{V} \). We know by Claim 4.3 that \( q \in M \). If \( q \in F \), we have by hypothesis that \( q \notin \partial M \). If \( q \notin F \), Claim 4.2 implies that \( N(q) \notin N(F) = \tilde{F}_{t_0} \).

Thus condition 3 in Lemma 4.2 implies that there exists some \( s_0 < t \leq t_0 \) such that \( N(q) \in \partial \tilde{F}_t \), hence \( N(q) \in (\partial \tilde{F}_t \cap (S^2 - N(F))) \). Since \( N : \tilde{V} \to \tilde{F}_{t_0} \) is a homeomorphism which preserves the length of curves, it follows from condition 4 in Lemma 4.2 that

\[
\forall \beta \in (\gamma_{N(q,t)}), \quad \nu_\beta \left((N|_V)^{-1} \circ \gamma_{N(q,t)}\right) \geq \nu_\beta(w, q).
\]

Thus we have that

\[
d_\beta(w, \partial M) \geq d_\beta(w, \partial M) - d_\beta(w, q) > 0,
\]

which implies that \( q \in M - \partial M \). Claim 4.5 is proved.

Now we may get a contradiction and prove Lemma 4.2. Set \( F_{t_0} = \tilde{V} \). We have that \( F \subset F_u \subset F_v \subset M - \partial M \) for \( s_0 \leq u \leq v \leq t_0 \). By Claim 4.3 the map \( N|_{F_{t_0}} : F_{t_0} \to \tilde{F}_{t_0} \) is a homeomorphism, hence the set \( F_{t_0} \) is compact. We conclude that \( t_0 \in Z \), which contradicts Claim 4.1 and proves Lemma 4.2. \( \square \)

**Lemma 4.3.** Consider a connected noncompact surface \( M \) and a \( C^1 \) local isometry \( N : (M, \beta) \to (S^2, \nu) \) such that there exists no point \( q \in \partial M \) at which \( \tilde{M} \) is concave. Fix \( w \in M - \partial M \). If \( \partial M \neq \emptyset \) assume further that \( d_\beta(w, \partial M) > \pi \). Then does not exist any unit speed \( \beta \)-geodesic \( \gamma : [0, \pi] \to M \) with \( \gamma (\pi/2) = w \).

**Proof.** Assume by contradiction that such a geodesic \( \gamma \) exists. Since \( N : (M, \beta) \to S^2 \) is a \( C^1 \) local isometry we have that \( N_{\gamma([0,\pi])} \) is injective. By a compactness argument there exists a small \( 0 < \delta < \pi/2 \) such that the set \( F = B_\delta(\gamma([0,\pi]), \delta) \) is a domain with \( C^1 \) boundary contained in \( M - \partial M \) and \( N \) is injective on an open set containing \( F \), hence \( N(F) \) is a domain in \( S^2 \) with \( C^1 \) boundary. The number \( \delta \) may be chosen sufficiently small such that \( N(F) = B_\delta(\gamma([0,\pi]), \delta) \).
Now we rotate $\tilde{\gamma} = (N \circ \gamma)$ in both directions fixing $\tilde{\gamma}(0)$ and $\tilde{\gamma}(\pi)$, obtaining unit speed $\nu$-geodesics $\tilde{\gamma}_s : [0, \pi] \to S^2$ such that $\tilde{\gamma}_0 = \tilde{\gamma}$ and the angle $\angle(\tilde{\gamma}_s(0), \tilde{\gamma}_s'(0)) = 2s$.

For $0 \leq t \leq \frac{\pi}{2}$, set

$$\hat{F}_t = \bigcup_{|s| \leq t} B_\nu(\tilde{\gamma}_s([0, \pi]), \delta) = B_\nu \left( \bigcup_{|s| \leq t} \tilde{\gamma}_s([0, \pi]), \delta \right).$$

Note that $\hat{F}_t = B_\nu(N(w), \frac{\pi}{2} + \delta)$. For $\frac{\pi}{2} \leq t < \pi - \delta$ set $\hat{F}_t = B_\nu(N(w), t + \delta)$.

We claim that the collection of compact domains $(\hat{F}_t)_{0 \leq t < \pi - \delta}$ satisfies the conditions of Lemma 4.2. Note that $\hat{F}_0 = B_\nu(g([0, \pi]), \delta)) = N(F)$, hence condition (1) in Lemma 4.2 is satisfied. Conditions (2) and (3) in Lemma 4.2 are also easily verified.

Now we will check that condition (4) in Lemma 4.2 is satisfied. If $0 < t < \frac{\pi}{2}$, the boundary $\partial \hat{F}_t$ is a union of an arc in $S_\nu(\tilde{\gamma}(0), \delta) \subset N(F)$, an arc in $S_\nu(\tilde{\gamma}(\pi), \delta) \subset N(F)$, whose points verify condition (4)-a) in Lemma 4.2. If $0 < t < \frac{\pi}{2} + \delta$ and an arc $C_t$ contained in $S_\nu(p_t, \frac{\pi}{2} + \delta)$ for some $p_t \in S^2$, such that $\exists \tilde{\gamma}_s(0)$ and $B_\nu(p_t, \frac{\pi}{2} + \delta)$ are locally on the same side of $C_t$, and an arc $C_{-t}$ contained in $S_{\nu}(p_{-t}, \frac{\pi}{2} + \delta)$ for some $p_{-t} \in S^2$ such that $\exists \tilde{\gamma}_s(\pi)$ and $B_\nu(p_{-t}, \frac{\pi}{2} + \delta)$ are locally on the same side of $C_{-t}$. The points in $C_t$ and $C_{-t}$ verify condition (4)-b). More precisely, let $E_t$ denote the equator containing the image of $\tilde{\gamma}$ and $H_t$ one of the two hemispheres determined by $E_t$ such that $\tilde{\gamma}_s([0, \pi]) \subset H_t$. Take $p_t \in H_t$ such that $d_\nu(p_t, E_t) = \frac{\pi}{2}$. Similarly we obtain a point $p_{-t}$ by using the equator given by the geodesic $\tilde{\gamma}_{-t}$. If $\frac{\pi}{2} \leq t < \pi - \delta$ we have that $\partial \hat{F}_t = S_\nu(N(w), t + \delta)$ and thus condition (4)-(b) is automatically verified.

If $\partial M \neq \emptyset$, let us check condition (1) in Lemma 4.2. We fix $0 < t < \pi - \delta$ and $z \in \partial \hat{F}_t$ such that $z \notin N(F)$. It suffices to show that there exists a piecewise smooth curve $\gamma_{zt} : [0, 1] \to \hat{F}_t$ from $N(w) = \tilde{\gamma}(0)$ to $z$ with $L_\nu(\gamma_{zt}) < \pi$. Assume first that $0 < t < \frac{\pi}{2}$. Since $\angle(\tilde{\gamma}_s(0), \tilde{\gamma}_s'(0)) = 2t < \pi$, the compact domain $D_t$ bounded by $\tilde{\gamma}_s([0, \pi])$ and $\tilde{\gamma}_{-s}([0, \pi])$ which contains $\tilde{\gamma}([0, \pi])$ is convex. Note that $\partial \hat{F}_t = S_\nu(D_t, \delta)$. Let $P(z)$ be the natural projection of $z$ onto $\partial D_t$. Then we may construct a piecewise smooth curve $\gamma_{zt}$ joining $N(w)$ to $z$, which first follows a minimal geodesic from $N(w)$ to $P(z)$, whose $\nu$-length is not greater then $\frac{\pi}{2}$, by an argument similar as in the proof of Lemma 3.3 and then follows a minimal geodesic from $P(z)$ to $z$. We will have that $L_\nu(\gamma_{zt}) \leq \frac{\pi}{2} + \delta < \pi$. Finally we consider $\frac{\pi}{2} \leq t < \pi - \delta$. Since $\partial \hat{F}_t = S_\nu(N(w), t + \delta)$, we define $\gamma_{zt}$ as a minimal geodesic from $N(w)$ to $z$, which satisfies $L_\nu(\gamma_{zt}) = t + \delta < \pi$. We conclude that condition (1) in Lemma 4.2 is satisfied.

Thus we may use Lemma 4.2 to obtain the existence of a connected set $U \subset M - \partial M$ containing $F$ such that $N|_U : U \to \bigcup_{0 \leq t < \pi - \delta} \hat{F}_t = B_\nu(N(w), \pi) = S^2 - \{N(w)\}$ is a bijection. By using the fact that $N$ is a $C^1$ local isometry we see that $N|_U$ is a homeomorphism which preserves lengths. Thus, by considering the isometry $dN_w : T_w M \to T_{N(w)} M$, we see that each unit speed $\beta$-geodesic starting at $w$ will be defined at least on $[0, \pi]$. This implies that $B_\beta(w, \pi)$ is a normal ball contained in $U$. Given a point $p \in U$, there exists a minimal $\nu$-geodesic $\sigma$ from $N(w)$ to $N(p)$ with $L_\nu(\sigma) < \pi$. Set $\gamma = (N|_U)^{-1} \circ \sigma$. We have that $L_\beta(\gamma) = L_\nu(\sigma) < \pi$. Since $\gamma$ joins $w$ and $p$ we obtain that $d_\beta(w, p) < \pi$, hence $U = B_\beta(w, \pi)$. Consider
divergent sequences $p_n, q_n$ in $U$. Since $N(p_n)$ and $N(q_n)$ converge to $(N(w))^*$, there exists a curve $\tilde{\tau}_n$ joining $N(p_n)$ and $N(q_n)$ in $S^2 - N(w)^*$ with $L_\nu(\tilde{\tau}_n) \to 0$, hence $L_\beta((N[U]^{-1} \circ \tau_n) \to 0$. Thus we have that $d_\beta(p_n, q_n) \to 0$. As a consequence we have that $\tilde{U} - U = \{q\}$ and $\tilde{N}(q) = (N(w))^*$.

We first consider the case that $q \in M$. Since $d_\beta(w, q) = \pi$ we have that $q \notin \partial M$, hence $\tilde{U}$ is a bounded surface without boundary, hence it is compact and agrees with $M$, which contradicts the fact that $(M, \beta)$ is not compact.

The second possibility is that $q \in \tilde{U} \cap \delta M$. Since $N|_U : U \to B_\nu(N(w), \pi)$ is a bijection and $\tilde{U} = U \cup \{q\}$, we conclude that $\tilde{N} : \tilde{U} \to S^2$ is a bijection. Take $y \in S^2$ with $0 < d_\nu(N(w), y) = r < \frac{\pi}{2}$. Then $\tilde{N}(q) \in S_\nu(y, \pi - r)$. Set $B = N^{-1}(B_\nu(y, \pi - r))$. Note that $\pi - r > \frac{\pi}{2}$ and that $N|_B : B \to S^2$ is injective, hence $M$ is concave in $q$, which contradicts our hypotheses. The proof of Lemma 4.3 is complete. \qed

**Lemma 4.4.** Consider a connected noncompact surface $M$ and a $C^1$ local isometry $N : (M, \beta) \to (S^2, \nu)$ such that there exists no point $q \in \delta M$ at which $M$ is concave. Fix $p \in M - \partial M$ and $\delta_0 > 0$. If $\delta M \neq \emptyset$, assume further that $0 < \delta_0 < \frac{L_\beta(p, \partial M)}{5}$. Then for any distinct points $z, u \in B = B_\beta(p, \delta_0)$, there exists a unique unit speed minimal geodesic $\gamma : [0, d] \to M - \partial M$ from $z$ to $u$ with $L_\beta(\gamma) < \pi$. In particular $N(z) \neq N(u)$, hence the map $N|_B : B \to S^2$ is injective.

**Proof.** By the triangle inequality we have that $\text{diam}_\beta(B) \leq 2\delta_0$. Fix distinct points $z, u \in B$. Since any metric ball in a Riemannian manifold is path-connected, there exists a continuous curve $\sigma : [0, a] \to B$ from $z$ to $u$. By compactness there exist $0 < r < \min \{\frac{\pi}{2}, \pi\}$ and a partition $0 = t_0 < t_1 < \cdots < t_k = a$ of the interval $[0, a]$ such that for all $0 \leq i \leq k - 1$ it holds that $B_i = B_\beta(\sigma(t_i), r)$ is a strongly convex normal ball in $M - \partial M$ whose interior $B_i$ contains $\sigma(t_{i+1})$. Without loss of generality we may assume that $\sigma(t) \neq z$ for $0 < t \leq a$.

We will show by induction that for any $1 \leq i \leq k$, there exists a unique unit speed minimal geodesic $\gamma_i : [0, s_i] \to M - \partial M$ from $z = \sigma(0)$ to $u_i = \sigma(t_i)$ with $L_\beta(\gamma_i) < \pi$. For $i = 1$ this assertion is trivial. Assume that for some $1 \leq i \leq k - 1$ there exists a unique unit speed minimal geodesic $\gamma_i : [0, s_i] \to M - \partial M$ from $z = \sigma(0)$ to $u_i = \sigma(t_i)$. Set $w_i = \gamma_i(\frac{r}{2})$. Since $\text{diam}_\beta(B) \leq 2\delta_0$ we have that

$$d_\beta(z, w_i) = \frac{d_\beta(z, u_i)}{2} = \frac{s_i}{2} \leq \delta_0.$$ \hfill (7)

If $\partial M \neq \emptyset$ we obtain from (7) that

$$d_\beta(w_i, \partial M) \geq d_\beta(p, \partial M) - d_\beta(p, z) - d_\beta(z, w_i) \geq d_\beta(p, \partial M) - 2\delta_0 \geq 3\delta_0. \hfill (8)$$

Since $B_0$ and $B_i$ are normal balls with the same radius $r$, we may extend $\gamma_i$ in both directions obtaining a unit speed $\beta$-geodesic $\xi_i : [-r, s_i + r] \to M$.

**Claim 4.6.** $s_i + 2r < \pi$, hence $s_i < \pi$ and $r < \frac{\pi}{2}$.

In fact, if $\partial M = \emptyset$, this inequality follows immediately from Lemma 4.3 applied to the geodesic $\xi_i$. If $\partial M \neq \emptyset$ we assume by contradiction that $s_i + 2r \geq \pi$. Since $r$ was chosen so that $r < \frac{\pi}{2}$, we obtain from (7) and (8) that $d_\beta(w_i, \partial M) \geq 3\delta_0 \geq s_i + \delta_0 > s_i + 2r \geq \pi$, hence $d_\beta(w_i, \partial M) > \pi$. By Lemma 4.3 we conclude that $s_i + 2r < \pi$, and this contradiction proves Claim 4.6.
Consider the set 
\[ \Omega = \Omega_i = \bar{B}_0 \cup \gamma_i([0, s_i]) \cup \bar{B}_i. \]

**Claim 4.7.** \( N \) is injective on \( \Omega \).

In fact, by Claim 4.6 we have that \( r < \frac{\pi}{2} \) and \( s_i < \pi \). Thus the fact that \( N \) is a \( C^1 \) local isometry implies that the maps \( N|_{\bar{B}_0} \), \( N|_{\bar{B}_i} \), and \( N|_{\gamma_i([0, s_i])} \) are injective. The map \( N \) is injective on \( \bar{B}_0 \cup \gamma_i([0, s_i]) \) and on \( \bar{B}_i \cup \gamma_i([0, s_i]) \), since \( r < \frac{\pi}{2} \) and then \( N(\bar{B}_0) \) and \( N(\bar{B}_i) \) are strongly convex. Thus, to show that \( N \) is injective on \( \Omega \) it suffices to show that \( N \) is injective on \( \bar{B}_0 \cup \bar{B}_i \). Fix \( x \in \bar{B}_0 \) and \( y \in \bar{B}_i \) such that \( v = N(x) = N(y) \). Since \( N \) is a \( C^1 \) local isometry we have that
\[ 2r \geq d_\beta(z, x) + d_\beta(u_i, y) \geq d_\nu(N(z), v) + d_\nu(N(u_i), v) \geq d_\nu(N(z), N(u_i)) = s_i. \]
The fact that \( 2r \geq s_i \) implies that \( w_i \in \bar{B}_0 \cap \bar{B}_i \). Since \( \bar{B}_0 \) and \( \bar{B}_i \) are strongly convex balls, there exist unit speed minimal \( \beta \)-geodesics \( \lambda_1 : [0, a_1] \to \bar{B}_0 \) from \( w_i \) to \( x \) and \( \lambda_2 : [0, a_2] \to \bar{B}_i \) from \( w_i \) to \( y \). We have that \( L_\nu(N \circ \lambda_j)(0) = N(w_i), (N \circ \lambda_j)(a_j) = v, \) for \( j = 1, 2 \). Thus we have that \( a_1 = a_2 \) and \( N \circ \lambda_1 = N \circ \lambda_2 \). Since \( N \) is a \( C^1 \) local isometry we obtain that \( \lambda_1'(0) = \lambda_2'(0) \), hence \( x = y \). Claim 4.7 is proved.

By compactness there exists \( \epsilon < r \) sufficiently small such that for any \( t \in [0, s_i] \) the ball \( \bar{B}_\beta(\gamma_i(t), \epsilon) \) is a normal ball in \( M - \partial M \) and \( N \) is injective on
\[ F = F_\epsilon = \bar{B}_0 \cup \bigcup_{t \in [0, s_i]} \bar{B}_\beta(\gamma_i(t), \epsilon) \cup \bar{B}_i = \bar{B}_0 \cup \bar{B}_\beta(\gamma_i([0, s_i]), \epsilon) \cup \bar{B}_i. \]

Set:
\[ \gamma_i = N \circ \gamma_i, \bar{B}_0 = N(\bar{B}_0) = \bar{B}_\nu(N(z), r), \bar{B}_i = N(\bar{B}_i) = \bar{B}_\nu(N(u_i), r), X = N(\Omega). \]

In the particular case that \( 2r > s_i \), we may use the fact that \( N \) is a \( C^1 \) local isometry, which is injective on the compact domain \( \Omega = \bar{B}_0 \cup \gamma_i([0, s_i]) \cup \bar{B}_i = \bar{B}_0 \cup \bar{B}_i \), to obtain that the number \( \epsilon > 0 \) as above may be chosen not so small such that \( \bar{B}_0 \cup \bar{B}_i \) is properly contained in \( \bar{B}_0 \cup \bar{B}_\beta(\gamma_i([0, s_i]), \epsilon) \cup \bar{B}_i \). More precisely, if \( 2r > s_i \) we set \( r_0 = d_\beta(w_i, \partial B_0 \cap \partial B_i) > 0 \). Since \( N \) is injective and a \( C^1 \) local isometry on \( \bar{B}_0 \cup \bar{B}_i \), each \( \bar{B}_\beta(\gamma_i(t), r_0) \) is a normal ball contained in \( \bar{B}_0 \cup \bar{B}_i \), for \( 0 \leq t \leq s_i \), and \( N \) is injective on \( \bar{B}_\beta(\gamma_i([0, s_i]), r_0) \). Thus by compactness there exists \( r_0 < \epsilon < r \) such that for any \( t \in [0, s_i] \) the ball \( \bar{B}_\beta(\gamma_i(t), \epsilon) \) is a normal ball in \( M - \partial M \) and \( N \) is injective on \( F_\epsilon = \bar{B}_0 \cup \bar{B}_\nu(\gamma_i([0, s_i], \epsilon)) \cup \bar{B}_i \).

The idea now is to construct the convex hull of \( X = N(\Omega) \) as the union \( \hat{F}_T \) of an increasing family of sets \( (\hat{F}_t)_{t \leq T} \) which satisfies the conditions in Lemma 4.2 for \( 0 \leq t < T \). Set \( \hat{F}_t = N(F_t) \). For \( \epsilon \leq t < r \), set \( \hat{F}_t = \bar{B}_0 \cup \bar{B}_\nu(\gamma_i([0, s_i], t)) \cup \bar{B}_i \).

To define \( \hat{F}_t \) for \( t > r \), we write \( t = s + r \) for convenience. Consider a unit speed \( \nu \)-geodesic \( \eta : \mathbb{R} \to S^2 \) orthogonal to \( \gamma_i \) at \( \eta(0) = N(w_i) = \gamma_i(\frac{\pi}{2}) \). For \( 0 \leq s \leq \frac{\pi}{2} \), set \( q_s = \eta(-\frac{s}{2} + s) \) and \( q^* = \eta(\frac{\pi}{2} - s) \). By symmetry we have that
\[ d_\nu(q_s, N(z)) = d_\nu(q_s, N(w_i)) = d_\nu(q^*, N(z)) = d_\nu(q^*, N(u_i)). \]

By Claim 4.6 we have that \( \frac{\pi}{2} < \frac{\pi}{2} \). Thus equation (4) implies that, if \( 0 \leq s \leq \frac{\pi}{2} \), it holds that \( d_s \leq \frac{\pi}{2} \). Hence by Claim 4.6 it holds that \( d_s + r \leq \frac{\pi}{2} + r < \pi \). As a consequence there exists a unique unit speed minimal \( \nu \)-geodesic \( \varphi_s : [0, d_s + r] \to S^2 \) satisfying \( \varphi_s(0) = q_s \) and \( \varphi_s(d_s) = N(z) \). Set \( \psi_s = R \circ \varphi_s : [0, d_s + r] \to S^2 \), where
$R$ is the reflection which fixes the image of $\eta$. Set $\varphi^s = S \circ \varphi_s$ and $\psi^s = S \circ \psi_s$, where $S$ is the reflection which fixes the equator containing the image of $\tilde{\gamma}_i$.

By (3) we have that $d_0 = \frac{\pi}{2}$ and that the map $s \in [0, \frac{\pi}{2}] \mapsto d_s$ is strictly decreasing, hence the number $d_s + r$ decreases from $\frac{\pi}{2} + r$ to $\frac{\pi}{2} + r$. By Claim 4.6 we have that $\frac{\pi}{2} + r < \frac{\pi}{2}$, hence there exists a unique $0 < s < \frac{\pi}{2}$ such that $d_s + r = \frac{\pi}{2}$.

Since $\varphi^s_i(d_s + r)$ is orthogonal to both $S_v(N(z), r)$ and $S_v(q_0, d_s + r)$, we obtain that $S_v(q_s, d_s + r)$ is tangent to both $\partial \tilde{B}_0 = S_v(N(z), r)$ and $\partial \tilde{B}_i = S_v(N(u_i), r)$. Similarly we obtain that $S_v(q^s, d_s + r)$ is tangent to $\partial \tilde{B}_0$ and $\partial \tilde{B}_i$.

For $0 \leq s \leq \bar{s}$, consider a $C^1$ piecewise smooth simple closed curve $\tau_s$ which follows $S_v(q_s, d_s + r)$ from $\varphi^s_i(d_s + r)$ to $\psi^s_i(d_s + r)$, then $\partial \tilde{B}_i$ from $\psi^s_i(d_s + r)$ to $\psi^s_i(d_s + r)$, then $S_v(q^s, d_s + r)$ from $\psi^s_i(d_s + r)$ to $\varphi^s_i(d_s + r)$, and then $\partial \tilde{B}_0$ from $\varphi^s_i(d_s + r)$ to $\varphi^s_i(d_s + r)$. For $0 < s < \bar{s}$, let $\tilde{F}_{s}^r$ be the domain which contains $X$ and is bounded by the image of $\tau_s$. Note that $\tilde{F}_{s}^r + r$ agrees with the convex hull of $X$. Set $T = \bar{s} + r$.

Now we will prove that the family $(\tilde{F}_t)_{0 \leq t \leq T}$ satisfies the conditions in Lemma 4.2. We will see that Conditions (2) and (3) in Lemma 4.2 hold even for $\epsilon \leq t \leq T$. Conditions (1) and (4) in Lemma 4.2 are trivially satisfied.

Claim 4.8. **Condition (3) in Lemma 4.2 holds for $\epsilon \leq t \leq T$.**

From the facts that $d_0 = \frac{\pi}{2}$ and that the image of $\tilde{\gamma}_i$ is contained in $S_v(q_0, \frac{\pi}{2}) = S_v(q^0, \frac{\pi}{2})$, we obtain easily that the image of $\tau_0$ agrees with $\partial \tilde{F}_r = S_v(\tilde{\gamma}_i([0, s]), r)$. This shows the continuous dependence of $\partial \tilde{F}_t$ on the parameter $\epsilon \leq t \leq T$ (note that the homotopy $s \mapsto \tau_s$ extends easily to $s = \bar{s}$). Thus to see that condition (3) in Lemma 4.2 holds it suffices to verify that each $\partial \tilde{F}_t$ is the image of a piecewise smooth simple closed curve with internal angles different from 0 and $2\pi$. This is clear for $r \leq t \leq T$, since each $\tau_s$ is $C^1$ and piecewise smooth. Fix $\epsilon \leq t < r$. The boundary $\partial \tilde{F}_t$ has 4 vertices. Let $x_1$ be the vertex contained in $S_v(\tilde{\gamma}_i(0), r) \cap S_v(q_0, \frac{\pi}{2} + t)$. Since the intersection between the circles $S_v(\tilde{\gamma}_i(0), r)$ and $S_v(q_0, \frac{\pi}{2} + t)$ contains exactly two points for $\epsilon \leq t < r$, then they intersect themselves transversely, relatively to the ambient space $S^2$. In particular the corresponding internal angle at the vertex $x_1$ is different from 0 and $2\pi$. By symmetry we conclude the same fact about the other 3 vertices. Thus condition (3) in Lemma 4.2 holds.

Claim 4.9. **Condition (2) in Lemma 4.2 holds for $\epsilon \leq t \leq T$.**

If $\epsilon \leq t \leq t' \leq r$, we have by construction that $\tilde{F}_t \subset \tilde{F}_{t'}$. For $r \leq t \leq t' \leq T$ write $t = r + s$ and $t' = r + s'$. Set $D_0 = \tilde{\gamma}_i([0, s])$ and

$$D_s = B_v(q_0, d_s) \cap B_v(q^s, d_s),$$

if $0 < s \leq \bar{s}$. Thus Claim 4.9 follows from Lemma 5.4.

If $\partial M \neq \emptyset$ we will verify that condition (5) in Lemma 4.2 holds. We will show that for any $\epsilon < t < T$ and any $x \in \partial \tilde{F}_t$ with $x \notin N(F)$, there exists a piecewise smooth curve $\gamma_{xt} : [0, 1] \to \tilde{F}_t$ joining $N(w_i)$ and $x$ such that $L_v(\gamma_{xt}) < d_{3\beta}(w_i, \partial M)$. First assume that $\epsilon < t \leq r$. In this case we have that $x \in S_v(q_0, \frac{\pi}{2} + t) \cup S_v(q^0, \frac{\pi}{2} + t)$. Let $P(x)$ be the natural projection of $x$ onto $\tilde{\gamma}_i([0, s])$. Let $\gamma_{xt}$ be the piecewise smooth curve which follows the image of $\tilde{\gamma}_i$ from $N(w_i)$ to $P(x)$ and then follows the minimal geodesic from $P(x)$ to $x$. Since $B_v(\tilde{\gamma}_i([0, s]), t) \subset \tilde{F}_t$, we have that the image of $\gamma_{xt}$ is contained in $\tilde{F}_t$. By using inequality (3) above, we
have that
\[ L_\nu(\gamma_{xt}) \leq \frac{s_i}{2} + t \leq \frac{s_i}{2} + r < \delta_0 + \frac{\delta_0}{2} = \frac{3\delta_0}{2} < 3\delta_0 \leq d_\beta(w_i, \partial M). \]

Now assume that \( t = r + s \) for some \( 0 < s < s_1 \). Thus \( x \in S_\nu(D_s, r) \). Let \( P_1(x) \) be the natural projection from \( x \) to \( \partial D_s \). We define some piecewise smooth curve \( \gamma_{xt} : [0, 1] \to \tilde{F}_t \), which follows a minimal geodesic \( \chi \) from \( N(w_i) \) to \( P_1(x) \) and then a minimal geodesic from \( P_1(x) \) to \( x \). Since \( D_s \) is strongly convex we have that the image of \( \gamma_{xt} \) is contained in \( \tilde{F}_t \). We claim that \( L_\nu(\chi) \leq \frac{\delta_0}{2} \). Indeed, similarly as in the proof of Lemma 3.4 we see that the map \( y \in \partial D_s \mapsto d_\nu(N(w_i), y) \) attains its maximum at \( N(z) \) and \( N(w_i) \), hence we have that
\[ L_\nu(\chi) = L_\nu(X) + r \leq d_\nu(N(w_i), N(z)) + r = \frac{s_i}{2} + r < \frac{3\delta_0}{2} < d_\beta(w_i, \partial M). \]

Thus we may use Lemma 4.2 to obtain that there exists a connected set \( U_i \) with \( F \subseteq U_i \subseteq M - \partial M \) such that \( N|_{U_i} : U_i \to \bigcup_{r \leq t < T} \tilde{F}_t \) is a bijection and a \( C^1 \) local isometry. From the continuity of the map \( r \leq s \leq s_1 \mapsto \tau_s \) we see that \( \text{int}(\tilde{F}_T) \subseteq \bigcup_{r \leq t < T} \tilde{F}_t \). The set \( \tilde{F}_T \) is strongly convex, hence we have that \( \text{int}(\tilde{F}_T) \) is also strongly convex, since the map \( x \in \tilde{F}_T \to d_\nu(x, \partial \tilde{F}_T) \) is concave (see Theorem 1.10 in [CC]). Since \( N|_{U_i} \) is a bijection which is a \( C^1 \) local isometry we obtain easily that \( W_i = (N|_{U_i})^{-1}(\text{int}(\tilde{F}_T)) \) is strongly convex. Since \( \sigma(t_{i+1}) \in B_i = B_\beta(u_i, r) \) we have that \( z \) and \( \sigma(t_{i+1}) \) belong to \( \text{int}(X) \subseteq W_i \). We conclude that there exists a unique unit speed \( \beta \)-geodesic \( \gamma_{i+1} \) from \( z \) to \( u_{i+1} = \sigma(t_{i+1}) \).

We claim that the image of each \( \gamma_{i+1} \) is contained in \( M - \partial M \). To show this we may assume that \( \partial M \neq \emptyset \). Given \( x \) in the image of \( \gamma_{i+1} \), if \( d_\beta(z, x) \leq d_\beta(x, u_{i+1}) \) we obtain an estimate as in inequality (4), obtaining that \( d_\beta(x, \partial M) \geq 3\delta_0 \). If \( d_\beta(z, x) \geq d_\beta(x, u_{i+1}) \) we obtain similarly that \( d_\beta(x, \partial M) \geq 3\delta_0 \), hence \( x \in M - \partial M \). Thus we have that the image of \( \gamma_{i+1} \) is contained in \( M - \partial M \).

By triangle inequality we have that \( L_\beta(\gamma_{i+1}) = L_\nu(N \circ \gamma_{i+1}) \leq s_i + r < \pi \), hence \( L_\beta(\gamma_{i+1}) < \pi \). We conclude by induction that there exists a unique minimal \( \nu \)-geodesic \( \gamma : [0, d] \to M - \partial M \) from \( z \) to \( \sigma(t_k) = u \) with length less than \( \pi \), whose image is contained in \( M - \partial M \). Since \( N \) is a \( C^1 \) local isometry and \( L_\beta(\gamma) < \pi \), we have that \( N \) is injective on the image of \( \gamma \). In particular we have that \( N(z) \neq N(u) \), hence \( N \) is injective on \( B_\beta(p, \delta_0) \). Lemma 1.3 is proved.

**Proof of the Efimov’s Theorem.** Assume by contradiction that there exists an immersion \( \varphi : M \to \mathbb{R}^3 \) as in the statement of the Efimov’s Theorem. Let \( \alpha \), respectively, \( \beta \) be the Riemannian metrics induced by \( \varphi \), respectively, by the Gauss map \( N \). Lemmas 3.3 and 4.4 imply that \( N \) is injective on \( M \) and that any two points in \( M \) are joined by a minimizing geodesic. In particular \( M \) is simply-connected, hence \( (M, \alpha) \) is a Hadamard surface. Thus we have that \( A_\alpha(M) = +\infty \) (see [MI]). Since \( |K| \geq \kappa > 0 \) we have that \( \int_M |K|dA_\alpha = +\infty \), hence by equation (1) we have that \( A_\beta(M) = +\infty \). However, since \( N \) is injective on \( M \) and a \( C^1 \) local isometry we have that \( A_\beta(M) \leq A_\nu(S^2) \), which gives us a contradiction and proves Efimov’s Theorem.

**Lemma 4.5.** Consider a connected noncompact surface \( M \) with compact boundary \( \partial M \neq \emptyset \) and a \( C^1 \) local isometry \( N : (M, \beta) \to (S^2, \nu) \) such that there exists no point \( q \in \delta M \) at which \( M \) is concave. Then \( M \) is bounded with respect to the metric \( \beta \). More precisely, \( M \subseteq B_\beta(\partial M, 5\pi) \).
Proof. Assume by contradiction that there exists a point $p \in M$ with $d_\beta(p, \partial M) \geq 5\pi$. Consider a continuous curve $\gamma : [0, 1] \to M$ with $\gamma(0) = p$ and $\gamma(1) \in \partial M$. Thus there exists a point $q$ in the image of $\gamma$ such that $d_\beta(p, q) = \pi$. By Lemma 4.4 there exists a unit speed minimal $\beta$-geodesic $\gamma : [0, \pi] \to M - \partial M$ from $p$ to $q$.

Set $w = \gamma \left( \frac{\pi}{2} \right)$. We have that

$$d_\beta(w, \partial M) \geq d_\beta(p, \partial M) - d_\beta(p, w) = d_\beta(p, \partial M) - \frac{\pi}{2} > \pi,$$

and this contradicts Lemma 4.3. Lemma 4.5 is proved.

Lemma 4.6. Consider a connected noncompact surface $M$ with compact boundary $\partial M \neq \emptyset$ and a $C^1$ local isometry $N : (M, \beta) \to (S^2, \nu)$ such that there exists no point $q \in \delta M$ at which $\tilde{M}$ is concave. Then $(M, \beta)$ is pre-compact.

Proof. Since $\partial M$ is a finite union of circles, there exists $0 < \epsilon_0 < \pi$ such that the set $C = B_\beta(\partial M, \epsilon_0)$ is a closed collar neighborhood of $\partial M$. Set $S_0 = S_\beta(\partial M, \epsilon_0)$.

Consider a sequence $p_n$ in $M$. We need to prove that there exists a Cauchy subsequence of $p_n$. By compactness of $C$, if $d_\beta(p_n, C) \to 0$ then $p_n$ has a convergent subsequence. Thus we may assume, by passing to a subsequence and using the fact that $M$ is bounded, that there exists some $\delta > 0$ such that $d_n = d_\beta(p_n, C) \to d$.

For each $n$, there exists a unit speed piecewise smooth curve $\sigma_n : [0, L_n] \to M$ with $\sigma_n(0) \in S_0$ and $\sigma_n(L_n) = p_n$ such that

$$L_n = L_\beta(\sigma_n) < d + 1.$$

By discarding a piece of the image of $\sigma_n$, if necessary, we may assume that $\sigma_n((0, L_n]) \subset M - C$. Fix $k_0 \in \mathbb{N}$ such that

$$\frac{d + 1}{k_0} < \frac{\epsilon_0}{10}.
$$

Consider a partition $0 = t_{n,0} = t_{n,1} < \cdots < t_{n,k_0} = L_n$, such that for any $0 \leq i < k_0 - 1$ it holds that

(9) $$L_\beta(\sigma_n|_{[t_{n,i}, t_{n,i+1}]} \leq \frac{L_n}{k_0} \leq \frac{d + 1}{k_0} < \frac{\epsilon_0}{10}.$$

Set $p_{n,i} = \sigma_n(t_{n,i})$. We will prove by induction that for each $0 \leq i \leq k_0$ it holds that $p_{n,i}$ has a Cauchy subsequence, hence $p_n = p_{n,k_0}$ has a Cauchy subsequence. For $i = 0$, the compactness of $S_0$ implies this easily. Assume that, for some $0 \leq i < k_0 - 1$, we have, by passing to a subsequence, that $p_{n,i}$ is a Cauchy sequence in $(M, \beta)$. We need to prove that $p_{n,i+1}$ has a Cauchy subsequence.

There exists $n_1 \in \mathbb{N}$ such that if $n \geq n_1$ then $d_\beta(p_{n,1}, \nu) < \frac{\epsilon_0}{10}$. Set $q_i = p_{n,1}$. By using (9), we obtain that for any $n \geq n_1$ it holds that

$$d_\beta(q_i, p_{n,i+1}) \leq d_\beta(q_i, p_{n,i}) + d_\beta(p_{n,i}, p_{n,i+1}) < \frac{\epsilon_0}{10} + \frac{\epsilon_0}{10} = \frac{\epsilon_0}{5}.$$

Thus for $n \geq n_1$ we have that $p_{n,i+1} \in B = B_\beta \left( q_i, \frac{\epsilon_0}{5} \right)$. Lemma 4.3 implies that the map $N|_B : B \to B_\nu \left( N(q_i), \frac{\epsilon_0}{5} \right)$ is injective and that, for any $m, n \geq n_1$, there exists a unique unit speed minimal $\beta$-geodesic $\eta_{mn}$ from $p_{n,i+1}$ to $p_{m,i+1}$ with $L_\beta(\eta_{mn}) < \pi$. This implies that $N \circ \eta_{mn}$ is the unique speed minimal $\nu$-geodesic from $N(p_{n,i+1})$ to $N(p_{m,i+1})$.

By passing to a subsequence, we may assume that $N(p_{n,i+1})$ is a Cauchy sequence in $(S^2, \nu)$. Given $\epsilon > 0$, there exists $n_0 \geq n_1$ such that if $m, n \geq n_0$ then $d_\nu(\eta_{mn}) < \epsilon$. In particular we have that
Lemma 4.7. Consider a connected noncompact surface \( M \) with compact boundary \( \partial M \neq \emptyset \) and a \( C^1 \) local isometry \( N : (M, \beta) \to (S^2, \nu) \) such that there exists no point \( q \in \delta M \) at which \( M \) is concave. Then \( A_\beta(M) \) is finite.

Proof. As in the proof of Lemma 4.6 there exists \( 0 < \epsilon_0 < \pi \) such that the set \( C = B_\beta(\partial M, \epsilon_0) \) is a closed collar neighborhood of \( \partial M \). Set \( S_0 = S_\beta(\partial M, \epsilon_0) \). Fix \( 0 < \epsilon < \frac{\pi}{3} \). Since \( M \) is pre-compact then \( D = (M - C) \cup S_0 \) is pre-compact. Thus there exist points \( q_1, \ldots, q_{n_0} \in D \) such that

\[
D \subset \bigcup_{1 \leq i \leq n_0} B_\beta(q_i, \epsilon).
\]

Set \( B_i = B_\beta(q_i, \epsilon) \). By Lemma 4.3 \( N|_{B_i} \) is injective, hence

\[
A_\beta(D) \leq \sum_{i=1}^{n_0} A_\beta(B_i) = \sum_{i=1}^{n_0} A_\nu(N(B_i)) \leq \sum_{i=1}^{n_0} A_\nu(B_\nu(N(q_i), \epsilon)) = n_0 A_\nu(B_\nu(N(q_1), \epsilon)),
\]

hence \( A_\beta(M) \leq A_\beta(C) + n_0 A_\nu(B_\nu(N(q_1), \epsilon)) \). Lemma 4.7 is proved.

Proof of Theorem A. Assume by contradiction that there exists an immersion \( \varphi : M \to \mathbb{R}^3 \) as in the statement of Theorem A. Let \( \alpha \), respectively, \( \beta \) be the Riemannian metrics induced by \( \varphi \), respectively, \( N \). By using Lemmas 4.1 and 4.7 we obtain that \( A_\beta(M) \) is finite, which contradicts equation (1) and proves Theorem A.

5. Appendix - Proof of Lemma 3.1

To prove Lemma 3.1 we first assume that \( d_{\text{int}} \) is a distance on \( D \). To show that \( d_\varphi \) and \( d_{\text{int}} \) induce the same topology on \( D \), we need to prove that, given \( q \in D \) and \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( p \in D \) with \( d_\varphi(p, q) < \delta \) then \( d_{\text{int}}(p, q) < \epsilon \). If \( q \in \text{int}(D) \) the proof is trivial. Thus we will assume that \( q \in \partial D \).

Fix \( q \in \partial D \) and \( \epsilon > 0 \). For some small \( \lambda > 0 \), there exists a curve \( \sigma : [-\lambda, \lambda] \to \partial D, \lambda > 0 \), parameterized by the \( g \)-arc length satisfying that \( \sigma(0) = q \) and such that \( \sigma|[-\lambda, 0] \) and \( \sigma|[0, \lambda] \) are smooth curves. Since \( D \) is a piecewise smooth surface with boundary and the angles at the vertices differ from 0 and \( 2\pi \), there exists a unit vector \( v \in T_qS \) pointing to \( \text{int}(D) \) and transversal to both \( \sigma'(0-) \) and \( \sigma'(0+) \). Let \( v_t \) be the parallel transport of \( v \) along the both directions on \( \sigma \). If \( \lambda \) is small enough, we may assume that \( v_t \) is transversal to \( \sigma'(t) \) and that \( v_t \) points to \( \text{int}(D) \).

Set \( \sigma_s(t) = \exp_{\sigma(t)} sv_t = \gamma_t(s) \). By smoothness of the geodesic flow there exists sufficiently small \( 0 < \eta < \min \{ \lambda, \frac{\pi}{3} \} \) such that:

1. for \( 0 < s \leq \eta \) and \( -\eta \leq t \leq \eta \), the point \( \sigma_s(t) \in \text{int}(D) \);
2. \( L_g(\sigma_\eta) < \frac{\eta}{5} \).

Given \( s_0 \in [0, \eta] \) and \( t_0 \in [-\eta, \eta] \), we will construct a piecewise smooth curve \( \xi = \xi_{s_0 t_0} : [0, 2\eta + |t_0| - s_0] \to D \) from \( q \) to \( \sigma_{s_0}(t_0) \) satisfying \( \xi((0, 2\eta + |t_0| - s_0)) \subset \text{int}(D) \) and \( L_g(\xi) < \epsilon \). From \( q \) to \( \sigma_{\eta}(0) = \gamma_0(\eta) \), let \( \xi_{[0, \eta]} \) coincide with the geodesic \( \gamma_0 : [0, \eta] \to D \). From \( \sigma_{\eta}(0) \) to \( \sigma_{\eta}(t_0) \) the curve \( \xi \) follows the curve \( \sigma_{\eta} \) in the direction that \( t \) is increasing if \( 0 \leq t_0 \), or in the other direction if \( t_0 < 0 \). More precisely, for \( 0 \leq s \leq |t_0| \), we define \( \xi(\eta + s) = \sigma_{\eta}(s) \), if \( 0 \leq t_0 \), and \( \xi(\eta + s) = \sigma_{\eta}(-s) \), if \( t_0 < 0 \).
Finally, from $\sigma_q(t_0)$ to $\sigma_s(t_0)$ the curve $\xi$ follows the geodesic $s \mapsto \gamma_{t_0}(\eta - s)$.
Namely, for $0 \leq s \leq \eta - s_0$ we define $\xi((\eta - s_0) + s) = \gamma_{t_0}(\eta - s) = \sigma((\eta - s))(t_0)$. By construction we have that $L_g(\xi) \leq \eta + L_g(\sigma_q) + (\eta - s_0) < \epsilon$. Given $s_1 \in [0, \eta]$ and $t_1 \in [-\eta, \eta]$, a similar construction as above shows that $\sigma_{q_2}(t_0)$ may be connected to $\sigma_{s_1}(t_1)$ by a piecewise smooth curve $\psi : [0, 1] \to D$ with $\psi((0, 1)) \subset \text{int}(D)$ and $L_g(\psi) < \epsilon$.

Set $X = \{ \sigma_s(t) \mid 0 \leq s \leq \eta, -\eta \leq t \leq \eta \}$. Since $X$ is a compact neighborhood of $q$ in $D$, we have that $\delta = d_g(q, D - X) > 0$. Now we take $p \in D$ with $d_g(p, q) < \delta$.
We have that $p \in X$, hence $p = \sigma_{s_0}(t_0)$ for some $s_0 \in [0, \eta]$ and $t_0 \in [-\eta, \eta]$. As a consequence we have that $d_{\text{int}}(p, q) \leq L_g(\xi_{s_0}) < \epsilon$.

Now we consider points $p, q, r \in D$ and we will show that $d_{\text{int}}(p, q) + d_{\text{int}}(q, r) \geq d_{\text{int}}(p, r)$. We will just consider the case that $q \in \partial D$, since the other is trivial. Fix $\epsilon > 0$ and consider piecewise smooth curves $\gamma : [0, 1] \to D$ from $p$ to $q$ with $\gamma((0, 1)) \subset \text{int}(D)$ and $L_g(\gamma) < d_{\text{int}}(p, q) + \epsilon$, and $\sigma : [0, 1] \to D$ from $q$ to $r$ with $\sigma((0, 1)) \subset \text{int}(D)$ and $L_g(\sigma) < d_{\text{int}}(q, r) + \epsilon$. By using a neighborhood $X$ of $q$ as above, it is easy to obtain a piecewise smooth curve $\varphi : [0, 1] \to D$ from $p$ to $r$ with $\varphi((0, 1)) \subset \text{int}(D)$ and $L_g(\varphi) < L_g(\gamma) + L_g(\sigma) + \epsilon$. In fact, take $0 < s_1 < 1$ such that $\gamma(s_1) \in X - \{ q \}$ and $0 < s_2 < 1$ such that $\sigma(s_2) \in X - \{ q \}$. We define a curve $\varphi$ which follows $\gamma$ from $t = 0$ to $t = s_1$ then follows a curve $\psi$ in $X \cap \text{int}(D)$ with $L_g(\psi) < \epsilon$, and then follows $\sigma$ from $t = s_2$ to $t = 1$. Thus we obtain that $d_{\text{int}}(p, q) + d_{\text{int}}(q, r) + 3\epsilon > d_{\text{int}}(p, r)$. By making $\epsilon \to 0$ we conclude the proof of Lemma 3.1.

Remark 4. It is not difficult to see that Lemma 3.1 may be improved to assume that internal angles are just different from 0, but this weaker assumption is not necessary for the proof of Theorem A.

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