Lorentz Symmetry Breaking and Causality

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Abstract. We investigate Lorentz symmetry (CPT even) breakdown without assuming that it is small. We look for constraints necessary for compatibility with causal structure in the context of scalar theories and QED.

1. Causality with Lorentz invariance
A simple model with Lorentz invariance is a scalar field theory with the action

\[ S = \int d^4x \Omega \frac{1}{2} \left( g^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) + m^2 \phi^2(x) \right), \]  
(1)

with \( \Omega = \sqrt{-\det g_{\mu\nu}} \). We assume that the metric is constant throughout space-time.

An observer is associated with a space-time plane that is space-like with respect to the metric \( g_{\mu\nu} \). Time evolution is associated with a space-like foliation of space-time. The foliations are given a time label by means of a time-like vector \( n^\mu \). This is illustrated in figure 1. Under Lorentz transformations \( g_{\mu\nu} \) is invariant as is the associated light-cone. The space-like foliation is changed corresponding to the associated change of observer.

![Figure 1](image1)

**Figure 1.** Example of a light-cone with space-like surface and time-like evolution direction.

2. Lorentz symmetry breaking
A model that exhibits Lorentz symmetry breaking is one with two fields [1] governed by an action
\[ S = \int d^4x \Omega \frac{1}{2} \{ g^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) + m^2 \phi^2(x) \} \]
\[ + \int d^4x \Omega \frac{1}{2} \{ \bar{g}^{\mu\nu} \partial_\mu \psi(x) \partial_\nu \psi(x) + \bar{m}^2 \psi^2(x) \}, \]

(2)

We are free to set \( \bar{g}_{\mu\nu} = -\Omega^2 \). If the two metrics are not identical then the model exhibits a breaking of Lorentz symmetry but with no restriction on its magnitude. Figure 2 indicates possible relationships between the two light-cones that are consistent with causal propagation because the two metrics share foliations of space-time that are space-like in both.

\[ \text{Figure 2. Examples of pairs of light-cones that share space-like surfaces. The first two examples represent situations in which there are space-time vectors that are time-like in both cones. In the third case each light-cone is seen as space-like from the other.} \]

It is formally possible for the two light-cones to be so rotated with respect to each other that there are no foliations of space-time that are space-like with respect to both metrics. We exclude such circumstances on the grounds that they do not permit simultaneous causal propagation of the two fields.

There have been many careful studies of the breakdown of Lorentz invariance [2, 3, 4] with the conclusion that observations constrain it to be a very small effect indeed [5]. Nevertheless it is interesting to consider what issues of principle are involved in dealing with quantum field theory in the context of a multi-metric space-time without any \textit{a priori} constraint on the relationship of the various metrics except for the general requirement of maintaining a causal structure for the theory. These issues of causality and multiple lightcone structure appear in other contexts [6, 7].

3. Mixing interaction
We encounter further restrictions on the relationship between the two metrics when we include an interchange term in the action [1] of the form

\[ S_I = \int d^4x \Omega \{ w^2 \phi(x) \psi(x) \}. \]

(3)

The equations of motion become

\[ g^{\mu\nu} \partial_\mu \partial_\nu \phi(x) + m^2 \phi(x) + w^2 \psi(x) = 0, \]
\[ \bar{g}^{\mu\nu} \partial_\mu \partial_\nu \psi(x) + \bar{m}^2 \psi(x) + w^2 \phi(x) = 0. \]

(4)

Plane wave solutions are \( \phi(x) = A e^{-iq_\mu x^\mu} \) and \( \psi(x) = \bar{A} e^{-iq_\mu x^\mu} \), where

\[ (g^{\mu\nu} q_\mu q_\nu - m^2)A - w^2 \bar{A} = 0, \]
\[ (\bar{g}^{\mu\nu} q_\mu q_\nu - \bar{m}^2)\bar{A} - w^2 A = 0. \]

(5)

For consistency we have the determinantal condition which yields the dispersion relation

\[ (g^{\mu\nu} q_\mu q_\nu - m^2)(\bar{g}^{\mu\nu} q_\mu q_\nu - \bar{m}^2) - w^4 = 0. \]

(6)
In the marginal case \( w^2 = m\bar{m} \) this becomes for low \( q_\mu \)
\[
(m^2 \hat{g}^{\mu\nu} + \bar{m}^2 \hat{g}^{\mu\nu})q_\mu q_\nu = 0. \tag{7}
\]
The effective light-cone at low \( q_\mu \) has the form
\[
\hat{g}^{\mu\nu}(u)q_\mu q_\nu = 0, \tag{8}
\]
where \( \hat{g}^{\mu\nu}(u) = u g^{\mu\nu} + (1 - u)\bar{g}^{\mu\nu} \) with \( u = m^2/(m^2 + \bar{m}^2) \). Hence \( \hat{g}^{\mu\nu}(u) \) interpolates between the two original metrics. To maintain causality it is important that \( \det \hat{g}^{\mu\nu}(u) \) does not vanish and change sign for \( 0 \leq u \leq 1 \). This requires the two original metrics not only to share space-like foliations but in addition to have in common some time-like vectors. Geometrically this means that example \((iii)\) in figure 2 is ruled out. Detailed arguments are presented in Ref. [1].

4. Quartic interaction
The above argument for the interchange interaction is suggestive rather than rigorous. However the same issue must be addressed in a model involving quartic interactions that include the term
\[
S_I = \int d^4x \Omega \left\{-\frac{1}{4}\sigma\phi^2(x)\psi^2(x)\right\}. \tag{9}
\]

Figure 3. Loop diagram resulting from the \( \phi^2\psi^2 \) interaction. Solid lines correspond to the \( \phi \)-propagator and dashed lines to the \( \psi \)-propagator.

In the computation of Green’s functions this leads to a Feynman diagram of the form shown in figure 3. Using the Schwinger representation of field propagators we obtain (at zero momentum) the result
\[
\int d^n k \int_0^1 du \int_0^\infty d\lambda \lambda \exp\{i\lambda(\hat{g}^{\mu\nu}(u) - um^2 - (1 - u)\bar{m}^2)\},
\]
where \( \hat{g}^{\mu\nu}(u) \) is given above. This result is proportional to
\[
\int_0^1 \frac{du}{\sqrt{-\det \hat{g}_{\mu\nu}(u)(um^2 + (1 - u)\bar{m}^2)^{n/2-2}}. \]

We see immediately in this case we must require that \( \det \hat{g}_{\mu\nu}(u) \neq 0 \) in the range \( 0 \leq u \leq 1 \) in order that the result is well defined.

5. Lorentz symmetry breaking in QED - photons
The most general gauge invariant action for electrodynamics is the pre-metric form [8]
\[
S = \int d^4x U^{\mu\nu\sigma\tau} F_{\mu\nu}(x) F_{\sigma\tau}(x). \tag{10}
\]
Here
\[
F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{11}
\]
The tensor density $U^{\mu\nu\sigma\tau}$ has the symmetry properties of the Riemann tensor.

\begin{align}
 U^{\mu\nu\sigma\tau} &= -U^{\nu\mu\sigma\tau} = U^{\sigma\tau\mu\nu}, \\
 U^{\mu\nu\sigma\tau} + U^{\mu\sigma\nu\tau} + U^{\mu\tau\nu\sigma} &= 0. 
\end{align}

Standard Lorentz invariant electrodynamics is associated with an (invariant) metric $g_{\mu\nu}$ for which

\begin{equation}
 U^{\mu\nu\sigma\tau} \propto g^{\mu\sigma} g^{\nu\tau} - g^{\nu\sigma} g^{\mu\tau}. \tag{14}
\end{equation}

Although pre-metric theory has apparently no associated metric, it does in fact have a preferred metric $-g_{\mu\nu}$ [9]. It can be defined by the requirement that it makes $g^{\mu\sigma} g^{\nu\tau} - g^{\nu\sigma} g^{\mu\tau}$ “closest to” $U^{\mu\nu\sigma\tau}$. We realise this idea by demanding that the overlap $F = U^{\mu\nu\sigma\tau}(g^{\mu\sigma} g^{\nu\tau} - g^{\nu\sigma} g^{\mu\tau})$ is stationary with respect to variations of $g_{\mu\nu}$ subject to the constraint $\det g_{\mu\nu} = -\Omega^2$. Introducing the Lagrange multiplier $\lambda$, we have

\begin{equation}
 \delta (F - \lambda \det g_{\mu\nu}) = 0, \tag{15}
\end{equation}

with as a result the eigenvalue-like equation [10]

\begin{equation}
 U^{\mu\nu\sigma\tau} g_{\nu\tau} + \lambda \Omega^2 g^{\mu\sigma} = 0. \tag{16}
\end{equation}

There is a standard decomposition of a tensor with the symmetry properties of $U^{\mu\nu\sigma\tau}$ namely

\begin{equation}
 U^{\mu\nu\sigma\tau} = \frac{U}{12} (g^{\mu\sigma} g^{\nu\tau} - g^{\nu\sigma} g^{\mu\tau}) \\
 + \frac{1}{2} (U^{\mu\sigma} g^{\nu\tau} + U^{\nu\tau} g^{\mu\sigma} - U^{\nu\sigma} g^{\mu\tau} - U^{\mu\tau} g^{\nu\sigma}) \\
 - C^{\mu\nu\sigma\tau}, \tag{17}
\end{equation}

where $U = U^{\mu\nu\sigma\tau} g_{\mu\sigma} g_{\nu\tau}$ and $U^{\mu\sigma} = U^{\mu\nu\sigma\tau} g_{\nu\tau} - (1/4)U g^{\mu\sigma}$. The residual tensor $C^{\mu\nu\sigma\tau}$ has the symmetry properties of $U^{\mu\nu\sigma\tau}$. In addition it is traceless,

\begin{equation}
 C^{\mu\nu\sigma\tau} g_{\nu\tau} = 0. \tag{18}
\end{equation}

That is $C^{\mu\nu\sigma\tau}$ has the properties of the Weyl tensor. We refer to it as a Weyl-like tensor (WLT). In the present case for our preferred metric we see that $U^{\mu\sigma} = 0$, so this decomposition reduces to

\begin{equation}
 U^{\mu\nu\sigma\tau} = \frac{U}{12} (g^{\mu\sigma} g^{\nu\tau} - g^{\nu\sigma} g^{\mu\tau}) - C^{\mu\nu\sigma\tau}, \tag{19}
\end{equation}

We can therefore use the standard Petrov classification of Weyl tensors [11] to classify Lorentz symmetry breaking in electrodynamics. We note that the gauge invariant equations of motion, for an appropriate normalisation of $U^{\mu\nu\sigma\tau}$, take the form

\begin{equation}
 (g^{\mu\sigma} g^{\nu\tau} - g^{\nu\sigma} g^{\mu\tau} - C^{\mu\nu\sigma\tau}) \partial_\mu \partial_\sigma A_\nu (x) = 0. \tag{20}
\end{equation}

6. Petrov classification of Weyl tensors

The Petrov classification, see Refs. [12, 13], is based on light-like vectors $v_\mu$ that satisfy

\begin{equation}
 v_\mu C_{\mu}[\nu \sigma[\tau v_\beta] \nu^\nu v^\sigma = 0. \tag{21}
\end{equation}

The 4-vector $v^\mu$ is referred to as a principal null direction - PND. There are four PNDS that are solutions though they may coincide, with various multiplicities. The geometrical significance of
the PNDs is most easily grasped using the Newman-Penrose spinor formalism but this requires a whole separate presentation see Refs. [12, 13]. More elementary accounts of this approach can be found in Refs. [14, 15].

The Petrov classification is based on the number and multiplicities of the PNDs. The extra conditions satisfied by PNDs of higher multiplicity can be found in Refs. [12, 13].

| Four distinct PNDs | - Class I |
| Two distinct PNDs each with multiplicity 2 | - Class D |
| Three distinct PNDs one with multiplicity 2 | - Class II |
| Two distinct PNDs with multiplicities 3 and 1 | - Class III |
| A single PND with multiplicity 4 | - Class N |

6.1. Newman-Penrose tetrad

The Newman-Penrose tetrad is useful for analysing the structure of WLTs. It comprises four null vectors \( l_\mu, n_\mu \) (real), \( m_\mu, \bar{m}_\mu \) (complex conjugate). The only nonvanishing scalar products are

\[ l_\mu n_\mu = 1 = -m_\mu \bar{m}_\mu. \]

Hence

\[ g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu. \]

6.2. Canonical forms - simplest case

Each Petrov class can be given a canonical form that reveals the structure of the WLT \( C^{\mu\nu\sigma\tau}. \)

The simplest case is class N. Choose the PND to be \( l_\mu \) of the NP tetrad and construct the anti-symmetric tensor \( A_{\mu\nu} = l_\mu m_\nu - m_\mu l_\nu. \) The canonical form is

\[ C_{\mu\nu\sigma\tau} = \lambda A_{\mu\nu} A_{\sigma\tau} + \text{c.c.} \] (22)

We can choose \( \lambda \) to be real. Other classes have more complicated canonical forms.

For plane wave solutions \( A_\mu(x) = \epsilon_\mu e^{-iq_\nu x^\nu} \) the gauge invariant equations of motion are, after imposing the gauge condition \( q_\mu \epsilon_\mu = 0 \),

\[ (q^2 g^{\tau\nu} - C^{\mu\nu\sigma\tau} q_\mu q_\sigma) \epsilon_\nu = 0. \] (23)

For Petrov class N this becomes

\[ (q^2 g^{\tau\nu} - \lambda(Q^\tau Q^\nu + \bar{Q}^\tau \bar{Q}^\nu)) \epsilon_\nu = 0. \] (24)

where \( Q^\nu = A^{\mu\nu} q_\mu. \) Using the results \( Q^2 = \bar{Q}^2 = 0, \) and \( Q_\mu \bar{Q} = -(l.q)^2 \) we find

\[ \left( \frac{q^2}{\lambda(l.q)^2} \right)^2 \left( \begin{array}{c} Q_\epsilon \\ Q_\bar{\epsilon} \end{array} \right) = 0. \] (25)

The consistency of these equations requires the determinantal condition \( (q^2)^2 - (\lambda(l.q)^2)^2 = 0 \) which factorises to yield two light-cones \( g^{(\pm)\mu\nu} q_\mu q_\nu = 0, \) where

\[ g^{(\pm)\mu\nu} = g_{\mu\nu} \pm \lambda l_\mu l_\nu. \]

The two light-cones correspond to different photon polarisations - hence birefringence. It is easy to check that the interpolating metric \( \hat{g}_{\mu\nu}(u) = u g^{(+)\mu\nu} + (1-u) g^{(-)\mu\nu} \) is never singular. The
light-cone structure is therefore consistent with causality in straightforward manner. See also Ref. [16].

The quartic dispersion relation for other Petrov classes may not factorise into two quadratic light-cones see Ref. [9]. For example class III (see Ref. [9]) yields

\[(q^2 - (l.q)^2)^2 - (l.q)^2(4m.q - l.q)(4\bar{m}.q - l.q) = 0.\]

Here the photon propagator does not have a Schwinger-style representation which results in further complication in analysing the causal structure in perturbation theory in QED.

7. QED - photons and electrons

QED requires us to introduce the electron-photon interaction and in particular to evaluate the vacuum polarisation diagram in figure 4.

![Figure 4. Vacuum polarisation diagram in QED.](image)

If Lorentz symmetry breaking arises because of a different metric, \(\bar{g}^{\mu\nu}\), for the electron then the vacuum polarisation tensor has a structure that reflects this new metric and exhibits a UV divergence of the form

\[\Sigma^{\mu\nu}(q) = \frac{e^2}{6\pi^2 n - 4} W^{\mu\alpha\nu\beta} q_\alpha q_\beta,\]

where

\[W^{\mu\alpha\nu\beta} = \bar{g}^{\mu\nu} \bar{g}^{\alpha\beta} - \bar{g}^{\mu\beta} \bar{g}^{\nu\alpha}.\]

The tensor coefficient of the UV divergence can be decomposed in the standard way to reveal a contribution \(V^{\mu\alpha\nu\beta}\) that is a WLT,

\[W^{\mu\alpha\nu\beta} = \hat{W}^{\mu\alpha\nu\beta} - V^{\mu\alpha\nu\beta}.\]

The presence of this \(V\)-tensor in the UV divergences of QED obliges us to include a counter term in the photon action of the kind discussed above:

\[\propto C^{\mu\alpha\nu\beta} F_{\mu\alpha}(x) F_{\nu\beta}(x)\]

The remaining \(\hat{W}\)-term relates to the renormalisation of the metric and other parameters.

We have then the following picture. If Lorentz invariance is broken with a new metric for the electron field, the resulting structure for the UV divergences forces explicit Lorentz symmetry breaking on the photon field with consequential birefringence.

Actually there exists an exception to this situation. The electron and photon metrics may both be invariant under a subgroup of the Lorentz group with an invariant 4-vector. The 4-vector may be time-like space-like or light-like [2, 9]. In these circumstances the \(V\)-tensor vanishes and there is no need to introduce a birefringence inducing term for the photon field. The lack of equality between the two metrics still gives rise to Lorentz symmetry breaking.
8. Conclusions
We examined simple models in which Lorentz symmetry was violated by the presence of different space-time metrics associated with different fields. In order to maintain a causal structure we showed in the case of two interacting scalar fields that the two light-cones associated with the two metrics must not only have in common a set of planes space-like in both but must also overlap in such a way as to share a set of vectors that are time-like in both. This makes possible the existence of Minkowski rotations applicable simultaneously to both metrics in a way consistent with causality.

We considered electrodynamics in a premetric formulation which is the most general model encoding Lorentz symmetry violation. By establishing a preferred metric it was possible to identify Lorentz symmetry violation with an associated Weyl-like tensor (WLT) and to use the Petrov classification to identify different types of violation and resulting birefringence. When the Lorentz violation is viewed as originating in a separate metric for the electron field we indicated how perturbative renormalisation in general requires the introduction of the type of counter-terms for the photon field corresponding to a WLT and hence birefringence. This can be avoided when the two metrics are invariant under an appropriate subgroup of the Lorentz group. There is still Lorentz symmetry violation but no birefringence.

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