GENERATING FUNCTION FOR GL$_n$-INVARIANT DIFFERENTIAL OPERATORS IN THE SKEW CAPPELLI IDENTITY

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ABSTRACT. Let Alt$_n$ be the vector space of all alternating $n \times n$ complex matrices, on which the complex general linear group GL$_n$ acts by $x \mapsto gxg^t$. The aim of this paper is to show that Pfaffian of a certain matrix whose entries are multiplication operators or derivations acting on polynomials on Alt$_n$ provides a generating function for the GL$_n$-invariant differential operators that play a role in the skew Capelli identity, with coefficients the Hermite polynomials.

1. INTRODUCTION

Let Alt$_n$ be the vector space consisting of all alternating $n \times n$ complex matrices, and C[Alt$_n$] the C-algebra of all polynomial functions on Alt$_n$. Then the complex general linear group GL$_n$ acts on Alt$_n$ by

$$g \cdot x := gxg^t \quad (g \in \text{GL}_n, x \in \text{Alt}_n),$$

from which one can define a representation $\pi$ of GL$_n$ on C[Alt$_n$] by

$$\pi(g)f(x) := f(g^{-1} \cdot x) \quad (g \in \text{GL}_n, f \in \text{C}[\text{Alt}_n]).$$

For $x = (x_{i,j})_{i,j=1,...,n} \in \text{Alt}_n$, with $x_{j,i} = -x_{i,j}$, let $M := (x_{i,j})_{i,j}$ and $D := (\partial_{i,j})_{i,j}$ be the alternating $n \times n$ matrices whose $(i, j)$-th entries are given by the multiplication operator $x_{i,j}$ and the derivation $\partial_{i,j} := \partial/\partial x_{i,j}$, respectively. Then the representation $d\pi$ of $\text{gl}_n$, the Lie algebra of GL$_n$, induced from $\pi$ is given by

$$d\pi(E_{i,j}) = -\sum_{k=1}^{n} x_{k,i} \partial_{k,i} \quad (i, j = 1, 2, \ldots, n)$$

where $E_{i,j}$ denotes the matrix unit of size $n \times n$ which is a basis for $\text{gl}_n$.

Let us denote the ring of differential operators on Alt$_n$ with polynomial coefficients by $\mathcal{D} \text{Alt}_n$, and its subring consisting of GL$_n$-invariant differential operators by $\mathcal{D} \text{GL}_n \text{Alt}_n$. Moreover, for a positive integer $n$, $[n]$ denotes the set $\{1, 2, \ldots, n\}$, and for a real number $x$, $\lfloor x \rfloor$ the greatest integer not exceeding $x$. Then the following fact is known:

**Theorem** ([4]). For $k = 0, 1, \ldots, \lfloor n/2 \rfloor$, let

$$\Gamma_k := \sum_{\ell \in \binom{[n]}{\ell}} \text{Pf}(x_{\ell}) \text{Pf}(\partial_{\ell}),$$

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where the summation is taken over all \( I \subset [n] \) such that its cardinality is \( 2k \), and \( x_i \) and \( \partial_i \) denote submatrices of \( M \) and \( D \) consisting of \( x_{i,j} \) and \( \partial_{i,j} \) with \( i, j \in I \), respectively. Then \( \{ \Gamma_k \}_{k=0,1,...,[n/2]} \) forms a generating system for \( \mathcal{P} \mathcal{D}(\text{Alt}_n)^{\text{GL}_n} \).

The aim of this paper is to find a generating function for \( \{ \Gamma_k \} \).

Following [6, 8], let us consider an alternating matrix with entries in \( \mathcal{P} \mathcal{D}(\text{Alt}_n) \) given by:

\[
\Phi(u) := \begin{bmatrix}
0 & x_{1,2} & \cdots & x_{1,n} & u \\
-x_{1,2} & 0 & \ddots & \vdots & u \\
\vdots & \ddots & \ddots & 0 & \ddots \\
-x_{1,n} & \cdots & -x_{n-1,n} & 0 & \ddots \\
-u & -u & \cdots & -u & 0 \\
-u & -\partial_{1,2} & \cdots & -\partial_{1,n} & 0 \\
-u & -\partial_{2,1} & \cdots & -\partial_{n,1} & 0 \\
-u & -\partial_{3,1} & \cdots & -\partial_{n,2} & 0 \\
\end{bmatrix}
\]

with \( u \in \mathbb{C} \) a parameter. We remark that the matrix \( \Phi(u) \) (or rather, \( \Phi(u) \) given below) naturally appears if we regard \( \text{GL}_n \) and \( \text{Alt}_n \) as a subgroup of the complex special orthogonal group \( \text{SO}_{2n} \) by the map (2.4) below and the holomorphic tangent space at the origin of the corresponding Hermitian symmetric space of noncompact type, respectively (see [2] for details, though we only deal with its commutative counterpart therein, i.e. the principal symbol).

Our main result of this paper is the following. Pfaffian \( \text{Pf}(\Phi(u)) \) of \( \Phi(u) \) (see the next paragraph for the definition of Pfaffian) provides a generating function for \( \{ \Gamma_k \} \), with coefficient being monic polynomial in \( u \) of degree \( n - 2k \), which is essentially equal to the Hermite polynomial, i.e.

\[
\text{Pf}(\Phi(u)) = \sum_{k=0}^{\lfloor n/2 \rfloor} \left(-\frac{i}{2}\right)^{n-2k} H_{n-2k}(iu) \Gamma_k,
\]

where \( i = \sqrt{-1} \) and \( H_m(x) \) denotes the Hermite polynomial of degree \( m \). Note that the minor summation formula of Pfaffian with commutative entries (cf. [3]) immediately implies that the principal symbol \( \sigma(\text{Pf}(\Phi(u))) \) of \( \text{Pf}(\Phi(u)) \) can be expanded as

\[
\sigma(\text{Pf}(\Phi(u))) = \sum_{k=0}^{\lfloor n/2 \rfloor} u^{n-2k} \gamma_k,
\]

where \( \gamma_k \) denotes the principal symbol of \( \Gamma_k \).

In general, for an associative algebra \( \mathcal{A} \) over a field \( \mathbb{K} \) of characteristic 0, which is not necessarily commutative, Pfaffian \( \text{Pf}(A) \) of an alternating matrix \( A = (A_{i,j}) \), \( A_{j,i} = -A_{i,j} \in \mathcal{A} \), is defined by

\[
\text{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \text{sgn}(\sigma) A_{\sigma(1),\sigma(2)}A_{\sigma(3),\sigma(4)} \cdots A_{\sigma(n-1),\sigma(n)}
\]
\[
\frac{1}{n!} \sum_{\sigma \in \Sigma_{2n} \atop \sigma(2i-1) < \sigma(2i)} \text{sgn}(\sigma) A_{\sigma(1),\sigma(2)} A_{\sigma(3),\sigma(4)} \cdots A_{\sigma(2n-1),\sigma(2n)}. 
\]
(c.f. [7]). If the algebra \( \mathcal{A} \) happens to be commutative, then this reduces to:

\[
Pf(A) = \sum_{\sigma} \text{sgn}(\sigma) A_{\sigma(1),\sigma(2)} A_{\sigma(3),\sigma(4)} \cdots A_{\sigma(2n-1),\sigma(2n)},
\]
where the summation is taken over those \( \sigma \in \Sigma_{2n} \) satisfying

\[
\sigma(2i-1) < \sigma(2i) \quad (i = 1, 2, \ldots, n) \quad \text{and} \quad \sigma(1) < \sigma(3) < \cdots < \sigma(2n-1).
\]
When dealing with Pfaffian, however, it is sometimes convenient to consider square matrices alternating along the anti-diagonal, which we call anti-alternating for short in this paper. Note that a \( 2n \times 2n \) matrix \( X \) is anti-alternating if and only if \( X J_{2n} \) is alternating, where \( J_{2n} \) denotes the nondegenerate \( 2n \times 2n \) symmetric matrix with \( 1 \)'s on the anti-diagonal and \( 0 \)'s elsewhere. We simply denote \( \text{Pf}(X J_{2n}) \) by \( \text{Pf}(X) \) when there is no danger of confusion. Moreover, adopting the convention that \( -i \) means \( 2n + 1 - i \) for \( i = 1, \ldots, 2n \), a square matrix \( X = (X_{i,j}) \) is anti-alternating if and only if \( X_{i,j} = -X_{j,-i} \) for all \( i, j \). Thus, we will consider the anti-alternating matrix given by

\[
\Phi(u) := \tilde{\Phi}(u) J_{2n}
\]
and calculate its Pfaffian in what follows.

The organization of this paper is as follows. In Section 2, we show that \( \text{Pf}(\Phi(u)) \) is invariant under the action of \( \text{GL}_n \). In Section 3, we calculate Pfaffian \( \text{Pf}(\Phi(u)) \) and show that it provides a generating function for \( \{\Gamma_k\} \) with coefficient essentially equal to the Hermite polynomial.

2. Invariant Differential Operators

As in the Introduction, let \( \pi \) denote the representation of \( \text{GL}_n \) on \( \mathbb{C}[\text{Alt}_n] \) defined by (1.2), and let \( M_{i,j} \) and \( D_{i,j} \) denote the multiplication operator by \( x_{i,j} \) and the derivation \( \partial_{i,j} \), respectively. The conjugation by \( \pi(g) \) of them are given by the following.

Lemma 2.1. We have

\[
\pi(g) D_{i,j} \pi(g)^{-1} = \sum_{a < b} \det(g_{i,j}^{a,b}) D_{a,b}, \quad (2.1)
\]
\[
\pi(g) M_{i,j} \pi(g)^{-1} = \sum_{a < b} \det((g^{-1})^{i,j}_{a,b}) M_{a,b} \quad (2.2)
\]
for all \( g = (g_{a,b})_{a,b} \in \text{GL}_n \), where \( g_{i,j}^{a,b} \) denotes a \( 2 \times 2 \) submatrix of \( g \) whose row- and column indices are in \( [a, b] \) and \( [i, j] \), respectively.

Proof. First, we note that

\[
g(E_{i,j} - E_{j,i}) g' = \sum_{a < b} \det(g_{i,j}^{a,b})(E_{a,b} - E_{b,a}). \quad (2.3)
\]
Therefore, setting \( x = \sum_{a < b} x_{a,b} (E_{a,b} - E_{b,a}) \), we have

\[
\pi(g) D_{i,j} \pi(g)^{-1} f(x) = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} f(x + \epsilon g(E_{i,j} - E_{j,i}) g').
\]
and hence obtain the first formula.

As for the multiplication operator $M_{i,j}$, it follows from (2.3) that the $(i, j)$-th entry of $g^{-1}x(g')^{-1}$ equals $\sum_{a<b} \det((g^{-1})_{a,b})x_{a,b}$. Therefore,

$$\pi(g)M_{i,j}\pi(g)^{-1}f(x) = \left(M_{i,j}\pi(g)^{-1}f\right)(g^{-1}x(g')^{-1})$$

$$= \sum_{a<b} \det((g^{-1})_{a,b})x_{a,b} \left(\pi(g)^{-1}f\right)(g^{-1}x(g')^{-1})$$

$$= \sum_{a<b} \det((g^{-1})_{a,b})M_{a,b}f(x).$$

This completes the proof. \hfill $\Box$

Henceforth, we will use $x_{i,j}$ and $\partial_{i,j}$ to denote $M_{i,j}$ and $D_{i,j}$ for simplicity.

For $g \in \text{GL}_n$ and $X = (X_{i,j}) \in \text{Mat}_{2n}(\mathbb{C}) \otimes \mathcal{P}(\text{Alt}_n)$, let us denote by $\text{Ad}_{\pi(g)}(X)$ the $2n \times 2n$ matrix whose $(i, j)$-th entry is given by $\pi(g)X_{i,j}\pi(g)^{-1}$ for $i, j = 1, \ldots, 2n$, following [7]. Furthermore, let $\text{SO}_{2n} := \{g \in \text{GL}_{2n}; g^tJ_{2n}g = J_{2n}, \det g = 1\}$, and $\iota$ the embedding of $\text{GL}_n$ into $\text{SO}_{2n}$ given by

$$\iota : g \mapsto \begin{bmatrix} g & 0 \\ 0 & J_n(g')^{-1}J_n \end{bmatrix}.$$  \hfill (2.4)

**Proposition 2.2.** Let $\Phi(u)$ be the matrix given by (1.6). Then we have

$$\text{Ad}_{\pi(g)}(\Phi(u)) = \iota(g')\Phi(u)\iota(g)^{-1}$$ \hfill (2.5)

for all $g \in \text{GL}_n$.

**Proof.** If we denote the $n \times n$ matrices $(\pi(g)\partial_{i,j}\pi(g)^{-1})_{i,j}$ and $(\pi(g)x_{i,j}\pi(g)^{-1})_{i,j}$ by $\tilde{D}$ and $\tilde{M}$, respectively, then the left-hand side of (2.5) can be written as

$$\text{Ad}_{\pi(g)}(\Phi(u)) = \begin{bmatrix} u1_n & \tilde{D}J_n \\ -J_n\tilde{M} & -u1_n \end{bmatrix}.$$  \hfill (2.5)

On the other hand, since the upper-right block and the lower-left block of $\Phi(u)$ can be written as $DJ_n$ and $-J_nM$, respectively, the right-hand side of (2.5) equals

$$\begin{bmatrix} g' & J_n g^{-1}J_n \\ J_n g^{-1}J_n & -J_nM & -u1_n \end{bmatrix} \begin{bmatrix} u1_n & DJ_n \\ -J_nM & -u1_n \end{bmatrix}$$

$$= \begin{bmatrix} u1_n & g'DgJ_n \\ -J_n g^{-1}M(g')^{-1} & -u1_n \end{bmatrix}.$$  \hfill (2.5)

Now, it follows from (2.3) that

$$g'Dg = \sum_{i<j} \partial_{i,j}g^t(E_{i,j} - E_{j,i})g$$

$$= \sum_{i<j_a<b} \det(g_{i,j}^a)\partial_{a,b}(E_{i,j} - E_{j,i}),$$
which equals the matrix $\hat{D}$ by Lemma 2.1. The same calculation shows that $g^{-1}M(g')^{-1} = \hat{M}$. Thus we obtain the proposition. 

As in the commutative case, the noncommutative Pfaffian transforms under the action of $GL_{2n}(K)$ as follows (see [7]).

**Lemma 2.3.** Let $X$ be an anti-alternating matrix with coefficient in $\mathcal{A}$. For $g \in GL_{2n}(K)$, we have

$$Pf(gXg'^t) = \det g \cdot Pf(X),$$

where we set $g'^t := J_{2n}g'J_{2n}$ for brevity.

By Proposition 2.2 and Lemma 2.3, we obtain the following.

**Corollary 2.4.** The Pfaffian $Pf(\Phi(u)) \in \mathcal{PD}(\text{Alt}_n)$ is invariant under the action of $GL_n$. Namely, we have

$$\pi(g)Pf(\Phi(u))\pi(g)^{-1} = Pf(\Phi(u))$$

for all $g \in GL_n$.

### 3. Generating Function

In this section, we show that Pfaffian $Pf(\Phi(u))$ of the matrix $\Phi(u)$ given by (1.6) provides a generating function for the invariant differential operators $\{\Gamma_k\}$ with coefficients the Hermite polynomials, which, combined with Corollary 2.4, implies that each $\Gamma_k$ is $GL_n$-invariant.

As is well known, Pfaffian is closely connected with the exterior algebra. Denoting by $[\pm n]$ the set $\{1, 2, \ldots, n, -n, \ldots, -2, -1\}$, let $V$ be a $2n$-dimensional vector space over $K$ with a basis $\{e_i\}_{i \in [\pm n]}$ and $\wedge^* V$ the exterior algebra associated to $V$. For $\omega, \theta \in \wedge^* V$, write the exterior product $\omega \wedge \theta$ as $\omega \theta$ for short. Furthermore, let $\wedge^* V \otimes \mathcal{A}$ be the exterior algebra with coefficient in $\mathcal{A}$, whose product is determined by

$$(\omega \otimes X)(\theta \otimes Y) := \omega \theta \otimes XY$$

for $\omega, \theta \in \wedge^* V$ and $X, Y \in \mathcal{A}$.

To an anti-alternating matrix $X = (X_{i,j})_{i,j \in [\pm n]}$ with $X_{i,j} \in \mathcal{A}$, we associate a 2-form $\Xi_X$ defined by

$$\Xi_X := \sum_{i,j \in [\pm n]} e_i e_{-j} \otimes X_{i,j} \in \wedge^2 V \otimes \mathcal{A}. \quad (3.1)$$

Then the Pfaffian $Pf(X)$ is the coefficient of the volume form $e_1 e_2 \cdots e_n e_{-n} \cdots e_{-1}$ in $\Xi_X$ divided by $2^n n!$:

$$\Xi_X^\otimes = 2^n n! e_1 e_2 \cdots e_n e_{-n} \cdots e_{-1} \otimes Pf(X). \quad (3.2)$$

Henceforth, to keep formulas concise, for a subset $I = \{i_1 < i_2 < \cdots < i_k\} \subset [n]$, put $-I := \{-i_k < \cdots < -i_2 < -i_1\}$ and write $e_I$ and $e_{-I}$ instead of $e_{i_1} e_{i_2} \cdots e_{i_k}$ and $e_{-i_1} \cdots e_{-i_k}$, respectively; for $\omega \in \wedge^* V$ and $X \in \mathcal{A}$, write $\omega X$ instead of $\omega \otimes X$.

Now take $\mathcal{A}$ to be $\mathcal{PD}(\text{Alt}_n)$, and define 2-forms $\tau, \Theta_- , \Theta_+ \in \wedge^2 V \otimes \mathcal{PD}(\text{Alt}_n)$ by

$$\tau := \sum_{i,j \in [n]} e_i e_{-j}, \quad \Theta_- := \sum_{i,j \in [n]} e_i e_{j} X_{i,j}, \quad \Theta_+ := \sum_{i,j \in [n]} e_{-j} e_{-i} \partial_{i,j}. \quad (3.3)$$
Then $\Omega := \Theta_- + 2\mu + \Theta_+$ is the 2-form corresponding to $\Phi(u)$ under (3.1), and $\text{Pf}(\Phi(u))$ is the coefficient of volume form $e_{[n]}e_{-[n]}$ in $\Omega^n$ divided by $2^n n!$.

**Lemma 3.1.** We have the following commutation relations among $\tau, \Theta_-$ and $\Theta_+$:

$$[\tau, \Theta_-] = [\tau, \Theta_+] = 0, \quad [\Theta_+, \Theta_-] = 2\tau^2. \quad (3.4)$$

**Proof.** These follow from easy calculation. For example, we see that

$$[\Theta_+, \Theta_-] = 4 \sum_{i < j, k < l} (e_{-j}e_{-l}e_k e_l \partial_{ij} x_{kl} - e_k e_l e_{-j} e_{-l} x_{ik} \partial_{ij})$$

is the coefficient of volume form $e_{[n]}e_{-[n]}$ for any $n$. We extend it to $\mathcal{P}(\text{Alt}_n)$ by definition, we obtain that

$$\Theta_+ \Theta_- = \sum_{k=0}^{m} \binom{m}{k} \Theta_+^k \Theta_-^{m-k} \quad (3.5)$$

for all $m \in \mathbb{N}$.

**Proposition 3.2.** Let $m$ be a nonnegative integer. Then we have

$$(\Theta_- + \Theta_+)^m = \sum_{k=0}^{\lfloor m/2 \rfloor} c_k(m)(2\tau^2)^k (\Theta_- + \Theta_+)^{m-2k}, \quad (3.6)$$

where $c_k(m)$ are given by

$$c_k(m) = \frac{m!}{2^k k! (m-2k)!} \quad (3.7)$$

for $k = 0, 1, 2, \ldots, \lfloor m/2 \rfloor$, and $c_k(m) = 0$ for $k < 0$ and $k > \lfloor m/2 \rfloor$.

We need the following lemma to prove the proposition, though we will only use the case where $a = 1$.

**Lemma 3.3.** For nonnegative integers $a$ and $b$, we have

$$\Theta_+^a \Theta_-^b = \sum_{k=0}^{\min(a, b)} a^k b^{k \downarrow} (2\tau^2)^k \Theta_+^{a-k} \Theta_-^{b-k}, \quad (3.8)$$

where, for $z \in \mathbb{C}$ and $k \in \mathbb{N}$, $z^k$ denotes the descending factorial $z(z-1) \cdots (z-k+1)$. Note that $z^k = 0$ if $z \in \mathbb{N}$ and $k > z$. 

Proof. In view of the convention about the descending factorial, we can assume that $a \leq b$ in (3.8). Now we use induction on $a$. It is trivial if $a = 0$. Suppose it is true for some $a \geq 0$. Then applying Lemma [3.1], we obtain that

$$
\Theta_+^{a+1} \Theta_-^b = \sum_{k=0}^{a} \binom{a}{k} b^k (2\tau^2)^k \Theta_+ \Theta_-^{a-k} \Theta_+^k
$$

$$
= \sum_{k=0}^{a} \binom{a}{k} b^k (2\tau^2)^k \left( \Theta_-^k \Theta_+ + [\Theta_+, \Theta_-^{a-k}] \right) \Theta_+^{a-k}
$$

$$
= \sum_{k=0}^{a} \binom{a}{k} b^k (2\tau^2)^k \left( \Theta_-^k \Theta_+^{a-k} + (b - k)2\tau^2 \Theta_+^{b-1-k} \Theta_-^{a-k} \right)
$$

$$
= \sum_{k=0}^{a} \binom{a}{k} b^k (2\tau^2)^k \Theta_-^k \Theta_+^{a+1-k} + \sum_{k=0}^{a} \binom{a}{k} b^k (2\tau^2)^k \Theta_-^{b-1-k} \Theta_+^{a-k}
$$

$$
= \sum_{k=0}^{a+1} \binom{a+1}{k} b^k (2\tau^2)^k \Theta_-^{b-k} \Theta_+^{a+1-k}.
$$

This completes the proof. \(\square\)

Proof of Proposition 3.2. Use induction on $m$. There is nothing to prove when $m = 0$. Suppose that (3.6) is true for some $m \geq 0$. Multiplying (3.6) by $\Theta_- + \Theta_+$ from the left, we obtain that

$$(\Theta_- + \Theta_+)^{m+1}$$

$$
= \sum_{k=0}^{[m/2]} c_k (m)(2\tau^2)^k \sum_{s=0}^{m-2k} \binom{m-2k}{s} \left( \Theta_-^{s+1} \Theta_+^{m-2k-s} + \Theta_+^s \Theta_-^{m-2k-s} \right)
$$

$$
= \sum_{k=0}^{[m/2]} c_k (m)(2\tau^2)^k \sum_{s=0}^{m-2k} \binom{m-2k}{s} \left( \Theta_-^{s+1} \Theta_+^{m-2k-s} + \Theta_+^s \Theta_-^{m+1-2k-s} + s2\tau^2 \Theta_-^{s-1} \Theta_+^{m-2k-s} \right).
$$

Now, in the inner summation, since $\binom{m-2k}{s-1} + \binom{m-2k}{s} = \binom{m+1-2k}{s}$, the first and second sums equal

$$
\sum_{s=0}^{m-2k} \binom{m-2k}{s} \left( \Theta_-^{s+1} \Theta_+^{m-2k-s} + \Theta_+^s \Theta_-^{m+1-2k-s} \right)
$$

$$
= \sum_{s=0}^{m+1-2k} \binom{m+1-2k}{s} \Theta_-^{s+1} \Theta_+^{m+1-2k-s}
$$

$$
= o \left( (\Theta_- + \Theta_+)^{m+1-2k} \right)
$$

while the last equals

$$
\sum_{s=0}^{m-2k} \binom{m-2k}{s} s2\tau^2 \Theta_-^{s-1} \Theta_+^{m-2k-s}
$$
\[(m - 2k)2^2 \sum_{s=0}^{m-2k} \binom{m - 1 - 2k}{s} \Theta^s_+ \Theta^{m-1-2k-s}_-\]
\[= (m - 2k)2^2 \Theta^s_+ (\Theta_+ \Theta^-)^{m-1-2k} \Theta^s_+ \Theta^{m-1-2k-s}_- \]

Thus
\[(\Theta_+ \Theta^-)^{m+1} = \sum_{k=0}^{\lfloor m/2 \rfloor} c_k(m)(2\tau^2)^k_0 (\Theta_+ \Theta^-)^{m+1-2k} \Theta^s_+ \Theta^{m-1-2k-s}_- \]
\[+ \sum_{k=1}^{\lfloor m/2 \rfloor + 1} (m + 2 - 2k)c_{k-1}(m)(2\tau^2)^{k-1}_0 (\Theta_+ \Theta^-)^{m+1-2k} \Theta^s_+ \Theta^{m-1-2k-s}_- \]

Therefore, it suffices to show that
\[c_k(m + 1) = c_k(m) + (m + 2 - 2k)c_{k-1}(m), \quad (3.9)\]
which follows immediately from the definition \[(3.7)\] of \(c_k(m)\). In fact, the right-hand side of \[(3.9)\] equals
\[
\frac{m!}{2^k k! (m - 2k)!} + \frac{(m + 2 - 2k)m!}{2^{k-1} (k - 1)! (m - 2k + 2)!}
\[= \frac{(m + 1)!}{2^k k! (m - 2k)!} = c_k(m + 1). \]
Hence \[(3.6)\] is true for \(m + 1\). \(\square\)

**Remark 3.4.** Proposition \[3.2\] holds true in a more general situation. Namely, let \(A\) be a noncommutative associative algebra over an arbitrary field of characteristic 0, and \(A, B\) two elements of \(A\) such that their commutator \([A, B] := AB - BA\) commutes with both \(A\) and \(B\):
\[ [A, [A, B]] = [B, [A, B]] = 0. \]
Then exactly the same argument as in the proposition yields the following formula:
\[ (A + B)^m = \sum_{k=0}^{\lfloor m/2 \rfloor} c_k(m) ([A, B])^{2k} \sum_{s=0}^{m-2k} \binom{m - 2k}{s} B^s A^{m-2k-s} \]
with \(c_k(m)\) given by \[(3.7)\].

Now we are ready.

**Theorem 3.5.** The Pfaffian \(\text{Pf}(\Phi(u))\) provides a generating function for the \(\text{GL}_n\)-invariant differential operators \(\{\Gamma_k\}\):
\[ \text{Pf}(\Phi(u)) = \sum_{k=0}^{\lfloor n/2 \rfloor} a_{n-2k}(u) \Gamma_k, \]
where \(a_m(u)\) are monic polynomials in \(u\) given by
\[ a_m(u) = \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{m!}{2^{2k}(m - 2k)!k!} u^{m-2k}. \]
for \(m = 0, 1, 2, \ldots \).
Proof. By Lemma 3.1 and Proposition 3.2, we have

\[
\Omega^n = \sum_{p=0}^{n} \binom{n}{p} (2u\tau)^{n-p}(\Theta_- + \Theta_+)^p
\]

\[
= \sum_{p=0}^{n} \sum_{q=0}^{\lfloor p/2 \rfloor} \frac{n!}{(n-p)! q! (p-2q)!} (2u\tau)^{n-p-2q} (\Theta_- + \Theta_+)^{p-2q}
\]

\[
= \sum_{p=0}^{n} \sum_{q=0}^{\lfloor p/2 \rfloor} \sum_{r,s=0}^{p-2q} \frac{n!}{(n-p)! k! r! s!} (2u\tau)^{p+2q} \Theta_+ \Theta_-^s.
\]  
(3.10)

Using the relations

\[
\Theta_- = 2^r! \sum_{I\in\binom{[n]}{r}} e_I \text{Pf}(x_I) \quad \text{and} \quad \Theta_+ = 2^s! \sum_{J\in\binom{[n]}{s}} e_J \text{Pf}(\partial_J)
\]

we obtain

\[
\Omega^n = \sum_{p=0}^{n} \sum_{q=0}^{\lfloor p/2 \rfloor} \sum_{r+s=p-2q} \frac{n!}{(n-p)! q!} 2^{p-2q} u^{n-p-2q} \sum_{I\in\binom{[n]}{r}, J\in\binom{[n]}{s}} e_{-I} e_J \text{Pf}(x_I) \text{Pf}(\partial_J).
\]  
(3.11)

With \(u^{n-p+2q}\) expanded as

\[
u^{n-p+2q} = (n-p+2q)! \sum_{K\in\binom{[n]}{n-p+2q}} e_K e_{-K},
\]

the only terms that survive in the summation \(\sum_{K,I,J}\) are those corresponding to \(I = J = [n] \setminus K\); in particular, \(r = s\) and \(p\) is even. Thus the sum \(\sum_{K,I,J}\) is equal to

\[
\sum_{I\in\binom{[n]}{r}} e_{[n]-I} e_{-[n]-I} e_I e_{-I} \text{Pf}(x_I) \text{Pf}(\partial_I)
\]

\[
= \sum_{I\in\binom{[n]}{r}} \text{sgn}\left([n]-I\right) e_{[n]} e_{-[n]-I} e_{-I} \text{Pf}(x_I) \text{Pf}(\partial_I)
\]

\[
= e_{[n]} e_{-[n]} \sum_{I\in\binom{[n]}{r}} \text{Pf}(x_I) \text{Pf}(\partial_I)
\]

since \(\text{sgn}\left([n]-I\right) = \text{sgn}\left(-[n]-I\right)\). Letting \(p = 2\nu\), we obtain that

\[
\text{Pf}(\Phi(u)) = \sum_{\nu=0}^{\lfloor n/2 \rfloor} u^{n-2\nu} \sum_{s=0}^{\nu} \frac{(n-2s)!}{(n-2\nu)! (\nu-s)! 2^{2(\nu-s)} \Gamma_s}
\]

\[
= \sum_{s=0}^{\lfloor n/2 \rfloor} \sum_{\nu=s}^{\lfloor n/2 \rfloor} \frac{(n-2s)!}{(n-2\nu)! (\nu-s)! 2^{2(\nu-s)} \Gamma_s} u^{n-2\nu} \Gamma_s
\]

\[
= \sum_{s=0}^{\lfloor n/2 \rfloor} \sum_{\nu=s}^{\lfloor n/2 \rfloor-s} \frac{(n-2s)!}{(n-2s-2r)! r! 2^r \Gamma_s} u^{n-2s-2r} \Gamma_s
\]
\[
\sum_{s=0}^{\lfloor n/2 \rfloor} a_{n-2s}(u) \Gamma_s.
\]

This completes the proof. \(\square\)

The polynomials \(a_m(u)\) are essentially equal to the Hermite polynomials given by
\[
H_m(x) = (-1)^m e^{x^2} \left( \frac{d}{dx} \right)^m e^{-x^2}.
\]
In fact, it is well known that the generating function for \(H_m(x)\) is given by
\[
e^{2tx-x^2} = \sum_{m=0}^{\infty} \frac{t^m}{m!} H_m(x),
\]
from which one can derive that
\[
H_m(x) = m! \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{(-1)^k 2^{m-2k}}{k!(m-2k)!} x^{m-2k}.
\]

Therefore,
\[
a_m(u) = \left(-\frac{\nu-1}{2}\right)^m H_m(\sqrt{-1}u),
\]
and we obtain (1.5).

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