Abstract

The T-product operation between two three-order tensors was invented around 2011 and it arises from many applications, such as signal processing, image feature extraction, machine learning, computer vision, and the multi-view clustering problem. Although there are many pioneer works about T-product tensors, there are no works dedicated to inequalities associated with T-product tensors. In this work, we first attempt to build inequalities at the following aspects: (1) trace function nondecreasing/convexity; (2) Golden-Thompson inequality for T-product tensors; (3) Jensen’s T-product inequality; (4) Klein’s T-product inequality. All these inequalities are related to generalize celebrated Lieb’s concavity theorem from matrices to T-product tensors. This new version of Lieb’s concavity theorem under T-product tensor will be used to determine the tail bound for the maximum eigenvalue induced by independent sums of random Hermitian T-product, which is the key tool to derive various new tail bounds for random T-product tensors. Besides, Qi et. al [1] introduces a new concept, named eigentuple, about T-product tensors and they apply this concept to study nonnegative (positive) definite properties of T-product tensors. The final main contribution of this work is to develop the Courant-Fischer Theorem with respect to eigentuples, and this theorem helps us to understand the relationship between the minimum eigentuple and the maximum eigentuple. The main content of this paper is Part I of a serious task about T-product tensors. The Part II of this work will utilize these new inequalities and Courant-Fischer Theorem under T-product tensors to derive tail bounds of the extreme eigenvalue and the maximum eigentuple for sums of random T-product tensors, e.g., T-product tensor Chernoff and T-product tensor Bernstein bounds.

Keywords: Hermitian T-product tensors, eigentuples, trace function, Lieb’s concavity for T-product tensors, Courant-Fischer theorem for T-product tensors.

AMS Subject Classification: 15A69; 65F10
1 Introduction

1.1 T-product Tensors

The T-product operation between two three order tensors was introduced by Kilmer and her collaborators in [2,3]. It has been shown as a powerful tool in many fields: signal processing [4,5], machine learning [6], computer vision [7,8], image processing [9], low-rank tensor approximation [10–12] etc. Due to wide applications of T-product, T-SVD and tubal ranks, Qi et. al [11] extend eigentuple concept first defined from [13] to study properties for symmetry of T-product tensors and positive (nonnegative) semidefiniteness of T-product tensors by defining a T-quadratic form, whose variable is an $m \times p$ matrix, and whose value is a $p$-dimensional vector. They further show that a T-quadratic form is positive semidefinite (definite) if and only if the smallest eigentuple of the corresponding T-symmetric tensor is nonnegative (positive). Besides T-quadratic form, general functions for T-product tensors and their properties are also studied based on T-SVD, see [14–17]. However, none of these works discussed further issues about inequalities associated with T-product tensors. The first part of this work about T-product tensors is to develop several new T-product tensors inequalities, which are the main topics discussed by this paper. In the matrix setting, there are many useful applications about these similar inequalities under the traditional matrix product, e.g., quantum information processing [18]. The second part of this work is to apply these new inequalities about T-product tensors to tail bounds estimation of the maximum eigenvalue and the maximum eigentuple for sums of random T-product tensors, e.g., Chernoff and Bernstein bounds. We will introduce these new inequalities about T-product tensors obtained at this Part I work at the next subsection.

1.2 New Inequalities about T-product Tensors

In this work, we define trace, denoted by $\text{Tr}$, as the summation of f-diagonal entries of a given symmetric T-product tensor $C \in \mathbb{C}^{m \times m \times p}$ and study properties of trace for T-product tensors. Our first main inequality about trace is following theorem:

**Theorem 1.1 (Monotonicity and Convexity of T-product Trace Function)** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with non-decreasing / convex / strictly convex properties, then so is the mapping $C \to \text{Tr} (f(C))$.

From trace definition, we will prove Golden–Thompson inequality for two Hermitian T-product tensors which will be utilized to prove T-product tensors martingale inequalities. A Hermitian T-product tensor is a tensor equal to its Hermitian transpose, which is defined by Eq. (11). Theorem 1.2 (Golden-Thompson inequality for T-product Tensors) Given two Hermitian T-product tensors $C, D \in \mathbb{C}^{m \times m \times p}$, we have

$$\text{Tr} (\exp(C + D)) \leq \text{Tr} (\exp(C) \star \exp(D)),$$

where $\star$ is the product operation between two T-product tensors defined by Eq. (15).

The next inequality we will show is Jensen’s operator inequality (positive semidefinite relation between two T-product tensors). A tensor with T-positive definite (or T-positive semi-definite) will be abbreviated as TPD (or TPSD), see Section 2.3 for its definition. If we have a TPSD relation between two T-product tensors $C$ and $D$ represented as $C \preceq D$, then the difference T-product tensor $(D - C)$ is a T-positive semi-definite tensor. Let $I_{mmp} \in \mathbb{C}^{m \times m \times p}$ be the identity tensor defined by Eq. (13).

**Theorem 1.3 (Jensen’s T-product Inequality)** For a continuous T-product tensor convex function $f$ defined on an interval $I$. The definition of T-product tensor convex is given by Eq. (51). we have the following
TPSD relation for each natural number $n$:

$$f \left( \sum_{i=1}^{n} C_{i}^{H} \ast \mathcal{X}_{i} \ast C_{i} \right) \preceq \sum_{i=1}^{n} C_{i}^{H} \ast f \left( \mathcal{X}_{i} \right) \ast C_{i},$$

(2)

where $\mathcal{X}_{i} \in \mathbb{C}^{m \times m \times p}$ are bounded, Hermitian T-product tensors with all eigenvalues in the interval $I$ and tensors $C_{i}$ satisfying $\sum_{i=1}^{n} C_{i}^{H} \ast C_{i} = I_{mmp}$.

The immediate application of Theorem 1.1 is to prove Klein’s inequality for T-product tensor.

**Theorem 1.4 (Klein’s T-product Inequality)** For all $C, D$ Hermitian T-product tensors and a differentiable convex function $f : \mathbb{R} \rightarrow \mathbb{R}$ or for all $C, D$ Hermitian T-product tensors and a differentiable convex function $f : (0, \infty) \rightarrow \mathbb{R}$, we have

$$\text{Tr} \left( f(C) - f(D) - (C - D) \ast f'(D) \right) \geq 0.$$  

(3)

In both situations, if $f$ is strictly convex, equality holds if and only if $C = D$.

Previous theorems will help us to establish the following main theorem of this paper about Lieb’s concavity for T-product tensors since tail bounds for sums of random T-product tensors will be derived based on such concavity relation.

**Theorem 1.5 (Lieb’s concavity theorem for T-product tensors)** Let $H$ be a Hermitian T-product tensor. Following map

$$A \rightarrow \text{Tr} e^{H + \log A}$$  

(4)

is concave on the positive-definite cone.

We are ready to present the theorem for the tail bound of the maximum eigenvalue induced by independent sums of random Hermitian T-product tensors and this theorem will play a key role to establish various new tail bounds of the maximum eigenvalue generated by independent sums of random T-product tensors.

**Theorem 1.6 (Master Tail Bound for Independent Sum of Random T-product Tensors for Eigenvalue)** Given a finite sequence of independent Hermitian T-product tensors $\{\mathcal{X}_{i}\}$, we have

$$\Pr \left( \lambda_{\text{max}} \left( \sum_{i=1}^{n} \mathcal{X}_{i} \right) \geq \theta \right) \leq \inf_{t > 0} \left\{ e^{-t\theta} \text{Tr} \exp \left( \sum_{i=1}^{n} \log E_{e^{t\mathcal{X}_{i}}} \right) \right\}.$$  

(5)

Similarly, we can generalize master tail bound for independent sum of random Hermitian T-product tensors with respect to eigenvalue from Theorem 1.6 to eigentuple version by the following theorem.

We begin with $\ominus$ operation defined in Proposition 2.1 from work [1].

Let $a = (a_{1}, a_{2}, \cdots, a_{p})^{T} \in \mathbb{C}^{p}$, then operator circ to the vector $a$ can be defined as

$$\text{circ}(a) \overset{\text{def}}{=} \begin{pmatrix} a_{1} & a_{p} & a_{p-1} & \cdots & a_{2} \\ a_{2} & a_{1} & a_{p} & \cdots & a_{3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{p} & a_{p-1} & a_{p-2} & \cdots & a_{1} \end{pmatrix},$$  

(6)

and $\text{circ}^{-1}(\text{circ}(a)) \overset{\text{def}}{=} a$. Suppose that $a, b \in \mathbb{C}^{p}$, we define

$$a \ominus b \overset{\text{def}}{=} \text{circ}(a) \cdot b,$$  

(7)

where $\cdot$ is the standard matrix and vector multiplication. Then, we are ready to present the following theorem.
Theorem 1.7 (Master Tail Bound for Independent Sum of Random T-product Tensors for Eigentuple)

Given a finite sequence of independent random Hermitian T-product tensors \( \{X^i\} \) such that \( X^i \in \mathbb{C}^{m \times m \times p} \), if \( \sum_{i=1}^{n} tX^i \) satisfies Eq. (98), we have

\[
\Pr \left( \max_{1 \leq i \leq n} \left( \sum_{i=1}^{n} X^i \right) \geq b \right) \leq \inf_{t>0} \min_{1 \leq j \leq p} \left\{ \frac{\text{Tr} \exp \left( \sum_{i=1}^{n} \log \mathbb{E}e^{tX^i} \right)}{\left( e^{tb} \right)_j} \right\},
\]

where \( e^{tb} \in \mathbb{C}^p \) is the exponential for the vector \( t \mathbf{b} \) with respect to \( \circ \) operation.

The last important theorem is the Courant-Fischer theorem for T-product tensors. This theorem will be used to figure out the relationship between the maximum eigentuple and the minimum eigentuple of a T-product tensor.

Theorem 1.8 (Courant-Fischer Theorem under T-product)

Let \( A \in \mathbb{C}^{m \times m \times p} \) be a Hermitian T-product tensor with eigentuples \( \mathbf{d}_1 \geq \mathbf{d}_2 \geq \cdots \geq \mathbf{d}_m \). Let \( \{U_j^{[l]}\} \in \mathbb{C}^{m \times p} \) be orthonormal matrices for \( 1 \leq j \leq m \) and \( 0 \leq l \leq p-1 \), \( S_k \) be the space spanned by \( \{U_j^{[l]}\} \) for \( 1 \leq j \leq k \) and \( 0 \leq l \leq p-1 \), and \( T_k \) be the space spanned by \( \{U_j^{[l]}\} \) for \( k \leq j \leq m \) and \( 0 \leq l \leq p-1 \). Then, we have

\[
d_k = \max_{S_k \subseteq \mathbb{C}^{m \times p}} \min_{\text{dim}(S_k)=k \times p} \left( \frac{\mathbf{X}^H \circ A \circ \mathbf{X}}{\circ} \right)_{(m-k+1) \times p}
= \min_{T_k \subseteq \mathbb{C}^{m \times p}} \max_{\text{dim}(T_k)=(m-k+1) \times p} \left( \frac{\mathbf{X}^H \circ A \circ \mathbf{X}}{\circ} \right)_{(m-k+1) \times p},
\]

where \( / \circ \) is the division (inverse operation) under \( \circ \) operation.

All these inequalities and maximum/minimum eigentuples relation about T-product tensors will be utilized to derive a series of new tail bounds for the extreme eigenvalue and eigentuple for sums of random T-product tensors. These new inequalities different from author Chang’s previous works about bounds for sums of random tensors based on Einstein product [19, 20].

1.3 Paper Organization

The rest of this paper is organized as follows. In Section 2, basic notions of T-product tensors are introduced. Lieb’s concavity theorem under T-product will be studied in Section 3. General tail bounds for random T-product tensors are provided in Section 4. Courant-Fischer Theorem under T-product is given in Section 5. Finally, conclusion will be drawn in Section 6.

Nomenclature: The sets of complex and real numbers are denoted by \( \mathbb{C} \) and \( \mathbb{R} \), respectively. The symbol \( \text{def} \) denotes mathematical definition.

2 T-product Tensors

In this section, we will review T-product operations briefly and discuss related properties in Sec. 2.1. The T-SVD decomposition of T-product tensors and T-Symmetric tensors will be presented in Sec. 2.2.
2.1 What are T-product Tensors

For a third order tensor $C \in \mathbb{C}^{m \times n \times p}$, we define $\text{bcirc}$ operation to the tensor $C$ as:

$$\text{bcirc}(C) \overset{\text{def}}{=} \begin{pmatrix} C^{(1)} & C^{(p)} & C^{(p-1)} & \ldots & C^{(2)} \\ C^{(2)} & C^{(1)} & C^{(p)} & \ldots & C^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C^{(p)} & C^{(p-1)} & C^{(p-2)} & \ldots & C^{(1)} \end{pmatrix}, \quad (10)$$

where $C^{(1)}, \ldots, C^{(p)} \in \mathbb{C}^{m \times n}$ are frontal slices of tensor $C$. The inverse operation of $\text{bcirc}$ is denoted as $\text{bcirc}^{-1}$ with relation $\text{bcirc}^{-1}(\text{bcirc}(C)) \overset{\text{def}}{=} C$.

For a third order tensor $C \in \mathbb{C}^{m \times m \times p}$, we define Hermitian transpose of $C$, denoted by $C^H$, as

$$C^H = \text{bcirc}^{-1}((\text{bcirc}(C))^H). \quad (11)$$

And a tensor $D \in \mathbb{C}^{m \times m \times p}$ is called a Hermitian T-product tensor if $D^H = D$. Similarly, we also define transpose of $C$, denoted by $C^T$, as

$$C^T = \text{bcirc}^{-1}((\text{bcirc}(C))^T) \quad (12)$$

And a tensor $D \in \mathbb{C}^{m \times m \times p}$ is called a Symmetric T-product tensor if $D^T = D$.

The identity tensor $I_{mp} \in \mathbb{C}^{m \times m \times p}$ can be defined as:

$$I_{mp} = \text{bcirc}^{-1}(I_{mp}), \quad (13)$$

where $I_{mp}$ is the identity matrix in $\mathbb{R}^{mp \times mp}$. A zero tensor, denoted as $O_{mp} \in \mathbb{C}^{m \times n \times p}$, is a tensor that all elements inside the tensor as 0.

In order to define the T-product operation, we need to define another kind of operation over a third order tensor. For a third order tensor $C \in \mathbb{C}^{m \times n \times p}$, we define unfold operation to the tensor $C$ as:

$$\text{unfold}(C) \overset{\text{def}}{=} \begin{pmatrix} C^{(1)} \\ C^{(2)} \\ \vdots \\ C^{(p)} \end{pmatrix}, \quad (14)$$

where $\text{unfold}(C) \in \mathbb{C}^{mp \times n}$, and the inverse operation of unfold is fold with the relation $\text{fold}(\text{unfold}(C)) \overset{\text{def}}{=} C$. Given $C \in \mathbb{C}^{m \times n \times p}$ and $D \in \mathbb{C}^{n \times k \times p}$, we define the T-product between $C$ and $D$ as

$$C \ast D \overset{\text{def}}{=} \text{fold}(\text{bcirc}(D)\text{unfold}(D)), \quad (15)$$

where $C \ast D \in \mathbb{C}^{m \times k \times p}$.

**Definition 1** Let $S = (s_{ijk}) \in \mathbb{C}^{m \times n \times p}$ be a $f$-diagonal tensor, i.e., each frontal slice of tensor $S$ is a diagonal matrix. Let $s_i = (s_{i1}, s_{i2}, \ldots, s_{ip})^T$ be the $i$-th tube of $S$ for $1 \leq i \leq \min\{m, n\}$. The $f$-diagonal tensor $S$ is in its standard form if $s_1 \geq s_2 \geq \cdots \geq s_{\min\{m, n\}}$, where $\geq$ is the elementwise comparison between two vectors.

We define the T-product tensor trace for a tensor $C = (c_{ijk}) \in \mathbb{C}^{m \times n \times p}$, denoted by $\text{Tr}(C)$, as following

$$\text{Tr}(C) \overset{\text{def}}{=} \sum_{i=1}^{m} \sum_{k=1}^{p} c_{iik}, \quad (16)$$

which is the summation of all entries in $f$-diagonal components. Then, we have following lemma about trace properties.
Lemma 1 For any tensors $C, D \in \mathbb{C}^{m \times m \times p}$, we have
\[
\text{Tr}(cC + dD) = c\text{Tr}(C) + d\text{Tr}(D),
\]
where $c, d$ are two constants. And, the transpose operation will keep the same trace value, i.e.,
\[
\text{Tr}(C) = \text{Tr}(C^T).
\]
Finally, we have
\[
\text{Tr}(C \ast D) = \text{Tr}(D \ast C).
\]

Proof: Eqs. (17) and (18) are true from trace definition directly.

From Eq. (15), the $i$-th frontal slice matrix of $D \ast C$ is
\[
D^{(i)} C^{(1)} + D^{(i-1)} C^{(2)} + \cdots + D^{(1)} C^{(i)} + D^{(m)} C^{(i+1)} + \cdots + D^{(i+1)} C^{(m)},
\]
similarly, the $i$-th frontal slice matrix of $C \ast D$ is
\[
C^{(i)} D^{(1)} + C^{(i-1)} D^{(2)} + \cdots + C^{(1)} D^{(i)} + C^{(m)} D^{(i+1)} + \cdots + C^{(i+1)} D^{(m)}.
\]
Because the matrix trace of Eq. (20) and the matrix trace of Eq. (21) are same for each slice $i$ due to linearity and invariant under cyclic permutations of matrix trace, we have Eq. (19) by summing over all frontal matrix slices.

Below, we will define the determinant of a T-product tensor $C \in \mathbb{C}^{m \times m \times p}$ and its associate properties. The determinant of a $m \times m \times p$ tensor $C$ is the $m$-linear alternating form defined as
\[
\det : (V_1, \ldots, V_m) \rightarrow \mathbb{C},
\]
where $V_i \in \mathbb{C}^{m \times p}$ is the $i$-th lateral matrix of the tensor $C$. Moreover, we require that $\det(I_{mmp}) = 1$. Given two tensors $C, D \in \mathbb{C}^{m \times m \times p}$, the determinant of $C \ast D$ is $\det(C \ast D) = \lambda \det(D)$ for some value $\lambda$.

If we set $D$ as $I_{mmp}$, we have
\[
\det(C \ast I_{mmp}) = \lambda \det(I_{mmp}) = \lambda = \det(C).
\]
Then, we have
\[
\det(C \ast D) = \det(C)\det(D)
\]

2.2 T-SVD Decomposition

Given a tensor $C \in \mathbb{C}^{m \times n \times p}$, Theorem 4.1 in [2] proposed a T-singular value decomposition (T-SVD) for $C$ as:
\[
C = \mathcal{U} \ast \mathcal{S} \ast \mathcal{V}^T,
\]
where $\mathcal{U} \in \mathbb{C}^{m \times m \times p}$ and $\mathcal{V} \in \mathbb{C}^{n \times n \times p}$ are orthogonal tensors, and $\mathcal{S} \in \mathbb{C}^{m \times n \times m}$ is a f-diagonal tensor. We also have $\mathcal{U}^T \ast \mathcal{U} = I_{mmp}$ and $\mathcal{V}^T \ast \mathcal{V} = I_{nmp}$. We define $\sigma(C)$ be the spectrum of $C$, i.e., the set of $s \in \mathbb{C}$, where $s$ are nonzero entries from tensor $\mathcal{S}$. We use $\|\cdot\|$ for the spectral norm, which is the largest singular value of a T-product tensor.

Given any integer $k$ and $B \in \mathbb{C}^{m \times n \times p}$, we define $B^k$ as
\[
B^k \overset{\text{def}}{=} \underbrace{B \ast B \ast B \ast \cdots \ast B}_{k \text{ terms of } B \text{ under T-product}}
\]
where $B^k \in \mathbb{C}^{m \times m \times p}$. Then, we have following corollary from T-SVD in Eq. (25).
Corollary 1 Suppose \( B \in \mathbb{C}^{m \times m \times p} \) is a Hermitian T-product tensor, and \( S^{-1} \) exists, where \( S \) is f-diagonal tensor obtained from the T-SVD of the tensor \( C \). Then, we have

\[
B^k = U \star S^k \star U^T. \tag{27}
\]

Then, we can define the T-product tensor exponential function and the T-product tensor logarithm function under T-product as below with tensor power.

Definition 2 Given a tensor \( \mathcal{X} \in \mathbb{C}^{m \times m \times p} \), the tensor exponential of the tensor \( \mathcal{X} \) is defined as

\[
e^\mathcal{X} \equiv \sum_{k=0}^{\infty} \frac{\mathcal{X}^k}{k!}, \tag{28}\]

where \( \mathcal{X}^0 \) is defined as the identity tensor \( I_{m \times m \times p} \). Given a tensor \( \mathcal{Y} \), the tensor \( \mathcal{X} \) is said to be a tensor logarithm of \( \mathcal{Y} \) if \( e^\mathcal{X} = \mathcal{Y} \).

From T-SVD in Eq. (25), we can express a Hermitian T-product tensor \( C \in \mathbb{C}^{m \times m \times p} \) as

\[
C = \sum_{i=1}^{m} \sum_{k=0}^{p-1} s_{iik} U_i^{[k]} \star (U_i^{[k]})^T, \tag{29}\]

where \( s_{iik} \) are eigenvalues of the tensor \( C \), and \( U_i^{[k]} \in \mathbb{C}^{m \times 1 \times p} \) is the \( i \)-th lateral slice (matrix) of the tensor \( U \) after \( k \) cyclic permutations. The matrix \( U_i^{[0]} \) is obtained from the \( i \)-th lateral slice (matrix) of the tensor \( U \) with column vectors as \( u_i^{(1)}, \ldots, u_i^{(p)} \), then we have

\[
U_i^{[k]} = \left( u_i^{(p+1-k) \mod p}, u_i^{(p+2-k) \mod p}, \ldots, u_i^{(p)}, u_i^{(1)}, \ldots, u_i^{(p-k)} \right). \tag{30}\]

Note that we have \( (U_i^{[k]})^H \star U_i^{[k]} = I_{1 \times p} \) and \( (U_i^{[k]})^H \star U_i^{[k']} = 0_{1 \times p} \) for \( i \neq i' \) or \( k \neq k' \). From Theorem 3.6 in [1], all values of \( s_{iik} \) are real and we define \( \lambda_{\text{max}} \equiv \max_{1 \leq i \leq m} \{ s_{iik} \} \), and \( \lambda_{\text{min}} \equiv \min_{0 \leq k \leq p-1} \{ s_{iik} \} \).

From Corollary [1] and Eq. (29), we can have following spectral mapping lemma.

Lemma 2 For any continuous function \( f : \mathbb{R} \to \mathbb{R} \) and any Hermitian T-product tensor \( C \), we have

\[
f(C) = \sum_{i=1}^{m} \sum_{k=0}^{p-1} f(s_{iik}) U_i^{[k]} \star (U_i^{[k]})^T. \tag{31}\]

2.3 Positive Semidefinite T-product Tensors

Given a Hermitian T-product tensor \( C \in \mathbb{C}^{m \times m \times p} \), and a tensor \( \mathcal{X} \in \mathbb{C}^{m \times 1 \times p} \) obtained from treating the matrix \( X \in \mathbb{R}^{m \times p} \) as a tensor with dimensions \( \mathbb{R}^{m \times 1 \times p} \). We define following quadratic form with respect to the matrix \( X \) as

\[
F_C(X) \equiv \mathcal{X}^T \star C \star \mathcal{X}, \tag{32}\]

and we say that a tensor \( C \) is T-positive definite (TPD) (or T-positive semi-definite (TPSD)) if \( F_C(X) > 0 \) (or \( F_C(X) \geq 0 \)) for any \( X \in \mathbb{C}^{m \times p} \), where \( 0 \) is a zero vector with size \( p \).
We now define eigentuples and eigenmatrices of a Hermitian T-product tensor which will be used to characterize TPD or TPSD for a given tensor. For a matrix \( X \in \mathbb{C}^{m \times p} = (x^{(1)}, \ldots, x^{(p)}) \), we define unfolding operation with respect to the matrix \( X \) columns, denoted by \( \text{cunfold}(X) \), as

\[
\text{cunfold}(X) \overset{\text{def}}{=} \begin{pmatrix}
x^{(1)} \\
x^{(2)} \\
\vdots \\
x^{(p)}
\end{pmatrix},
\]

where \( \text{cunfold}(X) \in \mathbb{C}^{mp} \). Then, suppose that \( C \in \mathbb{C}^{m \times m \times p} \) is a Hermitian T-product tensor, we define \( C \star X \) as

\[
C \star X = \text{fold}(\text{bcirc}(C)\text{cunfold}(X)),
\]

where \( C \star X \in \mathbb{C}^{m \times p} \). We also define a new product operation between a vector \( d = (d_1, d_2, \ldots, d_p)^T \) and a matrix \( X \in \mathbb{C}^{m \times p} \), denoted by \( \circ \), as

\[
d \circ X \overset{\text{def}}{=} X \cdot \begin{pmatrix}
d_1 & d_p & d_{p-1} & \cdots & d_2 \\
d_2 & d_1 & d_p & \cdots & d_3 \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
d_p & d_{p-1} & d_{p-2} & \cdots & d_1
\end{pmatrix},
\]

where \( \cdot \) is the standard matrix multiplication. Suppose that \( X \in \mathbb{C}^{m \times p} \) and \( X \neq O \), and \( d \in \mathbb{C}^p \), if we have

\[
C \star X = d \circ X,
\]

we call \( d \) as an eigentuple of \( C \), and \( X \) as an eigenmatrix of \( C \) corresponding to the eigentuple \( d \).

From Theorem 4.1 in [1], a T-square tensor \( C \in \mathbb{C}^{m \times m \times p} \) with eigentuples arranged as f-diagonal tensor \( S \) according to the standard form provided by Definition 1, i.e., \( s_1 \geq s_2 \geq \cdots \geq s_m \). Then \( C \) is TPD (or TPSD) if and only if the smallest eigentuple \( s_m > (\text{or} \geq) 0 \). We use \( \|C\|_{\text{vec}} \) to represent the spectral norm of eigentuple of the tensor \( C \), which is defined as

\[
\|C\|_{\text{vec}} \overset{\text{def}}{=} d_{\text{max}} \left( \sqrt{C^H \star C} \right).
\]

### 2.4 T-product Tensors Analysis

We will begin with monotonicity and convexity discussions of the trace function.

**Lemma 3** For a given continuous and non-decreasing function \( f : \mathbb{R} \rightarrow \mathbb{R} \), the associated trace function on a Hermitian T-product tensor \( C \) is given by

\[
C \rightarrow \text{Tr} \left( f(C) \right).
\]

Then we have

\[
C \succeq D \implies \text{Tr} \left( f(C) \right) \geq \text{Tr} \left( f(D) \right).
\]
Proof: We first assume that the function $f$ is differentiable, then the first derivative of $f$ is greater or equal than zero (monotonicity). We further define a trace function $g(t) \overset{\text{def}}{=} \text{Tr} \left( f \left( D + t(C - D) \right) \right)$. Then, we have

$$\text{Tr} \left( f(C) \right) - \text{Tr} \left( f(D) \right) = g(1) - g(0) = \int_0^1 g'(t) dt = \int_0^1 \text{Tr} \left( f' \left( (D + t(C - D)) \right) \ast (C - D) \right) dt$$

$$= \int_0^1 \text{Tr} \left( (C - D)^{1/2} \ast f' \left( (D + t(C - D)) \right) \ast (C - D)^{1/2} \right) \geq 0,$$

where we apply Lemma 1 at the last equality, and the last inequality comes from the nonnegative of $f'$. By applying the standard continuity argument, we can relax the requirement that $f$ is continuously differentiable to the requirement that $f$ is continuous.

The next lemma will be used to show the convexity of trace function on a Hermitian T-product tensor $C$.

Lemma 4 Let $C \in \mathbb{C}^{m \times m \times p}$ be a Hermitian T-product tensor, $f$ convex on $\mathbb{R}$, and $V_i^{[k]}$ for $1 \leq i \leq m$ and $0 \leq k \leq p - 1$ be any orthonormal base of $\mathbb{C}^{m \times p}$. Then, we have

$$\text{Tr} \left( f(C) \right) \geq \sum_{i=1}^m \sum_{k=0}^{p-1} f \left( \langle V_i^{[k]}, C \ast V_i^{[k]} \rangle \right),$$

(41)

where $\langle V_i^{[k]}, C \ast V_i^{[k]} \rangle$ is the Frobenius inner product between two matrices $V_i^{[k]}$ and $C \ast V_i^{[k]}$. There is an equality if each $V_i^{[k]}$ is an eigenmatrix of $C$ and it’s the only case if $f$ is strictly convex.

Proof: From spectral representation by Eq. (29), we have

$$\text{Tr} \left( f(C) \right) = \sum_{i=1}^m \sum_{k=0}^{p-1} \sum_{i'=1}^m \sum_{k'=0}^{p-1} f(s_{i'i'k'}) \left( \langle U_{i'}^{[k']}, U_{i'}^{[k']} \rangle \ast V_i^{[k]} \right)$$

$$= \sum_{i=1}^m \sum_{k=0}^{p-1} \sum_{i'=1}^m \sum_{k'=0}^{p-1} f(s_{i'i'k'}) \left\| \langle U_{i'}^{[k']}, U_{i'}^{[k']} \rangle \ast V_i^{[k]} \right\|^2$$

$$\geq \sum_{i=1}^m \sum_{k=0}^{p-1} \sum_{i'=1}^m \sum_{k'=0}^{p-1} f(s_{i'i'k'}) \left\langle V_i^{[k]}, \left( \langle U_{i'}^{[k']}, U_{i'}^{[k']} \rangle \ast V_i^{[k]} \right) \right\rangle$$

(42)

where the only inequality comes from the convexity of the function $f$. Since for each $i, k$, we have

$$\sum_{i'=1}^m \sum_{k'=0}^{p-1} \left\| \langle U_{i'}^{[k']}, U_{i'}^{[k']} \rangle \ast V_i^{[k]} \right\|^2 \leq \left\| V_i^{[k]} \right\|^2 = 1.$$  Note that each $V_i^{[k]}$ is an eigenmatrix of $C$ if and only if $\left\| \langle U_{i'}^{[k']}, U_{i'}^{[k']} \rangle \ast V_i^{[k]} \right\|^2 = 1$ for some $i', k'$, and is 0 otherwise, in which case the inequality in Eq. (41) is equality. When $f$ is strictly convex, equality in Eq. (41) can be true only if for each $i, k$, we have $\left\| \langle U_{i'}^{[k']}, U_{i'}^{[k']} \rangle \ast V_i^{[k]} \right\|^2 = 1$ for some $i', k'$, and is 0 otherwise.

From Lemma 3 and Lemma 4, we have the following theorem about convexity and monotonicity of a trace function. We recall theorem 1.1.
Theorem 1.1 (Monotonicity and Convexity of T-product Trace Function) Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function with non-decreasing / convex / strictly convex properties, then so is the mapping \( C \to \text{Tr} (f(C)) \).

**Proof:** Given two \( C, D \in \mathbb{C}^{m \times m \times p} \) Hermitian T-product tensors, \( f \) as a convex function, and \( V_i^{[k]} \) for \( 1 \leq i \leq m \) and \( 0 \leq k \leq p - 1 \) be an orthonormal basis of \( \mathbb{C}^{m \times p} \) consisting of eigenmatrices of \( \frac{C+D}{2} \). Then, from Lemma 4 we have

\[
\text{Tr} \left( f \left( \frac{C+D}{2} \right) \right) = \sum_{i=1}^{m} \sum_{k=0}^{p-1} f \left( \left( V_i^{[k]}, \frac{C+D}{2} \star V_i^{[k]} \right) \right) \\
= \sum_{i=1}^{m} \sum_{k=0}^{p-1} f \left( \frac{1}{2} \left( V_i^{[k]}, C \star V_i^{[k]} \right) + \frac{1}{2} \left( V_i^{[k]}, D \star V_i^{[k]} \right) \right) \\
\leq \sum_{i=1}^{m} \sum_{k=0}^{p-1} \left( \frac{1}{2} f \left( \left( V_i^{[k]}, C \star V_i^{[k]} \right) \right) + \frac{1}{2} f \left( \left( V_i^{[k]}, D \star V_i^{[k]} \right) \right) \right) \\
\leq \frac{1}{2} \text{Tr} (f(C)) + \frac{1}{2} \text{Tr} (f(D))
\]

where inequalities come from Lemma 4. This demonstrates that the map \( C \to \text{Tr} (f(C)) \) is midpoint convex.

For the strict convexity of \( f \) and \( \text{Tr} (f(\frac{C+D}{2})) = \frac{1}{2} \text{Tr} (f(C)) + \frac{1}{2} \text{Tr} (f(D)) \), we have \( \left( V_i^{[k]}, C \star V_i^{[k]} \right) = \left( V_i^{[k]}, D \star V_i^{[k]} \right) \) for each \( V_i^{[k]} \). From Lemma 4 the equality will be true when \( V_i^{[k]} \) are eigenmatrices for both tensors \( C \) and \( D \). Then, we have

\[
C \star V_i^{[k]} = \left( V_i^{[k]}, C \star V_i^{[k]} \right) V_i^{[k]} = \left( V_i^{[k]}, D \star V_i^{[k]} \right) V_i^{[k]} = D \star V_i^{[k]},
\]

which indicates that \( C = D \). An obvious continuity argument now shows that if \( f \) continuous as well as convex, \( C \to \text{Tr} (f(C)) \) is convex, and strictly convex so if \( f \) is strictly convex. Therefore, this Theorem is proved from Lemma 3 and above arguments. \( \square \)

From T-SVD, we have following relation for Hermitian T-product tensor:

\[
f(s) \leq g(s) \text{ for } s \in [a, b] \implies f(C) \leq g(C) \quad \text{when the eigenvalues of } C \text{ lie in } [a, b].
\]

Above Eq. (46) is named as transfer rule.

We have defined tensor exponential under Definition 3 and the exponential of an Hermitian T-product tensor is always TPD due to the spectral mapping Lemma 2. From transfer rule Eq. (46), the tensor exponential satisfies following relations for a Hermitian T-product tensor \( C \in \mathbb{C}^{m \times m \times p} \) that we will use at later theory development:

\[
\mathcal{I}_{mmp} + C \preceq \exp(C),
\]

and

\[
\cosh(C) \preceq \exp(C^2/2).
\]

From Theorem 1.1 and the monotonicity of the exponential function, we have

\[
C \preceq D \implies \text{Tr exp}(C) \leq \text{Tr exp}(D)
\]

Below, we want to prove the monotonicity and the concavity of the logarithm function. We will begin definitions about T-product tensor monotonicity and convexity first and present several lemmas used to
Lemma 7 We have following equivalent statements about a function \( f(x) : (0, \infty) \to (0, \infty) \):

1. \( f(x) \) is T-product tensor monotone function;

establish the monotonicity and the concavity of the logarithm function. Given two Hermitian T-product tensors \( \mathcal{C}, \mathcal{D} \in \mathbb{C}^{m \times m \times p} \), a function \( f : \mathbb{R} \to \mathbb{R} \) is said to be T-product tensor monotone if the following relation holds:

\[
\mathcal{C} \preceq \mathcal{D} \implies f(\mathcal{C}) \preceq f(\mathcal{D}).
\] (50)

A function \( f \) is said as a T-product tensor convex function if we have:

\[
f(t\mathcal{C} + (1-t)\mathcal{D}) \preceq tf(\mathcal{C}) + (1-t)f(\mathcal{D}),
\] (51)

where \( 0 \leq t \leq 1 \). Also, a function \( f \) is said as a T-product tensor concave function if \(-f\) is a T-product tensor convex function. The following derivation about the monotonicity and the concavity of the logarithm function is extended from matrices according to works in [21] and [22] to T-product tensors.

Lemma 5 For any \( \mathcal{C}, \mathcal{D} \in \mathbb{C}^{m \times m \times p} \), we have \( \sigma(\mathcal{C} \star \mathcal{D}) = \sigma(\mathcal{D} \star \mathcal{C}) \).

Proof: Since eigenvalues are roots of the characteristic polynomial, it is enough to show that \( \det(\lambda \mathcal{I}_{mmp} - \mathcal{C} \star \mathcal{D}) = \det(\lambda \mathcal{I}_{mmp} - \mathcal{D} \star \mathcal{C}) \). We first assume that \( \mathcal{C} \) has inverse, then from Eq. (24), we have

\[
\det(\lambda \mathcal{I}_{mmp} - \mathcal{C} \star \mathcal{D}) = \det(C^{-1} \ast (\lambda \mathcal{I}_{mmp} - \mathcal{D} \ast \mathcal{C}) \ast \mathcal{C}) = \det(\lambda \mathcal{I}_{mmp} - \mathcal{D} \ast \mathcal{C}).
\] (52)

This shows that \( \sigma(\mathcal{C} \star \mathcal{D}) = \sigma(\mathcal{D} \star \mathcal{C}) \).

If \( \mathcal{C} \) is not invertible, we choose a sequence \( \{\epsilon_n\} \) in \( \mathbb{C} \setminus \sigma(\mathcal{C}) \) with \( \epsilon_n \to 0 \), with property that all new tensors \( \mathcal{C}_n \) are invertible for each \( n \). Then,

\[
\det(\lambda \mathcal{I}_{mmp} - \mathcal{C} \star \mathcal{D}) = \lim_{n \to \infty} \det(\lambda \mathcal{I}_{mmp} - \mathcal{C}_n \star \mathcal{D}) = \lim_{n \to \infty} \det(\lambda \mathcal{I}_{mmp} - \mathcal{D} \ast \mathcal{C}_n) = \det(\lambda \mathcal{I}_{mmp} - \mathcal{D} \ast \mathcal{C}).
\] (53)

Lemma 6 For every tensor \( \mathcal{C} \in \mathbb{C}^{m \times m \times p} \) and every function \( f \) on \( \sigma(C^H \ast \mathcal{C}) \), we have

\[
\mathcal{C} \ast f(C^H \ast \mathcal{C}) = f(C \ast C^H) \ast \mathcal{C}.
\] (54)

Proof: Since \( \sigma(C^H \ast \mathcal{C}) = \sigma(\mathcal{C} \ast C^H) \) from Lemma 5 and \( \mathcal{C} \ast (C^H \ast \mathcal{C})^n = (\mathcal{C} \ast C^H)^n \ast \mathcal{C} \), for \( n \in \mathbb{N} \), this lemma is hold for \( f \) is a polynomial. If the function \( f \) is an arbitrary function on \( \sigma(C^H \ast \mathcal{C}) = [s_1, \cdots s_n] \), we define the Lagrange interpolation polynomial as

\[
p(x) \overset{\text{def}}{=} \sum_{i=1}^{n} f(s_i) \prod_{1 \leq j \leq n, i \neq j} \frac{x - s_j}{s_i - s_j},
\] (55)

where we have \( p(s_i) = f(s_i) \) for \( 1 \leq i \leq n \). Then, we also have

\[
\mathcal{C} \ast f(C^H \ast \mathcal{C}) = \mathcal{C} \ast p(C^H \ast \mathcal{C}) = p(\mathcal{C} \ast C^H) \ast \mathcal{C} = f(\mathcal{C} \ast C^H) \ast \mathcal{C},
\] (56)

and this Lemma is proved.

Following Lemma is adopted from Corollary 12 from [22].

Lemma 7 We have following equivalent statements about a function \( f(x) : (0, \infty) \to (0, \infty) \):

1. \( f(x) \) is T-product tensor monotone function;
2. $x / f(x)$ is T-product tensor monotone function;
3. $f(x)$ is T-product tensor concave function;
4. $1 / f(x)$ is T-product tensor convex function.

**Proof:** The proof is based on Corollary 12 from [22]. But those facts about using Theorem 2.5.2 and Theorem 2.5.3 from [21] should be modified from matrices settings to T-product tensors settings. With help from Lemma 6 and transfer rules provided by Eq. (46), the proof about Theorem 2.5.2 and Theorem 2.5.3 from [21] for T-product tensors is straightforward by replacing matrix multiplication operations to T-product operations.

We are ready to prove that the logarithmic function is T-product tensor monotone and concave function on $(0, \infty)$.

**Lemma 8** Given two TPD tensors $C, D \in \mathbb{C}^{m \times m \times p}$ with $O \preceq C \preceq D$, we have
\begin{equation}
\log(C) \preceq \log(D),
\end{equation}
and
\begin{equation}
t \log(C) + (1 - t) \log(D) \preceq \log(tC + (1 - t)D).
\end{equation}

**Proof:** We define a function $g(x) = \frac{x}{\log(x+1)}$ on $(0, \infty)$. Since $g(x)$ is the monotone function on $(0, \infty)$, Lemma 7 implies that $\log(1 + x)$ is T-product tensor monotone and concave on $(0, \infty)$. For each $\epsilon > 0$, $\log(\epsilon + x) = \log \epsilon + \log(1 + x/\epsilon)$ is T-product tensor monotone and concave on $(0, \infty)$. Let $\epsilon \to 0$, we achieve the desired result. □

In general, it is not practical to always working with Hermitian T-product tensor, we will apply dilations technique to expand any T-tensor into a Hermitian T-product tensor. For any tensor $C \in \mathbb{C}^{m \times n \times p}$, a dilation for the tensor $C$, denoted as $\mathcal{D}(C)$, will be
\begin{equation}
\mathcal{D}(C) = \begin{bmatrix} O & C \\ C^H & O \end{bmatrix},
\end{equation}
where $\mathcal{D}(C) \in \mathbb{C}^{(m+n) \times (m+n) \times p}$ and we have $(\mathcal{D}(C))^H = \mathcal{D}(C)$ (Hermitian T-product tensor after dilation). Also, we have
\begin{equation}
\mathcal{D}^2(C) = \begin{bmatrix} C \star C^H & O \\ O & C^H \star C \end{bmatrix}.
\end{equation}
Since Eq. (59) is zero trace, the largest eigenvalue of $\mathcal{D}(C)$ will be the same with the largest singular of $C$.

Since the expectation of a random T-product tensor can be considered as a convex combination, expectation preserves the TPSD order as:
\begin{equation}
\mathcal{X} \preceq \mathcal{Y} \text{ almost surely} \implies \mathbb{E}\mathcal{X} \preceq \mathbb{E}\mathcal{Y}.
\end{equation}

Also from Lemma 7, we know that the quadratic function $f(x) = x^2$ is T-product tensor convex, thus, we have
\begin{equation}
(\mathbb{E}C)^2 \preceq \mathbb{E}(C^2) .
\end{equation}

We will present one more theorem in this section about Golden-Thompson inequality for T-product tensors. We recall theorem 12.
Theorem 1.2 (Golden-Thompson inequality for T-product Tensors) Given two Hermitian T-product tensors $C, D \in \mathbb{C}^{m \times m \times p}$, we have

$$\text{Tr} \left( \exp(C + D) \right) \leq \text{Tr} \left( \exp (C) \ast \exp (D) \right),$$

where $\ast$ is the product operation between two T-product tensors defined by Eq. (15).

Proof: From T-SVD decomposition and Eq. (29), we can express the tensor $C$ as

$$C = \sum_\lambda \mathcal{P}_\lambda,$$

where $\lambda$ are eigenvalues and $\mathcal{P}_\lambda$ are corresponding projectors (T-product tensors) which are mutually orthogonal. Given $X \succeq O$, we define following mapping with respect to the tensor $C$ as:

$$\mathfrak{P}_C(X) : X \rightarrow \sum_\lambda \mathcal{P}_\lambda \ast X \ast \mathcal{P}_\lambda.$$  (64)

Then, we have following properties about mapping $\mathfrak{P}_C(X)$

1. $\mathfrak{P}_C(X)$ commutes with $C$;
2. $\text{Tr} \left( \mathfrak{P}_C(X) \ast C \right) = \text{Tr} \left( X \ast C \right)$;
3. $\mathfrak{P}_C(X) \succeq \frac{X}{|\text{sp}(C)|}$, where $\text{sp}(C) = \{\lambda_1, \lambda_2, \cdots, \lambda_{|\text{sp}(C)|}\}$.

The third property of $\mathfrak{P}_C(X)$ is true due to the following relation:

$$\mathfrak{P}_C(X) = \sum_{\lambda \in \text{sp}(C)} \mathcal{P}_\lambda \ast X \ast \mathcal{P}_\lambda$$

$$= \frac{1}{|\text{sp}(C)|} \sum_{x=1}^{\text{sp}(C)} \mathcal{U}_x \ast X \ast \mathcal{U}_x^H$$

$$\succeq \frac{X}{|\text{sp}(C)|},$$  (65)

where $\mathcal{U}_x = \sum_{i=1}^{\text{sp}(C)} \exp \left( \frac{-\sqrt{-\pi} x_i}{|\text{sp}(C)|} \right) \mathcal{P}_\lambda$. Let $A_1 = \exp(C)$ and $A_2 = \exp(D)$, we have

$$\log \text{Tr} \left( \exp (\log A_1 + \log A_2) \right) =_1 \frac{1}{n} \log \text{Tr} \left( \exp (\log A_1^{\otimes n} + \log A_2^{\otimes n}) \right)$$

$$\leq_2 \frac{1}{n} \log \text{Tr} \left( \exp \left( \log \mathfrak{P}_{A_1^{\otimes n}}(A_1^{\otimes n}) + \log A_2^{\otimes n} \right) \right)$$

$$+ \log \text{poly}(n)$$

$$=_3 \frac{1}{n} \log \left( \text{Tr} \left( \mathfrak{P}_{A_2^{\otimes n}}(A_2^{\otimes n}) \ast A_1^{\otimes n} \right) \right) + \log \text{poly}(n)$$

$$=_4 \log \text{Tr} \left( A_1 \ast A_2 \right) + \frac{\log \text{poly}(n)}{n}.$$  (66)

The equality $=_1$ comes from the fact that the trace is multiplicative under the Kronecker product. The inequality $\leq_2$ follows from inequality from the third property of $\mathfrak{P}_{A_2^{\otimes n}}(A_1^{\otimes n})$, the monotone of log and
and the number of eigenvalues of \( A_2^\otimes n \) growing polynomially with \( n \) due to the fact that the number of distinct eigenvalues of \( A_2^\otimes n \) is bounded by the number of different types of sequences of \( mp \) symbols of length \( n \), see Lemma II.1 in [23]. The equality (3) utilizes the commutativity property for tensors \( \mathcal{P}_{A_2^\otimes m} (A_1^\otimes m) \) and \( A_2^\otimes n \) based on the first property. Finally, the equality (4) applies trace properties from the second property of the mapping \( \mathcal{P}_{A_2^\otimes m} (A_1^\otimes m) \). If \( n \to \infty \), the result of this theorem is established.

\[ \square \]

### 3  Lieb’s Concavity Under T-product

In this section, we will extend several trace inequalities to T-product tensors: Jensen’s T-product tensor inequality in Section 3.1 and Klein’s T-product tensor inequality in Section 3.2. These new T-product tensor inequalities will play important roles in establishing a new version of Lieb’s concavity theorem under T-product in Section 3.3.

#### 3.1 Jensen’s T-product Inequality

In this subsection, we will derive Jensen’s T-product tensor inequality in Theorem 1.3. We begin with a lemma which will be used in later proof in Theorem 1.3.

Given two natural numbers \( m, n \), we define a T-product tensor \( \theta \in \mathbb{C}^{mn \times mn \times p} \) as

\[
\exp \left( 2\pi \sqrt{-1}/n \right) \times \mathcal{I}_{mmp}.
\]

Then, we can have tensor \( D \in \mathbb{C}^{mn \times mn \times p} \) obtained by

\[
D = \text{diag} \left( \theta, \theta, \ldots, \theta^{n-1}, \mathcal{I}_{mmp} \right),
\]

where \( \text{diag} \left( \theta, \theta, \ldots, \theta^{n-1}, \mathcal{I}_{mmp} \right) \) will be a matrix with entries as T-product tensors, and the diagonal part of this matrix is composed by tensors \( \theta, \theta, \ldots, \theta^{n-1}, \mathcal{I}_{mmp} \). Let \( D \in \mathbb{C}^{mn \times mn \times p} \) be another matrix of T-product tensor, i.e., entries \( d_{i,j} \) as T-product tensors. We define a new operation \( \oplus \) between two T-product tensors, \( A, B \) with dimensions belong to \( \mathbb{C}^{mn \times mn \times p} \) as:

\[
(A \oplus B)_{i,j} = \sum_{k=1}^{n} a_{i,k} \star b_{k,j},
\]

where both \( a_{i,k} \) and \( c_{k,j} \) are T-product tensors. Therefore, given any tensor \( C \in \mathbb{C}^{mn \times mn \times p} \), the \( i,j \)-th entry (a T-product tensor) of \( C \oplus D \) becomes \( \exp(2\pi \sqrt{-1}/n) \times c_{i,j} \), where \( c_{i,j} \in \mathbb{C}^{mn \times mn \times p} \) is a T-product tensor.

**Lemma 9** Given any tensor \( C \in \mathbb{C}^{mn \times mn \times p} \) and the tensor \( D \) defined by Eq. 67, we have

\[
\frac{1}{n} \sum_{k=1}^{n} D^{-k} \oplus C \oplus D^k = \text{diag} \left( c_{1,1}, c_{2,2}, \ldots, c_{n,n} \right)
\]

where \( D^k \) is the self-product of the tensor \( D \) by \( \oplus \) operation \( k \)-times.

**Proof:** By direct computation with \( \oplus \), we have following:

\[
\left( \frac{1}{n} \sum_{k=1}^{n} D^{-k} \oplus C \oplus D^k \right)_{i,j} = \frac{1}{n} \sum_{k=1}^{n} \left( \exp(2\pi \sqrt{-1}/j-n) \right)^{k} c_{i,j},
\]

where this summation is zero for \( i \neq j \), otherwise, it is \( c_{i,i} \).

We are ready to prove theorem 1.3.
**Theorem 1.3 (Jensen’s T-product Inequality)** For a continuous T-product tensor convex function \( f \) defined on an interval \( I \). The definition of T-product tensor convex is given by Eq. (51). we have the following TPSD relation for each natural number \( n \):

\[
f \left( \sum_{i=1}^{n} C_i^H \ast X_i \ast C_i \right) \leq \sum_{i=1}^{n} C_i^H \ast f (X_i) \ast C_i,
\]

(2)

where \( X_i \in \mathbb{C}^{m \times m \times p} \) are bounded, Hermitian T-product tensors with all eigenvalues in the interval \( I \) and tensors \( C_i \) satisfying \( \sum_{i=1}^{n} C_i^H \ast C_i = \mathcal{I}_{mmp} \).

**Proof:** Let us define a unitary tensor \( U = (u_{i,j}) \in \mathbb{C}^{mn \times mn \times p} \) for \( 1 \leq i,j \leq n \) as \( u_{i,j} = C_i \), \( D = \text{diag} (\theta, \theta^2, \ldots, \theta^{n-1}, I_{mmp}) \) defined by Eq. (67), and define the tensor \( \overline{X} \in \mathbb{C}^{mn \times mn \times p} \) as \( \text{diag} (X_1, \ldots, X_n) \). From Lemma 9 we have

\[
f \left( \sum_{i=1}^{n} C_i^H \ast X_i \ast C_i \right) = f \left( (U^H \oplus \overline{X} \oplus U)_{n,n} \right)
\]

\[
= f \left( \left( \sum_{i=1}^{n} \frac{1}{n} D^{-i} \oplus U^H \oplus \overline{X} \oplus U \oplus D^i \right)_{n,n} \right)
\]

\[
= f \left( \left( \sum_{i=1}^{n} \frac{1}{n} D^{-i} \oplus U^H \oplus \overline{X} \oplus U \oplus D^i \right)_{n,n} \right)
\]

\[
\leq \left( \frac{1}{n} \sum_{i=1}^{n} f \left( D^{-i} \oplus U^H \oplus \overline{X} \oplus U \oplus D^i \right) \right)_{n,n}
\]

\[
= \left( \frac{1}{n} \sum_{i=1}^{n} f \left( D^{-i} \oplus U^H \oplus \overline{X} \oplus U \oplus D^i \right) \right)_{n,n}
\]

\[
= (U^H \oplus f (\overline{X}) \oplus U)_{n,n}
\]

\[
= \sum_{i=1}^{n} C_i^H \ast f (X_i) \ast C_i,
\]

(71)

where the inequality comes from that the function \( f \) is a T-product tensor convex function. \( \square \)

### 3.2 Klein’s T-product Inequality

The immediate application of Theorem 1.1 is to prove Klein’s inequality for T-product tensor. We recall theorem 1.4.

**Theorem 1.4 (Klein’s T-product Inequality)** For all \( C, D \) Hermitian T-product tensors and a differentiable convex function \( f : \mathbb{R} \to \mathbb{R} \) or for all \( C, D \) Hermitian T-product tensors and a differentiable convex function \( f : (0, \infty) \to \mathbb{R} \), we have

\[
\text{Tr} \left( f(C) - f(D) - (C - D) \ast f'(D) \right) \geq 0.
\]

(3)

In both situations, if \( f \) is strictly convex, equality holds if and only if \( C = D \).
Proof: We define function $F(t)$ as
\begin{equation}
F(t) = \text{Tr} \left( f(\mathcal{D} + t(\mathcal{C} - \mathcal{D})) \right),
\end{equation}
where $t \in (0, 1)$. From Theorem [1.1], $F(t)$ is a convex function. Then, we have
\begin{equation}
F(0) + t(F(1) - F(0)) \geq F(t) \iff F(1) - F(0) \geq \frac{F(t) - F(0)}{t}
\end{equation}
By taking limit $t \to 0$ at $F(1) - F(0) \geq \frac{F(t) - F(0)}{t}$, we have
\begin{equation}
F(1) - F(0) \geq F'(0),
\end{equation}
then we obtain Klein’s inequality under T-product by rearrangement and substitution with Eq. (72). □

3.3 Lieb’s Concavity Theorem Under T-product

In this section, we will extend Lieb’s concavity theorem to T-product tensors and we begin with the definition about the relative entropy between two T-product tensors.

Definition 3 Given two TPD tensors $A \in \mathbb{C}^{m \times m \times p}$ and tensor $B \in \mathbb{C}^{m \times m \times p}$. The relative entropy between two T-product tensors $A$ and $B$ is defined as
\begin{equation}
D(A \parallel B) \overset{def}{=} \text{Tr} A \ast (\log A - \log B).
\end{equation}

We apply perspective function concept for T-product tensor convex and introduce the following lemma about the convexity of a T-product tensor convex function [24].

Lemma 10 Given $f$ as a convex function, two commuting tensors $X, Y \in \mathbb{C}^{m \times m \times p}$, i.e., $X \ast Y = Y \ast X$, and the existence of the $Y^{-1}$, then the following map $h$
\begin{equation}
h(X, Y) = f(X \ast Y^{-1}) \ast Y
\end{equation}
is jointly convex in the sense that, given $t \in [0, 1]$, if $X = tX_1 + (1-t)X_2$ and $Y = tY_1 + (1-t)Y_2$ with $X_1 \ast Y_1 = Y_1 \ast X_1$ and $X_2 \ast Y_2 = Y_2 \ast X_2$, we should have
\begin{equation}
h(X, Y) \leq th(X_1, Y_1) + (1-t)h(X_2, Y_2).
\end{equation}

Proof 1 Constructing tensors $A = (tY_1)^{1/2} \ast Y^{-1/2}$ and $B = ((1-t)Y_2)^{1/2} \ast Y^{-1/2}$, then we have
\begin{equation}
A^H \ast A + B^H \ast B = I_{mmp}
\end{equation}
Since we have
\begin{align*}
h(X, Y) &= f(X \ast Y^{-1}) \ast Y \\
&= Y^{1/2} \ast f(Y^{-1/2} \ast X \ast Y^{-1/2}) \ast Y^{1/2} \\
&= Y^{1/2} \ast f(A^H \ast X_1 \ast Y_1^{-1} \ast A + B^H \ast X_2 \ast Y_2^{-1} \ast B) \ast Y^{1/2} \\
&\leq_1 Y^{1/2} \ast (f(A^H \ast X_1 \ast Y_1^{-1}) \ast A \\
&\quad + f(X_2 \ast Y_2^{-1}) \ast B) \ast Y^{1/2} \\
&= (tY_1)^{1/2}f(X_1 \ast Y_1^{-1})(tY_1)^{1/2} + ((1-t)Y_2)^{1/2}f(X_2 \ast Y_2^{-1})((1-t)Y_2)^{1/2} \\
&= th(X_1, Y_1) + (1-t)h(X_2, Y_2)
\end{align*}
where $\leq_1$ is based on the condition provided by Eq. (78) and Theorem [23]
Following lemma is given to establish the joint convexity property of relative entropy for T-product tensors.

**Lemma 11 (Joint Convexity of Relative Entropy for T-product Tensors)** The relative entropy function of two TPD tensors is a jointly convex function. That is

\[
\mathbb{D}(tA_1 + (1-t)A_2 \parallel tB_1 + (1-t)B_2) \leq t\mathbb{D}(A_1 \parallel B_1) + (1-t)\mathbb{D}(A_2 \parallel B_2),
\]

where \( t \in [0,1] \) and all the following four tensors \( A_1, B_1, A_2 \) and \( B_2 \) are TPD tensors.

**Proof:** From the definition we wish to show the joint convexity of the function \( \mathbb{D}(A \parallel B) \) with respect to the tensors \( A, B \in \mathbb{C}^{m \times m \times p} \). Let us define tensor operators \( \mathcal{F}(\mathcal{X}) \defeq A \star \mathcal{X} \) and \( \mathcal{G}(\mathcal{X}) \defeq \mathcal{X} \star B \) for the variable tensor \( \mathcal{X} \in \mathbb{C}^{m \times m \times p} \). Then, we have \( \mathcal{F}(\mathcal{X}) \) and \( \mathcal{G}(\mathcal{X}) \) commuting on the inner product operation \( \langle \mathcal{F}(\mathcal{X}), \mathcal{G}(\mathcal{X}) \rangle \) defined as:

\[
\langle \mathcal{F}(\mathcal{X}), \mathcal{G}(\mathcal{X}) \rangle = \text{Tr}(\mathcal{F}^\dagger(\mathcal{X}) \star \mathcal{G}(\mathcal{X}))
\]

Then, we have \( \text{Tr}(\mathcal{F}^\dagger(\mathcal{X}) \star \mathcal{G}(\mathcal{X})) = \text{Tr}(\mathcal{G}^\dagger(\mathcal{X}) \star \mathcal{F}(\mathcal{X})) \). Since the function \( f(x) = x \log x \) is tensor convex, we apply Lemma to operators \( \mathcal{F}(\mathcal{X}), \mathcal{G}(\mathcal{X}) \) and the function \( h \) definition provided by Eq. (76) to obtain the following relation (\( \mathcal{I} = \mathcal{I}_{mnp} \) in this proof):

\[
\langle \mathcal{I}, h(\mathcal{F}(\mathcal{I}), \mathcal{G}(\mathcal{I})) \rangle = \langle \mathcal{I}, \mathcal{G}(\mathcal{I}) \star (\mathcal{F}(\mathcal{I}) \star \mathcal{G}^{-1}(\mathcal{I})) \log(\mathcal{F}(\mathcal{I}) \star \mathcal{G}^{-1}(\mathcal{I})) \rangle = \langle \mathcal{I}, \mathcal{F}(\mathcal{I})(\log \mathcal{F}(\mathcal{I}) - \log \mathcal{G}(\mathcal{I})) \rangle = \text{Tr}(\mathcal{A} \log \mathcal{A} - \mathcal{A} \log \mathcal{B}) = \mathbb{D}(A \parallel B),
\]

is jointly convex with respect to tensors \( A \) and \( B \).

Lieb’s concavity theorem is recalled below by theorem 1.5.

**Theorem 1.5 (Lieb’s concavity theorem for T-product tensors)** Let \( \mathcal{H} \) be a Hermitian T-product tensor. Following map

\[
\mathcal{A} \to \text{Tr}\mathcal{H} + \log \mathcal{A}
\]

is concave on the positive-definite cone.

**Proof:** From Klein’s inequality obtain from Theorem 1.4, the convexity of map \( t \to t \log t \) (which is strictly concave for \( t > 0 \)) and Hermitian T-tensors \( \mathcal{X}, \mathcal{Y} \), we have

\[
\text{Tr}\mathcal{Y} \geq \text{Tr}\mathcal{X} - \text{Tr}\mathcal{X} \log \mathcal{X} + \text{Tr}\mathcal{X} \log \mathcal{Y}.
\]

If we replace \( \mathcal{Y} \) by \( e^{\mathcal{H} + \log \mathcal{A}} \), we then have

\[
\text{Tr}e^{\mathcal{H} + \log \mathcal{A}} = \max_{\mathcal{X} \succ 0} \left\{ \text{Tr}\mathcal{X} \star \mathcal{H} - \mathbb{D}(\mathcal{X} \parallel \mathcal{A}) + \text{Tr}\mathcal{X} \right\}
\]

where \( \mathbb{D}(\mathcal{X} \parallel \mathcal{Y}) \) is the quantum relative entropy between two tensor operators. For real number \( t \in [0,1] \) and two positive-definite tensors \( A_1, A_2 \), we have

\[
\text{Tr}e^{\mathcal{H} + \log (tA_1 + (1-t)A_2)} = \max_{\mathcal{X} \succ 0} \left\{ \text{Tr}\mathcal{X} \star \mathcal{H} - \mathbb{D}(\mathcal{X} \parallel tA_1 + (1-t)A_2) + \text{Tr}\mathcal{X} \right\}
\geq t \max_{\mathcal{X} \succ 0} \left\{ \text{Tr}\mathcal{X} \star \mathcal{H} - \mathbb{D}(\mathcal{X} \parallel tA_1) + \text{Tr}\mathcal{X} \right\}
\]

\[
+ (1-t) \max_{\mathcal{X} \succ 0} \left\{ \text{Tr}\mathcal{X} \star \mathcal{H} - \mathbb{D}(\mathcal{X} \parallel (1-t)A_2) + \text{Tr}\mathcal{X} \right\}
= t\text{Tr}e^{\mathcal{H} + \log A_1} + (1-t)\text{Tr}e^{\mathcal{H} + \log A_2},
\]

17
where the first and last equalities are obtained based on the variational formula provided by Eq. (84), and the inequality is due to the joint convexity property of the relative entropy from Lemma 1.1. □

Based on Lieb’s concavity theorem for T-product tensors, we have the following corollary.

**Corollary 2** Let \( A \) be a fixed Hermitian T-product tensor, and let \( X \) be a random Hermitian T-product tensor, then we have

\[
\mathbb{E} \text{Tr} e^{A + X} \leq \text{Tr} e^{A + \log(\mathbb{E}e^X)}.
\]

**Proof 2** Define the random tensor \( Y = e^X \), we have

\[
\mathbb{E} \text{Tr} e^{A + X} = \mathbb{E} \text{Tr} e^{A + \log(Y)} \leq \text{Tr} e^{A + \log(\mathbb{E}Y)} = \text{Tr} e^{A + \log(\mathbb{E}e^X)},
\]

where the inequality is based on Lieb’s concavity theorem for T-product tensors obtained by Theorem 1.5 and Jensen’s T-product tensor inequality by Theorem 1.3

### 3.4 T-product Tensor Moments and Cumulants

Since the expectation of a random T-product tensor can be treated as convex combination, expectation will preserve the semidefinite order as

\[
\mathcal{X} \succ \mathcal{Y} \quad \text{almost surely} \quad \Rightarrow \quad \mathbb{E}(\mathcal{X}) \succ \mathbb{E}(\mathcal{Y}).
\]

From Jensen’s T-product tensor inequality by Theorem 1.3, we also have

\[
\mathbb{E}(\mathcal{X}^2) \succeq (\mathbb{E}(\mathcal{X}))^2.
\]

Suppose a random Hermitian T-product tensor \( \mathcal{X} \) having tensor moments of all orders, i.e., \( \mathbb{E}(\mathcal{X}^n) \) existing for all \( n \), we can define the tensor moment-generating function, denoted as \( M_{\mathcal{X}}(t) \), and the tensor cumulant-generating function, denoted as \( K_{\mathcal{X}}(t) \), for the tensor \( \mathcal{X} \) as

\[
M_{\mathcal{X}}(t) \overset{\text{def}}{=} \mathbb{E} e^{t\mathcal{X}}, \quad \text{and} \quad K_{\mathcal{X}}(t) \overset{\text{def}}{=} \log \mathbb{E} e^{t\mathcal{X}},
\]

where \( t \in \mathbb{R} \). Both the tensor moment-generating function and the tensor cumulant-generating function can be expressed as power series expansions:

\[
M_{\mathcal{X}}(t) = I + \sum_{n=1}^{\infty} \frac{t^n}{n!} \mathbb{E}(\mathcal{X}^n), \quad \text{and} \quad K_{\mathcal{X}}(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} \psi_n,
\]

where \( \psi_n \) is called tensor cumulant. The tensor cumulant \( \psi_n \) can be expressed as a polynomial in terms of tensor moments up to the order \( n \), for example, the first cumulant is the mean and the second cumulant is the variance:

\[
\psi_1 = \mathbb{E}(\mathcal{X}), \quad \text{and} \quad \psi_2 = \mathbb{E}(\mathcal{X}^2) - (\mathbb{E}(\mathcal{X}))^2.
\]

Finally, in this work, we also assume that all random variables are sufficiently regular for us to compute their expectations, interchange limits, etc.

### 4 Tail Bounds By Concatenation of Lieb’s Concavity

The goal of this section is to develop several important tools which will be applied intensively in the proof of probability inequalities for the extreme eigentuple (or eigenvalue) of a sum of independent random T-product tensors. The first tool is the Laplace transform method for T-product tensors discussed in Section 4.1, and the second tool is the tail bound for independent sums of random Hermitian T-product tensors presented by Section 4.2.
4.1 Laplace Transform Method for T-product Tensors

We extend the Laplace transform bound from matrices to T-product tensors based on [25]. Following lemma is given to establish the Laplace transform bound for the maximum eigenvalue of a T-product tensor.

**Lemma 12 (Laplace Transform Method for T-product Tensors: Eigenvalue Version)** Let $\mathcal{X}$ be a random Hermitian T-product tensor. For $\theta \in \mathbb{R}$, we have

$$
\mathbb{P}(\lambda_{\text{max}}(\mathcal{X}) \geq \theta) \leq \inf_{t > 0} \left\{ e^{-\theta t} \mathbb{E} \text{Tr} e^{t\mathcal{X}} \right\}
$$

(93)

**Proof 3** Given a fixed value $t$, we have

$$
\mathbb{P}(\lambda_{\text{max}}(\mathcal{X}) \geq \theta) = \mathbb{P}(\lambda_{\text{max}}(t\mathcal{X}) \geq t\theta) = \mathbb{P}(e^{\lambda_{\text{max}}(t\mathcal{X})} \geq e^{t\theta}) \leq e^{-t\theta} e^{\lambda_{\text{max}}(t\mathcal{X})}.
$$

(94)

The first equality uses the homogeneity of the maximum eigenvalue map, the second equality comes from the monotonicity of the scalar exponential function, and the last relation is Markov’s inequality. Because we have

$$
e^{\lambda_{\text{max}}(t\mathcal{X})} = \lambda_{\text{max}}(e^{t\mathcal{X}}) \leq \text{Tr} e^{t\mathcal{X}},
$$

(95)

where the first equality used the spectral mapping theorem from Lemma 2, and the inequality holds because the exponential of an Hermitian T-product tensor is TPD and the maximum eigenvalue of a TPD tensor is dominated by the trace from Eq. (16). From Eqs (93) and (95), this lemma is established.

The Lemma 12 helps us to control the tail probabilities for the maximum eigenvalue of a random Hermitian T-product tensor by utilizing a bound for the trace of the tensor moment-generating function introduced in Section 3.4.

Since T-product tensors also have notions about eigentuples, we then extend Lemma 12 from eigenvalues version to eigentuples version. We begin with the derivation of Markov’s inequality for random vectors.

**Lemma 13 (Markov’s inequality for Random Vector)** If $\mathbf{X} \in \mathbb{R}^p$ is a nonnegative random vector and $a > 0$, then the probability that $\mathbf{X}$ is at least $a = [a_i]$ can be bounded as:

$$
\mathbb{P}(\mathbf{X} \geq a) \leq \min_{i} \left\{ \frac{\mathbb{E}(\mathbf{X})_i}{a_i} \right\}
$$

(96)

where $1 \leq i \leq p$.

**Proof:** Because we have

$$
\mathbb{E}(\mathbf{X}) = \int_0^\infty x f(x) dx = \int_0^a x f(x) dx + \int_a^\infty x f(x) dx
$$

$$
\geq \int_a^\infty x f(x) dx \geq \int_a^\infty a f(x) dx = a \int_a^\infty f(x) dx
$$

$$
= a \mathbb{P}(\mathbf{X} \geq a),
$$

(97)

therefore, we have the desired inequality shown by Eq. (96). \qed

We are ready to present following lemma about Laplace transform method for T-product tensors with eigentuples.
Lemma 14 (Laplace Transform Method for T-product Tensors: Eigentuple Version) Let $X \in \mathbb{C}^{m \times m \times p}$ be a random T-positive definite (TPD) tensor and an all one vector $1_p = [1, 1, \cdots, 1]^T \in \mathbb{C}^p$. Suppose we have

$$\frac{1}{p} \lambda_p^{\max}(e^{tx}) + 1 - \frac{1}{p} \leq \text{Tr}(e^{tx}),$$

where $t > 0$. Then, for $b \in \mathbb{R}^p$, we obtain

$$P(d_{\max}(X) \geq b) \leq \inf_{t > 0} \min_i \left\{ \frac{E \left( \text{Tr} \left( e^{tx} \right) \right)} {e^{tb}} \right\},$$

where $d_{\max}$ is the maximum eigentuple of the TPD tensor $X$.

Proof: Given a fix value $t$, we have

$$P(d_{\max}(X) \geq b) = P(d_{\max}(tX) \geq tb) = P(e^{d_{\max}(tX)} \geq e^{tb}) \leq \min_i \left\{ \frac{E \left( e^{d_{\max}(tX)} \right)} {e^{tb}} \right\}.$$

The first equality uses the homogeneity of the maximum eigenvalue map, the second equality comes from the monotonicity of the exponential function with operation $\bigcirc$ defined in Proposition 2.1 from work [1], and the last relation is Markov’s inequality for random vector obtained from Lemma 13 since both $E \left( e^{d_{\max}(tX)} \right)$ and $e^{tb}$ are vectors with $p$ entries. Then, we have

$$e^{d_{\max}(tX)} \leq e^{\lambda_p^{\max}(e^{tx})1_p} \leq \text{Tr} \left( e^{tx} \right) 1_p,$$

where the first inequality comes from the relation that $d_{\max}(tX) \leq \lambda_p^{\max}(tX)1_p$, and the second inequality holds because $e^{\lambda_p^{\max}(tX)} = \lambda_p^{\max}(e^{tx})$ and the relation $\frac{1}{p} \lambda_p^{\max}(e^{tx}) + 1 - \frac{1}{p} \leq \text{Tr}(e^{tx})$. From Eqs (99) and (101), this lemma is established.

4.2 Tail Bounds for Independent Sums of Random T-product Tensors

This section will present the tail bound for the sum of independent random T-product tensors and several corollaries according to this tail bound for independent sums. We begin with the subadditivity lemma of tensor cumulant-generating functions.

Lemma 15 Given a finite sequence of independent random Hermitian T-product tensors $\{X_i\}$, where $X_i \in \mathbb{C}^{m \times m \times p}$, we have

$$E \text{Tr} \exp \left( \sum_{i=1}^n tX_i \right) \leq \text{Tr} \exp \left( \sum_{i=1}^n \log E e^{tx_i} \right), \text{ for } t \in \mathbb{R}.$$ (102)

Proof: We begin with the following definition for the tensor cumulant-generating function for $X_i$ as:

$$K_i(t) \overset{\text{def}}{=} \log(E e^{tx_i}).$$

\footnote{If we scale the random TPD tensor $X$ as the $\lambda_p^{\max}(e^{tx}) = 1$, then Eq. (98) is always valid.}

20
Then, we define the Hermitian T-product tensor $H_k$ as

$$H_k(t) = \sum_{i=1}^{k-1} tX_i + \sum_{i=k+1}^{n} \mathbb{K}_i(t).$$

(104)

By applying Eq. (104) to Theorem 1.5 repeatedly for $k = 1, 2, \cdots, n$, we have

$$E \text{Tr} \exp \left( \sum_{i=1}^{n} tX_i \right) =_1 E_0 \cdots E_{n-1} \text{Tr} \exp \left( \sum_{i=1}^{n-1} tX_i + tX_n \right)$$

$$\leq E_0 \cdots E_{n-2} \text{Tr} \exp \left( \sum_{i=1}^{n-1} tX_i + \log (E_{n-1} e^{tX_n}) \right)$$

$$= E_0 \cdots E_{n-2} \text{Tr} \exp \left( \sum_{i=1}^{n-2} tX_i + tX_{n-1} + \mathbb{K}_n(t) \right)$$

$$\leq E_0 \cdots E_{n-3} \text{Tr} \exp \left( \sum_{i=1}^{n-2} tX_i + \mathbb{K}_{n-1}(t) + \mathbb{K}_n(t) \right)$$

$$\cdots \leq \text{Tr} \exp \left( \sum_{i=1}^{n} \mathbb{K}_i(t) \right)$$

(105)

where the equality $=_1$ is based on the law of total expectation by defining $E_i$ as the conditional expectation given $X_1, \cdots, X_i$. □

We are ready to present the theorem for the tail bound of independent sums of random Hermitian T-product tensors with respect to the maximum eigenvalue. We recall theorem 1.6

**Theorem 1.6 (Master Tail Bound for Independent Sum of Random T-product Tensors for Eigenvalue)**

Given a finite sequence of independent Hermitian T-product tensors $\{X_i\}$, we have

$$\Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq \theta \right) \leq \inf_{t \geq 0} \left\{ e^{-t\theta} \text{Tr} \exp \left( \sum_{i=1}^{n} \log E e^{tX_i} \right) \right\}. \quad (5)$$

**Proof:** By substituting the Lemma 15 into the Laplace transform bound provided by the Lemma 12, this theorem is established. □

Several useful corollaries will be provided based on Theorem 1.6

**Corollary 3** Given a finite sequence of independent Hermitian random tensors $\{X_i\} \in \mathbb{C}^{m \times m \times p}$. If there is a function $f : (0, \infty) \to [0, \infty]$ and a sequence of non-random Hermitian T-product tensors $\{A_i\}$ with following condition:

$$f(t)A_i \geq \log E e^{tX_i}, \text{ for } t > 0.$$  \hspace{1cm} (106)

Then, for all $\theta \in \mathbb{R}$, we have

$$\Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq \theta \right) \leq mp \inf_{t > 0} \left\{ \exp \left[ -t\theta + f(t)\lambda_{\max} \left( \sum_{i=1}^{n} A_i \right) \right] \right\} \quad (107)$$
Proof: From the condition provided by Eq. (106) and Theorem 1.6, we have
\[
\Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq \theta \right) \leq e^{-\theta t} \text{Tr} \exp(f(t) \sum_{i=1}^{n} A_i) \\
\leq mpe^{-\theta \lambda_{\max}} \left( \exp(f(t) \sum_{i=1}^{n} A_i) \right) \\
= mpe^{-\theta \lambda_{\max}} \left( f(t) \lambda_{\max} \left( \sum_{i=1}^{n} A_i \right) \right),
\]
where the second inequality holds since we bound the trace of a TPD T-product tensor by the dimension size \(m \times p\) multiplied by the maximum eigenvalue; the last equality is based on the spectral mapping theorem since the function \(f\) is nonnegative. □

Corollary 4
Given a finite sequence of independent Hermitian random tensors \(\{X_i\} \in \mathbb{C}^{m \times m \times p}\). For all \(\theta \in \mathbb{R}\), we have
\[
\Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq \theta \right) \leq mp \inf_{t>0} \left\{ \exp \left[ -t \theta + n \log \lambda_{\max} \left( \frac{1}{n} \sum_{i=1}^{n} E_{e^{tX_i}} \right) \right] \right\}
\]
(109)

Proof: From T-tensor logarithm concavity property provided by Lemma 8, we have
\[
\sum_{i=1}^{n} \log E_{e^{tX_i}} = n \cdot \frac{1}{n} \sum_{i=1}^{n} \log E_{e^{tX_i}} \leq n \log \left( \frac{1}{n} \sum_{i=1}^{n} E_{e^{tX_i}} \right),
\]
and from the trace exponential monotone property provided by Lemma 3, we have
\[
\Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq \theta \right) \leq e^{-\theta t} \text{Tr} \exp \left( n \log \left( \frac{1}{n} \sum_{i=1}^{n} E_{e^{tX_i}} \right) \right) \\
\leq mp \inf_{t>0} \left\{ \exp \left[ -t \theta + n \log \lambda_{\max} \left( \frac{1}{n} \sum_{i=1}^{n} E_{e^{tX_i}} \right) \right] \right\},
\]
(111)
where the last inequality holds since we bound the trace of a positive-definite tensor by the dimension size \(m \times p\) multiplied by the maximum eigenvalue and apply spectral mapping theorem twice. □

Similarly, we can generalize master tail bound for independent sum of random Hermitian T-product tensors for eigenvalue version from Theorem 1.6 to master tail bound for independent sum of random Hermitian T-product tensors for eigentuple version by the following Theorem 1.7.

Theorem 1.7 (Master Tail Bound for Independent Sum of Random T-product Tensors for Eigentuple)
Given a finite sequence of independent random Hermitian T-product tensors \(\{X_i\}\) such that \(X_i \in \mathbb{C}^{m \times m \times p}\), if \(\sum_{i=1}^{n} tX_i\) satisfies Eq. (98), we have
\[
\Pr \left( d_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq b \right) \leq \inf_{t>0} \min_{1 \leq j \leq p} \left\{ \text{Tr} \exp \left( \frac{1}{n} \sum_{i=1}^{n} \log E_{e^{tX_i}} \right) \right\} \left( e^{tb} \circ \right)_j
\]
(8)
where \(e^{tb} \circ \in \mathbb{C}^p\) is the exponential for the vector \(tb\) with respect to \(\circ\) operation.
**Proof:** By substituting the Lemma 15 into the Laplace transform bound provided by the Lemma 14, this theorem is established.

Some useful corollaries will be provided based on Theorem 1.7.

**Corollary 5** Given a finite sequence of independent random Hermitian T-product tensors \( \{X_i\} \) with dimensions in \( \mathbb{C}^{m \times m \times p} \). If there is a function \( f : (0, \infty) \rightarrow [0, \infty] \) and a sequence of non-random Hermitian T-product tensors \( \{A_i\} \) with following condition:

\[
  f(t)A_i \geq \log \mathbb{E}e^{tX_i}, \text{ for } t > 0.
\]  

(112)

Then, for all \( b \in \mathbb{R}^p \) and \( \sum_{i=1}^{n} tX_i \) satisfying Eq. (98), we have

\[
  \Pr \left( d_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq b \right) \leq mp \inf_{t>0} \min_{1 \leq j \leq p} \left\{ \frac{\exp \left( f(t)\lambda_{\max} \left( \sum_{i=1}^{n} A_i \right) \right)}{e^{tb} \otimes_j} \right\}.
\]  

(113)

**Proof:** From the condition provided by Eq. (112) and Theorem 1.7, we have

\[
  \Pr \left( d_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq b \right) \leq \inf_{t>0} \min_{1 \leq j \leq p} \left\{ \frac{\text{Tr} \exp \left( f(t) \sum_{i=1}^{n} A_i \right)}{e^{tb} \otimes_j} \right\}
\]

\[
  \leq mp \inf_{t>0} \min_{1 \leq j \leq p} \left\{ \frac{\lambda_{\max} \left( \exp \left( f(t) \sum_{i=1}^{n} A_i \right) \right)}{e^{tb} \otimes_j} \right\}
\]

\[
  = mp \inf_{t>0} \min_{1 \leq j \leq p} \left\{ \frac{\exp \left( f(t) \lambda_{\max} \left( \sum_{i=1}^{n} A_i \right) \right)}{e^{tb} \otimes_j} \right\}
\]  

(114)

where the second inequality holds since we bound the trace of a TPD T-tensor by the eigenvalue size with \( m \times p \) multiplied by the maximum eigenvalue; the last equality is based on the spectral mapping theorem since the function \( f \) is nonnegative. This corollary is proved.

**Corollary 6** Given a finite sequence of independent random Hermitian T-product tensors \( \{X_i\} \) with dimensions in \( \mathbb{C}^{m \times m \times p} \), a real vector \( b \in \mathbb{R}^p \) and \( \sum_{i=1}^{n} tX_i \) satisfying Eq. (98), we have

\[
  \Pr \left( d_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq b \right) \leq mp \inf_{t>0} \min_{1 \leq j \leq p} \left\{ \frac{\exp \left( n \log \lambda_{\max} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}e^{tX_i} \right) \right)}{e^{tb} \otimes_j} \right\}
\]  

(115)

**Proof:** From T-tensor logarithm concavity property provided by Lemma 8, we have

\[
  \sum_{i=1}^{n} \log \mathbb{E}e^{tX_i} = n \cdot \frac{1}{n} \sum_{i=1}^{n} \log \mathbb{E}e^{tX_i} \leq n \log \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}e^{tX_i} \right),
\]  

(116)
and from the trace exponential monotone property provided by Lemma \ref{lem:exponential}, we also have

\[
\Pr \left( d_{\text{max}} \left( \sum_{i=1}^{n} \lambda_i \right) \geq b \right) \leq \inf_{t>0} \min_{1 \leq j \leq p} \left\{ \frac{\text{Tr} \exp \left( \sum_{i=1}^{n} \log |E_{i}(\lambda_i)| \right)}{e^{tb}} \right\}
\]

\[
\leq \inf_{t>0} \min_{1 \leq j \leq p} \left\{ \frac{\text{Tr} \exp \left( n \log \left( \frac{1}{n} \sum_{i=1}^{n} |E_{i}(\lambda_i)| \right) \right)}{e^{tb}} \right\}
\]

\[
\leq mp \inf_{t>0} \min_{1 \leq j \leq p} \left\{ \frac{\exp \left( n \log \lambda_{\text{max}} \left( \frac{1}{n} \sum_{i=1}^{n} |E_{i}(\lambda_i)| \right) \right)}{e^{tb}} \right\}
\]

(117)

where the last inequality holds since we bound the trace of a TPD tensor by the eigenvalue size with \(m \times p\) multiplied by the maximum eigenvalue; and spectral mapping theorem for \(\log\) and \(\exp\) functions. \(\square\)

5 Courant-Fischer Theorem under T-product Tensors and Minimum Eigentuple

In this section, Courant-Fischer theorem for T-product tensors will be proved and this theorem will be used to show the relationship between the maximum eigentuple and the minimum eigentuple of TPD T-product tensors. Let us recall theorem \ref{thm:Courant-Fischer}

**Theorem 1.8 (Courant-Fischer Theorem under T-product)** Let \(A \in \mathbb{C}^{m \times m \times p}\) be a Hermitian T-product tensor with eigentuples \(d_1 \geq d_2 \geq \cdots \geq d_n\). Let \(\{U_j^l\} \in \mathbb{C}^{m \times p}\) be orthonomal matrices for \(1 \leq j \leq m\) and \(0 \leq l \leq p - 1\), \(S_k\) be the space spanned by \(\{U_j^l\}\) for \(1 \leq j \leq k\) and \(0 \leq l \leq p - 1\), and \(T_k\) be the space spanned by \(\{U_j^l\}\) for \(k \leq j \leq m\) and \(0 \leq l \leq p - 1\). Then, we have

\[
d_k = \max_{S_k \subseteq \mathbb{C}^{m \times p}} \min_{X \in S_k} \frac{(X^H \ast A \ast X)}{\circ} \left( X^H \ast X \right)
\]

\[
= \min_{T_k \subseteq \mathbb{C}^{m \times p}} \max_{X \in T_k} \frac{(X^H \ast A \ast X)}{\circ} \left( X^H \ast X \right),
\]

(9)

where \(\circ\) is the division (inverse operation) under \(\otimes\).

**Proof:** We will just prove the first characterization of \(d_k\). The other can be proved similarly.

First, we wish to show that \(d_k\) is achievable. As \(S_k\) is the space spanned by \(\{U_j^l\}\) for \(1 \leq j \leq k\) and \(0 \leq l \leq p - 1\). For every \(X \in S_k\), we can express \(X\) as

\[
X = \sum_{j=1}^{k} \sum_{l=0}^{p-1} \alpha_j^l U_j^l. \quad (118)
\]
Then, we have
\[
\frac{(X^H \ast A \ast X)}{\bigoplus (X^H \ast X)} = \frac{\sum_{j=1}^{k} \sum_{l=0}^{p-1} (\alpha_{j}^{[l]})^2 d_j}{\sum_{j=1}^{k} \sum_{l=0}^{p-1} (\alpha_{j}^{[l]})^2 e} \geq \frac{\sum_{j=1}^{k} \sum_{l=0}^{p-1} (\alpha_{j}^{[l]})^2 d_k}{\sum_{j=1}^{k} \sum_{l=0}^{p-1} (\alpha_{j}^{[l]})^2 e} = d_k
\] (119)
where \( e = (1, 0, \cdots, 0)^T \in \mathbb{C}^p \).

To verify that this is the maximum eigentuple, as \( T_k \) is the space spanned by \( \{U_j^{[l]}\} \) for \( k \leq j \leq m \) and \( 0 \leq l \leq p - 1 \), for any \( S_k \) with dimension \( k \times p \) the intersection of \( S_k \) with \( T_k \) is non-empty. Then, we also have
\[
\min_{X \in S_k} \frac{(X^H \ast A \ast X)}{\bigoplus (X^H \ast X)} \leq \min_{X \in S_k \cap T_k} \frac{(X^H \ast A \ast X)}{\bigoplus (X^H \ast X)}. \tag{120}
\]
Any such \( X \) can be expressed as
\[
X = \sum_{j=k}^{m} \sum_{l=0}^{p-1} \alpha_{j}^{[l]} U_j^{[l]}, \tag{121}
\]
then, we have
\[
\frac{(X^H \ast A \ast X)}{\bigoplus (X^H \ast X)} = \frac{\sum_{j=k}^{m} \sum_{l=0}^{p-1} (\alpha_{j}^{[l]})^2 d_j}{\sum_{j=k}^{m} \sum_{l=0}^{p-1} (\alpha_{j}^{[l]})^2 e} \leq \frac{\sum_{j=k}^{m} \sum_{l=0}^{p-1} (\alpha_{j}^{[l]})^2 d_k}{\sum_{j=k}^{m} \sum_{l=0}^{p-1} (\alpha_{j}^{[l]})^2 e} = d_k, \tag{122}
\]
Therefore, all subspace of \( S_k \) with dimension \( k \times p \), we have
\[
\min_{X \in S_k} \frac{(X^H \ast A \ast X)}{\bigoplus (X^H \ast X)} \leq d_k. \tag{123}
\]
This theorem is proved since \( d_k \) is achievable and is the maximum eigentuple.\( \square \)

By applying Theorem 1.8 we have following relations:
\[
d_{\min}(\mathcal{X}) = -d_{\max}(-\mathcal{X}) \quad \text{and} \quad \lambda_{\min}(\mathcal{X}) = -\lambda_{\max}(-\mathcal{X}) \tag{124}
\]

6 Conclusion

In this Part I work, we try to establish following inequalities about T-product tensors: (1) trace function nondecreasing/convexity; (2) Golden-Thompson inequality for T-product tensors; (3) Jensen’s T-product inequality; (4) Klein’s T-product inequality. All these inequalities are used to generalize celebrated Lieb’s concavity theorem from matrices to T-product tensors. Then, this new version of Lieb’s concavity theorem under T-product tensor is utilized to build master tail bounds for the maximum eigenvalue and the maximum
eigentuple induced by independent sums of random Hermitian T-product. In order to find the relationship between the minimum eigentuple and the maximum eigentuple, we also extended the Courant-Fischer Theorem from matrices to T-product tensors. How these new inequalities and Courant-Fischer Theorem under T-product are used to derive new tail bounds of the extreme eigenvalue and eigentuple for sums of random T-product tensors is the main goal of our Part II paper.

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