BCS superfluidity in ultracold gases with unequal atomic populations

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Much progress has been made recently toward the experimental investigation of degenerate ultracold Fermi gases. Indeed the degenerate regime has already been reached in $^{40}$K by sympathetic cooling of two hyperfine states $^1$, whereas the possibility of cooling a mixture of $^6$Li and $^7$Li atoms in order to reach this range has been demonstrated $^2$. In addition to a number of very interesting physical effects linked to phase space restriction $^1$, one of the most fascinating prospect for experiment is the possibility $^2$ to reach the onset of Cooper pairs condensation and observe the resulting BCS superfluid, the equivalent of Bose Einstein condensates for Fermi gases. As we will see this phase is likely to present quite unusual features.

In addition to the problems raised by the use of evaporative cooling for Fermi gases a possible stumbling block in the path to the BCS superfluid is the need to achieve a near equality $^4$ between the number of atoms in the two hyperfine states assumed to form pairs. Indeed in the various schemes under investigation there is no fast relaxation mechanism which equalizes these two populations and the number of atoms in each state will be essentially conserved. Therefore in these ultracold atomic gases, one will necessarily have to deal with situations where the two populations of particles assumed to form pairs are not equal. Now the difference in chemical potential between these two populations acts as an effective field which tends to break pairs and the superfluid phase is destroyed when this difference is of order of the critical temperature. Hence this pair breaking effect becomes more important a problem if the critical temperature is low and, since there is presently some uncertainty $^5$ in the value of $T_c$, it is clearly of interest to investigate this question in detail. Moreover, independently of this possible experimental problem, this difference in chemical potential is an additional control parameter in the system quite interesting to play with and it is worth studying its effect, all the more since it is not trivial.

This problem of pairing with two unequal populations has already been considered a long time ago in the context of superconductivity. Here this is an applied magnetic field which tends to make the two spin populations unequal. Usually the critical field is limited by orbital effects. However the question of the limitation of the superconducting phase, when these orbital effects are small, has been considered very early and it was pointed out by Clogston $^6$ and Chandrasekhar $^7$ that the standard BCS phase could, at most, resist to a difference in chemical potential between the two populations of the order of the critical temperature (the so-called paramagnetic limit). Not long after, Fulde and Ferrell $^9$ (FF), and independently Larkin and Ovchinnikov $^10$ (LO), showed that pairing could somewhat adjust to the difference in chemical potential, instead of just resisting, by letting the pairs have a common momentum $K$ instead of having it equal to zero as for a standard superconductor. However the effect was actually found rather small. Indeed at zero temperature the standard BCS phase goes to the normal state by a first order transition $^6$ when the chemical potential difference $2\mu$ is equal to $\sqrt{2}\Delta_0$, where $\Delta_0 = 1.76 T_c$ is the zero temperature gap for equal populations. The FFLO phase goes to the normal state by a second order transition for $\mu = 0.754 \Delta_0$, which is not much beyond. Actually there is to date no undisputed observation of this phase in standard superconductors $^11$, most likely because it is quite sensitive to impurities. It is the purpose of the present paper to show that pairing can adjust even better than in the FFLO phase, and that this new phase can in some instances lead to quite a strong increase of the existence domain for the superfluid phase.

One can view the appearance of the FFLO phase in the following way. When the chemical potentials for the two spin populations are different, it becomes more costly in terms of kinetic energy to form $(k, -k)$ pairs in the standard BCS way because one can not pick the two particles very near the Fermi surfaces, since these surfaces do not match due to their size difference. In the FFLO phase this problem is remedied by taking a nonzero total momentum which amounts to shifting, in momentum space, one of the Fermi surface with respect to the other so that they almost match on some region. But naturally match-
ing gets worst on the opposite side. Nevertheless the total balance is barely favourable. Now one can think to improve this situation by making pairing stronger on the side where there is energy gain and weaker where there is energy loss. This means one looks for anisotropic pairing to find a lower energy ground state. Naturally this can not work if scattering is isotropic, as it will be essentially the case for a weakly interacting ultracold gas since p-wave scattering is negligible, and this will be indeed the situation in the dilute regime where the coupling constant \( \lambda = 2k_F \alpha / \pi \) is small. However, even if the bare scattering is isotropic, the renormalized interaction is not because of the existence of the Fermi surface. This effect is at the basis of the Kohn and Luttinger \[12\] paper, where they showed that, even with a repulsive interaction, Cooper pairs would necessarily form in high angular momentum. For our purpose the case of \(^6\)Li is of particular interest since the high density regime is bounded by an instability \[4\] occurring for \( \lambda \geq 1 \). In the vicinity of this instability the effective interaction will be strongly anisotropic, leading to a marked increase in the existence domain of the superfluid phase, compared to the FFLO phase. Note that most experiments are likely to be done in this range since it corresponds to higher critical temperature. We remark also that a similar phase with anisotropic order parameter should exist in clean superconductors with Pauli limited upper critical field and anisotropic interaction.

Before going into the effect of anisotropic pairing, it is of interest to show that this is the best possible choice for pairing with unequal particle number. Indeed assume that we look for the most general kind of pairing, where \( \mathbf{k} \uparrow \) is paired with \( f(\mathbf{k}) \downarrow \). On one hand we want pairing to be stable against scattering, which means that scattering must send this pair into another pair \( \mathbf{k}' \uparrow , f(\mathbf{k}') \downarrow \). On the other hand scattering conserves momentum which leads, for any \( \mathbf{k} \) and \( \mathbf{k}' \), to \( \mathbf{k} + f(\mathbf{k}) = \mathbf{k}' + f(\mathbf{k}') = \mathbf{K} \) where \( \mathbf{K} \) is a constant. This shows that \( f(\mathbf{k}) = \mathbf{K} - \mathbf{k} \), that is pairs with a total momentum \( \mathbf{K} \) is the most general solution. Note that this argument assumes translational invariance. This will not hold for a finite sample such as those obtained with trapped gases, which will induce surface effects. However these surface effects should be small if we do not want to have \( T_c \) drastically reduced \[13\] (more precisely we want the pair size to be small compared to the sample size). So it is reasonable to neglect size effects in a first step. Now the common momentum \( \mathbf{K} \) produces a breaking of rotational symmetry, and the standard symmetry analysis leading to pairs with a given angular momentum (s-wave pairs in the case of \(^6\)Li) is no longer valid. The general order parameter \( \Delta_\mathbf{k} \) will depend on the wavevector \( \mathbf{k} \). However we still have rotational invariance around \( \mathbf{K} \) and we can classify the solutions by their angular momentum \( m \). We will assume that the most stable pairing corresponds to \( m = 0 \), as it is likely to be so for a standard interaction. However other values of \( m \), corresponding to a breaking of the rotational symmetry around \( \mathbf{K} \), do not seem to be excluded from first principles and would be undoubtedly a very interesting situation.

We will explore now quantitatively the possibility offered by an anisotropic order parameter. However our purpose is more to demonstrate the importance of the effect than to perform an exact calculation. This last goal would require a perfect knowledge of the effective interaction, which is not available. We will consider specifically the case of \(^6\)Li and we restrict ourselves to the determination of the \( T = 0 \) critical difference in chemical potential \( 2\bar{\mu} = \mu \uparrow - \mu \downarrow \) above which superfluidity disappears. Since we want to go in the high density regime, we need to have an expression for the effective interaction in this range. For this purpose we take the paramagnon model which we have already used for an evaluation of \( T_c \) in the high density regime \[6\]. Actually we will not retain the full interaction of this model because most of the terms lead to a moderate anisotropy and they would produce a more complex calculation without adding much effect. The essential contribution of those terms is already included in the value \( T_{c0} \) of the critical temperature for equal populations. We will only retain the explicit attractive part coming from density fluctuations which produces near the instability a strong contribution for low momentum transfer and in this way lead to a strongly anisotropic interaction. Moreover we will for simplicity perform a weak coupling calculation, omitting all the frequency dependence and taking the zero frequency value of the interaction. To be coherent with this weak coupling approach we will also omit self-energy effects. This leads to the total effective attractive interaction \( V(\mathbf{k}, \mathbf{k}') \) \[6\] given by:

\[
N_f V(\mathbf{k}, \mathbf{k}') = \lambda + \frac{1}{2} \frac{\lambda^2 \tilde{\chi}_0(\mathbf{q})}{1 - \lambda \tilde{\chi}_0(\mathbf{q})} \tag{1}
\]

with \( \mathbf{q} = \mathbf{k} - \mathbf{k}' \). Here \( N_f = m k_F / 2 \pi^2 \) is the density of states at the Fermi surface for equal population and \( \tilde{\chi}_0(\mathbf{q}) \) is the reduced elementary bubble at zero frequency:

\[
2\tilde{\chi}_0(\mathbf{q}) = 1 + \frac{1}{y} (1 - \frac{y^2}{4}) \ln \frac{2 + y}{2 - y} \tag{2}
\]

with \( y = q / k_F \). The first term in Eq.(1) is the direct term and the last one is the indirect interaction due to density fluctuation exchange.

The difference in chemical potential \( 2\bar{\mu} \) has the effect of shifting the kinetic energies of the particle (measured from chemical potential) by \( \pm \bar{\mu} \). For the pair propagator this is just equivalent \[14\] to shift the frequency \( \omega \) by \( \bar{\mu} \). Similarly taking the momenta of the pair members to be \( \pm \mathbf{k} + \mathbf{K} / 2 \) instead of \( \pm \mathbf{k} \) produces a shift by \( \pm \mathbf{k}. \mathbf{K} / 2m \), leading to an overall shift in frequency by \( \bar{\mu} - \mathbf{k}. \mathbf{K} / 2m \). At finite temperature this means replacing the Matsubara frequency \( \omega_n = \pi T (2n + 1) \) by
\[ \omega_n - i\mu \hat{k}. \] On the other hand we expect the modification of the effective interaction caused by \( \hat{\mu} \) to be small, of order \( \mu / E_F \), and we can neglect it. This leads us finally to the following gap equation at finite temperature:

\[ \Delta_k = \frac{\lambda T}{N_f} \sum_n \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{\Delta_k}{\xi_k^2 + \Delta_k^2 + (\omega_n - i\mu \hat{k})^2} \left( 1 + \frac{1}{2} \frac{\lambda \chi_0(q)}{1 - \lambda \chi_0(q)} \right) \tag{3} \]

where \( \xi_k \) is the kinetic energy measured from the Fermi surface for \( \hat{\mu} = 0 \). Since we are looking for the critical \( \hat{\mu} \) and expect a second order phase transition as for the FFLO phase, we let \( \Delta_k \) go to zero in the denominator. Then, in order to get rid of the cut-off in the frequency summation, it is convenient to make use of \( \pi T \sum (1/\omega_n) = \lambda^{-1} + \ln(T_0^2/T) \), where \( T_0 \) is the critical temperature for \( \hat{\mu} = 0 \), \( K = 0 \) and in the absence of the indirect term in the interaction. When we specialize to \( T \to 0 \), the frequency summation can be performed by \( \pi T \sum \text{sgn}(\omega_n)/(\omega_n - i\mu k) - 1/\omega_n = \ln(0.88 T/|\mu k|) \). In this way we obtain the following equation for the critical difference in chemical potential \( \hat{\mu} \):

\[ \Delta_k = \int \frac{d\Omega_k}{4\pi} \Delta_0(1 + \lambda \ln 2|\hat{\mu}|/\mu \hat{k})/(1 + 2 \lambda \chi_0(q)) \tag{4} \]

where \( \Delta_0 \) is the zero temperature gap for \( \hat{\mu} = 0 \), \( K = 0 \) and in the absence of the indirect term in the interaction. When the indirect interaction is not present, \( \Delta_k \) is independent of \( \mathbf{k} \). Then the angular integration is easily performed, leading to \( \ln(\Delta_0/2\hat{\mu}) = (1/2)f_0(x) \), with \( x = K k_F/(2\mu \hat{k}) \). Here \( f_0(x) = \int_1^1 du \ln |1 + xu| \) which is negative and minimum for \( x \approx 1.200 \) leading to the standard FFLO result \( \hat{\mu}/\Delta_0 = 0.754 \Delta_0 \).

We consider now the effect of the indirect interaction. Since both \( \mathbf{k} \) and \( \mathbf{k}' \) are on the Fermi surface, \( q \) goes from 0 to \( 2k_F \) and \( \chi_0(q) \) goes from 1 to 1/2. Although it is not a problem to use numerically the exact expression for \( \chi_0(q) \), one can see that \( \chi_0(q) = 1 - a + a \cos \theta \), with \( a = 0.25 \) and \( q = 2k_F \sin(\theta/2) \) is a quite good approximation. Indeed it gives properly the two limits \( q = 0 \) and \( q = 2k_F \). Moreover it corresponds to keep only the first two terms in a Legendre polynomials expansion with slightly modified coefficients (the exact \( t = 0 \) coefficient is \( (1 + 2 \ln 2)/3 = 0.755 \) and the exact coefficient of \( \cos \theta \) is 0.232, the higher order terms being fairly small). This approximation allows to perform the azimuthal integration analytically, and since our calculation is anyway a model calculation it is quite reasonable to make this simplification, despite its slight inaccuracy in the vicinity of \( \theta = 0 \).

However, before proceeding to the solution of the resulting equation, it is interesting to consider first a slightly simplified version which allows to carry out the calculation completely explicitly. We merely replace the interaction term by the first two terms of its Legendre polynomial expansion \( V_0 + V_1 \cos \theta \), with \( V_0 = 0.5 + 0.25(a \lambda)^{-1} \log(1 + 2a \lambda/(1 - \lambda)) \) and \( V_1 = -1.5(a \lambda)^{-1} + 0.75(1 - 1 + a \lambda)(a \lambda)^{-2} \log(1 + 2a \lambda/(1 - \lambda)) \). In this case the solution of Eq.(4) has the form \( \Delta_k = 1 + \delta_1 \cos \alpha \) with \( \cos \alpha = \hat{k} \cdot \hat{K}. \) Then \( L \equiv \ln(2\hat{\mu}/\Delta_0) \) is solution of the second order equation \((L - 1 + \lambda + 1/\lambda V_0 + 0.5f_0(L - 1/\lambda + 3/\lambda V_1 + 1.5f_2) = 0.75f_2^2 \). We have set \( f_n(x) = \int_x^1 du u^n \ln(1 + xu) \) and \( f_1(x) \) and \( f_2(x) \) are both found to be largest for \( x \) in the range 1.1 - 1.2, and the largest \( L \) is also found for the same range of \( x \) for all values of \( \lambda \).

![FIG. 1. Full line : location of the second order transition \( \hat{\mu}/\Delta_0 \) to the normal state from Eq.(4) as a function of \( \lambda \). Dashed line : approximate solution with Legendre polynomial expansion. Long-dashed line : location of the first order transition \( \hat{\mu}/\Delta_0 = 1/\sqrt{2} \).](image-url)

Naturally the indirect interaction produces a trivial effect, namely the renormalization of the coupling constant \( \lambda \) into \( \lambda V_0 \), producing a corresponding change of the critical temperature and of the gap for equal population, which goes from \( \Delta_0 \) to \( \Delta_0 = \Delta_0 \exp(1/\lambda - 1/\lambda V_0) \). This effect is simply found by looking at the solution \( L_0 \) for \( x = 0 \), that is without FFLO phase, since in this case only the isotropic part of the interaction is relevant. In the present case this is given by \( L_0 = 1/\lambda - 1/\lambda V_0 \). We are only interested in \( L - L_0 \) which gives the increase in the superfluid domain due to the existence of our FFLO phase, in units of \( \Delta_0 \). In Fig.1 we have plotted as the dashed line \( \hat{\mu}/\Delta_0 \). We see that, roughly for \( \lambda < 0.6 \), there is essentially no change with respect to the standard FFLO result. And indeed the gap remains essentially isotropic in this range. On the other hand, for \( \lambda > 0.6 \), \( \hat{\mu} \) increases rapidly and for \( \lambda = 0.9 \) it is almost twice the standard FFLO result. It is also quite interesting to consider the anisotropy linked to \( \delta_1 \). As soon as \( \hat{\mu} \) starts to grow with respect to the standard FFLO result, the order parameter gets a sizeable anisotropy. For \( \lambda = 0.94 \) we obtain a node on the Fermi surface, and for...
larger $\lambda$ a change of sign over the Fermi surface with a nodal line.

Let us turn now to the results of the numerical solution of Eq.(4). They are given as the full line on Fig.1 for $\bar{\mu}/\Delta_0$ as a function of $\lambda$ (as above we have taken into account the renormalization of $\Delta_0$ into $\bar{\Delta}_0$ by calculating $L - L_0$). For small and intermediate $\lambda$ the result is essentially identical to the result of our simplified version. However when $\lambda$ approaches 1, $\bar{\mu}$ increases more rapidly. This is easy to understand by looking at Eq.(4). In this regime the effective interaction becomes very large for small $q$ and the log term favors wavevectors nearly along $K$ so it is better to have the gap function $\Delta_k$ peaked for wavevector along $K$ (see Fig.2). This can be seen more explicity by looking at the simplified form taken by this equation when $\lambda \to 1$. Taking into account the numerical evidence that in this limit the optimal $\bar{\mu}$ is obtained for $x = 1$, and making the change of variable $k K = u(1 - \lambda)/a$, we obtain for $F(u) \equiv \Delta_k$:

$$F(v) = -\frac{1}{4a} \int_0^\infty du \frac{M + \log u}{\sqrt{1 + 2(u + v) + (u - v)^2}}$$ \hspace{1cm} (5)

with $M = L + \log((1 - \lambda)/a) - 1$. This equation can be solved numerically. However the solution we are looking for behaves approximately as $\exp(-\mu u)$. Inserting this form into Eq.(5) and requiring that $F(v)$ is zero for large $v$ gives $\log \mu = M - C$ (for $a = 0.25$), where $C$ is the Euler constant. On the other hand requiring $F(0) = 1$ and making an asymptotic evaluation of the resulting integral gives $\mu = \exp( - \sqrt{2} ) \approx 0.24$ which is (surprisingly) not much different from the numerical result $\mu \approx 0.3$. This leads finally to the asymptotic evaluation $L \approx -1 - \log(1 - \lambda)$, which, together with $L_0 \approx 1 - 1/\log(0.5/(1 - \lambda))$, is in reasonable agreement with our direct solution of Eq.(4) found in Fig.1. Naturally the divergence of $\bar{\mu}$ itself is not to be taken seriously since our calculation requires $\bar{\mu}/E_F$ to be small anyway.

Coming back to Fig.1, we see that beyond $\lambda \approx 0.8$ the results from the full Eq.(4) get larger than those of the simplified equation. Hence we find a very large increase of the domain for our FFLO phase. When we take into account the fact that the critical temperature itself will increase rapidly in this range due to the indirect interaction, we see that looking in this region seems quite promising experimentally since the overall domain for the superfluid phase will be much increased. On the other hand the physical properties of this phase will be to a large extent quite different from those found for equal populations. A first reason is that this phase is gapless, just as the standard FFLO phase [9,10]. Next we find an important order parameter anisotropy. Indeed we have plotted in Fig.2, for various values of $\lambda$, the angular dependence of $\Delta_k$ in respect to $K$. As we have mentioned already, $\Delta_k$ gets more concentrated along the $K$ direction when $\lambda$ increases. The value of its minimum compared to its maximum is also plotted in Fig.2. We see that even for moderate values of $\lambda$ such as $\lambda \approx 0.4$ the anisotropy is quite sizeable. However when $\lambda$ is further increased the anisotropy becomes ultimately huge to the point that $\Delta_k \approx 0$ for a very large fraction of the Fermi surface. This is a highly unconventional situation and certainly quite unique among BCS superfluids. Another more well-known feature will further complicate the matter. We have in our system a degeneracy with respect to the direction of $K$, which will be in general lifted by a texture leading to a spatial inhomogeneity, that is yet another symmetry breaking, as investigated by LO [10] and more recently in Ref. [11]. Together with experimental inhomogeneity due to the trap this will lead to a remarkably complex physical situation.

In conclusion we have shown that in $^6$Li the indirect interaction due to density fluctuations exchange will lead to the appearance of a new BCS phase with anisotropic order parameter. Near the instability threshold, this results in a large increase of the superfluid domain as a function of the difference between the atom numbers in the two hyperfine states forming the Cooper pairs.

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