Superintegrable models on riemannian surfaces of revolution with integrals of any integer degree

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Abstract

We present a family of superintegrable systems defined on riemannian surfaces of revolution and which exhibit a linear integral and two integrals of any integer degree in the momenta. When this degree is 2 one recovers a metric due to Koenigs. The differential systems to be solved in their construction are shown to be driven by a linear ordinary differential equation of order $n$ which is homogeneous for even integrals and weakly inhomogeneous for odd integrals. Some globally defined examples are worked out which live either in $\mathbb{R}^2$ or in $\mathbb{H}^2$. 
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1 Introduction

The possibility of integrable or superintegrable (SI) dynamical systems with integrals of degree larger than 4 in the momenta is such a difficult problem that the conjecture that they could not exist was put forward, see for instance [1][p. 663]:

Conjecture 1 (Kozlov and Fomenko) On the two dimensional sphere, there are no riemannian metrics whose geodesic flows are integrable by means of an integral of degree \( n > 4 \) and do not admit integrals of degree \( \leq 4 \).

Up to now most of the integrable systems on riemannian surfaces, be explicit or not, do exhibit integrals of degree \( \leq 4 \) in the momenta. A first example of integrable system with integrals of sixth degree, emerging from astrophysics, was found by Gaffet [5] (discussed further in [10] and in [2]). More recently Tsiganov gave a new example of this kind [11]. Unfortunately none of these examples are globally defined on \( S^2 \). In fact Kiyohara [6] was the first to show the existence of integrable systems with globally defined riemannian metrics having integrals of arbitrary integer degree which, furthermore, are Zoll metrics. However the explicit form of these metrics is not known and it seems very difficult to disprove the Conjecture with respect to the non-existence of integrals of degree less than four because we do not know all the integrable systems with cubic and quartic integrals.

It seemed to us easier to consider the simpler situation of SI systems defined on riemannian surfaces of revolution, starting from the framework laid down by Matveev and Shevchishin [8] for cubic integrals. Their analysis, as we will show later on, can be generalized to integrals of any degree, starting from degree 2. In this last simple case it was proved in [14] that one recovers Koenigs metrics [7] studied and generalized in [9]. Koenigs systems exhibit the following integrals

\[
H, P_y, S_1, S_2
\]

where \( H, S_1, S_2 \) are of second degree in the momenta. In our generalization the integrals \( S_1 \) and \( S_2 \) will be of any integer degree.

Since the analysis required for even and odd degrees display some differences we have divided the article in two Parts.

In Part I we consider the case of integrals \( S_1 \) and \( S_2 \) of degree \( 2n \geq 2 \). Their local structure is constructed and shown to be determined by a linear and homogeneous ODE of order \( n \). Globally defined examples are given either on \( M = \mathbb{R}^2 \) or on \( M = \mathbb{H}^2 \).

In Part II the case of integrals \( S_1 \) and \( S_2 \) of degree \( 2n + 1 \geq 3 \) is considered. Their local structure is constructed and shown to be determined by a linear and weakly inhomogeneous ODE of order \( n \). Globally defined examples give rise to the same manifolds as in Part I. The last Section is devoted to some concluding remarks.
Part I

Integrals of even degree in the momenta

We will consider the cases for which the observables have for degree in the momenta $\#(Q) = 2n$ where $n \geq 2$. The case $n = 1$ is marginal since the corresponding SI models were discovered by Koenigs in [7] and generalized in [9].

Taking for hamiltonian

$$H = \Pi^2 + a(x) P^2_y \quad \Pi = a(x) P_x$$

we have the obvious result:

**Proposition 1** The system $(H, P_y)$ is integrable in Liouville sense.

**Proof:** We have $\{H, P_y\} = 0$ and since $H$ and $P_y$ are generically independent the proposition follows. □

This dynamical system will become SI if we can construct at least one more, generically independent integral, of degree $2n$ in the momenta. To this aim let us first define

$$G = \sum_{k=0}^{n} A_k H^k P^{2(n-k)}_y \quad A_n \neq 0 \quad \#(G) = 2n \quad n \geq 1$$

which defines a string of $n + 1$ real constants: $(A_0, A_1, \ldots, A_n)$.

The definition of the two observables $(S_1, S_2)$ by

$$S_1 = Q_1 + y G \quad S_2 = Q_2 + y Q_1 + \frac{y^2}{2} G \quad \#(Q_1) = \#(Q_2) = 2n$$

entail the relations

$$\{P_y, S_1\} = G \quad \{P_y, S_2\} = S_1.$$

These observables will become integrals if we impose:

**Proposition 2** The observables $S_1$ and $S_2$ are integrals iff

$$\{H, Q_1\} + 2a P_y G = 0 \quad \{H, Q_2\} + 2a P_y Q_1 = 0.$$

**Proof:** The first relation is nothing but $\{H, S_1\} = 0$ and the second one is $\{H, S_2\} = 0$. □

Let us observe that we get in fact two (maximally) SI systems which are

$$\mathcal{I}_1 = \{H, P_y, S_1\} \quad \& \quad \mathcal{I}_2 = \{H, P_y, S_2\}.$$

For further use let us define the sets

$$S^p_n = \{p, p + 1, \ldots, n\} \quad 0 \leq p \leq n$$

as well as the Pochhammer symbols

$$(z)_0 = 1 \quad \forall n \geq 1 \quad (z)_n = z(z + 1) \cdots (z + n - 1).$$
2 The local structure of the first integral

The first step will be

**Proposition 3** The most general form of $Q_1$ being

$$Q_1 = \sum_{k=1}^{n} b_k(x) \Pi^{2k-1} P_y^{2(n-k)+1}$$

the constraint $\{H, S_1\} = 0$ is equivalent to the differential system\footnote{A prime stands for a derivation with respect to the variable $x$.}:

$$0 = \frac{1}{2} a' b_1 - F$$
$$b'_k = (k + \frac{1}{2}) a' b_{k+1} - \frac{D_k^a F}{k!} \quad k \in \{1, 2, \ldots, n-1\}$$
$$b'_n = -\frac{D^n_a F}{n!},$$

where $F(a) = \sum_{k=0}^{n} A_k a^k$.

**Proof:** The first relation in (1.6) is

$$\{H, Q_1\} + 2a P_y G = 0.$$  

(2.3)

Expanding the left hand side leads to

$$\sum_{k=1}^{n-1} \left( b'_k - (k + 1/2) a' b_{k+1} + \frac{D_k^a F}{k!}\right) a \Pi^{2k} P_y^{2(n-k)+1}$$
$$+ \left( F - \frac{1}{2} a' b_1 \right) a P_y^{2n+1} + \left( b'_n + \frac{D^n_a F}{n!}\right) a \Pi^{2n} P_y = 0$$

which proves the Proposition. \hfill \Box

To reduce this system to a tractable form we will use now, instead of the coordinate $x$, the coordinate $a$. This is legitimate since our considerations are purely local. The hamiltonian becomes

$$H = \Pi^2 + a P_y^2 \quad \Pi = \frac{a}{x} P_a.$$  

(2.4)

Transforming the equations in Proposition\footnote{A prime stands for a derivation with respect to the variable $x$.} gives
Proposition 4 The constraint $\{H, S_1\} = 0$ is equivalent to the differential system:\[\begin{align*}
(a) \quad & b_1 = 2F \dot{x} \\
(b) \quad & \dot{b}_k = (k + \frac{1}{2}) b_{k+1} - \frac{D^k a^k F}{k!} \dot{x} \quad k \in \{1, 2, \ldots, n - 1\} \\
(c) \quad & \dot{b}_n = -\frac{D^n a^n F}{n!} \dot{x} = -A_n \dot{x}
\end{align*}\] (2.5)

Let us notice that we have $n + 1$ equations for $n$ unknown functions $\left(b_k(a), \ k \in S^1_n\right)$.

The definition will be useful:

**Definition 1** The linear differential operator $\text{Op}_n[F]$ is defined as

$$\text{Op}_n[F] = \sum_{s=0}^{n} \frac{F^{(n-s)}}{(n-s)! (1/2)_s} D^s a.$$ (2.6)

The Leibnitz formula gives for it the following property:

$$\text{Op}_n[FG] = \sum_{s=0}^{n} \frac{F^{(n-s)}}{(n-s)!} \text{Op}_s[G].$$ (2.7)

We are now in position to solve the system for the $b_k$:

**Proposition 5 (Linearizing ODE)** The local structure of $Q_1$ is given by

$$Q_1 = \sum_{k=1}^{n} b_k \Pi^{2k-1} P_y^{2(n-k)+1},$$ (2.8)

with

$$\forall k \in S^1_n \quad b_k[F] = \sum_{s=1}^{k} \frac{F^{(k-s)}}{(k-s)! (1/2)_s} D^s a x = \text{Op}_k[F] x - \frac{F^{(k)}}{k!} x$$ (2.9)

where $x$ is a solution of the linear and homogeneous ODE of order $n \geq 2$:

$$\text{Op}_n[F]x = 0.$$ (2.10)

**Proof:** To determine the functions $b_k$ we need to solve the differential system (2.5): we will proceed recursively. The relation (2.5)(a) gives

$$b_1 = F \frac{D_a x}{1/2}$$

\[A\text{ dot stands for a derivation with respect to the variable } a.\]
in agreement with (2.9) for \( k = 1 \). Let it be supposed that the relation for \( b_k \) in (2.9) is true for any \( k < n \) and let us prove that it is also true for \( k + 1 \leq n \). To this aim we will start from relation (2.5)(b) written

\[
(k + 1/2)b_{k+1} = \dot{b}_k + \frac{D_a^k F}{k!} \frac{\partial}{\partial b} + \sum_{s=1}^{k} \frac{D_a^{k+s} F}{(k-s)!} \frac{D_a^{s} x}{(1/2)_s} + \sum_{s=0}^{k} \frac{D_a^{k-s} F}{(k-s)!} \frac{D_a^{s+1} x}{(1/2)_s}.
\]

In the second sum let us shift \( s \to s' = s + 1 \) so that

\[
(k + 1/2)b_{k+1} = \sum_{s=1}^{k} \frac{D_a^{k+s} F}{(k-s)!} \frac{D_a^{s} x}{(1/2)_s} + \sum_{s=1}^{k+1} \frac{D_a^{k+s'} F}{(k+1-s')!} \frac{D_a^{s'} x}{(1/2)_{s'-1}}
\]

\[
= F \frac{D_a^{k+1} x}{(1/2)_k} + \sum_{s=1}^{k} \left[ \frac{1}{(1/2)_{s-1}} + \frac{k+1-s}{(1/2)_s} \right] \frac{D_a^{k+s} F}{(k+1-s)!} \frac{D_a^{s} x}{(1/2)_s},
\]

The relations

\[
\frac{1}{(1/2)_{s-1}} + \frac{k+1-s}{(1/2)_s} = \frac{k+1/2}{(1/2)_s} \quad (1/2)_{k+1} = (k+1/2)(1/2)_k
\]

give for final result

\[
b_{k+1} = \sum_{s=1}^{k+1} \frac{D_a^{k+s} F}{(k+1-s)!} \frac{D_a^{s} x}{(1/2)_s}
\]

which concludes the recurrence proof of the relation (2.9) for \( b_k \).

We are left with relation (2.5)(c) which integrates up to \( b_n = -A_n (x - x_0) \) and this last relation must agree with the \( b_n \) obtained from (2.5)(b) for \( k = n - 1 \). This produces the linear ODE

\[
\sum_{s=1}^{n} \frac{D_a^{n-s} F}{(n-s)!} \frac{D_a^{s} x}{(1/2)_s} + \frac{n!}{n!} (x - x_0) = 0.
\]

We can set \( x_0 = 0 \) because the metric does depend solely on \( \dot{x} \) and this proves (2.10). □

Remark: Let us observe that the relation (2.9) is valid for any choice of \( F \).

Let us define:

**Definition 2** We will say that \( F \) is simple if all of its zeroes are simple, with symbol \( \hat{F} \):

\[
\hat{F}(a) \equiv \sum_{k=0}^{n} A_k a^k = A_n \prod_{i=1}^{n} (a - a_i) \quad A_n \in \mathbb{R} \\{0\}.
\]

Then we have

**Proposition 6** For a simple \( F \), if one takes

\[
x = \sum_{i=1}^{n} \frac{\xi_i}{\sqrt{\Delta_i}}, \quad \Delta_i = \epsilon_i (a - a_i), \quad \epsilon_i^2 = 1, \quad a_i \in \mathbb{R}
\]

(2.11)
where the $\xi_i$ are $n$ arbitrary real parameters, the relation
\[
\text{Op}_n[\hat{F}] x = \sum_{i=1}^n \xi_i \frac{\Delta_i^{1/2}}{n!} D^n_a \left( \frac{\hat{F}}{\Delta_i} \right)
\]  \hspace{1cm} (2.12)
implies that $x$ is the general solution of the ODE (2.10).

\textbf{Proof:} The linearity of the ODE allows to check term by term
\[
\forall i \in S^n_1 \quad x_i = \Delta_i^{-1/2} \quad \Delta_i = \epsilon_i(a - a_i) \quad \epsilon_i^2 = 1,
\]
for which
\[
\forall s \in \mathbb{N} \quad \frac{D^n_a x_i}{(1/2)_s} = (-\epsilon_i)^s \Delta_i^{-s-1/2}. \quad (2.13)
\]
Hence
\[
\text{Op}_n[\hat{F}] x_i = \Delta_i^{1/2} \sum_{s=0}^n \frac{D^{n-s}_a \hat{F}}{(n-s)!} (-\epsilon_i)^s \Delta_i^{-s-1}
\]
so that using
\[
\forall s \in \mathbb{N} \quad (-\epsilon_i)^s \Delta_i^{-s-1} = \frac{1}{s!} D^n_a (\Delta_i^{-1})
\]
and Leibnitz formula we end up with
\[
\text{Op}_n[\hat{F}] x_i = \Delta_i^{1/2} \frac{D^n_a}{n!} \left( \frac{\hat{F}}{\Delta_i} \right).
\]
Adding all the terms gives (2.12). The proposition follows since each term in this sum does vanish. \Box

\textbf{Remarks:}

1. The previous formula is in fact valid for
\[
\forall k \in S^n_0 : \quad \text{Op}_k[\hat{F}] x_i = \Delta_i^{1/2} \frac{D^n_a}{k!} \left( \frac{\hat{F}}{\Delta_i} \right), \quad (2.14)
\]
which does vanish only for $k = n$.

2. The forms taken by $x(a)$ when $F$ is not simple are given in Appendix A.

As a bonus the metrics of constant scalar curvature are excluded:

\textbf{Proposition 7} The metric
\[
g = \frac{\dot{x}^2}{a^2} da^2 + \frac{dy^2}{a} \quad (2.15)
\]
is never of constant curvature if \( x(a) \) is a solution of the ODE (2.10). An embedding in \( \mathbb{R}^{2,1} \) is given by
\[
g = dX^2 + dY^2 - dZ^2
\]
where
\[
X = \frac{y}{\sqrt{a}} \quad Y - Z = -\frac{1}{\sqrt{a}} \quad Y + Z = \frac{y^2}{\sqrt{a}} + 2 \int \frac{\dot{x}^2}{\sqrt{a}} \, da.
\]

**Proof:** We have seen that the scalar curvature is
\[
2R = -\frac{2a \ddot{x} + \dot{x}}{(\dot{x})^3}
\]
so if we take \( R \) to be a constant, defining \( u = \frac{1}{\dot{x}^2} \) we have to solve
\[
a \dot{u} - u = 2R \quad \implies \quad u = K a - 2R.
\]
According to the value of the integration constant \( K \), and omitting an additive constant, we have
\[
K = 0 \quad \implies \quad x = \pm \frac{a}{\sqrt{-2R}}, \quad K \neq 0 \quad \implies \quad x = \pm \frac{2}{K} \sqrt{Ka - 2R}
\]
and both functions are never solutions of (2.10). The embedding formulas are easily checked. \( \square \)

For the next step we need

**Lemma 1** One has the following identity
\[
k \leq l : \quad \sum_{s=k}^{l} \frac{(-1)^s}{(l-s)!(s-k)!} = (-1)^k \delta_{kl}.
\]

**Proof:** For \( k = l \) this sum is just \((-1)^k\). For \( k < l \) defining \( N = l - k \) and \( t = s - k \) we have
\[
\sum_{s=k}^{l} \frac{(-1)^s}{(l-s)!(s-k)!} = \sum_{t=0}^{N} \frac{(-1)^{k+t}}{t!(N-t)!} = \frac{(-1)^k}{N!} \sum_{t=0}^{N} (-1)^t \binom{N}{t} = 0
\]
by the binomial theorem. \( \square \)

Let us first compute the coefficients \( b_k \):

**Proposition 8** For a simple \( F \) we have
\[
Q_1 = \sum_{k=1}^{n} b_k[\hat{F}] \Pi^{2k-1} P^{2(n-k)+1}_y,
\]
with
\[
\forall k \in \mathcal{S}_n \quad b_k[\hat{F}] = - \sum_{i=1}^{n} \epsilon_i \xi_i \frac{D_{a}^{k-1}}{(k-1)!} \left( \frac{\hat{F}}{\Delta_i} \right).
\]
Proof: Relation (2.9)

\[ b_k[\hat{F}] = \sum_{s=1}^{k} \frac{\hat{F}^{(k-s)}}{(k-s)!} \frac{D_s x}{(1/2)^s} \quad x = \sum_{i=1}^{n} \frac{\xi_i}{\sqrt{\Delta_i}} \]

and the identity (2.13) give first

\[ b_k[\hat{F}] = \sum_{i=1}^{n} \frac{\xi_i}{\sqrt{\Delta_i}} \sum_{s=1}^{k} \hat{F}^{(k-s)} \frac{(-\epsilon_i)^s \Delta_i^{-s}}{(k-s)!} \]

The change of index \( s = t + 1 \) gives

\[ b_k[\hat{F}] = -\sum_{i=1}^{n} \frac{\epsilon_i \xi_i}{\sqrt{\Delta_i}} \sum_{t=0}^{k-1} \frac{\hat{F}^{(k-1-t)}}{(k-1-t)!} (-\epsilon_i)^t \Delta_i^{-t-1} \]

Using the identity

\[ (-\epsilon_i)^t \Delta_i^{-t-1} = \frac{D_a \Delta_i^{-1}}{t!} \]

and Leibnitz formula, we are led to (2.21). □

Using this form of \( Q_1 \) it is rather difficult to obtain \( Q_2 \). To solve this problem we need to transform \( Q_1 \) according to

\[ Q_1 = \sum_{k=1}^{n} b_k[F] \Pi^{2k-1} P_y^{2(n-k)+1} = \sum_{k=1}^{n} \tilde{b}_k[F] H^{n-k} \Pi P_y^{2k-1}. \tag{2.22} \]

Let us determine the new coefficients \( \tilde{b}_k \):

**Proposition 9** In general we have

\[ \forall k \in S_n^1 : \quad \tilde{b}_k[F] = \sum_{s=1}^{n} \left( \frac{n-s}{k-s} \right) (-a)^{k-s} b_{n-s+1}[F] \tag{2.23} \]

and in the simple case \(^3\)

\[ \forall k \in S_n^1 : \quad \tilde{b}_k[\hat{F}] = (-1)^k A_n \sum_{i=1}^{n} \frac{\xi_i}{\sqrt{\Delta_i}} \sigma^{i-1}_{k-1}. \tag{2.24} \]

**Proof:** If, in the first form of \( Q_1 \), one uses \( \Pi^2 = H - a P_y^2 \) and interchanges the summations order, one gets the relation (2.23). Using formula (2.21) we have

\[ b_{n+1-s}[\hat{F}] = -\frac{1}{(n-s)!} \sum_{i=1}^{n} \frac{\epsilon_i \xi_i}{\sqrt{\Delta_i}} D_a^{n-s} \left( \frac{\hat{F}}{a-a_i} \right) \quad s \in S_n^1. \]

\(^3\)The symbol \( \sigma^{i-1}_{k-1} \) is defined in Appendix B.
Expanding the term inside the bracket using relation (3.3) one has
\[ b_{n+1-s}[\hat{F}] = -\frac{A_n}{(n-s)!} \sum_{i=1}^{n} \xi_i \sum_{l=1}^{s} (-1)^{l-1} \sigma_{l-1}^i \frac{(n-l)!}{(s-l)!} \alpha^{s-l}, \]
and inserting this formula in the definition, given above, of \( \tilde{b}_k \) we get
\[ \tilde{b}_k[\hat{F}] = (-1)^{k+1} A_n \sum_{i=1}^{n} \frac{\xi_i}{\sqrt{\Delta_i}} \sum_{s=1}^{k} \sum_{l=1}^{s} (-1)^{l-1+s} \frac{(n-l)!}{(k-s)!} \frac{a^{k-l}}{(n-k)!} \frac{\sigma_{l-1}^i}{(s-l)!}. \]
Reversing the first and the second summations we end up with
\[ \tilde{b}_k[\hat{F}] = (-1)^{k+1} A_n \sum_{i=1}^{n} \frac{\xi_i}{\sqrt{\Delta_i}} \sum_{l=1}^{k} \sum_{s=1}^{l} (-1)^{l-1} \sigma_{l-1}^i \frac{a^{k-l}}{(n-k)!} \frac{(n-l)!}{(k-s)!} \frac{\tau_{l-1}}{(s-l)!}, \]
and the identity (2.19) concludes the proof. □

It is interesting, in order to check the result obtained in Proposition 9, to write down the differential system for the \( \tilde{b}_k \). We have:

**Proposition 10** Defining
\[ \forall k \in \mathcal{S}_n^1 : \quad \tilde{b}_k[F] = A_n (-1)^k \beta_k[F], \quad (2.25) \]
for any choice of \( F \) we have the following relations valid
\[ \beta_1 = x \]
\[ 1 \leq k \leq n-1 : \quad \dot{\beta}_{k+1} = -a \dot{\beta}_k - \frac{1}{2} \beta_k + \sigma_k \dot{x} \quad (2.26) \]
\[ k = n : \quad 0 = -a \dot{\beta}_n - \frac{1}{2} \beta_n + \sigma_n \dot{x}. \]

For a simple \( F \) the formula obtained for \( \tilde{b}_k[\hat{F}] \) in Proposition 3 is indeed a solution of this system.

**Proof:** A routine computation of
\[ \{H, Q_1\} + 2 a P_y G = 0 \]
gives first
\[ \dot{\beta}_1 = \dot{x} \quad \implies \quad \beta_1 = x. \]
For \( k \in \{1, 2, \ldots, n-1\} \) we have
\[ \dot{\beta}_{k+1} = -a \dot{\beta}_k - \frac{1}{2} \beta_k + (-1)^{k} \frac{A_{n-k}}{A_n} \dot{x}, \]
while for $k = n$ we have
\[ 0 = -a \dot{\beta}_n - \frac{1}{2} \beta_n + (-1)^n \frac{A_0}{A_n} \dot{x} \]
which give (2.26) using the relation (B.2).

Let us now check that the formula
\[ \forall k \in S_n^1 \quad \beta_k[\hat{F}] = \sum_{i=1}^n \frac{\xi_i}{\Delta_i} \sigma_{k-1}^i \]
proved in Proposition 9 does solve this system.

For $k = 1$ we have
\[ \beta_1 = \sum_{i=1}^n \frac{\xi_i}{\Delta_i} \sigma_0^i = \sum_{i=1}^n \frac{\xi_i}{\sqrt{\Delta_i}} = x. \]

For $k \in S_n^2$, using the relation (B.4), we have
\[ -a \dot{\beta}_k - \frac{1}{2} \beta_k + \sigma_k \dot{x} = \sum_{i=1}^n \epsilon_i \frac{\xi_i}{2 \Delta_i^{3/2}} (a_i \sigma_{k-1}^i - \sigma_k) = - \sum_{i=1}^n \frac{\epsilon_i \xi_i}{2 \Delta_i^{3/2}} \sigma_k^i. \]

If $k \in \{1, 2, \ldots, n-1\}$ we do recover $\dot{\beta}_{k+1}$, while for $k = n$ the result vanishes. \( \square \)

As we will see now this last form of $Q_1$ will allow for a simple construction of $Q_2$.

3 The local structure of the second integral

Let us proceed with $Q_2$. From Proposition 2 we need to solve
\[ \{H, Q_2\} + 2a P_y Q_1 = 0. \]

Let us prove

**Proposition 11** For a simple $F$, the observable
\[ Q_2 = \sum_{k=1}^n \tilde{c}_k[\hat{F}] H^{n-k} P_y^{2k} \]
is given by
\[ \forall k \in S_n^1 \quad \tilde{c}_k[\hat{F}] = (-1)^{k+1} \frac{A_n}{2} \left( \sum_{i=1}^n \frac{\xi_i^2}{\Delta_i} \sigma_{k-1}^i + \sum_{i \neq j=1}^n \frac{\xi_i \xi_j}{\Delta_i \Delta_j} (\sigma_{k-1}^{ij} + a \sigma_{k-2}^{ij}) \right) \]
where the $\sigma_{k}^{ij}$ are defined in Appendix B.
Proof: An elementary computation gives
\[ \{H, Q_2\} + 2a P_y Q_1 = 2a \Pi \sum_{k=1}^{n} \left( \frac{D_a \tilde{c}_k}{\tilde{x}} + \tilde{b}_k \right) H^{n-k} P_y^{2k}. \]

Since we are working locally, this is equivalent to
\[ \forall k \in S_1 \quad D_a \tilde{c}_k = -\tilde{b}_k \tilde{x} \]
or explicitly
\[ D_a \tilde{c}_k = (-1)^k A_n \frac{1}{2} \sum_{i=1}^{n} \frac{\xi_i}{\sqrt{\Delta_i}} \sigma^i_{k-1} \sum_{j=1}^{n} \frac{\epsilon_j \xi_j}{(\Delta_j)^{3/2}}. \]

Expanding into
\[ D_a \tilde{c}_k = (-1)^k A_n \left( \sum_{i=1}^{n} \frac{\epsilon_i \xi^2_i}{(a - a_i)^2} \sigma^i_{k-1} + \sum_{i \neq j=1}^{n} \frac{\xi_i \xi_j \epsilon_j}{\sqrt{\Delta_i} (\Delta_j)^{3/2}} \sigma^i_{k-1} \right) \]
and integrating up to
\[ \tilde{c}_k = (-1)^k A_n \left( - \sum_{i=1}^{n} \frac{\xi^2_i}{\Delta_i} \sigma^i_{k-1} + 2 \sum_{i \neq j=1}^{n} \frac{\xi_i \xi_j(a - a_i)}{(a_i - a_j) \sqrt{\Delta_i \Delta_j}} \sigma^i_{k-1} \right). \]

The second piece can be written
\[ 2 \sum_{i > j=1}^{n} \frac{\xi_i \xi_j}{\sqrt{\Delta_i \Delta_j}} \frac{\sigma^i_{k-1}(a - a_i) - \sigma^j_{k-1}(a - a_j)}{a_i - a_j}. \]

Using the relation (B.4) gives
\[ \sigma^i_{k-1}(a - a_i) - \sigma^j_{k-1}(a - a_j) = a(\sigma^i_{k-1} - \sigma^j_{k-1}) + \sigma^i_k - \sigma^j_k \]
and thanks to (B.8) we obtain (3.2). □

Let us add an important algebraic relation:

**Proposition 12** The integrals \( S_1 \) and \( S_2 \) are algebraically related by
\[ S_1^2 - 2G S_2 = A_n^2 \sum_{k,l=1}^{n} Q_{kl} H^{2n-k-l} P_y^{2(k+l)} \quad (3.3) \]
where
\[ Q_{kl} = (-1)^{k+l+1} \sum_{i=1}^{n} \epsilon_i \xi^2_i \sigma^i_{k-1} \sigma^i_{l-1}. \quad (3.4) \]
Proof: We have first

\[ X \equiv S_1^2 - 2GS_2 = Q_1^2 - 2GQ_2. \]

Expanding this expression in powers of \( H \) and of \( P_y \) and upon use of the identities (B.4) and (B.10) leads, after some hairy computations to the given formula. □

Summarizing the results obtained up to now we have:

**Theorem 1** The hamiltonian

\[ H = \Pi^2 + a P_y^2 \quad \Pi = \frac{a}{x} P_a \quad a > 0 \quad (3.5) \]

for a simple \( F = \hat{F} \) and

\[ x = \sum_{i=1}^{n} \frac{\xi_i}{\sqrt{\Delta_i}} \quad \Delta_i = \epsilon_i(a - a_i) \quad \epsilon_i^2 = 1 \quad \xi_i \in \mathbb{R}, \quad (3.6) \]

exhibits two integrals

\[ S_1 = Q_1 + yG \quad S_2 = Q_2 + yQ_1 + \frac{y^2}{2}G \quad (3.7) \]

where

\[ G = \sum_{k=0}^{n} A_{n-k} H^{n-k} P_y^{2k} \quad Q_1 = \sum_{k=1}^{n} \tilde{b}_k H^{n-k} \Pi^{2k-1} \quad Q_2 = \sum_{k=1}^{n} \tilde{c}_k H^{n-k} P_y^{2k} \quad (3.8) \]

and

\[ \forall k \in S_1^n : \left\{ \begin{array}{l}
\tilde{b}_k = (-1)^k A_n \sum_{i=1}^{n} \frac{\xi_i}{\sqrt{\Delta_i}} \sigma^i_{k-1} \\
\tilde{c}_k = (-1)^{k+1} A_n \frac{2}{\sigma^i_{k-1}} \left( \sum_{i=1}^{n} \frac{\xi_i^2}{\Delta_i} \sigma^i_{k-1} + \sum_{i \neq j=1}^{n} \frac{\xi_i \xi_j}{\sqrt{\Delta_i \Delta_j}} (\sigma^i_{k-1} + a \sigma^i_{k-2}) \right) 
\end{array} \right. \quad (3.9) \]

These two integrals generate two maximally SI systems:

\[ \mathcal{I}_1 = \{ H, P_y, S_1 \} \quad \text{and} \quad \mathcal{I}_2 = \{ H, P_y, S_2 \}. \quad (3.10) \]

Proof: We just need to check the functional independence of the integrals. Let us define

\[ J_1 = dH \wedge dP_y \wedge dS_1 \quad J_2 = dH \wedge dP_y \wedge dS_2. \]

We have

\[ J_1 = dH \wedge dP_y \wedge dS_1 = y dH \wedge dP_y \wedge dQ_1 + Q_1 \frac{\partial H}{\partial P_x} dP_x \wedge dP_y \wedge dy \]

which cannot vanish everywhere due to the last term. Differentiating the identity (3.3) we have

\[ d(S_1)^2 - 2G dS_2 = 2dS_2 + dX(H, P_y) \]

which implies that

\[ 2G J_2 = dH \wedge dP_y \wedge (2G dS_2) = 2S_1 J_1 \]

does not vanish everywhere. □
Cascading

Let us first introduce some notations which make explicit the degree $2n$ of the integrals. We have first

$$G^{(n)} = \sum_{k=0}^{n} A_k H^k P_{y}^{2(n-k)} \quad F^{(n)} = \sum_{k=0}^{n} A_k a^k = A_n \prod_{i=1}^{n} (a - a_i) \quad (4.1)$$

and similarly for the symmetric functions of the roots:

$$\sigma^{(n)} \quad \sigma^{i(n)} \quad \sigma^{ij(n)} \quad (4.2)$$

The Hamiltonian is

$$H^{(n)} = (\Pi^{(n)})^2 + a P_y^2 \quad \Pi^{(n)} = \frac{a}{(\dot{x})^{(n)}} P_a \quad (4.3)$$

while the integrals are

$$S^{(n)}_1 = Q^{(n)}_1 + y G^{(n)} \quad S^{(n)}_2 = Q^{(n)}_2 + y Q^{(n)}_1 + \frac{y^2}{2} G^{(n)} \quad (4.4)$$

with

$$Q^{(n)}_1 = \sum_{k=1}^{n} \tilde{b}^{(n)}_k H^{n-k} \Pi P_{y}^{2k-1} \quad Q^{(n)}_2 = \sum_{k=1}^{n} \tilde{c}^{(n)}_k H^{n-k} P_{y}^{2k}. \quad (4.5)$$

We have the relations

$$\lim_{A_n \to 0} F^{(n)} = F^{(n-1)} = F^{(n-1)} = A_{n-1} \prod_{k=1}^{n-1} (a - a_k) \quad \lim_{A_n \to 0} G^{(n)} = P_y^2 G^{(n-1)}. \quad (4.6)$$

It follows from the ODE for $x(a)$ that

$$x^{(n)} = \sum_{i=1}^{n} \frac{\xi_i}{\sqrt{\Delta_i}} \to x^{(n-1)} = \sum_{i=1}^{n-1} \frac{\xi_i}{\sqrt{\Delta_i}} \quad \Rightarrow \quad H^{(n)} \to H^{(n-1)}. \quad (4.7)$$

Hence to take the limit properly we have to let $A_n \to 0$ and $\xi_n \to 0$. Let us denote this limit by the symbol LIM. We will now prove:

**Proposition 13** In the limit $A_n \to 0$ and $\xi_n \to 0$ the integrals $Q^{(n)}_1$ and $Q^{(n)}_2$ become reducible according to the relations:

$$\text{LIM } Q^{(n)}_1 = P_y^2 Q^{(n-1)}_1 \quad \text{LIM } Q^{(n)}_2 = P_y^2 Q^{(n-1)}_2. \quad (4.8)$$

Factoring out by $P_y^2$ the integrals, we have exhibited the cascading between the following SI systems:

$$\{H^{(n)}, P_y, Q^{(n)}_1\} \to \{H^{(n-1)}, P_y, Q^{(n-1)}_1\} \quad (4.9)$$

$$\{H^{(n)}, P_y, Q^{(n)}_2\} \to \{H^{(n-1)}, P_y, Q^{(n-1)}_2\}.$$
Proof: From Theorem 1 we have

\[ \text{LIM } Q_1^{(n)} = \sum_{k=1}^{n} \left( \text{LIM } \tilde{b}_k^{(n)} \right) \left( H^{(n-1)} \right)^{n-k} P_y^{2k-1}. \]

The first term in the sum vanishes:

\[ \tilde{b}_1^{(n)} = -A_n \sum_{i=1}^{n} \frac{\xi_i}{\sqrt{\Delta_i}} \implies \text{LIM } \tilde{b}_1^{(n)} = 0. \]

Substituting \( l = k - 1 \) in the remaining sum we get

\[ \text{LIM } Q_1^{(n)} = P_y^2 \sum_{l=1}^{n-1} \left( \text{LIM } \tilde{b}_{l+1}^{(n)} \right) \left( H^{(n-1)} \right)^{n-1-l} P_y^{2l-1} \]

Using relation (B.5) in Appendix B we have

\[ \lim_{A_n \to 0} A_n \sigma_i^{(n-1)} = (-1)^l \sum_{s=1}^{l} a_{l-s} A_{n-s} = (-1)^{l-1} \sum_{t=0}^{l-1} a_{l-1-t} A_{n-1-t} = -A_{n-1} \sigma_{l-1}^{(n-1)} \quad (4.10) \]

which leads to

\[ \text{LIM } \tilde{b}_{l+1}^{(n)} = (-1)^l \sum_{i=1}^{n-1} \frac{\xi_i}{\sqrt{\Delta_i}} A_{n-1} \sigma_{l-1}^{(n-1)} = \tilde{b}_l^{(n-1)} \implies \text{LIM } Q_1^{(n)} = P_y^2 Q_1^{(n-1)}. \]

From Theorem 1 we have also

\[ \text{LIM } Q_2^{(n)} = \sum_{k=1}^{n} \left( \text{LIM } \tilde{c}_k^{(n)} \right) \left( H^{(n-1)} \right)^{n-k} P_y^{2k}. \]

The first term vanishes again:

\[ \tilde{c}_1^{(n)} = \frac{A_n}{2} \left( \sum_{i=1}^{n} \frac{\xi_i}{\Delta_i} + \sum_{i \neq j=1}^{n} \frac{\xi_i \xi_j}{\sqrt{\Delta_i \Delta_j}} \right) \implies \text{LIM } \tilde{c}_1^{(n)} = 0. \]

Substituting \( l = k - 1 \) in the remaining sum we get

\[ \text{LIM } Q_2^{(n)} = P_y^2 \sum_{l=1}^{n-1} \left( \text{LIM } \tilde{c}_{l+1}^{(n)} \right) \left( H^{(n-1)} \right)^{n-1-l} P_y^{2l-1} \]

with

\[ \tilde{c}_{l+1}^{(n)}(a) = \frac{(-1)^l}{2} \left( \sum_{i=1}^{n} \frac{\xi_i^2}{\Delta_i} A_n \sigma_i^{(n)} + \sum_{i \neq j=1}^{n} \frac{\xi_i \xi_j}{\sqrt{\Delta_i \Delta_j}} \left( A_n \sigma_i^{(n)} + a A_n \sigma_l^{(n)} \right) \right) \quad (4.11) \]
Let use relation (B.8)

\[ A_n \sigma_{ij}^{(n)} = \frac{A_n \sigma_{i}^{(n+1)} - A_n \sigma_{j}^{(n+1)}}{a_i - a_j} \]

which, combined with (4.10), leads to

\[ \lim_{A_n \to 0} A_n \sigma_{ij}^{(n)} = \frac{A_{n-1} \sigma_{i}^{(n-1)} - A_{n-1} \sigma_{j}^{(n-1)}}{a_i - a_j} = -A_{n-1} \sigma_{ij}^{(n-1)}. \]

Plugging this last relation as well as (4.10) into (4.11) we get

\[ \text{LIM } \tilde{c}_{i+1}^{(n)} = \tilde{c}_{i}^{(n-1)} \implies \text{LIM } Q_2^{(n)} = P^2_y Q_2^{(n-1)} \]

which concludes the proof. \(\square\)

Remarks:

1. The cascading process reduces the degree of the integrals from \(2n\) to \(2(n-1)\).

2. Let us start from the iff equations (I.6) for integrals of degree \(2n\)

\[ \{ H, Q_1^{(n)} \} + 2a P_y G^{(n)} = 0 \]

\[ \{ H, Q_2^{(n)} \} + 2a P_y Q_1^{(n)} = 0. \]

The cascading substitutions \(Q_1^{(n)} \to P^2_y Q_1^{(n-1)}\) and \(Q_2^{(n)} \to P^2_y Q_2^{(n-1)}\) in these relations and factoring by \(P^2_y\) lead to iff equations for integrals of degree \(2(n-1)\).

3. We have given a direct proof of the cascading phenomenon based on the explicit form of the \(\tilde{b}_k^{(n)}\) and \(\tilde{c}_k^{(n)}\). This constitutes a check of the formulas obtained above for these coefficients.

5 Some globally defined examples

A look at the metric

\[ g = \left( \frac{\dot{x}}{a} \right)^2 \, da^2 + \frac{1}{a} \, dy^2 = \frac{dx^2}{a^2} + \frac{1}{a} \, dy^2 \quad (5.1) \]

shows that

Proposition 14 The metric is riemannian iff \(a > 0\).

In general \(a\) will take values in some interval \((a_m, a_M)\) with \(a_m > 0\). The end-points of this interval may be of two kinds:

1. True singularities, namely curvature singularities, which cannot be disposed of by some coordinates change and which do prevent the metric to be defined on a manifold. They can be detected from the behaviour in a neighbourhood of these points of the scalar curvature given by (2.18).
2. Apparent singularities, as for instance
\[ g \sim d\chi^2 + \chi^2 dy^2 \quad \chi \to 0^+ \quad y \in S^1 \]
which can be removed using cartesian coordinates
\[ x = \chi \cos y \quad y = \chi \sin y \quad \implies \quad g \sim dx^2 + dy^2 \]

Let us prove

**Proposition 15** We have the following possibilities:

a) If \( \dot{x}(a) \sim (a - a_1)^\alpha \) the point \( a = a_1 \neq 0 \) is a curvature singularity if \( \alpha > 1 \).

b) If \( \dot{x}(a) \sim a^\alpha \) then \( a = 0^+ \) is a curvature singularity if \( \alpha > 0 \).

c) If \( \dot{x}(a) \sim a^\alpha \) then \( a \to +\infty \) is a curvature singularity if \( \alpha \in (-\infty, -1/2) \cup -1/2, 0) \).

**Proof:** In the case a) the computation of the curvature gives
\[ -2R \sim \frac{2a_1 \alpha}{(a - a_1)^{2(\alpha - 1)}} \]
which proves the statement.

In the case b) we have
\[ -2R \sim \frac{(2\alpha + 1)}{a^{2\alpha}} \]
which is not continuous for \( a \to 0^+ \) if \( \alpha > 0 \).

In the case c) the same formula holds but now \( a \to +\infty \) and the curvature must remain bounded, which is excluded iff \( \alpha < 0 \) and \( \alpha \neq -1/2 \). The case \( \alpha = -1/2 \) needs a specific analysis for each metric. \( \square \)

As a first check let us consider the simplest case \( n = 1 \) for which the integrals are merely quadratic: we should recover one of the Koenigs metrics [14].

**Proposition 16** For \( n = 1 \), i. e. quadratic integrals, there is a single SI metric, globally defined (g. d.) on \( \mathbb{H}^2 \): it is the Koenigs metric of type 3.

**Proof:** Here we have \( F(a) = a - a_1 \) so we need to order the discussion according to the values taken by \( a_1 \). If \( a_1 > 0 \), we may take \( a_1 = 1 \) and \( \xi_1 = 1 \). Hence we have
\[ x(a) = \frac{1}{\sqrt{a + 1}} \quad a > 0, \]
and by Proposition 13 this metric is singular for \( a \to +\infty \). Indeed the coordinate change
\[ t = \sqrt{\frac{a}{a + 1}} \quad \implies \quad g = (1 - t^2) \frac{dt^2 + dy^2}{t^2} \quad t \in (0, 1) \quad y \in \mathbb{R} \]
shows that $t \to 1^-$ (i.e. $a \to +\infty$) is a curvature singularity of the metric because the conformal factor does vanish. The same argument applies if $a_1 = 0$.

If $a_1 < 0$, up to scalings, we may take $a_1 = -1$ and $\xi_1 = 1$. A first possible case is

$$x = \frac{1}{\sqrt{a - 1}} \quad a \in (1, +\infty)$$

and here too Proposition 13 shows that this metric is singular for $a \to +\infty$.

The last possible case is

$$x = \frac{1}{\sqrt{1 - a}} \quad a \in (0, 1) \quad \implies \quad g = \frac{1}{a} \left( \frac{da^2}{4a(1 - a)^2} + dy^2 \right).$$

The coordinate change $u = \sqrt{\frac{a}{1 - a}}$ shows that

$$g = (1 + u^2) \frac{du^2 + dy^2}{u^2} = (1 + u^2) g_0(H^2, \mathcal{P}) \quad u > 0 \quad y \in \mathbb{R}$$

where $g_0$ is the Poincaré half-plane model of $\mathbb{H}^2$. The resulting hamiltonian

$$H = \frac{u^2}{1 + u^2} \left( P_u^2 + P_y^2 \right)$$

is nothing but Koenigs metric of type 3 as it is written in Theorem 7 of [14] (setting $\xi = 0$) which was shown to be globally defined on the manifold $M \cong \mathbb{H}^2$.

This Koenigs metric of type 3 suggests the following generalization to SI systems with integrals of any even degree larger than 4:

**Proposition 17** Let us consider, for $n \geq 2$

$$F(a) = (a - 1) \hat{F}(a) \quad \hat{F}(a) = \prod_{i=2}^{n} (a - a_k) \quad 0 < a < 1 \quad (5.2)$$

with

$$a_i < 0 \lor a_i > 1 \quad i = 2, \ldots, n.$$

The associated SI system produces the metric

$$g = (1 + u^2) \frac{\mu^2(u) du^2 + dy^2}{u^2} \quad u \in (0, +\infty) \quad y \in \mathbb{R}, \quad (5.3)$$

where

$$\mu(u) = c + \sum_{i=2}^{n} \frac{\xi_i}{(1 + \rho_i u^2)^{3/2}} \quad (5.4)$$

with

$$c > 0 \quad \xi_i > 0 \quad \rho_i \in (0, 1) \cup (1, +\infty).$$

This metric and the related integrals $(S_1, S_2)$ are globally defined on $M \cong \mathbb{H}^2$. \quad 18
Proof: The choice made for $F(a)$ and Proposition imply that we may take

$$x(a) = \frac{c}{\sqrt{1 - a}} - \sum_{i=2}^{n} \frac{\epsilon_i (-\epsilon_i a_i)^{3/2}}{\sqrt{\epsilon_i (a - a_i)}}.$$  

(5.5)

It follows that, under the coordinate change

$$u = \sqrt{\frac{a}{1 - a}} \quad a \in (0, 1) \to u \in (0, +\infty), \quad \Rightarrow \quad \frac{dx}{\sqrt{a}} = \mu(u) du$$

where $\mu$ is given in (5.4). Under this substitution the metric takes the form given in the formula (5.3). The parameters $\rho_i$ which appear are given by

$$\rho_i = 1 - \frac{1}{a_i} \quad \Rightarrow \quad \rho_i \in (0, 1) \cup (1, +\infty).$$

Defining the coordinate change

$$t = u \Omega(u) \quad \Omega(u) = c + \sum_{i=2}^{n} \frac{\xi_i}{\sqrt{1 + \rho_i u^2}} : \quad u \in (0, +\infty) \to t \in (0, +\infty),$$

since $\frac{dt}{du} = \mu > 0$ it follows that the inverse function $u(t)$ is increasing and $C^\infty([0, +\infty))$. The metric becomes

$$g = (1 + u^2(t)) \Omega^2(t) \frac{dt^2 + dy^2}{t^2} = (1 + u^2(t)) \Omega^2(t) \ g_0(H^2, P)$$

and since the conformal factor $(1 + u^2(t)) \Omega^2(t)$ never vanishes we conclude that $M \cong \mathbb{H}^2$.

We have seen that the integrals are

$$S_1 = Q_1 + yG \quad S_2 = Q_2 + yQ_1 + \frac{y^2}{2}G \quad G = \sum_{k=0}^{n} A_n H^k P_y^{2(n-k)}. \quad (5.6)$$

The global structure of these integrals is easy to study because $(H, P_y, \Pi)$, hence $G$, are globally defined on $M$. Considering

$$Q_1 = \sum_{k=1}^{n} \tilde{b}_k H^{n-k} \Pi P_y^{2k-1} \quad \tilde{b}_k = (-1)^k A_n \left( \frac{c \sigma_{k-1}^1}{\sqrt{1 - a}} - \sum_{i \in I} \frac{\epsilon_i (-\epsilon_i a_i)^{3/2} \xi_i \sigma_{k-1}^i}{\sqrt{\Delta_i}} \right).$$

In terms of the coordinate $t$ we have

$$\forall k \in S_1^n : \quad \tilde{b}_k = (-1)^k A_n \sqrt{1 + u^2(t)} \left( c \sigma_{k-1}^1 + \sum_{i \in I} \frac{a_i \xi_i \sigma_{k-1}^i}{\sqrt{1 + \rho_i u^2(t)}} \right),$$

showing that all these coefficients are $C^\infty([0, +\infty))$ so that $Q_1$ hence $S_1$ are globally defined on $M \cong \mathbb{H}^2$.  

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Let us consider $Q_2$. We have

$$Q_2 = \sum_{k=1}^{n} \tilde{c}_k H^{n-k} P_g^{2k}$$

and the coefficients become

$$\forall k \in \mathcal{S}_n^1:\quad \tilde{c}_k = (-1)^{k-1} \frac{A_n}{2} (1 + u^2(t)) \left( c^2 \sigma_{k-1} + 2c \sum_{i \in I} \frac{a_i \xi_i}{\sqrt{1 + \rho_i u^2(t)}} \tau_{k-1}^{i} + \sum_{i \in I} \frac{a_i^2 \xi_i^2}{1 + \rho_i u^2(t)} \sigma_{k-1}^{i} + \sum_{i \neq j \in I} \frac{a_i a_j \xi_i \xi_j}{\sqrt{(1 + \rho_i u^2(t))(1 + \rho_j u^2(t))}} \tau_{k-1}^{ij} \right)$$

with

$$\tau_{k-1}^{ij} = \sigma_{k-1}^{ij} + \frac{u^2}{1 + u^2} \sigma_{k-2}^{ij}.$$ 

From this formula it follows that $Q_2$ hence $S_2$ are globally defined on $M \cong \mathbb{H}^2$. □

As a second example we have:

**Proposition 18** For $n \geq 2$ the choice

$$F(a) = (a - a_1)(a - a_2) \tilde{F}(a) \quad 0 < a_1 < a < a_2$$

with \[^4\]

$$\tilde{F}(a) = \prod_{i=3}^{n} (a - a_i) : \quad \left( a_i < a_1 \lor a_i > a_2 \quad i = 3, \ldots n \right)$$

leads to a SI system with the metric

$$g = \frac{1}{A(t)}(dt^2 + dy^2) \quad (t, y) \in \mathbb{R}^2 \quad (5.7)$$

globally defined on the manifold $M \cong \mathbb{R}^2$ as well as the integrals $S_1$ and $S_2$.

**Proof:** Let us consider \[^5\]

$$x(a) = -\frac{\tilde{\xi}_1}{\sqrt{a - a_1}} + \frac{\tilde{\xi}_2}{\sqrt{a_2 - a}} - \sum_{i=3}^{n} \frac{\epsilon_i \tilde{\xi}_i}{\sqrt{\epsilon_i (a - a_i)}}$$

with

$$\tilde{\xi}_1 = a_1 \sqrt{a_2 - a_1} \xi_1 \quad \tilde{\xi}_2 = a_2 \sqrt{a_2 + a_1} \xi_2 \quad \tilde{\xi}_i = a_i \sqrt{a_2 - a_1} \xi_i$$

and

$$\epsilon_i = +1 \quad a_i < a_1 \quad \& \quad \epsilon_i = -1 \quad a_i > a_2.$$
The coordinate change
\[ a = a_1 + (a_2 - a_1)s^2 \quad s \equiv \sin \theta : \quad a \in (a_1, a_2) \leftrightarrow \theta \in (0, \pi/2) \]
gives
\[ x(\theta) = -\frac{\xi_1 a_1}{s} + \frac{\xi_2 a_2}{\sqrt{1 - s^2}} - \sum_{i=3}^{n} \frac{\epsilon_i a_i \xi_i}{\sqrt{\rho_i + s^2}} \quad \rho_i = \frac{a_1 - a_i}{a_2 - a_1}. \]
So differentiating we get
\[ D_\theta x = \frac{\xi_1 a_1 c}{s^2} + \frac{\xi_2 a_2 s}{(1 - s^2)} + \sum_{i=3}^{n} \frac{a_i \xi_i s c}{(\rho_i + s^2)^{3/2}}. \]
These relations show that \( D_\theta x > 0 \).

Defining \( dt = \frac{dx}{\sqrt{a}} \) one gets
\[ t(s) = \sqrt{a_1 + (a_2 - a_1)s^2} \left( -\frac{\xi_1}{s} + \frac{\xi_2}{\sqrt{1 - s^2}} - \sum_{i=3}^{n} \frac{\epsilon_i \xi_i}{\sqrt{\rho_i + s^2}} \right). \]
It follows that \( \theta \in (0, \pi/2) \) is mapped into \( t \in \mathbb{R} \) and that the function \( t = h(s) \) is a \( C^\infty \) (increasing) bijection from \( \theta \in (0, \pi/2) \to \mathbb{R} \), hence its inverse function \( s = h^{-1}(t) \) is also a \( C^\infty \) (increasing) bijection.

We obtain the metric given by (5.7):
\[ g = \frac{1}{A(t)}(dt^2 + dy^2) \quad A = a \circ h^{-1} \quad (t, y) \in \mathbb{R}^2 \]
and since the conformal factor \( A(t) \) never vanishes, we conclude that the manifold is \( M \cong \mathbb{R}^2 \).

As in the previous case considered above, let us consider
\[ Q_1 = \sum_{k=1}^{n} \bar{b}_k \ H^{n-k} \prod_y P^{2k-1}_y \]
with the coefficients
\[ \bar{b}_k = (-1)^2 A_n \left( -a_1 \xi_1 \sigma_{k-1}^{1} / A(t) + a_2 \xi_2 \sigma_{k-1}^{2} / \sqrt{1 - A^2(t)} - \sum_{i=3}^{n} \frac{\epsilon_i \xi_i \sigma_{k-1}^{i}}{\sqrt{\rho_i + A^2(t)}} \right). \]
Since \( \theta \in (0, \pi/2) \) this proves that \( Q_1 \) is indeed globally defined. The check for \( Q_2 \) is similar. \( \square \)

Let us conclude with the following negative result:

**Proposition 19** The choice \( F = (a^2 + a_0^2)^n \) for \( n = 1, 2, \ldots \) never leads to a globally defined SI system.
Proof: In this case, setting $a_0 = 1$, we have

$$x(\theta) = \sum_{k=1}^{n} \left( \mu_+^k (\cos \theta)^{k-1/2} \cos((k-3/2)\theta) + \mu_-^k (\cos \theta)^{k-1/2} \sin((k-3/2)\theta) \right) \theta = \arctan a.$$ 

Since we have $\theta \in (0, \frac{\pi}{2})$, let us consider the metric behaviour for $\theta \to \frac{\pi}{2}$. For a fixed value of $k$, we have the following equivalent:

$$g \sim c_k^2 (\cos \theta)^{2k-1} d\theta^2 + \cos \theta dy^2.$$ 

Using the coordinate change $v = v_0 (1 - \sin \theta)^{(2k+1)/4}$ for an appropriate constant $v_0$ leads to

$$g \sim c_k^2 \left( dv^2 + v^{2/(2k+1)} dY^2 \right) \quad v \to 0^+ \quad k \in \{1, 2, \ldots, n\}$$

where $Y$ is merely homothetic to $y$. Even restricting $Y \in S^1$ the exponent of $v$ is never equal to 2 so we have a true singularity for $v \to 0^+$. □
Part II
Integrals of odd degree in the momenta

We will consider the cases for which the observables have for degree in the momenta \( \sharp(Q) = 2n + 1 \) where \( n \geq 1 \). The case \( n = 1 \) was first analyzed in [8] and [12].

The hamiltonian remains unchanged
\[
H = \Pi^2 + a P_y^2 \quad \Pi = \frac{a}{\dot{x}} P_y,
\]

while
\[
G = \sum_{k=0}^{n} A_k H^k P_y^{2n-k+1} \quad A_n \neq 0 \quad \sharp(G) = 2n + 1 \quad n \in S_n
\]
is still built up from the \( n + 1 \) constants: \( A_0, A_1, \ldots, A_n \). The SI stems from the two observables
\[
S_1 = Q_1 + y G \quad S_2 = Q_2 + y Q_1 + \frac{y^2}{2} G.
\]

Let us begin with the determination of \( Q_1 \).

6 The local structure of the first integral

Proposition 20 Taking
\[
Q_1 = \sum_{k=0}^{n} b_k(a) \Pi^{2k+1} P_y^{2n-k},
\]
the observable \( S_1 \) will be an integral iff
\[
\begin{cases}
(a) & b_0 = 2F \dot{x} \\
(b) & D_a b_{k-1} = (k + \frac{1}{2}) b_k - \frac{D_a^k F}{k!} \dot{x} \quad k = 1, \ldots, n \\
(c) & D_a b_n = 0
\end{cases}
\]

where \( F(a) = \sum_{k=0}^{n} A_k a^k \).

Proof: The equation
\[
\{H, S_1\} = \{H, Q_1\} + 2a P_y G = 0
\]
expands into
\[ \sum_{k=1}^{n+1} a_k b_{k-1} \Pi^{2k} P_y^{2(n-k+1)} - \sum_{k=0}^{n} (k + 1/2) a' b_k \right. \left. a \Pi^{2k} P_y^{2(n-k+1)} + \sum_{k=0}^{n} \frac{D^k F}{k!} a \Pi^{2k} P_y^{2(n-k+1)} = 0 \]
giving the differential system
\[
\begin{cases}
0 = \frac{1}{2} a' b_0 - F(a) \\
b'_{k-1} = (k + \frac{1}{2}) a' b_k - \frac{D^k F}{k!} \\
b_n = 0
\end{cases}
\]
for \( k = 1, \ldots, n \).

Switching to the new variable \( a \), instead of \( x \), gives (6.2). □

We can proceed to

**Proposition 21 (Linearizing ODE)** The differential system (6.2) has for (unique) solution
\[ b_k = \sum_{s=1}^{k+1} \frac{D^{k-s+1} F}{(k-s+1)! (1/2)_s} D_a x \quad b_n = \text{const} \in \mathbb{R} \setminus \{0\} \quad (6.3) \]
where \( x(a) \) is a solution of the ODE
\[ \text{Op}_n[F] x(a) = \left( n + \frac{1}{2} \right) b_n a + \beta_n \quad \beta_n \in \mathbb{R}. \quad (6.4) \]

Its solution in the simple case, up to an additive constant, is given by
\[ x = \frac{b_n}{2A_n} a + \sum_{i=1}^{n} \frac{\xi_i}{\sqrt{\Delta_i}} \quad \Delta_i = \epsilon_i (a - a_i) \quad \epsilon_i^2 = 1 \quad \xi_i \in \mathbb{R} \quad A_n \in \mathbb{R} \setminus \{0\} \quad (6.5) \]

**Proof:** The recursive proof giving \( b_k \) for any \( k \in \mathcal{S}_n \setminus \{n\} \) is similar to the one given for Proposition 5. Integrating equation (b) in (6.2) for \( k = n \) leads to
\[ b_{n-1} + \frac{D^n F}{n!} x = (n + 1/2) b_n a + \beta_n \]
which we combine with
\[ b_{n-1} = \sum_{s=1}^{n} \frac{1}{(1/2)_s (n-s)!} D^s_a x \]
to get (6.4). The homogeneous equation was already solved for in Proposition 6 and looking for an affine solution gives
\[ \frac{b_n}{2A_n} a + \frac{\beta_n}{A_n} - \frac{b_n A_{n-1}}{A_n^2} \]
in which the constant term may be deleted. □

**Remarks:**
1. The transition from integrals of degree $2n$ to $2n + 1$ is strikingly simple: one just adds a constant in $x$! This was observed in [12] for the cubic case but was not expected to be so general.

2. In Proposition 7 we have seen that if $x = \pm \frac{a}{\sqrt{-2R}}$ the metric is of constant negative curvature. In order to avoid such a case we must exclude the trivial possibility that all the $\xi_i$ be vanishing.

3. From now on we will define $\nu_n = b_n/A_n \in \mathbb{R}\{0\}$.

Let us conclude this section by giving a useful form of $Q_1$:

**Proposition 22** For a simple $F$ we have

$$Q_1 = \sum_{k=0}^{n} \tilde{b}_k[\hat{F}] H^{n-k} \Pi P_y^{2k}$$

(6.6)

with

$$\forall k \in S_0^n : \quad \tilde{b}_k[\hat{F}] = (-1)^k A_n \left( \nu_n \sigma_k + \sum_{i=1}^{n} \frac{\xi_i}{\sqrt{\Delta_i}} \sigma_{k-1}^i \right).$$

(6.7)

**Proof:** By the same argument used in Part I for $Q_1$ the coefficients $\tilde{b}_k[\hat{F}]$ are seen to be obtained from the coefficients $b_k[\hat{F}]$ by the relation

$$\tilde{b}_k[\hat{F}] = \sum_{s=0}^{k} \binom{n-s}{n-k} (-a)^{k-s} b_{n-s}[\hat{F}] = \binom{n}{n-k} (-a)^k b_n + \sum_{s=1}^{k} \binom{n-s}{n-k} (-a)^{k-s} b_{n-s}[\hat{F}]$$

where

$$b_{n-s}[\hat{F}] = \frac{b_n}{A_n} D_{a}^{n-s} \hat{F} + \sum_{l=1}^{n-s+1} \frac{1}{(1/2)l (n-s+1-l)!} D_{a}^{l} x^{(0)}$$

and $x^{(0)}$ is just the $\xi_i$ dependent part of $x$. So we have to compute two pieces:

$$\tilde{b}_k[\hat{F}] = \frac{b_n}{A_n} \sum_{s=0}^{k} \binom{n-s}{n-k} (-a)^{k-s} D_{a}^{n-s} F + \sum_{s=0}^{k} \binom{n-s}{n-k} (-a)^{k-s} b_{n-s+1}(even)$$

where $b_{n-s+1}(even)$ is the same as in the proof of the Proposition 5 and therefore gives the same result as in Proposition 9:

$$(-1)^k A_n \sum_{i=1}^{n} \xi_i \sigma_{k-1}^i.$$

The first piece gives

$$\nu_n \sum_{s=0}^{k} \sum_{l=n-s}^{n} A_l \binom{n-s}{n-k} (-a)^{k-s} \binom{l}{n-s} (-1)^{k-s} a^{k-l-n}.$$
Reversing the summations we conclude to

\[ \nu_n \sum_{l=n-k}^{n} \frac{A_l a^{k+l-n}}{(n-k)!} \sum_{s=n-l}^{k} \frac{(-1)^{s-k}}{(k-s)! (l+s-n)!} = \nu_n A_{n-k} = (-1)^k \nu_n A_n \sigma_k \]

and use of the identity (B.2) concludes the proof. □

As in Part I, let us check the result obtained for \( \tilde{b}_k[F] \) using its differential system. We have

**Proposition 23** Defining

\[ \forall k \in S_n^0 : \quad \tilde{b}_k[F] = A_n (-1)^k \beta_k[F] , \quad (6.8) \]

for any choice of \( F \) we have the following relations valid

\[ \begin{align*}
    k &= 0 : \quad \dot{\beta}_0 = 0 \\
    1 \leq k \leq n - 1 : \quad \dot{\beta}_{k+1} &= -a \dot{\beta}_k - \frac{1}{2} \beta_k + \sigma_k \dot{x} \quad (6.9) \\
    k &= n : \quad 0 = -a \dot{\beta}_n - \frac{1}{2} \beta_n + \sigma_n \dot{x} .
\end{align*} \]

For a simple \( F \) the formula obtained for \( \tilde{b}_k[F] \) in Proposition 22 is indeed a solution of this system.

**Proof:** The proof of (6.9) is a routine computation following the same lines as in the proof of (2.26). Let us check that for a simple \( F \) the relation (6.7) does give a solution of this differential system.

We have \( \beta_0 = \nu_n \) which is fine. Then computing

\[ -a \dot{\beta}_k - \frac{1}{2} \beta_k + \sigma_k \dot{x} = \frac{1}{2} \sum_{i=1}^{n} \epsilon_i \xi_i (a_i \sigma_{k-1}^i - \sigma_k) = -\frac{1}{2} \sum_{i=1}^{n} \epsilon_i \xi_i \Delta_i^{3/2} \sigma_k^i \]

So for \( 1 \leq k \leq n - 1 \) we get \( \dot{\beta}_{k+1} \) while for \( k = n \) it indeed vanishes. □

Now that \( Q_1 \) is fixed up let us construct \( Q_2 \).

7 The local structure of the second integral

As shown in Proposition 1, the structure of \( Q_2 \) follows from:

\[ \{ H, Q_2 \} + 2a P_y Q_1 = 0 \quad Q_2 = \sum_{k=0}^{n} \tilde{c}_k[F] H^{n-k} P_y^{2k+1} \quad (7.1) \]

and it is given by:
**Proposition 24** The observable $S_2$ is an integral iff $Q_2$ is determined from the differential system
\[
\forall k \in S_n^0: \quad D_a \tilde{c}_k = -\tilde{b}_k \dot{x}.
\] (7.2)

For a simple $F$ these coefficients are given by
\[
\forall k \in S_n^0: \quad \tilde{c}_k[F] = (-1)^{k+1} A_n \frac{\nu_n^2}{2} a \sigma_k + 2 \nu_n \sum_{i=1}^n \frac{\xi_i}{\sqrt{\Delta_i}} (\sigma^i_k + a \sigma^i_{k-1}) + \sum_{i=1}^n \frac{\xi_i^2}{\Delta_i} \sigma^i_{k-1} + \sum_{i \neq j=1}^n \frac{\xi_i \xi_j}{\sqrt{\Delta_i \Delta_j}} (\sigma^{ij}_{k-2} + a \sigma^{ij}_{k-1}) \right\}.
\] (7.3)

**Proof:** An elementary computation gives
\[
\{H, Q_2\} + 2 \nu_n^2 \frac{\nu}{2} \sum_{k=0}^n \left( \frac{D_a \tilde{c}_k}{\dot{x}} + \tilde{b}_k \right) H^{n-k} P_y^{2k+1}
\]
which implies (7.2).

In the computation of $D_a \tilde{c}_k$ there appears the constant term
\[
(-1)^{k+1} \frac{\nu_n^2 A_n}{2} \sigma_k
\]
while the terms linear in the $\xi_i$ give
\[
(-1)^{k+1} \nu_n A_n \left( \frac{1}{2} \sum_{i=1}^n \frac{\xi_i^2}{\Delta_i^{3/2}} + \sum_{i=1}^n \frac{\xi_i \sigma^i_{k-1}}{\sqrt{\Delta_i}} \right)
\]
and an integration yields
\[
(-1)^{k+1} \nu_n A_n \sum_{i=1}^n \frac{\xi_i}{\sqrt{\Delta_i}} (\sigma_k + (a - a_i) \sigma^i_{k-1}) = (-1)^{k+1} \nu_n A_n \sum_{i=1}^n \frac{\xi_i}{\sqrt{\Delta_i}} (\sigma^i_k + a \sigma^i_{k-1})
\]
after use of (B.4). The remaining terms are quadratic in the momenta and are easily seen to be the same as in Proposition 11. \qed

Let us add an important algebraic relation:

**Proposition 25** The integrals $S_1$ and $S_2$ are algebraically related by
\[
S_1^2 - 2 G S_2 = A_n^2 \left( \sum_{k,l=1}^n Q_{kl} H^{2n-k-l} P_y^{2(k+l+1)} + \nu_n^2 \sum_{k,l=0}^n (-1)^{k+l} \sigma_k \sigma_l H^{2n+1-k-l} P_y^{2(k+l)} \right)
\] (7.4)
where $Q_{kl}$ was already defined in (3.4).
Proof: Denoting the quantities defined in the Part 1 by a sharp subscript, we have

\[ G = P_y G_z \quad Q_1 = A + P_y Q_{1z} \quad Q_2 = B + P_y Q_{2z} \]

which implies

\[ X \equiv S_1^2 - 2G S_2 = Q_1^2 - 2G Q_2 = A^2 + 2AP_y Q_{1z} - 2BP_y G_z + P_y^2 X_z. \]

The first three terms, after several algebraic simplifications give the required formula while the last term is obvious. □

Summarizing the results obtained up to now we have

**Theorem 2** For a simple \( F \), the hamiltonian

\[ H = \Pi^2 + a P_y^2 \quad \Pi = \frac{a}{\dot{x}} P_a \quad a > 0 \quad (7.5) \]

where

\[ x = \frac{b_n}{2A_n} a + \sum_{i=1}^{n} \frac{\xi_i}{\Delta_i} \quad \Delta_i = \epsilon_i (a - a_i) \quad \epsilon_i^2 = 1, \quad (7.6) \]

with all the \( \xi_i \in \mathbb{R} \) and \( A_n \in \mathbb{R} \setminus \{0\} \), exhibits two integrals

\[ S_1 = Q_1 + y G \quad S_2 = Q_2 + y Q_1 + \frac{y^2}{2} G \quad (7.7) \]

where

\[ G = \sum_{k=0}^{n} A_{n-k} H^{n-k} P_y^{2k+1} \quad Q_1 = \sum_{k=0}^{n} \tilde{b}_k H^{n-k} \Pi^2 P_y^{2k} \quad Q_2 = \sum_{k=0}^{n} \tilde{c}_k H^{n-k} P_y^{2k+1} \quad (7.8) \]

with

\[ \forall k \in S_n^0 : \quad \tilde{b}_k [\hat{F}] = (-1)^{k} A_n \left( v_n \sigma_k + \sum_{i=1}^{n} \frac{\xi_i}{\sigma_{k-1}} \right) \quad (7.9) \]

and

\[ \forall k \in S_n^0 : \quad \tilde{c}_k [\hat{F}] = (-1)^{k+1} A_n \left\{ v_n^2 a \sigma_k + 2v_n \sum_{i=1}^{n} \frac{\xi_i}{\sigma_{k-1}} (\sigma_k^i + a \sigma_{k-1}^i) + \sum_{i=1}^{n} \frac{\xi_i^2 \sigma_{k-1}^i}{\Delta_i} + \sum_{i \neq j=1}^{n} \frac{\xi_i \xi_j}{\sqrt{\Delta_i \Delta_j}} (\sigma_k^{ij} + a \sigma_{k-2}^{ij}) \right\}. \quad (7.10) \]

These two integrals generate two possible (maximally) SI systems:

\[ \mathcal{I}_1 = \{ H, P_y, S_1 \} \quad \text{and} \quad \mathcal{I}_2 = \{ H, P_y, S_2 \}. \quad (7.11) \]

**Proof:** The functional independence proof is the same as for Theorem 1. □
8 Some globally defined examples

Let us begin with the case \( n = 1 \) (cubic integrals) for which the global structure was first analyzed in [12]. To compare most conveniently with our results let us first transform our metric

\[
g = \frac{\dot{x}^2}{a^2} da^2 + \frac{dy^2}{a}, \quad a > 0
\]  

under the coordinate change \( u = \sqrt{a} \). We get

\[
g = \frac{\mu^2}{u^2} du^2 + \frac{dy^2}{u^2} \quad \mu = 2\dot{x}(a = u^2).
\]  

So, for \( F(a) = a - a_1 \) we may take

\[
x(a) = \frac{\nu}{2} a + \frac{\epsilon c}{\sqrt{\epsilon(a - a_1)}} \quad \implies \quad \mu = \nu - \frac{c}{(\epsilon(u^2 - a_1))^{3/2}} \quad \epsilon = \pm 1 \quad \nu \neq 0
\]  

and since \( \nu \) cannot vanish we can set \( \nu = 1 \) showing that our local form of the metric is in perfect agreement with the local form given in [12]. It may be noticed that the single difference in the function \( \mu \) with respect to the Koenigs case (quadratic integrals) is just this constant \( \nu \).

However, as opposed to the Koenigs metrics, this form of \( \mu \) allows for a much larger number of globally defined cases. Let us prove:

**Proposition 26** The SI systems having for metric

\[
g = \frac{\mu^2(u)}{u^2} du^2 + \frac{dy^2}{u^2} \quad \mu(u) = 1 - \frac{c}{[\epsilon(u^2 - a_1)]^{3/2}}
\]  

are globally defined on \( M \cong \mathbb{H}^2 \) in the following cases:

| \( \mathcal{I} \) | \( \mu(u) \) | \( a_1 \) |
|-----------------|-----------------|---------|
| \( \mathcal{I}_{++} \) | \( 1 - \frac{1}{(u^2 - a_1)^{3/2}} \) | \( a_1 \in (-\infty, -1) \) |
| \( \mathcal{I}_{+-} \) | \( 1 + \frac{1}{(u^2 - a_1)^{3/2}} \) | \( a_1 \in (-\infty, 0) \) |
| \( \mathcal{I}_{-+} \) | \( 1 - \frac{1}{(a_1 - u^2)^{3/2}} \) | \( a_1 \in (0, 1) \) |
| \( \mathcal{I}_{--} \) | \( 1 - \frac{c}{(u^2 - a_1)^{3/2}} \) | \( a_1 \in (0, +\infty) \) |

**Proof:** The scalar curvature being

\[
R_g = -2 \frac{\mu + u \mu'}{\mu^3}
\]  

any singularity of it implies that the metric cannot be defined on any manifold.
In the case $I_{++}$, for $a_1 \geq 0$, $\mu$ vanishes for $u_0 = \sqrt{a_1 + 1} > \sqrt{a_1}$ leading to a curvature singularity. For $a < 0$ we have $u > 0$ and for $a_1 \geq -1$ the curvature is again singular for $u_0 = \sqrt{a_1 + 1}$. It remains to consider $a_1 < -1$. Defining

$$dt = \mu \, du \quad \implies \quad t = u \, \Omega(u) \quad \Omega(u) = 1 - \frac{1}{|a_1| \sqrt{u^2 + |a_1|}}$$

Since $\mu = \frac{dt}{du}$ never vanishes the inverse function $u(t)$ is $C^\infty([0, +\infty))$. The metric is now

$$G = \Omega^2(u(t)) \frac{dt^2 + dy^2}{t^2} = \Omega^2(u(t)) \, g_0(H^2, \mathcal{P})$$

and since $\Omega([0, +\infty)) = [1 - 1/|a_1|^{3/2}, 1)$, the conformal factor never vanishes showing that $M \cong \mathbb{H}^2$.

In the case $I_{+-}$, for $a_1 > 0$ we have $u > \sqrt{a_1}$ so that defining

$$dt = \mu \, du \quad \implies \quad t = u \, \Omega(u) \quad \Omega(u) = 1 - \frac{1}{a_1 \sqrt{u^2 - a_1}}$$

but this time $\Omega(u)$ vanishes for $u_0 = \sqrt{a_1 + 1/a_1^2} > \sqrt{a_1}$. Since $\Omega(u(t))$ appears as a conformal factor in the metric there can be no manifold.

For $a_1 = 0$ we have $u \in (0, +\infty)$. In this last case, defining $v = 1/3u^3$ the metric is

$$G \sim \frac{du^2}{u^8} + \frac{dy^2}{u^2} = dv^2 + (3v)^{2/3} \, dy^2$$

showing that $u \to 0+$ precludes any manifold.

For $a_1 < 0$, hence $u > 0$, the change of coordinate

$$dt = \mu \, du \quad \implies \quad t = u \, \Omega(u) \quad \Omega(u) = 1 + \frac{1}{|a_1| \sqrt{u^2 + |a_1|}}$$

implies for the metric

$$G = \Omega^2(u(t)) \frac{dt^2 + dy^2}{t^2} = \Omega^2(u(t)) \, g_0(H^2, \mathcal{P}),$$

where $u(t)$ is $C^\infty([0, +\infty))$ and $\Omega([0, +\infty)) = (1, 1 + 1/|a_1|^{3/2}]$ hence the manifold is again $\mathbb{H}^2$.

For $\epsilon = -1$ we must have $a_1 > 0$ and $u \in (0, \sqrt{a_1})$.

In the case $I_{--}$, for $a_1 \geq 1$, there is a curvature singularity for $u_0 = \sqrt{a_1 - 1}$ while for $0 < a_1 < 1$ the function $\mu(u)$ never vanishes. Let us define

$$dt = \mu \, du \quad \implies \quad t = u \, \Omega(u) \quad \Omega(u) = 1 - \frac{1}{a_1 \sqrt{a_1 - u^2}}.$$ 

The function $\Omega(u)$ is strictly decreasing with $\Omega([0, \sqrt{a_1}]) = (-\infty, -(1/a_1^{3/2} - 1)]$ and the metric becomes

$$G = \Omega^2(u(t)) \frac{dt^2 + dy^2}{t^2} = \Omega^2(u(t)) \, g_0(H^2, \mathcal{P}).$$
Since $\Omega(u(t))$ never vanishes we get $M = \mathbb{H}^2$.

In the last case $\mathcal{I}_{--}$, we can define
\[ dt = \mu(u) \, du \implies t = u \, \Omega(u) \quad \Omega(u) = 1 + \frac{1}{a_1 \sqrt{a_1 - u^2}} \]

where $\Omega(u)$ is strictly increasing with $\Omega([0, \sqrt{a_1}]) = [1 + 1/a_1^{3/2}, +\infty)$ and the metric becomes
\[ G = \Omega^2(u(t)) \frac{dt^2 + dy^2}{t^2} = \Omega^2(u(t)) \, g_0(H^2, \mathcal{P}). \]

Since $\Omega(u(t))$ never vanishes we get $M = \mathbb{H}^2$.

For the proof that the integrals are also globally defined on $\mathbb{H}^2$ the arguments presented in [12] do apply and need not be repeated. □

**Remark:** Our analysis does correct the Propositions 16 and 17 of the reference [12] but is in agreement with its Proposition 18.

Let us generalize the previous SI systems:

**Proposition 27** The SI systems $\mathcal{I}_{++}$, corresponding to $F(a) = (a - a_1)$, equipped with cubic integrals, do generalize to $F(a) = \prod_{i=1}^{n}(a - a_i)$ with integrals of degree $2n + 1$ (with $n \geq 2$) which remains globally defined on $M \cong \mathbb{H}^2$ under the following restrictions

\[ \mathcal{I}_{++} : \quad -\infty < a_i < a_1 < -1 \quad \& \quad \xi_i > 0 \quad \& \quad \frac{1}{|a_1|^{3/2}} + \sum_{i=2}^{n} \frac{\xi_i}{|a_i|^{3/2}} < 1, \quad (8.6) \]

\[ \mathcal{I}_{+-} : \quad -\infty < a_i < a_1 < 0 \quad \& \quad \xi_i > 0. \]

**Proof:** The first SI system, above $\mathcal{I}_{++}$, is generated by
\[ x(a) = \frac{a}{2} - \frac{1}{\sqrt{a - a_1}} + \sum_{i=2}^{n} \frac{\xi_i}{\sqrt{a - a_i}} \implies \mu(u) = 1 - \frac{1}{(u^2 - a_1)^{3/2}} - \sum_{i=2}^{n} \frac{\xi_i}{(u^2 - a_i)^{3/2}}. \]

The metric has the form (8.2). Noting that $\mu$ is strictly increasing from
\[ \mu(0) = 1 - \frac{1}{|a_1|^{3/2}} - \sum_{i=2}^{n} \frac{\xi_i}{|a_i|^{3/2}} > 0 \implies \mu(\infty) = 1. \]

Let us define
\[ dt = \mu(u) \, du \implies t = u \, \Omega(u), \quad \Omega(u) = 1 + \frac{1}{a_1 \sqrt{u^2 - a_1}} - \sum_{i=2}^{n} \frac{\xi_i}{a_i \sqrt{u^2 - a_i}}. \]

Since $D_u t > 0$ the inverse function $u(t)$ is $C^\infty([0, +\infty))$ and $\Omega(u)$ is strictly increasing with $\Omega([0, +\infty)) = [\mu(0), 1)$ and therefore never vanishes. The metric becomes
\[ g = \Omega^2(u(t)) \frac{dt^2 + dy^2}{t^2} = \Omega^2(u(t)) \, g_0(H^2, \mathcal{P}). \]
showing that \( M \cong \mathbb{H}^2 \).

Here we have
\[
Q_1 = \sum_{k=0}^{n} \tilde{b}_k H^{n-k} \Pi \Pi_{y}^{2k}
\]
with
\[
\tilde{b}_k = (-1)^k A_n \left( \frac{1}{2} \sigma_{k} - \frac{\sigma_{k-1}}{u^2(t) - a_1} + \sum_{i=2}^{n} \frac{\xi_i \sigma_{k-1}}{u^2(t) - a_i} \right)
\]
which are indeed \( C^\infty([0, +\infty)) \) hence \( Q_1 \) is globally defined on \( M \). The check for \( Q_2 \) is similar.

The second SI system, above \( I_{+, -} \), is generated by
\[
x(a) = \frac{a}{2} - \frac{1}{\sqrt{a - a_1}} - \sum_{i=2}^{n} \frac{\xi_i}{\sqrt{a - a_i}} \quad \Rightarrow \quad \mu(u) = 1 + \frac{1}{(u^2 - a_1)^{3/2}} + \sum_{i=2}^{n} \frac{\xi_i}{(u^2 - a_i)^{3/2}}.
\]
The function \( \mu \) is decreasing from
\[
\mu(0) = 1 + \frac{1}{|a_1|^{3/2}} + \sum_{i=2}^{n} \frac{\xi_i}{|a_i|^{3/2}} \quad \Rightarrow \quad \mu(+\infty) = 1
\]
hence it never vanishes. So we can define
\[
t = u \Omega(u) \quad \Omega(u) = 1 + \frac{1}{|a_1|\sqrt{u^2 - a_1}} + \sum_{i=2}^{n} \frac{\xi_i}{|a_i|\sqrt{u^2 - a_i}}.
\]
It follows that \( \Omega(u) \) decreases from \( \Omega(0) = \mu(0) \) to \( \Omega(+\infty) = 1 \) and never vanishes, showing by the same argument displayed above that \( M \cong \mathbb{H}^2 \).

The checks that \( Q_1 \) and \( Q_2 \) are globally defined are again elementary. □

Let us add:

**Proposition 28** The SI systems \( I_{-, +} \), corresponding to \( F(a) = (a - a_1) \), equipped with cubic integrals, do generalize to \( F(a) = \prod_{i=1}^{n} (a - a_i) \) with integrals of degree \( 2n + 1 \) (with \( n \geq 2 \)) which remains globally defined on \( M \cong \mathbb{H}^2 \) under the following restrictions
\[
I_{+, -} : \quad 0 < a_1 < +1, \quad a_i > 1 \quad \& \quad \xi_i > 0 \quad \& \quad \frac{1}{|a_1|^{3/2}} + \sum_{i=2}^{n} \frac{\xi_i}{|a_i|^{3/2}} > 1;
\]
\[
I_{-, -} : \quad 0 < a_1 < +1, \quad a_i > 1 \quad \& \quad \xi_i > 0.
\]

(8.7)

**Proof:** The first system, above \( I_{+, -} \), is generated by
\[
x(a) = \frac{a}{2} - \frac{1}{\sqrt{a_1 - a}} - \sum_{i=2}^{n} \frac{\xi_i}{\sqrt{a_i - a}} \quad \Rightarrow \quad \mu(u) = 1 - \frac{1}{(a_1 - u^2)^{3/2}} - \sum_{i=2}^{n} \frac{\xi_i}{(a_i - u^2)^{3/2}}.
\]
The function $\mu$ is decreasing from
\[
\mu(0) = 1 - \frac{1}{a_1^{3/2}} - \sum_{i=2}^{n} \frac{\xi_i}{a_i^{3/2}} < 0 \implies \mu(\sqrt{a_1}) = -\infty.
\]
So we can define
\[
t = u \Omega(u) \quad \Omega(u) = 1 - \frac{1}{a_1 \sqrt{u^2 - a_1}} - \sum_{i=2}^{n} \frac{\xi_i}{a_i \sqrt{u^2 - a_i}}.
\]
It follows that $\Omega(u)$ decreases from $\Omega(0) = \mu(0) < 0$ to $\Omega(\sqrt{a_1}) = -\infty$ and never vanishes, showing by the same argument given above that $M \cong \mathbb{H}^2$.

The second system, above $\mathcal{I}_{-+}$, is generated by
\[
x(a) = \frac{a}{2} + \frac{1}{\sqrt{a_1 - a}} + \sum_{i=2}^{n} \frac{\xi_i}{\sqrt{a_i - a}} \implies \mu(u) = 1 + \frac{1}{(a_1 - u^2)^{3/2}} + \sum_{i=2}^{n} \frac{\xi_i}{(a_i - u^2)^{3/2}}.
\]
The function $\mu$ is increasing from
\[
\mu(0) = 1 + \frac{1}{a_1^{3/2}} + \sum_{i=2}^{n} \frac{\xi_i}{a_i^{3/2}} < 0 \implies \mu(\sqrt{a_1}) = +\infty.
\]
So we can define
\[
t = u \Omega(u) \quad \Omega(u) = 1 + \frac{1}{a_1 \sqrt{a_1 - u^2}} + \sum_{i=2}^{n} \frac{\xi_i}{a_i \sqrt{a_i - u^2}}.
\]
It follows that $\Omega(u)$ increases from $\Omega(0) = \mu(0) > 0$ to $\Omega(\sqrt{a_1}) = +\infty$ and never vanishes, showing by the same argument as above that $M \cong \mathbb{H}^2$.

The checks that the integrals are globally defined are again elementary. \qed

Let us give another example which is a close cousin of the system considered in Proposition 18.

**Proposition 29** For $n \geq 2$ the choice
\[
F(a) = (a - a_1)(a - a_2) \hat{F}(a) \quad 0 < a_1 < a < a_2
\]
with
\[
\hat{F}(a) = \prod_{i=3}^{n} (a - a_i) : \left( a_i < a_1 \lor a_i > a_2 \quad i = 3, \ldots n \right)
\]
leads to a SI system with the metric
\[
g = \frac{1}{A(t)}(dt^2 + dy^2) \quad (t, y) \in \mathbb{R}^2 \quad (8.8)
\]
globally defined on the manifold $M \cong \mathbb{R}^2$ as well as the integrals $S_1$ and $S_2$.

\[\text{If } n = 2 \text{ we have } \hat{F}(a) = 1.\]
Proof: Let us consider

$$x(a) = \frac{\nu}{2}a - \frac{\hat{\xi}_1}{\sqrt{a - a_1}} + \frac{\hat{\xi}_2}{\sqrt{a - a_2}} - \sum_{i=3}^{n} \frac{\epsilon_i \hat{\xi}_i}{\sqrt{\epsilon_i(a - a_i)}}$$

with

$$\hat{\xi}_1 = \sqrt{a_2 - a_1} a_1 \xi_1 \quad \hat{\xi}_2 = \sqrt{a_2 + a_1} a_2 \xi_2 \quad \hat{\xi}_i = \sqrt{a_2 - a_1} a_i \xi_i$$

and

$$\epsilon_i = +1 \quad a_i < a_1 \quad \& \quad \epsilon_i = -1 \quad a_i > a_2.$$ 

The coordinate change

$$a = a_1 + (a_2 - a_1)s^2 \quad s \equiv \sin \theta : \quad a \in (a_1, a_2) \leftrightarrow \theta \in (0, \pi/2)$$

gives

$$x(\theta) = \frac{\nu}{2}(a_2 - a_1) s^2 - \frac{\xi_1 a_1}{s} + \frac{\xi_2 a_2}{s^2} - \sum_{i=3}^{n} \frac{\epsilon_i a_i \xi_i}{\sqrt{\epsilon_i(\rho_i + s^2)}} \quad \rho_i = \frac{a_1 - a_i}{a_2 - a_1}.$$ 

So differentiating we get

$$D_\theta x = \nu(a_2 - a_1) sc + \xi_1 \frac{a_1 c}{s^2} + \xi_2 \frac{a_2 s}{(1 - s^2)} + \sum_{i=3}^{n} \frac{a_i \xi_i s c}{(\rho_i + s^2)^{3/2}}$$

This relation show that $D_\theta x > 0$.

Defining $dt = \frac{dx}{\sqrt{a}}$ one gets

$$t(s) = \sqrt{a_1 + (a_2 - a_1)s^2} \left( \nu - \frac{\xi_1}{s} + \frac{\xi_2}{\sqrt{1 - s^2}} - \sum_{i=3}^{n} \frac{\epsilon_i \xi_i}{\sqrt{\rho_i + s^2}} \right).$$

From now on the proof follows exactly the same steps as in the proof of Proposition 18. □

9 Conclusion

Needless to say a lot of work is still necessary in the study of the models constructed here. Let us just mention a few items:

1. Determine the integrals for the non simple cases.

2. Find more globally defined systems with $\mathbb{R}^2$ or $\mathbb{H}^2$ for manifolds. It is possible that a general proof can be given that in this class of models (called “affine case” in [12]) the possible manifolds are restricted to $\mathbb{R}^2$ or $\mathbb{H}^2$.

\footnote{For $n = 2$ the last sum is absent.}
3. When the integrals are quadratic in the momenta we are back to a metric due to Koenigs. In this case it was shown in [9] that the integrals generate a quadratic algebra. Is there a generalization for integrals of higher degrees?

4. An even more difficult task would be to check whether the classical integrability survives to quantization in the sense of [3].

To end up the author is somewhat disappointed because the manifolds obtained here never meet $S^2$ and are of no use with respect to the Conjecture quoted in the Introduction. However, it was shown in [12] that the “hyperbolic case” is much more interesting since one is led either to teardrop orbifolds (Tannery orbifolds) or $S^2$. Unfortunately the analysis went through explicitly for cubic integrals but the generalization to higher degrees remains unknown, indicating a more difficult ground. However the reward could be here plenty of Zoll metrics (as is already the case for cubic integrals as shown in [13]) which are quite interesting geometric objects.

Appendices

A The cases of a non simple $F$

When $F$ is simple, the general solution of the ODE (2.10) was given in Proposition 6. This appendix will deal with all the remaining cases: either $F$ has a multiple real zero or it has multiple couples of complex conjugate zeroes.

A.1 A multiple real zero

Here we have $^8$

\[ F(a) = (a - a_1)^r \hat{F}(a) \quad 2 \leq r \leq n \]  

(A.1)

where $\hat{F}$ is simple.

One has:

**Proposition 30** For $F = (a - a_1)^r \hat{F}(a)$ with $2 \leq r \leq n$ and a generic $\hat{F}$, the solution of (2.10) is given by $^9$

\[ x = \sum_{k=1}^{r} \frac{\mu_k}{(\Delta_1)^{k-1/2}} + \sum_{i=r+1}^{n} \frac{\xi_i}{\sqrt{\Delta_i}} \quad \Delta_i = \epsilon_i(a - a_i). \]  

(A.2)

The general solution of (6.4) is simply obtained by adding to $x(a)$ the linear term $\frac{\nu_n}{2} a$.

---

$^8$If $r = n$ we take $\hat{F} = 1$.

$^9$If $r = n$ the sum over $i$ disappears.
**Proof:** Due to linearity we first check the ODE for
\[ x_k(a) = |\epsilon_1(a - a_1)|^{-k+1/2} \quad k \in S^1_k. \]
Since the variable is \( a \) we will denote the derivation order by a superscript. Using (2.7) and interchanging the summations order we have first
\[ \text{Op}_n[(a - a_0)^r \hat{F}] x_k = \sum_{l=0}^{n} \frac{\hat{F}^{(n-l)}}{(n-l)!} \text{Op}_l[(a - a_0)^r] x_k \]
and we will show that
\[ \forall l \in S^0_n \quad \forall k \in S^1_r \quad X_{l,k}^r \equiv \text{Op}_l[(a - a_0)^r] x_k = 0. \]
Computing the various derivatives and defining \( \sigma = s + r - l \) one gets
\[ X_{l,k}^r = (-\epsilon_1)^{l-r} \Delta_1^{r-l+1/2} \frac{\Gamma(k + l - r - 1/2)}{\Gamma(k - r + 1/2)} \sum_{\sigma=0}^{r} \frac{(-1)^\sigma r!}{\sigma!(r-\sigma)!} \frac{(k - 1/2)_{\sigma+l-r}}{(l/2)_{\sigma+l-r}}. \] (A.3)
Using the identities
\[ (a)^{N+\sigma} = (a)^N (a + N)^\sigma \quad (-r)_\sigma = (-1)^\sigma \frac{r!}{(r-\sigma)!} \quad \sigma \leq r \quad (-r)_\sigma = 0 \quad r > \sigma \] (A.4)
shows that the sum is proportional to
\[ \sum_{\sigma=0}^{\infty} \frac{(-r)_\sigma (k + l - r - 1/2)_\sigma}{\sigma!(l-r+1/2)_\sigma} \] (A.5)
which is a hypergeometric function [4][p. 61] at unity
\[ _2F_1 \left( \begin{array}{c} -r, k + l - r - 1/2 \\ l - r + 1/2 \end{array} ; 1 \right) = \frac{\Gamma(l-r+1/2) \Gamma(r-k+1)}{\Gamma(l+1/2) \Gamma(1-k)} \]
and does vanish for all \( k \in S^1_r \).

For the proof to be complete let us check now that \( x_i = \frac{1}{\sqrt{\epsilon_i(a - a_i)}} \) is also a solution of the ODE (2.10). We start from
\[ \text{Op}_n[(a - a_1)^r \hat{F}] x_i = \sum_{k=0}^{n} \frac{[(a - a_1)^r]^{(n-k)}}{(n-k)!} \text{Op}_k[\hat{F}] x_i \]
and use relation (2.14) which states that
\[ \forall k \in S^0_n : \quad \text{Op}_k[\hat{F}] x_i = \frac{\sqrt{\Delta_i}}{k!} D^k_a \left( \frac{\hat{F}}{\Delta_i} \right). \]
Inserting this second relation into the first one and using Leibnitz fromula we conclude to
\[ \text{Op}_n[(a - a_1)^r \hat{F}] x_i = \sqrt{\Delta_i} D^n_a \left( \frac{\hat{F}}{\Delta_i} \right) \]
which does vanish. \( \square \)
A.2 Multiple complex conjugate zeroes

In this case we have:

**Proposition 31** If $F$ has for structure

$$F(a) = (a^2 + a_0^2)^r \hat{F}(a) \quad 2 \leq 2r \leq n$$

where $\hat{F}$ is simple, then the solution of the ODE (2.10) becomes

$$x = \sum_{k=1}^{r} \left( \mu_k^+ P_k + \mu_k^- Q_k \right) + \sum_{k=2r+1}^{n} \frac{\xi_i}{\sqrt{\Delta_i}}. \quad (A.6)$$

Defining

$$a = a_0 \tan \theta \quad \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \quad a_0 > 0 \quad (A.7)$$

we have

$$\begin{cases} P_k = \cos \frac{\theta}{2} \mathcal{P} \\ Q_k = \sin \frac{\theta}{2} \mathcal{P} \end{cases} \quad \rightarrow \quad \mathcal{P} = (\cos \theta)^{k-1/2} \left[ U_{k-1}(\cos \theta) - U_{k-2}(\cos \theta) \right] \quad (A.8)$$

where the $U_n$ are the Tchebyshev polynomials of second kind supplemented with $U_{-1} = 0$. The general solution of (6.4) is again obtained by adding to $x(a)$ the linear term $\nu_n a$.

**Proofs:** Using the results of the previous proposition we know that if we start from

$$F(a) = (a - a_1)^r(a - a_2)^r \hat{F}(a) \quad 2 \leq 2r \leq n$$

the general solution is given by

$$x(a) = \sum_{k=1}^{r} \left( \frac{\lambda_k^+}{(a - a_1)^{k-1/2}} + \frac{\lambda_k^-}{(a - a_2)^{k-1/2}} \right) + \sum_{i=2r+1}^{n} \frac{\xi_i}{\sqrt{\Delta_i}}.$$

Since complex $a_i$ are allowed, let us take a pair of complex conjugate points $a_1 = ia_0$ and $a_2 = -ia_0$. The first piece in $x$ may be written

$$\sum_{k=1}^{r} \left( \frac{\mu_k^+}{(a_0 + ia)^{k-1/2}} + \frac{\mu_k^-}{(a_0 - ia)^{k-1/2}} \right)$$

so that the basis required is just made out of the real and imaginary parts of $(a_0 + ia)^{-k+1/2}$. This is most easily computed using

$$a = a_0 \tan \theta \quad \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \quad a_0 > 0.$$
We have
\[(1 + ia/a_0)^{-k+1/2} = (\cos \theta)^{k-1/2} e^{-i(k-1/2)\theta}\]
which gives a first expression for the real and imaginary parts:
\[P_k = (\cos \theta)^{k-1/2} \cos \left((k - 1/2)\theta\right) \quad Q_k = (\cos \theta)^{k-1/2} \sin \left((k - 1/2)\theta\right).\]
Recalling the defining relations of Tchebychev polynomials
\[T_n(\cos \theta) = \cos(n\theta) \quad U_n(\cos \theta) = \frac{\sin(n + 1)\theta}{\sin \theta}\]
we have
\[P_k = (\cos \theta)^{k-1/2} \cos \frac{\theta}{2} \left[T_k(\cos \theta) + (1 - \cos \theta)U_{k-1}(\cos \theta)\right].\]
Using the relation \[T_k(\cos \theta) - \cos \theta U_{k-1}(\cos \theta) = -U_{k-2}(\cos \theta)\]
gives for \(P_k\) the formula (A.8). A similar computation gives the required formula also for \(Q_k\).

Switching back to the variable \(a\) we have
\[P_k = \sqrt{\sqrt{a^2 + a_0^2} + a_0} \mathcal{R} \quad Q_k = \frac{a}{\sqrt{\sqrt{a^2 + a_0^2} + a_0}} \mathcal{R}\]
with
\[\forall k \in \mathcal{S}_r^1 \quad \mathcal{R} = (a^2 + a_0^2)^{-k/2} \left[U_{k-1}\left(\frac{a_0}{\sqrt{a^2 + a_0^2}}\right) - U_{k-2}\left(\frac{a_0}{\sqrt{a^2 + a_0^2}}\right)\right].\]
For \(k = 1\) we have merely
\[P_1(a) = \frac{\sqrt{a^2 + a_0^2} + a_0}{\sqrt{a^2 + a_0^2}} \quad Q_1(a) = \frac{a}{\sqrt{a^2 + a_0^2} \sqrt{a^2 + a_0^2} + a_0}.\] (A.9)

Let us give a second proof, which does not use the complexification argument given above, and which is similar to the one for Proposition 30 that the function \(x_k(a) = (1 + ia)^{-k+1/2}\) is indeed a solution of the ODE (2.10) when \(F = (a^2 + 1)^r \hat{F}\).

Using twice the formula (2.7) we have
\[
\text{Op}_n\left((a^2 + 1)^r \hat{F}\right) x_k = \sum_{s=0}^{n} \frac{\hat{F}^{n-s}}{(n-s)!} \sum_{l=0}^{s} \frac{[(1 - ia)^r]^{(s-l)}}{(s-l)!} \text{Op}_l((1 + ia)^r) x_k
\]
\[\text{Op}_n\left((a^2 + 1)^r \hat{F}\right) x_k = \sum_{s=0}^{n} \frac{\hat{F}^{n-s}}{(n-s)!} \sum_{l=0}^{s} \frac{[(1 - ia)^r]^{(s-l)}}{(s-l)!} \text{Op}_l((1 + ia)^r) x_k\]

\[\text{It is valid also for } k = 1 \text{ due to our convention that } U_{-1} = 0.\]
and we will prove that

\[ \forall l \in S^0_n \quad \forall k \in S^1_r \quad X^r\_{l,k} \equiv \text{Op}_l((1 + ia)^r) x_k = 0. \]

Computing the various derivatives one gets

\[ X^r\_{l,k} \propto \sum_{s=l-r}^{r} (-1)^s r! \frac{(k-1/2)_s}{(l-s)! (r-l+s)!} \times \sum_{\sigma=0}^{r} (-1)^\sigma r! (1/2)^{\sigma+l-r} \sigma!(r-\sigma)! (1/2)^{\sigma+l-r}. \]

This sum was already found in (A.3) and shown to vanish for all \( k \in S^1_r. \)

\[ \Box \]

### B Symmetric functions of the roots

Let us take a set of numbers \( \{a_1, a_2, \ldots, a_n\} \) and let us define

\[ P \equiv \prod_{k=1}^{n} (a - a_k) = \sum_{k=0}^{n} (-1)^k \sigma_k a^{n-k}, \tag{B.1} \]

where the \( \sigma_k \) are the symmetric functions of the roots for the monic polynomial \( P. \)

Restricting ourselves to real polynomials \( P \) this definition remains valid either if some root is multiple or if some complex conjugate couple of roots appear.

It follows that

\[ F = A_n \prod_{k=1}^{n} (a - a_k) = \sum_{k=0}^{n} A_k a^k \implies (-1)^k \sigma_k = \frac{A_{n-k}}{A_n} \quad \forall k \in S^0_n. \tag{B.2} \]

Omitting, in the polynomial \( P, \) a factor \((a - a_i)\) we will define

\[ \frac{P}{a - a_i} = \sum_{k=0}^{n+1} (-1)^{k-1} \sigma_{k-1}^i a^{n-k} \implies \sigma_{-1}^i = \sigma_{n}^i = 0. \tag{B.3} \]

Multiplying this last relation by \( a - a_i \) we get

\[ \forall k \in S^0_n : \quad \sigma_k = \sigma_k^i + a_i \sigma_{k-1}^i. \tag{B.4} \]

An easy recurrence, using (B.4) and (B.2), gives the relations

\[ \forall k \in S^1_n : \quad A_n \sigma_{k-1}^i = A_n \sum_{s=0}^{k-1} (-a_i)^{k-1-s} \sigma_s = (-1)^{k-1} \sum_{s=0}^{k-1} a_i^{k-1-s} A_{n-s}. \tag{B.5} \]

For \( i \neq j \) we can define

\[ \frac{P}{(a - a_i)(a - a_j)} = \sum_{k=0}^{n+2} (-1)^{k-2} \sigma_{k-2}^{ij} a^{n-k} \implies \sigma_{-2}^{ij} = \sigma_{-1}^{ij} = \sigma_{n-1}^{ij} = \sigma_{n}^{ij} = 0. \tag{B.6} \]
Multiplying this last relation by \((a - a_j)\) we get
\[
\forall k \in S^0_n : \quad \sigma_{k-1}^i = \sigma_{k-1}^{ij} + a_j \sigma_{k-2}^{ij} \tag{B.7}
\]
from which we deduce
\[
\forall k \in S^0_n \quad \sigma_{k-2}^{ij} = -\frac{\sigma_{k-1}^i - \sigma_{k-1}^j}{a_i - a_j}. \tag{B.8}
\]
A quadratic identity follows from the relation
\[
i \neq j : \quad \frac{P}{a - a_i} \frac{P}{a - a_j} = \frac{P}{(a - a_i)(a - a_j)}. \tag{B.9}
\]
As a preliminary remark let us observe that any product of the form
\[
AB = \sum_{k,l=1}^n A_k B_l H^{2n-k-l} P^2^{2(k+l)}
\]
after setting \(s = k + l\) and reversing the order of the summations becomes
\[
\sum_{s=2}^n U_s(A, B) H^{2n-s} P^2 s + \sum_{s=n+1}^{2n} V_s(A, B) H^{2n-s} P^2 s.
\]
with
\[
U_s(A, B) = \sum_{k=1}^{s-1} A_k B_{s-k}, \quad V_s = \sum_{k=s-n}^n A_k B_{s-k}.
\]
Taking this observation into account one obtains the relations
\[
\sum_{k=1}^{s-1} \sigma_{k-1}^i \sigma_{s-k-1}^j = \sum_{k=1}^{s-1} \sigma_k \sigma_{s-k-2}^{ij} + \sigma_{s-2}^{ij} \quad s \in \{2, 3, \ldots, n\} \tag{B.10}
\]
\[
\sum_{k=s-n}^n \sigma_{k-1} \sigma_{s-k-1}^j = \sum_{k=s-n}^n \sigma_k \sigma_{s-k-2}^{ij} \quad s \in \{n + 1, \ldots, 2n\}.
\]
To conclude, the symmetric functions needed by our analysis are therefore
\[
\begin{pmatrix}
\sigma_0 = 1 & \sigma_1 & \sigma_2 & \ldots & \sigma_{n-2} & \sigma_{n-1} & \sigma_n \\
\sigma_0^i = 1 & \sigma_1^i & \sigma_2^i & \ldots & \sigma_{n-2}^i & \sigma_{n-1}^i & \sigma_n^i = 0 \\
\sigma_0^{ij} = 1 & \sigma_1^{ij} & \sigma_2^{ij} & \ldots & \sigma_{n-2}^{ij} & \sigma_{n-1}^{ij} = 0 & \sigma_n^{ij} = 0
\end{pmatrix} \tag{B.11}
\]
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