Computing Information Agreement *

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Abstract

Agreement measures are useful to both compare different evaluations of the same diagnostic outcomes and validate new rating systems or devices. Information Agreement (IA) is an information-theoretic-based agreement measure introduced to overcome all the limitations and alleged pitfalls of Cohen’s Kappa. However, it is only able to deal with agreement matrices whose values are positive natural numbers. This work extends IA admitting also 0 as a possible value for the agreement matrix cells.

1 Basic Notions

Let $\mathcal{X}$ and $\mathcal{Y}$ be two raters that individually classify the instances of same non-empty data set $\mathcal{D}$ as belonging to one among $n$ possible classes, where $n$ is greater then 1. Their combined classifications produce an agreement matrix $A$ that is a $n \times n$-matrix whose cells $A[y|x]$ report how many instances of $\mathcal{D}$ were classified, at the same time, as belonging to the classes $y$ and $x$ by $\mathcal{Y}$ and $\mathcal{X}$, respectively.

Since $|\mathcal{D}| = \sum_{y=1}^{n} \sum_{x=1}^{n} A[y|x] > 0$, the probability for a randomly selected instance of $\mathcal{D}$ to be classified at the same time as belonging to the classes $y$ and $x$ by $\mathcal{Y}$ and $\mathcal{X}$, $p_{\mathcal{X},\mathcal{Y}}(y,x)$, equals $A[y|x]/S_A$ where $S_A$ is the sum of all the values in $A$, i.e., $S_A \overset{\text{def}}{=} \sum_{y=1}^{n} \sum_{x=1}^{n} A[y|x]$. Since any agreement matrix contains at least one positive value, $S_A$ must be greater than 0 too.

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The probability for an instance of $D$ to be put in the class $y$ by $\mathfrak{f}$ is denoted by $p_{\mathfrak{f}_A}(y)$ and it equals $S_A^Y(y)/S_A$, where $S_A^X(y)$ is the sum of all the values in the row $y$, i.e., $S_A^X(y) \overset{\text{def}}{=} \sum_{x=1}^n A[y][x]$. Analogously, the odd for the same instance to be classified in class $x$ by $X$ is $p_{X_A}(x) = S_A^X(y)/S_A$ where $S_A^X(x) \overset{\text{def}}{=} \sum_{y=1}^n A[y][x]$.

Let $Z$ and $W$ be two random variables. The Shannon entropy, $H(Z)$, of $Z$ \cite{4} evaluates the information carried by $Z$ itself. In the general case, it is formally defined as

$$H(Z) \overset{\text{def}}{=} -\sum_{z \in Z} p_Z(z) \log_2 p_Z(z)$$

where $p_Z(z)$ is the probability for $Z$ to get the value $z$ and $Z$ is the set of all the possible values for it. Without making any assumption on $p_Z(z)$, $H(Z)$ can be proved to belong to the closed interval $[0, \log_2 |Z|]$. It is worth to underline that, since $0$ is not included in the domain of the logarithmic function, $H(Z)$ is well-defined if and only if $p_Z(z) > 0$ for all $z \in Z$. Moreover, the following proposition holds.

**Proposition 1.** If $H(Z)$ is well-defined and $|Z| > 1$, $H(Z) > 0$.

*Proof. If $H(Z)$ is well-defined, then $p_Z(z) > 0$ for all $z \in Z$. Thus, $p_Z(z) \in (0, 1]$, $\log_2 p_Z(z)$ is non-positive, and so $p_Z(z) \cdot \log_2 p_Z(z)$ is. It follows that $H(Z)$ equals $0$ if and only if all its terms – i.e., $p_Z(z) \cdot \log_2 p_Z(z)$ – equal $0$, but this exclusively happens when $p_Z(z) = 1$. However, by definition of probability function, $\sum_{z \in Z} p_Z(z) = 1$. We can conclude that either $|Z| = 1$, which contradicts the proposition’s hypothesis, or $0 < p_Z(z) < 1$ for all $z \in Z$ and $H(Z) > 0$. $\square$*

The *conditional entropy of $W$ given $Z$* \cite{4} measures the quantity of information in $W$ when an insight of $Z$ is available and it is defined as

$$H(W/Z) \overset{\text{def}}{=} -\sum_{z \in Z} \sum_{w \in W} p_{ZW}(w, z) \log_2 \frac{p_{ZW}(w, z)}{p_Z(z)}$$

where $p_{ZW}(w, z)$ is the *joint probability* for both $Z$ and $W$ to get the values $z$ and $w$ at the same time and $W$ is the set of all the possible values for $w$.

The *mutual information $MI(Z,W)$* measures how far are $Z$ and $W$ from being independent, i.e., it gauges how much the values that they assume are related still being potentially different. $MI(Z,W)$ is formally defined as:

$$MI(Z,W) \overset{\text{def}}{=} \sum_{z \in Z} \sum_{w \in W} p_{ZW}(w, z) \log \frac{p_{ZW}(w, z)}{p_Z(z) \cdot p_W(w)}$$

and it is easy to prove that

$$MI(Z,W) = H(Z) + H(W) - H(ZW) = MI(W,Z) \geq 0$$

Given the probability distributions $P_{X_A} = \{p_{X_A}(x)\}_x$, $P_{Y_A} = \{p_{Y_A}(y)\}_y$, and $P_{X_A Y_A} = \{p_{X_A Y_A}(y,x)\}_{x,y}$, the entropy values for the so-called *marginal
random variables $X_A$ and $Y_A$—i.e., $H(X_A)$ and $H(Y_A)$, respectively—and for the random variable $X_A Y_A$—i.e., $H(X_A Y_A)$—can be computed as shown by Eq. 1. As a consequence, the mutual information between $X_A$ and $Y_A$ can be evaluated too. All these quantities are completely determined by the agreement matrix because $p_{X_A}(x)$, $p_{Y_A}(y)$, and $p_{X_A Y_A}(y, x)$ exclusively depend on $A$ itself. Moreover, it can be proved that $H(X_A) = H(Y_A^T)$, $H(Y_A) = H(X_A^T)$, and $H(X_A Y_A) = H(X_A Y_A^T)$ where $A^T$ denotes the transposed matrix of $A$, i.e., $A^T[y][x] = A[y][x]$ for all rows $y$ and for all columns $x$ in $A$.

The information agreement (IA) of $A$ [1] was introduced to gauge the agreement between the two raters $X$ and $Y$ on the data set $D$ by considering $A$. It is formally defined as follows:

$$IA(A) \overset{\text{def}}{=} \frac{MI(X_A, Y_A)}{\min\{H(X_A), H(Y_A)\}}.$$  

(5)

It is known that the information agreement is not well-defined for all the agreement matrices $A$. However, whenever $IA(A)$ is defined, its value belongs to the interval $[0, 1]$.

2 Extending IA

In its original form, the information agreement is not well-defined for all the possible agreement matrices $A$. In particular, since $IA$ is the ratio between $MI(X_A, Y_A)$ and $\min\{H(X_A), H(Y_A)\}$ (see Eq. 5) and $MI(X_A, Y_A)$ equals the sums and subtractions of entropies (see Eq. 4), $IA$ is not defined under two circumstances: when at least one entropy among $H(X_A)$, $H(Y_A)$, and $H(X_A Y_A)$ is not defined and when the minimum among $H(X_A)$ and $H(Y_A)$ is 0. According to what we noticed in Section 1, the former case exclusively occurs when there exist $x, y \in [1, n]$ such that either $p_{X_A}(x) = 0$, $p_{Y_A}(y) = 0$, or $p_{X_A Y_A}(y, x) = 0$. However, by definition of $p_{X_A}(x)$, $p_{Y_A}(y)$, and $p_{X_A Y_A}(y, x)$, this is equivalent to the existence of a value in $A$ that equals 0. As far as the latter case may concern, if both $H(X_A)$ and $H(Y_A)$ are well-defined, then both $H(X_A)$ and $H(Y_A)$ are greater than 0 by Prop. 1 because $n > 0$ by assumption. It follows that $IA(A)$ is well-defined if and only if all the values in $A$ are greater than 0.

Since the logarithmic function is defined and continuous in the interval $(0, +\infty)$, one possible solution to overcome the inability of computing $IA$ on an agreement matrix $A$ containing some 0 is to build a new symbolic agreement matrix $A_\epsilon$ that replaces all the occurrences of 0 in $A$ with a real variable $\epsilon$. The matrix $A_\epsilon$ is the 0-freed matrix and it is formally defined as follows:

$$A_\epsilon[y][x] \overset{\text{def}}{=} \begin{cases} A[y][x] & \text{if } A[y][x] \neq 0 \\ \epsilon & \text{otherwise} \end{cases}$$

where $\epsilon$ is a real variable assuming values in the open interval $(0, +\infty)$.

Because of their definitions, it is easy to see that $p_{X_{A_\epsilon} Y_{A_\epsilon}}(y, x)$, $p_{X_{A_\epsilon}}(x)$, and $p_{Y_{A_\epsilon}}(y)$ belong to the real interval $(0, 1)$ for all $x, y \in [1, n]$ and for all
$\epsilon \in (0, +\infty)$. It follows that $H(X_A, Y_A)$, $H(X_A)$, and $H(Y_A)$ are well-defined for any positive value of $\epsilon$ and so $IA(A_\epsilon)$ is. Thus, the limit for $IA(A_\epsilon)$ as $\epsilon$ tends to 0 from the right may be a reasonable estimation for $IA(A)$.

It is worth to underline that, while $IA(A)$, when defined, is a value, $IA(A_\epsilon)$ is a function on $\epsilon$ whose domain is open real interval $(0, +\infty)$ and, because of this, its limit as $\epsilon$ tends to 0 from the right may not exist. However, if this limit does exist, then it will be the extension-by-continuity of $IA$ over the matrix $A$. This limit is the Information Agreement extension by Continuity $IA_\epsilon$ and is formally defined as follows:

$$IA_\epsilon(A) \overset{\text{def}}{=} \lim_{\epsilon \to 0^+} IA(A_\epsilon) \quad (6)$$

In the following part of this section, we will prove that $IA_\epsilon(A)$ always exists and we show how to compute it. This achievement will be eased by the following proposition.

**Lemma 1.** Let $B$ be an $n \times n$-agreement matrix. For all $v, w \in [1, n]$, $p_{X_B}(v) = p_{Y_B\top}(v)$ and $p_{X_B Y_B}(w, v) = p_{X_B\top Y_B\top}(w, v)$.

**Proof.** By the definitions of $S_B^X(x)$ and $S_B$, $S_B^X(x) \overset{\text{def}}{=} \sum_{y=1}^n B[y][x]$ and $S_B \overset{\text{def}}{=} \sum_{x=1}^n \sum_{y=1}^n B[y][x]$. So, because of the definition of $B^\top$,

$$S_B^X(x) = \sum_{y=1}^n B[y][x] = \sum_{y=1}^n B^\top[x][y] = S_B^Y(x)$$

and, analogously,

$$S_B = \sum_{x=1}^n \sum_{y=1}^n B[y][x] = \sum_{x=1}^n \sum_{y=1}^n B^\top[x][y] = S_B^\top.$$

Since $p_{X_B}(x) \overset{\text{def}}{=} S_B^X(x)/S_B$ and $p_{Y_B\top}(x) \overset{\text{def}}{=} S_B^Y(x)/S_B\top$ by definition, it follows that $p_{X_B}(x) = p_{Y_B\top}(x)$.

Moreover, $B[y][x] = B^\top[x][y]$ for all $x, y \in [1, n]$ by definition of transposed matrix. Hence, $p_{X_B Y_B}(y, x) = p_{Y_B\top}(x, y)$ for all $x, y \in [1, n]$, because $p_{X_B Y_B}(x) \overset{\text{def}}{=} S_B^X(x)/S_B$ and $p_{X_B\top Y_B\top}(x, y) \overset{\text{def}}{=} p_{X_B\top Y_B\top}(x, y)/S_B\top$. \hfill \square

Thanks to Lemma 1 which unravels the relation between the probability function associated to an agreement matrix $B$ and that of $B^\top$, we can easily prove the following proposition about the entropy functions.

**Lemma 2.** Let $B$ be an $n \times n$-agreement matrix such that $B[y][x] > 0$ for all rows $y$ and for all columns $x$ in $B$. The following equalities hold:

1. $H(X_B) = H(Y_B\top)$;
2. $H(Y_B) = H(X_B\top)$;
3. $H(X_B Y_B) = H(X_B\top Y_B\top)$.
Proof. Let us prove the claim, point by point.

1. By Eq. 1 and by Lemma 1 it is immediate to see that

\[ H(X_B) = -\sum_{x=1}^{n} p_{X_B}(x) \log_2 p_{X_B}(x) \]
\[ = -\sum_{x=1}^{n} p_{Y_B^T}(x) \log_2 p_{Y_B^T}(x) = H(Y_B^T) \]

2. Let \( C \) be the matrix \( B^T \). So, \( H(X_C) = H(Y_C^T) \) by Point 1. However, it is easy to see that \( C^T = (B^T)^T = B \) and, thus, that \( H(Y_B) = H(Y_B^T) = H(X_C) = H(X_B^T) \).

3. Because of Lemma 1 we know that \( p_{X_A^A}(y, x) = p_{X_B^T}Y_B^T(x, y) \) for any \( x, y \in [1, n] \). It follows that

\[ H(X_A Y_A) = -\sum_{x=1}^{n} \sum_{y=1}^{n} p_{X_A Y_A}(y, x) \log_2 p_{X_A Y_A}(y, x) \]
\[ = -\sum_{x=1}^{n} \sum_{y=1}^{n} p_{X_B^T}Y_B^T(x, y) \log_2 p_{X_B^T}Y_B^T(x, y) = H(X_A X_Y) \]

This ends the proof of the claim.

From Lemma 2 trivially follows the following claim.

**Proposition 2.** Let \( B \) be an \( n \times n \)-agreement matrix such that \( B[y][x] > 0 \) for all rows \( y \) and for all columns \( x \) in \( B \). It holds that:

- \( MI(X_B^T, Y_B^T) = MI(X_B, Y_B) \);
- \( \min\{H(X_B), H(Y_B)\} = \min\{H(X_B^T), H(Y_B^T)\} \);
- \( IA(B) = IA(B^T) \).

**Proof.** Due of Lemma 2 and Eq. 3 it is easy to see that, for any \( n \times n \)-matrix \( B \) whose values are all positive, both \( MI(X_B^T, Y_B^T) \) equals \( MI(X_B, Y_B) \) and \( \min\{H(X_B), H(Y_B)\} \) equals \( \min\{H(X_B^T), H(Y_B^T)\} \). Moreover, both \( H(X_B) \) and \( H(Y_B) \) are well-defined because \( B[y][x] > 0 \) for all rows \( y \) and for all columns \( x \) in \( B \) by hypothesis. Hence, since \( n > 1 \) by assumption both \( H(X_B) \) and \( H(Y_B) \) are greater than 0 by Prop. 1 and so \( \min\{H(X_B), H(Y_B)\} \) is. Because of the definition of \( IA \) (see Eq. 5), the claim directly follows.

When the function \( IA(A_x) \) is studied, \( H(X_{A_x}) \) can be assumed to be smaller than or equal to \( H(Y_{A_x}) \) without any loss of generality. Indeed, if this is not the case — i.e., if \( H(Y_{A_x}) < H(X_{A_x}) \) —, the function \( IA(A_x^T) \), which equals \( IA(A_x) \) by Prop. 2 can be considered in place of \( IA(A_x) \) itself, and, by Lemma 2 we know that \( H(X_{A_x^T}) = H(Y_{A_x}) < H(X_{A_x}) = H(Y_{A_x^T}) \) will hold.
If \( H(X_A) \leq H(Y_A) \), then \( H(X_A) = \min\{H(X_A), H(Y_A)\} \). Thus, by Eq. 4 and Eq. 5 \( IA(A) = 1 + (H(Y_A) - H(X_A, Y_A))/H(X_A) \) and, because of continuity of + on \( \mathbb{R} \times \mathbb{R} \), if \( IA(\epsilon) \) exists, then
\[
IA(\epsilon) = 1 + \lim_{\epsilon \to 0^+} \frac{H(Y_A) - H(X_A, Y_A)}{H(X_A)}.
\]

In order to evaluate above formula, let us first introduce a function to restrict the domain of a generic random variable to those values that have probability greater than 0.

**Definition 1.** Let \( Z \) be a random variable getting values from \( Z \) and such that \( p_Z(z) \) is the probability for \( Z \) to have the value \( z \in Z \).

The refined random variable of \( Z \), denoted by \( \overline{Z} \), is a random variable getting values from the set \( \overline{Z} \) \( \defeq \{ z \in Z | p_Z(z) > 0 \} \) which contains all the values in \( Z \) that have non-null probability with respect to \( p_Z(\cdot) \).

It is worth to notice that \( p_Z(z) = p_{\overline{Z}}(z) \) for any value in the domain of \( \overline{Z} \).

The following proposition relates the entropy functions associated to \( X_A \), \( Y_A \), and \( X_A, Y_A \) to those associated to \( \overline{X}_A, \overline{Y}_A \), and \( \overline{X}_A \overline{Y}_A \), respectively.

**Proposition 3.** Let \( B \) an agreement matrix. The following equation holds:
\[
\bullet \quad \lim_{\epsilon \to 0^+} H(X_{B_\epsilon}) = H(\overline{X}_B)
\]
\[\bullet \quad \lim_{\epsilon \to 0^+} H(Y_{B_\epsilon}) = H(\overline{Y}_B)
\]
\[\bullet \quad \lim_{\epsilon \to 0^+} H(X_{B_\epsilon}, Y_{B_\epsilon}) = H(\overline{X}_B \overline{Y}_B)
\]

**Proof.** Let us focus on the first equation: the correctness of the other two equations can be proved in an analogous way. By definition,
\[
H(X_{B_\epsilon}) \defeq -\sum_{x=1}^{n} p_{X_{B_\epsilon}}(x) \log_2 p_{X_{B_\epsilon}}(x)
\]

Thus, by the continuity of both + and * on \( \mathbb{R} \times \mathbb{R} \),
\[
\lim_{\epsilon \to 0^+} H(X_{B_\epsilon}) = -\sum_{x=1}^{n} \lim_{\epsilon \to 0^+} (p_{X_{B_\epsilon}}(x) \log_2 p_{X_{B_\epsilon}}(x)).
\]

However, we know that \( p_{X_{B_\epsilon}}(x) \defeq S_{B_\epsilon}^X(x)/S_{B_\epsilon} \) and that \( S_{B_\epsilon}^X(x) \defeq \sum_{y=1}^{n} B_\epsilon[y, x] \) and \( S_{B_\epsilon} \defeq \sum_{y=1}^{n} \sum_{x=1}^{n} B_\epsilon[y, x] \). Since all the values in \( B \) are non-negative, all the non-symbolic values in \( B_\epsilon \) are positive by construction. Thus, because of the continuity of + on \( \mathbb{R} \times \mathbb{R} \), \( \lim_{\epsilon \to 0^+} S_{B_\epsilon} = S_B \) and, since we assumed that every agreement matrix contains at least one non-null value, \( S_B > 0 \). Analogously, \( \lim_{\epsilon \to 0^+} S_{B_\epsilon}^X(x) = S_B^X(x) \) and \( S_B^X(x) \geq 0 \). So, due to the continuity of / on \( \mathbb{R} \times \mathbb{R} > 0 \), \( \lim_{\epsilon \to 0^+} p_{X_{B_\epsilon}}(x) = 0 \) if and only if \( S_B^X(x) = n \epsilon + \) or, equivalently, if and only if \( S_B^X(x) = 0 \).
Lemma 3. Let \( p \) because of the continuity of both * on \( \mathbb{R} \times \mathbb{R} \) and log on \( \mathbb{R} \times \mathbb{R}_{>0} \). If instead \( S_B^p(x) = 0 \), it easy to prove, by using the de l’Hôpital’s rule, that the limit for \( p_{X_B^*}(x) \) as \( \epsilon \) tends to 0 from the right is 0.

It follows that

\[
\lim_{\epsilon \to 0^+} H(X_B) = -\sum_{x=1}^n \lim_{\epsilon \to 0^+} \left( p_{X_B^*}(x) \log_2 p_{X_B^*}(x) \right)
= -\left( \sum_{x \in \{1, n\}} p_{X_B^*}(x) \log_2 p_{X_B^*}(x) \right) - \sum_{x \in \{1, n\} \setminus \{1, n\}} 0,
\]

where \( \{1, n\} \) is the set \( \{ x \in [1, n] \mid p_{X_B^*}(x) > 0 \} \), and, by definition of \( X_B \),

\[
\lim_{\epsilon \to 0^+} H(X_B) = H(X_B^*).
\]

This concludes the proof for the first equation in the claim. The proof of the correctness of the remaining equations is analogous.

Thanks to the continuity of both * on \( \mathbb{R} \times \mathbb{R} \) and / on \( \mathbb{R} \times \mathbb{R}_{>0} \), Prop. \( \star \) proves that, whenever \( H(X_A^*) \) is greater than 0 and smaller than \( H(Y_A) \), \( IA_e(A) \) exists and it can be easily computed as \( IA_e(A) = 1 + (H(Y_A) - H(X_A^*) - H(X_A^*)/H(X_A)) \). This statement is summarized in the following theorem.

**Theorem 1.** Let \( B \) be an \( n \times n \)-agreement matrix. If \( 0 < H(X_B^*) \leq H(Y_B) \), then \( IA_e(B) \) exists and it equals:

\[
IA_e(B) = 1 + \frac{H(Y_B^*) - H(X_B^*Y_B^*)}{H(X_B^*)}.
\]

Intriguingly, Lemma \( \star \) can be extended to deal with refined random variables.

**Lemma 3.** Let \( B \) be an agreement matrix. The following equalities hold:

1. \( H(X_B^*) = H(Y_B^*) \);
2. \( H(Y_B^*) = H(X_B^*) \);
3. \( H(X_B^*Y_B) = H(Y_B^*X_B) \).

**Proof.** By Prop. \( \star \) \( H(X_B^*), H(Y_B^*), \) and \( H(X_B^*Y_B) \) equal the limits as \( \epsilon \) tends to 0 from the right for \( H(X_B^*), H(Y_B^*), \) and \( H(X_B^*Y_B^*) \), respectively.

However, by Lemma \( \star \) \( H(X_B^*), H(Y_B^*), \) and \( H(Y_B^*X_B^*) \) for any \( \epsilon > 0 \).

By Prop. \( \star \) \( H(X_B^*Y_B), H(Y_B^*X_B), \) and \( H(X_B^*Y_B^*) \) equal the limits as \( \epsilon \) tends to 0 from the right for \( H(Y_B^*), H(Y_B^*), \) and \( H(Y_B^*Y_B^*) \), respectively.

This concludes the proof of the claim. \( \square \)
Thanks to Lemma 3, it is easy to see that $IA(A) = IA(A^T)$. Moreover, if $H(X_A) > H(Y_A)$, then $H(X_A^T) < H(Y_A^T)$ by the same lemma. Hence, Theorem 4 deals with all the agreement matrices $A$ for which both $H(X_A)$ and $H(Y_A)$ are greater than 0.

In order to complete our analysis, it is worth to understand under which conditions $H(X_A)$ equals 0. By definition of entropy,

$$H(X_A) \overset{\text{def}}{=} -\sum_{x \in [1,n]} p_{X_A}(x) \log_2 p_{X_A}(x)$$

where $[1,n] \overset{\text{def}}{=} \{ x \in [1,n] \mid p_{X_A}(x) > 0 \}$. Since $\sum_{x \in [1,n]} p_{X_A}(x) = 1$ by definition of probability, $H(X_A) = 0$ if and only if $[1,n]$ contains exclusively one column $\overline{x}$ whose probability is 1, i.e., $p_X(\overline{x}) = p_{X_A}(\overline{x}) = 1$. Because of the definition of $p_{X_A}(x)$, this means that $\overline{x}$ is the only column in $A$ whose values are not all 0 or, equivalently, that $\overline{x}$ is the only non-null column in $A$. Thus, to prove the existence of $IA(A)$ for any agreement matrix $A$, we need to solve Eq. (7) when $A$ is a generic agreement matrix having exclusively one non-null column or row. As already observed above, the two cases are symmetrical and we can focus on one of the two cases. Let us consider an agreement matrix having exclusively one non-null column and $m$ non-null rows. For the sake of simplicity and without any loss in generality, we will impose that the values different from 0 are those contained in the column 1 and in the first $m$ rows as in the matrix $A^*$ depicted by Table 13. This assumption does not weaken the generality of the considered case because the entropy functions and, consequently, the information agreement do not take into account the position of classification events in the agreement matrix, but exclusively their probabilities.

Table 13 reports the 0-freed matrix of $A^*$. Since $S_{A^*}^X(x) \overset{\text{def}}{=} \sum_{y=1}^n A_{xy}^*$ and $S_{A^*}^Y(y) \overset{\text{def}}{=} \sum_{x=1}^n A_{xy}^*$ by definition, it is easy to see that

$$S_{A^*}^X(x) = \begin{cases} (n - m) \epsilon + \sum_{y=1}^m a_{xy} & \text{if } x = 1 \\ n \epsilon & \text{otherwise} \end{cases}$$

(8)
and, analogously,

\[ S^*_Y(y) = \begin{cases} 
(n - 1) \epsilon + a_y & \text{if } y \in [1, m] \\
 n \epsilon & \text{otherwise}
\end{cases} \tag{9} \]

As far as \( S^*_A \) may concern, it is easy to see that \( S^*_A = (n^2 - m) \epsilon + \sum_{y=1}^{m} a_y \).

The following preparatory lemma is meant to syntactically simplify Eq. 7.

**Lemma 4.** Let \( Z \) be a random variable that assumes values in \( Z \) and let \( p_Z(z) \) be the probability for \( Z \) to get the value \( z \).

If \( p_Z(z) = f(z)/c \), where \( c \in \mathbb{R} \setminus \{0\} \) is a constant value and \( f : Z \to \mathbb{R} \) is function such that \( \sum_{z \in Z} f(z) = c \), then the following equation holds:

\[ H(Z) = \log_2 c - \frac{1}{c} \sum_{z \in Z} f(z) \log_2 f(z). \tag{10} \]

**Proof.** Since \( H(Z) \overset{\text{def}}{=} -\sum_{z \in Z} p_Z(z) \log_2 p_Z(z) \) by definition, it holds that

\[
H(Z) = -\sum_{z \in Z} f(z) \log_2 \left( \frac{f(c)}{c} \right)
= -\sum_{z \in Z} f(z) \log_2 c
= \frac{1}{c} \left( \sum_{z \in Z} f(z) \log_2 c - \sum_{z \in Z} f(z) \log_2 f(z) \right)
= \frac{1}{c} \left( \sum_{z \in Z} f(z) \log_2 c - \sum_{z \in Z} f(z) \log_2 f(z) \right)
\]

However, \( \sum_{z \in Z} f(z) = c \) by hypothesis and, then,

\[ H(Z) = \log_2 c - \frac{1}{c} \sum_{z \in Z} f(z) \log_2 f(z) \]

This concludes the proof of the claim. \( \square \)

It is easy to see that if \( B \) is an \( n \times n \)-agreement matrix (potentially, also 0-freed), then the variable \( X_B, Y_B, X_B Y_B \) satisfy the conditions of Lemma 4 and the equations

\[ H(X_B) = \log_2 S_B - \frac{1}{S_B} \sum_{x=1}^{n} S^X_B(x) \log_2 S^X_B(x), \tag{11} \]
\[ H(Y_B) = \log_2 S_B - \frac{1}{S_B} \sum_{y=1}^{n} Y_B^y(y) \log_2 S_B^y(y), \] (12)

and

\[ H(X_BY_B) = \log_2 S_B - \frac{1}{S_B} \sum_{y=1}^{n} \sum_{x=1}^{n} B[y][x] \log_2 B[y][x], \] (13)

hold.

Let us introduce the shortcuts \( G^\uparrow(B) \triangleq (\ln 2) \cdot S_B \cdot (H(Y_B) - H(X_BY_B)) \), \( G_\downarrow(B) \triangleq (\ln 2) \cdot S_B \cdot H(X_B) \), and \( G(B) \triangleq G^\uparrow(B)/G_\downarrow(B) \). It is worth to notice that, for all \( \epsilon > 0 \), \( G(A^\epsilon) = (H(Y_{A^\epsilon}) - H(X_{A^\epsilon}Y_{A^\epsilon}))/H(X_{A^\epsilon}) \) because \( S_{A^\epsilon} > 0 \) for the same values of \( \epsilon \) and, thus,

\[ IA_\epsilon(A^\epsilon) = 1 + \lim_{\epsilon \to 0^+} \frac{H(Y_{A^\epsilon}) - H(X_{A^\epsilon}Y_{A^\epsilon})}{H(X_{A^\epsilon})} = 1 + \lim_{\epsilon \to 0^+} G(A^\epsilon) \] (14)

From Eq.12, Eq.13, and Eq.9 we can deduce that

\[
G^\uparrow(A^\epsilon) = (\ln 2) \cdot S_{A^\epsilon} \cdot (H(Y_{A^\epsilon}) - H(X_{A^\epsilon}Y_{A^\epsilon})) \\
= \sum_{y=1}^{n} \sum_{x=1}^{n} A^\epsilon[y][x] \ln A^\epsilon[y][x] - \sum_{y=1}^{n} S_{A^\epsilon}^y(y) \ln S_{A^\epsilon}^y(y) \\
= \left( \sum_{y=1}^{m} a_y \ln a_y \right) + (n^2 - m) \cdot \epsilon \cdot \ln \epsilon + \\
- \left( \sum_{y=1}^{m} (a_y + (n - 1) \cdot \epsilon) \ln (a_y + (n - 1) \cdot \epsilon) \right) + \\
- (n - m) \cdot (n \cdot \epsilon) \ln (n \cdot \epsilon) \\
= \left( \sum_{y=1}^{m} a_y \ln a_y \right) + (n - 1) \cdot m \cdot \epsilon \cdot \ln \epsilon + \\
- \left( \sum_{y=1}^{m} (a_y + (n - 1) \cdot \epsilon) \ln (a_y + (n - 1) \cdot \epsilon) \right) + \\
- (n - m) \cdot (n \cdot \ln n) \cdot \epsilon.
\]
Analogously, from Eq. 11 and Eq. 8 it follows that:
\[
G_{i}(A^*_i) = (\ln 2) \times S_{A^*_i} \times H(X_{A^*_i})
\]
\[
= S_{A^*_i} \times \left( \ln S_{A^*_i} - \frac{1}{S_{A^*_i}} \times \sum_{x=1}^{n} S_{A^*_i}(x) \times \ln S_{A^*_i}(x) \right)
\]
\[
= S_{A^*_i} \times \ln S_{A^*_i} - \sum_{x=1}^{n} S_{A^*_i}(x) \times \ln S_{A^*_i}(x)
\]
\[
= \left( n^2 - m \right) \times \left( \sum_{y=1}^{m} a_y \right) \times \ln \left( n^2 - m + \sum_{y=1}^{m} a_y \right) + \\
- \left( n - m \right) \times \left( \sum_{y=1}^{m} a_y \right) \times \ln \left( n - m + \sum_{y=1}^{m} a_y \right) + \\
- (n - 1) \times n \times \epsilon \times \ln n \times \epsilon
\]

Due to the continuity of + and $\times$ on $\mathbb{R} \times \mathbb{R}$ and that of log on $\mathbb{R} \times \mathbb{R}_{>0}$,
\[
\lim_{\epsilon \to 0^+} G^+_i(A^*_i) = \left( \sum_{y=1}^{m} \lim_{\epsilon \to 0^+} a_y \times \ln a_y \right) + (n - 1) \times m \times \lim_{\epsilon \to 0^+} \epsilon \times \ln \epsilon + \\
- \left( \sum_{y=1}^{m} \lim_{\epsilon \to 0^+} (a_y + (n - 1) \times \epsilon) \times \ln (a_y + (n - 1) \times \epsilon) \right) + \\
- (n - m) \times (n \times \ln n) \times \lim_{\epsilon \to 0^+} (\epsilon)
\]
\[
= \left( \sum_{y=1}^{m} a_y \times \ln a_y \right) + 0 - \left( \sum_{y=1}^{m} a_y \times \ln a_y \right) - 0 - 0
\]
\[
= 0
\]

and, in the same way,
\[
\lim_{\epsilon \to 0^+} G^+_i(A^*_i) = \lim_{\epsilon \to 0^+} \left( n^2 - m \right) \times \left( \sum_{y=1}^{m} a_y \right) \times \ln \left( n^2 - m + \sum_{y=1}^{m} a_y \right) + \\
- \lim_{\epsilon \to 0^+} \left( n - m \right) \times \left( \sum_{y=1}^{m} a_y \right) \times \ln \left( n - m + \sum_{y=1}^{m} a_y \right) + \\
- (n - 1) \times n \times \lim_{\epsilon \to 0^+} \epsilon \times \ln \epsilon - (n - 1) \times (n \times \ln n) \times \lim_{\epsilon \to 0^+} \epsilon
\]
The derivative of $G(A_\epsilon^*)$ is:

$$\frac{\partial G_i(A_\epsilon^*)}{\partial \epsilon} = (n^2 - m) \ln \left( (n^2 - m) * \epsilon + \sum_{y=1}^{m} a_y \right) + n^2 - m +$$

$$- (n - m) \ln \left( (n - m) * \epsilon + \sum_{y=1}^{m} a_y \right) - (n - m) +$$

$$- (n - 1) n \ln \epsilon - (n - 1) * n - (n - 1) * n * \ln n$$

$$= (n^2 - m) \ln \left( (n^2 - m) * \epsilon + \sum_{y=1}^{m} a_y \right) - (n^2 - n) * \ln \epsilon +$$

$$- (n - m) \ln \left( (n - m) * \epsilon + \sum_{y=1}^{m} a_y \right) - (n^2 - n) * \ln n$$

and the limit for it as $\epsilon$ tends to 0 is:

$$\lim_{\epsilon \to 0^+} \frac{\partial G_i(A_\epsilon^*)}{\partial \epsilon} = (n^2 - m) \lim_{\epsilon \to 0^+} \ln \left( (n^2 - m) * \epsilon + \sum_{y=1}^{m} a_y \right) +$$

$$- (n^2 - n) \lim_{\epsilon \to 0^+} \ln \epsilon - (n^2 - n) * \ln n$$

$$- (n - m) \lim_{\epsilon \to 0^+} \ln \left( (n - m) * \epsilon + \sum_{y=1}^{m} a_y \right) = \infty,$$
hence, we can apply the de l’Hôpital’s rule. The derivative of $G^\uparrow(A^*_*)$ on $\epsilon$ is

$$\frac{\partial G^\uparrow(A^*_*)}{\partial \epsilon} = 0 + (n - 1) \cdot m \cdot \ln \epsilon + (n - 1) \cdot m +$$

$$- \sum_{y=1}^{m} ((n - 1) \cdot \ln ((n - 1) + \epsilon + a_y) + (n - 1)) +$$

$$- (n - m) \cdot n \cdot \ln n$$

$$= (n - 1) \cdot m \cdot \ln \epsilon - (n - m) \cdot n \cdot \ln n +$$

$$- (n - 1) \cdot \sum_{y=1}^{m} \ln ((n - 1) + \epsilon + a_y).$$

The two derivatives do not share any common factor and they cannot be simplified. Moreover, the limit for $G(A^*_*)$ cannot be evaluated as the ratio between the limits of the derivatives of $G^\uparrow(A^*_*)$ and $G_\downarrow(A^*_*)$ because it has the form $-\infty/\infty$, which is indeterminate. As a matter of fact,

$$\lim_{\epsilon \to 0^+} \frac{\partial G^\uparrow(A^*_*)}{\partial \epsilon} = (n - 1) \cdot m \cdot \lim_{\epsilon \to 0^+} (\ln \epsilon) - (n - m) \cdot n \cdot \ln n +$$

$$- (n - 1) \cdot \sum_{y=1}^{m} \lim_{\epsilon \to 0^+} \ln ((n - 1) + \epsilon + a_y) = -\infty,$$

Luckily, de l’Hôpital’s rule can be applied again because the second derivative of $G_\downarrow(A^*_*)$ on $\epsilon$ is:

$$\frac{\partial^2 G_\downarrow(A^*_*)}{\partial \epsilon^2} = \frac{(n^2 - m)^2}{(n^2 - m) \cdot \epsilon + \sum_{y=1}^{m} a_y} - \frac{n^2 - n}{\epsilon} +$$

$$- \frac{(n - m)^2}{(n - m) \cdot \epsilon + \sum_{y=1}^{m} a_y}$$

$$= - \frac{(n^2 - n) \cdot \sum_{y=1}^{m} a_y}{\epsilon \cdot ((n^2 - m) \cdot \epsilon + \sum_{y=1}^{m} a_y) \cdot ((n - m) \cdot \epsilon + \sum_{y=1}^{m} a_y)},$$

and the limit for it as $\epsilon$ tends to 0 is:

$$\lim_{\epsilon \to 0^+} \frac{\partial^2 G_\downarrow(A^*_*)}{\partial \epsilon^2} = \lim_{\epsilon \to 0^+} \frac{(n^2 - m)^2}{(n^2 - m) \cdot \epsilon + \sum_{y=1}^{m} a_y} - \lim_{\epsilon \to 0^+} \frac{(n^2 - n)}{\epsilon} +$$

$$- \lim_{\epsilon \to 0^+} \frac{(n - m)^2}{(n - m) \cdot \epsilon + \sum_{y=1}^{m} a_y}$$

$$= \frac{(n^2 - m)^2}{\sum_{y=1}^{m} a_y} - \infty - \frac{(n - m)^2}{\sum_{y=1}^{m} a_y}$$

$$= -\infty.$$
So, there exists a right-neighbourhood of $\epsilon = 0$ such that none of its values is mapped in 0 through the second derivative of $G_i(A^*_i)$.

The ratio between $\frac{\partial^2 G_i(A^*_i)}{\partial \epsilon^2}$ and $\frac{\partial^2 G_j(A^*_j)}{\partial \epsilon^2}$ can be algebraically simplified because they both have $1/\epsilon$ as a factor. As a matter of fact,

$$
\frac{\partial^2 G_i(A^*_i)}{\partial \epsilon^2} = \frac{(n - 1) * m}{\epsilon} - (n - 1) * \sum_{y=1}^{m} \frac{n - 1}{(n - 1) * \epsilon + a_y}
$$

$$
= \frac{(n - 1) * m - \epsilon * \sum_{y=1}^{m} \frac{(n-1)^2}{(n-1)\epsilon+a_y}}{\epsilon}
$$

and

$$
\frac{\partial^2 G_j(A^*_j)}{\partial \epsilon^2} = -\frac{(n - 1) * m - \epsilon * \sum_{y=1}^{m} \frac{(n-1)^2}{(n-1)\epsilon+a_y}}{\epsilon} * \frac{\epsilon * \left(\frac{(n^2 - m) \epsilon + \sum_{y=1}^{m} a_y}{(n - 1) n \sum_{y=1}^{m} a_y} + \frac{(n - 1) m (n^2 - m) (n - m)}{(n - 1) n \sum_{y=1}^{m} a_y} + \frac{(n - 1) m (n^2 - m) \sum_{y=1}^{m} a_y}{(n - 1) n \sum_{y=1}^{m} a_y} + \frac{(n - 1) m (\sum_{y=1}^{m} a_y) (n - m)}{(n - 1) n \sum_{y=1}^{m} a_y} + \frac{(n - 1) m (n^2 - m) (n - m)}{(n - 1) n \sum_{y=1}^{m} a_y} + \frac{(n - 1) m (\sum_{y=1}^{m} a_y) (n - m)}{(n - 1) n \sum_{y=1}^{m} a_y}}{\epsilon^3} \sum_{y=1}^{m} a_y + 1}
$$

The first term of Eq. 16 equals $-m/n$, while each of the remaining terms
has instead the form
\[
\epsilon^c \cdot \left( \sum_{y=1}^{m} \frac{(n-1)^2}{(n-1)^{c+1} + n_y} \right)^d \cdot p(n, m, a_1, \ldots, a_m) \frac{(n-1) \cdot n \cdot \sum_{y=1}^{m} a_y}{n^2}
\]
for suitable natural numbers \(c \in \{1, 2, 3\}\) and \(d \in \{0, 1\}\) and fitting polynomial function \(p(n, m, a_1, \ldots, a_m)\). Since \(p(n, m, a_1, \ldots, a_m)\) is constant with respect \(\epsilon\) and, under the assumptions we made for \(A_\epsilon^*\), \((n-1) \cdot n \cdot \sum_{y=1}^{m} a_y\) is a positive real value, it is easy to see that the limit as \(\epsilon\) tends to 0 for each of the terms of Eq. 16, but the first one, is 0. It follows that,
\[
\lim_{\epsilon \to 0^+} G(A_\epsilon^*) = \lim_{\epsilon \to 0^+} \frac{\partial^2 G(A_\epsilon^*)}{\partial \epsilon^2} = -\frac{m}{n}
\]
and the following theorem holds.

**Theorem 2.** Let \(B\) be an \(n \times n\)-agreement matrix. If \(H(X_B) = 0\) and \(B\) accounts exactly \(m\) non-null rows, then \(IA_\epsilon(B)\) exists and it equals \((n - m)/n\).

**Proof.** The proof directly follows from both Eq. 14 and Eq. 17.

Since, whenever defined, \(IA\) is symmetric with respect to transposition, i.e., \(IAB = IAB^T\), we can prove the following corollary.

**Corollary 1.** Let \(B\) be an \(n \times n\)-agreement matrix. The information agreement extension by continuity of \(B\), \(IA_\epsilon(B)\), does exist. Moreover, if \(l\) and \(m\) are numbers of non-null columns and non-null rows in \(B\), respectively, then
\[
IA_\epsilon(B) = \begin{cases} 
\frac{n-l}{n} & \text{if } H(Y_B) = 0 \\
\frac{n-m}{n} & \text{if } H(X_B) = 0 \\
1 + \frac{H(Y_B) - H(X_B) Y_B}{H(X_B)} & \text{if } 0 < H(X_B) \leq H(Y_B) \\
1 + \frac{H(X_B) - H(X_B) Y_B}{H(Y_B)} & \text{if } 0 < H(Y_B) \leq H(X_B)
\end{cases}
\]

**Proof.** The proof of the claim directly follows from Lemma 3, Theorem 1, and Theorem 2.

### 3 Computing \(IA_\epsilon\)

Corollary 1 not only guarantees the existence of \(IA_\epsilon(A)\) for any agreement matrix \(A\), but also provides an effective way to compute it. Algorithm 1 is the algorithmic counterpart of Corollary 1 and the correctness of the former follows directly from the latter.
As far as the complexity of Algorithm 1 may concern, line 2 can certainly be assumed to take constant time with respect to the size of $A$. It is easy to figure out that lines 3, 7, and 11, which compute $H(X_A)$, $H(Y_A)$, and $H(X_AY_A)$, respectively, take time $\Theta(n^2)$, i.e., their execution times are upper-bounded and lower-bounded by functions proportional to $n^2$ in both best and worst-case scenarios (e.g., see [2]). If $A$ is an $n \times n$ matrix, then both lines 5 and 9 take time $O(n^2)$, i.e., in the worst-case scenario, their execution times are upper-bounded by functions proportional to $n^2$ (e.g., see [2]). All the remaining lines take constant time with respect to the input size. So, the overall cost of Algorithm 1 is $\Theta(n^2)$.

**Algorithm 1:** Computes $IA_e(A)$ for any agreement matrix $A$.

- **Input:** A generic agreement matrix $A$
- **Output:** The value $IA_e(A)$

```python
1 def getIAC(A):
2     n ← A.size  /* get the number of rows/cols in A */
3     HX_R ← H(refine(get_pX(A)))  /* compute $H(X_A)$ */
4     if HX_R = 0 then
5         m ← countNonNullRows(A)  /* count the non-null rows */
6         return (n − m)/n
7     HY_R ← H(refine(get_pY(A)))  /* compute $H(Y_A)$ */
8     if HY_R = 0 then
9         l ← countNonNullCols(A)  /* count the non-null cols */
10        return (n − l)/n
11     HXY_R ← H(refine(get_pXY(A)))  /* compute $H(X_AY_A)$ */
12     if HX_R < HY_R then
13         return 1 + (HY_R − HXY_R)/HX_R
14     else
15        return 1 + (HX_R − HXY_R)/HY_R
```

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