Weyl-Heisenberg Spaces for Robust Orthogonal Frequency Division Multiplexing

Zoran Cvetković\textsuperscript{a,*}, Vincent Sinn\textsuperscript{b}

\textsuperscript{a}Department of Informatics, King’s College London, Strand, London WC2R 2LS, UK, E-mail: zoran.cvetkovic@kcl.ac.uk, Phone: + 44 20 7848 2858, Fax: +44 20 7848 2932
\textsuperscript{b}Telecommunications Laboratory, University of Sydney, NSW 2006, Australia, E-mail: cvsinn@ee.usyd.edu.au

Abstract

Design of Weyl-Heisenberg sets of waveforms for robust orthogonal frequency division multiplexing (OFDM) has been the subject of a considerable volume of work. In this paper, a complete parameterization of orthogonal Weyl-Heisenberg sets and their corresponding biorthogonal sets is given. Several examples of Weyl-Heisenberg sets designed using this parameterization are presented, which in simulations show a high potential for enabling OFDM robust to frequency offset, timing mismatch, and narrow-band interference.

Keywords: OFDM, robustness, pulse-shape design.
1. Introduction

Orthogonal frequency division multiplexing (OFDM) is a communication technique which is very effective in reducing or completely eliminating intersymbol interference that arises in multipath propagation channels \[1-5\] and is rapidly emerging as a technology of choice for wireless applications. This robustness to multipath propagation is achieved by means of frequency multiplexing which allows extending intersymbol interval beyond the duration of the channel impulse response at the expense of a small level of redundancy. To that end, a symbol sequence \(a[n]\) is divided into subsequences, 
\[a_k[i] = a[k+iN], \quad k = 0, 1, \ldots, N-1,\]
which are multiplexed in frequency and transmitted in frames of \(N\) symbols with frame intervals of \(K\) samples, where \(K > N\). The transmitted signal thus has the form
\[
s[n] = \sum_{i=-\infty}^{\infty} \sum_{k=0}^{N-1} a_k[i] \varphi_k[n-iK],
\]
where waveforms \(\varphi_k[n]\), \(k = 0, 1, \ldots, N-1\), are commonly complex exponentials
\[
\varphi_k[n] = \frac{1}{\sqrt{N}} e^{j \frac{2\pi}{N} kn}, \quad 0 \leq n \leq N-1.
\]
Subsequences \(a_k[i]\) are demultiplexed at the receiver by projecting \(s[n]\) onto waveforms
\[
\psi_k[n] = \frac{1}{\sqrt{N}} e^{j \frac{2\pi}{N} kn}, \quad 0 \leq n \leq K-1,
\]
as \(a_k[i] = \langle \psi_k[n-iK], s[n] \rangle\). Each waveform \(\psi_k[n-iK]\) is orthogonal not only to all \(\varphi_l[n-jK]\), \(k \neq l, i \neq j\), but also to all their delayed versions \(\varphi_l[n-jK-d]\), for \(d \leq K - N\). In this manner intersymbol interference due to multipath propagation is completely eliminated if the impulse response of the channel does not exceed the guard interval, \(T_g = K - N\). Alternatively, one can use waveforms \(\psi_k[n]\) for the multiplexing and waveforms \(\varphi_k[n+N-K]\) for the demultiplexing. This alternative method is known in communications as the cyclic prefix scheme \(6\), since \(s[n+iK] = s[n+N+iK]\), for \(0 \leq n < K - N\), whereas the design described by \(2\) and \(3\) is referred to as the zero-padding scheme. These two methods are equivalent for all considerations in this paper.

Being so heavily optimized to make the transmission robust to multipath propagation, the scheme is very sensitive to strong narrowband interference, frequency offset and timing mismatch \(7\). The sensitivity to frequency offset and narrowband interference is caused by poor frequency localization of waveforms \(\varphi_k[n]\) and \(\psi_k[n]\), which is a consequence of their short duration and sharp transitions. These sharp transitions are also responsible for the sensitivity to timing mismatch.
Optimal design of OFDM waveforms has been a very active area of research [5, 8–29]. The topic is currently very relevant considering the role OFDM is likely to play in the next generation of wireless communication systems. One strand of work on OFDM waveform design is concerned with continuous-time waveforms, leading to many very insightful results, most importantly demonstrating that in the presence of Doppler spread or frequency offset, waveforms which extend over several frame intervals and attain high spectral containment achieve lower intersymbol (ISI) and interchannel (ICI) interference than short rectangular waveforms, pointing out that a performance can be improved if non-rectangular time-frequency transmission lattices are used [24], and concluding recently that with regard to practical design excellent time-frequency localization of waveforms is the most important requirement for low ISI/ICI [28]. The continuous-time approach, on the other hand, has not provided yet closed form solutions, and optimal or nearly optimal waveforms are designed using numerical procedures which are quite challenging optimization problems per se [29]. Furthermore, for discrete-time implementation these waveforms are sampled and truncated, and that causes a non-negligible departure from the orthogonality and a degradation of frequency localization [22].

These insights and limitations of the continuous-time analysis motivate a purely discrete-time approach pursued in [13, 15–19, 22, 25, 27]. Siohan, Siclet and Lacaille [22], as well as Bölcskei, Duhamel and Hleiss [25], consider offset OFDM with no redundancy, that is, the particular case when $K = N$, where owing to the offset multiplexing good time frequency localization of modulating waveforms is attainable; this is in contrast to the OFDM with no offset where good frequency localization is impossible to achieve unless some redundancy is introduced. Since the primary source of robustness of OFDM to multipath propagation is the redundancy inserted by making the frame interval larger than the number of symbols in a frame [15], herewith we focus on redundant OFDM schemes, i.e. cases with $K > N$. In the direction of designing waveforms for redundant OFDM robust to frequency offset and timing mismatch, a straightforward approach would be to perform optimization of their time-frequency localization under required orthogonality constraints. This would, however, be numerically very intensive and might have convergence problems in case of long waveforms [18]. Bölcskei therefore proposes to orthogonalize well localized waveforms using the discrete Zak transform [18]; the method does not guarantee that the resulting orthogonal waveforms would still have good time-frequency localization, but it gave very good results in examples
with small number of channels and relatively high redundancies. More recently Siclet, Siohan and Pinchon proposed a method for finding orthogonal modulating waveforms [27], and used those solutions to design waveforms via unconstrained optimization. The authors presented impressive design examples, but point out that there is no guarantee that their method works for arbitrary $K$ and $N$ and suggest that low-redundancy systems ($K$ close to $N$) may require a separate treatment.

In this paper, we present a complete parameterization, i.e. the complete set of solutions in a closed form, for waveforms

$$\varphi_k[n] = v[n]e^{j\frac{2\pi k}{N}n}$$

(4)

which satisfy orthogonality conditions

$$\langle \varphi_k[n - iK] , \varphi_l[n - jK] \rangle = \delta[k - l]\delta[i - j].$$

(5)

The parameterization is valid for any arbitrary pair of parameters $K$ and $N$, $K \geq N$, and imposes no restrictions on the length of $v[n]$. When there is redundancy in the system, $K > N$, given a modulating waveform $v[n]$ for which the corresponding waveforms $\varphi_k[n]$ satisfy the orthogonality conditions in (5), there exist infinitely many solutions for a modulating waveform $w[n]$ such that waveforms $\psi_k[n]$,

$$\psi_k[n] = w[n]e^{j\frac{2\pi k}{N}n},$$

(6)

are biorthogonal to corresponding waveforms $\varphi_k[n - iK]$,

$$\langle \psi_k[n - iK] , \varphi_l[n - jK] \rangle = \delta[k - l]\delta[i - j].$$

(7)

These waveforms $\psi_k[n]$ can also be used for perfect demultiplexing instead of waveforms $\varphi_k[n]$ themselves. We give a complete parameterization of such modulating waveforms $w[n]$, as this additional degree of design freedom might result in possible further improvements in system performance; the zero-padding and cyclic prefix schemes are particular cases of the complete set of solutions explored here, obtained when $v[n]$ is the $N$-sample rectangular waveform and $w[n]$ is the $K$-sample rectangular waveform. The parameterization of orthogonal waveforms $\varphi_k[n]$ presented here is a generalization of the idea proposed by Hleiss, Duhamel and Charbit [13], inspired by work of the first author and Vetterli on Weyl-Heisenberg frames in $\ell^2(\mathbb{Z})$ [30]. Hleiss, Duhamel and Charbit observe that the parameterization of tight Weyl-Heisenberg frames in $\ell^2(\mathbb{Z})$ [30] can be used to find solutions for OFDM waveforms and present some design example, but point out that proving...
a general result for arbitrary $K$ and $N$ is quite intricate [13]. The parameterization of OFDM waveforms and corresponding biorthogonal demultiplexing waveforms given here were previously outlined by the first author in conference publications [17, 19]. Considering the increasingly important role OFDM is playing in wireless communication technologies, there is a need for precise, complete, and detailed presentation of those solutions and provided it in this paper.

One may argue that imposing the orthogonality conditions in (5) may preclude exploring the complete set of modulating waveforms, since this orthogonality is not necessary for perfect demultiplexing. As long as waveforms $\varphi_k[n - iK]$ are linearly independent, which is satisfied under very mild conditions on $v[n]$, there exists a modulating waveform $w[n]$ such that its modulated translates $\psi_k[n - iK]$ are biorthogonal to waveforms $\varphi_k[n - iK]$ according to (7). Such waveforms $\psi_k[n]$ can be used for perfect demultiplexing in channels with no multipath propagation even when waveforms $\varphi_k[n - iK]$ are not mutually orthogonal, and this additional design freedom can be used to achieve a better joint time-frequency localization of $v[n]$ and $w[n]$ and thus reduce ISI/ICI. Besides, both orthogonality and biorthogonality are lost in multipath channels except in the particular case of zero-padding and cyclic prefix schemes. The idea of biorthogonal frequency division multiplexing (BFDM) in the continuous-time case was explored in detail in [14] and [28]. The biorthogonal waveforms designed to minimize ISI/ICI in [14] are in fact very close to orthogonal, and the conclusion of the theoretical analysis in [28] was that waveforms which minimize the interference should be very close to orthogonal. A question which arises is whether orthogonal rather than biorthogonal multiplexing is in fact optimal in terms of minimizing the interference and that design examples presented in [14] and [28] were close to orthogonal not because optimal waveforms are biorthogonal but because the numerical procedures approached optimal solutions but did not reach them exactly. Orthogonal multiplexing is also optimal in terms of robustness to additive white Gaussian noise, as it has been established already by Shannon [31], and also proved more recently by Kozek and Molisch [14]. This motivates the focus of this work on orthogonal waveforms, and additional design freedom is provided by solutions for biorthogonal demultiplexing, should there be some merit to it as it happens in the case of the cyclic prefix scheme, or should the transmultiplexer be optimized according to different noise and channel statistics as explored by Scaglione, Giannakis and Barbarossa in the case of waveforms up to $K$ samples long [15].

The parameterization of Weyl-Heisenberg sets for OFDM given here is based on a polyphase
representation of OFDM transmultiplexer, which is reviewed first in Section II. Further parameterization details, the particular form which OFDM waveforms have and the relationship between orthonormal Weyl-Heisenberg sets and tight Weyl-Heisenberg frames are discussed in Section III. Some waveform design issues are discussed in Section IV. Design examples and simulation results are presented in Section V.

2. Polyphase Representation of OFDM

2.1. Transmultiplexer Polyphase Representation

An OFDM multiplexer has the structure of a multirate synthesis filter bank. The filter bank implementing an \(N\)-channel multiplexer with \(K\)-sample frame interval, as shown in Figure 1a), performs \(K\)-point upsampling of input sequences, followed by linear filtering in each of its channels. The multiplexer output is given by

\[
s[n] = \sum_{i=-\infty}^{\infty} \sum_{k=0}^{N-1} a_k[i] \varphi_k[n - iK],
\]

where \(a_k[i]\) and \(\varphi_k[n]\) denote the input sequence and the impulse response of the filter in the channel \(k\), respectively. The treatment of orthogonal frequency division multiplexing in this paper is based on the techniques of polyphase analysis of multirate systems [8, 32].

![Figure 1: Filter banks for implementation of \(N\)-channel OFDM with \(K\)-sample frame interval. a) Multiplexer. b) Demultiplexer.](image)

For the analysis of multiplexers with \(K\)-point upsampling it is convenient to represent the multiplexed signal \(s[n]\) in terms of its \(K\) polyphase components as

\[
S(z) = \sum_{l=0}^{K-1} S_l(z^K)z^{-l},
\]

where

\[
S_l(z) = \sum_{n=-\infty}^{\infty} s[l + nK]z^{-n}.
\]

The polyphase components of \(s[n]\) are related to the input sequences as

\[
[S_0(z) \ldots S_{K-1}(z)]^T = M(z)[A_0(z) \ldots A_{N-1}(z)]^T,
\]

where \(M(z)\) is the \(K \times N\) multiplexer.
Towards finding a parameterization of OFDM pulse shapes, in the next subsection we consider the orthogonality conditions in (5) are equivalent to the paraunitarity of the adjoint of the multiplexer polyphase representation, $\phi_k[n]$ are filters which are complex-conjugated time-reversed versions of the multiplexer waveforms, and $\phi_k[n]$ is the impulse response of the filter in that channel. For $\phi_k[n] = \psi_k[n]$ the output sequence becomes $b_k[i] = \langle \psi_k[n-iK], s[n] \rangle$. To simplify the notation, all following considerations will be expressed in terms of impulse responses $\phi_k[n]$ rather than waveforms $\psi_k[n]$. Output sequences of the demultiplexer are related to the input signal $s[n]$ as $[B_0(z) \ldots B_{N-1}(z)]^T = D(z)[S_0(z) \ldots S_{K-1}(z)]^T$, where $D(z)$ is the polyphase representation of the demultiplexer, given by $[D(z)]_{k,l} = \sum_{n=-\infty}^{+\infty} \phi_k[-l+nK]z^{-n}$, $0 \leq k \leq N-1$, $0 \leq l \leq K-1$, and $B_k(z)$ is the $z$-transform of the sequence $b_k[i]$. Sequences $b_k[i]$ are perfectly demultiplexed sequences $a_k[i]$ if and only if the polyphase demultiplexer matrix is a left inverse of the polyphase multiplexer matrix, $D(z)M(z) = I$, where $I$ denotes the $N \times N$ identity matrix, or equivalently if and only if the following biorthogonality relationships hold \[8, 32\]:

$$\sum_n \phi_k[jK-n]|\varphi_l[n-iK] = \delta[k-l]\delta[i-j].$$

When the multicarrier scheme is redundant, $K > N$, and waveforms $\varphi_k[n-iK]$, $k = 0, 1, \ldots N - 1$, $i \in \mathbb{Z}$, are linearly independent, the polyphase multiplexer matrix $M(z)$ has infinitely many left inverses $D(z)$, that is, there exist infinitely many filters $\phi_k[n]$, $k = 0, 1, \ldots , N - 1$, which can be used for perfect demultiplexing. In the case when the multiplexer waveforms $\varphi_k[n]$ satisfy the orthogonality conditions in (5), one solution for filters which would achieve perfect demultiplexing are filters which are complex-conjugated time-reversed versions of the multiplexer waveforms, $\phi_k[n] = \varphi_k^*[n]$. In that case, the polyphase representation of the demultiplexer is the Hilbert adjoint of the multiplexer polyphase representation, $D(z) = \tilde{M}(z)$, which further means that the orthogonality conditions in (5) are equivalent to the paraunitarity of $M(z)$, $\tilde{M}(z)M(z) = I$. Towards finding a parameterization of OFDM pulse shapes, in the next subsection we consider the particular form $D(z)$ and $M(z)$ have in the case of OFDM.

---

\[1\] Throughout the paper, matrix rows and columns will be indexed starting with zero.

\[2\] $\tilde{M}(z)$ denotes the matrix obtained by transposing $M(z)$, conjugating all the coefficients of the polynomials in $M(z)$, and replacing $z$ by $z^{-1}$.
2.2. Polyphase Representation of OFDM Transmultiplexer

Waveforms in an OFDM multiplexer are modulated complex exponentials, 
\( \varphi_k[n] = v[n]e^{j2\pi kn} \), and then the polyphase multiplexer representation has a particular form established by the following theorem.

**Theorem 1.** Consider an \( N \)-channel OFDM multiplexer with \( K \)-sample frame interval, based on a modulating waveform \( v[n] \). Let \( M \) be the least common multiple of \( K \) and \( N \), and let \( V_j(z) \), \( j = 0, 1, \ldots, M - 1 \), be the components of the \( M \)-component polyphase representation of \( v[n] \),

\[
V_j(z) = \sum_{n=-\infty}^{\infty} v[j + nM]z^{-n}.
\] (9)

The polyphase representation of this multiplexer has the form

\[
M(z) = V(z)F_{N},
\] (10)

where \( F_{N} \) is the \( N \)-point discrete-Fourier transform matrix, \( [F_{N}]_{m,n} = e^{j2\pi mn} \), \( 0 \leq m \leq N-1 \), \( 0 \leq n \leq N-1 \), and \( V(z) \) is the \( K \times N \) matrix of the polyphase components of \( v[n] \) given by

\[
[V(z)]_{l,m} = 
\begin{cases} 
  z^{-p}V_{pK+iP+r}(z^J), & (l, m) = (iP + r, jP + r) \\
  0, & \text{otherwise}
\end{cases},
\] (11)

\( i = 0, 1, \ldots, L - 1 \), \( j = 0, 1, \ldots, J - 1 \), \( r = 0, 1, \ldots, P - 1 \), where \( P \) is the greatest common divisor of \( N \) and \( K \), \( J = N/P \), \( L = K/P \) and \( p \) is the integer such that

\[
j \equiv pL + i \pmod{J}.
\] (12)

**Proof:** To show that \( M(z) \) has this particular form, consider

\[
[M(z)]_{l,k} = \sum_{n=-\infty}^{+\infty} \varphi_k[l + nK]z^{-n} = \sum_{n=-\infty}^{+\infty} v[l + nK]e^{j2\pi k(l+nK)}z^{-n}.
\]

By representing \( n \) as \( n = qJ + p \), \( 0 \leq p \leq J - 1 \), \( q \in \mathbb{Z} \), we obtain

\[
[M(z)]_{l,k} = \sum_{p=0}^{J-1} e^{j2\pi k(l+pK)}z^{-p} \sum_{q=-\infty}^{+\infty} v[l + pK + qJK]z^{-qJ}.
\]

This gives

\[
[M(z)]_{l,k} = \sum_{p=0}^{J-1} e^{j2\pi k(l+pK)}z^{-p}V_{pK+l}(z^J).
\]
Consequently, we obtain
\[ [M(z)]_{l,k} = \sum_{p=0}^{J-1} e^{i\frac{2\pi}{N} kp(l,p)} z^{-p} V_{pK+l}(z^J) , \]
where \( m(l,p) \equiv pK + l \mod N \). Hence, the \( l \)-th row of \( V(z) \) has \( J \) nonzero entries, \( z^{-p} V_{pK+l}(z^J) \), \( p = 0,1, \ldots, J-1 \), positioned so that \( z^{-p} V_{pK+l}(z^J) \) is in the column \( m \equiv pK + l \mod N \), or equivalently, in the column \( m \) for which there exists an integer \( q \) such that
\[ qN + m = pK + l. \] (13)

By representing \( N = JP \) and \( K = LP \) the equality in (13) becomes \( qJP + m = pLP + l \) which is equivalent to \( m \equiv l \mod P \). Hence nonzero entries of \( V(z) \) appear only at locations \((l,m) = (iP + r, jP + r)\), and the entry at \((l,m) = (iP + r, jP + r)\) is \( z^{-p} V_{pK+l} = z^{-p} V_{pK+iP+r} \), where \( p \) is the integer which satisfies (13). By expressing \( l = iP + r, m = jP + r, N = JP \) and \( K = LP \) the congruence in (13) reduces to \( qJ + j = pL + i \) which is equivalent to \( j \equiv pL + i \mod J \).

Thus only entries \([V(z)]_{l,m} \) at locations \([V(z)]_{iP+r,jP+r} \), \( i = 0,1, \ldots, L-1 \), \( j = 0,1, \ldots, J-1 \), \( r = 0,1, \ldots, P-1 \) are different from zero. Hence each row of \( V(z) \) has \( J \) nonzero components, and each column has \( L \) nonzero entries. When \( N \) and \( K \) are coprime \( J = N \) and \( L = K \), and consequently \( V(z) \) is a full matrix. The structure of \( V(z) \) is illustrated in the following example.

**Example 1.** In the case \( N = 4, K = 6, V(z) \) has the form
\[
V(z) = \begin{bmatrix}
V_0(z^2) & 0 & z^{-1}V_6(z^2) & 0 \\
0 & V_1(z^2) & 0 & z^{-1}V_7(z^2) \\
z^{-1}V_8(z^2) & 0 & V_2(z^2) & 0 \\
0 & z^{-1}V_9(z^2) & 0 & V_3(z^2) \\
V_4(z^2) & 0 & z^{-1}V_{10}(z^2) & 0 \\
0 & V_5(z^2) & 0 & z^{-1}V_{11}(z^2)
\end{bmatrix}.
\]

Filters of a corresponding demultiplexer are also windowed complex exponentials, \( \phi_k[n] = w[n] e^{i\frac{2\pi}{N} kn} \). Analogously to the derivation of the polyphase representation of the multiplexer, it can be shown that the polyphase matrix of the demultiplexer has the form \( D(z) = F_N^* W(z) \), where \( F_N^* \) is the complex-conjugated transpose of \( F_N \), and \( W(z) \) is an \( N \times K \) matrix of polyphase
components of $w[n]$. To specify $W(z)$, consider the following set of polyphase components of $w[n]$: $W_j(z) = \sum_{n=-\infty}^{\infty} w[-j + nM] z^{-n}$, $j = 0, 1, \ldots M - 1$. In a manner analogous to the derivation of $V(z)$, it can be shown that

$$[W(z)]_{l,m} = \begin{cases} z^p W_{pK+jP+r}(z^j), & (l,m) = (iP + r, jP + r) \\ 0, & \text{otherwise} \end{cases}, \quad (14)$$

where $i \equiv pL + j \pmod{J}$.

Expressed in terms of $V(z)$ and $W(z)$, the condition for perfect transmultiplexing is that $W(z)$ is (modulo a multiplicative constant) a left inverse of $V(z)$: $W(z)V(z) = (1/N)I$. The left inverse of $V(z)$ is not unique when $V(z)$ is a $K \times N$ matrix where $K > N$, hence, given a multiplexing waveform $v[n]$, the corresponding waveform $w[n]$ which achieves perfect demultiplexing is not unique. Based on these polyphase description of the multiplexer and the demultiplexer, in the next subsection we will provide a complete parameterization of orthogonal Weyl-Heisenberg sets in $\ell^2(\mathbb{Z})$. This polyphase multiplexer description also leads to fast algorithms for implementation of OFDM based on long modulating waveforms [17].

3. Parameterization of Orthonormal Weyl-Heisenberg Sets in $\ell^2(\mathbb{Z})$

3.1. Orthogonal Weyl-Heisenberg Sets

From considerations in the previous section, it follows that modulating waveforms satisfy the orthogonality conditions in (15) if and only if the corresponding matrix $V(z)$, as defined in (10), is paraunitary:

$$\tilde{V}(z) V(z) = \frac{1}{N} I. \quad (15)$$

From the particular structure of $V(z)$, as described in the previous subsection it follows that $V(z)$ satisfies (15) if and only if a particular set of its submatrices satisfy the condition. This is illustrated by the following example.

**Example 2.** In the case given in Example [7], $V(z)$ is paraunitary if and only if its submatrices

$$V_0(z) = \begin{bmatrix} V_0(z^2) & z^{-1}V_6(z^2) \\ z^{-1}V_8(z^2) & V_2(z^2) \\ V_4(z^2) & z^{-1}V_{10}(z^2) \end{bmatrix} \quad V_1(z) = \begin{bmatrix} V_1(z^2) & z^{-1}V_7(z^2) \\ z^{-1}V_9(z^2) & V_3(z^2) \\ V_5(z^2) & z^{-1}V_{11}(z^2) \end{bmatrix} \quad (16)$$
are paraunitary. Further, matrices $V_i(z), \ i = 0, 1$ are paraunitary if and only if matrices

$$V_0(z) = \begin{bmatrix} V_0(z) & V_6(z) \\ V_5(z) & zV_3(z) \\ V_4(z) & V_{10}(z) \end{bmatrix} \quad V_1(z) = \begin{bmatrix} V_1(z) & V_7(z) \\ V_5(z) & zV_3(z) \\ V_4(z) & V_{11}(z) \end{bmatrix}$$

are paraunitary, where $V_0(z)^2 = \text{diag}(1, z, 1)V_i(z)\text{diag}(1, z), \ i = 0, 1$. □

A sufficient and necessary condition for the orthogonality relationships in (5) is established by the following theorem, from which a complete parameterization of orthonormal Weyl-Heisenberg sets in $\ell^2(\mathbb{Z})$ follows immediately.

**Theorem 2.** Consider a Weyl-Heisenberg set $\Phi = \{ \varphi_{k,i} : \varphi_{k,i}[n] = v[n-iK]e^{i2\pi k(n-iK)} \}_{k \in \mathbb{Z}, i \in \mathbb{Z}}$. Let $M$ be the least common multiple of $K$ and $N$, and let $V_j(z), \ j = 0, 1, \ldots, M - 1$, be the components of the $M$-component polyphase representation of $v[n]$ as given by (2). $\Phi$ is an orthonormal set if and only if the following matrices are paraunitary:

$$[V_r(z)]_{i,j} = z^{n(i,j)}V_{p(i,j)K+iP+r}(z),$$

$i = 0, 1, \ldots, L - 1, \ j = 0, 1, \ldots, J - 1, \ r = 0, 1, \ldots, P - 1$, where $P$ is the greatest common divisor of $N$ and $K$, $J = N/P$, $L = K/P$, $p(i,j)$ is the integer such that

$$j \equiv p(i,j)L + i \pmod{J}.$$

and $n(i,j) \in \{0,1\}$ is given by $n(i,j) = [p(i,0) + p(0,j) - p(i,j)]/J$.

**Proof:** It follows from (11) that $V(z)$ in (11) is paraunitary if and only if its submatrices $V_r(z), \ r = 0, 1, \ldots, P - 1,$

$$[V_r(z)]_{i,j} = [V(z)]_{iP+r,jP+r} = z^{-p(i,j)}V_{p(i,j)K+iP+r}(z^J),$$

where $p(i,j)$ is the integer which satisfies $j \equiv p(i,j)L + i \pmod{J}$, are paraunitary. Further, $V_r(z)$ is paraunitary if and only if $V'_r(z) = D_R(z)V_r(z)D_C(z)$ is paraunitary, where

$$D_L(z) = \text{diag}(z^{p(0,0)}, z^{p(0,1)}, \ldots, z^{p(0,L-1)}), \quad D_C(z) = \text{diag}(z^{p(0,0)}, z^{p(1,0)}, \ldots, z^{p(J-1,0)}) .$$

Matrices $V'_r(z)$ have the form

$$[V'_r(z)]_{i,j} = z^{p(i,0)+p(0,j)-p(i,j)}V_{p(i,j)K+iP+r}(z^J).$$
Corollary 1. The $M = \text{LCM}(K, N)$ polyphase components of a waveform $v[n]$ for which the Weyl-Heisenberg set $\Phi = \{\varphi_{k,i} : \varphi_{k,i}[n] = v[n - iK]e^{i\frac{2\pi}{N}k(n-iK)}\}_{k \in \mathbb{Z}, i \in \mathbb{Z}}$ is orthonormal are up to time delays entries of $P = \text{GCD}(N, K)$ paraunitary matrices of size $L \times J$, and vice versa, entries of $P$ arbitrary paraunitary $L \times J$ matrices are up to time delays the polyphase components of a waveform $v[n]$ for which $\Phi$ is an orthonormal set. \hfill \Box

Parameterizations of paraunitary matrices have been previously studied in the filter bank literature, and these combined with the special form of $V^o_r(z)$ described by (18) provide a complete parameterization of OFDM modulating waveforms. Details of this parameterization are illustrated by the following example.

Example 3. Consider a Weyl-Heisenberg set $\Phi$ underlying OFDM with $N = 128$-channels and $K = 160$-sample frame interval. The polyphase multiplexer matrix is paraunitary if and only if the following submatrices of the corresponding matrix $V(z)$ are paraunitary:

$$V_i(z) = \begin{bmatrix} V_{0:32+i}(z^4) & z^{-1}V_{5:32+i}(z^4) & z^{-2}V_{10:32+i}(z^4) & z^{-3}V_{15:32+i}(z^4) \\ z^{-3}V_{16:32+i}(z^4) & V_{1:32+i}(z^4) & z^{-1}V_{6:32+i}(z^4) & z^{-2}V_{11:32+i}(z^4) \\ z^{-2}V_{12:32+i}(z^4) & z^{-3}V_{17:32+i}(z^4) & V_{2:32+i}(z^4) & z^{-1}V_{7:32+i}(z^4) \\ z^{-1}V_{8:32+i}(z^4) & z^{-2}V_{13:32+i}(z^4) & z^{-3}V_{18:32+i}(z^4) & V_{3:32+i}(z) \\ V_{4:32+i}(z^4) & z^{-1}V_{9:32+i}(z^4) & z^{-2}V_{14:32+i}(z^4) & z^{-3}V_{19:32+i}(z^4) \end{bmatrix} , \quad i = 0, 1, \ldots 31 .$$

To obtain matrices of polyphase components of $v[n]$ with only unit delays, matrices $V_i(z)$ are multiplied from the right by $\text{diag}(1, z, z^2, z^3)$, the inverse delays of the first row of $V_i(z)$, and from the left by $\text{diag}(1, z^3, z^2, z, 1)$, the inverse delays of the first column of $V_i(z)$. This procedure gives matrices

$$V^o_i(z^4) = \text{diag}(1, z^3, z^2, z, 1)V_i(z)\text{diag}(1, z, z^2, z^3) ,$$
where

\[
V_i^o(z) = \begin{bmatrix}
V_{0.32+i}(z) & V_{5.32+i}(z) & V_{10.32+i}(z) & V_{15.32+i}(z) \\
V_{16.32+i}(z) & zV_{1.32+i}(z) & zV_{6.32+i}(z) & zV_{11.32+i}(z) \\
V_{12.32+i}(z) & V_{17.32+i}(z) & zV_{2.32+i}(z) & zV_{7.32+i}(z) \\
V_{8.32+i}(z) & V_{13.32+i}(z) & V_{18.32+i}(z) & zV_{3.32+i}(z) \\
V_{4.32+i}(z) & V_{9.32+i}(z) & V_{14.32+i}(z) & V_{19.32+i}(z)
\end{bmatrix}, \quad i = 0, 1, \ldots, 31. \tag{21}
\]

It follows that \(V_i(z)\) is paraunitary if and only if \(V_i^o(z)\) is paraunitary, and consequently, \(V(z)\) is paraunitary if and only if all \(V_i^o(z), i = 0, 1, \ldots, 31\) are paraunitary.

A modulating waveform for OFDM with \(N = 128\) and \(K = 160\) is then obtained as follows:
i) start with an arbitrary set of \(\text{GCD}(N, K) = 32\) paraunitary matrices \(V_i^o(z), i = 0, 1, \ldots, 31\) of size \(L \times J = 5 \times 4\), and express coefficients of polynomials in \(V_i^o(z)\) in terms of free parameters, e.g. angles of Givens rotations (see [8, 32]); ii) interleave entries of \(V_i^o(z)\) according to (21) to form \(v[n]\). An example of waveform obtained by applying an unconstrained optimization procedure to a solution \(v[n]\) obtained in this manner is shown in Figure [2b]. Matrices \(V_i^o(z)\) corresponding to this waveform are paraunitary matrices of zero degree monomials (i.e. unitary scalar matrices), so \(v[n]\) has 640 nonzero taps. However, the delay elements in \(V_i^o(z)\), that appear with \(V_{1.32+i}(z), V_{2.32+i}(z), V_{3.32+i}(z), V_{6.32+i}(z), V_{7.32+i}(z),\) and \(V_{11.32+i}(z)\), create zero taps, so the total waveform length is \(L_o = 1024\). The existence of these zero taps cannot be avoided in OFDM design, except when \(K/N\) is integer.

Another orthogonal OFDM waveform is the \(N\) samples long rectangular waveform. The corresponding polyphase matrices \(V_i^o(z)\) are all equal, and are given by

\[
V_i^o(z) = \frac{1}{\sqrt{128}} \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad i = 0, 1, \ldots, 31, \tag{22}
\]

where \(0 = [0 \ 0 \ 0 \ 0]\) and \(I\) is the \(4 \times 4\) identity matrix. \(\square\)

3.2. Biorthogonal Demultiplexing Waveforms

To parameterize all solutions for the demultiplexing waveforms, or equivalently, the complete set of inverses of \(V(z)\), consider a paraunitary \(V(z)\) and the associated paraunitary matrices \(V_i^o(z), i = 0, 1, \ldots, N/J\). The size of each \(V_i^o(z)\) is \(L \times J\), where \(L > J\). Each matrix \(V_i^o(z)\) can
be completed to a square \( L \times L \) paraunitary matrix \( V_i^s(z) = [V_i^o(z) \ V_i^c(z)] \). Let \( U_i(z) \) be an \( L \times J \) matrix given by
\[
U_i^o(z) = V_i^o(z) \begin{bmatrix} I \\
A_i(z) \end{bmatrix},
\]
where \( I \) is the \( J \times J \) identity matrix and \( A_i(z) \) is an \((L - J) \times J\) polynomial matrix. Then \((U_i^o(z^{-1}))^T\) is a left inverse of \( V_i^o(z) \). Conversely, any left inverse of \( V_i^o(z) \) has this form. A demultiplexing waveform \( w[n] \) is obtained by interleaving polynomials in matrices \( U_i^o(z) \) in the same manner polynomials in \( V_i^o(z), \ i = 0, 1, \ldots, N/J \) are interleaved to obtain \( v[n] \). Modulated versions \( \psi_k[n] \) of \( w[n] \) in (6) are then biorthogonal to waveforms \( \varphi_k[n] \) according to (7). Polynomial matrices \( A_i(z) \) are free parameters which can be optimized to achieve possible additional design requirements.

Note that the biorthogonal demultiplexing considered here is a generalized version of the cyclic prefix or zero padding schemes and it is different from biorthogonal frequency division multiplexing (BFDM), studied in [14] and [28], where multiplexing waveforms \( \varphi_k[n] \) are not mutually orthogonal according to (5). Some examples of biorthogonal waveforms designed in this manner are presented in [19]; below we illustrate the procedure for the case of rectangular waveforms \( v[n] \).

Example 4. Consider the rectangular window for \( N = 128 \) channel OFDM with \( K = 160\)-sample frame interval, as discussed in Example 3. Each of the corresponding matrices \( V_i^o(z) \) in (22) can be completed to the square paraunitary matrix
\[
V_i^s(z) = \frac{1}{\sqrt{128}} \begin{bmatrix} 0 & 1 \\
I & 0^T \end{bmatrix}, \ i = 0, 1, \ldots 31,
\]
where \( 0 = [0 \ 0 \ 0 \ 0] \) and \( I \) is the \( 4 \times 4 \) identity matrix. The corresponding matrices \( U_i^o(z) \) have the form
\[
U_i^o(z) = \frac{1}{\sqrt{128}} \begin{bmatrix} A_{i,1,1}(z) & A_{i,1,2}(z) & A_{i,1,3}(z) & A_{i,1,4}(z) \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{bmatrix}, \ i = 0, 1, \ldots 31.
\]
The \( K \) samples long rectangular demultiplexing window that is used in the zero-padding scheme is obtained for \( A_{i,1,1}(z) = z, A_{i,1,2}(z) = A_{i,1,3}(z) = A_{i,1,4}(z) = 0 \) for all \( i = 0, 1, \ldots 31 \). □
3.3. Equivalent Orthogonality Conditions and the Relationship with Weyl-Heisenberg Frames

The parameterization orthonormal Weyl-Heisenberg sets, presented in the previous subsection, allows for exact orthogonal design using unconstrained optimization procedures. Alternatively, if one wishes to pursue design using optimization under orthogonality constraints in (5), it is beneficial to find a minimal equivalent set of constraints expressed directly in terms of underlying prototype waveforms. To this end observe that the orthogonality conditions
\[ \sum_n \phi^*_n[n-kK] \phi_l[n-iK] = \delta[m-l] \delta[k-i] \]
can be written in terms of the prototype waveform, assuming it is real, as
\[ e^{j2\pi N (mk-l)i} \sum_{n=-\infty}^{\infty} v[n-kK] v[n-ik] e^{j2\pi (l-m)n} = \delta[m-l] \delta[k-i] . \]
This is equivalent to
\[ N^{-1} \sum_{n=0}^{N-1} V(i,k,n) e^{j2\pi (l-m)n} = \delta[m-l] \delta[k-i] , \tag{26} \]
where
\[ V(i,k,n) = \sum_{p=-\infty}^{\infty} v[n-iK+pN] v[n-kK+pN] . \]
Note further that (26) is equivalent to
\[ V(i,k,n) = \frac{\delta[k-i]}{N}, \quad n = 0, 1, \ldots, N-1, \]
which is satisfied if and only if
\[ \sum_i v[n+iN] v[n+iN+jK] = \frac{1}{N} \delta[j], \quad n = 0, 1, \ldots, N-1 . \tag{27} \]
Hence, the orthogonality relations in (5) are equivalent to the conditions in (27) expressed directly in terms of \( v[n] \).

One class of OFDM windows that follows immediately from (27) are windows which extend over only one frame interval, \( K \), for \( K \leq 2N \). With this restriction the system of constraints in (27) reduces to
\[ v[n] v[n] + v[n+N] v[n+N] = \frac{1}{N}, \quad 0 \leq n \leq K-N-1 \]
\[ v[n] v[n] = \frac{1}{N}, \quad K-N \leq n \leq N-1 , \tag{28} \]
and the complete set of solutions can be parameterized in terms of \( K-N \) angles \( \alpha_n \) as
\[ v[n] = \frac{1}{\sqrt{N}} \begin{cases} \cos(\alpha_n), & 0 \leq n \leq K-N-1 \\ 1, & K-N \leq n \leq N-1 \end{cases} . \tag{29} \]

The set of constraints in (27) is also identical to a set of sufficient and necessary conditions under which the set \( \{ \xi_{k,i} : \xi_{k,i}[n] = v[n-iN] e^{j2\pi k(n-iN)} \}_{k \in \mathbb{Z}, i \in \mathbb{Z}} \) forms a tight frame in \( \ell^2(\mathbb{Z}) \). Hence, the following theorem holds.
**Theorem 3.** A Weyl-Heisenberg set $\Phi = \{ \varphi_{k,i}[n] = v[n - iK]e^{j2\pi k(n-iK)} \}_{k \in \mathbb{Z}; i \in \mathbb{Z}}$ is orthonormal if and only if $\Xi = \{ \xi_{k,i}[n] = v[n - iN]e^{j2\pi k(n-iN)} \}_{k \in \mathbb{Z}; i \in \mathbb{Z}}$ is a tight frame in $\ell^2(\mathbb{Z})$. □

The result of Theorem 3 has been established previously for continuous-time Weyl-Heisenberg frames in [33] and [34] and also pointed out in the context of OFDM in [17, 18]. It can be proved along the lines of continuous-time proof, but that involves fairly sophisticated functional analysis and operator algebra. The proof provided here, on the other hand, draws only upon elementary results on filter banks.

Recently it was pointed out by Han and Zhang [40] that when the transmitted sequence $a[n]$ takes values from a finite alphabet, linear independence of waveforms $\varphi_{k,i}[n]$ is not necessary for perfect demultiplexing. Furthermore, Han and Zhang propose using Weyl-Heisenberg frames $\xi_{k,i}[n]$ instead of orthonormal families $\varphi_{k,i}[n]$ for transmission over time-frequency dispersive channels and demonstrate some merits of this approach. A complete parameterization of discrete-time tight Weyl-Heisenberg frames was given in [30].

### 4. OFDM Window Design Issues

An issue which is still a matter of debate is whether the exact (bi)orthogonality according to (5) and (7), as pursued in this paper and in [13, 14, 24, 27, 28], is really needed considering that it is lost due to multipath propagation, frequency offset, or timing mismatch, all of which occur simultaneously in a communication channel. It is reasonable to think that it would be sufficient to impose (bi)orthogonality of transmit and receive waveforms only at neighbouring locations in the time-frequency lattice, i.e.

$$\langle \psi_k[n - iK], \varphi_l[n - jK] \rangle = \delta[k - l]\delta[i - j], \ |k - l| < N_0, \ |i - j| < K_0,$$

for some small $N_0$ and $K_0$, and use the design freedom acquired by relaxing the remaining constraints to achieve higher spectral containment of the waveforms. This better frequency localization would in turn provide near-orthogonality at other lattice points and also improve robustness to the considered sources of degradation. Such an approach was considered in [10, 12]. A more radical strategy would be to abandon the orthogonality completely and suppress interchannel interference by further maximizing spectral containment of the waveforms, as proposed in [5, 21], or minimize
ISI/ICI explicitly if the channel or its statistics are known. Simulation results which compare a scheme with a partial orthogonality as specified in (30) to a scheme with the orthogonality across the whole time-frequency lattice reported in [14] showed a superior performance of the latter scheme. In the early phase of this research, we also considered a variant of the partial orthogonality approach, in particular $\langle \varphi_k[n-iK], \varphi_l[n-jK] \rangle = \delta[i-j]$ for $l = k$ only, but the results were inferior compared to the case of full orthogonality despite better frequency localization. Recently, Matz et al. showed in the continuous time case that optimal waveforms for transmission over time-frequency dispersive channels are very close to orthogonal [28]. In this section we provide some additional insight into why the exact orthogonality has merits in reducing the interference. Further, we show that, contrary to what is believed by many practitioners, tapering of the rectangular demultiplexing window in the zero-padding (or cyclic prefix) scheme cannot improve the robustness of OFDM to frequency offset.

Consider transmission of an OFDM signal through a multipath channel. The received signal $s_r[n]$ has the form $s_r[n] = \sum_{l=0}^{D} r_l s[n-d_l]$ where $s[n]$ is the transmitted signal as given by (1). Assume without loss of generality that $r_0 = 1$, $d_0 = 0$, and assume further that the channel does not change with time. Sequences $\tilde{a}_k[i]$ at the output of the demultiplexer, $\tilde{a}_k[i] = \sum_{n} \phi_k[iK-n] s_r[n]$, have the form $\tilde{a}_k[i] = a_k[i] F_{k,i} + \sum_{(p,q)\neq(0,0)} a_{k+p}[i+q] I_{k,i}(p,q)$, where $F_{k,i}$ and $I_{k,i}(p,q)$ are fading and intersymbol interference functions, respectively. These two functions can be expressed in terms of the crossambiguity function between the multiplexing and demultiplexing modulating waveforms, $A_{v,w}(x,y) = \sum_n v[n-x]w[n-y] e^{-j2\pi yn}$, as $F_{k,i} = 1 + \sum_{l=1}^{D} r_l e^{-j2\pi Kd_l} A_{v,w}(d_l,0)$, and $I_{k,i}(p,q) = e^{-j2\pi (k+p)qK} A_{v,w}(qK,p) + \sum_{l=1}^{D} r_l e^{-j2\pi (p+k)d_l} A_{v,w}(qK+d_l,p)$. Assuming that symbols $a_k[i]$ are independent, zero-mean, with variance 1, $E(a_k[i]\bar{a}_k^*[j]) = \delta[k-l]\delta[i-j]$, and that channel parameters $r_l$ are also statistically independent, $E(r_l r_{m}^*) = \delta[l-m]\sigma_r^2$, the expected squared value of the total interference $I = E(\sum_{(p,q)\neq(0,0)} |a_{k+p}[i+q]I_{k,i}(p,q)|^2)$, is given by

$$I = \sum_{(p,q)\neq(0,0)} \left( |A_{v,w}(qK,p)|^2 + \sum_{l=1}^{D} \sigma_r^2 |A_{v,w}(qK+d_l,p)|^2 \right).$$

Design methods which minimize this interference without imposing biorthogonality between multiplexing and demultiplexing waveforms and assuming that there is no frequency offset nor timing mismatch and statistics of the channel are known were studied in detail in [13, 16] for waveforms...
up to \( K \) samples long, and more recently for longer waveforms in [26]. When an OFDM system operates with a frequency offset \( \epsilon_f \) and a timing mismatch of \( \epsilon_t \) samples the expected squared value of intersymbol interference becomes

\[
I = \sum_{(p,q) \neq (0,0)} |A_{v,w}(qK - \epsilon_t, p + \epsilon_f)|^2 + \sum_{l=1}^{D} \sigma_l^2 |A_{v,w}(qK + d_l - \epsilon_t, p + \epsilon_f)|^2.
\]

To suppress this interference the waveforms should be designed so that \( A_{v,w}(x,y) \) has low magnitude in all regions \( E_{p,q} = (pK - \epsilon_t, pK + \Delta + \epsilon_t^+) \times (q - \epsilon_f^-, q + \epsilon_f^+) \) where \( (p,q) \) is a pair of integers different from \((0,0)\), \( \Delta \) is the maximal delay spread, and \( \epsilon_t^-, \epsilon_t^+, \epsilon_f^- \) and \( \epsilon_f^+ \) are bounds on possible timing mismatch and frequency offset. Ideally, \( A_{v,w}(x,y) \) should vanish in all regions of this form. However, with finite-length waveforms \( A_{v,w}(x,y) \) can have only a limited number of zeros within each region \( E_{m,n} \). For example, given a multiplexing waveform \( v[n] \) and assuming that \( v[n] \) and \( w[n] \) are of the same length, \( L \), the requirement that \( A_{v,w}(x,y) \) has \( n_z \) zeros within each \( E_{m,n} \) imposes \( n_z (2NL_v/K - N - 1) \) constraints on \( w[n] \). It follows that if the efficiency \( N/K \) is to be above 50\%, and the waveforms are at least \( 2N \) samples long, \( n_z \) cannot be more than 1, except in singular cases such as the zero-padding or cyclic prefix schemes. Hence, designing \( v[n] \) and \( w[n] \) so that: (i) \( A_{v,w}(pK, q) = 0 \), for all integer pairs \( (p,q) \neq (0,0) \),

\[
A_{v,w}(pK, q) = 0, \quad \text{for all integer pairs } (p,q) \neq (0,0),
\]

and (ii) they do not have sharp transitions, or have a somewhat flat shape within the main lobe in time, to avoid rapid changes of the crossambiguity function around its zeros, seems to be a good window design strategy. Note that the condition in (31) is equivalent to the biorthogonality between the multiplexing and demultiplexing waveforms in (7). This suggests that although it is not clear whether this strict biorthogonality is crucial for the effectiveness of OFDM, it is certainly a judicious design approach. In addition to the smoothness and the orthogonality requirements, in order to minimize interference between different OFDM channels and make the system robust to frequency offset, the waveforms should have adequate frequency characteristics, which basically means good spectral containment in the \([0, \frac{\pi}{N}]\) band. One way of balancing all these requirements is by designing the waveforms to optimize their time-frequency localization as concluded also in [28] for the continuous-time case.

In the case of the cyclic prefix or zero padding schemes the crossambiguity function \( A_{v,w}(x,y) \) is zero at all integer pairs \( (x,y) \) such that \( mK \leq x \leq mK + T_g \). Note that this is a singular
the modulating waveforms are constant within their support, and this time-invariance enables having the crossambiguity function with \( K - N \) zeros in all regions \( E_{m,n} \). However, \( A_{v,w} \) grows rapidly around its zeros and that makes these schemes very sensitive to timing mismatch and frequency offset. The problem of timing mismatch can be dealt with by introducing an adequate time-shift in the demultiplexing waveform at the expense of a reduction in the tolerable delay spread. The problem with the sensitivity to frequency offset, on the other hand, cannot be approached in such a straightforward manner. It might appear that using a demultiplexing waveform which is constant on an interval of length \( N + d_s \) and has a gradual decay toward zero would improve the robustness of these schemes to frequency offset owing to a better frequency localization of the waveform, while intersymbol interference would be still eliminated in the absence of frequency offset for delay spreads up to \( d_s \) samples. To demonstrate that this kind of modification cannot improve the robustness to frequency offset, consider a demultiplexing waveform \( w_r[n] \) given by

\[
w_r[n] = \begin{cases} r_1[n], & -K + \alpha + 1 \leq n \leq -N - d_s \\ \frac{1}{\sqrt{N}}, & -N - d_s + 1 \leq n \leq 0 \\ r_2[n], & 1 \leq n \leq \alpha \end{cases}
\]

(32)

where \( r_1[n] \) and \( r_2[n] \) are roll-off functions with support of \( \alpha = (K - N - d_s)/2 \) samples, so that the total duration of \( w_r[n] \) is \( K \) samples. The crossambiguity function, \( A_{v,w_r} \), between this waveform and the \( N \)-samples long rectangular waveform is \( A_{v,w_r}(x,y) = \frac{1}{N} \sum_{n=x}^{n=x+N} e^{j \frac{2\pi}{N} yn} \) for \( x \) in the range \( 0 \leq x \leq d_s \), and this is identical to the crossambiguity function obtained with the \( K \)-samples long rectangular demultiplexing waveform. Hence, this modification, even though it provides a better spectral containment of the demultiplexing waveform does not improve the robustness of the zero-padding scheme to frequency offset.

5. Design Examples and Experimental Results

In this section we present several examples of OFDM modulating waveforms. First, we present examples of windows designed for \( N = 128 \) channel OFDM and frame intervals of \( K = 160 \) and \( K = 192 \) samples. The waveforms are designed by minimizing a convex combination of their energy leakage out of the \([0, \pi/N]\) frequency band and their energy leakage outside of the main lobe in time. The minimization is performed as an unconstrained optimization using the parameterizations given in Section 3 combined with parameterizations of paraunitary matrices based on Given’s rotations.
Other optimization criteria can also be used as discussed in [22, 27, 28]. Waveform $v_{160}$ for $N = 128$ channel OFDM with $K = 160$-sample frame interval obtained in this manner, as discussed in Example 3 is shown in Figure 2a. Waveform $v_{192}$ shown in Figure 2c) is designed for 128 channel OFDM with $K = 192$-sample frame interval.

![Waveform](image1.png)

![Magnitude Response](image2.png)

![Waveform](image3.png)

![Magnitude Response](image4.png)

Figure 2: Modulating waveforms for $N = 128$-channel OFDM. a) Waveform $v_{160}$ for OFDM with $K = 160$-sample frame interval. b) Magnitude responses of $v_{160}$ and the 128-tap rectangular window. c) Waveform $v_{192}$ for OFDM with $K = 192$-sample frame interval. d) Magnitude responses of $v_{192}$ and the 128-tap rectangular window.

Second, we present examples of OFDM waveforms for upstream cable TV channels. Delay spread in cable TV channels is relatively short compared to frame intervals, so it does not cause considerable intersymbol interference. The major source of degradation in these channels is narrowband interference, ingress. The objective of OFDM modulation in such conditions is to enable
efficient transmission through the part of the spectrum which is not affected by ingress. The main OFDM waveform design objective is therefore to achieve a high spectral containment to limit the adverse effects of ingress only to tones directly affected. The two waveforms shown in Figure 3 are designed for transmission at 1440kbps (36 symbols per 25µs) through a 1600kHz cable channel, as specified in [36]. Both waveforms correspond to $N = 36$-channel OFDM with $K = 40$-sample frame interval. The waveform in Figure 3a) is symmetric with 360 nonzero taps, while the waveform in Figure 3c) is asymmetric, with 720 nonzero taps, and hence lower sidelobes.

![Waveforms for $N = 36$-channel OFDM with $K = 40$-sample frame interval. a) A symmetric waveform with 360 nonzero taps. b) Magnitude responses of the symmetric waveform and the 36-tap rectangular window. c) An asymmetric waveform with 720 nonzero taps. d) Magnitude responses of the asymmetric waveform and the 36-tap rectangular window.](image-url)
We simulated OFDM based on long waveforms proposed in this paper in some typical transmission scenarios and compared the results with cyclic prefix schemes which use same levels of redundancy. Orthogonal demultiplexing is used as opposed to biorthogonal demultiplexing in line with the observation made in [28] that optimal performance is obtained with close-to-orthogonal demultiplexing. Figures 4a) to 4i) show bit error rates as a function of $E_b/N_0$ values (signal-to-noise ratio) for $N = 128$-channel OFDM with $K = 160$-sample frame interval in the presence of multipath propagation, additive white Gaussian noise, frequency offset and timing mismatch. The error rates were measured for OFDM based on waveform $v_{160}$ shown in Figure 2a) and for the cyclic prefix scheme with $K - N = 32$ samples long cyclic prefix. The simulated channel was a $L = 33$-tap Rayleigh channel. The average powers of the taps in the ascending order were 0, $-1$, $-2$, $-3$, $-4$, $-5$, $-6$, $-7$, $-8$, $-9$, $-10$, $-11$, $-12$, $-13$, $-14$, $-15$, $-16$, $-17$, $-18$, $-19$, $-20$, $-21$, $-22$, $-23$, $-24$, $-1$, $-2$, $-3$, $-4$, $-5$, $-6$, $-7$, and $-8$ dB. The transmitted data symbols were encoded using a rate $R = 1/2$ convolutional code and then BPSK modulated. The one-tap equalizer was trained using one training frame, assuming the absence of noise during the transmission of this frame. Frequency offset and timing mismatch were, on the other hand, present during the training process.

The simulation results shown in Figure 4 were obtained for frequency offsets $\epsilon_f = 0, 0.05, 0.10, 0.15$ (i.e. up to the frequency offset equal to 15% of the bandwidth of one tone), and timing mismatch of $\epsilon_t = -8, -4, 0, 4, 8$ samples. While the error rates of the cyclic prefix scheme (CP-OFDM) increase significantly with impairments, OFDM based on $v_{160}$ (LW-OFDM) exhibits robust behavior. Some preliminary simulations, the results of which are not reported here, indicate that the discrepancy between the robustness of the cyclic prefix scheme and OFDM based on long waveforms becomes more pronounced as the redundancy increases.

The waveform shown in Figure 3a) was used in simulations of OFDM transmission in a narrowband interference scenario. Bit error rates were measured as a function of signal-to-interference ratio and are shown in Figure 5. We considered $N = 36$-channel OFDM with frame interval of $K = 40$ samples. Accordingly, the cyclic prefix was set to $K - N = 4$ samples. The simulated channel had $L = 3$ Rayleigh distributed taps. The average powers of the taps in the ascending order were 0, $-3$, and $-6$ dB. Encoding and channel estimation were performed in the same way as in the simulations reported in the above. We introduced also additive white Gaussian noise, the
level of which was kept at 11dB SNR, and in addition to that we placed a narrowband interferer with the bandwidth of one subcarrier in the middle between the 17th and 18th OFDM tone. The measured bit error rates of OFDM with the long waveform (LW) and the cyclic prefix scheme (CP) as a function of the $E_B/E_I$ values, where $E_I$ is the mean power of the interferer, are shown in Figure 5 for different frequency offset and timing mismatch values. The results demonstrate that the long waveform achieves a significantly lower bit error rates in the presence of a narrowband interferer than the cyclic prefix scheme. Furthermore, the larger the timing mismatch, frequency offset or the power of the narrowband interferer, the more the performance of the cyclic prefix scheme suffers. The scheme based on the long waveform, on the other hand, shows only a minor increase in the bit error rate when the impairments and interference aggravate.

6. Conclusion

This paper presented a complete parameterization of OFDM modulating waveforms, i.e. orthonormal Weyl-Heisenberg sets in $\ell^2(\mathbb{Z})$, and a complete parameterization of corresponding biorthogonal demodulating waveforms. Several design examples are provided and applied in typical transmission scenarios. Simulations demonstrated a significant potential of long waveforms parameterized here for improving the robustness of OFDM to frequency offset, timing mismatch and narrowband interference.

Acknowledgement

The authors are grateful to B. Vučetić for her support during the course of this project. The first author would also like to thank L. J. Cimini for teaching him principles of OFDM.

References

[1] R. W. Chang. Synthesis of Band-Limited Orthogonal Signals for Multichannel Data Transmission. Bell Syst. Tech. J. Vol. 45, pp. 1775-1796, Dec. 1966.

[2] L. J. Cimini. Analysis and Simulation of a Digital Mobile Channel Using Orthogonal Frequency Division Multiplexing. IEEE Trans. Commun. Vol. 33, No. 7, pp. 665–675, July 1985.

[3] T. Keller and L. Hanzo. “Adaptive Multicarrier Modulation: A Convenient Framework for Time-Frequency Processing in Wireless Communications,” Proc. IEEE, Vol. 88, No. 5, May 2000, pp. 611–640.

[4] R. van Nee and R. Prasad. OFDM for Wireless Multimedia Communications. Artech House Publishers, 2000.
[5] G. Cherubini, E. Eleftheriou, S. ¨Olcer, and J. M. Cioffi. Filter Bank Modulation Techniques for Very High Speed Digital Subscriber Line. *IEEE Commun. Magazine*, Vol. 38, pp. 98–104, May 2000.

[6] A. Peled and A. Ruiz. Frequency Domain Data Transmission Using Reduced Computational Complexity Algorithms, in *Proc. ICASSP*’80, Vol. 5, pp. 964–967, Apr. 1980.

[7] T. Pollet, M. van Bladel, and M. Moeneclaey. BER Sensitivity of OFDM Systems to Carrier Frequency Offset and Wiener Phase Noise. *IEEE Trans. Commun.*, Vol. 43, No. 2/3/4, pp. 191–193, Feb.–Apr. 1995.

[8] P. P. Vaidyanathan. *Multirate Systems and Filter Banks*. Prentice Hall, Englewood Cliffs, New Jersey, 1993.

[9] S. D. Sandberg and M. A. Tzannes, “Overlapped Discrete Multitone Modulation for High Speed Copper Wire Communications,” *IEEE J. Selected Areas in Communications*, Vol. 13, No. 9, pp. 1571–1585, Dec. 1995.

[10] A. Vahlín and N. Holte, “Optimal finite duration pulses for OFDM,” *IEEE Trans. Commun.* Vol. 44, No. 1, pp. 10–14, Jan. 1996.

[11] P. K. Remvik and N. Holte, “Carrier frequency offset robustness for OFDM systems with different pulse shaping filters,” *Proc. IEEE GLOBECOM*-97, Vol. 1, pp. 11-15, Nov. 1997.

[12] R. Haas and J. C. Belfiore, “A time-frequency well-localized pulse for multiple carrier transmission,” *Wireless Personal Commun.* Vol. 5, No. 1, pp. 1-18, July 1997.

[13] R. Hleiss, P. Duhamel, and M. Charbit, “Oversampled OFDM Systems,” in *Proc. DSP’97*, Vol. 1, pp. 329–332, July 1997.

[14] W. Kozek and A. F. Molisch, “Nonorthogonal pulseshapes for multicarrier communications in doubly dispersive channels,” *IEEE J. Sel. Areas Commun.*, Vol. 16, No. 8, pp. 1579–1589, Oct. 1998.

[15] A. Scaglione, G. B. Giannakis and S. Barbarossa, “Redundant filterbank precoders and equalizers part I: unification and optimal design,” *IEEE Trans. on Signal Processing*, Vol. 47, No. 7, pp. 1988–2006, July 1999.

[16] A. Scaglione, G. B. Giannakis and S. Barbarossa, “Redundant filterbank precoders and equalizers part II: blind channel estimation, synchronization and direct equalization,” *IEEE Trans. on Signal Processing*, Vol. 47, No. 7, pp. 2007–2022, July 1999.

[17] Z. Cvetković, “Modulating Waveforms for OFDM” in *Proc. ICASSP’99*, Vol. 5, pp. 2463–2466, Mar. 1999.

[18] H. Bölcskei, “Efficient design of pulse shaping filters for OFDM systems,” Proc. SPIE Wavelet Applications in Signal and Image Processing VII 1999, pp. 625-636.

[19] Z. Cvetković, “OFDM with Biorthogonal Demultiplexing,” in *Proc. ICASSP’00*, Vol. 5, pp. 2517–2520, June 2000.

[20] Y.-P. Lin and S.-M. Phoong, “ISI-Free Filterbank Transceivers for Frequency-Selective Channels,” *IEEE Trans. Signal Processing*, Vol. 49, No. 11, pp. 2648–2658, Nov. 2001.

[21] G. Cherubini, E. Eleftheriou, S. ¨Olcer, “Filtered Multitone Modulation for Very High-Speed Digital Subscriber Lines,” *IEEE J. Sel. Areas Commun.*, Vol. 20, No. 5, pp. 1016–1028, June 2002.

[22] P. Siohan, C. Siclet and N. Lacaille, “Analysis and Design of OFDM/QAM systems based on the filter banks theory”, *IEEE Trans. Signal Processing*, Vol. 50, No. 5, pp. 1170-1183, May 2002.

[23] N. Benvenuto, S. Tomasín, and L. Tomba. Equalization Methods in OFDM and FMT Systems for Broadband Wireless Communications. *IEEE Trans. Commun.*, Vol. 50, No. 9, pp. 1413–1418, Sep. 2002.

[24] T. Strohmer and S. Beaver, “Optimal OFDM design for time-frequency dispersive channels,” *IEEE Trans.*
[25] H. Bölcskei, P. Duhamel, and R. Hleiss, “Orthogonalization of OFDM/OQAM pulse shaping filters using the discrete Zak transform,” Signal Processing, vol. 83, pp. 1379-1391, July 2003.

[26] S.-M. Phoong, Y. Chang, and C.-Y. Chen, “DFT Modulated Filterbank Transceivers for Multipath Fading Channels,” IEEE Trans. Signal Processing, Vol. 53, No. 1, pp. 182–192, Jan. 2005.

[27] C. Siclet, P. Sohian, and D. Pinchon, “Perfect Reconstruction Conditions and Design of Oversampled DFT-Modulated Transmultiplexers,” EURASIP J. Applied Signal Processing, Vol. 2006, Article ID 15756, pp. 1–14, 2006

[28] G. Matz, D. Schafhuber, K. Gröchenig, M. Hartmann, F. Hlawatsch, “Analysis, Optimization, and Implementation of Low-Interference Wireless Multicarrier Systems,” IEEE Trans. Wireless Commun., Vol. 6, No. 5, pp. 1921–1931, May 2007.

[29] P. Jung and G. Wunder, “The WSSUS Pulse Design Problem in Multicarrier Transmission,” IEEE Trans. Commun., Vol. 55, No. 10, pp. 1918–1928, Oct. 2007.

[30] Z. Cvetković and M. Vetterli, “Tight Weyl-Heisenberg Frames in $\ell^2(\mathbb{Z})$,” IEEE Trans. Signal Processing, Vol. 46, No. 5, pp. 1256–1259, May 1998.

[31] C. Shannon, “Communication in the Presence of Noise,” Proc. IRE, Vol. 37, pp. 10–21, 1949.

[32] M. Vetterli and J. Kočačević, Wavelets and Subband Coding, Prentice-Hall, Englewood Cliffs, New Jersey, 1995.

[33] A. J. E. M. Janssen, “Duality and Biorthogonality for Weyl-Heisenberg Frames,” J. Fourier Analysis and Applications, Vol. 1, No. 4, pp. 403–436, Nov. 1995.

[34] I. Daubechies, H. J. Landau, and Z. Landau, “Gabor Time-Frequency Lattices and the Wexler-Raz Identity,” J. Fourier Analysis and Applications, Vol. 1, No. 4, pp. 403–436, Nov. 1995.

[35] Z. Cvetković, “On Discrete Short-Time Fourier Analysis,” IEEE Trans. Signal Processing, Vol. 48, No. 9, pp. 2628–2640, Sept. 2000.

[36] D. Andelman, “Variable Constellation Multitone (VCMT) Proposal,” Ultracom Communications Inc., IEEE 802.14a/98-013 document, June 1998.

[37] R. E. Blahut, Fast Algorithms for Digital Signal Processing, Addison-Wesley Publishing Company, Inc. 1985.

[38] H. Park. A Computational Theory of Laurent Polynomial Rings and Multidimensional FIR Systems, PhD Thesis, Department of Mathematics, UC Berkeley, May 1995.

[39] H. Park. Parahermitian Modules and Parauunitary Groups. preprint, 1998.

[40] F.-M. Han and X.-D. Zhang, “Wireless Multicarrier Transmission via Weyl-Heisenberg Frames over Time-Frequency Dispersive Channels,” IEEE Trans. Communications, Vol. 57, No. 6, pp. 1721–1733, June 2009.
Figure 4: Measured bit error rates as a function of $E_B/N_0$ values for $N = 128$-channel OFDM with $K = 160$-sample frame interval in transmissions over a 33-tap Rayleigh channel in the presence of frequency offset ($ef$) and timing mismatch ($et$). Two schemes are used: the cyclic prefix scheme (CP-OFDM) and OFDM based on waveform $v_{t60}$ shown in Figure 2a) (LW-OFDM). Frequency offset ($ef$) and timing mismatch ($et$) are set to the values given in the legend. a) $ef = 0$ - no frequency offset. b) $ef = 0.05$ - frequency offset is 5% of the bandwidth of one tone. c) $ef = 0.1$ - frequency offset is 10% of the bandwidth of one tone. b) $ef = 0.15$ - frequency offset is 15% of the bandwidth of one tone.
Figure 5: Measured bit error rates as a function of signal-to-interference ratios, $E_B/E_I$, for $N = 36$-channel OFDM with $K = 40$-sample frame interval in transmissions over a 3-tap Rayleigh channel in the presence of narrowband interference, timing mismatch ($e_t$), and frequency offset ($e_f$). Two schemes are used: the cyclic prefix scheme (CP) and OFDM based on the waveform shown in Figure 3b) (LW). Frequency offset ($e_f$) and timing mismatch ($e_t$) are set to the values given in the legend. a) $e_f = 0$ - no frequency offset. b) $e_f = 0.05$ - frequency offset is 5% of the bandwidth of one tone.