Products of locally cyclic groups

BERNHARD AMBERG and YAROSLAV SYSAK

Abstract. We consider groups of the form $G = AB$ with two locally cyclic subgroups $A$ and $B$. The structure of these groups is determined in the cases when $A$ and $B$ are both periodic or when one of them is periodic and the other is not. Together with a previous study of the case where $A$ and $B$ are torsion-free, this gives a complete classification of all groups that are the product of two locally cyclic subgroups. As an application, it is shown that the Prüfer rank of a periodic product of two locally cyclic subgroups does not exceed 3, and this bound is sharp. It is also proved that a product of a finite number of pairwise permutable periodic locally cyclic subgroups is a locally supersoluble group. This generalizes a well-known theorem of B. Huppert for finite groups.

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1. Introduction. Let the group $G = AB$ be the product of two subgroups $A$ and $B$, i.e. $G = \{ab \mid a \in A, b \in B\}$. It was proved by N. Itô that the group $G$ is metabelian if the subgroups $A$ and $B$ are abelian (see [1, Theorem 2.1.1]). This result laid the foundation for a systematic study of groups of the form $G = AB$ with various conditions on the subgroups $A$ and $B$. In particular, it follows directly from Itô’s result that every periodic group $G = AB$ with abelian subgroups $A$ and $B$ is locally finite. It is also well-known that the group $G = AB$ with cyclic subgroups $A$ and $B$ is supersoluble and abelian-by-finite ([1, Lemma 7.4.6]). Furthermore, a detailed description of the structure of the group $G = AB$ with torsion-free locally cyclic subgroups $A$ and $B$ was obtained by the second author in [9].

The aim of this paper is to describe the structure of groups which are products of two locally cyclic subgroups in the periodic and the mixed case. Altogether this gives a complete answer to [1, Question 15].
Theorem 1.1. Let the periodic group $G = AB$ be the product of two locally cyclic subgroups $A$ and $B$. Then $G$ contains uniquely determined locally cyclic normal subgroups $S$ and $T$ and a locally nilpotent subgroup $H = A^*B^*$ with $A^* \leq A$ and $B^* \leq B$ such that

$$G = (S \times T) \rtimes H = (S \times A^*)(T \times B^*)$$

where $\pi(S) \cap \pi(A^*) = \pi(T) \cap \pi(B^*) = \emptyset$, $S = [S, B^*]$, and $T = [T, A^*]$.

We recall that a group $G$ has finite Prüfer rank $r = r(G)$ (special rank in the sense of Mal’cev in Russian terminology) if each finitely generated subgroup of $G$ can be generated by $r$ elements and $r$ is the least positive integer with this property. Clearly a group is of rank 1 if and only if it is locally cyclic. It is also known that every finite $p$-group of the form $G = AB$ with cyclic subgroups $A$ and $B$ has rank at most 2 for $p$ odd \cite{5, Satz 8} and at most 3 for $p = 2$ \cite{6, Theorem 5.1}. The following consequence of Theorem 1.1 gives an exact upper bound for the Prüfer rank of the product $G = AB$ of two periodic locally cyclic subgroups $A$ and $B$.

Corollary 1.2. If $G = AB$ is a periodic group with locally cyclic subgroups $A$ and $B$, then the Prüfer rank of $G$ does not exceed 3.

It should be noted that this result is also new for arbitrary finite groups of the form $G = AB$ with cyclic subgroups $A$ and $B$.

Finally, the following theorem extends a well-known result of B. Huppert on the supersolubility of finite groups which are products of pairwise permutable cyclic subgroups (see \cite{5, Satz 34} or \cite{4, Satz VI.10.3}). This gives, in particular, an affirmative answer to \cite{2, Question 1}.

Theorem 1.3. Let the group $G = A_1A_2\cdots A_n$ be the product of finitely many pairwise permutable periodic locally cyclic subgroups $A_1, \ldots, A_n$. Then $G$ is a periodic locally supersoluble group.

2. Preliminaries. In what follows $G = AB$ is a group with locally cyclic subgroups $A$ and $B$.

Lemma 2.1. If $G = AB$ is an infinite $p$-group, then up to a permutation of the factors $A$ and $B$ the subgroup $A$ is quasicyclic and one of the following statements hold:

1. $G = A \times B$ with $B$ cyclic or quasicyclic;
2. $p = 2$ and $G = A \rtimes \langle b \rangle$ for some element $b \in G$ with $a^b = a^{-1}$ for all $a \in A$;
3. $p = 2$ and $G = A\langle b \rangle$ for some element $b \in G$ with $b^{2^n} = 1$ for some $n > 1$, $b^{2^n-1} \in A$ and $a^b = a^{-1}$ for all $a \in A$.

Proof. Clearly without loss of generality we may assume that the subgroup $A$ is infinite and so quasicyclic. Then the subgroup $B$ is either cyclic or quasicyclic. Since in the latter case the group $G$ is abelian by \cite[Lemma 7.4.4]{1}, the subgroup $A$ is complemented in $G$ and hence statement (1) holds.

Let the group $G$ be non-abelian. Then the subgroup $B$ is cyclic and so $A$ as a quasicyclic $p$-subgroup of finite index in $G$ must be normal and non-central in
G. In particular, B induces on A a non-trivial cyclic p-group of automorphisms. On the other hand, since quasicyclic p-groups have no automorphisms of order p > 2, it follows that p = 2. But then \( B = \langle b \rangle \) with \( b^{2n} = 1 \) for some \( n \geq 1 \) and b induces on A an automorphism of order 2 that inverts the elements of A. In particular, if \( A \cap B = 1 \), we obtain statement (2). In the second case, \( A \cap B = \langle b^{2n-1} \rangle \) and hence statement (3) holds. \( \square \)

**Corollary 2.2.** If \( G = AB \) is a p-group and C, D are subgroups of A and B, respectively, then \( CD = DC \) and so \( CD \) is a subgroup of G.

**Proof.** This is known if G is finite (see [5, Satz 3]), and follows from Lemma 2.1 in the general case. \( \square \)

**Lemma 2.3.** If the group \( G = AB \) is periodic and \( H \) is a finite subgroup of G, then \( H \) is contained in a finite subgroup \( E \) of G such that \( E = (A \cap E)(B \cap E) \). In particular, the group G is locally supersoluble.

**Proof.** Since the group H is finite, there exist finite subsets C of A and D of B such that H is contained in the set CD. Then the subgroups \( A_0 = \langle C \rangle \), \( B_0 = \langle D \rangle \), and \( \langle C, D \rangle \) are finite, because the group G is locally finite. Furthermore, it follows from [1, Lemma 1.2.3] that the normalizer \( N_G(\langle A_0, B_0 \rangle) \) contains a finite subgroup E such that \( \langle A_0, B_0 \rangle \leq E = (A \cap E)(B \cap E) \). Since the subgroup E is supersoluble by [1, Lemma 7.4.6] and \( H \subseteq CD \subseteq \langle C, D \rangle = \langle A_0, B_0 \rangle \), the lemma is proved. \( \square \)

As a direct consequence of this lemma, we have

**Corollary 2.4.** If the group \( G = AB \) is periodic, then there exists an ascending series of finite subgroups \( 1 = G_0 < G_1 < \cdots < G_n < \cdots G \) such that \( G_n = (A \cap G_n)(B \cap G_n) \) for each \( n > 0 \) and \( G = \bigcup_{n=1}^{\infty} G_n \).

If G is a periodic group and \( \pi \) is a set of primes, then a subgroup \( H \) of G is called a \( \pi \)-subgroup provided that all prime divisors of the order of any element of \( H \) are contained in \( \pi \). By a Sylow \( \pi \)-subgroup of G we simply mean a maximal \( \pi \)-subgroup \( G_{\pi} \) of G which will be denoted by \( G_{p} \) if \( \pi = \{ p \} \).

**Lemma 2.5.** Let \( G = AB \) be a periodic group and \( \pi \) a set of primes. Then the following statements hold.

1. If \( A_{\pi} \) and \( B_{\pi} \) are Sylow \( \pi \)-subgroups of A and B, respectively, then \( G_{\pi} = A_{\pi}B_{\pi} \) is a Sylow \( \pi \)-subgroup of G and
\[
N_G(G_{\pi}) = N_A(G_{\pi})N_B(G_{\pi}).
\]

2. If \( p, q \) are primes with \( p > q \), then a Sylow \( p \)-subgroup \( G_{p} \) is normalized by a Sylow \( q \)-subgroup \( G_{q} \). In particular, \( G_{\{p,q\}} = G_{p}G_{q} \) for any primes \( p \) and \( q \).

**Proof.** (1) It follows from [1, Lemma 1.3.2] that in the notation of Corollary 2.4 for each \( n \geq 1 \) the set \( (A_{\pi} \cap G_n)(B_{\pi} \cap G_n) \) is a Hall \( \pi \)-subgroup of \( G_n \). Therefore \( G_{\pi} = \bigcup_{n=1}^{\infty} (A_{\pi} \cap G_n)(B_{\pi} \cap G_n) \) is a Sylow \( \pi \)-subgroup of G. Since \( A_{\pi} = \bigcup_{n=1}^{\infty} (A_{\pi} \cap G_n) \) and \( B_{\pi} = \bigcup_{n=1}^{\infty} (B_{\pi} \cap G_n) \), this implies \( G_{\pi} = A_{\pi}B_{\pi} \). In addition, applying [1, Lemma 1.2.2], we have \( N_G(G_{\pi}) = N_A(G_{\pi})N_B(G_{\pi}) \).
(2) If \( \pi = \{p, q\} \) for some primes \( p > q \), then \( A_{\{p, q\}} = A_p \times A_q \), \( B_{\{p, q\}} = B_p \times B_q \), and \( G_{\{p, q\}} = (A_p \times A_q)(B_p \times B_q) \). As \( G \) and so its subgroup \( G_{\{p, q\}} \) is locally supersoluble by Lemma 2.3, the Sylow \( p \)-subgroup \( G_p = A_pB_p \) is normal in \( G_{\{p, q\}} \). Therefore, 
\[ G_{\{p, q\}} = (A_p \times A_q)(B_p \times B_q) = G_pA_qB_q = G_pG_q, \]
as claimed. \( \square \)

A Sylow basis of a periodic group \( G \) is defined to be a complete set \( S = \{G_p\} \) of Sylow \( p \)-subgroups of \( G \), one for each prime \( p \), such that \( G_pG_q = G_qG_p \) for all pairs \( p, q \) of primes, and \( G_\pi = \{G_p \mid p \in \pi\} \) is a Sylow \( \pi \)-subgroup of \( G \) for each set \( \pi \) of primes. As is well-known (see [3, Lemma 2.1]), every countable periodic locally soluble group possesses Sylow bases. The basis normalizer \( N_G(S) \) of a Sylow basis \( S \) of \( G \) is by definition the intersection \( N_G(S) = \bigcap_p N_G(G_p) \) of the normalizers \( N_G(G_p) \) of the Sylow \( p \)-subgroups \( G_p \) of \( S \) for all \( p \).

**Lemma 2.6.** Let the group \( G = AB \) be periodic and \( G_p = A_pB_p \) for each prime \( p \). Then \( S = \{G_p\} \) is a Sylow basis of \( G \). Moreover, if \( A^* = \bigcap_p N_A(G_p) \) and \( B^* = \bigcap_p N_B(G_p) \), then \( N_G(S) = A^*B^* \).

**Proof.** Indeed, by Lemma 2.5, the set \( S = \{G_p\} \) forms a Sylow basis of \( G \) and \( N_G(G_p) = N_A(G_p)N_B(G_p) \) for every \( p \) by [1, Lemma 1.2.2]. Therefore, \( N_G(S) = \bigcap_p N_G(G_p) = \bigcap_p N_A(G_p)N_B(G_p) \) and it is easy to check that 
\[ \bigcap_p N_A(G_p)N_B(G_p) = (\bigcap_p N_A(G_p))(\bigcap_p N_B(G_p)) = A^*B^* \] (see [1, Lemma 1.1.2]). Therefore \( N_G(S) = A^*B^* \). \( \square \)

The following lemma is a direct consequence of a well-known result of L. Kovacs (see [7, Theorem 2]).

**Lemma 2.7.** Let \( G \) be a finite soluble group, \( \pi \) a set of primes and \( H \) a Hall \( \pi \)-subgroup of \( G \). If for each \( p \in \pi \) the Prüfer rank of a Sylow \( p \)-subgroup of \( G \) does not exceed \( r \), then \( H \) is a subgroup of rank at most \( r + 1 \).

**Proof.** Indeed, it is obvious that if \( K \) is a subgroup of \( H \), then every Sylow subgroup of \( K \) is generated by \( r \) elements. Therefore, \( K \) can be generated by \( r + 1 \) elements by the result of Kovacs cited above. Thus every subgroup of \( H \) is generated by \( r + 1 \) elements and so \( H \) has rank at most \( r + 1 \). \( \square \)

### 3. Proof of Theorem 1.1

First of all, it follows from Lemma 2.1 that for each prime \( p \) every Sylow \( p \)-subgroup of \( G = AB \) satisfies the minimal condition for subgroups. Therefore, \( G \) satisfies the minimal condition for \( p \)-subgroups for all primes \( p \). Since the group \( G \) is metabelian by Ito’s theorem, the locally nilpotent residual \( R \) of \( G \) is contained in its derived subgroup \( G' \) and so it is abelian. It was proved by Hartley [3, Theorem 1] that in this case \( G = R \times H \), where \( H \) is any basis normalizer of \( G \). In particular, by Lemma 2.6, we can take \( H = A^*B^* \).

It is easy to see that the subgroup \( H \) is locally nilpotent and contains the center \( Z(G) \) of \( G \). Furthermore, \( G' = R \times H' \) and so \( H' \) is a normal subgroup of \( G \). Since \( R \) is abelian and \( N_G(H) = N_R(H) \times H \), it follows that \( N_R(H) \leq Z(G) \leq H \). Therefore, \( N_R(H) = 1 \) and hence \( H = N_G(H) \). We now show that the subgroup \( H = A^*B^* \) commutes with both subgroups \( A \) and \( B \).
Indeed, put \( S = R \cap \langle A, B^* \rangle \) and \( T = R \cap \langle A^*, B \rangle \). It is clear that \( S \) and \( T \) are normal subgroups of \( G \), \( \langle A, B^* \rangle = S \times H \), and \( \langle A^*, B \rangle = T \times H \). On the other hand, as \( G = AB \), we have also \( \langle A, B^* \rangle = AB_1 \) and \( \langle A^*, B \rangle = A_1B \) for some subgroups \( A_1 \) and \( B_1 \) such that \( A^* \leq A_1 \leq A \) and \( B^* \leq B_1 \leq B \). From here, we deduce \( AB_1 \cap A_1B = A_1B_1 = (S \cap T) \times H \). Moreover, passing to the factor group \( G/H' \), we may restrict ourselves to the case when the subgroup \( H = A^*B^* \) is abelian. Then the subgroups \( A^* \) and \( B^* \) centralize \( S \) and \( T \), respectively, and so the subgroup \( H \) centralizes the intersection \( S \cap T \). Since \( H = N_G(H) \), this implies \( S \cap T = 1 \). Thus \( A_1B_1 = H = A^*B^* \) and hence \( \langle A, B^* \rangle = AH = AB^* \) and \( \langle A^*, B \rangle = BH = A^*B \), as asserted.

Further, taking into account the equalities \( AB^* = S \times H \) and \( H = A^*B^* \), we conclude that the subgroup \( A^* \) centralizes \( S \), because \( [A^*, S] \leq H' \cap S = 1 \). Since in this case the normalizer \( N_S(B^*) \) is contained in \( N_G(H) \), we have \( N_S(B^*) = 1 \). Therefore, every element \( b \in B^* \) induces on \( S \) an automorphism leaving only the identity element fixed. But then every element of \( S \) can be written in the form \( b^{-1}s^{-1}bs \) with \( s \in S \) and hence \( S = [B^*, S] \). Similarly, using the equality \( A^*B = T \times H \), we derive \( T = [A^*, T] \).

Finally, we put \( A_0 = A \cap BS \) and \( B_0 = AT \cap B \). Clearly from the equalities \( G = AB \), \( AB^* = S \times H \), and \( A^*B = T \times H \), it follows that \( G = S \times A^*B = T \times AB^* \), \( A = A^* \times A_0 \), and \( B = B^* \times B_0 \). Therefore, \( S \times B = A_0B_0 \) and \( T \times A = AB_0 \). Furthermore, if \( S_p \) is a Sylow \( p \)-subgroup of \( S \), then \( S_p \times B = (A_0 \cap S_p)B \) and if \( p \neq 1 \), then \( A_0 \cap S_pB \neq 1 \). Since the subgroup \( A_0 \) is locally cyclic, this implies that \( S \) is locally cyclic and \( \pi(S) \) is contained in \( \pi(A_0) \). Moreover, as \( A^* \) and \( A_0 \) are subgroups of the locally cyclic subgroup \( A \), it also follows that \( [\pi(A^*), \pi(A_0)] = \emptyset \). Similarly, using the equality \( T \times A = AB_0 \), we obtain \( \pi(T) = \pi(B_0) \) and \( \pi(B^*) \cap \pi(B_0) = \emptyset \).

**Proof of Corollary 1.2.** By Corollary 2.4, we may restrict ourselves to the case in which the group \( G = AB \) is finite. By Theorem 1.1, \( G \) then contains cyclic normal subgroups \( S \) and \( T \) and a nilpotent subgroup \( H = A^*B^* \) with \( A^* \leq A \) and \( B^* \leq B \) such that \( G = (S \times T) \times H = (S \times A^*)(T \times B^*) \), where \( \pi(S) \cap \pi(A^*) = \pi(T) \cap \pi(B^*) = \emptyset \), \( S = [S, B^*] \), and \( T = [T, A^*] \). In particular, if for some prime \( p \) the subgroup \( H \) contains a non-cyclic Sylow \( p \)-subgroup \( P \), then \( S \) and \( T \) are \( p' \)-subgroups of \( G \).

Since \( P = A_pB_p \) with \( A_p = A \cap P \) and \( B_p = B \cap P \), both subgroups \( A_p \) and \( B_p \) are non-trivial and so \( p \notin \pi(S) \cup \pi(T) \). Therefore, if \( G_p \) is a non-cyclic Sylow \( p \)-subgroup of \( G \), \( S_p \times G_p \), and \( T_p = G_p \times T \), then up to conjugation \( G_p \) coincides with one of the following subgroups of \( G \): \( P = A_pB_p \), \( T_p \times A_p \), \( S_p \times B_p \), and \( S_p \times T_p \). In particular, the Sylow \( p \)-subgroups of \( G \) have rank at most 2 for \( p > 2 \) (see [4, Satz III.11.5]) and at most 3 for \( p = 2 \) (see [6, Theorem 5.1]). We now show that \( G \) is in fact a group of rank at most 3.

Indeed, suppose the contrary and let the group \( G \) contain a subgroup \( K \) whose minimal number of generators \( d(K) \) is at least 4. Since the Sylow subgroups of odd orders in \( K \) have rank at most 2 by what was noted above, each Sylow 2-subgroup \( Q \) of \( K \) must have rank 3 by Lemma 2.7. It is clear
that $Q = K \cap P$ for a Sylow 2-subgroup $P = A_2B_2$ of $G$. As the group $G$ is metabelian, the derived subgroup $P'$ is abelian and normal in $G$. Moreover, $P'$ has rank at most 2 by [6, Theorems 4.2 and 4.3(e)].

Put $N = P' \cap Q$. As $Q = K \cap P$, we have $N = K \cap P'$ and so $N$ is an abelian normal subgroup of $K$ with rank at most 2. In addition, $N \neq 1$, because otherwise the subgroup $Q$ is embedded in the factor group $P/P'$ whose rank is equal to 2. On the other hand, since the factor group $Q/N = Q/P' \cap Q$ is isomorphic to the factor group $QP'/P' \leq P/P'$, it is abelian of rank at most 2. Therefore, the Sylow 2-subgroups of the factor group $K/N$ have rank at most 2 and hence $K/N$ has rank at most 3 by Lemma 2.7. In particular, $d(K/N) < d(K) = 4$ and so $N$ is not contained in the Frattini subgroup $\Phi(K)$ of $K$, because otherwise $d(K/N) = d(K)$. Clearly, passing to the factor group $K/\Phi(K)$, we may assume that $\Phi(K) = 1$. Then the normal subgroup $N$ is complemented in $K$ and so in $Q$ by [4, Hilfsatz 3.4.4]. Moreover, since $\Phi(N) = 1$ by [4, Hilfsatz 3.3.b], the subgroup $N$ is elementary abelian of order at most 4.

Let $L$ be a complement to $N$ in $K$ and $M = Q \cap L$. Then $M$ is a Sylow 2-subgroup of $L$ and $Q = N \times M$, so that $M$ is abelian with $d(M) \leq 2$. Since the subgroup $K$ is supersoluble, its maximal subgroup $U$ of odd order is normal in $K$ and centralizes $N$. Therefore, $K = U \times Q = (U \times N) \rtimes M$ and hence the centralizer $C_N(M)$ is a non-trivial central subgroup of $K$. As $\Phi(K) = 1$, the subgroup $C_N(M)$ is complemented in $K$ and thus in $Q$. From this, it follows that $Q = M \times N$ is abelian and $N$ is a central subgroup of $K$. Therefore, $K = (U \rtimes M) \times N$ and the subgroup $U \rtimes M$ is three-generated by Lemma 2.7. This means that there exist elements $u, v, w$ of $U$ and $x, y, z$ of $M$ such that $U \rtimes M = \langle ux, vy, wz \rangle$. Then $M$ modulo $U$ is generated by $x, y, z$. In particular, if $d(M) = 1$, without loss of generality we may assume that $M = \langle x \rangle$ and $y = z = 1$, so that $U \rtimes M = \langle ux, v, w \rangle$. In the case $d(M) = 2$ we can take $M = \langle x, y \rangle$ and $z = 1$. Then $U \rtimes M = \langle ux, vy, w \rangle$.

Finally, since $Q = M \times N$ is a 2-subgroup of rank 3 as noted above, only two cases are possible: either $M = \langle x \rangle$ and $N = \langle a, b \rangle$ has order 4 or $M = \langle x, y \rangle$ with $x \neq 1 \neq y$ and $N = \langle a \rangle$ is of order 2. Therefore, $K = \langle ux, av, bw \rangle$ in the first case and $K = \langle ux, vy, aw \rangle$ in the second case. In both cases $d(K) < 4$ and this contradiction completes the proof. □

4. Products of a periodic and a torsion-free locally cyclic group. Recall that a group $G$ has finite torsion-free rank if it has a series of finite length whose factors are either periodic or infinite cyclic. The number $r_0(G)$ of infinite cyclic factors in such a series is an invariant of $G$ called its torsion-free rank. In this section, we describe the structure of the group $G = AB$ with locally cyclic subgroups $A$ and $B$, the first of which is periodic and the other non-trivial torsion-free. Clearly $r_0(B) = 1$ and we note first that $r_0(G) = 1$.

Lemma 4.1. Let $G = AB$ be a group with subgroups $A$ and $B$ such that $A$ is periodic abelian and $B$ is non-trivial torsion-free locally cyclic. Then $r_0(G) = 1$. 

Therefore \(G\) has the normal series \(A_0 < A_0B_0 < G\) in which \(A_0\) is the core of \(A\) in \(G\) and \(B_0\) is the core of \(B\) in \(G\) modulo \(A_0\). As is easily seen, the factors \(A_0\) and \(G/A_0B_0\) are periodic and the factor group \(A_0B_0/A_0\) is isomorphic to \(B_0\). Thus \(r_0(G) = r_0(B) = 1\), as claimed. \(\Box\)

The following lemma is a consequence of the well-known theorem of I. Schur on the finiteness of the derived subgroup of a group that is finite over its center (see [8, Corollary to Theorem 4.12]).

**Lemma 4.2.** If a group \(G\) contains a central subgroup \(Z\) such that the factor group \(G/Z\) is locally finite, then the derived subgroup of \(G\) is locally finite.

**Theorem 4.3.** Let the group \(G = AB\) be the product of two locally cyclic subgroups \(A\) and \(B\) such that \(A\) is periodic and \(B\) is non-trivial torsion-free. Then one of the following statements holds.

1. The subgroup \(A\) is normal in \(G\) and so \(G = A \rtimes B\);
2. \(A = A_1\langle a \rangle\) with \(a^2 \in A_1\), the subgroup \(A_1\) is normal in \(G\) and \(G = (A_1 \rtimes B)\langle a \rangle\) with \(b^a = b^{-1}\phi(b)\) for all \(b \in B\), where \(\phi : B \to A_1\) is a derivation of \(B\) into \(A_1\).

**Proof.** It is easy to see that each periodic normal subgroup \(H\) of \(G\) is contained in \(A\), because \(AH = A(AH \cap B)\) and \(AH \cap B = 1\). Therefore the core \(A_1 = \cap_{g \in G} A^g\) of \(A\) in \(G\) is the maximal periodic normal subgroup of \(G\).

Assume first that \(A_1 = 1\) and let \(B_1\) be the core of \(B\) in \(G\). Then \(B_1 \neq 1\) by the theorem of Zaitsev noted above and so the factor group \(G/B_1\) is periodic, because it is the product of two periodic subgroups \(AB_1/B_1\) and \(B/B_1\). Moreover, since the centralizer \(C_G(B_1)\) of \(B_1\) in \(G\) contains \(B\), the group \(G\) induces on \(B_1\) a periodic group of automorphisms which is isomorphic to the factor group \(A/C_A(B_1)\). As is well-known, a periodic group of automorphisms of any locally cyclic torsion-free group is of order 2. Therefore the order of \(A/C_A(B_1)\) does not exceed 2 and hence either \(A = C_A(B_1)\) or \(A = C_A(B_1)\langle a \rangle\) with \(a \in A\) and \(a^2 \in C_A(B_1)\).

On the other hand, since the centralizer \(C_G(B_1) = C_A(B_1)B\) is normal in \(G\) and periodic over \(B_1\), its derived subgroup \(C_G(B_1,)'\) is periodic by Lemma 4.2 and normal in \(G\). Therefore \(C_G(B_1,)' \leq A_1 = 1\) and hence \(C_G(B_1) = C_A(B_1) \times B\). But then again \(C_A(B_1)\) is normal in \(G\) and so \(C_A(B_1) = 1\). Thus in the case \(A_1 = 1\) we have either \(A = 1\) and \(G = B\) or \(A = \langle a \rangle\) with \(a^2 = 1\) and \(G = B \rtimes \langle a \rangle\) with \(b^a = b^{-1}\) for all \(b \in B\).

Finally, returning now to the general case, we derive that either \(G = A \rtimes B\) or \(G = (A_1 \times B)\langle a \rangle\) with \(b^a = \phi(b)b^{-1}\) for every \(b \in B\) and some element \(\phi(b) \in A_1\). Moreover, since \(\phi(bc)(bc)^{-1} = (bc)^a = b^a c^a = (\phi(b)b^{-1})(\phi(c)c^{-1}) = (\phi(b)\phi(c)) b^{-1}\), it follows that \(\phi(bc) = \phi(b)\phi(c) b\) for any \(b, c \in B\). The latter means in particular that the mapping \(\phi : B \to A_1\) is a derivation of \(B\) into \(A_1\), as claimed. \(\Box\)

5. Products of finitely many periodic locally cyclic groups. A well-known theorem of Huppert cited in the introduction says that every finite group of
Lemma 5.2. Let $G = A_1 A_2 \cdots A_n$ with pairwise permuting cyclic subgroups $A_i$ for $1 \leq i \leq n$ is supersoluble. This result was later extended to products of pairwise permuting locally cyclic Chernikov groups by Tomkinson [10]. He proved that in this case $G = A_1 A_2,\ldots, A_n$ is a locally supersoluble Chernikov group. In this section, we generalize this result to products of arbitrary periodic locally cyclic groups. Recall that a group is said to be hyperabelian (respectively, hypercyclic) if it has an ascending series of normal subgroups with abelian (respectively cyclic) factors.

**Lemma 5.1.** Let $G = A_1 A_2 \cdots A_n$ be the product of pairwise permutable periodic locally cyclic subgroups $A_i$. If the set $\pi = \bigcup_{i=1}^{n} \pi(A_i)$ is finite, $p$ is the largest prime in $\pi$, $P_i$ is the Sylow $p$-subgroup of $A_i$, and $Q_i$ is the $p$-complement to $P_i$ in $A_i$ for each $1 \leq i \leq n$, then $G$ is a $\pi$-group, $P = P_1 P_2 \cdots P_n$ is a normal Sylow $p$-subgroup of $G$, and $Q = Q_1 Q_2 \cdots Q_n$ is a $p$-complement to $P$ in $G$.

**Proof.** Since each of the $A_i$ is a subgroup of Prüfer rank 1, the group $G = A_1 A_2 \cdots A_n$ is hyperabelian of finite Prüfer rank by [2, Theorem 3.1]. Therefore, arguing by induction on $n$ and applying [1, Corollary 3.2.7], and [2, Lemma 3.2], we derive that $G$ is a $\pi$-group, $P = P_1 P_2 \cdots P_n$ is a Sylow $p$-subgroup of $G$ and $Q = Q_1 Q_2 \cdots Q_n$ is a complement to $P$ in $G$. Moreover, taking into account that the subgroups $A_i A_j$ are locally supersoluble by Lemma 2.3, we conclude that $P_i A_j \leq P$ for all $i, j$ and so $P$ is a normal subgroup of $G$. □

**Lemma 5.2.** Let $G = A_1 A_2 \cdots A_n$ be the product of pairwise permutable locally cyclic subgroups $A_i$. If the group $G$ is periodic and the set $\pi(G)$ is finite, then $G$ is locally supersoluble.

**Proof.** Since $\pi(G)$ is finite, every locally cyclic subgroup $A_i$ is a Chernikov group, i.e. a finite extension of a direct product of finitely many quasicyclic subgroups. Therefore $G$ is a locally supersoluble Chernikov group by [10, Theorem B]. □

**Proof of Theorem 1.3.** Let $G = A_1 A_2 \cdots A_n$ be the product of pairwise permutable periodic locally cyclic subgroups $A_i$. Then $G$ is a periodic group by Lemma 5.1. If the set $\pi(G)$ is finite, then the group $G$ is locally supersoluble hypercyclic by Lemma 5.2. In the other case the set $\pi(G)$ is infinite and thus it can be presented as a union $\pi(G) = \bigcup_{i=1}^{\infty} \pi_i$ of finite subsets $\pi_i$ such that $\pi_i \subset \pi_{i+1}$ for all $i \geq 1$. Let $P_{ij}$ be the Sylow $\pi_i$-subgroup of $A_j$ for $1 \leq j \leq n$ and $G_i = P_{i1} P_{i2} \cdots P_{in}$. Then $G_i$ is a Sylow $\pi_i$-subgroup of $G$ by Lemma 5.1 which is locally supersoluble as a group for each $i \geq 1$ by Lemma 5.2. Since $G = \bigcup_{i=1}^{\infty} G_i$, the group $G$ is also locally supersoluble, as claimed. □

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Bernhard Amberg
Institut für Mathematik
Johannes Gutenberg-Universität Mainz
55099 Mainz
Germany
e-mail: amberg@uni-mainz.de
YAROSLAV SYSAK
Institute of Mathematics
Ukrainian National Academy of Sciences
Kiev 01601
Ukraine
e-mail: sysak@imath.kiev.ua

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