Diffeomorphism groups of balls and spheres

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Abstract

In this paper we discuss the relationship between groups of diffeomorphisms of spheres and balls. We survey results of a topological nature and then address the relationship as abstract (discrete) groups. We prove that the identity component $\text{Diff}_0^\infty(S^{2n-1})$ of the group of $C^\infty$ diffeomorphisms of $S^{2n+1}$ admits no nontrivial homomorphisms to the group of $C^1$ diffeomorphisms of the ball $B^n$ for any $n$ and $m$. This result generalizes theorems of Ghys and Herman.

We also examine finitely generated subgroups of $\text{Diff}_0^\infty(S^n)$ and produce an example of a finitely generated torsion free group $\Gamma$ with an action on $S^1$ by $C^\infty$ diffeomorphisms that does not extend to a $C^1$ action of $\Gamma$ on $B^2$.

1 Introduction

Let $M$ be a manifold and let $\text{Diff}_0^r(M)$ denote the group of isotopically trivial $C^r$-diffeomorphisms of $M$. If $M$ has boundary $\partial M$, there is a natural map

$$\pi : \text{Diff}_0^r(M) \to \text{Diff}_0^r(\partial M)$$

given by restricting the domain of a diffeomorphism to the boundary. The map $\pi$ is surjective, as any isotopically trivial diffeomorphism $f$ of the boundary can be extended to a diffeomorphism $F$ of $M$ supported on a collar neighborhood $N \cong \partial M \times I$ of $\partial M$ by taking a smooth isotopy $f_t$ from $f$ to the identity, and defining $F$ to agree with $f_t$ on $\partial M \times \{t\}$.

One way to measure the difference between the groups $\text{Diff}_0^r(M)$ and $\text{Diff}_0^r(\partial M)$ is to ask whether $\pi$ admits a section. By section, we mean a map

$$\phi : \text{Diff}_0^r(\partial M) \to \text{Diff}_0^r(M)$$

such that $\pi \circ \phi$ is the identity on $\text{Diff}_0^r(\partial M)$. There are several categories in which to ask this, namely

i) **Topological**: Require $\phi$ to be continuous, ignoring the group structure.

ii) **(Purely) group-theoretic**: Only require $\phi$ to be a group homomorphism, ignoring the topological structure on $\text{Diff}_0^r(M)$.

iii) **Extensions of group actions**: In the case where no group-theoretic section exists, we ask the following local (in the sense of group theory) question. For which finitely generated groups $\Gamma$ and homomorphisms $\rho : \Gamma \to \text{Diff}_0^r(\partial M)$ does there exist a homomorphism $\phi : \Gamma \to \text{Diff}_0^r(M)$ such that $\pi \circ \phi = \rho$? If such a homomorphism exists, we say that $\phi$ extends the action of $\Gamma$ on $\partial M$ to a $C^r$ action on $M$.
In this paper, we treat the case of the ball \( M = B^{n+1} \) with boundary \( S^n \). Note in the category of homeomorphisms rather than diffeomorphisms, there is a natural way to extend homeomorphisms of \( S^n \) to homeomorphisms of \( B^{n+1} \). This is by “coning off” the sphere to the ball and extending each homeomorphism to be constant along rays. The result is a continuous group homomorphism

\[
\phi : \text{Homeo}_0(S^n) \to \text{Homeo}_0(B^{n+1})
\]

which is also a section of \( \pi : \text{Homeo}_0(B^{n+1}) \to \text{Homeo}_0(S^n) \) in the sense above. We will see, however, that the question of sections for groups of diffeomorphisms is much more interesting!

**Summary of results**

Our goal in this work is to paint a relatively complete picture of known and new results for the ball \( B^n \). Here is an outline.

- **Topological sections.** In Section 2 we give brief survey of known results on existence and nonexistence of topological sections, and the relationship between topological sections and exotic spheres. The reader may skip this section if desired; it stands independent from the rest of this paper.

- **Group-theoretic sections.** In contrast with the topological case, it is a theorem of Ghys that no group theoretic sections \( \phi : \text{Diff}^r_0(S^n) \to \text{Diff}^r_0(B^{n+1}) \) exist for any \( n \) or \( r \). A close reading of Ghys’ work in [7] produces finitely generated subgroups of \( \text{Diff}_0(S^{2n-1}) \) that fail to extend to \( \text{Diff}_0(B^{2n}) \) and we give an explicit presentation of such a group in Section 3. These examples rely heavily on the dynamics of finite order diffeomorphisms.

- **Extending actions of torsion free groups.** Building on Ghys’ work and using results of Franks and Handel involving distortion elements in finite groups, in Section 4 we explicitly construct a group \( \Gamma \) to prove the following.

**Theorem 1.1.** There exists a finitely generated, torsion-free group \( \Gamma \) and homomorphism \( \rho : \Gamma \to \text{Diff}^\infty(S^1) \) that does not extend to a \( C^1 \) action of \( \Gamma \) on \( B^2 \).

Note that in contrast to Theorem 1.1 any action of \( \mathbb{Z} \) of a free group, or any action of any group that is conjugate into the standard action of \( \text{PSL}(2, \mathbb{R}) \) on \( S^1 \) will extend to an action by diffeomorphisms on \( B^2 \).

- **Exotic homomorphisms.** In Section 5 we show that the failure of \( \pi : \text{Diff}_0^\infty(B^{n+1}) \to \text{Diff}_0^\infty(S^n) \) to admit a section is due (at least in the case where \( n \) is odd) to a fundamental difference between the algebraic structure of groups of diffeomorphisms of spheres and groups of diffeomorphisms of balls. We prove

**Theorem 1.2.** There is no nontrivial group homomorphism \( \text{Diff}_0^\infty(S^{2k-1}) \to \text{Diff}_0^1(B^m) \) for any \( m, k \geq 1 \).

This generalizes a result of M. Herman in [10]. Theorem 1.2 also stands in contrast to situation with homeomorphisms of balls and spheres – any continuous foliation of \( B^{n+1} \) by \( n \)-spheres can be used to construct a continuous group homomorphism \( \text{Homeo}_0(S^n) \to \text{Homeo}_0(B^{n+1}) \).
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2 Topological sections: known results

In order to contrast our work on group-theoretic sections with the (fundamentally different) question of topological sections, we present a brief summary of known results in the topological case.

Let $\text{Diff}(B^n)$ denote the group of smooth diffeomorphisms of $B^n$ that restrict to the identity on $\partial B^n = S^{n-1}$. The projection $\text{Diff}(B^n) \to \text{Diff}(S^{n-1})$ is a fibration with fiber $\text{Diff}(B^n \cap \partial)$. Hence, asking for a topological section of $\pi$ amounts to asking for a section of this bundle.

In low dimensions ($n \leq 3$), it is known that the fiber $\text{Diff}(B^n \cap \partial)$ is contractible, so a topological section exists. The $n = 2$ case is a well known result of Smale [12], and the (even more difficult) $n = 3$ case is due to Hatcher [9]. Incidentally, $\text{Diff}_0(B^3 \cap \partial)$ is also contractible and this is quite elementary – an element of $\text{Diff}(B^3 \cap \partial)$ is a nonincreasing or nondecreasing function of the closed interval, and we can explicitly define a retraction of $\text{Diff}(B^3 \cap \partial)$ to the identity via

$$r : \text{Diff}(B^1 \cap \partial) \times [0, 1] \to \text{Diff}(B^1 \cap \partial)$$

$$r(f, t)(x) = tf(x) + (1 - t)x.$$

Whether $\text{Diff}(B^4 \cap \partial)$ is contractible is an open question. To the best of the author's knowledge, whether $\text{Diff}_0(B^4) \to \text{Diff}_0(S^3)$ has a section is also open. However, in higher dimensions $\text{Diff}(B^n \cap \partial)$ is not always contractible, giving a first obstruction to a section. This is related to the existence of exotic smooth structures on spheres.

Exotic spheres

Let $f \in \text{Diff}(B^n \cap \partial)$ be a diffeomorphism. We can use $f$ to glue a copy of $B^n$ to another copy of $B^n$ along the boundary, producing a sphere $S^n_f$ with a smooth structure. If $f$ lies in the identity component of $\text{Diff}(B^n \cap \partial)$, then $S^n_f$ will be smoothly isotopic to the standard $n$-sphere $S^n$. If not, there is no reason that $S^n_f$ need even be diffeomorphic to $S^n$. In fact, it follows from the pseudoisotopy theorem of Cerf (in [9]) that for $n \geq 5$, the induced map from $\pi_0(\text{Diff}(B^n \cap \partial))$ to the group of exotic $n$-spheres is injective.

Moreover – and more pertinent to our discussion – it follows from Smale’s h-cobordism theorem that the map from $\pi_0(\text{Diff}(B^n \cap \partial))$ to exotic $n$-spheres is surjective. In particular, this means that in any dimension $n$ where exotic spheres exist, $\pi_0(\text{Diff}(B^n \cap \partial)) \neq 0$. Let us now return to the fibration $\pi : \text{Diff}(B^n) \to \text{Diff}(S^{n-1})$ and look at the tail end of the long exact sequence in homotopy groups. If we consider just the identity components $\text{Diff}_0(B^n) \to \text{Diff}_0(S^{n-1})$ we get

$$\ldots \to \pi_1(\text{Diff}_0(B^n)) \to \pi_1(\text{Diff}_0(S^{n-1})) \to \pi_0(\text{Diff}(B^n \cap \partial)) \to 0$$

Thus, whenever exotic spheres exist, the connecting homomorphism

$$\pi_1(\text{Diff}_0(S^{n-1})) \to \pi_0(\text{Diff}(B^n \cap \partial))$$
is nonzero, and so no section of the bundle exists.

**Question 2.1.** Does this bundle have a section in any dimensions \( n \geq 5 \) where exotic spheres do not exist?

We remark that for all \( n \geq 5 \), it is known that \( \text{Diff}(B^n \text{ rel } \partial) \) has some nontrivial higher homotopy groups. In fact the author learned from Allen Hatcher that recent work of Crowey and Schick \([5]\) shows that \( \text{Diff}(B^n \text{ rel } \partial) \) has infinitely many nonzero higher homotopy groups whenever \( n \geq 7 \).

### 3 Group-theoretic sections

Recall from the introduction that a *group-theoretic section* of \( \pi \) is a (not necessarily continuous) group homomorphism \( \phi : \text{Diff}^0_r(S^{n-1}) \to \text{Diff}^0_r(B^n) \) such that \( \pi \circ \phi \) is the identity. Recall also that, when \( \Gamma \) is a group and \( \rho : \Gamma \to \text{Diff}^0_r(S^{n-1}) \) specifies an action of \( \Gamma \) on \( S^{n-1} \), we say that \( \rho \) extends to a \( C^r \) action on \( B^n \) if there is a homomorphism \( \phi : \Gamma \to \text{Diff}^r_r(B^n) \) such that \( \pi \circ \phi = \rho \).

The question of existence of group-theoretic sections for spheres and balls is completely answered by the following theorem of Ghys.

**Theorem 3.1** (Ghys, \([7]\)). There is no section of \( \text{Diff}^1_0(B^{n+1}) \to \text{Diff}^1_0(S^n) \).

Moreover, there is no extension of the standard embedding of \( \text{Diff}^\infty_0(S^n) \) in \( \text{Diff}^1_0(S^n) \) to a \( C^1 \) action of \( \text{Diff}^\infty_0(S^n) \) on \( B^{n+1} \).

We ask to what extent the failure of sections holds *locally*, i.e. for finitely generated subgroups. At one end of the spectrum, if \( \Gamma \) is a free group, and \( \rho : \Gamma \to \text{Diff}^0_0(S^n) \) is any action, we can build an extension of \( \rho \) by taking arbitrary \( C^r \) extensions of the generators of \( \rho(\Gamma) \) – for instance, by using the collar neighborhood strategy sketched in the introduction. There are no relations to satisfy so this defines a homomorphism and gives a \( C^r \) action of \( \Gamma \) on \( B^{n+1} \).

At the other end, a careful reading of Ghys’ proof of Theorem 3.1 gives the following corollary of Theorem 3.1.

**Corollary 3.2.** For any \( n \), there exists a finitely generated subgroup \( \Gamma \subset \text{Diff}^\infty_0(S^{2n-1}) \) that does not extend to a subgroup of \( \text{Diff}^1_0(B^{2n}) \).

Although this follows directly from Ghys’ proof of Theorem 3.1, we outline the argument below in order to illustrate some of Ghys’ techniques. We pay special attention to the \( n = 1 \) case because we will use part of this construction in Section 4. The reader will note that the argument is unique to odd-dimensional spheres, so does not answer the following question.

**Question 3.3.** Is there a finitely generated group \( \Gamma \) and a homomorphism \( \rho : \Gamma \to \text{Diff}^\infty_0(S^{2n}) \) that does not extend to a \( C^1 \) (or even \( C^r \) for some \( 1 < r \leq \infty \)) action on \( B^{2n+1} \)?

**Sketch proof of Corollary 3.2.** In the \( n = 1 \) case, we can take \( \Gamma \) to be a two-generated group as follows. Any rotation of \( S^1 \) can be written as a commutator – a nice argument for this using some hyperbolic geometry appears in Proposition 5.11 of \([8]\) or Proposition 2.2 of \([7]\). So let \( f \) and \( g \) be such that their commutator \([f, g]\) is a finite order rotation, say a rotation of order 2. We may even take \( f \) and \( g \) to be hyperbolic.
The idea is to show that the finite order element $\tilde{f}$ and $\tilde{g}$ be lifts of $f$ and $g$ to diffeomorphisms of the threefold cover of $S^1$. Since $f$ and $g$ have fixed points, we can choose $\tilde{f}$ and $\tilde{g}$ to be the (unique) lifts that have fixed points. Then the commutator $[\tilde{f}, \tilde{g}]$ will be rotation of the threefold cover of $S^3$ by $\pi/3$. Since the threefold cover of $S^1$ is also $S^1$, we can consider $\tilde{f}$ and $\tilde{g}$ as diffeomorphisms of $S^1$. They then generate a subgroup $\Gamma$ of $\text{Diff}_{0}^\infty(S^1)$ satisfying the relations

i) $[\tilde{f}, \tilde{g}]^6 = 1$ and

ii) $[\tilde{f}, [\tilde{f}, \tilde{g}]^2] = [\tilde{g}, [\tilde{f}, \tilde{g}]^2] = 1$.

The second relation here comes from the fact that $[\tilde{f}, \tilde{g}]^2$ is the covering transformation. There may, incidentally, be other relations satisfied by this group, but this is of no importance to us.

We claim that $\Gamma$ does not extend to a subgroup of $\text{Diff}_{0}(B^2)$. To see this, we argue by contradiction. Assume that there is a homomorphism $\phi : \Gamma \to \text{Diff}_{0}^1(B^2)$ such that for any $\gamma \in \Gamma$, the restriction of $\phi(\gamma)$ to $\partial B^2 = S^1$ agrees with $\gamma$.

Let $r$ denote rotation of $S^1$ by $2\pi/3$, this is the element $[\tilde{f}, \tilde{g}]^2 \in \Gamma$ and so $\phi(r)$ is an order 3 diffeomorphism of the ball acting by rotation on the boundary. In particular, it follows from Kerekjarto’s theorem that $\phi(r)$ is conjugate to an order three rotation, hence has a unique interior fixed point $x$. (A reader unfamiliar with Kerekjarto’s theorem on finite order diffeomorphisms may wish to consult the very nice proof by Constantin and Kolev in [3].)

By construction, $\tilde{f}$ and $\tilde{g}$ both commute with $r$ so $\phi(\tilde{f})$ and $\phi(\tilde{g})$ commute with $\phi(r)$, hence fix $x$. The derivatives $D\phi(\tilde{f})_x$ and $D\phi(\tilde{g})_x$ commute with $D\phi(r)_x$ which acts as rotation by $2\pi/3$ on the tangent space. Moreover, $[D\phi(\tilde{f})_x, D\phi(\tilde{g})_x] = D\phi(r)_x$.

But the centralizer of rotation by $2\pi/3$ in $\text{SL}(2, \mathbb{R})$ is abelian, so writing $D\phi(r)_x$ as a commutator of elements in its centralizer is impossible. This is the desired contradiction, showing that no extension of the action of $\Gamma$ exists.

The case for $n > 1$ is similar. We consider $S^{2n-1}$ as the unit sphere

$$\{(z_1, ..., z_n) \in \mathbb{C}^n \mid \sum_{i=1}^{n} |z_i|^2 = 1\}.$$

The idea is to show that the finite order element

$$r : (z_1, ..., z_n) \mapsto (\lambda_1 z_1, ..., \lambda_n z_n)$$

where $\lambda_i$ are distinct $p^{th}$ roots of 1, can also be expressed as a product of commutators of elements $f_1, f_2, ... f_k$ that each commute with a power of $r$. Then we can take $\Gamma$ to be the subgroup generated by the diffeomorphisms $f_i$. Supposing again for contradiction that $\phi : \Gamma \to \text{Diff}_{0}^1(B^{2n})$ is a section, one can show with an argument using Smith theory that the diffeomorphism $\phi(r) \in \text{Diff}_{0}^1(B^{2n})$ has a single fixed point $x$. It follows in a similar way to the $n = 1$ case that the derivative of $\phi(r)$ at $x$ has abelian centralizer, giving a contradiction.

\[\square\]

4 \hspace{1em} Actions of torsion free groups

The proof of Corollary 3.2 relied heavily on finite order diffeomorphisms. Ghys’ proof of Theorem 4.1 – even in the case of even dimensional spheres – also hinges on the clever
use of finite order diffeomorphisms (and the tools that they bring: Smith theory, fixed sets, derivatives in \( SO(n) \), etc.). Thus, we ask the following refinement of Question 3.3.

**Question 4.1.** Is there a finitely generated, torsion-free group \( \Gamma \) and homomorphism \( \rho : \Gamma \to \text{Diff}^\infty_0(S^n) \) that does not extend to a smooth (or even \( C^r \), for some \( r \geq 1 \)) action on \( B^{n+1} \)?

The following theorem answers this question for \( n = 1 \).

**Theorem 1.1.** There exists a finitely generated, torsion-free group \( \Gamma \) and homomorphism \( \phi : \Gamma \to \text{Diff}^\infty_0(S^1) \) that does not extend to a \( C^1 \) action on \( B^2 \).

Our proof modifies Ghys’ construction by using a dynamical constraint based on algebraic structure to force a diffeomorphism to act by rotation at a fixed point. The algebraic structure in question is the notion of distorted elements and the constraint on dynamics follows from a powerful theorem of Franks and Handel. We provide a brief introduction in the following few paragraphs; a reader familiar with this work may wish to skip ahead to Corollary 4.3 and the proof of Theorem 1.1.

**Distorted elements**

Let \( \Gamma \) be a finitely generated group, and let \( S = \{ s_1, \ldots, s_k \} \) be a symmetric generating set for \( \Gamma \). For an element \( g \in \Gamma \), the word length (or \( S \)-word length) of \( g \) is the length of the shortest word in the letters \( s_1, \ldots, s_k \) that represents \( g \). We denote word length of \( g \) by \( |g| \).

We say that \( g \in \Gamma \) is distorted provided that \( g \) has infinite order and that

\[
\liminf_{n \to \infty} \frac{|g^n|}{n} = 0.
\]

Although the word length of \( g^n \) depends on the choice of generating set \( S \) for \( \Gamma \), it is not hard to see that whether \( g \) is distorted or not is independent of the choice of \( S \).

In [6], Franks and Handel prove a theorem about the dynamics of actions of distorted elements in finitely generated subgroups of \( \text{Diff}_0(\Sigma) \), where \( \Sigma \) is a closed, oriented surface. The following theorem is a consequence of their main result. We use the notation \( \text{fix}(g) \) for the set of points \( x \) such that \( g(x) = x \), and \( \text{per}(g) \) for the set of periodic points for \( g \).

**Theorem 4.2 (Franks-Handel, [6]).** Suppose that \( f \) is a distorted element in some finitely generated subgroup of \( \text{Diff}_0^1(S^2) \). Suppose also that for the smallest \( n > 0 \) such that \( \text{fix}(f^n) \neq \emptyset \), there are at least three points in \( \text{fix}(f^n) \). Then \( \text{per}(f) = \text{fix}(f^n) \).

We can derive a corresponding statement about actions on the disc.

**Corollary 4.3.** Suppose that \( f \) is a distorted element in some finitely generated subgroup of \( \text{Diff}_0^1(B^2) \) with a periodic point on the boundary of period \( k > 1 \). Then \( \text{fix}(f) \) consists of a single point.

**Proof.** Suppose \( f \) is distorted in \( \Gamma \subset \text{Diff}_0^1(B^2) \). By the Brouwer fixed point theorem, \( f \) has at least one fixed point. Since \( f \) has a periodic point on the boundary \( S^1 \), all fixed points for \( f \) lie in the interior of \( B^2 \). Double \( B^2 \) along the boundary to get the sphere, and double the action of \( \Gamma \). This can be smoothed to a \( C^1 \) action on \( S^2 \) using...
the techniques of K. Parkhe in [11]. The smoothing construction will not change the set of fixed or periodic points. Applying Theorem 4.2 to the action on $S^2$, we conclude that the doubled action on the sphere can have at most two fixed points (since there are non-fixed periodic points), so the original action of $f$ has a single fixed point.

With Corollary 4.3 as a tool, we are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1
Recall the group $\Gamma \subset \text{Diff}_0(S^1)$ from the proof of Corollary 3.2. It is generated by two elements $\hat{f}$ and $\hat{g}$, satisfying the relations $[\hat{f}, \hat{g}]^0 = 1$ and $[\hat{f}, [\hat{f}, \hat{g}]] = [\hat{g}, [\hat{f}, \hat{g}]] = 1$. Let $\hat{t}$ be the lift of $t$ to the universal central extension $\text{Diff}_0^\infty(\mathbb{R})$ of $\text{Diff}_0^\infty(S^1)$. This is a central extension of $\Gamma$ by an element $t$, satisfying the relation $t = [\hat{f}, \hat{g}]^2$. Explicitly, we can realize $\hat{t}$ as the group of all lifts of elements of $\Gamma$ to the infinite cyclic cover $\mathbb{R}$ of $S^1$. Since $\text{Diff}_0^\infty(\mathbb{R})$ is torsion free, $\hat{t}$ is as well. Finally, let $\hat{\Gamma}$ be the HNN extension of $\hat{t}$ obtained by adding a generator $a$ and relation $ata^{-1} = t^4$. HNN extensions of torsion free groups are torsion free, so $\hat{t}$ is torsion free also.

We now construct a homomorphism $\rho : \hat{\Gamma} \to \text{Diff}_0^\infty(S^1)$ and show that it does not admit an extension $\phi : \hat{\Gamma} \to \text{Diff}^2(B^2)$. The homomorphism $\rho$ will not be faithful (and in fact the image $\rho(\hat{\Gamma})$ will have torsion), but this is besides the point – the interesting part of this question is extending $\rho$ as an action of $\Gamma$. For example, a nonfaithful action (with torsion or not) of a free group $F$ on $S^1$ always extends to the disc as an action of a free group just by arbitrarily extending each generator.

To define $\rho$, set $\rho(a) = \text{id}$, and for all $\gamma \in \hat{\Gamma}$ let $\rho(\gamma)$ be the action of $\gamma$ on the quotient $\mathbb{R}/\mathbb{Z}$, i.e. the quotient action on the original circle $S^1$. In other words, the image of $\rho$ in $\text{Diff}_0^\infty(S^1)$ is the group $\Gamma$ of Corollary 3.2. Note that the fact that $\rho(t) = [\rho(\hat{f}), \rho(\hat{g})]$ is rotation by $2\pi/3$ ensures that the relation $\rho(a)\rho(t)\rho(a)^{-1} = \rho(t)^4$ is satisfied.

We claim that this action does not extend to a $C^2$ action on the disc. To see this, suppose for contradiction that some extension $\phi : \hat{\Gamma} \to \text{Diff}^2(B^2)$ exists. If $\phi(t)$ has finite order, then it must be rotation by $\pi/3$, and so has a unique fixed point $x$. Now we make the same argument (verbatim!) as in the proof of Corollary 3.2 since $\phi(t)$ commutes with $\phi(\hat{f})$ and $\phi(\hat{g})$, both $\phi(\hat{f})$ and $\phi(\hat{g})$ fix $x$ and have derivatives at $x$ in $\text{SO}(2)$. This contradicts the fact that $\phi(t)$ is the commutator of $\phi(\hat{f})$ and $\phi(\hat{g})$.

If instead $\phi(t)$ has infinite order, then it is a distorted element in $\phi(\hat{t})$. We know also that the restriction of $\phi(t)$ to the boundary is rotation by $2\pi/3$. Applying Corollary 4.3 we conclude that $\phi(t)$ has a single fixed point $x$. If the derivative $D\phi(t)_x$ were a nontrivial rotation of order at least 3, we could again look at derivatives at $x$ and give the same argument as in the finite order case to get a contradiction. Thus, it remains only to show that $D\phi(t)_x$ is a rotation of order at least 3. We show that it is rotation of order 3 exactly.

Lemma 4.4. The derivative $D\phi(t)_x$ is a rotation of order 3.

Proof. Since $t$ is central in $\Gamma$ and since $\rho(a)\rho(t)\rho(a)^{-1}x = \rho(t)^4x = x$ implies that $\rho(a)x = x$, the whole group $\phi(\hat{t})$ fixes $x$. Moreover, the derivatives of $\rho(t)$ and $\rho(a)$ at $x$ satisfy

$$D\phi(a)_xD\phi(t)_xD\phi(a)_x^{-1} = D\phi(t)_x^4.$$
This relation in $GL(2, \mathbb{R})$ implies that either $D\phi(t)_x$ has a fixed tangent direction or is an order 3 rotation. Our strategy to show that it is order 3 is to compare the “rotation number” of $\phi(t)$ at the fixed point and on the boundary.

Blow up the disc $B^2$ at $x$ to get a $C^0$ action of $\hat{\Gamma}$ on the closed annulus, $A$. The action of $\hat{\Gamma}$ on one boundary component of $A$ is the linear action on the space of tangent directions at $x$ (so $t$ either acts with a fixed point or as an order 3 rotation), and on the other boundary it is the original action on $\partial B^2$ as an order 3 rotation.

With this setup, we can apply the notion of “linear displacement” from [6] and conclude that since $\rho(t)$ is distorted, it must act on each boundary component of $A$ with the same rotation number and hence act as an order 3 rotation on both (See lemma 6.1 of [6]). But instead of defining “linear displacement” and “rotation number” here, it will be faster to give a complete, direct proof for our special case. The reader familiar with rotation numbers for circle homeomorphisms will see that it readily generalizes.

Suppose for contradiction that $t$ acts on one boundary component of $A$ with a fixed point. Let $\hat{A}$ denote the universal cover of $A$, identified with $\mathbb{R} \times [0, 1]$ with covering transformation $T : (x_1, x_2) \mapsto (x_1 + 1, x_2)$.

Let $\hat{t} \in \text{Homeo}_0(\hat{A})$ be the lift of the action of $t$ to $\hat{A}$ with a fixed point on one boundary component, without loss of generality say $(x_0, 1)$ is fixed. Then $\hat{t}$ acts on $\mathbb{R} \times \{0\}$ as translation by $m + 1/3$ for some integer $m$. Let $\hat{a}$ be any lift of the action of $a$.

Now $\hat{a}(\hat{t})^n\hat{a}^{-1}$ is a lift of $(\hat{t})^4^nT^l$ for some $l$. In particular, considering the displacement difference between the points $(x_0, 0)$ and $(x_0, 1)$ we have

$$\|\hat{a}(\hat{t})^n\hat{a}^{-1}(x_0, 1) - \hat{a}(\hat{t})^n\hat{a}^{-1}(x_0, 0)\| = \|(\hat{t})^4^n(x_0, 1) - (\hat{t})^4^n(x_0, 0)\|$$

$$= \|(x_0, 1) - (x_0 + (m + 1/3)^n, 1)\|$$

$$\sim (m + 1/3)^n$$

However, the distance $\|\hat{a}(\hat{t})^n\hat{a}^{-1}(x_0, 1) - \hat{a}(\hat{t})^n\hat{a}^{-1}(x_0, 0)\|$ grows linearly in $n$ – it is bounded by the maximum displacement of $\hat{a}$ and $\hat{t}$. Precisely, if

$$d = \max_{z \in \hat{A}} \{\max\{\|\hat{a}(z) - z\|, \|\hat{t}(z) - z\|\}\}$$

then we have

$$2(n + 2)d + 1 \leq \|\hat{a}(\hat{t})^n\hat{a}^{-1}(x_0, 1) - \hat{a}(\hat{t})^n\hat{a}^{-1}(x_0, 0)\|$$

and this is our desired contradiction.

\[\square\]

Remark 4.5. It is possible to modify the construction in the proof Theorem 1.1 to avoid finite order elements. The idea is to modify $\rho(\hat{f})$ slightly so that the diffeomorphism $\rho(t) := [\rho(\hat{f}), \rho(\hat{g})]$ is the composition of an order 3 rotation $r$ with an $r$-equivariant diffeomorphism $h$ supported on a collection of small intervals in $S^1$ that is conjugate to a translation on these intervals. We then modify $\rho(a)$ so that it is remains $r$-equivariant, but is conjugate to an expansion on the intervals of supp($h$) – i.e. so that $h$ and $\rho(a)$ act by a standard Baumslag-Solitar action on these intervals. Done correctly, $\rho(\hat{f})$, $\rho(\hat{g})$ and $\rho(a)$ will be infinite order diffeomorphisms, and will generate a subgroup of $\text{Diff}_r^\infty(S^1)$ satisfying the relations $[\rho(t), \rho(\hat{f})] = [\rho(t), \rho(\hat{g})] = 1$ and $\rho(t)\rho(a)\rho(t)^{-1} = \rho(a)^4$. We leave the details to the reader.
5 (Non-existence of) exotic homomorphisms

In [10], Michael Herman proved the following stronger version of Theorem 3.1 in the case where \( n = 1 \).

**Theorem 5.1** (Herman, [10]). Any group homomorphism \( \text{Diff}^\infty_0(S^1) \rightarrow \text{Diff}^1_0(B^2) \) is trivial.

Herman’s key tools are the deep fact that \( \text{Diff}^\infty_0(S^1) \) is simple, and the easy fact that \( S^1 \) is a finite cover of itself. We combine some of these ideas with the techniques of Ghys in [7] to prove a similar theorem for any odd dimensional sphere, with any group of diffeomorphisms of a ball as the target. This is Theorem 1.2 as stated in the introduction.

**Theorem 1.2**. There is no nontrivial group homomorphism \( \text{Diff}^\infty_0(S^{2k-1}) \rightarrow \text{Diff}^1_0(B^m) \) for any \( m, k \geq 1 \).

**Proof.** Let \( n = 2k - 1 \) and identify \( S^n \) with the unit sphere

\[ \{(z_1, ..., z_k) \in \mathbb{C}^n \mid \sum_{i=1}^{k} |z_i|^2 = 1\} \]

For any prime \( p \), there is a free \( \mathbb{Z}_p \)-action on \( S^k \) generated by the map

\[ f_p : (z_1, ..., z_k) \mapsto (\mu_1 z_1, ..., \mu_k z_k) \]

where \( \mu_i \) are any \( p^{th} \) roots of unity.

Suppose that we have a nontrivial homomorphism \( \phi : \text{Diff}^\infty_0(S^n) \rightarrow \text{Diff}_0(B^m) \). Since \( \text{Diff}^\infty_0(S^n) \) is a simple group (a deep result due to Mather and Thurston, see e.g. [1] for a proof), \( \phi \) must be injective. By the Brouwer fixed point theorem, \( \phi(f_p) \) must fix a point. Since \( f_p \) is a finite order diffeomorphism, the set \( \text{fix}(\phi(f_p)) \subset B^m \) of fixed points of \( \phi(f) \) is a submanifold of \( B^m \) (one way to see this is to average a metric so that \( f_p \) acts by isometries). That \( f_p \) is orientation preserving and of finite order further implies that \( \text{fix}(\phi(f_p)) \) has codimension at least 2, this is because any finite order diffeomorphism \( f \) is an isometry with respect to some metric, and if \( f \) is nontrivial its derivative at a fixed point is a nontrivial finite order element of \( O(n) \).

Let \( H \) be the group of isotopically trivial diffeomorphisms of \( S^n / \langle f_p \rangle \cong S^n \). We have an exact sequence

\[ 0 \rightarrow \mathbb{Z}_p \rightarrow H' \rightarrow H \rightarrow 1 \]

where \( H' \) is the group of all lifts of diffeomorphisms in \( H \) to \( f_p \)-equivariant diffeomorphisms of \( S^n \).

We claim now that \( \mathbb{Z}_p \) is the only normal subgroup of \( H' \). To see this, suppose that \( N \subset H' \) is a normal subgroup. Then the image of \( N \) in \( H \) must either be trivial or all of \( H \). If the image is trivial, then either \( N \) is trivial or \( N = \mathbb{Z}_p \) and we are done. If the image of \( N \) in \( H \) is all of \( H \), we consider a \( S^1 \times ... \times S^1 \) subgroup of \( H \), where the \( i^{th} \) \( S^1 \) factor is the norm 1 complex numbers mod \( \mu_i \). An element \( (\lambda_1, ..., \lambda_k) \in (S^1)^k / \langle \mu_1, ..., \mu_k \rangle \) acts on \( S^n / \langle f_p \rangle \) by pointwise multiplication,

\[ (z_1, ..., z_k) \mapsto (\lambda_1 z_1, ..., \lambda_k z_k) \]

Consider the extension \( \Gamma \) as in the diagram below.
We will show this is impossible. Indeed, it should already seem believable to the reader.

Specifically, $\Gamma$ is the group of all lifts of these actions $(z_1, ... z_k) \mapsto (\lambda_1 z_1, ... \lambda_k z_k)$ to $S^n$, the $p$-fold cover of $S^n/\langle f_p \rangle$. It may be helpful for the reader to consider the $n = 1$ case, in which case we are just working with rotations of $S^1$ and their lifts to a $p$-fold cover of $S^1$.

Note that $N \cap \Gamma$ is a normal subgroup of $\Gamma$ that projects to the full group $S^1 \times ... \times S^1$. In particular, since $(\mu_1^\frac{1}{p}, ... \mu_n^\frac{1}{p}) \in S^1 \times ... \times S^1$, we know that some diffeormorphism $g$ of the form

$$(z_1, ... z_k) \mapsto (\mu_1^{n_1 + \frac{1}{p}} z_1, ... \mu_k^{n_k + \frac{1}{p}} z_k), \quad n_i \in \mathbb{Z}$$

lies in $\Gamma$, hence in $H'$. It follows that $g^n = f_p$ is a generator of $\mathbb{Z}_p$, so $\mathbb{Z}_p \subset N$. Since $\mathbb{Z}_p \subset N$ and $N$ projects to $H$, it follows that $N = H'$, which is what we wanted to show.

Having shown that $\mathbb{Z}_p$ is the only normal subgroup of $H'$, we can conclude that the action of $\phi(H')$ on $\text{fix}(\phi(f_p)) \subset B^m$ is either faithful, trivial, or has kernel $\mathbb{Z}_p$. We already know that $\mathbb{Z}_p$ lies in the kernel – this is $\phi(f_p)$ acting on its fix set – so the action of $\phi(H')$ is not faithful. If the action is trivial, then for $x \in \text{fix}(\phi(f_p))$, we get a representation $D : H' \to \text{GL}(m, \mathbb{R}) \subset \text{GL}(m, \mathbb{C})$ by sending a diffeomorphism $f$ to the derivative of $\phi(f)$ at $x$. Since $\phi(f_p)$ has nontrivial derivative at any point, and $\mathbb{Z}_p = \langle f_p \rangle$ is the only normal subgroup of $H'$, the representation $D$ must be faithful. We will show this is impossible. Indeed, it should already seem believable to the reader that $H'$ is a “large” group and so is not linear. Here is a short, elementary argument to make this clear.

**Proof that $D$ cannot be a faithful representation.** Since $D\phi(f_p)(x)$ has order $p$, after conjugation in $\text{GL}(m, \mathbb{C})$ we may assume it is diagonal of the form

$$
\begin{bmatrix}
\alpha_1 I_{n_1} & 0 & \cdots & 0 \\
0 & \alpha_2 I_{n_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \alpha_k I_{n_k}
\end{bmatrix}
$$

where $\alpha_i$ are each distinct $p^{\text{th}}$ roots of unity, the distinct complex eigenvalues of $D\phi(f_p)(x)$, and $I_{n_i}$ is the $n_i \times n_i$ square identity matrix.

The centralizer of such a matrix in $\text{GL}(m, \mathbb{C})$ is the set of block diagonals of the form

$$
\begin{bmatrix}
A_{n_1} & 0 & \cdots & 0 \\
0 & A_{n_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{n_k}
\end{bmatrix}
$$
with $A_{n_i} \in \text{GL}(n_i, \mathbb{C})$. In other words, the centralizer is a subgroup isomorphic to \( \text{GL}(n_1, \mathbb{C}) \times \text{GL}(n_2, \mathbb{C}) \times \ldots \times \text{GL}(n_k, \mathbb{C}) \). In particular, (after conjugation) we may view $H'$ as a subgroup of $\text{GL}(n_1, \mathbb{C}) \times \text{GL}(n_2, \mathbb{C}) \times \ldots \times \text{GL}(n_k, \mathbb{C})$, with $f_p \in H$ a central element.

Since $D\phi(f_p)(x)$ has order $p$, at least one eigenvalue is not 1. Without loss of generality, assume $\alpha_1 \neq 1$. Now consider the homomorphism $H' \to \mathbb{R}$ given by projecting $\text{GL}(n_1, \mathbb{C}) \times \text{GL}(n_2, \mathbb{C}) \times \ldots \times \text{GL}(n_k, \mathbb{C})$ onto the first factor — i.e., onto $\text{GL}(n_1, \mathbb{C})$ — and then taking the determinant. We may assume that we chose $p > m$, so as to ensure that the image $\alpha_1^{p^i}$ of $f_p$ under this homomorphism is nontrivial. However, we showed above that the subgroup generated by $f_p$ was the only normal subgroup of $H'$. This means that this homomorphism to $\mathbb{R}$ must be faithful — but this is impossible since $H'$ itself is nonabelian. 

Thus, it remains only to deal with the case where $H'$ acts on $\text{fix}(\phi(f_p))$ with kernel $\mathbb{Z}_p$. In this case, we introduce an inductive argument. Consider the diffeomorphism

$$f_{p^2} : (z_1, \ldots, z_k) \mapsto (\nu_1 z_1, \ldots, \nu_k z_k)$$

where $\nu_i^2 = \mu_i$. Then $f_{p^2}$ is an order $p^2$ diffeomorphism acting freely on $S^n$, commuting with $f_p$ and so an element of $H'$. Since $f_{p^2} \notin \mathbb{Z}_p$, we know that $\phi(f_{p^2})$ acts nontrivially on $\text{fix}(\phi(f_p))$. Moreover, $\text{fix}(\phi(f_{p^2})) \subset \text{fix}(\phi(f_p))$, and is a nonempty submanifold of codimension at least two.

As before, we consider a group of diffeomorphisms of a quotient of $S^n$. Let $H_2$ be the group of isotopically trivial diffeomorphisms of $S^n/\langle f_{p^2} \rangle$. Since $S^n/\langle f_{p^2} \rangle$ is a compact manifold, $H_2$ is a simple group. Let $H'_2$ be the group of all lifts of elements of $H_2$ to $S^n$. The argument we gave above for $H$ works (essentially verbatim) to show that $\langle f_p \rangle \cong \mathbb{Z}_p$, and $\langle f_{p^2} \rangle \cong \mathbb{Z}_{p^2}$ are the only normal subgroups of $H'_2$.

Now consider the action of $H'_2$ on $\text{fix}(\langle f_{p^2} \rangle)$. If the action is trivial, we get as before a global fixed point and a linear representation $H'_2 \to \text{GL}(m, \mathbb{R})$. The argument using matrix centralizers above can be applied again in this case to derive a contradiction. Otherwise, the action of $H'_2$ on $\text{fix}(\phi(f_{p^2}))$ in nontrivial. In this case, we can proceed inductively by considering higher powers of $p$ and corresponding diffeomorphisms $f_{p^k}$. Each time we will reduce the dimension of the fix set (a finite process) or derive a contradiction.

\[\square\]

**Remark 5.2.** Note that our proof depended on the fact that $S^{2k-1}$ admits finite order diffeomorphisms that act freely, and so it does not readily generalize to odd dimensional spheres. We conclude with a natural follow-up problem.

**Problem 5.3.** Describe all homomorphisms $\text{Diff}_0^{\infty}(S^{2m}) \to \text{Diff}_0^1(B^m)$. Are any non-trivial?
References

[1] A. Banyaga The structure of classical diffeomorphism groups. Kluwer Academic, 1997.
[2] D. Calegari, M. Freedman Distortion in transformation groups. Geometry & Topology 10 (2006) 267-293.
[3] J. Cerf. La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie. Inst. Hautes Études Sci. Publ. Math. No. 39 (1970) 5-173.
[4] A. Constantin, B. Kolev. The theorem of Kerékjártó on periodic homeomorphisms of the disc and the sphere L’Enseignement Mathématique 40 (1994),193-193.
[5] D. Crowley, T. Schick The Gromoll filtration, KO-characteristic classes and metrics of positive scalar curvature. Preprint. arXiv:1204.6474v3.
[6] J. Franks, M. Handel Distortion Elements in Group actions on surfaces. Duke Math. J. 131, no. 3 (2006), 441-468
[7] E. Ghys Prolongements des difféomorphismes de la sphère. L’Enseignement Mathématique, 37 (1991) 45-59
[8] E. Ghys Groups acting on the circle. L’Enseignement Mathématique, 47 (2001) 329-407
[9] A. Hatcher A proof of the Smale conjecture Diff(S^3) ∼= O(4). Ann. Math. 117 no.3 (1983) 553-607.
[10] M. Herman An application of the simplicity of Diff^k(T^1). Unpublished.
    www.college-de-france.fr\media\jean-christophe-yoccoz\UPL61528_no_morphism.pdf
[11] K. Parkhe Smooth gluing of group actions and applications. Preprint. arXiv:1210.2325
[12] S. Smale. Diffeomorphisms of the 2-sphere. Proc. Amer. Math. Soc. 10 (1959) 621-626

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