TRO EQUIVALENT ALGEBRAS

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Abstract. In this work we study a new equivalence relation between $w^*$ closed algebras of operators on Hilbert spaces. The algebras $A$ and $B$ are called TRO equivalent if there exists a ternary ring of operators $M$ (i.e. $MM^*M \subset M$) such that $A = \text{span}(M^*BM)^{w^*}$ and $B = \text{span}(MAM^*)^{w^*}$. We prove that two reflexive algebras are TRO equivalent if and only if there exists a $*$ isomorphism between the commutants of their diagonals mapping the invariant projection lattice of the first algebra onto the lattice of the second one.

1. Introduction

A linear space $\mathcal{M}$ of operators between two Hilbert spaces satisfying

$$\mathcal{M}M^*\mathcal{M} \subset \mathcal{M}$$

is called a ternary ring of operators (TRO).

TRO’s were introduced in [12] and constitute a generalisation of selfadjoint operator algebras [11], [20]. They have many properties similar to $C^*$-algebras and von Neumann algebras. Recently, these objects have been studied from the point of view of operator space theory, in which they play an important role [5], [13], [19].

In [15], TRO’s were studied from a different angle, namely as normalisers of operator algebras:

If $A \subset B(H)$ and $B \subset B(K)$ are $w^*$ closed operator algebras, not necessarily selfadjoint, an operator $T \in B(H, K)$ is said to normalise the algebra $A$ into $B$ if $T^*BT \subset A$ and $TAT^* \subset B$. It is shown in [15] that such a normaliser $T$ defines a TRO $\mathcal{M}_T$ consisting of normalisers from $A$ into $B$:

$$\mathcal{M}_T^*BM_T \subset A \text{ and } \mathcal{M}_TAM_T^* \subset B.$$ 

In the present paper we are interested in a stronger situation, namely in the existence of a TRO $\mathcal{M}$ so that

$$A = \text{span}(M^*BM)^{w^*} \text{ and } B = \text{span}(MAM^*)^{w^*}.$$
In this case we call the algebras $\mathcal{A}$ and $\mathcal{B}$ TRO equivalent. Note that TRO equivalence is a generalisation of unitary equivalence. We show (section 2) that it is indeed an equivalence relation.

In the selfadjoint case, TRO equivalence coincides with the existence of an “equivalence bimodule” for the algebras (see section 2) thus TRO equivalence implies “Morita equivalence” in the sense of Rieffel [17]. This crucial observation partly motivated our work. In a companion paper we prove that TRO equivalence is the appropriate context in which Rieffel’s theory for Morita equivalence of $W^*$-algebras can be generalized to the class of possibly nonselfadjoint (abstract) dual operator algebras [6]. Using results of the present paper this theory is applied in [7] to the class of reflexive algebras.

Also TRO equivalence is related to the very important notion of stable isomorphism between dual operator algebras: In another paper [8] jointly written with V.I. Paulsen we prove that two unital dual operator algebras are stably isomorphic if and only if they have completely isometric normal representations with TRO equivalent images. These results are generalised, with V.I. Paulsen and I.G. Todorov, to dual operator spaces [9].

In the present paper we are concerned with the notion of TRO equivalence within the class of reflexive (not necessarily selfadjoint) algebras. We show (section 3) that two such algebras are TRO equivalent if and only if there exists a $\ast$ isomorphism between the commutants of their diagonals mapping the invariant projection lattice of the first algebra onto that of the second. This may be thought as a generalisation to the non-selfadjoint case of the remark of Connes [2] that two $W^*$ algebras are Morita equivalent in the sense of Rieffel if and only if they have faithful normal representations with isomorphic commutants.

In section 5 we specialise to the case of TRO equivalence of separably acting CSL algebras. Given the above criterion for TRO equivalence of reflexive algebras the problem is the following:

If $\mathcal{A}, \mathcal{B}$ are separably acting CSL algebras and $\phi : \text{Lat}(\mathcal{A}) \rightarrow \text{Lat}(\mathcal{B})$ is a lattice isomorphism, under what conditions does $\phi$ extend to a $\ast$ isomorphism between the generated von Neumann algebras $\text{Lat}(\mathcal{A})''$ and $\text{Lat}(\mathcal{B})''$? The interesting fact is that while $\phi$ always extends to a $\ast$ isomorphism between the generated $C^*$ algebras (Lemma 5.1), it does not always extend to the $w^*$ closures of these algebras (Remark 4.6).

In this paper we also consider an equivalence relation strictly weaker than TRO equivalence, which we call spatial Morita equivalence (section 4). Two $w^*$ closed operator algebras $\mathcal{A}, \mathcal{B}$ are called spatially Morita equivalent if there exist an $\mathcal{A}, \mathcal{B}$ bimodule $\mathcal{U}$ and a $\mathcal{B}, \mathcal{A}$ bimodule $\mathcal{V}$ such that $\mathcal{A} = \text{span}(\mathcal{UV})^{-w^*}$ and $\mathcal{B} = \text{span}(\mathcal{VU})^{-w^*}$. We show that two CSL algebras are spatially Morita equivalent if and only if they have isomorphic lattices. In this case if one of the algebras is “synthetic” then so is the other.

We present some definitions and concepts used in this work.
By an algebra $A$ we shall mean an algebra of operators on some Hilbert space; the diagonal of $A$ is $\Delta(A) = A \cap A^*$.  

A set of projections of a Hilbert space is called a lattice if it contains the zero and identity projections and is closed under arbitrary suprema and infima. If $A$ is a subalgebra of $B(H)$ for some Hilbert space $H$, the set

$$\text{Lat}(A) = \{L \in \text{pr}(B(H)) : L^\perp AL = 0\}$$

is a lattice. Dually if $\mathcal{L}$ is a lattice the space

$$\text{Alg}(\mathcal{L}) = \{A \in B(H) : L^\perp AL = 0 \ \forall \ L \in \mathcal{L}\}$$

is an algebra.

A lattice $\mathcal{L}$ such that $P \in \mathcal{L} \iff P^\perp \in \mathcal{L}$ is called an ortholattice. A commutative subspace lattice (CSL) is a projection lattice $\mathcal{L}$ whose elements commute; the algebra $\text{Alg}(\mathcal{L})$ is called a CSL algebra. A totally ordered CSL is called a nest.

An order-preserving bijection between two lattices is called a lattice isomorphism. If the lattices $\mathcal{L}_1, \mathcal{L}_2$ are ortholattices and $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ is a lattice isomorphism satisfying $\phi(P^\perp) = \phi(P)^\perp$ for all $P \in \mathcal{L}_1$ we call $\phi$ an ortholattice isomorphism.

Let $H_1, H_2$ be Hilbert spaces and $\mathcal{U}$ a subset of $B(H_1, H_2)$. The reflexive hull of $\mathcal{U}$ is defined [16] to be the space

$$\text{Ref}(\mathcal{U}) = \{T \in B(H_1, H_2) : Tx \in \text{span}(\mathcal{U}x) \text{ for each } x \in H_1\}.$$ 

Simple arguments show that

$$\text{Ref}(\mathcal{U}) = \{T \in B(H_1, H_2) \text{ for all projections } E, F : E\mathcal{U}F = 0 \Rightarrow ETF = 0\} = \{T \in B(H_1, H_2) \text{ for all operators } A, B : A\mathcal{U}B = 0 \Rightarrow ATB = 0\}.$$ 

A subspace $\mathcal{U}$ is called reflexive if $\mathcal{U} = \text{Ref}(\mathcal{U})$. A unital algebra is reflexive if and only if $A = \text{Alg}(\text{Lat}(A))$. CSL algebras are reflexive. Every CSL algebra contains a maximal abelian selfadjoint algebra (masa in the sequel). Hence we can view a CSL algebra as a masa bimodule.

Moreover, an algebra is a CSL algebra if and only if it is reflexive and contains a masa. If $\mathcal{U}$ is a reflexive masa bimodule, then there exists [3], [18] a smallest $w^*$ closed masa bimodule which is contained in $\mathcal{U}$ and whose reflexive hull is the space $\mathcal{U}$. We denote this space by $\mathcal{U}_{\text{min}}$. Whenever $\mathcal{U}_{\text{min}} = \mathcal{U}$ we call the space $\mathcal{U}$ synthetic. When $A$ is a separably acting CSL algebra, the space $A_{\text{min}}$ is an algebra which contains the diagonal of $A$ and whose lattice is $\text{Lat}(A)$ [1]. The first example of a nonsynthetic CSL algebra was given in [1].

Now we present some concepts introduced in [10].

Let $\mathcal{P}_i = \text{pr}(B(H_i)), i = 1, 2$. Define $\phi = \text{Map}(\mathcal{U})$ to be the map $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ which associates to every $P \in \mathcal{P}_1$ the projection onto the subspace $\text{span}(TPy : T \in \mathcal{U}, y \in H_1)^\perp$. The map $\phi$ is $\vee$-continuous (that is, it preserves arbitrary suprema) and 0 preserving.
Let $\phi^* = \text{Map}(U^*)$, $S_{1,\phi} = \{\phi^*(P) : P \in P_2\}$, $S_{2,\phi} = \{\phi(P) : P \in P_1\}$ and observe that $S_{1,\phi} = S_{2,\phi}^\perp$. Erdos proved that $S_{1,\phi}$ is $\wedge$-complete and contains the identity projection, $S_{2,\phi}$ is $\vee$-complete and contains the zero projection, while $\phi|_{S_{1,\phi}} : S_{1,\phi} \to S_{2,\phi}$ is a bijection. We call the families $S_{1,\phi}, S_{2,\phi}$ the semilattices of $U$.

In fact

$$\text{Ref}(U) = \{T \in B(H_1, H_2) : \phi(P)^\perp T P = 0 \text{ for each } P \in S_{1,\phi}\}. $$

When $\phi(I) = I$ and $\phi^*(I) = I$ we call the space $U$ essential.

In [15] it is proved that a TRO $\mathcal{M}$ is $w^*$ closed if and only if it is wot closed if and only if it is reflexive. In this case, if $\chi = \text{Map}(\mathcal{M})$

$$\mathcal{M} = \{T \in B(H_1, H_2) : T P = \chi(P)T \text{ for all } P \in S_{1,\chi}\}. $$

In the following theorem we isolate some consequences of [15, Theorem 2.10].

**Theorem 1.1.** (i) A TRO $\mathcal{M}$ is essential if and only if the algebras $\text{span}(\mathcal{M}^* \mathcal{M})^{-w^*}$, $\text{span}(\mathcal{M} \mathcal{M}^*)^{-w^*}$ contain the identity operators.

(ii) If $M$ is an essential TRO and $\chi = \text{Map}(\mathcal{M})$ then $S_{1,\chi} = \text{pr}((\mathcal{M}^* \mathcal{M})')$, $S_{2,\chi} = \text{pr}((\mathcal{M} \mathcal{M}^*)')$ and the map $\chi|_{S_{1,\chi}} : S_{1,\chi} \to S_{2,\chi}$ is an ortholattice isomorphism with inverse $\chi^*|_{S_{2,\chi}}$.

If $L \subset B(H)$ we denote by $L'$ the set of operators which commute with the elements of $L$ and the set of projections in $L$ by $\text{pr}(L)$.

2. TRO EQUIVALENT ALGEBRAS

**Definition 2.1.** Let $A, B$ be $w^*$ closed algebras acting on Hilbert spaces $H_1$ and $H_2$ respectively. If there exists a TRO $\mathcal{M} \subset B(H_1, H_2)$ such that $A = \text{span}(\mathcal{M}^* \mathcal{M} \mathcal{M}^-)^{-w^*}$ and $B = \text{span}(\mathcal{M} \mathcal{M}^* \mathcal{M}^-)^{-w^*}$ we write $A \overset{\mathcal{M}}{\sim} B$. We say that the algebras $A, B$ are **TRO equivalent** if there exists a TRO $\mathcal{M}$ such that $A \overset{\mathcal{M}}{\sim} B$.

A simple example of TRO equivalent, not necessarily selfadjoint algebras, is the following. Take a unital $w^*$ closed algebra $A \subset B(H)$ and let

$$B = \begin{bmatrix} A & A \\ A & A \end{bmatrix} \subset B(H \oplus H), \quad \mathcal{M} = \begin{bmatrix} \Delta(A) \\ \Delta(A) \end{bmatrix} \subset B(H,H \oplus H).$$

It is easy to see that $A \overset{\mathcal{M}}{\sim} B$.

**Proposition 2.1.** Let $A \subset B(H_1), B \subset B(H_2)$ be $w^*$ closed algebras. The following are equivalent:

(i) The algebras $A, B$ are TRO equivalent.

(ii) There exists an essential TRO $\mathcal{M} \subset B(H_1, H_2)$ such that $\mathcal{M}^* \mathcal{M} \subset A$ and $\mathcal{M} \mathcal{M}^* \subset B$.

If (ii) holds then $A \overset{\mathcal{M}}{\sim} B$. 
Observe that the space $M$ and $\chi$ information that it is essential, then Lemma 2.2. We need the following lemma.

**Lemma 2.2.** Let $S_1$ be a set of projections on the Hilbert space $H_i, i = 1, 2$, $\chi : S_1 \to S_2$ a map onto $S_2$, and $M = \{T \in B(H_1, H_2) : TL = \chi(L)T \text{ for all } L \in S_1\}$.

Observe that the space $M$ is a reflexive TRO \cite{15}. Moreover, if we have the information that it is essential, then

$$span(M^*M)^{-w^*} = (S_1)' \text{ and } span(MMM^*)^{-w^*} = (S_2)'$$

**Proof** Let $\phi = \text{Map}(M)$. We can observe that $M(S_1)' \subset M$, so if $P$ is a projection then $(S_1)'M^*P(H_2) \subset M^*P(H_1)$ therefore $(S_1)'\phi^*(P)(H_1) \subset \phi^*(P)(H_1)$. Since $S_1$ is selfadjoint, it follows that $\phi^*(P) \in (S_1)'$.

We proved that $S_2, \phi^* \subset (S_1)''$; thus $S_1, \phi \subset (S_1)''$ and so $(S_1, \phi)' \supset (S_1)'$. But $span(M^*M)^{-w^*} = (S_1, \phi)'$ (Theorem \cite{14}) since $M$ is an essential TRO.

We proved that $Span(M^*M)^{-w^*} \supset (S_1)'$.

Clearly, $M^*M \subset (S_1)'$ and so $span(M^*M)^{-w^*} = (S_1)'$.

Since $\chi$ maps onto $S_2$ we see that $M^*(S_2)' \subset M^*$ and similar arguments show that $span(MMM^*)^{-w^*} = (S_2)'$. \qed

**Theorem 2.3.** TRO equivalence is an equivalence relation.

**Proof** We only have to prove transitivity. Let $A, B, C$ be $w^*$ closed algebras, acting on the Hilbert spaces $H_1, H_2, H_3$ respectively, and $M, N$ be essential TRO’s such that $B \overset{M}{\supseteq} A$ and $B \overset{N}{\supseteq} C$. Thus

$$span(MBM*)^{-w^*} = A, \ span(NBN^*)^{-w^*} = C$$

and $$span(M^*AM)^{-w^*} = B = span(N^*CN)^{-w^*}.$$  

Define

$$S = pr((M^*M)' \cap (N^*N)')$$

and note that

$$S' = ((M^*M) \cup (N^*N))''.$$
Since $M^*BM^*M \subset B$ and similarly for $N$ it follows that $S'BS' \subset B$. Let $\chi = \text{Map}(M)$ and $\phi = \text{Map}(N)$. Define the TRO’s
\[ Z = \{ T \in B(H_2, H_1) : TL = \chi(L)T, L \in S \} \]
\[ Y = \{ T \in B(H_2, H_3) : TL = \phi(L)T, L \in S \}. \]
The map $\chi$ is an ortholattice isomorphism from $\text{pr}((M^*M)^*)'$ onto $\text{pr}((M^*M)^*)'$ (Theorem 1.1). Since $S \subset \text{pr}((M^*M)^*)'$ it follows that $M \subset Z$. Similarly $N \subset Y$ and thus both $Z$ and $Y$ are essential TRO’s. From the previous lemma we obtain
\[ \text{span}(Y^*Y)^{-w^*} = S' = \text{span}(Z^*Z)^{-w^*}. \]
We claim that
\[ \text{span}(Z^*AZ)^{-w^*} = B \text{ and } \text{span}(ZBZ^*)^{-w^*} = A. \]
Indeed since $Z^*Z \subset S'$ and $M \subset Z$ we have
\[ Z^*ZBZ^*Z \subset B \Rightarrow \text{MZ}^*(ZBZ^*)Z^*M^* \subset MBM^* \subset A \]
\[ \Rightarrow MM^*(ZBZ^*)MM^* \subset A. \]
Since $M$ is essential and $\text{span}(MM^*)$ (resp. $\text{span}(M^*M)$) is a $*$–algebra, one can find a bounded net in $\text{span}(MM^*)$ (resp. $\text{span}(M^*M)$) converging strongly to the identity operator. Since $A$ is $w^*$–closed it follows that $ZBZ^* \subset A$ and hence $\text{span}(ZBZ^*)^{-w^*} \subset A$.

On the other hand $A = \text{span}(MBM^*)^{-w^*} \subset \text{span}(ZBZ^*)^{-w^*}$ hence $A = \text{span}(ZBZ^*)^{-w^*}$. Therefore $Z^*AZ = Z^*\text{span}(ZBZ^*)^{-w^*}Z \subset B$ because $Z^*ZBZ^*Z \subset B$. It follows by Proposition 2.4 that $B = \text{span}(Z^*AZ)^{-w^*}$.

In the same way we have
\[ \text{span}(Y^*Y)^{-w^*} = B \text{ and } \text{span}(YBY^*)^{-w^*} = C. \]
Now put $L = \text{span}(Y^*Y)^{-w^*}$. Since $\text{span}(Y^*Y)^{-w^*} = S' = \text{span}(Z^*Z)^{-w^*}$ we have $Y^*Z^* \subset Y$. It follows that
\[ (Y^*Y^*)(Y^*Y) = Y^*Y^*Y^* \subset Y \]
since $Y$ is a TRO, hence $L^*L \subset L$. Thus the space $L$ is a TRO and it is essential because the spaces $Y$ and $Z^*$ are essential TRO’s.

To complete the proof it remains to show that
\[ L^*CL \subset A \text{ and } LAL^* \subset C. \]
Indeed, since $Y^*CY \subset B$ we have $\text{span}(Y^*CYZ^*) \subset ZBZ^* \subset A$ and thus $L^*CL \subset A$. On the other hand, $Y^*A^*Y \subset \text{span}(Y^*Y)^{-w^*} \subset C$ and therefore $LAL^* \subset C$. □

**Remark 2.4.** From the previous proof it follows that, if $A, B$ are TRO equivalent algebras and $B, C$ are also TRO equivalent algebras, then there exist essential TRO’s $Z, Y$ generating the same von Neumann algebra such that $B \cong A$, $B \nless C$ and the space $L = \text{span}(Y^*Y)^{-w^*}$ is an essential TRO satisfying $A \cong C$. 
Proposition 2.5. Let $A, B$ be $w^*$ closed algebras and $\mathcal{M}$ an essential TRO such that $A \overset{\mathcal{M}}{\sim} B$. Then $\Delta(A) \overset{\mathcal{M}}{\sim} \Delta(B)$.

Proof It is obvious that $\mathcal{M}^*\Delta(B)\mathcal{M} \subset \Delta(A)$ and $\mathcal{M}\Delta(A)\mathcal{M}^* \subset \Delta(B)$. By Proposition 2.1 follows that $\Delta(A) \overset{\mathcal{M}}{\sim} \Delta(B)$. $\square$

Lemma 2.6. Let $A, B$ be unital $w^*$ closed algebras, $\mathcal{M}$ be an essential TRO such that $A \overset{\mathcal{M}}{\sim} B$ and $\chi = \text{Map}(\mathcal{M})$. Then $\text{Ref}(A) \overset{\mathcal{M}}{\sim} \text{Ref}(B)$. Also the map $\chi : pr(\Delta(A)') \to pr(\Delta(B)')$ is an orthoisomorphism and $\chi(\text{Lat}(A)) = \text{Lat}(B)$.

Proof By the above proposition $\Delta(A) \overset{\mathcal{M}}{\sim} \Delta(B)$. From [15 Corollary 5.9] it follows that $\chi(pr(\Delta(A)')) = pr(\Delta(B)')$. Since $\mathcal{M}^*\mathcal{M} \subset \Delta(A)$ we have $pr((\mathcal{M}^*\mathcal{M})') \supset pr((\Delta(A)')')$. So by Theorem 1.1 the map $\chi : pr(\Delta(A)') \to pr(\Delta(B)')$ is an orthoisomorphism.

If $E, F$ are projections such that $EAF = 0$ then $EM^*BF = 0$ so $EM^*\text{Ref}(B)MF = 0$. It follows that $\mathcal{M}^*\text{Ref}(B)\mathcal{M} \subset \text{Ref}(A)$.

Similarly we can prove that $\mathcal{M}\text{Ref}(A)\mathcal{M}^* \subset \text{Ref}(B)$, hence $\text{Ref}(A) \overset{\mathcal{M}}{\sim} \text{Ref}(B)$. Since $\text{Lat}(\text{Ref}(A)) = \text{Lat}(A)$ and similarly for $B$, using again [15 Corollary 5.9] we have $\chi(\text{Lat}(A)) = \text{Lat}(B)$. $\square$

Remark 2.7. (i) Let $A, B$ be TRO equivalent $w^*$-closed algebras and suppose that the algebra $A$ is reflexive. Then the algebra $B$ is reflexive. Indeed if $\mathcal{M}$ is an essential TRO such that $A \overset{\mathcal{M}}{\sim} B$ as in the proof of Lemma 2.6 it follows that $A \overset{\mathcal{M}}{\sim} \text{Ref}(B)$, hence $\text{Ref}(A) \overset{\mathcal{M}}{\sim} \text{Ref}(B)$. Since $\text{Lat}(\text{Ref}(A)) = \text{Lat}(A)$ and similarly for $B$, using again [15 Corollary 5.9] we have $\chi(\text{Lat}(A)) = \text{Lat}(B)$. $\square$

(ii) An orthoisomorphism $\chi : pr(C) \to pr(D)$, where $C$ and $D$ are von Neumann algebras, does not necessarily extend to a $*$-homomorphism between the algebras. For example choose [14] nonabelian $*$-anti-isomorphic von Neumann algebras $C, D, \theta : C \to D$ a $*$-anti-isomorphism and let $\chi = \theta|_{pr(C)}$. Compare now Lemma 2.6 and Theorem 3.3.

Proposition 2.8. Let $A, B$ be unital $w^*$ closed algebras acting on the Hilbert spaces $H_1, H_2$ respectively and $\mathcal{M}$ be an essential TRO such that $A \overset{\mathcal{M}}{\sim} B$. Then there exists a TRO $\mathcal{N}$ which contains $\mathcal{M}$ such that $A \overset{\mathcal{N}}{\sim} B$ and $\Delta(A) = \text{span}(\mathcal{N}^*\mathcal{N})^{-w^*}, \Delta(B) = \text{span}(\mathcal{N}\mathcal{N}^*)^{-w^*}$.

Proof Let $\chi = \text{Map}(\mathcal{M})$. From Lemma 2.6 follows that $\chi(pr(\Delta(A)')) = pr(\Delta(B)')$. Let

$$\mathcal{N} = \{T \in B(H_1, H_2) : TL = \chi(L)T \text{ for all } L \in pr(\Delta(A)')\}.$$ 

Since $S_{1, \chi} = pr((\mathcal{M}^*\mathcal{M})') \supset pr(\Delta(A)')$ we have that $\mathcal{M} \subset \mathcal{N}$ so the TRO $\mathcal{N}$ is essential.

Using Lemma 2.2 we have $\Delta(A) = \text{span}(\mathcal{N}^*\mathcal{N})^{-w^*}, \Delta(B) = \text{span}(\mathcal{N}\mathcal{N}^*)^{-w^*}$. We shall show that $A = \text{span}(\mathcal{N}^*\mathcal{B}\mathcal{N})^{-w^*}$. Since $\mathcal{M}^*\mathcal{B}\mathcal{M} \subset \mathcal{N}^*\mathcal{B}\mathcal{N}$, we get
\[ A \subset \text{span}(N^*BN)\] Now we have
\[ NN^*BN^* \subset B \Rightarrow M^*NN^*BN^*M \subset M^*BM \subset A \]
\[ \Rightarrow M^*MN^*BNM^*M \subset A \]
hence \[ \text{span}(M^*M)\] - \[ w^* \] 
\[ N^*BN^* \subset A \]
therefore \[ A = \text{span}(N^*BN)\] - \[ w^* \].

It follows that \[ N^*BN \subset A \] therefore \[ A = \text{span}(N^*BN)\] - \[ w^* \].
We proved that \[ A \sim B \]. \[ \square \]

We isolate the following consequence of the above proposition:

**Corollary 2.9.** If the unital \( w^* \) closed algebras \( A, B \) are TRO equivalent then the diagonal algebras \( \Delta(A), \Delta(B) \) are Morita equivalent in the sense of Rieffel [17].

The following proposition says that if two non-unital \( w^* \) closed algebras are TRO equivalent, there exist TRO equivalent unital algebras which contain the previous algebras as ideals.

**Proposition 2.10.** Let \( A, B \) be non unital \( w^* \) closed algebras and \( M \) be an essential TRO such that \( A \sim^M B \). If \( A_M = \text{span}(A, M^*M)^{-w^*}, B_M = \text{span}(B, MM^*)^{-w^*} \), then

(i) The spaces \( A_M, B_M \) are unital algebras.
(ii) The algebra \( A \) (respectively \( B \)) is an ideal of \( A_M \) (respectively \( B_M \)).
(iii) \( A_M \sim^M B_M \).

(iv) There exists an essential TRO \( N \) which contains \( M \) such that \( A \sim^N B \), \( A_M \) = \( \text{span}(N^*N)^{-w^*}, B_M = \text{span}(NN^*)^{-w^*} \). (Observe that \( A_M = A_N \) and \( B_M = B_N \)).

**Proof** Claims (i),(ii) are consequences of the relations \( AM^*M \subset A, M^*MA \subset A, BM^*M \subset B, MM^*B \subset B \). Also, since \( MAM^* \subset B \) and \( M(M^*M)M^* \subset MM^* \) it easily follows that \( MAM^*M^* \subset BM \). Similarly we get \( MBM^*M \subset A_M \).

(iv) Since \( A_M \sim^M B_M \), by the previous proposition there exists an essential TRO \( N \) containing \( M \) such that \( A_M \sim^N B_M \) and
\[ \Delta(A_M) = \text{span}(N^*N)^{-w^*}, \Delta(B_M) = \text{span}(NN^*)^{-w^*} \].

It remains to show that \( A \sim^N B \).

Since \( NN^* \subset B_M \) and \( B \) is an ideal of \( B_M \) we have
\[ NN^*BNN^* \subset B \Rightarrow M^*NN^*BN^*M \subset M^*BM \subset A \]
\[ \Rightarrow M^*MN^*BNM^*M \subset A \Rightarrow \text{span}(M^*M)^{-w^*}NN^*BN \text{span}(M^*M)^{-w^*} \subset A \]
hence \( N^*BN \subset A \). Similarly we can prove that \( NA \subset B \). \[ \square \]
Proposition 2.11. Let $A, B$ be $w^*$ closed algebras and $M$ be an essential TRO such that $A \stackrel{M}{\sim} B$. If $F$ is a $w^*$ closed $A$-bimodule then the space $F(J) = \text{span}(MJM^*)^{-w^*}$ is a $B$-bimodule and $F(J) \stackrel{M}{\sim} F(J)$. The map $F$ is a bijection between $w^*$ closed bimodules of $A$ and those of $B$. Moreover the restriction of $F$ to the set of two sided $w^*$ closed ideals of $A$ maps onto those of $B$.

Proof Since $AM^*M \subset A$ we have $MAM^*M \subset MJM^*$. Hence $BF(J) \subset F(J)$. Similarly we have $F(J)B \subset F(J)$. If $F$ is a $w^*$ closed bimodule of $B$, the space $F(J) = \text{span}(M^*JM)^{-w^*}$ is a bimodule of $A$ and $F(J) = I$. So the map $F$ is onto. Clearly, $F$ is an injection. Also observe that if $F$ is a two sided $w^*$ closed ideal of $A$ then the space $F(J)$ is a two sided $w^*$ closed ideal of $B$. □

The following proposition is proved easily.

Proposition 2.12. Let $A, B$ be $w^*$ closed algebras and $M$ be an essential TRO such that $A \stackrel{M}{\sim} B$. We denote by $K(A)$ (respectively $K(B)$) the set of compact operators in $A$ (resp. $B$), by $F(A)$ (resp. $F(B)$) the set of finite rank operators in $A$ (resp. $B$), by $R_1(A)$ (resp. $R_1(B)$) the set of rank 1 operators in $A$ (resp. $B$). Then it follows

$$K(A)^{-w^*} \stackrel{M}{\sim} K(B)^{-w^*}, \quad F(A)^{-w^*} \stackrel{M}{\sim} F(B)^{-w^*},$$

$$\text{span}(R_1(A))^{-w^*} \stackrel{M}{\sim} \text{span}(R_1(B))^{-w^*}.$$ 

3. TRO EQUIVALENT REFLEXIVE ALGEBRAS

The goal of this section is to determine sufficient and necessary conditions for TRO equivalence of reflexive algebras. The following lemma is known. See for example [2, 8.5.32]. We include a proof for completeness.

Lemma 3.1. Let $C, E$ be von Neumann algebras acting on the Hilbert spaces $H_1, H_2$ respectively, $\theta : C \to E$ be a $*$ isomorphism and $M = \{T \in B(H_1, H_2) : TA = \theta(A)T \text{ for all } A \in \mathcal{C}\}$. Then the space $M$ is an essential TRO.

Proof Let $\mathcal{D} = \{A \oplus \theta(A) : A \in \mathcal{C}\}$. Since $\theta$ is $w^*$ continuous, as a $*$ isomorphism between von Neumann algebras [1, I.4.3, Corollaire 2], the space $\mathcal{D}$ is a von Neumann algebra. The commutant of $\mathcal{D}$ is the algebra

$$\left[ \begin{array}{cc} \mathcal{C}' & M^* \\ M & \mathcal{E}' \end{array} \right].$$

Let $\phi = \text{Map}(M)$. Since $\mathcal{E}'M \subset M$ we have that $\phi(I)^{\perp}E'\phi(I) = 0$, hence $\phi(I) \in \mathcal{E}$. Let $P = 0 \oplus \phi(I)^{\perp}$. Since $\phi(I)M = M$ and $\phi(I) \in \mathcal{E}$ we can verify that $P^{\perp}D'P = 0$, hence $P \in \mathcal{D}$. It follows that the projection $P$ is of the form $A \oplus \theta(A)$ for an operator $A \in \mathcal{C}$. Thus $\phi(I) = I$. Similarly we can prove that $\phi^*(I) = I$, so the space $M$ is an essential TRO. □
We give a new proof of Connes' remark (see the introduction) and also show that the isomorphism between the commutants extends the map of the Morita equivalence bimodule. This fact will be useful below.

**Theorem 3.2.** Let $\mathcal{A}, \mathcal{B}$ be von Neumann algebras acting on the Hilbert spaces $H_1, H_2$ respectively, $\mathcal{M}$ be an essential TRO such that $\mathcal{A} = \text{span}(\mathcal{M}^* \mathcal{M})^{-w^*}$, $\mathcal{B} = \text{span}(\mathcal{M} \mathcal{M}^*)^{-w^*}$ and $\chi = \text{Map}(\mathcal{M})$. Then there exists a $*$ isomorphism $\theta : \mathcal{A}' \to \mathcal{B}'$ which extends the map $\chi|_{\text{pr}(\mathcal{A}')}$. Conversely if the algebras $\mathcal{A}', \mathcal{B}'$ are $*$ isomorphic, the algebras $\mathcal{A}, \mathcal{B}$ are TRO equivalent.

**Proof** By Theorem 1.1,

$$\mathcal{M} = \{T \in B(H_1, H_2) : TL = \chi(L)T \text{ for all } L \in \text{pr}(\mathcal{A}')\}$$

Let $\mathcal{L} = \{L \oplus \chi(L) : L \in \text{pr}(\mathcal{A}')\}$. We can verify that

$$\mathcal{C} = \mathcal{L}' = \left[\begin{array}{cc} \mathcal{A} & \mathcal{M}^* \\ \mathcal{M} & \mathcal{B} \end{array}\right].$$

So the algebra $\mathcal{C}$ is a von Neumann algebra acting on the direct sum of the corresponding Hilbert spaces.

An easy calculation shows that the commutant of $\mathcal{C}$ is the set

$$\left\{\left[\begin{array}{cc} T & 0 \\ 0 & S \end{array}\right] : T \in \mathcal{A}', S \in \mathcal{B}' \text{ such that } SM = MT \forall M \in \mathcal{M}\right\}.$$ 

Let

$$\pi_1 : \mathcal{C}' \to \mathcal{A}' : \left[\begin{array}{cc} T & 0 \\ 0 & S \end{array}\right] \to T$$

$$\pi_2 : \mathcal{C}' \to \mathcal{B}' : \left[\begin{array}{cc} T & 0 \\ 0 & S \end{array}\right] \to S.$$ 

We shall show that the maps $\pi_1, \pi_2$ are surjective. Clearly $\pi_1(\mathcal{C}')$ is a von Neumann algebra, so it suffices to show that $\pi_1(\mathcal{C}')' \subset \mathcal{A}$.

If $A \in \pi_1(\mathcal{C}')'$ then $AT = TA$ for all $\left[\begin{array}{cc} T & 0 \\ 0 & S \end{array}\right] \in \mathcal{C}'$. Thus

$$\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right] \left[\begin{array}{cc} T & 0 \\ 0 & S \end{array}\right] = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right] \left[\begin{array}{cc} A & 0 \\ 0 & 0 \end{array}\right] \text{ for all } \left[\begin{array}{cc} T & 0 \\ 0 & S \end{array}\right] \in \mathcal{C}',$$

hence $\left[\begin{array}{cc} A & 0 \\ 0 & 0 \end{array}\right] \in \mathcal{C}$, and so $A \in \mathcal{A}$.

If $\left[\begin{array}{cc} T & 0 \\ 0 & S_1 \end{array}\right], \left[\begin{array}{cc} T & 0 \\ 0 & S_2 \end{array}\right] \in \mathcal{C}'$ then $S_1M = MT = S_2M$ for all $M \in \mathcal{M}$.

Since the TRO $\mathcal{M}$ is essential we have $S_1 = S_2$.

The conclusion is that we can define a map $\theta : \mathcal{A}' \to \mathcal{B}'$ such that

$$\theta(T) = S \Leftrightarrow \left[\begin{array}{cc} T & 0 \\ 0 & S \end{array}\right] \in \mathcal{C}'.$$ 

The map $\theta$ is a $*$ isomorphism. We shall show that $\theta$ is an extension of $\chi|_{\text{pr}(\mathcal{A}')}$. If $L \in \text{pr}(\mathcal{A}')$ we have $ML = \theta(L)M$ for all $M \in \mathcal{M}$. It follows $\theta(L)\perp \mathcal{M}L = \theta(L) - \chi(L)$.
0 and from this \( \chi(L) \leq \theta(L) \). Also \( \theta(L) M \perp = 0 \) hence \( \chi(L) \perp = \theta(L) \perp \).

By Theorem 1.1 \( \chi(L) \perp = \chi(L) \perp \) so \( \chi(L) = \theta(L) \).

Conversely, let \( \theta : A' \to B' \) be a * isomorphism and

\[ \mathcal{M} = \{ T : TA = \theta(A)T \text{ for all } A \in A' \}. \]

The space \( \mathcal{M} \) is an essential TRO by the previous lemma. It is obvious that \( \mathcal{M} \mathcal{B}^* \mathcal{M} \subset A \) and \( \mathcal{M} \mathcal{A} \mathcal{M}^* \subset B. \)

**Theorem 3.3.** Two unital reflexive algebras \( A, B \) are TRO equivalent if and only if there exists a * isomorphism \( \theta : \Delta(A)' \to \Delta(B)' \) such that \( \theta(\text{Lat}(A)) = \text{Lat}(B) \).

**Proof** Let \( A, B \) be TRO equivalent algebras, acting on the Hilbert spaces \( H_1, H_2 \) respectively. By Proposition 2.3 there exists an essential TRO \( \mathcal{M} \) such that \( A \mathcal{M} \mathcal{B} \) and \( \Delta(A) = \text{span}(\mathcal{M}^* \mathcal{M})^{-w^*}, \Delta(B) = \text{span}(\mathcal{M} \mathcal{A} \mathcal{M}^*)^{-w^*} \).

From the previous theorem there exists a * isomorphism \( \theta : \Delta(A)' \to \Delta(B)' \) which extends the map \( \chi|_{pr(\Delta(A)')} \). From Lemma 2.6 \( \theta(\text{Lat}(A)) = \text{Lat}(\mathcal{A}) = \text{Lat}(\mathcal{B}) \).

Conversely, let \( \theta : \Delta(A)' \to \Delta(B)' \) be a * isomorphism such that \( \theta(\text{Lat}(A)) = \text{Lat}(\mathcal{B}) \) and define

\[ \mathcal{M} = \{ T \in B(H_1, H_2) : TA = \theta(A)T \text{ for all } A \in \Delta(A)' \}. \]

By Lemma 3.1 the space \( \mathcal{M} \) is an essential TRO. It remains to show that \( A \mathcal{M} \mathcal{B} \). Let \( A, \mathcal{A}, L \in \text{Lat}(A), M_1, M_2 \in \mathcal{M} \) then \( M_1 \mathcal{A} M_2^* \theta(L) = M_1 \mathcal{A} L \mathcal{A} M_2^* = \theta(L) M_1 \mathcal{A} M_2^* \theta(L) \). Hence \( M_1 \mathcal{A} M_2^* \in \mathcal{B} \). We proved that \( \mathcal{M} \mathcal{A} \mathcal{M}^* \subset \mathcal{B} \). Similarly we can prove that \( \mathcal{M} \mathcal{B} \mathcal{M} \subset \mathcal{A} \).

**4. TRO equivalence and spatial Morita equivalence**

The following definition is due to I. Todorov (personal communication).

**Definition 4.1.** Let \( H_1, H_2 \) be Hilbert spaces, \( A \subset B(H_1), B \subset B(H_2) \) be w* closed algebras. If there exist linear spaces \( U \subset B(H_1, H_2), V \subset B(H_2, H_1) \) such that \( BuA \subset U, AVB \subset V, \text{span}(VU)^{-w^*} = A \) and \( \text{span}(U\mathcal{V})^{-w^*} = B \) we say that the algebras \( A, B \) are **spatially Morita equivalent** and the system \( (A, B, U, V) \) is a **spatial Morita context**.

As we prove in Theorem 4.3 and in remark 4.6 spatial Morita equivalence is strictly weaker relation than TRO equivalence.

**Theorem 4.1.** Let \( (A, B, U, V) \) be a spatial Morita context. Moreover we assume that the algebras \( A, B \) are unital. If \( \phi = \text{Map}(U) \) and \( \psi = \text{Map}(V) \) then

(i) \( S_1, \phi = \text{Lat}(A), S_2, \phi = \text{Lat}(B) \), so the map \( \phi : \text{Lat}(A) \to \text{Lat}(B) \) is a lattice isomorphism.

(ii) \( \psi|_{\text{Lat}(B)} = (\phi|_{\text{Lat}(A)})^{-1} \).
Proof Let \( \zeta_1 = \text{Map}(A) \) and \( \zeta_2 = \text{Map}(B) \). Since \( \text{span}(UV)^{-w^*} = B \) we get \( \zeta_2 = \phi \circ \psi \), hence \( \zeta_2(pr(B(H_2))) \subset S_{2,\phi} \) or equivalently \( \text{Lat}(B) \subset S_{2,\phi} \).

Since \( \zeta_1 = \psi \circ \phi \), if \( P \in pr(B(H_1)) \), then

\[
UV\phi(P)(H_2) \subset UV\psi(\phi(P))(H_1) = U\zeta_1(P)(H_1).
\]

Also

\[
UAP \subset UP \Rightarrow U\zeta_1(P)(H_1) \subset \phi(P)(H_2) \Rightarrow UV\phi(P)(H_2) \subset \phi(P)(H_2).
\]

We proved that \( \phi(P)\#UV\phi(P) = 0 \) and therefore \( \phi(P)\#B\phi(P) = 0 \) for all \( P \in pr(B(H_1)) \). It follows that \( S_{2,\phi} \subset \text{Lat}(B) \), hence we have equality.

Since \((A^*,B^*,U^*,V^*)\) is a spatial Morita context, using the previous arguments we have \( S_{1,\phi} = \text{Lat}(A) \).

Observe that \( \phi : \text{Lat}(A) \rightarrow \text{Lat}(B) \) is a bijection which preserves order. Since \( \psi \circ \phi = \zeta_1 \), which is the identity on \( \text{Lat}(A) \), it follows that \( \psi \circ \phi|_{\text{Lat}(A)} = Id|_{\text{Lat}(A)} \). Similarly \( \phi \circ \psi|_{\text{Lat}(B)} = Id|_{\text{Lat}(B)} \). The conclusion is that \( \psi|_{\text{Lat}(A)} = (\phi|_{\text{Lat}(A)})^{-1} \).

\[\square\]

Remark 4.2. If \( A, B \) are spatially Morita equivalent unital algebras and the algebra \( B \) is reflexive, the algebra \( A \) is reflexive too. Indeed, let \((A,B,U,V)\) be a spatial Morita context. If \( E, F \) are projections such that \( EBF = 0 \), we have \( EU(VU)VF = E(U\nu)(U\nu)F = 0 \) hence \( EUAVF = 0 \) and so \( EU\text{Ref}(A)VF = 0 \). We proved that \( U\text{Ref}(A)V \subset B \). But

\[
U\text{Ref}(A)V \subset B \Rightarrow \forall U\text{Ref}(A)VU \subset VBU \subset A \Rightarrow A\text{Ref}(A)A \subset A.
\]

Since the algebra \( A \) is unital we have \( \text{Ref}(A) = A \).

In the following theorem we prove that the converse of the above theorem is true for the case of CSL algebras.

Theorem 4.3. Two CSL algebras \( A \) and \( B \) are spatially Morita equivalent if and only if they have isomorphic lattices.

Proof By Theorem 4.1 it suffices to show that a lattice isomorphism between CSL’s induces spatial Morita equivalence of the corresponding algebras. Suppose that \( A \subset B(H_1) \) and \( B \subset B(H_2) \).

Let \( S_1 = \text{Lat}(A), S_2 = \text{Lat}(B) \), let \( \phi : S_1 \rightarrow S_2 \) be a lattice isomorphism and

\[
U = \{ T \in B(H_1,H_2) : \phi(L)^\#TL = 0 \text{ for all } L \in S_1 \},
\]

\[
V = \{ S \in B(H_2,H_1) : L^\#S\phi(L) = 0 \text{ for all } L \in S_1 \}.
\]

It is easily verified that \( \text{span}(VU) \) is an ideal of \( A \). Indeed if \( V \in V, U \in U \) and \( A \in A \) then for all \( L \in S_1 \) we have

\[
AV\phi(L) = ALV\phi(L) = LALV\phi(L)
\]

so \( L^\#AV\phi(L) = 0 \), hence \( AV \in V \). Similarly \( UA \in U \), showing that \( A\text{span}(VU)A \subset \text{span}(VU) \).

Also

\[
VUL = V\phi(L)UL = LV\phi(L)UL
\]
Remark 4.4. We do not know whether the ‘product’ we can prove that $B$ contains the space $\text{span} S$. We conclude that $S$.

Also since $W \subset \text{span} S$, hence $\text{span} S$ is a masa bimodule, so it contains the space $\mathcal{A}_{\text{min}}$. But the space $\mathcal{A}_{\text{min}}$ is a unital space and the space $\text{span}(\mathcal{V}U)^{-w^*}$ is an ideal of $A$. It follows that $A = \text{span}(\mathcal{V}U)^{-w^*}$. Similarly we can prove that $B = \text{span}(\mathcal{V}U)^{-w^*}$. □

**Remark 4.4.** We do not know whether the ‘product’ $\text{span}(\mathcal{V}U)^{-w^*}$ of two reflexive spaces, or even reflexive masa bimodules, is necessarily reflexive.

**Theorem 4.5.** If $A, B$ are $w^*$ closed TRO equivalent algebras then they are spatially Morita equivalent.

**Proof** Let $\mathcal{M}$ be an essential TRO such that $A \cong \mathcal{M}$ and put $A_{\mathcal{M}} = \text{span}(A, \mathcal{M}^* \mathcal{M})^{-w^*}$, $B_{\mathcal{M}} = \text{span}(B, \mathcal{M}^* \mathcal{M})^{-w^*}$. We recall from Proposition 2.10 that $A_{\mathcal{M}} = \sim B_{\mathcal{M}}$ and $A$, (respectively $B$) is an ideal of $A_{\mathcal{M}}$, (respectively $B_{\mathcal{M}}$).

Let $\mathcal{U} = \text{span}(\mathcal{B}\mathcal{M})^{-w^*}$ and $\mathcal{V} = \text{span}(\mathcal{M}\mathcal{B}\mathcal{M})^{-w^*}$. We shall show that the system $(\mathcal{A}, \mathcal{B}, \mathcal{U}, \mathcal{V})$ is a spatial Morita context.

(i) Since $B$ is an ideal of $B_{\mathcal{M}}$ we have $B(\mathcal{B}\mathcal{M})(\mathcal{M}\mathcal{B}\mathcal{M}) \subset \mathcal{B}\mathcal{M} \subset \mathcal{U}$. Since $\mathcal{U} = \text{span}(\mathcal{B}\mathcal{M})^{-w^*}$ and $A = \text{span}(\mathcal{M}\mathcal{B}\mathcal{M})^{-w^*}$, this implies that $\mathcal{B}\mathcal{U}A \subset \mathcal{U}$.

(ii) Similarly, the relation $(\mathcal{M}\mathcal{B}\mathcal{M})(\mathcal{M}\mathcal{B}\mathcal{M})B \subset \mathcal{M}\mathcal{B}$ implies $\mathcal{A} \mathcal{B} \subset \mathcal{V}$ since $\mathcal{V} = \text{span}(\mathcal{M}\mathcal{B}\mathcal{M})^{-w^*}$.

(iii) Observe that $\mathcal{M} \subset \mathcal{V}$, hence $\mathcal{M}\mathcal{B}\mathcal{M} \subset \mathcal{V}$. It follows that $\text{span}(\mathcal{V}U)^{-w^*} \subset \mathcal{A}$. Since $B$ is an ideal of $B_{\mathcal{M}}$ we have $\mathcal{M}\mathcal{B}\mathcal{M} \subset \mathcal{M}\mathcal{B}\mathcal{M} \subset \mathcal{A}$. But $\mathcal{V} = \text{span}(\mathcal{M}\mathcal{B}\mathcal{M})^{-w^*}$ and $\mathcal{U} = \text{span}(\mathcal{B}\mathcal{M})^{-w^*}$. It follows that $\mathcal{V} \mathcal{U} \subset \mathcal{A}$. Therefore $A = \text{span}(\mathcal{V}U)^{-w^*}$. 

and so $L^\perp VUL = 0$; hence $\mathcal{V}U \subset \mathcal{A}$. It follows that $\text{Ref}(\mathcal{V}U) \subset \mathcal{A}$. We shall prove that equality holds.

By Theorems 3.3, 4.4 in [10] we have that

$$
\mathcal{S}_{1,\text{Map}(\mathcal{U})} = \mathcal{S}_1, \quad \mathcal{S}_{2,\text{Map}(\mathcal{U})} = \mathcal{S}_2, \quad \text{Map}(\mathcal{U})|_{\mathcal{S}_1} = \phi,
$$
$$
\mathcal{S}_{1,\text{Map}(\mathcal{V})} = \mathcal{S}_2, \quad \mathcal{S}_{2,\text{Map}(\mathcal{V})} = \mathcal{S}_1, \quad \text{Map}(\mathcal{V})|_{\mathcal{S}_2} = \phi^{-1}.
$$

Let $\mathcal{W} = \text{Ref}(\mathcal{V}U)$ and $\zeta = \text{Map}(\mathcal{W})$. It follows that $\zeta = \text{Map}(\mathcal{V}) \circ \text{Map}(\mathcal{U})$.

Also since $\mathcal{W}^* = \text{Ref}(\mathcal{U}^* \mathcal{V}^*)$ we have $\zeta^* = \text{Map}(\mathcal{U}^*) \circ \text{Map}(\mathcal{V}^*)$, hence $\mathcal{S}_2,\zeta^* \subset \mathcal{S}_{2,\text{Map}(\mathcal{U}^*)} = (\mathcal{S}_{1,\text{Map}(\mathcal{U})})^\perp = (\mathcal{S}_1)^\perp$.

We conclude that $\mathcal{S}_1,\zeta \subset \mathcal{S}_1$. So if $L \in \mathcal{S}_1,\zeta$ we have $\text{Map}(\mathcal{U})(L) \in \mathcal{S}_2$ and

$$
\zeta(L) = \text{Map}(\mathcal{V}) \circ \text{Map}(\mathcal{U})(L) = \phi^{-1} \circ \phi(L) = L,
$$

hence

$$
\mathcal{W} = \{T : \zeta(L)^\perp TL = 0 \text{ for all } L \in \mathcal{S}_1,\zeta\} = \{T : L^\perp TL = 0 \text{ for all } L \in \mathcal{S}_1,\zeta\} = \mathcal{A}.
$$

Since $\mathcal{V}U \subset \mathcal{A}$ we obtain the equality $A = \text{Ref}(\mathcal{V}U)$.

Observe that the space $\text{span}(\mathcal{V}U)^{-w^*}$ is a masa bimodule, so it contains the space $\mathcal{A}_{\text{min}}$. But the space $\mathcal{A}_{\text{min}}$ is a unital space and the space $\text{span}(\mathcal{V}U)^{-w^*}$ is an ideal of $A$. It follows that $A = \text{span}(\mathcal{V}U)^{-w^*}$. Similarly we can prove that $B = \text{span}(\mathcal{V}U)^{-w^*}$. □
(iv) Since $B$ is an ideal of $B_M$, we have $B_M M^* B_M \subset B$ hence $UV \subset B$.

Now, observe that

$$M^* B_M \subset M V \Rightarrow \text{span}(M^* B_M) \subset \text{span}(M V) \Rightarrow$$

$$B_M \subset \text{span}(M V)^{-w^*}.$$  

Also since $UV \supset B M V$, it follows that

$$\text{span}(\text{span}(M V)^{-w^*}) \supset B M \subset \text{span}(M V)^{-w^*}.$$  

We proved that $B = \text{span}(U V)^{-w^*}$. The proof is complete. $\square$

**Remark 4.6.** Spatial Morita equivalence does not imply TRO equivalence. There exist multiplicity free nests $S_1, S_2$ which are isomorphic but the algebras $S_1', S_2'$ are not isomorphic. For an example, see [3, Example 7.19].

Thus isomorphism of the lattices does not guarantee TRO equivalence, even for multiplicity free nest algebras.

**Theorem 4.7.** Let $A \subset B(H_1), \quad B \subset B(H_2)$ be separably acting CSL algebras with isomorphic lattices. Then $A$ is synthetic if and only if $B$ is synthetic. In fact, if $\phi : \text{Lat}(A) \to \text{Lat}(B)$ is a lattice isomorphism and

$$U = \{ T \in B(H_1, H_2) : \phi(L)^* T L = 0 \text{ for all } L \in \text{Lat}(A) \}.$$  

Then $A$ (and $B$) is synthetic if and only if $U$ is synthetic.

**Proof** Let $S_1 = \text{Lat}(A), \quad S_1' = \{ \phi(L) \oplus L : L \in S_1 \}$

and

$$V = \{ S \in B(H_2, H_1) : L^* S \phi(L) = 0 \text{ for all } L \in S_1 \}.$$  

By Theorem 4.3 we have that

$$A = \text{span}(V U)^{-w^*} \quad \text{and} \quad B = \text{span}(U V)^{-w^*}.$$  

It is shown in [15, Proposition 4.2] that, if $C = \text{Alg}(L)$,

$$C = \begin{bmatrix} B & U \\ V & A \end{bmatrix} \quad \text{and} \quad C_{\min} \subset \begin{bmatrix} B_{\min} & U_{\min} \\ V_{\min} & A_{\min} \end{bmatrix}.$$  

We shall show that

$$C_{\min} = \begin{bmatrix} B_{\min} & U_{\min} \\ V_{\min} & A_{\min} \end{bmatrix}. \quad (1)$$  

Indeed if $W$ is any $w^*$ closed masa bimodule such that $\text{Ref}(W) = C$ and if $Q = 0 \oplus I$ then

$$\text{Ref}(Q^* W Q) = Q^* C Q = \begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix}.$$  

It follows that

$$\begin{bmatrix} 0 & U_{\min} \\ 0 & 0 \end{bmatrix}_{\min} \subset Q^* W Q \subset W.$$
Now taking $W = C_{\text{min}}$ we obtain
\[
\begin{bmatrix}
0 & U_{\text{min}} \\
0 & 0
\end{bmatrix} \subset C_{\text{min}}.
\]
Similarly we can prove that
\[
\begin{bmatrix}
0 & 0 \\
A_{\text{min}} & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 \\
V_{\text{min}} & 0
\end{bmatrix}, \begin{bmatrix}
B_{\text{min}} & 0 \\
0 & 0
\end{bmatrix} \subset C_{\text{min}}
\]
and (1) follows.

If $E, F$ are projections such that $EU_{\text{min}}VF = 0$ then $EU_{\text{min}}VF = 0$ hence $EBF = 0$. It follows that $B \subset \text{Ref}(U_{\text{min}}V)$ and therefore $B = \text{Ref}(U_{\text{min}}V)$.

Similarly we have that $A = \text{Ref}(V_{\text{min}}U)$.

Now suppose that the algebra $A$ is synthetic. Since $B = \text{Ref}(U_{\text{min}}V)$ we have that $B_{\text{min}} \subset \text{span}(U_{\text{min}}V)^{-w^*}$ so
\[
U \subset B_{\text{min}}U \subset \text{span}(U_{\text{min}}V)^{-w^*} \subset \text{span}(U_{\text{min}}A)^{-w^*} = \text{span}(U_{\text{min}}A_{\text{min}})^{-w^*}.
\]
Using (1) we have
\[
\begin{bmatrix}
0 & U_{\text{min}}A_{\text{min}} \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & U_{\text{min}} \\
0 & 0
\end{bmatrix} \begin{bmatrix}
0 & 0 \\
A_{\text{min}} & 0
\end{bmatrix} \subset C_{\text{min}}.
\]
It follows that $U_{\text{min}}A_{\text{min}} \subset U_{\text{min}}$, hence $U \subset U_{\text{min}}$ (from equation (2)), and so the bimodule $U$ is synthetic.

For the opposite direction we suppose that the bimodule $U$ is synthetic. Since
\[
\begin{bmatrix}
0 & 0 \\
0 & V_{\text{min}}U_{\text{min}}
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
V_{\text{min}} & 0
\end{bmatrix} \begin{bmatrix}
0 & U_{\text{min}} \\
0 & 0
\end{bmatrix},
\]
again using (1) we conclude that $V_{\text{min}}U_{\text{min}} \subset A_{\text{min}}$ and therefore $V_{\text{min}}U \subset A_{\text{min}}$. Since $UA \subset U$ it follows that $V_{\text{min}}U_{\text{min}}A_{\text{min}} \subset A_{\text{min}}$. But also $A_{\text{min}} \subset \text{span}(V_{\text{min}}U)^{-w^*}$ since $A = \text{Ref}(V_{\text{min}}U)$, and hence $A_{\text{min}}A \subset A_{\text{min}}$; therefore $A \subset A_{\text{min}}$ since $A_{\text{min}}$ is unital.

We have proved that the algebra $A$ is synthetic if and only if the bimodule $U$ is synthetic. Similarly one shows that $U$ is synthetic if and only if the algebra $B$ is synthetic. \qed

5. TRO equivalence and CSL algebras

In this section we assume that all Hilbert spaces are separable. Thus the $w^*$ topology on bounded sets of operators is metrisable. We are going to present some results on TRO equivalence of CSL algebras.

**Definition 5.1.** If $S$ is a CSL and $L \in S$ we denote by $L_\diamond$ the projection $\vee\{M \in S : M < L\}$. Whenever $L_\diamond < L$ we call the projection $L - L_\diamond$ an atom of $S$. If the CSL $S$ has no atoms we say that it is a continuous CSL. If the identity operator is the sum of the atoms we say that $S$ is totally atomic.
Lemma 5.1. Let $H_i, i = 1, 2$ be Hilbert spaces, $S_i, i = 1, 2$ be commutative lattices (not necessarily complete) containing zero and the identity and let $\theta : S_1 \to S_2$ be a lattice isomorphism. Then the map $\theta$ extends to a $*$ isomorphism $\rho : \text{span}(S_1)^{-\|\cdot\|} \to \text{span}(S_2)^{-\|\cdot\|}$.

Proof Using induction we shall prove that if $P_1, \ldots, P_n$ are projections of $S_1$ such that $\sum_{i=1}^n c_i P_i = 0$ where $c_i \neq 0$ for $1 \leq i \leq n$ then $\sum_{i=1}^n c_i \theta(P_i) = 0$.

The claim clearly holds for $n = 1, 2$. Assume that it holds for $k \in \{1, \ldots, n-1\}$.

Let $\sum_{i=1}^n c_i P_i = 0$ where $c_i \neq 0$ for $1 \leq i \leq n$ and put $A = \sum_{i=1}^n c_i \theta(P_i)$. It suffices to show that $\theta(P_k)A = 0$ for all $k \in \{1, \ldots, n\}$.

Let $B = \theta(P_n)A$. We shall show that $B = 0$.

Multiply the equation $\sum_{i=1}^n c_i P_i = 0$ with $P_1 \land P_n$. This gives

$$(c_1 + c_n)(P_1 \land P_n) + c_2 (P_2 \land P_1 \land P_n) + \ldots + c_{n-1} (P_{n-1} \land P_1 \land P_n) = 0.$$

By the inductive hypothesis we have

$$(c_1 + c_n)\theta(P_1 \land P_n) + c_2 \theta(P_2 \land P_1 \land P_n) + \ldots + c_{n-1} \theta(P_{n-1} \land P_1 \land P_n) = 0,$$

hence $\theta(P_1)B = 0$.

Similarly we can prove that $\theta(P_1)B = 0$ for $1 \leq i \leq n-1$.

Since $P_n = (c_n)^{-1} \sum_{i \neq n} c_i P_i$ it follows that $P_n \leq \lor_{i \neq n} P_i$ hence $\theta(P_n) \leq \lor_{i \neq n} \theta(P_i)$, so $\theta(P_n)B = 0$ and therefore $B = 0$. We proved that $\theta(P_n)A = 0$.

Using the same method we have $\theta(P_k)A = 0, k = 1, \ldots, n$. So the claim holds.

The conclusion is that the map

$$\rho : \text{span}(S_1) \to \text{span}(S_2) : \rho \left( \sum_{i=1}^n c_i P_i \right) = \sum_{i=1}^n c_i \theta(P_i)$$

is well defined, and it is clearly a $*$ isomorphism.

We shall show that $\rho$ is norm continuous. Let $T = \sum_{i=1}^n c_i P_i \in \text{span}(S_1)$ and let $c$ be in the spectrum $\sigma(\rho(T))$ of $\rho(T)$. Let $S_0$ be the smallest lattice containing the set $\{0, P_1, \ldots, P_n, I\}$. Then the space $\text{span}(S_0)$ is a $C^*$-algebra which is contained in $\text{span}(S_1)$. If $c$ is not in the spectrum $\sigma(T)$ of $T$, the operator $S = (cI - T)^{-1}$ is contained in $\text{span}(S_0)$ and hence in $\text{span}(S_1)$.

Since $S(cI - T) = (cI - T)S = I$ we have $\rho(S)(cI - \rho(T)) = (cI - \rho(T))\rho(S) = I$, contradicting $c \in \sigma(\rho(T))$.

We proved that $\sigma(\rho(T)) \subset \sigma(T)$. Therefore $\|\rho(T)\| \leq \|T\|$.

We conclude that the map $\rho$ extends to a $*$ isomorphism from the $C^*$-algebra $\text{span}(S_1)^{-\|\cdot\|}$ onto $\text{span}(S_2)^{-\|\cdot\|}$. \hfill $\square$

Lemma 5.2. Let $S_1, S_2$ be CSL’s, $\phi : S_1 \to S_2$ be a lattice isomorphism, $P$ be the sum of the atoms of $S_1$ and $Q$ be the sum of the atoms of $S_2$. Then there exists a $*$ isomorphism

$$\rho : S_1^{\prime\prime}|_P \to S_2^{\prime\prime}|_Q$$

such that $\rho(L|_P) = \phi(L)|_Q$ for all $L \in S_1$. 

Proof For any atom $A$ in a CSL $S$, we have
\[ A = \wedge\{L \in S : AL = A\} - \vee\{L \in S : AL = 0\}. \]
Letting $\mathcal{A}_i$ be the set of atoms of $S_i$, we then find that the lattice isomorphism $\phi : S_1 \to S_2$ induces a bijection $\rho_\alpha : \mathcal{A}_1 \to \mathcal{A}_2$, given by
\[ \rho_\alpha(A) = \wedge\{\phi(L) : L \in S : AL = A\} - \vee\{\phi(L) : L \in S : AL = 0\}. \]
Now $S''_i|_P$ and $S''_i|_Q$ are * isomorphic to $l^\infty(\mathcal{A}_1)$ and $l^\infty(\mathcal{A}_2)$ respectively; identify $S''_i|_P$ with $l^\infty(\mathcal{A}_1)$ and likewise identify $S''_i|_Q$ with $l^\infty(\mathcal{A}_2)$. The map $\rho_\alpha$ extends to an isomorphism $\rho : l^\infty(\mathcal{A}_1) \to l^\infty(\mathcal{A}_2)$, which satisfies our requirements, because any projection in $l^\infty(\mathcal{A}_i)$ is the sum of the atoms it contains.

The following theorem is a consequence of the above lemma.

Theorem 5.3. Let $S_1, S_2$ be totally atomic CSL’s. The algebras $\text{Alg}(S_1), \text{Alg}(S_2)$ are TRO equivalent if and only if the CSL’s $S_1, S_2$ are isomorphic.

Proof If $\text{Alg}(S_1)$ and $\text{Alg}(S_2)$ are TRO equivalent, then $S_1$ and $S_2$ are isomorphic by Theorem 3.3. Conversely, by the above lemma any lattice isomorphism $\phi : S_1 \to S_2$ extends to a * isomorphism from $S''_1 = (\Delta(\text{Alg}(S_1)))'$ onto $S''_2 = (\Delta(\text{Alg}(S_2)))'$. Using again Theorem 3.3 we conclude that the algebras $\text{Alg}(S_1)$ and $\text{Alg}(S_2)$ are TRO equivalent.

For the general case of CSL algebras we present the following theorem.

Theorem 5.4. Let $S_1, S_2$ be CSL’s acting on Hilbert spaces $H_1, H_2$ respectively, $P$ the sum of the atoms of $S_1$, $Q$ the sum of the atoms of $S_2$ and $A = \text{Alg}(S_1)$, $B = \text{Alg}(S_2)$, $A_0 = \text{span}(S_1)^{-\parallel \parallel}$, $B_0 = \text{span}(S_2)^{-\parallel \parallel}$. The following are equivalent:

(i) The algebras $A, B$ are TRO equivalent.

(ii) There exists a lattice isomorphism $\phi : S_1 \to S_2$ whose extension (Lemma 5.1) $\overline{\phi} : A_0 \to B_0$ is $w^*$-bicontinuous on the unit balls of $A_0, B_0$.

(iii) There exists a lattice isomorphism $\phi : S_1 \to S_2$ such that if $L = \{L \oplus \phi(L) : L \in S_1\}$ then
\[ L'' \cap (0 \oplus B_0^\perp) = 0, \quad L'' \cap (A_0'' \oplus 0) = 0. \]

(iv) There exists a lattice isomorphism $\phi : S_1 \to S_2$ such that if $L$ is as in (iii) then
\[ L'' \cap (0 \oplus B_0'' Q^\perp) = 0, \quad L'' \cap (A_0'' P^\perp \oplus 0) = 0. \]

Moreover if these conditions hold and
\[ \Delta(\phi) = \{T \in B(H_1, H_2) : TL = \phi(L)T \text{ for all } L \in S_1\} \]
then $A \overset{\Delta(\phi)}{\cong} B$.

Proof (i) $\Rightarrow$ (ii)
This is obvious by Theorem 3.3.
Lemma 10.1.10] the map \(\phi\) to a \(P\) case where \(B\in B\).

(ii) \(\Rightarrow\) (i)
Suppose that \(\phi : S_1 \to S_2\) is a lattice isomorphism whose extension by Lemma 5.1.10 \(\bar{\phi} : A_0 \to B_0\) is \(w^*\)-bicontinuous on the unit balls. By [14, Lemma 10.1.10] the map \(\bar{\phi}\) (respectively \(\bar{\phi}^{-1}\)) extends to a \(w^*\)-continuous homomorphism from \((S_1)''\) to \((S_2)''\) (respectively from \((S_2)''\) to \((S_1)''\)). One can check that the extensions are mutual inverses. (The assumption that the map \(\bar{\phi}\) is \(w^*\)-continuous doesn’t guarantee that its inverse is \(w^*\)-continuous. See exercise 10.5.30 in [14]).

Now Theorem 3.3 shows that the algebras \(A\) and \(B\) are TRO equivalent.

(i) \(\Rightarrow\) (iii)
If the algebras \(A, B\) are TRO equivalent, by Theorem 3.3 there exists a lattice isomorphism \(\phi : S_1 \to S_2\) which extends to a \(*\) isomorphism \(\rho : S'_1 \to S'_2\). We can verify that \(\mathcal{L}'' = \{A \oplus \rho(A) : A \in A''_0\}\) hence
\[
\mathcal{L}'' \cap (0 \oplus B''_0) = 0, \mathcal{L}'' \cap (A''_0 \oplus 0) = 0.
\]

(iii) \(\Rightarrow\) (iv)
This is obvious.

(iv) \(\Rightarrow\) (i)
It suffices to show that \(\phi\) extends to a \(*\) isomorphism from \(S''_1\) onto \(S''_2\). If it does not by (ii) one of the maps \(\bar{\phi} : A_0 \to B_0, \bar{\phi}^{-1} : B_0 \to A_0\) will not be \(w^*\) continuous on the unit ball. Suppose that \(\bar{\phi}\) is not \(w^*\) continuous on the unit ball. Then there exists a net \((A_i) \subset Ball(A_0)\) which converges in the \(w^*\) topology to 0 while the net \((\bar{\phi}(A_i))\) converges to a nonzero operator \(B \in B''_0\). Since the restriction of \(\phi\) on the lattice \(S_1|_P\) extends (Lemma 5.2) to a \(*\) isomorphism from \(A''_0|_P\) onto \(B''_0|_Q\) and the net \((A_iP)\) converges to 0 the net \((\bar{\phi}(A_i)Q)\) converges to 0 too. Therefore \(BQ = 0\).

Observe that \((L - L_0) \oplus (\phi(L) - \phi(L)_0) \in \mathcal{L}''\) for all \(L \in S_1\), hence \(P \oplus Q \in \mathcal{L}''\). It follows that \(A_iP^\perp \oplus \bar{\phi}(A_i)Q^\perp \in \mathcal{L}''\) for every index \(i\) and so \(0 \oplus BQ^\perp \in \mathcal{L}''\). This is a contradiction because \(BQ^\perp \neq 0\). The proof for the case where \(\phi^{-1}\) is not \(w^*\) continuous on the unit ball is similar.

Now suppose that conditions (ii) to (v) hold and let \(\rho : S''_1 \to S''_2\) be the extension of \(\phi\). By Lemma 3.1 the space
\[
\mathcal{M} = \{T \in B(H_1, H_2) : TA = \rho(A)T\text{ for all }A \in S''_2\}
\]
is an essential TRO. Since the space \(\Delta(\phi)\) contains \(\mathcal{M}\) it is essential too. We can easily verify that \(\Delta(\phi)^*B\Delta(\phi) \subset \mathcal{A}\) and \(\Delta(\phi)\mathcal{A}\Delta(\phi)^* \subset \mathcal{B}\). By Proposition 3.1 we have \(\mathcal{A} \overset{\Delta(\phi)}{\cong} \mathcal{B}\).

Remark 5.5. By Theorem 3.3 if the algebras \(\mathcal{A}\) and \(\mathcal{B}\) are TRO equivalent and \(\mathcal{A}\) is a CSL algebra then so is \(\mathcal{B}\).

In the special case of nest algebras we have the following result:

Theorem 5.6. All nest algebras with continuous nests are TRO equivalent.
**Proof** If $X$ is a subset of some $B(H)$ we denote by $X^\infty$ the set of all operators of the form $X^\infty = X \oplus X \oplus \ldots$, where $X \in X$, acting on $B(H^\infty)$. If $S_1, S_2$ are continuous nests, the nests $S_1^\infty, S_2^\infty$ are also continuous and of multiplicity $\infty$. It follows from [3, Theorem 7.24] that the nests $S_1^\infty, S_2^\infty$ are unitarily equivalent. So there exists a $*$ isomorphism from $(S_1^\infty)''$ onto $(S_2^\infty)''$ mapping $S_1^\infty$ onto $S_2^\infty$. Now taking compositions with the maps $S_1'' \rightarrow (S_1''')^\infty, X \rightarrow X^\infty, i = 1, 2$ we obtain a $*$ isomorphism from $S_1''$ onto $S_2''$ mapping $S_1$ onto $S_2$. The conclusion comes from Theorem 3.3. □

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