Comment on ‘Mean-field solution of structural balance dynamics in nonzero temperature’

Krzysztof Malarz1,∗ and Janusz A. Holyst2,3,†

1AGH University of Science and Technology, Faculty of Physics and Applied Computer Science, al. Mickiewicza 30, 30-059 Kraków, Poland
2Warsaw Technical University, Faculty of Physics, al. Koszykowa 75, 00-662 Warszawa, Poland
3ITMO University, 49 Kronverkskiy av., 197101, Saint Petersburg, Russia

(Dated: August 28, 2020)

In recent numerical and analytical studies, Rabbani et al. [Phys. Rev. E 99, 062302 (2019)] observed the first-order phase transition in social triads dynamics on complete graph with N = 50 nodes. With Metropolis algorithm they found critical temperature on such graph equal to 26.2. In this comment we extend their main observation in more compact and natural manner. In contrast to the commented paper we estimate critical temperature \( T^c \) for complete graph not only with \( N = 50 \) nodes but for any size of the system. We have derived formula for critical temperature \( T^c = (N-2)/a^2 \), where \( N \) is the number of graph nodes and \( a^2 \approx 1.71649 \) comes from combination of heat-bath and mean-field approximation. Our computer simulation based on heat-bath algorithm confirm our analytical results and recover critical temperature \( T^c \) obtained earlier also for \( N = 50 \) and for systems with other sizes. Additionally, we have identified—not observed in commented paper—phase of the system, where the mean value of links is zero but the system energy is minimal since the network contains only balanced triangles with all positive links or with two negative links. Such a phase corresponds to dividing the set of agents into two coexisting hostile groups and it exists only in low temperatures.

Keywords: Heider balance; mean-field approximation; the first order phase transition; heat-bath algorithm

There are several attempts to theoretical description and computational simulation of the Heider balance [1–3] in literature [4]. Very recently [5], the Heider’s dynamics has been enriched with social temperature \( T \) [6]. In Ref. 5 authors show that in the investigated system the first order phase transition from ordered to disordered state is observed.

In this Comment we show much more simpler theoretical approach leading to the same conclusions basing on heat-bath and mean-field approximation well supported with computer simulation.

Let us consider a network of \( N \) agents and let us assume strengths of links between two agents \( i \) and \( j \) are \( x_{ij} = \pm 1 \). The dynamics towards the Heider balance can be written as

\[
 x_{ij}(t + 1) = \text{sign}\left( \sum_{k} x_{ik}(t)x_{kj}(t) \right), \quad (1)
\]

where summation goes via \( M_{ij} \) common nearest neighbours of connected nodes \( i \) and \( j \), i.e. \( M_{ij} \) is the number of triangles that involve the link \( i,j \). It means that for a single triangle system presented in Fig. 1 the first and the third triangle are balanced in the Heider’s sense (as friend of my friend is my friend and enemy of my friend is my enemy), while the second and the fourth are not. In the latter case actors at triangles nodes either encounter the cognitive dissonance—as they cannot imagine how his/her friends can be enemies—or everybody hates everybody, what should lead to creation of two-against-one coalition.

\[
 x_{ij}(t + 1) = \begin{cases} 
 +1 & \text{with probability } p, \\
 -1 & \text{with probability } (1-p), 
\end{cases} \quad (2a)
\]

where

\[
 p = \frac{\exp(c)}{\exp(c) + \exp(-c)} \quad (2b)
\]

and

\[
 c = T^{-1} \sum_{k} x_{ik}(t)x_{kj}(t). \quad (2c)
\]

Here the positive variable \( T \) can be considered as a social temperature [6] and in the limit \( T \to 0^+ \) we have \( p \to 1 \).
The expected value \( \langle x_{ij}(t+1) \rangle \) in such an approach equals to

\[
\langle x_{ij}(t+1) \rangle = \tanh \left( T^{-1} \sum_{k} M_{ij} \langle x_{ik}(t)x_{kj}(t) \rangle \right),
\]

(3)

where \( \langle \cdots \rangle \) stands for a mean value related to the stochastic process defined by Eq. (2). The mean \( \langle x_{ij}(t+1) \rangle \) is a continuous variable that can be negative or positive and for \( T \to 0^+ \) the Eq. (3) reduces to Eq. (1).

In a mean-field-like approximation instead of Eq. (3) we have

\[
\{ \langle x_{ij}(t+1) \rangle \} \approx \tanh \left( \frac{1}{T} \sum_{k} M_{ij} \{ \langle x_{ik}(t) \rangle \} \{ \langle x_{kj}(t) \rangle \} \right),
\]

(4)

where \( \{ \cdots \} \) denotes average over all \( N(N-1)/2 \) available nodes’ pairs. Although the above approximation can be seen as a very crude approach below we give arguments why it is justified.

If the system consists only of balanced triads 1(a) and 1(c) then for \( x_{ij} = +1 \) we have always \( x_{ik}x_{kj} = +1 \) (see Fig. 2(a)). On the other hand when a link with a label \( x_{ij} = -1 \) belongs to a triad 1(c) then \( x_{ik}x_{kj} = -1 \). It follows that for such systems all components of the sum of RHS (3) are the same. Since we aim to consider systems close to equilibrium we assume that we can use this approach also for Eq. (4). Now let us assume—also in agreement with the spirit of mean-field approximation—that all averages are the same

\[
\{ \langle x_{ij} \rangle \} = \{ \langle x_{ik} \rangle \} = \{ \langle x_{kj} \rangle \} = x.
\]

(5)

It follows we get

\[
x(t + 1) = \tanh(ax^2(t)) ,
\]

(6)

where

\[
a = M/T,
\]

(7)

and \( M = \{ M_{ij} \} \) is the average number of common neighbours of agents \( i \) and \( j \) (it also the number of different triads containing the edge \( ij \)).

Considering Eq. (6) as one-dimension map one can study its fixed points. We immediately recognise \( x^0 = 0 \) as a stable fixed point for any value of \( a \) parameter and for \( a < 1 \) this is the only fixed point of Eq. (6). However for \( a > 1 \) there are two other fixed points, \( x^u < x^s \) corresponding to unstable \( x^u(a) \) and stable \( x^s(a) \) solutions. In fact \( x^u \) is a separatrix between domain of attractions of fixed points \( x^0 \) and \( x^s \). When \( a > 1 \) then \( x^u \approx 1/a \). When the parameter \( a \) diminishes from large values (it means the temperature \( T \) increases) then fixed points \( x^u \) and \( x^s \) coincide together to the point \( x^c \) for a certain value of \( a = a^c \) (see Fig. 3).

It means that for \( a > a^c \) the system is bi-stable and for \( a < a^c \) the system is mono-stable. The above values \( a^c, x^c \) can be received from a pair of transcendental algebraic relations describing the fixed point and its tangency condition, namely

\[
x^c = \tanh \left( a^c(x^c)^2 \right)
\]

(8a)

and

\[
2a^c x^c = \cosh^2 \left( a^c(x^c)^2 \right).
\]

(8b)

The solutions (see Fig. 3) are

\[
x^c_{\text{th}} \approx 0.796388
\]

(9a)
FIG. 4: The time evolution of average value of all links strengths \{x_{ij}\} for various values of social temperature $T$ and various system size $N$. The starting point of simulation is homogeneous state with \{x_{ij}\} = +1 and the scanning temperature step $\Delta T = 1$. The solid red line corresponds to $x_{\text{th}}$ given by Eq. (9a).

and

$$a_{\text{th}}^c = \tanh^{-1}(x_{\text{th}}^c)/(x_{\text{th}}^c)^2 \approx 1.71649.$$  \hspace{0.5cm} (9b)

Let us note that since $x_{\text{th}}^c > 0$ thus a system can express hysteresis phenomenon. It means also that one should not observe the values $0 < x < x_{\text{th}}^c$ as stable solutions.

To check the analytical results in computer simulation we directly apply Eq. (2) for time evolution of $x_{ij}$ for complete graph with $N$ nodes. For the complete graph the average number of nodes $ij$ pair neighbour is equal to $M = \{M_{ij}\} = N - 2$ and thus according to Eq. (7) one should expect

$$a_{\text{nu}}^c = (N - 2)/T_{\text{nu}}^c.$$  \hspace{0.5cm} (10a)

In order to find the value of $T_{\text{nu}}^c$ we start simulation
FIG. 5: (Color online). The dependence of (a, c) \( \langle x_{ij} \rangle \) and (b, d) \( \mathcal{E} \) vs. \( T \) for complete graph with \( N = 50 \) (a, b) and \( N = 200 \) (c, d) nodes. The averaging symbol \( \cdots \) stands for time average over the last \( \tau = 1000 \) time steps of simulation. The solid symbols correspond to the starting point \( \forall i, j : x_{ij} = +1 \), i.e. to the purple curves presented in Fig. 4. The open symbols stand for random initial state \( \{x_{ij}\} = 0 \). The solid blue lines indicate the mean-field approximation predictions \( \{x_{ij}\} = x \) (a, c) and \( \mathcal{E} = -x^3 \) (b, d). The solid black (dashed green) arrow marks the positions of the critical temperature \( T^c \) (\( T^d \)).

TABLE I: The numerically obtained values of \( T_{nu}^c \) and \( a_{nu}^c \) calculated basing on Eq. (10a) together with their estimated expanded uncertainties \( U(a_{nu}^c) \). The uncertainty of \( T_{nu}^c \) is \( u(T_{nu}^c) = 1/\sqrt{3} \), while \( U(a_{nu}^c) \) is calculated basing on Eq. (10b).

| \( N \) | \( T_{nu}^c \) | \( a_{nu}^c \) | \( U(a_{nu}^c) \) | \( a_{nu}^c - a_{\text{th}} \) |
|-------|-----------|----------|--------------|------------------|
| 25    | 11.5      | 2.00     | 0.30         | 0.2835           |
| 50    | 26.5      | 1.81     | 0.12         | 0.0935           |
| 100   | 55.5      | 1.766    | 0.055        | 0.0495           |
| 200   | 114.5     | 1.729    | 0.026        | 0.0125           |
| 400   | 230.5     | 1.726    | 0.013        | 0.0095           |
| 800   | 463.5     | 1.7217   | 0.0064       | 0.0052           |

with \( T = 0 \) and we scan the temperature \( T \) with step \( \Delta T \) and we look for such value of \( T^* \) for which \( \langle x_{ij} \rangle \) is positive but for \( T^* + \Delta T \) is zero. The true value of \( T_{nu}^c \) is hidden somewhere in interval \( [T^*, T^* + \Delta T] \). We assume that \( T^c \) value is uniformly distributed in the interval \( [T^*, T^* + \Delta T] \) what allow us for estimation of its uncertainty as \( u(T_{nu}^c) = \Delta T/\sqrt{3} \). The estimated value of \( T_{nu}^c = (T^* + T^* + \Delta T)/2 \). Basing on Eq. (7) we calculate the value of \( a_{nu}^c \) and we can estimate its expanded uncertainty as

\[
U(a_{nu}^c) = k \left| \frac{\partial u}{\partial T} \right|_{T = T^c} u(T_{nu}^c) = k \frac{N - 2}{(T_{nu}^c)^2} u(T_{nu}^c),
\]

with coverage factor \( k = 3 \) [8].

In Fig. 4 the time evolution of \( \{x_{ij}\} \) for various values of social temperature \( T \) and various system size \( N \) are presented. The starting point of simulation is homogeneous state with \( \{x_{ij}\} = +1 \) and the scanning temperature step is set to \( \Delta T = 1 \). The solid red line corresponds to \( x_{ij} \) given by Eq. (9b).

The obtained critical temperatures \( T_{nu}^c \), their uncer-
tainties $U(T^c_{nh})$ are collected in Tab. I. The obtained values of $a^c_{nu}$ coincide nicely with those obtained analytically (see Eq. (9b)) even under very crude assumptions given by Eq. (6). The values of $a^c_{nh}$ agree within expanded uncertainties $U(a^c_{nu})$ with its analytical partner $a^c_{ih}$.

In Figs. 5(a), (c) the dependences of $\{x_{ij}\}$ vs. $T$ for $N = 200$ and $N = 50$ are presented. The averaging symbol $\langle \cdots \rangle$ stands for time average over the last $\tau = 1000$ time steps of simulation, this time average should be approximately equal to the average $(\cdots)$ used in the Eq. (4), which comes from the ergodic theorem. The solid symbols corresponds to the starting point $\forall i, j : x_{ij} = +1$, while open symbols stand for random initial state $\{x_{ij}\} = 0$. The latter recovers $x^0 = 0$ mentioned earlier.

In Figs. 5(b), (d) the dependencies of the system energy

$$E = -\frac{\sum x_{ij} x_{jk} x_{ki}}{3} \tag{11}$$

are presented. There is a discontinuous change of mean system energy at the critical temperature, this corresponds to Fig. 4(b) in Ref. 5. According to our crude approximation (5) we expect $E = -x^3$, and this approximation is marked by solid blue line in Figs. 5(b), (d). Similarly to numerically obtained values of $\{x_{ij}\}$ also values of $\overline{E}$ agree fairly with proposed mean-field approximation.

We have observed also that for some range of temperature $0 \leq T < T^d$ ($T^d$ is another critical temperature, $T^d < T^c$) when we start from random initial condition $\{x_{ij}\} = 0$, then this average fluctuates around zero in time but the mean system energy is $E = -1$ (see Figs. 5(a)–(b) and 5(c)–(d)). In other words, this energy is the same as the system energy corresponding to paradise (with only positive links) state $\{x_{ij}\} = 1$. Although the fact, that the state $\{x_{ij}\} = 0$ is stable in time follows from Eq. (6) its nature can be seen strange since naively one would expect a picture of disordered system corresponding to the many unbalanced triangles with $E = 0$ and not $E = -1$. This could mean that ground state of the system is degenerated.

In order to understand this phenomenon we investigated frequencies $f$ of different triangles in the network when we start system evolution from various initial conditions. When we start with $\{x_{ij}\} = 0$ then for low temperatures (see Fig. 6(b)) there is a special distribution of the triangles types in the system, namely, there are only balanced triangles in the network. Let us note, that the energy of any balanced triangle is the same and equals to $-1$, thus the system mean energy is also $E = -1$ in this phase. Moreover, since the observed number of $(+++)\,$ triangles is exactly three times lower than number of $(+-+)\,$ triangles (see Fig. 6(b)) the mean link strength is equal zero. Such composition of balanced state has been already predicted by Cartwright and Harary [3] in middle ‘50 (see also e.g. Ref. 9) and it corresponds to forming in the system two groups of agents, where links among agents in the same group are friendly (positive) while links between members of these groups are hostile (negative). This special coexistence of two hostile groups is possible for $T < T^d$ when we have only numerical values of $T^d$ as presented by open symbols in Fig. 5. At $T = T^d$ the discontinuous phase transition takes place when the number of unbalanced triads abruptly increases and as results the system energy becomes zero (see dashed arrows connecting open symbols in Figs. 5(b) and 5(d)). This transition is not seen at values of $\{x_{ij}\}$, which is zero below and zero above $T^d$.

The ground state with $E = -1$ and $\{x_{ij}\} = 1$ is recognized for $0 \leq T < T^c$ when systems starts with homogeneous initial conditions [see Fig. 6(f)]. At $T = T^c$ the discontinuous phase transition changes $\{x_{ij}\}$ from

FIG. 6: Frequencies $f$ of appearing of triangles presented in Fig. 1 for $N = 50$ and various initial states.
\[ x^c \approx 0.796388 \] to zero (see Figs. 5(a), (c)) and at the same time the system energy changes from \( E = -(x^c)^3 \) to the value \( E = 0 \) (see Figs. 5(b), (d)). Let us stress that for \( T^d < T < T_c \) depending on the initial conditions the system energy is either close to the ground state or equal to zero (see Figs. 5(b), (d)).

In this Comment we present simple analytical approach which describes the first order phase transition observed in thermalised Heider’s balance system at the complete graph. Our mean-field approximation predicts that the system critical temperature equals to \( T_c = (N - 2)/a^c \), where \( N \) is the number of graph nodes and \( a^c \approx 1.71649 \) comes from combination of heat-bath and mean-field approximation. This temperature corresponds to transition from a system starting close to paradise, \( \{x_{ij}\} = x^c \approx 0.796388 \), to the disordered system with equal number of positive and negative links. The results of computer simulations agree within estimated uncertainties with our analytical calculations.

On the other hand, starting with randomly selected \( x_{ij} = \pm 1 \) we reach in simulation only the disordered state with \( \{x_{ij}\} = 0 \) corresponding to stable fixed point \( x^d \).

This indicates that heating the system from \( T = 0 \) to \( T = T_c \) decreases an average \( \{x_{ij}\} \) until \( x > x^c \), but above \( T_c \) we can only reach \( \{x_{ij}\} = 0 \). Cooling the system from \( T > T_c \) towards \( T \to 0^+ \) will never reproduce positive values of \( \{x_{ij}\} > 0 \) and in this sense the hysteresis-like loop may be observed in the system.

The mean-field approach should work perfectly on complete graph, and indeed, the computer simulations confirm our theoretical approach, as the obtained numerically values \( a^c \) agree within its expanded uncertainties with analytically calculated values \( a^c_{\text{th}} \). The direct implementation of Eq. (2) follows heat-bath algorithm [7, p. 505], which is equivalent to Metropolis scheme [10] utilised in Ref. 5.

The critical temperature \( T_c = 26.2 \) for complete graph with \( N = 50 \) nodes estimated in Ref. 5 agrees roughly with our estimation of \( T^d \) for \( N = 50 \). However, in contrast to Ref. 5 we show, that critical temperature \( T^d \) depends on system size as \( T_c = (N - 2)/a^c \), where \( a^c \) characterises the system independently on the system size \( N \).

Finally, we have identified—not mentioned in the commented paper—phase of the system, where values of links take random variables \( \{x_{ij}\} = 0 \) but the system stays in its ground state with \( E = -1 \). This ground state degeneration is observed for \( T < T^d \) with \( T^d < T_c \). This phase consists of special mixture of balanced triangles \((+++)\) and \((+-+)\), where density of the second type of triads is three times larger than the first one and it corresponds to dividing the set of agents into two coexisting hostile groups. Finding value of \( T^d \) analytically—where this phase disappears—is beyond predictions of our approach and it remains a challenge.

ACKNOWLEDGMENTS

We are grateful to Krzysztof Kulakowski, Krzysztof Suchecki and Piotr Górska for fruitful discussion and critical reading of the manuscript. This research has received funding as RENOIR Project from the European Unions Horizon 2020 research and innovation programme under the Marie Skłodowska–Curie grant agreement No. 691152, by Ministry of Science and Higher Education (Poland), grant Nos. W34/H2020/2016, 329025/PnH/2016, and by National Science Centre, Poland, Grant No. 2015/19/B/ST6/02612. J.A.H. was partially supported by the Russian Science Foundation, Agreement #17-71-30029 with co-financing of Bank Saint Petersburg.

[1] F. Heider, “Attitudes and cognitive organization,” The Journal of Psychology 21, 107–112 (1946).
[2] L. Festinger, A Theory of Cognitive Dissonance (Stanford University Press, Stanford, 1957).
[3] D. Cartwright and F. Harary, “Structural balance: A generalization of Heider’s theory,” Psychological Review 63, 277–293 (1956).
[4] T. Antal, P. L. Krapivsky, and S. Redner, “Dynamics of social balance on networks,” Physical Review E 72, 036121 (2005); P. Gawroński and K. Kulakowski, “Heider balance in human networks,” AIP Conference Proceedings 779, 93–95 (2005); P. Gawroński, P. Gronek, and K. Kulakowski, “The Heider balance and social distance,” Acta Physica Polonica B 36, 2549–2558 (2005); K. Kulakowski, P. Gawroński, and P. Gronek, “The Heider balance: A continuous approach,” International Journal of Modern Physics C 16, 707–716 (2005); P. Gawroński and K. Kulakowski, “A numerical trip to social psychology: Long-living states of cognitive dissonance,” Lecture Notes in Computer Science 4490, 43–50 (2007); K. Kulakowski, “Some recent attempts to simulate the Heider balance problem,” Computing in Science & Engineering 9, 80–85 (2007); S. A. Marvel, J. Kleinberg, R. D. Kleinberg, and S. H. Strogatz, “Continuous-time model of structural balance,” Proceedings of the National Academy of Sciences 108, 1771–1776 (2011); E. Estrada and M. Benzi, “Walk-based measure of balance in signed networks: Detecting lack of balance in social networks,” Physical Review E 90, 042802 (2014); R. Nishi and N. Masuda, “Dynamics of social balance under temporal interaction,” EPL (Europhysics Letters) 107, 48003 (2014); M. J. Krawczyk, M. del Castillo-Mussot, E. Hernandez-Ramirez, G. G. Naumis, and K. Kulakowski, “Heider balance, asymmetric ties, and gender segregation,” Physica A—Statistical Mechanics and its Applications 439, 66–74 (2015); P. Gawroński, M. J. Krawczyk, and K. Kulakowski, “Emerging communities in networks—A flow of ties,” Acta Physica Polonica B 46, 911–921 (2015); S. Wongkaew, M. Caponigro, K. Kulakowski,
and A. Borzì, “On the control of the Heider balance model,” European Physical Journal—Special Topics 224, 3325–3342 (2015); V. B. Kovchegov, “Modeling of human society as a locally interacting product-potential networks of automaton,” (2016), arXiv:1605.02377; F. Hassanibesheli, L. Hedayatifar, P. Gawroński, M. Stojkow, D. Zuchowska-Skiba, and K. Kulakowski, “Gain and loss of esteem, direct reciprocity and Heider balance,” Physica A—Statistical Mechanics and its Applications 468, 334–339 (2017); F. Hassanibesheli, L. Hedayatifar, H. Safdari, M. Ausloos, and G. R. Jafari, “Glassy states of aging social networks,” Entropy 19, 246 (2017); M. J. Krawczyk, S. Kaluzny, and K. Kulakowski, “A small chance of paradise—Equivalence of balanced states,” EPL (Europhysics Letters) 118, 58005 (2017); P. J. Górski, K. Kulakowski, P. Gawroński, and J. A. Hołyst, “Destructive influence of interlayer coupling on Heider balance in bilayer networks,” Scientific Reports 7, 16047 (2017); M. J. Krawczyk, M. Wołoszyń, P. Groniek, K. Kulakowski, and J. Mucha, “The Heider balance and the looking-glass self: Modelling dynamics of social relations,” Scientific Reports 9, 11202 (2019); K. Kulakowski, M. Stojkow, and D. Zuchowska-Skiba, “Heider balance, prejudices and size effect,” The Journal of Mathematical Sociology 44, 129–137 (2020).

[5] F. Rabbani, A. H. Shirazi, and G. R. Jafari, “Mean-field solution of structural balance dynamics in nonzero temperature,” Physical Review E 99, 062302 (2019).

[6] K. Kulakowski, “A note on temperature without energy—A social example,” (2008), arXiv:0807.0711 [physics.soc-ph].

[7] K. Binder, “Applications of Monte Carlo methods to statistical physics,” Reports on Progress in Physics 60, 487–559 (1997).

[8] B. N. Taylor and C. E. Kuyatt, Guidelines for Evaluating and Expressing the Uncertainty of NIST Measurement Results, Tech. Rep. (NIST, 1994).

[9] K. Malarz, M. Wołoszyń, and K. Kulakowski, “Towards the Heider balance with a cellular automaton,” Physica D 411, 132556 (2020).

[10] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller, “Equation of state calculations by fast computing machines,” The Journal of Chemical Physics 21, 1087–1092 (1953).