1. Introduction

The goal of this paper is to explain some phenomena arising in the realization of tractor bundles in conformal geometry as associated bundles. In order to form an associated bundle one chooses a principal bundle with normal Cartan connection (i.e., a normal parabolic geometry) corresponding to a given conformal manifold. We show that different natural choices can lead to topologically distinct associated tractor bundles for the same inducing representation. The nature of the choices is subtle, so we give a careful presentation of the relevant foundational material which we hope researchers in the field will find illuminating. The main considerations apply as well to more general parabolic geometries.

We focus particularly on tractor bundles associated to the standard representation of $O(p+1, q+1)$. The paper [BEG] gave a construction of a canonical tractor bundle and connection on any conformal manifold $(M, c)$ which are now usually called the standard tractor bundle and its normal tractor connection. This standard tractor bundle has other characterizations and realizations; these were studied in [CG2]. One of the realizations is as an associated bundle to a principal bundle over the conformal manifold; the standard tractor bundle is associated to the standard representation of $O(p+1, q+1)$. The complications arise because there are different ways to realize a given conformal manifold as a normal parabolic geometry, corresponding to different choices of structure group and lifted conformal frame bundle. Different such choices can give rise to different tractor bundles with connection associated to the standard representation, and for many natural choices one does not obtain the standard tractor bundle with its normal connection. For example, let $Q$ denote the model quadric for conformal geometry in signature $(p, q)$, consisting of the space of null lines for a quadratic form of signature $(p+1, q+1)$. If one takes the homogeneous space realization $Q = O(p+1, q+1)/P_{\text{line}}$, where $P_{\text{line}}$ denotes the isotropy group of a fixed null line, then for $pq \neq 0$ the bundle associated to the standard representation of $P_{\text{line}}$ is not the standard tractor bundle. Moreover its holonomy (which is trivial) is not equal to the conformal holonomy of $Q$ (which is $\{\pm I\}$). Recall that $Q$ is orientable if its dimension $n = p + q$ is even, so for $n$ even this phenomenon is not a consequence of failure of orientability of the conformal manifold. The issue is that this associated bundle does not have the correct topology.

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We begin by recalling in §2 the BEG construction of the standard tractor bundle and its normal connection. We then formulate a (slight variant of a) uniqueness theorem of [CG2] providing conditions on a bundle with auxiliary data on a conformal manifold which characterize it as the standard tractor bundle with its normal connection. Following [CG2], we review the construction of a tractor bundle and connection via the ambient construction and show using the Čap-Gover Uniqueness Theorem that this construction also produces the standard tractor bundle.

In §3 we review a fundamental prolongation result which we call the TMČS Theorem (for Tanaka, Morimoto, Čap-Schichl), which asserts an equivalence of categories between certain categories of parabolic geometries and categories of underlying structures on the base manifold. Our treatment is similar to that of [CSl] except that we parametrize parabolic geometries and underlying structures by triples \((g, P, \text{Ad})\), where \(g\) is a \(|k|\)-graded semisimple Lie algebra, \(P\) is a Lie group with Lie algebra \(p = g^0\), and \(\text{Ad}\) is a suitable representation of \(P\) on \(g\). Also we are more explicit about the choices involved in determining an underlying structure. We then illustrate the TMČS Theorem by showing how it can be used to represent general conformal manifolds and oriented conformal manifolds as parabolic geometries. In each case, in order to obtain a category of parabolic geometries one must make a choice of a Lie group \(P\) whose Lie algebra is the usual parabolic subalgebra \(p \subset \mathfrak{so}(p + 1, q + 1)\), and, depending on the choice of \(P\), also a choice of a lift of the conformal frame bundle. There are several choices, some of which are equivalent.

In §4 we describe the construction of tractor bundles and connections as associated bundles for general parabolic geometries, and then specialize to the parabolic geometries arising from conformal structures discussed in §3. We parametrize our associated bundles by suitably compatible \((g, P)\)-modules; as for our parametrization of parabolic geometries we find that this clarifies the dependence on the various choices. We make some observations about general tractor bundles as associated bundles for conformal geometry, and then we specialize to the question of which choices from §3 give rise to the standard tractor bundle when one takes the \((g, P)\)-module to be the standard representation. There are preferred choices for which one always obtains the standard tractor bundle: for conformal manifolds one should choose \(P^{\text{ray}}\), the subgroup of \(O(p + 1, q + 1)\) preserving a null ray, and for oriented conformal manifolds one should choose \(S P^{\text{ray}}\), the subgroup of \(SO(p + 1, q + 1)\) preserving a null ray. This is well-known and is often taken as the definition of the standard tractor bundle. What is novel in our discussion is the fact that so many other natural choices give bundles associated to the standard representation which are not the standard tractor bundle with its normal connection.

In §4 we also briefly discuss homogeneous models and conformal holonomy. We follow the usual convention of defining the conformal holonomy of a conformal manifold to be the holonomy of the standard tractor bundle with its normal connection, and show that for natural choices of principal bundles it often happens that the holonomy
of the tractor bundle with normal connection associated to the standard representation is not equal to the conformal holonomy. We conclude §4 with a brief discussion of analogous phenomena for the parabolic geometries corresponding to generic 2-plane fields on 5-manifolds, the consideration of which led us to become aware of these subtleties in the first place.

Throughout, our conformal structures are of signature \((p, q)\) on manifolds \(M\) of dimension \(n = p + q \geq 3\).

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2. Standard Tractor Bundle

The paper \([BEG]\) gave a concrete construction of a tractor bundle \(T\) on general conformal manifolds. \(T\) has rank \(n+2\), carries a fiber metric \(h\) of signature \((p+1, q+1)\), and has a null rank 1 subbundle \(T^1\) isomorphic to the bundle \(\mathcal{D}[-1]\) of conformal densities of weight \(-1\). We denote by \(\mathcal{D}[w]\) the bundle of conformal densities of weight \(w\) and by \(\mathcal{E}[w]\) its space of smooth sections. The bundle \(T\) was defined to be a particular conformally invariant subbundle of the 2-jet bundle of \(\mathcal{D}[1]\). It was then shown that a choice \(g\) of a representative of the conformal class induces a splitting

\[ T \cong \mathcal{D}[-1] \oplus TM[-1] \oplus \mathcal{D}[1], \]

where \(TM[w] = TM \otimes \mathcal{D}[w]\). With respect to this splitting, a section \(U \in \Gamma(T)\) is represented as a triple

\[ U = \begin{pmatrix} \rho \\ \mu^i \\ \sigma \end{pmatrix} \]

with \(\rho \in \mathcal{E}[-1], \mu^i \in \Gamma(TM[-1]), \sigma \in \mathcal{E}[1]\). Under a conformal change \(\hat{g} = e^{2\Upsilon}g\), the representations are identified by

\[ \begin{pmatrix} \hat{\rho} \\ \hat{\mu}^i \\ \hat{\sigma} \end{pmatrix} = \begin{pmatrix} 1 & -\Upsilon_j & -\frac{1}{2} \Upsilon_k \Upsilon^k \\ 0 & \delta^i_j & \Upsilon^i \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho \\ \mu^i \\ \sigma \end{pmatrix}. \]

Indices are raised and lowered using the tautological 2-tensor \(g \in \Gamma(S^2T^*M[2])\) determined by the conformal structure. The tractor metric \(h\) is defined by

\[ h(U, U) = 2\rho\sigma + g_{ij}\mu^i\mu^j. \]

The subbundle \(T^1\) is defined by \(\mu^i = 0, \sigma = 0\,\text{, and the map} \rho \mapsto \begin{pmatrix} \rho \\ 0 \\ 0 \end{pmatrix} \)
defines a conformally invariant isomorphism \(\mathcal{D}[-1] \cong T^1\).
Note for future reference that since conformal density bundles are trivial, $\mathcal{T}$ is isomorphic to $TM \oplus \mathbb{R}^2$ as a smooth vector bundle, where $\mathbb{R}^2$ denotes a trivial rank 2 vector bundle. It follows that $\mathcal{T}$ is orientable if and only if $M$ is orientable.

A connection $\nabla$ on $\mathcal{T}$ was defined in [BEG] directly in terms of the splitting and the chosen representative and the definition was verified to be conformally invariant and to give $\nabla h = 0$. The definition is:

$$\nabla_i \begin{pmatrix} \rho \\ \mu_j \\ \sigma \end{pmatrix} = \begin{pmatrix} \nabla_i \rho - P_{ik} \mu^k \\ \nabla_i \mu_j + \delta_i^j \rho + P_{ij} \sigma \\ \nabla_i \sigma - \mu_i \end{pmatrix}. $$

The occurrences of $\nabla_i$ on the right-hand side denote the connection induced by the representative $g$ on the density bundles, or that connection coupled with the Levi-Civita connection of $g$ in the case of $\nabla_i \mu^j$. $P_{ij}$ denotes the Schouten tensor of $g$.

A uniqueness theorem for such a tractor bundle was proven in §2.2 of [CG2]. We state the result assuming a conformal manifold, whereas in [CG2] the existence of the conformal structure was part of the conclusion. Let $(M, c)$ be a conformal manifold with tautological tensor $g \in \Gamma(S^2T^*M[2])$. Consider the following data. Let $\mathcal{T}$ be a rank $n + 2$ vector bundle over $M$ with metric $h$ of signature $(p + 1, q + 1)$ and connection $\nabla$ such that $\nabla h = 0$. Let $\mathcal{T}^1$ be a null line subbundle of $\mathcal{T}$ equipped with an isomorphism $\mathcal{T}^1 \cong \mathcal{D}[-1]$. If $v \in T_x M$ and $U \in \Gamma(\mathcal{T}^1)$, differentiating $h(U, U) = 0$ shows that $\nabla_v U \in (\mathcal{T}^1)^{\perp}$. The projection of $\nabla_v U$ onto $(\mathcal{T}^1)^{\perp}/\mathcal{T}^1$ is tensorial in $U$. Invoking the isomorphism $\mathcal{T}^1 \cong \mathcal{D}[-1]$, it follows that $v \otimes U \mapsto \nabla_v U + \mathcal{T}^1$ induces a bundle map $\tau : TM \otimes \mathcal{D}[-1] \to (\mathcal{T}^1)^{\perp}/\mathcal{T}^1$. The metric $h$ determines a metric $h_0$ of signature $(p, q)$ on $(\mathcal{T}^1)^{\perp}/\mathcal{T}^1$. The data $(\mathcal{T}, \mathcal{T}^1, h, \nabla)$ are said to be compatible with the conformal structure if $\tau^* h_0 = g$. We refer to [CG2] for the formulation of the curvature condition for $\nabla$ to be called normal.

**Čap-Gover Uniqueness Theorem.** Let $(M, c)$ be a conformal manifold. Up to isomorphism, there is a unique $(\mathcal{T}, \mathcal{T}^1, h, \nabla)$ as above compatible with the conformal structure with $\nabla$ normal.

Such a $\mathcal{T}$ is called a (or the) standard tractor bundle and $\nabla$ its normal tractor connection. Even though $(\mathcal{T}, \mathcal{T}^1, h, \nabla)$ is unique up to isomorphism, there are several different realizations. The tractor bundle and connection constructed in [BEG] satisfy the conditions and so provide one realization.

Another realization of the standard tractor bundle discussed in [CG2] is via the ambient construction of [FG1], [FG2]. Let $\mathcal{G} \to M$ be the metric bundle of $(M, c)$, i.e. $\mathcal{G} = \{(x, g_x) : x \in M, g \in c\} \subset S^2T^*M$. $\mathcal{G}$ carries dilations $\delta_s$ for $s > 0$ defined by $\delta_s(x, g) = (x, s^2 g)$, and the tautological 2-tensor $g$ can be viewed as a section $g \in \Gamma(S^2T^*\mathcal{G})$ satisfying $\delta_* g = s^2 g$. An ambient metric $\tilde{g}$ for $(M, c)$ is a metric of signature $(p + 1, q + 1)$ on a dilation-invariant neighborhood $\tilde{\mathcal{G}}$ of $\mathcal{G} \times \{0\}$ in $\mathcal{G} \times \mathbb{R}$ satisfying $\delta_* \tilde{g} = s^2 \tilde{g}$, $\iota_* \tilde{g} = g$, and a vanishing condition on its Ricci curvature. (In order to construct the standard tractor bundle and its normal connection, it suffices
that the tangential components of the Ricci curvature of \( \tilde{g} \) vanish when restricted to \( G \times \{0\} \). Here \( \iota : G \to G \times \mathbb{R} \) is defined by \( \iota(z) = (z, 0) \) and the dilations extend to \( G \times \mathbb{R} \) acting in the \( G \)-factor alone.

The ambient realization of \( \mathcal{T} \) is defined as follows. The fiber \( \mathcal{T}_x \) over \( x \in M \) is

\[
\mathcal{T}_x = \left\{ U \in \Gamma(T\tilde{G}|_{G_x}) : \delta_s^*U = s^{-1}U \right\},
\]

where \( G_x \) denotes the fiber of \( G \) over \( x \). The homogeneity condition implies that \( U \) is determined by its value at any single point of \( G_x \), so \( \mathcal{T}_x \) is a vector space of dimension \( n+2 \). The tractor metric \( h \) and the normal tractor connection \( \nabla \) can be realized as the restrictions to \( G \) of \( \tilde{g} \) and its Levi-Civita connection \( \tilde{\nabla} \). The null subbundle \( \mathcal{T}^1 \) is the vertical bundle in \( T\tilde{G} \subset T\tilde{G}|_G \). The infinitesimal dilation \( T \) defines a global section of \( \mathcal{T}^1 \), so determines the isomorphism \( \mathcal{T}^1 \cong \mathcal{D}[-1] \). It can be verified that the tractor bundle and connection defined this way satisfy the conditions above, so the uniqueness theorem implies that the ambient construction gives a standard tractor bundle with its normal connection. An isomorphism with the realization in \([\text{BEG}]\) is written down directly in \([\text{GW}]\) in terms of a conformal representative.

We mention in passing that the formulation of the ambient construction in \([\text{CG2}]\) appears to be more general than that above in that it allows an arbitrary ambient manifold \( \tilde{G} \) with a free \( \mathbb{R}_+ \)-action containing \( G \) as a hypersurface. But at least near \( G \) there is no real gain in generality: if \( \tilde{G} \) admits a metric \( \tilde{g} \) such that \( \iota^*\tilde{g} = g \), then the normal bundle of \( G \) in \( \tilde{G} \) is trivial so that near \( G, \tilde{G} \) is diffeomorphic to a neighborhood of \( G \times \{0\} \) in \( G \times \mathbb{R} \). This is because the 1-form dual to \( T \) with respect to \( \tilde{g} \) gives a global nonvanishing section of \( (T\tilde{G}/T\tilde{G})^* \).

The third usual construction of the standard tractor bundle is as an associated bundle to the Cartan bundle for the conformal structure. We postpone discussion of this construction to \( \S 4 \).

3. Tanaka-Morimoto-Čap-Schichl Theorem

A fundamental result in the theory of parabolic geometries asserts an equivalence of categories between parabolic geometries of a particular type \((g, P)\) and certain underlying structures. There are different forms of the result due to Tanaka \([\text{T}]\), Morimoto \([\text{M}]\), and Čap-Schichl \([\text{CSc}]\). We state a version which is a slight extension of Theorem 3.1.14 in \([\text{CSI}]\) and refer to it as the TMČS Theorem.

Let \( g = g_{-k} \oplus \cdots \oplus g_k \) be a \( |k| \)-graded semisimple Lie algebra with associated filtration \( g' = g_k + \cdots + g_0 \) and subalgebras \( p = g^0 \) and \( g_- = g_{-k} \oplus \cdots \oplus g_{-1} \). Let \( P \) be a Lie group with Lie algebra \( p \) and let \( \text{Ad} : P \to \text{Aut}_\text{filt}(g) \) be a representation of \( P \) as filtration-preserving Lie algebra automorphisms of \( g \) such that \( p \mapsto \text{Ad}(p)|_p \) is the usual adjoint representation of \( P \) on \( p \). Typically there is a Lie group \( G \) with Lie algebra \( g \) containing \( P \) as a parabolic subgroup with respect to the given grading, and \( \text{Ad} \) is the restriction to \( P \) of the adjoint representation of \( G \). But we assume neither that there exists such a \( G \) nor that we have chosen one. For fixed \( |k| \)-graded
groups which induces the identity on the common Lie algebra \( p \).

Given data \((g, P, \text{Ad})\) as above, the Levi subgroup \( P_0 \subset P \) is defined by

\[
P_0 = \{ p \in P : \text{Ad}(p)(g_i) \subset g_i, -k \leq i \leq k \}.
\]

We prefer the notation \( P_0 \) rather than the usual \( G_0 \) since we do not choose a group \( G \), and also to emphasize that \( P_0 \) depends on \( P \).

A parabolic geometry of type \((g, P, \text{Ad})\) (or just \((g, P)\) if the representation \( \text{Ad} \) is understood) on a manifold \( M \) is a \( P \)-principal bundle \( B \rightarrow M \) together with a Cartan connection \( \omega : TB \rightarrow g \). The definition of a Cartan connection depends only on the data \((g, P, \text{Ad})\); see, for example, [S]. We refer to [ˇCSl] for the conditions on the curvature of \( \omega \) for the parabolic geometry to be called regular and normal.

Next we formulate the notion of an underlying structure of type \((g, P, \text{Ad})\) on a manifold \( M \). The first part of the data consists of a filtration \( TM = T^{-k}M \supset \cdots \supset T^{-1}M \supset \{0\} \) of \( TM \) compatible with the Lie bracket such that at each point \( x \in M \) the induced Lie algebra structure on the associated graded \( \text{gr}(T_xM) \) (the symbol algebra) is isomorphic to \( g_+ \). We denote by \( \mathcal{F}(g_-, \text{gr}(TM)) \) the induced frame bundle of \( \text{gr}(TM) \) whose structure group is the group \( \text{Aut}_{\text{gr}}(g_-) \) of graded Lie algebra automorphisms of \( g_- \) and whose fiber over \( x \) consists of all the graded Lie algebra isomorphisms \( g_- \rightarrow \text{gr}(T_xM) \). The second part of the data is a \( P_0 \)-principal bundle \( E \rightarrow M \) equipped with a bundle map \( \Phi : E \rightarrow \mathcal{F}(g_-, \text{gr}(TM)) \) covering the identity on \( M \) which is equivariant with respect to the homomorphism \( \text{Ad} : P_0 \rightarrow \text{Aut}_{\text{gr}}(g_-) \) in the sense that \( \Phi(u.p) = \Phi(u).\text{Ad}(p) \) for \( p \in P_0, u \in E \). An underlying structure of type \((g, P, \text{Ad})\) on \( M \) is such a filtration of \( TM \) together with such a \( P_0 \)-principal bundle \( E \) and map \( \Phi \).

There are notions of morphisms of parabolic geometries and of underlying structures of type \((g, P, \text{Ad})\) which make these into categories. A morphism of parabolic geometries \( \mathcal{B}_1 \rightarrow M_1 \) and \( \mathcal{B}_2 \rightarrow M_2 \) of type \((g, P, \text{Ad})\) is a principal bundle morphism \( \phi : \mathcal{B}_1 \rightarrow \mathcal{B}_2 \) such that \( \phi^* \omega_2 = \omega_1 \). A morphism of underlying structures \( E_1 \rightarrow M_1 \) and \( E_2 \rightarrow M_2 \) of type \((g, P, \text{Ad})\) is a principal bundle morphism \( \phi : E_1 \rightarrow E_2 \) which covers a filtration-preserving local diffeomorphism \( f : M_1 \rightarrow M_2 \) and which is compatible with the maps \( \Phi_1, \Phi_2 \) in the sense that \( \Phi_2 \circ \phi = f_* \circ \Phi_1 \), where \( f_* : \mathcal{F}(g_-, \text{gr}(TM_1)) \rightarrow \mathcal{F}(g_-, \text{gr}(TM_2)) \) is the map on the frame bundles induced by the differential of \( f \). If \((g, P_1, \text{Ad}_1)\) and \((g, P_2, \text{Ad}_2)\) are equivalent from the point of view of the TMČS Theorem as defined above, then composition of the principal bundle actions with \( \gamma \) induces an equivalence of categories between the categories of parabolic geometries of types \((g, P_1, \text{Ad}_1)\) and \((g, P_2, \text{Ad}_2)\) and between the categories of underlying structures of types \((g, P_1, \text{Ad}_1)\) and \((g, P_2, \text{Ad}_2)\).

The simplest and most common situation is when \( \text{Ad} : P_0 \rightarrow \text{Aut}_{\text{gr}}(g_-) \) is injective. Then \( \Phi \) is a bijection between \( E \) and \( \Phi(E) \subset \mathcal{F}(g_-, \text{gr}(TM)) \), and \( \Phi(E) \) is a subbundle
of \( F(g_-, \text{gr}(TM)) \) with structure group \( \text{Ad}(P_0) \cong P_0 \). It is not hard to see that this association defines an equivalence between the category of underlying structures of type \((g, P, \text{Ad})\) and the category of reductions of structure group of the frame bundle \( F(g_-, \text{gr}(TM)) \) of filtered manifolds of type \( g_- \) to \( \text{Ad}(P_0) \subset \text{Aut}_{\text{gr}}(g_-) \). In general, an underlying structure determines the bundle \( \Phi(E) \), which is still a reduction of \( F(g_-, \text{gr}(TM)) \) to structure group \( \text{Ad}(P_0) \subset \text{Aut}_{\text{gr}}(g_-) \). But if \( \text{Ad} : P_0 \to \text{Aut}_{\text{gr}}(g_-) \) is not injective then the underlying structure contains more information. In all the cases we will consider, the kernel of \( \text{Ad} : P_0 \to \text{Aut}_{\text{gr}}(g_-) \) is discrete. Then \( \text{Ad} : P_0 \to \text{Ad}(P_0) \) and \( \Phi : E \to \Phi(E) \) are covering maps, and to fix an underlying structure one must also choose the lift \( \Phi : E \to \Phi(E) \) of the reduced bundle \( \Phi(E) \subset F(g_-, \text{gr}(TM)) \) to a \( P_0 \)-bundle.

**TMČS Theorem.** Let \( g \) be a \(|k|\)-graded semisimple Lie algebra such that none of the simple ideals of \( g \) is contained in \( g_0 \), and such that \( H^1(g_-, g)^1 = 0 \). Let \( P \) be a Lie group with Lie algebra \( p \) and let \( \text{Ad} : P \to \text{Aut}_{\text{filtr}}(g) \) be a representation of \( P \) as filtration-preserving Lie algebra automorphisms of \( g \) such that \( p \to \text{Ad}(p)|_p \) is the usual adjoint representation of \( P \) on \( p \). Then there is an equivalence of categories between normal regular parabolic geometries of type \((g, P, \text{Ad})\) and underlying structures of type \((g, P, \text{Ad})\) where \( g \) is a parabolic subgroup of \( G \). Our point in formulating parabolic geometries and underlying structures of type \((g, P, \text{Ad})\) rather than type \((G, P)\) is not really to extend the discussion to the case that such a \( G \) might not exist. There is such a group \( G \) for all the examples we care about. Rather, the point is to emphasize that the choice of a particular such \( G \) is irrelevant as far as the TMČS Theorem is concerned. The fact that the TMČS Theorem holds in the generality stated above has been communicated to us by Andreas Čap. The emphasis on \( g \) rather than \( G \) and further generalization in this direction are fundamental aspects of the work of Morimoto [M].

Here \( H^1(g_-, g)^1 \) denotes the 1-piece in the filtration of the first Lie algebra cohomology group. All the examples we will consider satisfy \( H^1(g_-, g)^1 = 0 \).

The discussion in [CSI] is in terms of categories of parabolic geometries and underlying structures of type \((G, P)\) and assumes from the outset that that \( P \) is a parabolic subgroup of \( G \). Our point in formulating parabolic geometries and underlying structures of type \((g, P, \text{Ad})\) rather than type \((G, P)\) is not really to extend the discussion to the case that such a \( G \) might not exist. There is such a group \( G \) for all the examples we care about. Rather, the point is to emphasize that the choice of a particular such \( G \) is irrelevant as far as the TMČS Theorem is concerned. The fact that the TMČS Theorem holds in the generality stated above has been communicated to us by Andreas Čap. The emphasis on \( g \) rather than \( G \) and further generalization in this direction are fundamental aspects of the work of Morimoto [M].

Consider the case of general conformal structures of signature \((p, q)\), \( p + q = n \geq 3 \). The filtration of \( TM \) is trivial, the frame bundle \( F \) is the full frame bundle of \( M \), and a conformal structure is equivalent to a reduction of the structure group of \( F \) to \( CO(p, q) = \mathbb{R}_+ O(p, q) \). The Lie algebra \( g \) is \( so(p + 1, q + 1) \) and we will consider various possibilities for \( P \). Take the quadratic form defining \( so(p + 1, q + 1) \) to be \( 2x^0 x^\infty + h_{ij} x^i x^j \) for some \( h_{ij} \) of signature \((p, q)\). Writing the matrices in terms of \( 1 \times n \times 1 \) blocks, the Levi subgroups \( P_0 \) will be of the form

\[
(3.1) \quad P_0 = \left\{ \begin{array}{c}
\lambda \quad 0 \quad 0 \\
0 \quad m \quad 0 \\
0 \quad 0 \quad \lambda^{-1}
\end{array} \right\}
\]
Theorem. \( P \) the other, so these choices of \( \lambda \) with various restrictions on \( P \) are equivalent from the point of view of the TMCS Theorem. This is the subgroup \( \mathfrak{g} \) block upper-triangular. \( \mathfrak{g}_- \cong \mathbb{R}^n \) consists of matrices of the form

\[
\begin{pmatrix}
0 & 0 & 0 \\
x^i & 0 & 0 \\
0 & -x_j & 0
\end{pmatrix}
\]

with \( x \in \mathbb{R}^n \), and \( h_{ij} \) is used to lower the index. Under the adjoint action, \( p \in P_0 \) acts on \( \mathfrak{g}_- \) by \( x \mapsto \lambda^{-1}mx \).

A natural first choice is to take \( P \) to be the subgroup \( \mathfrak{P}^{\text{ray}} \) of \( G = O(p + 1, q + 1) \) preserving the ray \( \mathbb{R}_+ e_0 \), with \( \text{Ad} \) the restriction of the adjoint representation of \( O(p + 1, q + 1) \). Then \( \mathfrak{P}^{\text{ray}}_0 \) is given by (3.1) with the restrictions \( \lambda > 0, m \in O(p, q) \). The map \( \text{Ad} : \mathfrak{P}^{\text{ray}}_0 \to CO(p, q) \) is an isomorphism, so an underlying structure is exactly a conformal structure.

There is another choice of \( P \) which is equivalent to \( \mathfrak{P}^{\text{ray}} \) from the point of view of the TMCS Theorem. Namely, consider the subgroup \( PP \) of \( PO(p + 1, q + 1) = O(p + 1, q + 1)/\{ \pm I \} \) preserving the line through \( e_0 \) in the projective action, with \( \text{Ad} \) induced by the inclusion of \( PP \) as a subgroup of \( G = PO(p + 1, q + 1) \). The coset projection \( \mathfrak{P}^{\text{ray}} \to PP \) is an isomorphism which maps one \( \text{Ad} \) representation to the other, so these choices of \( P \) are equivalent from the point of view of the TMCS Theorem.

If \( n \) is odd, there is yet another choice of \( P \) also equivalent from the point of view of the TMCS Theorem. This is the subgroup \( \mathfrak{S}^{\text{line}} P \) of \( SO(p + 1, q + 1) \) preserving the line through \( e_0 \), with \( \text{Ad} \) induced by the inclusion \( \mathfrak{S}^{\text{line}} P \subset O(p + 1, q + 1) \). Observe that \( \mathfrak{S}^{\text{line}} P_0 \) corresponds to \( \lambda \neq 0, m \in SO(p, q) \). If \( \lambda < 0, m \in SO(p, q) \), and \( n \) is odd, then \( \lambda^{-1}m \) has negative determinant, and one sees easily that \( \text{Ad} : \mathfrak{S}^{\text{line}} P_0 \to CO(p, q) \) is an isomorphism. The map \( A \mapsto \det(A)A \) is an isomorphism from \( \mathfrak{P}^{\text{ray}} \) to \( \mathfrak{S}^{\text{line}} P \) which maps the \( \text{Ad} \) representation of \( \mathfrak{P}^{\text{ray}} \) on \( \mathfrak{so}(p + 1, q + 1) \) to that of \( \mathfrak{S}^{\text{line}} P \). Thus \( \mathfrak{P}^{\text{ray}} \) and \( \mathfrak{S}^{\text{line}} P \) are equivalent from the point of view of the TMCS Theorem if \( n \) is odd.

The TMCS Theorem asserts an equivalence of categories between conformal structures and normal parabolic geometries of type \( (\mathfrak{so}(p + 1, q + 1), P^{\text{ray}}) \). (The regularity condition is automatic for conformal geometry since \( \mathfrak{so}(p + 1, q + 1) \) is \( |1| \)-graded.) Thus, for each conformal manifold, there is a \( P^{\text{ray}} \)-principal bundle \( \mathcal{B}^{\text{ray}} \) carrying a normal Cartan connection, and it is unique up to isomorphism. This is of course a classical result going back to Cartan. We will call this parabolic geometry of type \( (\mathfrak{so}(p + 1, q + 1), P^{\text{ray}}) \) the canonical parabolic geometry realization of the conformal manifold \( (M, c) \). One may equally well choose to represent the canonical parabolic geometry using the structure group \( PP \) (or \( \mathfrak{S}^{\text{line}} P \) if \( n \) is odd), since these categories of parabolic geometries are equivalent.

It is instructive to identify the principal bundles for model \( M \). For instance, suppose we take \( M = S^p \times S^q \) to be the space of null rays, with conformal structure determined by the metric \( g_{S^p} - g_{S^q} \). We have \( M = O(p + 1, q + 1)/P^{\text{ray}} \), so the canonical parabolic
geometry realization determined by the TMČS Theorem is \( \mathcal{B} = O(p + 1, q + 1) \) with Cartan connection the Maurer-Cartan form. Alternately we can choose \( M \) to be the quadric \( \mathcal{Q} = (S^p \times S^q)/\mathbb{Z}_2 \) embedded in \( \mathbb{P}^{n+1} \) as the set of null lines. We can realize \( \mathcal{Q} = PO(p + 1, q + 1)/PP \) and view this as a parabolic geometry for \( P^{\text{ray}} \) via the isomorphism \( P^{\text{ray}} \cong PP \). So the canonical parabolic geometry realization of the quadric is \( PO(p + 1, q + 1) \) with its Maurer-Cartan form as Cartan connection. Thus in this sense both \( PO(p + 1, q + 1)/PP \) and \( O(p + 1, q + 1)/P^{\text{ray}} \) are homogeneous models for the category of parabolic geometries of type \((\mathfrak{so}(p + 1, q + 1), P^{\text{ray}})\).

If \( n \) is odd, then there is an alternate homogeneous space realization of \( \mathcal{Q} \) using \( SP^{\text{line}} \); namely as \( \mathcal{Q} = SO(p + 1, q + 1)/SP^{\text{line}} \). The uniqueness assertion inherent in the TMČS Theorem implies that this realization must be isomorphic to that above. Indeed, the map \( A \mapsto \det(A)A \) determines an isomorphism \( PO(p + 1, q + 1) \rightarrow SO(p + 1, q + 1) \) of the two parabolic geometry realizations of \( \mathcal{Q} \).

Next let us choose \( P \) to be the subgroup \( P^{\text{line}} \) of \( O(p + 1, q + 1) \) preserving the line spanned by the first basis vector \( e_0 \), with \( \text{Ad} \) induced by the inclusion in \( G = O(p + 1, q + 1) \). The Levi factor \( P_0^{\text{line}} \) corresponds to the conditions \( \lambda \neq 0, m \in O(p, q) \). We have \( \text{Ad}(P_0^{\text{line}}) = CO(p, q) \), so an underlying structure includes the data of a conformal structure. But now \( \text{Ad} \) is not injective; its kernel is \( \{ \pm I \} \). So to determine an underlying structure we must additionally choose a lift \( E \) of the conformal frame bundle to a \( P_0^{\text{line}} \)-bundle. Such a lift always exists since \( P_0^{\text{line}} \) is a product: \( P_0^{\text{line}} \cong P_0^{\text{ray}} \times \{ \pm I \} \). If \( F_c \) denotes the conformal frame bundle of \( M \) with structure group \( CO(p, q) \cong P_0^{\text{ray}} \), then we can take \( E = F_c \times \{ \pm I \} \) with the product action of \( P_0^{\text{line}} \), and can take the map \( \Phi \) in the definition of underlying structures to be the projection onto \( F_c \). Since \( P^{\text{line}} \cong P^{\text{ray}} \times \{ \pm I \} \), if \( (B^{\text{ray}}, \omega^{\text{ray}}) \) denotes the canonical parabolic geometry realization of the conformal manifold, then the bundle \( B^{\text{line}} \) produced by the TMČS Theorem for this choice of \( E \) is just \( B^{\text{line}} = B^{\text{ray}} \times \{ \pm I \} \), and the Cartan connection is the pullback of \( \omega^{\text{ray}} \) to \( B^{\text{line}} \) under the obvious projection.

However, depending on the topology of \( M \), there may be a number of other inequivalent lifts \( E \) of the conformal frame bundle to a \( P_0^{\text{line}} \)-bundle, which will determine inequivalent \( P^{\text{line}} \)-principal bundles \( \mathcal{B} \) with normal Cartan connection via the TMČS Theorem. For instance, consider the quadric \( \mathcal{Q} \). The product bundle constructed in the previous paragraph gives rise to the realization \( \mathcal{Q} = (PO(p + 1, q + 1) \times \{ \pm I \})/P^{\text{line}} \), with Cartan bundle \( \mathcal{B} = PO(p + 1, q + 1) \times \{ \pm I \} \). On the other hand, the geometrically obvious realization of \( \mathcal{Q} \) as a homogeneous space for \( P^{\text{line}} \) is as \( \mathcal{Q} = O(p + 1, q + 1)/P^{\text{line}} \). If \( p = 0 \) or \( q = 0 \), then \( \mathcal{Q} = S^n \) is simply connected so there is only one lift. Indeed, \( O(n + 1, 1) \cong PO(n + 1, 1) \times \{ \pm I \} \), corresponding to the decomposition into time-preserving and time-reversing transformations. But if \( pq \neq 0 \), then \( O(p + 1, q + 1) \) and \( PO(p + 1, q + 1) \times \{ \pm I \} \) are inequivalent as \( P^{\text{line}} \)-principal bundles over \( \mathcal{Q} \), as we will see in the next section.

There are analogues of all these choices for oriented conformal structures. In this case the structure group reduction is to \( CSO(p, q) = \mathbb{R}_+SO(p, q) \subset CO(p, q) \). A natural choice is to take \( P \) to be \( SP^{\text{ray}} \), the subgroup of \( SO(p + 1, q + 1) \) preserving
the null ray, for which $SP^\text{ray}_0$ corresponds to $\lambda > 0$, $m \in SO(p, q)$. We have that $\text{Ad} : SP^\text{ray}_0 \to CSO(p, q)$ is an isomorphism, so in all dimensions and signatures underlying structures of type $(\mathfrak{so}(p+1, q+1), SP^\text{ray})$ are the same as oriented conformal structures. The parabolic geometry of type $(\mathfrak{so}(p+1, q+1), SP^\text{ray})$ determined by the TMČS Theorem is a reduction to structure group $SP^\text{ray}$ of the canonical parabolic geometry of type $(\mathfrak{so}(p+1, q+1), P^\text{ray})$ determined by the same conformal structure but forgetting the orientation.

If $n$ is even, for oriented conformal structures a choice of $P$ equivalent to $SP^\text{ray}$ from the point of view of the TMČS Theorem is the subgroup $PSP$ of $PSO(p+1, q+1)$ preserving the null line in the projective action. In all dimensions and signatures, a homogeneous model is $S^p \times S^q = SO(p+1, q+1)/SP^\text{ray}$. The quadric $Q$ is orientable if $n$ is even, and in this case it provides another homogeneous model for parabolic geometries of type $(\mathfrak{so}(p+1, q+1), SP^\text{ray})$: $Q = PSO(p+1, q+1)/PSP$.

For even $n$, a choice of $P$ for oriented conformal structures analogous to $P^\text{line}$ above is $SP^\text{line}$, since $\lambda^{-1}m$ remains orientation-preserving for $\lambda < 0$ if $n$ is even. For this choice the structure group reduction is to $\text{Ad}(SP^\text{line}_0) = CSO(p, q)$. But $\text{Ad}$ has kernel $\{\pm I\}$, so a lift of the oriented conformal frame bundle must be chosen to determine an underlying structure. One has the product decomposition $SP^\text{line}_0 \cong SP^\text{ray}_0 \times \{\pm I\}$, so one choice is always the product lift. But if $pq \neq 0$, the realization $Q = SO(p+1, q+1)/SP^\text{line}$ corresponds to an inequivalent lift.

4. Tractor Bundles as Associated Bundles

Let $M$ be a manifold with a parabolic geometry $(\mathcal{B}, \omega)$ of type $(\mathfrak{g}, P, \text{Ad})$. There is an associated vector bundle $\mathcal{V} \to M$ corresponding to any finite-dimensional representation $\rho : P \to GL(V)$. The sections of $\mathcal{V}$ can be identified with the maps $f : \mathcal{B} \to V$ which are $P$-equivariant in the sense that $R^*_pf = \rho(p^{-1})f$ for all $p \in P$. Suppose moreover that $(\mathcal{V}, \rho)$ is actually a $(\mathfrak{g}, P)$-representation, that is there is an action $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ on $V$ which is compatible with the $P$-action in the sense that the infinitesimal action of $\mathfrak{p}$ obtained by differentiating the action of $P$ agrees with the restriction of the action of $\mathfrak{g}$ to $\mathfrak{p}$. We will say that the $(\mathfrak{g}, P)$-module $(\mathcal{V}, \rho)$ is $\text{Ad}$-compatible if

$$\rho(\text{Ad}(p)(Z)) = \rho(p)\rho(Z)\rho(p^{-1}) \quad p \in P, \ Z \in \mathfrak{g}.$$

In this case there is an induced linear connection $\nabla$ on $\mathcal{V}$ defined as follows. Let $f$ be a section of $\mathcal{V}$ and let $X$ be a vector field on $M$. Choose a lift $\tilde{X}$ of $X$ to $\mathcal{B}$ and set

$$\nabla_X f = \tilde{X} f + \rho(\omega(\tilde{X}))f. \quad (4.1)$$

The fact that $\omega$ reproduces generators of fundamental vector fields, the equivariance of $f$, and the compatibility of the $(\mathfrak{g}, P)$-actions implies that the right-hand side is unchanged upon adding a vertical vector field to $\tilde{X}$. Thus $\nabla_X f$ is independent of the choice of lift $\tilde{X}$. So one may as well take $\tilde{X}$ to be $P$-invariant. Then $\tilde{X} f$ is clearly $P$-equivariant, and one checks easily that the $\text{Ad}$-compatibility implies that $\rho(\omega(\tilde{X}))f$ is
$P$-equivariant. Thus $\nabla_X f$ is a section of $\mathcal{V}$. The resulting map $(X, f) \mapsto \nabla_X f$ defines a connection on $\mathcal{V}$. We call $(\mathcal{V}, \nabla)$ the tractor bundle and tractor connection for the parabolic geometry $(\mathcal{B}, \omega)$ associated to the Ad-compatible $(\mathfrak{g}, P)$-module $(V, \rho)$.

As discussed in the previous section, typically one can find a Lie group $G$ with Lie algebra $\mathfrak{g}$ which contains $P$ as a parabolic subgroup and which induces the given Ad. If $(V, \rho)$ is any finite-dimensional representation of $G$, the induced representations of $\mathfrak{g}$ and $P$ define a $(\mathfrak{g}, P)$-module structure which is automatically Ad-compatible. So there is a tractor bundle and connection associated to any finite-dimensional representation of any such group $G$. We will call this the tractor bundle and connection associated to the restriction to $(\mathfrak{g}, P)$ of the $G$-module $(V, \rho)$.

We saw in §3 that one can choose different normal parabolic geometries corresponding to a given conformal manifold $(M, c)$. As we will see, different choices can give rise to different bundles associated to the same $O(p + 1, q + 1)$-module $(V, \rho)$. Recall that we have a canonical parabolic geometry corresponding to $(M, c)$: the normal parabolic geometry of type $(\mathfrak{so}(p + 1, q + 1), \mathfrak{pr}^{\mathrm{ray}})$. Applying the associated bundle construction for this choice gives a canonical tractor bundle and connection associated to any $O(p + 1, q + 1)$-module $(V, \rho)$.

We first compare tractor bundles and connections for the product parabolic geometry $(\mathcal{B}^{\mathrm{line}}, \omega^{\mathrm{line}})$ of type $(\mathfrak{so}(p + 1, q + 1), \mathfrak{pr}^{\mathrm{line}})$ with those for the canonical parabolic geometry. Recall that the product parabolic geometry was defined as follows. If $(\mathcal{B}^{\mathrm{ray}}, \omega^{\mathrm{ray}})$ is the canonical parabolic geometry, then $\mathcal{B}^{\mathrm{line}} = \mathcal{B}^{\mathrm{ray}} \times \{\pm I\}$ with the product action of $\mathfrak{pr}^{\mathrm{line}} \cong \mathfrak{pr}^{\mathrm{ray}} \times \{\pm I\}$, and $\omega^{\mathrm{line}}$ is the pullback of $\omega^{\mathrm{ray}}$ under the projection $\mathcal{B}^{\mathrm{line}} \to \mathcal{B}^{\mathrm{ray}}$. The bundle $\mathcal{B}^{\mathrm{line}}$ may alternately be described as the $\mathfrak{pr}^{\mathrm{line}}$-principal bundle associated to the $\mathfrak{pr}^{\mathrm{ray}}$-principal bundle $\mathcal{B}^{\mathrm{ray}}$ by the action of $\mathfrak{pr}^{\mathrm{ray}}$ on $\mathfrak{pr}^{\mathrm{line}}$ by left translation.

Let $(V, \rho^{\mathrm{ray}})$ be an Ad-compatible $(\mathfrak{so}(p + 1, q + 1), \mathfrak{pr}^{\mathrm{ray}})$-module. We will say that an Ad-compatible $(\mathfrak{so}(p + 1, q + 1), \mathfrak{pr}^{\mathrm{line}})$-module $(V, \rho^{\mathrm{line}})$ extends $(V, \rho^{\mathrm{ray}})$ if $\rho^{\mathrm{line}} = \rho^{\mathrm{ray}}$ on $\mathfrak{so}(p + 1, q + 1)$ and $\rho^{\mathrm{line}}|_{\mathfrak{pr}^{\mathrm{ray}}} = \rho^{\mathrm{ray}}$. For a given $(V, \rho^{\mathrm{ray}})$, there are always at least two choices of such $\rho^{\mathrm{line}}$; namely those determined by the two choices $\rho^{\mathrm{line}}(-I) = \pm I_V$. One checks easily that either choice of $\pm$ defines an Ad-compatible $(\mathfrak{so}(p + 1, q + 1), \mathfrak{pr}^{\mathrm{line}})$-module.

**Proposition 4.1.** Let $(M, c)$ be a conformal manifold. Let $(V, \rho^{\mathrm{ray}})$ be an Ad-compatible $(\mathfrak{so}(p + 1, q + 1), \mathfrak{pr}^{\mathrm{ray}})$-module and let $(V, \rho^{\mathrm{line}})$ be an Ad-compatible $(\mathfrak{so}(p + 1, q + 1), \mathfrak{pr}^{\mathrm{line}})$-module which extends $\rho^{\mathrm{ray}}$. Then the tractor bundle and connection associated to the $(\mathfrak{so}(p + 1, q + 1), \mathfrak{pr}^{\mathrm{line}})$-module $(V, \rho^{\mathrm{line}})$ for the product parabolic geometry $(\mathcal{B}^{\mathrm{line}}, \omega^{\mathrm{line}})$ are naturally isomorphic to the tractor bundle and connection associated to the $(\mathfrak{so}(p + 1, q + 1), \mathfrak{pr}^{\mathrm{ray}})$-module $(V, \rho^{\mathrm{ray}})$ for the canonical parabolic geometry $(\mathcal{B}^{\mathrm{ray}}, \omega^{\mathrm{ray}})$.

**Proof.** We first claim that the bundle associated to $(V, \rho^{\mathrm{line}})$ for $(\mathcal{B}^{\mathrm{line}}, \omega^{\mathrm{line}})$ is isomorphic to that associated to $(V, \rho^{\mathrm{ray}})$ for $(\mathcal{B}^{\mathrm{ray}}, \omega^{\mathrm{ray}})$. This is a special case of the following general fact, the proof of which is straightforward. Suppose that $P_1$ is a Lie subgroup of a Lie group $P_2$ and $\mathcal{B}_1$ is a $P_1$-principal bundle over a manifold $M$. Let
$\mathcal{B}_2$ be the $P_2$-principal bundle over $M$ associated to the action of $P_1$ on $P_2$ by left translation. Let $(V, \rho)$ be a $P_2$-module and $\mathcal{V}_2$ the vector bundle associated to $(V, \rho)$ for $\mathcal{B}_2$. Then $\mathcal{V}_2$ is naturally isomorphic as a smooth vector bundle to the vector bundle $\mathcal{V}_1$ associated to $(V, \rho|_{P_1})$ for $\mathcal{B}_1$.

It is clear from (4.1) that the tractor connections induced by the Cartan connections $\omega^{\text{ray}}$ and $\omega^{\text{line}}$ correspond under this isomorphism, since $\mathcal{B}^{\text{ray}}$ can be embedded as an open subset of $\mathcal{B}^{\text{line}}$ on which $\omega^{\text{line}}$ restricts to $\omega^{\text{ray}}$. $\square$

The following corollary is an immediate consequence of Proposition 4.1.

**Corollary 4.2.** If $(V, \rho)$ is an $O(p+1, q+1)$-module, then the tractor bundle and connection for the product parabolic geometry $(\mathcal{B}^{\text{line}}, \omega^{\text{line}})$ associated to the restriction to $(\mathfrak{so}(p+1, q+1), P^{\text{line}})$ of $(V, \rho)$ are naturally isomorphic to the tractor bundle and connection for the canonical parabolic geometry $(\mathcal{B}^{\text{ray}}, \omega^{\text{ray}})$ associated to the restriction to $(\mathfrak{so}(p+1, q+1), P^{\text{ray}})$ of $(V, \rho)$.

The tractor bundle and connection associated to a given $O(p+1, q+1)$-module for other normal parabolic geometry realizations of a conformal manifold may be different from those for the canonical parabolic geometry. The basic case is the standard representation $\mathcal{V}$ of $G = O(p+1, q+1)$, since any finite-dimensional $O(p+1, q+1)$-module is isomorphic to a submodule of a direct sum of tensor powers of the standard representation. Consider first the canonical parabolic geometry.

**Proposition 4.3.** Let $(M, c)$ be a conformal manifold. Let $(\mathcal{B}^{\text{ray}}, \omega^{\text{ray}})$ be the canonical parabolic geometry of type $(\mathfrak{so}(p+1, q+1), P^{\text{ray}})$. Let $(\mathcal{T}, \nabla)$ be the bundle and connection associated to the restriction to $(\mathfrak{so}(p+1, q+1), P^{\text{ray}})$ of the standard representation $\mathcal{V}$ of $O(p+1, q+1)$. Then $(\mathcal{T}, \nabla)$ is a standard tractor bundle with normal connection.

*Proof.* Since the action of $P^{\text{ray}} \subset O(p+1, q+1)$ preserves the quadratic form defining $O(p+1, q+1)$, it follows that the $S^2\mathcal{V}^*$-valued constant function on $\mathcal{B}$ whose value at each point is this quadratic form is $P^{\text{ray}}$-equivariant. So it defines a metric $h$ on $\mathcal{T}$. Since the quadratic form is annihilated by $\mathfrak{so}(p+1, q+1)$, the formula (4.1) for the connection shows that $\nabla h = 0$. Now $P^{\text{ray}}$ acts on $e_0$ by multiplication by $\lambda > 0$. It follows that $e_0$ determines a nonvanishing global section of $\mathcal{T}[1]$. This section determines a null rank 1 subbundle $\mathcal{T}^1$ of $\mathcal{T}$ together with an isomorphism $\mathcal{T}^1 \cong \mathcal{D}[-1]$. The compatibility of the data with the conformal structure and the normality of $\nabla$ follow from the fact that $\omega$ is the normal Cartan connection for the structure; see [CG1], [CG2]. Thus $(\mathcal{T}, \nabla)$ possesses the structure defining a standard tractor bundle with normal connection. $\square$

Recall that if $n$ is odd, then $P^{\text{ray}}$ and $SP^{\text{line}}$ are equivalent from the point of view of the TMCS Theorem. So by composing the principal bundle action on $\mathcal{B}^{\text{ray}}$ with the inverse of the isomorphism $A \mapsto (\det A)A$ from $P^{\text{ray}}$ to $SP^{\text{line}}$, the canonical parabolic geometry $(\mathcal{B}^{\text{ray}}, \omega^{\text{ray}})$ can be viewed as a parabolic geometry of type $(\mathfrak{so}(p+1, q+1), SP^{\text{line}})$. So there is a tractor bundle and connection associated to the restriction to
The tractor bundle can alternately be described as the bundle associated to the restriction to $(\mathfrak{so}(p+1, q+1), \mathcal{P}_{\text{ray}})$ of the representation $\det \otimes \mathcal{V}$ of $O(p+1, q+1)$ for the canonical parabolic geometry.

**Proposition 4.4.** Let $(M, c)$ be a nonorientable odd-dimensional conformal manifold. Let $(\mathcal{B}, \omega)$ be the corresponding normal parabolic geometry of type $(\mathfrak{so}(p+1, q+1), \mathcal{S}P_{\text{line}})$. The tractor bundle with normal connection associated to the restriction to $(\mathfrak{so}(p+1, q+1), \mathcal{S}P_{\text{line}})$ of the standard representation $\mathcal{V}$ of $O(p+1, q+1)$ is not a standard tractor bundle.

**Proof.** The group $\mathcal{S}P_{\text{line}}$ preserves a volume form on $\mathcal{V}$. There is an induced nonvanishing volume form for the associated bundle, so the tractor bundle is orientable. But we saw in §2 that the standard tractor bundle constructed in [BEG] is orientable if and only if $M$ is orientable. Since standard tractor bundles are unique up to isomorphism, it follows that the associated bundle is not a standard tractor bundle if $M$ is not orientable. 

Recall that the quadric $\mathcal{Q}$ is not orientable if $n$ is odd and $pq \neq 0$. Therefore we conclude:

**Corollary 4.5.** Let $n$ be odd and $pq \neq 0$. Represent $\mathcal{Q} = SO(p+1, q+1)/\mathcal{S}P_{\text{line}}$. The tractor bundle on $\mathcal{Q}$ associated to the standard representation of $\mathcal{S}P_{\text{line}}$ is not a standard tractor bundle.

Proposition [4.3] and Corollary [4.2] imply that for the product parabolic geometry of type $(\mathfrak{so}(p+1, q+1), \mathcal{P}_{\text{line}})$ on a general conformal manifold, the tractor bundle and connection associated to the standard representation are a standard tractor bundle with its normal connection. But as we saw in the last section, there may be other normal parabolic geometries of type $(\mathfrak{so}(p+1, q+1), \mathcal{P}_{\text{line}})$ corresponding to the same conformal structure, and the product parabolic geometry might not be the most geometrically natural choice. For other choices the associated tractor bundle need not be a standard tractor bundle.

**Proposition 4.6.** Represent $\mathcal{Q} = O(p+1, q+1)/\mathcal{P}_{\text{line}}$. If $pq \neq 0$, then the tractor bundle on $\mathcal{Q}$ associated to the standard representation of $\mathcal{P}_{\text{line}}$ is not a standard tractor bundle.

**Proof.** The null line subbundle $\mathcal{T}^1$ is associated to the action of $\mathcal{P}_{\text{line}}$ on the invariant subspace span$\{e_0\}$. It is easily seen that this associated bundle is the tautological bundle, whose fiber at a null line is the line itself. But the tautological bundle on $\mathcal{Q}$ is not trivial if $pq \neq 0$, so it cannot be isomorphic to $\mathcal{D}[-1]$. 

In Proposition [4.6] $\mathcal{T}^1$ is associated to the representation of $\mathcal{P}_{\text{line}}$ in which $p$ in $[3.1]$ acts by $\lambda$, while $\mathcal{D}[-1]$ is associated to the representation in which $p$ acts by $|\lambda|$. For this homogeneous space these associated bundles are not equivalent.
Remark 4.7. The tractor bundle on $Q$ in Proposition 4.6 is trivial since it is a bundle on a homogeneous space $G/P$ associated to the restriction to $P$ of a representation of $G$. In particular it is orientable. So if $n$ is odd, an alternate proof of Proposition 4.6 is to derive a contradiction to orientability of $T$ as in the proof of Proposition 4.4 and Corollary 4.5. But if $n$ is even, $Q$ is orientable and there is no contradiction to orientability of $T$. In this case the contradiction concerns the orientability (equivalently, the triviality) of $T^1$, not of $T$.

Observe also the following curious state of affairs in Proposition 4.6 when $n$ is odd. The standard tractor bundle on $Q$ is nontrivial but its distinguished null line subbundle is trivial. By contrast, the associated tractor bundle is trivial but its distinguished null line subbundle is nontrivial.

One encounters the same phenomena for oriented conformal structures. Recall from the previous section that oriented conformal structures are equivalent to normal parabolic geometries of type $(\mathfrak{so}(p+1,q+1), S^{\text{ray}})$. The same proof as in Proposition 4.3 shows that the associated bundle for the standard representation of $(\mathfrak{so}(p+1,q+1), S^{\text{ray}})$ is a standard tractor bundle. If $n$ is even, $S^{\text{line}}$ factors as $S^{\text{line}} = S^{\text{ray}} \times \{\pm I\}$, and the same proof as in Proposition 4.1 shows that the associated bundle for the product $S^{\text{line}}$-principal bundle is a standard tractor bundle. But if $n$ is even and the quadric is realized as $Q = SO(p+1,q+1)/S^{\text{line}}$, then the proof of Proposition 4.6 shows that the associated bundle to the standard representation of $S^{\text{line}}$ is not a standard tractor bundle if $pq \neq 0$.

In the theory of Cartan geometries one sometimes declares a particular connected homogeneous space $G/P$ to be the model, and a tractor bundle on $G/P$ to be a bundle associated to the restriction to $P$ of a representation of $G$. Such tractor bundles are necessarily trivial and the induced connection is the usual flat connection on a trivial bundle. For definite signature conformal structures the natural choice is to take the model to be the quadric $Q = S^n$, realized either as $O(n+1,1)/S^{\text{line}}$ or as $O_+(n+1,1)/S^{\text{ray}}$, where $O_+(n+1,1)$ denotes the time-preserving subgroup. The above discussion shows that for either realization the bundle associated to the standard representation of $G$ is the standard tractor bundle and its normal connection is the induced flat connection. For indefinite signature conformal structures, a natural choice is to take the model to be $S^p \times S^q = O(p+1,q+1)/S^{\text{ray}}$ and again the same statements hold. (The modification is necessary for definite signature since $O(n+1,1)/S^{\text{ray}}$ is not connected.) It is also possible to view the quadric as the homogeneous model in the case of indefinite signature. Since $PO(p+1,q+1)$ does not admit a standard representation, the realization $Q = PO(p+1,q+1)/PP$ does not admit a tractor bundle associated to the standard representation under the framework of this paragraph. But the realization $Q = O(p+1,q+1)/S^{\text{line}}$ does. The general results stated above of course remain true: the bundle associated to the standard representation of $O(p+1,q+1)$ is trivial and inherits the usual flat connection. But this is not the standard tractor bundle: the standard tractor bundle has no nontrivial parallel sections on $Q$. One must exercise similar care in interpreting other
results about homogeneous models. For instance, it is a general result [CSS] that on a homogeneous parabolic geometry $G/P$, a BGG sequence resolves the constant sheaf determined by the inducing representation of $G$. But one must keep in mind that the bundles in the BGG sequences are all defined as associated bundles. For example, for the BGG sequence associated to the standard representation $V$ on $\mathcal{Q} = O(p+1, q+1)/P^{\text{line}}$ with $pq \neq 0$, the constant sheaf $\mathcal{V}$ is realized as the global kernel of the first BGG operator $\text{tf}(\nabla^2 + P)$ acting not on the bundle of densities $\mathcal{D}[1]$, but on a twisted version thereof (the dual to the tautological bundle), and this twisted version arises as the projecting part of the associated bundle to $\mathcal{V}$, which is not the standard tractor bundle for the conformal structure on $\mathcal{Q}$. The global kernel of $\text{tf}(\nabla^2 + P)$ acting on $\mathcal{D}[1]$ for $\mathcal{Q}$ is trivial if $pq \neq 0$.

Similar issues arise in the consideration of conformal holonomy. We follow the usual practice of defining the conformal holonomy of a conformal manifold to be the holonomy of a standard tractor bundle with its normal tractor connection. This is well-defined by the Čap-Gover Uniqueness Theorem. But the above considerations demonstrate that one must be careful if one is realizing the standard tractor bundle as an associated bundle. The holonomy of a tractor bundle defined as an associated bundle to a standard representation might not equal the conformal holonomy if the principal bundle is not chosen correctly. This happens already for the quadric $\mathcal{Q}$ if $pq \neq 0$. If we realize $\mathcal{Q} = O(p+1, q+1)/P^{\text{line}}$, then the tractor bundle associated to the standard representation of $(\mathfrak{so}(p+1, q+1), P^{\text{line}})$ has trivial holonomy. But as discussed above, the standard tractor bundle of $\mathcal{Q}$ is the bundle associated to the standard representation of $(\mathfrak{so}(p+1, q+1), P^{\text{ray}})$ for the realization $\mathcal{Q} = PO(p+1, q+1)/PP$, viewed as a parabolic geometry for $(\mathfrak{so}(p+1, q+1), P^{\text{ray}})$ via the isomorphism $P^{\text{ray}} \simeq PP$. Its holonomy is $\{ \pm I \}$, since parallel translation in $S^p \times S^q$ to the antipodal point induces $-I$ on a fiber of the standard tractor bundle on $\mathcal{Q}$. Another instance of this is the following.

**Proposition 4.8.** Let $(M, c)$ be a nonorientable odd-dimensional conformal manifold. Recall from the TMČS Theorem that up to isomorphism there is a unique normal parabolic geometry of type $(\mathfrak{so}(p+1, q+1), SP^{\text{line}})$ corresponding to $(M, c)$. Let $(\mathcal{T}, \nabla)$ be the bundle and connection associated to the restriction to $(\mathfrak{so}(p+1, q+1), SP^{\text{line}})$ of the standard representation of $O(p+1, q+1)$. Then the holonomy of $(\mathcal{T}, \nabla)$ is not equal to the conformal holonomy.

**Proof.** The argument is similar to the proof of Proposition 4.4. The standard representation of $SP^{\text{line}}$ preserves a volume form, so there is an induced section of the associated bundle. The volume form is preserved also by $\mathfrak{so}(p+1, q+1)$, so this section is parallel. Thus the holonomy of the associated bundle for $SP^{\text{line}}$ is contained in $SO(p+1, q+1)$. But if the holonomy of the standard tractor bundle is contained in $SO(p+1, q+1)$, then the standard tractor bundle is orientable, so $M$ is orientable. \qed
These issues concerning standard tractor bundles as associated bundles arose in our work [GW] concerning conformal structures and ambient metrics of holonomy $G_2$ (by this we mean the split real form throughout this discussion). Nurowski [N] showed that a generic 2-plane field on a 5-manifold $M$ induces a conformal structure of signature $(2, 3)$ on $M$. The TMCS Theorem implies that generic 2-plane fields on oriented 5-manifolds are the underlying structures corresponding to normal regular parabolic geometries of type $(\mathfrak{g}_2, SQ^{\text{ray}})$, where $SQ^{\text{ray}}$ is the subgroup of $G_2$ preserving a null ray, analogous to $SP^{\text{ray}}$ above. For generic 2-plane fields on nonorientable manifolds one must change the group $P = SQ^{\text{ray}}$ to allow orientation-reversing transformations in $\text{Ad}(P_0)$. A first guess is to take $P$ to be $SQ^{\text{line}}$, the subgroup of $G_2$ preserving a null line. The TMCS Theorem implies that the category of generic 2-plane fields on general 5-manifolds is equivalent to the category of normal regular parabolic geometries of type $(\mathfrak{g}_2, SQ^{\text{line}})$. But just as in Proposition 4.4, the associated bundle to the restriction to $(\mathfrak{g}_2, SQ^{\text{line}})$ of the standard representation of $G_2$ need not be the standard tractor bundle for Nurowski’s induced conformal structure; in fact, it cannot be if $M$ is not orientable. By analogy with the situation above for general conformal structures, instead of $SQ^{\text{line}}$ one should use the subgroup $Q^{\text{ray}}$ of $\{\pm I\}G_2$ preserving a null ray. The TMCS Theorem again gives an equivalence of categories with normal regular parabolic geometries of type $(\mathfrak{g}_2, Q^{\text{ray}})$. And now, just as in Proposition 4.3, the tractor bundle associated to the restriction to $(\mathfrak{g}_2, Q^{\text{ray}})$ of the standard representation of $\{\pm I\}G_2$ need not be the standard tractor bundle of Nurowski’s conformal structure with its normal connection. $Q^{\text{ray}}$ is isomorphic to $SQ^{\text{line}}$, but they are embedded in $O(3, 4)$ differently, just as for $P^{\text{ray}}$ and $SP^{\text{line}}$ above. Using the realization of the standard tractor bundle as the associated bundle for $(\mathfrak{g}_2, Q^{\text{ray}})$, the same arguments as in [HS], [GW] for the orientable case now show that for general $M$, Nurowski’s conformal structures are characterized by having conformal holonomy contained in $\{\pm I\}G_2$, and in the real-analytic case the corresponding ambient metrics have metric holonomy contained in $\{\pm I\}G_2$.

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