Performance measures of two heterogeneous servers queueing models under trisectional fuzzy trapezoidal approach

R. Ramesh\(^1\)\(^*\) and R. Srinivasan\(^2\)

Abstract

This paper presents an analysis of performance measures for two heterogeneous servers queueing models under trisectional fuzzy trapezoidal approach. Ranking techniques are very notable in the fuzzy numbers system for defuzzification. Several authors have already proposed various techniques to find out the performance of fuzzy queues. It is possible to convert from fuzzy environment to crisp environment by our proposed ranking method in order to analyze the performance measures of fuzzy queues. Ultimately, the effectiveness and the accurate values of our proposed method have been successfully solved with example.

Keywords

Fuzzy sets, Fuzzy numbers, Heterogeneous Servers Queueing Models, Trisectional Fuzzy Trapezoidal ranking.

AMS Subject Classification

03E72.

1. Introduction

Queueing theory takes part in an important role in real life problems. Now-a-days, we face a lot of congesting problems in the queueing habitat such as ATM points, Medical shops, Reservation centers, Ration shops, Hospitals, Making calls in Telecommunications, etc. Emphasize the importance of time management is the ultimate aim of the researcher. At this juncture, queueing models take a very prominent role.

The basic preliminaries \cite{5, 7, 11}, and models of queueing are very crucial for our research purpose. In our day to day life situation, most of the time we apply the Fuzzy logic and applications \cite{12, 23}. Heterogeneous Server Queueing System \cite{13, 14} have applications in computers with job processing and in so many fields are modeled as a Poisson process with state dependent arrival rate. In computer and communication systems, the congestion control mechanisms is employed to prevent the formation of long queues by controlling the transmission rates of packets based on the queue length of packets at source or destination.

Many authors have so far applied various ranking techniques to measure the performances of the fuzzy queues. Area based ranking techniques \cite{4, 6, 9} are considered as very well known ranking techniques. Particularly some of the authors applied centroid based ranking techniques \cite{17, 18, 20}. Westman and Wang applied Ranking Fuzzy Numbers by Their Left and Right Wingspans \cite{22}. Our proposed ranking technique to measure the performance of Two Heterogeneous Servers Queueing Models. This is very easy method to compute the actual crisp values of the queueing models.
2. Preliminaries

2.1 Fuzzy Set
A Fuzzy Set \( \tilde{A} = \{ (x, \mu_{\tilde{A}}(x)) : x \in U \} \) is convinced by a membership function \( \mu_{\tilde{A}} \) mapping from elements of a universe of discourse \( U \) to the unit interval \([0,1]\).

(i.e) \( \mu_{\tilde{A}} : U \rightarrow [0,1] \) is a mapping called the degree of membership function of the fuzzy set \( \tilde{A} \) and \( \mu_{\tilde{A}}(x) \) is called the membership value of \( x \in U \) in the fuzzy set \( \tilde{A} \).

2.2 Triangular Fuzzy Number
A Triangular Fuzzy Number \( \tilde{A}(x) \) is represented by \( \tilde{A}(a_1, a_2, a_3) \) with the membership function

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
\frac{x-a_1}{a_2-a_1}, & a_1 \leq x < a_2 \\
1, & x = a_2 \\
\frac{x-a_3}{a_2-a_3}, & a_2 < x \leq a_3 \\
0, & \text{otherwise}
\end{cases}
\]

2.3 Trapezoidal Fuzzy Number
A Trapezoidal Fuzzy Number \( \tilde{A}(x) \) is represented by \( \tilde{A}(a_1, a_2, a_3, a_4) \) with the membership function

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
\frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\
1, & a_2 < x < a_3 \\
\frac{x-a_4}{a_3-a_4}, & a_3 \leq x \leq a_4 \\
0, & \text{otherwise}
\end{cases}
\]

2.4 Heterogeneous Servers Queueing Models
Let us consider the customers are arrived at single channel queue in respect of a Poisson process with an average fuzzy arrival rate \( \tilde{\lambda} \). The inter-arrival times are exponentially distributed with fuzzy parameter \( \tilde{\lambda} \). There are two servers and service times are exponentially distributed with fuzzy parameters \( \mu_1 \) and \( \mu_2 \) at server 1 and server 2 respectively, and \( \mu_1 > \mu_2 \). An arriving customer finding both the servers are busy, the customer waits in a line in order of arrival and the customer in front of queue occupies the server that falls vacant first. When only one of the servers is free, the arriving customer occupies the free server. When both the servers are free, the arriving customer chooses fast server with fuzzy probability \( \tilde{p}_1 \) and slow server with fuzzy probability \( \tilde{p}_2 \), such that \( \tilde{p}_1 + \tilde{p}_2 = 1 \). The capacity of the system is taken as finite, say \( N \). Each customer upon arriving in the queue will wait a certain length of time, say \( T \) for service to begin, if it does not begin by then it may renege with probability \( p \) or may remain in the queue for its service with probability \( q (= 1 - p) \) if certain customer retention strategy is applied. The reneging times \( T \) follow exponential distribution with fuzzy parameter \( \tilde{\xi} \).

Let \( P_n \) be the steady-state probability that there are \( n \) units in the system. \( P_{10} \) be the steady-state probability that server 1 is engaged and server 2 is free with no waiting line. \( P_{01} \) be the steady-state probability that server 1 is free and server 2 is engaged, there is no waiting line. \( P_{00} \) be the steady-state probability that there are no units in the system.

Also, \( P_2 = P_{11} \) and \( P_1 = P_{10} + P_{01} \).

Using the Markov chain theory, the steady-state equations of the model are

\[
\tilde{\lambda} P_{00} = \tilde{\mu}_1 P_{10} + \tilde{\mu}_2 P_{01} \tag{2.1}
\]

\[
P_{11} = \frac{1}{\mu_1} \left\{ \tilde{\lambda} P_{01} - \tilde{\lambda} \tilde{p}_1 P_{10} \right\} \tag{2.2}
\]

\[
P_{01} = \left( \frac{\tilde{\lambda} + (\mu_1 + \mu_2) \tilde{p}_1}{2 \tilde{\lambda} + (\mu_1 + \mu_2)} \right) P_{00} \tag{2.3}
\]

\[
P_{10} = \left( \frac{\tilde{\lambda} + (\mu_1 + \mu_2) \tilde{p}_1}{2 \tilde{\lambda} + (\mu_1 + \mu_2)} \right) P_{00} \tag{2.4}
\]

\[
P_1 = \left( \frac{\tilde{\lambda} + (\mu_1 \tilde{p}_2 + \mu_2) \tilde{p}_1}{2 \tilde{\lambda} + (\mu_1 + \mu_2)} \right) \left( \frac{\tilde{\lambda}}{\mu_1 \mu_2} \right) P_{00} \tag{2.5}
\]

\[
P_2 = \left( \frac{\tilde{\lambda} + (\mu_1 \tilde{p}_2 + \mu_2) \tilde{p}_1}{2 \tilde{\lambda} + (\mu_1 + \mu_2)} \right) \left( \frac{\tilde{\lambda}^2}{\mu_1 \mu_2} \right) P_{00} \tag{2.6}
\]

\[
P_2 = \left( \frac{\tilde{\lambda}}{\mu_1 + \mu_2} \right) P_1 \tag{2.7}
\]

\[
P_3 = \left( \frac{\tilde{\lambda}}{\mu_1 + \mu_2 + \tilde{\xi} p} \right) \left( \frac{\tilde{\lambda}}{\mu_1 + \mu_2} \right) P_1 \tag{2.8}
\]

\[
P_4 = \left( \frac{\tilde{\lambda}}{\mu_1 + \mu_2 + 2 \tilde{\xi} p} \right) \left( \frac{\tilde{\lambda}}{\mu_1 + \mu_2 + \tilde{\xi} p} \right) \left( \frac{\tilde{\lambda}}{\mu_1 + \mu_2} \right) P_1 \tag{2.9}
\]

Similarly

\[
P_n = \left( \prod_{k=2}^{n} \frac{\tilde{\lambda}}{\mu_1 + \mu_2 + (k-2) \tilde{\xi} p} \right) P_{1,2 \leq m \leq N} \tag{2.10}
\]

where

\[
P_{00} = \left[ 1 + \left( \frac{\tilde{\lambda} + (\mu_1 \tilde{p}_2 + \mu_2) \tilde{p}_1}{2 \tilde{\lambda} + (\mu_1 + \mu_2)} \right) \left( \frac{\tilde{\lambda}}{\mu_1 \mu_2} \right) + \sum_{n=2}^{N} \sum_{k=2}^{n} \left( \frac{\tilde{\lambda}}{\mu_1 + \mu_2 + (k-2) \tilde{\xi} p} \right) \left( \frac{\tilde{\lambda}}{2 \tilde{\lambda} + (\mu_1 + \mu_2)} \right) \right]^{-1}
\]

since \( \sum_{n=0}^{N} P_n = 1 \)

By the queuing theory concepts under above steady-state conditions we get

(i) The Expected Number of Customers in the System
(NS) = P_1 + nP_2(N_S) = P_1 + \sum_{n=2}^{N} nP_n
⇒ (NS) = \left\{ \left( \frac{\lambda + (\mu_1 \tilde{\xi}_2 + \mu_2 \tilde{\xi}_1)}{2\lambda + (\mu_1 + \mu_2)} \right) \left( \frac{\lambda (\mu_1 + \mu_2)}{\mu_1 \mu_2} \right) \right\} P_{00}
+ \sum_{N-2}^{N} \left\{ \left( \frac{\lambda (\mu_1 + \mu_2)}{\mu_1 \mu_2} \right) \sum_{k=2}^{N} \left( \frac{\lambda}{\mu_1 + \mu_2 + (k-2)\tilde{\xi}_p} \right) \right\} P_{00}

(ii) The Expected Number of Customers Served

(E(CS)) = \mu_1 P_1 + \mu_2 P_2 + (\mu_1 + \mu_2) \sum_{n=2}^{N} P_n
⇒ (E(CS)) = \mu_1 \left( \frac{\lambda + (\mu_1 + \mu_2) \tilde{\xi}_1}{2\lambda + (\mu_1 + \mu_2)} \right) \left( \frac{\lambda}{\mu_1} \right) P_{00}
+ \mu_2 \left( \frac{\lambda + (\mu_1 + \mu_2) \tilde{\xi}_2}{2\lambda + (\mu_1 + \mu_2)} \right) \left( \frac{\lambda}{\mu_2} \right) P_{00}
+ (\mu_1 + \mu_2) \times \sum_{n=2}^{N} \left\{ \prod_{k=2}^{n} \left( \frac{\lambda}{\mu_1 + \mu_2 + (k-2)\tilde{\xi}_p} \right) \right\} P_{01}

(iii) Average Reneging Rate

(Rr) = \sum_{n=2}^{N} (n-2)\tilde{\xi}_p P_n
⇒ (Rr) = \sum_{n=2}^{N} (n-2)\tilde{\xi}_p \left[ \prod_{k=2}^{n} \frac{\lambda}{\mu + (k-2)\tilde{\xi}_p} \right] P_{01}
⇒ (Rr) = \sum_{n=2}^{N} \left\{ (n-2)\tilde{\xi}_p \left[ \prod_{k=2}^{n} \frac{\lambda}{\mu + (k-2)\tilde{\xi}_p} \right] \times \left( \frac{\lambda + (\mu_1 \tilde{\xi}_2 + \mu_2 \tilde{\xi}_1)}{2\lambda + (\mu_1 + \mu_2)} \right) \left( \frac{\lambda (\mu_1 + \mu_2)}{\mu_1 \mu_2} \right) \right\} P_{00}

(iv) Average Retention Rate

( RR ) = \sum_{n=2}^{N} (n-2)\tilde{\xi}_q P_n
⇒ ( RR ) = \sum_{n=2}^{N} (n-2)\tilde{\xi}_q \left[ \prod_{k=2}^{n} \frac{\lambda}{\mu + (k-2)\tilde{\xi}_p} \right] P_{01}
⇒ ( RR ) = \sum_{n=2}^{N} \left\{ (n-2)\tilde{\xi}_q \left[ \prod_{k=2}^{n} \frac{\lambda}{\mu + (k-2)\tilde{\xi}_p} \right] \times \left( \frac{\lambda + (\mu_1 \tilde{\xi}_2 + \mu_2 \tilde{\xi}_1)}{2\lambda + (\mu_1 + \mu_2)} \right) \left( \frac{\lambda (\mu_1 + \mu_2)}{\mu_1 \mu_2} \right) \right\} P_{00}

2.5 Trisectional Fuzzy Trapezoidal Ranking

Let us consider an interval data \((P, Q)\). The trisection of this interval is taken as
\[d = \frac{Q - P}{3},\]
then the required trapezoidal fuzzy number is \([P, P + d, P + 2d, Q]\).
Consider a normal trapezoidal fuzzy number \(\tilde{A} = [a_1, a_2, a_3, a_4]\) represented in Fig. 1. Extend the line joining \(B(a_1, 0)\) and \(D(a_2, 1)\) as BDA and the line joining \(C(a_4, 0)\) and \(E(a_3, 1)\) as CEA. The intersection of extended lines BD and CE is \(A\). The coordinates for intersection point \(A\) is \((x, y)\), where
\[
x = \frac{a_1 a_3 - a_2 a_4}{a_3 - a_4},
\]
\[
y = \frac{x - a_2}{a_2 - a_1} + 1.
\]

The proposed ranking technique is based on the concept of in-center for triangle ABC and is defined as
\[
R(\tilde{A}) = ax + ba_1 + ca_4
\]
where
\[
a = a_4 - a_1,
\]
\[
b = \sqrt{y^2 + (a_4 - x)^2},
\]
\[
c = \sqrt{y^2 + (x - a_1)^2}.
\]
3. Numerical Example

Consider a Saloon shop with finite waiting space having two barbers who serve the customers with different service rates. Customers prefer to get service from the faster barber (server). The customers arrive in the saloon shop for their service, and if the service is not immediate they wait in the queue. If the waiting time of customers exceeds their expected wait, they may get reneged and leave the shop without getting service. If the shop employs some customer retention mechanism, there is a probability \( q = (1-p) \) that a reneging customer may remain in the queue for his service or he may leave the queue without getting service with complimentary probability \( p \). The customers are arrived by a fuzzy arrival rate \( \lambda \) and served by fuzzy service rates \( \mu_1 \) and \( \mu_2 \) with time distribution parameter \( \xi \).

Here we take the probabilities of the reneged Customers \( p \), retained Customers \( q \) as \( p = 0, 0.1, 0.2, \ldots, 0.9, 1 \), \( q = 1, 0.9, 0.8, \ldots, 0.1, 0 \) and \( N = 10 \). At this juncture, we are calculating the Expected Number of Customers in the System, the Expected Number of Customers Served, the Average Reneging Rate and Average Retention Rate.

Consider the arrival rate \( \lambda = [11, 12, 13, 14] \), the service rates \( \mu_1 = [3.5, 4, 4.5, 5] \) and \( \mu_2 = [3.3, 4, 4.5, 5] \), time distribution parameter \( \xi = [0.3, 0.4, 0.6, 0.6] \), fast server with fuzzy probability \( \pi_1 = [0.65, 0.7, 0.75, 0.8] \) and slow server with fuzzy probability \( \pi_2 = [0.2, 0.25, 0.3, 0.35] \), such that \( \pi_1 + \pi_2 = 1 \) per hour respectively.

Now the membership function of the trapezoidal fuzzy number \( [11, 12, 13, 14] \) is

\[
\mu_x(x) = \begin{cases} 
\frac{(x-11)}{12-11}, & 11 \leq x \leq 12 \\
1, & 12 \leq x \leq 13 \\
\frac{(x-14)}{13-14}, & 13 \leq x \leq 14 \\
0, & \text{otherwise}
\end{cases}
\]

Similarly we can proceed for all remaining trapezoidal fuzzy rates in this same way.

Now we apply the Trisectional Fuzzy Trapezoidal ranking Method to the trapezoidal fuzzy numbers.

\[
R(\lambda) = 12.5, R(\mu_1) = 4.25, R(\mu_2) = 3.75, R(\xi) = 0.45, R(\pi_1) = 0.725, R(\pi_2) = 0.275.
\]

By our calculations we tabulate the following numerical results for different measures of performance. Here we calculate the variation in performance measures with respect to \( q \).

The above result states that as we increase the probability of retaining a reneging customer there is a steady increase in the average number of customers in the system, average number of customers served and average retention rate. Here, the average reneging rates decrease subsequently on increasing \( q \). Thus, one can study the effect of different probabilities of retaining reneging customers on the different performance measures. When \( q = 0 \), there is no customer retention and therefore \( R_R = 0 \). Also, when \( q = 1 \), \( R_R = 0 \), that is, all reneging customers are retained.

| S. No | \( Q \) | \( N_q \) | \( E_{(CS)} \) | \( R_R \) | \( R_S \) |
|-------|-------|-------|-------|-------|-------|
| 1     | 0     | 1.389 | 0.171 | 0.237 | 0     |
| 2     | 0.1   | 1.396 | 0.171 | 0.216 | 0.062 |
| 3     | 0.2   | 1.401 | 0.171 | 0.195 | 0.092 |
| 4     | 0.3   | 1.407 | 0.170 | 0.175 | 0.073 |
| 5     | 0.4   | 1.415 | 0.170 | 0.153 | 0.098 |
| 6     | 0.5   | 1.422 | 0.170 | 0.131 | 0.157 |
| 7     | 0.6   | 1.430 | 0.169 | 0.110 | 0.177 |
| 8     | 0.7   | 1.439 | 0.169 | 0.089 | 0.202 |
| 9     | 0.8   | 1.448 | 0.169 | 0.066 | 0.235 |
| 10    | 0.9   | 1.457 | 0.168 | 0.037 | 0.269 |
| 11    | 1.0   | 1.468 | 0.168 | 0.001 | 0.293 |

4. Conclusion

In this paper, we have examined the new manner for finding the performance measures for Two Heterogeneous Servers Queueing Models Under Trisectional Fuzzy Trapezoidal Approach. We may use this technique for various Fuzzy Queues instead of using the existing techniques. This technique not only gives the crisp values but also gives more accuracy than the other values. This manner will be useful and helpful to all the researchers and inventors in the succeeding days.

References

[1] Allaviranloo, T. Jahantigh, M. A and Hajighasemi, S. A New Distance Measure and Ranking Method for Generalized Trapezoidal Fuzzy Numbers, Mathematical Problems in Engineering, 2013(2013), 6 pages.
[2] Ancker Jr., C.J., and Gafarian, A.V., Some Queueing Problems with Balking and Reneging I, Operations Research, 11(1963), 88–100.
[3] Ancker, Jr., C.J., and Gafarian, A.V., Some Queueing Problems with Balking and Reneging II, Operations Research, 11(1963), 928–937.
[4] Azman, F.N & Abdullah, L, Ranking Fuzzy Numbers by Centroid Method, Malaysian Journal of Fundamental and Applied Sciences, 8(3)(2012), 117–121.
[5] Bose, S. An Introduction to Queueing Systems, Klavar Academic/Plenum Publishers, New York, 2008.
[6] Chu, T.C. & Tsao, C.T., Ranking Fuzzy Numbers with an Area Between the Centroid Point and Original Point, Computers and Mathematics with Applications, 43(1–2)(2002), 111–117.
[7] Cooper, R. Introduction to Queueing Theory, 3rd Edition, CEE Press, Washington, 1990.
[8] Dat, L.Q., Yu, V.F. & Chou, S.Y., An Improved Ranking Method for Fuzzy Numbers Based on the Centroid Index, International Journal of Fuzzy Systems, 14(3)(2012), 413–419.
[9] Dinesh C.S. Bisht, Pankaj Kumar Srivastava, Trisectional Fuzzy Trapezoidal Approach to Optimize Interval Data
Performance measures of two heterogeneous servers queueing models under trisectional fuzzy trapezoidal approach

[10] Haight, F.A., Queueing with Reneging, *Metrika*, 2(1959), 186–197.

[11] Janos, S, *Basic Queueing Theory*, Globe Edit Publishers, Omniscriptum GMBH, Germany, 2016.

[12] Klir, G.J. and Yuvan, B., *Fuzzy Sets and Fuzzy Logic Theory and Applications*, Prentice Hall of India, 2005.

[13] Kumar, R. and Sharma, S, *Transient and Steady State Behavior of a Two-Heterogeneous servers Queueing System with Balking and Retention of Reneging Customers*, Performance prediction and analytics of fuzzy, reliability and queueing models, 2018, August, 251–264.

[14] Kumar, R. and Sharma, S., Transient solution of a Two-Heterogeneous servers Queueing System with Retention of Reneging Customers, *Bulletin of Malaysian Mathematical Sciences Society*, 42(1)(2019), 223–240.

[15] Parandin, N. and Araghi, M.A.F., Ranking of Fuzzy Numbers by Distance Method, *Journal of Applied Mathematics*, 15(19)(2008), 47–55.

[16] Ramesh, R. and Kumaraghuru, S., Cost Measures of Fuzzy Batch Arrival Queuing Model By Ranking Function Method, *International Journal of Scientific Research*, 4(1)(2015), 234–238.

[17] Ramesh, R. and Kumaraghuru, S., Analysis of Performance in Four Non-Preemptive Priority Fuzzy Queues by Centroid of Centroids Ranking Method, *International Journal of Computer Techniques*, 4(1)(2017), 12–20.

[18] Rao, P.P.B. and Shankar, N.R., Ranking Fuzzy Numbers with a Distance Method using Circumcenter of Centroids and an Index of Modality, *Advances in Fuzzy Systems*, 2011(2011), ID 178308.

[19] Rao, P.P.B. and Shankar, N.R., Ranking Generalized Fuzzy Numbers using Area, Mode, Spreads and Weights, *International Journal of Applied Science and Engineering*, 10(1)(2012), 41–57.

[20] Shankar, N.R., Sarathi, B.P. & Babu, S.S., Fuzzy Critical Path Method Based on a New Approach of Ranking Fuzzy Numbers using Centroid of Centroids, *International Journal of Fuzzy Systems & Applications*, 3(2)(2013), 16–31.

[21] Wang, Y.J. and Lee, H.S., The Revised Method of Ranking Fuzzy Numbers with an Area Between the Centroid and Original Points, *Computers and Mathematics with Applications*, 55(9), 2033–2042.

[22] Westman, L., and Wang, Z., Ranking Fuzzy Numbers by Their Left and Right Wingspans, *Joint IFSA world Congress and NAFIPS annual meeting*, Edmonton, (2013), 1039–1044.

[23] Zadeh, L.A., Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems*, 1(1978), 3–28.

**********

ISSN(O):2321 – 5666

Malaya Journal of Matematik