A comparison between Caputo and Caputo-Fabrizio fractional derivatives for modelling Lotka-Volterra differential equations

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Abstract

In this paper, we apply the concept of the fractional calculus to study three-dimensional Lotka-Volterra differential equations. Our goal is to compare the results of this system with respect to Caputo and Caputo-Fabrizio fractional derivatives. According to the existence of non-singular kernel in the definition of Caputo-Fabrizio operator, we analyze the stability of the system and try to improve a numerical method based on a corrected Adams-Bashforth method. Numerical results show that the behaviors of the Lotka-Volterra system depend on the fractional derivative order as well as the differential operators.

Keywords: Caputo-Fabrizio fractional derivative, Lotka-Volterra differential equations, Adam-Bashforth method, Stability.

1. Introduction

Recent years have seen an explosion in the use of fractional calculus in many fields of science and engineering. Nowadays, we can find a lot of content about the application of fractional calculus with a little search. In fact, having more degrees of freedom for differentiation is the excellent feature of fractional calculus in modeling of various phenomena \cite{1, 2}. Fractional derivatives are an efficient way for description of various processes, such as non-Gaussian \cite{3} and non-Markovian process \cite{4, 5}. In these processes, we can see the superiority of fractional derivatives in comparison with integer-order models.

There are different ways to define fractional derivatives. The Grunwald-Letnikov, Riemann-Liouville and Caputo definitions are commonly used to describe the several phenomena in science and engineering. In this paper, we focus on a new fractional derivative which is based on

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the Caputo definition. Caputo and Fabrizio, in their 2015 paper, proposed a new definition using Caputo derivative. In the new definition they replaced the singular kernel in the Caputo derivative with an exponential function. This new approach considered as the Caputo-Fabrizio (CF) fractional derivative. It has two representations for the temporal and spatial variables and it has used to modeling the behavior of diffusion-convection equation, advection-diffusion equation, fractional Nagumo equation, control the wave on a shallow water [6, 7]. The new operator has also successfully applied in cancer treatment, HIV/AIDS infection and tumor-obesity model [8, 9, 10].

The fractional derivatives have been also applied for the Lotka-Volterra systems which sometimes called predator-prey or parasite-host equations. Such a system plays a significant role in mathematical biology [11], and in financial systems, for example, biunivoc capital transfer from mother bank to subsiding bank and from subsiding bank to individuals or companies [12]. At first, these models introduced independently by Alfred J. Lotka and Vito Volterra as a simplified model of two species predator-prey population dynamics [13], in which the integer-order differential was presented. However, the classical differentiation is not always suitable in this case due to nonlocality of the interaction. In 2007, fractional-order Lotka-Volterra system introduced and described by Ahmed et al. [14]. In recent years, different types of this system have been studied by many researcher. For instance, in [15], authors studied a two-predator, one-prey generalization of the Lotka-Volterra system. A dynamical analysis of a prey-predator fractional order model using Caputo fractional derivative has been also investigated in [16].

Motivated by the above discussions, in this paper we consider three-dimensional Lotka-Volterra differential equations described by CF and Caputo fractional derivatives. Thus, the paper is organized as follows. In Sec. 2, some preliminaries of aforementioned operators are provided. In the subsequent, Sec. 3, we introduce a non-linear Lotka-Volterra differential equation, and the equilibrium points of the system are computed to assess stability of the equilibrium points. Then, by using a suitable numerical method described in Sec. 4, we solve the proposed system and survey the properties of Caputo and CF fractional derivatives and compare the results for both derivatives in Sec. 5.

2. Definitions and preliminaries

2.1. Fractional calculus

In this section, some basic definitions are presented for the fractional derivatives. Several definitions of fractional differential operator have been presented such as Grunwald-Letnikov,
Riemann-Liouville, Caputo. The Caputo derivative is widely used in mathematical analysis.

The Caputo definition of fractional derivative is defined as

$$C D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} f'(\tau) d\tau, \quad 0 < \alpha < 1, \ t > a. \quad (1)$$

The kernel \((t-\tau)^{-\alpha}\) in Eq. (1) cause a singularity at \(t = \tau\) that can be considered as a drawback in this definition. In 2015, Caputo and Fabrizio defined the following fractional derivative as

$$CF D_a^\alpha f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)f'(\tau) d\tau, \ t \geq a, \quad (2)$$

where \(f\) is a continuous and differentiable function on \(C^1[a,b]\) and \(M(\alpha)\) is a normalization function such that \(M(0) = M(1) = 1\).

For more details on the above-mentioned fractional operators, the readers are referred to [19, 18].

2.2. Stability of the fractional-order system

Consider the linear fractional-order autonomous system as follows:

$$D_a^\alpha x(t) = Ax(t), \quad (3)$$

where \(x(t) \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}, 0 < \alpha < 1\) and \(D_a^\alpha\) is one of Caputo or CF fractional derivatives.

**Definition 2.1.** The autonomous system, with \(x(t_0) = x_0\) is asymptotically stable if and only if

$$\lim_{t \to +\infty} \|x(t)\| = 0.$$ 

**Theorem 2.1.** The linear autonomous system (3) with Caputo fractional derivative for \(0 < \alpha < 1\) is asymptotically stable if and only if \(|\arg(\text{spec}(A))| > \frac{\alpha \pi}{2}\), spec\((A)\) is the spectrum (set of all eigenvalues) of \(A\) [20].

**Theorem 2.2.** The system (3) with CF derivative and \(0 < \alpha < 1\) is asymptotically stable if and only if the eigenvalues of matrix \(A\) satisfy \(\cos(\lambda(A)) < \|\lambda(A)\|(1-\alpha)\).

**Corollary 2.1.** Consider the eigenvalues of matrix \(A\) in the form \(\lambda(A) = a + ib\), the system (3) with CF derivative is asymptotically stable if and only if

$$a - (1-\alpha)(a^2 + b^2) < 0.$$ 

**Theorem 2.3.** The system (3) with CF derivative is asymptotically stable if eigenvalues \(\lambda(A)\) of the matrix \(A\) satisfy one of the following conditions
1) $\|\lambda(A)\| \geq \frac{1}{1-\alpha}, \lambda(A) \neq \frac{1}{1-\alpha}$.

2) $\text{Re}(\lambda(A)) > \frac{1}{1-\alpha}$.

3) $\text{Re}(\lambda(A)) < 0$.

4) $|\text{Im}(\lambda(A))| > \frac{1}{2(1-\alpha)}$.

Proof. The proof is straightforward with [21], and using Theorem 2.2 and Corollary 2.1.

3. Main results

3.1. Fractional prey-predator model

In this study, we consider 3-species prey-predator model as follows

$$\begin{align*}
D^\alpha_a x(t) &= x(t)(a_1 - a_2 x(t) - y(t) - z(t)) \\
D^\alpha_a y(t) &= y(t)((1 - a_3) + a_4 x(t)) \\
D^\alpha_a z(t) &= z(t)((1 - a_5) + a_6 x(t) + a_7 y(t))
\end{align*}$$

(4)

where $0 < \alpha \leq 1$ and $a_i > 0, i = 0, 1, \ldots, 7$ and $D^\alpha_a$ is one of the differential operators Caputo or CF with initial conditions

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0,$$

(5)

where $x_0, y_0, z_0 \in \mathbb{R}^+$. In this model $x(t) \geq 0$ represents the population of the prey, $y(t) \geq 0$ and $z(t) \geq 0$ represent the population of predators at the time $t$.

Now we consider system (4) in a compact form as follows:

$$\begin{align*}
D^\alpha_a u(t) &= F(u(t)) \quad 0 < t < \infty \\
u(0) &= u_0,
\end{align*}$$

(6)

where $u(t) = (x(t), y(t), z(t))^T \in \mathcal{L}[0,t']$, where $\mathcal{L}[0,t']$ be the set of all continuous vector $u(t)$ defined on the interval $[0,t']$ ($t' > 0$) and $F$ is a real-valued continuous vector function. Then the system (4) can be written in the form

$$D^\alpha_a u(t) = Au(t) + x(t)Bu(t) + y(t)Cu(t) + z(t)Du(t)$$

where

$$A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 1 - a_3 & 0 \\ 0 & 0 & 1 - a_5 \end{bmatrix}, \quad B = \begin{bmatrix} -a_2 & 0 & 0 \\ 0 & a_4 & 0 \\ 0 & 0 & a_6 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$
Theorem 3.1. For \( u(t) \in \mathscr{L}[0, t'] \), system (6) has a unique solution.

Proof. Let \( F(u(t)) = Au(t) + u_1(t)Bu(t) + u_2(t)Cu(t) + u_3(t)Du(t) \) and \( F(v(t)) = Av(t) + v_1(t)Bv(t) + v_2(t)Cv(t) + v_3(t)Dv(t) \) where \( F(u(t)), F(v(t)) \in \mathscr{L}[0, t'] \). Since \( u(t) = (u_1, u_2, u_3) \), \( v(t) = (v_1, v_2, v_3) \in \mathscr{L}[0, t'] \) such that \( u(t) \neq v(t) \). The following inequality holds

\[
\|F(u(t)) - F(v(t))\| = \left\|Au(t) + u_1(t)Bu(t) + u_2(t)Cu(t) + u_3(t)Du(t) - (Av(t) + v_1(t)Bv(t) + v_2(t)Cv(t) + v_3(t)Dv(t))\right\|
\leq \|A(u(t) - v(t))\| + \|u_1(t)(B(u(t) - v(t))\| + ||(u_1(t) - v_1(t))Bv(t)|| + \|u_2(t)C(u(t) - v(t))\|
+ \|u_3(t)D(u(t) - v(t))\| + \|(u_3(t) - v_3(t))Dv(t)||
\leq \left[\|A\| + \|B\| \|u_1(t)\| + \|v(t)\|\| + \|C\| \|u_2(t)\| + \|v(t)\|\| + \|D\| \|u_3(t)\| + \|v(t)\|\right] \times \|u(t) - v(t)\|
\]
then we have

\[
\|F(u(t)) - F(v(t))\| \leq L\|u(t) - v(t)\|
\]

where

\[
L = \|A\| + \|B\| + \|C\| + \|D\|)(M_1 + M_2) > 0,
\]

and \( M_1 \) and \( M_2 \) are positive constant and satisfy \( \|u\| \leq M_1, \|v\| \leq M_2 \) as a result of \( u, v \in \mathscr{L}[0, t'] \). It means that \( F(X(t)) \) is continuous and satisfying Lipschitz condition, then the initial value problem (6) has a unique solution. \( \square \)

3.2. Stability of Lotka-Volterra system

In this subsection, we discuss the stability of non-linear Lotka-Volterra differential equation (4) described by the CF derivative. In the case of non-linear systems, we study the local stability of equilibrium points and the following theorems are presented to investigate the local stability of equilibrium points. In order to determine the equilibrium points of system (4), let us consider

\[
D^\alpha_{a^+} x(t) = 0, \quad D^\alpha_{a^+} y(t) = 0, \quad D^\alpha_{a^+} z(t) = 0.
\]
The equilibrium points of system (4) are obtained and denoted as

\[
\begin{align*}
\epsilon_0 &= (0, 0, 0), \\
\epsilon_1 &= \left(\frac{a_1}{a_2}, 0, 0\right), \\
\epsilon_2 &= \left(\frac{a_5 - 1}{a_6}, 0, \frac{a_1a_6 - a_2(a_5 - 1)}{a_6}\right), \\
\epsilon_3 &= \left(\frac{a_3 - 1}{a_4}, \frac{a_1a_4 - a_2(a_3 - 1)}{a_4}, 0\right), \\
\epsilon_4 &= \left(\frac{a_3 - 1}{a_4}, \frac{a_4(a_5 - 1) - a_6(a_3 - 1)}{a_7a_4}, \frac{a_4(1 + a_1a_7 - a_5) + (a_6 - a_2a_7)(a_3 - 1)}{a_7a_4}\right).
\end{align*}
\]

To adjust the conditions for the actual situations, the equilibrium points must be nonnegative. In this regard, it is obvious that \(\epsilon_0\) and \(\epsilon_1\) always exist, and \(\epsilon_2\) exists when \(a_3 \geq 1\) and \(a_1a_4 \geq a_2(a_3 - 1)\), and it happens for \(\epsilon_3\) when \(a_5 \geq 1\) and \(a_1a_6 \geq a_2(a_5 - 1)\). Finally, the conditions \(a_3 \geq 1, a_4(a_5 - 1) \geq a_6(a_3 - 1)\) and \(a_4 \geq \frac{(a_2a_7 - a_6)(a_3 - 1)}{(1 + a_1a_7 - a_5)} \) (or if \(1 + a_1a_7 - a_5 < 0\) then \(a_4 \leq \frac{(a_6 - a_2a_7)(a_3 - 1)}{(1 + a_1a_7 - a_5)}\), else if \((1 + a_1a_7 - a_5) = 0\) then \(a_6 > a_2a_7\) are necessary for the existence of \(\epsilon_4\).

**Theorem 3.2.** Let \(\epsilon^*\) be an equilibrium point of the nonlinear system (4), with CF derivative, then equilibrium point \(\epsilon^*\) is asymptotically stable if eigenvalues of Jacobian matrix \(\lambda(J(\epsilon^*))\) satisfy one of the following conditions

1) \(\|\lambda(J(\epsilon^*))\| \geq \frac{1}{1 - \alpha}, \lambda(J(\epsilon^*)) \neq \frac{1}{1 - \alpha}\),

2) \(\text{Re}(\lambda(J(\epsilon^*))) > \frac{1}{1 - \alpha}\),

3) \(\text{Re}(\lambda(J(\epsilon^*))) < 0\),

4) \(\text{Im}(\lambda(J(\epsilon^*))) > \frac{1}{2(1 - \alpha)}\).

**Proof.** The proof follows from Theorem 2.3 and [22]. \(\square\)

To study the local stability of the equilibrium points such as \((x^*, y^*, z^*)\) for the system (4) we provide the Jacobian matrix \(J(x^*, y^*, z^*)\) as follows

\[
J(x^*, y^*, z^*) = \begin{bmatrix}
    a_1 - 2a_2x^* - y^* - z^* & -x^* & -x^* \\
    a_4y^* & 1 - a_3 + a_4x^* & 0 \\
    a_6z^* & a_7z^* & a_6x^* - a_5 + a_7y^* + 1
\end{bmatrix}.
\]
3.2.1. The first equilibrium

For \( \epsilon_0 \), the Jacobian can be expressed as

\[
J(\epsilon_0) = \begin{bmatrix}
a_1 & 0 & 0 \\
0 & 1 - a_3 & 0 \\
0 & 0 & 1 - a_5
\end{bmatrix},
\]

where eigenvalues are \( \lambda_1 = a_1 \), \( \lambda_2 = 1 - a_3 \), \( \lambda_3 = 1 - a_5 \). Since \( a_1 > 0 \), \( 1 - a_3 < 0 \), \( 1 - a_4 < 0 \), then \( \epsilon_0 \) is stable if \( a_1 > \frac{1}{1 - a} \).

3.2.2. The second equilibrium

For \( \epsilon_1 \), the Jacobian matrix is

\[
J(\epsilon_1) = \begin{bmatrix}
-a_1 & -\frac{a_1}{a_2} & -\frac{a_1}{a_2} \\
0 & 1 - a_3 + \frac{a_1a_4}{a_2} & 0 \\
0 & 0 & 1 - a_5 + \frac{a_1a_6}{a_2}
\end{bmatrix},
\]

where \( \lambda_1 = -a_1 < 0 \), \( \lambda_2 = 1 - a_3 + \frac{a_1a_4}{a_2} \), and \( \lambda_3 = 1 - a_5 + \frac{a_1a_6}{a_2} \). Thus, \( \epsilon_1 \) is asymptotically stable when

\[
a_1a_4 < a_2a_3 - a_2,
\]

\[
a_1a_6 < a_2a_5 - a_2,
\]

or

\[
a_2(1 - a_5)(1 - a) > a_2 - a_1a_4(1 - a),
\]

\[
a_2(1 - a_3)(1 - a) > a_2 - a_1a_6(1 - a).
\]

3.2.3. The third equilibrium

For \( \epsilon_2 \) the Jacobian matrix is

\[
J(\epsilon_2) = \begin{bmatrix}
\frac{a_2}{a_6}(1 - a_5) & 1 - a_5 & 1 - a_5 \\
0 & 1 - a_3 - \frac{a_4}{a_6}(1 - a_5) & 0 \\
\frac{a_1a_6}{a_2} + a_2(1 - a_5) & \frac{a_2}{a_6}(a_1a_6 + a_2(1 - a_5)) & 0
\end{bmatrix},
\]

we use the below notation

\[
J(\epsilon_2) = \begin{bmatrix}
A & B & C \\
0 & D & 0 \\
E & F & 0
\end{bmatrix}.
\]
The characteristic equation is as follows

\[ (\lambda - D)(\lambda^2 - A\lambda - CE) = 0, \]

by the condition \( \frac{a_5 - 1}{a_6} < \frac{a_1}{a_2} < \frac{a_3 - 1}{a_4} \) we get

\[ A < 0, C < 0, E > 0, D < 0, \]

therefore

\[ \lambda_1 = D < 0, \lambda_2 + \lambda_3 = A < 0, \lambda_2\lambda_3 = -CE > 0. \]

The eigenvalues are

\[ \lambda_1 = \left[ 1 - a_3 - \frac{a_4}{a_6}(1 - a_5) \right], \]

and

\[ \lambda_{2,3} = \frac{a_2(1 - a_5) \pm \sqrt{a_2^2(1 - a_5)^2 + 4a_6(1 - a_5)(a_1a_6 + a_2(1 - a_5))}}{2a_6}. \]

In this case, we can conclude that \( \varepsilon_2 \) is locally asymptotically stable. However, when the condition \( \frac{a_5 - 1}{a_6} < \frac{a_1}{a_2} < \frac{a_3 - 1}{a_4} \) was not available, \( \varepsilon_2 \) could be locally asymptotically stable when \( \lambda_1, \lambda_2, \lambda_3 > \frac{1}{1 - \alpha} \), which leads

\[ \left[ 1 - a_3 - \frac{a_4}{a_6}(1 - a_5) \right] (1 - \alpha) > 1, \]

and

\[ a_2(1 - a_4) \pm \sqrt{a_2^2(1 - a_4)^2 + 4a_6(1 - a_5)(a_1a_6 + a_2(1 - a_5))} (1 - \alpha) > 2a_6. \]

3.2.4. The forth equilibrium

Jacobian of \( \varepsilon_3 \) is

\[ J(\varepsilon_3) = \begin{bmatrix} \frac{a_2}{a_4}(1 - a_3) & 1 - a_3 & 1 - a_3 \\ a_1a_4 + a_2(1 - a_3) & 0 & 0 \\ 0 & 0 & w \end{bmatrix}, \]

where \( w = 1 - a_5 - \frac{a_6}{a_4}(1 - a_3) + \frac{a_7}{a_4}[a_1a_4 + a_2(1 - a_3)] \). Same as \( \varepsilon_2 \), we can provide the stability condition as

\[ \frac{a_3 - 1}{a_4} < \frac{a_1}{a_2} < \frac{a_5 - 1}{a_6}, \]
where \( \lambda_1 = 1 - a_4 - \frac{a_6}{a_4}(1 - a_3) + \frac{a_7}{a_4}[a_1a_4 + a_2(1 - a_3)] \) and

\[
\lambda_{2,3} = \frac{a_2(1 - a_3) \pm \sqrt{a_2^2(1 - a_3)^2 + 4a_4(1 - a_3)[a_1a_4 + a_2(1 - a_3)]}}{2a_4}.
\]

When the condition \( \frac{a_3 - 1}{a_4} < \frac{a_1}{a_2} < \frac{a_5 - 1}{a_6} \) is not available, \( \varepsilon_3 \) is locally asymptotically stable when \( \lambda_1(1 - \alpha) > 1, \lambda_2(1 - \alpha) > 1, \lambda_3(1 - \alpha) > 1. \)

3.2.5. The fifth equilibrium

For \( \varepsilon_4 \), Jacobian matrix is as follows

\[
J(\varepsilon_4) = \begin{pmatrix}
A & B & B \\
C & 0 & 0 \\
D & E & 0
\end{pmatrix},
\]

where

\[
A = \frac{a_2}{a_4}(1 - a_3), \\
B = \frac{1 - a_3}{a_4}, \\
C = -\frac{a_4 - a_6 + a_3a_6 - a_4a_5}{a_7}, \\
D = \frac{a_6(a_4 - a_6 + a_2a_7 + a_3a_6 - a_4a_5 + a_1a_4a_7 - a_2a_3a_7)}{a_4a_7}, \\
E = \frac{a_4 - a_6 + a_2a_7 + a_3a_6 - a_4a_5 + a_1a_4a_7 - a_2a_3a_7}{a_4}.
\]

To compute eigenvalues of the above matrix, we consider the characteristic polynomial, \( L(\lambda) = \lambda^3 + a\lambda^2 + b\lambda + c \), where \( a = -A, b = -B(C + D), c = -BCE \). It is obvious \( a, c > 0 \). If \( a_1a_2 > a_3 \) then Routh-Hurwitz criterion shows that the all roots of are negative. The equation \( a_1a_2 - a_3 = B[A(C + D) + CE] \) is positive if

\[
a_6 > \frac{a_2a_4(a_3 - 1)[w + a_2(a_3 - 1)]}{w(a_2 + a_4) + a_2a_4(a_3 - 1)},
\]

where

\[
w = a_4(1 + a_1a_7 - a_5) + (a_6 - a_2a_7)(a_3 - 1).
\]
To investigate the stability of the system in the sense of CF derivative we consider the following parameter for $L(\lambda)$ as

\[
p = b - \frac{a}{3},
\]

\[
q = \frac{2a^3}{27} - \frac{ab}{3} + c,
\]

\[
\Delta = \frac{q^2}{4} + \frac{p^3}{27}.
\]

If $\Delta > 0$, then we have only one real solution

\[
\lambda = \left(-\frac{q}{2} + \sqrt{\Delta}\right)^\frac{1}{3} + \left(-\frac{q}{2} - \sqrt{\Delta}\right)^\frac{1}{3} - \frac{q}{3}.
\]

(7)

If $\Delta = 0$, there are repeated roots

\[
\lambda_1 = -2\left(\frac{q}{2}\right)^\frac{1}{3} - \frac{q}{3}, \quad \lambda_2 = \lambda_3 = \left(\frac{q}{2}\right)^\frac{1}{3} - \frac{q}{3}.
\]

(8)

If $\Delta > 0$ then roots are same as below

\[
\lambda_1 = \frac{2\sqrt{p}}{\sqrt{3}} \sin\left(\frac{1}{3} \arcsin\left(\frac{3\sqrt{3}q}{2(\sqrt{p})^3}\right)\right) - \frac{a}{3},
\]

(9)

\[
\lambda_2 = -\frac{2\sqrt{p}}{\sqrt{3}} \sin\left(\frac{1}{3} \arcsin\left(\frac{3\sqrt{3}q}{2(\sqrt{p})^3} + \frac{\pi}{3}\right)\right) - \frac{a}{3},
\]

(10)

\[
\lambda_3 = \frac{2\sqrt{-p}}{\sqrt{3}} \cos\left(\frac{1}{3} \arcsin\left(\frac{3\sqrt{3}q}{2(\sqrt{-p})^3} + \frac{\pi}{6}\right)\right) - \frac{a}{3}.
\]

(11)

Consequently, $\varepsilon_4$ is locally asymptotically stable if in any case all of the eigenvalues satisfying these conditions

\[
\lambda_1, \lambda_2, \lambda_3 > \frac{1}{(1 - \alpha)}.
\]

We end this section by summarizing the stability conditions of all the equilibrium points for Caputo and CF fractional derivatives in the Table 1.

4. Numerical algorithm

The Predictor-Corrector method has been considered as a powerful numerical aproach can provide an accurate numerical solution of fractional differential equations (FDEs). For instance, in [23], a numerical method based on Adams-Bashforth methods is proposed for solving FDEs with Caputo derivative. In [24], authors have investigated a fractional Adams-Bashforth method for solving FDEs with CF fractional operator but their arguments are flawed. For this aim, in this part of paper we correct this method to solve the prey-predator system of Caputo and CF
Table 1: Stability conditions for Caputo and CF fractional derivatives.

| Equilibrium point | Caputo derivative | CF derivative |
|-------------------|-------------------|---------------|
| $\epsilon_0$     | Always saddle     | $a_1 > \frac{1}{1-\alpha}$ |
| $\epsilon_1$     | $a_1a_2 < a_2a_3 - a_2$ and $a_1a_2 < a_2a_3 - a_1$ | $a_1a_2 < a_2a_3 - a_2$ and $a_1a_2 < a_2a_3 - a_1$ or $\frac{a_1a_4 - a_3a_1}{a_4} > \frac{a_1}{1-\alpha}$ and $a_1a_6 - a_2a_5 > \frac{a_1}{1-\alpha}$ |
| $\epsilon_2$     | $\frac{a_5 - 1}{a_6} < \frac{a_3 - 1}{a_4} < \frac{a_5}{a_6}$ | $\frac{a_5 - 1}{a_6} < \frac{a_3 - 1}{a_4} < \frac{a_5}{a_6}$ or $\left\{ \begin{array}{l} 1 - a_3 - a_6 \frac{1 - a_3}{a_6} > \frac{1}{1 - \alpha} \\
\frac{a_2(1-a_3)^2}{a_2(1-a_3)^2 + 4a_6(1-a_5)(a_1a_6 + a_2(1-a_5))} > \frac{1}{1 - \alpha} \end{array} \right.$ |
| $\epsilon_3$     | $\frac{a_5 - 1}{a_4} < \frac{a_3 - 1}{a_6}$ or $1 - a_4 - a_6 \frac{1 - a_3}{a_4} + a_4 \frac{a_1a_4 + a_2(1 - a_3)}{a_4} > \frac{1}{1 - \alpha}$ and $\left\{ \begin{array}{l} \frac{a_2(1-a_3)^2}{a_2(1-a_3)^2 + 4a_6(1-a_5)(a_1a_6 + a_2(1-a_5))} > \frac{1}{1 - \alpha} \end{array} \right.$ |
| $\epsilon_4$     | $a_6 > a_2a_4(a_3 - 1)[w + a_2(a_3 - 1)]$ or $w(a_2 + a_4) + a_2a_4(a_3 - 1)$ |

operator and we compare the solutions using Caputo and CF derivative. Consider the following differential equation

$$CFD_0^\alpha f(x) = g(t, f(x)), \quad x \in [0, t'],$$

$$f^{(i)}(0) = f_0^i, \quad i = 0, 1, 2, \ldots, n - 1, \quad n = [\alpha],$$

which is equivalent to the following equation

$$f(x) = T_{n-1}(x) + \frac{1 - \alpha}{M(\alpha)(n - 2)!} \int_0^x (x - t)^{n-2} g(t, f(t))dt + \frac{\alpha}{M(\alpha)(n - 1)!} \int_0^x (x - t)^{n-1} g(t, f(t))dt$$

where $T_{n-1}(x)$ is the Taylor expansion of $f(x)$ centered at $x_0 = 0$ and $T_{n-1}(x) = \sum_{i=0}^{n-1} \frac{x^i}{i!} f_0^i$. 

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The corrector formula \( f_{k+1} \) can be written as follows

\[
f_{k+1} = T_{n-1}(x) + \frac{\alpha}{M(\alpha)(n-1)!} \left[ \sum_{i=0}^{k} b_{i,k+1}g(x_i, f_i) + b_{k+1,k+1}g(x_{k+1}, f_{k+1}^p) \right]
\]

\[
b_{i,k+1} = \frac{h^n}{n(n+1)} \begin{cases} 
  k^{n+1} - (k+1)^n(k-n), & i = 0 \\
  (k-i-2)^{n+1} - 2(k-i+1)^{n+1} + (k-i)^{n+1}, & 1 \leq i \leq k \\
  1, & i = k+1 
\end{cases}
\]

and by the fractional Adams-Bashforth-multon method [23], \( f_{k+1}^p \) is determined by

\[
f_{k+1}^p = T_{n-1}(x) + \frac{\alpha}{M(\alpha)(n-1)!} \sum_{i=0}^{k} d_{i,k+1}g(x_i, f_i)
\]

where

\[
d_{i,k+1} = \frac{h^n}{n} \left[ (k-i+1)^n - (k-i)^n \right]
\]

Now, consider the following fractional-order system involving CF derivative

\[
\begin{align*}
  \left\{ \begin{array}{l}
  CF D_{\alpha^+}^a x(t) = f_1(x, y, z), \\
  CF D_{\alpha^+}^a y(t) = f_2(x, y, z), \\
  CF D_{\alpha^+}^a z(t) = f_3(x, y, z).
  \end{array} \right.
\end{align*}
\]

We consider \( 0 \leq \alpha \leq 1 \) for simplicity and assume that \((x_0, y_0, z_0)\) is the initial point. Applying the above scheme, the system (15) can be discretized as follows

\[
x_{k+1} = x_0 + \frac{\alpha}{M(\alpha)(n-1)!} \left[ \sum_{i=0}^{k} b_{1,i,k+1}f_1(x_i, y_i, z_i) + b_{1,k+1,k+1}f_1(x_{k+1}, y_{k+1}, z_{k+1}) \right],
\]

\[
y_{k+1} = y_0 + \frac{\alpha}{M(\alpha)(n-1)!} \left[ \sum_{i=0}^{k} b_{2,i,k+1}f_2(x_i, y_i, z_i) + b_{2,k+1,k+1}f_2(x_{k+1}, y_{k+1}, z_{k+1}) \right],
\]

\[
z_{k+1} = z_0 + \frac{\alpha}{M(\alpha)(n-1)!} \left[ \sum_{i=0}^{k} b_{3,i,k+1}f_3(x_i, y_i, z_i) + b_{3,k+1,k+1}f_3(x_{k+1}, y_{k+1}, z_{k+1}) \right],
\]

where

\[
x_{k+1}^p = x_0 + \frac{\alpha}{M(\alpha)(n-1)!} \left[ \sum_{i=0}^{k} d_{1,i,k+1}f_1(x_i, y_i, z_i) \right],
\]

\[
y_{k+1}^p = y_0 + \frac{\alpha}{M(\alpha)(n-1)!} \left[ \sum_{i=0}^{k} d_{2,i,k+1}f_2(x_i, y_i, z_i) \right],
\]

\[
z_{k+1}^p = z_0 + \frac{\alpha}{M(\alpha)(n-1)!} \left[ \sum_{i=0}^{k} d_{3,i,k+1}f_3(x_i, y_i, z_i) \right],
\]

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and

\[ b_{j, i, k+1} = \frac{h^n}{n(n + 1)} \begin{cases} k^{n+1} - (k + 1)^n(k - n), & i = 0 \\ (k - i + 2)^{n+1} - 2(k - i + 1)^{n+1} + (k - i)^{n+1}, & 1 \leq i \leq 1 \\ 1, & i = k + 1 \end{cases} \]

\[ d_{j, i, k+1} = \frac{h^n}{n} [(k - i + 1)^n - (k - i)^n]. \]

In the following, we apply the suggested numerical technique for simulation the solutions of the system 4. It is worth mentioning that to obtain the numerical results in the sense of Caputo derivative we use the equivalent Adams-Bashforth-multon method described in [23].

5. Numerical implementation

In this part, we discuss the numerical results of the fractional order prey-predator model (4) with Caputo and CF derivatives, by using the numerical method described in Sec. 4. It is interesting to classify the numerical results according to the positions of the eigenvalues and discuss the behavior of the system in the sense of Caputo and CF derivative. To illustrate such a classification, we provide stability and unstability region for both operators in Fig. 1 and set four eigenvalues on the plane for different cases. The unstable domain of the system with Caputo derivative is an unbound region limited by two lines with angles \(-\frac{\alpha \pi}{2}\) and \(\frac{\alpha \pi}{2}\). On the other hand, the unstable domain of the system with CF derivative is a bounded closed circle centered at \((0, \frac{1}{2(1-\alpha)})\) with the radius \(\frac{1}{2(1-\alpha)}\). For both cases, it is clear that the stability of the system has an inverse relation with the order derivative; in fact, the less value of \(\alpha\), the more space for stabilitity, and vice versa. As it is shown in Fig 1, the eigenvalues can be located in four distinctive classes: \(\lambda_A\) is in an area where both systems are stable; the class of \(\lambda_B\) is where the system with Caputo derivatives is stable, but the system with CF derivatives is not stable; \(\lambda_C\) denotes a class of eigenvalues staying at where both systems are unstable; and finally, \(\lambda_D\) is where the system in the sense of CF derivative is stable but the system with Caputo derivatives is not stable.

We collect a summary of three examples in Table 2 to easily compare the behavior of the model with respect to the Caputo and CF fractional derivatives. In the following, we deliberate the elaboration of the examples.
Table 2: The summary of examples; C, CF, and $U(0)$ denote Caputo, Caputo-Fabrizio, and initial values, respectively, and the notation ✓ indicates the system is asymptotically stable, while ✗ implies unstability.

| Example 1 | $a_1 = 3$ | $a_2 = 0.5$ | $a_3 = 4$ | $a_4 = 3$ | $a_5 = 4$ | $a_6 = 9$ | $a_7 = 4$ |
|-----------|-----------|-------------|-----------|-----------|-----------|-----------|-----------|
| $U(0)$    | $x_0 = 0.5$ | $y_0 = 0.9$ | $z_0 = 0.1$ | $\varepsilon_0(0, 0, 0)$ | $\lambda_0(-3, -3.3)$ | ✗         |
| equilibrium |          |            |           | $\varepsilon_1(6, 0, 0)$ | $\lambda_1(-3, 15, 51)$ | ✓         |
| eigenvalues |          |            |           | $\varepsilon_2(0.33, 0, 2.83)$ | $\lambda_2(-0.083 - 2.914i, -0.083 + 2.914i, -2 + 0i)$ | ✓ ✓ ✓ ✓ |
|            |          |            |           | $\varepsilon_3(1, 2.5, 0)$ | $\lambda_3(-0.25 - 2.727i, -0.25 + 2.727i, 16 + 0i)$ | ✗ ✗ ✗ ✗ |
| Not Acceptable | |            |           | $\varepsilon_4(1, -1.5, 4)$ | $\lambda_4(-1.239 - 5.904i, -1.239 + 5.904i, 1.978 + 0i)$ | ✗ ✗ ✗ ✗ |

| Example 2 | $a_1 = 3$ | $a_2 = 0.5$ | $a_3 = 4$ | $a_4 = 3$ | $a_5 = 14$ | $a_6 = 9$ | $a_7 = 4$ |
|-----------|-----------|-------------|-----------|-----------|-----------|-----------|-----------|
| $U(0)$    | $x_0 = 2$ | $y_0 = 2$ | $z_0 = 3$ | $\varepsilon_0(0, 0, 0)$ | $\lambda_0(-13, -3.3)$ | ✗         |
| equilibrium |          |            |           | $\varepsilon_1(6, 0, 0)$ | $\lambda_1(-3, 15, 41)$ | ✓         |
| eigenvalues |          |            |           | $\varepsilon_2(1.44, 0, 2.28)$ | $\lambda_2(-0.361 - 5.429i, -0.361 + 5.429i, 1.333 + 0i)$ | ✗ ✗       |
|            |          |            |           | $\varepsilon_3(1, 2.5, 0)$ | $\lambda_3(-0.25 - 2.727i, -0.25 + 2.727i, 6 + 0i)$ | ✓         |
|            |          |            |           | $\varepsilon_4(1, 1, 1.5)$ | $\lambda_4(0.276 - 4.123i, 0.276 + 4.123i, -1.053 + 0i)$ | ✓ ✗       |

| Example 3 | $a_1 = 8$ | $a_2 = 0.5$ | $a_3 = 4$ | $a_4 = 1$ | $a_5 = 7$ | $a_6 = 9$ | $a_7 = 4$ |
|-----------|-----------|-------------|-----------|-----------|-----------|-----------|-----------|
| $U(0)$    | $x_0 = 0.5$ | $y_0 = 0.1$ | $z_0 = 5$ | $\varepsilon_0(0, 0, 0)$ | $\lambda_0(-6, -3.8)$ | ✗         |
| equilibrium |          |            |           | $\varepsilon_1(160, 0, 0)$ | $\lambda_1(-8, 157, 1434)$ | ✓         |
| eigenvalues |          |            |           | $\varepsilon_2(0.666, 0, 7.966)$ | $\lambda_2(-0.016 - 6.913i, -0.016 + 6.913i, -2.333 + 0i)$ | ✓ ✓       |
|            |          |            |           | $\varepsilon_3(3, 7.85, 0)$ | $\lambda_3(-0.075 - 4.852i, -0.075 + 4.852i, 52.4 + 0i)$ | ✓         |
|            |          |            |           | $\varepsilon_4(3, -5.25, 13.1)$ | $\lambda_4(-1.274 - 18.506i, -1.274 + 18.506i, 2.398 + 0i)$ | ✗         |

$\alpha = 0.98$ $\alpha \leq 0.66$ $\alpha = 0.6$ $\alpha = 0.4$
5.1. Example 1

As one can see in Table 2, the parameters of this example give five distinctive equilibrium points (and corresponding eigenvalues), while equilibrium $\epsilon_4$ is not acceptable since it has a negative value. Thus, we should expect negative-value solutions of the system when we do not impose any constraints on the components. There is a recommended paper [25] in order to avoid going toward such meaningless solutions and getting a certain solution.

Furthermore, this example is proposed to show that the stability of the equilibrium points depends on the value of the fractional order $\alpha$. As we expect from Fig. 1, the number of stable equilibrium points increases when we reduce the value of $\alpha$. In this case, when $\alpha$ is 0.98 for both operators, the system is asymptotically stable only at $\epsilon_2$ (see Table 2). Indeed, Fig. 2 (left) shows that the system gets steady at $\epsilon_2$, with different oscillations which are related to the definition of the operators. Nonetheless, the condition $\alpha \leq 0.66$ provides a larger area for the stability of the system so that three eigenvalues $\lambda_0$, $\lambda_1$, and $\lambda_3$ stay in the class of $\lambda_D$ (see Fig. 1 and Table 2). Hence, with appropriate initial values and differential orders, the system could converge to $\epsilon_3$, see Fig. 2(right).
5.2. Example 2

This example confirms the points mentioned in the previous example; by setting $\alpha = 0.6$, the equilibrium $\epsilon_4$ is the only stable equilibrium point in the sense of Caputo derivative, while the equilibrium points $\epsilon_0$, $\epsilon_1$ and $\epsilon_3$ are stable with respect to the CF derivative. But, regarding the intuitive depiction of stability in Fig. 1, it is interesting that we have here an eigenvalue, $\lambda_4$, in the class of $\lambda_B$ alongside the $\lambda_D$, where $\lambda_0$, $\lambda_1$, and $\lambda_3$ are established. As a result, Fig. 3 shows that the system can start from a point to converge asymptotically to the only equilibrium that is stable in the sense of Caputo, rather than CF derivative. Therefore, it could make a challenge for one who assumes a system having more stability region may lead more potential to achieve a steady state.

5.3. Example 3

This example can complete the discussion and make clear the substantial role of the initial values on the behavior of the very Lotka-Volterra model. Considering the information of Table 2 determining the location of four eigenvalues in the $\lambda_D$ class (Fig. 1), we expect that the system is mostly stable for CF derivative and noticeably unstable for Caputo derivative. Although Fig. 4 illustrates this expectation, it is not an absolute scenario when the evolution of the system is supposed to start from a point leading to equilibrium $\epsilon_3$. In fact, it depends on the domain of attraction that the initial values stay, and the corresponding eigenvalue which is in the class of $\lambda_B$ (see Fig. 5). Moreover, we suggest [25] for researchers who intend to know how to find the domain of attractions to specific equilibrium points.
Figure 3: (color online) The system with the parameters of the Example 2 and \((x_0, y_0, z_0) = (2, 2, 3)\) is asymptotically stable for Caputo (left) and unstable for CF (right).

Figure 4: (color online) The system with the parameters of the Example 3 and \((x_0, y_0, z_0) = (0.5, 0.1, 5)\) is asymptotically stable for both Caputo (left) and CF (right) at \(\epsilon_2\) with different oscillations.

Figure 5: (color online) The system with the parameters of the Example 3 and \((x_0, y_0, z_0) = (3, 8, 0)\) is asymptotically stable for CF (right) and unstable for Caputo (left).
6. Conclusion

In this paper, we study three-dimensional Lotka-Volterra differential equations with respect to the Caputo and CF fractional derivatives. Concerning the existence of non-singular kernel in the definition of CF operator, we investigated the stability of the system and tried to correct a numerical method based on Adams-Bashforth methods. This numerical scheme helped us to have acceptable and efficient results of the system for both Caputo and CF fractional derivatives. Numerical results showed that the behaviors of the Lotka-Volterra system depend on the fractional derivative order as well as the differential operators. In fact, various behaviors were exhibited by using different values of $\alpha$ and various derivative operators. Moreover, the CF fractional derivative provides quite different properties than the classical Caputo derivative which is advantageous in better understanding the complex behaviours of the real-world dynamical phenomena.

For future investigations, it would be interesting to know the behavior of Lotka-Volterra models not having commensurate fractional orders. In the real-world, there exist complex systems including memory with different powers which makes an anomalous behavior. Such a disorder distribution of memory across the system has an impact on the evolution of processes. Thanks to the fractional calculus, it can be simulated by considering a model with incommensurate fractional order derivatives. In this regard, finding the stability region of the studied model with incommensurate fractional orders in the sense of Caputo and CF derivatives as well as examining the domain of attractions are the future directions of this paper.

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