SOBOLEV SPACE THEORY FOR THE DIRICHLET PROBLEM
OF THE ELLIPTIC AND PARABOLIC EQUATIONS WITH THE
FRACTIONAL LAPLACIAN ON \(C^{1,1}\) OPEN SETS

JAE-HWAN CHOI, KYEONG-HUN KIM, AND JUNHEE RYU

ABSTRACT. We introduce a Sobolev space theory for the parabolic equation
\[ u_t = \Delta^{\alpha/2}u + f, \quad t > 0, \quad x \in D; \quad u(0, \cdot)|_D = u_0, \quad u|_{[0,T] \times \partial D} = 0 \]
as well as for the elliptic equation
\[ \Delta^{\alpha/2}u - \lambda u = f, \quad x \in D; \quad u|_D = 0. \]
Here \(\alpha \in (0, 2), \lambda \geq 0\) and \(D\) is a \(C^{1,1}\) open set. We prove uniqueness and existence of solutions in weighted Sobolev spaces, and obtain global Sobolev and Hölder estimates of solutions and their arbitrary order derivatives. We measure the Sobolev and Hölder regularities of solutions and their arbitrary derivatives using a system of weights consisting of appropriate powers of the distance to the boundary. The range of admissible powers of the distance to the boundary is sharp.

1. Introduction

We study the parabolic equation
\[
\begin{cases}
\partial_t u(t, x) = \Delta^{\alpha/2}u(t, x) + f(t, x), & (t, x) \in (0, T) \times D, \\
u(0, x) = u_0(x), & x \in D, \\
u(t, x) = 0, & (t, x) \in [0, T] \times D^c,
\end{cases}
\]
and the elliptic equation
\[
\begin{cases}
\Delta^{\alpha/2}u(x) - \lambda u(x) = f(x), & x \in D, \\
u(x) = 0, & x \in D^c,
\end{cases}
\]
where \(\alpha \in (0, 2)\) and \(D\) is either a half space or a bounded \(C^{1,1}\) open set. The fractional Laplacian \(\Delta^{\alpha/2}u\) is defined as
\[
\Delta^{\alpha/2}u(x) := c_d \lim_{\varepsilon \searrow 0} \int_{|y| > \varepsilon} \frac{u(x + y) - u(x)}{|y|^{d+\alpha}} dy,
\]
where \(c_d := \frac{2^{\alpha} \Gamma((d+\alpha)/2)}{\pi^{d/2} \Gamma(-\alpha/2)}\).

In recent few decades, there are growing interests in elliptic and parabolic equations having non-local operators. The need for such equations is natural since there are a considerable number of phenomena in various areas of science which involve jumps and therefore cannot be modeled with local operators. In the probabilistic point of view, equations (1.1) and (1.2) are related to a certain pure-jump process.
which is forced to assume undefined or killed state when it leaves the open set \( D \). The zero exterior condition describes that the influence of the jump process vanishes or is ignored when the process is outside of \( D \). See e.g. Section 2 for detail. In fact, the equations are ill-posed if only zero-boundary condition is assigned.

In this article, we develop a weighted Sobolev space theory for equations (1.1) and (1.2). The major novelty of our result is that it give sharp global estimates of arbitrary (real) order derivatives of solutions.

To give the reader a flavor of our results, we state an estimate for the solution of parabolic equation (1.1). For simplicity, let \( u_0 = 0 \). Denote \( \rho(x) := \text{dist}(x, \partial D) \), and assume

\[
-1 < \theta - d < p - 1. 
\]

Then, for any \( n = 0, 1, 2, \cdots \) (see Theorem 2.9), we have

\[
\sum_{k \leq n} \int_0^T \int_D \left( |\rho^{k-\alpha/2} D^k u|^p + |\rho^{k+\alpha/2} \Delta^{\alpha/2} D^k u|^p \right) \rho^\theta dx \, dt \\
\leq C \sum_{k \leq n} \int_0^T \int_D |\rho^{k+\alpha/2} D^k f|^p \rho^\theta dx \, dt. 
\]

(1.4)

In particular, if \( n = 0 \), then

\[
\int_0^T \int_D \left( |\rho^{-\alpha/2} u|^p + |\rho^{\alpha/2} \Delta^{\alpha/2} u|^p \right) \rho^\theta dx \, dt \\
\leq C \int_0^T \int_D |\rho^\alpha f|^p \rho^\theta dx \, dt. 
\]

(1.5)

We remark that condition (1.3) is sharp. Actually, even if \( f \in C^{\infty}_c((0,T) \times D) \), condition (1.3) is necessary to make the left-hand side of (1.5) finite (cf. Remark 2.6). Note that due to the presence of \( \rho^{\alpha/2} \) beside \( f \) in (1.4), the function \( f \) is allowed to blow up near the boundary of \( D \). Indeed, it can behave like \( \rho^{-\alpha/2} \) near \( \partial D \). Our results are general than (1.4) and include estimates of arbitrary real order derivatives of solutions. See Theorems 2.10 and 2.11 for our full Sobolev regularity results of the elliptic and parabolic equations.

We also prove a weighted Sobolev embedding theorem to obtain global space-time Hölder estimates of arbitrary derivatives of solutions (see Corollaries 2.13 and 2.14). One advantage of our results is that it gives Hölder estimates of solutions even when the free terms are quite rough. For instance, let \( D \) be bounded, \( u_0 = 0 \) and \( \rho^{\alpha/2} f \in L_p((0,T) \times D) \), then for the solution to parabolic equation (1.1) and for any \( 1/p < \nu \leq 1 \) satisfying \( \alpha(1 - \nu) - d/p =: \delta > 0 \), we have

\[
|\rho^{d/p+\alpha(\nu-1/2)} u|_{C^{\nu-1/p}(0,T);C(D))} \\
+|\rho^{d/p+\alpha(\nu-1/2)+\delta} u|_{C^{\nu-1/p}(0,T);C^\delta(D))} < \infty. 
\]

(1.6)

In particular, if \( \|\rho^{\alpha/2} f\|_{L_\infty((0,T) \times D)} < \infty \), then (1.6) yields

\[
\sup_{x \in D} |\rho^{\alpha/2-\varepsilon}(x) u(\cdot, x)|_{C^{1-\varepsilon'}([0,T])} < \infty, 
\]

(1.7)

and

\[
\sup_{x \in D} |\rho^{-\alpha/2+\varepsilon}(x) u(\cdot, x)|_{C^{\varepsilon'}([0,T])} + \sup_{t \in [0,T]} |\rho^{\alpha/2-\varepsilon'} u(t, \cdot)|_{C^{\alpha-\varepsilon'}(D)} < \infty 
\]

(1.8)

for any small \( \varepsilon, \varepsilon' > 0 \). See Remark 2.15 for detail. (1.7) and the second one in (1.8) give the maximal regularity with respect to the time and space variables.
respectively. The first one in (1.8) describes the decay rate near the boundary of $D$. See Remark 2.15 for the elliptic versions of (1.6), (1.7) and (1.8).

To position our results in the context of regularity theory, we give a description on related works below. Our focus lies in the results on domains. Accordingly, regarding the results on the whole space $\mathbb{R}^d$, we only refer e.g. to [3] [7] [19] [30] [40] for Hölder estimates and [18] [29] [31] [32] [44] [45] for $L_p$ estimates.

First, we describe Hölder estimates. As for elliptic equation (1.2), it was proved for non-local elliptic equations with singular kernels or general operators.

\[ f \in L_\infty(D) \implies u \in C^{\alpha/2}(\mathbb{R}^d), \quad \rho^{\alpha/2}u \in C^\alpha(D) \]

for some $s > 0$. Here, $\rho(x) := \text{dist}(x, \partial D)$. Higher order estimate

\[ |u|_{\beta,\alpha; D}^{(\alpha/2)} \leq C \left( |u|_{C^{\alpha/2}(\mathbb{R}^d)} + |f|_{\beta; D}^{(\alpha/2)} \right), \quad \beta > 0 \]

was also obtained in [47], where $| \cdot |_b^{(\alpha)}$ denotes the interior Hölder norm (see e.g. [24] or [33]). The result of [47] was generalized for elliptic equations with stable-like operators in [3] [35] [19]. We also refer to [41] [50] for the local result of the type

\[ f \in C^\beta(D) \implies u \in C^{\beta+\alpha}_{\text{loc}}(D), \quad \beta > 0 \]

proved for non-local elliptic equations with singular kernels or general operators. Also, see [13] [36] [48] for related works on non-linear elliptic equations. Now, we discuss the results on parabolic equation (1.1). In [22], it was proved that if $u_0 \in L_2(D)$ and $f \in L_\infty((0, T) \times D)$, then

\[ u \in C^{1-\varepsilon,\alpha/2}_{t,x}((t_0, T) \times D), \quad \rho^{\alpha/2}u \in C^{\frac{\alpha}{2}-\varepsilon}_{t,x}((t_0, T) \times D) \]  

for any $\varepsilon > 0$ and $t_0 \in (0, T)$. Note that this result is local with respect to the time variable. For a global estimate, we refer to [53], which in particular proved

\[ \sup_{t \leq T} \left( |u|_{\alpha+\gamma; D}^{(-\theta)} + |\Delta^{\alpha/2}u|_{\gamma; D}^{(-\theta)} \right) \leq C \left( |u_0|_{\alpha+\gamma; D}^{(-\theta)} + \sup_{t \leq T} |f|_{\gamma; D}^{(-\theta)} \right) \]

for any $\theta \in (0, \alpha/2)$ and $\gamma \in (0, 1)$. This estimate does not give Hölder regularity with respect to the space variable. Note that compared to (1.9) and (1.10), our result (1.10) gives global Hölder regularity with respect to both time and space variables.

Next, we describe results in $L_p$ spaces. The global summability and interior regularity results was studied e.g. in [1] [11] [8] [9] [17] [42] [44]. For instance, for elliptic equation (1.2), the inequality

\[ \|u\|_{L^p_{\text{loc}}(D)} + \|\Delta^{\alpha/4}u\|_{L^p_{\text{loc}}(D)} \leq C\|f\|_{L_p(D)}, \quad (1 < p < 2d/(d + \alpha)) \]  

was proved in [42], and the inequality

\[ \|\rho^{\beta-\alpha/2}\Delta^{\beta/2}u\|_{L_p(D)} \leq C\|f\|_{L_{m}(D)}, \]

was proved in [1] provided that

\[ \alpha/2 < \beta < \min\{1, \alpha\}, \quad 1 \leq m < d/(\alpha - \beta), \quad 1 \leq p < md/(d - m(\alpha - \beta)). \]

Also, the interior regularity

\[ f \in L_{p_*}(D) \implies u \in W^{\alpha/2,\alpha}_{\text{loc}}(D), \quad (p_* := \max\{p_d/(d + \alpha/2), 2\}) \]

was introduced in [46]. We also refer to [25] [26] for the regularity results in the $\mu$-transmission spaces $R^\mu_{\text{trans}}(D)$, and we refer to [11] [21] [28] for the results on Hilbert
spaces. Note that (1.11) and (1.12) do not cover the full regularity of solution, that is $\Delta^{\alpha/2}u$, and the free term and the solution lie in the different summability spaces.

As is briefly described above, Hölder regularity for parabolic equations is not fully satisfactory since they are either local in time or give only spatial Hölder regularities. Moreover, very few results in the literature cover the maximal $L_p$-regularity of solutions. To summarize our results briefly,

- we introduce a Sobolev space theory for both the elliptic and parabolic equations with the fractional Laplacian on domains, and obtain a sharp regularity result of arbitrary order derivatives of solutions;
- we introduce a new approach which can be applied for the Sobolev space theory of equations having more general non-local operators;
- we also obtain global Hölder estimates for the elliptic and parabolic equations; the free terms can be quite irregular and unbounded.

Now, we introduce the organization of this article. In Section 2, we introduce our main results, Sobolev space theory and Hölder estimates of solutions. In Section 3, we study the representation of solutions and estimate the zero-th order derivative of solutions. In Section 4, we prove higher regularity of solutions, and we give the proofs of main results in Section 5.

We finish the introduction with notations used in this article. We use “:=” or “=:” to denote a definition. $\mathbb{N}$ and $\mathbb{Z}$ denote the natural number system and the integer number system, respectively. We denote $\mathbb{N} := \mathbb{N} \cup \{0\}$, and as usual $\mathbb{R}^d$ stands for the Euclidean space of points $x = (x^1, \ldots, x^d)$,

$$B_r(x) = \{ y \in \mathbb{R}^d : |x - y| < r \}, \quad \mathbb{R}_+^d = \{(x^1, \ldots, x^d) \in \mathbb{R} : x^1 > 0\}.$$  

For nonnegative functions $f$ and $g$, we write $f(x) \approx g(x)$ if there exists a constant $C > 0$, independent of $x$, such that $C^{-1}f(x) \leq g(x) \leq Cf(x)$. For multi-indices $\beta = (\beta_1, \ldots, \beta_d)$, $\beta_i \in \mathbb{N}_+$, and functions $u(x)$ depending on $x$,

$$D_\beta u(x) := \frac{\partial^{\beta_1} u(x)}{\partial x^1} \cdot \cdots \cdot \frac{\partial^{\beta_d} u(x)}{\partial x^d}.$$  

We also use $D^n u$ to denote the partial derivatives of order $n \in \mathbb{N}_+$ with respect to the space variables. For an open set $U \subset \mathbb{R}^d$, $C(U)$ denotes the space of continuous functions $u$ in $U$ such that $|u|_{C(U)} := \sup_{x \in U} |u(x)| < \infty$. $C_0(U)$ is the set of functions in $C(U)$ satisfying $\lim_{|x| \to \infty} u(x) = 0$ and $\lim_{x \to \partial U} u(x) = 0$. By $C^\infty_b(U)$ we denote the space of functions whose derivatives of order up to 2 are in $C(U)$. For an open set $V \subset \mathbb{R}^m$, where $m \in \mathbb{N}$, by $C^\infty_c(V)$ we denote the space of infinitely differentiable functions with compact support in $V$. For a Banach space $F$ and $\delta \in (0, 1]$, $C^\delta(V; F)$ denotes the space of $F$-valued continuous functions $u$ on $V$ such that

$$|u|_{C^\delta(V; F)} := |u|_{C(V; F)} + |u|_{C^\delta(V; F)}$$

$$:= \sup_{x \in V} |u(x)|_F + \sup_{x,y \in V} \frac{|u(x) - u(y)|_F}{|x - y|^{\delta}} < \infty.$$  

Also, for $p > 1$ and a measure $\mu$ on $V$, $L_p(V, \mu; F)$ denotes the set of $F$-valued Lebesgue measurable functions $u$ such that

$$\|u\|_{L_p(V, \mu; F)} := \left( \int_V |u|^p_F \, d\mu \right)^{1/p} < \infty.$$
We drop $F$ and $\mu$ if $F = \mathbb{R}$ and $\mu$ is the Lebesgue measure. By $\mathcal{D}(U)$, where $U$ is an open set in $\mathbb{R}^d$, we denote the space of all distributions on $U$, and for given $f \in \mathcal{D}(U)$, the action of $f$ on $\phi \in C^\infty_c(U)$ is denoted by

$$(f, \phi)_U := f(\phi).$$

Finally, if we write $C = C(a, b, \cdots)$, then this means that the constant $C$ depends only on $a, b, \cdots$.

2. Main results

Throughout this article, $D$ is either a half space $\mathbb{R}^d_+$ or a bounded $C^{1,1}$ open set.

2.1. Notion of weak solutions, uniqueness, and existence. We introduce the uniqueness and existence of weak solutions in an optimal class of function spaces.

First, for suitable functions $f$ defined on $\mathbb{R}^d$ (e.g. $f \in C^2_0(\mathbb{R}^d)$), we define the fractional Laplacian $\Delta^{\alpha/2}$ as

$$\Delta^{\alpha/2} f(x) := c_d \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} \frac{f(x + y) - f(x)}{|y|^{d+\alpha}} dy,$$  \hspace{1cm} (2.1)

where $c_d = \frac{2^{\alpha/2}\Gamma(\frac{\alpha}{2})}{\pi^{\frac{d-\alpha}{2}}\Gamma(1-\frac{\alpha}{2})}$. Also, for functions $f, g$ defined on $E \subset \mathbb{R}^d$, we set

$$\langle f, g \rangle_E := \int_E fg \, dx.$$

**Definition 2.1.** (i) (Parabolic problem) For given $f \in L_{1,loc}([0, T] \times D)$ and $u_0 \in L_{1,loc}(D)$, we say that $u$ is a (weak) solution to the problem

$$
\begin{align*}
\begin{cases}
\partial_t u(t, x) = \Delta^{\alpha/2} u(t, x) + f(t, x), & (t, x) \in (0, T) \times D, \\
u(0, x) = u_0(x), & x \in D, \\
u(t, x) = 0, & (t, x) \in [0, T] \times D^c,
\end{cases}
\end{align*}
$$

if (a) $u = 0$ a.e. in $[0, T) \times D^c$, (b) $\langle u(t, \cdot), \phi \rangle_{\mathbb{R}^d}$ and $\langle u(t, \cdot), \Delta^{\alpha/2} \phi \rangle_{\mathbb{R}^d}$ exist for any $t \leq T$ and test function $\phi \in C^\infty_c(D)$, and (c) for any $\phi \in C^\infty_c(D)$ the equality

$$\langle u(t, \cdot), \phi \rangle_{\mathbb{R}^d} = \langle u_0, \phi \rangle_D + \int_0^t \langle u(s, \cdot), \Delta^{\alpha/2} \phi \rangle_{\mathbb{R}^d} ds + \int_0^t \langle f(s, \cdot), \phi \rangle_D ds$$  \hspace{1cm} (2.3)

holds for all $t \leq T$.

(ii) (Elliptic problem) Let $\lambda \in [0, \infty)$. For given $f \in L_{1,loc}(D)$, we say that $u$ is a (weak) solution to

$$
\begin{align*}
\begin{cases}
\Delta^{\alpha/2} u(x) - \lambda u(x) = f(x), & x \in D, \\
u(x) = 0, & x \in D^c,
\end{cases}
\end{align*}
$$

if (a) $u = 0$ a.e. in $D^c$, (b) $\langle u, \phi \rangle_{\mathbb{R}^d}$ and $\langle u, \Delta^{\alpha/2} \phi \rangle_{\mathbb{R}^d}$ exist for any test function $\phi \in C^\infty_c(D)$, and (c) for any $\phi \in C^\infty_c(D)$ we have

$$\langle u, \Delta^{\alpha/2} \phi \rangle_{\mathbb{R}^d} - \lambda \langle u, \phi \rangle_{\mathbb{R}^d} = \langle f, \phi \rangle_D.$$  \hspace{1cm} (2.5)

It is clear if $u(t, x)$ is a strong (or point-wise) solution to (2.2) and sufficiently regular, then $u$ becomes a weak solution in the sense of Definition 2.1.
For an explicit representation of weak solutions, we introduce some related stochastic processes. Let \( X = (X_t)_{t \geq 0} \) be a rotationally symmetric \( \alpha \)-stable \( d \)-dimensional Lévy process defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), that is, \( X_t \) is a Lévy process such that

\[
\mathbb{E} e^{\xi X_t} = e^{-|\xi|^\alpha}, \quad \forall \xi \in \mathbb{R}^d.
\]

Let

\[
\tau_D = \tau_D^\partial := \inf\{ t \geq 0 : x + X_t \notin D \}
\]

denote the first exit time of \( D \) by \( X \). We add an element, called a cemetery point, \( \partial \notin \mathbb{R}^d \) to \( \mathbb{R}^d \), and define the killed process of \( X \) upon \( D \) by

\[
X^D_t = X^{D,x}_t := \begin{cases} x + X_t & \text{for } t < \tau_D^\partial, \\ \partial & \text{for } t \geq \tau_D^\partial. \end{cases}
\]

The cemetery point \( \partial \) is introduced to define \( f(\partial) := 0 \) for any function \( f \) so that \( f(X^D_t) = 0 \) if \( t \geq \tau_D^\partial \). Let \( p^D(t, x, y) \) denote the transition density of \( X^D \), i.e., for any Borel set \( B \subset \mathbb{R}^d \),

\[
\mathbb{P}(X^D_t \in B) = \int_B p^D(t, x, y)dy.
\]

Recall \( \rho(x) = \text{dist}(x, \partial D) \). We denote \( L_{p,\theta}(D) := L_p(D, \rho^{\theta - d}dx) \) for any \( \theta \in \mathbb{R} \) and \( p > 1 \). In other words, \( L_{p,\theta}(D) \) is the set of functions \( u \) such that

\[
\| u \|_{L_{p,\theta}(D)} := \left( \int_D |u|^p \rho^{\theta - d}dx \right)^{1/p} < \infty.
\]

For \( T < \infty \), we also define the space

\[
L_{p,\theta}(D, T) := L_p((0, T); L_{p,\theta}(D))
\]

given with the norm

\[
\| u \|_{L_{p,\theta}(D, T)} = \left( \int_0^T \int_D |u|^p \rho^{\theta - d}dx dt \right)^{1/p}.
\]

Here are our uniqueness and existence results of the elliptic and parabolic equations. The proofs are given in Section 5.

**Theorem 2.2** (Parabolic case). Let \( \alpha \in (0, 2) \) and \( p \in (1, \infty) \). Assume \( \theta \in (d - 1, d - 1 + p) \), \( f \in L_{p,\theta + \alpha/2}(D, T) \) and \( u_0 \in L_{p,\theta - \alpha/2 + \alpha}(D) \).

(i) The function

\[
u(t, x) := \int_D p^D(t, x, y)u_0(y)dy + \int_0^t \int_D p^D(t - s, x, y)f(s, y)dy ds
\]

belongs to \( L_{p,\theta - \alpha/2}(D, T) \cap \{ u = 0 \text{ on } [0, T] \times D^c \} \) and is the unique weak solution to (2.2) in this function space.

(ii) For the solution \( u \), we have

\[
\| u \|_{L_{p,\theta - \alpha/2}(D, T)} \leq C(\| f \|_{L_{p,\theta + \alpha/2}(D, T)} + \| u_0 \|_{L_{p,\theta - \alpha/2 + \alpha}(D)}),
\]

where \( C \) is independent of \( u \) and \( T \).

**Theorem 2.3** (Elliptic case). Let \( \alpha \in (0, 2) \) and \( p \in (1, \infty) \). Assume \( \theta \in (d - 1, d - 1 + p) \) and \( f \in L_{p,\theta + \alpha/2}(D) \).
Due to Corollary 1.2, if $y - 1$ and consequently

\[\text{The right-hand side above is finite only if } \theta > -1. \text{ Therefore, the condition } \theta - d > -1 \text{ is needed to have } u \in L_{p,\theta-\alpha/2}(D).\]

(ii) Let $\lambda = 0$ and $D = \mathbb{R}^d$. Then $u^{(1/n)}$ converges weakly in $L_p(\mathbb{R}^d, \rho^{\theta-d-\alpha/2} dx)$, and the weak limit $u$ is the unique solution to equation (2.4) in the function space $L_{p,\theta-\alpha/2}(D) \cap \{ u = 0 \text{ on } D^c \}$.

(iii) For the solution $u$, we have

\[\|u\|_{L_{p,\theta-\alpha/2}(D)} \leq C\|f\|_{L_{p,\theta+\alpha/2}(D)},\]

where $C$ is independent of $u$ and $\lambda$.

**Remark 2.4.** By definition of the norm in $L_{p,\theta-\alpha/2}(D,T)$ and (2.7),

\[\|u\|^p_{L_{p,\theta-\alpha/2}(D,T)} = \int_0^T \int_D |\rho^{-\alpha/2} u|^p \rho^{\theta-d} dx dt < \infty\]

provided that $-1 < \theta - d < -1 + p$. This suggests that $u$ vanishes at certain rate near the boundary of $D$. The detailed behaviors of solutions and their derivatives will be handled in the following subsection.

**Remark 2.5.** The range $\theta \in (d-1, d-1+p)$ in Theorems 2.2 and 2.3 is sharp.

We demonstrate this with a simple example for the elliptic problem. The parabolic problem can be handled similarly.

Let $D = B_1(0)$ and $f$ be a (non-zero) nonnegative function in $C^\infty_c(D)$ so that $f \in L_{p,\theta}(D)$ for any $\theta \in \mathbb{R}$.

1. First, we show $\theta > d - 1$ is necessary. Denote

\[G^0_D(x,y) := \int_0^\infty \rho^D(t,x,y) dt \quad \text{and} \quad u(x) := \int_D G^0_D(x,y) f(y) dy.\]

Due to [14] Corollary 1.2], if $y \in \text{supp}(f)$ and $(r+1)/2 < |x| < 1$ where $r := 1 - \text{dist}(\text{supp}(f), \partial D) > 0$, then $G^0_D(x,y) \approx \rho(x)^\alpha/2$. Hence, for $(r+1)/2 < |x| < 1$,

\[u(x) \approx \rho(x)^\alpha/2 = (1 - |x|)^\alpha/2,\]

and consequently

\[\|u\|^p_{L_{p,\theta-\alpha/2}(D)} \geq C \int_{(r+1)/2}^1 (1 - s)^\theta - d s^{d-1} ds.\]

The right-hand side above is finite only if $\theta - d > -1$. Therefore, the condition $\theta - d > -1$ is needed to have $u \in L_{p,\theta-\alpha/2}(D)$.

2. Next, we show $\theta < d - 1 + p$ is also necessary. Suppose Theorem 2.3 holds for some $\theta \geq d - 1 + p$. Then,

\[\left\| \int_D G^0_D(\cdot,y) g(y) dy \right\|_{L_{p,\theta-\alpha/2}(D)} \leq C\|g\|_{L_{p,\theta+\alpha/2}(D)} \quad \forall g \in L_{p,\theta+\alpha/2}(D),\]
Now we take a function \( \Psi \) whose Fourier transform \( \hat{\Psi} \) all tempered distributions \( T \) satisfying

\[
\int_D u(x)g(x)dx = \int_D \left( \int_D G_D^0(x,y)g(x)dx \right) f(y)dy
\]

\[
\leq \left\| \int_D G_D^0(\cdot,y)g(y)dy \right\|_{L_p, \theta - \alpha p/2(D)} \|f\|_{L_{p', \theta' + \alpha p'/2}(D)}
\]

\[
\leq C\|g\|_{L_p, \theta + \alpha p/2(D)} \|f\|_{L_{p', \theta' + \alpha p'/2}(D)},
\]

where \( 1/p + 1/p' = 1 \) and \( \theta/p + \theta'/p' = d \). Since \( L_{p', \theta' - \alpha p'/2}(D) \) is the dual space of \( L_{p, \theta + \alpha p/2}(D) \) (cf. Lemma 2.7(iii)), this leads to

\[
\|u\|_{L_{p', \theta' - \alpha p'/2}(D)} \leq C\|f\|_{L_{p', \theta' + \alpha p'/2}(D)} < \infty. \tag{2.8}
\]

Note that \( \theta' \leq d - 1 \). As shown above, \((2.8)\) is not possible, and therefore we get a contradiction.

**Remark 2.6.** Since \( \Delta^{\alpha/2} \phi \) belongs to the dual space of \( L_{p, \theta - \alpha p/2}(D) \) for any \( \phi \in C_c^\infty(D) \) (cf. Lemma 1.3) and \( C_c^\infty(D) \) is dense in the dual space, we can replace \( \langle \cdot, \cdot \rangle_D \) and \( \langle \cdot, \cdot \rangle_{R^d} \) in (2.3) and (2.5) by \( \langle \cdot, \cdot \rangle_D \) for the solutions in Theorems 2.2 and 2.3.

### 2.2. Regularity of solution.

In this subsection, we present Sobolev regularity of solutions. We also obtain Hölder estimates of solutions based on a Sobolev embedding theorem. In particular, we give asymptotic behaviors of solutions and their ‘arbitrary’ order derivatives near the boundary of \( D \).

To describe such results, we first recall Sobolev and Besov spaces on \( R^d \). For \( p \in (1, \infty) \) and \( \gamma \in R \), the Sobolev space \( H^\gamma_p = H^\gamma_p(R^d) \) is defined as the space of all tempered distributions \( f \) on \( R^d \) satisfying

\[
\|f\|_{H^\gamma_p} := \|(1 - \Delta)^{\gamma/2}f\|_{L_p} < \infty,
\]

where

\[
(1 - \Delta)^{\gamma/2}f(x) := \mathcal{F}^{-1} \left( (1 + |\cdot|^2)^{\gamma/2} \mathcal{F}[f] \right)(x).
\]

Here, \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) denote the \( d \)-dimensional Fourier transform and the inverse Fourier transform respectively, i.e.,

\[
\mathcal{F}[f](\xi) := \int_{R^d} e^{-i\xi \cdot x} f(x)dx, \quad \mathcal{F}^{-1}[f](x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{R^d} e^{i\xi \cdot x} f(\xi)d\xi.
\]

As is well known, if \( \gamma \in N_+ \), then we have

\[
H^\gamma_p = W^\gamma_p := \{ f : D^\beta_x u \in L_p(R^d), |\beta| \leq \gamma \}.
\]

For \( T \in (0, \infty) \), define

\[
\mathbb{H}^\gamma_p(T) := L_p((0, T); H^\gamma_p), \quad L_p(T) := \mathbb{H}^0_p(T) = L_p((0, T); L_p).
\]

Now we take a function \( \Psi \) whose Fourier transform \( \mathcal{F}[\Psi] \) is infinitely differentiable, supported in an annulus \( \{ \xi \in R^d : \frac{1}{2} \leq |\xi| \leq 2 \} \), \( \mathcal{F}[\Psi] \geq 0 \) and

\[
\sum_{j \in Z} \mathcal{F}[\Psi](2^{-j}\xi) = 1, \quad \forall \xi \neq 0.
\]

For a tempered distribution \( f \) and \( j \in Z \), define

\[
\Delta_j f(x) := \mathcal{F}^{-1} \left[ \mathcal{F}[\Psi](2^{-j} \cdot) \mathcal{F}[f] \right](x), \quad S_0 f(x) := \sum_{j = -\infty}^0 \Delta_j f(x).
\]
The Besov space $B^\gamma_p = B^\gamma_p(\mathbb{R}^d)$, where $p > 1, \gamma \in \mathbb{R}$, is defined as the space of all tempered distributions $f$ satisfying

$$\|f\|_{B^\gamma_p} := \|S_0 f\|_{L^p} + \left( \sum_{j=1}^{\infty} 2^{\gamma pj} \|\Delta_j f\|_{L^p}^p \right)^{1/p} < \infty.$$ 

It is well known (see e.g. [51 Remark 2.5.12/2]) that if $\gamma = n + \delta$, where $n \in \mathbb{N}_+$ and $\delta \in (0, 1)$, then

$$\|f\|_{B^\gamma_p} \approx \|f\|_{H^\gamma_p} + \left( \sum_{|\beta| = n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|D_x^\beta f(x + y) - D_x^\beta f(x)|^p}{|y|^{d+\delta p}} dy dx \right)^{1/p}. \quad (2.9)$$

Moreover, for any $p > 1$, we have

$$H^{\gamma_2}_p \subset B^{\gamma_1}_p \quad \text{if} \quad \gamma_1 < \gamma_2. \quad (2.10)$$

Next, we introduce weighted Sobolev and Besov spaces on $D \subset \mathbb{R}^d$. Recall $\rho(x) = \text{dist}(x, \partial D)$ and $L_{p,\theta}(D) := L_p(D, \rho^{-\theta - d} dx)$. For any $\theta \in \mathbb{R}$ and $n \in \mathbb{N}_+$, define

$$H^{n,\theta}_{p,\theta}(D) = \{ u : u, \rho D_x u, \cdots, \rho^n D^n_x u \in L_{p,\theta}(D) \}.$$ 

The norm in this space is defined as

$$\|u\|_{H^{n,\theta}_{p,\theta}(D)} = \sum_{|\beta| \leq n} \left( \int_D |\rho^{\beta}| D_x^\beta u(x) \rho^{\theta - d} dx \right)^{1/p}. \quad (2.11)$$

To generalize this space and define $H^{n,\theta}_{p,\theta}(D)$ for any $\gamma \in \mathbb{R}$, we proceed as follows. We choose a sequence of nonnegative functions $\zeta_n \in C^\infty(D), n \in \mathbb{Z}$, having the following properties:

1. $\text{supp}(\zeta_n) \subset \{ x \in D : k_1 e^{-n} < \rho(x) < k_2 e^{-n} \}, \quad k_2 > k_1 > 0, \quad (2.12)$
2. $\sup_{x \in \mathbb{R}^d} |D_x^m \zeta_n(x)| \leq C(m) e^{mn}, \quad \forall m \in \mathbb{N}_+ \quad (2.13)$
3. $\sum_{n \in \mathbb{Z}} \zeta_n(x) > c > 0, \quad \forall x \in D. \quad (2.14)$

Such functions can be easily constructed by considering mollifications of indicator functions of the sets of the type $\{ x \in D : k_3 e^{-n} < \rho(x) < k_4 e^{-n} \}$. If the set $\{ x \in D : k_1 e^{-n} < \rho(x) < k_2 e^{-n} \}$ is empty, we just take $\zeta_n = 0$.

Now we define weighted Sobolev spaces $H^{n,\theta}_{p,\theta}(D)$ and weighted Besov spaces $B^{n,\theta}_p(D)$ for any $\gamma, \theta \in \mathbb{R}$ and $p > 1$. To understand these spaces, one needs to notice that if a distribution on $D$ has compact support in $D$, then it can be considered as a distribution on $\mathbb{R}^d$. In particular, for any distribution $f$ on $D$, $\zeta_{-n} f$ becomes a distribution on $\mathbb{R}^d$. Obviously, the action of $\zeta_{-n} f$ on $C^\infty_c(\mathbb{R}^d)$ is defined as

$$(\zeta_{-n} f, \phi)_{\mathbb{R}^d} = (f, \zeta_{-n} \phi)_D, \quad \phi \in C^\infty_c(\mathbb{R}^d). \quad (2.15)$$

By $H^{n,\theta}_{p,\theta}(D)$ and $B^{n,\theta}_p(D)$ we denote the sets of distributions $f$ on $D$ such that

$$\|f\|_{H^{n,\theta}_{p,\theta}(D)} := \sum_{n \in \mathbb{Z}} e^{n\theta} \|\zeta_{-n} f(e^n \cdot )\|_{L^p(D)}^p < \infty, \quad (2.16)$$

and

$$\|f\|_{B^{n,\theta}_p(D)} := \sum_{n \in \mathbb{Z}} e^{n\theta} \|\zeta_{-n} f(e^n \cdot )\|_{B^\gamma_p}^p < \infty,$$
Lemma 2.7. Let $\nu \in \mathbb{N}$ and $\bar{\omega}$ be a given weight function. Furthermore, for an equivalent norm in $B^\gamma(D)$, we have

$$
\sup_{x,y} |\rho(x) \cdot D^\gamma u(x) - \rho(y) \cdot D^\gamma u(y)| \leq C |\rho(x) - \rho(y)|^{1/p - 1/p'},
$$

and the reverse inequality holds for $\nu$. The similar statements hold in the space $B^\gamma(D)$ as well. Furthermore, if $\gamma = n \in \mathbb{N}$, then the norms defined in (2.11) and (2.16) are equivalent (cf. [43, Proposition 2.2]).

Obviously, by (2.10), we have for any $p > 1$ and $\theta \in \mathbb{R}$,

$$
H^\gamma(D) \subset B^\gamma(D) \quad \text{if} \quad \gamma_1 < \gamma_2.
$$

Furthermore, for an equivalent norm in $B^\gamma(D)$, we can apply (2.9) and prove the following: if $\gamma = n + \delta > 0$, where $n \in \mathbb{N}$, $\delta \in (0,1)$, and $\theta - d + \gamma \rho > -1$, then

$$
\|f\|_{B^\gamma(D)} \approx \|f\|_{H^\gamma(D)} + \left(\sum_{|\gamma| = \nu} \int_{\gamma} \int_{\gamma} |D^\delta u(x) - D^\delta u(y)|^p |x - y|^{\delta p} dy dx \right)^{1/p},
$$

where $\rho_{x,y} = \rho(x) \wedge \rho(y)$. The proof of (2.19) is left to the reader. Relation (2.19) will not be used elsewhere in this article.

Next, we choose (cf. [33]) an infinitely differentiable function $\psi$ in $D$ such that $\psi \approx \rho$ on $D$, and for any $m \in \mathbb{N}$

$$
\sup_D |\rho^m(x) D^\gamma \psi(x)| \leq C(m) < \infty.
$$

For instance, one can take $\psi(x) := \sum_{n \in \mathbb{Z}} e^{-n} \zeta_n(x)$.

Below we collect some other properties of the spaces $H^\gamma(D)$ and $B^\gamma(D)$. For $\nu \in \mathbb{R}$, we write $u \in \psi^\nu H^\gamma(D)$ (resp. $u \in \psi^\nu B^\gamma(D)$) if $\psi^\nu u \in H^\gamma(D)$ (resp. $\psi^\nu u \in B^\gamma(D)$).

**Lemma 2.7.** Let $\gamma, \theta \in \mathbb{R}$ and $p \in (1, \infty)$.

(i) The space $C^\infty(D)$ is dense in $H^\gamma(D)$ and $B^\gamma(D)$.

(ii) For $\delta \in \mathbb{R}$, $H^\gamma(D) = \psi^\delta H^\gamma(D)$ and $B^\gamma(D) = \psi^\delta B^\gamma(D)$. Moreover,

$$
\|u\|_{H^\gamma(D)} \approx \|\psi^{-\delta} u\|_{H^\gamma(D)}, \quad \|u\|_{B^\gamma(D)} \approx \|\psi^{-\delta} u\|_{B^\gamma(D)}.
$$

(iii) (Duality) Let $1/p + 1/p' = 1$, $\theta/p + \theta'/p' = d$.

Then, the dual spaces of $H^\gamma(D)$ and $B^\gamma(D)$ are $H^-\gamma(D)$ and $B^-\gamma(D)$, respectively.

(iv) (Sobolev-Hölder embedding) Let $\gamma - \frac{d}{p} \geq n + \delta$ for some $n \in \mathbb{N}$ and $\delta \in (0,1)$. Then for any $k \leq n$,

$$
|\psi^{\nu} D^\delta u|_{C^k(D)} + |\psi^{\nu} D^\delta u|_{C^k(D)} \leq C(d, \gamma, p, \theta) ||u||_{H^\gamma(D)}.
$$

**Proof.** The proofs for $B^\gamma(D)$ are similar to those for $H^\gamma(D)$, and we only consider the claims for $H^\gamma(D)$.

When $D$ is a half space, the claims are proved by Krylov in [39], and those are generalized by Lototsky in [43] for arbitrary domains. See Proposition 2.2, Theorems...
4.1 and 4.3 in [43]. Here we remark that those in [43] are still valid for bounded \(C^{1,1}\) open sets. The lemma is proved.

Now we define solution spaces for the parabolic equation. For \(T \in (0, \infty)\), denote
\[
\mathbb{H}^\gamma_{p,\theta}(D, T) := L_p((0, T); H^\gamma_{p,\theta}(D)).
\]
We write \(u \in \mathcal{D}^\gamma_{p,\theta}(D, T)\) if \(u \in \psi^{\alpha/2}\mathbb{H}^\gamma_{p,\theta}(D, T)\), \(u(0, \cdot) \in \psi^{\alpha/2-\alpha/p}B^\gamma_{p,\theta}(D)\), and there exists \(f \in \psi^{-\alpha/2}\mathbb{H}^{-\gamma-\alpha}_{p,\theta}(D, T)\) such that for any \(\phi \in C_c^\infty(D)\)
\[
(u(t, \cdot), \phi)_D = (u(0, \cdot), \phi)_D + \int_0^t (f(s, \cdot), \phi)_Dds, \quad \forall t \leq T.
\]
In this case, we write \(u_t := \partial_t u := f\). The norm in \(\mathcal{D}^\gamma_{p,\theta}(D, T)\) is defined as
\[
\|u\|_{\mathcal{D}^\gamma_{p,\theta}(D, T)} := \|\psi^{-\alpha/2}u\|_{\mathbb{H}^\gamma_{p,\theta}(D, T)} + \|\psi^{\alpha/2}u_t\|_{\mathbb{H}^{-\gamma-\alpha}_{p,\theta}(D, T)} + \|\psi^{-\alpha/2+\alpha/p}u(0, \cdot)\|_{B^\gamma_{p,\theta}(D)}.
\]

**Remark 2.8.** (i) The Banach space \(\mathcal{D}^\gamma_{p,\theta}(D, T)\) is a modification of the corresponding space defined for \(\alpha = 2\) (see e.g. [33] for \(C^1\) domains and [39] for a half space). The completeness of this space for \(\alpha \in (0, 2)\) can be proved by repeating the argument in [39] Remark 3.8.

(ii) The same argument in [38, Remark 5.5] shows that \(C_c^\infty([0, T] \times D)\) is dense in \(\mathcal{D}^\gamma_{p,\theta}(D, T)\).

The following two theorems address our Sobolev regularity results. The proofs are given in Section 4.

**Theorem 2.9** (Parabolic case). Let \(\gamma \in [0, \infty)\), and assume \(f \in \psi^{-\alpha/2}\mathbb{H}^\gamma_{p,\theta}(D, T)\) and \(u_0 \in \psi^{\alpha/2-\alpha/p}B^\gamma_{p,\theta}(D)\). The unique solution \(u\) in Theorem 2.3 belongs to \(\mathcal{D}^{\gamma+\alpha}_{p,\theta}(D, T)\), and for this solution we have
\[
\|u\|_{\mathcal{D}^{\gamma+\alpha}_{p,\theta}(D, T)} \leq C \left(\|\psi^{\alpha/2}f\|_{\mathbb{H}^\gamma_{p,\theta}(D, T)} + \|\psi^{-\alpha/2+\alpha/p}u_0\|_{B^\gamma_{p,\theta}}\right), \quad (2.21)
\]
where \(C\) depends only on \(d, p, \alpha, \gamma, \theta\) and \(D\).

**Remark 2.10.** Let \(\gamma + \alpha \geq n\), where \(n \in \mathbb{N}_+\). Then, (2.20) and (2.21) certainly yield
\[
\int_0^T \int_D \left(|\rho^{-\alpha/2}u|^{p} + |\rho^{1-\alpha/2}Du|^{p} + \cdots + |\rho^{n-\alpha/2}D^{n}u|^{p}\right)\rho^{\alpha-d}dxdt < \infty.
\]

**Theorem 2.11** (Elliptic case). Let \(\lambda \in [0, \infty)\) and assume \(f \in \psi^{-\alpha/2}H^\gamma_{p,\theta}(D)\). Then the unique solution \(u\) in Theorem 2.3 belongs to \(\psi^{\alpha/2}H^\gamma_{p,\theta}(D)\), and for this solution we have
\[
\lambda\|\psi^{\alpha/2}u\|_{H^\gamma_{p,\theta}(D)} + \|\psi^{-\alpha/2}u\|_{H^{-\gamma+\alpha}_{p,\theta}(D)} \leq C\|\psi^{\alpha/2}f\|_{H^\gamma_{p,\theta}(D)}, \quad (2.22)
\]
where \(C\) depends only on \(d, p, \alpha, \gamma, \theta\) and \(D\). In particular, it is independent of \(\lambda\).

For Hölder regularity of the solution to the parabolic equation, we use the following parabolic embedding.
Proposition 2.12. Let $\alpha \in (0, 2)$, $p \in (1, \infty)$, and $\gamma, \theta \in \mathbb{R}$. Then, for any $1/p < \nu \leq 1$,
\[
\left| \psi^{(\nu-1)/2} (u - u(0, \cdot)) \right|_{C^{\nu-1/p}([0,T]; H_{p,\theta}^{\gamma,\alpha}(D))} \leq C \|u\|_{H_{p,\theta}^{\gamma,\alpha}(D,T)},
\]
(2.23)
where $C$ depends only on $d$, $\nu$, $p$, $\theta$, $\alpha$ and $T$.

**Proof.** We repeat the argument in [39] which treats the case $\alpha = 2$. Considering $u - u_0$ in place of $u$, we may assume $u_0 = 0$. Let $u_t = f$. By (2.16) and Lemma 2.7 (ii),
\[
\left| \psi^{(\nu-1)/2} u \right|_{C^{\nu-1/p}([0,T]; H_{p,\theta}^{\gamma,\alpha}(D))} \leq C \sum_{n \in \mathbb{Z}} e^{p(n(\theta + \nu \alpha))} \|u(\cdot, e^n)\|_{H_{p,\theta}^{\gamma,\alpha}(D)} \|e^n\|_{H_{p,\theta}^{\gamma,\alpha}(D,T)}.
\]
(2.24)
Denote $v_n(t, x) = u(t, e^n x)$ and $\partial_t v_n(t, x) = f(t, e^n x)\zeta_n(e^n x)$. Thus, by Lemma A.5 with $a = e^{-\nu \alpha}$,
\[
e^{\nu \alpha} \|u(\cdot, e^n)\|_{H_{p,\theta}^{\gamma,\alpha}(D)} \leq C e^{\nu \alpha} \|u(\cdot, e^n)\|_{H_{p,\theta}^{\gamma,\alpha}(D,T)} + C e^{\nu \alpha} \|f(\cdot, e^n)\|_{H_{p,\theta}^{\gamma,\alpha}(D,T)}.
\]
Comming back to (2.23) and using (2.16),
\[
\left| \psi^{(\nu-1)/2} u \right|_{C^{\nu-1/p}([0,T]; H_{p,\theta}^{\gamma,\alpha}(D))} \leq C \|\psi^{-\alpha/2} u\|_{H_{p,\theta}^{\gamma,\alpha}(D,T)} + C \|\psi^{\alpha/2} f\|_{H_{p,\theta}^{\gamma,\alpha}(D,T)}.
\]
This and Lemma 2.7 (ii) prove (2.23).

Proposition 2.12 and Lemma 2.7 (iv) yield the following results.

**Corollary 2.13.** *(Hölder regularity for parabolic equation)* Let $u$ be taken from Theorem 2.4 $1/p < \nu \leq 1$, and
\[
\gamma + \alpha - \nu \alpha - \frac{d}{p} \geq n + \delta, \quad n \in \mathbb{N}_+, \quad \delta \in (0, 1).
\]
Then,
\[
\sum_{k=0}^n |\psi^{k+\frac{d}{p}} D_x^k u|_{C^{\nu-1/p}([0,T]; C(D))} + \sup_{t, s \in [0,T]} \frac{|\psi^{n+\delta+\frac{d}{p}} D_x^n (u(t, \cdot) - u(s, \cdot))|_{C^\delta(D)}}{|t - s|^{\nu-1/p}} \leq C \|u\|_{H_{p,\theta}^{\gamma,\alpha}(D,T)}.
\]

**Corollary 2.14.** *(Hölder regularity for elliptic equation)* Let $u$ be taken from Theorem 2.4 and
\[
\gamma + \alpha - \frac{d}{p} \geq n + \delta, \quad n \in \mathbb{N}_+, \quad \delta \in (0, 1).
\]
Then,
\[
\sum_{k=0}^n |\psi^{k+\frac{d}{p}} D_x^k u|_{C(D)} + |\psi^{n+\delta+\frac{d}{p}} D_x^n u|_{C^\delta(D)} \leq C \|\psi^{-\alpha/2} u\|_{H_{p,\theta}^{\gamma,\alpha}(D)}.
\]
Remark 2.15. Corollaries 2.13 and 2.14 give various Hölder estimates of solutions and their arbitrary order derivatives. Below we elaborate some special cases. We only consider $\gamma = 0, 1, 2, \cdots$ and $\theta = d$. Note $L_{p,d}(D) = L_p(D)$.

(i) Parabolic Hölder estimates when $\gamma = 0$. Let $u_0 = 0$ for simplicity, and assume $u^{\alpha/2}f \in \cap_{p>d/\alpha}L_{p,d}(D,T)$. Obviously this holds e.g. if $D$ is bounded and $u^{\alpha/2}f \in L_\infty([0,T] \times D)$. Taking $\nu \uparrow 1$ and $p \uparrow \infty$, from Corollary 2.13 we get

$$
\sup_{x \in D} |\psi^{\alpha/2-\delta}(x)u(\cdot, x)|_{C^{1-\delta}([0,T])} < \infty
$$

for any small $\delta, \varepsilon > 0$. This gives maximal regularity with respect to time variable.

Now, we take $p$ sufficiently large and $\nu$ sufficiently close to $1/p$ to get

$$
\sup_{x \in D} |\psi^{-\alpha/2+\delta'}(x)u(\cdot, x)|_{C^{\alpha/2-\delta}(0,\varepsilon)} + \sup_{\varepsilon \in [0,T]} |\psi^{-\alpha/2-\delta,u(\cdot, \varepsilon)}_{C^{\alpha/2-\delta}}(D) < \infty
$$

for any small $\delta', \varepsilon' > 0$. The second term above gives the maximal interior regularity with respect to space variable, and the first one gives a decay rate near the boundary of $D$. In particular,

$$
\sup_{\varepsilon \in [0,T]} |u(t, x)| \leq C(\delta')\psi^{\alpha/2-\delta'}(x), \quad \forall \delta' > 0.
$$

(ii) Elliptic Hölder estimates when $\gamma = 0$. Let $\psi^{\alpha/2}f \in \cap_{p>d/\alpha}L_{p,d}(D)$. Taking $p$ sufficiently large, from Corollary 2.13 we get

$$
|\psi^{\alpha/2}u|_{C^{\alpha-\varepsilon}(D)} + |u|_{C^{\alpha/2-\varepsilon}(D)} + |\psi^{-\alpha/2+\delta}u|_{C^{\alpha/2-\varepsilon}(D)} < \infty
$$

for any small $\delta, \varepsilon > 0$. In [47], it is proved that if $\lambda = 0$ and $f \in L_\infty(D)$, then

$$
|u|_{C^{\alpha/2}(D)} + |\psi^{-\alpha/2}u|_{C^{\alpha}(D)} < \infty
$$

for some $\beta > 0$. Thus, there is a slight gap between (2.25) and (2.26). However, our result holds even when $f$ blows up near the boundary since we assume (at most) $\psi^{\alpha/2}f$ is bounded.

(iii) Higher order estimates. Let $\gamma = n \in \mathbb{N}$. Then, the same arguments above show that all the claims in (i)-(ii) also hold for $\psi D_x u, \psi^2 D_x^2 u, \cdots, \psi^n D_x^n u$. That is, the estimates hold if one replaces $u$ by any of these functions. In particular, if $\psi^{\alpha/2}f, \psi^{\alpha/2+1}D_x f \in \cap_{p>d/\alpha}L_{p,d}(D)$, then, together with (2.23), we also have

$$
|\psi^{1+\alpha/2}D_x u|_{C^{\alpha-\varepsilon}(D)} + |\psi D_x u|_{C^{\alpha/2-\varepsilon}(D)} + |\psi^{1-\alpha/2+\delta}D_x u|_{C^{\alpha/2-\varepsilon}(D)} < \infty
$$

for any small $\delta, \varepsilon > 0$.

3. THE ZERO-TH ORDER DERIVATIVE ESTIMATES

In this section, we estimate the zero-th order derivative of the solutions to the parabolic equation

$$
\begin{cases}
\partial_t u(t, x) = \Delta^{\alpha/2}u(t, x) + f(t, x), & (t, x) \in (0, T) \times D, \\
u(0, x) = u_0(x), & x \in D, \\
u(t, x) = 0, & (t, x) \in [0, T] \times D^c.
\end{cases}
$$

(3.1)

as well as to the elliptic equation

$$
\begin{cases}
\Delta^{\alpha/2}u(x) - \lambda u(x) = f(x), & x \in D, \\
u(x) = 0, & x \in D^c.
\end{cases}
$$

(3.2)
3.1. Weak solutions for smooth data. Recall that $X = (X_t)_{t \geq 0}$ is a rotationally symmetric $\alpha$-stable $d$-dimensional Lévy process. Let $p(t, x) = p_d(t, x)$ denote the transition density function of $X$. Then, it is well known (e.g. [34 (3.6)]) that
\[
p_d(t, x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{-t|\xi|^\alpha} d\xi \\
\approx t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x|^{d+\alpha}} \approx \frac{t}{(t^{1/\alpha} + |x|)^{d+\alpha}}, \quad \forall (t, x) \in (0, \infty) \times \mathbb{R}^d.
\]
The equality above also implies that $p_d(t, \cdot)$ is a radial function and
\[
p_d(t, x) = t^{-\frac{d}{\alpha}} p_d(1, t^{-\frac{1}{\alpha}} x).
\]
Denote
\[
d_x = d_{D, x} := \begin{cases} p(x) : x \in D, \\ 0 : x \notin D. \end{cases}
\]
The following lemma gives an upper bound of $p^D(t, x, y)$.

**Lemma 3.1.** For any $x, y \in \mathbb{R}^d$,
\[
p^D(t, x, y) \leq \begin{cases} C \left( 1 \wedge \frac{d_x^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{d_y^{\alpha/2}}{\sqrt{t}} \right) p(t, x - y) & \text{if } D \text{ is a half space}, \\
Ce^{-ct} \left( 1 \wedge \frac{d_x^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{d_y^{\alpha/2}}{\sqrt{t}} \right) p(t, x - y) & \text{if } D \text{ is bounded}.
\end{cases}
\]
Here, $C, c > 0$ depend only on $d, \alpha$ and $D$.

**Proof.** See [10] Theorem 5.8 for the case $D = \mathbb{R}^d_+$. Let $D$ be bounded. Then, by [10] Theorem 4.5, there exist $C, c, r > 0$, depending only on $\alpha, d$ and $D$, such that for any $x, y \in D$
\[
p^D(t, x, y) \leq Ce^{-ct} \left( \frac{d_x^{\alpha/2}}{\sqrt{t} \wedge r^{\alpha/2}} \wedge 1 \right) \left( \frac{d_y^{\alpha/2}}{\sqrt{t} \wedge r^{\alpha/2}} \wedge 1 \right) p(t \wedge r^\alpha, x - y).
\]
This actually implies the claim of the lemma. Indeed, the case $t < r^\alpha$ is obvious, and if $t > r^\alpha$ then
\[
p(r^\alpha, x - y) = r^{-d} p(1, r^{-1}(x - y)) \leq r^{-d} p(1, t^{-\frac{1}{\alpha}}(x - y)) = r^{-d} t^{d/\alpha} p(t, x - y).
\]
This certainly proves the claim. The lemma is proved.

For $x \in \mathbb{R}^d$, we use $E_x$ and $P_x$ to denote the expectation and distribution of $x + X$. For instance, $P_x(X_t \in A) := P(x + X_t \in A)$. Recall that $f(\partial) := 0$ for any function $f$, where $\partial$ is the cemetery point.

Now, we introduce the probabilistic representation of equation (3.1) for smooth data.

**Lemma 3.2.** (i) Suppose $u_0 \in C^\infty_c(D)$ and $f \in C^\infty_c((0, T) \times D)$. Then,
\[
u(t, x) := E_x[u_0(X^D_t)] + \int_0^t E_x[f(s, X^D_{t-s})] ds
\]
\[
= \int_D p^D(t, x, y) u_0(y) dy + \int_0^t \int_D p^D(t - s, x, y) f(s, y) dy ds
\]
is a weak solution to (3.1) in the sense of Definition 2.1(i).
Lemma 3.3. Assume $Af$.

By the definition of the infinitesimal generator, $D$

Proof. Assume sup $u > 0$. Since $u \in C_0(D)$, there exists $x_0 \in D$ such that

$$u(x_0) = \sup_{x \in D} u(x).$$

By the definition of the infinitesimal generator,

$$Au(x_0) = \lim_{t \downarrow 0} \frac{E_x[u(X_t)] - u(x_0)}{t} \leq 0.$$
Thus, by Fubini’s theorem, 

\[ \text{(3.4)} \]

\( v(x) := \int_D G^L_D(x, y) f(y) dy. \)

By Lemma 3.4, \( G^L_D(x, y) \) is well defined if \( x \neq y \).

**Lemma 3.4.** Let \( D \) be a half space (resp. a bounded \( C^{1,1} \) open set) and \( \lambda > 0 \) (resp. \( \lambda \geq 0 \)). For \( f \in C(D) \), define

\[
(v(x) := \int_D G^L_D(x, y) f(y) dy. \quad (3.5)
\]

(i) \( v \in C_0(D) \), \( Av(x) \) exists for all \( x \in D \), and \( v \) is a strong(point-wise) solution to

\[
\begin{align*}
Av(x) - \lambda v(x) &= f(x), & x & \in D, \\
v(x) &= 0, & x & \in D^c.
\end{align*}
\]

(ii) \( v \) is a weak solution to \((3.2)\) in the sense of Definition 2.1(ii).

(iii) Let \( u \in C^\infty_c(D) \) and \( g := \Delta^{\alpha/2} u - \lambda u \). Then,

\[
u(x) = \int_D G^L_D(x, y) g(y) dy. \quad (3.7)
\]

**Proof.** (i) The claim follows from \([35, \text{Lemma 3.6}]\) if \( D \) is bounded and \( \lambda = 0 \). We repeat its proof for the case \( \lambda > 0 \). First, we show \( v \in C_0(D) \). By Lemma 3.1,

\[
\int_0^\infty \int_D e^{-\lambda t} p^D(t, x, y) |f(y)| dy dt \leq C \|f\|_{L^\infty(D)} \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda t} p(t, x - y) dy dt \\
= C \|f\|_{L^\infty(D)} \int_0^\infty e^{-\lambda t} dt < \infty.
\]

Thus, by Fubini’s theorem,

\[
v(x) = \int_0^\infty e^{-\lambda t} T^D_t f(x) dt.
\]

Since \( T^D_t f \in C_0(D) \) and \( \|T^D_t f\|_{L^\infty(D)} \leq \|f\|_{L^\infty(D)} \), the dominated convergence theorem easily yields \( v \in C_0(D) \).

Since \( \{T^D_t\}_{t \geq 0} \) is a Feller semigroup, for any \( x \in D \),

\[
A_D v(x) := \lim_{t \to 0^+} \frac{T^D_t v(x) - v(x)}{t} \quad (\text{provided that the limit exists})
\]

\[
= \lim_{t \to 0^+} \frac{1}{t} \left( T^D_t \int_0^\infty e^{-\lambda s} T^D_s f(x) ds - \int_0^\infty e^{-\lambda t} T^D_s f(x) ds \right)
\]

\[
= \lim_{t \to 0^+} \frac{1}{t} \left( \int_0^\infty e^{-\lambda s} T^D_{t+s} f(x) ds - \int_0^\infty e^{-\lambda t} T^D_s f(x) ds \right)
\]

\[
= \lim_{t \to 0^+} \frac{1}{t} \left( e^{\lambda t} \int_t^\infty e^{-\lambda s} T^D_s f(x) ds - \int_0^\infty e^{-\lambda t} T^D_s f(x) ds \right)
\]

\[
= \lim_{t \to 0^+} \frac{e^{\lambda t} - 1}{t} \int_t^\infty e^{-\lambda s} T^D_s f(x) ds + \lim_{t \to 0^+} \frac{1}{t} \left( \int_0^t e^{-\lambda s} T^D_s f(x) ds \right)
\]

\[
= \lambda v(x) + f(x).
\]
Since the limits exist, we conclude that $A_Dv$ exists, $A_Dv = Av$, and $v$ satisfies (3.6).

(ii) Let $\varphi \in C_c^\infty(D)$. Since $\|T_D^\varphi\|_{L_\infty(D)} \leq \|\varphi\|_{L_\infty(D)}$,
\[
\lim_{t \to \infty} e^{-\lambda T_D^\varphi} = 0.
\]

Since $\varphi \in C_c^\infty(D)$, we can use the relation $\partial_t T_D^\varphi = T_D^\Delta^{\alpha/2} \varphi$ (see [53 Lemma 8.4]) to get
\[
(v, \Delta^{\alpha/2} \varphi)_{\mathbb{R}^d} = \int_0^\infty (e^{-\lambda T_D^\varphi} f, \Delta^{\alpha/2} \varphi)_D dt
\]
\[
= \int_0^\infty (f, e^{-\lambda T_D^\varphi} \Delta^{\alpha/2} \varphi)_D dt
\]
\[
= \int_0^\infty (f, \Delta^{\alpha/2} \varphi)_D dt - \lim_{t \to \infty} \int_0^t (f, \lambda e^{-\lambda T_D^\varphi})_D dt
\]
\[
= (f, \varphi)_D + \lambda(v, \varphi)_D.
\]

(iii) Note that $f := \Delta^{\alpha/2} u - \lambda u \in C(D)$. Assume $\lambda > 0$ for the moment. Take $v(x)$ from (3.5). Then, since $u \in C^2_c(\mathbb{R}^d)$, we have $Av = \Delta^{\alpha/2} u$ (e.g. [6, Lemma 2.6]), and therefore both $u$ and $v$ satisfy the equation $Aw(x) - \lambda w(x) = f(x)$ for each $x \in D$. We conclude $u = v$ due to Lemma [53, 8.5, 8.6]. If $\lambda = 0$ and $D$ is a bounded $C^{1,1}$ open set, then the uniqueness result in [57 Theorem 3.10] easily yields (3.7). The lemma is proved.

3.2. Estimates of zero-th order of solutions. Denote
\[
T_D^0 u_0(t, x) := \int_D P^D(t, x, y) u_0(y) dy,
\]
\[
T_D f(t, x) := \int_0^t \int_D P^D(t - s, x, y) f(s, y) dy ds,
\]
\[
G_D^\lambda f(x) := \int_D G_D^\lambda(x, y) f(y) dy.
\]

In this subsection, we prove the operators
\[
T_D^0 : \psi^{\alpha/2 - \alpha/2} L_{p, \theta}(D) \to \psi^{\alpha/2} \mathbb{L}_{p, \theta}(D, T),
\]
\[
T_D : \psi^{-\alpha/2} L_{p, \theta}(D, T) \to \psi^{\alpha/2} L_{p, \theta}(D, T),
\]
\[
G_D^\lambda : \psi^{-\alpha/2} L_{p, \theta}(D) \to \psi^{\alpha/2} L_{p, \theta}(D)
\]
are bounded. Our proofs highly depend on the following lemma, which is proved in Lemma A.3.

Lemma 3.5. Let $\alpha \in (0, 2)$, $\gamma_0, \gamma_1 \in \mathbb{R}$. Suppose that
\[
-2 \frac{\alpha}{\alpha} < \gamma_0, -2 < \gamma_1 - \gamma_0 \leq 2 + \frac{\alpha}{\alpha}.
\]
Then, for any $(t, x) \in (0, \infty) \times \mathbb{R}^d$,
\[
\int_D P^D(t, x - y) \frac{dy^{\alpha/2}}{(\sqrt{t} + dy^{\alpha/2})^{\gamma_1}} dy \leq C(\sqrt{t} + dy^{\alpha/2})^{\gamma_0 - \gamma_1},
\]
where $C = C(d, \alpha, \gamma_0, \gamma_1, D)$. 

We first consider the operator $T_D$.

**Lemma 3.6.** Let $\alpha \in (0, 2)$ and $p \in (1, \infty)$. Suppose that
\[ d - 1 < \theta < d - 1 + p. \]
Then, there exists $C = C(d, \alpha, \theta, p, D)$ such that for any $f \in \psi^{-\alpha/2}L_{p, \theta}(D, T)$,
\[ \|\psi^{-\alpha/2}T_D f\|_{L_{p, \theta}(D, T)} \leq C\|\psi^{\alpha/2} f\|_{L_{p, \theta}(D, T)}. \]

**Proof.** By Lemma 2.7(11) and (12), it suffices to show
\[ \int_0^T \int_D d_x^{\mu-\alpha p/2}|T_D f(t, x)|^p\,dx\,dt \leq C \int_0^T \int_D d_x^{\mu+\alpha p/2}|f(t, x)|^p\,dx\,dt, \]
where $\mu := \theta - d$. For $p' = p/(p-1)$, since $\mu \in (-1, p-1)$, we can take $\beta_0$ satisfying
\[ \frac{2\mu}{p\alpha} + 1 - \frac{4}{p} < \beta_0 < \frac{2\mu}{p\alpha} + 1 + \frac{2}{p\alpha} \]
and
\[ -\frac{2(p-1)}{p} = -\frac{2}{p'} < \beta_0 < \left(2 + \frac{2}{\alpha}\right) \frac{1}{p'} = \left(2 + \frac{2}{\alpha}\right) \frac{p-1}{p}. \]
Since $1 - \frac{2}{p} < \frac{2\mu}{p\alpha} + 1 + \frac{2}{p\alpha} \frac{2}{p}$ and $\frac{2\mu}{p\alpha} + 1 < \frac{2(p-1)}{p\alpha} + 1$, we can take constants $\beta_1$ and $\beta_2$ such that
\[ 1 - \frac{2}{p} < \beta_0 - \beta_1 < \frac{2\mu}{p\alpha} + 1 + \frac{2}{p\alpha} - \frac{2}{p} \]
and
\[ \frac{2\mu}{p\alpha} + 1 < \beta_0 + \beta_2 < \frac{2(p-1)}{p\alpha} + 1. \]
Let $R_{t,x} := \frac{d_x^{\alpha/2}}{\sqrt{t+d_x^{\alpha/2}}}$. By Lemma 5.1 and Hölder’s inequality,
\[ |T_D f(t, x)| \leq C \left( \int_0^t \int_D p(t-s, x-y) d_y^{-\alpha p/2} R_{t-s,x}^{(1-\beta_1)p'} R_{t-s,y}^{(1-\beta_2)p'} dy dx \right)^{1/p'} \]
\[ \times \left( \int_0^t \int_D p(t-s, x-y) d_y^{\alpha p/2} R_{t-s,x}^{\beta_1 p} R_{t-s,y}^{\beta_2 p} f(s, y)^p dy dx \right)^{1/p} \]
\[ =: C \times I(t, x) \times \|f\|_{L_{p, \theta}(D, T)}. \]

By Lemma 3.5 with $\gamma_0 = (1-\beta_2)p' - \beta_0 p'$ and $\gamma_1 = (1-\beta_2)p'$, we have
\[ \int_D p(t-s, x-y) d_y^{-\alpha p/2} R_{t-s,y}^{(1-\beta_2)p'} dy \leq C (\sqrt{t-s} + d_x^{\alpha/2})^{-\beta_0 p'}. \]
Using this inequality and changing variables,
\[ I(t, x)p' \leq Cd_x^{(1-\beta_1)p'/2} \int_0^t (\sqrt{t-s} + d_x^{\alpha/2})^{-\beta_0 p'-(1-\beta_1)p'} ds \]
\[ \leq Cd_x^{\alpha p/2} \int_0^\infty d_x^{\alpha p} (\sqrt{s+1} - \beta_0 p'-(1-\beta_1)p') ds = Cd_x^{\alpha p/2}. \]
Lemma 3.7. Let $p \in C^{1,1}$ and Fubini’s theorem,

$$
\int_0^T \int_D d_x^{\mu_0} d_x^{-\alpha/2} T_{\alpha/2} f(t,x) |p| dx dt \leq C \int_0^T \int_D d_x^{\mu_0} d_x^{-\alpha/2 - \alpha \beta_0/2} II(t,x)^p dx dt
$$

where $p \in (0,2)$ and $p \in (1, \infty)$. Suppose that

$$
d - 1 < \theta < d - 1 + p + \left( \alpha(p - 1) \wedge \frac{3}{2} \right).
$$

Then, there exists $C = C(d, \alpha, \theta, p, D)$ such that for any $u_0 \in \psi^{-\alpha/2 + \alpha/p} L_{p,\theta}(D)$,

$$
\| \psi^{-\alpha/2} T_{\mu_0}^D u_0 \|_{L_{p,\theta}(D)} \leq C \| \psi^{-\alpha/2 + \alpha/p} u_0 \|_{L_{p,\theta}(D)}.
$$

Proof. As in the proof of Lemma 3.3, it is enough to prove

$$
\int_0^T \int_D d_x^{\mu_0} \phi(x) |T_{\mu_0}^D u_0(t,x) |p| dx dt \leq C \int_0^T \int_D d_x^{\mu_0} \phi(x) |u_0(x) |p| dx,
$$

where $\mu := \theta - d$. Since $\mu \in (-1, p - 1 + \frac{2}{\alpha p})$, we can choose $\beta_0$ satisfying

$$
\frac{2\mu}{p \alpha} - 1 - \frac{2}{p} < \beta_0 < \frac{2\mu}{p \alpha} - 1 + \frac{2}{p} + \frac{2}{p \alpha}
$$

and

$$
-\frac{2(p - 1)}{p} = -\frac{2}{p'} < \beta_0 < \left( \frac{2 + \frac{2}{\alpha}}{p'} \right) \frac{1}{p'} = \left( \frac{2 + \frac{2}{\alpha}}{p'} \right) \frac{p - 1}{p},
$$

where $p' = p/(p - 1)$. Also, since $\frac{2\mu}{p \alpha} - 1 + \frac{2}{p} < \frac{2}{p' \alpha} + 1$, we can choose $\beta_1$ satisfying

$$
\frac{2\mu}{p \alpha} - 1 + \frac{2}{p} < \beta_0 + \beta_1 < \frac{2}{p' \alpha} + 1.
$$
Let \( R_{t,x} := \frac{d^{\mu/2}}{\sqrt{t + d_x}} \). By Lemma 3.1 and Hölder’s inequality,
\[
|T^0_D u_0(t,x)| \leq C \left( \int_D p(t,x-y) d_y^{-\alpha \beta p'/2} R^{(1-\beta_1)}_{I_t,y} \right)^{1/p'} \\
\times \left( \int_D p(t,x-y) d_y^{\beta_0 p/2} R^{P}_{I_t,y} |u_0(y)|^p dy \right)^{1/p} \\
= C \times I(t,x) \times II(t,x),
\]
By Lemma 3.3 with \( \gamma_0 = (1-\beta_1)p' - \beta_0 p' \) and \( \gamma_1 = (1-\beta_1)p' \), we have
\[
I(t,x)^{p'} = \int_D p(t,x-y) d_y^{-\alpha \beta p'/2} R^{(1-\beta_1)}_{I_t,y} dy \leq C(\sqrt{t} + d_x^{p/2})^{-\beta_0 p'}
\]
Therefore, applying Fubini’s theorem,
\[
\int_0^T \int_D d_x^{\mu} |d_x^{-\alpha/2} T^0_D u_0(t,x)|^p dx dt \\
\leq C \int_0^T \int_D d_x^{\mu} (\sqrt{t} + d_x^{p/2})^{-\beta_0} II(t,x)^p dx dt \\
\leq C \int_D |u_0(y)|^p d_y^{\beta_0 p/2} K(T,y) dy,
\]
where
\[
K(T,y) := \int_0^T R^{\beta_1 p}_{I_t,y} \int_D p(t,x-y) d_x^{-\alpha \beta p'/2} (\sqrt{t} + d_x^{p/2})^{-\beta_0 p} R^P_{I_t,x} dx dt \\
= \int_0^T R^{\beta_1 p}_{I_t,y} \int_D p(t,x-y) d_x^{\mu}(\sqrt{t} + d_x^{p/2})^{-\beta_0 p} p dx dt.
\]
By Lemma 3.3 with \( \gamma_0 = 2\mu/\alpha \) and \( \gamma_1 = \beta_0 p + p \),
\[
K(T,y) \leq C \int_0^T R^{\beta_1 p}_{I_t,y} (\sqrt{t} + d_x^{p/2})^{2\mu/\alpha - \beta_0 p} dt \\
\leq Cd_y^{\mu - \alpha \beta_0 p/2 - \alpha/2 + \alpha} \int_0^\infty (\sqrt{t} + 1)^{2\mu/\alpha - \beta_0 p - \beta_1 p} dt \\
\leq Cd_y^{\mu - \alpha \beta_0 p/2 - \alpha/2 + \alpha}.
\]
This with (3.17) proves the lemma. \( \square \)

Finally, we consider the operator \( G^\lambda_D \) for elliptic equation (3.2).

**Lemma 3.8.** Let \( \alpha \in (0,2) \), \( p \in (1,\infty) \) and \( \theta \in (d-1,d-1+p) \). Suppose that \( D \) is a half space (resp. a bounded \( C^{1,1} \) open set) and \( \lambda > 0 \) (resp. \( \lambda \geq 0 \)). Then for any \( f \in \psi^{-\alpha/2} L_{p,\theta}(D) \),
\[
\|\psi^{-\alpha/2} G^\lambda_D f\|_{L_{p,\alpha}(D)} \leq C \|\psi^{\alpha/2} f\|_{L_{p,\alpha}(D)},
\]
where \( C = C(d,p,\alpha,\theta,D) \) is independent of \( \lambda \).

**Proof.** As before, we need to show
\[
\int_D d_x^{-\alpha p/2} |G^\lambda_D f(x)|^p dx \leq C \int_D d_x^{\mu + \alpha p/2} |f(x)|^p dx,
\]
(3.18)
where $\mu := \theta - d$. Take $\beta_0, \beta_1$ and $\beta_2$ satisfying (3.9)-(3.12). Let $R_{t,x} := \frac{g^{\alpha/2}}{\sqrt{t+d_y}}$.

By Lemma 3.1 and Hölder’s inequality,
\[
|G^\lambda_D f(x)| \leq C \left( \int_0^\infty \int_D p(t,x-y) d_y^{-\alpha\beta_0 p'/2} R_{t,x}^{(1-\beta_1)p'} dt \right)^{1/p'} \times \left( \int_0^\infty \int_D p(t,x-y) d_y^{\alpha\beta_2 p/2} R_{t,x}^{\beta_1 p} R_{t,y}^{\beta_2 p} |f(y)| dy dt \right)^{1/p}.
\]

Thus, we prove (3.18) and the lemma.

4. Higher order estimates

In this section we prove that one can raise regularity of solutions as long as the free terms are in appropriate functions spaces.

We first prepare some auxiliary results below. Let $\{\zeta_n : n \in \mathbb{Z}\}$ be a collection of functions satisfying (2.12)-(2.13) with $(k_1, k_2) = (1, e^2)$. We also take $\{\eta_n : n \in \mathbb{Z}\}$ satisfying (2.12)-(2.14) with $(k_1, k_2) = (e^{-2}, e^4)$ and
\[
\eta_n = 1 \text{ on } \{x \in D : e^{-n-1} < \rho(x) < e^{-n+3}\}.
\]

Consequently, $\eta_n = 1$ on the support of $\zeta_n$ and $\zeta_n \eta_n = \zeta_n$.

Lemma 4.1. For any $\gamma \in \mathbb{R}$, there exists a constant $C = C(d, \alpha, \gamma)$ such that for $u \in C^\infty_c(D)$ and $n \in \mathbb{Z}$,
\[
\begin{aligned}
&\left\| \Delta^{\alpha/2} \left( (u\zeta_n \eta_n)(e^{n\cdot}) - \zeta_n(e^{n\cdot}) \Delta^{\alpha/2} (u\eta_n)(e^{n\cdot}) \right) \right\|_{H^p_x} \\
&\leq C \left( \left\| \Delta^{\alpha/4} (u\eta_n)(e^{n\cdot}) \right\|_{H^p_x} + \left\| u(e^{n\cdot}) \eta_n(e^{n\cdot}) \right\|_{H^p_x} \right).
\end{aligned}
\]
Proof. By (2.1),
\[ \Delta^{\alpha/2}((u\zeta_n\eta_n)(e^n\cdot))(x) - \zeta_n(e^n x)\Delta^{\alpha/2}((u\eta_n)(e^n\cdot))(x) - u(e^n x)\eta_n(e^n x)\Delta^{\alpha/2}\zeta_n(e^n\cdot)(x) = C \int_{\mathbb{R}^d} H_n(x, y)|y|^{-d-\alpha} dy, \]
where
\[ H_n(x, y) := [(u\eta_n)(e^n(x + y)) - (u\eta_n)(e^n x)][\zeta_n(e^n(x + y)) - \zeta_n(e^n x)]. \]
In the virtue of (2.13), for any \( m \in \mathbb{N}_+ \),
\[ |D^m_x (\zeta_n(e^n(x + y)) - \zeta_n(e^n x))| \leq C(m)(1 \wedge |y|). \]
Thus, \( \zeta_n(e^n(x + y)) - \zeta_n(e^n x) \) becomes a point-wise multiplier in \( H^\gamma_p \) (see e.g. (37) Lemma 5.2), and therefore
\[ \|H_n(\cdot, y)\|_{H^\gamma_p} \leq C(1 \wedge |y|)\|(u\eta_n)(e^n(\cdot + y)) - (u\eta_n)(e^n \cdot)\|_{H^\gamma_p}. \]
By (37) Lemma 2.1, the above is bounded by
\[ C \left( \|u(e^n\cdot)\eta_n(e^n\cdot)\|_{H^\gamma_p} \wedge |y|^{\alpha/2+1} \right) \|\Delta^{\alpha/4}(u(e^n\cdot)\eta_n(e^n\cdot))\|_{H^\gamma_p}. \]
By Minkowski’s inequality and (4.2),
\[
\left\| \int_{\mathbb{R}^d} H_n(\cdot, y)|y|^{-d-\alpha} dy \right\|_{H^\gamma_p}
\leq C \|\Delta^{\alpha/4}(u(e^n\cdot)\eta_n(e^n\cdot))\|_{H^\gamma_p} \int_{|y| \leq 1} |y|^{-d-\alpha+1} dy
+ C \|u(e^n\cdot)\eta_n(e^n\cdot)\|_{H^\gamma_p} \int_{|y| > 1} |y|^{-d-\alpha} dy
\leq C \left( \|\Delta^{\alpha/4}((u\eta_n)(e^n\cdot))\|_{H^\gamma_p} + \|u(e^n\cdot)\eta_n(e^n\cdot)\|_{H^\gamma_p} \right). \]
On the other hand, by (2.1) and (2.13),
\[
|D^m_x \Delta^{\alpha/2}(\zeta_n(e^n\cdot))(x)| \leq C \|D^{m+2}_x \zeta_n(e^n\cdot)\|_{L^\infty} \int_{|y| \leq 1} |y|^{-d-\alpha+2} dy
+ C \|D^m_x \zeta_n(e^n\cdot)\|_{L^\infty} \int_{|y| > 1} |y|^{-d-\alpha} dy \leq C.
\]
Thus, again by (37) Lemma 5.2, we have
\[ \|u(e^n\cdot)\eta_n(e^n\cdot)\Delta^{\alpha/2}(\zeta_n(e^n\cdot))\|_{H^\gamma_p} \leq C\|u(e^n\cdot)\eta_n(e^n\cdot)\|_{H^\gamma_p}. \]
Combining (4.1), (4.3) and (4.4), we prove the lemma. \( \square \)

Lemma 4.2. Let \( d - 1 - \alpha p/2 < \theta < d - 1 + p + \alpha p/2 \). Then, for any \( \gamma \in \mathbb{R} \) and \( u \in C_{c}^{\infty}(D) \),
\[ \sum_{n \in \mathbb{Z}} e^{n(\theta - \alpha p/2)} \left\| \zeta_n(e^n\cdot)\Delta^{\alpha/2}((1 - \eta_n(e^n\cdot))u(e^n\cdot)) \right\|_{H^\gamma_p}^p \leq C \|\psi^{-\alpha/2}u\|_{L^p_{\psi, \theta}(D)}^p. \]
where \( C = C(d, p, \gamma, \alpha, \theta, D) \).
Proof. It is certainly enough to prove (4.3) for only \( \gamma = m \in \mathbb{N}_+ \).

By the choice of \( \{ \eta_n : n \in \mathbb{Z} \} \), we have \( \zeta_n(x) = \zeta_n(x) \eta_n(x) \) for all \( x \), and
\[
\zeta_{-n}(e^n x) (1 - \eta_n(e^n(x + y))) = 0\] if \( |y| < \delta_0 \),
where \( \delta_0 := 1 - e^{-1} \). Thus, by (4.1),
\[
F_n(x) := \zeta_{-n}(e^n x) \Delta^{\alpha/2} \left( [1 - \eta_n(e^n \cdot)] u(e^n \cdot) \right)(x)
= C \int_{|y| \geq \delta_0} u(e^n(x + y)) \zeta_{-n}(e^n x) [1 - \eta_n(e^n(x + y))] |y|^{-d - \alpha} dy
= C \int_{|x - y| \geq \delta_0} u(e^n y) \left( \zeta_{-n}(e^n x) [1 - \eta_n(e^n y)] |x - y|^{-d - \alpha} \right) dy.
\] (4.6)

Denote
\[ B_n := \text{supp}(\zeta_{-n}). \]

Then, since \( \zeta_{-n}(e^n x) (1 - \eta_n(e^n y)) = 0 \) for \( |x - y| < \delta_0 \), by (2.13), we have
\[
\left| D_x \left( \zeta_{-n}(e^n x) (1 - \eta_n(e^n y)) |x - y|^{-d - \alpha} \right) \right|
\leq C 1_{B_n} (e^n x) [1 - \eta_n(e^n y)] |x - y|^{-d - \alpha}
+ C |\zeta_{-n}(e^n x) (1 - \eta_n(e^n y)) |x - y|^{-d - \alpha - 1}
\leq C 1_{B_n} (e^n x) [1 - \eta_n(e^n y)] |x - y|^{-d - \alpha}.
\]

Similarly, for \( k \in \mathbb{N}_+ \),
\[
\left| D_x^k \left( \zeta_{-n}(e^n x) (1 - \eta_n(e^n y)) |x - y|^{-d - \alpha} \right) \right|
\leq C(k) 1_{B_n} (e^n x) [1 - \eta_n(e^n y)] |x - y|^{-d - \alpha}.
\]

It follows from (4.6) for each \( k \in \mathbb{N}_+ \),
\[
|D_x^k F_n(x)| \leq C(k) H_n(x),
\] (4.7)

where
\[
H_n(x) := 1_{B_n} (e^n x) \int_{|x - y| \geq \delta_0} |u(e^n y)| |1 - \eta_n(e^n y)| |x - y|^{-d - \alpha} dy.
\]

Since \( \| F_n \|_{H^m_p} \approx \sum_{k \leq m} \| D_x^k F_n \|_{L_p} \), from (4.7) we get
\[
\sum_{n \in \mathbb{Z}} e^n (\theta - \alpha p/2) \| \zeta_{-n}(e^n \cdot) \Delta^{\alpha/2} ([1 - \eta_n(e^n \cdot)] u(e^n \cdot)) \|_{H^m_p}^p
\leq C \sum_{n \in \mathbb{Z}} e^n (\theta - \alpha p/2) \| H_n \|_{L_p}^p.
\]

Therefore, to finish the proof of (4.3), we only need to show
\[
\sum_{n \in \mathbb{Z}} e^n (\theta - \alpha p/2) \| H_n \|_{L_p}^p \leq C \| \psi^{-\alpha/2} u \|_{L_{p,0}(D)}^p.
\] (4.8)
Case 1. Let \( d - 1 + \alpha p/2 < \theta < d - 1 + p + \alpha p/2 \). Observe that

\[
\int_{|y| \geq \delta_0} |u(e^n(x + y))(1 - \eta_n(e^n(x+y)))||y|^{-d-\alpha} \, dy \\
\leq \sum_{k=0}^{\infty} \int_{2^k \delta_0 \leq |y| < 2^{k+1} \delta_0} |u(e^n(x+y))||y|^{-d-\alpha} \, dy \\
= C(d) e^{n\alpha} \sum_{k=0}^{\infty} \int_{2^k \delta_0 \leq |y| < 2^{k+1} \delta_0} |u(e^n x + y)||y|^{-d-\alpha} \, dy \\
\leq C \sum_{k=0}^{\infty} 2^{-k\alpha} \frac{1}{e^{nd2d}} \int_{2^k \delta_0 \leq |y| < 2^{k+1} \delta_0} |u(e^n x + y)| \, dy \\
\leq C \sum_{k=0}^{\infty} 2^{-k\alpha} \mu(e^n x) = C \mu(e^n x),
\]

where \( \mu \) is the maximal function of \( u \) defined by

\[
\mu(x) = \sup_{x \in B_r(z) \setminus B_r(z)} \frac{1}{|B_r(z)|} \int_{B_r(z)} |u(y)| \, dy.
\]

Therefore, \( H_n(x) \leq C(1_{B_n} \mu)(e^n x) \). Since \( e^n \approx \rho \) on \( B_n \), by the change of variables,

\[
\sum_{n \in \mathbb{Z}} e^{n(\theta - \alpha p/2)} \|H_n\|_{L^p} \leq C \int_D \|\mu(x)\|^p \rho(x)^{\theta - d - \alpha p/2} \, dx.
\]

Due to \cite{20} Theorem 1.1, the function \( \rho^{\theta - \alpha p/2 - d} \) belongs to the class of Muckenhoupt \( A_p \)-weights, and therefore we can apply the Hardy-Littlewood Maximal inequality (\cite{23} Theorem 7.1.9) to get

\[
\sum_{n \in \mathbb{Z}} e^{n(\theta - \alpha p/2)} \|H_n\|_{L^p} \leq C \int_{\mathbb{R}^d} |u(x)|^p \rho(x)^{\theta - d - \alpha p/2} \, dx.
\]

This proves (4.8) if \( d - 1 + \alpha p/2 < \theta < d - 1 + p + \alpha p/2 \).

Case 2. Let \( d - 1 - \alpha p/2 < \theta < d + \alpha p/2 \). Then, we can choose \( \beta \in (0, \alpha) \) such that

\[-1 < \theta - d - \alpha p/2 + \beta p \leq 0.\]

By (4.6) and H\ölder’s inequality, for \( p' := p/(p-1) \),

\[
H_n(x) \leq 1_{B_n}(e^n x) \left( \int_{|x-y| \geq \delta_0} \frac{|u(e^n y)(1 - \eta_n(e^n y))|^p}{|x-y|^{d+\beta p}} \, dy \right)^{1/p} \\
\times \left( \int_{|x-y| \geq \delta_0} |x-y|^{-d-(\alpha-\beta)p'} \, dy \right)^{1/p'} \\
\leq C 1_{B_n}(e^n x) \left( \int_{|x-y| \geq \delta_0} \frac{|u(e^n y)|^p}{|x-y|^{d+\beta p}} \, dy \right)^{1/p} \leq C 1_{B_n}(e^n x) \left( \int_{|x-y| \geq \delta_0} \frac{|u(e^n y)|^p}{|x-y|^{d+\beta p}} \, dy \right)^{1/p}.
\]

(4.9)
By the change of variables and Fubini’s theorem,
\[
\sum_{n \in \mathbb{Z}} e^{n(\theta - \alpha p/2)} \|H_n\|_{L^p}^p \\
\leq \sum_{n \in \mathbb{Z}} e^{n(\theta - \alpha p/2)} \int_{\mathbb{R}^d} \int_{|x-y| \geq \delta_0} 1_{B_n}(e^n x) \frac{|u(e^n y)|^p}{|x-y|^{d+\beta p}} dy dx \\
= \int_{\mathbb{R}^d} \left( \sum_{n \in \mathbb{Z}} e^{n(\theta - d - \alpha p/2+\beta p)} \right) \int_{|x-y| \geq \delta_0} 1_{B_n}(x)|x-y|^{-d-\beta p} dx \left| |u(y)|^p dy. \right. \tag{4.10}
\]

In the virtue of (4.9) and (4.10), to prove (4.8), it suffices to show that for $y \in D$,
\[
\sum_{n \in \mathbb{Z}} e^{n(\theta - d - \alpha p/2+\beta p)} \int_{|x-y| \geq \delta_0} 1_{B_n}(x)|x-y|^{-d-\beta p} dx \leq Cd_y^{\theta-d-\alpha p/2}. \tag{4.11}
\]

For fixed $y \in D$, we take $n_0 = n_0(y) \in \mathbb{Z}$ such that
\[
e^{n_0+3} \leq d_y < e^{n_0+4}.
\]

If $n \leq n_0$ and $x \in B_n$, then $e^n < d_x < e^{n_0+2} < e^{n_0+3} \leq d_y$, and consequently $|x-y| \geq d_y - d_x \geq C e^{n_0}$. Thus,
\[
\sum_{n \leq n_0} e^{n(\theta - d - \alpha p/2+\beta p)} \int_{|x-y| \geq C e^n} 1_{B_n}(x)|x-y|^{-d-\beta p} dx \\
\leq C \int_{|x-y| \geq C e^n} \sum_{n \leq n_0} 1_{B_n}(x) d_x^{\theta-d-\alpha p/2+\beta p} |x-y|^{d+\beta p} dx \\
\leq C \int_{|x-y| \geq C e^n, d_x \leq d_y} |x-y|^{d+\beta p} dx \\
\leq C e^{-n_0 \beta} d_y^{\theta-d-\alpha p/2+\beta p} \leq C d_y^{\theta-d-\alpha p/2}, \tag{4.12}
\]

where $C$ is independent of $y$. For the second inequality above, we used $\sum_{n \leq n_0} 1_{B_n}(x) \leq C 1_{d_x \leq d_y}$, and for the third inequality, we used Lemma [A.4(ii)] with $\rho = Ce^{n_0}$ and $r = d_y$.

Next, we handle the summation for $n > n_0$. Since $\theta - \alpha p/2 - d < 0$,
\[
\sum_{n > n_0} e^{n(\theta - d - \alpha p/2+\beta p)} \int_{|x-y| \geq \delta_0 e^n} 1_{B_n}(x)|x-y|^{-d-\beta p} dx \\
\leq C \sum_{n > n_0} e^{n(\theta - d - \alpha p/2+\beta p)} \int_{|x-y| \geq \delta_0 e^n} |x-y|^{-d-\beta p} dx \\
= C \sum_{n > n_0} e^{n(\theta - d - \alpha p/2)} = Ce^{n_0(\theta-d-\alpha p/2)} \leq C d_y^{\theta-d-\alpha p/2}. \tag{4.13}
\]

Combining (4.12) and (4.13), we obtain (4.11). Thus, (4.8) and the lemma are proved. \qed
Lemma 4.3. Let \( d - 1 - \alpha p/2 < \theta < d - 1 + p + \alpha p/2 \) and \( \gamma \in \mathbb{R} \). Then, for any \( u \in C_c^\infty(D) \),
\[
\sum_{n \in \mathbb{Z}} e^{n(\theta-\alpha p/2)} \left\| \Delta^{\alpha/2} \left( u(e^n) \zeta_n(e^n) \right) - \zeta_n(e^n) \Delta^{\alpha/2} (u(e^n)) \right\|^p_{H^{\psi\alpha}(\gamma+\alpha/2)(D)},
\]
\[
\leq C \| \psi^{-\alpha/2} u \|^p_{H^{\psi\alpha}(\gamma+\alpha/2)(D)},
\]
where \( C = C(d, p, \alpha, \theta, \gamma, D) \).

Proof. Recall \( \eta_n \zeta_n = \zeta_n \). Thus, by the triangle inequality,
\[
\left\| \Delta^{\alpha/2} \left( u(e^n) \zeta_n(e^n) \right) - \zeta_n(e^n) \Delta^{\alpha/2} (u(e^n)) \right\|^p_{H^{\psi\alpha}(\gamma+\alpha/2)(D)} \leq C \| \psi^{-\alpha/2} u \|^p_{H^{\psi\alpha}(\gamma+\alpha/2)(D)},
\]
where \( C = C(d, p, \alpha, \theta, \gamma, D) \).

Lemma 4.4. Let \( d - 1 - \alpha p/2 < \theta < d - 1 + p + \alpha p/2 \), and \( \gamma \geq -\alpha \). Then, for any \( u \in C_c^\infty(D) \), we have \( \Delta^{\alpha/2} u \in \psi^{-\alpha/2} H^{\gamma, \theta}(D) \) and
\[
\| \psi^{\alpha/2} \Delta^{\alpha/2} u \|_{H^{\gamma, \theta}(D)} \leq C \| \psi^{-\alpha/2} u \|_{H^{\psi\alpha}(\gamma+\alpha/2)(D)},
\]
where \( C = C(d, p, \alpha, \theta, \gamma, D) \).

Proof. By Lemma 2.14(ii) and the relation \( \Delta^{\alpha/2} u(e^n x) = e^{-n\alpha} \Delta^{\alpha/2} u(e^n \cdot) \),
\[
\| \psi^{\alpha/2} \Delta^{\alpha/2} u \|_{H^{\gamma, \theta}(D)} \leq C \sum_{n} e^{n(\theta-\alpha p/2)} \| \zeta_n(e^n) \Delta^{\alpha/2} (u(e^n)) \|^p_{H^{\psi\alpha}(\gamma+\alpha/2)(D)}.
\]
By (4.1), the last term above is bounded by
\[
C \sum_{n} e^{n(\theta-\alpha p/2)} \left\| \Delta^{\alpha/2} \left( u(e^n) \zeta_n(e^n) \right) \right\|^p_{H^{\psi\alpha}(\gamma+\alpha/2)(D)} \leq C \| \psi^{-\alpha/2} u \|^p_{H^{\psi\alpha}(\gamma+\alpha/2)(D)}.
\]
The lemma is proved.

By Lemma 2.14 for any \( \gamma_0 \in \mathbb{R} \) and \( \phi \in C_c^\infty(D) \), \( \Delta^{\alpha/2} \phi \) belongs to the dual space of \( H^{\gamma_0+\alpha, \theta-\alpha p/2}(D) \) (see Lemma 2.7(iii)). Therefore, for \( u \in \psi^{\alpha/2} H^{\gamma_0+\alpha}(D) \), we can define \( \Delta^{\alpha/2} u \) as a distribution on \( D \) by
\[
(\Delta^{\alpha/2} u, \phi)_D := (u, \Delta^{\alpha/2} \phi)_D, \quad \phi \in C_c^\infty(D).
\]
Corollary 4.5. Let \( d - 1 - \alpha p/2 < \theta < d - 1 + p + \alpha p/2 \).

(i) Let \( \gamma \in \mathbb{R} \), \( u \in \psi^{\alpha/2} H^{\gamma, \theta}(D) \), and \( \Delta^{\alpha/2} u \) be defined as in (4.16). Then, \( \Delta^{\alpha/2} u \in \psi^{-\alpha/2} H^{\gamma, \theta}(D) \), and (4.15) holds.

(ii) If \( \gamma \geq -\alpha/2 \) and \( u \in H^{\gamma, \theta}(D) \), then the left-hand side of (4.14) makes sense, and inequality (4.13) holds.
Proof: If $\gamma \geq 0$, then (i) is a consequence of Lemma 4.3 and Lemma 2.7(i). If $\gamma < 0$, then by Lemmas 4.4 and 2.7(iii),
\[
\|\Delta^\alpha/2 u, \phi\|_D \leq C\|\psi^{-\alpha/2} u\|_{H^{\gamma+\alpha}_p(D)} \|\psi^{\alpha/2} \Delta^\alpha/2 \phi\|_{H^{\gamma-\alpha}_p(D)}
\]
\[
\leq C\|\psi^{-\alpha/2} u\|_{H^{\gamma+\alpha}_p(D)} \|\psi^{-\alpha/2} \phi\|_{H^{\gamma-\alpha}_p(D)},
\]
where $1/p + 1/p' = 1$ and $\theta/p + \theta'/p' = d$. This implies $\Delta^\alpha/2$ is a bounded linear operator from $\psi^{\alpha/2} H^{\gamma+\alpha}_p(D)$ to $\psi^{-\alpha/2} H^{\gamma-\alpha}_p(D)$. Thus, (i) is proved.

Next, we show (ii). The left-hand side of (4.14) makes sense due to (i) and (2.15). Now, the claim of (ii) follows from Lemma 3.3 and Lemma 2.7(i). The corollary is proved. \hfill $\square$

**Theorem 4.6** (Higher regularity for Parabolic equation). Let $0 \leq \mu \leq \gamma$, and $\theta \in (d - 1 - \frac{\alpha p}{2}, d - 1 + p + \frac{\alpha p}{2})$. Suppose that $f \in \psi^{-\alpha/2} H^\gamma_p(D,T)$, $u_0 \in \psi^{\alpha/2-\alpha/p} B^\gamma_p(D,T)$, and $u \in \psi^{\alpha/2} H^\gamma_p(D,T) \cap \{u = 0 \text{ on } [0,T] \times D^c\}$ is a weak solution to (2.2). Then, $u \in \psi^{\alpha/2} H^{\gamma_0}_p(D,T)$, and for this solution
\[
\|\psi^{-\alpha/2} u\|_{H^{\gamma_0}_p(D,T)} \leq C\left(\|\psi^{-\alpha/2 + \alpha/p} u_0\|_{B^\gamma_0_p(D)} + \|\psi^{\alpha/2} f\|_{H^{\gamma_0}_p(D,T)} + \|\psi^{-\alpha/2} u\|_{H^{\gamma_0}_p(D,T)}\right),
\] (4.17)
where $C = C(d, p, \alpha, \gamma, \mu, \theta, D)$.

**Proof.** 1. We first note that it is enough to consider the case $\gamma \leq \mu + \alpha/2$. Indeed, if the claim holds for the case $\gamma \leq \mu + \alpha/2$, then repeating the result with $\mu' = \mu + \alpha/2, \mu + 2\alpha/2, \cdots$ in order, we prove the lemma when $\gamma = \mu + k\alpha/2, k \in \mathbb{N}_+$. Now let $\gamma = \mu + k\alpha/2 + c$, where $k \in \mathbb{N}_+$ and $c \in (0, \alpha/2)$. Then, applying the previous result with $\mu' = \mu + k\alpha/2$, we prove the general case.

2. For each $n \in \mathbb{Z}$, denote
\[
\begin{align*}
u_n(t, x) &:= u(e^{\alpha t}, e^nx), & f_n(t, x) &:= f(e^{\alpha t}, e^nx), & u_{0n}(t, x) &:= u_0(e^nx).
\end{align*}
\]
Then, $u_n(\cdot)\zeta_n(e^n \cdot) \in H^\gamma_p(e^{-\alpha n}T)$ and it is a weak solution (or solution in the sense of distribution) to the equation
\[
\begin{align*}
\partial_t v_n(t, x) &= \Delta^\alpha/2 v_n(t, x) + F_n(t, x), & t, x &\in (0, e^{-\alpha n}T) \times \mathbb{R}^d \\
v_n(0, x) &= (u_{0n}(\cdot)\zeta_n(e^n \cdot))(x), & x &\in \mathbb{R}^d
\end{align*}
\]
where
\[
F_n(t, x) = e^{\alpha n} (f_n(\cdot, \cdot)\zeta_n(e^n \cdot))(t, x)
\]
\[
- \left(\Delta^\alpha/2 (u_n(\cdot, \cdot)\zeta_n(e^n \cdot))(t, x) - \zeta_n(e^n x)\Delta^\alpha/2 u_n(t, x)\right)
\]
\[
=: e^{\alpha n} (f_n(\cdot, \cdot)\zeta_n(e^n \cdot))(t, x) - G_n(t, x).
\]

3. By Corollary 4.3(ii) with $\gamma' = \mu - \alpha/2$, we have
\[
\begin{align*}
\sum_{n \in \mathbb{Z}} e^{\theta(\alpha - \alpha p/2)} \|G_n(e^{-\alpha n} \cdot, \cdot)\|_{H^\gamma_p(D)} &\leq C \sum_{n \in \mathbb{Z}} e^{\theta(\alpha - \alpha p/2)} \|G_n(e^{-\alpha n} \cdot, \cdot)\|_{H^{\gamma_0}_p(D)} \\
&\leq C \|\psi^{-\alpha/2} u(t, \cdot)\|_{H^{\gamma_0}_p(D)}.
\end{align*}
\] (4.18)
Therefore, due to \( f \in \psi^{-\alpha/2}H_{p,\theta}^{\gamma-\alpha}(D,T) \),
\[
F_n \in \psi^{-\alpha}(e^{-\alpha_n}T).
\]
Thus, we apply [43, Theorem 1] to conclude \( u_n\zeta_{-n}(e^{\alpha_n}) \in H_p^{\gamma}(e^{-\alpha_n}T) \), and
\[
\|\Delta^{\alpha/2}(u(\cdot, e^{\alpha_n})\zeta_{-n}(e^{\alpha_n}))\|^{p}_{H_p^{\gamma}(T)} = e^{\alpha_n}\|\Delta^{\alpha/2}(u(\cdot, e^{\alpha_n})\zeta_{-n}(e^{\alpha_n}))\|^{p}_{H_p^{\gamma-\alpha}(e^{-\alpha_n}T)} \\
\leq Ce^{\alpha_n}\|\zeta_{-n}(e^{\alpha_n})u_{0n}(\cdot)\|^{p}_{B_{p,\theta}^{\gamma-\alpha/2}} + Ce^{\alpha_n}\|G_n(\cdot, \cdot)\|^{p}_{H_p^{\gamma-\alpha}(e^{-\alpha_n}T)} \\
+ C\epsilon_{\alpha_n}\|\zeta_{-n}(e^{\alpha_n})u_{0n}(\cdot)\|^{p}_{B_{p,\theta}^{\gamma-\alpha/2/p}} + C\|G_n(\cdot, \cdot)\|^{p}_{H_p^{\gamma-\alpha}(T)} \\
+ C\epsilon_{\alpha_n}\zeta_{-n}(e^{\alpha_n})f(\cdot, e^{\alpha_n})\|^{p}_{H_p^{\gamma-\alpha}(T)}. \tag{4.19}
\]
By (4.18) and (4.19) (also see Lemma 2.7(ii)),
\[
\sum_{n \in \mathbb{Z}}e^{\alpha_n(\theta - \alpha_p/2)}\|\Delta^{\alpha/2}(u(\cdot, e^{\alpha_n})\zeta_{-n}(e^{\alpha_n}))\|^{p}_{H_p^{\gamma-\alpha}(T)} \\
\leq C\left(\|\psi^{-\alpha/2}\|^{p}_{B_{p,\theta}^{\gamma-\alpha/2/p}(D)} + \|\psi^{-\alpha/2}f\|^{p}_{H_p^{\gamma-\alpha}(D,T)} + \|\psi^{-\alpha/2}u\|^{p}_{H_p^{\gamma-\alpha}(D,T)}\right). \tag{4.20}
\]
Therefore, (4.20), Lemma 2.7(ii), and the relation
\[
\|u\|_{H_p^\gamma} \approx \left(\|u\|_{H_p^{\gamma-\alpha}} + \|\Delta^{\alpha/2}u\|_{H_p^{\gamma-\alpha}}\right)
\]
yield (4.14) for \( \gamma \leq \mu + \alpha/2 \). The theorem is proved. \( \square \)

**Theorem 4.7** (Higher regularity for elliptic equation). Let \( \lambda \geq 0 \), \( 0 \leq \mu \leq \gamma \), and \( \theta \in (d - 1 - \alpha_p, d - 1 + p + \alpha_p) \). Suppose that \( f \in \psi^{-\alpha/2}H_{p,\theta}^{\gamma-\alpha}(D) \), and \( u \in \psi^{\alpha/2}H_{p,\theta}^{\mu}(D) \cap \{u = 0 \text{ on } D^c\} \) is a solution to (2.4), then, \( u \in \psi^{\alpha/2}H_{p,\theta}^{\gamma}(D) \), and moreover
\[
\lambda\|\psi^{\alpha/2}u\|_{H_p^{\gamma-\alpha}(D)} + \|\psi^{-\alpha/2}u\|_{H_p^{\gamma-\alpha}(D)} \\
\leq C\left(\|\psi^{\alpha/2}f\|_{H_p^{\gamma-\alpha}(D)} + \|\psi^{-\alpha/2}u\|_{H_p^{\gamma-\alpha}(D)}\right),
\]
where \( C = C(d, p, \alpha, \gamma, \mu, \theta, D) \). In particular, \( C \) is independent of \( \lambda \).

**Proof.** We repeat the argument of the proof of Theorem 4.6. As before, we may assume \( \gamma \leq \mu + \alpha/2 \).

Let \( n \in \mathbb{Z} \). Since \( u \) is a weak solution to (2.4), \( u_n(x) := u(e^n x) \) and \( f_n(x) := f(e^n x) \) satisfy the following equation in weak sense;
\[
\Delta^{\alpha/2}(u_n(\cdot)\zeta_{-n}(e^{\alpha_n}))(x) - e^{\alpha_n}\lambda(u_n(\cdot)\zeta_{-n}(e^{\alpha_n}))(x) = F_n(x), \quad x \in \mathbb{R}^d, \tag{4.21}
\]
where
\[
F_n(x) = e^{\alpha_n}f_n(x)\zeta_{-n}(e^{\alpha_n}x) - G_n(x) \\
:= e^{\alpha_n}f_n(x)\zeta_{-n}(e^{\alpha_n}x) - \left(\Delta^{\alpha/2}(u_n(\cdot)\zeta_{-n}(e^{\alpha_n}))(x) - \zeta_{-n}(e^{\alpha_n}x)\Delta^{\alpha/2}u_n(x)\right).
\]
By Corollary 4.3 (ii), we get \( G_n \in H^{\gamma - \alpha}_p \) and
\[
\sum_{n \in \mathbb{Z}} e^{\alpha(\gamma - \alpha)p/2} \|G_n\|^{p}_{H^{\gamma - \alpha}_p} \leq C \sum_{n \in \mathbb{Z}} e^{\alpha(\gamma - \alpha)p/2} \|G_n\|^{p}_{H^{\gamma - \alpha}_p} \leq C \|\psi^{-\alpha/2} u\|^{p}_{H^{\gamma - \alpha}_{p,\theta}(D)}.
\] (4.22)

This implies that \( F_n \in H^{\gamma - \alpha}_p \).

If \( \lambda = 0 \), then the equality (4.21) easily yields
\[
\|\Delta^{\alpha/2}(u_n \cdot \zeta_n(\varepsilon^{n.}))\|^{p}_{H^{\gamma - \alpha}_p} = \|F_n\|^{p}_{H^{\gamma - \alpha}_p} \leq \|e^{\alpha p} f_n \cdot \zeta_n(\varepsilon^{n.})\|^{p}_{H^{\gamma - \alpha}_p} + \|G_n\|^{p}_{H^{\gamma - \alpha}_p}. \] (4.23)

Next, let \( \lambda > 0 \). Then, by [14, Theorem 1] (or [18, Theorem 2.1]), we have \( u_n \cdot \zeta_n(\varepsilon^{n.}) \in H^{\gamma - \alpha}_p \) and
\[
e^{\alpha p} f_n \cdot \zeta_n(\varepsilon^{n.})\|^{p}_{H^{\gamma - \alpha}_p} \leq C \left( \|e^{\alpha p} f_n \cdot \zeta_n(\varepsilon^{n.})\|^{p}_{H^{\gamma - \alpha}_p} + \|G_n\|^{p}_{H^{\gamma - \alpha}_p} \right). \] (4.24)

We multiply by \( e^{\alpha p (\gamma - \alpha)/2} \) to (4.23) and (4.24), then take sum over \( n \in \mathbb{Z} \). Finally, we use (4.22) and Lemma 2.7 (ii) to finish the proof of the theorem.

\[ \square \]

5. PROOF OF THEOREMS 2.2, 2.3, 2.9 AND 2.11

We only need to prove Theorems 2.2 and 2.3. This is because Theorem 2.10 is a consequence of Theorems 2.2 and 4.6 and Theorem 2.11 is a consequence of Theorems 2.3 and 4.7.

**Proof of Theorem 2.2**

1. Existence and estimate of solution.

First, assume \( u_0 \in C^\infty_c(D) \) and \( f \in C^\infty_c((0, T) \times D) \). Then, by Lemma 3.2, the function \( u \) defined in (2.6) becomes a weak solution to (2.2). Also, by Lemmas 3.6 and 3.7
\[
\|\psi^{-\alpha/2} u\|_{L^p(D)} \leq C \left( \|\psi^{-\alpha/2 + \alpha/p} u_0\|_{L^p(D)} + \|\psi^{-\alpha/2} f\|_{L^p(D)} \right). \]

Now we fix \( \gamma \in (0, \alpha/p) \). By (4.18), we have \( L_{p, \theta'}(D) \subset B^\gamma_{p, \theta'}(D) \) for any \( \theta' \in \mathbb{R} \), and therefore applying Theorem 4.6 with \( \mu = 0 \), we conclude \( u \in \psi^{\alpha/2} \mathbb{R}^\gamma_{p, \theta'}(D, T) \) and
\[
\|\psi^{-\alpha/2} u\|_{H^\gamma_{p, \theta'}(D)} \leq C \left( \|\psi^{-\alpha/2 + \alpha/p} u_0\|_{L^p(D)} + \|\psi^{-\alpha/2} f\|_{L^p(D)} \right). \]

Using this and Corollary 4.3 (ii), we have \( u_1 = \Delta^{\alpha/2} u + f \in \psi^{-\alpha/2} \mathbb{R}^\gamma_{p, \theta'}(D, T) \), \( u \in \mathcal{D}^\gamma_{p, \theta'}(D, T) \), and
\[
\|u\|_{\mathcal{D}^\gamma_{p, \theta'}(D, T)} \leq C \left( \|\psi^{-\alpha/2 + \alpha/p} u_0\|_{L^p(D)} + \|\psi^{-\alpha/2} f\|_{L^p(D)} \right). \] (5.1)

For general data, we take \( \{u_{0n}\}_{n \in \mathbb{N}} \subset C^\infty_c(D) \) and \( \{f_n\}_{n \in \mathbb{N}} \subset C^\infty_c((0, T) \times D) \) such that
\[
u_{0n} \to u_0 \quad \text{in} \quad \psi^{\alpha/2 - \alpha/p} B^\gamma_{p, \theta'}(D),
\]
\[
f_n \to f \quad \text{in} \quad \psi^{-\alpha/2} \mathbb{R}^\gamma_{p, \theta'}(D, T).
\]
Define $u_n$ (resp. $u$) by (2.0) with $u_{0n}$ (resp. $u_0$) and $f_n$ (resp. $f$). Then, by Lemmas 3.6 and 3.7, $u_n$ converges to $u$ in the space $L_{p,\theta - \alpha/2}(D, T)$. Also, considering the estimate (5.1) corresponding to $u_n - u_m$, we conclude $u_n$ is a Cauchy sequence in $\mathcal{H}_{p,\theta}^\gamma(D, T)$. Let $v$ denote the limit of $u_n$ in $\mathcal{H}_{p,\theta}^\gamma(D, T)$. Then, $v = u$ (a.e.) and therefore $u$ (or its version) is in $\mathcal{H}_{p,\theta}^\gamma(D, T)$.

Now we prove that (2.3) holds for all $t \leq T$. Since $u_n$ is a solution to (2.2) in the sense of Definition 2.1 taking $n \to \infty$ and using

$$\|\psi^{-\alpha/2}(u_n - u)\|_{\mathcal{H}_{p,\theta}^\gamma(D, T)} + \|\psi^{\alpha/2}(\Delta^{\alpha/2}u_n - \Delta^{\alpha/2}u)\|_{\mathcal{H}_{p,\theta}^{\gamma - \alpha/2}(D, T)} \to 0,$$

we find that (2.3) holds for $u$ almost everywhere on $[0, T]$. By Theorem 2.12, we know that $(u(t) - u_0, \phi)_D$ is a continuous in $t$, and therefore we conclude that (2.3) holds for all $t \leq T$. Thus, $u$ becomes a weak solution. (2.4) also follows from the estimates of $u_n$.

2. Uniqueness.

Let $u \in \psi^{-\alpha/2}L_{p,\theta}(D, T) \cap \{u = 0 \text{ on } [0, T] \times D^c\}$ be a weak solution to

$$\begin{cases}
\partial_t u(t, x) = \Delta^{\alpha/2}u(t, x), & (t, x) \in (0, T) \times D, \\
u(0, x) = 0, & x \in D, \\
u(t, x) = 0, & (t, x) \in [0, T] \times D^c.
\end{cases}$$

Then, by Theorem 4.6 with $\mu = 0$ and $\gamma > 0$, we have $u \in \psi^{-\alpha/2}\mathcal{H}_{p,\theta}^\gamma(D, T)$ for any $\gamma > 0$. This and Corollary 4.9(i) imply $u \in \mathcal{H}_{p,\theta}^{\gamma + \alpha}(D, T)$ for any $\gamma \in \mathbb{R}$.

Now we take a sequence $u_n \in C_c^\infty([0, T] \times D)$ (cf. Remark 2.8(ii)) such that $u_n \to u$ in $\mathcal{H}_{p,\theta}^{\gamma + \alpha}(D, T)$. In particular, $u_n \to u$, $u_n(0, \cdot) \to 0$, and $\partial_t u_n \to \partial_t u$ in their corresponding spaces. Define $f_n := \partial_t u_n - \Delta^{\alpha/2}u_n$, then $u_n$ trivially satisfies

$$\partial_t u_n - \Delta^{\alpha/2}u_n = f_n.$$

By Lemma 3.2(ii),

$$u_n(t, x) = E_x[u_n(0, X_t^D)] + \int_0^t E_x[f_n(s, X_{t-s}^D)]ds.$$

Also, by Lemmas 3.6 and 3.7 and Theorem 4.6

$$\|\psi^{-\alpha/2}u_n\|_{\mathcal{H}_{p,\theta}^{\gamma + \alpha}(D, T)} \leq C \left(\|\psi^{\alpha/p - \alpha/2}u_n(0, \cdot)\|_{B_{p,\theta}^{\gamma + \alpha - \alpha/p}(D)} + \|\psi^{\alpha/2}f_n\|_{\mathcal{H}_{p,\theta}^{\gamma}(D, T)}\right).$$

By Corollary 3.5(i), we have $\Delta^{\alpha/2}u_n \to \Delta^{\alpha/2}u$ in $\psi^{-\alpha/2}\mathcal{H}_{p,\theta}^\gamma(D, T)$, and therefore

$$f_n = \partial_t u_n - \Delta^{\alpha/2}u_n \to \partial_t u - \Delta^{\alpha/2}u = 0$$

as $n \to \infty$ in the space $\psi^{-\alpha/2}\mathcal{H}_{p,\theta}^\gamma(D, T)$. From (5.2), we conclude that $u = 0$. The uniqueness is also proved.

**Proof of Theorem 2.3**

If $\lambda > 0$ or $D$ is bounded, then it is enough to repeat the proof of Theorem 2.2. During the proof, one only needs to replace results for parabolic equations by their corresponding elliptic versions.

Therefore, we only consider the case when $\lambda = 0$ and $D$ is a half space.

1. A priori estimate and uniqueness.
We first prove the a priori estimate
\[ \| \psi^{-\alpha/2} u \|_{H^\alpha_p(D)} \leq C \| \psi^{\alpha/2} f \|_{L_p,0(D)} \]  
(5.3)
holds given that \( u \in L_{p,0-\alpha p/2}(D) \cap \{ u = 0 \text{ on } D^c \} \) is a weak solution to (2.4).

Note that by Theorem 4.7 we have \( u \in \psi^{\alpha/2} H^\alpha_{p,0}(D) \). Assume \( u \in C^\infty_c(D) \) for a moment. Then, for any \( \lambda > 0 \),
\[ \Delta^{\alpha/2} u - \lambda u = f - \lambda u \text{ on } D, \]
where \( f := \Delta^{\alpha/2} u \). Thus, applying (2.22) for \( \lambda > 0 \) and letting \( \lambda \downarrow 0 \), we get estimate (5.3) for \( \lambda = 0 \). For general case, we take \( u_n \in C^\infty_c(D) \) such that \( u_n \rightarrow u \) in \( \psi^{\alpha/2} H^\alpha_{p,0}(D) \). Then by Corollary 4.5(i), \( \Delta^{\alpha/2} u_n \rightarrow \Delta^{\alpha/2} u = f \) in \( \psi^{-\alpha/2} L_{p,0}(D) \).

Consequently, this leads to (5.3), which certainly implies
\[ \| \psi^{-\alpha/2} u \|_{L_{p,0}(D)} \leq C \| \psi^{\alpha/2} f \|_{L_{p,0}(D)}. \]
The uniqueness result of solution easily follows from this.

2. Weak convergence and existence.

Let \( u_n \in \psi^{\alpha/2} L_{p,0}(D) \cap \{ u = 0 \text{ on } D^c \} \) denote the solution to equation (2.4) corresponding to \( \lambda = \frac{1}{n} \). Then, by (2.22), \( \{ u_n \} \) is a bounded sequence in the space \( L_p(\mathbb{R}^d, \rho^{\beta-d-\alpha p/2} dx) \), and therefore there exists a subsequence \( \{ u_{n_i} \} \) which converges weakly to some \( u \in L_p(\mathbb{R}^d, \rho^{\beta-d-\alpha p/2} dx) \). Obviously, we have \( u = 0 \) (a.e.) on \( D^c \). By Lemma 1.4, for any \( \phi \in C^\infty_c(D) \), \( \Delta^{\alpha/2} \phi \) belongs to the dual space of \( \psi^{\alpha/2} H^\alpha_{p,0}(D) \). Therefore,
\[ (u_{n_i}, \phi)_{\mathbb{R}^d} = (u_{n_i}, \phi)_D \rightarrow (u, \phi)_D = (u, \phi)_{\mathbb{R}^d} \]
and
\[ (u_{n_i}, \Delta^{\alpha/2} \phi)_{\mathbb{R}^d} = (u_{n_i}, \Delta^{\alpha/2} \phi)_D \rightarrow (u, \Delta^{\alpha/2} \phi)_D = (u, \Delta^{\alpha/2} \phi)_{\mathbb{R}^d}, \]
as \( n_i \rightarrow \infty \). Thus, we conclude \( u \) is a weak solution to (2.4) in \( L_{p,0-\alpha p/2}(D) \cap \{ u = 0 \text{ on } D^c \} \). Now we prove the weak convergence. The above argument shows that any subsequence of \( u_n \) has a further subsequence which converges weakly in \( L_p(\mathbb{R}^d, \rho^{\beta-d-\alpha p/2} dx) \), and the limit becomes a solution to (2.4) in \( L_{p,0-\alpha p/2}(D) \cap \{ u = 0 \text{ on } D^c \} \). Due to the uniqueness of solution proved above, we conclude that this limit coincides with \( u \). This proves the weak convergence, and the theorem is proved.

APPENDIX A. Auxiliary results

Recall that \( p(t,x) = p_d(t,x) \) is the transition density function of a rotationally symmetric \( \alpha \)-stable \( d \)-dimensional Lévy process. It is a radial function and
\[ p_d(t,x) \approx t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x|^{d+\alpha}} \approx \frac{t}{(t^{1/\alpha} + |x|)^{d+\alpha}}, \]  
(A.1)
and
\[ p_d(t,x) = t^{-\frac{d}{\alpha}} p_d(1,t^{-1/\alpha} x). \]  
(A.2)

If \( f \) is a radial function, then we put \( f(r) := f(x) \) if \( r = |x| \).

Lemma A.1. (i) Let \( d \geq 2 \) and \( f \) be a nonnegative radial function on \( \mathbb{R}^d \). Then, for \( x_1 \neq 0 \),
\[ \int_{\mathbb{R}^{d-1}} f(x_1,x')dx' = C(d)|x_1|^{d-1} \int_0^\infty f(|x_1|(1 + s^2)^{1/2})s^{d-2}ds. \]  
(A.3)
(ii) Let \( d \geq 2 \). For any \( t > 0 \) and \( x^1 \neq 0 \),
\[
\int_{\mathbb{R}^{d-1}} p_d(t, x^1, x') \, dx' \approx p_1(t, x^1),
\]  
where the comparability relation depends only on \( d \) and \( \alpha \).

Proof. (i) By the change of variables,
\[
\int_{\mathbb{R}^{d-1}} f(x^1, x') \, dx' = \int_{\mathbb{R}^{d-1}} f(x^1, |x'|) |x^1|^{d-1} \, dx'
\]
\[
= |x^1|^{d-1} \int_{\mathbb{R}^{d-1}} f(|x^1|(1 + |x'|^2)^{1/2}) \, dx'
\]
\[
= C(d) |x^1|^{d-1} \int_0^\infty f(|x^1|(1 + s^2)^{1/2}) s^{d-2} \, ds.
\]

(ii) By \((A.1)\) and \((A.3)\), it suffices to prove that
\[
\int_0^\infty \frac{t^d |x^1|^{d-1} s^{d-2}}{(t^{1/\alpha} + |x^1|(1 + s^2)^{1/2})^{d+\alpha}} \, ds \approx \left( t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x^1|^{1+\alpha}} \right).
\]  
(A.5)

Let \( t|x^1|^{-\alpha} \leq 1 \), then
\[
\int_0^\infty \frac{t^d |x^1|^{d-1} s^{d-2}}{(t^{1/\alpha} + |x^1|(1 + s^2)^{1/2})^{d+\alpha}} \, ds \approx \int_0^\infty \frac{t^d |x^1|^{d-1} s^{d-2}}{|x^1|(1 + s^2)^{1/2}} \, ds
\]
\[
= C(d, \alpha) \frac{t}{|x^1|^{1+\alpha}}.
\]  
(A.6)

Now let \( t|x^1|^{-\alpha} \geq 1 \). We put
\[
\int_0^\infty \frac{t^d |x^1|^{d-1} s^{d-2}}{(t^{1/\alpha} + |x^1|(1 + s^2)^{1/2})^{d+\alpha}} \, ds = \int_0^{t^{1/\alpha}|x^1|^{-1}} \cdots + \int_0^\infty \int_0^{t^{1/\alpha}|x^1|^{-1}} \cdots =: I + II.
\]

Then,
\[
I \leq t^{-\frac{d}{\alpha}} |x^1|^{d-1} \int_0^{t^{1/\alpha}|x^1|^{-1}} s^{d-2} \, ds = C(d, \alpha) t^{-\frac{d}{\alpha}},
\]
\[
II \leq t|x^1|^{-1-\alpha} \int_0^{t^{1/\alpha}|x^1|^{-1}} \frac{s^{d-2}}{(1 + s^2)^{1/2}} \, ds = C(\alpha) t^{-\frac{d}{\alpha}}.
\]

Therefore, the left-hand side of \((A.5)\) is controlled by the right-hand side. Due to \((A.6)\), to prove \((A.5)\), we only need a proper lower bound of \( I \). Let \( t|x^1|^{-\alpha} \geq 1 \). By the changing variables \( s = t^{1/\alpha} |x^1|^{-1} l \),
\[
I \geq \int_0^{t^{1/\alpha}|x^1|^{-1}} \frac{t^d |x^1|^{d-1} s^{d-2}}{(t^{1/\alpha} + (t^{2/\alpha} + |x^1|^{-2} s^2)^{1/2})^{d+\alpha}} \, ds
\]
\[
= t^{-\frac{d}{\alpha}} \int_0^1 \frac{t^{d-2}}{(1 + (1 + t^2)^{1/2})^{d+\alpha}} \, dl = C(d, \alpha) t^{-\frac{d}{\alpha}}.
\]

The lemma is proved. \qed

Lemma A.2. Let \( \alpha \in (0, 2) \) and \( \gamma_0, \gamma_1 \in \mathbb{R} \). Suppose that
\[
-\frac{2}{\alpha} < \gamma_0, \quad -2 < \gamma_1 - \gamma_0 \leq 2 + \frac{2}{\alpha},
\]  
(A.7)
Then, for any \((t, x) \in (0, \infty) \times \mathbb{R}^d\),
\[
\int_{\mathbb{R}^d} p(t, x - y) \frac{|y|^{\gamma_0/2}}{(\sqrt{t} + |y|^{\alpha/2})_{\gamma_1}} dy \leq C(\sqrt{t} + |x|^{\alpha/2})^{\gamma_0 - \gamma_1}. \tag{A.8}
\]
where \(C = C(d, \alpha, \gamma_0, \gamma_1)\).

**Proof.** It suffices to prove (A.8) when \(t = 1\). Indeed, if it holds for \(t = 1\), then by (A.2),
\[
\int_{\mathbb{R}^d} p(t, x - y) \frac{|y|^{\gamma_0/2}}{(\sqrt{t} + |y|^{\alpha/2})_{\gamma_1}} dy
\]
\[
= C t^{\frac{\gamma_0 - \gamma_1}{\alpha}} \int_{\mathbb{R}^d} p(1, t^{-\frac{1}{2}} x - y) \frac{|y|^{\gamma_0/2}}{(1 + |y|^{\alpha/2})_{\gamma_1}} dy
\]
\[
\leq C t^{\frac{\gamma_0 - \gamma_1}{\alpha}} (1 + t^{-\frac{1}{2}} |x|^{\alpha/2})^{\gamma_0 - \gamma_1}
\]
\[
= C(\sqrt{t} + |x|^{\alpha/2})^{\gamma_0 - \gamma_1}.
\]
Thus, we may assume \(t = 1\). By (A.4) and (A.1),
\[
\int_{\mathbb{R}^d} p_d(1, x - y) \frac{|y|^{\gamma_0/2}}{(1 + |y|^{\alpha/2})_{\gamma_1}} dy
\]
\[
\approx \int_{\mathbb{R}} \left(1 + \frac{1}{|x - y|^{1+\alpha}}\right) \frac{|y|^{\gamma_0/2}}{(1 + |y|^{\alpha/2})_{\gamma_1}} dy =: I(x^1).
\]
Thus, it only remains to show for \(x^1 \in \mathbb{R}\),
\[
I(x^1) \leq C(1 + |x^1|^{\alpha/2})^{\gamma_0 - \gamma_1}. \tag{A.9}
\]

**Case 1.** Let \(|x^1| \leq 1\). Put
\[
I(x^1) = \int_{|y^1| \leq 2} \cdots dy^1 + \int_{|y^1| > 2} \cdots dy^1 =: I_1(x^1) + I_2(x^1).
\]
If \(|y^1| \leq 2\), then by (A.7),
\[
I_1(x^1) \leq C \int_{|y^1| \leq 2} |y^1|^{\gamma_0/2} dy^1 = C.
\]
If \(|y^1| > 2\), then \(|x^1 - y^1| \geq |y^1|/2\). Thus, by (A.7),
\[
I_2(x^1) \leq C \int_{|y^1| > 2} \frac{1}{|x^1 - y^1|^{1+\alpha}} \left(\frac{|y^1|^{\alpha/2}}{1 + |y^1|^{\alpha/2}}\right)^{\gamma_1} |y^1|^{\alpha(\gamma_0 - \gamma_1)/2} dy^1
\]
\[
\leq C \int_{|y^1| > 2} |y^1|^{\alpha(\gamma_0 - \gamma_1)/2 - \frac{\alpha}{2}} dy^1 = C.
\]
Therefore, \(I\) is bounded and (A.9) is proved for \(|x^1| \leq 1\).

**Case 2.** Let \(|x^1| > 1\). Put
\[
I(x^1) = \int_{|y^1| \geq 2} \cdots + \int_{|y^1| / 2 < |y^1| \leq 2|x^1|} \cdots + \int_{1/2 < |y^1| \leq |x^1| / 2} \cdots + \int_{|y^1| \leq 1/2} \cdots
\]
\[
=: J_1(x^1) + J_2(x^1) + J_3(x^1) + J_4(x^1).
\]
First, we estimate \(J_1\). Note that if \(r > 1\), then
\[
\frac{1}{2} \leq \frac{r^{\alpha/2}}{1 + r^{\alpha/2}} \leq 1. \tag{A.10}
\]
Combining this with (A.11), (A.12) and (A.13), we prove (A.9) for $|x| > 1$. The lemma is proved. □

Therefore, by (A.7) and (A.10),
\[
J_2(x^1) \leq C(1 + |x^1|^\alpha/2)^\gamma_0 - \gamma_1 \int_R p_1(1, x^1 - y^1) dy^1 = C(1 + |x^1|^\alpha/2)^\gamma_0 - \gamma_1. \tag{A.12}
\]

Next, we estimate $J_3$. If $1/2 \leq |y^1| \leq |x^1|/2$, then
\[
\frac{1}{3} \leq \frac{|y^1|^\alpha/2}{1 + |y^1|^\alpha/2} \leq 1, \quad |x^1 - y^1| \geq \frac{|x^1|}{2}.
\]
Hence, by (A.7) and (A.10),
\[
J_3(x^1) \leq \int_{1/2 \leq |y^1| \leq |x^1|/2} \frac{1}{|x^1 - y^1|^{1+\alpha}} \left( \frac{|y^1|^\alpha/2}{1 + |y^1|^\alpha/2} \right)^\gamma_1 |y^1|^{\alpha/2} \frac{dy^1}{x^1 - y^1} \leq C|x^1|^{-1-\alpha} \int_{1/2 \leq |y^1| \leq |x^1|/2} |y^1|^{\alpha/2} \frac{dy^1}{x^1 - y^1} \leq C|x^1|^{-1-\alpha} \int_{1/2 \leq |y^1| \leq |x^1|/2} |y^1|^{\alpha/2} \frac{dy^1}{x^1 - y^1} \leq C|x^1|^{\alpha/2} \frac{dy^1}{x^1 - y^1} \leq C(1 + |x^1|^\alpha/2)^\gamma_0 - \gamma_1. \tag{A.13}
\]

Lastly, we estimate $J_4$. If $|y^1| \leq 1/2$, then
\[
\frac{2}{3} \leq \frac{1}{1 + |y^1|^\alpha/2} \leq 1, \quad |x^1 - y^1| \geq \frac{|x^1|}{2}.
\]
Therefore, by (A.7) and (A.10),
\[
J_4(x^1) \leq \int_{|y^1| \leq 1/2} \frac{1}{|x^1 - y^1|^{1+\alpha}} \left( \frac{|y^1|^\alpha/2}{1 + |y^1|^\alpha/2} \right)^\gamma_1 |y^1|^{\alpha/2} dy^1 \leq C|x^1|^{-1-\alpha} \int_{|y^1| \leq 1} |y^1|^{\alpha/2} dy^1 \leq C|x^1|^{-1-\alpha} \leq C(1 + |x^1|^\alpha/2)^{-2/\alpha - 2} \leq C(1 + |x^1|^\alpha/2)^\gamma_0 - \gamma_1.
\]

Combining this with (A.11), (A.12) and (A.13), we prove (A.9) for $|x| > 1$. The lemma is proved. □
Lemma A.3. Let [A.7] hold for $\gamma_0, \gamma_1 \in \mathbb{R}$. Then, for $(t, x) \in (0, \infty) \times \mathbb{R}^d$,
\[
\int_D p(t, x - y) \frac{d_y^{\alpha \gamma_0/2}}{(\sqrt{t} + d_y^{\alpha/2})^{\gamma_1}} dy \leq C(\sqrt{t} + d_x^{\alpha/2})^{\gamma_0 - \gamma_1},
\]
where $C$ depends only on $d, \alpha, \gamma_0, \gamma_1$ and $D$.

Proof. Note that it is enough to assume $D$ is bounded. This is because if $D$ is a half space, the result follows from Lemma [A.2]

For $R > 0$, denote $D_R := \{x \in D : d_x \geq R\}$. Since $D$ is bounded, one can find $x_1, \ldots, x_n \in \partial D$ such that
\[
D \subset \left( \bigcup_{i=1}^n (D \cap B_{R/3}(x_i)) \right) \cup D_{R/6}.
\]
Therefore,
\[
\int_D p(t, x - y) \frac{d_y^{\alpha \gamma_0/2}}{(\sqrt{t} + d_y^{\alpha/2})^{\gamma_1}} dy \\
\leq \sum_{i=1}^n \int_{D \cap B_{R/3}(x_i)} p(t, x - y) \frac{d_y^{\alpha \gamma_0/2}}{(\sqrt{t} + d_y^{\alpha/2})^{\gamma_1}} dy \\
+ \int_{D_{R/6}} p(t, x - y) \frac{d_y^{\alpha \gamma_0/2}}{(\sqrt{t} + d_y^{\alpha/2})^{\gamma_1}} dy \\
= \sum_{k=1}^n I_k(t, x) + II(t, x).
\]

1. We estimate $I_k(t, x)$ for fixed $k \in \{1, 2, \ldots, n\}$.

First, assume $x \in B_R(x_k) \cap D$, then (by reducing $R$ if necessary) we can consider $C^{1,1}$-bijective (flattening boundary) map $\Phi = (\Phi^1, \ldots, \Phi^d)$ defined on $B_R(x_k)$ such that $\Phi(B_R(x_k) \cap D) \subset \mathbb{R}^d_+$ and $d_z \approx \Phi^1(z)$ on $B_R(x_k) \cap D$. Then one can easily handle $I_k$ using Lemma [A.2].

Second, assume $x \in D \setminus B_R(x_k)$. Since $r \to p(t, r)$ is nonincreasing, for any $y, z \in B_{R/3}(x_k)$, we have $|z - y| \leq 2R/3 < |x - y|$, which implies
\[
p(t, x - y) \leq p(t, z - y).
\]
If $\gamma_1 - \gamma_0 \geq 0$, choosing $z \in B_{R/3}(x_k) \cap D$ such that $d_z \leq C(D, R)d_x$ and using the result for the first case,
\[
I_k(t, x) \leq \int_{D \cap B_{R/3}(x_k)} p(t, z - y) \frac{d_y^{\alpha \gamma_0/2}}{(\sqrt{t} + d_y^{\alpha/2})^{\gamma_1}} dy \\
\leq C(\sqrt{t} + d_x^{\alpha/2})^{\gamma_0 - \gamma_1} \leq C(\sqrt{t} + d_x^{\alpha/2})^{\gamma_0 - \gamma_1}. \quad (A.14)
\]
If $\gamma_1 - \gamma_0 < 0$, by taking $z \in B_{R/3}(x_k) \cap D$ such that $d_z \leq d_x$, we also have $I_k(t, x)$.

2. We estimate $II(t, x)$.

We first consider the case $x \in D_{R/12}$. For $y \in D_{R/6}$, we have $d_x \approx d_y \approx 1$

\[
\left(\frac{\sqrt{t} + d_y^{\alpha/2}}{\sqrt{t} + d_x^{\alpha/2}}\right)^{\gamma_0 - \gamma_1} \leq C(diam(D), \gamma_0, \gamma_1, R, \alpha),
\]

\[
\leq C(\sqrt{t} + d_x^{\alpha/2})^{\gamma_0 - \gamma_1}.
\]
Using this, we get
\[ II \leq C(\sqrt{t} + d_x^{\alpha/2})^{\gamma_0 - \gamma_1} \int_{D_{R/6}} p(t,x-y) \left( \frac{d_y^{\alpha/2}}{\sqrt{t} + d_y^{\alpha/2}} \right)^{\gamma_0} \, dy. \]

Also, since \( d_y \approx 1 \) on \( y \in D_{R/6} \), it suffices to show that
\[ \int_{D_{R/6}} p(t,x-y) \left( \frac{1}{\sqrt{t} + 1} \right)^{\gamma_0} \, dy \leq C. \] (A.15)

Since (A.15) is obvious if \( t \leq 1 \) or \( \gamma_0 \geq 0 \). If \( t > 1 \) and \( \gamma_0 < 0 \), then by (A.1),
\[ \int_{D_{R/6}} p(t,x-y) \left( \frac{1}{\sqrt{t} + 1} \right)^{\gamma_0} \, dy \leq C \int_{D} t^{-d/\alpha - \gamma_0/2} \, dy \leq C. \]

Therefore, (A.15) is proved.

Next, we consider the case \( x \in D \setminus D_{R/6} \). Since \( d_y \approx 1 \), we have
\[ \frac{d_y^{\alpha/2}}{(\sqrt{t} + d_y^{\alpha/2})^{\gamma_1}} \approx \frac{1}{(\sqrt{t} + 1)^{\gamma_1}}. \]

Also note that \( |x-y| > R/12 \) for \( y \in D_{R/6} \). Thus, by (A.1),
\[ II \leq C_{t<1} \int_{|x-y| \geq R/12} \frac{t}{|x-y|^{d+\alpha}} \, dy + C_{t \geq 1} t^{-d/\alpha - \gamma_1/2} \leq C_{t<1} + C_{t \geq 1} t^{-\gamma_0/2 - d/\alpha} t^{(\gamma_0 - \gamma_1)/2} \leq C_{t<1} + C_{t \geq 1} t^{(\gamma_0 - \gamma_1)/2}. \]

Thus if \( \gamma_0 \geq \gamma_1 \), then by (A.17),
\[ II \leq C t^{(\gamma_0 - \gamma_1)/2} \leq C(\sqrt{t} + d_x^{\alpha/2})^{\gamma_0 - \gamma_1}. \]

Now let \( \gamma_0 < \gamma_1 \). Then \( 1 t<1 (\sqrt{t} + d_x^{\alpha/2}) \) is bounded above and \( t \approx (t + d_x^{\alpha/2}) \) if \( t > 1 \), we get
\[ II \leq C_{t<1} + C_{t \geq 1} t^{(\gamma_0 - \gamma_1)/2} \leq C(\sqrt{t} + d_x^{\alpha/2})^{\gamma_0 - \gamma_1} \]
provided that \( \gamma_0 < \gamma_1 \). The lemma is proved. \( \Box \)

Next, we provide some results for the distance function \( d_x \).

**Lemma A.4.** Let \( D \) be a half space or a bounded \( C^{1,1} \) open set.

(i) Let \( x_0 \in \partial D \) and \( r > 0 \). Then, for any \( \lambda > -1 \),
\[ \int_{B_r(x_0)} d_x^\lambda \, dx \leq C(d, \lambda, D) r^\lambda. \] (A.16)

(ii) Let \( y \in D \), \( r, \rho, \kappa_1 > 0 \) and \( -1 < \kappa_0 \leq 0 \). Suppose that \( r \leq c \rho \) for some \( c > 0 \).
Then, there exists a constant \( C = C(d, \kappa_1, \kappa_0, c, D) \) such that
\[ \int_{D_\rho(y) \cap D^r} \frac{d_x^{\kappa_0}}{|x-y|^{d+\kappa_1}} \, dx \leq C \rho^{-\kappa_1 + \kappa_0}, \]
where \( D_\rho(y) := \{ x \in D : |x-y| > \rho \} \) and \( D^r := \{ x \in D : d_x \leq r \} \).
Proof. (i) The result is trivial if $D$ is a half space. If $D$ is a bounded $C^{1,1}$ open set, then $\partial D$ is a $(d - 1)$-dimensional compact Lipschitz manifold. Thus, we have \([A.16]\) due to e.g. page 16 of [2].

(ii) Let $D$ be a half space.

Assume first $d \geq 2$. By the change of variables and Fubini’s theorem,

\[
\int_{|x-y| > \rho, |x| \leq r} \frac{|x|^\kappa_0}{|x-y|^{d+\kappa_1}} \, dx
= \int_{|x^1+y^1| \leq r} \frac{|x^1+y^1|^{\kappa_0}}{|x^1|^{|x^1-y^1|>(1+s^2)^{-1/2}\rho}1_{|x^1| \leq r}} \, dx^1 \, dx^1
= C \int_{|x^1+y^1| \leq r} \frac{|x^1|^\kappa_0}{|x^1|^{|x^1-y^1|>(1+s^2)^{-1/2}\rho}1_{|x^1| \leq r}} \, dx^1 \int_0^\infty \frac{s^{d-2}}{(1+s^2)(d+\kappa_1)/2} \, ds \, dx^1
= C \int_0^\infty \frac{s^{d-2}}{(1+s^2)(d+\kappa_1)/2} I(\rho, s, y^1, r) \, ds, \tag{A.17}
\]

where

\[
I(\rho, s, y^1, r) := \int_R \frac{|x|^\kappa_0}{|x-y^1|^{1+\kappa_1}} 1_{|x-y^1|>(1+s^2)^{-1/2}\rho}1_{|x^1| \leq r} \, dx^1.
\]

Take $p_0 = p_0(\kappa_0) > 1$ satisfying $-1 < p_0\kappa_0 \leq 0 < \kappa_1$, by Hölder’s inequality,

\[
I(\rho, s, y^1, r)
\leq \left( \int \frac{|x^1|^{\kappa_0}}{|x^1-y^1|^{1+\kappa_1}} \, dx^1 \right)^{1/p_0} \left( \int \frac{|x^1|^{-p_0f(\kappa_0)}1_{|x^1| > (1+s^2)^{-1/2}\rho}1_{|x^1| \leq r}} \, dx^1 \right)^{1/p_0'}
\leq C \rho^{\kappa_0 + \frac{1}{p_0'} \kappa_1} \frac{1}{r^\kappa_0} (1 + s^2)^\frac{(1+\kappa_1-1/p_0')}{2}. \tag{A.18}
\]

Combining \(A.17\) and \(A.18\), we have

\[
\int_{|x-y| > \rho, |x| \leq r} \frac{|x|^\kappa_0}{|x-y|^{d+\kappa_1}} \, dx
\leq C \rho^{-1-\kappa_1+\frac{1}{p_0'} \kappa_0 + \frac{1}{p_0}} \int_0^\infty \frac{s^{d-2}}{(1+s^2)(d+1/p_0')/2} \, ds
= C \rho^{-1-\kappa_1+\frac{1}{p_0'} \kappa_0 + \frac{1}{p_0}} \leq C \rho^{-\kappa_1+\kappa_0}.
\]

For $d = 1$, using \(A.18\), we get

\[
\int_{|x-y| > \rho, |x| \leq r} \frac{|x|^\kappa_0}{|x-y|^{1+\kappa_1}} \, dx = I(\rho, 0, y, r)
\leq C \rho^{\kappa_0 + \frac{1}{p_0'} \kappa_1} \frac{1}{r^\kappa_0} \leq C \rho^{-\kappa_1+\kappa_0}.
\]

2. Let $D$ be a bounded open set. We take $x_1, \ldots, x_n \in \partial D$ such that

\[
D^r \subset \bigcup_{i=1}^n B_{2r}(x_i).
\]
Therefore, by (i),
\[ \int_{D_r(y) \cap D_r'} \frac{d\rho_i}{|x-y|^{d+\kappa_i}} \, dx \leq \sum_{i=1}^n \int_{D_r(y) \cap B_{2r}(x_i)} \frac{d\rho_i}{|x-y|^{d+\kappa_i}} \, dx \leq C \rho^{-d-\kappa_i} r^{d+\kappa_0} \leq C \rho^{-\kappa_i} r^\kappa_0. \]

The lemma is proved. \( \square \)

We write \( u \in \mathcal{H}_p^{\gamma+\alpha}(T) \) if \( u \in \mathcal{H}_p^{\gamma+\alpha}(T) \), \( u(0,\cdot) \in B_{p}^{\gamma+\alpha-\alpha/p} \) and there exists \( f \in \mathcal{H}_p^{\gamma}(T) \) such that for any \( \phi \in C_c^\infty(\mathbb{R}^d) \),
\[
(u(t,\cdot),\phi)_{\mathbb{R}^d} = (u(0,\cdot),\phi)_{\mathbb{R}^d} + \int_0^t (f(s,\cdot),\phi)_{\mathbb{R}^d} \, ds, \quad \forall t \leq T.
\]

In this case, we write \( f = u_t \). The norm in \( \mathcal{H}_p^{\gamma+\alpha}(T) \) is defined as
\[
\|u\|_{\mathcal{H}_p^{\gamma+\alpha}(T)} := \|u\|_{\mathcal{H}_p^{\gamma+\alpha}(T)} + \|u_t\|_{\mathcal{H}_p^{\gamma}(T)} + \|u(0,\cdot)\|_{B_{p}^{\gamma+\alpha-\alpha/p}}.
\]

**Lemma A.5.** Let \( p \in (1, \infty) \), \( \alpha \in (0, 2) \), \( \gamma \in \mathbb{R} \) and \( 1/p < \nu \leq 1 \). For \( a > 0 \), \( 0 \leq s \leq t \leq T \) and \( u \in \mathcal{H}_p^{\gamma+\alpha}(T) \),
\[
\|u(t) - u(s)\|_{\mathcal{H}_p^{\gamma+\alpha-a}} \leq C |t-s|^\nu - 1/p a^{\nu-1} \left( a\|u\|_{\mathcal{H}_p^{\gamma+\alpha}(T)} + a^{-1}\|u_t\|_{\mathcal{H}_p^{\gamma}(T)} \right), \tag{A.19}
\]
where \( C = C(\alpha, p, \nu) \). In particular, \( C \) is independent of \( T \) and \( a \).

**Proof.** One can prove the lemma by following the proof of [39] Theorem 7.3, which treats the case \( a = 2 \). First, we note that due to the isometry \( (1-\Delta)^{\gamma/2} : \mathcal{H}_p^{\gamma} \to \mathcal{H}_p^{\gamma-\sigma} \), we only need to prove for any particular \( \gamma \in \mathbb{R} \), and therefore we assume \( \gamma = \nu \alpha - \alpha \). Second, since \( C_c^\infty([0, T] \times \mathbb{R}^d) \) is dense in \( \mathcal{H}_p^{\gamma+\alpha}(T) \), we may further assume \( u \in C_c^\infty([0, T] \times \mathbb{R}^d) \). Third, due to the scaling argument used at the beginning of the proof of [39] Theorem 7.3, it is enough to consider the case \( a = T = 1 \).

Finally, to prove \( (A.19) \) for the case \( a = T = 1 \), we just need to repeat the proof of [37] Theorem 7.2 word for word. Although [37] Theorem 7.2 handles the case \( a = 2 \), its proof works also for \( a \in (0, 2) \) thanks to [27] Lemma A.2. The lemma is proved. \( \square \)

**Declaration of interest**

Declarations of interest: none

**References**

[1] B. Abdellaoui, A. J. Fernández, T. Leonori, A. Younes, Global fractional Calderón-Zygmund regularity (2021), arXiv preprint, arXiv:2107.06535.
[2] H. Aikawa, Quasiadditivity of Riesz capacity, Math. Scand. 69 (1991), no.1, 15-30.
[3] A. Arapostathis, A. Biswas, L. Caffarelli, The Dirichlet problem for stable-like operators and related probabilistic representations, Commun. Partial Differ. Equ. 41 (2016), no.9, 1472-1511.
[4] P. Auscher, S. Bortz, M. Egert, O. Saari, Nonlocal self-improving properties: a functional analytic approach, Tunis. J. Math. 1 (2019), no.2, 151-183.
[5] J. Bae, M. Kassmann, Schauder estimates in generalized Hölder spaces (2015), arXiv preprint arXiv:1505.05498.
[6] B. Baeumer, T. Luks, M.M. Meerschaert, Space-time fractional Dirichlet problems. Math. Nachr. 291 (2018), no.17-18, 2516-2535.
[7] R.F. Bass, Regularity results for stable-like operators, J. Funct. Anal. 257 (2009), no.8, 2693-2722.
[8] U. Biccari, M. Warma, E. Zuazua, Local elliptic regularity for the Dirichlet fractional Laplacian. Adv. Nonlinear Stud. 17 (2017), no.2, 387-409.
[9] U. Biccari, M. Warma, E. Zuazua, Local Regularity for fractional heat equation in Recent advances in PDEs: analysis, numerics and control, SEMA SIMAI Springer Ser., Vol.17 (2018), Springer, Cham, 233–249.
[10] K. Bogdan, T. Grzywny, M. Ryznar, Dirichlet heat kernel for unimodal Lévy processes. Stoch. Process. Appl. 124 (2014), no.11, 3612-3650.
[11] K. Bogdan, T. Grzywny, K. Pietruska-Paluba, A. Rutkowski, Extension and trace for nonlocal operators, J. Math. Pures Appl. 137 (2020), 33-69.
[12] B. Böttcher, R.L. Schilling, J. Wang, Lévy-type processes: construction, approximation and sample path properties, Lecture Notes in Mathematics Vol. 2099 (vol. III of the “Lévy Matters” subseries), Springer, 2013.
[13] L. Caffarelli, L. Silvestre, Regularity theory for fully nonlinear integro-differential equations, Commun. Pure Appl. Math., 62 (2009), no.5, 597-638.
[14] Z.Q. Chen, P. Kim, R. Song, Heat kernel estimates for the Dirichlet fractional Laplacian, J. Eur. Math. Soc. 12 (2010), no.9, 1307-1329.
[15] K.L. Chung, Doubly-Feller process with multiplicative functional in Seminar on stochastic processes, 1985, Progr. Probab. Statist. Vol. 12, Birkhäuser Boston, 1986, 63-78.
[16] K.L. Chung, Z. Zhao, From Brownian motion to Schrödinger’s equation, A Series of Comprehensive Studies in Mathematics Vol. 312, Springer, 2012
[17] M. Cozzi, Interior regularity of solutions of nonlocal equations in Sobolev and Nikol’skii spaces, Ann. Mat. Pura Appl. 196 (2017), no.2, 555-578.
[18] H. Dong, D. Kim, On $L^p$-estimates for a class of non-local elliptic equations, J. Funct. Anal. 262 (2012), no.3, 1166-1199.
[19] H. Dong, D. Kim, Schauder estimates for a class of non-local elliptic equations, Discret. Contin. Dyn. Syst. 34 (2014), no.6, 2319-2347.
[20] B. Dyda, L. Ihnatsyeva, J. Lehrbäck, H. Tuominen, A.V. Vähäkangas, Muckenhoupt $A_p$-properties of Distance Functions and Applications to Hardy–Sobolev-type Inequalities. Potential Anal. 50 (2019), no.1, 83-105.
[21] M. Felsinger, M. Kassmann, P. Voigt, The Dirichlet problem for nonlocal operators, Math. Z. 279 (2015), no.3-4, 779-809.
[22] X. Fernández-Real, X. Ros-Oton, Regularity theory for general stable operators: parabolic equations, J. Funct. Anal. 272 (2017), no.10, 4165-4221.
[23] L. Grafakos, Classical Fourier Analysis, 3rd ed., Graduate Texts in Mathematics Vol. 249, Springer, New York, 2014.
[24] D. Gilbarg, N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
[25] G. Grubb, Local and nonlocal boundary conditions for μ-transmission and fractional elliptic pseudodifferential operators, Anal. PDE 7 (2014), no.7, 1649–1682.
[26] G. Grubb, Regularity in $L_p$ Sobolev spaces of solutions to fractional heat equations, J. Funct. Anal. 274 (2018), no.9, 2634-2660.
[27] B.S. Han, A regularity theory for stochastic partial differential equations driven by multiplicative space-time white noise with the random fractional Laplacians, Stoch. Partial Differ. Equ. Anal. Comput. 9 (2021), no.4, 940-983.
[28] W. Hoh, N. Jacob, On the Dirichlet problem for pseudodifferential operators generating Feller semigroups, J. Funct. Anal. 137 (1996), no.1, 19-48.
[29] I. Kim, K.H. Kim, P. Kim, Parabolic Littlewood-Paley inequality for $\phi(-\Delta)$-type operators and applications to stochastic integro-differential equations. Adv. Math. 249 (2013), 161-203.
[30] I. Kim, K.H. Kim, A Hölder regularity theory for a class of non-local elliptic equations related to subordinate Brownian motions, Potential Anal. 43 (2015), no.4, 653-673.
[31] I. Kim, K.H. Kim, An $L_p$-theory for a class of non-local elliptic equations related to nonsymmetric measurable kernels, J. Math. Anal. Appl. 434 (2016), no.2, 1302-1335.
[32] I. Kim, K.H. Kim, P. Kim, An $L_p$-theory for diffusion equations related to stochastic processes with non-stationary independent increment, *Trans. Am. Math. Soc.* **371** (2019), no.5, 3417-3450.

[33] K.H. Kim, N.V. Krylov, On the Sobolev space theory of parabolic and elliptic equations in $C^1$ domains, *SIAM J. Math. Anal.* **36** (2004), no.2, 618-642.

[34] K.H. Kim, D. Park, J. Ryu, An $L_p(L_p)$-theory for diffusion equations with space-time nonlocal operators, *J. Differ. Equ.* **287** (2021), 376-427.

[35] M. Kim, P. Kim, J. Lee, K.A. Lee, Boundary regularity for nonlocal operators with kernels of variable orders, *J. Funct. Anal.* **277** (2019), no.1, 279-332.

[36] M. Kim, K.A. Lee, Generalized Evans–Krylov and Schauder type estimates for nonlocal fully nonlinear equations with rough kernels of variable orders, *J. Differ. Equ.* **270** (2021), 883-915.

[37] N.V. Krylov, An analytic approach to SPDEs in *Stochastic Partial Differential Equations: Six perspectives*, Mathematical Surveys and Monographs Vol. 64, American Mathematical Society, Providence, 1999, 185-242.

[38] N.V. Krylov, Weighted Sobolev spaces and Laplace’s equation and the heat equations in a half space, *Commun. Partial Differ. Equ.* **24** (1999), no.9-10, 1611-1653.

[39] N.V. Krylov, Some properties of traces for stochastic and deterministic parabolic weighted Sobolev spaces, *J. Funct. Anal.* **183** (2001), no.1, 1-41.

[40] F. Kühn, Schauder estimates for equations associated with Lévy generators, *Integral Equ. Oper. Theory* **91** (2019), no.2, 1-21.

[41] F. Kühn, Interior Schauder estimates for elliptic equations associated with Lévy operators, *Potential Anal.* **56** (2022), no.3, 459-481.

[42] T. Leonori, I. Peral, A. Primo, F. Soria, Basic estimates for solutions of a class of nonlocal elliptic and parabolic equations, *Discret. Contin. Dyn. Syst.* **35** (2015), no.12, 6031-6068.

[43] S.V. Lototsky, Sobolev spaces with weights in domains and boundary value problems for degenerate elliptic equations, *Methods. Appl. Anal.* **7** (2000), no.1, 195-204.

[44] R. Mikulevičius, C. Phonsom, On $L^p$—theory for parabolic and elliptic integro-differential equations with scalable operators in the whole space, *Stoch. Partial Differ. Equ. Anal. Comput.* **5** (2017), no.4, 472–519.

[45] R. Mikulevičius, C. Phonsom, On the Cauchy problem for integro-differential equations in the scale of spaces of generalized smoothness, *Potential Anal.* **50** (2019), no.3, 467-519.

[46] S. Nowak, $H^{s,p}$ regularity theory for a class of nonlocal elliptic equations, *Nonlinear Anal.* **195** (2020), Article 111730.

[47] X. Ros-Oton, J. Serra, The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, *J. Math. Pures Appl.* **101** (2014), no.3, 275-302.

[48] X. Ros-Oton, J. Serra, Boundary regularity for fully nonlinear integro-differential equations, *Duke Math. J.* **165** (2016), no.11, 2079-2154.

[49] X. Ros-Oton, J. Serra, Regularity theory for general stable operators, *J. Differ. Equ.* **260** (2016), no.12, 8675-8715.

[50] X. Ros-Oton, E. Valdinoci, The Dirichlet problem for nonlocal operators with singular kernels: convex and nonconvex domains, *Adv. Math.* **288** (2016), 732-790.

[51] H. Triebel, *Theory of function spaces*, Modern Birkhäuser Classics, Birkhäuser/Springer Basel AG, Basel, 2010.

[52] X. Zhang, $L^p$-solvability of nonlocal parabolic equations with spatial dependent and non-smooth kernels (2012), arXiv preprint arXiv:1206.2709

[53] X. Zhang, G. Zhao, Dirichlet problem for supercritical nonlocal operators (2018), arXiv preprint arXiv:1809.05712

**Department of Mathematics, Korea University, 145 Anam-ro, Seongbuk-gu, Seoul, 02841, Republic of Korea**

*Email address: choi.jh1223@korea.ac.kr*

**Department of Mathematics, Korea University, 145 Anam-ro, Seongbuk-gu, Seoul, 02841, Republic of Korea**

*Email address: kyeonghun@korea.ac.kr*

**Department of Mathematics, Korea University, 145 Anam-ro, Seongbuk-gu, Seoul, 02841, Republic of Korea**

*Email address: junhryu@korea.ac.kr*