FERMAT HYPERSURFACES AND SUBCANONICAL CURVES

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Abstract. We show that any Fermat hypersurface of degree $s + 2$ is apolar to a $s$-subcanonical $(s + 2)$-gonal projectively normal curve, and vice versa.

Moreover, we extend the classical Enriques-Petri Theorem to $s$-subcanonical projectively normal curves, proving that such a curve is $(s + 2)$-gonal if and only if it is contained in a rational normal surface.

1. Introduction

A nice result by Macaulay [Mac94] is that an Artinian graded Gorenstein ring of socle dimension 1 and degree $s + 2$ can be realised as $A = \frac{\mathbb{C}[\partial_0, \ldots, \partial_N]}{F^\perp}$, where $F \in \mathbb{C}[x_0, \ldots, x_N]$ is a homogeneous polynomial of degree $s + 2$ and $F^\perp := \{ D \in \mathbb{C}[\partial_0, \ldots, \partial_N] | D(F) = 0 \}$, where $\mathbb{C}[\partial_0, \ldots, \partial_N]$ is the polynomial ring generated by the natural derivations over $\mathbb{C}[x_0, \ldots, x_N]$.

In this paper, we characterise those Artinian graded Gorenstein ring of socle degree $s + 2$ such that $F$ is a Fermat hypersurface.

To state our main results, we consider an embedding $j : C \to X \subset \mathbb{P}^{N+2}$ of a smooth projective curve $C$ such that the homogeneous coordinate ring $S_X$ of its image $X$ is integrally closed, that is we assume that $X$ is projectively normal. We say that $X$ is subcanonical if the dualising sheaf $\omega_X$ is isomorphic to $j^*\mathcal{O}_{\mathbb{P}^{N+2}}(s)$, where $s \in \mathbb{Z}$. The ring $S_X$ for projectively normal subcanonical curves works perfectly to produce Artinian graded Gorenstein ring of socle dimension 1. In fact it is not hard to show that, letting $\mathbb{P}^{N+2} := \text{Proj}(\mathbb{C}[\partial_0, \ldots, \partial_{N+2}])$, and fixing $\eta_1, \eta_2$ two general linear forms in $\mathbb{C}[\partial_0, \ldots, \partial_{N+2}]$, that can be assumed to be $\eta_1 = \partial_{N+1}$ and $\eta_2 = \partial_{N+2}$, then the ring $A := \frac{S_X}{(\eta_1, \eta_2)}$ is an Artinian graded Gorenstein ring of socle dimension 1. Hence, by the above mentioned Macaulay’s result, $A$ can be realised as $A = A^F = \frac{\mathbb{C}[\partial_0, \ldots, \partial_N]}{F^\perp}$ where $F \in \mathbb{C}[x_0, \ldots, x_N]$ is a homogeneous polynomial.

Hence the reader can see immediately that, given a projectively normal, subcanonical curve $X \subset \mathbb{P}^{N+2}$, it remains defined a rational map:

\begin{equation}
\alpha_X : \text{Gr}(N, N+2) \dashrightarrow \mathbb{H}_{N,s}
\end{equation}

where $\text{Gr}(N, N+2)$ is the Grassmannian of $N$-planes in $\mathbb{P}^{N+2}$ and $\mathbb{H}_{N,s}$ is the space of homogeneous polynomials of degree $s + 2$ in $\mathbb{P}^N$ modulo the action of $\text{PGL}(N, \mathbb{C})$, by the map $\langle \eta_1, \eta_2 \rangle \mapsto [F_{\eta_1, \eta_2}] := [F]$.

In [DZ], we have shown that a curve $C$ of genus $g \geq 3$ is either trigonal or isomorphic to a smooth plane quintic if and only if for every couple of general sections $\eta_1, \eta_2 \in H^0(C, \omega_C)$, $F_{\eta_1, \eta_2}$ is a Fermat cubic. This characterisation of Fermat hypercubics was obtained using the canonical embedding: i.e. we considered the case $s = 1$. Motivated by this result of [DZ], we say that a form $F$ is apolar to a subcanonical curve $X$ if there exist two sections $\eta_1, \eta_2$ of the $s$-subcanonical system such that $F = F_{\eta_1, \eta_2}$.

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In this paper, see Theorem 19 we prove:

**Theorem 1.** Let \((C, \mathcal{L})\) be a polarised curve, such that \(C \subset |\mathcal{L}| =: \mathbb{P}^N\) is a \(s\)-subcanonical projectively normal curve. Then \(C\) is \((s + 2)\)-gonal if and only if for every couple of general sections \(\eta_1, \eta_2 \in H^0(C, \mathcal{L})\), \(F_{\eta_1, \eta_2}\) is a Fermat \((s + 2)\)-tic.

and moreover, again see Theorem 19:

**Theorem 2.** Let \((C, \mathcal{L})\) be a polarised curve, such that \(C \subset |\mathcal{L}| =: \mathbb{P}^N\) is a \(s\)-subcanonical projectively normal curve. Then \(C\) is \((s + 2)\)-gonal if and only if \(C\) is contained in a rational normal surface \(S_{a_1, a_2}\) with \(a_2 \leq a_1 \leq (s + 1)a_2 + 2\) and \(N := a_1 + a_2 + 1\).

The above theorems fit in a long path started by classical algebraic geometers. In fact, Theorem 2 could be seen as a natural generalisation of the trigonal case of the Enriques-Petri Theorem (see [Enr19] for the original work of Enriques, completed by [Bab39], and [Pet23]; modern versions are in [Sok71] and [SD73]) to the case of \(s\)-subcanonical curves.

Instead Theorem 1, see the proof of Theorem 19, is an evidence of the actual possibility to rewrite in a more algebraic terms at least a part of the theory of curves using notions as Artinian Gorenstein rings, Fermat’s hypersurfaces, etc. as substitutes of canonical ring, varieties of minimal degree, etc. We note moreover that we could prove the classical Enriques-Petri Theorem with the same techniques used to prove 19.

We also study in Proposition 15 and in Proposition 18 large classes of projectively normal subcanonical curves to estimate the width of the class of curves to apply our theory.

Another aspect of our theory is to relate apolarity to plane curves. Proposition 11 and Theorem 12 are results in this direction.

Finally, we want to stress that in this paper we use standard techniques of algebraic geometry, but the link that apolarity establishes between the theory of Fermat hypersurfaces and the theory of \(n\)-gonal curves contained into varieties of minimal degree opens a range of ideas which are new and still not fully explored.

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2. **Preliminaries**

In this paper we will work with varieties and schemes over the complex field \(\mathbb{C}\). For us, a **variety** will always be irreducible.

2.1. **Arithmetic Gorenstein schemes.** Let us fix a closed subscheme \(X\) of \(\mathbb{P}^N\) of dimension \(n \geq 1\) and a system \(x_0, \ldots, x_N\) of projective coordinates. Moreover, let us—as usual—denote by \(\mathcal{I}_X\) the sheaf of ideals of \(X\) and by \(M^r(X) := \oplus_{t \in \mathbb{Z}} H^r(\mathcal{I}_X(t))\), \(1 \leq r \leq n\), the \(r\)-th Hartshorne-Rao (or deficiency) module of \(X\). We recall the following

**Definition 1.** Let \(X\) be as above. Then, \(X\) is said to be **arithmetically Cohen-Macaulay** (aCM for short) if \(N - n\) is equal to the length of a minimal free resolution of its homogeneous coordinate ring

\[ S_X := \frac{S}{M^0(X)} \]

as an \(S\)-module, where we have set \(S := \mathbb{C}[x_0, \ldots, x_N]\) and \(M^0(X) := \oplus_{t \in \mathbb{Z}} H^0(\mathcal{I}_X(t))\).

**Remark 1.** It can be proved that \(X\) is aCM if and only if all the Hartshorne-Rao modules \(M^r(X)\) vanish. (see for example [Mig98, 1.2.2 and 1.2.3]).
Then, we recall the fundamental

**Definition 2.** Let $X$ be as above. $X$ is said to be *arithmetically Gorenstein* ($aG$ for short) if it is $aCM$ and the last free module of a minimal free resolution of $S_X$ has rank 1.

Also we recall

**Definition 3.** Any subscheme $X \subset \mathbb{P}^N$ is said to be *subcanonical* if there exists an integer $s \in \mathbb{Z}$ such that $\omega_X \cong \mathcal{O}_X(s)$, where $\omega_X$ is the dualising sheaf of $X$.

We stress that the last two definitions are indeed equivalent for $aCM$ schemes; in fact, denoting by

$$K_X := \text{Ext}^{n-r}_S(S_X, S) = \text{Ext}^{n-r}_S(S_X, \mathcal{K}_{\mathbb{P}^N})$$

the canonical module of $X$, it holds:

**Proposition 3.** If $X$ is an $aCM$ closed subscheme of $\mathbb{P}^N$, then the following are equivalent:

1. $X$ is $aG$;
2. $S_X \cong K_X(-s)$ for some integer $s$;
3. the minimal free resolution of $S_X$ is self-dual, up to a twist.

**Proof.** See [Mig98, Proposition 4.1.1].

**Remark 2.** To point out the integer $s$ of Definition 3, we will say also that $X$ is $s$-subcanonical or $s$-arithmetically Gorenstein (and so, $s$-$aG$ for short).

**Definition 4.** A projective closed subscheme $X \subset \mathbb{P}^N$ is said to be *projectively normal* ($PN$ for short) if $S_X$ is integrally closed.

**Definition 5.** Let $X \subset \mathbb{P}^N$ be a projective closed subscheme. We say that $X$ is $j$-normal, with $j \in \mathbb{Z}$, $j \geq 0$, if the natural restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^N}(j)) \rightarrow H^0(\mathcal{O}_X(j))$$

is surjective.

**Remark 3.** It is obvious that $X$ is $j$-normal if and only if $h^1(\mathcal{I}_X(j)) = 0$. Moreover, $X$ is $PN$ if and only if it is $j$-normal for all $j \in \mathbb{Z}$, $j \geq 0$. In particular, an $aCM$ scheme is $PN$ by Remark 1.

Vice versa, it is not difficult to show that

**Proposition 4.** A $PN$ closed $n$-dimensional subvariety $X \subset \mathbb{P}^N$ is $aCM$ if and only if

$$h^i(\mathcal{O}_X(j)) = 0, \quad \text{for } 0 < i < n \text{ and } \forall j \in \mathbb{Z}.$$ 

In particular, a $PN$ curve is always $aCM$, and if the curve is subcanonical, then it is also $aG$.

**Proof.** See for example [BK05, 1.5].

2.2. **Apolarity.** Let $S := \mathbb{C}[x_0, \ldots, x_N]$ be the polynomial ring in $(N+1)$-variables. The algebra of the partial derivatives on $S$,

$$T := \mathbb{C}[\partial_0, \ldots, \partial_N], \quad \partial_i := \frac{\partial}{\partial x_i},$$

can act on the monomials in the following way:

$$\partial^a \cdot x^b = \begin{cases} a! \binom{b}{a} x^{b-a} & \text{if } b \geq a \\ 0 & \text{otherwise} \end{cases}$$
where $a, b$ are multiindices $\binom{b}{a} = \prod_i \binom{b_i}{a_i}$ etc.

We can think of $S$ as the algebra of partial derivatives on $T$ by defining

$$x^a \cdot \partial^b = \begin{cases} a! \binom{b}{a} \partial^b - a \binom{b}{a} & \text{if } b \geq a \\ 0 & \text{otherwise.} \end{cases}$$

These actions define a perfect paring between the homogeneous forms in degree $d$ in $S$ and $T$

$$S_d \times T_d \rightarrow \mathbb{C}.$$ Indeed, this is nothing but the extension of the duality between vector spaces: if $V := S_1$, then $T_1 = V^*$.

Moreover, the perfect paring shows the natural duality between $\mathbb{P}^N := \text{Proj}(S)$ and $\check{\mathbb{P}}^N = \text{Proj}(T)$. More precisely, if $(c_0, \ldots, c_N) \in \check{\mathbb{P}}^N$, this gives $f_c := \sum_i c_i x_i \in S_1$, and if $D \in T_a$,

$$D \cdot f^b_c = \begin{cases} a! \binom{b}{a} D(c) f^{b-a}_c & \text{if } b \geq a \\ 0 & \text{otherwise.} \end{cases}$$

in particular, if $b \geq a$

$$0 = D \cdot f^b_c \iff D(c) = 0.$$

**Definition 6.** We say that two forms, $f \in S$ and $g \in T$ are *apolar* if

$$g \cdot f = f \cdot g = 0.$$ Let $f \in S_d$ and $F := V(f) \subset \mathbb{P}^N$ the corresponding hypersurface; let us now define

$$F^\perp := \{ D \in T \mid D \cdot f = 0 \}$$

and

$$A^F := \frac{T}{F^\perp}.$$ **Lemma 5.** The ring $A^F$ is Artinian Gorenstein of socle of dimension one and degree $d$.

**Proof.** See [IK99, §2.3 page 67].

**Definition 7.** $A^F$ is called the *apolar* Artinian Gorenstein ring of $F$.

It holds the Macaulay Lemma, that is

**Lemma 6.** The map

$$F \mapsto A^F$$

is a bijection between the hypersurfaces $F \subset \mathbb{P}^N$ of degree $d$ and graded Artinian Gorenstein quotient rings

$$A := \frac{T}{I}$$

of $T$ with socle degree $d$.

**Proof.** See [IK99, Lemma 2.12 page 67].
2.2.1. Varieties of sum of powers. Consider a hypersurface \( F = V(f) \subset \mathbb{P}^N \) of degree \( d \).

**Definition 8.** A subscheme \( \Gamma \subset \mathbb{P}^N \) is said to be **apolar to** \( F \) if
\[
\mathcal{I}(\Gamma) \subset F^*.
\]

It holds the **Apolarity Lemma:**

**Lemma 7.** Let us consider the linear forms \( \ell_1, \ldots, \ell_s \in \mathbb{P}^N \) the corresponding points in the dual space. Then
\[
\Gamma \text{ is apolar to } F = V(f) \iff \exists \lambda_1, \ldots, \lambda_s \in \mathbb{C}^* \text{ such that } f = \lambda_1 \ell_1^d + \ldots + \lambda_s \ell_s^d
\]
where \( \Gamma := \{L_1, \ldots, L_s\} \subset \mathbb{P}^N \). If \( s \) is minimal, then it is called the **Waring number** of \( F \).

**Proof.** See [IK99, Lemma 1.15 page 12]. \( \square \)

2.3. Hypercubics and canonical sections. In [DZ], we studied the special case of the canonical curve \( C \subset \mathbb{P}^{g-1} \) of the map introduced in \([1]\). In fact it is a well-known result that \( C \) is PN (see [ACGH85, page 117]), and then, by Proposition \([1]\) the canonical curve \( C \) is aG. Therefore, if we take two general linear forms \( \eta_1, \eta_2 \in (R_C)_1 = H^0(\omega_C) \), then \( T := \langle \eta_1, \eta_2 \rangle \) is Artinian Gorenstein, and its values of the Hilbert function are \( 1, g - 2, g - 2, 1 \). In particular, the socledegree of \( T \) is 3, and by the Macaulay Lemma, this defines a hypercubic in \( \text{Proj}(T^*) \). In this way we obtain the rational map \( \alpha_C : \text{Gr}(g - 2, g) \longrightarrow H_{g - 3,3} \). In this paper we generalise this idea to the \( s \)-subcanonical curves.

3. **Subcanonical curves**

Let \( (C, \mathcal{L}) \) be a polarised curve, i.e. \( C \) is a smooth curve and \( \mathcal{L} \) is a line bundle on \( C \). We will suppose also that the complete linear system \( |\mathcal{L}| \) embeds \( C \) in \( \mathbb{P}^N = |\mathcal{L}|^* \), i.e. that \( \mathcal{L} \) is very ample. Therefore, \( (C, \mathcal{L}) \) is subcanonical if there exists an \( s \in \mathbb{Z} \) such that \( \mathcal{L}^s \cong \omega_C^s \).

**Remark 4.** Since we request \( \mathcal{L} \) to be very ample, if \( C \) is also subcanonical, \( C \) cannot be hyperelliptic.

Clearly, the only possible \( s \leq 0 \) are \( s = -2, -1 \) and \( C \) is rational, or \( s = 0 \), and \( C \) is elliptic. If \( s = 1 \), then \( C \) is a canonical curve. Therefore, from now on, for simplicity, we will say that \( (C, \mathcal{L}) \) is **subcanonical** if there exists an \( s \in \mathbb{N}, s > 1 \), such that \( \mathcal{L}^s \cong \omega_C^s \). If \( s = 2 \), we will talk of **half-canonical** curves. In other words, we restrict Definition \([3]\) to \( s \)-subcanonical curves with \( s \geq 2 \).

As we have observed in Subsection \([2,1]\), a subcanonical PN curve is always aG, therefore the construction made for the canonical curves can be extended to these curves.

Let us show this in the case of the half-canonical curves. We have supposed that \( \mathcal{L} \) is very ample and that embeds \( C \) in \( \mathbb{P}^N \), and we suppose also that \( C \) is PN, with this embedding. Clearly, by the theorem of Clifford (see for example [Har83 IV.5.4])
\[
N \leq \frac{g - 1}{s},
\]
with equality only if \( s = 1 \), i.e. \( \mathcal{L} = \omega_C \).

To ease reading we first consider the case \( s = 2 \).

**Proposition 8.** Let \( (C, \mathcal{L}) \) be a half canonical curve, embedded by the very ample, complete, linear system \( |\mathcal{L}| \) in \( \mathbb{P}^N \) and with image \( X \). Let \( \eta_1 \) and \( \eta_2 \) two general linear forms on \( \mathbb{P}^N \). Let us suppose that \( X \) is PN. Then, \( X \) is aG and the values
of the Hilbert function of the Artinian Gorenstein graded \( \mathbb{C} \)-algebra \( A := \frac{S_X}{\langle \eta_1, \eta_2 \rangle} \) are 
\( 1, N - 1, g - 1 - 2N, N - 1, 1 \). In particular, \( A \) is Artinian Gorenstein of socle of 
dimension one and degree 4.

**Proof.** We can write the homogeneous coordinate ring of \( X \) as 
\[
S_X = \frac{\text{Sym}(H^0(C, L))}{M^0(C)},
\]
and the Artinian Gorenstein algebra \( A \) as 
\[
A = \frac{S_X}{\langle \eta_1, \eta_2 \rangle},
\]
where \( \eta_1 \) and \( \eta_2 \) are two (general) linear forms.

Let us consider the following exact sequence of sheaves
\[
0 \to L^i \to \langle \eta_1, \eta_2 \rangle \otimes L^{i+1} \to L^{i+2} \to 0,
\]
with \( i \in \mathbb{Z}, i \geq 0 \). By the Base-Point-Free Pencil trick (see [ACGH85, page 126]) the sequence gives, in cohomology,

\begin{align*}
(2) \quad i = 0 : & \quad 0 \to H^0(\mathcal{O}_C) \to \langle \eta_1, \eta_2 \rangle \otimes H^0(L) \to H^0(\omega_C) \to H^1(\mathcal{O}_C) \to \langle \eta_1, \eta_2 \rangle \otimes H^1(L) \to H^1(\omega_C) \to 0 \\
(3) \quad i = 1 : & \quad 0 \to H^0(L) \to \langle \eta_1, \eta_2 \rangle \otimes H^0(\omega_C) \to H^0(L^{\otimes 2}) \to H^1(L) \to \langle \eta_1, \eta_2 \rangle \otimes H^1(\omega_C) \to 0 \\
(4) \quad i = 2 : & \quad 0 \to H^0(L^{\otimes 2}) \to \langle \eta_1, \eta_2 \rangle \otimes H^0(L^{\otimes 3}) \to H^0(L^{\otimes 4}) \to H^1(\omega_C) \to 0 \\
(5) \quad i > 2 : & \quad 0 \to H^0(L^{\otimes i}) \to \langle \eta_1, \eta_2 \rangle \otimes H^0(L^{\otimes i+1}) \to H^0(L^{\otimes i+2}) \to 0,
\end{align*}

with \( h^0(\omega_C) = g, h^1(L) = h^0(L) = N + 1 \). Moreover, the standard exact sequence of sheaves
\[
0 \to \mathcal{I}_X(i) \to \mathcal{O}_{\mathbb{P}^N}(i) \to \mathcal{O}_X(i) \to 0,
\]
with \( i \in \mathbb{N} \), gives rise to the following sequence in cohomology

\[
0 \to \mathcal{I}_X(i) \to \text{Sym}^i(H^0(L)) \to H^0(L^{\otimes i}) \to 0,
\]

Since \( X \) is PN by hypothesis. Now, let us write \( A \) as a graded algebra \( A = \bigoplus_i A_i \); then
\[
A_i = \frac{\text{Sym}^i(H^0(C, L))}{\langle \mathcal{I}_X(i), \langle \eta_1, \eta_2 \rangle \otimes \text{Sym}^{i-1}(H^0(L)) \rangle};
\]
we have clearly that
\[
A_0 \cong \mathbb{C}, \quad A_1 \cong \mathbb{C}[x_0, \ldots, x_N]/\langle \eta_1, \eta_2 \rangle.
\]
i.e. $A_1 \cong \mathbb{C}^{N-1}$. For the rest, putting together sequences (2), (3), (4) and (5), and (6) we obtain the commutative diagram, with $i \geq 0$

\[
\begin{array}{cccccc}
0 & \longrightarrow & I_X(i+2) & \longrightarrow & \text{Sym}^{i+2}(H^0(L)) & \overset{\varphi}{\longrightarrow} & H^0(L^{i+2}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \langle \eta_1, \eta_2 \rangle \otimes I_X(i+1) & \longrightarrow & \langle \eta_1, \eta_2 \rangle \otimes \text{Sym}^{i+1}(H^0(L)) & \longrightarrow & \langle \eta_1, \eta_2 \rangle \otimes H^0(L^{i+1}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \downarrow & & \\
0 & & & & & & H^0(L^{i+1}) & & \\
\end{array}
\]

Chasing in it, if we take an $\omega \in \text{Sym}^{i+2}(H^0(L))$, and denoting by $[\omega]$ its class in $A_{i+2}$, we have that $[\omega] \neq 0$ if and only if $p \circ \varphi(\omega) \in \text{coker}(\mu) \setminus \{0\}$. In other words, (8) $\text{coker}(\mu) \cong A_{i+2}$ in Diagram (7). It follows that, if $i > 2$, $A_{i+2} = \{0\}$, since $H^1(L^{i+1}) = \{0\}$.

If $i = 2$, we have that $\text{coker}(\mu) \cong H^1(\omega C)$, by (4), so $\dim \text{coker}(\mu) = 1$, and therefore $\dim(A_i) = 1$.

If $i = 1$, $A_3 \cong \mathbb{C}^{N-1}$ either by the Artinian property, or by Diagram (7) together with Sequence (3).

If $i = 0$, $A_2 \cong \mathbb{C}^{3-2N-1}$, again by Diagram (7) and Sequence (2).

Now we give the analogue statement of the above one for the general $s$-subcanonical curve, but, since we do not really need to compute the complete Hilbert function we can give a simpler proof suggested to us by G. Casnati.

**Theorem 9.** Let $(C, \mathcal{L})$ be an $s$-subcanonical curve, $s \geq 2$, embedded by the very ample, complete, linear system $|\mathcal{L}|$ in $\mathbb{P}^N$. Let $\eta_1$ and $\eta_2$ two general linear forms on $\mathbb{P}^N$. Let us suppose that $C$ is PN. Then, $C$ is aG and the Artinian Gorenstein graded $\mathbb{C}$-algebra $A := \frac{S_C}{(\eta_1, \eta_2)}$ is of socle dimension one and degree $s + 2$.

**Proof.** In fact, by Proposition 4 $S_C$ is aG (here and in what follows, by abuse of notation, we will call aG the homogeneous coordinate ring of an aG scheme).

Therefore, also $A = S_X/(\eta_1, \eta_2)$ is aG since $\eta_1, \eta_2$ is a regular sequence: see for example [BH93, Proposition 3.1.19(b)]. The ring $A$ is obviously graded, then by Proposition 3.1.19(b) it has symmetric Hilbert function since it is aG. By symmetry, the socle of $A$ is of dimension 1. Now, it remains to prove that the socle of $A$ is of degree $s + 2$. Let $K_A$ be the canonical model of $A$, see [BH93, Definition 3.6.8 page 139, also page 140] and let $a(A)$ be the $a$-invariant of $A$, see [BH93, Definition 3.6.13]. By Proposition 3.1.19(b), $K_C = S_C(s)$, then $K_A = A(s + 2)$ since [BH93, Corollary 3.6.14]. In particular, $a(A) = s + 2$ by [BH93, Corollary 3.6.14]. This means $A_{s+2} \neq 0$ and $A_i = 0$ for $i \geq s + 3$ (see the remark which follows [BH93, Theorem 3.6.19]).
Remark 5. With simple but tedious calculations we could find the values of the Hilbert function of $A$ of the preceding theorem.

3.1. Plane curves. There are various ways to obtain half-canonical (or—more generally—subcanonical) curves. One of these is via (smooth) plane curves and Veronese embeddings of the plane.

So, let us consider a (non-hyperelliptic) plane curve $C$ of odd degree $2n + 1$; therefore, $C \in [(2n+1)H]$, where $H$ is the hyperplane divisor on $\mathbb{P}^2$. By adjunction, we have, by abuse of notation

$$\omega_C = 2(n-1)H|_C.$$  

By the Clebsch formula, its genus is $g(C) = n(2n-1)$. 

Now, consider the $(n-1)$-th embedding of $\mathbb{P}^2$, $v_{n-1}: \mathbb{P}^2 \to \mathbb{P}^N$, where $N := \binom{n+1}{2} - 1$. $V_{n-1} := v_{n-1}(\mathbb{P}^2)$ is a Veronese surface, it is PN of degree $(n-1)^2$.

By the projective normality of $V_{n-1}$ it follows that $X := v_{n-1}(C) \subset V_{n-1}$ is PN also:

Proposition 10. Let $C$ be a smooth plane curve of odd degree $2n + 1$, and let $X$ be its $(n-1)$-tuple Veronese embedding $X := v_{n-1}(C) \subset V_{n-1} \subset \mathbb{P}^N$, with $N = \binom{n+1}{2} - 1$. Then, $X$ is PN and aG.

Proof. We need to show that $h^1(I_X(j)) = 0$ for all $j \in \mathbb{N}$.

By the above inclusions we can construct the following exact diagram of sheaves:

$$
\begin{array}{ccccccc}
0 & \to & I_{X,V_{n-1}}(j) & \to & 0 \\
& \uparrow & & \uparrow \\
& & I_X(j) & \to & \mathcal{O}_{\mathbb{P}^N}(j) & \to & \mathcal{O}_X(j) & \to & 0 \\
& & \uparrow & & \uparrow \\
0 & \to & I_{V_{n-1}}(j) & \to & \mathcal{O}_{\mathbb{P}^N}(j) & \to & \mathcal{O}_{V_{n-1}}(j) & \to & 0 \\
& & \uparrow & & \uparrow \\
& & 0 & \to & \mathcal{O}_{V_{n-1}}(j-X) & & 0 \\
& & \uparrow \\
& & 0 \\
\end{array}
$$

By the above diagram, $I_{X,V_{n-1}}(j) \cong \mathcal{O}_{V_{n-1}}(j-X)$ and, since $V_{n-1}$ is PN, then $h^1(I_{V_{n-1}}(j)) = 0$ for all $j \in \mathbb{N}$; therefore it is sufficient to show that $h^1(O_{V_{n-1}}(j-X)) = 0$ for all $j \in \mathbb{N}$.

Since $v_{n-1}: \mathbb{P}^2 \to V_{n-1} \subset \mathbb{P}^N$ is an embedding, we are reduced to show only that $h^1(O_{\mathbb{P}^2}(jn - 1 - 2n - 1)) = 0$ for all $j \in \mathbb{N}$, which is well-known (see [Hart83 III.5.1]).

Moreover, by (9), $X$ is also aG. 

By what we have just proven, we have that

Proposition 11. Let $C$ be a plane curve of degree $2n+1$. Let $A$ be the corresponding Artinian graded Gorenstein ring of the half-canonical curve $X := v_{n-1}(C)$. Then the Macaulay polynomial of $A$ is a quartic of Waring number at most $(n-1)^2$.
It is now obvious how to obtain similar results for s-subcanonical curves coming from plane curves:

**Theorem 12.** Let \( C \) be a plane curve of degree \( s(n - 1) + 3 \). Let \( A \) be the corresponding Artinian graded Gorenstein ring of s-subcanonical curve \( X := v_{n-1}(C) \). Then the Macaulay polynomial of \( A \) is an \((s + 2)\)-tic of Waring number at most \((n - 1)^2\).

### 3.2. Curves on Segre-Hirzebruch surfaces

#### 3.2.1. Rational normal scrolls.** For fixing some notation, we recall some basic facts about rational normal scrolls.

By definition, a rational normal scroll (RNS for short, in the following) of type \((a_1, \ldots, a_k)\), \( S_{a_1, \ldots, a_k} \), is the image of the \( \mathbb{P}^{k-1} \)-bundle \( \mathbb{P}(E) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_k)) \), via the embedding given by \( \mathcal{O}_{\mathbb{P}(E)}(1) \) in \( \mathbb{P}^N \), with \( N = \sum a_i + k - 1 \). Equivalently, one takes \( k \) disjoint projective spaces of dimension \( a_i \), \( \mathbb{P}^{a_i} \), and \( k \) rational normal curves \( C_i \subset \mathbb{P}^{a_i} \), together with isomorphisms \( \phi_i : \mathbb{P}^1 \to C_i \) (if \( a_i \neq 0 \), constant maps otherwise); then

\[
S_{a_1, \ldots, a_k} = \bigcup_{P \in \mathbb{P}^1} \langle \phi_1(P), \ldots, \phi_k(P) \rangle.
\]

We have also that

\[
\deg(S_{a_1, \ldots, a_k}) = \sum a_i = N - k + 1,
\]

and \( \dim(S_{a_1, \ldots, a_k}) = k \).

#### 3.2.2. Rational normal surfaces.** In the case \( k = 2 \), \( \mathbb{P}^{a_1 + a_2 + 1} \supset S_{a_1, a_2} \cong \mathbb{F}_e \), \( e := [a_1 - a_2] \) where \( \mathbb{F}_e \) is its minimal model, i.e. a Segre-Hirzebruch surface: \( \mathbb{F}_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e)) \). In fact, letting \( a_1 \geq a_2 \), \( S_{a_1, a_2} \) is the embedding of \( \mathbb{F}_e \) by the complete linear system \( |C_0 + a_1 f| \) where the zero section \( C_0 \) on \( \mathbb{F}_e \) is defined by \( \mathcal{O}_{\mathbb{F}_e}(C_0) = \mathcal{O}_{\mathbb{F}_e}(1) \) and \( f \) is the fibre of the scroll (see for example [Har83, V.2]). Moreover, we have \( C_0 \cdot C_0 = -e \), \( f \cdot f = 0 \), \( C_0 \cdot f = 1 \), and the Picard group of \( \mathbb{F}_e \) is generated by \( C_0 \) and \( f \). Geometrically \( C_0 \) corresponds in \( S_{a_1, a_2} \subset \mathbb{P}^{a_1 + a_2 + 1} \) to the unisecant \( a_2 \)-tic \( C_{a_2} \), while the section \( C_1 \sim C_0 + cf \) (where \( \sim \) denotes the linear equivalence of divisors) corresponds to the unisecant \( a_1 \)-tic curve. We recall moreover that:

\[
K_{S_{a_1, a_2}} \sim -2C_0 + (-2 - e)f.
\]

Finally, we prove the following

**Lemma 13.** Let \( C \subset S_{a_1, a_2} \subset \mathbb{P}^{a_1 + a_2 + 1} \) be a curve which is a very ample divisor, i.e. \( |C| \sim aC_0 + bf \) with \( a > 0 \) and \( b > 0 \) (see [Har83, Corollary V.2.18]). Then the natural restriction map \( \text{Pic}(S_{a_1, a_2}) \to \text{Pic}(C) \) restricted to the very ample divisors on \( S_{a_1, a_2} \) is injective.

**Proof.** Let \( D_1, D_2 \in \text{Pic}(S_{a_1, a_2}) \) be two very ample divisors; then, we can write

\[
D_i = \alpha_i C_0 + \beta_i f, \quad \alpha_i > 0, \quad \beta_i > \alpha_i e, \quad i = 1, 2.
\]

Let us suppose that \( D_1 \mid C \sim D_2 \mid C \); in particular \( D_1 C = D_2 C \), which means \( a((\beta_1 - \alpha_1 e) - (\beta_2 - \alpha_2 e)) + b(\alpha_1 - \alpha_2) = 0 \). This is equivalent to

\[
\alpha_1 - \alpha_2 = -\lambda a \\
(\beta_1 - \alpha_1 e) - (\beta_2 - \alpha_2 e) = \lambda b
\]

with \( \lambda \in \mathbb{C} \). If \( \lambda = 0 \), we are done. If \( \lambda > 0 \), then \( \alpha_1 < \alpha_2 \) and \( \beta_1 - \alpha_1 e > \beta_2 - \alpha_2 e \), but this contradicts (10); analogously if \( \lambda < 0 \). \( \square \)
3.2.3. **Half-canonical curves.** Let us consider now curves contained in rational normal surfaces; we look for conditions for these curves to be half-canonical. We will show that these are characterised to be 4-gonal. We start with the following:

**Proposition 14.** If a curve $C \subset S_{a_1,a_2} \subset \mathbb{P}^{a_1+a_2+1}$ (with $a_1 \geq a_2$) is half-canonical, then it is 4-gonal. More precisely, $C$ is linearly equivalent to $4C_0 + (3a_1 - a_2 + 2)f$, and therefore $p_a(C) = 3(a_1 + a_2 + 1)$ and $\deg(C) = 3a_1 + 3a_2 + 2$. $C$ is smooth if (and only if) $3a_2 + 2 \geq a_1$.

**Proof.** A curve $C \subset S_{a_1,a_2}$ determines a divisor $C \sim aC_0 + bf$. By adjunction, we have $K_C = ((a-2)C_0 + (b-2-e)f) |_C$, so $C$ is half-canonical with respect to the linear system $|C_0 + a_1f|$ if it holds that, by Lemma 15,

$$2C_0 + 2af = ((a-2)C_0 + (b-2-e)f).$$

This is equivalent to write: $a = 4$, $b = 3a_1 - a_2 + 2$. Since we want that $C$ is smooth, by [Har83, V.2.18] we must have $3a_1 - a_2 + 2 \geq 4(a_1 - a_2)$, that is $3a_2 + 2 \geq a_1$. Then the (arithmetic) genus $p_a(C)$ of $C$ is $p_a(C) = 3(a_1 + a_2 + 1)$, and its degree is $\deg(C) = 3a_1 + 3a_2 + 2$.

□

Next, we show that the smooth curves of the preceding proposition are PN:

**Proposition 15.** Let $a_1, a_2 \in \mathbb{N}$ such that $a_2 \leq a_1 \leq 3a_2 + 2$. Let $C \subset \mathbb{P}_e$ be a general member of the linear system $|4C_0 + (3a_1 - a_2 + 2)f|$ where $e = a_1 - a_2 \geq 0$. Then the image $X$ of $C$ by the embedding $\phi_{(C_0+a_1f)}: \mathbb{P}_e \to S_{a_1,a_2} \subset \mathbb{P}^{1+a_1+a_2}$ is PN.

**Proof.** Set $N := 1 + a_1 + a_2$, $S = S_{a_1,a_2} \subset \mathbb{P}^N$ and $X = \phi_{(C_0+a_1f)}(C)$. We only need to show that $h^1(I_X(j)) = 0$ for all $j \in \mathbb{N}$.

By the natural inclusions $X \subset S \subset \mathbb{P}^N$ we can construct the following exact diagram of sheaves:

\[
\begin{array}{cccccc}
& & & & & 0 \\
& & & & & 0 \\
0 & \longrightarrow & I_X(j) & \longrightarrow & \mathcal{O}_{\mathbb{P}^N}(j) & \longrightarrow & \mathcal{O}_X(j) & \longrightarrow & 0 \\
& & & & & 0 & \longrightarrow & \mathcal{O}_S(j) & \longrightarrow & 0 \\
& & & & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & I_S(j) & \longrightarrow & \mathcal{O}_{\mathbb{P}^N}(j) & \longrightarrow & \mathcal{O}_S(j) & \longrightarrow & 0 \\
& & & & & \uparrow & & \uparrow & & \uparrow \\
& & & & & 0 & \longrightarrow & \mathcal{O}_S(j-X) & \longrightarrow & 0 \\
& & & & & \uparrow & & \uparrow & & \uparrow \\
& & & & & 0 & \longrightarrow & & & 0 \\
\end{array}
\]

By the above diagram, $I_{X,S}(j) \cong \mathcal{O}_S(j-X)$ and, since $S$ is PN, then $h^1(I_S(j)) = 0$ for all $j \in \mathbb{N}$; therefore it is sufficient to show that $h^1(\mathcal{O}_S(j-X)) = 0$ for all $j \in \mathbb{N}$.

Since $\phi_{(C_0+a_1f)}: \mathbb{P}_e \to S \subset \mathbb{P}^N$ is an embedding, we are reduced to show only that $h^1(\mathcal{O}_{\mathbb{P}_e}(j(C_0 + af) - C)) = 0$ for all $j \in \mathbb{N}$.
Since 
\[ C \in 4C_0 + (3a_1 - a_2 + 2)f \]  and \( K_{\mathcal{F}} \sim -2C_0 + (-2 - e)f \),
then
\[ j(C_0 + a_1 f) - C \sim K_{\mathcal{F}} + (j-2)(C_0 + a_1 f). \]

If \( j \geq 3 \), then
\[
H^1(\mathcal{O}_{\mathcal{F}}((j(C_0 + a_1 f) - C)) = H^1(\mathcal{O}_{\mathcal{F}}(K_{\mathcal{F}} + (j-2)(C_0 + a_1 f)))
\]
\[ = \{0\} \]
by the Kodaira Vanishing theorem [Har83, III.7.15].

If \( j = 2 \), then \( h^1(\mathcal{O}_{\mathcal{F}}(K_{\mathcal{F}})) = 0 \) since \( \mathcal{F} \) is a rational surface.

If \( j = 1 \), by Serre duality
\[ H^1(\mathcal{O}_{\mathcal{F}}(K_{\mathcal{F}} - (C_0 + C))) \cong H^1(\mathcal{O}_{\mathcal{F}}(C_0 + a_1 f)))^\vee, \]
so we only need to show that \( h^1(\mathcal{O}_{\mathcal{F}}(C_0 + a_1 f))) = 0 \). Now, the general member \( H \in |C_0 + a_1 f| \) is a connected smooth rational curve and its degree is \( a_1 + a_2 \), since \( H \) corresponds to a hyperplane section of \( S \). Then \( h^1(H, \mathcal{O}_H(C_0 + a_1 f)) = 0 \) by Serre duality on the curve \( H \).

By the cohomology of
\[ 0 \to \mathcal{O}_{\mathcal{F}} \to \mathcal{O}_{\mathcal{F}}(C_0 + a_1 f) \to \mathcal{O}_H(C_0 + a_1 f) \to 0, \]
since \( h^1(\mathcal{O}_{\mathcal{F}}) = 0 \) (\( \mathcal{F} \) is a rational, hence regular, surface) and \( h^1(H, \mathcal{O}_H(C_0 + a_1 f)) = 0 \), it follows that \( h^1(\mathcal{O}_{\mathcal{F}}(C_0 + a_1 f)) = 0 \), which is the claim.

Finally, if \( j = 0 \), we have \( h^1(\mathcal{O}_{\mathcal{F}}(-C)) = 0 \) by Kodaira Vanishing.

In the hypothesis of Proposition [13], we can give the converse of Proposition [14] and therefore we can characterise the PN 4-gonal half-canonical curves, in the following

**Theorem 16.** Let \( (C, \mathcal{L}) \) be a polarised curve, such that \( C \subset |\mathcal{L}|^\vee = \mathbb{P}^N \) is a half-canonical PN curve. Then \( C \) is 4-gonal if and only if it is contained in a rational normal surface \( S_{a_1, a_2} \) with \( a_1 \leq 3a_2 + 2 \) and \( N := a_1 + a_2 + 1 \), if and only if for every couple of general sections \( \eta_1, \eta_2 \in H^0(C, \mathcal{L}) \), \( F_{\eta_1, \eta_2} \in \mathbb{C}[x_0, \ldots, x_{a_1+a_2-1}] \) is a Fermat quartic.

**Proof.** From Proposition [14] we deduce that if \( C \subset S_{a_1, a_2} \) and it is PN, then \( C \) is 4-gonal and \( a_2 \leq a_1 \leq 3a_2 + 2 \). If we take two general sections \( \eta_1, \eta_2 \in H^0(C, \mathcal{L}) \), the zero-dimensional scheme of length \( a_1 + a_2 \), \( \Gamma := S_{a_1, a_2} \cap V(\eta_1, \eta_2) \subset \mathbb{P}^{a_1+a_2-1} \) is apolar to a Fermat quartic hypersurface, \( F_{\eta_1, \eta_2} \in \mathbb{C}[x_0, \ldots, x_{a_1+a_2-1}] \) by the Apolarity Lemma [4] since \( S_{a_1, a_2} \) is ACM, and by Proposition [5].

Now, let us suppose that our half-canonical curve \( C \) is such that for every couple of general sections \( \eta_1, \eta_2 \in H^0(C, \mathcal{L}) \), there is a zero-dimensional scheme of length \( a_1 + a_2 \), \( \Gamma_{\eta_1, \eta_2} \subset \mathbb{P}^{a_1+a_2-1} := V(\eta_1, \eta_2) \) with \( I(\Gamma_{\eta_1, \eta_2}) \subset I(C) \), or, in other words, by the Apolarity Lemma [4] \( \Gamma_{\eta_1, \eta_2} \) is apolar to a Fermat quartic \( F_{\eta_1, \eta_2} \in \mathbb{C}[x_0, \ldots, x_{a_1+a_2-1}] \).

First of all, we can assume that \( \eta_1 = \partial_{N-1} \) and \( \eta_2 = \partial_N \) and \( F_{\partial_{N-1}, \partial_N} := x_0^4 + \cdots + x_{N-2}^4 \). We only need to find \( F_{\partial_{N-1}, \partial_N} \). It is easy to see that
\[
(12) \quad F_{\partial_{N-1}, \partial_N}(\partial_i \partial_j, \partial_i^4 - \partial_j^4), \quad i, j \in \{0, \ldots, N-2\}, \quad i \neq j.
\]
Then, the quadrics of \( Z(C) \) are of the form
\[
(13) \quad Q_{i,j} := \partial_i \partial_j + \partial_{N-1} L_{i,j} + \partial_N M_{i,j},
\]
where \( L_{i,j} \) and \( M_{i,j} \) are linear forms on \( \mathbb{P}^N \). Therefore, we have a set of \( \binom{N-2}{2} \) quadrics in \( Z(C) \), and \( Z(C) \) is not generated by quadrics.
Now, if the ideal 
\[ I := (Q_{i,j})_{0 \leq i < j \leq N-2} \]
defines a surface \( S = V(I) \), then we are done, since \( \Gamma := \mathcal{S} \cap \mathbb{P}^{N-2} - \mathcal{F} \) — where \( \mathbb{P}^{N-2} := V(\partial_{N-1}, \partial_N) \) — is a zero-dimensional scheme of length \( N - 1 \), in fact
\[ I(\Gamma) = (\partial_i \partial_j)_{0 \leq i < j \leq N-2}, \]
and therefore \( S \) is a surface of minimal degree, hence a rational normal surface.

Now, we have that
\[ I(C) = (Q_{i,j}, \partial_i^4 - \partial_j^4 + \partial_{N-1}N_{i,j} + \partial_N R_{i,j})_{0 \leq i < j \leq N-2} \]
with \( N_{i,j} \) and \( R_{i,j} \) homogeneous cubic forms of \( \mathbb{P}^{N} \), and by hypothesis, if \( \eta_1 \) and \( \eta_2 \) are two general linear forms, Equation (12) becomes
\[ F_{\eta_1, \eta_2} = (\ell_i \ell_j, \ell_i^4 - \ell_j^4), \quad i, j \in \{0, \ldots, N-2\}, \quad i \neq j. \]
with \( \ell_i \) linear forms, and
\[ Q_{i,j} := \ell_i \ell_j + \eta_1 L_{i,j} + \eta_2 M_{i,j}, \]
are the quadrics in \( I(C) \); but the only way to obtain these quadrics is from linear combinations of the \( Q_{i,j} \)'s, so
\[ I = (Q_{i,j})_{0 \leq i < j \leq N-2}. \]
Now, \( \Gamma' := S \cap \mathbb{P}^{N-2} - \mathcal{F} \) — where \( \mathbb{P}^{N-2} := V(\eta_1, \eta_2) \) — is a zero-dimensional scheme of length \( N - 1 \), in fact, again
\[ I(\Gamma') = (\ell_i \ell_j)_{0 \leq i < j \leq N-2}, \]
and therefore \( S \) is a surface of minimal degree, hence a rational normal surface, and the proof is complete.

\[ \square \]

Remark 6. It is not difficult to find that the quadrics \( Q_{i,j} \) of Equation (13) can be written in a particular form. In fact, if we choose two linear forms one of which is—for example—\( \eta_1 = \partial_i \), we deduce, in \( F_{\eta_1, \eta_2} \),
\[ \partial_{N-1}L_{i,j} + \partial_N M_{i,j} = \ell_i m_j, \]
where \( \ell_i \) and \( m_j \) are linear forms in the appropriate \( \mathbb{P}^{N-2} := V(\{\eta_1 = \eta_2 = 0\}) \).

By the generality of the two linear forms \( \eta_1, \eta_2 \), it follows, by Bertini Theorem, if \( N > 3 \), that the quadric \( \partial_{N-1}L_{i,j} + \partial_N M_{i,j} \) is reducible, and Equation (13) can be written as
\[ Q_{i,j} = \det \left( \begin{array}{cc} \partial_i & L_i \\ M_j & \partial_j \end{array} \right), \]
where \( L_i \) and \( M_j \) are linear forms in \( \mathbb{P}^N \) that determine \( \ell_i \) and \( m_j \) in Equation (14).

Actually, it is not hard to prove that, always by Equation (13), that \( L_i \) (or \( M_j \)) is a linear form of the type
\[ L_j = a\partial_{N-1} + b\partial_N. \]

3.2.4. s-subcanonical curves. Let us make the natural extension to the s-subcanonical curves contained in rational normal surfaces; from what we know from the last subsection, not surprisingly, we will see that these are characterised to be \( (s+2) \)-gonal.

We start with the following

**Proposition 17.** If a curve \( C \subset S_{a_1, a_2} \) (with \( a_1 \geq a_2 \)) is s-subcanonical, then it is \( s+2 \)-gonal; more precisely, \( C \) is linearly equivalent to \( (s+2)C_0 + (s(a_1 + a_2) + 2)f \), and therefore \( p_a(C) = \frac{s+1}{2}(s(a_1 + a_2) + 2) \) and \( \deg(C) = (s+1)(a_1 + a_2) + 2 \). \( C \) is smooth if (and only if) \( (s+1)a_2 + 2 \geq a_1. \)
Proof. We proceed as in the proof of Proposition 14. $C$ determines a divisor $C \sim aC_0 + bf$ and it is in $s$-subcanonical with respect to the linear system $|C_0 + a_1 f|$ if it holds that, by adjunction and Lemma 13,

$$sC_0 + sa_1 f = ((a - 2)C_0 + (b - 2 - e)f);$$

that is

$$a = s + 2$$
$$b = (s + 1)a_1 - a_2 + 2.$$ 

Since we want $C$ to be smooth, then $(s + 1)a_2 + 2 \geq a_1$. The (arithmetic) genus $p_a(C)$ of $C$ is given, by adjunction $p_a(C) = \frac{(s + 1)(a_1 + a_2 + 2)}{2}$, and its degree is

$$\deg(C) = (s + 1)(a_1 + a_2) + 2.$$ 

Next, we show that the smooth curves of the preceding proposition are PN:

**Proposition 18.** Let $a_1, a_2 \in \mathbb{N}$ such that $(s + 1)a_2 + 2 \geq a_1$. Let $C \in \mathbb{F}_e$ be a general member of the linear system $|(s + 2)C_0 + ((s + 1)a_1 - a_2 + 2)f|$ where $e = a_1 - a_2 \geq 0$. Then the image $X$ of $C$ by the embedding $\phi|_{C_0 + a_1 f}: \mathbb{F}_e \to S_{a_1, a_2} \subset \mathbb{P}^{1 + a_1 + a_2}$ is PN.

**Proof.** We proceed as in the proof of Proposition 15 with the same notations. From Diagram (11) it is sufficient to show that $h^1(O_{\mathbb{F}_e}(j - X)) = 0$ for all $j \in \mathbb{N}$, and then to show that $h^1(O_{\mathbb{F}_e}(j(C_0 + a_1 f) - C)) = 0$ for all $j \in \mathbb{N}$.

Now, we have

$$j(C_0 + a_1 f) - C \sim K_{\mathbb{F}_e} + (j - s)(C_0 + a_1 f).$$

If $j \geq s + 1$, then

$$H^1(O_{\mathbb{F}_e}(j(C_0 + a_1 f) - C)) = H^1(O_{\mathbb{F}_e}(K_{\mathbb{F}_e} + (j - s)(C_0 + a_1 f)))$$

$$= \{0\}$$ by the Kodaira Vanishing theorem [Har83 III.7.15].

If $j = s$, then $h^1(O_{\mathbb{F}_e}(K_{\mathbb{F}_e})) = 0$ since $\mathbb{F}_e$ is a rational surface.

If $j = s - 1$, by Serre duality

$$H^1(O_{\mathbb{F}_e}(K_{\mathbb{F}_e} - (C_0 + a_1 f))) \cong H^1(O_{\mathbb{F}_e}(C_0 + a_1 f)))^\vee,$$

so we only need to show that $h^1(O_{\mathbb{F}_e}(C_0 + a_1 f)) = 0$. Now, the general member $H \in |C_0 + a_1 f|$ is a connected smooth rational curve and its degree is $a_1 + a_2$, since $H$ corresponds to a hyperplane section of $S$. Then $h^1(H, \mathcal{O}_H(C_0 + a_1 f)) = 0$ by Serre duality on the curve $H$.

By the cohomology of

$$0 \to \mathcal{O}_{\mathbb{F}_e} \to \mathcal{O}_{\mathbb{F}_e}(C_0 + a_1 f) \to \mathcal{O}_H(C_0 + a_1 f) \to 0,$$

since $h^1(O_{\mathbb{F}_e}) = 0$ ($\mathbb{F}_e$ is a rational, hence regular, surface) and $h^1(H, \mathcal{O}_H(C_0 + a_1 f)) = 0$, it follows that $h^1(O_{\mathbb{F}_e}(C_0 + a_1 f)) = 0$, which is the claim.

If $j \leq s - 2$, we set $j = s - 2 - i$, with $0 \leq i \leq s - 2$. Again by Serre duality we have to show that $h^1(O_{\mathbb{F}_e}((i + 2)(C_0 + a_1 f))) = 0$.

As above the general member $H \in |(i + 2)(C_0 + a_1 f)|$ is a connected smooth curve of degree $(i + 2)(a_1 + a_2)$. Its genus $g$ is given by adjunction:

$$2g - 2 = (i + 2)(C_0 + a_1 f)((i + 2)(C_0 + a_1 f) + K_{\mathbb{F}_e})$$

$$= (i + 2)(i + 1)(a_1 + a_2) - 2(i + 2),$$

which means

$$g = \binom{i + 2}{2} \deg(S) - (i + 1).$$

Then, since $(i + 2)^2(\deg(S))^2 > 2g - 2$, we deduce $h^1(H, \mathcal{O}_H((i + 2)(C_0 + a_1 f))) = 0.$
Finally, considering the cohomology of

\[ 0 \rightarrow \mathcal{O}_A((i + 1)C_0 + a_1 f) \rightarrow \mathcal{O}_A((i + 2)(C_0 + a_1 f)) \rightarrow \mathcal{O}_H((i + 2)(C_0 + a_1 f)) \rightarrow 0, \]

since we have just proved that \( h^1(\mathcal{O}_A(C_0 + a_1 f)) = 0 \) and \( h^1(H, \mathcal{O}_H((i + 2)(C_0 + a_1 f))) = 0, \forall i, 0 \leq i \leq s - 2 \), we deduce, by induction on \( i \), that \( h^1(\mathcal{O}_A((i + 2)(C_0 + a_1 f))) = 0, \forall i, 0 \leq i \leq s - 2 \), and therefore the proposition is proved. \( \square \)

In the hypothesis of Proposition 18, we can give the converse of Proposition 17 and therefore we can characterise the PN \((s + 2)\)-gonal half-canonical curves, in the following

**Theorem 19.** Let \((C, \mathcal{L})\) be a polarised curve, such that \( C \subset |\mathcal{L}|^r := \mathbb{P}^N \) is a \( s \)-subcanonical PN curve. Then \( C \) is \((s + 2)\)-gonal if and only if it is contained in a rational normal surface \( S_{a_1,a_2} \) with \( a_2 \leq a_1 \leq (s + 1)a_2 + 2 \) and \( N := a_1 + a_2 + 1 \), if and only if for every couple of general sections \( \eta_1, \eta_2 \in H^0(C, \mathcal{L}), F_{\eta_1,\eta_2} \in \mathbb{C}[x_0, \ldots, x_{a_1+a_2-1}] \) is a Fermat \((s + 2)\)-tic.

**Proof.** From Proposition 17 we deduce that if \( C \subset S_{a_1,a_2} \) and it is PN, then \( C \) is \((s + 2)\)-gonal and \( a_2 \leq a_1 \leq (s + 1)a_2 + 2 \). If we take two general sections \( \eta_1, \eta_2 \in H^0(C, \mathcal{L}) \), the zero-dimensional scheme of length \( a_1 + a_2, \Gamma := S_{a_1,a_2} \cap V(\eta_1, \eta_2) \subset \mathbb{P}^{a_1+a_2-1} \) is apolar to a Fermat \((s + 2)\)-tic hypersurface, \( F_{\eta_1,\eta_2} \in \mathbb{C}[x_0, \ldots, x_{a_1+a_2-1}] \) by the Apolarity Lemma 7, since \( S_{a_1,a_2} \) is aCM, and by Theorem 9.

Now, let us suppose that our \( s \)-subcanonical curve \( C \) is such that for every couple of general sections \( \eta_1, \eta_2 \in H^0(C, \mathcal{L}) \), there is a zero-dimensional scheme of length \( a_1 + a_2, \Gamma_{\eta_1,\eta_2} \subset \mathbb{P}^{a_1+a_2-1} := V(\eta_1, \eta_2) \) with \( I(\Gamma_{\eta_1,\eta_2}) \subset I(C) \), or, in other words, by the Apolarity Lemma 7, \( \Gamma_{\eta_1,\eta_2} \) is apolar to a Fermat \((s + 2)\)-tic \( F_{\eta_1,\eta_2} \in \mathbb{C}[x_0, \ldots, x_{a_1+a_2-1}] \).

First of all, we can assume that \( \eta_1 = \partial_{N-1} \) and \( \eta_2 = \partial_N \) and \( F_{\partial_{N-1},\partial_N} := x_0^2 + \cdots + x_N^2 \). We only need to find \( F_{\partial_{N-1},\partial_N} \). It is easy to see that

\[ F_{\partial_{N-1},\partial_N} = (\partial_i \partial_j, \partial_i^{s+2} - \partial_j^{s+2}), \quad i, j \in \{0, \ldots, N - 2\}, \quad i \neq j. \]

Then, the quadrics of \( I(C) \) are of the form

\[ Q_{i,j} := \partial_i \partial_j + \partial_{N-1} L_{i,j} + \partial_N M_{i,j}, \]

where \( L_{i,j} \) and \( M_{i,j} \) are linear forms on \( \mathbb{P}^N \). Therefore, we have a set of \( (N-2) \choose 2 \) quadrics in \( I(C) \), and \( I(C) \) is not generated by quadrics.

Now, if the ideal

\[ I := (Q_{i,j})_{0 \leq i < j \leq N - 2} \]
defines a surface \( S = V(I) \), then we are done, since \( \Gamma := S \cap \mathbb{P}^{N-2} \) — where \( \mathbb{P}^{N-2} := V(\partial_{N-1}, \partial_N) \) — is a zero-dimensional scheme of length \( N - 1 \), in fact

\[ I(\Gamma) = (\partial_i \partial_j)_{0 \leq i < j \leq N - 2}, \]

and therefore \( S \) is a surface of minimal degree, hence a rational normal surface.

Now, we have that

\[ I(C) = (Q_{i,j}, \partial_i^{s+2} - \partial_j^{s+2} + \partial_{N-1} N_{i,j} + \partial_N R_{i,j})_{0 \leq i < j \leq N - 2} \]

with \( N_{i,j} \) and \( R_{i,j} \) homogeneous \( s \)-1-forms of \( \mathbb{P}^N \), and by hypothesis, if \( \eta_1 \) and \( \eta_2 \) are two general linear forms, Equation 15 becomes

\[ F_{\eta_1,\eta_2} = (\ell_i \ell_j, \ell_i^{s+2} - \ell_j^{s+2}), \quad i, j \in \{0, \ldots, N - 2\}, \quad i \neq j. \]

with \( \ell_i \) linear forms, and

\[ Q_{i,j} := \ell_i \ell_j + \eta_1 L_{i,j} + \eta_2 M_{i,j}, \]
are the quadrics in \( I(C) \); but the only way to obtain these quadrics is from linear combinations of the \( Q_{i,j} \)'s, so

\[
I = (Q_{i,j}')_{0 \leq i < j \leq N-2}.
\]

Now, \( \Gamma' := S \cap \mathbb{P}^{N-2} \) —where \( \mathbb{P}^{N-2} := V(\eta_1, \eta_2) \)—is a zero-dimensional scheme of length \( N-1 \), in fact, again

\[
I(\Gamma') = (\ell_i \ell_j)_{0 \leq i < j \leq N-2},
\]

and therefore \( S \) is a surface of minimal degree, hence a rational normal surface, and the proof is complete.

\[\Box\]

Remark 7. Clearly the observations done in Remark 6 on the quadrics of the ideal of the half-canonical curves hold in the general case of the quadrics of the ideal of the \( s \)-subcanonical curves.

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