Black holes in a bathtub

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Abstract. Many properties of black holes can be studied using acoustic analogues in the laboratory through the propagation of sound waves. We investigate in detail sound wave propagation in a rotating acoustic (2 + 1)-dimensional black hole, which corresponds to the “draining bathtub” fluid flow. We report results on i) quasinormal modes, ii) power-law tails, iii) superradiant amplification coefficients, iv) frequencies and growth times for the superradiant “black hole bomb” instability first studied by Press and Teukolsky.

At the classical level, black holes are the simplest objects that can be built out of gravity alone. Hawking showed that this simplicity is lost when quantum effects are taken into account [1]: black holes slowly evaporate by emitting an almost thermal radiation. Unfortunately it is very hard to experimentally verify this prediction. Prospects for detecting Hawking radiation changed only recently, when Unruh [2] realized that the basic ingredients of Hawking radiation can experimentally be reproduced in the laboratory.

Black hole evaporation is due to the existence of an apparent horizon, so the experimental setup should display the essential features that define apparent horizons in general relativity. Unruh considered precisely such a system: a fluid moving with a space-dependent velocity, for example water flowing through a nozzle. Where the fluid velocity exceeds the sound velocity we get the equivalent of an apparent horizon for sound waves. This is the acoustic analogue of a black hole, or a dumb hole. After Unruh’s “dumb hole” proposal many different kinds of analogue black holes have been devised, based on condensed matter physics, slow light and so on [3, 4]. At present the Hawking temperatures associated with these analogues are too low to be detectable, but the situation is likely to change in the near future [5].

A full understanding of the classical physics of analogue black holes is necessary to control the experimental setup and to increases chances of detecting Hawking radiation. Furthermore, some purely classical phenomena shed light on quantum aspects of (analogue and general-relativistic) black hole physics. For example, positive and negative norm mixing at the horizon leads to non-trivial Bogoliubov coefficients in the calculations of Hawking radiation [6]; superradiant instabilities of the Kerr metric are related to the quantum process of Schwinger pair production [7]; and more speculatively (classical) highly damped black hole oscillations could be related to area quantization [8].

Here we summarize recent results concerning classical wave propagation in a rotating acoustic (2+1)-dimensional black hole, which corresponds to the “draining bathtup” fluid flow introduced by Visser [4]. It is not possible to put the entire Kerr spacetime into perfect-fluid “acoustic” form. However one can find some vortex geometries which are formally equivalent to the equatorial slice of the Kerr metric [9]. The extension of our results to these geometries should be straightforward.
### 1. Sound waves in a draining bathtub

Consider a fluid having (background) density $\rho$. Assume the fluid to be locally irrotational (vorticity free), barotropic and inviscid. From the equation of continuity, the radial component of the fluid velocity satisfies $\rho v_r \sim 1/r$. Irrotationality implies that the tangential component of the velocity satisfies $v_\theta \sim 1/r$. By conservation of angular momentum we have $\rho v_\theta \sim 1/r$, so that the background density of the fluid $\rho$ is constant. In turn, this means that the background pressure $p$ and the speed of sound $c$ are constants. The acoustic metric describing the propagation of sound waves in this “draining bathtub” fluid flow is [4]:

$$ds^2 = -\left(c^2 - \frac{A^2 + B^2}{r^2}\right)dt^2 + \frac{2A}{r}drdt - 2Bd\phi dt + dr^2 + r^2d\phi^2.$$  \hspace{1cm} (1)

Here $A$ and $B$ are arbitrary real positive constants related to the radial and angular components of the background fluid velocity:

$$\vec{v} = \frac{-A\hat{r} + B\hat{\theta}}{r}. \hspace{1cm} (2)$$

In the non-rotating limit $B = 0$ the metric (1) reduces to a standard Painlevé-Gullstrand-Lemaître type metric [10]. The acoustic event horizon is located at $r_H = A/c$, an ergosphere forms at $r_{ES} = (A^2 + B^2)^{1/2}/c$ and the angular velocity of the black hole $\Omega = Bc/A^2$.

Unruh [2] first realized that the propagation of sound waves in a barotropic inviscid fluid with irrotational flow is described by the Klein-Gordon equation $\nabla_\mu \nabla^\mu \Psi = 0$ for a massless field $\Psi$ in a Lorentzian acoustic geometry, which in our case takes the form (1). In our acoustic geometry we can separate variables in the Klein-Gordon equation by the substitution

$$\Psi(t, r, \phi) = R(r)e^{i(m\phi - \omega t)}. \hspace{1cm} (3)$$

We can also introduce a tortoise coordinate $r_*$, defined as usual by the condition $dr_*/dr = \Delta$, where $\Delta \equiv (1 - A^2/c^2r^2)^{-1}$. To obtain a Schrödinger-like wave equation we simply set $R = ZH$ and specify $Z$ by the requirement that the coefficient of $H_{r_*}$ be zero [11]. The resulting wave equation can be recast in a more convenient form by the following rescaling: $\hat{r} = rA/c$, $\hat{\omega} = \omega A/c^2$, $\hat{B} = B/A$. The rescaling effectively sets $A = c = 1$ in the original wave equation, so that the acoustic horizon $\hat{r}_H = 1$. From now on we shall omit hats in all quantities. We get

$$H_{r_*} + \left\{\left(\omega - \frac{Bm}{r^2}\right)^2 - \left(\frac{r^2 - 1}{r^2}\right) \left[1 + \frac{1}{r^2} \left(m^2 - \frac{1}{4}\right) + \frac{5}{4r^4}\right]\right\} H = 0. \hspace{1cm} (4)$$

The rescaled wave equation (4) will be the starting point of our analysis of quasinormal modes (QNMs), late-time tails and superradiant phenomena.

### 2. A concrete setup: gravity waves in a shallow basin

The acoustic metric (1) can be realized using a simple experimental setup, that was described in detail by Schützhold and Unruh [12]. In this setup it no longer describes a sonic analogue but rather a shallow basin gravity wave analogue of a black hole. The idea is to use gravity waves in a viscosity free, incompressible liquid with irrotational flow: under appropriate circumstances, one can envisage the use of common fluids like water or mercury. Schützhold and Unruh assumed a shallow water, long wavelength approximation: the gravity waves’ amplitude $\delta h$, their wavelength $\lambda$ and the depth of the basin $h_B$ are such that $\delta h \ll h_B \ll \lambda$. Relaxing the assumptions that the bottom of the tank and the background flow surfaces are flat and parallel, Schützhold and Unruh showed that the most general rotationally symmetric and locally irrotational background flow profile can be described precisely by the draining bathtub metric.
(1) when the radial slope of the bottom of the tank – that in cylindrical coordinates \((z, r, \phi)\) will be described by some function \(f(r)\) – is small: \(f'(r) \ll 1\). In this gravity wave black hole analogue the constants \(A\) and \(B\) are proportional to the radial and tangential components of the background flow velocity: \(v^\phi = B/r^2\), \(v^r = -Ah_\infty/\left[ r(1 + f'(r)^2)^{1/2} \right]\), where \(h_\infty\) is the height of the tank far from the black hole, and the slope of the tank satisfies the relation \(f(r) = -\left( A^2 + B^2 \right) gr^2 \).

In this equation \(g\) is the gravitational acceleration, related to the constant \(c\) of the acoustic black hole metric (1) by \(c = \sqrt{gh_\infty}\). One of the main advantages of this acoustic black hole is apparent from this equation: the speed of the gravity waves can simply be tuned to one’s needs by adjusting the height of the basin \(h_\infty\). Another advantage is that the inclusion of surface tension and viscosity can be used to manipulate the waves’ dispersion relation.

3. Quasinormal modes and power-law tails

Quasinormal modes (QNMs) of a rotating acoustic black hole can be defined in the usual way. Close to the event horizon the solutions of equation (4) behave as \(H \sim e^{\pm i(\omega - Bm)r^*}\). Since (classically) only ingoing waves should be present at the horizon, we must choose the minus sign in the exponential. At spatial infinity \(H \sim e^{\pm i\omega r}\). Now we require that only outgoing waves (waves leaving the domain under study) should be present, and correspondingly choose the plus sign in the exponential. These boundary conditions are satisfied only for discrete complex frequencies \(\omega = \omega_R + i\omega_I\): the QNM frequencies.

QNMs play a very important role in black hole physics [13]. They govern the late time behavior of waves propagating outside the black hole: any perturbation of the black hole, after an initial transient, will damp exponentially in the so-called “ringdown phase”, the damping time being given by \(\tau = 1/\omega_I\). The frequencies and damping times of this ringdown signal depend only on the black hole parameters (mass and angular momentum), and can therefore be used to estimate these parameters from observational data. According to some questionable speculations, highly damped QNMs may even yield some information on black hole area quantization in quantum theories of gravity [8].

QNMs of the draining bathtub metric have first been computed by a simple WKB approximation [14]. Then the calculation has been refined using Leaver’s continued fraction method [15]. Figure 1 shows numerical results from [15]: there is no evidence for instabilities of the “draining bathtub” metric. In the limit of large \(m\) the QNM frequencies can be obtained analytically using WKB arguments:

\[
\omega \sim m/2 - i(2n + 1)/(2\sqrt{2}), \quad m \to \infty. \tag{6}
\]

For a \((2 + 1)\)-dimensional acoustic black hole, a monodromy calculation shows that there are no asymptotic QN frequencies - a result that does not support any of the recent conjectures. This is no surprise anyway. Hod’s original argument [8] relies heavily on black hole thermodynamics, and the very formulation of the laws of black hole thermodynamics for analogue black holes is a non-trivial matter [16].

After the exponential decay characteristic of the ringdown phase, black hole perturbations decay with a power-law tail [17] due to backscattering off the background curvature. Given the wave equation, the power-law decay can be computed using some general results derived in [18]. For the “draining bathtub” fluid flow the field falloff at very late times turns out to be \(\Psi \sim t^{-(2m+1)}\), where \(m\) is the angular quantum number. This time exponent is characteristic of any \((2 + 1)\)-dimensional flow, and not just of a black hole, being related to the peculiar character of the Green functions in odd-dimensional spacetimes [19].
Figure 1. Real and imaginary parts of the first three QNMs with $m = 1$ (continuous lines) and $m = -1$ (dashed lines). For $m > 0$, at large $B$ the frequency $\omega_R \propto mB$; the damping is only weakly dependent on $B$. For $m < 0$ the oscillation frequency $\omega_R$ crosses zero at some critical value of $B$ that depends on the mode order; it is hard to follow numerically the modes beyond this point. Even at large rotation parameters $B$, there is no evidence for unstable modes (zero crossings of $\omega_I$). If the fundamental mode goes unstable at all, it only does in the limit $B \to \infty$. Results for other values of $m$ are similar.

4. Superradiance and the acoustic black hole bomb

The rotating draining bathtub metric, possessing an ergoregion, can display the phenomenon of superradiance [11, 12]. This effect was discovered by Zel’dovich [20], who pointed out that a cylinder made of absorbing material and rotating around its axis with frequency $\Omega$ can amplify modes of scalar or electromagnetic radiation of frequency $\omega$, provided the condition $\omega < m\Omega$ (where $m$ is the azimuthal quantum number with respect to the axis of rotation) is satisfied. An excellent review of (inertial and non-inertial) superradiant phenomena can be found in [21].

Zel’dovich suggested that a rotating black hole whose angular velocity at the horizon is $\Omega$ can similarly amplify incident waves of frequency $\omega$ if $\omega < m\Omega$. Consider an incident plane wave of unit amplitude at infinity, frequency $\omega$ and azimuthal index $m$. Part of this wave will be absorbed by the black hole and part of it will be reflected back, the reflection coefficient being some (complex) number $R_m(\omega)$. Reflection coefficients can be computed by a straightforward numerical integration of the wave equation [14]; when $|R_m(\omega)|^2 > 1$ we have superradiant amplification. Results obtained from numerical integrations for $m = 1$ are shown in Figure 2. An interesting feature of acoustic geometries is that the acoustic black hole spin can be varied independently of the black hole mass. Therefore, at variance with the Kerr metric, the spin can be made very large, $B \gg 1$ (at least in principle: cf. [14]). The possibility of attaining high values of the spin means that rotational superradiance in acoustic black holes could be very efficient and experimentally observable.

Enclosing a rotating black hole by a reflecting mirror we can exploit superradiance to destabilize the system: after each reflection the wave is superradiantly amplified, so that the initial perturbation grows exponentially with time. This “acoustic black hole bomb”, first conceived by Press and Teukolsky [22], has recently been studied in more detail [23]. The calculation of oscillation frequencies and growing timescales for the black hole bomb is an eigenvalue problem similar to the QNM problem, except that now we have boxed boundary conditions (that is, we must impose a vanishing wavefunction) at the mirror radius $r_0$. The real part of these “boxed QNM frequencies” can be computed analytically: it is given by
Figure 2. Superradiant reflection coefficient for \( m = 1 \) and different values of \( B \leq 1 \) (left), \( B \geq 1 \) (right). \(|R|^2\) drops to zero at the critical frequency for superradiance, \( \omega_{SR} = mB \).

\[ \omega_R = j_{m,n}/r_0, \] where the \( j_{m,n} \)'s are zeros of the Bessel function of integer order \( m \). The instability switches off (\( \omega_I \) goes through zero) at the critical radius predicted by the superradiance condition:

\[ r_0 = r_{0,\text{crit}} \approx j_{m,n}/(mB). \]

The growth timescale of the instability grows with \( B \). In practice, experimental difficulties will set an upper limit on \( B \): for example, in a gravity-wave analogue we can hardly have \( B > 10 \) [14]. In any case, it should not be difficult to set up a gravity-wave analogue to trigger an acoustic black hole bomb in the lab: this would be, as far as we know, the first experimental observation of rotational (non-inertial) superradiance.

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