Bounds on the smallest strong nonlocality set of multipartite quantum states

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An orthogonal set of states in multipartite systems is called to be strong quantum nonlocality if it is locally irreducible under every bipartition of the subsystems [Phys. Rev. Lett. \textbf{122}, 040403 (2019)]. In this work, we study a subclass of locally irreducible sets: to preserving the orthogonality of the states, only trivial local measurement can be performed by each partite. We call set with this property is locally stable. We find that in the case of two qubits systems locally stable set coincide with locally indistinguishable set. Then we present a characterization of locally stable set via the dimensions of some states depended on the given set. Moreover, we give two constructions of strongly nonlocal sets in multipartite quantum systems with arbitrary local dimensions. As a consequence, the two results give lower bound and upper bounds on the size of the smallest set which is locally stable for each bipartition. Our results provide a complete answer to an open question raised by Halder et al. and a recent paper [Phys. Rev. A \textbf{105}, 022209 (2022)]. Compared with all previous relevant proofs, our proof here is quite concise. Our results show that this kind of strong quantum nonlocality is an universal property in quantum mechanics which does not depend on the number of participants and local dimensions.

I. INTRODUCTION

Quantum state discrimination problem is a fundamental problem in quantum information theory [1]. A quantum system is prepared in a state which is randomly chosen from a known set. The task is to identify the state of the system. If the known set is orthogonal, then taking a states dependent project measurement can finish this task perfectly. However, if the known set is non-orthogonal, it is impossible to distinguish the states perfectly [1]. Most of time, our quantum states are distributed in composite systems. So only local operations and classical communication (LOCC) are allowed in the distinguishing protocol. Under this setting, if the task can be accomplished perfectly, we say that the set is locally distinguishable, otherwise, locally indistinguishable. Bennett et al. [2] presented the first example of orthogonal product states in $\mathbb{C}^3 \otimes \mathbb{C}^3$ that are locally indistinguishable and they named such a phenomenon as quantum nonlocality without entanglement. The nonlocality here is in the sense that the information of the given set that can be inferred by using global measurement is strictly large than those obtained via LOCC. The results of local indistinguishability of quantum states have been practically applied in quantum cryptography primitives such as data hiding [3, 4] and secret sharing [5–7].

Since Bennett et al.’s result [2], the quantum nonlocality based on local discrimination has been studied extensively (see Refs. [8–42] for an incomplete list). Walgate and Hardy [8] derived a simple method to prove the local indistinguishability of a given set. Their proof is based on the following two observations: (1) In each protocol that can perfectly distinguished the set of states, the states must remain to be orthogonal to each other after every local measurement. Such kinds of measurement are called orthogonality preserving measurement. (2) In any local distinguishing protocol, there must exist one of the parties goes first and can be able to perform some nontrivial orthogonality preserving measurement. Here a measurement is called nontrivial if not all the POVM elements are proportional to the identity operator. Given a set of orthogonal states, if one can show that each partite could only start with a trivial local orthogonal preserving measurement to the states, then one can conclude that the set is locally indistinguishable. Since 2014, lots of the study (see Refs. [30–44]) on the locally indistinguishable set of quantum states are based on the two observations. Therefore, it is interesting to characterize the set whose quantum nonlocality can be verified via this method.

Recently, Halder et al. [45] introduced a stronger form of local indistinguishability based on the concept of local irreducibility. A set of multipartite orthogonal quantum states is said to be locally irreducible if it is not possible to eliminate one or more states from that set using orthogonality preserving local measurement. The triviality of every orthogonality preserving local measurement is still sufficient to indicate the local irreducibility. Based on this method, they gave two sets of product states in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ and $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ that are locally irreducible for each bipartition of the corresponding tripartite systems named such phenomenon as strong quantum nonlocality. More results via this method on this kind of strong quantum nonlocality can be found in Refs. [46–52]. All these results consider systems with at most five partites and local dimensions greater or equal than three. Therefore, it is interesting to consider whether this kind of strong nonlocality is a universal property in the sense that its existence does not depend on the number of partites and local dimensions. A recent work indicates that this property does not depend on the number of partites [53]. In this work, we will show that it does not depend
on the local dimensions.

In this work, we study a subclass of locally irreducible set which is called locally stable set here. We obtain a characterization of locally stable set by some algebraic quantity which leads to a lower bound on the smallest set which is locally stable for each bipartition of the subsystems. Then we give two contructions of strongly nonlocal sets which leads to an upper bound on the smallest set which is locally stable for each bipartition of the subsystems.

The rest of this article is organized as follows. In Sec. II, we review the concept of local indistinguishable set and introduce a special form of quantum nonlocality called local stable set. In Sec. III, we first give a complete descriptions of local stable set in two qubits systems. Then we give a characterization of locally stable sets. In Sec. IV, we give two contructions of strongly nonlocal sets in general multipartite quantum systems. In Sec. V, we study the strong nonlocality of $W$ type states in multi-qubit systems. Finally, we draw a conclusion and present some interesting problems in Sec. VI.

II. PRELIMINARIES

For any positive integer $d \geq 2$, we denote $\mathbb{Z}_d$ as the set $\{0, 1, \cdots, d - 1\}$. Let $\mathcal{H}$ be an $d$ dimensional Hilbert space. We always assume that $\{\{0\}, \{1\}, \cdots, \{d - 1\}\}$ is the computational basis of $\mathcal{H}$. A positive operator-valued measure (POVM) on $\mathcal{H}$ is a set of positive semidefinite operators $\{E_x\}_{x \in X}$ such that $\sum_{x \in X} E_x = I_{\mathcal{H}}$ where $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. Throughout this paper, we do not normalize states for simplicity.

Now we give a brief review of some concept related to local discrimination of quantum states.

**Definition 1 (Locally indistinguishable)** A set of orthogonal pure states in multipartite quantum system is said locally indistinguishable, if it is not possible to distinguish the states by using LOCC.

**Definition 2 (Locally irreducible)** An orthogonal set of pure states in multipartite quantum system is locally irreducible if it is not possible to eliminate one or more states from the set by orthogonality-preserving local measurements.

A measurement is trivial if all its POVM elements are proportional to the identity operator. Note that every LOCC protocol that distinguishes a set of orthogonal states is a sequence of orthogonality preserving local measurements (OPLM). There is a sufficient condition to prove that an orthogonal set is nonlocality (strongly nonlocality): each subsystem can only perform a trivial orthogonality preserving local measurement (the subsystems are corresponding to each bipartition of the original subsystems). This motivates us to introduce the following concept.

**Definition 3 (Locally stable)** An orthogonal set of pure states in multipartite quantum system is said to be locally stable if only trivial orthogonality preserving measurement can be performed for each local subsystems.

To show a set of states is locally indistinguishable or locally irreducible, it is sufficient to show that it is locally stable (See Fig. 1). Halder et. al. [45] introduce the concept of strong nonlocality for an orthogonal set in multipartite systems. That is, an orthogonal set $\mathcal{S}$ in multipartite systems is called to have strong nonlocality if it is locally irreducible under every bipartition of the subsystems. Therefore, to show that $\mathcal{S}$ is strong nonlocality it is sufficient to show that $\mathcal{S}$ is locally stable under every bipartition of the subsystems (this is coincide with the strongest nonlocality in Ref. [51]).

In this paper, we mainly consider whether a given orthogonal set of multipartite states is locally stable or is locally stable under every bipartition of the subsystems. Let $\mathcal{H} = \bigotimes_{i=1}^n \mathcal{H}_A_i$ whose local dimensions are $\dim_{\mathcal{H}} \mathcal{H}_A_i = d_i$. We denote $s(d_1, d_2, \cdots, d_n)$ the smallest cardinality of those orthogonal sets in $\mathcal{H}$ that are locally stable. And we denote $S(d_1, d_2, \cdots, d_n)$ the smallest cardinality of those orthogonal sets in $\mathcal{H}$ that are locally stable under every bipartition of the subsystems. In this work, we will give some bounds on the two quantities.

III. CHARACTERIZATION OF LOCALLY STABLE SET AND LOWER BOUNDS ON $s(d_1, d_2, \cdots, d_n)$ AND $S(d_1, d_2, \cdots, d_n)$

First, we give a complete description of the locally stable set in two qubits systems. The proof is very similar with that in Ref. [8] where they considered locally indistinguishable set.

**Theorem 1** Let $\mathcal{S} = \{|\Psi_i\rangle\}_{i=1}^{|\mathcal{S}|}$ be an orthogonal set of pure states in $\mathbb{C}^2 \otimes \mathbb{C}^2$. Then $\mathcal{S}$ is locally stable if and only if it is locally indistinguishable. In fact, $\mathcal{S}$ is locally stable if and only if $|\mathcal{S}| \geq 3$ and contains at least two entangled states.
Proof. If $S$ is locally stable, it is obviously local indistinguishable. If $S$ is not locally stable, then one of the partities may perform some nontrivial orthogonality preserving local measurement. Without loss of generality, we assume the first part can perform such a measurement. Hence there exist a semidefinite positive operator $E_x = M_x^*M_x$ which is not proportional to $I_2$ such that the elements in $S' := \{M_x \otimes I_2 | \psi_i \} |i = 1, \cdots , I\}$ are mutually orthogonal. By the singular value decomposition, there are two orthonormal sets \{v_{11}, v_{22}\} \{(v_{1j} | v_{2j}) = \delta_{i,j}\}$ and \{w_1, w_2\} \{(w_1 | w_2) = \delta_{i,j}\}$ such that

$$M_x = \lambda_1 |v_{11}\rangle \langle w_1| + \lambda_2 |v_{22}\rangle \langle w_2|$$

where $\lambda_1, \lambda_2 \geq 0$ are real numbers. As $E$ is not proportional to $I_2$, we have $\lambda_1 \neq \lambda_2$. Each $|\psi_i\rangle \ (1 \leq i \leq I)$ can be expressed as the following form

$$|\psi_i\rangle = |w_1\rangle \langle \psi_{i,1}| + |w_2\rangle \langle \psi_{i,2}|$$

here $|\psi_{i,j}\rangle$ may be unnormalized and even be zero vector. As both $S$ and $S'$ are orthogonal sets, we have

$$\langle \psi_{i,1} | \psi_{j,1}\rangle + \langle \psi_{i,2} | \psi_{j,2}\rangle = 0,
\lambda_1 \langle \psi_{i,1} | \psi_{j,1}\rangle + \lambda_2 \langle \psi_{i,2} | \psi_{j,2}\rangle = 0 \quad (1)$$

for $1 \leq i \neq j \leq I$. Therefore, we have $\langle \psi_{i,1} | \psi_{j,1}\rangle = \langle \psi_{i,2} | \psi_{j,2}\rangle = 0$ as $\lambda_1 \neq \lambda_2$. With this at hand, Alice and Bob can provide a local protocol to distinguish the set $S$. In fact, Alice can perform the measurement $\pi_A := \{v_{1j} | v_{1j} | v_{2j} | v_{2j}\}$ to the set $S$. For each outcome of this measurement, the possible states of Bob's part are orthogonal and hence can be distinguished by himself after he receiving the outcome of Alice.

The remaining result can be deduced from the known result (See Ref. [8]) that $S \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2$ is locally indistinguishable if and only if $|S| \geq 3$ and contains at least two entangled states. This completes the proof.

Before studying the locally stable set for general bipartite systems, we introduce some notation that may be used throughout this section.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert space of dimensional $d_1$ and $d_2$ respectively. Denote $L(\mathcal{H}_2, \mathcal{H}_1)$ by all the linear operations mapping $\mathcal{H}_2$ to $\mathcal{H}_1$. There is a nature linear correspondence between the two linear spaces $L(\mathcal{H}_2, \mathcal{H}_1)$ and $\mathcal{H}_1 \otimes \mathcal{H}_2$. Exactly, this correspondence is given by the linear mapping vec : $L(\mathcal{H}_2, \mathcal{H}_1) \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$ defined by the action on the basis vec$(|i\rangle |j\rangle) := |i\rangle |j\rangle$. More generally, if $|a\rangle := \sum_{i \in \mathbb{Z}_{d_1}} a_i |i\rangle \in \mathcal{H}_1$ and $|b\rangle := \sum_{j \in \mathbb{Z}_{d_2}} b_j |j\rangle \in \mathcal{H}_2$, then one can check that

$$\text{vec}(\langle a | b \rangle) = \langle a | \overline{b} \rangle \quad (2)$$

where $\overline{b} := \sum_{j \in \mathbb{Z}_{d_2}} \overline{b_j} |j\rangle$ and $\overline{b_j}$ is the complex conjugate of $b_j$. This mapping is also an isometry, in the sense that

$$\langle M, N \rangle = \langle \text{vec}(M), \text{vec}(N) \rangle. \quad (3)$$

where $M, N \in L(\mathcal{H}_2, \mathcal{H}_1)$. Here the inner product $\langle M, N \rangle$ is defined as $\text{Tr}[M^* N]$ for $M, N \in L(\mathcal{H}_2, \mathcal{H}_1)$ and $\langle |u\rangle | v \rangle$ is defined as $\langle u | v \rangle$ for $|u\rangle, |v\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$.

Let $\mathcal{H}_A \otimes \mathcal{H}_B$ be a composed bipartite systems whose local dimensions are $d_A$ and $d_B$ respectively. Suppose $\{|i\rangle_A | i \in \mathbb{Z}_{d_A}\}$ and $\{|j\rangle_B | j \in \mathbb{Z}_{d_B}\}$ be the computational bases of systems $A$ and $B$ respectively. Given an orthogonal set $S = \{|\psi_k\rangle \}_{k=1}^N$ of pure states in $\mathcal{H}_A \otimes \mathcal{H}_B$, our goal is to determine whether there is a nontrivial orthogonality preserving local measurement to this set.

For each $|\psi_k\rangle$, we can write it as the form $|\psi_k\rangle = \sum_{i \in \mathbb{Z}_{d_A}} |i\rangle_A |\psi_{k,i}\rangle_B$ where $|\psi_{k,i}\rangle_B$ may not be normalized and even may be equal to zero. If $M$ is a POVM element on subsystem $B$ that preserves the orthogonality relation, then we have the following equality $\langle \psi_k | A \otimes M | \psi_l \rangle = 0$, for $k \neq l$. Substituting the expression of $|\psi_k\rangle$ to these equations, one obtain that

$$\sum_{i \in \mathbb{Z}_{d_A}} \text{Tr}[M |\psi_{k,i}\rangle_B \langle \psi_{i,i}|B] = \sum_{i \in \mathbb{Z}_{d_A}} B |\psi_{k,i}\rangle_B \langle \psi_{i,i}|B) = 0. \quad (4)$$

Applying Eqs. (2) and (3) to the left hand side term of the above equation, one obtain that

$$(\text{vec}(M), \sum_{i \in \mathbb{Z}_{d_A}} |\psi_{k,i}\rangle_B \otimes |\psi_{i,i}|B) = 0 \quad (4)$$

whenever $k \neq l$. That is, for each pair $(k,l)$ with $1 \leq k \neq l \leq N$, the vector $\sum_{i \in \mathbb{Z}_{d_A}} |\psi_{k,i}\rangle_B \otimes |\psi_{i,i}|B$ is orthogonal to the vector vec$(M)$ in the linear space $\mathcal{H}_B \otimes \mathcal{H}_B$. Let $D_{A|B}(S)$ denotes the linear subspace of $\mathcal{H}_B \otimes \mathcal{H}_B$ spanned by $\sum_{i \in \mathbb{Z}_{d_A}} |\psi_{i,i}|B \otimes |\psi_{i,i}|B$ with $1 \leq k \neq l \leq N$. As one note that vec$(B)$ is a nonzero vector that satisfies all the relations in Eq. (4), we have $\text{dim}_C[D_{A|B}(S)] \leq d_B^2 - 1$. Moreover, as the map vec is an isometry, if $\text{dim}_C[D_{A|B}(S)] = d_B^2 - 1$, one can conclude that the POVM measurement $M$ that satisfies all the relations must be proportional to $I_B$. Similarly, we can define a subspace $D_{B|A}(S)$ of $\mathcal{H}_A \otimes \mathcal{H}_A$ in which case we use the decomposition of $|\psi_k\rangle = \sum_{j \in \mathbb{Z}_{d_B}} |j\rangle_B |\psi_{j,k}\rangle_A$ where $|\psi_{j,k}\rangle_A$ may not be normalized and even may be equal to zero.

**Theorem 2** Let $S$ be an orthogonal set of pure state in $\mathcal{H}_A \otimes \mathcal{H}_B$ whose local dimensions are $d_A$ and $d_B$. Let $D_{A|B}(S)$ and $D_{B|A}(S)$ be the linear spaces defined as above. Then the set $S$ is locally stable if and only if both of the following equalities are satisfied

$$\text{dim}_C[D_{A|B}(S)] = d_B^2 - 1 \text{ and } \text{dim}_C[D_{B|A}(S)] = d_A^2 - 1.$$  

**Proof.** Sufficiency. If $\text{dim}_C[D_{A|B}(S)] = d_B^2 - 1$, by the previous statement, the Bob's site can only start with a trivial orthogonality preserving measurement. If $\text{dim}_C[D_{B|A}(S)] = d_A^2 - 1$, so does Alice. Therefore, the set $S$ is locally stable by definition.

Necessity. Suppose not, without loss of generality, we could assume that $\text{dim}_C[D_{A|B}(S)] \leq d_B^2 - 2$ as by construction we always have $\text{dim}_C[D_{A|B}(S)] \leq d_B^2 - 1$. Because that the map vec is an isometry, we have the C-
Now we define $R$ with this note, we have $R$ completion space of $\mathcal{R}$. We claim that

$$\{ \sum_{i \in \mathbb{Z}_{+}} |\psi_{k,i} \rangle B |\psi_{l,i} \rangle | 1 \leq k \neq l \leq N \}$$

has the same dimension as $\mathcal{D}_{AB}(\mathcal{S})$. One note that for each $1 \leq k \neq l \leq N$, the set with two matrices $\{ \sum_{i \in \mathbb{Z}_{+}} |\psi_{k,i} \rangle B |\psi_{l,i} \rangle | \sum_{i \in \mathbb{Z}_{+}} |\psi_{l,i} \rangle B |\psi_{k,i} \rangle \}$ is linear equivalent to the set with the following two Hermitian matrices

$$
H_{kl} := \sum_{i \in \mathbb{Z}_{+}} |\psi_{k,i} \rangle B |\psi_{l,i} \rangle + |\psi_{l,i} \rangle B |\psi_{k,i} \rangle,
$$

$$
H_{kk} := \sum_{i \in \mathbb{Z}_{+}} i|\psi_{k,i} \rangle B |\psi_{l,i} \rangle - i|\psi_{l,i} \rangle B |\psi_{k,i} \rangle.
$$

With this note, we have

$$\mathcal{C}_{AB}(\mathcal{S}) = \text{span}_C \{ H_{kl} | 1 \leq k \neq l \leq N \}.$$ 

Now we define $\mathcal{R}_{AB}(\mathcal{S}) = \text{span}_R \{ H_{kl} | 1 \leq k \neq l \leq N \}$. We claim that

$$\text{dim}_R[\mathcal{R}_{AB}(\mathcal{S})] \leq \text{dim}_C[\mathcal{C}_{AB}(\mathcal{S})].$$

Clearly, $\mathcal{R}_{AB}(\mathcal{S}) \subseteq \mathcal{C}_{AB}(\mathcal{S})$ and all the matrices in $\mathcal{R}_{AB}(\mathcal{S})$ are Hermitian. To prove the above claim, it is sufficient to prove that if $H_1, H_2, \cdots, H_L \in \mathcal{R}_{AB}(\mathcal{S})$ are $R$-linear independence, then they are also $C$-linear independence. If not, there exists not all zero $x_j + iy_j \in \mathbb{C}$, $1 \leq j \leq L$ (here $x_j, y_j \in \mathbb{R}$ and we can always assume that some $x_j \neq 0$, otherwise multiplying both sides by $i$) such that

$$\sum_{j=1}^{L} (x_j + iy_j) H_j = 0.$$  \hspace{1cm} (5)

Take a complex conjugacy to both sides, we obtain

$$\sum_{j=1}^{L} (x_j - iy_j) H_j = 0.$$ \hspace{1cm} (6)

From Eqs. (5) and (6), we obtain that $\sum_{j=1}^{L} \text{det} H_j = 0$ which is contradicted with the assumption that $H_1, H_2, \cdots, H_L$ are $R$-linear independence. Therefore, we have

$$\text{dim}_R[\mathcal{R}_{AB}(\mathcal{S})] \leq \text{dim}_C[\mathcal{C}_{AB}(\mathcal{S})] \leq d_B^2 - 2.$$ 

We know that all the $d_B \times d_B$ Hermitian matrices form an $R$-linear space of dimensional $d_B^2$. As $1_{d_B}$ lies in the completion space of $\mathcal{R}_{AB}(\mathcal{S})$ and $\text{dim}_R[\mathcal{R}_{AB}(\mathcal{S})] \leq d_B^2 - 2$, there exists at least some other nonzero Hermitian matrix said $M_B$ which is orthogonal to the space $\text{dim}_R[\mathcal{R}_{AB}(\mathcal{S})]$ and the identity matrix $1_{d_B}$. Multiplying some nonzero real number, we can always assume that each eigenvalue $\lambda_i$ of $M_B$ satisfies $|\lambda_i| \leq 1/4$. Then we have both $M_1 := 1_{d_B}/2 + M_B$ and $M_2 := 1_{d_B}/2 - M_B$ are semidefinite positive and $M_1 + M_2 = 1_{d_B}$. Therefore, $\{ M_1, M_2 \}$ is a POVM. By definition, $M_1, M_2$ lie in the completion space of $\text{dim}_R[\mathcal{R}_{AB}(\mathcal{S})]$, hence it is an orthogonality preserving measurement with respect to the set $\mathcal{S}$. Moreover, it is easy to see that it is a nontrivial measurement. So this is contradicted to the condition that $\mathcal{S}$ is locally stable. Therefore, we must have $\text{dim}_C[\mathcal{D}_{AB}(\mathcal{S})] = d_B^2 - 1$.

This completes the proof.

As the subspace $\mathcal{D}_{BA}(\mathcal{S})$ (w.r.t. $\mathcal{D}_{AB}(\mathcal{S})$) is completely determined by $(N - 1)N$ generators, therefore, we can use an $(N - 1)N \times d_B^2$ matrix to represent it. And we denote the matrix as $\mathcal{D}_{BA}(\mathcal{S})$ (w.r.t. $\mathcal{D}_{AB}(\mathcal{S})$).

**Example 1** The set $\mathcal{S}_B$ with three Bell states is locally stable, where

$$\mathcal{S}_B := \{|\psi_{\pm} := |00 \rangle \pm |11 \rangle, |\phi_+ := |01 \rangle + |10 \rangle\}.$$ 

One can easily calculate out the matrix $\mathcal{D}_{AB}(\mathcal{S}_B)$ which we display in the following

$$\begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & 0 \\
0 & -1 & 1 & 0 \\
\end{bmatrix}.$$
dimC[δ̂_{A,i}|A_i⟩(S)] = d_i^2 - 1 if and only if the A_i part can only perform a trivial orthogonality preserving measurement. With these two equivalent relations, it is easy to recover the above two statements.

Using Theorem 3, we could derive a bound on the cardinality of a locally stable set in some given multipartite systems.

**Theorem 4 (Bounds on the size of locally stable set)**

Let S be an orthogonal set of pure state in \( ⊗_{i=1}^n H_{A_i} \), whose local dimension is \( \dim_{C}(H_{A_i}) = d_i \). We have the following statement:

(a) If the set S is locally stable, then \( |S| \geq \max_i\{d_i + 1\} \). Consequently,

\[
s(d_1, d_2, \ldots, d_n) \geq \max_i\{d_i + 1\}.
\]

(b) If the set S is locally stable for each bipartition of the subsystems, then \( |S| \geq \max_i\{d_i + 1\} \). Consequently,

\[
S(d_1, d_2, \ldots, d_n) \geq \max_i\{d_i + 1\}.
\]

Proof. (a) By Theorem 4, we have \( D_{A,i|A_i}(S) = d_i^2 - 1 \). And by the definition of \( D_{A,i|A_i}(S) \), we have

\[
d_i^2 - 1 = \dim_{C}[\delta_{A,i|A_i}(S)] \leq |S|^2 - |S|.
\]

Therefore, \( |S| > d_i \) for each \( 1 \leq i \leq n \).

(b) The proof is similar with the above proof.

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**IV. TWO CONSTRUCTIONS OF STRONGLY NONLOCAL SETS AND UPPER BOUNDS ON \( S(d_1, d_2, \ldots, d_n) \)**

Generally, it is difficult to show that the orthogonality preserving local measurement for each subsystem is trivial. We list two useful lemmas (developed in Ref. [50]) for verifying the trivialization of such measurement.

**Lemma 1 (Block Zeros Lemma)** Let an n × n matrix \( E = (a_{i,j})_{i,j \in Z_n} \) be the matrix representation of an operator \( E \) under the basis \( S = \{0\}, \{1\}, \ldots, \{n-1\} \).

Given two nonempty disjoint subsets \( S \) and \( T \) of \( B \), assume that \( \{|\psi_1\rangle\}_{i=0}^{-1}, \{|\phi_\ell\rangle\}_{\ell=0}^{l-1} \) are two orthogonal sets spanned by \( S \) and \( T \) respectively, where \( s = |S| \), and \( t = |T| \). If \( \langle \psi_1|E|\phi_\ell \rangle = 0 \) for any \( i \in S, j \in Z_n \), then \( \langle x|E|y \rangle = \langle y|E|x \rangle = 0 \) for \( |x \rangle \in S \) and \( \langle y \rangle \in T \).

**Lemma 2 (Block Trivial Lemma)** Let an n × n matrix \( E = (a_{i,j})_{i,j \in Z_n} \) be the matrix representation of an operator \( E \) under the basis \( S = \{0\}, \{1\}, \ldots, \{n-1\} \).

Given a nonempty subset \( S \) of \( B \), let \( \{|\psi_\ell\rangle\}_{\ell=0}^{l-1} \) be an orthogonal set spanned by \( S \). Assume that \( \langle \psi_1|E|\psi_\ell \rangle = 0 \) for any \( i \neq j \in Z_n \). If there exists a state \( |x \rangle \in S \), such that \( \langle x|E|y \rangle = 0 \) for all \( |y \rangle \in S \setminus \{|x\} \) and \( \langle x|\psi_\ell \rangle \neq 0 \) for any \( j \in Z_n \), then \( \langle y|E|z \rangle = 0 \) and \( \langle y|E|y \rangle = \langle z|E|z \rangle = 0 \) for all \( |y \rangle, |z \rangle \in S \) with \( \langle y \rangle \neq \langle z \rangle \).

In this section, we provide two constructions of strongly nonlocal sets: \( S \) (all but one state are genuinely entangled) and \( S_C \) (all but states are genuinely entangled).

Let \( \mathcal{H} := \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_N} \) be an N parties quantum systems with dimensional \( d_i \) for the ith subsystem. A string \( i = (i_1, i_2, \cdots, i_N) \) of \( C := Z_{d_1} \times Z_{d_2} \times \cdots \times Z_{d_N} \) is called weight \( k \) if there are exactly \( k \) nonzero \( i_j \)’s. And we denote the set of all weight \( k \) strings of \( C \) as \( C_k \) where \( 0 \leq k \leq N \). Set \( c_k \) be the cardinality of the set \( C_k \).

For each \( k \in \{0, 1, \ldots, N-1\} \), we define

\[
S_k := \{|\Psi_{k,i}\rangle \in \mathcal{H} \mid i \in C_k, \langle \Psi_{k,i}| := \sum \omega^{i_j k} |j\rangle |j\rangle \}.
\]

Here \( f_k : C_k \to Z_{c_k} \) is any fixed bijection and \( \omega := e^{2\pi i/N} \).

**Theorem 5** Let \( \mathcal{H} := \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_N} \) be an N parties quantum systems with dimensional \( d_i \) for the ith subsystem. The set \( S := \bigcup_{k=0}^{N-1} S_k \) is locally stable under each bipartition of the subsystems. The cardinality of the set is \( \prod_{n=1}^N d_n - \prod_{n=1}^N (d_n - 1) \). As a consequence,

\[
S(d_1, d_2, \ldots, d_n) \leq \prod_{n=1}^N d_n - \prod_{n=1}^N (d_n - 1).
\]

Proof. First, we show that \( A_1 = A_2 A_3 \cdots A_N \) can only perform a trivial orthogonality preserving measurement (OPM). Suppose \( \{M_{k} x \} \in S \) is an orthogonality preserving measurement with respect to the set \( S \) which is performed by \( C \), i.e., \( \langle \Psi |A_1 \otimes M_{k} \otimes \Phi \rangle = 0 \) for any two different \( \{|\Psi \rangle, \Phi \rangle \} \in S \). Set \( E := \mathbb{I}_{d_1} \otimes M_{k} X \). Let \( k, l \in Z_N \) and suppose \( k \neq l \). As \( C_k \cap C_l = \emptyset \), applying Block Zeros Lemma to the sets of base vectors corresponding to \( C_k \) and \( C_l \), we obtain that

\[
\langle i_k |E| j_l \rangle = \langle j_l |E| i_k \rangle = 0
\]

for any \( i_k \in C_k \) and \( j_l \in C_l \). Now we claim that for any \( k \in \{0, 1, \ldots, N-1\} \), if \( i_k, i_k' \) are two different strings of \( C_k \), then we also have

\[
\langle i_k |E| i_k' \rangle = \langle i_k' |E| i_k \rangle = 0.
\]

Moreover, \( \langle i_k |E| i_k \rangle = \langle 0 |E| 0 \rangle \). As \( C \leq N := \bigcup_{k=0}^{N-1} C_k \) contains \( \{0\} \times Z_{d_2} \times \cdots \times Z_{d_N} \), from the above relations, one could conclude that \( M_{k} \otimes X \times \mathbb{I}_{\mathcal{A}_1} \).

In the following, we will give a proof of the above claim by induction. First, the claim is true for \( k = 0 \). Now we assume that this claim is true for \( 0 \leq k < N - 1 \). Let \( l = k + 1 \) and fix any \( j_l = (j_1, j_2, \cdots, j_N) \in C_l \) such that \( j_1 \neq 0 \). For any \( i_l = (i_1, i_2, \cdots, i_N) \in C_l \) which is different from \( j_l \). If \( i_1 \neq j_1 \),

\[
\langle j_l |E| i_l \rangle = \langle j_l |E| i_l \rangle = 0.
\]

If \( i_1 = j_1 \), set \( j_k := (0, j_2, \cdots, j_N) \) and \( i_k := (0, i_2, \cdots, i_N) \), by definition, they are different strings of \( C_k \). Moreover,

\[
\langle j_l |E| i_l \rangle = \langle j_2 \cdots j_N |M_{k} \otimes M_{l} |i_2 \cdots i_N \rangle = \langle j_k |E| i_k \rangle = 0
\]
by induction. Applying Block Trivial Lemma to the set of base vectors corresponding to $C_l$, the set \{$(\langle i|, i\rangle)_{i \in \mathbb{Z}_{c_k}}$ and the vector $|j\rangle$, we obtain that for any different strings $i, i' \in C_l$,

$$\langle i|E|i\rangle = \langle i'|E|i\rangle = 0,$$

and $\langle j|E|j\rangle = \langle j|E|j\rangle$.

Note that $\langle j|E|j\rangle$ equals to

$$\langle j_2 \cdots j_N|M^*_EM_j|j_2 \cdots j_N\rangle = \langle j\rangle E|j\rangle = \langle 0|E|0\rangle.$$

This completes the proof of the claim. Therefore, the last $(N-1)$-parties could only start with a trivial OPM.

By the symmetric construction, one can also show that any $(N-1)$ parties could only start with a trivial OPM. This statement also implies that any $k$ (where $1 \leq k \leq N-1$) parties could only start with a trivial OPM.

Note that the elements in $S$ are not always with genuine entanglement. In fact, $|\Psi_{ij}\rangle = |0\rangle$ is fully product states. Now we claim that except this state, all others are with genuine entanglement. We only need to show that $|\Psi_{i,k,i}\rangle (1 \leq k \leq N-1, i \in \mathbb{Z}_{c_k})$ is entangled for any bipartition of the subsystems. We assume that the bipartition is $\{A_j\}|J|\{B_j\}$ where $I \cup J = \{1, 2, \cdots, N\}$, disjoint and $I \cup J = \{1, 2, \cdots, N\}$. Let $A$ and $B$ denote the computational bases of the systems $\{A_j\}$ and $\{B_j\}$ respectively. Suppose that $|\Psi_{i,k}\rangle = \sum_{a|A} \sum_{b|B} \psi_{a,b}|a\rangle \langle b|$. It sufficient to prove that the rank of the matrix $(\psi_{a,b})$ is greater than one. Clearly, $k$ can be expressed as two different forms such that $0 \leq k \leq |I|$ and $0 \leq t \leq |J|$. Suppose $k = s_1 + t_1 = s_2 + t_2$ such that $0 \leq s_1 < s_2 \leq |I|$ and $0 \leq t_2 < t_1 \leq |J|$. Choose any subsets $I_x \subseteq I \cap J \subseteq J$ such that $|I_x| = s_x$ for $x = 1, 2$ $(|J_y| = t_y$ for $y = 1, 2$).

We define

$$|I, I_x \rangle := \left((\otimes_{i \in I_x} 1_{A_i}) \otimes (\otimes_{i \in I \setminus I_x} 0_{A_i})\right) \in A,$$

$$|J, J_y \rangle := \left((\otimes_{j \in J_y} 1_{A_j}) \otimes (\otimes_{j \in J \setminus J_y} 0_{A_j})\right) \in B,$$

where $x, y \in \{1, 2\}$. The matrix $(\psi_{a,b})$ has the $2 \times 2$ minor

$$\begin{pmatrix} |I, I_x \rangle & |I, I_x \rangle & |J, J_y \rangle & |J, J_y \rangle \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\alpha \neq 0$. Therefore, the Schmidt rank of $|\Psi_{i,k}\rangle$ across this partition $\{A_j\}|J|\{B_j\}$ is greater than 1. Hence it is entangled.

In Theorem 5, if we replace the set $S_0$ by two states $|\Psi_{i,k}\rangle := \langle 0| \pm |1\rangle$ where $|0\rangle = \otimes_{i=1}^{n} |0\rangle_{A_i}$, $|1\rangle = \otimes_{i=1}^{n} |1\rangle_{A_i}$, and denote the new total set as $S_C$, then the set $S_C$ is genuinely entangled set that also has property of strong nonlocality. In fact, for each $1 \leq k \leq N-1, i \in \mathbb{Z}_{c_k}$, from the orthogonal relations

$$\langle \Psi_{i,k}|E|\Psi_{i,k}\rangle = 0,$$

we can deduce the orthogonal relations $\langle 0|E|\Psi_{i,k}\rangle = 0$. Therefore, the orthogonal relations of $S_C$ contains those from $S$ in Theorem 5. Using these relations, we could obtain that $S_C$ is also locally stable under each bipartition of the subsystems.

V. MORE EXAMPLES WITH STRONG NONLOCALITY

In this part, we try to use the algebraic quantities in section III to find more sets which has the property of strong nonlocality. The first result is out of our expectation. Using entangled states, three qubits is enough to show the strong nonlocality. We use the following Pauli gate operations $X, Y, Z,$ and the identity operation $I$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, the W state is $|100⟩ + |010⟩ + |001⟩$. Let $U(W)$ denote the matrix

$$X \otimes I \otimes I + Z \otimes X \otimes I + Z \otimes Z \otimes X.$$

Then the set $S_W := \{U(W)|ijk⟩ \mid i, j, k \in \mathbb{Z}_2\}$ (the states can be seen in the following table) is an orthogonal basis whose elements are all locally unitaly equivalent to the above W state [54].

| $\langle ijk|$ | $U(W)|ijk⟩$ | label |
|----------------|-----------------|-------|
| $|000⟩$ | $|000⟩ + |010⟩ + |001⟩$ | $|\psi_1⟩$ |
| $|001⟩$ | $|010⟩ + |011⟩ + |000⟩$ | $|\psi_2⟩$ |
| $|010⟩$ | $|110⟩ + |000⟩ - |111⟩$ | $|\psi_3⟩$ |
| $|011⟩$ | $|111⟩ + |010⟩ - |100⟩$ | $|\psi_4⟩$ |
| $|100⟩$ | $|000⟩ - |110⟩ - |110⟩$ | $|\psi_3⟩$ |
| $|101⟩$ | $|010⟩ - |100⟩ + |111⟩$ | $|\psi_7⟩$ |
| $|110⟩$ | $|011⟩ - |101⟩ + |110⟩$ | $|\psi_8⟩$ |

Using Matlab we can show that the ranks of the matrices $D_{R(R)}(S_W)$ $(R = A, B, C)$ are all 15. Therefore, by Theorem 3, the set $S_W$ is locally stable for every bipartition. Moreover, the statement holds also for any its subset with 6 elements.

Generally, we have an orthonormal basis whose elements are all locally unitaly equivalent to the generalized W state in n-qubit [54]. For any integer $n \geq 3$, we denote

$$U_n(W) := \sum_{i=1}^{n} Z_i \cdots Z_{i-1} X_i I_{i+1} \cdots I_n.$$

The set defined by $S_n^{(W)} := \{U_n(W)|i⟩ \mid i \in \mathbb{Z}_2^n\}$ is such a set.

Lemma 3 The set $S_n^{(W)}$ is an orthogonal set of pure states whose elements are all locally unitary equivalent to the multiqubit W state $|W_n⟩ := \sum_{i=1}^{n} |01\cdots00⟩ |01\cdots0⟩ |0⟩$ (see Fig. 2).
Proof. The prove that $S_n(W)$ is an orthogonal set, it is sufficient to prove that $U_n(W)$ is a unitary matrix up to a constant. This can be easily deduced when we notice that the anti-commutative relation $XZ = -ZX$. Therefore,

$$U_n(W)U_n(W)^\dagger = \sum_{k=1}^{n} \sum_{l=1}^{n} (Z_{k-1}X_kI_{k+1} \cdots I_n)(Z_l \cdots Z_{l-1}X_lI_{l+1} \cdots I_n)$$

$$= \sum_{k<l} I_l \cdots I_{k-1}(X_kZ_k + Z_kX_k)Z_{k+1} \cdots Z_{l-1}X_lI_{l+1} \cdots I_n$$

$$+ \sum_{k=1}^{n} I_l \cdots I_{k-1}I_kI_{k+1} \cdots I_n$$

$$= nI^{\otimes n}.$$

Clearly, $U_n(W)|0\rangle = |W_n\rangle$ which is exactly the $n$-qubit $W$ state. For any $i = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}_2^n$, the state $|\psi_i\rangle := U_n(W)|i\rangle$ can be written as

$$|\psi_i\rangle = U_n(W)X_1^{i_1}X_2^{i_2} \cdots X_n^{i_n}|0\rangle.$$

One can check that

$$X_1^{i_1}X_2^{i_2} \cdots X_n^{i_n}|\psi_i\rangle = \sum_{l=1}^{n} (-1)^{i_1+i_2+\cdots+i_{l-1}}|0_1 \cdots 0_{l-1}1_{l}0_{l+1} \cdots 0_n\rangle.$$

For each $l \in \{1, \cdots, n\}$, define $\theta_l := i_1 + \cdots + i_{l-1}$ and $Z(\theta) := |0\rangle\langle 0| + (-1)^{\theta}|1\rangle\langle 1|$. Then

$$\otimes_{i=1}^{n} (Z_l(\theta_l)X_l^{i_l})|\psi_i\rangle = |W_n\rangle.$$

That is, $|\psi_i\rangle$ is locally unitary equivalent to $|W_n\rangle$. \qed

In fact, by randomly choosing a subset of $S_n(W)$ with $2^n - 2^{n-2}$ elements, we find that it is locally stable for each bipartition for $3 \leq n \leq 8$. We conjecture that the above statement should indeed hold for all $n$-qubit systems provided $n \geq 3$. Moreover, we have obtained some numerical results on the smallest set that can show the local stability. Based on our numerical results (for systems with small dimension), we conjecture that the bound in Theorem 3 is even compact!

Conjecture 1 Let $\mathcal{H} = \otimes_{i=1}^{n}\mathcal{H}_{A_i}$ be a $n$-parties quantum systems whose local dimension $\dim_{C}(\mathcal{H}_{A_i}) = d_i \geq 2$. Then the following two statements hold

(a) There exists some orthogonal set $S$ of pure states in $\mathcal{H}$ such that it is locally stable and $|S| = \max_i\{d_i+1\}$. That is,

$$s(d_1, d_2, \ldots, d_n) = \max_i\{d_i + 1\}.$$

(b) There exists some orthogonal set $S$ of pure states in $\mathcal{H}$ such that it is locally stable for each bipartition of the subsystems and $|S| = \max_i\{d_i+1\}$. That is,

$$S(d_1, d_2, \ldots, d_n) = \max_i\{d_i + 1\}.$$

VI. CONCLUSION AND DISCUSSION

In this paper, we studied a special class of sets with quantum nonlocality, i.e., the locally stable sets. Locally stable sets are always locally indistinguishable set. And we found that the two concepts are coincide only in two qubits systems. We obtained an algebraic characterization of locally stable set. As a consequence, we obtained a lower bound of the cardinality on the locally stable set (and locally stable set under each bipartition of subsystems).

Moreover, we presented two constructions of sets that are locally stable set under each bipartition of subsystems. Its proof can be directly verified based on two basic lemmas developed in Ref. [50]. One of the set contains a fully product state. The other set contains only genuinely entangled states. Our result give a complete answer to an open question raised in Ref. [45] and Ref. [53]. This result gives an upper bound on the the smallest cardinality of those orthogonal sets in multipartite systems that are locally stable under every bipartition of the subsystems.

There are also some questions left to be considered. We conjectured that there is some set of cardinality $\max_i\{d_i + 1\}$ of orthogonal states in $\otimes_{i=1}^{n}\mathcal{H}_{A_i}$ (where $d_i = \dim_{C}(\mathcal{H}_{A_i})$) that is locally stable. We also conjecture that there is some set of cardinality $\max_i\{d_i + 1\}$ of orthogonal states in $\otimes_{i=1}^{n}\mathcal{H}_{A_i}$ (where $d_i = \dim_{C}(\mathcal{H}_{A_i})$) and $d_i = (\prod_{j=1}^{n} d_j)/d_i$ that is locally stable in every

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**FIG. 2.** This is the circuit which transfer the state $|\psi_i\rangle$ to $|W_n\rangle$.

Using the Matlab, we can calculate the rank of theirs corresponding quantities to check whether these sets are locally stable or not.

Example 2 The set $S_n(W) := \{U_n(W)|i\rangle \mid i \in \mathbb{Z}_2^n\}$ is locally stable under each bipartition of the subsystems for $3 \leq n \leq 8$.  

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**TABLE 1.** This is a table which shows the rank of theirs corresponding quantities to check whether these sets are locally stable or not.
bipartition of the subsystems. In addition, it is also to consider the smallest sets of product states that are locally stable. We hope that the study of locally stable sets will enrich our understanding of the quantum nonlocality.

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