CONFINEMENT OF DISLOCATIONS INSIDE A CRYSTAL WITH A PRESCRIBED EXTERNAL STRAIN

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Abstract. We study screw dislocations in an isotropic crystal undergoing antiplane shear. In the framework of linear elasticity, by fixing a suitable boundary condition for the strain (prescribed non-vanishing boundary integral), we manage to confine the dislocations inside the material. More precisely, in the presence of an external strain with circulation equal to $n$ times the lattice spacing, it is energetically convenient to have $n$ distinct dislocations lying inside the crystal. The novelty of introducing a Dirichlet boundary condition for the tangential strain is crucial to the confinement.

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1. Introduction

Starting with the pioneering work of Volterra [29], much attention has been drawn on dislocations in solids, as the ultimate cause of plasticity in crystalline materials [12, 22, 23, 27]. Observed experimentally for the first time in 1956 [15], dislocations are line defects in the lattice structure. The interest in dislocations became more and more evident as soon as it was understood that their presence can significantly influence the chemical and physical properties of the material. The measure of the lattice mismatch due to a dislocation is encoded in the Burgers vector, whose magnitude is of the order of one lattice spacing (see [18]). According to whether the Burgers vector is perpendicular or parallel to the dislocation line, ideal dislocations are classified as edge dislocations or screw dislocations, respectively. In nature, real dislocations come as a combination of these two types. For general treaties on dislocations, we refer the reader to [16, 18, 21].

In this paper we focus our attention on screw dislocations in a single isotropic crystal which occupies a cylindrical region $\Omega \times \mathbb{R}$ and which undergoes antiplane shear. According to the model proposed in [6], this allows us to study the problem in the cross section $\Omega \subset \mathbb{R}^2$. Throughout the work, we will assume that

$$\Omega$$ is a bounded convex open set with $C^1$ boundary. (H1)

We consider the lattice spacing of the material to be $2\pi$ and that all the Burgers vectors are oriented in the same direction (this is a simplification introduced in our model, but we expect that the results that we obtain also hold if dislocations with opposite Burgers vectors are allowed). Therefore, any dislocation line is directed along the axis of the cylinder, it is characterized by a Burgers vector of magnitude $2\pi$ along the same axis, and meets the cross section $\Omega$ at a single point. Moreover, we assume that an external strain acts on the crystal: we prescribe the tangential strain on $\partial \Omega$ to be a function

$$f \in L^1(\partial \Omega) \quad \text{with} \quad \int_{\partial \Omega} f(x) \, dx = 2\pi.$$ (H2)

This choice of the external strain will determine at most one dislocation inside $\Omega$, which we denote by $a$. Thus, the strain of the deformed crystal is represented by a field $F_a \in L^1(\Omega; \mathbb{R}^2) \cap L^2_{\text{loc}}(\Omega \setminus \{a\}; \mathbb{R}^2)$, solution to the following system:

$$\begin{cases}
\text{div } F_a = 0 & \text{in } \Omega, \\
\text{curl } F_a = 2\pi \delta_a & \text{in } \Omega, \\
F_a \cdot \tau = f & \text{on } \partial \Omega, 
\end{cases}$$ (1.1)

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where $\delta_\epsilon$ is the Dirac delta centered at $a$ and $\tau$ is the tangent unit vector to $\partial\Omega$. The divergence condition in (1.3) is an equilibrium condition in the framework of linearized elasticity. The condition on the curl encodes the presence of the dislocation inside the crystal, thus forbidding $F_\epsilon$ to be a gradient globally in $\Omega$; however, this is true locally in $\Omega \setminus \{a\}$. Both these conditions are intended in the sense of distributions in $\Omega$. The boundary condition $F_\epsilon \cdot \tau = f$ on $\partial\Omega$ must be intended in the sense of traces (see [9]), and has the consequence of confining the dislocation inside the domain. Notice also that the circulation of $F_\epsilon$ along any simple closed curve $\gamma$ enclosing the dislocation point $a$ is equal to $2\pi$: thus, the constant $2\pi$ in front of the delta in (1.1) can be interpreted as a compatibility condition with (1.2). The physical meaning of $f$ can be described as follows. Let us fix an arbitrary point $b \in \partial\Omega$; for any other $x \in \partial\Omega$ we can compute the line integral

$$\int_{\gamma_b^a} f(y) \, dy = \int_{\gamma_b^a} F_\epsilon(y) \cdot \tau \, dy,$$

where $\gamma_b^a$ is the counterclockwise arc of $\partial\Omega$ connecting $b$ to $x$. Since the curl of the field $F$ is concentrated in the point $a$, we see that the field $F$ is compatible and can be written locally as a deformation gradient $F = \nabla u$ (see Subsection 3.1 for the details), so that the integral above is equal to

$$\int_{\gamma_b^a} F_\epsilon(y) \cdot \tau \, dy = u(x) - u(b).$$

Hence, we recognize that fixing the boundary datum $f$ is equivalent to fixing, up to an additive constant, the Dirichlet datum of the deformation $u$. Notice that letting $x \to b$ in the integral above in such a way that $\gamma_b^a$ covers all of $\partial\Omega$, the value of the integral equals $2\pi$: this means that the deformation $u$ has a jump on the boundary, which physically corresponds to the lack of a layer of atoms on some line connecting $a$ to the boundary. We remark that the reconstruction of the deformation from an incompatible field $F$ is a classical problem dating back to Cesàro [8] (see also [19]). We refer the reader to [2] and the references therein where a boundary datum as in (1.2) is considered for problems involving dislocations in a different setting.

In principle, the position occupied by the dislocation $a$, at the equilibrium, is the one for which the Dirichlet energy

$$\frac{1}{2} \int_{\Omega} |F_\epsilon|^2 \, dx$$

is minimal, and will be influenced by the particular external strain acting on the crystal. Our aim is to study the dependence of the Dirichlet energy on the dislocation position $a \in \Omega$. Yet, due to the presence of a dislocation sitting at $a$, the field $F_\epsilon$ in (1.1) has infinite Dirichlet energy. Indeed, given $\epsilon > 0$, denoting by $B_\epsilon(a)$ the disk of radius $\epsilon$ centered at $a$ and by $\overline{B_\epsilon(a)}$ its closure, we have

$$\frac{1}{2} \int_{\Omega} |F_\epsilon|^2 \, dx = \lim_{\epsilon \to 0} \frac{1}{2} \int_{\Omega \setminus \overline{B_\epsilon(a)}} |F_\epsilon|^2 \, dx = \lim_{\epsilon \to 0} \pi |\log \epsilon| + O(1) = +\infty. \quad (1.2)$$

Notice that the rate of divergence to $+\infty$ in (1.2) is independent of $a$. This suggests to consider the energy far from the dislocation, that is to study the problem in $\Omega_\epsilon(a) := \Omega \setminus \overline{B_\epsilon(a)}$, thus neglecting the Dirichlet energy in the core $B_\epsilon(a)$. This strategy, called core radius approach, is well known and it is employed in different contexts such as linear elasticity (see, for instance, [21] [22] [25]; also [5] [9] for screw dislocations and [7] for edge dislocations), the theory of Ginzburg-Landau vortices (see, for instance, [3] [25] and the references therein), and liquid crystals (see, for instance, [13]). Since the core radius approach eliminates the non-integrability of $|F_\epsilon|^2$ around the dislocation, classical variational techniques can be used to tackle the problem.

The field $F_\epsilon$ can be approximated in $L^2_{\text{loc}}(\Omega \setminus \{a\}; \mathbb{R}^2)$ by a sequence of fields $F_\epsilon^\ast \in L^2(\Omega_\epsilon(a); \mathbb{R}^2)$, solutions to

$$\begin{cases}
\text{div} \ F_\epsilon^\ast = 0 & \text{in } \Omega_\epsilon(a), \\
\text{curl} \ F_\epsilon^\ast = 0 & \text{in } \Omega_\epsilon(a), \\
F_\epsilon^\ast \cdot \tau = f & \text{on } \partial\Omega \setminus \overline{B_\epsilon(a)}, \\
F_\epsilon^\ast \cdot \nu = 0 & \text{on } \partial B_\epsilon(a) \cap \Omega,
\end{cases} \quad (1.3)$$

where $\nu$ is the outer unit normal to $\Omega_\epsilon(a)$ (observe that in this case $\nu$ is the inner unit normal to $B_\epsilon(a) \cap \Omega$), see [5] and [7] Theorem 4.1. System (1.3) characterizes the minimizers of the
energy functional

\[ \mathcal{E}_\epsilon(a) := \min \left\{ \frac{1}{2} \int_{\Omega_{\epsilon}(a)} |F|^2 \, dx : F \in L^2(\Omega_{\epsilon}(a); \mathbb{R}^2), \operatorname{curl} F = 0, F \cdot \tau = f \text{ on } \partial \Omega \setminus \overline{B}_\epsilon(a) \right\} \]  

(1.4)

Notice that in the minimization problem (1.4) the zero-curl condition is a consequence of the optimality, but it is necessary to have some regularity on the field \( F \), making the boundary condition meaningful (see [9]). Moreover, if \( B_\epsilon(a) \subset \subset \Omega \), from Stokes’ Theorem the circulation of the minimizer on \( \partial B_\epsilon(a) \) is 2\( \pi \). Finally, to study the dependence of the Dirichlet energy (1.4) on \( a \), it is crucial to allow the disk \( B_\epsilon(a) \) to exceed the domain \( \Omega \), so that the dislocation \( a \) might be located on the boundary. Notice that (1.4) is well defined also if \( a \in \partial \Omega \). Indeed, by comparison, we will show that for \( \epsilon \) small enough (1.4) attains its minimum when the dislocation is in the interior of \( \Omega \), far away from the boundary. More precisely, we prove that the energy blows up when \( B_\epsilon(a) \) and \( \partial \Omega \) overlap (this makes other choices for obtaining the confinement effect – such as setting \( \mathcal{E}_\epsilon(a) = +\infty \) if \( a \in \partial \Omega \) – inadequate.)

In the first part of the paper, in view of (1.2), we study the asymptotic behavior of the functionals \( \mathcal{F}_\epsilon : \overline{\Omega} \to \mathbb{R} \) defined by

\[ \mathcal{F}_\epsilon(a) := \mathcal{E}_\epsilon(a) - \pi |\log \epsilon| \]  

(1.5)

as \( \epsilon \to 0 \), to some limit functional \( \mathcal{F} : \overline{\Omega} \to \mathbb{R} \cup \{+\infty\} \). To make the asymptotics precise, we resort to the notion of continuous convergence [10] Definition 4.7]: we say that the sequence of functionals \( \mathcal{F}_\epsilon \) continuously converge in \( \overline{\Omega} \) to \( \mathcal{F} \) as \( \epsilon \to 0 \) if, for any sequence of points \( a' \in \overline{\Omega} \) converging to \( a \in \overline{\Omega} \), the sequence (of real numbers) \( \mathcal{F}_\epsilon(a') \) converges to \( \mathcal{F}(a) \). It is well known that the notion of continuous convergence is equivalent to the \( \Gamma \)-convergence of \( \mathcal{F}_\epsilon \) and \( -\mathcal{F}_\epsilon \) respectively, see [10] Remark 4.9; we will often use this fact in the sequel. Let \( d(a) := \text{dist}(a, \partial \Omega) \), let \( b \in \partial \Omega \cap \partial B_{d(a)}(a) \), and let \( (\rho_a, \theta_a) \) be the polar coordinate system centered at \( a \) such that the point \( b \) has angular coordinate \( \theta_a(b) = 0 \). We denote by \( \rho_a \) and \( \theta_a \) the unit vectors associated with the polar coordinates. Finally let \( g : \partial \Omega \to \mathbb{R} \) be a primitive of \( f \) with a jump point at \( b \) (see (3.1)).

**Theorem 1.1.** Under the assumptions [H1] and [H2], as \( \epsilon \to 0 \) the functionals \( \mathcal{F}_\epsilon \) defined by (1.3) continuously converge in \( \overline{\Omega} \) to the functional \( \mathcal{F} : \overline{\Omega} \to \mathbb{R} \cup \{+\infty\} \) defined as

\[ \mathcal{F}(a) := \pi \log d(a) + \frac{1}{2} \int_{B_{d(a)}(a)} |K_a + \nabla v_a|^2 \, dx + \frac{1}{2} \int_{B_{d(a)}(a)} |\nabla v_a|^2 \, dx \]  

(1.6)

if \( a \in \Omega \), and \( \mathcal{F}(a) := +\infty \) otherwise. Here \( K_a(x) := \rho_a^{-1}(x) \theta_a(x) \) and \( v_a \) is the solution to

\[ \begin{align*}
\Delta v_a &= 0 & \text{in } \Omega, \\
v_a &= g - \theta_a & \text{on } \partial \Omega.
\end{align*} \]

In particular, \( \mathcal{F} \) is continuous over \( \overline{\Omega} \) and diverges to +\( \infty \) as the dislocation approaches the boundary, that is, \( \mathcal{F}(a) \to +\infty \) as \( d(a) \to 0 \). Thus, \( \mathcal{F} \) attains its minimum in the interior of \( \Omega \).

A consequence of Theorem 1.1 is that also the energies (1.4) attain their minimum in the interior of \( \Omega \).

**Corollary 1.2.** Under the assumptions [H1] and [H2], there exists \( \epsilon_1 > 0 \) such that, for every \( \epsilon \in (0, \epsilon_1) \), the infimum problem

\[ \inf\{ \mathcal{E}_\epsilon(a) : a \in \overline{\Omega} \}, \]  

(1.7)

admits a minimizer only in the interior of \( \Omega \). Moreover, if \( a' \in \Omega \) is a minimizer for (1.7), then (up to subsequences) \( a' \to a \) and \( \mathcal{F}_\epsilon(a') \to \mathcal{F}(a) \), as \( \epsilon \to 0 \), where \( a \) is a minimizer of the functional \( \mathcal{F} \) defined in (1.6). In particular, for \( \epsilon \) small enough, all the minimizers of (1.7) stay uniformly (with respect to \( \epsilon \)) far away from the boundary.

Section 3 is devoted to proving Theorem 1.1 and Corollary 1.2. For this case, we complement the analytical results by plots of the limiting energy \( \mathcal{F} \) for special choices of the boundary datum.

In the second part of the paper, we will make a substantial use of Theorem 1.1 to tackle the case of many dislocations. Indeed, having \( n \geq 2 \) dislocations in the crystal is equivalent to imposing a boundary datum with circulation equal to \( 2\pi n \), which we can obtain by taking the
tangential strain $F \cdot \tau$ to be $nf$ on $\partial \Omega$, with $f$ as in (1.12). Therefore, in analogy with (1.4), for $\epsilon > 0$, we define the energy of the $n$-tuple $(a_1, \ldots, a_n) \in \Omega^n$ as

$$E_\epsilon(a_1, \ldots, a_n) := \min \left\{ \frac{1}{2} \int_{\Omega} (F^2) \, dx : F \in \mathcal{X}, F \cdot \tau = nf \text{ on } \partial \Omega \setminus \bigcup_{i=1}^n \overline{B}_\epsilon(a_i) \right\}.$$  

Here $\Omega_\epsilon(a_1, \ldots, a_n) := \Omega \setminus \bigcup_{i=1}^n \overline{B}_\epsilon(a_i)$ and $\mathcal{X}$ is the space characterized by

$$F \in \mathcal{X} := \mathcal{X}_\epsilon(a_1, \ldots, a_n) \iff \begin{cases} F \in L^2(\Omega \setminus \bigcup_{i=1}^n \overline{B}_\epsilon(a_i); \mathbb{R}^2), \\ \text{curl } F = 0 \text{ in } \mathcal{D}'(\Omega \setminus \bigcup_{i=1}^n \overline{B}_\epsilon(a_i)), \\ \int_{\gamma} F \cdot \tau = 2\pi m, \end{cases}$$

where $\gamma$ is an arbitrary simple closed curve in $\Omega_\epsilon(a_1, \ldots, a_n)$ winding once counterclockwise around $m$ dislocations. In the sequel, for the sake of brevity, we shall omit the dependence on $\epsilon$ and the points $a_1, \ldots, a_n$ for the space $\mathcal{X}$. Notice that the spaces in (1.19) are encapsulated, namely, if $0 < \epsilon < \eta$ and if $F \in \mathcal{X}(\epsilon(a_1, \ldots, a_n))$, then its restriction to $\Omega_\eta(a_1, \ldots, a_n)$ belongs to $\mathcal{X}_\eta(a_1, \ldots, a_n)$, since $\Omega_\eta(a_1, \ldots, a_n) \subset \Omega_\epsilon(a_1, \ldots, a_n)$. In virtue of this, it follows that

$$E_\epsilon(a_1, \ldots, a_n) \geq E_\eta(a_1, \ldots, a_n).$$

A computation similar to (1.2) shows that the energy defined in (1.8) behaves asymptotically like $C |\log \epsilon|$ as $\epsilon \to 0$, where $C$ depends on the mutual positions of the dislocations. In particular, if the $a_i$’s are all distinct and inside $\Omega$, the energy diverges like $\pi n |\log \epsilon|$. This suggests to study the asymptotic behavior, as $\epsilon \to 0$, of the functionals $F_\epsilon : \Omega^n \to \mathbb{R} \cup \{+\infty\}$ defined by

$$F_\epsilon(a_1, \ldots, a_n) := E_\epsilon(a_1, \ldots, a_n) - \pi n |\log \epsilon|.$$  

In this context, we say that the sequence of functionals $F_\epsilon$ continuously converge in $\Omega^n$ to $F$ as $\epsilon \to 0$ if, for any sequence of points $(a'_1, \ldots, a'_n) \in \Omega^n$ converging to $(a_1, \ldots, a_n) \in \Omega^n$, the sequence (of real numbers) $F_\epsilon(a'_1, \ldots, a'_n)$ converges to $F(a_1, \ldots, a_n)$. Let $g : \partial \Omega \to \mathbb{R}$ be a primitive of $f$ with $n$ jump points $b_i \in \partial \Omega$ where the amplitude of each jump is $2\pi/n$ (see (4.1) for a precise definition), and for every $i \in \{1, \ldots, n\}$ set

$$d_i := \frac{\min_{j \neq i} \left\{ |a_i - a_j| / 2 \right\} \text{dist}(a_i, \partial \Omega)}{\log |a_i - a_j|}.$$  

Theorem 1.3. Let $n \geq 2$. Under the assumptions (H1) and (H2), as $\epsilon \to 0$ the functionals $F_\epsilon$ defined by (1.11) continuously converge in $\Omega^n$ to the functional $F : \Omega^n \to \mathbb{R} \cup \{+\infty\}$ defined as

$$F(a_1, \ldots, a_n) := \sum_{i=1}^n \pi \log d_i + \frac{1}{2} \int_{\Omega} \sum_{i=1}^n \left| \nabla a_{i-1} \cdot a_i \right|^2 \, dx + \sum_{i=1}^n \frac{1}{2} \int_{\Omega_\epsilon(a_i)} |K_{a_i}|^2 \, dx$$

$$+ \sum_{i=1}^n \int_{\Omega_\epsilon(a_i)} \nabla a_{i-1} \cdot K_{a_i} \, dx + \sum_{i<j} \int_{\Omega} K_{a_i} \cdot K_{a_j} \, dx,$$

if $(a_1, \ldots, a_n) \in \Omega^n$ with $a_i \neq a_j$ for every $i \neq j$, and $F(a_1, \ldots, a_n) := +\infty$ otherwise. Here $K_{a_i}(x) := \rho_{a_i}^{-1}(x) \theta_{a_i}(x)$ and $v_{a_1, \ldots, a_n}$ is the solution to

$$\begin{cases} \Delta v_{a_1, \ldots, a_n} = 0 & \text{in } \Omega, \\ v_{a_1, \ldots, a_n} = ng - \sum_{i=1}^n \theta_{a_i} & \text{on } \partial \Omega. \end{cases}$$

In particular, $F$ is continuous in $\Omega^n$ and diverges to $+\infty$ if either at least one dislocation approaches the boundary or at least two dislocations collide, that is, $F(a_1, \ldots, a_n) \to +\infty$ as $d_i \to 0$ for some $i$. Thus, $F$ attains its minimum in the interior of $\Omega^n$, at an $n$-tuple of distinct points.

We notice that (1.13) can be expressed in term of the functionals $F(a_i)$ defined in (1.6) for each single dislocation $a_i$, namely,

$$F(a_1, \ldots, a_n) = \sum_{i=1}^n F(a_i) + \sum_{i<j} \int_{\Omega} (K_{a_i} + \nabla v_{a_i}) \cdot (K_{a_j} + \nabla v_{a_j}) \, dx.$$  

A consequence of Theorem 1.3 is that also the energies (1.8) attain their minimum in the interior of $\Omega^n$ at an $n$-tuple of well separated points.
Corollary 1.4. Let \( n \geq 2 \). Under the assumptions \((H1)\) and \((H2)\), there exists \( \epsilon_2 > 0 \) such that, for every \( \epsilon \in (0, \epsilon_2) \), the infimum problem
\[
\inf\{\mathcal{E}_\epsilon(a_1, \ldots, a_n) : (a_1, \ldots, a_n) \in \Omega^n \},
\]
(1.14)
admits a minimizer only in the interior of \( \overline{\Omega} \), at an \( n \)-tuple of distinct points. Moreover, if \((a_1', \ldots, a_n') \in \Omega^n \) is a minimizer for \((1.14)\), then (up to subsequences) we have \((a_1', \ldots, a_n') \to (a_1, \ldots, a_n) \) and \( \mathcal{F}_\epsilon(a_1', \ldots, a_n') \to \mathcal{F}(a_1, \ldots, a_n) \), as \( \epsilon \to 0 \), where \((a_1, \ldots, a_n)\) is a minimizer of the functional \( \mathcal{F} \) defined in \((1.13)\). In particular, for \( \epsilon \) small enough, all the minimizers of problem \((1.7)\) are \( n \)-tuples of distinct points that stay uniformly (with respect to \( \epsilon \)) far away from the boundary and from one another.

Section 4 is devoted to proving Theorem 1.3 and Corollary 1.4.

We will always assume \((H1)\) and \((H2)\), even if it is not explicitly stated. We stress that the convexity and regularity assumptions on \( \Omega \) stated in \((H1)\) provide a uniform interior cone condition of angle between \( \pi/2 \) and \( \pi \), i.e.,

there exist \( \pi/2 < \alpha < \pi \) and \( \epsilon_\alpha > 0 \) such that for every \( b \in \partial \Omega \) the ball \( B_{\epsilon_\alpha}(b) \) meets \( \partial \Omega \) at two points \( b_1 \) and \( b_2 \) forming an angle at least \( \alpha \) with \( b \).

\[
(1.15)
\]
We point out that convexity and regularity play different roles: the former is conveniently assumed in order to simplify the exposition of the results (in fact, it can be removed without changing their essence); the latter, on the other hand, is fundamental in our proofs. Finally, we observe that the boundary condition of Dirichlet type \( F \cdot \tau = nf \), with \( f \) as in \((H2)\), is fundamental to keep the dislocations confined inside the material. In fact, the natural boundary conditions of Neumann type imply that the dislocations migrate to the boundary and leave the domain, since, in such a case, the Dirichlet energy of the system decreases as the dislocations approach \( \partial \Omega \) (see, e.g., [17]).

A key feature in our analysis is the rescaling introduced in \((1.5)\) and \((1.11)\) (see also [1]), which is related to the so-called Hadamard finite part of a divergent integral (see [14]). Such type of asymptotic analysis has the advantage of keeping into account the energetic dependence on the position of the dislocation \( a \in \Omega \), whereas it is well-known that the standard rescaling obtained by dividing the energy by \( |\log \epsilon| \) gives rise to an energy which only counts the number of dislocations in the bulk (see again [1]).

2. Notation and properties of \( K \)

We list here the notation used in the paper:

- \( B_r(x) \) denotes the open ball of radius \( r > 0 \) centered at \( x \in \mathbb{R}^2 \); \( \overline{B}_r(x) \) is its closure;
- for \( n \geq 1 \), and for \( x_1, \ldots, x_n \in \overline{\Omega} \), the symbol \( \Omega_r(x_1, \ldots, x_n) \) denotes the open set
\[
\Omega_r(x_1, \ldots, x_n) := \Omega \setminus \left( \bigcup_{i=1}^n \overline{B}_r(x_i) \right);
\]
\( \Omega_r(x_1, \ldots, x_n) \) is its closure;
- the function \( \mathbb{R}^2 \ni x \mapsto \text{dist}(x, E) \) denotes the distance of \( x \) from a set \( E \subset \mathbb{R}^2 \); in the particular case \( E = \partial \Omega \), we define \( d(x) := \text{dist}(x, \partial \Omega) \);
- we denote \( \Omega^{(r)} := \{ x \in \Omega : d(x) > r \} \) and by \( \overline{\Omega}^{(r)} \) its closure;
- \( \Omega^n \) denotes the cartesian product of \( n \) copies of \( \Omega \) and \( \overline{\Omega}^n \) its closure;
- for \( 0 < r < R \), \( A_r^R(x) := B_R(x) \setminus \overline{B}_r(x) \) denotes the open annulus of internal radius \( r \) and external radius \( R \) centered at \( x \in \mathbb{R}^2 \);
- \( \chi_E \) denotes the characteristic function of \( E \); \( \chi_E(x) = 1 \) if \( x \in E \); \( \chi_E(x) = 0 \) if \( x \notin E \);
- given a set \( E \) with piecewise \( C^1 \) boundary, \( \nu \) and \( \tau \) denote the outer unit normal and the tangent unit vectors to \( \partial E \), respectively;
- \( \text{diam} E \) denotes the diameter of a set \( E \subset \mathbb{R}^2 \);
- given \( x = (x_1, x_2) \in \mathbb{R}^2 \), we denote by \( x^\perp := (-x_2, x_1) \) the rotated vector;
- given \( x \in \mathbb{R}^2 \), we define \( \rho_x, \theta_x \) as the standard polar coordinate system centered at \( x \); \( \hat{\rho}_x \) and \( \hat{\theta}_x \) denote the corresponding unit vectors;
- given \( x \in \mathbb{R}^2 \), we denote by \( K_x \) the vector field \( K_x := \rho_x^{-1} \theta_x \): it is easy to see that \((1.2)\)
\[
\text{div} K_x = 0 \quad \text{in} \mathbb{R}^2, \\
\text{curl} K_x = 2\pi \delta_x \quad \text{in} \mathbb{R}^2,
\]
in the sense of distributions, and moreover,
\[
K_x \cdot \nu = 0 \quad \text{on} \partial B_r(x), \text{ for any } r > 0. \tag{2.2}
\]

Notice that \( K_x \) is the absolutely continuous part of the gradient of the function \( \theta_x \).
Moreover, the jump set of \( \theta_x \) is the half line starting from \( x \) and passing through \( y \) with \( [\theta_x] = 2\pi \) across it;
- given \( x \in \mathbb{R}^2 \), we denote by \( \phi_x \) the fundamental solution to the Laplacian, namely \( \phi_x(\cdot) = \log |\cdot - x| \). Notice that \( (\nabla \phi_x)^+ = K_x \);
- we denote by \( \omega \) the function \( \omega : \Omega \rightarrow \{1, 2\} \) that is equal to \( 1 \) inside \( \Omega \) and \( 2 \) on \( \partial \Omega \);
- \( \text{spt} \varphi \) denotes the support of the function \( \varphi \);
- given \( x \in \mathbb{R}^2 \), \( \delta_x \) is the Dirac measure centered at \( x \);
- \( \mathcal{H}^1 \) denotes the one-dimensional Hausdorff measure;
- the letter \( C \) alone represents a generic constant (possibly depending on \( \Omega \)) whose value might change from line to line.

We prove some properties of the vector field \( K \) defined in \((2.1)\).

**Lemma 2.1 (Properties of \( K \)).** Let \( y_1, y_2 \in \mathbb{R}^2 \). Set \( r := |y_1 - y_2|/2 \) and \( y \) the midpoint \((y_1 + y_2)/2\). Then the vector fields \( K_{y_1} \) and \( K_{y_2} \) satisfy the following properties:
\begin{enumerate}[(i)]
\item the scalar product \( K_{y_1} \cdot K_{y_2} \) is negative in the ball \( B_r(y) \), is zero on \( \partial B_r(y) \setminus \{y_1, y_2\} \), and is positive in \( \mathbb{R}^2 \setminus B_r(y) \). In particular, \( \int_{B_r(y)} K_{y_1} \cdot K_{y_2} \, dx \geq -2\pi \);
\item the following estimate holds: \( \int_{B_r(y)} K_{y_1} \cdot K_{y_2} \, dx \leq 2\pi, \text{ for } i = 1, 2 \).
\end{enumerate}
Moreover, let \( \ell \in \mathbb{N} \), and let \( y_1, \ldots, y_\ell \in \Omega \) be distinct points. Then,
\[ 2|\log \epsilon| + 2\pi \log d_1 \leq \int_{\mathbb{R}^2} |K_{y_1}|^2 \, dx \leq 2\pi |\log \epsilon| + 2\pi \log (\text{diam } \Omega); \tag{2.3} \]
\[ \alpha |\log \epsilon| + \alpha \log \epsilon_\alpha \leq \int_{\mathbb{R}^2} |K_{y_1}|^2 \, dx \leq \pi |\log \epsilon| + \pi \log (\text{diam } \Omega). \tag{2.4} \]

**Proof.** To prove (i) it is enough to notice that, given a point \( x \in \mathbb{R}^2 \setminus \{y_1, y_2\} \), the angle \( \hat{y}_1 x y_2 \) is larger than, equal to, or smaller than \( \pi/2 \), according to whether \( x \in B_r(y_1) \), \( x \in \partial B_r(y_1) \), \( x \in \mathbb{R}^2 \setminus B_r(y_1) \), respectively. To prove the estimate, we consider the ball \( B_r(y) \) whose diameter is the axis of the segment joining \( y_1 \) and \( y_2 \), and we define \( B_r(y)^+ \) the half of \( B_r(y) \) on the side of \( y_2 \). By symmetry, we have
\[
\int_{B_r(y)} K_{y_1} \cdot K_{y_2} \, dx = -2 \int_{B_r(y)^+} |K_{y_1} \cdot K_{y_2}| \, dx \geq -\frac{2}{r} \int_{B_r(y)^+} |K_{y_2}| \, dx \geq -\frac{2}{r} \int_{\pi/2}^{3\pi/2} \int_0^r \rho_2 d\theta_2 = -2\pi.
\]

To prove (ii), observe that \( |K_{y_1}| \leq 1/r \) in \( \mathbb{R}^2 \setminus B_r(y_1) \), so that
\[
\int_{B_r(y)} K_{y_1} \cdot K_{y_2} \, dx \leq \frac{1}{r} \int_{B_r(0)} |x|^{-1} \, dx = \pi.
\]

To prove (iii) and (iv) one integrates in polar coordinates centered at \( y_i \) over the sets \( A^\alpha_i(\Omega) \setminus \Omega \), \( A^\alpha_i(y_i) \cap \Omega \), \( A^\alpha_i(y_i) \cap \Omega \) with \( s := \min_{j \neq i} \{\epsilon_\alpha, |y_i - y_j|/2\} \), and uses the convexity assumption \([H1]\) and \([1.15]\).

3. One dislocation

In this section, we study the problem when the domain \( \Omega \) contains only one dislocation. We first introduce an equivalent formulation of \((1.4)\) in terms of a suitable displacement function, for which we prove a priori bounds; then we prove Theorem \([1.1]\) and Corollary \([1.2]\).
3.1. **Displacement formulation.** Thanks to the curl condition in (1.3), it is possible to write $F$ locally as a gradient. Choose a point $b \in \partial \Omega$ and define the function $g : \partial \Omega \to \mathbb{R}$ by

$$g(x) := \int_{\gamma_x^b} f(y) \, dy,$$

(3.1)

where $\gamma_x^b$ is the counterclockwise path in $\partial \Omega$ connecting $b$ and $x$. The assumption on $f$ implies that $g \in W^{1,1}(\partial \Omega \setminus \{b\})$, namely it is an absolutely continuous function outside $\{b\}$. In particular, from the Sobolev embedding, we recover also that $g \in H^{1/2}(\partial \Omega \setminus \{b\})$. Moreover, $g$ has a jump at $b$ of amplitude $[g](b) = 2\pi$. Let $\Sigma$ be a simple smooth curve in $\Omega$ connecting $a$ with $b$, intersecting $\partial B_r(a)$ at a single point and $\partial \Omega$ only at $b$. Notice that each connected component of $\Omega_-(a) \setminus \Sigma$ is simply connected. The condition curl $F$ is constant and equals the jump of $u$ Neumann condition on $\partial B_r(a)$ in $\Omega_+(a) \setminus \Sigma$. Therefore, the energy (1.4) can be expressed as

$$\mathcal{E}_c(a) = \min \left\{ \frac{1}{2} \int_{\Omega_+(a)} |\nabla u|^2 \, dx : u \in H^1(\Omega_+(a) \setminus \Sigma), \, u = g \text{ on } \partial \Omega \setminus \overline{B}_r(a) \right\}.$$  (3.2)

The energy above does not depend on the choice of the discontinuity point $b$, nor on the primitive of $g$, nor on the curve $\Sigma$, as we prove in Lemma 3.1 below. Hence, given $a \in \Omega$, we choose $b$ as one of the projections of $a$ on the boundary and $\Sigma$ as the segment joining them.

We denote by $u^\gamma_\phi \in H^1(\Omega_+(a) \setminus \Sigma)$ the minimizer of (3.2). From the Euler-Lagrange equation (1.3) of $\mathcal{E}_c(a)$ we obtain that $u^\gamma_\phi$ is harmonic, agrees with $g$ on $\partial \Omega \setminus \overline{B}_r(a)$, satisfies a homogeneous Neumann condition on $\partial B_r(a) \cap \Omega$, and its normal gradient is continuous across $\Sigma \cap \Omega_+(a)$. Moreover, as a consequence of Stokes Theorem, we obtain that the jump $[u^\gamma_\phi]$ across $\Sigma \cap \Omega_+(a)$ is constant and equals the jump of $g$: indeed, the circulation of $\nabla u^\gamma_\phi$ along an arbitrary closed curve intersecting $\Sigma$ only once and winding around $a$ only once is $2\pi$. Therefore $u^\gamma_\phi$ satisfies

$$\begin{cases}
\Delta u^\gamma_\phi = 0 & \text{in } \Omega_+(a) \setminus \Sigma, \\
[u^\gamma_\phi] = 2\pi & \text{on } \Sigma \cap \Omega_+(a), \\
u^\gamma_\phi = g & \text{on } \partial \Omega \setminus \overline{B}_r(a), \\
\partial u^\gamma_\phi / \partial \nu = 0 & \text{on } \partial B_r(a) \cap \Omega, \\
(\partial u^\gamma_\phi)^+ / \partial \nu = \partial u^\gamma_\phi^- / \partial \nu & \text{on } \Sigma \cap \Omega_+(a).
\end{cases}$$  (3.3)

In the last equation above, $\nu$ is a choice of the unit normal vector to $\Sigma$. Notice that, when $\text{dist}(a, \partial \Omega) \leq \epsilon$, the choice of $b \in \partial \Omega \cap B_r(a)$ implies that $\Omega_-(a) \setminus \Sigma = \Omega_+(a)$ and the jump condition of $u^\gamma_\phi$ across $\Sigma$ is empty.

**Lemma 3.1.** The functional $\mathcal{E}_c(a)$ does not depend on the choice of the discontinuity point $b$, nor on the primitive of $g$, nor on the cut $\Sigma$.

**Proof.** Let $g$ and $g'$ be two primitives of $f$ with the same discontinuity point. Then the two boundary data differ by a constant. The same holds true for the corresponding minimizers of (3.2), which have the same gradient and the same energy.

Let now assume that $g$ and $g'$ have two different discontinuity points, say $b$ and $b'$. Denote by $\Sigma$ and $\Sigma'$ the segments joining $a$ and $b$, and $a$ and $b'$, respectively. Let $\phi$ and $\phi'$ be the angles associated with the discontinuity points $b$ and $b'$, respectively, in the angular coordinate centered at $a$. It is not restrictive to assume that $\phi < \phi'$. Let $u$ be the solution to (3.3) associated with $g, b$, and $\Sigma$: define $v := u + 2\pi \chi_S$, where $S$ is the subset of $\Omega_+(a)$ in which $\theta_a \in (\phi, \phi')$. It is easy to see that $v$ is admissible for (3.2) associated with $g', b'$, and $\Sigma'$. In particular, $v$ is the solution to (3.3) associated with $g', b'$, and $\Sigma'$; moreover,

$$\frac{1}{2} \int_{\Omega_+(a)} |\nabla u|^2 \, dx = \frac{1}{2} \int_{\Omega_+(a)} |\nabla v|^2 \, dx.$$

This concludes the proof of the invariance with respect to the discontinuity point in the case of a straight cut.

For general curves $\Sigma$ and $\Sigma'$, the region in $\Omega_+(a)$ lying between $\Sigma$ and $\Sigma'$ is the union of some simply connected sets $S_i, i \geq 1$, bounded by portions of $\Sigma$ and $\Sigma'$. In this case, the same strategy applies, provided one adds or subtracts $2\pi \chi_{S_i}$, according to whether the portions of $\Sigma$ bounding $S_i$ precede or follows that of $\Sigma'$ in the positive orientation of the angular coordinate. This concludes the proof. \(\square\)
Remark 3.2. As already noticed in the Introduction, the functional (3.2) and the system (3.3) are well defined even in the case $a \in \partial \Omega$. In this case, Lemma 3.1 allows us to choose the discontinuity point $b = a$ and the cut $\Sigma = \emptyset$.

Given $a \in \Omega$, in view of Lemma 3.1 we take $b$ as one of the projection points of $a$ on the boundary $\partial \Omega$ and $\Sigma$ the segment joining $a$ and $b$. Let $u_a^o$ be the solution to (3.3). We decompose its gradient into regular plus a singular part, the latter being given by the field $K_a$ solution to (2.1). By adding and subtracting $\theta_a$ to $u_a^o$, we can write the following decomposition

$$\nabla u_a^o = K_a + \nabla \tilde{u}_a^o,$$

where $\tilde{u}_a^o \in H^1(\Omega_c(a))$ is the solution to

$$\begin{cases}
\Delta \tilde{u}_a^o = 0 & \text{in } \Omega_c(a), \\
\tilde{u}_a^o = g - \theta_a & \text{on } \partial \Omega \setminus B_c(a), \\
\partial \tilde{u}_a^o / \partial \nu = 0 & \text{on } \partial B_c(a).
\end{cases}$$

By Remark 3.2 system (3.3) is well defined also when $a \in \partial \Omega$. In this case, since we take $b = a$, we will denote by $u_a^o$ solutions to (3.3) and by $(\rho_b, \theta_b)$ the system of polar coordinates centered at $b$ such that $\theta_b = 0$ corresponds to any half line starting from $b$ and contained in $\mathbb{R}^2 \setminus \Omega$. Then we decompose $\nabla u_a^o$ as $\nabla \tilde{u}_a^o = 2K_b + \nabla \bar{u}_a^o$, where $\bar{u}_a^o \in H^1(\Omega_c(b))$ is the solution to

$$\begin{cases}
\Delta \bar{u}_a^o = 0 & \text{in } \Omega_c(b), \\
\bar{u}_a^o = g - 2\theta_b & \text{on } \partial \Omega \setminus B_c(b), \\
\partial \bar{u}_a^o / \partial \nu = 0 & \text{on } \partial B_c(b).
\end{cases}$$

The choice of the coefficient of $K_b$ is quite natural: roughly speaking, $\theta_b$ has a jump $\pi$ at $b$, therefore $[g] - 2[\theta_b] = 0$ on the whole boundary.

Remark 3.3. Notice that the functions $\tilde{u}_a^o$ and $\bar{u}_a^o$ that solve systems (3.5) and (3.6) are the minimizers of the Dirichlet energy on $\Omega_c(a)$ and $\Omega_c(b)$ with prescribed boundary data $g - \theta_a$ and $g - 2\theta_b$, respectively.

3.2. A priori bounds on harmonic functions. In this subsection we provide some estimates for the functions $\tilde{u}_a^o$ and $\bar{u}_a^o$ introduced above. To this aim, given $a \in \Omega$ we define $v_a$ as the solution to

$$\begin{cases}
\Delta v_a = 0 & \text{in } \Omega, \\
v_a = g - \theta_a & \text{on } \partial \Omega.
\end{cases}$$

and similarly, given $b \in \partial \Omega$, we define $v_b$ as the solution to

$$\begin{cases}
\Delta v_b = 0 & \text{in } \Omega, \\
v_b = g - 2\theta_b & \text{on } \partial \Omega.
\end{cases}$$

Notice that the boundary data $g - \theta_a$ and $g - 2\theta_b$ belong to $H^{1/2}(\partial \Omega)$.

Lemma 3.4. Let $a \in \Omega$ and $b \in \partial \Omega$. Then there exists a constant $C > 0$ such that

$$\frac{1}{2} \int_{\Omega_c(a)} |\nabla \tilde{u}_a^o|^2 \, dx \leq C \|g - \theta_a\|_{H^{1/2}(\partial \Omega)}^2,$$

and

$$\frac{1}{2} \int_{\Omega_c(b)} |\nabla \bar{u}_a^o|^2 \, dx \leq C \|g - 2\theta_b\|_{H^{1/2}(\partial \Omega)}^2.$$
**Lemma 3.5.** Let \( a \in \Omega \), \( b \in \partial \Omega \), and let \( \epsilon \in (0, d(a)/2) \). Then there exists a constant \( C > 0 \) independent of \( \epsilon \) such that for every \( x \in \partial B_{\epsilon}(a) \) we have
\[
|u_{\epsilon a}(x)| \leq |g - \theta_{a}|_{L^\infty(\partial \Omega)} \quad \text{and} \quad |
abla u_{\epsilon a}(x)| \leq C|g - \theta_{a}|_{L^\infty(\partial \Omega)}.
\]
(3.11)

and, for every \( x \in \partial B_{\epsilon}(b) \cap \Omega \), we have \( |u_{\epsilon b}(x)| \leq |g - 2\theta_{b}|_{L^\infty(\partial \Omega)} \).

The proof is standard and can be obtained by truncation arguments (see [11, Theorem 7, Section 2.2.c]).

The following result is an a priori bound on a suitable \( H^1 \) extension \( w^\epsilon \) of \( u_{\epsilon a} \) in \( B_{\epsilon}(a) \).

**Lemma 3.6.** Let \( a \in \Omega \) and let \( \epsilon \in (0, d(a)/2) \). Let \( w^\epsilon \) be the harmonic extension in \( B_{\epsilon}(a) \) of \( u_{\epsilon a} \). Then there exists a constant \( C > 0 \) independent of \( \epsilon \) such that
\[
\int_{B_{\epsilon}(a)} |\nabla w^\epsilon(x)|^2 \, dx \leq Ce |g - \theta_{a}|_{L^\infty(\partial \Omega)}^2.
\]
(3.12)

**Proof.** By applying Lemma 3.5, we obtain that \( u_{\epsilon a} \in W^{1,\infty}(\partial B_{\epsilon}(a)) \) and
\[
|u_{\epsilon a}|_{W^{1,\infty}(\partial B_{\epsilon}(a))} \leq C|g - \theta_{a}|_{L^\infty(\partial \Omega)}.
\]
(3.13)

It is a known fact in the theory of harmonic functions (see [11]) that there exists a constant \( C > 0 \) such that for every harmonic function \( \varphi \in H^1(B_1(0)) \)
\[
|\nabla \varphi|^2_{L^2(B_1(0))} \leq C|\varphi - m(\varphi)|^2_{H^{1/2}(\partial B_1(0))},
\]
where \( m(\varphi) \) is the average of \( \varphi \) on \( \partial B_1(0) \). Using the Poincaré inequality we have
\[
\|\varphi - m(\varphi)\|^2_{H^{1/2}(\partial B_1(0))} \leq C \left( \|\varphi - m(\varphi)\|^2_{L^2(\partial B_1(0))} + \int_{\partial B_1(0)} \int_{\partial B_1(0)} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^2} \, dx \, dy \right)
\]
\[
\leq C \left( \int_{\partial B_1(0)} \|\nabla \varphi(x)\|^2 \, dx + \int_{\partial B_1(0)} \int_{\partial B_1(0)} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^2} \, dx \, dy \right).
\]

Therefore, using this and employing the change of variables \( x = ex' + a \) and \( y = ey' + a \) we have
\[
\|\nabla w^\epsilon\|^2_{L^2(B_\epsilon(a))} = \int_{B_1(0)} |\nabla w^\epsilon(ex' + a')|^2 \, dx' \leq C \left( \int_{B_1(0)} \|\nabla w^\epsilon(ex' + a')\|^2 \, dx' + \int_{\partial B_1(0)} \int_{\partial B_1(0)} \frac{|w^\epsilon(ex' + a') - w^\epsilon(ey' + a')|^2}{|x' - y'|^2} \, dx' \, dy' \right)
\]
\[
= C \left( \epsilon \int_{\partial B_1(0)} \|\frac{\partial w^\epsilon(x)}{\partial \tau}\|^2 \, dx + \int_{\partial B_1(0)} \int_{\partial B_1(0)} \frac{|w^\epsilon(x) - w^\epsilon(y)|^2}{|x - y|^2} \, dx \, dy \right).
\]

Owing to the fact that \( w^\epsilon = u^\epsilon \) on \( \partial B_{\epsilon}(a) \) and by (3.13), we obtain (3.12).

3.3. The limit in the case of one dislocation. In this subsection we prove Theorem 1.1 and Corollary 1.2. For \( \epsilon > 0 \) and \( a \in \Omega \), let \( u_{\epsilon a} \) solve (3.3); by (3.2), the functional in (1.5) reads
\[
F_\epsilon(a) = \frac{1}{2} \int_{\Omega_{\epsilon a}(a)} |\nabla u_{\epsilon a}|^2 \, dx - \pi |\log \epsilon|.
\]
(3.14)

To prove the continuous convergence stated in Theorem 1.1 we will consider sequences \( a^\epsilon \to a \) treating the two cases \( a \in \Omega \) and \( a \in \partial \Omega \) separately.

**Proposition 3.7.** Let \( a \in \Omega \) and \( v_a \) satisfy (3.7). For every sequence \( a^\epsilon \to a \) as \( \epsilon \to 0 \) we have
\[
F_\epsilon(a^\epsilon) \to \pi \log d(a) + \frac{1}{2} \int_{\Omega_{\epsilon d(a)}(a)} |K_a + \nabla v_a|^2 \, dx + \frac{1}{2} \int_{B_{d(a)}(a)} |\nabla v_a|^2 \, dx.
\]
(3.15)

**Proof.** Since \( d(a^\epsilon) \to d(a) \) and \( d(a) > 0 \), we can take \( \epsilon \) so small that \( \epsilon < \min\{d(a^\epsilon), 1\} \). Then plugging the decomposition (3.4) into (3.14), \( F_\epsilon(a^\epsilon) \) reads
\[
F_\epsilon(a^\epsilon) = \frac{1}{2} \int_{\Omega_{\epsilon a^\epsilon}(a)} |\nabla u_{\epsilon a^\epsilon}|^2 \, dx + \frac{1}{2} \int_{\Omega_{\epsilon a^\epsilon}(a)} K_{\epsilon a^\epsilon} \cdot \nabla u_{\epsilon a^\epsilon} \, dx + \frac{1}{2} \int_{\Omega_{\epsilon a^\epsilon}(a)} |K_{\epsilon a^\epsilon}|^2 \, dx + \pi \log \epsilon.
\]
Set, for brevity, \( d := d(a) \), \( d^\epsilon := d(a^\epsilon) \), \( K := K_a \), and \( K_{\epsilon} := K_{\epsilon a^\epsilon} \). Writing
\[
\frac{1}{2} \int_{\Omega_{\epsilon a^\epsilon}(a)} |K_{\epsilon a^\epsilon}|^2 \, dx = \frac{1}{2} \int_{\Omega_{\epsilon a^\epsilon}(a)} |K_{\epsilon a^\epsilon}|^2 \, dx + \frac{1}{2} \int_{\partial \Omega_{\epsilon a^\epsilon}(a)} |K_{\epsilon a^\epsilon}|^2 \, dx = \frac{1}{2} \int_{\partial \Omega_{\epsilon a^\epsilon}(a)} |K_{\epsilon a^\epsilon}|^2 \, dx + \pi \log \frac{d^\epsilon}{d},
\]
we obtain
\[ F_\epsilon(a') = \pi \log d' + \frac{1}{2} \int_{\Omega_{d'(a')}} |K'|^2 dx + \int_{\Omega_{d'(a')}} K' \cdot \nabla u_{a'}^\epsilon dx \quad \text{and} \quad \frac{1}{2} \int_{\Omega_{d'(a')}} |\nabla u_{a'}^\epsilon|^2 dx. \quad (3.16) \]

If we prove that, as \( \epsilon \to 0 \),
\[ \pi \log d' + \frac{1}{2} \int_{\Omega_{d'(a')}} |K'|^2 dx \to \pi \log d + \frac{1}{2} \int_{\Omega_d(a')} |K|^2 dx, \]
(3.17a)
\[ \int_{\Omega_{d'(a')}} K' \cdot \nabla u_{a'}^\epsilon dx \to \int_{\Omega_d(a')} K \cdot \nabla v_a dx, \]
(3.17b)
\[ \frac{1}{2} \int_{\Omega_{d'(a')}} |\nabla u_{a'}^\epsilon|^2 dx \to \frac{1}{2} \int_{\Omega} |\nabla v_a|^2 dx, \]
(3.17c)
where \( v_a \) satisfies (3.7), then (3.15) follows.

Since \( d' \to d \), \( K' \chi_{B_\epsilon(a')} \) converges pointwise to \( K \chi_{B_d(a)} \), and \( K' \chi_{B_\epsilon(a')} \) are uniformly bounded for \( \epsilon \) small enough, (3.17a) follows by the Dominated Convergence Theorem.

To prove (3.17b), we integrate by parts to obtain
\[ \int_{\Omega_{d'(a')}} K' \cdot \nabla u_{a'}^\epsilon dx = \int_{\partial \Omega} (K' \cdot \nu)(g - \theta_{a'}) dx, \]
where we have used the first condition in (2.1), (2.2), and the fact that \( u_{a'}^\epsilon = g - \theta_{a'} \) on \( \partial \Omega \) (see (3.3)). Since \( K' \) and \( \theta_{a'} \) are uniformly bounded in \( \epsilon \) on the set \( \partial \Omega \), and converge pointwise to \( K \) and \( \theta_a \), respectively, by the Dominated Convergence Theorem, we have
\[ \int_{\Omega_{d'(a')}} K' \cdot \nabla u_{a'}^\epsilon dx \to \int_{\Omega} K \cdot \nabla v_a dx = \int_{\Omega_d(a')} K \cdot \nabla v_a dx, \]
which gives (3.17b). The last equality follows by the Divergence Theorem, combined with the first condition in (2.1) and property (2.2).

It remains to prove (3.17c). To do this, consider the harmonic extension \( w' \) of \( u_{a'}^\epsilon \) inside \( B_\epsilon(a') \). By applying Lemma 3.6 to \( w' \) with \( a' \) replaced by \( a' \), estimate (3.12) reads
\[ \int_{B_{d'(a')}} |\nabla w'(x)|^2 dx \leq C \epsilon^2 \|g - \theta_{a'}\|_{L^\infty(\partial \Omega)}^2, \]
(3.18)
which implies that
\[ \|w'\|_{H^1(B_{d'(a')})} \to 0, \quad \text{as} \quad \epsilon \to 0. \]
(3.19)
By combining (3.18) with (3.8), we have
\[ \int_{\Omega} |\nabla w'|^2 dx = \int_{\Omega_{d'(a')}} |\nabla u_{a'}^\epsilon|^2 dx + \int_{B_{d'(a')}} |\nabla w'|^2 dx \leq C \epsilon^2 \|g - \theta_{a'}\|_{L^2(\Omega)}^2 + C \epsilon^2 \|g - \theta_{a'}\|_{L^\infty(\partial \Omega)}^2. \]
Therefore, letting \( \epsilon \to 0 \), we obtain
\[ \limsup_{\epsilon \to 0} \int_{\Omega} |\nabla w'|^2 dx \leq C \epsilon \|g - \theta_a\|_{L^2(\Omega)}^2, \]
which, together with Poincaré inequality, implies that \( w' \) is uniformly bounded in \( H^1(\Omega) \). As a consequence there exists \( w \in H^1(\Omega) \) such that (up to subsequences) \( w' \to w \) weakly in \( H^1(\Omega) \).

Since \( w' = g - \theta_a \) on \( \partial \Omega \) for every \( \epsilon \), then \( w = g - \theta_a \) on \( \partial \Omega \). By the lower semicontinuity of the \( H^1 \) norm with (3.10) and (3.19), we obtain that
\[ \frac{1}{2} \int_{\Omega} |\nabla w|^2 dx \leq \liminf_{\epsilon \to 0} \frac{1}{2} \int_{\Omega} |\nabla w'|^2 dx \leq \limsup_{\epsilon \to 0} \frac{1}{2} \int_{\Omega} |\nabla v_a|^2 dx = \frac{1}{2} \int_{\Omega} |\nabla v_a|^2 dx, \]
(3.20)
which implies that \( w = v_a \), by the uniqueness of the minimizer \( v_a \). Therefore, all the inequalities in (3.20) are in fact equalities. This, together with (3.19), gives (3.17c) and completes the proof. \( \square \)

**Remark 3.8.** Let \( a' \to a \in \Omega \) as \( \epsilon \to 0 \) and consider the functions \( u_{a'}^\epsilon \) and \( v_a \) solutions to (3.5) and (3.7) associated with \( a' \), respectively. It is easy to see that
\[ \|\nabla u_{a'}^\epsilon - \nabla v_a\|_{L^2(\Omega,(\Omega,a))} \to 0. \]
(3.21)
Indeed, by the triangle inequality we have that
\[ \|\nabla u_{a'}^\epsilon - \nabla v_a\|_{L^2(\Omega,(\Omega,a))} \leq \|\nabla u_{a'}^\epsilon - \nabla v_a\|_{L^2(\Omega,(\Omega,a))} + \|\nabla v_a - \nabla v_a\|_{L^2(\Omega,(\Omega,a))}, \]
(3.22)
with \( v_a \) solution to (3.7). From the proof (3.17c) in Proposition 3.7, we obtain that the extension of \( u_{a,\varepsilon} \) converges strongly in \( H^1(\Omega) \) to \( v_a \), and therefore the first term in the right-hand side of (3.22) vanishes as \( \varepsilon \to 0 \). The second term converges to zero thanks to the continuity of the map that associates \( v_a \) with the boundary datum \( g - \theta_a \) (observe that \( g - \theta_a \to g - \theta_a \) in \( H^{1/2}(\partial \Omega) \)), so that (3.21) is proved.

**Proposition 3.9.** Let \( a \in \partial \Omega \) and \( a^\varepsilon \) be a sequence of points in \( \overline{\Omega} \) converging to \( a \) as \( \varepsilon \to 0 \).

Then there exist two constants \( C_1, C_2 > 0 \) independent of \( \varepsilon \), \( a^\varepsilon \), and \( a \), such that

\[
F_\varepsilon(a^\varepsilon) \geq C_1 \log(\max\{\varepsilon, d(a^\varepsilon)\}) + C_2, \tag{3.23}
\]

for every \( \varepsilon \) small enough. In particular, \( F_\varepsilon(a^\varepsilon) \to +\infty \) as \( \varepsilon \to 0 \).

**Proof.** Let \( \alpha \) and \( \epsilon_a \) be as in (1.15), let \( \varepsilon < \min\{\epsilon_a, 1\} \), and \( d' < \epsilon_a/2 \), where we set, for brevity, \( d := d(a) \) and \( d' := d(a^\varepsilon) \). We distinguish two possible scenarios: the slow collision \( \varepsilon < d' \) and the fast collision \( \varepsilon \geq d' \). In the former case \( \varepsilon < d' \), exploiting (3.16) we get

\[
F_\varepsilon(a^\varepsilon) \geq \pi \log d' + \frac{1}{2} \int_{\Omega^\varepsilon(a^\varepsilon)} |K_{a^\varepsilon}|^2 \, dx + \int_{\Omega^\varepsilon(a^\varepsilon)} K_{a^\varepsilon} \cdot \nabla \overline{u}_{a^\varepsilon} \, dx + \frac{1}{2} \int_{\Omega^\varepsilon(a^\varepsilon)} |\nabla \overline{u}_{a^\varepsilon}|^2 \, dx
\]

\[
= \pi \log d' + \frac{1}{2} \int_{\Omega^\varepsilon(a^\varepsilon)} |K_{a^\varepsilon} + \nabla \overline{u}_{a^\varepsilon}|^2 \, dx, \tag{3.24}
\]

where we have used that \( \int_{\Omega^\varepsilon(a^\varepsilon)} K_{a^\varepsilon} \cdot \nabla \overline{u}_{a^\varepsilon} \, dx = \int_{\Omega^\varepsilon(a^\varepsilon)} K_{a^\varepsilon} \cdot \nabla \overline{u}_{a^\varepsilon} \, dx \). As noted in Subsection 3.1, we may assume that the discontinuity point \( b^\varepsilon \) of the boundary datum \( g \) is one of the projections of \( a^\varepsilon \) on \( \partial \Omega \), so that \( b^\varepsilon \in \partial B_{d^\varepsilon}(a^\varepsilon) \cap \partial \Omega \). In particular, \( \Omega_{2d^\varepsilon}(b^\varepsilon) \subseteq \Omega(a^\varepsilon) \), so that

\[
\int_{\Omega(a^\varepsilon)} |K_{a^\varepsilon} + \nabla \overline{u}_{a^\varepsilon}|^2 \, dx \geq \int_{\Omega_{2d^\varepsilon}(b^\varepsilon)} |K_{a^\varepsilon} + \nabla \overline{u}_{a^\varepsilon}|^2 \, dx
\]

\[
\geq \inf \left\{ \int_{\Omega_{2d^\varepsilon}(b^\varepsilon)} |\nabla u|^2 \, dx : u \in H^1(\Omega_{2d^\varepsilon}(b^\varepsilon)), u = g \text{ on } \partial \Omega \setminus \overline{B_{2d^\varepsilon}(b^\varepsilon)} \right\}
\]

\[
= \inf \left\{ \int_{\Omega_{2d^\varepsilon}(b^\varepsilon)} |2K_{b^\varepsilon} + \nabla u|^2 \, dx : u \in H^1(\Omega_{2d^\varepsilon}(b^\varepsilon)), u = g - 2\theta_{b^\varepsilon} \text{ on } \partial \Omega \setminus \overline{B_{2d^\varepsilon}(b^\varepsilon)} \right\}
\]

\[
= \int_{\Omega_{2d^\varepsilon}(b^\varepsilon)} |2K_{b^\varepsilon} + \nabla \overline{u}|^2 \, dx,
\]

where \( \overline{u} \) solves

\[
\begin{align*}
\Delta \overline{u} &= 0 & \text{in } \Omega_{2d^\varepsilon}(b^\varepsilon), \\
\overline{u} &= g - 2\theta_{b^\varepsilon} & \text{on } \partial \Omega \setminus \overline{B_{2d^\varepsilon}(b^\varepsilon)}, \\
\partial \overline{u} \cdot \nabla \overline{u} &= -2K_{b^\varepsilon} \cdot \nabla \overline{u} & \text{on } \partial B_{2d^\varepsilon}(b^\varepsilon) \cap \Omega.
\end{align*}
\]

Since \( K_{b^\varepsilon} \cdot \nabla \overline{u} = 0 \) on \( \partial B_{2d^\varepsilon}(b^\varepsilon) \) by (2.2), it follows by uniqueness that \( \overline{u} = \overline{u}_{b^\varepsilon} \), where \( \overline{u}_{b^\varepsilon} \) solves (3.6) (see also Remark 3.3). Therefore, using Young’s inequality and (3.9), recalling that \( \varepsilon < d' \), we may bound (3.25) as follows:

\[
\frac{1}{2} \int_{\Omega_{2d^\varepsilon}(b^\varepsilon)} |2K_{b^\varepsilon} + \nabla \overline{u}|^2 \, dx \geq 2 \int_{\Omega_{2d^\varepsilon}(b^\varepsilon)} |K_{b^\varepsilon}|^2 \, dx + 2 \int_{\Omega_{2d^\varepsilon}(b^\varepsilon)} K_{b^\varepsilon} \cdot \nabla \overline{u} \, dx
\]

\[
\geq 2 \int_{\Omega_{2d^\varepsilon}(b^\varepsilon)} |K_{b^\varepsilon}|^2 \, dx - \lambda \int_{\Omega_{2d^\varepsilon}(b^\varepsilon)} |K_{b^\varepsilon}|^2 \, dx - \frac{1}{\lambda} \int_{\Omega_{2d^\varepsilon}(b^\varepsilon)} |\nabla \overline{u}|^2 \, dx
\]

\[
\geq (2 - \lambda) \int_{\Omega_{2d^\varepsilon}(b^\varepsilon)} |K_{b^\varepsilon}|^2 \, dx - \frac{C}{\lambda \max_{b^\varepsilon \in \partial \Omega} \|g - 2\theta_{b^\varepsilon}\|_{H^{1/2}(\partial \Omega)}}^2
\]

where \( C \) is a positive constant (see Lemma 3.4), and \( \lambda > 0 \) is a constant that will be chosen later.

Now, in view of the assumption on \( d' \) at the beginning of the proof, the set \( \Omega_{2d^\varepsilon}(b^\varepsilon) \) contains a sector, which, in polar coordinates centered at \( b^\varepsilon \), is the rectangle \( R := (2d^\varepsilon, \epsilon_a) \times (\phi_1, \phi_2) \) with \( \phi_1 - \phi_2 = \alpha \). Therefore, in the case \( \varepsilon < d' \), by combining (3.24) and (3.25) with (3.26), and using the estimate from below in (2.4), it follows that

\[
F_\varepsilon(a^\varepsilon) \geq (\pi - (2 - \lambda)\alpha) \log d' + (2 - \lambda)\alpha \log \frac{\epsilon_a}{d} \geq C \max_{b^\varepsilon \in \partial \Omega} \|g - 2\theta_{b^\varepsilon}\|_{H^{1/2}(\partial \Omega)}^2
\]

Recalling that \( \alpha > \pi/2 \) by (1.15), we can choose \( \lambda = (\alpha - \pi/2)/\alpha \), so that the inequality above can be written as

\[
F_\varepsilon(a^\varepsilon) \geq C_1 \log d' + C_2, \tag{3.27}
\]
with
\[ C_1 := \left(\alpha - \frac{\pi}{2}\right) \quad \text{and} \quad C_2 := \left(\alpha + \frac{\pi}{2}\right) \log \frac{\epsilon_0}{2} - \frac{C_\alpha}{\alpha - \pi/2} \max_{b \in \partial \Omega} \| g - 2\theta_b \|_{H^{1/2}(\partial \Omega)}. \] (3.28)

Moreover, in the case \( \epsilon \geq d^* \), we consider the projection \( b^* \) of \( a^* \) on \( \partial \Omega \), so that \( b^* \in \partial B_2(b) \cap \partial \Omega \), and the ball \( B_2(b_0) \) contains \( B_2(b) \). Arguing as in (3.25), (3.26), and using (2.4) with \( d^* \) replaced by \( \epsilon \), we can estimate (3.14) as follows
\[ F_\epsilon(a^*) \geq \int_{\Omega_2(b^*)} |2K_{\epsilon,b} + \nabla \bar{u}|^2 \, dx - \pi |\log \epsilon| \geq C_1 |\log \epsilon| + C_2, \] (3.29)
with the same constants \( C_1 \) and \( C_2 \) provided in (3.28).

Therefore, by (3.27) and (3.29) for every \( \epsilon < \min\{\epsilon_0, 1\} \), the thesis (3.23) follows. \( \square \)

**Proof of Theorem 1.1.** The proof of the continuous convergence is a straightforward consequence of Propositions 3.7 and 3.9. The continuity of \( \mathcal{F} \) follows from the relationship between continuous convergence and \( \Gamma \)-convergence (see [10, Remark 4.9]), which implies that \( \mathcal{F}(a) \) tends to +\( \infty \) as \( a \) approaches the boundary and that the infimum is therefore attained in the interior of \( \Omega \). \( \square \)

**Proof of Corollary 1.2.** Let \( \delta > 0 \). The energy functional \( \mathcal{E}(\cdot) \) is continuous over \( \overline{\Omega}^{(\delta)} \), for every \( \epsilon \in (0, \delta) \). In fact, given two points \( a_1 \) and \( a_2 \) at distance greater than \( \epsilon \) from \( \partial \Omega \), we can easily compare the two energies \( \mathcal{E}(a_1) \) and \( \mathcal{E}(a_2) \) exploiting the change of variables
\[ \bar{\Omega} \ni x \mapsto \Phi(x) := x + \zeta \left( \frac{x - a_1}{\epsilon} \right)(a_2 - a_1), \] (3.30)
being \( \zeta \) a suitable cut-off function. The diffeomorphism \( \Phi \) maps \( \overline{\Omega}(a_1) \) into \( \overline{\Omega}(a_2) \) and keeps the boundary \( \partial \Omega \) fixed. Choosing the curve \( \Sigma_2 \) in the definition of \( \mathcal{E}(a_2) \) as \( \Sigma_2 := \Phi(\Sigma_1) \) and exploiting the fact that \( \| D\Phi - I \|_{L^\infty(\Omega;R^{2\times 2})} \), \( \| \det D\Phi - 1 \|_{L^\infty(\Omega)} = o(|a_1 - a_2|) \), we get \( \mathcal{E}(a_1) = \mathcal{E}(a_2) + o(|a_1 - a_2|) \), which proves the continuity.

Fix now \( \epsilon > 0 \) and \( k \in \mathbb{N} \) and let \( a_{\epsilon,k} \) be such that
\[ \inf_{\bar{\Omega}} \mathcal{F}_\epsilon \leq \mathcal{F}_\epsilon(a_{\epsilon,k}) \leq \inf_{\bar{\Omega}} \mathcal{F}_\epsilon + 1/k. \] (3.31)

Without loss of generality, we can assume that the whole sequence \( a_{\epsilon,k} \) converges to some \( a^* \in \overline{\Omega} \), as \( k \to \infty \), and satisfies \( |a_{\epsilon,k} - a^*| < 1/k \). We claim that there exist \( \tau > 0 \) and \( \delta > 0 \) such that
\[ d(a^*) > \delta \quad \forall \epsilon \in (0, \tau). \] (3.32)
Assume by contradiction that there exists a subsequence of \( \epsilon \) (not relabeled) such that \( d(a^*) \to 0 \) as \( \epsilon \to 0 \). Let \( k_\epsilon \) be a sequence of natural numbers, increasing as \( \epsilon \) goes to zero, and let \( a \) be a cluster point of the family \( \{a_{\epsilon,k_\epsilon}\}_\epsilon \). In view of Theorem 1.1, thanks to [10, Corollary 7.20], we infer that \( a \) is a minimizer of the functional \( \mathcal{F} \). By the triangle inequality, we get
\[ d(a) \leq d(a^*) + |a^* - a_{\epsilon,k_\epsilon}| + |a_{\epsilon,k_\epsilon} - a| < d(a^*) + \frac{1}{k_\epsilon} + o(\epsilon) \to 0, \]
which implies that \( a \in \partial \Omega \), in contradiction with Theorem 1.1.

Let \( \epsilon_1 := \min\{\tau, \delta/2\} \) and \( \epsilon \in (0, \epsilon_1) \). In view of (3.32), the minimizing sequence \( \{a_{\epsilon,k}\}_k \) lies in the set \( \overline{\Omega}^{(\delta/2)} \), where the functional \( \mathcal{F}_\epsilon \) is continuous. Therefore, by (3.31), we conclude that
\[ \inf_{\bar{\Omega}} \mathcal{F}_\epsilon = \lim_{k \to \infty} \mathcal{F}_\epsilon(a_{\epsilon,k}) = \mathcal{F}_\epsilon(a^*), \]
amely \( a^* \) is a minimizer of \( \mathcal{F}_\epsilon \) and it is at distance at least \( \delta \) from the boundary.

Eventually, again by [10, Corollary 7.20], we infer that, up to a subsequence (not relabeled), \( a^* \) converges to a minimizer of \( \mathcal{F} \) so that \( \mathcal{F}_\epsilon(a^*) \to \mathcal{F}(a) \) as \( \epsilon \to 0 \). \( \square \)

### 3.4. Plots of the limiting energy.

In view of Corollary 1.2, for a fixed \( \epsilon > 0 \) small enough, the minima of \( \mathcal{F}_\epsilon \) are close to those of \( \mathcal{F} \), therefore, by (1.5), the energy landscape provided by \( \mathcal{F} \) is a good approximation of that of \( \mathcal{E}_\epsilon \).

Considering \( \Omega \) the unit ball centered at the origin \( O \), we present here the plots of the energy profile for two different boundary conditions \( f_1 = 1 \) on \( \partial \Omega \), and \( f_2 = 2 \) on \( \partial \Omega \cap \{ x > 0 \} \) and \( f_2 = 0 \) on \( \partial \Omega \cap \{ x < 0 \} \); the choices that we make for the numerics are \( g_1 = \theta_O \), and
\[ g_2 = \begin{cases} 2\theta_O & \text{in the first quadrant,} \\ \pi & \text{in the second and third quadrants,} \\ 2\theta_O - 2\pi & \text{in the fourth quadrant.} \end{cases} \]
From Figure 1 one can deduce that the energy profile associated with \( f_1 \) is radially symmetric and has a minimum in the center. The one for \( f_2 \) is no longer radially symmetric, but symmetric with respect to the \( y \) axis and has a minimum in the interior of \( \Omega \) which is located where \( f_2 \) is the largest (by symmetry, it is located along the \( x \)-axis, at about \( x = 0.65 \)).

4. Many dislocations

In this section we analyze the case with many dislocations. In analogy with Section 3 we first derive a displacement formulation for (1.8) in Subsection 4.1 and then we prove some a priori bounds in Subsection 4.2. In Subsection 4.3 we prove Theorem 1.3 and Corollary 1.4.

4.1. Displacement formulation.

**Definition 4.1** (multiplicity of a dislocation). Let \( n \in \mathbb{N} \) with \( n \geq 2 \) and let \( a_1, \ldots, a_n \in \overline{\Omega} \). We label the dislocations in such a way that the first \( \ell \) of them \( (\ell \leq n) \) are all distinct, so that \( a_i \neq a_k \) for all \( i, k \in \{1, \ldots, \ell\} \), \( i \neq k \), and, for every \( j \in \{\ell + 1, \ldots, n\} \) there exists \( i(j) \in \{1, \ldots, \ell\} \) such that \( a_j = a_{i(j)} \). For every \( i = 1, \ldots, \ell \) we say that a point \( a_i \) has multiplicity \( m_i \) if there are \( m_i - 1 \) points \( a_j \), with \( j \in \{\ell + 1, \ldots, n\} \) that coincide with \( a_i \). Clearly, \( \sum_{i=1}^{\ell} m_i = n \).

In analogy to Subsection 3.1 given \( n \) dislocations \( a_1, \ldots, a_n \in \overline{\Omega} \), we choose \( n \) (closed) segments \( \Sigma_1, \ldots, \Sigma_n \) joining the dislocations with the boundary, such that \( \Omega \setminus (\Sigma_1 \cup \cdots \cup \Sigma_n) \) is simply connected. One possible construction of such a family of segments is to fix a point \( a^* \notin \overline{\Omega} \) and take \( \Sigma_i \) as the portion of the segment joining \( a_i \) with \( a^* \) lying inside \( \overline{\Pi} \), for \( i = 1, \ldots, \ell \). With this construction, if \( a_i = a_j \), then \( \Sigma_i = \Sigma_j \). Moreover, we set \( b_i := \Sigma_i \cap \partial \Omega \), for \( i = 1, \ldots, \ell \). Given \( b \in \partial \Omega \setminus \{b_1, \ldots, b_{\ell}\} \), we denote by \( \gamma^x_b \) the counterclockwise path in \( \partial \Omega \) connecting \( b \) and \( x \).

Such parametrization induces an ordering on the points of the boundary: we say that \( x \) precedes \( y \) on \( \partial \Omega \), and we write \( x \preceq y \), if the support of \( \gamma^x_b \) contains that of \( \gamma^y_b \). Without loss of generality, we may relabel the \( b_i \)'s (and the \( a_i \)'s, accordingly) so that \( b \prec b_1 \preceq \ldots \preceq b_\ell \). According to this notation, for \( i = 1, \ldots, \ell \), we define \( g_{b_i} : \partial \Omega \setminus \{b_i\} \to \mathbb{R} \) as

\[
g_{b_i}(x) := \begin{cases} \int_{\gamma^x_{b_i}} f \, dt & \text{if } b \preceq x \prec b_i, \\ \int_{\gamma^x_{b_i}} f \, dt - 2\pi & \text{if } b_i \prec x \prec b. \end{cases}
\]

Moreover, we introduce the arithmetic mean

\[
g(x) := \frac{1}{n} \sum_{i=1}^{\ell} m_i g_{b_i}(x) = \int_{\gamma^x} f(y) \, dy - \frac{2\pi}{n} \sum_{i=0}^{\ell-1} m_i \quad \text{if } b_{j-1} \prec x \prec b_j,
\]

with \( j \in \{1, \ldots, \ell + 1\} \), \( b_0 = b_{\ell+1} := b \) and \( m_0 := 0 \). Note that, since \( g_{b_i} \) is continuous except at \( b_i \), where it has a jump of \( 2m_i \pi \), we have that the boundary datum \( g \) has a jump of \( 2\pi m_j/n \) at every \( b_j \) (whereas it is continuous at \( b \)).

In view of the construction above, for every \( \epsilon > 0 \), every connected component of \( \Omega_\epsilon(a_1, \ldots, a_n) \setminus (\Sigma_1 \cup \cdots \cup \Sigma_n) \) is simply connected. Therefore, in the same spirit of 3.2
and \([\text{(3.3)}]\), we may write

\[
\mathcal{E}_\epsilon(a_1, \ldots, a_n) = \frac{1}{2} \int_{\Omega(a_1, \ldots, a_n)} |\nabla u^\epsilon_{a_1, \ldots, a_n}|^2 \, dx,
\]

where \(u^\epsilon_{a_1, \ldots, a_n} \in H^1(\Omega(a_1, \ldots, a_n))\) is characterized by

\[
\begin{aligned}
\Delta u^\epsilon_{a_1, \ldots, a_n} &= 0 & \text{in } \Omega_\epsilon(a_1, \ldots, a_n) \setminus (\cup_{i=1}^n \Sigma_i), \\
[u^\epsilon_{a_1, \ldots, a_n}] &= 2\pi m_i & \text{on } \Sigma_i \cap \Omega_\epsilon(a_1, \ldots, a_n), \\
\partial u^\epsilon_{a_1, \ldots, a_n} / \partial \nu &= \sum_{i=1}^n m_i g_i = ng & \text{on } \partial \Omega \setminus (\cup_{i=1}^n \Sigma_i), \\
\partial \theta_{a_1, \ldots, a_n} / \partial \nu &= \partial (u^\epsilon_{a_1, \ldots, a_n})^- / \partial \nu & \text{on } (\cup_{i=1}^n \Sigma_i) \cap \Omega_\epsilon(a_1, \ldots, a_n),
\end{aligned}
\]

(\text{for more comments about the derivation of } \text{[4.3]} \text{ see Remark 4.2 below}).

Moreover, exploiting the decomposition \([\text{3.4]}\), we may write \(\nabla u^\epsilon_{a_1, \ldots, a_n}\) as the sum of a singular and a regular part:

\[
\nabla u^\epsilon_{a_1, \ldots, a_n} = \sum_{i=1}^n \omega(a_i) K_{a_i} + \nabla \tilde{v}^\epsilon_{a_1, \ldots, a_n}.
\]

Here we impose that the angular coordinate \(\theta_{a_i}\) centered at \(a_i\) which defines \(K_{a_i}\) has a jump of \(2\pi\) on the half line containing \(\Sigma_i\) if \(a_i \in \Omega\), and on the half line containing the outer normal to the boundary at \(a_i\) in case \(a_i \in \partial \Omega\). The function \(\tilde{v}^\epsilon_{a_1, \ldots, a_n} \in H^1(\Omega(a_1, \ldots, a_n))\) solves

\[
\begin{aligned}
\Delta \tilde{v}^\epsilon_{a_1, \ldots, a_n} &= 0 & \text{in } \Omega_\epsilon(a_1, \ldots, a_n), \\
\tilde{v}^\epsilon_{a_1, \ldots, a_n} &= \sum_{i=1}^n (g_i - \omega(a_i) \theta_{a_i}) & \text{on } \partial \Omega \setminus (\cup_{i=1}^n \Sigma_i), \\
\partial \tilde{v}^\epsilon_{a_1, \ldots, a_n} / \partial \nu &= -\sum_{i=1}^n K_{a_i} \cdot \nu & \text{on } \partial \Omega(a_1, \ldots, a_n). \\
\end{aligned}
\]

As already pointed out in \([\text{3.6]}\), the weight \(\omega(a_i)\) is necessary in order to balance the jump of \(g_i\) at \(b_i\). In particular, we deduce the representation formula

\[
\mathcal{E}_\epsilon(a_1, \ldots, a_n) = \frac{1}{2} \int_{\Omega(a_1, \ldots, a_n)} \left| \sum_{i=1}^n \omega(a_i) K_{a_i} + \nabla \tilde{v}^\epsilon_{a_1, \ldots, a_n} \right|^2 \, dx.
\]

Remark 4.2. Let us comment the second optimality condition in \([\text{4.3]}\). As already noticed, our choice of \(\Sigma\) makes every connected component of \(\Omega_\epsilon(a_1, \ldots, a_n) \setminus (\Sigma_1 \cup \ldots \cup \Sigma_n)\) simply connected. Therefore, on every connected component, the solution \(u^\epsilon_{a_1, \ldots, a_n}\) is unique up to an additive constant. Exploiting the fact that the circulation of an optimal field for \([\text{1.8]}\) along any simple closed curve which does not enclose any dislocation is zero, it is easy to see that the jump of \(u^\epsilon_{a_1, \ldots, a_n}\) is constant on every connected component of \((\Sigma_1 \cup \ldots \cup \Sigma_n) \cap \Omega_\epsilon(a_1, \ldots, a_n)\). In the simplest case when \(\Omega_\epsilon(a_1, \ldots, a_n) \setminus (\Sigma_1 \cup \ldots \cup \Sigma_n)\) is itself simply connected (see Fig. \([\text{2a]}\)), the condition \([u^\epsilon_{a_1, \ldots, a_n}] = 2\pi m_i\) follows from the circulation assumption on the admissible fields for \([\text{1.8]}\). If instead \(\Omega_\epsilon(a_1, \ldots, a_n) \setminus (\Sigma_1 \cup \ldots \Sigma_n)\) is not simply connected, which is the case when there are balls partially overlapping, cuts intersecting more than one ball, or balls touching the boundary not containing the corresponding cut (see Fig. \([\text{2b]}\)), the proof follows again by the
circulation assumption in (4.3) (which gives some constraints on the sum of the jumps) and by 
exploiting the fact that we are free to add to \( u^{a} \) a constant in every connected component
of \( \Omega \backslash (\Sigma_1 \cup \ldots \cup \Sigma_n) \) without changing the energy.

**Remark 4.3.** Notice that, similarly to Lemma 3.1, the energy (4.2) does not depend on the 
choice of the boundary points \( b_i \), nor on the curves \( \Sigma_i \), nor on the primitives \( g_i \) chosen. Moreover, we point out that the definition is consistent with the one given in (1.4); in particular, when 
\( a_i \equiv a \) for every \( i = 1, \ldots, n \), choosing \( b_i \equiv b \), the datum \( g \) in (1.1) agrees with the primitive of 
f (3.1), and we have \( E_c(a, \ldots, a) = n^2 E_c(a) \). Therefore, by (1.5) and (1.11), for every \( \epsilon \) small
enough
\[ F_\epsilon(a, \ldots, a) = n^2 F_\epsilon(a) + (n-1)\pi |\log \epsilon|, \]  
(4.5)

**Remark 4.4.** By linearity, the function \( \tilde{v}^{a_1, \ldots, a_n} \) introduced in (4.4) can be written as the superposition of the solutions \( \tilde{v}^{a_i} \) to
\[ \begin{cases} 
\Delta \tilde{v}^{a_i} = 0 & \text{in } \Omega(a_1, \ldots, a_n), \\
\tilde{v}^{a_i} = g_i - \omega(a_i) \theta_{a_i} & \text{on } \partial \Omega \setminus \bigcup_{j=1}^n B_{\epsilon}(a_j), \\
\partial \tilde{v}^{a_i} / \partial \nu = -K_{a_i} \cdot \nu & \text{on } \partial \Omega \setminus \bigcup_{j=1}^n \partial B_{\epsilon}(a_j),
\end{cases} \]  
(4.6)

namely \( \tau^{a_1, \ldots, a_n} = \sum_{i=1}^n \tau^{a_i} \).

**Remark 4.5.** Fix \( i \in \{1, \ldots, n\} \). By linearity, the solution \( \tilde{v}^{a_i} \) of (4.6) can be written as a superposition of one function satisfying an homogeneous Neumann boundary condition and the other an homogeneous Dirichlet boundary condition, namely
\[ \tilde{v}^{a_i} = \hat{u}^{a_i} + q^{a_i}, \]  
where \( \hat{u}^{a_i} \) and \( q^{a_i} \) solve, respectively,
\[ \begin{cases} 
\Delta \hat{u}^{a_i} = 0 & \text{in } \Omega(a_1, \ldots, a_n), \\
\hat{u}^{a_i} = g_i - \omega(a_i) \theta_{a_i} & \text{on } \partial \Omega \setminus \bigcup_{j=1}^n B_{\epsilon}(a_j), \\
\partial \hat{u}^{a_i} / \partial \nu = 0 & \text{on } \bigcup_{j=1}^n \partial B_{\epsilon}(a_j),
\end{cases} \]  
(4.7)

and
\[ \begin{cases} 
\Delta q^{a_i} = 0 & \text{in } \Omega(a_1, \ldots, a_n), \\
q^{a_i} = 0 & \text{on } \partial \Omega \setminus \bigcup_{j=1}^n B_{\epsilon}(a_j), \\
\partial q^{a_i} / \partial \nu = -K_{a_i} \cdot \nu & \text{on } \bigcup_{j=1}^n \partial B_{\epsilon}(a_j), \\
f_{\partial \Omega}(a_j) \partial q^{a_i} / \partial \tau = 0 & \text{for every } j = 1, \ldots, n.
\end{cases} \]  
(4.8)

**4.2. A priori bounds on harmonic functions.** We collect here some lemmas providing a
priori estimates that will be useful in the sequel. We start by noticing that Lemmas 3.4 and 3.5
can be extended to the case of a domain with multiple holes.

**Lemma 4.6.** Let \( a_1, \ldots, a_n \in \overline{\Omega} \), let \( h \in W^{1,1}(\partial \Omega) \), and let \( \hat{u} \) be the solution to
\[ \begin{cases} 
\Delta \hat{u} = 0 & \text{in } \Omega(a_1, \ldots, a_n), \\
\hat{u} = h & \text{on } \partial \Omega \setminus \bigcup_{j=1}^n B_{\epsilon}(a_j), \\
\partial \hat{u} / \partial \nu = 0 & \text{on } \partial \Omega(a_1, \ldots, a_n) \cap \left( \bigcup_{j=1}^n \partial B_{\epsilon}(a_j) \right).
\end{cases} \]  
(4.9)

Then \( \hat{u} \) is the minimizer of the Dirichlet energy in \( H^1(\Omega(a_1, \ldots, a_n)) \) with prescribed boundary data \( h \) on \( \partial \Omega \setminus \bigcup_{j=1}^n B_{\epsilon}(a_j) \), and the following estimates hold:
\[ |\hat{u}(x)| \leq C ||h||_{L^\infty(\partial \Omega)}, \quad \text{for all } x \in \overline{\Omega}(a_1, \ldots, a_n) \]  
(4.10)

and
\[ \int_{\Omega(a_1, \ldots, a_n)} |\nabla \hat{u}|^2 \, dx \leq C ||h||_{H^{1/2}(\partial \Omega)}^2 \]  
(4.11)

for some constant \( C > 0 \) independent of \( \epsilon \). Moreover, in the case \( a_1, \ldots, a_n \in \Omega \) are distinct points and \( \epsilon \ll \min_i d_i \), also the estimate on the gradient holds:
\[ |\nabla \hat{u}(x)| \leq C ||h||_{L^\infty(\partial \Omega)}, \quad \text{for all } x \in \bigcup_{j=1}^n \partial B_{\epsilon}(a_j). \]  
(4.12)

The minimality of \( \hat{u} \) follows from the uniqueness of the solution to (4.9); estimate (4.10) is a consequence of the maximum principle and Hopf’s Lemma, while (4.11) is obtained in a
similar way as in Lemma 3.4 by testing \( \hat{u} \) with the minimizer of the Dirichlet energy in \( H^1(\Omega) \) with prescribed boundary data \( h \) on \( \partial \Omega \setminus \bigcup_{j=1}^n B_{\epsilon}(a_j) \) and using the continuity of the map.
Since clearly have for every ary, such that we choose \( j \) and, by recursion, given \( j \) we choose \( i \), notice that, for this equality, we have used the maximum principle, combined with the Hopf’s Lemma, which associates the minimizer with the boundary datum. Finally, the proof of (4.12), which is analogous to the second inequality in (3.11), is standard.

**Lemma 4.7.** Let \( D_1, \ldots, D_\ell \subset \mathbb{R}^2 \) be open, bounded, simply connected sets with Lipschitz boundary, such that \( \mathcal{D}_1, \ldots, \mathcal{D}_\ell \) are pairwise disjoint, and intersecting \( \Omega \). Set \( \Omega' := \Omega \setminus \bigcup_{i=1}^\ell D_i \) and for every \( i = 1, \ldots, \ell \), let \( h_i \in C^1(\Omega') \). Consider the minimum problem

\[
\min_c \min_u \frac{1}{2} \int_{\Omega'} |\nabla u|^2 \, dx,
\]

with \( c = (c_1, \ldots, c_\ell) \in \mathbb{R}^\ell \) and \( u \) varying in the subset of functions in \( H^1(\Omega') \) satisfying

\[
u = h_i + c_i \quad \text{on } \partial D_i \cap \Omega, \quad \text{for } i = 1, \ldots, \ell.
\]

Then \( p \in H^1(\Omega') \) is a solution to (4.13) if and only if it satisfies

\[
\begin{cases}
\Delta p = 0 & \text{in } \Omega', \\
\partial p/\partial \nu = 0 & \text{on } \partial \Omega' \cap \partial \Omega, \\
\partial p/\partial \tau = \partial h_i/\partial \tau & \text{on } \partial D_i \cap \Omega, \quad \text{for } i = 1, \ldots, \ell, \\
\int_{\partial D_i \cap \partial \Omega} \partial p/\partial \nu \, d\mathcal{H}^1 = 0 & \text{for } i = 1, \ldots, \ell.
\end{cases}
\]

Moreover, a solution \( p \) satisfies

\[
\left| \max_{\Omega'} p - \min_{\Omega'} p \right| \leq \sum_{i=1}^\ell H^1(\partial D_i \cap \Omega) \max_{\partial D_i \cap \Omega} \left| \frac{\partial h_i(x)}{\partial \tau} \right|.
\]

**Proof.** First we remark that problem (4.13) is well posed, namely the class of admissible functions is not empty: indeed, the function \( u := \sum_{i=1}^\ell \zeta_i h_i \) satisfies \( 4.14 \), being \( \zeta_i \) suitable smooth functions in \( \Omega' \) which are identically 1 in a neighborhood of \( D_i \).

If \( p \) satisfies the minimum problem (4.13), then it is easy to obtain system (4.15) as the Euler-Lagrange conditions arising from minimality.

Conversely, assume that \( p \) satisfies (4.15). In order to show that \( p \) is optimal for (4.13), it is enough to show that the (4.15) has a unique solution (up to a constant). Take two solutions of (4.15); then their difference \( s \) is characterized by

\[
\begin{cases}
\Delta s = 0 & \text{in } \Omega', \\
\partial s/\partial \nu = 0 & \text{on } \partial \Omega' \cap \partial \Omega, \\
\partial s/\partial \tau = 0 & \text{on } \partial \Omega' \setminus \partial \Omega, \\
\int_{\partial D_i \cap \partial \Omega} \partial s/\partial \nu \, d\mathcal{H}^1 = 0 & \text{for } i = 1, \ldots, \ell.
\end{cases}
\]

The third condition implies that \( s \) is constant on the boundary of the holes (intersected with \( \Omega \)), namely \( s = s_i \) on \( \partial D_i \cap \Omega \) for some \( s_i \in \mathbb{R} \), for \( i = 1, \ldots, \ell \). In view of the maximum principle, we infer that either \( s \) attains its maximum on the boundary \( \partial \Omega' \setminus \partial \Omega \) or it is constant. The former case is excluded by Hopf’s Lemma combined with the second and fourth conditions in (4.17); therefore the latter case holds and the proof of the first statement is concluded.

We relabel the sets \( D_i \) so that the local minima of \( p \) on their boundary are ordered as follows:

\[
\min_{\partial D_i \cap \Omega} p \geq \min_{\partial D_j \cap \Omega} p \geq \cdots \geq \min_{\partial D_\ell \cap \Omega} p = \min_{\Omega'} p.
\]

Notice that, for the last equality, we have used the maximum principle, combined with the Hopf’s Lemma, which prevents the minimum of \( p \) to be on \( \partial \Omega \) (the same holds true for the maximum).

We now define an ordered subfamily of indices as follows: we choose

\[
j_1 \in \left\{ j : \max_{\partial D_i \cap \Omega} p = \max_{\Omega'} p \right\},
\]

and, by recursion, given \( j_i \) we choose

\[
j_{i+1} \in \left\{ j : \max_{\partial D_i \cap \Omega} p = \max_{\cup_{k=1}^{j_i} \partial D_k \cap \Omega} p \right\}.
\]

Since \( j_{i+1} > j_i \), in a finite number of steps, say \( i^* \), we find \( j_{i^*} = \ell \) and the procedure stops. We clearly have

\[
\max_{\Omega'} p = \max_{\partial D_{j_1} \cap \Omega} p \geq \max_{\partial D_{j_2} \cap \Omega} p \geq \cdots \geq \max_{\partial D_{j_{i^*}} \cap \Omega} p = \max_{\partial D_{i^*} \cap \Omega} p.
\]
Moreover, we claim that, for every \( i = 1, \ldots, i^* - 1 \)

\[
\min_{\partial D^i \cap \Omega} p \leq \max_{\partial D^{i'+1} \cap \Omega} p. \tag{4.18}
\]

Once proved the claim we are done, indeed

\[
\left| \max_{\Omega} - \min_{\Omega} p \right| = \max_{\partial D_{i_j} \cap \Omega} p - \min_{\partial D_{i_j} \cap \Omega} p
\]

\[
= \sum_{i = 1}^{i^*} \left( \max_{\partial D_i \cap \Omega} p - \min_{\partial D_i \cap \Omega} p \right) + \sum_{i = 1}^{i^* - 1} \left( \min_{\partial D_i \cap \Omega} p - \max_{\partial D_{i+1} \cap \Omega} p \right)
\]

\[
\leq \sum_{i = 1}^\ell \left( \max_{\partial D_i \cap \Omega} p - \min_{\partial D_i \cap \Omega} p \right) \leq \sum_{i = 1}^\ell \mathcal{H}^1(\partial D_i \cap \Omega) \max_{\partial D_i \cap \Omega} \left| \frac{\partial h_i(x)}{\partial \tau} \right|,
\]

where, in the last line, we have used the Mean Value Theorem. This proves (4.16).

In order to prove the claim (4.18), we argue by contradiction: we assume that there exists \( i \in \{1, \ldots, i^* - 1\} \) such that

\[
p := \min_{\partial D_i \cap \Omega} p > \bar{p} := \max_{\partial D_{i+1} \cap \Omega} p.
\]

Notice that, by construction, we have

\[
\min_{\cup_{i = 1}^{i^* - 1} \partial D_i \cap \Omega} p > \max_{\cup_{i = i^*}^\ell \partial D_i \cap \Omega} p.
\]

Consider the nonempty set \( E := \{ x \in \Omega^\prime : \ p > p(x) > \bar{p} \} \) and the function

\[
\psi(x) := \begin{cases} p(x) & \text{if } p(x) \leq \bar{p}, \\ \bar{p} & \text{if } x \in E, \\ p(x) + \bar{p} - \bar{p} & \text{if } p(x) \geq \bar{p}. \end{cases}
\]

The function \( \psi \) is an element of \( H^1(\Omega^\prime) \), it is admissible for the minimum problem (4.13), and it decreases the Dirichlet energy, contradicting the optimality of \( p \).

As a direct consequence of Lemma 4.7, we obtain the following corollary.

**Corollary 4.8.** Let \( a_1, \ldots, a_{\ell} \in \overline{\Omega} \) be distinct points and let \( \{a_1', \ldots, a_{\ell}'\} \) be a sequence of points in \( \Omega^\prime \) converging to \( (a_1, \ldots, a_{\ell}) \) as \( \epsilon \to 0 \), such that the balls \( B_{\epsilon}(a_j') \) are pairwise disjoint. Fix \( i \in \{1, \ldots, \ell\} \) and let \( p^\epsilon_{i'} \) be the solution (unique up to a constant) to

\[
\begin{aligned}
\Delta p^\epsilon_{i'} &= 0 & \text{in } \Omega \setminus \cup_{j=1}^i B_{\epsilon}(a_j') \\
\partial p^\epsilon_{i'}/\partial \nu &= 0 & \text{on } \partial \Omega \setminus \cup_{j=1}^i B_{\epsilon}(a_j') \\
\partial p^\epsilon_{i'}/\partial \tau &= -\partial \log(|x - a_j'|)/\partial \tau & \text{on } \partial B_{\epsilon}(a_j') \cap \Omega, \text{ for every } j \neq i, \\
\int_{\partial B_{\epsilon}(a_j') \cap \Omega} \partial p^\epsilon_{i'}/\partial \nu &= 0 & \text{on } \partial B_{\epsilon}(a_j') \cap \Omega, \\
\end{aligned}
\tag{4.19}
\]

Then, there exists a positive constant \( C \) independent of \( \epsilon \) such that

\[
\|p^\epsilon_{i'}\|_{L^\infty(\Omega \setminus \{a_1', \ldots, a_{\ell}'\})} \leq C \epsilon. \tag{4.20}
\]

Moreover,

\[
\int_{\Omega \setminus \{a_1', \ldots, a_{\ell}'\}} |\nabla p^\epsilon_{i'}|^2 \, dx \leq \frac{C}{|\log \epsilon|}. \tag{4.21}
\]

**Proof.** By applying Lemma 4.7 with \( D_j = B_{\epsilon}(a_j') \), and \( h_j = -\log(|x - a_j'|) \) for every \( j \in \{1, \ldots, \ell\} \setminus \{i\} \), \( h_i = 0 \), estimate (4.16) provides a positive constant \( C \) such that

\[
\max_{\Omega \setminus \{a_1', \ldots, a_{\ell}'\}} p^\epsilon_{i'} - \min_{\Omega \setminus \{a_1', \ldots, a_{\ell}'\}} p^\epsilon_{i'} \leq 2\pi C,
\]

which implies (4.20).

Let \( \zeta_\epsilon \) be a smooth cut-off function which is identically 1 inside \( B_{\epsilon}(0) \) and 0 outside \( B_{\epsilon/\tau}(0) \), and whose support vanishes as \( \epsilon \to 0 \). From the minimality of \( p^\epsilon_{i'} \), comparing with \( c_1 = \cdots = c_{\ell} = 0 \) and

\[
u(x) = -\sum_{j \neq i} \zeta_\epsilon(x - a_j) \log(|x - a_j'|)
\]
in (4.14), we obtain
\[ \int_{\Omega_\epsilon(a_1';...,a_n')} |\nabla p_{i\epsilon}|^2 \, dx \leq C \left( \int_{B_{1/\epsilon}(0)} |\nabla \zeta_i|^2 \, dx + |\text{spt} \, \zeta_i| \right), \] (4.22)
for some positive constant C independent of \( \epsilon \). By infimizing with respect to \( \zeta_i \in \{ \zeta \in C^\infty_c(B_{1/\epsilon}(0)); \zeta \equiv 1 \in B_{1/\epsilon}(0) \} \), the integral in the right-hand side of (4.22) is the capacity of a ball of radius \( \epsilon \) inside a ball of radius \( \sqrt{\epsilon} \), which is equal to \( 4\pi/|\log \epsilon| \), therefore we obtain
\[ \int_{\Omega_\epsilon(a_1';...,a_n')} |\nabla p_{i\epsilon}|^2 \, dx \leq C(\pi \left( \frac{4}{|\log \epsilon|} + \epsilon \right)), \]
which implies (4.21). The lemma is proved. \( \square \)

**Lemma 4.9.** Let \( a_1, \ldots, a_n \) be distinct points in \( \Omega \) and let \( (a_1';...,a_n') \) be a sequence of points in \( \Omega^n \) converging to \( (a_1,\ldots,a_n) \) as \( \epsilon \to 0 \). Then, for every \( i = 1, \ldots, n \), as \( \epsilon \to 0 \) we have
\[ \| \nabla \hat{v}_{i\epsilon} - \nabla v_{ai} \|_{L^2(\Omega,(a_1,...,a_n),\mathbb{R}^2)} \to 0, \] (4.23)
where \( \hat{v}_{i\epsilon} \) and \( v_{ai} \) are defined in (4.6) and (4.7), respectively.

**Proof.** Since the limit points \( a_1, \ldots, a_n \) are all distinct and far from the boundary, we may assume, without loss of generality, that \( \epsilon < d := \min_{i\in\{1,...,n\}} d_i \) (see (1.12)), so that, for \( \epsilon \) small enough, the balls \( B_i(a_i') \) are pairwise disjoint and do not intersect \( \partial \Omega \). Let now \( \epsilon \in \{1,\ldots,n\} \) be fixed. As noticed in Remark 4.5, the solution \( \hat{v}_{i\epsilon} \) of (4.6) can be written as \( \hat{v}_{i\epsilon} = \hat{u}_i + q_i' \), where \( \hat{u}_i \) and \( q_i' \) solve (4.7) and (4.8), respectively.

To prove (4.23), we will show that, for every fixed \( i \in \{1,\ldots,n\} \),
\[ \| \nabla \hat{u}_i - \nabla v_{ai} \|_{L^2(\Omega,(a_1,...,a_n),\mathbb{R}^2)} \to 0, \] (4.24a)
\[ \| q_i' \|_{L^2(\Omega,(a_1,...,a_n),\mathbb{R}^2)} \to 0. \] (4.24b)

Note that in (4.23) we can replace \( v_{ai} \) with \( v_{ai}' \) thanks to the continuity of the map \( C^0(\partial \Omega) \ni h \mapsto w_h \in H^1(\Omega) \), where \( w_h \) is harmonic in \( \Omega \) with boundary datum \( h \).

By Lemma 4.6, \( a_i' \) minimizes the Dirichlet energy in \( H^1(\Omega,(a_1',...,a_n')) \) with boundary datum \( g_{a_i'} - \theta_{a_i'} \) on \( \partial \Omega \), and by (4.11) we have
\[ \| \nabla \hat{u}_i \|_{L^2(\Omega,(a_1,...,a_n),\mathbb{R}^2)} \leq C \| g_{a_i'} - \theta_{a_i'} \|_{H^{1/2}(\partial \Omega)}. \]

Let us consider the extension of \( a_i' \) (not relabeled), which is harmonic in every \( B_i(a_i') \). By (4.10) and (4.12), an estimate similar to (3.12) holds: there exists \( C > 0 \) independent of \( \epsilon \) such that
\[ \| \nabla \hat{u}_i \|_{L^2(B_i(a_i'),\mathbb{R}^2)} \leq C \epsilon \| g_{a_i'} - \theta_{a_i'} \|_{L^2(\Omega)}, \]

Since all the points \( a_i' \) are in \( \Omega^{(d/2)} \), by the two estimates above, we obtain a uniform bound in \( H^1(\Omega) \) for \( \hat{u}_i' \), and therefore \( \hat{u}_i' \) has a subsequence that converges weakly to a function \( w_i \) in \( H^1(\Omega) \) as \( \epsilon \to 0 \). By the lower semicontinuity of the \( H^1 \) norm (arguing as in (3.20)) it turns out that \( w_i = v_{ai} \) and
\[ \lim_{\epsilon \to 0} \| \nabla \hat{u}_i \|_{L^2(\Omega,\mathbb{R}^2)} = \| \nabla v_{ai} \|_{L^2(\Omega,\mathbb{R}^2)}; \]
moreover, arguing as in Remark 3.8, we have
\[ \nabla \hat{u}_i' - \nabla v_{ai} \to 0 \quad \text{strongly in } L^2(\Omega;\mathbb{R}^2), \] (4.25)

hence (4.24a) follows.

To obtain an estimate for the gradient of \( q_i' \), we define \( P_i' := (\nabla q_i')^\perp = (-\partial_2 q_i', \partial_1 q_i') \) and notice that \( P_i' \) is a conservative vector field in \( \Omega \epsilon \), namely its circulation vanishes along any closed loop contained in \( \Omega \epsilon \). Therefore, there exists \( p_i' \in H^1(\Omega \epsilon) \) such that \( P_i' = \nabla p_i' \) and, in view of (4.8) with the fact that \( -K_{a_i'} \cdot v = \partial \log(|x - a_i'|)/\partial \nu \), \( p_i' \) is a solution to (4.19). Then by (4.21), we deduce that the gradient of \( p_i' \to 0 \) as \( \epsilon \to 0 \) and since the gradients of \( p_i' \) and \( q_i' \) have the same modulus, we obtain (4.24b). \( \square \)

**Remark 4.10.** From the proof of Lemma 4.9 we derive some important properties of the functions \( \hat{u}_i' \) and \( p_i' \) introduced in (4.7) and (4.19), respectively. Let \( a_1,\ldots,a_n \) be distinct points in \( \Omega \) and let \( (a_1',...,a_n') \) be a sequence of points in \( \Omega^n \) converging to \( (a_1,...,a_n) \) as \( \epsilon \to 0 \). Then, for every \( i = 1,\ldots,n \), as \( \epsilon \to 0 \), we have
Let $\partial B^A$ and $C > 0$ check (see (3.8) and (3.9)) that there exists a constant $\tilde{c}$.

Proof. We consider the domain $\bar{\Omega}$ for all $u$ decomposition (3.4), we write the energy (1.11) as

for all $\epsilon < \epsilon_0$ and for every $j \in \{1, \ldots, \ell\}$.

Proof. We consider the domain $E := \mathbb{R}^2 \setminus (\bigcup_{i=1}^n B_i(a)) \cup \bar{\Omega}$ and we extend $\tilde{v}_i^\epsilon$ on $E$ by defining it as the solution to the Dirichlet problem with datum $\varphi_{o} = \omega(a_i)\varphi_{a_i}$ on $\partial E \cap \partial \Omega$. It is easy to check (see (3.9)) that there exists a constant $C > 0$ such that

Let $\tilde{\epsilon} := 1 \min_{i,j} |a_i - a_j| : i,j \in \{1, \ldots, \ell\}, i \neq j$. For $\epsilon < \epsilon_0$, consider a family of functions $\zeta_{\epsilon} \in C^\infty(B_i(0))$ which are zero in a neighborhood of $B_i(\epsilon^2(0))$, equal to 1 in a neighborhood of $\partial B_i(0)$, and such that $\|\nabla \zeta_{\epsilon}\|_{L^2(B_i(0))}$ is uniformly bounded. We shall exploit the fact that $\tilde{v}_i^\epsilon$ is defined in the annulus $A_{2\epsilon}(a_j)$ for every $j \in \{1, \ldots, \ell\}$ to define its extension in $B_i(a_j)$. To this aim, consider the inversion function $\Gamma_{a_j}: C\backslash \{a_j\} \to C\backslash \{a_j\}$ given by $\Gamma_{a_j}(x) := \epsilon^2(x-a_j)/|x-a_j|^2$, and define

Also in this case, an easy check shows that

for some constant $C > 0$ independent of $\epsilon$, and for all $\epsilon < \tilde{\epsilon}$. The lemma is proved.

4.3. The limit in the case of many dislocations. This subsection is devoted to the study of the convergence of the rescaled energies (1.11) to the functional defined in (1.13). To this aim, recalling the definition (1.12) for the distances $d_i$’s, we distinguish two different scenarios:

- all the limit points are in the interior of $\bar{\Omega}$ and are all distinct, namely $\min_i d_i > 0$;
- either at least two limit points coincide or one limit point is on the boundary $\partial \bar{\Omega}$, namely $\min_i d_i = 0$.

4.3.1. The case $\min_i d_i > 0$. In this case, it is convenient to write the rescaled energy (1.11) as the sum of the rescaled energy of each dislocation plus a remainder term accounting for interactions: recalling the expression (1.5) for the rescaled energy of one dislocation and the decomposition (3.4), we write the energy (1.11) as

where

and

\[ G_{\epsilon}(a_i, a_j) := \int_{\partial \Omega(a_i, a_j)} (K_{a_i} + \nabla \tilde{v}_i^\epsilon)(K_{a_j} + \nabla \tilde{v}_j^\epsilon) \, dx, \]

$\tilde{u}_i^\epsilon$ and $\tilde{v}_i^\epsilon$ being solutions to (3.5) and (4.6) associated with $a_i$, respectively.
Proposition 4.12. Let \((a_1, \ldots, a_n) \in \Omega^n\) be an n-tuple of distinct points and let \((a'_1, \ldots, a'_n)\) be a sequence converging in \(\Omega^n\) to \((a_1, \ldots, a_n)\) as \(\epsilon \to 0\). Then, for every \(i = 1, \ldots, n\), we have \(R_i(a'_i) \to 0\) as \(\epsilon \to 0\).

Proof. For brevity, we define the \(d'_i\)'s as in (11.12), associated with the family \(\{a'_1, \ldots, a'_n\}\), and the \(d_i\)'s associated with the family \(\{a_1, \ldots, a_n\}\). By assumption \(d'_i \to d_i > 0\) for every \(i = 1, \ldots, n\); therefore, without loss of generality, we may take \(\epsilon < \min\{d_i, d_j\}\). In particular, the balls \(B_r(a'_i)\) are all contained in \(\Omega\) and are pairwise disjoint for every \(\epsilon \in (0, \min\{d_i, d_j\})\).

Fix now \(i \in \{1, \ldots, n\}\). The remainder \(I_{27}\) evaluated at \(a'_i\) can be written as \(R_i(a'_i) = R'_i(a'_i) - R''_i(a'_i)\), with
\[
R'_i(a'_i) := \frac{1}{2} \int_{\Omega \setminus \bigcup \Omega_i} (2K'_i + \nabla \psi'_i + \nabla \psi'_i) \cdot (\nabla \psi'_i - \nabla \psi'_i) \, dx,
\]
and
\[
R''_i(a'_i) := \frac{1}{2} \sum_{j \neq i} \int_{B_r(a'_j)} |K'_i + \nabla \psi'_i|^2 \, dx,
\]
where, for brevity, we have replaced the subscript \(a'_i\) with the subscript \(j\) coupled with the superscript \(\epsilon\). By the Divergence Theorem, (3.5), and (4.6), we have
\[
R'_i(a'_i) = \frac{1}{2} \int_{\Omega \setminus \bigcup \Omega_i} \left( -\frac{\partial \psi'_i}{\partial \nu} + \nabla \psi'_i \cdot (\nabla \psi'_i - \nabla \psi'_i) \right) \, dx
\]
and
\[
R''_i(a'_i) := \frac{1}{2} \sum_{j \neq i} \int_{B_r(a'_j)} |K'_i + \nabla \psi'_i|^2 \, dx.
\]
Then, \(R'_i(a'_i) \leq \|\nabla \psi'_i - \psi'_i\|_{L^2(\Omega \setminus \bigcup \Omega_i, \mathbb{R}^2)} + \|\nabla \psi'_i - \psi'_i\|_{L^2(\Omega \setminus \bigcup \Omega_i, \mathbb{R}^2)} \), which, in view of Lemma 4.10 and (3.21), converges to 0, as \(\epsilon \to 0\). Moreover, in view of Lemma 4.5 and the fact that \(\|K'_i\|_{L^\infty(B_r(a'_j), \mathbb{R}^2)} \leq 1/(2d'_i - \epsilon)\) for every \(j \neq i\), we may bound \(R''_i(a'_i)\) as follows:
\[
R''_i(a'_i) \leq \sum_{j \neq i} \int_{B_r(a'_j)} (|K'_i|^2 + |\nabla \psi'_i|^2) \, dx \leq \pi(n - 1) \epsilon^2 \left( \frac{1}{(2d'_i - \epsilon)^2} + C \|g_{\psi_i} - \theta'_i\|_{L^\infty(\Omega)}^2 \right),
\]
for some constant \(C > 0\) independent of \(\epsilon\). In particular \(R''_i(a'_i)\) tends to zero as \(\epsilon \to 0\). This concludes the proof of the proposition. \(\square\)

Proposition 4.13. Let \((a_1, \ldots, a_n) \in \Omega^n\) be an n-tuple of distinct points and let \((a'_1, \ldots, a'_n)\) be a sequence converging in \(\Omega^n\) to \((a_1, \ldots, a_n)\) as \(\epsilon \to 0\). Then, for every \(i, j = 1, \ldots, n\), with \(i \neq j\), we have
\[
G_i(a'_i, a'_j) \to \int_{\Omega} (K_{a_i} + \nabla v_{a_i}) \cdot (K_{a_j} + \nabla v_{a_j}) \, dx, \quad \text{as } \epsilon \to 0,
\]
(4.28)
v_{a_i} being the solution to (3.7) associated with \(a_i\).

Proof. For brevity, we define the \(d'_i\)'s as in (11.12), associated with the family \(\{a'_1, \ldots, a'_n\}\), and the \(d_i\)'s associated with the family \(\{a_1, \ldots, a_n\}\). Fix \(i, j \in \{a_1, \ldots, n\}, i \neq j\). By assumption \(d'_i \to d_i > 0\) and \(d'_j \to d_j > 0\); therefore, without loss of generality, we may take \(\epsilon < \min\{d_i, d_j\}\). Since the limit points are distinct, in view of Lemma 4.9 we have
\[
\chi_{\Omega \setminus \bigcup \Omega_i} \nabla \psi_{a'_i} \to \nabla v_{a_i}, \quad \chi_{\Omega \setminus \bigcup \Omega_i} \nabla \psi_{a'_j} \to \nabla v_{a_j}, \quad \text{strongly in } L^2(\Omega, \mathbb{R}^2),
\]
so that
\[
\int_{\Omega \setminus \bigcup \Omega_i} \nabla \psi_{a'_i} \cdot \nabla \psi_{a'_j} \, dx \to \int_{\Omega} \nabla v_{a_i} \cdot \nabla v_{a_j} \, dx.
\]
(4.29)
Setting for brevity \(d := \min\{d_i, d_j\}\), we decompose the domain of integration as
\[
\Omega \setminus \bigcup \Omega_i = (\Omega \setminus (a'_i, a'_j)) \cup (B_d(a'_i) \cup B_d(a'_j)) \setminus \bigcup_{k=1}^n B_r(a'_k).
\]
Since \(K_{a'_i} \to K_{a_i}\) and \(K_{a'_j} \to K_{a_j}\) a.e. in \(\Omega\), by the Dominated Convergence Theorem it is easy to see that, as \(\epsilon \to 0\),
\[
\int_{\Omega \setminus (a'_i, a'_j)} K_{a'_i} \cdot K_{a'_j} \, dx \to \int_{\Omega \setminus (a_i, a_j)} K_{a_i} \cdot K_{a_j} \, dx
\]
(4.30)
and
\[
\int_{B_\epsilon(a'_i)} K_{a'_i} \cdot K_{a'_j} \, dx = \int_{B_\epsilon(a'_i)} \frac{\hat{\theta}_{a_i} \cdot \hat{\theta}_{a'_j} - (a'_i - a_i)}{|x - a_i| |x - a'_j + (a'_j - a_i)|} \, dx \to \int_{B_\epsilon(a'_i)} K_{a_i} \cdot K_{a_j} \, dx \tag{4.31}
\]
(similarly, the same holds swapping the roles of \(i\) and \(j\).) Eventually, since the limit points \(a_1, \ldots, a_n\) are distinct, we have, for \(k \neq i, j\,
\[
\int_{B_\epsilon(a'_k)} |K_{a'_i} \cdot K_{a'_j}| \, dx \leq \frac{\pi \epsilon^2}{(2d_i - \epsilon)(2d_j - \epsilon)} \to 0, \quad \text{as } \epsilon \to 0, \tag{4.32}
\]
and
\[
\int_{B_\epsilon(a'_i)} |K_{a'_i} \cdot K_{a'_j}| \, dx \leq \frac{2\pi \epsilon}{2d_j - \epsilon} \to 0, \quad \text{as } \epsilon \to 0 \tag{4.33}
\]
(and again, the same holds swapping the roles of \(i\) and \(j\).) Notice that (4.32) and (4.33) are refined versions of Lemma 2.1(ii). By combining (4.30), (4.31), (4.32), and (4.33) we get
\[
\int_{\Omega(a'_i, \ldots, a'_n)} K_{a'_i} \cdot K_{a'_j} \, dx \to \int_{\Omega} K_{a_i} \cdot K_{a_j} \, dx, \quad \text{as } \epsilon \to 0. \tag{4.34}
\]
In order to study the asymptotic behavior as \(\epsilon \to 0\) of the \(L^2\) product of \(K_{a'_i}\) and \(\nabla \tilde{v}_{a'_j}\) we use the decomposition \(\tilde{v}_{a'_j} = \tilde{v}_j + q'_j\) introduced in Remark 4.5. We recall in particular Remark 4.10(i): \(\tilde{u}\) admits an \(H^1\) extension (not relabeled) that strongly converges to \(v_{a_1}\); thus, integrating by parts and exploiting again the Dominated Convergence Theorem, in the limit as \(\epsilon \to 0\) we get
\[
\int_{\Omega(a'_1, \ldots, a'_n)} K_{a'_i} \cdot \nabla \tilde{u}_j \, dx = \int_{\Omega(a'_i)} K_{a'_i} \cdot \nabla \tilde{u}_j \, dx - \int_{\bigcup_{k \neq i} B_\epsilon(a'_k)} K_{a'_i} \cdot \nabla \tilde{u}_j \, dx \\
= \int_{\partial \Omega} K_{a'_i} \cdot \nu (g_{b_j} - \theta_{a'_j}) \, dx + o(1) \tag{4.35}
\]
where we have used the fact that \(\tilde{u}_j\) has vanishing \(L^2\) norm in the balls \(B_\epsilon(a'_k)\) as \(\epsilon \to 0\); while \(K_{a'_i}\) is uniformly bounded in every ball \(B_\epsilon(a'_k)\) with \(k \neq i\), and satisfies \(K_{a'_i} \cdot \nu = 0\) on \(\partial B_\epsilon(a'_k)\). On the other hand, we recall that \((\nabla q'_j)^\perp = \nabla p'_j\) in the perforated domain, \(p'_j\) being a solution to (4.19). Since the solution to (4.19) is unique up to a constant, we choose \(p'_j\) such that \(p'_j = 0\) on \(\partial B_\epsilon(a'_k)\) Therefore, in view of Remark 4.10(ii), we infer that \(p'_j\) admits an extension in \(H^1(\Omega)\) (not relabeled) which is harmonic in every ball \(B_\epsilon(a'_k)\) and such that \(\|p'_j\|_{H^1(\Omega)} \to 0\) as \(\epsilon \to 0\). Therefore, by letting \(\phi'_j(x) := \log(|x - a'_k|)\), we have
\[
\int_{\Omega(a'_1, \ldots, a'_n)} K_{a'_i} \cdot \nabla q'_j \, dx = \int_{\Omega(a'_1, \ldots, a'_n)} \nabla \phi'_i \cdot \nabla p'_j \, dx = \int_{\bigcup_{k=1}^n \partial B_\epsilon(a'_k)} \phi'_i \nabla p'_j \cdot \nu \, dx \tag{4.36}
\]
and its absolute value can be estimated from above by
\[
\frac{(n - 1) \sqrt{\pi \epsilon}}{d_i - \epsilon} \|\nabla p'_j\|_{L^2(\Omega)} \to 0.
\]
Similarly, the same limits in (4.35) and (4.36) hold exchanging the roles of \(i\) and \(j\). The thesis (4.28) follows then by putting together (4.28), (4.34), (4.35), and (4.36).

4.3.2. The case \(d_i = 0\).

Lemma 4.14. Let \((a_1, \ldots, a_n) \in \overline{\Omega}\) and let \(0 < \epsilon < \eta\) be such that for every \(a_j \in \Omega\) we have \(B_\eta(a_j) \subseteq \Omega\) (i.e. \(d_j \geq \eta\)). Then there exists a positive constant \(C\), independent of \(\epsilon\) and \(\eta\), such that
\[
F_\epsilon(a_1, \ldots, a_n) \geq F_\eta(a_1, \ldots, a_n) - C. \tag{4.37}
\]
Proof. We start by comparing the energies $\mathcal{E}_\epsilon$ and $\mathcal{E}_\eta$. Recalling (1.10), it is easy to see that

$$\mathcal{E}_\epsilon(a_1, \ldots, a_n) \geq \mathcal{E}_\eta(a_1, \ldots, a_n) + \frac{1}{2} \int_{\partial A_\eta^\ell(a_i)} \sum_{j=1}^n \omega(a_j)K_{a_j} \cdot \nabla^\epsilon_{a_1, \ldots, a_n}^2 \, dx$$

$$= \mathcal{E}_\eta(a_1, \ldots, a_n) + J_1 + J_2 + J_3 + J_4,$$

where we have defined

$$J_1 := \frac{1}{2} \sum_{i,j=1}^\ell \int_{A_\eta^\ell(a_i)} \nabla^\epsilon_{a_1, \ldots, a_n}^2 \, dx,$$

$$J_3 := \sum_{i,j=1}^\ell m_j m_k \omega(a_j) \int_{A_\eta^\ell(a_i)} K_{a_j} \cdot K_{a_k} \, dx,$$

$$J_2 := \frac{1}{2} \sum_{i,j=1}^\ell m_j m_k \omega(a_j) \int_{A_\eta^\ell(a_i)} |K_{a_j}|^2 \, dx,$$

$$J_4 := \sum_{i,j=1}^\ell m_j \omega(a_j) \int_{A_\eta^\ell(a_i)} K_{a_j} \cdot \nabla^\epsilon_{a_1, \ldots, a_n} \, dx.$$

The term $J_1$ is strictly positive, and we can neglect it. In order to bound $J_2$ from below, we need to distinguish two cases, according to whether $d(a_i)$ is greater or smaller than $\eta$, namely whether or not the annulus $A_\eta^\ell(a_i)$ is contained in $\Omega$. In the former case, $\omega(a_i) = 1$ and a simple computation gives (see (2.3))

$$\frac{1}{2} \sum_{j=1}^\ell m_j^2 \omega(a_j) \int_{A_\eta^\ell(a_j)} |K_{a_j}|^2 \, dx \geq \frac{1}{2} \sum_{j=1}^\ell m_j^2 \omega(a_j) \int_{A_\eta^\ell(a_j)} |K_{a_j}|^2 \, dx = \pi m_j^2 \log(\eta/\epsilon).$$

In the latter, $A_\eta^\ell(a_i)$ is not completely contained in $\Omega$ and therefore, by hypothesis, we know that $a_i \in \partial \Omega$ and in particular $\omega(a_i) = 2$, so that

$$\frac{1}{2} \sum_{j=1}^\ell m_j^2 \omega(a_j) \int_{A_\eta^\ell(a_j)} |K_{a_j}|^2 \, dx \geq 2 \pi m_j^2 \omega(a_j) \int_{A_\eta^\ell(a_j)} |K_{a_j}|^2 \, dx > \pi m_j^2 \log \left( \frac{\min(\eta, \epsilon_a)}{\epsilon} \right),$$

where $\epsilon_a$ is given in (1.15) (see (2.4)). Therefore, for $\eta$ small enough, $\min(\eta, \epsilon_a) = \eta$ and (4.40) provides the same bound as (4.39), hence, summing over $i$, we obtain

$$J_2 \geq \sum_{i=1}^\ell \pi m_i^2 \log(\eta/\epsilon).$$

Recalling the definition (1.11) of $\mathcal{F}_\eta(a_1, \ldots, a_n)$, from (4.38) and (4.41) we obtain

$$\mathcal{F}_\epsilon(a_1, \ldots, a_n) \geq \mathcal{F}_\eta(a_1, \ldots, a_n) + \left( \sum_{i=1}^\ell m_i^2 - n \right) \pi \log(\eta/\epsilon) + J_3 + J_4 \geq \mathcal{F}_\eta(a_1, \ldots, a_n) + J_3 + J_4,$$

since $\eta > \epsilon$ and $\sum_{i=1}^\ell m_i^2 - n \geq 0$. We obtain the thesis (4.37) from (4.42), provided we bound $J_3$ and $J_4$ from below.

In view of Lemma 2.1(i), we have

$$J_3 = \sum_{i,j=1}^\ell m_j m_k \omega(a_j) \omega(a_k) \int_{A_\eta^\ell(a_i)} K_{a_j} \cdot K_{a_k} \, dx \geq -8\pi \ell n^2 \geq -8\pi n^3.$$

To estimate $J_4$, we start by splitting $\nabla^\epsilon_{a_1, \ldots, a_n}^2 = \nabla^\epsilon_{a_1, \ldots, a_n}^2 + \nabla^\eta_{a_1, \ldots, a_n}$, with $\nabla^\epsilon_{a_1, \ldots, a_n} = \sum_{k=1}^\ell \tilde{u}^\epsilon_{a_k}$ and $\nabla^\eta_{a_1, \ldots, a_n} = \sum_{k=1}^\ell \tilde{u}^\eta_{a_k}$, where $\tilde{u}^\epsilon_{a_k}$ and $\tilde{u}^\eta_{a_k}$ solve (4.7) and (4.8), respectively. Then

$$\sum_{i=1}^\ell \int_{A_\eta^\ell(a_i)} K_{a_j} \cdot \nabla^\epsilon_{a_1, \ldots, a_n} \, dx = \sum_{i=1}^\ell \int_{A_\eta^\ell(a_i)} K_{a_j} \cdot \nabla^\eta_{a_1, \ldots, a_n} \, dx + \sum_{i=1}^\ell \int_{A_\eta^\ell(a_i)} K_{a_j} \cdot \nabla^\eta_{a_1, \ldots, a_n} \, dx$$

and, by using the Divergence Theorem, (4.10), and the fact that $|K_{a_j}(x)| \leq |x|^{-1}$ on $\partial A_\eta^\ell(a_i)$, we can estimate

$$\sum_{i=1}^\ell \int_{A_\eta^\ell(a_i)} K_{a_j} \cdot \nabla^\eta_{a_1, \ldots, a_n} \, dx = \sum_{i=1}^\ell \int_{\partial A_\eta^\ell(a_i)} K_{a_j} \cdot \nabla^\eta_{a_1, \ldots, a_n} \, dx \geq -C \sum_{i=1}^\ell \int_{\partial A_\eta^\ell(a_i)} |K_{a_j} \cdot \nu| \, dx \geq -2\pi C n,$$

where $C = C(n)$. 


where \( \mathcal{C} := \| g - \sum_{k=1}^{n} \omega(a_k) \theta_{a_k} \|_{L^\infty(\Omega)}. \) Moreover

\[
\sum_{i=1}^{\ell} \int_{A_i} K_{a_j} \cdot \nabla q_{a_1,\ldots,a_n} \, dx = \sum_{i=1}^{\ell} \int_{A_i} \nabla \phi_{a_j} \cdot \nabla p_{a_1,\ldots,a_n} \, dx,
\]

with \( p_{a_1,\ldots,a_n} = \sum_{\ell=1}^{\ell} p_{a_{1,\ldots,a_n}} \), where \( p_{a_{1,\ldots,a_n}} \) solves (4.19). In particular, by using (4.20), we infer that the \( L^\infty \) norm of \( p_{a_1,\ldots,a_n} \) is bounded by \( Cn \), so that we can estimate

\[
\sum_{i=1}^{\ell} \int_{A_i} K_{a_j} \cdot \nabla q_{a_1,\ldots,a_n} \, dx \geq -2\pi Cn^2. \tag{4.45}
\]

Combining (4.44) with (4.45) and summing over \( j \), we obtain \( J_4 \geq -2\pi n^2(\mathcal{C} + Cn) \), which, together with (4.42) and (4.43), allows us to get estimate (4.37), with constant \( C = 2\pi n^2(\mathcal{C} + (C + 4)n) \). The lemma is proved.

**Proposition 4.15.** Let \((a_1,\ldots,a_n) \in \Omega^n \) and let \((a_{1,\epsilon},\ldots,a_{\epsilon,n})\) be a sequence of points in \( \Omega^n \) converging to \((a_1,\ldots,a_n)\) as \( \epsilon \to 0 \). If \( \min_{1 \leq i \leq n} d_i = 0 \) with \( d_i \) defined as in (1.12), then \( \mathcal{F}_{\epsilon}(a_{1,\epsilon},\ldots,a_{\epsilon,n}) \to \infty \), as \( \epsilon \to 0 \).

**Proof.** Our goal is to show that we can bound the energy \( \mathcal{F}_{\epsilon}(a_{1,\epsilon},\ldots,a_{\epsilon,n}) \) from below by a quantity that explodes in the limit as \( \epsilon \to 0 \). This will be achieved by applying the following iterative procedure, which is performed at \( \epsilon \) fixed.

**Step 0 (Labeling)** We start by relabeling in a more suitable way the limit dislocations and the approximating ones. According to Definition 4.1, we relabel the limit points so that the first \( \ell \) \((1 \leq \ell \leq n)\) are distinct. Moreover, we fix \( \delta > 0 \) such that the balls \( B_{\delta}(a_i) \) are pairwise disjoint for \( i = 1,\ldots,\ell \). For every \( i = 1,\ldots,\ell \), there are \( m_i \) points in \( \{a_{1,\epsilon},\ldots,a_{\epsilon,n}\} \) which converge to \( a_i \). We denote these points by \( a_{i,j} \), with \( j = 1,\ldots,m_i \). For \( \epsilon \) small enough we clearly have \( a_{i,j} \in B_{\delta}(a_i) \), for every \( j = 1,\ldots,m_i \) and \( i = 1,\ldots,\ell \).

At each iteration (from Step 1 to Step 3), we will replace the sequence (with respect to \( \epsilon > 0 \)) of families (indexed by \( i \in \{1,\ldots,\ell\} \)) \( \{a_{i,j}\}_{j=1}^{m_i} \) with a sequence of singletons \( c_i \), still converging to \( a_i \), as \( \epsilon \to 0 \), with multiplicity \( m_i \). Additionally, we will define a new core radius \( \eta(\epsilon) \) and in Step 4 we will compare the energies \( \mathcal{F}_{\epsilon}(a_{1,\epsilon},\ldots,a_{\epsilon,n}) \) and \( \mathcal{F}_{\eta(\epsilon)}(c_1,\ldots,c_n) \).

**Step 1 (Ordering)** Let \( i \in \{1,\ldots,\ell\} \) be fixed. According to Definition 4.1, we order the family \( \{a_{i,j}\}_{j=1}^{m_i} \) so that the first \( \ell_i \) are distinct, and we denote by \( m_{i,\epsilon} \) their multiplicities. Notice that \( \sum_{j=1}^{\ell_i} m_{i,j} = m_i \). With these positions, we clearly have

\[
\mathcal{F}_{\epsilon}(a_{1,\epsilon},\ldots,a_{\epsilon,n}) = \mathcal{F}_{\epsilon}(a_{1,1}, \ldots, a_{1,\ell_1}, a_{\ell_1+1}, \ldots, a_{\ell_1+\ell_2}, \ldots, a_{\ell_1+\ell_2+\ldots+\ell_{\ell-1}}, a_{\ell_1+\ell_2+\ldots+\ell_{\ell-1}+1}, \ldots, a_{\ell_1+\ell_2+\ldots+\ell_{\ell-1}+\ell_{\ell}}, a_{\ell+1,\epsilon}, \ldots, a_{\ell,n}). \tag{4.46}
\]

In case \( \ell_i > 1 \), for every \( j = 1,\ldots,\ell_i \), we associate to the distinct points \( a_{i,j} \) the following quantity: if \( a_{i,j} \in \partial\Omega \), we set

\[
s(a_{i,j}) := \min \left\{ \frac{|a_{i,j} - a_{i,k}|}{2} : k \in \{1,\ldots,\ell_i\} \setminus \{j\} \right\},
\]

whereas, if \( a_{i,j} \in \Omega \), we set

\[
s(a_{i,j}) := \min \left\{ \min \left\{ \frac{|a_{i,j} - a_{i,k}|}{2} : k \in \{1,\ldots,\ell_i\} \setminus \{j\} \right\} \right\}.
\]

Observe that, if the limit point \( a_i \in \Omega \), for \( \epsilon \) small enough \( d(a_{i,j}) \) is always greater than any mutual semidistance \( |a_{i,j} - a_{i,k}|/2 \), for all \( j, k \in \{1,\ldots,\ell_i\} \). Up to reordering the different \( \ell_i \) points \( \{a_{i,1},\ldots,a_{i,\ell_i}\} \) we can always suppose that

\[
0 < s(a_{i,1}) \leq \ldots \leq s(a_{i,\ell_i}).
\]

In case \( \ell_i = 1 \), namely when all the \( a_{i,j} \) coincide with \( a_{i,1} \), we set

\[
s(a_{i,1}) := \begin{cases} 0 & \text{if } a_i \in \Omega, \\ d(a_{i,1}) & \text{if } a_i \in \partial\Omega. \end{cases} \tag{4.47}
\]
Step 2 (Stop test) If \( s(a_{i,1}^*) = 0 \) for every \( i = 1, \ldots, \ell \), then we define \( c_i^* := a_{i,1}^*, \eta(\epsilon) := \epsilon, \) and we go to Step 4. Observe that by (4.47), \( c_i^* \in \partial \Omega \) if \( a_i \in \partial \Omega \). Otherwise, we define the following quantity

\[
\hat{s} = \hat{s}(\epsilon) := \min \left\{ s(a_{i,1}^*) > 0 : i \in \{1, \ldots, \ell\} \right\}
\]

and we go to Step 3. Observe that the set where the minimum is taken is not empty, hence \( \hat{s} \) is finite and strictly positive.

Step 3 (Iterative step) We compare \( \hat{s} \) with \( \epsilon \).

If \( \hat{s} > \epsilon \), we relabel \( a_{k,j}^* \) by \( \hat{a}_{k,j}^* \) and their multiplicities accordingly, set \( \hat{\epsilon} := \hat{s} \), and estimate \( F_{\epsilon}(a_{1}, \ldots, a_{n}) \) by means of (4.37) proved in Lemma 4.14, with \( \eta = \hat{\epsilon} \), to obtain

\[
F_{\epsilon}(a_{1}, \ldots, a_{n}) \geq F_{\hat{\epsilon}}(\ldots, \hat{a}_{k,j}^*, \ldots) - C.
\]

If \( \hat{s} \leq \epsilon \) we distinguish two cases:

1. \( \hat{s} \) is equal to \( |a_{i,1}^* - a_{i,2}^*|/2 \), for some \( i \): we replace the points \( a_{i,1}^* \) and \( a_{i,2}^* \) (and all those coinciding with either one of them) by \( \hat{a}_{i,1}^* \), the midpoint between \( a_{i,1}^* \) and \( a_{i,2}^* \), with multiplicity \( \hat{m}_{i,1}^* := m_{i,1}^* + m_{i,2}^* \). The replacement is performed simultaneously for all the indices \( i \) that realize the minimum in (4.48). For all the other indices, we simply relabel \( a_{k,j}^* \) by \( \hat{a}_{k,j}^* \) and their multiplicities accordingly.

2. \( \hat{s} \) is realized by \( d(i_{a_{i,1}^*}) \), for some \( i \): we replace the point \( a_{i,1}^* \) (and all those coinciding with it) by \( \hat{a}_{i,1}^* \), its projection to the boundary \( \partial \Omega \), with multiplicity \( \hat{m}_{i,1}^* := m_{i,1}^* \).

Setting \( \hat{\epsilon} := \epsilon + \hat{s} \) and recalling (1.10), (1.11), and (4.46), we have

\[
F_{\epsilon}(a_{1}, \ldots, a_{n}) \geq F_{\epsilon}(\ldots, \hat{a}_{k,j}^*, \ldots) - n \pi \log 2.
\]

Notice that \( \hat{\epsilon} \) satisfies the following bound:

\[
\hat{\epsilon} \leq \max\{2\epsilon, \hat{s}\} \leq \max\{2\epsilon, \hat{s}\}.
\]

We have obtained a new family \( \{\hat{a}_{k,j}^*\} \) and a new radius \( \hat{\epsilon} \) and we restart the procedure by applying Step 1 to these new objects.

Notice that the procedure ends after at most \( n^2 \) iterations. Indeed, when applying Step 3, we will always fall into case (1) after at most \( n \) iterations, and then the number of distinct points will decrease when we apply (1). In conclusion, since the number of distinct points is at most \( n \), we will reach the target situation after at most \( n^2 \) iterations of Step 3.

Step 4 (Estimates and conclusion) By combining the chain of inequalities obtained by applying Step 3 \( k(\leq n^2) \) times, estimates (4.50) and (4.49) give

\[
F_{\epsilon}(a_{1}, \ldots, a_{n}) \geq F_{\eta(\epsilon)}(\underbrace{c_1^{*}, \ldots, c_\ell^{*}}_{m_1\text{-times}}, \underbrace{c_{\ell+1}^{*}, \ldots, c_n^{*}}_{m_2\text{-times}}) - k(C + n \pi \log 2),
\]

where \( \eta(\epsilon) \) is a number depending on \( \epsilon \) obtained after \( k \) iterations of the procedure that defines \( \hat{\epsilon} \) in Step 3. Let us estimate \( \eta(\epsilon) \). At every iteration, the value \( \hat{s} \) can increase, but it is easy to see that it cannot grow larger than its double. Therefore, after \( k \) iterations of Step 3, by (4.51), we have

\[
\eta(\epsilon) \leq 2^k \max\{\epsilon, \hat{s}\} \leq 2n^2 \max\{\epsilon, \hat{s}\}.
\]

Since \( \hat{s} \) tends to 0 as \( \epsilon \to 0 \), we have \( \eta(\epsilon) \to 0 \) as \( \epsilon \to 0 \).

We now claim that the right-hand side of (4.52) tends to \( +\infty \) as \( \epsilon \to 0 \). Similarly to (4.26), we write the energy as the sum of three contributions:

\[
F_{\eta(\epsilon)}(\underbrace{c_1^{*}, \ldots, c_\ell^{*}}_{m_1\text{-times}}, \underbrace{c_{\ell+1}^{*}, \ldots, c_n^{*}}_{m_2\text{-times}}) = \sum_{i=1}^{\ell} F_{\eta(\epsilon)}(\underbrace{c_i^{*}}_{m_i\text{-times}}) + \sum_{i=1}^{\ell} m_i^2 R_{\epsilon}(c_i^{*}) + \sum_{i \neq j} m_i m_j G_{\epsilon}(c_i^{*}, c_j^{*})
\]

(4.53)
where, setting \( \omega_i := \omega(a_i) = \omega(c'_i) \) (see Step 3), and \( K'_i := K_{c'_i} \) for brevity,
\[
\mathcal{R}_e(c'_i) := \frac{1}{2} \int_{\Omega_{\eta(i)}(c'_i)} |\omega_i K'_i + \nabla \hat{v}'_i|^2 \, dx - \frac{1}{2} \int_{\Omega_{\eta(i)}(c'_i)} |\omega_i K'_i + \nabla \hat{u}'_i|^2 \, dx,
\]
(4.54) and
\[
\mathcal{G}_e(c'_i, c'_j) := \int_{\Omega_{\eta(i)}(c'_i) \cap \Omega_{\eta(j)}(c'_j)} \left( (\omega_i K'_i + \nabla \hat{v}'_i) \cdot (\omega_j K'_j + \nabla \hat{v}'_j) \right) \, dx,
\]
(4.55)
\( \hat{u}'_i \) and \( \hat{v}'_i \) being solutions to (3.5) (or (3.6)) and (4.6), respectively, associated with \( c'_i \) for \( i = 1, \ldots, \ell \) and core radius \( \eta(\epsilon) \).

Fix \( i \in \{1, \ldots, \ell\} \). If \( a_i \in \partial \Omega \), by using (4.5) again and (3.23), we have
\[
\mathcal{F}_{\eta(i)}(\underbrace{c'_i}_{m_i \text{-times}}) \geq C_1 m_i^2 |\log(\max\{\eta(\epsilon), d(i'_i)\})| + m_i (m_i - 1) \pi |\log(\eta(\epsilon))| + C_2 m_i^2
\]
where the equality follows from (4.5) and the inequality is a consequence of Proposition 3.7. If instead \( a_i \in \partial \Omega \), by using (4.5) again and (3.23), we have
\[
\mathcal{F}_{\eta(i)}(\underbrace{c'_i}_{m_i \text{-times}}) \geq C_1 m_i^2 |\log(\max\{\eta(\epsilon), d(i'_i)\})| + m_i (m_i - 1) \pi |\log(\eta(\epsilon))| + C_2 m_i^2
\]
and, since \( d(i'_i) = 0 \) as noticed in Step 2, we obtain
\[
\mathcal{F}_{\eta(i)}(\underbrace{c'_i}_{m_i \text{-times}}) \geq (C_1 m_i^2 + m_i (m_i - 1) \pi) |\log(\eta(\epsilon))| + C_2 m_i^2.
\]
(4.57)

By Lemma 4.11, the function \( \hat{v}'_i \) can be extended inside any ball \( B_{\eta(i)}(c'_i) \) with \( j \neq i \); therefore the remainder (4.53) can be rewritten as \( \mathcal{R}_e(c'_i) = \mathcal{R}'_e(c'_i) - \mathcal{R}''_e(c'_i) \) with
\[
\mathcal{R}'_e(c'_i) := \frac{1}{2} \int_{\Omega_{\eta(i)}(c'_i)} |\omega_i K'_i + \nabla \hat{v}'_i|^2 \, dx - \frac{1}{2} \int_{\Omega_{\eta(i)}(c'_i)} |\omega_i K'_i + \nabla \hat{u}'_i|^2 \, dx
\]
and
\[
\mathcal{R}''_e(c'_i) := \frac{1}{2} \sum_{j \neq i} \int_{B_{\eta(i)}(c'_i)} |\omega_j K'_j + \nabla \hat{v}'_j|^2 \, dx.
\]
Using the minimality of \( \hat{u}'_i \) in \( \Omega_{\eta(i)}(c'_i) \) it turns out that \( \mathcal{R}'_e(c'_i) \geq 0 \). Using that \( |K'_i| \leq 1/\eta(\epsilon) \) in \( B_{\eta(i)}(c'_i) \) for every \( j \neq i \) and Lemma 4.11 we obtain
\[
\mathcal{R}_e(c'_i) \geq -\mathcal{R}''_e(c'_i) \geq -\frac{1}{2} \sum_{j \neq i} \int_{B_{\eta(i)}(c'_i)} |\omega_j K'_j|^2 \, dx - \frac{1}{2} \sum_{j \neq i} \int_{B_{\eta(i)}(c'_i)} |\nabla \hat{v}'_j|^2 \, dx \geq -C.
\]
(4.58)

To estimate the interaction term (4.55), fix also \( j \in \{1, \ldots, \ell\} \setminus \{i\} \). Then, we can write
\[
\mathcal{G}_e(c'_i, c'_j) = \mathcal{G}'_e(c'_i, c'_j) + \mathcal{G}''_e(c'_i, c'_j) + \mathcal{G}'''_e(c'_i, c'_j)
\]
with
\[
\mathcal{G}'_e(c'_i, c'_j) := \omega_i \left( \int_{\Omega_{\eta(i)}(c'_i) \cap \Omega_{\eta(j)}(c'_j)} K'_i \cdot K'_j \, dx \right,
\]
(4.59a)
\[
\mathcal{G}''_e(c'_i, c'_j) := \omega_j \left( \int_{\Omega_{\eta(i)}(c'_i) \cap \Omega_{\eta(j)}(c'_j)} \nabla \hat{v}'_i \cdot K'_j \, dx + \omega_i \int_{\Omega_{\eta(i)}(c'_i) \cap \Omega_{\eta(j)}(c'_j)} \nabla \hat{v}'_j \cdot K'_i \, dx \right,
\]
(4.59b)
\[
\mathcal{G}'''_e(c'_i, c'_j) := \int_{\Omega_{\eta(i)}(c'_i) \cap \Omega_{\eta(j)}(c'_j)} \nabla \hat{u}'_i \cdot \nabla \hat{v}'_j \, dx.
\]
(4.59c)

Thanks to Lemma 2.4 (i), the functional \( \mathcal{G}'_e \) in (4.59a) is uniformly bounded from below by a constant. To estimate \( \mathcal{G}''_e \) and \( \mathcal{G}'''_e \) we recall the decomposition \( \hat{v}'_i = \hat{u}'_i + q'_i \) and the function \( \hat{p}'_i \), solution to (4.19), that we introduced in Remark 4.3, where \( k = i \) or \( k = j \). Here the functions \( \hat{u}'_i \) and \( q'_i \) are the solutions to the systems (4.7) and (4.8) with \( (a'_i, \ldots, a'_i) \) replaced by \( (c'_1, \ldots, c'_i) \), and the Dirichlet boundary condition for \( \hat{u}'_k \) given by \( g_{\partial \Omega} - \omega_k \theta_k \) on \( \partial \Omega \). Therefore, we can estimate (4.59c) as follows
\[
\mathcal{G}'''_e(c'_i, c'_j) \leq \int_{\Omega_{\eta(i)}(c'_i) \cap \Omega_{\eta(j)}(c'_j)} (|\nabla \hat{u}'_i|^2 + |\nabla \hat{u}'_j|^2 + |\nabla q'_i|^2 + |\nabla q'_j|^2) \, dx
\]
and since, the gradient of \( q'_k \) coincides in modulus with the gradient of \( p'_k \), by (4.11) and (4.21) we can bound \( \mathcal{G}'''_e(c'_i, c'_j) \) from below. The functional in (4.59b) is the most delicate to treat: if both \( a_i \) and \( a_j \) are in \( \Omega \), \( \mathcal{G}''_e(c'_i, c'_j) \) is bounded below by a constant (notice that in this case we
could have applied Proposition 4.13 to \( G_\varepsilon(c'_i, c''_j) \) itself and we would have concluded. In the
general case, by means of the above-mentioned decomposition, (4.59) reads
\[
G''_\varepsilon(c'_i, c''_j) = \omega_j \int_{\Omega_{\varepsilon}(c'_1, \ldots, c'_k)} \nabla u'_i \cdot K'_j \, dx + \omega_i \int_{\Omega_{\varepsilon}(c'_1, \ldots, c'_k)} \nabla u'_j \cdot K'_i \, dx
\]
\[
+ \omega_j \int_{\Omega_{n}(c'_1, \ldots, c'_k)} \nabla q'_i \cdot K'_j \, dx + \omega_i \int_{\Omega_{n}(c'_1, \ldots, c'_k)} \nabla q'_j \cdot K'_i \, dx.
\]
We first deal with the last two terms, involving the gradients of \( q'_i \) and \( q'_j \); from Hölder’s
inequality, recalling that the gradient of \( \epsilon \) chosen later. Finally, we can control the term
\( C \epsilon \) where \( \epsilon \) is a positive constant independent of \( \varepsilon \).

We combine the previous results to obtain
\[
\omega_j \int_{\Omega_{\varepsilon}(c'_1, \ldots, c'_k)} \nabla u'_i \cdot K'_j \, dx + \omega_i \int_{\Omega_{\varepsilon}(c'_1, \ldots, c'_k)} \nabla u'_j \cdot K'_i \, dx
\]
\[
\leq \frac{1}{2\lambda} \int_{\Omega_{\varepsilon}(c'_1, \ldots, c'_k)} (\omega_j |\nabla u'_i|^2 + |\nabla u'_j|^2) \, dx + \frac{\lambda}{2} \int_{\Omega_{\varepsilon}(c'_1, \ldots, c'_k)} (\omega_i |K'_i|^2 + \omega_j |K'_j|^2) \, dx
\]
\[
\leq \frac{C}{\lambda} \left( \|g_{ij} - \omega_i \theta_i\|^2_{H^{1/2}(\partial \Omega)} + \|g_{ij} - \omega_j \theta_j\|^2_{H^{1/2}(\partial \Omega)} \right) + 2\pi \lambda \log(\eta(\varepsilon)) + 2\pi \lambda \log(2\pi \lambda) \log(\Omega),
\]
where \( C \) is a positive constant independent of \( \varepsilon \) and \( \lambda > 0 \) is an arbitrary constant that will be
chosen later. Finally, we can control the term \( G_\varepsilon(c'_i, c''_j) \) as
\[
G_\varepsilon(c'_i, c''_j) \geq -2\pi \lambda |\log(\eta(\varepsilon))| - C_{ij}
\]
where all the terms independent of \( \varepsilon \) have been included in the constant \( C_{ij} \).

We can classify each point \( a_i \) according to whether it belongs to
1. the interior of \( \Omega \), with multiplicity \( m_1 = 1 \);
2. the interior of \( \Omega \), with multiplicity \( m_1 > 1 \);
3. the boundary \( \partial \Omega \).

For \( k = 1, 2, 3 \) we denote by \( I_k \) the set of indices corresponding to those points \( c'_i \to a_i \) belonging to
the \( k \)-th category. Therefore, combining (4.56), (4.57), (4.58), and (4.61), summing over \( i \) and \( j \), we obtain the following lower bound for the energy (4.53):
\[
F_{\varepsilon}(c'_1, \ldots, c'_k) \geq \pi(C_F - \lambda C_G) |\log \eta(\varepsilon)| + C,
\]
where
\[
C_F := \sum_{i \in I_2 \cup I_3} m_i (m_i - 1) + \sum_{i \in I_3} C_1 m_i^2, \quad C_G := 2n^2,
\]
and \( C \) is a constant independent of \( \varepsilon \). The assumption \( \min_{1 \leq i \leq n} d_i = 0 \) guarantees that \( C_F > 0 \),
since in this case either \( I_2 \neq \emptyset \) or \( I_3 \neq \emptyset \). Therefore, choosing \( \lambda \) in (4.61)
so that \( 0 < \lambda < C_F/C_G \), we obtain that the right-hand side of (4.52) tends to \( +\infty \) as \( \varepsilon \to 0 \). The proposition is proved. \( \square \)

4.3.3. **Proofs of Theorem 1.3 and of Corollary 1.4**

We combine the previous results to prove the main results in the case of \( n \) dislocations.

**Proof of Theorem 1.3**

Combining Propositions 3.7 and 3.9 with Propositions 4.12, 4.13, and 4.15
yields the continuous convergence of \( F_\varepsilon \) to \( F \). The continuity of \( F \) follows from the relationship
between continuous convergence and \( \Gamma \)-convergence (see [10, Remark 4.9]), which implies that
\( F(a_1, \ldots, a_n) \) tends to \( +\infty \) as either one of the \( a_i \)'s approaches the boundary or any two \( a_i \) and
\( a_j \) (with \( i \neq j \)) become arbitrarily close. Therefore \( F \) is minimized by \( n \)-tuples \( (a_1, \ldots, a_n) \) of
distinct points in \( \Omega^p \).

**Proof of Corollary 1.4**

It is a straightforward adaptation of the proof of Corollary 1.2; the same arguments work replacing the point \( a \) with the \( n \)-tuple \( (a_1, \ldots, a_n) \) and noticing that in this case every minimizer of the limit functional belongs to the set
\[
\{(x_1, \ldots, x_n) \in \Omega^p : d(x_i) > \delta \quad \forall i = 1, \ldots, n, \quad \|x_i - x_j\| > 4\delta \quad \forall i \neq j\},
\]
for a suitable \( \delta > 0 \), in which the functional \( E_\varepsilon \) defined in (1.8) is continuous (this can be proved
using a change of variables similar to (3.30)).
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