Lagrange-Poincaré Reduction for Optimal Control of Underactuated Mechanical Systems

Leonardo Colombo
Instituto de Ciencias Matemáticas, ICMAT (CSIC-UAM-UCM-UC3M)
leo.colombo@icmat.es

May 7, 2014

Keywords: Higher-order mechanics, Lagrange-Poincaré equations, optimal control, underactuated mechanical systems, Lagrangian reduction, variational principles.

Abstract

We deal with regular Lagrangian constrained systems which are invariant under the action of a symmetry group. Fixing a connection on the higher-order principal bundle where the Lagrangian and the (independent) constraints are defined, the higher-order Lagrange-Poincaré equations of classical mechanical systems with higher-order constraints are obtained from classical Lagrangian reduction. Higher-order Lagrange-Poincaré operator is introduced to characterize higher-order Lagrange-Poincaré equations.

Interesting applications are derived as, for instance, the optimal control of an underactuated Elroy’s Beanie and a snakeboard seen as an optimization problem with higher-order constraints.

Contents

1 Introduction .......................................................... 2
1.1 Motivation .......................................................... 3
1.2 Main contributions .................................................. 3
1.3 Organization of the paper .......................................... 4

2 Preliminaries on geometric mechanics ......................... 4
2.1 Higher-order tangent bundles ................................. 4
  2.1.1 Higher-order Euler-Lagrange equations .............. 5
  2.1.2 Higher-order Euler-Poincaré equations .............. 6
2.2 Connections and exterior calculus on adjoint bundles .......... 9
2.3 Higher-order quotient space .................................. 10
2.4 The reduced Lagrangian and constraints .................... 12
3 Higher-order Lagrange-Poincaré equations for systems with higher-order constraints

3.1 Higher-order Lagrange-Poincaré operator

3.1.1 Reduced higher-order Euler-Lagrange operator:

3.1.2 The reduced variations:

3.1.3 Lagrange-Poincaré operator:

3.2 Hamilton’s principle for systems with constraints

4 Optimal control of underactuated mechanical systems

4.1 Optimal control problem

4.2 Optimal control of an underactuated Elroy’s beanie

4.3 Optimal control of a Snakeboard

5 Conclusions and Future work

1 Introduction

As is well known, a standard tool in classical mechanics and fields theory is the reduction method (see for example [3],[5],[8],[7],[9],[10],[20],[21],[22],[25],[26] and reference therein). The aim of this paper is to characterize and obtain higher-order Lagrange-Poincaré equations by introduce the higher-order Lagrange-Poincaré operator where the configuration manifold is a higher-order principal bundle with structural group \(G\). Moreover, in this work we obtain higher-order Lagrange-Poincaré equations for systems with higher-order constraints using an extension of classical Lagrangian reduction from the perspective of reduction of variational principles [10] where the higher-order Lagrangian and higher-order constraints are \(G\)-invariants under the k-lift of an action of a Lie group.

In the first order case, the variational principle is formulated on a principal bundle \(\pi: Q \to Q/G\) where \(G\) is a Lie group and a principal connection \(\mathcal{A}\) is introduced on \(Q\). The connection yields a bundle isomorphism

\[
(TQ)/G \to T(Q/G) \oplus_{Q/G} \tilde{g}
\]

given by

\[
[v_q] \mapsto (T\pi(v_q), [q, \mathcal{A}(v_q)])_g,
\]

where the bracket is the standard Lie bracket on the Lie algebra \(g\) associated with the Lie group \(G\), (that is, \(g = T_eG\), where \(e\) is the identity element of the group \(G\)) and \(\tilde{g} := AdQ := (Q \times g)/G\).

A curve \(q(t) \in Q\) induces the two curves

\[
p(t) := \pi(q(t)) \in Q/G \quad \text{and} \quad \sigma(t) = [q(t), \mathcal{A}(\dot{q}(t))]_g \in \tilde{g}.
\]

Variational Lagrangian reduction [10] states that the Euler-Lagrange equations on \(Q\) for a \(G\) invariant Lagrangian \(L\) are equivalent to the Lagrange-Poincaré equations on \(TQ/G \cong T(Q/G) \oplus_{Q/G} \tilde{g}\) with reduced Lagrangian \(\tilde{L}\). The Lagrange-Poincaré equations reads

\[
\begin{align*}
\frac{D}{Dt} \frac{\partial \tilde{L}}{\partial \dot{\sigma}} - ad^*_\sigma \frac{\partial \tilde{L}}{\partial \sigma} &= 0, \\
\frac{\partial \tilde{L}}{\partial p} - \frac{D}{Dt} \frac{\partial \tilde{L}}{\partial \dot{p}} &= \left\langle \frac{\partial \tilde{L}}{\partial \sigma}, i_p \tilde{\mathcal{B}} \right\rangle,
\end{align*}
\]
where $\tilde{B}$ is the reduced curvature form associated to the connection $A$ and $D/Dt$ denotes suitable covariant derivative.

The one-form valued map $\mathcal{LP}(\tilde{L}) : T^{(2)}(Q/G) \times_{Q/G} \tilde{g} \to T^*(Q/G) \oplus \tilde{g}^*$ is defined in [10] to be the Lagrange-Poincaré operator. The decomposition of the range space for $\mathcal{LP}(\tilde{L})$ as a direct sum induce a decomposition of the Lagrange-Poincaré operator

$$\mathcal{LP}(\tilde{L}) = \text{Hor}(\mathcal{LP})\!(\tilde{L}) \oplus \text{Ver}(\mathcal{LP})\!(\tilde{L})$$

which define the horizontal Lagrange-Poincaré operator and the vertical Lagrange-Poincaré operator. The Lagrange-Poincaré equations are, by definition, the equations $\mathcal{LP}(\tilde{L}) = 0$.

In this work we present a general framework to obtain this class of reduced equations for higher-order Lagrangian systems. After a delicate study of the admissible variations, higher-order Lagrange-Poincaré operator is derived and we establish its relationship with the equations obtained from a variational principle for higher-order Lagrangian systems defined on higher-order reduced tangent bundles. Moreover, when the system is subject to higher-order constraints defined by a submanifold of $T^{(k)}Q/G \simeq T^{(k)}(Q/G) \oplus_{Q/G} k\tilde{g}$ is interesting for optimal control applications as we will seen along the paper.

### 1.1 Motivation

The motivation of this work is optimal control of underactuated mechanical systems, where the cost function is defined on a second-order reduced bundles. Control and optimal control of mechanical systems becoming now a principal research focus of nonlinear control theory and boundary values problems. In particular, there are an increasing interest in the control of underactuated mechanical systems [6, 18]. Underactuated mechanical systems are characterized by the fact that there are more degrees of freedom than actuators. Underactuated mechanical systems include spacecraft, underwater vehicles, mobile robots, helicopters, wheeled vehicles, underactuated manipulators...

In this paper we introduce an optimization strategy in an underactuated mechanical system, that is, we are interested in studying the implementation of devices in which a controlled quantity is used to influence the behavior of the undeactuated system in order to achieve a desired goal (control) using the most economical strategy (optimization).

Higher-order Lagrange-Poincaré operator appear, for example, in the study of methods for stabilizing mechanical Lagrangian systems, known as controlled Lagrangian systems [4],[12]. The reduction of the class of controlled Lagrangian systems with symmetries is a future application of the theory developed in this work.

### 1.2 Main contributions

In this paper, we accomplish a sequence of goals, each derived upon the previous. In particular:

1. In section 2, we define the higher-order Euler-Poincaré operator for to give the connection between the variational and differential-equations description of the evolution of a given higher-order Euler-Poincaré system.

2. In section 3, we define the higher-order Lagrange-Poincaré operator and derive from it higher-order Lagrange-Poincaré equations.

3. Moreover, we develop Lagrange-Poincaré variational reduction for systems subject to higher-order constraints in parallel with the existing theory for systems without constrains [25].
4. In section 4, we describe the optimal control problem for underactuated Lagrangian mechanical systems where the Lagrangian is defined on a reduce bundle. We transform the optimal control problem into a second-order variational problem with second-order constraints and study a few examples.

Combined, these items allow for a geometric and intrinsic study for the theory of controlled lagrangian systems understanding higher-order Lagrange-Poincaré operators and perhaps open the door to applications which were previously overlooked by geometric mechanicians.

1.3 Organization of the paper

The paper is organized as follows. In Section 2 we recall some geometric constructions and properties of higher-order tangent bundles and mechanics on higher-order tangent bundles: Euler-Lagrange and Euler-Poincaré equations for higher-order Lagrangians are derived after introduce higher-order Euler-Lagrange and higher-order Euler-Poincaré operators respectively. In Section 3, we obtain after a delicate study of the admissible variations higher-order Lagrange-Poincaré operator and consequently higher-order Lagrange-Poincaré equations. Finally, we derive the higher-order Lagrange-Poincaré equations for systems with higher-order constraints by classical variational Lagrangian reduction. We apply these techniques in Section 4, to solve an optimal control problem of an underactuated mechanical systems, e.g. the optimal control of an underactuated Elroy’s Beanie and the optimal control of a snakeboard. We reduce the original system (without controls and constraints) and solve an optimization problem in the reduced variables as a higher-order variational problem with higher-order constraints.

2 Preliminaries on geometric mechanics

In this section we recall some basic facts about the higher-order tangent bundle theory and higher-order mechanics. Along this section also we will particularize this construction to the case when the configuration space is a Lie group $G$ and a quotient bundle. For more details see [19, 32].

2.1 Higher-order tangent bundles

Let $Q$ be a differentiable manifold of dimension $n$; it is possible to introduce an equivalence relation in the set $C^k(\mathbb{R}, Q)$ of $k$-differentiable curves from $\mathbb{R}$ to $Q$. By definition, two given curves in $Q$, $\gamma_1(t)$ and $\gamma_2(t)$, where $t \in (-a, a)$ with $a \in \mathbb{R}$ have a contact of order $k$ at $q_0 = \gamma_1(0) = \gamma_2(0)$ if there is a local chart $(\varphi, U)$ of $Q$ such that $q_0 \in U$ and

$$\left. \frac{d^s}{dt^s} (\varphi \circ \gamma_1(t)) \right|_{t=0} = \left. \frac{d^s}{dt^s} (\varphi \circ \gamma_2(t)) \right|_{t=0},$$

for all $s = 0, \ldots, k$. This is a well defined equivalence relation in $C^k(\mathbb{R}, Q)$ and the equivalence class of a curve $\gamma$ will be denoted by $[\gamma]_{q_0}^{(k)}$. The set of equivalence classes will be denoted by $T^{(k)}Q$ and it is not hard to show that it has a natural structure of differentiable manifold. Moreover, $\tau_Q^k : T^{(k)}Q \to Q$ where $\tau_Q^k ([\gamma]_{q_0}^{(k)}) = \gamma(0)$ is a fiber bundle called the tangent bundle of order $k$ of $Q$. 
Given a differentiable function \( f : Q \to \mathbb{R} \) and \( l \in \{0, \ldots, k\} \), its \( l \)-lift \( f^{(l,k)} \) to \( T^{(k)}Q \), \( 0 \leq l \leq k \), is the differentiable function defined as

\[
f^{(l,k)}([\gamma]_0^{(k)}) = \frac{d}{dt} (f \circ \gamma(t)) \bigg|_{t=0}.
\]

Of course, these definitions can be applied to functions defined on open sets of \( Q \).

From a local chart \((q^i)\) on a neighborhood \( U \) of \( Q \), it is possible to induce local coordinates \((q^{(0)i}, q^{(1)i}, \ldots, q^{(k)i})\) on \( T^{(k)}U = (\tau^k_Q)^{-1}(U) \), where \( q^{(s)i} = (q^i)^{(s,k)} \) if \( 0 \leq s \leq k \). Sometimes, we will use the standard conventions, \( q^{(0)i} \equiv q^i \), \( q^{(1)i} \equiv \dot{q}^i \) and \( q^{(2)i} \equiv \ddot{q}^i \).

Given a vector field \( X \) on \( Q \), we define its \( k \)-lift \( X^{(k)} \) to \( T^{(k)}Q \) as the unique vector field on \( T^{(k)}Q \) satisfying the following identities

\[
X^{(k)}(f^{(l,k)}) = (X(f))^{(l,k)}, \quad 0 \leq l \leq k,
\]

for all differentiable function \( f \) on \( Q \). In coordinates, the \( k \)-lift of a vector field \( X = X^i \frac{\partial}{\partial q^i} \) is

\[
X^{(k)} = (X^i)^{(s,k)} \frac{\partial}{\partial q^{(s)i}}.
\]

### 2.1.1 Higher-order Euler-Lagrange equations

Now, we briefly review the main notions of variational calculus with higher-order constraints.

Let us consider a mechanical system whose dynamics is described by a Lagrangian \( L : T^{(k)}Q \to \mathbb{R} \) that depends on higher-order derivatives up to order \( k \). Given two points \( x, y \in T^{(k-1)}Q \) we define the infinite-dimensional manifold \( C^{2k}(x,y) \) of \( 2k \)-differentiable piecewise curves which connect \( x \) and \( y \) as

\[
C^{2k}(x,y) = \{ c : [0, T] \to Q \mid c \text{ is } C^{2k}, c^{(k-1)}(0) = x \text{ and } c^{(k-1)}(T) = y \}.
\]

Fixed a curve \( c \in C^{2k}(x,y) \), the tangent space to \( C^{2k}(x,y) \) at \( c \) is given by

\[
T_c C^{2k}(x,y) = \{ X : [0, T] \to TQ \mid X \text{ is } C^{2k-1}, X(t) \in T_{c(t)}Q, X^{(k-1)}(0) = 0 \text{ and } X^{(k-1)}(T) = 0 \}.
\]

Let us consider the action functional \( \mathcal{A} \) on \( C^{2k} \)-curves in \( Q \) given by

\[
\mathcal{A} : C^{2k}(x,y) \to \mathbb{R} \quad c \mapsto \int_0^T L(c^{(k)}(t)) \, dt.
\]  

**Definition 2.1. Hamilton’s principle.** A curve \( c \in C^{2k}(x,y) \) is a solution of the Lagrangian system determined by \( L : T^{(k)}Q \to \mathbb{R} \) if and only if \( c \) is a critical point of \( \mathcal{A} \).

In order to find the critical points of \( \mathcal{A} \), we need to characterize the curves \( c \) such that \( d\mathcal{A}(c)(X) = 0 \) for all \( X \in T_c C^{2k}(x,y) \). Taking a family of curves \( c_\epsilon \in C^{2k}(x,y) \) with \( c_0 = c \) and \( \epsilon \in (-b, b) \subset \mathbb{R} \), the stationary condition can be written as

\[
\frac{d}{d\epsilon} \bigg|_{\epsilon = 0} \mathcal{A}(c_\epsilon) = 0.
\]

Let us denote \( \delta^i = \frac{d}{d\epsilon} \bigg|_{\epsilon = 0} c_\epsilon^i \) and \( \delta^{(l)} c^i = \frac{d^l}{dt^l} \delta^i \) with \( l = 1, \ldots, k-1 \), the corresponding variations. Then a curve \( c : [0, T] \to \mathbb{R} \) is a critical point of the action among the curves whose first
(k − 1) derivatives are fixed at the endpoints if and only if c(t) is a solution of the higher-order Euler-Lagrange equations, locally written as

$$\sum_{i=0}^{k} (-1)^i \frac{d^i}{dt^i} \left( \frac{\partial L}{\partial q^{(i)}} \right) = 0 , \quad 1 \leq i \leq n.$$ 

(see also [13] and [39] for an intrinsic derivation of these equations).

**Higher-order Euler-Lagrange operator:** Let $L : T^{(k)}Q \to \mathbb{R}$ be a given Lagrangian and let

$$J(L)(q) = \int_0^T L(c^{(k)}(t))dt$$

be the action of $L$ defined over $C^{2k}$ – curves in $Q$. Then, there is an unique operator

$$\mathcal{E}L : T^{(2k)}Q \to T^*Q$$

such that for all variations of the form $\delta c_s \in T_c C^{2k}(x,y)$ with fixed endpoints we have that

$$\frac{d}{ds} \mathcal{A}(c(t))\bigg|_{s=0} = \int_0^T \mathcal{E}L(L)(c^{(2k)}(t)) \cdot \delta c(t) dt.$$

In local coordinates $\mathcal{E}L(L)$ is given by

$$\mathcal{E}L(L)_{\delta}(c^{(2k)})dc^i = \left( \sum_{i=0}^{k} (-1)^i \frac{d^i}{dt^i} \left( \frac{\partial L}{\partial q^{(i)}} \right) \right) dq^{(i)}.$$

The higher-order Euler-Lagrange equations can be written as $\mathcal{E}L(L)(c^{(2k)}) = 0$.

### 2.1.2 Higher-order Euler-Poincaré equations

Let $G$ be a Lie group. Consider the left and right-multiplication on itself

$$\mathcal{L} : G \times G \to G , \quad (g,h) \to \mathcal{L}_g(h) = gh,$$

$$\mathcal{R} : G \times G \to G , \quad (g,h) \to \mathcal{R}_g(h) = hg.$$

Obviously $\mathcal{L}_g$ and $\mathcal{R}_g$ are diffeomorphism.

This left (respect. right) multiplication allows us to trivialize the tangent bundle $TG$ as follows

$$TG \to G \times \mathfrak{g} , \quad (g,\dot{g}) \mapsto (g, g^{-1}\dot{g}) = (g, T_g \mathcal{L}_g^{-1}\dot{g}) = (g, \xi),$$

$$TG \to G \times \mathfrak{g} , \quad (g,\dot{g}) \mapsto (g, \dot{g}g^{-1}) = (g, T_g \mathcal{R}_g g^{-1}) = (g, \xi)$$

where $\mathfrak{g} = T_eG$ is the Lie algebra of $G$ and $e$ is the neutral element of $G$. In the same way, we have the identification $TTG \cong G \times 3\mathfrak{g}$.

In the case when the manifold $Q$ has a Lie group structure, we will denote $Q = G$ and we can also use the (left) trivialization (respect. right) to identify the higher-order tangent bundle $T^{(k)}G$ with $G \times k\mathfrak{g}$. That is, if $g : I \to G$ is a curve in $C^{(k)}(\mathbb{R}, G)$:

$$\Upsilon_L^{(k)} : T^{(k)}G \to G \times k\mathfrak{g}$$

$$[g]_0^{(k)} \mapsto (g(0), g^{-1}(0)\dot{g}(0), \frac{d}{dt} \bigg|_{t=0} (g^{-1}(t)\dot{g}(t)), \ldots, \frac{d^{k-1}}{dt^{k-1}} \bigg|_{t=0} (g^{-1}(t)\dot{g}(t)))$$
and 
\[ \Upsilon^{(k)}_R: T^{(k)}G \longrightarrow G \times k\mathfrak{g} \]
\[ [g]_0^{(k)} \longmapsto (g(0), \dot{g}(0)g^{-1}(0), \frac{d}{dt} \bigg|_{t=0} (\dot{g}(t)g^{-1}(t)), \ldots, \frac{d^{k-1}}{dt^{k-1}} \bigg|_{t=0} (\dot{g}(t)g^{-1}(t))). \]

It is clear that \( \Upsilon^{(k)}_L, \Upsilon^{(k)}_R \) are diffeomorphisms.

We will denote by \( \xi(t) = g^{-1}(t)\dot{g}(t) \in \mathfrak{g} \) (respect. \( \xi(t) = \dot{g}(t)g^{-1}(t) \in \mathfrak{g} \)). Therefore

\[ \Upsilon^{(k)}_L([g]^{(k)}_0) = (g, \xi, \dot{\xi}, \ldots, \xi^{(k-1)}) , \]

where

\[ \xi^{(l)}(t) = \frac{d^l}{dt^l}(g^{-1}(t)\dot{g}(t)), \quad 0 \leq l \leq k-1 \]

and \( g(0) = g, \xi^{(l)}(0) = \xi^{(l)} \). Respectively,

\[ \Upsilon^{(k)}_R([g]^{(k)}_0) = (g, \xi, \dot{\xi}, \ldots, \xi^{(k-1)}) , \]

where

\[ \xi^{(l)}(t) = \frac{d^l}{dt^l}(\dot{g}(t)g^{-1}(t)), \quad 0 \leq l \leq k-1. \]

We will indistinctly use the notation \( \xi^{(0)} = \xi, \xi^{(1)} = \dot{\xi} \), where there is not danger of confusion.

We may also define the surjective mappings \( \tau^{(l,k)}_G : T^{(k)}G \rightarrow T^{(l)}G \), for \( l \leq k \), given by

\[ \tau^{(l,k)}_G([g]^{(k)}_0) = [g]^{(l)}_0. \]

With the previous identifications we have that

\[ \tau^{(l,k)}_G(g(0), \xi(0), \dot{\xi}(0), \ldots, \xi^{(k-1)}(0)) = (g(0), \xi(0), \dot{\xi}(0), \ldots, \xi^{(l-1)}(0)) , \]

and it is easy to see that \( T^{(0)}G \equiv G, T^{(1)}G \equiv G \times \mathfrak{g} \) and \( \tau^{(0,k)}_G = \tau^k_G \).

Now, a higher-order Lagrangian \( L : T^{(k)}G \rightarrow \mathbb{R} \) is said to be left-invariant if its left trivialization \( L : G \times k\mathfrak{g} \rightarrow \mathbb{R} \) does not depend of the first entry. Respectively, the lagrangian is said to be right-invariant if its right trivialization \( L : G \times k\mathfrak{g} \rightarrow \mathbb{R} \) does not depend of the first factor.

The problem consists on finding the critical curves of the action functional

\[ \mathcal{J}(c(t)) = \int_0^T L(g, \xi, \dot{\xi}, \ldots, \xi^{(k-1)})dt \]

among all curves \( c(t) \in C^\infty(G \times k\mathfrak{g}) \) satisfying the boundary conditions for arbitrary variations \( \delta g = \frac{d}{dx} |_{x=0} g_\epsilon \), where, \( \epsilon \mapsto g_\epsilon \) is a smooth curve in \( G \) such that \( g_0 = g \).

The variations for the variable \( \xi \) and the corresponding time derivatives are given by

\[ \delta \xi^{(l)} = \frac{d^l}{dt^l}(\delta \xi), \]

for \( l = 1, \ldots, k-1 \); where the variation \( \delta \xi \) is induced by \( \delta g \) as \( \delta \xi = \dot{\eta} \pm [\xi, \eta] \) and where \( \eta \) is a curve on the Lie algebra with fixed endpoints and the sign depend of the chosen trivialization. Therefore, we can deduce by Hamilton’s principle, integrating \( k \) times by parts and using the boundary conditions \( \eta^{(l)}(0) = \eta^{(l)}(T) = 0 \), for \( l = 0, \ldots, k-1 \) the higher-order Euler-Lagrange equations for the Lagrangian \( L : G \times k\mathfrak{g} \rightarrow \mathbb{R} \) are.
\[
\left( \frac{d}{dt} - \text{ad}^*_\xi \right) \sum_{l=0}^{k-1} (-1)^l \frac{d^l}{dt^l} \left( \frac{\partial L}{\partial \xi} \right) = \mathcal{L}^*_g \left( \frac{\partial L}{\partial g} \right)
\]
(2.2)

as in [15] or

\[
\left( \frac{d}{dt} + \text{ad}^*_\xi \right) \sum_{l=0}^{k-1} (-1)^l \frac{d^l}{dt^l} \left( \frac{\partial L}{\partial \xi} \right) = \mathcal{R}^*_g \left( \frac{\partial L}{\partial g} \right)
\]
(2.3)

if we trivialize the tangent bundle \( T^{(k)}G \) to the right.

Now, if the Lagrangian is left-invariant (respectively, right-invariant) the right hand side of equations (2.2) and (2.3) vanishes and therefore we obtain the \( k^{th} - \text{Euler-Poincaré} \) equations given in [15],[23] and [24],

\[
\left( \frac{d}{dt} \mp \text{ad}^*_\xi \right) \sum_{l=0}^{k-1} (-1)^l \frac{d^l}{dt^l} \left( \frac{\partial L}{\partial \xi} \right) = 0.
\]
(2.4)

Obviously, when \( k = 1 \) we obtain the well know Euler-Poincaré equations [27, 28, 37].

**Higher-order Euler-Poincaré operator:** In analogous to the Euler-Lagrange operator, the Euler-Poincaré operator could be defined by the variational principle given in theorem 3.1 of [23]

**Theorem 2.2 (\( k^{th} \)-order Euler-Poincaré reduction [23]).** Let \( L : T^{(k)}G \to \mathbb{R} \) be a \( G \)-invariant Lagrangian and let \( \ell : k\mathfrak{g} \to \mathbb{R} \) be the associated reduced Lagrangian. Let \( g(t) \) be a curve in \( G \) and \( \xi(t) = g(t)^{-1} \dot{g}(t) \), be the reduced curve in the Lie algebra \( \mathfrak{g} \). Then the following assertions are equivalent.

(i) The curve \( g(t) \) is a solution of the \( k^{th} \)-order Euler-Lagrange equations for \( L : T^{(k)}G \to \mathbb{R} \).

(ii) Hamilton’s variational principle

\[
\delta J = \delta \int_0^T L \left( g, \dot{g}, ..., g^{(k)} \right) dt = 0
\]

holds upon using variations \( \delta g \) such that \( \delta g^{(j)} \) vanish at the endpoints for \( j = 0, ..., k - 1 \).

(iii) The \( k^{th} \)-order Euler-Poincaré equations for \( \ell : k\mathfrak{g} \to \mathbb{R} \):

\[
\left( \frac{d}{dt} - \text{ad}^*_\xi \right) \sum_{j=0}^{k-1} (-1)^j \frac{d^j}{dt^j} \frac{\partial \ell}{\partial \xi^{(j)}} = 0.
\]
(2.5)

(iv) The constrained variational principle

\[
\delta J_{\text{red}} = \delta \int_0^T \ell \left( \xi, \dot{\xi}, ..., \xi^{(k)} \right) = 0
\]

holds for constrained variations of the form \( \delta \xi = \dot{\eta} + [\xi, \eta] \), where \( \eta \) is an arbitrary curve in \( \mathfrak{g} \) such that \( \eta^{(j)} \) vanish at the endpoints, for all \( j = 0, ..., k - 1 \).
In addition there is a unique bundle map

\[ \mathcal{E}P(\ell) : 2k\mathfrak{g} \to \mathfrak{g}^* \]

such that for all variations vanishing at the endpoints we have

\[ dJ_{\text{red}}(\ell)(\xi, \dot{\xi}, \ldots, \xi^{(k)}) \cdot (\delta \xi, \ldots, \delta \xi^{(k)}) = \int_0^T \mathcal{E}P(\ell)(\xi, \dot{\xi}, \ldots, \xi^{(2k-1)}) \cdot \eta \, dt. \]

The map \( \mathcal{E}P(\ell) \) is dubbed \textit{higher-order Euler-Poincaré operator} and its expression is given by

\[ \mathcal{E}P(\ell)(\xi, \dot{\xi}, \ldots, \xi^{(2k-1)}) = \left( \frac{d}{dt} - \text{ad}_\xi^* \right) \sum_{j=0}^{k-1} (-1)^j \frac{d^j}{dt^j} \partial \ell \frac{\partial \ell}{\partial \xi^{(j)}}. \]

The \( k \)-th order Euler-Poincaré equations can be written as \( \mathcal{E}P(\ell)(\xi, \dot{\xi}, \ldots, \xi^{(2k-1)}) = 0. \)

\textbf{Remark 2.3.} After a right trivialization one can write the Euler-Poincaré operator as

\[ \mathcal{E}P(\ell)(\xi, \dot{\xi}, \ldots, \xi^{(2k-1)}) = \left( \frac{d}{dt} + \text{ad}_\xi^* \right) \sum_{j=0}^{k-1} (-1)^j \frac{d^j}{dt^j} \frac{\partial \ell}{\partial \xi^{(j)}}. \]

\section{2.2 Connections and exterior calculus on adjoint bundles}

Let \( \Phi : G \times Q \to Q \), \((g, q) \mapsto \Phi_g(q)\) be a left action, free and proper, of a Lie group \( G \) on a manifold \( Q \). Thus we get the principal bundle \( \pi : Q \to \hat{Q} := Q/G \), where \( \hat{Q} \) is endowed with the unique manifold structure for which \( \pi \) is a submersion.

To any element \( \xi \in \mathfrak{g} \) there corresponds a vector field \( \xi_Q \) on \( Q \), called the \textit{infinitesimal generator} and given by

\[ \xi_Q(q) := \left. \frac{d}{dt} \right|_{t=0} \Phi_{\exp(t\xi)}(q). \]

For any \( q \in Q \) these vector fields generate the \textit{vertical subspace} \( V_qQ := \{ \xi_Q(q) | \xi \in \mathfrak{g} \} = \ker(T_q\pi). \)

The \textit{adjoint bundle} is defined by \( \text{Ad}Q := Q \times \mathfrak{g} \to \hat{Q} \), where the quotient is taken relative to the action \((g, (q, \xi)) \mapsto (\Phi_g(q), \text{Ad}_{g^{-1}}(\xi))\). For \((q, \xi) \in Q \times \mathfrak{g}\) the corresponding element in \( \text{Ad}Q \) is denoted by \([q, \xi]_G\). Moreover, in each fiber \((\text{Ad}Q)_{\pi(q)}\) (depending smoothly for each \( x = \pi(q) \in \hat{Q} \)) there is a Lie bracket operation \([\cdot, \cdot]_{\pi(q)}\) given by

\[ [[q, \xi]_G, [q, \eta]_G]_{\pi(q)} := [q, [\xi, \eta]]_G \]

for \([q, \xi]_G, [q, \eta]_G \in \text{Ad}Q\).

Denoting by \( \Omega^1(Q, \mathfrak{g}) \) the space of \( \mathfrak{g} \)-valued 1-forms on \( Q \), we fix a connection \( \mathcal{A} \) on the principal bundle \( \pi : Q \to Q/G \), that is, a 1-form \( \mathcal{A} \in \Omega^1(Q, \mathfrak{g}) \) such that

\[ \mathcal{A}(\xi_Q(q)) = \xi, \text{ and } \Phi^*_g \mathcal{A} = \text{Ad}_g \circ \mathcal{A}, \]

where \( \xi_Q \) is the infinitesimal generator associated to \( \xi \in \mathfrak{g} \) for \( q \in Q \). A connection induce a splitting \( T_qQ = V_qQ \oplus H_qQ \) on the tangent space into the vertical and \textit{horizontal subspace} defined by

\[ H_qQ := \ker(\mathcal{A}(q)). \]
The covariant exterior differential associated to the principal connection \( \mathcal{A} \) is the map \( d^A : \Omega^k(Q, g) \to \Omega^{k+1}(Q, g) \) defined by

\[
d^A \alpha(q)(\beta_1, \ldots, \beta_k) := d\alpha(q)(\text{hor}_q(\beta_1), \ldots, \text{hor}_q(\beta_k)),
\]

where \( \text{hor}_q(\beta_j) \) is the horizontal part of \( \beta_j \in T_q Q \), \( j = 1, \ldots, k \) and \( \alpha \in \Omega^k(Q, g) \).

If we denote by \( \overline{\Omega}^k(Q, g) \) the subspace of \( \Omega^k(Q, g) \) consisting of \( g \)-valued \( k \)-forms \( \alpha \) such that \( \Phi^*_g \alpha = Ad_g \circ \alpha \) and \( \alpha(\beta_1, \ldots, \beta_k) = 0 \) if one of \( \beta_1, \ldots, \beta_k \in T_q Q \) is vertical; we have that \( d^A \alpha \in \overline{\Omega}^{k+1} \).

The real vector space \( \overline{\Omega}^k(Q, g) \) is isomorphic to \( \Omega^k(\hat{Q}, AdQ) \), the space of \( k \)-forms on \( \hat{Q} \) with values on the adjoint bundle \( AdQ \). Then, for each \( \alpha \in \overline{\Omega}^k(Q, g) \) there are a \( k \)-form \( \gamma \in \Omega^k(\hat{Q}, AdQ) \) given by

\[
\gamma(x)(v_1, \ldots, v_k) := [q, \alpha(q)(\beta_1, \ldots, \beta_k)]_G,
\]

for \( x \in \hat{Q}, x = \pi(q) \) where \( q \in Q, v_1, \ldots, v_k \in T_x \hat{Q} \) and \( \beta_j \in T_q Q \) such that \( T_q \pi(\beta_j) = v_j \) with \( j = 1, \ldots, k \).

Since \( \overline{\Omega}(Q, g) = \{ f : Q \to g | f \circ \Phi_g = Ad_g \circ f \} =: \mathcal{F}(Q, g) \) and \( \Omega^0(\hat{Q}, AdQ) = \Gamma(AdQ) \), the space of sections of \( AdQ \) we will use the notation \( \overline{\Omega}(Q, g) = \mathcal{F}(Q, g) \) and \( \Omega^0(\hat{Q}, AdQ) = \Gamma(AdQ) \) interchangeably. Moreover, there is a Lie algebra isomorphism between \( \mathcal{F}(Q, g) \) and \( \Gamma(AdQ) \), given by \( h(\pi(q)) = [q, f(q)]_G \) for \( f \in \mathcal{F}(Q, g) \) and \( h \in \Gamma(AdQ) \).

Then, for each function \( f \in \mathcal{F}(Q, g) \) and \( \alpha \in \overline{\Omega}T(Q, g) \) we have that

\[
d^A f(q)(\beta) = df(q)(\beta) + [\mathcal{A}(q)(\beta), f(q)]
\]

\[
d^A \alpha(q)(\beta, v) = d\alpha(q)(\beta, v) + [\mathcal{A}(q)(\beta), \alpha(q)(v)] - [\mathcal{A}(q)(v), \alpha(q)(\beta)]
\]

for any \( \beta, v \in T_q Q \).

A principal connection \( \mathcal{A} \) on \( Q \) induce an affine connection denoted by \( \nabla \) on the vector bundle \( AdQ \to \hat{Q} \) and a covariant derivative \( \frac{D}{dt} \) on \( (AdQ)^* \to \hat{Q} \) (see [30]).

The curvature \( \mathcal{B} \in \overline{\Omega}^2(\hat{Q}, g) \) associated with a connection \( \mathcal{A} \) is by definition, the Lie algebra valued two form on \( Q \) defined by

\[
\mathcal{B}(u, v) = d^A \mathcal{A}(\text{hor}_q(u_q), \text{hor}(v_q)).
\]

The Cartan structure equations state that for all vector fields \( u, v \in \mathfrak{X}(Q) \) the following identity holds

\[
\mathcal{B}(u, v) = d^A \mathcal{A}(u, v) - [\mathcal{A}(u), \mathcal{A}(v)]_g.
\]

The definition of curvature and exterior differential implies the Bianchi identity

\[
d^A \mathcal{B} = 0
\]

(see [1], [10] and [30] for more details about this topic).

### 2.3 Higher-order quotient space

Let \( f : M \to N \) be a smooth function. This induce the application \( T^{(k)}f : T^{(k)}M \to T^{(k)}N \) given by

\[
T^{(k)}f([\gamma]_{g_0}^{(k)}) := [f \circ \gamma]_{f(g_0)}^{(k)}.
\]
In particular, the action of a Lie group Φ is lifted to an action Φ(k) : G × T(k)Q → T(k)Q given by

\[ Φ_g^{(k)}([γ]_{q_0}) := T(g)Φ_g([γ]_{q_0}) = [Φ_g \circ γ]_{q_{Φ}(q_0)}. \]

If Φ is free and proper, we get a principal G–bundle \( T^{(k)}Q \to (T^{(k)}Q)/G \). The quotient \( (T^{(k)}Q)/G \) is a fiber bundle over \( Q/G \). The class of an element \([γ]_{q_0}^{(k)} \in T_{q_0}^{(k)}Q\) in the quotient is denoted \([γ]_{q_0}^{(k)}G\).

We can construct the vector bundle isomorphism

\[ \alpha_A : (TQ)/G \to T(Q/G) \oplus AdQ, \quad \alpha_A([q]_{q_0}^{(0)}G) := (Tπ([q]_{q_0})G, [q_0, A([q]_{q_0}^{(0)})G], \]

where the adjoint bundle \( AdQ \to Q/G \) is the vector bundle defined by \( AdQ := (Q \times g)/G = \tilde{g} \).

From [10] (see Lemma 2.3.4) we know that the covariant derivative of a curve \( σ(t) = [q(t), \xi(t)]_G \in AdQ \) relative to the principal connection \( A \) is given by

\[ \frac{D}{Dt}σ(t) = \frac{D}{Dt}[q(t), \xi(t)]_G = [q(t), \dot{ξ}(t) - [A(q(t), \dot{q}(t)), \xi(t)]]_G. \quad (2.7) \]

In the particular case when \( σ(t) = [q(t), A(q(t), \dot{q}(t))]_G \) we have,

\[ \frac{D}{Dt}σ(t) = [q(t), \dot{ξ}(t)]_G \]

\[ \frac{D^2}{Dt^2}σ(t) = \frac{D}{Dt}[q(t), \dot{ξ}(t)] = [q(t), \ddot{ξ}(t) - [ξ(t), \ddot{ξ}(t)]. \]

If we denote by \( ξ_1(t) = ξ(t), ξ_2(t) = ξ(t), ξ_3(t) = ξ(t) - [ξ(t, ξ_2(t)), ..., ξ_k(t) = ξ_{k−1}(t) - [ξ(t), ξ_{k−1}(t)], \) one can obtain that

\[ \frac{D^{k−1}}{Dt^{k−1}}σ(t) = [q(t), ξ_k(t)]_G, \]

where \( ξ_k \in \tilde{g} \). The bundle isomorphism that generalize \( α_A \) is given by

\[ α_A^{(k)} : (T^{(k)}Q)/G \to T^{(k)}(Q/G) \oplus kAdQ := T^{(k)}Q \oplus k\tilde{g}; \]

\[ α_A^{(k)}([q]_{q_0}^{(k)}G) = \left( T^{(k)}π([q]_{q_0}^{(k)}), \frac{D}{Dt}_{t=0} σ(t), \left. \frac{D^2}{Dt^2} σ(t) \right|_{t=0}, \left. \frac{D^{k−1}}{Dt^{k−1}} σ(t) \right|_{t=0} \right), \]

where \( σ(t) := [q(t), A(q(t), \dot{q}(t))]_G, q(t) \) is any curve representing \([q]_{q_0}^{(k)} \in T^{(k)}Q\) with \( q(0) = q_0 \) and \( k\tilde{g} := \tilde{g} \times \tilde{g} \times \ldots \times \tilde{g} \).

We use the following notation, as in [25], for the reduced variables:

\[ α_A^{(k)}([q]_{q_0}^{(k)}G) = (p, \dot{p}, \ddot{p}, ..., p^{(k)}, σ, \dot{σ}, \ddot{σ}, ..., σ^{(k−1)}), \]

where \( (p, \dot{p}, \ddot{p}, ..., p^{(k)}) \) are local coordinates on \( T^{(k)}(Q/G) \) and the dots denote the temporal derivative in a local chart; \( σ, \dot{σ}, \ddot{σ}, ..., σ^{(k−1)} \) are elements in \( \tilde{g} \), all seen as independent variables and \( σ^{(l)} := \frac{D^l}{Dt^l} σ(t) \) (covariant derivative).
2.4 The reduced Lagrangian and constraints

Let $L : T^{(k)}Q \to \mathbb{R}$, and $\phi^\alpha : T^{(k)}Q \to \mathbb{R}$ be a higher-order lagrangian and the higher-order (independent) constraints where $\alpha = 1, \ldots, m$, both $G$–invariants, then they induces a reduced Lagrangian $\tilde{L}$ and the reduced constraints $\tilde{\phi}$ on the quotient space $(T^{(k)}Q)/G$. Fixing a connection $\mathcal{A}$ then we can write the reduced lagrangian and constraints as,

$$\tilde{L} = \tilde{L}(p, \dot{p}, \ldots, p^{(k)}, \sigma, \dot{\sigma}, \ldots, \sigma^{(k-1)}): T^{(k)}(Q/G) \oplus k\mathfrak{g} \to \mathbb{R}$$

$$\tilde{\phi}^\alpha = \tilde{\phi}^\alpha(p, \dot{p}, \ldots, p^{(k)}, \sigma, \dot{\sigma}, \ldots, \sigma^{(k-1)}): T^{(k)}(Q/G) \oplus k\mathfrak{g} \to \mathbb{R}.$$  

Remark 2.4. If $Q$ is a Lie group, $Q = G$ the adjoint bundle is identify with $\mathfrak{g}$ via the isomorphism $\alpha^k_{\mathcal{A}} : (T^{(k)}G)/G \to k\mathfrak{g} \simeq k\mathfrak{g}$,

$$\alpha^k_{\mathcal{A}}([[g]^{(k)}_0]_G) := \left(g^{-1}(0)\dot{g}(0), \frac{d}{dt}\big|_{t=0} \xi(t), \ldots, \frac{d^{k-1}}{dt^{k-1}} \xi(t)\right),$$

where $\xi(t) = g^{-1}(t)\dot{g}(t)$. If we choose $g_0 = e$, that is, $[[g_0 g]^{(k)}]_G = [[g]^{(k)}_0]_G$, we can define the reduced Lagrangian and the reduced constraints given $\tilde{L} : k\mathfrak{g} \to \mathbb{R}$ and $\tilde{\phi} : k\mathfrak{g} \to \mathbb{R}$ (see [10]). Alternatively using a right trivialization, one can take the vector bundle isomorphism $\alpha^k_{\mathcal{A}} : (T^{(k)}G)/G \to k\mathfrak{g} \simeq k\mathfrak{g}$,

$$\alpha^k_{\mathcal{A}}([[g]^{(k)}_0]_G) := \left(\dot{g}(0)g^{-1}(0), \frac{d}{dt}\big|_{t=0} \xi(t), \ldots, \frac{d^{k-1}}{dt^{k-1}} \xi(t)\right),$$

where $\xi(t) = \dot{g}(t)g^{-1}(t)$.

3 Higher-order Lagrange-Poincaré equations for systems with higher-order constraints

In [25] was presented, after a generalization of variational reduction [10], higher-order Lagrange Poincaré equations. Given a reduced higher-order Lagrangian $\tilde{L} : T^{(k)}Q \oplus k\mathfrak{g} \to \mathbb{R}$ the higher-order Lagrange-Poincaré equations for a curve $c^{(k)}(t) = (p(t), \dot{p}(t), \ldots, p^{(k)}(t), \sigma(t), \dot{\sigma}(t), \ldots, \sigma^{(k-1)}(t)) \in C^\infty(T^{(k)}(Q/G) \oplus_{Q/G} k\mathfrak{g})$ are

$$\sum_{i=0}^{k}(-1)^i\frac{d}{dt} \left( \frac{\partial \tilde{L}}{\partial p^{(i)}} \right) = \left\langle \sum_{i=0}^{k-1}(-1)^i \frac{D}{Dt} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} \right) + \sum_{i=0}^{k-1}(-1)^i a_{\sigma^{(i)}\sigma^{(i-1-i)}} \frac{D}{Dt} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} \right) \right\rangle ; i\tilde{\mathcal{B}}$$

$$0 = \left( \frac{D}{Dt} - ad^*_\sigma \right) \sum_{i=0}^{k-1}(-1)^i \frac{D}{Dt} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} \right),$$

where $\tilde{\mathcal{B}}$ is the reduce curvature 2-form.

3.1 Higher-order Lagrange-Poincaré operator

In this subsection we introduce higher-order Lagrange-Poincaré operator and we will give the connection between the variational and differential-equation description of the evolution of a given system.
3.1.1 Reduced higher-order Euler-Lagrange operator:

The map $\mathcal{EL}(L) : T^{(2k)}Q \to T^*Q$, being $G$-invariant induce a quotient map

$$[\mathcal{EL}(L)]_G : (T^{(2k)}Q)/G \to T^*Q/G,$$

which depends only on the reduced lagrangian $\ell : T^kQ/G \to \mathbb{R}$, that is, we can identify $[\mathcal{EL}(L)]_G$ with an operator $\mathcal{EL}(\ell)$. We will call this operator, reduced higher-order Lagrange-Poincaré operator and it does not depend on the structure on the principal bundle $Q$.

3.1.2 The reduced variations:

Towards the construction of higher-order Lagrange-Poincaré operator the main point is to compute the variations of $(p, \dot{p}, \ldots, p^{(k)}, \sigma, \dot{\sigma}, \ldots, \sigma^{(k-1)}) \in T^{(k)}Q \oplus \tilde{g}$ where $\tilde{Q} = Q/G$ induced by the variations on $Q$, $\delta q(t) = \frac{d}{ds} \bigg|_{s=0} q(t, s)$ where $q(t) \in Q$.

In the study of the Lagrange-Poincaré equations and Lagrange-Poincaré operator we will use variations of curves in $\tilde{Q} \oplus \tilde{g}$, and a given arbitrary deformation $p(t, s) \oplus \sigma(t, s)$ with $p(t, 0) \oplus \sigma(t, 0) = p(t) \oplus \sigma(t)$ the corresponding covariant variation is, by definition,

$$\delta p(t) \oplus \delta \sigma(t) = \frac{\partial}{\partial s} \bigg|_{s=0} p(t, s) \oplus \frac{D}{Ds} \bigg|_{s=0} \sigma(t, s). \quad (3.1)$$

Since $(p, \dot{p}, \ldots, p^{(k)}) = T^k \pi([q]_{\tilde{g}}^{(k)}) := [\pi \circ q_p^{(k)}]$ implies that the variations $\delta p$ of $p(t)$ are arbitrary except in the extremes; that is, $\delta p^{(l)}(0) = \delta p^{(l)}(T) = 0$ for $l = 1, \ldots, k - 1; t \in [0, T]$. Then, $\delta(p, \dot{p}, \ldots, p^{(k)}) := (\delta p, \delta \dot{p}, \ldots, \delta p^{(k)})$ are the horizontal variations.

The difficult appear in the computation of the vertical variations (that is, for the variable $\sigma$). For this propose we need to use the connection $A$ since $\sigma(t) = [q(t), A(q(t), \dot{q}(t))]_G \in \tilde{g}$.

**Proposition 3.1.** [10] Let $\sigma(t) = [q(t), A(q(t), \dot{q}(t))]_G$ be a curve on the adjoint bundle $\tilde{g}$ where $A$ is a fix connection on the principal bundle $\pi : Q \to Q/G, A \in \Omega^1(Q, g)$. The covariant variations of $\sigma$ are given by,

$$\delta \sigma(t) = \frac{D}{Dt} [q(t), A(q(t), \dot{q}(t))]_G + [q(t), [A(q(t), \dot{q}(t)), A(q(t), \dot{q}(t))]_G] + [q(t), B(\delta q(t), \dot{q}(t))]_G$$

where $B$ is the curvature form given by the equality $B := dA - [A, A] \in \Omega^2(Q, g)$.

Now, following the last proposition; if we denote by $\eta(t) = [q(t), A(q(t), \dot{q}(t))]_G \in \tilde{g}$, thus,

$$\delta \sigma(t) = \frac{D}{Dt} \eta(t) + \frac{[\sigma(t), [A(q(t), \dot{q}(t))]_G, q(t), A(q(t), \dot{q}(t))]_G] + [q(t), B(\delta q(t), \dot{q}(t))]_G}$$

$$= \frac{D}{Dt} \eta(t) + [\sigma(t), \eta(t)] + [q(t), B(\delta q(t), \dot{q}(t))]_G$$

$$= \frac{D}{Dt} \eta(t) + [\sigma(t), \eta(t)] + \tilde{B}(p)(\delta p(t), \dot{p}(t))$$

where the last equality is given by Lemma 3.1.5 of [10], and $\tilde{B} \in \Omega^2(\tilde{Q}, \tilde{g})$ is the reduced curvature.

**Lemma 3.2.** [10] $\frac{D}{Dt} \frac{D}{Ds} \sigma(s, t) - \frac{D}{Ds} \frac{D}{Dt} \sigma(s, t) = -[\tilde{B}(p)(\dot{p}(t), \delta p(t)), \sigma(t)]$. 


Remark 3.4. Observe that under the hypotheses of the last theorem, \([\mathcal{E}L(L)]_G\) coincides with the operator \(\mathcal{L}P(\widetilde{L})\).
Definition 3.5. The 1-form valued bundle map

\[ \mathcal{LP}(\tilde{L}) : T^{(2k)}(Q/G) \times_{Q/G} 2k\mathfrak{g} \to T^*(Q/G) \oplus \mathfrak{g}^* \]

defined in previous theorem will be dubbed higher-order Lagrange-Poincaré operator.

There is a decomposition of the Lagrange-Poincaré operator as a decomposition of the range space for \( \mathcal{LP}(\tilde{L}) \) as a direct sum by

\[ \mathcal{LP}(\tilde{L}) = H(\mathcal{LP})(\tilde{L}) \oplus V(\mathcal{LP})(\tilde{L}) \]

which define the higher-order horizontal Lagrange-Poincaré operator and the higher-order vertical Lagrange-Poincaré operator respectively.

By definition the \( k \)th Lagrange-Poincaré equations are the equations \( \mathcal{LP}(\tilde{L}) = 0 \). The higher-order horizontal Lagrange-Poincaré equation and higher-order vertical Lagrange-Poincaré equations are

\[ H(\mathcal{LP})(\tilde{L}) = 0 \text{ and } V(\mathcal{LP})(\tilde{L}) = 0. \]

Theorem 3.6. Assume the hypothesis of Theorem (3.3), then the higher-order vertical and horizontal Lagrange-Poincaré operators are given by

\[ V(\mathcal{LP})(\tilde{L}) = \left( \frac{D}{Dt} - ad^*_\sigma \right) \sum_{i=0}^{k-1} (-1)^{(i)} \frac{D^{(i)}}{Dt^{(i)}} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} \right), \]

\[ H(\mathcal{LP})(\tilde{L}) = \sum_{i=0}^{k} (-1)^{(i)} \frac{d^{(i)}}{dt^{(i)}} \left( \frac{\partial \tilde{L}}{\partial p^{(i)}} \right) - \left\langle \sum_{i=0}^{k-1} \left( (-1)^{i} \frac{D^{(i)}}{Dt^{(i)}} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} \right) + \sum_{l=0}^{i-1} (-1)^{l} ad^*_{\sigma^{(i-1-l)}} \frac{D^{(l)}}{Dt^{(l)}} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} \right) \right) : i_p \tilde{\mathcal{B}} \right\rangle \]

respectively.

Proof:

Consider a variation \( \delta p \oplus \delta \sigma \) of a curve \( p(t) \oplus \sigma(t) \) corresponding to horizontal and vertical variations \( \delta q \) of a curve \( q \in \mathcal{C}^{(2k)}(t_0, q_T). \)

If we consider variations \( \delta p \oplus \delta \sigma \) corresponding to horizontal variations \( \delta q \) then we have, for all \( \delta p \) with \( \delta p(0) = \delta p(T) = 0 \) and \( \delta p^{(l)}(0) = \delta p^{(l)}(T) = 0 \) for \( l = 1, \ldots, k - 1; \)

\[ \delta \int_0^T \tilde{L}(c^k(t)) dt = \int_0^T \left( \sum_{i=0}^{k} \frac{\partial \tilde{L}}{\partial p^{(k)}} \delta p^{(i)} + \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} \delta \sigma^{(i)} \right) dt. \]

Using \( \sigma(t) = [q(t), A(q(t), \dot{q}(t))]_G \) and its higher-order derivatives, integrating k-times by parts and using Proposition (3.1)

\[ \delta \int_0^T \tilde{L}(c^k(t)) dt = \int_0^T \left( \sum_{i=0}^{k} (-1)^{(i)} \frac{d^{(i)}}{dt^{(i)}} \left( \frac{\partial \tilde{L}}{\partial p^{(i)}} \right) \right) (c^k(t)) \delta p \]

\[ - \sum_{i=0}^{k-1} \left( (-1)^{(i)} \frac{D^{(i)}}{Dt^{(i)}} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} \right) + \sum_{l=0}^{i-1} (-1)^{(i)} ad^*_{\sigma^{(i-1-l)}} \frac{D^{(l)}}{Dt^{(l)}} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} \right) \right) (c^k(t)) \tilde{\mathcal{B}}(p)(\dot{p}, \delta p) dt. \]
Therefore, arbitrariness of $\delta p$ minds
\[
H(\mathcal{LP})(\tilde{L})(c^k(t)) = \sum_{l=0}^{k} (-1)^{(l)} \frac{d}{dt(l)} \frac{\partial L}{\partial p^{(l)}}(c^k(t)) \\
- \left( \sum_{l=0}^{k-1} (-1)^{(l)} \frac{D}{Dt^{(l)}} \frac{\partial \tilde{L}}{\partial \sigma^{(l)}} + \sum_{l=1}^{l-1} (-1)^{(l)} a^{*}_{\sigma\alpha} D^{(l)}(i) \left( \frac{\partial \tilde{L}}{\partial \sigma^{(l)}} \right) (c^k(t)) \right) i_\beta \mathcal{B}(p).
\]

Now, consider variations $\delta \sigma$ of a curve $p(t) \oplus \sigma(t)$ corresponding to vertical variations $\delta q(t)$. We have,
\[
0 = \delta \int_{0}^{T} \tilde{L}(\mathcal{LP})(\tilde{L})(c^k(t)) dt = \int_{0}^{T} \sum_{l=0}^{k-1} \frac{\partial \tilde{L}}{\partial \sigma^{(l)}} \delta \sigma^{(l)} dt \\
= \int_{0}^{T} \left\langle \left( -\frac{D}{Dt} + a^{*}_{\sigma\alpha} \right) \sum_{l=0}^{k-1} (-1)^{(l)} \frac{D}{Dt^{(l)}} \frac{\partial \tilde{L}}{\partial \sigma^{(l)}}, \eta \right\rangle dt
\]
where $\eta \in g$ is a curve on the Lie algebra $g$ with $\eta(0) = \eta(T) = 0$. By arbitrariness of $\eta$ so we get
\[
V(\mathcal{LP})(\tilde{L}) = \left( \frac{D}{Dt} - a^{*}_{\sigma\alpha} \right) \sum_{l=0}^{k-1} (-1)^{(l)} \frac{D}{Dt^{(l)}} \frac{\partial \tilde{L}}{\partial \sigma^{(l)}}.
\]

Note that when $G = Q$ then the horizontal Lagrange-Poincaré operator is zero, $H(\mathcal{LP})(\tilde{L}) = 0$ and $V(\mathcal{LP})(\tilde{L})$ is the Euler-Poincaré operator. Moreover, when $G = e$ we have $V(\mathcal{LP})(\tilde{L}) = 0$ and therefore, the horizontal Lagrange-Poincaré operator is the Euler-Lagrangian operator.

### 3.2 Hamilton’s principle for systems with constraints

In this subsection we derive the higher-order Lagrange-Poincaré equations for regular systems with higher-order constraints following the ideas of classical Lagrangian reduction [10]. We can compute the higher-order Lagrange-Poincaré equations for systems with higher-order constraints from Hamilton’s principle for the higher-order reduce Lagrangian $\tilde{L} : T^{(k)}\hat{Q} \oplus k\mathfrak{g} \to \mathbb{R}$, and the higher-order reduce constraints $\tilde{\phi}^\alpha : T^{(k)}\hat{Q} \oplus k\mathfrak{g} \to \mathbb{R}$, for $\alpha = 1, \ldots, m$; where $\hat{Q} = Q/G$ and $\lambda_\alpha$ are the Lagrange multipliers.

**Hamilton’s principle**

\[
0 = \delta \int_{0}^{T} \tilde{L}(\hat{p}, \hat{\sigma}, \cdots, p^{(k)}, \sigma, \cdots, \sigma^{(k-1)}) + \lambda_\alpha \tilde{\phi}^\alpha(p, \hat{p}, \cdots, p^{(k)}, \sigma, \cdots, \sigma^{(k-1)}) dt \\
= \int_{0}^{T} \sum_{i=0}^{k} \left\langle \frac{\partial \tilde{L}}{\partial p^{(i)}}, \delta p^{(i)} \right\rangle + \sum_{i=0}^{k} \left\langle \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial p^{(i)}}, \delta p^{(i)} \right\rangle + \sum_{i=0}^{k} \left\langle \frac{\partial \tilde{L}}{\partial \sigma^{(i)}}, \delta \sigma^{(i)} \right\rangle + \sum_{i=0}^{k} \left\langle \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma^{(i)}}, \delta \sigma^{(i)} \right\rangle dt \\
= \sum_{i=0}^{k} \int_{0}^{T} \left\langle \frac{\partial \tilde{L}}{\partial p^{(i)}}, \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial p^{(i)}} \right\rangle dt + \sum_{i=0}^{k} \int_{0}^{T} \left\langle \frac{\partial \tilde{L}}{\partial \sigma^{(i)}}, \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma^{(i)}} \right\rangle dt.
\]
\( (I) \quad = \int_0^T \left\langle \sum_{i=0}^{k} (-1)^{(i)} \frac{d^{(i)}}{dt^{(i)}} \left( \frac{\partial \tilde{L}}{\partial p} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial p} \right), \delta p \right\rangle dt \quad \text{(*)} \\
\( (II) \quad = \int_0^T \left\langle \sum_{i=0}^{k-1} \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma^{(i)}}, \frac{D^{(i)}}{D t^{(i)}} \delta \sigma + \sum_{l=0}^{i-1} \frac{D^{(i)}}{D t^{(i)}} \left[ \tilde{B}(p(t), \delta p(t), \sigma^{(i-1-t)}(t)) \right] \right\rangle dt \\\n= \int_0^T \sum_{i=0}^{k-1} \left\langle \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma^{(i)}}, \frac{D^{(i)}}{D t^{(i)}} \delta \sigma \right\rangle dt \\
+ \int_0^T \sum_{i=0}^{k-1} \left\langle \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma^{(i)}} \sum_{l=0}^{i-1} \frac{D^{(i)}}{D t^{(i)}} [\tilde{B}(p(t), \delta p(t), \sigma^{(i-1-t)}(t))] \right\rangle dt \ .
\)

\( (1) \quad = \sum_{i=0}^{k-1} \int_0^T \left\langle \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma^{(i)}}, \frac{D^{(i)}}{D t^{(i)}} \delta \sigma \right\rangle dt = \sum_{i=0}^{k-1} \int_0^T \left\langle (-1)^{(i)} \frac{D^{(i)}}{D t^{(i)}} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma^{(i)}} \right), \delta \sigma \right\rangle dt.
\( (2) \quad = \sum_{i=0}^{k-1} \int_0^T \left\langle \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma^{(i)}}, \sum_{l=0}^{i-1} \frac{D^{(i)}}{D t^{(i)}} [\tilde{B}(p(t), \delta p(t), \sigma^{(i-1-t)})] \right\rangle dt \\
= - \sum_{i=0}^{k-1} \sum_{l=0}^{i-1} \int_0^T \left\langle (-1)^{(i)} \alpha \sigma^{(i-1-t)} \frac{D^{(i)}}{D t^{(i)}} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma^{(i)}} \right), \delta \sigma \right\rangle dt.
\)

Therefore,

\( (1) + (2) \quad = \sum_{i=0}^{k-1} \int_0^T \left\langle (-1)^{(i)} \frac{D^{(i)}}{D t^{(i)}} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma^{(i)}} \right), \delta \sigma \right\rangle \quad - \sum_{l=0}^{i-1} \int_0^T \left\langle (-1)^{(i)} \alpha \sigma^{(i-1-t)} \frac{D^{(i)}}{D t^{(i)}} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma^{(i)}} \right), \delta \sigma \right\rangle dt \\
= \sum_{i=0}^{k-1} \int_0^T \left\langle (-1)^{i} \frac{D^{(i)}}{D t^{(i)}} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma^{(i)}} \right), \frac{D}{D t} \eta(t) + [\sigma(t), \eta(t)] + \tilde{B}(\delta p(t), \dot{p}(t)) \right\rangle dt \\
\quad - \sum_{i=0}^{k-1} \sum_{l=0}^{i-1} \int_0^T \left\langle (-1)^{(i)} \alpha \sigma^{(i-1-t)} \frac{D^{(i)}}{D t^{(i)}} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma^{(i)}} \right), \delta \sigma \right\rangle dt \ .
\)

Finally,
(a) \[ = \sum_{i=0}^{k-1} \int_0^T \left\langle \left( -\frac{D}{Dt} + ad^*_p \right) (-1)^i \frac{D(i)}{Dt(i)} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma^{(i)}} \right) \right\rangle ; \eta \]

\[- \left\langle \left( -1 \right)^i \frac{D(i)}{Dt(i)} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma^{(i)}} \right) ; i_\tilde{B} \delta p \right\rangle dt.\]

(b) \[ = - \sum_{i=0}^{k-1} \sum_{l=0}^{i-1} \left\langle (-1)^l ad^*_\sigma(-1-l) \frac{D(i)}{Dt(i)} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma^{(i)}} \right) ; i_\tilde{B} \delta p \right\rangle dt\]

Then joining (a), (b) and \((*1)\) we obtain the equations of motion for the higher-order Lagrangian system with higher-order constraints determined by the reduced Lagrangian \(\tilde{L} : T^{(k)}\tilde{Q} \oplus k\tilde{g} \to \mathbb{R}\) and the reduced constraints \(\tilde{\phi}^\alpha : T^{(k)}\tilde{Q} \oplus k\tilde{g} \to \mathbb{R}\). These equations are

\[ \sum_{i=0}^k (-1)^i \frac{D(i)}{dt(i)} \left( \frac{\partial \tilde{L}}{\partial p^{(i)}} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial p^{(i)}} \right) = \left\langle \sum_{i=0}^{k-1} \left( (-1)^i \frac{D(i)}{Dt(i)} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma^{(i)}} \right) \right) \right\rangle + \sum_{l=0}^{i-1} (-1)^l ad^*_\sigma(-1-l) \frac{D(i)}{Dt(i)} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma^{(i)}} \right) \right\rangle ; i_\tilde{B}\]

\[ 0 = \left( \frac{D}{Dt} - ad^*_\sigma \right) \sum_{i=0}^{k-1} (-1)^i \frac{D(i)}{Dt(i)} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma^{(i)}} \right). \]

The following theorem summarize the results that has been obtained in this section:

**Theorem 3.7.** Let \(L : T^{(k)}Q \to \mathbb{R}\) be a \(G\)-invariant Lagrangian and \(\tilde{\phi}^\alpha : T^{(k)}Q \to \mathbb{R}, \alpha = 1, \ldots, m\) be the \(G\)-invariant constraints. Consider the principal \(G\)-bundle \(\pi : Q \to Q/G\) and choose a principal connection \(\mathcal{A}\) on \(Q\). Let \(\tilde{L} : T^{(k)}(Q/G) \oplus_{Q/G} k\tilde{g} \to \mathbb{R}\) and \(\tilde{\phi}^\alpha : T^{(k)}(Q/G) \oplus_{Q/G} k\tilde{g} \to \mathbb{R}\) be the reduced higher-order Lagrangian and the reduced higher-order constraints associated with the principal connection. Then a curve \(c(t) = (p(t), \dot{p}(t), \ldots, p^{(k)}(t), \sigma(t), \dot{\sigma}(t), \ldots, \sigma^{(k)}(t)) \in C^\infty(T^{(k)}(Q/G) \oplus_{Q/G} k\tilde{g})\) satisfies \(\delta S(c(t)) = 0\) for the action \(S : C^\infty(T^{(k)}(Q/G) \oplus_{Q/G} k\tilde{g}) \to \mathbb{R}\) given by

\[ S(c(t)) = \int_0^T \tilde{L}(p(t), \dot{p}(t), \ldots, p^{(k)}(t), \sigma(t), \dot{\sigma}(t), \ldots, \sigma^{(k)}(t)) dt \]

with respect to the horizontal variations \(\delta p\) satisfying the endpoints condition \(\delta p^{(j)}(0) = \delta p^{(j)}(T) = 0\) for \(j = 0, \ldots, k-1\) and the vertical variations

\[ \delta \sigma^{(j)} = \frac{D^j}{Dt^j} \delta \sigma + \sum_{l=0}^{j-1} \frac{D^l}{Dt^l} \left[ \tilde{B}(\dot{p}, \delta p) , \sigma^{(j-l-1)} \right] \]

with \(j = 0, \ldots, k-1\) and \(\delta \sigma = \frac{D}{Dt} \Xi + [\sigma, \Xi] + \tilde{B}(\dot{p}, \delta p)\) where \(\Xi\) is an arbitrary curve in \(\tilde{g}\) satisfying that \(\frac{D^j}{Dt^j} \Xi\) vanish at the endpoints for all \(j = 0, \ldots, k-1\); if and only if \(c(t)\) is a solution of the higher-order Lagrange-Poincaré equations with higher-order constraints given by
\[
\sum_{i=0}^{k} (-1)^{(i)} \frac{d^{(i)}}{dt^{(i)}} \left( \frac{\partial \tilde{L}}{\partial \dot{p}^{(i)}} + \lambda_{\alpha} \frac{\partial \tilde{\phi}^{\alpha}}{\partial \dot{p}^{(i)}} \right) = \left\{ \sum_{i=0}^{k-1} (-1)^{i} \frac{D^{(i)}}{dt^{(i)}} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} + \lambda_{\alpha} \frac{\partial \tilde{\phi}^{\alpha}}{\partial \sigma^{(i)}} \right) \right. \\
+ \left. \sum_{i=0}^{k-1} (-1)^{i} \left[ \sum_{l=0}^{i} (-1)^{l} \text{ad}_{\sigma}^{l} \right] \frac{D^{(i)}}{dt^{(i)}} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} + \lambda_{\alpha} \frac{\partial \tilde{\phi}^{\alpha}}{\partial \sigma^{(i)}} \right) \right\}_p \tilde{B} \right)
0 = \left( \frac{D}{dt} - \text{ad}_{\sigma}^{k} \right) \sum_{i=0}^{k-1} (-1)^{i} \frac{D^{(i)}}{dt^{(i)}} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} + \lambda_{\alpha} \frac{\partial \tilde{\phi}^{\alpha}}{\partial \sigma^{(i)}} \right),
0 = \tilde{\phi}^{\alpha}(p(t), \dot{p}(t), \ldots, p^{(k)}(t), \sigma(t), \dot{\sigma}(t), \ldots, \sigma^{(k)}(t)),
\]
for \(\alpha = 1, \ldots, m\); and where \(\lambda_{\alpha}\) are the Lagrange multipliers.

**Corollary 3.8.** If \(Q\) is a Lie group \(G\) then the motion of the system is given by higher-order Euler-Poincaré equations for systems with higher-order constraints [15],
\[
\begin{align*}
0 &= \left( \frac{d}{dt} - \text{ad}_{\sigma}^{k} \right) \sum_{i=0}^{k-1} (-1)^{i} \frac{d^{i}}{dt^{i}} \left( \frac{\partial \tilde{L}}{\partial \sigma} + \lambda_{\alpha} \frac{\partial \tilde{\phi}^{\alpha}}{\partial \sigma} \right), \\
0 &= \tilde{\phi}^{\alpha}(c(t)),
\end{align*}
\]
where \(c(t) = (\sigma(t), \dot{\sigma}(t), \ldots, \sigma^{(k-1)}(t)) \in C^{\infty}(k \mathfrak{g})\).

**Proof:** By remark (2.4) the adjoint bundle \(k \tilde{g}\) is identify with \(k \mathfrak{g}\) via the isomorphism \(\alpha^{(k)}_{\mathcal{A}} : T^{(k)}G/G \to k \mathfrak{g}\) and then one can obtain the higher-order reduced lagrangian and the higher-order reduced constraints \(\tilde{L} : k \mathfrak{g} \to \mathbb{R}\) and \(\tilde{\phi}^{\alpha} : k \mathfrak{g} \to \mathbb{R}; \alpha = 1, \ldots, m\). Applying Hamilton's principle for the extended Lagrangian \(\tilde{L} = \tilde{L} + \lambda_{\alpha} \tilde{\phi}^{\alpha}\) with the Lagrange multipliers \(\lambda_{\alpha}\) and choosing the variations \(\delta \sigma = \tilde{\Xi} + [\sigma, \tilde{\Xi}]\), where \(\tilde{\Xi}\) is a fix curve in the Lie algebra \(\mathfrak{g}\) with fixed endpoints and \(\delta \sigma^{(l)} = \frac{d^{l}}{dt^{l}} (\delta \sigma)\), for \(l = 1, \ldots, k - 1\); the critical points of the action integral defined by the Lagrangian \(\tilde{L}\) are the solutions of the higher-order Euler-Poincaré equations with higher-order constraints given by
\[
\begin{align*}
0 &= \left( \frac{d}{dt} - \text{ad}_{\sigma}^{k} \right) \sum_{i=0}^{k-1} (-1)^{i} \frac{d^{i}}{dt^{i}} \left( \frac{\partial \tilde{L}}{\partial \sigma} + \lambda_{\alpha} \frac{\partial \tilde{\phi}^{\alpha}}{\partial \sigma} \right), \\
0 &= \tilde{\phi}^{\alpha}(c(t)).
\end{align*}
\]

When the curvature is zero is also a very interesting case. We recover the higher-order Euler-Lagrange equations with higher-order constraints for the variables \((p, \dot{p}, \ldots, p^{(k)})\) and the higher-order Euler-Poincaré equations with higher-order constraints for the variables \((\sigma, \dot{\sigma}, \ldots, \sigma^{(k-1)})\) as in [13] and [15] and they are the higher-order Euler-Lagrange equations with higher-order constraints for higher-order Lagrangians with higher order constraints defined on a trivial principal bundle given by
Remark 3.9. Observe that in the first order case we obtain the Lagrange-Poincaré equations with higher-order Euler-Lagrange equations with higher-order constraints. When we take a right trivialization the higher-order Lagrange-Poincaré equations are

\[
\frac{\partial \tilde{L}}{\partial \dot{p}} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \dot{p}} - \frac{d}{dt} \left( \frac{\partial \tilde{L}}{\partial \dot{p}} \right) - \lambda_\alpha \dot{\lambda}_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial p} - \lambda_\alpha \frac{d}{dt} \left( \frac{\partial \tilde{\phi}^\alpha}{\partial p} \right) = \left\langle \frac{\partial \tilde{L}}{\partial \sigma} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma} ; i_\beta \tilde{B} \right\rangle
\]

In the second-order case the equations of motion, called, second order Lagrange-Poincaré equations with second-order constraints, are

\[
0 = \left( \frac{D}{Dt} - a^\sigma \right) \left( \frac{\partial \tilde{L}}{\partial \sigma} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma} - \frac{D}{Dt} \left( \frac{\partial \tilde{L}}{\partial \sigma} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma} \right) \right)
\]

Remark 3.10. When we take a right trivialization the higher-order Lagrange-Poincaré equations with higher-order constraints are written as

\[
\sum_{i=0}^{k} (-1)^{i} \frac{d^{i}(\tilde{L})}{dt^{i}} \left( \frac{\partial \tilde{L}}{\partial \dot{p}^{(i)}} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \dot{p}^{(i)}} \right) = \left\langle \sum_{i=0}^{k-1} (-1)^{i} \frac{D^{(i)}}{Dt^{(i)}} \left( \frac{\partial \tilde{L}}{\partial \sigma^{(i)}} + \lambda_\alpha \frac{\partial \tilde{\phi}^\alpha}{\partial \sigma^{(i)}} \right) \right\rangle
\]

with \( \alpha = 1, \ldots, m \); and where \( \lambda_\alpha \) are the Lagrange multipliers. Moreover, when \( G = \{e\} \) we obtain the higher-order Euler-Lagrange equations with higher-order constraints [13].
with \( \alpha = 1, \ldots, m \); and where \( \lambda_\alpha \) are the Lagrange multipliers since the variations of \( \sigma \) takes the form

\[
\delta \sigma^{(j)} = \frac{D^j}{Dt^j} \delta \sigma - \sum_{l=0}^{j-1} \frac{D^l}{Dt^l} \tilde{B}(\tilde{p}, \delta \tilde{p}), \sigma^{(j-l-1)}
\]

with \( j = 0, \ldots, k - 1 \).

## 4 Optimal control of underactuated mechanical systems

The proposal of this section is to study optimal control problems in the case of underactuated mechanical systems, that is, a regular Lagrangian control system such that the number of the control inputs is less than the dimension of the configuration space (“superarticulated mechanical system” following the nomenclature given in [2]).

We shall consider the configuration space is \( TQ/G \cong T(Q/G) \oplus \text{Ad}Q \) where, as in the previous section, \( G \) is a Lie group, \( Q \) is a \( n \)-dimensional differentiable manifold and \( \text{Ad}Q := \tilde{g} \) is the adjoint bundle associated with the principal bundle \( \pi : Q \rightarrow Q/G \). In what follows we assume that all the control systems are controllable, that is, for any two points \( q_0 \) and \( q_f \) in the configuration space, there exists an admissible control \( u(t) \) defined on some interval \([0, T]\) such that the system with initial condition \( q_0 \) reaches the point \( q_f \) in time \( T \) (see [3, 6] for more details).

Let \( A : TQ \rightarrow \mathfrak{g} \) be a fixed connection in the principal bundle \( \pi : Q \rightarrow Q/G \) and \( B \) be the curvature 2-form associated with the connection \( A \). Using the principal connection \( A \) one may identify the vector bundles \( TQ/G \rightarrow Q/G := \hat{Q} \) and \( T\hat{Q} \oplus \tilde{g} \rightarrow \hat{Q} \) via the isomorphism \( \alpha_A^{(1)} : TQ/G \rightarrow T\hat{Q} \oplus \tilde{g} \). Also using the principal connection \( A \), we can obtain a local basis of \( \Gamma(TQ/G) \cong \Gamma(T(Q/G) \oplus \tilde{g}) \cong \mathfrak{X}(Q/G) \oplus \Gamma(\tilde{g}) \) (for more details see [34]).

### 4.1 Optimal control problem

Define the control manifold \( U \subset \mathbb{R}^r \), \( r < n \) where \( u(t) \in U \) is the control parameter. Consider the reduce Lagrangian \( L : T(Q/G) \oplus \tilde{g} \rightarrow \mathbb{R} \), then the equations of motion of the system shall be considered the controlled Lagrange-Poincaré equations

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{p}^A} \right) - \frac{\partial L}{\partial p^A} + \left( \frac{\partial L}{\partial \sigma} \right) \dot{p} \frac{\partial}{\partial \sigma} + \eta \left( \frac{\partial L}{\partial \sigma} \right) \left[ \eta^a([q]_{q_0}) \right]_G,
\]

\[
\frac{D}{Dt} - \text{ad}^*_{\sigma} \frac{\partial L}{\partial \sigma} = u_a \eta^a \left( \frac{\partial L}{\partial \sigma} \right) \left[ [q]_{q_0} \right]_G,
\]

where we denote by \( \eta = \eta^a \oplus \tilde{\eta}^a \in \Gamma(T^*_{\sigma} \otimes (Q/G) \oplus \tilde{g}^*) \); \( \eta^a([q]_{q_0}) \in \Gamma_{\sigma}^{(0)}(Q/G); \tilde{\eta}^a([q]_{q_0}) \in \tilde{g}^*; a = 1, \ldots, r \); and \( A = 1, \ldots, n \). Here we assuming that \( \mathcal{B}^a = \{ (\eta^a, \tilde{\eta}^a) \} \), are independent elements of \( \Gamma(T^* \otimes (Q/G) \oplus \tilde{g}^*) \) and \( u_a \) are admissible controls.

Taking this into account, the optimal control problem can be formulated as: find a trajectory \( \gamma(t) = (p(t), \sigma(t), u(t)) \) of the state variables and control inputs satisfying (4.1), subject to initial conditions \( (p(0), \dot{p}(0), \sigma(0)) \) and final conditions \( (p(T), \dot{p}(T), \sigma(T)) \) and extremizing the functional

\[
\mathcal{J}(p, \dot{p}, \sigma, u) = \int_0^T C(p(t), \dot{p}(t), \sigma(t), u(t)) \, dt,
\]
where \( C : (T(Q/G) \oplus \tilde{g}) \times U \rightarrow \mathbb{R} \) is the cost function.

We can reformulate this optimal control problem as a higher-order order variational problem subject to higher-order constraints by the following procedure: we complete \( B^\alpha \) to be a basis of \( \Gamma(T^*(Q/G) \oplus \tilde{g}^*) \), namely \( \{ B^\alpha, B^\alpha_0 \} \), and take its dual basis \( \{ B_\alpha, B_\alpha_0 \} \) on \( \Gamma(T(Q/G) \oplus \tilde{g}) \). If we denote by \( B_\alpha = \{(\eta, \tilde{\eta})\} \in \Gamma(T(Q/G) \oplus \tilde{g}; B_\alpha = \{(\eta, \tilde{\eta})\} \in \Gamma(T(Q/G) \oplus \tilde{g}) \) where \( \eta, \tilde{\eta} \in \mathfrak{X}(Q/G); \tilde{\eta}, \tilde{\eta} \in \tilde{g} \); then the equations of motion with control are now rewritten as

\[
\begin{align*}
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{p}^A} \right) - \frac{\partial L}{\partial p^A} + \left( \frac{\partial L}{\partial \dot{\sigma}} ; i_p \tilde{B} \right) \eta^A_A + \left( \frac{D}{Dt} - ad^\sigma \right) \frac{\partial L}{\partial \dot{\sigma}} \tilde{\eta}^A_A &= u_a \\
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{p}^A} \right) - \frac{\partial L}{\partial p^A} + \left( \frac{\partial L}{\partial \dot{\sigma}} ; i_p \tilde{B} \right) \eta^A_A + \left( \frac{D}{Dt} - ad^\sigma \right) \frac{\partial L}{\partial \dot{\sigma}} \tilde{\eta}^A_A &= 0.
\end{align*}
\]

(4.3)

As mentioned before, the proposed optimal control problem is equivalent to a variational problem with second-order constraints (see [3] and reference therein), where we define the second-order Lagrangian \( \tilde{L} : T^{(2)}(Q/G) \times 2\tilde{g} \rightarrow \mathbb{R} \) given by

\[
\tilde{L}(p^A, \dot{p}^A, \ddot{p}^A, \sigma, \dot{\sigma}) = C(p^A, \dot{p}^A, \sigma, \dot{\sigma}),
\]

where \( C \) is the cost function considered in (4.2) and

\[
F_a(p^A, \dot{p}^A, \ddot{p}^A, \sigma, \dot{\sigma}) = \left( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{p}^A} \right) - \frac{\partial L}{\partial p^A} + \left( \frac{\partial L}{\partial \dot{\sigma}} ; i_p \tilde{B} \right) \right) \eta^A_A + \left( \frac{D}{Dt} - ad^\sigma \right) \frac{\partial L}{\partial \dot{\sigma}} \tilde{\eta}^A_A.
\]

(4.4)

The Lagrangian \( \tilde{L} \) is subjected to the second-order constraints:

\[
\Phi^\alpha(p^A, \dot{p}^A, \ddot{p}^A, \sigma, \dot{\sigma}) = \left( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{p}^A} \right) - \frac{\partial L}{\partial p^A} + \left( \frac{\partial L}{\partial \dot{\sigma}} ; i_p \tilde{B} \right) \right) \eta^A_A + \left( \frac{D}{Dt} - ad^\sigma \right) \frac{\partial L}{\partial \dot{\sigma}} \tilde{\eta}^A_A.
\]

(4.5)

**Remark 4.1.** It is possible to extend our analysis to systems with external forces \( f \) given by the following diagram

\[
\begin{array}{c}
T(Q/G) \oplus \tilde{g} \xrightarrow{f} T^*(Q/G) \oplus \tilde{g}^* \\
\downarrow Q/G
\end{array}
\]

just by adding the corresponding terms in the right hand side of (4.1). These equations are therefore rewritten as

\[
\begin{align*}
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{p}^A} \right) - \frac{\partial L}{\partial p^A} + \left( \frac{\partial L}{\partial \dot{\sigma}} ; i_p \tilde{B} \right) &= u_a \mu_A([q]_{q_0}) + f_A(p, \dot{p}, \sigma) \quad (4.4a) \\
\left( \frac{D}{Dt} - ad^\sigma \right) \frac{\partial L}{\partial \dot{\sigma}} &= u_a \tilde{\eta}_A([q]_{q_0}) + \tilde{f}_i(p, \dot{p}, \sigma) \quad (4.4b)
\end{align*}
\]

where

\[
f : T(Q/G) \oplus \tilde{g} \rightarrow T^*(Q/G) \oplus \tilde{g}^* \\
p, \dot{p}, \sigma \rightarrow (f(p, \dot{p}, \sigma), \tilde{f}(p, \dot{p}, \sigma)),
\]

such that \( f(p, \dot{p}, \sigma) = f_A(p, \dot{p}, \sigma)dp^A \) and \( \tilde{f}(p, \dot{p}, \sigma) \in \tilde{g}^* \).
4.2 Optimal control of an underactuated Elroy’s beanie

This system is probably the more simple example of a dynamical system with a non-Abelian Lie group symmetries. It consists in two planar rigid bodies attached at their centers of mass, moving freely in the plane (see [36] and [40]).

**Configuration space:** The configuration space is \( Q = SE(2) \times S^1 \) with coordinates \((x, y, \theta, \psi)\), where the three first coordinates describe the position and orientation of the center of mass of the first body and the last one the relative orientation between both bodies.

**Lagrangian function:** We consider the Lagrangian

\[
L(x, y, \theta, \psi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\psi}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2(\dot{\theta} + \dot{\psi})^2 - V(\psi)
\]

where \( m \) denotes the mass of the system and \( I_1 \) and \( I_2 \) are the inertias of the first body and the second body, respectively; additionally, we also consider a potential function of the form \( V(\psi) \).

**Symmetry:** The symmetry group we consider is \( G = SE(2) \). The group action (left-action) is given by the (left) multiplication:

\[
\Phi_g(q) = \begin{pmatrix}
z_1 + x \cos \alpha - y \sin \alpha \\
z_2 + x \sin \alpha + y \cos \alpha \\
\alpha + \theta \\
\psi
\end{pmatrix} \in SE(2)
\]

where \( g = (z_1, z_2, \alpha) \in SE(2) \) and \( q = (x, y, \theta, \psi) \in Q \). Therefore the lifted action is,

\[
T_q \Phi_g(\dot{q}) = \begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 & 0 \\
\sin \alpha & \cos \alpha & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta} \\
\dot{\psi}
\end{pmatrix} = \begin{pmatrix}
\dot{x} \cos \alpha - \dot{y} \sin \alpha \\
\dot{x} \sin \alpha + \dot{y} \cos \alpha \\
\dot{\theta} \\
\dot{\psi}
\end{pmatrix},
\]

and the dual map \( T_q^*\Phi_g \) is given by

\[
T_q^*\Phi_g = (T_q \Phi_g)^T = \begin{pmatrix}
\cos \alpha & \sin \alpha & 0 & 0 \\
-\sin \alpha & \cos \alpha & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Obviously the Lagrangian \( L \) is \( G \)-invariant since \( L(\Phi_gq, T_q \Phi_g\dot{q}) = \frac{1}{2}m((\dot{x} \cos \alpha - \dot{y} \sin \alpha)^2 + (\dot{x} \sin \alpha - \dot{y} \cos \alpha)^2) + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2(\dot{\theta} + \dot{\psi})^2 - V(\psi) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2(\dot{\theta} + \dot{\psi})^2 - V(\psi) = L(q, \dot{q}) \).

Moreover, if \( g = (y_1, y_2, \alpha) \) and \( h = (z_1, z_2, \beta) \)

\[
L_gh = gh = (y_1 + z_1 \cos \alpha - z_2 \sin \alpha, y_2 + z_1 \sin \alpha + z_2 \cos \alpha, \alpha + \beta),
\]

\[
R_gh = hg = (z_1 + y_1 \cos \beta - y_2 \sin \beta, z_2 + y_1 \sin \beta + y_2 \cos \beta, \alpha + \beta).
\]

After some computations

\[
\text{Ad}_g(\xi) = \begin{pmatrix}
\cos \alpha & -\sin \alpha & -y_2 \\
\sin \alpha & \cos \alpha & y_1 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\xi_1 \\
\xi_2 \\
\xi_3
\end{pmatrix},
\]
where \( \xi = (\xi_1, \xi_2, \xi_3) \in g = se(2) \). The Lie algebra \( se(2) \) is represented by matrices of the form

\[
\hat{\xi} = \begin{pmatrix} 0 & -\xi_3 & \xi_1 \\ \xi_3 & 0 & \xi_2 \\ 0 & 0 & 0 \end{pmatrix},
\]
with basis

\[
e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

The Lie algebra structure on \( se(2) \) is defined by

\[
[e_1, e_2] = 0, \quad [e_1, e_3] = e_2, \quad [e_2, e_3] = -e_1.
\]

Therefore, for \( v = (v_1, v_2, v_3) \in \mathbb{R}^3 \), then

\[
ad_v = \begin{pmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ 0 & 0 & 0 \end{pmatrix},
\]

where \( \hat{v} = \begin{pmatrix} 0 & -v_3 & v_1 \\ v_3 & 0 & v_2 \\ 0 & 0 & 0 \end{pmatrix} \).

The exponential mapping which takes elements in \( se(2) \) in elements in \( SE(2) \) is

\[
\exp(\xi) = \left( \frac{\xi_1}{\xi_3} \sin \xi_3 + \frac{\xi_2}{\xi_3} (\cos \xi_3 - 1), \frac{\xi_2}{\xi_3} \sin \xi_3 + \frac{\xi_1}{\xi_3} (1 - \cos \xi_3), \xi_3 \right) \in SE(2)
\]

and \( \exp(\xi) = (\xi_1, \xi_2, 0) \) if \( \xi_3 = 0 \).

Therefore, if \( q = (x, y, \theta, \psi) \) then

\[
\xi_Q(q) = \left. \frac{d}{dt} \right|_{t=0} \Phi_{\exp(\xi)}(q) = \frac{d}{dt} \left|_{t=0} \left( x + \frac{\xi_2}{\xi_3} \cos(\xi_3 t) + \frac{\xi_1}{\xi_3} - y \right) \sin(\xi_3 t) - \frac{\xi_2}{\xi_3},
\right.
\]

\[
\left. \left( y - \frac{\xi_1}{\xi_3} \right) \cos(\xi_3 t) + \left( \frac{\xi_2}{\xi_3} + x \right) \sin(\xi_3 t) + \frac{\xi_1}{\xi_3}, \theta + \xi_3 t, \psi \right) = (\xi_1 - \xi_3 y, \xi_2 + \xi_3 x, \xi_3, 0).
\]

In particular, \( (e_1)_Q = (1, 0, 0, 0) = \frac{\partial}{\partial x_1} \), \( (e_2)_Q = (0, 1, 0, 0) = \frac{\partial}{\partial x_2} \) and

\[
(e_3)_Q = (-y, x, 1, 0) = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + \frac{\partial}{\partial \theta}.
\]

The quotient space \( M = Q/G = (SE(2) \times S^1)/SE(2) \simeq S^1 \) is naturally parameterized by coordinate \( \psi \). The projection \( \pi : Q \longrightarrow M \) is given in coordinates by \( \pi(x, y, \theta, \psi) = \psi \).

In this case

\[
VQ = \text{Ker}(T\pi) \text{span} \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial \theta} \right\}.
\]
**Mechanical connection:** Since the lagrangian $L$ is of mechanical type then the kinetic energy comes from a riemannian metric on $Q$:

$$g = mdx \otimes dx + mdy \otimes dy + (I_1 + I_2)d\theta \otimes d\theta + I_2d\psi \otimes d\psi + I_2d\theta \otimes d\psi + I_2d\psi \otimes d\theta.$$ 

Consider the decomposition

$$TQ = VQ \oplus HQ,$$

where $HQ$ is the $g$-orthogonal complement of $VQ$. Observe that

$$HQ = \text{span} \left\{ \frac{\partial}{\partial \psi} - \frac{I_2}{I_1 + I_2} \frac{\partial}{\partial \theta} \right\}.$$

The corresponding connection one form $A_c(q) \in \mathfrak{se}(2)$ is

$$A_c(v_x, v_y, v_\theta, v_\psi) = \left( v_x + \frac{I_2}{I_1 + I_2} v_\psi y, v_y - \frac{I_2}{I_1 + I_2} v_\theta x, v_\theta + \frac{I_2}{I_1 + I_2} v_\psi \right),$$

or in matrix form

$$A_c = \begin{pmatrix} 1 & 0 & y \frac{I_2}{I_1 + I_2} y \\ 0 & 1 & -x \frac{I_2}{I_1 + I_2} x \\ 0 & 0 & 1 \frac{I_2}{I_1 + I_2} \end{pmatrix}.$$

We know that

$$A_c(\dot{q}) = \text{Ad}_g(g^{-1}\dot{q} + A(\psi)\dot{\psi}).$$

Therefore,

$$\text{Ad}_{g^{-1}}A_c = \begin{pmatrix} \cos \theta & \sin \theta & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & \frac{I_2}{I_1 + I_2} \end{pmatrix}$$

and

$$A(\psi) = \begin{pmatrix} 0 \\ 0 \\ \frac{I_2}{I_1 + I_2} \end{pmatrix}.$$

**Reduced lagrangian:** Given the connection $A_c$ then we define the following vector bundle isomorphism:

$$\rho_{A_c} : T(SE(2) \times S^1)/SE(2) \to TS^1 \oplus \widehat{SE(2)} [v_q]_G \to T\pi(v_q) \oplus [(q, A_c(v_q))]_G.$$ 

In coordinates,

$$\rho_{A_c}(\psi, \dot{\psi}, \xi) = (\psi, \dot{\psi}, \Omega) = (\psi, \dot{\psi}, \xi + A(\psi)\dot{\psi}).$$

Therefore, $\Omega_1 = \xi_1$, $\Omega_2 = \xi_2$ and $\Omega_3 = \xi_3 + \frac{I_2}{I_1 + I_2} \dot{\psi}$.

Then

$$l^R(\psi, \dot{\psi}, \xi) = \frac{1}{2}m(\xi_1^2 + \xi_2^2 + \frac{1}{2}I_1 \xi_3^2 + \frac{1}{2}I_2 (\xi_3 + \dot{\psi})^2) - V(\psi),$$

or in $(\psi, \dot{\psi}, \Omega)$—coordinates,

$$l^R(\psi, \dot{\psi}, \Omega) = \frac{1}{2}m(\Omega_1^2 + \Omega_2^2) + \frac{1}{2}(I_1 + I_2)\Omega_3^2 + \frac{1}{2}I_1 I_2 \dot{\psi}^2 - V(\psi).$$
Finally, in terms of the matrix $\hat{\Omega}$,

$$l^R(\psi, \dot{\psi}, \Omega) = \frac{1}{4} \text{Tr}(\hat{\Omega}\hat{\Omega}^T) + \frac{1}{2} \frac{I_1 I_2}{I_1 + I_2} \dot{\varphi}^2 - V(\psi),$$  

where

$$\mathbb{I} = \begin{pmatrix} I_1 + I_2 & 0 & 0 \\ 0 & I_1 + I_2 & 0 \\ 0 & 0 & 2m \end{pmatrix}. $$

Reduced equations: The reduced equations in $(\psi, \dot{\psi}, \xi)$ coordinates are:

$$\begin{align*}
\dot{\xi}_1 &= \xi_2 \xi_3 \\
\dot{\xi}_2 &= -\xi_1 \xi_3 \\
(I_1 + I_2) \dot{\xi}_3 + I_2 \ddot{\psi} &= 0 \\
I_2 (\dot{\xi}_3 + \ddot{\psi}) &= -\frac{\partial V}{\partial \psi} 
\end{align*}$$

(4.6)

and in $(\psi, \dot{\psi}, \Omega)$ coordinates:

$$\begin{align*}
\dot{\Omega}_1 &= \Omega_2 \Omega_3 - \frac{I_2}{I_1 + I_2} \dot{\psi} \Omega_2 \\
\dot{\Omega}_2 &= -\Omega_1 \Omega_3 + \frac{I_2}{I_1 + I_2} \dot{\psi} \Omega_1 \\
\dot{\Omega}_3 &= 0 \\
\frac{I_1 I_2}{I_1 + I_2} \ddot{\psi} &= -\frac{\partial V}{\partial \psi} + u_1 
\end{align*}$$

Reduced equations with controls: A reasonable reduced equations of motion with controls for the underactuated Elroy’s Beanie system are given by

$$\begin{align*}
\dot{\xi}_1 &= \xi_2 \xi_3 \\
\dot{\xi}_2 &= -\xi_1 \xi_3 \\
(I_1 + I_2) \dot{\xi}_3 + I_2 \ddot{\psi} &= 0 \\
I_2 (\dot{\xi}_3 + \ddot{\psi}) &= -\frac{\partial V}{\partial \psi} + u_1 
\end{align*}$$

(4.7)

and in $(\psi, \dot{\psi}, \Omega)$ coordinates:

$$\begin{align*}
\dot{\Omega}_1 &= \Omega_2 \Omega_3 - \frac{I_2}{I_1 + I_2} \dot{\psi} \Omega_2 \\
\dot{\Omega}_2 &= -\Omega_1 \Omega_3 + \frac{I_2}{I_1 + I_2} \dot{\psi} \Omega_1 \\
\dot{\Omega}_3 &= 0 \\
\frac{I_1 I_2}{I_1 + I_2} \ddot{\psi} &= -\frac{\partial V}{\partial \psi} + u_1 
\end{align*}$$

In the next we only use the second system of differential equations for our optimal control problem. The optimal control problem consist on finding a trajectory of the states variables and controls inputs, satisfying the previous equations given initial and final conditions and minimize the cost functional,

$$\text{extremize} \int_0^T \frac{1}{2} u_1^2 dt.$$
This optimal control problem is equivalent to solve the following second-order variational problem with second-order constraints given by minimize $\tilde{L} : T^{(2)}S^1 \oplus 2SE(2) \rightarrow \mathbb{R}$

$$\min \tilde{L}(\psi, \dot{\psi}, \ddot{\psi}, \Omega, \dot{\Omega}) = C \left( \psi, \dot{\psi}, \Omega, \frac{I_1 I_2}{I_1 + I_2} \ddot{\psi} + \frac{\partial V}{\partial \psi} \right),$$

subject the second-order constraints $\Phi^\alpha : T^{(2)}S^1 \oplus 2SE(2) \rightarrow \mathbb{R}$, $\alpha = 1, 2, 3$; 

$$\Phi^1(\psi, \dot{\psi}, \ddot{\psi}, \Omega, \dot{\Omega}) = \dot{\Omega}_1 + \Omega_2 \Omega_3 + \frac{I_2}{I_1 + I_2} \ddot{\psi} \Omega_2 = 0,$$

$$\Phi^2(\psi, \dot{\psi}, \ddot{\psi}, \Omega, \dot{\Omega}) = \dot{\Omega}_2 - \Omega_1 \Omega_3 - \frac{I_2}{I_1 + I_2} \ddot{\psi} \Omega_1 = 0,$$

$$\Phi^3(\psi, \dot{\psi}, \ddot{\psi}, \Omega, \dot{\Omega}) = \dot{\Omega}_3 = 0.$$

Now, we construct the augmented Lagrangian with the Lagrangian multipliers $\lambda_\alpha$, $\tilde{L} = \tilde{L} + \lambda_\alpha \Phi^\alpha$,

$$\tilde{L}(\psi, \dot{\psi}, \ddot{\psi}, \Omega, \dot{\Omega}) = \frac{1}{2} \left( \frac{I_1 I_2}{I_1 + I_2} \ddot{\psi} + \frac{\partial V}{\partial \psi} \right)^2 + \lambda_1 \left( \dot{\Omega}_1 + \Omega_2 \Omega_3 + \frac{I_2}{I_1 + I_2} \ddot{\psi} \Omega_2 \right)$$

$$+ \lambda_2 \left( \dot{\Omega}_2 - \Omega_1 \Omega_3 - \frac{I_2}{I_2 + I_3} \ddot{\psi} \Omega_1 \right) + \lambda_3 \dot{\Omega}_3.$$

The second-order Lagrange-Poincaré equations with second-order constraints for $\tilde{L}$ are

$$\left( \frac{D}{Dt} - ad^*_\Omega \right) \left( \lambda_2 (\ddot{\psi} + I_1 I \ddot{\psi}) + \dot{\lambda}_1, \lambda_1 (I_1 I \ddot{\psi} + \ddot{\psi}) - \lambda_2, \lambda_2 \Omega_1 + \dot{\lambda}_3 - \lambda_1 \Omega_2 \right) = 0$$

$$\left( I_1 I \ddot{\psi} + \frac{\partial V}{\partial \psi} \right) \frac{\partial^2 V}{\partial \psi^2} - \lambda_1 I \Omega_2 + \dot{\lambda}_2 I \Omega_1 - \lambda_1 I \dot{\Omega}_2 + \lambda_2 \left( I (\dot{\Omega}_1 + I \dot{\psi}) + \frac{D}{Dt} \left( \frac{\partial V}{\partial \psi} \right) ight)$$

$$= \langle \left( I_1 I \ddot{\psi} + \frac{\partial V}{\partial \psi} \right) \frac{\partial^2 V}{\partial \psi^2} - I \frac{D}{Dt} \left( \lambda_1 \Omega_2 - \lambda_2 \Omega_1 \right) + \lambda \times \Omega, t \dot{\psi} \dot{B} \rangle,$$

together

$$\dot{\Omega}_1 + \ddot{\Omega}_2 + I \ddot{\psi} \Omega_2 = 0,$$

$$\dot{\Omega}_2 - \ddot{\Omega}_1 - I \ddot{\psi} \Omega_1 = 0,$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $\frac{I_2}{I_1 + I_2} = I$ and $\ddot{\psi}$ is a constant given by integrate the equation $\dot{\Omega}_3 = 0$. Solving this set of four order differential equations it is possible to find the control that minimize the cost functional given initial and final conditions in terms of the configuration variables and its time derivatives.

### 4.3 Optimal control of a Snakeboard

In this example we will use nonholonomic reduction [11] for obtain the reduced equations of motion and after of it we will derive the associated optimal control problem for the snakeboard which will
be solved as a vakonomic system. We refer to [6], [31], [33] and [40] for a detailed description of this system.

The snakeboard is a modified version of a skateboard which consists on a rigid body (the board) with two sets of independently actuated wheels one on each end of the board and is allowed to spin about the vertical axis. The configuration manifold is the principal bundle \( \tilde{Q} = SE(2) \times S^1 \times S^1 \times S^1 \) with structure group \( SE(2) \) acting on the left with local coordinates \((x, y, \theta, \psi, \phi_1, \phi_2)\) where \((x, y, \theta)\) represents the position and orientation of the center of the board and the degree of freedom \( \psi \) move simultaneously the wheels with the proper phase relationship enables the rider to generate forward motions. For simplicity we assume that \( \phi_1 = -\phi_2 = \phi \). Therefore will be consider as configuration manifold \( Q = SE(2) \times T^2 \) with coordinates \((x, y, \theta, \psi, \phi)\).

### Lagrangian function:

The Lagrangian for the snakeboard consists only in kinetic energy terms. We take the simplest possible model for the various mass distribution and write the Lagrangian as

\[
L(x, y, \theta, \psi, \phi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\psi}, \dot{\phi}) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} (J + 2J_1) \dot{\theta}^2 + \frac{1}{2} J_0 (\dot{\psi} + \dot{\phi})^2 + \frac{1}{2} J_1 \dot{\phi}^2,
\]

where \( m \) denotes the the total mass of the board, \( J > 0 \) is the inertia of the board, \( J_0 > 0 \) the inertia of the rotor mounted on the center of mass of the body and \( J_1 > 0 \) is the inertia of each wheel axles. In the following we assume that \( J + J_0 + 2J_1 = mr^2 \) following [40]: it eliminates some terms in the derivation but does not affect the geometry of the problem.

The Lagrangian is independent of the board configuration and therefore it is invariant to all possible group actions.

### Constraints:

The rolling of the front and rear wheels of the snakeboard is modeled by using nonholonomic constraints which allows the wheels to spin about the vertical axis and roll in the direction that they are pointing. The wheels are not allowed to slide in the sideways directions. They are given by the one-forms at the point where \( q = (x, y, \theta, \psi, \phi) \)

\[
\begin{align*}
\omega_1(q) &= -\sin(\theta + \phi) dx + \cos(\theta + \phi) dy - r \cos \phi d\theta, \\
\omega_2(q) &= -\sin(\theta - \phi) dx + \cos(\theta - \phi) dy + r \cos \phi d\theta,
\end{align*}
\]

To avoid singularities of the distribution defined by the previous constraints we will assume, in the sequel, that \( \phi \neq \pm \pi/2 \). If we define the functions \( a(\theta, \phi) = -2r \cos^2 \phi \cos \theta, b(\theta, \phi) = -2r \cos^2 \phi \sin \theta, c(\phi) = \sin(2\phi) \), the constraint subbundle \( \mathcal{D} \) of \( TQ \) is

\[
\mathcal{D} = \text{span} \left\{ \frac{\partial}{\partial \psi}, \frac{\partial}{\partial \phi}, a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial \theta} \right\}.
\]

### Symmetry:

The symmetry group we consider is \( G = SE(2) \). The group action (left-action) is given by the (left) multiplication:

\[
\Phi_g(q) = \begin{pmatrix}
z_1 + x \cos \alpha - y \sin \alpha \\
z_2 + x \sin \alpha + y \cos \alpha \\
\alpha + \theta \\
\psi \\
\phi
\end{pmatrix}
\]
where \( g = (z_1, z_2, \alpha) \in SE(2) \) and \( q = (x, y, \theta, \psi, \phi) \in Q \). Therefore the lifted action is,

\[
T_q\Phi_g(\dot{q}) = \begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 & 0 & 0 \\
\sin \alpha & \cos \alpha & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{\phi} \\
\dot{\psi} \\
\dot{\phi}
\end{pmatrix} = \begin{pmatrix}
\dot{x} \cos \alpha - \dot{y} \sin \alpha \\
\dot{x} \sin \alpha + \dot{y} \cos \alpha
\end{pmatrix}
\]

and the dual map \( T^*_q\Phi_g \) is given by

\[
T^*_q\Phi_g = (T_q\Phi_g)^T = \begin{pmatrix}
\cos \alpha & \sin \alpha & 0 & 0 & 0 \\
-\sin \alpha & \cos \alpha & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

The Lagrangian \( L \) and the constraints \( \omega_i \) (\( i = 1, 2 \)) are \( SE(2) \)-invariant since

\[
L(\Phi_gq, T_q\Phi_g\dot{q}) = \frac{1}{2} m ((\dot{x} \cos \alpha - \dot{y} \sin \alpha)^2 + (\dot{x} \sin \alpha - \dot{y} \cos \alpha)^2)
+ \frac{1}{2} (J + 2J_1) \dot{\theta}^2 + \frac{1}{2} J_0 (\dot{\theta} + \dot{\psi})^2 + \frac{1}{2} J_1 \dot{\phi}^2
= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} (J + 2J_1) \dot{\theta}^2 + \frac{1}{2} J_0 (\dot{\theta} + \dot{\psi})^2 + \frac{1}{2} J_1 \dot{\phi}^2
= L(q, \dot{q}).
\]

**Reduced space and reduced lagrangian:** The quotient space \( M = Q/G = (SE(2) \times S^1 \times S^1)/SE(2) \simeq \mathbb{T}^2 \) is naturally parameterized by coordinates \((\psi, \phi)\). The projection \( \pi : Q \rightarrow M \) is given in coordinates by \( \pi(x, y, \theta, \psi, \phi) = (\psi, \phi) \).

In this case

\[
VQ = Ker(T\pi) = \text{span} \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial \theta} \right\}
\]

Since the lagrangian \( L \) is of mechanical type then the kinetic energy comes from a riemannian metric on \( Q \):

\[
G = m dx \otimes dx + m dy \otimes dy + m r^2 d\theta \otimes d\theta + J_0 d\psi \otimes d\psi + J_0 d\theta \otimes d\psi + J_0 d\psi \otimes d\theta + 2J_1 d\phi \otimes d\phi
\]

Consider the subbundle

\[
\mathcal{S} = D \cap VQ = \text{span} \left\{ a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial \theta} \right\}
\]

and therefore

\[
\mathcal{S}^\perp \cap \mathcal{D} = \text{span} \left\{ \frac{\partial}{\partial \psi}, \frac{\partial}{\partial \psi} - \frac{J_0 c}{k} \left( a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial \theta} \right) \right\}
\]

where \( k = m(a^2 + b^2 + c^2 r^2) = 4mr^2(\cos^2 \phi) \) away form \( \phi = \pm \pi/2 \).

Given the connection \( \mathcal{A}_c \) then we define the following vector bundle isomorphism:

\[
\rho_{\mathcal{A}_c} : T(SE(2) \times \mathbb{T}^2)/SE(2) \rightarrow T\mathbb{T}^2 \oplus \tilde{SE}(2)
\]

\[
[v_q]_G \mapsto T\pi(v_q) \oplus [(q, \mathcal{A}_c(v_q))]_G
\]
where the local form of the mechanical connection is given by

\[ \mathcal{A}_c = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{J_0}{mr^2} & 0 \end{pmatrix}. \]

In order to fully specify the motion of this mechanical system along the fiber, we need to add an equation governing the momentum. The third constraint needed is provided by the generalized momentum itself, since the motion of the system is required to flow along the unconstrained fiber directions in a manner consistent with this momentum. This is given by

\[ p = -2mr \cos^2 \phi \cos \theta \dot{x} - 2mr \cos^2 \phi \sin \theta \dot{y} + mr^2 \sin 2\phi \dot{\theta} + J_0 \sin 2\phi \dot{\psi}. \]

Now, in terms of the reduced variables \( \xi \), after some computations, we can rewrite the constraints as

\[ \xi = - \begin{pmatrix} \frac{-J_0}{2mr} \sin 2\phi & 0 \\ 0 & 0 \\ \frac{J_0}{mr^2} \sin^2 \phi & 0 \end{pmatrix} \begin{pmatrix} \dot{\psi} \\ \dot{\phi} \end{pmatrix} + \begin{pmatrix} -\frac{1}{2mr} \\ 0 \\ \frac{1}{2mr^2} \tan \phi \end{pmatrix} p. \]

Therefore the nonholonomic connection \( \mathcal{A} \) is

\[ \begin{pmatrix} -\frac{J_0}{2mr} \sin 2\phi & 0 \\ 0 & 0 \\ \frac{J_0}{mr^2} \sin^2 \phi & 0 \end{pmatrix} \]

according to [40].

Using the angular momentum \( \Omega \), we can rewrite the constrains as

\[ \begin{align*}
\xi_1 &= \frac{J_0}{2mr} \sin 2\phi \dot{\psi} - 2r \cos^2 \phi \Omega, \\
\xi_2 &= 0, \\
\xi_3 &= -\frac{J_0}{mr^2} \sin^2 \phi \dot{\psi} + \sin 2\phi \Omega.
\end{align*} \]

Therefore,

\[ l^R(\psi, \dot{\psi}, \phi, \dot{\phi}, \xi) = \frac{1}{2} m (\xi_1^2 + \xi_2^2) + \frac{1}{2} mr^2 \xi_3^2 + \frac{1}{2} J_0 \dot{\psi}^2 + J_1 \dot{\phi}^2 + J_0 \dot{\psi} \xi_3^2 \]

or in \((\psi, \dot{\psi}, \Omega)\)–coordinates

\[ l^R(\psi, \dot{\psi}, \phi, \dot{\phi}, \Omega) = -\frac{J_0^2}{2mr^2} \sin^2 \phi \dot{\psi}^2 + 2mr^2 \cos^2 \phi \Omega^2 + \frac{1}{2} J_0 \dot{\psi}^2 + J_1 \dot{\phi}^2. \]

**Reduced equations:** The reduced equations with controls in \((\psi, \dot{\psi}, \phi, \dot{\phi}, \xi)\)–coordinates are:

\[ 2J_1 \ddot{\phi} = u_\phi, \]

\[ \left( J_0 - \frac{J_0^2}{mr^2} \sin^2 \phi \right) \ddot{\psi} - \frac{J_0^2}{2mr^2} \sin(2\phi) \dot{\phi} \dot{\psi} - \frac{J_0}{2mr^2} \dot{\phi} (\sin(2\phi) (mr^2 \dot{\xi}_3 + J_0 \dot{\psi}) - 2mr \xi_1 \cos^2 \phi) = u_\psi, \]

\[ -\xi_1 \sin \phi + \xi_2 \cos \phi - r \xi_3 \cos \phi = 0, \]

\[ \xi_1 \sin \phi + \xi_2 \cos \phi + r \xi_3 \cos \phi = 0, \]

\[ 2J_0 \cos^2 \phi \dot{\phi} \dot{\psi} - \tan \dot{\phi} (\sin(2\phi) (mr^2 \dot{\xi}_3 + J_0 \dot{\psi}) - 2mr \xi_1 \cos^2 \phi) = 0. \]
The optimal control problem consists of finding a trajectory of the system described by the initial value problem for the set of four order ordinary differential equations given by

\[ \dot{\phi} = J_1 \phi, \]
\[ \dot{J}_1 = J_0^2 \sin^2(\phi), \]
\[ u_\psi = \left( J_0 - \frac{J_0^2}{mr^2} \sin^2(\phi) \right) \psi - \frac{J_0^2}{2} \sin(2\phi) \left( \dot{\phi} \sin(2\phi) \Omega + \frac{\dot{\phi} \psi}{mr^2} \right) - 4mr^2 \cos^4 \phi \Omega \]
\[ + J_0 \sin(2\phi)(\mathcal{K}(\phi) \dot{\phi} \sin \phi - \cos^2 \phi \dot{\psi}), \]
\[ 0 = \mathcal{K}(\phi)(\cos \phi \Omega + I), \]
\[ 0 = J_0 \cos^2 \phi \dot{\psi}(2\dot{\phi} - \sin(2\phi)) - J_0 \tan \phi \dot{\phi} \dot{\psi}(1 - \sin(2\phi) \sin^2 \phi) - mr^2 \Omega(4 \cos^4 \phi - \tan \phi \dot{\phi} \sin^2(2\phi)). \]

where \( I(\phi, \psi, \phi, \dot{\phi}) = \frac{J_0}{2mr^2} \psi \sin \phi \) and \( \mathcal{K}(\phi) = 2 \cos \phi \sin \phi - \sin(2\phi). \)

**Optimal control problem:** The optimal control problem consists of finding a trajectory of the states variables and controls inputs, satisfying the previous equations given initial and final conditions and minimizing the cost functional,

\[ \text{minimize} \int_0^T \frac{1}{2} \left( u_\phi^2 + u_\psi^2 \right) dt. \]

This optimal control problem is equivalent to solving the following second-order variational problem with second-order constraints given by minimizing \( \tilde{L} : T^{(2)}T^2 \oplus 2SE(2) \to \mathbb{R} \)

\[ \tilde{L}(\psi, \phi, \phi, \dot{\psi}, \phi, \dot{\phi}, \psi, \dot{\phi}, \Omega, \dot{\Omega}) = 2J_1^2 \dot{\phi}^2 + \frac{1}{2} u_\psi^2 \]

subject to the second-order constraints \( \Phi^\alpha : T^{(2)}T^2 \oplus 2SE(2) \to \mathbb{R}, \alpha = 1, 2; \)

\[ \Phi^1(\psi, \phi, \dot{\psi}, \dot{\phi}, \Omega, \dot{\Omega}) = \mathcal{K}(\phi)(\cos \phi \Omega + I), \]
\[ \Phi^2(\psi, \phi, \dot{\psi}, \dot{\phi}, \Omega, \dot{\Omega}) = J_0 \cos^2 \phi \dot{\psi}(2\dot{\phi} - \sin(2\phi)) - J_0 \tan \phi \dot{\phi} \dot{\psi}(1 - \sin(2\phi) \sin^2 \phi) - mr^2 \Omega(4 \cos^4 \phi - \tan \phi \dot{\phi} \sin^2(2\phi)). \]

Now, we construct the augmented Lagrangian with the Lagrangian multipliers \( \lambda_\alpha, \tilde{L} = \tilde{L} + \lambda_\alpha \Phi^\alpha; \)

\[ \tilde{L}(\phi, \psi, \dot{\phi}, \dot{\psi}, \dot{\phi}, \Omega, \dot{\phi}, \dot{\Omega}) = 2J_1^2 \dot{\phi}^2 + \frac{1}{2} u_\psi^2 + \lambda_1 \mathcal{K}(\phi)(\cos \phi \Omega + I) - \lambda_2 mr^2 \Omega(4 \cos^4 \phi - \tan \phi \dot{\phi} \sin^2(2\phi)) \]
\[ + \lambda_2 \left( J_0 \cos^2 \phi \dot{\psi}(2\dot{\phi} - \sin(2\phi)) - J_0 \tan \phi \dot{\phi} \dot{\psi}(1 - \sin(2\phi) \sin^2 \phi) \right). \]

Therefore, the second-order Lagrange-Poincaré equations with second-order constraints for the extended Lagrangian \( \tilde{L} \) give rise to the solution for the optimal control problem through solving the initial value problem for the set of four order ordinary differential equations given by Theorem (3.7).

### 5 Conclusions and Future work

In this paper, we have considered higher-order regular Lagrangians systems with higher-order constraints both invariant under a symmetry of a Lie group and we have derived the reduced higher-order Euler-Lagrange equations with higher-order constraints, so called higher-order Lagrange
Poincaré equations. Our approach is based on classical Lagrangian reduction \cite{10} variationally given by Hamilton’s principle for this kind of higher-order mechanical systems. Moreover, we obtain higher-order Lagrange-Poincaré equations by introduce the higher-order Lagrange-Poincaré operator. We solve an optimal control problem for underactuated mechanical systems as an optimization problem and we had illustrate our ideas by applying them to two interesting and well know examples: an underactuated Elroy’s beanie and the optimal control of a snakeboard.

This work is the beginning of new developments in geometric mechanics on Lie algebroids and optimal control problems of mechanical on Lie algebroids. More precisely, in a future work \cite{14} the author plans to give a geometric setting for this class of optimal control problems working on the Atiyah algebroid to obtain the higher-order Lagrange-Poincaré equations on Lie algebroids and solve an optimal control problem for nonholonomic mechanical control systems. Also, the study of higher-order variational principles on Lie algebroids will be developed as the first order case in \cite{29} and \cite{38}, for the treatment of underactuated mechanical systems and controlled Lagrangians system \cite{4}. Higher-order symplectic reduction for optimal control problems will be studied following the ideas given in \cite{35}.

Finally, in a future work, the author plans to extend this construction to the development of variational integrator as in \cite{16} to the case of non-trivial fiber bundles using a connection to split the reduced space and a retraction map to discretize the variational principle.

Acknowledgments

The author wishes to thank D. Martín de Diego and S. Grillo for fruitful discussions and comments.

This work has been supported by MICINN (Spain) Grant MTM2010-21186-C02-01, MTM 2011-15725-E, ICMAT Severo Ochoa Project SEV-2011-0087 and IRSES-project ”Geomech-246981”.

L.C also wants to thank CSIC and JAE program for a JAE-Pre grant.

References

\cite{1} Abraham R and Marsden JE. *Foundations of Mechanics*. Benjamin/Cummings Publishing Co. Inc.
Advanced Book Program, Reading, Mass., second edition, (1978)

\cite{2} J. Baillieul. *The geometry of controlled mechanical systems*. Mathematical control theory, 322–354, Springer, New York (1999).

\cite{3} A.M. Bloch. *Nonholonomic Mechanics and Control*. Interdisciplinary Applied Mathematics Series, 24, Springer-Verlag, New York (2003).

\cite{4} A. Bloch, D. Chang, N. Leonard, J. Marsden. *Controlled Lagrangians and the stabilization of mechanical systems. II. Potential shaping*. Automatic Control, IEEE Transactions. Volume: 46 , Issue: 10 1556-1571 (2001).

\cite{5} A.M. Bloch, P.S. Krishnaprasad, J.E. Marsden, R.M. Murray. *Nonholonomic mechanical systems with symmetry*, Arch. Rational Mech. Anal., 136, pp. 21–99, (1996).

\cite{6} F. Bullo, A. Lewis. *Geometric Control of Mechanical Systems: Modeling, Analysis, and Design for Simple Mechanical Control Systems*. Texts in Applied Mathematics, Springer Verlag, New York (2005).

\cite{7} M. Castrillón López, T. S. Ratiu. *Reduction in Principal Bundles: Covariant Lagrange-Poincaré Equations*. Communications in Mathematical Physics, vol. 236, no. 2, pp. 223-250 (2003).
[8] M. Castrillón, P.L. García, C. Rodrigo. Euler-Poincaré reduction in principal fibre bundles and the problem of Lagrange. Differential Geometry and its Applications Vol. 25, Issue 6, 585-593 (2007).

[9] H. Cendra, S. Ferraro and S. Grillo. Lagrangian reduction of generalized nonholonomic systems, Journal of Geometry and Physics, 58, 1271-1290 (2008).

[10] H. Cendra, J. Marsden, T. Ratiu. Lagrangian reduction by stages. Memoirs of the American Mathematical Society, 152 (722). pp. 1-108 (2001).

[11] H. Cendra, J. Marsden, T. Ratiu. Geometric mechanics, Lagrangian reduction and nonholonomic systems Mathematics Unlimited - 2001 and Beyond. Springer-Verlag, New York, pp. 221-273.

[12] D. Chang and J. Marsden, Reduction of Controlled Lagrangian and Hamiltonian Systems with Symmetry SIAM J. Control Optim., 43(1), 277-300 (2004).

[13] L. Colombo, D. Martín de Diego, M. Zuccalli. Optimal Control of Underactuated Mechanical Systems: A Geometrical Approach. Journal Mathematical Physics 51, 083519 (2010).

[14] L. Colombo and D. Martín de Diego. Second-order Lagrangian mechanics on Lie algebroids. (Preprint 2013).

[15] L. Colombo and D. Martín de Diego. On the geometry of higher-order variational problems on Lie groups. April 2011. Preprint available at arXiv:1104.3221v1.

[16] L. Colombo, F. Jimenez and D. Martín de Diego. Discrete Second-Order Euler-Poincaré Equations. An application to optimal control. International Journal of Geometric Methods in Modern Physics. Vol 9, N4 (2012).

[17] J. Cortés, M. de León, D. Martín de Diego, S. Martínez. Geometric description of vakonomic and nonholonomic dynamics, SIAM J. Control Optim. 41, no. 5, 1389–1412, (2002).

[18] J. Cortés. Geometric, Control and Numerical Aspects of Nonholonomic Systems, Lec. Notes in Math., 1793, Springer-Verlag, Berlin (2002).

[19] M. Crampin, W. Sarlet, F. Cantrijn. Higher order differential equations and higher order Lagrangian Mechanics. Math. Proc. Camb. Phil. Soc. 99, 565-587, (1986).

[20] M. Crampin, T. Mestag. Invariant Lagrangians, mechanical connections and the Lagrange-Poincaré equations. J. Phys. A: Math. Theor. 41 344015 doi:10.1088/1751-8113/41/34/344015, (2008).

[21] D. Ellis, F. Gay-Balmaz, D. Holm, T. Ratiu. Lagrange-Poincaré field equations, Journal of Geometry and physics, Vol:61, (2011).

[22] F. Gay-Balmaz, T. Ratiu. Reduced Lagrangian and Hamiltonian formulations of Euler-Yang-Mills fluids. J. Symplectic Geom. Volume 6, Number 2 (2008), 189-237.

[23] F. Gay-Balmaz, D. D. Holm, D. M. Meier, T. S. Ratiu, F.-X. Vialard. Invariant higher-order variational problems, Communications in Mathematical Physics, Vol:309, 413-458 (2012).

[24] F. Gay-Balmaz, D. D. Holm, D. M. Meier, T. S. Ratiu, F.-X. Vialard. Invariant higher-order variational problems II, Journal of Nonlinear Science, Volume 22, Number 4, 553-597, DOI: 10.1007/s00332-012-9137-2 (2012).

[25] F. Gay-Balmaz, D. Holm, T. Ratiu. Higher order Lagrange-Poincaré, and Hamilton-Poincaré, reductions, Bulletin of the Brazilian Mathematical Society, Vol:42, 579-606 (2011).
[26] S. Grillo, M. Zuccalli. Variational reduction of Lagrangian systems with general constraints. The Journal of Geometric Mechanics, Pages: 49 - 88, Issue 1 (2012).

[27] D. D. Holm: Geometric mechanics. Part I and II, Imperial College Press, London; distributed by World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, (2008).

[28] D. D. Holm, J. E. Marsden, T. S. Ratiu. The Euler-Poincaré equations and semidirect products with applications to continuum theories, Adv. Math. 137, no. 1, 1–81, (1998).

[29] D. Iglesias, J.C. Marrero, D. Martín de Diego, D. Sosa. Singular Lagrangian systems and variational constrained mechanics on Lie algebroids. Dyn. Syst. 23, no. 3, 351–397 (2008).

[30] Kobayashi S and Nomizu K. Foundations of Differential Geometry. Vol I and II. Wiley Classics Library. John Wiley and Sons Inc., New York, (1996)

[31] W-S. Koon: Reduction, Reconstruction and Optimal Control for Nonholonomic Mechanical Systems with Symmetry, PhD thesis, University of California, Berkeley (1997).

[32] M. de León, P. R. Rodrigues. Generalized Classical Mechanics and Field Theory, North-Holland Mathematical Studies 112, North-Holland, Amsterdam, (1985).

[33] M. de León, J.C. Marrero, D. Martín de Diego. Linear almost Poisson structures and Hamilton-Jacobi equation. Applications to nonholonomic mechanics. Journal Geometric Mechanics, Vol 2. Issue 2, 159 - 198 (2010).

[34] M. de León, J.C. Marrero, E. Martínez. Lagrangian submanifolds and dynamics on Lie algebroids, J. Phys. A: Math. Gen. 38 241308 (2005).

[35] M. de León, P. Pitanga, and P.R. Rodrigues. Symplectic reduction of higher order Lagrangian systems with symmetry, J. Math. Phys. 35(12) 65466556, (1994).

[36] A. Lewis. Reduction of simple mechanical systems. http://penelope.mast.queensu.ca/~andrew/

[37] J.E. Marsden, T. Ratiu. Introduction to Mechanics and Symmetry. Springer-Verlag, Text in Applied Mathematics, 17, Second Edition (1999).

[38] E. Martínez. Variational calculus on Lie algebroids. ESAIM: Control, Optimisation and Calculus of Variations, Volume14, Issue02, pp 356-380 (2008).

[39] P.D. Prieto-Martínez and N. Román-Roy, Lagrangian-Hamiltonian unified formalism for autonomous higher-order dynamical systems, J. Phys. A 44(38) (2011) 385203.

[40] J.P Ostrowski. The mechanics and control of undulatory robotic locomotion. PhD thesis, California Institute of Technology (1995).