Probing Extra Dimensions with Neutrino Oscillations

C.S. Lam

Department of Physics, McGill University

3600 University St., Montreal, Q.C., Canada H3A 2T8

Email: Lam@physics.mcgill.ca

Abstract

In the braneworld scenario, gravity and neutrino oscillation can both be used to detect the presence of an extra dimension. We argue that neutrino oscillation is particularly suitable if the size of the extra dimension is small, in which case the signature for the extra dimension is the disappearance of active neutrino fluxes into the bulk, caused by the destructive interference from the Kaluza-Klein states. We discuss a class of models to illustrate this general feature.

I. INTRODUCTION

This article is dedicated to Prof. Hiroshi Ezawa on the occasion of his seventieth birthday. I met Hiroshi in the early 1960’s, at the University of Maryland. Right away it was clear that I could learn much from him, both in physics and in mathematics. What I did not realize until later was his administrative talent and his superb quality in leadership, both amply demonstrated in his illustrious career. I would like to take this opportunity to wish Hiroshi a happy birthday, a happy retirement, and many many happy returns.

The possible existence of extra (spatial) dimensions beyond our three was first suggested by Theodor Kaluza in 1919, later modified by Oskar Klein in 1926. When superstring came along, consistency requires it to live in six extra dimensions. Unfortunately, there is no
experimental evidence to date for the presence of these extra dimensions. That may be due to the smallness of the extra dimensions, too small even for the largest accelerators to see. The lack of Kaluza-Klein excited states up to about 1 TeV places an upper bound on the size of the extra dimensions to be about $10^{-19}$ m.

Inspired by the discovery of higher-dimensional D-branes in non-perturbative string theories [1], where open strings are trapped, a braneworld scenario was proposed [2] in which the Standard-Model (SM) particles are confined to our three dimensional world, called a 3-brane. Only SM singlets such as gravitons and right-handed neutrinos may leave our world to roam in the extra-dimensional bulk. In that scenario, SM particles have no Kaluza-Klein (KK) excited states simply because they cannot get into the bulk, so the upper bound of $10^{-19}$ m placed on the size of the extra dimensions is no longer valid. In order to probe the presence and the property of the extra dimensions, we must use either gravity, or the right-handed neutrinos $\nu_R$.

A deviation from the inverse-square law of gravity can be used to detect the presence of an extra dimension. In a $3+n$ dimensional world, the surface area of a sphere of radius $r$ is proportional to $r^{2+n}$. Hence the gravitational force between two masses $m_1$ and $m_2$ is given by $G_n m_1 m_2 / r^{2+n}$, where $G_n$ is the Newtonian gravitational constant in $n$ extra dimensions. This is the case when $r$ is less than the size $R$ of the extra dimensions. Otherwise, the sphere is squashed in the extra dimensions to a size $R$, so the surface area of the squashed object is now proportional to $r^2 R^n$. The resulting gravitational force $G_n (\kappa / R^n) / r^2$ once again obeys the inverse-square law, where $\kappa$ is a computable geometrical factor. If a deviation from the inverse-square law is detected experimentally at $r < R$, then $R$ marks the size of the extra dimension. Present experiments found no deviation down to about 0.2 mm [3], so $R$ must be smaller than that. However, it could be as large as 0.1 mm. If so, and if there are at least two extra dimensions, then $G_n = G_0 R_n / \kappa$ is large enough for strong gravitational effects to be seen at TeV energies. This possibility led to a lot of excitement and many papers.

What if the size of the extra dimension is much smaller than 0.1 mm, or, there is only one extra dimension with such a large size? In that case gravity remains weak, and it is
powerless to yield any information in the near future on extra dimensions.

What about the other probe, the right-handed neutrinos $\nu_R$? They are assumed to be absent in the SM. In any case, they are SM singlets, hence *sterile*, in the sense that they experience none of the SM forces. How can they be detected even when they are present?

The answer is ‘mass’. If $\nu_R$’s exist, they would probably produce neutrino masses through the Dirac-mass coupling $\overline{\nu}_R \nu_L$ with the left-handed neutrinos $\nu_L$. A good indication of the presence of a right-handed neutrino is therefore the presence of a neutrino mass.

No mass has been detected in the tritium $\beta$-decay experiments. This places an upper bound of 2.2 eV on the mass [4] of the electron-antineutrino, $\overline{\nu}_e$. However, the pioneering neutrino experiments led by Davis and by Koshiba, the beautiful data coming from Super-Kamiokande, SNO, and KamLAND [5], support an oscillation explanation for the missing neutrinos they detected. It demands at least two of three neutrinos to have non-zero masses. Specifically, if $M_1, M_2, M_3$ are the masses of the three mass eigenstates, then \( \Delta M_\odot \equiv M_2^2 - M_1^2 \simeq (7.5 \times 10^{-3} \text{ eV})^2 \), and \( \Delta M_{\text{atm}}^2 \equiv |M_3^2 - M_2^2| \simeq (50 \times 10^{-3} \text{ eV})^2 \). There are also astrophysical evidence suggesting the neutrino masses to be bounded above by 0.23 eV, if they are degenerate [6].

Unlike the quarks, oscillation experiments discover that neutrinos mix strongly among themselves. Three rotation angles and one phase angle are needed to describe the mixing of three left-handed fermions. In the case of quarks, all three rotation angles are small. In the case of neutrinos, two of them ($\theta_{12}$ and $\theta_{23}$) are large and one of them ($\theta_{13}$) is small.

Now that we know the neutrinos have a mass, we shall assume the right-handed neutrinos to exist. Where do we find them? Neutrino mixing is large but quark mixing is small; neutrino masses are small but quark masses are at least a million times larger. These differences suggest that neutrinos are quite different from the quarks. Since the left-handed quarks and the left-handed neutrinos behave in much the same way under the SM, that indicates the right-handed quarks and the right-handed neutrinos are very different. Right-handed quarks are found at the SM energies, this difference may be telling us that the right-handed neutrinos should be found elsewhere.
Where? The popular scenario is to assume the right-handed neutrinos to live at very high energies. Through the seesaw mechanism [7], this scenario explains the smallness of the neutrino mass, though extra assumptions are needed to explain the large mixing of neutrinos this way.

With the braneworld scenario, there is another possibility. Quarks are confined to the 3-brane we live in, but $\nu_R$’s are free to roam in the bulk. That distinction might give rise to the qualitative difference between quarks and neutrinos. In the rest of this article, we shall examine this possibility more closely.

Each neutrino in the bulk yields an infinite tower of KK neutrinos in four dimensions, which we shall refer to as the bulk neutrinos, or the bulk states. These neutrinos are non-chiral, containing both the left-handed and the right-handed components. Since they originate from the bulk, they are sterile.

To proceed further, let us keep two general questions in mind. First, can the braneworld scenario explain the difference between quarks and neutrinos in a natural way? That is important because that is the raison d’être for going into extra dimensions. Second, do the data on neutrino oscillations even allow the extra-dimensions to exist? The second question is relevant because solar and atmospheric neutrino data demand the mixing with sterile neutrinos to be small, but in the braneworld scenario all the bulk neutrinos are necessarily sterile.

The answer to these questions depends to some extent on the size $R$ of the extra dimension. In most of the recent literature [8], the size is assumed to be large. A size of $R = 0.1$ mm corresponds to a characteristic energy of $2 \times 10^{-3}$ eV, putting it in the right ball park of the neutrino masses. Using perturbation theory, a weak coupling between the left-handed brane (the SM) neutrinos and the right-handed bulk neutrinos can be shown to produce a small neutrino mass as well as a small mixing with the sterile neutrinos. It however does not explain the large neutrino mixing in a natural way, though that can be arranged.

What if the size of the extra dimensions is much smaller than 0.1 mm? Then the weak-coupling assumption explains nothing, so neutrino oscillation is no longer a useful probe
for extra dimensions. Neither is gravity in that case. We are then back to the unenviable position of having no way to tell the presence or absence of an extra dimension.

That however is based on the assumption that the brane-bulk coupling is weak. We shall argue that it is more natural to expect that coupling to be strong, not weak. In that case, things are quite different, and neutrino oscillation can provide useful information even when the extra dimensions are small.

Strong coupling with the bulk may induce a large mixing between the brane neutrinos, thereby explaining naturally why neutrino mixing can be large while quark mixing is small. This answers the first question posted before in the affirmative. But why then is one of the three neutrino mixing angles small, instead of being all large? We shall show that the smallness of that particular mixing angle is intimately related to the smallness of the mass-gap ratio \( \Delta M^2_\odot / \Delta M^2_{\text{atm}} \).

What about the second question? With a strong coupling surely we expect a large amount of sterile neutrinos showing up in the the solar and atmospheric data, contrary to observation. Is there any way out of this fatal problem?

There is. The details will be discussed in the rest of this article, but let us summarize here what is involved.

The trick is to introduce an extra sterile neutrino on the brane, so that most of the mixings are between the sterile brane and bulk neutrinos, and not between the active and sterile neutrinos. In the strong coupling limit, the introduction of this sterile brane neutrino is forced on us by the dynamics; it is not arbitrary and artificial. This is so because one of the brane neutrinos is always absorbed by the KK tower of bulk neutrinos in the strong coupling limit. In other words, if we start with \( f \) flavor neutrinos in the brane, in the strong coupling limit there are only \( f - 1 \) mass eigenstates left on the brane.

Therefore, in order to have two separate mass gaps needed to explain the solar and atmospheric neutrino experiments, we need to have three mass eigenstates in the brane, and hence four flavor states to start out with. We know from the \( Z^0 \) width that there are only three active neutrinos, \( \nu_e, \nu_\mu, \) and \( \nu_\tau \), so the fourth one must be sterile. We shall denote it
by $\nu_s$.

With the extra dimensions small, the small mass of these neutrinos can no longer be explained in the usual way [8], so seesaw or some other mechanism must be invoked. In what follows we shall not ask the origin of these small masses, we simply introduce four parameters $m_a$ to describe the Majorana masses of the flavor neutrinos on the brane.

All masses from now on are understood to be measured in some unit $U$, so the parameters $m_a$ and later on $d_a$ are dimensionless.

In addition to the four brane neutrinos, a minimal model would consist of a single massless bulk neutrino in five spacetime dimensions. It decomposes into an infinite tower of four-dimensional KK neutrinos, to be called the bulk neutrinos, or bulk states. With $U$ properly chosen, the spectrum of this infinite tower can be taken to be the set of all integers.

A Dirac mass coupling of the type $d \nu_R \nu_L + h.c.$ is introduced to couple the left-handed brane neutrinos $\nu_L$ to the right-handed bulk neutrinos $\nu_R$. Since the KK tower comes from a single neutrino in the bulk, there is only one coupling constant $d_a$ per brane neutrino.

If the Dirac masses $d_a$ are comparable to the Dirac mass of any of the charged fermions, and if the Majorana masses $m_a$ are comparable to the neutrino masses, then $d_a$ is larger than $m_b$ by more than a million times. Defining the overall coupling strength to be $d^2 = \sum_{a=1}^{4} d_a^2$, and letting $e_a = d_a/d$, it is therefore likely that Nature is operating in the strong-coupling regime, where $d \gg m_a, e_a, 1$.

In summary, there are eight real parameters in the minimal theory. Four $m_a$’s, and four $d_a$’s. The flavor neutrinos are the states when $d = 0$; the mass eigenstates in the strong-coupling limit are the neutrinos when $d \to \infty$.

To get the mass eigenstates for $d \neq 0$, we have to diagonalize an infinite dimensional matrix, whose rows and columns are labeled by the four flavor neutrinos on the brane, and the infinite number of flavor neutrinos of the bulk. The neat thing is that the eigenvalues and the eigenvectors of this infinite dimensional matrix take on a very simple form in the strong coupling limit.

As mentioned before, one brane neutrino is absorbed into the KK tower of bulk neutrinos.
Taking this into account, the final mass spectrum in the strong coupling limit is as follows. The whole bulk spectrum is rigidly shifted by half a unit, so that the masses are now half integers. The three mass eigenvalues $M_1, M_2, M_3$ on the brane are sandwiched between the four Majorana masses $m_a$ of the flavor neutrinos. Namely, if we order the parameters according to $m_1 < m_2 < m_3 < m_4$, and $M_1 < M_2 < M_3$, then $m_1 < M_1 < m_2 < M_2 < m_3 < M_3 < m_4$. We shall refer to this inequality as the ordering relation. It is a crucial feature of the strong coupling model.

In the strong coupling limit, we are left with seven free parameters, four $m_a$’s and three independent $e_a$’s (because $\sum_{a=1}^{4} e_a^2 = 1$). It turns out that we can replace them by the four $m_a$’s and the three $M_i$’s, provided the ordering relation is maintained. Technically this is quite important because it is much simpler to deal with the latter set of parameters than the former set.

The eigenvalues, and hence the unitary overlapping matrix between the flavor and the mass eigenstates, can also be worked out.

One can then compute the probability amplitude of a flavor neutrino $\nu_a$ oscillating into a flavor neutrino $\nu_b$, after traversing a distance $L$. To do so, we must decompose the flavor neutrino $\nu_a$ into a linear combination of the eigenstates, because it is these normal states that propagate with a definite frequency. After a distance $L$, the mass eigenstates must all be converted back into the flavor state $\nu_b$ to get the probability amplitude.

When $d$ is large, the flavor states on the brane have only a tiny overlap with each of the bulk eigenstates. Nevertheless, since there are an infinite number of bulk eigenstates that a strongly coupled flavor brane neutrino can reach into, the total effect of the bulk is not negligible. The contribution from the infinite number of bulk states destructively interfere with one another, so completely that any active neutrino that oscillates into the bulk will not be able to come back. In other words, the bulk acts like an absorber to the active neutrinos in the brane.

Oscillations that go through the three brane eigenstates act just like ordinary oscillations without the presence of extra dimensions.
In other words, the effect of the extra dimensions is to cause part of the oscillating flux of active neutrinos to be lost in the bulk. This then is the signature of the presence of an extra dimension that we should look for.

We can now describe the physical significance of the seven parameters in the theory. \( M_1, M_2, M_3 \) are the eigenmasses of the three active brane neutrinos. Since the mass is actually \( M_i U \), it is only these products that can be determined experimentally. Different values of \( U \) corresponds to different size \( R \) of the extra dimension, because \( U \) was chosen to make the flavor mass of the bulk neutrinos to be an integer. Since \( U \) alone cannot be determined from the experiment, neither can \( R \).

This is not to say that we can use this method to probe an extra dimension no matter how small it is. The strong-coupling requirement \( d \gg 1 \) places a limit how small \( R \sim U^{-1} \) can be, if we assume the Dirac mass \( dU \sim d/R \) to be comparable to the Dirac mass of the charged fermions. The precise value of course depends on what we use for \( d \). Let us illustrate it with two extremes. If \( d/R \) is the electron mass 0.5 MeV, then we need to have \( R \gg 4 \times 10^{-13} \) m. If it is the top quark mass 175 GeV, then we can go down to an \( R \gg 10^{-18} \) m.

The other four parameters, \( m_1, m_2, m_3, m_4 \) can be used to fit the three mixing angles, and the amount of absorption by the bulk. Note that there is only one free parameter to describe the potential absorption for any \( \nu_a \) oscillating into any \( \nu_b \), so there are predictions that can be potentially falsified.

No absorption has been detected in the present data. Refined and precise data in the future may. It that happens, it is a good indication that an extra dimension exists.

In the minimal model, the absence of absorption can be achieved by letting \( m_4 \rightarrow \infty \). In that limit the mixing between the active neutrinos \( \nu_e, \nu_\mu, \nu_\tau \) and the sterile neutrino \( \nu_s \) also disappears. Then we revert to a situation indistinguishable from the case without extra dimensions. In principle, there is actually a way that might tell them apart, because the three parameters \( m_1, m_2, m_3 \) in the minimal model are constrained by the neutrino masses \( M_1 \) and \( M_2 \) through the ordering relation. As such we may not be able to use them to fit
the three experimentally measured mixing angles. In reality these constraints are fulfilled in the fit, so these two cases become indistinguishable.

When $M_1$ approaches $M_2$, the ordering constraint requires $M_1 = m_2 = M_2$. This pinching of $m_2$ implies $\theta_{13} = 0$. So in this model, one obtains the interesting prediction that $\Delta M^2_{\odot} = 0$ implies $\theta_{13} = 0$. The smallness of $\theta_{13}$ is then related to the smallness of the mass gap ratio $\Delta M^2_{\odot}/\Delta M^2_{\text{atm}}$.

This minimal model in five spacetime dimensions can be substantially generalized without changing any of the crucial features discussed above.

These descriptions and conclusions will be put into mathematical formulas in the next few sections. In Sec. 2, the mass matrix, its eigenvalue, and its eigenvectors of the minimal model are examined. They are then used to compute the oscillating amplitude in Sec. 3. Generalization beyond the minimal model will be discussed in Sec. 4.

II. THE MINIMAL MODEL AND ITS SOLUTION

The mathematical solution of the minimal model will be sketched here. For more details, please consult Refs. [9] and [10].

This model contains four flavor neutrinos in the brane, and a single massless flavor neutrino in a five dimensional spacetime. The latter decomposes into a KK tower of bulk neutrinos with integer masses, and the former are each given a Majorana mass $m_a$ ($a = 1, 2, 3, 4$). The brane neutrinos are coupled to the bulk neutrinos by a Dirac-mass coupling, with coupling strengths $d_a$. We assume all masses to be expressed in some common unit $U$, so that the eight real parameters $m_a$ and $d_a$ are dimensionless. Direct coupling between the brane neutrinos is assumed to be absent, and CP violation is ignored in this simple model.

The symmetric mass matrix of this model is

$$\mathcal{M} = \begin{pmatrix} m & D \\ D^T & B \end{pmatrix},$$

where $m = \text{diag}(m_1, m_2, m_3, m_4)$ is the $4 \times 4$ mass matrix of the brane neutrinos, and $B = \text{diag}(0, +1, -1, +2, -2, +3, -3, \cdots)$ is the infinite dimensional mass matrix of the bulk neutrinos.
neutrinos. The coupling between the two is supplied by $D$, a $4 \times \infty$ matrix in which every element of the $i$th row is equal to $d_a$.

The eigenvalue equation for the mass matrix is

$$M \begin{pmatrix} w \\ v \end{pmatrix} = \lambda \begin{pmatrix} w \\ v \end{pmatrix}, \quad (2)$$

where $w$ is a 4-dimensional column vector with components $w_a$, and $v$ is an $\infty$-dimensional column vector with components $v_n$. $\lambda$ is the mass eigenvalue. In component form, (2) reads

$$m_a w_a + d_a A = \lambda w_a, \quad (3)$$

$$b + (Bv)_n = \lambda v_n, \quad (4)$$

where

$$A = \sum_n v_n,$$

$$b = \sum_{a=1}^4 d_a w_a. \quad (5)$$

We shall choose the normalization of the eigenvectors by setting $b = 1$.

The eigenvector components can be solved from (3) and (4) to be

$$v_n = \frac{1}{\lambda - n},$$

$$w_a = A \frac{d_a}{\lambda - m_a} = (Ad) \frac{e_a}{\lambda - m_a}, \quad (6)$$

where

$$d^2 = \sum_{a=1}^4 d_a^2,$$

$$e_a \equiv d_a/d, \quad \Rightarrow$$

$$1 = \sum_{a=1}^4 e_a^2. \quad (7)$$

The constant $A$ may now be computed to be

$$A = \sum_n v_n = \sum_m \frac{1}{\lambda - m} = \frac{\pi}{\tan(\pi \lambda)}. \quad (8)$$
The eigenvalue equation is obtained from (8) and (6) and the normalization condition \( b = 1 \) to be

\[
1 = \sum_{a=1}^{4} d_a w_a = A d^2 \sum_{a=1}^{4} \frac{e_a^2}{\lambda - m_a}, \quad \Rightarrow
\]

\[
\frac{1}{\pi} \tan(\pi \lambda) = d^2 \sum_{a=1}^{4} \frac{e_a^2}{\lambda - m_a} \equiv d^2 r(\lambda). \tag{9}
\]

Let us solve this equation for the flavor eigenvalue \( d = 0 \), and for the mass eigenvalue in the strong coupling limit \( d \to \infty \). For \( d = 0 \), it follows from (9) that \( \tan(\pi \lambda) = 0 \), which implies \( \lambda \in \mathbb{Z} \), unless \( \lambda = m_a \) for some \( a \). These are the expected eigenvalues because they are simply the matrix elements of the diagonal matrix \( M \) when \( d = 0 \).

For \( d \to \infty \), we should have \( \tan(\pi \lambda) = \infty \), which implies \( \lambda = \mathbb{Z} + \frac{1}{2} \), unless \( r(\lambda) = 0 \). The former are the bulk eigenvalues, and the solutions of the latter are the brane eigenvalues \( M_i \) \((i = 1, 2, 3)\). Since \( r(\lambda) \) approaches \( \pm \infty \) when \( \lambda \to m_a + 0^\pm \), there is one zero of \( r(\lambda) \) between each successive pairs of \( m_a \)'s. In other words, the ordering relation \( m_1 < M_1 < m_2 < M_2 < m_3 < M_3 < m_4 \) mentioned in the Introduction is obeyed.

If \( d \) is large but not infinite, the eigenvalues will shift somewhat, but they are still bounded between consecutive \( m_a \)'s or consecutive \( n \)'s.

We will now show that the three independent parameters \( e_a^2 \) may be replaced by the three independent parameters \( M_i^2 \), provided the ordering relation holds. The argument is based on the simple observation that \( r(\lambda) \) is a meromorphic function of \( \lambda \), with four simple poles occurring at \( \lambda = m_a \), and three zeros occurring at \( \lambda = M_i \). Moreover, \( r(\lambda) \) approaches \( 1/\lambda \) when \( |\lambda| \to \infty \). Hence we can write \( r(\lambda) = \prod_{i=1}^{3} (\lambda - M_i) / \prod_{a=1}^{4} (\lambda - m_a) \). The residue at \( \lambda = m_b \) is then

\[
e_b^2 = \prod_{i=1}^{3} (m_b - M_i) / \prod_{a \neq b} (m_b - m_a). \tag{10}
\]

This formula determines \( e_a^2 \) once \( m_a \) and \( M_i \) are known. To keep \( e_a^2 > 0 \), the ordering relation has to be obeyed.

Let \( U_\lambda \) be the normalized eigenvector, with components \( U_{a \lambda} = w_a / N \) and \( U_{n \lambda} = v_n / N \). The norm \( N^2 \) of the original eigenvector \((w_a, v_n)\) is given by \( N^2 = (Ad)^2 s + T \), where
\[
s = \frac{1}{(Ad)^2} \sum_{a=1}^{4} w_a^2 = \sum_{a=1}^{4} \frac{e_a^2}{(\lambda - m_a)^2},
\]

\[
T = \sum_n v_n^2 = \frac{1}{(\lambda - n)^2}.
\]

(11)

III. OSCILLATION AMPLITUDE

Using (6) and (11), we can calculate the transition amplitude \( A_{ab} \) from a brane neutrino of flavor \( b \) and energy \( E \) (measured in units of \( U \)), to a brane neutrino of flavor \( a \) after it has traversed a distance \( L = 2E\tau \) (measured in units of \( U^{-1} \)). The transition amplitude is determined by the formula

\[
A_{ab}(\tau) = \sum_{\lambda} U_{a\lambda}^* U_{b\lambda} e^{-i\lambda^2 \tau} \equiv A_{ab}^S(\tau) + A_{ab}^K(\tau),
\]

(12)

where \( A^S \) is the contribution from the brane eigenvalues \( \lambda = M_1, M_2, M_3 \), and \( A^K \) is the contribution from the bulk eigenvalues \( \lambda \in \mathbb{Z} + \frac{1}{2} \).

When \( d \to \infty \), the quantities \( v_n, w_a/Ad, s \) and \( T \) are all of order 1, so the magnitude of \( w_a \) is determined by \( Ad \) and the magnitude of \( N^2 \) is determined by \( (Ad)^2 \). According to (9), \( Ad = 1/(dr) \). For bulk eigenvalues, \( r = O(1) \), so \( Ad = O(1/d) \). This implies \( N^2 \simeq T \) and \( U_{a\lambda} = O(1/d) \). In that case the bulk components of an eigenvector are much larger than the brane components. For brane eigenvalues, \( A = O(1) \), hence \( Ad = O(d) \) and \( w_a = O(d) \). In that case the brane components of an eigenvector dominate and \( N^2 \simeq (Ad)^2 s \).

Let us denote the large-\( d \) value of \( U_{aM_i} \) by \( V_{ai} \), the value of \( s \) at \( \lambda = M_i \) by \( s_i \), and \( 1/(M_i - m_a) \) by \( x_{ai} \). Then

\[
V_{ai} = e_a x_{ai} / \sqrt{s_i} \quad (1 \leq a \leq 4, \ 1 \leq i \leq 3),
\]

(13)

and

\[
A_{ab}^S(\tau) = \sum_{i=1}^{3} V_{ai}^* V_{bi} e^{-iM_i^2 \tau}.
\]

(14)

As it stands, \( V \) is a \( 4 \times 3 \) matrix, but we can make it into a square \( 4 \times 4 \) matrix by letting the last column to be \( V_{af} = e_a \). The meaning of this last column will be discussed later.
Note that we can write $V_{a4}$ in the same form as the other $V_{ai}$, namely, $V_{a4} = e_a x_{a4}/\sqrt{s_4}$, provided we let $\lambda = \infty$.

The resulting $4 \times 4$ matrix $V$ can be shown to be real orthogonal. It depends on the seven parameters, $m_a$ and $M_i$.

We have shown in (10) how to express $e_b^2$ in terms of these parameters. Similarly, it can be shown that

$$s_i = -\frac{3}{4} \prod_{k \neq i} (M_i - M_k)/\prod_{c=1}^4 (M_i - m_c). \quad (15)$$

We turn to the contribution from the bulk eigenvalues. Since $U_{a\lambda}$ is unitary, it follows from (12) that $A_{ab}(0) = \delta_{ab}$, hence

$$A^K_{ab}(0) = \delta_{ab} - A^S_{ab}(0). \quad (16)$$

Using (14) and the unitarity of the matrix $V$, we conclude that

$$A^S_{ab}(0) = \delta_{ab} - V^*_{af} V_{bf} = \delta_{ab} - e_a e_f. \quad (17)$$

Therefore

$$A^K_{ab}(0) = e_a e_b. \quad (18)$$

The contribution from the bulk eigenvalues can also be obtained directly from (12) and the paragraph following that equation to be

$$A^K_{ab}(\tau) = \sum_{\lambda \in \mathbb{Z} + \frac{1}{2}} \frac{1}{(dr)^2 T (\lambda - m_a)(\lambda - m_b)} e^{-i\lambda^2 \tau} \equiv e_a e_b F(\tau). \quad (19)$$

Both $r$ and $T$ are of order 1 as $d \to \infty$, so the contribution from each bulk eigenvalue to the sum is $O(1/d^2)$. Since there are an infinite number of bulk eigenvalues, the total contribution to the sum in (19) is not necessarily zero. In fact, we know from (18) that $F(0) = 1$ even at an infinite $d$.

It can be shown that $F(\tau) = g(K^2 \tau)$, where $K^2 = d^2(1 + \pi^2 d^2)$, and that

$$g(x) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-i a x}}{u^2 + 1} \quad (20)$$
is zero at \( x = 0 \), and decrease to 0 like \((1 - i)/\sqrt{2\pi x}\) for large \( x \). This means that \( A^K_{ab}(\tau) = e_a e_b g(K^2 \tau) \) is zero whenever \( \tau > 0 \), in the limit \( d \to \infty \). This function describes the absorption of the active neutrino flux into the bulk.

Therefore, in the strong coupling limit, we end up with

\[
A_{ab}(\tau) = A^S_{ab}(\tau) = \sum_{i=1}^{3} V_{ai}^* V_{bi} e^{-iM_i^2 \tau}.
\] (21)

In the limit \( m_4 \to \infty \), it follows from (10) that \( e_4^2 \to 1 \), and hence from (7) that \( e_b \to 0 \) for \( b = 1, 2, 3 \). Since the matrix \( V \) is real orthogonal, and \( e_4 = V_{44} \), it also follows that \( V_{a4} = 0 \) for \( a = 1, 2, 3 \). As a result, the active neutrinos \( \nu_e, \nu_\mu, \nu_\tau \) do not mix with the sterile neutrino \( \nu_s \), and the active neutrinos do not get absorbed by the bulk.

If furthermore \( M_1 = M_2 \), then the ordering relation forces \( m_2 = M_1 \). In that case \( V_{23} = 0 \). This means the mixing angle \( \theta_{13} = 0 \) if we identify the \( a = 2 \) flavor neutrino with \( \nu_e \). Hence a vanishing \( \Delta M^2_\odot/\Delta M^2_{atm} \) implies a vanishing \( \theta_{13} \).

\textbf{IV. GENERALIZATION OF THE MINIMAL MODEL}

The final result (21) remains valid for almost all \( B \) in (1). This is reasonable because (21) does not depend on the property of the absorptive bulk, which \( B \) affects. For a detailed argument, please consult Ref. [11].

This research is supported by NSERC and FRNT.
REFERENCES

[1] J. Polchinski, Phys. Rev. Lett. 75 (1995) 4724.

[2] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B429 (1998) 263; Phys. Rev. D59 (1999) 086004; I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B436 (1998) 257; L. Randall and R. Sundrum, Phys. Rev. Lett. 83(1999) 3370, 4690.

[3] J.C. Long, A.B. Churnside, and J.C. Price; C.D. Hoyle et. al., Phys. Rev. Lett. 86 (2001) 1418.

[4] J. Bonn et al., Nucl. Phys. B (Proc. Suppl.) 91 (2001) 273.

[5] http://www-sk.icrr.u-tokyo.ac.jp/doc/sk/;
   http://www.sno.phy.queensu.ca/;
   http://www.awa.tohoku.ac.jp/html/KamLAND/.

[6] H.V. Peiris et al., http://arxiv.org/ps/astro-ph/0302225.

[7] M.Gell-Mann, P.Ramond, R.Slansky, in ‘Supergravity’ (North-Holland, Amsterdam,1979); T.Yanagida, in ‘Proc.of the workshop on unified Theory and Baryon Number of the Universe’, (KEK, Japan,1979); R.N. Mohapatra, G. Senjanovic, Phys. Rev. Lett. 44 (1980) 912.

[8] N. Arkani-Hamed, S. Dimopoulos, G. Dvali, and J. March-Russell, hep-ph/9811448; K.R. Dienes, E. Dudas, and T. Gherghetta, Nucl. Phys. B557 (1999) 25; R.N. Mohapatra and A. Pérez-Lorenzana, Nucl. Phys. B576 (2000) 466; K.R. Dienes and I. Sarcevic, Phys. Lett. B500 (2001) 133; A. Lukas, P. Ramond, A. Romanino, and G.G. Ross, Phys. Lett. B495 (2000) 136; D.O. Caldwell, R.N. Mohapatra, and S. J. Yellin, Phys. Rev. Lett. 87 (2001) 041601, hep-ph/0101043, hep-ph/0102279; N. Cosme, J.-M. Frere, Y. Gouverneur, F.-S. Ling, D. Monderen, V. Van Elewyck, Phys. Rev. D63 (2001) 113018; R. Barbieri, P. Creminelli, and A. Strumia, Nucl. Phys. B585 (2000) 28; R.N.
Mohapatra and A. Pérez-Lorenzana, Nucl. Phys. B593 (2001) 451.

[9] C.S. Lam and J.N. Ng, Phys. Rev. D64 (2001) 113006 (http://arxiv.org/ps/hep-ph/0104129).

[10] C.S. Lam, Phys. Rev. D65 (2002) 053009 (http://arxiv.org/ps/hep-ph/0110142).

[11] D. Charuchittipan and C.S. Lam, to be published.