The Full Abstraction Problem for Higher Order Functional-Logic Programs

F.J. López-Fraguas and J. Rodríguez-Hortalá *
Departamento de Sistemas Informáticos y Computación
fraguas@fdi.ucm.es, juanrh@fdi.ucm.es

Abstract. Developing suitable formal semantics can be of great help in the understanding, design and implementation of a programming language, and act as a guide for software development tools like analyzers or partial evaluators. In this sense, full abstraction is a highly desirable property, indicating a perfect correspondence between the semantics and the observable behavior of program pieces. In this work we address the question of full abstraction for the family of modern functional logic languages, in which functions can be higher order and non-deterministic, and where the semantics adopted for non-determinism is call-time choice. We show that, with respect to natural notions of observation, any semantics based on extensional functions is necessarily unsound; in contrast, we show that the higher order version of CRWL, a well-known existing semantic framework for functional logic programming, based on an intensional view of functions, turns out to be fully abstract and compositional.

1 Introduction

Developing suitable formal semantics can be of great help in the understanding, design and implementation of a programming language, and acts as a guide for software development tools like analyzers or partial evaluators. In this sense, full abstraction is a highly desirable property, indicating a perfect correspondence between the semantics and the behavior of program pieces, according to a given criterion of observation.

The notion of full abstraction was introduced by Plotkin [19] in connection to PCF, a simple model of functional programming based on λ-calculus. He realized that the standard Scott semantics, in which expressions of functional types have classical mathematical functions as meanings, lacks full abstraction with respect to observing the value obtained in the evaluation of an expression. The reason lays in the impossibility of defining the function \textit{por} (parallel or) in PCF. Using this fact one can build two higher order (HO) expressions $e_1, e_2$ denoting two different mathematical functions $\varphi_1, \varphi_2$, both expecting boolean functions as arguments, such that $\varphi_1, \varphi_2$ only differ when applied to $\textit{por}$ as

* This work has been partially supported by the Spanish projects TIN2005-09207-C03-03, TIN2008-06622-C03-01, S-0505/TIC/0407 and UCM-BSCH-GR58/08-910502.
argument. Therefore $e_1, e_2$ have different Scott semantics but this difference cannot be observed. It is usually said that the semantics is too concrete. Notice, however, that Scott semantics for PCF is sound, that is, if two expressions have the same semantics, they cannot be observably distinguished. Unsoundness of a semantics can be considered a flaw, much more severe that being too concrete, which is more a weakness than a flaw.

Full abstraction for PCF was achieved in different technical ways (see e.g. [3]). But for our purposes it is more interesting to recall that the Scott semantics becomes fully abstract if PCF is enriched with the ‘missing’ por function (see e.g. [18]). The mainstream of functional logic programming (FLP, see [10]) is based rather in the theory of term rewriting systems than in $\lambda$-calculus; a consequence is that parallel or can be defined straightforwardly by an overlapping (almost orthogonal) rewriting system. So one could think of assigning to FLP languages a denotational semantics in the FP style. For instance, for a definition like $f x = 0$, one could assign to $f$ the meaning $\lambda x.0$. The next step of our discussion is taking into account that modern FLP languages like Curry [12] or Toy [16] also permit non-confluent and non-terminating programs that define non-deterministic non-strict functions. This suggests that the standard semantics should be modified in the sense that the meaning of a function would be some kind of set-valued function.

The starting motivation of this paper is that this roadmap cannot be followed anymore when non-determinism is combined with HO, at least when considering call-time choice [13, 9], which is the notion of non-determinism adopted in, e.g., Curry or Toy. The following example taken from [15] shows it:

**Example 1.** The following program computes with natural numbers represented by the constructors $0$ and $s$, and where $+$ is defined as usual. The syntax uses HO curried notation.

\[
\begin{align*}
  g \ X & \rightarrow 0 \\
  h \ X & \rightarrow s \ 0 \\
  f \ X & \rightarrow f \ X \\
  fadd \ F \ G \ X & \rightarrow (F \ X) + (G \ X) \\
  fdouble \ F & \rightarrow fadd \ F \ F
\end{align*}
\]

Here $f$ and $f'$ are non-deterministic functions that are (by definition of $f'$) extensionally equivalent. In a set-valued variant of Scott semantics, their common denotation would be the function $\lambda X.\{0, s \ 0\}$, or something essentially equivalent. But this leads to unsoundness of the semantics. To see why, consider the expressions $(fdouble \ f \ 0)$ and $(fdouble \ f' \ 0)$. In Curry or Toy, the possible values for $(fdouble \ f \ 0)$ are $0, s \ (s \ 0)$, while $(fdouble \ f' \ 0)$ can be in addition reduced to $s \ 0$. The operational reason to this situation is that $fdouble \ f \ 0$ is rewritten first to $fadd \ f \ f \ 0$ and then to $f \ 0 + f \ 0$; now, call-time choice enforces that evaluation of the two created copies of $f$ (which is an evaluable expression) must be shared. In the case of $f' \ 0 + f' \ 0$, since $f'$ is a normal form, the two occurrences of $f'$ $0$ evolve independently. We see then that $f$ and $f'$ can be put in a context able to distinguish them, implying that any semantics assigning $f$ and $f'$ the same denotation is necessarily unsound, and therefore not fully abstract.
The combination $\textit{HO} + \textit{Non-determinism} + \textit{call-time choice}$ was addressed in $\textit{HOCRWL}$ [7, 8], an extension to $\textit{HO}$ of $\textit{CRWL}$ [9], a semantic $\textit{FO}$ framework specifically devised for $\textit{FLP}$ with call-time choice semantics for non-determinism. $\textit{HOCRWL}$ adopts an \textit{intensional} view of functions, where different descriptions – in the form of $\textit{HO-patterns}$ – of the same extensional function are distinguished as different data. The intensional point of view of $\textit{HOCRWL}$ was an \textit{a priori} design decision, motivated by the desire of achieving enough power for $\textit{HO}$ programming while avoiding the complexity of higher-order unification of $\lambda$-terms modulo $\beta\eta$, followed in other approaches [17, 11]. The issues of soundness or full abstraction were not the (explicit nor implicit) concerns of [7, 8]; whether $\textit{HOCRWL}$ actually fulfils those properties or not is exactly the question considered in this paper. As we will get positive answers, an anticipated conclusion of our work is that one must take into account intensional descriptions of functions as sensible meanings of expressions in $\textit{HO}$ non-deterministic $\textit{FLP}$ programs, even if one does not want to explicitly program with $\textit{HO-patterns}$.

The rest of the paper is organized as follows. Next section recalls some essential preliminaries about applicative $\textit{HO}$ rewrite systems and the $\textit{HOCRWL}$ framework. We introduce also some terminology about semantics and extensionality needed for Sect. 3, where we examine soundness and full abstraction with respect to reasonable notions of observation based on the result of reductions. The section ends with a discussion of the problems encountered when programs have \textit{extra} variables, i.e., variables occurring in right, but not in left-hand sides of function defining rules. Finally Sect. 4 summarizes some conclusions and future work.

2 Higher-Order Functional-Logic Programs

2.1 Expressions, patterns and programs

We consider \textit{function} symbols $f, g, \ldots \in \textit{FS}$, \textit{constructor} symbols $c, d, \ldots \in \textit{CS}$, and \textit{variables} $X, Y, \ldots \in \textit{V}$; each $h \in \textit{FS} \cup \textit{CS}$ has an associated \textit{arity}, $\text{ar}(h) \in \mathbb{N}$; $\textit{FS}^n$ (resp. $\textit{CS}^n$) is the set of function (resp. constructor) symbols with arity $n$. The notation $o$ stands for tuples of any kind of syntactic objects $o$. The set of \textit{applicative expressions} is defined by $\textit{Exp} \ni e ::=$ $X$ | $h$ | $(e_1 e_2)$ . As usual, application is left associative and outer parentheses can be omitted, so that $e_1 e_2 \ldots e_n$ stands for $((e_1 e_2) \ldots e_n)$. The set of variables occurring in $e$ is written by $\textit{var}(e)$. A distinguished set of expressions is that of \textit{patterns} $t, s \in \textit{Pat}$, defined by: $t ::= X \mid c \ t_1 \ldots t_n \mid f \ t_1 \ldots t_m$, where $0 \leq n \leq \text{ar}(c), 0 \leq m < \text{ar}(f)$. Patterns are irreducible expressions playing the role of \textit{values}. \textit{FO-patterns}, defined by $\textit{FOPat} \ni t ::= X \mid c \ t_1 \ldots t_n \ (n = \text{ar}(c))$, correspond to $\textit{FO}$ constructor terms, representing ordinary non-functional data-values. Partial applications of symbols $h \in \textit{FS} \cup \textit{CS}$ to other patterns are $\textit{HO-patterns}$ and can be seen as truly data-values representing functions from an \textit{intensional} point of view. Examples of patterns with the signature of Ex. 1 are: 0, $s \ X, s, f' \ fadd \ f' \ f'$. The last three are $\textit{HO-patterns}$. Notice that $f, fadd \ f f$ are not patterns since $f$ is not a pattern ($\text{ar}(f) = 0$).
Contexts are expressions with a hole defined as $\text{Ctx} \ni C ::= [ ] \mid C \ e \mid e \ C$. Application of $C$ to $e$ (written $C[e]$) is defined by $[ ][e] = e$; $(C \ e')[e] = C[e] e'$; $(e' \ C)[e] = e' C[e]$. Substitutions $\theta \in \text{Subst}$ are finite mappings from variables to expressions; $[X_i/e_i, \ldots, X_n/e_n]$ is the substitution which assigns $e_i \in \text{Exp}$ to the corresponding $X_i \in \mathcal{V}$. We will mostly use pattern-substitutions (or simply $p$-substitutions) $P\text{Subst} = \{\theta \in \text{Subst} \mid \theta(X) \in \text{Pat}, \forall X \in \mathcal{V}\}$.

As usual while describing semantics of non-strict languages, we enlarge the signature with a new 0-ary constructor symbol $\perp$, which can be used to build the sets $\text{Expr}_\perp, \text{Pat}_\perp, P\text{Subst}_\perp$ of partial expressions, patterns and $p$-substitutions resp.

A HOCRWL-program (or simply a program) consists of one or more program rules of the form $f \ t_1 \ldots t_n \rightarrow r$ where $f \in FS^n$, $(t_1, \ldots, t_n)$ is a linear (i.e., variables occur only once) tuple of (maybe HO) patterns and $r$ is any expression. Notice that confluence or termination is not required. In the present work we restrict ourselves to programs not containing extra variables, i.e., programs for which $\text{var}(r) \subseteq \text{var}(f \ T)$ holds for any program rule. There are technical reasons for such limitation (see Sect. 3.2), whose practical impact is on the other hand mitigated by known extra-variables elimination techniques [4, 2].

HOCRWL-programs often allow also conditions in the program rules. However, programs with conditions can be transformed into equivalent programs without conditions; therefore we consider only unconditional rules.

Some FLP systems, like Curry, do not allow HO-patterns in left-hand sides of function definitions. We call left-FO programs to these special kind of HOCRWL-programs. We remark that all the notions and results in the paper are applicable to left-FO programs and we stress the fact that Ex. 1 is one of them.

### 2.2 The HOCRWL proof calculus [7]

The semantics of a program $\mathcal{P}$ is determined in HOCRWL by means of a proof calculus able to derive reduction statements of the form $e \rightarrow t$, with $e \in \text{Exp}_\perp$ and $t \in \text{Pat}_\perp$, meaning informally that $t$ is (or approximates to) a possible value of $e$, obtained by evaluation of $e$ using $\mathcal{P}$ under call-time choice.

The HOCRWL-proof calculus is presented in Fig. 1. We write $\mathcal{P} \vdash_{\text{HOCRWL}} e \rightarrow t$ to express that $e \rightarrow t$ is derivable in that calculus using the program $\mathcal{P}$. The HOCRWL-denotation of an expression $e \in \text{Exp}_\perp$ is defined as $[e]_{\text{HOCRWL}} = \{t \in \text{Pat}_\perp \mid \mathcal{P} \vdash_{\text{HOCRWL}} e \rightarrow t\}$. $\mathcal{P}$ and HOCRWL are frequently omitted in those notations.

Looking at in Ex. 1 we have $[[\text{fdouble } f \ 0]] = \{0, s \ (s \ 0), \perp, s \ \perp, s \ (s \ \perp)\}$ and $[[\text{fdouble } f' \ 0]] = \{0, s \ 0, s \ (s \ 0), \perp, s \ \perp, s \ (s \ \perp)\}$.

We will use the following result stating an important compositionality property of the semantics of HOCRWL-expressions: the semantics of a whole expression depends only on the semantics of its constituents, in a particular form reflecting the idea of call-time choice.

**Theorem 1 (Compositionality of HOCRWL semantics, [15]).** For any $e \in \text{Exp}_\perp$, $C \in \text{Ctx}$, $[C[e]] = \bigcup_{t \in [e]} [C[t]]$. 
The HOCRWL logic is related to several operational notions. In [7] a goal solving narrowing calculus was presented and its strong adequacy to HOCRWL shown. The operational semantics of [1] has been also used in many works in the field of FLP. Its equivalence with the first order version of HOCRWL was stated in [14], and it can be transferred to higher order through the results of [15, 1]. The formalization of graph rewriting of [5, 6] has been often used in FLP too, and although never formally proved, it is usually considered that it specifies the same behaviour. Finally, in [15] a notion of higher order rewriting with local bindings called HOlet-rewriting and its lifting to narrowing were proposed, and its adequacy to HOCRWL was formally proved. It can be summarized in the following result:

**Theorem 2 ([15]).** \( \forall e \in \text{Exp}, t \in \text{Pat}, t \in [e]^P \iff \mathcal{P} \vdash e \rightarrow t, \) where \( \rightarrow^* \) stands for the reflexive-transitive closure of the HOlet-rewriting relation.

Therefore, we can use the set of total values computed for an expression in HOCRWL as a characterization of the operational behaviour of that expression, as it has a strong correspondence, not only with its behaviour under HOlet-rewriting, but also under any of the operational notions mentioned above.

### 2.3 Extensionality

In order to achieve more generality and technical precision wrt. the discussion of Ex.1, we introduce here some new terminologies and notations about extensional equivalence and related notions that will be used later on. They can be expressed in terms of the HOCRWL semantics \([\_]\).

**Definition 1 (Extensional equivalence, extensional semantics).**

(i) Given \( n \geq 0 \), two expressions \( e, e' \in \text{Expr}_\perp \) are said to be \( n \)-extensionally equivalent (\( e \sim_n e' \)) iff \( [e e_1 \ldots e_n] = [e' e_1 \ldots e_n] \), for any \( e_1, \ldots, e_n \in \text{Expr}_\perp \).

(ii) Given \( n \geq 0 \), \( e \in \text{Expr}_\perp \), the \( n \)-extensional semantics of \( e \) is defined as:

\[
[e]_{\text{ext}} = \lambda t_1 \ldots t_n. [e t_1 \ldots t_n] \ (t_i \in \text{Pat}_\perp).
\]

We can establish some relationships between these notions:
Proposition 1.

(i) \( e \sim_n e' \Rightarrow e \sim_m e' \), for all \( m > n \).
(ii) \( e \sim_n e' \iff [ [ e \ ]_{1} \ldots t_{n} ] = [ [ e' \ ]_{1} \ldots t_{n} ] \), for any \( t_{1}, \ldots, t_{n} \in \text{Pat}_{\perp} \).
(iii) \( e \sim_n e' \iff [ [ e ]_{\text{ext}_n} ] = [ [ e' ]_{\text{ext}_n} ] \).

Proof. The proof is easy, thanks to compositionality of \([ [ \ ] ] \) (Th. 1).

(i) Assume \( e \sim_n e' \), \( m > n \), let \( e_{1} \ldots e_{m} \in \text{Expr}_{\perp} \). We must prove \([ [ e e_{1} \ldots e_{m} ] ] = [ [ e' e_{1} \ldots e_{m} ] ] \). We reason as follows:

\[
[ [ e e_{1} \ldots e_{m} ] ] = [ [ (e e_{1} \ldots e_{n}) e_{n+1} \ldots e_{m} ] ] = (by\ compositionality) \\
\bigcup_{t \in [ [ e e_{1} \ldots e_{n} ] ]} [ [ t e_{n+1} \ldots e_{m} ] ] = (since \ e \sim_n e') \\
\bigcup_{t \in [ [ e' e_{1} \ldots e_{n} ] ]} [ [ t e_{n+1} \ldots e_{m} ] ] = (by\ compositionality) \\
[ [ (e' e_{1} \ldots e_{n}) e_{n+1} \ldots e_{m} ] ] = [ [ e' e_{1} \ldots e_{m} ] ]
\]

(ii) Another direct use of compositionality
(iii) Consequence of (i),(ii) and definitions of \( \sim_n \), \([ [ \ ] ]_{\text{ext}_n} \).

3 CRWL and Full Abstraction

3.1 Full Abstraction

In this section we examine technically soundness and full abstraction of the \( \text{HOCRWL} \) semantics \([ [ \ ] ] \) and its extensional variants \([ [ \ ] ]_{\text{ext}_n} \). We can anticipate a positive answer for \([ [ \ ] ]_{\text{ext}_n} \) and negative for the others.

Full abstraction depends on a criterion of observability for expressions. In constructor based languages, like FLP languages, it is reasonable to observe the outcomes of computations, given by constructor forms reached by reduction. Here, we can interpret ‘constructor form’ in a liberal sense, including HO-patterns, or in a more restricted sense, only with FO-patterns. This leads to the following notions of observation.

**Definition 2 (observations).** Let \( \mathcal{P} \) be a program. We consider the following observations:

- \( \mathcal{O}^{\mathcal{P}} : \text{Expr} \mapsto \text{Pat} \) is defined as \( \mathcal{O}^{\mathcal{P}}(e) = \{ t \in \text{Pat} \mid \mathcal{P} \vdash e \rightarrow^{*} t \} \)
- \( \mathcal{O}^{\mathcal{P}_{0}} : \text{Expr} \mapsto \text{FOPat} \) is defined as \( \mathcal{O}^{\mathcal{P}_{0}}(e) = \{ t \in \text{FOPat} \mid \mathcal{P} \vdash e \rightarrow^{*} t \} = \mathcal{O}^{\mathcal{P}}(e) \cap \text{FOPat} \)

We remark that, due to the strong correspondence between reduction and semantics given by Th. 2, we also have \( \mathcal{O}^{\mathcal{P}}(e) = [ [ e ] ]^{P} \cap \text{Pat} \), implying in particular \( \mathcal{O}^{\mathcal{P}}(e) \subseteq [ [ e ] ]^{P} \) (and similar conditions hold for \( \mathcal{O}^{\mathcal{P}_{0}} \)).

Now we turn to the definition of full abstraction. In programming languages like PCF the condition for full abstraction is usually stated as:

\[
[ [ e ] ] = [ [ e' ] ] \Leftrightarrow \mathcal{O}(\mathcal{C}[e]) = \mathcal{O}(\mathcal{C}[e']), \text{for any context } \mathcal{C}
\]
where $O$ is the observation function of interest. Programs do not need to be mentioned, because programs and expressions can be identified by contemplating the evaluation of $e$ under $P$ as the evaluation of a big $\lambda$-expression or big let-expression embodying $P$ and $e$. Contexts pose no problems either. In our case, since programs are kept different from expressions, some care must be taken. It might happen that $P$ has not enough syntactical elements and rules to built interesting distinguishing contexts. For instance, if in Ex. 1 we drop the definition of $\text{fdouble}$, and we consider $O_{f\circ}$ as observation, then we cannot built a context that distinguishes $f$ from $f'$. This would imply that soundness or full abstraction would not be intrinsic to the semantics, but would greatly depend on the program. What we need is requiring the right part of (1) to hold for all contexts that might be obtained by extending $P$ with new auxiliary functions.

To be more precise, we say that $P'$ is a safe extension of $(P, e)$ if $P' = P \cup P''$, where $P''$ does not include defining rules for any function symbol occurring in $P$ or $e$. The following property of $\text{HOCRWL}$ regarding safe extensions will be crucial for full abstraction. The property is subtler than it appears to be, as witnessed by the fact that it fails to hold if programs have extra variables, as discussed in Sect. 3.2.

**Lemma 1.** $[e]^P = [e]^{P'}$ when $P'$ safely extends $(P, e)$.

**Proof.** As $P \subseteq P'$ then $[e]^P \subseteq [e]^{P'}$ trivially holds, as every $\text{HOCRWL}$-proof for $P \vdash e \rightarrow t$ is also a proof for $P' \vdash e \rightarrow t$.

On the other hand, to prove the inclusion $[e]^{P'} \subseteq [e]^P$ let us precisely formalize the notion of safe extension. For any program $P$, we write $\text{defs}(P)$ for the set of function symbols defined in $P$ (i.e., appearing at the root of some left-hand side of a program rule of $P$); for any expression $e$, we write $FS^e$ for the set of function symbols appearing in $e$; for any program $P$ and rule $(l \rightarrow r) \in P$ we define $FS^{(l \rightarrow r)} = FS^l \cup FS^r$ and $fs^P = \bigcup_{(l \rightarrow r) \in P} FS^{(l \rightarrow r)}$. Then $P'$ is a safe extension of $(P, e)$ iff $P' = P \cup P''$ such that $\text{defs}(P'') \cap (FS^e \cup FS^P) = \emptyset$.

Now we will see that for any proof for $P' \vdash a \rightarrow s$ if $\text{defs}(P'') \cap FS^P = \emptyset$ then $\text{defs}(P'') \cap FS^a = \emptyset$ and for any premise $a' \rightarrow s'$ appearing in that proof we have $\text{defs}(P'') \cap (FS^{a'} \cup FS^{s'}) = \emptyset$, by induction on the structure of $P' \vdash a \rightarrow s$. Let us do a case distinction over the rule applied at the root. If it was $B$ then the only statement is $a \rightarrow \bot$ for which the condition holds because $\bot \not\in FS$. If it was $RR$ then the only statement is $x \rightarrow x$, but $x \not\in FS$. If it was $DC$ then we apply the IH over each $e_i \rightarrow t_i$, because $\text{defs}(P'') \cap FS^{(h \ e_1 \ldots e_m)} = \emptyset$ implies $\text{defs}(P'') \cap FS^{t_i} = \emptyset$ for each $e_i$. All that is left is checking that $\text{defs}(P'') \cap FS^{(h \ t_1 \ldots t_m)} = \emptyset$. But $\text{defs}(P'') \cap FS^{t_i} = \emptyset$ for each $t_i$ by IH, and $h \in FS^{(h \ e_1 \ldots e_m)} \cap \text{defs}(P'') = \emptyset$ by hypothesis, so we are done. Finally, for $OR$ we apply the IH to $e_i \rightarrow p_i \theta$ and its premises, as we did in $DC$. Besides $f \in FS^{(f \ e_1 \ldots e_m \ a_1 \ldots a_m)} \cap \text{defs}(P'') = \emptyset$ by hypothesis, so $(f \ p_1 \ldots p_m \rightarrow r) \in P$, hence $\text{defs}(P'') \cap FS^{(f \ p_1 \ldots p_m \rightarrow r)} = \emptyset$, because $P''$ is a safe extension. Combining both facts with the absence of extra variables in program rules we get $FS_\theta \cap \text{defs}(P'') = \emptyset$. But $FS^{(f \ e_1 \ldots e_m \ a_1 \ldots a_m)} \cap \text{defs}(P'') = \emptyset$ by hypothesis, hence
Finally, assuming a proof \( \mathcal{P}' \vdash e \rightarrow t \) we may apply the property above because \( \text{defs}(\mathcal{P}'') \cap \text{FS} = \emptyset \), as \( \mathcal{P}'' \) is a safe extension. Therefore \( \mathcal{P}'' \) was not used in that proof and so it is also a proof for \( \mathcal{P} \vdash e \rightarrow t \), since \( \mathcal{P}' = \mathcal{P} \uplus \mathcal{P}'' \).

We can now define:

**Definition 3 (Full abstraction).**

(a) A semantics is **fully abstract** wrt \( \mathcal{O} \) iff for any \( \mathcal{P} \) and \( e, e' \in \text{Expr} \), the following two conditions are equivalent:

- (i) \( [e]^P = [e']^P \)
- (ii) \( \mathcal{O}^P(C[e]) = \mathcal{O}^{P'}(C[e']) \) for any \( \mathcal{P}' \) safely extending \( (\mathcal{P}, e), (\mathcal{P}, e') \) and any \( C \) built with the signature of \( \mathcal{P}' \).

(b) A notion weaker than full abstraction is: a semantics is **sound** wrt \( \mathcal{O} \) iff the condition (i) above implies the condition (ii).

For extensional semantics, our Ex. 1 (and obvious generalizations to arities \( k > 1 \)) constitutes a proof of the following negative result:

**Proposition 2.** For any \( k > 0 \), \([\llbracket \_ \rrbracket_{\text{ext}}\) is unsound wrt \( \mathcal{O}, \mathcal{O}_{fo} \). This remains true even if programs are restricted to be left-FO.

This contrast with the following:

**Theorem 3 (Full abstraction).** \( [\llbracket \_ \rrbracket] \) is fully abstract wrt \( \mathcal{O} \) and \( \mathcal{O}_{fo} \).

The proof for this theorem will be based on the compositionality of \( [\llbracket \_ \rrbracket] \) and the following result:

**Lemma 2.** Let \( \mathcal{P} \) be any program. Consider the transformation \( \hat{\_} : \text{Pat}_\bot \rightarrow \text{Pat} \) defined by:

\[
\hat{X} = X \quad \hat{\bot} = \text{bot} \quad h \hat{t_1} \ldots \hat{t_m} = h \hat{t_1} \ldots \hat{t_m}
\]

where \text{bot} is a fresh constant constructor symbol. Consider also the program \( \mathcal{P}' = \mathcal{P} \uplus \mathcal{P}_{g_t} \), where \( \mathcal{P}_{g_t} \) consists of the following rules defining some fresh symbols \( g_s \in \text{FS} \):

\[
g_{X} \rightarrow U \quad g_{\bot} \rightarrow \text{bot} \quad g(h \, t_1 \ldots t_m)(h \, X_1 \ldots X_m) \rightarrow h \, (g_{t_1}X_1) \ldots (g_{t_m}X_m)
\]

Then:

(i) \( \mathcal{P}' \) is a safe extension of \( (\mathcal{P}, e) \).

(ii) \( \hat{t} \in [e]^P \) iff \( \hat{t} \in [g_t \, e]^P \), for any \( e \in \text{Exp}_\bot, t \in \text{Pat}_\bot \) built with the signature of \( \mathcal{P} \).
Proof. It is clear that $\mathcal{P}'$ is a safe extension as it only defines new rules for fresh function symbols. The other equivalence holds by two simple inductions on the structure of $t$.

Proof (For Theorem 3). First of all we will prove the full abstraction wrt. $\mathcal{O}$. We will see that $[e]^P = [e']^P$ iff for any safe extension $\mathcal{P}'$ of $(\mathcal{P}, e)$ and $(\mathcal{P}, e')$, for any context $\mathcal{C}$ built with the signature of $\mathcal{P}'$ we have $\mathcal{O}'(\mathcal{C}[e]) = \mathcal{O}'(\mathcal{C}[e'])$. Concerning the left to right implication, assume $[e]^P = [e']^P$ and fix some safe extension $\mathcal{P}'$ and some context $\mathcal{C}$ built on it. First we will see that $\mathcal{O}'(\mathcal{C}[e]) \subseteq \mathcal{O}'(\mathcal{C}[e'])$. Assume some $t \in \mathcal{O}'(\mathcal{C}[e])$, then $t \in [\mathcal{C}[e]]^P'$ by definition and Th. 2. But then

$$t \in [\mathcal{C}[e]]^P' = \bigcup_{t \in [e]} [\mathcal{C}[t]]^P'$$

by Th. 1, as $\mathcal{P}'$ is a safe extension

$$= \bigcup_{t \in [e']} [\mathcal{C}[t]]^P'$$

by Lemma 1, as $\mathcal{P}'$ is a safe extension

$$= \bigcup_{t \in [e']} [\mathcal{C}[e]]^P'$$

by Lemma 1, as $\mathcal{P}'$ is a safe extension $[\mathcal{C}[e]]^P'$ by Th. 1

But then $t \in \mathcal{O}'(\mathcal{C}[e'])$ by definition and Th. 2. The other inclusion can be proved in a similar way.

Regarding the right to left implication, we will use the transformation $\hat{\cdot}$ of Lemma 2. We can also take the program $\mathcal{P}'$ of Lemma 2 which is a safe extension of $(\mathcal{P}, e)$ and $(\mathcal{P}, e')$ as it only defines new rules for fresh function symbols. Therefore we can assume $\mathcal{O}'(\mathcal{C}[e]) = \mathcal{O}'(\mathcal{C}[e'])$ for any $\mathcal{C}$ built on $\mathcal{P}'$. Besides, for any $t \in [e]^P$ we have $\bar{t} \in \mathcal{O}'(e)$ by Lemma 2, and so $\bar{t} \in \mathcal{O}'(g_t e) = \mathcal{O}'(g_t e')$ by definition, Th. 2 and hypothesis. But then $\bar{t} \in [g_t e]^P$ by definition and Th. 2, and so $t \in [e]^P$ by Lemma 2 again. The other inclusion of $[e]$ in $[e]$ can be proved in a similar way.

Now we will prove the full abstraction wrt. $\mathcal{O}_{f_o}$. The left to right implication can be proved in exactly the same way as we did for $\mathcal{O}$. Concerning the other implication we modify the transformation $\hat{\cdot}$ of Lemma 2 in the following way:

$$h \overline{t_1 \ldots t_m} = h_m \overline{t_1 \ldots t_m}$$

where $h_m$ is a fresh constructor symbol of arity $m$. Note that then $\forall t \in \operatorname{Pat}_{\bot}$ we have $\bar{t} \in \operatorname{FOPat}$. Besides it is still easy to prove that for any $e \in \operatorname{Exp}_{\bot}$, $\bar{t} \in \operatorname{Pat}_{\bot}$ built with the signature of $\mathcal{P}$, $t \in [e]^P$ iff $\bar{t} \in [g_t e]^P'$, where $\mathcal{P}' = \mathcal{P} \cup \mathcal{P}_{g_t}$, and that $\mathcal{P}'$ is a safe extension of $\mathcal{P}$, by a trivial modification of the proof for Lemma 2. With these tools the proof proceeds exactly like in the one for $\mathcal{O}$, but using these new definitions of $\hat{\cdot}$ and $g_t$.

3.2 Discussion: the case of extra variables

As pointed in Sect. 2, in this work we assume that our programs do not contain extra variables, i.e., $\operatorname{var}(r) \subseteq \operatorname{var}(f \bar{t})$ holds for any program rule $f \bar{t}_1 \ldots \bar{t}_n \to r$. 
This condition is necessary for the full abstraction results to hold, as we can see in the following example.

Example 2. Consider a signature such that \( FS = \{ f/1, g/1 \} \), \( CS = \{ 0/0, 1/0 \} \), and the program \( \mathcal{P} = \{ f X \rightarrow Y X \} \). Note the extra variable \( Y \) in the rule for \( f \).

Then we have \([ f \ 0 ]^\mathcal{P} = \{ \bot \} = [ f \ 1 ]^\mathcal{P} \), because any derivation of \( f \ 0 \rightarrow t \) using (OR) must have the form

\[
\begin{align*}
0 \rightarrow 0 & \quad \phi \ 0 \rightarrow t \\
\mathcal{P} \vdash f \ 0 \rightarrow t & \quad OR
\end{align*}
\]

where \( \phi \) can be any pattern \((f, g, 0, 1 \text{ or } \bot)\) and \( X \) can be (OR) or (B). In all cases the only possible value for \( t \) in \( \phi \ 0 \rightarrow t \) will be \( \bot \). A similar reasoning holds for \( f \ 1 \).

However, for \( \mathcal{P}' = \mathcal{P} \uplus \{ g \ 0 \rightarrow 1 \} \), which is a safe extension for \((\mathcal{P}, f \ 0)\) and \((\mathcal{P}, f \ 1)\) we can do:

\[
\begin{align*}
0 \rightarrow 0 & \quad 0 \rightarrow 0 \quad 1 \rightarrow 1 \\
\mathcal{P}' \vdash f \ 0 \rightarrow 1 & \quad OR
\end{align*}
\]

while for \( f \ 1 \) we can only do:

\[
\begin{align*}
1 \rightarrow 1 & \quad g \ 0 \rightarrow \bot \\
\mathcal{P}' \vdash f \ 1 \rightarrow \bot & \quad B
\end{align*}
\]

Hence the context \([ \ ]\) and the safe extension \( \mathcal{P}' \) yield different observations for \( f \ 0 \) and \( f \ 1 \).

The previous example can be discarded if we assume that we have at least one constructor for each arity, or at least for the maximum of the arities of function symbols. This is reasonable because it is like having tuples of any arity. With this assumption and the previous program and expression we do not have \([ f \ a ]^\mathcal{P} = [ f \ b ]^\mathcal{P} \) anymore, as \( c \ a \in [ f \ a ] \) and \( c \ b \in [ f \ b ] \), hence the hypothesis of the condition for full abstraction fails.

Nevertheless the following example shows that full abstraction fails even under the assumption of having a constructor for each arity.

Example 3. For \( \mathcal{P} = \{ f \ 1 \rightarrow 2, h X \rightarrow f \ (Y \ X) \} \) and \( FS = \{ f/1, h/1, g/1 \} \) we have \( \forall \theta \in PSubst_{\bot, 1} \notin \{ (\theta(Y)) \ 0 \} \mathcal{P} \cup \{ (\theta(Y)) \ 1 \} \mathcal{P} \), hence \([ h \ 0 ]^\mathcal{P} = \{ \bot \} = [ h \ 1 ]^\mathcal{P} \). But for \( \mathcal{P}' = \mathcal{P} \uplus \{ g \ 0 \rightarrow 1 \} \), which is a safe extension for \((\mathcal{P}, h \ 0)\) and \((\mathcal{P}, h \ 1)\), we have \( \mathcal{P}' \vdash h \ 0 \rightarrow 2 \) while \( \mathcal{P}' \vdash h \ 1 \not\rightarrow 2 \).

The point is that, if extra variables are allowed, for a fixed program \( \mathcal{P} \) and an expression \( e \) we cannot ensure that for any safe extension \( \mathcal{P}' \) for \((\mathcal{P}, e)\) it holds that \( [ e ]^\mathcal{P} = [ e ]^{\mathcal{P}'} \); i.e., Lemma 1 does not hold. We cannot even grant that \( [ e ]^\mathcal{P} = [ e ]^{\mathcal{P}'} \) implies that \( [ e ]^{\mathcal{P}'} = [ e ]^{\mathcal{P}'} \) for any safe extension \( \mathcal{P}' \), which in fact is what it is needed for full abstraction, and what we have exploited in
the (counter-)examples above. It is also relevant that both examples are left-FO programs, and therefore the problems do not come from the presence of higher order patterns in function definitions.

As a conclusion of this discussion, we contemplate the extension of this work to cope with extra variables as a challenging subject of future work.

4 Conclusions and Future Work

We have seen that reasoning extensionally in existing FLP languages with HO nondeterministic functions is not valid in general (Ex. 1, Prop. 2). In contrast, thinking in intensional functions is not an arbitrary exoticism, but rather an appropriate point of view for that setting (Th. 3). We stress the fact that adopting an intensional view of the meaning of functions is compatible with a discipline of programming in which programs are restricted to be left-FO, that is, the use of HO-patterns in left-hand sides of program rules is forbidden. This is the preferred choice by some people in the FLP community, mostly because HO-patterns in left-hand sides cause some problems to the type system. Our personal opinion is the following: since HO-patterns appear in the semantics even if they are precluded from programs, they could be freely permitted, at least as far as they are compatible with the type discipline. There are quite precise works [8] pointing out which are the problematic aspects, mainly opacity of patterns. Existing systems could incorporate restrictions, so that only type-safe uses of HO-patterns are allowed. More work could be done along this line.

We have seen in Sect. 3.2 how the presence of extra variables in programs destroys full-abstraction of the HOCRWL semantics. Recovering it for such family of programs is an obvious subject of future work. Another very interesting, and somehow related matter, is giving variables a more active role in the semantics. Certainly, the results in the paper are not restricted to ground expressions, but their interest for expressions having variables is limited by the fact that in the notions of semantics and observations considered in the paper, variables are implicitly treated as generic constants. For instance, the expressions $e_1 = X + X$ and $e_2 = X + 0$ do have the same semantics $[\bot]$, ([$e_1]_{\bot} = [e_2]_{\bot} = \{\bot\}$). Full abstraction of $[\bot]$ ensures that $O(C[e_1]) = O(C[e_2])$ for any context $C$. This is ok as far as one is only interested in possible reductions starting from $e_1$, $e_2$. If this is the case, certainly $e_1$ and $e_2$ have equivalent behavior (no successful reduction to a pattern can be done with any of them). However, in some sense $e_1$ and $e_2$ have different ‘meanings’, that are reflected in different behaviors; for instance, if $e_1$ and $e_2$ are subject to narrowing, or if $e_1$ and $e_2$ are used as right hand sides in a program rule.

Acknowledgments We are grateful to Rafa Caballero for his intense collaboration while developing this research.
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