HOLOMORPHIC CURVES IN BLOWN UP OPEN BOOKS

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Abstract. We use contact fiber sums of open book decompositions to define an infinite hierarchy of filling obstructions for contact 3-manifolds, called planar $k$-torsion for integers $k \geq 0$, all of which cause the contact invariant in Embedded Contact Homology to vanish. Planar 0-torsion is equivalent to overtwistedness, while every contact manifold with Giroux torsion also has planar 1-torsion, and we give examples of contact manifolds that have planar $k$-torsion for any $k \geq 2$ but no Giroux torsion, leading to many new examples of nonfillable contact manifolds. We show also that the complement of the binding of a supporting open book never has planar torsion. The technical basis of these results is an existence and uniqueness theorem for $J$-holomorphic curves with positive ends approaching the (possibly blown up) binding of an ensemble of open book decompositions.

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1. Introduction

1.1. Summary of main results. Suppose $(W, \omega)$ is a compact symplectic manifold with boundary $\partial W = M$, and $\xi$ is a positive, cooriented contact structure on $M$. We say that $(W, \omega)$ is a weak symplectic filling of $(M, \xi)$ if

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\[ \omega|_\xi > 0, \text{ and it is a strong symplectic filling if some neighborhood of } M \subset W \text{ admits a 1-form } \lambda \text{ with } d\lambda = \omega \text{ and } \ker \lambda|_M = \xi. \]

One of the foundational results of modern contact topology was the theorem of Gromov [Gro85] and Eliashberg [Eli90], that a closed contact 3-manifold admits no symplectic filling if it is overtwisted. In light of more recent developments, nonfillability in the overtwisted case can be viewed as a corollary of the vanishing of certain invariants, e.g. contact homology:

**Theorem** ([Yau06]). If \((M, \xi)\) is an overtwisted contact 3-manifold, then its contact homology is trivial.

Recall that contact homology (cf. [EGH00]) is a variation on the Floer homology of Hamiltonian systems, in which the chain complex is generated by closed orbits of a Reeb vector field in \((M, \xi)\) and the differential counts punctured holomorphic spheres in the symplectization \(\mathbb{R} \times M\) with one positive and finitely many negative ends. The resulting homology has the structure of a graded commutative algebra with unit, which is then trivial if and only if the unit is exact. In the appendix to [Yau06], Eliashberg sketches a proof of this vanishing result which uses the existence of a solid torus foliated by holomorphic planes with Morse-Bott asymptotic orbits of arbitrarily small period (Figure 1). After a nondegenerate perturbation of the Morse-Bott family, one finds a distinguished rigid plane that is the only holomorphic curve with its particular asymptotic orbit \(\gamma\) at \(+\infty\), giving a relation of the form \(\partial q_\gamma = 1\) in the contact homology chain complex.

In recent years, invariants based on holomorphic curves in symplectizations have not been applied much to symplectic filling questions, as there were fairly few existence and uniqueness results that could yield useful calculations, and in any case the analytical foundations of contact homology (and more generally Symplectic Field Theory) are still work in progress. In contrast, invariants based on Seiberg-Witten theory [KM97] and Heegaard Floer homology [OS05] have enjoyed great success in detecting new examples of nonfillable contact manifolds. A test case is provided by the question of Giroux torsion, which was suspected since the early 1990’s to furnish an obstruction to strong filling, but this was not proved until 2006, by David Gay [Gay06] using Seiberg-Witten theory. A second proof was produced a short time later, as a consequence of a computation of the Ozsváth-Szabó contact invariant. In particular:

**Theorem** ([GHVHM] and [GH]). If \((M, \xi)\) has Giroux torsion, then its Ozsváth-Szabó contact invariant vanishes. Moreover, if the Giroux torsion domain (see Definition 1.20) separates \(M\), then the Ozsváth-Szabó invariant with twisted coefficients also vanishes.

The vanishing result with twisted coefficients, proved in [GH], provides a new proof of the original Eliashberg-Gromov result for overtwisted contact manifolds, in addition to a large new class of tight contact 3-manifolds
that are not weakly fillable. Quite recently, Patrick Massot [Mas] has also produced examples of contact manifolds with no Giroux torsion for which
the untwisted Ozsváth-Szabó invariant vanishes.

Holomorphic curves recently reappeared in this subject through a third
proof of the Giroux torsion result, due to the author [Wen10b]. Though
it was not presented as such in that paper, the argument in [Wen10b]
can be viewed as a vanishing result for the $ECH$ contact invariant, i.e. the
distinguished class in Embedded Contact Homology that is conjectured to
be isomorphic to the Ozsváth-Szabó contact invariant, and recently proved
by Taubes [Tau] to be isomorphic to the corresponding object in Seiberg-
Witten Floer homology. Let us briefly recall the definition of $ECH$; a more
detailed explanation, including the version with twisted coefficients, will
be given in §3.2. Given a contact 3-manifold $(M, \xi)$ with nondegenerate
contact form $\lambda$ and generic compatible almost complex structure $J$ on its
symplectization, one defines a chain complex generated by so-called orbit
sets,

$$\gamma = ((\gamma_1, m_1), \ldots, (\gamma_n, m_n)),$$

where $\gamma_1, \ldots, \gamma_n$ are closed Reeb orbits and $m_1, \ldots, m_n$ are positive inte-
gers, called multiplicities. A differential operator $\partial$ is then defined by
counting a certain class of embedded rigid $J$-holomorphic curves in the
symplectization of $(M, \xi)$, which can be viewed as cobordisms between or-
bit sets. The homology of the resulting chain complex is the Embedded
Contact Homology $ECH_*(M, \lambda, J)$

\[
\partial \gamma = \emptyset,
\]

which follows from the existence of a unique embedded $J$-holomorphic
curve with positive ends approaching the orbits represented by $\gamma$, and no
negative ends. It should be clear why Eliashberg’s calculation of vanishing
contact homology fits into this framework, and the proof in [Wen10b] uses
a similar construction: instead of a family of planes, Giroux torsion gives
a family of holomorphic cylinders (Figure 2) which can be shown to be the
only cylinders asymptotic to their particular Reeb orbits.

The constructions in Figures 1 and 2 can be recast in purely contact geo-
metric terms by interpreting the embedded holomorphic curves as pages of

\[1\text{The chain complex obviously depends on } \lambda \text{ and } J, \text{ but due to the isomorphism with Seiberg-Witten Floer homology, it is now known that } ECH_*(M, \lambda, J) \text{ is an invariant of } (M, \xi), \text{ and actually even defines a topological invariant. Unfortunately it is not yet known how to prove this entirely within the context of } ECH.\]
Figure 1. An embedded $J$-holomorphic plane asymptotic to a Reeb orbit of small period in a Morse-Bott family (arrows indicate the Reeb vector field). Every overtwisted contact manifold contains a solid torus that can be foliated by a family of such planes, each of which is a subset of an overtwisted disk.

Figure 2. Any contact manifold with Giroux torsion contains a thickened torus foliated by $J$-holomorphic cylinders asymptotic to Morse-Bott Reeb orbits of small period.
open book decompositions that support the contact structure, where the binding circles have been “blown up,” i.e. replaced by 2-tori. In both cases there is also an extra region of “padding” just outside the blown up open book, which is also foliated by surfaces transverse to the Reeb vector field. From this perspective, one can imagine using similar ideas with more general open books to define much more general filling obstructions, and this will be the main subject in the present paper. In particular, we shall reinterpret both overtwistedness and Giroux torsion within the framework of an infinite hierarchy of new filling obstructions that we call planar torsion. Its definition combines two simple notions that are fundamental in contact topology: open book decompositions supporting contact structures, as introduced by Giroux [Gir], and the contact fiber sum along codimension 2 contact submanifolds, originally due to Gromov [Gro86] and Geiges [Gei97].

We will define a contact manifold to have planar \( k \)-torsion for some integer \( k \geq 0 \) if it contains a certain kind of subset called a planar \( k \)-torsion domain. Roughly speaking, the latter is a compact contact 3-manifold \((M, \xi)\), possibly with boundary, that contains a nonempty set of disjoint pre-Lagrangian tori dividing it into two pieces:

- A planar piece \( M^P \), which is disjoint from \( \partial M \) and looks like a connected open book with some binding components blown up and/or attached to each other by contact fiber sums. The pages must have genus zero and \( k + 1 \) boundary components.
- The padding \( M \setminus M^P \), which contains \( \partial M \) and consists of one or more arbitrary open books, again with some binding components blown up or fiber summed together.

The tori bounding \( M^P \) are themselves the result of a contact fiber sum operation that attaches two separate open books to each other along components of their binding. Some simple special cases of the form \( S^1 \times \Sigma \) are shown in Figure 3, and we will give a precise explanation of the general construction in \( \S 1.2 \). In a limited set of cases, summing open books together in this way leads to a contact manifold that may be strongly fillable, but whose fillings can be classified up to symplectic deformation in terms of diffeomorphism classes of Lefschetz fibrations—this classification will be examined in the follow-up paper [Wena]. Outside of these cases, the result is always a nonfillable contact manifold, and this situation will be our focus in all that follows. More precise versions of the following results will be explained in \( \S 1.2 \).

Summary of Main Results.

1. If \((M, \xi)\) has planar \( k \)-torsion for any integer \( k \geq 0 \), then it is not strongly fillable; in fact, it does not admit a contact type embedding into any closed symplectic 4-manifold (cf. Theorems 1 and 2).
Moreover, its untwisted ECH contact invariant vanishes (cf. Theorem 3).

(2) Under an extra topological assumption on a planar torsion domain, analogues of the above statements also hold for weak fillings and the twisted ECH contact invariant (cf. Theorem 4).

(3) \((M, \xi)\) has planar 0-torsion if and only if it is overtwisted, and every contact manifold with Giroux torsion also has planar 1-torsion (cf. Corollary 2).

(4) For every integer \(k \geq 2\), there exist contact manifolds that have planar \(k\)-torsion but no Giroux torsion (cf. Corollary 5 and Figure 7).
(5) If \((M, \xi)\) is supported by an open book decomposition with binding \(B \subset M\), then \(M \setminus B\) has no planar \(k\)-torsion for any \(k \geq 0\) (cf. Theorem \(\text{[3]}\)).

**Remark 1.1.** One can rephrase the above results by defining a contact invariant,  
\[
\text{PT}(M, \xi) := \sup \left\{ k \geq 0 \mid (M, \xi) \text{ has no planar } \ell\text{-torsion for any } \ell < k \right\},
\]
which takes values in \(\mathbb{N} \cup \{0, \infty\}\) and is infinite if and only if \((M, \xi)\) has no planar torsion. Then \(\text{PT}(M, \xi) < \infty\) always implies that \((M, \xi)\) is not strongly fillable, and one can think of manifolds with larger values of \(\text{PT}(M, \xi)\) as being “closer” to being fillable, a statement that can be made more precise in the algebraic setting of Symplectic Field Theory, cf. \([\text{LW}]\).

Our results thus show that \(\text{PT}(M, \xi) \leq 1\) whenever \((M, \xi)\) has Giroux torsion, \(\text{PT}(M, \xi) = 0\) if and only if \((M, \xi)\) is overtwisted, and there exist contact manifolds without Giroux torsion such that \(\text{PT}(M, \xi) < \infty\). One can show in fact that the examples treated in Corollary \(\text{[5]}\) have \(\text{PT}(M, \xi) = k\) for any given integer \(k \geq 2\), but this requires SFT methods which go beyond the scope of the present paper, see \([\text{LW}]\). It should be emphasized here that the scale defined by the invariant \(\text{PT}(M, \xi)\) measures something completely different from the standard quantitative measurement of Giroux torsion; the latter counts the maximum number of adjacent Giroux torsion domains that can be embedded in \((M, \xi)\), and can take arbitrarily large values while \(\text{PT}(M, \xi) \leq 1\). Likewise, \((M, \xi)\) has Giroux torsion zero whenever \(\text{PT}(M, \xi) \geq 2\).

Note that whenever the ECH contact invariant of \((M, \xi)\) vanishes, the Taubes isomorphism together with results in Seiberg-Witten theory \([\text{KM97}]\) imply the fact that \((M, \xi)\) is not fillable, but this makes for an extremely indirect proof of nonfillability. The right proof in our context should be one that uses only holomorphic curves, and for the case of planar torsion and strong fillings we will also do this, thus avoiding the use of Seiberg-Witten theory. There is a similarly more direct proof for the result on weak fillings, which is carried out in a separate paper of the author with Klaus Niederkrüger \([\text{NW}]\). The simplest examples of planar \(k\)-torsion with \(k \geq 2\) are manifolds of the form \(S^1 \times \Sigma\) with \(S^1\)-invariant contact structures: we will explain these in \(\S\text{1.3}\). The proofs of the results stated above and in \(\S\text{1.2}\) will then be given in \(\S\text{3}\) based on some more technical results in \(\S\text{2}\).

The technical work that makes these computations possible is a new existence and uniqueness theorem, proved in \(\S\text{2.2}\) for certain holomorphic curves in the presence of supporting open book decompositions. Let us briefly explain the main idea and its context.

We recall first some basic notions involving open books: if \(M\) is a closed and oriented 3-manifold, an *open book decomposition* is a fibration  
\[
\pi : M \setminus B \to S^1,
\]
where $B \subset M$ is an oriented link called the binding, and the closures of the fibers are called pages: these are compact, oriented and embedded surfaces with oriented boundary equal to $B$. An open book is called planar if the pages are connected and have genus zero, and it is said to support a contact structure $\xi$ if the latter can be written as $\ker \lambda$ for some contact form $\lambda$ (called a Giroux form) whose induced Reeb vector field $X_\lambda$ is positively transverse to the interiors of the pages and positively tangent to the binding. The latter definition is due to Giroux [Gir], who established a groundbreaking one-to-one correspondence between isomorphism classes of contact manifolds and their supporting open books up to right-handed stabilization.

Historically, open books appeared in contact topology much earlier via the existence theorem of Thurston and Winkelnkemper [TW75], which in modern terminology says that every open book decomposition on a 3-manifold supports a contact structure. In the 1990’s, they also appeared in dynamical applications due to Hofer, Wysocki and Zehnder [HWZ95b, HWZ98], whose work exploited the fact that certain families of punctured, asymptotically cylindrical $J$-holomorphic curves in the symplectization $\mathbb{R} \times M$ of a contact 3-manifold $M$ naturally give rise to the pages of supporting open books on $M$—indeed, whenever such a curve has an embedded projection to $M$, the nonlinear Cauchy-Riemann equation guarantees that this embedding is also positively transverse to $X_\lambda$. It is thus natural to ask whether every supporting open book can be lifted in this way to a family of holomorphic curves in the symplectization, typically referred to as a holomorphic open book.

Hofer observed in [Hof00] a fundamental problem with this idea in general: looking closely at the punctured version of the Riemann-Roch formula, one finds that the space of holomorphic curves with the necessary intersection theoretic properties to form open books has the correct dimension if and only if the genus is zero, and otherwise its virtual dimension is too low. Thus for planar open books there is no trouble—this fact was used by Abbas, Cieliebak and Hofer [ACH05] to solve the Weinstein conjecture for planar contact manifolds, and has since also found applications to the study of symplectic fillings [Wen10b]. Hofer suggested handling the higher genus case by introducing a “cohomological perturbation” into the nonlinear Cauchy-Riemann equation in order to raise the Fredholm index. This program has recently been carried out by Casim Abbas [Abb] (see also [vB]), though applications to problems such as the Weinstein conjecture are as yet elusive, as the compactness theory for the modified nonlinear Cauchy-Riemann equation is quite difficult (cf. [AHL]).

An alternative approach was introduced by the present author in [Wen10c], which produced a concrete construction of the planar holomorphic open books needed for the applications [ACH05] and [Wen10b], but also produced a non-generic construction that works for arbitrary open books, of
any genus. The non-generic construction breaks down under small pertur-
bations of the data, which may seem to limit its value, but in fact it is
quite useful: we will adapt it in \S2 to extend the results of [Wen10] to a
full classification of $J$-holomorphic curves in $\mathbb{R} \times M$ whose positive ends
approach the binding orbits of a supporting open book. We shall do this in
a more general context, involving not just one but potentially an ensemble
of open books with some of their binding orbits blown up and attached to
each other by contact fiber sums. The key fact making all of this possible
is that the specific almost complex structure for which holomorphic open
books always exist is compatible with an exact stable Hamiltonian struc-
ture, which admits a well behaved perturbation to a suitable contact form.
For the ensemble of open books provided by a planar torsion domain, this
result will make it easy to find an orbit set in the ECH chain complex that
satisfies $\partial \gamma = 0$.

The results in \S2 have further applications not treated in this paper, of
which the most obvious is the computation of a related filling obstruction
that lives in Symplectic Field Theory. The latter is a generalization of con-
tact homology introduced by Eliashberg, Givental and Hofer [EGH00], that
defines contact invariants by counting $J$-holomorphic curves with arbitrary
genus and positive and negative ends in symplectizations of arbitrary di-

mension. As shown in joint work of the author with Janko Latschev [LW],
SFT contains a natural notion of algebraic torsion, which one can define
as follows. In the picture developed by Cieliebak and Latschev in [CL09],
SFT associates to any contact manifold with appropriate choices of con-
tact form and almost complex structure a graded $BV_\infty$-algebra generated
by the symbols $q_\gamma$ (for each “good” closed Reeb orbit $\gamma$) and $\hbar$; the latter
can be regarded as part of the coefficient ring, but must be handled with
care since it has a nontrivial degree. The homology of this $BV_\infty$-algebra
is then an invariant of the contact structure, and it is natural to say that
$(M, \xi)$ has algebraic $k$-torsion if the homology satisfies the relation

$$[\hbar^k] = 0.$$  

For $k = 0$, this means $1 = 0$ and thus coincides with the notion of al-
gebraic overtwistedness (cf. [BN10]). It follows easily from the formalism
of SFT\footnote{For this informal discussion we are taking it for granted that SFT is well defined, which was not proved in [EGH00] and is quite far from obvious. The rigorous definition of SFT, including the necessary abstract perturbations to achieve transversality, is a large project in progress by Hofer-Wysocki-Zehnder, see for example [Hof06].} that algebraic torsion of any order gives an obstruction to strong symplectic filling; in fact it is stronger, since it also implies obstructions
to the existence of exact symplectic cobordisms between certain contact manifolds. We will not go into these matters any further here, except to
mention the following justification for keeping track of the integer $k \geq 0$
in planar $k$-torsion:
Theorem (LW). If \((M, \xi)\) has planar \(k\)-torsion, then it also has algebraic \(k\)-torsion.

The fact that planar torsion obstructs symplectic filling can thus also be viewed as a consequence of a computation in SFT. In [LW], we show that among the examples treated in §1.3, one can find contact manifolds \((M_k, \xi_k)\) for any integer \(k \geq 1\) that have algebraic torsion of order \(k\) but not \(k-1\): it follows that \(k\) is also the smallest integer for which \((M_k, \xi_k)\) has planar \(k\)-torsion. The formalism of SFT then implies that there is no exact symplectic cobordism with positive boundary \((M_k^+, \xi_k^+)\) and negative boundary \((M_k^-, \xi_k^-)\) if \(k^- > k^+\). Thus the enhanced algebraic structure of SFT leads to non-existence results that are not immediately apparent from any of the distinctly 3-dimensional invariants discussed above. Another obvious advantage of SFT is that one can sensibly define algebraic torsion for contact manifolds of arbitrary dimensions, not only dimension \(3\). In higher dimensions however, no examples are yet known except for 0-torsion, e.g. a contact manifold is algebraically overtwisted whenever it contains a plastiktufe [Nie06, BN] or is supported by a left-handed stabilization of an open book [BvK].

1.2. Planar torsion and its consequences. We now proceed to make the aforementioned definitions and results precise.

1.2.1. Summed open books. Assume \(M\) is an oriented smooth manifold containing two disjoint oriented submanifolds \(N_1, N_2 \subset M\) of real codimension 2, which admit an orientation preserving diffeomorphism \(\varphi : N_1 \to N_2\) covered by an orientation reversing isomorphism \(\Phi : \nu N_1 \to \nu N_2\) of their normal bundles. Then we can define a new smooth manifold \(M_\Phi\), the normal sum of \(M\) along \(\Phi\), by removing neighborhoods \(\mathcal{N}(N_1)\) and \(\mathcal{N}(N_2)\) of \(N_1\) and \(N_2\) respectively, then gluing together the resulting manifolds with boundary along an orientation reversing diffeomorphism
\[
\partial \mathcal{N}(N_1) \to \partial \mathcal{N}(N_2)
\]
determined by \(\Phi\). This operation determines \(M_\Phi\) up to diffeomorphism, and is also well defined in the contact category: if \((M, \xi)\) is a contact manifold and \(N_1, N_2\) are contact submanifolds with \(\varphi : N_1 \to N_2\) a contactomorphism, then \(M_\Phi\) admits a contact structure \(\xi_\Phi\), unique up to isotopy, which agrees with \(\xi\) away from \(N_1\) and \(N_2\) (cf. [Gei08 §7.4]). We then call \((M_\Phi, \xi_\Phi)\) the contact fiber sum of \((M, \xi)\) along \(\Phi\).

We will consider the special case of this construction where \(N_1\) and \(N_2\) are disjoint components\(^3\) of the binding of an open book decomposition
\[
\pi : M \setminus B \to S^1
\]
\(^3\)We use the word *component* throughout to mean any open and closed subset, i.e. a disjoint union of connected components.
that supports $\xi$. Then $N_1$ and $N_2$ are automatically contact submanifolds, whose normal bundles come with distinguished trivializations determined by the open book. In the following, we shall always assume that $M$ is oriented and the pages and binding are assigned the natural orientations determined by the open book, so in particular the binding is the oriented boundary of the pages.

**Definition 1.2.** Assume $\pi : M \setminus B \to S^1$ is an open book decomposition on $M$. By a *binding sum* of the open book, we mean any normal sum $M_\Phi$ along an orientation reversing bundle isomorphism $\Phi : \nu N_1 \to \nu N_2$ covering a diffeomorphism $\varphi : N_1 \to N_2$, where $N_1, N_2 \subset B$ are disjoint components of the binding and $\Phi$ is constant with respect to the distinguished trivialization determined by $\pi$. The resulting smooth manifold will be denoted by

$$M(\pi, \varphi) := M_\Phi,$$

and we denote by $\mathcal{I}(\pi, \varphi) \subset M(\pi, \varphi)$ the closed hypersurface obtained by the identification of $\partial N_1$ with $\partial N_2$, which we'll also call the *interface*. We will then refer to the data $(\pi, \varphi)$ as a *summed open book decomposition* of $M(\pi, \varphi)$, whose *binding* is the (possibly empty) codimension 2 submanifold

$$B_\varphi := B \setminus (N_1 \cup N_2) \subset M(\pi, \varphi).$$

The *pages* of $(\pi, \varphi)$ are the connected components of the fibers of the naturally induced fibration

$$\pi_\varphi : M(\pi, \varphi) \setminus (B_\varphi \cup \mathcal{I}(\pi, \varphi)) \to S^1;$$

if $\dim M = 3$, then these are naturally oriented open surfaces whose closures are generally immersed (distinct boundary components may sometimes coincide).

If $\xi$ is a contact structure on $M$ supported by $\pi$, we will denote the induced contact structure on $M(\pi, \varphi)$ by

$$\xi(\pi, \varphi) := \xi_\Phi$$

and say that $\xi(\pi, \varphi)$ is *supported* by the summed open book $(\pi, \varphi)$.

It follows from the corresponding fact for ordinary open books that every summed open book decomposition supports a contact structure, which is unique up to isotopy: in fact it depends only on the isotopy class of the open book $\pi : M \setminus B \to S^1$, the choice of binding components $N_1, N_2 \subset B$ and isotopy class of diffeomorphism $\varphi : N_1 \to N_2$.

Throughout this discussion, $M, N_1, N_2$ and the pages of $\pi$ are all allowed to be disconnected (note that $\pi : M \setminus B \to S^1$ will have disconnected pages if $M$ itself is disconnected). In this way, we can incorporate the notion of a binding sum of *multiple*, separate (perhaps summed) open books, e.g. given $(M_i, \xi_i)$ supported by $\pi_i : M_i \setminus B_i \to S^1$ with components $N_i \subset B_i$ for $i = 1, 2$, and a diffeomorphism $\varphi : N_1 \to N_2$, a binding sum of $(M_1, \xi_1)$
with \((M_2, \xi_2)\) can be defined by applying the above construction to the disjoint union \(M_1 \sqcup M_2\). We will generally use the shorthand notation

\[ M_1 \boxplus M_2 \]

to indicate manifolds constructed by binding sums of this type, where it is understood that \(M_1\) and \(M_2\) both come with contact structures and supporting summed open books, which combine to determine a summed open book and supported contact structure on \(M_1 \boxplus M_2\).

**Example 1.3.** Consider the tight contact structure on \(M := S^1 \times S^2\) with its supporting open book decomposition

\[ \pi : M \setminus (\gamma_0 \cup \gamma_\infty) \to S^1 : (t, z) \mapsto z/|z|, \]

where \(S^2 = \mathbb{C} \cup \{\infty\}\), \(\gamma_0 := S^1 \times \{0\}\) and \(\gamma_\infty := S^1 \times \{\infty\}\). This open book has cylindrical pages and trivial monodromy. Now let \(M'\) denote a second copy of the same manifold and

\[ \pi' : M' \setminus (\gamma'_0 \cup \gamma'_\infty) \to S^1 \]

the same open book. Defining the binding sum \(M \boxplus M'\) by pairing \(\gamma_0\) with \(\gamma'_0\) and \(\gamma_\infty\) with \(\gamma'_\infty\), we obtain the standard contact \(T^3\). In fact, each of the tight contact tori \((T^3, \xi_n)\), where

\[ \xi_n = \ker [\cos(2\pi n \theta) \, dx + \sin(2\pi n \theta) \, dy] \]

in coordinates \((x, y, \theta) \in S^1 \times S^1 \times S^1\), can be obtained as a binding sum of \(2n\) copies of the tight \(S^1 \times S^2\); see Figure 4.

**Example 1.4.** Using the same open book decomposition on the tight \(S^1 \times S^2\) as in Example 1.3 one can take only a single copy and perform a binding sum along the two binding components \(\gamma_0\) and \(\gamma_\infty\). The contact manifold produced by this operation is the quotient of \((T^3, \xi_1)\) by the contact involution \((x, y, \theta) \mapsto (-x, -y, \theta + 1/2)\), and is thus the torus bundle over \(S^1\) with monodromy \(-1\). The resulting summed open book on \(T^3/\mathbb{Z}_2\) has connected cylindrical pages, empty binding and a single interface torus of the form \(\mathcal{I}(\pi, \varphi) = \{2\theta = 0\}\), inducing a fibration

\[ \pi_\varphi : (T^3/\mathbb{Z}_2) \setminus \mathcal{I}(\pi, \varphi) \to S^1 : [(x, y, \theta)] \mapsto \begin{cases} y & \text{if } \theta \in (0, 1/2), \\ -y & \text{if } \theta \in (1/2, 1). \end{cases} \]

The following two special cases of summed open books are of crucial importance.

**Example 1.5.** An ordinary open book can also be regarded as a summed open book: we simply take \(N_1\) and \(N_2\) to be empty.

**Example 1.6.** Suppose \((M_i, \xi_i)\) for \(i = 1, 2\) are closed connected contact 3-manifolds with supporting open books \(\pi_i\) whose pages are diffeomorphic. Then we can set \(N_1 = B_1\) and \(N_2 = B_2\), choose a diffeomorphism \(B_1 \to B_2\)
Figure 4. Two ways of producing tight contact tori from $2n$ copies of the tight $S^1 \times S^2$. At left, copies of $S^1 \times S^2$ are represented by open books with two binding components (depicted here through the page) and cylindrical pages. For each dotted oval surrounding two binding components, we construct the binding sum to produce the manifold at right, containing $2n$ special pre-Lagrangian tori (the black line segments) that separate regions foliated by cylinders. The results are $(T^3, \xi_1)$ for $n = 1, 2$.

and define $M = M_1 \sqcup M_2$ accordingly. The resulting summed open book is called symmetric; observe that it has empty binding, since every binding component of $\pi_1$ and $\pi_2$ has been summed. A simple example of this construction is $(T^3, \xi_1)$ as explained in Example 1.3, and for an even simpler example, summing two open books with disk-like pages produces the tight $S^1 \times S^2$.

Remark 1.7. There is a close relationship between summed open books and the notion of open books with quasi-compatible contact structures, introduced by Etnyre and Van Horn-Morris [EVHM]. A contact structure $\xi$
is said to be \textit{quasi-compatible} with an open book if it admits a contact vector field that is positively transverse to the pages and positively tangent to the binding; if the contact vector field is also positively transverse to \( \xi \), then this is precisely the supporting condition, but quasi-compatibility is quite a bit more general, and can allow e.g. open books with empty binding. A summed open book on a 3-manifold gives rise to an open book with quasi-compatible contact structure whenever a certain orientation condition is satisfied: this is the result in particular whenever we construct binding sums of separate open books that are labeled with signs in such a way that every interface torus separates a positive piece from a negative piece. Thus the tight 3-tori in Figure 4 are examples, in this case producing an open book with empty binding (i.e. a fibration over \( S^1 \)) that is quasi-compatible with all of the contact structures \( \xi_n \). However, it is easy to construct binding sums for which this is not possible, e.g. Example 1.4.

1.2.2. Blown up open books and Giroux forms. We now generalize the discussion to include manifolds with boundary. Suppose \( M(\pi, \varphi) \) is a closed 3-manifold with summed open book \((\pi, \varphi)\), which has binding \( B_\varphi \) and interface \( I(\pi, \varphi) \), and \( N \subset B_\varphi \) is a component of its binding. For each connected component \( \gamma \subset N \), identify a tubular neighborhood \( N(\gamma) \) of \( \gamma \) with a solid torus \( S^1 \times D \), defining coordinates \((\theta, \rho, \phi)\) on \( S^1 \times D \), where \((\rho, \phi)\) denote polar coordinates\(^4\) on the disk \( D \) and \( \gamma \) is the subset \( S^1 \times \{0\} = \{\rho = 0\} \). Assume also that these coordinates are adapted to the summed open book, in the sense that the orientation of \( \gamma \) as a binding component agrees with the natural orientation of \( S^1 \times \{0\} \), and the intersections of the pages with \( N(\gamma) \) are of the form \( \{\phi = \text{const}\} \). This condition determines the coordinates up to isotopy. Then we define the \textit{blown up} manifold \( M(\pi, \varphi, \gamma) \) from \( M(\pi, \varphi) \) by replacing \( N(\gamma) = S^1 \times D \) with \( S^1 \times [0, 1] \times S^1 \), using the same coordinates \((\theta, \rho, \phi)\) on the latter, i.e. the binding circle \( \gamma \) is replaced by a 2-torus, which now forms the boundary of \( M(\pi, \varphi, \gamma) \). If \( \xi(\pi, \varphi) \) is a contact structure on \( M(\pi, \varphi) \) supported by \((\pi, \varphi)\), then we can define an appropriate contact structure \( \xi(\pi, \varphi, \gamma) \) on \( M(\pi, \varphi, \gamma) \) as follows. Since \( \gamma \) is a positively transverse knot, the contact neighborhood theorem allows us to choose the coordinates \((\theta, \rho, \phi)\) so that

\[
\xi(\pi, \varphi) = \ker (d\theta + \rho^2 d\phi)
\]

in a neighborhood of \( \gamma \). This formula also gives a well defined distribution on \( M(\pi, \varphi, \gamma) \), but the contact condition fails at the boundary \( \{\rho = 0\} \). We fix this by making a \( C^0 \)-small change in \( \xi(\pi, \varphi) \) to define a contact structure of the form

\[
\xi(\pi, \varphi, \gamma) = \ker [d\theta + g(\rho) \; d\phi],
\]

\(^4\)Throughout this paper, we use polar coordinates \((\rho, \phi)\) on subdomains of \( \mathbb{C} \) with the angular coordinate \( \phi \) normalized to take values in \( S^1 = \mathbb{R}/\mathbb{Z} \), i.e. the actual angle is \( 2\pi \phi \).
where \( g(\rho) = \rho^2 \) for \( \rho \) outside a neighborhood of zero, \( g'(\rho) > 0 \) everywhere and \( g(0) = 0 \).

Performing the above operation at all connected components \( \gamma \subset N \subset B_\phi \) yields a compact manifold \( M(\pi, \varphi, N) \), generally with boundary, carrying a still more general decomposition determined by the data \((\pi, \varphi, N)\), which we’ll call a \textit{blown up summed open book}. We define its \textit{interface} to be the original interface \( \mathcal{I}(\pi, \varphi) \), and its \textit{binding} is \( B(\varphi, N) = B_\varphi \setminus N \).

There is a natural diffeomorphism
\[
M(\pi, \varphi) \setminus B_\varphi = M(\pi, \varphi, N) \setminus \left( B(\varphi, N) \cup \partial M(\pi, \varphi, N) \right),
\]
so the fibration \( \pi_\phi : M(\pi, \varphi) \setminus \left( B(\varphi, N) \cup \mathcal{I}(\pi, \varphi) \right) \to S^1 \) carries over to \( M(\pi, \varphi, N) \setminus \left( B(\varphi, N) \cup \mathcal{I}(\pi, \varphi) \cup \partial M(\pi, \varphi, N) \right) \), and can then be extended smoothly to the boundary to define a fibration
\[
\pi(\varphi, N) : M(\pi, \varphi, N) \setminus \left( B(\varphi, N) \cup \mathcal{I}(\pi, \varphi) \right) \to S^1.
\]

We will again refer to the connected components of the fibers of \( \pi(\varphi, N) \) as the \textit{pages} of \((\pi, \varphi, N)\), and orient them in accordance with the coorientations induced by the fibration. Their closures are immersed surfaces which occasionally may have pairs of boundary components that coincide as oriented 1-manifolds, e.g. this can happen whenever two binding circles within the same connected open book are summed to each other.

Note that the fibration \( \pi(\varphi, N) : M(\pi, \varphi, N) \setminus \left( B(\varphi, N) \cup \mathcal{I}(\pi, \varphi) \right) \to S^1 \) is not enough information to fully determine the blown up open book \((\pi, \varphi, N)\), as it does not uniquely determine the “blown down” manifold \( M(\pi, \varphi) \). Indeed, \( M(\pi, \varphi) \) determines on each boundary torus \( T \subset \partial M(\pi, \varphi, N) \) a distinguished basis
\[
\{m_T, \ell_T\} \subset H_1(T),
\]
where \( \ell_T \) is a boundary component of a page and \( m_T \) is determined by the meridian on a small torus around the binding circle to be blown up. Two different manifolds \( M(\pi, \varphi) \) may sometimes produce diffeomorphic blown up manifolds \( M(\pi, \varphi, N) \), which will however have different meridians \( m_T \) on their boundaries. Similarly, each interface torus \( T \subset \mathcal{I}(\pi, \varphi) \) inherits a distinguished basis
\[
\{\pm m_T, \ell_T\} \subset H_1(T)
\]
from the binding sum operation, with the difference that the meridian \( m_T \) is only well defined up to a sign.

The binding sum of an open book \( \pi : M \setminus B \to S^1 \) along components \( N_1 \cup N_2 \subset B \) can now also be understood as a two step operation, where the first step is to blow up \( N_1 \) and \( N_2 \), and the second is to attach the resulting boundary tori to each other via a diffeomorphism determined by \( \Phi : \nu N_1 \to \nu N_2 \). One can choose a supported contact structure on the
blown up open book which fits together smoothly under this attachment to reproduce the construction of $\xi_{(\pi,\varphi,N)}$ described above.

**Definition 1.8.** A blown up summed open book $(\pi, \varphi, N)$ is called *irreducible* if the fibers of the induced fibration $\pi_{(\varphi,N)}$ are connected.

In the irreducible case, the pages can be parametrized in a single $S^1$-family, e.g., an ordinary connected open book is irreducible, but a symmetric summed open book is not. Any blown up summed open book can however be decomposed uniquely into *irreducible subdomains*

$$M_{(\pi,\varphi,N)} = M^1_{(\pi,\varphi,N)} \cup \ldots \cup M^\ell_{(\pi,\varphi,N)},$$

where each piece $M^i_{(\pi,\varphi,N)}$ for $i = 1, \ldots, \ell$ is a compact manifold, possibly with boundary, defined as the closure in $M_{(\pi,\varphi,N)}$ of the region filled by some smooth $S^1$-family of pages. Thus $M^i_{(\pi,\varphi,N)}$ carries a natural blown up summed open book of its own, whose binding and interface are subsets of $B_{\varphi}$ and $I_{(\pi,\varphi)}$ respectively, and $\partial M^i_{(\pi,\varphi,N)} \subset I_{(\pi,\varphi)} \cup \partial M_{(\pi,\varphi,N)}$. One can also write

$$M_{(\pi,\varphi,N)} = \tilde{M}^1_{(\pi,\varphi,N)} \boxplus \ldots \boxplus \tilde{M}^\ell_{(\pi,\varphi,N)},$$

where the manifolds $\tilde{M}^i_{(\pi,\varphi,N)}$ also naturally carry blown up summed open books and can be obtained from $M^i_{(\pi,\varphi,N)}$ by blowing down $\partial M^i_{(\pi,\varphi,N)} \cap I_{(\pi,\varphi)}$.

**Definition 1.9.** Given a blown up summed open book $(\pi, \varphi, N)$ on a manifold $M_{(\pi,\varphi,N)}$ with boundary, a *Giroux form* for $(\pi, \varphi, N)$ is a contact form $\lambda$ on $M_{(\pi,\varphi,N)}$ with Reeb vector field $X_\lambda$ satisfying the following conditions:

1. $X_\lambda$ is positively transverse to the interiors of the pages,
2. $X_\lambda$ is positively tangent to the boundaries of the closures of the pages,
3. $\ker \lambda$ on each interface or boundary torus $T \subset I_{(\pi,\varphi)} \cup \partial M_{(\pi,\varphi,N)}$ induces a characteristic foliation with closed leaves homologous to the meridian $m_T$.

We will say that a contact structure on $M_{(\pi,\varphi,N)}$ is *supported* by $(\pi, \varphi, N)$ whenever it is the kernel of a Giroux form. By the procedure described above, one can easily take a Giroux form for the underlying open book $\pi : M \setminus B \to S^1$ and modify it near $B$ to produce a Giroux form for the blown up summed open book on $M_{(\pi,\varphi,N)}$. Moreover, the same argument that proves uniqueness of contact structures supported by open books (cf. [Etn06, Prop. 3.18]) shows that any two Giroux forms are homotopic to each other through a family of Giroux forms. We thus obtain the following uniqueness result for supported contact structures.

**Proposition 1.10.** Suppose $M_{(\pi,\varphi,N)}$ is a compact 3-manifold with boundary, with a contact structure $\xi_{(\pi,\varphi,N)}$ supported by the blown up summed open book $(\pi, \varphi, N)$, and $(M_{(\pi,\varphi,N)}, \xi_{(\pi,\varphi,N)})$ admits a contact embedding
into some closed contact 3-manifold \((M', \xi')\). If \(\lambda\) is a contact form on \(M'\) such that

1. \(\lambda\) defines a Giroux form on \(M(\pi, \varphi, N) \subset M'\), and
2. \(\ker \lambda = \xi'\) on \(M' \setminus M(\pi, \varphi, N)\),

then \(\ker \lambda\) is isotopic to \(\xi'\).

1.2.3. Partially planar domains. We are now ready to state one of the most important definitions in this paper.

**Definition 1.11.** A blown up summed open book on a compact manifold \(M\) is called **partially planar** if \(M \setminus \partial M\) contains a planar page. A partially planar domain is then any contact 3-manifold \((M, \xi)\) with a supporting blown up summed open book that is partially planar. An irreducible subdomain

\[ M^P \subset M \]

that contains planar pages and doesn’t touch \(\partial M\) is called a **planar piece**, and we will refer to the complementary subdomain \(M \setminus M^P\) as the **padding**.

By this definition, every planar contact manifold is a partially planar domain (with empty padding), as is the symmetric summed open book obtained by summing together two planar open books with the same number of binding components (here one can call either side the planar piece, and the other one the padding). As we’ll soon see, one can also use partially planar domains to characterize the solid torus that appears in a Lutz twist, or the thickened torus in the definition of Giroux torsion, as well as many more general objects.

Definition 1.11 generalizes the notion of a partially planar contact manifold, which appeared in [ABW10]. There it was shown that not every contact 3-manifold is partially planar, because those which are can never occur as nonseparating contact hypersurfaces in closed symplectic manifolds. It follows easily that such a contact manifold also cannot admit any strong symplectic semifillings with disconnected boundary, as one could then produce a nonseparating contact hypersurface by attaching a symplectic 1-handle and capping off the result. By a generalization of the same arguments, we will prove the following in §3.1.

**Theorem 1.** Suppose \((M, \xi)\) is a closed contact 3-manifold that contains a partially planar domain and admits a contact type embedding \(\iota : (M, \xi) \hookrightarrow (W, \omega)\) into some closed symplectic 4-manifold \((W, \omega)\). Then \(\iota\) separates \(W\).

**Corollary 1.** If \((M, \xi)\) is a closed contact 3-manifold containing a partially planar domain, then it does not admit any strong symplectic semifilling with disconnected boundary.

This corollary generalizes similar results proved by McDuff for the tight 3-sphere [McD91] and Etnyre for all planar contact manifolds [Etn04].
Remark 1.12. Corollary 1 also follows from an algebraic fact that is a weaker version of the vanishing results (Theorems 3 and 4) stated below: namely if \((M, \xi)\) contains a partially planar domain, then its ECH contact invariant is in the image of the so-called \(U\)-map. The latter is an endomorphism

\[ U : \text{ECH}_*(M, \lambda, J) \to \text{ECH}_{*-2}(M, \lambda, J) \]

defined by counting embedded index 2 holomorphic curves through a generic point, and the same arguments that we will use to prove the vanishing results show that if this point is chosen inside the planar piece, then there is an admissible orbit set \(\gamma\) such that \(U[\gamma] = [\emptyset]\). One can then use recent results about maps on ECH induced by cobordisms (cf. [HT]) to show that any contact manifold satisfying this condition also satisfies Corollary 1. We omit the details of this argument since our proof will not depend on it.

We now come to the definition of a new symplectic filling obstruction.

Definition 1.13. For any integer \(k \geq 0\), a contact manifold \((M, \xi)\), possibly with boundary, is called a planar torsion domain of order \(k\) (or briefly a planar \(k\)-torsion domain) if it is supported by a partially planar blown up summed open book \((\pi, \varphi, N)\) with a planar piece \(M^P \subset M\) satisfying the following conditions:

1. The pages in \(M^P\) have \(k + 1\) boundary components.
2. The padding \(M \setminus M^P\) is not empty.
3. \((\pi, \varphi, N)\) is not a symmetric summed open book (cf. Example 1.6).

Definition 1.14. A contact 3-manifold is said to have planar torsion of order \(k\) (or planar \(k\)-torsion) if it admits a contact embedding of a planar \(k\)-torsion domain.

Example 1.15. Some simple examples of planar torsion domains of the form \(S^1 \times \Sigma\) with \(S^1\)-invariant contact structures are shown in Figures 3 and 7. More generally, blown up summed open books can always be represented by schematic pictures as in Figure 6 which shows two examples of planar torsion domains, each with the order labeled within the planar piece. Here each picture shows a surface \(\Sigma\) containing a multicurve \(\Gamma\): each connected component \(\Sigma_0 \subset \Sigma \setminus \Gamma\) then represents an irreducible subdomain with pages diffeomorphic to \(\Sigma_0\), and the components of \(\Gamma\) represent interface tori (labeled in the picture by \(I\)). Each irreducible subdomain may additionally have binding circles, shown in the picture as circles with the label \(B\). The information in these pictures, together with a specified monodromy map for each component of \(\Sigma \setminus \Gamma\), determine a blown up summed open book uniquely up to contactomorphism. If we take these particular

\[^5\text{For the special case where } (M, \xi) \text{ is planar, the corresponding result in Heegaard Floer homology has been proved by Ozsváth, Szabó and Stipsicz [OSS05].}\]
pictures with the assumption that all monodromy maps are trivial, then the first shows a solid torus $S^1 \times \mathbb{D}$ with an overtwisted contact structure that makes one full twist along a ray from the center (the binding $B$) to the boundary. The other picture shows the complement of a solid torus in the torus bundle $T^3/\mathbb{Z}_2$ from Example 1.4. More precisely, one can construct it by taking a loop $K \subset T^3/\mathbb{Z}_2$ transverse to the pages in that example, modifying the contact structure $\xi$ near $K$ by a full Lutz twist, and then removing a smaller neighborhood $N(K)$ of $K$ on which $\xi$ makes a quarter twist. Note that the appearance of genus in this picture is a bit misleading; due to the interface torus in the interior of the bottom piece, it has planar pages with three boundary components.

The next two theorems, which we will prove in §3.1 and §3.2 respectively, are the most important results of this paper. The first follows from the second, due to the combination of Theorem 1 with Taubes’ isomorphism of Embedded Contact Homology to Seiberg-Witten Floer homology [Tau], but we will give a separate proof using only $J$-holomorphic curves and thus avoiding the substantial overhead of Seiberg-Witten theory.

**Theorem 2.** If $(M, \xi)$ is a closed contact 3-manifold with planar torsion of any order, then it does not admit a contact type embedding into any closed symplectic 4-manifold. In particular, it is not strongly fillable.

**Theorem 3.** If $(M, \xi)$ is a closed contact 3-manifold with planar torsion of any order, then its ECH contact invariant vanishes.

This result assumes integer (non-twisted) coefficients, but as is well known in the world of Heegaard Floer homology (cf. [GH]), finer invariants can be obtained by allowing twisted, i.e. group ring coefficients. In ECH, this essentially means including among the coefficients terms of the form $e^A \in \mathbb{Z}[H_2(M)]$, where $e^A$ is an element of the graded ring $\mathbb{Z}[H_2(M)]$. 

---

**Figure 5.** Schematic representations of two planar torsion domains as described in Example 1.15.
where \( A \in H_2(M) \) and by definition \( e^A e^B = e^{A+B} \). The differential is then “twisted” by inserting factors of \( e^A \) where \( A \) describes the relative homology class of the holomorphic curves being counted (see §3.2.1 for more precise definitions). With these coefficients, it is no longer generally true that planar torsion kills the ECH contact invariant, but it becomes true if we insert an extra topological condition.

**Definition 1.16.** A contact 3-manifold is said to have fully separating planar \( k \)-torsion if it contains a planar \( k \)-torsion domain with a planar piece \( M^P \subset M \) that has the following properties:

1. There are no interface tori in the interior of \( M^P \).
2. Every connected component of \( \partial M^P \) is nullhomologous in \( H_2(M) \).

Observe that the fully separating condition is always satisfied if \( k = 0 \). The following will be proved in §3.2.

**Theorem 4.** If \((M, \xi)\) is a closed contact 3-manifold with fully separating planar torsion, then the ECH contact invariant of \((M, \xi)\) with twisted coefficients in \( \mathbb{Z}[H_2(M)] \) vanishes.

Appealing again to the isomorphism of [Tan] together with known facts [KM97] about Seiberg-Witten theory and weak symplectic fillings, we obtain the following consequence, which is also proved by a more direct holomorphic curve argument in [NW].

**Corollary 2.** If \((M, \xi)\) is a closed contact 3-manifold with fully separating planar torsion, then it is not weakly fillable.

As we will show in §1.3 these results yield many simple new examples of nonfillable contact manifolds.

**Remark 1.17.** One can refine the above result on twisted coefficients as follows: for a given closed 2-form \( \Omega \) on \( M \), define \((M, \xi)\) to have \( \Omega \)-separating planar torsion if it contains a planar torsion domain \( M_0 \) such that every interface torus \( T \subset M_0 \) lying in the planar piece satisfies \( \int_T \Omega = 0 \). Under this condition, our computation implies a similar vanishing result for the ECH contact invariant with twisted coefficients in \( \mathbb{Z}[H_2(M)/ \ker \Omega] \), with the consequence that \((M, \xi)\) admits no weak filling \((W, \omega)\) for which \( \omega|_{TM} \) is cohomologous to \( \Omega \). A direct proof of this is also given in [NW].

**Remark 1.18.** The fully separating condition in Corollary 2 is also sharp in some sense, as for instance, there are infinitely many tight 3-tori which have nonseparating Giroux torsion (and hence planar 1-torsion by Corollary 3 below) but are weakly fillable by a construction of Giroux [Gir94]. Further examples of this phenomenon are constructed in [NW] for planar \( k \)-torsion with any \( k \geq 1 \).

Let us now discuss the relationship between planar torsion and the most popular previously known obstructions to symplectic filling. Recall that a
contact 3-manifold \((M, \xi)\) is overtwisted whenever it contains an embedded closed disk \(D \subset M\) whose boundary is a Legendrian knot with vanishing Thurston-Bennequin number. Due to the powerful classification result of Eliashberg [Eli89], this is equivalent to the condition that \((M, \xi)\) admit a contact embedding of the following object:

**Definition 1.19.** A Lutz tube is the contact manifold \((S^1 \times D, \xi)\) with coordinates \((\theta, \rho, \phi)\), where \(\rho, \phi\) are polar coordinates on the disk and

\[
\xi = \ker [f(\rho) \, d\theta + g(\rho) \, d\phi]
\]

for some pair of smooth functions \(f, g\) such that the path

\[
[0, 1] \to \mathbb{R}^2 \setminus \{0\} : \rho \mapsto (f(\rho), g(\rho))
\]

makes exactly one half-turn (counterclockwise) about the origin, moving from the positive to the negative \(x\)-axis.

Similarly, we say that \((M, \xi)\) has Giroux torsion if it admits a contact embedding of the following:

**Definition 1.20.** A Giroux torsion domain is the contact manifold \(((0, 1] \times T^2, \xi)\) with coordinates \((\rho, \phi, \theta) \in [0, 1] \times S^1 \times S^1\) and \(\xi\) defined via these coordinates as in \((1.1)\), with the path \(\rho \mapsto (f(\rho), g(\rho))\) making one full (counterclockwise) turn about the origin, beginning and ending on the positive \(x\)-axis.

We now show that the known properties of overtwisted contact manifolds and Giroux torsion with respect to fillability are special cases of our more general results involving planar torsion.

**Proposition 1.21.** If \(L \subset M\) is a Lutz tube in a closed contact 3-manifold \((M, \xi)\), then any open neighborhood of \(L\) contains a planar 0-torsion domain. Similarly if \(L\) is a Giroux torsion domain, then any open neighborhood of \(L\) contains a planar 1-torsion domain.

**Proof.** Suppose \(L \subset M\) is a Lutz tube. Then for some \(\epsilon > 0\), an open neighborhood of \(L\) contains a region identified with

\[
L_\epsilon := S^1 \times D_{1+\epsilon},
\]

where \(D_r\) denotes the closed disk of radius \(r\) and \(\xi = \ker \lambda_\epsilon\) for a contact form

\[
\lambda_\epsilon = f(\rho) \, d\theta + g(\rho) \, d\phi
\]

with the following properties (see Figure 3 left):

1. \(f(0) > 0\) and \(g(0) = 0\),
2. \(f(1) < 0\) and \(g(1) = 0\),
3. \(f(\rho)g'(\rho) - f'(\rho)g(\rho) > 0\) for all \(\rho > 0\),
4. \(g'(1 + \epsilon) = 0\),
5. \(f(1 + \epsilon)/g(1 + \epsilon) \in \mathbb{Z}\).
Figure 6. The path $\rho \mapsto (f(\rho), g(\rho))$ used to define the contact form on $L_\epsilon$ (for the Lutz tube at the left and Giroux torsion domain at the right) in the proof of Prop. 1.21.

Setting $D(\rho) := f(\rho)g'(\rho) - f'(\rho)g(\rho)$, the Reeb vector field defined by $\lambda_\epsilon$ in the region $\rho > 0$ is

$$X_\epsilon = \frac{1}{D(\rho)} [g'(\rho)\partial_\theta - f'(\rho)\partial_\phi],$$

and at $\rho = 0$, $X_\epsilon = \frac{1}{f(0)}\partial_\theta$. Thus $X_\epsilon$ in these coordinates depends only on $\rho$ and its direction is always determined by the slope of the path $\rho \mapsto (f(\rho), g(\rho))$ in $\mathbb{R}^2$; in particular, $X_\epsilon$ points in the $-\partial_\phi$-direction at $\rho = 1 + \epsilon$, and in the $+\partial_\phi$-direction at some other radius $\rho_0 \in (0, 1)$. We can choose $f$ and $g$ without loss of generality so that these are the only radii at which $X_\epsilon$ is parallel to $\pm \partial_\phi$.

We claim now that $L_\epsilon$ is a planar 0-torsion domain with planar piece $L^P_\epsilon := S^1 \times D_{\rho_0}$. Indeed, $L^P_\epsilon$ can be obtained from the open book on the tight 3-sphere with disk-like pages by blowing up the binding: the pages in the interior of $L^P_\epsilon$ are defined by $\{\theta = \text{const}\}$. Similarly, the $\theta$-level sets in the closure of $L_\epsilon \setminus L^P_\epsilon$ form the pages of a blown up open book, obtained from an open book with cylindrical pages. The condition $f(1 + \epsilon)/g(1 + \epsilon) \in \mathbb{Z}$ implies that the characteristic foliation on $T := \partial L_\epsilon$ has closed leaves homologous to a primitive class $m_T \in H_1(T)$, which together with the homology class of the Reeb orbits on $T$ forms a basis of $H_1(T)$. Thus our chosen contact form $\lambda_\epsilon$ is a Giroux form for some blown up summed open book. (Note that the monodromy of the blown up open book in $L_\epsilon \setminus L^P_\epsilon$ is not trivial since the distinguished meridians on $\partial L_\epsilon$ and $\partial L^P_\epsilon$ are not homologous.)

The argument for Giroux torsion is quite similar, so we’ll only sketch it: given $L = [0, 1] \times T^2 \subset M$, we can expand $L$ slightly on both sides to create a domain

$$L_\epsilon = [-\epsilon, 1 + \epsilon] \times T^2,$$
with a contact form $\lambda_\varepsilon$ that induces a suitable characteristic foliation on $\partial L_\varepsilon$ and whose Reeb vector field points in the $\pm \partial \varphi$-direction at $\rho = -\varepsilon$, $\rho = 1 + \varepsilon$ and exactly two other radii $0 < \rho_1 < \rho_2 < 1$ (see Figure 6, right). This splits $L_\varepsilon$ into three pieces, of which $L_\varepsilon^p := \{ \rho \in [\rho_1, \rho_2] \}$ is the planar piece of a planar 1-torsion domain, as it can be obtained from an open book with cylindrical pages and trivial monodromy by blowing up both binding components. The padding now consists of two separate blown up open books with cylindrical pages and nontrivial monodromy. □

**Corollary 3.** Suppose $(M, \xi)$ is a closed contact 3-manifold. Then:

- $(M, \xi)$ has planar 0-torsion if and only if it is overtwisted.
- If $(M, \xi)$ has Giroux torsion then it has planar 1-torsion.

**Proof.** The only part not immediate from Prop. [L21] is the claim that $(M, \xi)$ must be overtwisted if it contains a planar 0-torsion domain $M_0$.

One can see this as follows: note first that if we write $M_0 = M_0^p \cup M_0'$, where $M_0^p$ is the planar piece and $M_0' = M_0 \setminus M_0^p$ is the padding, then $M_0'$ carries a blown up summed open book with pages that are not disks (which means $(M_0, \xi)$ is not the tight $S^1 \times S^2$). If the pages in $M_0'$ are surfaces with positive genus and one boundary component, then one can glue one of these together with a page in $M_0^p$ to form a convex surface $\Sigma \subset M_0$ whose dividing set is $\partial M_0^p \cap \Sigma$. The latter is the boundary of a disk in $\Sigma$, so Giroux’s criterion (see e.g. [Gei08, Proposition 4.8.13]) implies the existence of an overtwisted disk near $\Sigma$.

In all other cases the pages $\Sigma$ in $M_0'$ have multiple boundary components

$$\partial \Sigma = C^p \cup C',$$

where we denote by $C^p$ the connected component situated near the interface $\partial M_0^p$, and $C' = \partial \Sigma \setminus C^p$. We can then find overtwisted disks by constructing a particular Giroux form using a small variation on the Thurston-Winkelnkemper construction as described e.g. in [Etn06, Theorem 3.13]. Namely, choose coordinates $(s, t) \in (1/2, 1] \times S^1$ on a collar neighborhood of each component of $\partial \Sigma$ and define a 1-form $\lambda_1$ on $\Sigma$ with the following properties:

1. $d\lambda_1 > 0$
2. $\lambda_1 = (1 + s) \, dt$ near each component of $C'$
3. $\lambda_1 = (-1 + s) \, dt$ near $C^p$

Observe that all three conditions cannot be true unless $C'$ is nonempty, due to Stokes’ theorem. Now following the construction described in [Etn06], one can produce a Giroux form $\lambda$ on $M_0'$ which annihilates some boundary parallel curve $\ell$ near $\partial M_0^p$ in a page, and fits together smoothly with some Giroux form in $M_0^p$, so that $\ker \lambda$ is a supported contact structure and
is isotopic to $\xi$ by Prop. 1.10. Then $\ell$ is the boundary of an overtwisted disk.

Remark 1.22. We’ll see in Corollary 4 below that it is also easy to construct examples of contact manifolds that have planar torsion of any order greater than 1 but no Giroux torsion. It is not clear whether there exist contact manifolds with planar 1-torsion but no Giroux torsion.

By a recent result of Etnyre and Vela-Vick [EVV], the complement of the binding of a supporting open book never contains a Giroux torsion domain. In §3.1 we will prove a natural generalization of this:

Theorem 5. Suppose $(M, \xi)$ is a contact 3-manifold supported by an open book $\pi : M \setminus B \to S^1$. Then any planar torsion domain in $(M, \xi)$ must intersect $B$.

1.3. Some examples of type $S^1 \times \Sigma$. Suppose $\Sigma$ is a closed oriented surface containing a non-empty multicurve $\Gamma \subset \Sigma$ that divides it into two (possibly disconnected) pieces $\Sigma_+$ and $\Sigma_-$. We define the contact manifold $(M_\Gamma, \xi_\Gamma)$, where

$$M_\Gamma := S^1 \times \Sigma$$

and $\xi_\Gamma$ is the unique $S^1$-invariant contact structure that makes $\{\text{const}\} \times \Sigma$ into a convex surface with dividing set $\Gamma$. The existence and uniqueness of such a contact structure follows from a result of Lutz [Lut77]. We claim that $(M_\Gamma, \xi_\Gamma)$ is a partially planar domain if any connected component of $\Sigma \ \Gamma$ has genus zero. Indeed, a supporting summed open book can easily be constructed as follows: assign to each connected component of $\Sigma_+ \ \Gamma$ the orientation determined by $\Sigma$, and the opposite orientation to each component of $\Sigma_- \ \Gamma$. Then $S^1 \times \Gamma$ is a set of tori that divide $M_\Gamma$ into multiple connected components, on each of which $\xi_\Gamma$ is supported by a blown up open book whose pages are the $S^1$-families of pieces of $\Sigma_+$ or $\Sigma_-$, with trivial monodromy. Theorems 3 and 4 now lead to the following.

Corollary 4. Suppose $\Sigma \ \Gamma$ has a connected component $\Sigma_0$ of genus zero, and $\Sigma \ \Sigma_0$ is not diffeomorphic to $\Sigma_0$. Then $(M_\Gamma, \xi_\Gamma)$ has vanishing (untwisted) ECH contact invariant. Moreover, the invariant with twisted coefficients also vanishes if every component of $\partial \Sigma_0$ is nullhomologous in $\Sigma$.

Proof. The natural summed open book on $(M_\Gamma, \xi_\Gamma)$ is never an ordinary open book (because $\Gamma$ must be nonempty), and it is symmetric if and only if $\Gamma$ divides $\Sigma$ into two diffeomorphic connected components. Otherwise, if $\Sigma \ \Gamma$ contains a planar connected component $\Sigma_0$, then $(M_\Gamma, \xi_\Gamma)$ can be regarded as a planar torsion domain with planar piece $S^1 \times \Sigma_0$. □

Note that $(S^1 \times \Sigma, \xi_\Gamma)$ is always tight whenever $\Gamma$ contains no contractible connected components, as then any Giroux form for the summed open book has no contractible Reeb orbits. Whenever this is true, an argument due to
Giroux (see [Mas, Theorem 9]) implies that \((S^1 \times \Sigma, \xi_\Gamma)\) also has no Giroux torsion if no two connected components of \(\Gamma\) are isotopic. We thus obtain infinitely many examples of contact manifolds that have planar torsion of any order greater than 1 but no Giroux torsion:

**Corollary 5.** Suppose \(\Gamma \subset \Sigma\) has \(k \geq 3\) connected components and divides \(\Sigma\) into two connected components, of which one has genus zero and the other does not. Then \((S^1 \times \Sigma, \xi_\Gamma)\) does not have Giroux torsion, but has planar torsion of order \(k - 1\).

Some more examples of planar torsion without Giroux torsion are shown in Figure 7.

**Remark 1.23.** In many cases, one can easily generalize the above results from products \(S^1 \times \Sigma\) to general Seifert fibrations over \(\Sigma\). In particular, whenever \(\Sigma\) has genus at least four, one can find dividing sets on \(\Sigma\) such that \((S^1 \times \Sigma, \xi_\Gamma)\) has no Giroux torsion but contains a proper subset that is a planar torsion domain (see Figure 7). Then modifications outside of the torsion domain can change the trivial fibration into arbitrary nontrivial Seifert fibrations with planar torsion but no Giroux torsion. This trick reproduces many (though not all) of the Seifert fibered 3-manifolds for which [Mas] proves the vanishing of the Ozsváth-Szabó contact invariant.

1.4. **Outlook and open questions.** In light of the widely believed conjecture equating the ECH and Ozsváth-Szabó contact invariants, Theorems 3 and 4 would imply:

**Conjecture.** If \((M, \xi)\) is a closed contact 3-manifold with planar torsion of any order, then its untwisted Ozsváth-Szabó contact invariant vanishes, and the twisted invariant also vanishes whenever every boundary torus of the planar piece is nullhomologous in \(M\).

This would be a direct generalization of [GH]. Some partial results in this direction are proved in the recent paper by Patrick Massot [Mas].

Planar torsion carries an additional structure, namely the nonnegative integer valued order \(k\). This is detected in a natural way by an invariant defined via Symplectic Field Theory and known as algebraic torsion [LW], and it can be used to prove nonexistence results for exact symplectic cobordisms. As Michael Hutchings has explained to me, ECH can also detect \(k\), though in a less straightforward way: in [Hut09], Hutchings defines an integer \(J_+(u)\) that can be associated to any finite energy holomorphic curve \(u\) and, for the particular curves of interest, measures the same information as the exponent of \(\hbar\) in SFT. One can use it to decompose the ECH differential as \(\partial = \partial_0 + \partial_1 + \ldots\), where \(\partial_k\) counts curves \(u\) with \(J_+(u) = k\), and this leads to a spectral sequence. The order of planar torsion is then greater than or equal to the smallest integer \(k \geq 0\) for which the empty orbit set vanishes on page \((k+1)\) of this spectral sequence.
Figure 7. Some contact manifolds of the form $S^1 \times \Sigma$ that have no Giroux torsion but have planar torsion of orders 2, 2, 3 and 2 respectively. In each case the contact structure is $S^1$-invariant and induces the dividing set shown on $\Sigma$ in the picture. For the example at the upper right, Theorem 4 implies that the twisted ECH contact invariant also vanishes, so this one is not weakly fillable. In the bottom example, the planar torsion domain is a proper subset, thus one can make modifications outside of this subset to produce arbitrary nontrivial Seifert fibrations (see Remark 1.23).

Clearly, the order of planar torsion must be similarly detectable in Heegaard Floer homology, but as far as this author is aware, nothing is yet known about this.

**Question.** Can Heegaard Floer homology distinguish between two contact manifolds with vanishing Ozsváth-Szabó invariant but differing orders of
planar torsion? Does this imply obstructions to the existence of exact symplectic cobordisms?

It should also be mentioned that in presenting this introduction to planar torsion, we neither claim nor believe it to be the most general source of vanishing results for the various invariants under discussion. For the Ozsváth-Szabó invariant,\cite{Mas} produces vanishing results on some Seifert fibered 3-manifolds that fall under the umbrella of our Corollary 4 and Remark \ref{Rem} but also some that do not since there is no condition requiring the existence of a planar piece. It is likely however that the definition of planar torsion can be generalized by replacing the contact fiber sum with a more general “plumbing” construction that produces a notion of “higher genus binding.” Among its applications, this should allow a substantial generalization of Corollary 4 that encompasses all of the examples in \cite{Mas}; this will hopefully appear in a forthcoming paper.

And now the obvious question: what can be done in higher dimensions? It’s interesting to note that both of the crucial topological notions in this paper, supporting open books and contact fiber sums, make sense for contact manifolds of arbitrary dimension.

**Question.** Can binding sums of supporting open books be used to construct examples of nonfillable contact manifolds in dimensions greater than three?

As of this writing, I am completely in the dark about this. One would be tempted toward pessimism in light of the important role that intersection theory will play in the arguments of \S 2—on the other hand, the literature on holomorphic curves provides plenty of examples of beautiful geometric arguments that seem at first to make sense only in dimension four, yet generalize to higher dimensions surprisingly well.

## 2. Holomorphic summed open books

### 2.1. Technical background.

We begin by collecting some definitions and background results on punctured holomorphic curves that will be important for understanding the remainder of the paper.

A **stable Hamiltonian structure** on an oriented 3-manifold $M$ is a pair $\mathcal{H} = (\lambda, \omega)$ consisting of a 1-form $\lambda$ and 2-form $\omega$ such that $d\omega = 0$, $\lambda \wedge \omega > 0$ and $\ker \omega \subset \ker d\lambda$. Given this data, we define the cooriented 2-plane distribution $\xi = \ker \lambda$ and nowhere vanishing vector field $X$, called the **Reeb vector field**, which is determined by the conditions

$$\omega(X, \cdot) \equiv 0, \quad \lambda(X) \equiv 1.$$ 

The conditions on $\lambda$ and $\omega$ imply that $\omega|_\xi$ gives $\xi$ the structure of a symplectic vector bundle over $M$, and this distribution with its symplectic structure is preserved by the flow of $X$. As an important special case, if $\lambda$ is a contact form, then one can define a stable Hamiltonian structure in
the form $\mathcal{H} = (\lambda, h \, d\lambda)$ for any smooth function $h : M \to (0, \infty)$ such that $dh \wedge d\lambda \equiv 0$. Then $\xi$ is a positive and cooriented contact structure, and $X$ is the usual contact geometric notion of the Reeb vector field; we will often denote it in this case by $X_\lambda$, since it is uniquely determined by $\lambda$.

For the rest of this section, assume $\mathcal{H} = (\lambda, \omega)$ is a stable Hamiltonian structure with the usual attached data $\xi$ and $X$. We say that an almost complex structure $J$ on $\mathbb{R} \times M$ is compatible with $\mathcal{H}$ if it satisfies the following conditions:

1. The natural $\mathbb{R}$-action on $\mathbb{R} \times M$ preserves $J$.
2. $J \partial_t \equiv X$, where $\partial_t$ denotes the unit vector in the $\mathbb{R}$-direction.
3. $J(\xi) = \xi$ and $\omega(\cdot, J \cdot)$ defines a symmetric, positive definite bundle metric on $\xi$.

Denote by $J(\mathcal{H})$ the (nonempty and contractible) space of almost complex structures compatible with $\mathcal{H}$. Note that if $\lambda$ is contact then $J(\mathcal{H})$ depends only on $\lambda$; we will in this case say that $J$ is compatible with $\lambda$.

A periodic orbit $\gamma$ of $X$ is determined by the data $(x, T)$, where $x : \mathbb{R} \to M$ satisfies $\dot{x} = X(x)$ and $x(T) = x(0)$ for some $T > 0$. We sometimes abuse notation and identify $\gamma$ with the submanifold $x(\mathbb{R}) \subset M$, though technically the period is also part of the data defining $\gamma$. If $\tau > 0$ is the smallest positive number for which $x(\tau) = x(0)$, we call it the minimal period of this orbit, and say that $\gamma = (x, \tau)$ is a simple, or simply covered orbit. The covering multiplicity of an orbit $(x, T)$ is the unique integer $k \geq 1$ such that $T = k\tau$ for a simple orbit $(x, \tau)$.

If $\gamma = (x, T)$ is a periodic orbit and $\varphi_T^X$ denotes the flow of $X$ for time $t \in \mathbb{R}$, then the restriction of the linearized flow to $\xi_{x(0)}$ defines a symplectic isomorphism

$$(\varphi_T^X)_* : (\xi_{x(0)}, \omega) \to (\xi_{x(0)}, \omega).$$

We call $\gamma$ nondegenerate if 1 is not in the spectrum of this map. More generally, a Morse-Bott submanifold of $T$-periodic orbits is a closed submanifold $N \subset M$ fixed by $\varphi_T^X$ such that for any $p \in N$,

$$\ker ((\varphi_T^X)_* - 1) = T_p N.$$  

We will call a single orbit $\gamma = (x, T)$ Morse-Bott if it lies on a Morse-Bott submanifold of $T$-periodic orbits. Nondegenerate orbits are clearly also Morse-Bott, with $N \cong S^1$. We say that the vector field $X$ is Morse-Bott (or nondegenerate) if all of its periodic orbits are Morse-Bott (or nondegenerate respectively). Since $X$ never vanishes, every Morse-Bott submanifold $N \subset M$ of dimension 2 is either a torus or a Klein bottle. One can show (cf. [Wen10a, Prop. 4.1]) that in the former case, if $X$ is Morse-Bott, then every orbit contained in $N$ has the same minimal period.

To every orbit $\gamma = (x, T)$, one can associate an asymptotic operator, which is morally the Hessian of a certain functional whose critical points are the periodic orbits. To write it down, choose $J \in J(\mathcal{H})$, let $x : S^1 \to \mathbb{R}$.
$M : t \mapsto x(Tt)$, choose a symmetric connection $\nabla$ on $M$ and define
\[
A_{\gamma} : \Gamma(x^*\xi) \to \Gamma(x^*\xi) : \eta \mapsto -J(\nabla_{\partial t}\eta - T\nabla_{\eta}X).
\]

One can show that this operator is well defined independently of the choice of connection, and it extends to an unbounded self-adjoint operator on the complexification of $L^2(x^*\xi)$, with domain $H^1(x^*\xi)$. Its spectrum $\sigma(A_{\gamma})$ consists of real eigenvalues with multiplicity at most 2, which accumulate only at $\pm \infty$. It is straightforward to show that solutions of the equation $A_{\gamma}\eta = 0$ are given by $\eta(t) = (\varphi^{Tt}_X), \eta(0)$, thus $\gamma$ is nondegenerate if and only if $0 \not\in \sigma(A_{\gamma})$, and in general if $\gamma$ belongs to a Morse-Bott submanifold $N \subset M$, then
\[
\dim \ker A_{\gamma} = \dim N - 1.
\]

Choosing a unitary trivialization $\Phi$ of $(\xi, J, \omega)$ along the parametrization $x : S^1 \to M$ identifies $A_{\gamma}$ with a first-order differential operator of the form
\[
H^1(S^1, \mathbb{R}^2) \to L^2(S^1, \mathbb{R}^2) : \eta \mapsto -J_0 \dot{\eta} - S\eta,
\]
where $J_0$ denotes the standard complex structure on $\mathbb{R}^2 = \mathbb{C}$ and $S : S^1 \to \text{End}_2(\mathbb{R}^2)$ is a smooth loop of symmetric real 2-by-2 matrices. Seen in this trivialization, $A_{\gamma}\eta = 0$ defines a linear Hamiltonian equation $\dot{\eta} = J_0 S\eta$ corresponding to the linearized flow of $X$ along $\gamma$, thus its flow defines a smooth family of symplectic matrices
\[
\Psi : [0, 1] \to \text{Sp}(2)
\]
for which $1 \not\in \sigma(\Psi(1))$ if and only if $\gamma$ is nondegenerate. In this case, the homotopy class of the path $\Psi$ is described by its Conley-Zehnder index $\mu_{\text{CZ}}(\Psi) \in \mathbb{Z}$, which we use to define the Conley-Zehnder index of the orbit $\gamma$ and of the asymptotic operator $A_{\gamma}$ with respect to the trivialization $\Phi$,
\[
\mu^\Phi_{\text{CZ}}(\gamma) := \mu^\Phi_{\text{CZ}}(A_{\gamma}) := \mu_{\text{CZ}}(\Psi).
\]

Note that in this way, $\mu^\Phi_{\text{CZ}}(A)$ can be defined for any injective operator $A : \Gamma(x^*\xi) \to \Gamma(x^*\xi)$ that takes the form (2.1) in a local trivialization. In particular then, even if $\gamma$ is degenerate, we can pick any $\epsilon \in \mathbb{R} \setminus \sigma(A_{\gamma})$ and define the “perturbed” Conley-Zehnder index
\[
\mu^\Phi_{\text{CZ}}(\gamma - \epsilon) := \mu^\Phi_{\text{CZ}}(A_{\gamma} - \epsilon) := \mu_{\text{CZ}}(\Psi_\epsilon),
\]
where $\Psi_\epsilon : [0, 1] \to \text{Sp}(2)$ is the path of symplectic matrices determined by the equation $(A_{\gamma} - \epsilon)\eta = 0$ in the trivialization $\Phi$. It is especially convenient to define Conley-Zehnder indices in this way for orbits that are degenerate but Morse-Bott: then the discreteness of the spectrum implies that for sufficiently small $\epsilon > 0$, the integer $\mu^\Phi_{\text{CZ}}(\gamma \pm \epsilon)$ depends only on $\gamma$, $\Phi$ and the choice of sign.

The eigenfunctions of $A_{\gamma}$ are nowhere vanishing sections $e \in \Gamma(x^*\xi)$ and thus have well defined winding numbers $\text{wind}^\Phi(e)$ with respect to any
trivialization $\Phi$. As shown in [HWZ95a], all sections in the same eigenspace have the same winding, thus defining a function

$$\sigma(A_\gamma) \to \mathbb{Z} : \mu \mapsto \text{wind}^\Phi(\mu),$$

where we set $\text{wind}^\Phi(\mu) := \text{wind}^\Phi(e)$ for any nontrivial $e \in \ker(A_\gamma - \mu)$. In fact, [HWZ95a] shows that this function is nondecreasing and surjective: counting with multiplicity there are exactly two eigenvalues $\mu \in \sigma(A_\gamma)$ such that $\text{wind}^\Phi(\mu)$ equals any given integer. It is thus sensible to define the integers,

$$\alpha^-_\Phi(\gamma - \epsilon) = \max\{\text{wind}^\Phi(\mu) \mid \mu \in \sigma(A_\gamma - \epsilon), \mu < 0\},$$

$$\alpha^+_\Phi(\gamma - \epsilon) = \min\{\text{wind}^\Phi(\mu) \mid \mu \in \sigma(A_\gamma - \epsilon), \mu > 0\},$$

$$p(\gamma - \epsilon) = \alpha^+_\Phi(\gamma - \epsilon) - \alpha^-_\Phi(\gamma - \epsilon).$$

Note that the parity $p(\gamma - \epsilon)$ does not depend on $\Phi$, and it always equals either 0 or 1 if $\epsilon \not\in \sigma(A_\gamma)$. In this case, the Conley-Zehnder index can be computed as

$$(2.2) \quad \mu_{\text{CZ}}(\gamma - \epsilon) = 2\alpha^+_\Phi(\gamma - \epsilon) + p(\gamma - \epsilon) = 2\alpha^+_\Phi(\gamma - \epsilon) - p(\gamma - \epsilon).$$

Given $H = (\lambda, \omega)$ and $J \in \mathcal{J}(H)$, fix $c_0 > 0$ sufficiently small so that $(\omega + c \, d\lambda)|_\xi > 0$ for all $c \in [-c_0, c_0]$, and define

$$\mathcal{T} = \{\varphi \in C^\infty(\mathbb{R}, (-c, c)) \mid \varphi' > 0\}.$$

For $\varphi \in \mathcal{T}$, we can define a symplectic form on $\mathbb{R} \times M$ by

$$(2.3) \quad \omega_\varphi = \omega + d(\varphi \lambda),$$

where $\omega$ and $\lambda$ are pulled back through the projection $\mathbb{R} \times M \to M$ to define differential forms on $\mathbb{R} \times M$, and $\varphi : \mathbb{R} \to (-c, c)$ is extended in the natural way to a function on $\mathbb{R} \times M$. Then any $J \in \mathcal{J}(H)$ is compatible with $\omega_\varphi$ in the sense that $\omega_\varphi(\cdot, J \cdot)$ defines a Riemannian metric on $\mathbb{R} \times M$. We therefore consider punctured pseudoholomorphic curves

$$u : (\tilde{\Sigma}, j) \to (\mathbb{R} \times M, J)$$

where $(\Sigma, j)$ is a closed Riemann surface with a finite subset of punctures $\Gamma \subset \Sigma$, $\check{\Sigma} := \Sigma \setminus \Gamma$, and $u$ is required to satisfy the finite energy condition

$$(2.4) \quad E(u) := \sup_{\varphi \in \mathcal{T}} \int_{\check{\Sigma}} u^* \omega_\varphi < \infty.$$

An important example is the following: for any closed Reeb orbit $\gamma = (x, T)$, the map

$$u_\gamma : \mathbb{R} \times S^1 \to \mathbb{R} \times M : (s, t) \mapsto (Ts, x(Tt))$$

is a finite energy $J$-holomorphic cylinder (or equivalently punctured plane), which we call the trivial cylinder over $\gamma$. More generally, we are most interested in punctured $J$-holomorphic curves $u : \Sigma \to \mathbb{R} \times M$ that are
asymptotically cylindrical, in the following sense. Define the standard half cylinders
\[ Z_+ = [0, \infty) \times S^1 \quad \text{and} \quad Z_- = (-\infty, 0] \times S^1. \]
We say that a smooth map \( u : \hat{\Sigma} \to \mathbb{R} \times M \) is asymptotically cylindrical if the punctures can be partitioned into positive and negative subsets
\[ \Gamma = \Gamma^+ \cup \Gamma^- \]
such that for each \( z \in \Gamma^\pm \), there is a Reeb orbit \( \gamma_z = (x, T) \), a closed neighborhood \( U_z \subset \Sigma \) of \( z \) and a diffeomorphism \( \varphi_z : Z_\pm \to U_z \setminus \{z\} \) such that for sufficiently large \(|s|\),
\[
(2.5) \quad u \circ \varphi_z(s, t) = \exp(Ts, x(T)) h_z(s, t),
\]
where \( h_z \) is a section of \( \xi \) along \( u_{\gamma_z} \) with \( h_z(s, t) \to 0 \) for \( s \to \pm \infty \), and the exponential map is defined with respect to any choice of \( \mathbb{R} \)-invariant connection on \( \mathbb{R} \times M \). We often refer to the punctured neighborhoods \( U_z \setminus \{z\} \) or their images in \( \mathbb{R} \times M \) as the positive and negative ends of \( u \), and we call \( \gamma_z \) the asymptotic orbit of \( u \) at \( z \).

**Definition 2.1.** Suppose \( N \subset M \) is a submanifold which is the union of a family of Reeb orbits that all have the same minimal period. Consider an asymptotically cylindrical map \( u : \hat{\Sigma} \to \mathbb{R} \times M \) with punctures \( \Gamma^+ \cup \Gamma^- \subset \Sigma \) and corresponding asymptotic orbits \( \gamma_z \) with covering multiplicities \( k_z \geq 1 \) for each \( z \in \Gamma^\pm \). Then if \( k_N^\pm \geq 0 \) denotes the sum of the multiplicities \( k_z \) for all punctures \( z \in \Gamma^\pm \) at which \( \gamma_z \) lies in \( N \), we shall say that \( u \) approaches \( N \) with total multiplicity \( k_N^\pm \) at its positive or negative ends respectively.

Every asymptotically cylindrical map defines a relative homology class in the following sense. Suppose \( \gamma = \{(\gamma_1, m_1), \ldots, (\gamma_N, m_N)\} \) is an orbit set, i.e. a finite collection of distinct simply covered Reeb orbits \( \gamma_i \) paired with positive integers \( m_i \). This defines a 1-dimensional submanifold of \( M \),
\[ \bar{\gamma} = \gamma_1 \cup \ldots \cup \gamma_N, \]

together with homology classes
\[ [\gamma] = m_1[\gamma_1] + \ldots + m_N[\gamma_N] \]
in both \( H_1(M) \) and \( H_1(\bar{\gamma}) \). Given two orbit sets \( \gamma^+ \) and \( \gamma^- \) with \([\gamma^+] = [\gamma^-] \in H_1(M)\), denote by \( H_2(M, \gamma^+ - \gamma^-) \) the affine space over \( H_2(M) \) consisting of equivalence classes of 2-chains \( C \) in \( M \) with boundary \( \partial C \) in \( \gamma^+ \cup \gamma^- \) representing the homology class \([\gamma^+] - [\gamma^-] \in H_1(\gamma^+ \cup \gamma^-)\), where \( C \sim C' \) whenever \( C - C' \) is the boundary of a 3-chain in \( M \). Now, the projection of any asymptotically cylindrical map \( u : \hat{\Sigma} \to \mathbb{R} \times M \) to \( M \) can be extended as a continuous map from a compact surface with boundary (the circle compactification of \( \hat{\Sigma} \)) to \( M \), which then represents a relative homology class
\[ [u] \in H_2(M, \gamma^+ - \gamma^-) \]
for some unique choice of orbit sets \( \gamma^+ \) and \( \gamma^- \).

As is well known (cf. [Hof93, HWZ96a, HWZ96b]), every finite energy \( J \)-holomorphic curve with nonremovable punctures is asymptotically cylindrical if the contact form is Morse-Bott. Moreover in this case, the section \( h_z \) in (2.3), which controls the asymptotic approach of \( u \) to \( \gamma_z \) at \( z \in \Gamma^\pm \), either is identically zero or satisfies a formula of the form

\[
(2.6) \quad h_z(s, t) = e^{\mu s} (e^{\mu t} + r(s, t)),
\]

where \( \mu \in \sigma(A_\gamma) \) with \( \pm \mu < 0 \), \( e^{\mu} \) is a nontrivial eigenfunction in the \( \mu \)-eigenspace, and the remainder term \( r(s, t) \in \xi_{z(T)} \) decays to zero as \( s \to \pm \infty \). It follows that unless \( h_z \equiv 0 \), which is true only if \( u \) is a cover of a trivial cylinder, \( u \) has a well defined asymptotic winding about \( \gamma_z \),

\[
\text{wind}^\Phi(u) := \text{wind}^\Phi(e^\mu),
\]

which is necessarily either bounded from above by \( \alpha^-_\Phi(\gamma_z) \) or from below by \( \alpha^+_\Phi(\gamma_z) \), depending on the sign \( z \in \Gamma^\pm \). We say that this winding is extremal whenever the bound is not strict.

Denote by \( \mathcal{M}(J) \) the moduli space of unparametrized finite energy punctured \( J \)-holomorphic curves in \( \mathbb{R} \times M \): this consists of equivalence classes of tuples \( (\Sigma, j, \Gamma, u) \), where \( \hat{\Sigma} = \Sigma \setminus \Gamma \) is the domain of a pseudoholomorphic curve \( u : (\hat{\Sigma}, j) \to (\mathbb{R} \times M, J) \), and we define \( (\Sigma, j, \Gamma, u) \sim (\Sigma', j', \Gamma', u') \) if there is a biholomorphic map \( \varphi : (\hat{\Sigma}, j) \to (\hat{\Sigma}', j') \) such that \( u = u' \circ \varphi \). We assign to \( \mathcal{M}(J) \) the natural topology defined by \( C^\infty_{\text{loc}} \)-convergence on \( \hat{\Sigma} \) and \( C^0 \)-convergence up to the ends. It is often convenient to abuse notation by writing equivalence classes \( [(\Sigma, j, \Gamma, u)] \in \mathcal{M}(J) \) simply as \( u \) when there is no danger of confusion.

If \( u \in \mathcal{M}(J) \) has asymptotic orbits \( \{\gamma_z\}_{z \in \Gamma} \) that are all Morse-Bott, then a neighborhood of \( u \) in \( \mathcal{M}(J) \) can be described as the zero set of a Fredholm section of a Banach space bundle (see e.g. [Wen10a]). We say that \( u \) is Fredholm regular if this section has a surjective linearization at \( u \), in which case a neighborhood of \( u \) in \( \mathcal{M}(J) \) is a smooth finite dimensional orbifold. Its dimension is then equal to its virtual dimension, which is given by the index of \( u \),

\[
(2.7) \quad \text{ind}(u) := -\chi(\hat{\Sigma}) + 2c_1^\Phi(u^*\xi) + \sum_{z \in \Gamma^+} \mu_{\text{CZ}}^\Phi(\gamma_z - \epsilon) - \sum_{z \in \Gamma^-} \mu_{\text{CZ}}^\Phi(\gamma_z + \epsilon),
\]

where \( \epsilon > 0 \) is any small positive number, \( \Phi \) is an arbitrary choice of unitary trivialization of \( \xi \) along all the asymptotic orbits \( \gamma_z \), and \( c_1^\Phi(u^*\xi) \) denotes the relative first Chern number of the complex line bundle \( (u^*\xi, J) \to \hat{\Sigma} \), computed by counting the zeroes of a generic section of \( u^*\xi \) that is nonzero and constant at infinity with respect to \( \Phi \). We say that an almost complex

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\( ^6 \)The asymptotic formula (2.6) is a stronger version of a somewhat more complicated formula originally proved in [HWZ96a, HWZ96b]. The stronger version is proved in [Mor03], and another exposition is given in [Sie08].
structure $J \in \mathcal{J}(\mathcal{H})$ is Fredholm regular if all somewhere injective curves in $\mathcal{M}(J)$ are Fredholm regular. As shown in [Dra04] or the appendix of [Bon06], the set of Fredholm regular almost complex structures is of second category in $\mathcal{J}(\mathcal{H})$; one therefore often refers to them as generic almost complex structures.

It is sometimes convenient to have an alternative formula for $\text{ind}(u)$ in the case where $u$ is immersed. Indeed, the linearization of the Fredholm operator that describes $\mathcal{M}(J)$ near $u$ acts on the space of sections of $u^*\mathcal{T}(\mathbb{R} \times M)$, which then splits naturally as $\mathcal{T}\dot{\Sigma} \oplus N_u$, where $N_u \to \dot{\Sigma}$ is the normal bundle, defined so that it matches $\xi$ at the asymptotic ends of $u$. As explained e.g. in [Wen10a], the restriction of the linearization to $N_u$ defines a linear Cauchy-Riemann type operator $D_u^N : \Gamma(N_u) \to \Gamma(\text{Hom}_C(\mathcal{T}\dot{\Sigma}, N_u))$, called the normal Cauchy-Riemann operator at $u$, and the Fredholm index of this operator is precisely $\text{ind}(u)$. Thus whenever $u$ is immersed, we can compute $\text{ind}(u)$ directly from the punctured version of the Riemann-Roch formula proved in [Sch95]:

$$\text{ind}(D_u^N) = \chi(\dot{\Sigma}) + 2c_1^\phi(N_u) + \sum_{z \in \Gamma^+} \mu_{CZ}^\phi(\gamma_z - \epsilon) - \sum_{z \in \Gamma^-} \mu_{CZ}^\phi(\gamma_z + \epsilon).$$

Finally, let us briefly summarize the intersection theory of punctured $J$-holomorphic curves introduced by R. Siefring [Sie]. Given any asymptotically cylindrical smooth maps $u : \dot{\Sigma} \to \mathbb{R} \times M$ and $v : \dot{\Sigma}' \to \mathbb{R} \times M$, there is a symmetric pairing

$$u \ast v \in \mathbb{Z}$$

with the following properties:

1. $u \ast v$ depends only on the asymptotic orbits of $u$ and $v$ and the relative homology classes $[u]$ and $[v]$.
2. If $u$ and $v$ represent curves in $\mathcal{M}(J)$ with non-identical images, then their algebraic count of intersections $u \cdot v$ satisfies $0 \leq u \cdot v \leq u \ast v$.

In particular, $u \ast v = 0$ implies that $u$ and $v$ never intersect.

The first property amounts to homotopy invariance: it implies that $u_0 \ast v = u_1 \ast v$ whenever $u_0$ and $u_1$ are connected to each other by a continuous family of curves $u_\tau \in \mathcal{M}(J)$ with fixed asymptotic orbits. The second property gives a sufficient condition for two curves to have disjoint images, but this condition is not in general necessary: sometimes one may have $0 = u \cdot v < u \ast v$ if $u$ and $v$ have an asymptotic orbit in common, and one must then expect intersections to emerge from infinity under generic perturbations. The number $u \ast v$ can also be defined when $u$ and $v$ are holomorphic buildings in the sense of [BEH+03], so that it satisfies a similar continuity property under convergence of curves to buildings. The computation of $u \ast v$ is then a sum of the intersection numbers between corresponding levels,
plus some additional nonnegative terms that count “hidden” intersections at the breaking orbits.

Remark 2.2. The version of homotopy invariance described above assumes that $u$ and $v$ vary as asymptotically cylindrical maps with fixed asymptotic orbits, but if any of the orbits belong to Morse-Bott families, one can define an alternative version of $u \ast v$ that permits the orbits to move continuously. This more general theory is developed in [SW] and sketched in the last section of [Wen10a]. In general, the intersection number defined in this way is greater than or equal to $u \ast v$, because it counts additional nonnegative contributions for intersections that may emerge from infinity as the asymptotic orbits move. It’s useful to observe however that in the situation we will consider, both versions agree: in particular, if $u$ and $v$ are disjoint curves with $u \ast v = 0$ and a common positive asymptotic orbit that doesn’t intersect the images of $u$ and $v$, then no new intersections can appear under a perturbation that moves the orbit (independently for both curves). This follows from an easy computation of asymptotic winding numbers using the definitions given in [Wen10a].

Similarly, if $u \in \mathcal{M}(J)$ is somewhere injective, one can define the integer $\delta(u) \geq 0$, which algebraically counts the self-intersections of $u$ after perturbing away its critical points, but in the punctured case this need not be homotopy invariant. One fixes this by introducing the asymptotic contribution $\delta_\infty(u) \in \mathbb{Z}$, which is also nonnegative and counts “hidden” self-intersections that may emerge from infinity under generic perturbations. We then have

$$0 \leq \delta(u) \leq \delta(u) + \delta_\infty(u),$$

and the punctured version of the adjunction formula takes the form

$$u \ast u = 2 \left[ \delta(u) + \delta_\infty(u) \right] + c_N(u) + [\bar{\sigma}(u) - \#\Gamma],$$

(2.9)

where $\bar{\sigma}(u)$ is an integer that depends only on the asymptotic orbits and satisfies $\bar{\sigma}(u) \geq \#\Gamma$, and $c_N(u)$ is the constrained normal Chern number, which can be defined as

$$c_N(u) = c_1^\Phi(u^*\xi) - \chi(\Sigma) + \sum_{z \in \Gamma^+} \alpha_+^\Phi(\gamma_z + \epsilon) - \sum_{z \in \Gamma^-} \alpha_+^\Phi(\gamma_z - \epsilon).$$

(2.10)

Observe that $c_N(u)$ also depends only on the asymptotic orbits $\{\gamma_z\}_{z \in \Gamma}$ and the relative homology class $[u]$.

---

7The version of $c_N(u)$ defined in (2.10) is adapted to the condition that homotopies in $\mathcal{M}(J)$ are required to fix asymptotic orbits. A more general definition is given in [Wen10a] (see also Remark 2.2).
2.2. An existence and uniqueness theorem. We now prove a theorem on holomorphic open books which lies in the background of all the results that were stated in §1. The setup is as follows. Assume \((M', \xi)\) is a closed 3-manifold with a positive, cooriented contact structure, and it contains a compact 3-dimensional submanifold \(M \subset M'\), possibly with boundary, on which \(\xi\) is supported by a partially planar blown up summed open book \(\tilde{\pi} = (\tilde{\pi}, \tilde{\phi}, \tilde{N})\).

We will denote its binding and interface by \(B\) and \(I\) respectively, and denote the induced fibration by \(\pi : M \setminus (B \cup I) \to S^1\).

Denote the irreducible subdomains by \(M_i\) for \(i = 0, \ldots, N\), so

\[
M = M_0 \cup M_1 \cup \ldots \cup M_N
\]

for some \(N \geq 0\). If \(B_i\) and \(I_i\) denote the intersections of \(B\) and \(I\) respectively with the interior of \(M_i\), then the restriction of \(\pi\) to the interior of \(M_i \setminus (B_i \cup I_i)\) extends smoothly to its boundary as a fibration

\[
\pi_i : M_i \setminus (B_i \cup I_i) \to S^1.
\]

Denote by \(g_i \geq 0\) the genus of the fibers of \(\pi_i\), and assume without loss of generality that \(M_0\) is a planar piece, thus \(g_0 = 0\) and \(M_0 \cap \partial M = \emptyset\); in particular \(\partial M_0 \subset I\).

Definition 2.3. Given the above setup and an almost complex structure \(J\) compatible with some contact form on \((M', \xi)\), we shall say that a finite energy \(J\)-holomorphic curve \(u : \hat{\Sigma} \to \mathbb{R} \times M'\) is subordinate to \(\pi_0\) if all of the following conditions hold:

- \(u\) is not a cover of a trivial cylinder,
- All positive ends of \(u\) approach Reeb orbits in \(B_0 \cup I_0 \cup \partial M_0\),
- At its positive ends, \(u\) approaches each connected component of \(B_0 \cup \partial M_0\) with total multiplicity at most 1, and each connected component of \(I_0\) with total multiplicity at most 2.

Theorem 6. For any numbers \(\tau_0 > 0\) and \(m_0 \in \mathbb{N}\), the contact manifold \((M', \xi)\) with subdomain \(M \subset M'\) carrying the blown up summed open book \(\tilde{\pi}\) described above admits a Morse-Bott contact form \(\lambda\) and compatible Fredholm regular almost complex structure \(J\) with the following properties.

1. The contact structure \(\ker \lambda\) is isotopic to \(\xi\).
2. On \(M\), \(\lambda\) is a Giroux form for \(\tilde{\pi}\).
3. The components of \(I \cup \partial M\) are all Morse-Bott submanifolds, while the Reeb orbits in \(B\) are nondegenerate and elliptic, and their covers for all multiplicities up to \(m_0\) have Conley-Zehnder index 1 with respect to the natural trivialization determined by the pages.
(4) All Reeb orbits in $B_0 \cup \mathcal{I}_0 \cup \partial M_0$ have minimal period at most $\tau_0$, while every other closed orbit of $X_\lambda$ in $M'$ has minimal period at least 1.

(5) For each component $M_i$ with $g_i = 0$, the fibration $\pi_i : M_i \setminus (B_i \cup \mathcal{I}) \to S^1$ admits a $C^\infty$-small perturbation $\tilde{\pi}_i: M_i \setminus (B_i \cup \mathcal{I}) \to S^1$ such that the interior of each fiber $\tilde{\pi}_i^{-1}(\tau)$ for $\tau \in S^1$ lifts uniquely to an $\mathbb{R}$-invariant family of properly embedded surfaces $S_{\sigma,\tau}^{(i)} \subset \mathbb{R} \times M_i$, $(\sigma, \tau) \in \mathbb{R} \times S^1$, which are the images of embedded finite energy $J$-holomorphic curves $u_{\sigma,\tau}^{(i)} = (a_{\tau}^{(i)} + \sigma, F_\tau^{(i)}): \dot{\Sigma}_i \to \mathbb{R} \times M_i$, all of them Fredholm regular with index 2, and with only positive ends.

(6) Every finite energy $J$-holomorphic curve subordinate to $\pi_0$ parametrizes one of the planar surfaces $S_{\sigma,\tau}^{(i)}$ described above.

Remark 2.4. The uniqueness statement above suffices for our applications, but one could also generalize it in various directions, e.g. there is an obvious stronger result if $\tilde{\pi}$ has multiple planar subdomains. One could also weaken the definition of “subordinate to $\pi_0$” by allowing multiply covered orbits in $B_0 \cup \mathcal{I}_0$ up to some arbitrarily large (but finite) bound on the multiplicity. Some statement allowing for multiply covered orbits at $\partial M_0$ is probably also possible, but trickier to prove due to the appearance of multiply covered holomorphic curves in the limit of Lemma 2.16 below.

In addition to the applications treated in §3, Theorem 6 implies a wide range of existence results for finite energy foliations, e.g. it could be used to reduce the construction in [Wen08] to a few lines, after observing that every overtwisted contact structure is supported by a variety of summed open books with only planar pages. The proof of the theorem will occupy the remainder of §2.

A family of stable Hamiltonian structures. The first step in the proof is to construct a specific almost complex structure on $\mathbb{R} \times M$ for which all pages of $\tilde{\pi}$ admit holomorphic lifts. We will follow the approach in [Wen10c] and refer to the latter for details in a few places where no new arguments are required. The idea is to present each subdomain $M_i$ as an abstract open book that supports a stable Hamiltonian structure which is contact near $B \cup \mathcal{I} \cup \partial M$ and integrable elsewhere.

We must choose suitable coordinate systems near each component of the binding, interface and boundary. Choose $r > 0$ and let $D_r \subset \mathbb{R}^2$ denote the closed disk of radius $r$. For each binding circle $\gamma \subset B$, choose a small tubular neighborhood $\mathcal{N}(\gamma)$ and identify it with the solid torus $S^1 \times D_r$ with coordinates $(\theta, \rho, \phi)$, where $(\rho, \phi)$ denote polar coordinates on $D_r$. If
$r$ is sufficiently small then we can arrange these coordinates so that the following conditions are satisfied:

- $\gamma = S^1 \times \{0\}$, with the natural orientation of $S^1$ matching the coorientation of $\xi$ along $\gamma$
- $\pi(\theta, \rho, \phi) = \phi$ on $\mathcal{N}(\gamma) \setminus \gamma$
- $\xi = \ker(d\theta + \rho^2 \, d\phi)$

Similarly, for each connected component $T \subset \partial M$, let $\mathcal{N}(T) \subset M'$ denote a neighborhood that is split into two connected components by $T$, and denote $\mathcal{N}(T) = \mathcal{N}(T) \cap M$. Identify $\mathcal{N}(T)$ with $S^1 \times [-r, r] \times S^1$ with coordinates $(\theta, \rho, \phi)$ such that:

- $\mathcal{N}(T) = S^1 \times [0, r] \times S^1$
- For each $\phi_0 \in S^1$ the oriented loop $S^1 \times \{0, \phi_0\}$ in $T$ is positively transverse to $\xi$
- $\pi(\theta, \rho, \phi) = \phi$ on $\mathcal{N}(T)$
- $\xi = \ker(d\theta + \rho \, d\phi)$

Finally, we choose two coordinate systems for neighborhoods $\mathcal{N}(T)$ of each interface torus $T \subset \mathcal{I}$, assuming that $T$ divides $\mathcal{N}(T)$ into two connected components

$$\mathcal{N}(T) \setminus T = \mathcal{N}_+(T) \cup \mathcal{N}_-(T).$$

Choose an identification of $\mathcal{N}(T)$ with $S^1 \times [-r, r] \times S^1$ and denote the resulting coordinates by $(\theta_+, \rho_+, \phi_+)$, which we arrange to have the following properties:

- $T = S^1 \times \{0\} \times S^1, \mathcal{N}_+(T) = S^1 \times (0, r] \times S^1$ and $\mathcal{N}_-(T) = S^1 \times [-r, 0) \times S^1$
- For each $\phi_0 \in S^1$ the oriented loop $S^1 \times \{0, \phi_0\}$ in $T$ is positively transverse to $\xi$
- $\pi(\theta_+, \rho_+, \phi_+) = \phi_+$ on $\mathcal{N}_+(T)$ and $\pi(\theta_+, \rho_+, \phi_+) = -\phi_+ + c$ on $\mathcal{N}_-(T)$ for some constant $c \in S^1$
- $\xi = \ker(d\theta_+ + \rho_+ \, d\phi_+)$

Given these coordinates, it is natural to define a second coordinate system $(\theta_-, \rho_-, \phi_-)$ by

$$(\theta_-, \rho_-, \phi_-) = (\theta_+, \rho_+, -\phi_+ + c).$$

Then the coordinates $(\theta_-, \rho_-, \phi_-)$ satisfy minor variations on the properties listed above: in particular $\xi = \ker(d\theta_- + \rho_- \, d\phi_-)$ and $\pi(\theta_-, \rho_-, \phi_-) = \phi_-$ on $\mathcal{N}_-(T)$. In the following, we will use separate coordinates on the two components of $\mathcal{N}(T) \setminus T$, denoting both by $(\theta, \rho, \phi)$:

$$(\theta, \rho, \phi) := \begin{cases} (\theta_+, \rho_+, \phi_+) & \text{on } \mathcal{N}_+(T), \\ (\theta_-, \rho_-, \phi_-) & \text{on } \mathcal{N}_-(T). \end{cases}$$

Then $\pi(\theta, \rho, \phi) = \phi$ and $\xi = \ker(d\theta + \rho \, d\phi)$ everywhere on $\mathcal{N}(T) \setminus T$. Observe that these coordinates on $\mathcal{N}_+(T)$ or $\mathcal{N}_-(T)$ separately can be
extended smoothly to the closures $\overline{\mathcal{N}_\pm(T)}$ and $\overline{\mathcal{N}_-}(T)$, though in particular the two $\phi$-coordinates are different where they overlap at $T$.

**Notation.** For any open and closed subset $N \subset B \cup \mathcal{I} \cup \partial M$, we shall in the following denote by $\mathcal{N}(N)$ the union of all the neighborhoods $\mathcal{N}(\gamma)$ and $\mathcal{N}(T)$ constructed above for the connected components $\gamma, T \subset N$. Thus for example,

$$\mathcal{N}(B \cup \mathcal{I} \cup \partial M)$$

denotes the union of all of them.

The complement $M \setminus \mathcal{N}(B \cup \mathcal{I} \cup \partial M)$ is diffeomorphic to a mapping torus. Indeed, let $P$ denote the closure of $\pi^{-1}(0) \cap (M \setminus \mathcal{N}(B \cup \mathcal{I} \cup \partial M))$, a compact surface whose boundary components are in one to one correspondence with the connected components of $\mathcal{N}(B \cup \mathcal{I} \cup \partial P) \setminus \mathcal{I}$. The monodromy map of the fibration $\pi$ defines a diffeomorphism $\psi : P \to P$, which preserves connected components and without loss of generality has support away from $\partial P$, so we define the mapping torus

$$P_\psi = (\mathbb{R} \times P) / \sim,$$

where $(t + 1, p) \sim (t, \psi(p))$. This comes with a natural fibration $\phi : P_\psi \to S^1$ which is trivial near the boundary, so for a sufficiently small collar neighborhood $U \subset P$ of $\partial P$, a neighborhood of $\partial P_\psi$ can be identified with $S^1 \times U$. Choose positively oriented coordinates on each connected component of $U$

$$(\theta, \rho) : U \to [r - \delta, r + \delta) \times S^1$$

for some small $\delta > 0$. This defines coordinates $(\phi, \theta, \rho)$ on a collar neighborhood of $\partial P_\psi = S^1 \times \partial P$, so identifying these for $\rho \in (r - \delta, r]$ with the $(\theta, \rho, \phi)$ coordinates chosen above on the corresponding components of $\mathcal{N}(B \cup \mathcal{I} \cup \partial M) \setminus \mathcal{I}$ defines an attaching map, such that the union

$$P_\psi \cup \mathcal{N}(B \cup \mathcal{I} \cup \partial M)$$

is diffeomorphic to $M$, and the $\phi$-coordinate, which is globally defined outside of $B \cup \mathcal{I}$, corresponds to the fibration $\pi : M \setminus (B \cup \mathcal{I}) \to S^1$.

Choose a number $\delta' > \delta$ with $r - \delta' > 0$, and for each of the coordinate neighborhoods in $\mathcal{N}(B \cup \mathcal{I} \cup \partial M) \setminus \mathcal{I}$, define a 1-form of the form

$$\lambda_0 = f(\rho) \ d\theta + g(\rho) \ d\phi,$$

with smooth functions $f, g : [0, r) \to \mathbb{R}$ chosen so that

1. $\ker \lambda_0 = \xi$ on a smaller neighborhood of $B \cup \mathcal{I} \cup \partial M$.
2. For $\mathcal{N}(\mathcal{I}) \setminus \mathcal{I}$, $f(\rho)$ and $g(\rho)$ extend smoothly to $[-r, r]$ as even and odd functions respectively.
3. The path $[0, r] \to \mathbb{R}^2 : \rho \mapsto (f(\rho), g(\rho))$ moves through the first quadrant from the positive real axis to $(0, 1)$ and is constant for $\rho \in [r - \delta, r]$.
The function

\[ D(\rho) := f(\rho)g'(\rho) - f'(\rho)g(\rho) \]

is positive and \( f'(\rho) \) is negative for all \( \rho \in (0, r - \delta) \).

(5) \( g(\rho) = 1 \) for all \( \rho \in [r - \delta', r) \).

Some possible pictures of the path \( \rho \mapsto (f(\rho), g(\rho)) \in \mathbb{R}^2 \) (with extra conditions that will be useful in the proof of Lemma 2.8) are shown in Figure 8. Note that the functions \( f \) and \( g \) must generally be chosen individually for each connected component of \( \mathcal{N}(B \cup I \cup \partial M) \). Extend \( \lambda_0 \) over \( M' \setminus M \) so that \( \ker \lambda_0 = \xi \) on this region, and extend it over \( P_\psi \) as \( \lambda_0 = d\phi \). The kernel \( \xi_0 := \ker \lambda_0 \) is then a cofoliation on \( M' \): it is contact outside of \( M \) and near \( B \cup I \cup \partial M \), while integrable and tangent to the fibers on \( P_\psi \). In particular \( \lambda_0 \) is contact in the region \( \{ \rho < r - \delta \} \) near \( B \cup I \cup \partial M \), and its Reeb vector field here is

\[ X_0 = \frac{g'(\rho)}{D(\rho)}\partial_\theta - \frac{f'(\rho)}{D(\rho)}\partial_\phi, \]

which is positively transverse to the pages \( \{ \phi = \text{const} \} \) and reduces to \( \partial_\phi \) for \( \rho \in [r - \delta', r] \), which contains the region where \( P_\psi \) and \( \mathcal{N}(B \cup I \cup \partial M) \) overlap.

Proceeding as in [Wen10c], choose next a 1-form \( \alpha \) on \( P_\psi \) such that \( d\alpha \) is positive on the fibers and, in the chosen coordinates \( (\phi, \theta, \rho) \) near \( \partial P_\psi \), \( \alpha \) takes the form

\[ \alpha = (1 - \rho) \, d\theta, \]

where we assume \( r > 0 \) is small enough so that \( 1 - \rho > 0 \) when \( r \in [r - \delta, r + \delta] \). Then if \( \epsilon > 0 \) is sufficiently small, the 1-form

\[ \lambda_\epsilon := d\phi + \epsilon \alpha \]

is contact on \( P_\psi \). We extend it to the rest of \( M' \) by setting \( \lambda_\epsilon = \lambda_0 \) on \( M' \setminus M \), and on \( \mathcal{N}(B \cup I \cup \partial M) \),

\[ \lambda_\epsilon = f_\epsilon(\rho) \, d\theta + g_\epsilon(\rho) \, d\phi, \]

where the functions \( f_\epsilon, g_\epsilon : [0, r] \to \mathbb{R} \) satisfy

1. \( (f_\epsilon(\rho), g_\epsilon(\rho)) = (f(\rho), g(\rho)) \) for \( \rho \leq r - \delta' \),
2. \( g_\epsilon(\rho) = 1 \) and \( f'_\epsilon(\rho) < 0 \) for \( \rho \in [r - \delta', r - \delta] \),
3. \( (f_\epsilon(\rho), g_\epsilon(\rho)) = (\epsilon(1 - \rho), 1) \) for \( \rho \in [r - \delta, r] \),
4. \( f_\epsilon \to f \) and \( g_\epsilon \to g \) in \( C^\infty \) as \( \epsilon \to 0 \).

Now \( \lambda_\epsilon \) is a contact form everywhere on \( M' \), and \( \lambda_\epsilon \to \lambda_0 \) in \( C^\infty \) as \( \epsilon \to 0 \). Denote the corresponding contact structure by

\[ \xi_\epsilon = \ker \lambda_\epsilon. \]

The Reeb vector field \( X_\epsilon \) of \( \lambda_\epsilon \) is defined by the obvious analogue of (2.12) near \( B \cup I \cup \partial M \), is independent of \( \epsilon \) on \( M' \setminus M \), and on \( P_\psi \) is determined
uniquely by the conditions
\[ d\alpha(X_\epsilon, \cdot) \equiv 0, \quad d\phi(X_\epsilon) + \epsilon\alpha(X_\epsilon) \equiv 1. \]

It follows that as \( \epsilon \to 0 \), \( X_\epsilon \) converges to a smooth vector field \( X_0 \) that matches \([2.12]\) near \( B \cup I \cup \partial M \) and on \( P_\psi \) is determined by
\[ (2.13) \quad d\alpha(X_0, \cdot) \equiv 0 \quad \text{and} \quad d\phi(X_0) \equiv 1. \]

Observing that \( X_\epsilon \) is always positively transverse to the pages \( \{ \phi = \text{const} \} \), and applying Proposition \([1.10]\) we have:

**Lemma 2.5.** For \( \epsilon > 0 \) sufficiently small, \( \xi_\epsilon \) is a contact structure on \( M' \) isotopic to \( \xi \), and \( \lambda_\epsilon \) is a Giroux form for \( \pi \).

In order to turn \( \lambda_\epsilon \) into a stable Hamiltonian structure, we define an exact taming form as follows. For each coordinate neighborhood in \( N(B \cup I \cup \partial M) \setminus I \), fix a smooth function \( h : [r - \delta', r - \delta] \to \mathbb{R} \) such that \( h' < 0 \), \( h(\rho) = f(\rho) + c \) for \( \rho \) near \( r - \delta' \) and some constant \( c \geq 0 \), and \( h(\rho) = 1 - \rho \) for \( \rho \) near \( r - \delta \). For each interface torus \( T \subset I \) the function \( f(\rho) \) is the same on \( N_+(T) \) as on \( N_-(T) \), thus we may assume the same is true of \( h(\rho) \) and \( c \). Then
\[
F(\rho) := \begin{cases} 
1 - \rho & \text{for } \rho \in [r - \delta, r], \\
h(\rho) & \text{for } \rho \in [r - \delta', r - \delta], \\
f(\rho) + c & \text{for } \rho \in [0, r - \delta']
\end{cases}
\]
defines a smooth function on \([0, r)\) which, for components of \( N(I) \), has a smooth even extension to \([-r, r]\). By choosing \( f(\rho) \) appropriately on the components of \( N(\partial M) \), one can also arrange \( c = 0 \); it will be convenient (e.g. for Lemma \([2.8]\) below) to assume this for \( N(\partial M) \) but leave the choice of \( c \geq 0 \) and thus \( f(\rho) \) arbitrary everywhere else. Now there is a smooth 1-form \( \hat{\alpha} \) on \( M' \) such that
\[
\hat{\alpha} = \begin{cases} 
\alpha + d\phi & \text{on } P_\psi, \\
F(\rho) \, d\theta + g(\rho) \, d\phi & \text{on } N(B \cup I \cup \partial M), \\
\lambda_0 & \text{on } M' \setminus M,
\end{cases}
\]
and we use this to define an exact 2-form
\[ \omega = d\hat{\alpha}. \]

We claim that \((\lambda_0, \omega)\) defines a stable Hamiltonian structure on \( M' \). Indeed, outside \( M \) and in a sufficiently small neighborhood of \( B \cup I \cup \partial M \) this is clear since \( \lambda_0 \) is contact and \( \omega = d\lambda_0 \). On the subsets described in coordinates by \( r - \delta' \leq \rho < r - \delta \), \( \lambda_0 \) is still contact and \( \omega = -h'(\rho) \, d\theta \wedge d\rho = \frac{h'(\rho)}{F'(\rho)} \, d\lambda_0 \), thus \( \omega \) has maximal rank and its kernel is spanned by \( X_0 \). On \( P_\psi \), \( d\lambda_0 = 0 \) and \( \omega = d\alpha \) annihilates \( X_0 \) by \([2.13]\), so the claim is proved. In fact, for \( \epsilon > 0 \) sufficiently small, we still have \( \omega|_\epsilon > 0 \) and the kernel of \( \omega \) is still spanned by \( X_\epsilon \), thus we’ve proved:
Proposition 2.6. For sufficiently small $\epsilon \geq 0$,
\[ H_\epsilon := (\lambda_\epsilon, \omega) \]
defines a stable Hamiltonian structure on $M'$.

Definition 2.7. Any smooth family $H_\epsilon = (\lambda_\epsilon, \omega)$ of stable Hamiltonian structures on $M'$ defined for small $\epsilon \geq 0$ by the procedure above will be said to be adapted to $\check{\pi}$.

Lemma 2.8. There exists a number $\tau_1 > 0$ so that for any $\tau_0 > 0$ and $m_0 \in \mathbb{N}$, a family of stable Hamiltonian structures $H_\epsilon = (\lambda_\epsilon, \omega)$ on $M'$ adapted to $\check{\pi}$ can be constructed so as to satisfy the following additional conditions on the Reeb vector fields $X_\epsilon$:

1. The interface and boundary tori are Morse-Bott submanifolds, and all closed orbits in a neighborhood of $\mathcal{I} \cup \partial M$ are also Morse-Bott.
2. Each connected component $\gamma \subset B$ and all its multiple covers are nondegenerate elliptic orbits, and their covers up to multiplicity $m_0$ all have Conley-Zehnder index 1 with respect to the natural trivialization of $\xi$ along $\gamma$ determined by the coordinates.
3. All orbits in $B_0 \cup \mathcal{I}_0 \cup \partial M_0$ have minimal period at most $\tau_0$, while all other orbits have period at least $\tau_1$.

Moreover for each $\epsilon > 0$ sufficiently small, the contact form $\lambda_\epsilon$ admits a $C^\infty$-small perturbation to a globally Morse-Bott contact form whose Reeb vector field still satisfies the above conditions.

Proof. We first prove that the stated conditions can be established for $X_0$.

If $\gamma \subset B$ is a binding circle, then $\gamma$ and all its multiple covers can be made nondegenerate and elliptic by choosing the functions $f$ and $g$ so that
\[ f'(\rho)/g'(\rho) \in \mathbb{R} \setminus \mathbb{Q} \quad \text{for all } \rho > 0 \text{ sufficiently small}. \]
This implies that the slope of the curve $\rho \mapsto (f(\rho), g(\rho)) \in \mathbb{R}^2$ is constant for $\rho$ near 0, and this slope determines the Conley-Zehnder index of $\gamma$; in particular, the stated condition is satisfied whenever $f''(0)/g''(0)$ is a negative number sufficiently close to 0. Assume this from now on.

Similarly, we make every orbit in a neighborhood of $\mathcal{I} \cup \partial M$ Morse-Bott by assuming that in such a neighborhood, $\lambda_0 = f(\rho) \, d\theta + g(\rho) \, d\phi$ where $f$ and $g$ satisfy
\[ f'(\rho)g''(\rho) - f''(\rho)g'(\rho) > 0. \]
This means that the path $\rho \mapsto (f(\rho), g(\rho)) \in \mathbb{R}^2$ has nonzero inward angular acceleration as it winds (counterclockwise) about the origin; clearly for $\mathcal{N}(\mathcal{I})$ we can also still safely assume that $f$ and $g$ are restrictions of even and odd functions respectively on $[-r, r]$.

We now show that the periods of the orbits in $B_0 \cup \mathcal{I}_0 \cup \partial M_0$ can be made arbitrarily small compared to all other periods. Observe that by (2.12), the Reeb flow as we’ve constructed it preserves the concentric tori
\{ \rho = \text{const} \} \text{ in the neighborhood } \mathcal{N}(B_0 \cup I_0 \cup \partial M_0), \text{ thus it also preserves } M' \setminus \mathcal{N}(B_0 \cup I_0 \cup \partial M_0). \text{ Since the latter has compact closure, there is a positive lower bound for the periods of all closed orbits in } M' \setminus \mathcal{N}(B_0 \cup I_0 \cup \partial M_0), \text{ so it will suffice to leave } \lambda_0 \text{ fixed in this region and reduce the periods in } B_0 \cup I_0 \cup \partial M_0 \text{ while preserving a lower bound for all other orbits in } \mathcal{N}(B_0 \cup I_0 \cup \partial M_0).

Consider a binding orbit } \gamma \subset B_0: \text{ writing } \lambda_0 \text{ as } f(\rho) \, d\theta + g(\rho) \, d\phi \text{ near } \gamma, \text{ the period of } \gamma \text{ is } f(0) > 0. \text{ Choosing sufficiently small constants } \tau > 0 \text{ and } \epsilon_0 > 0, \text{ we impose the following additional conditions on } f \text{ and } g \text{ (see Figure 8, left):}

- (f(0), g(0)) = (\tau, 0),
- For all } \rho \in (0, r], \frac{g'(\rho)}{-f'(\rho)} \leq \frac{1}{\tau} + \epsilon_0 \in \mathbb{R} \setminus \mathbb{Q},

with equality for } \rho \leq 2r/3.
- For } \rho \in [2r/3, r], g(\rho) \geq 2/3 \text{ and } f(\rho) \leq \tau/3.

Since } f'(\rho)/g'(\rho) \text{ is irrational for } \rho \leq 2r/3, \text{ all closed orbits in } \mathcal{N}(\gamma) \setminus \gamma \text{ are outside this region. For any } \rho_0 \in [2r/3, r], \text{ (2.12) implies that a Reeb orbit in } \{ \rho = \rho_0 \} \text{ has its } \phi\text{-coordinate increasing at the constant rate of } -f'(\rho_0)/D(\rho_0). \text{ Its period is thus at least}

\begin{equation}
\left| \frac{D(\rho_0)}{f'(\rho_0)} \right| = \left| \frac{f(\rho_0)g'(\rho_0) - f'(\rho_0)g(\rho_0)}{f'(\rho_0)} \right| \geq |g(\rho_0)| - \frac{f(\rho_0)g'(\rho_0)}{f'(\rho_0)} \\
\geq \frac{2}{3} - \frac{\tau}{3} \left( \frac{1}{\tau} + \epsilon_0 \right) = \frac{2}{3} - \frac{1}{3} (1 + \tau \epsilon_0) > 0.
\end{equation}

We can therefore keep these periods bounded away from zero while shrinking } f(0) = \tau \text{ to make both the period at } \gamma \text{ and the ratio } -f'(\rho)/g'(\rho) \text{ near } \gamma \text{ arbitrarily small.}

The above requires only a small modification for the neighborhood of a torus } T \subset I_0 \cup \partial M_0: \text{ here we need } f \text{ and } g \text{ to extend over } \rho \in [-r, r] \text{ as even and odd functions respectively, so it is no longer possible to fix the slope } f'(\rho)/g'(\rho) \text{ throughout } \rho \in [0, 2r/3]. \text{ In fact } f'(0) \text{ must vanish, so we amend the above conditions by allowing them to hold for } \rho \in [r/3, r], \text{ but requiring the following for } \rho \in [0, r/3],

- } -g'(\rho)/f'(\rho) \geq 1/\tau + \epsilon_0,
- } f(\rho) \geq \tau(1 - \epsilon_0),
- } g(\rho) \leq \epsilon_0.
This modification is shown at the right of Figure 8. Now for \( \rho \leq r/3 \), the lower bound calculated in (2.14) becomes

\[
\frac{|D(\rho_0)|}{f'(\rho_0)} \geq \left| f(\rho_0)\frac{g'(\rho_0)}{f'(\rho_0)} - |g(\rho_0)| \right| \geq \tau(1 - \epsilon_0) \left( \frac{1}{\tau} + \epsilon_0 \right) - \epsilon_0 \\
= 1 + \epsilon_0 \left( \tau - 2 - \tau \epsilon_0^2 \right) > 0.
\]

Thus we can freely shrink \( f(0) = \tau \), the minimal period of the Morse-Bott family at \( T \), while bounding all other periods away from zero.

Since \( X_\epsilon \) is a small perturbation of \( X_0 \) outside a neighborhood of \( B \cup I \cup \partial M \), the same results immediately hold for \( X_\epsilon \): in particular, for any sequence \( \epsilon_k \to 0 \), \( M' \setminus \mathcal{N}(B_0 \cup I_0 \cup \partial M_0) \) cannot contain a sequence of orbits of \( X_{\epsilon_k} \) with periods below a certain threshold, as a subsequence of these would converge (by Arzelà-Ascoli) to an orbit of \( X_0 \). Similarly, this constraint on the periods will be satisfied by any sufficiently small perturbation of \( X_\epsilon \).

We can now choose such a perturbation to a globally Morse-Bott contact form as follows: let \( \mathcal{U} \subset M' \) denote a union of coordinate neighborhoods of the form \( \{|\rho| < r_0\} \) near each component of \( B \cup I \cup \partial M \), where \( r_0 > 0 \) is chosen such that all periodic orbits inside \( \mathcal{U} \) are Morse-Bott and none exist near \( \partial \mathcal{U} \) (because \( f'/g' \) is irrational). After a generic perturbation of \( \lambda_\epsilon \) in \( M' \setminus \mathcal{U} \), every Reeb orbit not fully contained in \( \mathcal{U} \) becomes nondegenerate (cf. the appendix of [ABW10]), which means all orbits outside \( \mathcal{U} \) are nondegenerate, while all the others (which are inside \( \mathcal{U} \)) are Morse-Bott by construction. \( \square \)

Remark 2.9. To satisfy the conditions stated in Theorem 6, we need a version of Lemma 2.8 with \( \tau_1 = 1 \). This can always be achieved by rescaling \( \lambda_\epsilon \) by a constant, and thus replacing \( \mathcal{H}_\epsilon = (\lambda_\epsilon, \omega) \) by \( (c\lambda_\epsilon, \omega) \) for some \( c > 0 \).
A symplectic cobordism. As a quick detour away from the proof of Theorem 6, we now explain a construction that will be useful for proving Theorem 5. Namely, we will need to know that the stable Hamiltonian structures $\mathcal{H}_0$ and $\mathcal{H}_\epsilon$ for some $\epsilon > 0$ can be related to each other by a cylindrical symplectic cobordism that looks standard near the binding.

To simplify the statement of the following result, let us restrict to the special case where $M = M'$ and $\pi : M \setminus B \to S^1$ is an ordinary (not summed or blown up) open book; this will suffice for the application we have in mind.

**Proposition 2.10.** There exists a family of stable Hamiltonian structures $\mathcal{H}_\epsilon = (\lambda_\epsilon, \omega)$ on $M$ adapted to the open book $\pi : M \setminus B \to S^1$ such that $[0, 1] \times M$ admits a symplectic form $\Omega$ with the following properties:

- $\Omega = \omega + d(t\lambda_0)$ near $\{0\} \times M$.
- $\Omega = d(\epsilon^t \lambda)$ near $\{1\} \times M$ for some contact form $\lambda$ with $\ker \lambda = \xi_\epsilon$ and some $\epsilon > 0$.
- $\Omega = d(\phi(t)\lambda_0)$ on $[0, 1] \times U$ for some neighborhood $U \subset M$ of $B$ on which $\lambda_\epsilon = \lambda_0$, and some smooth function $\phi : [0, 1] \to (0, \infty)$ with $\phi' > 0$.

**Remark 2.11.** We are not claiming that $\mathcal{H}_\epsilon$ in this result can be chosen to make the periods of binding orbits small as in Lemma 2.8 and Theorem 6. For our application we will not need this.

**Proof of Prop. 2.10.** In $(\theta, \rho, \phi)$-coordinates on $N(B)$, we can write $\lambda_0 = f(\rho) \, d\theta + g(\rho) \, d\phi$ with $f$ and $g$ chosen such that $f(\rho) = 1 - \rho$ for $\rho$ near $r - \delta'$. Then setting

$$F(\rho) = \begin{cases} 
1 - \rho & \text{for } \rho \in [r - \delta', r], \\
\beta(\rho) & \text{for } \rho \in [0, r - \delta']
\end{cases}$$

and defining $\hat{\alpha}$ and $\omega$ as before, we have $\omega \equiv d\hat{\alpha}$ where $\hat{\alpha} = \lambda_0$ on a neighborhood $U := \{\rho < r - \delta'\}$ of $B$.

With this stipulation in place, construct the family $\lambda_\epsilon$ as before. Next choose small numbers $\epsilon, \epsilon_1 > 0$ and a smooth function $\beta : [0, \infty) \to [0, \epsilon]$ such that

- $\beta(t) = 0$ for $t$ near $0$,
- $\beta(t) = \epsilon$ for $t \geq \epsilon_1$.

Define a 1-form $\hat{\lambda}$ on $[0, \infty) \times M$ by

$$\hat{\lambda}_{|\{t, p\}} = \lambda_{\beta(t)}|_p$$

for all $(t, p) \in [0, \infty) \times M$, and then define

$$\Omega = \omega + d(t\hat{\lambda})$$

on $[0, \infty) \times M$. Note that $\omega + d(t\lambda_0)$ is symplectic on $[0, \epsilon_1] \times M$ if $\epsilon_1 > 0$ is sufficiently small, and $\Omega$ is $C^\infty$-close to this if $\epsilon > 0$ is also small, implying
that $\Omega$ is also symplectic on $[0, \epsilon_1] \times M$. It is also obviously symplectic on $[\epsilon_1, \infty) \times M$ since it then equals

$$\omega + d(t\lambda_\epsilon)$$

for some $\epsilon > 0$, where $\lambda_\epsilon$ is contact and $\omega$ is $d\lambda_\epsilon$ multiplied by a smooth positive function. This construction thus gives a symplectic form on $[0, \infty) \times M$ which has the desired form already near $\{0\} \times M$ and on $[0, \infty) \times U$. To define a suitable top boundary for the cobordism, observe that $\Omega = d(\hat{\alpha} + t\hat{\lambda})$, thus the $\Omega$-dual vector field to $\hat{\alpha} + t\hat{\lambda}$ is a Liouville vector field $Y$:

$$\iota_Y \Omega := \hat{\alpha} + t\hat{\lambda}.$$ 

We claim that on the hypersurface $\{T\} \times M$ for $T > 0$ sufficiently large, $dt(Y) > 0$. Indeed, this is equivalent to the statement that $\hat{\alpha} + t\hat{\lambda}$ defines a positive contact form on $\{T\} \times M$, which is true if $T$ is large enough since its kernel is then a small perturbation of $\ker \lambda_\epsilon$. Thus fixing $T$ sufficiently large, $\{T\} \times M$ is a convex boundary component of $[0, T] \times M$. Moreover since the primitive of $\Omega$ is just $(1 + t)\lambda_0$ in $[\epsilon_1, \infty) \times U$, $Y$ takes the simple form $(1 + t)\partial_t$ in this region. Using the flow of $Y$ near $\{T\} \times M$, we can now identify a neighborhood of this hypersurface in $[0, T] \times M$ symplectically with a domain of the form

$$((1 - \epsilon_1, 1] \times M, d(e^t\lambda)),$$

where $\lambda$ is a constant multiple of the contact 1-form $\hat{\alpha} + T\lambda_\epsilon$, which defines a contact structure isotopic to $\xi_\epsilon$ due to Gray’s theorem. There is thus a diffeomorphism of $[0, T] \times M$ to $[0, 1] \times M$ that transforms $\Omega$ into the desired form. \(\square\)

Non-generic holomorphic curves and perturbation. Returning to the proof of Theorem 6 assume $\mathcal{H}_\epsilon = (\lambda_\epsilon, \omega)$ is a family of stable Hamiltonian structures adapted to the blown up summed open book $\hat{\pi}$ on $M \subset M'$ and satisfying Lemma 2.8. Choose any compatible almost complex structure $J_0 \in J(\mathcal{H}_0)$ which has the following properties in the coordinate neighborhoods $\mathcal{N}(B \cup I \cup \partial M)$:

- $J_0$ is invariant under the $T^2$-action defined by translating the coordinates $(\theta, \phi)$.
- $d\rho(J_0 \partial_\rho) \equiv 0$.

Observe that $\partial_\rho \in \xi_0$ always, so the second condition says that $J_0$ maps $\partial_\rho$ into the characteristic foliation defined by $\xi_0$ on the torus $\{\rho = \text{const}\}$. Note also that since $\xi_0$ is tangent to the fibers of $P_\psi$, these fibers naturally embed into $\mathbb{R} \times M'$ as $J_0$-holomorphic curves. The construction in [Wen10c, §3] now carries over directly to the present setting and gives the following result.
Proposition 2.12. For each \( i = 0, \ldots, N \), the interior of \( \mathbb{R} \times (M_i \setminus (B_i \cup \mathcal{I}_i)) \) is foliated by an \( \mathbb{R} \)-invariant family of properly embedded surfaces

\[
\{ S^{(i)}_{\sigma, \tau} \}_{(\sigma, \tau) \in \mathbb{R} \times S^1}
\]

with \( J_0 \)-invariant tangent spaces, where

\[
S^{(i)}_{\sigma, \tau} \cap (\mathbb{R} \times P_\psi) = \{ \sigma \} \times (\pi_i^{-1}(\tau) \cap P_\psi),
\]

and its intersection with each connected component of \( \mathbb{R} \times \mathcal{N} (B \cup \mathcal{I} \cup \partial M) \) can be parametrized in \((\theta, \rho, \phi)\)-coordinates by a map of the form

\[
[0, \infty) \times S^1 \to \mathbb{R} \times S^1 \times (0, r] \times S^1 : (s, t) \mapsto (a_i(s) + \sigma, t, \rho_i(s), \tau).
\]

Here \( a_i : [0, \infty) \to [0, \infty) \) is a fixed map with \( a_i(0) = 0 \) and \( \lim_{s \to \infty} a_i(s) = +\infty \), and \( \rho_i : [0, \infty) \to (0, r] \) is a fixed orientation reversing diffeomorphism.

Denote by \( \mathcal{F}^{(i)}_0 \) for \( i = 0, \ldots, N \) the resulting foliation on the interior of \( \mathbb{R} \times (M_i \setminus (B_i \cup \mathcal{I}_i)) \), whose leaves can each be parametrized by an embedded finite energy \( J_0 \)-holomorphic curve

\[
u^{(i)}_{\sigma, \tau} : \hat{\Sigma}_i \to \mathbb{R} \times M'.
\]

The collection of all these curves together with the trivial cylinders over their asymptotic orbits (which include all orbits in \( B \cup \mathcal{I} \cup \partial M \)) defines a \( J_0 \)-holomorphic finite energy foliation \( \mathcal{F}_0 \) of \( M \), as defined in [HWZ03, Wen08].

It’s important however to be aware that this foliation is not generally stable, due to the following index calculation. From now on we assume that \( \mathcal{H}_\epsilon \) has the properties specified in Lemma 2.8.

Proposition 2.13. \( \text{ind} \left( \nu^{(i)}_{\sigma, \tau} \right) = 2 - 2g_i \).

Proof. Let \( \Phi \) denote the natural trivialization of \( \xi_0 \) determined by the \((\theta, \rho, \phi)\)-coordinates along each of the asymptotic orbits of \( \nu^{(i)}_{\sigma, \tau} \). These orbits are in general a mix of nondegenerate binding circles \( \gamma \subset B_i \) with \( \mu^\Phi_{CZ}(\gamma) = 1 \) and Morse-Bott orbits that belong to \( S^1 \)-families foliating \( \mathcal{I} \cup \partial M \). If \( \gamma \) is one of the latter, then we observe that since \( \nu^{(i)}_{\sigma, \tau} \) doesn’t intersect \( \mathbb{R} \times (\mathcal{I} \cup \partial M) \), the asymptotic winding of \( \nu^{(i)}_{\sigma, \tau} \) as it approaches \( \gamma \) matches the winding of any nontrivial section in \( \ker A_\gamma \), which is zero in the chosen coordinates. Thus for sufficiently small \( \epsilon > 0 \), the two largest negative eigenvalues of \( A_\gamma - \epsilon \) both have zero winding, implying \( \alpha^\phi_\gamma (\gamma - \epsilon) = 0 \) and \( p(\gamma - \epsilon) = 1 \), hence by (2.12),

\[
\mu^\phi_{CZ}(\gamma - \epsilon) = 2\alpha^\phi_\gamma (\gamma - \epsilon) + p(\gamma - \epsilon) = 1.
\]

Since \( \nu^{(i)}_{\sigma, \tau} \) projects to an embedding in \( M' \), it is everywhere transverse to the complex subspace in \( T(\mathbb{R} \times M') \) spanned by \( \partial_\phi \) and \( X_0 \), though asymptotically \( \nu^{(i)}_{\sigma, \tau} \) becomes tangent to this space. We can thus define a sensible normal bundle \( N \to \hat{\Sigma}_i \) for \( \nu^{(i)}_{\sigma, \tau} \) as follows: let \( X \) denote the smooth vector field on \( M' \setminus (B \cup \mathcal{I} \cup \partial M) \) that equals \( \partial_\phi \) in every \((\theta, \rho, \phi)\)-coordinate
neighborhood (except at \( \{ \rho = 0 \} \), where this is not well defined), and \( X_0 \) everywhere outside of this. Then the \( J_0 \)-complex span of this vector field defines a bundle that extends smoothly over \( B \cup I \cup \partial M \), and we define the normal bundle \( N \to \dot{\Sigma}_i \) to be the restriction of this bundle to the image of \( u^{(i)}_{\sigma,\tau} \). From this construction it is clear that \( c_1^{\Phi}(N) = 0 \). Now since \( u^{(i)}_{\sigma,\tau} \) is embedded, its index is the index of the normal Cauchy-Riemann operator on the bundle \( N \to \dot{\Sigma}_i \), so by (2.8),

\[
\text{ind} \left( u^{(i)}_{\sigma,\tau} \right) = \chi(\dot{\Sigma}_i) + 2c_1^{\Phi}(N) + \sum_{\gamma} \mu^{\Phi}(\gamma - \epsilon) = \chi(\Sigma_i) = 2 - 2g_i,
\]

where the summation is over all the asymptotic orbits of \( u^{(i)}_{\sigma,\tau} \), whose Conley-Zehnder indices thus cancel out the terms in \( \chi(\dot{\Sigma}_i) \) resulting from the punctures. \( \square \)

From this calculation it follows that the higher genus curves in \( \mathcal{F}_0 \) will vanish under generic perturbations of the data. In contrast, the genus zero curves have exactly the right properties to apply the following useful perturbation result (cf. [Wen05, Theorem 4.5.44] or [Wenb]):

**Implicit Function Theorem.** Assume \( M \) is any closed 3-manifold with stable Hamiltonian structure \( \mathcal{H} = (\lambda, \omega) \), \( J \in \mathcal{J}(\mathcal{H}) \), and

\[
u = (u^{\mathbb{R}}, u^{M}) : \dot{\Sigma} \setminus \Gamma \to \mathbb{R} \times M
\]

is a finite energy \( J \)-holomorphic curve with positive/negative punctures \( \Gamma^\pm \subset \Sigma \) and the following properties:

1. \( \nu \) is embedded and asymptotic to simply covered periodic orbits at each puncture, and satisfies \( \delta_\infty(\nu) = 0 \).
2. \( \dot{\Sigma} \) has genus zero.
3. All asymptotic orbits \( \gamma_z \) for \( z \in \Gamma^\pm \) are either nondegenerate or belong to \( S^1 \)-parametrized Morse-Bott families foliating tori, and \( p(\gamma_z \mp \epsilon) = 1 \) for all \( z \in \Gamma^\pm \) and sufficiently small \( \epsilon > 0 \).
4. \( \text{ind}(\nu) = 2 \).

Then \( \nu \) is Fredholm regular and belongs to a smooth 2-parameter family of embedded curves

\[
u_{(\sigma,\tau)} = (u^{\mathbb{R}}_{\tau} + \sigma, u^{M}_{\tau}) : \dot{\Sigma} \to \mathbb{R} \times M, \quad (\sigma, \tau) \in \mathbb{R} \times (-1, 1)
\]

with \( \nu_{(0,0)} = \nu \), whose images foliate an open neighborhood of \( \nu(\dot{\Sigma}) \) in \( \mathbb{R} \times M \). Moreover, the maps \( u^{M}_{\tau} : \dot{\Sigma} \to M \) are all embedded and foliate an open neighborhood of \( u^{M}(\dot{\Sigma}) \) in \( M \), and if \( \gamma^\tau_z \) denotes a degenerate Morse-Bott asymptotic orbit of \( u_{(\sigma,\tau)} \) for some fixed puncture \( z \in \Gamma \), then the map \( \tau \mapsto \gamma^\tau_z \) parametrizes a neighborhood of \( \gamma^0_z \) in its \( S^1 \)-family of orbits.
Using this and a simple topological argument in [Wen10c], it follows that whenever \( g_i = 0 \), the family \( u_{\sigma,\tau}^{(i)} \) perturbs smoothly along with any sufficiently small perturbation of \( J_0 \). In particular, picking \( \epsilon > 0 \) small and \( J_\epsilon \in \mathcal{J}(\mathcal{H}_\epsilon) \) close to \( J_0 \), there is a corresponding family of \( J_\epsilon \)-holomorphic curves in \( \mathbb{R} \times M_i \) that project to a blown up summed open book on \( M_i \) that is \( C^\infty \)-close to the original one. Perturbing \( \lambda_\epsilon \) a little bit further outside a suitable neighborhood of \( B \cup \mathcal{I} \cup \partial M \), we can then also turn \( \lambda_\epsilon \) into a globally Morse-Bott contact form, and a corresponding perturbation of \( J_\epsilon \) makes the latter Fredholm regular. This proves the existence part of Theorem 6.

We will continue to denote the \( J_\epsilon \)-holomorphic pages constructed in this way by \( u_{\sigma,\tau}^{(i)} : \hat{\Sigma}_i \to \mathbb{R} \times M_i \), for all \( i = 0, \ldots, N \) with \( g_i = 0 \).

**Uniqueness.** Despite their obvious instability, the higher genus curves in the foliation \( \mathcal{F}_0 \) are useful due to the following uniqueness result based on intersection theory. Here \( m_0 \in \mathbb{N} \) denotes the multiplicity bound from Lemma 2.8, which we can assume to be arbitrarily large.

**Proposition 2.14.** Suppose \( v : \hat{\Sigma} \to \mathbb{R} \times M' \) is a somewhere injective finite energy \( J_0 \)-holomorphic curve that intersects the interior of \( \mathbb{R} \times M_i \) and has all its positive ends asymptotic to orbits in \( B \cup \mathcal{I} \cup \partial M \), where the orbits in \( B_i \) each have covering multiplicity at most \( m_0 \). Then \( v \) parametrizes one of the surfaces \( S_{\sigma,\tau}^{(i)} \).

**Proof.** We use the homotopy invariant intersection number \( u_* v \in \mathbb{Z} \) defined by Siefring [Sie] for asymptotically cylindrical maps \( u \) and \( v \). If \( v \) does not parametrize any leaf of \( \mathcal{F}_0^{(i)} \), then its intersection with \( \mathbb{R} \times M_i \) implies that it has at least one isolated positive intersection with some leaf \( S_{\sigma,\tau}^{(i)} \) with \( J_0 \)-holomorphic parametrization \( u_{\sigma,\tau}^{(i)} \), hence

\[ u_{\sigma,\tau}^{(i)} * v > 0. \]

By changing \( \tau \) slightly, we may assume without loss of generality that any ends of \( u_{\sigma,\tau}^{(i)} \) approaching Morse-Bott orbits in \( \mathcal{I} \cup \partial M \) are disjoint from the positive asymptotic orbits of \( v \). By homotopy invariance, we can also take advantage of the lack of negative ends for \( u_{\sigma,\tau}^{(i)} \) and \( \mathbb{R} \)-translate it until its image lies entirely in \( [0, \infty) \times M' \). We can likewise change \( v \) by a homotopy through asymptotically cylindrical maps so that its intersection with \( [0, \infty) \times M' \) lies entirely in the trivial cylinders over its positive asymptotic orbits, i.e. in \( [0, \infty) \times (B \cup \mathcal{I} \cup \partial M) \). An example of this kind of homotopy is shown in Figure 9. The intersection number above is then a sum of the form

\[ u_{\sigma,\tau}^{(i)} * v = \sum_\gamma u_{\sigma,\tau}^{(i)} * (\mathbb{R} \times \gamma), \]
where the summation is over some collection of orbits $\gamma$ in $B \cup \mathcal{I} \cup \partial M$, and we use $\mathbb{R} \times \gamma$ as shorthand for a $J_0$-holomorphic curve that parametrizes the trivial cylinder over $\gamma$. Note that $u^{(i)}_{\sigma,\tau}$ never has an actual intersection with $\mathbb{R} \times \gamma$, so the intersections counted by $u^{(i)}_{\sigma,\tau} \ast (\mathbb{R} \times \gamma)$ are asymptotic, i.e. they are hidden intersections that could potentially emerge from infinity under small perturbations of the data. Since we’ve arranged for $u^{(i)}_{\sigma,\tau}$ and $v$ to have no Morse-Bott orbits in common, the asymptotic intersections vanish except possibly for orbits $\gamma \subset B_i$ of covering multiplicity at most $m_0$. As explained in [Sie], these contributions can then be computed in terms of asymptotic winding numbers: in the natural trivialization $\Phi$ determined by the $(\theta, \rho, \phi)$-coordinates, each of the relevant orbits $\gamma$ has $\mu_{\mathcal{CZ}}(\gamma) = 1 = 2\alpha^\Phi(\gamma) + 1$, hence $\alpha^\Phi(\gamma) = 0$ using (2.2). By construction, the asymptotic winding of $u^{(i)}_{\sigma,\tau}$ as it approaches $\gamma$ is also zero, hence this winding is extremal, which implies

$u^{(i)}_{\sigma,\tau} \ast (\mathbb{R} \times \gamma) = 0$.

This is a contradiction. $\square$

The above proof also works for a $J_\epsilon$-holomorphic curve if it passes through a region that is foliated by $J_\epsilon$-holomorphic pages. In particular, since we’ve already shown this to be true in the planar piece $M_0$ for sufficiently small $\epsilon > 0$, we deduce the following parallel result:

**Proposition 2.15.** For all sufficiently small $\epsilon > 0$, the following holds: if $v : \Sigma \to \mathbb{R} \times M'$ is a somewhere injective finite energy $J_\epsilon$-holomorphic curve that intersects the interior of $\mathbb{R} \times M_0$ and has all its positive ends asymptotic to orbits in $B \cup \mathcal{I} \cup \partial M$, where the orbits in $B_0$ have covering multiplicity at most $m_0$, then $v$ is a reparametrization of one of the $J_\epsilon$-holomorphic pages $u^{(0)}_{\sigma,\tau}$. 
We can now prove the uniqueness statement in Theorem 6. Choose a sequence $\epsilon_k > 0$ converging to zero, denote $\lambda_k := \lambda_\epsilon_k$ and $\xi_k := \ker \lambda_k$, and choose generic almost complex structures $J_k \in \mathcal{J}(H_{\epsilon_k})$ with $J_k \to J_0$ in $C^\infty$. By small perturbations we can assume the forms $\lambda_k$ are all Morse-Bott and have the properties listed in Lemma 2.8: in particular the minimal periods of the orbits in $B_0 \cup I_0 \cup \partial M_0$ are bounded by an arbitrarily small number $\tau > 0$, while all others are at least 1, and the orbits in $B_0$ have Conley-Zehnder index 1. We can also assume that for sufficiently large $k$, planar $J_k$-holomorphic pages $u^{(i)}_{\sigma,\tau}$ in $\mathbb{R} \times M_i$ exist whenever $g_i = 0$, and hence Prop. 2.15 holds. Now arguing by contradiction, suppose that for every $k$, there exists a finite energy $J_k$-holomorphic curve $v_k : (\Sigma_k, J_k) \to (\mathbb{R} \times M', J_k)$ which is subordinate to $\pi_0$ and is (for large $k$) not equivalent to any of the planar curves $u^{(i)}_{\sigma,\tau}$. If $v_k$ has any positive end asymptotic to an orbit in $B_0$ or $I_0$, then it must intersect the interior of $\mathbb{R} \times M_0$ and Proposition 2.15 already gives a contradiction. We can therefore assume that the positive ends of $v_k$ approach simply covered orbits in distinct connected components of $\partial M_0$. This implies that they are all somewhere injective.

Lemma 2.16. A subsequence of $v_k$ converges to one of the $J_0$-holomorphic leaves of the foliation $\mathcal{F}_0$.

Proof. We proceed in three steps.

Step 1: Energy bounds. We use the stable Hamiltonian structure $H_{\epsilon_k} = (\lambda_k, \omega)$ to define the energy of $v_k$. To be precise, choose $c_0 > 0$ small enough so that $\omega + d(t \lambda_0)$ is symplectic on $[-c_0, c_0] \times M'$; the same is then true for all $\omega + d(t \lambda_k)$ with $k$ sufficiently large, so following (2.3) and (2.4), define

$$E_k(v_k) = \int_{\Sigma_k} v_k^* \omega + \sup_{\varphi \in \mathcal{T}} \int_{\Sigma_k} v_k^* d(\varphi \lambda_k),$$

where $\mathcal{T} = \{ \varphi \in C^\infty(\mathbb{R}, (-c_0, c_0)) \mid \varphi' > 0 \}$. Since $\omega$ is exact, $E_k(v_k)$ depends only on the asymptotic behavior of $v_k$. Now since the positive ends all approach simple orbits in distinct connected components of $\partial M_0$, the number of ends and sum of their periods are uniformly bounded, implying a uniform bound on $E_k(v_k)$.

Step 2: Genus bounds. After taking a subsequence we may assume that all the curves $v_k$ have the same number of positive and negative punctures. It is still possible however that the surfaces $\Sigma_k$ could have unbounded topology, i.e. their genus could blow up as $k \to \infty$. To preclude this, we apply the currents version of Gromov compactness, see [Tan98, Prop. 3.3] or [Hut02, Lemma 9.9]. The key fact is that since $E_k(v_k)$ is uniformly bounded, $H_k \to H_0$ and $J_k \to J_0$, $v_k$ as a sequence of currents has a convergent subsequence, and this implies in particular that the relative homology classes $[v_k]$ for this subsequence converge. We now plug this
into the adjunction formula (2.9) for punctured holomorphic curves, which implies
\[ v_k * v_k \geq 2 [\delta(v_k) + \delta_\infty(v_k)] + c_N(v_k) \geq c_N(v_k). \]
Both the right and left hand sides of this expression depend only on \([v_k]\) and on certain integer valued winding numbers of eigenfunctions at the asymptotic orbits of \(v_k\). As orbits vary in a Morse-Bott family that all have the same minimal period, these winding numbers remain constant, thus by the convergence of \([v_k]\), \(v_k * v_k\) converges to a fixed integer, implying an upper bound on \(c_N(v_k)\) for large \(k\). The latter can be written as \(c^I_1(v_k^* \xi_k) - \chi(\hat{\Sigma}_k)\) plus more winding numbers of eigenfunctions, thus every term other than \(\chi(\hat{\Sigma}_k)\) converges, and we obtain a uniform upper bound on \(-\chi(\hat{\Sigma}_k)\), or equivalently, an upper bound on the genus of \(\hat{\Sigma}_k\).

**Step 3: SFT compactness.** We can now assume the domains \(\hat{\Sigma}_k\) are a fixed surface \(\hat{\Sigma}\), so the sequence \(v_k\) with uniform energy bound \(E_k(v_k) < C\) satisfies the compactness theorem of Symplectic Field Theory [BEH+03].

There is one subtle point to be careful of here: since \(X_0\) is not a Morse-Bott vector field, it is not clear at first whether the SFT compactness theory can be applied as \(H_{\epsilon_k} \to H_0\). What saves us is the fact that \(v_k\) is asymptotic at \(+\infty\) to orbits with arbitrarily small period: then for energy reasons, we may assume the only orbits that can appear under breaking or bubbling are other orbits in \(B_0 \cup I_0 \cup \partial M_0\), all of which are Morse-Bott. With this observation, the proof of SFT compactness in [BEH+03] goes through unchanged. We can thus assume that \(v_k\) converges to a \(J_0\)-holomorphic building \(v_\infty\). The positive asymptotic orbits of \(v_\infty\) are all simply covered and lie in distinct connected components of \(\partial M_0\), thus the top level of \(v_\infty\) contains at least one somewhere injective curve \(v_+\) that is subordinate to \(\pi_0\). Then Prop. 2.14 implies that \(v_+\) parametrizes a leaf of the foliation \(\mathcal{F}_0\), so it has no negative ends. The same is true for every other top level component of \(v_\infty\) unless it is a trivial cylinder, and nontrivial curves must all be distinct since they approach distinct orbits at their positive ends. It follows that they do not intersect each other, so there is no possibility of nodes connecting them, and the building must be disconnected unless it consists of only a single component, namely \(v_+\).

We are now just about done: the implicit function theorem implies that if the limit \(v_\infty = \lim v_k\) has genus zero, then \(v_k\) is always one of the \(J_k\)-holomorphic pages \(u^{(i)}_{k,\tau}\) for sufficiently large \(k\). Likewise if \(v_\infty\) has genus \(g > 0\), then \(\text{ind}(v_k) = \text{ind}(v_\infty) = 2 - 2g \leq 0\) by Prop. 2.13, yet \(v_k\) must be Fredholm regular since \(J_k\) was chosen generically, and this also gives a contradiction. □
3. Proofs of the main results

3.1. Nonfillability. In preparation for the arguments that follow, we recall that every strong symplectic filling can be completed by attaching a cylindrical end. To be precise, assume $(M, \xi)$ is a closed, connected contact 3-manifold with positive contact form $\alpha$, and for any two smooth functions $f, g : M \to [-\infty, \infty]$ with $g > f$, define a subdomain of the symplectization $(\mathbb{R} \times M, d(e^t \alpha))$ by

$$S^g_f = \{(t, m) \in \mathbb{R} \times M \mid f(m) \leq t \leq g(m)\}.$$  

Here we include the cases $f \equiv -\infty$ and $g \equiv +\infty$ so that $S^g_f$ may be unbounded. Now suppose $M = \partial W$, where $(W, \omega)$ is a (not necessarily compact) symplectic manifold with contact type boundary, and $\lambda$ is a primitive of $\omega$ defined near $\partial W$ such that $\lambda|_M = e^f \alpha$ for some smooth function $f : M \to \mathbb{R}$. Then using the flow of the Liouville vector field $Y$ defined by $\iota_Y \omega = \lambda$, one can identify a neighborhood of $M$ in $(W, \omega)$ symplectically with a neighborhood of $\partial S^\infty_f$ in $(S^\infty_f, d(e^t \alpha))$. As a consequence, one can symplectically glue the cylindrical end $(S^\infty_f, d(e^t \alpha))$ to $(W, \omega)$ along $M$, giving a noncompact symplectic manifold

$$(W^\infty, \omega) := (W, \omega) \cup_M (S^\infty_f, d(e^t \alpha)),$$

which necessarily contains the half-symplectization $((T, \infty) \times M, d(e^t \alpha))$ whenever $T \in \mathbb{R}$ is sufficiently large.

Assume now that $M$ contains a partially planar domain $M_0 \subset M$. Its planar piece $M^P_0 \subset M_0$ comes with a planar blown up summed open book whose pages have closures in the interior of $M_0$. By Theorem \[\text{[1]}\] we can then find a Morse-Bott contact form $\alpha$ on $M$ and generic compatible almost complex structure $J_+$ such that the planar pages in $M^P_0$ lift to an $\mathbb{R}$-invariant foliation by properly embedded $J_+$-holomorphic curves in $\mathbb{R} \times M$, whose asymptotic orbits are simply covered and have minimal period less than an arbitrarily small number $\tau_0 > 0$, while all other closed orbits of $X_\alpha$ in $M$ have period at least 1. Assume that $\alpha$ is the contact form chosen for defining the symplectic cylindrical end in $(W^\infty, \omega)$.

Choose an almost complex structure $J$ on $W^\infty$ which is compatible with $\omega$, generic on $W \subset W^\infty$ and matches $J_+$ on $S^\infty_f \subset W^\infty$. Then every leaf of the $J_+$-holomorphic foliation in $\mathbb{R} \times M^P_0$ has an $\mathbb{R}$-translation that can be regarded as a properly embedded surface in $S^\infty_f \subset W^\infty$ parametrized by a finite energy $J$-holomorphic curve. The main idea used below is to show that these curves generate a moduli space of $J$-holomorphic curves that must fill the entirety of $W^\infty$, and leads to a contradiction in either of the situations considered by Theorems \[\text{[1]}\] and \[\text{[2]}\]. To prove this, we need a deformation result and a corresponding compactness result to show that the region filled by these curves is open and closed respectively. We shall
prove somewhat more general versions of these results than are immediately needed, as they are also useful for other applications (e.g. in [NW, Wena]).

3.1.1. Deformation and compactness results. For this section we generalize the above setup as follows: let \( u^+ : \tilde{\Sigma} \to W^{\infty} \) denote one of the \( J \)-holomorphic planar pages living in the cylindrical end of \((W^{\infty}, \omega)\), and pick any open neighborhood \( U \subset M \) and \( T > 0 \) such that

\[
u^+(\tilde{\Sigma}) \subset [T, \infty) \times U.
\]

Now choose any data \((\alpha', \omega', J')\) as follows:

- \( \alpha' \) is a Morse-Bott contact form on \( M \) that matches \( \alpha \) on \( U \cup N(B_0 \cup I_0 \cup \partial M_0) \) and has only Reeb orbits of period at least 1 outside of \( N(B_0 \cup I_0 \cup \partial M_0) \)
- \( \omega' \) is a symplectic form on \( W^{\infty} \) that matches \( d(e^t \alpha') \) on \( S^\infty_f \)
- \( J' \) is an \( \omega' \)-compatible almost complex structure on \( W^{\infty} \) that has an \( \mathbb{R} \)-invariant restriction

\[
J'_+ := J'|_{S^\infty_f}
\]

that is generic and compatible with \( \alpha' \) and matches \( J_+ \) on \( \mathbb{R} \times (U \cup N(B_0 \cup I_0 \cup \partial M_0)) \), and \( J' \) is generic on \( W \).

The advantage of this generalization is that fairly arbitrary changes to the data can be accommodated outside a neighborhood of a single page, which is useful for instance in the adaptation of these arguments for weak fillings (cf. [NW]). Let \( \mathcal{M}^*(J') \) denote the moduli space of all somewhere injective finite energy \( J' \)-holomorphic curves in \( W^{\infty} \), which is nonempty by construction since it contains \( u^+ \), and define

\[
\mathcal{M}^*_0(J') \subset \mathcal{M}^*(J')
\]

to be the connected component of this space containing \( u^+ \). The curves \( u \in \mathcal{M}^*_0(J') \) share all homotopy invariant properties of the planar \( J_+ \)-holomorphic pages in \( \mathbb{R} \times M \), in particular:

1. \( \text{ind}(u) = 2 \),
2. \( u \ast u = \delta(u) + \delta_\infty(u) = 0 \).

It follows that all curves in \( \mathcal{M}^*_0(J') \) are embedded. This situation is a slight variation on the setup that was considered in [ABW10, §4], only with the added complication that curves in \( \mathcal{M}^*_0(J') \) may have two ends approaching the same Morse-Bott Reeb orbit, which presents the danger of degeneration to multiply covered curves. The required deformation result is however exactly the same: it depends on the fact that a neighborhood of each embedded curve \( u \in \mathcal{M}^*_0(J') \) can be described by sections of its normal bundle which are nowhere vanishing, because they satisfy a Cauchy-Riemann type equation and have vanishing first Chern number with respect to certain special trivializations at the ends.
Proposition 3.1 ([ABW10 Theorem 4.7]). The moduli space $M^*_0(J')$ is a smooth 2-dimensional manifold containing only proper embeddings that never intersect each other: in particular they foliate an open subset of $W^\infty$.

The compactness result we need is a variation on [ABW10 Theorem 4.8], but somewhat more complicated due to the appearance of multiple covers. For the statement of the result, recall that the compactification in [BEH+03] for the space of finite energy holomorphic curves in an almost complex manifold with cylindrical ends consists of so-called stable holomorphic buildings, which have one main level and potentially multiple upper and lower levels, each of which is a (perhaps disconnected) nodal holomorphic curve. We will be considering sequences of curves in $W^\infty$ that stay within a bounded distance of the positive end, so there will be no lower levels in the limit. We shall use the term “smooth holomorphic curve” to mean a holomorphic building with only one level and no nodes. The following variation on Definition 2.3 will be convenient.

Definition 3.2. A $J'$-holomorphic curve $u : \Sigma \to W^\infty$ will be called subordinate to $\pi_0$ if it has only positive ends, all of which approach Reeb orbits in $B_0 \cup T_0 \cup \partial M_0$, with total multiplicity at most 1 for each connected component of $B_0 \cup \partial M_0$ and at most 2 for each connected component of $T_0$.

Observe that all the curves in $M^*_0(J')$ are subordinate to $\pi_0$. The intersection argument in the proof of Prop. 2.14 now implies:

Lemma 3.3. If $u \in M^*(J)$ is subordinate to $\pi_0$, then $u* u_+ = 0$.

Proposition 3.4. Choose an open subset $W_0 \subset W$ that contains $\partial W$ and has compact closure, and let $W^\infty_0 = W_0 \cup_\pi M^\infty_0$. Then there is a finite set of index 0 curves $\Theta(W_0) \subset M^*(J')$ subordinate to $\pi_0$ and with images in $\overline{W}^\infty_0$ such that the following holds. Any sequence of curves $u_k \in M^*_0(J')$ with images in $\overline{W}^\infty_0$ has a subsequence convergent (in the sense of [BEH+03]) to one of the following:

1. A curve in $M^*_0(J')$
2. A holomorphic building with empty main level and one nontrivial upper level consisting of a single connected curve that can be identified (up to $\mathbb{R}$-translation) with a curve in $M^*_0(J')$ with image in $S^\infty_1$
3. A $J'$-holomorphic building whose upper levels contain only covers of trivial cylinders, and main level consists of a connected double cover of a curve in $\Theta(W_0)$
4. A $J'$-holomorphic building whose upper levels contain only covers of trivial cylinders, and main level contains at most two connected components, which are curves in $\Theta(W_0)$.

Proof. Assume $u_k$ is a sequence of either index 2 curves in $M^*_0(J')$ or index 0 curves subordinate to $\pi_0$ with images in $\overline{W}^\infty_0$ and only simply covered asymptotic orbits. By [BEH+03], $u_k$ has a subsequence converging
to a stable $J'$-holomorphic building $u_{\infty}$. The main idea is to add up the indices of all the connected components of $u_{\infty}$ and use genericity to derive restrictions on the configuration of $u_{\infty}$. To facilitate this, we introduce a variation on the usual Fredholm index formula (2.7): for any finite energy holomorphic curve $v : \hat{\Sigma} \to \mathbb{R} \times M$ with positive and negative asymptotic orbits $\{\gamma_z\}_{z \in \Gamma^\pm}$, choose a small number $\epsilon > 0$ and trivializations $\Phi$ of the contact bundle along each $\gamma_z$ and define the constrained index

$$\hat{\text{ind}}(v) = -\chi(\hat{\Sigma}) + 2\mathcal{C}_T(v^*\xi) + \sum_{z \in \Gamma^+} \mu^\Phi_{\text{CZ}}(\gamma_z - \epsilon) - \sum_{z \in \Gamma^-} \mu^\Phi_{\text{CZ}}(\gamma_z - \epsilon).$$

The only difference here from (2.7) is that at the negative punctures we take $\mu^\Phi_{\text{CZ}}(\gamma_z - \epsilon)$ instead of $\mu^\Phi_{\text{CZ}}(\gamma_z + \epsilon)$, which geometrically means we compute the virtual dimension of a space of curves whose negative ends have all their Morse-Bott orbits fixed in place. So for curves without negative ends $\hat{\text{ind}}(v) = \text{ind}(v)$, and the constrained index otherwise has the advantage of being additive across levels, i.e. if the building $u_{\infty}$ has no nodes, then we obtain $\text{ind}(u_k) = \text{ind}(u_{\infty})$ if the latter is defined as the sum of the constrained indices for all its connected components. Observe that trivial cylinders over Reeb orbits always have constrained index 0. If $u_{\infty}$ does have nodes, the formula remains true after adding 2 for each node in the building, so we then take this as a definition of the index for a nodal curve or nodal holomorphic building. We now proceed in several steps.

**Step 1: Curves in upper levels.** We claim that every connected component of $u_{\infty}$ either has no negative ends or is a cover of a trivial cylinder (in an upper level). Indeed, curves in the main level obviously have no negative ends, and if $v$ is an upper level component with negative ends, the smallness of the periods in $B_0 \cup I_0 \cup \partial M_0$ constrains these to approach other orbits in $B_0 \cup I_0 \cup \partial M_0$, as otherwise $v$ would have negative energy. Then if $v$ does not cover a trivial cylinder, an intersection argument carried out in [ABW10] Proof of Theorem 4.8] implies that $v$ must intersect $u_+$, contradicting Lemma 3.3 above. The key idea here is to consider the asymptotic winding numbers that control holomorphic curves approaching orbits at $B_0 \cup I_0 \cup \partial M_0$, which differ for positive and negative ends at each of these orbits, and thus force $v$ to intersect $u_+$ in the projection to $M$. We refer to [ABW10] for the details.

**Step 2: Indices of connectors.** Borrowing some terminology from Embedded Contact Homology, we refer to branched multiple covers of trivial cylinders as connectors. These can appear in the upper levels of $u_{\infty}$, but can never have any curves above them except for further covers of trivial cylinders, due to Step 1. Since the positive ends of $u_{\infty}$ approach any given orbit in $B_0 \cup I_0 \cup \partial M_0$ with total multiplicity at most 2, only the following types of connectors can appear, both with genus zero:
• **Pair-of-pants** connectors: these have one positive end at a doubly covered orbit and two negative ends at the same simply covered orbit.

• **Inverted pair-of-pants** connectors: with two positive ends at the same simply covered orbit and one negative end at its double cover.

The second variety will be especially important, and we’ll refer to it for short as an *inverted connector*. As we computed in (2.15), all of the simply covered Morse-Bott orbits under consideration have \( \mu_{CZ}^\Phi(\gamma - \epsilon) = 1 \) in the natural trivialization, and in fact exactly the same argument produces the same result for their multiple covers. We thus find that the constrained Fredholm index is 0 for a pair-of-pants connector and 2 for the inverted variant.

**Step 3: Indices of multiple covers.** Suppose \( v \) is a connected component of \( u_\infty \) which is not a cover of a trivial cylinder: then it has no negative ends, and all its positive ends must approach orbits in \( B_0 \cup I_0 \cup \partial M_0 \) with total multiplicity at most 2. Thus if \( v \) is a \( k \)-fold cover of a somewhere injective curve \( v' \), we have \( k \in \{1, 2\} \), and all the asymptotic orbits of both \( v \) and \( v' \) have \( \mu_{CZ}^\Phi(\gamma - \epsilon) = 1 \) in the natural trivialization. Assume \( k = 2 \), and label the positive punctures of \( v \) as \( \Gamma = \Gamma_1 \cup \Gamma_2 \), where a puncture is defined to belong to \( \Gamma_2 \) if its asymptotic orbit is doubly covered, and \( \Gamma_1 \) otherwise. If \( \Gamma' = \Gamma'_1 \cup \Gamma'_2 \) denotes the corresponding punctures of \( v' \), whose orbits must all be distinct and simply covered, then \( \# \Gamma_2 = \# \Gamma'_2 \) and \( \# \Gamma_1 = 2 \# \Gamma'_1 \). Both domains must also have genus zero, so we have

\[
\begin{align*}
\text{ind}(v) &= -(2 - \# \Gamma) + 2c_1^\Phi(v) + \# \Gamma = -2 + 2(\# \Gamma'_2 + 2 \# \Gamma'_1) + 2k c_1^\Phi(v'), \\
\text{ind}(v') &= -(2 - \# \Gamma') + 2c_1^\Phi(v') + \# \Gamma' = -2 + 2(\# \Gamma'_2 + \# \Gamma'_1) + 2c_1^\Phi(v'),
\end{align*}
\]

hence

\[
(3.2) \quad \text{ind}(v) = k \text{ind}(v') + 2(k - 1)(1 - \# \Gamma_2).
\]

This formula also trivially holds if \( k = 1 \). This gives a lower bound on \( \text{ind}(v) \) since \( \text{ind}(v') \) is bounded from below by either 1 (in \( \mathbb{R} \times M \)) or 0 (in \( W^\infty \)) due to genericity. Now observe that whenever \( \Gamma_2 \) is nonempty, the doubly covered orbit must connect \( v \) to an inverted connector, whose constrained index is 2, so for \( k = 2 \) we have

\[
(3.3) \quad \text{ind}(v) + \sum_C \tilde{\text{ind}}(C) = k \text{ind}(v') + 2(k - 1) \geq 2,
\]

where the sum is over all inverted connectors that connect to \( v \) along doubly covered breaking orbits.

**Step 4: Indices of bubbles.** There may also be closed components in the main level of \( u_\infty \): these are \( J' \)-holomorphic spheres \( v \) which are either constant (ghost bubbles) or are \( k \)-fold covers of somewhere injective spheres \( v' \) for some \( k \in \mathbb{N} \). In the latter case, (3.2) also holds with \( \# \Gamma_2 = 0 \), implying \( \text{ind}(v) \geq 0 \), and the inequality is strict whenever \( k > 1 \).
If \( v \) is a ghost bubble, then \( \text{ind}(v) = -2 \), but then the stability condition implies the existence of at least three nodes connecting \( v \) to other components; let us refer to nodes of this type as ghost nodes. There is then a graph with vertices representing the ghost bubbles in \( u_\infty \) and edges representing the ghost nodes that connect two ghost bubbles together, and since \( u_\infty \) has arithmetic genus zero, every connected component of this graph is a tree. Let \( G \) denote such a connected component, with \( V \) vertices and \( E_i \) edges, which therefore satisfy \( V - E_i = 1 \), and suppose there are also \( E_e \) nodes connecting the ghost bubbles represented by \( G \) to non-constant components; we can think of these as represented by “external” edges in \( G \). By the stability condition, we have

\[
2E_i + E_e \geq 3V,
\]

which after replacing \( E_i \) by \( V - 1 \), becomes \( E_e - 2 \geq V \). Then the total contribution to \( \text{ind}(u_\infty) \) from all the ghost bubbles and ghost nodes represented by \( G \) is

\[
-2V + 2(E_i + E_e) = [-2V + (2E_i + E_e)] + E_e \geq V + (2 + V) = 2V + 2 \geq 4,
\]

unless \( u_\infty \) has no ghost bubbles at all.

Step 5: The total index of \( u_\infty \). We can now break down \( \text{ind}(u_\infty) \in \{0, 2\} \) into a sum of nonnegative terms and use this to rule out most possibilities. Ghost bubbles are excluded immediately due to (3.4). Similarly, there cannot be any multiply covered bubbles, because these imply the existence of at least one node and thus contribute at least four to \( \text{ind}(u_\infty) \). The only remaining possibility for multiple covers (aside from connectors) is a component with only positive ends, whose index together with contributions from attached inverted connectors is given by (3.3) and is thus already at least 2. In fact, if this component exists in an upper level, then the underlying simple curve must have index at least 1, implying an even larger lower bound in (3.3) and hence a contradiction. The remaining possibility, which occurs in the case \( \text{ind}(u_\infty) = 2 \), is therefore that the main level consists only of a connected double cover, and there are no nodes at all, nor anything other than trivial cylinders and connectors in the upper levels (Figure 10). The underlying simple curve in the main level has index 0 and has only simply covered asymptotic orbits, all in separate connected components of \( B_0 \cup I_0 \cup \partial M_0 \), thus it is subordinate to \( \pi_0 \).

Assume now that \( u_\infty \) contains no multiply covered components except possibly for connectors. If there is an upper level component \( v \) that is not a cover of a trivial cylinder, then genericity implies \( \text{ind}(v) \geq 1 \), and in fact the index must also be even since all the asymptotic orbits satisfy \( \mu_{cz}(\gamma - \epsilon) = 1 \). Then \( \text{ind}(u_\infty) = \text{ind}(v) = 2 \) and there are no nodes or inverted connectors; the latter implies that all positive asymptotic orbits of \( v \) must be simply covered. Then there also cannot be any doubly covered
Figure 10. The limit building $u_\infty$ in a case where all asymptotic orbits have total multiplicity two, so the main level may be a double cover of an index 0 curve, while the upper level includes connectors and trivial cylinders (the latter not shown in the picture). Superscripts indicate the covering multiplicities of the orbits $\gamma_1$, $\gamma_2$, and $\gamma_3$, and the numbers inside each component indicate the constrained index.

breaking orbits, leaving only the possibility that $v$ is the only nontrivial component in $u_\infty$.

Next assume there are only covers of trivial cylinders in the upper levels, in which case the main level is necessarily nonempty. Each component in the main level has a nonnegative even index, so there can be at most one node or one inverted connector in $u_\infty$, and only if $\text{ind}(u_\infty) = 2$. If the main level contains a component $v$ of index 2, then there are no nodes or inverted connectors. The latter precludes doubly covered breaking orbits, thus there are no connectors at all, and since $v$ cannot have negative ends, we conclude that $u_\infty = v$ (Figure 11). Otherwise all main level components in $u_\infty$ have index 0 and are subordinate to $\pi_0$. Examples of the possible configurations are shown in Figures 12, 13.

Step 6: Compactness for index 0 curves. If $\text{ind}(u_\infty) = 2$, then the somewhere injective index 0 curves that can appear in the building $u_\infty$ are all subordinate to $\pi_0$ and come in two types:

- Type 1: Curves with only simply covered asymptotic orbits.
- Type 2: Curves with exactly one doubly covered asymptotic orbit and all others simply covered, and satisfying $v \ast v = 0$.

Indeed, the second type can occur as the unique main level curve in $u_\infty$ if there is a single inverted connector in an upper level, attached along the doubly covered orbit (Figure 14). To see that $v \ast v = 0$ for such a curve, we use the continuity of the intersection number under convergence to buildings, and the fact that $u_k \ast u_k = 0$ since $u_k \in M^*(J)$; a computation shows that the contribution to $u_\infty \ast u_\infty$ from trivial cylinders and connectors
in the upper level plus breaking orbits adds up to 0. The index counting argument of the previous steps shows already that the curves of Type 1 form a compact and hence finite set. To finish the proof, we must show that the same is true for the Type 2 curves.

Suppose \( u_k \) is a sequence of Type 2 curves converging to a holomorphic building \( u_\infty \). Applying the index counting argument from the previous steps, \( u_\infty \) cannot contain any nodes or inverted connectors; the worst case scenario is that the upper levels contain only trivial cylinders and a single pair-of-pants connector, whose two negative ends connect to two main level components \( v_1^- \) and \( v_2^- \) that are both Type 1 curves (Figure 16). Since there are finitely many Type 1 curves, we may assume by genericity of \( J' \) that no two of them approach a common orbit in the Morse-Bott families \( I_0 \), but this must be the case for \( v_1^- \) and \( v_2^- \) as they are both attached to a connector over an orbit in \( I_0 \), so we conclude that both are the same curve,
Figure 16. A possible limit of the sequence $v_k$.

which we’ll call $v_-$. We can rule out this scenario by computing the self-intersection number $v_\infty * v_\infty$, which must a priori equal $v_k * v_k = 0$. Once more the connectors, trivial cylinders and breaking orbits contribute zero in total, so since the main level includes two copies of $v_-$, we deduce

\[ 0 = v_\infty * v_\infty = 4(v_- * v_-). \]

But we can also compute $v_- * v_-$ directly from the adjunction formula (2.9); indeed,

\[ v_- * v_- = 2[\delta(v_-) + \delta_\infty(v_-)] + c_N(v_-), \]

where we’ve dropped the last term in (2.9) since all the asymptotic orbits are simple. The constrained normal Chern number $c_N(v_-)$ is defined in (2.10) and can be deduced from the fact that $\text{ind}(v_-) = 0$: since all of the relevant orbits satisfy $\mu^\Phi_{\text{CZ}}(\gamma - \epsilon) = 1$ and $\alpha^\Phi(\gamma + \epsilon) = 0$, we find

\[ 2c^\Phi_1(v_-) = \text{ind}(v_-) + \chi(\Sigma) - \sum_{z \in \Gamma} \mu^\Phi_{\text{CZ}}(\gamma_z - \epsilon) = 2 - 2\#\Gamma, \]

\[ c_N(v_-) = c^\Phi_1(v_-) - \chi(\Sigma) + \sum_{z \in \Gamma^+} \alpha^\Phi(\gamma_z + \epsilon) = 1 - \#\Gamma - (2 - \#\Gamma) = -1. \]

This implies that $v_- * v_-$ is odd, and is thus a contradiction. \[ \square \]

3.1.2. Some proofs by contradiction. We are now in a position to prove the main results on symplectic fillings.

Proof of Theorem 1 and Corollary 1. Given Propositions 3.1 and 3.4 above, the result follows from the same argument as in [ABW10]. For completeness let us briefly recall the main idea: if $(M, \xi)$ embeds as a nonseparating contact type hypersurface into some closed symplectic 4-manifold $(W, \omega)$, then by cutting $W$ open along $M$ and gluing together an infinite chain of copies of the resulting symplectic cobordism between $(M, \xi)$ and itself, we obtain a noncompact but geometrically bounded symplectic manifold $(W, \omega)$ with contact type boundary $(M, \xi)$. Attaching a cylindrical end and considering the moduli space $M_0(J)$ that arises from a partially planar domain, one can use the monotonicity lemma to prevent the curves in $M_0(J)$ from escaping beyond a compact subset of $W$, thus the compactness result Prop. 3.4 applies. In combination with Prop. 3.1, this implies that outside a subset of codimension 2 (the images of finitely many curves
from Prop. 3.4, the set of all points in \( W \) filled by curves in \( M_0(J) \) must be open and closed, and is therefore everything; since those curves are confined to a compact subset, this implies \( W \) is compact and is thus a contradiction.

By a similar argument one can prove Corollary 1 independently of Theorem 1 for if \((W,\omega)\) is a strong filling with at least two boundary components \((M,\xi)\) and \((M',\xi')\), then the curves in \( M_0(J) \) emerging from the cylindrical end at \( M \) will foliate \( W^\infty \) except at a subset of codimension 2; yet they cannot enter the cylindrical end at \( M' \) due to convexity, and this is again a contradiction.

\[ \square \]

**Proof of Theorem 2.** Assume \((W,\omega)\) is a strong filling of \((M,\xi)\) and the partially planar domain \( M_0 \subset M \) is a planar torsion domain. It therefore has a planar piece \( M^P_0 \subset M_0 \), which is a proper subset of its interior. Combining Props. 3.1 and 3.4 as in the proof of Theorem 1 above, the curves in \( M_0(J) \) that emerge from \( M^P_0 \) in the cylindrical end of \( W^\infty \) form a foliation of \( W^\infty \) outside a subset of codimension 2. We can therefore pick a point \( p \in M \setminus M^P_0 \) and find a sequence of curves \( u_k \in M_0(J) \) for \( k \to \infty \) whose images contain \((k,p) \in [T,\infty) \times M \subset W^\infty \). Applying Prop. 3.4 again, these have a subsequence which converges to a \( J_+\)-holomorphic curve \( u' \) in \( \mathbb{R} \times M \), whose asymptotic orbits are in the same Morse-Bott families as the curves in \( M_0(J) \). The uniqueness statement in Theorem 6 then implies that \( u' \) is a lift of a page in the summed open book on \( M_0 \), which proves that \( M_0 = M \), and \( M_0 \setminus M^P_0 \) consists of a single family of pages diffeomorphic to the planar pages in \( M^P_0 \) and approaching the same Reeb orbits at their boundaries. In other words, \( M_0 \) is a symmetric summed open book, which contradicts the definition of a planar torsion domain.

\[ \square \]

**Proof of Theorem 3.** The idea is much the same as in the proof of Theorem 2, but instead of working in the compact context of a symplectic filling, we work in a noncompact symplectic cobordism diffeomorphic to \( \mathbb{R} \times M \), in which the negative end is “walled off” so that curves in \( M_0(J) \) cannot reach it. This wall is created by a family of holomorphic curves, namely a subset of the generally non-generic family arising from an open book decomposition (see Figure 17).

Specifically, suppose \( \pi : M \setminus B \to S^1 \) is an open book decomposition. Recall from Prop. 2.10 that there is a symplectic cobordism \((W,\Omega) = ([0,1] \times M, \Omega)\) where \( \Omega \) has the form \( \omega + d(t\lambda_0) \) near \(\{0\} \times M\), \( d(e^t\lambda) \) near \(\{1\} \times M\) and \( d(\varphi(t)\lambda_0) \) in a neighborhood of \([0,1] \times B\) for some positive increasing function \( \varphi \). Here \( \mathcal{H}_\epsilon = (\lambda_\epsilon, \omega) \) is a family of stable Hamiltonian structures adapted to the open book, so \( \xi_\epsilon = \ker \lambda_\epsilon \) for some small \( \epsilon > 0 \) is a supported contact structure and \( \lambda \) is a contact form for \( \xi_\epsilon \).

Arguing by contradiction, assume \((M,\xi_\epsilon)\) contains a planar torsion domain \( M_0 \) that is disjoint from \( B \). We can then find a neighborhood \( \mathcal{U} \subset M \) of \( B \) such that \( M_0 \subset M \setminus \mathcal{U} \) and \( \Omega = d(\varphi(t)\lambda_0) \) on \([0,1] \times \mathcal{U}\). Extend \( W \)
Figure 17. The symplectic cobordism used in the proof of Theorem 5, with the negative end “walled off” by holomorphic pages of an open book. The almost complex structure in the shaded region is a non-generic one for which holomorphic open books always exist.

To a noncompact symplectic manifold as follows: first attach to \( \{1\} \times M \) a positive cylindrical end that contains a half-symplectization of the form
\[
([T \times \infty) \times M, d(e^t \alpha)].
\]
Note that since \( \{1\} \times M \) is a convex boundary component of \((W, \Omega)\), we are free here to choose \( \alpha \) as any contact form with \( \ker \alpha = \xi \); in particular on \( M_0 \) we can assume it is the special Morse-Bott contact form provided by Theorem 6 and since \( M_0 \cap U = \emptyset \), we can also assume \( \alpha = \lambda_0 \) in \( U \) and \( \Omega = d(e^t \lambda_0) \) on \([1, \infty) \times U\). Secondly, attach to \( \{0\} \times M \) a negative cylindrical end of the form
\[
((-\infty, 0] \times M, \omega + d(\psi(t) \lambda_0)),
\]
where \( \psi : (-\infty, 0] \to \mathbb{R} \) is an increasing function with sufficiently small magnitude to make the form symplectic. Denote the resulting noncompact symplectic manifold by \((W^\infty, \omega)\).

Recall the special almost complex structure \( J_0 \in J(H_0) \) constructed in §2.2 for which all the pages of \( \pi \) admit \( J_0 \)-holomorphic lifts in \( \mathbb{R} \times M \). We now can choose an almost complex structure \( J \) on \((W^\infty, \omega)\) that has the following properties:
(1) $J$ is everywhere compatible with $\omega$
(2) $J = J_0$ on both $\mathbb{R} \times U$ and $(-\infty, 0] \times M$
(3) On $[T, \infty) \times M$, $J$ is the special almost complex structure compatible with $\alpha$ provided by Theorem 6.

Now the moduli space $\mathcal{M}_0(J)$ of $J$-holomorphic curves emerging from $M_0$ in the positive end can be defined as in the previous proof. The important new feature is that we also have $J$-holomorphic curves in $W^\infty$ coming from the $J_0$-holomorphic lifts of pages of the open book: in fact for some $T_0 \in \mathbb{R}$ sufficiently close to $-\infty$, every point in $(-\infty, T_0] \times M$ is contained in such a curve (see Figure 17). The leaves of the foliation in $[T, \infty) \times M_0$ obviously do not intersect these curves, so positivity of intersections implies that no curve in $\mathcal{M}_0(J)$ may intersect them. It follows that the curves in $\mathcal{M}_0(J)$ can never enter $(-\infty, T_0] \times M$, so the compactness result Prop. 3.4 applies, and we conclude as before that $\mathcal{M}_0(J)$ fills an open and closed subset of $W^\infty$ outside a subset of codimension 2. But this forces some curve in $\mathcal{M}_0(J)$ to enter the negative end eventually, and we have a contradiction.

Remark 3.5. For an arguably easier proof of Theorem 5, one can present it as a corollary of Theorem 2 by showing that whenever $(M, \xi)$ is supported by an open book $\pi : M \setminus B \to S^1$ and $U \subset M$ is a neighborhood of the binding, $(M \setminus U, \xi)$ can be embedded into a strongly fillable contact manifold. This can be constructed by a doubling trick using the binding sum: if $(M', \xi')$ is supported by an open book that has the same page $P$ as $\pi$ but inverse monodromy, then one can construct a larger contact manifold by summing every binding component in $M$ to a binding component in $M'$. The result is a symmetric summed open book which has a strong symplectic filling homeomorphic to $[0, 1] \times S^1 \times P$, in which the natural projection to $[0, 1] \times S^1$ forms a symplectic fibration. The details of this construction are carried out in [Wena].

3.2. Embedded Contact Homology.

3.2.1. Review of twisted and untwisted ECH. Before proving Theorems 3 and 4, we review the essential definitions of Embedded Contact Homology, mainly following the discussions in [HS06, §11] and [Tan]. Assume $(M, \xi)$ is a closed contact 3-manifold with nondegenerate contact form $\lambda$, and $J$ is a generic almost complex structure on $\mathbb{R} \times M$ compatible with $\lambda$. We will refer to Reeb orbits as even or odd depending on the parity of their Conley-Zehnder indices: in dynamical terms, an even orbit is always hyperbolic, while an odd orbit can be either elliptic or hyperbolic, the latter if and only if its double cover is even. In §2.1 we defined the notion of an orbit set $\gamma = \{(\gamma_1, m_1), \ldots, (\gamma_N, m_N)\}$, and we say that $\gamma$ is admissible if $m_i = 1$ whenever $\gamma_i$ is hyperbolic. Given $h \in H_1(M)$, choose a reference cycle, i.e. a 1-cycle $\rho_h$ in $M$ with $[\rho_h] = h$; without loss of generality we
can assume $\rho_h$ is represented by an embedded oriented knot in $M$ that is not contained in any closed Reeb orbit. Then adapting the definition of $H_2(M, \gamma^+ - \gamma^-)$ from §2.1, it makes sense to speak of relative homology classes in $H_2(M, \rho_h - \gamma)$ for any orbit set $\gamma$ with $[\gamma] = h$.

For any subgroup $G \subset H_2(M)$, define

$$\widetilde{C}_*(M, \lambda; h, G)$$

to be the free $\mathbb{Z}$-module generated by symbols of the form $e^A\gamma$, where $\gamma$ is an admissible orbit set with $[\gamma] = h$ and $A \in H_2(M, \rho_h - \gamma)/G$, meaning $A \sim A'$ whenever $A - A' \in G$. A differential $\partial : \widetilde{C}_*(M, \lambda; h, G) \to \widetilde{C}_*(M, \lambda; h, G)$ is defined by

$$\partial (e^A\gamma) = \sum_{\gamma', A'} \# \left( \frac{\mathcal{M}_{\text{adm}}(\gamma, \gamma', A')}{\mathbb{R}} \right) e^{A + A'}\gamma'$$

where the sum ranges over all admissible orbit sets $\gamma'$ and $A' \in H_2(M, \gamma - \gamma')/G$, and $\mathcal{M}_{\text{adm}}(\gamma, \gamma', A') \subset \mathcal{M}(J)$ is the oriented 1-manifold of (possibly disconnected) finite energy $J$-holomorphic curves $u : \hat{\Sigma} \to \mathbb{R} \times M$ satisfying the following conditions:

1. $\text{ind}(u) = 1$,
2. $[u] \sim A'$ in $H_2(M, \gamma - \gamma')/G$,
3. $u$ is admissible in the sense defined in [Hut02].

The definition of an admissible $J$-holomorphic curve given in [Hut02] is rather complicated, but for our purposes we will only need the following special case, which is immediate:

**Lemma 3.6** (see [Hut02, §4]). Suppose $u : \hat{\Sigma} \to \mathbb{R} \times M$ is a connected $J$-holomorphic curve of index 1 whose asymptotic orbits are all simply covered and distinct. Then $u$ is admissible if and only if it is embedded.

The orientation of $\mathcal{M}_{\text{adm}}(\gamma, \gamma', A')$ is chosen in accordance with [BM04], which requires first choosing an ordering for all the even orbits in $M$, then ordering the punctures of any $u \in \mathcal{M}_{\text{adm}}(\gamma, \gamma', A')$ accordingly. The signed count above is then finite due to the index inequality and compactness theorem in [Hut02].

These same results together with the gluing construction of [HT07, HT09] imply that $\partial^2 = 0$, and the resulting homology is denoted by $\widetilde{ECH}_*(M, \lambda; J; h, G)$. We have two natural choices for the subgroup $G$: if $G = H_2(M)$, then the terms $e^A$ are all trivial and we obtain the usual untwisted Embedded Contact Homology,

$$\widetilde{ECH}_*(M, \lambda; J; h) := \widetilde{ECH}_*(M, \lambda; J; h; H_2(M))$$

8The results in [Hut02] are stated only for a very special class of stable Hamiltonian structures arising from mapping tori, but they extend to the contact case due to the relative asymptotic formulas of Siefring [Sie08].
At the other end of the spectrum, taking $G$ to be the trivial subgroup leads to the fully twisted variant of ECH,
\[ \widetilde{ECH}_*(M, \lambda, J; h) := \widetilde{ECH}_*(M, \lambda, J; \{0\}). \]

Since every nontrivial finite energy $J$-holomorphic curve in $\mathbb{R} \times M$ has at least one positive puncture, the empty orbit set $\emptyset$ always satisfies $\partial \emptyset = 0$, and thus represents a homology class which we call the (untwisted) contact class,
\[ c(M, \lambda, J) = [\emptyset] \in ECH_*(M, \lambda, J; 0). \]

To define the twisted contact class, we note that for $h = 0$ there is a canonical choice of reference cycle $\rho_0$, namely the empty set, so $H_2(M, \rho_0 - \emptyset) = H_2(M)$ and it is natural to define
\[ \tilde{c}(M, \lambda, J) = [e^0 \emptyset] \in \widetilde{ECH}_*(M, \lambda, J; 0). \]

3.2.2. Proof of the vanishing theorems. We now prove Theorems 3 and 4. Assume $(M, \xi)$ contains a planar $k$-torsion domain $M_0$ with planar piece $M_0^P \subset M_0$. Note that for some planar torsion domains, there may be multiple subsets of $M_0$ that could sensibly be called the planar piece (e.g. $M_0$ could contain multiple planar open books summed together as in Figure 18), so whenever such an ambiguity exists, we choose $M_0^P$ to make $k$ as small as possible. Let $\lambda$ and $J$ denote the special Morse-Bott contact form and compatible Fredholm regular almost complex structure provided by Theorem 6. Then $\partial M_0^P$ is a nonempty union of tori
\[ \partial M_0^P = T_1 \cup \ldots \cup T_n \]
which are Morse-Bott families of Reeb orbits, and the interior of $M_0^P$ may also contain interface tori, which we denote by
\[ \mathcal{I}_0 = T_{n+1} \cup \ldots \cup T_{n+r}, \]
and binding circles
\[ B_0 = \beta_1 \cup \ldots \cup \beta_m. \]
The planar pages in $M_0^P$ have embedded $J$-holomorphic lifts to $\mathbb{R} \times M$, forming a family of curves,
\[ u_{\sigma, \tau} \in \mathcal{M}(J), \quad (\sigma, \tau) \in \mathbb{R} \times S^1, \]
which have no negative punctures and $m + n + 2r$ positive punctures, each asymptotic to simply covered orbits in $B_0 \cup \mathcal{I}_0 \cup \partial M_0^P$, exactly one in each connected component of $B_0 \cup \partial M_0^P$ and two in each component of $\mathcal{I}_0$. Moreover, other than these curves and the obvious trivial cylinders, there is no other connected finite energy $J$-holomorphic curve in $\mathbb{R} \times M$ with its positive ends approaching any subcollection of the asymptotic orbits of $u_{\sigma, \tau}$.

We now perturb $\lambda$ to a nondegenerate contact form $\lambda'$ by the scheme described in [Bon02], so that each of the original Morse-Bott tori $T_j \subset
Figure 18. A planar torsion domain for which the order is not uniquely defined: depending on the choice of planar piece, the order could be either 1 or 3.

\[ \mathcal{I}_0 \cup \partial M^P_0 \] admits exactly two nondegenerate Reeb orbits, one elliptic and one hyperbolic,

\[ \gamma^e_j \cup \gamma^h_j \subset T_j, \]

whose Conley-Zehnder indices with respect to the natural trivializations in the \((\theta, \rho, \phi)\)-coordinates are 1 and 0 respectively. Perturbing \( J \) to a generic \( J' \) compatible with \( \lambda' \), the family of curves \( u_{\sigma, \tau} \) gives rise to embedded \( J' \)-holomorphic curves (Figure 19) asymptotic to various combinations of these orbits and the components of \( B_0 \). If \( u : \hat{\Sigma} \to \mathbb{R} \times M \) is such a curve, then genericity implies \( \text{ind}(u) \geq 1 \), so we deduce from the index formula that such curves come in two types:

- \( \text{ind}(u) = 2 \) if all ends approaching \( \mathcal{I}_0 \cup \partial M^P_0 \) approach elliptic orbits,
- \( \text{ind}(u) = 1 \) if \( u \) has exactly one end approaching a hyperbolic orbit in \( \mathcal{I}_0 \cup \partial M^P_0 \).

In fact, up to \( \mathbb{R} \)-translation there is exactly one \( J' \)-holomorphic curve \( u_0 : \hat{\Sigma} \to \mathbb{R} \times M \) with all punctures positive and asymptotic to the orbits

\[ \gamma^h_1, \gamma^e_2, \ldots, \gamma^e_n, \gamma^e_{n+1}, \ldots, \gamma^e_{n+r}, \beta_1, \ldots, \beta_m. \]

Let us therefore define the orbit set

\[ \gamma_0 = \{ (\gamma^h_1, 1), (\gamma^e_2, 1), \ldots, (\gamma^e_n, 1), (\gamma^e_{n+1}, 2), \ldots, (\gamma^e_{n+r}, 2), (\beta_1, 1), \ldots, (\beta_m, 1) \}, \]

for which \([\gamma_0] = 0\), and define also the relative homology class

\[ A_0 = -[u_0] \in H_2(M, \rho_0 - \gamma_0). \]

The perturbation from \( J \) to \( J' \) creates some additional \( J' \)-holomorphic cylinders which arise from gradient flow lines along the Morse-Bott families of orbits, as described in [Bou02]. Namely for each \( j = 1, \ldots, n + r \), there are two embedded index 1 cylinders

\[ v^+_j, v^-_j : \mathbb{R} \times S^1 \to \mathbb{R} \times M, \]
each with positive end at $\gamma_j^e$ and negative end at $\gamma_j^h$; the images of these cylinders in $M$ are the two connected components of $T_j \setminus (\gamma_j^e \cup \gamma_j^h)$, thus after choosing the labels appropriately, we can assume their relative homology classes are related by

$$[v_j^+] - [v_j^-] = [T_j] \in H_2(M).$$

Now in the twisted ECH complex, the only curves other than $u_0$ counted by $\partial (e^{A_0 \gamma})$ are the disjoint unions of $v_j^\pm$ with collections of trivial cylinders for $j = 2, \ldots, n + r$. The negative ends of such a disjoint union give rise to the orbit set

$$\gamma_j := \{(\gamma_j^h, 1), (\gamma_j^e, 1), \ldots, (\gamma_{j-1}^e, 1), (\gamma_j^h, 1), (\gamma_{j+1}^e, 1), \ldots, (\gamma_n^e, 1),$$

$$(\gamma_{n+1}^e, 2), \ldots, (\gamma_{n+r}^e, 2), (\beta_1, 1), \ldots, (\beta_m, 1)\}$$

for $j = 1, \ldots, n$, and a similar expression for $j = n+1, \ldots, n+r$ which will appear twice due to the multiplicity attached to $\gamma_j^e$. Choosing appropriate
coherent orientations and adding all this together, we find
\[
\partial \left( e^{A_0} \gamma_0 \right) = e^0 \emptyset + \sum_{j=2}^{n} e^{A_0 + [v_j]} \left( e^{[T_j]} - 1 \right) \gamma_j
\]
\[
+ \sum_{j=n+1}^{n+r} 2e^{A_0 + [v_j]} \left( e^{[T_j]} - 1 \right) \gamma_j.
\]
We thus have \( \partial \left( e^{A_0} \gamma_0 \right) = e^0 \emptyset \) whenever \([T_j] = 0 \in H_2(M)\) for all \( j = 2, \ldots, n + r \), which proves Theorem 4. For untwisted coefficients, we divide the entire calculation by \( H_2(M) \) so that \( e^{[T_j]} - 1 = 0 \) always, thus \( \partial \gamma_0 = \emptyset \) holds with no need for any topological condition. With that, the proof of Theorem 5 is complete.

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