The Universe Expansion and Energy Problems

Moukaddem Nazih

Departement of Mathematics,
Lebanese University, Tripoli-Lebanon

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Abstract In this paper we first construct a mathematical model for the universe expansion that started up with the original Big Bang. Next, we discuss the problems of the mechanical and physical laws invariance regarding the spatial frame exchanges. We then prove the (theoretical) existence of a variable metric $g_t$, depending on time and satisfying a simplified Einstein equation, so that all free ordinary trajectories are geodesics. This is done by considering the classical Galileo–Newtonian space and time notions, by using generalized Newtonian principles and adding the approved physical new ones (as covariance principle, Mach principle, the Einstein equivalence principle . . . ) in order to establish a new cosmological model of the dynamical universe as being $(U(t))_{t>0} = (B_e(O, R(t)), g_t)_{t>0}$, where $B_e(O, R(t))$ is the Euclidean ball of radius $R(t)$ in $\mathbb{R}^3$ and $R(t) \sim t$ when $t \gg 0$ and $c = 1$. The cosmological metric $g_t$ is totally determined, at time $t$, by the mass–energy distribution $E_t(X)$ on $B_e(O, R(t))$. We study also the black holes phenomenon and we prove that the total and global cosmological energy distribution $E_t(X)$ satisfies a wave equation whose solutions are characterized by pseudo-frequencies depending on time and related to the spectrum of the Dirichlet problem on the unit ball $B_e(O, 1)$ for the Laplace–Beltrami operator $-\Delta$. Our model is consistent in the sense that all Newtonian and classical physical laws are valid as particular cases in classical situations. We end this construction by introducing, possibly, the most important feature of the expansion–time–energy triangle that is the temperature–pressure duality factor and so achieving the construction of our real physical model of the expanding universe. Then, we show that all basic results of modern Physics are still valid without using neither the second part of the special relativity second postulate nor the un-
certainty principle. Moreover, we give a mathematical model that explains the matter–antimatter duality and we conclude that there exist only two privileged fundamental forces. We then show that our model results in a well-posed initial value formulation for the most general Einstein’s equation and leads to a well-determined solution to this equation by using a constraint-free Hamiltonian system that reduces, according to our model, to six equations relating six independent unknown functions. We also adapt the Einstein’s general relativity theory to our setting thus freeing it from several obstacles and constraints and leading to the unification of general relativity with quantum Physics and Newton-Lagrange-Hamilton’s Mechanics. We end this paper by determining (within the framework of our model) the age, the size and the total energy of our universe and proving that only the energy $E$, the electromagnetic constant $ke^2$, the Boltzmann characteristic $K_B T$ and the speed of light $c$ (to which we add a quantum Statistics’ constant $A$) are time-independent universal constants. The other fundamental constants (such as $G$, $\hbar$, $K$, $\alpha$...) are indeed time dependent and naturally related to the previous ones proving, in that way, the unity of the fundamental forces and that of all Physics’ notions. This essentially is done by adapting the Einstein-de Sitter model (for the Hubble homogeneous and isotropic Cosmology) and the Einstein-Friedmann equations to our setting.

0 Introduction, Summary and Contents

In the beginning of the 21st century a crisis, which seems to be structural, reappears inside modern Physics. Physics seemed to have resolved, in the first half of the preceding century, all the problems that appeared at the end of the 19th century with the discovery of many phenomena and laws that were considered as being contradictory to the classical Galileo-Newtonian Mechanics and Physics. In our days, the modern cosmology is based on the Big Bang theory (universe expansion) supplied with multiple evidences. More recently, our comprehension of the matter is based on the atom-nucleus-electrons and nucleons-quarks model on one side, and on the hadrons-leptons classification and the matter-antimatter duality on the other side. Everything obeys a rigorous quantification of charges, energy levels and precise laws, in which the most important
are the energy and momentum conservation and the Pauli exclusion laws. With the Einstein’s famous formula $E = mc^2$, we better understood the equivalence of all energy forms. The quantization and unification of energy forms were also better understood after the discovery of the photon and the photo-electric effect, equally by Einstein. Also, the discovery of the quantum Mechanics and Schrödinger’s equations led to a great progress in the comprehension of the waving nature of matter and in the understanding of a large number of natural phenomena without giving any precise (theoretical and experimental) explanation.

On the light of these great discoveries, a part of which is due to the quantum theory and the other part to the relativity theory, two important questions above many others are raised:

1- Is there a compatibility or a complementarity between these two theories, which seem have contributed together in solving classical Physics impasses?

2- After the definitive comprehension of the electromagnetic phenomenon and the progress in the unification theory of the electromagnetic force with the weak and strong two interaction forces, can all the forces (including the gravitational one) be unified inside a global theory?

The answer to the first question seems to be negative. Many unification theories run into a reality that seems to be inexplicable. The mentioned reasons of failure in such attempts are various, such as, our technical incapability to execute infinitely small or infinitely large measures, the existence of imperceptible dimensions or the nonexistence of objective realities that would be governed by precise laws or, in case of such an existence, our incapability of understanding their true nature and real functioning.

With the help of our global model, we propose resolving a large number of open problems and removing the apparent contradictions related to the interpretation of the new results and facts, without contradicting the fundamental principles of classical and modern Mechanics or Physics, as long as they are scientifically (theoretically and experimentally) valid. Among these principles we can mention, as examples, the energy and momentum conservation laws, Maxwell laws, Mach-Einstein law on the equivalence between matter, energy and space curvature, the constancy of electromagnetic waves’ speed in the absolute vacuum, the wavy nature of the matter, the quantized nature of waves and finally, the indissociability of the expansion process,
temperature, pressure, interaction forces and energetic equilibriums’ notions. A certain number of these principles are reconfirmed, a posteriori, inside the framework of our model. However, our model excludes certain principles which have been introduced and used with the only justification (half-intellectual and half-experimental): getting rid of some apparent weaknesses in classical Physics. In fact, one can find at the base of our model the refutation (perfectly justified) of the special relativity second postulate which consists of supposing that the light speed does not depend on the inertial referential that is being used in order to measure it, without putting into question the light speed independence of the source movement. Another fundamental aspect in our model is to situate Schrödinger’s equations and the quantum Statistics into their proper context and within their fair limits. They actually consist on a sort of important approximate and predictive approach towards the studied phenomena and the explanation of the obtained experimental results; to this we associate the reinspection of the uncertainty principle. Evidently the reinspection of these postulates is based on logical reasoning and rigorous mathematics, offering at the same time a coherent alternative, in order to explain the phenomena whose apparent contradiction with better established physical principles was at the base of their adoption.

Concerning special relativity, we show that none of the experiments, (real or imaginary) which led to the spacetime relativistic notion, justifies the alteration of the natural (Galileo-Newtonian) space and time relationship. All these experiments admit coherent and simple explications. This is the case, for example, of the train, the two observers, the emitter and the mirror experiment, the experiment of Michelson-Morley or the one of the emitter in the middle of a truck with the two mirrors on both sides.... In addition, we show that the covariance law is totally respected by Maxwell’s equations by demonstrating that the wave equation is transformed, in a canonical way, for all inertial referential exchanges. This simply needs the use of a perfectly natural derivation notion that integrates the relative referential frame movement. In fact, a more general derivation notion serves to demonstrate the covariance properties (or tensoriality) for different types of moving frames and it coincides in habitual cases with the usual derivation notion. We also demonstrate that the arbitrary cleavage between the relativistic and non relativistic particles leads to evident contradictions.

Similarly, based on a hardly contestable mathematical logic, we show that
some experiments, like those of energetic particles arriving on a screen, after having passed into two thin slits slightly spaced, do not let us conclude that the fact of knowing which one of the two slits the particles passed in, is by itself sufficient for altering the real and objective physical result. These results can be altered uniquely by technical and circumstantial means used in order to arrive to this recognition. We demonstrate equally that the really noticed uncertainties anywhere in Nature are caused in fact by the dynamic, complex, and evolutive nature of natural phenomena (movement trajectories, interactions, energetic equilibriums...) and by the imperfection and limits of our technical means which are essentially circumstantial. These technical means are luckily more and more efficient and precise; which explains the permanent progress on the level of our understanding of the universe and the matter structure. Schrödinger’s equations give evidently a strong method to determine, for example, the probability of finding particles in a given space region and to explain phenomena that seemed to be unreachable, but this does not allow us to give quantum Statistics a theoretical or exact status.

Thus, we demonstrate by taking the simple pendulum example in a stable vertical equilibrium and the example of a ball at rest in a box (supposed to be at rest too) that the wavy nature of Matter does not allow us to talk neither about frequency nor about the wave length corresponding to oscillations in the space and thus we can not talk about their minimal energy that would not be null, in a flagrant contradiction with Newton’s principles. Consequently the fact of invoking the uncertainty principle has no place to be. In a similar way, the electron ground state energy in a hydrogen atom and its associated Bohr’s approximate radius are the results of an energy equilibrium between many forms of energy and many internal and external interaction forces and have nothing to do with the uncertainty principle. The total and the minimal potential energies have to be finite. The energy equilibrium is naturally traduced by the orbital clouds. We prove also that wavelength and frequency notions, when attributed to pointlike material particles into movement such as electrons inside atoms, lead to some contradictions.

Likewise, we consider that the use of quantum Mechanics and quantum Statistics methods is only justified for the analysis of infinitesimal subatomic cases where our capacity to carry out accurate (or even approximate) measurements with our presently technical means are so limited to make the analysis of these situations, within the framework of classical Mechanics, inefficient. The successful use of the quantum approach in order to obtain, in the macroscopic cases, nearly the same results as those obtained by Lagrange and Hamilton’s laws by the intermediate of Hamilton-Jacobi equations, must only
lead to a sort of a justification (or a legitimacy) for the use of these methods in the infinitesimal cases and does not allow us to conclude that Nature’s laws obey only the Quantum Mechanics rules. For our part, we think that the uncertainties that are inherent into these methods (and are in fact a legitimate consequences of them) reflect the approximate aspect of this approach and it is not excluded that, using other experimental or theoretical analysis means, we can better optimize these approximations and uncertainties.

We will give in the following some of the strong points of our model. Note that, to start, our model is based upon all the mathematical Physics laws and principles whose (theoretical and experimental) validity is unquestionable, together with submitting those who were partially and circumstantially admitted (in order to resolve some unexplainable problems) to an attentive examination. Those who have not resisted the mathematical logic have been abandoned, with all their consequences, after establishing the necessary justifications and the clearly more natural alternatives. After that, the model has been constructed on the base of some simple ideas which are far from being simplistic. We can resume them by the expansion theory (which is gaining ground since Hubble) and the use of a Riemannian metric, that is variable with time and position, reflecting Mach’s principle which was retaken by Einstein: matter = curvature; to which we add a scientific and philosophical principle that consists in the unity and coherence of many of Nature’s laws including: the conservation laws, the covariance laws, the equivalence laws, and the original conflicting unity of forces.

Thus, the universe at time $t \ (t>0)$ consists, according to our model, of a ball $B_{c}(O, R(t))$ of $\mathbb{R}^{3}$ (with $R(t) \sim t$ when $t \gg 0$) equipped with a Riemannian metric $g_{t}(X)$. This metric reflects at every instant, by the intermediate of its variable curvature, the energy distribution and all its effects. This metric contracts the distances and volumes around material agglomerations of high density level, and especially around the black holes, which are characterized by extremely high energy density level. On the other hand, this metric measures the distances with respect to our conventionally (Euclidean) scale in a place almost far from all matter influences (especially gravitational influence). All trajectories which describe free movements (i.e. under the action of natural forces only) in the universe would be (with respect to this metric) geodesic as the trajectories associated to free Newtonian movements (i.e. not submitted to exterior forces) which are straight lines covered in a constant velocity or, in other words, the geodesics relative to our flat Euclidean metric. The free fall from a reasonable distance from earth describes a geodesic $X(t)$
for a metric $g_t$ (i.e. $\nabla_{X'(t)}^g X'(t) = 0$) which can be determined easily in both cases; either we assume that the gravity is uniform or central. This notion helps in solving numerically the $n$ bodies’ problem for example.

However, instead of trying to determine the metric in question by the Einstein tensorial equation (which is reduced in our new context to six independent equations) we decided to follow another way. In fact, the dependence of this equation on a large number of factors, in addition to time, makes the resolution inextricable, even after all possible simplifications and reductions. Our way is progressive, beginning by a purely theoretical mathematical modeling of the virtual space expansion, followed by the progressive introduction of physical realities passing from the idealization to the regularization to the quasi-linearization, ending by integrating all the factors which make our real universe in an essentially simultaneous and non dissolable way.

In a first step, we can prove, using the (generalized) Newtonian fundamental principles of Mechanics, that the creation and the expansion of space in which lives the universe should (starting from a certain time) be produced at a quasi-constant speed which tends to $c$ (supposed to be 1). We introduce then for every $t$ the distribution of matter mass $m_t(X)$ and that of generalized potential energy $E_t(X)$ on the ball $B_e(O,R(t))$ to which we successively associate the measure $\mu_t := m_t(X)dX$ and $\nu_t := E_t(X)dX$, and we consider the measure $\mu_t$ associated to the physical metric $g_t$ determined by the distribution $E_t(X)$ by the following :

$$\mu_t = dv_{g_t} = v_t(X)dX = dX - \nu_t(X) = dX - E_t(X)dX$$

The measure $\nu_t$ measures the failure caused by the energy in order for the volume to be Euclidian and $\mu_t$ measures the real physical volume on the universe at time $t$, taking into consideration all energy manifestations. We consider then the semi-cone of time and space :

$$C' = \{(x,y,z,t) \in \mathbb{R}^4; x^2 + y^2 + z^2 \leq R^2(t); t \geq 0\} = \bigcup_{t \geq 0} B_e(O,R(t)) \times \{t\}$$

which we consider during our construction as being

$$C = \{(x,y,z,t) \in \mathbb{R}^4; x^2 + y^2 + z^2 \leq t^2; t \geq 0\} = \bigcup_{t \geq 0} B_e(O,t) \times \{t\}$$

in order to make it easier and simpler. This, in fact, means that we assume that the electromagnetic waves’ speed were always equal to the speed of light.
in the absolute vacuum (i.e. $c = 1$) and that the expansion speed were always the same as the empty geometric space one, which is determined according to our model, by the electromagnetic propagation. The general case will be discussed at the end of this paper.

The universe at time $t_0$ will be the intersection of this semi-cone with the plane of equation $t = t_0$ of $\mathbb{R}^4$ equipped with the Riemannian metric $g_{t_0}$. Then we apply the Stokes theorem on the semi-cone provided with the flat metric of Minkowsky (considering the empty virtual space in which the geometric space evolves with the time progress) on one side and on the same semi-cone equipped with the metric $h_t = dt^2 - g_t$ on the other side, in order to demonstrate that the generalized energy (covering the matter) $E$ verifies the canonical wave equation:

$$\Box E(t, X) = \frac{\partial^2}{\partial t^2} E(t, X) - \Delta E(t, X) = 0 \quad \text{for } X \in B_e(O, t)$$

with

$$E(t, X)|_{S_e(O, t)} = 0 \quad \text{for every } t,$$

whose solutions are pseudo-periodic functions admitting pseudo-frequencies decreasing with time. According to Planck-Einstein principle, we can write (along the propagation line):

$$E_\mu(t, X) = g_\mu(t) \psi\left(\frac{X}{t}\right) = h_\mu(t) f_\mu(t)$$

where $\psi$ and $\mu$ are respectively the eigenfunctions and the eigenvalues associated with the Dirichlet problem on the unit ball $B_e(O, 1)$, $f_\mu(t)$ is the frequency of the solution and $h_\mu(t)$ is a sort of a Planck’s constant.

Introducing the temperature factor, which is (with the pressure) non dissociable from the universe expansion, we can prove that for every free movement (geodesic for $g_t$) $X(t)$, the energy $E_\mu(t, X(t))$ is a decreasing function with respect to time (via the decreasing of cosmic temperature) and depends also on $\mu$ in a purely conventional manner.

Finally, we recover, in the framework of our model, the famous equation $E = mc^2 (= m)$ and we prove that, for every material particle of initial mass (at rest) $m_0 = m(0)$ circulating at a speed $v < 1$, the total energy $E(t)$ is equal to $m(t) \left(1 + \frac{v^2(t)}{2}\right)$ where $m(t) = \gamma(t)m_0$ is the mass of the particle at the instant $t$. The $\gamma(t)$ factor comes from the loss in mass energy by the intermediate of radiations depending on velocity fluctuation and temperature. This factor can be calculated theoretically or experimentally in many ways. We show that it decreases from 1 to 0 when the velocity goes from
Then we prove that our formulation concerning the energy and the momentum of particles are more coherent than the relativistic formulations. We compare, for instance, our expression for mass energy $E(t) = \gamma(t) m_0 c^2$ with the classical one $E = \gamma mc^2$ where $\gamma$ is the Lorentz factor.

We can continue in this way and reexamine all the modern Physics formulas and results (such as their validity is approved experimentally) for which one has used either the relativistic notions or the quantum statistics methods or also the uncertainty principle, in order to give them an interpretation that is more solid and (why-not) more precise, from the moment they do not give other than approximate results established from experiments. This naturally requires a collective hard and assiduous work. However, this reexamination needs the readjustment of some notions and the reestablishment of the time dependence for some notions and constants. We show for example that, the redshift phenomenon is explained by the increasing, with time and distance, of the wavelength and not by the velocity of the wave source.

Moreover, it is clear that within the framework of our model we can recover, even more precisely, all confirmed results in modern Cosmology that are based on Hubble and Friedmann works and on the Einstein-de Sitter model. Our model conforms with the second statement of the cosmological principle concerning the relative speed of galaxies but not with the first postulate: the universe can not actually look rigorously the same for any observer on any galaxy.

We continue our study by establishing a mathematical model that leads to a global classification of all (material and antimaterial) fundamental particles. This classification is achieved by using a wave equation where the Laplacian is replaced by the Dirac operator $D$ defined by the spinorial structure associated with the Riemannian space $(B(O,1), g_e)$ and we conclude that there exist only two privileged fundamental forces that are both essentially related to the original unity of the matter-energy, the original expansion movement and the natural unity of the universe, on one hand, and to the fundamental antagonistic aspects of the natural forces which essentially are related to the matter, i.e. attraction and repulsion, on the other hand. They are the gravitational and the electromagnetic forces.

In other respects we demonstrate that, within the framework of our modeling, the solution to the most general Einstein’s equation can be obtained by means of a well posed constraint free initial value formulation leading to
a well defined maximal Cauchy development. In the same manner, we prove
that the resolution of Einstein’s equation is equivalent to the resolution of a
constraint free Hamiltonian system that reduces to six equations of six inde-
pendent unknown functions \( g_{ij} \) and \( \pi^{ij} \) corresponding to an initial metric \( g_{t0} \)
defined on any Cauchy surface \( \Sigma_{t0} \) (or equivalently, on the universe \( B(O,t_0) \))
at an initial time \( t_0 \) such that its derivative components with respect to time
\( \dot{g}_{ij}(t_0) \) identify to twice the extrinsic curvature components \( K_{ij}(t_0) \) of \( (\Sigma_{t0},
g_{t0}) \) within the space - time manifold \( M = C(t) \) provided with its space -
time lorentzian metric, \( h_t = dt^2 - g_t \).

Finally, we notice that our model is totally consistent in a way that it
is compatible with classical Physics in the Newtonian and quasi-Newtonian
situations where the metric \( g_t \) becomes so close to \( g_e \) and the measure \( \mu_t \)
becomes so close to the Lebesgue measure as soon as the \( E(t,X) \) distribution
becomes approximately null in a given region of space. The metric \( g_t(X) \)
integrates and explains all the real and approximate situations as well as the
singular situations (black holes) and shows that physical reality is almost
continuous without being differentiable (except for the original singularity).

We end this study by readjusting the Einstein’s general relativity theory
to our model. For doing that, we use both the macroscopic model of the
homogeneous isotropic cosmology (considering the universe as a dust of ga-
laxies), which leads to the Friedmann - Einstein equations, together with
the presently reliable experimental values of some fundamental constants in
order to correctly determine the age, the size and the total energy of our uni-
verse. We then use some results originated in the quantum Statistics to show
that only the energy \( E \), the electromagnetic constant \( ke^2 \), the Boltzmann
characteristic \( \tilde{K}_B T \) and the speed of light \( c \) are universal time - independent
constants; the other fundamental constants (the gravitational constant \( G \),
the Planck constant \( \hbar \), the electromagnetic force factor \( \alpha \) and the curva-
ture parameter \( \tilde{K}(t) \)) are indeed time - dependent. This fact gives, by the
way, a new viewpoint on the quantization process showing its limits and
its relative character. Finally, we establish some relations involving all these
"constants" showing in this way the unity of all Physics theories : Electro-
magnetism, general relativity, quantum Physics, Thermodynamics and the
Newton - Lagrange - Hamilton’s Mechanics. These relationships lead also to
the unification of the fundamental forces. All of our results conform with
the well confirmed classical and modern Physics’ results. Nevertheless, we
note some deviations with respect to other approximate results that have
been expected in a general (and sometime hypothetical) way without being
rigourously established and which are far from making the unanimity of the
scientific community. Otherwise, our model clearly confirms that the universe
laws and the expansion process are well governed by the (slightly reviewed)
Einstein’s general relativity theory.

At the end, we think that, for understanding furthermore our universe,
we should combine the theory with practices, Mathematics with Physics and
adding some imagination, philosophy and confidence.

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## 1 Moving frames and Isometries

We start this paper by noticing that the three first sections are only de-
voted to establish some tensoriality properties concerning the moving frame
exchanges and to construct a (purely theoretical) mathematical model cor-
responding to the space creation. This geometrical space is filled up simulta-
neously by the physical real universe whose modeling will be achieved pro-
gressively through the remaining sections. The definitive model that is charac-
terized by the real physical metric $g_t(X)$ will be achieved in the seven$^{th}$ sec-
tion when we introduce the factor that consists on the temperature-pressure
We assume that there exists on $\mathbb{R}^3$ a family of Riemannian metrics $g_t$ continuously differentiable with respect to $t \in ]0, +\infty[$ (this will be the case of all mathematical objects indexed by $t$ in what follows) and that, for fixed $t_0 \geq 0$, there exists a continuous family of isometries $\varphi_{(t_0,t)} =: \varphi_t$ from $(\mathbb{R}^3, g_{t_0})$ onto $(\mathbb{R}^3, g_t)$. We suppose that $\mathbb{R}^3$ is provided with a referential frame $\mathcal{R}_0(t_0)$ and we consider a moving frame $\mathcal{R}(t)$ which coincides at $t = t_0$ with a frame $\mathcal{R}(t_0)$ having the same origin as $\mathcal{R}_0(t_0)$ and makes, in the same time, a family of linear transformations $A_t$ with respect to $\mathcal{R}_0(t_0)$. Let $a_0(t)$ denote the curve described by the origin of $\mathcal{R}(t)$ relatively to $\mathcal{R}_0(t_0)$. A model of this situation will be given together with many consequences in the second section of this paper. Finally, we consider a moving particle that coincides at $t = t_0$ with the origin of $\mathcal{R}_0(t_0)$ and we suppose that its trajectory is determined in $\mathcal{R}_0(t_0)$ by $x_0(t)$ for $t \geq t_0$.

For $t_1 > t_0$ and $t_0 \leq t \leq t_1$, let
\[
y_0(t) = \varphi_{t_1}(x_0(t)), \quad b_0(t) = \varphi_{t_1}(a_0(t)), \quad u_0(t) = \varphi_{t_1}(x_0(t)) \quad (\mathcal{R}_0)
\]
\[
o_0(t) = \varphi_{t_1}(a_0(t)).
\]

Next we denote by $x_1(t)$, $y_1(t)$, $\alpha_1(t)$ and $u_1(t)$ the new coordinates of the curves $x_0(t)$, $y_0(t)$, $\alpha_0(t)$ and $u_0(t)$ with respect to the frame $\mathcal{R}(t_1)$ whose origin is $b_0(t_1) = \alpha_0(t_1) = \varphi_{t_1}(a_0(t_1))$ (fig.1).

Then, for $t_0 \leq t \leq t_1$, we have:
\[
x_0(t) - b_0(t_1) = A_{t_1}.x_1(t),
\]
\[
y_0(t) - b_0(t_1) = A_{t_1}.y_1(t), \quad (\mathcal{R}_1)
\]
which yields
\[
y_1(t) - x_1(t) = A_{t_1}^{-1}(y_0(t) - x_0(t))
\]
and
\[
y_1(t_1) - x_1(t_1) = A_{t_1}^{-1}(y_0(t_1) - x_0(t_1)) = A_{t_1}^{-1}.u_0(t_1) - A_{t_1}^{-1}.x_0(t_1)
\]
and then
\[
u_1(t_1) = x_1(t_1) + A_{t_1}^{-1}.u_0(t_1) - A_{t_1}^{-1}.x_0(t_1)
\]
\[= x_1(t_1) + A_{t_1}^{-1}.u_0(t_1) - A_{t_1}^{-1}.b_0(t_1) - x_1(t_1)\]
\[= A_{t_1}^{-1}(u_0(t_1) - b_0(t_1)),\]

since (using \((\mathcal{R}_0)\) and \((\mathcal{R}_1)\)) we have: \(u_0(t_1) = y_0(t_1), \quad u_1(t_1) = y_1(t_1)\)
and \(x_0(t_1) = b_0(t_1) + A_{t_1}.x_1(t_1)\).

This equality can be written as
\[u_1(t_1) - \alpha_1(t_1) = A_{t_1}^{-1}(u_0(t_1) - \alpha_0(t_1))\]
since \(\alpha_1(t_1) = 0\).

Therefore we obtain, for \(t \geq t_0\), the following formula relating the coordinates in \(\mathcal{R}_0(t_0)\) to those in \(\mathcal{R}(t)\):
\[u(t) - \alpha(t) = A_t^{-1}(u_0(t) - \alpha_0(t)). \quad (1)\]

Here, \(u_0(t)\) and \(\alpha_0(t)\) respectively specify the trajectories (in \(\mathcal{R}_0(t_0)\)) of the particle and the origin of the moving frame \(\mathcal{R}(t)\) into the space \(\mathbb{R}^3\) provided (at any time \(t \geq t_0\)) with the variable metric \(g_t\), i.e. into \((\mathbb{R}^3, g_t)_{t \geq t_0}\), whereas \(u(t)\) and \(\alpha(t) = 0\) are the coordinate vectors of the trajectories \(u_0(t)\) and \(\alpha_0(t)\) in the moving frame \(\mathcal{R}(t)\). From equation (1), we deduce:
\[u'(t) - \alpha'(t) = A_t^{-1}(u'_0(t) - \alpha'_0(t)) + (A_t^{-1})'(u_0(t) - \alpha_0(t)) \quad (2)\]
or (using (1) again)
\[u'(t) = A_t^{-1}(u'_0(t) - \alpha'_0(t)) + (A_t^{-1})' \circ A_t(u(t)) \quad (2').\]

In particular, if the motion of the frame \(\mathcal{R}(t)\) is uniform with respect to \(\mathcal{R}_0(t_0)\) (i.e. \(\alpha'_0(t) = \overrightarrow{V}_0\) and \(A_t = A_{t_0} =: A\) for \(t \geq t_0\)), we get
\[u'(t) = A^{-1}(u'_0(t) - \overrightarrow{V}_0)\]
and if, in addition, we have \(\varphi_t = Id_{\mathbb{R}^3}\), we obtain
\[x'(t) = A^{-1}(x'_0(t) - \overrightarrow{V}_0)\]
and
\[x''(t) = A^{-1}.x''_0(t)\]
i.e.
\[\Gamma(t) = A^{-1}.\Gamma_0(t),\]

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where we have denoted by \( x(t) \) the coordinates of \( x_0(t) \) in the moving frame \( R(t) \).

On the other hand, if \( a_0(t) = 0 = \alpha_0(t) \) for \( t \geq t_0 \), then (using (2') and (1))

\[
u'(t) = A_t^{-1}.u'_0(t) + (A_t^{-1}').u_0(t), \quad (2'')
\]
and if, in addition, we take \( A_t \equiv A \), then we have \( R(t) = AR_0(t_0) \) for \( t \geq t_0 \)
and

\[
u'(t) = A^{-1}.u'_0(t),
\]
and finally for, \( \varphi_t = Id_{\mathbb{R}^3} \), we get

\[
x'(t) = A^{-1}.x'_0(t)
\]
as well as

\[
\Gamma(t) = A^{-1}.\Gamma_0(t).
\]

A new time derivation operator

In the general case, let

\[
v_0(t) = u_0(t) - \alpha_0(t),
\]

\[
v(t) = u(t) - \alpha(t) = u(t).
\]

So we have

\[
\frac{d}{dt}v_0(t) = \frac{d}{dt}u_0(t) - \frac{d}{dt}\alpha_0(t).
\]

Now we put

\[
\frac{d_1}{dt}v(t) := \frac{d}{dt}v(t) - (A_t^{-1})'(u_0(t) - \alpha_0(t)) \quad (d_1)
\]
(or equivalently, using (1)),

\[
\frac{d_1}{dt}v(t) = \frac{d}{dt}v(t) - (A_t^{-1})' \circ A_t.v(t). \quad (d'_1)
\]

Then, equation (2) shows that :

\[
\frac{d_1}{dt}v(t) = A_t^{-1}.\frac{d}{dt}v_0(t) = A_t^{-1}.\frac{d_1}{dt}v_0(t), \quad (3)
\]
since \( \frac{d}{dt} \) coincides with \( \frac{d}{dt} \) when using the coordinate vectors with respect to \( R_0(t_0) \).
This formula points out the tensoriality of the coordinate exchange of the
speed vector defined by this derivation that takes into account the speed of the moving frame origin and its rotation. For the acceleration vector defined when using the variable metric $g_t$ and this derivation, we have

$$\nabla g_t \frac{d}{dt} v(t) = \nabla g_t A_t^{-1} \frac{d}{dt} v_0(t)$$

$$= \nabla g_t A_t^{-1} \frac{d}{dt} v_0(t) = \nabla g_t A_{r(t)}^{-1} v'(t).$$

If $A_t = A$ for $t \geq t_0$, we get (using $(d_1)$)

$$\frac{d}{dt} v(t) = \frac{d}{dt} v(t)$$

and

$$\nabla g_t \frac{d}{dt} v(t) = \nabla g_t \frac{d}{dt} v(t) = \nabla g_t A_t^{-1} \frac{d}{dt} v_0(t) = \nabla g_t A_{r(t)}^{-1} v'(t).$$

If, in addition, we have $\alpha'(t) = 0$, we obtain

$$\frac{d}{dt} u(t) = A^{-1} \frac{d}{dt} u_0(t) = A^{-1} \frac{d}{dt} u_0(t)$$

and

$$\nabla g_t \frac{d}{dt} u(t) = \nabla g_t \frac{d}{dt} u(t) = \nabla g_t A^{-1} \frac{d}{dt} u_0(t)$$

or

$$\nabla g_t \frac{d}{dt} u(t) = \nabla g_t A_{r(t)}^{-1} u'(t).$$

If now we suppose that the metrics $g_t$ are flat, then the identities (4) and (5) yield successively:

$$v''(t) = \nabla g_t v'(t) = \nabla g_t (A^{-1} v_0(t))' = A^{-1} v_0'(t)$$

and

$$u''(t) = A^{-1} u''_0(t).$$

Finally, for $\varphi_t = Id_{\mathbb{R}^3}$, we get

$$\dot{x}(t) = \nabla g_t \frac{d}{dt} x(t) = \nabla g_t A_{r(t)}^{-1} x'(t) = A^{-1} x_0'(t),$$

i.e.

$$\Gamma(t) = A^{-1} \Gamma_0(t).$$
We notice that the equality (6) (resp. (7)) holds when we assume only that \( A_t \equiv A, g_t \) is flat for any \( t \) and \( \alpha_0(t) \) (resp. \( a_0(t) \)) is a geodesic. Therefore, if we assume in addition that \( u_0(t) \) (resp. \( x_0(t) \)) is a geodesic, then \( u(t) \) (resp. \( x(t) \)) itself is a geodesic.

More generally, let us consider two moving frames \( R_1(t) \) and \( R_2(t) \) coming from \( R_0(t_0) \) in the same manner as \( R(t) \). Using the following obvious notations

\[
v_0(t) = u_0(t) - \alpha_0(t),
\]

\[
w_0(t) = u_0(t) - \beta_0(t),
\]

\[
u_1(t) = v_1(t) \quad \text{(coord. of } v_0(t) \text{ in } R_1(t)),
\]

\[
w_1(t) = w_1(t) \quad \text{(coord. of } w_0(t) \text{ in } R_2(t)),
\]

\[
\frac{d_1}{dt} u_1(t) = \frac{d}{dt} u_1(t) - (A^{-1}_t)' \circ A_t . u_1(t),
\]

\[
\frac{d_2}{dt} u_2(t) = \frac{d}{dt} u_2(t) - (B^{-1}_t)' \circ B_t . u_2(t),
\]

we obtain

\[\frac{d_1}{dt} u_1(t) = A^{-1}_t . \frac{d}{dt} v_0(t)\]

and (using (3) which implies \( \frac{d}{dt} v_0(t) = A_t \frac{d}{dt} u_1(t) \))

\[
\frac{d_2}{dt} u_2(t) = B^{-1}_t \cdot \frac{d}{dt} w_0(t) = B^{-1}_t \cdot \frac{d}{dt} (u_0(t) - \beta_0(t)) = B^{-1}_t \cdot \frac{d}{dt} (v_0(t) + \alpha_0(t) - \beta_0(t))
\]

\[= B^{-1}_t \cdot \frac{d}{dt} v_0(t) + B^{-1}_t \cdot \frac{d}{dt} (w_0(t) - v_0(t)).\]

\[= B^{-1}_t \circ A_t . \frac{d_1}{dt} u_1(t) + B^{-1}_t . \frac{d}{dt} (w_0(t) - v_0(t)).\]

So, we have

\[\frac{d_2}{dt} u_2(t) = B^{-1}_t \circ A_t . \frac{d_1}{dt} u_1(t) + B^{-1}_t . \frac{d}{dt} (\alpha_0(t) - \beta_0(t)) \quad (8)\]

For \( \alpha'_0(t) = \beta'_0(t) \), we get

\[
\frac{d_2}{dt} u_2(t) = B^{-1}_t \circ A_t . \frac{d_1}{dt} u_1(t)
\]
\[ \nabla_{\frac{d^2}{dt^2}u_2(t)} u_2(t) = \nabla_{\frac{d}{dt}u_2(t)} B_{t^{-1} \circ A_t} \frac{d}{dt} u_1(t). \]

If furthermore we assume that \( A_t \equiv A \) and \( B_t \equiv B \), we obtain
\[ \frac{d}{dt} u_2(t) = \frac{d}{dt} u_1(t) \]
i.e.
\[ u_2'(t) = B^{-1} \circ A \cdot u_1'(t) \]
and
\[ \nabla_{\frac{d}{dt}u_2(t)} u_2'(t) = \nabla_{\frac{d}{dt}u_1(t)} (B^{-1} \circ A \cdot u_1(t))'. \]

If, in addition, \( g_t \) is flat, we get
\[ u_2''(t) = B^{-1} \circ A \cdot u_2''(t). \]
Finally, if we assume that \( A_t \equiv A, B_t \equiv B, \varphi_t \equiv Id_{\mathbb{R}^3} \) and \( a_0'(t) = b_0'(t) \), then we clearly obtain
\[ x_2'(t) = B^{-1} \circ A \cdot x_1'(t) \]
and
\[ x_2''(t) = B^{-1} \circ A \cdot x_1''(t) \]
i.e.
\[ \Gamma_2(t) = B^{-1} \circ A \cdot \Gamma_1(t), \]
where here, we have denoted by \( x_1(t) \) and \( x_2(t) \) respectively the coordinates of \( x_0(t) \) in the two moving frames \( \mathcal{R}_1(t) \) and \( \mathcal{R}_2(t) \). Notice that, in order to obtain the equality \( u_2''(t) = B^{-1} \circ A \cdot u_1''(t) \) (resp. \( x_2''(t) = B^{-1} \circ A \cdot x_1''(t) \)), it is sufficient that \( \alpha_0(t) \) and \( \beta_0(t) \) (resp. \( a_0(t) \) and \( b_0(t) \)) be geodesics for the flat metric. If furthermore \( u_1(t) \) (resp. \( x_1(t) \)) is a geodesic, then \( u_2(t) \) (resp. \( x_2(t) \)) itself is a geodesic.

The above relations show the tensoriality of the acceleration vector when exchanging two frames having the same speed vector or having both constant speed vectors that can be obtained from each other by means of a fixed linear transformation.

If now we assume that the transformations \( A_t \) and \( B_t \) are isometries of \((\mathbb{R}^3, g_t)\) and if \( \alpha_0'(t) = \beta_0'(t) \) for any \( t \), we obtain (using (8))
\[ \| \frac{d^2}{dt^2} u_2(t) \|_{g_t} = \| \frac{d}{dt} u_1(t) \|_{g_t}, \]

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and if furthermore we have $A_t \equiv A$ and $B_t \equiv B$, then we have necessarily

$$g_t = g_{t_0}$$

and

$$\|u_2'(t)\|_{g_{t_0}} = \|u_1'(t)\|_{g_{t_0}}$$

$$\|\tilde{\Gamma}_2(t)\|_{g_{t_0}} = \|\tilde{\Gamma}_1(t)\|_{g_{t_0}}$$

where

$$\tilde{\Gamma}_2(t) := u_2''(t)$$

and

$$\tilde{\Gamma}_1(t) := u_1''(t).$$

Finally, if in addition we take $\varphi_t = Id_{\mathbb{R}^3}$, then naturally we get

$$\|x_2'(t)\|_{g_{t_0}} = \|x_1'(t)\|_{g_{t_0}} = \|x_0'(t)\|_{g_{t_0}}$$

and

$$\|x_2''(t)\|_{g_{t_0}} = \|x_1''(t)\|_{g_{t_0}} = \|x_0''(t)\|_{g_{t_0}}$$

i.e.

$$\|\Gamma_2(t)\|_{g_{t_0}} = \|\Gamma_1(t)\|_{g_{t_0}} = \|\Gamma_0(t)\|_{g_{t_0}}.$$

2 Mathematical modeling of the expanding universe

We consider a function $\lambda \in C^0([0, \infty])$ that satisfies:

1. $\lambda \in C^2([0, \infty])$,
2. $\lambda(t) \neq 0$, for $t \in [0, \infty]$.

Next, we consider, for $t \geq 0$, the metric $g_t$ defined on $\mathbb{R}^3$ by:

$$g_t := \frac{1}{\lambda^2(t)} g_e,$$

and the function from $\mathbb{R}^3$ onto $\mathbb{R}^3$ defined by:

$$\varphi_t = \lambda(t) Id_{\mathbb{R}^3}.$$

So we have

$$t \varphi_t \circ g_t \circ \varphi_t = g_e.$$
and then $\varphi_t: (\mathbb{R}^3, g_e) \longrightarrow (\mathbb{R}^3, g_t)$ is an isometry for any $t \geq 0$.

Finally, we consider, in $(\mathbb{R}^3, g_e)$ provided with an orthonormal Euclidean frame $(O, \vec{i}, \vec{j}, \vec{k})$, the geodesic $exp_O(t\vec{V}_0)$, for $\vec{V}_0 \in \mathbb{R}^3$, which can be considered as the Ox coordinate axis parametrized by $t \longrightarrow x(t) = v_0 t$, where $v_0 = \|\vec{V}_0\|_{g_e}$. Let, for $t \geq 0$, $U(t) = \varphi_t(t\vec{V}_0)$, incorrectly written as $u(t) = \varphi_t(tv_0)$.

Assuming that $v_0 = 1$, we can consider

$$u(t) = \varphi_t(t) = t\lambda(t)$$

as a trajectory in $(\mathbb{R}^3_t, g_t)_{t \geq 0}$ of a particle lying, at $t = 0$, on the origin $O$.

Under these conditions, we have, for $t > 0$ :

$$u'(t) = t\lambda'(t) + \lambda(t)$$

and

$$\tilde{\Gamma}(t) := \nabla^{g_t}_{u'(t)} u'(t) = \frac{d}{dt} u'(t) = u''(t) = t\lambda''(t) + 2\lambda'(t).$$

Let $m$ be the mass of a fundamental material particle supposed to be, for a while, independent of time (although the volume depends obviously on $g_t$ and then on time).

**Remark** : We must however notice that a particle having a non vanishing mass $m$ (even for $m \ll 1$) can only move at a Euclidean speed $v < 1$ even though $v$ can be as close to 1 as we wish.

Let now $F(t) = m\tilde{\Gamma}(t)$ and $E(t)$ be a primitive of the function $g_t(F(t), u'(t))$ i.e.

$$\frac{dE}{dt} = E'(t) = g_t(m\tilde{\Gamma}(t), u'(t)).$$

We then consider the differential equation

$$g_t(m\tilde{\Gamma}(t), u'(t)) = g_e(m\Gamma(t), x'(t)) \equiv 0,$$

where $\Gamma(t) = x''(t) = 0$. This equation yields

$$\frac{1}{\lambda^2(t)} mu''(t)u'(t) \equiv 0$$

or

$$u''(t)u'(t) \equiv 0$$
(t\lambda'' + 2\lambda')(t\lambda' + \lambda) = 0 \quad (E)

Any solution \lambda of (E) satisfying the pre-requested conditions gives, in a way, a generalization to the space \( \mathbb{R}^3 \), provided with the variable metric \( g_t \), of the fundamental laws of classical Mechanics.

Let us prove that solutions of (E) actually are the set of all non zero constants. Indeed, the identity \( u''(t)u'(t) = 0 \) gives \( \frac{d}{dt}u'^2(t) = 0 \) which implies \( u'^2(t) = C^2 \) and \( |u'(t)| = C \). Now \( C \) can not be zero as

\[ u' = 0 \Rightarrow t\lambda' + \lambda = 0 \Rightarrow \frac{\lambda'}{\lambda} = -\frac{1}{t}, \text{ for } t > 0 \]

\[ \Rightarrow \ln \frac{\lambda}{C_1} = -\ln t = \ln \frac{1}{t} \Rightarrow \lambda = \frac{C_1}{t}, \]

which contradicts our hypothesis on the regularity at the origin for \( \lambda \). Equation (E) is therefore equivalent, on \([0, \infty[\), to

\[ t\lambda'' + 2\lambda' = 0 \quad (E'). \]

We take a local solution \( \lambda \) of (E') that is not identically equal to a non zero constant and we assume, for instance, that \( \lambda(1) = a > 0 \) and \( \lambda'(1) = b > 0 \). This solution is, in fact, analytical and defined on \([0, \infty[\) since, for \( t_0 > 0 \), (E') shows that \( \lambda'(t_0) = 0 \Leftrightarrow \lambda''(t_0) = 0 \) and then, if such a \( t_0 \) exists, the only solution of (E') on \([0, \infty[\) is \( \lambda \equiv \lambda(t_0) = \lambda(1) \) that obviously satisfies \( \lambda'(1) = 0 \), which is contradictory. This also proves that \( \lambda' \) and \( \lambda'' \) can not vanish at any \( t \in ]0, \infty[ \). Therefore (E') is equivalent on \([0, \infty[\) to

\[ t\lambda'' = -2\lambda' \Leftrightarrow \frac{\lambda''}{\lambda'} = \frac{-2}{t} \Leftrightarrow \int_1^t \frac{\lambda''}{\lambda'} ds = -\int_1^t \frac{2}{s} ds \]

\[ \Rightarrow \ln \lambda'(t) - \ln b = -2\ln t = \ln \frac{1}{t^2} \]

\[ \Rightarrow \lambda'(t) = \frac{b}{t^2} \Rightarrow \lambda(t) = \frac{-b}{t} + c \]

with \( c = a + b \).

Since \( b \) is \( \neq 0 \), this solution \( \lambda(t) \) can not extend, not only to a continuous function on \([0, \infty[\), but even to a distribution on \([0, \infty[ \). We obtain the same contradiction when we assume that \( \lambda'(1) = b < 0 \).
Therefore, the only families of \( g_t \) and \( \varphi_t \) satisfying the above conditions are defined by
\[
g_t = \frac{1}{\lambda^2} g_e \quad \text{and} \quad \varphi_t = \lambda \text{Id}_{\mathbb{R}^3}
\]
for \( t \geq 0 \) and a constant \( \lambda \neq 0 \).
Choosing \( \lambda = 1 \), we determine a Riemannian metric and a privileged scale so that \( g_t = g_e \) and \( \varphi_t = \text{Id}_{\mathbb{R}^3} \) for \( t \geq 0 \). Using this natural choice, we obtain \( u'(t) = 1 \) for \( t \geq 0 \) and then \( u' = H \) (Heaviside function).
Thus, it is shown that any Riemannian metric \( g \) on \( \mathbb{R}^3 \) in the conformal class of the Euclidean metric \( g_e \) that satisfies the fundamental laws of Mechanics (\( F = m\dot{\Gamma} \) and \( \frac{dE}{dt} = g(F, u') \)) is, up to a positive constant, the Euclidean metric itself. Choosing this constant equal to 1 amounts to fix a given scale for all physical objects. This metric will characterize all regions in the physical real universe that is assumed to be devoid of matter as well as of all its effects. Matter modifies the Euclidean distances and volumes and creates a physical (real) metric that contracts these latter. It will be constructed along the following sections.

**Consequences**

According to the preceding results, we can assimilate the spatial universe, at any time \( t_0 > 0 \) (when being theoretically considered as an empty geometrical space), to the Euclidean ball of radius \( R(t_0) = \int_0^{t_0} \| u'(r) \|_{g_e} dr = t_0 \), provided with the Euclidean metric \( g_e \):
\[
U(t_0) := (B(O, t_0), g_e).
\]
Then, we can take the Euclidean ball \( B(O, 1) \) equipped with the variable metric \( g_t := t^2 g_e := t^2 g_e \) as a mathematical model of the expanding universe (reduced to a virtual geometrical space) at the time \( t > 0 \):
\[
U_1(t) := (B(O, 1), t^2 g_e).
\]
Here, the function \( X \longrightarrow t_0X \) is an isometry from \( U_1(t_0) \) onto \( U(t_0) \), for any \( t_0 > 0 \).
We notice that, in order to study a motion taking place between time \( t_1 \) and time \( t_2 \), we can also use any of the following spatial models:
\[
(B(O, 1), t^2 g_e) \quad \quad t_1 \leq t \leq t_2,
\]
\[
\left( B(O, t_1), \frac{t^2}{t_1^2} g_e \right) \quad t_1 \leq t \leq t_2,
\]

\[
\left( B(O, t_2), \frac{t^2}{t_2^2} g_e \right) \quad t_1 \leq t \leq t_2.
\]

However, we notice that the real physical universe at the time \( t \) provided with its real curved metric is studied afterwards.

The properties \( u = t H, u' = H \) and \( \Gamma = u'' = \delta \) (Dirac measure on \( \mathbb{R}^+ \)) are generalized to \( \mathbb{R}^3 \), for \( X(t) = t \vec{V} \) where \( t \geq 0 \) and \( \vec{V} \in \mathbb{R}^3 \) with \( \| \vec{V} \|_{g_e} = 1 \), as

\[
t = |X(t)| := \| X(t) \|_{g_e} = d(O, X(t)),
\]

\[
 u' = 1 \text{ on the half-line } t \vec{V}, \ t \geq 0,
\]

\[
 u = Id_{t\vec{V}} \text{ and } u(t) = |X(t)| = t,
\]

\[
 \Gamma_{t\vec{V}} = \delta_{t\vec{V}} \text{(Dirac measure on the half-line } t \vec{V}).
\]

These properties can be stated as:

The time is the Euclidean distance.

The speed is the unit of time and distance.

The acceleration \( \Gamma_{t\vec{V}} \) is the potential of motion in the direction \( t \vec{V} \), concentrated at the origin of space and time.

\( E_0 = m_0 \) is the original potential mass energy (or the original inertial mass) that is perpetual and eternal.

\( F = m_0 \Gamma_{t\vec{V}} = m_0 \delta_{t\vec{V}} \) is the original energy (or mass) \( m_0 \) provided with the potential of motion in the \( \vec{V} \) direction.

Moreover, for an original hypothetical motion of a material particle of mass \( m \) described by \( X(t) = t \vec{V} \) with \( \| \vec{V} \|_{g_e} = V < 1 \), we have \( X'(t) = \vec{V} \) and \( X''(t) = 0 \) and so

\[
E(t) - E(0) = \int_0^t \frac{d}{ds}E(s)ds = \int_0^t g(mX''(s), X'(s))ds = 0.
\]

Therefore, we have

\[ E(t) = E(0) \text{ for any } t \]

and the particle energy for such a motion is unchanged.

Recall that, the above statements are a purely theoretical approach to the physical space expansion of the universe and can be used only as a macro-approximation of the Newtonian physical universe for large \( t \).
We also notice that all mechanical and physical quantities are essentially per-
ceptible and measurable when using moving frames with respect to a virtual
original one and with respect to each other. But, we can eliminate the first
kind of mobility in the following way:

If the motion of a particle, relatively to an original frame \((O, \vec{i}, \vec{j}, \vec{k})\), is de-
scribed by the vector \(\overrightarrow{OM} = X(t)\) and the motions of two other frames by
\(a(t)\) and \(b(t)\), then the motion of this particle is described in the first frame
\(R_1(t)\) by the vector \(\overrightarrow{O_1M} = Y(t) = X(t) - a(t)\) and in the second one \(R_2(t)\)
by the vector \(\overrightarrow{O_2M} = Z(t) = X(t) - b(t)\). So

\[
Y(t) = Z(t) + b(t) - a(t) = Z(t) + \overrightarrow{O_1O_2}.
\]

This can be used when we have to show that a mechanical or a physical law
does not depend on the choice of any one of these two frames; it is sufficient
to assume that one of them is at rest. So, we can use any Euclidean referen-
tial frame \((O_1, \vec{i}_1, \vec{j}_1, \vec{k}_1)\) where \(O_1\) is a fictive point that coincides at a given
time \(t_0\) (the present instant, for example) with a given point (on the earth,
for example).

Finally, let us consider a light emitter whose trajectory is described in \(\mathbb{R}^3\),
provided with a fixed frame \(R_0(t_0)\), by \(Y(t)\) whereas the receiver trajectory
is described by \(X(t)\) (fig.2). Then the light ray that reaches the receiver at
time \(t_1\) is, in fact, emitted by the emitter at a time \(t_0 < t_1\) from a point \(Y(t_0)\)
such as

\[
\|X(t_1) - Y(t_0)\|_{ge} = d = t_1 - t_0.
\]

**Space and time half-cone**

Let \(C\) be the half-cone defined by

\[
C = \{(x, y, z, t) \in \mathbb{R}^4; \quad x^2 + y^2 + z^2 \leq t^2, \quad t \geq 0\} = \bigcup_{t \geq 0} B(O, t) \times \{t\}.
\]

Let us say that, if \(B(O, t)\) is the geometrical space within which lives the real
physical universe at time \(t\), then \(C\) constitutes the geometrical virtual space
within which takes place the process of the expansion and the creation of the
physical real space intrinsically related to the progress of time.

For an event \(P_0 = (x_0, y_0, z_0, t_0) \in C\), \(U(t_0)\) is considered as the set of all
events \(P\) that are taking place simultaneously with \(P_0\), that is

\[
P = (x, y, z, t_0) \in U(t_0) \times \{t_0\} \subset C.
\]
The future of $P_0$, denoted by $\mathcal{F}(P_0)$, is the set of all events $Q \in U(t) \times \{t\}$ when $t \geq t_0$ that can be situated, at a moment $t \geq t_0$, on a trajectory of origin $P_0$. The set $\mathcal{F}(P_0)$ is (into our theoretical context) the upper half-cone with vertex $P_0$ having a span of $\frac{\pi}{2}$ (the light half-cone). The points $P'$ on the half-cone surface can be reached only by trajectories of speed 1 joining the point $P_0 = (x_0, y_0, z_0)_{t_0}$ of $B(O, t_0)$ to the point $P' = (x_1, y_1, z_1)_{t_1}$ of $B(O, t_1)$. A point $Q \in U(t_1) \times \{t_1\} \simeq B(O, t_1)$ within this half-cone can be joined to $P_0 \in B(O, t_0)$ only by trajectories $\gamma$ of Euclidean length (in $\mathbb{R}^3$), $l(\gamma)$, less than $t_1 - t_0$ (fig.3).

We denote by $\mathcal{P}(P_0)$ (the past of $P_0$) the set of events $Q \in U(t) \times \{t\}$, when $t \leq t_0$, that can be joined to $P_0$ by trajectories of origin $Q$. This set is the intersection of the lower half-cone of vertex $P_0$ with the half-cone of space and time $C$ (fig.3). Any trajectory $\gamma$ joining an event $Q \in U(t_2) \times \{t_2\} \simeq B(O, t_2)$ to $P_0 \in B(O, t_0)$, for $t_2 \leq t_0$, have to satisfy the inequality $l(\gamma) \leq t_0 - t_2$. Only the trajectories with speed 1 that pass through $P_0$ come from the half-cone surface of this intersection.

Let us now show that the experiment of the train, the mirror and the two observers can be naturally interpreted and does not justify the twisting of the natural pre-relativistic conception of the space and time. Indeed assume that we have a fixed virtual frame $(O_1, \vec{i}_1, \vec{j}_1, \vec{k}_1)$ and another frame $(O'_1, \vec{i}'_1, \vec{j}'_1, \vec{k}'_1)$ (the rest frame of the train) which is inertial with respect to the first one (i.e. moving uniformly in the direction $O_1x$ for example) and coinciding at $t = 0$ with it, just where the passenger was originally lying, as well as the emitter and the mirror above it (on the $O_1z$ axis) were lying. Then the ray, or more exactly the photon emitted at $t = 0$ that headed vertically (i.e. in the $O_1z$ direction) continued its way and finds itself at the instant $2t = 2h$ on a distance $2h$ on $O_1z$ ($h$ is the distance from the emitter to the mirror), because the mirror can be supposed as small and as distant as we wish. On the other hand, the photon emitted at the same instant ($t = 0$) that headed towards another direction in order to cross the mirror which, meanwhile, has moved horizontally (in the $O_1x$ direction) making the distance $d$, and then to reflect in order to recover the passenger, who has, meanwhile, moved horizontally to make a distance $2d$; this second photon has made the distance $2\sqrt{d^2 + h^2}$.

The passenger then recovers the second photon at the time $2t_1 = 2\sqrt{d^2 + h^2}$, when the first ray finds itself at the same distance $2t_1$ at the same time $2t_1$. If we have put initially a fictive fixed mirror (with respect to $(O_1, \vec{i}_1, \vec{j}_1, \vec{k}_1)$) a little higher, the first photon would have recovered its original position at the same time as the time at which the second one has recovered the passenger. Besides, if we suppose that the emission of light is strictly instantaneous (i.e. happening during a time less than any fraction of second) and unidirectional
(i.e. producing only one vertical beam) and if the mirror is sufficiently high and sufficiently small, then our passenger would never receive the reflected ray. The same situation occurs when we consider the fixed original frame and two other frames (on the earth for example) moving relatively to each other with a constant relative speed (the computation of distances becomes a little more complicated (fig.4)).

If now we suppose that both emitter and passenger are continuously located at the point $O_1'$, origin of the rest frame with respect to the passenger, and suppose that the mirror above them is macroscopic, then the photon that is vertically emitted at time $t = 0$ cross the mirror at a point that is not the same as the point which were originally located vertically above $O_1'$. Moreover the downward vertically reflected photon does not exactly reach the point $O_1'$ which meanwhile has moved horizontally. Thus, this photon has not been at any time located vertically above $O_1'$.

Moreover, concerning the passenger and its proper frame of origin $O_1'$, he can claim that the photon (or the signal) he has received (after reflection) at time $t$ has actually travelled the distance $2h$ only if the photon has been located, along its travel, vertically above him. Now, for a given speed $v$, the height $h$ is determined by the time $t_2$ of the crossing between the photon and the mirror and determines the direction of the emitted photon at $t = 0$ in order to satisfy the above property. Another photon that is emitted just after the first one in the same direction and does cross the mirror at time $t_2$ does not satisfy this property only if we modify adequately the height $h$. Similarly, another photon that is emitted at time $t = 0$ toward another direction needs also a modification of $h$ even though the cross point will not occur above the passenger. So, for $v$, $h$ and $t$ already given, the photon that would cross the mirror above the passenger has probably been emitted after the time $t = 0$ and would not probably be the same as that it would reach the passenger at time $t$. Can we then state that the light ray has travelled a distance $2h$ with respect to the proper passenger frame?

The introduction of the proper frame notion, which is moving with respect to another inertial frame, is canonically related to the flow of time notion and leads, in case of our experiment, to the following situation:

The relative passenger speed $v$ being arbitrary chosen as well as the mirror hight $h$ (supposed of being macroscopic), then the choice of the crossing time $t_2$ implies that a unique photon, having a well determined direction, can be all time vertically located above the passenger. All other photons of the light ray, having this same direction, as well as all photons of other rays of the beam, can not satisfy this property. Consequently all of these photons travel, with respect to the passenger proper frame, different distances. Therefore, this frame can not be canonically used for measuring the distances that are
travelled by photons and by light rays which are made up by an "infinite" succession of photons. Therefore, it can not be used for measuring neither the distance that is travelled by "light" nor the light propagation speed which is an intrinsic feature of light. The canonicity of the constancy of the light speed i.e. the speed of all photons of an arbitrary light ray will be established later on within a proper context i.e. with respect to a virtually fixed frame (using the derivative operator $\frac{\Delta x}{\Delta t}$) and to any other inertial frame (using the galilean transformation and the derivative operator $\frac{d}{dt}$).

**Remark on the relativistic spacetime notion**

It is well known that prerelativistic notions (adopted by Euclid, Descartes, Galileo and Newton between many others) confer to the three dimensional space and to the time, which is progressing continuously, an absolute character. The Euclidean distance $\Delta x$ between two punctual bodies at a given time $t_0$ and the time interval $\Delta t$ between two non simultaneous events have an intrinsic reality that is time and observer independent. We will show below that the introduction of the relativistic spacetime notion has no reason to take place.

For that, we consider two inertial observers $O_1$ and $O_2$ respectively located at the origin of two Euclidean referential frames $R_1 = (O_1, e_1, e_2, e_3)$ and $R_2 = (O_2, e_1, e_2, e_3)$ that coincide at a time $t_0$ and such as $O_2$ is moving on the $O_1x$ axis of $R_1$ with a constant relative speed $v$. The two observers would easily agree, at every time $t \geq 0$, on the measure of the Euclidean distance separating two arbitrary points $A_1$ and $A_2$ of the space $\mathbb{R}^3$. Indeed, we can assume (without loss of generality) that the frame $R_1$ is fixed, both points $A_1$ and $A_2$ are fixed on the $O_1x$ axis and that the frame $R_2$ is moving along the $O_1x$ axis with a constant velocity $v$ with respect to $R_1$. Thus, if $x_1$ and $x_2$ designate the abscissas of $A_1$ and $A_2$ in $R_1$, then the Euclidean length of the segment $A_1A_2$ is given by $x_2 - x_1$ when being measured by both observers using the two frames $R_1$ and $R_2$ at every time $t \geq 0$. Actually, at every time $t \geq 0$, the distance $A_1A_2$ measured by $R_2$ is always given by $(x_2 - tv) - (x_1 - tv) = x_2 - x_1$. So, a metallic bar, for instance, having $A_1$ and $A_2$ as extremities has a length $l = A_1A_2 = x_2 - x_1$ when being measured by the frame $R_1$ and by all inertial frames $R$ moving with any constant velocity $v$ with respect to $R_1$. Likewise, if the same bar is slipping on the $O_1x$ axis with a constant velocity $v_0$ with respect to $R_1$, then, at any time $t$, its length measured by $R_1$ is $x_2 + v_0t - (x_1 + v_0t) = x_2 - x_1$ and it is equal to
\[x_2 + v_0 t - vt - (x_1 + v_0 t - vt) = x_2 - x_1 = l \] when it is measured by any frame \( R_2 \) moving along the \( O_1x \) axis with an arbitrary constant velocity \( v \) with respect to \( R_1 \).

Moreover, when we consider the Galilean - Newtonian space-time \( \mathbb{R}^4 \) provided with the frame \( R'_1 = (O_1, e_1, e_2, e_3, e_4) \) where \( e_4 \) corresponds to the time axis \( O_1t \) that is orthogonal to the hyperplane which is determined by \( R_1 = (O_1, e_1, e_2, e_3) \), then the metal bar has always the same length, namely \( l = x_2 - x_1 \), when it is measured by both Euclidean frames \( R_1 \) and \( R'_1 \) at any arbitrary fixed time \( t_0 \). But, when we introduce the fourth dimension, i.e. the time, represented by the \( O_1t \) axis, the points \( A_1 \) and \( A_2 \) should be labelled in \( \mathbb{R}^4 \), at every time \( t \geq 0 \), by the frame \( R'_1 \) with the coordinates \((x_1, 0, 0, t)\) and \((x_2, 0, 0, t)\) which would be designated as \((x_1, t)\) and \((x_2, t)\) neglecting the two other dimensions. The length of the bar is then given in this frame by \( \sqrt{(x_2 - x_1)^2 + (t - t')^2} = x_2 - x_1 = \Delta x \). When the observer \( O_2 \) measures this length in the space-time at time \( t \), one should take into account the fact that \( O_2 \) would be then labelled by \( R'_1 \) with the coordinates \((vt, t)\) and that it would be itself located in the hyperplane of height \( t \) in \( \mathbb{R}^4 \), which is the same in which are located the points \( A_1 \) and \( A_2 \) at time \( t \), that is the points \((x_1, t)\) and \((x_2, t)\) in the frame \( R'_1 \). So, the observer \( O_2 \) obtains the same length \( \Delta x = l \) when using as well as its frame \( R_2 \) or its four-dimensional frame \( R'_2 = (O_2, e_1, e_2, e_3, e_4) \). The wrong issue that justified the introduction of the relativistic spacetime notion is the fact that the spatial interval \( \Delta x \) separating two non simultaneous events \( E_1 \) (having \((x_1, t_1)\) as coordinates in \( R'_1 \)) and \( E_2 \) (having \((x_2, t_2)\) as coordinates in \( R'_1 \)) at distinct times \( t_1 < t_2 \) depends on the inertial observers. Indeed, the event \( E_1 \) is labelled by \( O_2 \), using the frame \( R'_2 \) at time \( t_1 \), with \((x_1 - vt_1, t_1)\) and \( E_2 \) is labelled by \( O_2 \), using \( R'_2 \) at time \( t_2 \), with \((x_2 - vt_2, t_2)\) when we assume that \( O_2 \) is still lying on the \( O_1x \) axis. Let us denote, respectively, by \( A'_1 \), \( A'_2 \), \( A''_1 \) and \( A''_2 \) the points of coordinates \((x_1, t_1)\), \((x_2, t_1)\), \((x_1, t_2)\) and \((x_2, t_2)\) in the frame \( R'_1 \). The spatial interval from \( A'_1 \) to \( A'_2 \) measured by \( O_2 \), using the frame \( R'_2 \) at time \( t_1 \) is \( x_2 - vt_1 - (x_1 - vt_1) = x_2 - x_1 \) and the spatial interval from \( A''_1 \) to \( A''_2 \) measured by \( O_2 \), using the frame \( R'_2 \) at time \( t_2 \), is \( x_2 - vt_2 - (x_1 - vt_2) = x_2 - x_1 \). These results are the same as those obtained by this observer when using its frame \( R_2 \) when it is located in the space-time \( \mathbb{R}^4 \) respectively at the points \( O_2(t_1) = (O_2, t_1) \) and \( O_2(t_2) = (O_2, t_2) \) at times \( t_1 \) and \( t_2 \) (see fig.4).

Unlike the preceding results, the spatial interval \( \Delta_1 x \) between \( A'_1 \) and \( A''_2 \) measured by \( R'_1 \) is \( x_2 - x_1 \), and the spatial interval \( \Delta_2 x \) between \( A'_1 \) and \( A''_2 \) measured by \( R'_2 \) is \( x_2 - vt_2 - (x_1 - vt_2) \neq x_2 - x_1 \). Likewise, the Euclidean interval between \( A'_1 \) and \( A''_2 \) measured by \( R'_1 \) is

\[ I_1 = \sqrt{(x_2 - x_1)^2 + (t_2 - t_1)^2} \]

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and this same interval measured by \( R'_2 \) is

\[
I_2 = \sqrt{((x_2 - vt_2) - (x_1 - vt_1))^2 + (t_2 - t_1)^2}
\]

which implies \( I_2 \neq I_1 \).

Now, neither the \( \Delta x \) intervals nor the \( I_i \) intervals \((i = 1, 2)\) do correspond to a physical reality. Indeed, if we consider the bar \( A_1A_2 \) that lies at time \( t_1 \) at \( A'_1A'_2 \) in the space-time \( \mathbb{R}^4 \) and at \( A''_1A''_2 \) in the same space at time \( t_2 \), then its real spatial length measured by \( O_2 \) using the frame \( R'_2 \) is equal to \( x_2 - vt_1 - (x_1 - vt_1) = x_2 - x_1 \) at time \( t_1 \) and to \( x_2 - vt_2 - (x_1 - vt_2) = x_2 - x_1 \) at time \( t_2 \). Likewise, the two Euclidean intervals in the space-time \( \mathbb{R}^4 \) measured by \( O_2 \) using the frame \( R'_2 \) at times \( t_1 \) and \( t_2 \) also are equal :

\[
\sqrt{((x_2 - vt_1) - (x_1 - vt_1))^2 + (t_2 - t_1)^2} = \\
\sqrt{((x_2 - vt_2) - (x_1 - vt_2))^2 + (t_2 - t_1)^2} = x_2 - x_1
\]

The two intervals \( A'_1A''_1 \) and \( A''_1A'_1 \) measurements in the frames \( R'_1 \) and \( R'_2 \) do not have any physical meaning. They have nothing to do with the bar neither at time \( t_1 \) nor at time \( t_2 \). We can say the same thing concerning the area and the volume of any two dimensional or three dimensional body in the real space \( \mathbb{R}^3 \). Such a body is located entirely, at every time \( t \), in the hyperplane of hight \( t \) in the space-time \( \mathbb{R}^4 \). Lengths, areas and volumes are the same when measured by \( R_1 \) and \( R_2 \) in \( \mathbb{R}^3 \) or by \( R'_1 \) and \( R'_2 \) in \( \mathbb{R}^4 \) at any time \( t \).

In other respects, if the permanently expanding universe is represented by \( U(t_1) \) at time \( t_1 \) and by \( U(t_2) \) at time \( t_2 > t_1 \) and if we assume that it is provided at every time \( t \) with a metric \( g_t \) that evolves with time, we can not measure the distance of a point \( A_1 \) in \( U(t_1) \) to a point \( A_2 \) in \( U(t_2) \) because we can not adequately use neither \( g_{t_1} \) nor \( g_{t_2} \) in order to carry out this measurement. At any time \( t \), the universe is not static and the metric \( g_t \) is never the same at any distinct times \( t_1 \) and \( t_2 \). Consequently the problem of measuring the spatial distance \( \Delta x \) between two events \( E_1 \) and \( E_2 \) at different times, \( t_1 \) for \( E_1 \) and \( t_2 \) for \( E_2 \), or the Euclidean interval \( I \) in the space-time \( \mathbb{R}^4 \) between \( E_1 \) and \( E_2 \) should not be posed because it is physically meaningless.

We now suppose that the motion of a particle is described using two frames so that the relative speed of each of them with respect to the other is constant and that they can be obtained from each other by means of an orthogonal transformation \( A \). So, if \( \overrightarrow{OM} = X(t) \) and \( \overrightarrow{OO'} = a(t) \) in the first frame and \( \overrightarrow{OM'} = Y(t) \) in the second one, then we have \( \overrightarrow{O'M'} = \overrightarrow{O'O} + \overrightarrow{O'M} = \overrightarrow{OM} - \overrightarrow{OO'}, \) and so

\[
Y(t) = A^{-1}(X(t) - a(t))
\]
\[ Y'(t) = A^{-1}(X'(t) - a'(t)) \]
and
\[ Y''(t) = A^{-1}(X''(t) - a''(t)). \]

If, in addition we have \( a'(t) = 0 \), then
\[ Y'(t) = A^{-1}.X'(t) \]
and therefore
\[ \| Y'(t) \|_{g_e} = \| X'(t) \|_{g_e}. \]

If we have \( a''(t) = 0 \), then
\[ Y''(t) = A^{-1}.X''(t) \]
and therefore
\[ \| Y''(t) \|_{g_e} = \| X''(t) \|_{g_e}. \]

This yields, for instance,
\[
\begin{align*}
\Delta E_1(t) &:= \int_{t_0}^{t} X''(r).X'(r)dr = \frac{1}{2} \int_{t_0}^{t} \frac{d}{dr} \| X'(r) \|_{g_e}^2 dr \\
&= \frac{1}{2}(\| X'(t) \|_{g_e}^2 - \| X'(t_0) \|_{g_e}^2) \\
&= \Delta E_2(t) := \frac{1}{2} \int_{t_0}^{t} Y''(r).Y'(r)dr \\
&= \frac{1}{2}(\| Y'(t) \|_{g_e}^2 - \| Y'(t_0) \|_{g_e}^2).
\end{align*}
\]

### 3 General tensoriality with respect to the frame exchanges

We begin by noticing that, the hypothesis of the \( g_t \) family independence of the virtual position in \( \mathbb{R}^3 \) is perfectly justified when considering the procedure of modeling the theoretical universe expansion. Contrary to this situation, the study of a motion or a physical phenomenon taking place between two times \( t_1 \) and \( t_2 \) in the real dynamic space must take into account the existence of gravitational fields created by the matter distribution, the black holes, the energetic, electro-magnetic and quantum phenomena scattered in the universe at every time \( t \in [t_1, t_2] \). This explains, for instance, the deviation undergone by the light propagation with respect to classical
geodesics (straight lines).

So, we now assume that the real physical universe at time \( t \) is assimilated to

\[
U(t) = (B(O, t), g_t(X))
\]

where \( g_t(X) \) is a variable metric (depending on both position and time) which is determined entirely by all manifestations of the matter-energy that is filling the space at time \( t \).

In the following, we consider then the general case and some particular cases (real, approximate or virtual) concerning the families \( g_t, A_t \) and the curve \( a_0(t) \) (defined through the preceding sections) that will be specified later on. Indeed \( g_t \) can be depending on time or on position (locally or globally), \( a_0(t) \) can be a geodesic or not and \( A_t \) can be an isometry that depends or not on time....

So, let us consider within \( B(O, T) \), a trajectory \( X_0(t) \), for \( 0 < t < T \), with respect to a virtual fixed referential frame \( R_0 = (O_0, \vec{i}_0, \vec{j}_0, \vec{k}_0) \). In order to study the tensoriality of the speed vector and the acceleration vector relatively to frame exchanges, we consider two moving frames \( R_1(t) \) and \( R_2(t) \) so that their origins describe the trajectories \( a_0(t) \) and \( b_0(t) \) (with respect to \( R_0 \) and so that their passage matrices are given by \( A_t \) and \( B_t \). We notice that the choice of the frame \( R_0(t_0) \) does not have any influence on the nature of the results that will be established below since we aim to study the passage from one to another frame that are both moving with respect to \( R_0(t_0) \).

Let

\[
X_1(t) = A_t(X_0(t) - a_0(t))
\]

and

\[
X_2(t) = B_t(X_0(t) - b_0(t)),
\]

be, respectively, the expressions of the trajectory \( X_0(t) \) in \( R_1(t) \) and \( R_2(t) \).

**Another time derivation operator**

We now put

\[
\frac{d}{dt} X_1(t) := X'_1(t) + A_t.a'_0(t) - A'_t(X_0(t) - a_0(t)) = A_t.X'_0(t) \quad (d_*)
\]

and

\[
\Gamma_{1*}(t) = \nabla^{g_t}_{\frac{d}{dt} X_1(t)} = \nabla^{g_t}_{A_t.X'_0(t)} A_t.X'_0(t)
\]
as well as

\[ \frac{d}{dt} X_2(t) = B_t \cdot X'_0(t) \]

and

\[ \Gamma_2(t) := \nabla_{B_t \cdot X'_0(t)} B_t \cdot X'_0(t). \]

This derivation as well as the one we have denoted previously by \( \frac{d}{dt} \) takes into account the speed and the rotation of the moving frames. In these conditions, we have

\[ \frac{d}{dt} X_2(t) = B_t \circ A_t^{-1} \frac{d}{dt} X_1(t) \]

and

\[ \Gamma_2(t) := \nabla_{B_t \circ A_t^{-1}} B_t \circ A_t^{-1} \frac{d}{dt} X_1(t). \]

So, if \( B_t \) and \( A_t \) are isometries, then these notions depend only on the geometry (with respect to \( g_t \)) of the space and we have

\[ \| \frac{d}{dt} X_2(t) \|_{g_t} = \| \frac{d}{dt} X_1(t) \|_{g_t} \]

and

\[ \| \Gamma_2(t) \|_{g_t} = \| \Gamma_1(t) \|_{g_t}. \]

In the general case, if \( B_t = A_t \), we have

\[ \Gamma_2(t) = \Gamma_1(t). \]

On the other hand, if \( A_t \equiv A \) and \( B_t \equiv B \), we obtain

\[ \frac{d}{dt} X_1(t) = A \cdot X'_0(t), \]

\[ \frac{d}{dt} X_2(t) = B \cdot X'_0(t) = B \circ A^{-1} \frac{d}{dt} X_1(t) \]

and

\[ \Gamma_2(t) = \nabla_{B \circ A^{-1}} B \circ A^{-1} \frac{d}{dt} X_1(t). \]

If, in addition, \( g_t \) is flat, we have

\[ \Gamma_1(t) = \nabla_{(AX'_0(t))'} (AX'_0(t))' = AX''_0(t) = A\Gamma_0(t) \]

\[ \Gamma_2(t) = \nabla_{(BX'_0(t))'} (BX'_0(t))' = BX''_0(t) = B\Gamma_0(t) \]

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Furthermore, in the same conditions, we have
\[\Gamma_2(t) = B \circ A^{-1} \cdot \Gamma_1(t).\]

Furthermore, in the same conditions, we have
\[\Gamma_1(t) := \nabla^{g_1}_{X^1(t)} X_1'(t) = \nabla^{g_1}_{A(X_0(t) - a'_0(t))} A(X_0(t) - a'_0(t))\]
\[= X_1''(t) = A.(X_0(t) - a_0(t))''\]
and
\[\Gamma_2(t) := \nabla^{g_2}_{X_2(t)} X_2'(t) = X_2''(t) = B.(X_0(t) - b_0(t))''.\]

Therefore, if \(a_0(t)\) and \(b_0(t)\) are geodesics (or if \(a_0(t) = b_0(t) = 0\)), we obtain
\[\Gamma_{1*}(t) = \nabla^{g_1}_{A.X_0(t)} A.X_0'(t) = A.X_0''(t) = A.\Gamma_0(t) = \Gamma_1(t)\]
and
\[\Gamma_{2*}(t) = \nabla^{g_2}_{B.X_0(t)} B.X_0'(t) = B.X_0''(t) = B.\Gamma_0(t) = \Gamma_2(t)\]
and finally
\[\Gamma_2(t) = B \circ A^{-1} \cdot \Gamma_1(t).\]

All acceleration vectors are then null if \(X_0(t)\) is a geodesic.
If, in addition, \(A\) and \(B\) are isometries, then we have
\[\| \Gamma_2(t) \|_{g_0} = \| \Gamma_1(t) \|_{g_0}.\]

We conclude that, in the general case the speed vector transforms in a
tensorial manner in the following sense: If the vector \(X_0(t)\) is multiplied by a
function \(h(X_0(t)) =: h(t)\) where \(h\) is a differentiable function on a neigh-
borhood of the trajectory \(X_0(t)\), then the vectors \(\frac{d}{dt} X_1(t)\) and \(\frac{d}{dt} X_2(t)\)
are multiplied by the same function of \(t\). For the acceleration vector, we have
\[\nabla^{g_1}_{hX_0'(t)} hX_0'(t) = h(h\nabla^{g_1}_{X_0'(t)} X_0'(t) + dh(X_0(t)).X_0'(t)X_0'(t)\]
\[= h^2\Gamma_0(t) + \frac{1}{2} dh^2(X_0(t)).X_0'(t)X_0'(t)\]
\[= h^2\Gamma_0(t) + \frac{1}{2} \frac{d}{dt} h^2(X_0(t)) \frac{d}{dt} X_0(t)\]
and
\[\nabla^{g_1}_{h\frac{d}{dt}X}(t) h\frac{d}{dt}X_1(t) = \nabla^{g_1}_{hA.X_0'(t)} hA_{X_0'.t}X_0'(t) = h(h\nabla^{g_1}_{A.X_0'(t)} hA_{X_0'.t}X_0'(t)\]
\[= h^3\nabla^{g_1}_{X_0'(t)} A_{X_0'.t}X_0'(t) + h((A_{X_0'.t}).h)A_{X_0'.t}X_0'(t)\]
\[= h^3(t)\Gamma_{1*}(t) + h(dh(X_0(t)).(A_{X_0'.t})A_{X_0'.t}X_0'(t)\]
\[= h^3(t)\Gamma_{1*}(t) + \frac{1}{2} dh^2(X_0(t)).(A_{X_0'.t})A_{X_0'.t}X_0'(t)\]
\[= h^3(t)\Gamma_{1*}(t) + \frac{1}{2} dh^2(X_0(t)).(\frac{d}{dt} X_1(t)) \frac{d}{dt} X_1(t).\]
and
\[ \nabla_{hX_2(t)}^g h \frac{d}{dt} X_2(t) = h^2(t) \Gamma_2(t) + \frac{1}{2} \frac{dh}{dt}(X_0(t)) \left( \frac{d}{dt} X_2(t) \right) \frac{d}{dt} X_2(t). \]

If \( A_t \equiv A \), \( B_t \equiv B \) and \( a_0(t) = b_0(t) = 0 \), we obtain
\[ \nabla_{hX_0(t)}^g h X_0'(t) = h^2(t) \Gamma_0(t) + \frac{1}{2} \frac{dh}{dt}(X_0(t)) \frac{d}{dt} X_0(t), \]
\[ \nabla_{hX_1(t)}^g \frac{d}{dt} X_1(t) = \nabla_{hX_0(t)}^g h \frac{d}{dt} X_1(t) = \nabla_{hA.X_0(t)}^g h A .X_0'(t) \]
\[ = h^2 \nabla_{A.X_0(t)}^g AX_0'(t) + h X_0'(t)(h(X_0(t))AX_0'(t) \]
\[ = h^2 \nabla_{X_1(t)}^g X_1'(t) + h \frac{dh}{dt}(X_0(t))AX_0'(t)AX_0'(t) \]
\[ = h^2(t) \Gamma_1(t) + \frac{1}{2} \frac{dh}{dt}(X_0(t)) \frac{d}{dt} X_1(t) \frac{d}{dt} X_1(t) \]
and
\[ \nabla_{hX_2(t)}^g \frac{d}{dt} X_2(t) = h^2(t) \Gamma_2(t) + \frac{1}{2} \frac{dh}{dt}(X_0(t)) \frac{d}{dt} X_2(t) \frac{d}{dt} X_2(t). \]

These equalities show that the two vectors
\[ \nabla_{hX_1(t)}^g h \frac{d}{dt} X_1(t) \text{ and } \nabla_{hX_2(t)}^g h \frac{d}{dt} X_2(t) \]
transform in the same manner when we multiply the speed vectors by a given function \( h \). If, in addition, \( h \) is constant on \( X_0(t) \) \( (h(X_0(t)) \equiv h_0) \), then the two expressions of acceleration vector are multiplied by \( h_0^2 \). These properties imply a sort of tensoriality of the speed vector and the acceleration vector in the following sense:
Suppose that \( \chi(X) \) is a vector field on \( B(O, T) \) having the integral curves \( X_0(t) \) (i.e. \( X_0'(t) = \chi(X_0(t)) \)) for \( t_0 \leq t \leq T \). If we multiply \( \chi \) by a function \( h(X) \) supposed to be constant on the integral curves \( X_0(t) \) (i.e. \( h(X_0(t)) = h(x_0, y_0, z_0) \equiv h_0 \)) or equivalently if we carry out an affine reparametrization of them, then the two expressions of the speed vector are multiplied by the same constant \( h_0 \) and the two expressions of acceleration vector are multiplied by \( h_0^2 \).
When \( g_t \) is flat and \( a_0(t) \) and \( b_0(t) \) are geodesics (or \( a_0(t) = b_0(t) = 0 \)) then this tensoriality property is still valid for the ordinary acceleration vector
\[ \Gamma(t) = \frac{d}{dt^2} X(t) = X''(t). \]
Finally, let us notice that we can define (in the general case where \( g_t = g_t(X) \)),
globally or locally, all geometrical objects (i.e. depending only on the metric \( g_t \)) such as : Isometry groups, gradient and Hessian of functions, divergence of vector fields or differential forms, Laplacian of a function or a differential form (Hodge operator), Dirac operator....

4 Physical modeling of the expanding universe

We begin here by showing that the presumed invalidity of the Maxwell equation regarding inertial frames’ exchanges is only apparent and that the covariance principle is valid, with respect to such exchanges, for electromagnetism Maxwell’s equation.

Indeed, let us consider two inertial Euclidean frames \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) so that the motion of the second frame with respect to the first one is uniform. This situation can be brought back to suppose that, if \((x_1(t), y_1(t), z_1(t))\) is any trajectory in \( \mathcal{R}_1 \), then the trajectory in \( \mathcal{R}_2 \) is given by \((x_2(t), y_1(t), z_1(t))\) with \( x_2(t) = x_1(t) - vt \). Nevertheless these two frames (the study of the passage from each to other has led to the presumed invalidity of the covariance principle) are in fact both uniformly moving. Therefore, if \( v_1 \) and \( v_2 \) are respectively the constant speeds of these frames with respect to a virtual frame \( \mathcal{R}_0 \) supposed to be fixed (we can assume that the two speed vectors have both the same direction as the \( Ox \) axis of \( \mathcal{R}_0 \) and if \( \varphi \) is a function that can be supposed of the form \( \varphi(x_1, t) \) in the frame \( \mathcal{R}_1 \), then the wave equation is written in \( \mathcal{R}_1 \) as :

\[
\square_1 \varphi(x_1, t) := \frac{\partial^2 \varphi}{\partial t^2}(x_1, t) - \frac{\partial^2 \varphi}{\partial x_1^2}(x_1, t) = f(x_1, t). \tag{9}
\]

Putting \( \varphi(x_2, t) = \varphi(x_1 - vt, t) \), the form of this equation remains the same when written in the frame \( \mathcal{R}_2 \); that is it can be written as

\[
\square_2 \varphi(x_2, t) := \frac{\partial^2 \varphi}{\partial t^2}(x_2, t) - \frac{\partial^2 \varphi}{\partial x_2^2}(x_2, t) = f(x_2, t) \tag{10}
\]

in the following sense :

When we write \( \varphi(x_1, t) = \varphi(x_0 - v_1 t, t) \), the wave equation being written in \( \mathcal{R}_0 \) under its canonical form

\[
\square_0 \varphi(x_0, t) := \frac{\partial^2 \varphi}{\partial t^2}(x_0, t) - \frac{\partial^2 \varphi}{\partial x_0^2}(x_0, t) = f(x_0, t), \tag{11}
\]

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then equation (10) is obtained from equation (11) in the same way as equation (9) is obtained from the same equation (11). This is done by giving to the partial derivative \( \frac{\partial \varphi}{\partial t} \) in (10) the same meaning as we have given to \( \frac{\partial \varphi}{\partial t} \) in (9). Namely, we define the derivative \( \frac{\partial \varphi}{\partial t}(x_1, t) \) by replacing, into

\[
\frac{d}{dt} \varphi(x_0, t) = \partial_1 \varphi(x_0, t)x'_0(t) + \partial_2 \varphi(x_0, t),
\]

\( x_0(t) \) by \( x_1(t)(= x_0(t) - v_1t) \) and \( x'_0(t) \) by \( x'_1(t), \) supposed to be the same as \( x_0(t), \) which is not the same as taking

\[
\frac{\partial \varphi}{\partial t}(x_1, t) = \frac{d}{dt} \varphi(x_0 - v_1t, t) = \partial_1 \varphi(x_0 - v_1t, t)(x'_0(t) - v_1) + \partial_2 \varphi(x_0 - v_1t, t).
\]

The same operation is made for \( \frac{\partial \varphi}{\partial t}(x_2, t) \) by replacing, into \( \frac{d}{dt} \varphi(x_0(t), t), x_0(t) \) by \( x_2(t)(= x_0(t) - v_2t) \) and \( x'_0(t) \) by \( x'_2(t), \) supposed to be the same as \( x_0(t), \) which is not the same as

\[
\frac{\partial \varphi}{\partial t}(x_2, t) := \frac{d}{dt} \varphi(x_0(t) - v_2t, t).
\]

So, if we assume that \( \mathcal{R}_1 \) is at rest and \( \mathcal{R}_2 \) is inertial with respect to \( \mathcal{R}_1, \) we deduce (10) from (9) by using the partial derivative \( \frac{\partial \varphi}{\partial t}(x_2, t) \) obtained from

\[
\frac{d}{dt} \varphi(x_1, t) = \partial_1 \varphi(x_1, t)x'_1(t) + \partial_2 \varphi(x_1, t)
\]

by replacing \( x_1(t) \) by \( x_2(t)(= x_1(t) - vt) \) and \( x'_1(t) \) by \( x'_2(t), \) supposed to be the same as \( x'_1(t), \) which is not the same as

\[
\frac{\partial \varphi}{\partial t}(x_2, t) := \frac{d}{dt} \varphi(x_1(t) - vt, t).
\]

In other words, when taking the derivative with respect to the second variable \( t \) of the function \( \varphi(x_1 - vt, t), \) i.e. \( \frac{\partial \varphi}{\partial t}(x_1 - vt, t), \) we assume that the first variable is \( x_1(= x_2 + vt) \) and not \( x_1 - vt(= x_2) \). Briefly speaking, when making this operation, we consider the first variable as being \( x_1 \) instead of \( x_1 - vt \) (i.e. considering the time-dependence of the variable \( x_2 \) as being the same as that of the variable \( x_1 \)) and then we replace \( x_1 - vt \) by \( x_2 \).

More generally, we may briefly recall some general tensoriality properties regarding frame exchanges. Indeed, let \( \mathcal{R}_0 = (O_0, \vec{i}_0, \vec{j}_0, \vec{k}_0) \) be a virtual fixed Euclidean frame of \( \mathbb{R}^3 \) and let \( \mathcal{R}_1 = (O_1, \vec{i}_1, \vec{j}_1, \vec{k}_1) \) and \( \mathcal{R}_2 = (O_2, \vec{i}_2, \vec{j}_2, \vec{k}_2) \) be two other frames. We assume that \( O_1 \) and \( O_2 \) move respectively with relative constant velocities \( \vec{v}_1 \) and \( \vec{v}_2 \) with respect to \( \mathcal{R}_0. \) We furthermore suppose
that $\mathcal{R}_1$ and $\mathcal{R}_2$ are obtained from $\mathcal{R}_0$, for $t \geq 0$, by linear transformations $A_t$ and $B_t$. Finally, let $X_0(t) = (x_0(t), y_0(t), z_0(t))$, $X_1(t) = (x_1(t), y_1(t), z_1(t))$ and $X_2(t) = (x_2(t), y_2(t), z_2(t))$ be respectively the coordinates of a punctual particle trajectory in $\mathbb{R}^3$ with respect to these three frames. We then have, for $t \geq 0$,

$$X_1(t) = A_t(X_0(t) - t \vec{v}_1^t) =: A_t.Y_0(t),$$

$$X_2(t) = B_t(X_0(t) - t \vec{v}_2^t) =: B_t.Z_0(t)$$

and

$$X_2(t) = B_t.(A_t^{-1}.X_1(t) + t \vec{v}_1^t - t \vec{v}_2^t)$$

$$= B_t \circ A_t^{-1}.X_1(t) - B_t.t(\vec{v}_2^t - \vec{v}_1^t)$$

$$= B_t \circ A_t^{-1}.X_1(t) - B_t.t \vec{w}$$

where $\vec{w}$ is the relative velocity.

Now, we recall the definition of the two derivative notions $\frac{d}{dt}$ and $\frac{d^*}{dt}$ previously introduced:

$$\frac{d}{dt}X_1(t) := \frac{d}{dt}X_1(t) - A_t'(X_0(t) - t \vec{v}_1^t)$$

$$= \frac{d}{dt}X_1(t) - A_t'.Y_0(t)$$

$$= \frac{d}{dt}X_1(t) - A_t' \circ A_t^{-1}.X_1(t)$$

and

$$\frac{d^*}{dt}X_1(t) := \frac{d}{dt}X_1(t) + A_t.\vec{v}_1^t - A_t'(X_0(t) - t \vec{v}_1^t).$$

On the other hand, we have

$$\frac{d}{dt}X_1(t) = A_t \left( \frac{d}{dt}X_0(t) - \vec{v}_1^t \right) + A_t'(X_0(t) - t \vec{v}_1^t)$$

which is

$$\frac{d}{dt}X_1(t) = A_t.\frac{d}{dt}X_0(t) + A_t'.Y_0(t).$$

Therefore, we obtain

$$\frac{d}{dt}X_1(t) = A_t.\frac{d}{dt}Y_0(t) = A_t.\frac{d}{dt}X_0(t)$$

and

$$\frac{d^*}{dt}X_1(t) = A_t.\frac{d^*}{dt}X_0(t)$$

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since $\frac{d}{dt}$ and $\frac{d}{dt}^*$ are the same as $\frac{d}{dt}$ for the $\mathcal{R}_0$—coordinates. We notice that we have

$$\frac{d}{dt}X_2(t) := B_t \cdot \frac{d}{dt}Z_0(t) \quad \text{for} \quad Z_0(t) = X_0(t) - tv_2 = Y_0(t) - t\overrightarrow{v} \quad (d_2')$$

and

$$\frac{d}{dt}^*X_2(t) := B_t \cdot \frac{d}{dt}X_0(t)$$

which yields

$$\frac{d}{dt}^*X_2(t) = B_t \circ A_t^{-1} \frac{d}{dt}^*X_1(t).$$

Then, when $B_t \equiv A_t$, we get

$$\frac{d}{dt}^*X_2(t) = \frac{d}{dt}^*X_1(t).$$

Moreover, when $\overrightarrow{v}_1 = \overrightarrow{v}_2$ (i.e. $\overrightarrow{v} = 0$), we have (using $(d_2')$ and $(d_1')$)

$$\frac{d}{dt}X_2(t) = B_t \circ A_t^{-1} \frac{d}{dt}X_1(t)$$

and if we furthermore have $B_t \equiv A_t$, then

$$\frac{d}{dt}X_2(t) = \frac{d}{dt}X_1(t).$$

These derivative notions take into account, in a most natural way, the relative speeds and rotations of the moving frames. The first one coincides, for $A_t \equiv A$, with the classical derivation. The second one becomes, for $A_t \equiv A$,

$$\frac{d}{dt}X_1(t) = \frac{d}{dt}X_1(t) + A.\overrightarrow{v}_1.$$

The previous relations show some tensoriality properties for the speed vector regarding frame exchanges.

In the particular case where $A_t \equiv Id_{\mathbb{R}^3}$, we have

$$\frac{d}{dt}X_1(t) = X_1'(t) = \frac{d}{dt}Y_0(t) = Y_0'(t) = X_0'(t) - \overrightarrow{v}_1,$$

which is the classical Galilean derivation, and

$$\frac{d}{dt}X_1(t) = X_0'(t).$$
In the light of the preceding, we can clarify more precisely the Maxwell’s equation covariance problem. Indeed, after replacing \(X_0(t), X_1(t)\) and \(X_2(t)\) by \(x_0(t), x_1(t)\) and \(x_2(t)\) and the speed vectors \(\overrightarrow{v_1}, \overrightarrow{v_2}\) and \(\overrightarrow{v}\), simply by the scalar speeds \(v_1, v_2\) and \(v\), we can assert that this covariance validity needs only to make a slight modification of the spatial variable notion \((x \to x - vt)\) and also of the notion of the derivation with respect to the time variable \((\frac{d}{dt} \to \frac{d^*}{dt})\). These modifications constitute the two natural procedures for including the frames’ motions into the general motion within the universe, unlike the relativistic spacetime conception which leads to the deterioration of the natural relationship between space and time since, for us, distances are essentially proportional to time.

**Canonical form of the Maxwell equation**

Now, let us consider the classical time derivative
\[
\frac{d}{dt}\varphi(x_1, t) = \frac{d}{dt}\varphi(x_0 - v_1 t, t) = \partial_1\varphi(x_0 - v_1 t, t)(x_0'(t) - v_1) + \partial_2\varphi(x_0 - v_1 t, t).
\]

Then, taking the \(\frac{d^*}{dt}\) derivative instead of \(\frac{d}{dt}\), we obtain
\[
\frac{\partial^*_\varphi}{\partial t}(x_1, t) = \frac{\partial^*_\varphi}{\partial t}(x_0 - v_1 t, t)
\]
\[
= \partial_1\varphi(x_0 - v_1 t, t)\frac{d}{dt}x_1(t) + \partial_2\varphi(x_0 - v_1 t, t)
\]
\[
= \partial_1\varphi(x_0 - v_1 t, t)x_0'(t) + \partial_2\varphi(x_0 - v_1 t, t)
\]
\[
= \frac{\partial^*_\varphi}{\partial t}(x_1, t) = \frac{\partial^*_\varphi}{\partial t}(x_0 - v_1 t, t)
\]

where the latter derivative has the meaning previously specified as here we have \(\frac{d^*_\varphi}{\partial t}x_1(t) = x'_0(t)\) since \(A_t \equiv Id_{\mathbb{R}^3}\). It is made up by considering the time dependence of the first variable \(x_1(t) = x_0(t) - v_1 t\) as being the same as that of the variable \(x_0(t) = x_1(t) + v_1 t\), i.e. by neglecting the term \(v_1 t\) which comes from the relative motion of the frame \(R_1\). In other words, we have to take the derivative of the trajectory seen by \(R_1\) (i.e. \(x_1(t)\)) as being the derivative of this same trajectory when it is seen by \(R_0\) or by any other fixed frame \(R\) that is isometric to \(R_0\); in particular by \(R_1\) provided that this latter is considered as being at rest with respect to \(R_0\). In the same order of ideas we can write
\[
\frac{\partial^*_2\varphi}{\partial t^2}(x_1, t) = \frac{\partial^*_2\varphi}{\partial t^2}(x_0 - v_1 t, t).
\]
We can now maintain that the general intrinsic canonical form of the Maxwell equation is
\[ \Box^* \varphi(x, t) := \frac{\partial^2 \varphi}{\partial t^2}(x, t) - \frac{\partial^2 \varphi}{\partial x^2}(x, t) = 0 \]
which reduces to the classical Maxwell equation
\[ \Box \varphi(x, t) = \frac{\partial^2 \varphi}{\partial t^2}(x, t) - \frac{\partial^2 \varphi}{\partial x^2}(x, t) = 0 \]
when we consider the rest frame with respect to the motion of the origin. If we consider two other frames \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \), the canonical equation becomes respectively
\[ \Box^* \varphi(x_1, t) = \frac{\partial^2 \varphi}{\partial t^2}(x_1, t) - \frac{\partial^2 \varphi}{\partial x_1^2}(x_1, t) = 0 \]
and
\[ \Box^* \varphi(x_2, t) = \frac{\partial^2 \varphi}{\partial t^2}(x_2, t) - \frac{\partial^2 \varphi}{\partial x_2^2}(x_2, t) = 0 \]
which both reduce to the classical one when considering the respective rest frames regarding the respective origins' motions. However, when we consider the \( \mathcal{R}_1 \) to \( \mathcal{R}_2 \) frame exchange, the \( x_2 \) variable in the equation for \( x_2 \) is to be considered as \( x_2 = x_1 - vt \), where \( v \) is the relative speed of \( \mathcal{R}_2 \) with respect to \( \mathcal{R}_1 \), and this equation does not reduce to the classical form. The right hand side of this equation implies the meaning (specified above)
\[ \frac{\partial^2 \varphi}{\partial t^2}(x_1 - vt, t) - \frac{\partial^2 \varphi}{\partial x_1^2}(x_1 - vt, t). \]
This means that, in the derivative of \( \varphi(x_1 - vt, t) \) with respect to time, the first term is obtained by considering the time dependence of the first variable as coming only from the dependence on time of the variable \( x_1(t) \). The time dependence of \( x_2(t) = x_1(t) - vt \) by the intermediate of \( vt \) is to be neglected, which is very natural since the motion of the considered frame with respect to any other one can not be a canonical feature of the wave propagation or more generally of any real physical motion. If \( v = 0 \), both equations for \( x_1 \) and \( x_2 \) are identical and reduce to the classical one when written in any rest frame.

Obviously both operators \( \Box_1 \) and \( \Box_2 \), as they have been introduced using the operator \( \Box_0 \), are identical. Then, both of them reduce, when written each
in its appropriate rest frame, to the classical operator □. The study of these operators aimed only to prove that we can not privilege one inertial frame at the expense of all others. All such frames have the same right for expressing a physical law. Consequently, we can not choose one of them and deduce the non validity of the covariance principle relying on the only reason that the same physical law is expressed differently in another inertial frame. Therefore, only the operator □, is canonical and consequently it is the only one that can be used canonically in order to traduce a physical law in term of local coordinates.

Canonical constancy of the light speed

We can now conclude that the previous framework is the proper context that permits a correct interpretation of the principle of invariability of the light speed in vacuum with respect to all inertial frames. Indeed, if in an inertial frame \( R_0 \) the trajectory of a light beam is \( x_0(t) = ct \), then when we consider another frame \( R \), moving uniformly following the \( Ox \) axis of \( R_0 \) with a constant speed \( v \) (fig.5), we have

\[
x(t) = x_0(t) - vt = ct - vt
\]

where \( x(t) \) is the expression of the same trajectory in the frame \( R \) since here we have \( x(t) = A_t(x_0(t) - vt) \) and \( A_t = Id_R \). So we have

\[
\frac{dx}{dt}x(t) = \frac{d}{dt}x_0(t) = c (= 1),
\]

which is the canonical speed of light, and

\[
\frac{d_1}{dt}x(t) = \frac{d}{dt}x(t) = c - v
\]

which conforms to the Galileo-Newtonian notion of speed. This again shows the invalidity of the special relativity second postulate which states that the speed of light does not depend on the speed of the moving frame that is used to measure it. Naturally the fundamental principle of the light speed independence of the moving source speed is perfectly valid.

Remark : When we attest that the speed of light does depend on the inertial frame used for measuring it, we simply mean that :

When supposing that a light source is located at a supposed constant distance \( d \) on the \( x \) axis of a Euclidean frame \( R_1 = (O_1,e_1,e_2,e_3) \) and that
another frame $R_2 = (O_2, e_1, e_2, e_3)$ which coincides with $R_1$ at time $t = 0$, is moving along the $x$ axis with a uniform speed $v > 0$ (with respect to the frame $R_1$) towards the source, then we attest that the light that is emitted by the source at a time $t > 0$ reaches the observer that is located at $O_1$ a little later than the observer that is located at $O_2$. Similarly, if we suppose that $R_1$ is fixed and the two sources are located, at time $t = 0$, at the same distance $d$ from $O_1$ and if we suppose that they both are moving away from $O_1$, the first one with a uniform speed $v_1$ and the second one with a uniform speed $v_2 > v_1$ with respect to $R_1$, then the light that is emitted by the first source at a time $t > 0$ reaches the observer $O_1$ before the light that is emitted at the same time by the second source.

Besides, we have to notice that there is, for us, a real physical difference between the real motion (with respect to an initial referential frame) of the inertial frame and the real motion of a body or a particle. Actually, when we assume that the particle (for example) is accelerating with respect to the initial referential then it is accelerating with respect to the initial frame and the particle is (physically) radiating while if we assume that the particle is at rest (or animated with a uniform motion) with respect to the initial referential then whatever is the motion of the inertial frame it is obvious that the particle is not radiating.

On the other hand, we can imagine an experience which shows that the connection of a material movement to another can lead to a relative speed greater than 1. Indeed, assume that a sufficiently large metallic rod is fixed on a train which circulate at a uniform speed $v_1 > 0$ with respect to a virtual frame at rest. We next suppose that an electrical current generator is fixed at the origin of this rod and that the speed of propagation within the rod is $v_2$ when this rod is at rest. When the engine is in motion, the speed $v_1 + v_2$ of this propagation, with respect to the virtual frame, can achieve a speed $v > 1$. This kind of movement come out of the framework of the relativistic conception and out of the notion of proper time and clock delay induced by speed.

Moreover when we institute a space-time relationship based, essentially and theoretically, on the notion of inertial frames moving with a uniform speed $v \in [0, 1]$ relatively to each other, we can not neglect or exclude the situation that consists on considering three inertial frames such that the third frame is moving uniformly with respect to the second one which is itself moving in the same way with respect to the first one such as the addition of both relative speeds gives a number larger than 1 and leave such a phenomenon without any explanation.
We can also add to the preceding arguments that the fact of considering negative times \((t < 0)\) is fundamentally incompatible with the confirmed theory of the universe expansion which pre-assumes the existence of an original time or an origin for time \((t = 0)\), starting from which (or more precisely just after which) the time progresses homogeneously increasing \((t > 0)\).

**Alternative interpretation of Michelson-Morley experiment**

In order to prove the deficiency of the results obtained from the Michelson-Morley experiment we will describe first an experiment that can be used also for showing the interpretation deficiency of the emitter in the middle of a track (and the two mirrors on both sides) experiment. It also suggests the way for analyzing other experiments such as the emitter in a plane heading toward a given observer.

So, we consider an emitter of light that is at rest with respect to the earth and sending light rays parallel to the earth movement (supposed to be uniform) toward a mirror located at a distance \(L\) away from it. Let \(0 < v < 1\) and \(c = 1\) be respectively the speeds of the earth and the light with respect to a virtual fixed frame coinciding at \(t = 0\) with the emitter and having its \(Ox\) axis in the movement direction.

Let \(t_1\) and \(t_2\) be respectively the times taken by the beam (or more precisely by each photon) to reach the mirror and then to return to the emitter (fig.4’

We obviously have, by measuring both path lengths according to the fixed frame:

\[ L + t_1v = t_1 \]

and

\[ L - t_2v = t_2. \]

We then have

\[ t_1 = \frac{L}{1-v} \quad \text{and} \quad t_2 = \frac{L}{1+v}, \]

which yield

\[ t_1 + t_2 = \frac{2L}{1-v^2}. \]

If we assume, according to the second part of the special relativity second postulate, that the speed of light with respect to the frame for which the emitter is at rest is given by 1, we get

\[ t'_1 + t'_2 = 2L \]
where
\[ t'_i = \gamma (t_i - xv) \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1 - v^2}}. \]

We will now show that this equality yields a contradiction. Indeed

\[ 2L = t'_1 + t'_2 = \gamma [t_1 - (L + t_1v)v] \]
\[ + \gamma \{t_2 - [L + t_1v - t_2 - (L + t_1v)]v\} \]
\[ = \gamma (t_1 + t_2 - Lv - t_1v^2 + t_2v) \]
\[ = \gamma \left( \frac{2L}{1 - v^2} - Lv - \frac{L}{1 - v^2} + \frac{L}{1 + v} \right) \]

and then
\[ \gamma \left( \frac{2}{1 - v^2} - v - \frac{v^2}{1 - v} + \frac{v}{1 + v} \right) = 2 \]
that is
\[ \frac{2 - v + v^3 - v^2 - v^3 + v - v^2}{1 - v^2} = 2\sqrt{1 - v^2} \]
or
\[ \frac{2 - 2v^2}{1 - v^2} = 2\sqrt{1 - v^2} \]
which yields \( 1 = \sqrt{1 - v^2} \) and then \( v = 0 \), which is absurd.

Otherwise, if we use the classical Galilean transformation we have
\[ y(t) = x(t) - vt = t - vt \quad \text{for} \quad 0 < t < t_1 \]
and
\[ y(t) = x(t) - v(t-t_1) = t_1 - vt_1 - (t-t_1) - v(t-t_1) \quad \text{for} \quad t_1 < t < t_1 + t_2 \]
which give
\[ y(t_1) = t_1 - vt_1 \]
and
\[ y(t_1 + t_2) = t_1 - vt_1 - t_2 - vt_2. \]

Now, \( y(t_1 + t_2) = 0 \) gives
\[ t_1 - t_2 = v(t_1 + t_2) \]
and
\[ v = \frac{t_1 - t_2}{t_1 + t_2} \]
as it must be. Furthermore, we have
\[ y'(t) = 1 - v \quad \text{for} \quad 0 < t < t_1 \]
and
\[ y'(t) = -(1 + v) \quad \text{for} \quad t_1 < t < t_1 + t_2. \]
Then we get
\[ y'(t)t_1 + (-y'(t)t_2) = (1 - v)t_1 + (1 + v)t_2 = L + L = 2L \]
which agrees with the Galileo-Newtonian notions.

Now, in the light of the two previous experiments, we can show that the interpretations of experiments of type Michelson-Morley are erroneous. Indeed, suppose that the global at rest distance of the mirrors located on one side and the mirrors located on the opposite side is \( L \). Then when the light beam heads parallel and antiparallel to the earth’s movement we can evaluate the time taken to reach the interferometer as being
\[ t_1 = \frac{L}{c-v} + \frac{L}{c+v} = \frac{2L}{1-v^2}. \]
On the other side, we have for the beam heading in the other direction, which is “perpendicular” to the earth’s movement, either \( t_2 = 2L \) or \( t_2 = \frac{2L}{\sqrt{1-v^2}} \) whether if we consider the beam which is orthogonal to the first one or the other beam which goes in a slightly different direction in order to reach the interferometer when the used mirrors are macroscopic. So, when the apparatus is rotated by 90°, the situation is reversed and \( t_1 \) is replaced by one of the two previous expressions of \( t_2 \) and both \( t_2 \) are replaced by the expression of \( t_1 \), but \( |t_2 - t_1| \) remains unchanged. Consequently, in both cases the difference between the two paths’ lengths remains the same, which explains the unchanged interference pattern.

**Geometrization of the physical universe**

So, we propose in this paper to conserve the pre-relativistic (Galileo-Newtonian) conception of space and time and to consider the half-cone \( C = \{(x,y,z,t) \in \mathbb{R}^4; x^2 + y^2 + z^2 \leq t^2 \} \) as being the world of trajectories in \( \mathbb{R}^4 \), with some restrictions imposed on the real ones in order to conform
with the causality principle. Then, according to the preceding properties, we can deduce that all mechanical and physical laws, which are based on equalities implying speed vectors, acceleration vectors and vector fields (or more generally tensor fields), are invariant under frame exchanges carried out in \((\mathbb{R}^3, g_t)\) implicating relative constant speeds and constant isometries \((a'_0(t) = \vec{V}_1, b'_0(t) = \vec{V}_2, A, B \in O(g_t)\) for any \(t\)). We can mention for example: the fundamental laws of Mechanics, Maxwell’s equation, the conservation of energy principle, the least action principle....

Besides, we have already proved, in the general case of a variable metric (with variable curvature depending on time) \(g_t\), some properties of tensoriality with respect to frame exchanges and the conservation of physical laws with respect to some of them, notably those that involve isometries (for \(g_t\)) including all isometries and not only a sub-group of their global group.

We propose then to go forward in the direction of the physical reality of the universe taking into account the mass distribution, the gravitational phenomenon and other manifestations of matter such as electromagnetism and different forms of energy (excepting quantum phenomena and energy singularities which are, within our new framework, much easier to treat with).

Thus, let us first consider that an electrically neutral material particle of (nearly constant) inertial mass \(m_I\) is in a state of free fall within a uniform gravitational field (locally in \(\mathbb{R}^3\)) defined by \(\vec{g} = (0, 0, -g)\) in a fixed Euclidean frame. When this particle is observed with a Euclidean frame \((O, e_1, e_2, e_3)\) which follows itself a trajectory of free fall such that \(e_3 = (0, 0, 1)\), then (as Einstein showed it) the particle seems as being at rest in this new frame. Likewise, if this particle had an initial horizontal speed \(\vec{V}_0\) (in the fixed frame) at \(t = t_0\), then its motion seems, in the moving frame, as if it was uniform whereas it is in fact of parabolic shape in the fixed frame. The moving frame having uniform acceleration \(\vec{\Gamma} = \vec{g}\), in the fixed frame, plays then the role of an inertial frame regarding the two Newtonian inertia principles. This same property is valid for any other frame carrying out a free fall motion like our previous moving frame. We call such a frame \(\vec{g}\)-inertial frame. There exists (locally in \(\mathbb{R}^3\)) a dynamical metric tensor \(g_{ab}\) such that \(\nabla_{x'(t)} x''(t) = 0\) for any free fall movement described in the fixed frame by \(x(t)\), i.e. such that all trajectories corresponding to a free fall movement are geodesics for this metric (in other words, there exists theoretically a universal metric for which all natural motion are geodesics). Indeed, in favor of the presupposed symmetries of our (isolated) system and using the coordinates in the fixed frame,
we can assume that we have
\[ g_{ij} = dx_1^2 + dx_2^2 + b(t, x_3, g) dx_3^2. \]

Actually, we have, using this metric denoted by \( g_t(x) \):
\[ ||x||^2_{g_t(x)} = x_1^2 + x_2^2 + b(t, x_3, g) x_3^2 \]
and, if we take \( x = x_3 e_3 \), we obtain
\[ ||x_3 e_3||^2_{g_t(x)} = b(t, x_3, g) x_3^2. \]

Now the free fall trajectory is described by
\[ x_3(t) = a_3 - \frac{1}{2} g t^2 \quad \text{with} \quad x_3(0) = a_3, \ x_3(t_0) = 0 \quad \text{and} \quad a_3 = \frac{1}{2} g t_0^2. \]
So we get, for \( g > 0 \):
\[ t = \sqrt{\frac{2}{g}(a_3 - x_3(t))} \quad \text{and} \quad x_3'(t) = -gt. \]

Let us write the following equivalences :
\[ \nabla_{x_3(t)e_3}^{g_t} x_3'(t)e_3 = 0 \iff g^2 t \nabla_{e_3}^{g_t} te_3 = 0 \iff \]
\[ g^2 t^2 \nabla_{e_3}^{g_t} e_3 + g^2 t \frac{d}{dx_3} \sqrt{\frac{2}{g}(a_3 - x_3(t))} e_3 = 0 \iff \]
\[ \frac{d}{dx_3} b(t, x_3, g) = - \frac{d}{dx_3} \sqrt{\frac{2}{g}(a_3 - x_3(t))} \iff \]
\[ b(t, x_3, g) = \frac{k(t)}{\sqrt{\frac{2}{g}(a_3 - x_3(t))}} = \frac{k(t)}{t}. \]

For \( x(t) = x_3(t)e_3 \), we have
\[ ||x_3(t)e_3||^2 = b(t, x_3, g) x_3^2(t) = x_3^2(t)||e_3||_{g_t(t)} \]
and then
\[ b(t, x_3, g) = ||e_3||_{g_t(t)} = \frac{k(t)}{t}. \]
Taking a normal parametrization of this geodesic, we get
\[ 1 = \|x'(t)e_3\|_{g_3(t)} = \|gte_3\|_{g_3(t)} = gt\|e_3\|_{g_3(t)} \]
which yields
\[ \|e_3\|_{g_3(t)} = \frac{1}{gt} = b(t, x_3, g) \]
Therefore, we obtain
\[ g_{ij}(t) = dx_1^2 + dx_2^2 + \frac{1}{gt}dx_3^2 = dx_1^2 + dx_2^2 + \frac{1}{\sqrt{2g(a_3 - x_3(t))}}dx_3^2 \]
This metric obviously depends on the chosen level \( x_3(0) = a_3 \).

When the gravitational field \( \vec{g} \) is central and has a constant (Euclidean) norm \( g \) with a fixed center \( C \), then we can integrate (locally) this gravity within a metric \( g_{ab}(t, x) \) by means of its canonical connection \( \nabla_{g_{ab}} \) by defining all its geodesics by \( \nabla_{x(t)}^a x'(t) = 0 \), where \( x(t) \) denotes the coordinates of these curves in a Euclidean frame originated in \( C \) or any other fixed frame. In favor of symmetries (intuitive local homogeneity and isotropy principles), we can determine the metric \( g_{ab} \) using the normal Riemannian coordinates around \( C \) and then transforming them into spherical coordinates in order to obtain
\[ g_{ab}(t, r) = b(t, r, g)dr^2 + r^2d\sigma^2 \]
where \( b(t, r, g) \) decreases when \( t \) or \( g \) increases.

Thus, distances to the center and volumes are "inversely proportional" to the intensity of gravity. When the center of gravity \( C(t) \) is mobile, then the free fall motion does not occur following a straight line heading towards the center, but following a curve \( x(t) \) whose tangent vector is always pointing towards the moving center. However, a central gravitational field is never uniform; it always depends on the distance to the center (and then on the parametrization time of free trajectories). If \( C(t) \) is moving, then the field is radially constant with respect to a frame that is fixed at the center \( C(t) \). Nevertheless, if \( C \) is fixed, we can consider the gravitational field as being uniform for sufficiently distant neighboring objects. If we now assume that \( \|C'(t)\| << 1 \) and \( g_t << 1 \), we recover approximately the Euclidean metric. This is the case when we are located (locally) at a reasonable distance away from the earth surface. But, this is not, in general, the situation that corresponds to the physical reality. Indeed, although we can imagine an inertial frame so that the relative speed of a given star, for instance, is sufficiently small, the vector \( \vec{g} \) (or rather \( \vec{g}_t \)) depends strongly on the Euclidean distance of the body in movement to the center of this star. Nevertheless, in
a supposed isolated system, we can use empirical methods to determine the geodesics (trajectories of bodies corresponding to free fall motions) and study their speed vectors and then their relative acceleration vectors (by determining their relative deviations) and then deduce the Christoffel symbols along these curves, associated with the metric \((g_t)_{ab}\). We can also deduce the curvature tensor \(R_{abcd}(t)\) using the infinitesimal deviation equations ([4],p.3.3.18).

We may use, in addition, the identity \(\nabla^{(g_t)_{ab}}(g_t)_{ab} = 0\) as well as the possible symmetries.

The physical metric

These same considerations hold when we aim to determine (even locally) the global metric \(g_{ab}(t, x)\) of the expanding universe \(U(t) = (B(O, t), g_{ab}(t, x))_{t>0}\), taking into account the other phenomena (electromagnetism, radioactivity, energy singularities and quantum phenomena) in order to integrate them into the metric. This leads us to Einstein’s equation whose resolution gives an approximate metric \(g_{ab}(t, x)\) which could specify the trajectories associated with free motions, i.e. the geodesics of the dynamical universe \((\nabla^{g_{ab}} x'(t)=0)\). The metric \(g_{ab}(t,x)\) will be called the physical metric.

Indeed, let us denote (according to the weak equivalence principle) \(m_I = m_g\) the inertial or gravitational mass of a particle when it is located within a general gravitational field \(\vec{g}_t\), induced locally by a matter-energy distribution, that is expressed by the metric \(g_{ab}(t, x)\). Under these conditions we have, for any trajectory \(x(t)\),

\[\bar{g}_t = \bar{\Gamma}_t = \nabla^{ge}_{x'(t)} x'(t) = x''(t)\]

when using any virtual fixed Euclidean frame and

\[\nabla^{g_{ab}}_{x'(t)} x'(t) = 0\]

for any given free fall motion as we will show in the next section. So, the \(g_t\)-inertial mass \(m_I(t)\) depends on time by means of the metric \((g_t)_{ab}\) which reflects all sorts of energy forms and energy fluctuations.

If \(\mathcal{R}_0\) is a fixed Euclidean frame, \(\mathcal{R}_1\) is an identical frame to \(\mathcal{R}_0\) whose origin describes the curve \(a(t)\) with respect to \(\mathcal{R}_0\), and if \(x(t)\) is any trajectory (in \(\mathcal{R}_0\)), let us define

\[\bar{\Gamma}_0(x(t)) := \nabla^{g_\epsilon}_{x'(t)} x'(t)\]

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and
\[ \tilde{\Gamma}_{01}(x(t)) := \nabla^{g_{x(t)}}_{x'(t)} x'(t) - \nabla^{g_{a'(t)}}_{a'(t)} a'(t) = \tilde{\Gamma}_0(x(t)) - \tilde{\Gamma}_0(a(t)). \]

\( \tilde{\Gamma}_{01}(t) := \tilde{\Gamma}_{01}(x(t)) \) is in fact the dynamical acceleration of the trajectory \( x(t) \) with respect to the frame \( R_1 \) which has its proper dynamical acceleration \( \tilde{\Gamma}_0(a(t)) = : \tilde{\Gamma}_0(t) \). Then we have

\[ \tilde{\Gamma}_{01}(t) = \tilde{\Gamma}_0(x(t)) \text{ if and only if the frame } R_1 \text{ is } g_t - \text{inertial (i.e.} \nabla^{g_{a'(t)}}_{a'(t)} a'(t) = 0). \]

According to these notions and notations, we can state

\[ x(t) \text{ is a geodesic for the dynamical metric } \Leftrightarrow \tilde{\Gamma}_0(x(t)) = 0, \]

and then we have, for all such \( R_1 \):

\[ \tilde{\Gamma}_{01}(t) = 0 \Leftrightarrow a(t) \text{ is a geodesic}. \]

This constitutes a generalization of the first principle of inertia which states that

"a movement is uniform in an inertial frame if and only if it is uniform in any other inertial frame".

Notice that \( \nabla^{g_{x'(t)}}_{x'(t)} x'(t) - a'(t) = 0 \) is not necessarily null when \( \nabla^{g_{x'(t)}}_{x'(t)} x'(t) = \nabla^{g_{a'(t)}}_{a'(t)} a'(t) = 0 \) unlike the property \( \nabla^{g_{x'(t)}}_{x'(t)} x'(t) - a'(t) = 0 \) when \( x''(t) = a''(t) = 0 \).

In the light of the universe physical realities, the fundamental principle of Mach adopted by Einstein (Matter=curvature) and the general principle of modeling (Physics=Geometry), we have to deal with all aspects of matter-energy. This leads us to the symmetrical \((0,2)\)-tensor, defined on \( B(O,t) \subset \mathbb{R}^3 \), and modified in a sense that will be specified below, which is the (mass-energy) tensor \( T^*_{ab}(t) \) whose variable independent elements are only six.

We consider then the physical universe identified, at every time \( t > 0 \), to

\[ U(t) = (B(O,t), g_{ab}(t, x)) \]

where \( B(O,t) \) is the Euclidean ball of radius \( t \) and \( g_{ab}(t, x) \) is the (regularized) Riemannian metric associated to the physical universe at time \( t \). Recall that this metric is variable with the position and time and its curvature is, in general, positive and is itself variable with the position and time. This curvature is due to the distribution of mass-energy at any time \( t \) and then
essentially to the entire gravitational field and other manifestations and effects of matter.

The half-cone of space and time is constituted of all sections \( t = \text{const.} \), which are the ball-hypersurfaces of \( C \subset \mathbb{R}^4 \) (except for \( t = 0 \)). They are orthogonal to the time axis for the Minkowski metric on \( C \). Each of these balls, \( B(I, t_0) \) of center \( I(0, 0, 0, t_0) \) and Euclidean radius \( t_0 \), is provided with the Riemannian metric \( g_{\alpha\beta}(x) \) which depends on the position \( x \) and has a variable curvature. The Einstein’s equation is written (within our new context) as

\[
G_{\alpha\beta}(t) := R_{\alpha\beta}(t) - \frac{1}{2} R(t) g_{\alpha\beta}(t) =: T^*_{\alpha\beta}(t) \quad (\mathcal{E})
\]

where \( R_{\alpha\beta}(t) \) is the Ricci \((0,2)\)-tensor associated with the metric \( g_{\alpha\beta}(t) \) in \( U(t) \subset \mathbb{R}^3 \), \( R(t) \) is the scalar curvature of \( U(t) \) and \( T^*_{\alpha\beta}(t) \) is the symmetrical matter-energy tensor. This tensor depends naturally on the density \( \rho(t) \) of the mass distribution, the ambient gravitational field, the pressure \( P(t) \) resulting from phenomena typically associated to perfect fluids, the global electrical field \( \vec{E}(t) \) and the global magnetic field \( \vec{B}(t) \) and some other energy manifestations. All these tensorial objects depend on position and time according to whether space has, locally and at fixed time, a matter dominance or a radiational one. We notice that we have

\[
\nabla^\alpha T^*_{\alpha\beta}(t) = 0
\]

(according to the second Bianchi identity) where \( \nabla = \nabla(g_{\alpha\beta}(t)) \) is the Levi-Civita connection associated with the Riemannian metric \( g_{\alpha\beta}(t, x) \), insuring the validity of the conservation of energy law. Moreover, the trajectories of bodies which are only under the action of natural forces are substantially and theoretically geodesics for this metric. The crucial task is the approximate (local) resolution of Einstein equation on the base of a bank of dynamic data as precise as possible. This will be achieved in section 10.

On the other hand, we notice that the three bodies’ problem (or more generally any \( n \) bodies’ problem), whether they constitute or not an isolated system, has to be treated within the above setting. If \((g_t)_{\alpha\beta}\) is the (local) ambient metric and if \( x_i(t)(i = 1, 2, 3) \) describes the trajectory of any of these bodies in any virtual fixed frame, then we have

\[
\nabla_{x'_i(t)}^\alpha x'_i(t) = 0 \quad \text{for} \ i = 1, 2, 3.
\]

If we assume that these three bodies form an isolated system, then we can conversely use these equations together with all data of the problem to (locally) recover the metric \((g_t)_{\alpha\beta}\).
More generally, we can locally determine the metric \((g_t)_{ab}\) using empirical methods. If \(X(t)\) denotes the trajectory of a particle or a body moving only under the natural forces (i.e., forces originated by non-singular energy phenomena. Such a motion will be qualified as a free motion) with respect to any fixed frame, then we have

\[\nabla_{X'(t)}^g X'(t) = 0\] at every time \(t\).

So, the empirical or numerical determination of sufficiently many geodesics \(X(t)\) permits to determine approximately the Christoffel symbols and the metric \(g_t\) at each point of \(X(t)\). Therefore the metric \(g_t\) is determined either by means of its geodesics or by determining the tensor \(T^*_ab\) and then resolving the simplified Einstein’s equation.

We notice that, when \(T^*_ab = 0\) on a region \(D \subset B(O, t)\), then this means that, according to our definition of this tensor, the region \(D\) is not only devoid of Matter, but also it is apart from any energetic influence of matter and so we have \(g_{ab} = g_e\) on \(D\) as well as \(R_{ab} = 0\) and \(R = 0\). In the particular cases (perfect fluids, electromagnetic fields, scalar fields of Klein-Gordon), the asymptotical cases and quasi-newtonian cases, the resolution of equation \((E)\) is much easier, within our model, than within the framework of the standard general relativity. Some of these issues will be reviewed below.

**Remark**: Our tensor \(T^*_ab\) and our associated physical metric \(g_t\) incorporate, within their definition all matter-energy forms and effects which is not the case of the Einstein’s tensor and its associated spacetime metric. The Einstein’s tensors and metric reflect only the gravitational field that is caused by a given mass in the presence of a matter field (such as a perfect fluid with or without pressure) and possibly an electromagnetic field or in the vacuum. The vacuum Einstein’s equation

\[R_{ab} - \frac{1}{2} R g_{ab} = 0\]

is characterized by \(\rho = 0, T_{ab} = 0, R_{ab} = 0\) and \(R = 0\) and does not imply that the spacetime metric is the flat one. Moreover, when we introduce the cosmological constant \(\Lambda\), the vacuum Einstein’s equation becomes

\[R_{ab} = \frac{1}{2} R g_{ab} - \Lambda g_{ab}\]

and then we have \(R = 4\Lambda\) and if \(\Lambda \neq 0\) we have \(R \neq 0\) which implies that the vacuum geodesics are not the same as those of the flat space. Thus, in
the light of our model, $\Lambda$ reflects the influence of the cosmic matter on the regions where there is no matter (even though there is the intergalactic gravity, the cosmic radiations and the neutrinos) and logically it must depend on time and probably on space regions. For us a space region is in a state of absolute vacuum (i.e. $T^{*}_{ab}=0$) if and only if it is (nearly) a part of all matter manifestations as well as of their effects.

Finally, we mention that if $P_0$ is an isotropic observer (i.e. moving in the expansion direction approximately at the light speed), the horizon particle for him is roughly the union of all balls of center $P_1$ that are located on the isotropic line ($OP_0$) and therefore this horizon is the half-space located beyond the perpendicular plane at $P_0$ to this line (fig.6). Clearly, this horizon is purely theoretical because of the singularities (essentially the black holes) that are scattered in the real physical universe. An ordinary observer can only see (or interact with) a little part of the expanding universe.

**Comparison with the relativistic invariance of the speed of light**

Within the special relativity framework, the speed of light is independent of the Galilean observer which generally is represented into the relativistic spacetime ($\mathbb{R}^4, \eta$) (where $\eta$ is the Minkowsky metric $-dt^2 + dx_1^2 + dx_2^2 + dx_3^2$) by a timelike straight line $D$. Considering a normal parametrization $c : I \subset \mathbb{R} \rightarrow \mathbb{R}^4$ of $D$ (i.e. $\eta(c'(t), c'(t)) = -1$) and an arbitrary parametrization $\tilde{c} : J \subset \mathbb{R} \rightarrow \mathbb{R}^4$ of another Galilean observer $\tilde{D}$, we can write

$$\tilde{c}'(t) = \vec{k} + \alpha c'(t)$$

for $\vec{k} \in c'(t)\perp$ and $\alpha \in \mathbb{R}$, where $c'(t)\perp$ is the orthogonal hyperplane (for the metric $\eta$) of the vector $c'(t)$ which also is the set of simultaneous points of $c(t) \in D$ at time $t$. We then define the relative speed vector of the observer $\tilde{D}$ with respect to $D$ by

$$\vec{v}_{\tilde{D}/D} = \frac{\vec{k}}{\alpha}$$

and if $\tilde{c}$ is a normal parametrization with respect to $D$ of $\tilde{D}$, then we have $\alpha = 1$ and $\vec{v}_{\tilde{D}/D} = \vec{k}'$. One then proves that the relative speed of light with respect to an arbitrary Galilean observer $D$ is $c = 1$.

This is established by considering a lightlike line $L$ parametrized by $\tilde{c}(t) =$
\(M + tl\) with \(\eta(l, l) = 0\) and \(M\) is an arbitrary point of the timelike cone and taking \(A = c(t) \in D\) and \(B = \tilde{c}(t)\) the point of \(L\) that is simultaneous to \(c(t)\) for \(D\) and writing
\[
\tilde{c}'(t) = l = \frac{\vec{k}}{\alpha} + \alpha c'(t)
\]
where \(\alpha \in \mathbb{R}\) can be considered as being positive and \(\vec{k} \in c'(t)^\perp\) and finally by considering the relative speed vector as being
\[
\vec{v}_{L/D} = \frac{\vec{k}}{\alpha}
\]
that is assumed to represent the speed vector of light with respect to the observer \(D\).

Thus, the relation \(\eta(l, l) = \eta(\vec{k}, \vec{k}) + \alpha \eta(c'(t), c'(t)) = \eta(\vec{k}, \vec{k}) - \alpha^2 = 0\) implies
\[
\alpha = \sqrt{\eta(\vec{k}, \vec{k})} \quad \text{and} \quad \| \vec{v}_{L/D} \|_{\eta} = \sqrt{\eta(\frac{\vec{k}}{\alpha}, \frac{\vec{k}}{\alpha})} = \sqrt{\frac{\eta(\vec{k}, \vec{k})}{\alpha}} = 1.
\]

For us, this reasoning is only valid for a stationary observer i.e. when \(D\) identifies with the time axis where we have \(c(t) = N + t(1, 0, 0, 0)\), \(c'(t) = (1, 0, 0, 0)\), \(c'(t)^\perp\) is a hyperplane that is parallel to \((0, x_1, x_2, x_3)\) for \(x_1, x_2, x_3 \in \mathbb{R}\) and \(\tilde{c}(t) = M + t(1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})\) satisfies \(\eta(c'(t), \tilde{c}'(t)) = 0\).

When we consider an arbitrary Galilean observer, \(c'(t)^\perp\) is not necessarily (within the relativistic framework) the spacelike hyperplane previously considered.

The previously established relation for the Galilean observer \(D\) and \(\tilde{D}\) which led to the above definition of the relative speed \(\vec{v}_{D/D} = \vec{k}\) (within the special relativity framework) does not have for us any physical significance. This definition, as well as the relativistic spacetime notion, is only conceived in order to justify the special relativity second postulate that stipulates that the speed of light is the same for all Galilean observers.

For us (as it will be shown in section 5), the real physical universe actually is a part of \(\mathbb{R}^3\) that is modeled, at every time \(t\), by the Euclidean ball \(B(O, R(t))\) of \(\mathbb{R}^3\) provided with the physical metric \(g_t\) (that reflects the physical consistency of the universe) and the measure of any observer speed or any trajectory speed into \(B(O, R(t))\), that is parametrized into the space-time semi-cone
\[
C = \{(x, y, z, t); x^2 + y^2 + z^2 \leq R^2(t) \sim t^2, t > 0\} \simeq \bigcup_{t>0} B(O, t) \times \{t\}
\]
by \(c(t) = (t, x_1(t), x_2(t), x_3(t))\) and into \(B(O, R(t))\) by \(X(t) = (x_1(t), x_2(t), x_3(t))\), is given by \(\|X'(t)\|_{g_t}\). So, this speed is measured within \(B(O, R(t))\) using the
physical metric $g_t$ and not within the semi-cone $C$ of $\mathbb{R}^4$ and if we denote by 
\[ h_t := dt^2 - g_t \]
the metric on the semi-cone that is associated with $g_t$ at time $t$, we have 
\[ 0 < ||c'(t)||_{h_t} < 1 \quad \text{and} \quad ||X'(t)||_{g_t} < 1 \]
for any observer, meanwhile 
\[ ||c'(t)||_{h_t} = 0 \quad \text{and} \quad ||X'(t)||_{g_t} = 1 \]
for the trajectories that characterize the light rays (which actually are also characterized by $\nabla^{g_t} X'(t) = 0$).
Thus, when a part of the universe can be assimilated to the absolute vacuum, we then have (within this part) $g_t = g_e$ and $h_t = dt^2 - g_e = -\eta$ and we can consider two Galilean observers $D_1$ and $D_2$ that will have a relative speed with respect to a stationary observer $D_0$ (represented by the time axis or any vertical straight line) for each of them. So, if $c_1(t)$ and $c_2(t)$ are the normal parametrizations of $D_1$ and $D_2$, then we have 
\[ c'_1(t) = \vec{k}_1 + (1,0,0,0) \]
\[ c'_2(t) = \vec{k}_2 + (1,0,0,0) \]
where $\vec{k}_1$ and $\vec{k}_2$ are the parallel to the time-axis projections on $B(O,R(t))$ (which is an orthogonal projection) of $c'_1(t)$ and $c'_2(t)$ and the relative speed of $D_2$ with respect to $D_1$ is 
\[ \vec{v}_{D_2/D_1} =: \vec{k}_r = \vec{k}_2 - \vec{k}_1. \]
Similarly, the speed of light with respect to $D_0$ is in that case $||\vec{k}||_{g_e} = 1$ where $\vec{c}(t)$ designates the light trajectory and $\vec{c}'(t) = l$ is written as $l = \vec{k} + (1,0,0,0)$ with $h(l,l) := (dt^2 - g_e)(l,l) = 0$. Moreover, the relative speed of light with respect to the observer $D_1$ is $\vec{k} - \vec{k}_1$. Thus, if $D_0, D_1$ and $L$ are located in the same plane and if $v_{D_1/D_0} = v$, then the relative speed of light with respect to $D_1$ is $1 - v$.
Concerning the real physical universe $(B(O,R(t),g_t))$, we can only define the instantaneous relative speed vector of the light with respect to an observer $D_1$ at the intersection point of $D_1$ with the light trajectory $L$ by $\vec{k} - \vec{k}_1$ having the physical magnitude $||\vec{k} - \vec{k}_1||_{g_t}$ where $\vec{k}$ and $\vec{k}_1$ are the space-like associated vectors with $L$ and $D$. This magnitude reduces to $||\vec{k} - \vec{k}_1||_{g_e}$ within the absolute vacuum.
5 Matter-Energy, black holes and inertial mass

We begin this section by mentioning that the whole of this study is based on all main, seriously confirmed principles of Mechanics and Physics, established by Newton, Einstein, Hubble and many others, that coincide in all special cases with the codified, measured and verified laws by a large number of physicists as Maxwell, Lagrange, Hamilton, Shrödinger, Bohr and Planck and perfectly modelled by means of the work of Gauss, Euler and Riemann between many others.

Our starting line here is the following simplified and modified Einstein’s equation

\[ G_{ab} := R_{ab} - \frac{1}{2} R g_{ab} =: T^*_{ab} \]  

(12)

where all symmetrical tensors used here are defined on the ball \( B_e(O, R(t)) \subset \mathbb{R}^3 \) which will be considered as being the Euclidean ball \( B_e(O, t) \) for sake of simplicity. This reduces in fact to assume that the universe expansion speed is the same as the electromagnetic waves’ speed which is assumed of being always the same as the light speed in the absolute vacuum (i.e. \( c = 1 \)).

Recall that all tensors here describe physical and geometrical phenomena that are inherent in the real physical universe at time \( t \). The distribution of matter located in a region of the universe determines, at every time \( t \), a distribution \( m_t(X) \) of inertial mass which gives birth to a global gravitational field and other force fields. This inertial mass is subject to many different evolutive energy transformations (electromagnetic, thermonuclear and radioactive).

In addition to the inertial static aspect of matter (characterized by the inertial mass) and to this evolutive aspect, we must add the dynamical aspect, i.e. the creation and the transformation of kinetic energy by the movement of matter. We notice that the word energy here must be interpreted with a wide, total and unified sense.

The physical metric \( (g_t)_{ab} \) or \( g_t \) (depending on position and time) reflects, at every time \( t \), the permanent perturbation of the geometrical flat space \( (B_e(O, t), g_e) \). This perturbation is created, according to Mach-Einstein principle, by the matter - energy distribution; it is expressed by the creation of spatial curvature which is taken into account in the equation (12) through the Ricci tensor \( R_{ab}(t) \) and the non negative scalar curvature \( R(t) \). So the metric \( g_t \) measures in fact the effect of the matter contained into the space rather than the geometric volume of this space which is conventionally measured by the Euclidean metric \( g_e \). However, we notice that these tensors, contrary to Riemannian tensors, can admit singularities that are essentially related to
the phenomenon of collapsing of matter generating black holes. The equation (12) is written as

$$Rg_{ab} = 2(R_{ab} - T_{ab}^*) =: G_{ab}^*.$$  \hspace{1cm} (13)

It contains, at every time $t$, all geometrical, physical and cosmological features of the universe.

If we assume that $G_{ab}^*$ vanish on a ball $B := B(I, r) \subset B(O, t)$, then

$$R_{ab} = 0$$

is equivalent to $T_{ab}^* = 0$ on $B$, but, according to the definition of the tensor $T_{ab}^*$, the relation $T_{ab}^* = 0$ is equivalent to $g_{ab} = g_e$ on $B$ and then we have $R = 0$. If $R \neq 0$ on a neighborhood of $I$ in $B$, we have $g_{ab} = \frac{1}{R}G_{ab}^* = 0$ on this neighborhood and then $R = 0$, which is absurd. Likewise, $R$ can not vanish at any isolated point in $B$ (at $I$ for example) and even on any curve of empty interior in $B$ (which passes through $I$ for example) since, out of this point or this curve, we would have $R \neq 0$, which also is impossible. Therefore if $G_{ab}^* = 0$, $g_{ab}$ can not be a non vanishing $(0,2)$-tensor of class $C^2$ on $B$ with $R \neq 0$. However, $g_{ab}$ can then be considered as being a distribution whose support is included in $I$ (for symmetry reason) and having the form $k\delta_IG_e$ where

$$\delta_IG_e(X(I), Y(I)) = g_e(X(I), Y(I))$$

for any two tangent vectors $X(I)$ and $Y(I)$ at $I$ and

$$\delta_IG_e(X(P), Y(P)) = 0$$

for $P \in B$, $P \neq I$ and any two tangent vectors $X(P)$ and $Y(P)$ at $P$.

Actually, this situation corresponds to the fact of a formation of a black hole created by a complete concentric collapsing of a material agglomeration having a very large volume density of its inertial mass, which is expressed by a very large intensity of central self gravity. So $g_{ab}$ can not be null and therefore $supp(g_{ab})$ is reduced to the center of mass $I$ and we have (as we will show it below):

$$g_{ab} \simeq (V_e(B) - E)\delta_IG_e,$$

$$dv_g \simeq (V_e(B) - E)\delta_I,$$

$$dv_g(I) \simeq V_e(B) - E \quad \text{and} \quad E_t(X) \simeq E\delta_I$$

where $\delta_I$ is the Dirac mass at $I$ and $E$ is the equivalent mass energy to the global inertial mass $m$ of the agglomeration just before the last phase of the collapsing. The point $I$ is the center of a central gravity which absorbs all particles reaching $B$. Thus the black hole $B$ constitutes, in some meaning, a region of total absorption of matter as well as of electromagnetic waves.
(a region of no escape). In other words, $B$ is the Schwarzschild ball that is characterized by the fact that (roughly speaking) no signal can be emitted from it. The total potential mass energy concentrated in $I$ of a black hole $B$ is, approximately, $E \sim m$. At $I$, we can consider the matter-energy volume density and the curvature as being infinite. $I$ is a singularity of the physical space $B(O, t)$.

Moreover we neither can have $g_{ab}$ of class $C^2$ with $R \neq 0$ such as one of the eigenvalues $\lambda_i(t, X)$ ($i = 1, 2, 3$) of $G^*_{ab}$, being null (resp. two among them being null) on a ball $B(I, r)$ since we then would have $\text{vol}(B, g_{ab}) = 0$ and $g_{ab}$ would be, using a $g_e$-orthonormal basis of eigenvectors, of a diagonal form like

$$\frac{\lambda_1(t, X)}{R} dx_1^2 + \frac{\lambda_2(t, X)}{R} dx_2^2$$ (resp. $\frac{\lambda_1(t, X)}{R} dx_1^2$);

which constitutes a physical phenomenon that is incompatible with the (intuitive) local isotropy and homogeneity principles. So, each of these cases implies the vanishing of all eigenvalues on $B$ and leads again to the identity $G^*_{ab} = 0$ on $B$ and to the same absurdity. We are again in the situation of a static Schwarzschild black hole or other types of black holes.

Coming back to the universe $U(t)$, let us consider a material agglomeration filling a connected region included in $B(O, t)$. Let $m(t) \sim e(t)$ be its global inertial mass or equivalently its global potential matter energy. Now we propose to distinguish between two kinds of gravity inherent in this agglomeration of matter: the internal gravity (responsible together with other interaction forces, of the cohesion or the non dispersion of the agglomeration) defined within this region and the external gravity defined around this region. For the universe at $t = 0$, having the total inertial mass $M_0$ converted into the total energy $E_0$ (supposed to be finite as it is generally admitted by physicists), the totality of gravity before the Big Bang is internal. For the center $I$ of a static black hole $B(I, r)$, internal gravity is concentrated in $I$ and external one is defined, in extremal form, within $B$ and extended around $B$ as classical newtonian gravity. For an isolated material system such as, for instance, a galaxy with its significant gravitational extent, having a global inertial mass $m$, the global gravity of this system is essentially internal, whereas if we consider any star of this galaxy, then we have to distinguish between its internal gravity and its external one within this system. This is still true for all scale of material formations. We will show later on that internal gravity is strongly related to the binding energy and binding forces. Likewise, we must distinguish, in the dynamical universe, between kinetic energy and potential mass energy of a material system when it is moving...
with any speed \( v(t) < 1 \). The first one is in fact equal to

\[
E_k(t) = \frac{1}{2} m(t)v^2(t)
\]

where \( m(t) \) is its inertial mass at time \( t \). For the universe at \( t = 0 \), the kinetic energy is null and the potential mass energy is \( E_0 \sim M_0 \). For an isolated material system such as a galaxy moving with a speed \( v \), we have \( E_k = \frac{1}{2}mv^2 \) (where \( m \) is the global inertial mass) and the total energy is \( E = m(1 + \frac{1}{2}v^2) \).

Recall that, for any freely moving particle, we have (in a virtual fixed frame)

\[
\tilde{\Gamma}(t) = \nabla_{X'} g_{ab} X''(t) = 0,
\]

where \( \Gamma(t) = \nabla_{X'} X'(t) = X''(t) = 0 \) if and only if \( X(t) \) is the coordinate vector of any constant velocity trajectory in any inertial frame.

### Mass and Energy distributions

We assume now that the inertial mass distribution of matter in the universe \( B(O, t) \) is given by \( m_t(X) \) to which we associate the measure \( dm_t =: \rho_t \). We denote by \( g_t = g_{ab}(t, X) \) the Riemannian metric on \( B(O, t) \) which characterizes the real physical universe reflecting all effects of this distribution and by

\[
\mu_t := d\nu_{g_t} =: \nu_t(X)dX
\]

the measure of density \( \nu_t(X) \) with respect to the Lebesgue measure \( dX \) on \( B(O, t) \). The global inertial mass of the universe at every time \( t \) is given by

\[
M(t) = \int_{B(O,t)} \rho_t := \int_{B(O,t)} m_t(X)dX
\]

On the other hand, we denote by \( E(t, X) = E_t(X) \) the distribution of generalized potential energy which includes, by definition, all manifestations and effects of matter (material distribution \( m_t(X) \), pure energy of black holes, gravity, electromagnetism and interaction forces). Then we denote by \( \nu_t = E_t(X)dX \) the measure associated with this distribution. On the base of all preceding data, we can state

\[
\nu_t = dX - \mu_t.
\]

This equality expresses the fact that \( \nu_t \) measures the failure of the real physical volume of a domain in \( U(t) \), containing matter-energy distribution, to
be equal to the spatial Euclidean volume of this domain when it is supposed
to be empty. This equality can also be written as

$$\mu_t = dX - \nu_t \quad \text{or} \quad \nu_t(X) = 1 - E_t(X)$$

which, in that way, expresses that $\mu_t$ measures in fact the real physical volume
of this domain taking into account the modification of Euclidean distances
imposed by the metric $g_t$ which itself reflects the existence of matter in that
domain. We have naturally

$$\rho_t \leq \nu_t \leq dX \quad \text{or} \quad m_t(X) \leq E_t(X) \leq 1.$$

All of this is confirmed by our explicit calculation of the metrics corresponding
to both uniform and central gravitation studied in section 4. Moreover
these relations justify the metric $g_{ab}$ characterization previously established
for black holes.

The equivalence principle and the law of energy conservation give

$$E(t) = \int_{B(O,t)} E_t(X)dX = \int_{B(O,t)\setminus \bigcup_{\alpha \in A} B_{\alpha}} E_t(X)dX + \sum_{\alpha \in A} e_{\alpha}$$

$$= E(0) =: E_0 \sim M_0$$

where $e_{\alpha}$ denotes the $B_{\alpha}$ black hole energy for $\alpha \in A$.

Let us consider, for a while, the space and time half-cone

$$C = \{(x, y, z, t) \in \mathbb{R}^4; \ x^2 + y^2 + z^2 \leq t^2, \ t \geq 0\} = \bigcup_{t \geq 0} B(O, t) \times \{t\}$$

equipped with the metric $\eta$ given by

$$\eta(t, X) = dt^2 - g_e(X)$$

which is induced by the Minkowski metric defined on the virtual classical
space $\mathbb{R}^4$. Within this half-cone occurs the dynamic creation of real geometrical temporal space, expanding permanently, $B(O, t)$, and making up the
real physical space $U(t) = (B(O, t), g_t)$ always in expansion. We notice that,
in the interior of $C$, $\eta$ is a Riemannian metric.

We consider then the generalized potential energy function $E(t, X) = E_t(X)$,
for $X \in B(O, t)$ and we assume that $E$ is continuous on $C$ and that all its
partial derivatives of order $\leq 2$ exist and are continuous and bounded on
$C^* = C \setminus \{O\}$. The global potential energy of the universe at time $t$ is then written

$$E(t) = \int_{B(O,t)} E(t, X)dX = \int_0^t dr \int_{S(O,r)} E(r, X)d\sigma_r = E_0$$
then
\[ E'(t) = \int_{S(O,t)} E(t, X) d\sigma_t =: S(t) = 0, \]
for any \( t > 0 \). So, the function of time \( S \) is given by \( E_0 \delta_{\mathbb{R}^+} \) and we have
\[ E_0|_{S(O,t)} = 0 \text{ for } t > 0. \]

Notice that, in the interior of a black hole \( B(I, r) \subset B(O,t) \), when it exists, we have
\[ E_t(X) = e(I)\delta_I = m(I)\delta_I, \]
where \( e(I) \) is the potential mass energy which is equivalent to the inertial mass \( m(I) \). Thus \( e(I) \) is a part of the inertial energy \( E_0 \) of the original universe that has been reconcentrated at a given time (after the Big Bang) at \( I \); whereas \( E(t, X) = e(I)\delta_I \) denotes the generalized potential mass energy function on \( B(I, r) \). Notice also that the function \( E_t(X) \) is null in any region of the universe that can be considered as deprived of matter and of its effects. Finally we mention that, although the function \( E(t, X) \) is far from being of class \( C^2 \), we however can reasonably approach it by such a function (idealizing in such a way the universe) or consider it, as well as their partial derivatives of order \( \leq 2 \), in the distributional sense.

**Matter-Energy equation**

We now consider a part \( C_1 \) of \( C \) located between \( t = t_1 \) and \( t = t_2 \) for \( t_1 < t_2 \). We have \( \partial C_1 = B(O, t_1) \cup B(O, t_2) \cup S_1 \) where \( S_1 \) is the lateral boundary of \( C_1 \). We then consider the force field \( F_\eta \) defined on \( C^* \) and deriving from the global potential function \( E(t, X) \), i.e.
\[ F_\eta(t, X) := -\nabla^\eta E(t, X) := -\text{grad}_\eta E(t, X). \]

So, if \( u(t) = (t, X(t)) \) is a given trajectory in \( C \) and if \( F_\eta(u(t)) = 0 \), then we have
\[
\eta(F_\eta(u(t)), u'(t)) = 0 \iff \eta(\nabla^\eta E(u(t)), u'(t)) = 0 \\
\iff dE(u(t)).u'(t) = 0 \iff \frac{d}{dt} E(u(t)) = 0 \iff E(u(t)) = \text{const.}
\]
Likewise, the identity
\[ ||F_\eta(u(t))||_\eta = ||\nabla^\eta E(u(t))||_\eta = 0 \]
is equivalent to 
\[ dE(u(t)) \cdot \nabla^\eta E(u(t)) = 0 \]
and to 
\[ \frac{\partial E}{\partial t}(u(t))^2 = |\nabla^g E_t(X(t))|^2 = \sum_{i=1}^{3} \frac{\partial E}{\partial x_i}(u(t))^2. \]
On the other hand we have
\[ ||u'(t)||_{\eta} = 0 \Leftrightarrow ||(1, X'(t))||_{\eta} = 0 \Leftrightarrow |X'(t)| = 1, \]
which means that the Euclidean speed is equal to 1 and then we get
\[ |X(t) - X(t_0)| = t - t_0 \text{ for } t \geq t_0 \geq 0 \]
and the trajectory in \( C \) is reduced to a ray of a light cone surface.

Let us denote by \( d\eta \) the measure associated with \( \eta \) in \( C \) and by \( \Delta_\eta \) the Laplace-Beltrami operator on \( C \) (\( d\eta \) is a measure of density \( f(t, X) \geq 0 \) with respect to the Lebesgue measure on \( C \) with \( f(t, X) = 0 \) on \( \partial C \)). According to Stokes theorem, we have (where \( S_1 \) is the lateral frontier of \( C_1 \) and \( \vec{n} \) is the normal vector to \( S_1 \)) :
\[
\int_{C_1} \Delta_\eta E(t, X) d\eta = \int_{C_1} \text{div}_\eta(\nabla^\eta E(t, X)) d\eta \\
= \int_{B(O, t_2)} \eta(\nabla^\eta E(t_2, X), \frac{\partial}{\partial t}) f(t_2, X) dX \\
- \int_{B(O, t_1)} \eta(\nabla^\eta E(t_1, X), \frac{\partial}{\partial t}) f(t_1, X) dX + \int_{S_1} \eta(\nabla^\eta E(t, X), \vec{n}) d\eta \\
= \int_{B(O, t_2)} \frac{\partial E}{\partial t}(t_2, X) f(t_2, X) dX - \int_{B(O, t_1)} \frac{\partial E}{\partial t}(t_1, X) f(t_1, X) dX.
\]
Thus we obtain :
\[
\int_{C_1} \Delta_\eta E(t, X) d\eta = \int_{t_1}^{t_2} dt \int_{B(O, t)} \Delta_\eta E(t, X) f(t, X) dX \\
= \int_0^{t_2} dt \int_{B(O, t)} \Delta_\eta E(t, X) f(t, X) dX - \int_0^{t_1} dt \int_{B(O, t)} \Delta_\eta E(t, X) f(t, X) dX \\
= \int_{B(O, t_2)} \frac{\partial E}{\partial t}(t_2, X) f(t_2, X) dX - \int_{B(O, t_1)} \frac{\partial E}{\partial t}(t_1, X) f(t_1, X) dX = F(t_2) - F(t_1)
\]
where 
\[ F(t) := \int_{B(O, t)} \frac{\partial E}{\partial t}(t, X) f(t, X) dX \]
Therefore
\[ \int_{B(O,t_2)} \Delta \eta E(t_2, X)f(t_2, X) dX = F'(t_2) \]
and
\[ \int_{B(O,t_1)} \Delta \eta E(t_1, X)f(t_1, X) dX = F'(t_1) \]
Then we have
\[ \int_{B(O,t)} \Delta \eta E(t, X)f(t, X) dX = F'(t) \]
for \( t > 0 \); This equality can be written as
\[ \int_0^t dr \int_{S(O,r)} \Delta \eta E(r, X)f(r, X) d\sigma_r = \frac{d}{dt} \int_{B(O,t)} \frac{\partial E}{\partial t}(t, X)f(t, X) dX \]
which implies
\[ \int_{S(O,t)} \Delta \eta E(t, X)f(t, X) d\sigma_t = \frac{d^2}{dt^2} \int_{B(O,t)} \frac{\partial E}{\partial t}(t, X)f(t, X) dX \]
\[ = F''(t) = \frac{d^2}{dt^2} \int_0^t dr \int_{S(O,r)} \frac{\partial E}{\partial r}(r, X)f(r, X) d\sigma_r \]
\[ = \frac{d}{dt} \int_{S(O,t)} \frac{\partial E}{\partial t}(t, X)f(t, X) d\sigma_t = 0 \]
for \( t > 0 \).
So, we have
\[ F'(t) = \int_{B(O,t)} \Delta \eta E(t, X)f(t, X) dX = \text{const.} \]
which implies
\[ \int_{C_1} \Delta \eta E(t, X) d\eta = a(t_2 - t_1) \]
with \( a = F'(t) \) and then we get, for \( C(t) = \{ (x,y,z,r) \in \mathbb{R}^4 ; x^2 + y^2 + z^2 \leq r^2, \quad 0 \leq r \leq t \} \) :
\[ \int_{C(t)} \Delta \eta E(t, X) d\eta = at. \]
Using the change of variables
\[ (t, X) \to (\lambda s, \lambda X) \quad \text{for} \quad \lambda > 0 \quad \text{in} \ C(t), \]
we obtain
\[ \int_{C(s)} \Delta_\eta E(\lambda s, \lambda X) \lambda^4 d\eta = a\lambda s \]
or
\[ \int_{C(s)} \Delta_\eta E(\lambda(s, X)) d\eta = \frac{a}{\lambda^3} s. \]

Now, we have
\[ \int_{C(s)} \Delta_\eta E(s, X) d\eta = a \]
and then
\[ \int_{C(s)} \Delta_\eta E(s, X) d\eta = \lambda^3 \int_{C(s)} \Delta_\eta E(\lambda(s, X)) d\eta \]
which implies
\[ \int_{C(s)} \Delta_\eta E(s, X) d\eta = a s = 0 \]
and then
\[ a = 0, \quad F(t) = \text{const.} \quad \text{for } t > 0 \]
So, we finally have
\[ \int_{C_1} \Delta_\eta E(t, X) d\eta = 0 \quad \text{for any } t_1 \text{ and } t_2 \text{ satisfying } 0 < t_1 < t_2. \]

Let us now show that
\[ \Delta_\eta E(t, X) = 0 \quad \text{on } C^*. \]
Indeed, if we suppose that \( \Delta_\eta E(t_0, X_0) > 0 \), for example, then we would have \( \Delta_\eta E(t, X) > 0 \) on a neighborhood \( B \) of \( (t_0, X_0) \) in \( \{t_0\} \times B(O, t_0) \). Let us then consider the union \( C' \) of all future causal half-cones having their vertices on \( B \).

Now, by reasoning in the same way as above on a part \( C'_1 \) of \( C' \) located between \( t_1 \) and \( t_2 \) such that \( t_0 < t_1 < t_2 \), we can show that
\[ \int_{C'_1} \Delta_\eta E(t, x) d\eta = 0, \quad \text{for any } t_1 < t_2 \]
and then
\[ \int_{C'_1} \Delta_\eta E_B(t_0, X) f(t_0, X) dX = 0 \]
(the contribution of each half-cone being null) which is contradictory since \( \Delta_\eta E(t_0, X) \) is supposed to be continuous and positive on \( B \). Therefore we have
\[ \Delta_\eta E(t, X) = 0 \quad \text{on } C^* \quad (14) \]
Thus for $t > 0$, we obtain

$$\frac{\partial^2}{\partial t^2} E(t, X) - \Delta E(t, X) = 0 \quad (E^*)$$

This same result can be obtained, outside of the black holes or other singularities, supposing some regularity of the distribution $E(t, X)$ on a region of $C$ resulting from excluding small neighborhoods of black holes evolving with the expansion. If $B(I_\alpha(t), r)$ is such a black hole included in $B(O, t)$, for $t \in [t_1, t_2]$, and if we assume that $\sum_{\alpha \in A} c_\alpha(t)$ is negligible compared to $E_0(t)$ or, simply, is constant on $[t_1, t_2]$, we can apply the conservation of energy principle on this region of $C$ to obtain the equation $(E^*)$ on $B(O, t) \setminus \bigcup_{\alpha \in A} B_\alpha$.

We now consider the dynamical real universe $(U(t))_{t>0} = (B(O, t), g_t)_{t>0}$ and the half-cone $C$ equipped with the Riemannian metric $h$ defined by

$$h(t, X) = dt^2 - g_t(X) = dt^2 - g(t, X)$$

where $g_t$ is the metric $g_{ab}$ defined on $B(O, t)$ by equation $(\mathcal{E})$. We denote by $\beta$ the measure associated to $h$ in $C^*$ and we apply Stokes theorem on any part $C_1$ of $C^*$ (defined in the same way as before), then we can follow the same steps as above and use the light half-cones within $C$ whose rays are adapted to the metrics $g_t$ and $h$ (i.e. geodesics for $h$). After that, we can replace into the preceding results $d\eta$ by $d\beta$, $dX$ by $d\mu$, $\Delta_\eta$ by $\Delta_h$ and $F_\eta(t, X_i)$ by $F_h(t, X_i) := -\nabla^h E(t, X_i) := -\text{grad}_h E(t, X_i)$, for $X_i \in B(O, t)$, to obtain with the same hypotheses as above:

$$\int_{B(O, t)} \Delta_h E(t, X) k(t, X)dX = 0 \quad \text{for } t > 0,$$

with $k(t, X) \geq 0$ on $C$ and $k(t, X) = 0$ on $\partial C$,

$$\int_{C_1} \Delta_h E(t, X) d\beta = 0 \quad \text{for } 0 < t_1 < t_2,$$

$$\Delta_h E(t, X) = 0 \quad \text{on } C^*$$

and

$$\frac{\partial^2}{\partial t^2} E(t, X) - \Delta g_t E(t, X) = 0 \quad (15)$$

on $B(O, t)$ with $E(t, X)|_{S(O, t)} = 0$ for any $t > 0$.

We notice that the two equations $(E^*)$ and (15) are perfectly consistent.
Indeed the validity of the punctual equality $\Delta_{g_{t}}E_{i}(X) = \Delta_{g_{t}}E_{i}(X)$ is obvious as the left hand side is the very expression of the right hand side when using the normal Riemannian coordinates (with respect to the metric $g_{t}$) at any point $X \in B(O,t)$

**Force field, Acceleration and Geodesics**

Moreover, let us define

$$F_{g_{t}}(X) := -\nabla^{g_{t}}E_{i}(X)$$

and

$$F_{g_{e}}(X) := -\nabla^{g_{e}}E_{i}(X)$$

for $X \in B(O,t)$ and then consider a trajectory $X(t)$ in the dynamical universe $(U(t))_{t>0}$. Then

$$\frac{d}{dt}E(u(t)) = \frac{d}{dt}E(t,X(t)) = \frac{\partial E}{\partial t}(t,X(t)) + \frac{d}{ds}E_{t}(X(s))|_{s=t}$$

$$= \frac{\partial E}{\partial t}(t,X(t)) + dE_{t}(X(s)).X'(s)|_{s=t} = \frac{\partial}{\partial t}E(t,X(t)) + dE_{t}(X(t)).X'(t)$$

Moreover

$$\eta(\nabla^{h}E(u(t)), u'(t)) = \frac{d}{dt}E(u(t)) = h(\nabla^{h}E(u(t)), u'(t))$$

$$= \frac{\partial E}{\partial t}(u(t)) - g_{t}(\nabla^{g_{t}}E_{t}(X(t)), X'(t)) = \frac{\partial E}{\partial t}(u(t)) - g_{e}(\nabla^{g_{e}}E_{t}(X(t)), X'(t)).$$

Therefore

$$g_{t}(F_{g_{t}}(X(t)), X'(t)) = g_{e}(F_{g_{e}}(X(t)), X'(t))$$

$$= \frac{d}{dt}E(t,X(t)) - \frac{\partial E}{\partial t}(t,X(t)) = dE_{t}(X(t)).X'(t) = \frac{d}{ds}E_{t}(X(s))|_{s=t};$$

which implies, according to the generalized fundamental law of Dynamics,

$$F_{g_{e}}(X(t)) = -\nabla^{g_{e}}E_{i}(X(t)) = \nabla^{g_{e}}_{X'(t)}X'(t) = \Gamma(t) = X''(t)$$

and

$$F_{g_{t}}(X(t)) = -\nabla^{g_{t}}E_{i}(X(t)) = \nabla^{g_{t}}_{X'(t)}X'(t) = \tilde{\Gamma}(t),$$

the last two identical vectors being 0 for free motion. Thus, $F_{g_{e}}(X)$ is the global natural forces’ vector field. The force acting on a punctual material particle of mass $m_{t}$ located at $X$ actually is $m_{t}F_{g_{e}}(X) \equiv$
\(m_t \nabla^g E_t(X)\). For a moving material particle, the force is \(m(X(t)) F_{ge}(X(t)) = m(X(t)) \Gamma(t)\).

In particular,

\[X(t) \text{ is a geodesic for } g_e, \text{ with } ||X'(t)||_{ge} = 1\]

if and only if \(u(t)\) is a geodesic for \(\eta\) with \(||u'(t)||_{\eta} = 0\)

and then \(u(t)\) is a ray of a light cone surface in \((C, \eta)\), \(E(u(t)) = \text{const.}, \ F_{\eta}(u(t)) = 0\)

and

\[F_{ge}(X(t)) = -\nabla^g E_t(X(t)) = \Gamma(t) = X''(t) = 0.\]

Likewise,

\[X(t) \text{ is a geodesic for } g_t, \text{ with } ||X'(t)||_{gt} = 1\]

if and only if \(u(t)\) is a geodesic for \(h\) with \(||u'(t)||_{h} = 0\)

and then \(u(t)\) is a light ray in \((C, h)\) and we have \(E(u(t)) = \text{const.}, F_{h}(u(t)) = 0\)

and

\[F_{gt}(X(t)) = -\nabla^g E_t(X(t)) = \nabla^g_{X'(t)} X'(t) = \tilde{\Gamma}(t) = 0.\]

Recall that, the statement

\[F_{gt}(X(t)) = \tilde{\Gamma}(t) = 0\]

for any free motion in \((U(t))_{t>0}\) constitutes a generalization of the two fundamental Newtonian laws of inertia.

Moreover, we notice that, if we assume the virtual existence (purely theoretical) of the space \(\mathbb{R}^3\) and the half-cone of space and time before their real, physical and temporal existence starting with the Big Bang, which are modeled much later by Euclid and Descartes and afterwards by Galileo and Newton, then we can consider the original universe as being \(E_0 \delta_{\mathbb{R}^3}\) and \(E_0 \delta_C\).

Otherwise, after the Big Bang, we have, in the half-cone of space and time, \(E(t, X) = 0\) on \(\partial C^*\) and \(E(0, 0) = E_0\). Likewise, we have \(\Delta_{\eta} E(t, X) = 0\) on \(C^*\) and \(\Delta_{\eta} E = E_0 \Delta_{\eta} \delta_C\) where we have denoted, for a class \(C^2\) function \(\varphi(t, X)\) with compact support in \(C\) :

\[< \Delta_{\eta} \delta_C, \varphi(t, X) > = \lim_{t \to 0} \Delta_{\eta} \varphi(t, 0).\]
Conclusions

According to the preceding results, we conclude that our real, geometrical, physical and dynamical universe, modeled by \((U(t))_{t>0} = (B(O,t), g_t)_{t>0}\) is characterized by each of the equivalent following notions:

a) The material distribution of (the density of) the inertial mass \(m_t(X)\) together with the \(e_t(t)\delta_{t(t)}\) corresponding to the black holes scattered into \(B(O,t)\).

b) The regularized scalar field of the total potential energy represented by the distribution of generalized potential energy \(E_t(X)\).

c) The Riemannian regularized metric \(g_t\) on \(B(O,t)\).

d) The modified (matter-energy) tensor \(T^{*}_{ab}\).

e) The physical measure \(\mu_t\) on \(B(O,t)\) given by \(\mu_t = dV_t = v_t(X)dX\) where \(dX\) is the Lebesgue measure on \(B(O,t)\).

f) The measure \(\nu_t = E_t(X)dX\) of density \(E_t(X)\) with respect to the Lebesgue measure on \(B(O,t)\).

g) The vector field \(\nabla_h E_t(t,X)\) on \(C\), where \(h = dt^2 - g_t\) is the Riemannian metric on the half-cone of space and time.

h) The total and global vector field \(\nabla^g E_t(X)\) which is equal to the total and global force vectorfield \(F_{gt}(X)\) and is given by \(-\nabla^g E_t(X) = F_{gt}(X) = \Gamma_t = X''(t)\), for any material particle’s free motion into \((U(t))_{t>0}\).

k) The set of all dynamical geodesics \(X(t)\) for the metric \(g_t\) evolving with time (i.e. satisfying \(\tilde{\Gamma}(t) = \nabla_{X(t)} X'(t) = F_{gt}(X(t)) = -\nabla^g E_t(X(t)) = 0\)).

Let us now notice that our model is definitely consistent in the sense that, on one hand, it proves, a posteriori, the legitimacy of all mechanical and physical principles which are discovered and stated by all great scientists of the humanity, and on the other hand we have \(E_t(X) \equiv 0\) on a domain \(D\) of the dynamical universe between \(t = t_1\) and \(t = t_2 \Leftrightarrow \nu_t = 0 \Leftrightarrow T^{ab}_{\mu t} \equiv 0\) on the domains \(D_t\) in \(B(O,t)\) corresponding to \(D \Leftrightarrow g_t = g_{\nu t}\) in \(D_t \Leftrightarrow \mu_t = dX\) on \(D_t \Leftrightarrow h = \eta\) on the domain of \(C\) that corresponds to \(D \Leftrightarrow \Gamma_t = X''(t) \equiv 0\) for free motion in \(D\) \(\Leftrightarrow\) All trajectories corresponding to free motion in \(D\) are geodesics constituted of straight lines.

Remark

The property, noticed at p.46, of our physical metric \(g_t\) shows that our space-time provided with our metric \(h_t = dt^2 - g_t\) satisfies the three (slightly modified) postulates of the ”metric theories” which stipulate that:
(i) the space-time is provided with a metric,

(ii) the free falling bodies trajectories are geodesics.

(iii) In any local referential frame, the non gravitational Physics’ laws are the same as classical Physics’ laws (without special relativity considerations). This shows that our gravitational model satisfies the Einstein’s equivalence principle.

We end this section by noticing that, in our model, the geometrical space does not exist around the point of concentration of the original energy $E_0$ before the Big Bang, whereas the geometrical space $B(I, r)$ exists actually around the point of concentration $I$ of the energy of a black hole, but this space is equipped with a null metric outside $I$. Conversely, when a domain of the geometrical space of the universe $B(O, t)$ does not contain matter nor its effects, then this part of space has a very existence and is equipped with the Euclidean metric $g_e$. The gravitational field in this case is vanishing, but around the center $I$ of a black hole it is omnipresent.

6 Energy, Pseudo-waves and Frequencies

Let us now consider the wave equation (which is the matter-energy equation):

$$\Box E(t, X) = \frac{\partial^2 E}{\partial t^2}(t, X) - \Delta E(t, X) = 0 \quad (E^*)$$

and (using the separation of variables method) let us determine, for any $t > 0$, the solutions on $B(O, t)$ that satisfies $E(t, X)|_{S(O,t)} = 0$. So we consider the functions of the form

$$E_0(t, X) = f_0(t)F_0(X) \text{ for a fixed } t_0 > 0$$

satisfying

$$\Box f_0(t)F_0(X) = 0 \quad \text{with} \quad F_0(X)|_{S(O,t_0)} = 0 \quad (E_0)$$

It is well known that solutions $f_0$ and $F_0$ of equation ($E_0$) are obtained from the increasing sequence $\lambda_i(t_0)$ of eigenvalues of the Laplace-Beltrami
operator $-\Delta$ and the sequence of eigenfunctions $\varphi_{t_0,i}(X)$ that are associated with the Dirichlet problem on the ball $B(O,t_0)$ provided with the metric $g_e$. The corresponding solutions, $f_{0,i}(t)\varphi_{t_0,i}(X)$, are defined by

$$\Delta \varphi_{t_0,i}(X) = -\lambda_i(t_0)\varphi_{t_0,i}(X)$$

and

$$f''_{0,i}(t) + \lambda_i(t_0) f_{0,i}(t) = 0.$$

We pick out one of these solutions which will be denoted by

$$f_0(t)\varphi_{t_0}(X);$$

and then we get, for $0 \leq t \leq t_0$:

$$f''_0(t) + \lambda(t_0) f_0(t) = 0.$$ 

Moreover, if $\mu$ is the eigenvalue of the Dirichlet problem on the unit ball $B(O,1)$ having the same rank as $\lambda(t_0)$, we have

$$f''_0(t) + \frac{\mu}{t_0^2} f_0(t) = 0, \quad 0 \leq t \leq t_0 \quad (t_0 > 0).$$

The solution of this equation is obviously the periodic function

$$f_0(t) = f_0(0) \cos \frac{\sqrt{\mu}}{t_0} t + \frac{t_0}{\sqrt{\mu}} f'_0(0) \sin \frac{\sqrt{\mu}}{t_0} t.$$

The solution of $(E_0)$ corresponding to the eigenvalue $\lambda(t_0) = \frac{\mu}{t_0^2}$ is defined as

$$E_\mu(t,X) = (f_0(0) \cos \frac{\sqrt{\mu}}{t_0} t + \frac{t_0}{\sqrt{\mu}} f'_0(0) \sin \frac{\sqrt{\mu}}{t_0} t)\varphi_{t_0}(X) \quad (16)$$

for $X \in B(O,t_0)$, $t_0 > 0$ and $0 \leq t \leq t_0$.

Now, if $h_\mu(t)\psi_\mu(X)$ is the solution of the Dirichlet problem on the ball $B(O,1)$ associated with the eigenvalue $\mu$, we have, for $t_0 > 0$:

$$f_\mu(t) = h_\mu(\frac{t}{t_0}) \quad \text{and} \quad \varphi_{t_0,\mu}(X) = \psi_\mu(\frac{X}{t_0})$$

and then

$$E_\mu(t,X) = (h_\mu(0) \cos \frac{\sqrt{\mu}}{t_0} t + \frac{1}{\sqrt{\mu}} h'_\mu(0) \sin \frac{\sqrt{\mu}}{t_0} t)\psi_\mu(\frac{X}{t_0}).$$

$E_\mu(t,X)$ is then a periodic function of period $T_0 = 2\pi \frac{t_0}{\sqrt{\mu}}$. 

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Remark: We could also consider $E(t, X) = k_0(t)G_0(X)$ as a solution of the equation (15). One solution corresponding to an eigenvalue $\alpha(t_0)$ associated to the Laplace-Beltrami operator $-\Delta_{g_0}$ on the Riemannian manifold $(B(O,t_0),g_{t_0})$ would be then of the form

$$E(t, X) = k_0(t)\theta_{t_0}(X)$$

with

$$k_0''(t) + \alpha(t_0)k_0(t) = 0$$

and

$$\Delta_{g_0}\theta_{t_0}(X) = -\alpha(t_0)\theta_{t_0}(X)$$

However, in that case, we can not reduce the considered problem to the study of the Dirichlet problem on the space $(B(O,1),g_1)$ unless the function $X \mapsto \frac{X}{t_0}$ would be an isometry from $(B(O,t_0),g_{t_0})$ on $(B(O,1),g_1)$, which is, in the most favorable cases, a crude approximation.

In fact, the above solutions $E_{\mu}$ can not be assimilated to the solutions of our matter-energy equation on the dynamical universe $(U(t))_{t>0}$, i.e. equation $(E^*)$, but only on an interval of time, $t_1 \leq t < t_0$ with $t_0 > t_1$ and $t_1$ sufficiently (and relatively) close to $t_0$ in such a manner that the eigenfunctions $\varphi_t(X)$ and eigenvalues $\lambda(t)$ corresponding to $t \in [t_1, t_0]$ can be respectively considered as being reasonable approximations of $\varphi_{t_0}(X)$ and $\lambda(t_0)$. Moreover, in order to obtain a reasonable periodic approximation of equation $(E^*)$ on $(U(t))_{t_1 \leq t \leq t_0}$, it is necessary that the period $T_0 = 2\pi \frac{\sqrt{\mu}}{\sqrt{t_0-t_1}}$ be significantly less than $t_0 - t_1$, and then $\mu$ be significantly greater than $4\pi^2(\frac{t_0}{t_0-t_1})^2$.

By giving to $t_0$ a large number of convenable values $t_i$, we obtain approximate periodic solutions (of periods $T_i = 2\pi \frac{\sqrt{\mu}}{\sqrt{t_0}}$) to our problem on juxtaposed rings of $B(O,t)$ for $t = \sup_i t_i$. By replacing in (16) $t$ by $t_0$ and rewriting it as a function of the variable $t$ in place of $t_0$, we obtain the solution $E_{\mu}$ of $(E^*)$ defined on $B(O,t)$, for $t > 0$, by

$$E_{\mu}(t, X) = f_{\mu}(t)\varphi_{t,\mu}(X) = \left(f_{\mu}(0)\cos \sqrt{\mu} + \frac{t}{\sqrt{\mu}}f'_{\mu}(0)\sin \sqrt{\mu}\psi_{\mu}\left(\frac{X}{t}\right)\right)$$

where $f_{\mu}(0) = h_{\mu}(0)$ depends on $\mu$ and $f''_{\mu}(0)$ depends on $\mu$ and $t$. This solution can be approximated, on appropriate rings $B(O,t)\setminus B(O,t')$, by periodic functions of periods $T(t) = 2\pi \frac{\sqrt{\mu}}{\sqrt{t}}$ and frequencies $f(t) = \frac{1}{2\pi} \sqrt{\frac{\mu}{t}}$.

Therefore $E_{\mu}$ is a pseudo-wave (that will be incorrectly called wave) of
pseudo-period $T(t) = 2\pi \frac{1}{\sqrt{\mu}}$ and pseudo-frequency $f(t) = \frac{1}{2\pi} \frac{\sqrt{\mu}}{t}$ respectively (both depending on time $t$).

Thus, in order to assimilate these solutions to waves on significant intervals of time, we put $t = e^{\alpha}$, $\mu = 4\pi^2 e^{2\beta}$ and then noticing that we have $T = e^{\alpha - \beta}$, we have to take $e^{\alpha - \beta} << e^\alpha$. So we must take $\beta >> 0$ and therefore the eigenvalue $\mu >> 0$.

When $X(t)$ is a trajectory of a free motion on a given interval of time, i.e.

$$\tilde{\Gamma}(t) = \nabla^g_{\dot{X}(t)}X'(t) = 0 \quad \text{for} \quad t \in I,$$

the wave $E_\mu(t, X(t))$ corresponding to an eigenvalue $\mu$, satisfies the identity

$$-\nabla^g E_\mu(t, X(t)) = F_{g_\mu}(X(t)) = \tilde{\Gamma}(t) = 0.$$

Now, when the metric is Euclidean, the Newton’s inertia principle states that the trajectory $X(t)$ of an arbitrary particle is uniform (i.e. $X(t)$ is a geodesic and the speed $\|X'(t)\|_{g_\mu} = v$ is constant) if and only if the force field acting on the particle,

$$F_{g_\mu}(t) = -\nabla^g E(t, X(t)) = \nabla^g_{\dot{X}(t)}X'(t) = X''(t),$$

is null and then the particle energy is conserved along this trajectory, i.e.

$$E(t) = \text{const}.$$

In the case of our physical metric $g_t$, this same principle is generalized in the following way:

The particle trajectory $X(t)$ is a geodesic with respect to the metric $g_t$ (with $\|X'(t)\|_{g_t} = \text{const}$.) if and only if

$$-\nabla^g E(t, X(t)) = F_{g_t}(t) = \nabla^g_{\dot{X}(t)}X'(t) = \tilde{\Gamma}(t) = 0$$

and then the particle punctual energy is conserved along this geodesic, i.e.

$$E(t, X(t)) = \text{const}.$$

In our setting, we then have

$$E_\mu(t, X(t)) = f_\mu(t)\varphi_{t,\mu}(X(t)) = e(\mu)$$

(where $e(\mu)$ is a constant depending only on $\mu$) and

$$\Delta E_\mu(t, X(t)) = f_\mu(t)\frac{\mu}{t^2}\varphi_{t,\mu}(X(t)) = \frac{\mu}{t^2}e(\mu).$$
For original particles that propagate with the speed 1, we have $X(t) \in S(O,t)$ and $E_\mu(t, X(t)) = 0$; these are the original electromagnetic waves that made up the semi-cone of space and time. For a material particle that propagates along a geodesic (with respect to $g_t$) with a speed $v = \| X'(t) \|_{g_t} < 1$, we have

$$E_\mu(t, X(t)) = e_0(\mu) > 0.$$  

We notice that the photon energy other than the original ones equally are positif.

**Planck-Einstein Energy**

Moreover, by adapting the undulatory principle of Planck-Einstein to our setting, we conclude that, for any material or immaterial point $X$, moving freely in $(U(t))_{t>0}$, we have

$$E_\mu(t, X(t)) = h_\mu(t) f_\mu(t) = h_\mu(t) \frac{1}{2\pi} \frac{\sqrt{\mu}}{t} =: \overline{h}_\mu(t) \frac{\sqrt{\mu}}{t},$$

where $f_\mu(t)$ denotes here the frequency and $\overline{h}_\mu(t)$ replaces, in some way, the Planck constant. This equality implies

$$\overline{h}_\mu(t) = tc(\mu)$$

where $c(\mu)$ is a constant depending only on $\mu$; so

$$E_\mu(t, X(t)) = c(\mu) \sqrt{\mu} =: m(\mu)$$

**Remarks**: 1°) We notice that, as we will see in the next section, this constant indeed depends on time when being considered on a large scale of time. This is due to the perpetual and slight cooling of the cosmos.

2°) Contrary to the Planck constant (which is generally considered as a universal constant), our constant $\overline{h}_\mu(t)$ is proportional to time.

In other respects, let us consider a material agglomeration that is filling up a domain $D_t$ in $B(O, t)$. This domain is subdivided into some subdomains $D_{t,n}$, on each of them is defined an energy distribution $E_n(X_t)$ which coincides, for each domain $D_{t,n}$ that is filled up by a fundamental material particle, with a constant material distribution $m_n(X_t) =: m_n$ (the fundamental particles will be classified in section 8). The energy distributions $E_n(X_t)$ defined on all other subdomains are equally assumed to be constant. We then have

$$D_t = \bigcup_{1 \leq n \leq N} D_{t,n}$$

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with

\[ E_n(X_t) = e_n \sim m_n \text{ for } X_t \in D_{t,n}. \]

So, the energy of the domain \( D_t \) is

\[ E(D_t) = \sum_n \text{vol}(D_{t,n})m_n =: \sum_n V_n(t)m_n, \]

where \( \text{vol}(D_{t,n}) =: V_n(t) \) denotes here the Euclidean volume of the subdomain \( D_{t,n} \).

Obviously, even if \( D_t \) constitutes an isolated system such that \( \text{vol}(D_t) \) remains constant, \( V_n(t) \) evolves with time. This is due to several evolutive dynamical phenomena: Transformation matter-pure energy, radiations of all sorts, disintegration, collision, fission, fusion and chemical, nuclear and thermic interaction.

In the dynamical universe, we have to take into account the kinetic energy of matter into movement. However, we can speak about inertial mass, potential energy, kinetic energy, and linear momentum or quantity of movement (i.e. \( mv \)) only when considering a material point or a material domain moving with a speed inferior to 1.

The kinetic energy of a material point following a trajectory \( X(t) \) is in fact \( \frac{1}{2}m(X(t))X'(t)^2 \). The inertial mass or the potential mass energy of a material domain \( D_t \), at the time \( t \), is given by

\[ \rho_t(D_t) = \int_{D_t} m_t(X_t)dX_t. \]

Its kinetic energy is given by

\[ \frac{1}{2} \int_{D_t} m_t(X_t)v(X_t)^2dX_t. \]

When we are dealing with a domain devoid of matter and crossed everywhere by radiations as electromagnetic waves (visible or invisible light, X rays, \( \gamma \) rays) we can not speak neither about inertial mass nor about potential energy, kinetic energy or quantity of movement. We can only speak about the energy of propagated waves, beams and photons. Nevertheless, we can define, within our model, the linear momentum \( \overrightarrow{p} \) of an electromagnetic wave or photon as being \( p(t) = E(t, X(t)) \overrightarrow{X'(t)} \) and we then have \( p(t) = E(t) \) or \( p = E \) which agrees with the special relativity notion (here \( \|X'(t)\|_g = c = 1 \)).

Thus the kinetic energy of a material point \( X \) with \( m_0(X) = m_0 \), making up a trajectory \( X(t) \) is \( \frac{1}{2}m(X(t))X'(t)^2 \) and its total energy is \( m(t)(1 + \frac{1}{2}X'(t)^2) \) whenever its speed is \(<1 \). However, for a black hole \( B \) having an initial mass...
energy $e$ and moving with any speed $v$, we can consider its total energy as being $e(1+\frac{1}{2}v^2)$. Nevertheless, the distribution $E(t, X)$ is entirely determined by the distribution $m_t(X)$ together with black holes’ energies. That is also the case of the total force field $F_{ge}(X) = \nabla_{ge} E_t(X)$ and the acceleration vector field $\Gamma(X(t)) = X''(t)$ for any free motion. Therefore the metric $g_t$ which is intrinsically related to the distribution $E_t(X)$ and which satisfies $\Gamma(X(t)) = \nabla_{X'(t)} \Gamma'(X(t)) = 0$ (for free motions), takes into account all manifestations and effects of matter, including electromagnetic fields, interaction forces and the resulting binding forces, and not only the gravitational one.

Notice that, the kinetic energy of a system is not necessarily conserved neither globally nor locally as it is shown, for example, by the transformation of a part of the kinetic energy of a system into heat during a collision, for example. Conversely, the principle of the global momentum conservation for an isolated system such as the whole universe is valid. We then have at any time $t$:

$$\int_{B(O,t)} E_t(X_t) v(X_t) dX_t = 0.$$ 

where $v(X_t)$ denotes here the velocity vector.

In particular the center of gravity of the universe is fixed. This relation is written, for $B(O, t) = \bigcup_n D_n(t)$, as

$$\sum_n \int_{D_n(t)} m_n v(X_t) dX_t = 0.$$ 

When, for an isolated domain $D(t) = \bigcup_n D_n(t)$ we assign to the center of mass $G_n(t)$ of each $D_n(t)$ the resultant speed vector $v_n(t)$, we obtain

$$\sum_n V(D_n(t)) E_{t,n}(G_n(t)) v_n(t) = a.$$ 

or

$$\sum_n V_n(t) m_n v_n(t) = a.$$ 

where $a$ is a constant vector.

When a particle of mass $m$ is subject to constant exterior force and is filling a domain $D = \bigcup_n D_n(t)$ and moving at a speed vector $v(t)$, we have

$$\sum_n V_n(t) m_n v_n(t) = m v(t).$$ 

For an atom, for example, of mass $m$ and speed $v$ having $k_1$ electrons and $k_2$ quarks filling respectively the volumes $V_{1,i}$ and $V_{2,j}$ and having respectively
the speeds $v_{1,i}$ and $v_{2,j}$, we have
\[
\sum_{i=1}^{k_1} V_{1,i} m_{1,i} v_{1,i}(t) + \sum_{j=1}^{k_2} V_{2,j} m_{2,j} v_{2,j}(t) = m v.
\]

The solar system equally obeys the same scheme.

Indeed, let us assume that the solar system with its $N$ planets is isolated (which is not the case as it is inside the milky way) and designate the absolute speed vector of the system gravity center (i.e. with respect to a virtual fixed frame) by $v$. Likewise, let us designate the absolute speed vector of the sun (resp. of the $N$ planets) by $v_0$ (resp. by $v_i$, $i = 1,2,...N$). All trajectories are geodesic with respect to the cosmological metric $g$ (i.e. $\nabla_{v_i(t)} v_i(t) = 0$) and we have
\[
||v_i(t)||_{g} = v_i \quad \text{where } v_i \text{ is constant for } i = 0,1,...N.
\]

So, we have
\[
\sum_{i=0}^{N} m_i v_i = m_0 v_0 + \sum_{i=1}^{N} m_i = (\sum_{i=0}^{N} m_i) v = m_0 v + (\sum_{i=1}^{N} m_i) v.
\]

Now, for $i = 1,..., N$, we have
\[
v_i = v + u_i
\]
where $u_i$ is the relative speed vector with respect to the sun. Then
\[
m_0 v_0 + \sum_{i=1}^{N} m_i (v + u_i) = m_0 v + (\sum_{i=1}^{N} m_i) v.
\]

But the gravity center of the system is nearly the same as that of the sun and then we have $v_0 \simeq v$. Consequently, we get
\[
\sum_{i=1}^{N} m_i u_i = 0.
\]

Notice also that, apart from the whole universe, null other system is durably isolated (including galaxies, black holes and naturally all systems with planetary scale). However, it is the distribution (essentially local distribution) $E_t(X)$ that governs the free motion in all local or microlocal system. Thus, at the atom level, for instance, the free movement of electrons along their respective orbits or rather the trajectory of a material point of each electron,
is governed by the acceleration \( \Gamma(t) = \nabla g_E(X(t)) \) and satisfy the identity \( \tilde{\Gamma}(t) = \nabla g t X'(t) X'(t) = 0 \). However, after an exterior energetic support (thermic or electromagnetic, for example), the electron is submitted to possibly many energy transformations as a change of its orbital energy level or a pure separation from its original atom with a well determined kinetic energy. We notice that, in the inverse case, the energy conservation law is insured by emission of photons.

This is still valid for a nucleus which is, in addition, submitted to nuclear interaction forces that (under external stimulation or by a natural process) lead to matter-energy transformations such as: disintegration, fission, fusion, excitation and radiations and to chemical reactions leading to a transfer or liberation of energy that are governed by the energy conservation principle, the best energetic equilibrium rule and the mechanical principle of the least action and also by the Pauli exclusion principle.

We finally mention that each solution \( E(t, X(t)) \) of equation \( (E^*) \), propagating along \( X(t) \), is of the form \( E_{\mu_0}(X(t)) = f_{\mu_0}(t) \varphi_{t,\mu_0}(X(t)) \) and can not be equal to a linear combination of such solutions as we can verify this with the phenomenon of the diffraction of light. A light ray can not be diffracted into multiple rays that will have different energy than the incident ray.

Equality \( f_{\mu}(t) = \frac{1}{2\pi} \frac{\sqrt{\mu}}{t} \) shows that the undulatory frequency of the matter-energy increases with \( \mu \) and decreases with \( t \). So, just after the Big Bang (for \( t \) sufficiently small) all propagations have an undulatory character which is as pronounced as \( \mu \) is larger. For \( t \) sufficiently large, only the very large eigenvalues lead to waves that have a significant undulatory character. However, we have \( v(t) = \lambda(t) f_{\mu}(t) \) where \( \lambda(t) \) is the wavelength and \( v(t) \leq 1 \). Then, for \( t << 1 \), we have \( f_{\mu}(t) >> 1 \) for all \( \mu \) and so we have \( \lambda(t) << 1 \). Thus we can conceive intuitively that, when \( t = 0 \), we have \( \lambda = 0 \) and then there is neither material nor non material propagation. Moreover when \( v(t) \) is the speed of propagation of a wave starting at an infinitesimal time \( t \sim 0 \), we have

\[
v(t) = \frac{1}{2\pi} \frac{\sqrt{\lambda^2(t)\mu}}{t} \]

and then, for finite \( t > 0 \) when \( v(t) \sim 1 \), we have

\[
\frac{\lambda^2(t)\mu}{t^2} \leq 4\pi^2.
\]

So, for the wave of speed \( v = 1 \) (light, X rays, \( \gamma \) rays), we have \( \lambda^2(t)\mu = 4\pi^2 t^2 \) and \( \lambda(t) = \frac{2\pi t}{\sqrt{\mu}} \).

In order to come back to our starting point, we notice that these latter waves
are those that create, since the Big Bang until now, the geometrical space whose expansion occurs at a speed $v$ which is very near to 1 and theoretically must tend to 1. The material corpuscular universe is expanding at a speed inferior to 1 and its acceleration is submitted to the possibility of the material perception, which needs to be determined more and more precisely, although it theoretically must be nearly null.

**Remark**: If we have considered a block of matter of inertial mass $m$ with a very large density of mass (such as an ultradense neutron star) instead of the potential energy $e \sim m$ concentrated at a single point $I$ (as we have represented the potential energy of a black hole as $e(I)\delta I = m(I)\delta I$), then we would have not altered our mathematical model in virtue of the enormous extent of the virtual or the geometrical space. Thus, this fact allows us to avoid appealing to the infinity notion (Infinite density, infinite curvature, infinite pressure and infinite temperature).

## 7 Some repercussions on modern Physics

### Temperature and pressure

The major absent factor along our study until now is the temperature (intrinsically related to the pressure) factor. However, temperature is an inherent characteristic in the expansion operation: The universe is permanently expanding and cooling. Moreover, temperature is indissociable from all energy forms: heat, radiations (via thermal spectrum), chemical and nuclear reactions, internal energy of stars (via fusion and internal fluctuation pressure) and particularly, temperature is associated with the average kinetic energy of molecules in thermal equilibrium through the formula

$$<E_k> = \frac{3}{2}kT.$$

So we can briefly say that: temperature intervenes into and fashions equilibrium states of all systems’ energy. We then start by specifying that the relation used in the preceding sections, in order to describe the free trajectories $X(t)$ (i.e. $\nabla_{X'(t)} X'(t) = 0$) for punctual material particles and photons as

$$E_\mu(t, X(t)) = m(\mu),$$

is only valid on small time intervals where we can consider the temperature as being constant. Indeed, although the metric $g_t$ takes implicitly into account
the surrounding temperature factor, we have to recognize that the preceding
formula must be written as
\[ E_\mu(t, T(t), X(t)) = m(\mu, T(t)) \]

Beside of the dependence of the energy
\[ E_\mu(t, X(t)) = h_\mu(t)f_\mu(t) \]
on \( \mu \), we must add necessarily its dependence on \( T(t) \) through the dependence
of the metric itself on \( T(t) \). The energy \( E(t, X(t)) \) which is conveyed to us
by radiations from long time ago and long distance away is attenuated not
only because of collisions, but above all under the influence of the undulatory
universe cooling. The decrease of frequencies (i.e. the increase of wavelengths)
is counterbalanced by the increase of \( h(t) \). On the other hand, if the waves
are propagating along \( X(t) \) with a constant energy \( E(t, X(t)) \), then we can
not have
\[ \int_{B(0,t)} E_i dX_i = \text{const.} \]
for each \( t \).

If we adapt the ideal gas model to the whole universe, we can assume that
we have permanently
\[ P(t)V(t) = K(t)T(t); \]
which implies, for large enough positive \( t \) :
\[ P(t)t^3 = K'(t)T(t). \]

For \( t \ll 1 \), the situation could not be the same because then \( \text{vol}(B_\epsilon(O, R(t))) \)
could not be proportional to \( t^3 \). This is due to the fact that the speed of radiations’
propagation that verifies \( \nabla_{X(t)}^{g(t)} X'(t) = 0 \) could be originally less than
1 which is the speed of light in the vacuum. Indeed the metric \( g(t) \) contracts
the distances significantly at the early expansion because of the largeness of
the interaction forces’ magnitude (gravitational and other interaction forces)
at the origin as well as the largeness of the energy density and the pressure
and temperature intensities.

We notice that, we can avoid using the notion of a very large energy that is
concentrated in a point at the origin of time \( t = 0 \) with infinite pressure and
temperature and conceive starting our study by considering the quasi-original
universe as being reduced to a small ball of Euclidean radius \( r_0 \ll 1 \) at a
very small time \( t_0 > 0 \). The universe is then considered at time \( t_0 \) as being
a small ball, within it radiations are characterized by very large (but finite)
pressure and temperature. This situation evolves with the time’s progress
toward a state qualified as a soup of quarks and leptons before the formation
of hadrons followed by nucleons, atoms and galaxies that marks the passage
from a radiations’ dominated state to a matter dominated state. Temperature had certainly an important influence during this evolution which led to the current situation characterized by an approximate average temperature of 2.74\( K \). However, a large number of technical (theoretical or experimental) means are available for us to go forward in our investigations in order to discover more and more thoroughly the original states of our universe and the laws that govern its evolution.

Let us show now, using some examples, that we can recover some confirmed results in modern Physics without using neither the second part of the second postulate of special relativity nor the uncertainty principle.

**Remarks on relativistic formulas**

In this subsection, we will notice some remarks and establish some properties and results based on the refutation of the second part of the special relativity second postulate and the spacetime relativistic notion as well as on the canonicity of the Maxwell equation and the principle of the speed of light constancy that are established by using the derivations \( \frac{d}{dt} \) and \( \frac{d^*}{dt} \) which take into account the moving frame speed (see sections : 2,3 and 4.)

The speed actually is a continuous variable on \([0,1[\), then we have opted to do not make a sharp distinction, according to their speed, between relativistic and non relativistic particles concerning either their energy or their momentum. What is the critical speed starting from which we can use the relativistic formulas :

\[
p = \frac{mv}{\sqrt{1 - v^2}} =: \gamma mv, E = \sqrt{p^2 + m^2}, E = \gamma m \text{ and } v = \frac{p}{E}
\]

These formulas can not coincide with classical ones that give the particle energy at any non vanishing speed :

\[
E = m + \frac{1}{2} mv^2 = \gamma m \iff \gamma = 1 + \frac{v^2}{2} \iff \frac{1}{1 - v^2} = 1 + v^2 + \frac{v^4}{4}
\]

\[
\iff 1 = 1 - v^4 + \frac{v^4}{4} - \frac{v^6}{4} \iff 0 = -\frac{3v^4}{4} - \frac{v^6}{4}.
\]
On the other hand, when a particle with a non vanishing mass $m \ll 1$ have a speed near to 1, we would have in the relativistic framework $\gamma \gg 1$ and $p$ and $E$ would be very large whereas in classical Physics we have

$$E \simeq m + \frac{1}{2} m = \frac{3}{2} m \ll 1$$

Moreover, we have in classical Physics (for $v \ll 1$):

$$p = mv, \quad E_k = \frac{1}{2} mv^2 = \frac{1}{2} pv \quad and \quad E = m + \frac{1}{2} mv^2$$

whereas, in the relativistic framework, we have

$$p = \frac{h}{\lambda} = hf = E$$

for photons and $v = \frac{p}{E}$ for material particles. For these last particles, the two notions coincide only for $v = 0$. Indeed, if $v > 0$ then

$$v = \frac{p}{E} = \frac{mv}{E} \quad implies \quad E = m,$$

which is contradictory ($E = m \implies v = 0$).

**Remark**

We briefly notice that all results and formulas that are established by using the erroneous part of the second postulate can be established more precisely in a consistent manner. However, using relativistic formulas leads to very useful approximate results.

**The famous $E = mc^2$ statement**

Within our model, we have for material or immaterial points:

$$E_\mu(t, T(t), X(t)) = h_\mu(t, T(t)) f_\mu(t, T(t)) = m(\mu, T(t))$$

and, for any fundamental material particle with mass $m(t)$ that occupies a domain of volume $V(D_t) =: V(t)$ at time $t$ and for all speed $v < 1$, we
have

\[ E(t) = \int_{D_1} E_\mu(t, T(t), X(t))dX_t = \int_{D_1} h_\mu(t, T(t))f_\mu(t, T(t))dX_t \]
\[ = h_\mu(t, T(t))f_\mu(t, T(t))V(t) = m(\mu, T(t))V(t) \]

\[ p(t) = \int_{D_1} m(X_t)v(X_t)dX_t = m(t)v(t) \]

\[ E_k(t) = \frac{1}{2} \int_{D_1} m(X_t)v^2(X_t)dX_t = \frac{1}{2}m(t)v^2(t) \]

and

\[ E(t) = m(t) + \frac{1}{2}m(t)v^2(t) = m(t)(1 + \frac{1}{2}v^2(t)) =: \rho(t)V(t)(1 + \frac{1}{2}v^2(t)). \]

So we have

\[ h_\mu(t, T(t))f_\mu(t, T(t))V(t) = \rho(t)V(t)(1 + \frac{1}{2}v^2(t)) \]

and then

\[ E(t, X(t)) = h(t)f(t) = \rho(t)(1 + \frac{1}{2}v^2(t)) \]

where we have denoted \( E_\mu(t, T(t), X(t)) \) by \( E(t, X(t)) \), \( h_\mu(t, T(t)) \) by \( h(t) \) and \( f_\mu(t, T(t)) \) by \( f(t) \).

For \( v = 0 \) we get \( E(t_0, X(t_0)) = \rho(t_0) \) and so:

The punctual energy of matter at rest is equal to its punctual density, and then we recover the famous Einstein’s statement

\[ E_0 = m_0 \]

Thus, according to our model, the mass of a material particle moving at any speed \( v(t) < 1 \) is (contrary to what is adopted in special relativity theory) a decreasing function of time \( m(t) \). This decrease is naturally due to the mass energy loss by the radiation phenomenon which is related to speed and temperature fluctuations. Therefore, the total energy of a particle is

\[ E(t) = m(t)(1 + \frac{1}{2}v^2(t)) \]

where \( m(t) \) is the initial (at rest) mass \( m(0) = m_0 \) multiplied by a factor \( \gamma(t) \) depending on the speed and time

\[ m(t) = \gamma(t)m_0. \]
This factor can be determined experimentally or theoretically by calculating the amount of the energy loss through the radiation phenomenon as a function of acceleration, temperature and time.

However, writing \( m(t) = \gamma(t)m_0 \), one obtains

\[
E(t) = \gamma(t)m_0 \left(1 + \frac{v^2}{2}\right) \tag{17}
\]

\[
p(t) = \gamma(t)m_0 v \tag{18}
\]

and for \( v(t) \neq 0 \)

\[
\frac{E(t)}{p(t)} = \frac{1 + \frac{v^2}{2}}{v}.
\]

So, for \( v(t) \equiv 0 \), we get \( p(t) \equiv 0 \) and \( E(t) = \gamma(t)m_0 \), which yields, together with \( E(t) \equiv m_0 \), \( \gamma(t) \equiv \gamma(0) = 1 \).

For \( v \sim 1 \), we have \( E(t) \sim \frac{3}{2}p(t) \) and for \( v \sim 0 \), we have \( E(t) \sim m_0 \) and \( p(t) \sim 0 \).

Differentiating the equalities (17) and (18) with respect to \( t \), we obtain

\[
E'(t) = m_0 \gamma'(t) \left(1 + \frac{v^2}{2}\right) + m_0 \gamma(t) v v' \tag{19}
\]

\[
p'(t) = m_0 \gamma'(t) v + m_0 \gamma(t) v' \tag{20}
\]

and by differentiating them with respect to the speed we get

\[
\frac{dE}{dv} = m_0 \frac{d\gamma}{dv} \left(1 + \frac{v^2}{2}\right) + m_0 \gamma(t) v \tag{21}
\]

\[
\frac{dp}{dv} = m_0 \frac{d\gamma}{dv} v + m_0 \gamma(t) \tag{22}
\]

which yields after eliminating \( m_0 \gamma(t) \) in (19) and (22),

\[
E'(t) = m_0 \gamma'(t) \left(1 + \frac{v^2}{2}\right) + \left(\frac{dp}{dv} - m_0 \frac{d\gamma}{dv} v\right) v v'
\]

\[
= m_0 \gamma'(t) \left(1 + \frac{v^2}{2}\right) + p'(t) v - m_0 \gamma'(t)v^2
\]

\[
= m_0 \gamma'(t) \left(1 - \frac{v^2}{2}\right) + p'(t) v.
\]
So,
\[ \gamma'(t) = \frac{E'(t) - p'(t)v}{m_0(1 - \frac{v^2}{2})} \]
and
\[ \int_0^t \gamma'(s) ds = \frac{1}{m_0} \int_0^t \frac{E'(s) - p'(s)v}{1 - \frac{v^2}{2}} ds \]

Finally, we have
\[ \gamma(t) = 1 + \frac{1}{m_0} \int_0^t \frac{E'(s) - p'(s)v}{1 - \frac{v^2}{2}} ds \]
and
\[ m(t) = m_0 + \int_0^t \frac{E'(s) - p'(s)v}{1 - \frac{v^2}{2}} ds \]

So, we recover that for \( v = 0 \), we have \( E(t) = m(t) \equiv m_0 \).

Assuming \( v = v_0 \) for \( t_0 \leq t \leq t_1 \), we obtain
\[
m(t_1) - m(t_0) = \int_{t_0}^{t_1} \frac{E'(t)}{1 - \frac{v_0^2}{2}} dt - \int_{t_0}^{t_1} \frac{p'(t)v_0}{1 - \frac{v_0^2}{2}} dt
= \frac{1}{1 - \frac{v_0^2}{2}} (E(t_1) - E(t_0)) - \frac{v_0}{1 - \frac{v_0^2}{2}} (p(t_1) - p(t_0))
\]
and then
\[
\gamma(t_1) - \gamma(t_0) = \frac{1}{1 - \frac{v_0^2}{2}} (\gamma(t_1) - \gamma(t_0)) \left(1 + \frac{v_0^2}{2}\right) - \frac{v_0}{1 - \frac{v_0^2}{2}} (\gamma(t_1) - \gamma(t_0))
\]
which is possible only for \( \gamma(t_1) - \gamma(t_0) = 0 \).

Therefore
\[
\gamma(t_1) = \gamma(t_0) \quad \text{and} \quad m(t_1) = m(t_0)
\]

Consequently, we get
\[
\gamma(t) = \gamma(t_0) \quad m(t) = m(t_0) \quad E(t) = E(t_0) \quad p(t) = p(t_0)
\]
for every \( t \in [t_0, t_1] \).

In the same way, we obtain, by eliminating \( m_0 \gamma(t) \) in (20) and (21)
\[
\frac{dE}{dv} = m_0 \frac{d\gamma}{dv} \left(1 + \frac{v^2}{2}\right) + \frac{p'(t) - m_0 \gamma'(t)v}{v}
= m_0 \frac{d\gamma}{dv} \left(1 + \frac{v^2}{2}\right) + \frac{dp}{dv} v - m_0 \frac{d\gamma}{dv} v^2
= m_0 \frac{d\gamma}{dv} \left(1 + \frac{v^2}{2}\right) + \frac{dp}{dv} v
\]

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and then
\[
\frac{d\gamma}{dv} = \frac{\frac{dE}{dv} - \frac{dp}{dv}}{m_0(1 - \frac{v^2}{2})}
\]
and
\[
\int_0^v \frac{d\gamma}{dv} dv = \int_0^v \frac{\frac{dE}{dv} - \frac{dp}{dv}}{m_0(1 - \frac{v^2}{2})} dv
\]
which implies
\[
\gamma(v) = 1 + \int_0^v \frac{\frac{dE}{dv} - \frac{dp}{dv}}{m_0(1 - \frac{v^2}{2})} dv
\]
and
\[
m = m_0 + \int_0^v \frac{\frac{dE}{dv} - \frac{dp}{dv}}{1 - \frac{v^2}{2}} dv
\]
So, for \(v \sim 0\), we get \(m \simeq m_0 + E(v) - m_0\) and then \(m \simeq E\).

For \(v \simeq v_0 \neq 0\), we can write
\[
E(v) - E(v_0) \simeq \int_{v_0}^v \frac{E'(v) - v_0p'(v)}{1 - \frac{v^2}{2}} dv
\]
\[
= \frac{2}{2 - v_0^2}(E(v) - E(v_0)) - \frac{2v_0}{2 - v_0^2}(p(v) - p(v_0))
\]
which implies
\[
\left(\frac{2}{2 - v_0^2} - 1\right)(E(v) - E(v_0)) \simeq \frac{2v_0}{2 - v_0^2}(p(v) - p(v_0))
\]
and
\[
v_0^2(E(v) - E(v_0)) \simeq 2v_0(p(v) - p(v_0))
\]
that is
\[
E(v) - E(v_0) \simeq \frac{2}{v_0}(p(v) - p(v_0))
\]
So, we have
\[
\gamma(t)m_0 \left(1 + \frac{v_0^2}{2}\right) - \gamma(t_0)m_0 \left(1 + \frac{v_0^2}{2}\right) \simeq \frac{2}{v_0}(\gamma(t)m_0v - \gamma(t_0)m_0v_0)
\]
If \( m_0 \neq 0 \), we get
\[
\gamma(t) \left( 1 + \frac{v^2}{2} \right) - \gamma(t_0) \left( 1 + \frac{v_0^2}{2} \right) \simeq \frac{2}{v_0} (\gamma(t)v - \gamma(t_0)v_0)
\]
and then
\[
\gamma(t) \left( 1 + \frac{v^2}{2} \frac{2}{v_0} v \right) \simeq \gamma(t_0) \left( 1 + \frac{v_0^2}{2} - 2 \right)
\]
which implies
\[
\gamma(t) \simeq \gamma(t_0) \text{ and } m(t) \simeq m_0.
\]
For \( v \sim 1 \), we have \( \gamma(t) \simeq \gamma(t_0) \sim 0 \) and \( m(t) \sim 0 \).

We notice that a comparison of our formulas (18) and (17) respectively with the relativistic ones
\[
p = \gamma m_r v \quad \text{and} \quad E = \gamma m_r
\]
where \( \gamma = \frac{1}{\sqrt{1 - v^2}} \) is the Lorentz factor and \( m_r \) is the constant Einstein’s mass, gives, on one hand :
\[
m_r = \frac{\gamma(t)m_0}{\gamma} = \frac{m(t)}{\gamma}
\]
or
\[
\frac{\gamma}{\gamma(t)} = \frac{m_0}{m_r}
\]
and on the other hand :
\[
m_r = \frac{\gamma(t)m_0}{\gamma} \left( 1 + \frac{v^2}{2} \right) = \frac{m(t)}{\gamma} \left( 1 + \frac{v^2}{2} \right)
\]
or
\[
\frac{\gamma}{\gamma(t)} = \frac{m_0}{m_r} \left( 1 + \frac{v^2}{2} \right)
\]
which is only approximately possible for \( v \ll 1 \). We recall that the factor \( \gamma \) has first appeared with W.Kaufmann who called the expression \( \gamma m \) (which \( m ? \) the apparent mass. We notice also that a particle with mass \( m(t) \) can not achieve a final speed \( v = 1 \) and having a final mass \( m_f > 0 \) as, using then the relativistic formula, we would have
\[
E_f = \frac{m_f}{\sqrt{1 - v_f^2}} \sim +\infty
\]
which means that the energy needed for such a particle to achieve this speed is infinite.
Momentum, Kinetic energy and Mass

Within the framework of our model, we have privileged (as Einstein has originally done) the notion of the mass at rest $m_0$ of particles. However, we have adopted, for particles having significant speed, the notion of speed and time-dependent mass under the form

$$m(t) = \gamma(t)m_0$$

where $\gamma(t)$ decreases from 1 to 0 as the speed increases from 0 to 1. We have also adopted the expression

$$E_k(t) = \frac{1}{2} m(t)v^2(t) = \frac{1}{2} \gamma(t)m_0v^2(t)$$

for the kinetic energy and the expression

$$\vec{p}(t) = m(t)\vec{v}(t) = \gamma(t)m_0\vec{v}(t)$$

for the momentum of a moving particle.

The momentum of any particle is classically defined as

$$\vec{p} = m\vec{v} \quad \text{with} \quad \frac{d\vec{p}}{dt} = \vec{F}$$

where $\vec{F} = m\vec{\Gamma}$ is the force that acts on the particle, whereas the classical definition of the kinetic energy is

$$E_k = \frac{1}{2}mv^2.$$ 

Einstein has earlier showed that these definitions are erroneous for particles moving with high speeds. Indeed, a simple example ([2], p.112) shows that the classical definition of kinetic energy leads to the relation $v = \sqrt{\frac{2E_k}{m}}$ which contradicts the fundamental law of special relativity (and Physics in general) asserting that the speed of any material particle cannot exceed $c = 1$.

However, the second example ([2], p.113), which has been used for showing the non conservation of momentum during the collision of two particles $A$ and $B$ having the same mass $m$ and opposite speeds $\vec{v}$ and $-\vec{v}$ with respect to a given referential frame $S'$, do not permit to draw the same consequences that are established by using the relativistic formulas of the frame exchange. For us, the momentum is naturally conserved when using the rest frame $S$ for either $A$ or $B$. This is clearly showed by the diagrams of figure 11.
In other respects, it is clear that ([2], p.112) the classical formulas $p = mv$ and $F = \frac{dp}{dt}$ contradict the fundamental principle which stipulates that the speed of a body of a non vanishing mass can not exceed the speed of light. Likewise, the relation $F = \frac{dp}{dt}$, for $F \neq 0$, leads, within our framework, to a contradiction of the same nature as previously. Indeed, let us consider (for instance) an electron having at rest mass $m_0$ which is accelerated through an electrical field $E$ such as the exerted electrical force on the electron is a non vanishing constant.

When we write

$$F = m(t)\Gamma(t) = \gamma(t)m_0\Gamma(t)$$

and

$$p = m(t)v(t) = \gamma(t)m_0v(t),$$

for $m(t) \neq 0$ and then for $\gamma(t) \neq 0$ and $v(t) < 1$, we get

$$F = \frac{dp}{dt} \Leftrightarrow \gamma(t)m_0\Gamma(t) = \frac{d}{dt}(\gamma(t)m_0v(t)) \Leftrightarrow \frac{F}{m_0} = \frac{d}{dt}(\gamma(t)v(t)).$$

As $F$ is assumed to be constant, we obtain

$$\gamma(t)v(t) = \frac{F}{m_0} t + C = \frac{F}{m_0} t + \gamma(\tau)v(\tau) - \frac{F}{m_0},$$

for a $\tau > 0$.

Therefore, we have

$$\frac{d}{dt}(\gamma(t)v(t)) = \gamma'(t)v(t) + \gamma(t)\Gamma(t) = \frac{F}{m_0} = \gamma(t)\Gamma(t)$$

which implies

$$\gamma'(t) = 0 \quad \text{(since } v(t) \neq 0 \text{)}$$

and

$$\gamma(t) = \gamma = \text{const.} \quad \text{and} \quad m(t) = \gamma m_0 = \text{const.}$$

which is impossible since accelerated particles can not have constant mass.

Likewise, the above relation implies

$$v(t) = \frac{F}{m_0 \gamma} t + \frac{C}{\gamma} = \frac{F}{m} t + \frac{C}{\gamma} = \Gamma t + \frac{C}{\gamma}$$

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for strictly positive constant $\Gamma$, which is equally impossible since the speed of any moving particle of mass different from zero can not exceed 1. Besides, we notice that if the electrical field $E$ is kept constant, then the force $F = q(E + v \wedge B)$ can not physically kept constant for an accelerated electron into an electrical field.

Therefore the relation $\frac{dp}{dt} = F$ can only be approximately correct for minimal speeds where $\gamma'(t) \sim 0$, $\gamma(t) \sim 1$ and $m(t) \sim m_0$, provided that $m(t) \neq 0$. In this situation, we may write

$$\frac{dp}{dt} = \frac{d}{dt}m_0v(t) = m_0\Gamma(t) = F(t)$$

and, for $v = \text{const.}$, we have

$$\frac{dp}{dt} = \frac{d}{dt}m_0v = 0.$$ 

Recall that, within the framework of our model, we have

$$p(t) = m(t)v(t) = \gamma(t)m_0v(t)$$
$$E_k(t) = \frac{1}{2}m(t)v^2(t) = \frac{1}{2}\gamma(t)m_0v^2(t)$$
$$E(t) = m(t)(1 + \frac{1}{2}v^2(t)) = \gamma(t)m_0(1 + \frac{1}{2}v^2(t))$$

for $m(t) \neq 0$ and $v(t) < 1$.

These formulas conform with the two fundamental conservation laws (of energy and momentum).

Actually, the momentum conservation principle is clearly expressed, within the framework of our model, by

$$\nabla_{X'(t)}^{g_t}p(t) = \nabla_{X'(t)}^{g_t}\gamma(t)m_0X'(t) = m_0\nabla_{X'(t)}^{g_t}\gamma(t)X'(t) = m_0(\gamma(t)\nabla_{X'(t)}^{g_t}X'(t) + X'(t)\gamma(t)X'(t)) = m_0(\gamma(t)\nabla_{X'(t)}^{g_t}X'(t) + \gamma'(t)(X(t))X'(t))$$

So, if $X(t)$ is a geodesic, we get

$$\tilde{\Gamma}(t) = \nabla_{X'(t)}^{g_t}X'(t) = 0$$

and

$$\|X'(t)\|_{g_t} = v = \text{const.}$$

which gives $\gamma(t) = \text{const.}$ and then $\gamma'(t) = 0$ and

$$\nabla_{X'(t)}^{g_t}\tilde{p}(t) = 0.$$
This quantity is truthfully null for any free trajectory $X(t)$ i.e. for any geodesic with respect to the real physical metric which takes into account all of the natural forces acting on the particle. Therefore, we can state that the relation

$$\nabla^{g'}_{X(t)}p(t) = \overrightarrow{F}$$

is more consistent than the above classical one.

Finally, we mention that within the relativistic framework the definition of the momentum by $\overrightarrow{F} = \frac{d}{dt} \overrightarrow{P}$ leads to the relation ([2], (4.104))

$$\overrightarrow{\Gamma} = \frac{d}{dt} \overrightarrow{\gamma} = \frac{\overrightarrow{F} - \overrightarrow{\beta}(\overrightarrow{F}, \overrightarrow{\beta})}{mv\gamma}$$

which shows that, at large speed, the acceleration is not parallel to the force whereas within our model we have, for the force field along the trajectory $X(t)$:

$$F_{g'}(X(t)) = -\nabla^{g'}E_t(X(t)) = \overrightarrow{\Gamma}(t).$$

**Remarks on the quantum theory**

Concerning the uncertainty principle, it seems illogical that, after some experiments such as the one where particles hit a screen through two small slits slightly spaced, we conclude that the fact of knowing the origin of the particles hitting the screen could really alter the physical phenomenon. It is true that the means used in order to know this origin can alter the results by modifying the trajectories but this is only a technical and circumstantial phenomenon which does not allow us to conclude that our pure knowledge can transform the physical results that are determined objectively by the very physical conditions. Beside of that, our theoretical or practical capacity to discover any law of Nature does not influence the objective reality of this law. The very long history of discoveries in all domains shows the objectivity of natural laws independently from our circumstantial (theoretical, approximate, experimental or technical) capacity to discover them. The use of progressive energetic levels of particles and greater and greater frequencies (i.e. smaller and smaller wavelengths), for instance, has permitted the realization of important progress in the understanding of our physical universe through the ages as well as the understanding of matter (nucleons, quarks, hadrons, leptons) structure and also in the refinement of our knowledge on

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the quantization of both energy levels and angular as well as intrinsic atomic and nuclear momenta. The Schrödinger function and Quantum Statistics have also permitted a jump into our comprehension of the universe by giving effective methods to discover and interpret all results that are obtained by experimentation and led to powerful approximations for natural phenomena rules. These phenomena contain essentially a part of uncertainties due to the multitude of (energetic and dynamic) evolutive factors that govern all aspects of matter-energy: energy levels, trajectories, interactions...

Although these phenomena are far from being regular (differentiable), they are continuous. An electron, for instance, that changes its orbit (which is in permanent evolution) for a higher energy level (when absorbing a photon) or for a lower energy level (together with emission of photon), spends a very small fraction of time before achieving its final state. During this transition, continuity of trajectory and conservation of energy are both insured.

In other respects, we notice that the solution $\psi$ to the classical Schrödinger equation

$$
-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x),
$$

for example, which gives the probability of finding the particle per unit of $x$ using the distribution

$$
\frac{dP}{dx} = |\psi(x)|^2,
$$

is determined by experimental and predictive way. This function is completely different from our function $\psi(X)$ which comes from the resolution of the energy equation ($E^*$) and is given by

$$
E(t, X) = g(t)\psi\left(\frac{X}{t}\right)
$$

and determines both the energy and the frequency of a material or immaterial point as well as any punctual material particle at every given time $t$.

Moreover both functions must be distinguished from the trajectory $X(t)$ of this particle. So, for a simple pendulum or a quantum harmonic oscillator, for instance, the frequency of oscillation is not the same as the material points frequency. When the pendulum is located at the stable vertical equilibrium, the frequency of any material point is determined by its characteristic energy $E(t, X) = h(t)f(t)$; its potential energy and its kinetic energy are null and its mass energy is $E = mc^2 = m$. Nevertheless, we can not speak neither about its period nor its frequency $f = \frac{1}{T}$. So we can not assume that its frequency is different from zero and try to determine, using the uncertainty principle,
its ground state energy which would be different from zero (even if it was very small) contrary to Newtonian principles of classical Physics. This is also valid concerning a Tennis or any small ball, for example, being at rest into a box assumed to be also at rest.

Likewise, the minimal quantized energy level of the electron of the hydrogen atom corresponds to the orbit which is associated to the Bohr radius

\[ r = a_0 = \frac{4\pi\varepsilon_0\hbar^2}{me^2} \]

which is determined by the minimal energy

\[ E_m = \left( \frac{1}{2}mv^2 - \frac{e^2}{4\pi\varepsilon_0r} \right)_m = \left( \frac{\hbar^2}{2\pi r^2} - \frac{e^2}{4\pi\varepsilon_0r} \right)_m \]

where \( \hbar \) is here the classical Planck constant.

This result has nothing to do with the uncertainty principle; it is rather due to the fact that the minimal total energy that can have an electron inside a hydrogen atom (the ground state energy) is finite and characterized by both constants \( a_0 \) and \( \hbar \).

Let us finally specify that the uncertainty principle which can be written as \( \Delta x\Delta p_x > \frac{\hbar}{2} \) and \( \Delta E\Delta t > \frac{\hbar}{2} \), is only a legitimate consequence of using Schrödinger equations:

\[ \psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(k)e^{-ikx}dk \]

and

\[ g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(x)e^{ikx}dx \]

in order to find out the probability of localizing a given particle under some constraints in a given position. This principle is stated after using the probability distributions

\[ \frac{dP}{dx} = |\psi(x)|^2, \quad \frac{dP}{dk} = |g(k)|^2 \]

and their standard deviations \( \sigma_x \) and \( \sigma_k \) as well as the De Broglie relation \( p = \frac{\hbar}{2} \). Therefore this principle states simply that this particular approach and the use of this particular method hold within themselves the uncertainty so quantified. Nevertheless this does not mean that the position \( x \) of the
particle or its momentum $p_x$, at a given time, can not be well defined or can not be determined with more precision by a more efficient theoretical or experimental process.

Indeed, none can assert that one can not perform a technical device or a theoretical process that could be used in order to measure the width of a slit or the size of a particle that are much smaller then those which are reached at the present time by means of scattered particles. We maybe could use $\gamma-$rays having very much smaller wavelengths after inventing an intermediate device (or process) which makes their effects accessible to our sensitivity or our comprehension. Likewise, we can hope performing some new processes for measuring both position $x$ and momentum component $p_x$ of a given particle which could improve the uncertainties $\Delta x$ and $\Delta p_x$ as well as their product $\Delta x \Delta p_x$ which is limited presently by $\frac{\hbar}{2}$ when using our current process.

Obviously, we can use Schrödinger equations and quantum Statistics as effective approaches leading to qualitative and quantitative approximations (with naturally a margin of uncertainty) of the studied physical phenomena when real approximate measurements are difficult to achieve. These issues will be discussed and made more precise in the next sections.

Notice that, within the framework of our model, the notions of wavelength and frequency are characteristics of material or immaterial points but not of material particles even though they are considered physically as punctual particles such as electrons for example. This is precisely the context within which we have to understand and explain the undulatory character of matter. We then consider the fact of attributing a wavelength $\lambda = \frac{\hbar}{p}$ and a frequency $f = \frac{v}{\lambda}$ to a pointlike material particle as being a practical method for making approximate calculation and they do not correspond to a real periodic trajectory and can not be used for making exact calculation of relativistic or non relativistic particle’s energy by means of formulae such as :

$$E = \sqrt{(pc)^2 + (mc^2)^2} = \sqrt{p^2 + m^2} = \sqrt{\frac{h^2}{\lambda^2} + m^2} = \sqrt{\frac{h^2f^2}{v^2} + m^2},$$

for example (see section 8).

Likewise, when we use the relation $p = \gamma mv = \frac{h}{\lambda} = \frac{hf}{v}$ for a non relativistic punctual material particle, we get

$$E_T := hf = \gamma mv^2$$
which implies (according to relativistic formulas)

\[ \gamma m = \gamma mv^2 \]

and then \( v^2 = 1 \) which is absurd. However, the momentum \( p \) of a photon is really:

\[ p = E = hf = \frac{h}{\lambda} \]

and then we have \( \lambda = \frac{h}{p} \).

Likewise, the use of the relativistic momentum expression \( p = \frac{mv}{\lambda} \) for material particles leads to a flagrant contradiction. Indeed, let us write, for the electron of the Bohr atom model (for instance),

\[ \frac{<E_k>}{<E_p>} = \frac{<\frac{mv^2}{2}>}{ke^2 \frac{1}{\lambda}} = - \frac{<\frac{mv^2}{2}>}{ke^2 \frac{\lambda}{\pi}} \]

where we have used the formula ([2], p.139)

\[ \frac{1}{r} \sim \frac{2\pi}{\lambda} = \frac{p}{\hbar} \]

and the non relativistic expression of \( E_k \), which is legitimate for the energy levels of this atom. So, we have

\[ -\frac{1}{2} = -\frac{v\hbar}{2ke^2} = -\frac{v\hbar}{2\alpha hc} = -\frac{1}{2\alpha c}v, \]

which is contradictory as \( v \) decreases with \( r \).

On another side, when we compare our expression for the undulatory energy \( E(t) = h_\mu(t)f_\mu(t) \) to the De Broglie-Planck-Einstein one, which we will denote by \( E = h_P f_D \) (where \( h_P \) is the Planck constant and \( f_D \) is the De Broglie frequency), we get:

\[ h_\mu(t)f_\mu(t) = h_P f_D \]

which gives

\[ h_P = \frac{h_\mu(t)f_\mu(t)}{f_D(\mu,t)} = \frac{m(\mu,t)}{f_D(\mu,t)} \]

and

\[ f_D(\mu,t) = \frac{1}{h_P}h_\mu(t)f_\mu(t) = \frac{1}{h_P}m(\mu,t). \]

**Remark**: A complementary and more systematic study of the limits of Quantum theory is furnished in the next section.
Repercussions on some other notions

Finally, we notice that, in the light of our model, we can reexamine and make precise a large number of notions and factors having an important role in modern Physics (such as: Hubble law, radiation spectrum dependence on the powers of temperature, reunification of all forces problem...) without using neither the relativity second postulate nor the uncertainty principle. We have to take into account the dependence on time of some notions and constants. So, we can explain, for instance, the redshift and the blueshift phenomena by the dependence on time (or distance) and temperature of the frequency and not by the speed of the source. The only effect of the speed of the source is to make the emitter less or more distant from the receiver-analyser according to a given rate.

Likewise, we can show easily that, within the framework of our model, the relative speed of two isotropic galaxies moving in the direction of the expansion is proportional to their distance. This allows us to introduce the factor $R(t)$ that characterizes the expansion and to follow Hubble’s work by putting $r = r_0 R(t)$ and

$$H(t) = \frac{dR}{R} \quad \text{with} \quad H_0 = \left(\frac{dR}{dt}\right)_{t=t_0}$$

in order to obtain the Hubble’s law

$$v = \frac{dr}{dt} = H_0 r$$

Moreover, if $m$ is the total mass of a galaxie located at a sphere of sufficiently large radius $R(t)$ and $M$ is the total mass of the ball of radius $R(t)$, we have (J. W. Rohlf, p.552) the classical results

$$E_k = \frac{1}{2} m r_0^2 \left(\frac{dR}{dt}\right)^2$$

and

$$V = -\frac{4\pi m G r_0^2 R^2 \rho}{3}$$

where $V$ is the potential energy of the galaxy and $\rho$ is the mean density of the ball mass.

Next, following Einstein, we introduce the curvature parameter $K(t)$ which is, within our model, intrinsically related to the metric $g_t$ which, itself, reflects
the universe energy distribution. We then apply the conservation of energy law in order to obtain the Friedmann equation

$$\left( \frac{dR}{dt} \right)^2 = \frac{8\pi \rho G R^2}{3} - K$$

Although this equation could give useful information on the universe evolution, we mention that our model is different from the Einstein-de Sitter one as it is based on another conception of the space and time on one hand and, on the other hand, we recall that our curvature parameter $K$ depends on time and can not be 0. We notice also that our model does not conform with the first postulate of the cosmological principle which states that the universe would (thoroughly) look the same for any observer from any galaxy, although it conforms with the second one which states that the relative speed of galaxies (in the sense specified above) is proportional to their distance. However, the readjustment of Einstein’s general relativity theory, Hubble laws and Friedmann Einstein’s equations within our setting will be achieved in sections 10, 11 and 12.

We obviously have to use the quantum Statistics within its important significance and limits. An extensive reexamination of these notions and postulates and its resulting consequences requires naturally a long and laborious collective work.

All the preceding study invites us to believe that Physics is an exact science, but this science can be revealed to us only progressively and often approximately, meanly by joining experimentation to theory. This is essentially due to the complexity of natural phenomena (although the laws of Nature are essentially simple) and to the limits imposed by our technical and practical means and tools. However, all what remains in the Physics domain (not in the Methaphysics one) is governed by some number of principles although the majority of them has been discovered by experimental and theoretical means. The theoretical way uses essentially Mathematics which is initiated and dynamised by Physics and Technology, although they are purely theoretical and intellectual.

We conclude by saying that Mathematics and Physics are indissociable in the same way as (more generally) they are theory and practice (which is more experimental and utilitarian). This shows the fundamental need of imagination, philosophy and confidence in the collective humanity reason in order to go forward in the scientific discovery way in all domains.

**Remark :** Further fundamental repercussions on Modern physics will be
developed in the following sections.

8 The limits of Quantum theory

In this section we aim to specify the proper domain of the Quantum theory efficiency. We will show that the wave Quantum Mechanics (based on Schrödinger’s equations) and quantum Statistics constitute essentially experimental and approximate tools that result in a probabilistic and predictive approach for explaining physical phenomena. Consequently, they can not constitute a proper theoretical framework for instituting any intrinsic or canonical physical law. The De Broglie wavelength which is a canonical feature of electromagnetism is simply a practical object that is useful only for approximately studying the pointlike material particles behavior. Moreover the uncertainty principle is only a legitimate consequence of the Schrödinger probabilistic process and can not be considered as a universal principle. More fundamentally, we consider that the legitimacy of the wave quantum Mechanics (which is derived from classical Mechanics), is based on its ability to provide, in the macroscopic cases, approximate results that coincide with those given by Newton, Lagrange and Hamilton’s Mechanics. Classical Mechanics institutes laws for idealized physical situations (when sufficient data are known), whereas quantum Mechanics predicts and explains experimental observed results; the legitimacy of the latter is insured by the Bohr correspondence principle. Indeed, we show that quantum Statistics uniquely relies on the very physical characteristics of both the realized experiment and the involved particles (such as distances, symmetries, masses, charges, momenta and spins) as well as on mathematical Logics. Quantum Mechanics can and must be used in microscopic subatomic phenomena when our present means can not result in a theoretical formulation.

8.1 The De Broglie Wavelength

The early 20th century was marked by three fundamental discoveries: the photon by Einstein, the Planck constant and the Bohr model for the hydrogen atom. The Quantum period has begun. The quantized nature of light as well as of energy levels was clearly proved. Contemporaneously, many experiments and facts have shown that matter has also some wavy nature. The success of some new notions, the partial success of some others and the contestable success of special relativity theory led De Broglie to translate the notion of wavelength from electromagnetism to matter particles and bodies.
He then defined the wavelength of a particle as
\[ \lambda = \frac{h}{p} \]
where \( p \) is the relativistic momentum of the particle and \( h \) is the Planck’s constant. This was a practical and useful approximation for analyzing the energy, momentum and speed of particles. Nevertheless, this notion, joined to relativistic formulas, on one hand, and to the Bohr model, on the other hand, leads to some obvious contradictions. The wavy nature of matter has to be explained more generally and more precisely.

Remark 1. In the previous sections, we have proved that the second part of the special relativity second postulate is false. We also proved that the frequency \( f \) is a characteristic feature of a (material or immaterial) point that is extended only to fundamental particles (quarks and leptons). Integration of the relation \( E = hf \) over the domain occupied by a fundamental particle gives the famous relation
\[ E_0 = m_0c^2 := m_0 \]
for at rest matter and
\[ E = \gamma(t)m_0 \]
for a material particle into movement, where \( \gamma(t) \) is a decreasing function of the speed.

So, when we attribute a De Broglie wavelength \( \lambda = \frac{h}{p} \) and a frequency \( f = \frac{v}{\lambda} \) to a pointlike material particle, they do not correspond to a real periodic movement (or trajectory) and can not be used for making exact calculation of relativistic or non relativistic particle energy by means of formulas such as
\[ p = \gamma mv \quad , \quad E = \sqrt{p^2 + m^2} = \gamma m \quad \text{and} \quad v = \frac{p}{E}. \]

For non relativistic particles, we obtain in this way :
\[ p = \frac{h}{\lambda} = \frac{hf}{v} = \frac{E}{v} = \frac{\gamma m}{v} \]
which yields
\[ \gamma mv = \frac{\gamma m}{v} \]
and then \( v^2 = 1 \), which is absurd.
Moreover the Bohr model shows clearly that, for energy levels $E_n$ with $n$ sufficiently large, we have $f_{orb} \simeq f_{rad}$ (whereas, for lower $n$, we have $f_{orb} \neq f_{rad}$). So, when we consider two consecutive high levels $E_1$ and $E_2$ corresponding to frequencies $f_1$ and $f_2$, wavelengths $\lambda_1$ and $\lambda_2$ and speeds $v_1$ and $v_2$, we have (using the relation $E = \sqrt{p^2 + m^2} = \sqrt{\frac{\hbar^2}{\lambda^2} + m^2} = \sqrt{\frac{\hbar^2 f^2}{v^2} + m^2}$),

$$f_2 = \frac{\Delta E}{\hbar} = \frac{E_2 - E_1}{\hbar} = \frac{\sqrt{p_2^2 + m^2} - \sqrt{p_1^2 + m^2}}{\hbar} = \frac{\sqrt{\frac{\hbar^2}{\lambda^2_2} + m^2} - \sqrt{\frac{\hbar^2}{\lambda^2_1} + m^2}}{\hbar} = \sqrt{\frac{1}{\lambda^2_2} + \frac{m^2}{\hbar^2}} - \sqrt{\frac{1}{\lambda^2_1} + \frac{m^2}{\hbar^2}}.$$

Consequently, we obtain

$$\frac{v_2}{\lambda_2} = \frac{1}{\lambda_2} \sqrt{1 + m^2 \frac{\lambda^2_2}{\hbar^2}} - \frac{1}{\lambda_1} \sqrt{1 + m^2 \frac{\lambda^2_1}{\hbar^2}}$$

and

$$v_2 = \sqrt{1 + \frac{m^2}{p_2^2}} - \frac{\lambda_2}{\lambda_1} \sqrt{1 + \frac{m^2}{p_1^2}}$$

$$= \sqrt{1 + \frac{m^2}{p_2^2}} - \frac{p_1}{p_2} \sqrt{1 + \frac{m^2}{p_1^2}}$$

$$= \sqrt{1 + \frac{m^2}{p_2^2}} - \sqrt{\frac{p_1^2}{p_2^2} + \frac{m^2}{p_2^2}}$$

$$= \sqrt{1 + \frac{m^2}{p_2^2}} - \frac{v_1^2}{v_2} \sqrt{\frac{m^2}{p_2^2}}$$

$$= \sqrt{1 + \frac{m^2}{p_2^2}} - \frac{(n + 1) \lambda_2}{n \lambda_2} \frac{m^2}{p_2^2} < 0$$

which is absurd.

We obtain a similar contradiction when we use $f_1 = \frac{\Delta E}{\hbar}$.

The particle in a box case

We consider now a small ball (or particle) in a fixed box of length $L$. When we are looking for the ground state energy by using the De Broglie
wavelength notion $\lambda = \frac{h}{p}$ and the Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} = E\psi,$$

we arbitrarily exclude the case where the particle speed is $v_0 = 0$.

Now, the introduction of the speed notion implies necessarily the introduction of the time progress notion. Let then $E_0 = \frac{h^2}{8mL^2}$ be the quantum ground state energy that corresponds to the speed $v_0 \neq 0$. If $|v_0| = a_0$ (a positive constant), we obtain $|p_0| = ma_0$ when using classical Physics formulas and $|p_0| = \gamma_0 ma_0$ when using relativistic ones, where $\gamma_0 = \frac{1}{\sqrt{1 - v_0^2}}$. But $< p_0 > = 0$ implies $p_0(x) = \pm |p_0|$ and $v_0(x) = \pm a_0$ which constitute a physically and mathematically inconceivable phenomena ($v_0(x)$ can not pass from $-a_0$ to $+a_0$ instantaneously). So $|p_0|$ is time dependent which is contradictory as (according to generally accepted notions)

$$|p_0| = \frac{h}{\lambda_0} = \frac{h}{2L} = \text{const.}$$

Therefore, we have either $v_0 = 0$, which implies $E_0 = p_0 = 0$ (in accordance with the classical physics minimal energy) and $\lambda$ has no a real existence, or all quantities $v_0, p_0, E_0$ and (if we put $\lambda_0 = \frac{h}{p_0}$) $\lambda_0$ depend on time. In that case $L$ obviously depends on time, which is absurd unless the legitimacy of the theoretically exact measurements’ existence is thoroughly questioned.

**The pendulum and quantum harmonic oscillator case**

When we consider a pendulum, we assume either $v_0 = 0$, which corresponds to the stable vertical equilibrium state and implies $E_0 = p_0 = 0$ in accordance with classical Newtonian Physics, or $v_0 \neq 0$. In that case the ground state energy within the wave quantum Mechanics framework is

$$E_1 = \frac{h\omega_1}{2} = \frac{hf_1}{2},$$

which is a non vanishing constant, since the very physical nature of the pendulum notion imposes the attribution of a frequency $f_1 = \frac{1}{T_1}$ to the theoretically periodic movement of the pendulum.

Now, when we incorrectly identify the De Broglie wavelength $\lambda_1 = \frac{h}{p_1}$ with the wavelength $\lambda = \frac{2\pi}{f_1}$ of the periodic movement (where $p_1$ and $v_1$ are respectively the mean scalar momentum and speed), we obtain

$$E_1 = \frac{hf_1}{2} = \frac{h}{2} \lambda_1 f_1 = \frac{1}{2} p_1 v_1$$
which is the corresponding mean kinetic energy. The same results are va-
lid for a quantum harmonic oscillator. Furthermore, we obtain for the first
excited state $E_2 = 3E_1$ ([2], (7.121)). Now, it is physically and mathemati-
cally undeniable that mean speed, mean momentum and mean kinetic energy
depend continuously on the initial displacement of both pendulum and har-
monic oscillator. But displacement is a continuous variable; therefore the
energy levels can not be quantized by means of Schrödinger’s equation.

8.2 The uncertainty principle

We begin this subsection by noticing that it seems strongly illogical that,
after some experiments such as the one where particles hit a screen through
small slits slightly spaced, we conclude that the fact of knowing the origin of
the particles hitting the screen could really alter the physical phenomena. It
is true that the means used in order to know this origin can alter the results
by modifying the particles momenta and trajectories but this is only a tech-
nical and circumstantial phenomena that does not allow us to conclude that
our pure knowledge can transform the physical results that are determined
objectively by the real physical conditions. Beside of that, our theoretical or
practical capacity to discover any law of Nature does not influence the ob-
jective reality of this law. The very long history of discoveries in all domains
shows the objectivity of natural laws independently of our circumstantial
(theoretical, approximate, experimental or technical) capacity to discover
them. The use of progressively increasing energetic levels of particles (i.e. in-
creasing frequencies or decreasing wavelengths), for instance, has permitted
the realization of important progress in the understanding of our physical
universe through the ages as well as the understanding of matter (nucleons,
quarks, hadrons, leptons) structure and also the refinement of our knowledge
on the quantization of both energy levels inside atoms and angular (as well
as intrinsic) atomic and nuclear momentum.

The Schrödinger function and quantum Statistics have also permitted a jump
in our comprehension of the universe by giving effective methods for discov-
ring and interpreting all results that are obtained from experimentation and
led to powerful approximations for natural phenomena rules.

However, these phenomena contain essentially a part of uncertainties due to
the multitude of energetic and dynamic evolutive factors which govern all
aspects of matter-energy : energy levels, trajectories, interactions, transfor-
mations...

Although these phenomena are far from being regular (differentiable), they
are continuous. An electron, for instance, that changes its orbit (which is
permanently evolving) for a higher energy level (when absorbing a photon)
or for a lower energy level (together with photon emission), spends an infinitesimal time fraction before achieving its final state. During this transition, continuity of trajectory and energy conservation are both insured since we have

\[ E_0 \equiv E_e(t) \pm k(t)E_p \equiv E_e \pm E_p \]

where \( E_0 \) is the initial electron energy, \( E_e \) its final energy, \( E_p = hf \) is the photon energy and \( k(t) \) is a continuous function that increases from 0 to 1. Indeed, since photon is fundamentally a quantum object with a fixed wavelength, its existence is essentially related to time and distance. Its formation (and its absorption) takes an infinitesimal fraction of time and needs an infinitesimal extent of distance; moreover it cannot exist in a static state (i.e. independently of motion). Then, formation and existence of photon need time, distance, motion and speed notions. Its absorption and emission are necessarily related to time and energy change notions.

In the following, we will give some arguments aiming to show that the uncertainty principle, that can be written as \( \Delta x \Delta p_x \geq \frac{\hbar}{2} \) and \( \Delta E \Delta t \geq \frac{\hbar}{2} \), can not be of a canonical and universal nature. It is only a legitimate consequence of the use of the Schrödinger's equations

\[
\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(k)e^{-ikx}d\,k
\]

and

\[
g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(x)e^{ikx}d\,x
\]

in order to find out the probability of localizing a given particle, under some constraints, in a given position and determining its momentum. The uncertainty principle is stated after using the probability distributions

\[
\frac{dP}{dx} = |\psi(x)|^2 \quad \text{and} \quad \frac{dP}{dk} = |g(k)|^2
\]

and their standard deviations \( \sigma_x \) and \( \sigma_k \) as well as the De Broglie relation \( p = \frac{\hbar}{\lambda} \). Therefore, this principle states simply that this particular approach and the use of this particular method hold within themselves the uncertainty so quantized.

Moreover, the very definition of \( \sigma_x \) and \( \sigma_k \) only means that there is a large probability for the position \( x \) to be within a distance less than \( \sigma_x \) to the mean value \( \langle x \rangle \) and for the component \( p_x \) of the momentum \( p \) to be within an interval less than \( \sigma_k \) about the mean value \( \langle p_x \rangle \). However, it is obvious
that there is lesser probability of finding \( x \) within a distance lesser than \( \sigma_x \) to \( \langle x \rangle \) and a non negligible probability for \( x \) to be at a distance larger than \( \sigma_x \) to \( \langle x \rangle \).

Similarly, we can assert the same properties for \( p_x \) and \( \sigma_k \). So, \( \sigma_x \) and \( \sigma_k \) only determine a probabilistic estimation and they can not institute a sharp limiting for the uncertainty of both position and momentum and of their product.

The same reasoning can be produced when commenting on the Heisenberg relation \( \Delta A \Delta B \geq \frac{1}{2} |\langle M \rangle| \) for any two observables \( A \) and \( B \) where \( M \) is defined by \( \hat{M} = -i[\hat{A}, \hat{B}] \) ([3],11.021) and particularly for \( x, p_x \) and \(-i[\hat{x}, \hat{p}_x] = \hbar \).

Nevertheless, this does not mean that the position \( x \) of the particle or its momentum \( p_x \), at a given time, can not be well defined or can not be determined with more precision when more physical information are specified by more efficient theoretical or experimental processes.

### Scaling problem

More generally, the problem of determining the position, the trajectory and other characteristics (such as momentum and energy, for instance) of subatomic particles, which move with very large speed, was one basic problem in the heart of the foundation of quantum theory.

Indeed, in spite of our fantastic technical progress, we are, until now, incapable of visualizing or perceiving these minuscule particles and their movement and even of distinguishing between them. The time and distance scales that suit our perception actually are infinitely large regarding their infinitely small world. Our centimeters and grams and our seconds are really gigantic and inappropriate for analyzing this microworld (or rather this nano or femtoworld).

In spite of using the most sophisticated means, the electron motion around the nucleus appears for us as a foggy scene because of the infinitely small size of the electron orbit and the infinitely large speed of the electron. Not only we are incapable of determining its trajectory, but we are still at the stage of contenting ourself with determining the probability of finding it at such and such region of the minute space around the nucleus.

As for quarks, the ultrasophisticated means and the ultraclever methods are necessary in order to get some scanty information concerning their existence and their characteristics which are ultra-fluctuating and even ultra-
ephemeral. However, all that does not prevent us from conceding that the electron, for example, has, at every fixed time, a precise position and that it has a well defined speed and trajectory during an infinitesimal fraction of nanosecond in spite of all evolutions it may undergo.

In order to convince ourselves that this nanoworld respects mechanical and physical laws during infinitesimal time interval, we can imagine that a mini-creature (or a nanocreature) that is as intelligent as us but infinitely more sensitive than us regarding the infinitesimal distances and time-intervals making them (when living inside the nanoworld of atoms) capable of discerning (without using sophisticated technical means that would alter physical characteristics) between infinitesimal particles and noting the fractions of nanodistances between them as well as the fractions of nanoseconds separating two minute events and finally of perceiving the tiny transformations and fluctuations that occur within infinitesimal space and time. Moreover, we have to imagine that these intelligent creatures possess the means and the good will of communicating us their observations along infinitesimal time-intervals after registrating and schematizing them and above all after enlarging and rescaling them in order to make us capable of reading the slightest details concerning positions at very precise time and trajectories (during infinitesimal time intervals) of the nanoparticles of this nanoworld. This has to be done in such a manner that, for instance, the foggy scene of the electron motion transforms for us into interlacing lines. All that we need is to enlarge the distances and to slow down the motions.

In other respects, we can say, for instance, that the ground state energy of the hydrogen atom in the Bohr model is determined by the finiteness of the electron energy and has nothing to do with the uncertainty principle. Indeed, when an electron moves from an energy level corresponding to a $V_1$ potential energy to another level corresponding to a $V_0$ potential energy then it releases a photon $\gamma$ with $E(\gamma)$ energy. If $m_i(t)$ and $v_i(t)$ denote respectively the electron mass and speed that correspond to the $V_i$ levels, for $i = 0, 1$, then we must have

$$m_1(t)c^2 + \frac{1}{2}m_1(t)v_1^2(t) + V_1 - E(\gamma) = m_0(t)c^2 + \frac{1}{2}m_0(t)v_0^2(t) + V_0,$$

which yields

$$\Delta V = V_1 - V_0 = m_0(t)c^2 + \frac{1}{2}m_0(t)v_0^2(t) - m_1(t)c^2 - \frac{1}{2}m_1(t)v_1^2(t) + E(\gamma).$$
This shows that $\Delta V$ and consequently $V_0$ are finite.
Likewise, we can state that, when we use our proper expression for the kinetic energy $E_k$ of the Bohr atom electron (for instance), the kinetic energy can not exceed the absolute value of the potential energy because then we would have

$$\frac{ke^2}{r} \leq \gamma(t)m_0(1 + \frac{v^2}{2}) < \gamma(t)m_0(1 + \frac{e^2}{2})$$

which is impossible for sufficiently small $r$.
Therefore, there exists a finite minimal potential energy corresponding to a finite minimal energy level for the electron inside the hydrogen atom. This level is, as experiments show, $V_0 \simeq -13.6$ eV.

Besides, we notice that the inverse process to the above one takes place after an electromagnetic or a thermal energy absorption which leads the atom to an excited state and can even lead the electron to a pure "separation" from its original atom and even (occasionally) with a large kinetic energy. In that case we have (using obvious notations)

$$m_0(t)c^2 + \frac{1}{2}m_0(t)v_0^2(t) + \Delta E + V_0 = m_e c^2 + \frac{1}{2}m_e v^2.$$  

We mention that, for a non uniform movement, the mass $m_0(t)$ is variable because of the radiation phenomenon that comes with such a movement.

Theoretical and experimental measurements of the hydrogen atom ground state energy show that this energy is characterized by the planck constant $\hbar$ and the Bohr radius

$$r = a_0 = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$$

where $m$ is the electron mass corresponding to this energy level. The ground state energy actually is determined by the minimal value

$$E_m = \left(\frac{1}{2}mv^2 - \frac{e^2}{4\pi\varepsilon_0r}\right)_m = \left(\frac{\hbar^2}{2\pi r^2} - \frac{e^2}{4\pi\varepsilon_0 r}\right)_m.$$  

8.3 Classical versus quantum Mechanics

It is well known that classical Mechanics and Physics are based on some principles and laws that derive from a theoretical formulation essentially obtained from idealizing real physical systems and phenomena. This does not mean that observation and experiments are less important than theoretical formulations of classical Physics since these formulations stem from those observations and then are adopted and improved after many confrontations,
inspections and verifications. Quantum Mechanics consists of several predictive rules that derive from a huge number of experiments and ends up by founding the powerful probabilistic quantum Statistics. Some rules and results become postulates, principles or laws because none has observed exceptions that contradict them.

For our part, we maintain that the wave quantum theory structure, which leans upon Schrödinger equation, is essentially established with the (declared or undeclared) aim to be unified with the Lagrangian and Hamiltonian Mechanics by the intermediate of the Hamilton-Jacobi equation:

$$\mathcal{H} \left( q_j, \frac{\partial S}{\partial q_j}, t \right) + \frac{\partial S}{\partial t} = 0$$

where $S$ is the Hamilton’s principal function. This equation reduces, in the well known particular case, where the Hamiltonian is written as

$$\mathcal{H} = \frac{1}{2m} p^2 + V(r, t), \quad \text{with} \quad p = \nabla S \quad \text{and} \quad \mathcal{H} = -\frac{\partial S}{\partial t},$$

to

$$\frac{1}{2m} |\nabla S|^2 + V(r, t) + \frac{\partial S}{\partial t} = 0$$

that is

$$\mathcal{H} = \frac{1}{2m} |\nabla S|^2 + V(r, t).$$

Thus, the wave quantum theory is based, on one hand, upon the notion of Schrödinger’s wave functions (having the general form of $\Psi(r, t) = A_0(r, t) \exp(i\sigma(r, t))$), stationary waves, plane, quasiplane and packet waves and, on the other hand, upon the following eikonal equation (which is obtained when putting $S = \hbar \sigma$):

$$\frac{\hbar^2}{2m} |\nabla \sigma|^2 + V(r, t) + \hbar \frac{\partial \sigma}{\partial t} = 0$$

and finally upon the Schrödinger’s equation:

$$-\frac{\hbar^2}{2m} \nabla^2 \Psi + V(r, t) \Psi = i\hbar \frac{\partial \Psi}{\partial t}.$$  

This latter equation is written, for a time-independent potential $V$ and for $\Psi(r, t) = \psi(r) \exp(-i\omega t)$, as the classical time-independent Schrödinger’s equation

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi = \hbar \omega \psi = E\psi.$$  

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Then starts the mechanism that relates wave quantum Mechanics to Hermi-
tian operators (associated with Observables) and to expectation values by
means of relations such as
\[ \hat{p} = -i\hbar \nabla \quad , \quad \hat{\mathcal{H}}(\hat{q}_j, \hat{p}_j, t) \psi = i\hbar \frac{\partial \Psi}{\partial t}, \]
\[ \hat{\mathcal{H}}(\hat{q}_j, \hat{p}_j) \psi = E\psi \quad \text{for} \quad \mathcal{H} = E = \hbar \omega \]
and (as a particular case)
\[ \hat{\mathcal{H}} = -\frac{\hbar^2}{2m} \nabla^2 + V(r, t), \]
as well as the relations
\[ \langle r \rangle = \int \psi^* \hat{r} \psi d\tau \]
and
\[ \langle p \rangle = \int \psi^* \hat{p} \psi d\tau. \]
Moreover, when \( \psi \) is represented with the Hamiltonian eigenfunctions (i.e.
\( \hat{\mathcal{H}} \psi_n = E_n \psi_n \) for \( \psi = \sum \alpha_n \psi_n \)), we get
\[ \langle \mathcal{H} \rangle = \langle E \rangle = \sum |\alpha_n|^2 E_n. \]
All this is accompanied by the uncertainty principle and extended by the
Heisenberg matrix quantum theory.
It is very convenient to write down here the following quotation of [3] that
illuminates the preceding with a specific example:
"Attention is now directed to wave Mechanics and the immediate objective
is to derive the fundamentals of this branch of quantum theory in a way that
takes inspiration from one of Schrödinger’s lines of thought. As a specific
example, from which broader conclusions may be readily deduced, consider
an electron moving in a prescribed field characterized by a scalar potential
\( \varphi(r, t) \) and at most a negligible vector potential \( A(r, t) \). The wave which,
according to experimental evidence, is in some way associated with this electron
is called the wave function and is denoted \( \Psi(r, t) \). The program of deriva-
tion begin by assuming properties for the \( \Psi \)-wave such that, in a classical
situation, a packet of these waves moves according to the laws of Newtonian
mechanics and thereby ”explains” the motion of the electron. This is the
spirit of the correspondence principle since it expects as a first requirement
that the new Mechanics should predict, in a classical context, behavior ap-
propriate to that context. The hypotheses involved in this program are by
no means gratuitous but are suggested by Hamilton-Jacobi theory and by De Broglie’s results. Once the fundamental properties of the $\Psi$−wave have been determined in this way, it is an easy matter to derive the linear wave equation which $\Psi$ must obey. This equation stands at the apex of wave mechanics; from it an enormous number of deductions, some within the domain of classical Mechanics but most going far beyond that domain, can be made. It is of course, in the agreement between such deductions and the results of experimentation that the ultimate justification of the theory lies”.

The successful reconciliation between both theories has gone beyond the status of a justification process and has led to a hurried and non justified conclusion asserting that there exists, in fact, a unique Mechanics which is "naturally" the quantum Mechanics having two branches that are the wave and the matrix quantum Mechanics; the latter, initiated by Heisenberg, is considered as more general than the former. Moreover it is declared that classical Mechanics is a particular case of the quantum one and it has to be limited to macroscopic situations. For our part, we think that there is actually a unique theoretical Mechanics based upon well approved mechanical and physical laws, even though there are other ones to be discovered, checked and improved. Many fundamental laws have been established by Newton, Lagrange, Hamilton, Maxwell and his predecessors, Einstein, Planck and Bohr beside of a large number of physicists and mathematicians such as Gauss, Euler, Riemann, Fourier, Laplace, Hilbert, Schrödinger and many others. We have to admit that this Mechanics is not presently completely adapted for studying infinitesimal phenomena and therefore it must be superseded by quantum Mechanics as an efficient means for studying microscopic phenomena such as the dynamic behavior, the energy and the structure of particles. These phenomena are presently beyond the reach of our measurement means and tools and of our analyzing capacity. Until further decisive technological and theoretical progress, the analysis of these phenomena needs the predictive and probabilistic methods of the quantum Statistics guided by the quantum theory of Schrödinger, Heisenberg, Born, Fermi, Dirac, Pauli and many others. This theory was in fact inaugurated by Einstein, Planck and Bohr who have definitely proved the quantum nature of waves and energy levels beside of the quantization of electrical charges. The efficiency of these methods are fortunately increased by numerical methods progress and the presently huge capacity of empirical data treatment. However, we can state that, although some natural phenomena are quantized, there are a lot of others that are not. Electrical charges and energy levels inside the atom, for instance, are quantized. Electromagnetic waves are constituted with integer numbers of photons but wavelengths, speed, masses
and energies, for instance, are continuous variables evolving (themselves) continuously with the variable that essentially gives the continuity meaning: the time.

**Remark 2.** In the previous sections, we have established that the material and immaterial point energy is given by $E(t) = h(t)f(t)$ where $h(t)$ and $f(t)$ depend on time and $E(t)$ depends also on time by the intermediate of the temperature and environment. Frequency, wavelength and energy are then continuous mathematical objects.

**Schrödinger probability density and classical probability**

The general Schrödinger equation, where $\Psi(r, t) = A_0(r, t)\exp i\sigma(r, t)$, implies the following equation

$$\nabla \left( A_0^2 \frac{\nabla \sigma}{m}\right) + \frac{\partial A_0^2}{\partial t} = 0.$$

Comparison of this equation with the continuity equation of a substance of density $\rho$ having a current density $J = \rho v$ has led Born to identify $\Psi^*\Psi$ to an imaginary substance density $\rho$. Then, he interpreted $\Psi^*\Psi$ as being the probability of localizing the particle having $\Psi$ as its wave function. Namely, the probability of finding the particle at time $t$ in a given volume element $d\tau$ at position $r$ is

$$dP(r, t) = \Psi^*\Psi(r, t)d\tau.$$

Since $\Psi^*\Psi$ is interpreted as a probability density, it must obey the normalization condition

$$\int_{\mathbb{R}^3} \Psi^*\Psi d\tau = 1.$$

However, this fundamental notion joined to another fundamental one in Quantum theory which is the quantum measuring apparatus leads to a paradox which is clearly explained in the following quotation of [3]:

"Such an apparatus does not detect that a particular system is in a certain final state, rather it places the system in its final state and does so with a probability that depends upon the degree to which the final state was involved in the composition of the initial state!"

The basic paradox of quantum Mechanics exhibits itself here with unusual clarity; a distribution of measurements results is generated obeying a known calculus of probabilities without any apparent internal mechanism to explain how such a distribution comes into being. Many physicists accept this at face value, reasoning that the ultimate theory of the universe will probably contain
elements which are incomprehensible in terms abstracted from macroscopic experience; hence, if Quantum theory is the ultimate theory, it is not surprising that a paradox of the type just described should be incorporated in its makeup. Others, not satisfied with such a state of affairs, incline toward hidden variable theories. On this viewpoint, the pre-measurement systems of such apparatus, although quantum mechanically indistinguishable, are actually distinguishable in some yet more fundamental ways”.

Remark 3. In the next section, we will give a general classification of fundamental particles. Using Dirac operator, we show that there are originally two types of electrons that have two opposite “spins”. This classification gives a coherent explanation of the Stern-Gerlach experiment results which conversely give an argument that sustains it.

Apart from this paradox and this discussion, let us consider, as an example, the classical case of a particle in a box. If \( \psi_n \) denotes, for large \( n \), the stationary solution of the Schrödinger equation, then the probability distribution \( \frac{dP}{dx} = |\psi_n(x)|^2 \) can be compared to the classical probability which is in that case equal to \( \frac{1}{L} \). The reconciliation between these two notions increases with increasing \( n \) (c.f. [2]) and ends up by a sort of justification of the Bohr correspondence principle. Nevertheless, stationary solutions are generally considered as being highly improbable and essentially ephemeral and the utmost probable solutions are constituted with finite or infinite linear combination of such solutions. For our part, we think that only the limit cases (i.e. infinite linear combination of stationary solutions) reveal the real physical probability of finding the particle at a given position and this probability is the classical one.

Likewise, we consider that, for the harmonic oscillator, only the limit cases (taking parity into account) have genuine real value and they clearly give good approximate results as (using here and below the notations of [3]) :

\[
<x> = 0, \quad <F> = 0 \quad \text{and} \quad <E> = \frac{1}{2}KA_0^2.
\]

These results are naturally obtained, within idealized conditions, from the well established laws of classical Mechanics and Physics.

Wave packet and Born statistical interpretation

It is generally admitted that a wave mechanical packet represents the center of mass of a system of particles rather than a single particle since, in that case, there may be no particle present at the site of the packet. This
point of view which excludes the identification of a packet wave with a particle is called the Born statistical interpretation. However, when we attribute to a wave packet a definite centroid to which we associate the expectation values $< r >$ for the position and $< p >$ for the average momentum of all individual momenta of the packet wave components, we obtain, according to Ehrenfest’s theorem that $< p >$ is equal to the particle mass times the velocity of the centroid, and both $< r >$ and $< p >$ obey the laws of classical Mechanics. Contrary to the discussion about centroid of probability, hidden variables, multiple worlds or the real existence of particle entities, we maintain that what precedes gives only a new justification to the legitimacy of using wave Quantum approach when studying dynamical phenomena where classical Mechanics formulations are unreachable. For us Ehrenfest’s theorem states that statistical wave Quantum approach is, as well as the idealizing classical Physics one, just an approximate description of the real physical phenomenon.

**Relationship between wave functions and trajectories**

It is clear that a wave function $\Psi(r, t) = A(r, t) e^{i\sigma(r, t)}$ associated with a particle (such as an electron moving around a nucleus) that satisfies a Schrödinger’s equation is specified by its eikonal function $\sigma$ and its normalized amplitude $A$. The eikonal $\sigma$ which satisfies the eikonal equation determines the Hamilton’s principal function $S$ which (theoretically) determines the exact trajectory of (the center of mass of) the particle $(q_j(t))_j$. The particle trajectory can not be clearly perceived or specified with our present means. All we can perceive is its gross location at some fraction of time without discerning the particular line that is described by it because of the too many loops that are carried out by (the center of mass of) the particle during any fraction of time. So the role of the amplitude of the wave function $\Psi$ is to indicate the probability of finding the particle in a given region within the clouded region formed by the very swift particle into movement. Therefore $S$ is associated with the classical Newton-Lagrange-Hamilton Mechanics whereas $\Psi$ is associated with the quantum wave Mechanics and $\sigma$ is the connection between them.

Now, when we are dealing with two particles into movement, for instance, there are two wave functions $\Psi_1$ and $\Psi_2$, two Hamilton’s principal functions $S_1$ and $S_2$ and possibly a wave function $\Psi$ associated with the system formed with both particles and the Hamilton’s principal function $S$ associated with the center of mass of the system. If the two particles are distinguishable there are two trajectories and two probabilities and as usual the probability
of finding each of them inside two pre-indicated regions is the product of the two probabilities. If the two particles are indistinguishable bosons, then $\Psi$ is symmetrical and the two trajectories can be arbitrarily close to each other and they form a dense cloud which is more dense than the cloud formed by two indistinguishable fermions in virtue of the Pauli exclusion principle. This fact may explain the smaller probability of finding the (indifferently located) two fermions in a given region than that of finding two (indifferently located) bosons in a comparable region. Each of these probabilities is obtained by adding the amplitudes before squaring the resulting amplitude. It is normal that the new probabilities are related to the trajectory of each pair of particles as well as to the properties of each of them.

Finally, we can state that there is no antagonism between the results of quantum and classical Mechanics. The former deals only with microscopic physical situations (that roughly involve moderately small ”wavelengths”) where classical Physics is presently not efficient enough. Both quantum and classical Mechanics are applicable in macroscopic physical situations, roughly characterized by very small ”wavelengths”. In that case, if classical Physics which involves the Newtonian, Lagrangian and Hamiltonian confirmed idealizing laws is not easy to use, we can use quantum Mechanics which involves the wave packet approximate notion (with the imprecise De Broglie wavelength notion and the quantum Statistic approach).

Let us consider, for instance, the case of two rectangular barriers, one with relatively abrupt inclines and the other with a relatively gradual inclines for the potential levels (c.f. [3], p.170). The incident microscopic particle (or the packet wave) has a relatively large wavelength for the former barrier and a relatively small one for the second barrier. Classical Mechanics and quantum Mechanics both give a very little probability for the occurrence of tunneling phenomenon for the second barrier. For the first barrier type (if a sharp incline could really exist), classical Mechanics gives a null probability for both macroscopic and microscopic particles. When such a tunneling does exist, we can explain it, and other similar phenomena, by noticing that kinetic and mass energies transform easier into potential energy for abrupt potential energy inclines than for gradual ones, provided high energies are involved. This also means that such transformations are easier in smaller fraction of time. This proves, by the way, that if we have better knowledge of the particle physical situations, we can refine our probabilistic expectations. The uncertainty principle corresponds to the case where minimal physical conditions are known about particles and systems.
Conversely, determinism in Mechanics is achieved when all physical conditions are exactly known and practically realized. Initial conditions then imply, as Laplace asserted, a unique solution that extends wherever and whenever all conditions are known and satisfied. If, for example, we consider a ball that is in an actual stable equilibrium on a punctual vertex of a cone, then it must (in ideal conditions) stay indefinitely in that state. The solution is unique. If it is not really in a stable equilibrium (as it is probably the case), then it rolls downward along a cone ray in a given direction. In the idealized former case, only a given (yet infinitesimal) applied force can make it roll down in a given direction. This force can be determined, a posteriori, according to Newton’s laws. Then, we can not state in any case that a well determined initial conditions for a physical system can result, unexpectadly, on several solutions. The real problems is the possibility of defining entirely and exactly the initial conditions in order to predict the solution yet in an ideal surrounding situation.

**Conclusion**

We can now summarize the proceeding study by stating that:

- A particle is never reduced to a single point.
- Any particle has at any time $t$ a centroid.
- Even for a pointlike particle the centroid can not have a definite geometrical position inside the particle during any small time interval since a particle is permanently evolving due to internal and external interactions and energy transformations.
- If we use the Hamilton-Jacobi equation and a specific Schrödinger function $\Psi$ that determines an eikonal function $\sigma$ which is proportional to a Hamilton’s principal function $S$, we can determine $\sigma$ as the solution of the equation
  \[
  \frac{\hbar^2}{2m} |\nabla \sigma|^2 + V(r, t) + \hbar \frac{\partial \sigma}{\partial t} = 0.
  \]
  $S$ can then be theoretically determined. Therefore, if we assume that the particle centroid is fixed relatively to the particle and if the initial conditions are well defined as well as all surrounded conditions, then $S$ can be used to exactly determine the trajectory $q = (q_j)_j$ and the momentum $p = (p_j)_j$.
- Since such specifications are quasi impossible, we have to content ourselves
with using the probability density

\[ dP = \Psi^* \Psi d\tau \]

and then our knowledge of both position and momentum are limited by the uncertainty principle.

- Nevertheless, if we could have some specific information about the particle and some surrounding physical conditions, we can hope to set down some constraints on the centroid and the momentum. Then, when we take a given point as approximate centroid, we can determine a fictitious trajectory for this point and deduce, using some estimates, that the trajectory of the real centroid is within a space tube about the fictitious trajectory during a reasonable time interval. If this theoretically possible situation is realized in practice, then (using in a similar way some estimates for the momentum) we can obtain a smaller uncertainty than the limit given by the uncertainty principle.

- Finally, when we use a wave packet for a particle (in macroscopic cases or in the short wave limit and the Bohr’s correspondence principle case), we may have approximate values \(< r >\) for the position and \(< p >\) for the momentum of the particle but then, it is possible to use the idealizing classical Physics for getting better approximate values.

**8.4 Remarks on the quantum Statistics foundation**

The aim of this section is to show that only physical characteristics of an interference problem (particle types, momenta, distances, symmetries) determine the general quantum Statistics schemes.

**The two slits problem scheme**

Assume that a large number \(n\) of identical particles having a given momentum \(p\) are directed perpendicularly toward two parallel slits \(S_1\) and \(S_2\) extremely close to each other and having both the same infinitely small width. We further assume that we get, on a screen located behind the slits, only two possible outcomes \(E_1\) and \(E_2\) having respectively \(n_1\) and \(n_2\) events such as \(n = n_1 + n_2\). Finally, we assume that, between the \(n_1\) particles reaching \(E_1\), \(r_1\) particles originate from the slit \(S_1\) and \(s_1\) particles originate from the slit \(S_2\) and that between the \(n_2\) particles reaching \(E_2\), \(r_2\) particles originate from \(S_1\) and \(s_2\) particles originate from \(S_2\).

We then have

\[ n_1 = r_1 + s_1 \quad \text{and} \quad n_2 = r_2 + s_2. \]
Let $A_1$ and $A_2$ be two complex numbers such as

$$|A_1|^2 = \frac{n_1}{n_1 + n_2} \quad \text{and} \quad |A_2|^2 = \frac{n_2}{n_1 + n_2}$$

and $\alpha \in [0, 2\pi]$ such that

$$|A_1| = \cos \alpha = \sqrt{\frac{n_1}{n_1 + n_2}} = \sqrt{\frac{n_1}{n}}$$

and

$$|A_2| = \sin \alpha = \sqrt{\frac{n_2}{n_1 + n_2}} = \sqrt{\frac{n_2}{n}}.$$ 

We then have

$$A_1 = |A_1|e^{i\theta_1} = \cos \alpha e^{i\theta_1}$$

and

$$A_2 = |A_2|e^{i\theta_2} = \sin \alpha e^{i\theta_2}.$$ 

The probability that an $E_1$ event (resp. $E_2$ event) originates from $S_1$ is

$$|B_1|^2 = \frac{r_1}{r_1 + r_2} \quad \text{(resp.} \quad |B_2|^2 = \frac{r_2}{r_1 + r_2})$$

and the probability that an $E_1$ event (resp. $E_2$ event) originates from $S_2$ is

$$|C_1|^2 = \frac{s_1}{s_1 + s_2} \quad \text{(resp.} \quad |C_2|^2 = \frac{s_2}{s_1 + s_2})$$

for $B_1, B_2, C_1, C_2 \in \mathbb{C}$.

We then have

$$|B_1| = \sqrt{\frac{r_1}{r_1 + r_2}} = \cos \beta \quad \quad |B_2| = \sqrt{\frac{r_2}{r_1 + r_2}} = \sin \beta$$

$$|C_1| = \sqrt{\frac{s_1}{s_1 + s_2}} = \cos \gamma \quad \quad |C_2| = \sqrt{\frac{s_2}{s_1 + s_2}} = \sin \gamma$$

and

$$B_1 = |B_1|e^{i\beta_1} = \cos \beta e^{i\beta_1} \quad \quad B_2 = |B_2|e^{i\beta_2} = \sin \beta e^{i\beta_2}$$

$$C_1 = |C_1|e^{i\gamma_1} = \cos \gamma e^{i\gamma_1} \quad \quad C_2 = |C_2|e^{i\gamma_2} = \sin \gamma e^{i\gamma_2}.$$ 

Under these conditions, the relations

$$\begin{cases} 
|A_1|^2 = |B_1 + C_1|^2 \\
|A_2|^2 = |B_2 + C_2|^2 
\end{cases}$$
are equivalent to the system

\[
\begin{align*}
\cos^2 \alpha &= \cos^2 \beta + \cos^2 \gamma + 2 \cos \beta \cos \gamma \cos (\beta_1 - \gamma_1) \\
\sin^2 \alpha &= \sin^2 \beta + \sin^2 \gamma + 2 \sin \beta \sin \gamma \cos (\beta_2 - \gamma_2)
\end{align*}
\]

or also to the system

\[
\begin{align*}
n_1 &= \frac{r_1}{n} + \frac{s_1}{s_1 + s_2} + 2 \sqrt{\frac{r_1 s_1}{(r_1 + r_2)(s_1 + s_2)}} \cos (\beta_1 - \gamma_1) \\
n_2 &= \frac{r_2}{n} + \frac{s_2}{s_1 + s_2} + 2 \sqrt{\frac{r_2 s_2}{(r_1 + r_2)(s_1 + s_2)}} \cos (\beta_2 - \gamma_2).
\end{align*}
\]

When we exchange \(S_1\) and \(S_2\), the roles of \(n_1\) and \(n_2\), \(r_1\) and \(r_2\), \(s_1\) and \(s_2\), and \(r_1\) are exchanged and we obtain the same equations system.

If we assume a perfect symmetry of the physical system, we can state that

\[r_1 + r_2 = s_1 + s_2 = \frac{n}{2} =: m\]

and the above system is reduced to

\[
\begin{align*}
n_1 &= \frac{r_1}{2m} + \frac{s_1}{m} + 2 \sqrt{\frac{r_1 s_1}{m}} \cos (\beta_1 - \gamma_1) \\
n_2 &= \frac{r_2}{2m} + \frac{s_2}{m} + 2 \sqrt{\frac{r_2 s_2}{m}} \cos (\beta_2 - \gamma_2)
\end{align*}
\]

and, putting \(a = \cos (\beta_1 - \gamma_1)\) and \(b = \cos (\beta_2 - \gamma_2)\), to

\[
\begin{align*}
n_1 &= r_1 + s_1 + 2a \sqrt{r_1 s_1} \\
n_2 &= r_2 + s_2 + 2b \sqrt{r_2 s_2}
\end{align*}
\]

Adding these two equations, we get

\[2a \sqrt{r_1 s_1} + 2b \sqrt{r_2 s_2} = -m\]

or

\[2b \sqrt{(m - r_1)(m - s_1)} = -2a \sqrt{r_1 s_1} - m,
\]

which gives

\[4b^2(m - r_1)(m - s_1) = 4a^2 r_1 s_1 + m^2 + 4a \sqrt{r_1 s_1 m}\]
that is

\[(4b^2 - 1)m^2 - 4[(r_1 + s_1)b^2 + a\sqrt{r_1s_1}]m + 4(b^2 - a^2)r_1s_1 = 0.\]

As \(m\) is assumed to be an arbitrary large number, we obtain

\[
\begin{cases}
4b^2 - 1 = 0 \\
b^2(r_1 + s_1) + a\sqrt{r_1s_1} = 0 \\
b^2 = a^2
\end{cases}
\]

and then

\[
\begin{cases}
b = \pm \frac{1}{2} \\
a = \pm b \\
b^2(r_1 + s_1) + a\sqrt{r_1s_1} = 0.
\end{cases}
\]

These relations imply successively

\[
a = -\frac{r_1 + s_1}{\sqrt{r_1s_1}}b^2 = -\frac{r_1 + s_1}{\sqrt{r_1s_1}}a^2,
\]

\[
1 = -\frac{r_1 + s_1}{\sqrt{r_1s_1}}a,
\]

\[
a = -\frac{1}{2} \quad \text{and} \quad \frac{r_1 + s_1}{\sqrt{r_1s_1}} = 2
\]

and finally \(r_1 = s_1\) which implies

\[
r_1 = s_1 = r_2 = s_2 = \frac{m}{2} \quad n_1 = n_2 = m.
\]

Reciprocally, \(n_1 = n_2 = m\), with \(r_1 + r_2 = s_1 + s_2 = m\) and \(r_1 + s_1 = r_2 + s_2 = m\), is the only solution to the considered system, which is perfectly legitimate as we have considered a perfect symmetrical system.

We notice that, the solution of this problem can not be given by

\[
\begin{cases}
|A_1|^2 = |B_1|^2 + |C_1|^2 \\
|A_2|^2 = |B_2|^2 + |C_2|^2
\end{cases}
\]

or equivalently by

\[
\begin{cases}
\cos^2 \alpha = \cos^2 \beta + \cos^2 \gamma \\
\sin^2 \alpha = \sin^2 \beta + \sin^2 \gamma
\end{cases}
\]

since this latter leads to

\[
1 = 1 + 1.
\]

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We can sum up the preceding by noting that the above physical problem reduces to the determination of 8 Unknowns:

\[ |A_1|^2, |A_2|^2, |B_1|^2, |B_2|^2, |C_1|^2, |C_2|^2, \Re(B_1 \overline{C}_1) \text{ and } \Re(B_2 \overline{C}_2) \]

knowing 8 equations:

\[
|A_1|^2 + |A_2|^2 = |B_1|^2 + |B_2|^2 = |C_1|^2 + |C_2|^2 = 1,
|A_1|^2 = |B_1 + C_1|^2, \quad |A_2|^2 = |B_2 + C_2|^2, \quad |B_1 + C_1|^2 + |B_2 + C_2|^2 = 1,
\]

\[
r_1 + s_1 = n_1 \quad \text{and} \quad r_2 + s_2 = n_2.
\]

Obviously the previous 8 variables and 8 equations are not independent, the first three and the last two equations reduce the problem to the determination of 4 unknowns \((n_1, r_1, \Re(B_1 \overline{C}_1) \text{ and } \Re(B_2 \overline{C}_2)), \text{ for example})\). The remaining three equations and the assumed symmetry of the system determine these four unknowns.

We assume now that there are, as previously, two slits \(S_1\) and \(S_2\) and that only three possible outcomes \(E_1, E_2\) and \(E_3\). By processing as before, we notice that 12 variables are associated to this problem, which are (using obvious notations) the following:

\[
|A_1|^2, |A_2|^2, |A_3|^2, |B_1|^2, |B_2|^2, |B_3|^2, |C_1|^2, |C_2|^2, |C_3|^2, \\
\Re(B_1 \overline{C}_1), \Re(B_2 \overline{C}_2) \text{ and } \Re(B_3 \overline{C}_3).
\]

The equations relating them are only 10:

- \(|A_1|^2 + |A_2|^2 + |A_3|^2 = |B_1|^2 + |B_2|^2 + |B_3|^2 = |C_1|^2 + |C_2|^2 + |C_3|^2 = 1\) (which can be written as \(n_1 + n_2 + n_3 = n, \quad r_1 + r_2 + r_3 = m\) and \(s_1 + s_2 + s_3 = m\)).
- \(r_1 + s_1 = n_1, \quad r_2 + s_2 = n_2\) and \(r_3 + s_3 = n_3\).
- \(|A_1|^2 = |B_1 + C_1|^2, \quad |A_2|^2 = |B_2 + C_2|^2, \quad |A_3|^2 = |B_3 + C_3|^2 \) and \(|B_1 + C_1|^2 + |B_2 + C_2|^2 + |B_3 + C_3|^2 = 1\).

The first three equations permit to reduce the number of variables to 9 by writing, for example:

\[
n_3 = n - n_2 - n_1 \quad r_3 = m - r_2 - r_1 \quad s_3 = m - s_2 - s_1
\]

and, assuming that \(S_1\) and \(S_2\) are symmetrical, we reduce the variables \(n_1, n_2, r_1, r_2, s_1\) and \(s_2\) by 3 and the variables \(\Re(B_i \overline{C}_i), i = 1, 2, 3, \) by one. Five variables are left with the four equations:

\[
r_1 + s_1 = n_1 \quad |B_1 + C_1|^2 = |A_1|^2 \\
2|B_1 + C_1|^2 + |B_3 + C_3|^2 = 1 \quad |B_3 + C_3|^2 = |A_3|^2
\]
which permit the resolution of the problem using the fact that \( m \) is arbitrary. Thus, pushing this reasoning to the very end, we can show that the basis of the Probability and Statistics associated with the above problem is determined by its physical characteristics:

The symmetry of the slits, their distances, their width and the momentum of the particles which is traduced on the screen by the intrinsic characteristic of the particle energy (or momentum) \( \lambda = \frac{h}{p} \).

**Quantum Statistics versus classical Statistics**

Let us consider two particles \( P_1 \) and \( P_2 \) that can only occupy two independent physical states \( a_1 \) and \( a_2 \).

Several questions can be formulated concerning the occupation distribution and the answer to these questions fundamentally depends on the physical characteristics of these particles.

1°) If the two particles are distinguishable and each of them can occupy with the same probability each of both states, we can assert that the probability that \( P_1 \) be in \( a_1 \) and \( P_2 \) be in \( a_2 \) is equal to the probability that \( P_1 \) be in \( a_2 \) and \( P_2 \) be in \( a_1 \) which is equal to the probability that \( P_1 \) and \( P_2 \) be both in \( a_1 \) or in \( a_2 \). All these probabilities are then equal to \( \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \).

2°) Under the same conditions as previously, we can assert that if we know that particle \( P_1 \) is in state \( a_1 \), for example, then the probability that \( P_2 \) be in \( a_1 \) is equal to that of \( P_2 \) be in \( a_2 \) and both probabilities are equal to \( \frac{1}{2} \).

3°) Always under the same conditions, we can also assert that the probability that both particles be in the same state \( a_1 \) (or \( a_2 \)) is \( \frac{1}{2} \) and the probability that \( P_1 \) be in \( a_1 \) and \( P_2 \) in \( a_2 \) (or \( P_1 \) be in \( a_2 \) and \( P_2 \) in \( a_1 \)) is \( \frac{1}{4} \). Finally we can assert that the probabilities that both particles be in the same state (\( a_1 \) or \( a_2 \)) and that these two particles be in different states (\( a_1 \) and \( a_2 \) or \( a_2 \) and \( a_1 \)) are both equal to \( \frac{1}{2} \).

We now suppose that particles \( P_1 \) and \( P_2 \) are indistinguishable and that the question is:

What is the probability that both particles be in the same state (without specifying which of the two states)? The answer then depends on the physical nature of the particles.

4°) If both particles are bosons (i.e. of integer spin or also having symmetrical wave function), then experimentation shows that the probability that
both particles be in the same state \(a_1\) (or \(a_2\)) is twice the probability of being in two different states \((P_1 \text{ in } a_1 \text{ and } P_2 \text{ in } a_2\) or \(P_1 \text{ in } a_2 \text{ and } P_2 \text{ in } a_1\)). Thus the probability of each of the two first cases is \(\frac{1}{3}\) and that of each of the two last cases is \(\frac{1}{6}\). We can then state that, in accordance with Bose-Einstein Statistics, the probability that both particles be in the same state is \(\frac{2}{3}\) and the probability of being in two different states is \(\frac{1}{3}\).

5°) If both particles are fermions (i.e. of fractional spin or also having an antisymmetrical wave function) then experimentation shows that (in accordance with the Pauli exclusion principle) the probability that both particles be in the same state is null and the probability that \(P_1\) be in \(a_1\) and \(P_2\) in \(a_2\) is the same as the probability that \(P_1\) be in \(a_2\) and \(P_2\) in \(a_1\) which is \(\frac{1}{2}\). So, the probability that these two particles be in two different states is 1.

We consider now the triple experiment of collisions between particles \(^4\)He and \(^3\)He ([2], p.340) where we study the probability of the right angle scattering of these types of particles; the first is a boson and the second is a fermion.

1°) For the \(^4\)He - \(^3\)He scattering, we have two distinguishable particles and we naturally ask for determining the probability that the particle \(^4\)He be scattered upward and the particle \(^3\)He be scattered downward and vice versa. These two probabilities are equal and then we can as well ask for determining the global probability \(P_{34}\) of the right angle scattering. This problem must be resolved with classical Statistics. If \(P_{34}\) is the probability of right angle scattering that is observed after a large number of scattering experiments, we can conclude that the number of events that the particle \(^4\)He is scattered upward or downward is the same (this is due to the physical symmetry of the collision: An equal number of \(^4\)He particles comes slightly above or slightly under the collision axis) and we have

\[
P_{34} = a^2 + a^2.
\]

If we associate the amplitude \(A\) to each of these probabilities and the amplitude \(B\) to the global probability of the right angle scattering, we obtain

\[
P_{34} = |B|^2 = |A|^2 + |A|^2 = 2|A|^2.
\]

In other respects, for identical (indistinguishable) particles, the natural question is:

What is the probability of a right angle scattering of these particles independently of knowing the origin (from the right or the left) of those that are
scattered upward and those that are scattered downward?

2°) When we consider the $^3\text{He} - ^3\text{He}$ collision, experiments show that the probability of a right angle scattering is null and if the amplitude $A$ is associated with the probability of an hypothetical upward right angle scattering of $^3\text{He}$ particles coming from the right or from the left and the amplitude $B$ with the global probability of a right angle scattering, we have

$$P_{33} = |B|^2 = |A - A|^2 = 0.$$  

3°) Conversely, when we consider the $^4\text{He} - ^4\text{He}$ collision, we can attribute the positive number $a^2$ to the probability of the upward right angle scattering for $^4\text{He}$ particles coming from the right (or from the left) and we can conclude that the global probability of a right angle scattering is

$$P_{44} = a^2 + a^2 + a^2 + a^2 = 4a^2.$$  

If we associate the amplitude $A$ to the probability of an upward right angle scattering of particles coming from the right (or from the left) and the amplitude $B$ to the probability of the global right angle scattering, we obtain

$$P_{44} = |B|^2 = |A + A|^2 = 4|A|^2.$$  

Consequently, the physical reality of the experiment has determined that the probability of a right angle scattering of the $^4\text{He} - ^4\text{He}$ collision is twice the $^4\text{He}-^3\text{He}$ collision and this is independent of the fact of knowing or no the number of the particles that have followed any one of the possible trajectories. The fact of knowing such details can not obviously alter the answer to any question of the type:

What is the probability of a right angle scattering for two beams of particles having well defined physical characteristics (momentum, spin, charge, mass)? Only these characteristics hold the answer.

Concerning the above collisions, we notice that the charges, the masses and the spin have made the difference. The Pauli exclusion principle prevents the two particles $^3\text{He}$ to get sufficiently closer to each other in order to cause a right angle scattering. The masses inequality of the two particles $^3\text{He}$ and $^4\text{He}$ disadvantage them to get sufficiently closer to each other as to cause such a scattering.
**Interference Bragg condition**

Let us consider the light diffraction experiment through a slit of width $d$ ([2], p.5). The destructive interference is achieved for

$$n\lambda = d \sin \theta_n$$

where $\lambda$ is the wavelength of the used light and $\theta_n$ is given by

$$\sin \theta_n = \frac{\Delta L}{d/2} = \frac{2\Delta L}{d} = n\frac{\lambda}{d}$$

If we use the De Broglie relation $p = \frac{h}{\lambda}$, we can write it as

$$\frac{h}{p} = d \sin \theta_n$$

or

$$|\vec{p} \cdot \vec{d}| = nh.$$ 

The destructive (or constructive) interference condition is then traduced by a momentum quantization condition on the light photon. The momentum $p$ of the photon is inversely proportional to the real wavelength of the light ray. When we consider the Bragg scattering of $X$ rays through a given crystal ([2], p.142), we recover the same condition for a constructive interference, that is

$$n\lambda = 2d \sin \theta$$

where $\lambda$ is the wavelength of the used $X$ ray, $d$ is the distance between two adjacent layers of the crystal and $\theta$ is the angle that makes the ray with the plane of the crystal.

Again this condition can be written as

$$|\vec{p} \cdot \vec{d}| = n\frac{h}{2}$$

which is a sort of a quantization on the momentum $p = \frac{h}{\lambda}$ of the used photon.

We consider now the Davisson-Germer experiment. The obtained condition on the scattered electrons’ maxima is exactly the same as the previous one, namely

$$n\lambda = 2d \sin \theta$$

where $\lambda$ denotes here the De Broglie Wavelength attributed to the electron: $\lambda := \frac{h}{p}$.
We have already seen that this practical and useful notion is derived from the fundamental notion that characterizes the used electrons (as well as all other particles) namely the momentum $p$.

The constructive interference intrinsic condition, which is traduced here by the reflected electrons’ maxima in some directions is in fact the quantization relation

$$|\vec{p} \cdot \vec{d}| = n\frac{\hbar}{2}.$$

A similar interpretation can be furnished concerning the Thomson-Reid experiment. This phenomenon contributes to consider that electrons possess a wavy nature similar to electromagnetic waves. This is true in a certain sense but we do not have to deduce that the interference phenomenon here is identical to that of the electromagnetic waves and that every particle possesses a real wavelength identical to that of the electromagnetic wave. The only two fundamental common points between particles and waves (or more exactly photons) is the momentum and its quantization which is associated with the experiment physical characteristics. Recall that material particles possess other characteristics (mass, charge, spin) that photon does not possess.

The interference problem during a scattering from crystal is related, beside of the physical nature of the electron, to the atomic structure of the crystal and to the layout of the energy bands within the crystal and to the Fermi gap of the material as it is shown by the fact of recovering the Bragg condition when analyzing the wave numbers

$$k = \pm \frac{2\pi}{\lambda} = \pm \frac{n\pi}{a}$$

where $a = n\frac{\lambda}{2}$ characterizes the gaps between the crystal energy bands ([2], p.373). This clearly shows that the electrons scattering (or their reflection similar to the electromagnetic wave reflection) is advantaged for some angles that are determined by a given momentum of the electrons, the energy bands of the crystal and the Fermi gap that characterizes the material taking into account that all three factors are readily quantized.

Finally, we notice one of the numerous contradictions to which leads the formula $\lambda = \frac{2\pi}{p}$ when stated for material particles. Indeed, when we attribute to the electron inside the hydrogen atom (in accordance with the Bohr model) the wavelength $\lambda$ that satisfies

$$\lambda \simeq 2\pi < r >,$$
we get ([2],p.139)
\[ \frac{1}{r} \simeq \frac{2\pi}{\lambda} = \frac{p}{\hbar} \]
Thus, using the formula
\[ E_k = \frac{1}{2}mv^2 \]
we obtain

\[ -\frac{1}{2} = \frac{\langle E_k \rangle}{\langle V \rangle} = -\frac{1}{2}\frac{m}{ke^2} < \frac{v^2}{r} > \simeq -\frac{1}{2}\frac{m}{ke^2} \left[ \frac{m<v^2>}{\hbar} \right] \]
\[ = -\frac{<v>^2}{2ke^2} = -\frac{<v>^2}{2\alpha \hbar c} = -\frac{1}{2\alpha c} < v > \]
which gives
\[ \frac{1}{\alpha c} < v > \simeq 1 \]
or
\[ < v > \simeq \alpha c. \]
This approximate relation is obtained independently of the electron mass (in both cases: constant or depending on the speed) and independently of the energy levels, the momentum and the mean radius \(< r >\).
But, we know that \(\alpha \simeq \frac{1}{137}\) is quasi-constant for the significative energy scale of the hydrogen atom. Nevertheless, this relation is correct only for the ground state of the hydrogen atom.

### 9 Matter, antimatter and fundamental forces

Let us consider the dynamical universe \(U(t)\) as being the Riemannian space \((B_e(O,t), g_t)\), where \(g_t\) is the physical metric at time \(t > 0\), and the Laplace operator \(-\Delta\) on \((B_e(O,t), g_t)\). If \(E(t,X)\) is the universe matter-energy distribution at time \(t\), then \(E\) satisfies the matter-energy equation :

\[ \square E(t,X) = \frac{\partial^2}{\partial t^2} E(t,X) - \Delta E(t,X) = 0 \quad \text{for} \quad X \in B(O,t) \quad (E^*) \]

with \[ E(t,X)|_{S(O,t)} = 0 \quad \text{for every} \quad t > 0. \]

Let \(E_\mu(t,X)\) be a solution of \((E^*)\) written as

\[ E_\mu(t,X) = g_\mu(t)\psi \left( \frac{X}{t} \right) \quad \text{for} \quad X \in B_e(O,t), \]
where $\mu$ is an eigenvalue associated to the Dirichlet problem on $B_e(O, 1)$ with respect to $-\Delta$ and $\psi$ is the associated eigenfunction and let $D$ be the Dirac operator defined by the spinorial structure of the space $(B_e(O, t), g_e)$. The spinor fields are, in this case, the sections

$$\Phi : B_e(O, t) \rightarrow B_e(O, t) \times \Sigma_3$$

where $\Sigma_3 \simeq \mathbb{C}^2[\mathbb{R}^4] = \mathbb{C}^2$. These spinor fields are then identified with the functions

$$\Phi : B_e(O, t) \rightarrow \mathbb{C} \simeq \mathbb{R}^4$$

$$X \rightarrow (\Phi_1(X), \Phi_2(X)) = (\varphi_1(X) + i\varphi_2(X), \varphi_3(X) + i\varphi_4(X)) \simeq (\varphi_1(X), \varphi_2(X), \varphi_3(X), \varphi_4(X))$$

and, in that case, we have :

\begin{align*}
\text{i) } & D^2 := D \circ D = - \begin{pmatrix} \Delta & 0 & 0 & 0 \\ 0 & \Delta & 0 & 0 \\ 0 & 0 & \Delta & 0 \\ 0 & 0 & 0 & \Delta \end{pmatrix} \simeq - \begin{pmatrix} \Delta & 0 & 0 & 0 \\ 0 & \Delta & 0 & 0 \\ 0 & 0 & \Delta & 0 \\ 0 & 0 & 0 & \Delta \end{pmatrix} \\
\text{ii) } & D \text{ is an elliptic operator formally selfadjoint of order 1.}
\end{align*}

Consequently the set of solutions to the equation

$$\frac{\partial^2}{\partial t^2} \vec{E}(t, X) - D \vec{E}(t, X) = 0 \quad \text{with} \quad \vec{E}_t(X)|_{S_e(O, 1)} = 0 \quad (D)$$

determines a hilbertian space having a hilbertian basis $\Phi_p = (\varphi_1^p, \varphi_2^p, \varphi_3^p, \varphi_4^p)$, for $p \in \mathbb{Z}$, of eigenvectors associated to the Dirichlet problem defined by using the Dirac operator instead of the Laplace-Beltrami operator on the unit ball $B_e(O, 1)$.

Moreover, for a simple eigenvalue $\mu_n > 0$ for $-\Delta$, $\lambda_n = \sqrt{\mu_n}$ is an eigenvalue for $D$ to which is associated the eigenvector

$$\Phi^n = (\varphi_1^n, \varphi_2^n, \varphi_3^n, \varphi_4^n)$$

and we have

$$D \Phi^n = \lambda_n \Phi^n$$

and

$$D \circ D \Phi^n = \Delta \Phi^n = \mu_n \Phi^n.$$
Therefore, we have

\[
D \circ D(\varphi^n_1, \varphi^n_2, \varphi^n_3, \varphi^n_4) = -(\Delta \varphi^n_1, \Delta \varphi^n_2, \Delta \varphi^n_3, \Delta \varphi^n_4)
\]

\[
= \mu_n(\varphi^n_1, \varphi^n_2, \varphi^n_3, \varphi^n_4);
\]

which implies that \(\varphi^n_i, i = 1, 2, 3, 4,\) is an eigenfunction associated with the eigenvalue \(\mu_n\) of the classical Dirichlet problem on \(B_e(O, 1)\) and we have:

\[
\varphi^n_2 = a_n \varphi^n_1, \quad \varphi^n_3 = b_n \varphi^n_1, \quad \varphi^n_4 = c_n \varphi^n_1.
\]

We notice that, when we deal with the Laplace-Beltrami operator \(-\Delta_{g_t}\) on \(B_e(O, t)\), we can not have a simple relationship between the eigenvalues of \(-\Delta_{g_t}\) and those of the Dirac operator \(D_{g_t}\). This is only possible when the scalar curvature associated with \(g_t\) is constant, which could be the case of the very early universe.

Now, we fix three simple eigenvalues \(\mu_1, \mu_2\) and \(\mu_3\) of the Dirichlet problem on the unit ball \(B_e(O, 1)\) with respect to the Laplace-Beltrami operator \(-\Delta\) and we put \(\lambda_i = \sqrt{\mu_i}\) for \(i = 1, 2, 3\). We notice that we think that the following mathematical modeling aiming to classify the different types of matter and antimatter does not depend of the choice of these three eigenvalues.

We think that this choice corresponds to the instoration of a measure scale concerning all fundamental notions related to the matter-energy, the time and the distances. We assume then that our choice corresponds to the international system (I.S.) which leads to the classical universal constants of Physics.

According to this choice, we can reconsider the following equalities (previously and successively stated) concerning the energy \(E(t, X(t))\):

\[
E_\mu(t, X(t)) = h_\mu(t)f_\mu(t) = c(\mu)\sqrt{\mu} = m(\mu),
\]

\[
E_\mu(t, X(t), T(t)) = h_\mu(t, T(t))f_\mu(t, T(t)) = c(\mu, t)\sqrt{\mu} = m(\mu, t),
\]

and

\[
h_P = \frac{m(\mu, t)}{f_D(\mu, t)} = \frac{E_\mu(t, X(t))}{f_D(\mu, t)}
\]

where \(h_P\) is the Planck constant and \(f_D\) is the De Broglie frequency. Then, with our fixed choice of \(\mu\), we can write the second relation as

\[
E(t) := E(t, X(t)) = h(t)f(t)
\]

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where $h$, $f$ and $E$ depend on time by the temperature intermediary. The last equality can be written as

$$h_P(t) = \frac{E(t, X(t))}{f_D(t)}$$

or

$$E(t, X(t)) = h_P(t)f_D(t)$$

which is the classical relation

$$E = h_P f_D$$

The only difference is that $E$ and $f_D$ depend on time through the temperature $T(t)$ dependence. The variations of this dependence is really negligible at our temporal and cosmic scale.

We notice also that, according to our model, $h(t)$ and $f(t)$ change, at fixed temperature, with distance (or time) but their product keeps constant. However $h(t)$, $f(t)$ and $h(t)f(t) = E(t)$ change with temperature. Moreover $f_D$ changes with energy and therefore with temperature. Otherwise, we think that what we measure in most experiences are in fact the quantities $f(t)$ and $\lambda(t)$ and not the classical De Broglie’s quantities $f_D$ and $\lambda_D$.

### Classification of matter, antimatter and energy

We consider now the eigenvector subspace $E_{\lambda_1}$, which is generated by

$$\Phi_1 = (\varphi_1, a_1 \varphi_1, b_1 \varphi_1, c_1 \varphi_1) =: (\psi_1, \psi_2, \psi_3, \psi_4)$$

and we put

$$\overline{\Phi}_1 = (\overline{\varphi}_1, a_1 \overline{\varphi}_1, b_1 \overline{\varphi}_1, c_1 \overline{\varphi}_1) := (-\varphi_1, -a_1 \varphi_1, -b_1 \varphi_1, -c_1 \varphi_1)
=: (\overline{\psi}_1, \overline{\psi}_2, \overline{\psi}_3, \overline{\psi}_4)$$

We then take the vectors of $\mathbb{R}^8$

$$\Gamma_1 = (\psi_1, \overline{\psi}_2, \psi_3, \overline{\psi}_4, \overline{\psi}_1, \psi_2, \overline{\psi}_3, \overline{\psi}_4)$$

and

$$\Gamma_2 = (\overline{\psi}_1, \overline{\psi}_2, \overline{\psi}_3, \overline{\psi}_4, \psi_1, \psi_2, \psi_3, \psi_4).$$

Thus, we obtain

$$D \times D \Gamma_1 = \lambda_1 \Gamma_1,$$

$$D \times D \Gamma_2 = \lambda_1 \Gamma_2$$
and
\[ \Delta \psi_1 = \mu_1 \psi_1, \quad \Delta \bar{\psi}_1 = \mu_1 \bar{\psi}_1. \]

Now, we replace in \( \Gamma_1 \), \( \psi_1 \) by \( e_{-1/2}^-, \psi_2 \) by \( \nu_e \), \( \psi_3 \) by \( u_{1/2} \), \( \psi_4 \) by \( d_{1/2} \), \( \bar{\psi}_1 \) by \( e_{+1/2}^- \), \( \bar{\psi}_2 \) by \( \bar{\nu}_e \), \( \bar{\psi}_3 \) by \( \bar{u}_{-1/2} \) and \( \bar{\psi}_4 \) by \( \bar{d}_{-1/2} \). Likewise, in \( \Gamma_2 \) we replace \( \psi_1 \) by \( e_{-1/2}^- \), \( \psi_2 \) by \( \nu_e \), \( \psi_3 \) by \( u_{-1/2} \), \( \psi_4 \) by \( d_{-1/2} \), \( \bar{\psi}_1 \) by \( e_{+1/2}^+ \), \( \bar{\psi}_2 \) by \( \bar{\nu}_e \), \( \bar{\psi}_3 \) by \( \bar{u}_{1/2} \) and \( \bar{\psi}_4 \) by \( \bar{d}_{1/2} \). By rearranging the components of the vectors \( \Gamma_1 \) and \( \Gamma_2 \), we obtain the two energy vectors
\[ \Gamma_1 = \left( e_{1/2}^-, e_{-1/2}^+, \bar{\nu}_e, \nu_e, u_{1/2}, \bar{u}_{-1/2}, d_{1/2}, \bar{d}_{-1/2} \right) \]
and
\[ \Gamma_2 = \left( e_{1/2}^+, e_{-1/2}^-, \bar{\nu}_e, \nu_e, \bar{u}_{1/2}, u_{-1/2}, \bar{d}_{1/2}, d_{-1/2} \right) \]
each of which represents one of the two polarizations of the same electromagnetic wave or the same photon.

In that way, we have associated to the Dirichlet problem solution \( \psi_1 \) the electron \( e_{1/2}^- \) in \( \Gamma_1 \) and \( e_{-1/2}^- \) in \( \Gamma_2 \), to the solution \( \psi_2 \) we have associated the neutrinos \( \nu_e \), to the solution \( \psi_3 \) (resp. \( \psi_4 \)) we have associated the quark \( u_{1/2} \) in \( \Gamma_1 \) and the quark \( u_{-1/2} \) in \( \Gamma_2 \) (resp. the quark \( d_{1/2} \) in \( \Gamma_1 \) and \( d_{-1/2} \) in \( \Gamma_2 \)) and finally to each solution \( \bar{\psi}_1 \), we have associated the antiparticle of the particle associated with \( \psi_i \) with an opposite spin to the particle spin.

Moreover, we think that if we fix, in \( \Gamma_1 \) (resp. \( \Gamma_2 \)), the couple \( (e_{1/2}^-, e_{-1/2}^-) \) (resp. \( (e_{1/2}^+, e_{-1/2}^-) \)) in the first case, then all the couples involving the neutrinos-antineutrinos and the quarks-antiquarks can be located in any one of the other cases of \( \mathbb{R}^8 \simeq \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \).

This possibility evokes the colors symmetry of the standard model and could explain the existence of each quark and antiquark under three distinct variants. The existence of three colors attributed to each flavor of quarks is well confirmed by the rate of the hadrons’ formation during the electron-positron annihilation experiments. This also could explain the existence of several kinds of mixed colored gluons.

By processing in the same way with the eigenvalues \( \mu_2 \) and \( \lambda_2 \) as well as with \( \mu_3 \) and \( \lambda_3 \), we obtain the pure energy vectors
\[ \Gamma'_1 = \left( \mu_{1/2}^-, \mu_{-1/2}^+, \bar{\nu}_\mu, \nu_\mu, s_{1/2}, \bar{s}_{-1/2}, c_{1/2}, \bar{c}_{-1/2} \right) \]
\[ \Gamma'_2 = \left( \mu_{1/2}^+, \mu_{-1/2}^-, \bar{\nu}_\mu, \nu_\mu, s_{1/2}, \bar{s}_{-1/2}, c_{1/2}, \bar{c}_{-1/2} \right) \]
\[ \Gamma''_1 = \left( \tau_{1/2}^-, \tau_{-1/2}^+, \bar{\nu}_\tau, \nu_\tau, b_{1/2}, \bar{b}_{-1/2}, t_{1/2}, \bar{t}_{-1/2} \right) \]
\[ \Gamma''_2 = \left( \tau_{1/2}^+, \tau_{-1/2}^-, \bar{\nu}_\tau, \nu_\tau, b_{1/2}, \bar{b}_{-1/2}, t_{1/2}, \bar{t}_{-1/2} \right) \].
Thus, we obtain (within the framework of our model) that the fundamental particles are precisely 24, twelve particles:

\[ e^{-}, \mu^{-}, \tau^{-}, \nu_{e}, \nu_{\mu}, \nu_{\tau}, u, d, s, c, b \text{ and } t \]

to which are associated their antiparticles. Each of the three former particles (and antiparticles) exists under two variants that correspond to two opposite spins. Each of the three following particles exists only with a negative spin \(-1/2\) (left particles). Each of the six flavors of quarks exists with two opposite spins and under three variants which correspond to three colors.

Anyone of these particles and antiparticles (except probably the neutrinos and antineutrinos) is formed with a distribution \( E(t, X) \) on a domain \( D_{t} \) giving the particle (or the antiparticle) a material (or antimaterial) mass \( m(t) \) with a given density \( \rho(t) \).

Thus, the fundamental particles are (according to our model) originated by the solutions \( \Phi^{\alpha} \) of the Dirichlet problem \( (D) \) associated to the Dirac operator on the unit ball \( B_{e}(O, 1) \), each of them determines 4 solutions of the Dirichlet problem associated with the Laplace-Beltrami operator on \( B_{e}(O, 1) \). These solutions determine, on one hand, the \( \Gamma \) energy vectors which are associated to the three generation of leptons and quarks and, on the other hand, determine the solutions of the matter-energy equation \( (E^{*}) \) giving birth to particular distributions \( E(t, X) \) on domains \( D_{t} \) constituting in that way all fundamental particles (and antiparticles) having given mass-energies that evolves with time, temperature and all sorts of interaction.

All of these particles (except the neutrinos) undergo interactions between themselves and with photons. All of them can annihilate with their antiparticles in order to create photons and gluons. Conversely, photons and gluons can give birth to pairs of particles.

Only the electron, the neutrinos and the \( u \) quark are absolutely stable against all disintegrations. The five more massive quarks (\( t, b, c, s \) and \( d \)) can transform by means of natural disintegration or weak or electromagnetic interactions in order to give birth to less massive quarks. The less massive of all quarks, namely the \( u \) quark, need to be supplied with energy to transform into another quark. This supply can be achieved during interactions with electrons or antineutrinos (for instance). The \( u \) quark transformation can also be achieved during a nuclear transformation resulting on a binding energy increase. The leptonic families \( \mu \) and \( \tau \) always give birth to electrons or positrons together with neutrinos or antineutrinos. Neutrinos are created during annihilations, disintegrations and weak interactions between all types of particles. They are absolutely stable and electrically neutral and we do not no if they are material (or antimaterial) particles or a type of a pure energy particles which, for us, is more probable. The infinitesimal mass of
these particles (in case where it really exists) is not yet precisely determined. We notice that the only absolutely stable fundamental particles (i.e. having an illimited lifetime) are the electron, the quark $u$ and the neutrinos. To these stable elementary particles we must add the only composite material particle that is considered until now as being stable against all sorts of disintegration (or more precisely having a lifetime that is more then $10^{32}$ years) : the proton. The immaterial particles (i.e. having a null mass) of pure energy, namely the photons (and gluons) are equally stable. They hold by themselves the pre-existence of matter and the capacity of creating all sorts of matter and antimatter and interacting with all sorts of matter and the capacity to transform into all sorts of energy.

**Natural forces**

Concerning natural forces, we deduce from our global study that there essentially exist two fundamental forces that are inherent into the creation, the formation and the evolution of the universe. The first one is the electromagnetic force which is originated in the formation of two sorts of matter (and antimatter) since they were differently charged. The arithmetical addition of the electron and the six quarks charges leads to the equality

$$1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{2}{3} + \frac{2}{3} + \frac{2}{3}$$

The two differently charged particles attraction recalls the original unity of matter-energy. The two likewise charged particles repulsion recalls the expansion, the original movement and the original dispersion of matter. This force has an unlimited range and is proportional to the product of charges and to the inverse of the distance squared. It takes place between all types of charged matter (and antimatter) : quark-quark, quark-lepton, lepton-lepton and particularly electron-electron, electron-proton, proton-proton inside atoms and between atoms and, more generally, between all charged material agglomerations. The electromagnetic force results from electrical and electromagnetic fields which naturally exist around charged static matter and charged matter into movement. It did not need to be conveyed by any type of intermediaries. The electromagnetic waves are simply radiations that are emitted by accelerated charges or by electron changes of energy levels (to speak only about atomic radiations without speaking about nuclear ones) regardless of the possible receivers of these radiations. The energy exchange, conveyed by electromagnetic waves (exchange of a large number of photons), leads only to the modification of the surrounding electromagnetic field which modifies the surrounding force field that is the gradient field of the generalized energy.
distribution $E_t(X)$. This distribution, as we have seen before, is reflected by the real physical metric $g_t$ which characterizes the physical universe, permanently evolving.

The second fundamental force is the generalized gravity which is based on the tendency of material particles to attract each other and matter to contract in order to recover its quasi-original ultracondensed state as the neutron stars or to transform into black holes. This tendency is counterbalanced by the electromagnetic repulsion and the fermionic exclusion, on one hand, and the kinetic energy related to the movement that has been initiated by the Big Bang on the other hand. The gravitational force takes place naturally for all material particles or bodies: quark-quark, nucleon-nucleon, nucleus-nucleus, nucleus-electron, atom-atom, star-meteorite, star-star, galaxy-galaxy, etc. It is proportional to the product of masses and to the inverse of the distance squared. It has an unlimited range and did not need to be conveyed neither by gravitons nor by gluons. A given mass is surrounded by a gravitational field which attracts proportionally any other mass located around it. It is the same for the invisible black holes which probably constitute a large part of the mass equivalent of the universe.

Although these two forces take place simultaneously and do not exclude each other (both fields exist around charged matter), the study of laws that govern them shows that the electromagnetic force is predominant over the gravitational one except on the cosmic level and in the case of two large masses weakly charged (such as an apple a few meters above the earth). However, we notice that both of static formulas that characterize these two forces, namely $F = K q_1 q_2 / r^2$ and $F = G m_1 m_2 / r^2$, are inadequate at subatomic scale. Indeed, concerning the gravitational force, we have to take into account the important (vibrational and rotational) movement of all very energetic subparticles that are extremely close each to other and introduce a correction (generally qualified as relativistic) to the above formula for gravity involving speeds, accelerations, frequencies, momenta and mass evolutions. For the electromagnetic force, the corrections that have to be operated to the Coulomb formula are decisive. Inside atoms and nucleons, we have to use Lorentz formula, namely $F = q(E + v \times B)$. The global (orbital and intrinsic) magnetic field $B$ produces the indispensable correction to the attractive and repulsive forces inside atoms, nucleus and nucleons taking into account the important fact that moving charges, even if they are both positive or negative, give birth to force fields that can be attractive or repulsive according to the movement directions. The association of these force fields (including the screen effect) with gravitational force (corrected by dynamic effects and their
multi-energetic consequences) leads to the internal force field at all levels of material particles to which we have to add external forces and the internal and external interactions.

**Strong and weak interactions**

Let us, for instance, consider a neutron constituted with a bound state of two negatively charged quarks and a positively charged one symbolized by $udd$. Two fundamental reasons push these quarks to form such a bound state owning some cohesion inside this particle: the original unity of matter and the fractional charges of these quarks. An individual quark does not exist in Nature a little time after the Big Bang which induces the confinement effect. It is the same for a particle with fractional charge. The internal force field, that prevents the particle against the dispersion of these three quarks or against the separation of one of them, comes from the corrected tripolar gravity field and the total electromagnetic one. The global field is responsible of the neutron inner cohesion. The binding energy of this bound state is equal to

$$m_u c^2 + 2m_d c^2 - m_n c^2$$

This bound state is not static (as well as for all hadrons); it is not stable too (as well as for all hadrons except the proton). Indeed, there is, inside the neutron, a permanent exchange of pure energy particles (i.e. massless particles), called gluons, in the same manner as the permanent exchange of photons inside atoms. Furthermore, there is an electromagnetic and gravitational interaction between each of the quarks inside the neutron and the neighboring quarks inside nucleons of neighboring nucleus in the same manner as there is an electromagnetic interaction between protons and electrons of two neighboring atoms inside molecules. To all these forces is added an electromagnetic attraction between nucleons inside the nucleus, doubled with the electromagnetic interactions between the nucleus and electrons inside the atom and between all particles inside and outside the atom. Electromagnetic interactions can lead to a neutron decay (qualified as weak decay) that transforms neutrons into protons by the $\beta^-$ decay process symbolized by

$$n \rightarrow p + e^- + \bar{\nu}_e$$

This decay actually is realized by the emission, from the quark $d$, of a virtual (for us a real) particle denoted by $W^{*-}$ which gives birth to an electron and an antineutrinos symbolized by

$$W^{*-} \rightarrow e^- + \bar{\nu}_e.$$
However, we notice that in a large number of interaction and decay processes (called weak) three particles denoted by $W^+$, $W^-$ and $Z^0$ are emitted; they are material bosons having well determined mass energies (although the values of which are approximately determined essentially because of their dynamic and evolutive nature) and very short lifetimes. We produce below (fig.7), as representative examples, some schemes (called Feynman diagrams) of interactions and decays in which are involved quarks, leptons and neutrinos beside of these three particles acting as weak interaction intermediates (c.f.[2]).

The previous description of neutron bound state is also valid for a proton bound state $uud$ inside a nucleus of a given atom. The electrical repulsion effect between two positively charged quarks inside a proton or between two protons inside an atom nucleus is counterbalanced by the attractive forces generated by the electromagnetic and gravitational fields resulting in an energetic equilibrium at every time (even though evolving permanently) between mass, potential and kinetic energies. The binding energy of the proton bound state is

$$2m_u c^2 + m_d c^2 - m_p c^2$$

and inside the nucleus is

$$\Sigma m_p c^2 + \Sigma m_n c^2 - m_N c^2$$

We notice that, for light nucleus, the nucleus stability against strong decay, such as, for instance, natural fission and $\alpha$- decay requires an approximate equality of the atomic numbers $Z$ and $A - Z$ whereas, for heavy nucleus, the stability requires a significantly larger number of neutrons than protons. The repulsion effect is then more important when $Z$ is close to $A - Z$ and it cannot be durably counterbalanced by the electromagnetic attractive component together with gravity. The $\beta$ (called weak) decay and the $X$ - rays emission are always possible in a natural or artificial ways (by means of weak or electromagnetic interactions) except for protons as it is generally admitted. We notice however that the so called strong interactions are actually conveyed by the gluons’ intermediary which are massless bosons, exchanged (emitted and received) at short range inside nucleons and nucleus contrary to the $\alpha$, $\beta$ and $\gamma$ decays which are of a different nature.

Do we have to consider gluons as being fundamentally different from electromagnetic photons?

Our answer is No : virtual or real, involved into strong interactions such as those represented in fig.8 or into interactions such as $g \leftrightarrow q + \bar{q}$, they are not essentially different from photons. Indeed, photons are involved, in a similar way, into many interaction, exchange and annihilation processes. We
can invoke, for instance, annihilations such as $\gamma \leftrightarrow e^+ + e^-$, annihilations with pair production such as those represented in fig.9, or jet phenomena produced by electron-positron’s collision represented by

$$e^+ + e^- \rightarrow q + \bar{q} + g$$

The difference between these two pure energy particles consists on the fact that gluons are only exchanged by particles with extremely short range and lifetime while photons are exchanged within any range but also can be emitted in many circumstances independently of any potential receiver.

Consequently, we do not think that there exists a third fundamental force that would be independent of the two previously depicted fundamental forces, based on strong charges (or colors) inherent in quarks and conveyed by intermediary gluons (which are for us, as the photons, intermediary particles for strong interaction that do not transmit charges). Actually, each quark does exist with three different colors in much the same manner as same particles exist with two opposite spins. These colors only constitute three different kinds of the same particle without (until now) any detectable physical difference between them contrary to the clear physical difference between the two opposite spins of the same particle. Our judgment is however sustained by the fact that there is no significantly stable elementary particle (such as a nucleon, for instance) which is constituted with similarly charged quarks nor a nucleon, for example, that is constituted with uniquely two protons or two neutrons. Gravity and electromagnetic forces having not permitted such a bound state, we can wonder why the ”strong charges” or the ”strong forces” would not have played a favorable role?

Thus, we can maintain that, although strong forces and charges do not really exist, strong interactions and bounds do exist. Strong bounds, organically related to the two fundamental forces, contribute together (within all energy rules’ agreement) to form all bound states between quarks inside hadrons and between hadrons inside nucleus. The stability of these bound states is directly related to the stability of the energy equilibrium (which is less or minus temporary) between the (electromagnetic and gravitational) potential energy, the vibrational and rotational (thermodynamic) kinetic energy and the masses’ energy of all components. The strong bounds (and interactions) essentially rely on the transformation matter-energy and reciprocally. These transformations are conveyed by intermediary gluons that are exchanged between quarks (involving many energy transformations) which change their nature and then the nature of hadrons containing them. We can have changes of masses, of binding energy and of electrical charges (which are
accompanied with emission of electrons or positrons, for example) leading to all sorts of nuclear transformations. That is what explain the extremely short range and lifetime of gluons contrary to the photons which are emitted without any prior exchange condition.

Likewise, all weak interactions involving the $W^+, W^-$ and $Z^0$ bosons (as those we have showed in fig. 7) and weak decays (when studied parallel to interactions such as

\[ e^+ + e^- \rightarrow \psi' \rightarrow \pi^+ + \pi^- + \psi \quad \text{with} \quad \psi \rightarrow e^+ + e^- , \]
\[ e^+ + n \rightarrow \pi_e + p , \quad e^- + p \rightarrow n + \nu_e , \]
\[ p + p \rightarrow d + e^+ + \nu_e \quad \text{and} \quad \gamma + d \rightarrow n + p \]
do not indicate the existence of another fundamental force (called weak force) that would be based on weak charges carried by hadrons and leptons and would be conveyed by the $W$ and $Z$ bosons. These particles have a very short lifetime, an extremely short range and are, essentially, the weak interactions’ intermediates between quarks and leptons at extremely short distances. Electromagnetic interactions, conservation laws and the Pauli exclusion law are sufficient for explaining all these phenomena as well as some others such as the formation of deuteron in the early universe from neutron and proton and the nonexistence of a bound state of two protons or two neutrons, for instance. They are also sufficient for explaining the (solar) cycle :

\[ p + p \rightarrow d + e^+ + \nu_e \]
\[ p + d \rightarrow ^3\text{He} + \gamma \]
\[ ^3\text{He} + ^3\text{He} \rightarrow p + p + \alpha \]

and the interaction

\[ n + ^3\text{He} \rightarrow \alpha + \gamma , \]

for instance.

**Remark** : Other discussions on the bound states and fundamental forces will be given in sections 11 and 13.
Binding energy and Matter-Energy

The global energy of the universe $E_0$ was concentrated, before the Big Bang, at a singular point of the space-time (i.e. at the vertex of the space-time semi cone) where we can consider that all of the energy was a sort of a binding energy. The appearance, after the Big Bang, of the matter under the form of hadrons with their internal binding energy and particularly of protons and neutrons which (later on) formed the nuclei with their proper binding energy was not the only transformation of the initial energy $E_0$. Indeed, we have to add to the mass energy of these particles and agglomerations of particles and to their binding energy that is inherent in their formation the kinetic energy of the matter into movement and their interaction potential energy which essentially consists on the gravitational and electromagnetic potential energies. These two sorts of potential energy are in fact two forms of binding energy. For an atom (for instance), the quantity

$$(m_N + \Sigma m_i - m_a)c^2$$

where $m_N, \Sigma m_i$ and $m_a$ are respectively the nucleus, the electrons and the atoms’ masses, is the binding energy of the atom (i.e. the mass difference $(m_N + \Sigma m_i) - m_a$ that is transformed into energy). This energy is tightly related to the electromagnetic potential energy inside the atom (the gravitational potential energy inside the atom is negligible). The passage of an electron from a potential level to another necessarily is accompanied with a change of the binding energy inside the atom. The passage of the electron’s hydrogen atom from the ground state to an excited state is expressed by a weaker attractive electromagnetic potential energy and a weaker binding energy in such a way that

$$e'_l := (m'_p + m'_e - m'_H)c^2 < (m_p + m_e - m_H)c^2 =: e_l$$

We notice that the mass $m'_H$ of the atom and the mass $m'_e$ of the electron inside the atom have increased; the electron’s energy increase is due to the speed and the kinetic energy decreases.

The absorption of a photon with energy $e_p$ by the electron increases its (negative) potential energy and slightly increases its mass by decreasing its speed and its kinetic energy. However, the total electron’s energy (potential + kinetic) has increased and obviously the global mass of the atom has increased by $\frac{e_p}{c^2}$. As the electron has absorbed the photon energy $e_p$, the increase of the atom mass originates from the transformation of the binding energy dif-
ference $e_l - e'_l$ into mass:

$$\frac{e_l - e'_l}{c^2} = m'_H - m_H = \frac{e_p}{c^2}$$

The gravitational potential energy can be interpreted in the same manner in terms of binding energy that is related to the transformation of mass into energy and vice versa.

For a planet that has a stable orbit around gravitational pole (such as a star), there is a stable equilibrium between the mass energies, the kinetic energies, the gravitational potential energy and the binding energy when neglecting the thermic and gravitational radiations. When the planet orbit becomes closer to the pole, the gravitational potential energy becomes more and more negative, the orbital speed and the kinetic energy of the planet continually increase; the binding energy also increases whereas the planet mass decreases as well as the mass of the system constituted of the gravitational source and the planet. Part of the mass difference transforms into additional binding energy of the system that is caused by the increasing intensity of the gravitational field; the other part transforms into kinetic energy. In the extreme case where a planet is swallowed by a black hole, the mass energy is entirely transformed into a binding energy traduced by the black hole energy increase. A similar phenomenon happens when the orbits of a binary system are slightly and continuously reduced; the absolute value of the negative gravitational potential energy increases, the kinetic energy also increases and the total mass of the system decreases whereas the binding energy increases.

We also notice that the passage of the hydrogen atom’s electron from an orbit to another that is closer to the nucleus displays a similar scheme where the electromagnetic potential energy plays the role of the gravitational potential one and the (temporary) energetic equilibrium is again insured by the intermediate of the new orbital motion after the emission of a photon with a well determined energy: the electron’s total energy decreases, its electromagnetic potential energy decreases, its speed and kinetic energy increase, its mass decreases the energy and the mass of the atom decrease whereas the binding energy increases. A similar process happens inside nuclei where the binding energy between nucleons is created at the expense of the decrease of all individual masses. In that case, the electromagnetic and gravitational potential energies are caused by the charged constituents of the nuclei, i.e. the quarks, although each nucleon forms a bounded state between three quarks having different flavor, spin or color.
To sum up, the universe initial energy $E_0$ has, after the Big Bang, the following forms: Mass energy, kinetic energy, binding or interaction energy having gravitational or electromagnetic potential energy character and the pure energy (that can be considered as a sort of binding energy) of the black holes. However, the mass energy of the visible matter is only a small part of the total universe’s mass energy and we think that what is generally called dark matter or dark energy is constituted with black holes (whose energy has a gravitational mass energy character), neutron stars, brown dwarves and other invisible stars associated with a large number of binary systems and finally with invisible ordinary matter such as planets. Nevertheless, the classification of the background energy associated with the neutrinos still is enigmatic. Neutrinos have probably no mass; although they have no electromagnetic interactions they significantly contribute to the radiational energy of the universe.

**Brief description of the universe**

Finally, we notice that our global model is compatible with the classical description of the different stages of the universe evolution and the evolution of matter, antimatter and energies.

1. At the beginning of the expansion (for $t \ll 1$), the infinitely small sized universe was dominated by ultra-energetic radiations, in perfect thermal equilibrium state, having infinitely large frequencies (i.e. infinitely small wavelengths) with infinitely large radiations’ density under infinitely large pressure and at an infinitely large temperature; all of them decreasing very quickly.

2. Then began the stage qualified as quark-leptons soup (without any doubt, with their antiparticles) having two opposite sorts of electrical charges, followed by the formation of protons, neutrons and (without doubt) neutrinos with their antiparticles. This formation has become possible thanks to the relative attenuation of the gigantic original radiations energy and original pressure and temperature.

3. The conservation and exclusion rules permitted then for some electromagnetic interactions to take place more than others resulting in a fall of neutrinos’ formation (neutrinos’ freeze) and favored the progressive disappearance of antimatter (such as positrons) for the benefit of more protons than neu-
trons’ production. All this is governed by energetic equilibriums involving stability effect and lifetime disparity.

4. After that occurred the formation of light stable nucleus and other, more or less stable, material particles coming with the decrease of radiations’ density for the benefit of matter density, so permitting the formation of all material agglomerations from atoms and molecules to galaxies. From this stage qualified as photons’ freeze, the tendency toward matter density predominance over the radiation one is increasing although the general density, the general pressure and the cosmic temperature are decreasing because of the continual expansion which occurs at a speed very close to its limit 1.

Finally, according to our previous global study, we can suppose that we have, for each \( t > 0 \), the formula

\[
v(t) = \lambda(t)f(t) = \frac{1}{2\pi} \frac{\sqrt{\mu}}{t} \lambda(t)
\]

where \( v(t) \), \( \lambda(t) \) and \( f(t) = \frac{1}{2\pi} \frac{\sqrt{\mu}}{t} \) are respectively the expansion speed, the wavelength and the frequency of the expansion waves creating the universe geometrical space. We can then assume that

\[
\lambda(t) = \frac{2\pi}{\sqrt{\mu}} t \frac{a(t)}{b(t)} \quad \text{and} \quad v(t) = \frac{a(t)}{b(t)}
\]

with

(i) \( a(t) \) is an increasing function satisfying the following properties:

\[
\lim_{t \to 0^+} a(t) = 0 \quad \text{and} \quad \lim_{t \to +\infty} a(t) = +\infty.
\]

(ii) \( b(t) \) is also increasing and satisfies the properties:

\[
\lim_{t \to 0^+} b(t) = b > 0 \quad \text{and} \quad \lim_{t \to +\infty} b(t) = +\infty.
\]

(iii) \( v(t) = \frac{a(t)}{b(t)} \) is equally increasing and satisfies the properties:

\[
\lim_{t \to 0^+} \frac{a(t)}{b(t)} = 0 \quad \text{and} \quad \lim_{t \to +\infty} \frac{a(t)}{b(t)} = 1.
\]

(Such a function exists; indeed we can obviously take

\[
v(t) = \frac{\ln(1 + t)}{\ln(1 + \alpha + t)} \quad \text{with} \quad \ln(1 + \alpha) = b).
\]
In order to determine approximately \( a(t) \), \( b(t) \) and \( b \), we have to collect a large number of extremely precise measures using many sophisticated technical means such as ultra-powerful telescopes and ultra-high-energy nuclear accelerators in order to go forward in the knowledge of the most early universe and the understanding of the original matter-energy structure. However, we notice that the value of the real number \( b \) is decisive: For small or infinitely small \( b \), the universe age is close to what is now generally admitted; whereas, if \( b \) is large or infinitely large, then the universe age is larger then what we are presuming (this age will be approximately determined at section 11) and its evolution until the earliest stage that we can scrutinize at present has taken a long time. In this case, the time \( T_0 \) (expressed in seconds) needed for the universe to attain the size \( B_\ast(0, 1) \) (where 1 represents here \( 3 \times 10^8 \text{m} \)), which would correspond to a significant expansion speed (becoming later on close to 1), is rather large and it could even be very large (\( T_0 \gg 1 \)). Therefore, if we assume that the present universe size is approximately \( t \times (3 \times 10^8 \text{m}) \), then the effective universe age, starting from \( t = 0 \), would be nearly \( t + T_0 \). Conversely, if we assume that the time interval from the Big Bang until now is \( t \), then the universe size would be close to \( t - T_0 \).

This eventuality is supported by the validity of general relativity theory concerning the electromagnetic waves - gravitational field interaction. The influence of gravitational force on waves is confirmed by many natural phenomena’s observation and many experiments such as the Pound and Rebka one. This influence is moreover reconfirmed by our model as the gravitational field curves geodesics and contracts distances. If \( X(t) \) describes an electromagnetic wave trajectory (with respect to a virtual fixed frame), we have \( \nabla_t^{g_t} X'(t) = 0 \) and \( \| X'(t) \|_{g_t} = c = 1 \) according to special relativity first postulate of Einstein whereas we have, inside a significative gravitational field, \( \| X'(t) \|_{g_t} < 1 \). Thus, although the Pound-Rebka experiment interpretation is, according to our model, different from the generally accepted one, it proves the existence of the gravitational force action on photons. Indeed, according to Pound and Rebka, if the gravity does not cause a blueshift \( \Delta_1 E = gL \) when the photon \( \gamma \) is propagating toward the earth and if the temperature is actually constant and the vacuum is actually absolute, then we would obtain an optimal resonance for a fixed emitter. The fact that such a resonance is obtained when moving the emitter implies that, according to them, it is necessary to cause a redshift \( \Delta_0 E = -\beta E \) with \( \beta = \frac{v}{c} \) and the movement has to be upward. When the photon is propagating upward, we have
then to move the emitter upward too in order to cause a blueshift $\Delta_0 E = \beta E$ which would balance the redshift $\Delta_1 E = -gL$.

Our interpretation coincides partially with the Pound and Rebka one. If there is no a blueshift (or a redshift) $\Delta_1 E = gL$ and if the temperature is absolutely constant and the absolute vacuum is perfectly respected, then the photon energy does not change and the resonance phenomenon would happen without moving the emitter. The photon energy $E$ would be, according to our model, constant and it is given, for all distances (under the same conditions), by $E = h(t)f(t)$, even though $h$ and $f$ are depending on time. This quantity is moreover equal to $h_p f_D$ as it has been noticed previously. To summarize, as $E$ and $f_D$ are constant, then Pound and Rebka would not need to move the source. Conversely, if the previous conditions are not respected, then even if the gravitational blueshift (or redshift) does not exist, we would need to cause a blueshift in both cases. Finally, since there is an energy change $\Delta_1 E = \pm gL$, we need to cause an appropriate $\Delta_0 E$. For a fixed position of the photon source, this could be supplied (in case of propagation toward the earth) by temperature fluctuations and the lake of absolute vacuum, but, in the other case, this can not be achieved since both $\Delta_1 E(i = 0, 1)$ would have the same sign. We will need then to operate a correction $\Delta_0 E$ (which is necessarily a blueshift in the case of upward propagation) which, according to our model, can be produced only by speed fluctuations (accelerations) or distance variations. The distance variation implies a variation of the temperature effect and of the lake of absolute vacuum influence.

**General modeling of the universe**

We notice also that our study could have been entirely reconstructed by taking as starting point the universe $U(t_0) = (B_e(O, R(t_0)), g_{t_0})$, for any fixed $t_0$, provided that we can recover back this privileged instant and that we have a sufficient knowledge on how the universe goes back and forward in time.

Let us consider, for instance, the universe at time $t > 0$ as being the ball $B_e(O, R(t))$ equipped with the physical metric $g_t(X)$ for $|X| \leq R(t)$. This metric is determined by the generalized energy distribution $E_t(X) = E(t, X)$. The matter-energy equation ($E^*$) on the ball $B_e(O, R(t))$ becomes

$$\square E(t, X) = \frac{\partial^2 E}{\partial t^2}(t, X) - \Delta E(t, X) = 0$$

with

$$E(t, X)|_{S_e(O, R(t))} = 0.$$
When we bring back the resolution of this problem to that of the Dirichlet problem on the unit ball $B_e(O, 1)$ and when we choose a particular eigenvalue $\mu$, we obtain the pseudo-periodic solution

$$E_\mu(t, X) = \left( f_\mu(0) \cos \sqrt{\mu} \frac{t}{R(t)} + \frac{R(t)}{\sqrt{\mu}} f'_\mu(0) \sin \sqrt{\mu} \frac{t}{R(t)} \right) \psi_\mu \left( \frac{X}{R(t)} \right)$$

having $T_\mu(t) = 2\pi \frac{R(t)}{\sqrt{\mu}}$ and $f_\mu(t) = \frac{1}{2\pi} \sqrt{\mu} \frac{R(t)}{R(t)}$ as pseudo-period and pseudo-frequency respectively. This function satisfies, for any geodesic (relatively to $g_t$) trajectory $X(t)$, the relation

$$E(t) := E_\mu(t, X(t)) = h_\mu(t) f_\mu(t) = h(t) f(t) = h_P f_D$$

where $h_P$ is the Planck constant and $f_D$ is the De Broglie frequency which depends on time naturally. All our above formulas which have been established in simplified contexts can be adapted to the definitive setting when we consider the dynamical universe $U(t)$ as being, for every time $t$, modelized by the Riemannian space $(B_e(O, R(t)), g_t)$ where $R(t)$ is the actual radius of the universe and $g_t$ is the physical real metric which is determined by the material distribution, the time and the temperature (or the pressure).

Finally, we can state that, for every time $t > 0$, we have, for any fixed $t_0 > 0$ :

$$R(t) = R(t_0) + k(t)(t - t_0)$$

with $k(t) \sim 1$ for $t \gg 1$, $k(t)$ is increasing and

$$\lim_{t \to +\infty} k(t) = 1.$$

According to our study, we can notice finally that the space and time semi-cone is rather of the form sketched in fig.10 unless the speed of electromagnetic waves’ propagation and that of the space expansion were always equal to 1 and then the semi-cone of space and time would have really the form sketched in fig.3.

Our progressive approach has been (subjectively) imposed for us in order to avoid some complications (that could result from a large number of factors which are involved) along the construction of our model and also in order to achieve it with most possible simplicity and clarity. However, we are perfectly aware that there are inside this study many points to be detailed, specified, clarified, added and discovered.
10 The reviewed Einstein’s General Relativity Theory

In this section, we propose to prove that our dynamical universe is overall described by readjusting general relativity theory to our model. Once this adaptation is achieved, a large number of problems related to theoretical Physics and Cosmology, for instance, can be unequivocally well posed and clearly resolved in a most simpler way.

We start by noticing that our mathematical and physical background along this section will essentially be the results established in the previous sections on one hand and on the great classical work on general relativity theory of R. Wald ([4], General relativity, 1984).

Preliminaries

Our model briefly consists of considering the physical universe as being, at every time \( t > 0 \), the Riemannian space \( (B(O,R(t)), g_t) \), where \( R(t) \sim t \) for \( t \gg 0 \), \( B(O,R(t)) \) is the ball of Euclidean radius \( R(t) \) and \( g_t \) is the regularized Riemannian metric that reflects, at time \( t \), the whole physical consistence of the universe. This consistence is entirely specified by the generalized matter-energy distribution \( E_t(X) \) on \( B(O,R(t)) \), which includes the matter distribution \( m_t(X) \) as well as all manifestations of matter-energy. This distribution is reflected by the curvature of the position and time dependent metric \( g_t \). We then define on \( B(O,R(t)) \) the measure

\[ \nu_t(X) = E_t(X)dX \]

i.e. the measure of density \( E_t(X) \) with respect to the Lebesgue measure \( dX \) on \( B(O,R(t)) \). We then have

\[ dv_{g_t}(X) =: \mu_t(X) = dX - \nu_t(X) = (1 - E_t(X))dX \]

This property expresses that \( \mu_t \) measures the physical volume of a domain in \( \mathbb{R}^3 \) that contains a distribution of matter-energy and \( \nu_t \) measures the failure of the physical volume to be equal to the volume of the same domain when supposed being empty. This latter is conventionally measured by the (Euclidean) Lebesgue measure. Thus the matter-energy filling the space contracts distances and volumes. The dynamical feature of the permanently expanding universe whose expansion occurs nearly at the speed of light in the vacuum \( c (=1) \) is described by the semi-cone of space-time:

\[ C = \{ (x, y, z, t); x^2 + y^2 + z^2 \leq R^2(t), t > 0 \} = \bigcup_{t>0} B(O, R(t)) \times \{ t \} \]
which will be considered (for sake of simplicity and clarity) as being

$$C = \{ (x, y, z, t); x^2 + y^2 + z^2 \leq t^2, t > 0 \} = \bigcup_{t > 0} B(O, t) \times \{ t \}$$

equiped with the Riemannian metric

$$h = dt^2 - g_t$$

The universe will then be represented, at any time \( t_0 \), as the hypersurface-intersection of \( C \) with the hyperplane of equation \( t = t_0 \) in \( \mathbb{R}^4 \). This hypersurface will be denoted by \( \Sigma_{t_0} \) and \( C \) by \( M \). Then, within the framework of our modeling, \( \Sigma_t \) is a compact Cauchy surface and our space-time manifold \( (M, h_{ab}) \) is asymptotically flat. The vector field \( (\frac{\partial}{\partial t})^a \) on \( M \) is \( \Sigma_t \)-hypersurface orthogonal. These properties shall considerably simplify the foundation of the general relativity theory as well as its use for explaining the universe dynamics.

**Einstein’s tensor equation - Lagrangian formulation**

Our starting point is the equation

$$(3) G_{ab}(t) := (3) R_{ab}(t) - \frac{1}{2} (3) R (3) g_{ab}(t) =: (3) T^*_{ab}(t)$$  \hspace{1cm} (23)

defined on \( (B(O, t), g_t) \), where \( (3) R_{ab} \) and \( (3) R \) respectively designate the Ricci curvature and the scalar curvature associated with the metric \( (3) g_{ab}(t) := g_t \) and \( (3) T^*_{ab}(t) \) is the matter-energy tensor that specifies, at every time \( t > 0 \), the physical consistency of the universe. This consistency is fundamentally related to the very existence of the permanently evolving matter-energy which is filling the virtual empty space \( B(O, t) \). According to our definitions, \( (3) T^*_{ab}(t) \equiv 0 \) on a domain \( D \subset B(O, t) \) means that \( D \) is absolutely empty (i.e. \( D \) is quasi-free of all matter-energy manifestations and effects : gravity, electromagnetism, thermodynamic effects...) and then we have

$$g_{ab}(t) \equiv g_e \quad \text{on} \; D.$$

Next, we consider on \( (M, h_{ab}) \), the Einstein’s equation :

$$(4) G_{ab} := (4) R_{ab} - \frac{1}{2} (4) R h_{ab} = (4) T^*_{ab}$$  \hspace{1cm} (24)
where \(^{(4)}R_{ab}\) and \(^{(4)}R\) are respectively the Ricci and scalar curvatures associated with \(h_{ab}\) on \(M\). Here, \(^{(4)}T_{ab}^*\) represents (up to a constant) the generalized Einstein’s stress-energy tensor. Thus, with the previous notations, we have

\[
^{(3)}T_{ab}^* = 0 \Rightarrow g_t = g_e \Rightarrow h = dt^2 - g_e = \eta,
\]

where \(\eta\) is, up to sign, the flat Minkowsky metric on the semi-cone \(C\) and then we have \(^{(4)}T_{ab}^* = 0\). We notice that our manifold \(M\) is evolving with time since we have, for every time \(t_0\):

\[
M = C(t_0) = \{(x, y, z, t); x^2 + y^2 + z^2 \leq t^2, 0 < t \leq t_0\}.
\]

In the following, we shall intensively use the results presented in [4] for sake of greatly valuable economy. Thus, for further facility and clarity we shall slightly modify our notations in order to conform to those used in this capital reference. Consequently, our metric on the dynamical space-time \(M = C(t)\) will be denoted by \(g_{ab}\). It is chosen to be of a lorentzian signature and is defined by

\[
^{(4)}g_{ab} = -dt^2 + ^{(4)}h_{ab}
\]

where \(^{(4)}h_{ab}\) denotes here the induced metric on \(\Sigma_t\) by \(^{(4)}g_{ab}\) so that our physical metric on the universe \(B(O, t)\), previously denoted as \(g_t\), identifies with the metric \(^{(3)}h_{ab}\) obtained by restricting \(^{(4)}h_{ab}\) on \(\Sigma_t\).

Now, we are ready to obtain the adequate Lagrangian and Hamiltonian formulations which will be well adapted to the new context within which we shall present the general relativity theory that globally describes the general laws of our dynamical, permanently evolving universe. Naturally, these laws correspond to an idealization of the universe by the intermediate of the metric \(g_t\) regularization on \(B(O, t)\). Indeed, the real physical metric is far from being of class \(C^2\) because of singularities that are essentially reduced to black holes. In the light of these adaptations, our two equations (24) and (23) become

\[
^{(4)}R_{ab} - \frac{1}{2} (^{(4)}R) g_{ab} = ^{(4)}T_{ab}^* \quad (E)
\]

and

\[
^{(3)}R_{ab} - \frac{1}{2} (^{(3)}R) h_{ab} = ^{(3)}T_{ab}^* \quad (E^*)
\]

with

\[
^{(3)}T_{ab}^* = 0 \iff ^{(3)}h_{ab} = ^{(3)}g_e \iff ^{(4)}g_{ab} = \eta_{ab}
\]
where $\eta_{ab}$ is the Minkowsky metric.
Thus $B(O,t)$ is the virtual vacuum space within which leaves our real physical universe and $C(t)$ is the virtual space within which evolves the dynamical universe. We notice that equation (E) (using an orthonormal basis with respect to $h_{ab}$ together with the vector $(\frac{\partial}{\partial t})^a = (1,0,0,0)$) implies

$$(4) \, R = - (4) T^a : = - (4) T_{a}^a$$

and equation $(E^*)$ implies

$$(3) \, R = -2 (3) T^a : = -2 (3) T_{a}^a.$$ We notice also that the volume form $\varepsilon_{abc} = : (3) \varepsilon$ associated with $h_{ab}$ is equal to $\sqrt{h} \varepsilon_{abc}$, where $e_{abc} = : (3) e$ is the canonical (Euclidean) volume form of $\mathbb{R}^3$. In the same way, the volume form $\varepsilon_{abcd} = : (4) \varepsilon$ associated with $g_{ab}$ is equal to $\sqrt{-g} \varepsilon_{abcd} = \sqrt{h} \varepsilon_{abcd}$, where $e_{abcd} = : (4) e$ is the canonical (Euclidean) volume form associated with the Minkowski space-time (here $g$ and $h$ are respectively the determinants of the matrices associated with $g_{ab}$ and $h_{ab}$ when expressed by using the canonical bases of $\mathbb{R}^4$ and $\mathbb{R}^3$). We further have

$$(3) \varepsilon = i \frac{\partial}{\partial t} (4) \varepsilon$$

as, in the framework of our model, the unitary orthogonal vector field to the hypersurfaces $\Sigma_t$ is $\nu = (\frac{\partial}{\partial t})^a$. Finally, we notice that the flow of time function is nothing but the forth coordinate $t$ since, within our model, there is no a relativistic proper time notion.

Now, following R.Wald, the Hilbert action associated with the Einstein’s vacuum equation :

$$(4) \, R_{ab} - \frac{1}{2} (4) R \, g_{ab} = 0 \quad (E_0)$$

is given by

$$S_{G} [g^{ab}] = \int_M \mathcal{L}_G \, (4) e$$

where

$$\mathcal{L}_G = \sqrt{-g} \, (4) R = \sqrt{h} \, (4) R.$$ 

We have then (for a one parameter family $(g_{ab})_\lambda$ ([4], E.1.18)) :

$$\frac{dS_{G}}{d\lambda} = \int \frac{d\mathcal{L}_G}{d\lambda} (4) e = \int \nabla^a v_a \sqrt{-g} \, (4) e + \int (4) R_{ab} - \frac{1}{2} (4) R \, g_{ab} \delta g^{ab} \sqrt{-g} \, (4) e$$

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and, neglecting the first term of the right hand side of this equation, we obtain (E.1.19):

$$\frac{\delta S_G}{\delta g_{ab}} = \sqrt{-g} \left( ^{(4)} R_{ab} - \frac{1}{2} ^{(4)} R \ g_{ab} \right),$$

where $$\delta g_{ab} = \frac{d(g_{ab})}{dx} \bigg|_{\lambda=0}$$ and $$\nu_a = \nabla^b (\delta g_{ab}) - g^{cd} \nabla_a (\delta g_{cd})$$; but, when we take into account the contribution of the boundary term of this equation, the real action must be modified in order to become (E.1.24):

$$S'_G = S_G + 2 \int_{\hat{U}} K.$$  

Here, $$\hat{U}$$ is the boundary of the part of the space-time semi-cone $$C$$ located between two Cauchy hypersurfaces $$\Sigma_{t_1}, \Sigma_{t_2}$$ and the lateral boundary. On the last part, the curvature vanishes and the action $$S'_G$$ becomes

$$S'_G = S_G + 2 \int_{\Sigma_{t_2}} K^{(3)} \varepsilon - 2 \int_{\Sigma_{t_1}} K^{(3)} \varepsilon.$$  

We recall that (E.1.39) here $$K$$ is

$$K = K^a_a = h^a_b \nabla_a \left( \frac{\partial}{\partial t} \right)^b$$

where

$$K_{ab} = \nabla_a \left( \frac{\partial}{\partial t} \right)_b = B_{ba}$$

$$= \frac{1}{2} \mathcal{L}_{\frac{\partial}{\partial t}} \ g_{ab} = \frac{1}{2} \mathcal{L}_{\frac{\partial}{\partial t}} \ h_{ab} = \frac{1}{2} \dot{h}_{ab}$$

is the extrinsic curvature tensor of the hypersurface $$\Sigma_t$$ (c.f. 9.3.19 and 9.3.20). This implies that the scalar extrinsic curvature of $$\Sigma_t$$ is

$$K = \frac{1}{2} \dot{h} = \frac{1}{2} \dot{g}$$  

$$(K(t) = \frac{1}{2} \dot{h}(t) = \frac{1}{2} \dot{g}(t))$$

where $$\dot{h}$$ and $$\dot{g}$$ are the common trace of $$^{(4)} \dot{g}_{ab} = ^{(4)} \dot{h}_{ab}$$ or $$^{(3)} \dot{h}_{ab}$$.

### Hamiltonian formulation

We now consider the Hamiltonian formulation associated with the equation ($E_0$). Using the notations of [4], we notice that, in the context of our model, we have

$$N = 1$$  

(the lapse function is 1)
and
\[ N^a = 0 \quad \text{(there is no shift vector).} \]

We also have \((E.2.28)\) and \((E.2.29)\)

\[ (4) \, R_{ab} \left( \frac{\partial}{\partial t} \right)^a \left( \frac{\partial}{\partial t} \right)^b = K^2 - K_{ac} K^{ac} \]

\[ L_G = \sqrt{h} \left( (3) R + K_{ab} K^{ab} - K^2 \right) \]

and always
\[ K_{ab} = \frac{1}{2} \dot{h}_{ab} \]

By defining the canonically conjugate momentum to \( h_{ab} \) by \((E.2.31)\)

\[ \Pi^{ab} = \frac{\partial L_G}{\partial \dot{h}_{ab}} = \sqrt{h} (K^{ab} - K h^{ab}) \]

and the configuration space as the set of all Riemannian metrics on \( \Sigma_t \), we define the Hamiltonian density by \((E.2.32)\)

\[ H_G = \Pi^{ab} \dot{h}_{ab} - L_G = \sqrt{h} \left( -(3) R + \frac{1}{h} \Pi_{ab} \Pi_{ab} - \frac{1}{2} h \Pi^2 \right) \]

where
\[ \Pi = \Pi^{a} \]

The Hamiltonian \( H \) is then the function defined, for each \( \Sigma_t \), by

\[ H(g_{ab}, \Pi^{ab}) = \int_{\Sigma_t} H_G (3) \varepsilon. \]

The Hamiltonian formulation resulting from variations of \( h_{ab} \) that satisfy \( \delta h_{ab} = 0 \) on the \( \Sigma_t \) hypersurfaces is equivalent to \((E_0)\). The metrics \( h_{ab} \) being asymptotically flat, the solutions of \((E_0)\) are the solutions of the following constraint free Hamiltonian system :

\[ \dot{h}_{ab} = \frac{\delta H_G}{\delta \Pi^{ab}} = \frac{2}{\sqrt{h}} (\Pi_{ab} - \frac{1}{2} \Pi \, h_{ab}) \]

\[ \dot{\Pi}^{ab} = -\frac{\delta H_G}{\delta h_{ab}} = -\sqrt{h} \left( (3) R_{ab} - \frac{1}{2} (3) R \, h^{ab} \right) \]

\[ + \frac{1}{2} \, \frac{1}{\sqrt{h}} h^{ab} (\Pi_{cd} \Pi^{cd} - \frac{1}{2} \Pi^2) \]

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This system reduces, within our framework, to six equations of six independent unknowns. A solution $h_{ab}$ of this system is nothing but our original metric $g_t$ on $B(O,t)$ in the (virtual) particular case when the tensor $(3)G^*_{ab} = (3)T^*_{ab}$ describes uniquely the gravity effect.

**Remark**: Both constraints (E.2.33) and (E.2.34) originate from the variations of $H_G$ with respect to $N$ and $N_a$. However, we can get rid of constraint (E.2.34) by using the notion of wheeler’s superspace. Within the framework of our model, both constraints have no existence. Actually, they constitute the heritage of special relativity into the standard general relativity theory. Besides, even for this theory, $N$ and $N_a$ do not constitute genuine dynamical variables.

In order to include all other effects of the matter-energy distribution, filling the universe, within a realistic Lagrangian formulation, we have to consider a Lagrangian density $\mathcal{L}$ given by

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_M.$$ 

This density determines an action

$$S = S_G + S_M$$

whose extremization is equivalent to the resolution of $(E)$. In the particular case of a coupled gravitational field and Klein-Gordon scalar field, then $\mathcal{L}$, $\mathcal{L}_G$, $\mathcal{L}_{KG}$, $T^{KG}_{ab}$ and $S_{KG}$ are explicitly given and related by the relations (E.1.22) and (E.1.24)-(E.1.26) of [4]. For the coupled Einstein-Maxwell equation we can refer to the relations (E.1.23) - (E.1.26) of [4].

More generally, we can state that the idealized physical universe is equivalently characterized by:

1. The matter-energy distribution $E_t(X)$ for all couples $(t, X)$ such that $t > 0$ and $X \in B(O,t)$.

2. The tensor $(3)T^*_{ab}$ defined on $B(O,t)$ for $t > 0$.

3. The Riemannian metric $g_t$ defined, for every $t > 0$, on $B(O,t)$.

4. The distribution $E(t, X) = E_t(X)$ for $t > 0$ and $X \in \Sigma_t$.  

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5. The metric \((4)g_{ab}\) defined on \(C\) by

\[ (4)g_{ab} = -dt^2 + (4)h_{ab} \]

where

\[ (4)h_{ab}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = 0 \]

\[ (4)h_{ab}(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_i}) = 0 \text{ for } i = 1, 2, 3 \]

and

\[ (4)h_{ab} = g_t \text{ on } \Sigma_t = B(O, t) \times \{t\} \]

6. The Einstein’s stress-energy tensor \((4)T_{ab}\) satisfying the Einstein’s equation:

\[ (4)R_{ab} - \frac{1}{2} (4)R (4)g_{ab} = 8\pi (4)T_{ab} \]

7. The Lagrangian density

\[ \mathcal{L} = \mathcal{L}_G + \mathcal{L}_M \]

where

\[ \mathcal{L}_G = \sqrt{-g} (4)R \]

and \(\mathcal{L}_M\) is the Lagrangian density corresponding to all fields apart from the gravitational one, which in that way guarantees that the extremization of the action

\[ S'_G = S_G + 2 \int_{\Sigma_t} K - 2 \int_{\Sigma_{t_0}} K \]

with respect to variations satisfying \(\delta g_{ab} = 0\) or \(\delta h_{ab} = 0\) on both \(\Sigma_t\) and \(\Sigma_{t_0}\) gives the solutions of \((E)\) and \((E^*)\).

8. The Hamiltonian density \(\mathcal{H}\) and the Hamiltonian \(H = \int_{\Sigma_t} \mathcal{H}\) defined by means of the Lagrangian \(\mathcal{L}\) by

\[ \mathcal{H} = \Pi^{ab} h_{ab} - \mathcal{L} \]

where

\[ \Pi^{ab} = \frac{\partial \mathcal{L}}{\partial h_{ab}} \]

\(h_{ab}\) is then the solution of the Hamiltonian system
\[ \dot{h}_{ab} = \frac{\delta H}{\delta \Pi^{ab}} \]
\[ \ddot{\Pi}^{ab} = -\frac{\delta H}{\delta h_{ab}}. \]

**General features of the solution**

Consequently, our modeling leads to a determinist solution of the space-time Einstein’s equation. Our space-time will be, for every time \( t \), the Riemannian manifold \((C(t), (4)g_{ab})\) realized as the Cauchy maximal development associated to the initial conditions

\[ (3)h_{ab}(t_0) \quad \text{and} \quad \frac{1}{2} (3)\dot{h}_{ab}(t_0) \]

defined on an arbitrary Cauchy surface \( \Sigma_{t_0} \) of \( C(t) \) for \( t_0 < t \) in such a manner that the Riemannian metric \((3)h_{ab}(t_0)\) on \( \Sigma_{t_0} \) identifies to the physical metric \( g_{t_0} \) on \( B(O,t_0) \) of the previous sections and \( \frac{1}{2} (3)\dot{h}_{ab}(t_0) = \frac{1}{2} (g_{t_0})_{ab} \) constitutes the extrinsic curvature tensor of \( \Sigma_{t_0} \) into the space-time \((C(t), (4)g_{ab})\). This solution could be a good approximation of the real metric on a time interval as large as our approximation and regularization of the initial conditions \((3)h_{ab}(t_0)\) and \( \frac{1}{2} (3)\dot{h}_{ab}(t_0) \) on \( \Sigma_{t_0} \) are close to the physical reality of our universe at time \( t_0 \). A permanent readjustment of initial conditions based on the enlargement and refinement of available data is necessary.

Our model contains the right degree of liberty number and does not undergo the gauge freedom notion. Indeed, the physical nature of the universe forces it, in virtue of the constancy of the propagation speed of electromagnetic waves and its isotropic character, to exist, at every time \( t \), under the form of a ball having a Euclidean radius \( t \) and to evolve within the space-time semi-cone \( C \). This forces any gauge diffeomorphism \( \psi \) to transform \( C(t) \) into a semi-cone \( C(t') \) and to be of the form \( \psi = (t, \varphi_t) \) where \( \varphi_t \) is a diffeomorphism of \( B(O,t) \) on \( B(O,t') \) satisfying \( g_t = \varphi_t^* g_{t'} \). This simply reduces to a purely conventional rescaling unless \( \psi \) being an isometric transformation of \( C \) in \( \mathbb{R}^4 \) and \( \varphi \) being an isometric transformation of \( B(O,t) \) in \( \mathbb{R}^3 \). In that case \( \psi \) constitutes a trivial gauge diffeomorphism:

\[ \psi^* g_{ab} = g_{ab} \quad \text{and} \quad \varphi_t^* g_t = g_t \]

**Remarks**

According to our model, we can state the following properties:
1. The real physical metric of the universe cannot be globally determined by a linearization process: the real perturbations of the Minkowsky metric on $C$ and the Euclidean metric on $B(O,t)$, due to the matter-energy effects, are far from being "small", especially around black holes.

2. Our universe is evidently not homogeneous nor isotropic. There indeed exists a foliation family $(\Sigma_t)_{t>0}$ of the space-time, but for arbitrary $p,q \in \Sigma_t$, there cannot exist an isometry of $\Sigma_t$ which transforms $p$ into $q$. Furthermore, there does not necessarily exist an isometry of $C$ that leaves $p \in \Sigma_t$ fixed and transforms a unitary spatial vector at $p$ into another vector having the same properties. Consequently, our model is totally different from the $K = \mp 1$ cases of the Robertson-Walker’s model even though it is, in case of extreme idealization, very similar to the case $K = 0$ of this model. Indeed, the corresponding metric to this latter case reduces, within the framework of our model, to

$$(4)\ g = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

where $dx^2 + dy^2 + dz^2$ is the restriction to the ball $B(O,R(t))$ of the euclidean metric on $\mathbb{R}^3$.

Moreover, the evolution equations for the homogeneous isotropic Cosmology are written down (c.f. (5.2.14) and (5.2.15) of [4]) as

$$3\frac{\dot{a}^2}{a^2} = 8\pi \rho - \frac{3K}{a^2}$$

and

$$3\frac{\ddot{a}}{a} = -4\pi(\rho + 3P),$$

where $K$ is the curvature parameter (which is, contrary to our model, a constant), $\rho$ is the average matter density of the universe and $P$ is the average pressure associated with the massless thermic radiations filling the universe, which together constitute the stress-energy tensor of Einstein. This implies (5.2.18)

$$\dot{\rho} + 3(\rho + P)\frac{\dot{a}}{a} = 0$$

which, for standard dust model ($P = 0$) leads to

$$\rho_m a^3 = cte$$

(where $\rho_m$ is the matter mean density for this model) and, for standard radiations model ($P = \frac{\rho}{3}$), leads to

$$\rho_r a^4 = cte,$$
(where $\rho_r$ is the radiation mean density for this last model).

Thus, we have, for the standard dust model (5.2.21):

$$a^2 - \frac{C}{a} = 0 \quad \text{with} \quad C = \frac{8}{3}\pi \rho a^3$$

and, for the standard radiation model, we have (5.2.22):

$$a^2 - \frac{C'}{a} = 0 \quad \text{with} \quad C' = \frac{8}{3}\pi \rho a^4.$$ In the first case, which corresponds to our present universe, we have (Table 5.1)

$$a(t) = \left(\frac{9C}{4}\right)^{\frac{1}{3}} t^{\frac{2}{3}}$$

and in the second one, which corresponds to the early universe just after the (hot) Big Bang, we have ((5.4.1) and (5.4.2)):

$$a(t) = (4C')^{\frac{1}{3}} t^{\frac{1}{2}}$$

and

$$\rho_r(t) = \frac{3}{32\pi G t^2}.$$ Then, using quantum statistics, we find that, for infinitely small $t$, the temperature $T$ of the universe is proportional to $\rho_r^{\frac{1}{4}}$ and to $\frac{1}{a}$.

Besides, we have, according to our model (for the very early universe), after a crude simplification within the context of the homogeneous isotropic Cosmology:

$$g_t = a^2(t)(dx^2 + dy^2 + dz^2)$$

on $B(O,R(t))$, which implies

$$\int_{B(O,R(t))} dv_g = \int_{B(O,R(t))} a^3(t)dv_{ge} = \int_{B(O,R(t))} dX - E = \frac{4\pi R^3(t)}{3} - E$$

where $E$ is the total energy of the universe. This allows us to calculate, for instance, $C'$ as a function of $E$ and $R(t)$ for $t \ll 1$ and as a function of $E$ and $t$ when assuming $R(t) \simeq t$.

Moreover, we also could have some information on the Hubble’s factor (i.e. the time dependent Hubble’s constant) $H(t) = \frac{\dot{a}}{a}$. 3. The universe is not stationary. The translation of time vector field $(\frac{\partial}{\partial t})^a$
is not a killing vector field. The universe is not static too, even though the family $\Sigma_t$ is orthogonal to $(\frac{\partial}{\partial t})^a$ and we have

$$g_{ab} = -dt^2 + h_{ab}.$$  

4. Our universe is not spherically symmetric nor axisymmetric. Hence the Schwarzschild’s solution can not be other than an idealization of the universe reducing it to a gravitational field resulting from a material spherical and static core.

5. **(Cosmological constant)** When we write down the Einstein’s tensor equation with a non vanishing cosmological constant, we get:

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab} - \Lambda g_{ab}.$$  

The Einstein’s vacuum equation becomes

$$R_{ab} - \frac{1}{2}Rg_{ab} = -\Lambda g_{ab}$$  

which (by contracting) gives

$$R = 4\Lambda.$$  

Then the curvature of the Einstein’s vacuum, characterized by $\rho = 0$ and $T_{ab} = 0$, is non vanishing. In our setting, the identity $T^*_{ab} = 0$ implies $T_{ab} = 0$ but $T_{ab} = 0$ does not imply $T^*_{ab} = 0$. This comparison shows that our condition $T^*_{ab} = 0$ (and then $(3)T^*_{ab} = 0$ and $g_t = g_e$) is very restrictive and, in fact, idealistic. It corresponds to an absolute vacuum region whose existence is highly improbable.

The $\Lambda$ ”constant” appears into our cosmological setting as being the result of the influence of the cosmic matter and cosmic radiations and gravity on regions that are deprived of matter and are not lying within a direct gravitational and electromagnetic field i.e. the regions of $B(O,t)$ characterized by $(3)T^*_{ab} = \Lambda g_e$. Our field tensorial equation then is

$$(3)R_{ab} - \frac{1}{2}(3)Rg_t = \Lambda g_e$$  

where $\Lambda$ probably depends on the considered region and the time.

Nevertheless, a large number of results obtained from such hypothesis, reductions and idealizations still continue to be qualitatively and quantitatively (far or less) valid. This is particularly true for the results obtained by
means of what is qualified as Newtonian limit, homogeneous isotropic cosmology or, simply, spatially homogeneous cosmology and their consequences on some features related to the universe evolution and to its causal structure. It is the same for some consequences of the Kruskal extension of the Schwarzschild solution concerning stationary black holes in the vacuum and charged Kerr black holes associated to Einstein-Maxwell equation as well as for thermodynamicslike properties of black holes.

Naturally, there is always a gigantic work to be done about these important problems. We can talk also about the necessary modifications to be made for adapting the Einstein-de Sitter-Friedmann cosmological model to our framework. All this demands avoiding some erroneous and unjustified postulates and principles and adapting everything to the dynamical expansion reality of our universe which is, even though gigantic, perpetually finite.

We can add that our dynamical model makes possible the construction of a quantum theory of the general relativity after getting rid of useless constraints previously imposed on the Hamiltonian system specifying the dynamical evolution of the universe as well as of the constraints imposed to the initial values leading, in this way, to a well posed initial values formulation for general relativity theory.

11 Introduction to a reviewed cosmology

In this section, we will adapt the (Einstein - Friedmann - Hubble - de Sitter) homogeneous isotropic Cosmology to our model in order to specify some approximate results, that have been established half theoretically and half experimentally but not rigorously.

We start by recalling that the universe is assimilated, at any time \( t \gg 0 \), to the Riemannian space \( U(t) = (B(O,t),g_t) \) where \( g_t \) is a position and time-dependent metric. We then adopt the global macroscopic cosmological model within which the present universe reduces to a dust of galaxies that is distributed in a homogeneous and isotropic manner into the ball \( B(O,t) \) (although this is not rigorously exact). So, we will follow the Hubble - Friedmann’s work (c.f. [2]) by putting

\[
    r = r_0 R(t)
\]

where \( R(t) \) denotes the expansion parameter.
This equation becomes, within our framework,

$$ct = ct_0 R(t)$$

where $c$ is the speed of light in the vacuum. We then have

$$R(t) = \frac{t}{t_0}$$

and

$$\frac{dR}{dt} = \frac{1}{t_0}$$

and

$$H := \frac{dR}{R} = \frac{1}{t}.$$ 

This leads to the Friedmann’s equation

$$\left( \frac{dR}{dt} \right)^2 = \frac{8\pi G \rho R^2}{3c^2} - K(t)$$

(25)

where here $K(t)$ denotes the time-dependent curvature parameter associated with the space curvature which is originated in the matter distribution on the universe and reflected by the Riemannian metric $g_t$. This process is made possible thanks to our homogeneity and isotropy hypothesis which is valid when we are dealing with general macroscopic results.

Remark : In our model, the curvature parameter that appears in the Friedmann equation ([2], 19.58)

$$\left( \frac{dR}{dt} \right)^2 = \frac{8\pi G \rho R^2}{3c^2} - K$$

depends on time (as well as does the gravitational constant $G$). The fact of considering $K$ as being an absolute constant (i.e. independent of time) as it is generally accepted, leads to flagrant contradictions. Actually, in order to establish this equation, Friedmann has taken a ball of the universe of radius $r =: r_0 R(t)$ having the mass $M$ and a galaxy that is located on the corresponding sphere having the mass $m$ and established the relation ([2],19.56)

$$E = -\frac{K m r_0^2}{2}.$$ 

Now, the constant $K$ as introduced here necessarily depends on the arbitrary chosen instant $t_0$ (i.e. $K = K(t_0)$). Otherwise, if we choose another privileged
instant \( t_1 \neq t_0 \) to which corresponds another radius \( r_1 \neq r_0 \), we obtain by proceeding similarly the relation

\[
E = -\frac{Kmr_1^2}{2}
\]

which, in case of taking \( K \) as an absolute constant, contradicts the energy conservation law.

In other respects, the fact of considering \( K \) and \( G \) as being absolute constants leads to a contradiction between Friedmann equation and the second cosmological principle which stipulates that the relative speed of galaxies is proportional to their relative distance. Indeed, as \( \frac{dr}{dt} \) is proportional to \( \frac{v_r}{R} \) and as \( \rho \) is proportional to \( \frac{1}{R^3} \), then this equation implies that \( \frac{dR}{dt} \) decreases with increasing \( R \) as \( \frac{2}{cR} \) and consequently \( v_r \) is decreasing as \( \frac{\beta}{\sqrt{r}} \). Similarly, the relations (19.66) and (19.67) of [2], namely

\[
\frac{dR}{dt} = \sqrt{\frac{8\pi G \rho_c}{3c^2}} R^{-\frac{3}{2}}
\]

where \( \rho_c \) is the critical density (19.65) of [2] and

\[
R = \left( \frac{3}{2} \right)^{\frac{3}{4}} \frac{8\pi G \rho_c}{3c^2} t^{\frac{3}{4}}
\]

show clearly the deficiency of the Einstein - de Sitter - Friedmann model. Indeed, these relations imply

\[
\frac{dr}{dt} \propto \frac{dR}{dt} \propto \frac{1}{R^2} \propto \frac{1}{t^{\frac{3}{2}}}
\]

which shows that \( r \) is increasing with time whereas \( v_r = \frac{dr}{dt} \) is decreasing.

The equation (25) can be written as

\[
\frac{1}{t_0^2} = \frac{8\pi G}{3} \frac{E}{c^2 \sqrt{\rho_c}} R^2 - K
\]

\[
= \frac{2GE}{c^5 t^3} R^2 - K = \frac{2GE}{c^5 t_0^3} t_0^2 - K.
\]

So, we get

\[
\frac{2GE}{c^5 t} = 1 + Kt_0^2
\]

and then

\[
E = \frac{c^5 t (1 + Kt_0^2)}{2G} = \frac{c^5 (t_0 + K_0t_0^2)}{2G_0} = \frac{c^5 C_0}{2G_0}
\]

(26)
where $G_0$ is the gravitational constant calculated at the time $t_0$ and

$$C_0 = t_0 + K_0 t_0^3 = \frac{2G_0E}{c^5}.$$ 

Furthermore, we can write, within a significant time interval about $t_0$:

$$t(1 + K t_0^2) \simeq C_0.$$ 

By differentiation, we obtain

$$1 + K t_0^2 + t_0^2 t K' = 0$$

which yields

$$(K + K' t) t_0^2 = -1$$

i.e.

$$K + K' t = -\frac{1}{t_0^2}$$

and then

$$(Kt)' = -\frac{1}{t_0^2}.$$ 

Therefore, we have

$$Kt = -\frac{t}{t_0^2} + b$$

with

$$b = K_0 t_0 + \frac{1}{t_0}.$$ 

So, we get

$$Kt = -\frac{t}{t_0^2} + K_0 t_0 + \frac{1}{t_0}$$

and then

$$K = -\frac{1}{t_0^2} + \frac{1 + K_0 t_0^2}{t_0 t}.$$  \hspace{1cm} (27) 

Now, we write down the equality

$$\rho(t) = \rho_m(t) + \rho_r(t)$$

where $\rho_m(t)$ and $\rho_r(t)$ respectively denote the mean mass energy and radiation energy densities.
Besides, our model implies
\[ \rho(t) = \frac{E}{\text{vol}(B(O, ct))} = \frac{E}{\frac{4}{3} \pi (ct)^3} = \frac{3E}{4\pi c^3 t^3} \]
and the homogeneous cosmology leads, for \( t \gg 1 \), to
\[ \rho_m(t) = \frac{c^2}{6\pi G t^2} \]
Actually, this value can be considered as an approximate value for \( \rho_m(t) \) within the framework of our model since it has been obtained by using the relation ([4], (5.2.19))
\[ \rho_m a^3 = \text{cte} \]
(which is actually the case within our model as it will be shown below) and by replacing \( \rho_m \) by \( \frac{\alpha}{a^3} \) into equation (5.2.14) when neglecting the term \( \frac{3K}{a^2} \).
This gives
\[ 3 \frac{\dot{a}^2}{a^2} = 8\pi \frac{\alpha}{a^3} \]
or
\[ \dot{a}^2 = \frac{8\pi \alpha}{3} \frac{1}{a} = \frac{8\pi \rho_m a^3}{3} \frac{1}{a} \]
which can be written as (5.2.21)
\[ \dot{a}^2 = \frac{C}{a} \]
with
\[ C = \frac{8\pi \rho_m a^3}{3} \]
The solution of this equation is ([4], Table (5.1))
\[ a = \left( \frac{9C}{4} \right)^{\frac{2}{3}} t^{\frac{2}{3}} \]
where we have replaced \( \tau \) by \( t \) as there is no proper time within our model. After reintroducing the constant \( c \) and \( G \), the preceding gives the announced approximate result.

Therefore, the above density equation gives
\[ \frac{3E}{4\pi c^3 t^3} = \frac{c^2}{6\pi G t^2} + \frac{a}{t^3} \]
that is
\[ \frac{3E}{4\pi c^3} = \frac{c^2}{6\pi G} t + a \]
with
\[ a = -\frac{c^2}{6\pi G} t + \frac{3E}{4\pi c^3}. \]
When \( t \) goes to infinity, our model implies that
\[ a \sim -\frac{c^2}{6\pi G} t \]
and, for \( t \gg 1 \), we have
\[ \rho_r(t) = \frac{a}{t^3} = -\frac{c^2}{6\pi G} \frac{1}{t^2} \]
\[ + \frac{3E}{4\pi c^3} \frac{1}{t^3} = -\rho_m(t) + \frac{3E}{4\pi c^3} \frac{1}{t^3}. \] (\( \rho \))
Writing the density equation (\( \rho \)) for \( t = t_0 \gg 1 \), we obtain
\[ \frac{3E}{4\pi c^3 t_0^3} = \frac{c^2}{6\pi G_0 t_0^2} + \frac{a_0}{t_0^3} \]
and then
\[ \frac{3c^5}{4\pi c^3 t_0^3} \times 2G_0 = \frac{c^2}{6\pi G_0 t_0^2} + \frac{a_0}{t_0^3} \]
that is
\[ \frac{3c^2}{8\pi G_0} (K_0 + \frac{1}{t_0^2}) = \frac{c^2}{6\pi G_0 t_0^2} + \frac{a_0}{t_0^3}. \]
By putting
\[ K_0 t_0^3 + t_0 = \frac{4}{9} t_0 + b_0, \]
we obtain
\[ \frac{3c^2}{8\pi G_0} \left( \frac{4}{9} t_0^2 + \frac{b_0}{t_0^3} \right) = \frac{c^2}{6\pi G_0 t_0^2} + \frac{a_0}{t_0^3} \]
which yields
\[ \frac{3c^2}{8\pi G_0} t_0^3 = \frac{a_0}{t_0^3} \]
and
\[ b_0 = \frac{8\pi G_0 a_0}{3c^2}. \]
We then have
\[ C_0 = K_0 t_0^3 + t_0 = \frac{4}{9} t_0 + \frac{8\pi G_0 a_0}{3c^2} \]
and

$$K_0 = -\frac{5}{9t_0^2} + \frac{8\pi G_0a_0}{3c^2t_0^3} \quad (28)$$

Likewise, we have

$$\frac{2G_0E}{c^5} = C_0 = \frac{4}{9}t_0 + \frac{8\pi G_0a_0}{3c^2} \quad (29)$$

and then we recover the (previously obtained) relation

$$a_0 = (\frac{2G_0E}{c^5} - \frac{4}{9}t_0) \times \frac{3c^2}{8\pi G_0}$$

$$= \frac{3E}{4\pi c^3} - \frac{c^2t_0}{6\pi G_0} \quad (30)$$

**Energy, age and size of the universe**

Now, we take for $t_0$ the present time and we use the Stefan - Boltzmann law and the generally accepted estimation of the neutrinos’ contribution to the present universe density for setting

$$\rho_r(t_0) = \frac{a_0}{t_0^3} = \frac{3E}{4\pi c^3t_0^3} - \frac{c^2}{6\pi G_0t_0^2}$$

$$= -0.4 \times 10^6 eV/m^3 = -6.4 \times 10^{-14} J/m^3.$$ 

By writing down the second Einstein - Friedmann’s equation

$$\frac{d^2R}{dt^2} = -\frac{4\pi G}{3c^2} (\rho + 3P),$$

where $P$ is the mean pressure, we obtain (within the framework of our model) :

$$\rho + 3P = 0.$$ 

But

$$\rho = \rho_m + \rho_r = \rho_m + 3P$$

and then

$$\rho_m + \rho_r + \rho_r = 0$$

which yields

$$\rho_m = -2\rho_r \quad \text{and} \quad \rho = -\rho_r.$$
This clearly shows that when we study the general relativity we have to consider the mean radiational density and the mean pressure of the universe as being negative, which is very normal as their effect is antigravitational.

Thus, we have
\[ \rho_0 = -\frac{a_0}{t_0^3} = 6.4 \times 10^{-14} \text{ J/m}^3. \]

Consequently equation (29) gives
\[ E = \frac{c^5}{2G_0} \left( \frac{4}{9} t_0 - \frac{8\pi G t_0^3}{3c^2} \right) \times 6.4 \times 10^{-14} \]
and
\[ \frac{E}{\rho_0} = \frac{4\pi c^3 t_0^3}{3} = \frac{c^5}{2G_0} \times \frac{4}{9} t_0 - \frac{8\pi G t_0^3}{3c^2} \times 6.4 \times 10^{-14} \]

Therefore, we obtain
\[ \frac{4\pi c^3 t_0^3}{3} = \frac{2c^5 t_0}{9G_0 \pi \times 6.4 \times 10^{-14}} - \frac{8\pi G c^3 t_0^3}{6G_0} \]
which yields
\[ \frac{8\pi}{3} t_0^2 = \frac{2c^2 \times 10^{14}}{9G_0 \times 6.4} \]
and
\[ t_0^2 = \frac{3}{8\pi} \times \frac{2c^2 \times 10^{14}}{9G_0 \times 6.4} \]
\[ = \frac{3 \times 2 \times 9 \times 10^{16} \times 10^{14}}{8\pi \times 9 \times 6.67 \times 10^{-11} \times 6.4} \approx 5.595 \times 10^{38}. \]
Finally, we have
\[ t_0 \approx 2.365 \times 10^{19} \text{ s}. \]
This same result could have been obtained directly by using only the second Friedmann’s equation. Indeed this equation gives
\[ \rho_m + \rho_r + \rho_r = 0 \]
which leads to
\[ \rho_m = -2\rho_r \]
that is
\[ \rho_m = \frac{c^2}{6\pi G_0 t_0^2} = 12.8 \times 10^{14} \text{ J/m}^3 \]
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and then
\[ t_0^2 = \frac{c^2}{6\pi G_0 \times 12.8 \times 10^{-14}} = \frac{9 \times 10^{16} \times 10^{14}}{6 \times 3.14 \times 6.67 \times 10^{-11} \times 12.8} \approx 5.595 \times 10^{38} \]
which gives again
\[ t_0 = 2.365 \times 10^{19} \text{s}. \]
Thus, the universe radius is
\[ r_0 = c t_0 \simeq 7.1 \times 10^{27} \text{m}. \]
The total universe energy is
\[ E = \frac{4\pi c^3 t_0^3 \rho_0}{3} = \frac{2c^5 t_0}{9G_0} - \frac{4\pi c^3 t_0^{3/2}}{3} \times 6.4 \times 10^{-14} \]
\[ = 9.57 \times 10^{70} J (= 19.147 \times 10^{70} - 9.57 \times 10^{70}) J. \]
The above value of \( \rho_m \) conforms with its value deduced from the density equation (\( \rho \)) which gives
\[ \rho_m(t_0) = \frac{3E}{2\pi c^2 t_0^2}. \]
The Hubble’s parameter is
\[ H_0 = \frac{1}{t_0} \simeq 4.228 \times 10^{-20}. \]
The radiation energy density is presently
\[ \rho_r = -0.4 \times 10^6 \text{eV/m}^3. \]
The mass energy density is
\[ \rho_m = 0.8 \times 10^6 \text{eV/m}^3. \]
The present matter mass (including the black holes) is
\[ M = \frac{\rho_m}{c^2} \times \frac{4\pi c^3 t_0^3}{3} = \rho_m \times \frac{4\pi ct_0^3}{3} \]
\[ = 2.126 \times 10^{54} Kg. \]
The equivalent mass of the total energy is
\[ M_e := \frac{E}{c^2} = 1.063 \times 10^{54} Kg. \]
Comparison with the Einstein - de Sitter model

According to the standard homogeneous isotropic model (with $k = 0$), we have (c.f.[2], p.555-557)

$$R(t) \propto t^2 \quad H(t) \propto \frac{2}{3t}$$

$$\lambda \propto \frac{1}{T} \propto R$$

and then (using the well confirmed Stefan - Boltzmann’s result)

$$\rho_r(t) \propto T^4 \propto \frac{1}{R^3} \propto \frac{1}{t^2}$$

$$\rho_m(t) \propto T^3 \propto \frac{1}{R^3} \propto \frac{1}{t}$$

So, we obtain

$$\frac{\rho_r}{\rho_m} \propto \frac{1}{t^2}$$

which is inexact according to this same model.

On the other hand, our model clearly shows the fundamental property

$$\rho \propto \frac{1}{R^3} \propto \frac{1}{t^3}$$

which conforms with our result

$$\rho_r \propto \frac{1}{t^3}$$

that gives (joined to the Stefan - Boltzmann result)

$$T^4 \propto \frac{1}{t^3} \quad \text{or} \quad T \propto \frac{1}{t^{3/4}}$$

$$\lambda \propto t \propto \frac{1}{T^{3/4}}$$

and

$$\rho \propto \rho_r \propto \rho_m \propto T^4 \propto \frac{1}{t^{3/4}}$$

which is obviously more consistent.
In other respects, the Einstein’s tensor $T$ is written, in the framework of the homogeneous isotropic cosmology, as ([4],5.2.1)

$$T_{ab} = \rho_m u_a u_b$$

where $\rho_m$ is the mean density of the mass energy. Moreover, the vector field $u^a$ becomes, in the framework of our model, the coordinate vector field $(\frac{\partial}{\partial t})^a$ and then we have

$$T_{ab} = \rho_m dt^2.$$

On the other hand, the expression of the total mass of the universe is established as being, within the framework of this cosmology, ([4],11.2.10)

$$M = \frac{1}{4\pi} \int_{\Sigma} R_{ab} n^a \xi^b dV = 2 \int_{\Sigma} (T_{ab} - \frac{1}{2} T g_{ab}) n^a \xi^b dV$$

So, by adapting this expression to our model, $\Sigma$ becomes $B(O,t)$, $g_{ab}$ becomes $h_t = dt^2 - g_t$ and $n^a$ identifies with $(\frac{\partial}{\partial t})^a$ which also constitutes a reasonable approximation of $\xi^a$. Consequently, we have $T_{ab} n^a \xi^b = \rho_m, T = \rho_m$ and $g_{ab} n^a \xi^b = 1$ and then we obtain

$$M = 2 \int_{B(O,t)} (\rho_m - \frac{1}{2} \rho_m) dV = \int_{B(O,t)} \rho_m dV$$

which is, for us, the global mass of the universe including the black holes and the invisible matter mass.

**Comparison with Newtonian gravity**

Comparing our matter equation

$$\frac{\partial^2}{\partial t^2} E(t, X(t)) - \Delta E(t, X(t)) = 0$$

and our identities

$$X''(t) = \Gamma(t) = -\nabla g \cdot E(t, X(t))$$

with the two equations that characterize Newtonian gravity (c.f.[4],(4.4.17) and (4.4.21))

$$\Delta \varphi = 4\pi \rho \quad \text{(Poisson equation)}$$

$$X'' = \Gamma = -\nabla g \cdot \varphi,$$

we obtain (by identification)

$$\varphi(X(t)) = E(t, X(t))$$

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\[ \Delta \varphi(X(t)) = \Delta E(t, X(t)) = 4\pi \rho(X(t)). \]

Consequently, we have

\[ \frac{\partial^2 E}{\partial t^2}(t, X(t)) = \Delta E(t, X(t)) = 4\pi \rho(X(t)). \]

Now, the total energy of the idealized universe within the Newtonian gravity context is given by (c.f. [4], (11.2.2))

\[ E_N = \frac{1}{4\pi} \int_{S(O, R)} \nabla \varphi \cdot \vec{n} dS. \]

Thus, we get

\[ E_N = \frac{1}{4\pi} \int_{S(O, R)} \nabla E \cdot \vec{n} dS \]

\[ = \frac{1}{4\pi} \int_{B(O, R)} \Delta E dX = \frac{1}{4\pi} \int_{B(O, R)} 4\pi \rho dX \]

\[ = \frac{1}{4\pi} \int_{B(O, R)} \rho dX = E \]

where \( \rho \) denotes here the mean energy density of the universe at time \( t \) and \( E \) is the total energy of the universe according to our model.

Moreover, by integrating on a mean unit ball of the universe at time \( t \), we obtain

\[ \int_{B(I, 1)} \frac{\partial^2 E}{\partial t^2}(t, X_t) dX_t = \frac{\partial^2}{\partial t^2} \int_{B(I, 1)} E(t, X_t) dX_t \]

\[ = 4\pi \int_{B(I, 1)} \rho(X_t) dX_t = \frac{16\pi}{3} \rho(t) = \frac{16\pi}{3} E(t) \]

where \( \rho(t) = E(t) \) denote the mean energy density of the universe at time \( t \).

Then we have

\[ E''(t) = \frac{16\pi}{3} E(t) \quad \text{or} \quad \rho''(t) = \frac{16\pi}{3} \rho(t) \]

which implies

\[ E(t) = \rho(t) = Ce^{-4\sqrt{\pi}t} \]

where \( C \) is a constant that can be determined when using the known values \( t_0 \) and \( \rho_0 \). Indeed, the relation

\[ E(t_0) = Ce^{-4\sqrt{\pi}t_0} = \rho_0 \]

implies

\[ C = \rho_0 e^{4\sqrt{\pi}t_0}. \]

So we have

\[ \rho(t) = \rho_0 e^{4\sqrt{\pi}(t_0 - t)} \]
Remarks

The deviations of the above values (for $t_0$, $r_0$, $E$,...) with respect to the classical approximate values expected by the Einstein - de Sitter’s standard model, for instance, can be essentially explained by the two following factors:

1°) The generally accepted value of the present Hubble’s parameter $H_0$ is incorrect for two reasons:
The first one is the use of some relativistic notions, the deficiency of which has been already showed, such as the proper time and the relativistic formulas used in order to determine the redshift parameter $z$.
The second reason is the fact of being based on measurements established when considering visible galaxies whose positioning, mutual distances and relative speeds are far from accurately representing the expansion parameter. We are not at the center of the universe and there exists a large number of galaxies that are lying and moving outside of our horizon.
The huge difference between the two estimations of the mean matter energy densities is due to the large difference between the two estimations of the universe size.

2°) The curvature parameter generally used in the first Friedmann’s equation is assumed to be constant. But this is absurd since the fundamental notion of curvature (which reflects and characterizes the matter - energy distribution which is the real universe essence) is essentially dynamic and evolutive (locally and globally). Indeed, according to the well confirmed expansion theory, the universe does not reduce to $\mathbb{R}^3$ nor to a fixed domain in $\mathbb{R}^3$, but (according to our model) to a ball $B(O,t)$ always expanding. This is the reason that pushed us to start by using the first equation of Friedmann in order to show the necessity of using a time-dependent (macroscopic and global) curvature parameter $K(t)$ that is associated with a (both local and global) time-dependent metric $g_t$, although the determination of $t_0$ can be achieved by using only the second Friedmann’s equation.

3°) We also notice that the sign difference between the densities $\rho_m$ and $\rho_r$ is fundamentally due to the fact that the first one is associated to the attractive force of gravity while the second one is associated with the pressure who gives rise to a force having an essentially opposed nature. These are the two fundamental forces of Nature, namely the gravitational and the radiational pressure (or equivalently the electromagnetic) forces. Thus, the equation
\[
\rho + 3P = \rho_m + \rho_r + 3P = 0
\]
adds a new dimension to the cosmological energy conservation problem. Indeed, if we qualify the quantity \(3P\) as (mean density of) negative energy and the quantity \(\rho + 3P\) as (mean density of) generalized global energy, we can state

The generalized global energy is eternally null.

This principle recalls the momentum conservation principle when applied to the whole universe:

The global momentum of the universe is eternally null.

4°) It is well known that in the framework of the Newtonian gravity, the gravitational field density is defined by

\[-\frac{1}{8\pi}|\nabla g^\varphi|^2.\]

Conversely, the definition of a gravitational energy density within the framework of the standard general relativity seems to have serious problems (see [4], p. 286). However, it appears that, in the framework of our model, we can define the general energy density of the universe including the gravitational field when using our metric \(g_t\) (which reflects all of the physical consistence of the universe characterized by our global matter-energy Tensor \(T^*_a b\)) by

\[-\frac{1}{8\pi}|\nabla g^t E_t|,\]

\(\nabla g^t E_t\) here is the gradient, with respect to \(g_t\), of the matter-energy distribution \(E_t(X)\) on \(B(O, t)\) at time \(t\).

Nevertheless, we notice that, according to our model, we can only speak of pressure and negative energy, and then of null generalized global energy (when assuming that the universe is originally reduced to an energy \(E_0\) concentrated at a given point), after the Big Bang. Consequently, our model does not support the theories that sustain that the universe is coming out from literally nothing. Also, we notice that the meaning of the term negative energy used here is completely different from that to which is generally attributed inside the inflation theory.
12 Fundamental constants of modern Physics

We will here establish some relations involving several fundamental physical constants showing that a large number of them are dependent on time (and temperature) and leading to the unification of the fundamental forces as well as to the unification of all branches of Physics: General relativity (i.e. Cosmology) Quantum theory, Electromagnetism, Thermodynamics and Newton - Lagrange - Hamilton Mechanics.

Recall that we have showed in the previous section that

\[ E = \frac{c^5 C_0}{2 G_0} \]

where

\[ C_0 = K_0 t_0^3 + t_0 = \frac{4}{9} t_0 + \frac{8 \pi G_0 a_0}{3 c^2} \]

This implies

\[ G_0 = \frac{c^5 C_0}{2 E} = \frac{c^5}{2 E} \left( \frac{4}{9} t_0 + \frac{8 \pi G_0 a_0}{3 c^2} \right) \]
\[ = \frac{2 c^5 t_0}{9 E} + \frac{4 \pi c^3 G_0 a_0}{3 E} \]
\[ = \frac{2 c^5 t_0}{9 E} + \frac{4 \pi c^3 G_0 t_0^3 a_0}{3 E t_0^3} \]
\[ = \frac{2 c^5 t_0}{9 E} - \frac{4 \pi c^3 t_0^3}{3 G_0 E} \rho_0 \]
\[ = \frac{2 c^5 t_0}{9 E} - G_0. \]

Consequently, we have

\[ 2 G_0 = \frac{2 c^5 t_0}{9 E} \]

and

\[ G_0 = \frac{c^5 t_0}{9 E} \left( = \frac{243 \times 10^{40} \times 2.365 \times 10^{19}}{9 \times 9.57 \times 10^{70}} \approx 6.67 \times 10^{-11} \right). \]

This equality being valid for an arbitrary \( t_0 \gg 1 \), we obtain, for \( t \gg 1 \):

\[ G = \frac{c^5 t}{9 E} \quad \text{and} \quad E = \frac{c^5 t}{9 G} \quad (31) \]

Besides, we have

\[ \rho_m = -2 \rho_r = 2 \rho = \frac{6 E}{4 \pi c^3 t^3}. \]
which gives by using quantum Statistics (c.f. [4] p.108)

\[
\frac{6E}{4\pi c^3 t^3} = \sum_{i=1}^{n} \alpha_i g_i \frac{\pi^2 (K_B T)^4}{30 \hbar^3 c^5}
\]

where here the Boltzmann’s constant is denoted by \(K_B\).

Thus, we have

\[
\frac{3E}{2\pi c^3 t^3} = \sum_{i=1}^{n} \alpha_i g_i \frac{\pi^2}{30c^5} \frac{(K_B T)^4}{\hbar^3}
\]

which gives

\[
\frac{3c^5 t}{2\pi c^3 t^3} = \frac{c^2}{6\pi G t^2} = \left(\sum_{i=1}^{n} \alpha_i g_i \frac{\pi^2}{30c^5} \frac{(K_B T)^4}{\hbar^3}\right)
\]

that is

\[
\frac{5c^7}{\pi G t^2} = \left(\sum_{i=1}^{n} \alpha_i g_i \pi^2 \frac{(K_B T)^4}{\hbar^3}\right)
\]

or

\[
\frac{1}{G t^2} = \left(\sum_{i=1}^{n} \alpha_i g_i \pi^2 \frac{5c^7}{\hbar^3}\right) \frac{(K_B T)^4}{\hbar^3} =: A \frac{(K_B T)^4}{\hbar^3}.
\]

We then have

\[
G = \frac{1}{A} \frac{\hbar^3}{(K_B T)^4 t^2}
\]

(32)

where

\[
A = \frac{1}{G} \frac{\hbar^3}{(K_B T)^4 t^2}
\]

can be calculated by using the presently determined values of the involved constants corresponding to \(t_0 = 2.365 \times 10^{19}\).

Then, equation (31) gives

\[
\frac{c^5 t}{9E} = \frac{1}{A} \frac{\hbar^3}{(K_B T)^4 t^2} \frac{1}{t^2}
\]

which yields

\[
\frac{\hbar^3}{(K_B T)^4} = \frac{Ac^5}{9E} \frac{1}{t^3} = AG t^2.
\]

(33)
In other respects, if we denote by \( \alpha \) the classical electromagnetic force factor and by \( K_E \) the electromagnetic constant denoted usually by \( k \), we obtain

\[
\alpha = \frac{K_E e^2}{\hbar c} = \frac{K_E e^2}{c} \times \left( \frac{9E}{(K_B T)^4 A c^5 t^3} \right)^{\frac{1}{3}}
\]

\[
= K_E e^2 \left( \frac{9E}{Ac^8} \right)^{\frac{1}{3}} (K_B T)^{-\frac{4}{3}} t^{-1}
\]

and

\[
\alpha = \frac{K_E e^2}{c} ((K_B T)^4 AGt^2)^{-\frac{1}{4}} = \frac{K_E e^2}{c A^{\frac{1}{4}}} (K_B T)^{-\frac{4}{3}} G^{-\frac{1}{3}} t^{-\frac{2}{3}}
\]

(34)

Using the above relations, we can obtain several relations that specify the dependence of the fundamental constants on each other in addition to time. Indeed, (32) implies

\[
\frac{G(K_B T)^4}{\hbar^3} = \frac{1}{A} \frac{1}{t^2} = \frac{C_1}{t^2}
\]

where \( C_1 = \frac{1}{A} \) (35)

and (33) implies

\[
\frac{\hbar}{(K_B T)^3} = \frac{A c^5}{9E} t^3 = C_2 t^3
\]

with \( C_2 = \frac{A c^5}{9E} \)

or

\[
\frac{\hbar}{(K_B T)^{\frac{3}{2}}} = C_2^{\frac{1}{2}} t
\]

(36)

Likewise, (34) implies

\[
\alpha t = ke^2 \left( \frac{9E}{Ac^8} \right)^{\frac{1}{3}} (K_B T)^{-\frac{4}{3}} = \frac{C_3}{(K_B T)^{\frac{4}{3}}}
\]

where

\[
C_3 = ke^2 \left( \frac{9E}{Ac^8} \right)^{\frac{1}{3}}
\]

and

\[
\alpha (K_B T)^{\frac{4}{3}} = \frac{C_3}{t}
\]

(37)

which, added to the relation

\[
G = \frac{c^5}{9E} t =: C_0 t \quad \text{or} \quad t = \frac{G}{C_0}
\]

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gives
\[ \alpha(K_BT) = \frac{C_3C_0}{G} \]
and then
\[ \alpha G(K_BT) = C_3C_0 =: C_5 \] (38)
where
\[
C_5 = ke^2 \left( \frac{9E}{Ac^3} \right)^{\frac{1}{4}} \frac{c^5}{9E} = ke^2 (9E)^{\frac{1}{4}} \times (9E)^{-1} \times (Ac^3)^{-\frac{1}{4}} c^5
\]
\[
= \frac{ke^2 c^5}{A^{\frac{1}{4}}(9E)^{\frac{1}{4}}}
\]
which, added to (36), implies
\[ \alpha G \frac{h}{C_2 t} = C_5 \]
and then
\[ \alpha G \frac{h}{C_2 t} = C_5 C_2^\frac{1}{3} t = C_6 t \] (39)
with
\[ C_6 = \frac{ke^2 c^4}{9E} \]
(verification : \( \alpha h = C_6 \frac{t}{G} = C_6 \frac{9E}{c^5} = \frac{ke^2 c^4 \cdot 9E}{c^5} = \frac{ke^2}{c} \)).

Finally, (35) and (38) imply
\[ \alpha G \frac{h}{A(K_BT)^4 t^2} = \alpha \left( \frac{h}{K_BT} \right)^4 \frac{1}{At^2} \]
and
\[ \alpha G \frac{h}{9E} = \frac{ke^2 c^4 t}{9E} \]
Therefore
\[ \alpha \left( \frac{h}{K_BT} \right)^4 = \frac{A}{9E} ke^2 c^4 t^3 \] (40)
(verification : \( \alpha \left( \frac{h}{K_BT} \right)^4 = \frac{1}{A} \frac{h^3}{G(K_BT)^4} \frac{1}{t^2} \times \frac{1}{9E} ke^2 c^4 t^3 = \frac{1}{9E} ke^2 c^4 \frac{h^3}{G(K_BT)^4} t \))

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which gives
\[ \alpha h = \frac{1}{9E} \frac{ke^2}{G} c^4 t = ke^2 \frac{c^4 t}{9E} \frac{1}{9E} = \frac{ke^2}{c}. \]

To sum up, we write down the following fundamental relations:
\[
T \propto \frac{1}{t^4}, \quad \rho \propto T^4 \propto \frac{1}{t^8} \propto \frac{1}{V^{3/2}} \quad \text{and} \quad T \propto \frac{1}{V^{3/2}}
\]
(where \( V \) is the volume of the universe at time \( t \))
\[
\alpha h = \frac{K_E e^2}{c},
\]
\[
\frac{\overline{\hbar}^3}{(K_B T)^4} \propto t^3 \quad \text{and} \quad \overline{\hbar} = (K_B T)^{\frac{1}{2}} \propto K_B^{\frac{1}{2}}
\]
\[
E = \frac{c^5 t}{9G} \quad \text{and} \quad G = \frac{c^5}{9E} \propto \frac{1}{T^2}
\]
Moreover, concerning the curvature parameter \( K(t) \), we have, using the relations (28) and (30),
\[
K(t) = -\frac{5}{9t^2} + \frac{8\pi G a}{3c^2 t^3}
\]
\[
= -\frac{5}{9t^2} + \frac{8\pi}{3c^2 t^3} \left( \frac{c^2}{6\pi} t + \frac{3E}{4\pi c^3} \right)
\]
\[
= -\frac{5}{9t^2} + \frac{8\pi}{3c^2 t^3} \left( \frac{c^2}{6\pi} t + \frac{3}{4\pi c^3} \right)
\]
\[
= -\frac{5}{9t^2} + \frac{8\pi}{3c^2 t^3} \left( \frac{c^2}{6\pi} t + \frac{c^2}{12\pi} \right)
\]
\[
= -\frac{5}{9t^2} - \frac{2}{9t^2} = -\frac{7}{9t^2}
\]
Finally, we have on one hand
\[
\alpha = \frac{K_E e^2}{\overline{\hbar}^4} = \frac{K_E e^2}{c K_B^4 t}
\]
and on the other hand
\[
\alpha = \frac{K_E e^2}{c \overline{\hbar}} \propto \frac{K_E e^2}{c (K_B T)^{\frac{1}{2}} t} = \frac{K_E e^2}{c (K_B T)^{\frac{1}{2}}} \times \frac{c^5}{9GE} = \frac{c^4 K_E e^2}{9(K_B T)^{\frac{3}{2}} GE}
\]
or

\[ \alpha G \propto \frac{c^4 K_E e^2}{9(K_B T)^{\frac{3}{2}} E}. \]

Let us notice that, since \( \hbar \propto t \), we get (using (36)) \( K_B T \) is constant and it is easy to show that the preceding relations imply

\[ \alpha \propto \frac{1}{t}, \quad G \propto t \quad \text{and} \quad K_B \propto t^2. \]

At the end, we notice that we have (within the framework of our model)

\[ R_{t_0}(t) = \frac{t}{t_0} \quad \text{and} \quad \frac{dR_{t_0}}{dt} = \frac{1}{t_0} \]

and then the Friedmann equation becomes

\[ \frac{1}{t^2} = \frac{8\pi \rho G^2}{3c^2} - K(t) \]

which yields, for \( t = t_0 \)

\[ \frac{1}{t_0^2} = \frac{8\pi \rho_0 G_0}{3c^2} + \frac{7}{9t_0^2} = \frac{8\pi}{3c^2} \frac{3E}{4\pi c^3 t_0^3} \frac{e^5 t_0}{9E} + \frac{7}{9t_0^2} \]

\[ = \frac{2}{9t_0^2} + \frac{7}{9t_0^2} \]

This again shows the validity of our model and the legitimacy of the time-dependence of the constants \( K \) and \( G \).

**Remarks**

1°) The above relations show that only the fundamental constants \( E, A, c, K_B T \) and \( K_E e^2 \) are independent of time, the other ones \( G, \hbar, K \) and \( \alpha \) depend on time and are related between them and to the first ones.

By taking \( c = 1 \) and taking into account the other relations, we can state that only \( E, K_E e^2, K_B T \) and \( t \) have an intrinsic existence which shows that the universe reduces to three basic elements:

The original energy, electromagnetism and time.

This last factor, which is at the same time distance and extent, determines (with the second factor) the dynamic expansion. So, the universe is essentially the original energy always expanding.

2°) Let us denote \( K_B T = \frac{2}{3} \langle E_K \rangle \), where \( \langle E_K \rangle \) is the generalized mean kinetic energy, by \( E_* \) and replace \( K_E \) by \( k \), then the previous relations lead, for \( c = 1 \), to the following ones:

\[ \alpha \hbar = ke^2 \]

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\[ \hbar = E_4^3 t \quad \text{and} \quad \alpha E_4^3 t = ke^2 \]
\[ G = \frac{t}{9E} \quad \text{and} \quad \hbar = 9E_4^4 GE \]
\[ \alpha = \frac{ke^2}{9E_4^4 EG} \quad \text{or} \quad \alpha GE_4^4 = \frac{ke^2}{9E}. \]

These relations prove, particularly, the unity of the fundamental forces of Nature.

3º) The relation \[ A = \frac{1}{2 \sqrt{K_B T}} \] shows the ultimate powerful of the quantum Statistics. The other ones show, on one hand, that the quantization of every thing is not justified (\( G \) and \( \hbar \), for instance, are continuously depending on time) and, on the other hand, that the quantization that is lying on a long and precise experimentation recovers its global legitimacy by its unification with the theoretical Physics and Mechanics. In the present case we recover the unification of the quantum Physics with the reviewed general relativity which are both unified with the Newton - Lagrange - Hamilton Mechanics. In particular, it is showed that the universe evolution is thoroughly described by the (reviewed) Einstein’s general relativity theory.

4º) These same relations joined to the results of section 9 and the other sections institute the bases of the unification of all Physics’ branches: Electromagnetism, general Relativity (i.e. Cosmology), Thermodynamics, quantum Physics and Mechanics, Particles’ Physics with the (Newton - Lagrange - Hamilton) Mechanics.

13 Commentaries and open issues

Our physical and mathematical global model permits to give many answers, clarifications and precisions concerning open problems of modern Physics. Moreover, it permits to reformulate some other problems and to draw some new perspectives for their resolution.

The crucial points which led to these possibilities are:

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1°) The theoretical, mathematical and physical refutation of the second part of the second special relativity postulate and the replacement of the relativistic approximate formulas by alternative formulas that reflect more precisely the Nature’s laws although the precision of which could be improved and more exactly quantified (see sections 4 and 7). These modifications should be generalized to all recesses of modern Physics.

2°) The logical and mathematical clarification of the experimental and theoretical Quantum theory limits and the specification of the circumstantial nature of the Heisenberg uncertainty principle despite of the practical efficiency of these tools headed by Schrödinger’s equations (sections 7 and 8). Nevertheless, the Quantum theory must be used in order to resolve all problems that does not fall in with an idealized modeling that permits their resolution by means of classical Mechanics and Physics.

3°) The geometrization of the three dimensional universe by means of the physical metric $g_t$ and the setting up of the Matter - Energy equation $(E')$ as well as the geometrization of the four dimensional space - time by means of the evolution metric $h = dt^2 - g_t$, after the restoration of the natural space and time notions. Our metrics ensure that the evolution of all quantities with time is taken into account (sections 4, 5 and 6). The Einstein’s vacuum spacetime metric takes into account only the gravitational field and the Einstein’s tensor characterizes some particular matter fields and electromagnetic fields. Our metric and tensor take into account all forms and effects of matter-energy : Global gravitational field (included that of black holes), all forms of matter field, global electromagnetic field, global cosmic radiations, pressure and temperature, all energy evolutions, interactions and singularities.

4°) The logical (and, a posteriori, physical) refutation of the first Hubble’s cosmological principle according to which any galaxy can be considered as lying at the center of the universe. This refutation is solidly supported by our model which reveals its consistence and its extreme compatibility with all definitely established physical and mechanical laws. When joined to the first point, this refutation has permitted to modify the Einstein’s general relativity theory, the Friedmann - Einstein’s equations and the Einstein - Hubble - de Sitter - Friedmann’s Cosmology in such a way that (one time adapted to our setting) it is clearly showed that these theories and equations describe correctly the universe evolution (i.e. the universe expansion). This adaptation, joined to the demonstration of the time - dependence of all fundamental constants, apart from $E$, $k e^2$, $K_B T$, $c$ and $A$ (which gives a new dimension
to the quantization process), leads to the unification of all Physics’ branches (sections 10, 11 and 12).

5°) The matter-energy equation and the use of the Dirac operator have permitted to yield a new approach to the Physics of particles by instoring a new classification of the fundamental particles that has a chance (if being sustained by experimental results) to lie at the root of this branch of Physics which is presently deeply related to the Cosmology and has many beneficial applications for the humanity future (section 9).

Along this study, we have given clear answers to many open questions (such as, for instance, those listed at the end of [2] or those invoked throughout the excellent panoramic book directed by Paul Davies and wisely entitled: The New Physics) and some possible answers to other open problems. We have also formulated other questions and emitted some hypotheses. Nevertheless, we will, in the following, give some properties and remarks concerning the above last point.

1°) According to our model, the fundamental material particles are only nine (to which we add the three neutrinos):

e  u  d  s  c  b  t  μ  τ  ν_e  ν_μ  ν_τ,

each of them intrinsically exists with two different spins and each of the six quarks intrinsically exists with three different colors.

They constitute, with their antiparticles, the three pure energy vectors represented by Γ_1, Γ_2, Γ_3 and other Γ vectors; each of them having two distinct polarizations. These latter potentially hold all fundamental particles to which they give birth. The fundamental particles form all bounded states called hadrons (baryons and mesons) that are more or less stable.

The neutrinos are the direct or undirect partners of a lot of (weak, strong and electromagnetic) interactions, decays, collisions, nuclear syntheses and annihilations. They are (originally and continually) produced during these interactions essentially in order to make them possible and compatible with the conservation laws. Some significative examples of production and interactions involving these particles are schematically listed below (c.f. [2]) :

\begin{align*}
W^+ & \rightarrow e^+ + ν_e \\
W^- & \rightarrow e^- + \overline{ν_e} \\
π^+ & \rightarrow μ^+ + ν_μ \\
π^- & \rightarrow μ^- + \overline{ν_μ} \\
K^+ & \rightarrow μ^+ + ν_μ \\
K^- & \rightarrow μ^- + \overline{ν_μ}
\end{align*}
These interactions and many others show that the two leptonic families $\mu$ and $\tau$ are produced by the interactions of neutrinos with several types of fundamental (and other) particles as well as by other interactions and by several decays. They are produced with a very short lifetimes. We consider that these families are potentially composite particles, although they are fundamental particles (i.e. solutions to the wave equation associated with the Dirac operator). They have electrical charges and their very fast decays give birth to stable neutrinos and unstable hadrons beside of stable electrons or positrons (which interact and annihilate very quickly). This is roughly the case of the five more massive quarks which are fundamental particles although they are not absolutely stable and they end up by giving birth to the $u$ quark.
2°) It is well known that neutrinos are left handed particles (i.e. have a negative helicity) and that antineutrinos are right handed particles. These properties have to be related to the two different polarizations of the electromagnetic waves and to the existence of two electron’s types with opposite spins $e_{\frac{1}{2}}$ and $e_{-\frac{1}{2}}$. They can be considered as intrinsic characteristics of these particles. The $\beta^-$ decay of the $W^-$ particle produces a negatively polarized electron and an antineutrino that can not be other than positively polarized. The collision - annihilation $p - \overline{p}$ or rather $q - \overline{q}$ of a left handed quark with a right handed antiquark produces (by the intermediate of a $W$ particle) one right handed positron and a neutrino that can not be other than left handed. Can we then talk about parity violation?

3°) Our model is based on the notion of field theory which does not require the existence of strong charges (in the same manner as gravity does not require gravitational charge) nor the existence of intermediate charge conveyors. Gluons are in fact similar particles to photons and like them do not carry charges; they only are mediators for quarks and nucleons’ strong interactions (in the same manner as photons are mediators for electromagnetic interactions between charged particles).

Unlike photons which, on one hand, form a permanently evolving sea inside atoms and, on the other hand, propagate throughout the universe, gluons only form a sea of particles (among other particles) inside hadrons (and nuclei) that evolve and transform permanently bearing and causing all sorts of interactions. Otherwise, gluons are fundamentally different from $W$ and $Z^0$ massive particles which are at the origin of weak interactions from their formation until their decays and their effects. Besides, we notice that neither gravitons nor balls of glu exist within our model. Gravitational field does exist; it curves the space and need neither gravitational charges nor intermediary particles for conveying ”gravitational force” and it has an unlimited range inside the universe. The strong interactions exist only inside hadrons and nuclei and are conveyed by gluons.

4°) The only non hadronic stable particles inside our model are electrons, photons, neutrinos and the $u$ quark. The only presumably stable hadron is the proton. The other hadrons (except the neutron) have ephemeral existence. This fact sustains our hypothesis about the non existence of particular strong forces. The hadronic bound states are achieved by the ephemeral and episodic attractions caused by the gravitation and electromagnetic attractions. This fact is supported by comparing, for instance, the following three bound
states: the proton state $uud$, the neutron state $udd$ and the $W^{++}$ state
$uuu$. The first two states have a spin of $\frac{1}{2}$ and the third one has a spin of $\frac{3}{2}$. This shows that only two constituent quarks have aligned spins for the former states whereas all three spins are aligned for the latter one. Moreover, the first bound state is formed with two positively charged quarks ($+\frac{2}{3}$) and one negatively charged one ($-\frac{1}{3}$), the second state is formed with two negatively charged quarks ($-\frac{1}{3}$) and one positively charged one ($+\frac{2}{3}$) whereas the third state is formed with three positively charged quarks ($+\frac{2}{3}$). The above two factors contribute to make the $\Delta^{++}$ particle extremely unstable, compared to the other two states, with its infinitely short lifetime. The extremely larger lifetime of the proton compared to that of the neutron could be explained by the fact that the quark which is antialigned with the two others inside the proton is the quark $u$ which has a larger electrical charge than the quark $d$ which is antialigned with the other two quarks inside the neutron and so the electromagnetic attraction inside the proton is larger than that inside the neutron (whereas it does not exist inside the $W^{++}$ particle). The mass difference between the quarks $d$ and $u$ and consequently between the neutron and the proton has a determinant role concerning the stability difference; the neutron has the possibility of decaying in order to give birth to a proton, the decaying of which is apparently forbidden. The transformation of the quark $d$ into the quark $u$ inside the neutron has no major constraints. The quark $u$ can not naturally give birth to another quark. We can conclude that the energy equilibrium between the (gravitational and electromagnetic) potential energy, the (vibrational and rotational) kinetic energy and the mass energy which is established (at very short distances) between the three quarks $u, u$ and $d$ inside the proton is extremely more stable than that established between the quarks $u, d$ and $d$ inside the neutron which itself is extremely more stable than between quarks and antiquarks inside all other hadrons.

$5^0$) The non existence of three different strong charges and three or eight differently charged gluons (even though the existence of each quark with three different colors has been proved and the existence of several differently colored gluons is not excluded) does not prevent the possibility that strong interactions could be associated with the symmetry $SU(3)$. Likewise, the weak interactions could also be associated with the symmetry $SU(2)$. Moreover the weak interactions and the electromagnetic interactions could be unified by means of a gauge theory associated with the group $SU(2) \times U(1)$ inside an electroweak theory. It is equally possible to construct a grand unification theory based on the group $SU(5)$ and to ask for the specific conditions concerning the different symmetry breakings. All this could allow us to construct the table of all particles which could ever exist starting from our 24 funda-
mental particles and to characterize all particles that are qualified as gauge particles, including particles of Higgs’ type.

On other respects, we could give in to the mathematical charm of the supersymmetry theory and associate to it the supergravity theory. But the existence of garvitons, Higgs particles and their supersymmetric partners (gravitinos and Higgsinos) is excluded inside our model. Likewise, we have to admit that the Kaluza - Klein compactification of the five dimensional space (four of them constitute the spacetime related to general relativity and the fifth invisible compactified one corresponds to Electromagnetism) and that the theory of everything (TOE) attempted by Cremmer and Julia and its association with the supersymmetry $N = 8$ theory (with its pyrgons) do not have anything to do with our model.

There is also the ten dimensional supersymmetric string theory which is associated with a symmetry group that owns a gauge group $G$ of rank 16 (which could be $SO(32)/\mathbb{Z}_2$ or $E8 \times E8$) that reproduces 496 Yang - Mills gauge particles, 480 of which are solitons. The association of the beautiful topological results of Witten and other mathematical ingredients with this theory does not constitute for us other than a fascinating intellectual gymnastic exceptionally esthetic which, after long theoretical and experimental confrontations and appropriate modifications could contribute to enlighten the last part of the knowledge path leading to the ultimate laws of Nature.

We finally notice that inflation and vacuum energy and its polarization have no place in our model. The vacuum that is crowded with all sorts of real (material or immaterial) particles which are created and are interacting, annihilating and decaying more or less fastly is not a real vacuum. These particles can really exist and participate to many energy transformations and can also be specifically polarised but could we then speak about polarised vacuum? The vacuum negative energy may only be explained by the radiational pressure (resulting from a radiational sea when it exists) that act in the expansion direction. This certainly is not a matter of a negative gravity associated with a false vacuum.

6°) When we admit the existence of the original energy that was concentrated at some point, our model explains the universe creation and formation process by the following splittings :

- The energy splitting into two polarizations at the propagation time.
- The matter-antimatter splitting together with the electrical charge splitting.
– The spin splitting of the electron and the other two-spins particles.

These three phenomena are tightly related to the temperature: Temperature = Energy (via frequencies) = (radiational) pressure. The propagation-expansion, the interactions and the subsequent evolution can be well explained and obey precise laws.

We can briefly state that the universe formation process essentially reduces to

1. The propagation (related to the polarization splitting) of electromagnetic waves starting from the original energy.

2. The matter-antimatter formation and particularly the electron-positron and the u quark-antiquark creation. The other quarks-antiquarks and leptons-antileptons (with the neutrinos-antineutrinos) creation has led to the state qualified as quarks and leptons soup that preceded the hadrons (and the others) formation.

This scheme indicates that there exist only three privileged ultra-fundamental particles (with their antiparticles): The photon (with its two polarizations) that is essentially an energy particle into movement, the electron (with its two opposite spins) and the u quark (with its two spins and three colors).

The other fundamental particles end up all by giving birth to these three particles and to neutrinos.

The real universe is essentially made up with the most stable five particles: Photons, Electrons, Neutrinos, Protons and Neutrons. The last two particles are made up by the u and d quarks. The uud bound state being more stable than the udd state because the d quark can naturally transform into the u quark. The other leptons and hadrons are extremely less stable and have an ephemeral existence. The bound state uu does not exist and the state uuu can exist only during an insignificant infinitesimal time. Moreover, the disappearance of the antimatter could be explained by the extreme unstability of all particles that are partially made up with antimaterial fundamental particles on one hand and, on the other hand, by the absorption of antineutrinos by a large number of “absolutely” stable protons in order to give birth to the relatively unstable neutrons which are yet less numerous than protons. All antiparticles have quickly annihilated giving birth to photons and gluons.

According to the preceding, we conclude that the discovery and the understanding of the Nature’s laws and the subsequent evolution of the energy-matter as well as the universe evolution constitutes the Sciences’ domain,
the existence of the original energy and the original splitting and movement’s reason constitutes the Metaphysics’ domain, whereas the reflection of the consequences of these laws and this evolution on the humanity constitutes the Philosophy and the human reason’s domain.

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e.mail: hmoukadem@hotmail.com