ON THE DIMENSION OF THE HILBERT SCHEME OF CURVES

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Abstract. Consider a component of the Hilbert scheme whose general point corresponds to a degree $d$ genus $g$ smooth irreducible and nondegenerate curve in a projective variety $X$. We give lower bounds for the dimension of such a component when $X$ is $\mathbb{P}^3$, $\mathbb{P}^4$ or a smooth quadric threefold in $\mathbb{P}^4$ respectively. Those bounds make sense from the asymptotic viewpoint if we fix $d$ and let $g$ vary. Some examples are constructed using determinantal varieties to show the sharpness of the bounds for $d$ and $g$ in a certain range. The results can also be applied to study rigid curves.

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1. Introduction

In this section, we briefly recall some basic facts about Hilbert schemes, and state the main results of this paper.

Let $P$ be the Hilbert polynomial of a subscheme in $\mathbb{P}^r$. We can ask if there exists a good parameter space $\mathcal{H}_{P,r}$ parametrizing all the subschemes that have $P$ as their Hilbert polynomial. Grothendieck proved the following fundamental result on the existence of $\mathcal{H}_{P,r}$.

Theorem 1.1. There exists a fine moduli space $\mathcal{H}_{P,r}$. Moreover, it is a projective scheme.

Very few facts about the global properties of $\mathcal{H}_{P,r}$ have been obtained. However, the connectedness of $\mathcal{H}_{P,r}$ has been proved in Hartshorne’s thesis.

Theorem 1.2. The Hilbert scheme $\mathcal{H}_{P,r}$ is connected for any $P$ and $r$.

Here curves are our main interests. The Hilbert polynomial $P$ of a curve is a linear function with leading coefficient $d$ and constant term $1 - g$, where $d$ and $g$ are the degree and genus of the curve. In this case, we use the notation $\mathcal{H}_{d,g,r}$ in stead of $\mathcal{H}_{P,r}$. Sometimes we will also simply use $\mathcal{H}$ when there is no confusion.

Consider the dimension of $\mathcal{H}$. We have the following result. See, for instance [7, Section 1.E], for related references.
Theorem 1.3. Let $C$ be a 1-dimensional subscheme in $\mathbb{P}^r$ such that $[C] \in \mathcal{H}_{d,g,r}$. The tangent space of $\mathcal{H}$ at $[C]$ can be identified as

$$T_{[C]}\mathcal{H} = H^0(\mathcal{N}_{C/\mathbb{P}^r}),$$

where $\mathcal{N}_{C/\mathbb{P}^r}$ is the normal sheaf of $C$ in $\mathbb{P}^r$. Moreover, if $C$ is a local complete intersection, then

$$h^0(\mathcal{N}_{C/\mathbb{P}^r}) - h^1(\mathcal{N}_{C/\mathbb{P}^r}) \leq \dim_{[C]}\mathcal{H} \leq h^0(\mathcal{N}_{C/\mathbb{P}^r}).$$

Let $U$ be a component of $\mathcal{H}_{d,g,r}$ whose general point corresponds to a smooth irreducible and nondegenerate curve $C$. Also let $\ell_{d,g,r}$ be the lower bound for the dimension of all such components $U$. Our aim is to estimate $\ell_{d,g,r}$. Define a number $h_{d,g,r} = \chi(\mathcal{N}_{C/\mathbb{P}^r}) = h^0(\mathcal{N}_{C/\mathbb{P}^r}) - h^1(\mathcal{N}_{C/\mathbb{P}^r}) = (r+1)d - (r-3)(g-1)$. By Theorem 1.3 we know that $\ell_{d,g,r} \geq h_{d,g,r}$. However, this bound $h_{d,g,r}$ may not be good in many cases.

For the beginning case $r = 3$, $h_{d,g,3} = 4d$ is independent of $g$. If we fix $d$ and let $g$ vary, the genus of a degree $d$ irreducible and nondegenerate curve in $\mathbb{P}^3$ can be as large as the Castelnuovo bound $\pi(d,3) = \frac{d^2}{4} + O(d)$. One can refer to [6, Section 3] for a good introduction on the Castelnuovo theory and related results. When $g$ approaches $\pi(d,3)$, we can compute $\ell_{d,g,3}$ explicitly. Roughly speaking, $\ell_{d,g,3}$ is asymptotically equal to $g$, which is much larger than $4d$. Therefore, it would be nice if we can come up with an improved lower bound.

Theorem 1.4. Define an integer-valued function $\mu(d,g)$ in the range $g^2 \geq d^3$ as follows,

$$\mu(d,g) = 1 + \frac{2d^2 - 3d - 2g}{g + d + \sqrt{g^2 - d^3 + 4dg + 4d^2}},$$

where $\lfloor \cdot \rfloor$ is the floor function. Then for any $d \geq 3$ and $g \leq \pi(d,3)$, we have

$$\ell_{d,g,3} \geq \begin{cases} 4d, & \text{if } g^2 < d^3; \\ 4d + g - 1 - \mu(d,g)d, & \text{if } g^2 \geq d^3. \end{cases}$$

The function $\mu$ involved in Theorem 1.4 may look confusing. But let us analyze this new bound a little bit. If $g^2 \geq d^3$, we always have $g - 1 - \mu(d,g)d > 0$. Moreover, if we fix $d$, $g - 1 - \mu(d,g)d$ is an increasing function of $g$. It implies that in the range $g^2 \geq d^3$, $\ell_{d,g,3}$ is strictly larger than the expected dimension $4d$. It actually goes up to $g$ when the genus approaches the Castelnuovo bound $\pi(d,3)$, which has been already predicted by the Castelnuovo theory.

We can also present an example to show the power of this bound. Suppose $d = 100$ and $g$ can vary from 0 to the Castelnuovo bound $\pi(100,3) = 2401$. Pick $g = 1100$, which is large but not close to the Castelnuovo bound. The bound $4d$ only tells us that $\ell_{100,1100,3} \geq 400$. However, by Theorem 1.4 we get $\ell_{100,1100,3} \geq 1099$, which is much better.

Now consider the case $r \geq 4$. The number $h_{d,g,r} = (r+1)d - (r-3)(g-1)$ could be negative if $g$ is larger than $d$. So it makes sense to find at least a positive bound for $\ell_{d,g,r}$. Furthermore, it may help answer a question about rigid curves.

A rigid curve in $\mathbb{P}^r$ is a smooth irreducible and nondegenerate curve that does not have any deformation except those induced from the automorphisms of $\mathbb{P}^r$. Apparently, rational normal curves are rigid. To the author’s best knowledge, people have not found any other rigid curves. In [7], Harris and Morrison conjectured that there does not exist a rigid curve except rational normal curves. One way to
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To attack this conjecture, we bound $l_{d,g,r}$. For instance, if the equality $l_{d,g,r} > \dim \text{PGL}(r) = r^2 + 2r$ holds, there cannot exist a degree $d$ genus $g$ rigid curve in $\mathbb{P}^r$. In fact, this is one of our motivations to study $l_{d,g,r}$.

For the case $r = 4$, we have the following result.

**Theorem 1.5.** Let $C$ be a degree $d$ genus $g$ smooth irreducible and nondegenerate curve in $\mathbb{P}^4$. Fix $d$ and let $g$ vary. If $g > 3d\sqrt{d} + O(d)$, then $C$ is not rigid.

Here we could be more precise on the range of $d$ and $g$ as we have done in Theorem 1.4. However, we choose to only focus on the asymptotic behavior, since the order $d\sqrt{d}$ seems to be important. Currently we have not been able to extend the result to $r \geq 5$. But combining the results in [2], we expect the following conjecture to hold in general.

**Conjecture 1.6.** For $r \geq 5$, there always exists a constant $\lambda_r$ such that if $g \geq \lambda_r d\sqrt{d} + O(d)$, a degree $d$ genus $g$ smooth irreducible and nondegenerate curve in $\mathbb{P}^r$ is not rigid.

In addition to projective spaces, we can also study the deformation of curves on a hypersurface. The beginning case would be a smooth quadric threefold in $\mathbb{P}^4$. Since all the smooth quadrics in $\mathbb{P}^4$ are isomorphic, we fix one and denote it by $Q$. Let $\mathcal{H}_{d,g}(Q)$ be the union of components of the Hilbert scheme whose general point parameterizes a degree $d$ genus $g$ smooth irreducible and nondegenerate curve on $Q$. Here a nondegenerate curve means that it is not contained in a $\mathbb{P}^3$. For a curve $[C] \in \mathcal{H}_{d,g}(Q)$, as in Theorem 1.3, $\chi(N_C/Q) = h^0(N_C/Q) - h^1(N_C/Q) = 3d$ provides a lower bound for the dimension of any component in $\mathcal{H}_{d,g}(Q)$. We can still ask how good this lower bound would be. A similar result as Theorem 1.5 can be established as follows.

**Theorem 1.7.** If $g > \frac{1}{\sqrt{2}}d\sqrt{d} + O(d)$, then the dimension of any component of $\mathcal{H}_{d,g}(Q)$ is strictly greater than the expected dimension $3d$. On the other hand, if $g < \frac{2}{15\sqrt{5}}d\sqrt{d} + O(d)$, then there always exists a component of $\mathcal{H}_{d,g}(Q)$ whose dimension equals $3d$.

Again, we only focus on the asymptotic behavior. The coefficients of $d\sqrt{d}$ might be improved by refining our techniques, but it seems hard to obtain a better order than $d\sqrt{d}$.

Throughout the paper, we work over the complex number field. A degree $d$ genus $g$ curve in a projective space means a 1-dimensional subscheme that has $dm + 1 - g$ as its Hilbert polynomial. Most of the time we will only consider smooth irreducible and nondegenerate curves.

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2. THE HILBERT SCHEME OF CURVES IN $\mathbb{P}^3$

In this section, we will verify Theorem 1.3. Let us briefly describe the outline of the proof. Fix $d$ and let $g$ vary. On one hand, we construct some components of the Hilbert scheme with the expected dimension $4d$ when $g$ is relatively small. On the other hand, if $g$ is quite large, the curve must lie on a low degree surface. We can estimate the dimension of the deformation of the curve on that surface, which would provide a better bound than $4d$. 
2.1. Determinantal curves in \( \mathbb{P}^3 \). As mentioned before, we want to construct some components of the Hilbert scheme that have \( 4d \) as their dimension.

For a curve \( C \) in \( \mathbb{P}^3 \), let \( \mathcal{I}_C = \mathcal{I}_{C/P^3} \) denote the ideal sheaf of \( C \), and let \( \mathcal{N}_C \) be the normal sheaf \( \mathcal{N}_{C/P^3} \). Firstly, let us look at an example constructed in [3].

Consider a curve \( C \) whose ideal sheaf has resolution as follows,

\[
0 \to \mathcal{O}_{\mathbb{P}^3}(-s-1) \to \mathcal{O}_{\mathbb{P}^3}(s+1)(-s) \to \mathcal{I}_C \to 0.
\]

It is easy to derive the determinantal model for such a curve from this resolution. Pick an \( s \times (s+1) \) matrix \( A \) whose entries are general linear forms. Then the ideal sheaf of the curve defined by the determinants of all the \( s \times s \) minors of \( A \) has the above resolution. Tensor the exact sequence \( \mathcal{I} \) with \( \mathcal{O}_{\mathbb{P}^3}(k) \), and we get \( \mathrm{Ext}^1(\mathcal{I}_C(k)) = 0 \) for any \( k \). Hence, \( C \) is projectively normal. We can also get the Hilbert polynomial of \( C \). Actually, when \( k \) is large enough, we have

\[
\mathrm{h}^0(\mathcal{I}_C(k)) = (s+1) \cdot \mathrm{h}^0(\mathcal{O}_{\mathbb{P}^3}(k-s)) - s \cdot \mathrm{h}^0(\mathcal{O}_{\mathbb{P}^3}(k-s-1)) = \frac{1}{6}(k-s+2)(k-s+1)(k+2s+3).
\]

The Hilbert polynomial of \( C \) equals

\[
\mathrm{h}^0(\mathcal{O}_{\mathbb{P}^3}(k)) - \mathrm{h}^0(\mathcal{I}_C(k)) = \frac{1}{2}(s^2+s)k - \frac{1}{6}(2s^3-3s^2-5s).
\]

So immediately we obtain the degree and genus of \( C \),

\[
d = \frac{1}{2}s(s+1),
\]

\[
g = 1 + \frac{1}{6}(2s^3-3s^2-5s).
\]

If we take all possible linear forms as entries of \( A \), by the above construction, we get an irreducible component \( U \) in the Hilbert scheme whose general point \( [C] \) corresponds to a smooth irreducible and nondegenerate curve. By counting parameters, the dimension of \( U \) is

\[
4s(s+1) - 1 - \dim \text{PGL}_s - \dim \text{PGL}_{s+1} = 2s^2 + 2s = 4d.
\]

Actually \( U \) is smooth at \([C]\), due to the fact that \( \mathrm{H}^1(\mathcal{N}_C) = 0 \). By [9, Remark 2.2.6], we have,

\[
\mathrm{H}^0(\mathcal{N}_C) = \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{I}_C),
\]

\[
\mathrm{H}^1(\mathcal{N}_C) = \mathrm{Ext}^2(\mathcal{I}_C, \mathcal{I}_C).
\]

Apply the functor \( \text{Hom}(\_, \mathcal{I}_C) \) to the exact sequence \( \mathcal{I} \). We get a long exact sequence

\[
0 \to \text{Hom}(\mathcal{I}_C, \mathcal{I}_C) \to \text{Hom}(\mathcal{O}_{\mathbb{P}^3}(s+1)(-s), \mathcal{I}_C) \to \text{Hom}(\mathcal{O}_{\mathbb{P}^3}(s+1)(-s-1), \mathcal{I}_C)
\]

\[
\to \text{Ext}^1(\mathcal{I}_C, \mathcal{I}_C) \to \text{Ext}^1(\mathcal{O}_{\mathbb{P}^3}(s+1)(-s), \mathcal{I}_C) \to \text{Ext}^1(\mathcal{O}_{\mathbb{P}^3}(s+1)(-s-1), \mathcal{I}_C)
\]

(3) \( \to \text{Ext}^2(\mathcal{I}_C, \mathcal{I}_C) \to \text{Ext}^2(\mathcal{O}_{\mathbb{P}^3}(s+1)(-s), \mathcal{I}_C) \to \cdots \)

Note that \( \text{Ext}^2(\mathcal{O}_{\mathbb{P}^3}(-s), \mathcal{I}_C) = \mathrm{H}^2(\mathcal{I}_C(s)) \). Twist \( \mathcal{I} \) by \( \mathcal{O}_{\mathbb{P}^3}(s) \), and we get \( \mathrm{H}^2(\mathcal{I}_C(s)) = 0 \). Moreover, \( \text{Ext}^1(\mathcal{O}_{\mathbb{P}^3}(-s-1), \mathcal{I}_C) = \mathrm{H}^1(\mathcal{I}_C(s+1)) = 0 \), since \( C \) is projectively normal. Then from \( \mathcal{I} \) and \( \mathcal{I} \), it follows that \( \mathrm{H}^1(\mathcal{N}_C) = 0 \) as we expect.

Let us look at the values of \( d \) and \( g \) obtained above. One observation is that \( g^2 \sim \frac{d^3}{3} \) asymptotically. It implies that the ratio \( \frac{g^2}{d^3} \) might be an important index.
More precisely, we want to find a number $\lambda$ such that if $\frac{g^2}{d^3} \leq \lambda$ asymptotically, then there exists a component of the Hilbert scheme whose dimension is close to $4d$. In this case, the lower bound $4d$ is still good. Actually, we will show that $\lambda = 1$ is almost the best.

Continue to consider determinantal curves. Modify the entries of the matrix $A$ by using degree $t$ homogeneous polynomials instead of linear forms. Then the ideal sheaf of $C$ has resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-t - ts) \to \mathcal{O}_{\mathbb{P}^3}(-(s+1)t) \to \mathcal{I}_C \to 0.$$  

Compute the Hilbert polynomial as before. We obtain the degree and genus of $C$ as follow,

$$d = \frac{1}{2} s(s + 1)t^2,$$
$$g = 1 + \frac{1}{6} s(s + 1)(2s + 1)t^3 - s(s + 1)t^2.$$  

By counting parameters, the dimension of the component of such curves is

$$4s(s + 1) \left( \frac{t + 3}{3} \right) - 1 - \dim \text{PGL}_s - \dim \text{PGL}_{s+1}$$
$$= \frac{1}{6} s(s + 1)(t^3 + 6t^2 + 11t - 6).$$  

We denote the above value by $l$.

Note that the ratio $\frac{g^2}{d^3}$ satisfies the inequality

$$\frac{g^2}{d^3} < \frac{2}{9} \left( 4 + \frac{1}{s^2 + s} \right).$$  

So asymptotically $\frac{g^2}{d^3} \leq 1$. Moreover, the ratio tends to 1 if and only if $s = 1$, that is, when $C$ is a complete intersection of two degree $t$ surfaces.

Another interesting fact is that when $t \leq 3$ we always have $l = 4d$ for any $s$. But as $t$ increases, $l$ will get much larger than $4d$. We have already discussed the case $t = 1$. Now we take a look at $t = 2$ and $3$.

If $t = 2$, we have

$$d = 2s(s + 1),$$
$$g = 1 + \frac{8}{3}(s - 1)s(s + 1).$$

Asymptotically $\frac{g^2}{d^3}$ goes to $\frac{8}{9}$.

If $t = 3$, we have

$$d = \frac{9}{2} s(s + 1),$$
$$g = 1 + \frac{9}{2} s(s + 1)(2s - 1).$$

Asymptotically $\frac{g^2}{d^3}$ goes to $\frac{8}{9}$ as well, still less than 1.

Next, we further modify the matrix $A$ by allowing the entries at different rows to have different degrees. Suppose $A = (F_{ij})$, $1 \leq i \leq s$, $1 \leq j \leq s + 1$, and the degree of $F_{ij}$ is $k_i$. Let $t = \sum_{i=1}^s k_i$. Then the ideal sheaf of $C$ has resolution

$$0 \to \bigoplus_{i=1}^s \mathcal{O}_{\mathbb{P}^3}(-t - k_i) \to \mathcal{O}_{\mathbb{P}^3}(-(s+1)t) \to \mathcal{I}_C \to 0.$$
We can obtain the degree and genus of $C$,

\[ d = \frac{1}{2}(t^2 + \sum_{i=1}^s k_i^2), \]
\[ g = 1 + \frac{1}{6}(2t^3 - 6t^2 + 3(\sum_{i=1}^s k_i^3)t + \sum_{i=1}^s (k_i^3 - 6k_i^2)). \]

In this case, the dimension estimate by counting parameters depends on how many $k_i$'s may have the same value. However, we are more interested in if $g^2/d$ can approach a better upper bound, e.g., much larger than 1.

Let $u = \sum_{i=1}^s k_i^2$, then $\sum_{i=1}^s k_i^3 \leq tu$. Hence, we have

\[ g < \frac{1}{6}(2t^2 + 4ut), \]
\[ \frac{g^2}{d^3} < \frac{8}{9} \frac{(1 + 2\alpha)^2}{(1 + \alpha)^3}, \]

where $\alpha = \frac{u}{t}$. When $\alpha = \frac{1}{2}$, the right hand side of the last inequality has the maximum $\frac{256}{243}$, which is still close to 1.

The above examples partially explains why we separate the case $g^2 < d^3$ in Theorem 1.4. The reader may also wonder why we do not construct other examples in addition to determinantal curves to see if we can get $g^2/d$ much larger than 1 and at the same time keep the dimension of the component relatively low. In fact, this is impossible. Next subsection explains the different situation when $g^2 \geq d^3$.

2.2. Curves on a surface in $\mathbb{P}^3$. In this subsection, we want to show if $g$ is much larger than $d$, then a curve must be contained in a relatively low degree surface in $\mathbb{P}^3$. Moreover, we can estimate the deformation of the curve on that surface, which provides a proof for the second part of Theorem 1.4.

Firstly, we cite a result originally mentioned by Halphen and proved later by Gruson and Peskine [5].

**Theorem 2.1.** Let $C$ be a connected smooth curve of degree $d$ and genus $g$ in $\mathbb{P}^3$. $s$ is a positive integer such that $s(s-1) < d$. If $g$ satisfies

\[ g > \frac{d}{2}(s + \frac{d}{s} - 4) - \frac{r(s-r)(s-1)}{2s}, \]

where $0 \leq r < s, d + r \equiv 0 \pmod{s}$, then $C$ must lie on a surface of degree less than $s$.

Note that if $s \sim \sqrt{d}$, then the right hand side of $\text{(6)} \sim \sqrt{d^3}$. Hence, Theorem 2.1 can help us deal with the case $g^2 > d^3$.

Since we only want asymptotic results, Theorem 2.1 can be slightly modified for our convenience.

**Proposition 2.2.** Let $C$ be a connected smooth curve of degree $d$ and genus $g$ in $\mathbb{P}^3$. $s$ is a positive integer such that $s(s+1) < d$. If $g$ satisfies

\[ g > \frac{d}{2}(s + \frac{d}{s+1} - 3), \]

then $C$ must lie on a surface of degree $k \leq s$. 

\[ \text{Here the integer } k \text{ is the number of } \alpha_i \text{ with } k_i = k. \]
For fixed \( d \) and \( g \) in the range \( g^2 \geq d^3 \), consider the smallest positive integer \( s \) satisfying \( s(s+1) < d \) and the inequality (7). Then there exists a surface \( S \) of degree \( k \leq s \) such that \( S \) contains \( C \). Let \( \mathcal{H}_{d,g}(S) \) be the Hilbert scheme parameterizing degree \( d \) and genus \( g \) curves on \( S \). \( \mathcal{H}_{d,g}(S) \) can be viewed as a subscheme of \( \mathcal{H}_{d,g,3} \).

We want to estimate \( \dim_{[C]} \mathcal{H}_{d,g,3} \). If \( S \) is smooth, then \( \chi(\mathcal{N}_{C/S}) \) provides a lower bound for \( \dim_{[C]} \mathcal{H}_{d,g,3} \). We have the exact sequence

\[
0 \to \mathcal{N}_{C/S} \to \mathcal{N}_{C/P^3} \to \mathcal{N}_{S/P^3} \otimes \mathcal{O}_C \to 0.
\]

By adjunction formula, \( \mathcal{N}_{S/P^3} \otimes \mathcal{O}_C = \mathcal{O}_C(k) \). Then we can compute \( \chi(\mathcal{N}_{C/S}) \) by the exact sequence (8) and Riemann-Roch,

\[
\chi(\mathcal{N}_{C/S}) = \chi(\mathcal{N}_{C/P^3}) - \chi(\mathcal{N}_{S/P^3} \otimes \mathcal{O}_C) = 4d - \chi(\mathcal{O}_C(k)) = 4d + g - 1 - kd \geq 4d + g - 1 - sd.
\]

So we have

\[
\dim_{[C]} \mathcal{H}_{d,g,3} \geq \dim_{[C]} \mathcal{H}_{d,g}(S) \geq 4d + g - 1 - sd.
\]

Therefore, we get a lower bound for the dimension of \( \mathcal{H}_{d,g,3} \):

\[
l_{d,g,3} \geq 4d + g - 1 - sd.
\]

The advantage of (9) is because in the range \( g^2 \geq d^3 \), as \( g \) increases, \( s \) decreases, and \( 4d + g - 1 - sd \) is more dominated by \( g \). For instance, if we fix \( d \) and let \( g \) approach the Castelnuovo bound \( \pi(d,3) \), then the dimension of \( \mathcal{H}_{d,g,3} \) tends to \( g \). But at this moment \( s \) is very small. Therefore, the estimate (9) does not lose much information from the asymptotic viewpoint.

Now we can finish the proof of Theorem 1.4 easily.

**Proof.** 4d is the classical lower bound for any \( d, g \). Moreover, in the range \( g^2 \geq d^3 \), the smallest integer \( s \) satisfying \( s(s+1) < d \) and \( g > \frac{d}{2}(s + \frac{d}{2} - 3) \) is given by \( s = \mu(d, g) \). Apply the lower bound \( 4d + g - 1 - sd \) obtained in (9). It then completes the proof. \( \square \)

In the above argument, there is one gap we need to fix, that is, when the surface \( S \) is singular and \( C \) passes through singular points of \( S \). In that case we cannot simply apply cohomology to estimate the dimension of the deformation of \( C \) on \( S \). Instead, we have to use Ext groups. Before doing that, we will prove a simple result, which shows that the situation is not very bad even if \( S \) is singular.

**Lemma 2.3.** Let \( S_{\text{sing}} \) denote the singular locus of a surface \( S \). Under the assumption of Proposition 2.2, if \( C \cap S_{\text{sing}} \) is not empty, then it is \( 0 \)-dimensional.

**Proof.** If the dimension of \( S_{\text{sing}} \) is 0, then the statement is trivial. Otherwise the dimension of \( S_{\text{sing}} \) is 1. By Bézout, the degree of \( S_{\text{sing}} \) is at most \( k(k-1) \leq s(s-1) < d \). Hence, \( C \) cannot be contained in \( S_{\text{sing}} \). \( \square \)

By Lemma 2.3, we can apply the following result from [10] Lemma 2.13, Theorem 2.15].

For fixed \( d \) and \( g \) in the range \( g^2 \geq d^3 \), consider the smallest positive integer \( s \) satisfying \( s(s+1) < d \) and the inequality (7). Then there exists a surface \( S \) of degree \( k \leq s \) such that \( S \) contains \( C \). Let \( \mathcal{H}_{d,g}(S) \) be the Hilbert scheme parameterizing degree \( d \) and genus \( g \) curves on \( S \). \( \mathcal{H}_{d,g}(S) \) can be viewed as a subscheme of \( \mathcal{H}_{d,g,3} \).
Proposition 2.4. Keep the above notation. If \( C \cap S_{\text{sing}} \) is 0-dimensional, then \( C \subset S \) is generically unobstructed and the dimension of every irreducible component of \( \mathcal{H}_{d,g}(S) \) at \( [C] \) is at least
\[
\dim \text{Hom}_C(\mathcal{I}_{C/S}/\mathcal{I}_{C/S}^2, \mathcal{O}_C) = \dim \text{Ext}_C^1(\mathcal{I}_{C/S}/\mathcal{I}_{C/S}^2, \mathcal{O}_C).
\]

If \( S \) is smooth, the value of \( (10) \) is just \( \mathcal{X}(\mathcal{N}_{C/S}) \). When \( S \) is singular, we need to verify some exact sequences of Kähler differentials. We will do it in a more general setting since the results can be applied to many other cases.

Proposition 2.5. Suppose \( C \) is a smooth connected curve, \( X \) is an \((n-k)\)-dimensional local complete intersection, and \( C \subset X \subset \mathbb{P}^n \), \( n \geq 3, 1 \leq k \leq n-2 \). If \( C \cap X_{\text{sing}} \) is 0-dimensional, we have the following exact sequences
\[
\begin{align*}
(11) & \quad 0 \to \mathcal{I}_{C/X}/\mathcal{I}_{C/X}^2 \xrightarrow{d} \Omega_X \otimes \mathcal{O}_C \to \mathcal{O}_C \to 0, \\
(12) & \quad 0 \to (\mathcal{I}_X/\mathcal{I}_X^2) \otimes \mathcal{O}_C \xrightarrow{d} \Omega_{\mathbb{P}^n} \otimes \mathcal{O}_C \to \Omega_X \otimes \mathcal{O}_C \to 0.
\end{align*}
\]

Note that if \( X \) is smooth, those results are well-known. When \( X \) is singular, the above sequences are still exact except the left hand sides may not be injective, cf. [8 II 8].

Proof. It suffices to verify that the map to the middle term is always injective for each sequence. Since the question is local, we only need to work on a local affine chart \( U \). Suppose \( x_1, \ldots, x_n \) are the local coordinates, and \( f_1, \ldots, f_k \) locally cut out \( X \) in \( U \). We have \( \Omega_X(U) = \Omega_{\mathbb{P}^n} \otimes \mathcal{O}_X(U)/(df_1, \ldots, df_k) \).

Firstly, let us verify \((11)\). Pick an element \( g \in \mathcal{I}_{C/X}(U) \). Suppose we have
\[
dg = \sum_{j=1}^n \frac{\partial g}{\partial x_j} dx_j = 0 \in \Omega_X \otimes \mathcal{O}_C(U).
\]
There also exist \( a_1, \ldots, a_k \in \mathcal{O}_C(U) \) such that restricted on \( C \),
\[
\frac{\partial g}{\partial x_j} = \sum_{i=1}^k a_i \frac{\partial f_i}{\partial x_j}, \quad 1 \leq j \leq n.
\]
It follows that \( d(g - \sum_{i=1}^k a_i f_i) = 0 \) on \( C \). Since \( C \) is smooth, the vanishing of \( g - \sum_{i=1}^k a_i f_i \) and its differential on \( C \) tell us that \( g - \sum_{i=1}^k a_i f_i \in \mathcal{I}_C^2(U) \), which implies \( g = g - \sum_{i=1}^k a_i f_i = 0 \) as elements in \( \mathcal{I}_{C/X}/\mathcal{I}_{C/X}^2 \).

Next, let us verify the exactness of \((12)\). Take an element \( h = \sum_{i=1}^k b_i f_i \in \mathcal{I}_X(U) \). If \( dh = 0 \) restricted on \( C \), since \( f_1, \ldots, f_k \) vanish on \( C \), we have
\[
\sum_{i=1}^k b_i \frac{\partial f_i}{\partial x_j} dx_j = 0, \quad 1 \leq j \leq n
\]
on \( C \). Note that \( X_{\text{sing}} \cap U \) consists of those points where the matrix
\[
\left( \frac{\partial f_i}{\partial x_j} \right)_{1 \leq i \leq k, 1 \leq j \leq n}
\]
drops rank. Since \( C \cap X_{\text{sing}} \) consists of at most finitely many points, \( b_1, \ldots, b_k \) must vanish at a non empty open subset of \( C \cap U \), which forces that they vanish completely on \( C \cap U \). Hence, \( h \otimes 1 = \sum_{i=1}^k f_i \otimes b_i = 0 \in (\mathcal{I}_X/\mathcal{I}_X^2) \otimes \mathcal{O}_C(U) \). \( \square \)

Now consider the deformation of \( C \) on \( X \). We have the following result.
Proposition 2.6. Keep the above assumption. If \( C \cap X_{\text{sing}} \) is 0-dimensional, the dimension of every component of \( \mathcal{H}_{d,g}(X) \) at \([C]\) is at least
\[
\mathcal{X}(\mathcal{N}_{C/P^n}) - \mathcal{X}(\mathcal{N}_{X/P^n}|_C).
\]
Moreover, suppose \( X \) is a complete intersection cut out by hypersurfaces \( F_1, \ldots, F_k \), \( \deg F_i = d_i, i = 1, \ldots, k \). The above lower bound can be written explicitly as
\[
(n + 1 - \sum_{i=1}^k d_i)d + (k - n + 3)(g - 1).
\]

Proof. By the assumption, \( C \subset X \) is generically unobstructed, so we can apply the result from [9] Lemma 2.13, Theorem 2.15. The local dimension of any component of \( \mathcal{H}_{d,g}(X) \) at \([C]\) is at least
\[
\dim \text{Hom}(\mathcal{I}_{C/X}/T^2_{C/X}, \mathcal{O}_C) - \dim \text{Ext}^1(\mathcal{I}_{C/X}/T^2_{C/X}, \mathcal{O}_C).
\]
Note that if \( X \) is smooth, the value of \([13]\) is \( \mathcal{X}(\mathcal{N}_{C/X}) \), which equals \( \mathcal{X}(\mathcal{N}_{C/P^n}) - \mathcal{X}(\mathcal{N}_{X/P^n}|_C) \) due to the well-known exact sequence
\[
0 \rightarrow \mathcal{N}_{C/X} \rightarrow \mathcal{N}_{C/P^n} \rightarrow \mathcal{N}_{X/P^n}|_C \rightarrow 0.
\]
If \( X \) is singular, apply the functor \( \text{Hom}(\cdot, \mathcal{O}_C) \) to \([14]\). Then we get a long exact sequence
\[
0 \rightarrow \text{Hom}(\Omega_C, \mathcal{O}_C) \rightarrow \text{Hom}(\Omega_X \otimes \mathcal{O}_C, \mathcal{O}_C) \rightarrow \text{Hom}(\mathcal{I}_{C/X}/T^2_{C/X}, \mathcal{O}_C) \\
\rightarrow \text{Ext}^1(\Omega_C, \mathcal{O}_C) \rightarrow \text{Ext}^1(\Omega_X \otimes \mathcal{O}_C, \mathcal{O}_C) \rightarrow \text{Ext}^1(\mathcal{I}_{C/X}/T^2_{C/X}, \mathcal{O}_C)
\]
\( (14) \rightarrow 0. \)
The last term is zero, because \( \text{Ext}^2(\Omega_C, \mathcal{O}_C) = H^2(\mathcal{T}_C) = 0. \)
Moreover, apply the functor \( \text{Hom}(\cdot, \mathcal{O}_C) \) to \([12]\), we get another long exact sequence
\[
0 \rightarrow \text{Hom}(\Omega_X \otimes \mathcal{O}_C, \mathcal{O}_C) \rightarrow \text{Hom}(\Omega_{P^n} \otimes \mathcal{O}_C, \mathcal{O}_C) \rightarrow \text{Hom}(\mathcal{I}_X/T^2_X, \mathcal{O}_C) \\
\rightarrow \text{Ext}^1(\Omega_X \otimes \mathcal{O}_C, \mathcal{O}_C) \rightarrow \text{Ext}^1(\Omega_{P^n} \otimes \mathcal{O}_C, \mathcal{O}_C) \rightarrow \text{Ext}^1(\mathcal{I}_X/T^2_X, \mathcal{O}_C) \\
(15) \rightarrow 0.
\]
The last term is zero, because \( \text{Ext}^2(\Omega_{P^n} \otimes \mathcal{O}_C, \mathcal{O}_C) = H^2(\mathcal{T}_{P^n}|_C) = 0. \)
Note that \( C \) is smooth, so \( \text{Ext}^1(\Omega_C, \mathcal{O}_C) = H^1(\mathcal{T}_C) \) and \( \text{Ext}^1(\Omega_{P^n} \otimes \mathcal{O}_C, \mathcal{O}_C) = H^1(\mathcal{T}_{P^n}|_C) \) for any \( i \). From \([12]\), we know \( \mathcal{I}_X/T^2_X \otimes \mathcal{O}_C \) is locally free, so \( \text{Ext}^1((\mathcal{I}_X/T^2_X) \otimes \mathcal{O}_C, \mathcal{O}_C) = H^1(\mathcal{N}_{X/P^n}|_C) \). Then by \([14]\) and \([15]\), we have
\[
\dim \text{Hom}(\mathcal{I}_{C/X}/T^2_{C/X}, \mathcal{O}_C) - \dim \text{Ext}^1(\mathcal{I}_{C/X}/T^2_{C/X}, \mathcal{O}_C) \\
= \mathcal{X}(\mathcal{T}_{P^n}|_C) - \mathcal{X}(\mathcal{N}_{X/P^n}|_C) - \mathcal{X}(\mathcal{T}_C) - \dim \text{Ext}^2(\Omega_X \otimes \mathcal{O}_C, \mathcal{O}_C) \\
= \mathcal{X}(\mathcal{N}_{C/P^n}) - \mathcal{X}(\mathcal{N}_{X/P^n}|_C) - \dim \text{Ext}^2(\Omega_X \otimes \mathcal{O}_C, \mathcal{O}_C).
\]
\( \mathcal{X}(\mathcal{N}_{C/P^n}) \) equals \( h_{d,g,n} = (n + 1)d - (n - 3)(g - 1) \). If \( X \) is a complete intersection cut out by \( F_1, \ldots, F_k \), the normal sheaf \( \mathcal{N}_{X/P^n} \) splits into \( \bigoplus_{i=1}^k \mathcal{O}_X(d_i) \). Therefore, in this case we can compute \( \mathcal{X}(\mathcal{N}_{X/P^n}|_C) \) explicitly as \( \mathcal{X}(\bigoplus_{i=1}^k \mathcal{O}_C(d_i)) = \sum_{i=1}^k (1 - g + d_i). \)
Now the theorem follows if we can show that \( \text{Ext}^2(\Omega_X \otimes \mathcal{O}_C, \mathcal{O}_C) = 0. \) In case \( X \) is smooth, we have the well-known exact sequence
\[
0 \rightarrow \mathcal{T}_X \otimes \mathcal{O}_C \rightarrow \mathcal{T}_{P^n} \otimes \mathcal{O}_C \rightarrow \mathcal{N}_{X/P^n} \otimes \mathcal{O}_C \rightarrow 0.
\]
If $X$ is singular, the last map may not be surjective. Instead, we have
\[ 0 \to T_X \otimes \mathcal{O}_C \to T_{\mathbb{P}^n} \otimes \mathcal{O}_C \to N_{X/\mathbb{P}^n} \otimes \mathcal{O}_C \to \mathcal{F} \to 0, \]
where $\mathcal{F}$ is a sheaf supported at some points of $C \cap X_{\text{sing}}$. Split the above sequence into two short exact sequences
\begin{align*}
(16) & \quad 0 \to T_X \otimes \mathcal{O}_C \to T_{\mathbb{P}^n} \otimes \mathcal{O}_C \to \mathcal{E} \to 0 \\
(17) & \quad 0 \to \mathcal{E} \to N_{X/\mathbb{P}^n} \otimes \mathcal{O}_C \to \mathcal{F} \to 0.
\end{align*}
Since $H^2(T_X \otimes \mathcal{O}_C) = 0$, then from (16), the map $H^1(T_{\mathbb{P}^n} \otimes \mathcal{O}_C) \to H^1(\mathcal{E})$ is surjective. Moreover, $\mathcal{F}$ is only supported at finitely many points on $C$, so $H^1(\mathcal{F}) = 0$. From (17), the map $H^1(\mathcal{E}) \to H^1(N_{X/\mathbb{P}^n} \otimes \mathcal{O}_C)$ is also surjective. Hence, we get a surjective map $H^1(T_{\mathbb{P}^n} \otimes \mathcal{O}_C) \to H^1(N_{X/\mathbb{P}^n} \otimes \mathcal{O}_C)$, i.e., a surjective map $\text{Ext}^1(\mathcal{O}_{\mathbb{P}^n} \otimes \mathcal{O}_C, \mathcal{O}_C) \to \text{Ext}^1((I_C/I_C^2) \otimes \mathcal{O}_C, \mathcal{O}_C)$. Then from (15), it follows that $\text{Ext}^2(\Omega_X \otimes \mathcal{O}_C, \mathcal{O}_C) = 0$. \hfill \Box

Now, apply Proposition 2.4 and 2.6 to our situation when $X = S$ is a surface in $\mathbb{P}^3$. The bound $4d + g - 1 - sd$ is still valid as a lower bound for $l_{d,g,3}$. Now we have completely finished the proof of Theorem 1.5.

At the end of this section, we want to show that the new bound in Theorem 1.5 makes sense. Using determinantal curves, we already constructed components with the expected dimension $4d$ and the corresponding values of $g$ and $d$ satisfy $\frac{2d}{3} \sim \frac{g}{3}$. Actually, if $d$ is large and $g \leq \frac{1}{6\sqrt{2}}d\sqrt{d} + k_1d + k_2\sqrt{d} + k_3$, where $k_1, k_2, k_3$ are some constants, there always exists a component of $\mathcal{H}_{d,g,3}$ with the expected dimension $4d$, cf. [11]. Moreover, we can construct those components up to $g \sim \frac{2}{3\sqrt{2}}d\sqrt{d}$ asymptotically, cf. [1] and [13]. Therefore, in the range $g^2 < d^3$, $4d$ is almost the best lower bound for $l_{d,g,3}$. On the other hand, in the range $g^2 > d^3$ we always have $\mu(d, g) < \sqrt{d}$. Furthermore, as $g$ increases, $\mu(d, g)$ decreases and $4d + g - 1 - \mu(d, g)d$ is dominated by $g$. In fact, we know the dimension of a component of $\mathcal{H}_{d,g,3}$ whose general points correspond to smooth irreducible and nondegenerate curves is always less than or equal to $4d + g$ for any $d, g$, cf. [6] 2.b). Therefore, the result of Theorem 1.5 does not lose much information from the asymptotic perspective. More importantly, it is better than the expected dimension $4d$ if $g$ is much larger than $d$.

3. The Hilbert scheme of curves in $\mathbb{P}^4$

In this section we will prove Theorem 1.5. The idea of the proof is simple. We will show that if $g$ is large enough, a degree $d$ genus $g$ smooth irreducible and nondegenerate curve $C$ in $\mathbb{P}^4$ must be contained in a surface $S$ such that $S$ is a complete intersection and $C$ is not contained in its singular locus $S_{\text{sing}}$. By estimating the dimension of the deformation of $C$ on $S$, we can derive the desired result.

For the first step, let us recall some basic results from the Castelnuovo theory.

**Theorem 3.1.** Let $C$ be a degree $d$ genus $g$ reduced irreducible and nondegenerate curve in $\mathbb{P}^r$. Then $g$ has an upper bound $\pi(d, r) = \frac{d^2}{2(r-1)} + O(d)$.

For the precise definition of $\pi(d, r)$ and the proof of the theorem, cf. e.g., [6].

By the above theorem, it is easy to find a low degree threefold $F$ that contains $C$. 
Lemma 3.2. Let \( k \) be a positive integer and \( N = \binom{k+4}{4} - 1 \). If \( g \) satisfies
\[
(18) \quad g > \pi(dk, N),
\]
then \( C \) is contained in an irreducible threefold \( F \) of degree \( a \leq k \).

Proof. Embed \( \mathbb{P}^4 \) into \( \mathbb{P}^N \) by the Veronese map of degree \( k \). Then the image \( C' \) of \( C \) is a curve of degree \( dk \) and genus \( g \). Since \( g \) is larger than the Castelnuovo bound \( \pi(dk, N) \), \( C' \) must be contained in a hyperplane in \( \mathbb{P}^N \). That is, \( C \) is contained in a degree \( k \) threefold in \( \mathbb{P}^4 \). Then we take an irreducible component \( F \) of this threefold that contains \( C \). \( F \) has degree \( a \leq k \).

Fix \( F \) and its degree \( a \). Our next goal is to find another threefold that contains \( C \) as well.

Lemma 3.3. Suppose \( l \) is an integer and \( l \geq a \). Let \( M = \binom{l+4}{4} - \binom{l-a+4}{4} - 1 \). If \( g \) satisfies
\[
(19) \quad g > \pi(dl, M),
\]
then we can find a degree \( b \) irreducible threefold \( G \) containing \( C \) such that \( b \leq l \) and the surface \( S = F \cap G \) is a complete intersection.

Proof. Embed \( \mathbb{P}^4 \) into \( \mathbb{P}^N \) by the Veronese map of degree \( l \). By a similar argument as before, we can show that \( C \) is contained in at least \( \binom{l-a+4}{4} + 1 \) independent degree \( l \) threefolds in \( \mathbb{P}^4 \). Notice that there are at most \( \binom{l-a+4}{4} \) independent degree \( l \) threefolds containing \( F \) as a component, since \( F \) is irreducible. Hence, we can find a degree \( l \) threefold containing \( C \) but not \( F \). Take an irreducible component \( G \) of this threefold that contains \( C \). \( G \) has degree \( b \leq l \) and \( S = F \cap G \) is a complete intersection. \( \square \)

In order to apply standard deformation theory for \( C \subset S \), we should avoid the situation \( C \subset S_{\text{sing}} \).

Lemma 3.4. Let \( S \) be a surface in \( \mathbb{P}^4 \) cut out by two threefolds of degree \( a \) and \( b \) respectively. If \( S_{\text{sing}} \) is 1-dimensional, its degree has an upper bound \( \frac{1}{2}ab(a+b-2) \).

Proof. Take a general hyperplane section \( X = H \cap S \) in \( \mathbb{P}^4 \). \( X \) is a curve of degree \( ab \) and arithmetic genus \( \frac{1}{2}ab(a+b-4) + 1 \) in \( H \cong \mathbb{P}^3 \). Even though \( X \) might be reducible, the total number of its singularities is at most \( ab + \frac{1}{2}ab(a+b-4) + 1 - 1 = \frac{1}{2}ab(a+b-2) \). Since \( H \cap S_{\text{sing}} \subset X_{\text{sing}} \), we get \( \deg S_{\text{sing}} \leq \deg X_{\text{sing}} \leq \frac{1}{2}ab(a+b-2) \).

By this lemma, we immediately get the following consequence.

Lemma 3.5. Keep the above assumption. If the degree \( d \) of the curve \( C \) satisfies
\[
(20) \quad d > \frac{1}{2}ab(a+b-2),
\]
then \( C \cap S_{\text{sing}} \) is either empty or 0-dimensional.

Consider the deformation of \( C \) on \( S \). Since \( S \) is a complete intersection and \( C \not\subset S_{\text{sing}} \), we can apply Proposition 2.6 to derive the following result.

Lemma 3.6. The dimension of the deformation of \( C \) on \( S \) is at least \( 5d + g - 1 - (a + b)d \).

Now we have all the ingredients to prove Theorem 1.5.
Proof. By an elementary calculation, if \( g > 3d\sqrt{d} + O(d) \), we can find integers \( k, a, l, b \) successively in the above setting such that they satisfy the inequalities \( (18), (19) \) and \( (20) \). Therefore, by Lemma 3.2, 3.3 and 3.5, we know that \( C \) lies in a complete intersection surface \( S \) of type \( (a, b) \) and \( C \not\subset S_{\text{sing}} \). Moreover, we can check that \( (a + b)d < g \). Then by Lemma 3.6 the dimension of the deformation of \( C \) on \( S \geq 5d + g - 1 - (a + b)d \geq 5d > 24 = \dim \text{PGL}(5) \). \( \square \)

It is possible to enlarge the range \( g > 3d\sqrt{d} + O(d) \) by refining the results in Lemma 3.2, 3.3 and 3.4. However, it seems that only the leading coefficient could be improved rather than the exponent \( d^{3/2} \). So when \( g \) is slightly bigger than \( d \), the situation remains mysterious to us. On the other hand, by the result of [2], Conjecture 1.6 mentioned in the introduction section sounds highly possible and might be handled by an analogous argument. We state the conjecture again as the end of this section.

**Conjecture 3.7.** For \( r \geq 5 \), there always exists a constant \( \lambda_r \) such that if \( g \geq \lambda_r d\sqrt{d} + O(d) \), a degree \( d \) genus \( g \) smooth irreducible and nondegenerate curve in \( \mathbb{P}^r \) is not rigid.

4. The Hilbert scheme of curves on a quadric threefold

In this section we will prove Theorem 1.7. Recall that \( \mathcal{H}_{d,g}(Q) \) parameterizes degree \( d \) genus \( g \) smooth irreducible and nondegenerate curves on a smooth quadric \( Q \) in \( \mathbb{P}^4 \). For \( [C] \in \mathcal{H}_{d,g}(Q) \), \( \mathcal{X}(\mathcal{N}_{C/Q}) = 3d \) is a lower bound for the dimension of any component of \( \mathcal{H}_{d,g}(Q) \). Theorem 1.7 provides a further analysis for the sharpness of this bound. Its proof consists of two steps.

Firstly, if \( g \) is large enough, \( C \) must lie on another threefold \( F \) of low degree. Consider the deformation of \( C \) on the surface \( X = Q \cap F \). We can easily derive the first part of Theorem 1.7. For the second part, we use a similar method as in [11]. A component whose general element represents a curve as the intersection of \( Q \) and a determinantal surface has dimension \( 3d \). Then we apply the smoothing technique in [12] to enlarge the range of the pair \( (d, g) \) to cover the case when \( g < \frac{2}{15\sqrt{2}}d\sqrt{d} + O(d) \).

By the main result of [11], we can verify the first step easily.

**Lemma 4.1.** If \( g > \frac{1}{\sqrt{2}}d\sqrt{d} + O(d) \), the dimension of the deformation of \( C \) on \( Q \) is bigger than \( 3d \).

**Proof.** When \( d \) and \( g \) satisfy the above inequality, we can find an integer \( k \) such that \( d > 2k(k-1) \) and \( g > \frac{d^2}{4k} + \frac{1}{2}kd \). By the result of [11], there exists an integral surface \( X \in |\mathcal{O}_Q(a)| \) containing \( C \), where \( a \leq k \). Since \( d > 2k(k-1) \) and \( X \) is of degree \( 2a \), \( C \not\subset X_{\text{sing}} \). By Proposition 2.6 \( \mathcal{X}(\mathcal{N}_{C/X}) = 3d + g - ad - 1 \) provides a lower bound for the dimension of the deformation of \( C \) on \( X \). A simple calculation shows that \( 3d + g - ad - 1 \geq 3d + g - kd - 1 > 3d \). \( \square \)

The second step is harder. We still want to construct a component of the Hilbert scheme that parameterizes certain determinantal curves. But the curves should be contained in the quadric \( Q \). A natural idea is to take the intersection of a determinantal surface with \( Q \).

Let \((H_{ij})\) be a \( t \times (t+1) \) matrix. The entry \( H_{ij} \) is a general linear form in \( \mathbb{P}^4 \). Those \( t \times t \) minors define a determinantal surface \( S \). The ideal sheaf of \( S \) has the
following resolution
\[ 0 \to \mathcal{O}^{\otimes t}_C(-t - 1) \to \mathcal{O}^{\otimes (t+1)}_C(-t) \to \mathcal{I}_S \to 0. \]

By Bertini, if we take a general quadric threefold \( Q \), \( C = Q \cap S \) is smooth. It is not hard to get the degree and genus of \( C \),
\[ d = t(t + 1), \]
\[ g = \frac{2}{3}t^3 - \frac{1}{2}t^2 - \frac{7}{6}t + 1. \]

Note that asymptotically \( g \sim \frac{2}{3}d\sqrt{d} \).

Let us count parameters. The dimension of the component parameterizing curves generated in the above way is \( 5(t+1) - 1 - \dim \text{PGL}(t) - \dim \text{PGL}(t+1) = 3t(t+1) = 3d \). In order to show that this is a real component of \( H_{d,g}(Q) \), we have to check that for \( C = S \cap Q \), \( H^1(N_C/Q) = 0 \). Actually for general \( S \) and \( Q \), the ideal sheaf \( \mathcal{I}_{C/Q} \) has the resolution
\[ 0 \to \mathcal{O}^{\otimes t}_Q(-t - 1) \to \mathcal{O}^{\otimes (t+1)}_Q(-t) \to \mathcal{I}_{C/Q} \to 0. \]

By [9, Remark 2.2.6], we know that \( H^1(N_C/Q) = \text{Ext}^2_Q(\mathcal{I}_{C/Q}, \mathcal{I}_{C/Q}) \). Apply the functor \( \text{Hom}_Q(-, \mathcal{I}_{C/Q}) \) to the exact sequence. Then it is easy to derive the conclusion \( H^1(N_C/Q) = 0 \).

The above construction is nice. But it has strong restriction on the values of \( d \) and \( g \). We really want to extend the result to more general values of \( d \) and \( g \). Here we will follow the methods in [11] and [12]. The idea works as follows. Take a smooth determinantal curve \( \Gamma \) constructed as above and a smooth rational curve \( \gamma \) on \( Q \) such that they meet transversely. Further assume that \( H^1(N_{\Gamma/Q}) = H^1(N_{\gamma/Q}) = 0 \). Then the nodal curve \( \Gamma \cup \gamma \) can be smoothed out in \( Q \). Moreover, the vanishing property of \( H^1(N_{\gamma}) \) is locally preserved under this smoothing process. Then after smoothing the nodal curve, we may get the degree and genus in a more general range.

Firstly, let us introduce an important smoothing technique used in [12].

**Lemma 4.2.** Let \( \Gamma' = \Gamma \cup \gamma \) be a nodal union of two smooth irreducible curves on the quadric threefold \( Q \). Suppose the singularities of \( C \) are only nodes. Then \( \mathcal{I}_{C/Q} \) is a torsion sheaf supported on each node of \( C \). Furthermore, if \( H^1(N_{\Gamma'/Q}) = 0 \), by the argument of [12] Proposition 1.6, \( C \) is smoothable in \( Q \).

Proof. Let us first set up some notation. For a connected reduced curve \( C \) on \( Q \), denote \( N_{\Gamma/Q}^C \) as the cokernel of the map \( T_C \to T_Q/C \) and let \( T_{C/Q}^1 \) be the cotangent sheaf of \( C \) in \( Q \). \( T_{C/Q}^1 \) can be defined as the cokernel of the map \( N_{C/Q}^C \to N_{C/Q} \). Suppose the singularities of \( C \) are only nodes. Then \( T_{C/Q}^1 \) is a torsion sheaf supported on each node of \( C \). Furthermore, if \( H^1(N_{\Gamma'/Q}^C) = 0 \), by the argument of [12] Proposition 1.6, \( C \) is smoothable in \( Q \).

Now in our case, the ideal sheaves \( \mathcal{I}_{\Gamma'/Q} \cong \mathcal{O}_Q(-P_1 - \ldots - P_3) \) and \( \mathcal{I}_{\gamma'/Q} \cong \mathcal{O}_Q(-P_1 - \ldots - P_3) \). As in [12] Lemma 5.1, we can also establish two exact sequences of sheaves on \( \Gamma' \),
\[ 0 \to \mathcal{I}_{\Gamma'/Q} \otimes N_{\Gamma'/Q} \to N_{\Gamma'/Q} \to \mathcal{N}_{\Gamma'/Q} \to 0, \]
\[ 0 \to \mathcal{N}_{\gamma/Q}(-P_1 - \ldots - P_3) \to \mathcal{I}_{\Gamma'/Q} \otimes N_{\Gamma'/Q} \to T_{\Gamma'/Q}^1 \to 0. \]

By the assumption and the fact that \( H^1(T_{\Gamma'/Q}^1) = 0 \), we get \( H^1(N_{\Gamma'/Q}) = 0 \) from the long exact sequences of cohomology. Hence, \( \Gamma' \) is smoothable in \( Q \). Moreover,
the map $\mathcal{N}_{\gamma/Q} \to \mathcal{N}_{\Gamma/Q}$ is injective and its cokernel $T_{\gamma/Q}^{\Gamma}$ is supported at the nodes. So $H^1(\mathcal{N}_{\gamma/Q}) = 0$ implies that $H^1(\mathcal{N}_{\Gamma/Q}) = 0$. \hfill \qed

We still need another source curve $\gamma$. Here we will consider rational normal curves in $\mathbb{P}^3$ that lie on the quadric $Q$.

**Lemma 4.3.** Let $R = Q \cap H$ be a general hyperplane section of $Q$. $P_1, \ldots, P_m \in R$ are $m \geq 6$ points in general position. For any integer $\delta \leq 4$, there exists a rational normal curve $\gamma \subset R$ such that $\gamma$ passes through exactly $\delta$ points of $P_1, \ldots, P_m$. Furthermore, suppose those points on $\gamma$ are $P_1, \ldots, P_\delta$. Then we have $H^1(\mathcal{N}_{\gamma/Q}(-P_1 - \ldots - P_\delta)) = 0$.

**Proof.** $R$ is a smooth quadric surface in $H \cong \mathbb{P}^3$. It is easy to find a degree 3 smooth rational curve $\gamma$ on $R$ that passes through $\delta$ general points, say, $P_1, \ldots, P_\delta$. By the exact sequence

$$0 \rightarrow \mathcal{N}_{\gamma/R} \rightarrow \mathcal{N}_{\gamma/Q} \rightarrow \mathcal{N}_{R/Q} \otimes \mathcal{O}_\gamma \rightarrow 0,$$

we can also get $H^1(\mathcal{N}_{\gamma/Q}(-P_1 - \ldots - P_\delta)) = 0$. \hfill \qed

Now we have all the ingredients to prove the second part of Theorem 1.2.

**Proof.** Take a determinantal curve $\Gamma \subset Q$ whose degree $d_\Gamma = t(t + 1)$ and genus $g_\Gamma = \frac{4}{3}t^3 - \frac{1}{2}t^2 - \frac{7}{6}t + 1$. Consider a general hyperplane section of $\Gamma$. We get $d_\Gamma$ points in general position. By Lemma 4.2 and 4.3, we can pick a suitable degree 3 rational curve $\gamma$, such that $\Gamma \cap \gamma$ consists of $\delta$ reduced points for any $\delta \leq 4$ and the nodal union $\Gamma \cup \gamma$ is smoothable. Hence, starting from the pair $(d_\Gamma, g_\Gamma)$, we can get a new pair $(d', g' = g_\Gamma + \delta - 1)$ where the Hilbert scheme $\mathcal{H}_{d', g'}(Q)$ also has a component of expected dimension $3d'$. Do the same step again and eventually it covers every pair $(d, g)$ in the form $(d_\Gamma + 3k, g_\Gamma + h)$, $0 \leq h \leq 3k$.

Now we fix $d$. Note that $d_\Gamma = t(t + 1) \equiv 0$ or $2$ (mod 3). So if $d \equiv 0$ (mod 3), by the above construction, the range of $g$ for which $\mathcal{H}_{d, g}(Q)$ has a component of dimension $3d$ contains the following,

$$\frac{1}{6}(4t^3 - 3t^2 - 7t + 6) \leq g \leq \frac{1}{6}(4t^3 - 3t^2 - 7t + 6) + 3 \frac{d - t(t + 1)}{3},$$

for any $t(t + 1) \leq d$ and $t \equiv 0$ or $2$ (mod 3). In order to cover the case $d \equiv 1$ or $2$ (mod 3), we can use a suitable line $l$ on $Q$ instead of the rational curve $\gamma$ in Lemma 4.3 such that $l$ intersects the source curve only at one point $P$. One can easily check that $H^1(\mathcal{N}_{l/Q}) = H^1(\mathcal{N}_{l/Q}(-P)) = 0$ hold. Then after smoothing the nodal union of $l$ and the source curve, this construction provides $(d - 1, g) \rightarrow (d, g)$. So if $d \not\equiv 0$ (mod 3), we can always consider $d - 1$ or $d - 2$ instead. In sum, the desired range of genus includes

$$L(t) = \frac{1}{6}(4t^3 - 3t^2 - 7t + 6) \leq g \leq \frac{1}{6}(4t^3 - 3t^2 - 7t + 6) + d - t(t + 1) - 2 = R(t),$$

where $t(t + 1) \equiv 0$ (mod 3). Since $t \equiv 0$ or $2$ (mod 3), each time $t$ increases by $1$ or $2$. In order to avoid that the value of $g$ jumps for a fixed $d$, we have to require that $L(t + 2) \leq R(t)$. Solving this inequality and plugging the upper bound of $t$ into $R(t)$, we get the desired range of $g$ up to $\frac{2}{15\sqrt{3}}d\sqrt{d} + O(d)$. \hfill \qed
Remark 4.4. We can obtain a similar result for the Hilbert scheme of curves on a general cubic threefold $Y$. It is easy to check that the curve $C$ cut out by a determinantal surface and $Y$ also satisfies $H^1(N_{C/Y}) = 0$. However, when we resume the process to quartic threefolds, the determinantal model does not work any longer. Another long standing problem is about quintic threefolds, since the expected dimension of the Hilbert scheme is 0 in that case. Even for rational curves, the famous Clemens’ conjecture has been only solved when the degree of the curve is small. If we consider threefolds of higher degree, things become further unclear. To the author’s best knowledge, we even do not know if a general threefold of degree $k > 5$ in $\mathbb{P}^4$ contains an irreducible curve whose degree is not divisible by $k$. In sum, the Hilbert scheme of curves on a threefold of higher degree remains mysterious to us.

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