The big projective module as a nearby cycles sheaf

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Abstract

We give a new geometric construction of the big projective module in the principal block of the BGG category \(\mathcal{O}\), or rather the corresponding \(\mathcal{D}\)-module on the flag variety. Namely, given a one-parameter family of nondegenerate additive characters of the unipotent radical of a Borel subgroup which degenerate to the trivial character, there is a corresponding one-parameter family of Whittaker sheaves. We show that the unipotent nearby cycles functor applied this family yields the big projective \(\mathcal{D}\)-module.

1 Introduction

Let \(k\) be an algebraically closed field of characteristic zero and \(G\) a reductive group over \(k\). Fix a Borel subgroup \(B \subset G\) with unipotent radical \(N \subset B\), and choose a splitting \(T := B/N \to B\), so that we may speak of the opposite Borel \(B^-\) and its unipotent radical \(N^-\). Write \(\Delta\) for the set of simple roots of \(G\) with respect to \(B\).

We study the abelian category \(\mathcal{D}(G/B)^N\) of \(N\)-equivariant holonomic \(\mathcal{D}\)-modules on the flag variety \(G/B\), which is equivalent to the principal block of the BGG category \(\mathcal{O}\) by the Bernstein-Beilinson localization theorem. The category \(\mathcal{D}(G/B)^N\) has finitely many simple objects, labeled by elements of the Weyl group \(W\), which we will denote by \(\mathcal{L}_w\) for \(w \in W\). Write \(C_w = NwB/B\) for the Schubert cell corresponding to \(w\), and \(X_w = C_w^\circ\) for the Schubert variety. Recall that \(\mathcal{L}_w\) is the IC (intersection cohomology) sheaf on \(X_w\), pushed forward to \(G/B\). Let \(\mathcal{M}_w\) and \(\mathcal{M}_w^\ast\) be the !- and \(\ast\)-pushforwards to \(G/B\), respectively, of the IC sheaf on \(C_w\). Denote by \(\mathcal{P}_w\) a projective cover of \(\mathcal{L}_w\), i.e. an indecomposable projective object of \(\mathcal{D}(G/B)^N\) which maps nontrivially to \(\mathcal{L}_w\). Recall that \(\mathcal{P}_w\) is unique up to non-canonical isomorphism.

Consider \(\mathcal{L}_e\), which is the delta sheaf at the closed \(N^-\)-orbit in \(G/B\). Its projective cover \(\mathcal{P}_e\) is the longest indecomposable projective object of \(\mathcal{D}(G/B)^N\); it is often referred to as the big projective.

It is well-known that \(\mathcal{P}_e\) can be constructed by averaging as follows. Fix a nondegenerate additive character \(\psi^- : N^- \to \mathbb{G}_a\) and write \(e^{\psi^-} := (\psi^-)^!e^x[1 - \dim N^-]\), where \(e^x\) is the exponential \(\mathcal{D}\)-module on \(\mathbb{G}_a\). The Whittaker sheaf \(\mathcal{W}(\psi^-)\) is obtained by pushing \(e^{\psi^-}\) forward along the canonical open embedding \(N^- \to G/B\) (the \(\ast\)- and !-pushforwards are the same, i.e. the extension is clean). Then \(\mathcal{P}_e\) is isomorphic to the sheaf obtained by averaging \(\mathcal{W}(\psi^-)\) against the action of \(N\) (see e.g. Proposition 14.3.1 in [2] and its proof). The !- and \(\ast\)-averaging functors give the same result thanks to the Verdier self-duality of \(\mathcal{P}_e\), although the canonical morphism from from the !-average to the \(\ast\)-average is not an isomorphism.

Our construction also begins with a Whittaker sheaf and yields \(\mathcal{P}_e\), but proceeds in a different fashion (the author does not know how to formally link the two constructions). Fix a one-parameter family of nondegenerate additive characters of \(N\) (not \(N^-\)) degenerating to the trivial character, which gives rise to a one-parameter family of Whittaker sheaves on \(G/B\). We will prove that the nearby cycles sheaf of such a family is isomorphic to \(\mathcal{P}_e\).

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Notations and conventions. For any variety \(X\) over \(k\) we write \(\mathcal{D}(X)\) for the abelian category of holonomic \(\mathcal{D}\)-modules on \(X\). If \(f : X \to Y\) is a smooth morphism, we write

\[f^\Delta : f^![\dim X - \dim Y] : \mathcal{D}(Y) \to \mathcal{D}(X)\]

for the normalized pullback functor.
2 Whittaker sheaves

Let \( H \) be an algebraic group over \( k \) and write \( \mu : H \times H \to H \) for the group operation. A character sheaf (sometimes called a multiplicative local system) on \( H \) is a line bundle with connection \( \chi \) on \( H \) that satisfies \( \mu^* \chi \cong \chi \otimes \chi \). If \( \varphi : K \to H \) is a homomorphism, then \( \varphi^* \chi \) is a character sheaf on \( K \).

On \( \mathbb{G}_a \) the basic character sheaf is the exponential \( \mathcal{P}\text{-}\text{module} e^\varphi \). We can exponentiate an additive character \( \varphi : H \to \mathbb{G}_a \) to obtain a character sheaf \( e^\varphi := \varphi^* e^x \) on \( H \).

If \( H \) acts on a variety \( X \) we denote \( \mathcal{P}(X)^x \) by the abelian category of \( \chi \)-equivariant holonomic \( \mathcal{P}\)-modules on \( X \); when \( \chi \) is the trivial character sheaf this is the \( H \)-equivariant category \( \mathcal{P}(X)^H \). An object of \( \mathcal{P}(X)^x \) consists of \( \mathcal{F} \in \mathcal{P}(X) \) together with an isomorphism \( \chi \otimes \mathcal{F} \to a^* \mathcal{F} \) satisfying the usual cocycle condition on \( H \times H \times X \) (here \( a : H \times X \to X \) is the action map).

All of the above also makes sense in families, i.e. over other bases schemes than \( \text{Spec} k \). We will only need to work with families over \( \mathbb{A}^1 \). For example, if \( p : \mathbb{G}_{a, \mathbb{A}^1} \to \mathbb{G}_a \) denotes the projection, then \( p^* e^x \) is a character sheaf on the constant group scheme \( \mathbb{G}_{a, \mathbb{A}^1} \).

Resume the notation of the introduction. We fix an additive character

\[
\psi : N \to \mathbb{G}_a
\]

which is nondegenerate in the sense that the induced homomorphism

\[
N/[N, N] = \bigoplus_{a \in \Delta} N_a \to \mathbb{G}_a
\]

is nontrivial on every simple root group \( N_a \).

Recall that the big cell \( C_{w_0} \) is an \( N \)-torsor that is trivialized by our choice of maximal torus \( T \to B \). Write \( j : N \to G/B \) for the resulting inclusion.

**Definition 2.1.** The Whittaker sheaf attached to \( \psi \) is defined by \( \mathcal{W}(\psi) := j_* e^\psi \).

It is well-known that the canonical morphism \( j_* e^\psi \to \mathcal{W}(\psi) \) is an isomorphism, i.e. the extension is clean.

Now we extend \( \mathcal{W}(\psi) \) to a one-parameter family of character sheaves using a choice of dominant regular coweight \( \gamma : \mathbb{G}_m \to T \). The assumption of dominance implies that the \( \gamma \)-conjugation action \( (s, n) \mapsto \text{ad}_{\gamma}(n) \) of \( \mathbb{G}_m \) on \( N \) extends uniquely to an action of the multiplicative monoid \( \mathbb{A}^1 \). Thus \( \gamma \) induces a homomorphism of constant group schemes \( \psi_\gamma : N_{\mathbb{A}^1} \to \mathbb{G}_{a, \mathbb{A}^1} \); given on points by

\[
\psi_\gamma(n, s) = (\text{ad}_{\gamma}(n), s).
\]

Now exponentiation yields the character sheaf \( e^{\psi_\gamma} := \psi_\gamma^* p^* e^x \) on \( N_{\mathbb{A}^1} \), where as before \( p : \mathbb{G}_{a, \mathbb{A}^1} \to \mathbb{G}_a \) is the projection. Since \( \gamma \) was chosen regular the \( ! \)-fiber over \( 0 \in \mathbb{A}^1 \) of \( e^{\psi_\gamma} \) is the trivial character sheaf on \( N \).

Finally, we obtain the desired one-parameter family \( \mathcal{W}(\psi, \gamma) := (j \times \text{id})_* e^{\psi_\gamma} \), an \( e^{\psi_\gamma} \)-equivariant \( \mathcal{P}\)-module on \( (G/B)_{\mathbb{A}^1} \). The extension is clean away from \( 0 \in \mathbb{A}^1 \), but the \( ! \)-restriction of \( \mathcal{W}(\psi, \gamma) \) to the fiber over \( 0 \) is (up to shift) \( \mathcal{W}_{w_0} \).

3 Nearby cycles

Let \( X \) be a variety over \( k \). Write \( X_{\mathbb{A}^1} = X \times \mathbb{A}^1, X^0 = \mathbb{A}^1 \setminus \{0\} \), and \( X^0 = X \times \mathbb{A}^1 \). Recall the existence of the unipotent nearby cycles functor

\[
\Psi : \mathcal{P}(X^0) \to \mathcal{P}(X),
\]

which has the following properties:

1. \( \Psi \) is exact,
2. \( \Psi \) is compatible with restriction to open subvarieties and pushforward along proper morphisms,
3. if \( \mathcal{F} \in \mathcal{P}(X_{\mathbb{A}^1}) \) is lisse, then \( \Psi(\mathcal{F}|_{X^0}) = i^! \mathcal{F}[1] \) where \( i : X \to X_{\mathbb{A}^1} \) is the inclusion of the fiber over \( 0 \),
4. $\Psi$ naturally lifts to a functor taking values in the category whose objects are pairs $(\mathcal{F}, m)$ where $\mathcal{F}$ is a $\mathcal{D}$-module on $X$ and $m$ is a nilpotent endomorphism of $\mathcal{F}$.

For any $\mathcal{F} \in \mathcal{D}(X_{\tilde{\mathcal{A}}^1})$ the resulting nilpotent endomorphism of $\Psi(\mathcal{F})$ is called its monodromy.

See \cite{loc. cit.} for a construction of $\Psi$, which allows for nonconstant families. We will only need the case of a constant family with $X = G/B$ or a subvariety of $G/B$.

Now fix an algebraic group $H$ acting on $X$ and a character sheaf $\chi$ on the constant group scheme $H_{\mathcal{A}^1}$.

Write $\tilde{\chi}$ for the restriction of $\chi$ to $H_{\mathcal{A}^1}$ and denote by $\chi_0$ the $!$-restriction of $\chi$ to the fiber over 0. Then it follows from Beilinson’s construction in \cite{loc. cit.} that $\Psi_X$ lifts to a functor of equivariant $\mathcal{D}$-modules

$$\Psi: \mathcal{D}(X_{\tilde{\mathcal{A}}^1}) \tilde{\rightarrow} \mathcal{D}(X)^{\chi_0}.$$  

4 Theorem

In our situation, we can view unipotent nearby cycles as a functor

$$\Psi: \mathcal{D}((G/B)_{\mathcal{A}^1})^{e^\chi} \rightarrow \mathcal{D}(G/B)^N$$

by first restricting to $(G/B)_{\mathcal{A}^1}$, which we omit from the notation.

Theorem 4.1. We have $\Psi(\mathcal{W}(\psi, \gamma)) \cong \mathcal{P}_e$.

Remark 4.2. The theorem shows that the $\mathcal{D}$-module $\Psi(\mathcal{W}(\psi, \gamma))$ does not depend on $\gamma$, but in fact $\gamma$ determines the monodromy endomorphism. Namely, note that $\mathcal{P}_e$ is weakly $T$-equivariant and so $\mathfrak{h} := \text{Lie} T$ acts on $\mathcal{P}_e$ by endomorphisms. Then the action of $\gamma$, viewed as an element of $\mathfrak{h}$, on $\mathcal{P}_e$ corresponds to the monodromy endomorphism of $\Psi(\mathcal{W}(\psi, \gamma))$.

Example 4.3. Consider the case $G = \text{SL}_2$, so that $G/B = \mathbb{P}^1$. Since $N = \mathbb{G}_a$ we can take $\psi = \text{id}$, so that $\mathcal{W}(\psi) = j_* e^x$ where $j: \mathcal{A}^1 \rightarrow \mathbb{P}^1$ is the inclusion. Moreover $T = \mathbb{G}_m$ and $\gamma$ is the $n$th power map on $\mathbb{G}_m$.

Let $s = w_0 \in W = S_2$ be the nontrivial (and longest) element of the Weyl group. Then $\mathcal{L}_s \cong \mathcal{P}_s$ is the IC sheaf on $\mathbb{P}^1$ and $\mathcal{L}_e$ is the delta sheaf at $\infty$. The sheaf $\mathcal{P}_e$ has a composition series $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 = \mathcal{P}_e$ with $\mathcal{F}_1 \cong \mathcal{L}_e, \mathcal{F}_2/\mathcal{F}_1 \cong \mathcal{L}_s$, and $\mathcal{P}_e/\mathcal{F}_2 \cong \mathcal{L}_e$. The algebra of endomorphisms of $\mathcal{P}_e$ is canonically identified with $k[e]/(e^2)$, and the monodromy endomorphism of $\Psi(\mathcal{W}(\psi, \gamma))$ corresponds to $ne$.

Proof of the theorem. We will write $\Psi := \Psi(\mathcal{W}(\psi, \gamma))$ for brevity. The proof proceeds in two steps: we will first show that for all $w \neq e$ we have $\text{Ext}^i(\Psi, \mathcal{L}_w) = 0$ for all $i$, then prove that $\text{Hom}(\Psi, \mathcal{L}_e) = k$ and $\text{Ext}^i(\Psi, \mathcal{L}_e) = 0$ for $i \neq 0$.

We claim that for any $w \neq e$ there exists a simple parabolic subgroup $P \subset G$ (in particular, $P \neq B$) such that $\mathcal{L}_w \cong \pi^* \mathcal{F}$ for some object $\mathcal{F} \in \mathcal{D}(G/P)^N$, where $\pi: G/B \rightarrow G/P$ is the projection. Indeed, if $\alpha$ is a simple root such that $\ell(w w_0) = \ell(w) - 1$, then $NwB$ is stable under the right action of $P = B \cup Bs_B B$. The claim follows because $\mathcal{L}_w$ is pushed forward from the IC sheaf on $X_w$.

In this paragraph we use the de Rham pushforward functor $\pi_* = \pi_1$, which takes values in the derived category of $\mathcal{D}(G/P)$. We will show that $\pi_* \Psi = 0$, so that

$$\text{Ext}^i(\Psi, \mathcal{L}_w) \cong \text{Ext}^i(\pi_* \Psi, \mathcal{F}[-1]) = 0$$

for all $i$. Since $\Psi$ commutes with pushforward along proper morphisms, we need only prove that $\pi_* \mathcal{W}(\psi, \gamma) = 0$, where $\pi: (G/B)_{\mathcal{A}^1} \rightarrow (G/P)_{\mathcal{A}^1}$ is the projection (again we omit the restriction to $(G/B)_{\mathcal{A}^1}$ from our notation). Because $\mathcal{W}(\psi, \gamma)$ is cleanly extended from $C_{w_0}^{w_0, \mathcal{A}^1}$, the sheaf $\pi_* \mathcal{W}(\psi, \gamma)$ is $*$-extended from the $N_{w_0}^{w_0}$-orbit $(Nw_0 P/P)_{\mathcal{A}^1}$. Thus by the $e^\psi$-equivariance it suffices to check that $i_{w_0}^* \pi_* \mathcal{W}(\psi, \gamma) = 0$, where $i_{w_0}: \mathcal{A}^1 \rightarrow (G/P)_{\mathcal{A}^1}$ is the constant section $w_0 P$. One has $P^- \cap N = N_\alpha$, so the desired vanishing follows from base change. This is because for any $t \neq 0$, the character $\text{ad}_{\psi(t)}(\psi)$ is nontrivial when restricted to $N_\alpha$, and a nontrivial exponential $\mathcal{D}$-module has vanishing de Rham cohomology.
Finally, we show that $\text{Ext}^i(\Psi, L_e) = 0$ if $i \neq 0$ and $\text{Hom}(\Psi, L_e) = k$. Recall that there is a surjection $\mathcal{M}_{w_0} \to L_e$ whose kernel does not have $L_e$ as a subquotient, which implies that

$$\text{Ext}^i(\Psi, \mathcal{M}_{w_0}) \to \text{Ext}^i(\Psi, L_e)$$

for all $i$. But $\mathcal{M}(\psi, \gamma)$ restricts to the IC sheaf on $C_{w_0, \mathbb{A}^1}$, so $\Psi|_{C_{w_0}}$ is the IC sheaf on $C_{w_0}$, whence the claim.

\[\square\]

References

[1] A. Beilinson, “How to glue perverse sheaves.” In Lecture Notes in Mathematics Volume 1289, pp. 42-51. Springer, 1987.

[2] Edward Frenkel and Dennis Gaitsgory. “Local geometric Langlands correspondence and affine Kac-Moody algebras.” In Algebraic geometry and number theory, pp. 69-260. Springer, 2006.