Non-commutative branes in D-brane backgrounds

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ABSTRACT

We study Myers world-volume effective action of coincident D-branes. We investigate a system of $N_0$ D0-branes in the geometry of D$p$-branes with $p = 2$ or $p = 4$. The choice of coordinates can make the action simplified and tractable. For $p = 4$, we show that a certain point-like D0-brane configuration solving equations of motion of the action can expand to form a fuzzy two-sphere via magnetic moment effect without changing quantum numbers. We compare non-commutative D0-brane configurations with dual spherical D($6 - p$)-brane systems. We also discuss the relation between these configurations and giant gravitons in 11-dimensions.

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1 Introduction

Effective action of $N_p$ coincident $D_p$-branes has been extended to include couplings to Ramond-Ramond field strength $F^{(n)}$ of any form degree $n$ other than that of $n \leq p + 2$ \[1, 2\]. The couplings of the $D_p$-branes with $F^{(n)}$ ($n \geq p + 4$) induce dielectric effects (Myers effects) \[1\] or their magnetic analogues (magnetic moment effects) \[3, 4\]. The effects are important in understanding the nature of non-abelian $D$-brane systems and have been studied in various contexts \[1, 5, 6, 7\]. In particular, they play an important role \[8, 9, 10, 11, 12\] in the context of AdS/CFT correspondence \[13\].

Most analyses given so far concern flat spacetime where the RR field strength $F^{(n)}$ is put as an external field \[1, 5, 6, 7\]. Such backgrounds ignore the back reaction of $F^{(n)}$, i.e., they do not solve supergravity equations of motion. For example, consider the effective action of $D_0$-branes in the background with flat metric and $F^{(4)}_{\mu 123} = \text{const.}$ \[1\]. For $\mu = 0$, a static non-commutative configuration of $D_0$-branes other than the usual point-like configuration can solve equations of motion of the action due to the dielectric effect \[1\]. Similarly for $\mu = 4$, $D_0$-branes can expand via magnetic moment effect \[4\] and there are a point-like and a non-commutative configuration with a constant momentum $P_4$ as solutions of the action. In each of these cases, the energy of the non-commutative configuration of $N_0$ $D_0$-branes is less than that of corresponding commutative configuration. This means that the true ground state of the model is given by the configuration of expanding $D_0$-branes. On the other hand, if we consider to put $F^{(6)}_{\mu 12345} = \text{const.}$ in flat background, a non-commutative configurations of $D_0$-branes is unstable and a stable configuration is given by point-like $D_0$-branes \[7\].

If we deal with a supergravity background with some remaining supersymmetries, we expect to have a BPS configuration of commutative $D_0$-branes. Thus if there exists an expanding configuration with the same number of supersymmetries, it would have the same energy as the commutative configuration. One of our purposes of this paper is to see if such a structure certainly exists in $D_p$-brane backgrounds. Note that ref.\[4\] gives similar discussion in the near horizon geometry of $D_4$-branes. There it was shown that substitution of certain non-commutative matrices $X^i$ to the non-abelian action of $D_0$-branes yields the same terms as the expansion of the dual spherical $D_2$-brane action. We shall try to argue the problem from the microscopic point of view by explicitly solving the equations of motion of $D_0$-branes in $D_p$-brane backgrounds. We consider the coupling of $D_0$-branes with the magnetic RR field strength $F^{(8-p)}$ associated with the $D_p$-branes. Thus we only deal with magnetic moment effect.

One of the difficulties of dealing with the action in general backgrounds is that it is coordinate dependent due to the appearance of commutators of $U(N_0)$ adjoint scalars $X^i$.
and the non-abelian Taylor expansions. We shall see that in some cases we can avoid the difficulties and give explicit form of the action by choosing coordinate system properly.

We explicitly consider the geometry of Dp-branes with \( p = 2 \) or \( p = 4 \) and analyze the magnetic moment effect. For such a background, we show that we can choose appropriate coordinate system and can write the non-abelian action of D0-branes explicitly. In particular, for D4-brane background, we find two solutions with the same energy and momentum by solving equations of motion of the action explicitly if we take near horizon limit. One of them is point-like and the other is fuzzy spherical. We also see that each of these systems is related to the spherical D\((6-p)\)-branes with an appropriate U(1) field \( F_{ab} \) on the branes.

This paper is organized as follows. In section 2, we briefly review Myers non-abelian action of D-branes and consider the application to general backgrounds. In section 3, we analyze the action in the geometry of Dp-branes with \( p = 2 \) or \( p = 4 \) and try to solve the equations of motion. We compare the non-abelian action with that of spherical D\((6-p)\)-brane action. Section 4 gives summary and discussion. In particular, we briefly discuss the relation between expanding D0-brane via Myers effects and rotating spherical M-branes (giant gravitons) in \( AdS_m \times S^n \) in 11-dimensions.

## 2 Effective action of coincident D-branes

The world-volume effective action of \( N_p \) coincident Dp-branes in type IIA or IIB theory is given by Myers [1]. It is constructed from the D9-brane action in type IIB theory by applying T-duality transformation along \( 9 - p \) space coordinates \( x^{p+1}, \ldots, x^9 \). The action involves a U\((N_p)\) gauge field \( A_a \) and \( 9 - p \) adjoint scalars \( X^i \) (and their superpartners). The field strength of \( A_a \) is \( F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b] \) and the covariant derivative of \( X^i \) is \( D_a X^i = \partial_a X^i + i[A_a, X^i] \) as usual. We choose static gauge \( x^a = \sigma^a \) \((a = 0, 1, \ldots, p)\) for spacetime coordinates \( x^\mu \) and worldvolume coordinates \( \sigma^a \). Suppose that Dp-branes are put in the background of string frame metric \( ds^2 = G_{\mu \nu} dx^\mu dx^\nu \), NS-NS 2-form potential \( B_{\mu \nu} \) and \( n \)-form RR potentials \( C^{(n)} \) for \( n = \ldots, p-1, p+1, \ldots \). Then the proposed action is written as sum of the Born-Infeld action \( S_{BI} \) and the Chern-Simons action \( S_{CS} \). Bosonic part of \( S_{BI} \) is given as

\[
S_{BI} = -T_p \int d^{p+1}\sigma \text{Tr} \left[ e^{-\phi} \sqrt{-\det(P[E_{ab} + E_{ai}(Q^{-1} - \delta)^i_k E^{kj} E_{jb}] + \lambda F_{ab}) \det(Q^i_j)} \right]
\]

where \( \lambda = 2\pi l_s^2 \), \( E_{\mu \nu} = G_{\mu \nu} + B_{\mu \nu} \), \( T_p = 2\pi / g_s (2\pi l_s)^{p+1} \) is the tension of the Dp-brane and

\[
Q^i_j = \delta^i_j + i\lambda^{-1}[X^i, X^k] E_{kj}.
\]

\[
(1)
\]

\[
(2)
\]
The pull-back \( P[\cdots] \) is defined as
\[
P[Z_{a_1\cdots a_n}] = Z_{\mu_1\cdots \mu_n} D_{a_1} X^{\mu_1} \cdots D_{a_n} X^{\mu_n}.
\] (3)

The field \( G_{\mu\nu} \) (or \( B_{\mu\nu}, \phi \)) in the above action is a functional of \( X^i \) and the explicit form would be given by a non-abelian Taylor expansion of the corresponding background fields as [14], e.g.,
\[
G_{\mu\nu}(X^i(\sigma^a), \sigma^a) = \sum_{n=0}^{\infty} \frac{1}{n!} X^{i_{i_1}} \cdots X^{i_n} \partial_{i_1} \cdots \partial_{i_n} G_{\mu\nu}(x^i, \sigma^a)|_{x^i=0}.
\] (4)

In the above expression we assume that the D\( p \)-brane is temporarily put on \( x^i = 0 \) and the scalar fields \( X^i \) represent fluctuation around \( x^i = 0 \). Chern-Simons action is given as
\[
S_{CS} = T_p \int \text{Tr} \left( P \left[ e^{iX^i} \sum C^{(n)} e^B \right] e^{\lambda F} \right).
\] (5)

Here \( i_X \) is an interior product which reduces the form degree \(-1\) as e.g.,
\[
i_X i_X C^{(2)} = \frac{1}{2} C^{(2)}_{ij} [X^j, X^i].
\] (6)

This interior product induces the coupling of the D-branes to the RR potential of higher degree \( n = p + 3, p + 5, \cdots \). Note that we interpret \( \text{Tr}(\ ) \) as symmetrized trace: we take the traces after we symmetrize all \( F_{ab}, \ D_a X^i, \ [X^i, X^j] \) and each \( X^i \) appearing in the non-abelian Taylor expansions.

In practice, the action eq.(1) or eq.(5) can be justified a priori only when there is an isometry along each \( x^i \) since it is constructed from the D9-brane action via T-duality transformation [1]. However, as was described above, the action can formally be defined for any coordinate system \( \{x^\mu\} \) in any spacetime if we use non-abelian Taylor expansion of background fields \( E_{\mu\nu}, \phi \) and \( C^{(n)} \). For such a non-trivial background, the action does not have general covariance anymore and the meaning of the action is not clear enough. In spite of the subtlety, we would like to consider the action of a set of coincident D\( p \)-branes in a particular background without an isometry along \( x^i \) (\( i = p + 1, \cdots, 9 \)). We have to define \( G_{\mu\nu}, B_{\mu\nu} \) and \( C^{(n)} \) by using non-abelian Taylor expansion as described above if they are not constants. Thus in general, the action can describe the behavior of the D-branes only around \( x^i \sim 0 \). It is not suitable enough to deal with expanding brane configurations.

We would like to determine the Chern-Simons action without using infinite series of Taylor expansion since it is important to obtain expanding brane configurations. We seek for a coordinate system where the field \( C^{(n)} \) becomes independent of \( X^i \) or is represented as a finite polynomial of \( X^i \). Even if there is no such coordinate system, we can consider a
model that some of the adjoint scalar fields $X^i$ are set to be diagonal from the beginning: $X^i = x^i 1$. In some cases, such a model does not need to use non-abelian Taylor expansion if we choose an appropriate coordinate system. In practice, we use such method to analyze the action of D0-branes in the Dp-brane background in the next section.

Now we especially consider the action of $N_0$ D0-branes for our future purpose. Each term of the Chern-Simons action can be expanded by polynomials of $X^i$ around $X^i = 0$. For example, couplings of D0-branes with the background $C^{(3)}$ and $C^{(5)}$ in $\mathcal{L}_{CS}$ is respectively given as

\[
\mathcal{L}_{CS}^3 = \frac{i T_0}{\lambda} \text{Tr} P[(i x^i x^i) C^{(3)}] = \frac{i T_0}{2\lambda} \text{Tr} \left( C^{(3)}_{ij} (X, \sigma) [X^j, X^i] + C^{(3)}_{ijk} (X, \sigma) [X^k, X^j] \partial_i X^i \right) = \frac{i T_0}{3\lambda} \text{Tr} (X^i X^j X^k) F_{tijk} - i \frac{T_0}{4\lambda} \text{Tr} (X^i X^j X^k X^l) F_{ijkl} + \frac{T_0}{4\lambda} \text{Tr} (X^i X^j X^k X^l) \partial_i F_{tijk} - \frac{T_0}{30\lambda} \text{Tr} (X^i X^j X^k X^l \dot{X}^m) \left[ 4 \partial_i F_{tjkln} - 2 \partial_j F_{klim} \right] + \frac{T_0}{30\lambda} \text{Tr} (X^i X^j X^k X^l \dot{X}^m) \left[ \partial_i \partial_j F_{tklm} + 2 \partial_i \partial_j F_{tklm} \right] + O(X^6), \tag{7}
\]

\[
\mathcal{L}_{CS}^5 = \frac{-T_0}{2\lambda^2} \text{Tr} P[(i x^i x^i)^2 C^{(5)}] = \frac{-T_0}{8\lambda^2} \text{Tr} \left( C^{(5)}_{ijkl} (X, \sigma) [X^j, X^i] [X^l, X^i] + C^{(5)}_{ijklm} (X, \sigma) [X^j, X^i] [X^l, X^k] \partial_i X^m \right) = \frac{T_0}{10\lambda^2} \text{Tr} (X^i X^j X^k X^l X^m) F_{tijklm} - \frac{T_0}{12\lambda^2} \text{Tr} (X^i X^j X^k X^l X^m \dot{X}^n) F_{tijklmn} + \frac{T_0}{12\lambda^2} \text{Tr} (X^i X^j X^k X^l X^m X^n) \partial_n F_{tijklm} + O(X^7), \tag{8}
\]

where each $F^{(n)}$ or its derivative in the above equations denotes the value at $X^i = 0$: e.g., $F^{(n)}|_{X^i = 0}$. Here we use the symmetrized trace prescription. For some other definition of trace, not all terms are collected in the form of field strength $F^{(n+1)}$. In practice, gauge invariance of $\mathcal{L}_{CS}$ is proven when symmetrized trace prescription is applied \[15\].

The Born-Infeld action for coincident D0-branes reduces to

\[
S_{BI} = -T_0 \int dt \text{Tr} \left( e^{-\phi} \sqrt{-[E_{00} + (Q^{-1})_k X^k X^i E_{ij}] \det(Q^{ij})} \right) \tag{9}
\]

when we fix the gauge $E_{0i} = 0$ and $A_0 = 0$. The matrix $(Q^{-1})^i_j$ which is inverse to $Q^i_j$ is defined by polynomial expansion of $X^i$. It is known that this form of the action can be reliable only up to fourth order in $F_{ab}$ (or $D_a X^i$, $[X^i, X^j]$) \[16, 17\]. The problem is not critical for our analysis since we mainly deal with up to second order of the commutators $[X^i, X^j]$. 


3 D0-branes in the geometry of Dp-branes

In this section we consider the non-abelian action of \( N_0 \) D0-branes in supergravity backgrounds. Since we expect to have a configuration of branes that are expanding into \( S^2 \) or \( S^4 \) via Myers effect, we deal with a background which has SO(3) or SO(5) symmetry. Here we consider the geometry of Dp-branes with \( p = 4 \) or \( p = 2 \).

3.1 D0-branes in the Dp-brane geometry

Geometry of Dp-branes is described by the string frame metric\[18,19\]
\[
\begin{align*}
\eta_{\mu\nu} dx^\mu dx^\nu + H^{n} \delta_{mn} dx^m dx^n \\
e^\phi = H^{3-p}
\end{align*}
\] (10)
where \( r = \sqrt{x^m x_m}, \mu, \nu \in \{0, \cdots, p\} \), \( m, n \in \{p + 1, \cdots, 9\} \) and
\[
H = 1 + \frac{k}{r^{7-p}}
\] (12)
with
\[
k = \frac{N}{7-p} \frac{2\pi}{T_{6-p} V_{8-p}}.
\] (13)
Here \( V_q \) is the volume of unit \( q \)-sphere:
\[
V_q = \frac{2\pi^{\frac{q+1}{2}}}{\Gamma\left(\frac{q+1}{2}\right)}.
\] (14)

There is a non-zero \((p+2)\)-form field strength \( F^{(p+2)} \) corresponding to the electric RR-potential \( C^{(p+1)} \) of \( N \) Dp-branes. Here we rather characterize the background by dual magnetic \((8-p)\)-form field strength \( F^{(8-p)} \) as
\[
F^{(8-p)}_{m_1 m_2 \cdots m_{8-p}} = -\epsilon_{m_1 \cdots m_{8-p} n} \partial_n H.
\] (15)
We see that infinite series of non-abelian Taylor expansion is needed if we write down the Chern-Simons term by using this expression of \( F^{(8-p)} \) and the non-commutative coordinates \( X^i \). Fortunately, we find an appropriate coordinate system for which \( F^{(8-p)} \) becomes constant. The new coordinates \( \{r, \phi, z_{p+3}, \cdots, z_9\} \) are given from \( \{x^{p+1}, \cdots, x^9\} \) by the relation
\[
\begin{align*}
x_{p+1} &= r \sqrt{1 - (z_{p+3}^2 + \cdots + z_9^2)} \cos \phi, \\
x_{p+2} &= r \sqrt{1 - (z_{p+3}^2 + \cdots + z_9^2)} \sin \phi, \\
x_{p+3} &= r z_{p+3}, \\
\vdots &= \vdots \\
x_9 &= rz_9.
\end{align*}
\] (16)
The corresponding metric component is rewritten as
\[
\delta_{mn}dx^m dx^n = dr^2 + r^2[1 - (z_{p+3}^2 + \cdots + z_9^2)]d\phi^2 + r^2(dz_{p+3}^2 + \cdots + dz_9^2)
\]
\[\quad + \frac{r^2}{1 - (z_{p+3}^2 + \cdots + z_9^2)}(z_{p+3}dz_{p+3} + \cdots + z_9dz_9)^2. \tag{17}\]

The explicit form of \(F^{(8-p)}\) is given as
\[
F^{(8-p)}_{\mu_1\mu_2\cdots\mu_{8-p}} = \begin{cases} 
-\frac{2\pi}{\ell_{6-p}v_{8-p}} N\epsilon_{\mu_1\mu_2\cdots\mu_{8-p}} & \text{for } \mu_1, \mu_2 \cdots, \mu_{8-p} \in \{ \phi, z_{p+3}, \cdots, z_9 \} \\
0 & \text{otherwise}
\end{cases} \tag{18}
\]
where \(\epsilon_{\phi z_{p+3} \cdots z_9} = 1\). By using this expression, the Chern-Simons action can be represented unambiguously without using non-abelian Taylor expansion for each \(p\).

In order to give the Born-Infeld action of \(N_0\) coincident D0-branes explicitly, we first assume that the coordinates \(r\) and \(\phi\) remain to be commutative fields on the D0-branes. This restriction is necessary for the action to be tractable and also can be understood by the fact that the meaning of matrix generalization of a radial or an angular coordinate is less clear than that of flat-like coordinate \(Z^i\) or \(X^a\). Furthermore, we assume \(X^a = 0\) \((a = 1, \cdots, p)\) for simplification. The other \(7-p\) transverse coordinates become adjoint scalar fields \(Z^{p+3}, \cdots, Z^9\) on the branes. Then, the action is written as
\[
\mathcal{L}_{BI} = -T_0 \text{Tr} \left( H^{\frac{p+4}{2}} \sqrt{1 - H\dot{r}^2 - Hr^2(1 - Z^iZ^j)\dot{\phi}^2 - H^4(Q^{-1})^{j_k\hat{k}} \hat{Z}^j \hat{Z}^k G_{ij}} \right) det(Q^{ij}) \tag{19}
\]
where \(\hat{i} = p + 3, \cdots, 9\). We expand the action in terms of \([Z^i, \hat{Z}^j]\) or \(\hat{Z}^i\) and take the leading contribution:
\[
\sqrt{det(Q^{ij})} = 1 - \frac{1}{4\lambda^2} Hr^4([Z^i, \hat{Z}^j]^2) + \cdots, \tag{20}
\]
\[\quad (Q^{-1})^{j_k\hat{k}} \hat{Z}^j \hat{Z}^k = \hat{Z}^j \hat{Z}^j + \cdots. \tag{21}\]

We also consider the non-relativistic limit. Then the action becomes
\[
\mathcal{L}_{BI} = -N_0 T_0 H^{\frac{p+4}{2}} \left\{ 1 - \frac{1}{2N_0} H^2 \text{Tr}(\hat{Z}^i \hat{Z}^j G_{ij}) - \frac{1}{2} H\dot{r}^2 - \frac{1}{4N_0\lambda^2} Hr^4 \text{Tr}([Z^i, Z^j]^2) \right. \\
\left. - \frac{1}{2} Hr^2 \left[ 1 - \frac{1}{N_0} \text{Tr}(Z^iZ^i) \right] \dot{\phi}^2 \right\}. \tag{22}\]

### 3.2 \(p = 4\)

We explicitly consider the \(p = 4\) case. The background four-form field strength is
\[
F^{(4)}_{\phi z_7 z_8 z_9} = -\frac{2\pi}{T_2V_4} N. \tag{23}
\]
By substituting this into eq. (7) and use partial derivative operation properly, the Chern-Simons part of the Lagrangian is given as

\[ L_{CS} = \frac{i}{2} N \text{Tr}(\hat{Z}^i \hat{Z}^j \hat{Z}^k) \dot{\phi} \epsilon_{ijk}. \] (24)

Combining this with eq. (22), the total Lagrangian becomes

\[ L = -N_0 T_0 + \frac{1}{2} N_0 T_0 H r^2 + \frac{1}{2} N_0 T_0 H r^2 \left[ 1 - \frac{1}{N_0} \text{Tr}(\hat{Z}^i \hat{Z}^i) \right] \dot{\phi}^2 + \frac{T_0}{2} H^2 \text{Tr}(\hat{Z}^i \hat{Z}^i G_{ij}) \]
\[ + \frac{T_0}{4 \lambda^2} H r^4 \text{Tr}([\hat{Z}^i, \hat{Z}^j]^2) - \frac{i}{2} N \text{Tr}(\hat{Z}^i \hat{Z}^j \hat{Z}^k) \dot{\phi} \epsilon_{ijk}. \] (25)

Note that \( H = 1 + N \lambda / 2 T_0 r^3 \) from eq. (13).

Equations of motion with respect to \( \hat{Z}^i \) and \( \phi \) are obtained as

\[ T_0 \frac{d}{dt} (\dot{Z}^i G_{ij} H^2) + \frac{r N}{\lambda} \left[ \frac{\lambda T_0}{N} r H \dot{\phi}^2 Z^i - \frac{T_0}{\lambda N} r^3 H [[\hat{Z}^i, Z^i], Z^i] + \frac{3}{4} i \left( \frac{\lambda}{r} \dot{\phi} \right) [Z^i, Z^k] \epsilon_{ijk} \right] = 0, \] (26)
\[ P_{\phi} \equiv N_0 T_0 H r^2 \dot{\phi} \left[ 1 - \frac{1}{N_0} \text{Tr}(\hat{Z}^i \hat{Z}^i) \right] - \frac{i}{2} N \text{Tr}(\hat{Z}^i \hat{Z}^j \hat{Z}^k) \epsilon_{ijk} = \text{constant}. \] (27)

We assume that each \( \hat{Z}^i \) is a constant matrix. There is a pointlike solution satisfying \( \hat{Z}^i = 0 \) and \( H r^2 \dot{\phi} = \text{constant} \). Motion along \( r \) is determined by solving equation of motion with respect to \( r \). There is no other solution found in general in this class.

If we restrict the geometry to be near horizon region of D4-branes, situation becomes more interesting. In this case, since \( H = N \lambda / 2 T_0 r^3 \), \( \dot{\phi} / r \) is restricted to be constant from eq. (27) if we assume \( \hat{Z}^i \) to be a constant matrix. Then eq. (26) is explicitly solved by

\[ \hat{Z}^i = w \alpha^i \] (28)

with

\[ w = 0, \quad \frac{\lambda \dot{\phi}}{2 r}, \quad \text{or} \quad \frac{\lambda \dot{\phi}}{4 r}. \] (29)

Here \( \alpha^i (\hat{i} = 7, 8, 9) \) is \( N_0 \) dimensional representation of SU(2):

\[ [\alpha^i, \alpha^j] = 2i \epsilon_{ijk} \alpha^k. \] (30)

For each constant \( w (\neq 0) \), the solution represents a fuzzy two-sphere of effective radius measured by \( z^i \)

\[ r_z = w N_0 \sqrt{1 - \frac{1}{N_0^2}} \sim w N_0 \] (31)

where \( r_z^2 \equiv \frac{1}{N_0} \text{Tr}(\hat{Z}^i \hat{Z}^i) \). Behavior of \( r \) is determined by solving equation of motion for \( r \). To see the energy of the solutions, we calculate the Hamiltonian of this system for \( \hat{Z}^i \)
satisfying eq.(28):

\[
\mathcal{H} = T_0 N_0 + \frac{r^3}{\lambda N N_0} P_r^2 + \frac{r}{\lambda N N_0} \left\{ P_\phi^2 + \frac{w^2 (N_0^2 - 1)}{1 - w^2 (N_0^2 - 1)} (P_\phi - w N N_0)^2 \right\}. \tag{32}
\]

For a given \( P_\phi \) and a particular motion along \( r \), Hamiltonian has two degenerate minima

\[
w = 0, \quad \frac{P_\phi}{N N_0}.
\]

Note that the relation

\[
P_\phi = \frac{\lambda N N_0 \dot{\phi}}{2r}
\]

is satisfied for both values of \( w \) and they correspond to two of the solutions eq.(29):

\[
w = 0, \quad \frac{\lambda \dot{\phi}}{2r}.
\]

The other solution is at unstable point of the energy which is double-well form as in the case of giant gravitons \([3, 4, 23, 24]\). To summarize, we have two configurations of \( N_0 \) D0-branes: one is point-like and the other is fuzzy \( S^2 \) with the radius \( r_z = w N_0 \) for \( N_0 \to \infty \). Definition of the coordinates \( z_i \) restricts the radius as \( 0 \leq r_z \leq 1 \). Then for the expanding configuration, \( P_\phi \) cannot exceed \( N \):

\[
P_\phi \leq N. \tag{35}
\]

To discuss the problem beyond the expansion eq.(22), we substitute eq.(28) with an arbitrary \( w = w(t) \) into the original Born-Infeld action eq.(19). The resulting action has a simple form

\[
S_{BI} \left( Z^i = w \alpha^i \right) = -T_0 N_0 \int dt \sqrt{1 - H \dot{r}_z^2 - H r_z^2 (1 - r_z^2) \dot{\phi}^2 - H r_z^2 \frac{1}{1 - r_z^2} \dot{r}_z^2} \\
\times \sqrt{1 + \frac{4 H r_z^4 r^4 z_z^4}{\lambda^2 (N_0^2 - 1)}}. \tag{36}
\]

Note that here the only approximation we use is that we take the leading contribution of \((Q^{-1})^i_k = \delta^i_k + \cdots\). Remarkably, if we take \( N_0 \to \infty \) limit, this action precisely coincides with that of a spherical D2-brane with \( N_0 \) D0-branes bound on it. This can be seen by considering a spherical D2-brane of worldvolume \((t, \theta, \psi)\) on which U(1) field strength

\[
F_{\theta \psi} = \frac{N_0}{2} \sin \theta
\]

describing \( N_0 \) D0-branes exists. Here

\[
dz_i dz_i = dr_z^2 + r_z^2 (d\theta^2 + \sin^2 \theta d\psi^2). \tag{38}
\]
Assuming that \( r = r(t) \) and \( r_z(\equiv \sqrt{z_i z_i}) = r_z(t) \), the D2-brane action is \[ S^{D2} = -T_2 \int dt d\theta d\psi e^{-\phi} \sqrt{-\det(P[G_{ab} + \lambda F_{ab}])} + T_2 \int P[C^{(3)}] \]
\[ = -4\pi T_2 \int dt \sqrt{1 - H \dot{r}^2 - H r^2 (1 - \dot{r}_z^2)} \frac{1}{1 - \dot{r}_z^2} \sqrt{r^4 r_z^4 H + \frac{N_0^2 \lambda^2}{4}} + N \int dr_z^2 \dot{\phi} \]
which is equivalent to eq. (36) up to \( 1/N_0^2 \) correction.

3.3 \( p = 2 \)

Next we consider \( N_0 \) D0-branes in the geometry of \( N \) D2-branes. D0-branes couple to the background six-form field strength

\[
F_{\phi z_5 z_6 z_7 z_8 z_9}^{(6)} = -\frac{2\pi}{T_4 V_6} N.
\]

Then from eq. (8), Chern-Simons term is calculated as

\[
\mathcal{L}_{CS} = -\frac{3}{4} N \text{Tr}(Z^i Z^j Z^k Z^l Z^m) \dot{\phi} \epsilon_{ijklm}.
\]

Combining with \( \mathcal{L}_{BI} \) of eq. (22), we have

\[
\mathcal{L} = -N_0 T_0 H^{-\frac{1}{2}} \left\{ 1 - \frac{1}{2} H \dot{r}^2 - \frac{1}{2} H r^2 \left[ 1 - \frac{1}{N_0} \text{Tr}(Z^i Z^i) \right] \dot{\phi}^2 - \frac{1}{2 N_0} H^\frac{1}{2} \text{Tr}(Z^i \dot{Z}^i G_{ij}) \right. \\
- \frac{1}{4 N_0 \lambda^2} H r^4 \text{Tr}([Z^i, Z^i]^2) \right\} - \frac{3}{4} N \text{Tr}(Z^i \dot{Z}^i Z^k \dot{Z}^l Z^m) \dot{\phi} \epsilon_{ijklm}
\]

where \( H = 1 + \frac{k}{r^4} \) with \( k = 3N \lambda^2 / 2T_0 \). If we assume \( Z^i = 0 \), then the motion of \( \phi \) and \( r \) is determined from eq. (12). As in the previous subsection, we expect to find a constant non-commutative \( Z^i \neq 0 \) which solves equations of motion and has the same motion as for the commutative solution. In this case, we cannot obtain such a solution even in the near horizon limit.

On the other hand, we can find a solution with \( Z^i = Z^i(t) \) though such a configuration is not related to the point-like configuration. In particular, we will concentrate on a configuration which is related to a spherical D4-brane of radius \( r_z = r_z(t) \). For this aim, we need to represent a fuzzy four-sphere of radius \( r_z \) by \( Z^i \):

\[
r_z^2 = \frac{1}{N_0} \text{Tr}(Z^i Z^i).
\]

This has been constructed in the context of longitudinal 5-brane of Matrix theory \[20\] as in ref. \[21, 22\]. According to ref. \[21\], fuzzy \( S^4 \) is represented by \( N_0 \times N_0 \) matrices \( G_i^{(n)} \).
\(i = 1, \cdots, 5\) that are given by \(n\)-fold symmetric product of \(\text{SO}(5)\) 4 \(\times\) 4 gamma matrices \(\gamma^i\):

\[
G_i^{(n)} = (\gamma_i \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \gamma_i \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \gamma_i)_{\text{sym}}.
\] (44)

Here \(\text{sym}\) denotes that the tensor product space is restricted to be completely symmetric and the dimension of the matrices \(G_i^{(n)}\) is \((n + 3)!/3!n! = (n + 1)(n + 2)(n + 3)/6\). This means that the number of D0-branes is restricted to \(N_0 = (n + 1)(n + 2)(n + 3)/6\). Let us take

\[
Z^i = w G_i^{(n)}.
\] (45)

Then from the property of \(G_i^{(n)}\), we find that \(\Phi\)

\[
r_z^2 \equiv \sum_i (Z^i)^2 = w^2 n^2 \left(1 + \frac{4}{n}\right) 1_{N_0}.
\] (46)

This means that the radius of non-commutative \(S^4\) measured by \(Z^i\) is \(r_z = wn\) for large \(n\). There are some useful relations:

\[
\epsilon^{ijkln} Z^i Z^j Z^k Z^l = (8n + 16)w^3 Z^m,
\] (47)

\[
\text{Tr}[Z^i, Z^j]^2 = -16n(n + 4)w^4 N_0,
\] (48)

\[
[[Z^i, Z^j], Z^k] = -16w^2 Z^i.
\] (49)

We take an ansatz

\[
Z^i = w(t) G_i^{(n)}.
\] (50)

Substitution of this in the action eq.(42) yields

\[
\mathcal{L} = -N_0 T_0 H^{-\frac{1}{2}} \left[1 - \frac{1}{2} H \dot{r}_z^2 - \frac{1}{2} H r_z^2 (1 - r_z^2) \dot{\phi}^2 - \frac{1}{2} H \frac{r_z^2}{1 - r_z^2} \dot{r}_z^2 + \frac{2}{3N_0 \lambda^2} H r_z^4 r_z^4 \right] - n N r_z^5 \dot{\phi}.
\] (51)

in the \(N_0(\sim n^3/6) \rightarrow \infty\) limit. By solving equations of motion, we can determine the classical motion of the fuzzy 4-spherical D0-branes.

Next, we will see from the dual D4-brane picture that this system is corresponding to \(n\) coincident spherical D4-branes with \(N_0\) D0-branes on them when \(N_0 \gg 1\). To show this, we put \(n\) coincident spherical D4-branes of radius \(r_z\) in the same background. We transform coordinate from \(\{z_5, \cdots, z_9\}\) to \(\{r_z, \theta_1, \theta_2, \theta_3, \psi\}\) as

\[
\begin{align*}
z_5 &= r_z \cos \theta_1, \quad z_6 = r_z \sin \theta_1 \cos \theta_2, \quad z_7 = r_z \sin \theta_1 \sin \theta_2 \cos \theta_3, \\
z_8 &= r_z \sin \theta_1 \sin \theta_2 \sin \theta_3 \cos \psi, \quad z_9 = r_z \sin \theta_1 \sin \theta_2 \sin \theta_3 \sin \psi.
\end{align*}
\] (52)
We assume that the worldvolume of spherical D4-branes is labeled by \( \{ t, \theta_1, \psi \} \) and the transverse U(\( n \)) scalar fields are all commutative with \( x^1 = x^2 = 0, r = r(t), \phi = \phi(t) \) and \( r_z = r_z(t) \). Background five-form potential \( C^{(5)} \) is written by new coordinates by choosing a gauge:

\[
C^{(5)}_{\phi \theta_1 \theta_2 \theta_3 \psi} = \frac{2\pi N}{5T_4 V_6} r_z^5 \sin^3 \theta_1 \sin^2 \theta_2 \sin \theta_3.
\] (53)

The Chern-Simons action for \( n \) coincident (anti) D4-branes is

\[
S_{CS}^{nD4} = -nT_4 \int dt \left( \frac{2\pi N}{5T_4 V_6} r_z^5 \right) \int S^4 d\theta_1 d\theta_2 d\theta_3 d\psi \dot{\phi} \sin^3 \theta_1 \sin^2 \theta_2 \sin \theta_3
\]

\[
= -nN \int dt \dot{\phi} r_z^5.
\] (54)

Next we consider Born-Infeld action. The self-dual field strength \( F_{ab} \) satisfying

\[
F = \ast_4 F, \quad \frac{1}{8\pi^2} \int S^4 \text{Tr}_n F \wedge F = N_0
\] (55)

is on the D4-branes. This describes that \( N_0 \) D0-branes are bound on D4-branes. The normalization is confirmed by the fact that the coupling \( C^{(0)} \wedge F \wedge F \) in \( S_{CS} \) reproduces the coupling to \( N_0 \) D0-branes as

\[
T_4 \int_{S^4 \times \{ t \}} \frac{\chi^2}{2} \text{Tr}_n (C^{(0)} \wedge F \wedge F) = T_4 N_0 \int_{\{ t \}} C^{(0)}.
\] (56)

Then the action is

\[
S_{BI}^{nD4} = -T_4 \int dt \int S^4 e^{-\phi} \text{Tr}_n \sqrt{\det(P[G_{ab} + \lambda F_{ab}])}
\]

\[
= -T_4 \int dt \frac{H^{\frac{1}{2}}}{H r_z} \sqrt{1 - H r_z^2 - H \frac{r^2}{1 - r_z^2} r_z^2 - H (1 - r_z^2) r^2 \dot{\phi}^2}
\]

\[
\times \int S^4 \epsilon_4 \left( nH r_z^4 + \frac{\chi^2}{4} \text{Tr}_n F_{ab} F^{ab} \right)
\]

\[
= -T_4 \int dt \frac{H^{\frac{1}{2}}}{H r_z} \sqrt{1 - H r_z^2 - H \frac{r^2}{1 - r_z^2} r_z^2 - H (1 - r_z^2) r^2 \dot{\phi}^2}
\]

\[
\times \left( nV_4 r_z^4 + 4\pi^2 N_0 \lambda^2 \right)
\] (57)

where \( \epsilon_4 = \sin^3 \theta_1 \sin^2 \theta_2 \sin \theta_3 \) is volume element of \( S^4 \). If there is no D0-brane on the D4-branes, it is known that there exist two degenerate configurations \( r_z = 0 \) and \( r_z^3 = P_\phi / N \) for a constant \( P_\phi \). Existence of D0-brane charge makes the expanding configuration unstable. Note that this property is different from the case of a spherical D2-brane in the D4-brane background where the D0-brane charge on a spherical D2-brane does not affect the stability of the radius \( r_z \) of the expanding configuration.

\footnote{We take anti D4-branes to obtain the same sign as eq.(51).}
At this point, we see that there is a non-trivial correspondence between this picture from spherical D4-branes and that from non-commutative D0-branes. Expansion of $S_{BI}^{nD4} + S_{CS}^{nD4}$ includes all terms of D0-brane action eq.(51) and the coefficients all agree with each other. Missing terms in eq.(51) would be obtained if we consider beyond the leading contribution of $[Z^i, Z^j]$ in $det(Q_{ij})$. This property supports the fact that a fuzzy 4-sphere configuration of D0-branes in the D2-brane background certainly represents a bound state of $n$ coincident spherical D4-branes and $N_0$ D0-branes on them.

4 Summary and Discussion

We have investigated the non-abelian worldvolume effective action of coincident D0-branes in general backgrounds. In particular, we considered the action in the D$p$-brane geometry with $p = 2$ or $p = 4$. For such a spacetime, we have shown that coupling of $N_0$ D0-branes with background $(8-p)$-form field strength in the Chern-Simons action can be written explicitly if we choose a particular coordinate system. By using the expression, the non-abelian action of D0-branes can be analyzed as a microscopic theory. Especially for $p = 4$ or $p = 2$, it was shown that fuzzy $(6-p)$-sphere configuration is certainly regarded as a bound state of $D(6-p)$-branes and D0-branes.

Moreover, if we consider near horizon geometry of $p = 4$, we explicitly solved the equations of motion of the action written in particular coordinates and have found two solutions that have same properties, e.g., momentum and charge. One is point-like and the other is expanding into a form of a spherical D2-brane. It seems that the two degenerate configurations of D0-branes may be BPS states preserving some of the supersymmetries. The precise relation between the non-abelian D0-branes with $X^i = w\alpha^i$ and the dual spherical D2-brane on which U(1) field strength lives has been clarified. Since there are two degenerate configurations with the same quantum numbers in the system, it is reminiscent of giant gravitons [3, 23, 24]. On the other hand, for $p = 2$, we could not find the corresponding structure of expanding brane configurations as in $p = 4$. This may be related to the fact that no supersymmetries remain if $N_0$ D0-branes and $N$ D2-branes both exist.

Now we briefly discuss the relation between expanding brane configurations via Myers effect and 11-dimensional giant gravitons. Giant graviton (or dual giant graviton) in $AdS_m \times S^n$ with $(m, n) = (7, 4)$ or $(4, 7)$ is known as a spherical $M(n-2)$-brane (or $M(m-2)$-brane) shaped object expanding into $S^{n-2} \subset S^n$ (or $S^{m-2} \subset S^m$) with non-zero angular momentum $P_{\phi}$ along $S^n$ [3, 23, 24]. We cannot obtain a non-singular 10-dimensional configuration by explicit compactification of these systems. However, as was discussed
in ref. [4, 25], there are some D0-brane systems which are considered as ten-dimensional counterparts of giant gravitons in a sense that each framework resembles each other in the mechanism of expansion of D0-branes or gravitons. For example, compactification of the system of dual giant graviton expanding into spherical M2-brane in $AdS_4 \times S^7$ along $\phi$ direction corresponds to dielectric D0-branes expanding into spherical D2-brane under the RR field $F_{0123}^{(4)} \neq 0$. 11-dimensional momentum $P_\phi$ along $\phi$ changes into D0-branes by the relation $P_\phi = N_0 / g \kappa$. Similarly, 10-dimensional counterpart of giant graviton expanding into $S^2(\subset S^4)$ in $AdS_7 \times S^4$ is $N_0$ D0-branes expanding into spherical D2-brane by the background NS-NS field strength $H_{789}$. This relation is obtained when we presumably set that the 11-dimensional direction is along $\phi$. We can relate this giant graviton system to that of D0-branes expanding via magnetic moment effect under $F_{6789}^{(4)} \neq 0$. To see this, we first transform coordinates of $AdS_7$ by using a particular isometry of the Anti de Sitter space. With the transformation of time coordinate $t \to t'$, angular momentum changes as $P_\phi \to P_\phi' + P_{\psi'}$ for some of the new coordinates $\psi' \in AdS_7$. Compactification of the system along $\psi'$ indicates the configuration of D0-branes expanding via magnetic moment effect since $P_\phi'$ remains to be angular momentum and $P_{\psi'}$ changes to D0-brane charges.

Next, consider (dual) giant gravitons expanding into spherical M5-branes. We expect that there is a similar relation between these configurations and expanding D0-brane configurations in 10-dimensions. If there exist corresponding configurations in 10-dimensions, coincident D0-branes must couple to $\ast H^{(3)}_{\muijklmn}$ and $F^{(6)}_{ijklmn}$ by

$$\text{Tr}(\ast H^{(3)}_{\muijklmn} \dot{X}^\mu \dot{X}^i \dot{X}^j \dot{X}^k \dot{X}^l \dot{X}^m \dot{X}^n) \quad (58)$$

and

$$\text{Tr}(F^{(6)}_{ijklmn} X^i X^j X^k X^l X^m X^n) \quad (59)$$

respectively. We see that the action eqs. (1) and (3) does not have corresponding terms. They may appear in the non-perturbative action of D0-branes. In practice, it was discussed in ref. [25] that similar terms can be considered at strong coupling. Note that D0-branes would expand into spherical NS5-branes and not into D4-branes by these couplings.

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