Book Embeddings of \( k \)-Map Graphs *

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Abstract. A map is a partition of the sphere into regions that are labeled as countries or holes. A map graph has the countries of a map as its vertices and there is an edge if and only if the countries are adjacent and meet in at least one point. For a \( k \)-map graph, at most \( k \) countries meet in a point. A graph is \( k \)-planar if it can be drawn in the plane with at most \( k \) crossings per edge.

A \( p \)-page book embedding of a graph is a linear ordering of the vertices and an embedding of the edges to \( p \) pages, such that there is no conflict in any page, that is any two embedded edges do not twist or cross. The book thickness of a graph is the minimum number of pages in all book embeddings.

We show that any \( k \)-map graph with \( n \) vertices admits a book embedding in \( 6\lfloor k/2 \rfloor + 5 \) pages, that can be computed in \( O(nk) \) time from its map. On the other hand, there are \( k \)-map graphs that need \( \lfloor 3k/4 \rfloor \) pages. In passing, we obtain an improved upper bound of eleven pages for 1-planar graphs and of 17 pages for optimal 2-planar graphs.

1 Introduction

A \( p \)-page book embedding of a graph consists of a linear ordering of the vertices, which is defined by placing them from left to right, and an embedding of the edges in \( p \) pages, such that there is no conflict in any page. For two vertices \( u \) and \( v \), let \( u < v \) if \( u \) precedes \( v \) in the linear ordering and let \( u \leq v \) if \( u < v \) or \( u = v \). If \( u \leq w \), then two edges \((u, v)\) and \((x, y)\) twist or cross if \( u < x < v < y \). They nest if \( u \leq x < y \leq v \) and are disjoint if \( u < v \leq x < y \). There is a conflict in a page if any two edges twist that are embedded in the page. For sets of vertices \( U \) and \( W \) let \( U < W \) if \( u < w \) for all \( u \in U \) and \( w \in W \). An interval \([u, w]\) consists of all vertices \( v \) with \( u \leq v \leq w \). Vertex \( v \) is outside the interval if \( v \leq u \) or \( v \geq w \). Thus the vertices on the boundary are both in and outside. Obviously, two edges do not twist if there is an interval such that both vertices of one of them are in and the vertices of the other edge are outside the interval.

If \( U \) is a set of vertices, then let \( [U] \) be the interval that contains exactly the vertices of \( U \). For \( w < U \) let \([w, U]\) denote the interval \( w \leq v \leq u \) for \( u \in U \). An interval \([U, w]\) is defined accordingly.

The book thickness of a graph \( G \) is the minimum number of pages in all book embeddings of \( G \). Book thickness is also known as stacknumber or pagenumber \([19, 23]\).

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The book thickness of \( n \)-vertex graphs with \( m \) edges is at most \( \sqrt{m} \) and at least \( \lceil \frac{m-2}{n} \rceil \). The complete graph \( K_n \) has book thickness \( \lceil n/2 \rceil \). Every nondiscrete outerplanar graph has book thickness one. A graph has book thickness at most two if and only if it is a subgraph of a planar graph with a Hamiltonian cycle. Every planar graph has book thickness at most four. The upper bound has been shown by Yannakakis by a linear time algorithm that constructs a 4-page book embedding of a planar graph. Recently, Bekos et al. and Yannakakis have shown that some planar graphs need four pages, so that the bound is tight.

There are several approaches to extend the planar graphs, for example by drawings on surfaces of higher genus, forbidden minors, drawings in the plane with restrictions on crossings, or generalized adjacency relations. Graphs with bounded genus have constant book thickness. Also minor-closed graphs, e.g., graphs with constant tree-width, have constant book thickness.

A graph is \((g,k)\)-planar if it can be drawn on a surface of Euler genus at most \( g \) with at most \( k \) crossings per edge. Clearly, \((0,0)\)-planar graphs are the planar graphs and \((0,k)\)-planar graphs are known as \( k \)-planar graphs. An \( n \)-vertex \((g,k)\)-planar graph with fixed \( g \) and \( k \) has book thickness \( O(\log n) \). For \( k \)-planar graphs, this improves the \( O(\sqrt{n}) \) bound from \( [25] \) to \( O(\log n) \). Bekos et al. have shown that the book thickness of 1-planar graphs is constant and that 39 pages suffice.

Recently, Bekos et al. have introduced \( k \)-framed graphs that consist of a planar graph with faces of degree at most \( k \), such that there are crossed edges in the interior of each face. They allow crossed multi-edges but no multi-edges in the frame. The latter admit smaller faces, for example for 1-planar graphs. A framed multigraph is maximal if every face \( f \) of degree \( k \) induces a \( k \)-clique by the edges in the boundary and the crossed edges in the interior of \( f \). Bekos et al. state a bound of \( 6\lceil k/2 \rceil + 5 \) pages for any \( k \)-framed graph, which shall be raised to \( 6\lceil k/2 \rceil + 7 \) to correct an error in the earlier versions.

A \( k \)-map is a partition of the sphere into disc homeomorph regions that are labeled as countries or holes, such that at most \( k \) countries meet in a point. It generalizes planar duality by holes and an adjacency in a point. The latter admits large cliques. A \( k \)-map defines a \( k \)-map graph with the countries as vertices and an edge if and only if two countries meet in at least one point. Map graphs have been introduced by Chen et al., who have shown that any \( n \)-vertex \( d \)-map graph has at most \( k(n-2) \) edges and admits a clique of size \( \lceil 3k/2 \rceil \). If \( k \) is small, then \( k \)-map graphs are related to \( k \)-planar graphs. Chen et al. have observed that the 2- and 3-map graphs are the planar graphs. The 4-map graphs are the kite-augmented 1-planar graphs and the 5-map graphs are the clique-augmented 2-fan-crossing graphs. A graph is kite-augmented 1-planar if it has a drawing such that every edge is crossed at most once and a pair of crossed edges induces a \( K_4 \). A graph is clique-augmented 2-fan-crossing if every edge is crossed at most twice. Moreover, if an edge is crossed by two edges, then the crossing edges are incident to a common vertex, such that there
is a $K_5$ induced by the vertices of the edges involved in the crossings. Bekos et al. [5–7] have shown that any $k$-map graph is a subgraph of a $2k$-framed graph. Hence, one obtains an upper bound of $6k + 7$ for the book thickness of $k$-map graphs by their approach. Map graphs are simple, but their representations allow multi-edges. We show that a graph is the simplification of a maximal $k$-framed multigraph if and only if it is a $k$-map graph. Hence, $k$-framed multigraphs and $k$-map graphs have the same book thickness, since multi-edges don’t matter for book embeddings.

Our contribution. We establish improved upper and lower bounds on the book thickness of $k$-map graphs. The lower bound of $\lceil k/2 \rceil$ is raised to $\lfloor 3k/4 \rfloor$ using larger cliques. For the upper bound, we first show that any $k$-map graph is a $k$-framed multigraph consisting of a planar multigraph with faces of degree at most $k$. Then we use a modification of Yannakakis’ algorithm for the embedding of planar graphs. We introduce block-expansions as a new method for the computation of the vertex ordering. The embedding of edges is done in two phases. For any 2-level framed multigraph, first, only the edges of the outer cycle, the inner edges, and all edges that are incident to the first outer vertex of a face are embedded in three pages. The remaining edges of any face of degree $k$ are embedded in a set of at most $\lfloor k/2 \rfloor$ pages. There is an outerplanar and consistent face-conflict graph, such that two remaining edges from any two faces do not twist if the faces have the same color. As outerplanar graphs are 3-colorable, any 2-level $k$-framed multigraph can be embedded in $3\lfloor k/2 \rfloor + 3$ pages. Twice this number suffices for $k$-framed multigraphs, where one page can be saved as in Yannakakis’ 5-page algorithm [28], so that we obtain an upper bound of $6\lfloor k/2 \rfloor + 5$. In addition, we show that the book thickness of 1-planar graphs is at most eleven.

In the remainder of this paper, we introduce basic notions in Section 2 and establish the relationship between framed multigraphs and map graphs. The book embedding of 2-level framed multigraphs is described in Section 3. We study the composition and applications in Section 4 and state some open problems in Section 5.

2 Preliminaries

We consider undirected multigraphs $G = (V, E)$ with sets of $n \geq 1$ vertices $V$ and edges $E$, some of which have multiple copies. Self-loops are excluded. An undirected edge is denoted by a pair $(u, v)$, since it is later oriented from $u$ to $v$. A graph is $k$-planar ($k \geq 0$) if it admits a drawing (or a topological embedding) such that every edge is crossed at most $k$ times. Clearly, 0-planar graphs are planar. A planar drawing has faces, which are hole-free regions if the planar graph is connected. A face is specified by the set $V(f)$ of vertices in its boundary. If $V(f)$ contains $k \geq 2$ vertices, then $f$ has degree $k$ and is called a $k$-face. For convenience, we use (mixed) sets of vertices and edges to specify a face, e.g., a vertex $a$ and an edge $(b, c)$ for a triangle $(a, b, c)$. 

A multi-edge between two vertices consists of several copies of an edge, one of which is the original or 0-copy. All but one copy is removed if a multigraph is simplified. A simplified multigraph is a (topologically) simple graph. Multi-edges shall be non-redundant, such that there is no 2-face with two copies of an edge as its boundary. Hence, there are vertices in the interior and the exterior of a 2-cycle formed by two copies of an edge. For convenience, we shall not distinguish between a planar multigraph and a planar drawing (embedding), such that we speak of vertices, edges, and faces of a multigraph.

![Fig. 1](image)

**Fig. 1.** (a) A 4-framed multigraph, that is a 1-planar graph consisting of three W-configurations. The graph is a 7-framed graph if two copies of edges (drawn blue and dashed) and the crossed edges are removed from the inner face. (b) The 1-planar crossed cube. (c) The 5-framed crossed dodecahedron graph.

There is a close relationship between \( k \)-map graphs and \( k \)-framed graphs, that consist of a planar frame with faces of degree at most \( k \) and of a set of crossed edges in the interior of each face. Bekos et al. [5–7] have shown that any \( k \)-map graph is a subgraph of a maximal 2\( k \)-framed graph. We show that multi-edges help to obtain smaller faces. Multiple adjacencies between countries are natural for \( k \)-maps. Chen et al. [12] have shown that any \( k \)-map graph admits a representation by a planar graph [12]. Let \( W = (V, P, L) \) be a planar bipartite graph, whose first set is in one-to-one correspondence to the set of vertices of a graph \( G \). Each vertex of the second set \( P \) is called a point, that is used to establish edges. Set \( L \) consists of 2-sets \( \{v, p\} \) with \( v \in V \) and \( p \in P \), called a link. The half-square of \( W \) is a graph \( G = H^2(W) \) with vertex set \( V \) such that there is an edge \((u, v)\) in \( G \) if and only if there is a point \( p \) and links \((u, p)\) and \((v, p)\) in \( W \). Then \( W \) is called a witness of \( G \). Graph \( W \) is a \( k \)-witness if any point has degree at most \( k \). In particular, there are points of degree two, called 2-points. If \( p \) is a 2-point of \( W \) with links \((u, p)\) and \((v, p)\), then \( p \) subdivides edge \((u, v)\) of \( G \). Conversely, there is an edge contraction at a 2-point in the half-square.

Chen et al. [12] have shown that there is a one-to-one correspondence between points in maps and witnesses, that leads to the following characterization.
Proposition 1. A graph $G$ is a $k$-map graph if and only if $G = H^2(W)$ for a planar $k$-witness $W$

We wish to normalize any $k$-map graph, similar to a triangulation of a planar graph. It is often easier to work with normalized graphs than with general ones. We do so by adding multi-edges that shall be uncrossed in a drawing. Note that an edge $(u, v)$ of a $k$-map graph is uncrossed if the countries for $u$ and $v$ in a map meet in a segment and not just in a point.

A witness $W$ is called planar-maximal if (1) each face in a planar drawing of $W$ is a quadrangle or a hexagon, (2) there is a 2-point $p'$ with links $(u, p')$ and $(p', v)$ if $p$ is a point of degree $d \geq 3$, such that $u$ and $v$ are consecutive neighbors at $p$, and (3) there are no quadrangular faces with two vertices and two 2-points in the boundary. Note that the added 2-points are redundant in the sense of [12], since they define edges that are defined by $p$. The planar skeleton $P(W)$ is the subgraph of a planar-maximal witness, in which all $d$-points for $d \geq 3$ are removed. A $k$-map graph $G$ is planar-maximal if $G = H^2(W)$ for a planar-maximal witness $W$ with $d$-points for $d \leq k$. Its planar skeleton $P(G)$ is $H^2(P(W))$, which is a planar multi-graph with multiple copies of an edge $(u, v)$, one for each 2-path $(u, p, v)$ in the planar skeleton of $W$ consisting of a 2-point and two links. This is relevant for the definition of faces if there are separation pairs, and it extends $k$-framed graphs. Multiple copies of an edge are ignored for the book embedding. The restriction to quadrangles and hexagons implies that $G$ is a hole-free map graph, that is, graph $G$ admits a map without holes [13]. Then $G$ is 2-connected [13]. A point of a witness is redundant if all pairs of its neighbors can also be connected through other points [12]. In particular, there may be many 2-points connecting two vertices $u$ and $v$. Then the size of the set of points of a witness is no longer related to the size of its set of vertices. This resembles the situation of multi-edges in (planar) graphs. We avoid this situation by the exclusion of duplicate 2-points. Two 2-points are duplicates if there is a quadrangular face with two 2-points and two vertices. Chen et al. [12] have shown that a witness without redundant points has at most $3n - 6$ points and $O(kn)$ edges, which was improved to $kn - 2k$ [11], if it has $n$ vertices and points of degree at most $k$. Hence there are $O(kn)$ 2-points if duplicates are excluded. Thus we can assume that a planar-maximal witness for an $n$-vertex $k$-map graph has $O(kn)$ points, and that the (planar-maximal) half-square has $O(kn)$ edges. Note that $k = (n - 1)/2$ if $G$ is an $n$-clique.

There is a normal form for planar-maximal $k$-map graphs, which generalizes the normal form for 1-planar graphs by Alam et al. [1]. It is obtained via its witness, that is augmented similar to a triangulation of a planar graph.

Lemma 1. (i) If $G$ is a planar-maximal $k$-map graph, then there is a planar graph $G'$ with multi-edges and $d$-faces with $d \leq k$, such that $G$ is obtained by expanding each $d$-face to a $d$-clique, and then removing multi-edges.

(ii) For any $k$-map graph $G = (V, E)$ there is a planar-maximal $k$-map supergraph $G' = (V', E')$ with $V' = V$ and $E \subseteq E'$, that can be constructed in linear time.
Clearly, remove duplicate 2-points by merging 2-points in quadrangles with two vertices. multi-edges in the half-square if there are different routes for the 2-paths. Finally, \( u, p, v \) routed close to the 2-path (bors at point \( p \)).

\(|P(G)| = 2 - \text{faces of } G\) and \(|P(G')| = 2d\)-faces of \( P(W) \), since \( P(W) \) does not admit quadrangular faces with two 2-points. For any \( 2d \)-face of \( P(W) \) there is a \( d \)-point that is adjacent to the vertices of the face. Hence, it creates a \( d \)-clique in the \( d \)-face of \( G' \). By assumption, \( k \)-map graphs are simple, such that there are no multi-edges.

(ii) Assume that \( W \) is a witness without redundant points. Then it has \( O(n) \) points \([12]\). A planar-maximal augmentation \( W^+ \) of a witness \( W \) can be constructed in linear time in the size of the set of vertices of \( W \) if there are no duplicate 2-points. For the construction of \( W^+ \), first partition the \( d \)-faces of \( W \) with \( d \geq 8 \) by 2-paths such that only \( d' \)-faces with \( d' \leq 6 \) remain. This operation creates new links for the half-square, including new edges and thereby multiple copies of an edge for \( G \). It generalizes the triangulation of planar graphs and the augmentation of 1-planar graphs to kite-augmented ones \([10]\). Then add a 2-point \( p' \) and links \((u,p')\) and \((p',v)\) if vertices \( u \) and \( v \) are consecutive neighbors at point \( p \) if \( p \) has degree at least three. Links \((u,p')\) and \((p',v)\) can be routed close to the 2-path \((u,p,v)\), such that they are uncrossed. This creates multi-edges in the half-square if there are different routes for the 2-paths. Finally, remove duplicate 2-points by merging 2-points in quadrangles with two vertices. Clearly, \( H^2(W^+) \) is a supergraph of \( H^2(W) = G \). Clearly, all taken steps can be done in linear time in the size \( n \) of the set of vertices of \( W \) if \( W \) has no duplicate 2-points.

\[ \square \]

A \( k \)-framed multigraph \( G \) consists of a frame \( F(G) \) and of sets of crossed edges. The frame is a spanning planar subgraph of \( G \) with nonredundant multi-edges. A face of the frame is a \( d \)-face with \( 3 \leq d \leq k \) that contains a set of crossed edges. The crossed edges are drawn in the interior of the face, see Figure\([1]\). There are also crossed multi-edges if there are copies in different faces. The set of edges \( E(f) \) of face \( f \) consists of the edges in the boundary of \( f \) and the crossed edges in its interior. Any face has a distinguished vertex, called its first outer vertex, that is denoted by \( \alpha(f) \). So \( E(f) \) is partitioned into the set \( E_\alpha(f) \) of edges incident to the first outer vertex and the remainder \( E^-(f) \). If \( f \) is a \( d \)-face and \( E(f) \) induces a \( d \)-clique, then \( E^-(f) \) induces a \((d - 1)\)-clique.

We assume that framed multigraphs are biconnected, since the book thickness of a graph is the maximum book thickness of its biconnected components \([8]\). In addition, we assume that the frame is biconnected, which is useful later on.

**Lemma 2.** For any biconnected \( k \)-framed multigraph \( G \) there is a \( k \)-framed multigraph \( G' \) on the same set of vertices and with the same set of crossed edges in each face, such that the frame of \( G' \) is biconnected and is a planar supergraph of the frame of \( G \).

**Proof.** If the frame of \( G \) is disconnected, then a face \( f \) has a hole with an inner component \( M \). There are crossed edges between vertices in the boundary of \( f \), in the outer face of \( M \), and between vertices of \( f \) and \( M \). If \( M \) consists of a
single vertex, then connect it to an edge in the boundary of $f$ such that there is a triangle. Otherwise, consider an edge $e$ in the outer face of $M$ and an edge $e'$ in the boundary of $f$. Create an internally triangulated quadrangle with $e$ and $e'$ on opposite sides. Then $M$ is biconnected to the component with face $f$. Every crossed edge can be routed in the interior of the new face $f'$, whose boundary consists of the boundary of $f$ and the outer face of $M$. Finally, create a triangle with $v$, its predecessor in one component and its successor in the other component if there is a cutvertex $v$ in the frame of $G$.

For convenience, we assume that framed multigraphs are maximal such that any face of degree $d$ induces a $d$-clique. Then there may be crossed multi-edges in the interior of faces that may not be adjacent. Clearly, the drawing of a framed multigraph can be augmented to a maximal one, first by establishing 2-connectivity of the frame and then by filling the interior of each face such that there is a clique. In addition, we assume that the outer face is a triangle (or that there are no crossed edges in the outer face), which is obtained as before when establishing 2-connectivity in Lemma 2. However, there are no crossed edges incident to the vertices of the outer triangle.

A separation pair $\langle s, t \rangle$ of graph $G$ is such that $G - \{s, t\}$ partitions into at least two connected components. It is an inner separation pair if vertex $t$ (or $s$) is not in the outer face of a given drawing of $G$. A component without vertices in the outer face is called an inner component. In general, there are several inner components that share exactly vertices $s$ and $t$.

Note that the book thickness of a graph is bounded by the book thickness of any augmentation by vertices and edges. Hence, we consider maximal $d$-framed multigraphs for our study of an upper bound on the book thickness.

There is a close relationship between framed multigraphs and map graphs.

**Theorem 1.** Any $k$-map graph is the simplification of a maximal $k$-framed multigraph.

**Proof.** Chen et al. [12] have shown that a graph $G$ is a $k$-map graph if and only if it is the half-square of a $k$-witness $W$ such that $G = H^2(W)$. A witness admits the construction of a frame as follows. Consider a planar drawing of $W$. For any point $p$ of $W$, add a cycle of 2-paths around $p$. A 2-path consist of a point $t$ of degree two and edges $(u, t)$ and $(t, v)$ for vertices $u$ and $v$ that are consecutive at $p$. There is a multi-edge between $u$ and $v$ if there are 2-paths around several points. We assume that there is no face in a drawing of the augmentation of $W$ containing two 2-points (and two vertices), similar to nonredundant multi-edges. By the half-square there is an uncrossed edge in $G$ from every 2-path in $W$. Hence, every $k$-point for $k \geq 3$ is in a face formed by the 2-paths of its neighbors, which is a $k$-face in $F(G)$. It defines a $k$-clique. Every other edge of $G$ is a crossed edge in the interior of a face of $F(G)$, that is the edge is created by the 2-path between two neighbors of a point.

**Lemma 3.** The simplification of any maximal $k$-framed multigraph is a $k$-map graph.
Proof. Construct a $k$-witness $W$ from a maximal $k$-framed multigraph $G$ as follows. First, subdivide every edge of the frame by a point, which it taken as a 2-point of $W$. Then add a $k$-point in each $k$-face and connect it to the vertices in the boundary. This creates a $k$-clique for each face of degree $k$, which is feasible, since $G$ is maximal. Clearly, any edge of the frame is represented in $W$ by the 2-path with the added 2-points, and conversely, and any crossed edge in the interior of a face is represented via the inserted $k$-point, and conversely. As $k$-map graphs are simple, that is $H^2(W)$, we must simplify the given maximal $k$-framed multigraph. \qed

Corollary 1. Any simple subgraph of a $k$-framed multigraph is a subgraph of a $k$-map graph.

Note that a subgraph of a $k$-map graph is not necessarily a $k$-map graph. In fact, the removal of an edge from a 4-map graph may result in a non 4-map graph, as shown by Chen et al. [12]. This fact is due to the need for an augmentation, such as kite-augmented 1-planar [10] and clique-augmented 5-planar graphs [9].

3 Two-Level Graphs

We recall basic notions from [28] and extend them for our needs. Familiarity with Yannakakis approach for 2-level planar graphs will be helpful. Basically, we traverse distinguished sets of blocks by Yannakakis nested method and treat them as a single X-block.

The peeling technique, introduced by Heath [22], has been used in all later approaches on upper bounds for the book thickness of generalized planar graphs [2, 5, 18, 28]. It decomposes a graph into 2-level graphs and computes a leveling of the vertices of a graph, such that there are layered separators [17]. So the computation of a book embedding of a graph is reduced to that of its 2-level subgraphs. The peeling technique generalizes canonically to planar multigraphs.

The vertices in the outer face of a planar drawing are at level zero. Vertices are in level $\ell+1$ if they are in the outer face, when all vertices at levels at most $\ell$ are removed. So there are no edges between vertices in levels $i$ and $j$ if $|i - j| > 1$ if the peeling technique is used, both for a planar multi-graph and a $k$-map or $k$-framed multigraph. In consequence, the book embedding of such graphs can be composed of the book embedding of its 2-level subgraphs at odd and even levels, so that the book thickness of a graph is at most twice the book thickness of its 2-level subgraphs.

3.1 Planar 2-Level Multigraphs

A planar 2-level (multi-) graph is the subgraph induced by a cycle $C$ of level $\ell$ vertices, called outer vertices, and of the level $\ell+1$ vertices in its interior, called inner vertices, see Figure 2. The subgraph $I$ in the interior is composed of blocks. A block is the cycle of outer vertices of a $2$-connected component. It will be the outer cycle at the next level. It may consist of an edge with its vertices or
of a single vertex, which is called an elementary block. Two blocks may share a vertex, which is a cutvertex of \( I \). These vertices are distinguished as the leader of blocks. A connected component of \( I \) is called a block-tree. It is a cactus consisting of blocks with branches at cutvertices. Two block trees are separated by chords between outer vertices or a face that can contain such a chord. For example, the frame of the graph in Figure 1(a) has three block trees, each consisting of a single quadrangle. By Lemma 2 we can assume that planar 2-level multigraphs are biconnected.

Yannakakis [28] has simplified the problem of embedding a planar 2-level graph into a 3-page book by the assumption that the graph is triangulated and that the inner subgraph is connected. Connectivity can be achieved at the expense of planarity. There are outer chords if a planar 2-level graph is triangulated and the inner subgraph is not connected. Then two connected components can be connected by an additional edge, which crosses the outer chords that separate them.

For our book embedding of planar 2-level multigraphs, we follow the block oriented description by Yannakakis [28], see also [2]. The one by Bekos et al. [5] can be regarded as face oriented. The edges of a planar 2-level multigraph are outer edges on the outer cycle \( C \), outer chords between non-consecutive outer vertices in the interior of \( C \), binding edges between inner and outer vertices (in this direction), that are classified into forward and backward binding, and inner edges between two vertices that are consecutive for a block. There are no inner
chords between two non-consecutive inner vertices and no copies of inner edges. Such edges are flipped into the interior of a block and are considered at the next level.

The faces of a planar 2-level multigraph are the faces between $C$ and $I$ in the interior of $C$. The outer face and the faces in the interior of blocks are discarded. Each face contains at least one outer vertex. It may contain one or two outer chords. Faces may contain vertices and edges from blocks in several block-trees. A face has $r$ binding edges, where $r \geq 0$ is even by the alternation between outer and inner vertices.

Each block $B$ has a least vertex $\lambda(B)$, called the leader, which is the cutvertex of $B$ and its parent in $I$ if $B$ is not the root in a block tree. For $B = b_0, \ldots, b_q$ with $q \geq 0$ let $b_0 = \lambda(B)$ and traverse $B$ in ccw-order. A cutvertex may be the leader of several blocks, that are ordered clockwise like the outer cycle. The leader of a block plays a special role, see also [28]. Any inner vertex is in a single block, except if it is a cutvertex or the first vertex of a block tree. The root of a block tree is special. A block tree $T$ has a first face $f_T$, which is the least face containing a vertex of $T$. The least outer vertex in this face is denoted by $\alpha(T)$, and is also called the first outer vertex of $T$. By 2-connectivity, $f_T$ has a last binding edge $(a_0, v_s)$ between a vertex $a_0$ of any block of $T$ and an outer vertex $v_s$. Vertex $v_s$ is searched by a ccw-traversal of $f_T$ from its least outer vertex. Vertex $a_0$ is set to be the leader of the root of $T$ and is called the first vertex of $T$, denoted $\lambda(T)$. Vertex $v_s$ is called the last outer vertex of $T$, denoted $\omega(T)$. Hence, the first outer face $f_T$ of any block-tree $T$ contains the vertices $\alpha(T), \lambda(T)$ and $\omega(T)$. Observe that a face may contain the root of several block-trees and vertices $\alpha(T_i)$ and $\lambda(T_i)$ for $i = 1, \ldots, r$ and $r \geq 0$, that all have the same first outer vertex. However, any face contains the first vertex $\lambda(T)$ of at most one block-tree, since there is the edge $(\lambda(T), \omega(T))$ and another binding edge between a vertex of $T$ and an outer vertex by biconnectivity, so that there is a closed curve through $\omega(T)$ and $\lambda(T)$, that separated $T$ from the remainder of $H$. A face may contain the first and last outer vertex of several block-trees, since we allow multi-edges.

Any inner vertex is in a single block except if its is a cutvertex or the first vertex of a block-free. For uniqueness, we assign each vertex to the block that is closest to the root in its block tree, and we denote the set of vertices assigned to $B$ by $V(B)$, see [28]. Hence, $b_0 \notin V(B)$ if $B = b_0, b_1, \ldots, b_q$ with $q \geq 0$, in general. Vertices $b_1$ and the first and $b_q$ the last vertex of $B$ and edges $(b_0, b_1)$ and $(b_0, b_q)$ are the first and the last edge of $B$, respectively. The least outer vertex in the face containing the last edge $(b_0, b_q)$ of block $B$ is called the dominator of $B$, denoted $\alpha(B)$ if $B$ has at least two vertices. Vertex $\alpha(B)$ sees $B$ according to [28]. Similarly, there is a last outer vertex $\omega(B)$, which is the least outer vertex in the face containing the first edge $(b_0, b_1)$. Note that there may be edges $(b_0, v)$ with outer vertices $v$ such that $v < \alpha(B)$ and $v > \omega(B)$, respectively. If $B$ is the root of block-tree $T$, then its first outer vertex $\alpha(T)$ does not necessarily see $B$. Then the first vertex of $T$, that is $\lambda(T)$, is an elementary block and the root of $T$. Recall that $\lambda(T)$ is connected to the last outer vertex of $T$ by an edge.
Yannakakis [28] assumes that $\alpha(B) < V(B) < \omega(B)$ for any block $B$. We need a generalization, since there are inner separation pairs and multi-edges. We say that block $B$ is covered by the outer vertex $v$ if $v = \alpha(B) = \omega(B)$. By biconnectivity, if there is a binding edge between a vertex of $B$ and an outer vertex if $B$ is an extreme block of $T$, that is it has no parent. Hence, any binding edge is incident to $v$ if $B$ is covered by $B$ and the binding is incident to a vertex of $V(B)$. In consequence, there is a close relationship between covered blocks and separation pairs.

The following observation by Yannakakis [28], also stated in [2], describes the structure of the inner subgraph.

**Lemma 4.** Let $H$ be a 2-connected planar 2-level multigraph with outer cycle $C = v_0, \ldots, v_t$. Then the following hold.

(i) Any block $B$ has a first and a last outer vertex $\alpha(B)$ and $\omega(B)$, such that $\alpha(B) \leq \omega(B)$.

(ii) Block $B$ is in an inner component at an inner separation pair $\langle \lambda(B), \omega(B) \rangle$ if and only if $\alpha(B) = \omega(B)$, that is $B$ is covered by $v$.

(iii) If $B$ is an uncovered block with leader $b_0$, $\alpha(B) = v_i$, $\omega(B) = v_j$ and $v_i \neq v_j$, then $H - \{v_i, b_0, v_j\}$ partitions into a right part $H_1$ and a left part $H_2$, such that $H_2$ contains the vertices $v_{i+1}, \ldots, v_{j-1}$ and the vertices of $B$. $H_1$ is the other part. $H - \{v_i, v_j\}$ partitions similarly with the vertices $v_{i+1}, \ldots, v_{j-1}$ in the left part $H_2$ if $(v_i, v_j)$ is an outer chord.

**Proof.** Every face has an outer vertex and thus a first and last outer vertex. The vertices $\alpha(B)$ and $\omega(B)$ of block $B$ are in the face containing the last and the first edge of $B$, respectively, if $B$ is nonelementary. We have $\alpha(B) \leq \omega(B)$ since the outer cycle and blocks are traversed in opposite directions. If $B$ is elementary, then $\alpha(B)$ and $\omega(B)$ are taken from the first face of the block-tree containing $B$, such that $\alpha(B) < \omega(B)$.
For (ii), if $v = \alpha(B) = \omega(B)$ for some block $B$, then there is a single outer vertex that can see all vertices and edges of $B$. There are faces with $v$ and the first and last edge of $B$, respectively. Hence, $(\lambda(B), v)$ is a separation pair. Conversely, $v = \alpha(B) = \omega(B)$ if $(b, v)$ is an inner separation pair such that $b$ is the leader of $B$ and $B$ is in an inner component.

For (iii), if $B$ is not the root of a block tree, then its leader $b_0$ is a cutvertex of the inner subgraph that is partitioned by the removal of $b_0$. Similarly, the outer cycle $C$ is partitioned by the removal of $v_i = \alpha(B)$ and $v_j = \omega(B)$. There is a curve $\Gamma$ from $v_i$ via $b_0$ to $v_j$ that partitions the planar drawing of $H$. The curve first follows a binding edge incident to $v_i$, which must exist since $v_i = \alpha(B)$. Then it goes along the boundary of the face containing $v_i$ and $b_0$, which is the boundary of blocks. It passes $b_0$ and then follows the boundary of the face containing $b_0$ and $v_j$. The boundary consists of edges from blocks and a final binding edge incident to $v_j$, which exists, since $v_j = \omega(B)$. We close $\Gamma$ in the outer face. There is a shortcut for $\Gamma$ using edges $(b_0, v_i)$ and $(b_0, v_j)$ that can be added uncrossed in the respective faces of $H$. Now part $H_2$ is in the interior of $\Gamma$ and $H_1$ is outside. Similarly, there is a partition of $H - \{v_i, v_j\}$ if $B$ is a root of a block tree or if $(v_i, v_j)$ is an outer chord, which proves (iii).

We say that a partition as in Lemma 4(iii) is induced by an uncovered block $B$ or an outer chord $(v, v_j)$. An inner separation pair $(b, v)$ is maximal if the inner vertex $b$ is in a block that is not covered by $v$. Then there is no inner vertex $a$ such that $(a, v)$ is a separation pair whose inner components contain the inner components of $(b, v)$. The inner components form a block-subtree with leader $b$. The set of blocks of the inner components is a maximal inner separation pair is called a super-block and is denoted by $B^+$. The cutvertex $b$ is the leader of $B^+$ and is not assigned to it, similar to blocks. We order the blocks either clockwise or counterclockwise at $b$, depending on the later use. Then the boundary of $B^+$ is traversed in clockwise or counterclockwise order. The first and last edge of $B^+$ are defined accordingly. Clearly, any block of $B^+$ is covered by $v$, so that $v = \alpha(B^+) = \omega(B^+)$, that is any super-block is covered. Also, any binding edge incident to a vertex of $B^+$ is incident to $v$. There is not necessarily a binding edge $(c, v)$ if $c$ is a cut-vertex in $B^+$ or $c = b$. However, there is a binding multi-edge $(c, v)$ if faces are triangulated from $v$.

### 3.2 Vertex Ordering

By the peeling technique, the vertex ordering or linear layout of a planar multi-graph $G$ is composed of the vertex ordering of its 2-level subgraphs. Yannakakis [28] has proposed two methods for the traversal of blocks of a planar 2-level graph $H$. Suppose the outer cycle is traversed in cw-order. In the consecutive method, any block is traversed individually in counterclockwise order from its leader, such that its vertices are consecutive at this moment, except for the cutvertex, that is in the parent block, in general. Blocks in the same block-tree and with the same dominator are visited in cw-order, whereas block-trees are visited in ccw-order. In the nested method, the set of uncovered blocks of a block-tree with
the same dominator is traversed by depth-first search [14]. Each vertex is listed exactly once at its first appearance. The traversed blocks form a block-subtree. The nested method partitions the set of vertices of a block into many segments between two cutvertices for children, such that each segment can be assigned to an interval in the vertex ordering. In particular, there is an interval for the vertices of any block-subtree.

We use the consecutive method at uncovered blocks and super-blocks, as it admits a simpler description, whereas the nested method must be used at block-expansions, as it treats a set of blocks like a single one. If $B$ is an uncovered block with dominator $v_i$, then traverse $B$ in ccw-order from its leader and place it to the right of $v_i$. Blocks of the same block-tree are ordered clockwise at $v_i$ if they are dominated by $v_i$. If $B^+$ is a super-block at a maximal inner separation pair $\langle a_i, v \rangle$, such that $v < \omega(T)$, use copies of $v$ after $v$ such that each block of $B^+$ is dominated by a copy of its own. Then lay out the vertices of the blocks of $B^+$ as before and remove the copies of $v$ (or keep them as placeholders, which are isolated vertices).

The block-expansion of an uncovered block or super-block $A$ at its vertex $a_i$ by a super-block $B^+$ is obtained traversing the boundary of $B^+$ in postorder [14], that is any block is traversed in ccw-order, blocks with the same cutvertex are visited in ccw-order, and the cutvertex is listed last, after the vertices of the blocks of the block-subtree. The obtained sequence of vertices is inserted right before $a_i$. The block-expansion of $A$ is obtained by expanding it at any of its vertices by super-blocks that are covered by the last outer vertex $\omega(T)$, and is denoted by $A^*$. The block-expansion of a planar 2-level multigraph $H$ is obtained by expanding all uncovered blocks in all block-trees.

The boundary of an expanded block no longer a simple cycle, which does not matter for our further investigations. Note that there are one or two edges incident to $a_i$ from the first block of any inner component at $\langle a_i, v \rangle$, such that there are several edges between $a_i$ and vertices of $B^+$. In fact, there is a similarity between a block-expansion and an elementary root. We treat an expanded block like an ordinary one and let $\alpha(A^*) = \alpha(A), \omega(A^*) = \omega(A)$ and $\lambda(A^*) = \lambda(A)$, where $\omega(A) = \omega(T)$. We use block-expansions to capture the case of “small faces” in [5–7]. It leads to a new vertex ordering.

For the vertex ordering $L(H)$ of a planar 2-level multi-graph $H$, we first compute all super-blocks and all block-expansions. Blocks that are contained in any super-block $B^+$ or block-expansion $A^*$ are discarded for a moment. The remaining blocks, super-blocks, and block-expansions, are called $X$-blocks. Now we use Yannakakis [28] consecutive method for X-blocks. We obtain the vertex ordering as in [28] with the consecutive method if there are no covered blocks. As a remainder, choose a vertex $v_0$ in the outer face of $G$, which is set to be the least outer vertex. Then traverse the outer cycle $C = v_0, \ldots, v_t$ from $v_0$ in clockwise order (cw-order), such that the vertex ordering is $v_0 < \ldots < v_t$. Blocks and expanded blocks are traversed counterclockwise (ccw-order). The roles of cw-order and ccw-order switch form level to level. By induction on the levels, let $C$ be the outer cycle of a planar 2-level multigraph $H$. For any outer vertex $v_i$,
place the X-blocks dominated by \( v_i \) just right of \( v_i \), where X-blocks from the same block-tree are ordered clockwise and X-blocks from different block-trees are ordered counterclockwise at \( v_i \). In addition, if the dominator of a block is the leader of the outer cycle, that is \( v_0 \), then place the vertices immediately to the left of \( v_1 \), as in [25].

The ordering of \( C \) implies an ordering for the inner vertices, blocks, X-blocks, block-trees, outer chords, and faces of \( H \) and an orientation of the edges according to the ordering of its vertices. Each face \( f \) has outer vertices and thus a first outer vertex \( \alpha(f) \) and a last outer vertex \( \omega(f) \), which are the least and last outer vertex in the boundary of \( f \). Clearly, \( \alpha(f) = \omega(f) \) is possible. Faces are ordered according to their first outer vertex and in ccw-order if faces have the same first outer vertex. For the computation of the vertex ordering, a triangulation of each face from its first outer vertex may be helpful, as the ordering of X-blocks that are dominates by \( v_i \) coincides with the ordering of the incident edges and triangulation edges, where there may be more multi-edges, for example, if there are inner separation pairs. As an example, consider Figure 2.

**Lemma 5.** Let \( H \) be a 2-connected planar 2-level multigraph with outer cycle \( C = v_0, \ldots, v_i \). If \( H \) is partitioned into parts \( H_1 \) and \( H_2 \) induced by an uncovered X-block \( B \) with \( v_i = \alpha(B) \neq \omega(B) = v_j \) or by an outer chord \( (v_i, v_j) \), as described in Lemma 4, then the vertex ordering satisfies \( V_i < v_i < U < V_2 < v_j < V_r \), where \( V_2 \) is the set of vertices of part \( H_2 \), \( U \) is the set of vertices of X-blocks dominated by \( v_i \) in part \( H_1 \), \( V_i \) is the set of vertices of X-blocks dominated by vertices \( v < v_i \) and \( V_r \) is the set of vertices of X-blocks dominated by vertices \( v \geq v_j \).

If \( B \) is a covered X-block, then it is a super-block that is covered by some outer vertex \( v_i \) if \( v_i < \omega(T) \). Now \( L(H) \) satisfies \( V_i < v_i < U < V_2 < V_r \), where \( V_i, U \) and \( V_r \) are as before, and \( V_2 \) is the set of vertices of \( B \).

**Proof.** The statement extends Lemmas 1 and 2 in [25], which prove the partition of the set of vertices and the vertex ordering in case of a connected inner subgraph and no block expansions, see also [2].

We proceed by induction on the dominators of X-blocks and the first vertex of an outer chord. All outer vertices \( v < v_i \) and the X-blocks dominated by \( v \) precede \( v_i \) in the vertex ordering. This includes vertices from blocks that are merged into another block by a block-expansion. Similarly, all outer vertices \( v' > v_j \) and the X-blocks dominated by \( v' \) succeed \( v_j \). So vertices of blocks that are covered by the last outer vertex of a block-tree may move to \( V_i \). Consider the X-blocks dominated by \( v_i \). If \( B_1 \) and \( B_2 \) are dominated by \( v_i \) such that \( B_1 \in H_1 \) and \( B_2 \in H_2 \), then \( B_1 \) precedes \( B_2 \) in the vertex ordering, since \( B_1 \) is in the right and \( B_2 \) is in the left part. In fact, if \( B_1 \) and \( B_2 \) are in the same block tree, then there is a path from \( B_2 \) to \( B_1 \) in their block tree, as shown by Yannakakis [25] if there are no blocks that are covered by \( v_i \). If \( B_1 \) and \( B_2 \) are in different block trees, then the block trees are separated by an outer chord or a face that can contain an outer chord in its interior. Now vertex \( v_i \) dominates the root of the block tree \( T_2 \) containing \( B_2 \), such that \( v_i < w \) for any vertex \( w \) in \( T_2 \). The block
tree $T_1$ containing $B_1$ precedes $T_2$, such that blocks from $T_1$ precede those from $T_2$ if they are dominated by $v_j$. Hence, $V_1 < v_i < U < V_2 < v_j$, where $V_2 < v_j$ is clear from [28]. The case with an outer chord $(v_i, v_j)$ is similar. Any block $B$ that is dominated by the last outer vertex $\omega(T)$ of its block-tree $T$ is covered by $\omega(T)$, since $\omega(B) \leq \omega(T)$. It is a priori merged into an extended block. Hence, $\omega(T)$ does not dominate X-blocks of $T$. The blocks from an inner component at a separation pair $\langle a, v_i \rangle$ are merged into a single super-block $B^+$ that is placed to the right of $v_i$ and to the left of $v_{i+1}$. There is an interval exactly for the vertices of $B^+$. By the vertex ordering at $v_i$, X-blocks are ordered clockwise at $v_i$ if they are dominated by $v_i$, such that $U$ contains the vertices of all X-blocks that precede $B^+$. The inner components are ordered like block-trees and the vertices from any single inner component are placed to the right of $v_i$ and to the left of $v_{i+1}$. Hence, the stated properties hold.

We now return to the original set of blocks. If $B$ is an uncovered block with $\omega(B) = v_j$ or $e = (v_i, v_j)$ is an outer chord, then there are no vertices of part $H_2$ to the right of $v_j$. If $B$ is covered by $v_i$, then the vertices of $B$ are in an interval that is exclusive for $B$. The interval is placed between $v_i$ and $v_{i+1}$ if $v_i \neq \omega(T)$ for the block-tree containing $B$ and to the left of the cutvertex $a_i$ of block $A$ if $v_i = \omega(T)$, such that there is a block-expansion for $A$. In the latter case, $B$ is part of a super-block that is traversed in postorder, similar to the nested method.

### 3.3 Embedding of Edges

Yannakakis [28] has used three pages for the embedding of planar 2-level graphs. All outer edges, all outer chords and all backward binding edges (between vertices of a block and its dominator) are embedded in page $P_1$. The inner edges of a block, in particular, the first and the last edge, are embedded in a single page $P_2$ or $P_3$, and the forward binding edges are embedded in the other page. These pages alternate between a block and its parent, that is at an odd and an even distance between a block and the root of its block-tree. We adopt this embedding after a triangulation from the first outer vertex of each face. Any triangulation edge is a crossed edge of the given maximal 2-level framed multigraph. We obtain a planar 2-level multi-graph $H^+$ with a set edges $E^+$ that includes the edges of $H$. For any face $f$, let $E^-(f)$ be the set of remaining crossed edges and let $E^- = \cup_f E^-(f)$.

Clearly, the vertex ordering $L(H)$ coincides with the vertex ordering of $L(H^+)$, since vertex $v$ dominates a block in $L(H)$ if and only if $v$ is the least outer vertex in a face containing $v$ and the last edge of a non-elementary block if and only if there is a triangle in $H^+$ with $v$ and the last edge of the block, and similarly for the first face of a block-tree or any block-expansion. First, we show that the edges of $H^+$ can be embedded in three pages using our vertex ordering $L(H)$. Then one vertex per face is done, since its incident edges are embedded. All these edges can be disregarded subsequently, which is important, in particular
for Lemma 12. Hence, only \( k - 1 \) vertices remain for a \( k \)-face. If face \( f \) contains vertices of a covered block, then it contains a single outer vertex, such that \( V^-(f) \) consists only of inner vertices. This even simplifies the situation. There is a face-conflict graph that represents a possible conflict between two remaining edges of any two faces. We show that the face-conflict graph is outerplanar, such that it is 3-colorable. Moreover, it represents conflicts, such that there is no crossing of remaining edges from two faces with the same color. The arguments are similar to the planar case. In total, the edges of a 2-level \( k \)-framed multigraph can be embedded in \( 3 \lfloor (k - 1)/2 \rfloor + 3 \) pages.

**Lemma 6.** Any triangulated planar 2-level \( k \)-framed multigraph \( H^+ \) can be embedded in three pages if the vertex ordering \( L(H) \) is used.

**Proof.** Yannakakis [28] has proved that all outer edges, all outer chords and all backward binding edges of a triangulated planar 2-level graph can be embedded in page \( P_1 \). The edges of block \( B \) are embedded in page \( P_2 \) and forward binding edges incident vertices of \( B \) in page \( P_3 \). Pages \( P_2 \) and \( P_3 \) alternate for \( A \) if block \( A \) is the parent of \( B \). Yannakakis excludes outer chords and covered blocks. The extension is proved in the same way using X-blocks.

If \( B^+ \) is a super-block that is covered by some vertex \( v_i < \omega(T) \) for the block-tree \( T \) containing \( B^+ \), then its vertices are placed in an interval \([B^+]\) to the right of \( v_i \) that exclusively contains the vertices of \( B^+ \). For any vertex \( b \) in \( B^+ \), \((b,v)\) is a backward binding edge and is embedded in page \( P_1 \). The edges form a fan at \( v_i \), such that they do not twist mutually. They do not twist other edges in \( P_1 \), since all vertices between \( v_i \) and (the left boundary of) \([B^+]\) are from X-blocks that are dominated by \( v_i \). If X-block \( A^+ \) contains the leader of \( B^+ \), then embed the inner edges of the blocks of \( B^+ \) in page \( P_3 \) if the forward binding edges incident to vertices of \( A^+ \) are embedded in \( P_3 \).

Suppose the uncovered block \( A \) is expanded at its vertex \( a_i \) by \( B^+ \). Then there are forward binding edges \((a_i,\omega(T))\) and \((b,\omega(T))\) for any vertex \( b \) in \( B^+ \), since \( \omega(T) \) is the last outer vertex that may be incident to such binding edges. These edges are embedded in pages \( P_2 \) or \( P_3 \), opposite to the page for the inner edges of \( A \) [28]. The inner edges of \( B^+ \) are embedded in the same page as the inner edges of \( A \), since the nested method is used. The vertices of \( B^+ \) are immediately to the left of \( a_i \), and there are no other vertices in \([B^+,a_i]\), such that binding edges with a vertex in \([B^+,a_i]\) do not twist mutually. In particular, if \( a_i \) is a cut-vertex and the leader of blocks that succeed \( A \), then the edges incident \( a_i \) and these blocks can be embedded in the page opposite to the page used for the inner edges of \( A \). If \( A \) is expanded at vertices \( a_i \) and \( a_j \) with \( a_i < a_j \), then all binding edges incident to a vertex \( a \) of \( A \) with \( a_i < a < a_j \) are incident to \( \omega(T) \). They are all embedded in the same page. If blocks \( A \) and \( B \) are expanded at vertices \( a_i \) and \( b_j \), respectively, then the binding edges are embedded by the above rule. Suppose that blocks \( A \) and \( B \) are uncovered such that \( B \) is the first child of \( A = a_0, \ldots, a_p \) in ccw-order. Then vertex \( a_m \) with \( 0 \leq m \leq q \) is the leader of \( B \) and \( a_m \) is minimal. Now all vertices with an edge incident to a vertex of \( B \) are in the interval \([a_m,\omega(T)]\). Block \( A \) is expanded
only at vertices \(a_i\) with \(a_0 \leq a_i \leq a_m\). If \(e\) is an edge from a block-expansion at \(A\), then \(e = (u, \omega(T))\) with \(u \leq a_m\), whereas the vertices of any edge \(e'\) incident to a vertex of \(B\) is in \([a_m, \omega(T)]\), such that \(e\) and \(e'\) can be embedded in the same page. By induction, we obtain that the edges from all block-expansions, all uncovered blocks and all super-blocks can be embedded in pages \(P_2\) and \(P_3\) without creating a conflict. Hence, all edges of \(H^+\) can be embedded in three pages. \(\square\)

We now consider the sets of remaining edges \(E^-(f)\) of the faces. There is no need to distinguish covered and uncovered blocks, since the edges incident to the dominator are in the set \(E_\alpha\). Hence, all backward binding edges between a block and its dominator are disregarded. In particular, if \(e\) is remaining edge with a vertex in a covered block, then both vertices are inner vertices, that are in the interior of specified interval.

**Definition 1.** A face \(f\) of a planar 2-level multigraph is called bad for block \(B\) if \(f\) has degree at least four and

- (i) \(B\) is non-elementary and \(f\) contains the last edge of \(B\) in its boundary and
- (a) either \(B\) is uncovered or (b) \(B\) is a covered block in a super-block or
- (ii)(a) \(B\) is a covered block that is merged into an expanded block and \(f\) also contains the last edge of \(B\) and the successor of the leader of \(B\) in its block or
- (b) \(B\) is an elementary root of a block-tree \(T\) and \(f\) is the first face of \(T\) that also contains an outer vertex \(v\) with \(\alpha(T) < v < \omega(T)\) or an inner vertex.

A face is bad if it is bad for any block \(B\), and good, otherwise.

For an example, see Figure 2. In particular, face \(f\) is good if it does not contain any cutvertex in its boundary. Clearly, a face can be bad for several blocks, namely if they have the same dominator. In return, there are blocks without a bad face, for example, if there is a triangle containing the last edge of a block. As any non-elementary block has a last edge and any elementary one a first face, then following is clear.

**Lemma 7.** For any block \(B\) of a planar 2-level multigraph there is at most one face \(f\) such that \(f\) is bad for \(B\).

Hence, the number of bad faces is bounded by the number of blocks.

Next we define a conflict between two faces via bad faces, and then prove in Lemma 12 that there is no conflict between two remaining edges if the faces are not in conflict. Note that our notion of conflict is different from the one in [5–7], since we disregard all edges that are incident to the first outer vertex in any face. This restriction is important.

The last edge \((b_0, b_q)\) and any forward binding edge incident to a vertex \(b_i\) for \(0 < i < q\) of block \(B = b_0, \ldots, b_q\) and \(q \geq 2\) twist, as observed by Yannakakis [28]. Similarly, crossed edges incident to \(b_0, b_q\) and \(b_i\) may twist, such that they shall be embedded in different sets of pages. Also the vertex of an elementary block and the cutvertex of a block-subtree at a block-expansion behave in the same
way, since they are spanned by crossed edges between vertices from the bad face. For a block-tree \( T \), e say that face \( f \) is on the front side if \( v = \omega(T) \) for any outer vertex of \( f \) and \( f \) contains an inner vertex from any block of \( T \). The faces on the front side are ordered clockwise at \( \omega(T) \), the last of which contains the first vertex of \( T \). Let \( A \) be an uncovered block in block-tree \( T \), such that \( A \) is expanded at its vertex \( a \), by some super-block \( B^+ \). Let \( x \) be any cutvertex of \( B \) or \( x = a_i \). Then the vertices of the block-subtree of \( B^+ \) with leader (cutvertex) \( x \) are in an interval \([B^+, x]\) immediately to the left of \( x \). Hence, the remaining vertices from any face with a vertex in the block-subtree are in this interval, except for the first and last face in ccw-order at \( \omega(T) \). The first face is bad if it contains vertices that span \( x \). The last face contains \( x \), the first vertex of the last sibling at \( x \) in cw-order, and probably the predecessor of \( x \) in its block. There are at most \( 2m \) faces in the front side with \( x \) in their boundary if the block-subtree has \( m \) siblings at \( x \). Any of these faces may contain a crossed edge \( e \) that is incident to \( x \) such that \( e \) twists any edge from the bad face that spans \( x \). If block \( A \) is expanded to \( A^* \), then all vertices from the block-expansion at its vertex \( a_i \) are in the interval \([a_{i-1}, a_i]\). Hence, any remaining edge \( e \) from the bad face of \( A \) does not twist any remaining edge from a face in the front side with a vertex in the expansion at a vertex of \( A \), since the vertices of \( e \) are outside \([a_{i-1}, a_i]\).

**Definition 2.** Two faces \( f \) and \( f' \) of a planar 2-level multigraph \( H \) are in conflict if (i) \( f \) is bad for a non-elementary block \( B \) that is uncovered or in a super-block and \( f' \) contains a vertex of \( B \) except if \( f' \) is on the front side or (ii) \( f \) is bad for a covered block or an elementary block and \( f' \) contains the leader of \( B \).

The face-conflict graph \( H^\times \) has the faces of \( H \) as its vertices. There is an edge \((f, f')\) in \( H^\times \) if \( f \) and \( f' \) are in conflict.

The following is obvious.

**Lemma 8.** Two faces contain vertices of a single block including its leader if they are in conflict.

**Lemma 9.** Any two edges \( e \in E^{-}(f) \) and \( e' \in E^{-}(f') \) do not twist if faces \( f \neq f' \) do not both contain vertices assigned to any block \( B \).

**Proof.** First, assume that \( f \) and \( f' \) do not contain inner vertices of the same block-tree. Then there is a partition of \( H \) induced by an outer chord \((v_i, v_j)\), that may be added by a triangulation, as described in Lemma 4 such that all vertices of \( V^{-}(f) \) are in part \( H_2 \) and all vertices of \( V^{-}(f') \) are in part \( H_1 \). As shown in Lemma 5, the vertices of \( V^{-}(f) \) are in the interval \([v, v_j]\) with \( v_i \leq v \) and those of \( V^{-}(f) \) are outside, or vice versa. Note that \( v_i \) is disregarded if it is the first outer vertex of \( f \) or \( f' \). Clearly, \( e \) and \( e' \) do not twist in this case.

Next, assume that \( B > B' \) in a block-tree \( T \). If \( B \) is uncovered, then there is a partition of \( H \) induced by \( B \) such that part \( H_2 \) contains \( B \) and \( B' \) is in part \( H_1 \). As before, the vertices of \( V^{-}(f) \) are in the interval \([v, v_j]\), where \( v_j = \omega(B) \) for the least block \( B \) with vertices in \( V^{-}(f) \), and those of \( V^{-}(f') \) are outside. Similarly, if \( B \) is covered by some outer vertex \( v \neq \omega(T) \), then the vertices
of $V(B)$ are in an interval to the right of the interval for the vertices of $B'$.
Similar to the case of forward binding edges, the interval for the vertices of $f'$ is contained in the interval $[\lambda(B), B]$ if $B'$ is dominated by some vertex $v' \leq v$ and $\lambda(B') < \lambda(B)$ are assigned to the same block, the interval for $V^-(f')$ precedes the one for $V^-(f)$ if $f' < f$ and it includes the interval for $V^-(f)$ if $f' > f$.

At last, if $B$ is merged into an expanded block $A^*$, then its vertices are in an interval $[a_{j-1}, a_j]$, where $a_j$ is the cut-vertex in an uncovered block $A$, such that it is disjoint from the interval for the vertices of $B'$ and $f'$ or properly nests within that interval if $B'$ is not merged into $A^*$, too. Now the vertices of $B$ are in an interval to the left of $\lambda(B)$ that is either disjoint from the interval for the vertices of $B'$, that is to the left of $\lambda(B')$, or it is a subinterval.

Hence, in any case, there are disjoint or intervals for the remaining vertices of $f$ and $f'$ or one is a subinterval of the other, such that edges $e$ and $e'$ cannot twist.

Hence, if $e$ and $e'$ twist and are remaining edges in two faces $f$ and $f'$, then the faces are close. In particular, edges cannot twist if their faces are separated by an outer chord of the frame. Clearly, a face may contain vertices from different block-trees and it may be bad for several blocks, namely if the blocks have the same dominator, and it may contain the root of several block-trees the leader of several blocks that are merged into an expanded block, as Figure 2 illustrates.

**Lemma 10.** The face-conflict graph of a planar 2-level multigraph is outerplanar.

**Proof.** The face-conflict graph $H^\times$ is a subgraph of the planar dual $H$ from which the outer face and the faces inside non-trivial blocks are removed. Then all faces of $H^\times$ are in the outer face, since each face has at least one outer vertex. A face of $H^\times$ is a cutvertex if it has an outer chord in its boundary or may contain an outer chord in its interior. It is isolated if it has no inner vertices.

Orient the edges of $H^\times$ away from bad faces. There are two cases. In case (i), if face $f$ is bad for a non-elementary block $B$ that is uncovered or in a super-block, then route an edge $(f, f')$ from $f$ through $B$ to $f'$, such that any two edges incident to $f$, do not cross, and similarly for $f'$. Edge $(f, f')$ enters $B$ through
the last edge \((b_0, b_q)\) of \(B = b_0, \ldots, b_q\). Block \(B\) is entered only by edges incident to \(f\), since \(B\) has at most one bad face. Block \(B\) is left through any vertex \(b\) assigned to \(B\), except if \(b\) is on the front side, that is all faces with \(b\) in their boundary are on the front side. Also the leader of \(B\) is excluded. There may be multi-edges \((f, f')\) if \(f\) is bad for several blocks. Multiple copies can be removed. In case (ii), if \(B\) is the elementary root of its block-tree \(T\), or \(B\) is merged into an expanded block \(A^\ast\) and \(f\) is bad for \(B\), then route an edge \((f, f')\) from \(f\) through \(b_0\) to \(f'\) if \(f'\) contains \(b_0\) in its boundary.

Consider two edges \((f_1, f'_1)\) and \((f_2, f'_2)\) of the face-conflict graph, as illustrated in Figure 4. Let \(f_i\) be bad for \(B_i\) for \(i = 1, 2\), such the \(B_i\) is the least such block. First, assume that case (i) holds for both blocks. If the edges are adjacent, then they do not cross. In particular, if \(f'_1 = f_2\), then \(B_1\) is the parent of \(B_2\) if \(B_1\) and \(B_2\) are in the same block tree, such that \(\lambda(B_2)\) is a vertex of \(B_1\). Then \((f_1, f'_1)\) enters \(f'_1\) through \(\lambda(B_2)\) and \((f_2, f'_2)\) enters \(B_2\) through the last edge of \(B_2\), such that they do not cross. Otherwise, assume \(f_1 < f_2\). Then \(B_1 < B_2\), since edges and blocks are ordered clockwise. Consider a curve \(\Gamma\) from \(\alpha(B_2)\) through \(\lambda(B_2)\) to \(\omega(B_2)\), where \(\omega(B_2)\) is the last outer vertex in a face with vertices of \(B_2\) in its boundary. Then \(\omega(B'_2) = \omega(B_2)\) if there is a binding edge \((\lambda(B_2), \omega(B_2))\) in \(H\). Otherwise, \(\omega(B'_2)\) is the last outer vertex after \(\omega(B_2)\) in a face that sees the first edge of \(B_2\), such that \(\omega(B'_2) \geq \omega(B_2)\). Route \(\Gamma\) such that it first follows the binding edge incident to \(\alpha(B_2)\) to some block \(D_1\) and then it follows the blocks \(D_1, \ldots, D_r\) on the side of the dominator \(\alpha(B_2)\) to the leader \(\lambda(B_2)\) in \(D_r\). There is no binding edge between \(\alpha(B_2)\) and any vertex of \(D_i\) for \(i > 1\). Next, \(\Gamma\) follows the inner boundary of \(f'_1\), that is blocks \(D'_1, \ldots, D'_r\) up to the binding edge incident to \(\omega(B'_2)\) in the boundary of \(f'_2\). Then \(D_r = D'_1\) and \(D_i \neq D'_j\), since \(B_2\) is the least block. Curve \(\Gamma\) is routed along uncrossed edges of the frame. It is completed to a closed curve by a part of the outer cycle between \(\alpha(B_2)\) and \(\omega(B'_2)\). The faces \(f_1, f'_1\) and \(f_2, f'_2\) are on opposite sides of \(\Gamma\), such that the edges \((f_1, f'_1)\) and \((f_2, f'_2)\) are on opposite sides of \(\Gamma\). Hence, the edges cannot cross.

Next, suppose that case (ii) holds for both blocks. Then the bad face for \(f_i\) contains the leader of \(B_i\) is its boundary, \(i = 1, 2\). There is at most one bad face next to \(b_0\), such that \(b_0\) is not passed by other edges of the conflict graph. If \(B_1\) and \(B_2\) are in different block-trees or in block-expansions such that \(B_2\) is not in the block-subtree with root \(B_1\), then \((f_1, f'_1)\) and \((f_2, f'_2)\) are separated as follows. There is an outer chord or a binding edge incident to \(\omega(T)\) in \(H\), that separates \(B_1\) and \(B_2\) in \(H\) as described in Lemma 4. Then edges \((f_1, f'_1)\) and \((f_2, f'_2)\) can meet in a common face, for example \(f'_1 = f_2\), but they cannot cross in the dual. If \(B_1\) is an ancestor of \(B_2\) in a block-subtree that is part of a block-expansion, then \(f_1 < f'_1 \leq f'_2 \leq f_2\) or \(f'_1 < f_2 \leq f'_2 < f'_1\), such that the edges do not cross.

Consider the mixed case. If \(f_1\) is the first face of block-tree \(T_1\), then the blocks with vertices in the boundary of \((f_1, f'_1)\) and \((f_2, f'_2)\) are separated by an outer chord if \(B_2\) is not in \(T_1\) and by a curve through \(\alpha(B_2), \lambda(B_2), \omega(B_2)\), as described in Lemma 4, such that \((f_1, f'_1)\) and \((f_2, f'_2)\) can meet but do not cross.
In particular, if the leader of $B_2$ is the first vertex of $T_1$, that is $B_1$, then there is a triangle $(f_1, f_2, f_2')$, where $f_1' = f_2$ or $f_1' = f_2'$.

At last suppose that block $B_2$ is merged into an expanded block $A^*$. If block $B_1$ is not in $A^*$, then $(f_1, f_1')$ and $(f_2, f_2')$ can be separated as described before. Let $B_1 = A$ be the uncovered block that is expanded to $A^*$ and suppose that the leader of $B_2$ is vertex $a_i$ of $A$. All other cases are similar. Then vertex $a_i$ is on the front side, such that it blocks any edge from the bad face of $A$. In fact, vertices $a_0, \ldots, a_m$ are blocked if they are on the front side and $a_m$ the first vertex of $A$ that is the leader of a block that is not covered by $\omega(T)$ or $a_m$ is the least vertex of $A$ in the boundary of the bad face for $B_2$, in which case we have $f_1' = f_2$. Otherwise, $(f_1, f_1')$ and $(f_2, f_2')$ can be separated as described before.

Hence, any two edges of the conflict graph do not cross, so that $H^*$ is outer-planar.

The next lemma includes all edges of a face. If can be restricted to the remaining set of edges if the faces have degree at least five.

**Lemma 11.** There are edges $e \in E(f)$ and $e' \in E(f')$ that twist if faces $f \neq f'$ are in conflict.

*Proof.* Assume that $f$ is bad for block $B$. If $B = b_0, \ldots, b_q$ is non-elementary and is not merged into an expanded block, then $f'$ contains a vertex $b$ of $B$ and some outer vertex $v'$ such that $(b, v')$ is a forward binding edge. Let $u$ be a forth vertex in $f$. Now edges $(b_0, b_q)$ and $(b, v')$ twist. Similarly, if $B$ is elementary or is merged into an expanded block $A^*$, then $\lambda(B)$ is spanned by a crossed edge $(a_j, a_{j+1})$ of $f$, whereas the further inner vertices of $f'$ are to the left of $a_j$. □

The converse of Lemma 12 is true when restricted to the remaining edges. It completes the correctness proof for our algorithm. For the proof, we use the computed vertex ordering $L(H)$ and the fact that edges incident to the first outer vertex of each face are excluded. It resembles the case for the last edge of a block and forward and backward binding edges incident to its vertices from Lemma 9.

**Lemma 12.** Edges $e \in E^-(f)$ and $e' \in E^-(f')$ do not twist if faces $f \neq f'$ are not in conflict.

*Proof.* By Lemma 9, $e$ and $e'$ do not twist if faces $f$ and $f'$ do not share vertices of a block $B$. If all shared blocks are covered, then the remaining vertices of at least one of $f$ and $f'$ are inner vertices, which simplifies the situation. As in the proof of Lemma 10, we must distinguish between the cases from Definition 4. Let $f < f'$ and assume that block $B$ is non-elementary and is uncovered or in a super-block. Then $f$ does not contain the last edge of $B$. If $b_f$ is the least vertex of $B$ in $f$ and $v_f$ its last outer vertex, then the remaining vertices of $f$ are in the interval $[b_f, v_f]$, where $v_f$ is the last inner vertex of $f$ if it has a single outer vertex. Then the outer vertices of $f$ precede those of $f'$ if $f < f'$. The inner vertices of block $B$ in $f'$ precede those of $f$ and inner vertices in $f'$ succeed the
Fig. 5. Illustration to the proof of Lemma 12. Faces $f$ and $f'$ are not in conflict and they do not contain vertices of the same block.

last vertex of $B$, since they are in blocks $B' > B$. Hence, edge $e$ does not twist $e'$. Note that face $f'$ may be the first face for a block-tree with inner vertices from block $B$ that is in a different (earlier) block-tree.

If $f$ is the first face of a block-tree $T$ and $f$ and $f'$ contain vertices of the root of $T$, then the first face is not bad for $B$, for example it is a triangle or contains only outer vertices $v \geq \omega(T)$ besides the first outer vertex of $T$. Then $T$ has an elementary root $b_0$, the vertices of $V^- (f')$ are in an interval $[b_0, \omega(T)]$ and the vertices of $V^- (f)$ are outside this interval.

At last, assume that $B$ is covered by $\omega(T)$, such that it is merged into an expanded block $A^*$. Then $V^- (f)$ contains only inner vertices from blocks that are merged into $A^*$. Then the vertices of $V^- (f')$ are in an interval that excludes the leader $\lambda(B)$ if $f$ is bad for $B$, such that the intervals for $V^- (f)$ and $V^- (f')$ are disjoint, or if $\lambda(B) \in V^- (f) \cup V^- (f')$, then the intervals for $V^- (f)$ and $V^- (f')$ share vertex $\lambda(B)$ and $[V^- (f')]$ is a subinterval of $[V^- (f)]$ if $f < f'$. All other cases are similar.

Hence, edge $e$ does not span exactly one vertex of $e'$ or vice versa, such that $e$ and $e'$ do not twist.

We comprise the above Lemmas to the following result:

**Theorem 2.** Any 2-level framed multigraph $H$ can be embedded in $3K + 3$ pages if the set of remaining edges $E^- (f)$ of every face can be embedded in $K$ pages using the vertex ordering $L(H)$. In particular, $H$ can be embedded in $3 \lfloor k/2 \rfloor + 3$ pages if $H$ is $k$-framed.

**Proof.** Graph $H$ consists of a planar $k$-framed multigraph and of sets of crossed edges for the faces. The sets from the triangulated planar multigraph can be embedded in three pages, as shown in Lemma 6. Then the edges of the frame and the edges incident to the first outer vertex of each face are done. By Lemma 12 edges from different sets $E^- (f)$ and $E^- (f')$ can be embedded in the same set of pages if $f$ and $f'$ are not in conflict, that is $f$ and $f'$ have the same color in the face-conflict graph. By Lemma 10 the face-conflict graph is outerplanar, such that it is 3-colorable. Then three sets of $K$ pages each suffice for $E^-$. Since any edge of $H$ is in $E_\alpha$ or $E^-$ (or both), all edges of $H$ are embedded in $3K + 3$ pages. If $H$ is $k$-framed, then $K \leq \lceil (k-1)/2 \rceil$. □
Corollary 2. The sets of crossed edges from all good faces of a 2-level $k$-framed multigraph of $a$ can be embedded in $\lceil (k - 1)/2 \rceil + 3$ pages.

Proof. The face-conflict graph is discrete if all bad faces are discarded. Then one color suffices. $\Box$

3.4 Composition

As observed by Yannanakis [28], the vertices of a block at level $\ell + 1$ including the leader are placed between two consecutive level $\ell - 1$ vertices. Here it is assumed that vertices from blocks are placed just right of the second outer vertex if the blocks are dominated by the first outer vertex. The assumption is adopted from [28] and has no side effects for the embedding of edges at any level. The vertices of a block at level $\ell$ are consecutive in $L(H)$, whereas there is an interval between $a_j$ and $a_{j+1}$ if there is a block-expansion at $a_j$. The vertices in the interval are incident to other vertices in the interval, to $a_k$ and by binding edges to $\omega(T)$ if $T$ is the block containing (the block of) $a_j$. Hence, it does not matter that the vertices of a block at level $\ell + 1$ are not consecutive in $L(H)$, similar to the nested method. In consequence, the same set of pages can be reused for all odd (even) levels, such that twice the number of pages for 2-level graphs suffices for a book embedding of framed multigraphs.

The page for the inner edges of block $B$ at level $\ell$ can be reused for the embedding of the backward binding edges and chords in its interior at the next level $\ell + 1$, as observed by Yannakakis [28] for his 5-page algorithm.

Theorem 3. Any $k$-framed multigraph can be embedded in $6\lfloor k/2 \rfloor + 5$ pages. The book embedding can be computed in linear time in the size of $G$ (number of vertices and edges).

Proof. The planar frame is recursively decomposed into 2-level graphs, which are used to compute the linear ordering. Every 2-level subgraph of a framed multigraph has a book embedding in $3\lfloor k/2 \rfloor + 3$ pages, as shown in Theorem 2. These pages can be reused for all odd levels, and another set of $3\lfloor k/2 \rfloor + 2$ pages is used for the even levels. There are no edges between any two vertices at levels $i$ and $j$ with $|i - j| \geq 2$ in the planar frame. Since crossed edges are in the interior of faces of the frame, this also holds for the edges of a framed multigraph. One edge is saved, as described before [28].

Concerning the running time, the frame of a framed multigraph with $n$ vertices has at most $3n - 6$ edges including multiplicities for multi-edges, since there are vertices on either side of a 2-cycle by a multi-edge. It thus has $O(n)$ faces. The vertex ordering can be computed in linear time in the number of vertices, both for 2-level multigraphs and the frame. Similarly, the 3-coloring of an outerplanar 2-level face-conflict graph is computable in linear time, such that the coloring of all faces takes $O(n)$ time. Finally, any edge of $G$ can be embedded in constant time. $\Box$
4 Applications

For any even \( k \), a \( n \)-clique with \( n = 3k/2 \) can be represented by a \( k \)-map [12]. Then three points of degree \( k \) support all adjacencies, for example \( k = 4 \) for \( K_6 \). Hence, we obtain an improved lower bound on the book thickness of \( k \)-map graphs. In improved upper bound is obtained from Theorems 1 and 3.

**Theorem 4.** The book thickness of \( k \)-map graphs is at most \( 6\lfloor k/2 \rfloor + 5 \).

**Corollary 3.** For any \( k \geq 3 \), there are \( k \)-map graphs (or \( k \)-framed multigraphs) with book thickness at least \( \lceil 3k/4 \rceil \).

**Proof.** The book thickness of \( K_n \) is \( \lceil n/2 \rceil \) [8] and \( K_n \) is a \( k \)-map graph for \( n \leq \lfloor 3k/2 \rfloor \) [12]. \( \square \)

Chen et al. [13] have observed that any triangulated 1-planar graph is a 4-map graph. The drawing of a triangulated 1-planar graph consists of triangles and quadrangles, which contain a pair of crossed edges, as shown by Alam et al. [1]. Hence, any triconnected 1-planar graph is a 4-framed graph. In general, there are W-configurations [27] with a pair of crossed edges in the outer face of a component at a separation pair, see Figure 1(a). Now multi-edges come into play, such that any 2-connected 1-planar graph is a subgraph of a 4-framed multigraph. Clearly, each 1-planar graph can be augmented to a 2-connected 1-planar multigraph.

From Theorem 4 we obtain a bound of 17 for the book thickness of 1-planar graphs, which improves the previous bounds of 39 in [2] and 29 that can be obtained from [5]. We can do even better.

**Theorem 5.** Any 1-planar graph can be embedded in eleven pages.

**Proof.** Any 1-planar graph is a subgraph of a 4-framed multigraph whose faces are triangles or quadrangles and there is a pair of crossed edges in each quadrangle. Consider a 2-level graph. If \( f \) is a triangle, then its edges are embedded in pages \( P_1, P_2 \) and \( P_3 \) by Lemma 6. If \( f \) is a quadrangle, then only one crossed edge remains for the set \( V^-(f) \). By Theorem 2, all remaining crossed edges can be embedded in three pages. As observed in [28], one page can be saved at the composition, such that eleven pages suffice for the book embedding of any 1-planar graph. \( \square \)

The crossed cube, as shown in Figure 1(b), is a 4-framed graph, whose frame is a (planar) cube, such that their is a pair of crossed edges in each face. It is a 1-planar graph with 8 vertices and 24 edges. It has book thickness four, since the vertex ordering is taken from a Hamiltonian cycle of the frame, such that four pages suffice, and it needs four pages as shown in [8].

Bekos et al. [4] have characterized optimal 2-planar graphs and have shown that edges are uncrossed or crossed twice and edges that are crossed twice can be grouped or caged to \( K_5 \) if \( n \)-vertex graphs are optimal and have \( 5n - 10 \) edges,
see Figure 1(c). In consequence, if an edge is crossed twice, then its crossing edges are incident to a common vertex and the vertices of these three edges form $K_5$. In consequence, an optimal 2-planar graph is a 5-map graph $\mathcal{K}$, such that it is a 5-framed graph. Bekos et al. [5] have shown that the book thickness of optimal 2-planar graphs is 23, which can be improved.

**Corollary 4.** Any clique augmented 2-planar graph, and in particular, any optimal 2-planar graph, can be embedded in 17 pages.

## 5 Conclusion

We extend Yannakakis algorithm [28] on the book embedding of planar graphs by block expansions and generalize the approach by Bekos et al. [5] from framed graphs to framed multigraphs. Multi-edges help to obtain smaller faces, which leads to fewer pages for the book embedding. Maximal framed multigraphs coincide with map graphs if restricted to simple graphs. Thus we improve the upper bound for the book thickness of $d$-map graphs of $O(\log n)$ by Dujmović and Frati [18] and $6\lceil d/2 \rceil + 5$ (claimed) by Bekos et al. [5] to $6\lfloor d/2 \rfloor + 5$. In particular, we show that the book thickness of 1-planar graphs is at most eleven.

There are several other classes of beyond-planar graphs [15], such as $k$-planar, fan-planar, fan-crossing, 1-fan-bundle, fan-crossing free, and quasi-planar graphs, for which the book thickness has not yet been investigated in detail. It is unlikely that they are framed multigraphs, such that new techniques are needed for upper bounds on the book thickness.

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