Gradient Estimate for Solutions to Poisson Equations in Metric Measure Spaces

Renjin Jiang

Abstract. Let $(X,d)$ be a complete, pathwise connected metric measure space with a locally Ahlfors $Q$-regular measure $\mu$, where $Q > 1$. Suppose that $(X,d,\mu)$ supports a (local) $(1,2)$-Poincaré inequality and a suitable curvature lower bound. For the Poisson equation $\Delta u = f$ on $(X,d,\mu)$, Moser-Trudinger and Sobolev inequalities are established for the gradient of $u$. The local Hölder continuity with optimal exponent of solutions is obtained.

1 Introduction

Let $M$ be an $n$-dimensional ($n \geq 2$) complete, connected Riemannian manifold with Riemannian metric $\rho$. Denote by $\Delta, \nabla$ the Laplace-Beltrami operator and the gradient on $M$, respectively. Assume that the Ricci curvature is bounded from below by a constant $K \in \mathbb{R}$, i.e.,

\[ \text{Ric}_x (X,X) \geq -K |X|^2, \quad \forall x \in M, X \in T_x M. \]

(1.1)

Let $p$ and $\{P_t\}_{t>0}$ be the heat kernel and heat semigroup of the Laplace-Beltrami operator on $M$, respectively. In 1986, a breakthrough was made by Li and Yau in [25], where they obtained pointwise estimates on $p$ and the gradient of $\nabla p$. When $M$ has non-negative Ricci-curvature, their estimates read as:

\[ \frac{C}{V(x, \sqrt{t})} \exp \left\{ -\frac{\rho(x,y)^2}{ct} \right\} \leq p(x,y,t) \leq \frac{C}{V(x, \sqrt{t})} \exp \left\{ -\frac{\rho(x,y)^2}{ct} \right\}, \]

\[ |\nabla_x p(x,y,t)| \leq \frac{C}{\sqrt{t}V(x, \sqrt{t})} \exp \left\{ -\frac{\rho(x,y)^2}{ct} \right\}, \]

where $V(x, \sqrt{t})$ denotes the volume of the metric ball $B(x, \sqrt{t})$. Li-Yau type estimates have turned out to be powerful tools in many branches of modern mathematics, see, for example, [27, 39] for applications to Poisson equation on Riemannian manifold with non-negative Ricci curvature.

On the other hand, Gross [15] derived the remarkable Gaussian Sobolev inequality

\[ \int_{\mathbb{R}^n} |f(x)|^2 \ln |f(x)| \, d\nu(x) \leq \int_{\mathbb{R}^n} |\nabla f(x)|^2 \, d\nu(x) + \|f\|_{L^2(\nu)}^2 \ln \|f\|_{L^2(\nu)}, \]

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where \( \nu \) denotes the Gaussian measure on \( \mathbb{R}^n \), which is also referred to as the logarithmic Sobolev inequality. While the classical Sobolev inequality highly depends on the dimension \( n \), the logarithmic Sobolev inequality is uniform in all dimension \( n \), which enables one to extend it to infinite dimension. Moreover, when passing from Euclidean spaces to Riemannian manifolds, the logarithmic Sobolev inequality (in different forms) even reflects some deep geometric properties.

Recall that “square of the length of the gradient”, which is due to Bakry and Emery [3], is defined as
\[
\Gamma_2(u, u) = \frac{1}{2} \Delta(|\nabla u|^2) - \langle \nabla \Delta u, \nabla u \rangle, \quad u \in C^\infty(M).
\]
The diffusion semigroup is said to have curvature greater or equal to some \( K \in \mathbb{R} \), if
\[
\Gamma_2(u, u) \geq -K \langle \nabla u, \nabla u \rangle, \quad \forall u \in C^\infty(M).
\]
(1.2)

It is well known that (1.2) is equivalent to (1.1). Moreover, they are all equivalent to:
\[
P_t(u^2) - (P_t u)^2 \leq \frac{e^{2Kt} - 1}{K} P_t(|\nabla u|^2), \quad \forall t \geq 0, \forall u \in C^\infty(M),
\]
(1.3)

\[
P_t(u^2 \log u^2) - (P_t u^2) \log(P_t u^2) \leq \frac{2(e^{2Kt} - 1)}{K} P_t(|\nabla u|^2), \quad \forall t \geq 0, \forall u \in C^\infty_c(M),
\]
see [2]. Wang [37] showed that (1.1) is also equivalent to the so-called dimension-free Harnack inequality; see also [38].

Our main aim in this paper is to provide a semigroup approach via the logarithmic Sobolev inequality (1.3), instead of Li-Yau type estimates for the gradient of the heat kernel, to study the local behavior of solutions to the Poisson equation \( \Delta u = f \). Taking a Riemannian manifold that satisfies (1.3) as a guiding example, we will single out the crucial assumptions necessary for our semigroup approach, by formulating the arguments in an abstract metric space. Our results indicate that already the logarithmic Sobolev inequality (1.3) together with a 2-Poincaré inequality (see (1.4) below) is sufficient to guarantee Euclidean type local behavior of solutions to Poisson equation.

Let us now describe the metric setting. Let \((X, d)\) be a complete, pathwise connected metric measure space. Suppose that \((X, d)\) is endowed with a locally \( Q \)-regular measure \( \mu \), \( Q > 1 \), where local \( Q \)-regularity means that there exist constants \( C_Q \geq 1 \) and \( R_0 \in (0, \infty] \) such that for every \( x \in X \) and all \( r \in (0, R_0) \),
\[
C_Q^{-1} r^Q \leq \mu(B(x, r)) \leq C_Q r^Q.
\]

The reader interested in Riemannian manifolds should here think \( X \) to be a weighted Riemannian manifold.

By the work of Buser [7], each complete Riemannian manifold with Ricci-curvature bounded from below admits a local 2-Poincaré inequality. Correspondingly, we assume a (weak) 2-Poincaré inequality on \((X, d, \mu)\). That is, there exist \( C_P > 0 \) and \( \lambda \geq 1 \) such that for all Lipschitz functions \( u \) and each ball \( B_r(x) = B(x, r) \) with \( r < R_0 \),
\[
\int_{B_r(x)} |u - u_{B_r(x)}| \, d\mu \leq C_P r \left( \int_{B_{\lambda r}(x)} [\text{Lip} u]^2 \, d\mu \right)^{1/2},
\]
(1.4)
where and in what follows, for each ball $B \subset X$, $u_B = \int_B u \, d\mu = \mu(B)^{-1} \int_B u \, d\mu$, and

$$\text{Lip } u(x) = \lim_{r \to 0} \sup_{d(x,y) \leq r} \frac{|u(x) - u(y)|}{r}.$$ 

Although our results work for $\lambda > 1$ as well, we will assume throughout the paper, that $\lambda = 1$, for simplicity. See [20, 18, 22] for more about the Poincaré inequality on metric measure spaces.

For a locally Lipschitz continuous function $u$, define its $H^{1,p}(X)$ norm $(p > 1)$ by

$$\|u\|_{H^{1,p}(X)} := \|u\|_{L^p(X)} + \|\text{Lip } u\|_{L^p(X)}.$$ 

Then the Sobolev space $H^{1,p}(X)$ is defined to be the completion of the set of all locally Lipschitz continuous functions $u$ with $\|u\|_{H^{1,p}(X)} < \infty$. By the work of Cheeger [9], we can assign a derivative to each Lipschitz function $u$. In what follows, let $D$ be a Cheeger derivative operator in $(X, d, \mu)$. It is shown in [9] that $|Du|$ is comparable to $\text{Lip } u$ for each locally Lipschitz continuous function $u$, and $D$ satisfies the Leibniz rule; see Section 2 for details. Actually, the construction of $D$ is irrelevant for our approach as long as $D$ has the properties above and comes with an associated inner product, with $Du \cdot Du$ comparable to the square of $\text{Lip } u$. In the Riemannian setting, we simply consider $\nabla u$ with the Riemannian inner product $\langle \nabla u, \nabla \phi \rangle$. The local Sobolev space $H^{1,p}_{\text{loc}}(X)$ is defined as usual. For an open set $U \subset X$, the space $H^{1,p}_{\text{loc}}(U)$ is defined to be the closure in $H^{1,p}(X)$ of Lipschitz functions with compact support in $U$.

Let $\Omega \subseteq X$ be a domain. As in the Riemannian setting, a Sobolev function $u \in H^{1,2}(\Omega)$ is called a solution of $\Delta u = g$ in $\Omega$, if

$$- \int_{\Omega} Du(x) \cdot D\phi(x) \, d\mu(x) = \int_{\Omega} g(x)\phi(x) \, d\mu(x), \quad \forall \phi \in H^{1,2}_{0}(\Omega).$$

Biroli and Mosco [5] studied the Poisson equation by assuming that $\mu$ is doubling and that a 2-Poincaré inequality holds. In their paper, the Green function, existence of solutions and Hölder continuity of solutions are studied. We remark that the Hölder continuity in [5] is obtained from Moser iteration and the exponent of Hölder continuity is not of exact form. For potential theory on metric spaces, we refer to [6].

Our main aim is to establish a Moser-Trudinger type inequality and Sobolev inequality for the gradients of solutions. Thus, modelling (1.3), we assume the following curvature condition. Assume that there exists a nonnegative function $c_s(T)$ on $(0, \infty)$ such that for each $0 < t < T$ and every $g \in H^{1,2}(X)$, we have

$$\int_X g(y)^2 p(t, x, y) \, d\mu(y) \leq (2t + c_s(T)T^2) \int_X |Dg(y)|^2 p(t, x, y) \, d\mu(y)$$

+ $\left( \int_X g(y)p(t, x, y) \, d\mu(y) \right)^2$  

for almost every $x \in X$, where $p(t, x, y)$ refers to the heat kernel associated to the Dirichlet form $\int_X Df \cdot Dg \, d\mu$, see Section 2 for details. In the Riemannian setting, $p$ is the usual heat kernel. The function $c_s(T)$ should be viewed as a consequence of some abstract lower curvature bound $-\kappa$, and it is non-decreasing as one can deduce from the assumption. Many examples in the classical smooth setting can be found in [2, 3, 10, 15, 37, 38].
Further examples include compact Alexandrov spaces with curvature bounded from below. It is well known that the (local) Poincaré inequality (1.4) holds on Alexandrov spaces with curvature bounded from below; see, for instance, [40]. Very recently, Gigli et al verified that (1.6) holds on them, see [13, Theorem 4.3].

Lott and Villani ([26]) and Sturm ([35, 36]) independently introduced and analyzed Ricci curvature in measure metric spaces via optimal mass transportation. On a metric space with Ricci curvature (in the sense of Lott-Sturm-Villani) bounded from below that additionally satisfies a local angle condition, a semi-concavity condition and that the pointwise Lipschitz constant coincides with the length of the gradient, (1.6) holds by results of Koskela and Zhou [24, Corollary 6.2] (that employ the contraction property of the gradient flow of entropy due to Savaré [30]).

Koskela et al [23] established the Lipschitz regularity of Cheeger-harmonic (i.e. $\Delta u = 0$) functions under the above assumptions. They also showed for the space $(X_\alpha, |\cdot|, dx)$, where $|\cdot|$ denotes the Euclidean metric, $dx$ the Lebesgue measure, $\alpha \in (\pi, 2\pi)$,

$$X_\alpha = \{(r \cos \phi, r \sin \phi) \in \mathbb{R}^2 : \phi \in [0, \alpha], r \geq 0\},$$

that (1.6) does not hold and that there exists a Cheeger-harmonic function which is not locally Lipschitz continuous. On the other hand, the space $(X_\alpha, |\cdot|, dx)$ with $\alpha \in (0, \pi]$ satisfies our assumptions. Under the same assumptions, for the Poisson equation $\Delta u = g$, the local Lipschitz continuity of solutions $u$ is established when $g \in L^p$ with $p \geq Q$ in [19].

We are in position to state our first gradient estimate.

Theorem 1.1. Let $Q \in (1, \infty)$ and assume that (1.4) and (1.6) hold. Then there exist $c, C > 0$ such that for all $u \in H^{1,2}(8B)$ and $g \in L^Q(8B)$ that satisfy $\Delta u = g$ in $8B$, where $B = B_R(y_0)$ with $256R < R_0$,

$$\int_B \exp \left( \frac{c|Du(x)|}{(1 + \sqrt{c_\alpha(R^2)R}C(u, g))^{\alpha/4}} \right)^{\frac{p}{\alpha}} \mu(x) \leq C,$$

where $C(u, g) = R^{-Q/2-1}\|u\|_{L^{2^*}(8B)} + \|g\|_{L^Q(8B)}$.

The technical requirement $8B$ and $R < R_0/256$ can certainly be relaxed. The point is that, in the abstract setting, when dealing with an equation that $\Delta u = g$ in $\lambda B$ for some $\lambda > 1$, we need to consider an auxiliary equation in a ball bigger than $\lambda B$; see our arguments in Section 4.

Let us consider the Poisson equation $\Delta u = g$ with $g \in L^p_{\text{loc}}(X)$ and $p < Q$. Since $u$ belongs to $H^{1,2}_{\text{loc}}(X)$ by definition, it is then natural to restrict $p \in (2_*, Q) \cap (1, Q)$, where $2_* = \frac{Q}{Q+2}$. Notice that $2_* < 1$ only for $Q < 2$. We have the following result.

Theorem 1.2. Let $Q \in (1, \infty)$, $p \in (2_*, Q) \cap (1, Q)$ and assume that (1.4) and (1.6) hold. Then there exists a constant $C$ such that for all $u \in H^{1,2}(8B)$ and $g \in L^p(8B)$ that satisfy $\Delta u = g$ in $8B$,

$$\left( \int_B |Du|^p \, d\mu \right)^{1/p} \leq C(1 + \sqrt{c_\alpha(R^2)R}) \left\{ R^{-1} \left( \int_{8B} |u|^2 \, d\mu \right)^{1/2} + R \left( \int_{8B} |g|^p \, d\mu \right)^{1/p} \right\},$$

where $B = B_R(y_0)$ with $R < R_0/256$ and $p^* = \frac{Qp}{Q-2}$. 

R. Jiang
How to prove the above results? As mentioned above, we use a semigroup approach. This method was introduced in [8] in the Euclidean setting to study variable coefficient parabolic equations, and was applied in [23] to Lipschitz continuity of Cheeger-harmonic functions; see Section 3 below. By using this method, for the auxiliary equation $\Delta v = g\chi_{\Omega}$ in $256B$, we obtain a pointwise estimate for the gradient of $v$ by generalized Riesz potentials based on the heat semigroup. By using the mapping properties of the generalized Riesz potentials, we then establish the above two theorems for the solutions of the auxiliary equations. Then, for general solutions of the Poisson equation, Theorem 1.1 and Theorem 1.2 follow by using density arguments and the theory of Cheeger-harmonic functions.

As a corollary to Theorem 1.2, we have the following Hölder-continuity estimate.

**Corollary 1.1.** Let $Q \in (1, \infty)$, $p \in (\frac{Q}{2}, Q) \cap (1, Q)$ and assume that (1.4) and (1.6) hold. Suppose that $u \in H^{1,2}_{\text{loc}}(\Omega)$ satisfies $\Delta u = g$ with $g \in L^p_{\text{loc}}(\Omega)$, where $\Omega \subseteq X$ is a domain. Then $u$ is locally Hölder continuous with exponent $2 - \frac{Q}{p}$ in $\Omega$.

The paper is organized as follows. In Section 2, we give some basic notation and notions for Cheeger derivatives, Dirichlet forms and Orlicz spaces. Several auxiliary results regarding Poisson equations are also given in Section 2. Section 3 is devoted to introducing the method and some estimates. We study auxiliary equations in Section 4 and prove Theorem 1.1 and Theorem 1.2 for the solutions of the auxiliary equations. The main results are proved in Section 5.

Finally, we make some conventions. Throughout the paper, we denote by $C, c$ positive constants which are independent of the main parameters, but which may vary from line to line. The symbol $B_R(x) = B(x, R)$ denotes an open ball with center $x$ and radius $R$ and $B_{CR}(x) = CB_R(x) = B(x, CR)$. For $p \in (1, Q)$, denote $\frac{Qp}{Q-p}$ by $p^*$, and for $p \in (1, \infty)$, denote $\frac{Qp}{Q+p}$ by $p_*$.

## 2 Preliminaries

In this section, we give some basic notation and notions and several auxiliary results.

### 2.1 Cheeger Derivative in metric measure spaces

Let $(X,d,\mu)$ be a metric measure space with $\mu$ Ahlfors $Q$-regular for some $Q > 1$. Cheeger [9] generalized Rademacher’s theorem of differentiability of Lipschitz functions on $\mathbb{R}^n$ to metric measure spaces. Precisely, the following theorem provides us the differential structure.

**Theorem 2.1.** Assume that $(X,\mu)$ supports a weak $p$-Poincaré inequality for some $p > 1$ and that $\mu$ is doubling. Then there exists $N > 0$, depending only on the doubling constant and the constants in the Poincaré inequality, such that the following holds. There exists a countable collection of measurable sets $U_\alpha$, $\mu(U_\alpha) > 0$ for all $\alpha$, and Lipschitz functions $X^{\alpha}_1, \ldots, X^{\alpha}_{k(\alpha)} : U_\alpha \to \mathbb{R}$, with $1 \leq k(\alpha) \leq N$ such that $\mu(X \setminus \cup_{\alpha=1}^\infty U_\alpha) = 0$, and for all $\alpha$ the following holds: for $f : X \to \mathbb{R}$ Lipschitz, there exist $V_\alpha(f) \subseteq U_\alpha$ such that $\mu(U_\alpha \setminus V_\alpha(f)) = 0$, and Borel functions $b_1^\alpha(x, f), \ldots, b_{k(\alpha)}^\alpha(x, f)$ of class $L^\infty$ such that if $x \in V_\alpha(f)$, then

$$\text{Lip}(f - a_1X^{\alpha}_1 - \cdots - a_{k(\alpha)}X^{\alpha}_{k(\alpha)})(x) = 0$$
if and only if \((a_1, \cdots, a_{k(\alpha)}) = (b_1^\alpha(x, f), \cdots, b_{k(\alpha)}^\alpha(x, f))\). Moreover, for almost every \(x \in U_{a_1} \cap U_{a_2}\), the “coordinate functions” \(X_i^{\alpha}\) are linear combinations of the \(X_i^{\alpha'}\)’s.

By Theorem 2.1, for each Lipschitz function \(u\) we can assign a derivative \(Du\), which we call Cheeger derivative following [23]. For each locally Lipschitz function \(f\), we define \(\text{lip} f\) by

\[
\text{lip} f(x) = \liminf_{r \to 0} \sup_{d(x, y) \leq r} \frac{|f(x) - f(y)|}{r}.
\]

By [9], under the assumptions of Theorem 2.1, for each locally Lipschitz \(f\), \(\text{Lip} f\) and \(\text{lip} f\) coincide with the minimal upper gradient \(g_u\) of \(u\) almost everywhere, and they all are comparable to \(|Du|\). See also [21].

By [31] and [9], the Sobolev spaces \(H^{1, p}(X)\) are isometrically equivalent to the Newtonian Sobolev spaces \(N^{1, p}(X)\) defined in [31] for \(p \geq 2\). Franchi et al [11] further showed that the differential operator \(D\) can be extended to all functions in the corresponding Sobolev spaces. A useful fact is that the Cheeger derivative satisfies the Leibniz rule, i.e., for all \(u, v \in H^{1, 2}(X)\),

\[
D(\alpha)(uv)(x) = u(x)Dv(x) + v(x)Du(x).
\]

### 2.2 Dirichlet forms and heat kernels

Having defined the Sobolev spaces \(H^{1, p}(X)\) and the differential operator \(D\), we now consider Dirichlet forms on \((X, \mu)\). Define the bilinear form \(\mathcal{E}\) by

\[
\mathcal{E}(f, g) = \int_X Df(x) \cdot Dg(x) \, d\mu(x)
\]

with the domain \(D(\mathcal{E}) = H^{1, 2}(X)\). It is easy to see that \(\mathcal{E}\) is symmetric and closed. Corresponding to such a form there exists an infinitesimal generator \(A\) which acts on a dense subspace \(D(A)\) of \(H^{1, 2}(X)\) so that for all \(f \in D(A)\) and each \(g \in H^{1, 2}(X)\),

\[
\int_X g(x)Af(x) \, d\mu(x) = -\mathcal{E}(g, f).
\]

Now let us recall several auxiliary results established in [23].

**Lemma 2.1.** If \(u, v \in H^{1, 2}(X)\), and \(\phi \in H^{1, 2}(X)\) is a bounded Lipschitz function, then

\[
\mathcal{E}((\phi)(uv)) = \mathcal{E}((\phi)u, v) + \mathcal{E}(\phi, uv) - 2 \int_X \phi Du(x) \cdot Dv(x) \, d\mu(x).
\]

Moreover, if \(u, v \in D(A)\), then we can unambiguously define the measure \(A(\alpha)(uv)\) by setting

\[
A(\alpha)(uv) = uAu + vAv + 2Du \cdot Dv.
\]

Also, associated with the Dirichlet form \(\mathcal{E}\), there is a semigroup \(\{T_t\}_{t \geq 0}\), acting on \(L^2(X)\), with the following properties (see [12, Chapter 1]):

1. \(T_t \circ T_s = T_{ts}, \forall t, s > 0\),
2. \(\int_X |T_tf(x)|^2 \, d\mu(x) \leq \int_X f(x)^2 \, d\mu(x), \forall f \in L^2(X, \mu)\) and \(\forall t > 0\).
3. $T_t f \to f$ in $L^2(X, \mu)$ when $t \to 0$.
4. if $f \in L^2(X, \mu)$ satisfies $0 \leq f \leq C$, then $0 \leq T_t f \leq C$ for all $t > 0$,
5. if $f \in D(A)$, then $\frac{1}{t}(T_t f - f) \to Af$ in $L^2(X, \mu)$ as $t \to 0$, and
6. $AT_t f = \frac{\partial}{\partial t} T_t f$, $\forall t > 0$ and $\forall f \in L^2(X, \mu)$.

A measurable function $p : \mathbb{R} \times X \times X \to [0, \infty]$ is said to be a heat kernel on $X$ if

$$T_t f(x) = \int_X f(y)p(t, x, y) \, d\mu(y)$$

for every $f \in L^2(X, \mu)$ and all $t \geq 0$, and $p(t, x, y) = 0$ for every $t < 0$. Let the measure on $X$ be doubling (i.e., $\mu(2B) \leq C_d \mu(B)$ for each ball $B$) and assume that the 2-Poincaré inequality (1.4) holds. Sturm ([34]) proved the existence of a heat kernel and a Gaussian estimate for the heat kernel, which in our settings reads as: there exist positive constants $C, C_1, C_2$ such that

$$(2.1) \quad C^{-1} t^{-\frac{d}{2}} e^{-\frac{d|\log t|^2}{Ct}} \leq p(t, x, y) \leq Ct^{-\frac{d}{2}} e^{-\frac{d|\log t|^2}{Ct^2}}.$$ 

Moreover, the heat kernel is proved in [33] to be a probability measure, i.e., for each $x \in X$ and $t > 0$,

$$(2.2) \quad T_t 1(x) = \int_X p(t, x, y) \, d\mu(y) = 1.$$ 

The following lemma was established in [23].

**Lemma 2.2.** Let $T > 0$. Then for $\mu$-almost every $x \in X$, $D_x p(\cdot, x, \cdot) \in L^2([0, T] \times X)$ and there exists a positive constant $C_{T,x}$, depending on $T$ and $x$, such that

$$\int_0^T \int_X |D_x p(t, x, y)|^2 \, d\mu(y) \, dt \leq C_{T,x}.$$ 

By a slight modification to the proof of [23, Lemma 3.3], we deduce the following estimate.

**Lemma 2.3.** There exist $c, C > 0$ such that for every $x \in X$,

$$\int_0^s \int_{B(x) \times B(x)} |D_x p(t, x, y)|^2 \, d\mu(y) \, dt \leq CR^{-2c} e^{-cR^2/2s},$$

whenever $R > 0$ and $s \in (0, R^2]$.

### 2.3 Orlicz and Zygmund spaces

A continuous, strictly increasing function $\Phi : [0, \infty] \to [0, \infty]$ with $\Phi(0) = 0$ and $\Phi(\infty) = \infty$ is called an Orlicz function. If $\Phi$ is also convex, then $\Phi$ is called a Young function. The Orlicz space $\Phi(X)$ is then defined to be the space of all measurable functions $f$ with $\int_X \Phi(f) \, d\mu < \infty$. For $f \in \Phi(X)$, we define its Luxemburg norm as

$$\|f\|_{\Phi(X)} := \inf \left\{ \lambda > 0 : \int_X \Phi \left( \frac{|f|}{\lambda} \right) \, d\mu \leq 1 \right\}.$$
For a Young function $\Phi$, the space $\Phi(X)$ is then a Banach space; see [28].

Functions of the type

$$\Phi_\alpha(t) = t \log^\alpha(e + t)$$

with $\alpha > 0$ are of particular importance for us. For such functions, the spaces $\Phi_\alpha(X)$ are also called Zygmund spaces. The complementary function of $\Phi_\alpha$, $\Psi_{1/\alpha}$, is equivalent to $\exp t^{1/\alpha} - 1$. Moreover, we have the Orlicz-Hölder inequality

$$\|fg\|_{L^1(X)} \leq C\|f\|_{\Phi_\alpha(X)}\|g\|_{\Psi_{1/\alpha}(X)},$$

(2.3)

where $C$ depends only on $Q$ and $\alpha$; see [28, 1].

Since our aim is to prove a Moser-Trudinger type inequality, of the form

$$\int_{B_R(y_0)} \exp (c|f|) \frac{\mu}{R^Q} \, d\mu \leq C,$$

in what follows, we modify the Orlicz function $\Psi_\alpha(t) = \exp t^\alpha - 1$ to the new function

$$\Psi_{R,\alpha}(t) = \frac{e^{\frac{t^\alpha}{R^Q}} - 1}{R^Q},$$

where $\alpha, R \in (0, \infty)$. Then the complementary function $\Phi_{R,1/\alpha}(t)$ of $\Psi_{R,\alpha}$ is equivalent to $t[\log(e + R^Q t)]^{1/\alpha}$. Moreover, $\Psi_{R,\alpha}$ and $\Phi_{R,1/\alpha}$ satisfy the Orlicz-Hölder inequality

$$\|fg\|_{L^1(X)} \leq C\|f\|_{\Psi_{R,\alpha}(X)}\|g\|_{\Phi_{R,1/\alpha}(X)},$$

(2.4)

2.4 Several auxiliary results

We first recall the Sobolev-Poincaré inequalities, which follow from the Poincaré inequality, see [4, 16, 17, 29]. There exist positive constants $c$, $C$, only depending on $C_P$ and $C_Q$, such that for all $u \in H^{1,2}_0(B_r(x))$ with $r \leq R_0$

$$\|u\|_{L^2(B_r(x))} \leq C\|Du\|_{L^2(B_r(x))},$$

(2.5)

when $Q > 2$; while

$$\int_{B_r(x)} \exp \left(\frac{c|u|}{\|Du\|_{L^2(B_r(x))}}\right)^2 \, d\mu \leq C$$

(2.6)

for $Q = 2$; and for $Q \in (1, 2)$

$$\|u\|_{L^\infty(B_r(x))} \leq Cr^{1-Q/2}\|Du\|_{L^2(B_r(x))},$$

(2.7)

Lemma 2.4. Let $Q \in (1, \infty)$ and $p \in (\frac{Q}{2}, \infty] \cap (1, \infty]$. Then there exists $C > 0$ such that for all $u \in H^{1,2}_0(B)$ and $g \in L^p(B)$ that satisfy $\Delta u = g$ in $B$, where $B = B_R(y_0)$ with $R < R_0$,

$$\|u\|_{L^\infty(B)} \leq CR^2\mu(B)^{-1/p}\|g\|_{L^p(B)}.$$

(2.8)
Proof. We note that [5, Theorem 4.1] states that the above inequality holds for $p > \max\{\frac{Q}{2}, 2\}$, assuming that the measure is doubling. As the proof is similar to that of [5, Theorem 4.1], we here give a sketch of proof to indicate the difference of the range of $p$.

For $k \in \mathbb{N}$, let

$$\zeta_k(u) := \max\{|u - k, 0\} - \min\{|u + k, 0\},$$

and $\Lambda(k) := \{x \in B: |u| > k\}$. Then we have $\zeta_k(u) \in H_0^{1,2}(B)$. Taking a truncation argument as in [5, p.146], we arrive at

$$\int_B |D\zeta_k(u)|^2 \, d\mu \leq \int_B g\zeta_k(u) \, d\mu.$$

Let us first assume that $Q > 2$. Then by the Sobolev inequality and the Hölder inequality, we obtain

$$\int_B |D\zeta_k(u)|^2 \, d\mu \leq \left(\int_{A(k)} |g|^{2^*} \, d\mu\right)^{1/2^*} \|\zeta_k(u)\|_{L^{2^*}(B)} \leq C \mu(A(k))^{1/2^* - 1/p} \|g\|_{L^p(B)} \|D\zeta_k(u)\|_{L^{2^*}(B)},$$

hence, $\|D\zeta_k(u)\|_{L^{2^*}(B)} \leq C \mu(A(k))^{1/2^* - 1/p} \|g\|_{L^p(B)}$. Applying the Sobolev inequality again, we conclude that

$$\left(\int_B |\zeta_k(u)|^2 \, d\mu\right)^{1/2} \leq C \left(\int_B |D\zeta_k(u)|^2 \, d\mu\right)^{1/2} \leq C \mu(A(k))^{1/2^* - 1/p} \|g\|_{L^p(B)}.$$

From this inequality, we further deduce that for $h > k > 0$, we have

$$(h - k)\mu(A(h))^{1/2^*} \leq \left(\int_B |\zeta_k(u)|^2 \, d\mu\right)^{1/2^*} \leq C \mu(A(k))^{1/2^* - 1/p} \|g\|_{L^p(B)},$$

and hence,

$$\mu(A(h)) \leq (C\|g\|_{L^p(B)})^{2^*} \frac{\mu(A(k))^{1/2^* - 1/p} (h - k)^{2^*}}{(h - k)^{2^*}}.$$

By the fact that $(\frac{1}{2^*} - \frac{1}{p})2^* > 1$ and an argument as [5, p.147], we conclude that $\mu(A(d)) = 0$, for $d = CR^2 \mu(B)^{-1/p} \|g\|_{L^p(B)}$. Hence, we obtain that $\|u\|_{L^\infty(B)} \leq CR^2 \mu(B)^{-1/p} \|g\|_{L^p(B)}$.

The proof of $Q = 2$ is similar to the above argument, except when applying the Sobolev inequality, we need to choose a sufficient large exponent, depending on $p$, to substitute for $2^*$. We omit the details.

When $Q \in (1, 2)$, by (2.7) and the Hölder inequality, we have

$$\|u\|_{L^\infty(B)}^2 \leq CR^2 \mu(B)^{-1/p} \|g\|_{L^p(B)}^2 \leq CR^2 \mu(B)^{-1/p} \|g\|_{L^p(B)} \|u\|_{L^\infty(B)},$$

proving the lemma. $\square$

Recall that $\Phi_{R,1/\alpha}(t) = t[\log(e + R^2 t)]^{1/\alpha}$ and $\Psi_{R,\alpha}(t) = \frac{1}{R^2}(e^{Rt} - 1)$. 

---
Lemma 2.5. Let $Q \in (1, \infty)$ and $p \in [1, \infty]$. Then there exists $C > 0$, depending on $p, Q$, such that for all $u \in H_0^{1,2}(B)$ and $g \in L^p(B)$ that satisfy $\Delta u = g$ in $B$, where $B = B_R(y_0)$ with $R < R_0$:

(i) when $Q > 2$ and $p = 2^*$, $\|Du\|_{L^2(B)} \leq C\|g\|_{L^2(B)}$;

(ii) when $Q = 2$, for any $p > 1$, $\|Du\|_{L^2(B)} \leq C\mu(B)^{1-1/p}\|g\|_{L^p(B)}$;

(iii) when $Q \in (1,2)$, $\|Du\|_{L^2(B)} \leq CR^{1-Q/2}\|g\|_{L^1(B)}$.

Proof. By using the Hölder inequality and (2.5), we conclude that

$$\int_B |Du(x)|^2\,dx - \int_B g(x)u(x)\,dx \leq \|g\|_{L^{1^-}(B)}\|u\|_{L^{2^*}(B)} \leq C\|g\|_{L^2(B)}\|Du\|_{L^2(B)}.$$

Hence, $\|Du\|_{L^2(B)} \leq C\|g\|_{L^2(B)}$, which proves (i).

For (ii), by (2.6), we see that for any $q \geq 1$,

$$\|u\|_{L^q(B_i(x))} \leq C\mu(B)^{1/q}\|Du\|_{L^2(B_i(x))}.$$

From this and the Hölder inequality, we deduce that

$$\int_B |Du(x)|^2\,dx - \int_B g(x)u(x)\,dx \leq \|g\|_{L^1(B)}\|u\|_{L^{2}(B)} \leq C\mu(B)^{1-1/p}\|g\|_{L^p(B)}\|Du\|_{L^2(B)},$$

which implies $\|Du\|_{L^2(B)} \leq \mu(B)^{1/p-1}\|g\|_{L^p(B)}$.

For (iii), by (2.7), we have

$$\int_B |Du(x)|^2\,dx = -\int_B g(x)u(x)\,dx \leq \|g\|_{L^1(B)}\|u\|_{L^{\infty}(B)} \leq C\|g\|_{L^1(B)}R^{1-Q/2}\|Du\|_{L^2(B)}$$

proving the lemma. \hfill \Box

Lemma 2.6. Let $Q \in (1, \infty)$, $p \in (Q, \infty] \cap (1, \infty)$ and $B = B_R(y_0)$ with $R < R_0$. For every $g \in L^p(B)$, there exists $u \in H_0^{1,2}(B)$ such that $\Delta u = g$ in $B$.

Proof. For each $k \in \mathbb{N}$, let $g_k = g\chi_{B\cap|g|\leq k}$. Then by [4, p.131], there exists $u_k \in H_0^{1,2}(B)$ such that $\Delta u_k = g_k$ in $B$. Moreover, by Lemma 2.4 and Lemma 2.5, we have

$$\|u_k - u\|_{L^2(B)} + \|D(u_k - u)\|_{L^2(B)} \leq C\|g_k - g\|_{L^p(B)} \to 0,$$

as $k, j \to \infty$. Hence $\{u_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $H_0^{1,2}(B)$, and there exists $u \in H_0^{1,2}(B)$ such that $\lim_{k \to \infty} u_k = u$ in $H_0^{1,2}(B)$. Moreover, for each $\phi \in H_0^{1,2}(B)$, we have

$$-\int_B Du(x) \cdot D\phi(x)\,dx = -\lim_{k \to \infty} \int_B Du_k(x) \cdot D\phi(x)\,dx = \lim_{k \to \infty} \int_B g_k(x)\phi(x)\,dx = \int_B g(x)\phi(x)\,dx,$$

proving the lemma. \hfill \Box

Combining Lemma 2.4 and Lemma 2.6, we deduce the following estimate.
Lemma 2.7. Let \( Q \in (1, \infty) \) and \( p \in (\frac{Q}{2}, \infty) \cap (1, \infty) \). Then there exists a positive constant \( C \) such that for all \( u \in H^{1,2}_{\text{loc}}(X) \) and \( g \in L^p_{\text{loc}}(X) \) that satisfy \( \Delta u = g \) in \( 2B \), where \( B = B_R(y_0) \) with \( R < R_0/2 \),

\[
\|u\|_{L^\infty(B)} \leq C[R^{-Q/2}\|u\|^R_{L^2(B)} + R^{2-Q/p}\|g\|_{L^p(B)}].
\]

Proof. By Lemma 2.6, there exists \( \bar{u} \in H^{1,2}_{0}(2B) \) such that \( \Delta \bar{u} = g \) in \( 2B \). Then from Lemma 2.4, we deduce that

\[
\|\bar{u}\|_{L^\infty(B)} \leq CR^2\mu(B)^{-1/p}\|g\|_{L^p(B)}.
\]

Now \( u - \bar{u} \) is Cheeger-harmonic in \( 2B \), which together with [5, Theorem 5.4] implies that

\[
\|u - \bar{u}\|_{L^\infty(B)} \leq CR^{-Q/2}\|u - \bar{u}\|_{L^2(B)}.
\]

The above two estimates give the desired results. \( \Box \)

We also need the Hölder continuity of the solutions.

Lemma 2.8. Let \( Q \in (1, \infty) \) and \( p \in (\frac{Q}{2}, \infty) \cap (1, \infty) \). Then there exist \( C > 0 \) and \( \gamma \in (0, 1) \) such that for all \( u \in H^{1,2}_{\text{loc}}(X) \) and \( g \in L^p_{\text{loc}}(X) \) that satisfy \( \Delta u = g \) in \( 4B \), where \( B = B_R(y_0) \) with \( R < R_0/4 \), and almost all \( x, y \in B \),

\[
|u(x) - u(y)| \leq C \left( R^{-Q/2}\|u\|^R_{L^2(4B)} + R^{2-Q/p}\|g\|_{L^p(4B)} \right) \left( \frac{d(x, y)}{R} \right) \gamma.
\]

Proof. Let \( M_2 = \sup_{B_{2R}(y_0)} u \), \( m_2 = \inf_{B_{2R}(y_0)} u \), \( M_1 = \sup_{B_R(y_0)} u \) and \( m_1 = \inf_{B_R(y_0)} u \). By Lemma 2.6, there exists \( \bar{u} \in H^{1,2}_{0}(2B_R(y_0)) \) such that \( \Delta \bar{u} = g \) in \( B_{2R}(y_0) \).

Let \( M_R = \|\bar{u}\|_{L^\infty(B_{2R}(y_0))} \). Applying [5, Theorem 1.1] to \( M_2 + M_R - (u - \bar{u}) \) and \( (u - \bar{u}) - m_2 + M_R \) respectively, we obtain that

\[
M_2 - m_1 \leq \sup_{B_R(y_0)} M_2 + M_R - (u - \bar{u}) \leq C_3 \inf_{B_R(y_0)} [M_2 + M_R - (u - \bar{u})] \leq C_3 [M_2 - M_1 + 2M_R],
\]

\[
M_1 - m_2 \leq \sup_{B_R(y_0)} (u - \bar{u}) - m_2 + M_R \leq C_3 \inf_{B_R(y_0)} [(u - \bar{u}) - m_2 + M_R] \leq C_3 [m_1 - m_2 + 2M_R].
\]

Adding the last two inequalities, we deduce that

\[
(C_3 + 1)(M_1 - m_1) \leq (C_3 - 1)(M_2 - m_2) + 4C_3 M_R.
\]

By Lemma 2.4, we conclude that for each \( p \in \left( \frac{Q}{2}, \infty \right) \cap (1, \infty) \),

\[
\text{osc}(u, B_R(y_0)) \leq \frac{C_3 - 1}{C_3 + 1} \text{osc}(u, B_{2R}(y_0)) + CR^{2-Q/p}\|g\|_{L^p(B_{2R}(y_0))},
\]

which together with a standard iteration as in [14, p.201] and Lemma 2.7 yields the desired estimate. \( \Box \)

By Lemma 2.7, similarly to the proof of [19, Lemma 2.2], we have the following Caccioppoli inequality.

Lemma 2.9. Let \( Q \in (1, \infty) \) and \( p \in (\frac{Q}{2}, \infty) \cap (1, \infty) \). Then there exists a positive constant \( C \) such that for all \( u \in H^{1,2}_{\text{loc}}(X) \) and \( g \in L^p_{\text{loc}}(X) \) that satisfy \( \Delta u = g \) in \( B_R(y_0) \), where \( r < R < R_0 \),

\[
\| Du \|^R_{L^2(B_r(y_0))} \leq CR^{1+Q(\frac{1}{2}-\frac{1}{p})}\|g\|^R_{L^p(B_R(y_0))} + \frac{C}{(R-r)}\|u\|^R_{L^2(B_R(y_0))}.
\]
3 Poisson equation

Let \( B = B_R(y_0) \subset \Omega \) satisfy \( 8B \subset \subset \Omega \). Let \( \psi \) be a Lipschitz function such that \( \psi = 1 \) on \( B_{2R}(y_0) \), \( \text{supp} \psi \subset B_{4R}(y_0) \) and \( |D\psi| \leq \frac{C}{R} \). For all \( x, x_0 \in 8B \), set \( w_{x_0}(t, x) := u\psi(x) - T_t(u\psi)(x_0) \). Then \( Dw_{x_0}(t, x_0) = D(u\psi)(x_0) = Du(x_0) \) for every \( x_0 \in B_{2R}(y_0) \).

The following functional is the main tool for us; see [8, 23, 19]. Let \( x_0 \in B = B_R(y_0) \). For all \( t \in (0, R^2) \), define

\[
J(t) := \frac{1}{t} \left\{ \int_0^t \int_X |Dw_{x_0}(s, x)|^2 p(s, x_0, x) \, d\mu(x) \, ds \right. \\
\left. + \int_0^t \int_X w_{x_0}(s, x)\psi(x)Au(x)p(s, x_0, x) \, d\mu(x) \, ds \right\}.
\]

(3.1)

The main aim of this section is to prove the following estimate.

**Theorem 3.1.** Let \( Q \in (1, \infty) \), \( p \in (\frac{Q}{2}, \infty) \cap (1, \infty) \) and assume that (1.4) and the curvature condition (1.6) hold. Then there exists \( C > 0 \) such that for all \( u \in H^{1,2}_{loc}(X) \) and \( g \in L^p_{loc}(X) \) that satisfy \( \Delta u = g \) in \( 8B \), where \( B = B_R(y_0) \) with \( R < R_0/8 \), and almost every \( x_0 \in B \),

\[
(3.2) |Du(x_0)|^2 \leq C(1 + c_p(R^2)R^2)C(u, g)^2 + \int_0^{R^2} \frac{1}{t} \int_X |w_{x_0}(t, x)\psi(x)g(x)| \, p(t, x_0, x) \, d\mu(x) \, dt,
\]

where \( C(u, g) = R^{-Q/2-1}\|u\|_{L^2(B)} + R^{1-Q/p}\|g\|_{L^p(B)} \).

**Remark 3.1.** In this paper, the curvature condition (1.6) is only employed once, in the proof of Theorem 3.1; see the proof at the end of this section.

Notice that \( w_{x_0}(0, x_0) = 0 \). We use the Hölder continuity of \( u \) to obtain the Hölder continuity of \( w_{x_0}(t, x) \) at \((0, x_0)\).

**Lemma 3.1.** Let \( Q \in (1, \infty) \) and \( p \in (\frac{Q}{2}, \infty) \cap (1, \infty) \). Then there exists \( C > 0 \) such that for all \( u \in H^{1,2}_{loc}(X) \) and \( g \in L^p_{loc}(X) \) that satisfy \( \Delta u = g \) in \( 8B \), where \( B = B_R(y_0) \) with \( R < R_0/8 \), and almost all \( x_0 \in B \), \( x \in 2B \) and all \( t \in (0, R^2) \),

\[
|w_{x_0}(t, x)| = |u\psi(x) - T_t(u\psi)(x_0)| \leq CC(u, g)R^{1-\gamma}(d(x, x_0)^\gamma + t^{\gamma/2}),
\]

where \( C(u, g) = R^{-Q/2-1}\|u\|_{L^2(B)} + R^{1-Q/p}\|g\|_{L^p(B)} \) and \( \gamma \in (0, 1) \) is as in Lemma 2.8.

**Proof.** In the following proof, we will repeatedly use the fact that for fixed \( \beta, \delta \in (0, \infty) \), \( \beta \theta^{-\beta}e^{-\delta \theta} \)
and \( t^{p/2}\theta^{-\beta}e^{-\delta \theta} \) are bounded on \((0, \infty)\).

By Lemma 2.8, we see that for almost all \( x_0 \in B \), \( x \in 2B \),

\[
|u(x) - u(x_0)| \leq CRC(u, g) \left( \frac{d(x, x_0)}{R} \right)^\gamma,
\]

where \( C \) and \( \gamma \) are independent of \( u, g \) and \( B \). Thus for almost all \( x_0 \in B, x \in 2B \) and all \( t \in (0, R^2) \),

by Lemma 2.7, we have

\[
|w_{x_0}(t, x)| = |u(x)\psi(x) - T_t(u\psi)(x_0)|
\]
Proposition 3.1. \( \text{Poisson Equation} \)

\[ |u(x)\psi(x) - u(x_0)\psi(x_0)| + u(x_0)\psi(x_0) - T_j(u)(x_0)| \]
\[ \leq CC(u, g)R^{1-\gamma}d(x, x_0)^\gamma + \int_{2B} |u(x_0)\psi(x_0) - u(y)\psi(y)|p(t, x_0, y) \, d\mu(y) \]
\[ + \int_{X \setminus 2B} |u(x_0)\psi(x_0) - u(y)\psi(y)|p(t, x_0, y) \, d\mu(y) \]
\[ \leq CC(u, g)R^{1-\gamma}d(x, x_0)^\gamma + CC(u, g)R^{1-\gamma} \int_{2B} d(y, x_0)^\gamma t^{-\frac{Q}{2}} e^{-\frac{Qy^2}{2 t^{1/2}}} - \frac{Qy^2}{2 t^{1/2}} \, d\mu(y) \]
\[ + e^{-cR^2/|t|} \int_{L^2(4B)} t^{-\frac{Q}{2}} e^{-\frac{Qy^2}{2 t^{1/2}}} \, d\mu(y) \]
\[ \leq CRC(u, g) \left[ R^{-\gamma}(d(x, x_0)^\gamma + t^{\gamma/2}) + e^{-cR^2/|t|} \right] \int_{X} p(t, x_0, x) \, d\mu(x) \]
\[ \leq CRC(u, g) \left[ R^{-\gamma}(d(x, x_0)^\gamma + t^{\gamma/2}) \right] \int_{X} p(t, x_0, x) \, d\mu(x) \]

where \( l = \frac{2c}{c_2^2} \), as desired. \( \square \)

The following result shows the motivation for using the functional \( J \).

Proposition 3.1. Let \( Q \in (1, \infty) \), \( p \in (\frac{Q}{2}, \infty) \cap (1, \infty) \) and \( B = B_R(y_0) \) with \( R < R_0/8 \). Suppose that \( u \in H^{1, 2}_{loc}(X) \) and \( g \in L^p_{loc}(X) \) satisfy \( \Delta u = g \) in \( 8B \). Then, for almost every \( x_0 \in B \), \( \lim_{t \to 0} J(t) = |Du(x_0)|^2 \).

Proof. By Lemma 2.2, for almost every \( x_0 \in B \), \( D_y p(s, x_0, \cdot) \in L^2(X) \). From this together with the fact that for almost every \( s \), \( w_{x_0}(s, \cdot) \), \( p(s, x_0, \cdot) \) are bounded functions and belong in \( H^{1, 2}_{loc}(X) \), \( \text{supp} \psi \subset 4B \), we see that \( w_{x_0, \psi}p \in H^{1, 2}_{loc}(B(y_0, 4R)) \). Thus, we conclude that

\[ (3.3) \int_0^t \int_X w_{x_0}(s, x)\psi(x)Au(x)p(s, x_0, x) \, d\mu(x) = \int_0^t \int_{4B} w_{x_0}(s, x)\psi(x)g(x)p(s, x_0, x) \, d\mu(x). \]

By Lemma 3.1, \( |w_{x_0}(s, x)| \leq CC(u, g)R^{1-\gamma}(d(x, x_0)^\gamma + s^{\gamma/2}) \) for some \( \gamma \in (0, 1) \) and almost every \( x \in 2B \). This further implies that

\[ \left| \int_0^t \int_X w_{x_0}(s, x)p(s, x_0, x)\psi(x)Au(x) \, d\mu(x) \, ds \right| \]
\[ \leq CC(u, g)R^{1-\gamma} \int_0^t \int_{2B} (d(x, x_0)^\gamma + s^{\gamma/2})s^{-\frac{Q}{2}} e^{-\frac{d(x, x_0)^2}{c t^{1/2}}} \, |g(x)| \, d\mu(x) \, ds \]
\[ + C||u||_{L^\infty(4B)} \int_0^t \int_{4B \setminus 2B} s^{-\frac{Q}{2}} e^{-cR^2/|t|} \, |g(x)| \, d\mu(x) \, ds \]
\[ \leq CC(u, g)R^{1-\gamma} \int_0^t s^{\gamma/2} \int_X s^{-\frac{Q}{2}} e^{-\frac{d(x, x_0)^2}{c t^{1/2}}} \, |g(x)| \, d\mu(x) \, ds + C t^2 \|u\|_{L^\infty(4B)} R^{-Q/2} \|g\|_{L^1(4B)} \]
\[ \leq CC(u, g)R^{1-\gamma} \int_0^t s^{\gamma/2} T_i_s(|g|)(x_0) \, ds + C t^2 \|u\|_{L^\infty(4B)} R^{-Q/2} \|g\|_{L^1(4B)}, \]
where \( l = \frac{C_1}{2C_2} \). By the fact that \( T_t - I \to 0 \) in the strong operator topology as \( t \to 0 \), we obtain

\[
\lim_{t \to 0^+} \left| \frac{1}{t} \int_0^t \int_X w_{x_0}(s, x)p(s, x_0, x)\psi(x)Au(x) \, d\mu(x) \, ds \right|
\leq \lim_{t \to 0^+} \left\{ CC(u, g)R^{1-\gamma} \frac{1}{t} \int_0^s \gamma^{7/2} T_{ls}(|g|(x_0)) \, ds + C\|u\|_{L^\infty(4B)}R^{-\gamma/2} \|g\|_{L^1(4B)} \right\}
\]

(3.4) \quad = CC(u, g)R^{1-\gamma} \lim_{s \to 0^+} \gamma^{7/2} T_{ls}(|g|(x_0)) = 0,

for almost every \( x_0 \in B_R(y_0) \), which implies that

\[
\lim_{t \to 0^+} J(t) = \lim_{s \to 0^+} T_s(|Du\psi|)^2(x_0) = |Du(x_0)|^2
\]

for almost every \( x_0 \in B_R(y_0) \), proving the proposition. \( \square \)

By Lemma 3.1, similarly to [23, (24)] and [19, (3.5)], we deduce the following equality. We omit the details.

**Lemma 3.2.** Let \( Q \in (1, \infty) \), \( p \in (\frac{Q}{2}, \infty] \cap (1, \infty) \) and \( B = B_R(y_0) \) with \( R < R_0/8 \). Suppose that \( u \in H^{1,2}_{10}(X) \) and \( g \in L^p_{\text{loc}}(X) \) that satisfy \( \Delta u = g \) in \( 8B \). Then for almost every \( x \in B \) and all \( t \in (0, R^2) \),

\[
\int_0^t \int_X \left( A + \frac{\partial}{\partial s} \right) w_{x_0}^2(s, x)p(s, x_0, x) \, d\mu(x) \, ds = \int_X w_{x_0}^2(t, x)p(t, x_0, x) \, d\mu(x).
\]

We now begin to estimate the functional \( J(t) \).

**Proposition 3.2.** Let \( Q \in (1, \infty) \) and \( p \in (\frac{Q}{2}, \infty] \cap (1, \infty) \). Then there exists \( C > 0 \) such that for all \( u \in H^{1,2}_{10}(X) \) and \( g \in L^p_{\text{loc}}(X) \) that satisfy \( \Delta u = g \) in \( 8B \), where \( B = B_R(y_0) \) with \( R < R_0/8 \), and almost every \( x_0 \in B \),

\[
J(R^2) \leq C \left( \frac{\|u\|_{L^2(8B)}}{R^{Q/2+1}} + \frac{\|g\|_{L^p(8B)}}{R^{Q/p-1}} \right)^2.
\]

**Proof.** Since \( w_{x_0}(t, x) = u(x)\psi(x) - T_t(u\psi(x))(x_0) \), we have

\[
|Du\psi|^2 = |Dw_{x_0}|^2 = \frac{1}{2} Aw_{x_0}^2 - m(x_0)(Au + uA\psi + 2 Du \cdot D\psi)
\]

in the weak sense of measures. Also, in what follows we extend \( A \) formally to all of \( H^{1,2}(X) \) by defining

\[
\int_X v(x)Au(x) \, d\mu(x) = - \int_X Dv(x) \cdot Du(x) \, d\mu(x) = \int_X Av(x)u(x) \, d\mu(x).
\]

Moreover, we set \( m(t) = T_t(u\psi)(x_0) \). Then \( \frac{\partial}{\partial t} w_{x_0}^2 = 2 w_{x_0} \frac{\partial}{\partial t} w_{x_0} = -2 w_{x_0} m'(t) \), which further implies that

\[
|Dw_{x_0}|^2 = \frac{1}{2} \left( A + \frac{\partial}{\partial t} \right) w_{x_0}^2 - m(x_0)(Au + uA\psi + 2 Du \cdot D\psi - m'(t)).
\]
in the weak sense of measures. Thus, we obtain

\[
\int_0^t \int_X |Dw_{x_0}(s, x)|^2 p(s, x_0, x) \, d\mu(x) \, ds
\]

\[
= \frac{1}{2} \int_0^t \int_X \left(A + \frac{\partial}{\partial s}\right) w_{x_0}^2(s, x) p(s, x_0, x) \, d\mu(x) \, ds
\]

\[
- \int_0^t \int_X w_{x_0}(s, x)[\psi Au + uA\psi + 2Du \cdot D\psi - m'(s)] p(s, x_0, x) \, d\mu(x) \, ds.
\]

(3.5)

Recall that for each \( s > 0 \) and \( x_0 \in X, T_s(1)(x_0) = 1 \). We then have

\[
\int_0^t \int_X w_{x_0}(s, x)m'(s)p(s, x_0, x) \, d\mu(x) \, ds = \int_0^t \int_X m'(s)T_s(1)(x_0)(1 - T_s(1)(x_0)) \, ds = 0.
\]

We now estimate the second term in (3.5). Recall that \( \psi = 1 \) on \( 2B = 2B_R(0) \) and \( \text{supp} \, \psi \subseteq 4B \).

By Lemma 2.7, Lemma 2.9, Lemma 2.3 and the Hölder inequality, we obtain

\[
\left| \int_0^t \int_X w_{x_0}(s, x)u(x)A\psi(x)p(s, x_0, x) \, d\mu(x) \, ds \right|
\]

\[
= \left| \int_0^t \int_X D(w_{x_0}(s, \cdot)u(p(s, x_0, \cdot))(x) \cdot D\psi(x) \, d\mu(x) \, ds \right|
\]

\[
\leq Ct^{1/2}R^{-1+s} \|u\|^2_{L^\infty(4B)} \left( \int_0^t \int_{S_B(x_0)} \right)\right)^{1/2}
\]

\[
+ CtR^{-1+s}e^{-\frac{R^2}{2}\tau} \|u\|^2_{L^\infty(4B)} \left( \int_0^t \int_{2B} \right)\right)^{1/2}
\]

\[
\leq Ct^{1/2}R^{-1+s}e^{-\frac{R^2}{2}\tau} \|u\|^2_{L^\infty(4B)} + CtR^{-1+s}e^{-\frac{R^2}{2}\tau} \|u\|^2_{L^\infty(4B)} \|Du\|^2_{L^2(4B)}
\]

\[
\leq Ct e^{-cR^2/\tau}(R^{-1+s} \|u\|^2_{L^2(8B)} + R^{1-s} \|g\|^2_{L^p(8B)})^2.
\]

Similarly, we have

\[
\left| \int_0^t \int_X w_{x_0}(s, x)p(s, x_0, x)Du(x) \cdot D\psi(x) \, d\mu(x) \, ds \right|
\]

\[
\leq Ct e^{-cR^2/\tau}(R^{-1+s} \|u\|^2_{L^2(8B)} + R^{1-s} \|g\|^2_{L^p(8B)})^2.
\]

Combining the above estimates, by (3.5) and Lemma 3.2, we obtain that

\[
tJ(t) \leq \frac{1}{2} \left| \int_0^t \int_X \left(A + \frac{\partial}{\partial s}\right) w_{x_0}^2(s, x) p(s, x_0, x) \, d\mu(x) \, ds \right|
\]

\[
+ \left| \int_0^t \int_X w_{x_0}(s, x)[u(x)A\psi(x) + 2Du(x) \cdot D\psi(x)] p(s, x_0, x) \, d\mu(x) \, ds \right|
\]

\[
\leq \frac{1}{2} \int_X w_{x_0}^2(t, x)p(t, x_0, x) \, d\mu(x) + Ct e^{-cR^2/\tau}(\frac{\|u\|^2_{L^2(8B)}}{R^{2s-1}} + \frac{\|g\|^2_{L^p(8B)}}{R^{p-1}})^2.
\]

(3.6)
Hence, by Lemma 2.7 again, we conclude that

\[
J(R^2) \leq \frac{1}{2R^2} \int_X w_{x_0}^2(R^2, x)p(R^2, x_0, x) \, d\mu(x) + C \left( \frac{\|u\|_{L^2(8B)}^2}{R^{Q/2+1}} + \frac{\|g\|_{L^p(8B)}^2}{R^{Q/p-1}} \right)
\]

\[
\leq \frac{1}{2R^2} \|u\|_{L^\infty(4B)}^2 \int_X p(R^2, x_0, x) \, d\mu(x) + C \left( \frac{\|u\|_{L^2(8B)}^2}{R^{Q/2+1}} + \frac{\|g\|_{L^p(8B)}^2}{R^{Q/p-1}} \right)
\]

\[
\leq C \left( \frac{\|u\|_{L^2(8B)}^2}{R^{Q/2+1}} + \frac{\|g\|_{L^p(8B)}^2}{R^{Q/p-1}} \right)^2,
\]

which completes the proof of Proposition 3.2. \hfill \Box

We use the Hölder continuity (Lemma 3.1) of \( w_{x_0}(t, x) \) to deduce the following estimate.

**Proposition 3.3.** Let \( Q \in (1, \infty) \) and \( p \in (\frac{Q}{2}, \infty] \cap (1, \infty] \). Then there exists \( C > 0 \) such that for all \( u \in H^1_{\text{loc}}(X) \) and \( g \in L^p_{\text{loc}}(X) \) that satisfy \( \Delta u = g \) in \( 8B \), where \( B = B_{R}(y_0) \) with \( R < R_0/8 \), and almost every \( x_0 \in B \),

\[
\int_0^{R^2} \frac{1}{t} \int_X w_{x_0}^2(t, x)p(t, x_0, x) \, d\mu(x) \, dt \leq CR^2(C(u, g))^2,
\]

where \( C(u, g) = R^{-Q/2-1}\|u\|_{L^2(8B)} + R^{1-Q/p}\|g\|_{L^p(8B)} \).

**Proof.** By Lemma 3.1, we deduce that

\[
\int_X w_{x_0}^2(t, x)p(t, x_0, x) \, d\mu(x) = \int_{2B} w_{x_0}^2(t, x)p(t, x_0, x) \, d\mu(x) + \int_{X\setminus 2B} w_{x_0}^2(t, x)p(t, x_0, x) \, d\mu(x)
\]

\[
\leq C(C(u, g)^{1-\gamma})^2 \int_{2B} (d(x, x_0)^2 + t^{\gamma/2})^2 t^{-\frac{\gamma}{2}} e^{-\frac{d(x, x_0)^2}{2ct^\gamma}} \, d\mu(x)
\]

\[
+ C\|u\|_{L^\infty(4B)}^2 \int_{X\setminus 2B} t^{-\frac{\gamma}{2}} e^{-\frac{d(x, x_0)^2}{2ct^\gamma}} \, d\mu(x)
\]

\[
\leq C(C(u, g)^{-2\gamma})^2 \int_X p(t, x_0, x) \, d\mu(x) + Ce^{-e^{-cR^2/t}} \|u\|_{L^\infty(4B)}^2 \int_{X\setminus 2B} p(t, x_0, x) \, d\mu(x)
\]

\[
\leq CR^2(C(u, g)^2[R^{-2\gamma}t^{\gamma} + e^{-cR^2/t}]) \int_X p(t, x_0, x) \, d\mu(x)
\]

\[
\leq CR^2(C(u, g)^2R^{-2\gamma}t^{\gamma}),
\]

where \( l = \frac{2c}{c^\gamma} \) and we used the fact that \( e^{-cR^2/t} \leq C(\frac{R}{t})^\gamma \). From this, we further conclude that

\[
\int_0^{R^2} \frac{1}{t} \int_X w_{x_0}^2(t, x)p(t, x_0, x) \, d\mu(x) \leq \int_0^{R^2} CC(u, g)^2R^{-2\gamma}t^{\gamma-1} \, dt \leq CR^2(C(u, g))^2,
\]

which completes the proof of Proposition 3.3. \hfill \Box
We are now in position to prove the main result of this section.

**Proof of Theorem 3.1.** Let us first estimate the derivative $J'(t) = \frac{d}{dt} J(t)$. By (3.3), (3.1) and (3.6), we deduce that

$$
\frac{d}{dt} J(t) = -\frac{1}{t^2} J(t) + \frac{1}{t} \int_X |Dw_{x_0}(t, x)|^2 p(t, x_0, x) d\mu(x)
$$

$$
+ \frac{1}{t} \int_X w_{x_0}(t, x) \psi(x) g(x) p(t, x_0, x) d\mu(x)
$$

$$
\geq \frac{1}{t} \left( \int_X |Dw_{x_0}(t, x)|^2 p(t, x_0, x) d\mu(x) - \frac{1}{2t} \int_X w_{x_0}^2(t, x) p(t, x_0, x) d\mu(x) \right)
$$

$$
- \frac{C}{t} e^{-cR^2/t} C(u, g)^2 + \frac{1}{t} \int_X w_{x_0}(t, x) \psi(x) g(x) p(t, x_0, x) d\mu(x).
$$

For each fixed $t \in (0, R^2)$, either

$$
\int_X |Dw_{x_0}(t, x)|^2 p(t, x_0, x) d\mu(x) \geq \frac{1}{2t} \int_X w_{x_0}^2(t, x) p(t, x_0, x) d\mu(x)
$$

or

$$
\int_X |Dw_{x_0}(t, x)|^2 p(t, x_0, x) d\mu(x) < \frac{1}{2t} \int_X w_{x_0}^2(t, x) p(t, x_0, x) d\mu(x).
$$

In the first case, we have

$$
(3.7) \quad \frac{d}{dt} J(t) \geq -\frac{C}{t} e^{-cR^2/t} C(u, g)^2 - \frac{1}{t} \int_X w_{x_0}(t, x) \psi(x) g(x) p(t, x_0, x) d\mu(x).
$$

In the second case, by the curvature condition (1.6) with $T = R^2$, we deduce that

$$
\frac{d}{dt} J(t) \geq -c_k(R^2) \int_X |Dw_{x_0}(t, x)|^2 p(t, x_0, x) d\mu(x) - \frac{C}{t} e^{-cR^2/t} C(u, g)^2
$$

$$
+ \frac{1}{t} \int_X w_{x_0}(t, x) \psi(x) g(x) p(t, x_0, x) d\mu(x)
$$

$$
\geq -c_k(R^2) \int_X w_{x_0}^2(t, x) p(t, x_0, x) d\mu(x) - \frac{C}{t} e^{-cR^2/t} C(u, g)^2
$$

$$
+ \frac{1}{t} \int_X w_{x_0}(t, x) \psi(x) g(x) p(t, x_0, x) d\mu(x).
$$

(3.8)

From (3.7) and (3.8), we see that (3.8) holds in both cases. Integrating over $(0, R^2)$ and applying Proposition 3.3 we conclude that

$$
\int_0^{R^2} J'(t) dt \geq -\int_0^{R^2} \left\{ \frac{c_k(R^2)}{2t} \int_X w_{x_0}^2(t, x) p(t, x_0, x) d\mu(x) - \frac{C}{t} e^{-cR^2/t} C(u, g)^2 \right\} dt
$$

$$
+ \int_0^{R^2} \frac{1}{t} \int_X w_{x_0}(t, x) \psi(x) g(x) p(t, x_0, x) d\mu(x) dt
$$

$$
\geq -C(1 + c_k(R^2) R^2) C(u, g)^2 + \int_0^{R^2} \frac{1}{t} \int_X w_{x_0}(t, x) \psi(x) g(x) p(t, x_0, x) d\mu(x) dt.
$$
Combining Proposition 3.1 and Proposition 3.2, we obtain that for almost every \( x_0 \in B \),

\[
|Du(x_0)|^2 = J(R^2) - \int_0^{R^2} \frac{d}{dt} J(t) \, dt \\
\leq C(1 + c_s(R^2)R^2)C(u, g)^2 + \left| \int_0^{R^2} \frac{1}{t} \int_X w_{x_0}(t, x)\psi(x)g(x)p(t, x_0, x) \, d\mu(x) \, dt \right|,
\]

which completes the proof of Theorem 3.1. \( \square \)

We end this section by using Theorem 3.1 to obtain an \( L^\infty \)-estimate for \( |Du| \) when \( g \in L^\infty \).

**Lemma 3.3.** Let \( Q \in (1, \infty) \) and \( B = B_R(y_0) \) with \( R < R_0/8 \). Suppose that \( u \in H^{1,2}_{\text{loc}}(X) \) and \( g \in L^\infty_{\text{loc}}(X) \) that satisfy \( \Delta u = g \) in \( 8B \). Then \( ||Du||_{L^\infty(B)} < \infty \).

**Proof.** By Theorem 3.1, we have that for almost every \( x_0 \in B \),

\[
|Du(x_0)|^2 \leq C(1 + c_s(R^2)R^2)C(u, g)^2 + \left| \int_0^{R^2} \frac{1}{t} \int_X w_{x_0}(t, x)\psi(x)g(x)p(t, x_0, x) \, d\mu(x) \, dt \right|,
\]

where \( C(u, g) = R^{-Q/2-1}||u||_{L^Q(8B)} + R^{1-Q/p}||g||_{L^p(8B)} \). Applying Lemma 3.1, similarly to the proof of Proposition 3.3, we further deduce that

\[
|Du(x_0)|^2 \leq C(1 + c_s(R^2)R^2)C(u, g)^2 + CC(u, g)||g||_{L^\infty(8B)},
\]

which implies that \( ||Du||_{L^\infty(B)} < \infty \), proving the lemma. \( \square \)

## 4 Auxiliary equations

Suppose that \( \Delta u = g \) in \( 8B \). From Section 3, we have the following pointwise boundedness of \( |Du| \): for almost every \( x_0 \in B \),

\[
|Du(x_0)|^2 \leq C(1 + c_s(R^2)R^2)C(u, g)^2 + \int_0^{R^2} \frac{1}{t} \int_X |w_{x_0}(t, x)\psi(x)g(x)| p(t, x_0, x) \, d\mu(x) \, dt,
\]

where \( C(u, g) = R^{-Q/2-1}||u||_{L^Q(8B)} + R^{1-Q/p}||g||_{L^p(8B)} \) and \( p \in \left( \frac{Q}{2}, \infty \right] \cap (1, \infty] \). Hence, the main problem left is to estimate the second term on the right-hand side. We do not know how to estimate it for general \( g \), but we can estimate it provided that we assume that the support of \( g \) is contained in \( \lambda B \) for some \( \lambda \in (0, 1) \).

Thus, in this section, we study the auxiliary equation that for a ball \( B = B_R(y_0) \) with \( R < R_0/8 \),

\[
- \int_{8B} Du(x) \cdot D\phi(x) \, d\mu(x) = \int_{8B} g(x)\phi(x) \, d\mu(x), \quad \forall \phi \in H^{1,2}_0(8B),
\]

where \( u \in H^{1,2}_0(8B) \) and \( g \in L^\infty(X) \) with \( \text{supp} \, g \subset B/4 \).

The main aim of this section is to prove Theorem 1.1 and Theorem 1.2 when \( u \) and \( g \) are as above.
Theorem 4.1. Let \( Q \in (1, \infty) \) and suppose that (1.4) and (1.6) hold. Then there exists \( c, C > 0 \) such that for all \( u \in H^1_0(8B) \) and \( g \in L^\infty(X) \) with \( \text{supp } g \subset B/4 \) that satisfy \( \Delta u = g \) in \( 8B \), where \( B = B_{R}(y_0) \) with \( R < R_0/8 \):

(i)

\[
\int_B \exp \left\{ \frac{c|Du(x_0)|}{(1 + c_s(R^2)R)||g||_{L^Q(B/4)}} \right\} \frac{Q}{\mu} \, d\mu(x_0) \leq C;
\]

(ii) for \( p \in (\frac{Q}{2}, Q) \cap (1, \infty) \),

\[
\left( \int_B |Du|^p \, d\mu \right)^{1/p} \leq C(1 + c_s(R^2)R)^{1-Q/p} \mu \left( \int_B |g|^p \, d\mu \right)^{1/p}.
\]

Using our assumption that the support of \( g \) lies in \( B/4 \), we deduce following estimate on \( |Du(x_0)| \) for \( x_0 \in B \setminus \frac{3}{8}B \).

Lemma 4.1. For \( p \in (\frac{Q}{2}, Q) \cap (1, \infty) \), we have

\[
||Du||_{L^\infty(B\setminus\frac{3}{8}B)} \leq C(1 + c_s(R^2)R)^{1-Q/p} ||g||_{L^p(B/4)}.
\]

Proof. By Theorem 3.1, we have that for almost every \( x_0 \in B \),

\[
|Du(x_0)|^2 \leq C(1 + c_s(R^2)R^2)C(u, g)^2 + \int_0^{R^2} \frac{1}{t} \int_X |w_{x_0}(t, x)\psi(x)g(x)| p(t, x_0, x) \, d\mu(x) \, dt,
\]

where \( C(u, g) = R^{-Q/2-1} ||u||_{L^2(8B)} + R^{1-Q/p} ||g||_{L^p(B/4)} \). By Lemma 2.4, we have that \( ||u||_{L^\infty(8B)} \leq CR^{2-Q/p} ||g||_{L^p(B/4)} \), and hence,

\[
|Du(x_0)|^2 \leq C(1 + c_s(R^2)R^2)[R^{1-Q/p} ||g||_{L^p(B/4)}]^2 + \int_0^{R^2} \frac{1}{t} \int_{B/4} |w_{x_0}(t, x)g(x)|p(t, x_0, x) \, d\mu(x) \, dt.
\]

(4.1)

For every \( x_0 \in B \setminus \frac{3}{8}B \), since \( \text{supp } g \subset B/4 \), we have \( d(x, x_0) > R/8 \) for each \( x \in B/4 \). Hence, by the H"older inequality and Lemma 2.4, we deduce that

\[
\int_0^{R^2} \frac{1}{t} \int_{B/4} |(w\psi)(x) - T_t(w\psi)(x_0)||g(x)|p(t, x_0, x) \, d\mu(x) \, dt
\]

\[
\leq C \int_0^{R^2} \frac{1}{t} \int_{B/4} |(w\psi)(x) - T_t(w\psi)(x_0)||g(x)| \frac{1}{t^{Q/2}} e^{-R^2/ct} \, d\mu(x) \, dt
\]

\[
\leq C ||u||_{L^\infty(8B)} ||g||_{L^1(B/4)} \int_0^{R^2} \frac{1}{t^{Q/2+1}} \left( \frac{t}{R^2} \right)^{Q/2+1} \, dt
\]

\[
\leq C [R^{1-Q/p} ||g||_{L^p(B/4)}]^2
\]

which together with (4.1) proves the lemma. \( \square \)
Lemma 4.2. (i) There exists $C > 0$ such that for all $x_0 \in \frac{3}{8}B$ and $x \in \frac{1}{2}B$,

$$|u(x_0) - u(x)| \leq C d(x_0, x) \log^{1/1^*} \left( \frac{eR}{d(x_0, x)} \right) ||Du||_{\Psi_{R, 1^*}}(B).$$

(ii) Let $p \in (\frac{Q}{2}, Q) \cap (1, Q)$. There exists $C > 0$ such that for all $x_0 \in \frac{3}{8}B$ and $x \in \frac{1}{2}B$,

$$|u(x_0) - u(x)| \leq C d(x_0, x)^{2 - \frac{Q}{p}} ||Du||_{L^p(B)}.$$

Proof. Notice that by Lemma 3.3, we have $||Du||_{L^\infty(B)} < \infty$. Thus we may assume that $u$ is (Lipschitz) continuous in $B$.

For all $x_0 \in \frac{3}{8}B$ and $x \in B/2$, $d(x, x_0) < 14R/8$. We first consider the case that $d(x, x_0) \leq R/8$. Let $B_1 = B(x_0, d(x, x_0))$ and $B_0 = B(x, 2d(x, x_0))$. For $j \geq 2$ and $i \geq 1$ set $B_j = 2^{-1}B_{j-1}$ and $B_{-j} = 2^{-1}B_{-i+1}$ inductively. Further,

$$|u(x) - u(x_0)| \leq \sum_{j=-\infty}^{\infty} |u_{B_j} - u_{B_{j+1}}|,$$

where for each $j \geq 0$, the Poincaré inequality yields that

$$|u_{B_j} - u_{B_{j+1}}| \leq C\text{diam}(B_j) \left( \frac{1}{\mu(B_j)} \int_{B_j} |Du|^2 \, d\mu \right)^{1/2}.$$

Applying the Orlicz-Hölder inequality (2.4), we have

$$\int_{B_j} |Du|^2 \, d\mu \leq C ||Du||_{\Psi_{R, 1^*/2(B)}(B)} ||\chi_{B_j}||_{\Phi_{R, 2/1^*(X)}} = C ||Du||_{\Psi_{R, 1^*/2}(B)} ||\chi_{B_j}||_{\Phi_{R, 2/1^*(X)}},$$

where

$$||\chi_{B_j}||_{\Phi_{R, 2/1^*(X)}} = \inf \left\{ \lambda > 0 : \int_{B_j} \frac{1}{\lambda} \log^{2/1^*} \left( e + \frac{R^Q}{\lambda} \right) \, d\mu \leq 1 \right\} = \inf \left\{ \lambda > 0 : \frac{1}{\lambda} \log^{2/1^*} \left( e + \frac{R^Q}{\lambda} \right) \leq \mu(B_j)^{-1} \right\} \leq C (2^{-j}d(x_0, x))^Q \log^{2/1^*} \left( \frac{eR}{2^{-j}d(x_0, x)} \right).$$

Hence, we obtain that

$$|u_{B_j} - u_{B_{j+1}}| \leq C 2^{-j}d(x_0, x) \log^{1/1^*} \left( \frac{eR}{2^{-j}d(x_0, x)} \right) ||Du||_{\Psi_{R, 1^*/2}(B)}.$$
Similarly, for each \( j < 0 \),

\[
|u_{B_j} - u_{B_{j+1}}| \leq C 2^j d(x_0, x) \log^{1/1'} \left( \frac{eR}{2d(x_0, x)} \right) \|Du\| \Psi_{R, 1'}(B).
\]

Hence, for all \( x_0 \in \frac{3}{8} B \) and \( x \in B/2 \) with \( d(x, x_0) \leq R/8 \), we obtain

\[
|u(x_0) - u(x)| \leq \sum_{j=-\infty}^{\infty} |u_{B_j} - u_{B_{j+1}}| \leq C d(x_0, x) \log^{1/1'} \left( \frac{eR}{d(x_0, x)} \right) \|Du\| \Psi_{R, 1'}(B).
\]

For all \( x_0 \in \frac{3}{8} B \) and \( x \in B/2 \) with \( d(x, x_0) \geq R/8 \), by applying a similar approach as in the case \( d(x, x_0) \leq R/8 \) to the pairs \((x, y_0)\) and \((x_0, y_0)\), respectively, we obtain

\[
|u(x_0) - u(x)| \leq |u(x) - u(y_0)| + |u(x_0) - u(y_0)| \leq C d(x_0, x) \log^{1/1'} \left( \frac{eR}{d(x_0, x)} \right) \|Du\| \Psi_{R, 1'}(B)
\]

for all \( x_0 \in \frac{3}{8} B \) and \( x \in B/2 \), proving (i).

By the fact that \( p' > 2 \) for \( p \in \left( \frac{Q}{2}, Q \right) \cap (1, Q) \) and the Hölder inequality, we have

\[
|u_{B_j} - u_{B_{j+1}}| \leq C \text{diam}(B_j) \left( \int_{B_j} |Du|^2 \, d\mu \right)^{1/2} \leq C \text{diam}(B_j) \left( \int_{B_j} |Du|^{p'} \, d\mu \right)^{1/p'}.
\]

Using this inequality instead of (4.2) in the “telescope” approach above, we see that (ii) holds, proving the lemma. \( \square \)

**Proposition 4.1.** (i) For \( p = Q > 1 \), there exists \( C > 0 \) such that for almost every \( x_0 \in B \),

\[
|Du(x_0)|^2 \leq C (1 + c \sqrt{2 R^2}) \|g\|_{L^Q(B/4)}^2 + C \left\| g \right\|_{L^Q(B/4)} + \|Du\| \Psi_{R, 1'}(B) \int_{B/4} \log^{1/1'} \left( \frac{eR}{\mu(B/4)} \right) \left| g(x) \right| \frac{d\mu(x)}{d(x, x_0)^{Q'-1}}.
\]

(ii) For \( p \in \left( \frac{Q}{2}, Q \right) \cap (1, Q) \), there exists \( C > 0 \) such that for almost every \( x_0 \in B \),

\[
|Du(x_0)|^2 \leq C (1 + c \sqrt{2 R^2}) [R^{1-Q/p} \|g\|_{L^p(B/4)}]^2 + C \left\| g \right\|_{L^p(B/4)} \|Du\| \Psi_{R, 1'}(B) \int_{B/4} \left| g(x) \right| \frac{d\mu(x)}{d(x, x_0)^{Q'-2+Q/p}}.
\]

**Proof.** By (4.1), we have that for almost every \( x_0 \in B \) and \( p \in \left( \frac{Q}{2}, Q \right) \cap (1, Q) \),

\[
|Du(x_0)|^2 \leq C (1 + c \sqrt{2 R^2}) [R^{1-Q/p} \|g\|_{L^p(B/4)}]^2 + \int_0^1 \frac{1}{t} \int_{B/4} |w_{x_0}(t, x)g(x)| p(t, x_0, x) \, d\mu(x) \, dt,
\]

where \( w_{x_0}(t, x) = (u\psi)(x) - T(t(u\psi))(x_0) \).

Let us first prove (i). By Lemma 4.1, we have that \( \|Du\| \leq C \|g\|_{L^Q(B)} \). Thus, assume \( x_0 \in \frac{3}{8} B \).
Now by the fact $T_1 = 1$, we write
\[
\int_{B/4} |w_{x_0}(t,x)g(x)| p(t,x_0,x) \, d\mu(x) \leq \int_{B/4} |u\psi(x) - u\psi(x_0)||g(x)|p(t,x_0,x) \, d\mu(x) + \int_{B/4} |T_i(u\psi(x_0) - u\psi)(x_0)||g(x)|p(t,x_0,x) \, d\mu(x)
\]
\[
=: H_1 + H_2.
\]

By Lemma 4.2 (i), we have
\[
H_1 \leq \int_{B/4} |u(x) - u(x_0)||g(x)|p(t,x_0,x) \, d\mu(x)
\]
\[
\leq \int_{B/4} C d(x_0,x) \log^{1/1^+}\left(\frac{eR}{d(x_0,x)}\right) \|Du\|_{\mathcal{L}_{r,1}^{1}(B)} \|g(x)|p(t,x_0,x) \, d\mu(x)
\]
\[
\leq C \|Du\|_{\mathcal{L}_{r,1}^{1}(B)} \log^{1/1^+}\left(\frac{eR}{t}\right) \int_{B/4} \|g(x)|e^{-\frac{d(x_0,x)^2}{2\xi t}} \, d\mu(x).
\]

Notice that for $x \notin B/2$ and $x_0 \in 3B/8$, we have $d(x,x_0) > R/8$. For the term $H_2$, by Lemma 4.2(i) again, we have
\[
|T_i(u\psi(x_0) - u\psi)(x_0)|
\]
\[
\leq \int_{X,B/2} |u\psi(x) - u\psi(x_0)||p(t,x_0,x) \, d\mu(x) + \int_{B/2} |u\psi(x) - u\psi(x_0)||p(t,x_0,x) \, d\mu(x)
\]
\[
\leq C \|u\|_{L^{\infty}(B)} e^{-R^2/ct} \int_{X} p(t,x_0,x) \, d\mu(x)
\]
\[
+ \int_{B/2} C d(x_0,x) \log^{1/1^+}\left(\frac{eR}{d(x_0,x)}\right) \|Du\|_{\mathcal{L}_{r,1}^{1}(B)} p(t,x_0,x) \, d\mu(x)
\]
\[
\leq CR \|g\|_{L^{2}(B/4)}^{1/2} + C t^{1/2} \log^{1/1^+}\left(\frac{eR^2}{t}\right) \|Du\|_{\mathcal{L}_{r,1}^{1}(B)} \int_{X} p(t,x_0,x) \, d\mu(x)
\]
\[
\leq C \left[\|g\|_{L^{2}(B/4)} + \|Du\|_{\mathcal{L}_{r,1}^{1}(B)}\right] t^{1/2} \log^{1/1^+}\left(\frac{eR^2}{t}\right),
\]
where $l = \frac{C_2}{2C_1}$. By this estimate, we further obtain
\[
H_2 \leq C \left[\|g\|_{L^{2}(B/4)} + \|Du\|_{\mathcal{L}_{r,1}^{1}(B)}\right] t^{1/2} \log^{1/1^+}\left(\frac{eR^2}{t}\right) \int_{B/4} \|g(x)|p(t,x_0,x) \, d\mu(x)
\]
\[
\leq C \left[\|g\|_{L^{2}(B/4)} + \|Du\|_{\mathcal{L}_{r,1}^{1}(B)}\right] \log^{1/1^+}\left(\frac{eR^2}{t}\right) \int_{B/4} \|g(x)|e^{-\frac{d(x_0,x)^2}{2\xi t}} \, d\mu(x).
\]

Combining the estimates for $H_1$ and $H_2$, we conclude that
\[
\int_{B/4} |(u\psi)(x) - T_i(u\psi)(x_0)g(x)| \, p(t,x_0,x) \, d\mu(x)
\]
end, we establish the following boundedness of Riesz potentials.

Theorem 4.2. Let \( Q \) be the Hardy-Littlewood maximal function on every \( B \in \mathcal{B} \). Let \( \beta \) be a non-negative measurable function \( f \) on \( B_{R}(y_{0}) \) and \( x \in B_{R}(y_{0}) \), define its Riesz potential \( \mathcal{R}_{a,\beta}f \) by

\[
\mathcal{R}_{a,\beta}f(x) = \int_{B_{R}(y_{0})} \frac{\log \left( \frac{eR}{d(x,y)} \right)^{\beta}}{d(x,y)^{\beta-1}} f(y) \, d\mu(y).
\]

It is easy to see that Riesz potential \( \mathcal{R}_{a,\beta}f \) is well defined for \( f \in L^{\alpha}(B) \). Recall that \( \mathcal{M} \) denotes the Hardy-Littlewood maximal function on \( X \).

**Theorem 4.2.** Let \( Q \in (1, \infty) \), \( \alpha \in (0, Q) \) and \( \beta \in [0, \infty) \). Then there exist \( c, C > 0 \) such that for every \( B_{0} = B_{R}(y_{0}) \subset X \) with \( R < R_{0} \):

(i) For \( p = Q/\alpha \),

\[
\int_{B_{0}} \exp \left( \frac{c\mathcal{R}_{a,\beta}(f)}{\|f\|_{L^{Q/\alpha}(B_{0})}} \right)^{q_{0}} \|f\|_{L^{Q/\alpha}(B_{0})}^{q_{0} - 1} \, d\mu \leq C;
\]

(ii) For \( \beta = 0 \) and \( p \in (1, Q/\alpha) \),

\[
\|\mathcal{R}_{a,0}(f)\|_{L^{p}(B_{0})} \leq C\|f\|_{L^{p}(B_{0})}.
\]

**Proof.** Let us prove (i). Let \( \phi(r) = r^{\alpha-\beta}(\log \frac{2R}{r})^{\beta} \). For \( r \in (0, 2R) \), write

\[
\mathcal{R}_{a,\beta}f(x) = \int_{B_{0} \cap B_{r}(x)} \phi(d(x,y)) f(y) \, d\mu(y) + \int_{B_{0} \setminus B_{r}(x)} \phi(d(x,y)) f(y) \, d\mu(y).
\]

The desired estimate follows.

Using Lemma 4.2 (ii) instead of Lemma 4.2 (i) in the argument above, we see that (ii) holds as well, proving the proposition. \( \square \)

Now the main problem is reduced to estimating the Riesz potentials in Proposition 4.1. To this end, we establish the following boundedness of Riesz potentials.

Let \( \beta \in (0, Q) \) and \( \alpha \in (0, \infty) \). For a non-negative measurable function \( f \) on \( B_{R}(y_{0}) \) and \( x \in B_{R}(y_{0}) \), define its Riesz potential \( \mathcal{R}_{a,\beta}f \) by

\[
\mathcal{R}_{a,\beta}f(x) = \int_{B_{R}(y_{0})} \frac{\log \left( \frac{eR}{d(x,y)} \right)^{\beta}}{d(x,y)^{\beta-1}} f(y) \, d\mu(y).
\]

It is easy to see that Riesz potential \( \mathcal{R}_{a,\beta}f \) is well defined for \( f \in L^{\alpha}(B) \). Recall that \( \mathcal{M} \) denotes the Hardy-Littlewood maximal function on \( X \).

**Theorem 4.2.** Let \( Q \in (1, \infty) \), \( \alpha \in (0, Q) \) and \( \beta \in [0, \infty) \). Then there exist \( c, C > 0 \) such that for every \( B_{0} = B_{R}(y_{0}) \subset X \) with \( R < R_{0} \):

(i) For \( p = Q/\alpha \),

\[
\int_{B_{0}} \exp \left( \frac{c\mathcal{R}_{a,\beta}(f)}{\|f\|_{L^{Q/\alpha}(B_{0})}} \right)^{q_{0}} \|f\|_{L^{Q/\alpha}(B_{0})}^{q_{0} - 1} \, d\mu \leq C;
\]

(ii) For \( \beta = 0 \) and \( p \in (1, Q/\alpha) \),

\[
\|\mathcal{R}_{a,0}(f)\|_{L^{p}(B_{0})} \leq C\|f\|_{L^{p}(B_{0})}.
\]

**Proof.** Let us prove (i). Let \( \phi(r) = r^{\alpha-\beta}(\log \frac{2R}{r})^{\beta} \). For \( r \in (0, 2R) \), write

\[
\mathcal{R}_{a,\beta}f(x) = \int_{B_{0} \cap B_{r}(x)} \phi(d(x,y)) f(y) \, d\mu(y) + \int_{B_{0} \setminus B_{r}(x)} \phi(d(x,y)) f(y) \, d\mu(y).
\]
In what follows, for a ball $B = B_r(z)$ and $k \in \mathbb{Z}$, let $U_k(B) := B_{2^k r}(z) \setminus B_{2^{k-1} r}(z)$. If $\alpha \in (0, Q)$, then

\[
\int_{B_0 \cap B_r(x)} \phi(d(x, y)) f(y) \, d\mu(y) \leq \sum_{k \geq 0} \int_{U_k(B_r(x))} \phi(d(x, y)) f(y) \, d\mu(y) \\
\leq \sum_{k \leq 0} (2^k r)^{\alpha - Q} \left( \log \frac{eR}{2^k r} \right)^{\beta} \int_{U_k(B_r(x))} f(y) \, d\mu(y) \\
\leq C \sum_{k \leq 0} (2^k r)^{\alpha} \left( k \log \frac{eR}{r} \right)^{\beta} \int_{B_{2^k r}(x)} f(y) \, d\mu(y) \\
\leq C r^\alpha \left( \log \frac{eR}{r} \right)^{\beta} M(f)(x).
\]

On the other hand, by the Hölder inequality, we obtain

\[
\int_{B_0 \setminus B_r(x)} \phi(d(x, y)) f(y) \, d\mu(y) \\
\leq \|f\|_{L^{Q/\alpha}(B_0)} \left\{ \int_{B_0 \setminus B_r(x)} d(x, y)^{-Q} \left( \log \frac{eR}{d(x, y)} \right)^{\frac{\rho_0}{Q-\alpha}} \, d\mu(y) \right\}^{\frac{Q-\alpha}{Q}} \\
\leq \|f\|_{L^{Q/\alpha}(B_0)} \left\{ \sum_{1 \leq k \leq 2 \log_2 R/r} \int_{U_k(B_r(x))} (2^k r)^{-Q} \left( \log \frac{eR}{2^k r} \right)^{\frac{\rho_0}{Q-\alpha}} \, d\mu(y) \right\}^{\frac{Q-\alpha}{Q}} \\
\leq C \|f\|_{L^{Q/\alpha}(B_0)} \left( \log \frac{eR}{r} \right)^{\beta + \frac{\rho_0}{Q-\alpha}}.
\]

By letting $r^\alpha = \min\{R^\alpha, \sqrt[Q-\alpha]{\|f\|_{L^{Q/\alpha}(B_0)}} / M(f)(x)\}$, we obtain that

\[
\mathcal{R}_{\alpha, \beta, f}(x) \leq C \|f\|_{L^{Q/\alpha}(B_0)} \max \left\{ 1, \left( \log \frac{eR^\alpha M(f)(x)}{\|f\|_{L^{Q/\alpha}(B_0)}} \right)^{\frac{\rho_0}{Q-\alpha}} \right\}.
\]

Hence, by the Hölder inequality, we obtain

\[
\int_B \exp \left\{ e^{\mathcal{R}_{\alpha, \beta, f}(f)} \right\} \frac{\mu}{\|f\|_{L^{Q/\alpha}(B_0)}} \, d\mu \leq C \int_B e^{\mathcal{R}_{\alpha, \beta, f}(f)} \frac{\mu}{\|f\|_{L^{Q/\alpha}(B_0)}} \, d\mu \\
\leq \frac{C}{\mu(B) \|f\|_{L^{Q/\alpha}(B_0)}} R^\alpha \mu(B) \frac{\rho_0}{Q-\alpha} \|M(f)\|_{L^{Q/\alpha}(B)} \leq C,
\]

proving (i).

The case (ii) follows similarly, the theorem is proved.$\square$

As an application of the mapping properties of the Riesz potential, we obtain the main result of this section.
Proof of Theorem 4.1. By Proposition 4.1, we have that for almost every $x_0 \in B$,

$$|Du(x_0)|^2 \leq C(1 + c_s(R^2)R^2)||g||^2_{L^2(B/4)} + C \left( ||g||_{L^2(B/4)} + C ||g||_{L^2(B/4)} ||Du||_{\Psi_{k,1}^1(B)} \right) \int_{B/4} \frac{\log^{1/1'} \left( \frac{eR}{d(x,x_0)} \right) |g(x)|}{d(x,x_0)^{2-1}} d\mu(x).$$

Recall that for $R, \gamma > 0$, $\Psi_{R,\gamma}(t) = \frac{1}{R^\gamma}(e^t - 1)$. By Theorem 4.2 with $\alpha = 1$ and $\beta = 1/1'$, we see that

$$G(x_0) := \int_{B/4} \frac{\log^{1/1'} \left( \frac{eR}{d(x,x_0)} \right) |g(x)|}{d(x,x_0)^{2-1}} d\mu(x) \in \Psi_{R,1/2}^1(B),$$

with

$$\int_B \left[ \exp \left( \frac{G(x_0)}{C ||g||_{L^2(B/4)}} \right)^{\frac{\gamma}{\gamma - 1}} - 1 \right] d\mu(x_0) \leq 1.$$

Thus, we deduce that

$$\int_B \left[ \exp \left( \frac{|Du(x_0)|^2}{C(1 + c_s(R^2)R^2)||g||^2_{L^2(B/4)} + C ||g||_{L^2(B/4)} ||Du||_{\Psi_{k,1}^1(B)}} \right)^{\frac{\gamma}{\gamma - 1}} - 1 \right] d\mu(x_0) \leq 1,$$

which implies that

$$||Du||^2_{\Psi_{k,1}^1(B)} \leq C(1 + c_s(R^2)R^2)||g||^2_{L^2(B/4)} + C ||g||_{L^2(B/4)} ||Du||_{\Psi_{k,1}^1(B)} \leq C(1 + c_s(R^2)R^2)||g||^2_{L^2(B/4)} + \frac{1}{2} ||Du||^2_{\Psi_{k,1}^1(B)},$$

and hence,

$$\int_B \left[ \exp \left( \frac{|Du(x_0)|}{c(1 + \sqrt{c_s(R^2)R})||g||_{L^2(B/4)}} \right)^{\frac{\gamma}{\gamma - 1}} d\mu(x_0) \leq C,$$

proving (i).

Now for $p \in (\frac{Q}{Q}, Q) \cap (1, Q)$, by Proposition 4.1, we have that for almost every $x_0 \in B$,

$$|Du(x_0)|^2 \leq C(1 + c_s(R^2)R^2)[R^{1-Q/p}||g||_{L^2(B/4)}]^2 + C \left( ||g||_{L^2(B/4)} + ||Du||_{L^2(B)} \right) \int_{B/4} \frac{|g(x)|}{d(x,x_0)^{2-2+Q/p}} d\mu(x).$$

According to Theorem 4.2 (ii), we have that

$$\overline{G}(x_0) := \int_{B/4} \frac{|g(x)|}{d(x,x_0)^{2-2+Q/p}} d\mu(x) \in L^{\frac{Q}{Q'}}(B),$$
which implies that

\[ \| Du \|^2_{L^\infty(B)} \leq C(1 + c_k(R^2)R^2)\| g \|_{L^p(B)}^2 \mu(B) \frac{2(2-p)}{2-p} + C \| g \|_{L^p(B)} \| G \|_{L^{\frac{2p}{2-p}}(B)} \]

Thus, we obtain that

\[ \| Du \|_{L^p(B)} \leq C(1 + \sqrt{c_k(R^2)R})\| g \|_{L^p(B)} \], proving the theorem. \( \square \)

5 Proofs of the main results

In this section, we prove the main results of this paper. By Theorem 4.1, our proofs of Theorem 1.1 and Theorem 1.2 are reduced to approximation arguments and use of Cheeger-harmonic functions.

We first prove Theorem 1.1.

Proof of Theorem 1.1. For each \( k \in \mathbb{N} \), let \( g_k = g\chi_{8B(\mathcal{Q} \cap [k])} \). Then, by Lemma 2.6, there exist \( u_k \in H^{1,2}_0(256B) \) such that \( \Delta u_k = g_k \) in \( 256B \). By Theorem 4.1, we obtain

\[ \int_{2B} \exp \left\{ \frac{|Du_k(x)|}{c(1 + \sqrt{c_k(R^2)})\| g_k \|_{L^p(8B)}} \right\} \frac{\phi^2}{\phi^2} \, d\mu(x) \leq C. \]

Moreover, By Lemma 2.5 and the Sobolev inequality, we have

\[ \| u_k - u \|_{L^2(256B)} + \| D(u_k - u) \|_{L^2(256B)} \leq C_k \| g_k - g \|_{L^2(8B)} \to 0, \]

as \( k, j \to \infty \). Hence \( \{u_k\}_{k \in \mathbb{N}} \) is a Cauchy sequence in \( H^{1,2}_0(256B) \), and there exists \( \bar{u} \in H^{1,2}_0(256B) \) such that \( \lim_{k \to \infty} u_k = \bar{u} \) in \( H^{1,2}_0(256B) \) and \( \Delta \bar{u} = g\chi_{8B} \) in \( 256B \). By Theorem 4.1 (i) again, we further deduce that

\[ \| Du_k - D\bar{u} \|_{W^{1,r}(32B)} \leq C(1 + \sqrt{c_k(R^2)})\| g_k - g \|_{L^2(8B)} \to 0 \]

as \( k, j \to \infty \), which implies that

\[ \| D\bar{u} \|_{W^{1,r}(32B)} \leq C(1 + \sqrt{c_k(R^2)})\| g \|_{L^2(8B)}. \] (5.1)

On the other hand, since

\[ \int_{8B} D\bar{u}(x) \cdot D\phi(x) \, d\mu(x) = -\int_{8B} g(x)\phi(x) \, d\mu(x) = \int_{8B} Du(x) \cdot D\phi(x) \, d\mu(x), \quad \forall \phi \in H^{1,2}_0(8B), \]

we see that \( u - \bar{u} \) is Cheeger-harmonic in \( 8B \). By [23] or Theorem 3.1 with \( g = 0 \), we have

\[ \| D(u - \bar{u}) \|_{L^p(B)} \leq C(1 + \sqrt{c_k(R^2)}) \frac{\| u - \bar{u} \|_{L^2(8B)}}{\mathcal{Q}/2 + 1} \leq C(1 + \sqrt{c_k(R^2)}) \left( \frac{\| u \|_{L^2(8B)}}{\mathcal{Q}/2 + 1} + \| g \|_{L^2(8B)} \right), \]
which together with (5.1) implies that
\[ \int_B \exp \left( \frac{|Du(x_0)|}{c(1 + \sqrt{c_5(R^2)R})C(u, g)} \right) \frac{e^t}{t} d\mu(x_0) \leq C, \]
where \( C(u, g) = \frac{||u||_{L^2(B)}}{R^2} + ||g||_{L^0(B)} \), completing the proof of Theorem 1.1.

Observe that in Theorem 4.1, the range of \( p \) lies in \((\frac{Q}{2}, Q) \cap (1, Q)\). Thus, to obtain the results for all \( p \in (2, Q) \cap (1, Q) \), we need some extra estimates. Notice that \((\frac{Q}{2}, Q) \cap (1, Q) \neq (2, Q) \cap (1, Q)\) only for \( Q > 2 \).

We want to use the interpolation theory to study the case of \( p \in (2, \frac{Q}{2}) \) when \( Q > 2 \). To this end, let us recall the nonincreasing rearrangement function. For a measurable function \( f \), let \( \sigma_f \) denote its distribution function; then its nonincreasing rearrangement function, \( f^* \), is defined by letting for all \( t > 0 \), \( f^*(t) = \inf \{ s : \sigma_f(s) \leq t \} \).

We also need the following Hardy’s inequalities; see [32, p.196].

**Lemma 5.1.** Let \( q \geq 1 \) and \( g \) be a nonnegative function defined on \((0, \infty)\). Then
(i) \((\int_0^\infty \int_0^1 g(u) du \right)^{q/r-1} dt)^{1/q} \leq (q/r)\int_0^\infty [ug(u)]^{q/r-1} u^r du^{1/q} \)
(ii) \((\int_0^\infty \int_0^\infty g(u) du \right)^{q/r-1} dt)^{1/q} \leq (q/r)\int_0^\infty [ug(u)]^{q/r-1} u^r du^{1/q} \)

**Proposition 5.1.** Let \( Q > 2 \) and \( p \in (2, \frac{Q}{2}) \). Suppose that \( u \in H^{1,2}_0(256B) \), \( g \in L^\infty(X) \) with \( \text{supp } g \subset 8B \), and \( \Delta u = g \) in \( 256B \), where \( B = B_R(y_0) \) with \( 256B \subset \subset \Omega \). Then \( |Du| \in L^p(25B) \) with
\[ ||Du||_{L^p(25B)} \leq C(1 + \sqrt{c_5(R^2)R})||g||_{L^0(8B)}. \]

**Proof.** For \( t > 0 \), define
\[ g^t(x) := \begin{cases} g(x) & \text{if } |g(x)| > g^*(t); \\ 0 & \text{if } |g(x)| \leq g^*(t) \end{cases} \]
and \( g_t := g - g^t \). We then have
\[ (g^t)^*(s) \leq \begin{cases} g^*(s) & \text{if } s \in (0, t); \\ 0 & \text{if } s \geq t \end{cases} \]
and
\[ (g_t)^*(s) \leq \begin{cases} g^*(t) & \text{if } s \in (0, t); \\ g^*(s) & \text{if } s \geq t. \end{cases} \]
Notice here that, for \( t \geq \mu(8B) \), \( g^t = g \) and \( g_t = 0 \).

Let \( G \) be the Green function on \( 256B \) such that for each \( h \in L^\infty(256B) \), \( v := \int_{256B} Gh d\mu \in H^{1,2}_0(256B) \) and \( \Delta v = h \) in \( 256B \); see [5]. Write
\[ u = \int_{256B} G g d\mu = \int_{256B} G g^t d\mu + \int_{256B} G g_t d\mu =: u_1 + u_2. \]

Fix a \( q \in (\frac{Q}{2}, Q) \). By using Theorem 4.1 (ii) and Lemma 2.5, we obtain
\[ ||Du||_{L^p(32B)} \]
By Lemma 2.5 and the Sobolev inequality, we have
\[
\|Du_1\| + \|Du_2\| \leq C \left( \int_0^\infty \|Du_1 \chi_{32B}\|^p(t) + \|Du_2 \chi_{32B}\|^p(t) \, dt \right)^{1/p'}
\]
\[
\leq C \left( \int_0^\infty \left[ \frac{1}{t^{1/p}} \|g\|_{L^2(8B)} \right]^{p'} dt \right)^{1/p'} + C(1 + \sqrt{c_4(R^2)R}) \left( \int_0^\infty \left[ \frac{1}{t^{1/p}} \|g\|_{L^p(8B)} \right]^{p'} dt \right)^{1/p'}
\]
\[=: H_1 + H_2.\]

By the assumption that \(p^* > 2\) and Hardy’s inequality (Lemma 5.1(i)), we obtain
\[
H_1 \leq C \left( \int_0^\infty \frac{1}{t^{1/p}} \left( \int_0^\infty \|g'(s)\|^{2p} ds \right)^{p'/2} dt \right)^{1/p'}
\]
\[
\leq C \left( \int_0^\infty \left[ \frac{1}{t^{1/p}} \|g\|_{L^2(8B)} \right]^{p'} dt \right)^{1/p'} \leq C \left( \int_0^\infty \left[ \frac{1}{t^{1/p}} \|g\|_{L^p(8B)} \right]^{p'} dt \right)^{1/p'}
\]
\[
\leq C \left( \int_0^\infty \left[ \frac{1}{t} \|g\|_{L^2(8B)} \right]^{p'} dt \right)^{1/p'} \leq C \|g\|_{L^p(8B)}.
\]

Similarly, we have \(H_2 \leq C(1 + \sqrt{c_4(R^2)R})\|g\|_{L^p(8B)}\) (see [32]), and the desired estimate follows, proving the proposition.

We now are in position to prove Theorem 1.2. The proof is similar to that of Theorem 1.1. We give it for completeness.

**Proof of Theorem 1.2.** For each \(k \in \mathbb{N}\), let \(g_k = g\chi_{8B\cap \{|\xi|<8\}}\). Then there exists \(u_k \in H_0^{1,2}(256B)\) such that \(\Delta u_k = g_k\) in \(256B\). By Theorem 4.1 (ii) and Proposition 5.1, we obtain that for all \(p \in (2^*, Q) \cap (1, Q),\)
\[
\|Du_k\|_{L^p(32B)} \leq C(1 + \sqrt{c_4(R^2)R})\|g_k\|_{L^p(8B)}.
\]

By Lemma 2.5 and the Sobolev inequality, we have
\[
\|u_k - u\|_{L^2(256B)} + \|Du_k - u_j\|_{L^2(256B)} \leq C\|g_k - g_j\|_{L^p(8B)} \to 0,
\]
as \(k, j \to \infty\). Hence \(\{u_k\}_{k \in \mathbb{N}}\) is a Cauchy sequence in \(H_0^{1,2}(256B)\), and there exists \(\bar{u} \in H_0^{1,2}(256B)\) such that \(\lim_{k \to \infty} u_k = \bar{u}\) in \(H_0^{1,2}(256B)\) and \(\Delta \bar{u} = g\chi_{8B}\) in \(256B\). By Theorem 4.1 (ii) and Proposition 5.1 again, we further deduce that
\[
\|Du_k - Du_j\|_{L^p(32B)} \leq C(1 + \sqrt{c_4(R^2)R})\|g_k - g_j\|_{L^p(8B)} \to 0
\]
as \(k, j \to \infty\), which implies that
\[
(5.2) \quad \|D\bar{u}\|_{L^p(32B)} \leq C(1 + \sqrt{c_4(R^2)R})\|g\|_{L^p(8B)}.
\]

By the fact that \(\Delta \bar{u} = g\chi_{8B}\) in \(256B\), we deduce that
\[
\int_{8B} D\bar{u} \cdot D\phi \, d\mu = - \int_{8B} g\phi \, d\mu = \int_{8B} Du \cdot D\phi \, d\mu, \quad \forall \phi \in H_0^{1,2}(8B),
\]
which implies that $u - \bar{u}$ is Cheeger-harmonic in $8B$. By [23] or Theorem 3.1 with $g = 0$, we have
\[
\| \nabla (u - \bar{u}) \|_{L^\infty(B)} \leq C(1 + \sqrt{c_k(R^2)}R) \left( \frac{\| u - \bar{u} \|_{L^2(8B)}}{R^{Q/2+1}} + \frac{R^{1-Q/p} \| g \|_{L^p(8B)}}{R} \right),
\]
which together with (5.2) implies that
\[
\left( \int_B |Du|^{p^*} \, d\mu \right)^{1/p^*} \leq C\| \nabla (u - \bar{u}) \|_{L^\infty(B)} + C(1 + \sqrt{c_k(R^2)}R)\mu(B)^{-1/p^*} \| g \|_{L^p(8B)}.
\]
This completes the proof of Theorem 1.2. \hfill \Box

At last, we use Theorem 1.2 to prove the Hölder continuity of solutions to Poisson equations.

**Proof of Corollary 1.1.** For almost all $x, y \in B = B_r(y_0)$ with $256B \subset \subset \Omega$, by Theorem 1.2 and the Poincaré inequality, similarly to the “telescope” approach in Lemma 4.2, we have that for almost all $x, y \in B$,
\[
|u(x) - u(y)| \leq Cd(x, y)^{2-Q/p} \left( R^{-1} \left( \int_{8B} |u|^2 \, d\mu \right)^{1/2} + R \left( \int_{8B} |g|^p \, d\mu \right)^{1/p} \right).
\]
From this, we conclude that $u$ can be extended to a locally Hölder continuous function in $\Omega$, which completes the proof of Corollary 1.1. \hfill \Box

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