Log-Precision Transformers are Uniform Threshold Circuits

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Abstract

We prove that transformer neural networks with logarithmic precision in the input length (and where the feedforward subnetworks are computable using linear space in their input length) can be simulated by constant-depth uniform threshold circuits. Thus, such transformers only recognize formal languages in $\text{TC}^0$, the class of languages defined by constant-depth, poly-size threshold circuits. This demonstrates a connection between a practical claim in NLP and a theoretical conjecture in computational complexity theory: “attention is all you need” (Vaswani et al., 2017), i.e., transformers are capable of all efficient computation, only if all efficiently computable problems can be solved with log space, i.e., $L = \text{P}$. We also construct a transformer that can evaluate any constant-depth threshold circuit on any input, proving that transformers can follow instructions that are representable in $\text{TC}^0$.

1 Introduction

This work aims to characterize the computational model implicit in a transformer, a highly effective neural network architecture (Vaswani et al., 2017). In other words, what computational primitives can a transformer layer implement, and what problems can these primitives be combined to solve? These questions are important for interpreting transformer models in a principled way, as well as understanding potential limitations on the reasoning capabilities of large transformer-based language models.

A strand of work at the intersection of NLP and theoretical computer science has provided tools to analyze the computational abilities of transformers in terms of the theory of circuit complexity (Hahn, 2020; Hao et al., 2022; Merrill et al., 2022). In particular, Merrill et al. (2022) showed that a simplified class of “saturated” transformers with a floating point datatype can be simulated by non-uniform threshold circuits. This implies that saturated transformers over floats can only recognize formal languages in the class non-uniform $\text{TC}^0$. However, Merrill et al. (2022) left open two important follow-up questions:

What about soft attention? Saturated attention is a simplifying assumption on the attention patterns in transformers. It would be nice to have a result that holds for general transformers with any valid attention patterns. If so, we could provably upper bound the computational abilities of any transformer, rather than appealing to the observation that some large transformer language models are approximately saturated (Merrill et al., 2021) to justify a full saturation assumption.

Does the bound obey uniformity? Non-uniform $\text{TC}^0$ is a strange class: it contains some uncomputable languages, and is thus incomparable to the Chomsky hierarchy. To deal with this, circuit complexity theorists (cf. Arora and Barak, 2009) have proposed uniform variants of circuit complexity classes like $\text{TC}^0$ that are efficiently computable. These uniform classes have known relationships to more linguistically meaningful formal language classes, like the regular languages and logCFL\(^1\). Thus, it would be informative to derive a uniform version of Merrill et al. (2022)’s bound to understand whether transformers are capable of recognizing classes of formal languages theorized to have similar structure to natural languages.

In this paper, we resolve both these questions, under the mild assumption that all values in the transformer have $O(\log n)$ precision on sequences of length $n$, and that transformer’s subnetworks

\(^1\text{Languages reducible to context-free parsing in log-space.}\)
are similarly computable in $O(\log n)$ space. Log precision is enough to represent the positional encodings at the input layer of the transformer, and encode pointers to all other positions in the sequence at later transformer layers. Assuming log precision across all layers captures the idea that all values in the transformer will have similar precision, and that, on long sequences, this precision will not be enough to losslessly encode the full input sequence into a single vector. Instead, the processing of the sequence must somehow be distributed in parallel.

This simple log precision assumption also allows us to dispense with defining the details of the underlying binary representation of numbers, e.g., floats vs. rationals, which was necessary in the setup of prior work (Merrill et al., 2022). It thus improves the theory by making the underlying assumptions more parsimonious.

**Upper Bound** Our main contribution is proving that log-precision transformers can be simulated by a uniform constant-depth threshold circuits. Thus, such transformers can only recognize formal languages in $\text{TC}^0$, a set of languages recognizable with log-space overhead. This result allows comparison of the expressiveness of transformers to other neural nets, as well as to formal models of natural language grammar. Strikingly, it is dramatically reduced compared to the Turing-completeness of infinite-precision transformers. Since we believe log-precision is more realistic for practical transformers than infinite precision, we conclude that transformers are not Turing-complete in practice.

Our upper bound also connects the power of transformers to several open questions in complexity theory. Since $\text{TC}^0 \subseteq \text{L}$, transformers can recognize any language in $\text{P}$ only if $\text{TC}^0 = \text{NC}^1 = \text{L} = \text{P}$. Assuming any one of these equalities is false immediately implies limits on the practical computational power of transformers. While the answer to $\text{TC}^0 = \text{NC}^1$ may be unclear, it is the opinion of the authors that $\text{TC}^0 = \text{P}$ is unlikely, and thus it is unlikely that log-precision transformers can recognize any language in $\text{P}$.

Our upper bound also motivates a paradigm for fast hypothesis testing in complexity theory. If a problem is conjectured to separate $\text{TC}^0$ and $\text{NC}^1$ or $\text{L}$ and $\text{P}$, a transformer could be trained on synthetic data for the problem. If it succeeds and its solution can be verified, this provides an existence proof that the problem is in $\text{TC}^0$, falsifying the conjecture.

**Instruction Following and Advice Transformers** We also consider an instruction following setting (Brown et al., 2020; Finlayson et al., 2022) where the transformer is provided the description of a task along with an input $x$ on which to execute the instruction. We construct a practically parameterizable transformer that can execute instructions perfectly if they are provided in the form of $\text{TC}^0$ circuits. This complements recent work that studies transformers’ ability to follow other forms of instructions such as regular expressions.

In summary, our findings reveal new insights on both the abilities and the limitations of transformers, and bring out bounded precision and threshold computations as key notions for understanding the implicit computational model of transformers in practice.

## 2 Definitions and Notation

Let $\{0,1\}^*$ denote the set of all finite strings over $\{0,1\}$. For $x \in \{0,1\}^*$, let $|x|$ be its length. We define a boolean function as a function that maps $\{0,1\}^*$ to (potentially infinite) strings over $\{0,1\}$. Boolean functions can implement arithmetic functions, if we define a semantics for binary strings as numbers. We imagine some such encoding scheme is used to cast the arithmetic operations in the transformer as boolean functions, but the details of the encoding scheme used will be unimportant.

## 3 Circuits

Circuits are a model of computation for computing boolean functions of fixed-length binary strings.\(^3\) Formally, a circuit is a directed acyclic computation graph. The leaf nodes represent binary variables and their negations. The internal nodes represent functions in some set $G$, and the directed

\(^2\)All of these equalities remain unproven, but hold as $\subseteq$.\(^3\)For a mini-tutorial on circuit complexity theory and its relevance to transformers, see Merrill et al. (2022).
edges represent the flow of function outputs into inputs of other functions. One or more nodes in the circuit are marked such that their value is the output of the circuit.

**Definition 1** For a set of functions $G$, a $G$-circuit is a directed acyclic computation graph where the internal nodes have labels from $G$.

**Complexity Measures** The size of a circuit is the total number of gates of any type, including negation. The depth of a circuit is the length of the longest path from any input node to any output node.

**Circuit Families** A circuit family generalizes a circuit to take variable-length binary strings as input. Formally, a circuit family is a sequence of circuits $C_n : \{0, 1\}^n \rightarrow \{0, 1\}$ for $n \in \mathbb{N}$. A circuit family implicitly recognizes a formal language defined as follows:

**Definition 2** A circuit family $C_n$ recognizes $L \subseteq \{0, 1\}^*$ if and only if, for all $x \in \{0, 1\}^*$,

$$x \in L \iff C_{|x|}(x) = 1.$$  

We now define classes of languages by constraining the complexity of the circuit families needed to recognize them:

**Definition 3** Let non-uniform $\text{AC}^0$ be the set of $L \subseteq \{0, 1\}^*$ such that $L$ is recognizable by a polynomial size, constant-depth $\{\land, \lor\}$-circuit family.

For $k \in \mathbb{N}$, let $\theta_{\leq k}$ be a threshold gate that takes $m$ input bits and returns whether

$$\sum_{i=1}^{m} x_i \leq k.$$  

We define $\theta_{\geq k}$ analogously.

**Definition 4** Let $\text{TC}^0$ be the set of $L \subseteq \{0, 1\}^*$ such that $L$ is recognizable by a polynomial size, constant-depth $\{\land, \lor\}_k \in \mathbb{N}$-circuit.

The gates $\lnot$, $\land$, and $\lor$ are all just special cases of thresholds, so we can imagine $\text{TC}^0$ circuits to have access to these as well. Thus, $\text{TC}^0$ circuits can implement $\text{AC}^0$ circuits.

**Circuit Serialization** We identify a circuit with its serialization in a formal language that identifies each node’s label and adjacency list. We will adopt a specific grammar for concreteness, but our construction can be adapted to other string representations of circuits.

We define a circuit serialization as its traversal starting at the leaf nodes, followed by the other nodes, ordered by some topological sort. Each node is written in a Polish-notation adjacency list representation, where its arguments are pointers encoded in unary. To serialize $\{\land, \lor\}$-circuits, we use the following grammar, where the $i$ parameter is passed through $\text{Gate}[i]$ nonterminals to track the index of the gate in left-to-right order:

- Circuit $\rightarrow$ Gate[1]\, Gate[2] \cdots \, Gate[g]
- Gate[i] $\rightarrow$ X
- Gate[i] $\rightarrow$ NOT\, Arg[i]
- Gate[i] $\rightarrow$ Op\, Arg[i]^*
- Arg[i] $\rightarrow$ $\sigma\, \overline{1}^j$
- Op $\rightarrow$ AND \mid OR

where we add the constraint $i < j$ to enforce gates an only point to arguments earlier in the serialization. Thus, the span $\sigma\, \overline{1}^j$ is a unary encoding of a pointer to the $j$-th node in the circuit as an argument of the current node $i$.

It is a bit more complicated to serialize threshold circuits. Formally, a threshold circuit serialization is generated by the following grammar,

- Circuit $\rightarrow$ Gate[1]\, Gate[2] \cdots \, Gate[g]
- Gate[i] $\rightarrow$ X
- Gate[i] $\rightarrow$ Dir\, $1^k\, \overline{m-k}^\ast\, \text{Arg}[i]^m$
- Arg[i] $\rightarrow$ $\sigma\, \overline{1}^j$
- Dir $\rightarrow$ <= \mid >=

where the $i < j$ constraint applies as before, and where $m$ is the arity of the gate and $k \leq m$ is its threshold. The span $1^k\, \overline{m-k}^\ast$ after Dir can be interpreted semantically as a unary encoding of the parameter $k$ for a threshold gate, padded by 0’s to the number of total arguments of gate $i$.

For simplicity, we imagine $\lnot$ gates are represented in this circuit as $\theta_{\leq 0}$ gates applied to one argument. Thus, the circuit $\theta_{\geq 1}(x_1, \lnot x_2)$ would be represented as\footnote{Spaces below (and in the grammar) added for readability. We will ignore these spaces when passing circuit serializations as transformer inputs in §7.}$^d$

\begin{align*}
X \times \times & \leftarrow 00 \ & \& \leftarrow & 10 \ & \& \leftarrow & \overline{11}
\end{align*}
other. We formalize this (cf. Arora and Barak, 2009) by saying that there exists a resource-constrained Turing machine that maps the input $1^n$ to a serialization of circuit $C_n$.

**Definition 5** A language $L$ is $(S(n), I(n))$-space uniformly computable by a circuit model $M$ iff there exists a Turing machine that, for all $m, n \geq 0$, uses $S(n)$ space to map $1^n$ to an $M$-circuit recognizing $L$ on inputs of size $I(n)$.

This notion of uniformity is more general than the standard notion in that the input size $I(n)$ is a function of the problem complexity $n$. The reason for this is that we will apply uniformity to subcomputations with different input sizes $I(n)$ within a larger computation of input size $n$. The standard notion of uniformity can be recovered by setting $I(n) = n$.

Furthermore, we will refer to a circuit family as uniform if it uniformly computable with $S(n) = O(\log n)$ (cf. Arora and Barak, 2009). We can define uniform versions of $\text{AC}^0$ and $\text{TC}^0$ by adapting the previous definitions exactly, but also enforcing uniformity. For the rest of the paper we will clarify whether we mean the uniform or non-uniform variant of $\text{TC}^0$ where it is not clear from context, since both classes are relevant to our results.

### 4 Bounded-Precision Transformers

A transformer (Vaswani et al., 2017) is a neural network architecture made up of a constant number of transformer layers. A transformer layer is a module that computes self-attention over a sequence followed by an elementwise transformation of the output vectors.

#### 4.1 Precision and Space

We will assume that each transformer is resource bounded in terms of the precision of each value it computes and, for some of our results, the space it uses for the computation of key operations such as embedding, attention, and activation. Specifically, we will assume precision $p$, i.e., the values at all layers, as well as the outputs of all key intermediate operations in it (attention, activation, arithmetic operators, etc.), are represented using $p$ bits. This is a realistic assumption as, in practice, today’s transformers are typically limited to the 64-bit precision of the underlying hardware. Formally, we define $p$-precision as follows:

**Definition 6** A $k$-ary function $f : x_1, \ldots, x_k \mapsto y$ is $p$-precision if $x_1, \ldots, x_k, y \in \{0, 1\}^p$.

Note that $p$-precision arithmetic operations can be computed by interpreting the intermediate values according to some numerical semantics for binary strings, and using some $p$-precision approximation of those semantics. The details of the underlying semantics are unimportant; our result holds for any possible definition of $p$-precision addition and multiplication that can be computed using uniform $\text{TC}^0$ circuits.

For float-like semantics of bitstrings, adding $n$ numbers of size $c \log n$ losslessly can blow up the precision of their sum. For example, imagine adding the floating points $1 \cdot 2^0 + 1 \cdot 2^c$. We obtain $(2^c + 1) \cdot 2^0$, whose mantissa takes $c + 1$ bits to represent. In practice, computers do not preserve full precision in such situations: instead, small terms like $1 \cdot 2^0$ are discarded. In general, assuming the same floating point representation, we could imagine each float is first mapped to an integer, each integer is trimmed to size $c \log n$ by discarding lower order bits if necessary, these integers are added up, the result trimmed again if needed, and then that integer is converted back to a float of precision $c \log n$.

We aim to capture this notion of lossy addition of floats at a high level of generality—that is, without making specific assumptions about the underlying datatype or the binary representation of numbers. To do this, we assume the transformer data type permits a bounded-precision addition operator defined as follows:

**Definition 7** A $p$-precision addition operator $\oplus$ is a function that maps values $x_1, \ldots, x_n$ (with each $x_i \in \{0, 1\}^p$) to $s = \bigoplus_{i=1}^n x_i \in \{0, 1\}^p$ such that computing $\oplus$ can be reduced via uniform $\text{TC}^0$ circuits to adding $n$ integers of size $p$.

As discussed above, we introduce this definition to capture natural floating-point arithmetic schemes at a high level of generality without the need to specify a concrete datatype.

#### 4.2 Transformer Definition

The building block of a transformer is a transformer layer. We define this at a high level of abstraction as follows:

**Definition 8** (Transformer layer) A $p$-precision, $s$-space transformer layer is a pair $(f, g)$ of $p$-precision functions, each computable in space $s$, where $f$ is a binary attention function and $g$ is a unary activation function.
The function computed by a transformer layer can be described as follows.

**Definition 9** (Transformer layer computation) Let \( Z_i = \sum_{j=1}^n f(h^i_j, h^f_j) \). A transformer layer \( \langle f, g \rangle \) computes a function mapping \( h^i \) to \( h^{i+1} \) as

\[
   h^{i+1}_j = g \left( \bigoplus_{j=1}^n f(h^i_j, h^f_j) / Z_i \right),
\]

where \( \bigoplus \) is a \( c \log n \)-precision approximate addition operator over the transformer datatype as in Def. 7.

Finally, we define a transformer of depth \( d \) as a cascade of \( d \) transformer layers:

**Definition 10** (Precision/space-bounded transformer) A \( p \)-precision, \( s \)-space transformer of depth \( d \) over alphabet \( \Sigma \) is a pair consisting of (1) an \( s \)-space-computable position embedding function \( \phi : \Sigma \times \mathbb{N} \to \{0,1\}^c \) for \( c \leq p \) and (2) a \( d \)-tuple of \( p \)-precision, \( s \)-space transformer layers.

For a position embedding function \( \phi \) and \( w \in \Sigma^n \), let \( \phi_i(w) \) denote the position-wise broadcasted embedding of \( w \), i.e., for \( 1 \leq i \leq n \),

\[
   \phi_i(w) \triangleq \phi(w_i, i).
\]

**Definition 11** (Transformer computation) A transformer \( \phi, \langle L^1, \cdots, L^d \rangle \) computes the following function of a string \( w \in \Sigma^* \):

\[
   T(w) = (L^d \circ L^{d-1} \circ \cdots \circ L^1)(\phi(w)).
\]

We will use \( n \) to denote the length of the input sequence to the transformer. We will take the depth \( d \) to be fixed with respect to \( n \).

As noted earlier, 64-bit precision is common in practice, irrespective of input size \( n \). However, we will use a richer model that allows \( p \) to grow with \( n \), which is necessary to have a meaningful position embedding for \( n \)-bit inputs and for the ability of a value in the transformer to point to other values. Specifically, we will consider \( O(\log n) \)-precision transformers (“log-precision”), which have enough precision to represent positional encodings, but not enough precision to pool the full input into one position.

Similarly, we will adopt \( O(\log n) \)-space transformers (“log-space”) as a reasonable model of practical transformers. Note that \( \phi, f, g \) have inputs of size \( O(\log n) \) by log precision, so they are really computable in linear space in the size of their own inputs (but log space in the size of the global input sequence).

**Relationship to actual transformers** Our transformers do not enforce that \( f, g \) take the form of a feedforward neural net. But a feedforward net whose primitive operations (e.g., scalar multiplication) have \( O(\log n) \) precision can be computed in \( O(\log n) \) space. Thus, bounded-precision practical transformers (Vaswani et al., 2017) are a special case of our model of transformers. This makes our setup appropriate for proving upper bounds on transformers, which is our main contribution here.

Any lower bounds (e.g., in §7) will technically need to show that the functions constructed for \( \phi, f, g \) can be parameterized in practice by a feedforward net. However, due to the universal approximation of feedforward neural networks (Cybenko, 1989), Weiss et al. (2021) find that constructions derived without explicitly expressing \( \phi, f, g \) as a feedforward net are often learnable in practical transformers.

**5 Simulating Log-Precision Transformers with Non-Uniform Threshold Circuits**

We start with a simple proof that log-precision transformers can be simulated by non-uniform threshold circuits, before presenting the more technical uniform version of the results later in §6. This first result extends the findings of Merrill et al. (2022), who showed that saturated attention transformers\(^5\) can be simulated in \( TC^0 \).

Here, we remove the simplifying saturated attention assumption. Instead, we show that our log-precision assumption is enough to prove that a transformer can be simulated in \( TC^0 \) with any attention function.

First, we use a lemma of Hao et al. (2022) showing that any boolean function of \( O(\log n) \) bits can be computed by a circuit of size polynomial in \( n \):

**Lemma 1** (adapted from Hao et al., 2022) Let \( f : \{0,1\}^* \to \{0,1\} \) be a boolean function. For all \( c \in \mathbb{R}^+ \) and \( n \in \mathbb{N} \), there exists a circuit of size at most \( n^c + c \log n + 1 \) and depth 3 that computes \( f \) on inputs of size \( c \log n \).

This can be achieved via a circuit that encodes the DNF representation of \( f \) on inputs of size \( c \log n \). The DNF formula has at most \( 2^c \log n = n^c \)

\(^5\)Saturated attention is uniform attention over a subset of the prior layer nodes.
terms. The circuit has a NOT gate for each input bit, an AND gate for each DNF term, and an OR gate combining the outputs of all AND gates.

Following from Def. 7, also show that approximate addition of \( n \) transformer values with \( c \log n \)-precision can be done in uniform \( \text{TC}^0 \).

**Lemma 2** Let \( \oplus \) be a \( c \log n \)-precision addition operator. Given \( n \) strings \( x_1, \ldots, x_n \in \{0,1\}^{c \log n} \), the following quantity can be computed in uniform \( \text{TC}^0 \): \( s = \bigoplus_{i=1}^{n} x_i \).

**Proof.** By Def. 7, computing \( s \) is reducible to adding \( n \) integers of \( c \log n \) precision. Since this integer addition problem is itself in uniform \( \text{TC}^0 \) (Kayal, 2015), we conclude that computing \( s \) can be done in uniform \( \text{TC}^0 \).

We now use Lem. 1 and Lem. 2 to prove the following:

**Theorem 1** (Non-uniform) Any log-precision transformer can be simulated by a constant-depth threshold circuit family.

**Proof.** Let \( n \) be the input size of an \( O(\log n) \)-precision transformer. We show by induction that we can construct a composition of constant depth circuits to compute a layer of this transformer. Thus, any constant-depth transformer will be computable by a constant-depth threshold circuit.

In the base case, we compute \( h^0_i = \phi(w_i, i) \) using Lem. 1, yielding a polynomial size depth-3 circuit.

In the inductive case, each vector input to the layer \( h^j_i \) has size (at most) \( c \log n \) because of the log-precision assumption. The output of \( f \) thus also has \( c \log n \) bits. Applying Lem. 1 once for each of the \( c \log n \) output bits, we can compute \( f(h^j_i, h^j_j) \) with \( c \log n \) polynomial size depth-3 circuits. Now, via Lem. 2 we can construct a \( \text{TC}^0 \) circuit to compute the normalizing constant \( Z_j \). Since \( f(h^j_i, h^j_j), Z_j \), and \( h^j_i \) all have \( c \log n \) bits, we can compute \( \frac{f(h^j_i, h^j_j)}{Z_j} h^j_i \) with a polynomial number of \( \text{AC}^{0} \) circuits using Lem. 1.\(^6\) Now, we compute the summation over \( j \) by again applying Lem. 2. We then apply Lem. 1 to compute \( g \) with a polynomial size depth-3 circuit. This completes the inductive step.

\(^6\)This may seem counterintuitive since multiplication of two \( n \)-precision numbers is outside \( \text{AC}^0 \). But, here, we can leverage the fact that the precision is \( c \log n \).

### 6 Simulating Log-Precision Log-Space Transformers with Uniform Threshold Circuits

We will now extend the argument from the last section to show that \( O(\log n) \)-precision transformers can be simulated by uniform constant-depth threshold circuits if we make one additional assumption—namely, \( \phi, f, \text{ and } g \) are computable in \( O(\log n) \) space.\(^7\) The overall proof idea is similar, but due to the uniformity condition, the proof becomes substantially more technical. We must not just show the existence of a threshold circuit family computing a transformer, but also show that this circuit family can be generated by a log-space Turing machine.

We begin by extending Lem. 1 to respect uniformity:

**Lemma 3** Let \( f : \{0,1\}^{*} \rightarrow \{0,1\} \) be a linear-space computable boolean function and \( c \in \mathbb{R}^+ \). There exists a Turing machine that, for all \( n \in \mathbb{N} \), uses \( O(\log n) \) space to map input \( 1^n \) to a circuit of size at most \( n^c + c \log n + 1 \) and depth 3 that computes \( f \) on inputs of size \( c \log n \).

**Proof.** We give the proof in the form of an algorithm for the Turing machine to implement. First, we print out \( 2c \log n \) nodes representing the unnegated and negated input nodes.

Now, we need to show how to construct nodes corresponding to \( n^c \) DNF terms. To this end, we loop over all inputs \( x \in \{0,1\}^{c \log n} \) by maintaining the \( c \log n \) bit binary representation of \( x \) (initialized with \( (\log n) \)) and incrementing it by 1 at each step of the loop. By our linear-space computability assumption and because \( x \) has \( c \log n \) bits, we can compute \( f(x) \) as a subroutine in \( O(\log n) \)-space. If \( f(x) = 1 \), we create a new \( \wedge \) node \( i \) with \( c \log n \) arguments, defined as follows. For \( j \in [c \log n] \), we create an argument pointer to node \( j \) if \( x_j = 1 \), and to node \( c \log n + j \) otherwise. If \( f(x) = 0 \), we instead output a node \( x_1 \wedge \neg x_1 \) that is always false. Looping over all \( x \)’s creates \( n^c \) nodes total.

Finally, by looping over \( i \in [c \log n] \), we create a \( \lor \) gate that, for each \( i \), has an argument \( 2c \log n + i \).

\(^7\)Note that since the inputs to \( \phi, f, \text{ and } g \) have size \( O(\log n) \) due to our log-precision assumption, the additional assumption on space amounts to these functions being linear-space computable in the size of their input.
We show that this Turing machine maps input \( n \) to a serialized circuit computing \( f \) on inputs of size \( n \). In the first layer, each \( x \) that satisfies \( f \) produces a DNF term. On the other hand, each \( x \) that does not satisfy \( f \) produces a term that is always false. Thus, taking the disjunction of these terms in the last layer computes \( f(x) \).

**Log Space.** To complete the proof, we justify that \( M \) uses \( O(\log n) \) space. Looping over \( x \in \{0, 1\}^{c \log n} \) is accomplished by treating \( x \) as a binary number initialized to 0 and incrementing it each step. Thus, tracking the loop state only requires storing \( c \log n \) bits. Looping over \( i \in [n^c] \) similarly uses a counter of size \( c \log n \).

We conclude \( M \) uses \( O(\log n) \) space to map \( 1^n \) to a circuit of size at most \( n^c + c \log n + 1 \) and depth 3 that computes \( f \) on inputs of size \( c \log n \). \( \square \)

We can leverage this lemma to derive the uniform analog of Thm. 1, as follows. Note that the “log” in log-precision and log-space below refers to \( O(\log n) \)-precision and \( O(\log n) \)-space.

**Theorem 2** (Uniform) Any log-precision, log-space transformer can be simulated by a constant-depth uniform threshold circuit family.

**Proof.** By assumption, there is some fixed \( c \in \mathbb{R}^+ \) such that the precision of the transformer is \( c \log n \) on inputs of size \( n \). We will provide a proof by induction over transformer layers \( \ell \) that there is a Turing machine \( M \) operating in \( O(\log n) \) space that, on input \( 1^n \), outputs a circuit that simulates the transformer’s computation on inputs of size \( n \).

In the base case, we can apply Lem. 3 to construct a Turing machine that maps \( 1^n \) to a constant-depth threshold circuit computing \( h_0 = \phi(w_1, i) \).

In the inductive case, we assume we can output in \( O(\log n) \) space a circuit computing every value in the previous layer \( \ell \). We will show that we can, in \( O(\log n) \) space, now output a circuit computing every value in layer \( \ell + 1 \). Recall that the next transformer layer \( h_{\ell+1} \) is the following function of the previous layer \( h_\ell \):

\[
h_{\ell+1}^{i,j} = g \left( \sum_{j=1}^{n} \frac{f(h_\ell^{i,j})}{Z_i} h_j \right).
\]

By Lem. 3, we can generate a depth-3 circuit of size at most \( z = n^{c'} + c' \log n + 1 \), where \( c' = 2c \) (since the input to \( f \) is of size \( 2c \log n \)) that computes \( f(h_\ell^{i,j}) \) for specific \( i, j \). We do this sequentially for \( 1 \leq j \leq n \), padding each circuit with unused nodes so that each one has size exactly \( z \), and the \( z \)-th node corresponds to the output. Thus, the indices of the output nodes for each of the columns will be \( w_l + zj \) for \( 1 \leq j \leq n \), where \( w_j \) is the index of the last output node \( h_n^\ell \) of the previous layer.

At this point, we use the fact that for \( p = c \log n \), the \( p \)-precision approximate sum of \( n \) \( p \)-precision numbers can be computed by a uniform threshold circuit via Lem. 2. We can thus use a Turing machine as a sub-routine to generate, on input \( 1^n \), a threshold circuit of size \( z' \) that computes a \( p \)-precision “sum” gate over \( n \) items of precision \( p \) each. We set the inputs of this gate to be nodes \( w_l + zj \) for \( 1 \leq j \leq n \). By construction, this computes a \( p \)-precision approximation \( s_i \) of the sum \( Z_i = \sum_{j=1}^{n} f(h_\ell^i, h_j) \), whose value is located at the node at index \( w_l + zn + z' \).

Using \( p \)-precision arithmetic operator circuits, we can now also generate a circuit to compute \( \frac{f(h_\ell^i, h_j)}{s_i} \) for each \( 1 \leq i \leq n \), by using index \( w_l + zj \) as before for the value of \( f(h_\ell^i, h_j) \) and index \( w_l + zn + z' \) for the value of \( s_i \). Here too we use circuits of identical size \( z'' \), making \( w_l + zn + z' + z'' \) the index of the output nodes of these \( n \) circuits. Next, we again employ a \( p \)-precision approximate summation circuit of size \( z' \), similar to the computation of \( s_i \), to compute the sum of these \( n \) values. Finally, we compute \( h_{\ell+1}^{i,j} \) by applying \( g \) via Lem. 3.

Note that this requires keeping only \( \ell, i, \) and \( n \) in memory, each of which takes \( O(\log n) \) bits.

We repeat this process for all \( 1 \leq i \leq n \) to compute the entire \( \ell + 1 \) layer, which finishes the inductive step: if we can output a circuit computing layer \( \ell \) in \( O(\log n) \) space, then we can do the same for layer \( \ell + 1 \). \( \square \)

**7 Lower Bounds for Instruction Following and Advice Transformers**

We now define a circuit evaluation task, where a transformer receives a serialized circuit \( C \) (as described in §3), followed by an input string \( x \in \{0, 1\}^n \), and is asked to return the value of \( C(x) \). It is known that LSTMs cannot evaluate boolean formulae like this (Merrill, 2020). In contrast, we will show that transformers can.

This task is closely related to the instruction learning and instruction following tasks considered by Brown et al. (2020) and Finlayson et al. (2022). For instance, the latter provide a trans-
form former two inputs, a regular expression $r$ as an “instruction” and a 0-1 string $s$, and ask it to return whether $s$ belongs to the regular language represented by $r$. Viewing from this lens, the setup we consider in this section can be seen as asking the following question: Can transformers follow instructions provided in the form of a circuit? We will prove that the answer is yes for all constant depth threshold circuits. This, to the best of our knowledge, provides the first non-trivial lower bound for transformers in the instruction learning setting.

We note that while it is unknown whether the class of regular languages is contained in $\text{TC}^0$, the other side is known: there are problems computable by $\text{TC}^0$ circuits that are not computable by a regular language. These include problems involving counting and arithmetic, which are beyond regular languages. Our results thus expand the understanding of the kinds of instructions transformers are able to follow.

To demonstrate the practicality of our lower bound construction, we will not just prove the existence of transformers that can follow $\text{TC}^0$ instructions but also specify concrete choices for the positional embedding scheme and the class of attention functions that are sufficient to do so.

Let $x, y$ be the concatenation of vectors $x, y$, or, in the case where $y$ is a scalar, the vector with $y$ appended to $x$.

**Fractional Positional Embeddings** For $\sigma \in \Sigma$, let $v(\sigma)$ be the one-hot encoding of $\sigma$ into $\mathbb{R}^{[2]}$. We define the fractional positional embedding as

$$\phi(w, i) = \langle v(\sigma), i/n \rangle.$$

**Saturated Attention** We imagine $f(h^i, h^f)$ is computed via saturated attention (cf. Merrill et al., 2022), which provides a simple model of the types of attention we can expect to be learned in transformers (Merrill et al., 2021). First, queries are computed as $q_i = Qh^i$, and then keys $k_j = Kh^j$, where

$$f(h^i, h^f) = \begin{cases} 1 & \text{if } s_{ij} = \max_k s_{ik} \\ 0 & \text{otherwise.} \end{cases}$$

After normalization, saturated attention creates a distribution that is uniform over a subset of positions. Thus, it is capable of parameterizing hard attention, uniform attention over the full sequence, and various attention patterns in between.

**Simple Pooling Functions** For simplicity, we assume pooling functions $g$ are thresholded linear functions of their inputs. Thus, they could be implemented by a feedforward neural net. Without loss of generality, we also imagine that attention heads have a value function, which can be folded into the pooling function from the last layer.

Now, we are ready to present the main result. Our construction below is specific to the circuit serialization scheme discussed in §3, but can be extended to other serializations as well.

**Lemma 4** For all $d$, there exists a transformer with fractional positional embeddings, saturated attention, thresholded linear pooling functions, and depth $2d$ that, for any threshold circuit $C$ of depth $d$, maps input $\langle C, x \rangle$ to the value $C(x)$.

**Proof.** We will construct a pair of two transformer layers that evaluate all the nodes at depth $\ell$ in the threshold circuit, for any $\ell$. It follows that a transformer of depth $2d$ can compute the value $C(x)$.

We refer to a token of type $X$ as an *input node*. Similarly, we will call a token of semi-terminal type Dir a *gate node*. Finally we will call a token of type $\delta$ an *argument*.

**Base Case: Input Nodes.** We construct one attention layer that attends uniformly over all positions whose value returns 1 if $w_i = X$ and 0 otherwise. Thus, this head computes $\#(X)/n$, where $\#(X)$ is the number of occurrences of $X$ in $w$. We then define a second layer that, at input node $i$, retrieves the token at position $j$ defined by

$$j = \frac{1 - \#(X) + i}{n}.$$

$j$ is the index of the $i$th input value. Thus, after this layer, each input node $i$ stores its value $x_i$.

In the base case, we also construct an attention head that, at the $i$th gate node, counts the fraction of nodes (out of $n$) that are gate nodes to the left of the current node. Thus, this head computes $i/n$. Also at gate node $i$, we construct an attention head that counts the fraction of nodes to its right before the next $\delta$ node that have value 1. This head thus has value $k_i/m_i$ where $k_i$ is the threshold value of the $i$-th gate and $m_i$ is its arity. We apply the same construction at each argument $\delta$ to count the 1’s that follow until the next non-1 symbol.

Finally, using the first attention layer, we have each $J$ node attend to the first argument symbol $\delta$ to its left and retrieve its index $j/n$. Then, in
the second attention layer, each argument attends uniformly over all nodes with values \( j/n \). The net effect is for each argument node to store \( j/n \), i.e., the pointer it is encoding in unary as \( \& 1^j \).

**Inductive Case: Gate Nodes.** By our inductive assumption over prior layers, all tokens corresponding to circuit nodes at depth \( \leq \ell \) contain their appropriate value. We now construct 2 transformer layers to evaluate gate nodes at depth \( \ell + 1 \).

In the first attention layer, each argument \( j \) attends to the the closest gate node \( i \) to its left, which is the gate it belongs to. Recall from the base case that argument \( j \) already stores \( j/n \). Each argument \( \& 0^j \) attends with position key \( j/n \) to gate node \( j \) and retrieves its value in the previous layer.

The second attention layer applies at gate nodes, not arguments. At gate \( i \) of arity \( m_i \), we set the attention \( s(i,j) \) to indicate whether argument \( j \) belongs to gate node \( i \), which holds for exactly \( m \) arguments. We set the attention value to be the binary value of the referent of argument \( j \). Thus, the attention head computes \( c_i/m_i \), where \( c_i \) is the number of arguments of node \( i \) that are 1. We repeat this for all gate nodes.

At this point, for the \( i \)-th gate node, we have computed both \( c_i/m_i \) and \( k_i/m_i \). Thresholding \((c_i - k_i)/m_i \) at 0 allows us to decide, based on whether Dir is \( \leq \) or \( \geq \), whether the current gate node should output a 0 or a 1. Repeating this for all gates at layer \( \ell + 1 \) completes the inductive step that we can evaluate all gate nodes in this layer.

This shows that transformers with simple position embeddings, attention, and pooling functions can simulate any instruction provided in the form of a TC\(^0\) circuit.

Formally, an instruction \( I \) is any description of a function \( f_I \) of \( \{0,1\}^* \). We say a transformer correctly follows an instruction \( I \) if, for all \( x \in \{0,1\}^* \), it correctly computes \( f_I(x) \) on input \( \langle I, x \rangle \). A non-uniform instruction description is a family of length-specific descriptions \( \{I_n\}_{n=1}^{\infty} \). We say a transformer correctly follows a non-uniform instruction family \( \{I_n\} \) if, for all \( n \) and all \( x \in \{0,1\}^n \), it correctly computes \( f_I(x) \) on input \( \langle I_n, x \rangle \). The non-uniform description \( \{I_n\} \) may take any form. When it forms a TC\(^0\) circuit family, we refer to it as a TC\(^0\) instruction description. It follows from Lemma 4 that:

**Theorem 3** Depth-2d transformers can correctly follow any depth-\( d \) TC\(^0\) instruction description.

### 7.1 Advice Transformers

We can also view circuit evaluation abilities of transformers (Lem. 4) from the lens of advice taking Turing machines which, in addition to their usual input, are also provided an input length dependent (but input independent) advice string. For instance, P/\( \text{poly} \) is the class of problems decidable in polynomial time when the Turing machine is additionally given an advice string of size polynomial in the input length (cf. Arora and Barak, 2009).

In the same vein, let T/\( \text{poly} \) be the class of log-precision, constant-depth transformers with polynomial advice strings. In other words, on an input of size \( n \), we allow the transformer to receive an additional \( \text{poly}(n) \) bits of input that cannot depend on the standard input. Now let \( \{C_n\}_{n=1}^{\infty} \) be a circuit family demonstrating that a problem is in non-uniform TC\(^0\). Then, by passing the description of \( C_n \) as advice for input length \( n \), it immediately follows from Lem. 4 that advice transformers can simulate non-uniform TC\(^0\).

**Theorem 4** Non-uniform TC\(^0\) \( \subseteq \) T/\( \text{poly} \).

Since non-uniform TC\(^0\) even contains some undecidable languages (Aaronson et al., 2022), T/\( \text{poly} \) is clearly a very powerful class and a strict superset of T, the class of decision problems recognized by transformers (which are all decidable). Thus, a problem in T/\( \text{poly} \) cannot always be solved by a transformer on its own. However, if given a description of how to do so (an “advice”) in the form of a TC\(^0\) circuit, our result shows that a transformer would be able to solve that problem.

### 8 Conclusion

Answering two open questions from Merrill et al. (2022), we prove log-precision transformers with soft attention can be simulated by uniform constant-depth threshold circuits. This establishes majority as a fundamental operation for understanding the computational model of transformers: any log-precision transformer can be re-expressed as a polynomial number of majority computations with constant depth. This result also establishes potential limits on the computational power of transformers: if L \( \subset \) P, transformers cannot compute all poly-time functions. The intuition at the heart of this result is that forcing
a model to be parallelizable may sacrifice its expressiveness. Since parallelism seems essential to pretraining any massive model at scale, any large language model—transformer or otherwise—may suffer from a similar tradeoff.

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