On the convergence to a statistical equilibrium for the wave equations coupled to a particle

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Abstract

We consider a linear Hamiltonian system consisting of a classical particle and a scalar field describing by the wave or Klein-Gordon equations with variable coefficients. The initial data of the system are supposed to be a random function which has some mixing properties. We study the distribution $\mu_t$ of the random solution at time moments $t \in \mathbb{R}$. The main result is the convergence of $\mu_t$ to a Gaussian probability measure as $t \to \infty$. The mixing properties of the limit measures are studied. The application to the case of Gibbs initial measures is given.

Key words and phrases: a wave field coupled to a particle; Cauchy problem; random initial data; mixing condition; Volterra integro-differential equation; compactness of measures; characteristic functional; convergence to statistical equilibrium; Gibbs measures

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1 Introduction

The paper concerns problems of long-time convergence to an equilibrium distribution for a coupled system consisting of a field and a particle. For one-dimensional chains of harmonic oscillators, the results have been established by Spohn and Lebowitz in [36], and by Boldrighini et al. in [2]. Ergodic properties of one-dimensional chains of anharmonic oscillators coupled to heat baths were studied by Jakšić, Pillet and others (see, e.g., [23, 14]). In [6, 7, 8, 10], we studied the convergence to equilibrium for the systems described by partial differential equations. Later on, similar results were obtained in [9] for $d$-dimensional harmonic crystals with $d \geq 1$, and in [11] for a scalar field coupled to a harmonic crystal.

Here we treat the linear Hamiltonian system consisting of the scalar wave or Klein–Gordon field $\varphi(x)$, $x \in \mathbb{R}^d$, coupled to a classical particle with position in $q \in \mathbb{R}^d$, $d \geq 3$. The Hamiltonian functional of the coupled system reads

$$H(\varphi, \pi, q, p) = H_A(q, p) + H_B(\varphi, \pi) + q \cdot \langle \nabla \varphi, \rho \rangle.$$ \hspace{1cm} (1.1)

Here “$\cdot$” stands for the standard Euclidean scalar product in $\mathbb{R}^d$, $\langle \cdot, \cdot \rangle$ denotes the inner product in the real Hilbert space $L^2(\mathbb{R}^d)$ (or its extensions), $H_A$ is the Hamiltonian of the particle,

$$H_A(q, p) = \frac{1}{2} \left( |p|^2 + \omega^2 |q|^2 \right), \text{ with some } \omega > 0,$$

and $H_B$ denotes the Hamiltonian for the wave or Klein-Gordon field. We suppose that

$$H_B(\varphi, \pi) = \frac{1}{2} \int_{\mathbb{R}^d} \left( \sum_{i,j=1}^d a_{ij}(x) \nabla_i \varphi(x) \nabla_j \varphi(x) + a_0(x) |\varphi(x)|^2 + |\pi(x)|^2 \right) dx$$

in the case of the wave field (WF), and

$$H_B(\varphi, \pi) = \frac{1}{2} \int_{\mathbb{R}^d} \left( \sum_{j=1}^d \left( |\nabla_j - iA_j(x)|\varphi(x)|^2 + m^2 |\varphi(x)|^2 + |\pi(x)|^2 \right) dx, \text{ with some } m > 0,$$

in the case of the Klein-Gordon field (KGF). We impose the conditions A1–A5 below on the coefficients $a_{ij}(x)$, $a_0(x)$ and $A_j(x)$. In particular, the functions $a_{ij}(x) - \delta_{ij}$, $a_0(x)$ and $A_j(x)$ vanish outside a bounded domain.

We assume that the initial data $Y_0 := (\varphi_0, \pi_0, q_0, p_0)$ are a random element of a real functional space $\mathcal{E}$ consisting of states with finite local energy, see Definition 2.1 below. The distribution of $Y_0$ is a probability measure $\mu_0$ of mean zero satisfying conditions S1–S3 below. In particular, we assume that the initial measure $\mu_0$ satisfies a mixing condition. Roughly speaking, it means that

$$Y_0(x) \text{ and } Y_0(y) \text{ are asymptotically independent as } |x - y| \to \infty.$$

We study the distributions $\mu_t$, $t \in \mathbb{R}$, of the random solution $Y_t := (\varphi_t, \pi_t, q_t, p_t)$ at time moments $t \in \mathbb{R}$. Our main objective is to prove the weak convergence of the measures $\mu_t$ to an equilibrium measure $\mu_\infty$,

$$\mu_t \to \mu_\infty \text{ as } t \to \infty,$$ \hspace{1cm} (1.2)

where the limit measure $\mu_\infty$ is a Gaussian measure on $\mathcal{E}$. We derive the explicit formulas for the limiting correlation functions of $\mu_\infty$. The similar convergence holds for $t \to -\infty$ because
our system is time-reversible. We prove that the dynamic group is mixing (and, in particular, ergodic) with respect to the limit measures \( \mu_\infty \). Moreover, we extend results to the case of non translation-invariant initial measures \( \mu_0 \) and give an application to the case of the Gibbs initial measures.

Let us outline the strategy of the proof. When the field variables \((\varphi_t, \pi_t)\) are eliminated from the equations of the coupled system, the particle evolves according to a linear Volterra integro-differential equation of a form (see Eqn (3.2) below)

\[
\ddot{q}_t = -\omega^2 q_t + \int_0^t D(t - s)q_s \, ds + F(t), \quad t \in \mathbb{R},
\]

where \( D(t) \) is a matrix-valued function depending on the coupled function \( \rho \), \( F(t) \) is a vector-valued function depending on the initial field data \((\varphi_0, \pi_0)\). Therefore, our first objective is to study the long-time behavior of the solutions to Eqn (1.3). We prove that for the solutions \( q_t \) of Eqn (1.3) with \( F(t) \equiv 0 \), the following bound holds

\[
|q_t| + |\dot{q}_t| \leq C\varepsilon_F(t),
\]

where \( \varepsilon_F(t) = e^{-\delta|t|} \) with some \( \delta > 0 \) for the WF, and \( \varepsilon_F(t) = (1 + |t|)^{-3/2} \) for the KGF (see Theorem 3.1 below).

The deterministic dynamics of the equations with delay has been extensively studied by many authors under some restrictions on the kernel \( D(t) \): Myshkis [31], Grossman and Millet [17], Driver [4] and others. For details on the first results and problems in the theory of equations with delay, we refer to the survey paper by Corduneanu and Lakshmikantham [3]. For further development of the theory, see the monograph by Grippenberg, Londen and Staffans [16]. The stability properties for Volterra integro-differential equations can be found in the papers by Murakami [30], Hara [18], and Kordonis and Philos [26].

The linear stochastic Volterra equations of convolution type have been treated also by many authors, see, e.g., Appleby and Freeman [1], the survey article by Karczewska [24] and the references therein.

Note that in the literature frequently the asymptotic behavior of the solutions of Eqn (1.3) is studied assuming that \( F(t) \) is a Gaussian with noise or (and) that the kernel \( D(t) \) has the exponential decay or is of one sign. However, in our case, \( F(t) \) is not Gaussian white-noise, in general. Moreover, in the case of the KGF, the decay of \( D(t) \) is like \((1 + |t|)^{-3/2}\).

In recent years the nonlinear generalized Langevin equation, i.e., the equation of a form (cf. Eqn (A.20) below)

\[
\ddot{q}_t = -\nabla V(q_t) - \int_0^t \Gamma(t - s)\dot{q}_s \, ds + F(t), \quad t \in \mathbb{R},
\]

with a stationary Gaussian process \( F(t) \) and with a smooth (confining or periodic) potential \( V(q) \), has been investigated also extensively, see, e.g., [22, 32, 35, 41]. In particular, the ergodic properties of (1.5) were studied by Jakšić and Pillet in [22], the qualitative properties of solutions to Eqn (1.5) were established by Ottobre and Pavliotis in [32]. Rey-Bellet and Thomas [33] have investigated a model consisting of a chain of non-linear oscillators coupled to two heat reservoirs. The nonlinear stochastic integro-differential equations were studied also in Mao’ works (see, e.g., [27, 28]).
In this paper, we study a linear "field-particle" model. However, we do not assume that the initial distribution of the system is a Gibbs measure or absolutely continuous with respect to a Gibbs measure. Therefore, in particular, the force $F(t)$ in Eqn (1.3) is non-Gaussian, in general.

The key step in our proof is the derivation of the asymptotic behavior for the solutions $Y_t$ of the coupled field-particle system. Using bound (1.4), we prove the following asymptotics in mean (see Corollary 5.2 below):

$$\langle Y_t, Z \rangle \sim \langle W_t(\varphi_0, \pi_0), \Pi(Z) \rangle, \quad t \to \infty,$$

where $W_t$ is a solving operator to the Cauchy problem for the wave or Klein-Gordon equations (2.13), $(\varphi_0, \pi_0)$ is a initial state of the field, and the function $\Pi(Z)$ is defined in (2.24). This asymptotics allows us to apply the results from [8, 10], where the weak convergence of the statistical solutions has been proved for wave and Klein–Gordon equations with variable coefficients. We divide the proof of (1.2) into two steps: we first establish the weak compactness of the measures family $\{\mu_t, t \in \mathbb{R}\}$ (see Section 4), and then we prove the convergence of the characteristic functionals of the measures $\mu_t$ (Section 6).

In conclusion, note that convergence (1.2) remains true for a linear Hamiltonian system consisting of $N$ wave fields coupled to a single particle. In this case, the Hamiltonian is

$$\sum_{k=1}^{N} H_B(\varphi_k, \pi_k) + H_A(q, p) + q \cdot \sum_{k=1}^{N} \langle \nabla \varphi_k, \rho_k \rangle.$$

The paper is organized as follows. In Section 2 we describe the model, impose the conditions on the coupled function $\rho$ and on the initial measures $\mu_0$ and state the main results. The limit behavior for solutions of Eqn (1.3) is studied in Section 3. In Section 4 we prove the compactness of the measures family $\{\mu_t, t \in \mathbb{R}\}$. The asymptotics (1.6) is proved in Section 5. In Section 6 we establish the convergence of characteristic functionals of $\mu_t$ to a limit and complete the proof of the main result. In Section 7 we study the mixing properties of the dynamics with respect to the limit measures $\mu_\infty$. In Section 8 we extend the results to the case of non translation–invariant initial measures. Appendix A concerns the case of Gibbs initial measures. The existence of the solutions of the coupled system is proved in Appendix B.
2 Main Results

2.1 Model

After taking formally variational derivatives in (1.1), the coupled dynamics becomes

$$
\dot{\varphi}_t(x) = \pi_t(x), \quad \dot{\pi}_t(x) = L_B \varphi_t(x) + q_t \cdot \nabla \rho(x), \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R},
$$

$$
\dot{q}_t = p_t, \quad \dot{p}_t = -\omega^2 q_t + \int_{\mathbb{R}^d} \nabla \rho(x) \varphi_t(x) \, dx.
$$

(2.1)

Here $L_B$ is a differential operator of one of two types:

$$
L_B = \begin{cases} 
L_W := \sum_{i,j=1}^d \nabla_i (a_{ij}(x) \nabla_j) - a_0(x), \\
L_{KG} := \sum_{j=1}^d (\nabla_j - iA_j(x))^2 - m^2,
\end{cases}
$$

(2.2)

where $\nabla_i = \partial/\partial x_i, i = 1, \ldots, d; d \geq 3$, and $d$ is odd in the case when $L_B = L_W$. For simplicity of exposition, we consider the case $d = 3$ only.

We study the Cauchy problem for the system (2.1) with initial data

$$
\varphi_t(x)|_{t=0} = \varphi_0(x), \quad \pi_t(x)|_{t=0} = \pi_0(x), \quad x \in \mathbb{R}^3, \quad q_t|_{t=0} = q_0, \quad p_t|_{t=0} = p_0.
$$

(2.3)

Write $\phi_t = (\varphi_t(\cdot), \pi_t(\cdot)), \xi_t = (q_t, p_t), Y_t = (\phi_t, \xi_t)$. Then the system (2.1)–(2.3) becomes

$$
\dot{Y}_t = \mathcal{L}(Y_t), \quad t \in \mathbb{R}; \quad Y_t|_{t=0} = Y_0.
$$

(2.4)

We assume that the coefficients of $L_B$ satisfy the following conditions A1–A5.

A1. $a_{ij}(x), a_0(x), A_j(x)$ are real $C^\infty$-functions.

A2. $a_{ij}(x) = \delta_{ij}, a_0(x) = 0, A_j(x) = 0$ for $|x| > R_a$, where $R_a \leq \infty$. Then

$$
L_B \varphi_t(x) = (\Delta - m^2) \varphi_t(x) \quad \text{for} \quad |x| > R_a.
$$

Here $m > 0$ in the case of the Klein–Gordon field (KGF), i.e., $L_B = L_{KG}$, and $m = 0$ in the case of the wave field (WF), i.e., $L_B = L_W$.

In the WF case, we impose the next conditions A3 and A4.

A3. $a_0(x) \geq 0$, and the hyperbolicity condition holds: there exists a constant $\alpha > 0$ such that

$$
\sum_{i,j=1}^3 a_{ij}(x) k_i k_j \geq \alpha |k|^2, \quad x, k \in \mathbb{R}^3.
$$

(2.5)

A4. A non-trapping condition [39]: for $(x(0), k(0)) \in \mathbb{R}^3 \times \mathbb{R}^3$ with $k(0) \neq 0,$

$$
|x(t)| \to \infty \quad \text{as} \quad t \to \infty,
$$

(2.6)

where $(x(t), k(t))$ is a solution to the Hamiltonian system

$$
\dot{x}(t) = \nabla_k h(x(t), k(t)), \quad \dot{k}(t) = -\nabla_x h(x(t), k(t)), \quad \text{with} \quad h(x, k) = \frac{1}{2} \sum_{i,j=1}^3 a_{ij}(x) k_i k_j.
$$
Example. In the WF case, A1–A4 hold for the acoustic equation with constant coefficients
\[ \ddot{\phi}(x) = \Delta \phi(x), \quad x \in \mathbb{R}^3. \]
For instance, A4 follows because \( \hat{k}(t) \equiv 0 \Rightarrow x(t) \equiv k(0)t + x(0). \)

Write \( M_a = \max_{x \in \mathbb{R}^3} \max_{i,j} |a_{ij}(x) - \delta_{ij}|, |a_0(x)| \), or \( M_a = \max_{x \in \mathbb{R}^3} |A_j(x)| \).

A5. \( M_a \) is sufficiently small (we will specify this condition in the proof of Lemma 3.3).

Now we formulate the conditions R1–R3 on \( \rho(x) \) and \( \omega > 0 \).
R1. In the case of the WF, we assume that \( \|\nabla \rho\|_{L_2}^2 < \alpha \omega^2 \) with \( \alpha \) from condition (2.5). In the KGF case, \( \|\nabla \rho\|_{L_2}^2 < m^2 \omega^2 \).
R2. The function \( \rho(x) \) is a real-valued smooth function, \( \rho(-x) = \rho(x), \rho(x) = 0 \) for \( |x| \geq R_\rho \).
R3. For any \( k \in \mathbb{R}^3 \setminus \{0\} \), \( \hat{\rho}(k) = \int e^{ik \cdot x} \rho(x) dx \neq 0 \).

Remark. Condition R1 implies that the Hamiltonian \( H(\phi_i, \xi_i) \) is nonnegative for finite energy solutions (see Appendix B). In the case of the constant coefficients, i.e., \( L_B = \Delta - m^2 \), condition R1 can be weakened as follows.

R1’. The matrix \( \omega^2 I - K_m \) is positive definite, where \( K_m = (K_{m,ij})_{i,j=1}^3 \) stands for the \( 3 \times 3 \) matrix with matrix elements \( K_{m,ij} \).
\[ K_{m,ij} := (2\pi)^{-3} \int_{\mathbb{R}^3} \frac{k_i k_j |\hat{\rho}(k)|^2}{k^2 + m^2} dk, \quad m \geq 0. \] (2.7)

However, to prove the main result in the case of the KGF, we need a stronger condition than R1’. Namely, the matrix \( (\omega^2 - m^2) I - K_m \) is positive definite. This condition is fulfilled, in particular, if \( \|\nabla \rho\|_{L_2}^2 < m^2(\omega^2 - m^2) \).

2.2 Phase space for the coupled system

We introduce a phase space \( \mathcal{E} \).

Definition 2.1 (i) Choose a function \( \zeta(x) \in C_0^\infty(\mathbb{R}^3) \) with \( \zeta(0) \neq 0 \). Denote by \( H^s_{\text{loc}}(\mathbb{R}^3), \) \( s \in \mathbb{R} \), the local Sobolev spaces, i.e., the Fréchet spaces of distributions \( \varphi \in D'(\mathbb{R}^3) \) with the finite seminorms \( \|\varphi\|_{s,R} := \|\Lambda^s(\zeta(x)/R)\varphi\|_{L_2(\mathbb{R}^3)}, \) where \( \Lambda^s \) stands for the pseudodifferential operator with the symbol \( \langle k \rangle^s \), i.e.,
\[ \Lambda^s \psi := F_{k \rightarrow x}^{-1}(\langle k \rangle^s \hat{\psi}(k)), \quad \langle k \rangle := \sqrt{|k|^2 + 1}, \]
and \( \hat{\psi} \) is the Fourier transform of the tempered distribution \( \psi \).

(ii) \( \mathcal{H} \equiv H^1_{\text{loc}}(\mathbb{R}^3) \oplus H^0_{\text{loc}}(\mathbb{R}^3) \) is the Fréchet space of pairs \( \phi \equiv (\varphi(x), \pi(x)) \) with real valued functions \( \varphi(x) \) and \( \pi(x) \), which is endowed with the local energy seminorms
\[ \|\phi\|_{R}^2 = \int_{|x|<R} (|\varphi(x)|^2 + |\nabla \varphi(x)|^2 + |\pi(x)|^2) dx < \infty, \quad R > 0. \]
In the case of the KGF, we assume that \( \varphi(x) \) and \( \pi(x) \) are complex valued functions.

(iii) \( \mathcal{E} \equiv \mathcal{H} \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \) is the Fréchet space of vectors \( Y \equiv (\phi(x), q, p) \) with the local energy seminorms
\[ \|Y\|_{R}^2 = \|\phi\|_{R}^2 + |q|^2 + |p|^2, \quad R > 0. \] (2.8)

(iv) For \( s \in \mathbb{R} \), write \( \mathcal{H}^s \equiv H^1_{\text{loc}}(\mathbb{R}^3) \oplus H^0_{\text{loc}}(\mathbb{R}^3) \) and \( \mathcal{E}^s \equiv \mathcal{H}^s \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \). In particular, \( \mathcal{H} \equiv \mathcal{H}^0, \mathcal{E} \equiv \mathcal{E}^0 \).
Using the standard technique of pseudodifferential operators and Sobolev’s Theorem (see, e.g., [19]), one can prove that $\mathcal{E}^0 \equiv \mathcal{E} \subset \mathcal{E}^{-\varepsilon}$ for every $\varepsilon > 0$, and the embedding is compact.

**Proposition 2.2** Let conditions A1–A3, R1 and R2 hold. Then

(i) For every $Y_0 \in \mathcal{E}$, the Cauchy problem (2.4) has a unique solution $Y_t \in C(\mathbb{R}, \mathcal{E})$.

(ii) For any $t \in \mathbb{R}$, the operator $S_t : Y_0 \mapsto Y_t$ is continuous on $\mathcal{E}$. Moreover, for any $T > 0$ and $R > \max\{R_\mu, R_\lambda\}$, 

$$
\sup_{|t| \leq T} \|S_t Y_0\|_{\mathcal{E}, R} \leq C(T) \|Y_0\|_{\mathcal{E}, R+T}.
$$

This proposition can be proved using a similar technique as in [25, Lemma 6.3] and [12, Proposition 2.3], and the proof is based on Lemma 2.3 below (cf. [12, Lemma 3.1]). Introduce a Hilbert space $H^1_F(\mathbb{R}^3)$ as follows. For the KGF, $H^1_F(\mathbb{R}^3)$ is the Sobolev space $H^1(\mathbb{R}^3)$. In the case of the WF, $H^1_F(\mathbb{R}^3)$ stands for the completion of real space $C_0^\infty(\mathbb{R}^3)$ with norm $\|\nabla \varphi\|_{L^2}$. Denote by $E$ the Hilbert space $H^1_F(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ with finite norm

$$
\|Y\|_E^2 = \int_{\mathbb{R}^3} (|\nabla \varphi(x)|^2 + m^2|\varphi(x)|^2 + |\pi(x)|^2) \, dx + |q|^2 + |p|^2 \quad \text{for } Y = (\varphi(x), \pi(x), q, p),
$$

where $m > 0$ for the KGF case, and $m = 0$ for the WF case.

**Lemma 2.3** Let conditions A1–A3, R1 and R2 be valid. Then the following assertions hold.

(i) For every $Y_0 \in E$, the Cauchy problem (2.7) has a unique solution $Y_t \in C(\mathbb{R}, E)$.

(ii) For $Y_0 \in E$, the energy is conserved, finite and nonnegative, $H(Y_t) = H(Y_0) \geq 0$, $t \in \mathbb{R}$.

(iii) For every $t \in \mathbb{R}$, the operator $S_t : Y_0 \mapsto Y_t$ is continuous on $E$. Moreover,

$$
\|Y_t\|_E \leq C\|Y_0\|_E \quad \text{for } t \in \mathbb{R}.
$$

We outline the proof of Lemma 2.3 and Proposition 2.2 in Appendix B.

### 2.3 Conditions on the initial measure

Let $(\Omega, \Sigma, P)$ be a probability space with expectation $\mathbb{E}$ and $\mathcal{B}(\mathcal{E})$ denote the Borel $\sigma$-algebra in $\mathcal{E}$. We assume that $Y_0 = Y_0(\omega, x)$ in (2.4) is a measurable random function with values in $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$. In other words, the measurable map $(\omega, x) \mapsto Y_0(\omega, x)$ is a measurable map $\Omega \times \mathbb{R}^3 \to \mathbb{R}^8$ with respect to the (completed) $\sigma$-algebra $\Sigma \times \mathcal{B}(\mathbb{R}^3)$ and $\mathcal{B}(\mathbb{R}^8)$. Then $Y_t = S_t Y_0$ is also a measurable random function with values in $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$, by Proposition 2.2. Denote by $\mu_0(dY_0)$ the Borel probability measure in $\mathcal{E}$ giving the distribution of $Y_0$. Without loss of generality, we may assume that $(\Omega, \Sigma, P) = (\mathcal{E}, \mathcal{B}(\mathcal{E}), \mu_0)$ and $Y_0(\omega, x) = \omega(x)$ for $\mu_0(d\omega) \times dx$-almost all $(\omega, x) \in \mathcal{E} \times \mathbb{R}^3$.

Set $\mathcal{D} = \mathcal{D}_0 \oplus \mathbb{R}^2 \oplus \mathbb{R}^3$, $\mathcal{D}_0 := [C_0^\infty(\mathbb{R}^3)]^2$, and

$$
\langle Y, Z \rangle := \langle \phi, f \rangle + q \cdot u + p \cdot v \quad \text{for } Y = (\phi, q, p) \in \mathcal{E} \quad \text{and } Z = (f, u, v) \in \mathcal{D}.
$$

For a probability measure $\mu$ on $\mathcal{E}$, denote by $\hat{\mu}$ the characteristic functional (the Fourier transform)

$$
\hat{\mu}(Z) = \int \exp(i \langle Y, Z \rangle) \mu(dY), \quad Z \in \mathcal{D}.
$$

A measure $\mu$ is called Gaussian (with zero expectation) if its characteristic functional is of the form $\hat{\mu}(Z) = \exp\{- (1/2)Q(Z, Z)\}$, $Z \in \mathcal{D}$, where $Q$ is a real, nonnegative, quadratic form on $\mathcal{D}$. A measure $\mu$ is called translation-invariant if $\mu(T_h B) = \mu(B)$ for any $B \in \mathcal{B}(\mathcal{E})$ and $h \in \mathbb{R}^3$, where $T_h Y(x) = Y(x - h)$.
We assume that the initial measure \( \mu_0 \) has the following properties S0–S3.

**S0** \( \mu_0 \) has zero expectation value, \( \mathbb{E}Y_0(x) \equiv \int Y_0(x) \mu_0(dy) = 0 \) for \( x \in \mathbb{R}^3 \).

**S1** \( \mu_0 \) has finite mean energy density, i.e., \( \mathbb{E}(|q_0|^2 + |p_0|^2) < \infty \), and

\[
\mathbb{E}\left(|\varphi_0(x)|^2 + |\nabla \varphi_0(x)|^2 + |\pi_0(x)|^2\right) \leq c_0 < \infty. \tag{2.10}
\]

Write \( \mu_0^B := P_\mu_0 \), where \( P : (\varphi_0, q_0, p_0) \in \mathcal{E} \rightarrow \phi_0 \in \mathcal{H} \). Now we impose conditions S2 and S3 on the measure \( \mu_0^B \). For simplicity of exposition, we assume that \( \mu_0^B \) has translation-invariant correlation matrices (the case of non translation-invariant measures \( \mu_0^B \) is considered in Section S).

**S2** The correlation functions of the measure \( \mu_0^B \),

\[
Q_0^{ij}(x, y) := \int \phi_0^j(x)\phi_0^j(y) \mu_0^B(dy), \quad x, y \in \mathbb{R}^3, \quad \phi_0 = (\phi_0^0, \phi_0^1) \equiv (\phi_0, \pi_0),
\]

are translation-invariant, i.e., \( Q_0^{ij}(x, y) = q_0^{ij}(x - y), i, j = 0, 1 \).

Now we formulate the mixing condition for the measure \( \mu_0^B \).

Let \( \mathcal{O}(r) \) be the set of all pairs of open convex subsets \( \mathcal{A}, \mathcal{B} \subset \mathbb{R}^3 \) at distance \( d(\mathcal{A}, \mathcal{B}) \geq r \), and let \( \sigma(\mathcal{A}) \) be the \( \sigma \)-algebra in \( \mathcal{H} \) generated by the linear functionals \( \phi \mapsto \langle \phi, f \rangle \), where \( f \in |C_0^\infty(\mathbb{R}^3)|^2 \) with supp \( f \subset \mathcal{A} \). Define the Ibragimov mixing coefficient of a probability measure \( \mu_0^B \) on \( \mathcal{H} \) by the rule (cf [20, Def. 17.2.2])

\[
\varphi(r) \equiv \sup_{(\mathcal{A}, \mathcal{B}) \in \mathcal{O}(r)} \sup_{A \in \sigma(\mathcal{A}), B \in \sigma(\mathcal{B}), \mu_0^B(A \cap B) > 0} \frac{|\mu_0^B(A \cap B) - \mu_0^B(A)\mu_0^B(B)|}{\mu_0^B(B)}. \tag{2.11}
\]

**Definition 2.4** We say that the measure \( \mu_0^B \) satisfies the strong uniform Ibragimov mixing condition if \( \varphi(r) \rightarrow 0 \) as \( r \rightarrow \infty \).

**S3** The measure \( \mu_0^B \) satisfies the strong uniform Ibragimov mixing condition, and

\[
\int_0^{+\infty} r^{d_F} \varphi^{1/2}(r)dr < \infty, \tag{2.12}
\]

where \( d_F = d - 1 \) for the KGF, and \( d_F = d - 2 \) for the WF, \( d \) is dimension of the space.

**Remark 2.5** (i) The examples of the measures \( \mu_0^B \) with zero mean satisfying conditions (2.10), S2 and S3 are given in [6, Section 2.6].

(ii) Instead of the strong uniform Ibragimov mixing condition, it suffices to assume the uniform Rosenblatt mixing condition [34] together with a higher degree (> 2) in the bound (2.10), i.e., to assume that there exists a \( \delta, \delta > 0 \), such that

\[
\mathbb{E}\left(|\varphi_0(x)|^{2+\delta} + |\nabla \varphi_0(x)|^{2+\delta} + |\pi_0(x)|^{2+\delta}\right) < \infty.
\]

In this case, the condition (2.12) needs the following modification: \( \int_0^{+\infty} r^{d_F} \alpha^p(r)dr < \infty \), where \( p = \min(\delta/(2 + \delta), 1/2) \), \( \alpha(r) \) is the Rosenblatt mixing coefficient defined as in (2.11) but without \( \mu_0^B(B) \) in the denominator.
2.4 Convergence to equilibrium for Klein-Gordon equations

We first consider the Cauchy problem for the wave (or Klein–Gordon) equation,

\[
\begin{cases}
\ddot{\varphi}_t(x) = L_B \varphi_t(x), & t \in \mathbb{R}, \ x \in \mathbb{R}^3, \\
\varphi_t(x)|_{t=0} = \varphi_0(x), \ \dot{\varphi}_t(x)|_{t=0} = \pi_0(x).
\end{cases}
\] (2.13)

Lemma 2.6 follows from [29] Thms V.3.1, V.3.2 as the speed of propagation for Eqn (2.13) is finite.

**Lemma 2.6** Let conditions A1–A4 hold. Then (i) for any \(\phi_0 = (\varphi_0, \pi_0) \in \mathcal{H}\), there exists a unique solution \(\phi_t = (\varphi_t(x), \dot{\varphi}_t(x)) \in C(\mathbb{R}, \mathcal{H})\) to the Cauchy problem (2.13).

(ii) For any \(t \in \mathbb{R}\), the operator \(W_t : \phi_0 \mapsto \phi_t\) is continuous on \(\mathcal{H}\), and for any \(T > 0\), \(R > R_a\),

\[
\sup_{|t| \leq T} \|W_t \phi_0\|_R \leq C(T) \|\phi_0\|_{R+T}.
\]

Let \(E_m(x)\) be the fundamental solution of the operator \(-\Delta + m^2\), i.e., \((-\Delta + m^2)E_m(x) = \delta(x)\). Since \(d = 3\), \(E_m(x) = e^{-m|x|}/(4\pi|x|)\). For almost all \(x, y \in \mathbb{R}^3\), introduce the matrix-valued function \(Q^B_\infty(x, y) = q^B_\infty(x - y)\), where

\[
q^B_\infty = \frac{1}{2} \begin{pmatrix}
q_0^{00} + E_m * q_0^{11} & q_0^{01} - q_0^{10} \\
q_0^{10} - q_0^{01} & q_0^{11} + (-\Delta + m^2)q_0^{00}
\end{pmatrix}.
\] (2.14)

Here \(q_{ij}^0\), \(i, j = 0, 1\), are correlation functions of \(\mu_0^B\) (see condition S2), * stands for the convolution. We can rewrite \(q^B_\infty\) in the Fourier transform as

\[
\hat{q}^B_\infty(k) = \frac{1}{2} \left( q_0(k) + \hat{\mathcal{C}}(k) \hat{q}_0(k) \hat{\mathcal{C}}^T(k) \right),
\] (2.15)

where \((\cdot)^T\) denotes a matrix transposition, and

\[
\hat{\mathcal{C}}(k) := \begin{pmatrix}
0 & \omega^{-1}(k) \\
-\omega(k) & 0
\end{pmatrix}, \quad \omega(k) = \sqrt{|k|^2 + m^2}.
\] (2.16)

**Remark 2.7** Conditions S0, (2.10), S2 and S3 imply, by [29] Lemma 17.2.3], that the derivatives \(D^\alpha q^B_{ij}\) are bounded by the mixing coefficient:

\[
|D^\alpha q^B_{ij}(z)| \leq C e_0 \varphi^{1/2}(|z|), \text{ for any } z \in \mathbb{R}^3, \ |\alpha| \leq 2 - i - j, \ i, j = 0, 1.
\]

Therefore, \(D^\alpha q_{ij}^0 \in L^p(\mathbb{R}^3), \ p \geq 1\) (see [6] p.16]). Hence, \((q^B_\infty)^{ij} \in L^1(\mathbb{R}^3)\) if \(m \neq 0\) by (2.14). If \(m = 0\), then the bound (2.12) implies the existence of the convolution \(E_m * q_0^{11}\) in (2.14).

Denote by \(Q^{B,0}_\infty(f, f)\) the real quadratic form on \(\mathcal{D}_0 \equiv [C_0^\infty(\mathbb{R}^3)]^2\) defined by

\[
Q^{B,0}_\infty(f, f) = \langle Q^{B}_\infty(x, y), f(x) \otimes f(y) \rangle = \langle q^B_\infty(x - y), f(x) \otimes f(y) \rangle.
\] (2.17)

**Definition 2.8** \(\mu^B_t\) is a Borel probability measure in \(\mathcal{H}\) which gives the distribution of \(\phi_t\):

\[
\mu^B_t(A) = \mu^B_t(W_t^{-1} A), \text{ for any } A \in \mathcal{B}(\mathcal{H}) \text{ and } t \in \mathbb{R}.
\]

For the measures \(\mu^B_t\), the following result was proved in [5], [7].
Theorem 2.9 Let conditions A1–A4 hold and let the measure \( \mu_0^B \) have zero mean and satisfy conditions (2.10), S2 and S3. Then (i) the measures \( \mu_t^B \) weakly converge as \( t \to \infty \) on the space \( \mathcal{H}^{-\varepsilon} \) for each \( \varepsilon > 0 \). This means the convergence
\[
\int F(\phi) \mu_t^B(d\phi) \to \int F(\phi) \mu_\infty^B(d\phi) \quad \text{as} \quad t \to \infty
\] for any bounded continuous functional \( F(\phi) \) on \( \mathcal{H}^{-\varepsilon} \).
(ii) The limit measure \( \mu_\infty^B \) is a Gaussian measure on \( \mathcal{H} \). The characteristic functional of \( \mu_\infty^B \) is of the form
\[
\hat{\mu}_\infty^B(f) = \exp \left\{ - (1/2) \mathcal{Q}_\infty^B(f, f) \right\}.
\]
where \( \mathcal{Q}_\infty^B(f, f) = \mathcal{Q}_\infty^{B_0}(\Omega f, \Omega f), \quad f \in \mathcal{D}_0, \)
where \( \Omega' \) is a linear continuous operator, and \( \Omega' = I \) in the case of the constant coefficients (see Remark 2.10 below).
(iii) The correlation matrices of \( \mu_t^B \) converge to a limit, i.e., for any \( f_1, f_2 \in \mathcal{D}_0, \)
\[
\int \langle \phi, f_1 \rangle \langle \phi, f_2 \rangle \mu_t^B(d\phi) \to \mathcal{Q}_\infty^B(f_1, f_2) \quad \text{as} \quad t \to \infty.
\]
(iv) \( \mu_\infty^B \) is invariant, i.e., \( W_t^* \mu_\infty^B = \mu_\infty^B, \quad t \in \mathbb{R} \). Moreover, the flow \( W_t \) is mixing w.r.t. \( \mu_\infty^B, \)
i.e., the convergence (1.1) holds.

Remark 2.10 Now we explain the sense of the operator \( \Omega' \) in (2.19). To prove (2.18) in the case of variable coefficients, we constructed in [6, 7] a version of the scattering theory for solutions of infinite global energy. Namely, in the case of the WF, we introduce appropriate spaces \( \mathcal{H}_\gamma \) of the initial data. By definition, \( \mathcal{H}_\gamma, \gamma > 0, \) is the Hilbert space of the functions \( \phi = (\varphi, \pi) \in \mathcal{H} \) with finite norm
\[
\| \phi \|^2 = \int e^{-2\gamma |x|} \left( |\pi(x)|^2 + |\nabla \varphi(x)|^2 + |\varphi(x)|^2 \right) dx < \infty.
\]
It follows from (2.10) that \( \mu_0^B \) is concentrated in \( \mathcal{H}_\gamma \) for all \( \gamma > 0, \) since
\[
\int \| \phi_0 \|^2 \mu_0^B(d\phi_0) \leq e_0 \int \exp(-2\gamma |x|) dx < \infty.
\]
Denote by \( W_t \) the dynamical group of Eqn (2.13), while \( W_t^0 \) corresponds to the ‘free’ equation, with \( L_B = \Delta - m^2 \). In the WF case, the following long-time asymptotics holds (see [7])
\[
W_t \phi_0 = \Omega W_t^0 \phi_0 + r_t \phi_0, \quad t > 0,
\]
where \( \Omega \) is a ‘scattering operator’. \( \Omega : \mathcal{H}_\gamma \to \mathcal{H}_\gamma \) is a linear continuous operator for sufficiently small \( \gamma > 0 \). The remainder \( r_t \) is small in local energy seminorms \( \| \cdot \|_R, \forall R > 0: \)
\[
\| r_t \phi_0 \|_R \to 0, \quad t \to \infty.
\]
The representation (2.21) is based on our version of the scattering theory for solutions of finite energy,
\[
(W_t)' f = (W_t^0)' \Omega' f + r_t' f, \quad t > 0,
\]
where (\( W_t \)) and (\( W_t^0 \)) are ‘formal adjoint’ to the groups \( W_t \) and \( W_t^0, \) respectively, see (2.23). \( \Omega', r_t' : \mathcal{H}_\gamma \to \mathcal{H}_\gamma, \| r_t' f \|_\gamma \to 0 \) as \( t \to \infty, \) where \( \| \cdot \|_\gamma \) denotes the norm in the Hilbert space \( \mathcal{H}_\gamma \) dual to \( \mathcal{H}_\gamma \). In particular, for \( f \in \mathcal{D}_0, \) \( \Omega' f \in \mathcal{H}_\gamma \) and the quadratic form \( \mathcal{Q}_\infty^{B_0} \) from (2.19) is continuous in \( \mathcal{H}_\gamma' \) (for details, see Theorem 8.1 in [7, p.1245]).
In the case of the KGF, we derived in [6] the dual representation (2.22), where the remainder \( r_t' \) is small in mean: \( \mathbb{E} \langle \phi_0, r_t' f \rangle^2 \to 0, \ t \to \infty. \) Moreover, \( \Omega' f \in \mathcal{H}_m' \equiv L^2(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3) \) for \( f \in \mathcal{D}_0, \) and the quadratic form \( \mathcal{Q}_\infty^{B_0} \) is continuous in \( \mathcal{H}_m'. \)
2.5 Convergence to equilibrium for the coupled system

To formulate the main result for the coupled system we introduce the following notations. Let $W'_t$ denote the operator adjoint to $W_t$:

$$\langle \phi, W'_t f \rangle = \langle W_t \phi, f \rangle, \quad \text{for } f \in [S(\mathbb{R}^3)]^2, \quad \phi \in \mathcal{H}, \quad t \in \mathbb{R}. \quad (2.23)$$

Let $Z = (f, u, v) \in \mathcal{D}$, i.e., $f \in [C_0^\infty(\mathbb{R}^3)]^2$, $(u, v) \in \mathbb{R}^3 \times \mathbb{R}^3$. Write

$$\Pi(Z) := f_s(x) + \alpha(x) \cdot u + \beta(x) \cdot v. \quad (2.24)$$

Here $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3)$, where

$$\alpha_i(x) := \sum_{r=1}^{+\infty} \int_0^{+\infty} \mathcal{N}(s) W'_s \nabla \rho_0 ds, \quad \text{with } \rho_0 := (\rho, 0), \quad (2.25)$$

$$\beta_i(x) := \sum_{r=1}^{+\infty} \int_0^{+\infty} \mathcal{N}(s) W'_s \nabla \rho_0^0 ds, \quad \text{with } \rho_0^0 := (0, \rho), \quad i = 1, 2, 3, \quad (2.26)$$

the matrix-valued function $\mathcal{N}(s) = (\mathcal{N}(s))_{i,j=1}^3$, $s > 0$, is introduced in Corollary 3.2 and

$$f_s(x) := f(x) + \sum_{i=1}^{+\infty} \int_0^{+\infty} (W'_s \alpha_i)(x) \langle W_s \nabla \rho_0^0, f \rangle ds. \quad (2.27)$$

**Definition 2.11** $\mu_t$ is a Borel probability measure in $\mathcal{E}$ which gives the distribution of $Y_t$: $\mu_t(B) = \mu_0(S_t^{-1}B), \forall B \in \mathcal{B}(\mathcal{E}), \forall t \in \mathbb{R}$.

Our main result is as follows.

**Theorem 2.12** Let conditions A1–A5, R1–R3 and S0–S3 hold. Then

(i) the measures $\mu_t$ weakly converge in the Fréchet spaces $\mathcal{E}^{-\varepsilon}$ for each $\varepsilon > 0$,

$$\mu_t \xrightarrow{\mathcal{E}^{-\varepsilon}} \mu_\infty \quad \text{as } \quad t \to \infty, \quad (2.28)$$

where $\mu_\infty$ is a limit measure on $\mathcal{E}$. This means the convergence

$$\int F(Y) \mu_t(dY) \to \int F(Y) \mu_\infty(dY) \quad \text{as } \quad t \to \infty$$

for any bounded continuous functional $F(Y)$ on $\mathcal{E}^{-\varepsilon}$.

(ii) The limit measure $\mu_\infty$ is a Gaussian equilibrium measure on $\mathcal{E}$. The limit characteristic functional is of the form $\tilde{\mu}_\infty(Z) = \exp\{- (1/2) Q_\infty(Z, Z)\}$, $Z \in \mathcal{D}$. $Q_\infty(Z, Z)$ denotes the real quadratic form on $\mathcal{D}$,

$$Q_\infty(Z, Z) = Q_{\infty}^B(\Pi(Z), \Pi(Z)) = Q_{\infty}^{\beta, 0}(\Omega' \Pi(Z), \Omega' \Pi(Z)), \quad (2.29)$$

where $Q_{\infty}^{\beta, 0}$ is defined in (2.17), and $\Pi(Z)$ is defined in (2.24).

(iii) The correlation functions of $\mu_t$ converge to a limit, i.e., for any $Z_1, Z_2 \in \mathcal{D}$,

$$\int \langle Y, Z_1 \rangle \langle Y, Z_2 \rangle \mu_t(dY) \to Q_\infty(Z_1, Z_2) \quad \text{as } \quad t \to \infty. \quad (2.30)$$
(iv) The measure $\mu_\infty$ is invariant, i.e., $S_t^*\mu_\infty = \mu_\infty$, $t \in \mathbb{R}$.
(v) The flow $S_t$ is mixing w.r.t. $\mu_\infty$, i.e., $\forall F,G \in L^2(\mathcal{E},\mu_\infty)$ the following convergence holds,

$$\lim_{t \to \infty} \int F(S_t Y) G(Y) \mu_\infty(dY) = \int F(Y) \mu_\infty(dY) \int G(Y) \mu_\infty(dY).$$

The assertions (i) and (ii) of Theorem 2.12 follow from Propositions 2.13 and 2.14 below.

**Proposition 2.13** The family of the measures $\{\mu_t, t \geq 0\}$ is weakly compact in $\mathcal{E}^{-\varepsilon}$ with any $\varepsilon > 0$.

**Proposition 2.14** For any $Z \in \mathcal{D}$,

$$\hat{\mu}_t(Z) \equiv \int \exp(i\langle Y, Z \rangle) \mu_t(dY) \to \exp\{-\frac{1}{2}Q_\infty(Z,Z)\}, \quad t \to \infty.$$  

Proposition 2.13 (Proposition 2.14) provides the existence (the uniqueness, resp.) of the limit measure $\mu_\infty$. Proposition 2.13 is proved in Section 4. Proposition 2.14 and the assertion (iii) of Theorem 2.12 are proved in Section 6. Theorem 2.12 (iv) follows from (2.28) since the group $S_t$ is continuous in $\mathcal{E}$ by Proposition 2.2 (ii). The assertion (v) is proved in Section 7.

### 3 Long-time behavior of the solutions

Using the operator $W_t$, we rewrite the system (2.1) in the form

$$\phi_t = W_t \phi_0 + \int_0^t q_s \cdot W_{t-s} \nabla \rho^0 \, ds, \quad (3.1)$$

$$\dot{q}_t = -\omega^2 q_t + \langle \nabla \rho_0, \phi_t \rangle = -\omega^2 q_t + \int_0^t D(t-s) q_s \, ds + F(t), \quad (3.2)$$

where $\phi_t = (\varphi_t(\cdot), \pi_t(\cdot))$, $\rho^0 = (0, \rho)$, $\rho_0 = (\rho, 0)$, $F(t)$ denotes the vector-valued function, $D(t) = \langle \nabla \rho_0, W_t \phi_0 \rangle$, $D(t)$ stands for the matrix-valued function with entries

$$D_{ij}(t) := \langle \nabla_i \rho_0, W_t \nabla_j \rho^0 \rangle, \quad \dot{D}_{ij}(t) := \langle \nabla_i \rho_0, W_t \nabla_j \rho^0 \rangle, \quad i,j = 1,2,3. \quad (3.3)$$

Note that in the case of the constant coefficients, i.e., $a_{ij}(x) \equiv \delta_{ij}$ and $a_{0}(x) \equiv 0$ or $A_{ij}(x) \equiv 0$,

$$D_{ij} = (2\pi)^{-3} \int_{\mathbb{R}^3} k_i k_j \frac{\sin \omega(k)t}{\omega(k)} |\hat{\rho}(k)|^2 \, dk, \quad \omega(k) = \sqrt{|k|^2 + m^2}, \quad m \geq 0. \quad (3.4)$$

In sections 3 and 5 we study the long-time behavior of the solutions $Y_t = (\phi_t, \xi_t)$ of problem (2.4) by the following way. In Section 3.1 we prove the time decay for the solutions $q_t$ of (3.2) with $F(t) \equiv 0$. Then we establish the time decay for the solutions $Y_t$ of (2.4) in the case when the initial data of the field vanish for $|x| \geq R_0$ (Section 3.2). Finally, for any initial data $Y_0 \in \mathcal{E}$, we derive the long-time asymptotics of the solution $Y_t$ in the mean (Section 5).

At first, consider the Cauchy problem for Eqn (3.2) with $F(t) \equiv 0$, i.e.,

$$\dot{q}_t = -\omega^2 q_t + \int_0^t D(t-s) q_s \, ds, \quad t > 0, \quad (3.5)$$

$$q_t|_{t=0} = q_0, \quad \dot{q}_t|_{t=0} = p_0. \quad (3.6)$$

For the solutions of problem (3.5)–(3.6), the following assertion holds.
Theorem 3.1 Let conditions A1–A5 and R1–R3 be satisfied. Then \( |q_t| + |\dot{q}_t| \leq C \varepsilon_{F}(t)(|q_0| + |p_0|) \) for any \( t \geq 0 \). Here
\[
\varepsilon_{F}(t) = \begin{cases} 
 e^{-\delta t} & \text{with a } \delta > 0, \text{ for the WF,} \\
 (1 + t)^{-3/2}, & \text{for the KGF.}
\end{cases}
\] (3.7)

Corollary 3.2 Denote by \( V(t) \) a solving operator of the Cauchy problem (3.5), (3.6). Then the variation constants formula gives the following representation for the solution of problem (3.2), (3.6):
\[
\begin{pmatrix} q_t \\ \dot{q}_t \end{pmatrix} = V(t) \begin{pmatrix} q_0 \\ p_0 \end{pmatrix} + \int_0^t V(s) \begin{pmatrix} 0 \\ F(t - s) \end{pmatrix} ds, \quad t > 0.
\]

Evidently, \( V(0) = I \). The matrix \( V(t) \), \( t > 0 \), is called the resolvent or principal matrix solution for Eqn (3.2). Theorem 3.1 implies that
\[
|q_t| + |\dot{q}_t| \leq C_1 \varepsilon_{F}(t)(|q_0| + |p_0|) + C_2 \int_0^t \varepsilon_{F}(s)|F(t - s)| ds, \quad \text{for } t \geq 0.
\] (3.8)

Moreover, the matrix \( V(t) \) has a form
\[
\begin{pmatrix} \dot{N}(t) & N(t) \\ \dot{N}(t) & \ddot{N}(t) \end{pmatrix},
\]
with matrix-valued entries satisfying the bound:
\[
|N^{(j)}(t)| \leq C \varepsilon_{F}(t), \quad t > 0, \quad j = 0, 1, 2.
\] (3.9)

In next subsection, we prove Theorem 3.1 for the WF case. In the case of the KGF, Theorem 3.1 can be proved combining the technique of [21] and [12, Appendix], where Theorem 3.1 was proved for the Klein-Gordon equation with constant coefficients, the methods of Section 3.1, where the result is established in the case of the wave equations with variable coefficients, and Vainberg’ results [38] for Klein-Gordon equations with variable coefficients.

3.1 Exponential stability of the zero solution in the WF case

To prove Theorem 3.1 we solve the Cauchy problem (3.5), (3.6) by using the Laplace transform,
\[
\tilde{q}(\lambda) = \int_0^{+\infty} e^{-\lambda t} q_t dt, \quad \Re \lambda > 0.
\]

Then Eqn (3.5) becomes
\[
\lambda^2 \tilde{q}(\lambda) = -\omega^2 \tilde{q}(\lambda) + \tilde{D}(\lambda)\tilde{q}(\lambda) + p_0 + \lambda q_0.
\] (3.10)

Let \( H^s \equiv H^s(\mathbb{R}^3) \) denote the Sobolev space with norm \( \| \cdot \|_s \). Denote by \( R_\lambda : H^0 \rightarrow H^2, \ \Re \lambda > 0 \), an operator such that \( R_\lambda f = \varphi_\lambda(x) \) is a solution to the following equation
\[
(\lambda^2 - L_B)\varphi_\lambda(x) = f(x).
\]

Then the entries of \( \tilde{D}(\lambda) \) are
\[
\tilde{D}_{ij}(\lambda) = \langle \nabla_i \rho, R_\lambda(\nabla_j \rho) \rangle, \quad i, j = 1, 2, 3.
\] (3.11)
Denote by $R_\lambda^0$, $\Re\lambda > 0$, the operator $R_\lambda$ in the case when $L_B = \Delta$. As shown in [37], Lemma 3], the operator $R_\lambda^0(R_\lambda^0, \Re\lambda > 0$, is analytic (finite-meromorphic, resp.) depends on $\lambda$. By conditions A1–A3, the operator $R_\lambda$, with $\Re\lambda > 0$, has not poles and equals $R_\lambda f = \int_0^\infty e^{-\lambda t} \varphi_\lambda(x) \, dt$, where $\varphi_\lambda(x)$ is the solution to the Cauchy problem (2.13) with initial data $\varphi_0 \equiv 0$, $\pi_0 = f \in H^0(\mathbb{R}^3)$. Moreover, by energy estimates, the following bound holds (see [37, Theorem 2]),

$$\|R_\lambda f\|_1 + |\lambda|\|R_\lambda f\|_0 \leq C\|f\|_0.$$  (3.12)

We rewrite Eqn (3.10) as

$$\tilde{q}(\lambda) = \left[(\lambda^2 + \omega^2)I - \tilde{D}(\lambda)\right]^{-1}(p_0 + \lambda q_0) \equiv \tilde{N}(\lambda)(p_0 + \lambda q_0),$$

where $\tilde{N}(\lambda)$ stands for the $3 \times 3$ matrix of the form

$$\tilde{N}(\lambda) = A^{-1}(\lambda), \text{ with } A(\lambda) := (\lambda^2 + \omega^2)I - \tilde{D}(\lambda) \text{ for } \Re\lambda > 0.$$  (3.13)

We first study properties of $A(\lambda)$. Write $\mathbb{C}_\beta := \{\lambda \in \mathbb{C} : \Re\lambda > \beta\}$ for $\beta \in \mathbb{R}$.

**Lemma 3.3** Let conditions A1–A5 and R1–R3 hold. Then

(i) $A(\lambda)$ admits an finite-meromorphic continuation to $\mathbb{C}$; and there exists a $\delta > 0$ such that $A(\lambda)$ has not poles in $\mathbb{C}_{-\delta}$;

(ii) for every $\beta \in (0, \delta)$, $\exists N_\beta > 0$ such that $v \cdot A(\lambda)v \geq C|\lambda|^2|v|^2$ for $\lambda \in \mathbb{C}_{-\beta}$ with $|\lambda| \geq N_\beta$ and for every $v \in \mathbb{R}^3$.

(iii) There exists a $\delta_* > 0$ such that $v \cdot A(\lambda)v \neq 0$ for $\lambda \in \mathbb{C}_{-\delta_*}$ and for every $v \neq 0$.

In the case when $\rho(x) = \rho_r(|x|)$ and $L_B = \Delta$, Lemma 3.3 was proved in [25, Lemma 7.2] (see also [12, Lemma 4.3] in the case of the constant coefficients).

**Proof** Let $\psi$ be a smooth positive function which is like $e^{-|x|^2}$ as $|x| \to \infty$. By $\hat{R}_\lambda$ $(\hat{R}_\lambda^0)$ we denote the operator $R_\lambda$ $(R_\lambda^0)$, resp.) which is considered as an operator from $H^p_0$ to $H^p_\psi$, where $H^p_0 = \{f \in H^s : f(x) = 0 \text{ for } |x| \geq b\}$ with a norm $\|\cdot\|_{s,b}$, $H^s_\psi$ is the space with a norm $\|\varphi\|_{s,\psi} = \|\psi\varphi\|_s$. We choose a $b$ such that $b \geq \max\{R_p, R_a\}$ (see conditions A2 and R2).

Now we state properties (V1)–(V4) of the operator $\hat{R}_\lambda$ which follow from Vainberg’s results [37, 39].

(V1) (see [37, Theorem 3]) The operator $\hat{R}_\lambda^0$ admits an analytic continuation on $\mathbb{C}$, and for any $\gamma > 0$,

$$\|\hat{R}_\lambda^0 f\|_{0,\psi} + |\lambda|\|\hat{R}_\lambda^0 f\|_{0,\psi} \leq C(\gamma)\|f\|_{0,b}, \quad |\Re\lambda| < \gamma.$$  

The operator $\hat{R}_\lambda$ admits a finite-meromorphic continuation on $\mathbb{C}$, and for any $\gamma > 0$ there exists $N = N(\gamma)$ such that in the region $M_{\gamma,N} := \{\lambda \in \mathbb{C} : |\Re\lambda| \leq \gamma, |\Im\lambda| \geq N\}$ the following estimate holds: $\|\hat{R}_\lambda f\|_{j,\psi} \leq 2\|\hat{R}_\lambda^0 f\|_{j,\psi}$, $j = 0, 1$, for $f \in H^\gamma_0$ (see [37, Theorem 4]).

(V2) For any $\gamma > 0$, $\hat{R}_\lambda$ has at most a finite number of poles in the domain $\mathbb{C}_{-\gamma}$.

(V3) $\hat{R}_\lambda$ has not poles for $\Re\lambda \geq 0$, by conditions A1–A3.

(V4) There exist constants $C, T, \alpha, \beta > 0$ such that for any $f \in H^\gamma_0$,

$$\|\hat{R}_\lambda f\|_{0,b} \leq C|\lambda|^{-1}e^{T|\Re\lambda|}\|f\|_{0,b}, \quad \text{for } \lambda \in U_{\alpha,\beta} = \{\lambda \in \mathbb{C} : |\Re\lambda| < \alpha \ln |\Im\lambda| - \beta\}.$$  

We return to the proof of Lemma 3.3.
(i) In the case of the constant coefficients, i.e., when \( L_B = \Delta \), \( \tilde{D}_{ij}(\lambda) = \langle \nabla_i \rho, R^j_k(\nabla_j \rho) \rangle \)

admits an analytic continuation to \( \mathbb{C} \). Therefore, in this case, \( A(\lambda) \) admits an analytic continuation to \( \mathbb{C} \). In the general case, item (i) of Lemma 3.3 follows from (V1)–(V3).

(ii) By (3.11) and (3.12), \( \tilde{D}_{ij}(\lambda) \to 0 \) as \( |\lambda| \to \infty \) with \( \Re \lambda > 0 \). On the other hand, property (V1) implies that for any \( \gamma > 0 \) there exists \( N = N(\gamma) > 0 \) such that

\[
|\tilde{D}_{ij}(\lambda)| \leq C(\gamma)|\lambda|^{-1} \quad \text{for} \quad \lambda \in M_{\gamma,N}.
\]

Hence, there exists a \( \beta > 0 \) such that \( |\tilde{D}_{ij}(\lambda)| \leq C|\lambda|^{-1} \to 0 \) as \( |\lambda| \to \infty \) with \( \lambda \in \mathbb{C}_{-\beta} \). This implies the assertion (ii) of Lemma 3.3.

(iii) Note first that \( \det A(\lambda) \neq 0 \) for \( \Re \lambda > 0 \), by (2.20). Further, the matrix \( A(\lambda) \) is positive definite for \( 3\lambda = 0 \). Indeed, let \( \lambda = \mu \in \mathbb{R} \setminus 0 \), and put \( f = \nabla \rho \cdot v \) with \( v \in \mathbb{R}^3 \). Then \( f \in H^0_\psi \) and \( f[k]^{-1} \in H^0 \). Denoting \( \varphi_\mu = R_\mu f \in H^2 \), we obtain

\[
\langle f, R_\mu f \rangle = \langle \varphi_\mu, (\mu^2 - L_B)\varphi_\mu \rangle \geq \alpha \| \nabla \varphi_\mu \|_0^2 = \alpha \| \nabla (R_\mu f) \|_0^2;
\]

by condition A3. On the other hand, \( \langle f, R_\mu f \rangle \leq \| \nabla (R_\mu f) \|_0 \cdot \| F^{-1}(|k|^{-1}f) \|_0 \). Hence,

\[
\| \nabla (R_\mu f) \|_0 \leq \frac{1}{\alpha} \| F^{-1}(|k|^{-1}f) \|_0.
\]

Therefore, for any \( \mu > 0 \) and \( v \in \mathbb{R}^3 \setminus \{0\} \), we obtain

\[
v \cdot \tilde{D}(\mu)v = \langle f, R_\mu f \rangle \leq \frac{1}{\alpha} \left\| F^{-1}(|k|^{-1}f) \right\|_0^2 = \frac{1}{\alpha(2\pi)^3} \left\| |k|^{-1}f \right\|_0^2
\]

\[
= \frac{1}{\alpha(2\pi)^3} \int \left( \frac{k \cdot v}{|k|^2} \right)^2 |\hat{\rho}(k)|^2 dk < \omega^2 |v|^2,
\]

by condition R1. In the case \( \mu = 0 \), put \( \tilde{R}_0 f := \lim_{\varepsilon \to +0} \tilde{R}_\varepsilon f \), where the limit is understood in the space \( H^0_\psi \). Then \( \langle f, \tilde{R}_0 f \rangle < \omega^2 |v|^2 \) by (3.15). Hence, for any \( v \in \mathbb{R}^3 \setminus \{0\} \) and \( \mu \in \mathbb{R} \),

\[
v \cdot A(\mu)v = (\mu^2 + \omega^2)|v|^2 - v \cdot \tilde{D}(\mu)v > 0.
\]

Moreover, there exists a \( \delta_0, \delta_0 > 0 \), such that

\[
v \cdot A(\lambda)v \neq 0 \quad \text{for} \quad |\lambda| < \delta_0 \quad \text{and for any} \quad v \in \mathbb{R}^3 \setminus \{0\}.
\]

Now let \( \lambda = iy + 0 \) with \( y \in \mathbb{R} \), and put again \( f = \nabla \rho \cdot v \in H^0_\psi \). By property (V1), there exists \( N_0 > 0 \) such that \( v \cdot A(iy)v \sim (\omega^2 - y^2)|v|^2 + C|v|^2/|y| \neq 0 \) for \( |y| \geq N_0 \) and \( v \neq 0 \). Hence, to prove the assertion (iii) of Lemma 3.3, it suffices to show that

\[
\det A(iy + 0) = \det \left[ (\omega^2 - y^2)I - \tilde{D}(iy + 0) \right] \neq 0 \quad \text{for} \quad \delta_0 \leq |y| \leq N_0.
\]

In [12], we have proved that in the case when \( L_B = \Delta \), condition R3 and the Plemelj formula [15] yield

\[
\Im \langle f, \hat{R}^{0}_{iy+0}f \rangle = -\frac{\pi}{2} y^3(2\pi)^{-3} \int_{|\theta|=1} (v \cdot \theta)^2 |\hat{\rho}(|y|\theta)|^2 dS_\theta \neq 0 \quad \text{for any} \quad v, y \in \mathbb{R}^3 \setminus \{0\},
\]

where \( \hat{R}^{0}_{iy+0}f := \lim_{\varepsilon \to +0} \hat{R}^{0}_{iy+\varepsilon}f \). In the case when \( L_B = L_W \), we can choose \( M_a \) so small that

\[
v \cdot \Im \tilde{D}(iy + 0)v \equiv \Im \langle f, \hat{R}^{0}_{iy+0}f \rangle \neq 0 \quad \text{for all} \quad v \in \mathbb{R}^3 \setminus \{0\} \quad \text{and} \quad |y| \in (\delta_0, N_0)
\]
(see condition A5). In fact, we split \( \langle f, \dot{R}_{iy+0} f \rangle \) into two terms
\[
\langle f, \dot{R}_{iy+0} f \rangle = \langle f, \dot{R}_{iy+0}^0 f \rangle + \langle f, (\dot{R}_{iy+0} - \dot{R}_{iy+0}^0) f \rangle.
\] (3.18)
Since \( f = \nabla \rho \cdot v \), then there exists a constant \( C_0 > 0 \) such that for \( |y| \in (\delta_0, N_0) \) we have
\[
|\langle f, (\dot{R}_{iy+0} - \dot{R}_{iy+0}^0) f \rangle| = |\langle f, \dot{R}_{iy+0} (L_B - \Delta) \dot{R}_{iy+0}^0 f \rangle| \leq C_0 \|\rho\|^2_1 \|v\|^2 M_a,
\] (3.19)
where \( M_a = \max_{x \in \mathbb{R}^3} \{|a_{ij}(x) - \delta_{ij}|, |a_0(x)|\} \). Hence, (3.16) and (3.19) imply that if \( M_a \) is enough small, then (3.17) holds. For example, assume that
\[
M_a \leq \frac{M}{2C_0 \|\rho\|^2_1}, \quad \text{with } M = \min_{\delta_0 \leq |y| \leq N_0} \min_{\nu \neq 0} \frac{|v \cdot S(y) v|}{|v|^2} > 0,
\]
where \( S(y) \), \( y \in \mathbb{R}^3 \), stands for the \( 3 \times 3 \) matrix with the entries \( S_{ij}(y) \),
\[
S_{ij}(y) = \frac{\pi}{2} y^3 (2\pi)^{-3} \int_{|\theta| = 1} \theta_i \theta_j |\rho(|y| \theta)|^2 dS_{\theta}, \quad i, j = 1, 2, 3.
\]
Hence, for \( |y| \in (\delta_0, N_0) \), \( |\nabla \langle f, \dot{R}_{iy+0} f \rangle| = |v \cdot S(y) v| \geq 2C_0 \|\rho\|^2_1 \|v\|^2 M_a \). Therefore, (3.18) and (3.19) imply bound (3.17). Finally, \( v \cdot \nabla A(iy + 0) = -v \cdot \nabla \tilde{D}(iy + 0) v \neq 0 \). Therefore, there exists \( \delta_* > 0 \) such that \( v \cdot \nabla A(iy + x) v \neq 0 \) for \( |x| \leq \delta_* \). Lemma 3.3 is proved.

For any \( \delta < \delta_* \), denote by \( \mathcal{N}(t) \) the inverse Laplace transformation of \( \tilde{N}(\lambda) \),
\[
\mathcal{N}(t) = \frac{1}{2\pi i} \int_{-\infty - \delta}^{\infty - \delta} e^{\lambda t} \tilde{N}(\lambda) d\lambda, \quad t > 0.
\]

Lemma 3.4 Let \( L_B = L_W \) and conditions A1–A5, R1–R3 hold. Then, for \( j = 0, 1, \ldots \) and any \( \delta < \delta_* \),
\[
|\mathcal{N}^{(j)}(t)| \leq C e^{-\delta t}, \quad t > 1.
\] (3.20)

Proof By Lemma 3.3 the bound on \( \mathcal{N}(t) \) follows. To prove the bound for \( \tilde{N}(t) \), we consider \( \lambda \tilde{N}(\lambda) \) and prove the bound
\[
|v \cdot (\lambda \tilde{N}(\lambda))' v| \leq \frac{C|v|^2}{1 + |\lambda|^2} \quad \text{for } \lambda \in \mathbb{C}_{\delta}.
\] (3.21)
Therefore,
\[
|t\tilde{N}(t)| = C \left| \int_{\mathbb{R} \lambda = -\delta} e^{\lambda t} (\lambda \tilde{N}(\lambda))' d\lambda \right| \leq C_1 e^{-\delta t},
\]
and bound (3.20) for \( \tilde{N}(t) \) follows. By Lemma 3.3 (ii), to prove bound (3.21), it suffices to show that \( |\tilde{N}^{(j)}_i(\lambda)| \leq C(1 + |\lambda|)^{-3} \). Since \( R'_f f = -2\lambda^2 R^2 f \), then by formulas (3.11), (3.12) and property (V1), we have
\[
|\tilde{D}^{(j)}_i(\lambda)| \leq 2|\lambda| (\nabla_i \rho, R^2 \nabla_j \rho) \leq C < \infty \quad \text{as } |\lambda| \to \infty \quad \text{with } \lambda \in \mathbb{C}_{\delta}.
\]
Therefore, (3.13) and Lemma 3.3 imply that, for \( i, j = 1, 2, 3 \),
\[
|\tilde{N}^{(j)}_i(\lambda)| \leq \frac{C_1}{1 + |\lambda|^3} \quad \text{as } |\lambda| \to \infty.
\]
This yields (3.21). Bound (3.20) with \( j \geq 2 \) can be proved in a similar way.

Corollary 3.5 The solution of the Cauchy problem (3.3)–(3.6) is \( q_t = \tilde{N}(t)q_0 + \mathcal{N}(t)p_0 \).

Therefore, in the case of the WF, Lemma 3.4 implies Theorem 3.1 with any \( \delta < \delta_* \).
3.2 Time decay for $Y_t$ when $\phi_0(x) = 0$ for $|x| \geq R_0$

For the solution $Y_t$ of (2.4), the following bound holds.

**Lemma 3.6** Let conditions A1–A5 and R1–R3 hold and let $Y_0 \in \mathcal{E}$ be such that

$$\varphi_0(x) = \pi_0(x) = 0 \quad \text{for} \quad |x| > R_0,$$

(3.22)

with some $R_0 > 0$. Then for every $R > 0$ there exists a constant $C = C(R, R_0) > 0$ such that

$$\|Y_t\|_{\mathcal{E},R} \leq C \mathcal{E}_R(t) \|Y_0\|_{\mathcal{E},R_0}, \quad t \geq 0.$$  

(3.23)

Here $\mathcal{E}_R(t) = (1 + t)^{-3/2}$ for the KGF. In the case of the WF, $\mathcal{E}_R(t) = e^{-\delta t}$ with a $\delta \in (0, \min(\delta_*, \gamma))$, where constants $\delta_*$ and $\gamma$ are introduced in Lemma 3.3 (iii) and in bound (3.24), respectively.

**Proof** Step (i): At first, we prove bound (3.23) for $\xi_t = (q_t, p_t)$. In the case of the WF, condition (3.22) and the Vainberg bounds (see [39] or [7, Proposition 10.1]) imply that, for any $R > 0$, there exist constants $\gamma = \gamma(R, R_0) > 0$ and $C = C(R, R_0) > 0$ such that

$$\|W_t\|_{R} \leq C e^{-\gamma t} \|\phi_0\|_{R_0}, \quad t \geq 0.$$  

(3.24)

Therefore, bound (3.23) with $F(t) \equiv \langle \nabla \rho_0, W_t \phi_0 \rangle = -\langle \rho_0, \nabla W_t \phi_0 \rangle$ and condition R2 yield

$$|\xi_t| \leq C_1 e^{-\delta t} |\xi_0| + C(\rho) \int_0^t e^{-\delta s} \|\nabla(W_{t-s} \phi_0^0)\|_{L^2(B_{R_0})} ds \leq C e^{-\delta t} \|Y_0\|_{\mathcal{E},R_0},$$

(3.25)

with any $\delta < \min(\delta_*, \gamma)$. If $L_B = L_{KG}$, then we apply the Vainberg bound [38]:

$$\|W_t\|_{R} \leq C(1 + t)^{-3/2} \|\phi_0\|_{R_0}, \quad t \geq 0.$$  

(3.26)

Hence, $|F(t)| \leq C(1 + t)^{-3/2} \|\phi_0\|_{R_0}$, and bound (3.23) for $\xi_t$ follows from (3.8).

Step (ii): Now we prove bound (3.23) for $\phi_t$. In the case of the WF, Eqn (3.1), condition (3.22), bounds (3.24) and (3.25) yield

$$\|\phi_t\|_{R} \leq C_1 e^{-\gamma t} \|\phi_0\|_{R_0} + C_2 \int_0^t e^{-\delta s} \|Y_0\|_{\mathcal{E},R_0} e^{-\gamma (t-s)} ds \leq C e^{-\delta t} \|Y_0\|_{\mathcal{E},R_0}, \quad t \geq 0,$$

with any $\delta < \min(\delta_*, \gamma)$. For the KGF, the bound $\|\phi_t\|_{R} \leq C(1 + t)^{-3/2} \|Y_0\|_{\mathcal{E},R_0}$ follows from Eqn (3.1), bound (3.26), and estimate (3.23) for $q_t$. This proves Lemma 3.6.

4 Compactness of the measures $\mu_t$

Proposition 2.13 can be deduced from bound (4.1) below by the Prokhorov Theorem [40, Lemma II.3.1] using the method of [10, Theorem XII.5.2], since the embedding $\mathcal{E} \equiv \mathcal{E}^0 \subset \mathcal{E}^{-\varepsilon}$ is compact for every $\varepsilon > 0$.

**Lemma 4.1** Let conditions A1–A5, R1–R3 and S0–S2 hold. Then

$$\sup_{t \geq 0} \mathbb{E} \|S_t Y_0\|_{\mathcal{E},R}^2 \leq C(R) < \infty, \quad \forall R > 0.$$  

(4.1)
Proof Let \( \rho \equiv 0 \). In this case, we denote by \( S_t^0 \) the solving operator \( S_t \). Note first that

\[
\sup_{t \geq 0} \mathbb{E}\|S_t^0 Y_0\|_{\mathcal{E},R}^2 \leq C(R), \quad \forall R > 0. \tag{4.2}
\]

Indeed, by the notation (2.8), \( \|S_t^0 Y_0\|_{\mathcal{E},R}^2 = \|W_t \phi_0\|_{\mathcal{E},R}^2 + |q_t^0|^2 + |\dot{q}_t^0|^2 \), where \( q_t^0 \) is a solution to the Cauchy problem

\[
\dot{q}_t^0 + \omega^2 q_t^0 = 0, \quad t \in \mathbb{R}, \quad (q_t^0, \dot{q}_t^0)|_{t=0} = (q_0, p_0).
\]

Hence, \( |q_t^0| + |\dot{q}_t^0| \leq C(|q_0| + |p_0|) \). By [6 bound (11.2)] and [7 bound (9.2)], we have

\[
\sup_{t \in \mathbb{R}} \mathbb{E}\|W_t \phi_0\|_{\mathcal{E},R}^2 \leq C(R), \quad \forall R > 0. \tag{4.3}
\]

This implies (4.2). Further, we represent the solution to problem (2.4) as

\[
S_t Y_0 = S_t^0 Y_0 + \int_0^t S_{t-\tau} BS_t^0 Y_0 d\tau,
\]

where, by definition, \( BY = (0,0,q \cdot \nabla \rho, \langle \varphi, \nabla \rho \rangle) \) for \( Y = (\varphi, q, \pi, p) \). Hence, condition A2, (3.28), and (4.2) yield

\[
\mathbb{E}\|S_t Y_0\|_{\mathcal{E},R}^2 \leq \mathbb{E}\|S_t^0 Y_0\|_{\mathcal{E},R}^2 + \mathbb{E} \int_0^t \|S_{t-\tau} BS_t^0 Y_0\|_{\mathcal{E},R}^2 d\tau \leq C(R) + \int_0^t \mathbb{E}\|S_t^0 Y_0\|_{\mathcal{E},R}^2 d\tau \leq C_1(R) < \infty. \tag{5.4}
\]

5 Asymptotic behavior for \( Y_t = (\phi_t, q_t, p_t) \) in mean

Proposition 5.1 Let conditions A1–A5, R1–R3 and S0–S2 be satisfied.

(i) The following bounds hold,

\[
\mathbb{E}|q_t - \langle W_t \phi_0, \alpha \rangle|^2 \leq C\tilde{\varepsilon}_F(t), \tag{5.1}
\]
\[
\mathbb{E}|p_t - \langle W_t \phi_0, \beta \rangle|^2 \leq C\tilde{\varepsilon}_F(t), \quad t > 0, \tag{5.2}
\]

where the functions \( \alpha \) and \( \beta \) are defined in (2.23) and (2.26), \( \tilde{\varepsilon}_F(t) = (1 + t)^{-1} \) for the KGF, and \( \tilde{\varepsilon}_F(t) = \varepsilon_F^2(t) = e^{-2\delta t} \) with a \( \delta > 0 \) for the WF.

(ii) Let \( f \in [C_0^\infty(\mathbb{R}^3)]^2 \) with \( \text{supp} f \subset B_R \). Then, for \( t \geq 1 \),

\[
\mathbb{E}|\langle \phi_t, f \rangle - \langle W_t \phi_0, f_* \rangle|^2 \leq C\tilde{\varepsilon}_F(t), \tag{5.3}
\]

where the function \( f_* \) is defined in (2.24).

Proof (i) At first, Theorem 3.1 and Corollary 3.2 yield

\[
\mathbb{E}\left| q_t - \int_0^t \mathcal{N}(s) \langle W_{t-s} \phi_0, \nabla \rho_0 \rangle \, ds \right|^2 \leq C\tilde{\varepsilon}_F^2(t) \tag{5.4}
\]
with $\varepsilon_F(t)$ from (3.7). Further,

$$
\mathbb{E} \left| \int_{t}^{+\infty} \mathcal{N}_r(s) \langle W_{t-s} \phi_0, \nabla_r \rho_0 \rangle \, ds \right|^2 = \int_{t}^{+\infty} \mathcal{N}_r(s) \, ds = \int_{s_1}^{+\infty} \int_{t}^{+\infty} \mathcal{N}_r(s_2) \mathbb{E} \left( \langle W_{t-s}, \phi_0, \nabla_r \rho_0 \rangle \langle W_{t-s_2} \phi_0, \nabla_r \rho_0 \rangle \right) \, ds_2.
$$

For any $t, s_1, s_2 \in \mathbb{R}$,

$$
\left| \mathbb{E} \left( \langle W_{t-s_1} \phi_0, \nabla_r \rho_0 \rangle \langle W_{t-s_2} \phi_0, \nabla_r \rho_0 \rangle \right) \right| \leq C \sup_{\tau \in \mathbb{R}} \mathbb{E} |\langle W_\tau \phi_0, \nabla_r \rho_0 \rangle|^2 \leq C_1 \sup_{\tau \in \mathbb{R}} \mathbb{E} \|W_\tau \phi_0\|_{R^*}^2 \leq C_2 < +\infty
$$

by bound (4.3). Hence, using (3.9), we obtain

$$
\mathbb{E} \left| \int_{t}^{+\infty} \mathcal{N}(s) \langle W_{t-s} \phi_0, \nabla_r \rho_0 \rangle \, ds \right|^2 \leq \left( \int_{t}^{+\infty} \varepsilon_F(s) \, ds \right)^2 = C_1 \varepsilon_F(t).
$$

Therefore, (5.1) follows from (5.4), (5.5) and (2.25) because

$$
\langle W_{t-s} \phi_0, \nabla_r \rho_0 \rangle = \langle W_t \phi_0, W_s' \nabla_r \rho_0 \rangle.
$$

The bound (5.2) can be proved in a similar way.

(ii) Let $f \in [C^\infty_0(\mathbb{R}^3)]^2$ with $\text{supp } f \subset B_R$. By Eqn (5.1), we have

$$
\langle \phi_t, f \rangle = \langle W_t \phi_0, f \rangle + \int_0^t q_{t-s} \cdot \langle W_s \nabla \rho^0, f \rangle \, ds.
$$

Using Vainberg’s bounds [38, 39], we obtain

$$
\langle W_s \nabla \rho^0, f \rangle = \begin{cases} 
O(e^{-\gamma |s|}) & \text{with } \gamma > 0 \text{ if } L_B = L_W, \\
O((1 + |s|)^{-3/2}) & \text{if } L_B = L_{KG}.
\end{cases}
$$

If $L_B = L_W$ we put $\varepsilon_F(t) = \varepsilon_F^2(t) = e^{-2\delta t}$ with any $\delta < \min(\delta_s, \gamma)$, see Lemma 3.6. Applying the Parseval inequality and bounds (5.1) and (5.7), we get

$$
\mathbb{E} \left| \int_0^t \left( q_{t-s} - \langle W_{t-s} \phi_0, \alpha \rangle \right) \cdot \langle W_s \nabla \rho^0, f \rangle \, ds \right|^2 \leq \left( \int_0^t (\varepsilon_F(t-s))^{1/2} \|W_s \nabla \rho^0, f\| \, ds \right)^2 \leq C \varepsilon_F(t).
$$

Write $I(t) := \mathbb{E} \left| \int_t^{+\infty} \langle W_{t-s} \phi_0, \alpha \rangle \cdot \langle W_s \nabla \rho^0, f \rangle \, ds \right|^2$. Then

$$
|I(t)| \leq C \varepsilon_F(t).
$$
This follows from (5.7) and from the following estimate:
\[
\mathbb{E}|\langle W_t \phi_0, \alpha \rangle|^2 = \sum_{i=1}^3 \sum_{j=1}^3 \int_0^{+\infty} \mathcal{N}_{ir}(s) \langle W_{r-s} \phi_0, \nabla_r \rho_0 \rangle \, ds \leq C < \infty, \quad \text{for } \tau \in \mathbb{R},
\]
by (4.3) and (3.9). Relation (5.6) and bounds (5.8) and (5.9) imply (5.3).

**Corollary 5.2** Let \( Z = (f, u, v) \in \mathcal{D} = [C_0^\infty(\mathbb{R}^3)]^2 \times \mathbb{R}^3 \times \mathbb{R}^3 \). Then
\[
\langle Y_t, Z \rangle = \langle W_t \phi_0, \Pi(Z) \rangle + r(t),
\]
where \( \Pi(Z) \) is defined in (2.24), \( \langle Y_t, Z \rangle = \langle \phi_t, f \rangle + q_t \cdot u + p_t \cdot v \), \( Y_t = (\phi_t, q_t, p_t) \) is a solution to the Cauchy problem (2.4), and \( \mathbb{E} (|r(t)|^2) \leq C \varepsilon_F(t) \).

## Convergence of characteristic functionals and correlation functions

**Proof of Proposition 2.14** By the triangle inequality,
\[
\left| \mathbb{E} e^{i(Y_t, Z)} - e^{-\frac{1}{2} Q_{\alpha}(Z, Z)} \right| \leq \left| \mathbb{E} (e^{i(Y_t, Z)} - e^{i(W_t \phi_0, \Pi(Z))}) \right| + \left| \mathbb{E} e^{i(W_t \phi_0, \Pi(Z))} - e^{-\frac{1}{2} Q_{\alpha}(Z, Z)} \right|.
\]
Applying Corollary 5.2 we estimate the first term in the r.h.s. of (6.1) by
\[
\mathbb{E} |\langle Y_t, Z \rangle - \langle W_t \phi_0, \Pi(Z) \rangle| \leq \mathbb{E} |r(t)| \leq \left( \mathbb{E} |r(t)|^2 \right)^{1/2} \leq C \varepsilon_m^{1/2}(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty.
\]
It remains to prove the convergence \( \mathbb{E} (\exp(i \langle W_t \phi_0, \Pi(Z) \})) \equiv \hat{\mu}_t^B(\Pi(Z)) \) to a limit as \( t \rightarrow \infty \).

In [6, 7], we have proved the convergence of \( \hat{\mu}_t^B(f) \) to a limit for \( f \in \mathcal{D}_0 \equiv [C_0^\infty(\mathbb{R}^3)]^2 \). However, \( \Pi(Z) \notin \mathcal{D}_0 \) in general. Consider the cases of the WF and KGF separately.

In the WF case, \( \Pi(Z) \in \mathcal{H}_\gamma \) if \( Z \in \mathcal{D} \), for sufficiently small \( \gamma > 0 \), where \( \mathcal{H}_\gamma \) is introduced in Remark 2.10. This follows from formulas (2.24)–(2.27), from the bound (3.9), and from the estimate
\[
\| W_t' f \|_{\gamma}' \leq C \varepsilon^{1/3} \| f \|_{\gamma}, \quad t \in \mathbb{R}, \quad \text{for any } \gamma \in \mathcal{H}_\gamma.
\]
The last estimate can be proved in a similar way as the same estimate for \( (W_0') \) in [7, Lemma 8.2] using the energy estimates.

**Lemma 6.1** Let \( L_B = L_W \). Then the quadratic forms \( Q_t^B(f, f) := \int (\phi_0, f)^2 \mu_t^B(d\phi_0), t \in \mathbb{R}, \) and the characteristic functionals \( \hat{\mu}_t^B(f), t \in \mathbb{R}, \) are equicontinuous on \( \mathcal{H}_\gamma \).

**Proof** In the case when \( L_B = \Delta, \) Lemma 6.1 was proved in [7, Corollary 4.3]. In the general case, i.e., when \( L_B = L_W \), this lemma can be proved by a similar way and the proof is based on the bound \( \mathbb{E} W_t \phi_0 \|_{\gamma}^2 \leq C < \infty \) for any \( \gamma > 0 \). Now we prove this bound. By (4.3), we have
\[
e_t := \mathbb{E} |\varphi_t(x)|^2 + |\nabla \varphi_t(x)|^2 + |\pi_t(x)|^2 \leq C < \infty,
\]
and since \( \mathbb{E} W_t \phi_0 \|_{\gamma}^2 = e_t |B_R| \), where \( |B_R| \) denotes the volume of the ball \( B_R = \{ x \in \mathbb{R}^3 : |x| \leq R \} \). Hence, the bound (6.2) implies, similarly to (2.20), that for any \( \gamma > 0 \) there is a constant \( C = C(\gamma) > 0 \) such that
\[
\mathbb{E} W_t \phi_0 \|_{\gamma}^2 = e_t \int \exp(-2\gamma|x|) \, dx \leq C < \infty. \quad \blacksquare
\]
In the case of KGF, we write $H'_m = L^2(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3)$. Then $\Pi(Z) \in H'_m$ if $Z \in \mathcal{D}$. This follows from formulas \textbf{(2.24)} and \textbf{(2.27)} and the bound \textbf{(3.9)}.

**Lemma 6.2** Let $L_B = L_{KG}$. Then (i) the quadratic forms $Q_t^B(f, f)$, $t \in \mathbb{R}$, are equicontinuous on $H'_m$, (ii) the characteristic functionals $\hat{\mu}_t^B(f)$, $t \in \mathbb{R}$, are equicontinuous on $H'_m$.

**Proof** (i) It suffices to prove the uniform bound

$$\sup_{t \in \mathbb{R}} |Q_t^B(f, f)| \leq C \|f\|^2_{H'_m} \quad \text{for any } f \in H'_m. \quad \text{(6.3)}$$

At first, note that $Q_t^B(f, f) = \langle Q_0(x, y), W_t^0 f(x) \otimes W_t^0 f(y) \rangle$. On the other hand, by conditions S0, S2 and S3, the correlation functions $Q_0^B(x, y)$ of the measure $\mu_0^B$ satisfy the following bound: for $\alpha, \beta \in \mathbb{Z}^3$, $|\alpha| = 1$, $|\beta| = 1$, $i, j = 0, 1$,

$$|D_{x,y}^{\alpha, \beta} Q_t^B(x, y)| \leq C e_0 \varphi^{1/2}(|x - y|), \quad x, y \in \mathbb{R}^3, \quad \text{(6.4)}$$

according to [20] Lemma 17.2.3. Therefore, by (6.12),

$$\int_{\mathbb{R}^3} |D_{x,y}^{\alpha, \beta} Q_t^B(x, y)|^p dy \leq C e_0^p \int_{\mathbb{R}^3} \varphi^{p/2}(|x - y|) dy \leq C_1 e_0^p \int_0^\infty r^2 \varphi^{1/2}(r) dr < \infty, \quad p \geq 1.$$

Hence, by the Shur lemma, the quadratic form $\langle Q_0(x, y), f(x) \otimes f(y) \rangle$ is continuous in $[L^2(\mathbb{R}^3)]^2$. Therefore,

$$\sup_{t \in \mathbb{R}} |Q_t^B(f, f)| \leq \sup_{t \in \mathbb{R}} |\langle Q_0(x, y), W_t^0 f(x) \otimes W_t^0 f(y) \rangle| \leq C \sup_{t \in \mathbb{R}} \|W_t^0 f\|^2_{L^2} \leq C \|f\|^2_{H'_m}.$$

The last inequality follows from the energy conservation for the Klein–Gordon equation.

(ii) By the Cauchy-Schwartz inequality and (6.3), we obtain

$$|\hat{\mu}_t^B(f_1) - \hat{\mu}_t^B(f_2)| = \left| \int \left( e^{i(\phi_0 f_1)} - e^{i(\phi_0 f_2)} \right) \mu_t^B(d\phi_0) \right| \leq \int \left| e^{i(\phi_0 f_1 - f_2)} - 1 \right| \mu_t^B(d\phi_0) \leq \sqrt{\int |\langle \phi_0, f_1 - f_2 \rangle|^2 \mu_t^B(d\phi_0)}$$

$$= \sqrt{Q_t^B(f_1 - f_2, f_1 - f_2)} \leq C \|f_1 - f_2\|_{H'_m}.$$

We return to the proof of Proposition \textbf{2.14} By [8] Proposition 2.3 (or [10] Proposition 3.3), and by Lemmas 6.1 and 6.2, the characteristic functionals $\hat{\mu}_t^B(\Pi(Z))$ converge to a limit as $t \to \infty$. This completes the proof of Proposition \textbf{2.14} and Theorem \textbf{2.12} (i)–(ii).

**Lemma 6.3** Let all assumptions of Theorem \textbf{2.12} be satisfied. Then convergence \textbf{(2.30)} holds.

**Proof** It suffices to prove the convergence of $\int |\langle Y, Z \rangle|^2 \mu_t(dY) = E|\langle Y_t, Z \rangle|^2$ to a limit as $t \to \infty$. It follows from Corollary \textbf{5.2} that for $Z \in \mathcal{D}$,

$$E|\langle Y_t, Z \rangle|^2 = E|\langle W_t \phi_0, \Pi(Z) \rangle|^2 + o(1) = Q_t^B(\Pi(Z), \Pi(Z)) + o(1), \quad t \to \infty,$$

where $\Pi(Z)$ is defined in \textbf{(2.24)}. Therefore, by the results from [8] [10] and by Lemmas 6.1 and 6.2, the quadratic forms $Q_t^B(\Pi(Z), \Pi(Z))$ converge to a limit as $t \to \infty$. Formula \textbf{(2.29)} implies \textbf{(2.30)}.■
7  Ergodicity and mixing for the limit measures

Denote by $E_{\infty}(E^B_{\infty})$ the integral w.r.t. $\mu_{\infty}$ ($\mu^B_{\infty}$, respectively). In [5], we have proved that $W_t$ is mixing w.r.t. $\mu^B_{\infty}$, i.e., for any $f,g \in L_2(\mathcal{H},\mu^B_{\infty})$, the following convergence holds,

$$
E^B_{\infty}(f(W_t\phi)g(\phi)) \to E^B_{\infty}(f(\phi))E^B_{\infty}(g(\phi)) \quad \text{as } t \to \infty. \quad (7.1)
$$

Recall that the limit measure $\mu_{\infty}$ is invariant by Theorem 2.12 (iv). Now we prove that the flow $S_t$ is mixing w.r.t. $\mu_{\infty}$. This mixing property means that the convergence (7.28) holds for the initial measures $\mu_0$ that are absolutely continuous w.r.t. $\mu_{\infty}$, and the limit measure coincides with $\mu_{\infty}$.

**Theorem 7.1** The phase flow $S_t$ is mixing w.r.t. $\mu_{\infty}$, i.e., for any $F,G \in L_2(\mathcal{E},\mu_{\infty})$ we have

$$
E_{\infty}(F(S_tY)G(Y)) \to E_{\infty}(F(Y))E_{\infty}(G(Y)) \quad \text{as } t \to \infty.
$$

In particular, the flow $S_t$ is ergodic w.r.t. $\mu_{\infty}$, i.e., for any $F \in L_2(\mathcal{E},\mu_{\infty})$,

$$
\lim_{T \to \infty} \frac{1}{T} \int_0^T F(S_tY) \, dt = E_{\infty}(F(Y)) \quad (\text{mod } \mu_{\infty}).
$$

To prove Theorem 7.1 we introduce new notations. Represent $Y \in \mathcal{E}$ as $Y = (Y^0,Y^1)$ with $Y^0 = (\varphi(\cdot),q) \in H^1_{\text{loc}}(\mathbb{R}^d) \times \mathbb{R}^d$, $Y^1 = (\pi(\cdot),p) \in L_2^{\text{loc}}(\mathbb{R}^d) \times \mathbb{R}^d$, and $Z \in \mathcal{D}$ as $Z = (Z^0,Z^1)$ with $Z^0 = (f^0(\cdot),u^0)$, $Z^1 = (f^1(\cdot),u^1) \in C_0^\infty(\mathbb{R}^d) \times \mathbb{R}^d$. For $t \in \mathbb{R}$, introduce a "formal adjoint" operator $S'_t$ on the space $\mathcal{D}$ by the rule

$$
\langle S_tY,Z \rangle = \langle Y,S'_tZ \rangle, \quad Y \in \mathcal{E}, \quad Z \in \mathcal{D}. \quad (7.2)
$$

**Lemma 7.2** For $Z \in \mathcal{D}$,

$$
S'_tZ = (\tilde{f}_t(\cdot),\bar{u}_t,f_t(\cdot),u_t), \quad (7.3)
$$

where $(f_t(x),u_t)$ is the solution of system (2.1) with the initial data (see (2.3))

$$(\varphi_0,q_0,\pi_0,p_0) = (f^1,u^1,f^0,u^0).$$

**Proof** Differentiating (7.2) in $t$ with $Y,Z \in \mathcal{D}$, we obtain $\langle \dot{S}_tY,Z \rangle = \langle Y,\dot{S}'_tZ \rangle$. The group $S_t$ has the generator

$$
\mathcal{L} = \begin{pmatrix} 0 & 1 \\ \mathcal{A} & 0 \end{pmatrix}, \quad \text{with } \mathcal{A} \begin{pmatrix} \varphi \\ q \end{pmatrix} = \begin{pmatrix} L_B\varphi + q \cdot \nabla \rho \\ -\omega^2q + \langle \nabla \rho, \varphi \rangle \end{pmatrix}. \quad (7.4)
$$

The generator of $S'_t$ is the conjugate operator $\mathcal{L}' = \begin{pmatrix} 0 & \mathcal{A} \\ 1 & 0 \end{pmatrix}$. Hence, (7.3) holds with

$$
\begin{pmatrix} \tilde{f}_t(x) \\ \bar{u}_t \end{pmatrix} = \mathcal{A} \begin{pmatrix} f_t(x) \\ u_t \end{pmatrix}. \quad \blacksquare
$$

Since the limit measure $\mu_{\infty}$ is Gaussian with zero mean, the proof of Theorem 7.1 reduces to that of the following convergence.
Lemma 7.3 For any $Z_1, Z_2 \in \mathcal{D}$,
\[
\mathbb{E}_{\infty}\left(\langle S_t Y, Z_1 \rangle \langle Y, Z_2 \rangle\right) \to 0, \quad t \to \infty. \tag{7.5}
\]

Proof First we note that, by relation (2.29),
\[
\mathbb{E}_{\infty}\left(\langle Y, Z_1 \rangle \langle Y, Z_2 \rangle\right) = \mathbb{E}_{\infty}^B\left(\langle \phi, \Pi(Z_1) \rangle \langle \phi, \Pi(Z_2) \rangle\right),
\]
where $\Pi(Z)$ is defined in (2.24). Secondly, for fixed $t$, we have $S_t^* Z \in \mathcal{D}$. Further,
\[
\mathbb{E}_{\infty}\left(\langle S_t Y, Z_1 \rangle \langle Y, Z_2 \rangle\right) = \mathbb{E}_{\infty}\left(\langle Y, S_t^* Z_1 \rangle \langle Y, Z_2 \rangle\right) = \mathbb{E}_{\infty}^B\left(\langle \phi, \Pi(S_t^* Z_1) \rangle \langle \phi, \Pi(Z_2) \rangle\right)
\]
\[
= \mathbb{E}_{\infty}^B\left(\langle \phi, (\Pi S_t^* - W_t^* \Pi) Z_1 \rangle \langle \phi, \Pi(Z_2) \rangle\right) + \mathbb{E}_{\infty}^B\left(\langle \phi, W_t^* \Pi(Z_1) \rangle \langle \phi, \Pi(Z_2) \rangle\right)
\]
\[
= I_1 + I_2. \tag{7.6}
\]
Note that $\langle \phi, \Pi(Z) \rangle \in L_2(\mathcal{H}, \mu_\infty^B)$ for all $Z \in \mathcal{D}$. Indeed, by (2.29),
\[
\mathbb{E}_{\infty}^B|\langle \phi, \Pi(Z) \rangle|^2 = Q_{\infty}^B(\Pi(Z), \Pi(Z)) = Q_\infty(Z, Z) < \infty.
\]
Therefore, the convergence (7.1) implies
\[
I_2 \equiv \mathbb{E}_{\infty}^B\left(\langle \phi, W_t^* \Pi(Z_1) \rangle \langle \phi, \Pi(Z_2) \rangle\right)
\]
\[
= \mathbb{E}_{\infty}^B\left(\langle W_t \phi, \Pi(Z_1) \rangle \langle \phi, \Pi(Z_2) \rangle\right) \to \mathbb{E}_{\infty}^B\left(\langle \phi, \Pi(Z_1) \rangle\right) \mathbb{E}_{\infty}^B\left(\langle \phi, \Pi(Z_2) \rangle\right), \quad t \to \infty.
\]
On the other hand, $\mathbb{E}_{\infty}^B(\phi, \Pi(Z_i)) = \mathbb{E}_{\infty}(Y, Z_i) = 0$, for $Z_i \in \mathcal{D}$, because $\mu_\infty$ has zero mean. Therefore,
\[
I_2 \to 0, \quad t \to \infty. \tag{7.7}
\]
Now we prove that $\mathbb{E}_{\infty}^B|\langle \phi, \Pi(S_t^* Z) - W_t^* \Pi(Z) \rangle|^2 = 0$ for all $t > 0$. This yields
\[
I_1 \equiv \mathbb{E}_{\infty}^B\left(\langle \phi, \Pi(S_t^* Z_1) - W_t^* \Pi(Z_1) \rangle \langle \phi, \Pi(Z_2) \rangle\right) = 0. \tag{7.8}
\]
Indeed, by Corollary 5.2
\[
\mathbb{E}|\langle S_{\tau+t} Y, Z \rangle - \langle W_{\tau+t} \phi, \Pi(Z) \rangle|^2 \to 0, \quad \tau \to \infty.
\]
On the other hand, since $\langle S_{\tau+t} Y, Z \rangle = \langle S_t Y, S_{\tau+t} Z \rangle$, we have, for all $t > 0$,
\[
\mathbb{E}|\langle S_t Y, S_{\tau+t}^* Z \rangle - \langle W_t \phi, \Pi(S_{\tau+t}^* Z) \rangle|^2 \to 0, \quad \tau \to \infty.
\]
Therefore, by the triangle inequality,
\[
A := \mathbb{E}|\langle W_{\tau+t} \phi, \Pi(S_{\tau+t}^* Z) \rangle - \langle W_{\tau+t} \phi, \Pi(Z) \rangle|^2 \to 0, \quad \tau \to \infty.
\]
Since $\langle W_{\tau+t} \phi, \Pi(Z) \rangle = \langle W_{\tau} \phi, W_{\tau+t}^* \Pi(Z) \rangle$, we obtain
\[
A = \mathbb{E}|\langle W_{\tau} \phi, W_{\tau+t}^* \Pi(Z) \rangle|^2 \to 0, \quad \tau \to \infty.
\]
Hence, by Theorem 2.9 (iii) and Lemmas 6.1 and 6.2
\[
\mathbb{E}_{\infty}^B|\langle \phi, \Pi(S_{\tau+t}^* Z) - W_{\tau+t}^* \Pi(Z) \rangle|^2 = \lim_{\tau \to \infty} \mathbb{E}|\langle W_{\tau} \phi, \Pi(S_{\tau+t}^* Z) - W_{\tau+t}^* \Pi(Z) \rangle|^2 = 0 \quad \text{for all} \quad t > 0.
\]
Finally, (7.6)–(7.8) imply the convergence (7.5). Theorem 7.1 is proved. 

8 Non translation invariant initial measures

In this section we extend the results of Theorem 2.12 to the case of non translation-invariant initial measures. Note that the proof of Theorem 2.12 is based on two assertions. We first derive the asymptotic behavior of solutions $Y_t$ in mean: $⟨Y_t, Z⟩ \sim ⟨W_t\phi_0, Π(Z)⟩$ as $t → ∞$ (see Corollary 5.2). This asymptotics allows us to reduce the convergence analysis for the coupled system to the same problem for the wave (or Klein-Gordon) equation. The second assertion is the weak convergence of the measures $µ^B_t = W_t^\ast µ^B_0$ to a limit as $t → ∞$ (see Theorem 2.9). However, the weak convergence of $µ^B_t$ holds under weaker conditions on $µ^B_0$ than $S_2$ and $S_3$. Now we formulate these conditions (see [8] for $L_B = L_W$ and [10] for $L_B = L_{KG}$).

8.1 Conditions on $µ^B_0$

In the case of the KGF, we assume that $µ^B_0$ has zero mean, satisfies a mixing condition $S_3$ and has a finite mean energy density (see (2.10)), i.e.,

$$
\int \left( |φ_0(x)|^2 + |∇φ_0(x)|^2 + |π_0(x)|^2 \right) µ^B_0(dφ_0) = Q_{00}^0(x,x) + |∇_x Q_{00}^0(x,y)|_{x=y} + Q_{11}^0(x,x) ≤ e_0 < ∞.
$$

(8.1)

However, condition $S_2$ of translation invariance for $µ^B_0$ can be weakened as follows.

$S_2'$ The correlation functions of the measure $µ^B_0$ have the form

$$
Q_{ij}^0(x,y) = q_{ij}^0(x-y)ζ_-(x_1)ζ_-(y_1) + q_{ij}^0(x-y)ζ_+(x_1)ζ_+(y_1), \; i,j = 0,1.
$$

(8.2)

Here $q_{ij}^0(x-y)$ are the correlation functions of some translation-invariant measures $µ_±^B$ with zero mean value in $H$, $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{R}^d$, the functions $ζ_± \in C^\infty(\mathbb{R})$ such that

$$
ζ_\pm(s) = \begin{cases} 
1, & \text{for } ± s > a, \\
0, & \text{for } ± s < -a,
\end{cases}
$$

(8.3)

and $a > 0$. The measure $µ^B_0$ is not translation-invariant if $q_{ij}^0 ≠ q_{ij}^0$.

In the case of WF, instead of $S_2$ and $S_3$ we impose the following conditions $S_2'$ and $S_3'$.

$S_2'$ The correlation functions of $µ^B_0$ have the form

$$
Q_{ij}^0(x,y) = \begin{cases} 
q_{ij}^0(x-y), & x_1, y_1 < -a, \\
q_{ij}^0(x-y), & x_1, y_1 > a,
\end{cases}
$$

(8.4)

with some $a > 0$ and $q_{ij}^0$ as in (8.2). However, in the WF case, instead of (8.1) we impose a stronger condition (8.5). Namely, the following derivatives are continuous and the bounds hold,

$$
|D_{x,y}^{\alpha,\beta} Q_{ij}^0(x,y)| ≤ \left\{ \begin{array}{ll} 
Cν_\kappa(|x-y|) & \text{if } \kappa = 0,1, \ldots, d - 2 \\
Cν_{d-1}(|x-y|) & \text{if } \kappa = d - 1, d, d + 1
\end{array} \right\} \kappa = i + j + |\alpha| + |\beta|,
$$

(8.5)

with $|\alpha| ≤ (d-3)/2 + i, |\beta| ≤ (d-3)/2 + j, i,j = 0,1$. Here $ν_\kappa ∈ C[0,∞)$ ($\kappa = 0, \ldots, d - 1$) denote some continuous nonnegative nonincreasing functions in $[0,∞)$ with the finite integrals

$$
\int_0^\infty (1 + r)^{\kappa-1}ν_\kappa(r)dr < ∞. \; \text{Moreover, for } d ≥ 5, \int_0^\infty (1 + r)^{d-4+\kappa}ν_\kappa(r)dr < ∞ \text{ with } \kappa = 0,2.
$$
Let $O(r)$ be the set of all pairs of open convex subsets $A, B \subset \mathbb{R}^d$ at distance $d(A, B) \geq r$, and let $\alpha = (\alpha_1, \ldots, \alpha_d)$ with integers $\alpha_i \geq 0$. Denote by $\sigma_{i\alpha}(A)$ the $\sigma$-algebra of the subsets in $H$ generated by all linear functionals

$$\phi_0 = (\phi_0^0, \phi_0^1) \to (D^i\phi_0^i, f), \quad \text{with} \quad |\alpha| \leq 1 - i, \quad i = 0, 1,$$

where $f \in C_c^\infty(\mathbb{R}^d)$ with $\sup f \subset A$. For $\kappa = 0, 1$, let $\sigma_{\kappa}(A)$ be the $\sigma$-algebra generated by $\sigma_{i\alpha}(A)$ with $i + |\alpha| \geq \kappa$, i.e., $\sigma_{\kappa}(A) \equiv \bigvee_{i+|\alpha|\geq\kappa} \sigma_{i\alpha}(A)$. We define the (Ibragimov) mixing coefficient of $\mu^B_0$ on $H$ as (cf. (2.11))

$$\varphi_{\kappa_1, \kappa_2}(r) \equiv \sup_{(A, B) \in O(r)} \sup_{A \in \sigma_{\kappa_1}(A), B \in \sigma_{\kappa_2}(B)} \frac{|\mu^B(A \cap B) - \mu^B(A)\mu^B(B)|}{\mu^B(B)}, \quad \kappa_1, \kappa_2 = 0, 1.$$

We assume that the measure $\mu^B_0$ satisfies the strong uniform Ibragimov mixing condition, i.e., for any $\kappa_1, \kappa_2 = 0, 1$, $\varphi_{\kappa_1, \kappa_2}(r) \to 0, r \to \infty$. Moreover,

$$\varphi_{\kappa_1, \kappa_2}(r) \leq C\nu^2_\kappa(r), \quad \text{where} \quad \kappa = \kappa_1 + \kappa_2, \quad \kappa_1, \kappa_2 = 0, 1.$$

**Remark 8.1**

(i) In [8, 10], we have constructed the generic examples of the initial measures $\mu^B_0$ satisfying all assumptions imposed.

(ii) Condition S3 and the bound (8.1) imply the bound (6.4).

(iii) Condition (8.5) implies (8.1). Condition S3’ implies estimates (8.5) with $i + |\alpha| \leq 1, j + |\beta| \leq 1$. The mixing condition S3’ is weaker than condition S3. On the other hand, the estimates (8.5) with $\kappa > 2$ are not required for translation-invariant initial measures $\mu^B_0$ or in the KGF case.

(iv) The conditions S2 and S3 admit various modifications. We choose the variant which allows an application to the case of the Gibbs measures $\mu^B_\pm$ (see Section A.3 below).

### 8.2 Convergence to equilibrium

**Theorem 8.2** (see [8, 10]) Let conditions A1–A4 and all conditions imposed on $\mu^B_0$ in Section 8.1 be satisfied. Then the assertions of Theorem 2.12 remains true with the matrix $Q^B_\infty(x, y) = q^B_\infty(x - y)$ of the following form. In the Fourier transform, $\hat{q}^B_\infty(k) = \hat{q}^+_\infty(k) + \hat{q}^-_\infty(k)$, where (cf. (2.13))

$$\hat{q}^+_\infty(k) = \frac{1}{2}\left(\hat{q}^+(k) + \hat{C}(k)\hat{q}^+(k)\hat{C}^T(k)\right),$$

$$\hat{q}^-_\infty(k) = i\text{sgn}(k)\frac{1}{2}\left(\hat{C}(k)\hat{q}^-(k) - \hat{q}^-\hat{C}^T(k)\right),$$

with $q^+ = (q_+ + q_-)/2, q^- = (q_+ - q_-)/2$, and $\hat{C}(k)$ from (2.16).

**Theorem 8.3** Let conditions A1–A5, R1–R3, S0, S1, and all assumptions imposed on $\mu^B_0$ be satisfied. Then the assertions of Theorem 2.12 hold.

This theorem can be proved in a similar way as Theorem 2.12 (see Sections 4, 6).

In Appendix A we will give an application of Theorems 8.2 and 8.3 to the case when the measures $\mu^B_\pm$ from condition S2’ are Gibbs measures with different temperatures $T_+ \neq T_-$.  

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Appendix A: Gibbs measures

Here we study the case $L_B = \Delta - m^2$ only. Consider first the 'free' wave (or Klein–Gordon) equation,

\begin{equation}
\begin{aligned}
\psi_t(x) &= (\Delta - m^2)\varphi_t(x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^d, \\
\varphi_t(x)|_{t=0} &= \varphi_0(x), \quad \dot{\varphi}_t(x)|_{t=0} = \pi_0(x),
\end{aligned}
\end{equation}

where $m \geq 0$, $d \geq 3$, and $d$ is odd if $m = 0$. Denoting $\phi_t = (\varphi_t, \pi_t)$, $t \in \mathbb{R}$, we rewrite (A.1) in the form

\begin{equation}
\dot{\phi}_t = \mathcal{L}_B(\phi_t), \quad t \in \mathbb{R}, \quad \phi_t|_{t=0} = \phi_0,
\end{equation}

with $\mathcal{L}_B = \begin{pmatrix} 0 & \Delta - m^2 & 0 \\ \Delta - m^2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. In the Fourier transform representation, system (A.1) becomes

\begin{equation}
\dot{\hat{\phi}}_t(k) = \hat{\mathcal{L}}_B(k)\hat{\phi}_t(k), \quad \text{hence} \quad \hat{\phi}_t(k) = \hat{\mathcal{G}}_t(k)\hat{\phi}_0(k), \quad \text{where} \quad \hat{\mathcal{G}}_t(k) = \exp(\hat{\mathcal{L}}_B(k)t).
\end{equation}

Here we denote

\begin{equation}
\hat{\mathcal{L}}_B(k) = \begin{pmatrix} 0 & \omega^2(k) & 0 \\ -\omega^2(k) & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{\mathcal{G}}_t(k) = \begin{pmatrix} \cos \omega t & \sin \omega t & \omega t \\ -\omega \sin \omega t & \cos \omega t & 0 \end{pmatrix},
\end{equation}

with $\omega \equiv \omega(k) = \sqrt{|k|^2 + m^2}$. Hence, the solution of (A.2) is $\phi_t = W_t^0\phi_0 = \mathcal{G}_t(\cdot)\ast \phi_0$, where $\mathcal{G}_t(x) = F_{k \rightarrow x}^{-1}[\hat{\mathcal{G}}_t(k)]$. For simplicity of exposition, we omit below the index 0 in the notation of the group $W_t^0$.

A.1 Phase space

We define the weighted Sobolev spaces with any $s, \alpha \in \mathbb{R}$.

**Definition A.1** (i) $H^{s}_\alpha(\mathbb{R}^d)$ is the Hilbert space of the distributions $\varphi \in S'(\mathbb{R}^d)$ with finite norm

\begin{equation}
\|\varphi\|_{s,\alpha} \equiv \|\langle x \rangle^\alpha \Lambda^s \varphi\|_{L^2(\mathbb{R}^d)} < \infty, \quad \Lambda^s \varphi \equiv F^{-1}\{ |k|^s \hat{\varphi}(k) \}, \quad s, \alpha \in \mathbb{R}.
\end{equation}

(ii) $H^s_\alpha \equiv H^{s+1}_{\alpha+1}(\mathbb{R}^d) \oplus H^s_\alpha(\mathbb{R}^d)$ is the Hilbert space of pairs $\phi \equiv (\varphi(x), \pi(x))$ with finite norm

\begin{equation}
\|\phi\|_{s,\alpha} = \|\varphi\|_{s+1,\alpha} + \|\pi\|_{s,\alpha}, \quad s, \alpha \in \mathbb{R}.
\end{equation}

(iii) $\mathcal{E}^s_\alpha \equiv H^s_\alpha \oplus \mathbb{R}^d \oplus \mathbb{R}^d$ is the Hilbert space of vectors $Y \equiv (\phi(x), q, p)$ with finite norm

\begin{equation}
\|Y\|_{s,\alpha} = \|\phi\|_{s,\alpha} + |q| + |p|, \quad s, \alpha \in \mathbb{R}.
\end{equation}

Note that $H^s_\alpha \subset H^s_\alpha$ (and also $\mathcal{E}^s_\alpha \subset \mathcal{E}^s_\alpha$) if $\bar{s} > s$ and $\bar{\alpha} > \alpha$, and this embedding is compact. Moreover, for any $\alpha$, $H^0_\alpha \subset \mathcal{H}$, $\mathcal{E}^0_\alpha \subset \mathcal{E}$ (see Definition 2.1).

**Lemma A.2** Let $L_B = \Delta - m^2$, $s, \alpha \in \mathbb{R}$, and conditions A1' and A2 hold. Then (i) for every $Y_0 \in \mathcal{E}^s_\alpha$, the Cauchy problem (2.4) has a unique solution $Y_t \in C(\mathbb{R}, \mathcal{E}^s_\alpha)$.

(ii) For every $t \in \mathbb{R}$, the operator $S_t: Y_0 \mapsto Y_t$ is continuous on $\mathcal{E}^s_\alpha$. Moreover, there exist positive constants $C_1, C_2 > 0$ such that $\|S_t Y_0\|_{s,\alpha} \leq C_1(t)C_2\|Y_0\|_{s,\alpha}$.

This lemma can be proved by the similar technique from [23], where the nonlinear "wave field–particle" system was studied.
A.2 Gibbs measures for the Klein-Gordon equation

Write \( \phi = (\varphi, \pi) \). We introduce the (normalized) Gibbs measures \( g^B_\beta \) on the space \( \mathcal{H}^s_\alpha \). Formally,

\[
g^B_\beta(d\phi) = \frac{1}{Z_B} e^{-\beta H_B(\phi)} \prod_{x \in \mathbb{R}^d} d\phi(x), \quad H_B(\phi) = \frac{1}{2} \int (|\nabla \varphi(x)|^2 + m^2|\varphi(x)|^2 + |\pi(x)|^2) \, dx.
\]

Now we adjust the definition of the Gibbs measures \( g^B_\beta \). Write \( \phi = (\phi^0, \phi^1) \equiv (\varphi, \pi) \), and denote by \( Q^{ij}(x,y) \), \( i, j = 0, 1 \), the correlation functions of \( g^B_\beta \),

\[
Q^{ij}(x,y) = \int \phi^i(x)\phi^j(y) g^B_\beta(d\phi) = q^{ij}(x-y), \quad x, y \in \mathbb{R}^d.
\]

We will define the Gibbs measures \( g^B_\beta \) as the Gaussian measures with the correlation functions

\[
q^{00}(x-y) = T \mathcal{E}_m(x-y), \quad q^{11}(x-y) = T \delta(x-y), \quad q^{01}(x-y) = q^{10}(x-y) = 0, \quad (A.5)
\]

where \( T = 1/\beta \), \( \mathcal{E}_m(x) \) is the fundamental solution of the operator \(-\Delta + m^2\). The correlation functions \( q^{ij} \) do not satisfy condition (8.1) because of singularity at \( x = y \). The singularity means that the measures \( g^B_\beta \) are not concentrated in the space \( \mathcal{H} \).

**Definition A.3** For \( \beta > 0 \), define the Gibbs measures \( g^B_\beta(d\phi) \) as the Borel probability measures \( g^B_\beta(d\phi) = g^0_\beta(d\varphi) \times g^1_\beta(d\pi) \) in \( \mathcal{H}^s_\alpha = H^{s+1}_\alpha(\mathbb{R}^d) \otimes H^s_\alpha(\mathbb{R}^d) \), \( s, \alpha < -d/2 \), where \( g^0_\beta(d\varphi) \) and \( g^1_\beta(d\pi) \) are Gaussian Borel probability measures in spaces \( H^{s+1}_\alpha(\mathbb{R}^d) \) and \( H^s_\alpha(\mathbb{R}^d) \), respectively, with characteristic functionals

\[
\begin{align*}
\hat{g}^0_\beta(f) &= \int \exp\{i \langle \varphi, f \rangle \} g^0_\beta(d\varphi) = \exp \left\{ -\frac{1}{2\beta} \langle (-\Delta + m^2)^{-1} f, f \rangle \right\} \quad f \in C_0^\infty(\mathbb{R}^d), \quad (A.6) \\
\hat{g}^1_\beta(f) &= \int \exp\{i \langle \pi, f \rangle \} g^1_\beta(d\pi) = \exp \left\{ -\frac{1}{2\beta} \langle f, f \rangle \right\}
\end{align*}
\]

By the Minlos theorem, the Borel probability measures \( g^0_\beta \) and \( g^1_\beta \) exist in the spaces \( H^{s+1}_\alpha(\mathbb{R}^d) \) and \( H^s_\alpha(\mathbb{R}^d) \), respectively, because formally

\[
\int \|\varphi\|^2_{s+1,\alpha} g^0_\beta(d\varphi) < \infty, \quad \int \|\pi\|^2_{s,\alpha} g^1_\beta(d\pi) < \infty, \quad s, \alpha < -d/2. \quad (A.7)
\]

We verify (A.7). Definition (A.3) implies, for \( \varphi \in H^s_\alpha(\mathbb{R}^d) \),

\[
\|\varphi\|^2_{s,\alpha} = (2\pi)^{-2d} \int_{\mathbb{R}^d} \langle x \rangle^{2\alpha} \left( \int_{\mathbb{R}^d} e^{-i(x-k') \cdot \varphi(k)} dk \right) dk' dx. \quad (A.8)
\]

Let \( g(d\varphi) \) be a translation invariant measure in \( H^s_\alpha(\mathbb{R}^d) \) with a correlation function \( Q(x,y) = q(x-y) \). Let us introduce the following correlation function

\[
C(k,k') \equiv \int \hat{\varphi}(k)\overline{\hat{\varphi}(k')} g(d\varphi)
\]

in the sense of distributions. Since \( \varphi(x) \) is real-valued, we have

\[
C(k,k') = F_{x \to k} F_{x' \to -k'} Q(x,x') = (2\pi)^d \delta(k-k')\hat{q}(k).
\]

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Then, integrating (A.8) with respect to the measure \( g(d\varphi) \), we obtain the formula
\[
\int \|\varphi\|^2_{s,\alpha} g(d\varphi) = (2\pi)^{-d} \int_{\mathbb{R}^d} \langle x \rangle^{2\alpha} dx \int_{\mathbb{R}^d} \langle k \rangle^{2\alpha} \hat{q}(k) dk.
\]
Substituting \( \hat{q}(k) = T \) (see (A.5)) we obtain the second bound in (A.7). To obtain the first bound in (A.7) we replace \( s \) into \( s + 1 \) and put \( \hat{q}(k) = T \hat{E}_m(k) = T(|k|^2 + m^2)^{-1} \).

Below for spaces \( \mathcal{E}_\alpha^s \) and \( \mathcal{H}_\alpha^s \), we put \( s, \alpha < -d/2 \). Definition A.3 implies the following lemma (cf. the convergence (7.1)).

**Lemma A.4** The Gibbs measures \( g_\beta^B \) are invariant w.r.t. \( W_t \). Moreover, the flow \( W_t \) is mixing w.r.t. \( g_\beta^B \).

In Section A.4, we will define the Gibbs measures \( g_\beta \) for the coupled system and check the mixing property for the dynamics \( S_t \) w.r.t. \( g_\beta \).

### A.3 Application of Theorems 8.2 and 8.3 to Gibbs measures \( \mu_{\pm}^B \)

Let \( \mu_{\pm}^B \) (see condition S2') be the Gibbs measures \( g_{\pm}^B \equiv g_{\pm}^{B,\pm} \) (with \( \beta_{\pm} = 1/T_{\pm} \)) corresponding to different positive temperatures \( T_\pm \neq T_\mp \). We define the Gibbs measures \( g_{\pm}^B \) in the space \( \mathcal{H}_\alpha^s \) (see Definition A.3) as the Gaussian measures with the correlation functions (cf. (A.5))
\[
q_{\pm}^{00}(x - y) = T_{\pm} \mathcal{E}_m(x - y), \quad q_{\pm}^{11}(x - y) = T_{\pm} \delta(x - y), \quad q_{\pm}^{01}(x - y) = q_{\pm}^{10}(x - y) = 0, \quad (A.9)
\]
where \( x, y \in \mathbb{R}^d \).

Let us introduce \( (\phi_-, \phi_+) \) as a unit random function in the probability space \( (\mathcal{H}_\alpha^s \times \mathcal{H}_\alpha^s, g_\pm^B \times g_\pm^B) \). Then \( \phi\pm \) are Gaussian independent vectors in \( \mathcal{H}_\alpha^s \). Define a Borel probability measure \( \mu_0^B \equiv g_0^B \) on \( \mathcal{H}_\alpha^s \) as the distribution of the random function
\[
\phi_0(x) = \zeta_-(x_1)\phi_-(x) + \zeta_+(x_1)\phi_+(x),
\]
where functions \( \zeta_\pm \) are introduced in (8.3). Then correlation functions of \( g_0^B \) are of the form (8.2) with \( q^{ij}_\pm \) from (A.9). Hence, the measure \( g_0^B \) has zero mean and satisfies condition (8.2) or (8.4). However, \( g_0^B \) does not satisfy (8.1) because of singularity at \( x = y \). Therefore, Theorem 8.2 cannot be applied directly to \( \mu_0^B \equiv g_0^B \). The embedding \( \mathcal{H}_\alpha^s \subset \mathcal{H}^s \) is continuous by the standard arguments of pseudodifferential equations, [19]. The next lemma follows by Fourier transform and the finite speed of propagation for the wave and Klein-Gordon equation.

**Lemma A.5** The operators \( W_t : \phi_0 \mapsto \phi_t \) allow a continuous extension \( \mathcal{H}^s \mapsto \mathcal{H}^s \).

Let \( \phi_0 \) be the random function with the distribution \( g_0^B \). Hence \( \phi_0 \in \mathcal{H}_\alpha^s \) a.s. Denote by \( g_t^B \) the distribution of \( W_t \phi_0 \). For the measures \( g_t^B \), the following result was proved in [8, Theorem 3.1] and [10, Section 4].

**Lemma A.6** Let \( s < -d + 1/2 \). Then there exists a Gaussian Borel probability measure \( g_\infty^B \) on the space \( \mathcal{H}^s \) such that
\[
g_t^B \xrightarrow{\mathcal{H}^s} g_\infty^B, \quad t \to \infty.
\]
The correlation matrix $Q^B_\infty(x,y) = (q^B_{\infty ij}(x-y))_{i,j=0,1}$ of the limit measure $g^B_\infty$ has a form

$$
\begin{align}
q^B_{\infty,00}(x-y) &= \frac{1}{2}(T_+ + T_-)\mathcal{E}_m(x-y), \\
q^B_{\infty,10}(x-y) &= -q^B_{\infty,01}(x-y) = \frac{1}{2}(T_+ - T_-)\mathcal{P}(x-y), \\
q^B_{\infty,11}(x-y) &= \frac{1}{2}(T_+ + T_-)\delta(x-y), \\
\end{align}
$$

(A.11)

where $\mathcal{P}(x) = -iF^{-1}_{k->x}[\text{sgn}(k_1)/\omega(k)]$. In particular, the limiting mean energy current density is formally

$$
\nabla q^B_{\infty,10}(0) = \frac{T_+ - T_-}{2} \nabla \mathcal{P}(0) = -\frac{T_+ - T_-}{2(2\pi)^d} \int_{\mathbb{R}^d} \frac{k \text{sgn}(k_1)}{\sqrt{|k|^2 + m^2}} dk = -\infty \cdot (T_+ - T_-, 0, \ldots, 0).
$$

The infinity means the 'ultraviolet divergence'.

Denote by $g_0$ a Borel probability measure on $\mathcal{E}_\alpha^s$ such that $P g_0 = g_0^B$, where $P : (\phi_0, q_0, p_0) \in \mathcal{E}_\alpha^s \to \phi_0 \in \mathcal{H}_\alpha^s$, and $g_0^B$ is the probability measure on $\mathcal{H}_\alpha^s$ constructed above. Then, by the asymptotic behavior of $Y_t$ (see Section 5) and by Lemma A.6 the following result holds (cf. Theorems 2.12 and 8.3).

**Lemma A.7** Let $s < -d + 1/2$. Then the measures $g_t = S_t^* g_0$ weakly converge to a limit measure $g_\infty$ as $t \to \infty$ on the space $\mathcal{E}^s$. The limit measure $g_\infty$ is Gaussian and its characteristic functional is $\hat{g}_\infty(Z) = \exp\{-1/2 Q_\infty(Z, Z)\}$, where $Q_\infty(Z, Z) = \langle q^B_\infty(x-y), \Pi(Z) \otimes \Pi(Z) \rangle$ with $q^B_\infty$ from (A.11).

### A.4 Gibbs measure for the coupled system

For $\beta > 0$, we introduce the (normalized) Gibbs measures $g_\beta$ on the space $\mathcal{E}_\alpha^s$. Formally,

$$
g_\beta(d\phi d\xi) = \frac{1}{Z} e^{-\beta H(\phi, \xi)} \prod_{x \in \mathbb{R}^d} d\phi(x) d\xi.
$$

**Definition A.8** For $\beta > 0$, define the Gibbs measures $g_\beta(d\phi d\xi)$ in $\mathcal{E}_\alpha^s$, $s, \alpha < -d/2$, as

$$
g_\beta(d\phi d\xi) = \frac{1}{Z} e^{-\beta q(\rho, \nabla \phi)} g^B_\beta(d\phi) \times g^A_\beta(d\xi). 
$$

(A.12)

Here $\beta = 1/T$ is an inverse temperature, $g^B_\beta(d\phi)$ is defined in Definition A.3 and $g^A_\beta$ is the Gibbs measure on $\mathbb{R}^d \times \mathbb{R}^d$,

$$
g^A_\beta(d\xi) = \frac{1}{Z_A} e^{-\beta H_A(\xi)} d\xi, \quad H_A(\xi) = \frac{1}{2} (|p|^2 + \omega^2 |q|^2).
$$

(A.13)

In Section A.6 we will prove the invariance of the Gibbs measures $g_\beta$ w.r.t. the group $S_t$.

**Lemma A.9** The flow $S_t$ is mixing w.r.t. $g_\beta$, i.e., for any functions $F_1, F_2 \in L_2(\mathcal{E}_\alpha^s, g_\beta)$, we have

$$
\int F_1(S_t Y) F_2(Y) g_\beta(dY) \to \int F_1(Y) g_\beta(dY) \int F_2(Y) g_\beta(dY) \quad \text{as} \ t \to \infty.
$$
Proof It suffices to check that for any \( Z_1, Z_2 \in \mathcal{D} \),
\[
\int \langle S_t Y_0, Z_1 \rangle \langle Y_0, Z_2 \rangle \; g_\beta(dY_0) \to 0, \quad t \to \infty.
\] (A.14)

Let \( Z_1 = (f, u, v) \in \mathcal{D} \). By Corollary 3.2 and formulas (2.25)–(2.27), we obtain
\[
|q_t - \langle W_t \phi_0, \alpha \rangle| + |p_t - \langle W_t \phi_0, \beta \rangle| \leq C_1 \varepsilon_m(t)|\xi_0| + C_2 \sqrt{\varepsilon_m(t)} \sup_{\tau \in \mathbb{R}} |\langle W_\tau \phi_0, \nabla \rho_0 \rangle|,
\]
and
\[
|\langle \phi_t, f \rangle - \langle W_t \phi_0, f_s \rangle| \leq C_1 \varepsilon_m(t)|\xi_0| + C_2 \sqrt{\varepsilon_m(t)} \left( \sup_{\tau \in \mathbb{R}} |\langle W_\tau \phi_0, \nabla \rho_0 \rangle| + \sup_{\tau \in \mathbb{R}} |\langle W_\tau \phi_0, \alpha \rangle| \right).
\]

These bounds can be proved similarly to Proposition 5.1. Hence, to prove (A.14) it suffices to verify that
\[
\int \langle \phi_0, W_t \chi \rangle \langle Y_0, Z_2 \rangle \; g_\beta(dY_0) \to 0 \quad \text{as} \quad t \to \infty,
\] (A.15)
with \( \chi = \alpha, \beta, f_s \). Since
\[
F_{x \to k}[W_t f] = \begin{pmatrix}
\cos \omega(k)t & -\omega(k) \sin \omega(k)t \\
\omega^{-1}(k) \sin \omega(k)t & \cos \omega(k)t
\end{pmatrix} \begin{pmatrix}
\hat{f}^0(k) \\
\hat{f}^1(k)
\end{pmatrix},
\]
then Definition [A.8] equalities [A.5], and the Lebesgue–Riemann theorem imply (A.15). \( \blacksquare \)

A.5 Effective Hamiltonian

To prove the invariance of the Gibbs measures \( g_\beta \) we use notations introduced by Jakšić and Pillet in [23]. At first, we rewrite the system (3.1)–(3.2) in new variables. Introduce an effective potential by
\[
V_{eff}(q) = \frac{1}{2} (\omega^2 |q|^2 - q \cdot K_m q),
\] (A.17)
where \( K_m \) is the 'coupling constant matrix' defined in (2.7). By condition R1', \( V_{eff}(q) \geq 0 \). Define \( \mathbb{R}^d \)-valued function \( h(x) \),
\[
h(x) = (\Delta - m^2)^{-1} \nabla \rho(x), \quad x \in \mathbb{R}^d,
\] (A.18)
where \( \rho \) is the coupled function, and put \( h_0 = (h, 0) \in \mathbb{R}^d \times \mathbb{R}^d \). Then the fist equations in (2.1) become
\[
\dot{\phi}_t = \mathcal{L}_B \phi_t + q_t \cdot \nabla \rho_0 = \mathcal{L}_B (\phi_t + q_t \cdot h_0), \quad \text{with} \quad \mathcal{L}_B = \begin{pmatrix}
0 & 1 \\
\Delta - m^2 & 0
\end{pmatrix},
\] (A.19)
because \( \mathcal{L}_B h_0 = (0, \nabla \rho) \). Define a \( \mathbb{R}^2 \)-valued function \( \psi \equiv \psi(x), \; x \in \mathbb{R}^d \), where
\[
\psi = (\psi^0, \psi^1) : \quad \psi^0 = \varphi + q \cdot h, \quad \psi^1 = \pi.
\]
Then (A.19) becomes \( \dot{\psi}_t = \mathcal{L}_B \psi_t + \dot{q}_t \cdot h_0 \). Recall that \( \mathcal{L}_B \) is the generator of the group \( W_t \). Hence, in new variables \( (\psi_t, \xi_t) \) the system (3.1)–(3.2) becomes
\[
\psi_t = W_t \psi_0 + \int_0^t W_{t-s} h_0 \cdot \dot{q}_s \, ds, \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R},
\]
\[
\dot{\psi}_t = -\nabla V_{eff}(q_t) - \int_0^t \Gamma(t-s) \dot{q}_s \, ds + \mathcal{F}(t),
\] (A.20)
where $F(t) := \langle \nabla \rho_0, W_t \psi_0 \rangle$, $\nabla V_{\text{eff}}(q_t) = (\omega^2 I - K_m)q_t$, the matrix $K_m$ is defined in (2.7), its entries are

$$K_{m,ij} = -\langle \nabla_i \rho_0, h_0^j \rangle = (2\pi)^{-d} \int \frac{k_i k_j |\hat{\rho}(k)|^2}{k^2 + m^2} dk,$$

and $\Gamma(t)$ stands for the $\mathbb{R}^d \times \mathbb{R}^d$ matrix with entries $\Gamma_{ij}(t)$,

$$\Gamma_{ij}(t) := -\langle \nabla_i \rho_0, W_t h_0^j \rangle = (2\pi)^{-d} \int k_i k_j \frac{\cos \omega(k)t}{\omega^2(k)} |\hat{\rho}(k)|^2 dk, \quad i, j = 1, \ldots, d. \quad (A.21)$$

The equation (A.20) is called the generalized or retarded Langevin equation with the random force $F(t)$ and with the memory kernel $\Gamma(t)$.

**Remark A.10** (i) By (A.21), we have $\Gamma(0) = K_m$. Moreover, $\dot{\Gamma}_{ij}(t) = -D_{ij}(t)$, where $D_{ij}(t)$ are the entries of the matrix $D(t)$ defined in (3.4).

(ii) $\rho_0(x) = h_0(x) = 0$ for $|x| \geq R_\rho$, by condition R2. Hence, $\Gamma(t) = 0$ for $|t| > 2R_\rho$ if $m = 0$ due to a strong Huyghen’s principle, and $|\Gamma(t)| \leq C(1 + |t|)^{-d/2}$ if $m \neq 0$.

(iii) It follows from (A.5), (A.16) and (A.21) that $\int (F(t) \otimes F(s)) g_{\beta}^B(\psi) = (1/\beta) \Gamma(t - s)$ (fluctuation–dissipation relation).

(iv) The force $F(t)$ equals $F(t) - \Gamma(t)q_0$ with $F(t)$ from (3.2).

Introduce an effective Hamiltonian $H_{\text{eff}}^A(\xi) = |p|^2/2 + V_{\text{eff}}(q)$. Hence, by (1.1),

$$H(\phi, \xi) = H_B(\psi) + H_{\text{eff}}^A(\xi).$$

**Definition A.11** (i) Define a map $T$ on $\mathcal{E}^s_\alpha$ by the rule

$$T : (\phi, \xi) \to (\psi, \xi), \quad \psi = \phi + q \cdot h_0.$$

(ii) Denote $g^T_{\beta}(d\psi d\xi) := g_{\beta}(T^{-1}(d\psi d\xi))$. Then, $g^T_{\beta}(d\psi d\xi) = g^B_{\beta}(d\psi) \times g_{\text{eff}}^A(d\xi)$, where $g^B_{\beta}$ is defined in Definition A.3. $g_{\text{eff}}^A$ is a Gaussian measure defined by $g_{\text{eff}}^A(d\xi) = (1/Z)e^{-\beta H_{\text{eff}}^A(\xi)} d\xi$.

**A.6 Invariance of Gibbs measures $g_\beta$**

**Proposition A.12** Let conditions R1 and R2 hold. Then the Gibbs measures $g_\beta$, $\beta > 0$, are invariant with respect to the dynamics, i.e.

$$S_t^s g_\beta(\omega) := g_\beta(S_t^{-1}s^s\omega) = g_\beta(\omega), \quad \text{for } \omega \in \mathcal{B}(\mathcal{E}^s_\alpha) \text{ and } t \in \mathbb{R}. \quad (A.22)$$

Here $\mathcal{B}(\mathcal{E}^s_\alpha)$ is the Borelian $\sigma$-algebra of subsets in $\mathcal{E}^s_\alpha$.

**Proof** For simplicity, we omit indices $\alpha, s$ in notations $\mathcal{E}^s_\alpha$ and $\mathcal{H}^s_\alpha$. The invariance (A.22) is equivalent to the identity:

$$\frac{d}{dt} \int_\mathcal{E} F(S_t Y) g_\beta(dY) = 0, \quad t \in \mathbb{R}, \quad (A.23)$$

for any bounded continuous functional $F(Y)$ on $\mathcal{E}$, i.e., $F(Y) \in C_b(\mathcal{E})$. It suffices to prove (A.23) with $t = 0$ only. Indeed, since $S_{t+\tau} = S_t S_\tau$, we have

$$\frac{d}{dt} \int_\mathcal{E} F(S_{t}S_\tau Y) g_\beta(dY) = \frac{d}{d\tau} \int_\mathcal{E} F(S_{t+\tau} Y) g_\beta(dY) = \frac{d}{dt} \int_\mathcal{E} F(S_t S_\tau Y) g_\beta(dY). \quad (A.24)$$
Let \( \frac{d}{d\tau} \int_{\mathcal{E}} F(S_{rY})g_{\beta}(dY) \bigg|_{\tau=0} = 0 \). Since \( F(S_{1Y}) \in C_b(\mathcal{E}) \), then \( \frac{d}{d\tau} \int_{\mathcal{E}} F(S_{rY})g_{\beta}(dY) \bigg|_{\tau=0} = 0 \) for any fixed \( t \in \mathbb{R} \). Hence, (A.24) implies
\[
0 = \frac{d}{dt} \int_{\mathcal{E}} F(S_{rY})g_{\beta}(dY) \bigg|_{t=0} = \frac{d}{dt} \int_{\mathcal{E}} F(S_{1Y})g_{\beta}(dY),
\]
and (A.23) follows. Moreover, it suffices to verify (A.23) with \( t = 0 \) and \( F(Y) = \exp(iY, Z) \) for every \( Z = (f_0(x), f_1(x), u, v) \in \mathcal{D} \). Then, by (2.4), identity (A.23) with \( t = 0 \) becomes
\[
\frac{d}{dt} \int_{\mathcal{E}} e^{i(S_{rY}; Z)}g_{\beta}(dY) \bigg|_{t=0} = \int_{\mathcal{E}} e^{i(Y; Z)}i(\mathcal{L}(Y), Z)g_{\beta}(dY) = 0,
\]
where
\[
\mathcal{L}(\varphi, \pi, q, p) = (\pi, (\Delta - m^2)\varphi + q \cdot \nabla \rho, p, -\omega^2 q + \langle \nabla \rho, \varphi \rangle).
\]
Now we prove (A.25). Denote by \( I \) the integral
\[
I := \int_{\mathcal{E}} e^{i(Y; Z)}i(\mathcal{L}(Y), Z)g_{\beta}(dY),
\]
and check that \( I = 0 \). Definition \( \text{A.11} \) implies \( g_{\beta}(dY) = g_{\beta}^T(TdY) \). Hence,
\[
\int_{\mathcal{E}} F(Y)g_{\beta}(dY) = \int_{\mathbb{R}^{2d}} g_{\beta}^{eff}(d\xi) \int_{\mathcal{H}} F(\psi - q \cdot h_0, \xi)g_{\beta}^B(d\psi).
\]
(A.27)
Using (A.26), (A.27), and (A.18), we rewrite \( I \) in the form
\[
I = \int_{\mathbb{R}^{2d}} e^{i(uq + vp)}g_{\beta}^{eff}(d\xi) \int_{\mathcal{H}} e^{i(\psi^0 - q \cdot h_0, f_0) + i(\psi^1, f_1)} \left( i(\psi^1, f_0) + i((\Delta - m^2)\psi^0, f_1) + iu \cdot p + iv \cdot [-\omega^2 q + \langle \nabla \rho, \psi^0 - q \cdot h \rangle] \right) g_{\beta}^{0}(d\psi^0)g_{\beta}^{1}(d\psi^1).
\]
Integrals over Gaussian measures \( g_{\beta}^{0}(d\psi^0) \) and \( g_{\beta}^{1}(d\psi^1) \) can be represented as variational derivatives of their characteristic functionals \( g_{\beta}(f_0) \) and \( \hat{g}_{\beta}(f_1) \):
\[
\int e^{i(\psi, f_1)}i(\psi, \cdot)g_{\beta}^{i}(d\psi) = \left\langle \frac{\delta}{\delta f} \hat{g}_{\beta}(f), \cdot \right\rangle, \quad i = 0, 1, \quad f \in C_0^\infty(\mathbb{R}^d).
\]
Then
\[
I = \int_{\mathbb{R}^{2d}} e^{i(uq + vp)}e^{-i\psi \cdot (h_0, f_0)} \left( \frac{\delta}{\delta f_1} f_0 + ((\Delta - m^2)\frac{\delta}{\delta f_0}, f_1) + iu \cdot p + v \cdot [-\omega^2 q + \langle \nabla \rho, \cdot \cdot \rangle - i\langle \nabla \rho, q \cdot h \rangle] \right) g_{\beta}^{0}(f_0)g_{\beta}^{1}(f_1)g_{\beta}^{eff}(d\xi).
\]
(A.28)
Using (A.6), we calculate
\[
\left\langle \frac{\delta}{\delta f} \hat{g}_{\beta}^0(f), \cdot \right\rangle = -\frac{1}{\beta} e^{-\frac{1}{2\beta}((-\Delta + m^2)^{-1}f, f)} \frac{\delta}{\delta f}((-\Delta + m^2)^{-1}f, \cdot) \quad \left\langle \frac{\delta}{\delta f} \hat{g}_{\beta}^1(f), \cdot \right\rangle = -\frac{1}{\beta} e^{-\frac{1}{2\beta}(f, f)}(f, \cdot) \quad \left\langle \frac{\delta}{\delta f} \hat{g}_{\beta}^2(f), \cdot \right\rangle = -\frac{1}{\beta} e^{-\frac{1}{2\beta}(f, f)}(f, \cdot)
\]
\( f \in C_0^\infty(\mathbb{R}^d) \).
Therefore, we reduce (A.28) to the following integral

\[
I = C \int_{\mathbb{R}^{2d}} e^{i(u-q-v-p)} e^{-i\omega \langle h, f_0 \rangle} \left( iu \cdot p - iv \cdot \nabla V_{\text{eff}}(q) + \frac{1}{\beta} v \cdot (f_0, h) \right) e^{-\beta \mathcal{H}_{P}^f(\xi)} d\xi
\]

\[
= C_1 \int_{\mathbb{R}^{2d}} e^{i(u-q-v-p)} (u \cdot \nabla p - v \cdot \nabla_q) \left[ e^{-i\omega \langle h, f_0 \rangle} - \beta \mathcal{H}_{P}^f(q, p) \right] dq dp,
\]

by (A.17) and (A.18). Partial integration in \( q \) and \( p \) leads to

\[
I = C_2 \int_{\mathbb{R}^{2d}} e^{i(u-q-v-p)} (-u \cdot (iv) + v \cdot (iu)) e^{-i\omega \langle h, f_0 \rangle} - \beta \mathcal{H}_{P}^f(q, p) dq dp = 0. \quad \blacksquare
\]

### Appendix B: Existence of solutions

Proposition 2.2 can be proved by using the methods of [25, Lemma 6.3]. In this section, we outline the proof of this proposition.

**Proof of Lemma 2.3.** Step (i) If \( \rho = 0 \), then the existence and uniqueness of the solution \( Y_t \in C(\mathbb{R}, E) \) to problem (2.4) is well-known (see, for example, [29]). Represent the solution \( Y_t \) as the pair of the functions \( (Y^0, Y^1) \), where \( Y^0_t = (\varphi_t, q_t), Y^1_t = (\pi_t, p_t) \). Therefore, problem (2.4) for \( Y_t \in C(\mathbb{R}, E) \) is equivalent to

\[
Y_t = e^{L_0 t} Y_0 + \int_0^t e^{L_0 (t-s)} B Y_s ds,
\]

(B.1)

where \( Y_0 = (\varphi_0, q_0, \pi_0, p_0) \in E = H^1_\lambda(\mathbb{R}^3) \otimes \mathbb{R}^3 \otimes L^2(\mathbb{R}^3) \otimes \mathbb{R}^3, \)

**L_0 = \begin{pmatrix} 0 & I \\ A_0 & 0 \end{pmatrix}, \quad \mathcal{A}_0 \left( \begin{array}{c} \varphi \\ q \end{array} \right) = \left( \begin{array}{c} L_B \varphi \\ -\omega^2 q \end{array} \right),

B(Y^0, Y^1) = (0, R Y^0), \quad R Y^0 = \left( q \cdot \nabla \rho, \left( \varphi, \nabla \rho \right) \right) \)

(\text{cf (7.4)}). Note that \( \|e^{L_0 t} Y_0\|_E \leq C\|Y_0\|_E \); and the second term in (B.1) is estimated by

\[
\sup_{|t| \leq T} \left\| \int_0^t e^{L_0 (t-s)} B Y_s ds \right\|_E \leq C T \sup_{|s| \leq T} \|Y_s\|_E.
\]

This bound and the contraction mapping principle imply the existence and uniqueness of the local solution \( Y_t \in C([-\varepsilon, \varepsilon], E) \) for some \( \varepsilon > 0. \)

**Step (ii)** To prove the energy conservation

\[
H(Y_t) = H(Y_0) \quad \text{for} \quad t \in \mathbb{R},
\]

we first assume that \( \phi_0 = (\varphi_0, \pi_0) \in C^3(\mathbb{R}^3) \times C^2(\mathbb{R}^3) \) and \( \phi_0(x) = 0 \) for \( |x| \geq R_0 \). Then \( \varphi_t(x) \in C^2(\mathbb{R}^3_x \times \mathbb{R}_t) \) and

\[
\varphi_t(x) = 0 \quad \text{for} \quad |x| \geq |t| + \max\{R_0, R_a, R_p\}
\]

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by the integral representation (B.1) and conditions A2 and R2. Therefore, for such initial data, relation (B.3) can be proved by integrating by parts. Hence, for \( Y_0 \in E \), (B.3) follows from the continuity of \( S_t \) and from the fact that \( C_0^3(\mathbb{R}^3) \oplus \mathbb{R}^3 \oplus C_0^2(\mathbb{R}^3) \oplus \mathbb{R}^3 \) is dense in \( E \).

**Step (iii)** In the case of WF, we apply condition A3 and obtain

\[
\frac{1}{2} \int \left( \sum_{ij} \nabla_i \varphi(x) a_{ij}(x) \nabla_j \varphi(x) + 2q \cdot \nabla \varphi(x) \rho(x) \right) \, dx \\
\geq \frac{1}{2} \int \left( \alpha |\nabla \varphi(x)|^2 + 2q \cdot \nabla \varphi(x) \rho(x) \right) \, dx = \frac{\alpha}{2} \| \nabla \varphi \| + \frac{q \rho}{\alpha} \| \rho \|^2 - \frac{1}{2\alpha} |q|^2 \| \rho \|^2,
\]

where \( \| \cdot \| \) stands for the norm in \( L^2 \). In the case of KGF,

\[
\frac{1}{2} \int \left( m^2 |\varphi(x)|^2 - 2 \varphi(x) q \cdot \nabla \rho(x) \right) \, dx \geq \frac{1}{2} m^2 \| \varphi \| - \frac{q \cdot \nabla \rho}{m^2} \| \rho \|^2 - \frac{1}{2m^2} |q|^2 \| \nabla \rho \|^2.
\]

Hence, the Hamiltonian functional \( H(Y) \) is nonnegative. Indeed, in the case of WF,

\[
H(Y) \geq \frac{1}{2} \int \left( |\pi(x)|^2 + \alpha |\nabla \varphi(x) + \frac{q \rho(x)}{\alpha}|^2 + a_0(x) |\varphi(x)|^2 \right) \, dx \\
+ \frac{1}{2} \left( \omega^2 - \frac{1}{\alpha} \| \rho \|^2 \right) |q|^2 + \frac{1}{2} |p|^2 \geq 0
\]

by condition R1. In the case of KGF,

\[
H(Y) \geq \frac{1}{2} \int \left( |\pi(x)|^2 + \sum_j |(\nabla_j - iA_j(x)) \varphi(x)|^2 + m^2 |\varphi(x) - \frac{q \cdot \nabla \rho(x)}{m^2}|^2 \right) \, dx \\
+ \frac{1}{2} \left( \omega^2 - \frac{1}{m^2} \| \nabla \rho \|^2 \right) |q|^2 + \frac{1}{2} |p|^2 \geq 0
\]

by condition R1. Moreover, by (B.3), (B.4) and (B.5), we obtain

\[
\| Y_1 \|_{L^2}^2 \leq C H(Y_1) = C H(Y_0).
\]

On the other hand, in the case of KGF, we have

\[
H(Y) \leq \frac{1}{2} \left\{ \sum_j \| (\nabla_j - iA_j(x)) \varphi \|^2 + \| \nabla \varphi \|^2 + \| \pi \|^2 + \| \nabla \rho \|^2 \right\} \\
\leq C \| Y \|_{L^2}^2,
\]

since \( |q \cdot (\nabla \varphi, \rho)| \leq (\| \nabla \varphi \|^2 + \| q \|^2 \| \rho \|^2) / 2 \). In the WF case,

\[
H(Y) \leq C \left( \| \nabla \varphi \|^2 + \| \pi \|^2 + (\omega^2 + \| \rho \|^2) |q|^2 + |p|^2 + \int a_0(x) |\varphi(x)|^2 \, dx \right).
\]

Since \( Y \in E \), \( \varphi \in H^1_F \). For the WF case, \( H^1_F \) is the completion of real space \( C_0^\infty(\mathbb{R}^3) \) with the norm \( \| \nabla \varphi \| \). Therefore, \( H^1_F = \{ \varphi \in L^6(\mathbb{R}^3) : |\nabla \varphi| \in L^2 \} \) by Sobolev’s embedding theorem. Hence,

\[
\int a_0(x) |\varphi(x)|^2 \, dx \leq C \| \varphi \|_{L^6}^2 \leq C_1 \| \nabla \varphi \|^2.
\]
Using (B.6) and (B.7), we obtain the \textit{a priori} estimate
\[ \|Y_t\|_E \leq C_1 \|Y_0\|_E \quad \text{for } t \in \mathbb{R}. \] (B.8)

Therefore, properties (i)–(iii) of Lemma 2.3 for arbitrary \( t \in \mathbb{R} \) follow from bound (B.8).

We return to the proof of Proposition 2.2. Let us choose \( R > \max\{ R_a, R_\rho \} \) with \( R_a \) and \( R_\rho \) from conditions A2 and R2. Then, by the integral representation (B.1), the solution \( Y_t \) for \( |x| < R \) depends only on the initial data \( Y_0(x) \) with \( |x| < R + |t| \). Thus, the continuity of \( S_t \) in \( E \) follows from the continuity in \( E \).

For every \( R > 0 \), define the local energy seminorms by
\[ \|Y\|_{E(R)}^2 := \int_{|x|<R} \left( (\nabla \varphi(x))^2 + m^2 |\varphi(x)|^2 + |\pi(x)|^2 \right) dx + |q|^2 + |p|^2, \quad Y = (\varphi, \pi, q, p), \]
where \( m > 0 \) for the KGF case, and \( m = 0 \) for the WF case. By estimate (B.8), we obtain the following local energy estimates:
\[ \|S_t Y_0\|_{E(R)}^2 \leq C \|Y_0\|_{E(R+|t|)}^2 \quad \text{for } R > \max\{ R_\rho, R_a \} \quad \text{and } t \in \mathbb{R}. \]

Hence, for any \( T > 0 \) and \( R > \max\{ R_\rho, R_a \} \),
\[ \sup_{|t| \leq T} \|S_t Y_0\|_{E, R} \leq C(T) \|Y_0\|_{E, R+T}. \]
References

[1] Appleby, J.A. and Freeman, A., "Exponential asymptotic stability of linear Itô–Volterra equations with damped stochastic perturbations," Electronic J. of Probability 8, 1-22 (2003).

[2] Boldrighini, C., Pellegrinotti, A., and Triolo, L., "Convergence to stationary states for infinite harmonic systems," J. Stat. Phys. 30, 123-155 (1983).

[3] Corduneanu, C. and Lakshmikantham, V., "Equations with unbounded delay: a survey," Nonlinear Analysis, TMA 4 (5), 831-877 (1980).

[4] Driver, R.D., Ordinary and Delay Differential Equations (Springer–Verlag, New York, 1977).

[5] Dudnikova, T.V. and Komech, A.I., "Ergodic properties of hyperbolic equations with mixing," Theory Probab. Appl. 41 (3), 436-448 (1996).

[6] Dudnikova, T.V., Komech, A.I., Kopylova, E.A., and Suhov, Yu.M., "On convergence to equilibrium distribution, I. The Klein-Gordon equation with mixing," Commun. Math. Phys. 225 (1), 1-32 (2002). e-print: ArXiv:math-ph/0508042.

[7] Dudnikova, T.V., Komech, A.I., Ratanov, N.E. and Suhov, Yu.M., "On convergence to equilibrium distribution, II. The wave equation in odd dimensions, with mixing," J. Stat. Phys. 108 (4), 1219-1253 (2002). e-print: ArXiv:math-ph/0508039.

[8] Dudnikova, T.V., Komech, A.I., and Spohn, H., "On a two-temperature problem for wave equation," Markov Processes and Related Fields 8, 43-80 (2002). e-print: ArXiv:math-ph/0508044.

[9] Dudnikova, T.V., Komech, A.I., and Spohn, H., "On the convergence to statistical equilibrium for harmonic crystals," J. Math. Phys. 44 (6), 2596-2620 (2003). ArXiv: math-ph/0210039.

[10] Dudnikova, T.V. and Komech, A.I., "On a two-temperature problem for the Klein-Gordon equation," Teor. Veroyatn. Ee Primen. 50, 675-710 (2005) [Russian] (English translation: Theory Prob. Appl. 50 (4), 582-611 (2006)).

[11] Dudnikova, T.V. and Komech, A.I., "On the convergence to a statistical equilibrium in the crystal coupled to a scalar field," Russian J. Math. Phys. 12 (3), 301-325 (2005). e-print: ArXiv:math-ph/0508053.

[12] Dudnikova, T.V., "Convergence to equilibrium distribution. The Klein-Gordon equation coupled to a particle," Russian J. Math. Phys. 17 (1), 77-95 (2010). e-print: arXiv:0711.1091.

[13] Egorov, Yu.V., Komech, A.I., and Shubin, M.A., Elements of the Modern Theory of Partial Differential Equations (Springer, Berlin, 1999).

[14] Eckmann, J.-P., Pillet, C.-A., and Rey-Bellet, L., "Non-equilibrium statistical mechanics of anharmonic chains coupled to two heat baths at different temperatures," Commun. Math. Phys. 201, 657-697 (1999).
[15] Gel’fand, I.M. and Shilov, G.E., Generalized Functions. Vol.I: Properties and Operations (Academic Press, New York, 1964).

[16] Gripenberg, G., Londen, S.-O., and Staffans, O., Volterra Integral and Functional Equations, vol. 34, in: Encyclopedia of Mathematics and its Applications (Cambridge University Press, Cambridge, 1990).

[17] Grossman, S.I. and Miller, R.K., "Nonlinear Volterra integrodifferential systems with $L^1$-kernels," J. Differential Equations 13, 551–566 (1973).

[18] Hara, T., "Exponential asymptotic stability for Volterra integrodifferential equations of nonconvolution type," Funkcialaj Ekvacioj 37, 373-382 (1994).

[19] Hörmander, L., The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators (Springer-Verlag, 1985).

[20] Ibragimov, I.A. and Linnik, Yu.V., Independent and Stationary Sequences of Random Variables (Wolters-Noordhoff, Groningen, 1971).

[21] Imaikin, V., Komech, A., and Vainberg, B., "On scattering of solitons for the Klein–Gordon equation coupled to a particle," Comm. Math. Phys. 268 (3), 321-367 (2006).

[22] Jakšić, V. and Pillet, C.-A., "Ergodic properties of the non-Markovian Langevin equation," Lett. Math. Phys. 41(1), 49-57 (1997).

[23] Jakšić, V. and Pillet, C.-A., "Ergodic properties of classical dissipative systems. I," Acta Math. 181 (2), 245-282 (1998).

[24] Karczewska, A., "Convolution type stochastic Volterra equations," - Torun: Juliusz Schauder Center for Nonlinear Studies; Nicolaus Copernicus Univ. (2007).– (Lecture Notes in Nonlinear Analysis 10). e-print: arXiv:0712.4357 (2007).

[25] Komech, A., Spohn, H., and Kunze, M., "Long-time asymptotics for a classical particle interacting with a scalar wave field," Comm. Partial Diff. Equations 22, 307-335 (1997).

[26] Kordonis, I.-G.E. and Philos, Ch.G., "The behavior of solutions of linear integro-differential equations with unbounded delay," Computers and Mathematics with Appl. 38, 45–50 (1999).

[27] Mao, X., "Stability of stochastic integro-differential equations," Stochastic Analysis and Appl. 18 (6), 1005–1017 (2000).

[28] Mao, X. and Riedle, M., "Mean square stability of stochastic Volterra integro-differential equations," Systems & Control Letters 55, 459–465 (2006).

[29] Mikhailov, V.P., Partial Differential Equations (Mir, Moscow, 1978).

[30] Murakami, S., "Exponential asymptotic stability for scalar linear Volterra equations," Differential and Integral Equations 4, 519–525 (1991).

[31] Myshkis, A.D., Linear Differential Equations with Retarded Argument (Russian, 2-nd edition, Nauka, Moscow, 1972).
[32] Ottobre, M. and Pavliotis, G.A., ”Asymptotic analysis for the generalized Langevin equation,” Nonlinearity, 24(5), 1629-1653 (2011).

[33] Rey-Bellet, L. and Thomas, L.E., ”Exponential convergence to non-equilibrium stationary states in classical statistical mechanics,” Commun. Math. Phys. 225, 305-329 (2002).

[34] Rosenblatt, M.A., ”A central limit theorem and a strong mixing condition,” Proc. Nat. Acad. Sci. U.S.A. 42 (1), 43-47 (1956).

[35] Snook, I., The Langevin and Generalized Langevin Approach to the Dynamics of Atomic, Polymeric and Colloidal Systems (Elsevier, 2006).

[36] Spohn, H. and Lebowitz, J., ”Stationary non-equilibrium states of infinite harmonic systems,” Comm. Math. Phys. 54 (2), 97-120 (1977).

[37] Vainberg, B.R., ”Behavior of the solution of the Cauchy problem for a hyperbolic equation as $t \to \infty$,” Math. of the USSR-Sbornik 7 (4), 533-568 (1969); trans. Mat. Sb. 78 (4), 542-578 (1969).

[38] Vainberg, B.R., ”Behaviour for large time of solutions of the Klein-Gordon equation,” Trans. Moscow Math. Soc. 30, 139-158 (1974).

[39] Vainberg, B.R., Asymptotic Methods in Equations of Mathematical Physics (Gordon and Breach, New York, 1989).

[40] Vishik, M.I. and Fursikov, A.V., Mathematical Problems of Statistical Hydromechanics (Kluwer Academic Publishers, 1988).

[41] Zwanzig, R., ”Nonlinear generalized Langevin equations,” J. Stat. Phys. 9, 215-220 (1973).