Conformal invariance and the Lundgren-Monin-Novikov equations for vorticity fields in 2D turbulence: Refuting a recent claim

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February 19, 2018

Abstract
The recent claim by Grebenev \textit{et al.} \cite{Grebenev2017} that the inviscid 2D Lundgren-Monin-Novikov (LMN) equations on a zero vorticity characteristic naturally would reveal local conformal invariance when only analyzing these by means of a classical Lie-group symmetry approach, is invalid and will be refuted in the present comment. To note is that within this comment the (possible) existence of conformal invariance in 2D turbulence is not questioned, only the conclusion as is given in Grebenev \textit{et al.} (2017) and their approach how this invariance was derived is what is being criticized and refuted herein. In fact, the algebraic derivation for conformal invariance of the 2D LMN vorticity equations in Grebenev \textit{et al.} (2017) is flawed. A key constraint of the LMN equations has been wrongly transformed. Providing the correct transformation instead will lead to a breaking of the proclaimed conformal group. The corrected version of Grebenev \textit{et al.} (2017) just leads to a globally constant scaling in the fields and not to a local one as claimed. In consequence, since in Grebenev \textit{et al.} (2017) only the first equation within the infinite and unclosed LMN chain is considered, also different Lie-group infinitesimals for the one- and two-point probability density functions (PDFs) will result from this correction, replacing thus the misleading ones proposed.

Keywords: Statistical Physics, Conformal Invariance, Turbulence, Probability Density Functions, Lie Groups, Symmetry Analysis, Integro-Differential Equations, Closure Problem

PACS: 47.10.-g, 47.27.-i, 05.20.-y, 02.20.Qs, 02.20.Tw, 02.30.Rz, 02.50.Cw

1. Summary of the key results obtained in Grebenev \textit{et al.} (2017)

Considered is the first equation in the unclosed chain of the inviscid 2D Lundgren-Monin-Novikov (LMN) vorticity equations (Eq. [3])

\[
\frac{\partial f_1(x, \omega, t)}{\partial t} - \frac{\partial}{\partial x^1} \int d^2x'd\omega' \frac{x^2 - x'^2}{2\pi|x - x'|^2} f_2(x, \omega, x', \omega', t) + \frac{\partial}{\partial x^2} \int d^2x'd\omega' \frac{x^1 - x'^1}{2\pi|x - x'|^2} f_2(x, \omega, x', \omega', t) = 0, \tag{1.1}
\]

describing the dynamics of the 1-point probability density function (PDF) $f_1$ in terms of the (unclosed) 2-point PDF $f_2$, where $\omega$ and $\omega'$ denote the sample space variables of the single vorticity component at the space-time points $(x,t)$ and $(x',t)$, respectively. This equation (1.1) is supplemented by the two normalization constraints for the PDFs (Eq. [4])

\[
\int d\omega f_1 = 1, \quad \int d\omega f_2 = f_1. \tag{1.2}
\]

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The relations (1.1) and (1.2) form the complete set of equations which were subjected to a systematic Lie-group symmetry analysis in Grebenev et al. (2017). By introducing the following vector of independent variables (Eq. [5])

\[ (y^0, y) = (t, y) = (t, x, \omega, x', \omega') = (t, x^1, x^2, \omega, x'^1, x'^2, \omega'), \]  

(1.3)

and by using this \( y \) notation interchangeably with the original \( x \) notation, this governing system of equations (1.1)-(1.2) can be equivalently rewritten as (Eqs. [6-7])

\[
E_1: \quad \frac{\partial J^0}{\partial y^0} + \frac{\partial J^1}{\partial y^1} + \frac{\partial J^2}{\partial y^2} = 0, \\
E_2: \quad J^1 + \frac{1}{2\pi} \int d^2x' d\omega' \omega' \frac{x^2 - x'^2}{|x - x'|^2} f_2 = 0, \\
E_3: \quad J^2 - \frac{1}{2\pi} \int d^2x' d\omega' \omega' \frac{x^1 - x'^1}{|x - x'|^2} f_2 = 0, \\
E_4: \quad 1 - \int d\omega f_1 = 0, \\
E_5: \quad f_1 - \int d\omega f_2 = 0, 
\]  

(1.4)-(1.8)

where \( J^0 := f_1 \). The Lie-group symmetry analysis for the above system was performed successively in Grebenev et al. (2017), first for equation \( E_1 \), then by including \( E_2 \) and \( E_3 \) into the analysis, to then finally restrict the obtained symmetry result by \( E_4 \) and \( E_5 \). To note is that only the first step, i.e. the symmetry analysis for \( E_1 \) was discussed generally, while all subsequent steps were performed under a specific Lie-point symmetry ansatz to explicitly bring forward a local conformal invariance for this system.

In infinitesimal form, the most general Lie-point symmetry admitted by \( E_1 \) (1.4), being itself an equation in continuity form with four independent and three dependent variables, is given as (Eqs. [27-30])

\[
X = \xi^0(t, x, \omega) \frac{\partial}{\partial t} + \xi^1(t, x, \omega) \frac{\partial}{\partial x} + \xi^2(t, x, \omega) \frac{\partial}{\partial x^2} + \xi^3(\omega) \frac{\partial}{\partial \omega} \\
+ \eta^0(t, x, \omega, \mathbf{J}) \frac{\partial}{\partial J^0} + \eta^1(t, x, \omega, \mathbf{J}) \frac{\partial}{\partial J^1} + \eta^2(t, x, \omega, \mathbf{J}) \frac{\partial}{\partial J^2},
\]  

(1.9)

with

\[
\eta^i(t, x, \omega, \mathbf{J}) = a^i_k(t, x, \omega) J^k + b^i(t, x, \omega), \quad i, k = 0, 1, 2,
\]  

(1.10)

where the coefficients \( a^i_k \) have the specified form

\[
a^i_k = \xi^i_k - \delta^i_k (\xi^0_0 + \xi^1_1 + \xi^2_2 + C(\omega)), \quad \xi^i_k := \frac{\partial \xi^i}{\partial y^k},
\]  

(1.11)

and where the \( b^i \) are arbitrary solutions of \( E_1 \) (1.4). Note that while the three infinitesimals \( \xi^0, \xi^1 \) and \( \xi^2 \) are arbitrary (1-point) space-time functions, the generating infinitesimal for the vorticity \( \xi^3 \), however, is independent of space and time; it is an arbitrary function only of its own defining variable \( y^3 = \omega \). This result stems from the fact that the variable \( y^3 = \omega \) is not an active part of equation \( E_1 \) (1.4) with the effect then that a symmetry analysis identifies it as a hidden parameter that can only be arbitrarily re-parametrized. In (1.11), the function \( C \) is also only a function of \( \omega \) not depending on space and time. The above result (1.9)-(1.11) has been independently validated by using the computer algebra package DESOLV-II of Vu et al. (2012),
matching the result given in Grebenev et al. (2017) by Eqs. [27-30], up to the minor misprint† in the dependencies of the infinitesimals (instead of \( y \) only \( x, \omega \)) and the missing constraint \( \xi_0^0 = 0 \) in Eq. [30].

Based on this result (1.9)-(1.11) for \( E_1 \), the second step in Grebenev et al. (2017) includes the equations \( E_2 \) (1.5) and \( E_3 \) (1.6) into the symmetry analysis, however, not generally, but rather with the following specifically chosen ansatz for the infinitesimals (Eqs. [A.35-A.38])‡

\[
\begin{align*}
\xi^0 &= 0, \\
\xi^1 &= c^{11}(x)x^1 + c^{12}(x)x^2 + d^1(x), \\
\xi^2 &= c^{21}(x)x^1 + c^{22}(x)x^2 + d^2(x), \\
\xi^4 &= c^{11}(x)x^1 + c^{12}(x)x^2 + d^4(x), \\
\xi^5 &= c^{21}(x)x^1 + c^{22}(x)x^2 + d^2(x),
\end{align*}
\]

along with the constraints (Eq. [A.40] leading to [A.41])

\[
c^{22}(x) = c^{11}(x) \quad \text{and} \quad c^{21}(x) = -c^{12}(x),
\]

which, in overall function, already represents the structure of a local conformal invariance for the combined system \( E_1-E_3 \). Note that according to notation (1.3), the functions \( \xi^4 \) (1.15) and \( \xi^5 \) (1.16) represent the infinitesimals for the independent variables \( y^1 = x^1 \) and \( y^5 = x^2 \), respectively.

With the ansatz (1.12)-(1.17) and the result (1.9)-(1.11) for \( E_1 \), a combined symmetry analysis for \( E_1-E_3 \) inevitably leads to the relations (Eqs. [40-45])

\[
\begin{align*}
d^1(x) &= 2c^{11}(x) - c^{11}(x)x^1 - c^{12}(x)x^2, \\
d^2(x) &= -c^{11}(x)x^1 - c^{12}(x)x^2, \\
d^1(x) &= c^{12}(x)x^1 - c^{11}(x)x^2, \\
d^2(x) &= 2c^{11}(x) + c^{22}(x)x^1 - c^{11}(x)x^2,
\end{align*}
\]

with

\[
3c^{11} = -c^{12}, \quad 3c^{12} = c^{11}, \quad \text{and hence:} \quad c^{11} + c^{12} = 0, \quad c^{12} + c^{22} = 0,
\]

and the further results (Eq. [39] and Eq. [52])‡‡

\[
\begin{align*}
\xi^6 &= 2c^{11}(x)\omega', \\
\eta_{f_2} &= -\left(8c^{11}(x) + C(\omega)\right)f_2 + b'(t, y),
\end{align*}
\]

where \( b' \) is an arbitrary solution to the equations \( E_2 \) and \( E_3 \) in correspondence to the two arbitrary solutions \( b^1 \) and \( b^2 \) given in (1.10) for \( E_1 \), i.e.,

\[
\begin{align*}
b^1(t, x, \omega) + \frac{1}{2\pi} \int d^2x' d\omega' \omega' \frac{x^2 - x'^2}{|x - x'|^2} b'(t, y) &= 0, \\
b^2(t, x, \omega) - \frac{1}{2\pi} \int d^2x' d\omega' \omega' \frac{x^1 - x'^1}{|x - x'|^2} b'(t, y) &= 0.
\end{align*}
\]

†If claimed not to be a misprint, then it is definitely a mistake in Grebenev et al. (2017) to denote the dependencies of the infinitesimals in Eqs. [27-30] with \( y \) instead of \( (x, \omega) \). The reason is that if the \( J^i \) are formally identified as functions of \( y \), then equation \( E_1 \) (1.4) has to be augmented by the 9 constraints \( J^0 = 0, \) for \( k = 4, 5, 6 \), to indicate and to provide the relevant information that all \( J^i \) are 1-point and not 2-point functions.

‡Note that the infinitesimal for the time variable \( \xi^0 \) in Grebenev et al. (2017) has been ultimately put to zero, not during the symmetry analysis itself, which was explicitly performed in Appendix A, but later when discussing the result in Section 3 on p. 8; see Eq. [33]. Hence, for convenience, \( \xi^0 \) is considered herein throughout as zero.

‡‡Note that \( C_1 + C_2 \) in Grebenev et al. (2017) corresponds exactly to \( C(\omega) \) in (1.11); see p. 16 where “the constant \( C \) [in Eq. 29 or A.4] was presented as a sum of the two constants \( C = C_1 + C_2 \).”
To note is that the dependency structure of the infinitesimals \( \xi^3, \eta^0, \eta^1 \) and \( \eta^2 \) in (1.9)-(1.10) stays unchanged after this analysis, i.e., augmenting the symmetry analysis for \( \mathbf{E}_1 \) by including \( \mathbf{E}_2 \) and \( \mathbf{E}_3 \) does not restrict these infinitesimals any further; they simply are not effected and thus remain unchanged by this extension. Employing the specific ansatz (1.12)-(1.17) and the result (1.18)-(1.21), the latter three infinitesimals can at least be explicitly written out as (Eq. [51])

\[
\begin{align*}
\eta^0 &= -\left(6c^{11}(x) + C(\omega)\right)f_1 + b^0(t, x, \omega), \\
\eta^1 &= -\left(3c^{11}(x) + C(\omega)\right)J^1 + c^{12}(x)J^2 + b^1(t, x, \omega), \\
\eta^2 &= -c^{12}(x)J^1 - \left(3c^{11}(x) + C(\omega)\right)J^2 + b^2(t, x, \omega).
\end{align*}
\]

(1.26)

Now, by including also the last two remaining equations into the symmetry analysis, namely the two consistency conditions \( \mathbf{E}_4 \) (1.7) and \( \mathbf{E}_5 \) (1.8), further restrictions for the infinitesimals can be expected. Including first the latter equation \( \mathbf{E}_5 \), we just obtain the trivial restriction (Eq. [53])

\[
b^0(t, x, \omega) = \int d\omega b^0(t, y),
\]

(1.27)

meaning that equation \( \mathbf{E}_5 \) (1.8) already transforms as an invariant under the generating transformations (1.12)-(1.26). In other words, when taking along the constraint (1.27), equation \( \mathbf{E}_5 \) is fully compatible to the already determined symmetries of subsystem \( \mathbf{E}_1-\mathbf{E}_3 \); no symmetries are broken when augmenting this system by \( \mathbf{E}_5 \).

When including equation \( \mathbf{E}_4 \) (1.7), however, the situation is different: Besides the trivial restriction (Eq. [53])\footnote{Note that the explicit form of the restrictions (1.28) and (1.29) can also be represented differently, for example, when splitting the arbitrary function \( C(\omega) \) additively into two separate ones \( C(\omega) = C_1(\omega) + C_2(\omega) \), as has been done in Grebenev et al. (2017). For instance, if \( C_2 \) is linked to \( b^0 \) and \( C_1 \) to \( \xi_3 \), then (1.28) and (1.29) can also be equivalently written as \( \int d\omega b^0(t, x, \omega) = \int d\omega C_2(\omega) \) and \( \xi_3 = 6c^{11}(x) + C_1(\omega) \), respectively.}

\[
\int d\omega b^0(t, x, \omega) = 0,
\]

(1.28)

we also obtain the crucial restriction\footnote{Restriction (1.29) guarantees that the for the combined system \( \mathbf{E}_1-\mathbf{E}_5 \) determined symmetry transformation is universally valid for all possible solutions of the 1-point PDF \( f_1 \), which essentially is also the purpose of every symmetry analysis: To find transformations that leave equations invariant independent of the particular structure they may give as solutions for the dependent variables. See the next section for the derivation of (1.29) and for a more detailed discussion on that issue.}

\[
\xi_3^2 = 6c^{11}(x) + C(\omega),
\]

(1.29)

which forces the function \( c^{11} \) to be a constant now not depending on the spatial coordinate \( x \), simply because the left-hand side \( \xi_3^2 \) is according to the result \( \xi^3 = \xi^3(\omega) \), which itself was obtained in (1.9), only a function of \( \omega \). Hence, since this result is globally valid for all \( \omega \in \mathbb{R} \), including the case \( \omega = 0 \), the above restriction (1.29) is equivalent to the combined restriction\footnote{Note that according to result (1.22), the constraint (1.30) also restricts its dual function \( c^{12}(x) \) to be a constant: \( c^{12}(x) = 0 \) and \( c^{12}(x) = 0 \).}

\[
c^{11}(x) = 0 \quad \text{and} \quad c^{11}(x) = 0, \quad \forall \omega \in \mathbb{R},
\]

(1.30)

which contradicts the ansatz made for \( \xi_3^2 \) (Eq. [38]) in Grebenev et al. (2017), where also for the particular case of zero vorticity \( \omega = 0 \) this function is prescribed to be non constant: \( c^{11}(x) \neq 0 \) and \( c^{11}(x) \neq 0 \).

Hence, due to the constraint (1.29), or equivalently due to (1.30), the local conformal invariance for \( \mathbf{E}_1-\mathbf{E}_5 \) (1.4)-(1.8), which itself as a combined system represents the first equation in the infinite and unclosed hierarchy of LMN vorticity equations, cannot be confirmed as claimed in Grebenev et al. (2017), also not for the particular case \( \omega = 0 \). In the next section, this wrong and thus misleading conclusion in Grebenev et al. (2017) will be examined more closely and also be viewed from different perspectives.
2. Revealing and correcting the mistake in Grebenev et al. (2017)

2.1. First perspective

When performing a Lie-group symmetry analysis on the governing equation \( E_1 \) (1.4), it provides us with two strong results: (i) The infinitesimal \( \xi^3 \) (1.9) is only a function of its defining vorticity variable: \( \xi^3 = \xi^3(\omega) \), and (ii) the infinitesimals \( \eta^i \) (1.10) for the dependent variables can be supplemented by a function \( C(1.11) \), depending also only on the vorticity variable: \( C = C(\omega) \).

In particular, when augmenting the symmetry analysis by also including the remaining equations \( E_2-E_5 \), this twofold result from \( E_1 \) is of global nature, meaning that \( \xi^3 \) and \( C \) should and may not depend on the spatial variable \( x \) for all \( \omega \in \mathbb{R} \), including also the zero vorticity case \( \omega = 0 \).

In Grebenev et al. (2017), however, the following unexplained and misleading ansatz for \( \xi^3 \) is made (Eq. [38]):

\[
\xi^3 = \left( 6c^{11}(x) + C_1 \right) \omega, \quad \text{for} \quad c^{11}_1(x) \neq 0 \quad \text{and} \quad c^{11}_2(x) \neq 0, \tag{2.1}
\]

which obviously, as explained above, is not compatible with the symmetry result as stipulated by the governing equation \( E_1 \) (1.4). The argument in Grebenev et al. (2017), however, is that for the specific case \( \omega = 0 \) this conflict is resolved. Although this argument itself is correct, since a zero infinitesimal \( \xi^3 = 0 \) is inherently independent of any variables whatsoever, they overlooked the fact that their local condition, which just only holds for the single value \( \omega = 0 \), cannot be employed to transform the non-local constraint equation \( E_4 \) (1.7).\(^1\) The reason is that \( E_4 \) is a global relation that sums over all vorticity values \( \omega \in \mathbb{R} \), and not only locally for \( \omega = 0 \).

To therefore correctly transform this global constraint \( E_4 \)

\[
\int d\omega f_1 = 1, \tag{2.2}
\]

where \( \omega \) needs to be varied by \( d\omega \) over the whole (infinite) integration range, one has to use a transformation rule for \( \omega \) that is globally valid for all values, and not only for the fixed value \( \omega = 0 \). Hence, to invariantly transform (2.2) in line with \( E_1 \), the transformation rule (2.1) is not the correct choice, since it is only valid for \( \omega = 0 \) and thus not being variable — as already said, for \( \omega \neq 0 \) the rule (2.1) has to be discarded, simply because it breaks the symmetry condition \( \xi^3 = \xi^3 = 0 \), that means the condition \( \xi^3 = \xi^3(\omega) \) for \( E_1 \) to be invariant.

It is clear that only the following globally valid ansatz (up to order \( \mathcal{O}(\epsilon^2) \) in the group parameter \( \epsilon \))

\[
\bar{\omega} = \omega + \epsilon \cdot \xi^3(\omega), \quad \forall \omega \in \mathbb{R}, \tag{2.3}
\]

will invariantly transform the global constraint \( E_4 \) (2.2) in line with \( E_1 \). The associated infinitesimal constraint that will be induced as result then has the form

\[
0 = 1 - \int d\bar{\omega} \, \bar{f}_1 = 1 - \int d\omega \left( \frac{\partial f_1}{\partial \omega} \right) = (f_1 + \epsilon \eta^0 + \mathcal{O}((\epsilon^2)^0)) = 1 - \int d\omega \left( 1 + \epsilon \xi^3 \right) (f_1 + \epsilon \eta^0) + \mathcal{O}(\epsilon^2) \tag{2.3}
\]

\[
= 1 - \int \omega (1 + \epsilon \xi^3) (f_1 + \epsilon \eta^0) + \mathcal{O}(\epsilon^2) = 1 - \int d\omega (f_1 + \epsilon (\eta^0 + \xi^3 f_1)) + \mathcal{O}(\epsilon^2) \tag{2.3}
\]

\[
= 1 - \int d\omega (f_1 + \epsilon (-6c^{11} f_1 - C f_1 + b^0 + \xi^3 f_1)) + \mathcal{O}(\epsilon^2) \tag{2.26}
\]

\[
= (1.28) \& (2.2) \int d\omega (6c^{11} + C - \xi^3) f_1 + \mathcal{O}(\epsilon), \tag{2.4}
\]

\(^1\)The same mistake has also been made in Sec. 3.3 in Grebenev et al. (2017), which, if corrected, invalidates their claim that the probability measure \( \mu(t, x, \omega) = f_1(t, x, \omega) d\omega \) is local-conformally invariant. On the one side their mistake is that Eq. [38] for \( \omega \neq 0 \) may not be used to transform \( \mu \) since it is inconsistent to the symmetry transform of the governing Eq. [6], and on the other side their mistake is that Eq. [38] for \( \omega = 0 \) cannot be used to transform \( \mu \) since \( \omega \) is then rigidly fixed and thus not variable anymore.
which, if we seek for a symmetry transformation that is valid for all possible solutions \( f_1 \), is equivalent to the constraint (1.29)

\[
6c^{11}(x) + C(\omega) - \xi^3(\omega) = 0,
\]

that now forces \( c^{11} \) to be a constant,\(^1\) as already discussed before in the previous section due to that \( \xi^3 \), according to the rule (2.3), is only a function of \( \omega \) not depending on \( x \).

As exercised in Grebenev et al. (2017), which again refers to Ibragimov et al. (2002), the determining equation (2.5) can also be derived alternatively by noting that the non-local determining symmetry equation (2.4) can be split with respect to group variable \( f_1 \) using variational differentiation. Since the bracketed term in (2.4) does not depend on \( f_1 \), taking the variational or functional derivative \( (\delta/\delta f_1(\omega)) \) of this equation then leads to the same local result (2.5):

\[
0 = \frac{\delta}{\delta f_1(\omega)} \int d\omega(6c^{11} + C - \xi^3) f_1
\]

\[
= \int d\omega(6c^{11} + C - \xi^3)(\omega - \bar{\omega}) = 6c^{11}(x) + C|_{\omega=\bar{\omega}} - \xi^3|_{\omega=\bar{\omega}},
\]

(2.6)

Note here that it is valid to take the variational derivative of equation (2.4) since \( f_1 \) can be \textit{continuously} varied to still satisfy equation (2.4) by just choosing the non-constant coefficient or pre-factor of \( f_1 \) appropriately, trivially of course as (2.5). In contrast of course to the defining equation (2.2) itself, which determines or fixes \( f_1 \) and which thus cannot be continuously varied:

An arbitrary non-zero functional variation of \( f_1 \) will violate the constraint (2.2), as can be clearly seen by taking the variational derivative \( (\delta/\delta f_1(\omega)) \) of this constraint

\[
0 = \frac{\delta}{\delta f_1(\omega)}(-1 + \int d\omega f_1) = \int d\omega(\omega - \bar{\omega}) = 1,
\]

(2.7)

turning the constraint (2.2) thus into the contradiction \( 1 = 0 \). This conflict just tells us that equation (2.2) cannot be functionally varied simply because it defines and determines the function \( f_1 \), similar as in the usual variation for real numbers if we would fix a variable to a certain value, say \( x = 1 \), then any variation on it would be meaningless, since \( x \) is defined or determined strictly as 1. Evidently, taking the variation of \( x = 1 \) leads to the same conflict

\[
0 = \frac{\partial}{\partial x}(-1 + x) = 1,
\]

(2.8)
as in (2.7) for the functional variation of the defining and determining equation (2.2) for \( f_1 \).

Coming back to the general result (2.5) by choosing \( \xi^3 = 0 \) as it would be the case in Grebenev et al. (2017) when applying in Eq. [38] their necessary zero-vorticity constraint \( \omega = 0 \) (Eq. [32]), we note that their results for the infinitesimals \( y^0 \) and \( y^1 \) as given by Eqs. [51-52] are incorrect. Instead of an unrestricted \( C = C_1 + C_2 \), the restriction \( C = -6c^{11} \) has to be used in their results in order to be consistent with \( \xi^3 = 0 \), exactly as it is required by (2.5). Hence, for \( \xi^3 = 0 \), the correct generating infinitesimals for \( f_1 \) and \( f_2 \) are given by (1.26) \& (1.24)

\[
\eta_{f_1} \equiv y^0 = b^0(t, x, \omega), \quad \eta_{f_2} \equiv y^1 = b^{11}f_2 + b^1(t, y), \quad \forall \omega \in \mathbb{R}, \quad c_1^{11} = c_2^{11} = 0,
\]

(2.9)

and not by Eqs. [51-52] as proposed in Grebenev et al. (2017). To note is that the above result (2.9) is globally valid for all vorticity values \( \omega \), including the case \( \omega = 0 \). Regarding Sec. 3.3 in Grebenev et al. (2017), it is clear that for the above choice \( \xi^3 = 0, \forall \omega \in \mathbb{R} \), and its resulting transform (2.9), the probability measure \( \mu = f_1 d\omega \) (Eq. [68]) remains to be invariant, but not local-conformally anymore as claimed since \( c^{11} \), according to (2.9), is a spatial constant now.

\(^1\)Another independent but equivalent argument that \( c^{11} \) needs to be a constant is to recognize that the non-local determining equation (2.4) is living in a jet space where \( x \) and \( f_1 \) are jet coordinates defined by the underlying symmetry analysis of the governing system (1.1)-(1.2), i.e., a particular designed space where \( x \) and \( f_1 \) are defined as independent coordinates: \( \partial_s f_1 = \partial_x f_1 = 0 \). Then, in writing (2.4) out as \( 6c^{11} + \int d\omega(C - \xi^3_1) f_1 = 0 \) (since \( c^{11} \) by construction is independent of \( \omega \)) and by taking the spatial derivatives on both sides, one directly obtains the overall consistent result \( c_1^{11} = c_2^{11} = 0 \), simply due to that \( C \) and \( \xi^3 \) on the one side are independent functions of \( x \) and on the other that \( f_1 \) is a jet variable with respect to \( x \). — On jet spaces in general, see e.g. Olver (1993).
2.2. Second perspective

It is clear that when analyzing any system of equations on symmetries, as for example for the case $E_4$-$E_5$ (1.4)-(1.8) considered herein, the symmetry result should not depend on the choice which subsystem is considered first and in which order it is being evaluated, i.e., no matter which direction in evaluation one takes, a symmetry analysis should always give exactly the same result, otherwise a consistent analysis is not guaranteed. For example, let us first consider the following approach: Before starting any analysis, we already specify the coordinate $\omega$ of $(J^{i}, f_2)$ by their corresponding functions $(\eta^{i}, \eta_2)$ in (2.12). The initial system $E_4$ is related to $J^{i}$ and $f_2$, $f_1$ from (2.15). Formally, the four equations $E_1^*$-$E_3^*$ & $E_3^*$ can now be identified as a system being independent of the jet coordinate $f_2$. Performing a symmetry analysis on this reduced subsystem for the ansatz (1.12)-(1.17), one yields the same results (1.18)-(1.27) as before in just replacing the $\omega$-dependent infinitesimals $(\eta^{i}, \eta_2)$ in (2.15) and (1.24) by their corresponding $\omega$-independent infinitesimals $(\eta^{*, i}, \eta_2^*)$:

\[
\begin{align*}
\eta^{*, 0} &= -(6c^{11}(x) + C^{*}) f_1^* = \eta_1^*, \\
\eta^{*, 1} &= -(3c^{11}(x) + C^{*}) J^{*1} + c^{12}(x) J^{*2}, \\
\eta^{*, 2} &= -c^{12}(x) J^{*1} - (3c^{11}(x) + C^{*}) J^{*2}, \\
\eta_2^* &= -(8c^{11}(x) + C^{*}) f_2^*, \\
\end{align*}
\]

where $C^{*}$ is any arbitrary constant and where, for simplicity and convenience, the solutions $b^*$ and $b^{*i}$ were chosen as the trivial zero solutions, simply because of not being relevant here for the present discussion on consistency. Now, in using the defining relation (2.15), we can read off the unreduced infinitesimal $\eta^{0}$ for the jet coordinate $f_1$ from (2.16) as

\[
\eta^{0} = -(6c^{11}(x) + C(\omega)) f_1 \equiv \eta f_1, \tag{2.17}
\]

which is necessary now in order to find the infinitesimal $\xi^{3} \equiv \xi_{\omega}$ from the last and remaining equation $E_4$ (2.13) in this system. Obviously, since (2.17) matches the result (2.16), the symmetry

$^1$C* in (2.16) is then defined as the evaluation $C(\omega)|_{\omega=\omega^*} \equiv C^{*}$. 

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analysis of $E_4$ (2.13) is identical to the one performed in the previous section (2.4), with the same result (2.5), however, now without any constraints on the infinitesimal $\xi^3$:

$$6c^{11}(x) + C(\omega) - \xi_3^3(t, x, \omega) = 0. \quad (2.18)$$

Solving (2.18) explicitly for $\xi^3$, the only solution that is in accordance or in line with subsystem $E_1^* - E_3^*$ & $E_5^*$ is given by the particular solution

$$\xi_3^3(t, x, \omega) = \left(6c^{11}(x) + C(\omega)\right)\omega - \omega^* - \int_{\omega^*}^{\omega} \frac{dC(\tilde{\omega})}{d\tilde{\omega}}(\tilde{\omega} - \omega^*) d\tilde{\omega}, \quad (2.19)$$

since for $\omega = \omega^*$ this infinitesimal turns zero $\xi_3^3|_{\omega=\omega^*} = 0$, with the effect then that under this transformation the value $\omega = \omega^*$ gets mapped to the same value again: $\omega^* = \omega \mapsto \tilde{\omega} = \omega^*$, and therefore keeping subsystem $E_1^* - E_3^*$ & $E_5^*$ thus invariant.

Hence, it seems that the analysis of Grebenev et al. (2017) just got generalized to arbitrary vorticity isolines, since the result (2.19) is not restricted to the particular value of a zero-vorticity isoline $\omega^* = 0$, as in Eq. [38] in Grebenev et al. (2017). Hence we could say that we have shown local conformal invariance for the 2D LMN vorticity equations (up to second order in the LMN chain of equations) on all its vorticity isolines, that is, for all values of $\omega \in \mathbb{R}$. But, unfortunately, that is not the case. Because, when looking again at the determining equation (2.18) when evaluated at $\omega = \omega^*$,

$$\xi_3^3|_{\omega=\omega^*} = 6c^{11}(x) + C(\omega^*), \quad \text{where } \xi_{31}^3 \neq 0, \xi_{32}^3 \neq 0, \quad (2.20)$$

which in this section was obtained by first putting $\omega$ to a fixed value $\omega^*$ and then by performing a symmetry analysis, this equation (2.20) only constitutes an overall consistent equation if the same result is also obtained when reversing this procedure: First by performing a symmetry analysis and then by specifying $\omega = \omega^*$. For this reverse direction, however, the governing equations are $E_1^* - E_5^*$ (1.4)-(1.8), for which the corresponding result to (2.20) is then given by (2.5)

$$\xi_3^3|_{\omega=\omega^*} = 6c^{11}(x) + C(\omega^*), \quad \text{where } \xi_{31}^3 = \xi_{32}^3 = 0. \quad (2.21)$$

The decisive difference between these two equations (2.20) and (2.21) is that their left-hand sides show different dependencies: While the left-hand side of (2.20) depends in general on $x$, the left-hand side of (2.21) is strictly independent of $x$. Hence, in order to obtain a consistent result for $\xi_3^3$, the spatial function $c^{11}(x)$ has to be reduced to a constant, i.e., $c_1^{11} = c_2^{11} = 0$; only then can the two equations (2.20) and (2.21) be matched. This completes the second independent proof, demonstrating again that local conformal invariance cannot be confirmed as proclaimed in Grebenev et al. (2017), de facto disproving their claim not only for the zero-vorticity isoline but also for all non-zero ones.

### 2.3. Third perspective

In this approach we assume that we have a solution for the 1-point and 2-point PDF $(f_1, f_2)$, obtained, for example, by a direct numerical simulation (DNS) of the (inviscid) deterministic Navier-Stokes equations for some specific unbounded flow configuration. Obviously, this solution $(f_1, f_2)$ will then satisfy identically its defining PDF equations $E_1^* - E_5^*$ (1.4)-(1.8). The question now is whether this solution remains to be solution of this system $E_1^* - E_5^*$ (1.4)-(1.8) when being transformed on a zero-vorticity isoline according to the local conformal rule as proposed in Grebenev et al. (2017) (Eqs. [33-45,51-52]). The answer is obtained by augmenting the defining system $E_1^* - E_5^*$ by certain integral consequences. For example, it is trivial to conclude that if the

\[\text{Note that since } \xi^3 \text{ is associated to the 1-point quantity } f_1 \text{ there is no dependence on any 2-point coordinates.}\]
existing and available solution \((f_1, f_2)\) satisfies the differential equation \(E_1\) (1.4) along with the constraint \(E_4\) (1.7) identically, then it also satisfies identically its integral consequence

\[
0 = \int d\omega \left( \frac{\partial J^0}{\partial y^0} + \frac{\partial J^1}{\partial y^1} + \frac{\partial J^2}{\partial y^2} \right) = \frac{\partial}{\partial y^0} \int d\omega J^0 + \int d\omega \left( \frac{\partial J^1}{\partial y^1} + \frac{\partial J^2}{\partial y^2} \right) = \frac{\partial}{\partial y^1} M^1 + \frac{\partial}{\partial y^2} M^2, \tag{2.22}
\]

where the \(M^i\) are defined as: \(M^i = \int d\omega J^i, \ i = 1, 2\). Hence, in the following we will consider the following augmented system

\[
\begin{align*}
E_1' &: \frac{\partial J^0}{\partial y^0} + \frac{\partial J^1}{\partial y^1} + \frac{\partial J^2}{\partial y^2} = 0, \tag{2.23} \\
E_{11} &: \frac{\partial M^1}{\partial y^1} + \frac{\partial M^2}{\partial y^2} = 0, \tag{2.24} \\
E_{12} &: M^1 - \int d\omega J^1 = 0, \tag{2.25} \\
E_{13} &: M^2 - \int d\omega J^2 = 0, \tag{2.26} \\
E_2 &: J^1 + \frac{1}{2\pi} \int d^2\mathbf{x}' d\omega' \omega' \frac{x^2 - x'^2}{|\mathbf{x} - \mathbf{x}'|^2} f_2^* = 0, \tag{2.27} \\
E_3 &: J^2 - \frac{1}{2\pi} \int d^2\mathbf{x}' d\omega' \omega' \frac{x^1 - x'^1}{|\mathbf{x} - \mathbf{x}'|^2} f_2^* = 0, \tag{2.28} \\
E_4 &: -1 - \int d\omega f_1 = 0, \tag{2.29} \\
E_5 &: f_1^* - \int d\omega f_2^* = 0, \tag{2.30}
\end{align*}
\]

which consistently extends the initial system of equations \(E_1\)-\(E_5\) (1.4)-(1.8) without changing the associated solution space. Note that we here proceed as in the previous section (viz. the second perspective), where already before the upcoming symmetry analysis the above subsystem \((E_1, E_2, E_3, E_5)\) is formally reduced to the \(\omega\)-independent subsystem \((E_1', E_2', E_3', E_5')\) in that the coordinate \(\omega\) got again specified to some arbitrary but fixed value \(\omega = \omega^*\) where \(\omega^* \in \mathbb{R}\), including thus also again the zero-value choice \(\omega^* = 0\) as in Grebenev et al. (2017).

Now, knowing that a symmetry analysis of this reduced system \((E_1', E_2', E_3, E_5)\) results to (2.16), we can read off again from the underlying consistency relation (2.15) the corresponding unreduced infinitesimals as

\[
\begin{align*}
\eta^0 &= -\left(6 c^{11}(\mathbf{x}) + C(\omega)\right) f_1, \\
\eta^1 &= -\left(3 c^{11}(\mathbf{x}) + C(\omega)\right) J^1 + c^{12}(\mathbf{x}) J^2, \\
\eta^2 &= -c^{12}(\mathbf{x}) J^1 - \left(3 c^{11}(\mathbf{x}) + C(\omega)\right) J^2,
\end{align*} \tag{2.31}
\]

\(^1\text{It is obvious that (2.31) matches again the result (1.26). For simplicity and convenience, the solutions } b' \text{ and } b' \text{ were chosen as the trivial zero solutions, simply because they are not relevant for the discussion to be demonstrated herein.}\)
which are necessary now to determine the infinitesimals $\xi^3$ and $\eta_{M'}$ from the four remaining equations $E_{11} - E_{13}$ (2.24)-(2.26) and $E_4$ (2.29). Demanding their invariance, one obtains the following determining equations for $\eta_{M'}$, $\eta_{M''}$, and $\xi^3$:

$$
\begin{align*}
\frac{\partial \eta_{M'}}{\partial y^1} + \frac{\partial \eta_{M''}}{\partial y^2} &= 0, \\
\frac{\partial \eta_{M'}}{\partial M'} - \frac{\partial \xi^1}{\partial y^1} - \frac{\partial \eta_{M''}}{\partial M''} + \frac{\partial \xi^2}{\partial y^2} &= 0, \\
\frac{\partial \eta_{M'}}{\partial M'} - \frac{\partial \xi^1}{\partial y^1} &= 0, \\
\frac{\partial \eta_{M''}}{\partial M''} - \frac{\partial \xi^2}{\partial y^2} &= 0, \\
\frac{\partial \xi^1}{\partial y^1} + \frac{\partial \xi^2}{\partial y^2} &= 0,
\end{align*}
$$

(2.32)

where (2.32) results from equation $E_{11}$ (2.24), and (2.33) from $E_{12}$ (2.25), $E_{13}$ (2.26) and $E_4$ (2.29), respectively. In line with the already obtained symmetry result (2.31) of subsystem $(E_1^*, E_2^*, E_3^*, E_5^*)$, the last equation in (2.33) induces the already well-known relation (2.18)

$$
\xi^3 = 6c_{11}^1(x) + C(\omega),
$$

(2.34)

which, when inserted along with (2.31) into the two former equations of (2.33), then gives the solution for the infinitesimals $\eta_{M'}$, explicitly as:

$$
\eta_{M'} = \int d\omega(\eta^1 + \xi^3 J^1) = \int d\omega\left(-3c_{11}^1(x) + C(\omega)\right)J^1 + c_{12}^1(x)J^2 + (6c_{11}^1(x) + C(\omega))J^1 = \int d\omega\left(3c_{11}^1(x)J^1 + c_{12}^1(x)J^2\right),
$$

(2.25) & (2.26)

$$
\eta_{M''} = \int d\omega(\eta^2 + \xi^3 J^2) = \int d\omega\left(-c_{12}^1(x)J^1 - 3c_{11}^1(x) + C(\omega)\right)J^2 + (6c_{11}^1(x) + C(\omega))J^2 = \int d\omega\left(-c_{12}^1(x)J^1 + 3c_{11}^1(x)J^2\right),
$$

(2.25) & (2.26)

However, this result is inconsistent to the determining equations given by (2.32), as can be seen by evaluating already the first equation

$$
0 = \frac{\partial \eta_{M'}}{\partial y^1} + \frac{\partial \eta_{M''}}{\partial y^2} = 3c_{11}^1M^1 + c_{12}^1M^2 - c_{12}^2M^1 + 3c_{12}^1M^2 = 6c_{11}^1M^1 + 6c_{12}^1M^2 \neq 0.
$$

(1.22)

The above relation can only be made consistent if the spatial function $c_{11}^1(x)$ is reduced to a global constant, i.e., if $c_{11}^1 = c_{12}^1 = 0$, thus eventually breaking the local conformal invariance of the subsystem $(E_1^*, E_2^*, E_3^*, E_5^*)$, where this breaking occurs not only for the specific value $\omega^* = 0$, but for all real values $\omega^* \in \mathbb{R}$.

This result finally also answers our question stated in the beginning, namely whether the local conformal transformation as proposed in Grebenev et al. (2017) will map a given solution on a zero-vorticity isoline to a new solution. The answer is clearly no,

1In fact, it is only the first equation in (2.32) that is inconsistent. The four other equations evaluate identically to zero for the considered ansatz (1.12)-(1.22).

2In particular, if the set of functions $M' = \int d\omega J'$ is a solution, then the local-conformally transformed set $\tilde{M}' = \int d\omega J'$ by Grebenev et al. (2017) is not a solution anymore, since the governing equation $E_{11}$ (2.24) does not stay invariant under this transformation; only for the reduced case $c_{11}^1 = c_{12}^1 = 0$ it will stay invariant.

3In particular, if the set of functions $M' = \int d\omega J'$ is a solution, then the local-conformally transformed set $\tilde{M}' = \int d\omega J'$ by Grebenev et al. (2017) is not a solution anymore, since the governing equation $E_{11}$ (2.24) does not stay invariant under this transformation; only for the reduced case $c_{11}^1 = c_{12}^1 = 0$ it will stay invariant. 
2.4. Final remarks

Important to note in this overall discussion is that the invariant transformation examined in this investigation (1.12)-(1.30)\(^1\) is only an equivalence and not a true symmetry transformation, simply due to that we are dealing here with an unclosed system of equations (1.1)-(1.2) where the dynamical rule for the 2-point PDF \(f_2\) is not known beforehand. In contrast to a true symmetry transformation, which maps a solution of a specific (closed) equation to a new solution of the same equation, an equivalence transform acts in a weaker sense in that it only maps an (unclosed) equation to a new (unclosed) equation of the same class.\(^2\) Of course, it is trivial and goes without saying that if once a real solution for \(f_2\) is known, then the equivalence (1.24) turns into a symmetry transformation and \(f_2\) gets mapped to a new solution \(\tilde{f}_2 = f_2 + \epsilon \cdot \eta f_2 + \mathcal{O}(\epsilon^2)\). But since this is not the case here, any invariant transformation of (1.12)-(1.30) will thus at this stage only map between equations and not between solutions, where \(f_2\) then is the unknown source or sink term, or collectively the unknown constitutive law of these equations.

Hence, for the global invariant scaling determined herein \((c_1^{11} = c_2^{11} = 0)\) we cannot expect any information about the inner solution structure of the 1-point PDF equation as long as the dynamical equation for the 2-point PDF \(f_2\) is not modeled. Without empirical modeling it is clear that the closure problem of turbulence cannot be circumvented by just employing the method of a Lie-group symmetry analysis. For more details on this issue, see e.g. Frewer et al. (2014a) and the references therein.

Finally, to end this comment, it should not go unmentioned that Grebenev et al. (2017) is not the first article from the group of Oberlack et al. dealing with symmetries and the LMN equations which is flawed. The previously published comments by Frewer et al. (2014b, 2015a,b, 2016, 2017) and Frewer (2016) clearly prove this. Nor should it be ignored that the present flawed result of Grebenev et al. (2017) forms a basic building block of a recently granted 3-year DFG project (Gepris, No. 385665358). A detailed critical discussion of this project is given in ResearchGate. In this regard please also visit https://zenodo.org/communities/turbsym/.

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\(^1\)Note that only the reduced case \(c_1^{11} = c_2^{11} = 0\) constitutes an invariant transformation.

\(^2\)Equivalence transformations can be successfully applied for example to classify unclosed differential equations according to the number of symmetries they admit when specifying the unclosed terms (see e.g. Meleshko (2002); Khabirov & ¨Unal (2002a,b); Chirkunov (2012); Meleshko & Moyo (2015); Bihlo & Popovych (2017)). A typical task in this context sometimes is to find a specification of the unclosed terms such that the maximal symmetry algebra is gained. Once the equation is closed by such a group classification, invariant solutions can be determined. But in how far these equations and their solutions are physically relevant and whether they can be matched to empirical data is not clarified a priori by this approach, in particular if such a pure Lie-group-based type of modelling is performed fully detached from empirical research. In this regard, special attention has to be given to the unclosed statistical equations of turbulence as considered herein, since the unclosed 2-point PDF in (1.1)-(1.2) is only an analytical and theoretical unknown, but not an empirical one since it is fully determined by the underlying deterministic Navier-Stokes equations, which again are well-known for to break statistical symmetries in turbulence within intermittent events (see e.g. Frisch (1995)). Hence extra caution has to be exercised when employing a pure symmetry-based modeling to turbulence.
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