EIGENSYSTEM OF AN $L^2$-PERTURBED HARMONIC OSCILLATOR IS AN UNCONDITIONAL BASIS

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Abstract. We prove the following. For any complex valued $L^p$-function
$b(x)$, $2 \leq p < \infty$ or $L^\infty$-function with the norm $\|b\|_{L^\infty} < 1$, the spec-
trum of a perturbed harmonic oscillator operator $L = -d^2/dx^2 + x^2 +
b(x)$ in $L^2(\mathbb{R})$ is discrete and eventually simple. Its SEAF (system of
eigen- and associated functions) is an unconditional basis in $L^2(\mathbb{R})$.

1. Introduction

In this paper we consider the perturbed operator

(1) $L = L^0 + B$

where

$L^0 := -d^2/dx^2 + x^2$

is a harmonic oscillator and $B$ is multiplication by a complex-valued function,

$B\phi(x) = b(x)\phi(x)$

or maybe a more general linear operator. The spectrum of $L^0$ is the set

$\text{Sp}(L^0) = \{\lambda_k^0 = 2k + 1 : k \in \mathbb{Z}_+ = 0, 1, 2, \ldots\}$.

The corresponding eigenfunctions are the Hermite functions $h_k(x), k \in \mathbb{Z}_+$
(see for example [15, Ch. 6, Sect 34]), [27, Ch. 5, Sect. 4]; $\|h_k\|_2 = 1$.

Define the Banach space

(2) $V = \{\phi \in L^2_{\text{loc}}(\mathbb{R}) : \|\phi h_k\|_2 < \infty \quad \forall k \in \mathbb{Z}_+ \quad \text{and} \quad \lim_{k \to \infty} \|\phi h_k\|_2 = 0\}$,

with the norm

(3) $\|\phi\| = \sup\{\|\phi h_k\|_2\}$.

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form, unconditional basis.
Our proofs in this paper use in an essential way the condition $b \in V$. In Section 5 we will use known estimates for Hermite functions to prove that the following spaces are embedded in $V$:

\[ L(p, \alpha) = \{ \phi : (1 + |x|^2)^{\alpha/2}|\phi(x)| \in L^p(\mathbb{R}) \}, \quad \alpha/2 + t(p) \leq 0 \]

where $t$ is defined in (54) and

\[ L_0^\infty(\mathbb{R}) = \{ \phi \in L^\infty(\mathbb{R}) : \text{ess sup}_{|s| \geq t} |\phi(s)| \to 0 \quad \text{as} \quad t \to \infty \}. \]

We can claim as a special case of (4) that the spaces $L^p(\mathbb{R})$, $2 \leq p < \infty$ are embedded in $V$ - see Lemma 8 and Proposition 9.

We will now state our main results. Their proofs are given in Sections 3 and 4. Put

\[ \Pi(a, b) := \{ x + iy \in \mathbb{C} : |x| < a, |y| < b \}, \]

\[ D(p, r) := \{ z \in \mathbb{C} : |z - p| < r \} \]

and

\[ S(n) := \Pi(2n, Y) \cup \left( \bigcup_{k=n}^\infty D(\lambda^0_k, 1/16) \right) \]

with

\[ Y = 8 \left( \|b\| + 2\pi \|b\|^2 \right). \]

For $z \notin \text{Sp}(L)$ (resp. $z \notin \text{Sp}(L^0)$) put $R(z) := (z - L)^{-1}$ (resp. $R^0(z) := (z - L^0)^{-1}$).

**Proposition 1.** Suppose $b \in V$. The spectrum of $L$ is discrete and there exists $N_* \in \mathbb{N}$ such that

\[ \text{Sp}(L) \subset S := S(N_*) \]

and for each $n \geq N_*$ the disk $D(\lambda^0_n, 1/16)$ contains exactly one eigenvalue $\lambda_n$ of $L$.

Put

\[ S_* := \frac{1}{2\pi i} \int_{\partial \Pi(2N_*, Y)} R(z)dz, \]

\[ P^0_k := \frac{1}{2\pi i} \int_{\partial D(\lambda^0_k, 1)} R^0(z)dz, \quad \text{for} \ k \in \mathbb{Z}_+ \]

and

\[ P_k := \frac{1}{2\pi i} \int_{\partial D(\lambda^0_k, 1)} R(z)dz, \quad \text{for} \ k \geq N_. \]
By Proposition 1 all the integrals in (9-11) are well defined. Of course, $P_k^0 f = \langle f, h_k \rangle h_k \forall k$ and $\dim P_k^0 = 1$.

**Proposition 2.** The constant $N_s$ from Proposition 1 can be chosen in such a way that

$$ \dim(P_k) = \dim(P_k^0) = 1 \forall k \geq N_s \tag{12} $$

and

$$ \dim(S_s) = \dim(S_s^0) = N_s \text{ where } S_s^0 = \sum_{j=0}^{N_s-1} P_j^0. \tag{13} $$

Also,

$$ \|R(z)\| \leq 32 \forall z \notin S \in \mathbb{R}, \tag{14} $$

$$ \|P_n\| \leq 32, \tag{15} $$

and

$$ \left\| \frac{1}{2\pi i} \int_{|z-z_0|=1} \frac{R(z)}{z-z_n} dz \right\| \leq 35 \tag{16} $$

whenever $n \geq N_s$.

Let us notice that for us Propositions 1 and 2 have a limited purpose; they give an accurate construction of spectral projections and the system of eigenfunctions and associated functions (SEAF) of the operator $L \in \mathbb{R}$ so we can talk about spectral decompositions (17) in our main theorem – Theorem 3.

More deliberate analysis would give asymptotics of $L$’s eigenvalues ($\lambda_k$). Such asymptotics - at least for real-valued $b(x)$ - could be found in [3] (see references there as well). Another group of questions - inverse problems - for real valued $b(x)$ such that $b'(x)$ and $xb(x)$ are in $L^2$ as well is considered in a series of papers [9]-[13].

**Theorem 3.** Suppose $b \in V$; the operator (1) generates the spectral decompositions

$$ f = S_s f + \sum_{k \geq N_s} P_k f \text{ for all } f \in L^2(\mathbb{R}) \tag{17} $$

where $S_s, P_k$ are defined by (9, 11) and $N_s$ is from Proposition 1. These series converge unconditionally.
Equation (17) could be written as

\[ f = S_\ast f + \sum_{k \geq N_\ast} \langle f, \psi_k \rangle \varphi_k \quad \text{for all } f \in L^2(\mathbb{R}) \]

since \( P_k \) are 1-dimensional projections for \( k \geq N_\ast \). In (30) we give conditions for \( N_\ast \) which guarantee (12). \( S_\ast \) is an \( N_\ast \)-dimensional projection and \( E_\ast = \text{Image}(S_\ast) \) is an invariant subspace for \( L \), but we cannot say more about the structure of \( L|E_\ast \). It is likely that \( L|E_\ast \) has Jordan subspaces, and \( \mu \in \text{Sp}(L|E_\ast) \) could have any algebraic multiplicity \( m \geq 2 \) but of course its geometric multiplicity \( \leq 2 \) (really, \( \leq 1 \) as an elementary analysis of the Wronskian shows). It would be interesting to get an analog of V. Tkachenko’s results for

\[ M = M_0 + V \]
\[ M_0 = -d^2/dx^2, \quad Vf = v(x)f(x), \quad v \in L^2(I) \]

on a finite interval \( I = [0, \pi] \). In [38] Proposition 1.2 it is shown that for every finite set \( K = \{k_1, \ldots, k_t\} \) of pairwise distinct points in \( \mathbb{C} \) and every set of \( Q(K) = \{q_1, \ldots, q_t\} \) of positive integers there exists a Sturm-Liouville operator (19) for which \( \{k_1, \ldots, k_t\} \) are points of the Dirichlet spectrum (or of the periodic spectrum, or of the anti-periodic spectrum) with algebraic multiplicities equal to respective numbers from \( Q(K) \).

Many questions about properties of SEAF of operators \( L \in (1) \) with \( b(x) = u(x) + iv(x), \ u(x) = u(-x), \ v(-x) = -v(-x) \) have been raised in the context of the PT-operator theory (see [7], [39], [6], [32], [8]). In [4] it is shown (Section 6 and Theorem 5.8) that if \( u(x) = 0 \) and \( |v(x)| \leq M < 2/\pi \) then \( L \) is similar to a self-adjoint operator with discrete spectrum. It happens this is true even with \( M < 1 \).

In Section 6 we consider bounded potentials \( b(x) \in L^\infty \) or, more generally, perturbations in (1) with \( B \) being a bounded operator in \( L^2(\mathbb{R}) \). V. Katsnelson’s approach [25]-[26] to analysis of SEAF of dissipative operators leads to claims (see Theorem 12) that such a perturbation is “good”, i.e., an analog of Theorem 3 holds if \( \|B\| < 1 \). The constant 1 is sharp as the special (counter) examples in Section 6 show.

In the multi-dimensional case see M. Agranovich’s surveys [1], [2] Ch. 5] on basis properties of eigensystems of weak perturbations of self-adjoint elliptic pseudodifferential operators on closed manifolds. In particular the results of [1] Sect. 6.2].
2. Technical preliminaries: the discrete Hilbert transform

2.1. Let $G$ be the discrete Hilbert transform

\[(G\xi)_n = \sum_{k=0}^{\infty} \frac{\xi_k}{k-n}, \quad \text{for} \quad \xi \in \ell^2(\mathbb{Z}_+).\]  

The operator $G$ is a bounded mapping from $\ell_2$ to $\ell_2$ - see for example [20, Sect. 8.12, statement 294], and [40, Ch. 4, Thm. 9.18].

Given a positive weight sequence $\{W(k)\}_{k \geq 0}$ define

\[\ell^2(W) = \{\xi : \sum_0^{\infty} |\xi_k|^2 W(k) < \infty\}\]  

and denote $w(k) = W(k)^{-1}$. We will use this convention throughout the paper. If $W(k) = (k + 1)^\alpha$ with $\alpha < 1$ then $G$ is a bounded mapping from $\ell^2(W)$ into itself. That is,

\[\sum_{n=0}^{\infty} (n+1)^\alpha \left| \sum_{k \neq n} \frac{\xi_k}{k-n} \right|^2 \leq C \sum_{n=0}^{\infty} (n+1)^\alpha |\xi_n|^2.\]  

Furthermore, given any weight sequence $\psi(k) \to \infty$ there exists another weight sequence

\[W_\psi(k) \uparrow \infty \quad \text{with} \quad W_\psi(k) \leq \psi(k)\]  

(so that $\ell^2(\psi) \subset \ell^2(W_\psi)$) such that $G$ is a bounded mapping from $\ell^2(W_\psi)$ into itself, that is,

\[\sum_{n=0}^{\infty} W_\psi(n) \left| \sum_{k \neq n} \frac{\xi_k}{k-n} \right|^2 \leq C \sum_{n=0}^{\infty} W_\psi(n) |\xi_n|^2.\]  

For a proof of these facts we refer to Appendix, Corollary 19.

2.2. Define a perturbed Hilbert transform $G_\tau$ by

\[(G_\tau\xi)_n = \sum_{k=0}^{\infty} \frac{\xi_k}{k+\tau - n}.\]  

An analog of inequalities (22),(24) still holds for small $\tau = (\tau_k)_{k=0}^\infty$.

**Lemma 4.** Suppose $W$ is a positive sequence and the following hold:

(a) $G \in \ell^2(W)$ is a bounded map from $\ell^2(W)$ to $\ell^2(W)$,

(b) $\sum_{n=0}^{\infty} W(n) |\xi_n|^2 < \infty$.

Then

\[\sum_{n=0}^{\infty} W(n) \left| \sum_{k \neq n} \frac{\xi_k}{k-n} \right|^2 \leq C \sum_{n=0}^{\infty} W(n) |\xi_n|^2.\]  

For a proof of these facts we refer to Appendix, Corollary 19.
(b) \( \tau \) is a sequence such that \( |\tau_k| \leq 1/16 \),

(26) \[ \sum_{n=0}^{\infty} \frac{r(n)}{(1+n)^2} < \infty \] where \( r(n) = \sup \{ W(i+n)w(i) : i \geq -n \} \).

Then \( G_\tau \) is a bounded map from \( \ell^2(W) \) to \( \ell^2(W) \).

Proof. With the boundedness of \( G \), it suffices to show that the difference \( G - G_\tau \) is a bounded map from \( \ell^2(W) \) to \( \ell^2(W) \). The matrix entries satisfy inequalities

\[ |(G - G_\tau)_{k,j}| = \left| \frac{1}{k-j} - \frac{1}{k-j+\tau_k} \right| \leq \frac{1}{(k-j)^2}, \quad k \neq j, \]

so it suffices to show that any operator \( A \) with matrix entries \( A_{k,j} \),

\[ A_{j,j} = 0; \quad |A_{k,j}| \leq \frac{1}{(k-j)^2}, \quad k \neq j \]

is a bounded map from \( \ell^2(W) \) to \( \ell^2(W) \). Indeed, decompose \( A \) over its diagonals

\[ A = \sum_{t=-\infty}^{\infty} A^t \text{ with } A^t_{i,i+j} = \delta(t,j)A_{i,i+t}, \quad \forall i \in \mathbb{Z}_+, \quad j \geq -i. \]

So \( \|A^t\|_{2,W} \leq \max_{i \in \mathbb{Z}_+} \left( |A^t_{i,i+t}| (|W(i+t)||w(i)|) \right) \leq 2t^{-2}r(t) \). Hence, by (26)

\[ \|A\|_{2,W} \leq 2 \sum_{t=-\infty \atop t \neq 0}^{\infty} t^{-2}r(t) < \infty \]

and the lemma follows. \( \square \)

2.3. Define the space

\[ \ell^2(W, L^2(\mathbb{R})) = \{ (\xi_k(x))_{k=0}^{\infty} : \xi_k(x) \in L^2(\mathbb{R}) \quad \forall k \in \mathbb{Z}_+ \text{ and } \{ ||\xi_k(x)||_2 \}_{k=0}^{\infty} \in \ell^2(W) \} \]

of \( L^2(\mathbb{R}) \)-valued sequences with the inner product \( \langle \xi, \eta \rangle = \sum_{k=0}^{\infty} W(k) \langle \xi_k(x), \eta_k(x) \rangle \).

Lemma 5. Suppose \( G_\tau \) is a bounded map from \( \ell^2(W) \) into itself. The perturbed Hilbert transform \( \tilde{G}_\tau \) defined in the space of \( L^2(\mathbb{R}) \)-valued sequences by

\[ (\tilde{G}_\tau(\xi_k(x)))_n = \sum_{j=0 \atop j \neq n}^{\infty} \frac{\xi_k(x)}{k-n+\tau_k} \]
is a bounded operator from $\ell^2(W, L^2(\mathbb{R}))$ into itself with

\[(27) \quad \|\tilde{G}_\tau\|_W \leq \|G_\tau\|_W.\]

**Proof.** Suppose $\xi = (\xi_j(x))_{j=0}^\infty \in \ell^2(W, L^2(\mathbb{R}))$ with $\xi_j(x) = \sum_{k=0}^\infty \xi_j^{(k)} h_k(x) \in L^2(\mathbb{R})$. We have

\[
\|\tilde{G}_\tau\xi\|_W^2 = \sum_{n=0}^\infty \|G_\tau\xi_n\|_{L^2}^2 W(n) = \sum_{n=0}^\infty \left| \sum_{j=0}^\infty \xi_j h_k(x) \right|_{L^2}^2 W(n) = \sum_{k=0}^\infty \sum_{n=0}^\infty |G_\tau \xi_n^{(k)}|^2 W(n)
\]

Define the ellipsoid in $\ell_2(L^2(\mathbb{R}))$:

\[
\tilde{E}(W) := \left\{ \xi \in \ell^2(W, L^2(\mathbb{R})) : \sum_{k=0}^\infty \|\xi_k(x)\|_{L^2}^2 W(n) \leq 1 \right\}.
\]

Under the conditions of Lemma 4 we have

\[(28) \quad \tilde{G}_\tau(\tilde{E}(W)) \subset C(W) \tilde{E}(W).\]

**Remark.** In [30], [31] analysis of the spectra of 1D periodic Dirac operators used in an essential way the discrete Hilbert transform to prove localization of $\text{Sp}(L)$ in small discs around $n \in \mathbb{Z}$ for $|n|$ large enough.

### 3. Proof of Propositions 1 and 2

**Proof.** Because $b \in V$ we may choose $J \in \mathbb{Z}_+$ with

\[(29) \quad \|h_kb\|_2 \leq \frac{1}{68} \text{ whenever } k \geq J.\]

Recall $\|b\| = \sup\{\|h_kb\|_2\}$. Choose $N_* \in \mathbb{N}$ with

\[(30) \quad N_* \geq \frac{(2J + 4\|b\|\sqrt{J} + 1)}{2}\]

so

\[(31) \quad \frac{\|b\|^2 J}{(2(N_* - J) - 1)^2} \leq \frac{1}{16}.\]
Fix $z \notin S$, with $S = S(N_*)$ from [8]. We will show $z \notin \text{Sp}(L)$. It is enough to show $\|BR^0(z)\| \leq 1/2$ since then

\begin{equation}
R(z) = R^0(z)(I - BR^0(z))^{-1}.
\end{equation}

Let $f = \sum f_j h_j \in L^2(\mathbb{R})$ with $\|f\|_2 = 1$. We have, by (29) and Cauchy’s inequality,

\[
\|BR^0(z)f\|_2^2 = \left\| \sum_{k=0}^{\infty} f_k b h_k \right\|_2^2 \leq \left( \sum_{k=0}^{\infty} |f_k| \frac{\|bh_k\|_2}{|z - \lambda_k^0|} \right)^2 \\
\leq \sup_{j < J}\{\|h_j b\|_2^2\} S_1 + \sup_{j \geq J}\{\|h_j b\|_2^2\} S_2
\]

where

\[
S_1 = \sum_{k=0}^{J-1} |z - \lambda_k^0|^{-2} \quad \text{and} \quad S_2 = \sum_{k=J}^{\infty} |z - \lambda_k^0|^{-2}.
\]

If $|\text{Re}(z)| < 2N_*$ then $\text{Im}(z) \geq Y$. Thus,

\[
S_1 \leq \sum_{k=0}^{J-1} \left[ (\text{Re}(z) - \lambda_k^0)^2 + Y^2 \right]^{-1} \leq Y^{-2} \sum_{k \in \mathbb{Z}} \left[ \frac{(\text{Re}(z) - (2k + 1))^2}{Y} + 1 \right]^{-1} \\
\leq Y^{-2} \left( 2 + \int_{-\infty}^{\infty} \frac{1}{(2x/Y)^2 + 1} dx \right) \leq Y^{-2} \left( 2 + \frac{\pi}{2} \right).
\]

So (7) implies $\sup_{j < J}\{\|h_j b\|_2^2\} S_1 \leq 1/16$.

Now suppose $|\text{Re}(z)| > 2N_*$. We have

\[
S_1 = \sum_{k=0}^{J-1} \left[ (\text{Re}(z) - \lambda_k^0)^2 + \text{Im}(z)^2 \right]^{-1} \leq \frac{J}{(2(N_* - J) - 1)^2}.
\]

So (31) implies $\sup_{j < J}\{\|h_j b\|_2^2\} S_1 \leq 1/16$.

Finally, because

\[
S_2 = \sum_{j=J}^{\infty} \frac{1}{|z - \lambda_j^0|^2} \leq \left( 16^2 + 1 + \frac{\infty}{2} \sum_{j=1}^{\infty} \frac{1}{(2j)^2} \right) \leq 17^2,
\]

the condition (29) implies $\sup_{j \geq J}\{\|h_j b\|_2^2\} S_2 \leq 1/16$.

By proving

\begin{equation}
\|BR^0(z)\| \leq 1/2, \quad \text{for} \quad z \notin S
\end{equation}

we have shown $\text{Sp}(L) \subset S$. So we have proven [8].
Consider the family
\[ L(t) = L^0 + tB, \quad 0 \leq t \leq 1. \]

For \( \xi \notin S \) we have
\[ R(L(t),\xi) = R^0(\xi)(I - tBR^0(\xi))^{-1} \]
so (33) implies that \( R(L(t),\xi) \) is well-defined and depends continuously on \( t \in [0,1] \).

Thus
\[ S_*(t) := \frac{1}{2\pi i} \int_{\partial \Pi(2N_*,\gamma)} R(L(t),z)dz \]
and
\[ P_k(t) := \frac{1}{2\pi i} \int_{\partial D(\lambda^*_k,1)} R(L(t),z)dz, \quad \text{for} \ k \geq N_* \]
also depend continuously on \( t \in [0,1] \). Since \( \text{Sp}(L(0)) \) contains only simple eigenvalues, it follows from Lemma 4.10 in Chapter 1 of [24] that
\[ \dim(S_*(t)) = \dim(S_*(0)) = N_* \]
and
\[ \dim(P_k(t)) = \dim(P_k(0)) = 1 \quad \text{for} \ k \geq N_* \]
whenever \( t \in [0,1] \).

So, \( \dim(S_*) = \dim(S_*(1)) = N_* \), \( \dim(P_k) = \dim(P_k(1)) = 1 \), \( k \geq N_* \)
therefore the spectrum of \( L = L(1) \) is discrete and contains exactly one (simple) eigenvalue in each \( D(\lambda^*_k,1/16) \) for each \( k \geq N_* \). Because \( z \notin S \) implies \( \|R^0(z)\| \leq 16 \), (32) and (33) imply (14), (15) and (16).

\[ \square \]

4. Proof of Theorem 3

The following is a lemma from [23], [24, Ch5, Lemma 4.17a].

**Lemma 6.** Let \( \{Q^0_k\}_{j \in \mathbb{Z}_+} \) be a complete family of orthogonal projections in a Hilbert space \( X \) and let \( \{Q_k\}_{j \in \mathbb{Z}_+} \) be a family of (not necessarily orthogonal) projections such that \( Q_j Q_k = \delta_{j,k}Q_j \). Assume that
\[ \dim(Q^0_0) = \dim(Q_0) = m < \infty \]
\[ \sum_{j=1}^{\infty} \|Q^0_j(Q_j - Q^0_j)u\|^2 \leq c_0\|u\|^2, \quad \text{for every} \quad u \in X \]
where \( c_0 \) is a constant smaller than 1. Then there is a bounded operator \( W: X \to X \) with bounded inverse such that \( Q_j = W^{-1}Q^0_jW \) for \( j \in \mathbb{Z}_+ \).
Remark. The Bari-Markus criterion is often given with more restrictive conditions for norms of deviations
\[ \sum_{j=0}^{\infty} \|Q_j - Q_j^0\|^2 < \infty \]
and with an algebraic assumption
\[ \dim(Q_j) = \dim(Q_j^0) < \infty, \quad j = 0, 1, \ldots \]
-see [28], [29] and [19] Ch. 6, Sect. 5.3, Theorem 5.2.

These conditions have been proven for Sturm-Liouville operators on \( I = [0, \pi] \) with a singular potential \( v \in H^{-1} \) for periodic, antiperiodic or Dirichlet boundary value problems [16], [34], or for 1-dimensional Dirac operators on \( [0, \pi] \) - see [17].

We now go directly to the proof of Theorem 3.

Proof. We first show that there exists an integer \( N \geq N_* \) with \( N_* \) from Proposition 1 such that
\[ \sum_{k \geq N_1} \|P_k^0(P_k - P_0^0)f\|_2^2 \leq \frac{1}{2}, \quad \text{for each } f \in L^2(\mathbb{R}); \quad \|f\|_2 = 1. \]
(36)

Suppose \( n \geq N_* \). If we write \( R \) and \( R^0 \) as
\[ R^0(z) = \frac{P_n^0}{z - \lambda_n^0} + \Phi_n^0(z) \quad \text{and} \quad R(z) = \frac{P_n}{z - \lambda_n} + \Phi_n(z) \]
then \( \Phi_n \) and \( \Phi_n^0 \) are analytic, operator-valued functions in \( D(\lambda_n^0, 1) \).

Set
\[ \Psi_n^0 := \Phi_n^0(\lambda_n) \quad \text{and} \quad \Psi_n := \Phi_n(\lambda_n). \]

From the identity
\[ P_n - P_n^0 = \frac{1}{2\pi i} \int |z - \lambda_n^0| = 1/4 (R(z) - R^0(z))dz \]
\[ = \frac{1}{2\pi i} \int |z - \lambda_n^0| = 1/4 R(z)BR^0(z)dz \]
we have \( \frac{1}{2} \)

\[ P_n - P_n^0 = P_nB\Psi_n^0 + \Psi_nBP_n^0. \]

\( ^1 \text{This representation is essentially from [24] Eqn. 4.38, Ch. 5]; but there the terms of positive degree from the Laurent expansions of } R(\xi,T) \text{ and } R(\xi,S) \text{ are not taken into account.} \)
Let $f = \sum_{j=0}^{\infty} f_j h_j(x) \in L^2(\mathbb{R})$ with $\|f\|_2 = 1$.

Then

$$\sum_{n \geq N} \|P_0^0 (P_n^0 - P_n) f\|_2^2 = \sum_{n \geq N} \|P_0^0 P_n B \Psi_0^0 f + P_0^0 \Psi_n B P_0^0 f\|_2^2 \leq 2 \sum_{n \geq N} \left( \|P_0^0 P_n B \Psi_0^0 f\|_2^2 + \|P_0^0 \Psi_n B P_0^0 f\|_2^2 \right).$$

Let $W_\psi$ be the sequence from (23) with $b \neq 0$

$$\psi(k) = [\sup\{\|h_j b\|_2 : j \geq k\}]^{-1}
\tag{37}$$

and let $C(W_\psi)$ be from (28), (44). We will now show that if $N$ is chosen with

$$N \geq \min\{n : W_\psi(n)^{-1} \leq C(W_\psi)/64\}
\tag{38}
\text{and } N \geq \min\{n : \|h_n b\|_2 \leq (70)^{-1}\}
\tag{39}

then (36) holds.

To come to this claim it suffices to show

$$\sum_{n \geq N} \|P_0^0 P_n B \Psi_0^0 f\|_2^2 \leq 1/4
\tag{40}$$

and

$$\sum_{n \geq N} \|P_0^0 \Psi_n B P_0^0 f\|_2^2 \leq 1/4
\tag{41}$$

Proof of (40):

By (15)

$$\sum_{n \geq N} \|P_0^0 P_n B \Psi_0^0 f\|_2^2 \leq 32^2 \sum_{n \geq N} \|B \Psi_0^0 f\|_2^2.$$

Now,

$$\sum_{n \geq N} \|B \Psi_0^0 f\|_2^2 = \sum_{n \geq N} \left\| \sum_{k \neq n} \frac{BP_0^0 f}{\lambda_k^0 - \lambda_n} \right\|_2^2 = \sum_{n \geq N} \left\| \sum_{k \neq n} \frac{f_k b(x) h_k(x)}{\lambda_k^0 - \lambda_n} \right\|_2^2.$$

We have

$$\sum_{n \geq N} \left\| \sum_{k \neq n} \frac{f_k b(x) h_k(x)}{\lambda_k^0 - \lambda_n} \right\|_2^2 = \|I_N \tilde{G}_r \xi\|_2^2
\tag{42}$$
where
\[ \xi = (\xi_k(x))_{k=0}^{\infty}, \quad \xi_k = f_k b(x) h_k(x), \]
\[ \tau_j = \lambda_j^0 - \lambda_j, \quad \text{and} \]
\[ (I_j \eta)_k = \begin{cases} \eta_k & \text{if } k \geq j, \\ 0 & \text{if } k < j \end{cases} \]
or - more generally - for \( F \subset \mathbb{Z}_+ \)
\[ (I(F) \eta)_k = \begin{cases} \eta_k & \text{if } k \in F, \\ 0 & \text{if } k \notin F. \end{cases} \]

If for \( b \) we take \( \psi_k \) as in (37), the vector valued sequence \( \xi \) belongs to \( \tilde{E}(W_\psi, L^2(\mathbb{R})) \) – see (23). So, by (28)
\[ \tilde{G}_\tau \xi \in C(W_\psi) \tilde{E}(W_\psi). \]

By (38), (42), and (44)
\[ 32^2 \sum_{n \geq N} \left\| \sum_{k \neq n} f_k b(x) h_k(x) \alpha^0_k - \alpha_n \right\|_2^2 \leq 32^2 C(W_\psi)^2 W_\psi(N)^{-2} \leq 1/4. \]

So (40) holds.

**Proof of (41):**

By (16) and (39)
\[ \sum_{n \geq N} \| P_n^0 \Psi_n B P_n^0 f \|_2^2 \leq \| \Psi_n \|^2 \sum_{n \geq N} \| B P_n^0 f \|_2^2 \leq 35^2 \sum_{n \geq N} |f_n|^2 \| bh_n \|_2^2 \leq 1/4. \]

By justifying (40) and (41) we have shown how to choose \( N \) with
\[ \sum_{k \geq N} \| P_k^0 (P_k - P_k^0) f \|_2^2 \leq \frac{1}{2} \| f \|_2^2, \quad \forall f \in L^2(\mathbb{R}). \]

To complete the proof of Theorem 3, it suffices to apply Lemma 6 to the orthogonal collection of projections
\[ \{ S_N^0, P_N^0, P_{N+1}^0, \ldots \} \]
where
\[ S_N^0 = P_0^0 + P_1^0 + \ldots + P_{N-1}^0. \]
and the (not necessarily orthogonal) collection of projections
\[ \{S_N, P_N, P_{N+1}, \ldots \} \]
where
\[ S_N = S_* + P_{N_*} + P_{N_*+1} + \ldots + P_{N-1}. \]
Inequality (46) and Lemma 6 imply that the series
\[ f = S_N + \sum_{k \geq N} P_k f \quad \text{for all } f \in L^2(\mathbb{R}) \]
converge unconditionally. By (47) it follows that the series (17) converge unconditionally as well. \( \square \)

5. Example spaces

The main hypothesis in Propositions 1-2 and Theorem 3 was \( b \in V \). In this section we will use known estimates for the Hermite functions \( \{h_k\} \) to prove that the spaces (45) are embedded in \( V \). The following lemma is essentially [36, Formula 8.91.10] and [18, Formula 6.11] together with Theorem B and the table on p. 700 in [5]. See also [35].

Lemma 7. Let \( N = 2n + 1 \). There are constants \( 2C, \gamma > 0 \) such that
\[ |h_n(x)| \leq \begin{cases} C(N^{1/3} + |x^2 - N|)^{-1/4} & \text{if } x^2 \leq 2N \\ C \exp(-\gamma x^2) & \text{if } x^2 \geq 2N \end{cases} \]
for all \( n \in \mathbb{Z}_+ \).

As Szegő’s book indicates, there is a long interesting history of estimates for Hermite, Laguerre and other orthogonal polynomials (see for example [14], [21] and the paper of Erdelyi [18]). For later developments see [33] and the references there. Inequalities (48) imply that
\[ |h_n(x)| \leq C(1 + n)^{-1/12}, \quad x \in \mathbb{R}, n \in \mathbb{Z}_+. \]
An alternative proof of (49) is given in [3] Lemma 4.

If \( b \in L(p, \alpha) \) then
\[ \|b(x)h_n(x)\|_2 = \|b(x)(1 + |x|^2)^{-\alpha/2}h_n(x)(1 + |x|^2)^{\alpha/2}\|_2 \leq \|b(x)(1 + |x|^2)^{-\alpha/2}\|_p \|h_n(x)(1 + |x|^2)^{\alpha/2}\|_q \]
\[ \tag{50} \]
\footnote{In what follows we use the letter \( C \) as a generic positive absolute constant.}
where $1/p + 1/q = 1/2$. Following the procedure outlined in [37, Sect. 1.5, p.27] (48) can be used to bound

$$
\| h_n(x)(1 + |x|^2)^{\alpha/2} \|_q = 2^{1/q} \left( \int_0^\infty |h_n(x)|^q (1 + |x|^2)^{q\alpha/2} dx \right)^{1/q}.
$$

Breaking the integral into two parts and applying (48) gives

$$
\int_{\sqrt{N}/2}^{3\sqrt{N}/2} |h_n(x)|^q (1 + |x|^2)^{q\alpha/2} dx \leq C n^{\alpha/2 - 1/2}
$$

and

$$
\int_{3\sqrt{N}/2}^{2\sqrt{N}} |h_n(x)|^q (1 + |x|^2)^{q\alpha/2} dx \leq C n^{\alpha/2 - 1/12} \int_0^{N^{2/3}} (1 + y)^{-q/4} dy.
$$

We omit the details. By (50) and (51-52) we have the following.

**Lemma 8.** Let $b \in L(p, \alpha)$ with $2 < p < \infty$ and $p \neq 4$. Then

$$
\| bh_n \|_2 \leq C(n + 1)^{\alpha/2 + t(p)}
$$

with

$$
t(p) = \begin{cases} 
-1/2 \left( 2 - \frac{2}{p} \right) & \text{if } 2 \leq p < 4 \\
-1/2p & \text{if } 4 < p < \infty
\end{cases}
$$

If $p = 4$, then

$$
\| bh_n \|_2 \leq C n^{\alpha/2 - 1/8} \log(n + 2).
$$

**Remark.** The estimates (53), (55) are sharp in the following sense:

$$
\| bh_n \|_2 \geq c(n + 1)^{\alpha/2 + t(p)} \quad \text{if } 2 \leq p < \infty, \ p \neq 4
$$

and

$$
\| bh_n \|_2 \geq c n^{\alpha/2 - 1/8} \log(n + 2) \quad \text{if } p = 4.
$$

**Proposition 9.** With $t$ as in (54), the spaces $L(p, \alpha)$ are embedded in $V$ whenever $\alpha/2 + t(p) \leq 0$ and $(p, \alpha) \neq (4, 1/4)$. Also, $L_0^\infty(\mathbb{R})$ is embedded in $V$.

**Proof.** If $\alpha/2 + t(p) < 0$ the result follows from Lemma 8.

Suppose now that $b \in Z$, where $Z = L_0^\infty(\mathbb{R})$ or $Z = L(p, \alpha)$ with $\alpha/2 + t(p) = 0$. In either case the map $\Phi : Z \to \ell^\infty(L^2(\mathbb{R}))$ defined by

$$
\Phi(b) = \{ b(x)h_k(x) \}
$$
is bounded. If
\[ b \in Z_0 = \{ \phi \in Z : \phi \text{ has compact support} \} \]
then
\[ \Phi(b) \in \ell^\infty_0(L^2(\mathbb{R})) = \{ \eta \in \ell^\infty(L^2(\mathbb{R})) : \|\eta_k\|_2 \to 0 \}. \]
But \( Z_0 \) is a dense subset of \( Z \) and \( \ell^\infty_0(L^2(\mathbb{R})) \) is a closed subspace of \( \ell^\infty(L^2(\mathbb{R})) \) so we conclude that \( \Phi(Z) \subset \ell^\infty_0(L^2(\mathbb{R})) \). That is, \( \|bh_k\|_2 \to 0 \) whenever \( b \in Z \). □

6. Bounded potentials

6.1. Propositions \( \Box \) and Theorem \( \Box \) succeed in dealing with \( L^\infty_0(\mathbb{R}) \)-potentials but the methods of the previous sections based on the property \( \|bh_k\| \to 0 \) of a potential \( b \) cannot be used to analyze an arbitrary \( L^\infty(\mathbb{R}) \)-potential. For example, if
\[ m(x) = (-1)^n M, \quad n \leq x < n + 1, \quad n \in \mathbb{Z} \]
then \( \|mh_k\|_2 = M \) for all \( k \geq 0 \). Of course, if \( \|bL^\infty\| = \rho < 1 \) or \( B \) is a bounded operator in \( L^2(\mathbb{R}) \) with \( \|B\| = \rho < 1 \) the resolvent
\[ R(z) = (I - R^0(z)B)^{-1}R^0(z) \]
\[ = R^0(z)(I - BR^0(z)) \]
is well-defined outside of the union of disks
\[ U_\rho = \cup_{k=0}^\infty D(2k + 1; \rho). \]
Indeed, for any \( z \in \mathbb{C}\setminus U_\rho \)
\[ \|R^0(z)\| = \left( \min_k |z - (2k + 1)| \right)^{-1} \]
\[ = 1/r(z), \quad r(z) > \rho \]
and
\[ \|R^0(z)B\|, \|BR^0(z)\| \leq \rho/r(z) < 1. \]
Therefore, the following is true.

**Proposition 10.** Let \( B \) be a bounded operator in \( L^2(\mathbb{R}) \) with \( \|B\| = \rho < 1 \). Then the operator
\[ L = -d^2/dx^2 + x^2 + B \]
has a discrete spectrum \( SpL; SpL \subset U_\rho \in \Box \) and \# \((SpL \cap D(2k + 1; \rho)) = 1 \) with a simple eigenvalue \( \lambda_k = 2k + 1 + \xi_k, \quad |\xi_k| \leq \rho, \quad k = 0, 1, 2, \ldots \).
As in Section 1 Proposition 1, (11), one could define projections
\[ P_n = \frac{1}{2\pi i} \int_{|z-(2n+1)|=\rho} R(z)dz, \quad \rho < r < 1 \quad \text{for all} \quad n = 0, 1, 2, \ldots; \]
\[ \dim(P_n) = 1, \quad P_n f = \langle f, \psi_n \rangle \phi_n. \]

But now certainly there are no associated functions.

6.2. The next step, i.e., the proof of an analog of Theorem 3, is done by a direct application of (a special case with multiplicity 1) V. Katsnelson’s theorem [25, Thms. 2,3]:

**Proposition 11.** Let \( A \) be a dissipative operator in a Hilbert space \( H \) with a discrete spectrum \( \text{Sp}A = \{\mu_n\}_0^\infty \), each \( \mu_n \) being a simple eigenvalue: \( A\phi_n = \mu_n\phi_n, \quad n \in \mathbb{Z}_+ \).

If
\[
\sup_{0 \leq j < \infty} \sum_{\substack{k=0 \atop k \neq j}}^{\infty} \frac{\text{Im}\mu_j \text{Im}\mu_k}{|\mu_j - \mu_k|^2} < \infty
\]
and
\[
\sup_{0 \leq j, k < \infty} \frac{4\text{Im}\mu_j \text{Im}\mu_k}{|\mu_j - \mu_k|^2} < 1
\]
then \( \{\phi_n\}_0^\infty \) is an unconditional basis in \( H \), i.e., for some bounded invertible operator \( U : H \to H \) the system \( e_n = U\phi_n, \quad n = 0, 1, \ldots \) is an orthogonal basis in \( H \).

It leads to the following.

**Theorem 12** (à la folklore c. 1970). Let \( B \) and \( L \) be as in Proposition 10 and
\[
L\phi_k = \lambda_k \phi_k, \quad \lambda_k \in D_k, \quad k = 0, 1, \ldots.
\]
The system of eigenfunctions \( \{\phi_k\}_0^\infty \) is an unconditional basis in \( L^2(\mathbb{R}) \).

**Proof.** To have a dissipative operator we consider
\[
\tilde{L} = L + i\rho = -d^2/dx^2 + x^2 + (B + i\rho)
\]
so if \( B = B_1 + B_2, \quad B_j \) selfadjoint, \( j = 1, 2 \), then \( \rho + B_2 \geq 0 \). The eigenvalues are shifted as well so
\[
\mu_k = \lambda_k + i\rho = 2k + 1 + \xi_k + i\rho,
\]
and \( \text{Im}\mu_k \geq 0, \quad |\xi_k| \leq \rho, \quad k = 0, 1, \ldots \).
Therefore, $2k + 1 - \rho \leq \Re \mu_k = \Re(\lambda_k) \leq 2k + 1 + \rho$, and

$$0 \leq \Im \mu_k = \Im \lambda_k + \rho \leq 2\rho.$$ 

The condition

$$\sup_{k \neq j} \frac{4\Im \mu_k \cdot \Im \mu_j}{|\mu_k - \bar{\mu}_j|^2} < 1$$

holds as it follows from the following estimates. This ratio is the largest when $k$ and $j$ are neighbors, say, $j = k + 1$.

Let $z = 2k + 1 + x + iy$ \quad $w = 2k + 3 + u + iv$; \quad $|x|, |u| \leq \rho < 1$ and $0 \leq y, v \leq h = 2\rho$. Then

$$\zeta^2 = \frac{4yv}{(2 + x - u)^2 + (y + v)^2} \leq \frac{(y + v)^2}{(2(1 - \rho))^2 + (y + v)^2} \leq \frac{4\rho^2}{(1 - \rho)^2 + 4\rho^2} < 1.$$ 

Another condition (60) to check is

$$s^* = \sup_k \sum_{j \neq k} \frac{1}{|\mu_k - \bar{\mu}_j|^2} < \infty.$$ 

By (63)

$$|\mu_k - \bar{\mu}_j| = |2(k - j) + \xi_k - \bar{\xi}_j| \geq 2|k - j| - 2\rho \geq 2(1 - \rho)|k - j|$$

and

$$s^* \leq \frac{1}{4(1 - \rho)^2} \cdot 2 \cdot \frac{\pi^2}{6} \leq \frac{1}{(1 - \rho)^2} < \infty.$$ 

Proposition 11 implies that \{\phi_k\} is an unconditional basis.

**Remark.** For any bounded operator $B$, with $R^0(z)$, \quad $\Im z \neq 0$, being of the Schatten class $S_r$, \quad $r > 1$, the products $R^0(z)B$, $BR^0(z)$ are in $S_r$ as well so we can use M. Keldysh’s theorem [28, Thm. 4.3] to claim: \quad the spectrum $\text{Sp}(L^0 + B)$ is discrete and its SEAF is complete.

But this SEAF is not necessarily an unconditional basis.

6.3. We will now construct examples of bounded operators $B$, \quad $\|B\| = 1$, such that the perturbation (59) has a discrete spectrum, all points of $\text{Sp}(L^0 + B)$ are simple eigenvalues and (62) holds, the system \{\phi_k\} is complete but it is not a basis in $L^2(\mathbb{R})$. 

Special 2-dimensional blocks play an important role in this construction. For $0 < t < 1$, $0 < s = 1 - k^2 < 1$, $k > 0$ we define a $2 \times 2$ matrix

$$b = \begin{bmatrix} 1 - t & t \\ -st & -1 + t \end{bmatrix}$$

(64)

**Lemma 13.** As an operator in $\mathbb{C}^2$ with the Euclidean norm

$$1 - t \leq \|b\| \leq 1;$$

(65)

moreover,

$$1 - \frac{1}{2}tk^2 \leq \|b\| \leq 1 - \frac{1}{2}t(1 - t)k^2.$$  

(66)

**Proof.** Of course, $\|b\|$ is larger than any of its entries so $1 - t \leq \|b\|$. Then

$$b = (1 - t) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + t \begin{bmatrix} 0 & 1 \\ -s & 0 \end{bmatrix}$$

and $\|b\| \leq 1 - t + t = 1$ so (65) holds. To get more accurate estimates let us take $c = (1 1)^T$, then

$$\|b\|^2 \geq \frac{\|be\|^2}{\|e\|^2} = \frac{1}{2} \left( 1 + 1 - 2tk^2 + t^2k^4 \right)$$

(67)

$$1 - tk^2 + \frac{1}{2}t^2k^4 \geq \left( 1 - \frac{1}{2}tk^2 \right)^2$$

On the other side, for $p = (x \ y)^T$, $|x|^2 + |y|^2 \leq 1$

$$\|bp\|^2 = |(1 - t)x + ty|^2 + |stx + (1 - t)y|^2$$

$$\leq (1 - t)^2(|x|^2 + |y|^2) + t^2|y|^2 + s^2t^2|x|^2 + 2t(1 - t)(1 + s)|x||y|$$

$$\leq (1 - t)^2 + t^2 + t(1 - t)(2 - k^2)$$

$$= 1 - t(1 - t)k^2 \leq \left( 1 - \frac{1}{2}t(1 - t)k^2 \right)^2.$$  

So (67) and (68) justify (66). \qed

**Lemma 14.** Let

$$\ell^0 = \begin{bmatrix} E & 0 \\ 0 & E + 2 \end{bmatrix}$$

and $\ell = \ell^0 + b$. 

(69)
EIGENSYSTEM OF AN $L^2$-PERTURBED HARMONIC OSCILLATOR

Then
\[ \ell = (E + 1) + tc \quad \text{where} \quad c = \begin{bmatrix} -1 & 1 \\ -1 + k^2 & 1 \end{bmatrix} \]

and
\[ cg^\pm = \pm kg^\pm, \quad g^\pm = (1 \ 1 \pm k)^T, \]
\[ \ell g^\pm = (E + 1) \pm tk. \]

Proof. As soon as these formulas are written they could be checked by direct substitution. □

Lemma 15. The angle $\alpha$ between eigenvectors $g^+$ and $g^-$ from (71) is determined by the equation
\[ (\cos \alpha)^2 = 1 - \frac{k^2}{1 + \frac{1}{4}k^4} > 1 - k^2 \]
so $\sin \alpha < k$. With the basis decomposition
\[ f = \Phi_0(f)u_0 + \Phi_1(f)u_1, \quad u_0 = \frac{g^+}{\|g^+\|}, \quad u_1 = \frac{g^-}{\|g^-\|}. \]

in $C^2$ we have
\[ \|\Phi_0\| = \|\Phi_1\| = \frac{1}{\sin \alpha} > \frac{1}{k}. \]

Proof. Again, direct evaluation shows
\[ (\cos \alpha)^2 = \frac{(g^+, g^-)^2}{\|g^+\|^2\|g^-\|^2} = \frac{(1 + (1 + k)(1 - k))^2}{(1 + (1 + k)^2)(1 + (1 - k)^2)} \]
\[ = 1 - \frac{k^2}{1 + \frac{1}{4}k^4} > 1 - k^2 \quad \text{and} \quad \sin \alpha < k. \]

(73) comes from straight-forward calculations. □

6.4. The special 2-dimensional blocks of this subsection give us a diagonal representation of 2-dimensional elements of a "bad" perturbation $B$, $\|B\| = 1$.

Proposition 16. Let $H = \ell^2(\mathbb{Z}_+)$ and $L^0e_n = (2n + 1)e_n, \quad n = 0, 1, 2, \ldots$. Put $B = \{b(m)\}_0^\infty$, $b(m) \in [0, 1]$ with fixed $t$, $0 < t < 1$, and $s = s(m) = 1 - k^2(m), \quad k(m) = 2^{-m-1}, \quad \text{i.e.,}$
\[ Be_{2m} = (1 - t)e_{2m} + te_{2m+1} \]
\[ Be_{2m+1} = -s(m)te_{2m} - (1 - t)e_{2m+1}, \quad m = 0, 1, 2, \ldots. \]
Then \( \|B\| = 1 \); for \( L = L^0 + B \)

\[
\text{Sp}(L) = \{4m + 2 \pm t \cdot k(m), \ m \in \mathbb{Z}_+\},
\]
each point \( \lambda \in \text{Sp}L \) is a simple eigenvalue, the system of eigenvectors \( \{\phi_m, \psi_m\} \)

\[
\phi_m = e_{2m} + (1 + k(m))e_{2m+1}, \\
\psi_m = e_{2m} + (1 - k(m))e_{2m+1}
\]  
(75)
is complete in \( H \) but it is not a basis in \( H \).

Of course, the same is true for any \( k(m) \), \( 0 < k(m) < 1 \), \( \lim_{m \to \infty} k(m) = 0 \).

Proof. The proof comes directly from Lemmas 13-15. We have \( \|B\| = \sup \|b(m)\| \) so

\[
1 \geq \|B\| \geq \sup \left\{1 - \frac{1}{2}tk^2(m) : m \in \mathbb{Z}_+\right\} = 1,
\]
and \( \|B\| = 1 \).

The 2-dimensional subspaces

\[
Y_m = \text{span}\{e_{2m}, e_{2m+1}\} = \text{span}\{\phi_m, \psi_m\}
\]
are invariant for \( B \) and \( L = L^0 + B \). If \( Q_m \) is an orthogonal projection onto \( Y_m \) then

\[
f = \sum_0^\infty Q_m f, \quad \forall f \in \ell^2(\mathbb{Z}),
\]
and of course these series converge unconditionally. But \( Q_m f = \Phi_m(f)\phi_m + \Psi_m(f)\psi_m, \Phi_m(g) = \Psi_m(g) = 0 \) if \( g \perp Y_m \), and by (73)

\[
\|\Phi_m\| = \|\Psi_m\| \geq \frac{1}{k(m)} = 2^{m+1} \to \infty.
\]
So the system (75) is not a basis. \( \square \)

The examples of Proposition 16 show that the constant 1 is sharp in the hypothesis \( \|B\| = \rho < 1 \) (see Proposition 10 and Theorem 12); even a non-strict inequality would not be good. However we do not have counterexamples where \( Bf = b(x)f(x), \ b \in L^\infty(\mathbb{R}) \).
7. Appendix

We introduced the weighted spaces $\ell^2(W)$ in Section 2 formula (21). In this appendix we will explain or prove statements and inequalities (22 - 24) of Section 2.1. The following sufficient (and necessary) conditions, \textit{“condition A$_2$,}” for boundedness of $G$ in $\ell^2(W)$ in terms of the weight are given in [22 Thm. 10]. Recall the convention $w(k) := W(k)^{-1}$.

**Proposition 17.** Let $\{W(k)\}_{k=0}^{\infty}$ be a positive sequence. Then $G : \ell^2(W) \to \ell^2(W)$ is a bounded operator if and only if

$$\sup_{k,n \geq 0} \sigma^+(k,n) \sigma^-(k,n) = M = M(W) < \infty \tag{76}$$

where $\sigma^+(k,n) = \frac{1}{1+n} \sum_k W(j), \quad \sigma^-(k,n) = \frac{1}{1+n} \sum_k w(j)$.

This proposition could be used to prove (23-24) but in this appendix we will provide a self-contained (elementary) proof of all statements on discrete Hilbert transform (or its perturbations) used in Sections 2-4.

Let $T_0 = 0 < T_1 < \ldots$ be a sequence of integers such that

$$t_k = T_k - T_{k-1}, \quad k = 1, 2, \ldots \tag{77}$$

and

$$t_{k+1} \geq Rt_k \quad k = 1, 2, \ldots; R > 2. \tag{78}$$

Define $W : \mathbb{Z}_+ \to \mathbb{R}_+$ a monotone increasing weight sequence by

$$W(j) = 2^k, \quad T_k \leq j < T_{k+1}, \quad k = 0, 1, \ldots \tag{79}$$

**Proposition 18.** Under the assumptions (77), (78), (79) $G$ is a bounded map from $\ell^2(W)$ into itself.

**Proof.** Notice that (78) implies

$$t_q/t_p \geq R^{q-p} \quad \text{if} \quad q \geq p \tag{80}$$

and

$$T_k \geq t_k \geq R^{k-1}t_1 > R^{k-1}, \quad k \geq 1. \tag{81}$$

Therefore

$$s = \sum_{j=0}^{\infty} W(j)(1+j)^{-2} < \infty. \tag{82}$$
Indeed,
\[
  s = \sum_{k=0}^{\infty} 2^k \sum_{j=T_k}^{T_{k+1}-1} \frac{1}{(1+j)^2} < t_1 + \sum_{k=1}^{\infty} 2^k T_k^{-1}
  < t_1 + R \sum_{k=1}^{\infty} (2/R)^k = t_1 + \frac{2R}{R - 2}.
\]

It suffices to prove
\[
  \|G a\|_W^2 = \sum_{n=0}^{\infty} W(n) \left| \sum_{j=0}^{\infty} \frac{a_j}{j - n} \right|^2 < \infty
\]
and
\[
  \sum_{n=1}^{\infty} W(n) \left| \sum_{j=1}^{\infty} \frac{a_j}{j - n} \right|^2 \leq C \sum_{k=0}^{\infty} W(k)|a_k|^2
\]
for some constant $C > 0$ independent of $a$ just for $a = \{a_k\}_{k=0}^{\infty}$ which have finite support, say $a_k = 0$ if $k > K$ and $\sum W(j)|a_j|^2 = 1$. For such $a$ guarantees that all series in (83) and the left side of (84) converge absolutely.

Indeed, if $n \geq 2K$ then
\[
  \left| \sum_{j=n+1}^{\infty} \frac{a_j}{j - n} \right|^2 = \sum_{j=0}^{K} \frac{1}{(n-j)^2} \leq \frac{4K}{n^2}
\]
and the left side in (84)
\[
  \leq \sum_{0}^{2K} + 4K \left( \sum_{2K+1}^{\infty} \frac{W(n)}{n^2} \right)
\]
by (82).

We now prove (84).

\[
  \|G a\|_W^2 = \sum_{N=1}^{\infty} \sum_{n \in J_N} W(n) \left| \sum_{j \in J(N)} \frac{a_j}{j - n} + \sum_{p=2}^{\infty} \sum_{j \in J_N^{\pm p}} \frac{a_j}{j - n} \right|^2
\]
where
\[
  J_N = [T_N, T_{N+1}), N \geq 0, \ J_N = \emptyset, \ N < 0 \quad \text{and} \quad J(N) = J_{N-1} \cup J_N \cup J_{N+1}.
\]

Let $\tilde{R}$ be a constant satisfying
\[
  1 < \tilde{R} < R/2
\]
(for example $\tilde{R} = 1/2 + R/4$ satisfies (86) in since $R > 2$ in (78)). Then by Cauchy’s inequality:

\[(87)\]
\[
\left| \sum_{j \in J(N)} a_j \frac{j - n}{j - n} + \sum_{p=2}^{\infty} \sum_{j \in J_N \pm p} a_j \frac{j - n}{j - n} \right|^2 \leq \left( \frac{1}{R} - 1 \right) \left[ \tilde{R} \left| \sum_{j \in J(N)} a_j \frac{j - n}{j - n} \right|^2 + \sum_{p=2}^{\infty} \tilde{R}^p \left| \sum_{j \in J_N \pm p} a_j \frac{j - n}{j - n} \right|^2 \right],
\]

which is, by another application of Cauchy’s inequality

\[(88)\]
\[
\leq \left( \frac{1}{R} - 1 \right) \left[ \tilde{R} \left| \sum_{j \in J(N)} a_j \frac{j - n}{j - n} \right|^2 + \sum_{p=2}^{\infty} \tilde{R}^p \left( \sum_{k \in J_N \pm p} W(k)|a_k|^2 \right) \left( \sum_{j \in J_N \pm p} w(j)(j - n)^{-2} \right) \right].
\]

Combining (85) with (87-88) we have

\[(89)\]
\[
\|Ga\|^2_W \leq S_0 + S_- + S_+
\]

where

\[(90)\]
\[
S_0 = \sum_{N=1}^{\infty} \sum_{n \in J_N} W(n) \tilde{R} \left| \sum_{j \in J(N)} a_j \frac{j - n}{j - n} \right|^2
\]

\[(91)\]
\[
S_- = \sum_{N=1}^{\infty} \sum_{p=2}^{\infty} \left( \sum_{k \in J_N \pm p} W(k)|a_k|^2 \right) \tilde{R}^p \sum_{n \in J_N} W(n) \left( \sum_{j \in J_N \pm p} w(j)(j - n)^{-2} \right)
\]

\[(92)\]
\[
S_+ = \sum_{N=1}^{\infty} \sum_{p=2}^{\infty} \left( \sum_{k \in J_N \pm p} W(k)|a_k|^2 \right) \tilde{R}^p \sum_{n \in J_N} W(n) \left( \sum_{j \in J_N \pm p} w(j)(j - n)^{-2} \right)
\]

To bound $S_0 \in (90)$ we use the fact that the canonical Hilbert transform $G : \ell^2 \to \ell^2$ is bounded and all projections $I(F) \in (43)$ are of the norm 1
To bound $S_{\pm} \in (91)$

\[
\sum_{N=1}^{\infty} \sum_{p=2}^{\infty} \sum_{k \in J_{N \pm p}} W(k) |a_k|^2 \left( \sum_{n \in J_N} W(n) \left( \sum_{j \in J_{N-p}} w(j)(j-n)^{-2} \right) \right)
\]

\[
\leq \sum_{N=1}^{\infty} \sum_{p=2}^{\infty} \sum_{k \in J_{N \pm p}} W(k) |a_k|^2 \left( \frac{2\tilde{R}/R}{1 - 2\tilde{R}/R} \right) \sum_{k=1}^{N} W(k) |a_k|^2.
\]

A similar argument applied to $S_{\pm}$ yields:

\[
\sum_{N=1}^{\infty} \sum_{p=2}^{\infty} \sum_{k \in J_{N \pm p}} W(k) |a_k|^2 \left( \sum_{n \in J_N} W(n) \left( \sum_{j \in J_{N+p}} w(j)(j-n)^{-2} \right) \right)
\]

\[
\leq \left( \frac{2\tilde{R}/R}{1 - 2\tilde{R}/R} \right) \sum_{k=1}^{\infty} W(k) |a_k|^2.
\]
So combining (89) with (93), (95), and (96) we have

$$
\|Ga\|_W^2 \leq \left[ \tilde{R}\|G|\ell^2|\right]^2 \left( \frac{1}{1 - 2\tilde{R}/R} \right) \|a\|_W^2
$$

\[\square\]

Remark. We can ease the hypothesis (78) and assume a weaker condition

$$
\lim \inf t_{k+1}/t_k = R > 2.
$$

The proof could be adjusted; we omit the details.

**Corollary 19.** (a) Inequality (22) holds.
(b) Given any weight sequence $\psi(k) \to \infty$ there exists another weight sequence $W$ which satisfies (22) and (24).

**Proof.** (a) For any $\alpha > 0$ define

$$
T_0 = 0, \quad T_k = \left[ \frac{2^k}{\alpha} \right] + k, \quad k = 1, 2, \ldots
$$

and

$$
U(j) = 2^k, \quad T_k \leq j < T_{k+1}.
$$

Then with $W(n) = (n + 1)^\alpha$, $n \in \mathbb{Z}_+$ the two sequences $U$ and $W$ are equivalent, i.e., for some constants $0 < c(\alpha), C(\alpha)$

$$
0 < c(\alpha) \leq U(j)w(j) \leq C(\alpha) < \infty, \forall j
$$

so $\ell^2(U) = \ell^2(W)$ and the norms (21) with weights $U$ and $W$ are equivalent. With (99) for $t_k = T_k - T_{k-1}$ we have:

$$
\lim_{k \to \infty} t_{k+1}/t_k = 2^{1/\alpha}.
$$

If $\alpha < 1$, (101) implies that

$$
t_{k+1} \geq Rt_k \quad \text{for} \quad k \geq K_\alpha, \quad R = \frac{1}{2}(2 + 2^{1/\alpha}) > 2.
$$

Now Proposition 18 and Remark 7 formula (98) imply (22).

(b) Choose $R > 2$. Define

$$
T_0 = 0, \quad T_2 = \min\{t : \psi(t) \geq 2, t \geq 1\},
$$

$$
T_{k+1} = \min\{t + T_k : \psi(t + T_k) \geq 2^{k+1}, \quad t \geq Rt_k\}, \quad k = 1, 2, \ldots
$$

Then $W \in (79)$ is good by Proposition 18 and (22), (24) hold.

To support Lemma 4 and its use in Sections 2-3 we now prove the following.
Lemma 20. Under the assumptions (77), (78), (79) the condition (26) holds, i.e.,

\[
\sum_{n=0}^{\infty} \frac{r(n)}{(1+n)^2} < \infty \quad \text{where} \quad r(n) = \sup \{ W(i+n)w(i) : i \geq -n \}.
\]

Proof. With given \( k, n \geq 0 \) define \( p, q \) by \( T_p \leq k < T_{p+1}, T_q \leq k + n < T_{q+1} \).
Then \( q \geq p \geq 0; \) if \( q \leq p + 1 \) then \( w(k)W(k+n) = 2^{-p} \cdot 2^q \leq 2 \).
If \( q \geq p + 2 \) by (78) and (80)

\[
n > T_q - T_{p+1} = \sum_{p+2}^{q} t_k \geq t_{p+2} \sum_{0}^{q-p-2} R^j \geq t_2 \cdot \frac{R^{q-p-1} - 1}{R - 1} \geq \frac{1}{\beta} R^{q-p}
\]

where \( \beta = R^2/t_2 \).
Therefore, with \( \gamma = \log 2/\log R < 1 \)

\[
2^{q-p} = (R^{q-p})^\gamma \leq (\beta n)^\gamma
\]

and for any \( k, n \), \( w(k)W(k+n) = 2^{q-p} \leq 2 + (\beta n)^\gamma \). So (102) holds. \( \square \)

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References

[1] M. Agranovich, Elliptic Operators on Closed Manifolds. Encyclopedia Math. Sci., vol. 63, Berlin, Springer-Verlag, 1994, p. 1-130.
[2] M. Agranovich, B. Katsenelenbaum, A. Sivov, N. Voitovich, Generalized method of eigenoscillations in diffraction theory. Wiley (1999).
[3] E. Akhmerova, The asymptotics of the spectrum of nonsmooth perturbations of a harmonic oscillator. (Russian) Sibirsk. Mat. Zh. 49 (2008), no. 6, 1216–1234; translation in Sib. Math. J. 49 (2008), no. 6, 968–984
[4] S. Albeverio, A. Motovilov, A. Shkalikov, Bounds on variation of spectral subspaces under \( J \)-self-adjoint perturbations, Integral Equations Operator Theory 64 (2009), no. 4, 455–486.
[5] R. Askey, S. Wainger, Mean convergence of expansions in Laguerre and Hermite series, American Journal of Mathematics, Vol. 87, No.3 (Jul. 1965), pp. 695-708.
[6] C. Bender. Making sense of non-Hermitian Hamiltonians. Rep. Progr. Phys. 70 (2007), no. 6, 947–1018.
[7] C. Bender, S. Boettcher, Real spectra in non-Hermitian Hamiltonians having PT symmetry, Phys. Rev. Lett. 80 (1998), 5243-5246.
[8] E. Caliceti, F. Cannata, S. Graffi, PT symmetric Schrodinger Operators: reality of the perturbed eigenvalues. SIGMA 6 (2010).
[9] D. Chelkak, Approximation in the space of spectral data of a perturbed harmonic oscillator. Nonlinear problems and function theory. J. Math. Sci. (N. Y.) 117 (2003), no. 3, 4260–4269.
[10] D. Chelkak, P. Kargaev, E. Korotyaev, Inverse problem for harmonic oscillator perturbed by potential, characterization. Comm. Math. Phys. 249 (2004), no. 1, 133–196.
[11] D. Chelkak, P. Kargaev, E. Korotyaev, An inverse problem for an harmonic oscillator perturbed by potential: uniqueness. Lett. Math. Phys. 64 (2003), no. 1, 7–21.
[12] D. Chelkak, P. Kargaev, E. Korotyaev, Inverse problem for harmonic oscillator perturbed by potential. Inverse problems and spectral theory, 93–102, Contemp. Math., 348, Amer. Math. Soc., Providence, RI, 2004.
[13] D. Chelkak, E. Korotyaev, The inverse problem for perturbed harmonic oscillator on the half-line with a Dirichlet boundary condition. Ann. Henri Poincaré 8 (2007), no. 6, 1115–1150.
[14] H. Cramer: On some classes of series used in mathematical statistics. Den Sjette Skandinaviske Matematiker Kongres i Kobenhavn. Kongressberetningen, Copenhagen (1926), 399-425.
[15] P. Dirac, The principles of quantum mechanics, Oxford University Press, 1958.
[16] P. Djakov, B. Mityagin, Bari-Markus property for Riesz projections of Hill operators with singular potentials. Functional analysis and complex analysis, 59–80, Contemp. Math., 481, Amer. Math. Soc., Providence, RI, 2009.
[17] P. Djakov, B. Mityagin, Bari-Markus property for Riesz projections of 1D periodic Dirac operators, arXiv:0901.0856 or Math. Nach. 283, No. 3, 443-462 (2010).
[18] A. Erdelyi, Asymptotic solutions of differential equations with transition points or singularities, Journal of Mathematical Physics, vol. 1 (1960), pp. 16–26.
[19] I. C. Gohberg, M. G. Krein, Introduction to the theory of linear non-self-adjoint operators, vol. 18 (Translation of Mathematical Monographs). Providence, Rhode Island, American Mathematical Society 1969.
[20] G. H. Hardy, D.E. Littlewood, G. Polya, Inequalities, Cambridge University Press, Cambridge (1988). Inequalities, Cambridge Univ. Press, Cambridge, 1952.
[21] E. Hille, A class of reciprocal functions. Ann. of Math. (2) 27 (1926), no. 4, 427–464.
[22] R. Hunt, B. Muckenhoupt, and R. Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform. Trans. Amer. Math. Soc. 176 (1973), 227-251.
[23] T. Kato, Similarity for sequences of projections. Bull. Amer. Math. Soc. 73 1967 904–905.
[24] T. Kato, Perturbation theory for linear operators, Springer Verlag, Berlin, 1980.
[25] V. Katsnelson, Conditions for a system of root vectors of certain classes of operators to be a basis. (Russian) Funkcional. Anal. i Prilozen. 1 1967 no. 2, 39–51.
28 JAMES ADDUCI AND BORIS MITYAGIN

[26] S.V. Khrushchev, N.K. Nikol’skii, B.S. Pavlov, Unconditional bases of exponentials and of reproducing kernels, in “Complex Analysis and Spectral Theory”, Lecture Notes in Math. Vol. 864, Springer-Verlag, Berlin/Heidelberg, 1981, pp.214-335.

[27] B.M. Levitan and I. S. Sargsjan, Introduction to spectral theory: selfadjoint ordinary differential operators. AMS, (1973).

[28] A. S. Markus, Introduction to the spectral theory of polynomial operator pencils. Translations of mathematical monographs, AMS. Vol 71. (1988).

[29] A.S. Markus, A basis of root vectors of a dissipative operator. Dokl. Akad. Nauk SSSR 132 524-527 (Russian); translated as Soviet Math. Dokl. 1 1960 599-602.

[30] B. Mityagin, Convergence of expansions in eigenfunctions of the Dirac operator. (Russian) Dokl. Akad. Nauk 393 (2003), no. 4, 456–459.

[31] B. Mityagin, Spectral expansions of one-dimensional periodic Dirac operators. Dyn. Partial Differ. Equ. 1 (2004), no. 2, 125–191.

[32] A. Mostafazadeh, Pseudo-Hermitian Quantum Mechanics, arXiv:0810.5643 (2008).

[33] P. Nevai, Exact bounds for orthogonal polynomials associated with exponential weights. J. Approx. Theory 44 (1985), no. 1, 82–85.

[34] Savchuk, A. M.; Shkalikov, A. A. Sturm-Liouville operators with distribution potentials. (Russian) Tr. Mosk. Mat. Obs. 64 (2003), 159–212; translation in Trans. Moscow Math. Soc. 2003, 143–192

[35] H. Skovgaard, ”Asymptotic forms of Hermite polynomials,” Technical Report 18, Contract Nonr-220(11), Department of Mathematics, California Institute of Technology 24 (1956).

[36] G. Szegö, Orthogonal Polynomials. AMS, 1939.

[37] S. Thangavelu, Lectures on Hermite and Laguerre Expansions, Math. Notes 42, Princeton University Press, 1993.

[38] V. Tkachenko, Non-selfadjoint Sturm-Liouville operators with multiple spectra. Interpolation theory, systems theory and related topics (Tel Aviv/Rehovot, 1999), 403–414, Oper. Theory Adv. Appl., 134, Birkhuser, Basel, 2002.

[39] M. Znojil, Non-Hermitian supersymmetry and singular, PT-symmetrized oscillators. J. Phys. A 35 (2002), no. 9, 2341–2352.

[40] A. Zygmund, Trigonometric Series, London, Cambridge Univ. Press, 1968.

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