An analytic study towards instabilities of the glasma

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Strong longitudinal color flux fields will be created in the initial stage of high-energy nuclear collisions. We investigate analytically time evolution of such boost-invariant color fields from Abelian-like initial conditions, and next examine stability of the boost-invariant configurations against rapidity dependent fluctuations\cite{1}. We find that the magnetic background field has an instability induced by the lowest Landau level whose amplitude grows exponentially. For the electric background field there is no apparent instability although pair creations due to the Schwinger mechanism should be involved.

I. INTRODUCTION

In the high-energy limit of nucleus-nucleus collisions, the relevant degrees of freedom of the incident nucleus are the small-$x$ partons described in the framework of the Color Glass Condensate (CGC)\cite{2} with the typical momentum scale, or saturation scale $Q_s(x)$. On the other hand, success of hydrodynamic models in the data analyses at RHIC seems to imply the formation of quark-gluon plasma (QGP) in local equilibrium in a very early stage ($\tau < 1 \text{ fm}/c$) of the event. Then it must be a decisive challenge to understand the physical mechanism for such a rapid equilibration from the CGC initial condition.

Plasma instabilities\cite{3}, especially the Weibel type, are intensively investigated so far, where one analyzes coupled equations of motion for the gauge fields and the hard particles with anisotropic momentum distribution. Despite that many intriguing phenomena have been reported, the kinetic description for particles with momentum $p_\perp \simeq Q_s$ should be applicable only after particle formation $\tau > 1/Q_s$. In the very early stage, $\tau \simeq 1/Q_s$, the system is so dense that the field description may be more appropriate. Here we shall investigate the instability problem in the classical Yang-Mills (YM) equations.

The dense pre-equilibrium system appearing between the first impact and the equilibrated QGP was recently named Glasma\cite{4}. The glasma, produced from the CGC initial condition, is still well-described by strong coherent YM field and can be treated as a weakly coupled system. Reflecting the CGC initial condition, this coherent field is boost-invariant in the high energy limit, and thus the time-evolution towards a thermalized QGP should involve the process exciting rapidity-dependent degrees of freedom. If rapidity-dependent unstable modes exist in the glasma, they will drive the system into an isotropic/thermalized state more efficiently. In fact, numerical simulations for the glasma have shown instability against the rapidity-dependent perturbations\cite{5}. Here we present an analytic attempt for understanding the dynamics of the glasma instability\cite{1}.

II. EXPANDING COLOR FLUX TUBE

In the CGC picture for the nuclear collisions, the random color sources are set on the light-cone, $x^\pm = (t \pm z)/\sqrt{2}$, and cross each other at the origin (Fig. 1). The two sources are accompanied with pure gauge fields $\alpha_{1,2}^i(x_\perp)$ ($i = x, y$) behind, which uniquely fix the initial condition for the gauge field in the forward light-cone as ($\tau = \sqrt{2 x^+ x^-}$ and $\eta = \frac{1}{2} \ln(x^+/x^-)$)\cite{6}:

\begin{align}
A_\eta &= -\tau^2 \alpha(\tau = 0, x_\perp) = \tau^2 \frac{ig}{2} [\alpha_1(x_\perp), \alpha_2(x_\perp)], \\
A_i &= \alpha^i(\tau = 0, x_\perp) = \alpha_1^i(x_\perp) + \alpha_2^i(x_\perp).
\end{align}
Here we use the Fock-Schwinger gauge, $A_\tau = 0$. The corresponding initial field strengths are

$$
E^z|_{\tau=0^+} = -ig[\alpha_1^i, \alpha_2^j], \quad B^z|_{\tau=0^+} = ig\epsilon_{ij}[\alpha_1^i, \alpha_2^j].
$$

with all transverse components vanishing. Note that this setup is perfectly longitudinal-boost invariant and has no $\eta$ dependence while the transverse coherence length of the fields should be of order $1/Q_s$, reflecting the CGC structures of the incident nuclei. Thus we have the picture of multiple tubes of the boost-invariant color flux of size $1/Q_s$ distributed in the transverse plane.

One can compute the gluon production, in principle, by solving the source-free YM equations in the forward light-cone from this initial condition. Event averaging corresponds to averaging over the initial color source distribution in the CGC framework. Note, however, that we don’t take this averaging, but presume that thermalization should occur in each event. In particular, we study here the time-evolution of a single isolated magnetic/electric color flux tube.

**Magnetic flux tube**: A magnetic flux tube is the unique object in the glasma configuration. This can be realized with

$$
\alpha(\tau, x_\perp) = 0, \quad \alpha_i(\tau, x_\perp) \neq 0.
$$

For simplicity, we assume that the color direction of $\alpha_i$ and $B^z$ are the same and constant, and then one can ignore the commutators. The Gauss constraint is trivially satisfied in this case, and the YM equations reduce to the Bessel equation\[1\]. For definiteness, we take the Gaussian profile in the transverse plane for the initial condition:

$$
\tilde{\alpha}_i^{\text{init}}(k_\perp) \propto -i\frac{\epsilon_{ij}k_j}{k^2} \ e^{-\frac{k^2}{4Q_s^2}}.
$$

The solution is written in terms of the modified Bessel functions $I_{0,1}(z)$:

$$
B^z(\tau \geq 0, r) = B_0 e^{-Q_s^2r^2} e^{-Q_s^2\tau^2} I_0(2Q_s^2r\tau),
$$

$$
E^T(\tau \geq 0, r) = B_0 e^{-Q_s^2r^2} e^{-Q_s^2\tau^2} I_1(2Q_s^2r\tau).
$$

where $E^T \equiv \sqrt{(E^x)^2 + (E^y)^2}$ and $r = |x_\perp|$. This describes the outward expansion of the flux tube as seen in Fig. 2 (left and middle). In the right panel of Fig. 2, we show the energy densities of the flux tube integrated over the transverse plane, $(B^z)^2$ and $(E^T)^2$ as a function of time $\tau$. Remarkably, this time dependence is quite similar to the numerical results in \[4\].
$B_z^2$ ($E_z^2$)  

$E^T (B^T)$  

$Q_s x_{\perp}$  

FIG. 2: Spatial profile of the longitudinal magnetic field $B_z$ (left) and the transverse electric field $E_T$ (middle), and time dependence of the averaged field strengths, $(B^z)^2$ and $(E^T)^2$ (right) for a single flux tube. The spatial profiles are plotted at five different times $Q_s \tau = 0$ (solid), 0.5 (longest dash), 1.0, 1.5, 2.0 (shortest dash) with setting $B_0 = 1$. The initial value of $(B^z)^2(\tau)$ is also set to 1. Notice that $E^T = 0$ at $\tau = 0$. These plots are true for the electric flux tube after exchanging $E$ and $B$.

**Electric flux tube**: In contrast, an electric flux tube is realized with $\alpha(\tau, x_{\perp}) \neq 0$, $\alpha_i(\tau, x_{\perp}) = 0$. We assume again that the color orientations of $\alpha$ and $E_z$ are the same and constant. Then the YM equation reduces to the Bessel equation. With the Gaussian initial condition the following results are completely dual to the magnetic case by exchanging $E$ and $B$.

**III. INSTABILITIES OF THE GLASMA**

The outward expansion of the boost-invariant flux tubes never achieves the isotropization of the system, because its longitudinal pressure is never positive ($\propto (B^z)^2 - (E^T)^2$ for the magnetic case). This is where the rapidity-dependent fluctuations play the leading part.

We introduce the small fluctuations $a_{i,\eta}$ as

$$A_i = A_i(\tau, x_{\perp}) + a_i(\tau, \eta, x_{\perp}), \quad A_\eta = A_\eta(\tau, x_{\perp}) + a_\eta(\tau, \eta, x_{\perp}),$$

where $A_i$ and $A_\eta$ are boost-invariant background fields in the previous section. We perform stability analysis of this system against the perturbations $a_{i,\eta}$ [7]. Note that coupling between $A_{i,\eta}$ and $a_{i,\eta}$ is uniquely fixed in the non-Abelian gauge theory. For simplicity, we replace the background fields by $\tau$-independent and spatially constant electric/magnetic fields, and consider SU(2) group.

**Constant magnetic field**: Setting the constant magnetic field $B^z$ to the third color direction, we use the gauge, $\alpha = 0$ and $\alpha_i^a = \delta^{a3}(B/2)(y\delta_{i1} - x\delta_{i2})$. By inspection, pure $a_i$ fluctuation is found stable, and now we study the pure $a_i$ fluctuations. After some algebra with help of the Gauss law constraint, one can derive for $a_i^{(\pm)}$ with the third color charge ($\pm$) and positive spin $+$ [1, 7]

$$\frac{1}{\tau} \partial_\tau(\tau \partial_\tau \tilde{a}_+^{(\pm)}) + \left(\frac{\nu^2}{\tau^2} + mgB \pm 2gB\right) \tilde{a}_+^{(\pm)} + \left(-\partial_\perp^2 + \frac{g^2 B^2 r^2}{4}\right) \tilde{a}_+^{(\pm)} = 0,$$

where $m$ and $\nu$, respectively, are the orbital angular momentum and the momentum conjugate to the rapidity (and a similar equation for negative spin $-$). Note that the term $\pm 2gB$ stems from the anomalous magnetic moment. Replacing the $x_{\perp}$-dependent part with the 2D harmonic oscillator wavefunction (Landau levels), the latter two terms in Eq. (10) are combined and yields $(\frac{\nu^2}{\tau^2} - gB) \tilde{a}_+^{(\pm)}$ for the lowest mode. At $\sqrt{gB} \tau_{\text{wait}} = \nu$, this “spring constant” becomes negative:
FIG. 3: \( \text{Re}[I_{i\nu}(z)] \) for \( \nu=8 \) (dashed) and 12 (solid), normalized at \( z=0.1 \). The lowest mode of Eq. (10) is given with the modified Bessel function \( I_{i\nu}(\sqrt{gB}\tau) \).

\textit{Nielsen-Olesen instability}\[1, 8\]. The typical scale \( \sqrt{gB} \sim Q_s \). Interestingly, this means that the mode with \( \nu \) is stable until \( \tau_{\text{wait}} = \nu/Q_s \), and then it grows up within \( \tau_{\text{grow}} = 1/Q_s \). Conversely, the maximum value \( \nu_{\text{max}} \) for the unstable modes at \( \tau \) satisfies \( Q_s\tau = \nu_{\text{max}} + 1 \sim \nu_{\text{max}} \). This may be related to the linear \( \tau \)-dependence of \( \nu_{\text{max}} \) found in \[3\].

Constant electric field: Setting the constant electric field \( E^z \) to the third color direction, we take the gauge, \( \alpha^a = -(1/2)E\delta^a_3 \) and \( \alpha_i = 0 \). In the case of \( a_\eta \neq 0 \) and \( a_i = 0 \), \( a_\eta \) is found stable. Let us consider the opposite case: \( a_\eta = 0 \) and \( a_i \neq 0 \). Then the YM equations reduce to

\[
\frac{1}{\tau} \partial_\tau (\tau \partial_\tau \tilde{b}^{(\pm)}) + \left\{ k_\perp^2 + \frac{1}{\tau^2} \left( \nu \pm \frac{gE\tau^2}{2} \right)^2 \right\} \tilde{b}^{(\pm)} = 0,
\]

where \( \tilde{b}(\tau, \nu, k_\perp) \) is related to the transverse part of \( \tilde{a}_i \). One sees that the spring constant is non-negative and stable: no exponentially growing mode. From the physics point of view, we expect, and are to clarify, a certain relation of our formulation to the Schwinger mechanism for the particle pair creation.

IV. SUMMARY

We have studied the time-evolution of the boost-invariant color flux tube in a simplified Abelian-like setup, and found the outward expansion of the tube whose energy density evolves similarly to the numerical simulations\[4\]. In the stability analysis we have found the unstable mode on the magnetic background but no apparent unstable mode in the electric case \[1, 8\]. To be more realistic, we need to treat the time-dependent background field, interactions among multi flux tubes, back-reactions of the fluctuations on the background field, and so on.

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