A Diagrammatic Construction of Third Homology Classes of Knot Quandles

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Abstract

We construct elements of the third quandle homology groups of knot quandles, which are called the shadow fundamental classes. They play the same roles for the shadow quandle cocycle invariants of knots as the fundamental classes of knot quandles does for the quandle cocycle invariants. As an application of the shadow fundamental classes, we show the relation between the shadow quandle cocycle invariants and the based shadow quandle cocycle invariants.

Moreover, we will show, for a prime knot, that any third quandle homology classes are considered as images of the shadow fundamental classes of some links.

1 Introduction.

Knot quandles, introduced by Joyce [J], are algebraic concepts, which are related to the peripheral systems of knots. Their (co)homology theories are defined in [CKLS], from which we can obtain many invariants of classical knots and of surface knots. The cocycle invariants in that paper and the shadow cocycle invariants in [CKS1] are examples of those invariants.

Eisermann [E] determined the second (co)homology groups of knot quandles. Furthermore, as a consequence of his result, we have homological interpretation of the cocycle invariants. Our purpose comes directly from these results, that is, we will try to determine the third (co)homology groups of knot quandles, and to give a homological explanation to the shadow cocycle invariants.

The arcs of a diagram of a knot $K$ can be considered as generators of the knot quandle $Q(K)$ of $K$. Thus we can regard the diagram as coloured by $Q(K)$. Since any coloured diagram on $S^2$ is shadow colourable, the knot diagram has a $Q(K)$-shadow colouring, which gives a third homology class of $Q(K)$. We call such a homology class a shadow diagram class as an element of the rack homology group of $Q(K)$, and also call it a shadow fundamental class as that of the quandle homology group. As for these homology classes, we have:

**Theorem 1.1** *(Theorem 4.2).* Let $L$ be a non-trivial $n$-component link.
a) The shadow diagram classes are non-zero elements of the third rack homology group $H^3_R(Q(L); \mathbb{Z})$ for all diagrams of $L$.

b) There exist $n$ shadow fundamental classes of $L$ in the third quandle homology group $H^3_Q(Q(L); \mathbb{Z})$.

c) The third quandle homology group $H^3_Q(Q(L); \mathbb{Z})$ splits into the direct sum

$$(\bigoplus \mathbb{Z}[L_i]) \oplus \left( H^3_Q(Q(L); \mathbb{Z})/\left( \bigoplus \mathbb{Z}[L_i] \right) \right),$$

where $[L_1], \ldots, [L_n]$ are distinct shadow fundamental classes of $L$.

Also we show the relation of the shadow fundamental classes to the shadow cocycle invariants, and apply it to generalise a result in [S]. In the proof of the result, Satoh used the fact that the set of $\mathbb{Z}_p$-shadow colourings becomes a module. We show that Satoh’s result derives essentially from the connected-ness of $\mathbb{Z}_p$ by using the shadow fundamental classes, or, in other words, we prove the following theorem in general.

**Theorem 1.2** (Theorem 4.3). Let $X$ be a connected finite quandle and $\phi$ a quandle 3-cocycle of $X$. For any link $L$, there holds an equation

$$\Phi_\phi(L) = |X| \cdot \Phi^*_\phi(L)$$

between the shadow cocycle invariant $\Phi_\phi(L)$ and the based shadow cocycle invariant $\Phi^*_\phi(L)$.

In respect of the first motivation, by the method in [CKS1] we obtain a generalised knot diagram on a closed manifold which represents the given third homology class. Considering the possible surgeries on these diagrams, we can obtain a knot diagram in usual sense as a representative of the given homology class when a knot $K$ is prime, that is, we conclude:

**Theorem 1.3** (Theorem 5.1). Let $K$ be a prime knot. For any quandle 3-cycle $c \in C^3_Q(Q(K); \mathbb{Z})$, there exists a pair of a link $L$ and a homomorphism $f : Q(L) \rightarrow Q(K)$ such that $[c] = f_*[L_{sh}]$, where $[L_{sh}]$ is one of the shadow fundamental classes of $L$.

This theorem indicates one way to represent elements of the third quandle homology groups. We expect that the diagrammatic representation is useful for complete determination of the third homology groups of quandles.

## 2 Preliminaries.

In this section, we will define the concepts of racks and quandles, their (co)homology theories, diagrams and their (shadow) colourings, and knot quandles.
2.1 Racks and Quandles.

Quandles are introduced by Joyce [J] for characterising unframed knots. He proved that the quandles associated with knots completely determine the knot types up to orientation, and explained Fox’s tricolourability of knots in the viewpoint of quandles. Racks introduced by Fenn-Rourke [FR] are proved to be complete invariants of framed knots. Both quandles and racks obey the axioms of right invertibility and of right distributivity, but only quandles satisfy the idempotency.

A rack \( R \) is a non-empty set with two binary operations \( \triangleright \) and \( \blacktriangleleft \) satisfying two axioms

\[
\begin{align*}
(R1) & \quad (a \triangleright b) \blacktriangleleft b = a = (a \blacktriangleleft b) \triangleright b \\
(R2) & \quad (a \triangleright b) \blacktriangleleft c = (a \blacktriangleleft c) \triangleright (b \blacktriangleleft c)
\end{align*}
\]

for any \( a, b \) and \( c \in R \). In addition, if a rack \( Q \) satisfies an axiom

\[
(Q) \quad a \blacktriangleleft a = a = a \triangleright a
\]

for any \( a \in Q \), it is called a quandle. For two racks \( R \) and \( R' \), a map \( f: R \to R' \) is called a (rack) homomorphism when it obeys

\[
(RH) \quad f(a \triangleright b) = f(a) \triangleright f(b)
\]

for any \( a \) and \( b \in R \). Notice that the other right-distributive laws for \( \blacktriangleleft \) and \( \triangleright \) can be derived from the axioms (R1) and (R2), and that the compatibility of a homomorphism for \( \blacktriangleleft \) is a consequence of (R1) and (RH). Note that sometimes we write \( \triangleright^{+1} \) and \( \blacktriangleleft^{-1} \) instead of \( \triangleright \) and \( \blacktriangleleft \), respectively.

The most important example of quandles is the conjugation quandle of a group. A group \( G \) can be viewed as a quandle with operations defined by

\[
g \blacktriangleleft h = h^{-1}gh \quad \text{and} \quad g \triangleright h = hgh^{-1}
\]

for \( g \) and \( h \in G \). We denote by \( G_{\text{conj}} \) the conjugation quandle derived from \( G \).

On the other hand, we can associate a group \( G_Q \) with a quandle \( Q \). The group \( G_Q \) is generated by \( Q \) with relations

\[
b^{-1}ab = a \blacktriangleleft b \quad \text{and} \quad bab^{-1} = a \triangleright b
\]

for each \( a \) and \( b \in Q \). We call \( G_Q \) the associated group of \( Q \). There is a canonical homomorphism from \( Q \) to \( (G_Q)_{\text{conj}} \) which maps \( a \in Q \) to \( a \in (G_Q)_{\text{conj}} \).

A quandle \( Q \) is called trivial if \( a \blacktriangleleft b = a \) holds for any \( a \) and \( b \in Q \). We denote by \( T_n \) the trivial quandle with \( n \) elements. Notice that the conjugation quandle of a commutative group \( G \) is trivial.

For two elements \( a \) and \( b \) in a rack \( R \), they are said to be connected if there exist finite sequences \( (x_1, \ldots, x_n) \) and \( (\epsilon_1, \ldots, \epsilon_n) \), where \( x_i \) is in \( R \) and \( \epsilon_i \) is in \( \{\pm 1\} \) for each \( i \), such that
\[
(\cdots (a \triangleleft x_1) \cdots) \triangleleft x_n = b
\]
holds. We call the set of elements connected to \( a \) the orbit of \( a \). If all elements of \( R \) are connected to one another, the rack \( R \) itself is called to be connected.

For example, the trivial quandle has no pair of connected elements. On the other hand, the dihedral quandle \( \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} \), which has operations defined by

\[
a \triangleleft b = a \blacktriangledown b = 2b - a,
\]
is connected when \( p \) is an odd prime.

### 2.2 Homology and cohomology theories

Here we define (co)homology theories of a rack or a quandle. We will follow the definitions in [CJKS1] except the coefficients of boundary maps. It is because, under this modification of boundary maps, the shifting homomorphisms in \( \mathbb{Z} \) become easily explained diagrammatically. Applications and computations of (co)homology theories can be found in [CJKS2] or in the other papers of the same authors.

Let \( R \) be a rack and \( A \) a ring with unit. We denote, by \( C_n^R(R; A) \), a free \( A \)-module \( A[R^n] \) for a positive integer \( n \) and, by \( C_0^R(R; A) = A \). Then, \( C_n^R(R; A) \) is a chain complex with boundary maps \( \partial_n: C_n^R(R; A) \to C_{n-1}^R(R; A) \) defined by linearly extending a map on the basis

\[
\partial_n(x_1, \ldots, x_n)
= \sum_{i=1}^n (-1)^{n-i} \{(x_1 \triangleleft x_i, \ldots, x_{i-1} \triangleleft x_i, x_{i+1}, \ldots, x_n)
- (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\}
\]
for \( n \geq 2 \) and \( \partial_1(x) = 1 \), where \( x_1, \ldots, x_n \) and \( x \) are in \( R \). As usual, we set \( Z_n^R(R; A) = \ker \partial_n \) and \( B_n^R(R; A) = \text{im} \partial_{n+1} \). The \( n \)-th rack homology \( H_n^R(R; A) \) of \( R \) is defined as the quotient module \( Z_n^R(R; A)/B_n^R(R; A) \). The elements of \( C_n^R(R; A) \), \( Z_n^R(R; A) \) and \( B_n^R(R; A) \) are called, respectively, rack \( n \)-chains, cycles and boundaries. If the difference of two rack chains \( c_1 \) and \( c_2 \) is a rack boundary, then they are said to be rack homologous.

Considering the dual concepts \( C_n^R(R; A) = C_n^R(R; A)^* \) and

\[
\delta^n = \partial_n^*: C_n^R(R; A) \to C_n^R(R; A),
\]
we obtain a cochain complex \( C_n^R(R; A) \). The \( n \)-th rack cohomology \( H_n^R(R; A) \) of \( R \) is the quotient \( Z_n^R(R; A)/B_n^R(R; A) \), where \( Z_n^R(R; A) \) denotes \( \ker \delta^{n+1} \) and \( B_n^R(R; A) \) \( \text{im} \delta^n \). We call the elements of \( C_n^R(R; A) \), \( Z_n^R(R; A) \) and \( B_n^R(R; A) \) rack \( n \)-cochains, cocycles and coboundaries, respectively. Also we call two rack cochains \( \phi_1 \) and \( \phi_2 \) rack cohomologous when \( \phi_1 - \phi_2 \) is a rack coboundary. Since \( C_n^R(R; A) \) is a free module, rack \( n \)-cochains can be considered as maps from \( R^n \) to \( A \).
For a quandle $Q$, we can define other homology and cohomology theories of $Q$. Let $C^n_D(Q;A)$ be the submodule of $C^n_R(Q;A)$ generated by $n$-tuples $(x_1, \ldots, x_n)$ such that $x_i = x_{i+1}$ for some $i$, and let $C^n_Q(Q;A)$ be the quotient module $C^n_R(Q;A)/C^n_D(Q;A)$. Their boundary maps are obtained naturally from $\partial_n$, which make $C^n_D(Q;A)$ and $C^n_Q(Q;A)$ chain complexes. Thus we have homology theories $H^n_W(Q;A) = Z^n_W(Q;A)/B^n_W(Q;A)$, where $W$ denotes $D$ or $Q$. They are called the degeneracy homology for $W = D$ and the quandle homology for $W = Q$.

In the case of cohomology theories, we first define $C^n_Q^*(Q;A)$. The submodule $C^n_D^*(Q;A)$ of $C^n_R^*(Q;A)$ consists of all rack $n$-cochains whose values on $C^n_D(Q;A)$ are constantly zero, and $C^n_Q^*(Q;A)$ is the quotient module $C^n_R^*(Q;A)/C^n_D^*(Q;A)$. Their coboundary maps are also derived from $\Delta^n$, and the cochain complex $C^*_W(Q;A)$ induces a cohomology theory $H^*_W(Q;A)$, which we call the quandle cohomology and the degeneracy cohomology, respectively, for $W = Q$ and for $W = D$.

Similarly as in the rack case, we use terms such as quandle (co)chains, quandle (co)cycles, and so on. For simplicity, we will omit notations of the coefficient rings of (co)homology groups, when we are concerned with the integral coefficient case.

Directly from their definitions, there exist short exact sequences of complexes

$$0 \rightarrow C^*_Q(Q;A) \xrightarrow{\iota} C^*_R(Q;A) \xrightarrow{\rho} C^*_D(Q;A) \rightarrow 0,$$

and

$$0 \rightarrow C^*_Q(Q;A) \xrightarrow{\iota} C^*_R(Q;A) \xrightarrow{\rho} C^*_D(Q;A) \rightarrow 0.$$

Hence we have long exact sequences of (co)homology theories

$$\cdots \rightarrow H^*_Q(Q;A) \xrightarrow{\iota} H^*_R(Q;A) \xrightarrow{\rho} H^*_D(Q;A) \xrightarrow{\Delta} H^*_{Q-1}(Q;A) \rightarrow \cdots$$

and

$$\cdots \rightarrow H^*_Q(Q;A) \xrightarrow{\iota} H^*_R(Q;A) \xrightarrow{\rho} H^*_D(Q;A) \xrightarrow{\Delta} H^*_{Q+1}(Q;A) \rightarrow \cdots,$$

where $\Delta_n$ and $\Delta^n$ denote the connecting homomorphisms.

In [CJKS1], it is conjectured that all the connecting homomorphisms are zero maps. This conjecture is affirmatively proved by Litherland-Nelson:

**Theorem 2.1** (Litherland-Nelson [LN]). For an arbitrary quandle $Q$, all the connecting homomorphisms $\Delta$ are zero maps. Moreover, the resulting short exact sequence $0 \rightarrow H^*_Q(Q;A) \rightarrow H^*_R(Q;A) \rightarrow H^*_D(Q;A) \rightarrow 0$ is splittable, that is, there holds

$$H^*_R(Q;A) \cong H^*_Q(Q;A) \oplus H^*_D(Q;A).$$

In the proof of this theorem, Litherland-Nelson constructed a splitting homomorphism of the short exact sequence of chain complexes. We will see the detail of this homomorphism in §1.1 which plays an important role in §2.3.

Notice that, by the duality, we have a corollary as to cohomology theories:
Corollary 2.2. All the connecting homomorphisms $\Delta^*$ in the long exact sequence of the cohomology theories of a quandle $Q$ are zero maps. Moreover, the rack homology group $H^0_R(Q; A)$ splits as follows:

$$H^0_R(Q; A) \cong H^0_Q(Q; A) \oplus H^0_D(Q; A).$$

2.3 Diagrams.

Carter-Kamada-Saito [CKS1] defined generalised knot diagrams for representing rack homology classes. Their definition of generalised knot diagrams works for arbitrary dimensions, but in this paper we only need those of dimension at most three. Furthermore, the definition in [CKS1] contains exceptional diagrams such as endpoints, branch points and hemmed crossings, which are not necessary for us. So our definition below is a simplified one.

To avoid complication, we only say “diagrams” instead of generalised knot diagrams. A diagram is a pair $(M, D)$ of an oriented compact manifold $M$ and its subspace $D$ which can be decomposed into unit diagrams. To begin with, we define unit diagrams of each dimension.

The unit 0-diagram is a pair $E^0_0 = (\{0\}, \emptyset)$ with signature $+1$ or $-1$. This diagram is of use only when we consider the boundaries of 1-diagrams.

Let $I$ be the interval $[-1, 1]$ and suppose that $I$ is oriented as usual. We have two types of the unit 1-diagrams, that is, pairs $E^1_1 = (I, \{0\})$ with signature $+1$ or $-1$ and $E^1_0 = (I, \emptyset)$ with signature $0$. The diagrams $E^1_1$ and $E^1_0$ are drawn in Figure 1, where the first one is $E^1_1$ with signature $+1$, the second is $E^1_1$ with $-1$, and the third is $E^1_0$. The signatures of $E^1_1$'s are depicted with the arrows at the origin.

The unit 0-diagram is a pair $E^0_0 = (\{0\}, \emptyset)$ with signature $+1$ or $-1$. This diagram is of use only when we consider the boundaries of 1-diagrams.

Let $S$ be a square $I^2$, and let $l_1$ and $l_2$ be lines $I \times \{0\}$ and $\{0\} \times I$ in $S$, respectively. The orientation of $l_1$ is fixed as in Figure 1, but that of $l_2$ is not. The unit 2-diagrams are of three types; pairs $E^2_2 = (S, l_1 \cup l_2)$, $E^2_1 = (S, l_1)$ and $E^2_0 = (S, \emptyset)$. The diagram $E^2_2$ has its signature $+1$ or $-1$ as $E^1_1$, but the signatures of $E^2_1$ and $E^2_0$ are 0. The left two diagrams in Figure 3 shows $E^2_2$ with signature $+1$ and with $-1$. The orientation of $l_2$ in $E^2_2$ depends on the signature of the diagram. We call $l_1$ and $l_2$ the higher and the lower lines, respectively, and cut the lower line near the origin to distinguish them. The unit 2-diagrams
Figure 2: Lines and their orientations.

Figure 3: Unit 2-diagram.

$E_2^2$ and $E_0^2$ are, respectively, the right two diagrams in Figure 3.

Let $B$ be a cube $I^3$. Denote, by $s_1$, $s_2$ and $s_3$, sheets $I^2 \times \{0\}$, $I \times \{0\} \times I$ and $\{0\} \times I^2$ in $B$, respectively. The orientations of sheets $s_1$ and $s_2$ are given as in

Figure 4: Sheets.

Figure 4, but that of $s_3$ is variable. There are four types of the unit 3-diagrams, denoted by $E_3^3$, $E_2^3$, $E_1^3$ and $E_0^3$. They are defined, respectively, by

$$
E_3^3 = (B, s_1 \cup s_2 \cup s_3), \ E_2^3 = (B, s_1 \cup s_2), \ E_1^3 = (B, s_1) \text{ and } E_0^3 = (B, \emptyset).
$$

In this case, the signatures of $E_2^3$, $E_1^3$ and $E_0^3$ are all 0. Only $E_3^3$ has its signature $+1$ or $-1$, which determines the orientation of $s_3$ as in Figure 4. Each diagram is drawn in Figure 5, where the signature of $E_3^3$ is disregarded. We call $s_1$, $s_2$, and $s_3$ in $E_3^3$, respectively, the highest, the middle and the lowest sheets, and
also call \( s_1 \) and \( s_2 \) in \( E_2^3 \), respectively, the higher and the lower ones. The lower two sheets \( s_2 \) and \( s_3 \) are drawn separated as in Figure 5 to distinguish one sheet from the others.

For a unit diagram \( E = E_n^k \), we will call a pair \((n, k)\) its type. The points on the intersection of two lines or of just two sheets are called double points. The origin of \( E_3^3 \) is said to be a triple point. We say multiple points for both double and triple points.

For \( n \neq 0 \), the boundary of a unit \( n \)-diagram consists of \( 2n \) faces in the usual sense. Obviously, each face is one of unit \((n-1)\)-diagrams. For example, the boundary of \( E_3^2 \) consists of six faces as depicted in Figure 6. Two of them are \(+E_2^2\) and \( -E_2^2\) and the rest are \( E_1^2\)’s. Diagrams in general are obtained by attaching these unit diagrams between their faces. Let \( F_1 = (B_1, P_1) \) and \( F_2 = (B_2, P_2) \) be two faces of the same type with opposite signatures (they may belong to the same diagram). An attaching map \( f: F_1 \to F_2 \) is an orientation reversing homeomorphism \( B_1 \to B_2 \) such that \( f \) maps \( P_1 \) to \( P_2 \) with preserving the levels of each components. As for the orientations of components, we assume that \( f \) preserves them when the faces are odd dimensional, but it reverses them otherwise. We show examples of attaching maps in Figure 7. In the upper
line, two $E_2^1$'s with opposite signatures are attached, where the faces are odd dimensional. On the other hand, an attaching map between $E_2^3$ and $E_3^3$ are depicted in the lower line.

Figure 7: Attaching two diagrams.

2.4 Colourings and shadow colourings.

Let $D$ be a diagram on a manifold $M$. Regarding the line $l_2$ and the sheets $s_2$ and $s_3$ of unit diagrams as separated in fact, we can consider that $D$ consists of connected components. Denote by $\mathcal{C}(D)$ the set of all connected components of $D$, and denote by $\mathcal{R}(D)$ the set of all connected components of the exterior $M \setminus D$. We call elements of $\mathcal{C}(D)$ and of $\mathcal{R}(D)$, respectively, components and regions of $D$. For example, the unit diagram $E_2^2$ has three components and four regions.

Let $p$ be a point of $D$. If $p$ is not a multiple point, one component and two regions are found in its neighbourhood. Denote by $c_p$ the component where $p$
exists, and by $r_p^{\text{ini}}$ and $r_p^{\text{ter}}$ the regions so that the normal vector of $c_p$ points to $r_p^{\text{ter}}$. These regions $r_p^{\text{ini}}$ and $r_p^{\text{ter}}$ are called, respectively, the initial and the terminal region of $p$ (or of $c_p$).

On the other hand, if $p$ is a double point, there are three components of $D$ in the neighbourhood of $p$. Denote by $o_p$ the component in the higher level, or the over-arc. The other two components are in the lower level. We denote them by $u_p^{\text{ini}}$ and $u_p^{\text{ter}}$ so that the normal vector of $o_p$ points to $u_p^{\text{ter}}$. They are also called the initial and the terminal under-arc of the crossing $p$. Additionally, for a multiple point $p$, there exists a unique region $r_p^{\text{ini}}$ adjacent to $p$ such that $r_p^{\text{ini}}$ is the initial region of all components adjacent to both $p$ and $r_p^{\text{ini}}$. Figure 8 shows these notations.

Let $R$ be a rack. An $R$-colouring $C$ is a map $C(D) \to R$ satisfying

\[(C) \quad C(u_p^{\text{ini}}) \triangleleft C(o_p) = C(u_p^{\text{ter}})\]

for each double point $p$. A pair of an $R$-colouring $C$ and a map $C' : R(D) \to R$ is called an $R$-shadow colouring if it satisfies

\[(SC) \quad C'(r_p^{\text{ini}}) \triangleleft C(c_p) = C'(r_p^{\text{ter}}),\]

where $p \in D$ is not a multiple point. We call $C(c)$ the colour of a component $c$, and $C'(r)$ the shadow colour of a region $r$. When a pair of a diagram and its $R$-(shadow) colouring is given, we call it an $R$-(shadow) coloured diagram. An example of (shadow) coloured diagrams is drawn in Figure 9. The diagram is (shadow) coloured with the dihedral quandle $\mathbb{Z}_3$. In Figure 9 and in figures below, the letters in boxes signify the shadow colours of regions.

Assume that homology theories are with integral coefficients. We will show the correspondence between $n$-chains of $R$ and $R$-coloured $n$-diagrams or be-
Figure 9: Example of (shadow) coloured diagram.

tween \((n+1)\)-chains and \(R\)-shadow coloured \(n\)-diagrams for \(n = 0, 1, 2\) and 3.

First we consider \(R\)-(shadow) coloured unit diagrams. In each dimension, unit diagrams \(E_k^n\) correspond to 0 if \(k < n\). Figure 10 shows the correspondence between \(R\)-coloured unit diagrams \(E_k^n\) and rack \(n\)-chains, where each \(\epsilon\) denotes the signature of the corresponding diagram. On the other hand, Figure 11 shows how \(R\)-shadow coloured unit diagrams \(E_k^n\) correspond to rack \((n + 1)\)-chains. The shadow colour \(\alpha\) in Figure 11 is that of \(r_{\text{ini}}^p\) of the origin \(p\). For an \(R\)-(shadow) coloured unit diagram \(E_k^n\), denote the corresponding chain by \(\langle E_k^n \rangle\).

We recall that a diagram \(D\) is the union of the images of unit diagrams \(E_1\),

![Diagram with labels and arrows](image)

Figure 10: Coloured diagrams and corresponding chains.

\[
\begin{align*}
E_1^1 & \quad \cdots \quad \epsilon(x) \\
E_2^2 & \quad \epsilon(x, y) \\
E_3^3 & \quad \epsilon(x, y, z)
\end{align*}
\]
\[ E_1 \quad (\alpha, x) \quad \mapsto \quad -(\alpha, x) \]
\[ E_2 \quad (\alpha, x, y) \quad \mapsto \quad -(\alpha, x, y) \]
\[ E_3 \quad (\alpha, x, y, z) \quad \mapsto \quad -(\alpha, x, y, z) \]

Figure 11: Shadow coloured diagrams and corresponding chains.

\[ E_k \quad \text{Suppose that } D \text{ is } R-(\text{shadow}) \text{ coloured. Since the (shadow) colouring of } D \text{ induces that of each unit diagram, we can consider a rack chain} \]
\[ \langle D \rangle = \sum_{i=1}^{k} \langle E_i \rangle \]

as the corresponding chain of \( D \). For a rack chain \( c \), a (shadow) coloured diagram \( D \) is said to represent \( c \) if the corresponding chain \( \langle D \rangle \) equals to \( c \).

The boundary \( \partial D = (\partial M, \partial M \cap D) \) of a (shadow) coloured diagram \( D \) is also a (shadow) coloured diagram. We easily show that the corresponding chain \( \langle \partial D \rangle \) of \( \partial D \) is the image of \( \langle D \rangle \) via the boundary map of \( C^R_\ast (R) \). In Figure 12 we draw a picture of the boundary of \( E_3 \). We can easily check that
\[ \partial(x, y, z) = (y, z) - (y, z) - (x \lhd y, z) + (x, z) + (x \lhd z, y \lhd z) - (x, y) \]
or
\[ \partial(\alpha, x, y, z) = (\alpha \lhd x, y, z) - (\alpha, y, z) - (\alpha \lhd y, x \lhd y, z) + (\alpha, x, z) + (\alpha \lhd z, x \lhd z, y \lhd z) - (\alpha, x, y) \]
holds in the figure. Therefore, if a diagram \( D \) is on a closed manifold \( M \), the corresponding chain \( \langle D \rangle \) is a rack cycle. We have the inverse:
Theorem 2.3 (Carter-Kamada-Saito [CKS1]). For $n = 1, 2$ and 3, let $c$ be a rack $n$-cycle of $R$. There exists a coloured $n$- or shadow coloured $(n-1)$-diagram $D$ on a closed manifold $M$ such that $D$ represents $c$.

Proof. We prove this only for the case that a rack 3-cycle is represented by a shadow coloured 2-diagram. See [CKS1] for the precise proof.

A 3-cycle $c$ is written in the form of

$$\sum_{i=1}^{k} \epsilon_i (\alpha_i, x_i, y_i),$$

where $\epsilon_i$ is +1 or −1, and $\alpha_i, x_i$ and $y_i \in R$. Let $E_1, \ldots, E_k$ be copies of the unit 2-diagram $E_2$ and give them $R$-shadow colourings and signatures so that each $E_i$ represents $\epsilon_i (\alpha_i, x_i, y_i)$. There are $4k$ faces of $E_1, \ldots, E_k$. Since $\partial c = 0$, we can name the faces as $F_1, \ldots, F_{2k}, F'_1, \ldots, F'_{2k}$ such that $\langle F_j \rangle + \langle F'_j \rangle = 0$ for each $j$. Therefore, by attaching $F_j$ and $F'_j$ canonically, we obtain a diagram $D$ on an oriented closed surface $M$ such that $\langle D \rangle = c$ holds.

If an $R$-(shadow) coloured diagram $D$ represents a rack cycle, we denote by $[D]$ the rack homology class which $\langle D \rangle$ belongs to.

2.5 Knot quandles and their presentations.

Let $D$ be a regular projection of an oriented link $L$ on $S^2$, or a link diagram of $L$. It is clear that $D$ is a diagram on $S^2$ in the sense of [2.3].

Joyce [J] defined the knot quandles topologically, but he proved in the same paper that the knot quandles can be defined through Wirtinger presentations.
We will use the second definition of knot quandles. The symbols here follow that in [E].

The knot quandle $Q(L)$ of a link $L$ is generated by $C(D)$ with relations

$$u_p^{\text{ini}} \circ o_p = u_p^{\text{ter}}$$

for each double point (i.e., crossing) $p$. It is well known that $Q(L)$ completely determines the unoriented link type of $L$. Directly from the definition, the diagram $D$ can be considered as coloured by $Q(L)$. We call this colouring the canonical colouring of a link diagram $D$.

Since the knot group $\pi(L)$ of $L$ has the Wirtinger presentation obtained from that of $Q(L)$ by replacing $u_p^{\text{ini}} \circ o_p$ with $o_p^{-1} u_p^{\text{ini}} o_p$, the associated group of $Q(L)$ is equivalent to $\pi(L)$. Notice that the canonical homomorphism in this case is induced by the identity map on the generating set $C(D)$.

In [E], the structure of the second quandle (co)homology group of a knot quandle is determined:

**Theorem 2.4** (Eisermann [E]). Let $L$ be an $n$-component link and let $m$ be the number of non-trivial components of $L$. The knot quandle $Q(L)$ has its second quandle (co)homology groups

$$H_2^Q(Q(L); \mathbb{Z}) \cong H_2^Q(Q(L); \mathbb{Z}) \cong \mathbb{Z}^m.$$  

**Remark 2.1.** As noticed, since $D$ is a $Q(L)$-coloured 2-diagram on $S^2$, it represents a 2-cycle $\langle D \rangle$ of $Q(L)$. We call the second homology class $\langle D \rangle$ the diagram class of $D$, and call its image $[L]$ via $\rho_*: H_2^R(Q(L)) \to H_2^Q(Q(L))$ the fundamental class of $L$.

It is shown in [E] that the fundamental class is uniquely determined, and, if $L$ is non-trivial, $[L]$ is proved to be a non-zero element of $H_2^R(Q(L))$. Moreover, when $L$ is a non-trivial knot, $H_2^R(Q(L))$ is shown to be $\mathbb{Z}[L]$.

### 3 Lemmas on coloured diagrams.

Though our purpose is to construct third homology classes of knot quandles, §3 is devoted to lemmas on 1-cycles and on 2-chains of quandles in general, which play important roles in the later sections.

#### 3.1 Shadow colourability of 1-diagrams.

Let $Q$ be a quandle. Though we are concerned with rack 1-cycles, it is useful to suppose $Q$ to be a quandle here, for the associated group $G_Q$ is considered. As already mentioned (Theorem 2.3), a rack 1-cycle $c \in Z_1^R(Q)$ can be represented by a $Q$-coloured 1-diagram on a closed 1-manifold, that is, by a diagram on a disjoint union of copies of $S^1$. Shadow colourability can be considered on each circle independently. So it is sufficient only to consider diagrams on $S^1$.  

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Let $D$ be a $Q$-coloured 1-diagram on $S^1$. By reading colours and signatures of $D$ along $S^1$ starting from its basepoint, we obtain two sequences $(x_1, \ldots, x_n)$ of elements of $Q$ and $(\epsilon_1, \ldots, \epsilon_n)$ of $\pm 1$. Denote by $\Pi(D)$ the product $x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$ in the associated group $G_Q$.

**Lemma 3.1.** Let $\alpha$ be an element of $Q$. If a $Q$-coloured 1-diagram $D$ on $S^1$ has a $Q$-shadow colouring such that the shadow colour of the base-region is $\alpha$, then $\Pi(D)$ commutes with $\alpha$.

Moreover, when the canonical map $Q \to G_Q$ is injective, the inverse holds.

**Proof.** Suppose that $D$ is $Q$-shadow coloured as in the statement. Obviously, we obtain an equation

$$(\cdots (\alpha \triangleleft^{x_1} x_1) \cdots) \triangleleft^{x_n} x_n = \alpha$$

(see Figure 13). It follows that $(x_1^{\epsilon_1} \cdots x_n^{\epsilon_n})^{-1} \alpha(x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}) = \alpha$ holds in $G_Q$, that is, $\Pi(D)$ commutes with $\alpha$.

![Figure 13: Shadow coloured 1-diagram.](attachment://figure13.png)

Inversely, if $\Pi(D)$ commutes with $\alpha$ in $G_Q$, the elements of $Q$, $(\cdots (\alpha \triangleleft^{x_1} x_1) \cdots) \triangleleft^{x_n} x_n$ and $\alpha$, map to the same element of $G_Q$. Therefore, when the canonical map $Q \to G_Q$ supposed to be injective, $(\cdots (\alpha \triangleleft^{x_1} x_1) \cdots) \triangleleft^{x_n} x_n = \alpha$ holds also in $Q$. Then, by giving shadow colours to regions along $S^1$, we obtain a $Q$-shadow colouring of $D$ without contradiction.

We denote by $D_\alpha$ the $Q$-shadow coloured diagram as in Lemma 3.1. In other dimensional cases, if a whole manifold $M$ is connected, we also denote by $D_\alpha$ the $Q$-shadow coloured diagram which is obtained from a $Q$-coloured diagram $D$ by colouring the base-region with $\alpha$. A $Q$-coloured diagram $D$ is called to be freely $Q$-shadow colourable if $D_\alpha$ exists for any $\alpha \in Q$. As a consequence of Lemma 3.1, we obtain:

**Corollary 3.2.** If a $Q$-coloured 1-diagram $D$ on $S^1$ is freely $Q$-shadow colourable, $\Pi(D)$ belongs to the centre of $G_Q$.

We have seen the facts on the shadow colourability of 1-diagrams. Before dealing with the shadow colourability of 2-diagrams in §3.2, we focus on the way to construct a coloured 2-diagram which connects rack homologous 1-cycles.
Lemma 3.3. Let $D$ and $D'$ be $Q$-coloured 1-diagrams on $S^1$. There exists a $Q$-coloured 2-diagram $\tilde{D}$ on an annulus $S^1 \times I$ such that $\partial \tilde{D} = D \cup (-D')$, if and only if $\Pi(D)$ and $\Pi(D')$ are conjugate in $G_Q$.

Proof. As in Lemma 3.1, let $(x_1, \ldots, x_n)$ and $(\epsilon_1, \ldots, \epsilon_n)$ be the colours and the signatures of $D$, and let $(y_1, \ldots, y_m)$ and $(\delta_1, \ldots, \delta_m)$ be those of $D'$. If $\Pi(D)$ and $\Pi(D')$ are conjugate, two words $x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$ and $y_1^{\delta_1} \cdots y_m^{\delta_m}$ are connected by a finite sequence of replacing operations each of which is one of three types shown later. There exist subdiagrams on $S^1 \times I$ that correspond to these operations, and by gluing them we obtain a new diagram $\tilde{D}$. As the orientation of its boundary $S^1 \times \{\pm 1\}$ is considered, $\partial \tilde{D}$ is the disjoint union of two diagrams $D$ and $-D'$.

To prove “only if” part, we recall the way to decompose tangles or braids into fundamental diagrams. By a similar argument, we obtain decomposability of 1-diagrams on $S^1 \times I$ into subdiagrams as depicted in Figures 14, 15 and 16. Therefore, directly from this fact, we see that $\Pi(D)$ and $\Pi(D')$ are conjugate.

Now we will construct subdiagrams which correspond to the replacing operations. In the figures, $w_1$, $w_2$ and $w$ are words, and $a$ and $b$ are elements of $Q$. Each rectangle is considered to be an annulus as identified the left and the right edges.

1) Figure 14 shows an operation $w_1 a^{\pm 1} w_2 \longleftrightarrow w_1 w_2$. The indices $\pm 1$ of $a$’s are their signatures, which are determined by the orientation of the arc adjacent to them. We also allow the diagram turned upside down.

![Figure 14: Replacing operation of type I.](image)

2) The operation of type II comes from the definition of associated group. Replacing such as

$$w_1 (a \smallfrown b)^{\pm 1} w_2 \longleftrightarrow w_1 b^{\epsilon} a^{\pm 1} b^{\epsilon} w_2$$

is considered, where $\epsilon = \pm 1$. The corresponding subdiagram is depicted in Figure 15 where $\epsilon$ is the signature of the crossing.

3) The last operation is conjugation, that is, $wa^{\pm 1} \longleftrightarrow a^{\pm 1} w$. It corresponds to a subdiagram where a component crosses a line $\{\ast\} \times I \subset S^1 \times I$ as drawn in Figure 16.

Thus we have completed the proof of Lemma 3.3. \qed
Remark 3.1. We can exchange the operation of type II with
\[ w_1 b^\epsilon (a <^\epsilon b)^{\pm 1} w_2 \longleftrightarrow w_1 a^{\pm 1} b^\epsilon w_2 \]
and
\[ w_1 (a <^\epsilon b)^{\pm 1} b^{-\epsilon} w_2 \longleftrightarrow w_1 b^{-\epsilon} a^{\pm 1} w_2. \]

Figure 17 shows the subdiagram corresponding to the first operation, and shows its decomposition into the product of subdiagrams of types I and II.

For a trivial diagram \((S^1, \emptyset)\), the product \(\Pi(S^1, \emptyset)\) is equal to \(e\). Thus, supposing \(D'\) in Lemma 3.3 to be trivial, we have:
Corollary 3.4. A $Q$-coloured 1-diagram $D$ on $\mathbb{S}^1$ is a boundary of some $Q$-coloured 2-diagram $\tilde{D}$ on $\mathbb{D}^2$, if and only if $\Pi(D) = e$.

Proof. Let $D'$ be a trivial diagram on $\mathbb{D}^2$. Since $\Pi(D) = \Pi(\partial D') = e$, Lemma 3.3 shows that there exists a $Q$-coloured diagram $\tilde{D}'$ on $\mathbb{S}^1 \times I$ such that $\partial \tilde{D}' = D \cup (-\partial D')$. Attaching $D'$ to $\tilde{D}'$ along $\partial D'$, we obtain $\tilde{D}$ on $\mathbb{D}^2$.

3.2 Shadow colourability of 2-diagrams.

We consider here only the case that 2-diagrams are on $\mathbb{D}^2$ or on $\mathbb{S}^2$, for both $\mathbb{D}^2$ and $\mathbb{S}^2$ have trivial fundamental groups. Shadow colourability of 2-diagrams on a manifold $M$ is closely related to representations of $\pi_1(M)$ on $Q$.

Let $D$ be a 2-diagram on a square $I^2 \simeq \mathbb{D}^2$. We suppose that $\partial D$ is a set of points on $I \times \{\pm 1\}$. By the assumption, we can decompose $D$ into subdiagrams as drawn in Figure 19. As for these subdiagrams, if the shadow colour of the left-end region is given, we can decide the colour of each region from left to right order. Since the diagram $D$ is obtained by arranging subdiagrams from up to down, we have:

Lemma 3.5. If $D$ is a $Q$-coloured 2-diagram on $\mathbb{D}^2$, $D$ is freely $Q$-shadow colourable.

For a diagram $D$ on $\mathbb{S}^2$, by removing a sufficiently small disk from $\mathbb{S}^2$, we can regard $D$ as a diagram on $\mathbb{D}^2$. Therefore, directly from Lemma 3.5, we obtain:

Corollary 3.6. If $D$ is a $Q$-coloured 2-diagram on $\mathbb{S}^2$, $D$ is freely $Q$-shadow colourable.

Figure 18: Attaching or removing a trivial diagram on a disk.
As shown in the construction of $Q$-shadow colouring of $D$, the whole shadow colouring of $D$ is uniquely determined when a shadow colour $\alpha$ to the base-region is given. We denote this $Q$-shadow coloured diagram by $D_\alpha$ as in §3.1.

**Lemma 3.7.** Let $D$ be a $Q$-coloured 2-diagram on $S^2$, and let $\alpha$ and $\beta$ be connected elements of $Q$. Two shadow coloured diagrams $D_\alpha$ and $D_\beta$ represent rack homologous 3-cycles.

**Proof.** Since $\alpha$ and $\beta$ are connected, there exists a sequence $(x_1, \epsilon_1), \ldots, (x_n, \epsilon_n)$ such that $(\cdots (\alpha \triangleleft_{\epsilon_1} x_1) \cdots) \triangleleft_{\epsilon_n} x_n = \beta$. We prove this lemma by constructing a $Q$-shadow coloured 3-diagram $\tilde{D}$ on $S^2 \times I$ such that $\partial \tilde{D} = D_\alpha \cup (\neg D_\beta)$ holds. The diagram $\tilde{D}$ consists of $D \times I$ and $n$ parallel sheets $S_i = S^2 \times p_i$, where $-1 < p_1 < \cdots < p_n < 1$ (See Figure 20). Suppose that every sheet $S_i$ is in the lowest level. The intersection of $S_i$ with $D \times I$ is a diagram on $S_i \simeq S^2$, which is equivalent to $D$ (See Figure 21). By Corollary 3.6, we can regard each sheet $S_i$ as a $Q$-shadow coloured diagram with the shadow colour of the base-region being $x_i$. Also we assume that $S_i$ is oriented as its normal vector is in the same direction as that of $\{*\} \times I$ when the signature $\epsilon_i$ is +1, or in the opposite when $\epsilon_i$ is −1.

On these conditions, the regions between two sheets $S_i$ and $S_{i+1}$ can be shadow coloured with $Q$ by the similar argument in Corollary 3.6 (See the right one of Figure 21). So the whole diagram $\tilde{D}$ is $Q$-shadow colourable.

Clearly, when we colour the regions in $S \times [-1, p_1]$ by $Q$ so that $S \times \{-1\}$ becomes $D_\alpha$, the whole diagram $\tilde{D}$ has $D_\alpha \cup (\neg D_\beta)$ as its boundary. Therefore,
\[ \langle D_\alpha \rangle - \langle D_\beta \rangle = \partial \langle \tilde{D} \rangle \] holds. Since \( \langle D_\alpha \rangle \) and \( \langle D_\beta \rangle \) are rack 3-cycles, it completes the proof.

Directly from Lemma 3.7, we have:

**Corollary 3.8.** When \( Q \) is connected, a \( Q \)-coloured 2-diagram \( D \) on \( S^2 \) determines a unique third rack homology class of \( Q \).
### 3.3 Deformations of 2-diagrams.

In the rest of §3, we consider deformations of 2-diagrams which preserve the rack (or quandle) homology classes represented by the diagrams.

Let $D$ be a 2-diagram on a surface $M$. **Reidemeister deformations** are operations to replace a disk on $M$ with a new disk. There are three types of Reidemeister deformations, called R-I, R-II and R-III. In Figures 22, 23 and 24 there are drawn the disks that each deformation removes from and attaches to $M$.

![Figure 22: R-II deformation.](image)

![Figure 23: R-III deformation.](image)

Lemmas 3.9 and 3.10 are easily seen from the figures below. See [CKS1] for a detailed proof.

**Lemma 3.9.** Let $D$ and $D'$ be $Q$-(shadow) coloured 2-diagrams on $M$. If $D'$ can be obtained from $D$ by an R-II or R-III deformation, then chains $\langle D \rangle$ and $\langle D' \rangle$ are rack homologous.

On the other hand, an R-I deformation changes the represented rack homology class, but it keeps the quandle homology class:

**Lemma 3.10.** If $D'$ can be obtained from $D$ by an R-I deformation, then chains $\langle D \rangle$ and $\langle D' \rangle$ are quandle homologous.
4 Shadow diagram classes of a link.

In this section, we will construct some elements of the third homology groups of the quandle $Q(L)$ of a link $L$. These homology classes are derived from the concepts of the diagram classes and of the shadow colourings, thus we call them the shadow diagram classes of $L$. The construction is motivated by the shadow cocycle invariants in [CKS1]. We also show the relation between the shadow cocycle invariants and the shadow diagram classes. As an application, we generalise the result in [S] using the shadow diagram classes.

4.1 Shifting and splitting homomorphisms.

Before constructing shadow diagram classes, we introduce two homomorphisms of rack homology groups: the one is a rack version of the “shifting homomorphism” defined in [CJKS3], and the another is the splitting homomorphism in [LN].

For a rack $R$, define a homomorphism $\sigma_n: C_n^R(R; A) \to C_{n-1}^R(R; A)$ by linearly extending a map on its basis

$$(x_1, x_2, \ldots, x_n) \mapsto (x_2, \ldots, x_n).$$

By easy calculation, we can see that

$$\partial_{n-1} \circ \sigma_n(x_1, x_2, \ldots, x_n) = \partial_{n-1}(x_2, \ldots, x_n)$$

$$= \sum_{i=2}^n (-1)^{(n-1)-(i-1)} \left\{ \left( x_2 \triangleleft x_i, \ldots, x_{i-1} \triangleleft x_i, x_{i+1}, \ldots, x_n \right) - \left( x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \right) \right\}$$

$$= \sigma_{n-1} \left( \sum_{i=1}^n (-1)^{n-1} \left\{ \left( x_1 \triangleleft x_i, \ldots, x_{i-1} \triangleleft x_i, x_{i+1}, \ldots, x_n \right) - \left( x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \right) \right\} \right)$$

$$= \sigma_{n-1} \circ \partial_n(x_1, x_2, \ldots, x_n)$$

for each $n$, thus $\sigma_n$ is a chain map $C^R_n(R; A) \to C^R_{n-1}(R; A)$, which induces a homomorphism $H^R_n(R; A) \to H^R_{n-1}(R; A)$. We call this the shifting homomorphism and denote it also by $\sigma_n$. 

---

Figure 24: R-I deformation.
For a quandle $Q$, Litherland-Nelson constructed an endomorphism $\alpha_*$ of $C^R_n(Q;A)$ which gives a splitting of $C^R_n(Q;A)$ into the direct sum of $C^D_n(Q;A)$ and $\alpha_*C^R_n(Q;A)$ as chain complexes. To begin with, we introduce a product of $\bigoplus C^R_n(Q;A)$ defined on its bases as

$$(x_1, \ldots, x_n) \bullet (y_1, \ldots, y_m) = (x_1, \ldots, x_n, y_1, \ldots, y_m).$$

Through this product, we can obtain a series of endomorphisms of $C^R_n(Q;A)$ defined as

$$\alpha_n: (x_1, x_2, \ldots, x_n) \mapsto x_1 \bullet (x_2 - x_1) \bullet \cdots \bullet (x_n - x_{n-1})$$

on its basis, where, in the R.H.S., $x_1, x_2 - x_1, \ldots, x_n - x_{n-1}$ are all considered as elements of $C^R_n(Q;A)$. It is proved in [LN] that $\alpha_*$ becomes a chain map from $C^R_n(Q;A)$ to itself. Through this chain map, we have:

**Theorem 4.1** (Litherland-Nelson [LN]). For an arbitrary quandle $R$, the rack chain complex splits into the direct sum of the degeneracy chain complex and the image of $\alpha_*$, that is,

$$C^R_n(Q;A) = C^D_n(Q;A) \oplus \alpha_*C^R_n(Q;A)$$

holds, and therefore $C^Q_n(Q;A)$ is isomorphic to $\alpha_*C^R_n(Q;A)$.

Now we have a projection $H^R_n(Q;A) \to H^Q_n(Q;A)$, denoted also by $\alpha_*$, and call it the splitting homomorphism. Theorem 2.1 and Corollary 2.2 are direct consequences of this theorem.

### 4.2 Two homomorphisms on diagrams.

At first, we give a diagrammatic translation of the shifting homomorphisms. Let $R$ be a rack. As noticed in §2.4 every rack $n$-chain $c$ of $R$ can be represented by an $R$-shadow coloured $(n-1)$-diagram $D$. The first factor of each basis of $c$ corresponds to the shadow colour of some region, therefore, we can see that the $(n-1)$-chain $\sigma_n(c)$ is represented by the $R$-coloured $(n-1)$-diagram obtained by disregarding the shadow colours of $D$. We denote this diagram by $\sigma_n D$.

Considering the inverse operation, that is, to give shadow colours to regions of a coloured diagram, we can make some higher homology classes from a lower one. In §3 we have already seen sufficient conditions for a diagram to have an additional shadow colouring.

Splitting homomorphism is more complicated to be translated diagrammatically. For each cycle $c \in C^Q_n(Q)$ of a quandle $Q$, let $\hat{c}$ be an element of $C^R_n(Q)$ such that $\hat{c}$ is mapped to $c$ through the canonical projection $\rho: C^R_n(Q) \to C^Q_n(Q)$. Similarly as in Theorem 2.3 we have a shadow coloured diagram $D$ which represents $\hat{c}$. Though the diagram representing a rack cycle can be supposed to be on some closed manifold, $D$ may be a diagram with boundary in this case. Since $\partial_n(c) = 0$ holds in $C^Q_n(Q) = C^R_n(Q)/C^D_n(Q)$, $\partial D$ can be supposed to represent the degeneracy chain $\partial_n(\hat{c})$. 
Our purpose here is to realise $\alpha_*$ diagrammatically as an operation to make a diagram on a closed manifold representing a given quandle cycle. Obviously, from Theorem 2.1, each quandle homology class can be represented by some rack cycle, and thus it can be represented by a (shadow) coloured diagram on a closed manifold. However, since we are using a diagram to obtain higher homology classes, it is necessary to examine that every diagram representing the quandle cycle can be transformed to another on a closed manifold.

We are concerned only with cases of $Q$-shadow coloured 1- or 2-diagrams. In these cases, splitting homomorphisms are explicitly in the following forms:

\[
\alpha_2(x, y) = x \bullet (y - x) = (x, y) - (x, x),
\]
\[
\alpha_3(x, y, z) = x \bullet (y - x) \bullet (z - y)
= (x, y, z) - (x, y, y) + (x, x, y) - (x, x, z).
\]
As for quandle 2-cycles, since $C_1^Q(Q)$ is a zero module, they are also rack cycles in themselves. Thus a quandle 2-cycle can be represented by a shadow coloured 1-diagram on a circle or a disjoint union of finitely many circles. So it is sufficient to add some points on this diagram as shown in Figure 25. The new diagram $D'$ obtained by this operation obviously represents $\alpha_2(\hat{c})$.

Before considering the case of 3-cycles, we will discuss the diagrammatic translation of $\alpha_2$ by coloured diagrams without shadow colourings. Also in this case, we can suppose a 2-diagram representing a quandle 2-cycle to be on a closed surface or a disjoint union of closed surfaces. To add degeneracy terms $(x, x)$ of $\alpha_2(x, y)$, we modify the diagram at each crossing by RI transformations, as drawn in Figure 26. In this figure, the initial under-arc is twisted to generate a term $(x, x)$.

Fix a quandle 3-cycle $c \in Z_3^Q(Q)$ and $Q$-shadow coloured 2-diagram $D$ which represents $\hat{c}$. At first, we modify the diagram $D$ so that $\sigma D$ represents $\alpha_2(D)$ as
Next, we will parallelise the arcs of $D$ such that the additional arcs are supposed to be in the lowest level and to be smoothened near each crossing, as drawn in Figure 27. For simplicity, the trivial components to appear are removed there.

Easily, we can see that the new diagram $D'$ represents $\alpha_3(\hat{c})$, but it still has its boundary. Since $\partial_3(\hat{c})$ is a degeneracy 2-chain, each additional point on $\partial D'$ lies next to an original point with the same colour and with the opposite signature. Thus, by connecting these pairs with arcs, we obtain another diagram $D''$ with trivially coloured boundary, which can be capped off by disks.

### 4.3 Construction of shadow diagram classes.

Let $L$ be a link and $D$ a diagram of $L$ on $S^2$. As mentioned in §2.5, the diagram $D$ is canonically coloured by $Q(L)$, thus $D$ is freely $Q(L)$-shadow colourable by Corollary 3.6. Denote by $D_\alpha$ a $Q(L)$-shadow coloured diagram obtained from $D$ by colouring the base-region with $\alpha \in Q(L)$. The rack 3-chain $\langle D_\alpha \rangle$ is a cycle, for it is represented by a diagram on a closed manifold. Thus, we have a rack homology class $[D_\alpha] \in H_{\text{R}3}(Q(L))$ from $D_\alpha$, called a shadow diagram class of $L$.

Now, we will show one of the main theorems:

**Theorem 4.2.** Let $L$ be a non-trivial $n$-component link.

a) The shadow diagram classes $[D_\alpha]$ are non-zero elements of $H_{\text{R}3}(Q(L))$ for each diagram $D$ of $L$ and $\alpha \in Q(L)$.

b) There exist linearly independent $n$ shadow fundamental classes of $L$ in $H_{\text{Q}3}(Q(L))$.

c) The third quandle homology group $H_{\text{Q}3}(Q(L))$ splits into the direct sum

$$\bigoplus [L_i] \oplus \left( H_{\text{Q}3}(Q(L))/\bigoplus [L_i] \right),$$

where $[L_1], \ldots, [L_n]$ are distinct shadow fundamental classes of $L$.

**Remark 4.1.** If $L$ has more than one component, the shadow ‘fundamental’ class is not unique. It seems to be slightly confusing, but, as far as the shadow cocycle invariants concern, all the shadow fundamental classes work in the same way.

**Proof.** We have already noticed that the shifting homomorphism is an operation to disregard shadow colours of diagrams. Thus, it is clear that $\sigma_3(D_\alpha) = (D)$, which concludes $\rho_\ast \sigma_3(D_\alpha) = \rho_\ast [D] = [L]$. By Theorem 2.4, the fundamental class $[L]$ is non-zero since $L$ is non-trivial. It completes the proof of (a).
If \( \alpha \) and \( \beta \in Q(L) \) are connected, \( D_\alpha \) and \( D_\beta \) represent rack homologous 3-cycles by Lemma 3.7 so \([D_\alpha] = [D_\beta]\) holds in \( H^3_3(Q(L)) \).

Let \( D \) and \( D' \) be diagrams of \( L \) on \( S^2 \). We can obtain \( D' \) from \( D \) by finitely many Reidemeister deformations. Through these deformations, a \( Q(L) \)-shadow coloured diagram \( D_\alpha \) becomes another \( Q(L) \)-shadow coloured diagram \( D'_\beta \). Clearly, \( \beta \) is connected with \( \alpha \). Therefore, Lemmas 3.7, 3.9 and 3.10 show that \( \langle D'\rangle \), \( \langle D'_\beta \rangle \) and \( \langle D_\alpha \rangle \) are all quandle homologous.

Thus we conclude that \([L_\alpha]\) is determined independently of the choice of diagrams and of the shadow colours of the base-region as long as the shadow colours are connected. It is well known that the knot quandle of \( n \)-component link has \( n \) orbits. So, it follows that there exist at most \( n \) shadow fundamental classes of \( L \).

Let \( \alpha_1, \ldots, \alpha_n \) be elements of \( Q(L) \) which are not connected each other. To prove that \([D_{\alpha_1}], \ldots, [D_{\alpha_n}]\) are linearly independent, we use some evaluation maps. For a 2-cocycle \( \phi \in Z^2_2(Q(L)) \) and an element \( \beta \in Q(L) \), define a rack 3-cocycle \( \tilde{\phi}_\beta \) by

\[
\tilde{\phi}_\beta(\alpha, a, b) = \begin{cases} 
\phi(a, b) & \text{if } \alpha \text{ and } \beta \text{ are in the same orbit}, \\
0 & \text{otherwise}.
\end{cases}
\]

Easy computation shows that \( \tilde{\phi}_\beta \) is also a cocycle. Denote by \( S \) a linear combination \( \sum c_i[D_{\alpha_i}] \). Clearly by the definition, we have

\[
\langle S, [\tilde{\phi}_{\alpha_i}] \rangle = c_i\langle [D], [\phi] \rangle.
\]

Thus, if we can choose a non-trivial cocycle \( \phi \) such that \( \langle [D], [\phi] \rangle \) is non-zero, then the classes \([D_{\alpha_1}], \ldots, [D_{\alpha_n}]\) are linearly independent. In fact, Eisermann’s results in [E] says that there exists such a cocycle \( \phi \) of \( Q(L) \) when \( L \) is non-trivial.

By the definition, the shadow fundamental class \([L_\alpha]\) can be written in the form of \( \rho_\alpha[D_\alpha] \) for some knot diagram \( D \) of \( L \). As shown in \( \S 4.2 \), \( \alpha_\ast \rho_\ast[D_\alpha] \) can be represented by some shadow coloured diagram on a closed surface. By observing the operations in \( \S 4.2 \) precisely, we can see that the resulting diagram \( D'_\alpha \) from \( D_\alpha \) is also on a sphere \( S^2 \). Moreover, the additional arcs of \( D'_\alpha \) are all simple closed curves and in the lowest level of the diagram. Since such curves on \( S^2 \) can be removed via Reidemeister deformations II and III, the 3-chain \( \langle D'_\alpha \rangle \) is rack homologous to \( \langle D_\alpha \rangle \) from Lemma 3.9 which concludes that \([L_\alpha]\) is equal to one of the shadow diagram classes. Our proof of (b) is completed.

Fix \( \alpha \in Q(L) \). From the fact above, if a diagram \( D \) with the canonical colouring represents the fundamental class \([L]\) in \( H^3_3(Q(L)) \), then the shadow coloured diagram \( D_\alpha \) represents the shadow fundamental class \([L_\alpha]\) in \( H^3_3(Q(L)) \). Thus we have two homomorphisms

\[
H^Q_2(Q(L)) \cong \mathbb{Z}[L] \cong \mathbb{Z}[L_\alpha] \subset H^3_3(Q(L))
\]

and
H^R_3(Q(L)) \cong H^R_2(Q(L)) \cong H^Q_2(Q(L)) \cong \mathbb{Z}[L].

Easily, we can check diagrammatically that the composition of them becomes the identity on \( \mathbb{Z}[L] \). Thus, with Theorem 2.1, \( H^Q_3(Q(L)) \) is proved to split as in the statement of (c).

\[ \qed \]

4.4 Relations to shadow cocycle invariants.

Shadow cocycle invariants \( \Phi_\phi \) defined in [CJKS] are invariants of links computed with a quandle 3-cocycle \( \phi \) of a finite quandle \( X \). Satoh [S] introduced based shadow cocycle invariants \( \Phi_\phi^{\ast} \) of links, and he proved that, if \( X \) is a dihedral quandle \( \mathbb{Z}_p \) for a prime odd \( p \), an equation \( \Phi_\phi = |X| \cdot \Phi_\phi^{\ast} \) holds for the two invariants. Here, we will prove this equation for a connected quandle \( X \) in general, by using the concept of shadow fundamental classes.

Let \( L \) be a link and \( D \) a diagram of \( L \). At first, we will define the \textbf{shadow cocycle invariant} \( \Phi_\phi(L) \) and the \textbf{based shadow cocycle invariant} \( \Phi_\phi^{\ast}(L) \) of \( L \). Fix a quandle 3-cocycle \( \phi \in \mathbb{Z}^3_Q(X; A) \) of a finite quandle \( X \), where the operation of \( A \) is supposed to be written as multiplication. When \( C \) is an \( X \)-shadow colouring of \( D \), we define the \textbf{Boltzmann weight} \( w(c) \) of each crossing \( c \) by

\[ w(c) = \phi(C(r_{\text{ini}}^c), C(u_{\text{ini}}^c), C(o_c))^c. \]

The symbols above follow in \S\ 2.4. Then, we define the \textbf{whole weight} \( W(C) \) of \( C \) by the product \( \prod w(c) \) of weights of all crossings.

Both the unbased and the based shadow cocycle invariants are of the state-sum type. They use \( X \)-shadow colourings as states, but they differ on the colourings which they allow as states.

In the case of shadow cocycle invariants, all \( X \)-shadow colourings are allowed to be states of \( D \). Therefore, \( \Phi_\phi(L) \) is the sum \( \sum W(C) \), where \( C \) ranges all \( X \)-shadow colourings.

On the other hand, when we are concerned with the based shadow cocycle invariants, we fix a point \( p \) of an arc of \( D \) which is not a crossing. An \( X \)-shadow colouring is allowed to be a state when the regions \( r_{\text{ini}}^p \) and \( r_{\text{iter}}^p \) have as same a colour as \( c_p \) (See Figure 29). It is proved in [S] that the sum \( \Phi_\phi^{\ast}(L) \) of whole weights of all colourings that satisfy the condition as above is invariant of \( L \).

\textbf{Remark 4.2.} The invariant \( \Phi_\phi^{\ast} \) has a name as “based”, for the chosen point \( p \) is called a “basepoint” in [S]. Since we are necessary to consider the basepoint of a manifold where a diagram lies, it is confusing to call \( p \) a basepoint. Thus, we use the term “chosen point” instead of basepoint.

\textbf{Theorem 4.3.} If \( X \) is a connected finite quandle, an equation

\[ \Phi_\phi(L) = |X| \cdot \Phi_\phi^{\ast}(L) \]

holds between the unbased and the based shadow cocycle invariants.
Proof. Let $p$ be a chosen point of a diagram $D$ of $L$ and set $k = |X|$. Fix an $X$-colouring $C$ of $D$. From Corollary 3.6, when we give a shadow colour $\alpha$ to the region $r_p$, we have a whole shadow colouring of $D$. Thus there are $k$ shadow colourings $C_1, \ldots, C_k$ extending $C$. Clearly, only one of them, say $C_1$, is an allowed colouring, others are not. Since $X$ is connected, Corollary 3.8 says that $[D_1] = \cdots = [D_k]$, where $D_i$ denotes a shadow coloured diagram $D$ with $C \cup C_i$. Therefore we have an equation

$$\sum_i \langle [D_i], \phi \rangle = |X| \cdot \langle [D_1], \phi \rangle.$$ 

This lemma is a direct conclusion of the equation. \qed

5 Topological realisation of 3-cycles.

5.1 Surgeries on coloured diagrams.

When the homology theory with integral coefficients concerns, every 3-cycle $c$ in $Z^3_\mathbb{Z}(Q(K))$ can be represented by a $Q(K)$-shadow coloured diagram $D$ on some closed surface $M$. We prove, under some conditions, that $D$ can be chosen as a diagram on $S^2$.

Theorem 5.1. Let $K$ be a prime knot. For any quandle 3-cycle $c \in Z^3_\mathbb{Z}(Q(K))$, there exists a pair of a link $L$ and a homomorphism $f : Q(L) \to Q(K)$ such that $[c] = f_*[L_{sh}]$, where $[L_{sh}]$ is one of the shadow fundamental classes of $L$.

Proof. As already seen in §4.2, there exists a $Q(K)$-shadow coloured 2-diagram $D$ on a closed surface $M$ such that $D$ represents $[c]$. Denote by * the basepoint of $M$ and denote by $a$ the shadow colour of the base-region. Throughout the proof, closed curves on $M$ are supposed to be based, that is, they start from *. A closed curve $C$ on $M$ is called to be in generic position when $C$ does not pass any crossings of $D$ and it crosses over $D$ transversely where it intersects with $D$. Any closed curve $C$ on $M$ can be transformed homotopically into a new curve
$C'$ in generic position, thus we will omit the notation about genericity. Also we
notice that the intersection of a curve $C$ with the diagram $D$ makes $(C, D \cap C)$
a $Q(K)$-shadow coloured 1-diagram. We denote this diagram also by $C$.

Suppose that there exists an essential curve $C$ on $M$ such that $\Pi(C) = e \in \pi(K)$. By Corollary 3.24 and Lemma 3.23 there exists a $Q(K)$-shadow coloured $2$-diagram $D_1$ on $D^2$ such that $\partial D_1 = C$. Therefore, by cutting $M$ along $C$, and by attaching $D_1$ to $C$ and $-D_1$ to $-C$, as depicted in Figure 30, we have

a new diagram $D'$ on a new surface $M'$ with genus less than that of $M$. It is
clear that $D'$ represents $[D] + [D_1] - [D_1] = [c]$.

After repeating the above argument, we can suppose that, for any essential
curve $C$ on $M$, $\Pi(C)$ is not trivial. Since $\Pi(C)$ has $Q(K)$-shadow colouring,
Lemma 3.41 says that $\Pi(C)$ commutes with $a$. We notice that $\Pi(C)$ and $a$ are elements of $\pi(K)$, thus they are represented by some loops in the exterior $E(K)$
of $K$. We denote the loops by the same symbols $\Pi(C)$ and $a$, respectively.
Choose a homotopy $H$: $\Pi(C) \cdot a \simeq a \cdot \Pi(C)$. Clearly, $H$ is an image of a torus
in $E(K)$, via some continuous map. Let $\lambda$ be a longitude of $K$ such that $a$ and $\lambda$ gives the peripheral system of $K$. By the fact that $a$ bounds some meridian
disk and the assumption that $K$ is prime, we conclude that $\Pi(C)$ is written in
the form of $\lambda^i a^j$.

We can change the word presentation of $\Pi(C)$ with preserving the homology
class which $D$ represents. By Lemma 3.3 for two word presentations of $\Pi(C)$,
we have a shadow coloured diagram $D_2$ on $S^1 \times I$. As drawn in Figure 31
by cutting $M$ along $C$ and attaching two diagrams $D_2$ and $-D_2$, we obtain a new
curve $C'$ on a diagram where $\Pi(C') = \Pi(C)$ is presented by another word. Thus
we can suppose that the word presentation of $\lambda$ is uniform.

Cut $M$ into a $2n$-gon $N$ along $n$ essential curves $C_1, \ldots, C_n$. Obviously,
$\Pi(\partial N)$ is the product of $\Pi(C_1)$, $\ldots$, $\Pi(C_n)$ and $\Pi(C_1)^{-1}$, $\ldots$, $\Pi(C_n)^{-1}$ in some order. On the boundary $\partial N$, there exist the same numbers of $\lambda$- and $\lambda^{-1}$-
segments, and some $a$-segments as drawn in Figure 32. We will first remove
Figure 31: Deforming word presentations.

Figure 32: Boundary of 2n-gon $N$.

the $\lambda^{\pm 1}$-segments from $\partial N$. When $\lambda$- and $\lambda^{-1}$-segments are adjacent, we easily connect them together. This surgery is shown in the left two of Figure 33. If an $a$-segment exists between $\lambda^{\pm 1}$-segments, we attach a checker-board diagram there, which is depicted in the right two of the same figure. The latter surgery does not preserve the rack homology classes. However, since the checker-board diagram represents a degeneracy 3-chain, the quandle homology classes of the diagrams are invariant under this surgery.

Therefore, by repeating those surgeries, we have a new polygon $N'$ such that there are only $a$-segments on $\partial N'$. Since $\Pi(\partial N') = \Pi(\partial N) = e$ holds and since all the endpoints of arcs are coloured with $a$, there are the same numbers of points with signature $+1$ and with $-1$. Connecting adjacent points with opposite signatures by an arc make the diagram $N'$ simpler. After finitely many times of surgeries as such, a diagram on $D^2$ with trivially coloured boundary is obtained. Finally, by capping off the diagram, we have a diagram $\tilde{D}$ on $S^2$ such that a 3-cycle $\langle \tilde{D} \rangle$ is quandle homologous to $c$.

The diagram $\tilde{D}$ can be regarded as a diagram of a link $L$. Obviously, a $Q(K)$-colouring of a link diagram of $L$ gives a homomorphism $f: Q(L) \rightarrow Q(K)$, which completes the proof. \qed
5.2 Examples of computation.

Since our motivation is to determine the third quandle homology group of knot quandle $Q(K)$, it is natural to consider whether the quotient $H^Q_3(Q(K))/\mathbb{Z}[K_{sh}]$ is trivial or not.

By Theorem 5.1 for a prime knot $K$, it is sufficient to consider the images of the shadow fundamental classes of some links via homomorphisms between knot quandles.

Regrettably, we could not found any elements of $H^Q_3(Q(K))/\mathbb{Z}[K_{sh}]$. Thus, further research is expected.

We show two examples of computations. Let $K$ be a knot $4_1$. The homomorphisms which we use are due to the tables in [KS]. At the first, we compute the shadow fundamental class of $K$. By Figure 34 we obtain

$$[K_{sh}] = (4,1,4) - (1,3,1) - (1,1,3) + (1,3,2) = (4,1,4) - (1,3,1) + (1,3,2).$$

**Example 5.1.** Let $L$ be a knot $9_{37}$. We have a surjective homomorphism $f$ from $Q(L)$ to $Q(K)$ defined by

$$1 \mapsto 2, \ 2 \mapsto 3, \ 3 \mapsto 4, \ 4 \mapsto 3, \ 5 \mapsto 1, \ 6 \mapsto 4 < 1, \ 7 \mapsto 4, \ 8 \mapsto 1, \ 9 \mapsto 4.$$

The induced shadow colouring of $L$ is shown in Figure 35. We disregard the shadow colours which are not used in the calculus below. Now we have the
image of $[L_{sh}]$ via $f_\ast$ as follows:

$$f_\ast[L_{sh}] = -(1, 3, 1) - (1, 1, 3) - (1 \blacktriangleleft 4, 4 \blacktriangleleft 1, 4)$$

$$+ (4, 1, 3) + ((1 \blacktriangleleft 4) \blacktriangleleft 1, 4 \blacktriangleleft 1, 4) - (1 \blacktriangleleft 4, 1, 4 \blacktriangleleft 1)$$

$$+ (1, 1, 2) + ((1 \blacktriangleleft 4) \blacktriangleleft 1, 4, 4 \blacktriangleleft 1) - (1 \blacktriangleleft 4, 4, 1)$$

$$= -(1, 3, 1) + (4, 1, 4) + (1, 3, 2)$$

$$- (4, 1, 4) - (1, 3, 2) - (1 \blacktriangleleft 4, 4 \blacktriangleleft 1, 4)$$

$$+ (4, 1, 3) + ((1 \blacktriangleleft 4) \blacktriangleleft 1, 4 \blacktriangleleft 1, 4) - (1 \blacktriangleleft 4, 1, 4 \blacktriangleleft 1)$$

$$+ (1, 1, 2) + ((1 \blacktriangleleft 4) \blacktriangleleft 1, 4, 4 \blacktriangleleft 1) - (1 \blacktriangleleft 4, 4, 1)$$

$$= [K_{sh}]$$

$$- ((1 \blacktriangleleft 4) \blacktriangleleft 1, 4, 4) - (1 \blacktriangleleft 4, 4 \blacktriangleleft 1, 1) - (1 \blacktriangleleft 4, 4, 1)$$

$$+ (4, 1, 3) + ((1 \blacktriangleleft 4) \blacktriangleleft 1, 4 \blacktriangleleft 1, 4) - (1 \blacktriangleleft 4, 1, 4 \blacktriangleleft 1)$$

$$+ (1, 1, 2) + ((1 \blacktriangleleft 4) \blacktriangleleft 1, 4, 4 \blacktriangleleft 1) - (1 \blacktriangleleft 4, 4, 1)$$

$$= [K_{sh}]$$

$$+ \{-(1 \blacktriangleleft 4, 4 \blacktriangleleft 1, 1) + (4, 1, 3)\}$$
Example 5.2. Let $L$ be a knot $10_{59}$. A homomorphism $f: Q(L) \to Q(K)$ defined by

$$
1 \mapsto 1, \quad 2 \mapsto 1, \quad 3 \mapsto 4, \quad 4 \mapsto 1, \quad 5 \mapsto 1,
6 \mapsto 3, \quad 7 \mapsto 4 \triangleright 1, \quad 8 \mapsto 4, \quad 9 \mapsto 3, \quad 10 \mapsto 2
$$

is surjective. Figure 36 shows the induced shadow colouring of $L$. In this case, we have $f_*[L_{sh}] = -2[K_{sh}]$.

Figure 36: Shadow coloured diagram of $10_{59}$.

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