Abstract

We study the non-parametric estimation of the value $\vartheta(f)$ of a linear functional evaluated at an unknown density function $f$ with support on $\mathbb{R}^+$ based on an i.i.d. sample with multiplicative measurement errors. The proposed estimation procedure combines the estimation of the Mellin transform of the density $f$ and a regularisation of the inverse of the Mellin transform by a spectral cut-off. In order to bound the mean squared error we distinguish several scenarios characterised through different decays of the upcoming Mellin transforms and the smoothnes of the linear functional. In fact, we identify scenarios, where a non-trivial choice of the upcoming tuning parameter is necessary and propose a data-driven choice based on a Goldenshluger-Lepski method. Additionally, we show minimax-optimality over Mellin-Sobolev spaces of the estimator.

Keywords: Linear functional model, multiplicative measurement errors, Mellin-transform, Mellin-Sobolev space, minimax theory, inverse problem, adaptation

AMS 2000 subject classifications: Primary 62G05; secondary 62F10, 62C20,

1 Introduction

In this paper we are interested in estimating the value $\vartheta(f)$ of a linear functional evaluated at an unknown density $f : \mathbb{R}^+ \to \mathbb{R}^+$ of a positive random variable $X$, when $Y = XU$ for...
some multiplicative positive error term $U$ is only observable. We assume that $X$ and $U$ are independent of each other and that $U$ has a known density $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$. In a multiplicative measurement errors model the density of $f_Y : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ of the observable $Y$ is thus given by

$$f_Y(y) = [f \ast g](y) = \int_0^\infty f(x)g(y/x)x^{-1}dx, \quad y \in \mathbb{R}^+,$$

such that $\ast$ denotes multiplicative convolution. Therefore, the estimation of $f$ and hence $\vartheta(f)$ using an i.i.d. sample $Y_1, \ldots, Y_n$ from $f_Y$ is called a multiplicative deconvolution problem, which is an inverse problem.

Vardi [1989] and Vardi and Zhang [1992] introduce and study intensively multiplicative censoring, which corresponds to the particular multiplicative deconvolution problem with multiplicative error $U$ uniformly distributed on $[0, 1]$. Multiplicative censoring is a common challenge in survival analysis as explained and motivated in Van Es et al. [2000]. The estimation of the cumulative distribution function of $X$ is considered in Vardi and Zhang [1992] and Agharjian and Wolfson [2005]. Series expansion methods are studied in Andersen and Hansen [2001] treating the model as an inverse problem. The density estimation in a multiplicative censoring model is considered in Brunel et al. [2016] using a kernel estimator and a convolution power kernel estimator. Assuming a uniform error distribution on an interval $[1-\alpha, 1+\alpha]$ for $\alpha \in (0, 1)$ Comte and Dion [2016] analyze a projection density estimator with respect to the Laguerre basis. Belomestny et al. [2016] study a beta-distributed error $U$.

The multiplicative measurement error model covers all those three variations of multiplicative censoring. It was considered by Belomestny and Goldenshluger [2020] for the point-wise density estimation. The key to the analysis of multiplicative deconvolution is the multiplication theorem, which for a density $f_Y = f \ast g$ and their Mellin transforms $\mathcal{M}[f_Y], \mathcal{M}[f]$ and $\mathcal{M}[g]$ (defined below) states $\mathcal{M}[f_Y] = \mathcal{M}[f] \mathcal{M}[g]$. Exploiting the multiplication theorem Belomestny and Goldenshluger [2020] introduce a kernel density estimator of $f$ allowing more generally $X$ and $U$ to take also negative values. Moreover, they point out that transforming the data by applying the logarithm is a special case of their estimation strategy. Note that by applying the logarithm the model $Y = XU$ writes $\log(Y) = \log(X) + \log(U)$, and hence multiplicative convolution becomes (additive) convolution for the log-transformed data. As a consequence, first the density of $\log(X)$ is eventually estimated employing usual strategies for non-parametric (additive) deconvolution problems (see for example Meister [2009]) and then secondly transformed back to an estimator of $f$. Thereby, regularity conditions commonly used in (additive) deconvolution problems are imposed on the density of $\log(X)$, which however is difficult to interpret as regularity conditions on the density of $f$. Furthermore, the analysis of a global risk of an estimator $f$ using this naive approach is challenging as Comte and Dion [2016] point out.

The global estimation of the density under multiplicative measurement errors is considered in
Brenner Miguel et al. [2021] using the Mellin transform and a spectral cut-off regularization of its inverse to define an estimator for the unknown density \( f \). Brenner Miguel [2021] studies the global density estimation under multiplicative measurement errors for multivariate random variables while the global estimation of the survival function can be found in Brenner Miguel and Phandoidaen [2021]. In this paper we estimate the value \( \vartheta(f) \) of a known linear functional of the unknown density \( f \) plugging in the estimator of \( f \) proposed by Brenner Miguel et al. [2021]. In additive deconvolution linear functional estimation has been studied for instance by Butucea and Comte [2009], Mabon [2016] and Pensky [2017] to mention only a few. In the literature, the most studied examples for estimating linear functionals is point-wise estimation of the unknown density \( f \), the survival function, cumulative distribution function (c.d.f.) or the Laplace transform of \( f \). These examples are particular cases of our general setting. More precisely, we show below, that in each of those examples the quantity of interest can be written as linear functional in the form

\[
\vartheta(f) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(-t) M[f](t) dt,
\]

where \( \Psi : \mathbb{R} \to \mathbb{C} \) is a known function and \( M[f] \) denotes the Mellin transform of \( f \).

Exploiting properties of the Mellin transform we characterize the underlying inverse problem and natural regularity conditions which borrow ideas from the inverse problems community (see e.g. Engl et al. [2000]). More precisely, we identify conditions on the decay of the Mellin transform of \( f \) and \( g \) and of the function \( \Psi \) to ensure that our estimator is well-defined. We illustrate those conditions by different scenarios. The proposed estimator, however, involves a tuning parameter and we specify when this parameter has to be chosen non-trivially. For that case, we propose a data-driven choice of the tuning parameter inspired by the work of Goldenshluger and Lepski [2011] who consider data-driven bandwidth selection in kernel density estimation. We establish an oracle inequality for the plug-in spectral cut-off estimator under fairly mild assumptions on the error density \( g \). Moreover we show that uniformly over Mellin-Sobolev spaces the proposed estimator is minimax-optimal.

The paper is organized in the following way: in section 2 we develop the data-driven plug-in estimator and introduce our basic assumptions. We state an oracle type upper bound for the mean squared error of the plug-in spectral cut-off estimator with fully-data driven choice of the tuning parameter. In section 3 we state a maximal upper bound over Mellin-Sobolev spaces mean squared error of the spectral cut-off estimator for the plug-in spectral cut-off estimator with optimal tuning parameter realising a squared-bias-variance trade-off and lower bounds for the point-wise estimation of the unknown density \( f \), the survival function and the c.d.f. The proofs can be found in the appendix.
2 Data-driven estimation

We begin this section by introducing the Mellin transform and collecting some of its properties. We define for a measurable weight function \( \omega : \mathbb{R} \rightarrow \mathbb{R}^+ \), a constant \( p \in \mathbb{R}^+ \) and a measurable set \( \Omega \subseteq \mathbb{R} \) the weighted \( L^p_\Omega(\omega) \)-norm of any measurable function \( h : \Omega \rightarrow \mathbb{C} \) by
\[
\|h\|_{L^p_\Omega(\omega)} := \int_\Omega |h(x)|^p \omega(x) \, dx.
\]
Denote by \( L^p_\Omega(\omega) \) the set of all measurable functions from \( \Omega \) to \( \mathbb{C} \) with finite \( \| \cdot \|_{L^p_\Omega(\omega)} \)-norm. Let \( C \) denote by \( \| \cdot \|_2 \) for measurable set properties. We define for a measurable weight function \( \omega : \mathbb{R} \rightarrow \mathbb{R}^+ \)
\[
\int_\mathbb{R} \omega(x) \, dx < \infty \quad \text{for} \quad \omega \in C.
\]

Let \( \Omega \) be a well-defined isomorphism and denote by \( \Phi : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \) the corresponding weighted scalar product. Using a slight abuse of notation \( x^a \) with \( a \in \mathbb{R} \) denotes the weight function \( x \mapsto x^a \), and we write \( \| \cdot \|_a := \| \cdot \|_{L^2_\Omega(x^a)} \), respectively \( \langle \cdot, \cdot \rangle_{x^a} := \langle \cdot, \cdot \rangle_{L^2_\Omega(x^a)} \). Further we use the abbreviation \( L^p_{\mathbb{R}} = L^p_{\mathbb{R}}(\omega) \) for the unweighted \( L^p_{\mathbb{R}} \) space with \( \omega(x) = 1 \) for all \( x \in \mathbb{R} \). For a measurable function \( h : \mathbb{R} \rightarrow \mathbb{C} \) let us denote by \( \|h\|_{\infty,\omega} \) the essential supremum of the function \( x \mapsto h(x)\omega(x) \).

**Mellin transform** Let \( c \in \mathbb{R} \). For two functions \( h_1, h_2 \in L^1_{\mathbb{R}^+}(x^{c-1}) \) and any \( y \in \mathbb{R} \) we have
\[
\int_0^\infty |h_1(x)h_2(y/x)x^{-1}| \, dx < \infty
\]
which allows us to define their multiplicative convolution
\[
(h_1 \ast h_2)(y) = \int_0^\infty h_1(y/x)h_2(x)x^{-1} \, dx, \quad y \in \mathbb{R}.
\]
(2.1)

For a proof sketch of \( h_1 \ast h_2 \in L^1_{\mathbb{R}^+}(x^{c-1}) \) and the following properties we refer to Brenner Miguel [2021]. If in addition \( h_1 \in L^2_{\mathbb{R}^+}(x^{2c-1}) \) (respectively \( h_2 \in L^2_{\mathbb{R}^+}(x^{2c-1}) \)) then \( h_1 \ast h_2 \in L^2_{\mathbb{R}^+}(x^{2c-1}) \), too. For \( h \in L^1_{\mathbb{R}^+}(x^{c-1}) \) we define its Mellin transform \( \mathcal{M}_c[h] : \mathbb{R} \rightarrow \mathbb{C} \) at the development point \( c \in \mathbb{R} \) by
\[
\mathcal{M}_c[h](t) := \int_0^\infty x^{c-1+it} h(x) \, dx, \quad t \in \mathbb{R}.
\]
(2.2)

One key property of the Mellin transform, which makes it so appealing for multiplicative deconvolution problems, is the multiplication theorem, which for \( h_1, h_2 \in L^1_{\mathbb{R}^+}(x^{c-1}) \) states
\[
\mathcal{M}_c[h_1 \ast h_2](t) = \mathcal{M}_c[h_1](t)\mathcal{M}_c[h_2](t), \quad t \in \mathbb{R}.
\]
(2.3)

Making use of the Fourier transform, the domain of definition of the Mellin transform can be extended to \( L^2_{\mathbb{R}^+}(x^{2c-1}) \). Therefore, let \( \varphi : \mathbb{R} \rightarrow \mathbb{R}^+ \), with \( x \mapsto \exp(-2\pi x) \) and denote by \( \varphi^{-1} : \mathbb{R}^+ \rightarrow \mathbb{R} \) its inverse. Note that the diffeomorphisms \( \varphi, \varphi^{-1} \) map Lebesgue null sets on Lebesgue null sets. Consequently, the map \( \Phi_c : L^2_{\mathbb{R}^+}(x^{2c-1}) \rightarrow L^2_{\mathbb{R}} \), with \( h \mapsto \varphi^c \cdot (h \circ \varphi) \) is a well-defined isomorphism and denote by \( \Phi_c^{-1} : L^2_{\mathbb{R}} \rightarrow L^2_{\mathbb{R}^+}(x^{2c-1}) \) its inverse. For \( h \in L^2_{\mathbb{R}^+}(x^{2c-1}) \) the Mellin transform \( \mathcal{M}_c[h] : \mathbb{R} \rightarrow \mathbb{C} \) developed in \( c \in \mathbb{R} \) is defined through
\[
\mathcal{M}_c[h](t) := (2\pi)\mathcal{F}[\Phi_c[h]](t) \quad \text{for any} \ t \in \mathbb{R}.
\]
Here, \( \mathcal{F} : \mathbb{L}^2_{\mathbb{R}} \rightarrow \mathbb{L}^2_{\mathbb{R}} \) with \( H \mapsto (t \mapsto \mathcal{F}[H](t) := \lim_{k \to \infty} \int_{-k}^{k} \exp(-2\pi itx)H(x)dx) \) denotes the Plancherel-Fourier transform where the limit is understood in a \( \mathbb{L}^2_{\mathbb{R}} \) convergence sense. Due to this definition several properties of the Mellin transform can be deduced from the well-known Fourier theory. In particular for any \( h \in \mathbb{L}^1_{\mathbb{R}^+}(x^{c-1}) \cap \mathbb{L}^2_{\mathbb{R}^+}(x^{2c-1}) \) we have

\[
\mathcal{M}_c[h](t) = \int_0^{\infty} x^{c-1+it}h(x)dx \quad \text{for any } t \in \mathbb{R},
\]

which coincides with the common definition of a Mellin transform given in Paris and Kaminski [2001].

**Example 2.1.** Now let us give a few examples of Mellin transforms of commonly considered distribution families.

(i) **Beta Distribution** admits a density \( g_b(x) := 1_{[0,1]}(x)b(1-x)^{b-1} \) for \( b \in \mathbb{N} \) and \( x \in \mathbb{R}^+ \). Then, we have \( g_b \in \mathbb{L}^2_{\mathbb{R}^+}(x^{2c-1}) \cap \mathbb{L}^1_{\mathbb{R}^+}(x^{c-1}) \) for any \( c > 0 \) and

\[
\mathcal{M}_c[g_b](t) = \prod_{j=1}^{b} \frac{j}{c - 1 + j + it}, \quad t \in \mathbb{R}.
\]

(ii) **Scaled Log-Gamma Distribution** given by its density \( g_{\mu,a,\lambda}(x) = \frac{\exp(\lambda x)}{\Gamma(a)}x^{-\lambda-1}(\log(x) - \mu)^{a-1}1_{(e^{\lambda},\infty)}(x) \) for \( a, \lambda, x \in \mathbb{R}^+ \) and \( \mu \in \mathbb{R} \). Then, for \( c < \lambda + 1 \) hold \( g_{\mu,a,\lambda} \in \mathbb{L}^2_{\mathbb{R}^+}(x^{2c-1}) \cap \mathbb{L}^1_{\mathbb{R}^+}(x^{c-1}) \) and

\[
\mathcal{M}_c[g_{\mu,a,\lambda}](t) = \exp(\mu(c - 1 + it))(\lambda - c + 1 - it)^{-a}, \quad t \in \mathbb{R}.
\]

Note that \( g_{\mu,1,\lambda} \) is the density of a Pareto distribution with parameter \( e^\mu \) and \( \lambda \) and \( g_{0,a,\lambda} \) is the density of a Log-Gamma distribution.

(iii) **Gamma Distribution** admits a density \( g_d(x) = \frac{x^{d-1}}{\Gamma(d)}\exp(-x)1_{\mathbb{R}^+}(x) \) for \( d, x \in \mathbb{R}^+ \). Then, for \( c > -d + 1 \) we have \( g_d \in \mathbb{L}^2_{\mathbb{R}^+}(x^{2c-1}) \cap \mathbb{L}^1_{\mathbb{R}^+}(x^{c-1}) \) and

\[
\mathcal{M}_c[g_d](t) = \frac{\Gamma(c + d - 1 + it)}{\Gamma(d)}, \quad t \in \mathbb{R}.
\]

(iv) **Weibull Distribution** admits a density \( g_m(x) = mx^{m-1}\exp(-x^m)1_{\mathbb{R}^+}(x) \) for \( m, x \in \mathbb{R}^+ \). For \( c > -m + 1 \), \( \mathcal{M}_c[g_m] \) is well-defined and

\[
\mathcal{M}_c[g_m](t) = \frac{(c - 1 + it)}{m}\Gamma\left(\frac{c - 1 + it}{m}\right), \quad t \in \mathbb{R}.
\]

(v) **Lognormal Distribution** admits a density \( g_{\mu,\lambda}(x) = \frac{1}{\sqrt{2\pi}\lambda x}\exp(-\log(x) - \mu)^2/2\lambda^2)1_{\mathbb{R}^+}(x) \) for \( \lambda, x \in \mathbb{R}^+ \) and \( \mu \in \mathbb{R} \). \( \mathcal{M}_c[g_{\mu,\lambda}] \) is well-defined for any \( c \in \mathbb{R} \) and it holds

\[
\mathcal{M}_c[g_{\mu,\lambda}](t) = \exp(\mu(c - 1 + it))\exp\left(\frac{a^2(c - 1 + it)^2}{2}\right), \quad t \in \mathbb{R}.
\]
By construction the operator $\mathcal{M}_c : \mathbb{L}^2_{\mathbb{R}^+}(x^{2c-1}) \to \mathbb{L}^2_{\mathbb{R}}$ is an isomorphism and we denote by $\mathcal{M}_c^{-1} : \mathbb{L}^2_{\mathbb{R}} \to \mathbb{L}^2_{\mathbb{R}^+}(x^{2c-1})$ its inverse. If $H \in \mathbb{L}^1_{\mathbb{R}} \cap \mathbb{L}^2_{\mathbb{R}}$ then the inverse Mellin transform is explicitly expressed through

$$
\mathcal{M}_c^{-1}[H](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x^{-c-it} H(t) dt, \quad \text{for any } x \in \mathbb{R}^+.
$$

Furthermore, a Plancherel-type equation holds for the Mellin transform. Precisely, for all $h_1, h_2 \in \mathbb{L}^2_{\mathbb{R}^+}(x^{2c-1})$ we have

$$
\langle h_1, h_2 \rangle_{x^{2c-1}} = (2\pi)^{-1} \langle \mathcal{M}_c[h_1], \mathcal{M}_c[h_2] \rangle_{\mathbb{L}^2_{\mathbb{R}}} \quad \text{and} \quad \|h_1\|_{x^{2c-1}}^2 = (2\pi)^{-1} \|\mathcal{M}_c[h_1]\|_{\mathbb{L}^2_{\mathbb{R}}}^2.
$$

**Linear functional** In the following paragraph we introduce the linear functional, motivate it through a collection of examples and determine sufficient conditions to ensure that the considered objects are well-defined. We then define an estimator based on the empirical Mellin transform and the multiplication theorem for Mellin transforms. Let $c \in \mathbb{R}$ and $f \in \mathbb{L}^2_{\mathbb{R}^+}(x^{2c-1})$. In the sequel we are interested in estimating the linear functional

$$
\vartheta(f) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(-t) \mathcal{M}_c[f](t) dt
$$

for a function $\Psi : \mathbb{R} \to \mathbb{C}$ with $\overline{\Psi(t)} = \Psi(-t)$ for any $t \in \mathbb{R}$ and such that $\Psi \mathcal{M}_c[f] \in \mathbb{L}^1_{\mathbb{R}}$. The slattern is fulfilled, if $\Psi \in \mathbb{L}^2_{\mathbb{R}}$. Nevertheless a more detailed analysis of the decay of $\mathcal{M}_c[f]$ and $\Psi$ allows to ensure the integrability in a less restrictive situation. Before we present an estimator for $\vartheta(f)$ let us briefly illustrate our general approach by typical examples.

**Illustration 2.2.** We study in the sequel point-wise estimation at a given point $x_o \in \mathbb{R}^+$ in the following four examples.

(i) **Density:** Introducing the evaluation $f(x_o)$ of $f$ at the point $x_o$, if $\mathcal{M}_c[f] \in \mathbb{L}^1_{\mathbb{R}}$ then we have $f(x_o) = \mathcal{M}_c^{-1}[\mathcal{M}_c[f]](x_o) = \vartheta(f)$ with $\Psi(t) := x_o^{-c+it}$, $t \in \mathbb{R}$, satisfying $\overline{\Psi(t)} = \Psi(-t)$ for all $t \in \mathbb{R}$.

(ii) **Cumulative distribution function:** Considering the evaluation $F(x_o) = \int_0^{x_o} f(x) dx$ of the c.d.f. $F$ at the point $x_o$ define for $c < 1$ the function $\psi(x) := x^{1-2c}\mathbb{1}_{(0,x_o)}(x)$, $x \in \mathbb{R}^+$, which belongs to $\mathbb{L}^1_{\mathbb{R}^+}(x^{c-1}) \cap \mathbb{L}^2_{\mathbb{R}^+}(x^{2c-1})$. Setting

$$
\Psi(t) := \mathcal{M}_c[\psi](t) = \int_0^{x_o} x^{-c+it} dx = (1 - c + it)^{-1} x_o^{1-c+it}
$$

we get $\vartheta(f) = F(x_o)$ by an application of the Plancherel equality.

(iii) **Survival function:** Introducing the evaluation $S(x_o) = \int_{x_o}^{\infty} f(x) dx$ of the survival function $S$ at the point $x_o$ define the function $\psi(x) := x^{1-2c}\mathbb{1}_{(x_o,\infty)}(x)$, $x \in \mathbb{R}^+$, which for
$c > 1$ belongs to $L_{R^+}^1(x^{-c}) \cap L_{R^+}^2(x^{2c-1})$. Setting
\[
\Psi(t) := \mathcal{M}_c[\psi](t) = \int_{x_0}^{\infty} x^{-c+it} dx = -(1 - c + it)^{-1} x_o^{-c+it}
\]
we get $\vartheta(f) = S(x_o)$ by an application of the Plancherel equality.

(iv) Laplace transform: Given the evaluation $L(x_o) = \int_{0}^{\infty} \exp(-x_o x) f(x) dx$ of the Laplace transform $L$ at the point $x_o$ define for $c < 1$ the function $\psi(x) := x^{1-2c} \exp(-t_o x)$, $x \in \mathbb{R}^+$, which belongs to $L_{R^+}^1(x^{-c}) \cap L_{R^+}^2(x^{2c-1})$. Setting
\[
\Psi(t) := \mathcal{M}_c[\psi](t) = \int_{0}^{\infty} x^{-c+it} \exp(-x_o x) dx = x_o^{-c+it} \Gamma(1 - c + it)
\]
we get $\vartheta(f) = L(x_o)$ by an application of the Plancherel equality.

It is worth stressing out that in all four examples introduced in Illustration 2.2, the quantity of interest is independent of the choice of the model parameter $c \in \mathbb{R}$. However, the conditions on $c \in \mathbb{R}$ given Illustration 2.2 and the assumption $f \in L_{R^+}^2(x^{2c-1})$ ensure that the representation $\vartheta(f)$ is well-defined, and hence are essential for our estimation strategy. Consequently, we present the upcoming theory for almost arbitrary choices of $c \in \mathbb{R}$.

**Remark 2.3.** Consider Illustration 2.2. Since $S = 1 - F$ there is an elementary connection between the estimation of the survival function and the estimation of the c.d.f.. For example, we eventually deduce from a c.d.f. estimator $\hat{F}(x_o)$ a survival function estimator $\hat{S}(x_o)$ through $\hat{S}(x_o) := 1 - \hat{F}(x_o)$ with same risk, that is $E_{f_Y}((\hat{S}(x_o) - S(x_o))^2) = E_{f_Y}((\hat{F}(x_o) - F(x_o))^2)$.

Thus we can define for any $c \neq 1$ a survival function (respectively c.d.f.) estimator using the results of (ii) and (iii) in Illustration 2.2.

**Estimation strategy** To define an estimator of the quantity $\vartheta(f)$ we make use of the multiplication theorem (2.3) as it is common for deconvolution problems. To do so, let $f \in L_{R^+}^2(x^{2c-1}) \cap L_{R^+}^1(x^{-c})$ and $g \in L_{R^+}^1(x^{-c})$ then we deduce $\mathcal{M}_c[f_Y](t) = \mathcal{M}_c[f](t) \mathcal{M}_c[g](t)$ for all $t \in \mathbb{R}$ by application of the multiplication theorem. Under the mild assumption that $\mathcal{M}_c[g](t) \neq 0$ for all $t \in \mathbb{R}$ we conclude that $\mathcal{M}_c[f](t) = \mathcal{M}_c[f_Y](t)/\mathcal{M}_c[g](t)$ for all $t \in \mathbb{R}$ and rewrite (2.7) into
\[
\vartheta(f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(-t) \frac{\mathcal{M}_c[f_Y](t)}{\mathcal{M}_c[g](t)} dt.
\]

A naive approach is to replace in (2.8) the quantity $\mathcal{M}_c[f_Y]$ by its empirical counterpart $\hat{\mathcal{M}}_c(t) := n^{-1} \sum_{j=1}^{n} Y_j^{-1+it}$, $t \in \mathbb{R}$. However, the resulting integral is not well-defined, since
$\Psi \hat{M}_c / M_c[g]$ is generally not integrable. We ensure integrability introducing an additional spectral cut-off regularisation which leads to the following estimator

$$\hat{\vartheta}_k := \frac{1}{2\pi} \int_{-k}^{k} \Psi(-t) \frac{\hat{M}_c(t)}{M_c[g](t)} dt \quad \text{for any } k \in \mathbb{R}^+. \quad (2.9)$$

The following proposition shows that the estimator is consistent for suitable choice of the cut-off parameter $k \in \mathbb{R}^+$. We denote by $E^n_f$ the expectation corresponding to the distribution $P^n_{\nu}$ of $(Y_1, \ldots, Y_n)$ and use the abbreviation $E_f := E^1_f$. Analogously, we define $E^n_{\nu}$ and $E_f$.

**Proposition 2.4.** For $c \in \mathbb{R}$ assume that $f \in \mathbb{L}^2_{\mathbb{R}^+} (x^{2c-1})$, $\Psi M_c[f] \in \mathbb{L}^1_{\mathbb{R}}$ and $E_{f\nu}(Y_1^{2(2c-1)}) < \infty$. Then for any $k \in \mathbb{R}^+$ holds

$$E^n_{f\nu}( (\hat{\vartheta}_k - \vartheta(f))^2) \leq \|I_{[k,\infty)} \Psi M_c[f]\|^2_{\mathbb{L}^2_{\mathbb{R}}} + \frac{E_{f\nu}(Y_1^{2(2c-1)})}{n} \|I_{[-k,k]} \Psi / M_c[g]\|^2_{\mathbb{L}^1_{\mathbb{R}}} \quad (2.10)$$

If additionally $\|g\|_{\infty, x^{2c-1}} < \infty$ holds, we get

$$E^n_{f\nu}( (\hat{\vartheta}_k - \vartheta(f))^2) \leq \|I_{[k,\infty)} \Psi M_c[f]\|^2_{\mathbb{L}^2_{\mathbb{R}}} + \frac{\|g\|_{\infty, x^{2c-1}}}{n} E_f(X_1^{2(2c-1)}) \Delta_{\vartheta,g}(k) \quad (2.11)$$

where $\Delta_{\vartheta,g}(k) := \frac{1}{2\pi} \int_{-k}^{k} \left| \frac{\Psi(t)}{M_c[g](t)} \right|^2 dt$.

Choosing now a sequence of spectral cut-off parameters $(k_n)_{n \in \mathbb{N}}$ such that $k_n \to \infty$ and $\|I_{[-k_n,k_n]} \Psi / M_c[g]\|^2_{\mathbb{L}^1_{\mathbb{R}}} n^{-1} \to 0$ (respectively $\Delta_{\vartheta,g}(k_n) n^{-1} \to 0$) implies that $\hat{\vartheta}_{k_n}$ is a consistent estimator of $\vartheta(f)$, that is $E^n_{f\nu}( (\hat{\vartheta}_{k_n} - \vartheta(f))^2) \to 0$ for $n \to \infty$. We note that the additional assumption, $\|g\|_{\infty, x^{2c-1}} < \infty$, is fulfilled by many error densities and thus rather weak.

**Remark 2.5.** Despite the fact, that the first bound (2.10) only requires a finite second moment of $Y_1^{2c-1}$, we have in many cases $\Delta_{\vartheta,g}(k) \|I_{[-k,k]} \Psi / M_c[g]\|^{-2}_{\mathbb{L}^1_{\mathbb{R}}} \to 0$ for $k \to \infty$, implying that the bound of the variance term in (2.11) increases slower in $k$ than the bound presented in (2.10). It is worth stressing out, that there exist cases where the opposite effect occurs. For instance let the error $U$ be lognormal-distribution with parameter $\mu = 0, \lambda = 1$, see Example 2.1. Then $\sup_{y} E_f(y^{2c-1} g(y) = \sup_{z \in \mathbb{R}} \frac{\exp(2(2c-1)z)}{\sqrt{2\pi}\lambda} \exp(-(z-\mu)^2/2\lambda^2)) < \infty$. Thus if $E(X_1^{2(2c-1)}) < \infty$ both bounds are finite and following the argumentation of Butucea and Tsybakov [2008] one can see, that in the special case of point-wise density estimation, the inequality presented in (2.10) is more favourable than the inequality presented in (2.11).

For the upcoming theory, we will focus on the second bound of Proposition 2.4. Assuming that $\|g\|_{\infty, x^{2c-1}} < \infty$, allows us to state that the growth of the second summand, also referred as variance term, is determined by the growth of $\Delta_{\vartheta,g}(k)$ as $k$ going to infinity.
The parametric case  In this paragraph we determine when Proposition 2.4 implies a parametric rate of the estimator. To be precise, there are two scenarios only which occur.

(P) If $\sup_{k \in \mathbb{R}^+} \Delta_{\psi,g}(k) = \|\Psi \mathcal{M}_c[g]^{-1}\|_{L^2(\mathbb{R})}^2 < \infty$, i.e. the second summand in (2.11) is uniformly bounded in $k$ and hence of order $n^{-1}$. Then for all sufficiently large values of $k \in \mathbb{R}^+$, the bias term is negligible with respect to the parametric rate $n^{-1}$.

(NP) If $\sup_{k \in \mathbb{R}^+} \Delta_{\psi,g}(k) = \|\Psi \mathcal{M}_c[g]^{-1}\|_{L^2(\mathbb{R})}^2 = \infty$, i.e. the second summand is unbounded and hence necessitates an optimal choice of parameter $k \in \mathbb{R}^+$ realising to squared-bias-variance trade-off.

Our aim is now to characterise when the case (P) occur. To do so, we start by introducing a typical characterisation of the decay of the error density and the decay of the function $\Psi$, starting with the error density. Let us first revisit Example 2.1 to analyse the decay of the presented densities.

Example 2.6 (Example 2.1 continued).

(i) Beta Distribution: For $c > 0$ and $b \in \mathbb{N}$ we have $\mathcal{M}_c[g_b](t) = \prod_{j=1}^{b} \frac{j}{c-1+j+it}$ for $t \in \mathbb{R}$ and thus

$$c_{g,c}(1 + t^2)^{-b/2} \leq |\mathcal{M}_c[g_b](t)| \leq C_{g,c}(1 + t^2)^{-b/2} \quad t \in \mathbb{R}$$

where $c_{g,c}, C_{g,c} > 0$ are positive constants only depending on $g$ and $c$.

(ii) Scaled Log-Gamma Distribution: For $\lambda, a \in \mathbb{R}^+$, $\mu \in \mathbb{R}$ and $c < \lambda + 1$ we have $\mathcal{M}_c[g_{\mu,a,\lambda}](t) = \exp(\mu(c - 1 + it))(\lambda - c + 1 - it)^{-a}$ for $t \in \mathbb{R}$.

$$c_{g,c}(1 + t^2)^{-a/2} \leq |\mathcal{M}_c[g_{\mu,a,\lambda}](t)| \leq C_{g,c}(1 + t^2)^{-a/2} \quad t \in \mathbb{R}$$

where $c_{g,c}, C_{g,c} > 0$ are positive constants only depending on $g$ and $c$.

(iii) Gamma Distribution: For $d \in \mathbb{R}^+$ and $c > -d + 1$ we have $\mathcal{M}_c[g_d](t) = \frac{\Gamma(c+d-1+it)}{\Gamma(d)}$ for $t \in \mathbb{R}$ and thus

$$c_{g,c}(1 + t^2)^{(c+d-1.5)/2} \exp(-|t|\pi/2) \leq |\mathcal{M}_c[g_d](t)| \leq C_{g,c}(1 + t^2)^{(c+d-1.5)/2} \exp(-|t|\pi/2)$$

for $t \in \mathbb{R}$ where $c_{g,c}, C_{g,c} > 0$ are positive constants only depending on $g$ and $c$.

(iv) Weibull Distribution: Let $m \in \mathbb{R}^+$ and $c > -m + 1$ we have $\mathcal{M}_c[g_m](t) = \frac{\Gamma(c-1+it)}{m} \Gamma\left(\frac{c-1+it}{m}\right)$ for $t \in \mathbb{R}$ and thus

$$c_{g,c}(1 + t^2)^{2c-2-2m/2m} \exp(-|t|\pi/2m) \leq |\mathcal{M}_c[g_m](t)| \leq C_{g,c}(1 + t^2)^{2c-2-2m/2m} \exp(-|t|\pi/2m)$$

for $t \in \mathbb{R}$ where $c_{g,c}, C_{g,c} > 0$ are positive constants only depending on $g$ and $c$. 


(v) **Lognormal Distribution:** Let $\lambda \in \mathbb{R}^+$, $\mu \in \mathbb{R}$ and $c \in \mathbb{R}$ we have $\mathcal{M}_c[g_{\mu,\lambda}](t) = \exp(\mu(c - 1 + i t)) \exp \left( \frac{\lambda^2(c-1+it)^2}{2} \right)$ for $t \in \mathbb{R}$ and thus

$$c_{g,c} \exp(-\lambda^2 t^2/2) \leq |\mathcal{M}_c[g_{\mu}](t)| \leq C_{g,c} \exp(-\lambda^2 t^2/2)$$

for $t \in \mathbb{R}$ where $c_{g,c}, C_{g,c} > 0$ are positive constants only depending on $g$ and $c$.

Motivated by Example 2.1 we distinguish between smooth error and supersmooth error densities staying in the terminology of Fan [1991], Belomestny and Goldenshluger [2020] or Brenner Miguel et al. [2021]. An error density $g$ is called smooth if there exists a $\gamma, c_{g,c}, C_{g,c} > 0$ such that

$$c_{g}(1 + t^2)^{-\gamma/2} \leq |\mathcal{M}_c[g](t)| \leq C_{g,c}(1 + t^2)^{-\gamma/2}, \quad t \in \mathbb{R} \quad \text{([G1])}$$

and it is referred to as super smooth if there exists $\lambda, \rho, c_{g,c}, C_{g,c} > 0$ and $\gamma \in \mathbb{R}$ such that

$$c_{g,c}(1 + t^2)^{-\gamma/2} \exp(-|\lambda| t^\rho) \leq |\mathcal{M}_c[g](t)| \leq C_{g,c}(1 + t^2)^{-\gamma/2} \exp(-|\lambda| t^\rho), \quad t \in \mathbb{R}. \quad \text{([G2])}$$

On the other hand to calculate the growth of $\Delta_{\Psi,g}$ we specify the decay of $\Psi$. Similar to the error density $g$ we consider the case of a smooth $\Psi$, i.e. there exists $\epsilon_{\Psi,c}, \Psi_{\epsilon,c} > 0$ and $p \geq 0$ such that

$$c_{\Psi,c}(1 + t^2)^{-\epsilon/2} \leq |\Psi(t)| \leq C_{\Psi,c}(1 + t^2)^{-\epsilon/2}, \quad t \in \mathbb{R}, \quad \text{([\Psi 1])}$$

and a super smooth $\Psi$, i.e. there exists $\mu, R, c_{\Psi,c}, \Psi_{\epsilon,c} > 0$ and $p \in \mathbb{R}$ such that

$$c_{\Psi,c}(1 + t^2)^{-\epsilon/2} \exp(-\mu |t|^R) \leq |\mathcal{M}_c[\Psi](t)| \leq C_{\Psi,c}(1 + t^2)^{-\epsilon/2} \exp(-\mu |t|^R), \quad t \in \mathbb{R}. \quad \text{([\Psi 2])}$$

As we see in the following Illustration the examples of $\Psi$ considered in Illustration 2.2 do fit into these two cases.

**Illustration 2.7 (Illustration 2.2 continued).**

(i) **Point-wise density estimation:** We have that $|\Psi(t)| = x_o^{-\epsilon}$ and thus $p = 0$ in sense of $[\Psi 1]$.

(ii) **Point-wise cumulative distribution function estimation:** We have that $|\Psi(t)| = \frac{x_o^{1-\epsilon}}{\sqrt{(1-\epsilon)^2+t^2}}$

and thus $p = 1$ in sense of $[\Psi 1]$.

(iii) **Point-wise survival function estimation:** We have that $|\Psi(t)| = \frac{x_o^{1-\epsilon}}{\sqrt{(1-\epsilon)^2+t^2}}$ and thus $p = 1$ in sense of $[\Psi 1]$.

(iv) **Laplace transform estimation:** We have that $|\Psi(t)| = t_o^{c-1}|\Gamma(1-c+it)|$ and thus $p = 1 - 2c, \mu = \pi/2$ and $R = 1$ in the sense of $[\Psi 2]$. 

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After the introduction of the typical terminology for deconvolution settings we can state when the function $\Delta_{\Psi,g}$ is bounded. We summarize the collection of scenarios in the following Proposition.

**Proposition 2.8.** Assume that for a $c \in \mathbb{R}$ holds $f \in L_2^{2c-1}(x^{2c-1})$, $\Psi_M[f] \in L^{1}_{\mathbb{R}}$, $\sigma = \mathbb{E}_f(X_1^{2c-1}) < \infty$ and $\|g\|_{\infty,x^{2c-1}} < \infty$. Then for the cases

(i) $[\Psi 1]$ and $[G1]$ with $2\rho - 2\gamma > 1$;

(ii) $[\Psi 2]$ and $[G1]$ or

(iii) $[\Psi 2]$ and $[G2]$ with $\gamma < 1/2$ and $\rho > 1$, $\rho = 1$, $\lambda < \pi/2$ or $\rho = 1$, $\mu = \pi/2$, $\gamma < -c$.

we get that $\sup_{k \in \mathbb{R}^+} \Delta_{\Psi,g}(k) < \infty$. Furthermore, for all $k \in \mathbb{R}$ sufficiently large we have

$$\mathbb{E}_n((\hat{\vartheta}_k - \vartheta(f))^2) \leq \frac{C(\Psi, g, \sigma)}{n}.$$  

The proof of Proposition 2.8 is a straightforward calculus and thus omitted. For our four examples of $\Psi$ we get a parametric rate for the estimation of the survival function and cumulative distribution function if the error density fulfills $[G1]$ with $\gamma < 1/2$ and a parametric rate for the estimation of the Laplace transform if the error density fulfills $[G1]$ with $\gamma > 0$ or if $g$ fulfills $[G2]$ with $(\rho < 1)$, $(\rho = 1, \lambda < \pi/2)$, $(\rho = 1$ or $\lambda = \pi/2, \gamma < -c)$.

The non-parametric case We now focus on the case (NP), that is $\sup_{k \in \mathbb{R}^+} \Delta_{\Psi,g}(k) = \infty$, which occurs in several situations. In this scenario the first summand of Proposition 2.4 is decreasing in $k$ while the second summand is increasing and unbounded. A choice of the parameter $k \in \mathbb{R}^+$ realizing an optimal trade-off is thus non-trivial. We therefore define a data-driven procedure for the choice of the parameter $k \in \mathbb{R}^+$ inspired by the work of Goldenshluger and Lepski [2011].

In fact, let us reduce the set of possible parameters to $K_n := \{k \in \mathbb{N} : \|g\|_{\infty,x^{2c-1}} \Delta_{\Psi,g}(k) \leq n, k \leq n^{1/2}(\log n)^{-2}\}$ and denote $K_n = \max K_n$. We further introduce the variance term up to a $(\log n)$-term

$$V(k) := \chi\|g\|_{\infty,x^{2c-1}} \sigma \Delta_{\Psi,g}(k)(\log n)n^{-1}$$

where $\chi > 0$ is a numerical constant which is specified below and $\sigma := \mathbb{E}_f(X_1^{2c-1})$. Based on a comparison of the estimators constructed above an estimator of the bias term is given by

$$A(k) := \sup_{k' \in [k, K_n]} ((\hat{\vartheta}_{k'} - \hat{\vartheta}_k)^2 - V(k'))_+$$

where $[a, b] := (a, b) \cap \mathbb{N}$ for $a, b \in \mathbb{R}^+$. Analogously, we define $\lfloor a, b \rfloor = [a, b] \cap \mathbb{N}$ and $\lfloor a, b \rfloor = [a, b] \cap \mathbb{N}$. Since the term $\sigma$ in $V(k)$ depends on the unknown density $f$, and hence it
is itself unknown, we replace it by the plug-in estimator \( \hat{\sigma} := \frac{1}{n} \sum_{j=1}^{n} \frac{Y_j^{2(c-1)}}{E(U_1^{2(c-1)})} \). Summarising we estimate \( V(k) \) and \( A(k) \) by

\[
\hat{V}(k) := 2\chi \| g \|_{\infty, x^{2c-1}} \hat{\Delta}_{\Psi, g}(k) \log(n) n^{-1} \quad \text{and} \quad \hat{A}(k) := \sup_{k' \in [k, K_n]} ((\hat{\vartheta}_{k'} - \hat{\vartheta}_k)^2 - \hat{V}(k'))_+.
\]

Below we study the fully data-driven estimator \( \hat{\vartheta}_{k} \) of \( \vartheta(f) \) with

\[
\hat{k} := \arg \min_{k \in K_n} (\hat{A}(k) + \hat{V}(k)).
\]

**Theorem 2.9.** For \( c \in \mathbb{R} \) assume that \( f \in L^2_{\mathbb{R}^+}(x^{2c-1}) \), \( \Psi \mathcal{M}_c[f] \in L^1_{\mathbb{R}}, \mathbb{E}_{f^y}(Y_1^{8(c-1)}) < \infty \) and \( \| g \|_{\infty, x^{2c-1}} < \infty \). Then for \( \chi \geq 72 \) holds

\[
\mathbb{E}_{f^y}((\vartheta(f) - \hat{\vartheta}_{\hat{k}})^2) \leq C_1 \inf_{k \in K_n} (\| \mathbb{I}_{[k, \infty)} \Psi \mathcal{M}_c[f] \|_{L^1_{\bar{k}}}^2 + V(k)) + \frac{C_2}{n}
\]

where \( C_1 \) is a positive numerical constant and \( C_2 \) is a positive constant depending on \( \Psi, g, \mathbb{E}_{f^y}(Y_1^{8(c-1)}) \).

The proof of Theorem 2.9 is postponed to the appendix. Let us shortly comment on the moment assumptions of Theorem 2.9. For \( c \in \mathbb{R} \) close to one, the apparently high moment assumption \( \mathbb{E}_{f^y}(Y_1^{8(c-1)}) < \infty \) is rather weak. For the point-wise density estimation, compare Illustration 2.2, this assumption is always true if \( c = 1 \). For the point-wise survival function estimation (respectively. cumulative distribution function estimation), \( c = 1 \) cannot be full-filled but arbitrary values of \( c \in \mathbb{R} \) close to one are possible. As already mentioned, for the pointwise density estimation the assumption \( \Psi \mathcal{M}_1[f] \in L^1_{\mathbb{R}} \) implies that \( \mathcal{M}_1[f] \in L^1_{\mathbb{R}} \). For \( c = 1 \), we see that \( \| g \|_{\infty, x} < \infty \) is fullfilled for many examples of error densities.

## 3 Minimax theory

In the following section we develop the minimax theory for the plug-in spectral cut-off estimator under the assumptions \([G1]\) and \([\Psi1]\). Over the Mellin-Sobolev spaces we derive an upper for all linear functional satisfying assumption \([\Psi1]\). We state a lower bound for each of the cases (i)-(iii) of Illustration 2.2 separately, that is point-wise estimation of the density \( f \), the survival function \( S \) and the cumulative distribution function \( F \). We finish this section, by motivating the regularity spaces through their analytically implications.

**Upper bound** Let us restrict to the scenario where \([G1]\) and \([\Psi1]\) holds for \( 2p - 2\gamma \leq 1 \). Here one can state that there exist a constant \( C_{\Psi, g} > 0 \) such that \( \Delta_{\Psi, g}(k) \leq C_{\Psi, g} k^{2\gamma - 2p + 1} \).
Now let us consider the bias term. To do so, we introduce we Mellin-Sobolev spaces at the development point \( c \in \mathbb{R} \) by
\[
\mathcal{W}_c^s(\mathbb{R}^+) := \{ h \in L^2_{\mathbb{R}^+}(x^{2c-1}) : |h|_{s,c}^2 := \| (1 + t^2)^{s/2} \mathcal{M}_c[h] \|_{L^2_{\mathbb{R}^+}} < \infty \}
\] (3.1)
with corresponding ellipsoids \( \mathcal{W}_c^s(L) := \{ h \in \mathcal{W}_c^s(\mathbb{R}^+) : |h|_{s,c}^2 < L \} \). We denote the subset of densities by
\[
\mathcal{D}_{c}^{s,c,L} := \{ f \in \mathcal{W}_c^s(\mathbb{R}^+) : f \text{ is a density, } \mathcal{E}_f(X_l^{2c-2}) \leq L \}. \tag{3.2}
\]

Using this construction we get the following result as a direct consequence.

**Theorem 3.1.** Assume that for a \( c \in \mathbb{R} \) \([G1]\) holds for \( g \) and \([\Psi 1]\) for \( \Psi \). Additionally, assume that \( \|g\|_{\infty, x^{2c-1}} < \infty \). Setting for any \( s > 1/2 - p \) the cut-off parameter to \( k_n := n^{1/(2s+2\gamma)} \) implies then
\[
\sup_{f \in \mathcal{D}_{c}^{s,c,L}} \mathbb{E}_f \{ (\hat{\varphi}_{k_n} - \varphi(f))^2 \} \leq C_{L,s,c,L} n^{-(2s+2p-1)/(2s+2\gamma)}
\]
where \( C_{L,s,c,L} > 0 \) is a constant depending on \( L, s, p, \gamma, c \) and \( \|g\|_{\infty, x^{2c-1}} \).

**Proof of Theorem 3.1.** Evaluating the upper bound in Proposition 2.4 under \([G1]\) and \([\Psi 1]\) we have \( \|g\|_{\infty, x^{2c-1}} \sigma \Delta_{\varphi,g}(k)n^{-1} \leq C_{g,L} \frac{k^{2s-2p+1}}{n} \) and
\[
\left( \int_k^\infty |\Psi(t)\mathcal{M}_c[f](t)|dt \right)^2 \leq C_{L,c} \int_k^\infty |\Psi(t)|^2(c^2 + t^2)^{-s}dt \leq C_{L,c,s}\psi k^{2s-2p+1}.
\]
Now choosing \( k_n := n^{1/(2s+2\gamma)} \) balances both term leading to the rate \( n^{-(2s+2p-1)/(2s+2\gamma)} \). \( \Box \)

The assumption \( s > 1/2 - p \) implies that \( \Psi \mathcal{M}_c[f] \in L^1_{\mathbb{R}^+} \) by a simple calculus which can be found in proof of Theorem 3.1 in the appendix. Before considering the lower bounds let us illustrate the last Theorem using our examples (i) to (iii) of Illustration 2.2.

**Illustration 3.2.**

(i) **Point-wise density estimation:** Since \( p = 0 \) we assume that \( s > 1/2 = 1/2 - p \). In this scenario Theorem 3.1 implies
\[
\sup_{f \in \mathcal{D}_{c}^{s,c,L}} \mathbb{E}_f \{ (\hat{\varphi}_{k_n} - \varphi(f))^2 \} \leq C_{L,s,c} n^{-(2s-1)/(2s+2\gamma)}.
\]

(ii) **Point-wise cumulative distribution function estimation:** We have \( p = 1 \) and hence for any \( s \geq 0 \) holds \( s > 1/2 - p \). Recall that for \( \gamma < 1/2 \) we are in the parametric case where we choose \( k \in \mathbb{R}_+ \) sufficiently large. For \( \gamma \geq 1/2 \) we deduce from Theorem 3.1 for any \( c < 1 \) that
\[
\sup_{f \in \mathcal{D}_{c}^{s,c,L}} \mathbb{E}_f \{ (\hat{\varphi}_{k_n} - \varphi(f))^2 \} \leq C_{L,s,c} n^{-(2s-1)/(2s+2\gamma)}.
\]
(iii) **Point-wise survival function estimation:** We have \( p = 1 \) and hence for any \( s \geq 0 \) holds \( s > 1/2 - p \). Recall that for \( \gamma < 1/2 \) we are in the parametric case where we choose \( k \in \mathbb{R}_+ \) sufficiently large. For \( \gamma \geq 1/2 \) we deduce from Theorem 3.1 for any \( c < 1 \) that

\[
\sup_{f \in \mathbb{D}^s_{k+L}} \mathbb{E}_{f_n}(\hat{\vartheta}_k - \vartheta(f))^2 \leq x_{o}^{-2c}C_{L,s,\varphi,g,c} n^{-(2s-1)/(2s+2\gamma)}.
\]

In example (i) the sign of \( c \) has a strong impact on the upper bound. In fact, for \( c > 0 \) it appears that the estimation in a point \( x_o \) close to 0 is harder than for bigger values of \( x_o \). The case for \( c < 0 \) has an opposite effect. Further in (ii) and (iii), i.e. estimating the survival function and the c.d.f. estimation, the estimator of the c.d.f. seems to have a better behaviour close to 0 than the survival function estimator. We stress out, that in Illustration 2.2 we already mention that one can use an estimator for the survival function to construct an estimator for the c.d.f and vice versa. The results of Illustration 3.2 suggests to estimate the survival function directly or using the c.d.f. estimator, according if \( x_o \in \mathbb{R}^+ \) is close to 0 or not.

**Remark 3.3.** Belomestny and Goldenshluger [2020] derive for point-wise density estimation a rate of \( n^{-2s/(2s+2\gamma+1)} \) under similar assumptions on the error density \( g \). However, they consider Hölder-type regularity classes rather than Mellin-Sobolev spaces which are of a global nature. Even if the rates in Illustration 3.2 seem to be less sharp compared to Belomestny and Goldenshluger [2020], they cannot be improved as shown by the lower bounds below.

Additionally, if \( \gamma \geq 1 \) we have that \( k_n := n^{1/(2s+2\gamma)} \leq k^{1/2} \) and thus \( k_n \in \mathcal{K}_n \). We can deduce the following Corollary using the similar arguments of the proof of Theorem 3.1 on Theorem 2.9. We therefore omit its proof.

**Corollary 3.4.** Assume that for a \( c \in \mathbb{R} \) holds [G1] holds for \( g \) and [Ψ1] for \( \Psi \). Further let \( \mathbb{E}_{f_n}(Y_1^{-8(c-1)}) \|g\|_{\infty,\varphi} < \infty \) and \( f \in \mathbb{D}^{s,c,L}_{k+L} \) for any \( s > 1/2 - p \). Then

\[
\mathbb{E}_{f_n}(\hat{\vartheta}_k - \vartheta(f))^2 \leq C_{f,g,\psi} \log(n) n^{-(2s+2p-1)/(2s+2\gamma)}
\]

where \( C_{f,g,\psi} > 0 \) is a constant depending on \( L,s,p,\varphi,\psi,\gamma \), \( \mathbb{E}_{f_n}(Y_1^{-8(c-1)}) \) and \( \|g\|_{\infty,\varphi} \).

To state that the presented rates of Theorem 3.1 cannot be improved over the whole Mellin-Sobolev ellipsoids, we give a lower bound result for the cases (i)-(iii) in the following section.

**Lower bound** For the following part, we will need to have an additionally assumption on the error density \( g \). In fact, we will assume that \( g \) has bounded support, that is \( g(x) = 0 \) for \( x > d, d \in \mathbb{R}^+ \). For the sake of simplicity we will say that \( d = 1 \). Further we assume that there exists \( c_g, c'_g \in \mathbb{R}^+ \) such that

\[
c'_{g}(1 + t^2)^{-\gamma/2} \leq |\mathcal{M}_{1/2}[g](t)| \leq c'_g(1 + t^2)^{-\gamma/2} \text{ for } |t| \to \infty.
\]
For technical reasons we will restrict ourselves to the case of $c > 1/2$.

**Theorem 3.5.** Let $s, \gamma \in \mathbb{N}$, assume that $[G1]$ and $[G1']$ holds. Then there exist constants $C_{g,x,o,i}, L_{s,g,x,o,c,i} > 0, i \in [3]$, such that

(i) **Point-wise density estimation:** for all $L \geq L_{s,g,x,o,c,1}$, $n \in \mathbb{N}$ and for any estimator $\hat{f}(x_o)$ of $f(x_o)$ based on an i.i.d. sample $(Y_j)_{j \in [1,n]}$,

$$\sup_{f \in \mathbb{D}_{s,c,L}} \mathbb{E}^n_{f,L}(\hat{f}(x_o) - f(x_o))^2 \geq C_{g,x,o,1} n^{-(2s-1)/(2s+2\gamma)}.$$

(ii) **Point-wise survival function estimation:** for all $L \geq L_{s,g,x,o,c,2}$, $n \in \mathbb{N}$ and for any estimator $\hat{S}(x_o)$ of $S(x_o)$ based on an i.i.d. sample $(Y_j)_{j \in [1,n]}$,

$$\sup_{f \in \mathbb{D}_{s,c,L}} \mathbb{E}^n_{f,L}(\hat{S}(x_o) - S(x_o))^2 \geq C_{g,x,o,2} n^{-(2s+1)/(2s+2\gamma)}.$$

(iii) **Point-wise cumulative distribution function estimation:** for all $L \geq L_{s,g,x,o,c,3}$, $n \in \mathbb{N}$ and for any estimator $\hat{F}(x_o)$ of $F(x_o)$ based on an i.i.d. sample $(Y_j)_{j \in [1,n]}$,

$$\sup_{f \in \mathbb{D}_{s,c,L}} \mathbb{E}^n_{f,L}(\hat{F}(x_o) - F(x_o))^2 \geq C_{g,x,o,3} n^{-(2s+1)/(2s+2\gamma)}.$$

We want to stress out that in the multiplicative censoring model, the family $(g_k)_{k \in \mathbb{N}}$ of Beta$(1,k)$ densities fulfills both assumption $[G1]$ and $[G1']$.

**Regularity assumptions** While in the theory of inverse problems the definition of the Mellin-Sobolev spaces is quite natural, we want to stress out that elements of these spaces can be characterised by their analytical properties. In Brenner Miguel et al. [2021] one can find a characterisation of $\mathbb{W}_1^s(\mathbb{R}^+)$. Since the generalisation for the spaces $\mathbb{W}_c^s(\mathbb{R}^+)$ is straightforward, we only state the result while the proof for the case $c = 1$ can be found in Brenner Miguel et al. [2021].

**Proposition 3.6.** Let $s \in \mathbb{N}$. Then $f \in \mathbb{W}_c^s(\mathbb{R}^+)$ if and only if $f$ is $s - 1$-times continuously differentiable where $f^{(s-1)}$ is locally absolutely continuous with derivative $f^{(s)}$ and $\omega^j f^{(j)} \in L_{s+2}^2(\omega^{2c-1})$ for all $j \in [0, s]$.
Appendix

A Proofs of section 2

Usefull inequality The next inequality was is state in the following form in Comte [2017] based on a similar formulation in Birgé and Massart [1998].

Lemma A.1. (Bernstein inequality) Let $X_1, \ldots, X_n$ independent random variables and $T_n(X) := \sum_{j=1}^n (X_i - \mathbb{E}(X_i))$. Then for $\eta > 0$,

$$
\mathbb{P}(|T_n(X) - \mathbb{E}(T_n(X))| \geq n\eta) \leq 2 \exp(-\frac{n\eta^2}{2v^2 + b\eta}) \leq 2 \max(\exp(-\frac{m\eta^2}{4v^2}), \exp(-\frac{m\eta}{4b}))
$$

if $n^{-1} \sum_{i=1}^n \mathbb{E}(|X_i^n|) \leq \frac{m}{2} v^2 \eta^2 m^{-2}$ for all $m \geq 2$. If the $X_i$ are identically distributed, the previous condition can be replaced by $\mathbb{V}ar(X_1) \leq v^2$ and $|X_1| \leq b$.

Proof of Proposition 2.4. Let us denote for any $k \in \mathbb{R}^+$ the expectation $\vartheta_k := \mathbb{E}_{f_y}[\hat{\vartheta}_k]$ which leads to the usual squared bias-variance decomposition

$$
\mathbb{E}_{f_y}((\hat{\vartheta}_k - \vartheta(f))^2) = (\hat{\vartheta}_k - \vartheta(f))^2 + \mathbb{V}ar_{f_y}(\hat{\vartheta}_k).
$$

Consider the first summand in (A.1)- An application of the Fubini-Tonelli theorem implies

$$
(\hat{\vartheta}_k - \vartheta(f))^2 = \left( \frac{1}{2\pi} \int_{[-k,k]^c} \Psi(-t)M_c[f](t)dt \right)^2 \leq \|1_{[k,\infty]}\Psi M_c[f]\|^2_{L^2_{\mathbb{R}}}
$$

Study the the second term in (A.1). Then the bound in (2.10) follows then by the following inequality

$$
\mathbb{V}ar_{f_y}(\hat{\vartheta}_k) \leq \frac{1}{n} \mathbb{E}_{f_y}((\frac{1}{2\pi} \int_{-k}^k |\Psi(t)| \frac{Y_{1}^{c-1}}{M_c[g](t)} dt)^2) = \mathbb{E}_{f_y}(Y_{1}^{2(c-1)}) \left( \int_{-k}^k \left| \frac{\Psi(t)}{M_c[g](t)} \right|^2 dt \right)^2.
$$

To show (2.11) we see that

$$
\mathbb{V}ar_{f_y}(\hat{\vartheta}_k) \leq \frac{1}{n} \mathbb{E}_{f_y}((\frac{1}{2\pi} \int_{-k}^k \Psi(-t) \frac{Y_{1}^{c-1+it}}{M_c[g](t)} dt)^2)
$$

$$
= \frac{1}{n} \int_0^\infty f_Y(y) \left| \frac{1}{2\pi} \int_{-k}^k \Psi(-t) \frac{g^{c-1+it}}{M_c[g](t)} dt \right|^2 dy
$$

$$
\leq \|f_Y\|_{L^\infty,x^{2c-1}} \frac{1}{2\pi n} \int_{-k}^k \left| \frac{\Psi(t)}{M_c[g](t)} \right|^2 dt.
$$

Furthermore we have for any $y > 0$ that

$$
g^{2c-1} f_Y(y) = \int_0^\infty f(x) x^{2c-2} g(y/x) \frac{y^{2c-1}}{x^{2c-1}} dx \leq \|g\|_{L^\infty,x^{2c-1}} \mathbb{E}_f(X_1^{2c-2}).
$$

□
Proof of Theorem 2.9. Let us set $\vartheta := \vartheta(f)$. By the definition of $\hat{k}$ follows for any $k \in K_n$

$$(\vartheta - \hat{k}_n)^2 \leq 2(\vartheta - \hat{k}_n)^2 + 2(\vartheta_k - \hat{k}_n)^2 \leq 2(\vartheta - \hat{k}_n)^2 + 2(\vartheta_k - \hat{k}_n)^2 + 2(\vartheta_{k\wedge k^n} - \hat{k}_n)^2$$

$$(\vartheta - \hat{k}_n)^2 \leq 2(\vartheta - \hat{k}_n)^2 + 2(\hat{A}(k) + \hat{V}(k) + \hat{A}(k) + \hat{V}(k)) \leq 2(\vartheta - \hat{k}_n)^2 + 4(\hat{A}(k) + \hat{V}(k)).$$

Consider $\hat{A}(k)$ we have by a straight forward calculus $\hat{A}(k) \leq A(k) + \sup_{k' \in K_n} (V(k') - \hat{V}(k'))_+$ and thus

$$A(k) \leq \sup_{k' \in [k, K_n]} \left(3(\hat{k}_k - \vartheta_k)^2 + 3(\vartheta_k - \hat{k}_k)^2 + V(k')_+\right) + 3 \max_{k' \in [k, K_n]} (\vartheta_k - \vartheta_{k'}).$$

By the monotonicity of $V(k)$ we deduce that for $k < k'$ holds $V(k) \geq \frac{1}{2} V(k) + \frac{1}{2} V(k')$ which simplifies the term to

$$A(k) \leq 6 \sup_{k' \in [k, K_n]} \left(3(\hat{k}_k - \vartheta_{k'})^2 - \frac{1}{6} V(k')\right)_+ \leq 3 \max_{k' \in [k, K_n]} (\vartheta_k - \vartheta_{k'}).$$

while the latter summand can be bounded for any $k' \in [k, K_n]$ by

$$(\vartheta_k - \vartheta_{k'})^2 \leq \left(\frac{1}{2\pi} \int_{[-k', k']} |\Psi_{(t)}(\mathcal{M}_c[f])| \right) dt \leq \frac{1}{\pi^2} \int_k^\infty |\Psi_{(t)}(\mathcal{M}_c[f])| dt^2.$$

Further we have that $(\vartheta - \hat{k}_n)^2 \leq 2(\vartheta - \hat{k}_n)^2 + 2(\vartheta_k - \hat{k}_n)^2$ which implies

$$(\vartheta - \hat{k}_n)^2 \leq \frac{16}{\pi^2} \left(\int_k^\infty |\Psi_{(t)}(\mathcal{M}_c[f])| dt^2 + V(k)\right) + 4 \sup_{k' \in K_n} (V(k') - \hat{V}(k')_+$$

$$+ 26 \sup_{k' \in [k, K_n]} \left((\hat{k}_k - \vartheta_{k'})^2 - \frac{1}{6} V(k')\right)_+.$$

To control the last term we split the centred arithmetic mean $\hat{k}_k - \vartheta_k$ into two terms, applying at one term a Bernstein inequality, cf lemma A.1, and standard techniques on the other term. For a positive sequence $(\nu_{c_n})_{n \in \mathbb{N}}$ and $t \in \mathbb{R}$ introduce

$$\mathcal{M}_c(t) = n^{-1} \sum_{j=1}^n (Y_j c_{-1} + \nu_{(c_n)})(Y_j c_{-1} + \nu_{(c_n, \infty)})(Y_j c_{-1}) := \mathcal{M}_c,1(t) + \mathcal{M}_c,2(t).$$

Split the centred arithmetic mean $\hat{k}_k - \vartheta_k = \nu_{k, 1} + \nu_{k, 2}$ where $\nu_{k, i} := \frac{1}{2\pi} \int_k^\infty \frac{\Psi_{(t)}(\mathcal{M}_c,[t])}{\mathcal{M}_c(t)} (\mathcal{M}_c,1(t) - \mathbb{E}_{\nu_{k}'} (\mathcal{M}_c,1(t)) ) dt$. Thus we have

$$(\vartheta - \hat{k}_n)^2 \leq \frac{16}{\pi^2} \left(\int_k^\infty |\Psi_{(t)}(\mathcal{M}_c[f])| dt^2 + V(k)\right) + 4 \sup_{k' \in K_n} (\hat{V}(k') - V(k')_+$$

$$+ 52 \sup_{k' \in K_n} \left((\nu_{k', 1} - \frac{1}{12} V(k')\right)_+ + 52 \sup_{k' \in K_n} \nu_{k', 2}^2.$$

The claim of the theorem follows thus by the following lemma.
Lemma A.2. Under the assumptions of Theorem 2.9 with \( c_n := \sqrt{n^{1/2} \sigma \|g\|_{\infty,x^{2e-1}} \log(n) / 42} \) hold

\[
\begin{align*}
(i) & \quad \E_{f_r} \left( \sup_{k' \in K_n} \left( \nu_{k',1}^2 - \frac{1}{12} V(k') \right) \right) \leq C \left( \sigma, \|g\|_{\infty,x^{2e-1}} \right) \frac{1}{n}, \\
(ii) & \quad \E_{f_r} \left( \sup_{k' \in K_n} \nu_{k',2}^2 \right) \leq \frac{C(\Psi, g, \sigma)}{n} \text{ and} \\
(iii) & \quad \E_{f_r} \left( \sup_{k' \in K_n} (V(k') - \hat{V}(k'))_+ \right) \leq \frac{C(\E(\chi_1^{4(c-1)}), \sigma)}{n}.
\end{align*}
\]

Proof of Lemma A.2. To prove (i), we see that

\[
\E_{f_r} \left( \sup_{k' \in K_n} \left( \nu_{k',1}^2 - \frac{1}{12} V(k') \right)_+ \right) \leq \sum_{k \in K_n} \E_{f_r} \left( \left( \nu_{k,1}^2 - \frac{1}{12} V(k) \right)_+ \right) \\
\leq \sum_{k \in K_n} \int_0^\infty \P_{f_r} \left( \left( \nu_{k,1}^2 - \frac{1}{12} V(k) \right)_+ \geq x \right) dx \\
\leq \sum_{k \in K_n} \int_0^\infty \P_{f_r} \left( |\nu_{k,1}| \geq \sqrt{\frac{V(k)}{12} + x} \right) dx.
\]

Now our aim is to apply the Bernstein inequality Lemma A.1. To do so, defining for \( y > 0 \) the function \( h_k(y) := \frac{1}{2\pi} \int_{-k}^k \frac{\Psi(t)}{|\chi_c(0,0)|} y^t dt \) leads to

\[
\nu_{k,1} = \frac{1}{n} \sum_{j=1}^n Y_j^{c-1} 1_{(0,c_n)}(Y_j^{c-1} h_k(Y_j) - \E_{f_r}(Y_1^{c-1} 1_{(0,c_n)}(Y_1) h_k(Y_1)))
\]

where \( |h_k(y)| \leq (2\pi)^{-1} \int_{-k}^k \left| \frac{\Psi(t)}{|\chi_c(0,0)|} \right| dt \leq k^{\Delta_{\Psi,g}(k)} \) implying \( |Y_j^{c-1} 1_{(0,c_n)}(Y_j^{c-1}) h_k(Y_j)| \leq c_n k^{\Delta_{\Psi,g}(k)} =: \nu. \) Further,

\[
\Var_{f_r}(Y_1^{c-1} 1_{(0,c_n)}(Y_1) h_k(Y_1)) \leq \E_{f_r}(Y_1^{2c-2} h_k^2(Y_1)) \leq \|x^{2c-1} g\|_{\infty,\sigma} \Delta_{\Psi,g}(k) =: \nu.
\]

Therefore the Bernstein inequality yields, for any \( x > 0 \)

\[
\P_{f_r}( |\nu_{k,1}| \geq \sqrt{\frac{V(k)}{12} + x} ) \leq 2 \max(\exp(-\frac{n}{4v}(\frac{V(k)}{12} + x)), \exp(-\frac{n}{8\nu}(\sqrt{\frac{V(k)}{12} + \sqrt{x})))
\]

using the concavity of the square root. We have thus to bound the 4 upcoming terms. In fact

\[
\frac{n}{4v} \frac{V(k)}{12} = \frac{\chi}{48} \log(n) \geq \frac{3}{2} \log(n) \quad \text{and} \quad \frac{n}{4\nu} \geq \frac{1}{4\sigma}
\]

\[
18
\]
for $\chi \geq 72$ which implies $\exp(-\frac{n}{4\sigma} (V(k) + x)) \leq n^{3/2} \exp(x/4\sigma)$. Moreover we have

$$n \frac{\sqrt{V(k)}}{8b} = n \sqrt{\sigma \|g\|_{\infty, x^{2c-1}} \Delta_{\psi,k}(k) \chi \log(n)} n^{-1}$$

by definition of $K_n$ and $c_n = n^{1/2} \sigma \|g\|_{\infty, x^{2c-1}} \log(n)/2$. In analogy we can show that

$$n \frac{\sqrt{V(k)}}{8b} = \frac{42n^{3/4}}{\sigma \|g\|_{\infty, x^{2c-1}} \log(n)} \log(n)$$

implying that $\exp(-\frac{n}{8b} (\sqrt{V(k)} + \sqrt{x})) \leq n^{-3/2} \exp(-18\sqrt{x} \log(n)(\sigma \|g\|_{\infty, x^{2c-1}})^{-1})$. Thus we conclude

$$\mathbb{E}_{f_y} (\sup_{k' \in K_n} (2V(k') - \frac{1}{6} V(k'))_+) \leq \sum_{k \in K_n} n^{-3/2} \int_0^\infty \exp(-x \min(\frac{1}{4\sigma}, 18 \sqrt{x} \log(n)(\sigma \|g\|_{\infty, x^{2c-1}})^{-1})) dx$$

$$\leq C(\sigma, \|g\|_{\infty, x^{2c-1}}) \sum_{k \in K_n} n^{-3/2} \leq C(\sigma, \|g\|_{\infty, x^{2c-1}}) \frac{n^{-3/2}}{n}.$$

For part (ii) we have $|\nu_{k',2}| \leq (2\pi)^{-1} \int_{k'}^{k} |\Psi(t)||\mathcal{M}_c(g)(t)|^{-1} |\hat{\mathcal{M}}_{c,2}(t) - \mathbb{E}_{f_y}(\hat{\mathcal{M}}_{c,2}(t))| dt$ implying with the Cauchy Schwartz inequality that

$$\mathbb{E}_{f_y} (\sup_{k' \in K_n} \nu_{k',2}^2) \leq \mathbb{E}_{f_y} (\left( \frac{1}{2\pi} \int_{-K_n}^{K_n} \frac{|\Psi(t)|^2}{|\mathcal{M}_c(g)(t)|^2} |\hat{\mathcal{M}}_{c,2}(t) - \mathbb{E}_{f_y}(\hat{\mathcal{M}}_{c,2}(t))| dt \right)^2)$$

$$\leq \frac{K_n}{2\pi} \Delta_{\psi,g}(K_n) n^{-1} \mathbb{E}_{f_y} (Y_{2c-2}^2 \mathbf{1}_{(c_n, \infty)}(Y_c^{-1}))$$

$$\leq C_{\psi,g} n^{1/2} \mathbb{E}_{f_y} (Y_{c-1}(2+u)) c^{-u}.$$ 

for any $u \in \mathbb{R}^+$. Choosing $u = 6$ leads to $\mathbb{E}_{f_y} (\sup_{k' \in [k, K_n]} \nu_{k',2}^2) \leq C_{\psi,g,\sigma} \mathbb{E}_{f_y} (Y_{c-1}^8) n^{-1}$. To show inequality (iii), we first define the event $\Omega := \{ |\hat{\sigma} - \sigma| < \frac{\sigma}{2} \}$. Then on $\Omega$ we have $\frac{\sigma}{2} \leq \hat{\sigma} \leq \frac{3}{2} \sigma$. Which implies that $V(k) \leq \hat{V}(k) \leq 3V(k)$ and

$$\mathbb{E}_{f_y} (\sup_{k' \in K_n} (V(k') - \hat{V}(k'))) \leq 2\chi \mathbb{E}_{f_y} (|\sigma - \hat{\sigma}| \mathbf{1}_{\Omega'}) \leq 2\chi \frac{\mathbb{V}ar_{f_y}(\hat{\sigma})}{\sigma}$$

by application of the Cauchy-Schwartz and the Markov inequality. This implies the claim.  

### B Proofs of section 3

**Proof of Theorem 3.5.** First we outline here the main steps of the proof. We will construct propose two densities $f_o, f_1$ in $D_k^{\infty, c, L}$ by a perturbation with a small bump, such that the difference $(\vartheta(f_1) - \vartheta(f_2))^2$ and the Kullback-Leibler divergence of their induced distributions can
be bounded from below and above, respectively. The claim follows then by applying Theorem 2.5 in Tsybakov [2008]. We use the following construction, which we present first.

We set \( f_o(x) := \exp(-x) \) for \( x \in \mathbb{R}^+ \). Let \( C_\infty^\infty(\mathbb{R}^+) \) be the set of all infinitely differentiable functions with compact support in \( \mathbb{R}^+ \) and let \( \psi \in C_\infty^\infty(\mathbb{R}^+) \) be a function with support in \([-1, 1]\), \( \int_{-1}^{1} \psi (x) dx = 0 \), \( \psi^{(-1)}(0) \neq 0 \). We define for \( j \in \mathbb{N} \) the finite constant \( C_{j, \infty} := \max(\|\psi(\cdot\|_{\infty, x, l}, l \in [0, j]) \). For each \( x_o \in \mathbb{R}^+ \) and \( h \in (0, x_o/2) \) (to be selected below) we define the bump-function \( \psi_{h, x_o}(x) := \psi(\frac{x-x_o}{h}), x \in \mathbb{R} \). Let us further define the operator \( f \in Tsybakov [2008] \). We use the following construction, which we present first.

The corresponding survival function \( S_o \) of \( f_o \) is given by \( S_o(x) = \exp(-x) \), for \( x \in \mathbb{R}^+ \), while \( F_o(x) = 1 - \exp(-x) \). The resulting survival function and cumulative distribution functions \( F_1 \) and \( S_1 \) of \( f_1 \) are then given by

\[
S_1(x) = S_o(x) + \delta h^{\gamma+1} \psi_{\gamma, h, y_o}(x), \quad x \in \mathbb{R}^+
\]

\[
F_1(x) = F_o(x) - \delta h^{\gamma+1} \psi_{\gamma, h, y_o}(x), \quad x \in \mathbb{R}^+.
\]

To ensure that \( S_1 \), respectively \( F_1 \), is a survival function, respectively a cumulative distribution function, it is sufficient to show that \( f_1 \) is a density.

**Lemma B.1.** For any \( 0 < \delta < \delta_o(\gamma, x_o) := \exp(-3x_o/2)(3x_o/2)^{-\gamma}(C_{\gamma, \infty}c_{\gamma})^{-1} \) the function \( f_1 \), defined in eq. B.1, is a density, where \( c_{\gamma} = \sum_{i=1}^{\gamma} c_{i, \gamma} \).

Further one can show that these functions all lie inside the ellipsoids \( \mathbb{D}_{s,c,L}^{s,c,L} \) for \( L \) big enough. This is captured in the following lemma.

**Lemma B.2.** Let \( s \in \mathbb{N} \) and \( c > 1/2 \). Then, for all \( L \geq L_{s,c,\gamma, \delta, \psi, x_o} > 0 \) holds \( f_o \) and \( f_1 \), as in (B.1), belong to \( \mathbb{D}_{s,c,L}^{s,c,L} \).

For sake of simplicity we denote for a function \( \varphi \in \mathbb{L}_{L^2}^2 \) the multiplicative convolution with \( g \) by \( \bar{\varphi} := [\varphi \ast g] \).

**Lemma B.3.** Let \( h \leq h_o(\psi) \). Then

1. \( (S_1(x_o) - S_o(x_o))^2 = (F_1(x_o) - F_o(x_o))^2 \geq \frac{c_{\gamma}^{-2}}{2} \delta^2 \psi^{(-1)}(0) h^{2s+1} \)

2. \( (f_1(x_o) - f_o(x_o))^2 \geq \frac{c_{\gamma}^2}{2} \delta^2 \psi^{(-1)}(0) h^{2s-1} \)
3. \( KL(\tilde{f}_1, \tilde{f}_0) \leq C(g, x_o, f_o) \|\psi\|^2 \delta^2 h^{2s+2\gamma} \) where \( KL \) is the Kullback-Leibler-divergence.

Selecting \( h = n^{-1/(2s+2\gamma)} \), it follows

\[
\frac{1}{M} \sum_{j=1}^{M} KL(\tilde{f}_1^{(n)}, \tilde{f}_0^{(n)}) = \frac{n}{M} \sum_{j=1}^{M} KL(\tilde{f}_1, \tilde{f}_0) \leq C^{(2)}_{g,y_o,\psi,f_o,\delta}
\]

where \( C^{(2)}_{g,y_o,\psi,f_o,\delta} < 1/8 \) for all if \( \delta \leq \delta_1(g, y_o, \psi, f_o) \). Thereby, we can use Theorem 2.5 of Tsybakov [2008], which in turn for any estimator \( \hat{f} \) of \( f \) implies

\[
\sup_{f \in \mathcal{D}_{R^+}^{s,L}} \mathbb{P}_n^n((\hat{f}(x_o) - f(x_o))^2 \geq \frac{C^{(1)}_{\psi,\delta,\gamma}}{2} n^{-(2s-1)/(2s+2\gamma)}) \geq c > 0;
\]

\[
\sup_{f \in \mathcal{D}_{R^+}^{s,L}} \mathbb{P}_n^n((\hat{S}(x_o) - S(x_o))^2 \geq \frac{C^{(1)}_{\psi,\delta,\gamma}}{2} n^{-(2s-1)/(2s+2\gamma)}) \geq c > 0 \quad \text{and}
\]

\[
\sup_{f \in \mathcal{D}_{R^+}^{s,L}} \mathbb{P}_n^n((\hat{F}(x_o) - F(x_o))^2 \geq \frac{C^{(1)}_{\psi,\delta,\gamma}}{2} n^{-(2s+1)/(2s+2\gamma)}) \geq c > 0.
\]

Note that the constant \( C^{(1)}_{\psi,\delta,\gamma} \) does only depend on \( \psi, \gamma \) and \( \delta \), hence it is independent of the parameters \( s, L \) and \( n \). The claim of Theorem 3.5 follows by using Markov’s inequality, which completes the proof.

Proofs of the lemmata

Proof of Lemma B.1. For any \( h \in C_c^\infty(\mathbb{R}^+) \) holds \( S[h] \in C_c^\infty(\mathbb{R}^+) \) and thus \( S^j[h] \in C_c^\infty(\mathbb{R}) \) for any \( j \in \mathbb{N} \). Further for \( h \in C_c^\infty(\mathbb{R}^+) \) holds \( \int_{-\infty}^{\infty} h^{(1)}(x)dx = 0 \) which implies that for any \( \delta > 0 \) and we have \( \int_0^\infty f_1(x)dx = 1 \).

By construction (B.1) the function \( \psi_{h,x_o} \) has support \( \supp(\psi_{h,x_o}) \in [x_o/2, 3x_o/2] \). Since \( \supp(S[h]) \subseteq \supp(h) \) for all \( h \in C_c^\infty(\mathbb{R}^+) \) the function \( \psi_{\gamma,h,x_o} \) has support in \([x_o/2, 3x_o/2]\) too. First, for \( x \notin [x_o/2, 3x_o/2] \) holds \( f_1(x) = \exp(-x) \geq 0 \). Further for \( x \in [x_o/2, 3x_o/2] \) holds

\[
f_1(x) = f_o(x) + \delta h^{s+\gamma-1/2}x^{-1}\psi_{\gamma,y_o}(x) \geq \exp(-3x_o/2) - \delta(3x_o/2)^\gamma C_{\gamma,\infty}\gamma
\]

since \( \|\psi_{j,h,x_o}\|_\infty \leq (3x_o/2)^j C_{j,\infty}\gamma h^{-j} \) for any \( s \geq 1 \) and \( j \in \mathbb{N} \) where \( c_j := \sum_{i=1}^j c_{i,j} \).

Now choosing \( \delta \leq \delta_0(\psi, \gamma) := \exp(-3x_o/2)(3x_o/2)^\gamma (C_{\gamma,\infty}\gamma)^{-1} \) ensures \( f_1(x) \geq 0 \) for all \( x \in \mathbb{R}^+ \).

Proof of Lemma B.2. Our proof starts with the observation that for all \( t \in \mathbb{R} \) and \( c > 0 \) that

\[
\mathcal{M}_c[f_o](t) \sim t^{-1/2} \exp(-\pi/2|t|), \quad |t| \geq 2,
\]

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by applying the Stirling formula, compare Belomestny and Goldenshluger [2020]. Thus for every \( s \in \mathbb{N} \) there exists \( L_{s,c} \) such that \( |f_o|_{s,c}^2 \leq L \) for all \( L \geq L_{s,c} \).

Next we consider \( |f_o - f_1|_{s,c}^2 \). We have \( |f_o - f_1|_{s,c}^2 = \delta^2 h^{2s+2\gamma-1} |\psi_{\gamma,h,x_o}|_{s,c}^2 \) where \( |.|_{s,c} \) is defined in (3.1). Now since \( \text{supp}(\psi_{\gamma,h,x_o}) \subset [x_o/2, 3x_o/2] \) and \( \psi_{\gamma,h,x_o} \in C_c^\infty(\mathbb{R}^+) \) we have that its Mellin transform is well-defined for any \( c \in \mathbb{R} \). By a integration by parts we see that for any \( \phi \in C_c^\infty(\mathbb{R}^+) \) and \( t, c \in \mathbb{R} \) holds

\[
\mathcal{M}_c[S[\phi]](t) = (c + it)\mathcal{M}_c[\phi](t)
\]

and thus \( |\mathcal{M}_{c-1}[\psi_{\gamma+s,h,x_o}]|_{s,c}^2 = ((c - 1)^2 + t^2)^s |\mathcal{M}_{c-1}[\psi_{\gamma,h,x_o}]|_{s,c}^2 \) and thus

\[
|\omega^{-1}\psi_{\gamma,h,x_o}|_{s,c}^2 \leq C \int_{-\infty}^{\infty} \mathcal{M}_{c-1}[\psi_{\gamma+s,h,x_o}]|_{s,c}^2 dt = C \int_{x_o/2}^{3x_o/2} x^{2c-3} |\psi_{\gamma+s,h,x_o}(x)|^2 dx
\]

by the Parseval formula, cf eq. 2.6, which implies that \( |\omega^{-1}\psi_{\gamma,h,y_o}|_{s,c}^2 \leq C_{c,x_o} |\psi_{\gamma,s,h,x_o}|^2 \). Now applying the Jensen inequality leads to

\[
\|\psi_{\gamma+s,h,x_o}\| \leq C_{\gamma,s} \sum_{j=1}^{\gamma+s} h^{-2j} \int_{x_o-h}^{x_o+h} x^{2j} \psi(j)(x-x_o)^2 dx \leq C_{\gamma,s,x_o} h^{-2\gamma-2s+1} C_{\infty,\gamma,s}.
\]

Thus \( |f_o - f_1|_{s,c}^2 \leq C_{c,s,\gamma,\delta,\psi,x_o} \) and \( |f_1|_{s,c}^2 \leq 2 |f_0 - f_1|_{s,c}^2 + |f_1|_{s,c}^2 \leq 2(C_{c,s,\gamma,\delta,\psi} + L_{s,c}) =: L_{s,c,\gamma,\delta,\psi,x_o,1} \). Now let us consider the moment condition. First we see that \( \int_0^\infty x^{2(c-1)} f_o(x) = C_c \). Further since \( h < x_o/2 \) that

\[
\delta h^{s+\gamma-1/2} \int_0^\infty x^{2(c-1)} \psi_{\gamma,h,x_o}(t) \leq C_{\gamma,\delta} \sum_{j=1}^\gamma h^{s+\gamma+1/2-j} \int_{x_o/2}^{3x_o/2} x^{2(c-1)+j} \psi(j)(x-x_o) dx \\
\leq C_{s,c,\gamma,\delta,\psi,x_o}
\]

Thus we have \( \mathbb{E}_{f_o}(X^{2c-2}) = \mathbb{E}_{f_1}(X^{2(c-1)}) \leq C_c + C_{s,c,\gamma,\delta,\psi,x_o} =: L_{s,c,\gamma,\delta,\psi,x_o,2} \) Choosing now \( L_{s,c,\gamma,\delta,\psi,x_o} = \max(L_{s,c,\gamma,\delta,\psi,x_o,1}, L_{s,c,\gamma,\delta,\psi,x_o,2}) \) shows the claim.

**Proof of Lemma B.3.**

First we see that \( (S_o(x_o) - S_1(x_o))^2 = (F_o(x_o) - F_1(x_o))^2 = \delta^2 h^{2s+2\gamma-1} (\psi_{\gamma-1,h,x_o}(x_o))^2 \) and that \( (\psi_{\gamma-1,h,x_o}(x_o))^2 = \sum_{j=1}^{\gamma-1} c_{i,j-1} h^{-(i+j)} \psi(i)(0)\psi(j)(0) =: \sum c_{i-1,j-1} h^{-2\gamma+2\psi(-1)(0)^2} \).

For \( h \) small enough we thus

\[
(S_o(x_o) - S_1(x_o))^2 \geq \frac{c_{i,j}^2}{2} \delta^2 h^{2s+2\gamma-1} \psi(-1)^2 = c_{\gamma,\psi} h^{2s+1}
\]

for \( h < h_o(\gamma, \psi) \). In analogy, we can show that

\[
(f_o(x_o) - f_1(x_o))^2 = \delta^2 h^{2\gamma+2s-1} (\psi_{\gamma,h,x_o}(x_o))^2 \geq c_{\gamma,\psi} h^{2s-1}.
\]
For the second part we have by using $\text{KL}(\tilde{f}_1, \tilde{f}_o) \leq \chi^2(\tilde{f}_1, \tilde{f}_o) := \int_{\mathbb{R}^+} g(x) x^{-1} f_o(x) dx$ it is sufficient to bound the $\chi$-squared divergence. We notice that $\tilde{f}_0 - \tilde{f}_o$ has support in $[0, 3x_o/2]$ since $f_1 - f_o$ has support in $[x_o/2, 3x_o/2]$ and $g$ has support in $[0, 1]$ In fact for $x > 3x_o/2$ holds $\int \chi f_0(y) - \tilde{f}_o(y) = \int_0^\infty (f_0 - f_o)(x)^{-1} g(x/y) dx = 0$. Since $f_o$ is monotone decreasing we can deduce that $\tilde{f}_o$ is monotone decreasing since for $x_2 \geq x_1 \in \mathbb{R}^+$ holds

$$\tilde{f}_o(x_2) = \int_0^1 g(x) x^{-1} f_o(x_2/x) dx \leq \int_0^1 g(x) x^{-1} f_o(x_1/x) dx = \tilde{f}_o(x_1)$$

since the integrand is strictly positive. We conclude therefore that there exists a constant $c_{f_o,x_o,g} > 0$ such that $\tilde{f}_o(x) \geq c_{f_o,x_o,g} > 0$ for all $x \in (0, 3x_o/2)$. Thus

$$\chi^2(\tilde{f}_1, \tilde{f}_o) \leq \tilde{f}_o(3x_o/2)^{-1} ||\tilde{f}_1 - \tilde{f}_o||^2 = \tilde{f}_o(3x_o/2)^{-1} \delta^2 h^{2s+2\gamma-1} ||\omega^{-1}\psi_{\gamma,h,x_o}||^2.$$ 

Let us now consider $||\omega^{-1}\psi_{\gamma,h,x_o}||^2$. In the first step we see by application of the Plancherel, cf. 2.6, that $||\omega^{-1}\psi_{\gamma,h,x_o}||^2 = \frac{1}{2\pi} \int_{\mathbb{R}^+} |\mathcal{M}_{1/2}[\omega^{-1}\psi_{\gamma,h,x_o}](t)|^2 dt$. Now for $t \in \mathbb{R}$, we see by using the multiplication theorem for Mellin transforms that $\mathcal{M}_{1/2}[\omega^{-1}\psi_{\gamma,h,x_o}](t) = \mathcal{M}_{1/2}[\gamma](t)^\gamma \mathcal{M}_{1/2}[\psi_{h,x_o}](t)$. Again we have $\mathcal{M}_{1/2}[\omega^{-1}\psi_{\gamma,h,x_o}](t) = (-1/2+it)^\gamma \mathcal{M}_{1/2}[\psi_{h,x_o}](t)$. Together with assumption [G1'] we get

$$||\omega^{-1}\psi_{\gamma,h,y_o}||^2 \leq \frac{C_1(g)}{2\pi} \int_{-\infty}^{\infty} |\mathcal{M}_{1/2}[\psi_{h,y_o}](t)|^2 dt = C_1(g) ||\omega^{-1}\psi_{h,y_o}||^2 \leq C(g, y_o) h||\psi||^2.$$ 

Since $M \geq 2^K$ we have thus $\text{KL}(\tilde{f}_0(\theta), \tilde{f}_0(\theta_0)) \leq \frac{C_1(g, y_o)||\psi||^2}{f_o(3y_o/2)} \delta^2 h^{2s+2\gamma}$. 

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