Word Measures on Symmetric Groups

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Abstract

Fix a word \( w \) in a free group \( F \) on \( r \) generators. A \( w \)-random permutation in the symmetric group \( S_N \) is obtained by sampling \( r \) independent uniformly random permutations \( \sigma_1, \ldots, \sigma_r \in S_N \) and evaluating \( w(\sigma_1, \ldots, \sigma_r) \). In \([\text{Pud14}, \text{PP15}]\) it was shown that the average number of fixed points in a \( w \)-random permutation is \( 1 + \theta (N^{1-\pi(w)}) \), where \( \pi(w) \) is the smallest rank of a subgroup \( H \leq F \) containing \( w \) as a non-primitive element. We show that \( \pi(w) \) plays a role in estimates of all stable characters of symmetric groups. In particular, we show that for all \( t \geq 2 \), the average number of \( t \)-cycles is \( \frac{1}{t} + O(N^{-\pi(w)}) \). As an application, we prove that for every \( s \), every \( \varepsilon > 0 \) and every large enough \( r \), Schreier graphs with \( r \) random generators depicting the action of \( S_N \) on \( s \)-tuples, have second eigenvalue at most \( 2\sqrt{2r-1} + \varepsilon \) asymptotically almost surely. An important ingredient in this work is a systematic study of not-necessarily connected Stallings core graphs.

Contents

1 Introduction ................................................................. 2
  1.1 Main theorem ......................................................... 2
  1.2 General “stable” class functions .................................. 4
  1.3 Stable irreducible characters ....................................... 5
  1.4 Expansion of random Schreier graphs .......................... 6
  1.5 Overview of the paper .............................................. 7
  1.6 Similar phenomena in other families of groups ............... 9

2 From words to subgroups ................................................. 13

3 Multi core graphs ......................................................... 14
  3.1 Core graphs ......................................................... 14
  3.2 Multi core graphs and their morphisms ........................ 15

4 Free and algebraic morphisms .......................................... 18
  4.1 Free morphisms ..................................................... 18
  4.2 Algebraic morphisms ............................................... 20
  4.3 The algebraic-free decomposition of morphisms .............. 21

5 \( B \)-Surjective morphisms and norms of morphisms ............ 22

6 Möbius inversions and the leading terms of \( \Phi \) ................. 25
  6.1 Basis dependent Möbius inversions ............................... 26
  6.2 Algebraic Möbius Inversion ....................................... 30

7 The proof of Theorem 1.3 ................................................ 32
  7.1 Maximal Euler characteristic ..................................... 34
  7.2 The set of critical morphisms .................................... 35
1 Introduction

Fix \( r \in \mathbb{Z}_{\geq 1} \). Throughout this paper we let \( \mathbf{F} \) denote the free group on \( r \) generators. A word \( w \in \mathbf{F} \) induces a map on any finite group, \( w : G^r \to G \), by substituting the letters of \( w \) with elements of \( G \). This map defines a distribution on the group \( G \): the push forward of the uniform distribution on \( G^r \). Equivalently, this distribution is the normalized number of times each element in \( G \) is obtained by a substitution in \( w \). We call this distribution the \( w \)-measure on \( G \). For example, if \( w = xyxy^{-2} \), a \( w \)-random element in \( G \) is \( ghgh^{-2} \) where \( g, h \) are independent, uniformly random elements of \( G \).

More concretely, we study the expected value with respect to word measures of certain class functions (functions invariant under conjugation). Given a class function \( f : G \to \mathbb{R} \), we analyze \( E_w[f] \), the average value of this function under the \( w \)-measure on \( G \). Word measures are constant on conjugacy classes of \( G \), i.e. are themselves class functions on the group \( G \). Therefore, the expressions \( E_w[f] \), running over a suitable family of class functions (for example, all irreducible characters of \( G \)), uniquely determine the \( w \)-measure on \( G \). Several papers studying word measures on various groups are motivated by questions from the field of free probability, where the asymptotic statistics of such measures on families of groups was analyzed. In recent years, different works found more refined and deeper structure in these measures. We mention some of these works in Section 1.6. The current work is the first one where non-trivial bounds are given on all “natural” families of class functions on a given family of groups, as we now explain.

Our focus in this paper is on word measures on the symmetric groups \( S_N \), and especially on the following class functions. For every \( k \in \mathbb{Z}_{\geq 1} \), denote

\[
\xi_k(\sigma) \overset{\text{def}}{=} \#\text{fix}\left(\sigma^k\right) \tag{1.1}
\]

where \( \#\text{fix}(\tau) \) is the number of fixed points of the permutation \( \tau \). We study the expected value under word measures of products of the \( \xi_k \)'s in the form of \( \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_k^{\alpha_k} \) with \( k \in \mathbb{Z}_{\geq 1} \) and \( \alpha_1, \ldots, \alpha_k \in \mathbb{Z}_{\geq 0} \) with \( \sum \alpha_k \geq 1 \). When we write \( E_w[\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}] \) there is a suppressed parameter \( N \), namely, \( E_w[\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}] \) is a map \( \mathbb{Z}_{\geq 1} \to \mathbb{Q} \), where \( N \) is mapped to the average value of this class function under the \( w \)-measure on \( S_N \).

1.1 Main theorem

For every word \( w \in \mathbf{F} \) and \( \alpha_1, \ldots, \alpha_k \) as above, the expectation \( E_w[\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}] \) is a rational function of \( N \), for large enough \( N \): this is essentially a result of [Nic94], and see also Section 4 and especially Remark 31 in [LP10]. (It also follows from the analysis in the current paper – see Corollary 6.9.) For example, \( E_{xyx^{-1}y^{-1}}[\xi_1 \xi_2] = 3 + \frac{4N^4 - 9N^3 + 23N^2 - 13N - 12}{N(N - 1)(N - 2)(N - 3)(N - 4)} \) for all \( N \geq 6 \). In particular, for large enough \( N \), \( E_w[\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}] \) can be written as a Laurent series in \( N \). Our main goal in this paper is to estimate the leading terms of this Laurent series expansion. The special case of \( \xi_1 = \#\text{fix}(\sigma) \), the average number of fixed points, was studied in [Pud14, PP15]. These papers show a connection between \( E_w[\xi_1] \) and invariants of \( w \) as an element of the free group.

In order to explain these invariants, we need a few notions from the realm of free groups. A basis of a free group is a free generating set (or, equivalently for finitely generated free groups, a generating set of minimal size). An element \( w \in \mathbf{F} \) is called primitive if it belongs to a basis of \( \mathbf{F} \). The rank of the
free group $F$, denoted $rkF$, is the size of a basis of $F$. The classical Nielsen-Schreier theorem states that subgroups of free groups are free. The primitivity rank of a word, which plays an important role in this paper, was first introduced in [Pud14]:

**Definition 1.1.** The primitivity rank $\pi(w)$ of a word $w \in F$ is the minimal rank of a subgroup $H \leq F$ containing $w$ as a non-primitive element. If there are no such subgroups, set $\pi(w) = \infty$. We also consider the set of critical subgroups of $w$ defined as

$$\text{Crit}(w) = \{ H \leq F \mid \text{rk}H = \pi(w), H \ni w \text{ and } w \text{ non-primitive in } H \}.$$

For example, $\pi(w) = 0 \iff w = 1$ as the trivial word is contained in the trivial subgroup but not as a primitive element. Words with $\pi(w) = 1$ are precisely proper powers and if $w \in F$ is not a proper power and $m \geq 2$, then $\text{Crit}(w^m) = \{ \langle u^d \rangle \mid d \mid m, 1 \leq d < m \}$. Finally, $\pi(w) = \infty$ if and only if $w$ is primitive in $F$, and in any other case $\pi(w) \leq r = rkF$ [Pud14, Lemma 4.1]. The set $\text{Crit}(w)$ is always finite [PP15, Section 4]. We can now state the aforementioned result from [PP15].

**Theorem 1.2.** [PP15, Theorem 1.8] For every word $w \in F$

$$\mathbb{E}_w [\#\text{fix}(\sigma)] = 1 + \frac{|\text{Crit}(w)|}{N^\pi(w) - 1} + O \left( \frac{1}{N^\pi(w)} \right).$$

Since the expected number of fixed point in a uniformly random permutation is 1, the theorem can be restated as

$$\mathbb{E}_w [\xi_1] = \mathbb{E}_{\text{unif}} [\xi_1] + \frac{|\text{Crit}(w)|}{N^\pi(w) - 1} + O \left( \frac{1}{N^\pi(w)} \right),$$

where $\mathbb{E}_{\text{unif}} [f]$ is the expectation of the function $f$ with respect to the uniform distribution on $S_N$. The main result of this paper is the following generalization of Theorem 1.2. The quantity $\langle \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}, \xi_1 - 1 \rangle$ appearing in the statement is defined on page 4 below.

**Theorem 1.3.** For every non-power $w \in F$, and for every $k \in \mathbb{Z}_{\geq 1}$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{Z}_{\geq 0}$, there exists a positive integer $C_{\alpha_1, \ldots, \alpha_k} \in \mathbb{Z}_{\geq 1}$ such that

$$\mathbb{E}_w [\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}] = \mathbb{E}_{\text{unif}} [\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}] + C_{\alpha_1, \ldots, \alpha_k} \cdot \frac{|\text{Crit}(w)|}{N^\pi(w) - 1} + O \left( \frac{1}{N^\pi(w)} \right). \ (1.2)$$

Moreover, the constant $C_{\alpha_1, \ldots, \alpha_k}$ is equal to $\langle \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}, \xi_1 - 1 \rangle$.

In particular, $\mathbb{E}_w [\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}] \geq \mathbb{E}_{\text{unif}} [\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}]$ for all large enough $N$. Note that the exclusion of powers in the statement of the theorem is necessary: for example, let $x \in F$ be a basis element. While $\mathbb{E}_{x^3} [\xi_2] = 4$ for $N \geq 6$, we have $\pi(x^3) = 1$, $\text{Crit}(x^3) = \{ \langle x \rangle \}$, $\mathbb{E}_{\text{unif}} [\xi_2] = 2$ for $N \geq 2$ and $\langle \xi_2, \xi_1 - 1 \rangle = 1$, and so (1.2) would give in this case $2 + 4 \cdot \frac{1}{N} + O \left( \frac{1}{N^2} \right) = 3 + O \left( \frac{1}{N} \right)$, which is incorrect. However, these expected values can still be understood. Indeed, $\langle \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}, (\sigma) \rangle = \langle \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}, (\sigma^t) \rangle$. Hence, we can still obtain an approximation for the expected value of $\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}$ under a power-word, and see also Corollary 1.6. In Remark 7.3 we provide a combinatorial formula for $\mathbb{E}_{\text{unif}} [\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}]$.

**Remark.** Throughout this paper, $N^{-\infty}$ should be interpreted as zero. In particular, the results in this paper, Theorems 1.2 and 1.3 included, hold for primitive words as well, for which, as mentioned above, $\pi(w) = \infty$. For example, in this case (1.2) becomes $\mathbb{E}_w [\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}] = \mathbb{E}_{\text{unif}} [\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}]$. Indeed, primitive words induce the uniform measure on every finite group [PP15, Observation 1.2].

We perceive our results as interesting and elegant for their own sake. We do, however, have further motivation for this study. One piece of motivation comes from consequences of Theorem 1.3 to the expansion of random Schreier graphs of $S_N$ as detailed in Section 1.4 below. More motivation comes from
a growing body of evidence to an interplay between word measures in general and primitivity rank of words in particular on the one hand, and seemingly unrelated questions and challenges in combinatorial and geometric group theory on the other hand. This is illustrated by the works [PP15, HMP20] dealing with “profinite rigidity” of words, and by the papers [LW22, LW21] where the primitivity rank of a word is shown to have crucial consequences for the one-relator group this word defines. We add here further evidence: in Section 1.6 we explain how some of the ideas in this paper are related to the notion of stable commutator length, our proof of Theorem 1.3 in Section 7 suggests a deep connection between the “dependence theorems” of [Lou13, LW22] and word measures on $S_N$, and in Appendix A we use our main result to get a new and simple proof of the conjugacy separability of free groups.

1.2 General “stable” class functions

Consider the abstract polynomial ring $A = \mathbb{Q}[\xi_1, \xi_2, \ldots]$ in countably many variables. Every element $f \in A$ corresponds to a class function in $S_N$ for all $N$. The elements analyzed in Theorem 1.3 are precisely the monomials in $A$, and thus give a linear basis for $A$. For every class functions $f, g \in A$ and every $N \in \mathbb{Z}_{\geq 1}$, we have the ordinary inner product in $S_N$ defined as

$$\langle f, g \rangle_{S_N} \overset{\text{def}}{=} \frac{1}{N!} \sum_{\sigma \in S_N} f(\sigma) \cdot g(\sigma).$$

For every $f, g \in A$, this inner product stabilizes for large enough $N$ – see Proposition B.1. We denote this constant value by $\langle f, g \rangle$. In particular, note that $\langle f, 1 \rangle = \mathbb{E}_{\text{unif}}[f]$ for every large enough $N$. The following corollary thus follows immediately from Theorem 1.3. As above, for $f \in A$ we denote by $\mathbb{E}_w[f]$ a function $\mathbb{Z}_{\geq 1} \to \mathbb{Q}$ which maps $N$ to the average value of $f$ in $S_N$ under the $w$-measure.

**Corollary 1.4.** For every class function $f \in A$ and every non-power $w \in F$,

$$\mathbb{E}_w[f] = \langle f, 1 \rangle + \langle f, \xi_1 - 1 \rangle \cdot \frac{|\text{Crit}(w)|}{N^{\pi(w)-1}} + O\left(\frac{1}{N^{\pi(w)}}\right).$$

Some elements of the ring $A$ coincide with characters of families of representations: such a family consists of a representation of $S_N$ for every large enough $N$. These families are precisely the families of stable representations of $\{S_N\}$, in the sense of [CF13]. These families were studied in [CEF15] and subsequent works.

An interesting special case of Corollary 1.4 deals with statistics of short cycles in $S_N$. For $t \in \mathbb{Z}_{\geq 1}$, let $a_t(\sigma)$ denote the number of cycles of length $t$ in the permutation $\sigma$. This is an element in $A$: for example, $a_2 = \frac{2 - \xi_1}{2}$. For $N \geq t$, the expected number of $t$-cycles in a uniformly random permutation in $S_N$ is $\frac{1}{t}$. For $t \geq 2$, $\langle a_t, \xi_1 - 1 \rangle = 0$ (see Appendix B for more details). Therefore,

**Corollary 1.5.** Let $t \geq 2$. For every non-power $w \in F$,

$$\mathbb{E}_w[a_t] = \frac{1}{t} + O\left(\frac{1}{N^{\pi(w)}}\right).$$

Another special case of Theorem 1.3 we mention explicitly is that of the functions $\xi_d$, as they relate to a general conjecture about word measures. This conjecture asks whether two words $w_1$ and $w_2$ in $F$ inducing the same measure on every finite group are necessarily in the same orbit of Aut$F$. It appears, for example, as [AV11, Question 2.2] and [Sha13, Conjecture 4.2], and see also [CMP20]. (The converse, that two words in the same orbit induce the same measure on every finite group, is a simple observation.) The special case of this conjecture when $w_1$ is primitive, namely, in the orbit of the word $x$, was settled in [PP15]: it follows from Theorem 1.2 that if $w_2$ induces the same measure as $x$, namely, uniform measure, on $S_N$ for all $N$, then $w_2$ must be primitive too. This was generalized in [HMP20, Theorem 1.4] to show
that the conjecture is true when \( w_1 = x^d \) or \( w_1 = [x, y]^d \) for arbitrary \( d \in \mathbb{Z}_{\geq 1} \), and in [Wil21] to all surface words \( w_1 = [x_1, y_1] \cdots [x_g, y_g] \) or \( w_1 = x_1^2 \cdots x_g^2 \) and their powers.

Consider the case \( w_1 = x^d \). The proof in [HMP20, Theorem 1.4] has two steps: first, it can be shown that if \( w_2 \) induces the same measures as \( x^d \) then \( w_2 = u^d \) is a \( d \)-th power of some non-power word \( u \). Then it is shown that if \( w_2 \) is not in the orbit of \( x^d \), then for every large enough \( N \), the average number of fixed points in a \( w_2 \)-random permutation in \( S_N \) is strictly larger than that of \( x^d \). Our results here give a quantitative version of this step. For every \( d \in \mathbb{Z}_{\geq 1} \), \( \langle \xi_d, 1 \rangle = \tau \left( d \right) \), where \( \tau \left( d \right) \) is the number of positive divisors of \( d \), and \( \langle \xi_d, \xi_1 - 1 \rangle = 1 \). Hence,

**Corollary 1.6.** Assume \( 1 \neq u \in \mathbb{F} \) is a non-power and let \( w = u^d \) for some \( d \in \mathbb{Z}_{\geq 1} \). Then,

\[
\mathbb{E}_w \left[ \xi_1 \right] = \mathbb{E}_u \left[ \xi_d \right] = \tau \left( d \right) + \frac{\left| \text{Crit} \left( u \right) \right|}{N\pi(u)-1} + O \left( \frac{1}{N\pi(u)} \right).
\]

### 1.3 Stable irreducible characters

Arguably, the most important elements in the ring \( \mathcal{A} \) of class functions are the elements corresponding to families of irreducible characters \( \chi = \{ \chi_N \}_{N \geq N_0} \) (\( \chi_N \) being an irreducible character of \( S_N \)): these are precisely the characters of stable families of irreducible representations. For a partition \( \lambda = (\lambda_1 \geq \ldots \geq \lambda_\ell > 0) \), denote \( |\lambda| = \sum_{i=1}^\ell \lambda_i \), so \( \lambda \vdash |\lambda| \). For every \( N \geq |\lambda| + \lambda_1 \), consider the partition

\[
\lambda \cup \{ N - |\lambda| \} = (N - |\lambda| \geq \lambda_1 \geq \ldots \geq \lambda_\ell) \vdash N.
\]

These partitions give rise to a family of irreducible characters \( \chi = \{ \chi_N \}_{N \geq |\lambda| + \lambda_1} \), one for every \( N \geq |\lambda| + \lambda_1 \). This family corresponds to an element of \( \mathcal{A} \) – see Appendix B. Table 1 describes the four “simplest” families of irreducible characters in \( \mathcal{A} \).

| Young Diagram of \( \chi \) | \( \lambda \) | Element of \( \mathcal{A} \) | Poly in the \( a_i \)'s | Dimension of \( \chi_N \) |
|---------------------------|--------|------------------|-----------------|-----------------|
| \[
\begin{array}{cccc}
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\end{array}
\] | \( \emptyset \) | 1 | 1 | 1 |
| \[
\begin{array}{cccc}
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\end{array}
\] | \( 1, \xi_1 - 1 \) | \( a_1 - 1 \) | \( N - 1 \) |
| \[
\begin{array}{cccc}
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\end{array}
\] | \( 2, \left( \frac{\xi_1^2 + \xi_2}{2} - 2 \xi_1 \right) \) | \( \left( \frac{a_1(a_1 - 3)}{2} \right) + a_2 \) | \( \frac{N(N-3)}{2} \) |
| \[
\begin{array}{cccc}
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\end{array}
\] | \( 1, 1 \) | \( \left( \frac{\xi_1^2 - \xi_2}{2} - \xi_1 + 1 \right) \) | \( \left( \frac{(a_1 - 1)(a_1 - 2)}{2} \right) - a_2 \) | \( \frac{(N-1)(N-2)}{2} \) |

**Table 1:** Some families of irreducible characters belonging to \( \mathcal{A} \)

**Corollary 1.7.** Let \( \chi \in \widehat{S}_\infty \) so that \( \chi \neq 1, \xi_1 - 1 \). Then, for non-powers \( w \in \mathbb{F} \),
\[ \mathbb{E}_w [\chi] = O \left( \frac{1}{N^{\pi(w)}} \right). \]

In fact, the elements of \( \hat{S}_\infty \) form, too, a linear basis of letters, then the \( \mu \)-function of Schur’s Lemma gives \( \pi \) hand, \( \pi \in \mathbb{C} \). Moreover, given two class functions \( \mu \in \mathbb{C} \). Finally, \( \mu \in \mathbb{C} \).\[ \text{Fro}99 \] showed that \( \mu \in \mathbb{C} \). Another known special case of this conjecture is the commutator \( [x, y] = x y x^{-1} y^{-1} \); indeed, \( \pi ([x, y]) = 2 \) and already in 1896, Frobenius [Fro96] showed that \( \mathbb{E}_{[x, y]} [\chi] = \frac{1}{\dim \chi} \) for every finite group \( G \) and every irreducible character \( \chi \) of \( G \). Moreover, given two class functions \( \mu_1, \mu_2 : G \to \mathbb{R} \) and an irreducible character \( \chi \) of \( G \), a simple application of Schur’s Lemma gives \( \langle \mu_1 \ast \mu_2, \chi \rangle_G = \frac{\langle f_{\mu}, \chi \rangle_G}{\dim \chi} \). If \( w_1 \in F \{ x_1, \ldots, x_k \} \), \( w_2 \in F \{ x_{k+1}, \ldots, x_r \} \) are two words generated by disjoint sets of letters, then the \( w_1 w_2 \)-measure on \( G \) is the convolution of the \( w_1 \)- and the \( w_2 \)-measures, and by the corollary of Schur’s Lemma, \( \mathbb{E}_{w_1, w_2} [\chi] = \frac{\mathbb{E}_{w_1} [\chi] \cdot \mathbb{E}_{w_2} [\chi]}{\dim \chi} \). On the other hand, \( \mu(w_1 w_2) = \mu(w_1) + \mu(w_2) \) [Pud14, Lemma 6.8]. Hence, knowing the conjecture for two such words implies the claim for their product. In particular, this implies the conjecture for every product of disjoint commutators and powers, that is, for every word of the form \( w = [x_1, y_1] \cdot [x_2, y_2] \cdot \ldots \cdot [x_r, y_r] \cdot z_1^{k_1} \cdots z_m^{k_m} \in F \{ x_1, \ldots, x_r, y_1, \ldots, y_r, z_1, \ldots, z_m \} \), with \( r, m \in \mathbb{Z}_{\geq 0} \) and \( k_1, \ldots, k_m \in \mathbb{Z} \). See Section 1.6 for generalizations of Conjecture 1.8 for other families of groups.

### 1.4 Expansion of random Schreier graphs

As an application of our results, we prove expansion properties of random Schreier graphs of the symmetric group. If \( G \) is a \( d \)-regular graph on \( n \) vertices, its adjacency matrix has eigenvalues

\[ d = \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq -d, \]

with \( \mu_1 = d \) considered as a trivial eigenvalue. Denote by \( \mu (G) \) the largest absolute value of a non-trivial eigenvalue of the graph \( G \). Namely, \( \mu (G) = \max (\mu_2, -\mu_n) \). An expander graph is a sparse graph with high connectivity. One standard way to measure expansion is with \( \mu (G) \) – smaller \( \mu (G) \) implies better expansion (see [HLW06] for a survey). Here we study random Schreier graphs of the groups \( S_N \).

**Definition 1.9.** Given an action of a group \( G \) on a set \( X \), and a tuple \( g_1, \ldots, g_r \in G \), the corresponding Schreier graph is the \( 2r \)-regular graph with vertex set \( X \) and an edge \( x \sim g_i(x) \) for every \( x \in X \) and \( i \in [r] \). Note that we allow multiple edges as well as loops.

The group \( S_N \) acts naturally on the set of \( s \)-tuples of distinct elements in \( [N] \). Choosing uniformly at random a tuple of permutations \( \sigma_1, \ldots, \sigma_s \in S_N \), consider the (random) Schreier graph corresponding to this action. Denoting \( d = 2r \), this is a random \( d \)-regular graph on \( (N)_s \) vertices. The fact that for a fixed \( s \), this family of random \( d \)-regular graphs has a uniform spectral
gap with high probability is known since the work [FJR+98]. Theorem 2.1 therein states that for every \(\varepsilon > 0\),
\[
|\mu(G) - (1 + \varepsilon) \sqrt{\frac{2d-1}{d}}| \leq \frac{2\sqrt{d-1}}{d} \left(\begin{array}{c} 2\sqrt{d-1} - 1 + \varepsilon \\ \sqrt{d-1} \end{array}\right)^{1/(s+1)}
\]
asymptotically almost surely (namely, with probability tending to 1 as \(N \to \infty\); a.a.s. in short).

For \(s = 1, 2\) much stronger bounds are known. Friedman [Fri08] famously proved a conjecture of Alon and showed that for \(s = 1\), a random \(d\)-regular graph in this model is nearly Ramanujan in the strong sense that for every \(\varepsilon > 0\), a.a.s. \(\mu(G) < 2\sqrt{d-1} + \varepsilon\). More recently, following Bordenave’s new proof of Friedman’s theorem [Bor20], Bordenave and Collins [BC19] proved the same result for Schreier graphs on pairs of elements, namely, for \(s = 2\). It is conjectured that the same result holds for any fixed value of \(s\) — see, for instance, [RS19, Conjecture 1.6]\(^1\). This conjecture, and progress towards it, may serve as steps towards answering an even harder question: whether or not random Cayley graphs of \(S_N\) (namely, Cayley graphs with respect to a random set of elements of some fixed size) are a.a.s. nearly Ramanujan. In fact, it is not even known whether these random Cayley graphs are uniformly expanders. See [BL22] for a recent survey.

In [Pud15], using a different approach, the special case of the action of \(S_N\) on \([N]\) was studied. It was proved that a.a.s. \(\mu(G) < 2\sqrt{d-1} + 0.84\). The same approach was later improved in [FP21] to give a.a.s.
\[
\mu(G) < 2\sqrt{d-1} \cdot \exp\left(\frac{2}{e^2(d-1)}\right) < 2\sqrt{d-1} + \frac{0.6}{\sqrt{d-1}}.
\]
Here we generalize this method and prove the following bound for all values of \(s\):

**Theorem 1.10.** Fix \(s, r \in \mathbb{Z}_{\geq 1}\) and let \(d = 2r\). Let \(G\) be a random \(d\)-regular Schreier graph depicting the action of \(r\) random permutations on \(s\)-tuples of distinct elements from \([N]\). Then a.a.s. as \(N \to \infty\),
\[
|\mu(G) - (1 + \varepsilon) \sqrt{\frac{2d-1}{d}}| \leq \frac{2\sqrt{d-1}}{d} \left(\begin{array}{c} 2\sqrt{d-1} - 1 + \varepsilon \\ \sqrt{d-1} \end{array}\right)^{1/(s+1)}
\]
For fixed \(s\) and growing \(d\), this bound is
\[
|\mu(G) - (1 + \varepsilon) \sqrt{\frac{2d-1}{d}}| \leq \frac{2\sqrt{d-1}}{d} \left(\begin{array}{c} 2\sqrt{d-1} - 1 + \varepsilon \\ \sqrt{d-1} \end{array}\right)^{1/(s+1)}
\]
In particular, for every fixed \(s\) and every \(\varepsilon > 0\), if \(d\) is large enough, (1.4) gives that a.a.s. \(\mu(G) < 2\sqrt{d-1} + \varepsilon\).

**Remark 1.11.** The bound (1.4) is achieved by optimizing our method for large values of \(r\) (and \(d\)) and fixed \(s\). For specific, small values of \(r\), the method gives better bounds. For example, for \(r = 2\) (so \(d = 4\)) and \(s = 1\), (1.4) gives a bound of \(\approx 3.735\), while the method can actually yield a bound of \(\approx 3.596\) (compare with \(2\sqrt{3} \approx 3.464\)).

**Remark 1.12.** Conjecture 1.8, if true, yields that the bound (1.3) holds as is also for the Schreier graphs in Theorem 1.10, namely, a bound which is independent of \(s\). See Remark 8.7 for more details.

### 1.5 Overview of the paper

**Outline of the proof of Theorem 1.3**

Let \(1 \neq w \in F\) be a non-power. Consider the function \(\mathbb{E}_w [\xi_{1\alpha_1} \cdots \xi_{k\alpha_k}] : \mathbb{Z}_{\geq 1} \to \mathbb{Q}\) from Theorem 1.3. As this function is invariant under replacing \(w\) with a conjugate (automorphic image, in fact), we may assume that \(w\) is cyclically reduced. A natural approach to study this function is to consider \(\alpha_1 + \ldots + \alpha_k\)

\(^1\)It is plausible that the method of proof in [BC19] can be used to prove this conjecture in full.
cycles describing powers of \( w \): \( \alpha_1 \) copies of \( w \), \( \alpha_2 \) copies of \( w^2 \) and so on. This graph is denoted \( \Gamma_{\alpha_1, \ldots, \alpha_k}^w \) (see Example 3.6) and illustrated in Figure 3.1. Then \( \mathbb{E}_w \left[ \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k} \right] (N) \) counts the average number of labelings of the vertices of \( \Gamma_{\alpha_1, \ldots, \alpha_k}^w \) which agree with the independent uniformly random permutations represented by the \( r \) generators of \( F \) (\( x \) and \( y \) in Figure 3.1). The fact that \( \mathbb{E}_w \left[ \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k} \right] (N) \) is given by a function in \( Q (N) \) (Corollary 6.9) follows by a simple argument (see also [Pud14, Section 5] for a more straightforward explanation of the technique).

It follows from (the arguments in) [Nic94, LP10] that \( \mathbb{E}_w \left[ \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k} \right] = \mathbb{E}_{\text{unif}} \left[ \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k} \right] + O \left( \frac{1}{N} \right) \). Our goal is to give a more precise estimate of the \( O \left( \frac{1}{N} \right) \) term. First, we imitate the proof in [PP15] of Theorem 1.2. This part starts with generalizing the function \( \mathbb{E}_w \left[ \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k} \right] \) as follows. Consider the graph-morphism \( \eta^{w}_{\alpha_1, \ldots, \alpha_k} : \Gamma_{\alpha_1, \ldots, \alpha_k}^w \to X_B \) where \( X_B \) is the bouquet (see Figure 3.1): \( \mathbb{E}_w \left[ \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k} \right] (N) \) is then also equal to the average number of lifts of this morphism to a random \( N \)-covering of \( X_B \) (Proposition 3.7). The same average can be defined to any morphism \( \eta \) between finite graphs: and this is the essence of the map \( \Phi_\eta \) in Definition 3.4. In particular, \( \Phi_\eta^{w_{\alpha_1, \ldots, \alpha_k}} = \mathbb{E}_w \left[ \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k} \right] \). The graphs we consider here have directed edges labeled by a fixed basis \( B \) of \( F \) and each connected component is a Stallings core graph: we call such graphs multi core graphs – see Section 3 for the precise definition. They correspond to multisets of conjugacy classes of f.g. subgroups of \( F \).

As in [PP15], we introduce Möbius inversions of the function \( \Phi \) (Definitions 6.6 and 6.13). While the left inversion \( L^\Phi \) is quite natural (corresponds to injective lifts of the morphism rather than arbitrary lifts in \( \Phi \) – see Proposition 6.8), the other two inversions \( R^\Phi \) and \( C^\Phi \) are more mysterious. However, the crux of introducing these Möbius inversions is that their analysis proves some non-trivial cancellations in the computation of \( \Phi \), culminating in Theorem 6.2. The flow of ideas in this part is very similar to [PP15], and the reader is advised to read the overview there [PP15, Section 2].

Explaining the content of Theorem 6.2 requires first to describe the notion of algebraic morphism. For two free groups \( H \leq J \), we say that \( J \) is an algebraic extension of \( H \) if there is no intermediate subgroup of \( J \) which is a proper free factor of \( J \). This notion was coined in [MVW07] and it gives a notion of algebraicity of morphisms between connected core graphs. In Section 4 we generalize this notion to morphisms between general multi core graphs. In Sections 4 and 5 we also introduce pullbacks in the category of multi core graphs, consider \( B \)-surjective morphisms and define norms of morphisms – all these are required to prove Theorem 6.2. Many of the definitions around the category of multi core graphs are not obvious and we think this part of the paper may be of independent interest.

Theorem 6.2 considers an arbitrary morphism \( \eta : \Gamma \to \Delta \) of multi core graphs. It follows from Theorem 4.7(1) that there are finitely many decompositions \( \Gamma \xrightarrow{\eta_1} \Sigma \xrightarrow{\eta_2} \Delta \) of \( \eta \) with \( \eta_1 \) algebraic. We let \( \chi_{\max} (\eta) \) denote the maximal Euler characteristic of \( \Sigma \) of \( \Sigma \) in such a decomposition with \( \eta_1 \) algebraic and non-isomorphism, and \( \text{Crit} (\eta) \) denote the set of such decompositions with \( \chi (\Sigma) \) maximal – see Definition 6.1. Then Theorem 6.2 states that

\[
\Phi_\eta (N) = N \chi (\Gamma) + |\text{Crit} (\eta)| \cdot N \chi_{\max} (\eta) + O \left( N \chi_{\max} (\eta) - 1 \right) .
\]  

(1.5)

When \( \eta \) is the morphism from \( \Gamma_1^w \), the core graph of \( \langle w \rangle \), to the bouquet \( X_B \), (1.5) is precisely Theorem 1.2 – the main result of [PP15]. However, when applied to the morphism \( \eta^{w}_{\alpha_1, \ldots, \alpha_k} : \Gamma_{\alpha_1, \ldots, \alpha_k}^w \to X_B \) with \( \sum i \alpha_i \geq 2 \), (1.5) does not yield anything new: it only recovers earlier results from [Nic94, LP10]. Indeed, in these cases there is an algebraic morphism \( \Gamma_{\alpha_1, \ldots, \alpha_k}^w \to \Gamma_1^w \), so \( \chi_{\max} (\eta^{w}_{\alpha_1, \ldots, \alpha_k}) = 0 \), and (1.5) only gives information about the free term of the Laurent expansion of \( \mathbb{E}_w \left[ \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k} \right] \).

Section 7, using arguments from combinatorial and geometric group theory, strengthens (1.5) for the morphisms \( \eta^{w}_{\alpha_1, \ldots, \alpha_k} \) and completes the proof of Theorem 1.3. First, we handle separately algebraic morphisms from \( \Gamma_{\alpha_1, \ldots, \alpha_k}^w \) with codomain of Euler characteristic 0, and define \( \chi_{\max} (\eta^{w}_{\alpha_1, \ldots, \alpha_k}) \) to be the maximal negative Euler characteristic of the (codomain of) an algebraic morphism from \( \Gamma_{\alpha_1, \ldots, \alpha_k}^w \) and \( \text{Crit}_{\alpha_1, \ldots, \alpha_k} (w) \) accordingly (see Definition 7.1). Theorem 7.2, which follows readily from our analysis of the algebraic Möbius inversions of \( \Phi \), gives the following estimate for \( \mathbb{E}_w \left[ \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k} \right] \):

\[
\Phi_\eta^{w_{\alpha_1, \ldots, \alpha_k}} (N) = \mathbb{E}_{\text{unif}} \left[ \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k} \right] + |\text{Crit}_{\alpha_1, \ldots, \alpha_k} (w)| \cdot N \chi_{\max} (\eta^{w}_{\alpha_1, \ldots, \alpha_k}) (w) + O \left( N \chi_{\max} (\eta^{w}_{\alpha_1, \ldots, \alpha_k}) (w) - 1 \right) .
\]
It remains to show that $\chi_{\alpha_1,\ldots,\alpha_k}^{\max}(w) = 1 - \pi(w)$ and that $|\text{Crit}_{\alpha_1,\ldots,\alpha_k}(w)| = \langle \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}, \xi_1 - 1 \rangle \cdot |\text{Crit}(w)|$. The former equality is the content of the short Proposition 7.5. The latter equality is the content of Section 7.2. It is quite straightforward to see that every critical subgroup of $w$ corresponds to $\langle \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}, \xi_1 - 1 \rangle$ distinct critical morphisms in $\text{Crit}_{\alpha_1,\ldots,\alpha_k}(w)$. The hard part is to show that these are the only elements of $\text{Crit}_{\alpha_1,\ldots,\alpha_k}(w)$. To this end, we use a “dependence” theorem of Louder [Lou13] – Theorem 7.6 below – concerning free quotients of certain graphs of groups, from which we conclude that an algebraic morphism from $\Gamma_{\alpha_1,\ldots,\alpha_k}$ which, roughly, does not factor non-trivially through $\Gamma_1^w$ must be of Euler characteristic strictly smaller then $1 - \pi(w)$. In fact, this type of dependence theorems of Louder (see also Louder and Wilton [LW22]) fits so well into our proof here, that it suggests a deeper connection between these theorems and Conjecture 1.8. See also Section 7.2.1 for more points of intersection between this paper and works of Louder and Wilton.

**Paper organization**

We end the introduction in Section 1.6, which describes some fascinating evidence that the phenomena we prove and the phenomena we conjecture regarding the symmetric group are, in fact, more universal. We stress that Section 1.6 is completely orthogonal to the remaining sections and the reader interested solely in our proven results may safely skip it.

Sections 2 through 5 and Appendix C are devoted to the study of multi core graphs. We begin with the equivalent category of multisets of conjugacy classes of finitely generated subgroups of $F$, introduced in Section 2. Section 3 introduces the geometric counterpart of the latter: multi core graphs and their morphisms, and also the above-mentioned function $\Phi_\eta$. Section 4 defines free and algebraic morphisms of multi core graphs, as well as pullbacks (also known as fiber products). Section 5 and Appendix C deal with surjective morphisms and with the norms of morphisms, generalizing analogous concepts from [Pud14]. Section 6 introduces the Möbius inversions of the function $\Phi_\eta$ and proves the above mentioned Theorem 6.2 – a naive analogue of Theorem 1.2. In Section 7 we strengthen Theorem 6.2 and prove our main result, Theorem 1.3.

Finally, Section 8 contains the proof of Theorem 1.10 about random Schreier graphs of $S_N$, in Appendix A we use our results in order to obtain a new and simple proof of the well-known conjugacy separability of free groups, and Appendix B develops more formally the ring $A$ of class functions introduced above. We end with a glossary of the main notation used along the paper.

### 1.6 Similar phenomena in other families of groups

Some of the phenomena discussed above regarding word measures on the symmetric group have parallels, at least partially, in other families of groups. The mere fact that for every natural family of class functions $f$ and every word $w \in F$, the expectation $E_w[f]$ is a rational function in the running parameter (usually $N$) of the family of groups, is true not only for symmetric groups [Nic94, LP10], but also for unitary groups$^2$ [Rād06, MŚS07], for orthogonal and compact symplectic groups [MP19b], for natural families of class functions of $\text{GL}_N(F_q)$, where $F_q$ is a fixed finite field [EWPS21], and for generalized symmetric groups [MP21, Ord20, Sho21].

However, it seems there are deeper universal phenomena which are common to all these families of groups. In each of the above-mentioned families, there are also natural families of irreducible characters defined analogously to those in $S_N$: these are the characters of stable families of irreducible representations in the sense of [CF13]. We elaborate a bit below in the sequel of this subsection. It seems that the primitivity rank of a word plays a role in all the above mentioned families of groups. More precisely, we conjecture the following generalization of Conjecture 1.8:

---

$^2$Given a compact group $G$, the $w$-measure on $G$ is the pushforward, via the word map $w: G^* \to G$, of the Haar measure on $G^*$. 

9
Conjecture 1.13. Let \( G = \{G(N)\}_N \) be a natural family of groups as those mentioned above, and let \( \chi = \{\chi_N\}_{N \geq N_0} \) be a stable irreducible character with \( \chi_N \in G(N) \). Then for any word \( w \in F \), as \( N \to \infty \),

\[
E_w [\chi] = O \left( (\dim \chi)^{1 - \pi(w)} \right).
\]

Here the implied constant may depend on \( w \), on \( G \) and on \( \chi \).

As stated, this conjecture may sound a bit vague, but it has a very concrete meaning in each of the above mentioned families of groups. Before elaborating on what this means for each of these families, we mention that the conjecture is trivial for \( w = 1 \), and is true for proper powers, namely, \( E_w [\chi_N] = O (1) \), in all cases studied in the works mentioned above. The conjecture is also true for \( w = [x, y] \) by [Fro96], and thus also for any word of the form \( w = [x_1, y_1] \cdot [x_2, y_2] \cdot \ldots \cdot [x_r, y_r] \cdot z_1^{k_1} \cdot \ldots \cdot z_m^{k_m} \), as explained on page 6. In fact, if true, Conjecture 1.13 may be seen as a generalization of Frobenius’ result about the commutator word \([x, y] \).

**Unitary groups**

Consider the unitary groups \( U(N) \). By analogy to \( \xi_k \) from (1.1), define \( \zeta_k: U(N) \to \mathbb{C} \) by \( \zeta_k (B) = \text{tr} (B^k) \), only here \( k \in \mathbb{Z} \) may also be negative, and define \( A^U = \mathbb{Q} [\ldots , \zeta_{-2}, \zeta_{-1}, \zeta_1, \zeta_2, \ldots] \). In the case of \( U(N) \), the natural families of irreducible characters referred to in Conjecture 1.13 are those corresponding to elements in \( A^U \). In terms of highest weight vectors\(^3\), one starts with an arbitrary highest weight vector of length \( N_0 \) and adds \( N - N_0 \) zeros to obtain a character of \( U(N) \) for all \( N \geq N_0 \).

The expected value of monomials in the \( \zeta_k \)’s under word measures is the main object of study in [MP19a], where these values are given a topological interpretation in terms of surfaces and mapping class groups. In particular, the defining character of \( U(N) \) is \( \zeta_1 \), and it satisfies [MP19a, Corollary 1.8]

\[
E_w [\zeta_1] = E_w [\text{tr} (B)] = O \left( N^{1 - 2 \cdot \text{cl}(w)} \right),
\]

where \( \text{cl}(w) \) is the commutator length of \( w \):

\[
\text{cl}(w) \overset{\text{def}}{=} \min \{g \mid w = [u_1, v_1] \cdots [u_g, v_g] \text{ with } u_i, v_i \in F\}.
\]

(If \( w \notin [F, F] \), we say that \( \text{cl}(w) = \infty \).) Note that if \( w = [u_1, v_1] \cdots [u_g, v_g] \), then \( w \) is non-primitive in the subgroup \( \langle u_1, v_1, \ldots, u_g, v_g \rangle \), hence \( \pi(w) \leq 2g \). Thus

\[
\pi(w) \leq 2 \cdot \text{cl}(w),
\]

and (1.6) yields that Conjecture 1.13 holds for the irreducible character \( \zeta_1 \).

Moreover, there is a nice relation between general *polynomial* characters of \( U(N) \) and an important invariant of words called *stable commutator length*, which is defined as

\[
\text{scl}(w) \overset{\text{def}}{=} \lim_{m \to \infty} \frac{\text{cl}(w^m)}{m}.
\]

Indeed, from [MP19a, Theorem 1.7] it follows that for every \( w \in F \), \( \ell > 0 \) and \( j_1, \ldots, j_\ell \in \mathbb{Z} \),

\[
E_w [\zeta_{j_1} \cdots \zeta_{j_\ell}] = Tr_{w^{j_1}, \ldots, w^{j_\ell}} (N) = O \left( N^{\chi_{\max} (w^{j_1}, \ldots, w^{j_\ell})} \right),
\]

where \( \chi_{\max} (w^{j_1}, \ldots, w^{j_\ell}) \) is the maximal Euler characteristic of a surface admissible for \( w^{j_1}, \ldots, w^{j_\ell} \) [MP19a, Definition 1.2]. This is all very much related to Calegari’s works on stable commutator length. First, [Cal09, Lemma 2.6] yields that if a surface \( \Sigma \) is admissible for \( w^{j_1}, \ldots, w^{j_\ell} \), then

\[
\text{scl}(w) \leq \frac{-\chi(\Sigma)}{2 |j_1 + \ldots + j_\ell|},
\]

\(^3\)For the theory of highest weight vectors see, e.g., [Bum04, Chapters 24,25].
so \( \chi (\Sigma) \leq -2 \cdot \scl (w) \cdot |j_1 + \ldots + j_\ell| \), which, combined with (1.7), gives

\[
E_w [\zeta_{j_1} \cdots \zeta_{j_\ell}] = O \left( N^{-2\cdot\scl(w)} \cdot |j_1 + \ldots + j_\ell| \right). \tag{1.9}
\]

Now consider the subring \( \mathcal{A}^U_{\text{poly}} \overset{\text{def}}{=} \mathbb{Q} [\zeta_1, \zeta_2, \ldots] \) of \( \mathcal{A}^U \) generated by traces of positive powers of the matrices in \( U (N) \). The irreducible characters corresponding to elements of \( \mathcal{A}^U_{\text{poly}} \) are families of characters of polynomial irreducible representations of \( U (N) \). In the language of highest weight vectors, these are irreducible characters with non-negative weights. By the representation theory of \( U (N) \), every such character corresponds to some partition \( \mu \) (these are the positive weights). Let \( \eta_\mu \in \mathcal{A}^U_{\text{poly}} \) be the family of polynomial irreducible characters corresponding to the partition \( \mu \). There is a simple formula expressing \( \eta_\mu \) as a linear combination of the monomials in \( \mathcal{A}^U_{\text{poly}} \). For a partition \( \lambda = (1^{\alpha_1} 2^{\alpha_2} \ldots k^{\alpha_k}) \), define the monomial \( \zeta_\lambda \overset{\text{def}}{=} \zeta_1^{\alpha_1} \cdots \zeta_k^{\alpha_k} \). In addition, let

\[
z_\lambda \overset{\text{def}}{=} \prod_{r} r^{\alpha_r} \cdot \alpha_r !.
\]

Note that \( \frac{1}{z_\lambda} \) is the probability that a random permutation in \( S_{|\lambda|} \) has cycle structure \( \lambda \). Finally, given two partitions \( \lambda, \rho \) of \( m \), denote the value of \( \chi^\rho \) (the irreducible character of \( S_m \) corresponding to \( \rho \)) on a permutation with cycle structure \( \lambda \) by \( \chi^\rho (\lambda) \). The formula for the polynomial character \( \eta_\mu \) is

\[
\eta_\mu = \sum_{\lambda \vdash |\mu|} \frac{1}{z_\lambda} \chi^\mu (\lambda) \, \zeta_\lambda \tag{1.10}
\]

(this is basically a special case of \([\text{Sta99, Corollary 7.17.5}]\)). For example, \( \eta_{1,1,1} = \frac{1}{6} \zeta_1^3 - \frac{1}{2} \zeta_2 \zeta_1 + \frac{1}{3} \zeta_3 \). In particular, (1.10) yields that \( \dim \eta_\mu \) is a polynomial in \( N \) of degree \( |\mu| \). We conclude from (1.9) that for every family \( \eta_\mu \) of polynomial irreducible characters,

\[
E_w [\eta_\mu] = O \left( N^{-2\cdot\scl(w)} \cdot |\mu| \right) = O \left( (\dim \eta_\mu)^{-2\cdot\scl(w)} \right). \tag{1.11}
\]

More strikingly, in the same paper \([\text{Cal09}]\), Calegari also proves that every word \( w \) admits extremal surfaces: these are surfaces admissible for \( w^{j_1}, \ldots, w^{j_\ell} \) for some \( \ell > 0 \) and \( j_1, \ldots, j_\ell > 0 \), such that there is equality in (1.8). In particular, \( \scl (w) \) is rational for every \( w \), which is the main result of \([\text{Cal09}]\). As explained in \([\text{MP19a, Section 5.1}]\), for such values of \( j_1, \ldots, j_\ell > 0 \) admitting extremal surfaces, (1.7) becomes

\[
E_w [\zeta_{j_1} \cdots \zeta_{j_\ell}] = \# \{ \text{extremal surfaces} \} \cdot N^{-2\cdot\scl(w)(j_1+\ldots+j_\ell)} \left( 1 + O \left( N^{-2} \right) \right). \tag{1.12}
\]

Now consider \( \eta_k \), the irreducible polynomial character of \( U (N) \) corresponding to the partition \( (k) \). In this case, \( \chi^{(k)} \) is the trivial character of \( S_k \), and so (1.10) becomes

\[
\eta_k = \sum_{\lambda \vdash k} \frac{1}{z_\lambda} \zeta_\lambda.
\]

Because the coefficients here are all positive, the positive contribution of extremal surfaces to \( E_w [\eta_k] \) cannot be balanced out. So, if \( w \) admits extremal surfaces with \( j_1 + \ldots + j_\ell = k \), then\(^4\)

\[
E_w [\eta_k] = \Theta \left( (\dim \eta_k)^{-2\cdot\scl(w)} \right). \tag{1.12}
\]

From (1.11) and (1.12) we conclude that in the case of families of polynomial irreducible characters of \( U (N) \), Conjecture 1.13 is equivalent to the following one.

\(^4\)We use the notation \( f = \Theta (g) \) if these are two functions of \( N \in \mathbb{Z}_{\geq 1} \) satisfying \( f = O (g) \) and \( g = O (f) \).
Conjecture 1.14. For any \( w \in F \),
\[
\pi(w) \leq 2 \cdot \text{scl}(w) + 1.
\]

This conjecture was verified numerically for various words – see, for instance, [CH21, Proposition 4.4]. In fact, Heuer arrived to the exact same conjecture independently [Heu19, Conjecture 6.3.2], based entirely on computer experiments!

We stress that [MP19a] does not provide such sharp bounds for general, non-polynomial, characters of \( U(N) \). For example, the irreducible character with highest weight vector \((1, 0, \ldots, 0, -1)\) is of dimension \( N^2 - 1 \) and is equal to \( \zeta_1 \zeta_{-1} - 1 \). One can infer from the analysis of admissible surfaces of Euler characteristic zero that for non-powers, \( E_w[\zeta_1 \zeta_{-1} - 1] = O(N^{-2}) \). Yet Conjecture 1.13 says, in this case, that it should be of order \( O(N^{2(1-\pi(w))}) \), which is open when \( \pi(w) \geq 3 \).

Orthogonal and symplectic groups

In the case of the orthogonal group \( O(N) \) and compact symplectic group \( \text{Sp}(N) \), the defining standard representation (\( N \)-dimensional in the case of \( O(N) \), \( 2N \)-dimensional for \( \text{Sp}(N) \)) has real trace, and for a matrix \( B \) in the defining representation, \( \text{tr}(B^{-k}) = \text{tr}(B^k) \). So here the ring of class functions is \( \mathbb{Q} [\zeta_1, \zeta_2, \ldots] \) with \( \zeta_k(B) \overset{\text{def}}{=} \text{tr}(B^k) \). The paper [MP19b] studies monomials in the \( \zeta_k \)'s and describes their expected value under word measures in terms of, again, surfaces and mapping class groups. There is one case where the general result there translates into a concrete algebraic bound: the standard character \( \zeta_1 \). Corollary 1.10 in [MP19b] states that for both \( O(N) \) and \( \text{Sp}(N) \),
\[
E_w[\zeta_1] = O\left(N^{1-\min(\text{scl}(w), 2 \cdot \text{cl}(w))}\right),
\]
where
\[
\text{scl}(w) \overset{\text{def}}{=} \min \left\{ g \mid w = u_1^2 \cdots u_g^2 \text{ with } u_i \in F \right\}.
\]
As argued above, this shows that Conjecture 1.13 holds for this character. We do not have significant evidence towards conjecture 1.13 in the case of other characters.

Generalized symmetric group

Consider either the groups \( \{C_m \wr S_N\}_N \) where \( C_m \) is a fixed cyclic group of order \( m \geq 2 \), or \( \{S^1 \wr S_N\}_N \) where \( S^1 = \mathbb{R}/\mathbb{Z} \). One can define here too natural families of irreducible characters. The standard character \( \zeta_1 \), that of the standard \( N \)-dimensional representation, is irreducible. In [MP21, Theorem 1.11], it is shown that
\[
E_w[\zeta_1] = \begin{cases} 
D^m_w \cdot N^{\chi_m(w)} + O\left(N^{\chi_m(w)-1}\right) & \text{if } G(N) = C_m \wr S_N \\
D^\infty_w \cdot N^{\chi_\infty(w)} + O\left(N^{\chi_\infty(w)-1}\right) & \text{if } G(N) = S^1 \wr S_N.
\end{cases}
\]
Here, \( \chi_m(w) \) is the maximal Euler characteristic\(^5\) of a subgroup \( H \leq F \) such that \( w \in \ker(H \to C_m^{rkH}) \), and \( D^m_w \) is the number of such subgroups of maximal Euler characteristic. Similarly, \( \chi_\infty(w) \) is the maximal Euler characteristic of a subgroup \( H \leq F \) such that \( w \in [H, H] \), and \( D^\infty_w \) is the number of such subgroups of maximal Euler characteristic. If \( w \in \ker(H \to C_m^{rkH}) \) or \( w \in [H, H] \), then \( w \) is a non-primitive element of \( H \). Thus, in all these cases \( E_w[\zeta_1] = O\left(N^{1-\pi(w)}\right) \) and, again, Conjecture 1.13 holds. More evidence towards Conjecture 1.13 in these families of groups is found in [Ord20, Sho21].

\(^5\chi(H) = 1 - \text{rk}H.\)
Matrix Groups over finite fields

Finally, fix a finite field $\mathbb{F}_q$ and consider a family of groups such as $\{\text{GL}_N(\mathbb{F}_q)\}_N$. The ring of class functions corresponding to this family of groups can be constructed as follows. For every positive integer $k$ and $A \in \text{GL}_k(\mathbb{F}_q)$, define $\xi_A : \text{GL}_N(\mathbb{F}_q) \to \mathbb{Z}_{\geq 0}$ by

$$\xi_A(B) = \# \{ M \in \text{Mat}_{N \times k}(\mathbb{F}_q) \mid BM = MA \}.$$  

This is indeed a class function. For example, if $A = (1) \in \text{GL}_1(\mathbb{F}_q)$, then $\xi_A(B)$ counts the number of fixed point in the action of $B$ on $\mathbb{F}_q^N$. If $A \sim A'$ are conjugates in $\text{GL}_k(\mathbb{F}_q)$, then $\xi_A = \xi_{A'}$. These class functions are analogous to the monomials $\xi_1^{a_1} \cdots \xi_k^{a_k}$ defined on symmetric groups, and they linearly span the ring of class functions for this family of groups:

$$\mathcal{A} \defeq \text{span}_\mathbb{C} \left( 1, \{ \xi_A \}_{k \in \mathbb{Z}_{\geq 1}, A \in \text{GL}_k(\mathbb{F}_q)} \right).$$

Consider elements of $\mathcal{A}$ corresponding to irreducible characters of $\{\text{GL}_N(\mathbb{F}_q)\}_N$ (for every large enough $N$). Such families of characters are the ones Conjecture 1.13 relates to in this case. As an example, one such family of irreducible characters is given by the permutation-representation given by the action of $\text{GL}_N(\mathbb{F}_q)$ on the projective space $\mathbb{P}^{N-1}(\mathbb{F}_q)$ minus the trivial representation. This is a $\frac{q^N - q}{q - 1}$-dimensional representation. As an element of $\mathcal{A}$, it is given by $\chi = \left( \frac{1}{q^N - 1} \sum_{A \in \text{GL}_1(\mathbb{F}_q) \cong \mathbb{F}_q} \xi_A \right) - 2$. Conjecture 1.13 says that in this case one should have $E_w[\chi] = O((q^N)^{1-\pi(w)})$. In [EWPS21] it is shown that $E_w[f]$ is equal to a rational expression in $q^N$ for every $f \in \mathcal{A}$, and partial evidence is given towards Conjecture 1.13 in the case of $\{\text{GL}_N(\mathbb{F}_q)\}_N$.

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2 From words to subgroups

We now consider a few generalizations of our object of study that will be crucial for the remainder of the paper. The quantities we wish to study are of the form

$$E_w[\xi_1^{a_1} \cdots \xi_k^{a_k}] = E_{\sigma_1, \ldots, \sigma_r \in S_N} \left[ \# \text{fix}(w(\sigma_1, \ldots, \sigma_r))^{a_1} \cdots \# \text{fix}(w^k(\sigma_1, \ldots, \sigma_r))^{a_k} \right].$$

Assume that $w$ is written in the ordered basis $B = \{ b_1, \ldots, b_r \}$ of $F$. Choosing a uniformly random $r$-tuple of permutations from $S_N$ is the same as choosing at random a homomorphism $\varphi : F \to S_N$, as $\varphi(b_1), \ldots, \varphi(b_r)$ is a uniformly random $r$-tuple of permutations. Replacing the letters of $w$ by the permutations $\varphi(b_1), \ldots, \varphi(b_r)$, we obtain the permutation $\varphi(w)$. Hence,

$$E_w[\xi_1^{a_1} \cdots \xi_k^{a_k}] = E_{\varphi \in \text{Hom}(F, S_N)} \left[ \# \text{fix}(\varphi(w))^{a_1} \cdots \# \text{fix}(\varphi(w^k))^{a_k} \right].$$  \hspace{1cm} (2.1)

Following [PP15], the first step in our analysis is to generalize the function we study. This generalization is crucial for the next steps. The most straightforward generalization is to consider quantities of the form

$$E_{\varphi \in \text{Hom}(F, S_N)} \left[ \# \text{fix}(\varphi(w_1)) \cdots \# \text{fix}(\varphi(w_t)) \right],$$  \hspace{1cm} (2.2)

for arbitrary words $w_1, \ldots, w_t \in F$. Next, we generalize from fixed points of a word to common fixed points of several words, or, equivalently, to common fixed points of subgroups: note that given a finite set of words
and arbitrary orientations and with wedge point \( B \). Let \( B \leq \mathbf{F} \) be a multiplication of the expectation by subgroups also means that the Euler characteristic of multi core graphs (see Definition 3.3 below) is always empty set as an element in codomain and if the trivial group is excluded. In addition, non-surjective morphisms allow us to have the neatlly as stated on page 19 only if the image of a morphism may avoid some of the components in its \( \eta \) (see Definition 5.2) is zero if and only if it is an isomorphism. However, pullbacks exist as simply and statement of Proposition 4.3(3) as well as of Definition 5.2, and imply that the norm \( \# \text{fix} (\varphi) \) of a morphism \( \varphi \) of these objects, which we denote \( \text{MOCC} \), of the map \( H \rightarrow J \). Given \( f \): \( [\ell] \rightarrow [m] \), two different choices of representatives as in Definition 2.1 may yield equivalent morphisms. We defer the exact definition of this equivalence to the next section, where we give a geometric description of the category \( \text{MOCC} \) in terms of multi core graphs.

\[ \mathbf{E}_{\varphi \in \text{Hom}(\mathbf{F},S_N)} [\# \text{fix} (\varphi(H_1)) \cdots \# \text{fix} (\varphi(H_\ell))] , \]  

where \( H_1, \ldots, H_\ell \leq \mathbf{F} \) are f.g. (finitely generated) subgroups of \( \mathbf{F} \).

If \( H, H' \leq \mathbf{F} \) are conjugate subgroups then \( \# \text{fix} (\varphi(H)) = \# \text{fix} (\varphi(H')) \). Therefore, (2.3) depends, in fact, on a multiset of conjugacy classes of non-trivial f.g. subgroups of \( \mathbf{F} \). We shall work in the category of these objects, which we denote \( \text{MOCC} \). Finally, assume that there are two multisets of non-trivial f.g. subgroups \( H_1, \ldots, H_\ell \leq \mathbf{F} \) and \( J_1, \ldots, J_m \leq \mathbf{F} \), and that there is a map \( f: [\ell] \rightarrow [m] \), such that \( H_i \leq J_{f(i)} \) for all \( 1 \leq i \leq \ell \). Let \( \{ \varphi_j: J_j \rightarrow S_N \}^m_{j=1} \) be independent, uniformly random homomorphisms. Our final generalization of the object of study is to

\[ \mathbf{E}_{\{\varphi_j \in \text{Hom}(J_j,S_N)\}^m_{j=1}} [\# \text{fix} (\varphi_{f(1)}(H_1)) \cdots \# \text{fix} (\varphi_{f(\ell)}(H_\ell))] . \]  

In the following section we will use the following formal definition of morphisms in the category \( \text{MOCC} \), which arises naturally from the above-mentioned generalizations of our object of study:

**Definition 2.1.** Let \( \mathcal{H} = \{ H_1^\mathbf{F}, \ldots, H_\ell^\mathbf{F} \} \) and \( \mathcal{J} = \{ J_1^\mathbf{F}, \ldots, J_m^\mathbf{F} \} \) be two elements of \( \text{MOCC} \). A morphism \( \eta: \mathcal{H} \rightarrow \mathcal{J} \) consists of a map \( f: [\ell] \rightarrow [m] \) and a choice of representatives \( \overline{H_1} \in H_1^\mathbf{F}, \ldots, \overline{H_\ell} \in H_\ell^\mathbf{F}, \overline{J_1} \in J_1^\mathbf{F}, \ldots, \overline{J_m} \in J_m^\mathbf{F} \) so that \( \overline{H_i} \leq \overline{J_{f(i)}} \) for all \( i \in [\ell] \).

Given \( f: [\ell] \rightarrow [m] \), two different choices of representatives as in Definition 2.1 may yield equivalent morphisms. We defer the exact definition of this equivalence to the next section, where we give a geometric description of the category \( \text{MOCC} \) in terms of multi core graphs.

**Remark 2.2.** We made several non-obvious choices in our definitions in the current section of the category \( \text{MOCC} \), and in the equivalent categories \( \text{MuGCC}_B \) defined in the next section. For example, one could consider multi core graphs with \( k \) ordered basepoints for some fixed \( k \). Even in the category of multi core graphs without basepoints, we made the non-obvious choices of excluding the trivial subgroup of \( \mathbf{F} \), and not demanding in Definition 2.1 that the map \( f: [\ell] \rightarrow [m] \) be surjective. There are good arguments for not making these choices. For example, restricting to surjective maps \( [\ell] \rightarrow [m] \) would simplify the statement of Proposition 4.3(3) as well as of Definition 5.2, and imply that the norm \( \| \eta \| \) of a morphism \( \eta \) (see Definition 5.2) is zero if and only if it is an isomorphism. However, pullbacks exist as simply and neatly as stated on page 19 only if the image of a morphism may avoid some of the components in its codomain and if the trivial group is excluded. In addition, non-surjective morphisms allow us to have the empty set as an element in \( \text{MOCC} \) with a unique morphism to any other element. Avoiding trivial subgroups also means that the Euler characteristic of multi core graphs (see Definition 3.3 below) is always non-positive, which is convenient. Notice that the affect of adding a trivial subgroup to the multiset in (2.3) would be a multiplication of the expectation by \( N \).

## 3 Multi core graphs

We use core graphs, and more generally multi core graphs, as a geometric picture of the multisets of subgroups considered above.

### 3.1 Core graphs

Let \( B = \{ b_1, \ldots, b_r \} \) be a basis of \( \mathbf{F} \), and consider the bouquet \( X_B \) of \( r \) circles with distinct labels from \( B \) and arbitrary orientations and with wedge point \( o \). Then \( \pi_1(X_B,o) \) is naturally identified with \( \mathbf{F} \). The notion of \((B\text{-labeled})\) core graphs, introduced in [Sta83], refers to finite\(^6\), connected graphs with every

\(^6\)One may include also infinite core graphs in the definition, but these are not needed in the current paper.
vertex having degree at least two (so no leaves and no isolated vertices), that come with a graph morphism to $X_B$ which is an immersion, namely, locally injective. In other words, this is a finite connected graph with at least one edge and no leaves, with edges that are directed and labeled by the elements of $B$, such that for every vertex $v$ and every $b \in B$, there is at most one incoming $b$-edge and at most one outgoing $b$-edge at $v$. We stress that multiple edges between two vertices and loops at vertices are allowed.

There is a natural one-to-one correspondence between finite $B$-labeled core graphs and conjugacy classes of non-trivial f.g. subgroups of $F$. Indeed, given a core graph $\Gamma$ as above, pick an arbitrary vertex $v$ and consider the “labeled fundamental group” $\pi_1^{lab}(\Gamma, v)$: closed paths in a graph with oriented and $B$-labeled edges correspond to words in the elements of $B$. In other words, if $p: \Gamma \to X_B$ is the immersion, then $\pi_1^{lab}(\Gamma, v)$ is the subgroup $p_*(\pi_1(\Gamma, v))$ of $\pi_1(X_B, o) = F$. The conjugacy class of $\pi_1^{lab}(\Gamma, v)$ is independent of the choice of $v$ and is the conjugacy class corresponding to $\Gamma$. We denote it by $\pi_1^{lab}(\Gamma)$.

Conversely, if $H \leq F$ is a non-trivial f.g. subgroup, the conjugacy class $H^F$ corresponds to a finite core graph, denoted $\Gamma_B(H^F)$, which can be obtained in different manners. For example, let $\Upsilon$ be the topological covering space of $X_B$ corresponding to $H^F$, which is equal in this case to the Schreier graph depicting the action of $F$ on the right cosets of $H$ with respect to the generators $B$. Then $\Gamma_B(H^F)$ is obtained from $\Upsilon$ by ‘pruning all hanging trees’, or, equivalently, as the union of all non-backtracking cycles in $\Upsilon$. One can also construct $\Gamma_B(H^F)$ from any finite generating set of $H$ using “Stallings foldings” – see [Sta83, KM02, Pud14, PP15] for more details about foldings and about core graphs in general.

### 3.2 Multi core graphs and their morphisms

Here, we consider core graphs which are not necessarily connected:

**Definition 3.1.** Let $B$ be a basis of $F$. A $B$-labeled multi core graph is a disjoint union of finitely many core graphs. In other words, this is a finite graph, not necessarily connected, with no leaves and no isolated vertices, and which comes with an immersion to $X_B$. We denote the set of $B$-labeled multi core graphs by $\mathcal{MCGB}(F)$.

Because a connected core graph corresponds to a conjugacy class of non-trivial f.g. subgroups of $F$, a multi core graph corresponds to a multiset of such objects. Therefore, every basis $B$ of $F$ gives rise to a one-to-one correspondence

$$
\mathcal{MCGB}(F) = \left\{ \begin{array}{c}
B\text{-labeled} \\
n\text{multi core graphs}
\end{array} \right\} \leftrightarrow \left\{ \text{finite multisets of conjugacy classes} \right\}.
$$

(3.1)

For a multi core graph $\Gamma \in \mathcal{MCGB}(F)$ we let $\pi_1^{lab}(\Gamma)$ denote the corresponding multiset in $\mathcal{MOCC}(F)$, and for a multiset $\mathcal{H} \in \mathcal{MOCC}(F)$ we let $\Gamma_B(\mathcal{H})$ denote the corresponding multi core graph.

**Definition 3.2.** A morphism $\eta: \Gamma \to \Delta$ between $B$-labeled multi core graphs is a graph-morphism which commutes with the immersions $p, q$ to $X_B$.

$$
\begin{array}{c}
\Gamma \\
\eta
\end{array} \quad \xymatrix{ \Gamma \ar[r]^\eta \ar[d]^p & \Delta \ar[l]^q \\
X_B
} \quad \xymatrix{ \Gamma \ar[r]^\eta & \Delta \ar[l]^q \\
X_B
}
$$

In particular, the morphism $\eta$ is itself an immersion, and it preserves the orientations and labels of the edges. To get a description of $\eta$ in terms of subgroups à la Definition 2.1, assume that $\Gamma$ consists of $\ell$ components $\Gamma_1, \ldots, \Gamma_\ell$ and that $\Delta$ consists of $m$ components $\Delta_1, \ldots, \Delta_m$. Let $f: [\ell] \to [m]$ be the induced map on connected components, so $\eta(\Gamma_i) \subseteq \Delta_{f(i)}$. For every $i \in [\ell]$, pick an arbitrary vertex $v_i \in \Gamma_i$ and let $H_i = \pi_1^{lab}(\Gamma_i, v_i)$. As $\eta$ is an immersion, it induces injective maps at the level of fundamental groups:
indeed, any non-backtracking cycle in $\Gamma$ is mapped to a non-backtracking cycle in $\Delta$. Therefore, $\eta$ can be thought of as the embedding, for all $i \in [\ell]$,

$$H_i \hookrightarrow \pi_1 \left( \Delta_{f(i)}, \eta(v_i) \right). \tag{3.2}$$

We still need to conjugate the images in (3.2) so that they all sit in the same subgroups in the conjugacy class of subgroups of $\Delta_j$. Formally, pick an arbitrary vertex $p_k \in \Delta_k$ for all $k \in [m]$ and let $J_k = \pi_1 (\Delta_k, p_k)$. For every $i \in [\ell]$, let $u_i \in F$ satisfy $u_i \left[ \pi_1 \left( \Delta_{f(i)}, \eta(v_i) \right) \right] u_i^{-1} = J_{f(i)}$. So now $u_i H_i u_i^{-1} \leq J_{f(i)}$, and we get a morphism as in Definition 2.1.

Conversely, every embedding of subgroups $H \hookrightarrow J$ of $\mathbf{F}$ gives rise to a morphism of core graphs $\Gamma_B (H) \to \Gamma_B (J)$. Indeed, if one considers the entire covering space $\Upsilon_H$ of $X_B$ corresponding to $H$, there is certainly a morphism to $\Upsilon_J$, the one corresponding to $J$. Because every non-backtracking closed path in $\Upsilon_H$ is mapped to a non-backtracking closed path in $\Upsilon_J$ (being non-backtracking is a local property), we see that the image of $\Gamma_B (H)$ is contained in $\Gamma_B (J)$. Thus any morphism in $\text{MOCC} (\mathbf{F})$ as in Definition 2.1, gives rise to a morphism of the corresponding multi core graphs. We say that two morphisms in $\text{MOCC} (\mathbf{F})$ are identical if they induce the same morphism of multi core graphs, up to a post-composition by an isomorphism of the codomain. This equivalence of morphisms in $\text{MOCC} (\mathbf{F})$ can also be defined in completely algebraic terms\textsuperscript{7}, and, in particular, it does not depend on the basis $B$.

Throughout the text we use the following three important invariants of multi core graphs.

**Definition 3.3.** Let $\Gamma \in \text{MuCG}_B (\mathbf{F})$ be a multi core graph and $\mathcal{H} = \pi_1^{\text{lab}} (\Gamma) = \{ H^F_1, \ldots, H^F_m \}$ the corresponding multiset in $\text{MOCC} (\mathbf{F})$. We denote by $\text{rk} \mathcal{H} = \text{rk} \Gamma$ the sum of ranks of $H_1, \ldots, H_{\ell}$, by $\chi (\Gamma) = \chi (\mathcal{H}) \overset{\text{def}}{=} \# V (\Gamma) - \# E (\Gamma)$ the Euler characteristic of $\Gamma$, and by $c (\Gamma) = c (\mathcal{H})$ the number of connected components of $\Gamma$ (which is $\ell$ in the current notation). These three quantities are related by $\text{rk} \Gamma + \chi (\Gamma) = c (\Gamma)$. Note that as we excluded the trivial subgroup, $\chi (\Gamma) \leq 0$ for all $\Gamma \in \text{MuCG}_B (\mathbf{F})$.

We are now able to define another form of the function $\Phi$ as in (2.4), which depends on a morphism of multi core graphs:

**Definition 3.4.** Let $\eta : \Gamma \to \Delta$ be a morphism of multi core graphs. Assume that $\pi_1^{\text{lab}} (\Gamma) = \{ H^F_1, \ldots, H^F_m \}$ and $\pi_1^{\text{lab}} (\Delta) = \{ J^F_1, \ldots, J^F_m \}$. As above, let $f : [\ell] \to [m]$ correspond to $\eta$, and assume that $u_i H_i u_i^{-1} \leq J_{f(i)}$ for some $u_i \in \mathbf{F}$ for all $i \in [\ell]$ be the embedding corresponding to $\eta$. Let $\{ \varphi_k : J_k \to S_N \}_{k=1}^m$ be independent, uniformly random homomorphisms. Define

$$\Phi_\eta (N) \overset{\text{def}}{=} \mathbb{E}_{\{ \varphi_k \in \text{Hom}(J_k, S_N) \}} \left[ \# \text{fix} \left( \varphi_{f(1)} (u_1 H_1 u_1^{-1}) \right) \cdot \ldots \cdot \# \text{fix} \left( \varphi_{f(\ell)} (u_\ell H_{\ell} u_\ell^{-1}) \right) \right].$$

**Example 3.5.** For $\text{id} : \Gamma \to \Gamma$, we have $\Phi_\text{id} (N) = N^{\chi (\Gamma)}$. Indeed, the value of $\Phi$ is multiplicative with respect to the different connected components of the codomain. In a component $\Gamma_0$ of rank $k$, the probability that $k$ independent permutations fix some $i \in [N]$ is $\frac{1}{N^k}$, so the expected number of common fixed points is $N^{1-k} = N^{\chi (\Gamma_0)}$.

**Example 3.6.** To illustrate, we present the geometric picture, in terms of multi core graphs, of the object of study this paper began with $- \mathbb{E}_w [\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}]$. Here there is a multiset $\mathcal{H} \in \text{MOCC} (\mathbf{F})$ of size $\alpha_1 + \ldots + \alpha_k$, with $\alpha_1$ copies of $\langle w \rangle^F$, $\alpha_2$ copies of $\langle w^2 \rangle^F$, and so on. The corresponding multi core graph is denoted $\Gamma^w_{\alpha_1, \ldots, \alpha_k} \overset{\text{def}}{=} \Gamma_B (\mathcal{H})$. It consists of $\alpha_1$ disjoint copies of a cycle of length $|w|_c$ depicting $w$, together with $\alpha_2$ disjoint copies of a cycle of length $|w^2|_c$ depicting $w^2$, and so on, where $|w|_c$ denotes the length of the cyclic reduction of $w$. The second multiset $J \in \text{MOCC} (\mathbf{F})$ is the singleton $\{ \mathbf{F}^F \} = \{ \{ \mathbf{F} \} \}$, corresponding to the (multi) core graph $X_B$. There is only one possible morphism between the two - the

\textsuperscript{7}This equivalence is the generalization of the fact that if $H \leq J$ then so does $jHj^{-1} \leq J$ for all $j \in J$ and the corresponding morphism of core graphs is identical. We do not elaborate further because we anyway use here only the “geometric” description of this equivalence, which is more straightforward.
Figure 3.1: Let $\mathbb{F}_2$ have basis $B = \{ x, y \}$, and let $w = xyyxy^{-2} \in \mathbb{F}_2$. The multi core graph in the top part of the figure is $\Gamma = \Gamma_B(\mathcal{H})$ where $\mathcal{H} = \{ \langle w \rangle^{\mathbb{F}_2}, \langle w^2 \rangle^{\mathbb{F}_2}, \langle w^3 \rangle^{\mathbb{F}_2} \}$. It is denoted $\Gamma_{2,1,1}^w$ in the notation from Example 3.6. The bottom part shows the bouquet $X_B$. There is a single morphism of multi core graphs between these two, and we denote it by $\eta_{2,1,1}^w$. We have $\Phi_{\eta_{2,1,1}^w} = E_w [\xi_1^2 \xi_2 \xi_3]$, where $\Phi_{\eta_{2,1,1}^w}$ is defined in Definition 3.4.

As explained in [PP15, Section 6] for the simpler case analyzed there, $\Phi_{\eta}(N)$ can also be given the following completely geometric interpretation. Let $\hat{\Delta}_N$ be a random $N$-sheeted covering space of $\Delta$, defined as follows. Its vertex-set is $V(\Delta) \times [N]$. For every directed edge $e = (u, v) \in E(\Delta)$, choose uniformly at random a permutation $\sigma_e \in S_N$, and introduce in $\hat{\Delta}_N$ a directed edge $(u, i) \rightarrow (v, \sigma_e(i))$ with the same label as $e$ for every $i \in [N]$. This is indeed an $N$-sheeted covering of $\Delta$ with the projection $(u, i) \mapsto u$ and $((u, i), (v, \sigma_e(i))) \mapsto e$.

**Proposition 3.7.** Let $\eta: \Gamma \rightarrow \Delta$ be a morphism of multi core graphs. Then $\Phi_{\eta}(N)$ is equal to the average number of lifts of $\eta$ to the random $N$-covering $\hat{\Delta}_N$.

\[
\Phi_{\eta_{\alpha_1,\ldots,\alpha_k}^w} = E_w [\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}].
\]

**Proof.** By multiplicativity of $\Phi_{\eta}$, it suffices to prove the claim assuming that $\Delta$ is connected. Then the proposition practically reduces to [PP15, Lemma 6.2].
4 Free and algebraic morphisms

A subgroup $H$ of a free group $F$ is called a free factor of $F$, and $F$ a free extension of $H$, denoted $H \leq F$, if it is generated by some subset of a basis of $F$. Equivalently, this means that there is another subgroup $K \leq F$, such that $F = H \ast K$. The useful notion of an algebraic extensions of free groups is defined as follows (see [MVW07] for a survey):

**Definition 4.1.** Let $H$ be a subgroup of the free group $F$. Then $F$ is an algebraic extension of $H$, denoted $H \leq_{alg} F$, if there is no intermediate proper free factor of $F$. Namely, if whenever $H \leq J \leq F$, we have $J = F$.

Given a morphism of connected core graphs, we may say it is free (algebraic) if the induced map in the level of fundamental groups gives a free (algebraic, respectively) extension of groups. A crucial ingredient of our argument is to find the right generalizations of these notions to morphisms of multi core graphs. We start with free morphisms.

**4.1 Free morphisms**

**Definition 4.2.** If $H_1, \ldots, H_\ell$ are subgroups of the free group $J$, we say that $J$ is a free extension of the multiset $\{H_1, \ldots, H_\ell\}$, denoted $\{H_1, \ldots, H_\ell\} \leq J$, if $J$ decomposes as a free product

$$J = \left( *_{i=1}^\ell j_iH_i j_i^{-1} \right) \ast K$$

for some conjugate subgroup $j_iH_i j_i^{-1}$ of $H_i$ (so $j_i \in J$) and some subgroup $K \leq J$.

Now let $\eta : \Gamma \rightarrow \Delta$ be a morphism of multi core graphs with $\Delta$ connected. As explained in Section 3.2, one can pick an arbitrary subgroup $J$ in the single conjugacy class in $\pi_1^{lab}(\Delta)$, and for every component $\Gamma_1, \ldots, \Gamma_\ell$ of $\Gamma$, a suitable subgroup $H_i$ so that $H_i \leq J$. Say that $\eta$ is a free morphism if $\{H_1, \ldots, H_\ell\} \leq J$.

Finally, say that a general morphism $\eta : \Gamma \rightarrow \Delta$ of multi core graphs is free if $\eta|_{\eta^{-1}(\Delta')} : \eta^{-1}(\Delta') \rightarrow \Delta'$ is free for every connected component $\Delta'$ of $\Delta$.

The definition of a free morphism does not depend on any of the choices made: not on the choice of $J$ and not on the choice of the $H_i$'s. The following theorem states some properties of free morphisms. In particular, it shows that the set of multi core graphs together with free morphisms form a valid category. By an injective morphism we mean a morphism which is both edge-injective and vertex-injective.

**Proposition 4.3.** 1. Every injective morphism of multi core graphs is free. In particular, the identity morphism is free.

2. The composition of two free morphisms is free.

3. If $\eta : \Gamma \rightarrow \Delta$ is a free morphism of multi core graphs, then $\chi(\Delta) \leq \chi(\Gamma)$, with equality if and only if (i) $\eta$ induces an isomorphism between $\Gamma$ and the connected components of $\Delta$ meeting $\text{Im}(\eta)$, and (ii) the remaining connected components of $\Delta$ are cycles.

4. If $\xymatrix{ \bullet \ar@<1ex>[r]^{\eta} & \bullet \ar@<1ex>[l]^{\psi} \ar@<1ex>[r]^{\varphi} & \bullet }$ is a composition of morphisms with $\psi$ free, then $\Phi_{\varphi} = \Phi_{\eta}$.

**Proof.** Item 1 is a generalization of the fact that if $(\Gamma_1, v_1) \hookrightarrow (\Gamma_2, v_2)$ is an embedding of connected, pointed graphs, then $\pi_1(\Gamma_1, v_1) \leq \pi_1(\Gamma_2, v_2)$ – the proof in the case that several connected components in the domain are mapped to a single component in the codomain is straightforward (the simple idea of the proof also appears in the proof of Lemma 4.4 below). Item 2 is a straightforward generalization of the
transitivity of free extensions in free groups. Item 3 follows from the fact that if \( \{ H_1, \ldots, H_\ell \} \leq J \), then 
\[ \sum_{i=1}^\ell \text{rk} H_i \leq \text{rk} J, \]
and so
\[ \chi (\Gamma_B (\{ H_1^F, \ldots, H_\ell^F \})) = \ell - \sum_{i=1}^\ell \text{rk} H_i \geq 1 - \text{rk} J = \chi (\Gamma_B (J^F)), \]
with equality if and only if \( \ell = 1 \) and \( H_1 = J \), or \( \ell = 0 \) and \( \text{rk} J = 1 \). Finally, item 4 is a straightforward generalization of the corresponding claim for connected core graphs – see, e.g., [PP15, Remark 5.2].

We end this subsection with two lemmas concerning free morphisms that we need in Section 4.2. Lemma 4.4 generalizes the fact that injective morphisms are free. Lemma 4.5 generalizes the fact that if \( H \leq F \) and \( K \leq F \) are all free groups, then \( H \cap K \leq H \) (e.g., [PP15, Claim 3.9]).

**Lemma 4.4.** Let \( \Gamma \) be a multi core graph. Let \( \mathcal{P} \) be a partition of a subset of the edge-set of \( \Gamma \). For every block \( \beta \in \mathcal{P} \), consider the multi core graph \( \Sigma_\beta \) obtained by deleting from \( \Gamma \) the edges outside \( \beta \), and then recursively pruning all leaves and deleting isolated vertices. Let \( \Sigma = \bigcup_{\beta \in \mathcal{P}} \Sigma_\beta \) be the multi core graph obtained as the disjoint union of the \( \Sigma_\beta \)'s. The embeddings \( \Sigma_\beta \hookrightarrow \Gamma \) give rise to a morphism \( \eta: \Sigma \to \Gamma \). Then \( \eta \) is free.

**Proof.** Because freeness of morphisms is tested in every connected component of the codomain separately, we may assume \( \Gamma \) is connected. Fix a basepoint \( \odot \) and a spanning tree \( T \) in \( \Gamma \). Let \( J = \pi_1^{\text{lab}} (\Gamma, \odot) \). An arbitrary orientation of the edges of \( \Gamma \) outside \( T \) standardly gives rise to a basis of \( J \). We shall construct a similar basis which shows the freeness of \( \eta \).

Let \( \Lambda \) be an arbitrary connected component of \( \Sigma \), embedded in \( \Gamma \). Note that \( T \cap \Lambda \) is a forest inside \( \Lambda \), which can be extended to a spanning tree \( T_\Lambda \) of \( \Lambda \). Fix \( T_\Lambda \) for every \( \Lambda \). For every edge \( e \in (\bigcup_\Lambda T_\Lambda) \setminus T \), orient \( e \) arbitrarily, and let \( j_e = u_1 e u_2 \in J \), where \( u_1 \in F \) corresponds to the path through \( T \) from \( \odot \) to the tail of \( e \) and \( u_2 \in F \) to the path through \( T \) from the head of \( e \) to \( \odot \). Let \( C_\Lambda \) be the set of all such elements of \( J \) obtained from all edges in \( (\bigcup_\Lambda T_\Lambda) \setminus T \).

For all \( \Lambda \), let \( \odot_\Lambda \) be a fixed basepoint of \( \Lambda \) and \( u_\Lambda \in F \) be the path from \( \odot \) to \( \odot_\Lambda \) through \( T \). Construct a basis \( C_\Lambda' \) for \( H_\Lambda \) using \( T_\Lambda \), and note that \( C_\Lambda \) is a basis for \( u_\Lambda H_\Lambda u_\Lambda^{-1} \), which is a subgroup of \( J \). Now \( C_\Lambda \cup \bigcup_\Lambda C_\Lambda \) is a basis of \( J \) which shows that indeed
\[ \{ H_\Lambda \}_{\Lambda \leq J}, \]

namely, \( \eta \) is free.

There is a natural notion of pullback in the equivalent categories \( \mathcal{MOCC} (F) \) and \( \mathcal{MOCGB} (F) \). It is very similar to the well-established notion of pullback in the case of connected core graphs (e.g., [Sta83, Sections 1.3 and 5.5]). If \( \eta_1: \Gamma_1 \to \Delta \) and \( \eta_2: \Gamma_2 \to \Delta \) are morphisms, the pullback is the multi core graph \( \Sigma \) defined as follows: begin with the graph \( \Sigma' \) with vertex-set
\[ \{(u_1, u_2) \mid u_i \text{ a vertex of } \Gamma_i, \ \eta_1 (u_1) = \eta_2 (u_2)\}, \]
and edge-set defined analogously, where the edge \( (e_1, e_2) \) begins at the pair of tails and ends at the pair of heads. Then, recursively remove all leaves and isolated vertices to obtain \( \Sigma \). There are natural morphisms \( \sigma_i: \Sigma \to \Gamma_i, i = 1, 2 \), defined as the projection to the \( i \)-th coordinate. This pullback satisfies the universal property of pullbacks: for every pair of morphisms \( \gamma_1: \Lambda \to \Gamma_1, \gamma_2: \Lambda \to \Gamma_2 \) such that \( \eta_1 \circ \gamma_1 = \eta_2 \circ \gamma_2 \), there is a unique \( \overline{\gamma}: \Lambda \to \Sigma \) such that the following diagram commutes:
In algebraic terms, the connected component of \((u_1, u_2)\) in the pullback \(\Sigma\) corresponds to the conjugacy class in \(\pi_1^{\text{lab}}(\Delta, \eta_1(u_1))\) of \(\pi_1^{\text{lab}}(\Gamma_1, u_1) \cap \pi_1^{\text{lab}}(\Gamma_2, u_2)\). In total, for every connected components \(G_1\) of \(\Gamma_1\) and \(G_2\) of \(\Gamma_2\) which are mapped to same component \(D\) of \(\Delta\), let \(J\) be a representative of the conjugacy class \(\pi_1^{\text{lab}}(D)\), and let \(H_i\) be a representative of \(\pi_1^{\text{lab}}(G_i)\) such that \(H_1, H_2 \leq J\) agree with the morphisms \(\eta_1, \eta_2\). Then there is one connected component in the pullback \(\Sigma\) for every non-trivial conjugacy class of subgroups of \(J\) in the set \(\{H_1 \cap jH_2j^{-1}\}_{j \in J}\). Most importantly, the pullback construction can be completely defined in the category \(\mathcal{MOCC}(F)\), and is thus basis-independent.\(^8\)

**Lemma 4.5.** In the above notation, if \(\eta_1: \Gamma_1 \rightarrow \Delta\) and \(\eta_2: \Gamma_2 \rightarrow \Delta\) are morphisms with \(\eta_2\) free and \((\Sigma, \sigma_1, \sigma_2)\) is the pullback, then \(\sigma_1: \Sigma \rightarrow \Gamma_1\) is also free.

\[
\begin{array}{ccc}
\Sigma & \xrightarrow{\sigma_1} & \Gamma_1 \\
\sigma_2 & \downarrow & \downarrow \eta_1 \\
\Gamma_2 & \xrightarrow{\sigma_2} & \Delta \\
\end{array}
\] (4.1)

**Proof.** Note that in the diagram (4.1), there is no interaction between components of \(\Gamma_1, \Gamma_2\) and \(\Sigma\) which are mapped to different components of \(\Delta\), and recall that freeness is tested independently in every connected component of the codomain. Thus, we may assume that \(\Delta\) is connected. Because the pullback can be constructed in pure algebraic terms, we may assume \(J = \pi_1^{\text{lab}}(\Delta, v)\) is the ambient free group (\(v\) is some arbitrary vertex in \(\Delta\)), and pick a basis \(Q\) of \(J\) which extends a basis for the components of \(\Gamma_2\). Namely, every component of \(\Gamma_2\) corresponds to the conjugacy class of the subgroup generated by some subset of \(Q\), and the different subsets are disjoint. The geometric picture in this basis is that \(\Delta\) and every component of \(\Gamma_2\) are bouquets (a single vertex with several loops).

But now, for every component of \(\Gamma_2\), let \(\beta \subseteq Q\) be the corresponding subset of basis elements. This gives rise to a partition of a subset of the edge-set of \(\Gamma_1\), with one block consisting of all edges colored by elements from \(\beta\), for every component of \(\Gamma_2\). It is easy to see that the pullback \(\Sigma\) is identical to the construction described in Lemma 4.4 with respect to this partition of the edges of \(\Gamma_2\), and thus \(\sigma_1\) is free. \(\square\)

### 4.2 Algebraic morphisms

We turn to defining our generalization of the notion of algebraic extensions.

**Definition 4.6.** Let \(\eta: \Gamma \rightarrow \Delta\) be a morphism of multi core graphs. We say that \(\eta\) is algebraic if whenever \(\Gamma \xrightarrow{\eta_1} \Sigma \xrightarrow{\eta_2} \Delta\) is a decomposition of \(\eta\) with \(\eta_2\) free, we have that \(\eta_2\) is an isomorphism.

Because the definition of a free morphism is basis-independent, so is the definition of an algebraic morphism. The following theorem lists some important properties of algebraic morphisms. In particular, it shows that the set of multi core graphs together with algebraic morphisms form a valid category.

---

\(^8\)Note that it is very much possible that the pullback \(\Sigma\) be empty, and recall that the empty multiset is an element of \(\mathcal{MOCC}(F)\).
Theorem 4.7. 1. Every algebraic morphism of multi core graphs is surjective.

2. The identity morphism is algebraic.

3. The composition of two algebraic morphisms is algebraic.

Proof. Every \( \eta : \Gamma \to \Delta \) decomposes as \( \Gamma \xrightarrow{\eta_1} \Sigma \xrightarrow{\eta_2} \Delta \), where \( \Sigma \) is the image of \( \eta \) and \( \eta_2 \) is its embedding in \( \Delta \). Note that \( \Sigma \) may contain multiple components which are embedded in the same component of \( \Delta \). By Proposition 4.3(1), \( \eta_2 \) is free. Thus \( \eta \) cannot be algebraic unless \( \eta_2 \) is an isomorphism, namely, \( \eta \) is surjective. This shows item 1.

For item 2, note that any decomposition of an identity morphism \( \text{id} : \Gamma \to \Gamma \) is through an injective and surjective morphisms: \( \Gamma \xrightarrow{\eta_1} \Sigma \xrightarrow{\eta_2} \Gamma \). Then \( \eta_1 \) is free by Proposition 4.3(1). If we assume that \( \eta_2 \) is also free, we obtain, by Proposition 4.3(3), that \( \chi(\Sigma) = \chi(\Gamma) \), and that both \( \eta_1 \) and \( \eta_2 \) are isomorphisms.

Finally, assume that \( \Lambda \xrightarrow{\eta_1} \Sigma \xrightarrow{\eta_2} \Gamma \xrightarrow{\eta_3} \Delta \) is a chain of two algebraic morphisms. Assume that there is a decomposition of \( \eta_2 \circ \eta_1 \) as \( \Lambda \xrightarrow{\gamma_1} \Gamma \xrightarrow{\gamma_2} \Delta \), with \( \gamma_2 \) free. Let \( \Sigma \) be the pullback of \( \eta_2 \) and \( \gamma_2 \) and \( \gamma : \Lambda \to \Sigma \) the unique morphism so that Diagram (4.2) commutes. By Lemma 4.5, \( \sigma_1 \) is free. In the notation of Diagram (4.2), we obtain that \( \Lambda \xrightarrow{\gamma_1} \Sigma \xrightarrow{\sigma_1} \Gamma \xrightarrow{\eta_2} \Delta \) is a decomposition of the algebraic \( \eta_1 \), hence \( \sigma_1 \) is an isomorphism.

Remark 4.8. It is easy to come up with surjective morphisms in \( \text{MuCG}_B(\mathbf{F}) \) that are not algebraic: consider, for instance, the morphism from \( \bullet \xrightarrow{x} \bullet \) to \( \bullet \xrightarrow{x} \circ \bullet \xrightarrow{y} \). Theorem 4.7(1) says that if \( \eta : \mathcal{H} \to \mathcal{J} \) is an algebraic morphism in \( \text{MOCC}(\mathbf{F}) \), then it is surjective in \( \text{MuCG}_B(\mathbf{F}) \) for any basis \( B \) of \( \mathbf{F} \). It is a subtle matter to understand if this has some converse – see [MVW07, PP14, Kol21, VM21].

4.3 The algebraic-free decomposition of morphisms

Theorem 4.9. Let \( \eta : \Gamma \to \Delta \) be a morphism of multi core graphs. Then there is a decomposition

\[
\eta = \varphi \circ \psi \text{ such that } \varphi \text{ is algebraic and } \psi \text{ is a free. This decomposition is unique in the sense that if } \Gamma \xrightarrow{\varphi'} \Sigma' \xrightarrow{\psi'} \Delta \text{ is another such decomposition of } \eta, \text{ then there is an isomorphism of } \Sigma \text{ and } \Sigma' \text{ which commutes with the two decompositions of } \eta.
\]

Moreover, this decomposition is universal in the following sense: for every other decomposition \( \Gamma \xrightarrow{\varphi'} \Sigma' \xrightarrow{\psi'} \Delta \) of \( \eta \),

1. if \( \varphi' \) is algebraic, \( \varphi \) factors through \( \varphi' \) (namely, \( \exists \theta \) with \( \varphi = \theta \circ \varphi' \)), and
2. If \( \psi' \) is free, \( \psi \) factors through \( \psi' \) (namely, \( \exists \theta' \) with \( \psi = \psi' \circ \theta' \)).

\[
\begin{tikzcd}
\Sigma' \ar[r, shift right=2pt, \text{alg}] \ar[dr, \text{alg}, bend left] \ar[dd, \exists \theta'] & \Delta \\
\Sigma \ar[r, \psi] \ar[dr, \exists \theta' \circ \psi', bend left] & \Delta' \ar[dl, \psi', bend left] \\
\Sigma'' \ar[rr, \psi', bend right] & &
\end{tikzcd}
\]

Note that by definitions, in the first case \( \theta \) must be algebraic, and in the second case \( \theta' \) must be free.

**Proof.** Assume we work in the category \( \mathcal{MuCG}_B(F) \). Consider all decompositions of \( \eta \) as \( \Gamma \xrightarrow{\varphi} \Sigma \xrightarrow{\psi} \Delta \) with \( \psi \) free and \( \varphi \) surjective (in the terminology of Section 5, \( \varphi \) is \( B \)-surjective). There is at least one such decomposition: the decomposition of \( \eta \) to a surjective and an injective morphisms. Euler characteristics of multi core graphs are non-positive, and so among such decompositions, we may pick one with \( \chi(\Sigma) \) maximal. We fix this triple of \( \Sigma, \varphi, \psi \). By Proposition 4.3(3), \( \varphi \) is algebraic. This proves the existence of the algebraic-free decomposition.

Assume that \( \Gamma \xrightarrow{\varphi'} \Sigma' \xrightarrow{\psi'} \Delta \) is another such decomposition. Let \( \Lambda \) be the pullback of \( \psi \) and \( \psi' \) as in the following diagram. By Lemma 4.5, \( \sigma \) and \( \sigma' \) are free morphisms. But \( \varphi \) and \( \varphi' \) are algebraic, and so \( \sigma \) and \( \sigma' \) must be isomorphisms. This proves the uniqueness of the algebraic-free decomposition.

Now let \( \Gamma \xrightarrow{\varphi'} \Sigma' \xrightarrow{\psi'} \Delta \) be another decomposition of \( \eta \) with \( \varphi' \) algebraic, but \( \psi' \) not necessarily free. Let again \( (\Lambda, \sigma, \sigma') \) be the pullback of \( \psi \) and \( \psi' \) as in Diagram (4.3). As \( \psi \) is free, so is \( \sigma' \), by Lemma 4.5. But \( \varphi' \) is algebraic and so \( \sigma' \) is an isomorphism, and \( \sigma \circ (\sigma')^{-1} \) is the required morphism \( \Sigma' \to \Sigma \).

Finally, if \( \Gamma \xrightarrow{\varphi'} \Sigma' \xrightarrow{\psi'} \Delta \) is another decomposition of \( \eta \) with \( \psi' \) free, but \( \varphi' \) not necessarily algebraic, then by Lemma 4.5 again, \( \sigma \) (and \( \sigma' \)) are free. Let \( \overline{\Lambda} \) be the image of \( \theta \) in \( \Lambda \) and \( \overline{\sigma} \) the restriction of \( \sigma \) to \( \overline{\Lambda} \). The embedding \( \overline{\Lambda} \hookrightarrow \Lambda \) is free by Proposition 4.3(1), and so \( \overline{\sigma} \), which is the composition of this embedding with the free morphism \( \sigma \), is free as well by 4.3(2). As \( \varphi \) is surjective, so is \( \overline{\sigma} \). By the maximality of \( \chi(\Sigma) \) and Proposition 4.3(3), \( \overline{\sigma} \) must be an isomorphism, and so \( \sigma' \circ \overline{\sigma}^{-1} \) is the sought-after morphism from \( \Sigma \) to \( \Sigma' \).

![Diagram](image)

5  **\( B \)-Surjective morphisms and norms of morphisms**

Theorem 4.7(1) already manifested the importance of surjective morphisms of core graphs. Yet, surjective morphisms play an even bigger role in this work, and in the current section we develop some concepts that will be useful in the following sections. Surjective morphisms of multi core graphs is the analogue of the partial order “\( B \)-cover” defined in [PP15, Definition 3.3]. There, core graphs are connected and have basepoints which must be mapped to basepoints by morphisms, and so there is at most one morphism between two given subgroups \( H, J \leq F \), which exists if and only if \( H \leq J \). Thus, one can define a
partial order on subgroups of $\mathcal{F}$, which holds whenever $H \leq J$ and the corresponding morphism between core graphs is surjective. In contrast, in the current paper, because multi core graphs are not necessarily connected and do not have basepoints, there may be several different morphisms between any two of them. Thus, being surjective is a property of a morphism, and not of a pair of multi core graphs. As illustrated in Remark 4.8, the property of being surjective depends on the basis $B$. If one considers elements in the category $\mathcal{MOCC}(\mathcal{F})$, one can say that a morphism $\eta: \mathcal{H} \to \mathcal{J}$ is $B$-surjective if the corresponding morphism in $\mathcal{MuCG}_B(\mathcal{F})$ is surjective.

Let $\eta: \Gamma \to \Delta$ be a surjective morphism of multi core graphs. Note that $\eta$ determines a partition $P$ of $V(\Gamma)$: two vertices $v_1,v_2$ are equivalent if and only if $\eta(v_1) = \eta(v_2)$. This partition uniquely determines $\Delta$ and $\eta$: the vertices of $\Delta$ correspond to the blocks of the partitions, and there is a $b$-edge from block $\beta_1$ to block $\beta_2$ if and only if there is a $b$-edge in $\Gamma$ from some vertex in $\beta_1$ to some vertex in $\beta_2$. It is clear how $\eta$ is defined.

However, not every partition of the vertex set of $\Gamma$ defines a morphism, as the graph resulting from the above procedure may have multiple edges of the same directed label incident to some vertex. Yet, given such a partition $P$ of $V(\Gamma)$, we may define the “generated” multi core graph $\Delta$ and surjective morphism $\eta : \Gamma \to \Delta$ via Stallings foldings, as follows. We start the procedure with the $B$-labeled directed graph formed as above by gluing together the vertices of $\Gamma$ according to the blocks of $P$ and drawing edges between the blocks as above. Given two edges of the same label and the same head, one may identify their tails (if different) and the two edges. Similarly, given two edges of the same label and same tail-vertex, one may identify their heads and the two edges. Applying these identifications iteratively, a finite number of times, yields a multi core graph $\Delta$, which is independent of the choices made throughout the folding process\(^9\) (see [Sta83]). This procedure yields a new partition of the vertex set of $\Gamma$, which is coarser than $P$. There is also only one reasonable way to define the morphism $\Gamma \to \Delta$, by mapping every vertex to the block in the new partition it belongs to, and every edge to the equally-labeled and equally-directed edge between the corresponding blocks.

**Definition 5.1.** Let $P$ be a partition of a finite set $X$. We define the norm of the partition as

$$||P|| \overset{\text{def}}{=} \sum_{\beta \in P} (|\beta| - 1).$$

This is the minimal number of identifications of pairs of elements of $X$ required in order to generate the partition.

Given a $B$-surjective morphism of multi core graphs $\eta: \Gamma \to \Delta$, we define its $B$-norm, denoted $||\eta||_B$, to be the smallest norm of a partition generating it:

$$||\eta||_B \overset{\text{def}}{=} \min \{ ||P|| \mid P \text{ is a partition of } V(\Gamma) \text{ generating } \eta \}.$$ \hspace{1cm} (5.1)

One can also think of the $B$-norm as follows. Define a merging-step of a multi core graph to be the gluing of two vertices of this graph followed by folding. Then, $||\eta||_B$ is equal to the smallest number of merging-steps which lead from $\Gamma$ to $\Delta$ to create $\eta$.

We use here the notation $\|\cdot\|_B$ so that we can use this notation also when $\eta$ is a ($B$-surjective) morphism of two multisets of conjugacy classes of subgroups, in which case the basis is not pre-determined. Because folding steps cannot decrease the Euler characteristic of a graph and the gluing of two vertices decreases the Euler characteristic by one, we obtain

$$||\eta||_B \geq \chi(\Gamma) - \chi(\Delta).$$ \hspace{1cm} (5.2)

There is also an “algebraic”, basis-independent version of a norm of morphisms in the categories $\mathcal{MOCC}(\mathcal{F})$ and $\mathcal{MuCG}_B(\mathcal{F})$. We describe it gradually in the following lines.

\(^9\)In our situation we never introduce leaves nor isolated vertices in the process.
**Definition 5.2.** Given a multiset $\mathcal{H} = \{ H_1^F, \ldots, H_\ell^F \}$ of conjugacy classes of subgroups of the free group $F$ and a subgroup $J \leq F$ such that $H_i \leq J$ for all $i = 1, \ldots, \ell$, let $\eta: \mathcal{H} \to \{ J^F \}$ be the corresponding morphism in the category $\text{MOCC}(F)$. Consider the two following types of morphisms which we call immediate morphisms:

1. Adding a generator from $J$ to one of the classes, namely, let $\mathcal{H}'$ be identical to $\mathcal{H}$ except for that some $H_i^F$ is replaced by $\langle H_i, j \rangle^F$ for some $j \in J$. The corresponding morphism $\varphi: \mathcal{H} \to \mathcal{H}'$ maps $H_i$ to $\langle H_i, j \rangle$ and every other $H_k$ to itself.

2. Merging together two of the classes, namely, let $\mathcal{H}'$ be identical to $\mathcal{H}$ except for that for some $i \neq k$, $H_i^F$ and $H_k^F$ are replaced by $\langle jH_i^{-1}, H_k \rangle^F$ for some $j \in J$. The corresponding morphism $\varphi: \mathcal{H} \to \mathcal{H}'$ maps $jH_i^{-1}$ and $H_k$ to $\langle jH_i^{-1}, H_k \rangle$ and every other $H_l$ to itself.

Note that in both cases $\eta$ factors through $\varphi$ by a unique morphism $\eta': \mathcal{H}' \to J$. Define the norm of $\eta$, denoted $\|\eta\|$, to be the smallest number of immediate morphisms whose compositions gives $\eta$. If $\mathcal{H} = \emptyset$ is the empty multiset, we set $\|\emptyset\| = -\chi(J) = \text{rk} J - 1$.

Now let $\eta: \Gamma \to \Delta$ be a morphism of $B$-labeled multi core graphs with $\Delta$ connected. Define $\|\eta\|$ to be the norm of the corresponding morphism $\pi_1^{\text{lab}}(\Gamma) \to \pi_1^{\text{lab}}(\Delta)$. Finally, for the general case where $\Delta$ is not necessarily connected, let $\Delta_1, \ldots, \Delta_m$ be the connected components of $\Delta$, and for $k \in [m]$, let $\eta_k$ denote the morphism $\eta^{-1}(\Delta_k) \to \Delta_k$ obtained by restricting $\eta$ to $\eta^{-1}(\Delta_k)$. Define $\|\eta\| = \sum_{k=1}^m \|\eta_k\|$.

Note that $\|\eta\|$ is a non-negative integer, but may be zero even if $\eta$ is not an isomorphism. Indeed, $\|\eta\| = 0$ whenever $\eta: \Gamma \to \Delta$ induces an isomorphism between $\Gamma$ and the connected components of $\Delta$ meeting the image of $\eta$ and the remaining connected components of $\Delta$ correspond to cyclic subgroups. One can give an equivalent definition for free morphisms using the norm:

**Lemma 5.3.** Let $\eta: \Gamma \to \Delta$ be a morphism of multi core graphs. Then $\|\eta\| \geq \chi(\Gamma) - \chi(\Delta)$, with equality if and only if $\eta$ is free.

**Proof.** Any component $D$ of $\Delta$ not meeting $\eta(\Gamma)$ is trivially a free extension of its preimage, and contributes $-\chi(D)$ to $\|\eta\|$. Thus we may assume every component of $\Delta$ meets $\eta(\Gamma)$. Now, the two types of identifications from Definition 5.2 cannot decrease the Euler characteristic of the element of $\text{MOCC}(F)$ by more than one. Hence $\|\eta\| \geq \chi(\Gamma) - \chi(\Delta)$ with equality if and only if there is a set of identifications each decreasing the Euler characteristic by exactly one. The first identification decreases the Euler characteristic by one if and only if $\langle H_i, j \rangle = H_i \ast \langle j \rangle$. The second identification decreases the Euler characteristic by one if and only if $\langle jH_i^{-1}, H_k \rangle = jH_i^{-1} \ast H_k$. In both cases, the corresponding morphism describing the step is free. By transitivity of free morphisms, the claim follows.

It is easy to see that a combinatorial merging-step as above where we glue together two vertices from the same component, corresponds to a step of the first kind from Definition 5.2, and two vertices from different components to a step of the second kind. Therefore, if $\eta: \Gamma \to \Delta$ is a surjective morphism in $\text{MuCG}_B(F)$ then

$$\|\eta\| \leq \|\eta\|_B. \quad (5.3)$$

An extension of the arguments from [Pud14, Section 3] leads to the following stronger statement:

**Theorem 5.4.** Let $\eta: \Gamma \to \Delta$ be a morphism of $B$-labeled multi core graphs. Let $\Sigma = \text{Image}(\eta)$ denote the image of $\eta$ in $\Delta$, and let $\Gamma \xrightarrow{\eta} \Sigma \xrightarrow{\iota} \Delta$ be the decomposition of $\eta$ to a surjective and an injective morphisms. Then

$$\|\eta\| = \|\eta\|_B + [\chi(\Sigma) - \chi(\Delta)].$$

In particular, if $\eta$ is $B$-surjective, then

$$\|\eta\| = \|\eta\|_B.$$
Theorem 5.4 has some nice corollaries we mention next. However, the theorem and its corollaries are not needed for proving our main results from Section 1, and so we defer its proof to Appendix C. The first corollary is immediate from Theorem 5.4 together with Lemma 5.3 and Proposition 4.3(1).

**Corollary 5.5.** Let \( \eta : \Gamma \to \Delta \) be a morphism of multi core graphs. Then, in the notation of Theorem 5.4,

\[
\eta \text{ is free } \iff \|\eta\| = \chi(\Gamma) - \chi(\Delta) \iff \|\eta\|_B = \chi(\Gamma) - \chi(\Sigma) \iff \eta_1 \text{ is free.}
\]

The second corollary concerns computability:

**Corollary 5.6.** Given a morphism \( \eta : \Gamma \to \Delta \) of multi core graphs, there is an algorithm to compute its norm \( \|\eta\| \), and to determine whether it is free and whether it is algebraic.

**Proof.** By Theorem 5.4, it is enough to compute \( \|\eta\|_B \) to obtain \( \|\eta\| \), and the \( B \)-norm of a morphism is obviously computable. By Lemma 5.3, it is straightforward to determine whether \( \eta \) is free given \( \|\eta\| \).

Finally, if \( \Gamma \xrightarrow{\eta_1} \Sigma \xrightarrow{\eta_2} \Delta \) is a decomposition of \( \eta \) with \( \eta_2 \) free, we may assume without loss of generality that \( \eta_1 \) is surjective (otherwise, replace \( \eta_1(\Gamma) \to \Sigma \) is free as well). Because there are only finitely many surjective morphisms from \( \Gamma \), let alone decompositions \( \Gamma \xrightarrow{\eta_1} \Sigma \xrightarrow{\eta_2} \Delta \) of \( \eta \) with \( \eta_1 \) surjective, we may go through all of them (such that \( \eta_2 \) is not an isomorphism) and test whether \( \eta_2 \) is free. Then \( \eta \) is algebraic if and only if there is no such decomposition with \( \eta_1 \) surjective and \( \eta_2 \) free. \( \square \)

We end this section with an upper bound on the \( B \)-norm of morphisms, to be used in Section 8.

**Proposition 5.7.** The \( B \)-norm of a \( B \)-surjective morphism \( \eta : \Gamma \to \Delta \) satisfies

\[
\|\eta\|_B \leq [c(\Gamma) - c(\Delta)] + \text{rk}(\Delta) = c(\Gamma) - \chi(\Delta).
\]

**Proof.** Clearly, the \( B \)-norm is additive if we consider the different components of \( \Delta \), and so is the bound we give. So it is enough to prove that when \( \Delta \) is connected, \( \|\eta\|_B \leq c(\Gamma) - 1 + \text{rk}(\Delta) \). We prove this by induction on \( c(\Gamma) \). If \( c(\Gamma) = 1 \), we are in the situation of [Pud14, Lemma 3.3] which says that \( \|\eta\|_B \leq \text{rk}(\Delta) \). If \( c = c(\Gamma) \geq 2 \), then because \( \eta \) is onto, there must be a vertex in \( \Delta \) which is in the image of at least two different components of \( \Gamma \). Merge such two vertices in two different components of \( \Gamma \) to obtain a graph \( \Gamma' \) with \( c - 1 \) components and \( \eta' : \Gamma' \to \Delta \). By induction,

\[
\|\eta\| \leq 1 + \|\eta'\| \leq 1 + (c - 1 - 1) + \text{rk}(\Delta) = c - 1 + \text{rk}(\Delta).
\]

\( \square \)

## 6 Möbius inversions and the leading terms of \( \Phi \)

Recall Definition 3.4 of \( \Phi_\eta(N) \), and that our goal is to estimate \( \Phi_\eta(N) \) for certain morphisms of multi core graphs as in Example 3.6. The main result of this section is Theorem 6.2.

**Definition 6.1.** Let \( \eta : \Gamma \to \Delta \) be a morphism of multi core graphs. Denote

\[
\chi^\text{max}(\eta) \overset{\text{def}}{=} \max \left\{ \chi(\Sigma) \bigg| \begin{array}{l} \Gamma \xrightarrow{\eta_1} \Sigma \xrightarrow{\eta_2} \Delta \text{ is a decomposition of } \eta \\ \text{with } \eta_1 \text{ algebraic and non-isomorphism} \end{array} \right\}. \quad (6.1)
\]

Every decomposition as in (6.1) and with \( \chi(\Sigma) = \chi^\text{max}(\eta) \) maximal, is called critical. Let \( \text{Crit}(\eta) \) denote the set of critical decompositions of \( \eta \) up to equivalence as in Theorem 4.9 (or as in Definition 6.4 below).

If \( \Gamma \) is connected, \( H \leq F \) a representative of the conjugacy class \( \pi^{\text{lab}}(\Gamma) \), and \( \eta : \Gamma \to X_B \), then \( \chi^\text{max}(\eta) \) is equal to \( 1 - \pi(H) \), where \( \pi(H) \) is the primitivity rank of \( H \) ([PP15, Definition 1.7]). Because algebraic morphisms are surjective (Theorem 4.7(1)) and there are finitely many surjective morphisms with domain \( \Gamma \), \( \text{Crit}(\eta) \) is always a finite set.
Theorem 6.2. Let \( \eta : \Gamma \rightarrow \Delta \) be a morphism of multi core graphs. Then

\[
\Phi_\eta (N) = N^{\chi(\Gamma)} + |\text{Crit}(\eta)| \cdot N^{\max(\eta)} + O \left( N^{\max(\eta)-1} \right).
\]

Theorem 6.2 is proven at the end of this Section 6. When \( \Gamma \) and \( \Delta \) are connected, Theorem 6.2 reduces to [PP15, Theorem 1.8] which is written in purely algebraic terms. As in the argument in [PP15], the remaining ingredient of the proof is the definition of certain Möbius inversions of the function \( \Phi \) and the study of their properties. The current section is devoted to an extension of the arguments of [PP15] to multi core graphs. In Section 6.1, we study Möbius inversions based on decompositions of \( B \)-surjective morphisms. While an extension of the Möbius inversions in [PP15], this is a bit unusual as Möbius inversions are usually defined in terms of posets (partially ordered sets) – and see Remark 6.7. In Section 6.2 we introduce Möbius inversions in decompositions in the category of algebraic morphisms, which has no analogue in [PP15].

Remark 6.3. Note that the number \( \chi^{\max}(\eta) \) and the set \( \text{Crit}(\eta) \) from Definition 6.1 are algorithmically computable. Indeed, every algebraic morphism is surjective (Proposition 6.15(1)) and so it is enough to go over the finitely many decompositions \( \Gamma \xrightarrow{\eta_1} \Sigma \xrightarrow{\eta_2} \Delta \) of \( \eta \) with \( \eta_1 \) surjective. By Corollary 5.6, it is possible to determine whether \( \eta_1 \) is algebraic. Given the finite set of such decompositions with \( \eta_1 \) algebraic, it is straightforward to compute \( \chi^{\max}(\eta) \) and \( \text{Crit}(\eta) \). See also Remark 7.9 for some more information about algebraic morphisms from \( \Gamma_{\alpha_1,\ldots,\alpha_k}^w \).

6.1 Basis dependent Möbius inversions

Definition 6.4. Let \( \eta : \Gamma \rightarrow \Delta \) be a \( B \)-surjective morphism. Denote by \( \text{Decomp}_B(\eta) \) the set of decompositions of \( \eta \) into two surjective morphisms \( \Gamma \xrightarrow{\eta_1} \Sigma \xrightarrow{\eta_2} \Delta \), where the latter decomposition is considered identical to \( \Gamma \xrightarrow{\eta'_1} \Sigma' \xrightarrow{\eta'_2} \Delta \) if there is an isomorphism \( \Sigma \cong \Sigma' \) which commutes with both decompositions.

Similarly, let \( \text{Decomp}^3_B(\eta) \) denote the set of decompositions \( \Gamma \xrightarrow{m_1} \Sigma_1 \xrightarrow{n_2} \Sigma_2 \xrightarrow{n_3} \Delta \) of \( \eta \) into three surjective morphisms. Again, two such decompositions are considered equivalent (and therefore the same element in \( \text{Decomp}^3_B(\eta) \)) if there are isomorphisms \( \Sigma_i \cong \Sigma'_i \), \( i = 1, 2 \), which commute with the decompositions.

Note that \( \text{Decomp}_B(\eta) \) and \( \text{Decomp}^3_B(\eta) \) are finite sets, as the multi core graph \( \Gamma \) is finite. In another point of view, \( \text{Decomp}_B(\eta) \) can be thought of as the set of all partitions of \( V(\Gamma) \), the vertex-set of \( \Gamma \), which give rise (without folding) to valid multi core graphs, and which are finer than (or equal to) the partition induced by \( \eta \). We remark that two distinct decompositions of \( \eta \) may have isomorphic \( \Sigma \)'s, as the morphisms \( \Gamma \rightarrow \Sigma \) could be distinct. Moreover, two distinct decompositions may be equivalent up to an automorphism of \( \Gamma \), as the following example illustrates.

Example 6.5. Let \( H \leq F \) be any non-trivial f.g. subgroup. Consider the multi core graphs \( \Gamma_n \) consisting of \( n \) disjoint copies of \( \Gamma_B(H) \). Denote by \( \eta_n \) the unique morphism \( \Gamma_n \rightarrow X_B \). There are at least \( \binom{n}{2} \) distinct decompositions in \( \text{Decomp}_B(\eta_n) \) with intermediate multi core graph \( \Gamma_{n-1} \), corresponding to a choice of a pair of components of \( \Gamma_n \) which are identified to a single component in \( \Gamma_{n-1} \) (there may be more if \( H \leq uHu^{-1} \), or equivalently \( H = uHu^{-1} \), for some \( u \in F \setminus H \)). All of these decompositions are related by an automorphism of \( \Gamma_n \), permuting the connected components of \( \Gamma_n \), but they are distinct elements of \( \text{Decomp}_B(\eta_n) \).
Similarly, we define the right Möbius inversion morphism \( \eta \) equation that holds for every function \( \Phi \)

**Definition 6.6.** We define the left inversion of \( \Phi \) on \( B \)-surjective morphisms, denoted \( L^B \), by the following equation that holds for every \( B \)-surjective morphism \( \eta \),

\[
\Phi_\eta = \sum_{(\eta_1, \eta_2) \in \text{Decomp}_B(\eta)} L^B_{\eta_2}. 
\] (6.2)

Similarly, we define the right Möbius inversion \( R^B \) by the following equation holding for every \( B \)-surjective morphism \( \eta \)

\[
\Phi_\eta = \sum_{(\eta_1, \eta_2) \in \text{Decomp}_B(\eta)} R^B_{\eta_1}. 
\] (6.3)

Finally, the following equation for all \( B \)-surjective \( \eta \) defines the two-sided inversion \( C^B \) of \( \Phi \):

\[
\Phi_\eta = \sum_{(\eta_1, \eta_2, \eta_3) \in \text{Decomp}_B^3(\eta)} C^B_{\eta_2}. 
\] (6.4)

Indeed, (6.2) well defines a map \( L^B: \mathbb{Z}_{\geq 1} \rightarrow \mathbb{Q} \) for every \( B \)-surjective morphism \( \eta \) by induction on the size of \( \text{Decomp}_B(\eta) \). The base case is \( L^B_{\text{id}}(N) = \Phi_{\text{id}}(N) = N x^I \). For a general \( B \)-surjective \( \eta \), note that \( (\text{id}, \eta) \in \text{Decomp}_B(\eta) \) and so

\[
L^B_\eta = \Phi_\eta - \sum_{(\eta_1, \eta_2) \in \text{Decomp}_B(\eta) \setminus \{\text{id}, \eta\}} L^B_{\eta_2}. 
\] (6.5)

For every element \( (\eta_1, \eta_2) \in \text{Decomp}_B(\eta) \) other than \( (\text{id}, \eta) \), \( \eta_1 \) is not an isomorphism, and so \( |\text{Decomp}_B(\eta_2)| < |\text{Decomp}_B(\eta)| \). Thus, the summation on the right hand side of (6.5) is on morphisms with a strictly smaller set of decompositions, and the terms are well-defined by the induction hypothesis.

A similar argument shows that (6.3) and (6.4) well define \( R^B \) and \( C^B \), respectively. Note that \( C^B \) is the right inversion of \( L^B \) and the left inversion of \( R^B \):

\[
L^B_\eta = \sum_{(\eta_1, \eta_2) \in \text{Decomp}_B(\eta)} C^B_{\eta_1} \quad R^B_\eta = \sum_{(\eta_1, \eta_2) \in \text{Decomp}_B(\eta)} C^B_{\eta_2}. 
\] (6.6)

**Remark 6.7.** One could define a partial order on \( \text{Decomp}_B(\eta) \) by setting \( (\eta_1, \eta_2) \leq (\eta'_1, \eta'_2) \) whenever there is a (necessarily surjective) morphism \( \theta: \Sigma \rightarrow \Sigma' \) which makes the following diagram commute.

![Diagram](image)

Using this partial order, one could define the maps \( L^B, R^B, C^B \) as Möbius inversions of a map defined on pairs of comparable elements in a locally-finite poset. This is the ordinary manner of defining Möbius inversions. We chose a different language here which seems more elegant.

We turn to the study of \( \Phi \) using the three Möbius inversions \( L^B, R^B, C^B \). Recall the geometric interpretation of \( \Phi \) in Proposition 3.7. This gives rise to a similar interpretation for \( L^B \). Recall that \( (N)_s = N(N-1) \ldots (N-s+1) \).
Proposition 6.8. Let $\eta: \Gamma \to \Delta$ be a $B$-surjective morphism. In the notation of Proposition 3.7, $L^B_\eta(N)$ is equal to the average number of injective lifts $\hat{\eta}: \Gamma \to \hat{\Delta}_N$ of $\eta$. Moreover, for every large enough $N$,

$$L^B_\eta(N) = \frac{\prod_{v \in V(\Delta)} (N)_{|\eta^{-1}(v)|}}{\prod_{e \in E(\Delta)} (N)_{|\eta^{-1}(e)|}}.$$  

(6.7)

Proof. By Proposition 3.7, $\Phi_\eta(N)$ is equal to the average number of lifts $\hat{\eta}: \Gamma \to \hat{\Delta}_N$ of $\eta$. Every such lift can be written as the composition of a surjection and an embedding.

Note that the image $Im\hat{\eta}$ is a multi core graph. Because $(p \circ \iota) \circ \overline{\eta}$ is a decomposition of the surjective $\eta$, the morphism $(p \circ \iota)$ is surjective too, and so $(\overline{\eta}, p \circ \iota) \in \text{Decomp}_B(\eta)$. There is thus a one-to-one correspondence between the lifts $\hat{\eta}$ of $\eta$, and the union over all decompositions $(\eta_1, \eta_2) \in \text{Decomp}_B(\eta)$ of injective lifts of $\eta_2$. Therefore,

$$\Phi_\eta(N) = \mathbb{E}[\# \text{ lifts of } \eta] = \sum_{(\eta_1, \eta_2) \in \text{Decomp}_B(\eta)} \mathbb{E}[\# \text{ injective lifts of } \eta_2],$$

and we conclude that, indeed, $L^B_\eta(N)$ is equal to the number of injective lifts of $\eta_2$ to $\hat{\Delta}_N$.

It remains to prove the right hand side of (6.7) gives the average number of injective lifts of $\eta: \Gamma \to \Delta$. First we embed the vertices of $\Gamma$ in $\hat{\Delta}_N$. For every $v \in V(\Delta)$, the fiber $\eta^{-1}(v)$ should be embedded in the fiber $p^{-1}(v)$ which is of size $N$, and there are $(N)_{|\eta^{-1}(v)|}$ choices for such an embedding. Second, for every edge $e \in E(\Delta)$, we obtain $\eta^{-1}(e)$ restrictions on the permutation $\sigma_e$. Such a set of conditions occurs with probability $\frac{(N-|\eta^{-1}(v)|)!}{N!} \frac{1}{(N)_{|\eta^{-1}(v)|}}$. This implies the claim. \qed

Corollary 6.9. If $\eta: \Gamma \to \Delta$ is a $B$-surjective morphism, then $\Phi_\eta$, $L^B_\eta$, $R^B_\eta$ and $C^B_\eta$ are all rational functions in $N$ for every large enough $N$.

Proof. For $L^B_\eta$ this follows directly from Proposition 6.8. The other three functions are equal to finite sums of $L^B_\psi$ with certain $B$-surjective morphisms $\psi$. \qed

We now develop an alternate expression for the right hand side of (6.7), in order to obtain an expression for the double sided Möbius inversion $C^B_\eta$.

Definition 6.10. Let $X$ be a finite set. Define the norm $||\sigma||$ of a permutation $\sigma \in \text{Sym}(X)$ as the minimal length of a product of transpositions which gives $\sigma$. Equivalently, $||\sigma|| = \sum_c \text{len}(c) - 1$, the sum being on the cycles of $\sigma$. Also, $||\sigma||$ is equal to the norm (as in Definition 5.1) of the partition of $X$ induced by the cycles of $\sigma$.

If, in addition, $Y$ is also a set and $\varphi: X \to Y$ some map, let

$$[X]_j^\varphi \overset{\text{def}}{=} |\{\sigma \in \text{Sym}(X) \mid \varphi \circ \sigma = \varphi, ||\sigma|| = j\}|$$

denote the number of $\varphi$-preserving permutations of $X$ of norm $j$. Note that this number depends only on the partition induced by $\varphi$ on $X$. 

28
Proposition 6.11. Let $\eta: \Gamma \to \Delta$ be a $B$-surjective morphism. Then

$$L^B_\eta(N) = \sum_{t \geq 0} \sum_{j_0 \geq 0 \atop j_1, \ldots, j_r \geq 1} (-1)^t \sum_{i=0}^{j_0} [V(\Gamma)]_{j_0} [E(\Gamma)]_{j_1} \cdots [E(\Gamma)]_{j_r} N^{\chi(\Gamma)-\sum j_i} \quad (6.8)$$

Proof. This is the same as [PP15, Section 7.1] – we repeat here briefly a sketch of the argument. We use the identity $(N)_k = N^k \sum_{j=0}^k (-1)^j [k]_j N^{-j}$, where $[k]_j$ denotes the number of permutations in $S_k$ with norm $j$. Multiplying this identity for the sets $\eta^{-1}(v)$, we obtain

$$\prod_{v \in V(\Delta)} (N)_{|\eta^{-1}(v)|} = N^{|V(\Gamma)|} \sum_{j=0}^{\chi(\Gamma)} (-1)^j [V(\Gamma)]_j N^{-j},$$

since an $\eta$-preserving permutation decomposes uniquely as a product of permutations in the $\eta$-fibers. Similarly,

$$\prod_{e \in E(\Delta)} (N)_{|\eta^{-1}(e)|} = N^{|E(\Gamma)|} \sum_{j=0}^{\chi(\Gamma)} (-1)^j [E(\Gamma)]_j N^{-j}.$$

Combined with (6.7), we get

$$L^B_\eta(N) = N^\chi(\Gamma) \frac{\sum_{j=0}^{\chi(\Gamma)} (-1)^j [V(\Gamma)]_{1+j} N^{-j}}{\sum_{j=0}^{\chi(\Gamma)} (-1)^j [E(\Gamma)]_{1+j} N^{-j}}.$$

Using the fact that

$$1 + \sum_{i \geq 1} a_i N^{-i} = \sum_{t \geq 0} \left( -\sum_{i} a_i N^{-i} \right)^t = \sum_{t \geq 0} (-1)^t \sum_{j_1, \ldots, j_r \geq 1} a_{j_1} \cdots a_{j_r} N^{-\sum j_i},$$

the claim follows. \qed

We use this expression in order to obtain a combinatorial interpretation for $C^B_\eta(N)$, which then implies the following Theorem.

Theorem 6.12. Let $\eta: \Gamma \to \Delta$ be a $B$-surjective morphism. Then

$$C^B_\eta(N) = O \left( N^{\chi(\Gamma)-\|\eta\|B} \right).$$

Proof. We give a sketch of the analysis carried out in more detail in [PP15, Section 7.1]. Recall that $C^B$ is the right Möbius inversion of $L^B$. Our starting point is the expression (6.8) for $L^B_\eta$. A permutation of the vertex set $V(\Gamma)$ induces a partition of the vertex set. Identifying the blocks of this partition and folding, gives rise to a $B$-surjective morphism $\eta_1: \Gamma \to \Sigma$. If the permutation is $\eta$-preserving, $\eta_1$ defines a partition which refines the partition of $\eta$, and then there is a $B$-surjective morphism $\eta_2: \Sigma \to \Delta$ with $\eta = \eta_2 \circ \eta_1$.

Similarly, an $\eta$-preserving permutation of the edge-set $E(\Gamma)$ induces a natural $B$-surjective morphism which is the first half of a decomposition of $\eta$. This can be seen by gluing every two edges in the same cycle of the permutation, and then folding. This gluing is equivalent to gluing together the origins of the two edges, or equivalently their termini, since the permutation is $\eta$-preserving and in particular preserves edge labels and directions.

Given both a vertex permutation and a sequence of edge permutations of $\Gamma$, we may glue along all of these permutations, and then fold in order to obtain a $B$-surjective morphism $\eta_1: \Gamma \to \Sigma$ corresponding to a partition refining the $\eta$-partition of $V(\Gamma)$. This implies that every term in (6.8) can be attributed
respectively, by analogy with Definition 6.6. For instance, \( \Phi \) and central Möbius inversions of algebraic morphisms, with the same identifications as in Definition 6.4. We also define the algebraic left, right Proposition 6.14.

denote by \( \psi \) and \( \Theta \), respectively, by analogy with Definition 6.6. For instance, \( L^\text{alg} \) is defined by

\[
\Phi_\eta = \sum_{(\eta_1, \eta_2) \in \text{Decomp}_\text{alg}(\eta)} L^\text{alg}_{\eta_2}.
\]

Recall, by Theorem 4.7(1), that if \( \eta \) is algebraic, then \( \text{Decomp}_\text{alg}(\eta) \subseteq \text{Decomp}_B(\eta) \).

**Proposition 6.14.** The right Möbius inversion \( R^B \) is supported on algebraic morphisms, and on those it is equal to \( R^\text{alg} \) (in particular, it is independent of the basis \( B \)).

**Proof.** We prove all claims together by induction on the size of \( \text{Decomp}_B(\eta) \). Note that the base case is \( \text{id}: \Gamma \to \Gamma \), which is algebraic, and \( R^B_{\text{id}}(N) = R^\text{alg}_{\text{id}}(N) = \Phi_{\text{id}}(N) = N\chi(\Gamma) \), which is basis independent. For the general case,

\[
R^B_\eta(N) = \Phi_\eta(N) - \sum_{(\eta_1, \eta_2) \in \text{Decomp}_B(\eta): \eta_2 \text{ non-isomorphism}} R^B_{\eta_1}(N)
\]

where the second equality is by the induction hypothesis. If \( \eta \) is algebraic, the pairs \((\eta_1, \eta_2)\) in the summation in right hand side of (6.9) are exactly those in \( \text{Decomp}_\text{alg}(\eta) \setminus \{(\eta, \text{id})\} \) and so the entire summation is equal to \( R^\text{alg}_{\eta_1}(N) \). Finally, assume that \( \eta \) is not algebraic. Consider the unique decomposition \( \Gamma \xrightarrow{\varphi} \sum_{\psi} \psi \to \Delta \) of \( \eta \) into an algebraic morphism \( \varphi \) and a free one \( \psi \), as in Theorem 4.9. As \( \eta \) is \( B \)-surjective, so is \( \psi \). By the same Theorem 4.9, for every \((\eta_1, \eta_2) \in \text{Decomp}_B(\eta)\) with \( \eta_1 \) algebraic, there is (a unique) \( \overline{\eta_2} \) so that

\[
L^B_\eta(N) = \sum_{(\eta_1, \eta_2) \in \text{Decomp}_B(\eta)} \tilde{C}^B_{\eta, \eta_1}(N).
\]

We will prove that \( C^B_{\eta, \eta_1}(N) \) depends only on \( \eta_1 \) (and not on \( \eta \)), implying that \( C^B_{\eta, \eta_1} \) is the right Möbius inversion of \( L^B \) as in (6.6), and therefore equal to the central inversion \( C^B_\eta \). On the other hand, the expression \( \tilde{C}^B_{\eta, \eta_1}(N) \) was obtained as a signed sum of expressions of the form \( N\chi(\Gamma) - \sum_j j_i \), where \( \sum j_i \geq ||\eta_1||_B \), since this number of identifications yields \( \eta_1 \). This will imply the claim.

It remains to prove that \( C^B_{\eta, \eta_1}(N) \) is indeed independent of \( \eta \). This \( C^B_{\eta, \eta_1}(N) \) was obtained as a sum over sequences of \( \eta \)-preserving vertex and edge permutations generating \( \eta_1 \) (with Stallings foldings). Note that such a sequence of permutations is then also \( \eta_1 \)-preserving. This implies that we can equivalently describe this set of permutations as sequences of \( \eta_1 \)-preserving permutations generating \( \eta_1 \) after folding. Hence, the expression depends only on \( \eta_1 \).

### 6.2 Algebraic Möbius Inversion

We also work with Möbius inversion based on algebraic morphisms. This has no direct parallel in [PP15].

**Definition 6.13.** For an algebraic morphism \( \eta: \Gamma \to \Delta \) in \( \text{MutCG}_B(\mathbf{F}) \) or, equivalently, in \( \text{MOCC}(\mathbf{F}) \), denote by \( \text{Decomp}_\text{alg}(\eta) \) and \( \text{Decomp}_3^\text{alg}(\eta) \) the set of decompositions of \( \eta \) into two (three, respectively) algebraic morphisms, with the same identifications as in Definition 6.4. We also define the left, right and central Möbius inversions of \( \Phi \) (restricted to algebraic morphisms), denoted \( L^\text{alg}_\eta(N), R^\text{alg}_\eta(N), C^\text{alg}_\eta(N) \), respectively, by analogy with Definition 6.6. For instance, \( L^\text{alg} \) is defined by
Otherwise, Proof.

If \( \eta \) be an algebraic morphism. Then for every basis \( B \),

\[
C^\text{alg}_\eta(N) = \sum_{(\eta_1, \eta_2) \in \text{Decomp}_B(\eta): \eta_1 \text{ is free}} C^B_{\eta_2}(N),
\]

By Proposition 6.15.

\[
R^B_{\eta}(N) = \Phi_{\eta}(N) - \sum_{(\eta_1, \eta_2) \in \text{Decomp}_\text{alg}(\phi)} R^\text{alg}_{\eta_1}(N) = 0.
\]

Proposition 6.15. Let \( \eta \) be an algebraic morphism. Then for every basis \( B \),

\[
C^\text{alg}_\eta(N) = \sum_{(\eta_1, \eta_2) \in \text{Decomp}_B(\eta): \eta_1 \text{ is free}} C^B_{\eta_2}(N).
\]

Proof. Denote the right hand side of the above equality by \( F_{\eta}(N) \) for the course of the proof. For a morphism \( \gamma \) denote by \( \text{alg}(\gamma) \) and \( \text{free}(\gamma) \) the morphisms in the unique decomposition of \( \gamma \) into algebraic and free morphisms given by Theorem 4.9. Let \( \eta \) be algebraic. For every \( (\eta_1, \eta_2) \in \text{Decomp}_B(\eta) \), consider the decomposition \( \bullet \overset{\text{alg}(\eta_1)}{\longrightarrow} \overset{\text{free}(\eta_1)}{\longrightarrow} \overset{\text{alg}(\eta_2)}{\longrightarrow} \overset{\text{free}(\eta_2)}{\longrightarrow} \bullet \) of \( \eta \). Because \( \eta \) is algebraic, so is \( \beta = \eta_2 \circ \text{free}(\eta_1) \). Thus,

\[
R^\text{alg}_{\eta}(N) = \sum_{(\alpha, \beta) \in \text{Decomp}_\text{alg}(\eta)} \left[ \sum_{(\eta_1, \eta_2) \in \text{Decomp}_B(\eta): \eta_2 \circ \text{free}(\eta_1) = \beta} C^B_{\eta_2}(N) \right] = \sum_{(\alpha, \beta) \in \text{Decomp}_\text{alg}(\eta)} F_{\beta}(N).
\]

This implies the claim, by definition of the Möbius inversion.

Corollary 6.16. Let \( \eta : \Gamma \to \Delta \) be an algebraic morphism. Then

\[
C^\text{alg}_\eta(N) = \begin{cases} 
N^\chi(\Gamma) & \text{if } \eta \text{ is an isomorphism}, \\
O\left( N^\chi(\Gamma) - \|\eta\| \right) & \text{otherwise}.
\end{cases}
\]

Proof. If \( \eta \) is an isomorphism, then \( C^\text{alg}_\eta(N) = C^\text{id}_\eta(N) = \Phi_{\text{id}}(N) = N^\chi(\Gamma) \), as noted in Example 3.5. Otherwise,

\[
C^\text{alg}_\eta(N) = \sum_{(\eta_1, \eta_2) \in \text{Decomp}_B(\eta): \eta_1 \text{ free}} C^B_{\eta_2}(N)
\]

by Proposition 6.15.

\[
\sum_{(\eta_1, \eta_2) \in \text{Decomp}_B(\eta): \eta_1 \text{ free}} O\left( N^\chi(\text{Im}(\eta_1)) - \|\eta_2\| \right)
\]

by Theorem 6.12.

\[
\sum_{(\eta_1, \eta_2) \in \text{Decomp}_B(\eta): \eta_1 \text{ free}} O\left( N^\chi(\Gamma) - \|\eta_1\| - \|\eta_2\| \right)
\]

by Lemma 5.3.

\[
\sum_{(\eta_1, \eta_2) \in \text{Decomp}_B(\eta): \eta_1 \text{ free}} O\left( N^\chi(\Gamma) - \|\eta_1\| - \|\eta_2\| \right)
\]

by (5.3).

(6.10)
Finally, by the very definition of the basis-independent norm of morphisms in Definition 5.2, it is clear that \(\|\eta\| \leq \|\eta_1\| + \|\eta_2\|\) for every \((\eta_1, \eta_2) \in \text{Decomp}_B(\eta)\). Hence every term in (6.10) is at most \(O \left( N^{\chi(\Gamma)} - \|\eta\| \right)\), which is at most \(O \left( N^{\chi(\Delta) - 1} \right)\) by Lemma 5.3 as \(\eta\) is not free. This completes the proof as the summation in (6.10) is finite.

We can now prove the main result of this section. Recall that \(\eta: \Gamma \to \Delta\), and we ought to show that

\[
\Phi_\eta(N) = N^{\chi(\Gamma)} + |\text{Crit}(\eta)| \cdot N^{\chi_{\max}(\eta)} + O \left( N^{\chi_{\max}(\eta) - 1} \right).
\]  

(6.11)

**Proof of Theorem 6.2.** We may assume without loss of generality that \(\eta\) is algebraic. Otherwise, the decomposition of \(\eta\) into an algebraic morphism \(\varphi\) and a free morphism \(\psi\) given by Theorem 4.9, satisfies that \(\Phi_\eta = \Phi_\varphi \psi\) by Proposition 4.3(4), and that \(\chi_{\max}(\eta) = \chi_{\max}(\varphi)\) and \(|\text{Crit}(\eta)| = |\text{Crit}(\varphi)|\) by Theorem 4.9.

So assume that \(\eta\) is algebraic. We have \(\Phi_\eta(N) = \sum_{(\eta_1, \eta_2, \eta_3) \in \text{Decomp}_B(\eta)} C^\text{alg}_{\eta_2}(N)\). If \(\eta_2 = \text{id}\) the contribution is \(C^\text{alg}_{\text{id}: \text{Im}(\eta_1) \to \text{Im}(\eta_1)}(N) = N^{\chi(\text{Im}(\eta_1))}\), and these contributions give rise to the first two terms in (6.11) plus \(O \left( N^{\chi_{\max}(\eta) - 1} \right)\). In any other decomposition \((\eta_1, \eta_2, \eta_3) \in \text{Decomp}_B(\eta)\), Corollary 6.16 yields that

\[
C^\text{alg}_{\eta_2}(N) = O \left( N^{\chi(\text{Im}(\eta_2)) - 1} \right) = O \left( N^{\chi(\text{Im}(\eta_2 \circ \eta_1)) - 1} \right) = O \left( N^{\chi_{\max}(\eta) - 1} \right).
\]

We end this section with the following full analysis of algebraic and \(B\)-surjective morphisms in rank one free group.

**Lemma 6.17.** Let \(F_1 \cong \mathbb{Z}\) with basis \(B = \{b\}\). Let \(\eta: \mathcal{H} \to \mathcal{J}\) be a morphism in \(\text{MOCC}(F_1)\) such that the image of \(\eta\) meets every element of the multiset \(\mathcal{J}\). Then \(\eta\) is algebraic, and for all large enough \(N\),

\[
C^\text{alg}_\eta(N) = \begin{cases} 
1 & \text{if } \eta = \text{id}, \\
0 & \text{otherwise}, 
\end{cases}
\]

(6.12)

\(L^\text{alg}_\eta(N) = 1\) and \(\Phi_\eta(N) = |\text{Decomp}_\text{alg}(\eta)| = |\text{Decomp}_B(\eta)|\).

**Proof.** Recall that the elements in the multisets in \(\text{MOCC}(F_1)\) are conjugacy classes of non-trivial subgroups. Every non-trivial subgroup of \(F_1\) is of rank 1. Hence, by Proposition 4.3(3), there are no free morphisms in \(\text{MOCC}(F_1)\) in which the image meets every component of the codomain, except for isomorphisms. The definition of algebraic morphisms now implies that every morphism in \(\text{MOCC}(F_1)\) with image meeting every component of the codomain is algebraic. As every algebraic morphism is \(B\)-surjective, we obtain that \(\text{Decomp}_B(\eta) = \text{Decomp}_\text{alg}(\eta)\) and so \(L^\text{alg}_\eta(N) = L^B_\eta(N)\) and \(C^\text{alg}_\eta(N) = C^B_\eta(N)\).

Next we prove that \(L^B_\eta(N) = 1\). By Proposition 6.8, \(L^B\) is multiplicative on the elements of \(\mathcal{J}\) and it is thus enough to prove that \(L^B_\eta(N) = 1\) when \(\mathcal{J}\) is a singleton. But then \(\mathcal{J} = \{\{b^j\}^{F_1}\}\) for some \(j\), and every component of \(\mathcal{H}\) is \(\{b^{jm}\}^{F_1}\) for some \(m \in \mathbb{Z}_{\geq 1}\). In particular, the morphism of \(B\)-labeled multi core graphs is a topological covering map, and every vertex and every edge in \(\Gamma_B(\mathcal{J})\) have fiber of the same size. It now follows from (6.7) that \(L^B_\eta(N) = 1\).

That \(\Phi_\eta(N) = |\text{Decomp}_\text{alg}(\eta)| = |\text{Decomp}_B(\eta)|\) follows immediately, and (6.12) follows by considering \(C^\text{alg}\) as the right Möbius inversion of \(L^\text{alg}\) and a simple induction on \(|\text{Decomp}_B(\eta)|\).

7 **The proof of Theorem 1.3**

Throughout this section, we fix a non-power \(1 \neq w \in F\). Recall from Example 3.6 that the function \(E_w \left[ \xi_1^{a_1} \cdots \xi_k^{a_k} \right]\) in the center of Theorem 1.3 is equal to \(\Phi_{\eta_{a_1}, \ldots, a_k}^{w}\), where \(\eta_{a_1}, \ldots, a_k: \Gamma_{a_1, \ldots, a_k}^w \to X_B\).
When \( k = 1 \) and \( \alpha_1 = 1 \), this is the special case from Theorem 1.2, that was proven in [PP15], and is immediate from Theorem 6.2. However, in any other case, Theorem 6.2 as is does not teach us anything new. Indeed, if \( \alpha_1 + 2\alpha_2 + \ldots + k\alpha_k \geq 2 \), then there is \((\varphi_1, \varphi_2) \in \text{Decomp}_B (\eta_{\alpha_1,\ldots,\alpha_k}^{w})\) with \( \varphi_1 \) algebraic (by Lemma 6.17) and non-isomorphism, so that \( \text{Im} (\varphi_1) = \Gamma_B (\langle w \rangle^F) \). In particular, \( \chi^{\max} (\eta_{\alpha_1,\ldots,\alpha_k}^{w}) = 0 \), so Theorem 6.2 says only that

\[
\mathbb{E}_w [\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}] = \left[ 1 + \text{Crit} (\eta_{\alpha_1,\ldots,\alpha_k}^{w}) \right] + O \left( N^{-1} \right).
\]

This agrees with the statement of Theorem 1.3: as we explain below, \( \langle \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}, 1 \rangle = 1 + \text{Crit} (\eta_{\alpha_1,\ldots,\alpha_k}^{w}) \).

But this is only the easier part of this theorem (that also follows from [Nic94, LP10]). To establish Theorem 1.3 in full, we need some more machinery. We start with the following twist on Definition 6.1 of \( \chi^{\max} \) and of Crit, which considers only negative Euler characteristics. Because the codomain of \( \eta_{\alpha_1,\ldots,\alpha_k}^{w} \) is \( X_B \), any morphism \( \Gamma_{\alpha_1,\ldots,\alpha_k}^{w} \to \Sigma \) is part of a decomposition of \( \eta_{\alpha_1,\ldots,\alpha_k}^{w} \).

**Definition 7.1.** For a non-power \( 1 \neq w \in F \) and \( k \geq 1, \alpha_1, \ldots, \alpha_k \geq 0 \), let \( \Gamma_{\alpha_1,\ldots,\alpha_k}^{w} \) and \( \eta_{\alpha_1,\ldots,\alpha_k}^{w} \) denote the corresponding multi core graph and morphism as above, and let

\[
\chi^{\max}_{\alpha_1,\ldots,\alpha_k} (w) \overset{\text{def}}{=} \max \left\{ \chi (\Sigma) \mid \Gamma_{\alpha_1,\ldots,\alpha_k}^{w} \to \Sigma \text{ is algebraic with } \chi (\Sigma) < 0 \right\}.
\]

Denote by \( \text{Crit}_{\alpha_1,\ldots,\alpha_k} (w) \) the set of algebraic morphisms with domain \( \Gamma_{\alpha_1,\ldots,\alpha_k}^{w} \) and codomain of Euler characteristic \( \chi^{\max}_{\alpha_1,\ldots,\alpha_k} (w) \), up to equivalence as in Theorem 4.9.

The proof of Theorem 1.3 consists of (i) the following theorem which is an analogue of Theorem 6.2, (ii) showing that \( \chi^{\max}_{\alpha_1,\ldots,\alpha_k} (w) = \chi^{\max} (\eta_{\alpha_1,\ldots,\alpha_k}^{w}) = 1 - \pi (w) \) – this is done in Section 7.1, and (iii) showing that \( |\text{Crit}_{\alpha_1,\ldots,\alpha_k} (w)| = \langle \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}, 1 \rangle \cdot |\text{Crit} (w)| \), which is done in Section 7.2.

**Theorem 7.2.** Let \( 1 \neq w \in F \) be a non-power. Then

\[
\mathbb{E}_w [\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}] = \mathbb{E}_\text{unif} [\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}] + |\text{Crit}_{\alpha_1,\ldots,\alpha_k} (w)| \cdot N^{\chi^{\max}_{\alpha_1,\ldots,\alpha_k} (w)} + O \left( N^{\chi^{\max}_{\alpha_1,\ldots,\alpha_k} (w) - 1} \right).
\]

**Proof.** Recall that \( \eta_{\alpha_1,\ldots,\alpha_k}^{w} : \Gamma_{\alpha_1,\ldots,\alpha_k}^{w} \to X_B \), and that our goal is to prove the stated approximation of \( \Phi_{\eta_{\alpha_1,\ldots,\alpha_k}^{w}} (N) = \mathbb{E}_w [\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}] \) to an algebraic \( \varphi \) and a free \( \psi \) from Theorem 4.9. Because \( \eta_{\alpha_1,\ldots,\alpha_k}^{w} \) has a decomposition \((\omega_1, \omega_2)\) in \( \text{Crit}_{\alpha_1,\ldots,\alpha_k} \), and in particular with \( \text{Im} (\omega_1) \) connected, \( \Sigma \) must be connected as well (by Theorem 4.9(1)). Let \( J \) be the group in the conjugacy class \( \pi \text{lab} (\Sigma) \) to which \( \langle w \rangle \) is mapped as a subgroup. We have again that \( w \) is a non-power in \( J \), and as in the proof of Theorem 6.2, the quantities \( \chi^{\max}_{\alpha_1,\ldots,\alpha_k} (w) \) and \( \text{Crit}_{\alpha_1,\ldots,\alpha_k} (w) \) are the same in \( J \) as in \( F \). Thus we may assume without loss of generality that \( \Sigma = X_B \) and \( J = F \), namely, that \( \eta_{\alpha_1,\ldots,\alpha_k}^{w} \) is algebraic.

Using the algebraic Möbius Inversions from Section 6.2 we have

\[
\Phi_{\eta_{\alpha_1,\ldots,\alpha_k}^{w}} (N) = \sum_{(\beta_1, \beta_2, \beta_3) \in \text{Decomp}_3^{\text{alg}} (\eta_{\alpha_1,\ldots,\alpha_k}^{w})} C_{\beta_2}^{\text{alg}} (N).
\]

By definition of \( \chi^{\max}_{\alpha_1,\ldots,\alpha_k} (w) \), any summand in (7.1) satisfies \( \chi (\text{Im} (\beta_2)) = 0 \) or \( \chi (\text{Im} (\beta_2)) \leq \chi^{\max}_{\alpha_1,\ldots,\alpha_k} (w) \). Consider the following cases:

**Case I:** \( \chi (\text{Im} (\beta_2)) = 0 \) \( \quad \) As \( w \) is a non-power, the only cyclic subgroups of \( F \) containing \( w^m \) are \( \langle w^d \rangle \) with \( d \mid m \). This shows that \((\beta_1, \beta_2)\) is part of a decomposition of \( \omega_1 \), and everything takes place inside the ambient group \( \langle w \rangle \cong \mathbb{Z} \). Conversely, every decomposition \((\beta_1, \beta_2, \beta_3) \in \text{Decomp}_3^{\text{alg}} (\omega_1) \) satisfies \( \chi (\text{Im} (\beta_2)) = 0 \). The entire category of algebraic morphisms inside \( \text{MOCC} (\langle w \rangle) \) are identical to that inside \( \text{MOCC} (\mathbb{Z}) \). And so

\[
\sum_{(\beta_1, \beta_2, \beta_3) \in \text{Decomp}_3^{\text{alg}} (\eta_{\alpha_1,\ldots,\alpha_k}^{w}) : \chi (\text{Im} (\beta_2)) = 0} C_{\beta_2}^{\text{alg}} (N) = \Phi_{\omega_1} (N) = \mathbb{E}_x [\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}] = \mathbb{E}_\text{unif} [\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}].
\]
Case II: \( \chi(\text{Im}(\beta_2)) = \chi_{\alpha_1,\ldots,\alpha_k}^{\max}(w) \) and \( \beta_2 \) is an isomorphism  

By Corollary 6.16, we have in this case \( C_{\beta_2}^{alg}(N) = N\chi(\text{Im}(\beta_2)) = N\chi(\text{Im}(\beta_1)) = N\chi_{\alpha_1,\ldots,\alpha_k}^{\max}(w) \). There is exactly one such decomposition in \( \text{Decomp}_{alg}^3(\eta_{\alpha_1,\ldots,\alpha_k}) \) for every \( \beta_1 \in \text{Crit}_{\alpha_1,\ldots,\alpha_k}(w) \), and so the total contribution of these summands in (7.1) is \( |\text{Crit}_{\alpha_1,\ldots,\alpha_k}(w)| \cdot N\chi_{\alpha_1,\ldots,\alpha_k}^{\max}(w) \).

All remaining terms in (7.1): In every other case, either \( \chi(\text{Im}(\beta_2)) < \chi_{\alpha_1,\ldots,\alpha_k}^{\max}(w) \) or \( \chi(\text{Im}(\beta_2)) = \chi_{\alpha_1,\ldots,\alpha_k}^{\max}(w) \) but \( \beta_2 \) is not an isomorphism, and Corollary 6.16 yields that \( C_{\beta_2}^{alg}(N) = O\left(N\chi_{\alpha_1,\ldots,\alpha_k}(w)^{-1}\right) \).

This completes the proof of the theorem.

\( \square \)

Remark 7.3. The analysis above readily leads to a precise formula for \( E_{\text{unif}}[\xi_{s_1}^{\alpha_1} \cdots \xi_{s_k}^{\alpha_k}] = E_x[\xi_{s_1}^{\alpha_1} \cdots \xi_{s_k}^{\alpha_k}] \).

Denote by \( \eta_{s_1}^{\alpha_1} \cdots \eta_{s_k}^{\alpha_k} \) the morphism corresponding to the single-letter word \( x \). By Lemma 6.17,

\[
E_x[\xi_{s_1}^{\alpha_1} \cdots \xi_{s_k}^{\alpha_k}] = \Phi_{\eta_{s_1}^{\alpha_1} \cdots \eta_{s_k}^{\alpha_k}}(N) = |\text{Decomp}_{alg}(\eta_{\alpha_1,\ldots,\alpha_k})|.
\]

Every such decomposition induces a partition on the components of \( \Gamma_{\alpha_1,\ldots,\alpha_k} \) (by their image in the intermediate multi core graph). If a block in the partition consists of cycles corresponding to \( \langle x^d \rangle \) for some \( d \mid \gcd(k_1, \ldots, k_m) \), and there are \( d^{m-1} \) non-equivalent morphisms to a cycle. So if \( S \) is a multiset with \( \alpha_1 \)'s, \( \alpha_2 \)'s and so on, then

\[
E_{\text{unif}}[\xi_{s_1}^{\alpha_1} \cdots \xi_{s_k}^{\alpha_k}] = \sum_{\mathcal{P} \in \text{Partitions}(S)} \left[ \prod_{A \in \mathcal{P}} \left[ \sum_{d|\gcd\{k \in A\}} d^{|A|-1} \right] \right].
\]

7.1 Maximal Euler characteristic

**Lemma 7.4.** Let \( J \leq F \) be a f.g. subgroup and let \( u \in F \). If \( \text{rk}(\langle J, u \rangle) \leq \text{rk}J \), then \( J \leq_{alg} \langle J, u \rangle \).

**Proof.** This is [MVW07, Corollary 3.14], but we give the short proof here for completeness. Assume by contradiction that \( J \leq L \leq \langle J, u \rangle \). Then \( \langle L, u \rangle = \langle J, u \rangle \) and \( \text{rk}L + 1 \leq \text{rk}(J, u) \) and hence \( L \ast \langle u \rangle = \langle J, u \rangle \).

Since \( J \) is a subgroup of \( L \), and this is a free product, it follows that \( J = L \). Therefore, \( J \ast \langle u \rangle = \langle J, u \rangle \), which contradicts the rank inequality.

\( \square \)

**Proposition 7.5.** Let \( w \in F \) be a non-power as above. Then for every \( k \geq 1 \) and \( \alpha_1, \ldots, \alpha_k \geq 0 \) not all zeros,

\[
\chi_{\alpha_1,\ldots,\alpha_k}(w) = \chi_{\max}(\eta_1^w) = 1 - \pi(w).
\]

**Proof.** We start by proving that \( \chi_{\max}(\eta_1^w) \geq \chi_{\max}(\eta_1^w) \). Because \( w \) is not a power, \( \pi(w) \geq 2 \), namely, \( \chi_{\max}(\eta_1^w) < 0 \). Let \( \beta: \Gamma_1^w \to \Sigma \) be a critical morphism of \( \eta_1^w \) (as in Definition 6.1), so \( \chi(\Sigma) = \chi_{\max}(\eta_1^w) < 0 \).

By Lemma 6.17, the natural morphism \( \Gamma_{\alpha_1,\ldots,\alpha_k}^w \to \Gamma_1^w \) is algebraic. Therefore, the composition \( \Gamma_{\alpha_1,\ldots,\alpha_k}^w \to \Gamma_{\alpha_1,\ldots,\alpha_k}^w \to \Sigma \) is also algebraic, and so \( \chi_{\max}(\eta_1^w) \leq \chi(\Sigma) = \chi_{\max}(\eta_1^w) \).

To prove the converse inequality, let \( \Gamma_{\alpha_1,\ldots,\alpha_k}^w \to \Sigma \) be algebraic with \( \chi(\Sigma) < 0 \). We need to prove that \( \chi(\Sigma) \leq \chi_{\max}(\eta_1^w) \). Let us restrict our attention to a component \( \Sigma_0 \) of \( \Sigma \) with negative Euler characteristic. This gives an algebraic extension of the corresponding powers of \( w \), i.e., the components of \( \Gamma_{\alpha_1,\ldots,\alpha_k}^w \) mapped to \( \Sigma_0 \), and it is enough to show that \( \chi(\Sigma_0) \leq \chi_{\max}(\eta_1^w) \). So we assume without loss of generality that \( \Sigma \) is connected.

Assume that \( \pi_{lab}(\Sigma) = M^F \) for some f.g. \( M \leq F \). Assume that \( \Gamma_{\alpha_1,\ldots,\alpha_k}^w \) has \( s = \sum \alpha_i \) connected components corresponding to \( \langle w^{k_1} \rangle^F, \ldots, \langle w^{k_s} \rangle^F \), and that the morphism \( \varphi \) maps \( \langle w^{k_i} \rangle \to u_i M u_i^{-1} \) for some \( u_i \in F \) with \( u_1 = 1 \) (conjugate \( M \) if needed). Showing that \( \chi(\Sigma) \leq \chi_{\max}(\eta_1^w) \) is equivalent to that \( \text{rk}M \geq \pi(w) \).
Consider the subgroups $J_1 \leq J_2 \leq \ldots \leq J_{s+1}$ defined by gradually adding $u_2, \ldots, u_s, w$ to $M$:

\[
J_1 \overset{\text{def}}{=} \langle M \rangle, \quad J_2 \overset{\text{def}}{=} \langle J_1, u_2 \rangle, \quad \ldots, \quad J_s = \langle J_{s-1}, u_s \rangle, \quad J_{s+1} = \langle J_s, w \rangle.
\]

Note that the extensions $J_i \leq J_{i+1}$ are not free. Indeed, for $i = 1, \ldots, s-1$, $u_{i+1}M u_{i+1}^{-1}$ and $M$ both contain $u_{k,m(k_1,k_{i+1})}$, so $J_{i+1} = \langle J_i, u_{i+1} \rangle$ is not a free extension of $J_i$. For $i = s$, $J_{s+1} = \langle J_s, w \rangle$, but $w$ has powers contained in $J_s$, so once again this is not a free extension. Hence $\text{rk} J_{i+1} \leq \text{rk} J_i$ for $i = 1, \ldots, s$ and so $\text{rk} J_{s+1} \leq \text{rk} M$. By Lemma 7.4, these are all algebraic extensions, and by transitivity of algebraic extensions, $M \leq_{\text{alg}} J_{s+1}$, corresponding to an algebraic morphism $\Sigma \xrightarrow{\psi} \Delta \overset{\text{def}}{=} \Gamma_B (J_{s+1})$. Note that the composition $\Gamma^w_{\alpha_1, \ldots, \alpha_k} \xrightarrow{\varphi} \Sigma \xrightarrow{\psi} \Delta$ maps the subgroups $\langle w^{k_1} \rangle, \ldots, \langle w^{k_s} \rangle$ to $J_{s+1}$ itself (the latter contains, in particular, $w$). Hence this composition factors through $\Gamma^w_1$, as in the following diagram:

\[
\begin{array}{ccc}
\Gamma^w_{\alpha_1, \ldots, \alpha_k} & \xrightarrow{\varphi} & \Sigma \\
\omega_1 \downarrow_{\text{alg}} & & \downarrow_{\text{alg}} \\
\Gamma^w_1 & \xrightarrow{\alpha} & \Delta
\end{array}
\]

Because $\psi \circ \varphi$ is algebraic, so is $\alpha$. As $M$ was not cyclic, $J_{s+1}$ is not cyclic, so $\chi (\Delta) < 0$. We deduce that $\langle w \rangle \leq_{\text{alg}} J_{s+1}$. Hence $\pi (w) \leq \text{rk} J_{s+1} \leq \text{rk} M$. \hfill \Box

### 7.2 The set of critical morphisms

The previous results already show that for a non-power $w$,

\[
E_w [\xi^1_1 \cdots \xi^k_k] = \langle \xi^1_1 \cdots \xi^k_k, 1 \rangle + |\text{Crit}_{\alpha_1, \ldots, \alpha_k} (w)| \cdot N^{1-\pi(w)} + O \left( N^{-\pi(w)} \right). \tag{7.2}
\]

In order to prove Theorem 1.3, it remains to show that all morphisms in $\text{Crit}_{\alpha_1, \ldots, \alpha_k} (w)$ are obtained from $\text{Crit} (w)$, or, equivalently, from $\text{Crit} (\eta^w_1)$, in the following straightforward way. If $\varphi : \Gamma^w_{\alpha_1, \ldots, \alpha_k} \rightarrow \Sigma$ is a critical morphism in $\text{Crit}_{\alpha_1, \ldots, \alpha_k} (w)$, then $\Sigma$ has one component $\Sigma_0$ with $\chi (\Sigma_0) = \chi^{\max} (\eta^w_1) = 1 - \pi (w)$, and the remaining components are cycles corresponding to powers of $w$. Let $\varphi_0 : \Gamma_0 \rightarrow \Sigma_0$ be the restriction of $\varphi$ to the disjoint union $\Gamma_0$ of components of $\Gamma^w_{\alpha_1, \ldots, \alpha_k}$ mapped to $\Sigma_0$.

Proposition 7.7 below shows that $\varphi_0$ can always be factored through a unique $\beta : \Gamma^w_{\alpha_1, \ldots, \alpha_k} \rightarrow \Sigma_0$ in $\text{Crit} (\eta^w_1)$.

On the other hand, it is clear that the number of $\varphi \in \text{Crit}_{\alpha_1, \ldots, \alpha_k} (w)$ corresponding to a given $\beta \in \text{Crit} (\eta^w_1)$ as above, depends only on $\alpha_1, \ldots, \alpha_k$ and not on $w$ nor on $\beta$. With notation as in Remark 7.3, this number is equal to

\[
\sum_{\mathcal{P} \in \text{Partitions}(S)} \left[ \sum_{A \in \mathcal{P}} \left[ \prod_{B \in \mathcal{P} \setminus \{ A \}} \sum_{d \mid \gcd \{ k_i \in B \}} d^{|B|-1} \right] \right],
\]

and in Proposition 7.8 below, we prove it is equal to $\langle \xi^1_1 \cdots \xi^k_k, 1 - 1 \rangle$.

Proposition 7.7 crucially relies on the following theorem of Louder, as stated in [LW22, page 553].

**Theorem 7.6.** [Lou13] Consider the following graph of groups $\Delta$ in the shape of a star, with $\ell$ vertices around a central vertex, and $m_i \geq 1$ edges between the center and the $i$-th peripheral vertex – see Figure 7.1. The center vertex-group is $\mathbb{Z}$ with generator denoted $w$. The peripheral vertex-groups are free groups $H_1, \ldots, H_{\ell}$ and all edge-groups are infinite cyclic. For every $i \in [\ell], j \in [m_i]$, there is an element $v_{i,j} \in H_i$ and a positive integer $n_{i,j}$ so that the $j$-th edge between $H_i$ and $\langle w \rangle$ attaches $v_{i,j}$ to $w^{m_{i,j}}$. We assume further that

1. For all $i, j$, $v_{i,j} \neq 1$ and is not a proper power.
2. For all $i$, $\langle v_{i,1} \rangle^{H_i}, \ldots, \langle v_{i,m_i} \rangle^{H_i}$ are distinct conjugacy classes.

3. There exists no free splitting of any of the groups $H_i$, $H_i = H_i' \ast \langle v_{i,k} \rangle$, such that all the remaining elements $v_{i,j}, j \neq k$ are conjugate into $H_i'$.

4. $\sum_{i,j} n_{i,j} \geq 2$.

Let $\pi_1(\Delta)$ denote the corresponding group. Namely,

$$\pi_1(\Delta) = \left\langle H_1, \ldots, H_\ell, w, \{t_{i,j}\}_{i\in[\ell], j\in[m_i]} \mid t_{i,j}v_{i,j}t_{i,j}^{-1} = w^{n_{i,j}}, \{t_{i,1} = 1\}_{i\in[\ell]} \right\rangle.$$

Then, if $f: \pi_1(\Delta) \rightarrow J$ is a surjective homomorphism onto a free group $J$, and $f|_{H_i}$ is injective for every $i$, then

$$\text{rk} J - 1 < \sum_{i=1}^\ell (\text{rk} H_i - 1).$$

**Proposition 7.7.** Let $1 \neq w \in F$ be a non-power. Assume, as above, that $\varphi_o: \Gamma^w_{\alpha_1,\ldots,\alpha_k} \rightarrow \Sigma_o$ is a critical morphism in $\text{Crit}_{\alpha_1,\ldots,\alpha_k}(w)$ with $\Sigma_o$ connected and $\chi(\Sigma_o) < 0$ (and so $\chi(\Sigma_o) = \chi^\max(\eta^w) = 1 - \pi(w)$). Then $\varphi_o$ factors through a unique $\beta: \Gamma^w_1 \rightarrow \Sigma_o$ in $\text{Crit}(\eta^w)$.

**Proof.** Because $w$ is not a power, there is a unique morphism $\omega_1: \Gamma^w_{\alpha_1,\ldots,\alpha_k} \rightarrow \Gamma^w_1$, and therefore at most one decomposition of $\varphi_o$ through some $\beta: \Gamma^w_1 \rightarrow \Sigma_o$ in $\text{Crit}(\eta^w)$. It remains to show such a factorization exists.

We freely use notation from the proof of Proposition 7.5. Recall the notion of pullback in the category $\mathcal{MOCC}(F)$ from page 19. Let $(\Lambda, \sigma_1, \sigma_2)$ be the pullback of $\psi_1: \Sigma_o \rightarrow X_B$ and of $\psi_2: \Gamma^w_1 \rightarrow X_B$, and let $\beta: \Gamma^w_{\alpha_1,\ldots,\alpha_k} \rightarrow \Lambda$ be the unique morphism making the following diagram commute.

As every pullback involving $\Gamma^w_1$, the multi core graph $\Lambda$ is a union of cycles corresponding to powers of $w$. Our goal is to show that the image of $\beta$ meets a sole component of $\Lambda$ which is isomorphic to $\Gamma^w_1$, and so $\sigma_1$, restricted to this component, is in $\text{Crit}(\eta^w)$. Assume to the contrary that this is not the case, so the image of $\beta$ consists of some $\Gamma^w_{\alpha_1,\ldots,\alpha_k}$ with $\sum \alpha_i \geq 2$. Without loss of generality, assume the image is $\Gamma^w_{\alpha_1,\ldots,\alpha_k}$ itself and $\sum i\alpha_i \geq 2$, and so $\sigma_1 = \varphi_o$. In particular, we assume that $\left(\Gamma^w_{\alpha_1,\ldots,\alpha_k}, \varphi_o, \omega_1\right)$ is itself the pullback.
Let $s = \sum \alpha_i$ and $k_1, \ldots, k_s$ be the powers of $w$ in the components of $\Gamma_{\alpha_1, \ldots, \alpha_k}^w$. Let $M \leq F$ be a f.g. subgroup with $M^F = \pi_{1, \text{lab}}(\Sigma_o)$, such that $\varphi_o$ is given by $w^{k_1}, u_2w^{k_2}u_2^{-1}, \ldots, u_sw^{k_s}u_s^{-1} \in M$, for some $u_2, \ldots, u_s \in F$. In the notation of Louder’s Theorem 7.6 with $\ell = 1$, consider the graph of groups $\Delta$ with two vertices: $H_1 = M$ and $\langle w \rangle$, and $m_1 = s$ parallel edges with $Z$ as edge-group between them. We denote $v_i, w_i, n_i, t_i$, and set $v_1 = w^{k_1}, v_2 = u_2w^{k_2}u_2^{-1}, \ldots, v_s = u_sw^{k_s}u_s^{-1}$, and $n_i = k_i$ for $i \in [s]$. We claim that these choices satisfy the four assumptions in Theorem 7.6. Indeed:

- If some $v_i = t^d$ was a proper power in $M$ (so $d \geq 2$), then this equation is also valid in $F$, so that $d \mid k_i$. But then both $\varphi$ and $\varphi_o$ factor through the morphism $\Gamma_{\alpha_1, \ldots, \alpha_k}^w \to \Delta$ which maps $\langle w^{k_i} \rangle$ to $\langle w^{k_i/d} \rangle$ (and leaves all other elements unchanged), in contradiction to $(\Gamma_{\alpha_1, \ldots, \alpha_k}^w, \varphi_o, \varphi)$ being the pullback.

- Assume that $\langle v_i \rangle$ and $\langle v_j \rangle$ are conjugate subgroups of $M$ for some $i \neq j$, say without loss of generality that $v_i = mv_jm^{-1}$ (the other possibility being $v_i = mv_j^{-1}m^{-1}$) with $m \in M$. Then both $\varphi$ and $\varphi_o$ factor through the morphism $\Gamma_{\alpha_1, \ldots, \alpha_k}^w \to \Delta$ which maps $\langle w^{k_i} \rangle$ isomorphically to $\langle w^{k_i} \rangle$, and $\langle w^{k_i} \rangle$ to the same component, as

  $$\langle w^{k_i} \rangle = \langle u_i^{-1}v_iu_i \rangle = \langle u_i^{-1}mv_jm^{-1}u_i \rangle = u_i^{-1}m \langle v_j \rangle m^{-1}u_i.$$

  This, again, contradicts our assumption that $(\Gamma_{\alpha_1, \ldots, \alpha_k}^w, \varphi_o, \varphi_1)$ is itself the pullback.

- Assume there is a free splitting $M = M' \ast \langle v_i \rangle$ with all other $v_j$’s conjugate into $M'$. But then $\varphi_o$ factors through the free factor $\{M', \langle v_i \rangle \} \to \{M^F\}$, in contradiction to $\varphi_o$ being algebraic.

- Finally, our assumption that $\sum io_i \geq 2$ is equivalent to $\sum ni \geq 2$.

Let $\pi_1(\Delta)$ be the fundamental group of this graph of groups, namely,

$$\pi_1(\Delta) = \left\langle M, w, t_1, \ldots, t_s \mid t_iw_i t_i^{-1} = w^{k_i}, t_1 = 1 \right\rangle.$$

Consider, as in the proof of Proposition 7.5, the extension $J_{s+1} = \langle M, u_2, \ldots, u_s, w \rangle$ of $M$ inside $F$. The same argument as in that proof applies to show that $J_{s+1}$ is an algebraic extension of $M$. In particular, the composition of $\varphi$ with this algebraic extension $M \leq \pi_1(\Delta)$ gives an algebraic morphism from $\Gamma_{\alpha_1, \ldots, \alpha_k}^w$ to $\Gamma_{\alpha_1, \ldots, \alpha_k}^w$, which also factors through $\omega_1 : \Gamma_{\alpha_1, \ldots, \alpha_k}^w \to \Gamma_{\alpha_1, \ldots, \alpha_k}^w$. Thus $J_{s+1}$ is a proper algebraic extension of $w$, and hence $\text{rk} J_{s+1} \geq \pi(w)$.

On the other hand, $J_{s+1}$ is a free quotient of $\pi_1(\Delta)$ which extends an embedding of $M$: this is obtained by mapping $w$ to itself, $t_1 \mapsto 1$ and $t_i \mapsto u_i^{-1}$ for $i \geq 2$. By Louder’s Theorem 7.6, we have $\text{rk} J_{s+1} < \text{rk} M$, and so $\text{rk} M > \text{rk} J_{s+1} \geq \pi(w)$. This contradicts our assumption that $\text{rk} M = \pi(w)$.

**Proposition 7.8.** Let $1 \neq w \in F$ be a non-power. For every $H \in \text{Crit}(w)$, there are $\langle \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}, \xi_1 - 1 \rangle$ distinct critical morphisms in $\text{Crit}_{\alpha_1, \ldots, \alpha_k}(w)$ mapping a non-empty subset of the powers of $w$ to $w$ and then to $H$, and the remaining powers of $w$ to cyclic subgroups of $\langle w \rangle$.

As we already know from Proposition 7.7 that every morphism in $\text{Crit}_{\alpha_1, \ldots, \alpha_k}(w)$ corresponds to exactly one $H \in \text{Crit}(w)$, Proposition 7.8 yields that $|\text{Crit}_{\alpha_1, \ldots, \alpha_k}(w)| = \langle \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}, \xi_1 - 1 \rangle \cdot |\text{Crit}(w)|$.

**Proof.** One can give a direct argument, but we like the following one better. In the notation of Section 1, let $\kappa(f)$ denote the constant corresponding to the class function $f \in A$ in the equality

$$E_w[f] = \langle f, 1 \rangle + \kappa(f) \cdot \frac{|\text{Crit}(w)|}{N^{\pi(w)-1}} + O\left(\frac{1}{N^{\pi(w)}}\right).$$

We know such equality holds with some $\kappa(f) \in \mathbb{Q}$ because we already know by (7.2) that such constants exist for every $f = \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}$. 

37
By a theorem of Frobenius [Fro96], already mentioned on page 6, for any irreducible character \( \chi \) of any finite group \( G \), \( E_{[x,y]}[\chi] = \frac{1}{\dim \chi} \). By Proposition B.2, every class function \( f \in A \) is of the form

\[
f = \sum_{\chi \in \hat{S}_\infty} \langle f, \chi \rangle \chi
\]

with finitely many non-vanishing terms. Reverse-engineering Theorem 1.2 for \( w = [x, y] \) gives \( \pi ([x, y]) = 2 \) and \(|\text{Crit} ([x, y])| = 1 \). Hence,

\[
\sum_{\chi \in \hat{S}_\infty} \frac{\langle f, \chi \rangle}{\dim \chi} = E_{[x,y]}[f] = \langle f, 1 \rangle + \frac{\kappa(f)}{N} + O \left( \frac{1}{N^2} \right).
\]

As explained on page 6, for every \( \chi \in \hat{S}_\infty \), the dimension \( \dim \chi \) is a polynomial function of \( N \), which has degree \( \geq 2 \) if \( \chi \neq 1, \xi_1 - 1 \). Subtracting from the left hand side the summands corresponding to \( \chi = 1 \) and to \( \chi = \xi_1 - 1 \) leaves \( O \left( N^{-2} \right) \). Thus

\[
O \left( \frac{1}{N^2} \right) = \frac{\kappa(f)}{N} - \frac{\langle f, \xi_1 - 1 \rangle}{N - 1} = \frac{\kappa(f) - \langle f, \xi_1 - 1 \rangle}{N} + O \left( \frac{1}{N^2} \right).
\]

Thus \( \kappa(f) = \langle f, \xi_1 - 1 \rangle \).

This completes the proof of our main result, Theorem 1.3.

7.2.1 Some remarks

Remark 7.9. Proposition 7.8 teaches us that if \( \sum i\alpha_i \geq 2 \), then every algebraic morphism from \( \Gamma_{\alpha_1, \ldots, \alpha_k}^w \) with codomain of EC (Euler characteristic) \( 1 - \pi(w) \), actually originates from an algebraic morphism from \( \Gamma_1^w \). Similarly, there may be algebraic morphisms from \( \Gamma_{\alpha_1, \ldots, \alpha_k}^w \) originating from algebraic morphisms from a different \( \Gamma_{\beta_1, \ldots, \beta'_1}^w \) if there is a morphism from a subset of the connected components of \( \Gamma_{\alpha_1, \ldots, \alpha_k}^w \) to \( \Gamma_{\beta_1, \ldots, \beta'_1}^w \). Note that such \( \beta_1, \ldots, \beta'_t \) satisfy \( \sum j\beta_j < \sum i\alpha_i \). It is natural to consider algebraic morphisms from \( \Gamma_{\alpha_1, \ldots, \alpha_k}^w \) which cannot be constructed in this way. Namely, these are algebraic morphisms not having any connected component of EC 0 in their codomain and which do not factor through a (non-identity) morphism from \( \Gamma_{\alpha_1, \ldots, \alpha_k}^w \) with codomain of EC 0. For the sake of the current Section 7.2.1, call such algebraic morphisms \( (w; \alpha_1, \ldots, \alpha_k) \)-pure.

Let \( \eta: \Gamma_{\alpha_1, \ldots, \alpha_k}^w \to \Sigma \) be an algebraic morphism. Every connected component \( \Sigma_o \) of \( \Sigma \) with \( \eta|_{\eta^{-1}(\Sigma)} \) not an isomorphism, has the property that every edge of \( \Sigma_o \) is covered by at least two edges from \( \Gamma_{\alpha_1, \ldots, \alpha_k}^w \). This shows that \( (w; \alpha_1, \ldots, \alpha_k) \)-pure morphisms are reducible, in the sense of [LW17]. Thus, [LW17, Theorem 1.2] says that every \( (w; \alpha_1, \ldots, \alpha_k) \)-pure morphism has codomain of EC at most \( -\sum i\alpha_i \). In light of Conjecture 1.8 and its connection to algebraic morphisms as in (7.1), it is plausible to conjecture that a tight bound here should be \( (1 - \pi(w)) \cdot \sum i\alpha_i \). This yields the conjecture when \( \pi(w) = 2 \). In fact, this conjecture about \( (w; \alpha_1, \ldots, \alpha_k) \)-pure morphisms is precisely [LW21, Conjecture 1.5].

Remark 7.10. Louder’s Theorem 7.6 is strengthened in [LW22, Theorem 1.11] to the fact that under the same assumptions (except for \( \sum n_{ij} \geq 2 \) which can be discarded), the following inequality holds:

\[
\text{rk}J - 1 \leq \sum_i (\text{rk}H_i - 1) - \left( \sum_{i,j} n_{ij} - 1 \right).
\]

This implies that algebraic morphisms \( \Gamma_{\alpha_1, \ldots, \alpha_k}^w \to \Sigma \) such that \( \Sigma \) is connected with \( \chi(\Sigma) < 0 \), and where \( \Gamma_{\alpha_1, \ldots, \alpha_k}^w \) is itself the pullback of \( \Sigma \to \Sigma_0 \) and \( \Gamma_{\alpha_1, \ldots, \alpha_k}^w \to \Gamma_{\alpha_1, \ldots, \alpha_k}^w \), satisfy that \( \text{rk} \Sigma \geq \pi(w) - 1 + \sum_i i\alpha_i \). This may be relevant to strengthening Theorem 1.3 towards Conjecture 1.8.
8 Expansion of random Schreier graphs: the proof of Theorem 1.10

Fix \( s \in \mathbb{Z}_{\geq 1} \) and assume throughout that \( N \geq s \). Also, fix a basis \( B = \mathbb{F} = \mathbb{F}_r \). Let \( \sigma_1, \ldots, \sigma_r \in S_N \) be independent, uniformly random permutations, and let \( G = G(\sigma_1, \ldots, \sigma_r) \) be the \( d = 2r \)-regular Schreier graph depicting the action of \( S_N \) on \( ([N])_s \), the set of \( s \)-tuples of distinct elements in \( [N] \), with respect to \( \sigma_1, \ldots, \sigma_r \). This is a graph with \( (N)_s = N(N-1)\cdots(N-s+1) \) vertices. In this section we prove Theorem 1.10, stating that the random graph \( G \) is a.a.s. an expander with a spectral bound as given in (1.4). Namely, the largest absolute value of a non-trivial eigenvalue of \( A_G \), the adjacency matrix of \( G \), satisfies a.a.s. \( \mu(G) \leq 2\sqrt{d-1} \cdot \exp \left( \frac{2s^2}{\pi^2(d-1)} \right) \).

We follow the strategy laid out in [Pud15] and its addendum [FP21]. The strategy is based on the trace method together with the results from [PP15]. As in [FP21], instead of analyzing directly the regular adjacency operator, we analyze the non-backtracking spectrum and only at the end of the argument deduce a bound on \( \mu(G) \).

Denote by \( \overrightarrow{E} \) the set of oriented edges of \( G \), namely, each edge of \( G \) appears twice in this set, once with every possible orientation, so \( |\overrightarrow{E}| = (N)_s \cdot d \). For \( e \in \overrightarrow{E} \), we denote by \( \overrightarrow{e} \) the same edge with the reverse orientation, and by \( h(e) \) and \( t(e) \) the head and tail of \( e \), respectively. The Hashimoto or non-backtracking matrix \( B = B_G \) is a \( (\overrightarrow{E}) \times (\overrightarrow{E}) \) 0-1 matrix with rows and columns indexed by the elements of \( \overrightarrow{E} \). The \( e, f \) entry is defined by

\[
B_{e,f} = \begin{cases} 
1 & \text{if } t(e) = h(f) \text{ and } f \neq \overrightarrow{e}, \\
0 & \text{otherwise}.
\end{cases}
\]

The Ihara-Bass formula gives a dictionary between the spectrum of \( B_G \) and that of the adjacency matrix \( A_G \). Every eigenvalue \( \lambda \in \text{Spec}(A_G) \) with \( |\lambda| \geq 2\sqrt{d-1} \) gives rise to two real eigenvalues of \( B_G \) in \( (-d-1, 1] \cup [1, d-1] \), while every eigenvalue with \( |\lambda| < 2\sqrt{d-1} \) corresponds to two non-real eigenvalues lying on the circle of radius \( \sqrt{d-1} \) around 0 in \( \mathbb{C} \). In both cases, the two eigenvalues of \( B_G \) are given by \( \frac{\lambda \pm \sqrt{\lambda^2 - 4(d-1)}}{2} \in \text{Spec}(B_G) \). In particular, the trivial eigenvalue \( d \in \text{Spec}(A_G) \) corresponds to \( 1, d-1 \in \text{Spec}(B_G) \), which are considered to be trivial eigenvalues of \( B_G \). In addition, there are \( (d-2) \cdot (N)_s \) additional \( \pm 1 \) eigenvalues. For more details see [FP21, Section 2] and the references therein.

Order the eigenvalues of \( B_G \) by their absolute value to get

\[
d - 1 = |\nu_1| \geq |\nu_2| \geq \ldots \geq |\nu_{2(N)_s+1}| = \ldots = |\nu_{d(N)_s}| = 1.
\]

We let \( \nu(G) \overset{\text{def}}{=} |\nu_2| \) denote the largest absolute value of a non-trivial eigenvalue. If \( (N)_s \geq 2 \) then \( \nu(G) \in [\sqrt{d-1}, d-1] \). Notice that if \( \nu(G) > \sqrt{d-1} \), in which case \( \nu_2 \) is real, then

\[
\mu(G) = \nu(G) + \frac{d - 1}{\nu(G)}.
\]

Our immediate goal is to bound \( \nu(G) \) from above. The trace of the \( t \)-th power \( B_G^t \) of the Hashimoto matrix \( B_G \) is equal to the number of cyclically non-backtracking closed walks of length \( t \) in \( G \). As every edge in \( G \) is directed and corresponds to one of \( \sigma_1, \ldots, \sigma_r \), a cyclically non-backtracking closed oriented path of length \( t \) corresponds to a cyclically reduced word of length \( t \) in \( \{ \sigma_1\pm 1, \ldots, \sigma_r\pm 1 \} \). In other words, every such path corresponds to \( w(\sigma_1, \ldots, \sigma_r) \) where \( w \in \mathbb{F} = \mathbb{F}_r \) is cyclically reduced and of length \( t \). Denote the set of cyclically reduced words of length \( t \) in \( \mathbb{F}_r \) by \( \mathcal{CR}_t(\mathbb{F}_r) \). The number of closed paths corresponding to a given such \( w \) is equal to the number of \( s \)-tuples in \( ([N])_s \) fixed by \( w(\sigma_1, \ldots, \sigma_r) \in S_N \). Denote by \( \chi_s \) the (reducible) character corresponding to this permutation-representation of \( S_N \). Then the number of closed paths in \( G \) corresponding to \( w \) is \( \chi_s(w(\sigma_1, \ldots, \sigma_r)) \). We have:

\[
\sum_{i=1}^{d(N)_s} \nu_i^t = \text{tr}(B_G^t) = \sum_{w \in \mathcal{CR}_t(\mathbb{F}_r)} \chi_s(w(\sigma_1, \ldots, \sigma_r)).
\]

(8.3)
There is an exact formula for the number of such words:

**Proposition 8.1.** [Riv10, Theorem 1.1] The number of cyclically reduced words of length \( t \) in \( \mathbb{F}_r \) is

\[
|\mathfrak{C}_t(\mathbb{F}_r)| = (2r - 1)^t + r + (-1)^t (r - 1).
\]

(This is also Proposition 17.2 in [Man11].) So if \( t \) is even, \( |\mathfrak{C}_t(\mathbb{F}_r)| = (d - 1)^t + (d - 1) \). In addition, if \( t \) is even, for every real eigenvalue \( \nu \), the summand \( \nu^t \) is positive. Since every non-real eigenvalue \( \nu \) lies on \( \{ z \in \mathbb{C} : |z| = \sqrt{d - 1} \} \), the summand \( \nu^t \) in this case has real part at least \(-\sqrt{d - 1}^t\). Recall also that there is a trivial eigenvalue \( \nu_1 = d - 1 \) and that (at least) \((N)_{s} \cdot (d - 2) + 1\) out of the \((N)_{s} \cdot d\) eigenvalues are \( \pm 1 \). Hence, for \( t \) even we have

\[
\operatorname{tr} \left( B_C^t \right) = (d - 1)^t + \sum_{i=2}^{2(N)_{s} - 1} \nu_i^t + (N)_{s} \cdot (d - 2) + 1.
\]

Taking real parts we have

\[
\operatorname{Re} \left[ \nu_2 (\Gamma)^t \right] = \operatorname{tr} \left( B_C^t \right) - (d - 1)^t - \sum_{i=3}^{2(N)_{s} - 1} \operatorname{Re} \left[ \nu_i^t \right] - (N)_{s} \cdot (d - 2) - 1
\]

\[
\leq \left[ \sum_{w \in \mathfrak{C}_t(\mathbb{F}_r)} \chi_s \left( w \left( \sigma_1, \ldots, \sigma_r \right) \right) \right] - (d - 1)^t + 2(N)_{s} \sqrt{d - 1}^t - (N)_{s} (d - 2) - 1
\]

Prop. 8.1

\[
\leq \left[ \sum_{w \in \mathfrak{C}_t(\mathbb{F}_k)} \left[ \chi_s \left( w \left( \sigma_1, \ldots, \sigma_r \right) \right) - 1 \right] \right] + d - 1 + 2(N)_{s} \sqrt{d - 1}^t - (N)_{s} (d - 2) - 1
\]

\[
\leq \left[ \sum_{w \in \mathfrak{C}_t(\mathbb{F}_k)} \left[ \chi_s \left( w \left( \sigma_1, \ldots, \sigma_r \right) \right) - 1 \right] \right] + 2(N)_{s} \sqrt{d - 1}^t.
\]

Taking expectations we obtain

\[
\mathbb{E} \left[ \operatorname{Re} \left[ \nu_2 (\Gamma)^t \right] \right] \leq \left[ \sum_{w \in \mathfrak{C}_t(\mathbb{F}_k)} \left( \mathbb{E}_w \left[ \chi_s \right] - 1 \right) \right] + 2(N)_{s} \sqrt{d - 1}^t. \tag{8.4}
\]

We can finally use our main results from the current paper. For \( N \geq s \), the action of \( S_N \) on \((\mathbb{N})_{s}\) is transitive, and so the expected number of fixed points is \( \langle \chi_s, 1 \rangle = 1 \). Corollary 1.4 therefore gives

\[
\mathbb{E}_w \left[ \chi_s \right] - 1 = \langle \chi_s, \xi_1 - 1 \rangle \cdot \left[ \frac{\left| \operatorname{Crit} (w) \right|}{N \pi(w) - 1} + O \left( \frac{1}{N \pi(w)} \right) \right]. \tag{8.5}
\]

To proceed, we estimate the number of words in \( \mathfrak{C}_t(\mathbb{F}_r) \) of a given primitivity rank, and then provide a bound of the big-\( O \) term in (8.5) in a uniform manner across all words of a given length and a given primitivity rank. The first of these tasks is given by [Pud15]:

**Theorem 8.2.** [Pud15, Proposition 4.3 and Theorem 8.2] For every \( r \geq 2 \) and \( m \in \{1, \ldots, r\} \),

\[
\limsup_{t \to \infty} \left[ \sum_{w \in \mathfrak{C}_t(\mathbb{F}_r): \pi(w) = m} |\operatorname{Crit}(w)| \right]^{1/t} = \max \left( \sqrt{2r - 1}, 2m - 1 \right). \tag{8.6}
\]
Remark 8.3. These counting results in [Pud15] are stated for reduced, but not necessarily cyclically reduced, words. However, the proof also applies to the slightly smaller set of cyclically reduced words, and, besides, we only use here the inequality \( \leq \) which obviously follows from the original statements in [Pud15]. We also remark that for \( m \in \{2, \ldots, r\} \), the equality (8.6) holds with ordinary limit instead of limsup, and for \( m = 1 \), it holds with ordinary limit on even values of \( t \).

Theorem 8.2 does not cover the cases \( \pi(w) = 0 \) and \( \pi(w) = \infty \). But \( \pi(w) = 0 \) if and only if \( w = 1 \) so this is irrelevant in \( C^R_t(F_r) \). The other extreme, \( \pi(w) = \infty \), holds if and only if \( w \) is primitive, in which case \( E_w[\chi_s] = E_{unif}[\chi_s] = 1 \), so these words contribute nothing to the summation (8.4). (The exponential growth rate of primitive words is \( 2r - 3 \) – see [PW14].)

The second task, of a uniform bound on the big-\( O \) term in Corollary 1.4 and in (8.5), is given by the following proposition.

**Proposition 8.4.** Let \( f \in A \) be a class function. Then there are constants \( A, D \geq 1 \) such that for every word of length \( t \), and any \( N > (At)^2 \),

\[
\left| E_w[f] - \langle f, 1 \rangle - \langle f, \xi_1 - 1 \rangle \cdot \frac{|\text{Crit}(w)|}{N^{\pi(w) - 1}} \right| \leq \frac{(A \cdot t)^{2(\pi(w) + 2D)}}{N^{\pi(w) - 1} \cdot (N - (At)^2)}.
\] (8.7)

We first prove a version of Proposition 8.4 for the monomials \( \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k} \) – see Lemma 8.6. Using the notation of Section 7, recall that \( E_w[\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}] = \Phi_{\alpha_1, \ldots, \alpha_k} \), and that \( \Phi_{\eta} = \sum_{(n_1, n_2) \in \text{Decomp}_{\eta}(B)} L^B_{n_2}. \) By Proposition 6.8 and Corollary 6.9, the functions \( \Phi_{\eta} \) and \( L^B_{\eta} \) are equal to a rational expression in \( N \) (with rational coefficients) for every large enough \( N \). In particular, these functions are given by power series in \( \frac{1}{N} \). We shall use the following lemma.

**Lemma 8.5.** Let \( \eta : \Gamma \to X_B \) be a \( B \)-surjective morphism of multi core graphs. Assume that \( |V(\Gamma)| \leq T \) and that \( |E(\Gamma)| \leq T \). Then the coefficient of \( N^{\chi(\Gamma) - p} \) in the power series expansion of \( L^B_{\eta}(N) \) is bounded in absolute value by \( T^{2p} \).

**Proof.** By Proposition 6.11,

\[
L^B_{\eta}(N) = \sum_{t \geq 0} \sum_{j_0 \geq 0; j_1, \ldots, j_t \geq 1} (-1)^{t + \sum_{i=0}^t j_i} [V(\Gamma)]_{j_0}^n \cdot [E(\Gamma)]_{j_1}^n \cdot \ldots \cdot [E(\Gamma)]_{j_t}^n \cdot N^{\chi(\Gamma) - \sum_{i=0}^t j_i}.
\]

The coefficient of \( N^{\chi(\Gamma) - p} \) is thus

\[
b_p \overset{\text{def}}{=} \sum_{t \geq 0} \sum_{j_0 \geq 0; j_1, \ldots, j_t \geq 1: \sum j_i = p} (-1)^{t + \sum j_i} [V(\Gamma)]_{j_0}^n \cdot [E(\Gamma)]_{j_1}^n \cdot \ldots \cdot [E(\Gamma)]_{j_t}^n.
\] (8.8)

We proceed by induction on \( p \) and ignore the signs in (8.8). For \( p = 0 \), \( b_0 = 1 \). Note that

\[
[V(\Gamma)]_1^j, [E(\Gamma)]_1^j \leq \left( \frac{T}{2} \right)^j \leq \left( \frac{T^{2j}}{2^j} \right),
\]

since any permutation counted by these numbers is the product of \( j \) cycles of length 2, and the number of vertices and edges of the graph is bounded by \( T \). Therefore, the \( t = 0 \) term of (8.8) is bounded by \( T^{2p} / 2^{2p} \).

For \( t \geq 1 \) we put aside the term \([E(\Gamma)]_{jt}^n\) to obtain

\[
b_p = \frac{T^{2p}}{2^{2p}} + \sum_{j=1}^{p} \left[ E(\Gamma) \right]_1^j \cdot \sum_{t \geq 1} \sum_{j_0 \geq 0; j_1, \ldots, j_{t-1} \geq 1: \sum j_i = p-j} (-1)^{t+j+\sum j_i} [V(\Gamma)]_{j_0}^n \cdot [E(\Gamma)]_{j_1}^n \cdot \ldots \cdot [E(\Gamma)]_{j_{t-1}}^n
\]

\[
= \frac{T^{2p}}{2^{2p}} + \sum_{j=1}^{p} \left[ E(\Gamma) \right]_1^j \cdot (-1)^j b_{p-j}.
\]
Lemma 8.6. Let \( w \in \mathfrak{C}(F_r) \) and fix \( k \geq 1 \) and \( \alpha_1, \ldots, \alpha_k \geq 0 \), not all zeros. Let \( T = t \cdot \sum \alpha_i \) denote the number of edges and the number of vertices in the multi core graph \( \Gamma_{\alpha_1, \ldots, \alpha_k}^w \) and let \( D = \sum \alpha_i \) denote the number of connected components in this graph. Then, for all \( N > T^2 \),

\[
|\mathbb{E}_w[\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}] - \langle \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}, 1 \rangle - \langle \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}, \xi_1 - 1 \rangle \cdot \frac{|\text{Crit}(w)|}{N^\pi(w) - 1} | \leq \frac{T^{2\pi(w) + 2D}}{N^\pi(w) - 1 \cdot (N - T^2)}.
\]

Proof. By Proposition 6.8 and Corollary 6.9, \( \mathbb{E}_w[\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}] \) is equal to a rational expression in \( N \) with rational coefficients for large enough \( N \) (in fact, \( N \geq T^2 \) suffices), and by Theorem 1.3 its degree is zero. In particular, it is equal to a power series in \( \frac{1}{N^p} \). Denote

\[
\mathbb{E}_w[\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}] = \sum_{p=0}^{\infty} \frac{a_p}{N^p}.
\]

By Theorem 1.3, \( a_0 = \langle \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}, 1 \rangle \), \( a_1 = \ldots = a_{\pi(w) - 2} = 0 \), and \( a_{\pi(w) - 1} = \langle \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}, \xi_1 - 1 \rangle \cdot \frac{|\text{Crit}(w)|}{N^\pi(w) - 1} \). In this notation, our goal is to bound \( \sum_{p=\pi(w)}^{\infty} \frac{a_p}{N^p} \). We claim that for every \( p \geq \pi(w) \) we have \( |a_p| \leq T^{2p + 2D} \). Assuming this inequality,

\[
\sum_{p=\pi(w)}^{\infty} \frac{a_p}{N^p} \leq \sum_{p=\pi(w)}^{\infty} \frac{T^{2p + 2D}}{N^p} = \frac{T^{2\pi(w) + 2D}}{N^\pi(w) - 1} \cdot \frac{1}{1 - \frac{T^2}{N}} = \frac{T^{2\pi(w) + 2D}}{N^\pi(w) - 1 \cdot (N - T^2)},
\]

as required. It remains to prove that \( |a_p| \leq T^{2p + 2D} \). Assume without loss of generality that \( \eta_{\alpha_1, \ldots, \alpha_k}^w : \Gamma_{\alpha_1, \ldots, \alpha_k}^w \rightarrow X_B \) is onto – otherwise, work in a free factor of \( F_r \), generated by a suitable subset of \( B \). As

\[
\mathbb{E}_w[\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}] = \sum_{(\eta_1, \eta_2) \in \text{Decomp}_B(\eta_{\alpha_1, \ldots, \alpha_k}^w)} L_{\eta_2}^B(\eta_{\alpha_1, \ldots, \alpha_k}^w),
\]

for every decomposition \( (\eta_1, \eta_2) \) as in (8.9), \( \text{Im}(\eta_1) \) has at most \( T \) vertices and at most \( T \) edges, so by Lemma 8.5,

\[
L_{\eta_2}^B(N) \leq N^{\chi(\text{Im}(\eta_1))} \sum_{q=0}^{\infty} \frac{T^{2q}}{N^q}.
\]

In particular, the coefficient of \( N^{-p} \) is \( T^{2p + 2\chi(\text{Im}\eta_1)} \) or 0 if \( \chi(\text{Im}\eta_1) < -p \). Summing these over all decompositions gives

\[
|a_p| \leq \sum_{c=-p}^{0} \sum_{\eta_{\alpha_1, \ldots, \alpha_k}^w \in \text{Decomp}_B(\eta_{\alpha_1, \ldots, \alpha_k}^w)} \chi(\text{Im}(\eta_1)) = c \cdot T^{2p + 2c}.
\]

\[
\leq \sum_{c=-p}^{0} \left( \frac{T}{2} \right)^{D-c} \cdot T^{2p + 2c} \leq \sum_{c=-p}^{0} \frac{T^{2D - 2c}}{2^{D-c}} \cdot T^{2p + 2c} = T^{2D + 2p} \cdot \sum_{c=-p}^{0} 2^c \leq T^{2D + 2p},
\]

where the second inequality is by Proposition 5.7 and the fact that \( \Gamma_{\alpha_1, \ldots, \alpha_k}^w \) has \( D \) components. \( \square \)
Proof of Proposition 8.4. By definition, every \( f \in \mathcal{A} \) is a finite linear combination of the form
\[
f = \sum_{k, \alpha_1, \ldots, \alpha_k} \beta_{\alpha_1, \ldots, \alpha_k} \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}
\] (8.10)
with \( \beta_{\alpha_1, \ldots, \alpha_k} \in \mathbb{R} \). All the terms in the bounded expression in (8.7) are linear, so it is bounded by the corresponding linear combinations of bounds from Lemma 8.6. Set \( D \) to be the maximal value of \( \sum \alpha_i \) over the non-vanishing monomials in (8.10), and \( A_0 \) to be the maximal value of \( \sum i \alpha_i \). Then
\[
\left| \mathbb{E}_w [f] - \langle f, 1 \rangle - \langle f, \xi_1 - 1 \rangle \cdot \frac{\text{Crit}(w)}{N^{\pi(w)-1}} \right| \leq \sum_{k, \alpha_1, \ldots, \alpha_k} |\beta_{\alpha_1, \ldots, \alpha_k}| \cdot \frac{(A_0 t)^{2\pi(w)+2D}}{N^{\pi(w)-1} \left( N - (At)^2 \right)}
\]
\[
\leq \frac{(At)^{2\pi(w)+2D}}{N^{\pi(w)-1} \left( N - (At)^2 \right)}
\]
with \( A = A_0 \cdot \max \left( \sum_{k, \alpha_1, \ldots, \alpha_k} |\beta_{\alpha_1, \ldots, \alpha_k}|, 1 \right) \).

We now have all the ingredients needed to prove Theorem 1.10.

Proof of Theorem 1.10. Recall our notation of \( \chi_s \) from the beginning of this section and that \( \langle \chi_s, 1 \rangle = 1 \). The value of \( \langle \chi_s, \xi_1 - 1 \rangle \) is \( s \): this is a suitable Kostka number [Sta99, Proposition 7.18.7], but any constant suffices for our needs. From Proposition 8.4 it now follows that there are \( A, D \geq 1 \) with
\[
\mathbb{E}_w [\chi_s] \leq 1 + \frac{1}{N^{\pi(w)-1}} \left( s \cdot \text{Crit}(w) + \frac{(At)^{2r+2D}}{N - A^2t^2} \right).
\]
We now bound the summation from (8.4) (see also Remark 8.3):
\[
\sum_{w \in \mathcal{C}_t(F_k)} (\mathbb{E}_w [\chi_s] - 1) = \sum_{m=1}^{r} \sum_{w \in \mathcal{C}_t(F_k): \pi(w)=m} (\mathbb{E}_w [\chi_s] - 1)
\]
\[
\leq \sum_{m=1}^{r} \frac{1}{N^{\pi(w)-1}} \sum_{w \in \mathcal{C}_t(F_k): \pi(w)=m} \left( s \cdot \text{Crit}(w) + \frac{(At)^{2r+2D}}{N - A^2t^2} \right)
\]
\[
\leq \sum_{m=1}^{r} \sum_{w \in \mathcal{C}_t(F_k): \pi(w)=m} s \cdot \text{Crit}(w) \left( 1 + \frac{(At)^{2r+2D}}{N - A^2t^2} \right)
\]
\[
= \left( 1 + \frac{(At)^{2r+2D}}{N - A^2t^2} \right) s \sum_{m=1}^{r} \frac{1}{N^{\pi(w)-1}} \sum_{w \in \mathcal{C}_t(F_k): \pi(w)=m} \text{Crit}(w)
\]
Thm 8.2
\[
\leq \left( 1 + \frac{(At)^{2r+2D}}{N - A^2t^2} \right) s \sum_{m=1}^{r} \frac{1}{N^{\pi(w)-1}} \left[ \max (\sqrt{2r-1}, 2m-1) + \varepsilon \right]^t,
\]
where the last inequality holds for every \( \varepsilon > 0 \) and every large enough \( t \). So under the same assumptions on \( \varepsilon \) and \( t \), we get from (8.4)
\[
\mathbb{E} \left[ \text{Re} \left[ \nu_2 (G)^t \right] \right] \leq 2 \left( N s \right) \sqrt{d - 1} + \left( 1 + \frac{(At)^{2r+2D}}{N - A^2t^2} \right) s \sum_{m=1}^{r} \frac{1}{N^{\pi(w)-1}} \left[ \max (\sqrt{2r-1}, 2m-1) + \varepsilon \right]^t. \quad (8.11)
\]
We will soon take \( t \) to be a function of \( N \) so that as \( N \to \infty \), \( N^{1/t} \to c \) for a constant \( c \) specified below. Then for every \( \varepsilon > 0 \) and every large enough \( N \),
\[
\left( 1 + \frac{(At)^{2r+2D}}{N - A^2t^2} \right) s \cdot 2 (r + 1) \leq (1 + \varepsilon)^t.
\]
Because the right hand side of (8.11) is at most \((r + 1)\) times the maximal summand (among the \(r + 1\) summands), we get that for every \(\varepsilon > 0\) and large enough \(N\),

\[
E \left[ \text{Re} \left[ \nu_2 (G)^t \right] \right] \leq \left[ (1 + \varepsilon) \cdot \max \left( \left\{ N^{s/t} \sqrt{d-1} \right\} \cup \left\{ \frac{2m - 1}{N^{(m-1)/t}} \mid 2m - 1 \in \left[ \sqrt{d-1}, d - 1 \right] \right\} \right) \right]^t,
\]  
(8.12)

where we used the observation that if \(2m - 1 < \sqrt{d-1}\) then the term corresponding to \(m\) in (8.11) is \(\frac{2m - 1}{N^{(m-1)/t}}\), and is thus strictly smaller than the first term \(2 (N)_s \sqrt{d-1}\). A simple analysis yields that, at least for large values of \(d\), the optimal value of \(t = t (N)\) is such that

\[
N^{1/t} \to \frac{2}{\sqrt{d-1}}
\]
as \(N \to \infty\). Whenever \(2m - 1 \in \left[ \sqrt{d-1}, d - 1 \right]\), write \(m = \beta \sqrt{d-1}\) with \(\beta > \frac{1}{2}\). If \(N^{1/t} \to \frac{2}{\sqrt{d-1}}\) then

\[
\frac{2m - 1}{N^{(m-1)/t}} = \frac{2\beta \sqrt{d-1} - 1}{(N^{1/t})^{\beta \sqrt{d-1} - 1}} < N^{1/t} \sqrt{d-1} \cdot \frac{2\beta}{(N^{1/t})^{\beta \sqrt{d-1}}}
\]

\[
\approx \frac{N^{1/t} \sqrt{d-1} \cdot \frac{2\beta}{e^{2\beta}}} \leq N^{1/t} \sqrt{d-1},
\]

where the last inequality follows as \(\frac{2\beta}{e^{2\beta}} \leq 1\) with equality if and only if \(\beta = e/2\). Therefore, with this value of \(t\), we obtain from (8.12) that for every \(\varepsilon > 0\),

\[
E \left[ \text{Re} \left[ \nu_2 (G)^t \right] \right] \leq \left[ (1 + \varepsilon) \cdot \sqrt{d-1} \cdot N^{s/t} \right]^t \approx \left[ (1 + \varepsilon) \cdot \sqrt{d-1} \cdot e^{\frac{2\beta}{e}} \right]^t
\]

for every large enough \(N\). Recall that if \(\nu_2 (G)\) is non-real, then it has absolute value \(\sqrt{d-1}\), and so we always have \(\text{Re} \left[ \nu_2 (G)^t \right] \geq -\sqrt{d-1}^t\) for \(t\) even. Therefore, for \(x = \frac{2s}{e \sqrt{d-1}}\),

\[
\text{Prob} \left\{ \nu (G) \geq (1 + 2\varepsilon) \cdot e^x \sqrt{d-1} \cdot \left[ (1 + 2\varepsilon) e^x \sqrt{d-1} \right]^t \right\} \leq \text{Prob} \left\{ \nu (G) \geq (1 + 2\varepsilon) \cdot e^x \sqrt{d-1} \right\} \leq \left[ (1 + \varepsilon) e^x \sqrt{d-1} \right]^t,
\]

which yields that for every \(\varepsilon > 0\)

\[
\text{Prob} \left\{ \nu (G) \geq (1 + 2\varepsilon) \cdot \sqrt{d-1} \cdot e^x \right\} \to 0 \quad \text{as} \quad N \to \infty.
\]

Finally, by (8.2), when \(\nu (G) > \sqrt{d-1}\), we have that \(\mu (G) = \nu (G) + \frac{d - 1}{\nu (G)}\), and as \(e^x + e^{-x} < 2e^{x^2/2}\) for \(x > 0\), we conclude that

\[
\text{Prob} \left\{ \mu (G) \geq 2\sqrt{d-1} \cdot e^{\frac{2 \varepsilon^2}{(d-1)}} \right\} \to 0 \quad \text{as} \quad N \to \infty.
\]

\[\square\]

**Remark 8.7.** For a fixed value of \(r\) (or equivalently \(d\)), our bound on \(\mu (G)\) gets weaker as \(s\) grows. Assuming Conjecture 1.8, we could improve this bound to be independent of \(s\). Indeed, we could decompose the character \(\chi_s\) into a sum of irreducible characters. Each character of degree \(\theta (N^m)\) corresponds to \(O(N^m)\) eigenvalues, and if indeed \(E_w [\chi] = O(N^m (1 - \pi(w)))\), we could choose \(t\) separately for each \(m\), such that \(N^{m/t} \to e^{\frac{2\varepsilon^2}{(d-1)}}\), and obtain the bound \(\nu (G) < \sqrt{d-1} \cdot e^{\frac{2\varepsilon^2}{(d-1)}}\) a.a.s. as \(N \to \infty\).
A Conjugacy separability of free groups

We use our results to give a simple proof of the known fact that free groups are conjugacy separable.

**Definition A.1.** A group $G$ is said to be **conjugacy separable** if for every two distinct conjugacy classes $g^G \neq h^G$ of elements of $G$ there exists some homomorphism $G \xrightarrow{\phi} Q$ to a finite group $Q$ such that $\phi(g)^Q \neq \phi(h)^Q$.

It is known that finitely generated free groups are conjugacy separable – see, for example, [Bau65, Page 278] or [LS77, Proposition I.4.8]. We give another simple proof of this fact.

**Proposition A.2.** Finitely generated free groups are conjugacy separable.

**Proof.** Assume that $u, v \in F = F_r$ are conjugate under every homomorphism to a finite group $Q$. We write $u, v$ as maximal powers in the free group, that is, $u = u_0^k, v = v_0^m$, where $u_0, v_0$ are non-powers. Recall that $\tau(k)$ marks the number of positive divisors of $k$. We assume without loss of generality that $\tau(k) \geq \tau(m)$.

Using $Q = S_N$, we see that for every $r$ permutations $\sigma_1, \ldots, \sigma_r \in S_N$, the resulting permutations $u(\sigma_1, \ldots, \sigma_r)$ and $v(\sigma_1, \ldots, \sigma_r)$ are conjugate. In particular, they have the same number of fixed points. It follows that

$$\Phi\{\langle u \rangle^F, \langle v \rangle^F\}_F = \mathbb{E}_{\sigma_1, \ldots, \sigma_r \in S_N} \left[ \#\text{fix}(u) \cdot \#\text{fix}(v) \right] = \mathbb{E} \left[ \#\text{fix}(u)^2 \right] = \Phi\{\langle u \rangle^F, \langle u \rangle^F\}_F.$$

As a simple consequence of Theorem 7.2 we obtain\(^{11}\),

$$\Phi\{\langle u \rangle^F, \langle u \rangle^F\}_F = \tau(k)^2 + \sum_{d|k} d + O \left( \frac{1}{n} \right) \geq \tau(k)^2 + 1 + O \left( \frac{1}{n} \right). \quad (A.1)$$

On the other hand, the decompositions of $\{\langle u \rangle^F, \langle v \rangle^F\} \rightarrow H \rightarrow F$ with $\chi(H) = 0$ and $c(H) = 2$ (two connected components) are in one to one correspondence with pairs of positive roots of $u$ and $v$. Therefore, there are $\tau(k) \cdot \tau(m) \leq \tau(k)^2$ such decompositions. It follows that there is at least one decomposition $\{\langle u \rangle^F, \langle v \rangle^F\} \rightarrow H \rightarrow F$ with $\chi(H) = 0$ and $c(H) = 1$, namely $H = \{\langle w \rangle^F\}$ for some $1 \neq w \in F$. Therefore, some conjugates of $u$ and $v$ belong to $\langle w \rangle$, which yields that $u_0$ and $v_0$ are conjugate. It remains to show that $k = m$. Indeed, it follows that

$$\Phi\{\langle u \rangle^F, \langle v \rangle^F\}_F = \tau(k) \cdot \tau(m) + \sum_{d|\gcd(k,m)} d + O \left( \frac{1}{n} \right),$$

and therefore,

$$\tau(k)^2 + \sum_{d|k} d = \tau(k) \cdot \tau(m) + \sum_{d|\gcd(k,m)} d.$$

It follows that $k = m$, and so $u$ and $v$ are conjugate.

**Remark A.3.** This proof is even simpler for non-powers: in this case there are no non-trivial decompositions of the morphisms $\{\langle u \rangle \rightarrow F, \{\langle v \rangle^F\} \rightarrow F$ with Euler characteristic 0.

---

\(^{11}\)In fact, while (A.1) can be derived directly from Theorem 7.2, it is a much simpler fact. Using left Möbius inversion, one can easily see that for $\eta : \Gamma \rightarrow \Delta, L_\eta^N = N^{\chi(\Gamma)}(1 + O(\frac{1}{n}))$, and therefore one simply needs to count decompositions of $\{\langle u \rangle^F, \langle u \rangle^F\} \rightarrow F$ as $\{\langle u \rangle^F, \langle u \rangle^F\} \rightarrow H \rightarrow F$ with $\chi(H) = 0$. A similar argument applies to computing free coefficient of $\Phi\{\langle u \rangle^F, \langle v \rangle^F\}_F$. The method for solving such counting problems, albeit not explicitly with multi core graphs involving (powers of) different words, appears already in [Nic94] and especially in [LP10].
B The ring of class functions

Recall that for any $N$ and any permutation $\sigma \in S_N$, $\xi_k(\sigma)$ denotes the number of fixed points of $\sigma^k$, and $a_t(\sigma)$ denotes the number of $t$-cycles in $\sigma$. As in Section 1, we consider $A = \mathbb{Q}[\xi_1, \xi_2, \ldots]$, the ring of formal polynomials in the countably many variables $\xi_k$. Every element of $A$ is a class function defined on $S_N$ for every $N$. Note that as class functions on $S_N$,

$$\xi_k = \sum_{t \mid k} t \cdot a_t.$$ 

Therefore, $A$ can be equivalently defined as $A = \mathbb{Q}[a_1, a_2, \ldots]$. The following proposition shows that for any $f, g \in A$, the inner product $\langle f, g \rangle_{S_N}$ stabilizes for large enough $N$, and therefore our definition in Section 1 of $\langle f, g \rangle$ (as the constant value obtained for large $N$) makes sense.

**Proposition B.1.** For every two class functions $f, g \in A$, and for all large enough $N$, $\langle f, g \rangle_{S_N}$ is independent of $N$.

**Proof.** By the previous paragraph, $f$ and $g$ are equal to polynomials in the $a_t$’s. Thus, it is enough to prove the proposition when $f$ and $g$ are monomials in the $a_t$’s. Note that $\langle f, g \rangle_{S_N} = \langle f, g, 1 \rangle_{S_N}$, so it is enough to show that for every monomial $m$ in the $a_t$’s, $\langle m, 1 \rangle_{S_N}$ stabilizes. But [DS94, Theorem 7] states that for every $b_1, \ldots, b_k \in \mathbb{Z}_{\geq 0}$, and for every $^{12} N \geq \sum_{t=1}^{k} t b_t$

$$\left\langle a_{t_1}^{b_1} \cdots a_{t_k}^{b_k}, 1 \right\rangle_{S_N} = \prod_{t=1}^{k} \mathbb{E} \left[ Z_t^{b_t} \right],$$

where $Z_t$ is Poisson with parameter $\frac{1}{t}$. In particular, $\left\langle a_{t_1}^{b_1} \cdots a_{t_k}^{b_k}, 1 \right\rangle_{S_N}$ is constant for $N \geq \sum_{t=1}^{k} t b_t$. \qed

It is well known that there is a natural correspondence between partitions $\lambda$ of $N$ and irreducible representations of $S_N$. We denote the character corresponding to $\lambda$ by $\chi^\lambda$. Recall our notation from Section 1 and particularly Section 1.6 of $|\lambda|$ (the sum of blocks in $\lambda$), of $\chi^\lambda(\rho)$ where $\rho \vdash |\lambda|$ (the value of $\chi^\lambda$ on permutations with cycle structure $\rho$) and of $z_{\lambda} \overset{\text{def}}{=} \prod_{r} \alpha_r^{\lambda_r}$, where $\lambda$ has $\alpha_r$ parts of size $r$. Also recall from Section 1 that every partition $\lambda$ gives rise to a family of irreducible characters $\chi = \{\chi_N\}_{N \geq |\lambda| + \lambda_1}$, and that $\hat{S}_\infty$ denotes the family of such families of irreducible characters.

**Proposition B.2.** Every $\chi \in \hat{S}_\infty$ corresponds to an element of $A$, namely, $\chi_N$ and this element of $A$ coincide as class functions on $S_N$ for every $N \geq |\lambda| + \lambda_1$. Moreover, the elements of $A$ corresponding to the elements of $\hat{S}_\infty$ constitute a linear basis of $A$.

**Proof.** Our proof relies on results from [Mac98]. For a partition $\lambda$, denote by $\ell(\lambda)$ the number of parts in $\lambda$. If $\rho \vdash k$ and $\sigma \vdash m$, denote by $\rho \cup \sigma$ the partition of $m + k$ obtained at the disjoint union of parts of $\rho$ and $\sigma$. Also denote

$$\left( \frac{a}{\lambda} \right) \overset{\text{def}}{=} \prod_{r} \left( \frac{a_r}{\alpha_r(\lambda)} \right) = \prod_{r} a_r \cdot \left( a_r - 1 \right) \cdots \left( a_r - \alpha_r(\lambda) + 1 \right) \frac{1}{\alpha_r(\lambda)!},$$

where $\alpha_r(\lambda)$ is the number of parts of size $r$ in $\lambda$. Now let $\chi = \{\chi_N\}_{N \geq |\lambda| + \lambda_1} \in \hat{S}_\infty$ be the family of irreducible characters corresponding to the partition $\lambda$. According to [Mac98, Example I.7.14], for every $N \geq |\lambda| + \lambda_1$, the class function $\chi_N$ on $S_N$ is equal to

$$\chi_N = \sum_{\rho, \sigma : \rho \vdash |\rho| + |\sigma| = |\lambda|} (-1)^{f(\sigma)} \cdot \chi^\lambda(\rho \cup \sigma) \cdot \frac{\left( \frac{a}{\rho} \right)}{z_{\sigma}}, \quad (B.1)$$

$^{12}$There is a typo in the original statement of [DS94, Theorem 7], where it says $N \geq \sum_{t=1}^{k} t a_t$ instead.
where the sum is over all partitions $\rho$ and $\sigma$, including the empty partitions (of 0), with $|\rho| + |\sigma| = |\lambda|$. See Example B.4 below. In particular, (B.1) shows that indeed $\chi$ coincides with a certain element of $A$ for every $N \geq |\lambda| + \lambda_1$.

Now fix $k \in \mathbb{Z}_{\geq 1}$, and consider all partitions $\{\lambda \vdash q | 0 \leq q \leq k\}$. The number of such partitions is $p(0) + p(1) + \ldots + p(k)$. For large enough $N$, all these partitions give rise to distinct irreducible characters of $S_N$, and are, in particular, linearly independent class functions. On the other hand, the formula (B.1) shows that as elements in $A$, they are spanned by the monomials

$$a_1^{m_1} a_2^{m_2} \cdots a_k^{m_k}$$

with $m_1 + 2m_2 + \ldots + km_k \leq k$. These are precisely the possible cycle-structures of elements in $S_0, S_1, \ldots, S_k$, and therefore there are $p(0) + p(1) + \ldots + p(k)$ such monomials, which span a linear subspace of $A$ of dimension $p(0) + p(1) + \ldots + p(k)$. We conclude that this subspace is spanned by the $\chi \in \widetilde{S}_\infty$ corresponding to $\lambda$ with $|\lambda| \leq k$, and thus that the elements in $\widetilde{S}_\infty$ form a linear basis of $A$. \hfill \Box

**Remark B.3.** Proposition B.2 yields that every class function $f \in A$ is a linear combination of the elements of $\widetilde{S}_\infty$. Together with the orthogonality of irreducible characters, this gives another proof of Proposition B.1.

We end this appendix by illustrating how the formula (B.1) works.

**Example B.4.** Consider the partition $\lambda = (1)$, of a single element. It gives rise to the family of irreducible characters in the second row of Table 1. The character $\chi^1$ is the trivial character on the trivial group $S_1$. In the sum (B.1), $(\rho; \sigma)$ are either $(1; \emptyset)$ or $(\emptyset; 1)$, and

$$\chi_N = \left(\frac{-1}{1}\right)^0 \cdot \frac{1}{1} \left(\frac{a}{1}\right) = a_1 - 1 = \xi_1 - 1.$$  

Next, consider the partition $\lambda = (1, 1)$. It gives rise to the family of irreducible characters in the fourth row of Table 1. The character $\chi^{1,1}$ is the sign character on $S_2$. In the sum (B.1), $(\rho; \sigma)$ are either $(2; \emptyset)$, $(1, 1; \emptyset)$, $(1, 1)$, $(\emptyset; 1, 1)$ or $(\emptyset; 2)$, and so

$$\chi_N = \left(\frac{-1}{1}\right)^0 \cdot \frac{-1}{1} \left(\frac{a}{1}\right) + \left(\frac{-1}{1}\right)^1 \cdot \frac{1}{1} \left(\frac{a}{1}\right) + \left(\frac{-1}{1}\right)^1 \cdot \frac{1}{1} \left(\frac{a}{1}\right) + \left(\frac{-1}{1}\right)^2 \cdot \frac{1}{2} \left(\frac{a}{0}\right) + \left(\frac{-1}{1}\right)^1 \cdot \frac{-1}{2} \left(\frac{a}{0}\right)$$

$$= -\left(\frac{a_2}{1}\right) + \left(\frac{a_1}{1}\right) + \frac{1}{2} \cdot \frac{1}{2} = \frac{(a_1 - 1)}{2} = \frac{(a_1 - 1)(a_1 - 2)}{2} - a_2.$$  

**C Norm of morphisms: the proof of Theorem 5.4**

In this final section we prove Theorem 5.4 which shows that the norm and the $B$-norm of a morphism are, in principle, identical. More concretely, if $\eta: \Gamma \to \Delta$ is a morphism of $B$-labeled multi core graphs, $\Sigma = \text{Im}(\eta)$ and $\Gamma \overset{\eta}{\longrightarrow} \Sigma \overset{\iota}{\longrightarrow} \Delta$ is the decomposition of $\eta$ to a surjective and an injective morphisms, then

$$\|\eta\| = \|\eta\|_B + |\chi(\Sigma) - \chi(\Delta)|. \quad \text{(C.1)}$$

The following proof generalizes the ideas in [Pud14, Section 3], which dealt only with connected core graphs.

**Proof of Theorem 5.4.** Clearly, any component $\Delta'$ of $\Delta$ which does not meet $\eta(\Gamma)$, adds $-\chi(\Delta')$ to both sides of (C.1), so we may ignore such components altogether and assume that $\text{Im}(\eta)$ meets every component of $\Delta$.

Recall Definition 5.2, and consider the set of all possible sequences

$$\left(\beta_1, \ldots, \beta_{\|\eta\|}\right): \beta_{\|\eta\|} \circ \cdots \circ \beta_1 = \eta \quad \text{(C.2)}$$
of length $\|\eta\|$ of immediate morphisms, such that the composition of the sequence gives $\eta$. For each such sequence, consider a sequence of non-negative integers
\[(h_1, \ldots, h_{\|\eta\|}),\]
defined as the number of edges in the codomain of $\beta_i$ not covered by edges from its domain. Namely, if $\beta_i: \Sigma_{i-1} \to \Sigma_i$, then $h_i = |E(\Sigma_i)| - |E(\text{Im}\beta_i)|$. There are two observations to be made now:

- $h_i = 0$ if and only if $\beta_i$ is $B$-surjective, and if and only if $\beta_i$ also corresponds to a “merging step” corresponding to the $B$-norm from Definition 5.1.

- $h_i > 0$ if and only if $\beta_i$ is $B$-injective.

In the case of an immediate morphism of the first type (of the two types described in Definition 5.2), these observations are explained in [Pud14, Section 3]. The reasoning in the case of immediate morphisms of the second type is very similar.

Among all sequences as in (C.2), consider one with minimal corresponding integer sequence with respect to the lexicographic order. We will next prove that in such a sequence it is impossible to have $h_i > 0$ and $h_{i+1} = 0$. This will imply that $\eta$ can be obtained as a sequence $(\beta_1, \ldots, \beta_k)$ of merging-steps from Definition 5.1, followed by a sequence $(\beta_{k+1}, \ldots, \beta_{\|\eta\|})$ of embeddings. This break-up of $\eta$ thus exactly corresponds to the decomposition $\Gamma \overset{\eta}{\longrightarrow} \Sigma \overset{\iota}{\longrightarrow} \Delta$ of $\eta$ to a surjective $\eta$ and an injective $\iota$. So $\|\eta\|_B \leq k = \|\eta\|$, and knowing the converse inequality from (5.3), we get $\|\eta\|_B = k$. As $\iota$ is injective, it is also free (Proposition 4.3(1)), and by Lemma 5.3, $\|\iota\| = \chi(\Sigma) - \chi(\Delta)$. All in all
$$\|\eta\| = k + \|\iota\| = \|\eta\|_B + \chi(\Sigma) - \chi(\Delta),$$
as required.

It remains to prove that in the minimal sequence $(h_1, \ldots, h_{\|\eta\|})$, we cannot have $h_i > 0$ and $h_{i+1} = 0$. Let $\beta_i: \Sigma_{i-1} \to \Sigma_i$ be an immediate morphism and $h_i = |E(\Sigma_i)| - |E(\text{Im}\beta_i)|$ the corresponding integer. The two types of immediate morphisms from Definition 5.2 have the following geometric realizations. A step of the first type, where $H^F$ is replaced by $\langle H, j \rangle^F$, is obtained geometrically by adding a cycle spelling the word $j$ at the vertex $v$ at which $H$ is based (so $\pi_1^{\text{lab}}(\Sigma, v) = H$), and folding. A step of the second type, where $H^F$ and $H'^F$ are replaced by $\langle jH_j^{-1}, H' \rangle^F$, is obtained geometrically by adding a path spelling the word $j$ starting at $v'$ and ending at $v$, and folding. In both cases, if $h_i > 0$, the excessive edges of $\Sigma_i \setminus \text{Im}\beta_i$ form either a path, a cycle or a balloon. If $h_i = 0$, this process can also be obtained by gluing together two suitable vertices of $\Sigma_{i-1}$ and folding.

Now assume that $h_i > 0$ and $h_{i+1} = 0$, denote by $p = \Sigma_i \setminus \text{Im}\beta_i$ the (open) path, cycle or balloon in $\Sigma_i \setminus \text{Im}\beta_i$, and consider a pair of vertices of $\Sigma_i$ that are glued together to obtain $\beta_{i+1}$ (with folding). We claim that we may “exchange” the order of these two steps and get a pair of integers which is lexicographically smaller that $(h_i, h_{i+1})$. Indeed, this is certainly true if both vertices are not on $p$, in which case we may first merge them and only then add the path/cycle corresponding to $\beta_i$: this results in the pair $(0, h')$ (with $h' < h_i$, although this is immaterial). If one or two of the merged vertices are on $p$, we can easily find a step which, algebraically, is equivalent to merging them, and which can be performed on $\Sigma_{i-1}$ with corresponding $h$ strictly smaller than $h_i$ (and then perform the step corresponding algebraically to $\beta_i$). This is illustrated in Figure C.1.
Figure C.1: In every pair of figures, the one on the left shows the path $p$ of length $h_i > 0$ which corresponds to the immediate morphism $\beta_i$, and two vertices whose merging corresponds to the immediate morphism $\beta_{i+1}$ (with $h_{i+1} = 0$). The right figure in every pair shows how the same final result can be obtained by first performing a step which is equivalent to merging the two vertices and only then performing a step equivalent to $\beta_i$. Making this change results in a lexicographically smaller pair of integers.

### Glossary

| Term | Definition | Reference | Remarks |
|------|------------|-----------|---------|
| $F$  | free group of rank $r$ | | |
| $E_w$ | expectation w.r.t. the $w$-measure | | |
| $E_{\text{unif}}$ | expectation w.r.t. the uniform measure | | |
| $\xi_k(\sigma)$ | number of fixed points in the permutation $\sigma^k$ | Equation (1.1) | |
| $a_t(\sigma)$ | number of $t$-cycles in the permutation $\sigma$ | | |
| $\pi(w)$ | primitivity rank of $w$ | Definition 1.1 | |
| $\text{Crit}(w)$ | set of critical subgroups of $w$ | Definition 1.1 | |
| $A$ | the algebra $\mathbb{Q}[\xi_1, \xi_2, \ldots]$ | page 4 | |
| $\langle f, g \rangle$ | stable value of $\langle f, g \rangle_{S_N}$ | page 4 | $f, g \in A$ |
| $S_\infty$ | stable irreducible characters of $\{S_N\}_N$ | page 5 | subset of $A$ |
| $B = \{b_1, \ldots, b_r\}$ | a fixed basis of $F$ | | |
| $\mathcal{MOCC}(F)$ | the category of multisets of conjugacy classes of non-trivial f.g. subgroups of $F$ | for the morphisms see Definition 2.1 | |
| $\mathcal{MuCG}_B(F)$ | the category of $B$-labeled multi core graphs | Definition 3.1 | for the morphisms see Definition 3.2 |
| $\pi_1^{\text{lab}}(\Gamma)$ | the multiset in $\mathcal{MOCC}(F)$ corresponding to the multi core graph $\Gamma$ | Section 3.2 | |
| $\Gamma_B(\mathcal{H})$ | the multi core graph corresponding to the multiset $\mathcal{H} \in \mathcal{MOCC}(F)$ | Section 3.2 | |
| Term | Definition/Description | Reference | Remarks |
|------|------------------------|-----------|---------|
| rkΓ = rkH, χ(Γ) = χ(H), c(Γ) = c(H) | sum of ranks of subgroups in H, Euler characteristic of Γ, | Definition 3.3 | \(H = \pi^\text{lab}(\Gamma); \text{rk} + \chi = c\) |
| Φ_η(N) | the expected number of lifts of η to a random \(N\)-cover of Δ | Definition 3.4, Proposition 3.7 | \(\eta: \Gamma \rightarrow \Delta\) is a morphism in \(\mathcal{M} \mathcal{U} \mathcal{C} \mathcal{G}_B(F)\) |
| \(X_B\) | the bouquet in \(\mathcal{M} \mathcal{U} \mathcal{C} \mathcal{G}_B(F)\) representing \(\{F^F\}\) | | |
| \(\Gamma \xrightarrow{\psi} \Delta\) | a free morphism of multi core graphs | Definition 4.2 | |
| \(\|\eta\|_B\) | \(B\)-norm of the \(B\)-surjective morphism \(\eta\) | Equation (5.1) | |
| \(\|\eta\|\) | norm of the morphism \(\eta\) | Definition 5.2 | |
| \(\chi_{\text{max}}(\eta)\) | maximal \(\chi(\Sigma)\) of all decompositions of \(\eta: \Gamma \rightarrow \Delta\) as \(\Gamma \rightarrow_{\text{alg}} \Sigma \rightarrow \Delta\) | Definition 6.1 | |
| \(\text{Crit}(\eta)\) | set of critical decompositions of \(\eta\) | Definition 6.1 | |
| \(\text{Decomp}_B(\eta), \text{Decomp}_B^3(\eta)\) | decompositions of \(\eta\) to pairs/triples of \(B\)-surjective morphisms | Definition 6.4 | \(\eta\) is \(B\)-surjective |
| \(L^B, R^B, C_B\) | Möbius inversions of \(\Phi\) in the category of \(B\)-surjective morphisms | Section 6.1 | |
| \(\text{Decomp}_{\text{alg}}(\eta), \text{Decomp}_{\text{alg}}^3(\eta)\) | decompositions of \(\eta\) to pairs/triples of algebraic morphisms | Definition 6.4 | \(\eta\) is algebraic |
| \(L_{\text{alg}}, R_{\text{alg}}, C_{\text{alg}}\) | Möbius inversions of \(\Phi\) in the category of algebraic morphisms | Section 6.2 | |
| \(\Gamma_{\alpha_1,\ldots,\alpha_k}^w\) | the multiset of cycles corresponding to \(\xi_1^\alpha_1 \cdot \cdots \cdot \xi_k^\alpha_k\) in \(\mathcal{M} \mathcal{U} \mathcal{C} \mathcal{G}_B(F)\) | Example 3.6 | |
| \(\eta_{\alpha_1,\ldots,\alpha_k}^w\) | the morphism from \(\Gamma_{\alpha_1,\ldots,\alpha_k}^w\) to \(X_B\) | Example 3.6 | |
| \(\chi_{\text{max}}^{\alpha_1,\ldots,\alpha_k}\) | maximal negative \(\chi(\Sigma)\) of all algebraic morphisms \(\Gamma_{\alpha_1,\ldots,\alpha_k}^w \rightarrow \Sigma\) | Definition 7.1 | |
| \(\text{Crit}_{\alpha_1,\ldots,\alpha_k}\) | critical morphisms realizing \(\chi_{\alpha_1,\ldots,\alpha_k}^{\text{max}}\) | Definition 7.1 | |

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