THE CONVERGENCE-THEORETIC APPROACH TO G-FIRST COUNTABLE AND SYMMETRIZABLE SPACES

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1. Introduction

The convergence-theoretic approach to general topology as developed by S. Dolecki and his collaborators has reached a level of maturity allowing for classical general topology to be revisited systematically in a textbook-like format [5]. After re-interpreting topological notions in terms of functorial inequalities in the category of convergence spaces and continuous maps, many classical results are unified and refined with proofs that are reduced to an algebraic calculus of inequalities.

This article fits in this context and serves as yet another illustration of the power of the method. More specifically, we spell out convergence-theoretic characterizations of the notions of weak base, g-first-countable space, semi-metrizable space, and symmetrizable spaces. We should note that these easy characterizations are probably part of the folklore of convergence space theory, but we couldn’t find them explicitly stated in the literature. With the help of the already established similar characterizations of the notions of Fréchet-Ursyohn, sequential [14][2], and accessibility spaces [6], we give a simple algebraic proof of a classical result regarding when a symmetrizable (respectively, g-first-countable, respectively sequential) space is semi-metrizable (respectively first-countable, respectively Fréchet) that clarifies the situation for non-Hausdorff spaces. Using additionally the results of [4] and [16] on the commutation of the topologizer with product, we obtain simple algebraic proofs of various results (of Y. Tanaka) on the stability under product of symmetrizability and weak first-countability, and we obtain the same way a (as far as we know, entirely new) characterization of spaces whose product with every metrizable topology is g-first-countable, respectively symmetrizable.

2. Preliminaries: Generalities on convergences

We use notations and terminology consistent with the recent book [5]. This section is included only to make the article self-contained, and we refer the reader to [5] for a comprehensive treatment of convergence spaces.

If $X$ is a set, we denote by $2^X$ its powerset, by $[X]^{<\infty}$ the set of finite subsets of $X$ and by $[X]^{\omega}$ the set of countable subsets of $X$. If $\mathcal{A} \subset 2^X$, we write

\[ A^\uparrow := \{ B \subset X : \exists A \in \mathcal{A}, A \subset B \} \]

\[ A^\cap := \left\{ \bigcap_{S \in \mathcal{F}} S : \mathcal{F} \in [A]^{<\infty} \right\} \]

\[ A^\# := \{ B \subset X : \forall A \in \mathcal{A}, A \cap B \neq \emptyset \}. \]

A family $\mathcal{F}$ of non-empty subsets of $X$ is called a filter if $\mathcal{F} = \mathcal{F}^\cap = \mathcal{F}^\uparrow$. We denote by $\mathcal{F}^X$ the set of filters on $X$. Note that $2^X$ is the only family $\mathcal{A}$ satisfying
\( \mathcal{A} = \mathcal{A}^\dagger = \mathcal{A}^0 \) that has an empty element. Thus we sometimes call \( \{\emptyset\}^\dagger = 2^X \) the degenerate filter on \( X \). The set \( \mathcal{F} X \) is ordered by \( \mathcal{F} \leq \mathcal{G} \) if for every \( \mathcal{F} \in \mathcal{F} \) there is \( \mathcal{G} \in \mathcal{G} \) with \( \mathcal{G} \subset \mathcal{F} \). Maximal elements of \( \mathcal{F} X \) are called ultrafilters and \( \mathcal{F} \in \mathcal{F} X \) is an ultrafilter if and only if \( \mathcal{F} = \mathcal{F}^\# \). We denote by \( \mathbb{U} X \) the set of ultrafilters on \( X \).

A convergence \( \xi \) on a set \( X \) is a relation between \( \mathcal{F} X \) and \( X \), denoted
\[
x \in \lim_{\xi} \mathcal{F}
\]
whenever \( (x, \mathcal{F}) \in \xi \) (and we then say that \( \mathcal{F} \) converges to \( x \) for \( \xi \)), satisfying the following two conditions:
\[
\forall \mathcal{F}, \mathcal{G} \in \mathcal{F} X, \ \mathcal{F} \leq \mathcal{G} \implies \lim_{\xi} \mathcal{F} \subseteq \lim_{\xi} \mathcal{G}
\]
\[
\forall x \in X, \ x \in \lim_{\xi} \{x\}^\dagger.
\]

The pair \( (X, \xi) \) is then called a convergence space. We denote by \( \lim_{\xi} (x) \) the set of filters that converge to \( x \) for \( \xi \).

A map \( f \) between two convergence spaces \( (X, \xi) \) and \( (Y, \sigma) \) is continuous if for every \( \mathcal{F} \in \mathcal{F} X \) and \( x \in X \),
\[
x \in \lim_{\xi} \mathcal{F} \implies f(x) \in \lim_{\sigma} f(\mathcal{F}),
\]
where
\[
f(\mathcal{F}) := \{ B \subset Y : f^{-1}(B) \in \mathcal{F} \} = \{ f(\mathcal{F}) : \mathcal{F} \in \mathcal{F} \}^\dagger.
\]

Consistently with [3], we denote by \( |\xi| \) the underlying set of a convergence \( \xi \), and, if \( (X, \xi) \) and \( (Y, \sigma) \) are two convergence spaces we often write \( f : |\xi| \to |\sigma| \) instead of \( f : X \to Y \) even though one may see it as improper since many different convergences have the same underlying set. This allows to talk about the continuity of \( f : |\xi| \to |\sigma| \) without having to repeat for what structure.

Every topology can be seen as a convergence. Indeed, if \( \tau \) is a topology and \( \mathcal{N}_\tau(x) \) denotes the neighborhood filter of \( x \) for \( \tau \), then
\[
x \in \lim \mathcal{F} \iff \mathcal{F} \geq \mathcal{N}_\tau(x)
\]
defines a convergence that completely characterizes \( \tau \). Moreover, a function between two topological space is continuous in the topological sense if and only if it is continuous in the sense of convergences. Hence, we do not distinguish between a topology \( \tau \) and the convergence it induces, thus embedding the category \( \textbf{Top} \) of topological spaces and continuous maps as a full subcategory of the category \( \textbf{Conv} \) of convergence spaces and continuous maps.

A convergence is called Hausdorff if the cardinality of \( \lim \mathcal{F} \) is at most one, for every filter \( \mathcal{F} \). Of course, a topology is Hausdorff in the usual topological sense if and only if it is in the convergence sense. A point \( x \) of a convergence space \( (X, \xi) \) is isolated if \( \lim_{\xi} (x) = \{\{x\}^\dagger\} \). A prime convergence is a convergence with at most one non-isolated point.

Given two convergences \( \xi \) and \( \theta \) on the same set \( X \), we say that \( \xi \) is finer than \( \theta \) or that \( \theta \) is coarser than \( \xi \), in symbols \( \xi \geq \theta \), if the identity map from \( (X, \xi) \) to \( (X, \theta) \) is continuous, that is, if \( \lim_{\xi} \mathcal{F} \subseteq \lim_{\theta} \mathcal{F} \) for every \( \mathcal{F} \in \mathcal{F} X \). With this order, the set \( \mathcal{C}(X) \) of convergences on \( X \) is a complete lattice whose greatest element is the discrete topology and least element is the antidiscrete topology, and for which, given \( \Xi \subset \mathcal{C}(X) \),
\[
\lim_{\bigwedge \Xi} \mathcal{F} = \bigcap_{\xi \in \Xi} \lim_{\xi} \mathcal{F} \quad \text{and} \quad \lim_{\bigvee \Xi} \mathcal{F} = \bigcup_{\xi \in \Xi} \lim_{\xi} \mathcal{F}.
\]
In fact, \( \text{Top} \) is a reflective subcategory of \( \text{Conv} \) and the corresponding reflector \( T \), called topologizer, associates to each convergence \( \xi \) on \( X \) its topological modification \( T\xi \), which is the finest topology on \( X \) among those coarser than \( \xi \) (in \( \text{Conv} \)). Concretely, \( T\xi \) is the topology whose closed sets are the subsets of \( |\xi| \) that are \( \xi \)-closed, that is, subsets \( C \) satisfying

\[
C \in \mathcal{F}^\# \implies \lim_\xi \mathcal{F} \subset C.
\]

A subset \( O \) of \( |\xi| \) is \( \xi \)-open if its complement is closed, equivalently if

\[
\lim_\xi \mathcal{F} \cap O \neq \emptyset \implies O \in \mathcal{F}.
\]

Given a map \( f : |\xi| \to Y \), there is the finest convergence \( f\xi \) on \( Y \) making \( f \) continuous (from \( \xi \)), and given \( f : X \to |\sigma| \), there is the coarsest convergence \( f^{-}\sigma \) on \( X \) making \( f \) continuous (to \( \sigma \)). The convergences \( f\xi \) and \( f^{-}\sigma \) are called final convergence for \( f \) and \( \xi \) and initial convergence for \( f \) and \( \sigma \) respectively. Note that

\[
f : |\xi| \to |\sigma| \text{ is continuous } \iff \xi \geq f^{-}\sigma \iff f\xi \geq \sigma.
\]

If \( A \subset |\xi| \), the induced convergence by \( \xi \) on \( A \), or subspace convergence, is \( i^-\xi \), where \( i : A \to |\xi| \) is the inclusion map. If \( \xi \) and \( \tau \) are two convergences, the product convergence \( \xi \times \tau \) on \( |\xi| \times |\tau| \) is the coarsest convergence on \( |\xi| \times |\tau| \) making both projections continuous, that is,

\[
\xi \times \tau := p^-_\xi \xi \lor p^-_\tau \tau,
\]

where \( p^-_\xi : |\xi| \times |\tau| \to |\xi| \) and \( p^-_\tau : |\xi| \times |\tau| \to |\tau| \) are the projections defined by \( p^-_\xi(x, y) = x \) and \( p^-_\tau(x, y) = y \) respectively.

Because \( \text{Top} \) is a reflective subcategory of \( \text{Conv} \), \( \text{Top} \) is closed under initial constructions so that a subspace of a topological convergence space is topological and a product of topological convergence spaces is topological. However, the convergence induced by \( T\xi \) on a subset \( A \) and \( T(i^-\xi) \) do not need to coincide, and similarly, the topological modification of a product generally does not coincide with the product of the topological modifications.

Given a convergence \( \xi \) and \( x \in |\xi| \), the filter

\[
\mathcal{V}_\xi(x) := \bigwedge_{\mathcal{F} \in \lim_\xi \mathcal{F}} \mathcal{F}
\]

is called the vicinity filter of \( x \) for \( \xi \). In general, \( \mathcal{V}_\xi(x) \) does not need to converge to \( x \) for \( \xi \). If \( x \in \lim_\xi \mathcal{V}_\xi(x) \) for all \( x \in |\xi| \), we say that \( \xi \) is a pretopology or is pretopological.

The category \( \text{PrTop} \) of pretopological spaces and continuous maps is a (full) reflective subcategory of \( \text{Conv} \). The corresponding reflector \( S_0 \) associates to each convergence \( \xi \) its pretopological modification \( S_0\xi \), which is the finest among the pretopologies coarser than \( \xi \). Explicitly, \( x \in \lim_{S_0}\xi \mathcal{F} \) if \( \mathcal{F} \geq \mathcal{V}_\xi(x) \) so that \( \mathcal{V}_\xi(x) = \mathcal{V}_{S_0\xi}(x) \). Topologies are in particular pretopological, so that \( S_0 \geq T \). Moreover,

\[
O \text{ is } \xi \text{-open } \iff O \in \bigcap_{x \in O} \mathcal{V}_\xi(x),
\]

and \( \xi \) is topological if and only if \( \mathcal{V}_\xi(x) \) has a filter-base composed of open sets, in which case \( \mathcal{V}_\xi(x) = \mathcal{N}_\xi(x) \).

In contrast to \( T \), the reflector \( S_0 \) does commute with initial convergence (so that the pretopological modification of an induced convergence is the convergence
induced by the pretopological modification), but not with suprema, hence not with products.

A sequence \( \{x_n\}_{n=1}^{\infty} \) on a set \( X \) induces a filter

\[
(x_n)_n := \{ \{x_n : n \geq k\} : k \in \omega \}^\uparrow.
\]

A filter that is induced by some sequence is called sequential filter. We denote by \( EX \) the set of sequential filters on \( X \). To a convergence \( \xi \), we associate its sequentially based modification \( Seq \xi \) defined by

\[
\lim_{\text{Seq}\xi} F = \bigcup_{E \in E\xi, E \leq F} \lim_{\xi} E.
\]

A useful immediate consequence of [3, Corollary 10] (that can be traced all the way back to [13], but [3] gives a formulation more consistent with our terminology) is:

**Proposition 1.** If \( \sigma \) is Hausdorff pretopology, then

\[
Seq \sigma = Seq T \sigma.
\]

In the same vain, letting \( F_1 X \) denote the set of filters with a countable filter-base, we can associate to each convergence \( \xi \) its first-countable modification \( I_1 \xi \) defined by

\[
\lim_{I_1\xi} F = \bigcup_{H \in F_1\xi, H \leq F} \lim_{\xi} H.
\]

Both \( Seq \) and \( I_1 \) are concrete coreflectors from \( Conv \) to the full categories of sequentially based and first-countable convergences respectively.

\( T \), \( S_0 \), \( Seq \) and \( I_1 \) are concrete endofunctors of \( Conv \), and as such each satisfy the following properties of a modifier \( F \) acting on convergence spaces (for all \( \xi, \theta \) and \( f \)):

\[
|F\xi| = F|\xi|
\]

\[
\xi \leq \theta \implies F\xi \leq F\theta
\]

\[
F(f^{-}\theta) \geq f^{-}(F\theta)
\]

All four are also idempotent, that is, satisfy \( F(F\xi) = F\xi \) for all \( \xi \). The reflectors \( T \) and \( S_0 \) are additionally contractive \( (F\xi \leq \xi \) for all \( \xi \) and the coreflectors \( Seq \) and \( I_1 \) are additionally expansive \( (F\xi \geq \xi \) for all \( \xi \)).

Recall that a topological space is sequential if every sequentially closed subset (that is, subset that contains the limit point of every sequence on it) is closed. It is easily seen that a subset of a topology \( \xi \) is sequentially closed if and only if \( \xi = Seq \xi \)-closed, so that, \( \xi \) is sequential if and only if \( \xi = T Seq \xi \). As \( \sigma \leq Seq \sigma \) for every convergence \( \sigma \), the inequality \( \xi \leq T Seq \xi \) is true for every topology \( \xi \). Hence if \( \xi = T \xi \) then

\[
\xi \text{ is sequential } \iff \xi \geq T Seq \xi,
\]

and it turns out that

\[
(2.2) \quad \xi \geq T Seq \xi \iff \xi \geq T I_1 \xi
\]

and we take one or the other of these inequalities as a definition of a sequential convergence.
A topological space $X$ is Fréchet if for every $x \in X$ and $A \subset X$, if $x \in \text{cl}A$ then there is a sequence on $A$ converging to $x$. It is easily seen (e.g., [14][2]) that if $\xi$ is a topology then
\[
\text{Fréchet} \iff \xi \geq S_0 \text{Seq} \xi \iff \xi \geq S_0 I_1 \xi,
\]
and we take either one of these inequalities as a definition of a Fréchet convergence.

Note that a pretopology is Fréchet if and only if each vicinity filter is a Fréchet filter, in the following sense:

\[
F \in \mathcal{F}_X \text{ is Fréchet if } F = \bigwedge_{E \in \mathcal{E}_X} F \leq E.
\]

Of course, every countably based filter is in particular a Fréchet filter.

Several other topological properties can be characterized with the help of a functorial inequality; see [2, 5].

A subset $K$ of a convergence space is $\xi$-compact if $\lim_{\xi} U \cap K \neq \emptyset$ for every ultrafilter $U$ on $K$, and countably compact if every countably based filter $H$ with $K \in H^*$, there is an ultrafilter $U \geq H$ with $\lim U \cap K \neq \emptyset$. Given two convergences $\xi$ and $\sigma$ on the same set, we say that $\xi$ is locally (countably) $\sigma$-compact if every $\xi$-convergent filter has a (countably) $\sigma$-compact element.

The reflector $Epi_{I_1}$ associates to a convergence $\xi$ the coarsest among convergences $\theta$ on $|\xi|$ satisfying
\[
\forall \tau = I_1 \tau, \ T(\xi \times \tau) = T(\theta \times \tau).
\]

We invite the reader to consult [16], [15] or [5] for various characterizations and applications of this functor. For our purpose, it is enough to know [15] that $\xi \geq Epi_{I_1} \xi \geq T \xi$ for every $\xi$ and that $Epi_{I_1} \xi$ is characterized by
\[
\theta \geq Epi_{I_1} \xi \iff \forall \tau = I_1 \tau, \ \theta \times \tau \geq Epi_{I_1}(\xi \times \tau)
\]
\[
\iff \forall \tau = I_1 \tau \text{ prime topology, } \theta \times \tau \geq T(\xi \times \tau).
\]

Topological spaces whose product with every first-countable space is sequential were characterized [15] as those topologies $\tau$ satisfying
\[
\tau \geq Epi_{I_1} I_1 \tau = Epi_{I_1} \text{Seq} \tau.
\]

A topology (or convergence) satisfying (2.5) is called strongly sequential. We note that:

**Proposition 2.** [17] Proposition 4] A regular sequential locally countably compact topology is strongly sequential.

3. **Weak base and g-first-countable spaces**

Recall the classical topological notion of weak base.

**Definition 3.** Let $X$ be a topological space. A family $B$ of subsets of $X$ is called a weak base for $X$ if

\[
B = \bigcup_{x \in X} B_x
\]

where $B_x$ be a filter-base such that $B_x \leq \{x\}$, for each $x \in X$, and $U \subset X$ is open if and only if for every $x \in U$ there is $V \in B_x$ with $x \in V \subset U$.

A space $X$ is $g$-first-countable if it has a weak base such that $B_x$ is countable, for each $x \in X$. This notion was introduced in [11] under the name weakly first-countable and renamed by Siwiec in [20].
Proposition 4. Let $(X, \tau)$ be a topological space.

(1) If $\xi = S_0 \xi$ with $T \xi = \tau$ then if for each $x \in X$, $B_x$ is a filter-base of $V_\xi(x)$ then $B = \bigcup_{x \in X} B_x$ is a weak base for $X$.

(2) If $B = \bigcup_{x \in X} B_x$ is a weak base for $\tau$ then $B$ defines a pretopology $\xi$ by $V_\xi(x) := B_x^+$ such that $\tau = T \xi$.

Proof. (1): Assume $\xi = S_0 \xi$ with $T \xi = \tau$ and $B_x$ is a filter-base of $V_\xi(x)$ for each $x \in X$. Let $U \in O_\tau$ with $x \in U$. Because $\xi \geq \tau$, $V_\xi(x) \geq N_\tau(x)$ so that there is $V \in V_\xi(x)$ with $x \in V \subset U$. If conversely, $U$ is such that for every $x \in U$ there is $V \in V_\xi(x)$ with $x \in V \subset U$, then $U \subseteq \lim_{\xi} V$ and thus $U$ is $\xi$-open, hence $\tau$-open.

(2): If $B = \bigcup_{x \in X} B_x$ is a weak base for $\tau$ and $\xi$ is a pretopology defined by $V_\xi(x) := B_x^+$, then $T \xi = \tau$. Indeed, $U$ is $\xi$-open if and only if $U \in \bigcap_{x \in U} V_\xi(x)$, which is here the weak base condition. □

Corollary 5. A topological space $(X, \tau)$ is g-first-countable if and only if there is a first-countable pretopology $\sigma$ with $T \sigma = \tau$.

In particular, in view of (2.2), g-first-countable spaces are sequential.

A map $d : X \times X \to \mathbb{R}^+$ such that $d(x, x) = 0$ for all $x \in X$ induces a topology $\tau_d$ on $X$ for which $O \tau$ is $\tau_d$-open if for every $x \in O$, there is $\epsilon > 0$ with $B(x, \epsilon) \subset O$ where $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$. A topological space $(X, \tau)$ is o-metrizable (in the sense of [18]) if there is $d$ with $\tau = \tau_d$.

Theorem 6. The following are equivalent for a topology $\tau$:

(1) $\tau$ is o-metrizable;

(2) $\tau$ is g-first-countable;

(3) there is $\sigma = S_0 \sigma = I_1 \sigma$ with $T \sigma = \tau$.

Proof. (2) $\iff$ (3) is Corollary 5. (1) $\implies$ (3) : If $\tau$ is o-metrizable by $d$ then $\{V_d(x) : x \in |\tau|\}$ defines a first countable pretopology whose topological modification is $\tau$.

(3) $\implies$ (1): For each $x \in |\sigma|$, let $\{V_n(x) : n \in \mathbb{N}\}$ be a decreasing filter-base of $V_\sigma(x)$. Let

$$d : |\sigma| \times |\sigma| \to \mathbb{R}^+$$

$$(x, y) \mapsto \begin{cases} 0 & \text{if } y \in \bigcap_{n \in \mathbb{N}} V_n(x) \\ \frac{1}{n} & \text{if } y \in V_n(x) \setminus V_{n+1}(x) \\ 1 & \text{otherwise} \end{cases}$$

Note that $d(x, y) \leq \frac{1}{n}$ if and only if $y \in V_n(x)$ so that $V_n(x) = \bar{B}(x, \frac{1}{n})$.

Of course, $d(x, x) = 0$ for all $x$, and $O \in O_\tau = O_\sigma$ if and only if $O = \bigcap_{x \in O} V_\sigma(x)$, that is, if and only if for every $x \in O$, there is $n_x$ with $x \in V_{n_x}(x) \subset O$. □

More generally, we will call a convergence $\xi$ g-first-countable if there is $\sigma = S_0 \sigma = I_1 \sigma$ with $\sigma \geq \xi \geq T \sigma$.

In the Hausdorff case, there is a unique choice for the pretopology $\sigma$:

Proposition 7. A Hausdorff convergence $\xi$ is g-first-countable convergence if and only if $I_1 \xi$ is a pretopology and $\xi$ is sequential.

In this case, $I_1 \xi$ is the only first-countable pretopology $\sigma$ with $\sigma \geq \xi \geq T \sigma$. 
Proof. If $\xi$ is $g$-first-countable then $\xi$ is in particular sequential, and there is $\sigma = S_0 \sigma = I_1 \sigma$ with $\sigma \geq \xi \geq T \sigma$. In particular, $\sigma \geq I_1 \xi$. Moreover, to see the reverse inequality, note that if $x \in \lim_{\alpha, \xi} F$ then there is $H \in F_1$ with $H \subseteq F$ and $x \in \lim_\xi H$. For every $\epsilon \in \mathbb{E}[\xi]$ with $\epsilon \geq H$,

$$x \in \lim_{\alpha, \xi} \epsilon \subseteq \lim_{\alpha, \xi} T x \epsilon = \lim_{\alpha, \xi} \sigma \epsilon$$

by Proposition 1, so that $x \in \lim_{\alpha} \epsilon$. As $\sigma = S_0 \sigma$,

$$x \in \lim_\sigma \left( \bigwedge_{\epsilon \in \mathbb{E}[\xi], \epsilon \geq H} \epsilon, \right)$$

and $\bigwedge_{\epsilon \in \mathbb{E}[\xi], \epsilon \geq H} \epsilon = H$ because $H \in F_1$, hence is a Fréchet filter, so that $x \in \lim_\sigma H \subseteq \lim_{\alpha, \xi} F$. Thus $\sigma = I_1 \xi$ is pretopological and is the only $\sigma$ witnessing the definition of $g$-first-countable.

Conversely, if $I_1 \xi$ is a pretopology and $\xi$ is sequential, that is, $\xi \geq T I_1 \xi$, then $\sigma = I_1 \xi$ witnesses the definition of $g$-first-countability of $\xi$. \[\square\]

As a result, if $\xi$ is $g$-first-countable, Fréchet and Hausdorff, then $\xi = S_0 I_1 \xi$ and $I_1 \xi$ is pretopological, so that $\xi = I_1 \xi$ is first-countable. This is a classical result that we will refine in the next section.

4. Accessibility spaces, semi-metrizable spaces and symmetrizable spaces

A map $d : X \times X \rightarrow \mathbb{R}^+$ such that for every $x$ and $y$ in $X$, $d(x, y) = d(y, x)$ and $d(x, y) = 0$ if and only if $x = y$, is called a semi-metric on $X$. Traditionally, e.g., [5], a topological space $(X, \tau)$ is called semi-metrizable if there is a semi-metric $d$ on $X$ for which, for every $x \in X$ and $A \subseteq X$,

$$(4.1) \quad x \in \text{cl}_\tau A \iff d(x, A) := \inf_{a \in A} d(x, a) = 0,$$

and symmetrizable if there is a semi-metric $d$ on $X$ for which $O$ is $\tau$-open if and only if for every $x \in O$, there is $\epsilon > 0$ with $B(x, \epsilon) \subseteq O$. This can easily be reformulated in terms more directly comparable to (4.1): $\tau$ is symmetrizable if and only if $F \subseteq X$ is $\tau$-closed if and only if $d(x, F) > 0$ for every $x \in X \setminus F$. Hence a semi-metrizable space is symmetrizable.

Note that a semi-metric $d$ induces a pretopology $\tilde{d}$ defined by the vicinity filters

$$V_{\tilde{d}}(x) = \{B(x, \epsilon) : \epsilon > 0\}^\uparrow = \left\{ B\left( x, \frac{1}{n} \right) : n \in \omega \right\}^\uparrow.$$

Of course, $\tilde{d}$ is a first-countable pretopology, that is,

$$\tilde{d} = S_0 \tilde{d} = I_1 \tilde{d}.$$

Theorem 8. Let $(X, \tau)$ be a topological space.

1. $(X, \tau)$ is semi-metrizable if and only if there is a semi-metric $d$ on $X$ for which $\tau = d$;

2. $(X, \tau)$ is symmetrizable if and only if there is a semi-metric $d$ on $X$ for which $\tau = T d$.

Proof. (1) If $\tau$ is semi-metrizable there is a semi-metric $d$ satisfying (4.1). Then $\tilde{d} \geq \tau$ because $V_{\tilde{d}}(x) \geq N_{\tau}(x)$. Indeed, if $O$ is a $\tau$-open set and $O \notin V_{\tilde{d}}(x)$, then $X \setminus O \in (V_{\tilde{d}}(x))^\#$ so that $d(x, X \setminus O) = 0$, hence $x \in \text{cl}_\tau (X \setminus O) = X \setminus O$ by (4.1).
that is, \( x \notin O \). Conversely, \( \tau \geq \tilde{d} \) because \( \mathcal{N}_\tau(x) \geq \mathcal{V}_{\tilde{d}}(x) \). Indeed, if there is \( \epsilon > 0 \) with \( B(x, \epsilon) \notin \mathcal{N}_\tau(x) \) then \( X \setminus B(x, \epsilon) \in (\mathcal{N}_\tau(x))^\# \) so that \( x \in \text{cl}_\tau(X \setminus B(x, \epsilon)) \).

By (4.1), \( d(x, X \setminus B(x, \epsilon)) = 0 \) which is not possible, for \( d(x, y) \geq \epsilon \) for every \( y \notin B(x, \epsilon) \).

(2) The topology \( \tau \) is symmetrizable by a semi-metric \( d \) if and only if \( O \) is \( \tau \)-open exactly if for every \( x \in O \), there is \( \epsilon > 0 \) with \( B(x, \epsilon) \subset O \), that is, if and only if \( O \in \bigcap_{x \in O} \mathcal{V}_{\tilde{d}}(x) \), equivalently, if \( O \) is \( \tilde{d} \)-open. In other words, \( \tau \) is symmetrizable by \( d \) if and only if \( \tau \)-open sets and \( \tilde{d} \)-open sets coincide, that is, if and only if \( \tau = T \tilde{d} \).

Note that we can now more generally define a convergence \( \xi \) to be semi-metrizable if \( \xi = \tilde{d} \) for some semi-metric \( d \) on \( |\xi| \) (which of course imposes that \( \xi \) be a pretopology) and symmetrizable if

\[
\tilde{d} \geq \xi \geq T \tilde{d}
\]

for some semi-metric \( d \) on \( |\xi| \).

In view of Theorem 8, Theorem 6 and the characterizations (2.3) and (2.2) of Fréchet and sequential spaces, we have the following immediate relations between the notions at hand:

\[
\begin{align*}
\text{Fréchet} & \implies \text{sequential} \\
\xi \geq S_0 I_1 \xi & \implies \xi \geq T I_1 \xi \\
\uparrow & \uparrow \\
\text{first-countable (pretop.)} & \implies \text{g-first-countable} \\
\xi = S_0 \xi = I_1 \xi & \exists \sigma = S_0 \sigma = I_1 \sigma : \sigma \geq \xi \geq T \sigma \\
\uparrow & \uparrow \\
\text{semi-metrizable} & \implies \text{symmetrizable} \\
\exists d : \xi = \tilde{d} & \exists d : \tilde{d} \geq \xi \geq T \tilde{d}
\end{align*}
\]

Figure 4.1. Relations among the notions considered

Analogously to Proposition 7, we have:

**Corollary 9.** If \( \xi \) is a Hausdorff convergence, then it is symmetrizable if and only if \( \xi \) is sequential and \( I_1 \xi \) is a semi-metrizable pretopology.

**Proof.** If \( \xi \) is symmetrizable, it is in particular g-first-countable so that \( \sigma = I_1 \xi \) is the only first-countable pretopology with \( \sigma \geq \xi \geq T \sigma \). Since there is a semi-metric \( d \) with \( \tilde{d} \geq \xi \geq T \tilde{d} \), we conclude that \( \tilde{d} = I_1 \xi \).

Conversely, if \( \xi \) is sequential, that is, \( \xi \geq T I_1 \xi \) and \( I_1 \xi = \tilde{d} \) for some semi-metric \( d \), then \( \xi \) is symmetrizable. \( \square \)

Of course, if \( \xi \) is Fréchet Hausdorff and symmetrizable then \( \xi \geq S_0 I_1 \xi = I_1 \xi = \tilde{d} \) is semi-metrizable.

In other words, the horizontal implications in the diagram above can be reversed when the space is a Hausdorff Fréchet space. This is well known, and easy to see as we already have proved. The observation can be traced all the way back to [19], it was included without proof in [9], and a proof can be found in the popular

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ON CONVERGENCE AND SYMMETRIZABILITY

Yet approaches to its proof have remained the subject of more research, e.g., [10, 11] without yielding any refinement. The convergence-theoretic viewpoint provides as a consequence such a slight refinement by clarifying what condition is needed if separation is dropped, with an immediate and transparent proof.

The relevant notion is that of an accessibility space as introduced by Whyburn in [23]. Recall that a topological space \((X, \tau)\) is an accessibility space if for each \(x_0 \in X\) and every \(H \subset X\) with \(x_0 \in \text{cl}_\tau(H \setminus \{x_0\})\), there is a closed subset \(F\) with \(x_0 \in \text{cl}(F \setminus \{x_0\})\) and \(x_0 \notin \text{cl}_\tau(F \setminus H \setminus \{x_0\})\). It is immediate that

**Proposition 10.** A Fréchet topological space in which sequences have unique limits, in particular a Fréchet Hausdorff spaces, is accessibility.

Kannan offered [12] the following convergence-theoretic characterization (see also [7] for an extensive discussion and an extension to general convergences):

**Theorem 11.** [12] A topology \(\tau\) is accessibility if and only if it is topologically maximal (within the class of pretopologies), that is,

\[ \sigma = S_0 \sigma \geq \tau \text{ and } T \sigma = T \tau \implies \sigma = \tau. \]

In Figure 4.1 above, \(\xi = T \xi\) is a topology, and in the rows bottom to top \(\tilde{d}, \sigma\) and \(S_0 I_1 \xi\) are pretopologies finer than \(\xi\) with \(\xi = T \tilde{d}, \xi = T \sigma\) and \(\xi = T(S_0 I_1 \xi)\) respectively. Thus, it is an immediate consequence of Theorem 11 that if \(\xi\) is additionally an accessibility space, then \(\xi = \tilde{d}, \xi = \sigma\) and \(\xi = S_0 I_1 \xi\) respectively, by topological maximality:

**Theorem 12.** In Figure 4.1, the two columns coincide if the space is an accessibility space. In particular an accessibility sequential space is Fréchet; an accessibility g-first-countable space is first-countable; an accessibility symmetrizable space is semi-metrizable.

In view of Proposition 10, this refines the classical result by relaxing the condition Fréchet Hausdorff to accessibility.

Note also that Proposition 7 and Corollary 9 delineate the circumstances under which the vertical implications can be reversed among Hausdorff convergences: the upper two exactly when \(I_1 \xi\) is a pretopology, the lower two exactly when \(I_1 \xi\) is moreover semi-metrizable.

### 5. STRONGLY G-FIRST-COUNTABLE AND STRONGLY SYMMETRIZABLE SPACES

A product of a g-first-countable (respectively symmetrizable) topology with a metrizable topology does not need to be g-first-countable (respectively symmetrizable), as shows [21, Example 4.5], which is an example of a symmetrizable (hence g-first-countable) space and a metrizable space whose product is not even sequential.

As far as we know, the spaces whose product with every metrizable topology is g-first-countable, respectively symmetrizable, have not been characterized. We set out to do this in this section.

Let us call strongly g-first-countable a convergence \(\xi\) for which there is \(\sigma = I_1 \sigma = S_0 \sigma\) with

\[ \sigma \geq \xi \geq \text{Epi}_1 \sigma. \]

**Proposition 13.** A Hausdorff convergence \(\xi\) is strongly g-first-countable if and only if it is strongly sequential and g-first-countable.
Proof. If $\xi$ is strongly $g$-first-countable then there is a first-countable pretopology $\sigma$ with $\xi \geq Epi_{I_1}\sigma \geq T\sigma$ and thus $\xi$ is $g$-first-countable. Since $\xi$ is also Hausdorff, $\sigma = I_1\xi$ by Proposition [7] so that $\xi \geq Epi_{I_1}I_1\xi$ and $\xi$ is strongly sequential.

Conversely, if $\xi$ is $g$-first-countable and Hausdorff, then $I_1\xi$ is a pretopology by Proposition [7]. If moreover $\xi \geq Epi_{I_1}I_1\xi$, then $\sigma = I_1\xi$ witnesses the definition of strongly $g$-first-countable for $\xi$. □

Theorem 14. The following are equivalent for a Hausdorff convergence $\xi$:

(1) $\xi$ is strongly $g$-first-countable;

(2) $\xi \times \tau$ is strongly $g$-first-countable for every first-countable pretopology $\tau$;

(3) $\xi \times \tau$ is $g$-first-countable for every prime first-countable topology $\tau$.

Proof. (1) $\implies$ (2): If $\xi \geq Epi_{I_1}\sigma$ where $\sigma = I_1\sigma = S_0\sigma \geq \xi$ and $\tau = I_1\tau = S_0\tau$ then $\xi \times \tau \geq Epi_{I_1}(\sigma \times \tau)$

and $\sigma \times \tau$ is a first-countable pretopology finer that $\xi \times \tau$, so that $\xi \times \tau$ is strongly $g$-first-countable.

(2) $\implies$ (3) is clear.

(3) $\implies$ (1): If conversely, $\xi \times \tau$ is $g$-first-countable for every prime first-countable topology $\tau$, then in particular, taking a singleton for $\tau$, $\xi$ must be $g$-first-countable. As $\xi$ is Hausdorff and prime topologies are Hausdorff, we conclude that $\xi \times \tau$ is Hausdorff and $g$-first-countable whenever $\tau$ is a prime first-countable topology. In view of Proposition [7],

$$I_1(\xi \times \tau) = I_1\xi \times \tau$$

is the only first-countable pretopology witnessing the definition of $g$-first-countability of $\xi \times \tau$. Hence

$$\xi \times \tau \geq T(I_1\xi \times \tau)$$

for every prime first-countable topology $\tau$ and thus $\xi \geq Epi_{I_1}I_1\xi$ by [7]. Hence $\xi$ is strongly $g$-first-countable. □

Similarly, we call strongly symmetrizable a convergence $\xi$ for which there is a semi-metric $d$ on $|\xi|$ with

$$d \geq \xi \geq Epi_{I_1}d,$$

and we obtain with similar arguments that

Proposition 15. A Hausdorff convergence $\xi$ is strongly symmetrizable if and only if it is strongly sequential and symmetrizable.

Theorem 16. The following are equivalent for a Hausdorff convergence $\xi$:

(1) $\xi$ is strongly symmetrizable;

(2) $\xi \times \tau$ is strongly symmetrizable for every semi-metrizable pretopology $\tau$;

(3) $\xi \times \tau$ is symmetrizable for every prime metrizable topology $\tau$.

In the diagram below, we omit “Fréchet”, for a Fréchet space does not need to be strongly sequential. We could replace it with strongly Fréchet, which does imply strongly sequential, but this is of little importance here.
6. More product theorems

Theorems 6 and 8 also shed light on results of Y. Tanaka on stability under product of g-first-countable and symmetrizable spaces:

**Theorem 17.** [21] Let $X$ be a regular Hausdorff topological space.

1. If $X$ is a locally countably compact symmetrizable space then $X \times Y$ is symmetrizable for every symmetrizable space $Y$.
2. If $X$ is locally countably compact and g-first-countable, then $X \times Y$ is g-first-countable for every g-first-countable space $Y$.

Both results are in fact instances of results of [4] characterizing the relation between two convergences $\xi$ and $\sigma$ on the same underlying set for which

\[(6.1) \quad \xi \times T \tau \geq T(\sigma \times \tau).\]

for every $\tau$ in a specified class of convergence. The full scope of results studying (6.1) is not needed here and we invite the interested reader to consult [4] for an account of the many topological applications of variants of (6.1). For our present purpose, we will only need this very particular case:

**Theorem 18.** [4, Theorem 9.10] Let $T \sigma$ be a regular Hausdorff topology. The following are equivalent:

1. $T \sigma$ is locally countably $Epi_{I_1} \sigma$-compact;
2. $T \sigma \times T \tau \geq T(\sigma \times \tau)$ for every $\tau \geq T I_1 \tau$;
3. $T \sigma \times T \tau \geq T(\sigma \times \tau)$ for every $\tau = I_1 T \tau$.

We will need the following technical lemma:

**Lemma 19.** Let $\xi$ be a regular Hausdorff $g$-first-countable topology, so that $\xi = T \sigma$ for some $\sigma = I_1 \sigma = S_0 \sigma$. If $\xi$ is locally countably compact, then $\xi = Epi_{I_1} \sigma$, and thus $\xi$ is locally countably $Epi_{I_1} \sigma$-compact.

**Proof.** Because $\xi = T I_1 \sigma \geq T I_1 T \sigma = T I_1 \xi$, $\xi$ is in particular sequential and regular. If $\xi$ is also locally countably compact, then by Proposition 2 it is also strongly sequential so that in view of Proposition 13 $\xi$ is strongly $g$-first-countable, and thus $\xi = Epi_{I_1} \alpha$ for some first-countable pretopology $\alpha$. By Proposition 7 $\alpha = I_1 \xi = \sigma$, which concludes the proof. $\Box$
Proof of Theorem 17. We prove (2) and (1) follows the same argument replacing the role of $\sigma$ and $\tau$ by pretopologies of the form $\tilde{d}$ for semi-metrics.

Let $\xi = T \sigma$ for $\sigma = I_1 \sigma = S_0 \sigma$ and $\theta = T \tau$ for $\tau = I_1 \tau = S_0 \tau$. In view of Lemma 19, $\xi$ is locally countably Epi-$I_1\sigma$-compact. Therefore Theorem 18 applies to the effect that

$$\xi \times \theta = T \sigma \times T \tau \geq T(\sigma \times \tau)$$

and $\sigma \times \tau$ is a first countable pretopology finer than $\xi \times \theta$. Hence $\xi \times \theta$ is g-first-countable.

In fact, the converse of each item in Theorem 17 is also true [22], as could be obtained from a slight variation of Theorem 18 in which the inequality only need to be tested for first-countable pretopologies. As this was not shown or formulated this way in [4], we omit the converse.

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