Encoded Universality for Generalized Anisotropic Exchange Hamiltonians

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Quantum computation [1, 2] is known to be universal as long as arbitrary single-qubit and non-local (entangling) two-qubit unitary operations can be applied in an arbitrarily structured sequence (a quantum circuit) [3, 4]. These operations are generated by Hamiltonian control fields, which may be either directly related to interactions intrinsic to the physical system of qubits, or may be imposed as additional, external, fields. Universality is thus determined fundamentally by the physical structure of the qubit implementation and by control of this. Early studies of universality sought on the one hand to establish specific examples of universal gate sets [5], and on the other hand to determine whether there were any restrictions on the underlying interactions. This was nicely summarized by the pleasing result that “almost any”, i.e., a generic two-body interaction could provide universal operations [6, 7].

Now that experimental efforts to implement quantum computation are being undertaken, the question of just which interactions to use in a particular physical implementation becomes relevant. In particular, there is often a significant distinction between controlling an interaction that is intrinsic to the system, and introducing a new interaction with an external control field. In many of the physical implementations that have been suggested to date, the inherent physical interactions do not suffice to generate the universal set of quantum computing operations over physical qubits and must be supplemented by such additional, external Hamiltonian terms. This may introduce demanding nanoscale engineering constraints as well as additional unwanted sources of decoherence. Consequently, the question of whether and how we can use a particular physical system containing some very specific, non-generic interactions, for universal quantum computation has become very relevant, now that experimental efforts to implement small scale quantum logic are underway.

Recently, Whaley and coworkers [8, 9, 10, 11] have established a new paradigm of “encoded universality”, within which the limitations of non-generic intrinsic physical interactions can be overcome by a suitable encoding of the states to be used for quantum logic into a subspace of the system Hilbert space that is derived from two or more physical qubits. This is achieved with a general algebraic approach that allows the appropriate encoding to be determined from analysis of the properties of Lie algebras that are generated by the interaction Hamiltonian. To date, this general algebraic encoded universality approach has successfully been applied to two kinds of interactions, both of which are variants of the two-particle exchange interaction. The isotropic (Heisenberg) exchange interaction, i.e., $J^x = J^y = J^z \equiv J$ (for meaning of the symbols $J$, see [1]), was proved universal on encodings of three physical qubits and higher [12, 13]. Efficient solutions were found for the smallest encoding, when the logical qubit is encoded into three physical qubits and the interaction can be implemented between any pair of neighboring physical qubits (arranged in a chain or two-dimensional lattice) [12]. More recently, the symmetric anisotropic exchange, $J^x = J^y \neq J^z = 0$ has also been demonstrated to be universal [8, 14]. Like the Heisenberg case, the smallest encoding here is also three physical qubits. In the case of the symmetric anisotropic exchange, it was shown that this could be used to form either an encoded qubit or an encoded qutrit [14].

These instances of encoded universality derived from a single physical interaction are to be distinguished from related results of Levy [15] and of Wu and Lidar [16, 17] for exchange interactions that are necessarily supplemented by an additional single-qubit energy spectrum, $\sigma^z$. With this additional interaction, whether imposed statically or coupled to an external control field as in the earlier paradigm of controllable one- and two-qubit interactions, Wu and Lidar found a two-qubit encoding that was universal. These authors have applied this approach incorporating the additional $\sigma^z$ interaction to more general situations described by Hamiltonians containing exchange interactions with coupling coefficients $J_{ij}^{\alpha}$ having different values [18]).

The significance of these encoded universality schemes for quantum computation lies in the fact that they require
active manipulation only of two-particle exchange interactions, and hence can be generically referred to “exchange-only computation”. They are closely related to numerous proposals for quantum computation in solid state systems in which the exchange interaction is a common feature. These can be summarized as follows. (I) The case of isotropic exchange, $J^\alpha = J$ for $\alpha = x, y, z$, is represented by spin-coupled quantum dots [13, 21] and by donor atom nuclear/electron spins [22]. (II) The symmetric anisotropic exchange, $J^\alpha = J$ for $\alpha = x, y$, includes quantum dots in cavities [23], atoms in cavities [24] exciton coupled quantum dots [25] and quantum Hall systems [26]. (III) A more general anisotropic exchange, $J^\alpha = J \neq J^y \neq 0$ for $\alpha = x, y$, is represented by the proposal to use electrons on helium as qubits [27]. While the general two-body exchange Hamiltonian $\hat{H}_{ij} = \sum^\alpha J^\alpha_{ij} \hat{\sigma}^\alpha_i \hat{\sigma}^\beta_j$, where $\alpha = x, y$ and $\alpha$ and $\sigma^\alpha$ are the Pauli matrices, is applicable to these theoretical solid state proposals, we note that this exchange Hamiltonian does not contain any cross product terms $\hat{\sigma}^\alpha_i \hat{\sigma}^\beta_j$, $\alpha \neq \beta$ which may result for instance from the Dzyaloshinski-Moriya term of the spin-orbit interaction [28, 29].

In this paper we develop the encoded universality representation for the generalized anisotropic interaction that results from allowing asymmetry in the exchange tensor, *i.e.*, $J_z = 0$, $J^x \neq J^y$, and that also incorporates additional cross-terms, $\hat{\sigma}^\alpha_i \hat{\sigma}^\beta_j$, $\alpha \neq \beta$, in the Hamiltonian. This interaction becomes relevant to some of the recent proposals for solid state implementation of quantum computation when additional physical effects such as symmetry breaking perturbations [17], originating e.g., from surface and interface effects, spin-orbit coupling [28, 29], dipole-dipole coupling in the spin-spin interaction, and anisotropy in exciton exchange interaction in quantum dots [30, 31, 32] are taken into account. The asymmetric anisotropic interaction including these cross-terms is also part of a more general model considered by Terhal and DiVincenzo [33] in the framework of fermionic quantum computation [34, 35].

The structure of the remainder of this paper is as follows. In Sections II and III we use the algebraic approach developed by Kempe et al. [8, 9, 11] to establish code spaces for universal encoding of the generalized anisotropic exchange interaction. These are found to be two four-dimensional subspaces of the original Hilbert space for three physical qubits, characterized by the parity of the bit string (even or odd number of logical values 1). A logical qubit can be formed within any of these four-dimensional code spaces. In Section IV we consider how to perform single-qubit operations on these encoded qubits. It is shown that the asymmetry of the anisotropic interaction is removed via the commutation relations of two-body interaction Hamiltonians within this subspace. The symmetric anisotropic interaction including these cross-terms is also part of a more general model considered by Terhal and DiVincenzo [33] in the framework of fermionic quantum computation [34, 35].

II. GENERALIZED ANISOTROPIC EXCHANGE INTERACTION

A. Asymmetric Anisotropic Exchange Interaction

The anisotropic exchange interaction between two physical qubits, $i$ and $j$, is described by the following Hamiltonian operator,

$$\hat{H}_{ij} = \frac{1}{2} \sum_{\alpha=x,y} J^\alpha_{ij} \hat{\sigma}_i^\alpha \hat{\sigma}_j^\alpha = \frac{1}{2}(J^x_{ij} \hat{\sigma}_i^x \hat{\sigma}_j^x + J^y_{ij} \hat{\sigma}_i^y \hat{\sigma}_j^y),$$

where $J^\alpha_{ij}$ is the coupling strength between the qubits, the upper index $\alpha$ corresponds to either the $xx$ or $yy$ term of the exchange interaction, and $\sigma^\alpha$ are the Pauli matrices. If both coupling strengths are identical, the Hamiltonian describes the symmetric anisotropic interaction often referred to as the XY model. The encoded universality [8, 11, 13] for this case was studied by Kempe et al. [8, 11, 13]. The asymmetric anisotropic exchange interaction is defined when $J^x_{ij} \neq J^y_{ij}$. This can be reexpressed as a sum of symmetric ($s$) and antisymmetric ($a$) terms,

$$\hat{H}_{ij} = \hat{H}_{ij}^s + \hat{H}_{ij}^a = \frac{1}{2}(J^x_{ij} \sigma^x_i \sigma^x_j + \sigma^y_i \sigma^y_j) + J^a_{ij}(\sigma^x_i \sigma^x_j - \sigma^y_i \sigma^y_j),$$

where $J^s_{ij} = \frac{1}{2}(J^x_{ij} + J^y_{ij})$ and $J^a_{ij} = \frac{1}{2}(J^x_{ij} - J^y_{ij})$. This asymmetric anisotropic Hamiltonian can be seen to split into two distinct parts that act on orthogonal two-dimensional sectors of the four-dimensional Hilbert space if the
symmetric term is reexpressed as proportional to \((\sigma^+_i \sigma^-_j + \sigma^-_i \sigma^+_j)\), and the antisymmetric component as proportional to \((\sigma^+_i \sigma^-_j - \sigma^-_i \sigma^+_j)\), where \(\sigma^+ = (\sigma^x + i\sigma^y)/2\) and \(\sigma^- = (\sigma^x - i\sigma^y)/2\) are raising and lowering operators of the system. These sectors are characterized by the parity of the bit string which refers to even or odd occupation number defined as the number of 1’s in the bit string. In particular, the symmetric term \(\hat{H}^s_{ij}\) operates in the subspace spanned by \(\{|01\}, |10\}\), and the antisymmetric term \(\hat{H}^a_{ij}\) in the subspace spanned by \(\{|00\}, |11\}\). We explicitly point out that the symmetric term preserves the occupation number, while the antisymmetric changes this occupation number by two. If these pairs of two-qubit states are taken to form logical qubit states, then it is easily verified that both \(\hat{H}^s_{ij}\) and \(\hat{H}^a_{ij}\) act as \(\sigma^z\) on these pairs of two-qubit states.

The origin of the asymmetry in the anisotropic interaction can be understood as a consequence of energy non-conserving process similar to the anti-rotating wave terms arising in the interaction of a two-level system with semiclasical radiation, but happening now in a correlated way on both coupled physical qubits. We may assume that asymmetry in the anisotropic exchange interaction between physical systems is a consequence of the system complexity when numerous mechanisms of mutual coupling take place simultaneously. An example of similar symmetry breaking in the case of the isotropic (Heisenberg) exchange interaction between quantum dots derives from the spin-orbit or a, usually weaker, dipole-dipole coupling.

### B. Cross-Product Terms

In general, it has been recognized recently that anisotropy in the exchange interaction may be accompanied by cross-product terms in the two-body Hamiltonian \[33\]. These can arise, for instance, from spin-orbit coupling \[36\] as noted above. The total interaction can then be described as follows,

\[
\hat{H}_{ij} = \frac{1}{2} \sum_{\alpha=x,y} J_{ij}^{\alpha} \sigma_\alpha^i \sigma_\alpha^j + \frac{1}{2} \sum_{\alpha \neq \beta = x,y} J_{ij}^{\alpha \beta} \sigma_\alpha^i \sigma_\beta^j
\]  

(3)

This Hamiltonian can be reexpressed in a form that emphasizes the effect of its various terms on subspaces of different parity (the upper index \(s\) for odd and \(a\) for even):

\[
\hat{H}_{ij} = \hat{H}^s_{ij} + \hat{H}^a_{ij} = \hat{H}^s_{ij} + \hat{H}^a_{ij}
\]

\[
= \frac{1}{2} [J_{ij}^x (\sigma^x_i \sigma^x_j + \sigma^y_i \sigma^y_j) + J_{ij}^y (\sigma^x_i \sigma^y_j - \sigma^y_i \sigma^x_j)] + \frac{1}{2} [K_{ij}^x (\sigma^x_i \sigma^x_j - \sigma^y_i \sigma^y_j) + K_{ij}^y (\sigma^x_i \sigma^y_j + \sigma^y_i \sigma^x_j)],
\]  

(4)

where \(K_{ij}^x = \frac{1}{2} (J_{ij}^{xy} + J_{ij}^{yx})\) and \(K_{ij}^y = \frac{1}{2} (J_{ij}^{xy} - J_{ij}^{yx})\). We note that the cross-product terms \(\hat{H}^s_{ij}\) and \(\hat{H}^a_{ij}\) act on the subspace spanned by the basis states \(\{|01\}, |10\}\) and \(\{|00\}, |11\}\), respectively. Both terms are seen to act as a \(\sigma^y\) operation on these states.

These subspaces are seen to be spanned by basis sets characterized by the bit-string parity \(B^s = \{|01\}, |10\}\) and \(B^a = \{|00\}, |11\}\). The action of the total Hamiltonian is simultaneous in both subspaces. In particular, the symmetric component of the interaction (indexed \(s\)) acts only in \(B^s\), and the antisymmetric part (indexed \(a\)) only in \(B^a\). In each of the two subspaces the interaction is characterized by the expression \(J_{ij}^{\alpha} \sigma^\alpha_{Bk} + K_{ij}^{\alpha} \sigma^\alpha_{Bk}\), where \(k\) is either \(s\) or \(a\). This can be reformulated as \(\hat{J}_{ij}^{s} \sigma^s_{Bk} + \hat{J}_{ij}^{a} \sigma^a_{Bk}\), where the effective coupling now becomes a complex number,

\[
\hat{J}^{s}_{ij} = J^{s}_{ij} - iK^{s}_{ij}.
\]  

(5)

The operators \(\sigma^s_{Bk}, \sigma^a_{Bk}\) and \(\sigma^\alpha_{Bk}\) now apply to the pairs of states within any of the two-dimensional subspaces \(B^s\) or \(B^a\).

### III. Algebraic Aspects of the Interaction

The set of asymmetric anisotropic exchange Hamiltonians between neighboring physical qubits, given by \[33\], \(\{\hat{H}_{ij}, 1 \leq i < j \leq n\}\), generates the Lie algebra \(L\), where \(n\) is the total number of physical qubits. We consider first the Hamiltonian without cross terms. The effect of the cross-product terms will be considered below. We follow
here the algebraic approach due to Kempe et al. [8, 9, 11] and first study the properties of the algebra commutant \( \mathcal{L}' \). Our goal is to identify suitable encoding of quantum information and so we do not here exploit the potential of the algebraic approach in providing a constructive proof for such encodings for general \( n \). Since the symmetric anisotropic interaction has been proved to be universal over three-qubit and higher encodings [8, 9, 13], we shall examine here only the minimal case where \( n = 3 \) to see whether analogous results hold for asymmetric anisotropic exchange.

We identify two operators as the elements of the commutant:

\[
\hat{Z} = \bigotimes_{k=1}^{n} \sigma_z^k, \quad \hat{X} = \bigotimes_{k=1}^{n} \sigma_x^k.
\]  

(6)

Both of these operators commute for even \( n \), and anticommute for odd \( n \). We remark that \( \hat{Z} \) is a parity operator. This commutes with the generalized anisotropic exchange Hamiltonian which preserves the parity. We point out that in the case of the symmetric anisotropic (XY) interaction, the commutant becomes larger, represented now by the operators \( \hat{X} \) and \( \hat{S}_z = \bigoplus_{k=1}^{n} \sigma_z^k \) [8]. In the case of the isotropic exchange interaction these are further expanded to \( \hat{S}_x \) and \( \hat{S}_z \) [9]. In the present work we focus on the universality properties and do not address the decoherence-free aspects of the encoding. Also, we note that including the cross-product terms into the Hamiltonian operator changes the commutant structure, since \( \hat{X} \) is no longer an element of the resulting commutant. We show explicitly in Section IV that the proposed codes derived from the algebraic analysis without cross terms are nevertheless also universal for the general case including the cross-product terms. The remainder of this Section will therefore continue to deal with the algebraic analysis for (2) alone.

We assume that the algebra \( \mathcal{M} \) generated from these operators by linear combination and multiplication is identical to the commutant \( \mathcal{L}' \). Then, the splitting of \( \mathcal{M} \) into the irreducible representations \( J \in \mathcal{J} \),

\[
\mathcal{L}' = \mathcal{M} = \bigoplus_{J \in \mathcal{J}} \mathcal{I}_{n,J} \otimes M(C^{d_J})
\]  

(7)

translates into the structure of the irreducible representations of the Lie algebra generated by the Hamiltonian operators

\[
\mathcal{L} \cong \bigoplus_{J \in \mathcal{J}} \mathcal{L}_j(n,J) \otimes \mathcal{I}_{d_J}
\]  

(8)

over the Hilbert space

\[
\mathcal{H} \sim \sum_{J \in \mathcal{J}} C^{n,J} \otimes C^{d_J},
\]  

(9)

where \( n_J \) and \( d_J \) are the dimension and degeneracy of irreducible representation \( J \), respectively.

For the case of three physical qubits, \( n = 3 \) (note that \( N = 2^n \)), the operators \( \hat{X} \) and \( \hat{Z} \) are 8x8 and possess a block diagonal structure of four 2x2 blocks

\[
\hat{X}_B = \bigoplus_{k=1}^{N/2} \sigma_k^x, \quad \hat{Z}_B = \bigoplus_{k=1}^{N/2} \sigma_k^z
\]  

(10)

when expressed in the basis set \( B \) obtained by a suitable permutation of the standard basis:

\[
B = \{|000\}, |111\}, |110\}, |001\}, |101\}, |010\}, |011\}, |100\}\}
\]  

(11)

The commutation relation taken over these two operators generates \( \hat{Y}_B = \bigoplus_{k=1}^{N/2} \sigma_k^y \), and hence the complete \( \text{su}(2) \) algebra over the 2x2 blocks.

The algebra \( \mathcal{M} \) is now expressed as the tensor product \( \mathcal{I}_4 \otimes M(C^2) \). Consequently, its commutant \( \mathcal{M}' \) - associated with the Lie algebra \( \mathcal{L} \) by our assumption that \( \mathcal{M} \) is identical to the Lie algebra commutant \( \mathcal{L}' \) - splits as \( M(C^2) \otimes \mathcal{I}_2 \). The Hilbert space of this system splits accordingly into two four dimensional subspaces, \( \mathcal{H}^2 = \mathcal{H}^1 \oplus \mathcal{H}_4 \), which
are characterized by different bit-string parities. As expected, these subspaces are not mixed by the interaction Hamiltonian \( H \), which preserves the bit-string parity. The four-dimensional subspaces thus define two independent codes that are spanned by the following two sets of code words:

\[
\begin{align*}
(I) & \quad \{ |000\rangle, |110\rangle, |101\rangle, |011\rangle \} \\
(II) & \quad \{ |111\rangle, |001\rangle, |010\rangle, |100\rangle \}
\end{align*}
\]

(12)

These states are used as convenient basis sets for representing \( H \). In the following they will be referred to the order above, i.e. states 1, 2, 3, and 4 reading from left to right. Before we define the qubit encoding onto these subspaces, we first examine the effect of the asymmetric anisotropic exchange interaction Hamiltonian on the code words.

**IV. SINGLE-QUBIT OPERATIONS**

**A. Asymmetric Anisotropic Exchange**

As shown above, the symmetric and antisymmetric component of the exchange Hamiltonian (2) act simultaneously on two orthogonal two-dimensional subspaces spanned respectively by \( B^s = \{ |01\rangle, |10\rangle \} \) and \( B^a = \{ |00\rangle, |11\rangle \} \).

We now apply this Hamiltonian to the pairs of physical qubits 1-2, 1-3, and 2-3, in the three qubit codes given by (13). We emphasize that the effect of this interaction is the same for both codes, i.e., for (I) and for (II). In fact, the Hamiltonian acts simultaneously and identically on both subspaces \( H^s \), without mixing them, and it can therefore be expressed as a direct sum of two 4x4 matrices in the basis of the codes (I) and (II). We now analyze the action of the Hamiltonian on these codes. In the code basis (12), the effect of the asymmetric anisotropic exchange interaction, schematically summarized in Figure 1, applied to any pair of these qubits possess the following forms:

\[
H_{12} = \begin{pmatrix}
0 & J^a & 0 & 0 \\
J^s & 0 & 0 & 0 \\
0 & 0 & 0 & J^s \\
0 & 0 & J^s & 0
\end{pmatrix},
\]

\[
H_{13} = \begin{pmatrix}
0 & 0 & J^a & 0 \\
0 & 0 & 0 & J^s \\
0 & J^s & 0 & 0 \\
J^a & 0 & 0 & 0
\end{pmatrix},
\]

\[
H_{23} = \begin{pmatrix}
0 & 0 & 0 & J^a \\
0 & 0 & J^s & 0 \\
J^s & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

(13)

Here the lower index indicates between which physical qubits, \( i \) and \( j \), the interaction is turned on, and \( J^a = J_{ij}^a / 2 \), \( J^s = J_{ij}^s / 2 \) are the coupling strengths for the symmetric and antisymmetric parts respectively of (2), for physical qubits \( i \) and \( j \). For instance, for a triangular arrangement (shown in Figure 3 below) the coupling strengths \( J^s \) and \( J^a \) are the same in each of the Hamiltonian matrices (13), since they are derived from equivalent nearest neighbor interactions. This is necessary for the elimination of the antisymmetric component that is accomplished below via use of commutation relations, and it therefore affects the architecture of a potential qubit array. This aspect is discussed further in Section VI below. In order to clarify the notation, we provide an example, noting that, e.g., the matrix \( H_{12} \) represents the coupling between the physical qubits 1 and 2 which simultaneously transforms the logical qubits 1 and 2 (in each of the code spaces (I) or (II)) via the antisymmetric component of the Hamiltonian (\( J^a \)), and the logical qubits 3 and 4 through its symmetric component (\( J^s \)).

Let us now consider the action of these three matrices and of their commutators. We start with \( H_{12} = (J^a \sigma^z) \oplus (J^s \sigma^z) \). From (13) it is evident that the symmetric component of \( H_{12} \), i.e., the lower right 2x2 block, acts as a \( \sigma^z_{34} \) operation over the code words 3 and 4 from (12), i.e., the states \( |101\rangle \) and \( |011\rangle \) from (I), with coupling strength \( J^s \). It has the same effect over the states \( |100\rangle \) and \( |001\rangle \) from (II), i.e. it acts as encoded \( \sigma^x \) on the states in both (I) and (II). The antisymmetric component of \( H_{12} \), which is the top left 2x2 block of this matrix, acts on the other two orthogonal states from the code, namely \( |000\rangle \) and \( |110\rangle \) from (I), or on \( |111\rangle \) and \( |001\rangle \) from (II). This also results in an encoded \( \sigma^x \) operation but with coupling strength \( J^a \). This is the first element required for an encoded SU(2) operation. The effect of these interactions is summarized schematically in Figure 3.

The second element is an encoded \( \sigma^y \) operation. These operations are generated through the commutator of a pair of Hamiltonian operators from (13). For instance, taking the commutator of interactions between physical qubits 1-3 and qubits 2-3 yields \( [H_{13}, H_{23}] = i[(J^a)^2 - (J^s)^2] \sigma^y_{34} \), where \( \sigma^y_{34} \) acts exclusively on the states \( |101\rangle \) and \( |011\rangle \). Since all other elements of the resulting 4x4 matrix are equal to zero, this commutation relation results exclusively in an encoded \( \sigma^y \) operator between the code words 3 and 4, i.e.,

\[
[H_{13}, H_{23}] = i \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & i[(J^a)^2 - (J^s)^2] \\
0 & -i[(J^a)^2 - (J^s)^2] & 0 & 0
\end{pmatrix},
\]

(14)
The third and last element required for an encoded SU(2) operation is encoded \( \bar{\sigma}^z \). It can be easily verified that these operations are now obtainable from a second level commutator, namely of the Hamiltonian (13) with the encoded \( \bar{\sigma}^y \) operations. For example, \([H_{12}, \sigma^y_{34}] = i2J^z\sigma^z_{34}\).

Together, these three encoded \( \bar{\sigma}^x, \bar{\sigma}^y, \) and \( \bar{\sigma}^z \) operations ensure that any arbitrary SU(2) operation may be performed on the encoded qubits. The interactions underlying these operations and the combinations just described are summarized schematically in Figure 2. We note that the Hamiltonian matrices always act simultaneously on both sets of orthogonal subspaces \((I)\) and \((II)\). We can use the encoded operations described above to generate additional encoded \( \bar{\sigma}^x \) operations that do not simultaneously act on the orthogonal subspaces from the code, by forming the commutator between the \( \bar{\sigma}^y \) and \( \bar{\sigma}^z \) operators.

Analogous sets of operators can be defined starting from the other two exchange Hamiltonians, i.e., \( H_{13} \) and \( H_{23} \). The connections resulting from all of Hamiltonian interactions and their commutators are equivalent in each case to those illustrated in Figure 2. In total therefore, we have three distinct ways of defining the logical qubit from each of the subspaces \((I)\) and \((II)\), with arbitrary SU(2) operations possible on any of these six possible sets of qubits. From the subspaces \((I)\) the possible encodings are \({\{|110\rangle, |011\rangle\}}\), or \({\{|110\rangle, |101\rangle\}}\), or \({\{|101\rangle, |011\rangle\}}\). From the subspace \((II)\) the possible qubit encodings are \({\{|001\rangle, |100\rangle\}}\), and \({\{|001\rangle, |010\rangle\}}\), and \({\{|010\rangle, |100\rangle\}}\).

Examination of the commutators derived from each of the three different starting exchange Hamiltonians shows that the resulting encoded operations, e.g., (13), are in each case characterized by zeros in the corresponding locations where the antisymmetric terms appear in (13). This means that by making use of the commutation relations defined by the interactions of two-physical qubits from the three qubit codes, we have completely eliminated the effect of that part of the interaction which, as noted before, changes the occupation number of the code by two, and which corresponds to the antisymmetric component of the Hamiltonian, \( J^{\text{xy}} \) (see (2)). Up to a numerical factor given by the product of coupling strengths that are accumulated in the course of applying the commutation relations, the problem then reduces to that of the symmetric anisotropic exchange solved previously in \( \text{(3)} \). We note that in the limit \( J^{\text{xy}} \to 0 \), the results of Kempe and Whaley obtained for encoding into three physical qubits \( \text{(13)} \) are reconstructed. The corresponding three-qubit codes, spanned by \({\{|110\rangle, |011\rangle\}}\) or \({\{|001\rangle, |010\rangle, |100\rangle\}}\), define a logical qutrit, and the Hamiltonian operators over the code become 3x3 matrices generating the complete su(3) Lie algebra \( \text{(13)} \). The fact that SU(2) is a subgroup of SU(3) then further implies existence of the truncated qubit representation within the three-qubit code that was established in \( \text{(3)} \). In the truncated qubit representation, one of the physical qubits is kept constant, and is used merely as an auxiliary element for generation of the necessary commutation relations.

We also remark that the spin-orbit generated anisotropy can be alternatively eliminated to the first order in the spin-orbit coupling by suitable shaping of the pulsed interaction between physical qubits as recently proposed \( \text{(30, 37)} \).

### B. Cross-Product Terms

The inclusion of the cross-product terms transforms the Hamiltonian operators given by (13) into hermitian matrices of the same structure whose coupling coefficients \( J^n = J_{ij}^n/2 \) and \( J^z = J_{ij}^z/2 \) are now complex (see \( \text{(4)} \)). In fact, the situation captured in the Hamiltonian of (4) is the most general anisotropic exchange form containing asymmetry in all terms including the cross-products. It provides a generalization of the usual symmetric anisotropic exchange referred to as an XY model.

Under these circumstances, application of the commutation relations between the Hamiltonian matrices (13) is still capable of generating the su(2) algebra for single qubit operations. The result of the commutation relation is again proportional to the \( \bar{\sigma}^y \) operation. For instance, \([H_{13}, \hat{H}_{23}] = i(J^y)^2 - (J^z)^2)\sigma^y_{34} \). On the other hand, elementary matrix algebra shows that now only two of three possible commutation relations between pairs of complex Hamiltonian matrices (13) of the three-qubit code can eliminate the coupling between states of different occupation number and thereby generate this encoded \( \bar{\sigma}^y \). The commutation relation which does not generate this transformation is \([H_{12}, \hat{H}_{23}] \). This fact limits which two of three possible logical qubit encodings should be considered as universal out of the codes listed in \( \text{(2)} \). If only the antisymmetric cross-product term, \( \hat{H}_{ij}^\text{xy} \), in \( \text{(4)} \) is considered (i.e., \( K_{ij}^a = 0 \)) this limitation is removed and all three commutation relations can result in cancellation of the antisymmetric component of the interaction. The other relations generating encoded \( \bar{\sigma}^x \) and \( \bar{\sigma}^z \) hold accordingly.

In conclusion, the commutation relations suffice to completely remove asymmetry in the most general anisotropic interaction, including the cross-product terms \( \sigma^x_j \sigma^y_i \). The encodings proposed here provide a direct route to elimination of these terms. In Section \( \text{VI} \) we show how to implement the commutation relations efficiently.
V. TWO-QUBIT OPERATIONS

An entangling two-qubit gate - namely the controlled-Z (C(Z)) operation - is obtained via the following sequence of encoded $\bar{\sigma}^z$ operations:

$$\hat{U}_{C(Z)} = e^{i\bar{\sigma}^z_1 \pi/4} e^{i\bar{\sigma}^z_2 \pi/4} e^{-i(\bar{\sigma}^z_1 \otimes \bar{\sigma}^z_2) \pi/4}. \quad (15)$$

The crucial element of this sequence is the last term on the right-hand side. This is enacted by applying the encoded $\bar{\sigma}^z$ operation onto the triplet of physical qubits 2-3-4 that connects two logical qubits within the triangular architecture (see Figure 3). To illustrate this C(Z) sequence, we focus on an example with the following encoding of logical qubit: $|0_L\rangle = |110\rangle$, $|1_L\rangle = |011\rangle$. The logical two-qubit configurations are then given as

$$
\begin{align*}
|0_L0_L\rangle &= |10\rangle_1 0\rangle_2 0\rangle_3 \\
|0_L1_L\rangle &= |10\rangle_1 0\rangle_2 1\rangle_3 \\
|1_L0_L\rangle &= |01\rangle_1 0\rangle_2 0\rangle_3 \\
|1_L1_L\rangle &= |01\rangle_1 0\rangle_2 1\rangle_3,
\end{align*}
$$

where the boxes indicate those physical qubits which are 'bridging' two logical qubits. Via commutation relations of the exchange Hamiltonians between the physical qubits 2-4 and 3-4 within the triangular architecture (Figure 3) we generate the $\sigma^y_{2,3}$ operation which, when commuted further with the exchange interaction between the qubits 2 and 3, results in the corresponding $\sigma^z$ operation. Turning this $\sigma^z$ operation on for the duration $t = \pi/2$ results in a phase transformation of the states, such that $|0_L0_L\rangle = |10\rangle_1 0\rangle_2 0\rangle_3 \rightarrow e^{-i\pi/2} |0_L0_L\rangle$ and $|0_L1_L\rangle = |10\rangle_1 0\rangle_2 1\rangle_3 \rightarrow e^{i\pi/2} |0_L1_L\rangle$. The other two states are not addressed by the encoded operation and remain intact. The resulting diagonal transformation over the two-qubit logical states, characterized by diagonal elements $\{-i, i, 1, 1\}$, has provided the desired entanglement between the logical qubits. We emphasize that we needed one double commutator to obtain this transformation. In order to illustrate that this suffices to generate the controlled-Z operation, we first apply an encoded $\bar{\sigma}^z$ onto the second logical qubit for duration $t = \pi/4$. This further transforms the relative phase relations between the states of two logical qubits to $\{-i, i, 1, -i\}$ (up to an overall phase $e^{i\pi/4}$). This result is equivalent to the unitary transformation $e^{-i(\bar{\sigma}^z_1 \otimes \bar{\sigma}^z_2) \pi/4}$ in (13). This transformation, when supplemented by the encoded single qubit $\bar{\sigma}^z$ rotations on both logical qubits, results in the desired controlled-Z operation, C(Z) [2].

VI. IMPLEMENTATION ISSUES

We now turn our attention to practical aspects of implementation of universal quantum computation with generalized anisotropic exchange interactions. Our goal is now to translate the theoretical development of encoded universality with this class of Hamiltonians into an appropriate quantum circuit. So far, we have employed the commutation relations between the interaction Hamiltonians (13) to generate an su(2) algebra over a suitably selected code subspace [12]. A practical question is how to implement the commutation relations. In principle, this can always be carried out via the Baker-Hausdorff-Campbell operator expansion [9]. However this does not necessarily provide the efficiency required in practical implementation. A useful approach in the present context is based on conjugation by unitary operations considered previously by Kempe et al. [3, 14] and by Lidar and Wu [17]. Conjugation was recently applied to the case of symmetric anisotropic exchange interactions by Kempe and Whaley [13]. The key observation here was that in the three-qubit encoding of a logical qutrit, the complete SU(3) Lie group can be obtained through conjugating the evolution operators generated by the symmetric anisotropic Hamiltonians over the physical qubits.

A general feature of conjugation operations that we would like to stress in the present context, is that they can provide the same effect over the encoded qubit as exponentiated commutation relations. The goal is therefore to find a conjugating condition under which this equivalence holds. In the present case, the antisymmetric term in the Hamiltonian (2) complicates the situation, since the conjugating sequence of the unitary evolutions in general mixes different states of the code space, and may also result in leakage of the encoded qubit population into the orthogonal part of the code subspace. We note however that the mixing effect of the symmetric and antisymmetric term of the Hamiltonian (2) can be eliminated by choosing a suitable duration of the exchange interaction. Since $[\hat{H}^s_{ij}, \hat{H}^a_{ij}] = 0$ for any $i$ and $j$, the unitary evolution operator generated by the Hamiltonian (2) splits into a product $U(\tau) = \exp(-i\hat{H}^s_{ij}\tau)\exp(-i\hat{H}^a_{ij}\tau)$. For a suitably chosen time duration, one of the terms can always be made to
generate the identity from the interaction Lie group, if \( J^s_i \neq J^a_i \). At the same time, the effect of the other term can be tuned to provide desired transformation.

### A. Single qubit operations

We now illustrate this possibility of turning off the mixing effect of the antisymmetric terms in the evolution operator by judicious choice of conjugation operations, with a specific example. For instance, the unitary evolution generated by the \( \sigma_y^{ij} \) operator, resulting from the commutation relation \([H_{13}, H_{23}]\), can be obtained from the following conjugation:

\[
\hat{U}(\sigma_y^{ij}, \phi) = e^{-i\sigma_y^{ij}\phi} = e^{iH_{13}\theta}e^{iH_{23}\phi^*}e^{-iH_{13}\theta},
\]

where \( \phi' = \phi/J^s \), and \( \theta \) is the time duration satisfying simultaneously the two conjugation conditions

\[
\theta = 0 (mod \pi)/J^a = \pi/2 (mod \pi)/J^s.
\]

Due to the asymmetry of the exchange coupling terms \((J^a \neq J^s)\) and to the unitarity of the quantum evolution, this condition can easily be fulfilled, as long as the ratio of \( J^s \) and \( J^a \) is not a rational number. We note that rational numbers create a subset of measure zero within the set of real numbers, and hence it is very unlike that we would meet such a situation in experimental implementations.

In order to further elucidate the effect of conjugation operations, we focus on analysis of the conjugating sequence expressed by \( (17) \). The timing condition \( (18) \) sets the unitary conjugation operator into the following matrix form in the code basis \( (12) \):

\[
\hat{U}(H_{13}, \theta) = e^{iH_{13}\theta} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & i & 0 \\
0 & i & 0 & 0 \\
i & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & i \\
i & 0 \\
0 & 0
\end{pmatrix} \oplus \begin{pmatrix}
1 & 0 \\
0 & i \\
i & 0 \\
0 & 0
\end{pmatrix}
\]

\[
\hat{P}_{24} = (\hat{S} \oplus \hat{S}) \hat{P}_{24}.
\]

Here \( \hat{P}_{24} \) is the permutation matrix exchanging the basis states \(|110\rangle\) and \(|011\rangle\) of the code \((I)\), or \(|001\rangle\) and \(|100\rangle\) of the code \((II)\). The operator \( \hat{S} \) is an operator inducing the shift of the relative phase by \( \pi/2 \). We emphasize that the antisymmetric term of the Hamiltonian \( H_{13} \) results in identity, while the symmetric term results in exchange between two code words that are phase-shifted by \( i \).

The identity \( \hat{U}e^{-\hat{U}H_{13}^\dagger}\hat{U}^\dagger = e^{-\hat{U}H_{13}\hat{U}^\dagger} \) allows us to reduce the act of conjugation of the unitary operator to the conjugation of its generator. Since the Hamiltonian matrices, \( (19) \), express interactions between different pairs of physical qubits within three qubit codewords, they are related to each other by permutation operations. For instance, the matrix \( H_{12} = (J^a \sigma^x) \oplus (J^s \sigma^x) \) can be expressed as \( \hat{P}_{24} H_{23} \hat{P}_{24} \). We can now express the desired conjugation in the following form,

\[
e^{iH_{13}\theta}H_{23}e^{-iH_{13}\theta} = (\hat{S} \oplus \hat{S})\hat{P}_{24} H_{23} \hat{P}_{24} (\hat{S}^\dagger \oplus \hat{S}^\dagger) = (J^a \sigma^x \hat{S}^\dagger) \oplus (J^s \sigma^x \hat{S}^\dagger) = (J^a \sigma^y) \oplus (J^s \sigma^y),
\]

Via conjugation, we have now obtained the 2x2 block diagonal matrix whose blocks are now proportional to the Pauli matrix \( \sigma^y \). This procedure provides a generalization of the well-known conjugation of the Pauli matrices: \( e^{i\sigma_z \pi/4}e^{i\sigma_x \phi}e^{-i\sigma_z \pi/2} = e^{i\sigma_y \phi} \). It allows us to generate the full \( su(2) \) algebra in each block, via conjugation with the Hamiltonian \( H_{12} \). The block-diagonal structure ensures that the antisymmetric component of the interaction does not mix with the symmetric one. In contrast to the effect of commutation relations between the Hamiltonian matrices \( (13) \), which eliminate the coupling between states of different occupation number within the codes \( (12) \), each block in \( (20) \) can now be used for single qubit operations over the corresponding code states. For the sake of simplicity, we choose the same qubit encodings as emerged from the commutation relations in Section \( IV \). The conjugation can alternatively be formulated to generate the \( \sigma^y \) transformations corresponding to the other two Hamiltonian operators in \( (12) \).

Development of a conjugating procedure for the case of the general Hamiltonian \( (I) \), containing the cross-product terms, is possible within the same framework. However the relevant timing conditions have to reflect that the
coupling coefficients $J^a$ and $J^s$ may now be complex numbers. Just as in the previous case, the goal is to generate the desired conjugating unitary transformation by exponentiating the appropriate general Hamiltonian operator, where the symmetric part of the interaction leads to exchange between the two coupled code words phase-shifted by $i$, and the antisymmetric term results in identity (see (19)). We first illustrate new timing conditions derived from focusing only on the antisymmetric term in the generalized anisotropic exchange.

The antisymmetric coupling acts on the state with even bit-string parity, $B^{\alpha} = \{ |00\rangle, |11\rangle \}$. It can be reformulated as the sum $J^a \sigma_0^a + K^a \sigma_0^y$ where $\sigma_0^a$ and $\sigma_0^y$ refer only to the even parity states. In order to establish the conjugating condition, this operator is exponentiated and factorized into the product of three unitary operators $e^{-i J^a \sigma_0^a} e^{-i K^a \sigma_0^y} e^{i J^a \sigma_0^a \sigma_0^y \theta / 2}$. The condition for attaining the identity is then:

$$\Theta = 0 \text{ (mod } \pi) / J^a = 0 \text{ (mod } \pi) / K^a = 0 \text{ (mod } \pi)/(J^a K^a / 2) \text{.}$$

Considering now in addition that the coupling coefficient $J^s$ is complex and its imaginary part is also to be eliminated, an analogous timing condition can easily be formulated.

The second conjugation needed for $\sigma^z$ operations, implementing the double commutator (Section V), is carried out in a similar fashion:

$$\hat{U}(\sigma^z_{34}, \phi) = e^{-i \sigma^z_{34} \phi} = e^{i H_{12} \theta} \hat{U}(\sigma^y_{34}, \phi) e^{-i H_{12} \theta},$$

where $\hat{U}(\sigma^z_{34}, \phi)$ is the result of the first conjugation given by (17). In our example, the condition for the time duration of the second conjugating operation generated by $H_{12}$ reads as $\theta = 0 \text{ (mod } \pi) / J^a = \frac{\pi}{4} \text{ (mod } \pi) / J^s$.

### B. Two qubit operations

We now focus on specific aspects of implementation of the two-qubit gates via conjugation. The entangling part of the controlled-Z gate, described in Section V above, is obtained as a conditional effect of the $\sigma^z$ operation on the physical qubits of both logical qubits (on the 'bridging' qubits). The conjugation however complicates the situation, due to its antisymmetric component which affects also the states $|1_L 0_L\rangle = |110\rangle$ and $|1_L 1_L\rangle = |111\rangle$.

However, the effect of the antisymmetric term in the interaction can be completely eliminated by imposing an additional timing condition for the conjugated operation. In our specific example, this operation was generated by $H_{23}$, and the timing condition is then given as follows:

$$\phi' = 0 \text{ (mod } 2\pi) / J^a = \phi \text{ (mod } 2\pi) / J^s.$$ (23)

We emphasize that this condition has to be satisfied only up to an arbitrary global phase.

### C. Efficiency

The present approach based on conjugation is much more effective than application of the Baker-Hausdorff-Campbell formula

$$e^{i[\hat{A}, \hat{B}]} = \lim_{n \to \infty} e^{-i \hat{A} \sqrt{n}} e^{i \hat{B} / \sqrt{n}} e^{i \hat{A} / \sqrt{n}} e^{-i \hat{B} / \sqrt{n}}$$

whose asymptotic character translates into a sequence of a large number of elementary operations. In contrast, the conjugation requires only three gates for implementation of the encoded $\sigma^y$ operation, emulating a single commutation relation, and five gates for encoded $\sigma^z$, corresponding to a double commutator. The entangling two-qubit operation, i.e. the controlled-Z up to the local transformations, is based on generating $\sigma^z$, and hence requires also just five discrete gates. This is the same as in the case of the symmetric anisotropic interaction studied previously [13]. The timing conditions, expressed in number of gates (18) and (21), translate into a prolonged transformation of the conjugating unitaries. It should be pointed out that the duration of the conjugating operation, given by the ratio of the coupling coefficients $J^a$ and $J^s$ in (18) for instance, does not change if larger number of logical qubits defined with this three-qubit encoding are addressed with these gate sequences. Therefore this approach scales well with size, having only a linear cost in terms of computational complexity as the number of encoded qubits increases.

An alternative to the present analytical approach based on conjugation is a numerical optimization of gate sequences in order to generate the desired quantum computing operations [12].
VII. ARCHITECTURE

Since the $\sigma^y$ interactions for a given Hamiltonian are defined through the commutation relations with the other two available couplings among three physical qubits, the most suitable architecture is triangular. This is summarized in Figure 2. An equilateral triangular architecture then ensures that $J^x$ and $J^y$ are the same within any pair of physical qubits taken from a three qubit code. To additionally accommodate also two-qubit logical operations, it is convenient to arrange triangles of physical qubits into a linear chain with alternating triangle orientations. This layout is shown in Figure 3. Other layouts, such as a hexagonal lattice, may also be employed.

The implementation of commutation relations between exchange Hamiltonians via unitary conjugation allows for a number of other architecture structures than equilateral triangle. The change in the coupling strengths between physical qubits, which may result from other architectures, would be reflected in the timing conditions for conjugating operations discussed above. In fact, this flexibility is an important aspect of the implementation of the Lie algebra of the generalized anisotropic exchange via unitary conjugation, because it allows one to relax the requirement of an equilateral triangular architecture, to a lattice of a rectangular or any other structure in order to accommodate physical and experimental requirements.

VIII. CONCLUSION

In the present work we have demonstrated that encoded universality may be achieved for generalizations of the anisotropic exchange interaction that remove the symmetry between exchange components acting in the $x$ and $y$ directions, and that also incorporate cross-product terms in the Hamiltonian. Using the algebraic approach due to Kempe et al. [8, 9, 10, 11], we find that the Lie algebra generated by asymmetric anisotropic exchange interaction within encoding into three physical qubits splits into two irreducible representations that act correspondingly on two invariant four-dimensional subspaces of the Hilbert space. Their basis sets are used to define two sets of four code words each. Analysis of actions of generalized exchange interactions and their commutation relations within three physical qubits results in generation of the full $su(2)$ algebra over a single logical qubit. The most suitable architecture, capturing both the physical properties of the code and the interactions among its elements, is a chain of equilateral triangles of alternating orientations. Application of encoded operations within physical qubits connecting two logical qubits is shown to result in an entangling two-qubit operation, namely the controlled-Z.

Implementation issues, related to the efficient implementation of the commutation relations among exchange interactions, were studied in connection with the properties of unitary conjugation. It was shown that the effect of the commutation relation between a pair of physical interactions is perfectly mimicked by suitably timed conjugation of the unitary operations that are generated by these Hamiltonians. The timing conditions, explicitly formulated here, result in significant improvement of implementation efficiency, compared to both the asymptotic approach based on the Baker-Hausdorff-Campbell formula and to recent numerically optimized gate sequences for exchange Hamiltonian [12]. The results were found to be valid also in the presence of cross-product terms $\sigma_i^x \sigma_j^y$ in the generalized Hamiltonian. Within the implementation of unitary operations via unitary conjugation, proposed here, the proposed equilateral triangular architecture may be relaxed according to the experimental situation.

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FIG. 1: Actions of the asymmetric anisotropic exchange interactions over the three qubit code spaces. The solid line transforms code words via the symmetric component of the Hamiltonian while the dashed line through its antisymmetric part. The earlier changes the bit-string parity and preserves the occupation number; the latter changes the occupation number by two while conserving the parity. Indexes indicate which physical qubits are coupled.
FIG. 2: The commutator algebra between the exchange interaction within the three qubit encoding generates the full su(2) over the encoded logical qubit. The commutation relations generating this single qubit operations dictate that the appropriate architecture be an equilateral triangle.
FIG. 3: The layout of the scalable architecture. The two qubit entangling operation, a controlled-Z gate, is implemented using the physical qubits connecting two logical qubits within the layout, e.g. physical qubits 2-3-4.
FIG. 4: A quantum circuit for generation of the encoded $\sigma^y$ operation via unitary conjugation.