COUPLING CHIRAL BOSONS TO GRAVITY

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Abstract

The chiral boson actions of Floreanini and Jackiw (FJ), and of McClain, Wu and Yu (MWY) have been recently shown to be different representations of the same chiral boson theory. MWY displays manifest covariance and also a (gauge) symmetry that is hidden in the FJ side, which, on the other hand, displays the physical spectrum in a simple manner. We make use of the covariance of the MWY representation for the chiral boson to couple it to background gravity showing explicitly the equivalence with the previous results for the FJ representation.

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In recent years there has been a great deal of attention devoted to the quantization of chiral scalar fields. The main motivation is that chiral bosons are the basic objects of two of the most interesting opened problems of present days theoretical physics. They appear in the construction of many string models [1] where some symmetries are manifest before chiral boson fermionization, but not after. Technically, it is advantageous to keep chiral bosons, instead of their fermionic counterparts, because it suffices to compute lower loop graphs on the world-sheet. Furthermore, in the description of quantum Hall effect [2], chiral bosons play an important role since there they appear as the edge-states of the Hall fluid, which are believed to be the only gapless excitation of the sample.

To obtain a chiral boson one usually eliminates one half of the degrees of freedom from the scalar field by means of a chiral constraint \( \partial_\pm \phi \approx 0 \). The problem in following this route is that the chiral constraint is second-class by Dirac’s classification scheme [3]. Therefore one is not allowed to gauge away its associated Lagrange multiplier field that therefore acquires a dynamical character. Siegel [4] proposed to covariantize the chiral constraint, i.e., to transform it from second to first-class, by squaring it, or what is equivalent, setting one chiral component of the energy-momentum tensor to zero as a constraint. Siegel’s action has a reparametrization symmetry, at tree level, that becomes anomalous at one-loop level (i.e., becomes second-class again after quantization) [3, 5] due to the existence of a central extension in the conformal algebra of the energy-momentum tensor. Later on, Hull [7] has shown how to cancel the conformal anomaly of Siegel’s model introducing auxiliary fields on the zero-mode sector, the so called no-movers fields. Nevertheless, to square a second-class constraint to make it first-class results in a theory presenting (infinitely) reducible constraints [8]. Independently, Floreanini and Jackiw [14] proposed an action where the chiral constraint appears from the equations of motion and therefore does not involve any Lagrange multiplier field. This (first-order) chiral boson formulation introduces however some spurious solutions to (second-order) field equations that need careful boundary conditions adjustments to be eliminated. Another drawback in FJ proposal is the lack of manifest Lorentz covariance, which makes the coupling to gauge and gravitational fields difficult [15].

More recently, McClain, Wu and Yu [11] have shown that an action containing infinite scalar fields, coupled by combinations of right and left chiral constraints, carefully adjusted to be first-class, possess the spectrum of a single chiral boson. In

\[3\] The idea of using infinite auxiliary scalar fields to covariantize second-class constraints has been introduced earlier in the literature by Mikovic et al [12] in the context of the relativistic super-particle.
fact, MWY and FJ are just two different representations of the same chiral boson theory \[13\], each of which displaying a different feature of the very same problem: in the MWY side not only the Lorentz covariance is manifest, but it also displays a symmetry that is hidden in the other representation, while the FJ side, on the other hand, presents the spectrum in a simpler manner. Depending on one’s interests, one can pass from one representation to the other either transforming, iteratively, the second-class constraint of the FJ model into first-class, à la Faddeev-Shatashvili, with the introduction of infinite Wess-Zumino fields, or by resolving iteratively the MWY constraints by means of the Faddeev-Jackiw technique for first-order constrained systems.

In this paper we shall make use of the manifest covariance of the MWY representation to couple it to background gravity. The results we obtain are shown to be consistent with the ones previously obtained for the FJ representation by Sonnenchein \[10\] and by Bastianelli and van Nieuwenhuizen \[15\], which are quickly derived in an appendix, in a form slightly different than their original formulation.

The MWY action is a representation of a chiral boson in terms of a sum over infinite scalar fields

\[
S^{MWY} = \int d^2x \sum_{k=0}^{\infty} (-)^k \frac{1}{2} \partial_\mu \phi_k \partial^\mu \phi_k
\]  

chirally coupled to each other by a set of (infinite) irreducible constraints \(T^{(\pm)}_k\) or \(T^{(-)}_k\) each one corresponding to one of the chiralities

\[
T^{(\pm)}_k = \Omega^{(\pm)}_k - \Omega^{(\mp)}_{k+1}
\]

with \(\Omega^{(\pm)}_k\) being right and left chiral constraints \(\Omega^{(\pm)}_k = \pi_k \pm \phi'\) that satisfy two uncoupled Kac-Moody algebra:

\[
\begin{align*}
\big\{\Omega^{(\pm)}_k(x), \Omega^{(\pm)}_m(y)\big\} &= (\pm)2\delta_{km}\delta'(x-y) \\
\big\{\Omega^{(+)}_k(x), \Omega^{(-)}_m(y)\big\} &= 0
\end{align*}
\]

Here the curly brackets \(\{A(x), B(y)\}\) represents the Poisson bracket of the fields \(A(x)\) and \(B(x)\). One can verify that MWY constraints \(T^{(\pm)}_k(x)\) closes an (infinite) first-class Abelian algebra under the Poisson bracket operation

\[
\big\{T^{(\pm)}_k(x), T^{(\pm)}_m(y)\big\} = 0
\]
In view of the manifest covariance presented by the model, coupling to gravity is straightforward, and reads

\[ S^{MWY} = \frac{1}{2} \int d^2 x \sqrt{-g} \sum_{k=0}^{\infty} (-)^k g^{\mu \nu} \partial_\mu \phi_k \partial_\nu \phi_k \]

where \( g_{\mu \nu} \) is the background metric tensor and \( g = \det g_{\mu \nu} = g_{00} g_{11} - g_{01}^2 \). We adopt the usual notation where \( \dot{\phi} = \partial_0 \phi \) and \( \phi' = \partial_1 \phi \), and \( x^0 = \tau \) and \( x^1 = \sigma \) are the two-dimensional world-sheet variables. To obtain the gauged Floreanini-Jackiw counterpart we have to reduce the MWY-constraints (4) as explained above. In order to effect such a reduction we make use of the Faddeev-Jackiw symplectic technique. To this end we rewrite the MWY action in its first-order form, by introducing the momentum \( \pi_k \) conjugate to \( \phi_k \),

\[ S_{\pm}^{MWY} = \int d^2 x \sum_{k=0}^{\infty} \left\{ \pi_k \dot{\phi}_k - \mathcal{H}^{MWY} + \lambda_k T_k^{(\pm)} \right\} \]

where \( \mathcal{H}^{MWY} \) is the canonical Hamiltonian density

\[ \mathcal{H}^{MWY} = \frac{1}{2} \sum_{k=0}^{\infty} \left[ \frac{(-)^k}{g^{00}} \left( \frac{\pi_k^2 + \phi_k'^2}{\sqrt{-g}} \right) - \frac{g^{01}}{g^{00}} \pi_k \phi_k' \right] \]

Now we eliminate the momentum \( \pi_k \), in an iterative fashion, making use of the MWY-constraints. Implementing the first constraint, \( \pi_0 = \pi_1 \mp \phi_0' \mp \phi_1' \) results, after its substitution, in the following Lagrangian density

\[ \mathcal{L}_{\pm}^{MWY} = \mp \dot{\phi}_0 \phi_0' - \mathcal{G}_\pm \phi_0'^2 - \phi_1' \left( \dot{\phi}_0 + \mathcal{G}_\pm \phi_0' \right) + \pi_1 \left[ \dot{\phi}_0 + \dot{\phi}_1 + \mathcal{G}_\pm (\phi_0' + \phi_1') \right] + \sum_{k=2}^{\infty} \left[ \pi_k \dot{\phi}_k - \frac{1}{2} \frac{(-)^k}{g^{00}} \left( \frac{\pi_k^2 + \phi_k'^2}{\sqrt{-g}} \right) + \frac{g^{01}}{g^{00}} \pi_k \phi_k' \right] \]

where

\[ \mathcal{G}_\pm = \frac{1}{g^{00}} \left( \frac{1}{\sqrt{-g}} \pm \frac{g^{01}}{g^{00}} \right) \]
We repeat this procedure for the constraint \( \pi_1 = \pi_2 \mp \phi_1' \mp \phi_2' \), in this way eliminating the momentum \( \pi_1 \), and so on. After all the remaining constraints have been implemented we find the following effective action

\[
L_{\text{MWY}}^{\pm} = \sum_{k=0}^{\infty} \left[ \mp \phi_k \phi_k' - G_{\pm} \phi_k'^2 - 2 \left( \phi_k + G_{\pm} \phi_k' \right) \sum_{m=k+1}^{\infty} \phi_m' \right] \quad (10)
\]

It is a simple algebraic manipulation to rewrite this action as

\[
L_{\text{MWY}}^{\pm} = \sum_{k=0}^{\infty} \left[ \mp \phi_k \phi_k' - G_{\pm} \phi_k'^2 - 2 \phi_k' \sum_{m=1}^{k-1} \left( \dot{\phi}_m + G_{\pm} \phi_m' \right) \right] \quad (11)
\]

which shows that all the MWY scalar fields have decoupled from each other. To make this point clearer, we rewrite the action (11) as a double series function

\[
S_{\text{MWY}}^{\pm} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \int d^2 x \left( \mp \phi_k \phi_m' - G_{\pm} \phi_k' \phi_m' \right) \quad (12)
\]

and introduce a new (collective) variable as

\[
\Phi = \sum_{k=0}^{\infty} \phi_k \quad (13)
\]

The MWY action for the collective field \( \Phi \) assumes the form of a Floreanini-Jackiw action coupled to the gravitational field by the factor \( G_{\pm} \)

\[
S_{\text{MWY}}^{\pm} = \int d^2 x \left( \mp \Phi \Phi' - G_{\pm} \Phi'^2 \right) \quad (14)
\]

An interesting feature of the interacting action (14) is the reparametrization symmetry that the coupling \( G_{\pm} \) induces on the system, that reads

\[
\delta_{\epsilon} \Phi = \epsilon \Phi' \quad \delta_{\epsilon} G_{\pm} = (\epsilon + \epsilon G_{\pm}') - \epsilon' G_{\pm} \quad (15)
\]

It is also interesting to note that for each of the chiral fields, \( \phi_+ \) or \( \phi_- \), there will exist a class of metrics for which they propagate as in flat space. This will be the case whenever \( G_+ = 1 \) or \( G_- = 1 \), respectively. These two conditions corresponds, as can be seen from (14), to respectively, \( g^{00} + g^{11} \pm 2g^{01} = 0 \), or in terms of light cone coordinates, to:
\begin{align*}
g^{--} &= 0 \\
g^{++} &= 0.
\end{align*}
A symmetric two dimensional metric can be written in terms of the light cone components in the general form

\begin{equation}
g_{\mu\nu} = \frac{1}{2} \begin{pmatrix}
g^{++} + 2g^{+-} & g^{+-} - g^{--} \\
\frac{1}{2}g^{++} - g^{--} & g^{++} + g^{+-} - 2g^{+-}
\end{pmatrix}
\end{equation}

Therefore, in metrics of the form

\begin{equation}
g_{\mu\nu} = \frac{1}{2} \begin{pmatrix}
g^{++} + g^{+-} & g^{+-} \\
g^{++} & g^{++} - g^{+-}
\end{pmatrix}
\end{equation}

which gives $\mathcal{G}_+ = 1$, the chiral field $\phi_+$ propagates as in flat space. Similarly, for metrics whose general form reads

\begin{equation}
g_{\mu\nu} = \begin{pmatrix}
g^{--} + g^{+-} & g^{+-} - g^{--} \\
g^{--} - g^{+-} & g^{--} - g^{+-}
\end{pmatrix}
\end{equation}

implying $\mathcal{G}_- = 1$, the chiral field $\phi_-$ remain uncoupled. Stated differently, chiral bosons, when immersed in a curved background, select to couple to a special combination of the metric elements $\mathcal{G}_\pm$, in a similar way as it happens when coupling to gauge fields. Under conditions (16) the metrics (18) and (19) become, in a sense, “chiral metrics”. Consequently when immersed in a chiral background with chirality opposite to its own, chiral boson just do not experiment the curvature. It should be noted that, contrarily to the case of gauge fields, where the coupling is additive, when conditions (16) are not satisfied each of the chiralities of the field $\phi$ couples to the corresponding $\mathcal{G}_\pm$ that depends on the whole metric and not on (18) or (19).

Conditions (16) correspond to a sort of selfduality and anti-selfduality over the metric tensor, justifying us to call them as chiral metrics. Indeed, if we define the “dual metric” with respect to one index, as

\begin{equation}
^{*}g_{\mu\nu} = \epsilon^{\mu\lambda}g_{\lambda\nu}
\end{equation}

with $\epsilon^{10} = -\epsilon^{01} = 1$, and take the trace, then conditions (16) will read

\begin{equation}
Tr\ (g_{\mu\nu}) = \pm Tr\ (^{*}g_{\mu\nu})
\end{equation}
Concluding, we have shown explicitly that the MWY representation for the chiral boson can be coupled to an external background metric in a standard way, making use of the manifest covariance. The incorporation of the series of infinite chiral constraints in the model was shown to lead to the same result as the one previously obtained for the FJ model. We have also pointed out the classes of metrics for which one of the chiralities propagate as in free space.

A Appendix

In reference [15] it was shown that a version of the Floreanini Jackiw action, coupled to gravity can be obtained beginning with a scalar field coupled covariantly to gravity then writing it as a first order action and imposing a (non covariant) constraint that selects one of the (chiral) solutions of the classical equation of motion. Here we will derive the same result using the Mandelstan [16] decomposition for a scalar field into chiral bosons. Let us begin, as [15], with a scalar field coupled covariantly to gravity:

$$\mathcal{L} = \frac{1}{2} \sqrt{-g} g_{\mu \nu} \partial_\mu \phi \partial_\nu \phi$$  \hspace{1cm} (A.1)$$

The introduction of the auxiliary variable $p$ makes it possible to write this Lagrangian density in a first order form

$$\mathcal{L} = \frac{\sqrt{-g}}{2} \left[ -g^{00} p^2 + 2 g^{00} p \dot{\phi} + 2 g^{01} \dot{\phi} \phi' + g^{11} \phi' \phi' \right]$$  \hspace{1cm} (A.2)$$

Decomposing the scalar field in its (Mandelstan) components:

$$\phi = \phi_+ + \phi_-$$  \hspace{1cm} (A.3)$$

one is able to associate each of this fields with one of the (chiral) solutions of the classical equation of motion by imposing[17]:

$$p = \frac{g^{01}}{g^{00}} (\phi'_+ + \phi'_-) + \frac{1}{\sqrt{-g} g^{00}} (\phi'_- - \phi'_+)$$  \hspace{1cm} (A.4)$$

Inserting (A.3) and (A.4) in (A.2) we get

$$\mathcal{L} = -\dot{\phi}_+ \phi'_+ - \mathcal{G}_+ \phi'_+ \phi'_+$$
$$+ \dot{\phi}_- \phi'_- - \mathcal{G}_- \phi'_- \phi'_-$$  \hspace{1cm} (A.5)$$
showing explicitly the decomposition of the scalar field in chiral components, each of
them coupling with the metric exactly as in [15].

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