ON THE $\gamma$-FILTRATION OF ORIENTED COHOMOLOGY OF COMPLETE SPIN-FLAGS

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Abstract. We study the characteristic map of algebraic oriented cohomology of complete spin-flags and the ideal of invariants of formal group algebra. As an application, we provide an annihilator of the torsion part of the $\gamma$-filtration. Moreover, if the formal group law determined by the oriented cohomology theory is congruent to the additive formal group law modulo 2, then at degree 2 and 3, the $\gamma$-filtration of complete spin-flags is torsion free.

1. Introduction

Oriented cohomology theories [LM] of algebraic varieties over base field $k$ are cohomology theories generalized from the Chow group $\text{CH}$ and the Grothendieck group $\text{K}_0$. They are algebraic analogue of cohomology theories of complex manifolds. In particular, each oriented cohomology theory $h$ determines a one-dimensional formal group law $F$ over the coefficient ring $R = h(\text{Spec } k)$. For example, $\text{CH}$ (resp. $\text{K}_0$) determines the additive formal group law $F_a$ (resp. the multiplicative formal group law $F_m$).

Given a split simple simply connected linear algebraic group $G$ with the variety of complete flags $X$ and a fixed maximal torus $T$, let $W$ be its Weyl group and $\Lambda$ be the weight lattice with respect to $T$. For arbitrary oriented cohomology $h$ and corresponding formal group law $F$, Calmès-Petrov-Zainoulline [CPZ] construct a formal group algebra $R[[\Lambda]]_F$ and a characteristic map $c_F : R[[\Lambda]]_F \to h(X)$. These constructions generalize those of Demazure for the Chow group [Dem73] and for the Grothendieck group [Dem74]. They provide algebraic tools to study oriented cohomology of homogeneous varieties. For instance, the $\gamma$-filtration of $h(X)$ is defined using $c_F$, and the associated quotients $\gamma^d h(X)$ are studied in [MZZ]. More precisely, it shows in loc.it. that $\gamma^d h(X)$ is torsion free, provided that the torsion index $t$ of $G$ is invertible in $R$. This does not include the case when 2 is not invertible in $R$ and $G$ is of type $B_n$ and $D_n$. The goal of this paper is to study this case. More precisely, our main result is

1.1 Theorem. Let $G$ be split, simple simply connected of type $B_n$ with $n \geq 3$ or of type $D_n$ with $n \geq 4$, and let $X$ be its variety of complete flags. Let $h$ be a weakly birationally invariant oriented cohomology theory with coefficients in $R$ satisfying Assumption [3.4]. Suppose that 2 is regular in $R$ but $\frac{1}{2} \not\in R$. Let $F$ be the corresponding formal group law over $R$, and let $d \geq 2$.

(i) If $R$ has characteristic zero, then the torsion part of $\gamma^d h(X)$ is annihilated by $\zeta_d^2 \eta_d^2$, where the integers $\zeta_d$ and $\eta_d$ are defined in [4.7].

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Let $d = 2$ or $3$ and $F \equiv F_a \mod 2$. Then $\gamma^{(d)} h(X) \cong \gamma^{(d)} \text{CH}(X; R)$. In particular, if $R$ has characteristic zero, then $\gamma^{(d)} h(X)$ is torsion free.

Note that the annihilator we obtained depends only on the filtration degree $d$. It does not depend on the rank of $G$, nor on the specific cohomology theory $h$. The cohomology theories satisfying the hypothesis of Theorem 1.1 include any oriented cohomology theory over a field $k$ of characteristic zero (see §3.5) such that 2 is regular but not invertible in $h(k)$, e.g., the algebraic cobordism.

For $d = 1$, following the argument in [MZZ, Corollary 8.8], it is easy to see that $\gamma^{(1)} h(X) \cong \gamma^{(1)} \text{CH}(X)$, so it is always torsion free (when the characteristic of $R$ is zero). That’s the reason why we restrict that $d \geq 2$. On the other hand, similar result was proved in [BZZ, Theorem 6.1] for Chow group of twisted flag varieties of type $B_n$ and $D_n$. Note that for $h = K_0$ and $R = \mathbb{Z}$, the formal group law is $F(x, y) = x + y - xy$, so it does not satisfy the hypothesis of Theorem 1.1(ii). Therefore, our result does not contradict [GZ, Theorem 3.1], which says that the torsion part of $\gamma^{(2)} K_0(X)$ is $\mathbb{Z}/2$. Corollary 6.3 provides more precise application to the Grothendieck group.

To prove Theorem 1.1, we study $\ker c_F$ and the ideal $I_W^F$ of $R[[\Lambda]]_F$ generated by nonconstant $W$-invariants. The ideal $I_W^F$ itself has classical meanings. For example, $I_W^{F_a}$ is generated by the basic polynomials invariants [Hum], and a theorem of Chevalley says that $I_W^{F_m}$ is generated by the fundamental representations of $G$. On the other hand, $I_W^F \subset \ker c_F$, and they coincide when the torsion index of $G$ is invertible in $R$. We study the generators of $I_W^F$ and the index of the embedding of $I_W^F$ in $\ker c_F$. We then use the deformation map [MZZ] between formal group algebras of two distinct $F$ and $F'$ to define a map between $\gamma$-filtrations of corresponding oriented cohomologies $h$ and $h'$. Such map enables us to compare arbitrary $h$ with $\text{CH}$.

This paper is organized as follows: In Section 2 we recall the definition of the formal group algebra $R[[\Lambda]]_F$ and the deformation map. In Section 3 we recall the definition of characteristic map and $\gamma$-filtration. In Section 4 we study the generators of $I_W^F$. In Section 5 we provide an upper bound of the index of the embedding of $I_W^F$ in $\ker c_F$. In Section 6 we use the deformation map and the results in Sections 4 and 5 to define a map between the $\gamma$-filtrations of different oriented cohomologies, and prove Theorem 1.1.

Through this paper, we adopt:

- $R$ is a commutative ring with identity such that 2 is regular but not invertible.
- $G$ is a split simple simply connected linear algebraic group of classical Dynkin type $B_n$ with $n \geq 3$ or type $D_n$ with $n \geq 4$.
- $t$ is the torsion index of $G$, which is a power of 2 in this case ([Dem73, Tot]).
- $X$ is the variety of complete flags of $G$.
- $W$ is the Weyl group of $G$.
- $\Lambda$ is the group of characters of a maximal torus of $G$, which corresponds to the weight lattice of $G$.
- $\{\omega_1, ..., \omega_n\}$ is the set of fundamental weights, which is a basis of $\Lambda$.
- $\Sigma$ is the set of roots with a fixed set of simple roots $\Pi = \{\alpha_1, ..., \alpha_n\}$.
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2. THE FORMAL GROUP ALGEBRA AND THE DEFORMATION MAP

In this section we recall the definition of formal group algebra in [CPZ] and the deformation map in [MZZ]. Recall that a one-dimensional commutative formal group law $F$ over $R$ is a power series

$$F(x, y) = x + y + \sum_{i,j \geq 1} a_{ij} x^i y^j$$

with $a_{ij} \in R$ satisfying the following conditions:

$$F(x, F(y, z)) = F(F(x, y), z), \quad F(x, y) = F(y, x), \quad F(x, 0) = x.$$

We use the notations $x +_F y = F(x, y)$, $2 \cdot_F x = F(x, x)$ and $3 \cdot_F x = F(x, 2 \cdot x)$, etc.

**Example.**

1. The *additive formal group law* $F_0$ is defined by $F_0(x, y) = x + y$.

2. The *multiplicative formal group law* $F_m$ is defined by $F_m(x, y) = x + y - axy$ with $a \in R^\times$.

3. The *Lorentz formal group law* is defined by

$$F_l(x, y) = \frac{x + y}{1 + \beta xy} = (x + y) \sum_{i=0} (\beta xy)^i, \quad \beta \neq 0 \in R.$$

4. [Sil] §IV.1] Let $E$ be the elliptic curve defined by

$$y = x^3 + a_1 x y + a_2 x^2 y + a_3 y^2 + a_4 x y^2 + a_6 y^3,$$

then the *elliptic formal group law* over $R = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ is defined by

$$F_e(x, y) = x + y - a_1 x y - a_2 (x^2 y + xy^2) - 2a_3 (x^3 y + xy^3) + (a_1 a_2 - 3a_3) x^2 y^2 + \ldots.$$

**Definition.** Let $F$ be a formal group law over $R$. Consider the polynomial ring $R[x_\lambda]$ in the variables $x_\lambda$ with $\lambda \in \Lambda$. Let

$$\epsilon : R[x_\lambda] \to R, \quad x_\lambda \mapsto 0$$

be the augmentation map, and let $R[[x_\lambda]]$ be the (ker $\epsilon$)-adic completion of $R[x_\lambda]$. Let $\mathcal{J}_F$ be the closure of the ideal of $R[[x_\lambda]]$ generated by $x_0$ and elements of the form $x_{\lambda_1 + \lambda_2} - F(x_{\lambda_1}, x_{\lambda_2})$ for all $\lambda_1, \lambda_2 \in \Lambda$. Here $x_0 \in R[x_\lambda]$ is the element determined by the zero element of $\Lambda$. The *formal group algebra* $R[[\Lambda]]_F$ is defined to be the quotient

$$R[[\Lambda]]_F = R[[x_\lambda]] / \mathcal{J}_F.$$

The augmentation map induces a ring homomorphism $\epsilon : R[[\Lambda]]_F \to R$ with kernel $\mathcal{I}_F$. Then we have a filtration of $R[[\Lambda]]_F$:

$$R[[\Lambda]]_F = \mathcal{I}_F^0 \supseteq \mathcal{I}_F^1 \supseteq \mathcal{I}_F^2 \supseteq \cdots$$

and the associated graded ring

$$Gr_R(\Lambda, F) \overset{def}{=} \bigoplus_{i=0}^\infty \mathcal{I}_F^i / \mathcal{I}_F^{i+1}.$$

**Example.** By [CPZ] Lemma 4.2, $Gr_R(\Lambda, F)$ is isomorphic to the symmetric algebra $S^*_R(\Lambda)$. The isomorphism maps $\prod x_\lambda_i$ to $\prod \lambda_i$. Indeed, $R[[\Lambda]]_F$ is non-canonically isomorphic to $R[[x_{\omega_1}, \ldots, x_{\omega_n}]]$. 

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**Example.** Consider the polynomial given by

$$y = x^3 + a_1 x y + a_2 x^2 y + a_3 y^2 + a_4 x y^2 + a_6 y^3,$$
The action of the Weyl group $\mathcal{W}$ on $\Lambda$ induces a $W$-action on $R[[\Lambda]]_F$. Let $I^{W}_{F}$ be the ideal of $R[[\Lambda]]_F$ generated by the subset of constant-free $W$-invariants $R[[\Lambda]]^{W}_{F} \cap I_{F}$. For $d \geq 0$, let

$$I^{(d)}_{F} = I_{F}^{d}/I_{F}^{d+1},$$

$$(R[[\Lambda]]^{W}_{F})^{(d)} = (R[[\Lambda]]^{W}_{F} \cap I^{d}_{F})/(R[[\Lambda]]^{W}_{F} \cap I^{d+1}_{F}),$$

$$(I^{W}_{F})^{(d)} = (I^{W}_{F} \cap I^{d}_{F})/(I^{W}_{F} \cap I^{d+1}_{F}).$$

Then $I^{(d)}_{F}$ is a free $R$-module generated by $x_{\omega_{1}}^{m_{1}} \cdots x_{\omega_{n}}^{m_{n}}$ with $\sum_{i=1}^{n} m_{i} = d$.

2.4 For any two formal group laws $F$ and $F'$ over $R$, there is an $R$-algebra isomorphism, called the deformation map from $F$ to $F'$

$$\Phi_{F \to F'} : R[[\Lambda]]_{F} \to R[[\Lambda]]_{F'},$$

defined as follows: firstly, one defines $\Phi_{F \to F'}(x_{\omega_{i}}) = x_{\omega_{i}} \in R[[\Lambda]]_{F'}$. For any $\lambda = \sum_{i=1}^{n} m_{i} \omega_{i} \in \Lambda$, we have $x_{\lambda} = x_{\sum_{i=1}^{n} m_{i} \omega_{i}} \in R[[\Lambda]]_{F}$. Then we define

$$\Phi_{F \to F'}(x_{\lambda}) = (m_{1} \cdot F x_{\omega_{1}}) + F \cdots (m_{n} \cdot F x_{\omega_{n}}) \in R[[\Lambda]]_{F'}.$$  

Clearly $\Phi_{F' \to F} \circ \Phi_{F \to F'} = id_{R[[\Lambda]]_{F}}$, so it is an isomorphism. It maps $I^{d}_{F}$ into $I^{d}_{F'}$, hence induces an isomorphism of $R$-modules

$$\Phi_{d}^{F \to F'} : I_{F}^{(d)} \to I_{F'}^{(d)}.$$  

A key property is that for any $\Pi_{i=1}^{d} x_{\lambda_{i}} \in I_{F}^{(d)}$, we have

$$\Phi_{d}^{F \to F'}(\Pi_{i=1}^{d} x_{\lambda_{i}}) = \Pi_{i=1}^{d} x_{\lambda_{i}} \in I_{F'}^{(d)},$$

so $\Phi_{d}^{F \to F'}$ is $W$-equivariant. We then have

$$\Phi_{d}^{F \to F'} : (I_{F}^{(d)})^{W} \xrightarrow{\cong} (I_{F'}^{(d)})^{W},$$

but in general $\Phi_{d}^{F \to F'}((I_{F}^{W})^{(d)})$ is not contained in $(I_{F'}^{W})^{(d)}$. One of the main interests of [MZZ] [8] and Section 4 of this paper is the difference between $I_{F}^{W}$ and $I_{F'}^{W}$, i.e., to determine the smallest integer $\tau_{d}^{F \to F'}$ such that

$$\tau_{d}^{F \to F'}(I_{F}^{W})^{(d)} \subset \Phi_{d}^{F \to F'}((I_{F'}^{W})^{(d)}).$$

If $R$ is a UFD, such integer exists and is called the $d$-th exponent of the $W$-action from $F$ to $F'$. In particular, by [MZZ], $\tau_{d}^{F \to F_{n}}$ coincides with the exponent $\tau_{d}$ defined in [BNZ], so $\tau_{d}^{F \to F_{n}}|2$ if $G$ is of type $B_{n}$ (resp. $D_{n}$) and $d \leq 2n - 1$ (resp. $d \leq 2n - 3$) by [BZZ].

3. The $\gamma$-filtration of oriented cohomology theory

In this section we recall the definition of characteristic map and the $\gamma$-filtration of oriented cohomology theory of variety of complete flags [MZZ].

3.1 An algebraic oriented cohomology theory $h$ (in the sense of Levine–Morel) is a contravariant functor from the category of smooth projective varieties over a field $k$ to the category of commutative (graded) $R$-algebras such that $h(\text{Spec} \ k) = R$. It is characterized by the axioms in [MA] [1.1]. For instance, there exists push-forward for projective morphism, and the projective bundle property and the extended homotopy property hold.
A cohomology theory is *birationally invariant* [CPZ Definition 8.7] if for any proper birational morphism \( f : X \to Y \) of smooth projective varieties, the push-forward of the fundamental class \( f_* (1_X) \) is \( 1_Y \in h(Y) \), and is *weakly birationally invariant* if \( f_* (1_X) \) is invertible in \( h(Y) \). The Chow ring \( CH \) over arbitrary base field is birationally invariant, and by [LM Theorem 4.3.9], the connective \( K \)-theory defined over a field of characteristic 0 is universal among all birationally invariant theories. Moreover, if the base field has characteristic 0, all oriented cohomology theories in the sense of Levine–Morel are weakly birationally invariant [CPZ Corollary 8.10].

Each oriented cohomology theory determines characteristic classes, that is, a collection of maps
\[
c^h_i : K_0(X) \to h(X), \quad i \geq 1
\]
characterized by properties [LM Definition 1.1.2]. In particular, for any two line bundles \( L_1 \) and \( L_2 \) over \( X \) one has
\[
c^h_i (L_1 \otimes L_2) = F(L_1, L_2) \in h(X)
\]
for some formal group law \( F \) over \( R \). This defines a map from the set of oriented cohomology theories to the set of one-dimensional commutative formal group laws. For example, \( F_0 \) corresponds to the Chow group \( CH \) and \( F_m \) corresponds to the Grothendieck group \( K_0 \).

**3.2 From now on, let \( X \) be the variety of complete flags, and fix a Borel subgroup \( B \) of \( G \). If \( G \) is of type \( B_n \) \((n \geq 3)\) or of type \( D_n \) \((n \geq 4)\), then the torsion index \( t \) is a power of 2 (see [Dem73] for definition and [Tot] for computations).

Let \( F \) be the formal group law corresponding to the oriented cohomology \( h \), then there is a characteristic map, which is an \( R \)-algebra homomorphism
\[
c^\gamma_F : R[[\Lambda]]_F \to h(X)
\]
defined by \( c^\gamma_F (x_\lambda) = c^\gamma_i (L(\lambda)) \). Here \( L(\lambda) \) is the line bundle over \( X \) corresponding to the character \( \lambda \).

**3.3 Definition.** [MZZ p.9] The \( \gamma \)-filtration of \( h(X) \) is defined as follows: \( \gamma^d h(X) \) is defined to be the \( R \)-submodule of \( h(X) \) generated by
\[
c^\gamma_i (L_1) \cdot \cdots \cdot c^\gamma_i (L_m)
\]
with \( m \geq d \) and \( L_1, \ldots, L_m \) line bundles over \( X \). Define
\[
\gamma^{(d)} h(X) = \gamma^d h(X)/\gamma^{d+1} h(X).
\]

By definition, \( c^\gamma_F \) induces maps
\[
c^\gamma_F : T^d_F \to \gamma^d h(X) \quad \text{and} \quad c^{(d)}_F : T^{(d)}_F \to \gamma^{(d)} h(X).
\]

The Bruhat decomposition gives \( X = \sqcup_{w \in \mathcal{W}} BwB/B \), i.e., \( X \) is a disjoint union of affine spaces. The closure of \( BwB/B \) is denoted by \( X_w \) and is called a Schubert variety. For any simple root \( \alpha_i \), let \( P_i \) be the minimal parabolic subgroup corresponding to \( \alpha_i \). For any \( w \in \mathcal{W} \) and \( I_w = \{i_1, \ldots, i_r\} \) a reduced decomposition of \( w \), the Bott–Samelson variety is defined as:
\[
X_{I_w} := P_{i_1} \times_B \cdots \times_B P_{i_r}.
\]

The multiplication map induces \( q_{I_w} : X_{I_w}/B \to X \) which factors through \( X_w \):
\[
q_{I_w} : X_{I_w}/B \to X_w \to X
\]
where the first map is surjective and birational, and the second one is a closed embedding. Denote \( \zeta_w := (\varphi_L), (1) \in h(X) \).

3.4 Assumption. \[ \text{Assumption 13.2} \] For each \( w \in W \), let \( I_w \) be a reduced decomposition of \( w \). The set \( \{ \zeta_w \}_{w \in W} \) forms a \( R \)-basis of \( h(X) \).

Remark:assumption

3.5 For example, according to \[ \text{Lemma 13.3} \], \( \text{CH} \) and \( K_0 \) in \( \mathbb{Z} \) or \( \mathbb{Z}/m \) coefficients over arbitrary base field satisfy this assumption, and so does any oriented cohomology theory over a field of characteristic zero. If, in addition, \( h \) is weakly birationally invariant and \( t \) is regular in \( R \), then by \[ \text{Theorem 13.12} \], \( c_F \) coincides with the characteristic map defined in \[ \text{[CPZ], §6} \] (one can view the latter map as the algebraic replacement of \( c_F \)). In this case, \( T^W_F \subset \ker c_F \). Furthermore, if the torsion index \( t \) is invertible in \( R \), then \( c_F \) is surjective with \( \ker c_F = T^W_F \).

Throughout this paper, we always assume that \( h \) is weakly birationally invariant and satisfies Assumption 3.4, for example, \( h \) can be any oriented cohomology theory over \( k \) with characteristic zero.

4. The invariants

In this section, we study the generators of \( T^W_F \), and prove Lemma 4.6 and 4.8 concerning the invariants \( \Theta_d \). The “only if” parts of the two lemmas are proved in Lemmas 8.3, 8.4, 8.5 of \[ \text{MZZ} \].

First, we prove some property of formal group law. Let

\[
F(x, y) = x + y + \sum_{m=1}^{\infty} a_{mm} x^m y^m + \sum_{l=3}^{\infty} \sum_{j+k=l, j<k} a_{jk} x^j y^k + x^k y^j.
\]

We use \( \varphi_F(x) \in R[[\Lambda]]_F \) to denote the (formal) inverse of \( x \in R[[\Lambda]]_F \), and \( O(s) \) to denote a sum of terms of degree \( \geq s \).

4.1 Definition. We say that a formal group law \( F \) is even if

\[
F(x, y) \equiv x + y \mod 2.
\]

4.2 Example. (1) If \( \frac{a}{b} \in R \), then the Lorentz formal group law \( F_1(x, y) = \frac{x+y}{1+\frac{a}{b}xy} \) is even.

(2) If all the elements \( a_1, a_2, a_3, a_4 \) and \( a_6 \) in Example 2.1 (4) are even integers, then the elliptic formal group law \( F_e \) is even. This follows from the fact that all the coefficients of \( F_e(x, y) \) (except for those of \( x \) and \( y \)) are combinations of \( a_i \), \( i = 1, 2, 3, 4, 6 \).

4.3 Lemma. If the formal group law \( F \) satisfies that \( 2|a_{mm} \) for \( 1 \leq m < s \), then

\[
\varphi_F(x) \equiv x + a_{ss} x^{2s} + O(2s + 1) \mod 2.
\]

Consequently, if \( 2|a_{ss} \) for all \( s \), then \( \varphi_F(x) \equiv x \mod 2 \).

Proof. In general, we have

\[
\varphi_F(x) = -x + a_{11} x^2 + O(3),
\]

so the lemma holds for \( s = 1 \).

We proceed by induction on \( s \). Assume it holds for \( s = k - 1 \), i.e., if \( 2|a_{mm} \) for \( m < k - 1 \), then

\[
\varphi_F(x) \equiv x + a_{k-1,k-1} x^{2k-2} + b_0 x^{2k-1} + b_1 x^{2k} + O(2k + 1) \mod 2.
\]
Now assume \( s = k \), i.e., assume in addition that \( 2|a_{k-1,k-1} \). By the induction assumption,
\[
\tau_F(x) \equiv x + b_0x^{2k-1} + b_1x^{2k} + O(2k+1) \mod 2.
\]
It suffices to show that \( b_0 \equiv 0 \) and \( b_1 \equiv a_{kk} \mod 2 \). Modulo 2 and \( O(2k+1) \), we have
\[
0 \equiv F(x, \tau_F(x)) \\
= x + (x + b_0x^{2k-1} + b_1x^{2k}) + a_{kk}x^k(x + b_0x^{2k-1} + b_1x^{2k})^k \\
+ \sum_{i=3}^{2k} \sum_{i < j, i + j = l} a_{ij} (x^i + b_0x^{2k-1} + b_1x^{2k})^i x^j (x + b_0x^{2k-1} + b_1x^{2k})^j.
\]

Now, modulo \( O(2k+1) \), we have \( x^k(x + b_0x^{2k-1} + b_1x^{2k})^k \equiv x^{2k} \) and for each \( i + j \geq 3 \), we have
\[
x^i(x + b_0x^{2k-1} + b_1x^{2k})^j \equiv x^i \sum_{j_1+j_2+j_3 = j} \binom{j}{j_1,j_2,j_3} (b_0x^{2k-1})^{j_1} (b_1x^{2k})^{j_2} (x)^{j_3} \equiv x^{i+j}.
\]
Therefore, modulo 2 and \( O(2k+1) \), we have
\[
0 \equiv F(x, \tau_F(x)) \equiv b_0x^{2k-1} + b_1x^{2k} + a_{kk}x^{2k}.
\]
Hence, \( b_0 \equiv 0 \mod 2 \) and \( b_1 \equiv a_{kk} \mod 2 \).

### 4.4 Notation: weight

We now define some elements of \( \mathcal{I}_F^W \) which are possible candidates of the generators of \( \mathcal{I}_F^W \). Let \( \{e_i\}_{i=1}^n \) be the standard basis of \( \mathbb{R}^n \) that defines the root system of \( G \). The element \( e_i \) belongs to \( \Lambda \), hence can be written as a linear combination of \( \omega_i \)'s. If \( G \) is of type \( B_n \) with \( n \geq 3 \), then
\[
e_1 = \omega_1, \ e_i = \omega_i - \omega_{i-1} \text{ for } 2 \leq i \leq n - 1, \text{ and } e_n = 2\omega_n - \omega_{n-1}.
\]
If \( G \) is of type \( D_n \) with \( n \geq 4 \), then
\[
e_1 = \omega_1, \quad e_i = \omega_i - \omega_{i-1} \text{ for } 2 \leq i \leq n - 2, \\
\quad e_{n-1} = \omega_{n-1} - \omega_{n-2}, \quad \text{and } e_n = \omega_n + \omega_{n-1} - \omega_{n-2}.
\]
For \( d = 1, \ldots, n \), define the \( W \)-invariant element \( \Theta_d \in R[[\Lambda]]_F^W \cap \mathcal{I}_F \) together with a positive integer \( r_d \) as follows:

1. If \( G \) is of type \( B_n \) with \( n \geq 3 \), define \( \Theta_d^B = \sum_{i=1}^n x_{e_i}^d x_{-e_i}^d \). Since the Weyl group \( W \) acts on \( \{e_i\}_{i=1}^n \) by permutations and by sign changes, we see that \( \Theta_d^B \in R[[\Lambda]]_F^W \). Let \( r_d = 2 \) if \( d \) is a power of 2 and \( r_d = 1 \) otherwise.

2. If \( G \) is of type \( D_n \) with \( n \geq 4 \), define \( \Theta_d^D = \Theta_d^B \) for \( d = 1, \ldots, n - 1 \) and \( \Theta_n^D = \prod_{i=1}^n (x_{e_i} - x_{-e_i}) \). Since \( W \) acts by permutations of \( e_i \)'s and by sign changes of even numbers of \( e_i \)'s, we see that \( \Theta_d^D \in R[[\Lambda]]_F^W \). Let \( r_n = 2^n \). For \( d = 1, \ldots, n - 1 \), let \( r_d = 2 \) if \( d \) is a power of 2, and \( r_d = 1 \) otherwise.

#### 4.5 Example

By [Hum 3.12] and [Mac Remark 2 in page 19 and Ch. I. (2.4)], if \( F = F_0 \), then the coefficients of the polynomials \( \Theta_d \in R[[x_{\omega_1}, \ldots, x_{\omega_n}]] \) are integers with g.c.d. \( r_d \), and
\[
R[[\Lambda]]_F^W = R[[\frac{1}{r_1}\Theta_1, \ldots, \frac{1}{r_n}\Theta_n]].
\]
But this may fail if \( F \neq F_n \). In the following lemma, we will provide a necessary and sufficient condition for this to hold. The idea of the proof is to express \( x_{e_i} \) in terms of \( x_{\omega_j} \) using the relations in \([4.4]\) and study their coefficients via the non-canonical isomorphism \( R[[\Lambda]]_F \cong R[x_{\omega_1}, ..., x_{\omega_n}] \).

**Lemma.**

1. Let \( G \) be of type \( B_n \) with \( n \geq 3 \) (resp. of type \( D_n \) with \( n \geq 4 \)) and let \( d \leq n \) (resp. \( d < n \)) be a positive power of 2, then \( F \) is even if and only if \( \frac{\Theta^B_d}{2^d} \in \mathcal{I}_F \) for some \( d \) (hence for all \( d \)).

2. Let \( G \) be of type \( D_n \), then \( 2 | a_{mm} \) for all \( m \geq 1 \) if and only if \( \frac{\Theta^B_{2^m}}{2^m} \in \mathcal{I}_F \).

**Proof.** (1) The “only if” part was proved in [MZZ], so we only prove the “if” part. Suppose \( G \) is of type \( B_n \). Let \( \nu_0 = 0, \nu_i = e_1 + \cdots + e_i = \omega_i \) for \( i = 1, ..., n-1 \) and \( \nu_n = 2\omega_n \). We show that if \( F \) is not even, then \( 2 \nmid \Theta^B_d \) for any \( d \). Since \( F \) is not even, then \( 2 \nmid a_{ss} \) for some \( s \) or \( 2 \nmid a_{jk} \) for some \( j < k \).

First, assume that \( s \) is the smallest integer such that \( 2 \nmid a_{ss} \). For any \( \lambda_1, \lambda_2 \in \Lambda \), let \( x_{\lambda_1} - x_{\lambda_2} = \sum_{k=1}^{\infty} f_k(x_{\lambda_1}, x_{\lambda_2}) \), where \( f_k(x, y) \) is a homogeneous polynomial of degree \( k \) in \( R[x, y] \). For instance, \( f_1(x, y) = x - y \). Since the binomial formula satisfies \( (z_1 + z_2)^d \equiv z_1^d + z_2^d \mod 2 \), by Lemma 4.3 modulo 2 and \( \mathcal{I}_F^{2d+2d} \) we obtain

\[
\Theta^B_d = \sum_{i=1}^{n} a_{e_i}^d x_{e_i}^d = \sum_{i=1}^{n} a_{e_i}^d x_{e_i}^d (x_{e_i} + a_{ss} x_{e_i})^{2d}
\]

\[
= \sum_{i=1}^{n} a_{e_i}^d (x_{e_i} + a_{ss} x_{e_i})^{2d} = \sum_{i=1}^{n} (x_{e_i} + a_{ss} x_{e_i})^{2d}
\]

\[
= \sum_{i=1}^{n} \left[ \left( \sum_{k=1}^{\infty} f_k(x_{\nu_i}, x_{\nu_i-1}) \right)^{2d} + a_{ss} \left( \sum_{k=1}^{\infty} f_k(x_{\nu_i}, x_{\nu_i-1}) \right)^{2sd+d} \right]
\]

Notice that \( f_k(x_{\nu_i}, x_{\nu_i-1})^{2d} \) is a homogeneous polynomial of degree \( 2kd \). Therefore, the degree \( (2s+1)d \) term of \( \Theta^B_d \) is given by

\[
\sum_{i=1}^{n} a_{ss}^d f_1(x_{\nu_i}, x_{\nu_i-1})^{2sd+d} = \sum_{i=1}^{n} a_{ss}^d (x_{\nu_i} - x_{\nu_i-1})^{2sd+d}.
\]

Since \( 2ds + d \) is not a power of 2, by Lucas’ Theorem, \( 2 \nmid \binom{2ds + d}{a} \) for some \( 0 < a < 2ds + d \), so \( 2 \nmid (x_{\nu_i} - x_{\nu_i-1})^{2sd+d} \) for all \( i \). Since \( 2 \nmid a_{ss} \), so we have \( 2 \nmid \Theta^B_d \) in \( \mathcal{I}_F / \mathcal{I}_F^{2sd+d} \), which implies that \( 2 \nmid \Theta^B_d \) in \( \mathcal{I}_F \).

Suppose that \( 2 | a_{ss} \) for all \( s \geq 1 \) and \( l_0 \) is the smallest integer such that \( 2 \nmid a_{j_0, l_0 - j_0} \) for some \( j_0 < l_0/2 \). Then we can write

\[
F(x, y) \equiv x + y + \sum_{l=l_0}^{l_0} \sum_{j<k, j+k=l} a_{jk}(x^j y^k + x^k y^j) \mod 2.
\]
4.7 If \( f \in T^d_F \setminus T^{d+1}_F \), we say that \( \text{deg} \ f = d \). Then for \( d = 1, \ldots, n \), we have \( \text{deg} \ \Theta^{B}_{d} = 2d \). For the type \( D_{n} \), \( \text{deg} \ \Theta^{B}_{d} = 2d \) for \( d = 1, \ldots, n-1 \) and \( \text{deg} \ \Theta^{B}_{n} = n \). Given a \( n \)-tuple \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with \( \alpha_i \in \mathbb{Z}_{\geq 0} \), let \( r_{\alpha} = \prod_{i=1}^{n} r_{\alpha_i}^{\alpha_i} \), \( \Theta(\alpha) = \prod_{i=1}^{n} \Theta^{B}_{\alpha_i} \) and \( |\alpha| = \sum_{i=1}^{n} \alpha_i \cdot \text{deg} \ \Theta_{i} \).

Let \( \nu_{2}(m) \) be the 2-adic valuation of \( m \). To simplify the notations, we define a collection of integers \( \{\zeta_{d}, \eta_{d}\}_{d \geq 1} \) which depends on the Dynkin type of \( G \).
4.8 Lemma. Let \( G \) be of type \( B_n \) with \( n \geq 3 \) or of type \( D_n \) with \( n \geq 4 \) and let \( d \geq 2 \).

1. We have

\[
\zeta_d \cdot (R[\Lambda])_{\mathbb{K}}^{W}(d) \subseteq (\Theta(\alpha))_{|\alpha|=d} \subseteq (R[\Lambda])_{\mathbb{K}}^{W}(d).
\]

Moreover, \( F \) is even if and only if for some \( d \) (hence for all \( d \)),

\[
(R[\Lambda])_{\mathbb{K}}^{W}(d) = \left\{ \frac{1}{r_{\alpha}} \Theta(\alpha) \right\}_{|\alpha|=d}.
\]

2. We have

\[
\zeta_d \cdot (I_{\mathbb{K}}^{W})_{\mathbb{K}}^{F}(d) \subseteq \left\{ \sum_{\deg \Theta_i \leq d} g_i \Theta_i | g_i \in I_{\mathbb{K}}^{(d-deg \Theta_i)} \right\} \subseteq (I_{\mathbb{K}}^{W})_{\mathbb{K}}^{F}(d).
\]

Moreover, \( F \) is even if and only if for some \( d \) (hence for all \( d \)),

\[
(I_{\mathbb{K}}^{W})_{\mathbb{K}}^{F}(d) = \left\{ \sum_{\deg \Theta_i \leq d} g_i \Theta_i | g_i \in I_{\mathbb{K}}^{(d-deg \Theta_i)} \right\}.
\]

Proof. (1) The first statement and the “only if” part of the second statement were proved in [MZZ] Lemma 8.4. For the “if” part of the second statement, note that the assumption

\[
(R[\Lambda])_{\mathbb{K}}^{W}(d) = \left\{ \frac{1}{r_{\alpha}} \Theta(\alpha) \right\}_{|\alpha|=d}
\]

for some \( d \) implies that \( \frac{1}{r_{\alpha}} \Theta_1 \in I_{\mathbb{K}}^{F} \). By Lemma 4.6, \( F \) is even.

(2) The first statement and the “only if” part of the second statement were proved in [MZZ] Lemma 8.5. The proof of the “if” part is similar to that of (1). \( \square \)

4.9 Remark. [MZZ] Lemmas 8.4, 8.5, Theorem 8.6] Indeed, in Lemma 4.8 if one replaces the condition that \( F \) is even by the condition that \( \frac{1}{r_{\alpha}} \in R \), then we have

\[
(R[\Lambda])_{\mathbb{K}}^{W}(d) = (\Theta(\alpha))_{|\alpha|=d}, \quad (I_{\mathbb{K}}^{W})_{\mathbb{K}}^{F}(d) = \left\{ \sum_{\deg \Theta_i \leq d} g_i \Theta_i | g_i \in I_{\mathbb{K}}^{(d-deg \Theta_i)} \right\},
\]

and \( (I_{\mathbb{K}}^{W})_{\mathbb{K}}^{F}(d) = \Phi_d^{F \rightarrow F'} ((I_{\mathbb{K}}^{W})_{\mathbb{K}}^{F}(d)) \) for arbitrary \( F, F' \) and \( d \geq 2 \). Similarly, if both \( F \) and \( F' \) are even, then one still has \( (I_{\mathbb{K}}^{W})_{\mathbb{K}}^{F}(d) = \Phi_d^{F \rightarrow F'} ((I_{\mathbb{K}}^{W})_{\mathbb{K}}^{F}(d)) \). For general \( F \) and \( F' \), one has \( \zeta_d \cdot (I_{\mathbb{K}}^{W})_{\mathbb{K}}^{F}(d) \subset \Phi_d^{F \rightarrow F'} ((I_{\mathbb{K}}^{W})_{\mathbb{K}}^{F}(d)) \). We will use these facts in the next section.
5. The kernel of the characteristic map

In this section we compute an upper bound of the index of embedding $$(\mathcal{I}_F^W)^{(d)}$$ in $$\ker c_F^{(d)}$$, which will be used in Section 6 to prove the main result.

Let $$\hat{R} = R[\frac{1}{2}]$$. Let $$\hat{\mathcal{I}}_F \subset \hat{R}[\Lambda]^W_F$$ (resp. $$\hat{c}_F$$) be the corresponding augmentation ideal (resp. the characteristic map). Let $$c_F^{(d)}$$ and $$\tilde{c}_F^{(d)}$$ be the characteristic maps on the corresponding subquotients on $$\mathcal{I}_F^{(d)}$$ and $$\hat{\mathcal{I}}_F^{(d)}$$, respectively. By [3.5] $$(\mathcal{I}_F^W)^{(d)} = \ker c_F^{(d)}$$. By [CPZ, Proposition 6.5], there is a commutative diagram

(2) \[ \begin{array}{ccc} \ker c_F^{(d)} & \longrightarrow & \mathcal{I}_F^{(d)} \\ \downarrow & & \downarrow \gamma^{(d)} \\ (\mathcal{I}_F^W)^{(d)} & \longrightarrow & \ker \tilde{c}_F^{(d)} \end{array} \]

For any $$y \in \ker c_F^{(d)}$$ we have $$y \in \ker \tilde{c}_F^{(d)} = (\mathcal{I}_F^W)^{(d)}$$, so by Remark 4.9

(3) \[ y = \sum_{\deg \Theta_i \leq d} g_i \Theta_i, \quad g_i \in \mathcal{I}_F^{(d-\deg \Theta_i)}. \]

The following two lemmas generalize [GZ, §1B], [BNZ, Lemma 6.4] and [BZZ, Proposition 4.5] from $$F_n$$ to general $$F$$. One also notes that if $$F = F_m$$, then $$\ker c_F = \mathcal{I}_F^W$$.

**Lemma.**

1. Let $$G$$ be of type $$B_n$$ with $$n \geq 3$$ or of type $$D_n$$ with $$n \geq 4$$ and let $$d = 2$$ or 3. Then $$2 \cdot \ker c_F^{(d)} \subseteq (\mathcal{I}_F^W)^{(d)}$$. If $$F$$ is even, then $$\ker c_F^{(d)} = (\mathcal{I}_F^W)^{(d)}$$.

2. Let $$G$$ be of type $$B_n$$ with $$n \geq 3$$ or of type $$D_n$$ with $$n \geq 5$$. Let $$d = 4$$. We have $$4 \cdot \ker c_F^{(d)} \subseteq (\mathcal{I}_F^W)^{(d)}$$. If $$F$$ is even, then $$2 \cdot \ker c_F^{(d)} \subseteq (\mathcal{I}_F^W)^{(d)}$$.

**Proof.**

1. Suppose that $$G$$ is of type $$B_n$$. For any $$y \in \ker c_F^{(2)}$$, by Equation (3), we have

\[ y = u \cdot \Theta_1 \in (\mathcal{I}_F^W)^{(2)} \]

for some $$u \in \hat{R}$$. That is, $$u \cdot \Theta_1 = y$$ in $$\mathcal{I}_F^{(2)}$$, so both sides are polynomials of degree 2 in $$R[x_{\omega_1}, ..., x_{\omega_n}]$$. Note that

\[ \Theta_1 = 2 \sum_{i=1}^{n-2} (x_{\omega_i}^2 - x_{\omega_i} x_{\omega_{i+1}}) + 2x_{\omega_n-1}^2 - 4x_{\omega_n-1} x_{\omega_n} + 4x_{\omega_n}^2. \]

The g.c.d. of the coefficients of $$\Theta_1 \in (\mathcal{I}_F^W)^{(2)}$$ is 2, so $$2u \in R$$. Therefore, $$2y = 2u \cdot \Theta_1 \in (\mathcal{I}_F^W)^{(2)}$$.

If $$F$$ is even, then by Lemma 4.6, $$\Theta_1 \in R[\Lambda]^W_F$$, so $$y = \frac{\Theta_1}{2} \cdot 2u \in (\mathcal{I}_F^W)^{(2)}$$.

Now let $$d = 3$$. For any $$y \in \ker c_F^{(3)}$$, by Equation (4), we have

(4) \[ y = \Theta_1 \cdot f_1 \in \mathcal{I}_F^{(3)} \]
for some $f_1 \in \mathcal{I}_F^{(1)}$. We show that $2f_1 \in \mathcal{I}_F^{(1)}$. Suppose that $f_1 = \sum_{i=1}^n a_i x_\omega$, with $a_i \in \mathcal{R}$. Write

$$y = \sum_{i=1}^n a_{ii} x_{\omega_i}^3 + \sum_{i<j} (a_{iij} x_{\omega_i} x_{\omega_j} + a_{ijj} x_{\omega_i} x_{\omega_j}) + \sum_{i<j<k} a_{ijk} x_{\omega_i} x_{\omega_j} x_{\omega_k} \in \mathcal{I}_F^{(3)}$$

with $a_{ii}, a_{iij}, a_{ijj}, a_{ijk} \in \mathcal{R}$. For any $i < n$, by comparing the coefficients of $x_{\omega_i}^3$ in Equation (4), we see that $2a_i = a_{iii} \in \mathcal{R}$. By comparing the coefficients of $x_{\omega_i}^2 x_{\omega_j}$, we have $2a_n = a_{11n} \in \mathcal{R}$. Hence, $2f_1 \in \mathcal{I}_F^{(1)}$ and $2y = 2f_1 \Theta_1 \in (\mathcal{I}_F^{W})^{(3)}$.

If $F$ is even, then by Lemma 4.6, $\frac{\Theta_2}{2} \in R[[\Lambda]]^{W}$, so $y = \frac{\Theta_2}{2} \cdot 2f_1 \in (\mathcal{I}_F^{W})^{(3)}$.

If $G$ is of type $D_n$ with $n \geq 4$, the proof is similar, since the generator involved in this case is $\Theta_1$ only.

(2) Let $G$ be of type $B_n$. For $y \in \ker c_F^{(4)}$, by Equation (3),

$$y = f_0 \Theta_2 + f_2 \Theta_1$$

for some polynomials $f_i \in \mathcal{I}_F^{(i)}$. Notice that there exists positive integer $b$ such that the polynomials $2^b f_0 \in \mathcal{I}_F$ and $2^b f_2 \in \mathcal{I}_F$. Let $b_0$ be the smallest among these integers, and we claim that $b_0 \leq 2$. If not, then $b_0 \geq 3$. It implies that

$2^{b_0} f_0 \Theta_2 + 2^{b_0} f_2 \Theta_1 = 2^{b_0} y \equiv 0 \mod 8$

with $2^{b_0} f_i \in \mathcal{I}_F^{(i)}$. Since $\frac{\Theta_2}{2} \in \mathcal{I}_F^{(2)}$ and $\frac{\Theta_2}{2} \in \mathcal{I}_F^{(4)}$, so in $\mathcal{I}_F^{(4)}$, we have

$$2^{b_0} f_0 \frac{\Theta_2}{2} + 2^{b_0} f_2 \frac{\Theta_1}{2} \equiv 0 \mod 4.$$

By Example 2.3, $\mathcal{I}_F^{(4)} \cong \mathcal{I}_a^{(4)}$. By the proof of [BNZ] Lemma 6.4, this implies that $g.c.d.\{2^{b_0} f_0, 2^{b_0} f_2\} = 2$, therefore, $2^{b_0 - 1} f_i \in \mathcal{I}_F^{(i)}$. This contradicts to the minimality assumption of $b_0$. Hence $b_0 \leq 2$ and $4y = 4f_0 \Theta_2 + 4f_2 \Theta_1 \in (\mathcal{I}_F^{W})^{(4)}$.

If $F$ is even, then $\frac{\Theta_1}{2} \in \mathcal{I}_F^{(2)}$, therefore, $2y = 4f_0 \cdot \frac{\Theta_2}{2} + 4f_2 \frac{\Theta_1}{2} \in (\mathcal{I}_F^{W})^{(4)}$.

If $G$ is of type $D_n$ with $n \geq 5$, the proof is similar, since the only generators of $(R[[\Lambda]])^{(4)}$ are $\Theta_1$ and $\Theta_2$. \(\square\)

**5.2 Lemma.** Let $G$ be of type $B_n$ with $n \geq 3$ or of type $D_n$ with $n \geq 4$, then $\eta_d \cdot \ker c_F^{(d)} \subseteq (\mathcal{I}_F^{W})^{(d)}$, where the integer $\eta_d$ was defined in (4.7).

**Proof.** Let $G$ be of type $B_n$. The case of type $D_n$ is similar. For $d \leq 4$, it is proved in Lemma 6.1. So let $d \geq 5$. For any $y \in \ker c_F^{(d)}$, by Equation (3),

$$y = \sum_{\deg \Theta_i \leq d} f_{d-2i} \Theta_i \in \mathcal{I}_F^{(d)} , f_{d-2i} \in \mathcal{I}_F^{(d-2i)}.$$

The polynomials $f_{d-2i} \in \mathcal{I}_F^{(d-2i)}$ are non-uniquely determined by $y$, and there exists positive integer $b$ (determined by $\{f_{d-2i}\}$) such that $2^b f_{d-2i} \in \mathcal{I}_F^{(d-2i)}$ for all $i$. Suppose that $b_0$ is the smallest among these integers and among $\{f_{d-2i}\}$ satisfying Equation (5). We claim that $2^{b_0} | \eta_d$. If not, then $2\eta_d | 2^{b_0}$. Then in $\mathcal{I}_F^{(d)}$, we have

$$2^{b_0} y = \sum_{\deg \Theta_i \leq d} 2^{b_0} f_{d-2i} \Theta_i \in \mathcal{I}_F^{(d)} , 2^{b_0} f_{d-2i} \in \mathcal{I}_F^{(d-2i)}.$$
By Example 2.3, \( I_F^{(d)} \cong I_{\gamma}^{(d)} \). By the proof of [BZZ Proposition 4.5], we know that there exists \( g_{d-2i} \in I_F^{(d-2i)} \) such that

\[
y = \sum_{\deg \Theta_i \leq d} g_{d-2i} \Theta_i \in I_F^{(d)}
\]

with \( 2^{b_0} g_{d-2i} \in I_F^{(d-2i)} \). This contradicts the minimality assumption of \( b_0 \). Therefore, \( 2^{b_0} | \eta_d \) and \( \eta_d y = \sum \eta_d f_{d-2i} \Theta_i \in (I_W^{(d)}) \).

6. Comparison of \( \gamma \)-filtrations

In this section we apply the computation in Sections 4 and 5 to compare \( \gamma \)-filtrations of different oriented cohomology theories, and prove the main result of this paper.

**Lemma.** Let \( G \) be of type \( B_n \) with \( n \geq 3 \) or of type \( D_n \) with \( n \geq 4 \). Let \( h \) and \( h' \) be two weakly birationally invariant oriented cohomology theories satisfying Assumption 3.4. Let \( F \) and \( F' \) be the corresponding formal group laws, respectively. Then:

(a) The map \( \zeta_d h \cdot \Phi_d^{F \rightarrow F'} : I_F^{(d)} \rightarrow I_{F'}^{(d)} \) induces a map \( \gamma^{(d)} h(X) \rightarrow \gamma^{(d)} h'(X) \).

(b) If \( F \) and \( F' \) are even, then one can replace \( \zeta_d \) in (a) by \( \eta_d \).

**Proof.** (a) Suppose \( G \) is of type \( B_n \). The case of type \( D_n \) can be proved similarly. We have the following diagram

\[
\begin{array}{c}
(I_W^{(d)})^c \xrightarrow{\ker c_F^{(d)}} I_F^{(d)} \xrightarrow{c_F^{(d)}} I_{F'}^{(d)} \xrightarrow{\Phi_d^{F \rightarrow F'}} \gamma^{(d)} h(X) \\
\end{array}
\]

It suffices to show that \( \zeta_d h \cdot \Phi_d^{F \rightarrow F'} \) maps \( \ker c_F^{(d)} \) into \( \ker c_{F'}^{(d)} \). For any \( y \in \ker c_F^{(d)} \), by Lemma 5.2, \( \eta_d \cdot y \in (I_W^{(d)})^c \). By Lemma 1.3,

\[
\zeta_d h \cdot y = \sum_{\deg \Theta_i \leq d} g_i \Theta_i
\]

for some \( g_i \in I_{F'}^{(d-2i)} \). By Equation (11), we have

\[
\Phi_d^{F \rightarrow F'} (\zeta_d h \cdot y) = \sum_{\deg \Theta_i \leq d} g_i \Theta_i \in (I_W^{(d)})^c \subseteq \ker c_F^{(d)}.
\]

Therefore, \( \zeta_d h \cdot \Phi_d^{F \rightarrow F'} \) induces a map \( \gamma^{(d)} h(X) \rightarrow \gamma^{(d)} h'(X) \).

(b) If \( F \) and \( F' \) are even, then for any \( y \in \ker c_F^{(d)} \), by Lemma 5.2, \( \eta_d \cdot y \in (I_W^{(d)})^c \).

By Remark 1.9, \( \Phi_d^{F \rightarrow F'} (I_W^{(d)}) = (I_{F'}^{(d)})^c \). Hence,

\[
\Phi_d^{F \rightarrow F'} (\eta_d \cdot y) \in (I_W^{(d)})^c \subseteq \ker c_{F'}^{(d)}.
\]

Therefore, \( \eta_d \cdot \Phi_d^{F \rightarrow F'} \) induces a map \( \gamma^{(d)} h(X) \rightarrow \gamma^{(d)} h'(X) \). □

We are now ready to prove the main result of this paper.
Proof of Theorem 1.1. We only consider the $B_n$ case, since the $D_n$ case is similar.

(i) By Lemma 6.1, there is a commutative diagram

\begin{equation}
\begin{array}{cc}
\mathcal{I}_F^{(d)} & \gamma^{(d)} \mathfrak{h}(X) \\
\zeta_d \eta_d \Phi_d^{F \to F_a} & \downarrow \\
\mathcal{I}_a^{(d)} & \gamma^{(d)} \text{CH}(X; R).
\end{array}
\end{equation}

Given any torsion element $u \in \gamma^{(d)} \mathfrak{h}(X)$, since $\gamma^{(d)} \text{CH}(X; R) \subseteq \text{CH}^d(X; R)$ is torsion free, so $u$ is mapped to 0 in $\gamma^{(d)} \text{CH}(X; R)$. Lift $u$ to an element $v \in \mathcal{I}_F^{(d)}$, and look at its image $\zeta_d \eta_d \Phi_d^{F \to F_a}(v) \in \mathcal{I}_a^{(d)}$. Since $c_a^{(d)}(v) = 0$, so $\zeta_d \eta_d \Phi_d^{F \to F_a}(v) \in \ker c_a^{(d)}$, hence by Lemma 5.2, $\Phi_d^{F \to F_a}(v) \in (\mathcal{I}_a^{(d)})^\sigma$. Therefore, the isomorphism $\Phi_d^{F \to F_a}$ restricted to $\ker c_a^{(d)}$ induces an isomorphism

$$
\ker c_a^{(d)} \cong \ker c_a^{(d)}.
$$

(ii) Let $d = 2$ or 3. Since $F$ and $F_a$ are even, so by Lemma 5.1

$$(\mathcal{I}_F^{(d)})^\sigma = \ker c_F^{(d)} \quad \text{and} \quad (\mathcal{I}_a^{(d)})^\sigma = \ker c_a^{(d)}.$$

By Remark 1.1, we know that $\Phi_d^{F \to F_a}((\mathcal{I}_F^{(d)})^\sigma) = (\mathcal{I}_a^{(d)})^\sigma$. Therefore, the isomorphism $\Phi_d^{F \to F_a}$ restricted to $\ker c_F^{(d)}$ induces an isomorphism

$$
\gamma^{(d)} \mathfrak{h}(X) \cong \gamma^{(d)} \text{CH}(X; R).
$$

\[ \square \]

6.2 Remark. In Theorem 1.1 (i), if $F$ is even, then one can use Lemma 6.1 (b) to replace $\zeta_d \eta_d^2$ by $\zeta_d \eta_d$.

6.3 Corollary. If $F$ is the corresponding formal group law for $\mathfrak{h}$, then the map $\zeta_d \cdot \Phi_d^{F \to F}$ induces a map $\gamma^{(d)}K_0(X) \to \gamma^{(d)} \mathfrak{h}(X)$. In particular, if $R = \mathbb{Z}$, then the torsion part of $\gamma^{(d)}K_0(X)$ is annihilated by $\zeta_d \eta_d$.

Proof. The proof is similar to those of Lemma 6.1 (a) and Theorem 1.1 (i) by using the fact that $\ker c_{F_m} = (\mathcal{I}_F^{(d)})^\sigma$.

\[ \square \]

6.4 Remark. This corollary can be used to refine the upper bound in [BZZ] of the annihilator of Chow group of twisted flag varieties.

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ON THE $\gamma$-FILTRATION OF ORIENTED COHOMOLOGY OF COMPLETE SPIN-FLAGS

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