Research Article

The Structure of EAP-Groups and Self-Autopermutable Subgroups

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A subgroup $H$ of a given group $G$ is said to be autopermutable, if $HH^\alpha = H^\alpha H$ for all $\alpha \in \text{Aut}(G)$. We also call $H$ a self-autopermutable subgroup of $G$, when $HH^\alpha = H^\alpha H$ implies that $H^\alpha = H$. Moreover, $G$ is said to be EAP-group, if every subgroup of $G$ is autopermutable. One notes that if $\alpha$ runs over the inner automorphisms of the group, we obtain the notions of conjugate-permutability, self-conjugate-permutability, and ECP-groups, which were studied by Foguel in 1997, Li and Meng in 2007, and Xu and Zhang in 2005, respectively. In the present paper, we determine the structure of a finite EAP-group when its centre is of index $4$ in $G$. We also show that self-autopermutability and characteristic properties are equivalent for nilpotent groups.

1. Introduction

Let $H$ be a subgroup of a given group $G$. Then we call $H$ to be autopermutable, if $HH^\alpha = H^\alpha H$ for all $\alpha \in \text{Aut}(G)$. The subgroup $H$ is said to be self-autopermutable, if $HH^\alpha = H^\alpha H$ implies that $H^\alpha = H$. Moreover, we call the group $G$ to be an EAP-group if every subgroup of $G$ is autopermutable. Clearly, if $\alpha$ runs over the inner automorphisms of the group, we obtain the notions of conjugate-permutability [1], self-conjugate-permutability [2], and ECP-groups [3], respectively. One notes that the subgroup $H = \langle b \rangle$ of the Dihedral group $D_n = \langle a, b : a^2 = b^2 = 1, ab = a^{-1} \rangle$ is conjugate-permutable, which is not autopermutable. To see this, consider the automorphism $\alpha$ which sends $a$ and $b$ into $a$ and $ab$, respectively. Clearly $HH^\alpha$ is not a subgroup of $G$, which means that $HH^\alpha \neq H^\alpha H$. It is easily seen that similar examples can be obtained by taking a direct product of $D_n$ with any other group. Also, every noncharacteristic normal subgroup of a given group is an example for a self-conjugate-permutable subgroup which is not self-autopermutable. Moreover, $D_8$ is an ECP-group, which is not an EAP-group.

In the present paper, we determine the structure of a finite EAP-group, when its centre is of index $4$. We also prove that self-autopermutability and characteristic properties are equivalent in nilpotent groups.

2. Finite EAP-Groups

In this section, we determine the structure of finite EAP-groups, when their centres are of index $4$. In fact we prove the following theorem.

**Theorem 1.** Let $G$ be a finite group with the centre of index $4$. Then $G$ is an EAP-group if and only if the Sylow $2$-subgroup of $G$ is one of the following forms:

(i) $Q_8$;
(ii) $\langle a, b \mid a^{2n+1} = b^2 = 1, a^b = a^{2n+1} \rangle, n \geq 3$;
(iii) $\mathbb{Z}_2 \times Q_8$;
(iv) $\langle a, b, c, d \mid a^2 = b^2 = c^4 = d^4 = 1, b \in Z(G), [a, c] = [a, d] = c^3, [c, d] = 1 \rangle$;

In the present paper, we determine the structure of a finite EAP-group, when its centre is of index $4$. We also prove that self-autopermutability and characteristic properties are equivalent in nilpotent groups.
We remark that a nonabelian group is said to be Hamiltonian, if all of its subgroups are normal. The following result gives our claim, when G is a 2-group with cyclic centre of index 4.

**Theorem 2.** Let G be a finite 2-group with cyclic centre of index 4. Then G is an EAP-group if and only if G ≅ Qₘ or \( a^2 = b^2 = c^2 = d^2 = e^4 = 1, a, b, c, d, e \in Z(G) \), for all n ≥ 3.

**Proof.** Consider the group G to be Qₘ. Since Qₘ is Hamiltonian group, the result follows easily. Now assume G = \( \langle a, b | a^{2^{m+1}} = b^2 = 1, a^b = a^{2^{m+1}} \rangle \), for all n ≥ 3. One can easily check that G contains exactly three proper subgroups of orders 2, for 1 ≤ i ≤ n + 1. We also observe that the subgroups of orders 2 are automorphisms and as the subgroups of orders 2ⁿ+1 are normal, they are also automorphisms. Now, one can check that there are exactly two cyclic and one noncyclic subgroups of orders 2, 2 ≤ i ≤ n, so that one of the cyclic subgroups is central and hence all the subgroups of G satisfy the required property.

Conversely, assume that G is an EAP-group, Z(G) = \( \langle x | x^{2^{m+1}} = 1 \rangle \), G/Z(G) = \( \langle Z(G), aZ(G), bZ(G), abZ(G) \rangle \), where \( a^2, b^2 \in Z(G) \) and so \( |a|, |b| = 2^{m+1} \). In case n = 1, then the group G is either Dₘ or Qₘ. As explained before, Dₘ cannot be an EAP-group and hence G ≅ Qₘ. Now suppose n > 1 and the elements a and b are both of order 2. Then every element \( y \in G \) has the following form (as G is nilpotent of class 2):

\[
y = a^ib^jx^k, \quad 0 \leq i, j < 2, \quad 0 \leq k < 2^n. \tag{1}
\]

Clearly, the map \( \alpha \) given by \( \alpha(y) = b^ja^ib^jx^k \) is an automorphism of G, which sends a into b. Thus HH⁴⁻⁺ and HH⁻⁺⁻ for the subgroup H = \( \langle b \rangle \), which contradicts the assumption. Now, if |a|, |b| < 2ⁿ⁺⁺ we may replace a and b by the elements ax⁻¹ and bx⁻¹, both of which are of order 2. This reduces to the previous case. Therefore we must have a and b of order 2ⁿ⁺⁺. Then G has a cyclic subgroup of order 2ⁿ⁺⁺ and so G is of order 2ⁿ⁺⁺ with the centre of index 4. Hence, by [4, 5.3.4], the group G has the following presentation:

\[
G = \langle a, b | a^{2^{m+1}} = b^2 = 1, a^b = a^{2^{m+1}} \rangle, \quad n \geq 3. \tag{2}
\]

This is an EAP-group and so the proof is completed. □

The following result considers the case when G is a 2-group with noncyclic centre of index 4.

**Theorem 3.** Let G be a finite 2-group with noncyclic centre of index 4. Then G is an EAP-group if and only if G is one of the following forms:

(i) G = \( \mathbb{Z}_2 \times Qₘ \);

(ii) G = \( \langle a, b, c, d, e | a^2 = b^2 = c^2 = d^2 = e^4 = 1, a, b, c, d, e \in Z(G) \rangle \),

(iii) G = \( \langle a, b, c, d, e | a^2 = b^2 = c^2 = d^2 = e^4 = 1, a, b, c \in Z(G) \rangle \),

(iv) G = \( \langle a, b, c, d, e | a^2 = b^2 = c^2 = d^2 = e^4 = 1, a, b, c, d \in Z(G) \rangle \).

**Proof.** The sufficient condition is obvious. We only need to prove the necessity condition. Let G be an EAP-group and G/Z(G) = \( \langle Z(G), aZ(G), bZ(G), abZ(G) \rangle \), where a², b² ∈ \( Z(G) \). Assume that Z(G) is not an elementary abelian 2-group. Since Z(G) is the direct product of its cyclic subgroups, by the same argument as in Theorem 2, there are no EAP-groups in this case. Now, assume that Z(G) is an elementary abelian 2-group. Clearly G must be a group of order either 16 or 32. The structure of such groups is given as follows in [5]. If |G| = 16, then

(i) G = \( \mathbb{Z}_2 \times D₄ \);

(ii) G = \( \langle a, b | a^4 = b^4 = 1, bab⁻¹ = a^3 \rangle \approx \mathbb{Z}_4 \times \mathbb{Z}_4 \);

(iii) G = \( \mathbb{Z}_2 \times Q₄ \);

(iv) G = \( \langle a, b, c, d | a^2 = b^2 = c^2 = d^2 = 1, a, b, c, d, [c, d] = 1 \rangle \).

As D₄ is not an EAP-group, hence the group of form (i) cannot be an EAP-group. For the group of form (ii) we can consider H = \( \langle b \rangle \) and \( \alpha \in \text{Aut}(G) \) which sends a and b into a and ab, respectively. Clearly, HH⁴⁻⁺ and HH⁻⁺⁻ and hence G cannot be an EAP-group. Thus when |a| = |b|, then G is of the form given in either (iii) or (iv).

Assume |G| = 32. Then such groups in the list of small groups with elementary abelian centres of index 4 are only of the following forms:

(i) G = \( \langle a, b, c | a^4 = b^4 = c^2 = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle \);

(ii) G = \( \langle a, b, c, d, e | a^2 = b^2 = c^2 = d^2 = e^4 = 1, a, b, c, d \in Z(G) \rangle \);

(iii) G = \( \langle a, b, c, d, e | a^2 = b^2 = c^2 = d^2 = e^4 = 1, a, b, c, d, e \in Z(G) \rangle \);

(iv) G = \( \langle a, b, c, d, e | a^2 = b^2 = c^2 = d^2 = e^4 = 1, a, b, c, d, e \in Z(G) \rangle \).

For the group of form (i) we may consider the cyclic subgroup \( H = \langle b \rangle \) and \( \alpha \in \text{Aut}(G) \) which sends a, b, c, and d into a, ab, ac, and ed, respectively. In case the group G is of form (ii), we consider \( H = \langle e \rangle \) and \( \alpha \in \text{Aut}(G) \) which sends a, b, c, d, and e into a, ab, ac, be, and ed, respectively. Also if the group is considered to be of form (iii), one may consider \( H = \langle e \rangle \) and \( \alpha \in \text{Aut}(G) \) which sends a, b, c, d, and e into e, b, c, d, and a, respectively. Now, one can easily check that in these cases HH⁴⁻⁺⁺ and so G cannot be an EAP-group. Hence, when |G| = 32, then G is of either form (iv) or form (v). The proof is complete. □

**Proof of Theorem 1.** The necessity condition is obvious and Theorems 2 and 3 establish the result, when G is a 2-group. If G is not a 2-group, then we may write G = \( S_1 \times S_2 \cdots \times S_k \),
in such a way that $S_1$ is a Sylow $p_1$-subgroup and $S_i$ is an abelian Sylow $p_i$-subgroup, where $p_i$ is an odd prime number, for $2 \leq i \leq k$. Clearly, $\text{Aut}(G) \equiv \text{Aut}(S_1) \times \text{Aut}(S_2) \times \cdots \times \text{Aut}(S_k)$ and for any subgroup $H$ of $G$, $H \equiv H_1 \times H_2 \times \cdots \times H_k$, where $H_i \leq S_i$ for $1 \leq i \leq k$. Thus $H$ is an autopermutable subgroup of $G$ if $H_1$ is an autopermutable subgroup of $S_1$. This completes the proof.

3. Self-Autopermutable Subgroups in Nilpotent Groups

We call a subgroup $H$ of a given group $G$ to be weakly characteristic, when $H^\alpha \leq N_G(H)$ implies that $H^\alpha = H$ for all $\alpha \in \text{Aut}(G)$. Also, given the subgroups $H$ and $K$, then $H$ satisfies the subcharacteriser condition, if $H \trianglelefteq K$ implies that $N_{\text{Aut}(G)}(K) \leq N_{\text{Aut}(G)}(H)$, where $N_{\text{Aut}(G)}(K) = \{\alpha \in \text{Aut}(G) ; K^\alpha = K\} \leq \text{Aut}(G)$. Clearly, if one considers the inner automorphisms of the group then weakly normal and normaliser condition properties are obtained.

The following result of [6] shows that self-conjugate-permutable, weakly normal property, and subnormaliser condition are equivalent for $p$-subgroups of a given group.

Theorem 4 (see [6], Proposition 3.3). Let $H$ be a $p$-subgroup of a group $G$. Then the following properties are equivalent:

(i) $H$ is a self-conjugate-permutable subgroup;
(ii) $H$ is a weakly normal subgroup;
(iii) $H$ satisfies the subnormaliser condition.

In this section, it is shown that self-autopermutable subgroups in nilpotent groups are always characteristic.

Proposition 5. Let $H$ be a subgroup of a group $G$.

(i) If $H$ is self-autopermutable, then $H$ is weakly characteristic in $G$.
(ii) If $H$ is weakly characteristic, then $H$ satisfies the subcharacteriser condition in $G$.

Proof. (i) If $H^\alpha \leq N_G(H)$, as $H \trianglelefteq N_G(H)$, we have $HH^\alpha = H^\alpha H$. Applying the condition that $H$ is self-autopermutable subgroup of the group $G$, we get $H^\alpha = H$. By definition, $H$ is weakly characteristic.

(ii) Let $K \leq G$, such that $H \trianglelefteq K$. We have $H^\alpha \leq K^\alpha = K \leq N_{\text{Aut}(G)}(K)$ for every $\alpha \in N_{\text{Aut}(G)}(K)$. Since $H$ is weakly characteristic in $G$, we have $H^\alpha = H$. Thus $\alpha \in N_{\text{Aut}(G)}(H)$ and the result is obtained.

The following theorem is one of the main results in this section.

Theorem 6. Let $H$ be a subgroup of a nilpotent finite group $G$. If $H$ satisfies the subcharacteriser condition then $H$ is characteristic in $G$.

Proof. Write $G \equiv P_1 \times P_2 \times \cdots \times P_t$, where $P_i$ is a Sylow $p_i$-subgroup of $G$, for $1 \leq i \leq t$. We may also write $H \equiv H_1 \times H_2 \times \cdots \times H_t$, with $H_i \leq P_i$, $1 \leq i \leq t$. Since $H$ satisfies the subcharacteriser condition in $G$, one can easily see that $H_i$ satisfies the subcharacteriser condition in $P_i$. Therefore $H_i \trianglelefteq P_i$ implies that $H_i$ is characteristic in $P_i$, which proves the result.

Finally, we show that self-autopermutability, weakly characteristic, and subcharacteriser conditions are equivalent, for every subgroup of a nilpotent group.

Corollary 7. Let $H$ be a subgroup of a finite nilpotent group $G$. Then

(i) $H$ is a self-autopermutable;
(ii) $H$ is a weakly characteristic;
(iii) $H$ satisfies the subcharacteriser condition in $G$.

Proof. The result follows by Proposition 5 and Theorem 6.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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