HELCITY-CONSERVATIVE FINITE ELEMENT DISCRETIZATION FOR MHD SYSTEMS

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Abstract. We construct finite element methods for the magnetohydrodynamics (MHD) system that precisely preserve magnetic and cross helicity, the energy law and the magnetic Gauss law at the discrete level. The variables are discretized as discrete differential forms fitting in a de Rham complex. We present numerical tests to show the properties of the algorithm.

Keywords: magnetohydrodynamics, helicity, divergence-free, structure-preserving, finite element.

1. Introduction

Numerical simulation for the magnetohydrodynamics (MHD) system is important in plasma physics. There have been a lot of efforts in designing stable and efficient numerical methods for solving the MHD equations.

There are various conserved quantities in the MHD system. Among them, the energy law and the magnetic Gauss law \((\nabla \cdot \mathbf{B} = 0)\) have been proved crucial both for the MHD physics and for computation, c.f., [8]. Moreover, topology of magnetic and fluid fields plays an important role in many applications of MHD. The linking and knot structures of the magnetic field are rearranged in magnetic reconnection and this fact leads to a number of consequences in physics. The helicity of a divergence-free vector field is a standard measure for the extent to which the field lines wrap and coil around one another [10], which is conserved in non-dissipative systems (ideal flows). Fluid and MHD helicity is known to be important in turbulence regime as discussed in for example, [11, 37]. Helicity also provides a local lower bound for the energy [4, p. 122], i.e., a topological obstruction of energy relaxation. We refer to [4, 5, 33, 34, 35] and the references therein for more discussions on MHD helicity, and [40] for discussions on knots in plasma physics.

In many algorithms, these quantities are approximated up to a discretization error, rather than conserved. It is therefore of great interest to construct numerical methods that precisely preserve the helicity up to a machine precision, together with other conservative quantities including the energy and the magnetic Gauss law. These conservation laws are related to each other. For example, to obtain well defined magnetic helicity \(\int \mathbf{A}_h \cdot \mathbf{B}_h \, dx\) at the discrete level, where \(\mathbf{A}_h\) is any vector potential of the magnetic potential \(\mathbf{B}_h\) satisfying \(\nabla \times \mathbf{A}_h = \mathbf{B}_h\), the discrete magnetic field \(\mathbf{B}_h\) has to be precisely divergence-free. As a consequence, it is necessary to use algorithms preserving the magnetic Gauss law.

Without an attempt to provide a complete review of a huge literature in MHD simulations, we only mention some early work on finite element methods [16, 39], and recent work on finite element methods that preserve the energy law and the magnetic Gauss law at the discrete level [20, 22]. However, it seems unlikely that either magnetic or cross helicity is preserved in the schemes proposed in [20] or [22] even in a semi-discretization with continuous time [21]. We refer to Section 3.1 below for an analysis of the artificial helicity pollution in one of the schemes.
On the other hand, we mention some existing efforts on helicity-preserving schemes. Liu and Wang [30] studied helicity-preserving finite difference methods for axisymmetric Navier-Stokes and MHD flows. Rebholz [27, 38] constructed energy- and helicity-preserving finite element methods for the Navier-Stokes equations. See [36] for further discussions on, e.g., turbulence models. Kraus and Maj [25] studied helicity-preserving schemes for the MHD system based on discrete exterior calculus. However, to the best of our knowledge, it remains open to construct helicity-preserving finite element methods for MHD.

The goal of this paper is thus to construct finite element methods for the MHD system preserving the energy, the magnetic Gauss law and the magnetic and cross helicity at the discrete level in the ideal MHD limit. Actually, the proposed scheme preserves the local helicity as well, meaning that the helicity is conserved on any subdomain of \( \Omega \) if the field has certain vanishing conditions on its boundary. This in turn provides a lower bound for the local discrete energy, i.e., a topological obstruction of energy relaxation at the discrete level. As demonstrated by the numerical results, the proposed algorithm also shows a significant improvement on the approximation of helicity even for the resistive MHD models, compared to popular schemes that are not constructed with an emphasis on structure-preservation.

To preserve several conserved quantities at the discrete level, we adopt a discrete differential form point of view. Comparing to recent efforts [12, 13] on structure-preserving discretization for the fluid mechanics with \( H(\text{div}) \)-conforming velocity, we use an \( H(\text{curl}) \)-based formulation for the fluid part to preserve the helicity. A similar discretization for the Navier-Stokes equations based on the Nédélec edge element can be found in [14]. However, helicity-preservation was not addressed there. Comparing with existing works for the MHD discretization, e.g., [20, 22], we introduce the discrete Hodge star (\( L^2 \) projections) of the magnetic field and the vorticity as independent variables in proper spaces. This is the key to obtain the helicity conservation. As a summary, we use (\( u_h, \omega_h, j_h, E_h, H_h, B_h, p_h \)) \( \in [H^b_0(\text{curl}, \Omega)]^5 \times H^b_0(\text{div}, \Omega) \times H^b_0(\text{grad}) \) as variables, and the discrete variational form (3.4 below) naturally follows from the continuous level. The algorithm is valid for unstructured meshes on general domains and can be of arbitrary order in the framework of the finite element exterior calculus (FEEC) [2, 3].

The resulting system thus has more variables than the scheme in, e.g., [22], but is still easy to solve. In the numerical tests, we use iterative methods as solvers.

The rest of the paper will be organized as follows. In Section 2 we provide preliminaries and notation. In Section 3 we present our algorithm that preserves the helicity. In Section 4 we present numerical results on the convergence and helicity-preserving properties of our algorithms. In Section 5 we give some concluding remarks.

2. Preliminaries

2.1. Notation. Helicity can be defined on 3D contractible domains. One can extend the definition to nontrivial topology and different space dimensions [4, Chapter 3]. However, for simplicity of presentation, in this paper we assume that \( \Omega \subset \mathbb{R}^3 \) is a contractible bounded Lipschitz domain.

We use the standard notation for the inner product and the norm of the \( L^2 \) space

\[
(u, v) := \int_{\Omega} u \cdot v \, dx, \quad \| u \|_0 := \left( \int_{\Omega} |u|^2 \, dx \right)^{1/2}.
\]

Define the following \( H(D, \Omega) \) space with a given linear operator \( D \):

\[
H(D, \Omega) := \{ v \in L^2(\Omega), Dv \in L^2(\Omega) \},
\]

and

\[
H_0(D, \Omega) := \{ v \in H(D, \Omega), \forall \nu = 0 \text{ on } \partial \Omega \},
\]
where \( t_D \) is the trace operator:
\[
t_D v := \left\{ \begin{array}{ll} v, & D = \text{grad}, \\ v \times n, & D = \text{curl}, \\ v \cdot n, & D = \text{div}. \end{array} \right.
\]
We also define:
\[
L^2_0(\Omega) := \left\{ v \in L^2(\Omega) : \int_\Omega v \, dx = 0 \right\}.
\]
By definition, \( H_0(\text{grad}, \Omega) \) coincides with \( H^1_0(\Omega) \).

The de Rham complex in three space dimensions with vanishing boundary conditions reads:
\[
0 \rightarrow H_0(\text{grad}, \Omega) \xrightarrow{\text{grad}} H_0(\text{curl}, \Omega) \xrightarrow{\text{curl}} H_0(\text{div}, \Omega) \xrightarrow{\text{div}} L^2_0(\Omega) \rightarrow 0
\]
The sequence \( \text{(2.1)} \) is exact on contractible domains, meaning that \( \mathcal{N}(\text{curl}) = \mathcal{R}(\text{grad}) \) and \( \mathcal{N}(\text{div}) = \mathcal{R}(\text{curl}) \), where \( \mathcal{N} \) and \( \mathcal{R} \) denote the kernel and range of an operator, respectively.

The main idea of the discrete differential forms \( \text{[2, 3]} \) or the finite element exterior calculus \( \text{[7, 19]} \) is to construct finite element discretizations for the spaces in \( \text{(2.1)} \) such that they fit into a discrete sequence:
\[
0 \rightarrow H^h_0(\text{grad}, \Omega) \xrightarrow{\text{grad}} H^h_0(\text{curl}, \Omega) \xrightarrow{\text{curl}} H^h_0(\text{div}, \Omega) \xrightarrow{\text{div}} L^2_{0,h}(\Omega) \rightarrow 0
\]
The discrete de Rham sequences can be of arbitrary order \( \text{[3, 6]} \). Figure 1 shows the finite elements of the lowest order (Whitney forms). We use \( Q^\text{curl}_{h} \), the \( L^2 \)-projection to

\[
\text{Figure 1. DOFs of the finite element de Rham sequence of lowest order}
\]

\( H^h_0(\text{curl}, \Omega) \):
\[
(2.1) \quad Q^\text{curl}_{h} : \left[ L^2(\Omega) \right]^3 \rightarrow H^h_0(\text{curl}, \Omega),
\]
and the discrete curl operator \( \nabla_h \times : H^h_0(\text{div}, \Omega) \rightarrow H^h_0(\text{curl}, \Omega) \) defined by the following relation:
\[
(2.2) \quad (\nabla_h \times \mathbf{U}, \mathbf{V}) = (\mathbf{U}, \nabla \times \mathbf{V}), \quad \forall (\mathbf{U}, \mathbf{V}) \in H^h_0(\text{div}, \Omega) \times H^h_0(\text{curl}, \Omega).
\]
We shall also use the \( L^2 \)-adjoint operator of the discrete gradient, i.e., the discrete divergence operator \( \nabla_h \cdot : H^h_0(\text{curl}) \rightarrow H^h_0(\text{grad}) \) defined by
\[
(2.3) \quad (\nabla_h \cdot \mathbf{v}_h, \phi_h) := - (\mathbf{v}_h, \nabla \phi_h), \quad \forall \phi_h \in H^h_0(\text{grad}).
\]

2.2. MHD equations and conserved quantities. Consider the following system of equations in \( \Omega \times (0,T) \):
\[
\partial_t \mathbf{u} - \mathbf{u} \times \omega - R_e^{-1} \nabla \times \nabla \times \mathbf{u} - c \mathbf{j} \times \mathbf{B} + \nabla \left( \frac{1}{2} |\mathbf{u}|^2 + p \right) = f, \quad (2.4a)
\]
\[
j - \nabla \times \mathbf{B} = 0, \quad (2.4b)
\]
\[
\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \quad (2.4c)
\]
\[
R_m^{-1} j - (\mathbf{E} + \mathbf{u} \times \mathbf{B}) = 0, \quad (2.4d)
\]
\[
\nabla \cdot \mathbf{B} = 0, \quad (2.4e)
\]
\[
\nabla \cdot \mathbf{u} = 0, \quad (2.4f)
\]
where $\partial_t u = \partial u / \partial t$ and $\partial_t B = \partial B / \partial t$ are time derivatives of $u$ and $B$; $u$ and $p$ are the fluid velocity and pressure, respectively; $j$, $B$ and $E$ are the volume current density, the magnetic field and the electric field, respectively. The fluid momentum equation (2.4a) is in the Lamb form [26] with the vorticity $\omega := \nabla \times u$. In (2.4a), $j \times B$ is called the Lorentz force, the force that the magnetic field acts on the conducting fluid, and $c$ is the coupling number. The parameters $R_e$ and $R_m$ are the fluid and magnetic Reynolds numbers, respectively. Throughout this paper, we will refer to the version of (2.4a) without $R_m^{-1} j$ and $R_e^{-1} \nabla \times \nabla \times u$ (formally, $R_e = R_m = \infty$), as the ideal MHD system.

We consider the following boundary conditions for (2.2) on $\partial \Omega \times (0,T)$:

$$(5) \quad u \times n = 0, \quad P := p + \frac{1}{2}|u|^2 = 0, \quad B \cdot n = 0, \quad \text{and} \quad E \times n = 0,$$

where $n$ is the unit outer normal vector. The initial conditions for the fluid velocity, magnetic field are given for any $x \in \Omega$

$$(6) \quad u(x,0) = u_0(x), \quad B(x,0) = B_0(x).$$

In fact, the boundary conditions for $u$ and $P$ in (2.5) can be seen as a vorticity boundary condition since $u \times n = 0$ implies $(\nabla \times u) \cdot n = 0$ on $\partial \Omega$. We refer to [14, 23] for similar boundary conditions for the Navier-Stokes equations. As we shall see below, these boundary conditions are the ones that lead to the helicity conservation of ideal MHD systems.

We briefly review some conserved quantities of (2.5) below. First of all, MHD equation (2.4) preserves the energy.

**Theorem 1.** The MHD system (2.4) with the boundary condition (2.5) satisfies the following energy identity:

$$(7) \quad \frac{1}{2} \frac{d}{dt} \|u\|^2_0 + \frac{c}{2} \frac{d}{dt} \|B\|^2_0 + R^{-1}_e \|
abla \times u\|^2_0 + cR^{-1}_m \|j\|^2_0 = (f,u).$$

The energy law (2.7) is well known (see, for example, [29]) and the key of the proof is a cancelation between the Lorentz force term $(j \times B, u) = ((\nabla \times B) \times B, u)$ obtained by multiplying $u$ to the equation (2.4a) and the magnetic advection $(u \times B, \nabla \times B)$ obtained by multiplying $B$ to (2.4c). This cancelation reflects symmetry in the operator structure of the MHD system (c.f., [31] (4.7)).

We now discuss the helicity conservation for (2.4). There are two kinds of helicity in the MHD system: the magnetic helicity $H_m$ and the cross helicity $H_c$, which are defined, respectively, as follows:

$$H_m := \int \Omega B \cdot A dx, \quad \text{and} \quad H_c := \int \Omega B \cdot u dx.$$ 

Here $A$ is any potential such that $\nabla \times A = B$. The definition of $H_m$ is gauge invariant, i.e., not depending on the choice of the magnetic potential $A$ since $\int_\Omega B \cdot \nabla \phi dx = 0$ for any scalar field $\phi$.

In ideal MHD systems, the magnetic and cross helicity is conserved. We state a slightly more general helicity identity as follows. The proofs (usually with vanishing boundary conditions and $R_e = R_m = \infty$) can be found in, e.g., [32].

**Lemma 1.** For the MHD system (2.4), the following identity holds:

$$(8) \quad \frac{d}{dt} H_m = -2 \int_{\partial \Omega} n \cdot (A_t + 2E) \times A ds - 2R^{-1}_m \int \Omega B \cdot \nabla \times B dx,$$

$$(9) \quad \frac{d}{dt} H_c = \int_{\partial \Omega} \left( u \times B \times u \right) - \left( \frac{1}{2} |u|^2 + p \right) B \cdot n ds$$

$$+ \int \Omega f \cdot B dx - (R^{-1}_e + R^{-1}_m) \int \Omega \nabla \times B \cdot \nabla \times u dx.$$
Here the first term on the right hand side of (2.8) is due to the contribution of boundary terms and the second term reflects the effect of diffusion. With the boundary condition $\mathbf{B} \cdot \mathbf{n} = 0$ as (2.5), we can choose $\mathbf{A}$ such that $\mathbf{A} \times \mathbf{n} = 0$ and thus the first term vanishes. In ideal MHD systems, the second term also vanishes and therefore the magnetic helicity is conserved. Similarly, with the boundary conditions in (2.5) and $\mathbf{f}$ being any gradient field, (1) vanishes in the ideal MHD limit and therefore the cross helicity is conserved. This observation is summarized in the following theorem.

**Theorem 2.** In the ideal MHD systems with the boundary conditions (2.5) with $\mathbf{f}$ being a gradient field, the magnetic helicity and the cross helicity are conserved in the evolution:

$$\frac{d}{dt} \mathcal{H}_m = \frac{d}{dt} \mathcal{H}_c = 0.$$ 

### 3. Helicity-preserving numerical discretization

In this section, we construct finite element methods that preserve both magnetic and cross helicity. To motivate the scheme, in Section 3.1 we show that extra terms pollute the helicity in some existing numerical schemes, e.g., the algorithms in [22]. We present our new finite element methods in Section 3.2 and prove their properties.

#### 3.1. Helicity-pollution in non-conservative schemes.

Let $\mathbf{V}^h \subset [H^1_0]^3$ and $Q^h \subset L^2_0$ be an inf-sup stable Stokes finite element pair and $Q^V_h : [L^2(\Omega)]^3 \to \mathbf{V}^h$ be the $L^2$ projection. We then consider the operator form of the finite element schemes in [22] adapted to the Lamb form of the MHD system (2.4): Find $(\mathbf{u}_h, p_h, \mathbf{B}_h) \in \mathbf{V}^h \times Q^h \times H^1_0(\text{div})$ such that

$$\partial_t \mathbf{u}_h - Q^V_h(\mathbf{u}_h \times (\nabla \times \mathbf{u}_h)) - R^{-1}_m \Delta \mathbf{u}_h + \tilde{\nabla}_h p_h - c Q^V_h [(\nabla \times \mathbf{B}_h) \times \mathbf{B}_h] = Q^V_h \mathbf{f}, \quad (3.1a)$$

$$\partial_t \mathbf{B}_h - R^{-1}_m \nabla \times \left[\nabla_h \times \mathbf{B}_h - Q^\text{curl}_h (\mathbf{u}_h \times \mathbf{B}_h)\right] = 0, \quad (3.1b)$$

where $\tilde{\nabla}_h : Q^h \to \mathbf{V}^h$ is the $L^2$ adjoint of the divergence operator, i.e.,

$$(\tilde{\nabla}_h p_h, \mathbf{v}_h) = -(p_h, \nabla \cdot \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}^h.$$ 

and

$$(\Delta \mathbf{u}_h, \mathbf{v}_h) := (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}^h.$$ 

To see the pollution of the magnetic helicity, from (3.1b) we have: for any $\mathbf{A}_h \in H^1_0(\text{curl})$ satisfying $\nabla \times \mathbf{A}_h = \mathbf{B}_h$,

$$\frac{d}{dt} (\mathbf{A}_h, \mathbf{B}_h) = 2((\mathbf{A}_h)_t, \mathbf{B}_h) = -2R^{-1}_m (\nabla \times \mathbf{A}_h, \nabla_h \times \mathbf{B}_h) + 2(\mathbf{u}_h \times \mathbf{B}_h, Q^\text{curl}_h \mathbf{B}_h)$$

$$= -2R^{-1}_m (\nabla \times \mathbf{A}_h, \nabla_h \times \mathbf{B}_h) + 2(\mathbf{u}_h \times \mathbf{B}_h, (Q^\text{curl}_h - \text{id}) \mathbf{B}_h).$$

Here the first term on the right hand side is due to the magnetic diffusion which is consistent with the continuous level (2.8), while the second term is nonphysical due to the numerical scheme.
To see the pollution of the cross helicity, we test the first equation of \((3.1)\) by \(B_h\) (equivalently, by \(Q_h V B_h\)) and test the second equation of \((3.1)\) by \(u\), and obtain:

\[
\frac{d}{dt}(u_h, B_h) = ((u_h)_t, B_h) + ((B_h)_t, u_h)
\]

\[
= (u_h \times (\nabla \times u_h), Q_h^V B_h) - (u_h \times (Q_h^\text{curl} \nabla \times u_h), B_h)
\]

\[
+ c((\nabla_h \times B_h) \times B_h, (Q_h^V - \text{id}) B_h) - (p_h, \nabla \cdot Q_h^V B_h) + (f, B_h)
\]

\[
- R^{-1}_m (\nabla_h \times B_h, \nabla \times u_h) - R^{-1}_e (\nabla u_h, \nabla \nabla_h^V B_h)
\]

\[
= (u_h \times (\nabla \times u_h), (Q_h^V - \text{id}) B_h) + (u_h \times ((\text{id} - Q_h^\text{curl}) \nabla \times u_h), B_h)
\]

\[
+ c((\nabla_h \times B_h) \times B_h, (Q_h^V - \text{id}) B_h) - (p_h, \nabla \cdot Q_h^V B_h) + (f, B_h)
\]

\[
- R^{-1}_m (\nabla_h \times B_h, \nabla \times u_h) - R^{-1}_e (\nabla u_h, \nabla \nabla_h^V B_h),
\]

where the last three terms are due to source or diffusion terms as the continuous level, and the remaining terms are nonphysical due to the numerical discretization.

### 3.2. Full-discrete finite element formulation

We present a helicity-preserving full discretization for the MHD system. We use the Crank-Nicolson method as the temporal scheme. Similar conclusions also hold for semi-discretization with continuous time. We begin our discussion by defining

\[
X_h = [H^5_0(\text{curl}, \Omega) \times H^h_0(\text{div}, \Omega) \times H^h_0(\text{grad})].
\]

Below we use \(u_h, B_h, \omega_h, p_h, j_h, H_h, E_h\) to denote the variables evaluated at the mid-point of the time interval \([t_n, t_{n+1}]\). For \(u_h\) and \(B_h\), this means

\[
u_h := \frac{u_{h + 1} + u_h}{2} \quad \text{and} \quad B_h := \frac{B_{h + 1} + B_h}{2}.
\]

For other variables whose time derivatives do not appear in the equations, one defines, e.g., \(p_h := p_{h + 1}^{n+1/2}\) as an independent variable without referring to \(p_{h}^{n}\) or \(p_{h}^{n+1}\), i.e., one uses stagger grids in the time direction.

We also denote

\[
D_t \int_{\Omega} a \cdot b \, dx := \int_{\Omega} a^{n+1} \cdot b^{n+1} \, dx - \int_{\Omega} a^{n} \cdot b^{n} \, dx,
\]

as the difference of the inner product at two successive time steps.

The main scheme can be written as follows.
Algorithm 1 Main algorithm

Given \( (u^0, B^0) \).

for \( n = 0, 1, \cdots, N \) do

Find \( (u_h, \omega_h, j_h, E_h, H_h, B_h, p_h) \) such that for all \( (v_h, \mu_h, k_h, F_h, G_h, C_h, q_h) \in X_h: \)

\[
\begin{align*}
(D_t u_h, v_h) - (u_h \times \omega_h, v_h) + \nabla^{-1}(\nabla \times u_h, \nabla \times v_h) \\
+ (\nabla p_h, v_h) - c(j_h \times H_h, v_h) &= (f, v_h), \\
(D_t B_h, C_h) + (\nabla \times E_h, C_h) &= 0, \\
(R^{-1}_m j_h - [E_h + u_h \times H_h], G_h) &= 0, \\
(\omega_h, \mu_h) - (\nabla \times u_h, \mu_h) &= 0, \\
(j_h, k_h) - (B_h, \nabla \times k_h) &= 0, \\
(B_h, F_h) - (H_h, F_h) &= 0, \\
(u_h, \nabla q_h) &= 0,
\end{align*}
\]

where

\[
(D_t u_h) := \frac{u_h^{n+1} - u_h^n}{\Delta t} \quad \text{and} \quad (D_t B_h) := \frac{B_h^{n+1} - B_h^n}{\Delta t}.
\]

end for

Figure 2 summarizes the choice of variables and spaces and their relation.

Remark 1. There are a number of variables in (3.4). However, several of them can be obtained easily as projections of other variables. For example, from (3.4) we have

\[
E_h = R^{-1}_m \nabla \times B_h - \mathcal{Q}_h^{\text{curl}}(u_h \times H_h), \quad \omega_h = \mathcal{Q}_h^{\text{curl}}(\nabla \times u_h), \quad j_h = \nabla \times B_h, \quad B_h = \mathcal{Q}_h^{\text{curl}} H_h.
\]

In particular, \( \omega_h \) and \( H_h \) can be obtained by local projections of \( \nabla \times u_h \) and \( B_h \), respectively. Therefore in the numerical tests, one can solve the system (3.4) efficiently.

In the analysis below, we will need the following basic fact about the Crank-Nicolson scheme.

Lemma 2. For any given \( a \in \mathbb{R}^3 \), we have

\[
(D_t a, a) = \frac{1}{2\Delta t} (\|a^{n+1}\|^2 - \|a^n\|^2).
\]

Furthermore, for any pair of vectors \( (a, b) \in \mathbb{R}^3 \times \mathbb{R}^3 \), we have the following identity:

\[
(D_t \int a \cdot b \, dx) = \int D_t a \cdot b \, dx + \int a \cdot D_t b \, dx,
\]

where

\[
D_t \int a \cdot b \, dx := \frac{1}{\Delta t} \left( \int a^{n+1} \cdot b^{n+1} \, dx - \int a^n \cdot b^n \, dx \right).
\]

The Gauss law \( \nabla \cdot B = 0 \) is automatically preserved in (3.4). Namely, we have
Theorem 3. If the initial data satisfies $\nabla \cdot B_h^0 = 0$, then we have
\begin{equation}
\nabla \cdot B_h^n = 0 \quad \forall n \geq 0.
\end{equation}
The proof is the same as in [22]. For completeness, we include the proof here.

Proof. From the equation
\begin{equation}
(D_t B_h, C_h) + (\nabla \times E_h, C_h) = 0 \quad \forall C_h \in H_0^1(\text{div}, \Omega).
\end{equation}
We have that $D_t B_h = -\nabla \times E_h$. Taking divergence, we obtain that
\begin{equation}
D_t \nabla \cdot B_h = 0.
\end{equation}
This completes the proof. $\square$

We now show the energy law for (3.4):

Theorem 4. The discrete energy law holds:
\begin{equation}
((D_t u_h, u_h) + c(D_t B_h, B_h)) + R_c^{-1}\|\nabla \times u_h\|^2 + c R_m^{-1}\|j_h\|^2 = (f, u_h),
\end{equation}
and
\begin{equation}
1/2(\|u_h^{n+1}\|^2 + c\|B_h^{n+1}\|^2) \leq 1/2(\|u_h^n\|^2 + c\|B_h^n\|^2) - \frac{1}{2} R_c^{-1} \sum_{j=0}^{n} (\Delta t)\|\nabla \times u_h^{j+1/2}\|^2
- c_p R_m^{-1} \sum_{j=0}^{n} (\Delta t)\|\nabla \times B_h^{j+1/2}\|^2 + (\Delta t) R_c \sum_{j=0}^{n} \|f^{j+1/2}\|^2,
\end{equation}
where $c_p$ is the constant in the Poincaré inequality $\|u_h\| \leq c_p\|\nabla \times u_h\|$ for $u_h$ satisfying $\nabla \cdot u_h = 0$.

Proof. Taking $v_h = u_h$ in the momentum equation of (3.4), we obtain
\begin{equation}
(D_t u_h, u_h) + R_c^{-1}\|\nabla \times u_h\|^2 - c([\nabla_h \times B_h] \times Q_h^{\text{curl}} B_h, u_h) = (f, u_h).
\end{equation}
Moreover, we have that
\begin{align*}
(D_t B_h, B_h) &= -(\nabla \times E_h, B_h) = -(\nabla \times [R_m^{-1} j_h - u_h \times Q_h^{\text{curl}} B_h], B_h) \\
&= -R_m^{-1} (j_h, \nabla_h \times B_h) + (u_h \times Q_h^{\text{curl}} B_h, \nabla_h \times B_h) \\
&= -R_m^{-1}\|j_h\|^2 - (u_h \times Q_h^{\text{curl}} B_h, j_h).
\end{align*}
Therefore, we have
\begin{equation}
-c(u_h \times Q_h^{\text{curl}} B_h, j_h) = c(D_t B_h, B_h) + c R_m^{-1}\|j_h\|^2.
\end{equation}
This shows the equality (3.12). Then (3.13) follows from a sum and the estimate
\begin{equation}
|(f, u_h)| \leq \|f\|\|v_h\| \leq 1/2 R_c^{-1} c_p^{-2}\|u_h\|^2 + R_c\|f\|^2 \leq 1/2 R_c^{-1}\|\nabla \times u_h\|^2 + R_c\|f\|^2.
\end{equation}
$\square$

We now discuss the magnetic and cross helicity for the discrete MHD system. The following theorems can be similarly stated and proved for any contractible subdomain if the variables satisfy the boundary conditions (2.5) on the boundary of it. Therefore we obtain identities for both local and global helicity. For simplicity of presentation, we focus on the helicity on $\Omega$, i.e., the global helicity.

Theorem 5. For any solution of the discrete ideal MHD system (3.4), the following identity of the magnetic helicity holds:
\begin{equation}
D_t \int_{\Omega} B_h \cdot A_h \, dx = R_m^{-1} \int_{\Omega} H_h \cdot j_h \, dx,
\end{equation}
where $A_h$ is any vector potential of $B_h$ satisfying $\nabla \times A_h = B_h$ in $\Omega$. 

The discrete energy law holds:

Theorem 4. The discrete energy law holds:
Proof. Since the magnetic Gauss law is precisely preserved, there exists \( A_h \in H_0^1(\text{curl}, \Omega) \), such that \( \nabla \times A_h = B_h \) in \( \Omega \). Hence

\[
Dt \int_{\Omega} B_h \cdot A_h \, dx = \int_{\Omega} \nabla \times Dt A_h \cdot A_h \, dx + \int_{\Omega} Dt A_h \cdot B_h \, dx
\]

\[
= \int_{\Omega} Dt A_h \cdot \nabla \times A_h \, dx + \int_{\Omega} Dt A_h \cdot B_h \, dx = 2 \int_{\Omega} Dt A_h \cdot B_h \, dx.
\]

Note that \( Dt B_h = -\nabla \times E_h \) and \( Dt A_h = \nabla \times Dt A_h \). Therefore, there exists \( \phi \in H_0^1(\text{grad}, \Omega) \) such that \( Dt A_h = -E_h - \nabla \phi \). This means

\[
(3.15) \quad \int_{\Omega} Dt A_h \cdot B_h \, dx = -\int_{\Omega} (E_h + \nabla \phi) \cdot B_h \, dx = -\int_{\Omega} E_h \cdot B_h \, dx.
\]

However, \( E_h = -Q_{h}^{\text{curl}}(u_h \times Q_{h}^{\text{curl}} B_h) + R_m^{-1} Q_{h}^{\text{curl}} j_h \) by \((3.4)\) and \((3.4)\). Therefore,

\[
(3.16) \quad (B_h, E_h) = R_m^{-1}(B_h, Q_{h}^{\text{curl}} j_h) = R_m^{-1}(Q_{h}^{\text{curl}} B_h, j_h).
\]

This completes the proof. \( \square \)

We now show identities for the cross helicity.

**Theorem 6.** The following identity holds for the cross helicity:

\[
(3.17) \quad Dt \int_{\Omega} u_h \cdot B_h \, dx = -R_e^{-1}(\nabla \times u_h, \nabla \times H_h) - R_m^{-1}(\nabla \times u_h, j_h) + (f, H_h).
\]

Proof. Taking \( v_h = Q_{h}^{\text{curl}} B_h \), we have from \((3.4)\):

\[
(D_t u_h, B_h) + (Q_{h}^{\text{curl}} \nabla \times u_h) \times u_h, Q_{h}^{\text{curl}} B_h \rangle + (\nabla p_h, B_h)
\]

\[
+ R_e^{-1}(\nabla \times u_h, \nabla \times Q_{h}^{\text{curl}} B_h) = (f, Q_{h}^{\text{curl}} B_h).
\]

We also note by \((3.4)\) and \((3.4)\) that

\[
E_h = -Q_{h}^{\text{curl}}(u_h \times Q_{h}^{\text{curl}} B_h) + R_m^{-1} Q_{h}^{\text{curl}} j_h.
\]

On the other hand, we have that

\[
D_t B_h = -\nabla \times E_h = \nabla \times Q_{h}^{\text{curl}}(u_h \times Q_{h}^{\text{curl}} B_h) - R_m^{-1} \nabla \times Q_{h}^{\text{curl}} j_h.
\]

Consequently,

\[
D_t \int_{\Omega} u_h \cdot B_h \, dx = (D_t u_h, B_h) + (u_h, D_t B_h)
\]

\[
= -((Q_{h}^{\text{curl}} \nabla \times u_h) \times u_h, Q_{h}^{\text{curl}} B_h) - (\nabla p_h, B_h) - R_e^{-1}(\nabla \times u_h, \nabla \times Q_{h}^{\text{curl}} B_h)
\]

\[
+ (u_h, \nabla \times Q_{h}^{\text{curl}}(u_h \times Q_{h}^{\text{curl}} B_h)) - R_m^{-1} (\nabla \times u_h, Q_{h}^{\text{curl}} j_h) + (f, Q_{h}^{\text{curl}} B_h)
\]

\[
= -R_e^{-1}(\nabla \times u_h, \nabla \times H_h) - R_m^{-1}(u_h, j_h) + (f, H_h).
\]

From Theorem [5] and Theorem [6] we see that the discrete magnetic helicity and the discrete cross helicity are both conserved in the ideal MHD limit with suitable boundary conditions. We summarize this result as follows.

**Theorem 7.** Assume that \((f, H_h) = 0\). Then we have the helicity conservation in the ideal MHD limit:

\[
D_t \int_{\Omega} B_h \cdot A_h \, dx = 0, \quad D_t \int_{\Omega} B_h \cdot u_h \, dx = 0,
\]

i.e.,

\[
\int_{\Omega} B_h^0 \cdot A_h^0 \, dx = \cdots = \int_{\Omega} B_h^0 \cdot A_h^0 \, dx, \quad \int_{\Omega} B_h^n \cdot u_h^0 \, dx = \cdots = \int_{\Omega} B_h^0 \cdot u_h^0 \, dx.
\]
The helicity provides a lower bound for the energy [4]. Thanks to the discrete de Rham complex and its properties, this bound can be carried over to the discrete level, supplying a control of the (local) discrete energy from below. We focus on the magnetic helicity, although a similar result holds for any divergence-free field.

**Lemma 3.** There exists a positive constant $C$ such that

\begin{equation}
H_m := \int_\Omega B_h \cdot A_h \, dx \leq C \| B_h \|^2.
\end{equation}

**Proof.** Choose $A_h \in H^1_0(\text{curl}, \Omega)$ such that $\nabla \times A_h = B_h$, $(A_h, \nabla \psi_h) = 0$, $\forall \psi_h \in H^1_0(\text{grad}, \Omega)$, and $A_h \times n = 0$ on $\partial \Omega$. By the discrete Poincaré inequality [3, 19], there exists a universal positive constant $C$ such that $\| A_h \| \leq C \| \nabla \times A_h \| = \| B_h \|$. Consequently,

\begin{equation}
H_m = \int_\Omega A_h \cdot B_h \, dx \leq \| A_h \| \| B_h \| \leq C \| B_h \|^2.
\end{equation}

\[ \square \]

4. Numerical Experiments

We report a couple of numerical tests of the convergence and helicity conservation of the proposed scheme. In particular, we investigate and compare helicity changes with various Reynolds numbers and in different algorithms. The implementation is based on the FEniCS project [4].

4.1. Convergence of Algorithm. In this section, we carry out a 3D convergence test with the following form of solutions on the domain $\Omega = (0, 1)^3$. Let

\begin{equation}
p = h(x)h(y)h(z),
\end{equation}

where $h(\mu) = (\mu^2 - \mu)^2$. Further, we let

\begin{equation}
g_1(t) = 4 - 2t, \quad g_2(t) = 1 + t \quad \text{and} \quad g_3(t) = 1 - t.
\end{equation}

We now introduce analytic velocity and magnetic field that satisfy the boundary conditions. Namely,

\[ u = \begin{pmatrix} -g_1h'(x)h(y)h(z) \\ -g_2h'(x)h(y)h(z) \\ -g_3h'(x)h(y)h(z) \end{pmatrix} \quad \text{and} \quad B = \omega = \nabla \times u. \]

With this setting, $u \times n = 0$, $B \cdot n = 0$ and the modified pressure $P = |u|^2/2 + p$ satisfies the boundary condition. Furthermore, it holds that $\nabla \cdot B = 0$.

Before presenting the convergence results, we first make some remarks on solvers of the coupled system. As we shall see, the coupled system is easy to solve, even though it has more independent variables than existing schemes, e.g., those in [22].

In the tests below, we will solve the coupled system [3.4] in an iterative process, referred to as the outer iteration. In each outer iteration, the first step is to solve $u_h, p_h$ by treating other terms in the momentum equation explicitly, i.e., solving the following problem: find $(u_h, p_h) \in H^1_0(\text{curl}, \Omega) \times H^1_0(\text{grad}, \Omega)$ for a given $f$ and $g$, such that

\begin{align}
(\Delta t)^{-1}(u_h, v_h) + (\nabla p_h, v_h) &= (f, v_h), \quad \forall v_h \in H^1_0(\text{curl}, \Omega) \\
(u, \nabla q_h) &= (g, q_h), \quad \forall q_h \in H^1_0(\text{grad}, \Omega).
\end{align}

If $f = 0$, the above system boils down to a Poisson equation for $p$. We solve (4.3) by an AMG-preconditioned minimum residual iterative method. Table 1 shows the uniform convergence with respect to the mesh size. After obtaining $u_h$ and $p_h$ from solving (4.3), we update other variables in (3.4) by simple operations. For example, $\omega_h$ and $E_h$ are updated from (3.4d) and (3.4e) by $L^2$ projections of $\nabla \times u_h$ and $-u_h \times B_h$, respectively.
The outer iterations typically takes about 4 to 5 iterations to achieve the appropriate tolerance, e.g., $L^2$ norm difference between two consecutive iterates divided by the time step size is smaller than $10^{-7}$. As $R_e$ and $R_m$ become smaller, the convergence takes more nonlinear (outer) iterations.

| Mesh size          | Iteration Numbers |
|--------------------|-------------------|
| $h_x = h_y = h_z = 2^{-2}$ | 11                |
| $h_x = h_y = h_z = 2^{-3}$ | 11                |
| $h_x = h_y = h_z = 2^{-4}$ | 13                |
| $h_x = h_y = h_z = 2^{-5}$ | 13                |
| $h_x = h_y = h_z = 2^{-6}$ | 13                |

Table 1. Number of iterations of preconditioned MINRES to achieve relative tolerance $10^{-10}$ for solving (4.3), $\Delta t = 1$.

Convergence results are shown in Table 2.

4.2. Tests for Helicity conservation. In this section, we investigate the helicity behavior of our algorithms with various Reynolds numbers. We also compare the algorithm to another discretization based on existing schemes [39].

In the tests below, we use the following initial conditions for $u_0^h = (u_1, u_2, u_3)^T$:

$$u_1 = -\sin(\pi(x - 1/2)) \cos(\pi(y - 1/2)) z(z - 1)$$
$$u_2 = \cos(\pi(x - 1/2)) \sin(\pi(y - 1/2)) z(z - 1)$$

and $u_3 = 0$.

For the magnetic field, we provide the following initial condition:

$$B_0^h = (-\sin(\pi x) \cos(\pi y), \cos(\pi x) \sin(\pi y), 0)^T$$

Figure 3 shows the initial conditions for $u_h$ and $B_h$. We note that the desired boundary conditions are satisfied:

$$u_h^0 \times n = 0 \quad \text{and} \quad B_h^0 \cdot n = 0 \text{ on } \partial \Omega.$$  

Furthermore, we have $\nabla \cdot B_h^0 = 0$ for the initial data. We can show that the helicity vanishes, i.e., $\mathcal{H}_m = \mathcal{H}_c = 0$.

To evaluate the magnetic helicity of our algorithm, we obtain $B_h$ and compute the potential $A_h$ by solving the following equation: find $A_h \in H^0_0(\text{curl})$ such that

$$(\nabla \times A_h, \nabla \times C_h) = (B_h, \nabla \times C_h), \quad \forall C_h \in H^0_0(\text{curl}).$$  

Since curl has a nontrivial kernel, (4.5) is a singular system. However, this non-uniqueness does not affect the helicity. In the implementation, we apply the Krylov space method, i.e., GMRES with ILU preconditioners, to solve (4.5), which is known to converge for consistent singular problems [24] [28].

| $h$  | $\|B - B_h\|_0$ | order | $\|u - u_h\|_0$ | order |
|------|-----------------|-------|-----------------|-------|
| $2^{-2}$ | 1.60E-3 | x   | 4.56E-4 | x     |
| $2^{-3}$ | 7.80E-4 | 1.03 | 2.25E-4 | 1.02  |
| $2^{-4}$ | 3.39E-4 | 1.20 | 1.06E-4 | 1.09  |
| $2^{-5}$ | 1.62E-4 | 1.06 | 5.31E-5 | 1.00  |

Table 2. Convergence results for the MHD system. The error is computed at the time level $T = 1$ with the Crank-Nicolson time stepping with $\Delta t = 0.01$. $R_e = R_m = 10^4$. 


We now discuss the effect of resistivity on the cross and magnetic helicity. Figure 4 shows the evolution of helicity in Algorithm 1 with various choices of $R_e$ and $R_m$. As $R_e$ and $R_m$ increase, the helicity is closer to be conservative. This is consistent with Theorem 7 stating that both the magnetic and the cross helicity are conserved in the ideal MHD limit.

To compare the helicity from Algorithm 1 and other algorithms, we consider another finite element algorithm based on the scheme proposed in [39] for solving the stationary incompressible MHD system.

The finite element scheme presented in [39] has $B_h$ in the Nédélec space with any stable Stokes pair for $u_h$ and $p_h$. To show the effect of the discretization for the magnetic part and adapt the scheme to the vorticity boundary conditions (2.5), we shall only use the scheme for the magnetic part of the algorithm and consider the time-dependent setting.
Algorithm 0. Find \((u_h, \omega_h, B_h, p_h) \in H^0_0(\text{curl}) \times H^0_0(\text{curl}) \times H^0_0(\text{curl}) \times H^0_0(\text{grad})\) such that for all \((v_h, \mu_h, C_h, q_h) \in H^0_0(\text{curl}) \times H^0_0(\text{curl}) \times H^0_0(\text{curl}) \times H^0_0(\text{grad})\),

\[
(D_t u_h, v_h) - (u_h \times \omega_h, v_h) + R_e^{-1} (\nabla \times u_h, \nabla \times v_h) + (\nabla p_h, v_h) - c ((\nabla \times B_h) \times B_h, v_h) = 0, \quad (4.6a)
\]

\[
(\omega_h, \mu_h) - (\nabla \times u_h, \mu_h) = 0, \quad (4.6b)
\]

\[
(D_t B_h, C_h) - (u_h \times B_h, \nabla \times C_h) + R_m^{-1} (\nabla \times B_h, \nabla \times C_h) = 0, \quad (4.6c)
\]

\[
(u_h, \nabla q_h) = 0. \quad (4.6d)
\]

In Algorithm 0 we have used the Crank-Nicolson time stepping as Algorithm 1. In [39] there is a Lagrange multiplier to impose the weak divergence-free condition for the magnetic field, i.e.,

\[
(B_h, \nabla z_h) = 0, \quad \forall z_h \in H^0_0(\text{grad}). \quad (4.7)
\]

However, we may drop this constraint in the above time dependent formulation because from \((4.6c)\) we conclude that

\[
D_t B_h - \nabla_h \times \mathbb{Q}^\text{curl}_h (u_h \times B_h) + R_m^{-1} \nabla_h \times \nabla \times B_h = 0.
\]

Taking a discrete divergence on both sides, we have that \(D_t (\nabla_h \cdot B_h) = 0\), i.e., if the initial data satisfies \((4.7)\), then the solution satisfies \((4.7)\) at any time step.

Figure 5 compares the cross and magnetic helicity produced in Algorithm 0 and Algorithm 1, respectively. In fact, for Algorithm 0 we do not even have a precise definition of the magnetic helicity since the discrete magnetic field is not divergence-free. The curve in Figure 5 for \(H_m\) demonstrates a discrete helicity computed by projecting the magnetic field to the divergence-free Raviart-Thomas space.

Figure 5 shows that even for resistive MHD systems, our asymptotic-helicity-conservative scheme Algorithm 1 shows a significant difference in the helicity behavior compared to Algorithm 0 which is not designed with an emphasis on helicity-preservation.

We also plot the snapshot of the velocity component in the time evolution in Figure 6 to verify the stability of our computation.

5. Conclusion

We constructed finite element methods that preserve the discrete energy law, the magnetic Gauss law and the magnetic, cross helicity precisely at the discrete level. The construction relies on discrete de Rham complexes and mathematical properties of the MHD system. In
Figure 6. Magnitude of \( u_h \) and streamline obtained with Algorithm 1 with \( R_e = R_m = 1000, c = 0.05, h = 1/16 \) and \( \Delta t = 1/1000 \) at time level \( T = 1 \).

In particular, the Lorentz force term and the magnetic advection term cancel with each other in the proof of the energy law, and the fluid and magnetic advection terms cancel with each other in the proof of the cross helicity conservation. These cancelations reflect symmetry in the operator structures of the MHD system [31], carried over from the continuous level to the discrete level, and can be important for the construction of efficient solvers [31].

For the spatial discretization, we used finite element differential forms, e.g., the Nédélec and Raviart-Thomas elements, in the numerical tests. The discussions in this paper also hold with other discrete de Rham complex, e.g., spline spaces [9].

To preserve the helicity and energy in the full discretization, we used the Crank-Nicolson scheme as the temporal discretization, c.f., [38]. One can choose other temporal schemes that preserve quadratic invariants, c.f., [17].

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Appendix: Existence and uniqueness of solutions

In this appendix, we discuss the existence and uniqueness of solutions to the nonlinear scheme. For technical issues and for simplicity, we will actually work on a slightly modified system other than (3.4). We first introduce some notation.

Define

\[
Z_h := \{ v \in H^h_0(\text{curl}) : \nabla_h \cdot v_h = 0 \}, \quad H^h_0(\text{div } 0) := \{ C_h \in H^h_0(\text{div}) : \nabla \cdot C_h = 0 \}.
\]

Lemma 4. For \( B_h \in H^h_0(\text{div } 0) \), we have \( \nabla_h \cdot (Q_h \text{curl } B_h) = 0 \).

Proof. Denote \( H_h := Q_h^{\text{curl}} B_h \).

\[
(\nabla_h \cdot H_h, \nabla_h \cdot H_h) = -(\nabla \nabla_h \cdot H_h, H_h) = -(\nabla \nabla_h \cdot H_h, B_h) = (\nabla_h \cdot H_h, \nabla \cdot B_h) = 0.
\]

For functions in \( Z_h \) and \( H^h_0(\text{div } 0) \), we recall the following Gaffney type inequalities [18]: there exist positive constants \( C \) such that

\[
\| \nabla \times v_h \|_{L^{3+\delta}} \leq C \| \nabla \times v_h \|, \quad \forall v_h \in Z_h, \tag{5.1}
\]

\[
\| B_h \|_{L^{3+\delta}} \leq C \| \nabla \times B_h \|, \quad \forall B_h \in H^h_0(\text{div } 0). \tag{5.2}
\]
Here $\delta \in (0, 3]$ is a positive number depending on the regularity of the domain $\Omega \subset \mathbb{R}^3$. If $\Omega$ is convex or has $C^{1,1}$ boundary, we can choose $\delta = 3$. In the sequel, we assume that $\Omega$ is such that we can choose $\delta = 1$, i.e., $Z_h \hookrightarrow L^4(\Omega)$.

In the analysis below, we slightly modify the diffusion term in equations (3.4) and assume that all the variables at the $n$-th time step are zero to avoid dealing with the cross terms between the $n$-th and the $(n+1)$-th time steps. The same analysis works for, e.g., a backward Euler time discretization.

Consider the following variational form: find $(u_h, B_h) \in Z_h \times H_0^1(\text{div} 0)$, such that for any $(v_h, C_h) \in Z_h \times H_0^1(\text{div} 0)$,

\begin{align}
(\Delta t)^{-1} (u_h, v_h) - (u_h \times Q_h^{\text{curl}}(\nabla \times u_h), v_h) + R_{\epsilon}^{-1} (\nabla \times u_h, \nabla \times v_h) \\
- c((\nabla \times B_h) \times Q_h^{\text{curl}} B_h, v_h) = (F, v_h),
\end{align}

(5.3)

\begin{align}
(\Delta t)^{-1} (B_h, C_h) - (u_h \times Q_h^{\text{curl}} B_h, \nabla \times C_h) + 1/2R_m^{-1}(\nabla \times Q_h^{\text{curl}} B_h, \nabla \times Q_h^{\text{curl}} C_h) \\
+ 1/2R_m^{-1}(\nabla \times B_h, \nabla \times C_h) = (G, C_h).
\end{align}

(5.4)

Here $F$ and $G$ denote some general source terms.

Comparing with (3.4), we modified the magnetic diffusion term in (5.4) by changing $1/2R_m^{-1}(\nabla \times B_h, \nabla \times C_h) = 1/2R_m^{-1}(\nabla \times Q_h^{\text{curl}} B_h, \nabla \times Q_h^{\text{curl}} C_h)$.

We write (5.3), (5.4) in the following standard form: find $(u_h, B_h) \in Z_h \times H_0^1(\text{div} 0)$, such that for any $(v_h, C_h) \in Z_h \times H_0^1(\text{div} 0)$,

\begin{align}
a((u_h, B_h); (v_h, C_h)) = ((F, cG), (v_h, C_h)),
\end{align}

(5.5)

where the trilinear form $a(\cdot; \cdot, \cdot)$ is defined by

\begin{align}
a((u_h, B_h); (v_h, C_h), (w_h, K_h)) := (\Delta t)^{-1}(u_h, v_h) - (w_h \times Q_h^{\text{curl}}(\nabla \times u_h), v_h) \\
+ R_{\epsilon}^{-1}(\nabla \times u_h, \nabla \times v_h) - c((\nabla \times B_h) \times Q_h^{\text{curl}} K_h, v_h) + (\Delta t)^{-1}c(B_h, C_h) \\
- c(w_h \times Q_h^{\text{curl}} B_h, \nabla \times C_h) + 1/2cR_m^{-1}(\nabla \times B_h, \nabla \times C_h) \\
+ 1/2cR_m^{-1}(\nabla \times Q_h^{\text{curl}} B_h, \nabla \times Q_h^{\text{curl}} C_h).
\end{align}

Introduce the following norm:

\begin{align}
\|(u_h, B_h)\|_V := (\Delta t)^{-1}\|u_h\|^2 + (\Delta t)^{-1}c\|B_h\|^2 + R_{\epsilon}^{-1}\|\nabla \times u_h\|^2 \\
+ cR_m^{-1}\|\nabla \times B_h\|^2 + cR_m^{-1}\|\nabla \times Q_h^{\text{curl}} B_h\|^2.
\end{align}

From Lemma 4 (5.3) and (5.2), we can further bound the following terms by the $\|\cdot\|_V$ norm:

\begin{align}
\|u_h\|_{L^4} + \|B_h\|_{L^4} + \|Q_h^{\text{curl}} B_h\|_{L^4} &\leq C\|(u_h, B_h)\|_V.
\end{align}

We include the existence theorem for nonlinear variational forms, which is given in, for example, [15]. Since we focus on the discrete level, we only state the results for finite dimensional problems.

**Theorem 8.** Assume that $V$ is a finite dimensional vector space, and there exists a positive constant $\alpha$ such that a bounded trilinear form $a(\cdot; \cdot, \cdot)$ on $V$ satisfies

\begin{align}
a(v; v, v) \geq \alpha\|v\|^2, \quad \forall v \in V.
\end{align}

Then the problem: given $f \in V^*$, find $u \in V$, such that for all $v \in V$,

\begin{align}
a(u; u, v) = f(v),
\end{align}

has at least one solution.

**Lemma 5.** The trilinear form $a(\cdot; \cdot, \cdot)$ is bounded, i.e., there exists a positive constant $C$ such that

\begin{align}
|a((u_h, B_h); (v_h, C_h), (w_h, K_h))| \leq C\|(u_h, B_h)\|_V\|(v_h, C_h)\|_V\|(w_h, K_h)\|_V.
\end{align}
Proof. It suffices to bound the nonlinear terms:
\[
\left| (w_h \times Q_h^{\text{curl}}(\nabla \times u_h), v_h) \right| \leq \|\nabla \times u_h\| L^4 \|v_h\| L^4 \\
\leq \|\nabla \times u_h\| \|\nabla \times w_h\| \|\nabla \times v_h\|,
\]
\[
\left| (\nabla_h \times B_h) \times Q_h^{\text{curl}} K_h, v_h \right| \leq \|\nabla_h \times B_h\| \|Q_h^{\text{curl}} K_h\| L^4 \|v_h\| L^4 \\
\leq \|\nabla_h \times B_h\| \|\nabla \times Q_h^{\text{curl}} K_h\| \|\nabla \times v_h\|,
\]
and the estimate for \((w_h \times Q_h^{\text{curl}} B_h, \nabla_h \times C_h)\) is the same.
\[
\]
Lemma 6. The trilinear form \(a(\cdot,\cdot,\cdot)\) is coercive, i.e., there exists a positive constant \(\alpha\) such that
\[
a((u_h, B_h), (u_h, B_h)) \geq \alpha \|(u_h, B_h)\|^2.
\]
Proof.
\[
a((u_h, B_h), (u_h, B_h)) = (\Delta t)^{-1} \|u_h\|^2 + (\Delta t)^{-1} \|B_h\|^2 \\
+ 1/2cR_m^{-1} \|\nabla_h \times B_h\|^2 + 1/2cR_m^{-1} \|\nabla \times Q_h^{\text{curl}} B_h\|^2.
\]
\[
\]
We are now in a position to state the existence of the discrete variational form.

Theorem 9. For any \((F, G) \in (Z_h)^* \times (H_h^0(\text{div} 0))^*\), there exists at least one solution for (5.5).

The uniqueness of solutions of (5.5) with small data follows from standard argument, c.f., [15].

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