Variable sampling processing and frequency domain analysis of sound signals

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Abstract. This paper describes the transition from analog to digital signals, and the relationship between the angular frequency of continuous-time signals and the digital angular frequency of discrete-time signals. In this paper, the frequency domain expression of the signal after the upper sampling (interpolation) and down sampling (decimation) of the digital signal is strictly derived theoretically. In the frequency domain, the spectrum of the L-sampled signal can be obtained by shrinking the original signal spectrum by L times, the amplitude is constant, and the image frequency is generated within the original period length. Under the premise that no signal distortion occurs under M times, the frequency domain is widened by M times relative to the original signal frequency domain, and the amplitude changes nonlinearly. In this paper, MATLAB is used to perform up-and down-sampling and distortion processing on sound signals to verify the correctness of theoretical frequency domain changes.

1. Introduction
A digital signal is a digital representation of a discrete time signal and is typically obtained from an analog signal. An analog signal is a set of data that changes over time. Some analog signals can be represented by mathematical functions, where time is the independent variable and the signal itself is the dependent variable.

The discrete time signal is the sampling result of the analog signal. The value of a discrete signal is meaningful only at certain fixed points in time (not defined elsewhere), and does not have a continuous value on the time axis like an analog signal.

If the value of the discrete time signal at each sampling point is only an approximation of the value of the original analog signal, then we can use a finite word length to represent the value of all sampling points. Such a discrete time signal is called a digital signal.

With the rapid development of electronic technology, the application of digital signals has become increasingly widespread. Many modern media processing tools, especially those that need to be connected to a computer, have changed from the original analog signal representation to the digital signal representation. Common examples of our daily life include cell phones, video or audio players, and digital cameras.

This paper first reviews the sampling problem of ideal impulse sampling, then briefly describes the principle and concept of up sampling and down sampling, and then theoretically rigorously deduces the digital signal after sampling (interpolation) and down sampling (extraction). The frequency domain expression of the signal is finally processed by MATLAB to verify the correctness of the theoretical derivation.
2. Sampling, extraction and interpolation
Sampling acts as a bridge between continuous-time signals and discrete-time signals. Under certain conditions, a continuous-time signal can be completely recovered from its samples. This allows us to use discrete-time system techniques to implement continuous-time systems and process continuous-time signals: first, convert a continuous-time signal into a discrete-time signal, and then use a discrete-time system to process the discrete-time signal, and then transform it back into continuous time. This process is shown in Figure 1.

2.1. Impulse string sampling
The impulse string sampling is performed by a sampling method in which one cycle impulse string is multiplied by a continuous time signal to be sampled.

The sampling signal is set to \( x(t) \), the sampling period is \( T \), and the periodic impulse string \( p(t) \) is a sampling function. When sampling, there are:

\[
p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)
\]

\[
x_p(t) = x(t) \cdot p(t) = \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT)
\]

\[
X_p(j\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) P(j(\Omega - \theta))d\theta = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\Omega - k\Omega_c)) \tag{1}
\]

Then calculate the Fourier transform of the discrete time signal,

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}
\]

\[
x[n] = x(t) \mid_{t=nT} = x(nT) \tag{2}
\]

\[
X_p(j\Omega) = \sum_{n=-\infty}^{\infty} x_p(nT) e^{-j\Omega nT}
\]

It can be derived that the relationship between the continuous time signal and the Fourier transform of the discrete time signal is:

\[
X(e^{j\omega}) = X(j\Omega)|_{\Omega=\omega/T} \tag{3}
\]

And the relationship between the two is shown in Figure 2.
2.2. Spectrum analysis

We now analyse the spectral changes of discrete time signals after up sampling and down sampling. Derived their frequency domain formulas separately.

Let the research signal be $x[n]$, the original sampling rate is $F_s$, and its Fourier transform is $X(e^{j\omega})$.

2.2.1. Upper sampling frequency domain analysis. The signal after sampling is set to, according to the DTFT conversion relationship:

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{n=+\infty} x[n]e^{-j\omega n}
\]

\[
X_p(e^{j\omega}) = \sum_{n=-\infty}^{n=+\infty} x_p[n]e^{-j\omega n}
\]

Since $x_p[n]$ is only non-zero when $n=kL$, there are:

\[
X_p(e^{j\omega n}) = \sum_{k=-\infty}^{+\infty} x_p[kL]e^{-j\omega kL}
\]

Since $x_p[kL] = x[k]$, so

\[
X_p(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x[k]e^{-j(L\omega)k}
\]

By observation, we can conclude that:

\[\Omega = \frac{\omega}{T}\]
\[ X_p(e^{j\omega}) = X(e^{jL\omega}) \] (8)

Therefore, in the frequency domain, the signal spectrum after the up sampling can be obtained by shrinking L times on the frequency spectrum of the original signal spectrum, and the amplitude is unchanged. Considering the periodicity of the discrete-time Fourier transform, under the condition of satisfying the Nyquist sampling theorem, the spectrum of the up sampled signal will not only exhibit a compressed state, but also a mirror component will appear in the period.

2.2.2. Down-sampling frequency domain analysis. Using the z-transform for the down sampled signal is available:

\[ X_b(z) = \sum_{n=-\infty}^{+\infty} x[Mn]z^{-n} \] (9)

The definition \( x_{int}[n] \) is as follows:

\[ x_{int}[n] = \begin{cases} x[n], & n = 0, \pm M, \pm 2M ... \\ 0, & \text{else} \end{cases} \] (10)

So, we can rewrite \( X_b(z) \) as:

\[ X_b(z) = \sum_{n=-\infty}^{+\infty} x[Mn]z^{-n} = \sum_{n=-\infty}^{+\infty} x_{int}[n]z^{-n} = \sum_{n=-\infty}^{+\infty} x_{int}[n]z^{-n} = X_{int}(z^{M}) \] (11)

Then define another sequence \( c[n] \):

\[ c[n] = \begin{cases} 1, & n = 0, \pm M, \pm 2M \\ 0, & \text{else} \end{cases} \] (12)

And \( x_{int}[n] = x[n]c[n] \), at the same time, consider a unit circle, by the nature of polar coordinates, you can rewrite \( c[n] \) as:

\[ c[n] = \frac{1}{M} \sum_{k=0}^{M-1} e^{-j \frac{2\pi kn}{M}} \] (13)
\[ X_{in1}(z) = \sum_{n=-\infty}^{\infty} x_{in1}[n] z^{-n} = \sum_{n=-\infty}^{\infty} c[n] x[n] z^{-n} \]

\[ = \frac{1}{M} \sum_{n=-\infty}^{\infty} \left[ \left( \sum_{k=0}^{M-1} e^{-j \frac{2\pi kn}{M}} \right) x[n] z^{-n} \right] \]

\[ = \frac{1}{M} \sum_{k=0}^{M-1} \left[ \sum_{n=-\infty}^{\infty} \left( e^{-j \frac{2\pi kn}{M}} x[n] z^{-n} \right) \right] \]

\[ = \frac{1}{M} \sum_{k=0}^{M-1} X \left( e^{-j \frac{2\pi k}{M} z} \right) \]  

(14)

Bringing equation (17) into equation (14), there are:

\[ X_b(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X \left( e^{-j \frac{2\pi k}{M} e^{j\omega} \frac{1}{M}} \right) \]

\[ = \frac{1}{M} \sum_{k=0}^{M-1} X \left( e^{-j \frac{\omega + 2\pi k}{M}} \right) \]  

(15)

It can be concluded that if M[x] is extracted for x[n], the time domain compression, its frequency domain \( X_b(e^{j\omega}) \) is broadened by M times relative to the \( X(e^{j\omega}) \) frequency domain, and the amplitude changes nonlinearly.

3. Variable sampling simulation of sound signals

Now we use MATLAB to sample the sound signal up and down to observe the waveform and frequency domain changes. The range of all the spectrograms below is (negative semi-axis symmetry, 2 is the period). The sound signal input MATLAB is 48000Hz. We have performed 3 times extraction and 3 times interpolation respectively, and finally performed 10 times down sampling to observe the occurrence of mixing. Stacking time effect.

The original waveform and spectrum of the sound signal are as follows:
3.1.1. Times up sampling. We now perform 3 times up sampling (interpolation) of the sound signal. The current sampling rate is 48000*3=144000 Hz. The waveform is shown in Figure 4(a) and the spectrum is shown in Figure 4 when the observation is the same as the original signal. (b).
In contrast to Figure 4, on the waveform, the up sampled sound waveform stretches three times, and the sound sounds slower and lower.

In addition, since the Fourier transform of the discrete-time signal is in the period of 2, the compression of the spectrogram also causes the image of the next cycle to enter the range of 0~, and the image frequency appears.

4. Conclusion
This paper first reviews the transition from analog to digital, and the angular frequency of the continuous-time signal and the digital angular frequency of the discrete-time signal are \( \Omega = \omega / T \).
Next, the frequency domain expression of the signal after the up sampling (interpolation) and down sampling (decimation) of the digital signal is strictly derived from the definition of the Fourier transform of the discrete time signal. Among them, the derivation of the sampling frequency domain expression specifically mentions a wrong derivation method: that $X_e(e^{j\omega})$ cannot be derived by Fourier transform, and needs to use z transform.

Finally, this paper uses MATLAB simulation to process the vocal signal to verify the correctness of the theoretical derivation, and finds that the discrete time signal is broadened in the time domain, the frequency domain shrinks; the time domain shrinks and the frequency domain broadens. The spectrogram of the distorted signal does not reflect the signal information of the original signal.

References

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