$SU(N_c)$ gauge theories for all $N_c$.

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Abstract

We show that $SU(N_c)$ gauge theories in 2+1 dimensions are close to $N_c = \infty$ for $N_c \geq 2$. The dimensionful coupling, $g^2$, is proportional to $1/N_c$, at large $N_c$, confirming the usual diagram-based expectation. Preliminary calculations in 3+1 dimensions indicate that the same is true there.
1 Introduction.

The idea of considering $SU(N_c)$ gauge theories as perturbations in powers of $1/N_c$ around $N_c = \infty$ is an old one. It has been buoyed by the fact that, if one assumes confinement for all $N_c$, then the phenomenology of the $SU(\infty)$ quark-gluon theory is strikingly similar to that of (the non-baryonic sector of) QCD [1, 2]. This makes it conceivable that the physically interesting $SU(3)$ theory could be largely understood by solving the much simpler $SU(\infty)$ theory (for a review see [3]). The fact that the lattice $SU(\infty)$ theory can be re-expressed as a single plaquette theory [4], provided the basis of a number of interesting computational explorations of that theory (for a review see [5]). Unfortunately this latter scheme is limited by the fact that it does not tell us anything about the size of the corrections to the $N_c = \infty$ limit.

Over the last few years I have investigated $D = 2 + 1$ $SU(N_c)$ gauge theories by Monte Carlo simulation. These theories are similar to the corresponding $D = 3 + 1$ theories: the interaction strength vanishes at short distances, while at long distances there appears to be a non-perturbative linear confining potential. The coupling sets the overall mass scale in both cases: in $D = 2 + 1$ $g^2$ has dimensions of mass while in $D = 3 + 1$ the scale invariance is anomalous, so the coupling runs and the rate at which it runs introduces a mass scale (i.e. the $\Lambda$ parameter). On the other hand the computational requirements are obviously much less in $D = 2 + 1$ than in $D = 3 + 1$. This has enabled me to obtain a detailed continuum mass spectrum that is much more accurate than that available in 4 dimensions.

The $C = +$ sector of the light mass spectrum turned out be quite similar for both $SU(2)$ [6] and $SU(3)$ [7]. (This also appears to be the case in $D = 3 + 1$, although the comparison there is weakened by the much larger errors.) If the reason for this is that both are close to the $N_c = \infty$ limit, then this provides an economical understanding of the spectra of $SU(N_c)$ gauge theories for all $N_c$, i.e. there is a common spectrum with small corrections.

At the same time, many theoretical approaches are simpler in the large $N_c$ limit. For example, there has been recent progress in calculating the large $N_c$ mass spectrum using light front quantisation techniques [8].

Motivated by all this I have investigated the $N_c \to \infty$ limit in the most direct fashion: by performing some $SU(4)$ calculations. In $D = 2 + 1$ I have found that the $SU(3)$ and $SU(4)$ spectra are almost identical in both $C = +$ and $C = -$ sectors. As we shall
see in the next section, we can describe the variations between the \( N_c = 2, 3 \) and \( 4 \) mass spectra by a simple \( O(1/N_c^2) \) correction. This is the expected form for the leading large-\( N_c \) correction in pure gauge theories \cite{3}. In the final section I present some preliminary results for the \( SU(4) \) theory in \( 4 \) dimensions. The light masses considered, and the topological susceptibility, indicate that here too there is a rapid approach to the large-\( N_c \) limit.

2 2+1 dimensions.

Consider the string tension. Using smeared Polyakov loops in the standard fashion \cite{10}, I have calculated the continuum string tension, \( \sigma \), in units of \( g^2 \) for \( SU(2) \) \cite{10}, \( SU(3) \) and \( SU(4) \) \cite{7}, gauge theories in \( D = 2 + 1 \). The values of \( \sqrt{\sigma/g^2} \) are plotted in Fig. 1 against the number of colours, \( N_c \). We see that there is an approximate linear rise with \( N_c \). We obtain a good fit with

\[
\frac{\sqrt{\sigma}}{g^2} = 0.1974(12)N_c - \frac{0.120(8)}{N_c}.
\]

We would obtain a similar behaviour with other light glueball masses: we focus on the string tension since it is the most accurate quantity in our calculations.

Some observations.

• For large \( N_c \), \( \sqrt{\sigma} \propto g^2 N_c \). That is to say, the overall mass scale of the theory, call it \( \mu \), is proportional to \( g^2 N_c \). So in units of the mass scale of the theory,

\[
g^2 \propto \frac{\sqrt{\mu}}{N_c}.
\]

This is the usual diagram-based expectation, but here the argument is entirely non-perturbative.

• Our correction term has the theoretically-expected functional form, i.e. it is \( O(1/N_c^2) \) relative to the leading term. If we try a fit with a \( O(1/N_c) \) correction instead, we obtain an unacceptably poor \( \chi^2 \). (But higher powers are not necessarily excluded by our ‘data’.) These expectations apply not only to the continuum limit, but also to lattice corrections, and our lattice values do indeed show that the leading lattice correction is \( O(\alpha g^2 N_c) \).

• The coefficients of the leading term and the first correction are comparable, suggesting a well-behaved expansion in powers of \( 1/N_c \).

We have also fitted the lightest glueball masses, with some cases shown in Fig. 2. For example we find for the lightest glueball

\[
\frac{m_{0^{++}}}{\sqrt{\sigma}} = 4.046(70) + \frac{2.67(28)}{N_c^2}
\]
and for its first excitation
\[ \frac{m^*_{0^{++}}}{\sqrt{\sigma}} = 6.28(16) + \frac{2.16(86)}{N_c^2}. \]  
(5)

This provides us with mass predictions not just for \( N_c = \infty \) but for all \( N_c \).

3 3+1 dimensions.

What we know about 4 dimensional gauge theories is currently much less precise. As far as continuum properties are concerned, reasonably accurate quantities include the string tension, the lightest scalar and tensor glueballs and the deconfining transition. As in the \( D = 2 + 1 \) case, the \( SU(2) \) and \( SU(3) \) values are within \( \sim 20\% \) of each other. This encourages an investigation of the \( SU(4) \) theory.

In \( D = 3 + 1 \), \( SU(4) \) calculations are slow and we have only performed a very preliminary study.

We use the standard plaquette action, and so our first potential hurdle is the presence of the well-known bulk transition that occurs as we increase \( \beta \) from the strong coupling regime towards weak coupling. We have performed a scan on a \( 10^4 \) lattice which locates this transition at \( \beta_c = 10.4 \pm 0.1 \). This turns out to be a rather strong coupling and so it does not complicate our subsequent calculations.

We performed 4000,6000 and 3000 sweeps respectively on \( 10^4,12^4 \) and \( 16^4 \) lattices at \( \beta = 10.7, 10.9 \) and 11.1. We carried out correlator measurements every 5’th sweep.

- We obtain string tensions \( a\sqrt{\sigma} = 0.296(14), 0.229(7) \) and \( 0.196(7) \) at \( \beta = 10.7, 10.9 \) and 11.1 respectively. (Here \( a \) is the lattice spacing.) Comparing with \( SU(2) \) and \( SU(3) \) [8], and using \( \beta = 2N_c/g^2 \), we see that the bare coupling, \( g^2(a) \), for the same lattice spacing (in physical units) varies roughly as \( g^2(a) \propto 1/N_c \). A better coupling to use is the mean-field improved one \[ g_I^2(a), \] obtained from \( g^2(a) \) by dividing it by \( \langle \frac{1}{N_c} TrU_p \rangle \). For example \( a\sqrt{\sigma} \) is approximately the same at \( \beta = 2.47, 6.0, 11.0 \) for \( N_c = 2, 3, 4 \) respectively. This corresponds to \( g_I^2(a) \simeq 2.509, 1.684, 1.244 \) respectively. These vary almost exactly as \( 1/N_c \), confirming the usual diagrammatic expectation.

- In Fig. 3 I plot the ratios of scalar and tensor glueball masses to \( \sqrt{\sigma} \). The \( N_c = 2, 3 \) values are continuum extrapolations, while the two \( N_c = 4 \) values are simply those obtained at \( \beta = 10.9, 11.1 \) since our calculations are not accurate enough for a continuum extrapolation. Although the \( N_c = 4 \) errors are large, we are certainly consistent with the \( N_c \) variation being given by a simple \( 1/N_c^2 \) correction.

- We have also calculated the topological susceptibility, \( \chi_t \equiv \langle Q^2 \rangle / vol \), using the
standard cooling method. In Fig. 4 I show the dimensionless ratio $\chi_1^{1/4}/\sqrt{\sigma}$ for $\beta = 10.9$ and 11.1, and compare these values to the $N_c = 2, 3$ continuum values. (Warning: for such a short Monte Carlo run the errors are certain to be underestimated.) Again there is consistency with a simple $1/N_c^2$ correction. We remark that for $SU(4)$ one expects, semiclassically, very few small instantons and this is confirmed in our cooling calculations. This has the advantage that the lattice ambiguities that arise when instantons are not much larger than $a$ are reduced compared to $SU(3)$ and dramatically reduced as compared to $SU(2)$. The large-$N_c$ physics of topology (and the related meson physics) is of obvious interest.

4 Conclusions.

We have calculated the mass spectra of gauge theories with $N_c = 2, 3, 4$ in 3 dimensions. We have found that the string tension and lightest masses are accurately given by a $O(1/N_c^2)$ correction to the $N_c = \infty$ limit. This provides a unified understanding of all these theories in terms of just one theory with modest corrections to it. In practical terms it means that we know the corresponding masses for all values of $N_c$.

Preliminary calculations in 4 dimensions suggest that the situation is the same in that case. Here we have investigated not only the lightest masses but the topological susceptibility as well. We remark that the lattice calculation of $Q$ becomes rapidly cleaner as $N_c$ increases.

Our work provides explicit confirmation of the old, bold conjecture [2] that, in the context of the $1/N_c$ expansion, 3 is a large number. Moreover, we provide a significant extension of that conjecture: 2 is also a large number.

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Figure 1: Continuum string tension versus number of colours in $D = 2 + 1$. Line is the fit in eqn(1).

Figure 2: Some continuum glueball masses, in $D = 3$, for 2,3,4 colours: $0^{++}$(•), $0^{+*}$(×), $2^{++}$(∗), $0^{-+}$(○) and linear fits.
Figure 3: Lightest scalar (●) and tensor (○) glueball masses in $D = 4$. Continuum values for $N_c = 2, 3$ and lattice values ($\beta = 10.9$ and $\beta = 11.1$) for $N_c = 4$.

Figure 4: The topological susceptibility: continuum values for $N_c = 2, 3$ and averaged lattice value for $N_c = 4$. 