An Enhanced Symmetry for the $p$-adic Wavelets

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Abstract

Wavelet analysis has been extended to the $p$-adic line $\mathbb{Q}_p$. The $p$-adic wavelets are complex valued functions with compact support. As in the case of real wavelets, the construction of the basis functions is recursive, employing scaling and translation. Consequently, wavelets form a representation of the affine group generated by scaling and translation. In addition, $p$-adic wavelets are eigenfunctions of a pseudo-differential operator, as a result of which they turn out to have a larger symmetry group. The enhanced symmetry of the $p$-adic wavelets is demonstrated.
1 Introduction

Wavelet analysis may be considered as one of the cornerstones of modern day multi-resolution analysis
for linear and non-linear processes. It is a useful method in the digital signal processing and pattern
recognition. However, wavelet analysis also finds applications in pure mathematics [1], statistical
data analysis [2], quantum field theory [3], nonlinear dynamics [4], to name a few. It is extremely
useful in cases where traditional Fourier analysis is not very efficient. This is because wavelets are
localised in both time and frequency, a property that resembles real life signals. In this approach,
functions are expanded in terms of a set of basis functions known as wavelet functions, which are
compactly supported in frequency and time domain. This allows for local processing of functions
independently at different scales.

The mathematical description of wavelet theory begins with the representation of a square inte-
grable function in terms of a complete set of orthonormal basis (although in practical applications,
it is often more efficient to use a nonorthogonal set of functions). The first basis of this type was
introduced by Haar [5] for $L^2(\mathbb{R})$ long before the notion of wavelet transform was developed. The
Haar wavelets, as they are called now, consist of dyadic translation and dilatation of a basic comp-
actly supported piecewise constant function, known as the (Haar) mother wavelet function. Morlet
and Grossmass’s pioneering work on wavelet analysis [6] was based on seismic wave data analysed
by Morlet and collaborators and used smooth basis functions with compact support. Later work
by Mallat, Meyer, Daubechies [7–10] and others developed the modern theory of wavelet transform
(see [11] for a brief historical review).

The basis functions in wavelet analysis are constructed by application of scaling and translation
on a mother wavelet, as a result of which all the basis functions have the same shape. Let $\Psi_M(x)$ be
the mother wavelet. Then the scaled and translated functions are

$$\Psi_{a,b}(x) = \frac{1}{\sqrt{a}} \Psi_M \left( \frac{x-b}{a} \right), \quad a \in \mathbb{R}^+ \text{ and } b \in \mathbb{R} \quad (1)$$

The functions $\Psi_{a,b}(x)$ may be thought to have resulted from $\Psi_M(x)$ due to the action of the elements
\[ g(a, b), \text{ which satisfy} \]
\[ g(a_1, b_1) g(a_2, b_2) = g(a_1 a_2, b_1 + a_1 b_2) \]
(2)

and thus form the affine group ‘ax + b’, a semi-direct product of the group of scaling and translation.

In this notation \( \Psi_M(x) = \Psi_{1,0}(x) \), which for the Haar wavelet is

\[ \Psi_{M,0}^{\text{Haar}}(x) = \begin{cases} 1 & \text{for } 0 \leq x < \frac{1}{2} \\ -1 & \text{for } \frac{1}{2} \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \]

An orthonormal basis may be obtained by restricting the scaling parameter \( a \) to, say, the dyadic numbers \( 2^n \) (\( n \in \mathbb{Z} \)), and the translation parameter \( b \in \mathbb{Z} \). These are the original Haar wavelets.

One may also get an orthonormal set of basis wavelets in the interval \([0, 1]\) by restricting \( n \) to negative integers and translations appropriately. The bases have the structure of a binary rooted tree with the mother wavelet \( \Psi_{1,0} \) at the base. In the first generation, we have \( \Psi_{\frac{1}{2},0} \) and \( \Psi_{\frac{1}{2},1} \). In the second, \( \Psi_{\frac{1}{2^2},0} \) and \( \Psi_{\frac{1}{2^2},1} \) are connected to \( \Psi_{\frac{1}{2},0}, \) and \( \Psi_{\frac{1}{2},2} \) and \( \Psi_{\frac{1}{2},3} \) are connected to \( \Psi_{\frac{1}{2},1}, \) and so on.

A straightforward generalisation of the dyadic case would start from the mother wavelet that assumes the values \( \{1, \omega, \omega^2, \cdots, \omega^{p-1}\} \) (where \( \omega \) is a primitive \( p \)-th root of unity) in the \( p \) segments \( \left[ \frac{\ell}{p}, \frac{\ell+1}{p} \right] \) (\( \ell = 0, \cdots, p-1 \)) of the interval \([0, 1]\). In order to get an orthonormal basis \( \left\{ \Psi_{\frac{1}{p^n}}^{(p)} \right\} \), we now restrict the scaling parameter to \( p^n \) (\( n \in \mathbb{Z} \)) and the translations \( m \in \mathbb{Z} \).

Closely related to the real numbers is the field of \( p \)-adic numbers \( \mathbb{Q}_p \) obtained by the Cauchy completion of the rationals by a notion of distance derived from the non-archimedean \( p \)-adic norm \([12–14]\). This leads to a very different topology on \( \mathbb{Q}_p \). The ultrametric field \( \mathbb{Q}_p \) and its extensions have found many applications in physics, from glassy systems (see Ref. \([15]\) for a review) to quantum mechanics, quantum field theory (see the monograph \([13]\) and references therein), string theory (e.g., Refs. \([16–20]\) and citations therein) and more recently in a discrete approach to holographic duality \([21–25]\). Wavelet analysis on \( \mathbb{Q}_p \) was pioneered by Kozyrev \([26]\), who proposed a basis of complex valued compactly supported wavelet functions for \( L^2(\mathbb{Q}_p) \). Kozyrev’s construction may be considered as a generalization of the Haar type wavelets to the \( p \)-adic case. This was further generalised in Refs. \([27–29]\) (a recent review is Ref. \([30]\)).

Interestingly Kozyrev’s wavelets are eigenfunctions \([26]\) of the (generalised) Vladimirov derivative \([13]\), a (family of) pseudo-differential operator(s) on \( \mathbb{Q}_p \). Together with the natural shift operators that relate wavelets at different scales, this suggests a possibility of extending (the scaling part of) the affine symmetry of the wavelets to a larger one. We recall that various extensions of (the scaling part of) the affine symmetry of wavelets on \( \mathbb{R} \) have been suggested is Refs. \([31, 32]\). These are all deformations of \( \mathfrak{sl}(2) \). Indeed, we also find a symmetry algebra that is closely related to \( \mathfrak{sl}(2) \). In fact, this is not surprising given the fact that in the context of \( p \)-adic adS/CFT, the conformal field theories (CFT) on \( \mathbb{Q}_p \), or its algebraic extensions, are indeed analogues of one-dimensional CFTs \([25]\) on the real line. An \( \text{SL}(2, \mathbb{R}) \) symmetry is associated with the latter \([33]\) (see also \([34]\)). In this approach, \( \mathbb{Q}_p \) (or its algebraic extension) is the asymptotic boundary of the Bruhat-Tits tree, which plays the analogue of the anti-de Sitter bulk space.
2 A few results from \( p \)-adic analysis

Let us recapitulate a few facts about the field \( \mathbb{Q}_p \) and aspects of \( p \)-adic analysis. More details are available in, e.g., Refs. [12–14]. First, let us fix a prime \( p \). The \( p \)-adic norm of an integer \( n \) is \( |n|_p = p^{-\text{ord}_p(n)} \), where \( \text{ord}_p \) is the highest power of \( p \) that divides \( n \). Clearly, \( \text{ord}_p \) behaves like a logarithm since \( \text{ord}_p(n_1 n_2) = \text{ord}_p(n_1) + \text{ord}_p(n_2) \). Hence the \( p \)-adic norm \( |\cdot|_p \) of a rational number \( m/n \) is defined as \( |m/n|_p = p^{\text{ord}_p(n) - \text{ord}_p(m)} \). The field \( \mathbb{Q}_p \) is the Cauchy completion obtained by including all the (suitably defined equivalence classes of) limits of converging sequences of rational numbers with respect to the \( p \)-adic norm. This norm has the ultrametric property leading to a stronger than usual triangle inequality \( |\xi - \xi'|_p \leq \max(|\xi|_p, |\xi'|_p) \).

An element \( \xi \in \mathbb{Q}_p \) admits a Laurent series expansion in \( p \):

\[
\xi = p^N (\xi_0 + \xi_1 p + \xi_2 p^2 + \cdots) = p^N \sum_{n=0}^{\infty} \xi_n p^n
\]

where \( N \in \mathbb{Z} \) and \( \xi_n \in \{0, 1, \cdots, p-1\} \), \( \xi_0 \neq 0 \). (There is, however, nothing special about this choice; one may work with other representative elements for the coefficients \( \xi_n \).) The expansion in \( p \) above is convergent in the \( p \)-adic norm. The subset \( \mathbb{Z}_p \) consisting of elements with \( N \geq 0 \) in Eq. (4) (i.e., elements with norm less than equal to 1) is a subring known as the \( p \)-adic integers \( \mathbb{Z}_p \).

The finite part of the series Eq. (4), \( \sum_{n=N}^{-1} \xi_n - N p^n \), consisting only of the negative powers of \( p \), is called the fractional part \( \{\xi\}_p \) of \( \xi \). The rest, an infinite series in general, is the integer part \( [\xi]_p \). Let us define the complex valued function \( \chi : \mathbb{Q}_p \rightarrow \mathbb{C} \), as

\[
\chi^{(p)}(\xi) = \exp (2\pi i \xi) = \exp (2\pi i \{\xi\}_p)
\]

As in the real case, the contribution to Eq. (5) from the integer part \( [\xi]_p \) is trivial and only the fractional part matters. Since \( \chi^{(p)}(\xi + \xi') = \chi^{(p)}(\xi) \chi^{(p)}(\xi') \), it is called an additive character of \( \mathbb{Q}_p \). The totally disconnected nature of the \( p \)-adic space means that the (complex valued) continuous functions on \( \mathbb{Q}_p \) are locally constant. One such function is the indicator function

\[
\Omega^{(p)}(\xi - \xi') = \begin{cases} 1 & \text{for } |\xi - \xi'|_p \leq 1 \\ 0 & \text{otherwise} \end{cases}
\]

and similarly for other open sets.

Using the additive character, one can define the Fourier transform of a complex valued function

\[
\tilde{f}(\omega) = \int d\xi \chi^{(p)}(-\omega \xi) f(\xi)
\]

where \( d\xi \) is the translationally invariant Haar measure on \( \mathbb{Q}_p \) normalized as \( \int_{\mathbb{Z}_p} d\xi = 1 \). The inverse
Fourier transformation is given by

\[ f(\xi) = \int d\omega \chi^{(p)}(\omega \xi) \tilde{f}(\omega) \]  

(8)

The proof, available in the references cited, will be omitted. The Fourier transform of an indicator function is also an indicator function. In this sense, these are analogues of the Gaussian functions on the real line.

The totally disconnected topology of \( \mathbb{Q}_p \) does not allow for the usual definition of a derivative. The generalised Vladimirov derivative, therefore, is defined as an integral kernel

\[ D^\alpha f(\xi) = \frac{1 - p^n}{1 - p^{-1-\alpha}} \int d\xi' \frac{f(\xi') - f(\xi)}{|\xi' - \xi|^{1+\alpha}} \]  

(9)

This expression is in fact meaningful for any \( \alpha \in \mathbb{C} \) [29]. Furthermore, \( D^{\alpha_1} D^{\alpha_2} = D^{\alpha_2} D^{\alpha_1} = D^{\alpha_1 + \alpha_2} \).

In the following, we shall need a map from \( \mathbb{Q}_p \) to \( \mathbb{R} \), which was introduced in Ref. [35]. Following Kozyrev [26], we modify the Monna map slightly to define

\[ \mu : \mathbb{Q}_p \rightarrow \mathbb{R}_+ \]

\[ \sum_{m=N}^{\infty} \xi_m p^m \mapsto \sum_{m=N}^{\infty} \xi_m p^{-m-1} \]  

(10)

This map, although, continuous, is not one-to-one. However, the induced map \( \mu : \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \mathbb{N} \cup \{0\} \), is one-to-one.

3 A brief review of Kozyrev wavelets on \( \mathbb{Q}_p \)

The notion of Haar wavelets was generalised to the case of \( \mathbb{Q}_p \) by Kozyrev in Ref. [26]. The set of functions

\[ \psi^{(p)}_{n,m,j}(\xi) = p^{-\frac{n}{2}} \chi^{(p)}(jp^{n-1}\xi) \Omega^{(p)}(|p^n\xi - m|_p), \quad \xi \in \mathbb{Q}_p \]  

(11)

where \( n \in \mathbb{Z} \), \( m \in \mathbb{Q}_p/\mathbb{Z}_p \) and \( j \in \{1, 2, 3, \cdots, p - 1\} \). These functions satisfy

\[ \int_{\mathbb{Q}_p} \psi^{(p)}_{n,m,j}(\xi) d\xi = 0 \]  

(12)

therefore belong to the set of mean zero locally constant functions \( \mathbb{D}_0(\mathbb{Q}_p) \). The wavelets provide an orthonormal basis for \( L^2(\mathbb{Q}_p) \)

\[ \int_{\mathbb{Q}_p} \psi^{(p)}_{n,m,j}(\xi) \psi^{(p)}_{n',m',j'}(\xi) d\xi = \delta_{n,n'} \delta_{m,m'} \delta_{j,j'} \]  

(13)

These properties are exactly analogous to the Haar wavelets on \( \mathbb{R} \).
The mother wavelet $\psi^{(p)}_{0,0,1}(\xi)$ is supported on $\mathbb{Z}_p$. Its value in this subset is 1 everywhere except on $S_1$, i.e., $p$-adic numbers with norm exactly equal to 1. The open set $S_1$, is a union of open sets labelled by the value of $\xi_0$ in Eq. (4), where it assumes the values $\omega_p^{j_0}$ ($\omega_p$ is a primitive $p$-th root of unity). After scaling $\psi^{(p)}_{n,0,j}(\xi)$ is supported on $|\xi|_p \leq p^n$, where it is $p^{-n/2}$ for all $|\xi|_p < p^n$, and takes the values $p^{-n/2} \omega_p^{j \xi_0}$ in the set $|\xi|_p = p^n$, depending on the value of $\xi_0$

$$
\psi^{(p)}_{n,0,j}(\xi) = \begin{cases} 
p^{-n/2} & \text{for } |\xi|_p \leq p^n \\
p^{-n/2} \omega_p^{j \xi_0} & \text{for } |\xi|_p = p^n \\
0 & \text{for } |\xi|_p > p^n
\end{cases} \tag{14}
$$

The effect of translation is more involved. Moreover, the actual transformation depends on the choice of representative. Since the parameter of translation $m$ takes values in $\mathbb{Q}_p/\mathbb{Z}_p$, let us take, e.g., $m = m_0 p^{-1} + \cdots$ (where the terms denoted by the dots, being equivalent to zero, will not matter). The wavelet $\psi^{(p)}_{0,m,j}(\xi)$ is supported on the subset labelled by $\xi_0 = m_0$ of the open set $|\xi|_p = p$. This subset is a union of subsets labelled by the values on $\xi_1$, where the function assumes the values $\omega_p^{j m_0} \omega_p^{j \xi_1}$:

$$
\psi^{(p)}_{0,m \in p^{-1} \mathbb{Z}_p,j}(\xi) = \begin{cases} 
\omega_p^{j \xi_0} \omega_p^{j \xi_1} & \text{for } |\xi|_p = p \text{ and } \xi_0 = m_0 \\
0 & \text{for } |\xi|_p < p, |\xi|_p > p \text{ and } |\xi|_p = p, \text{ but } \xi_0 \neq m_0
\end{cases} \tag{15}
$$

Notice that the measure of the support of a wavelet does not change under translation. The form of the translated wavelet with other parameters can similarly be worked out. For example, for $m \in p^{-2} \mathbb{Z}_p$, the translated wavelet is supported in the subset of the set $|\xi|_p = p^2$ defined by $\xi_0 = m_0$ and $\xi_1 = m_1$. A schematic graph of the wavelets is shown in the Fig. 1.

![Figure 1: A schematic representation of the wavelets. The sets are ordered by the values $|\xi|_p = p^n$. (Colour code as follows: grey = 1, black = 0, other colours correspond to primitive roots of unity.)](image)

The close relation between the generalised Haar wavelets on $\mathbb{R}$ and the Kozyrev wavelets\(^1\) on $\mathbb{Q}_p$

\(^1\)The additional freedom in choosing the phase through $j$ in Eq. (11) will not be important, therefore, we set $j = 1$ in the rest of this section and the next one, to avoid clutter. Also the labels for the Haar wavelets are changed from the ones used in Section 1 to conform to the labels for the Kozyrev wavelets.
can be established via the Monna map Eq. (10). As in [26]), it lets us define

\[ \mu : \psi_n^{(p)} \mapsto \Psi_{n,m}^{(H)} \]  

(16)

where \( \Psi_{n,m}^{(H)} \) are

\[ \Psi_{n,m}^{(H)}(x) = p^{\frac{n}{2}} \sum_{\ell=0}^{p-1} e^{\frac{2\pi i \ell}{p} \Omega[ n \ell + m, m+1 ]} \]  

(17)
in which \( \Omega[a,b] \) is the indicator function for the interval \([a, b] \), i.e., \( \Omega[a,b] = 1 \) for \( x \in [a, b] \) and vanishes otherwise.

An important property of the Kozyrev functions is that they are eigenfunctions of the Vladimirov derivative [26]

\[ D_\alpha \psi^{(p)}_{n,m,j}(\xi) = p^{(1-n)\psi^{(p)}_{n,m,j}(\xi)} \]  

(18)

with eigenvalue \( p^{(1-n)} \). This property of the Kozyrev wavelets suggests that they behave like homogeneous functions on \( \mathbb{R} \) with definite scaling property. In order to make it apparent let us rewrite the above as

\[ D_\alpha \psi^{(p)}_{n,m}(a\xi + b) = |a| p^{(1-n)} \psi^{(p)}_{n,m}(a\xi + b) \]  

(19)

where we have emphasised the scaling behaviour that will play an important role in the next section.

4 Enhanced symmetry of the Kozyrev wavelets

By their construction, the Kozyrev wavelets on \( \mathbb{Q}_p \), like the (generalised) Haar wavelets on \( \mathbb{R} \) (or on \([0, 1] \in \mathbb{R} \), are organised by the affine group \( 'ax + b' \). Specifically, one can define raising and lowering operators such that

\[ a_{\pm} \psi_n^{(H)}(x) = \psi_{n+1,m}^{(H)}(x), \quad x \in \mathbb{R} \]  

\[ a_{\pm}^{(p)} \psi_n^{(p)}(\xi) = \psi_{n+1,m}^{(p)}(\xi), \quad \xi \in \mathbb{Q}_p \]  

(20)

As we have seen, in the \( p \)-adic case, there is an additional operator that acts naturally on the wavelets. It is the generalised Vladimirov derivative, the action of which is given in Eq. (18). It is easy to see, by a straightforward adaptation of the proof in [26], that in the limit \( ^2 \alpha \to 0 \), one gets

\[ (\log_p D) \psi_{n,m}^{(p)}(\xi) = (1 - n) \psi_{n,m}^{(p)}(\xi) \]  

(21)

Combining the actions of \( \log_p D \) and the raising and lowering operators \( a_{\pm} \), let us define the shift operators \( J_{\pm}^{(p)} = a_{\pm} \log_p D \). They act as

\[ J_{\pm}^{(p)} \psi_{n,m}^{(p)}(\xi) = (1 - n) \psi_{n\pm1,m}^{(p)}(\xi), \quad \xi \in \mathbb{Q}_p \]  

(22)

on the Kozyrev wavelets.

\^2 Our definition of \( \log_p D = \ln D / \ln p \), where \( \ln D = \lim_{\alpha \to 0} \frac{1}{\alpha} (D^\alpha - 1) \) is standard.
The property Eq. (19) is in fact analogous to the action of the scaling operator $p^{-x\partial_x}$ on homogeneous functions on $\mathbb{R}$. Let us consider the elementary homogeneous function $x^s$, which, for simplicity we shall restrict to the interval $[0, 1] \in \mathbb{R}$. We define the map $\rho$ that takes the set of Kozyrev wavelets $\psi_{n,m}^{(p)}(\xi)$ to the set of monomials $x^{n-1}$ defined on the interval $[0, 1]$

$$\rho \left( \psi_{n,m}^{(p)}(\xi) \right) = x^{n-1}$$  \hspace{1cm} (23)

where we recognise the exponent as $(n - 1) = -2 \log_p \left( \max|\psi_{n,m}^{(p)}| \right) - 1$.

Applying this map on Eq. (18), we see that

$$\rho \left( D^\alpha \psi_{n,m}^{(p)}(\xi) \right) = \rho^{\alpha(1-n)} \rho \left( \psi_{n,m}^{(p)}(\xi) \right)$$

$$\rho \circ D^\alpha \circ \rho^{-1} \left( x^{n-1} \right) = \rho^{\alpha(1-n)} x^{n-1}$$  \hspace{1cm} (24)

Therefore, we conclude that

$$\rho \circ D^\alpha \circ \rho^{-1} = p^{-\alpha x\partial_x} = \exp \left( -\alpha \ln p \frac{\partial}{\partial \ln x} \right)$$  \hspace{1cm} (25)

In other words, the Vladimirov operator on $\mathbb{Q}_p$ maps to the scaling operator on the monomials on $[0, 1] \in \mathbb{R}$.

Taking a cue from this, it is natural to make the following association

$$\rho \circ J_{\pm}^{(p)} \circ \rho^{-1} = -x^2 \frac{\partial}{\partial x} \equiv L_+ \quad \text{and} \quad \rho \circ J_{\pm}^{(p)} \circ \rho^{-1} = -\frac{\partial}{\partial x} \equiv L_-$$  \hspace{1cm} (26)

These operators, together with $L_0 = -x\partial_x$, satisfy $[L_0, L_{\pm}] = \mp L_{\pm}$ and $[L_+, L_-] = 2L_0$ to generate the $\mathfrak{sl}(2, \mathbb{R})$ algebra.

From their action on the Kozyrev wavelets, we find the following algebra for the operators $J_{\pm}^{(p)}$ and $D^\alpha$:

$$\left[ D^\alpha, J_{\pm}^{(p)} \right] = (1 - p^{\pm\alpha}) D^\alpha J_{\pm}^{(p)}$$

$$\left[ J_{\pm}^{(p)} \right] = 2 \ln p D,$$  \hspace{1cm} (27)

In the second line we have the $\mathfrak{sl}(2, \mathbb{R})$ algebra. This is consistent with what we would get for the operators $L_0$ and $L_{\pm}$ using the map $\rho$ on the operators on the $p$-adic side.

The commutator in the first line can also be rewritten as

$$p^{\mp \frac{\alpha}{2}} D^\alpha J_{\pm}^{(p)} - p^{\pm \frac{\alpha}{2}} J_{\pm}^{(p)} D^\alpha = 0$$  \hspace{1cm} (28)

which is a kind of a deformed commutator $[A, B]_{\text{def}} = qAB - q^{-1}BA$, with $q = p^{\pm\alpha}$. These commutators define a symmetry that is larger than the scaling part of the affine group. We would also like to point to Ref. [31] where several possible extensions of the affine symmetry of the wavelets on the real line was proposed. Although the symmetry is enhanced in both cases, the algebraic structure of the enhanced symmetry is quite different. However, both are deformations of $\mathfrak{sl}(2)$. This is expected
due to the exponential nature of scaling on the wavelet basis [31].

We shall now provide some evidence in favour of the assertions above. Let us begin by expressing the monomial in the basis of generalised Haar wavelets on $[0, 1]$ (this restriction is only to stay within the set of square integrable functions)

$$x^{n-1} = \sum_{n'=\infty}^{0} \sum_{m'=0}^{(n')-1/p^{n'}} c_{n', m'}^{(n-1)} \Psi^{(H)}_{n', m'}(x) \in [0, 1]$$

where, the coefficients $c_{n', m'}^{(n-1)}$ are determined by using the orthonormality of the Haar wavelets:

$$c_{n', m'}^{(n-1)} = \frac{1}{p^\frac{n}{n'}} \int_0^1 x^{n-1} \Psi^{(H)}_{n', m'}(x) dx = \frac{\frac{p-1}{n}}{n} \sum_{j=0}^{p-1} e^{-\pi i \frac{n'}{p} j} \int_{(m'+j)p^{n'}} x^{n-1} dx$$

$$= \frac{p^{-(n-2)p^{n'}}}{n} \sum_{j=0}^{p-1} e^{-\pi i \frac{n'}{p} j} [(m' + j + 1)^n - (m' + j)^n]$$

Let us first consider $L_-$, which acts as a derivative, therefore, $L_-(x^{n-1}) = (n-1)x^{n-2}$. We want to show that

$$(n - 1) \int_0^1 dx \Psi_{n', m'}(x) \frac{d}{dx} x^{n-1} = (n - 1) \int_0^1 dx \Psi_{n', m'}(x)x^{n-2}.$$  

The RHS essentially repeats the computation above:

$$(n - 1) \int_0^1 x^{n-2} \Psi_{n', m'}^{(H)}(x) dx = (n - 1) \int_0^1 x^{n-2} \Psi_{n', m'}^{(H)}(x) dx$$

$$= \left( \int_{m'}^{m'+p} 1 + \int_{m'+p}^{m' + p^{n'}} e^{\frac{2\pi i}{p} j} + \cdots + \int_{m'+p^{n'-1}}^{m'+p^{n'}} e^{\frac{2\pi i(p-1)}{p} j} \right) (n - 1)x^{n-2} dx$$

$$= p^{-(n-2)p^{n'}} \left[ m^{n-1} + (m' + 1)^{n-1} \left( 1 - e^{\frac{2\pi i}{p}} \right) + \cdots \right.$$

$$+ (m' + p - 1)^{n-1} e^{\frac{2\pi (p-2)}{p}} \left. \left( 1 - e^{\frac{2\pi i}{p}} \right) + e^{\frac{2\pi (p-1)}{p}} (m' + p)^{n-1} \right]$$

The LHS, on the other hand, is

$$\int_0^1 dx \Psi_{n', m'}^{(H)}(x) \frac{d}{dx} x^{n-1} = \left[ \Psi_{n', m'}^{(H)}(x)x^{n-1} \right]_0^1 - \int dx \frac{d\Psi_{n', m'}^{(H)} x^{n-1}}{dx}$$

$$= -\int dx x^{n-1} p^{n'} \sum_{j=0}^{p-1} e^{\frac{2\pi i}{p} j} \left[ \delta \left( x - \frac{j + m'}{p^{n'}} \right) - \delta \left( x - \frac{j + m' + 1}{p^{n'}} \right) \right]$$

$$= -p^{n'} \sum_{j=0}^{p-1} e^{\frac{2\pi i}{p} j} \left[ \left( \frac{j + m'}{p^{n'}} \right)^{n-1} - \left( \frac{j + m' + 1}{p^{n'}} \right)^{n-1} \right]$$
Modulo the boundary term\(^{3}\) (which we have omitted in the last two lines), this is exactly the same as the RHS of Eq. (31). The boundary term at \(x = 0\) vanishes, so it does at \(x = 1\), unless the support of the Haar wavelet extends to this boundary. In that case, it is not properly defined, and one may take it to be either zero or one, or anything in between. We choose to take it to be zero. This is our justification for ignoring the boundary term, however, there may well be a better argument.

Let us now consider the effect of dilatation, under which \(x \to \lambda x\) and we would like to choose \(\lambda = p^{-\alpha}\). There is, however, a problem, since the interval \([0, 1]\) \(\to [0, \lambda]\) after scaling. Consequently, \(p^{-\alpha} \partial_x \Psi(x^{n-1}) = e^{-\alpha \ln p \cdot \frac{d}{dx}} \cdot e^{(n-1) \ln x} = p^{\alpha(1-n)} x^{n-1}\), is now supported on \([0, p^{-\alpha}]\).

\[
(p^{-\alpha} x)^{n-1} = \sum_{n', m'} e_{n', m'}^{(n-1)} \Psi_{n', m'}^{(H)}(p^{-\alpha} x)
\]

\[
= \frac{p^{-\left(n-\frac{1}{2}\right)n'}}{n} \sum_{j=0}^{p-1} e^{\frac{2\pi i j}{p^n}} [(m' + j + 1)^n - (m' + j)^n] \frac{p^{\alpha(1-n)} x^{n-1}}{p^\alpha} \sum_{k=0}^{p-1} e^{\frac{2\pi i k}{p^{\alpha}}} \prod_{\substack{\ell = 0 \\ \ell \neq \alpha}}^{\alpha-1} \frac{1}{p^{n' - \alpha}} x^{n-1}
\]

\[
= p^{\alpha(1-n)} \sum_{n', m} e_{n'-\alpha, m}^{(n-1)} \Psi_{n'-\alpha, m}^{(H)}(x) = p^{\alpha(1-n)} x^{n-1}
\]

The scaling transformation is meaningful for any \(\alpha > 0\), however, the shift in subscripts (labels of the Haar wavelets) seems strange. To understand it better, let us restrict \(\alpha \in \mathbb{Z}\), and consider for definiteness \(\alpha = \ell\). Depending on whether \(\ell \in \mathbb{Z}^\pm\), the unit interval is shrunk (respectively expanded) to \([0, p^{-\ell}]\). As a result one needs fewer (respectively, more) basis elements. This is ensured by the shift of the labels.

One can consider the limit \(\alpha \to 0\) as well. In this case, the interval remains the same, and we need to prove that \(\int_0^1 dx \Psi_{n', m'}^{(H)}(x) \frac{d}{dx} x^{n-1} = (n-1) \int_0^1 dx \Psi_{n', m'}^{(H)}(x) x^{n-1}\). The RHS is again similar to Eq. (31), and on the LHS we can perform an integration by parts following the steps in Eq. (32), to verify the equation above. The action of \(L_+\) on the monomials can be checked in the same way.

## 5 Translations

So far we have considered only the scaling part of the affine symmetry of the wavelets. Just like the real wavelets, the orbits of the \(p\)-adic wavelets under the action of translations and dilatations form the affine group \([28, 30]\). The parameter of translation is restricted to \(m \in \mathbb{Q}_p / \mathbb{Z}_p\). It is clear from the definition of the Vladimirov derivative Eq. (9) and the form of the wavelet function Eq. (11),

\[^{3}\text{There is some ambiguity in the boundary terms. Instead of the way we have defined it above one may also take } \int_0^1 dx \Psi_{n', m'}^{(H)}(x) \frac{d}{dx} \varphi(x) = \int_0^{\frac{1}{p^n}} dx \Psi_{n', m'}^{(H)}(x) \frac{d}{dx} \varphi(x) = \left[ \Psi_{n', m'}^{(H)}(x) \varphi(x) \right]_{x = 0}^{x = \frac{1}{p^n}} + \cdots, \text{ which are many more terms. Since the Haar wavelets are step functions, the boundary values are not well defined.}\]
that the action of $D^\alpha$ commutes with translation.

The action of $J^{[p]}_m$ scales the argument of the wavelet, and effect of translation is a semi-direct product with this scaling. Let us consider translation by $m = p^{-1} (m_0 + \cdots) \in p^{-1} \mathbb{Z}_p$ of the wavelet Eq. (14) obtained from the mother wavelet by a scaling by $p^n$. We get

$$\psi^{(p)}_{n,0,j}(\xi) = \begin{cases} 
  p^{-\frac{n}{2}} \omega_p^{j\xi_0} \omega_p^{j\xi_1} & \text{for } |\xi|_p = p^{n+1} \text{ and } \xi_0 = m_0 \\
  0 & \text{for } |\xi|_p \leq p^n, |\xi|_p \geq p^{n+2} \text{ and }
  \text{ but } \xi_0 = m_0
\end{cases} \quad (34)$$

On the other hand, the translated wavelet $\psi^{(p)}_{0,m\in p^{-1}\mathbb{Z}_p,j}(\xi)$ Eq. (15), after a scaling is $\psi^{(p)}_{n,p^m,j}(\xi)$. The form of this function depends on whether $n$ is negative or positive. For $n < 0$, it is

$$\psi^{(p)}_{n,p^m,j}(\xi - m) = \begin{cases} 
  0 & \text{for } |\xi|_p \leq 1, |\xi|_p \geq p^2 \text{ and } |\xi|_p = p \\
  p^{\frac{n}{2}} & \text{for } |\xi|_p = p \text{ and } \xi_i = m_i \forall i = 0,1,\cdots,|n| + 1 \\
  p^{\frac{n}{2}} \omega_p^{j(\xi_{|n|+1}-m_{|n|+1})} & |\xi|_p = p \text{ and } \xi_i = m_i \forall i = 0,1,\cdots,|n| \text{ but } \xi_{|n|+1} \neq m_{|n|+1}
\end{cases} \quad (35)$$

For $n > 0$, the final functions are different for $n = 1$ and $n \geq 2$. (This is because the translation parameter $m$ is chosen to be in $p^{-1}\mathbb{Z}_p$.)

$$(n = 1) \quad \psi^{(p)}_{1,p^m,j}(\xi - m) = \begin{cases} 
  p^{-\frac{1}{2}} \omega_p^{-j m_0} & |\xi|_p \leq 1 \\
  p^{-\frac{1}{2}} \omega_p^{j(\xi_0-m_0)} & |\xi|_p = p \text{ and } \xi_0 \neq m_0 \\
  p^{-\frac{1}{2}} & |\xi|_p = p \text{ but } \xi_0 = m_0 \\
  0 & |\xi|_p \geq p^2
\end{cases} \quad (36)$$

$$(n \geq 2) \quad \psi^{(p)}_{n,p^m,j}(\xi - m) = \begin{cases} 
  p^{-\frac{n}{2}} \omega_p^{-j m_0} & |\xi|_p < p^n \\
  p^{-\frac{n}{2}} \omega_p^{j\xi_0} & |\xi|_p = p^n \\
  0 & |\xi|_p > p^n
\end{cases}$$

After a scaling by a positive power of $p$, the (scaled) translation parameter $p^m$ is valued in $\mathbb{Z}_p$, and hence is equivalent to no translation. The form of the resultant wavelets Eq. (36) are, therefore, the same (upto a phase) as the scaled wavelets Eq. (14).

6 Conclusions

Kozyrev’s wavelets on the $p$-adic line $\mathbb{Q}_p$ are a natural generalisation of the Haar wavelets on the real line. By their recursive construction using scaling and translation, the wavelet basis form a representation of these transformations, specifically, the group generated by their semi-direct product. This is the affine group, or the ‘$ax+b’ group. However, the Kozyrev functions are also eigenfunctions of the generalised Vladimirov derivative, which is a pseudo-differential operator. We have argued that when combined with the action of the Vladimirov derivative, the scaling part of the symmetry is enhanced to a deformation of $\mathfrak{so}(2,\mathbb{R})$. Indeed there is a one parameter family of symmetries, labelled

---

4In the following, we have reinstated the phase label $j$ once again.
by the exponent $\alpha \in \mathbb{C}$ of the Vladimirov derivative $D$. The logarithm of the Vladimirov operator, $\log_p D$ can be defined through a limiting procedure $\alpha \to 0$. With this operator, we find an $\mathfrak{sl}(2, \mathbb{R})$ algebra.

The behaviour of the Kozyrev wavelets as eigenfunctions of $D$, are similar to homogeneous functions on $\mathbb{R}$ with definite scaling property. In fact, we have proposed a map from the wavelets to the monomials $x^{n-1}$, ($n \in \mathbb{N}$). This identification has been justified by expanding the monomials in terms of generalised Haar wavelets. It would be instructive to verify with other wavelet basis.

The enhanced symmetries of the $p$-adic wavelets, and their natural connection with the generalised Haar wavelets on $\mathbb{R}$ suggests that the latter system should also have a larger symmetry. At a formal level, this follows from combining the map in Eq. (16). Specifically,

$$\mu \circ a^{(p)} \circ \mu^{-1} = a_\pm$$

where, $a_\pm$ are raising and lowering operators in Eq. (22). Similarly,

$$\left( \mu \circ D^\alpha \circ \mu^{-1} \right) \Psi_{n,m}^{(H)} = p^{\alpha(1-n)} \Psi_{n,m}^{(H)}$$

in which the $n$ in the exponent, as in Eq. (23), is $n = -2 \log_p \left( \max |\psi_{n,m}^{(p)}| \right)$. The idea of enlargement of symmetry of real wavelets have been explored in [31, 32]. On a wavelet system consisting of smooth functions, it should be possible to realise the symmetry operators as differential operators. A generalisation of these results to wavelets on multi-dimensional spaces $\mathbb{Q}_p^n$ should also be possible.

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