Smooth and peaked solitons of the CH equation

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Abstract
The relations between smooth and peaked soliton solutions are reviewed for the Camassa–Holm (CH) shallow water wave equation in one spatial dimension. The canonical Hamiltonian formulation of the CH equation in action-angle variables is expressed for solitons by using the scattering data for its associated isospectral eigenvalue problem, rephrased as a Riemann–Hilbert problem. The dispersionless limit of the CH equation and its resulting peakon solutions are examined by using an asymptotic expansion in the dispersion parameter.

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1. Introduction

The Camassa–Holm (CH) equation introduced in [11] takes the form
\[ u_t - u_{xxt} + 2\omega u_x + 3u u_x - 2u_x u_{xx} - uu_{xxx} = 0, \] (1)
where \( \omega \) is a real constant. Its soliton solution behaviour for the initial value problem with \( \omega \neq 0 \) is shown in figure 1. CH is a model of unidirectional propagation of shallow water waves over a flat bottom [11, 24, 37, 38]. Remarkably, it also arises as a model of axially symmetric waves in a hyperelastic rod [21]. The analysis of CH has created a confluence of scientific endeavours that summons ideas from water waves, integrable systems, PDE analysis, asymptotics, geometry and Lie groups. This confluence of endeavours has sustained the interest in CH and provided opportunities for contributions to its analysis from many different fields in mathematics. The interest in CH may be measured by noting that the original CH paper [11] has acquired well over a thousand citations on Google Scholar.

This paper aims to review some of the geometric highlights of recent work on the CH equation. It is certainly not exhaustive. It mainly focuses on comparing the soliton theory for smooth CH solutions with the peakon theory for its singular solutions that arise in the
Section 2 briefly explains the application of the inverse scattering transform (IST) method for obtaining the CH soliton solutions. The set of scattering data is introduced and the formulation of inverse scattering as a Riemann–Hilbert problem (RHP) is outlined. The solution is expressed via the scattering data in a form that admits the peakon limit in the sections that follow.

In section 3 the map between the action-angle variables (expressed via the scattering data) and the momentum of the CH solution is formulated as a momentum map from the symplectic action-angle variables to the dual of the Lie algebra of smooth vector fields on the real line. This is a Poisson map, but the noncanonical bi-Hamiltonian structure of the CH equation which led to the discovery of its isospectral problem in [11] and its geometrical significance as geodesic flow on the diffeomorphism group are not discussed here.

In section 4 we introduce the peakons as singular solutions of CH that appear in its dispersionless limit. The N-peakon solution is governed by a finite-dimensional integrable dynamical system.

Section 5 presents the multi-peakon solution as a limiting case of the CH multi-soliton solution and points out the similarity between the dynamics of the peakon system and the well-known Toda lattice.

In section 6 we comment briefly on the existence of additional integrals of motion of the peakon system, a property known as superintegrability. The two-peakon system is analysed explicitly.

Section 7 comments on the higher dimensional generalization of the dispersionless CH equation, known as EPDiff [33, 34].

Section 8 mentions some of the limitations of our present discussion and points out three open problems for further research.

2. Soliton solutions of the CH equation from inverse scattering

2.1. Inverse scattering for the KdV equation

One of the most significant results in the theory of nonlinear partial differential equations was the discovery by Gardner, Greene, Kruskal and Miura (GGKM) [28, 29] of a method for obtaining the exact solution of the initial-value problem for the KdV equation. Prior to their work the only known exact solutions of KdV were the travelling wave solutions. The GGKM method is based on representing the KdV equation

\[ u_t + 6uu_x + u_{xxx} = 0, \tag{2} \]

as a compatibility condition for a system of two equations comprising an eigenvalue problem

\[ L \psi = \lambda \psi \] and an evolution equation for its eigenfunctions:

\[ \psi_{xx} + u \psi = \lambda \psi \tag{3} \]

\[ \psi_t = (u_x + \gamma) \psi - (4\lambda + 2u)\psi_x, \tag{4} \]

where \( \gamma \) is an arbitrary constant. Imposing compatibility \( \psi_{xx,t} = \psi_{xt,xx} \) for a time-independent spectral parameter \( \lambda \) implies KdV. The GGKM method—known as the IST—is conceptually analogous in many ways to the Fourier transform method for solving linear equations. It was recast by Lax [41] into a general framework that allows the IST to be used for solving other nonlinear PDEs. An important consequence is the proportionality between the discrete eigenvalues of the isospectral problem (3) and the asymptotic speeds of the solitons that emerge from the initial value problem for the KdV equation (2). The hallmark feature of
Figure 1. A Gaussian initial condition for the CH equation (1) breaks up into an ordered train of solitons as time evolves (the time direction being vertical). The soliton train eventually wraps around the periodic domain, thereby allowing the leading solitons to overtake the slower emergent solitons from behind in collisions that cause phase shifts, as discussed in [11].

A soliton collision is the preservation of their identities as coherent solitary wave structures (asymptotically in time) even after suffering fully nonlinear collision interactions. For example, the collision of two KdV solitons only results in a phase shift of their space-time trajectories from the positions where they would have been at a given time without the interaction.

The KdV equation was formulated as a completely integrable Hamiltonian system in a work by Faddeev and Zakharov [54]. The Hamiltonian form was also noted by Gardner [27]. Reviews of the IST may be found, for example, in [1, 23, 45]. For discussions of other related bi-Hamiltonian equations, see [22].

2.2. Inverse scattering for CH solitons with dispersion

In this section we outline the application of the IST for the CH equation (1), which admits a Lax pair formulation [11]

\[
\begin{align*}
\Psi_{xx} &= \frac{1}{4} (1 - \lambda (m + \omega)) \Psi, \\
\Psi_t &= - \left( \frac{2}{\lambda} + u \right) \Psi_x + \frac{\mu_x}{2} \Psi + \gamma \Psi,
\end{align*}
\]

where \(\gamma\) is an arbitrary constant. We will use this freedom for a proper normalization of the eigenfunctions. The compatibility condition of the Lax equations is the CH equation (including the constraint \(m = u - u_{xx}\)).

In our further considerations, the variable \(m\) will be a Schwartz class function, \(\omega > 0\) and we take \(m(x, 0) + \omega > 0\). Then \(m(x, t) + \omega > 0\) for all \(t\) [13]. Let us introduce a new spectral
parameter \( k \) such that
\[
\lambda(k) = \frac{1}{\omega}(1 + 4k^2). \tag{7}
\]

The spectrum of the problem (5) under these conditions is described in [12]. The continuous spectrum in terms of \( k \) corresponds to real \( k \). The discrete spectrum (in the upper half-plane) consists of finitely many points \( k_n = i\kappa_n, \ n = 1, \ldots, N \), where \( \kappa_n \) is real and \( 0 < \kappa_n < 1/2 \).

For all real \( k \neq 0 \), a basis in the space of solutions of (5) can be introduced, \( \psi(x, k) \) and \( \bar{\psi}(x, \bar{k}) \), fixed by its asymptotic behaviour when \( x \to \infty \) [12]:
\[
\psi(x, k) = e^{-ikx} + o(1), \quad x \to \infty. \tag{8}
\]

Another basis can be introduced, \( \varphi(x, k) \) and \( \bar{\varphi}(x, \bar{k}) \), fixed by its asymptotic when \( x \to -\infty \):
\[
\varphi(x, k) = e^{-ikx} + o(1), \quad x \to -\infty, \tag{9}
\]

and the relation between the two bases on the continuous spectrum (real \( k \)) is [15]
\[
\varphi(x, k) = a(k)\psi(x, k) + b(k)\bar{\psi}(x, k), \tag{10}
\]

where
\[
|a(k)|^2 - |b(k)|^2 = 1. \tag{11}
\]

The quantity \( R(k) := b(k)/a(k) \) is called the reflection coefficient.

All of the required information about the scattering, i.e. \( a(k) \) and \( b(k) \), is provided by the reflection coefficient \( R(k) \) for \( k > 0 \) [17]. It is sufficient to know \( R(k) \) only on the half line \( k > 0 \), since \( a(-k) = a(k) \), \( b(-k) = b(k) \) and thus \( R(-k) = R(k) \).

The constant \( \gamma \) in (6) can be chosen for each eigenfunction in such a way that \( a(k) \) does not depend on \( t \) and is a generating function of the integrals of motion [17]. Detailed analysis confirms that \( \varphi(x, k) \) admits an analytic continuation for \( k \) in the upper complex plane, and the same for \( \psi(x, k) \) for \( k \) in the lower complex plane. At the points of the discrete spectrum, the quantity \( a(k) \) has simple zeros [12]. This means that \( \varphi \) and \( \bar{\psi} \) are linearly dependent (10):
\[
\varphi(x, i\kappa_n) = b_n \bar{\psi}(x, -i\kappa_n). \tag{12}
\]

In other words, the discrete spectrum is simple, there is only one (real) eigenfunction \( \varphi^{(n)}(x) \), corresponding to each eigenvalue \( i\kappa_n \), and we can take this eigenfunction to be
\[
\varphi^{(n)}(x) \equiv \psi(x, i\kappa_n). \tag{13}
\]

The asymptotic behaviour of \( \varphi^{(n)} \), according to (9) and (12), is
\[
\varphi^{(n)}(x) = e^{\kappa_n x} + o(e^{\kappa_n x}), \quad x \to -\infty, \tag{14}
\]
\[
\varphi^{(n)}(x) = b_n e^{-\kappa_n x} + o(e^{-\kappa_n x}), \quad x \to \infty. \tag{15}
\]

The sign of \( b_n \) obviously depends on the number of the zeros of \( \varphi^{(n)} \). Suppose that
\[
0 < \kappa_1 < \kappa_2 < \cdots < \kappa_N < 1/2. \tag{16}
\]

Then from the oscillation theorem for the Sturm–Liouville problem [7], \( \varphi^{(n)} \) has exactly \( n - 1 \) zeros. Therefore,
\[
b_n = (-1)^{n-1}|b_n|. \tag{17}
\]

The set
\[
S \equiv \{ R(k) \ (k > 0), \kappa_n, R_0 = b_n/ia'(i\kappa_n), n = 1, \ldots, N \} \tag{18}
\]
is called the scattering data. The Hamiltonians for the CH equation in terms of the scattering data are presented in [17]. The time evolution of the scattering data is [15]

\[ R(k, t) = R(k, 0) e^{-\frac{4i}{\lambda_n} t}, \]

\[ R_n(t) = R_n(0) \exp \left( \frac{4\kappa_n}{\lambda_n} t \right), \]

where \( R_n(t) \) is always a positive quantity [15] and \( \lambda_n = \lambda(\kappa_n) \).

The scattering coefficient \( a(k) \) is analytic for \( \text{Im} k > 0 \) with asymptotic behaviour [17]

\[ e^{ik\beta} a(k) \to 1, \quad |k| \to \infty, \]

where \( \beta \) is a constant. The quantity \( \beta \) is an integral of motion:

\[ \beta = \int_{-\infty}^{\infty} \left( \sqrt{1 + m(x) - 1} \right) dx. \]

Moreover, (23) is analytic for \( \text{Im} k < 0 \), and (24) is analytic for \( \text{Im} k > 0 \) [15].

From (10) with (23) and (24) we obtain

\[ \psi(x, k) e^{i\beta a(k)} = X_0(x) + \sum_{n=1}^{N} R_n(t) e^{2\kappa_n y / \sqrt{\omega}} \psi(x, -i\kappa_n), \quad \text{Im} k < 0, \]

\[ \psi(x, k) e^{i\beta a(k)} = X_0(x) - \sum_{n=1}^{N} R_n(t) e^{2\kappa_n y / \sqrt{\omega}} \psi(x, -i\kappa_n), \quad \text{Im} k > 0. \]

From (27) one has a linear system for the quantities \( \psi(x, -i\kappa_n, t) \) whose solution is

\[ \psi(x, -i\kappa_n, t) = X_0(x) \sum_{p=1}^{N} A^{-1}_{np} \left[ y, t \right], \quad n = 1, \ldots, N, \]
where
\[ A_{pn}[y,t] \equiv \delta_{pn} + \frac{R_n(t) e^{-2\kappa_n y/\sqrt{\omega}}}{\kappa_p + \kappa_n}. \] (30)

Taking \( k = -i/2 \) in (27) produces
\[ e^{-\frac{i}{2}(x - \frac{y}{\sqrt{\omega}})} \equiv \psi(x, -i/2) = X_0(x) \left( 1 - \sum_{n,p=1}^{N} \frac{R_n(t) e^{-2\kappa_n y/\sqrt{\omega}}}{\kappa_n + \frac{1}{2}} A_{np}^{-1}[y,t] \right). \] (31)

The substitution \( k = i/2 \) in (28) with \( a(i/2) = 1 \) gives
\[ e^{\frac{i}{2}(x - \frac{y}{\sqrt{\omega}})} \equiv \phi(x, i/2) e^{-\beta/2 a(i/2)} = X_0(x) \left( 1 - \sum_{n,p=1}^{N} \frac{R_n(t) e^{-2\kappa_n y/\sqrt{\omega}}}{\kappa_n - \frac{1}{2}} A_{np}^{-1}[y,t] \right). \] (32)

From (31) and (32) there follows a parametric representation
\[ x = X(y,t) \equiv \frac{y}{\sqrt{\omega}} + \ln f_+ \mp \ln f_-, \] (33)
\[ f_\pm \equiv 1 - \sum_{n,p=1}^{N} \frac{R_n(t) e^{-2\kappa_n y/\sqrt{\omega}}}{\kappa_n \mp \frac{1}{2}} A_{np}^{-1}[y,t]. \] (34)

2.3. Parametric form of the dispersive CH soliton solution

From (25) and (33) one may compute the solution in the parametric form
\[ u(X(y,t),t) = X_t(y,t), \quad x = X(y,t), \] (35)
where \( X(y,t) \) is given in terms of the scattering data in (33) and (34).

Upon introducing the following new notation:
\[ \xi_j = 2\kappa_j \left( -\frac{y}{\sqrt{\omega}} + \frac{2t}{\lambda_j} + x_{j0} \right), \] (36)
\[ x_{j0} = \frac{1}{2\kappa_j} \ln \frac{R_j(0)}{2\kappa_j}, \] (37)
\[ \phi_j = \ln \frac{1 - 2\kappa_j}{1 + 2\kappa_j}, \] (38)
\[ \gamma_{ij} = \ln \left( \frac{\kappa_i - \kappa_j}{\kappa_i + \kappa_j} \right)^2, \] (39)
one may rewrite the expression for \( f_\pm \) (34) in the form [43, 46–48]
\[ f_\pm = \sum_{\mu=0,1} \exp \left[ \sum_{i=1}^{N} \mu_i (\xi_i \mp \phi_i) + \sum_{1 \leq i < j \leq N} \mu_i \mu_j \gamma_{ij} \right]. \] (40)

The solution for \( m \) can be obtained from (35). First we note that
\[ \frac{\partial u(X(y,t),t)}{\partial y} = u_x(X(y,t),t)X_y, \]
and also
\[ \frac{\partial u(X(y, t), t)}{\partial y} = \frac{\partial X_t}{\partial y} = X_{ty}. \]
Thus, \( u_x(X(y, t), t) = X_{ty}/X_y \). Similarly, \( u_{xx}(X(y, t), t) = 1/X_y (X_{ty}/X_y) \)
and
\[ m(X(t, y), t) = u(X(t, y), t) - u_{xx}(X(t, y), t) = X_t - \frac{1}{X_y} \left( \frac{X_{ty}}{X_y} \right)_y. \]
Finally,
\[ m(x, t) = \int_{-\infty}^{\infty} P(y, t) \delta(x - X(y, t)) \, dy, \tag{41} \]
with
\[ P(y, t) = X_t X_y - \left( \frac{X_{ty}}{X_y} \right)_y, \tag{42} \]
\[ u(x, t) = \frac{1}{2} \int_{-\infty}^{\infty} P(y, t) \exp(-|x - X(y, t)|) \, dy. \tag{43} \]
As shown in [33], this representation is also useful in the study of multidimensional solutions.

2.4. Relation to KdV hierarchy

The spectral problem (5) is gauge equivalent to a standard Sturm–Liouville problem, well known from the KdV hierarchy, cf (3), with a short notation \( q = m + \omega \):
\[ -\Phi_{yy} + U(y) \Phi = \mu \Phi, \quad \mu = \frac{\lambda}{4} - \frac{1}{4\omega}, \tag{44} \]
\[ \Phi(y) = q^{1/4} \Psi, \quad \frac{dy}{dx} = \sqrt{q}, \]
\[ U(y) = \frac{1}{4q(y)} + \frac{q_{yy}(y)}{4q(y)} - \frac{3q^2(y)}{16q^2(y)} - \frac{1}{4\omega}. \tag{45} \]
Note that (44) leads to two possible expressions for the change of the variables in the Liouville transformation
\[ y = \sqrt{\omega} x + \int_{-\infty}^{x} (\sqrt{q(x')} - \sqrt{\omega}) \, dx' + \text{const}, \tag{46} \]
\[ y = \sqrt{\omega} x + \int_{-\infty}^{x} (\sqrt{q(x')} - \sqrt{\omega}) \, dx' + \text{const}. \tag{47} \]
These two possibilities, (46) and (47) are only consistent if
\[ \int_{-\infty}^{\infty} (\sqrt{q(x)} - \sqrt{\omega}) \, dx = \text{constant}, \]
which is always the case, since the integral under question is (up to a multiplier) the Casimir function \( \beta \) (22).

The matching of the CH hierarchy to the KdV hierarchy requires solving the Ermakov–Pinney equation (45) [12], which is not straightforward and leads to the same solution (35) in the parametric form.
3. Momentum map formulation with action-angle variables

The canonical Poisson brackets for the scattering data of the CH equation are computed in [17] where the action-angle variables are also expressed in terms of the scattering data. Let us consider the action variable for the $N$-soliton solution. (These considerations can easily be extended to the variables of the continuous spectrum.) The angle variable is

$$/Phi_1n = \ln R_n(t);$$

it is linear in time $t$ and

$$\dot{/Phi}_1n = 4/\kappa_n/\lambda_n.$$ 

Let us introduce the functional

$$/Lambda_1n := 4/\kappa_n/\lambda_n$$

into Hamilton's principle

$$\delta S[u, /Phi_1n, /Pi_1n] = \int \left( \ell[u] + \sum_{n=1}^{N} /Pi_n(/Phi_n - /Lambda_n[u]) \right) dt.$$ 

Here the Lagrange multiplier $/Pi_n$ enforces the action-angle relation for the CH scattering data as a $T^N$ shift of the angles $/Phi_1n$ at constant angular frequencies $/Lambda_1n$, with $n = 1, \ldots, N$. The stationary variation of this constrained action principle yields

$$0 = \delta S = \int \left( \left( \frac{\delta \ell}{\delta u} - \sum_{n=1}^{N} /Pi_n \frac{\delta /Lambda_n}{\delta u} \right) \delta u + \sum_{n=1}^{N} (/Phi_n - /Lambda_n[u]) \delta /Pi_n - \sum_{n=1}^{N} /Pi_n \delta /Phi_n \right) dt.$$ 

Since by definition $m = \delta \ell/\delta u$ is the momentum, $\delta S = 0$ implies the Eulerian representation

$$m = \frac{\delta \ell}{\delta u} = \sum_{n=1}^{N} /Pi_n \frac{\delta /Lambda_n}{\delta u}, \quad \text{with}$$

$$/Phi_n = /Lambda_n[u] \quad \text{and} \quad /Pi_n = 0.$$ 

Relation (48) is the momentum map

$$\Phi_n, /Pi_n \in T^* T^N \rightarrow m \in X^*$$

for the cotangent lift of the toral $T^N$ action (49) on the angles $/Phi_n$ at constant angular frequencies $/Lambda_n$. This momentum map from the action-angle scattering variables $T^*(T^N)$ to the flow momentum $X^*(\mathbb{R})$ (dual to the smooth vector fields $X(\mathbb{R})$ on the real line) provides the Eulerian representation of the N-soliton solution of CH in terms of the scattering data and squared eigenfunctions of its isospectral eigenvalue problem. Momentum maps for Hamiltonian dynamics are reviewed in [42], for example.

By using the spectral quantities of the $N$-soliton solution (recall: $\lambda_n = (1 - 4/\kappa_n^2)/\omega$) one may express the variation of the spectrum with respect to the CH solution in terms of the squared eigenfunctions of the isospectral problem as [17]

$$\frac{\delta /Lambda_n}{\delta m(x, t)} = \frac{1 + 4/\kappa_n^2}{2\omega /\kappa_n \lambda_n} R_n(t) [\psi(x, -i/\kappa_n, t)]^2,$$

where $\psi(x, -i/\kappa_n, t)$ is the eigenfunction that belongs to the eigenvalue $\lambda_n$, see (12).

On the other hand, the expansion of $u(x,t)$ over squares of eigenfunctions is given by [16]

$$u(x, t) = \sum_{n=1}^{N} \frac{4/\kappa_n}{\omega /\lambda_n^2} R_n(t) [\psi(x, -i/\kappa_n, t)]^2.$$ 

Consequently,

$$m(x, t) = \sum_{n=1}^{N} \frac{4/\kappa_n}{\omega /\lambda_n^2} R_n(t) (1 - \partial^2) [\psi(x, -i/\kappa_n, t)]^2,$$
or

\[ m(x, t) = \sum_{n=1}^{N} \Pi_n J_n(x, t), \]

where \( \Pi_n \) and \( J_n(x, t) \) denote explicitly

\[ \Pi_n = \frac{8\kappa_n^2}{\lambda_n(1 + 4\kappa_n^2)} = \frac{2\Lambda_n \kappa_n}{1 + 4\kappa_n^2}, \quad (50) \]

\[ J_n(x, t) \equiv \frac{\delta \Lambda_n}{\delta u(x, t)} = \frac{(1 + 4\kappa_n^2)}{2\omega \kappa_n \lambda_n} R_n(t)(1 - \partial^2)[\bar{\psi}(x, -i\kappa_n, t)]^2. \quad (52) \]

Thus, the momentum map (48) from the action-angle variables undergoing dynamics (49) to the Eulerian representation of the momentum for the CH solution is expressed in terms of the scattering data and squared eigenfunctions of its \( N \)-soliton isospectral eigenvalue problem. Perhaps not unexpectedly, this momentum map may be applied to the action-angle representation of the solution of any integrable Hamiltonian PDE.

4. Peakons

4.1. Peakons: the singular solution ansatz

Camassa and Holm [11] discovered the ‘peakon’ solitary travelling wave solution of the CH equation (1) for a shallow water wave

\[ u(x, t) = c e^{-|x - ct|/\alpha}, \quad (53) \]

whose fluid velocity \( u \) is a function of position \( x \) on the real line and time \( t \). The peakon travelling wave moves at a speed equal to its maximum height, at which it has a sharp peak (jump in derivative). Peakons are an emergent phenomenon, obtained in solving the initial value problem for a partial differential equation derived by an asymptotic expansion of Euler’s equations using the small parameters of shallow water dynamics. Peakons are nonanalytic solitons, which superpose as

\[ u(x, t) = \frac{1}{2} \sum_{a=1}^{N} p_a(t) e^{-|x - q_a(t)|/\alpha} = \frac{1}{2} \sum_{a=1}^{N} p_a(t) g(x - q_a(t))/\alpha), \quad (54) \]

for sets \( \{p\} \) and \( \{q\} \) satisfying the canonical Hamiltonian dynamics. Peakons satisfy the CH equation (1) that arises for unidirectional shallow water waves in the limit of zero linear dispersion. Peakons also generalize to higher dimensions, as shown in [33]. We explain how peakons were derived in the context of shallow water asymptotics and describe some of their remarkable mathematical properties.

Peakons were found in [11] to arise in the absence of linear dispersion. That is, they arise when \( \omega = 0 \) in (1). Each term in sum (54) is a soliton with a sharp peak at its maximum, and hence named ‘peakon’. Expressed using its momentum, \( m = (1 - \alpha^2 \partial_x^2) u \), the peakon velocity solution (54) of dispersionless CH becomes a sum over a delta functions, supported on a set of points moving on the real line. Namely, the peakon velocity solution (54) implies

\[ m(x, t) = \alpha \sum_{a=1}^{N} p_a(t) \delta(x - q_a(t)), \quad (55) \]
because of the relation \((1 - \alpha^2 a^2) e^{-|x|/\alpha} = 2\alpha \delta(x)\). These solutions satisfy equation (1) when \(\omega = 0\). As discussed in [33], the peakon momentum relation (55) is again a momentum map.

### 4.2. Integrable peakon dynamics of CH

Substituting the peakon solution ansatz (54) and (55) into the dispersionless CH equation

\[
m_t + um_x + 2mu_x = 0, \quad \text{with} \quad m = u - \alpha^2 u_{xx},
\]

yields Hamilton’s canonical equations for the dynamics of the discrete set of peakon parameters \(p_a(t)\) and \(q_a(t)\):

\[
\dot{q}_a(t) = \frac{\partial h_N}{\partial p_a} \quad \text{and} \quad \dot{p}_a(t) = -\frac{\partial h_N}{\partial q_a},
\]

for \(a = 1, 2, \ldots, N\), with Hamiltonian given by [11],

\[
h_N = \frac{1}{4} \sum_{a, b=1}^{N} p_a p_b e^{-|q_a - q_b|/\alpha}.
\]

Or explicitly,

\[
\dot{q}_a = \frac{1}{2} \sum_{b=1}^{N} p_b e^{-|q_a - q_b|/\alpha}
\]

\[
\dot{p}_a = \frac{p_a}{2\alpha} \sum_{b=1}^{N} p_b e^{-|q_a - q_b|/\alpha} \text{sgn}(q_a - q_b).
\]

Thus, one finds that the points \(x = q_a(t)\) in the peakon solution (54) move with the flow of the fluid velocity \(u\) at those points, since \(u(q_a(t), t) = \dot{q}_a(t)\). In terms of fluid dynamics, this means the \(q_a(t)\) are the Lagrangian coordinates. Moreover, the singular momentum solution (55) is the Lagrange-to-Euler map for an invariant manifold of the dispersionless CH equation (56). On this finite-dimensional invariant manifold for the partial differential equation (56), the dynamics is canonically Hamiltonian.

With Hamiltonian (58), the canonical equations (57) for the \(2N\) canonically conjugate peakon parameters \(p_a(t)\) and \(q_a(t)\) were interpreted in [11] as describing geodesic motion on the \(N\)-dimensional Riemannian manifold whose co-metric is \(g^{ab}(\{q\}) = e^{-|q_a - q_b|/\alpha}\). Moreover, the canonical geodesic equations arising from Hamiltonian (58) comprise an integrable system for any number of peakons \(N\). This integrable system was studied in [11] for solutions on the real line, and in [3, 18] and references therein, for spatially periodic solutions.

The integrals generated by the action variables in terms of the coordinates can be recovered as \(\text{tr}(L^2)\), where \(L\) is the Lax operator for the peakon system [11].

Being a completely integrable Hamiltonian soliton equation, the CH equation (1) has an associated isospectral eigenvalue problem, discovered in [11] for any value of its dispersion parameter \(\omega\). When \(\omega = 0\), this isospectral eigenvalue problem has a purely discrete spectrum. Moreover, in this case, each discrete eigenvalue corresponds precisely to the time-asymptotic velocity of a peakon. This discreteness of the CH isospectrum in the absence of linear dispersion implies that only the singular peakon solutions (55) emerge asymptotically in time, in the solution of the initial value problem for the dispersionless CH equation (56). This is borne out in numerical simulations of the dispersionless CH equation (56), starting from a smooth initial distribution of velocity [26, 36].
Figure 2 shows the emergence of peakons from an initially Gaussian velocity distribution and their subsequent elastic collisions in a periodic one-dimensional domain under dispersionless CH dynamics. This figure demonstrates that singular solutions dominate the initial value problem for dispersionless CH dynamics and, thus, that it is imperative to go beyond smooth solutions for the CH equation; the situation is similar for the EPDiff equation.

**Peakons as mechanical systems.** Governed by canonical Hamiltonian equations, each \( N \)-peakon solution can be associated with a mechanical system of moving particles. Calogero et al [10] further extended the class of mechanical systems of this type. The \( r \)-matrix approach was applied to the Lax pair formulation of the \( N \)-peakon system for CH by Ragnisco and Bruschi [50], who also pointed out the connection of this system with the classical Toda lattice. A discrete version of the Adler–Kostant–Symes factorization method was used by Suris [51] to study a discretization of the peakon lattice, realized as a discrete integrable system on a certain Poisson submanifold of \( gl(N) \) equipped with an \( r \)-matrix Poisson bracket. Beals et al [6] used the Stieltjes theorem on continued fractions and the classical moment problem for studying multi-peakon solutions of the CH equation. Generalized peakon systems are described for any simple Lie algebra by Alber et al [3].

**5. Peakon limit of the CH soliton solutions**

The limit \( \omega \to 0 \) in the \( N \)-soliton solution \( u(x, t) \) produces the \( N \)-peakon solution (54). The limiting procedure is described in detail in [44]. Due to (7) one can write for the discrete eigenvalues \( \kappa_n = i\kappa_n \)

\[
2\kappa_j = (1 - \omega \lambda_j)^{1/2} = 1 - \frac{1}{2}\omega \lambda_j + \cdots.
\]  

(61)

Solution (33) depends explicitly on \( \kappa_j \) (61) and the limit can be computed with (61) by taking \( \omega \to 0 \) and keeping the eigenvalue \( \lambda_j \) constant. The result is expression (54) with
\[ p_i = \frac{4D^{(0)}_{N-i+1}D^{(2)}_{N-i}}{D^{(1)}_{N-i}D^{(1)}_{N-i}} \quad (i = 1, 2, \ldots, N) \]  
\[ q_i = \alpha \ln \left[ \frac{2D^{(0)}_{N-i+1}D^{(2)}_{N-i}}{D^{(1)}_{N-i}} \right] \quad (i = 1, 2, \ldots, N) \]

where

\[ D^{(m)}_n = \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq N} \Delta_n(i_1, i_2, \ldots, i_n)(\lambda_{i_1}\lambda_{i_2} \cdots \lambda_{i_n})^m R_{i_1}R_{i_2} \cdots R_{i_n}, \quad n = 1, 2, \ldots, N, \]

\[ \Delta_n(i_1, i_2, \ldots, i_n) = \prod_{1 \leq l < m \leq n} (\lambda_{i_l} - \lambda_{i_m})^2 \quad (n \geq 2) \]

\[ R_i(t) = R_i(0) e^{\xi_i t}, \quad (\xi_i = \ln R_i(0)). \]

The quantities \( D^{(m)}_n \) are called Hankel determinants. By definition, \( D^{(m)}_0 = 1 \). In general, the Hankel determinant is a determinant of an \( n \times n \) matrix of the form \( D^{(m)}_n \equiv \det (a^{(m)}_{ij}) \), where \( a_{ij} \) are the elements of a sequence, i.e.

\[ a^{(m)}_{ij} = A_{i+j+m-2}. \]

In this particular case

\[ A_l = \sum_{i=1}^{N} \lambda_i^l R_i(t). \]  

**Similarity of peakon lattice and Toda lattice.** Hankel determinants appear in the solutions of other integrable systems, e.g., in the Toda lattice, see [31]. The Toda equation [52]

\[ \frac{dp_n}{dt} = e^{q_{n+1} - q_n} - e^{q_n - q_{n-1}}, \]

\[ \frac{dq_n}{dt} = p_n, \quad n \in \mathbb{Z}, \]

is one of the most-studied integrable systems. A finite Toda chain with \( N \) nodes is obtained by imposing fixed boundary conditions at the ends, by setting \( q_0 = -q_{N+1} = \infty \). The Toda solution is in the form

\[ q_n(t) = q_1(0) + \ln \frac{D^{(0)}_n}{D^{(0)}_{n-1}}; \]  
\[ q_1(0) = -\frac{1}{N} \ln D^{(0)}_N(0) \quad \text{is a constant.} \]

The Hankel determinants are obtained by a similar sequence

\[ A_l = \sum_{i=1}^{N} \lambda_i^l R_i(t), \]  
\[ R_i(t) = R_i(0) e^{-\lambda_i t}, \]

where \( \lambda_i \) are \( N \) different constants (eigenvalues of the Lax matrix) and the quantities \( R_i(0), i = 1, \ldots, N \) represent another set \( N \) of constants.
Because of its simple form, the $N$-peakon solution can be used as an approximation of the $N$-soliton CH solution when the dispersion term is small (so the term $2\omega u_x u$ can be neglected). Similarly, the Toda chain with complex dynamical variables (the so-called complex Toda chain (CTC)) provides an approximation for the $N$-soliton solution of the nonlinear Schrödinger equation (NLS)

$$iu_t + \frac{i}{2} u_{xx} + |u|^2 u = 0,$$

see [30–32] for more details. Such an approximation is called an adiabatic approximation and means that the $N$-soliton solution consists of $N$ well-separated solitons

$$u(x, t) \approx \sum_{k=1}^{N} 2\nu_k e^{i[2\mu_k (x - \xi_k(t)) + \delta_k(t)]} \cosh(2\nu_k (x - \xi_k(t)),$$

i.e. the overlap of the solitons is small. The variables $q_n(t)$ of the CTC are related to the NLS solitons parameters by

$$q_k(t) = -2v_0 \xi_k(t) + i (2\mu_0 \xi_k(t) - \delta_k(t)) + \text{const},$$

where $\xi_k, \delta_k, \mu_k = \frac{1}{2} \xi_k$ and $v_k = \left(\frac{1}{2} \xi_k^2 - \mu_k^2\right)^{1/2}$ characterize the centre-of-mass position, the phase, velocity and amplitude, respectively, of the $k$th soliton in the chain. The values $v_0$ and $\mu_0$ are, respectively, the average amplitude and velocity of the soliton train. The quantities $\xi_k$ and $\delta_k$ can be obtained as the real and imaginary parts of $q_k(t)$. Such soliton trains and their asymptotic behaviour have important applications for soliton-based fibre optics communications.

6. Superintegrability of the peakon system

An integrable Hamiltonian system in $2N$-dimensional phase space can be represented in terms of its action-angle (canonical) variables as

$$\dot{\Lambda}_n = 0, \quad \dot{\Phi}_n = \Lambda_n, \quad n = 1, 2, \ldots, N. \quad (71)$$

If a Poisson bracket exists such that

$$\{\Phi_n, \Lambda_l\} = \frac{1}{2} \delta_{nl}, \quad \{\Phi_n, \Phi_l\} = \{\Lambda_n, \Lambda_l\} = 0, \quad (72)$$

then the system is Hamiltonian, with a Hamiltonian function

$$h_N = \Lambda_1^2 + \cdots + \Lambda_N^2. \quad (73)$$

The integrals $\Lambda_n, n = 1, 2, \ldots, N$, are functionally independent and in involution under the bracket. This guarantees the integrability of the system. There is however another set of integrals:

$$I_j = (\Phi_j - \Phi_{j+1})(\Lambda_1 + \cdots + \Lambda_N) - (\Lambda_j - \Lambda_{j+1})(\Phi_1 + \cdots + \Phi_N) \quad j = 1, 2, \ldots, N - 1. \quad (74)$$

If the ‘action’ variables $\Lambda_n$ are all different, set (74) is functionally independent from the set $\Lambda_n, n = 1, 2, \ldots, N$. In addition, integrals (74) form another set of $N$-integrals in involution together with $H$. Systems that possess two sets of functionally independent integrals in involution are said to be superintegrable.

An example of such a system is the Toda lattice, see the discussion in [2]. The peakon system is also superintegrable. The canonical variables in terms of the scattering data for CH

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3 The coefficient $1/2$ in the definition of the bracket appears in order to match it to the Poisson bracket used in the CH Hamiltonian formulation.
equation can be used in the peakon limit: \( \Lambda_n = 2/\kappa_n \), \( \Phi_n = \ln R_n(t) \). Hamiltonian (73) is also a peakon limit (\( \omega \to 0 \)) of the \( N \)-soliton Hamiltonian [17]

\[
H_N(\omega) = \omega^2 \sum_{n=1}^{N} \left( \ln \frac{1 - 2\kappa_n}{1 + 2\kappa_n} + \frac{4\kappa_n (1 + 4\kappa_n^2)}{(1 - 4\kappa_n^2)^2} \right).
\]

The Poisson bracket for the CH peakon solution is

\[
\{A, B\} \equiv -\int_{-\infty}^{\infty} \frac{\delta A}{\delta m(m \partial_1 + \partial_m)} \frac{\delta B}{\delta m} \, dx,
\]

and the scattering data satisfy (72) with respect to (75), see [17] for the details.

More interesting are the integrals (74). From (63) and (62) one can recover these integrals in the coordinate form. For example, when \( N = 2 \) we have

\[
I_1 = \ln \frac{\sqrt{J + p_1 - p_2}}{\sqrt{J - p_1 + p_2}} + \frac{\sqrt{J}}{p_1 + p_2} \left( \frac{q_1 + q_2}{\alpha} + \ln \frac{p_1}{p_2} \right),
\]

where \( J = (p_1 - p_2)^2 + 4 p_1 p_2 e^{-|q_1 - q_2|/\alpha} \).

Note that \( I_1 \) depends on both combinations \( q_1 + q_2 \) and \( q_1 - q_2 \) as well as the momentum variables. The Hamiltonian \( h_2 \) (which depends only on \( q_1 - q_2 \) and the momentum variables) and \( I_1 \) form a complete system of integrals in involution. The integration of the 2-peakon system with these integrals may be performed as follows. First, one may express \( q_1 \) and \( q_2 \) in terms of \( I_1, h_2 \) and the momentum variables: \( q_i = q_i(p_1, p_2, I_1, h_2) \). Next, one computes

\[
\dot{q}_i = \frac{\partial q_i}{\partial p_1} \dot{p}_1 + \frac{\partial q_i}{\partial p_2} \dot{p}_2.
\]

The substitution of \( \dot{q}_i, \dot{p}_i \) from (59), (60) to (78) produces an algebraic equation that gives, say, \( p_2 \) as a function of \( p_1 \). Then (60) is an ODE for \( p_1 \) of the form \( \dot{p}_1 = f(p_1, I_1, h_2) \). This calculation demonstrates the superintegrability of the 2-peakon system. From a practical viewpoint, it turns out to be much more convenient to work with the other system of integrals in involution: \( h_2 \) and the conserved momentum \( P = p_1 + p_2 \). In fact, \( J = 4h_2 - 3P^2 \) is itself an integral.

7. CH generalizations in more dimensions

In [36, 40], weakly nonlinear analysis and the assumption of columnar motion in the variational principle for Euler’s equations are found to produce a two-dimensional generalization of the dispersionless CH equation (56). This generalization is the Euler–Poincaré (EP) equation [34] for the Lagrangian consisting of the kinetic energy

\[
\ell = \frac{1}{2} \int \left[ ||u||^2 + \alpha^2(\text{div } u)^2 \right] \, dx \, dy,
\]

where the fluid velocity \( u \) is a two-dimensional vector. Evolution generated by the kinetic energy in Hamilton’s principle results in geodesic motion, with respect to the velocity norm \( ||u|| \), which is provided by the kinetic energy Lagrangian. For ideal incompressible fluids governed by Euler’s equations, the importance of geodesic flow was recognized by Arnold [4] for the \( L^2 \) norm of the fluid velocity. The EP equation generated by any choice of kinetic energy norm without imposing incompressibility is called ‘EPDiff,’ for ‘Euler–Poincaré equation for
geodesic motion on the diffeomorphisms.’ EPDiff is given by [34]
\[
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) m + \nabla u^T \cdot m + m(\text{div } u) = 0,
\]
(80)
with the momentum density \( m = \delta \ell / \delta u \), where \( \ell = \frac{1}{2} \| u \|^2 \) is given by the kinetic energy, which defines a norm in the fluid velocity \( \| u \| \), yet to be specified. By design, this equation has no contribution from either the potential energy, or the pressure. It conserves the velocity norm \( \| u \| \) given by the kinetic energy. Its evolution describes geodesic motion on the diffeomorphisms with respect to this norm [34]. An alternative way of writing the EPDiff equation (80) in either two, or three dimensions is
\[
\frac{\partial}{\partial t} m - u \times \text{curl } m + \nabla (u \cdot m) + m(\text{div } u) = 0.
\]
(81)
This form of EPDiff involves all three differential operators, curl, gradient and divergence. For the kinetic energy Lagrangian \( \ell \) given in (79), which is a norm for irrotational flow (with \( \text{curl } u = 0 \)), we have the EPDiff equation (80) with the momentum \( m = \delta \ell / \delta u = u - \alpha^2 \nabla (\text{div } u) \).

EPDiff (80) may also be written intrinsically as
\[
\frac{\partial}{\partial t} \delta \ell / \delta u = -\text{ad}^* \frac{\delta \ell}{\delta u},
\]
(82)
where \( \text{ad}^* \) is the \( L^2 \) dual of the ad-operation (commutator) for vector fields. See [5, 33, 42] for additional discussions of the beautiful geometry underlying this equation.

Building on the peakon solutions (55) for the CH equation and the pulsons for its generalization to other travelling wave shapes in [26], Holm and Staley [36] introduced the following measure-valued singular momentum solution ansatz for the \( n \)-dimensional solutions of the EPDiff equation (80):
\[
m(x, t) = \sum_{a=1}^{N} \int P^a(s, t) \delta(x - Q^a(s, t)) \, ds.
\]
(83)
As shown in [33] formula (83), regarded as a map from the space of \( \{Q(s)\}, \{P(s)\} \) to the space of \( m \)'s, defines an equivariant momentum map \( J_{\text{Sing}} : T^* \text{Emb}(S^k, \mathbb{R}^n) \to \mathcal{X}(\mathbb{R}^n)^* \) under the left action of the diffeomorphisms on \( k \)-dimensional subspaces smoothly embedded in \( \mathbb{R}^n \). These singular momentum solutions, called ‘diffeons’, are vector density functions supported in \( \mathbb{R}^n \) on a set of \( N \) surfaces (or curves) of codimension \( (n-k) \) for \( s \in \mathbb{R}^k \) with \( k < n \). They may, for example, be supported on sets of points (vector peakons, \( k = 0 \)), one-dimensional filaments (strings, \( k = 1 \)), or two-dimensional surfaces (sheets, \( k = 2 \)) in three dimensions.

The EPDiff equation (80) has many further interpretations beyond fluid applications. For instance, in [35], it was shown that shapes in computational anatomy, represented by geometrical structures such as landmarks and image outlines, can also be described by singular solutions of the EPDiff equation. As examples of recent discussions of uses of EPDiff in computational anatomy, see [20, 53].

8. Three open problems

(1) Throughout this discussion the solutions \( u(x, t) \) were assumed to be the functions in the Schwartz class, \( \omega > 0 \). The situation when the condition \( m(x, 0) + \omega > 0 \) on the initial data does not hold is more complicated and requires separate analysis [12, 14, 39]. In general, it leads to wave-breaking [14]. An attempt at developing the inverse scattering theory for this
case has been made by Kaup [39], who suggested applying the inverse scattering approach separately in each interval where \( m(x, t) + \omega \) is of the same sign. The problem posed by this approach is: How to join solutions that are valid in different intervals?

(2) The peakon solution (62) was obtained from the soliton solution under the assumption \( m(x, 0) + \omega > 0 \). Thus, all \( p_k \) are of the same sign, since all the eigenvalues \( \lambda_n \) in this case are positive. However, one can formally use the same solution with the eigenvalues of various signs to model \( p_k \) of various signs (mixture of peakons and ‘anti-peakons’) and thus to study peakon–anti-peakon interactions, see e.g. [49]. The result is that the multi-peakon interaction (including anti-peakons) in general decomposes into a sequence of pairwise collisions [26]. The multi-peakon interaction was studied rigorously in the recent papers [8, 9] and was shown to be the building block for the construction of unique global weak solutions after wave-breaking, constructed in a conservative, respectively dissipative framework. The collision of a antisymmetric peakon, anti-peakon pair was studied analytically already in [11]. For an early numerical study, see e.g. [26]. When the eigenvalues are of mixed signs, the Hankel determinants in the denominator of (62) may develop singularities for finite values of \( t \). This ‘peakon-breaking’ phenomenon is apparently the analogue of the wave-breaking mentioned earlier when \( \omega \to 0 \). This outstanding topic needs to be investigated further.

(3) Stability for EPDiff singular momentum solutions, that is, proving their stability analytically, remains an outstanding problem. (The stability of the peakon in one dimension was proved in [19] and that of separated multi-peakons in [25].)

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