ON THE SMALLEST LAPLACE EIGENVALUE FOR NATURALLY REDUCTIVE METRICS ON COMPACT SIMPLE LIE GROUPS

EMILIO A. LAURET

Abstract. Eldredge, Gordina and Saloff-Coste recently conjectured that, for a given compact connected Lie group $G$, there is a positive real number $C$ such that $\lambda_1(G, g) \text{diam}(G, g)^2 \leq C$ for all left-invariant metric $g$ on $G$. In this short note, we establish the conjecture for the small subclass of naturally reductive left-invariant metrics on a compact simple Lie group.

For an arbitrary compact homogeneous Riemannian manifold $(M, g)$, Peter Li [Li80] proved that
\begin{equation}
\lambda_1(M, g) \geq \frac{\pi^2/4}{\text{diam}(M, g)^2}.
\end{equation}
Here, $\lambda_1(M, g)$ denotes the smallest positive eigenvalue of the Laplace–Beltrami operator on $(M, g)$ and $\text{diam}(M, g)$ is the diameter of $(M, g)$, that is, the maximum Riemannian distance between two points in $M$. This lower bound has been recently improved by Judge and Lyons [JL17, Thm. 1.3].

In contrast, there is no uniform upper bound for the term $\lambda_1(M, g) \text{diam}(M, g)^2$ among all compact homogeneous Riemannian manifolds. For instance, the product $(M_n, g_n)$ of $n$-dimensional round spheres of constant curvature one satisfies $\lambda_1(M_n, g_n) = d$ but its diameter clearly goes to infinity when $n \to \infty$.

Eldredge, Gordina and Saloff-Coste have recently claimed the existence of a uniform upper bound valid on a restricted space of homogeneous Riemannian manifolds.

Conjecture 1. [EGS18, (1.2)] Given $G$ a compact connected Lie group, there exists $C > 0$ such that
\begin{equation}
\lambda_1(M, g) \leq \frac{C}{\text{diam}(M, g)^2}
\end{equation}
for all left-invariant metric $g$ on $G$.

Among many other results, they confirm its validity for $SU(2)$ in [EGS18, Thm. 8.5]. Explicit values of $C$ for $SU(2)$ and $SO(3)$ can be found in [La19, Thm. 1.4].

The main goal of this article is to give a simple and short proof of the validity of the weaker conjecture after restricting to naturally reductive left-invariant metrics on a compact connected simple Lie group $G$. The reader should not consider this result as a strong evidence of Conjecture 1.

Theorem 2. Let $G$ be a compact connected simple Lie group. There exists $C > 0$ such that
\begin{equation}
\lambda_1(G, g) \leq \frac{C}{\text{diam}(G, g)^2}
\end{equation}
for all naturally reductive left-invariant metric $g$ on $G$. 

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Proof. It is well known that the space of left-invariant metrics on $G$ is in correspondence with the space of inner products on its Lie algebra $\mathfrak{g}$. Let $K$ be a closed subgroup of $G$ and let $\mathfrak{p}$ denote the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$ with respect to the Killing form $B_{\mathfrak{g}}$ of $\mathfrak{g}$. For $h$ a bi-invariant metric on $K$ and $\alpha$ a positive real number, we define the left-invariant metric $g_{h,\alpha}$ on $G$ induced by the inner product on $\mathfrak{g}$ given by

\[(4) \quad g_{a,h}(X_{i} + Y_{i}, X_{j} + Y_{j}) = h(X_{i}, Y_{j}) + \alpha (B_{\mathfrak{g}})(X_{i}, X_{j}) \quad \text{for} \quad X_{i} \in \mathfrak{t}, \ Y_{i} \in \mathfrak{p} \ (i = 1, 2).\]

We recall that $B_{\mathfrak{g}}$ is a negative definite bilinear form on $\mathfrak{g}$. D’Atri and Ziller proved that any naturally reductive metric on $G$ is isometric to one of the above form for some $K$, $h$ and $\alpha$ (see [DZ79] Thm. 3]).

Since the term $\lambda_{1}(M, g) \text{diam}(M, g)^{2}$ is invariant under positive scaling of $g$, we can assume without loosing generality that $\alpha = 1$. We abbreviate $g_{h} = g_{h,1}$.

We consider the sub-Riemannian homogeneous manifold $(G, \mathfrak{p}, -B_{\mathfrak{g}}|_{\mathfrak{p}})$. Any curve on it is restricted to move only in the directions of $\mathfrak{p}$. It follows that

\[(5) \quad \text{diam}(G, g_{h}) \leq \text{diam}(G, \mathfrak{p}, -B_{\mathfrak{g}}|_{\mathfrak{p}}).\]

If $\mathfrak{t} = \mathfrak{g}$, then $g_{h} = -B_{\mathfrak{g}}$ in the only possibility. If $\mathfrak{t} \neq \mathfrak{g}$, then $\mathfrak{p}$ is bracket generating in $\mathfrak{g}$ since $G$ is simple. Thus, Chow’s Theorem ensures that $\text{diam}(G, \mathfrak{p}, -B_{\mathfrak{g}}|_{\mathfrak{p}}) < \infty$.

We now consider the Riemannian submersion with totally geodesic fibers (see [BBS2] §2.2)) given by

\[(6) \quad (K, h) \longrightarrow (G, g_{h}) \xrightarrow{\pi} (G/K, -B_{\mathfrak{g}}|_{\mathfrak{p}}).\]

If $f$ is an eigenfunction of the Laplace–Beltrami operator $\Delta_{h}$ of the base space $(G/K, -B_{\mathfrak{g}}|_{\mathfrak{p}})$ with associated eigenvalue $\lambda$, then $f \circ \pi$ is an eigenfunction of the Laplace–Beltrami operator $\Delta_{g_{h}}$ of $(G, g_{h})$ with associated eigenvalue $\lambda$. Consequently,

\[(7) \quad \lambda_{1}(G, g_{h}) \leq \lambda_{1}(G/K, -B_{\mathfrak{g}}|_{\mathfrak{p}}).\]

There is a finite collection $\mathcal{K}$ of closed subgroups of $G$ such that, for any naturally reductive left-invariant metric $g$ on $G$, there are $K \in \mathcal{K}$, $\alpha > 0$, and $h$ a bi-invariant metric on $\mathfrak{t}$ such that $(G, g)$ is isometric to $(G, h|_{\mathfrak{t}} \oplus \alpha (-B_{\mathfrak{g}})|_{\mathfrak{p}})$ (see [GS10] Cor. 3.7)). The proof follows since the upper bounds in (5) and (7) depends only on $K$. More precisely, taking

\[(8) \quad C = \max_{K \in \mathcal{K}} \lambda_{1}(G/K, -B_{\mathfrak{g}}|_{\mathfrak{p}}) \text{diam}(G, \mathfrak{p}, -B_{\mathfrak{g}}|_{\mathfrak{p}})^{2},\]

we conclude that (3) holds for all naturally reductive left-invariant metric on $G$. □

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INMABB, CONICET AND UNIVERSIDAD NACIONAL DEL SUR, BAHÍA BLANCA, ARGENTINA.

E-mail address: emiliolauret@gmail.com