SPECTRAL ASYMPTOTICS FOR A SINGULARLY PERTURBED FOURTH ORDER LOCALLY PERIODIC SELF-ADJOINT ELLIPTIC OPERATOR

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Abstract. We consider the homogenization of a singularly perturbed self-adjoint fourth order elliptic equation with locally periodic coefficients, stated in a bounded domain. We impose Dirichlet boundary conditions on the boundary of the domain. The presence of large parameters in the lower order terms and the dependence of the coefficients on the slow variable give rise to the effect of localization of the eigenfunctions. We show that the $j$th eigenfunction can be approximated by a rescaled function that is constructed in terms of the $j$th eigenfunction of fourth or second order effective operators with constant coefficients, depending on the large parameters.

1. Introduction and problem statement

We study the spectral asymptotics of a self-adjoint fourth order elliptic equation with locally periodic coefficients. The problem is stated in a bounded domain, and we impose Dirichlet boundary conditions on the boundary of the domain. The problem is a combination of homogenization and singular perturbation: because of the rapidly varying coefficients, homogenization arguments should be applied after a proper rescaling of the equation. As a result, we obtain an effective problem stated in the whole space, which will be of fourth or second order, depending on the choice of the large parameters ($\alpha, \beta$ in (1)). We focus only on those cases when the localization of eigenfunctions is observed.

Similar problems for second order locally periodic elliptic operators, that are closely related to the present paper, were studied in [23, 20]. Dependence of the problem on a slow variable (in the coefficients or in the geometry) together with the presence of a large parameter in the equation give rise to the effect of localization of eigenfunctions. These results correspond to the so-called subcritical case, when eigenfunctions can be approximated by scaled exponentially decaying functions (eigenfunctions of a harmonic oscillator operator).

A second order locally periodic elliptic operator with large potential was studied in [4]. Homogenization of periodic elliptic systems with large potential was treated in [5]. In both cases, under a generic assumption on the ground state of
an auxiliary cell problem, it was proved that the solution can be approximately factorized as the product of a fast oscillating cell eigenfunction and of a slowly varying solution of a scalar second-order equation. These two cases correspond to the so-called critical case.

There is a vast literature devoted to the homogenization of elliptic systems and higher order elliptic equations in domains with microstructure. For the homogenization of linear elliptic systems in the we refer to [1, 24, 18, 7]. Homogenization of boundary value problems for higher order equations in domains with fine-grained boundary were studied in [11, 12, 16, 19]. Homogenization of linear higher order equations in perforated domains were studied in [8, 22, 13]; nonlinear higher order equations in perforated domains were considered in [14, 10]. In [17] a spectral asymtotics for a fourth order elliptic operator with rapidly oscillating coefficients was obtained. A spectral asymptotics for a biharmonic operator in a domain with a deeply indented boundary was constructed in [15].

We turn to the formulation of the problem. Let Ω be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary. We consider the following Dirichlet spectral problem for a fourth order self-adjoint uniformly elliptic operator:

\[
\begin{cases}
\partial_{ij}(a_{ijkl}^\varepsilon \partial_k u^\varepsilon) - \frac{1}{\varepsilon^\alpha} \partial_i(b_{ij}^\varepsilon \partial_j u^\varepsilon) + \frac{1}{\varepsilon^\beta} c^\varepsilon u^\varepsilon = \lambda^\varepsilon u^\varepsilon, & x \in \Omega, \\
 u^\varepsilon = \nabla u^\varepsilon \cdot n = 0, & x \in \partial \Omega,
\end{cases}
\]  

(1)

where $n$ denotes the exterior unit normal to $\partial \Omega$. We use summation convention over repeated indices and use “$\cdot$” for the standard scalar product in $\mathbb{R}^d$; $\varepsilon > 0$ is a small parameter; $\alpha, \beta$ are positive parameters.

Our main assumptions are:

(H1) The coefficients are real and of the form $a_{ijkl}^\varepsilon(x) = a_{ijkl}(x, \frac{x}{\varepsilon})$, $b_{ij}^\varepsilon(x) = b_{ij}(x, \frac{x}{\varepsilon})$, where the functions $a_{ijkl}(x, y), b_{ij}(x, y) \in C(\overline{\Omega}; L^\infty(\mathbb{T}^d))$, $c(x, y) \in C^2(\overline{\Omega}; L^\infty(\mathbb{T}^d))$ are periodic in $y$; $\mathbb{T}^d$ is the unit torus. We denote $|a_{ijkl}|, |b_{ij}| \leq \Lambda^{-1}$, $\Lambda > 0$.

(H2) Symmetry condition: $a_{ijkl} = a_{klij}, b_{ij} = b_{ji}$.

(H3) The coefficients $a_{ijkl}(x, y)$ satisfy the uniform ellipticity condition in $\Omega \times \mathbb{T}^d$: there is $\Lambda > 0$ such that, almost everywhere,

$$
a_{ijkl}(x, y)\xi_i\xi_j\xi_k \geq \Lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^{d \times d}.
$$

(H4) The function $c(x, y)$ is assumed to be strictly positive almost everywhere in $\Omega \times \mathbb{T}^d$, and its local average

$$
\overline{c}(x) = \int_{\mathbb{T}^d} c(x, y) \, dy
$$

has a unique global minimum at $x = 0 \in \Omega$, with a non-degenerate Hessian $H = \nabla\nabla \overline{c}(0)$:

$$
\overline{c}(x) = \overline{c}(0) + \frac{1}{2} H x \cdot x + o(|x|^2).
$$

(H5) The coefficients $b_{ij}(x, y)$ satisfy the uniform ellipticity condition in $\Omega \times \mathbb{T}^d$: there is $\Lambda' > 0$ such that, almost everywhere,

$$
b_{ij}(x, y)\xi_i\xi_j \geq \Lambda'|\xi|^2, \quad \forall \xi \in \mathbb{R}^d.
$$
For domains \( \mathcal{O} \) in \( \mathbb{R}^d \) we denote the \( L^2(\mathcal{O}) \)-norm of \( u \) by \( ||u||_{2,\mathcal{O}} = \int_{\mathcal{O}} |u|^2 \, dx \), and for bounded Lipschitz domains \( \Omega \),

\[
H^2_0(\Omega) = \{ v \in H^2(\Omega) : v = \nabla v \cdot n = 0 \text{ on } \partial \Omega \}.
\]

We consider the following bilinear form corresponding to (1)

\[
A_\varepsilon(u,v) = \int_{\Omega} a_{ijkl}^\varepsilon \partial_{ij} u \partial_{kl} v \, dx + \frac{1}{\varepsilon^\alpha} \int_{\Omega} b_{ij}^\varepsilon \partial_i u \partial_j v \, dx + \frac{1}{\varepsilon^\beta} \int_{\Omega} c^\varepsilon uv \, dx.
\]

(2)

The weak form of (1) reads: Find \( \lambda_\varepsilon \in \mathbb{C} \) and nonzero \( u_\varepsilon \in H^2_0(\Omega) \) such that

\[
A_\varepsilon(u_\varepsilon,v) = \lambda_\varepsilon \int_{\Omega} u_\varepsilon v \, dx,
\]

for all \( v \in H^2_0(\Omega) \).

By the Riesz-Schauder, Hilbert-Schmidt theorems and the minmax principle \((\text{[9, 21]}\)), for each \( \varepsilon \) small enough, we have the following classical result.

**Lemma 1.1.** Suppose that (H1)–(H4) are satisfied. Then for all sufficiently small \( \varepsilon > 0 \), the eigenvalues \( \lambda^\varepsilon_i \) of (1) are real and such that

\[
0 < \lambda^\varepsilon_1 \leq \lambda^\varepsilon_2 \leq \cdots \leq \lambda^\varepsilon_i \to \infty \text{ as } i \to \infty,
\]

where each eigenvalue is counted as many times as its multiplicity. The eigenfunctions \( u^\varepsilon_i \) form an orthonormal basis in \( L^2(\Omega) \). All eigenvalues are of finite multiplicity and are characterized by the variational principle:

\[
\lambda^\varepsilon_i = \min_{\mathcal{V}} \frac{A_\varepsilon(v,v)}{\int_{\Omega} v^2 \, dx},
\]

where the minimum is taken over all nonzero functions \( v \in H^2_0(\Omega) \) that are orthogonal in \( L^2(\Omega) \) to the first \( i-1 \) eigenfunctions \( u^\varepsilon_1, \ldots, u^\varepsilon_{i-1} \).

The goal of this paper is to describe the asymptotic behavior of the eigenpairs \( (\lambda^\varepsilon_k, u^\varepsilon_k) \), as \( \varepsilon \to 0 \). We restrict ourselves to the values of the parameters \( \alpha \geq 0, \beta > 0 \) (singular perturbation) such that \( \alpha < \beta \) (concentration effect is observed) and \( \beta < 4 \) (subcritical case). The result is presented in the three theorems 2.1, 3.1, 4.1.

The case \( \alpha = \beta \) is classical, and the standard two-scale convergence can be applied to describe the asymptotics. Depending on the value of \( \alpha \) one gets either fourth order limit operator without second order terms \((0 < \alpha = \beta < 2)\), or fourth order with second order terms \((\alpha = \beta = 0, \alpha = \beta = 2)\), or just a second order limit operator \((\alpha = \beta > 2)\).

The case \( \beta = 4, \alpha < 4 \) is the critical case, when the oscillations of the eigenfunctions are expected to be of order \( \varepsilon \). As it is seen from [5], the technique to be used is different, and this case is to be considered elsewhere. In addition, the values of \( \alpha, \beta \) such that \( 3 \leq \beta < 4 \) and \( \alpha < \beta - 2 \) are not covered by the present paper (the hatched region in Figure 1), because the error coming from Lemma 2.3 while passing to the limit does not vanish, as \( \varepsilon \to 0 \). Another argument is to be applied, and this case will be considered elsewhere.
To describe the asymptotic behavior of eigenpairs \((\lambda_k^\varepsilon, u_k^\varepsilon)\) as \(\varepsilon \to 0\), we divide the domain for the parameters \((\alpha, \beta)\) into the following regions (see Figure 1):

\[
R_1 = \{ (\alpha, \beta) : 0 \leq \alpha < 1, \ 3\alpha < \beta < 3 \}, \\
R_2 = \{ (\alpha, \beta) : 0 < \alpha < 1, \ \beta = 3\alpha \}, \\
R_3 = \{ (\alpha, \beta) : 0 < \alpha < 2, \ \alpha < \beta < 3\alpha, \ \beta < \alpha + 2 \}, \\
R_4 = \{ (\alpha, \beta) : \alpha = 2, \ 2 < \beta < 4 \}, \\
R_5 = \{ (\alpha, \beta) : 2 < \alpha < 4, \ \alpha < \beta < 4 \}.
\]

The reason for distinguishing these regions is that we get different asymptotics in each case. In short, in \(R_1\) we get a fourth order equation in the limit without second order terms; in \(R_2\) the limit equation contains both fourth and second order terms; in \(R_3, R_4, R_5\) the limit equations are of the second order. We choose to consider in details one case \(\beta = 3\alpha = 1\), corresponding to region \(R_2\), since all the terms contribute in the limit (see Theorem 2.1). The spectral asymptotics in the other cases are described in Sections 3, 4.

![Figure 1. The partition of the parameter region for \((\alpha, \beta)\).](image.png)
with $a_{ijkl}^{\text{eff}}$ defined by (18), $\tilde{b}_{ij}(0) = \int_{T_d} b_{ij}(0,y)dy$, and $H = \nabla \nabla \tilde{c}(0)$.

The proof of Theorem 2.1 will occupy the rest of this section and is given for the case $\beta = 3\alpha = 1$. The argument used is the same for the other values of $(\alpha, \beta) \in R_2$.

2.1. Estimates for eigenvalues of the original problem. To motivate the change of variables we will make in the next subsection we prove the following a priori estimates for the eigenvalues and the eigenfunctions of problem (1).

Lemma 2.2. Suppose that (H1)–(H4) are satisfied. Let $(\lambda_i^{(1)}, u_i^{(1)})$ be the $i$th eigenpair of (1) with $\beta = 3\alpha = 1$, normalized by $\|u_i^{(1)}\|_{2,\Omega} = 1$. Then there exist positive constants $C_1, C_2(i)$ such that for all sufficiently small $\varepsilon > 0$,

$$-\frac{C_1}{\varepsilon^{2/3}} \leq \lambda_i^{(1)} - \tilde{c}(0) \leq \frac{C_2(i)}{\varepsilon^{2/3}},$$

$$\|\Delta u_i^{(1)}\|_{2,\Omega} \leq \frac{C}{\varepsilon^{1/3}}.$$

To prove Lemma 2.2 we will use the following estimate for integrals of oscillating functions.

Lemma 2.3. Let $g(x, y) \in C^2(\bar{\Omega}; L^\infty(T^d))$ be such that $\int_{T_d} g(x, y) dy = 0$ for all $x \in \bar{\Omega}$. Then there exists a positive constant $C$ such that

$$\left| \int_{\Omega} g(x, \frac{x}{\varepsilon}) uv dx \right| \leq C\varepsilon^2(\|\Delta u\|_{2,\Omega}\|v\|_{2,\Omega} + \|u\|_{2,\Omega}\|\Delta v\|_{2,\Omega}),$$

for all $u, v \in H^1_0(\Omega)$.

Proof. Let $\Psi(x, y) \in C^2(\bar{\Omega}; C^1(T^d))$, periodic in $y$, be defined by

$$\Delta_y \Psi(x, y) = g(x, y), \ y \in T^d,$$

Since the local average of $g(x, y)$ is zero, $\Psi$ is well-defined. By the Green formula,

$$\int_{\Omega} g(x, \frac{x}{\varepsilon}) u(x)v(x) dx = \int_{\Omega} (\Delta_y \Psi)(x, \frac{x}{\varepsilon}) u(x)v(x) dx$$

$$= \varepsilon^2 \int_{\Omega} \left( \Delta \Psi(x, \frac{x}{\varepsilon}) - 2 \text{div}(\nabla_x \Psi)(x, \frac{x}{\varepsilon}) + (\Delta_x \Psi)(x, \frac{x}{\varepsilon}) \right) u(x)v(x) dx$$

$$= \varepsilon^2 \int_{\Omega} \left( \Psi(x, \frac{x}{\varepsilon}) \Delta(uv) + 2(\nabla_x \Psi)(x, \frac{x}{\varepsilon}) \cdot \nabla(uv) + (\Delta_x \Psi)(x, \frac{x}{\varepsilon}) uv \right) dx.$$

After an application of the Green formula to the $\nabla u \cdot \nabla v$ term coming from $\Delta(uv)$, by the H"older and triangle inequalities, we have

$$\left| \int_{\Omega} g(x, \frac{x}{\varepsilon}) u(x)v(x) dx \right|$$

$$\leq \varepsilon^2 \left( \|\Psi\|_{L^\infty(\Omega \times T^d)}(\|\Delta u\|_{2,\Omega}\|v\|_{2,\Omega} + \|u\|_{2,\Omega}\|\Delta v\|_{2,\Omega}) + 2\|\nabla_x \Psi\|_{L^\infty(\Omega \times T^d)}(\|\nabla u\|_{2,\Omega}\|v\|_{2,\Omega} + \|u\|_{2,\Omega}\|\nabla v\|_{2,\Omega}) + \|\Delta_x \Psi\|_{L^\infty(\Omega \times T^d)}\|u\|_{2,\Omega}\|v\|_{2,\Omega} \right).$$

The estimate follows from the regularity of $\Psi$ and the Poincaré inequality. □
**Proof of Lemma 2.2.** Let \( v \in C_0^\infty(\mathbb{R}^d) \) be such that \( \|v\|_{2,\mathbb{R}^d} = 1 \), and set \( v^\varepsilon(x) = v(\varepsilon^{-1/6}x) \). We assume that \( \varepsilon \) is small enough such that \( \text{supp} v^\varepsilon \subset \Omega \). By the variational principle,

\[
\lambda_1^\varepsilon \leq \frac{\int_{\Omega} a_{ijkl} \partial_{ij} v^\varepsilon \partial_{kl} v^\varepsilon \, dx + \varepsilon^{-1/3} \int_{\Omega} b_{ijkl} \partial_i v^\varepsilon \partial_j v^\varepsilon \, dx}{\int_{\Omega} (v^\varepsilon)^2 \, dx} + \frac{1}{\varepsilon} \int_{\Omega} c(v^\varepsilon)^2 \, dx.
\]

(4)

By the boundedness of the coefficients, the first fraction in (4) is bounded by \( C\varepsilon^{-2/3} \). The second fraction in (4) is estimated using (H4) and Lemma 2.3:

\[
\frac{1}{\varepsilon} \int_{\Omega} c(x, x^\varepsilon)(v^\varepsilon)^2 \, dx = \tilde{c}(0) + \frac{1}{\varepsilon} \int_{\mathbb{R}^d} (\varepsilon^{1/6}x - \tilde{c}(0))v^2 \, dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^d} (\varepsilon^{1/6}x - \tilde{c}(\varepsilon^{1/6}x)/\varepsilon^5) \varepsilon^2 \, dx \\
\leq \frac{\tilde{c}(0)}{\varepsilon} + C\varepsilon^{-2/3},
\]

for sufficiently small \( \varepsilon > 0 \).

In order to obtain an estimate from below for the first eigenvalue \( \lambda_1^\varepsilon \), we need to estimate the second derivatives of the corresponding eigenfunctions. Let \( u_1^\varepsilon \) denote any eigenfunction corresponding to \( \lambda_1^\varepsilon \), normalized by \( \|u_1^\varepsilon\|_{2,\Omega} = 1 \). Then, by (3),

\[
\lambda_1^\varepsilon - \frac{1}{\varepsilon} \int_{\Omega} c(x, x^\varepsilon)(u_1^\varepsilon)^2 \, dx = \int_{\Omega} a_{ijkl} \partial_{ij} u_1^\varepsilon \partial_{kl} u_1^\varepsilon \, dx + \frac{1}{\varepsilon^{1/3}} \int_{\Omega} b_{ijkl} \partial_i u_1^\varepsilon \partial_j u_1^\varepsilon \, dx.
\]

On the one hand, by the ellipticity of \( a_{ijkl} \) and the boundedness of the coefficients,

\[
\lambda_1^\varepsilon - \frac{1}{\varepsilon} \int_{\Omega} c(x, x^\varepsilon)(u_1^\varepsilon)^2 \, dx \geq \Lambda \|\nabla \nabla u_1^\varepsilon\|_{2,\Omega}^2 - C \frac{1}{\varepsilon^{1/3}} \int_{\Omega} |\nabla u_1^\varepsilon|^2 \, dx \\
\geq \Lambda \|\nabla \nabla u_1^\varepsilon\|_{2,\Omega}^2 - C_2 \gamma \|\Delta u_1^\varepsilon\|_{2,\Omega}^2 - C_2 \varepsilon^{-2/3} \|u_1^\varepsilon\|_{2,\Omega}^2 \\
\geq C (\|\nabla \nabla u_1^\varepsilon\|_{2,\Omega}^2 - \varepsilon^{-2/3}),
\]

(5)

where \( \gamma > 0 \) in the Cauchy inequality is chosen small enough such that the resulting constant \( C \) is positive. Note that one can choose \( \gamma \) that depends just on the ellipticity constant of \( a_{ijkl} \) and the upper bound for \( b_{ijkl} \).

On the other hand, from the upper estimate for \( \lambda_1^\varepsilon \),

\[
\lambda_1^\varepsilon - \frac{1}{\varepsilon} \int_{\Omega} c(x, x^\varepsilon)(u_1^\varepsilon)^2 \, dx \\
\leq \frac{\tilde{c}(0)}{\varepsilon} - \frac{1}{\varepsilon} \int_{\Omega} c(x, x^\varepsilon)(u_1^\varepsilon)^2 \, dx + C\varepsilon^{-2/3} \\
= \frac{1}{\varepsilon} \int_{\Omega} (\tilde{c}(0) - \tilde{c}(x))(u_1^\varepsilon)^2 \, dx + \frac{1}{\varepsilon} \int_{\Omega} (\tilde{c}(x) - c(x, x^\varepsilon))(u_1^\varepsilon)^2 \, dx + C\varepsilon^{-2/3} \\
\leq C\varepsilon \|\Delta u_1^\varepsilon\|_{2,\Omega} + \varepsilon^{-2/3},
\]

(6)

where we in the third step used that 0 is a minimum point for \( \tilde{c}(x) \) by (H4), and Lemma 2.3.

Combining (5) and (6), by the Cauchy inequality, we deduce that,

\[
\|\nabla \nabla u_1^\varepsilon\|_{2,\Omega}^2 \leq C\varepsilon^{-2/3}.
\]

(7)
We proceed with the estimate from below for $\lambda_1^\varepsilon$. By (9) and (7) we have
\[
\lambda_1^\varepsilon \geq \frac{1}{\varepsilon} \int_\Omega c(x,\frac{x}{\varepsilon})(u_1^\varepsilon)^2 \, dx - \frac{C}{\varepsilon^{2/3}} \\
= \frac{\tilde{c}(0)}{\varepsilon} + \frac{1}{\varepsilon} \int_\Omega (\tilde{c}(x) - \tilde{c}(0))(u_1^\varepsilon)^2 \, dx + \frac{1}{\varepsilon} \int_\Omega (c(x,\frac{x}{\varepsilon}) - \tilde{c}(x))(u_1^\varepsilon)^2 \, dx - \frac{C}{\varepsilon^{2/3}}.
\]
By (H4) and Lemma 2.3 combined with (7)
\[
\lambda_1^\varepsilon \geq \frac{\tilde{c}(0)}{\varepsilon} - \frac{C}{\varepsilon^{2/3}},
\]
for all sufficiently small $\varepsilon > 0$. In this way we have obtained the required estimates for the first eigenvalue $\lambda_1^\varepsilon$. Since $\lambda_1^\varepsilon$ is the smallest eigenvalue, the estimate from below for $\lambda_i^\varepsilon$, $i > 1$, follows from the corresponding estimate for $\lambda_1^\varepsilon$.

To estimate $\lambda_i^\varepsilon > \lambda_1^\varepsilon$ for $i > 1$, one can use as a test function the projection of $v^\varepsilon$ onto the orthogonal complement of the span of the first $i - 1$ eigenvectors, with respect to the $L^2(\Omega)$ inner product. Since the span is finite dimensional this projection is nonzero for all sufficiently small $\varepsilon > 0$.

Let $m_1$ be the multiplicity of the first eigenvalue $\lambda_1^\varepsilon = \lambda_2^\varepsilon = \cdots = \lambda_{m_1}^\varepsilon$. We estimate from above $\lambda_{m_1}^\varepsilon$, similar arguments can be applied to estimate other eigenvalues.

For $v \in C_0^\infty(R^d)$, we introduce $v^\varepsilon(x) = v(\frac{x}{\varepsilon})$ and denote $\pi_{\varepsilon,k} = \int_\Omega v^\varepsilon u_k^\varepsilon \, dx$, $k = 1,\ldots,m_1$. Then $V^\varepsilon(x) = v^\varepsilon(x) - \sum_k \pi_{\varepsilon,k} u_k^\varepsilon(x)$ is orthogonal in $L^2(\Omega)$ to $\text{span}\{u_1^\varepsilon,\ldots,u_{m_1}^\varepsilon\}$. For convenience we assume the normalization condition
\[
\|V^\varepsilon\|_{L^2(\Omega)}^2 = \|v^\varepsilon\|_{L^2(\Omega)}^2 - \sum_{k=1}^{m_1} \pi_{\varepsilon,k}^2 = \varepsilon^{d/6}. \tag{8}
\]
Using $V^\varepsilon$ as a test function in the variational principle, we deduce that
\[
\lambda_{m_1+1}^\varepsilon \leq \varepsilon^{-d/6} (A_\varepsilon(v^\varepsilon, v^\varepsilon) - \lambda_1^\varepsilon \sum_k \pi_{\varepsilon,k}^2). \tag{9}
\]
By (H1)–(H3) and Lemma 2.3 we get
\[
A_\varepsilon(v^\varepsilon, v^\varepsilon) - \lambda_1^\varepsilon \sum_k \pi_{\varepsilon,k}^2 \leq \frac{\tilde{c}(0)}{\varepsilon} \|v^\varepsilon\|_{L^2(\Omega)}^2 - \lambda_1^\varepsilon \sum_k \pi_{\varepsilon,k}^2 + C \varepsilon^{d/6} \varepsilon^{-2/3}.
\]
Due to (8), (9) and the estimate from below for $\lambda_1^\varepsilon$,
\[
\lambda_{m_1+1}^\varepsilon \leq \frac{\tilde{c}(0)}{\varepsilon} + \frac{C_1 \sum_k \pi_{\varepsilon,k}^2}{\varepsilon^{d/6} \varepsilon^{2/3}} + \frac{C \varepsilon^{d/6} \varepsilon^{-2/3}}{\varepsilon^{2/3}}.
\]
Due to the normalization condition for $u_1^\varepsilon$,
\[
\sum_k \pi_{\varepsilon,k}^2 \leq m_1 \varepsilon^{d/6} \|v\|_{L^2(R^d)}^2,
\]
thus
\[
\lambda_{m_1+1}^\varepsilon \leq \frac{\tilde{c}(0)}{\varepsilon} + \frac{C_2}{\varepsilon^{2/3}},
\]
and the estimate is proved. \qed
2.2. Rescaling the problem and computing the asymptotics. Led by Lemma 2.3, we shift the spectrum of (1) by \( \bar{c}(0)/\varepsilon \) and rescale such that the eigenvalues become bounded. Let
\[
z = \frac{x}{\varepsilon^{1/6}}, \quad v^\varepsilon(z) = u^\varepsilon(\varepsilon^{1/6} z), \quad \eta^\varepsilon = \varepsilon^{2/3}(\lambda^\varepsilon - \bar{c}(0))/\varepsilon, \quad \Omega_\varepsilon = \varepsilon^{-1/6} \bar{\Omega}.
\]
Then (3) takes the form
\[
\int_{\Omega_\varepsilon} \hat{a}^\varepsilon_{ijkl} \partial_i V^\varepsilon \partial_k \varphi dz + \int_{\Omega_\varepsilon} \hat{b}^\varepsilon_{ij} \partial_i V^\varepsilon \partial_j \varphi dz + \frac{1}{\varepsilon^{1/3}} \int_{\Omega_\varepsilon} (\hat{c}^\varepsilon - \bar{c}(0)) v^\varepsilon \varphi dz
= \eta^\varepsilon \int_{\Omega_\varepsilon} v^\varepsilon \varphi dz,
\]
for any \( v \in H^2_0(\Omega_\varepsilon) \), where
\[
\hat{a}^\varepsilon_{ijkl}(z) = a(\varepsilon^{1/6} z, \frac{z}{\varepsilon^{5/6}}), \quad \hat{b}^\varepsilon_{ij}(z) = b(\varepsilon^{1/6} z, \frac{z}{\varepsilon^{5/6}}), \quad \hat{c}^\varepsilon(z) = c(\varepsilon^{1/6} z, \frac{z}{\varepsilon^{5/6}}).
\]
For the rest of \( R_2 \), one uses \( z = \varepsilon^{-\alpha/2} x \).

In order to describe the asymptotic behavior of eigenpairs \( (\eta^\varepsilon_k, v^\varepsilon_k) \) of (11), we consider the Green operator
\[
G_\varepsilon : L^2(\Omega_\varepsilon) \to L^2(\Omega_\varepsilon),
\]
where \( V^\varepsilon \in H^2_0(\Omega_\varepsilon) \) is the unique solution to the boundary-value problem
\[
\begin{aligned}
&\partial_i (\hat{a}^\varepsilon_{ijkl} \partial_k V^\varepsilon) + \partial_i (\hat{b}^\varepsilon_{ij} \partial_j V^\varepsilon) + \frac{\hat{c}^\varepsilon - \bar{c}(0)}{\varepsilon^{1/3}} V^\varepsilon + \mu V^\varepsilon = f_\varepsilon, \quad z \in \Omega_\varepsilon, \\
&V^\varepsilon = \nabla V^\varepsilon \cdot n = 0, \quad z \in \partial \Omega_\varepsilon.
\end{aligned}
\]
Here \( \mu > 0 \) is a large enough constant, but depending just the ellipticity constant \( \Lambda \). The operator \( G_\varepsilon \) can be considered as an operator from \( L^2(\mathbb{R}^d) \) into itself by extending the corresponding solution \( V^\varepsilon \) by zero outside \( \Omega_\varepsilon \). The existence and uniqueness is ensured by the Riesz-Fr\'echet representation theorem since the corresponding symmetric quadratic form
\[
A_\varepsilon(V^\varepsilon, V^\varepsilon) = \int_{\Omega_\varepsilon} \hat{a}^\varepsilon_{ijkl} \partial_i V^\varepsilon \partial_k V^\varepsilon dz + \int_{\Omega_\varepsilon} \hat{b}^\varepsilon_{ij} \partial_i V^\varepsilon \partial_j V^\varepsilon dz + \frac{1}{\varepsilon^{1/3}} \int_{\Omega_\varepsilon} (\hat{c}^\varepsilon - \bar{c}(0)) (V^\varepsilon)^2 dz + \mu \int_{\Omega_\varepsilon} (V^\varepsilon)^2 dz
\]
is coercive. Indeed, by (H1)–(H4) and Lemma 2.3 we have:
\[
A_\varepsilon(V^\varepsilon, V^\varepsilon) \geq \Lambda \int_{\Omega_\varepsilon} |\nabla \nabla V^\varepsilon|^2 dz - \Lambda^{-1} \int_{\Omega_\varepsilon} |\nabla V^\varepsilon|^2 dz
- C_0 \varepsilon^{4/3} \|V^\varepsilon\|_{2, \Omega_\varepsilon} \|\Delta V^\varepsilon\|_{2, \Omega_\varepsilon}
+ \frac{1}{\varepsilon^{1/3}} \int_{\Omega_\varepsilon} (\hat{c}(\varepsilon^{1/6} z) - \bar{c}(0)) (V^\varepsilon)^2 dz + \mu \|V^\varepsilon\|_{2, \Omega_\varepsilon}^2.
\]
Since the Hessian matrix \( H \) is positive definite, there exist a positive constant \( K_1 \) such that
\[
\hat{c}(\varepsilon^{1/6} z) - \bar{c}(0) \geq K_1 [\varepsilon^{1/6} z]^2.
\]
Due to the Dirichlet boundary conditions,
\[ \| \nabla V^\varepsilon \|_{2,\Omega}^2 \leq \| V^\varepsilon \|_{2,\Omega} \| \Delta V^\varepsilon \|_{2,\Omega}. \] (15)
Applying the Cauchy inequality \( \| V^\varepsilon \| \| \Delta V^\varepsilon \| \leq \delta \| V^\varepsilon \|^2 + \| \Delta V^\varepsilon \|^2/\delta \) with \( \delta = \Lambda/3 \) in (14) we get
\[
A_\varepsilon (V^\varepsilon, V^\varepsilon) \geq \frac{\Lambda}{3} \int_{\Omega_\varepsilon} |\nabla\nabla V^\varepsilon|^2 \, dz + K_1 \int_{\Omega_\varepsilon} |z|^2 (V^\varepsilon)^2 \, dz
+ \left( \mu - \frac{3}{\Lambda^3} - \frac{2C_0 \varepsilon^{4/3}}{\Lambda} \right) \| V^\varepsilon \|^2_{2,\Omega_\varepsilon}.
\]
For \( \varepsilon \) small enough, we can choose \( \mu \) depending just on \( \Lambda \) and \( |\Omega| \) such that, for some positive constant \( \tilde{C} \), we have
\[
A_\varepsilon (V^\varepsilon, V^\varepsilon) \geq \tilde{C} (\| \nabla\nabla V^\varepsilon \|_{2,\Omega_\varepsilon}^2 + \int_{\Omega_\varepsilon} |z|^2 (V^\varepsilon)^2 \, dz + \| V^\varepsilon \|_{2,\Omega_\varepsilon}^2). \] (16)
Even though \( A_\varepsilon (V^\varepsilon, V^\varepsilon) \geq C \| \Delta V^\varepsilon \|_{2,\Omega_\varepsilon} \) is enough for coercivity of the quadratic form, we will make use of the last inequality. The addition of the constant \( \mu \) has the effect of shifting the entire spectrum of (11) by \( \mu \).

We introduce also the limit Green operator
\[
G : L^2 (\mathbb{R}^d) \rightarrow L^2 (\mathbb{R}^d),
\]
where \( V \in H^2 (\mathbb{R}^d) \) is the unique solution to the equation
\[
\partial_{ij} \left( a^{\text{eff}}_{ijkl} (\partial_k V) \right) - \partial_i (\overline{b}_{ij} (0) \partial_j V) + \frac{1}{2} (Hz \cdot z) V + \mu V = f, \quad z \in \mathbb{R}^d. \] (17)
Here \( H \) is the Hessian matrix of \( \mathcal{E} \) at \( x = 0 \) (see (H4)), and the effective coefficients are defined by
\[
a^{\text{eff}}_{ijkl} = \int_{\mathbb{T}^d} \left( a_{ijmn} (0, y) \partial_m n_{kl}(y) + a_{ijkl} (0, y) \right) dy, \] (18)
where the periodic functions \( n_{kl} \in H^2 (\mathbb{T}^d)/\mathbb{R} \) solve the following cell problems:
\[
\partial_{ij} (a_{ijmn} (0, y) \partial_m n_{kl}(y)) = -\partial_{ij} a_{ijkl} (0, y), \quad y \in \mathbb{T}^d. \] (19)
Due to the periodicity of \( a_{ijkl} \) in \( y \), the above problem is well-posed, the solution \( n_{kl} \) is unique up to an additive constant.

**Lemma 2.4.** Under the assumptions (H1)–(H4), \( a^{\text{eff}} \) defined by (18) is coercive on \( \mathbb{R}^{d \times d} \), i.e. there is a positive constant \( C \) such that \( a^{\text{eff}}_{ijkl} \xi_{ij} \xi_{kl} \geq C |\xi|^2 \) for all \( \xi \in \mathbb{R}^{d \times d} \).

**Proof.** Using (H1)–(H4) gives a well-defined \( a^{\text{eff}} \). We rewrite (18) as
\[
a^{\text{eff}}_{ijkl} = \int_{\mathbb{T}^d} \delta_{pi} \delta_{qj} a_{pqrs} (0, y) (\delta_{rk} \delta_{sl} + \partial_{rs} n_{kl}) dy.
\]
Using \( n_{ij} \) as a test function in equation (19) for \( n_{kl} \) gives
\[
\int_{\mathbb{T}^d} a_{pqrs} (0, y) (\delta_{rk} \delta_{sl} + \partial_{rs} n_{kl}) \partial_{pq} n_{ij} \, dy = 0.
\]
Thus
\[ a_{ijkl}^{\text{eff}} = \int_{T^d} a_{pqrs}(0, y)(\delta_{rk}\delta_{sl} + \delta_{rs}\delta_{kl})(\delta_{pi}\delta_{qj} + \partial_{pq}N_{ij}) \, dy. \]

The last equation shows that \( a^{\text{eff}} \) is symmetric by (H2): \( a_{ijkl}^{\text{eff}} = a_{klji}^{\text{eff}} \). Moreover, with \( \xi \in \mathbb{R}^{d \times d} \) we obtain
\[
a_{ijkl}^{\text{eff}}\xi_{ij}\xi_{kl} = \int_{T^d} a_{pqrs}(0, y)(\xi_{kl}(\delta_{rk}\delta_{sl} + \delta_{rs}\delta_{kl})(\xi_{ij}(\delta_{pi}\delta_{qj} + \partial_{pq}N_{ij})) \, dy
\geq \Lambda \sum_{p,q} \int_{T^d} |\xi_{ij}(\delta_{pi}\delta_{qj} + \partial_{pq}N_{ij})|^2 \, dy,
\]
by the coerciveness of \( a(0, y) \) guaranteed by (H3). The last inequality implies that \( a^{\text{eff}} \) is positive definite. Indeed, if \( \xi \in \mathbb{R}^{d \times d} \) is such that \( a_{ijkl}^{\text{eff}}\xi_{ij}\xi_{kl} = 0 \), then \( |\xi_{ij}(\delta_{pi}\delta_{qj} + \partial_{pq}N_{ij})| = 0 \) for all \( p, q \). It follows that \( \nabla\xi_{ij}(y_i\delta_{qj} + \partial_{q}N_{ij}) = 0 \) for all \( q \). Therefore, \( \xi_{ij}(y_i\delta_{qj} + \partial_{q}N_{ij}) \) is constant and so necessarily \( \xi_{iq}y_i \) is periodic by the periodicity of \( N_{ij} \). Hence \( \xi_{iq} = 0 \) for all \( i, q \). We conclude that for all \( \xi \in \mathbb{R}^{d \times d} \setminus \{0\}, \)
\[ a_{ijkl}^{\text{eff}}\xi_{ij}\xi_{kl} > 0. \]

By the compactness of the unit ball in \( \mathbb{R}^{d \times d} \), there is a positive constant \( C \) such that \( a_{ijkl}^{\text{eff}}\xi_{ij}\xi_{kl} \geq C|\xi|^2 \) for all \( \xi \in \mathbb{R}^{d \times d} \).

The bilinear form corresponding to (17) takes the form
\[ A_{\text{eff}}(u, v) = \int_{\mathbb{R}^d} \left( a_{ijkl}^{\text{eff}}\partial_{kl}u\partial_{ij}v + b_{ij}\partial_{j}u\partial_{i}v + \frac{1}{2}(Hz \cdot z) uv + \mu uv \right) \, dz, \]
and it is coercive. Namely, there exists a positive constant \( \hat{C} \) such that
\[ A_{\text{eff}}(V, V) \geq \hat{C} \left( \|\nabla V\|^2_{2, \mathbb{R}^d} + \|zV\|^2_{2, \mathbb{R}^d} + \|V\|^2_{2, \mathbb{R}^d} \right). \]
Thus, by the Riesz-Fréchet representation theorem, the Green operator is well-defined. Using Lemma 2.4 we see that the operator \( G \) is self-adjoint. Moreover, due to the compact embedding of \( H^2(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, |z|^2 dz) \) in \( L^2(\mathbb{R}^d) \), the operator \( G \) is compact as an operator in \( L^2(\mathbb{R}^d) \): \( L^2(\mathbb{R}^d, |z|^2 dz) \) is a weighted \( L^2 \)-space with the weight \( |z|^2 \). As a direct consequence, we have the following result.

**Lemma 2.5.** The spectrum of the limit problem
\[ \partial_{ij}(a_{ijkl}^{\text{eff}}\partial_{kl}v) - \partial_{i}(b_{ij}\partial_{j}v) + \frac{1}{2}(Hz \cdot z) v = \eta v, \quad z \in \mathbb{R}^d, \tag{20} \]
is real, discrete, and consists of a countably infinite number of eigenvalues, each of finite multiplicity:
\[ \eta_1 \leq \eta_2 \leq \cdots, \quad \eta_j \to \infty \text{ as } j \to \infty. \]
The corresponding eigenfunctions \( v_j^c \) form an orthonormal basis in \( L^2(\mathbb{R}^d) \).

We proceed to the proof of the convergence of spectra. We will prove that the Green operator \( G^c \) converges uniformly to \( G \) in \( L(L^2(\mathbb{R}^d)) \). Then we apply the following result, the proof of which can be found in [3, Lemma 2.6] (see also [4]), to conclude the desired convergence of eigenvalues and eigenfunctions.
Lemma 2.6. Let $G_ε$ be a sequence of compact self-adjoint operators acting in $L^2(\mathbb{R}^d)$. Assume that $G_ε$ converges uniformly to a compact self-adjoint operator $G$. Let $ζ_ε^k$ and $ζ_k$ be the $k$th eigenvalues of the operators $G_ε$ and $G$, respectively; $v_ε^k, v_k$ are eigenfunctions corresponding to $ζ_ε^k, ζ_k$. Then as $ε → 0$,

(i) $ζ_ε^k → ζ_k$,

(ii) up to a subsequence, $v_ε^k$ converges strongly in $L^2(\mathbb{R}^d)$ to $v_k$.

The uniform convergence of the Green operators is a straightforward consequence of the convergence of the solutions to the corresponding boundary value problems with weakly converging data in $L^2(\mathbb{R}^d)$, as has been pointed out in [3 Theorem 2.2].

Lemma 2.7. Let $f_ε$ be a sequence converging weakly to $f$ in $L^2(\mathbb{R}^d)$, and let $V^ε$ be the unique solution of (12). Then $V^ε$ converges weakly in $H^2(\mathbb{R}^d)$ and strongly in $L^2(\mathbb{R}^d)$ to the unique solution $V$ of the effective problem (17). Moreover,

\[ \nabla V^ε \overset{2}{\rightarrow} \nabla V(z) \text{ two-scale in } L^2(\mathbb{R}^d), \]
\[ \nabla \nabla V^ε \overset{2}{\rightarrow} \nabla \nabla V(z) + \nabla \nabla \left( N_{kl}(ζ) \partial_{kl} V(z) \right) \text{ two-scale in } L^2(\mathbb{R}^d), \]

where $N_{kl} ∈ H^2(\mathbb{T}^d)/R, k, l = 1, \ldots, d$, solve problem (19).

Proof. The proof consists of two parts. First, we derive a priori estimates for $V^ε$. Second, we pass to the two-scale limit in order to obtain the effective problem.

The estimates for $V^ε$ follows from (16):

\[ \| \nabla \nabla V^ε \|_{2,Ω_ε}^2 + \| \nabla V^ε \|^2_{2,Ω_ε} + \| z V^ε \|_{2,Ω_ε}^2 + \| V^ε \|_{2,Ω_ε}^2 \leq C \| f_ε \|_{2,Ω_ε}^2. \]

Having in hand the a priori estimate (21), we deduce (see, for example, Proposition 1.14 in [2]) that in $L^2(\mathbb{R}^d)$ we have the following two-scale convergences:

\[ V^ε \overset{2}{\rightarrow} V(z), \quad \nabla V^ε \overset{2}{\rightarrow} \nabla V(z), \quad \nabla \nabla V^ε \overset{2}{\rightarrow} \nabla \nabla V(z) + \nabla \nabla W(z, ζ), \]

where $W(z, ζ) ∈ L^2(\mathbb{R}^d; H^2(\mathbb{T}^d))$. The strong convergence of $V^ε$ to $V$ in $L^2(\mathbb{R}^d)$ follows also from (21), namely from the boundedness of weighted $L^2$-norm, which gives compactness. We are going to pass to the limit in the weak formulation of (12):

\[
\int_{Ω_ε} \bar{a}^ε_{ijkl} \partial_{kl} V^ε \partial_{ij} Φ^ε dz + \int_{Ω_ε} \bar{b}^ε_{ij} \partial_i V^ε \partial_j Φ^ε dz + \frac{1}{ε^{1/3}} \int_{Ω_ε} (\tilde{c}^ε - \tilde{c}(ε^{1/6} z)) V^ε Φ^ε dz \\
+ \frac{1}{ε^{1/3}} \int_{Ω_ε} (\tilde{c}(ε^{1/6} z) - \tilde{c}(0)) V^ε Φ^ε dz + \mu \int_{Ω_ε} V^ε Φ^ε dz = \int_{Ω_ε} f_ε Φ^ε dz, \]

where $Φ^ε(z) = Φ(z, z/ε^{1/6}), Φ(z, ζ) ∈ C(\overline{Ω_ε}; L^∞(\mathbb{T}^d))$. Note that the term containing $\tilde{c}(ε^{1/6} z)$ is added and subtracted for convenience, since we are going to use Lemma 2.3 when passing to the limit. Due to the regularity assumptions (H1), the coefficients can be regarded as a part of a test function.
First we take a test function \( \Phi_\varepsilon = \varepsilon^{5/3} \varphi(z) \psi(\frac{z}{\varepsilon^{5/6}}) \), with \( \varphi \in C_0^\infty(\mathbb{R}^d), \psi \in C^\infty(\mathbb{T}^d) \). Then (23) transforms into

\[
\int_{\Omega_\varepsilon} \hat{a}_{ijkl} \partial^{1/3} V^\varepsilon(z) \partial_{ij} \psi(\zeta) + \varepsilon^{5/6} \partial_{ij} \varphi(z) \partial_i \psi(\zeta) + \varepsilon^{5/6} \partial_i \varphi(z) \partial_j \psi(\zeta) + \varepsilon^{5/3} \partial_i \varphi(z) \psi(\zeta) \bigg|_{\zeta = \frac{z}{\varepsilon^{5/6}}} \, dz \\
+ \varepsilon^{5/3} \int_{\Omega_\varepsilon} \hat{b}_{ij} \partial_i V^\varepsilon(z) \partial_{ij} \psi(\zeta) + \varepsilon^{5/3} \psi(\zeta) \partial_i \varphi(z) \bigg|_{\zeta = \frac{z}{\varepsilon^{5/6}}} \, dz \\
+ \varepsilon^{4/3} \int_{\Omega_\varepsilon} (\tilde{e} \varepsilon^2 + \bar{c}(\varepsilon^{1/6} z)) V^\varepsilon(z) \varphi(z) \psi(\frac{z}{\varepsilon^{5/6}}) \, dz \\
+ \varepsilon^{4/3} \int_{\Omega_\varepsilon} (\tilde{c}(\varepsilon^{1/6} z) - \bar{c}(0)) V^\varepsilon(z) \varphi(z) \psi(\frac{z}{\varepsilon^{5/6}}) \, dz \\
+ \mu \varepsilon^{5/3} \int_{\Omega_\varepsilon} V^\varepsilon(z) \varphi(z) \psi(\frac{z}{\varepsilon^{5/6}}) \, dz = \varepsilon^{5/3} \int_{\Omega_\varepsilon} f(z) \varphi(z) \psi(\frac{z}{\varepsilon^{5/6}}) \, dz.
\]

Using Lemma 2.3 and (22) we may pass to the limit, as \( \varepsilon \to 0 \), using the two-scale convergence, and obtain

\[
\int_{\mathbb{R}^d} \int_{\mathbb{T}^d} a_{ijmn}(0, \zeta) \partial_{\zeta_m} W(z, \zeta) \partial_{ij} \psi(\zeta) \varphi(z) \, d\zeta \, dz \\
= - \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} a_{ijkl}(0, \zeta) \partial_i V(z) \partial_{ij} \psi(\zeta) \varphi(z) \, d\zeta \, dz.
\]

From the last identity we deduce that \( W(z, \zeta) = N_{kl}(\zeta) \partial_k V(z) \), where the periodic functions \( N_{kl}(\zeta) \) solve (19).

Now we take \( \Phi_\varepsilon = \varphi(z) \in C_0^\infty(\mathbb{R}^d) \) as a test function in (23), and passing to the limit get the weak formulation of the effective problem (17):

\[
\int_{\mathbb{R}^d} \int_{\mathbb{T}^d} (a_{ijmn}(0, \zeta) \partial_m N_{kl}(\zeta) + a_{ijkl}(0, \zeta)) \partial_{ij} \varphi(z) \, d\zeta \, dz \\
+ \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} b_{ij}(0, \zeta) \partial_i \partial_j V(z) \partial_i \varphi(z) \, dz + \frac{1}{2} \int_{\mathbb{R}^d} \partial_i (Hz \cdot z) V(z) \varphi(z) \, dz \\
+ \mu \int_{\mathbb{R}^d} V(z) \varphi(z) \, dz = \int_{\mathbb{R}^d} f(z) \varphi(z) \, dz,
\]

for any \( \varphi \in C_0^\infty(\mathbb{R}^d) \). Lemma 2.7 is proved. \( \square \)

**Lemma 2.8.** The Green operator of (12) converges uniformly, in \( \mathcal{L}(L^2(\mathbb{R}^d)) \), to the Green operator of (17), as \( \varepsilon \to 0 \).

**Proof.** Let \( \Phi_\varepsilon \in L^2(\mathbb{R}^d) \) be a maximizing sequence for \( \sup_{\|f\|_{L^2(\mathbb{R}^d)} = 1} \| (G^\varepsilon - G) f \|_{L^2(\mathbb{R}^d)} \).

By compactness there is a subsequence \( \Phi_\varepsilon \) weakly converging to some \( f \) in \( L^2(\mathbb{R}^d) \).

By Lemma 2.7 \( G_{\varepsilon} f_\varepsilon \to G f \) strongly in \( L^2(\mathbb{R}^d) \), and by the compactness of \( G, Gf_\varepsilon \to Gf \) strongly in \( L^2(\mathbb{R}^d) \). Hence

\[
\| G_{\varepsilon} - G \| \leq \| G_{\varepsilon} f_\varepsilon - G f \|_{L^2(\mathbb{R}^d)} + \| G f_\varepsilon - G f \|_{L^2(\mathbb{R}^d)} + o(1),
\]

as \( \varepsilon \to 0 \), and the convergence along a subsequence follows. Since the limit \( G_{\varepsilon} f_\varepsilon \) is unique by Lemma 2.7, the whole sequence converges. \( \square \)
Due to the Lemma 2.8, the sequence of the Green operators of the rescaled problem (12) converges uniformly to the Green operator of the effective problem (17). Lemma 2.6 applied to the Green operators ensures the convergence of spectrum of the rescaled problem (11).

Lemma 2.9. Let \((\eta^\varepsilon_k, v^\varepsilon_k)\) be kth eigenpair of the rescaled spectral problem (11). Then under the assumptions (H1)–(H4): As \(\varepsilon \to 0\),

(a) \(\eta^\varepsilon_k \to \eta_k\), where \(\eta_k\) is the kth eigenvalue of the effective problem (20),

(b) along a subsequence \(v^\varepsilon_k\) converges weakly in \(H^2(\mathbb{R}^d)\) and strongly in \(L^2(\mathbb{R}^d)\) to \(v_k\), where \(v_k\) is the eigenfunction corresponding to \(\eta_k\) under a proper orthonormalisation.

The last lemma combined with (10) yields Theorem 2.1.

In the next sections we consider the cases when \((\alpha, \beta)\) belong to \(R_1, R_3, R_4, R_5\).

3. The case \((\alpha, \beta) \in R_1\)

Recall that

\[ R_1 = \{ (\alpha, \beta) : 0 \leq \alpha < 1, \ 3\alpha < \beta < 3 \}. \]

Theorem 3.1. Let \((\alpha, \beta) \in R_1\) and let \((\lambda^\varepsilon_k, u^\varepsilon_k)\) be the kth eigenpair of (1) normalized by \(\|u^\varepsilon_k\|^2_{L^2(\Omega)} = \varepsilon^{d\beta/6}\). Suppose that the conditions (H1)–(H4) are satisfied. Then we have the following representation:

\[ u^\varepsilon_k = v^\varepsilon_k \left( \frac{x}{\varepsilon^{\beta/6}} \right), \quad \lambda^\varepsilon_k = \frac{\bar{c}(0)}{\varepsilon^\beta} + \frac{\eta^\varepsilon_k}{\varepsilon^{2\beta/3}}, \]

where \((\eta^\varepsilon_k, v^\varepsilon_k)\) are such that as \(\varepsilon \to 0\),

(i) \(\eta^\varepsilon_k \to \eta_k\),

(ii) up to a subsequence, \(v^\varepsilon_k\) converges to \(v_k\) weakly in \(H^2(\mathbb{R}^d)\) and strongly in \(L^2(\mathbb{R}^d)\),

where \(\eta_k\) is the kth eigenvalue, and \(v_k\) is an eigenfunction corresponding to \(\eta_k\) normalized by \(\|v_k\|^2_{L^2(\mathbb{R}^d)} = 1\), of the uniformly elliptic effective spectral problem

\[ \partial_{ij}(a_{ijkl}^\text{eff} \partial_{kl} v) + \frac{1}{2}(Hz \cdot z)v = \eta v, \quad z \in \mathbb{R}^d, \]

with \(a_{ijkl}^\text{eff}\) defined by (18), and \(H = \nabla \nabla \bar{c}(0)\).

Proof. We shift the spectrum by \(\bar{c}(0)/\varepsilon^\beta\) and make the following change of variables:

\[ \gamma = \frac{\beta}{6}, \quad z = \frac{x}{\varepsilon^\gamma}, \quad v(z) = u(\varepsilon^\gamma z), \quad \eta^\varepsilon = \varepsilon^{2\beta/3}\left( \lambda^\varepsilon - \frac{\bar{c}(0)}{\varepsilon^\beta} \right), \quad z \in \Omega_\varepsilon = \varepsilon^{-\gamma} \Omega. \]

Then we obtain the rescaled problem

\[
\begin{cases}
\hat{A}^\varepsilon v^\varepsilon = \eta^\varepsilon v^\varepsilon, & z \in \Omega_\varepsilon, \\
v^\varepsilon = \nabla v^\varepsilon \cdot n = 0, & z \in \partial \Omega_\varepsilon,
\end{cases}
\]

where

\[
\hat{A}^\varepsilon v^\varepsilon = \partial_{ij}(\hat{a}_{ijkl}^\varepsilon \partial_{kl} v^\varepsilon) - \varepsilon^{-\alpha+\beta/3} \partial_l(\hat{b}_{ij}^\varepsilon \partial_j v^\varepsilon) + \varepsilon^{-\beta/3}(\hat{c}^\varepsilon - \bar{c}(0))v^\varepsilon,
\]

(25)
a compact, self-adjoint and positive operator on \(L^2\).

Thus, up to a subsequence, sequence of solutions \(V\).

By (H1)–(H4), for all sufficiently small \(\varepsilon > 0\), the boundary value problem

\[
\eta = c, \quad \xi = c \quad \text{in } (0, 1),
\]

is coercive on \(R^d\), W(z, ζ) ∈ \(L^2(R^d; H^2(T^d))\). Passing to the limit in the variational formulation of (26) we find that \(V \in H^2(R^d) \cap L^2(R^d, |z|^2dz)\) is the unique solution to the equation

\[
A_1^{\text{eff}} V + \mu V = f, \quad z \in R^d,
\]

where

\[
A_1^{\text{eff}} = \partial_{ij}(a_{ijkl}^{\text{eff}} \partial_{kl} V) + \frac{1}{2}(Hz \cdot z)V,
\]

and \(a_{ijkl}^{\text{eff}}\) is coercive on \(R^{d \times d}\) and given by (18). Note that the second order term vanishes because of the hypothesis \(-\alpha + \beta/3 > 0\) and the boundedness of \(\nabla V\). The Green operator of (27), as an operator on \(L^2(R^d)\), is well-defined, is compact, self-adjoint, and positive. Due to the uniqueness of the solution to (27), the whole sequence \(V\) converges to \(V\).

In this way the Green operator of (26) converges uniformly to the Green operator of (27), as \(\varepsilon \to 0\). By Lemma 2.6, the spectrum of (24) converges to the spectrum of the limit operator (28) in the sense of Kuratowsky convergence of subsets of \(R\). Changing back the variables yields the desired result. ☐

4. The cases \((\alpha, \beta) \in R_3, R_4, R_5\)

Recall that

\[
R_3 = \{(\alpha, \beta) : 0 < \alpha < 2, \ \alpha < \beta < 3\alpha, \ \beta < \alpha + 2\},
\]

\[
R_4 = \{(\alpha, \beta) : \alpha = 2, \ 2 < \beta < 4\},
\]

\[
R_5 = \{(\alpha, \beta) : 2 < \alpha < 4, \ \alpha < \beta < 4\}.
\]

For these regions, the limit problems are of second order and have the same form, but the effective coefficients and the corresponding cell problems are different. Note that we should assume the coerciveness of the matrix \(b(x, y)\) so that the effective problems are well-posed. We gather the results for these cases in the following theorem.
Let the effective coefficients be defined by
\[
b_{ij}^{\text{eff}} = \begin{cases} 
\int_{\mathcal{D}} b_{ij}(0, y) \, dy, & (\alpha, \beta) \in R_3, \\
\int_{\mathcal{D}} b_{ik}(0, y) (\delta_{kj} + \partial_k M_j) \, dy, & (\alpha, \beta) \in R_4, \\
\int_{\mathcal{D}} b_{ik}(0, y) (\delta_{kj} + \partial_k N_j) \, dy, & (\alpha, \beta) \in R_5.
\end{cases}
\]  

(29)

where $M_n \in H^2(\mathbb{T}^d)/\mathbb{R}$ and $N_n \in H^1(\mathbb{T}^d)/\mathbb{R}$ are the unique solutions to the respective cell problems
\[
\partial_{y_i y_j} (a_{ijkl}(0, y) \partial_{y_k y_l} M_n) - \partial_{y_i} (b_{ij}(0, y) \partial_{y_j} M_n) = \partial_{y_i} b_{ni}(0, y), \quad y \in \mathbb{T}^d; 
\]
\[
- \partial_{y_i} (b_{ij}(0, y) \partial_{y_j} N_n) = \partial_{y_i} b_{ni}(0, y), \quad y \in \mathbb{T}^d.
\]  

(30)

(31)

Theorem 4.1. Let $(\alpha, \beta) \in R_3 \cup R_4 \cup R_5$ and let $(\lambda_k^{\varepsilon}, v_k^{\varepsilon})$ be the $k$th eigenpair of (1) normalized by $\|u_k^{\varepsilon}\|_{L^2}^2 = \varepsilon^d (\beta - \alpha)/4$. Suppose that the conditions (H1)–(H5) are satisfied. Then the following representation holds:
\[
u_k^{\varepsilon}(x) = v_k^{\varepsilon} \left( \frac{x - x_0}{\varepsilon^{\gamma}} \right), \quad \lambda_k^{\varepsilon} = \frac{c(0)}{\varepsilon^{\beta}} + \frac{\eta_k^{\varepsilon}}{\varepsilon^{\alpha}},
\]

where $(\eta_k^{\varepsilon}, v_k^{\varepsilon})$ are such that as $\varepsilon \to 0$,

(i) $\eta_k^{\varepsilon} \to \eta_k$,

(ii) up to a subsequence, $v_k^{\varepsilon}$ converges to $v_k$ weakly in $H^1(\mathbb{R}^d)$ and strongly in $L^2(\mathbb{R}^d)$,

where $\eta_k$ is the $k$th eigenvalue, and $v_k$ is an eigenfunction corresponding to $\eta_k$ normalized by $\|v_k\|_{L^2} = 1$, of the harmonic oscillator problem
\[-\partial_i (b_{ij}^{\text{eff}} \partial_j v) + \frac{1}{2} (H z \cdot z) v = \eta v, \quad z \in \mathbb{R}^d,\]

(32)

with $b_{ij}^{\text{eff}}$ defined by (29), and $H = \nabla \nabla c(0)$.

Proof. We shift the spectrum by $c(0)/\varepsilon^\beta$ and make the following change of variables:
\[
\gamma = \frac{\beta - \alpha}{4}, \quad z = \frac{x}{\varepsilon^{\gamma}}, \quad v(z) = u(\varepsilon^\gamma z), \quad \eta^{\varepsilon} = \frac{\varepsilon^{\alpha+\beta}}{2} \left( \lambda - \frac{c(0)}{\varepsilon^\beta} \right), \quad z \in \Omega_{\varepsilon} = \varepsilon^{-\gamma} \Omega.
\]

Then we obtain the rescaled problem
\[
\begin{cases}
\hat{A}_2^{\varepsilon} \nu^{\varepsilon} = \eta^{\varepsilon} \nu^{\varepsilon}, & z \in \Omega_{\varepsilon}, \\
\nu^{\varepsilon} = \nabla \nu^{\varepsilon} \cdot n = 0, & z \in \partial \Omega_{\varepsilon},
\end{cases}
\]

(33)

where
\[
\hat{A}_2^{\varepsilon} v^{\varepsilon} = \varepsilon^{\alpha - 2 \gamma} \partial_{ij} (\hat{a}^{\varepsilon}_{ijkl} \partial_{kl} v^{\varepsilon}) - \partial_i (\hat{b}^{\varepsilon}_{ij} \partial_j v^{\varepsilon}) + \varepsilon^{\alpha - \beta + 2 \gamma} (\hat{c}^{\varepsilon} - c(0)) v^{\varepsilon},
\]

(34)

and
\[
\hat{a}^{\varepsilon}_{ijkl} = a_{ijkl}(\varepsilon^\gamma z, \frac{z}{\varepsilon^1 - \gamma}), \quad \hat{b}^{\varepsilon}_{ij} = b_{ij}(\varepsilon^\gamma z, \frac{z}{\varepsilon^1 - \gamma}), \quad \hat{c}^{\varepsilon} = c(\varepsilon^\gamma z, \frac{z}{\varepsilon^1 - \gamma}).
\]

As above, to describe the asymptotic behavior of the eigenpairs $(\eta_k^{\varepsilon}, v_k^{\varepsilon})$, as $\varepsilon \to 0$, we prove the uniform convergence of the corresponding Green operators and then use Lemma [2.6]
Let $\mu > 0$ and $f_\varepsilon$ be a sequence converging weakly in $L^2(\mathbb{R}^d)$ to $f$. Consider the boundary value problem
\begin{equation}
\begin{cases}
\hat{A}_\varepsilon V^\varepsilon + \mu V^\varepsilon = f_\varepsilon, & z \in \Omega_\varepsilon, \\
V^\varepsilon = \nabla V^\varepsilon \cdot n = 0, & z \in \partial \Omega_\varepsilon.
\end{cases}
\tag{35}
\end{equation}
By (H1)–(H5), for all sufficiently small $\varepsilon > 0$, the Green operator of $\hat{A}_\varepsilon$ is a compact, self-adjoint and positive operator in $L^2(\mathbb{R}^d)$. Moreover, for the sequence of solutions $V^\varepsilon$ to $\hat{A}_\varepsilon$ we have
\begin{equation}
\varepsilon^\frac{3\alpha-\beta}{4} \| \nabla V^\varepsilon \|_{2,\mathbb{R}^d} + \| \nabla V^\varepsilon \|_{2,\mathbb{R}^d} + \| V^\varepsilon \|_{2,\mathbb{R}^d} + \| |z| V^\varepsilon \|_{2,\mathbb{R}^d} \leq C.
\end{equation}
We proceed by dividing into the cases:
\begin{equation}
\frac{3\alpha-\beta}{4} \begin{cases}
< 1 - \gamma, & (\alpha, \beta) \in R_3, \\
= 1 - \gamma, & (\alpha, \beta) \in R_4, \\
> 1 - \gamma, & (\alpha, \beta) \in R_5,
\end{cases}
\end{equation}
where $1 - \gamma$ will be the scale in the two-scale convergence.

4.1. $(\alpha, \beta) \in R_3$. Up to a subsequence,
\begin{equation}
V^\varepsilon \to V(z), \quad \nabla V^\varepsilon \xrightarrow{2} \nabla V(z), \quad \varepsilon^\frac{3\alpha-\beta}{4} \nabla V^\varepsilon \xrightarrow{2} W(z, \zeta),
\end{equation}
where $V \in H^1(\mathbb{R}^d)$, $W(z, \zeta) \in L^2(\mathbb{R}^d; H^2(\mathbb{T}^d))$. Passing to the limit in the variational formulation of $\hat{A}_\varepsilon$ we find that $V \in H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, |z|^2dz)$ is the unique solution to the equation
\begin{equation}
-\partial_t (b_{ij}^{\text{eff}} \partial_j V) + \frac{1}{2} (Hz \cdot z)V + \mu V = f, \quad z \in \mathbb{R}^d,
\end{equation}
where $b_{ij}^{\text{eff}} = \tilde{b}_{ij}(0) = \int_{\mathbb{T}^d} b_{ij}(0, y) dy$ is coercive on $\mathbb{R}^d$ by (H5).

4.2. $(\alpha, \beta) \in R_4$. In this case $\frac{3\alpha-\beta}{4} = 1 - \gamma$. Up to a subsequence,
\begin{equation}
V^\varepsilon \to V(z), \quad \nabla V^\varepsilon \xrightarrow{2} \nabla V(z) + \nabla \zeta W(z, \zeta), \quad \varepsilon^\frac{3\alpha-\beta}{4} \nabla V^\varepsilon \xrightarrow{2} \nabla \zeta \nabla \zeta W(z, \zeta),
\end{equation}
in $L^2(\mathbb{R}^d)$, where $V \in H^1(\mathbb{R}^d)$, $W(z, \zeta) \in L^2(\mathbb{R}^d; H^2(\mathbb{T}^d))$. Passing to the limit in the variational formulation of $\hat{A}_\varepsilon$ we find that $V \in H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, |z|^2dz)$ is the unique solution to the same equation $\hat{A}_\varepsilon$ with
\begin{equation}
b_{ij}^{\text{eff}} = \int_{\mathbb{T}^d} b_{ik}(0, y)(\delta_{kj} + \partial_k M_j) dy,
\end{equation}
where $M_j \in H^2(\mathbb{T}^d)/\mathbb{R}$ solves $\hat{A}_\varepsilon$. By the periodicity of $b(x, y)$, $M_j$ is well-defined.

Lemma 4.2. Under the assumptions (H1)–(H3) and (H5), $b_{ij}^{\text{eff}}$ defined by (37) is coercive on $\mathbb{R}^d$.

Proof. By the definition of $b_{ij}^{\text{eff}}$,
\begin{equation}
b_{ij}^{\text{eff}} = \int_{\mathbb{T}^d} \delta_{ij} b_{kl}(0, y)(\delta_{kj} + \partial_k M_j) dy.
\end{equation}
By using $M_i$ as a test function in the equation $\hat{A}_\varepsilon$ for $M_j$ we have
\begin{equation}
\int_{\mathbb{T}^d} b_{kl}(0, y) \partial_{y_l} M_i(\delta_{kj} + \partial_k M_j) dy = -\int_{\mathbb{T}^d} a_{pqr}(0, y) \partial_{y_p} M_j \partial_{y_q} M_i dy.
\end{equation}
Therefore,
\[
b_{ij}^{\text{eff}} = \int_{T^d} b_{kl}(0, y)(\delta_{li} + \partial_{y_i} M_l)(\delta_{kj} + \partial_{y_j} M_j) \, dy \\
+ \int_{T^d} a_{pqrs}(0, y)(\partial_{y_p} y_s, M_j)(\partial_{y_q} y_r, M_l) \, dy,
\]
which shows that \( b_{ij}^{\text{eff}} \) is symmetric. Moreover, for \( \xi \in \mathbb{R}^d \), by the last equation, (H3), and (H5), we have
\[
b_{ij}^{\text{eff}} \xi_i \xi_j \geq C \sum_l \int_{T^d} |\xi_i (\delta_{li} + \partial_{y_i} M_l)|^2 \, dy,
\]
which shows that \( b_{ij}^{\text{eff}} \) is nonnegative definite. If \( \xi \in \mathbb{R}^d \) is such that \( b_{ij}^{\text{eff}} \xi_i \xi_j = 0 \) we have by the last inequality that \( |\xi_i (\delta_{li} + \partial_{y_i} M_l)| = 0 \) for all \( l \). In particular, \( \nabla_y (\xi_i (\delta_{li} + \partial_{y_i} M_l)) = 0 \) which by the periodicity of \( M_i \) is only possible if \( \xi = 0 \). Thus \( b_{ij}^{\text{eff}} \) is positive definite on \( \mathbb{R}^d \). By the compactness of the unit ball in \( \mathbb{R}^d \), \( b_{ij}^{\text{eff}} \) is coercive on \( \mathbb{R}^d \). \( \square \)

4.3. \((\alpha, \beta) \in R_5\). Up to a subsequence,
\[
V^\varepsilon \rightarrow V(z), \quad \nabla V^\varepsilon \xrightarrow{\ast} \nabla V(z) + \nabla \xi \partial_j V(z), \quad \varepsilon^{\alpha - \beta} \nabla \nabla V^\varepsilon \xrightarrow{\ast} W_1(z, \xi),
\]
in \( L^2(\mathbb{R}^d) \), where \( V \in H^1(\mathbb{R}^d) \), \( W_1(z, \xi) \in L^2(\mathbb{R}^d \times T^d) \), and \( N_j \in H^1(T^d) / \mathbb{R} \) solves (31). Passing to the limit in the variational formulation of (35) we find that \( V \in H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, |z|^2 \, dz) \) is the unique solution to (36) with
\[
b_{ij}^{\text{eff}} = \int_{T^d} b_{kl}(0, y)(\delta_{kl} + \partial_{y_k} N_l) \, dy.
\]
By the similar argument used in Lemma 4.2, \( b_{ij}^{\text{eff}} \) is coercive on \( \mathbb{R}^d \).

In this way, in all the three cases, the Green operator of (35) converges uniformly to the Green operator of (36), as \( \varepsilon \to 0 \). By Lemma 2.6, the spectrum of (33) converges to the spectrum of the limit operator (32). Changing back the variables yields the desired result. \( \square \)

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