THE HORN PROBLEM AND PLANAR NETWORKS

ANTON ALEKSEEV, MASHA PODKOPAEVA, AND ANDRAS SZENES

Abstract. The problem of determining the set of possible eigenvalues of 3 Hermitian matrices that sum up to zero is known as the Horn problem. The answer is a polyhedral cone, which, following Knutson and Tao, can be described as the projection of a simpler cone in the space of triangular tableaux (or hives) to the boundary nodes of the tableau.

In this paper, we introduce a combinatorial problem defined in terms of certain weighted planar graphs giving rise to exactly the same polyhedral cone. In our framework, the values at the inner nodes of the triangular tableaux receive a natural interpretation. Other problems of linear algebra fit into the same scheme, among them the Gelfand–Zeitlin problem. Our approach is motivated by the works of Fomin and Zelevinsky on total positivity and by the ideas of tropicalization.

1. Introduction

This article is motivated by the following classical problem of linear algebra: under which conditions do three $n$-tuples of ordered real numbers $\lambda_1 \geq \cdots \geq \lambda_n$, $\mu_1 \geq \cdots \geq \mu_n$ and $\nu_1 \geq \cdots \geq \nu_n$ serve as the sets of eigenvalues of three Hermitian $n$-by-$n$ matrices, $A$, $B$, and $C$, related by the equality $C = A + B$?

This problem has a long history (see [8, 5] for background). The first nontrivial necessary condition, $\nu_1 \leq \lambda_1 + \mu_1$, was already known in the nineteenth century. Beginning with the work of Weyl in 1912 [16], different sets of inequalities of this type were found. Finally, in 1962, Horn put forward a very complex, recursively defined set of conditions of the form

$$
\sum_{k \in K} \nu_k \leq \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j,
$$

where $I, J, K \subset \{1, 2, \ldots, n\}$, which, together with the obvious equality $\text{Tr}(A) + \text{Tr}(B) = \text{Tr}(C)$, he conjectured to be necessary and sufficient.

Horn’s conjecture was proved by Klyachko [13] and by Knutson and Tao [14], [15]. Since then several other proofs were given (e.g. [12]), and the results were extended to other problems of similar type (cf. [1], [3], [4]).

In this paper, we will use the “hive model” of Horn’s polyhedral cone due to Knutson and Tao (see [14, 6]). Let $T_n$ be the $n$th order regular triangulation of the equilateral triangle. For real $n$-tuples $\lambda_1 \geq \cdots \geq \lambda_n$, $\mu_1 \geq \cdots \geq \mu_n$, and $\nu_1 \geq \cdots \geq \nu_n$ satisfying

$$
\sum_i \lambda_i + \sum_j \mu_j = \sum_k \nu_k,
$$

we associate real numbers to the boundary nodes of $T_n$ in the way demonstrated in Figure 1 for the case of $n = 3$ (see Section 2 for details).
The Knutson–Tao theorem states that the three $n$-tuples may be realized as ordered sets of eigenvalues of Hermitian matrices $A$, $B$, and $C = A + B$ if and only if they satisfy the **hive condition**:

*There exists a concave function $f$ defined on the equilateral triangle, linear on each small triangle of the triangulation, and whose values at the boundary nodes of $T_n$ coincide with the values we ascribed to these nodes above.*

Naturally, such a function $f$ is uniquely determined by its values on the nodes of $T_n$. The condition of concavity translates into a set of inequalities parametrized by the internal edges, or, equivalently, by elementary rhombi of the triangulation. An example is the inequality

\[(2) \quad x \leq \lambda_1 + \nu_1,\]

where $x$ is the value of $f$ at the central node in Figure 1.

In the present paper motivated by constructions in the theory of cluster algebras and total positivity (see [7]), we introduce a combinatorial framework where the inequalities of the hive model arise in a natural way. Instead of Hermitian matrices, we consider certain oriented planar graphs, called *planar networks*, whose edges are weighted by real (or tropical) numbers. An example of a weighted planar network is shown in Figure 2.

\[\text{Figure 2. A planar network. The maximal path is shown in thick red.}\]

In this setup, the eigenvalues of a matrix correspond to certain piecewise linear functions of the weights. The analogue of the top eigenvalue $\lambda_1$, for example, will be the maximum of the weights of oriented paths extending from the left to the right end of the network (where the weight of a path is the sum of weights of the edges contained in the path). For the weighted network in Figure 2 we obtain
\[ \lambda_1 = 1 + 1 + 2 + 0 = 4. \] The definition of the other \( \lambda \)'s is given in Section 3; by analogy, we will call these quantities the *eigenvalues* of the weighted network.

The addition of matrices is replaced by concatenation of planar networks. In Figure 3 we give an example of a pair of networks \( \Gamma \) and \( \Delta \) concatenated to form the network \( \Gamma \circ \Delta \). Providing \( \Gamma \) and \( \Delta \) with weights will then allow us to fill in the boundary values of the tableau in Figure 1: the value of \( \nu_1 \), for example, will be the maximum of the weights of paths extending from left to right, while \( \lambda_1 \) will be the maximum taken over paths extending from left to the middle line \( L \).

Figure 3. Concatenation of networks \( \Gamma \) and \( \Delta \). A path is shown in thick red.

Now we can formulate our first correspondence result, Theorem 4: for any pair of weighted planar networks \( \Gamma \) and \( \Delta \), the eigenvalues of \( \Gamma \), \( \Delta \), and \( \Gamma \circ \Delta \) satisfy the hive condition and the trace condition (1).

A key feature of our construction is that there is a natural explicit definition of the values of the function \( f \) at the internal nodes of \( T_n \), and this makes the proofs rather straightforward. For example, the value \( x \) of \( f \) assigned to the middle node in Figure 1 is the maximal sum of the weights of two disjoint oriented paths in \( \Gamma \circ \Delta \), one of which extends from left to right, while the other one from left to the middle line \( L \). An example of such a pair is shown in Figure 3. Note that inequality (2) is now obvious: on the right-hand side of (2), the maximum is taken over all pairs of paths, one going from left to right and the other one from left to the middle, while on the left-hand side of (2), the maximum is taken only over a subset of such pairs, namely, over all pair of disjoint paths.

A natural question is to compare the set defined by these eigenvalue analogs with the set of eigenvalues of triples of Hermitian matrices. Theorem 5 states that any triple of ordered \( n \)-tuples satisfying the hive condition is the triple of the sets of eigenvalues of \( \Gamma \), \( \Delta \), and \( \Gamma \circ \Delta \) for some weighted networks \( \Gamma \) and \( \Delta \). For \( n = 3 \), the graphs underlying \( \Gamma \) and \( \Delta \) can be chosen as shown in Figure 3.

We begin our paper with a planar network interpretation of the precursor of the Horn problem: the interlacing inequalities for eigenvalues of a Hermitian matrix and its principal submatrices (see [10]). These latter inequalities play a prominent role in the description of the Gelfand–Zeitlin integrable system (see [9]). Then we proceed to prove our main results, Theorems 4 and 5.

One can regard the eigenvalue problem for planar networks as a rather non-trivial tropicalization of the eigenvalue problem for Hermitian matrices. From this perspective, it is natural to expect but not very easy to prove that both problems are governed by the same set of inequalities. In the forthcoming paper [2], we will prove this “detrropicalization” correspondence principle for the Horn problem, thus providing a new proof of the theorem of Knutson and Tao.
Acknowledgements. We are grateful to S. Fomin, A. Knutson, E. Meinrenken, M. Vergne, J. Weitsmann, C. Woodward, and A. Zelevinsky for inspiring discussions and comments.

2. Gelfand–Zeitlin and Horn problems

In this section, we recall two classical problems of linear algebra: the Gelfand–Zeitlin and Horn problems.

2.1. The Gelfand–Zeitlin problem. Let $\mathcal{H}_n$ be the set of $n$-by-$n$ Hermitian matrices. For $A \in \mathcal{H}_n$ and $1 \leq k \leq n$, denote by $A^{(k)}$ the principal submatrix of $A$ of size $k$, i.e., the $k$-by-$k$ submatrix sitting in the upper left corner of $A$. Let $(\lambda^{(k)}_1 \geq \cdots \geq \lambda^{(k)}_k)$ be the sequence of ordered eigenvalues of $A^{(k)}$; this way we obtain a set of functions

$$\lambda^{(k)}_i : \mathcal{H}_n \to \mathbb{R}, \quad 0 < i \leq k \leq n.$$  

It is a classical result of linear algebra (see, e.g., [10]) that these functions satisfy the following interlacing inequalities:

$$\lambda^{(k+1)}_i \geq \lambda^{(k)}_i \geq \lambda^{(k+1)}_{i+1}, \quad 0 < i < n.$$  

The converse is also true: any set of numbers satisfying the interlacing inequalities appears as the eigenvalues of a Hermitian matrix and its principal submatrices.

One can recast inequalities (4) in the following form. Let $T = T_n$ be a regular triangulation of the equilateral triangle (Figure 4) with the set of nodes $V_T$ parametrized by the indices $0 \leq i \leq k \leq n$; thus we have $|V_T| = (n+1)(n+2)/2$. To each node of the triangulation, we associate a coordinate variable $t^k_i : \mathbb{R}^{|V_T|} \to \mathbb{R}$.

![Figure 4. Triangulation $T_n$.](image)

Denote by $\mathcal{ET}$ the set of horizontal edges of the triangulation $T$ parameterized by pairs $(i, k)$ satisfying $0 < i \leq k \leq n$; again, to each horizontal edge, we associate coordinate functions on $\mathbb{R}^{|\mathcal{ET}|}$, $h^{(k)}_i : \mathbb{R}^{|\mathcal{ET}|} \to \mathbb{R}, \quad 0 < i \leq k \leq n$. Note that this index set coincides with the index set of the eigenvalues of the principal submatrices of an Hermitian matrix [11].
Definition 1. The cone $C_2 \subset \mathbb{R}^{VT}$ is the polyhedral cone defined by the system of inequalities
\begin{align*}
t_i^{k+1} + t_i^{k-1} & \geq t_i^{k+1} + t_i^k, \\
t_i^{k+1} + t_i^k & \geq t_i^{k+1} + t_i^{k-1},
\end{align*}
for $0 < i \leq k < n$.

The horizontal boundary map is the map
\[ \tilde{\partial} : \mathbb{R}^{VT} \to \mathbb{R}^{\bar{ET}} : \{t_i^k\}_{0 < i \leq k \leq n} \mapsto \{h_i^{(k)} = t_i^k - t_i^{k-1}\}_{0 < i \leq k \leq n}. \]

Note that inequalities \((5)\) are parametrized by the internal non-horizontal edges of the triangulation $T$, or, alternatively, by rhombi of two types having these edges as short diagonals (see Figure 5): for each rhombus, the corresponding inequality states that the sum of the two numbers assigned to the endpoints of the short diagonal is greater than or equal to the sum of the two numbers assigned to the endpoints of the long diagonal.

![Figure 5. The two types of rhombi.](image)

Observe that
- the set $C_2$ and the linear map $\tilde{\partial}$ are invariant under the transformations $t_i^k \mapsto t_i^k + c_k$ for $(c_k, \ldots, c_0) \in \mathbb{R}^{k+1},$
- the linear map $\tilde{\partial}$ establishes a linear isomorphism between $\{t_i^k = 0\mid k = 0, \ldots, n\} \subset \mathbb{R}^{VT}$ and $\mathbb{R}^{\bar{ET}}$.

Then we have the following characterization of the interlacing inequalities.

Proposition 1. Let $C_{GZ} \subset \mathbb{R}^{\bar{ET}}$ be the cone defined by the interlacing inequalities \((4)\). Then
\[ C_{GZ} = \tilde{\partial}(C_2). \]
In fact, the linear map $\tilde{\partial}$ establishes an isomorphism of polyhedral cones:
\begin{equation}
\tilde{\partial} : C_2 \cap \{t_i^k = 0\mid k = 0, \ldots, n\} \to C_{GZ}.
\end{equation}

Proof. Let $t = \{t_i^k\}_{i,k}$ be in $C_2$. Then we can rewrite inequalities \((5)\) as
\begin{align*}
t_i^{k+1} - t_i^{k-1} & \geq t_i^{k+1} - t_i^k, \\
t_i^k - t_i^{k-1} & \geq t_{i+1}^{k+1} - t_i^{k+1}.
\end{align*}
Therefore, the image $h = \{h_i^{(k)}\}_{i,k}$ of $t$ under the map $\tilde{\partial}$ satisfies the condition
\[ h_i^{k+1} \geq h_i^k \geq h_{i+1}^{k+1}, \]
i.e., the interlacing inequalities.

Conversely, given a point $h = \{h_i^{(k)}\}_{i,k}$ in $C_{GZ}$, define $t_i^k = h_i^k + \cdots + h_i^{k+1}$ for all $i$ and $k$. Then $h = \partial t$, and the interlacing inequalities for $h_i^k$ imply precisely the inequalities \((5)\) for $t_i^k$. \qed
2.2. The Horn problem. The Horn problem is related to the eigenvalues of triples of Hermitian matrices that add up to zero. More formally, consider the eigenvalue map

\[ \Lambda^3 : \mathcal{H}_n \times \mathcal{H}_n \times \mathcal{H}_n \to \mathbb{R}^{3n} \]

listing the eigenvalues of a triple of Hermitian matrices of rank \( n \), where the eigenvalues of each matrix are listed in decreasing order. The Horn cone is defined as the image

\[ \mathcal{C}_{\text{Horn}} = \Lambda^3 \left( \left\{ (A, B, C) \in \mathcal{H}_n^3 \mid A + B = C \right\} \right) \subset \mathbb{R}^{3n}. \]

Note that \( A + B = C \) implies \( \text{Tr}(A) + \text{Tr}(B) = \text{Tr}(C) \) since the trace is a linear functional. For the Horn cone, this means that if

\[ (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_n) \in \mathcal{C}_{\text{Horn}}, \]

then

\[ \sum_{i=1}^n \lambda_i + \sum_{i=1}^n \mu_i = \sum_{i=1}^n \nu_i. \]

In [11], Horn gave a rather complicated set of inequalities that would conjecturally define \( \mathcal{C}_{\text{Horn}} \); in particular, he suggested that \( \mathcal{C}_{\text{Horn}} \) is a polyhedral cone inside the hyperplane given by Equation (7). His conjecture was proved more than 30 years later [13, 14]. In the present paper, we will use the following elegant description of \( \mathcal{C}_{\text{Horn}} \) due to Knutson and Tao [14].

Consider again the triangulation \( T = T_n \) and define the subcone \( \mathcal{C}_3 \subset \mathcal{C}_2 \) cut out from \( \mathcal{C}_2 \) by the inequalities corresponding to the third set of rhombi (see Figure 6).

![Figure 6. Third type of rhombi.](image)

More precisely, we define the polyhedral cone \( \mathcal{C}_3 \subset \mathbb{R}^{VT} \) by the following set of inequalities:

\[
\begin{align*}
t_{i+1}^k &+ t_{i-1}^{k+1} &\geq& n t_i^k + n t_{i-1}^{k-1} \\
t_{i+1}^k &+ t_i^{k+1} &\geq& t_{i-1}^{k} + t_i^{k-1} \\
t_i^k &+ t_{i-1}^{k-1} &\geq& t_{i+1}^k + t_{i-1}^{k-1}
\end{align*}
\]

for \( 0 < i \leq k < n \).

**Remark 1.** We note that \( \mathcal{C}_3 \) is invariant under the translations \( t_i^k \to t_i^k + c \) for \( c \in \mathbb{R} \).
Now we define the map \( \partial : \mathbb{R}^V T \to \mathbb{R}^{3n} \) by the formula
\[
\partial : \{ t_i^k, 0 \leq i \leq k \leq n \} \mapsto \{ \lambda_i = t_i^k - t_i^{k-1}, \mu_i = t_i^n - t_i^{n-1}, \nu_i = t_i^j - t_i^{j-1}, 1 \leq i \leq n \}.
\]
In fact, \( \partial \) is simply the cohomological boundary operator restricted to the outer edges of the triangulation \( T_n \). Then Horn’s conjecture may be formulated as follows.

**Theorem 1** (Knutson–Tao). \( \partial(C_3) = C_{\text{Horn}} \).

**Remark 2.** Since the operator \( \partial \) is invariant under the translation mentioned in Remark 1, we can normalize \( t_0^0 = 0 \), and then the theorem may be equivalently stated as follows:
\[
(9) \quad \partial(C_3 \cap \{ t_0^0 = 0 \}) = C_{\text{Horn}}.
\]

3. **Planar networks**

In this section, we introduce the notion of planar networks, the key tool for the rest of the paper. There are a number of possible definitions of planar networks. We chose the one that was the most convenient for our purposes, but the proofs could be adapted to the other definitions as well.

**Definition 2.** A planar network is the following data:
- a finite graph \( \Gamma \) with vertex set \( V \Gamma \) and edge set \( E \Gamma \),
- a pair of reals, \( a < b \),
- an embedding of \( \Gamma \) into the strip \( \{ a \leq x \leq b \} \subset \mathbb{R}^2 \) such that the image of each edge is a segment of a straight line, which is not parallel to the \( y \)-axis.

We will call the vertices on the line \( \{ x = a \} \) sources and the vertices on the line \( \{ x = b \} \) sinks of \( \Gamma \); the other vertices will be called internal.

Observations:
- A subnetwork of a planar network \( \Gamma \) is naturally defined as a subgraph of \( \Gamma \), with the rest of the data unchanged. The sources and sinks of the subnetwork thus have to be subsets of the sources and sinks of \( \Gamma \).
- A planar network \( \Gamma \) is naturally oriented (from left to right), and we will use this orientation in what follows.
- Using this orientation, one can characterize the vertices of \( \Gamma \) by the number of incoming and outgoing edges. A source, for example, is always a vertex of degree \((0, d)\) for some nonnegative integer \( d \).

A crucial role in our analysis will be played by paths.

**Definition 3.** A multipath in \( \Gamma \) is a subnetwork whose every vertex that is an internal vertex of \( \Gamma \) is of degree \((1, 1)\).

A multipath has the same number of sources and sinks; a multipath with \( k \) sources is called a \( k \)-path. Each \( k \)-path is simply the union of \( k \) disjoint paths of \( \Gamma \) connecting a source with a sink. The set of \( k \)-paths in \( \Gamma \) will be denoted by \( P_k \Gamma \).

**Definition 4.** Let \( \mathbb{T} = \mathbb{R} \cup \{ -\infty \} \) be the semifield of tropical numbers. A weighting of a planar network \( \Gamma \) is an assignment of a tropical number to each edge of \( \Gamma \). Identifying the set of weightings of \( \Gamma \) with \( \mathbb{T}^{E \Gamma} \), we can introduce
- the coordinate function \( w_e : \mathbb{T}^{E \Gamma} \to \mathbb{T} \) for each edge \( e \in E \Gamma \), and
• the weight functional

\[ w_\alpha : \mathcal{T}^{\mathcal{E}_\Gamma} \to \mathbb{T}, \quad w_\alpha = \sum_{e \in E_\alpha} w_e \]

for each subgraph \( \alpha \) of \( \Gamma \), in particular, for each multipath in \( \Gamma \). When \( \alpha \) has no edges, we set \( w_\alpha = 0 \).

3.1. The maximum functionals. A certain collection of piecewise linear functions on the space \( \mathcal{T}^{\mathcal{E}_\Gamma} \) of weightings of a planar network \( \Gamma \) will play an important role in what follows. For a planar network \( \Gamma \), \( i > 0 \), and a weighting \( \epsilon \in \mathcal{T}^{\mathcal{E}_\Gamma} \), we define

\[ l_i \Gamma : \mathcal{T}^{\mathcal{E}_\Gamma} \to \mathbb{T} \]

by

\[ l_i \Gamma(\epsilon) = \max\{w_\alpha(\epsilon) | \alpha \in P_i \Gamma\} \]

if the set \( P_i \Gamma \) is nonempty; otherwise, we set \( l_i \Gamma = -\infty \). By definition, we put \( l_0 \Gamma = 0 \) and denote by \( \Gamma \) the \((n + 1)\)-tuple \((l_0 \Gamma, \ldots, l_n \Gamma)\).

Remark 3. If, for a weighting \( \epsilon \), we have \( l_i \Gamma(\epsilon) \in \mathbb{R} \) for all \( i \), then we can define the eigenvalues associated to \( \epsilon \) by the formula

\[ \lambda_i = l_i \Gamma(\epsilon) - l_{i-1} \Gamma(\epsilon), \]

so as \( l_i \Gamma(\epsilon) = \lambda_1 + \cdots + \lambda_i \).

Example 2. The simplest example is a planar network \( \Gamma \) that contains no edges (Figure 7). Then \( \mathcal{T}^{\mathcal{E}_\Gamma} \) is a single point, and the image of \( \Gamma \) is the point \((0, -\infty, \ldots, -\infty)\).

Example 3. The next example is a planar network with exactly \( n \) edges \( e_i \) connecting the vertices \((a, i)\) and \((b, i)\) (Figure 8). Denote the corresponding weights by \( w_i \) and let \((\varpi_1, \varpi_2, \ldots, \varpi_n)\) be the permutation of the \( n \)-tuple \((w_1, w_2, \ldots, w_n)\) such that \( \varpi_i \geq \varpi_{i+1} \) for all \( i = 1, \ldots, n-1 \). Then

\[ l_i \Gamma = \varpi_1 + \cdots + \varpi_i. \]

The image of \( \Gamma \) is the closure in \( \mathbb{T}^n \) of the polyhedral cone defined by the inequalities \( \varpi_i \geq \varpi_{i+1} \).

![Figure 7](image-url)

![Figure 8](image-url)

Lemma 4. (1) If \( \Gamma' \) is a subnetwork of \( \Gamma \), then \( \text{im}(\Gamma') \subset \text{im}(\Gamma) \).

(2) If the network \( \Gamma' \) can be obtained from \( \Gamma \) by insertion of a new vertex splitting an existing edge, then \( \text{im}(\Gamma') = \text{im}(\Gamma) \).
Proof. 1. Set the weights of all edges \( e \in E \setminus E' \) equal to \(-\infty\). The image of this subset of weightings under \( l^\Gamma \) coincides with the image of \( l^{\Gamma'} \).

2. Let \( s : T^{E'} \to T^{E \setminus E'} \) be the map assigning to the split edge the sum of the weights of the two edges obtained by the insertion of the new vertex. Then \( s \) preserves the functional \( l : l^{\Gamma}(se) = l^{\Gamma'}(e) \). This implies \( \text{im}(l^{\Gamma'}) = \text{im}(l^{\Gamma}) \). \( \square \)

Our main goal is to study the image of the piecewise linear map \( l^{\Gamma} \) for a planar network \( \Gamma \). We will need the following consequence of Lemma 4.

**Corollary 5.** By allowing embedded edges that are unions of intervals, we can always replace a planar network \( \Gamma \) by another planar network \( \Gamma' \) having no vertices of degree \((1,1)\) so that \( \text{im}(l^{\Gamma}) = \text{im}(l^{\Gamma'}) \).

### 4. Main results

In this section, we state the main results of the paper. We establish a correspondence principle between the functionals \( l^{\Gamma} \) for planar networks \( \Gamma \) and eigenvalues of Hermitian matrices. In particular, we show that the Gelfand–Zeitlin cone and the Horn cone appear as images of natural piecewise linear functions on the space of weightings of planar networks.

#### 4.1. Planar networks and Gelfand–Zeitlin

In this section, we will assume that the planar network \( \Gamma \) has precisely \( n \) sources and sinks. Without loss of generality, we can assume that the set of \( y \)-coordinates of the sources and sinks is the set of the first \( n \) integers \( \{1, 2, \ldots, n\} \). We will say that such a network \( \Gamma \) is a planar network of rank \( n \).

For a planar network of rank \( n \), we denote by \( \Gamma(k) \) the maximal subgraph of \( \Gamma \) that does not contain the sinks or sources with \( y \) coordinates above the line \( \{y = k\} \); these are the vertices 

\[(a, k + 1), (b, k + 1), \ldots, (a, n), (b, n)\].

Then \( \Gamma(k) \) is a planar network of rank \( k \).

The collection of maps \( l_i^{\Gamma(k)} \), \( 0 \leq i \leq k \leq n \), defines a map from the set of weightings of \( \Gamma \) to the set of triangular tableaux (Figure 4) filled by tropical numbers:

\[L^\Gamma : T^{E \setminus E'} \to T^{VT} \]

Note that each row of the tableau gives the map \( l^{\Gamma(k)} \) for the appropriate \( k \).

**Theorem 2.** Let \( \Gamma \) be a planar network of rank \( n \). Then

\[\text{im}(L^\Gamma) \subset \overline{C^2} \cap \{t_0^k = 0 \mid k = 0, \ldots, n\} \].

Here and below the closure is taken in \( T^N \).

The image of \( L^\Gamma \) depends on the planar network \( \Gamma \). There are networks, however, for which this image is maximal. Let \( \Gamma_0 = \Gamma_0[n] \) be the network in Figure 9.

**Remark 4.** Clearly, eliminating from a rank-\( n \) planar network \( \Gamma \) all vertices (with the adjacent edges) that cannot be reached from a source, we do not change the image of the functional \( L^\Gamma \). Combining this with Corollary 5 we see that we can replace the subnetwork \( \Gamma^{(k)}_0 \subset \Gamma_0 \) with the network \( \Gamma_0[k] \) (see Figure 10).

**Theorem 3.**

\[\text{im}(L^\Gamma_0) = \overline{C^2} \cap \{t_0^k = 0 \mid k = 0, \ldots, n\} \].
4.2. Planar networks and Horn. We will call two planar networks \((\Gamma, [a, b])\) and \((\Delta, [a', b'])\) composable if \(a' = b\) and the set of sources of \(\Delta\) is a subset of the set of sinks of \(\Gamma\). The network \((\Gamma \circ \Delta, [a, b'])\) is then defined in the obvious manner.

**Definition 5.** We say that a subnetwork of \(\Gamma \circ \Delta\) is a \(\Gamma \Delta\)-path if it is the union of the composable multipaths \(\Gamma\) and \(\Delta\). Such a subnetwork belongs to a set

\[
P^k_i \Gamma \Delta = \{ \alpha = \gamma \cup \delta | \gamma \in P_k \Gamma, \delta \in P_i \Delta \text{ a composable pair}\},
\]

for some \(k \geq i \geq 0\).

We have the following characterization of these subnetworks.

**Lemma 6.** A subnetwork \(\alpha\) of \(\Gamma \circ \Delta\) is a \(\Gamma \Delta\)-path if it has only vertices of degrees \((0, 1), (1, 0)\), and \((1, 1)\) and

- the vertices of degree \((0, 1)\) are sources of \(\Gamma\),
- the vertices of degree \((1, 0)\) are sinks of \(\Gamma\) or \(\Delta\).

Using this type of subnetworks, we can fill in the values at the nodes of the triangulation \(T = T_n\) as follows. For \(n \geq k \geq i \geq 0\) and a weighting \(\epsilon : E(\Gamma \circ \Delta) \to T\), we define the piecewise linear function

\[
m^k_i \Gamma \Delta(\epsilon) = \max_{\alpha \in P^k_i \Gamma \Delta} w_\alpha(\epsilon).
\]

Putting these together, we obtain a piecewise linear map \(M\Gamma\Delta : \mathbb{T}^E(\Gamma \circ \Delta) \to \mathbb{T}^V T\) given by \(t^k_i = m^k_i \Gamma \Delta(\epsilon)\) for \(n \geq k \geq i \geq 0\).

**Theorem 4.** Let \(\Gamma\) and \(\Delta\) be two composable networks of rank \(n\). Then the piecewise linear map \(M\Gamma\Delta\) satisfies

\[
\text{im}(M\Gamma\Delta) \subset \overline{C_3} \cap \{t^0_0 = 0\},
\]

where \(C_3\) is the polyhedral cone defined by inequalities \(\mathbf{8}\).
Similarly to the Gelfand–Zeitlin case, we have the following completeness result.

**Theorem 5.** For planar networks $\Gamma_0$ shown in Figure 2 and $\Delta_0$ shown in Figure 8 we have

$$\text{im}(M\Gamma_0\Delta_0) = \overline{C_3 \cap \{f_0 = 0\}},$$

and consequently

$$\text{im}(\partial \circ M\Gamma_0\Delta_0) = \overline{C_{\text{Horn}}}.$$  

In other words, the image of the map $\partial \circ M\Gamma_0\Delta_0$ coincides with the closure of the Horn cone describing the eigenvalues of triples of Hermitian matrices.

5. Proofs

5.1. Proof of Theorem 2. We observe that a path in $P_1\Gamma$ can also be described as the graph of a continuous function $f : [a, b] \to \mathbb{R}$ such that $\text{gr}(f) = \{(x, f(x)), x \in [a, b]\} \subset \Gamma$. In these terms, we can describe $P_k\Gamma$ as

$$P_k\Gamma = \{(f_1, \ldots, f_k) : [a, b] \to \mathbb{R} \text{ continuous, } \text{gr}(f_i) \subset \Gamma, f_i < f_{i+1}, i = 1, \ldots, k\}.$$  

Here and below we use the notation $f < g$ as a shorthand for $f(x) < g(x), x \in [a, b]$. To make our notation more readable, for a $k$-tuple $f = (f_1, \ldots, f_k) \in P_k\Gamma$, we will write

$$w[f]$$

for the functional $w_{\epsilon, \text{gr}(f)} : \mathbb{T}^{E(\Gamma)} \to \mathbb{T}$.

Theorem 2 is equivalent to the inequalities

\begin{equation}
\begin{aligned}
l_i\Gamma(k)(\epsilon) + l_{i-1}\Gamma(k-1)(\epsilon) &\geq l_{i-1}\Gamma(k)(\epsilon) + l_i\Gamma(k-1)(\epsilon), \\
l_i\Gamma(k)(\epsilon) + l_{i+1}\Gamma(k-1)(\epsilon) &\geq l_{i+1}\Gamma(k)(\epsilon) + l_{i-1}\Gamma(k-1)(\epsilon),
\end{aligned}
\end{equation}

for $\epsilon \in \mathbb{T}^{E(\Gamma)}$.

Consider the first of the two inequalities. Our method of proof is to show that, for $f \in P_{i-1}\Gamma(k)$ and $g \in P_i\Gamma(k-1)$, there exist $\tilde{f} \in P_{i-1}\Gamma(k)$ and $\tilde{g} \in P_i\Gamma(k)$ such that

\begin{equation}
w[\tilde{f}] + w[\tilde{g}] = w[f] + w[g].
\end{equation}

For a positive integer $N$, we consider the map $\sigma = (\sigma_1, \ldots, \sigma_N) : \mathbb{R}^N \to \mathbb{R}^N$ sending each vector to the vector with the same coordinates but listed in decreasing order: $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N$. Thus $\sigma_1 : \mathbb{R}^N \to \mathbb{R}$ is simply the largest coordinate of a vector. Clearly, $\sigma$ is a piecewise linear, continuous map. Now, for a planar network $\Gamma$ and $2 \leq i \leq n$, we can define a map

$$P_{i-1}\Gamma \times P_i\Gamma \to P_{i-1}\Gamma \times P_i\Gamma$$

in the following way. For an $(i-1)$-path $f = (f_1, \ldots, f_{i-1})$ and an $i$-path $g = (g_1, \ldots, g_i)$, we set $f \cup g = (f_1, \ldots, f_{i-1}, g_1, \ldots, g_i)$. We observe that, for $x \in [a, b]$, no real number can occur more than twice in the sequence

$$(f_1(x), \ldots, f_{i-1}(x), g_1(x), \ldots, g_i(x))$$

of length $(2i - 1)$. Define the function $\sigma_j(f \cup g)$ by

$$\sigma_j(f \cup g)(x) = \sigma_j(f_1(x), \ldots, f_{i-1}(x), g_1(x), \ldots, g_i(x)).$$

We have $\sigma_{j-1}(f \cup g) < \sigma_{j+1}(f \cup g)$ for all $j$. Now we can define our map by sending the pair $(f, g)$ to the pair $(\text{even}(f, g), \text{odd}(f, g))$, where

$$\text{even}(f, g) = [(\sigma_2(f \cup g), \sigma_4(f \cup g), \ldots, \sigma_{2i-2}(f \cup g))] \in P_{i-1}\Gamma$$

and

$$\text{odd}(f, g) = [(\sigma_1(f \cup g), \sigma_3(f \cup g), \ldots, \sigma_{2i-1}(f \cup g))] \in P_i\Gamma.$$
This map has the following two properties:

- if \( f \in P_{i-1} \Gamma \) and \( g \in P_1 \Gamma \), then
  \[
  w[\text{even}(f, g)] + w[\text{odd}(f, g)] = w[f] + w[g],
  \]

- if \( f \in P_{i-1} \Gamma^{(k)} \) and \( g \in P_1 \Gamma^{(k-1)} \), then
  \[
  \text{even}(f, g) \in P_{i-1} \Gamma^{(k-1)} \text{ and } \text{odd}(f, g) \in P_1 \Gamma^{(k)}.
  \]

Thus, these two multipaths can be chosen as \( \tilde{f} \) and \( \tilde{g} \) in (12), which implies the first inequality in (11).

The second inequality is proved in a similar manner. We define a map

\[
P_{i+1} \Gamma \times P_{i-1} \Gamma \to P_i \Gamma \times P_i \Gamma
\]

by sending a pair of \( i \)-paths \((f, g)\) of length 2\( i \) to the pair \((\text{even}(f, g), \text{odd}(f, g))\), where

\[
\text{even}(f, g) = [(\sigma_2(f \cup g), \sigma_4(f \cup g), \ldots, \sigma_{2i}(f \cup g))] \in P_i \Gamma
\]

and

\[
\text{odd}(f, g) = [(\sigma_1(f \cup g), \sigma_3(f \cup g), \ldots, \sigma_{2i-1}(f \cup g))] \in P_i \Gamma.
\]

As in the previous case, we observe that

\[
w[\text{even}(f, g)] + w[\text{odd}(f, g)] = w[f] + w[g],
\]

and if \( f \in P_{i+1} \Gamma^{(k)} \) and \( g \in P_{i-1} \Gamma^{(k-1)} \), then \( \text{even}(f, g) \in P_i \Gamma^{(k-1)} \) and \( \text{odd}(f, g) \in P_i \Gamma^{(k)} \), which implies the second inequality in (11).

5.2. Proof of Theorem 3. We will prove a somewhat stronger statement: there exists a graph and a collection of multipaths for which the image of \( L \Gamma \) is the full cone \( C_G \). In this section, we return to the functional \( w_\alpha : T^{E_T} \to \mathbb{T} \), for \( \alpha \in E_T \), defined as \( w_\alpha = \sum_{e \in \alpha} w_e \).

We will call a choice of multipaths \( \alpha(k, i) \in P_1 \Gamma^{(k)} \), \( k = 1, \ldots, n, i = 1, \ldots, k \) a collection. Then to each collection \( A = \{\alpha(k, i)\}_{i,k} \), we can associate the linear function \( w_A : T^{E_T} \to T^{E_T} \cap \{t_k^i = 0, k = 0, 0, \ldots, n\} \) defined by \( t_k^i = w_\alpha(k, i) \) for all \( i \) and \( k \).

We say that the collection \( A \) non-degenerate if \( w_A \) is surjective.

Let \( \Gamma \) be a planar network, and let \( \{\alpha(k, i)\} \) be a non-degenerate collection of multipaths in \( \Gamma \). Denote by \( B_{GZ}(A) \subset T^{E_T} \) the subset of those weightings of \( \Gamma \) for which the collection \( A \) satisfies the interlacing inequalities

\[
B_{GZ}(A) = \{ \epsilon \in T^{E_T} | (t_k^i = w_\alpha(k, i) (\epsilon), 0 < i \leq k \leq n) \in \mathbb{C}_T \} =
\]

\[
= w_A^{-1}(\mathbb{C} \cap \{t_k^0 = 0 | k = 0, \ldots, n\}),
\]

and let \( B_{\max}(A) \) be the subset of weightings for which each of the multipaths in \( A \) is maximal:

\[
B_{\max}(A) = \{ \epsilon \in T^{E_T} | l_i \Gamma^{(k)} (\epsilon) = w_\alpha(k, i) (\epsilon) \}.
\]

In these terms, Theorem 2 is equivalent to the statement that, for any collection \( A \), we have \( B_{\max}(A) \subset B_{GZ}(A) \).

Lemma 7. Suppose that a planar network \( \Gamma \) has a non-degenerate collection \( A \) such that \( B_{\max}(A) \supset B_{GZ}(A) \). Then, in fact,

\[
L \Gamma(B_{\max}(A)) = L \Gamma(B_{GZ}(A)) = \mathbb{C}_T \cap \{t_k^0 = 0 | k = 0, \ldots, n\}
\]
Proof. Indeed, we have $B_{GZ}(A) = w_A^{-1}(C_2 \cap \{t_k^0 = 0 \mid k = 0, \ldots, n\})$, and since $\Gamma$ is nondegenerate, we also have $w_A(B_{GZ}(A)) = C_2 \cap \{t_k^0 = 0 \mid k = 0, \ldots, n\}$. On the other hand, the restrictions $L\Gamma(B_{\text{max}}(A))$ and $w_A(B_{\text{max}}(A))$ coincide, and this, combined with $B_{\text{max}}(A) \supset B_{GZ}(A)$ implies (13). □

Now we consider the planar network $\Gamma_0$ shown in Figure 9. Given a decreasing sequence of integers $a = (a_1, \ldots, a_i)$ and an increasing sequence $b = (b_1, \ldots, b_i)$, we will say that a multipath $\alpha \in P_i \Gamma_0$ is of type $[a, b]$ if its sources are given by the list $a$ and its sinks are given by the list $b$. It is easy to verify that there is a single multipath in $P_i \Gamma_0$ of type $[(k, \ldots, k - i + 1), (1, \ldots, i)]$. Denote this multipath by $\alpha(k, i)$ and consider the collection $A = \{\alpha(k, i), 0 < i \leq k \leq n\}$ (see Figure 11).

![Figure 11](image1.png)

**Figure 11.** $\alpha(4, 2)$ for $n = 6$, shown in thick red.

**Lemma 8.** The collection $A$ is non-degenerate.

Proof. As shown in Figure 12, we introduce the following notation for the weights of the edges of $\Gamma$:

- we denote by $h_i$ the weights of the horizontal edges adjacent to the sinks,
- we denote by $a_{i,j}$ the weights of the slanted edges,
- the weights of the rest of the edges we put equal to 0.

![Figure 12](image2.png)

**Figure 12.** Weighting of the edges of $\Gamma_0$.

Then we can regard the map $w_A$ as a linear map from $\Gamma_{n(n+1)/2}$ to itself, and in order to prove its surjectivity it is sufficient to show that the Jacobian of $w_A$ is nonzero.

We have $\frac{\partial w_A(\alpha(k,i))}{\partial h_k} = 0$ for $i < k$, since these multipaths $\alpha(k,i)$ do not contain the edge weighted $h_k$ and $\frac{\partial w_A(\alpha(k,k))}{\partial h_k} = 1$. Similarly, the first multipath of our collection $A$ containing the edge with weight $a_{r,s}$ is the multipath $\alpha(r+1,s)$, therefore we have $\frac{\partial w_A(\alpha(k,i))}{\partial a_{k-1,i}} = 1$ and $\frac{\partial w_A(\alpha(k,i))}{\partial a_{r,s}} = 0$ for $k \leq r$. Thus, with an appropriate ordering...
of the variables $h_i$ and $a_{i,j}$, the Jacobi matrix of $w_A$ is triangular with 1’s on the diagonal.

We prove Theorem 2 in two steps:

• we begin with a graphical description of the interlacing inequalities describing $\mathcal{B}_{GZ}(A)$ in terms of cells in the complement of $\Gamma_0$,

• then, again, using a graphical device, we show that should these inequalities hold, our collection $A$ will consist of maximal multipaths.

Thus, the theorem will be proved if we show that the conditions of Lemma 7 hold for our collection $A$.

5.2.1. Graphical representation of the interlacing inequalities. We enumerate the cells formed by the connected components of the complement of $\Gamma_0$ by the symbols $[k, i]$, $k = 0, \ldots, n - 1$, $i = 0, \ldots, k$ as shown in Figure 13.

![Figure 13. Enumeration of the cells of $\Gamma_0$.](image)

To the cell $[k, i]$, we associate the functional $c_{[k, i]} : T^{E\Gamma_0} \to \mathbb{T}$ that is the sum of the signed weights along the clockwise oriented boundary of the cell, where the sign depends on whether the orientation of the boundary coincides with the orientation of the edge or not (Figure 14). The same applies to the unbounded cells.

![Figure 14. $c_{[k, i]} = -w_a - w_b - w_c + w_e + w_f + w_d$.](image)

Next, we define the functionals

$$r_{[k, i]}(\epsilon) = c_{[k, i]} + c_{[k-1, i-1]} + \cdots + c_{[0, 0]}$$

and

$$r_{[k, i]}(\epsilon) = c_{[k, i]} + c_{[k-1, i]} + \cdots + c_{[0, 0]}$$

which correspond to the boundary of the shaded regions shown in Figures 15 and 16 below.

**Lemma 9.** The polyhedral cone $\mathcal{B}_{GZ}(A)$ is given by the weightings $\epsilon \in T^{E\Gamma_0}$ satisfying

$$r_{[k, i]}(\epsilon) \geq 0, \quad 0 \leq i < k < n,$$

$$r_{[k, i]}(\epsilon) \leq 0, \quad 0 < i \leq k < n.$$
Proof. By Definition 1 that \(B_{GZ}(A)\) is defined by the inequalities

\[
w_\alpha(k+1,i) + w_\alpha(k,i-1) - w_\alpha(k+1,i-1) - w_\alpha(k,i) \geq 0
\]

and

\[
w_\alpha(k+1,i+1) + w_\alpha(k,i-1) - w_\alpha(k+1,i) - w_\alpha(k,i) \leq 0,
\]

for 0 < \(i\) ≤ \(k\) < \(n\).

It is not difficult to identify these inequalities with inequalities (14). In Figures 17 and 18 we present a graphical proof of this equivalence for the case of the first inequality in (14) when \(k + 1 = n = 5\) and \(i = 2\):

- the thick (red) lines show \(\alpha(k+1,i)\) and \(\alpha(k,i)\),
- the dashed lines show \(\alpha(k+1,i-1)\) and \(\alpha(k,i-1)\),
- the sum of weights along the boundary of the shaded area represents the difference in the caption.

One then notes that the difference of the two shaded regions is the union of the cells \([k,i], \ldots, [k-i,0]\), which, in view of the definition of \(r_{[k,i]}\), completes the proof. The proof of the equivalence of the other pair of inequalities is analogous. \(\square\)
5.2.2. Maximality of the collection $A$. Now we are ready to prove Theorem\textsuperscript{3}. We will call $\beta \in P_1 \Gamma_0$ a maximal multipath for the weighting $\epsilon$ if $l_1 \Gamma_0 = w_\beta(\epsilon)$. By Lemmas\textsuperscript{7} and \textsuperscript{9} we may assume that we are given a weighting $\epsilon$ of $\Gamma_0$ satisfying $r_{[k,i]}^\epsilon(\epsilon) \geq 0$ and $r_{[k,i]}^\epsilon(\epsilon) \leq 0$. We must prove that for this weighting, the paths of the collection $A$ are maximal. We fix this $\epsilon$ for the rest of the section and will often drop it from our notation for brevity.

For any $k$-tuple $a = (a_1, \ldots, a_k)$, we will denote the sum $a_1 + \cdots + a_k$ by $\sum a$.

**Lemma 10.** Let $\beta \in P_1 \Gamma_0$ be a maximal multipath of type $[a, b]$ for the weighting $\epsilon \in \mathcal{B}_{GZ}$ such that the value of $\sum b$ is minimal among all maximal multipaths in $P_1 \Gamma_0$. Then $b = (1, 2, \ldots, i)$.

**Proof.** First consider the case $i = 1$. If $\beta$ ends at $b > 1$, then there is a cell $[b-1, b-1]$ whose upper edge is a part of $\beta$. Then $w_\beta \leq w_{\tilde{\beta}} = w_\beta - r_{[k,b-1]}^{\epsilon}$ for some $k$, where $\tilde{\beta}$ is a path ending at $b - 1$ (see Figure 19). This contradicts the assumption on $\beta$, and so the $i = 1$ case is proved.

Now let $\beta \in P_1 \Gamma_0$ be a maximal multipath for $\epsilon$ with the smallest possible value of $b_1 + \cdots + b_i$. Clearly, we can apply the above argument to the lowest path in $\beta$ and conclude that $b_1 = 1$. If $b_2 > 2$, then, as before, we can subtract the appropriate $r_{[k,b_2-1]}^{\epsilon}$ from $w_\beta$, and obtain a multipath with a strictly lower sum of end-values. This contradicts the assumption on $\beta$, hence $b_2 = 2$. Repeating this argument, we can show that $b_j = j$ for $j = 1, \ldots, i$, which completes the proof. \hfill $\square$

Now we consider the sources of maximal multipaths.

**Lemma 11.** Let $\beta \in P_1 \Gamma_0$ be a maximal multipath of type $[a, (1, 2, \ldots, i)]$ for a weighting $\epsilon \in \mathcal{B}_{GZ}$ for which the value of $\sum a = a_1 + \cdots + a_i$ is the largest among all the maximal multipaths in $P_1 \Gamma_0$. Then $a = (n, n-1, \ldots, n-i+1)$.

**Proof.** We begin with the case $i = 1$. Suppose that a path $\beta \in P_1 \Gamma_0$ connecting $a < n$ with the sink $1$ satisfies the conditions of the lemma. Then $w_\beta \leq w_{\tilde{\beta}} = w_\beta + r_{[a,0]}^{\epsilon}$ for a path $\tilde{\beta}$ that starts at $a + 1$. This contradicts our assumptions on $\beta$, and thus $a = n$.

If $i > 1$ and $\beta \in P_1 \Gamma_0$ satisfies the conditions of the lemma, then we can still show as above that necessarily $a_1 = n$. Now assume that there is a gap between the sources of $\beta$, i.e. that, for some $j < n$, $j$ is a term of the sequence $a$, but $j + 1$ is not. If the first vertex $v_{j+1,1}$ of $\Gamma_0$ on the line $y = j + 1$ is not in $\beta$, then we can again add $r_{[j,0]}^{\epsilon}$ to $w_\beta$ and obtain a new multipath with larger $\sum a$.\hfill $\square$

![Figure 19. $w_\beta$ (solid) = $w_{\tilde{\beta}}$ (dashed) − $r_{[k,b-1]}^{\epsilon}$ (shaded).](image-url)
If \( v_{j+1,1} \in \beta \), then \( \beta \) has an elbow at \( v_{j+1,1} \), by which we mean that the two edges in \( \beta \) containing \( v_{j+1,1} \) are the downward incoming edge and the horizontal outgoing edge. The other two edges are excluded by our assumptions. Now consider the sequence of vertices \( v_{j+1,1}, v_{j+2,1}, \ldots, v_{j+s,1} \in \beta \) such that \( v_{j+s+1,1} \notin \beta \). It is easy to verify that \( \beta \) has necessarily an elbow at all of the vertices of this sequence.

![Figure 20. Graphical representation of \( w_\beta = w_\beta + r_{[j+s-1,s-1]} \).](image)

This implies that \( w_\beta \leq w_\beta + r_{[j+s-1,s-1]} \) for a multipath \( \tilde{\beta} \) with the source at height \( j \) moved up to \( j + 1 \). We give an example of this operation in Figure 20, where
- \( i = 3 \) and \( s = 2 \),
- \( \beta \) is marked with thick (red) lines,
- \( \tilde{\beta} \) is marked with dashed lines,
- the shaded area is the domain corresponding to \( r_{[j+s-1,s-1]} \).

This contradicts our assumption that \( \beta \) is a maximal multipath with the largest \( \sum a \), thus there can be no gaps among the sources of \( \beta \), which completes the proof.

Now we can quickly finish the proof of Theorem 3. We must show that \( \text{im}(L_\Gamma) = C_2 \cap \{ t_k = 0 \mid k = 0, \ldots, n \} \). We consider the non-degenerate collection \( A = \{ \alpha(k,i) \} \). According to Lemma 7, it is sufficient to show that if the weights \( w_\alpha(\epsilon) \) satisfy the interlacing inequalities for a weighting \( \epsilon \) of \( \Gamma_0 \), then they are also maximal. This obviously follows from Lemmas 10 and 11.

5.3. **Proof of Theorem 4**. Let \( \Gamma \) and \( \Delta \) be two planar networks of rank \( n \). We must prove the inequalities

\[
\begin{align*}
m_i^{k+1} &\Gamma_\Delta + m_{i-1}^{k+1} \Gamma_\Delta &\geq m_i^{k+1} \Gamma_\Delta + m_{i-1}^{k+1} \Gamma_\Delta, \\
m_i^{k+1} &\Gamma_\Delta + m_{i-1}^{k+1} \Gamma_\Delta &\geq m_i^{k+1} \Gamma_\Delta + m_{i-1}^{k+1} \Gamma_\Delta, \\
m_i^{k+1} &\Gamma_\Delta + m_{i-1}^{k+1} \Gamma_\Delta &\geq m_i^{k+1} \Gamma_\Delta + m_{i-1}^{k+1} \Gamma_\Delta.
\end{align*}
\]

We begin with the first inequality. As in the proof of Theorem 2 we will show that for \( \alpha \in P^{k+1}_i \Gamma_\Delta \) and \( \beta \in P^{k+1}_i \Gamma_\Delta \), one can find \( \tilde{\alpha} \in P^{k+1}_i \Gamma_\Delta \) and \( \tilde{\beta} \in P^{k+1}_i \Gamma_\Delta \) such that

\[ w_\alpha(\epsilon) + w_\beta(\epsilon) = w_{\tilde{\alpha}}(\epsilon) + w_{\tilde{\beta}}(\epsilon), \]

for any weighting \( \epsilon \in \mathcal{T}^E(\Gamma_\Delta) \).

We can informally describe this problem as follows. Imagine that we have a group of tourists with \( k \) men and \( k - 1 \) women; \( i - 1 \) of the men and \( i \) of the women are
fit. The group is planning an excursion where the fit tourists go from one town to a neighboring town and the unfit tourists only go from the first town to a park halfway to the second town. The organizer of the excursion devises a route for each member of the group under the following special condition: the paths of the men should not intersect, and similarly, the paths of the women should not intersect. Just before the trip it turns out that there will be \(i\) fit men, and \(i - 1\) fit women, with the total numbers of men and women unchanged. Can the organizer redraw the routes under the same special condition, and so that if a certain segment was used only by one person in the original plan, then this segment will be used by only one person in the new plan as well?

To prove the theorem it will be convenient to consider a slightly more general notion of a planar network in which we allow multiple edges (in fact, we will need only double edges). If we have a double edge emanating from a vertex, then this edge will contribute 2 to the out-degree of this vertex; similarly, this edges will contribute 2 to the in-degree of the vertex to which it points. We will also consider planar networks with a different number of sources and sinks. We will call such a planar network of type \([k_1, k_2]\) if the sum of out-degrees of all its sources is \(k_1\) and the sum of all in-degrees of its sinks is \(k_2\).

The sum of multipaths \(\Theta = \alpha \cup \beta\) is naturally such a generalized planar network of type \([2k - 1, 2i - 1]\) with the edges \(e \in E\alpha \cap E\beta\) having multiplicity 2. We will consider such a double edge as two separate edges. The end-line of \(\Gamma\), which is also the start-line of \(\Delta\), will be called the middle line of \(\Theta\). According to our strategy, the inequality

\[
m^k_i \Gamma \Delta + m^{k-1}_i \Gamma \Delta \geq m^k_{i-1} \Gamma \Delta + m^{k-1}_{i} \Gamma \Delta
\]

will follow if we can decompose \(\Theta\) as the union of multipaths. one from \(P^{k-1}_{i-1} \Gamma \Delta\) and the other from \(P^k_i \Gamma \Delta\). This will be shown in Proposition 14 below.

More generally, we consider decompositions of \(\Theta\) into two multipaths from \(P^p_q \Gamma \Delta\) and \(P^{p'}_{q'} \Gamma \Delta\) for some integers \(p, q, p', q'\). Clearly, we will have \(q + q' = 2i - 1\) and \(p + p' = 2k - 1\). To such a decomposition, we can associate a coloring of the edges of \(\Theta\) in two colors, say, red and green. We will call such a coloring valid. Let us classify all valid colorings.

First we note that just as in the proof of Lemma 4, the operation of eliminating a vertex of degree \((1, 1)\) and replacing its two adjacent edges by a single edge will not influence the image of \(M \Gamma \Delta\), so we will assume that \(\Theta\) has no vertices of degree \((1, 1)\).

Recall that a closed path in an unoriented graph is a connected subgraph whose every vertex has degree 2; an open path has, in addition, two vertices of degree 1.

**Lemma 12.** Assume that \(\Theta\) has no vertices of degree \((1, 1)\) and consider the equivalence relation on the edges of \(\Theta\) generated by the following relation: two edges are related if they are either both incoming edges of a vertex of \(\Theta\) or both outgoing edges of a vertex of \(\Theta\). Then the resulting equivalence classes are (possibly closed) unoriented paths in \(\Theta\) with edges having alternating orientations.

We will call this decomposition of \(\Theta\) the canonical path decomposition of \(\Theta\) (see Figure 21).

**Proof.** The fact that \(\Theta\) is the union of two multipaths implies that

- the sources of \(\Theta\) are of degree \((0, 1)\) or \((0, 2)\),
Figure 21. Canonical path decomposition of $\Theta$ of type $[5, 3]$ ($k = 3$ and $i = 2$).

- the degrees of all the other vertices have degrees from the following list:
  
  $(2, 2), (2, 1), (2, 0), (1, 1), (1, 0)$;

- moreover, the vertices that are neither sources nor sinks and that are not on the middle-line of $\Theta$ can have only degrees $(2, 2)$ or $(1, 1)$.

The statement of the lemma clearly follows from the fact that no vertex of $\Theta$ has in- or out-degree greater than two. □

Remark 5. Because of the alternating orientation of the edges, the following paths of the canonical path decomposition have an even number of edges:

- closed paths,
- open paths beginning and ending at a source,
- open paths beginning and ending at a sink,
- open paths beginning and ending at the middle line.

Now we define an alternating coloring of $\Theta$ as a coloring of the edges of $\Theta$ in two colors in such a way that the consecutive edges of each path of its canonical path decomposition are colored differently (see Figure 22). By Remark 5, such a coloring always exists.

Figure 22. An alternating coloring of $\Theta$.

Lemma 13. Assume that $\Theta$ has no vertices of degree $(1, 1)$, and denote the elements of its path decomposition by $Q$.

(1) Then there are precisely $2^{|Q|}$ alternating colorings of $\Theta$, corresponding to a coloring of each of the $|Q|$ paths chosen independently.
The alternating colorings of $\Theta$ coincide with the valid colorings of $\Theta$.

Proof. Clearly, every path has precisely two alternating colorings, and the colorings of different paths are independent of each other. This implies the first statement. A coloring of the edges of $\Theta$ is valid if and only if

- at any vertex having in-degree 2, the two incoming edges are colored differently,
- at any vertex having out-degree 2, the two outgoing edges are colored differently. In particular, the two edges of a double edge have different colors.

These conditions coincide with the definition of an alternating coloring. □

Now we prove a somewhat strengthened version of the decomposition statement, which will imply our theorem:

**Proposition 14.** Let $\Theta$ be a generalized planar network with the following properties:

- $\Theta$ is of type $[2k - 1, 2i - 1]$,
- the sources of $\Theta$ have degrees $(0, 1)$ or $(0, 2)$,
- all vertices of $\Theta$, apart from the sources, have degrees $(d_1, d_2)$ with $2 \geq d_1 \geq d_2$.

Then there is $\tilde{\alpha} \in \mathcal{P}_k \Theta$ and $\tilde{\beta} \in \mathcal{P}_{k-1} \Theta$ such that $\tilde{\alpha} \cup \tilde{\beta} = \Theta$.

Proof. We can partition the set $Q$ of path components of $\Theta$ as follows:

$$Q = Q_{00} \cup Q_{L0} \cup Q_{0R} \cup Q_{LR} \cup Q_{LL} \cup Q_{RR} \cup Q_{cl},$$

where $Q_{cl}$ consists of all closed paths and the two indices of the rest of the $Q$’s indicate the beginning and the end of the path with the convention that

- $L$ stands for a source of $\Theta$,
- $R$ stands for a sink of $\Theta$,
- $0$ stands for an internal vertex of $\Theta$.

We must show that among the $2^{|Q|}$ alternating colorings of the paths in $Q$, there is at least one for which precisely $k$ edges emanating from a source (source-edges), and precisely $i$ edges ending in a sink (sink-edges) are red. It is clear that in order to specify the coloring of a path, it is sufficient to color its source-edge or sink-edge whenever the path has one of these.

We note that the contribution of each of $Q_{00}$, $Q_{LL}$, $Q_{RR}$, and $Q_{cl}$ to the source-degree and the sink-degree is even, so their coloring is not essential. The total contribution of $Q_{L0}$ and $Q_{LR}$ to the source-degree and the total contribution of $Q_{0R}$ and $Q_{LR}$ to the sink-degree are odd. Therefore the coloring algorithm is as follows:

- if $|Q_{LR}|$ is even, then $|Q_{0R}|$ is odd and $|Q_{L0}|$ is odd. Then we color in red
  - the sink-edges of half of the paths in $Q_{LR}$ and of $(|Q_{0R}| + 1)/2$ paths in $Q_{0R}$,
  - the source-edges of $(|Q_{L0}| + 1)/2$ paths in $Q_{L0}$.
- if $|Q_{LR}|$ is odd, then $|Q_{0R}|$ and $|Q_{L0}|$ are even. Then we color in red
  - the sink-edges of $(|Q_{LR}| + 1)/2$ of the paths in $Q_{LR}$ and of half of the paths in $Q_{0R}$,
  - the source-edges of half of paths in $Q_{L0}$. 

This coloring algorithm ensures that the number of paths whose source-edges are colored in red is greater by 1 than the number of paths with source-edges colored in green. The same is true for the sink-edges. Thus we indeed obtain an element of $P_i^k \Theta$ colored in red and an element of $P_{i-1}^{k-1} \Theta$ colored in green.

As we explained above, Proposition 14 implies Inequality (10).

The other two inequalities from (15) are proved in a similarly. Consider the second inequality. Let $\alpha \in P_{i-1}^{k-1} \Gamma \Delta$ and $\beta \in P_i^k \Gamma \Delta$. Then $\Theta = \alpha \cup \beta$ is a generalized planar network of type $[2k - 1, 2i]$. As before, we obtain a path decomposition $Q = Q_{00} \cup Q_{0L} \cup Q_{0R} \cup Q_{LR} \cup Q_{LL} \cup Q_{RR} \cup Q_{\alpha}$ of $\Theta$ with $2^{|Q|}$ alternating colorings. Among these colorings, we must find one with precisely $k$ source-edges and precisely $i$ sink-edges colored in red. For this, we use the following algorithm:

- if $|Q_{LR}|$ is even, then $|Q_{0R}|$ is even while $|Q_{L0}|$ is odd. Then we color in red
  - the sink-edges of half of the paths in $Q_{LR}$ and in $Q_{0R}$,
  - the source-edges of $(|Q_{L0}| + 1)/2$ paths in $Q_{L0}$.

- if $|Q_{LR}|$ is odd, then $|Q_{0R}|$ is odd and $|Q_{L0}|$ is even. Then we color in red
  - the sink-edges of $(|Q_{LR}| + 1)/2$ of the paths in $Q_{LR}$ and of $(|Q_{0R}| - 1)/2$
  - the source-edges of half of the paths in $Q_{L0}$.

This coloring algorithm ensures that the number of paths with red source-edges is greater by 1 than the number of paths with green source-edges, whereas the number of paths with red sink-edges is equal to the number of green sink-edges, and so we indeed obtain an element of $P_i^k$ colored in red and an element of $P_{i-1}^{k-1}$ colored in green.

Finally, for the third inequality, we have a generalized planar network $\Theta$ of type $[2k, 2i - 1]$. The coloring procedure is similar to the other two cases, and will be omitted.

We have thus proved inequalities (15), which completes the proof of the theorem.

**Remark 6.** A similar proof can be given for Theorem 2 which is the analog of Theorem 4 in the Gelfand–Zeitlin case.

### 5.4. Proof of Theorem 5

The strategy of the proof of Theorem 5 is analogous to that of Theorem 3. The steps of the proof are as follows:

1. We identify the special collection $B = \{ \beta(k, i); 0 \leq i \leq k \leq n \}$. Denote by $\tau(k)$ the $k$th horizontal line of $\Gamma_0 \circ \Delta_0$. Then

$$\beta(k, i) = \alpha(n - i, k - i) \cup \bigcup_{j=n-i+1}^{n} \tau(j).$$

The following counterpart of Lemma 7 is valid: the theorem follows from the inclusion

$$w_B^{-1}(\tilde{C}_3) \cap \{ t_0^0 = 0 \} \subset \{ \epsilon \in \mathbb{T}^E(\Gamma_0 \circ \Delta_0) | m_i^k \Gamma_0 \Delta_0(\epsilon) = w_{\beta(k, i)}(\epsilon), 0 \leq i \leq k \leq n \};$$

in words, we must show that if, for some weighting, the weights of our special collection satisfy the inequalities defining $C_3$ (where $t_0^0 = 0$), then the multipaths of our collection are maximal for this weighting.

2. We identify the regions corresponding to the inequalities (15) for $m_i^k = w_{\beta(k, i)}$. We denote the cells of $\Gamma_0$ as before and the cells of $\Delta_0$ by $[i, i + 1]$ for
Figure 23. $\beta(4,2)$ for $n=6$, shown in solid red.

$i = 0, \ldots, n$. The first inequality corresponds to $r_{[k,i]}^{\gamma}$, the third to $r_{[k,i]}^{\delta}$, while the second inequality corresponds to

$$r_{[k,i]}^{\tau} = c[k,i] + c[k,i+1] + \cdots + c[k,k] + c[k,k+1]$$

(see Figure 24).

Figure 24. $r_{[k,i]}^{\tau}$.

We thus obtain the graphical representation:

$$w_B(\epsilon) \in C_3 \iff r_{[k,i]}^{\tau}(\epsilon) \geq 0, r_{[k,i]}^{\gamma}(\epsilon) \leq 0, r_{[k,i]}^{\delta}(\epsilon) \geq 0.$$  

3. Now we must show that if condition (18) holds for a weighting $\epsilon$ of $\Gamma_0 \circ \Delta_0$, i.e., if $r_{[k,i]}^{\tau}, -r_{[k,i]}^{\gamma}, r_{[k,i]}^{\delta} \geq 0$ for all $i \leq k < n$, then the collection $B$ is maximal for $\epsilon$.

From now on, we assume that we have a fixed weighting $\epsilon$ of $\Gamma_0 \circ \Delta_0$ satisfying (18), and we will often drop $\epsilon$ from the notation. When we say that $\alpha \in P_k^\Gamma \Gamma_0 \Delta_0$ is maximal, we mean that $w_\alpha(\epsilon) = m_k^\Gamma \Gamma_0 \Delta_0(\epsilon)$.

Note that every $\alpha \in P_k^\Gamma \Gamma \Delta$ has two decompositions:

$$\alpha = \gamma \cup \delta = \alpha' \cup \alpha''$$

$\gamma \in P_k \Gamma_0$, $\delta \in P_k \Delta_0$, $\alpha' \in P_\Gamma(\Gamma_0 \circ \Delta_0)$, $\alpha'' \in P_{k-i} \Gamma_0$.

We record the endpoints of these multipaths as follows (see Figure 25):

- denote by $a = (a_1, \ldots, a_k)$ the decreasing sequence of sources of $\gamma$;
- denote by $b = (b_1, \ldots, b_{k-1})$ the increasing sequence of sinks of $\alpha''$;
- denote by $b' = (b'_1, \ldots, b'_k)$ the increasing sequence of sinks of $\gamma$. Note that $b$ is a subsequence of $b'$.
- denote by $c = (c_1, \ldots, c_i)$ the decreasing sequence of sinks of $\alpha'$ (or $\delta$).
When necessary, we will indicate the index $\alpha$ explicitly by writing, for example, $b_\alpha$ instead of $b$.

Now, similarly to Lemmas 10 and 11, we can formulate a sequence of lemmas normalizing the form of a maximal path that eventually lead to the statement that $\beta(k, i)$ is maximal for the weighting $\epsilon$.

**Lemma 15.** Let $\alpha \in P^k\Gamma_0\Delta_0$ be a maximal path for which $\sum b$ minimal. Then $b_{k-i} = b'_{k-i}$, which means that $\alpha''$ consists of the lowest $k - i$ paths of $\gamma$.

**Proof.** To prove this, we observe that
\begin{equation}
 c_{[j,j+1]}(\epsilon) \geq 0 \text{ for } 1 \leq j \leq n - 1,
\end{equation}

since $c_{[j,j+1]} = r_{[j,j]}^\rightarrow - r_{[j,j]}^\leftarrow$. Assume, contrary to the statement of the lemma, that for some $j \leq k - i$, we have $b_j > b'_{j}$ (see Figure 26). Then we obtain
\[
 w_{\tilde{\alpha}}(\epsilon) = w_\alpha(\epsilon) + c_{[b',b'+1]}(\epsilon) + \cdots + c_{[b_{j-1},b_j]}(\epsilon) \geq w_\alpha(\epsilon),
\]

for a multipath $\tilde{\alpha}$. Clearly, $\tilde{\alpha}$ is maximal with $\sum b_{\tilde{\alpha}} < \sum b_\alpha$, which contradicts our assumption on $\alpha$.

**Lemma 16.** Let $\alpha \in P^k\Gamma_0\Delta_0$ be a maximal path for which $b_{k-i} = b'_{k-i}$ and the value of $a_1$ is maximal. Then $a_1 = n$.

The proof of this statement is identical to the first part of the proof of Lemma 11 (see Figure 27).
Lemma 17. Let \( i > 0 \) and let \( \alpha \in P^k \Gamma_0 \Delta_0 \) be a maximal multipath for which \( b_{k-i} = b'_{k-i} \), \( a_1 = n \) and the value of \( c_1 \) is maximal. Then \( c_1 = n \), and hence \( \tau(n) \subset \alpha \).

Proof. Since \( b_{k-i} = b'_{k-i} \) for \( i > 0 \), we know that the highest path of \( \alpha \) is a path in \( \Gamma_0 \circ \Delta_0 \). Assume that \( c_1 \neq n \). Then, as shown in Figure 28 for an appropriate sequence \( q_j, j = c_1, \ldots, n - 1 \), we have

\[
 w_{\tilde{\alpha}} = w_\alpha + \sum_{j=c_1}^{n-1} r_{\rightarrow j} \geq 0,
\]

where \( \tilde{\alpha} \in P^k \Gamma_0 \Delta_0 \). Again, this contradicts the assumptions of the lemma, hence \( c_1 = n \). \( \square \)

Now, using the proof of Theorem 3, we can quickly finish the proof of Theorem 5 by induction on \( n \).

Indeed, Theorem 3 is trivial for \( n = 1 \). Assume that the theorem has already been proved for the case of rank \( n - 1 \), and that \( \epsilon \) is a weighting satisfying (18). Then the fact that \( \beta(k,0) \) is maximal for \( k = 1, \ldots, n \) is a special case of Theorem 3. If \( k \geq i > 0 \), then the sequence of lemmas above shows that there is at least one maximal \( \alpha \in P^k \Gamma_0 \Delta_0 \) containing \( \tau(n) \). Proving that \( \beta(k,i) \) is maximal is then equivalent to showing that \( \beta(k-1,i-1) \) is maximal for the restriction of \( \epsilon \) to the planar network obtained by removing \( \tau(n) \) from \( \Gamma_0 \circ \Delta_0 \). This, however, follows from our inductive assumption, which completes the proof.
REFERENCES

[1] S. Agnihotri, C. Woodward, Eigenvalues of products of unitary matrices and quantum Schur
 bert calculus, Math. Res. Lett. 5 (1998), no. 6, 817–836.
[2] A. Alekseev, M. Podkopaeva, A. Szenes, Tropical correspondence principle for matrix prod-
 ucts, in preparation.
[3] P. Belkale, S. Kumar, Eigenvalue problem and a new product in cohomology of flag vari-
 eties, Invent. Math. 166 (2006), 185–228.
[4] A. Berenstein, R. Sjamaar, Coadjoint orbits, moment polytopes, and the Hilbert–Mumford
criterion, J. Amer. Math. Soc. 13 (2000), no. 2, 433–466.
[5] M. Brion, Restriction de repr´esentations et projections d’orbites coadjointes. S´eminaire Bour-
baki, 64`me ann´ee, 2011-2012, no 1043.
[6] A. S. Buch, The saturation conjecture (after A. Knutson and T. Tao). With an appendix by
William Fulton. Enseign. Math. (2) 46 (2000), no. 1–2, 43–60.
[7] S. Fomin, A. Zelevinsky, Total positivity: tests and parametrizations, Math. Intelligencer 22
(2000), no. 1, 23–33.
[8] W. Fulton, Eigenvalues, invariant factors, highest weights, and Schubert calculus. Bull. Amer.
Math. Soc. (N.S.) 37 (2000), no. 3, 209–249.
[9] V. Guillemin, S. Sternberg, The Gelfand–Cetlin system and quantization of the complex flag
manifolds, J. Funct. Anal. 52 (1983), no. 1, 106–128.
[10] R. A. Horn, C. R. Johnson, Matrix analysis, Cambridge University Press, Cambridge, 1985.
[11] A. Horn, Eigenvalues of sums of Hermitian matrices, Pacific J. Math. 12 (1962), 225–241.
[12] M. Kapovich, B. Leeb J. J. Millson, The generalized triangle inequalities in symmetric spaces
and buildings with applications to algebra (English summary), Mem. Amer. Math. Soc. 192
(2008), no. 896.
[13] A. A. Klyachko, Stable vector bundles and Hermitian operators, Selecta Math. (N.S.) 4
(1998), 419–445.
[14] A. Knutson, T. Tao, The honeycomb model of $GL_n(\mathbb{C})$ tensor products I: proof of the satu-
ration conjecture, Journal of the AMS 12 (1999), 1055–1090.
[15] A. Knutson, T. Tao, C. Woodward, The honeycomb model of $GL_n(\mathbb{C})$ tensor products II:
Puzzles determine facets of the Littlewood-Richardson cone, J. Amer. Math. Soc. 17 (2004),
19–48.
[16] H. Weyl, Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differential-
gleichungen, Math. Ann. 71 (1912) 441–479.

ANTON ALEKSEEV, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GENEVA, 2-4 RUE DU
LIÈVRE, C.P. 64, 1211 GENÈVE 4, SWITZERLAND
E-mail address: Anton.Alekseev@unige.ch

MARIA PODKOPEVA, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GENEVA, 2-4 RUE DU
LIÈVRE, C.P. 64, 1211 GENÈVE 4, SWITZERLAND
E-mail address: Maria.Podkopaeva@unige.ch

ANDRAS SZENES, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GENEVA, 2-4 RUE DU LIÈVRE,
C.P. 64, 1211 GENÈVE 4, SWITZERLAND
E-mail address: Andras.Szenes@unige.ch