Higher Order Spin-dependent Terms in D0-brane Scattering from the Matrix Model

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Abstract

The potential describing long-range interactions between D0-branes contains spin-dependent terms. In the matrix model, these should be reproduced by the one-loop effective action computed in the presence of a nontrivial fermionic background $\psi$. The $\frac{v^3 \psi^2}{r^8}$ term in the effective action has been computed by Kraus and shown to correspond to a spin-orbit interaction between D0-branes, and the $\frac{\psi^4}{r^11}$ term in the static potential has been obtained by Barrio et al. In this paper, the $\frac{v^2 \psi^4}{r^9}$ term is computing in the matrix model and compared with the corresponding results of Morales et al obtained using string theoretic methods. The technique employed is adapted to the underlying supersymmetry of the matrix model, and should be useful in the calculation of spin-dependent effects in more general Dp-brane scatterings.
1 Introduction

Recent developments in superstring duality owe much to the recognition of the role of Dirichlet-branes as BPS states which act as sources for Ramond-Ramond fields [1]. The long-distance interactions of D-branes via the exchange of a closed string have been studied, and the static potential vanishes [1], as is characteristic for BPS states. The leading term in the long range velocity-dependent potential for a pair of slowly moving D0-branes behaves as \( \frac{v^4}{r^7} \), where \( v \) is the relative speed and \( r \) is the separation [2, 3]. This result can be reproduced from the low energy effective theory describing the dynamics of D0-branes in terms the states of open strings which end on the D0-branes. The low energy effective theory is the dimensional reduction of 10D supersymmetric Yang Mills theory to 1+0 dimensions [4, 5, 6, 7], and the potential follows from the one loop effective action computed in the presence of a nontrivial background containing information about the relative motion of the D0-branes. These results also provide important tests of the conjecture by Banks, Fischler, Shenker and Susskind (BFSS) that the dynamical degrees of freedom of eleven-dimensional M-theory in the infinite momentum frame are D0-branes [8]. Low energy D0-brane scattering amplitudes computed using the dimensionally reduced super Yang-Mills theory (a quantum mechanical matrix model) reproduce tree level scattering amplitudes for eleven-dimensional supergravitons.

As BPS states, D0-branes belong to a shortened supersymmetry multiplet, and the leading term in the velocity-dependent potential is the same for all spin states in the supermultiplet. Spin dependent terms in the long range potential were considered by Harvey [9], who pointed out that they should be reproduced in the matrix model by calculation of the one loop effective action in the presence of a fermionic background \( \psi \). Comparison of these results with the spin dependence of supergraviton scattering in eleven dimensions will be important tests of the BFSS conjecture. On dimensional grounds, the spin dependence of the potential was argued in [8] to be coded in an effective action of the form

\[
\frac{v^4}{r^7} + \frac{v^3 \psi^2}{r^8} + \frac{v^2 \psi^4}{r^9} + \frac{v \psi^6}{r^{10}} + \frac{\psi^8}{r^{11}}. \tag{1}
\]

The order \( \psi^2 \) term in this expansion has been computed in the matrix model by Kraus [10], who showed that it reproduces the spin-orbit interaction for a D0-brane probe moving in the linearized metric of a spinning D0-brane. This
interaction has also been computed using string theoretic methods by Morales et al \[14, 15\].

The matrix model calculation carried out in \[10\] was done by treating the fermionic part of the background as a perturbation about a purely bosonic background. The order $\psi^8$ term in the static potential has been obtained by similar means in \[11\]. Here, we carry out the computation of the order $\psi^4$ term in the matrix model effective action \( \Pi \) in a manner which is better adapted to the underlying supersymmetry. This is achieved by recognizing that the one loop effective action in the presence of background fields is related to the superdeterminant of the operator which appears in the part of the action quadratic in the quantum fields. In the case where there are no fermionic background fields, this superdeterminant factorizes into a number of ordinary determinants, which are easily computed using Schwinger's proper time formalism \[7\]. In the presence of a nontrivial fermionic background, the superdeterminant no longer factorizes. It can still conveniently be computed using the Schwinger proper time formalism, but one has to work a little harder than in the case of a purely bosonic background.

The outline of the paper is as follows. In §2, the one-loop effective action for the matrix model in the presence of a fermionic background is formulated in terms of a superdeterminant. The approach to be taken to evaluate the one-loop effective action is illustrated in §3 by reconsideration of the well known spin-independent terms in the potential between two D0-branes. The extension of this formalism in the case of a superdeterminant is carried out in §4, and §5 gives the form of the one-loop effective action to order $\psi^4$. This is compared with string theoretic calculations by Morales et al \[15\] in §6. The paper concludes with a discussion of the relevance of the techniques employed in this paper to calculations of more general Dp-brane scattering amplitudes. A number of spinor identities and details of computations are contained in two appendices.

2 The One-loop Effective Action as a Superdeterminant

The matrix model provides a description of the low energy dynamics of D0-branes in terms of a quantum mechanical model obtained by the dimensional reduction of 10D supersymmetric Yang-Mills theory to 1+0 dimensions.
In particular, for gauge group SU(2) and an appropriate choice of background fields, the one-loop effective action for this theory yields the long-range potential between a pair of D0-branes. In the presence of a fermionic background, this one-loop effective action is a superdeterminant, which we manipulate into a convenient form in this section.

The ten-dimensional supersymmetric Yang-Mills action (using the conventions of [12]) is

\[ S = \int d^{10}x \, \text{Tr} \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \bar{\psi} \Gamma^\mu D_\mu \psi \right), \]

where \( A_\mu \) and \( \psi \) take values in the Lie algebra of SU(2), and \( \psi \) is a sixteen-dimensional Majorana-Weyl spinor (spinor conventions are detailed in Appendix A). In computing the effective action, the fields are decomposed as a sum of background and quantum pieces,

\[ A_\mu = A_\mu^{(b)} + A_\mu^{(q)}, \quad \psi = \psi^{(b)} + \psi^{(q)}. \]

Since the terms (1) of interest contain no derivatives of the background spinor fields, they will be chosen to be constant. Only the piece of the action quadratic in the quantum fields is relevant in computing the one-loop effective action. Choosing background field gauge, and including quantum ghost fields \( c^{(q)} \) and \( c^{*(q)} \) but no ghost background, this can be written (after rotation to a Euclidean metric)

\[ S_{\text{quad}} = \int d^{10}x \left( \frac{1}{2} A_\mu^{(q)} \left( D_\rho^{(b)} D^\rho_{\mu} \delta_{\mu\nu} - 2i F_{\mu\nu}^{(b)} \right) A_{\nu}^{(q)} + \frac{i}{2} \bar{\psi}^{(q)} \Gamma^\mu D_\mu \psi^{(q)} \right) \]

\[ - \frac{1}{2} A_\mu^{(q)} \bar{\psi}^{(b)} \Gamma_\mu \psi^{(q)} - \frac{1}{2} \bar{\psi}^{(q)} \Gamma_\nu \psi^{(b)} A_{\nu}^{(q)} + c^{(q)} D_\mu^{(b)} D_\mu^{(b)} c^{(q)} \right). \]  

In (2), there is no trace because the background fields are taken to be matrices in the adjoint representation of SU(2), and are contracted with quantum fields which in turn are vectors in the adjoint representation. The covariant derivatives with respect to background fields are defined by \( D_\mu^{(b)} \psi^{(q)} = \partial_\mu \psi^{(q)} - i A_\mu^{(b)} \psi^{(q)} \), and \( F_{\mu\nu}^{(b)} = \partial_\mu A_\nu^{(b)} - \partial_\nu A_\mu^{(b)} - i [A_\mu^{(b)}, A_\nu^{(b)}] \). Upon reduction to \((1+0)\) dimensions with coordinate \( \tau \), this piece of the action becomes (with \( i = 1, \ldots, 9 \)):

\[ S_{\text{quad}} = \int d\tau \left( \frac{1}{2} \Phi^* \Delta_1 \Phi + c^{*(q)} \left( D_\tau^{(b)} D_\tau^{(b)} - A_i^{(b)} A_i^{(b)} \right) c^{(q)} \right). \]

where

\[ \Phi^* = (A_\mu^{(q)}, \bar{\psi}^{(q)}), \quad \Phi = \begin{pmatrix} A_\mu^{(q)} \\ \bar{\psi}^{(q)} \end{pmatrix}, \]
and $\Delta_1$ is the operator

$$
\begin{bmatrix}
(D^{(b)}_\tau D^{(b)}_\tau - A^{(b)}_i A^{(b)}_i)\delta_{\mu\nu} - 2i F_{\mu\nu} & -\bar{\psi}^{(b)}(D^{(b)}_\tau) - i\Gamma_i A^{(b)}_i \\
-\Gamma_\nu\psi^{(b)} & i(\Gamma_0 D^{(b)}_\tau - i\Gamma_i A^{(b)}_i)
\end{bmatrix}.
$$

Integrating out the quantum fields, the one-loop partition function is

$$
Z_1 = \int [dA^{(q)}] [d\psi^{(q)}] [dc^{(q)}] [dc^{(q)}] \exp(-S_{quad})
= \frac{\det(D^{(b)}_\tau A^{(b)}_i A^{(b)}_i)}{(\text{sdet } \Delta_1)^{1/2}},
$$

where “det” and “sdet” denote the functional determinant and functional superdeterminant respectively. Since it is proposed to compute these from the functional trace and supertrace of the heat kernels associated with the relevant operators, it is necessary to convert the operator appearing in the superdeterminant into one which is of Laplace type. We make use of the definition

$$
\text{sdet} \begin{pmatrix} A & \chi \\ \Sigma & B \end{pmatrix} = \frac{\det(A - \chi B^{-1}\Sigma)}{\det B}
$$

to write

$$
\text{sdet } \Delta_1 = (\det i\Gamma.D)^{-1} \det(D^2\delta_{\mu\nu} - 2i F_{\mu\nu} - \bar{\psi}\Gamma_\mu \frac{1}{i(\Gamma.D)}\Gamma_\nu\psi),
$$

where the superscript “(b)” on background fields has been dropped since all fields which appear from now on will be backgrounds fields, $D^2$ is shorthand for $D^2 - A_i A_i$, and $\Gamma.D$ is shorthand for $(\Gamma_0 D_\tau - i\Gamma_i A_i)$. Using $(\Gamma.D)^2 = -D^2 1_{16} - \frac{i}{2} \Gamma_\rho F_{\rho\sigma} \equiv -D^2 1_{16} - \frac{i}{2} \Gamma.F$ (with $\Gamma_\rho F_{\rho\sigma} = \frac{1}{2} [\Gamma_\rho, \Gamma_\sigma]$), and $\{\Gamma_\rho, \Gamma_\sigma\} = -2\delta_{\rho\sigma} 1_{16}$ in the Euclidean metric), this can be rewritten

$$
\text{sdet } \Delta_1 = \left(\det(D^2 1_{16} + \frac{i}{2} \Gamma.F)\right)^{-\frac{1}{2}} \det(D^2\delta_{\mu\nu} - 2i F_{\mu\nu}
- \bar{\psi}\Gamma_\mu i(\Gamma.D) \frac{1}{D^2 1_{16} + \frac{i}{2} \Gamma.F} \Gamma_\nu\psi)
= \left(\det(D^2 + \frac{i}{2} \Gamma.F)\right)^{\frac{1}{2}} \text{sdet } \Delta_2,
$$

where $\Delta_2$ is the operator

$$
\Delta_2 = \begin{pmatrix}
D^2\delta_{\mu\nu} - 2i F_{\mu\nu} & \bar{\psi}\Gamma_\mu i(\Gamma.D) \\
\Gamma_\nu\psi & D^2 1_{16} + \frac{i}{2} \Gamma.F
\end{pmatrix}.
$$
Putting the results (3) and (4) together, the one loop effective action is

\[- \ln Z_1 = - \ln \det D^2 + \frac{1}{4} \ln \det (D^2 1_{16} + \frac{i}{2} \Gamma.F) + \frac{1}{2} \ln \text{sdet} \Delta_2.\]

Because all the operators are of Laplace type, this can be computed as

\[- \ln Z_1 = \int_0^\infty \frac{ds}{s} \left( \text{Tr} e^{sD^2} - \frac{1}{4} \text{Tr} e^{s(D^2 1_{16} + i \Gamma.F/2)} - \frac{1}{2} \text{Str} e^{s\Delta_2} \right), \tag{6}\]

where \(\text{Tr}\) and \(\text{Str}\) denote the functional trace and functional supertrace respectively, as well as traces over the gauge indices.

The background fields can be decomposed as

\[A_\mu = A_\mu^+ T_+ + A_\mu^3 T_3 + A_\mu^- T_-, \quad \psi = \psi^+ T_+ + \psi^3 T_3 + \psi^- T_-, \]

where \((T_+, T_3, T_-)\) are the \(3 \times 3\) matrix generators of the adjoint representation of \(\text{SU}(2)\). For the case of a scattering of \(D0\)-branes with relative speed \(v\) and impact parameter \(b\), the relevant supersymmetric Yang-Mills background is \(A_3^+ = v \tau, A_3^3 = b, \) and \(\psi_3^3\) constant, with all other components of \(A_\mu\) and \(\psi\) vanishing [4]. The only nontrivial components of the Yang-Mills field strength are thus \(F_{01} = - F_{10} = v T_3\), and \(\Gamma.F = 2v \Gamma_0 \Gamma_1 T_3\). Due to the fact that all of the background fields are proportional to \(T_3\), the operators in the effective action (3) decompose into two decoupled pieces with respect to their gauge indices. The operator acting on the quantum fields \(A_\mu^{(q)}, \psi^{3(q)}\) and \(c^{3(q)}\) is \(\partial_\tau^2\), so these are massless free fields which decouple [4]; their contribution to the effective action (6) is

\[\int_0^\infty \frac{ds}{s} (1 - \frac{1}{4} 16 - \frac{1}{2} 10 + \frac{1}{2} 16) \text{Tr} e^{s\partial_\tau^2},\]

which vanishes as a result of a cancellation between bosonic and fermionic (including ghost) degrees of freedom due to supersymmetry. The other piece is a \(2 \times 2\) block with respect to gauge indices, corresponding to the quantum fields \((A_\mu^{+(q)}, A_\mu^{-(q)}), (\psi^{+(q)}, \psi^{-(q)})\) and \((c^{+(q)}, c^{-(q)})\). So the trace over gauge indices in the expression (3) can be restricted to this \(2 \times 2\) block, in which the generator \(T_3\) accompanying the background fields is represented by the matrix

\[T_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}\]

With this simplification, determining the one-loop effective action reduces to the calculation of the functional traces and supertraces in the expression (3) in the presence of the background fields appropriate to the description of \(D0\)-brane scattering.
3 Evaluation of Functional Traces

Although the results of computation of the functional traces \( \text{Tr} e^{sD^2} \) and \( \text{Tr} e^{s(D^21_{16} + i\Gamma.F/2)} \) are well known [7], they are rederived here to illustrate the method that is to be employed to compute the more complicated functional supertrace \( \text{Str} e^{s\Delta_2} \). Concentrating on \( \text{Tr} e^{s(D^21_{16} + i\Gamma.F/2)} \), this is

\[
\int d\tau \lim_{\tau \to \tau'} \text{tr} e^{s(\partial^2\tau^2 - \tau^2 + \Gamma.F)} e^{s(X^2 - v^2\tau^2)1_{16}^{16}} \delta(\tau - \tau'),
\]

where \( 1_{16}^{16} \) denotes the tensor product of the unit matrices in the two-dimensional representation of the gauge group and the sixteen dimensional spinor representation, and “tr” is the trace over these representations. Representing the delta function in the form \( \int \frac{dk}{2\pi} e^{ik(\tau - \tau')} \), this can be written

\[
\int d\tau \int \frac{dk}{2\pi} \text{tr} \left( e^{s(-b^21_{16}^{16} + ivT_3\Gamma_0\Gamma_1) e^{s(X^2 - v^2\tau^2)1_{16}^{16}}} \right),
\]

where \( X = \partial_\tau + ik \). Performing the traces over the gauge and spinor indices yields

\[
\text{Tr} e^{s(D^21_{16} + i\Gamma.F/2)} = 32 \int d\tau \cosh sv e^{-sb^2} K_0(s), \quad (8)
\]

where

\[
K_0(s) \equiv \int \frac{dk}{2\pi} e^{s(X^2 - v^2\tau^2)}.
\]

Similarly,

\[
\text{Tr} e^{sD^2} = 2 \int d\tau e^{-sb^2} K_0(s). \quad (9)
\]

Thus one is left to evaluate \( K_0(s) \). To achieve this, we employ a method introduced in [13], which has the advantage that it extends naturally to the situation to be encountered when evaluating the functional supertrace \( \text{Str} e^{s\Delta_2} \) in (7). Noting that

\[
\frac{dK_0(s)}{ds} = K_2(s) - v^2\tau^2 K_0(s) \quad (10)
\]

with

\[
K_2(s) \equiv \int \frac{dk}{2\pi} X^2 e^{s(X^2 - v^2\tau^2)}
\]

the aim is to express \( K_2(s) \) in terms of \( K_0(s) \) so that the differential equation (10) for \( K_0(s) \) can be solved. To do this, one uses the fact that

\[
\int \frac{dk}{2\pi} \frac{\partial}{\partial k} \left( X e^{s(X^2 - v^2\tau^2)} \right) = 0.
\]
Performing the derivative gives

\[ 0 = iK_0(s) + \int \frac{dk}{2\pi} X \int_0^1 du e^{us(X^2-v^2\tau^2)} 2isX e^{(1-u)s(X^2-v^2\tau^2)} \]

\[ = iK_0(s) + 2i \sum_{n=0}^{\infty} \frac{s^{n+1}}{(n+1)!} \int \frac{dk}{2\pi} X \, \text{ad}^{(n)}(X^2 + v^2\tau^2)(X) e^{s(X^2-v^2\tau^2)}, \]

where \( \text{ad}^{(n)}(A)(B) \) stands for the \( n \) commutators \([A, [A, \cdots [A, B] \cdots]]\). The commutators are easily computed and the series summed to give

\[ 0 = K_0(s) + \frac{\sinh 2sv}{v} K_2(s) + \tau (\cosh 2sv - 1) K_1(s) + (\cosh 2sv - 1) K_0(s), \]

where \( K_1(s) \equiv \int \frac{dk}{2\pi} X e^{s(X^2-v^2\tau^2)} \). The latter is computed in terms of \( K_0(s) \) from the identity

\[ 0 = \int \frac{dk}{2\pi} \frac{\partial}{\partial k} e^{s(X^2-v^2\tau^2)}, \]

which yields

\[ K_1(s) = -\tau v \left( \frac{\cosh 2sv - 1}{\sinh 2sv} \right) K_0(s). \]

Solving (11) for \( K_2(s) \) and substituting into (10) gives the differential equation

\[ \frac{d \ln K_0(s)}{ds} = -v^2\tau^2 + \frac{v^2\tau^2 (\cosh 2sv - 1)^2}{\sinh^2 2sv} - v \frac{\cosh 2sv}{\sinh 2sv}, \]

which is solved (with the boundary condition that \( K_0(s) = (4\pi s)^{-\frac{1}{2}} \) in the limit \( v \to 0 \)) by

\[ K_0(s) = \left( \frac{v}{2\pi \sinh 2sv} \right)^{\frac{1}{2}} e^{-v\tau^2 \tanh sv}. \]

Substituting this result into (8) and (9) yields the standard expressions for the functional traces \( \text{Tr} e^{s(D^2 + \frac{i}{2} F/F^2)} \) and \( \text{Tr} e^{sD^2} \).

## 4 Evaluating the Functional Supertrace

The procedure in the previous section will be adapted to evaluate \( \text{Str} e^{s\Delta_2} \) in the effective action (6). Using (5), and replacing the delta function in the functional supertrace by its Fourier representation,

\[ \text{Str} e^{s\Delta_2} = \int d\tau \int \frac{dk}{2\pi} \, \text{str} e^{s\Delta} \equiv \int d\tau \, \tilde{K}_0(s), \]

where \( \text{str} \) denotes the supertrace.
where
\[
\Delta = \left( \begin{array}{c}
(X^2 - v^2 \tau^2 - b^2)1_{10}1_{2} - 2iF \quad i\bar{\psi}\Gamma(\Gamma_0X - i\Gamma_1v\tau T_3 - i\Gamma_2bT_3) \\
\Gamma\psi 
\end{array} \right) 
\]
with \( X = \partial_\tau + ik \). The supertrace “str” involves an ordinary trace over gauge indices, and a supertrace over Lorentz indices in the vector and spinor representations (which have been suppressed). We choose to write \( \Delta \) in the form
\[
\Delta = \tilde{X}^2 - (v^2 \tau^2 + b^2)1 + F.A + Y.
\]
Here, \( 1 \) is the tensor product of unit matrices in the gauge and Lorentz representations,
\[
1 = \left( \begin{array}{cc}
1_{10}1_{2} & 0 \\
0 & 1_{16}1_{2}
\end{array} \right).
\]
Also,
\[
F.A = \left( \begin{array}{cc}
-2iF_{\mu\nu} & 0 \\
0 & \frac{i}{2}F_{\rho\sigma}\Gamma_{\rho\sigma}
\end{array} \right)
\]
is the contraction of the Yang-Mills field strength with a supermatrix containing the generators of the vector and (Weyl) spinor representation of the Lorentz group in its diagonal blocks. The operator
\[
\tilde{X} = (\partial_\tau + ik)1 + N
\]
contains the supermatrix
\[
N = \left( \begin{array}{cc}
0 & \frac{i}{2}\bar{\psi}\Gamma_{\mu}\Gamma_{0} \\
0 & 0
\end{array} \right). \tag{15}
\]
It can be viewed as a supercovariant derivative (shifted by \( ik \)). Finally, \( Y \) is the off-diagonal supermatrix
\[
Y = \left( \begin{array}{cc}
0 & \bar{\psi}\Gamma_{\mu}\Gamma_{i}A_{i} \\
\Gamma_\nu\psi & 0
\end{array} \right). \tag{16}
\]
with \( \Gamma_iA_i = \Gamma_1v\tau T_3 + \Gamma_2bT_3 \) (note that we have used the fact that \( N^2 = 0 \), which eliminates a potential contribution to \( Y \)).

As in the case of the functional trace evaluated at the start of this section, the quantity \( \tilde{K}_0(s) = \int \frac{dk}{2\pi} e^{s\Delta} \) in (14) is computed using the fact that it satisfies the differential equation
\[
\frac{d\tilde{K}_0(s)}{ds} = \tilde{K}_2(s) + (-v^2 \tau^2 - b^2 + F.A + Y)\tilde{K}_0(s), \tag{17}
\]
where $\tilde{K}_n(s) = \int \frac{dk}{2\pi} \tilde{X}^n e^{i\Delta}$. As before, the requirement is to express $\tilde{K}_2(s)$ in terms of $\tilde{K}_0(s)$ to allow this differential equation to be solved. $\tilde{K}_2(s)$ is computed with the aid of the identity

$$\int \frac{dk}{2\pi} \tilde{X} e^{i\Delta} = i\tilde{K}_0(s) + 2i \sum_{n=0}^{\infty} \frac{s^{n+1}}{(n+1)!} \int \frac{dk}{2\pi} \tilde{X} \text{ad}^{(n)}(\Delta)(\tilde{X}) e^{i\Delta}. \tag{18}$$

Recalling that we are interested in the term in (1) of fourth order in the background fermions $\psi$, it is easily established that the commutators have the following structure to this order:

$$\text{ad}^{(2n)}(\Delta)(\tilde{X}) = 2^n v^{2n} \tilde{X} + N_{2n} \tilde{X} + Y_{2n}$$

and

$$\text{ad}^{(2n+1)}(\Delta)(\tilde{X}) = 2^{n+1} v^{2n+2} \tau \mathbf{1} + N_{2n+1} \tilde{X} + Y_{2n+1}, \tag{19}$$

with

$$N_1 = 0, \quad N_n = 2[\tilde{X}, Y_{n-1}] + [F.A + Y, N_{n-1}] \quad (n \geq 2) \tag{20}$$

and

$$Y_1 = [F.A + Y, \tilde{X}], \quad Y_2 = [F.A + Y, Y_1],$$

$$Y_{2n+1} = 2\tau v^2 N_{2n} + 2^n v^{2n} Y_1 + N_{2n} Y_1 + [F.A + Y, Y_{2n}],$$

$$Y_{2n+2} = 2\tau v^2 N_{2n+1} + N_{2n+1} Y_1 + [F.A + Y, Y_{2n+1}] \quad (n \geq 1). \tag{21}$$

The general form of the supermatrices $N_n$ and $Y_n$ in powers of $\psi$, $\tau$ and $b$ can be established inductively to be

$$N_n = \begin{pmatrix} O(\psi^4) & O(\psi^3) \\ 0 & O(\psi^4) \end{pmatrix}, \quad Y_n = \begin{pmatrix} O(\psi^2) + A_1 O(\psi^4) & O(\psi) + A_1 O(\psi^3) \\ O(\psi^3) & O(\psi^2) + A_1 O(\psi^4) \end{pmatrix}. \tag{22}$$

As a result, $[\tilde{X}, [\tilde{X}, Y_n]] = 0$ and $[\tilde{X}, N_n] = 0$ to order $\psi^4$, eliminating some potential additional contributions to (18), (20) and (21). The explicit form of the matrices $N_n$ and $Y_n$ is required only for $N_1, N_2, N_3$ and $Y_1$, and these are presented in Appendix B.

Using these results, (18) then yields

$$0 = \left( \frac{\sinh 2sv}{v} + 2N(s) \right) \tilde{K}_2(s) + \left( \tau(\cosh 2sv - 1) + 2[\tilde{X}, N(s)] + 2Y(s) \right) \tilde{K}_1(s) + \left( \cosh 2sv + 2[\tilde{X}, (Y(s))] \right) \tilde{K}_0(s), \tag{23}$$
where
\[ N(s) = \sum_{n=2}^{\infty} \frac{s^{n+1}}{(n+1)!} N_n, \quad Y(s) = \sum_{n=1}^{\infty} \frac{s^{n+1}}{(n+1)!} Y_n. \]

In order to solve for \( \tilde{K}_2(s) \) in terms of \( \tilde{K}_0(s) \), it is necessary to have an expression for \( \tilde{K}_1(s) \) in terms of \( \tilde{K}_0(s) \). This again follows from the identity
\[
0 = \int \frac{dk}{2\pi} \partial_k e^{k\Delta},
\]
which yields
\[
\tilde{K}_1(s) = - \left( \frac{\sinh 2sv}{v} + 2N(s) \right)^{-1} \left( \tau(\cosh 2sv - 1) + 2Y(s) \right) \tilde{K}_0(s).
\]
Substituting into (23),
\[
\tilde{K}_2(s) = v^2 \tau^2 \frac{(\cosh 2sv - 1)^2}{\sinh^2 2sv} \left( 1 + \frac{2v}{\sinh 2sv} N(s) \right)^{-1} \left( 1 + \frac{2}{\tau(\cosh 2sv - 1)} \left( [\tilde{X}, N(s)] + 2Y(s) \right) \right) \tilde{K}_0(s) - v \cosh 2sv \left( 1 + \frac{2v}{\sinh 2sv} N(s) \right)^{-1} \left( 1 + \frac{2}{\cosh 2sv} \left[ \tilde{X}, Y(s) \right] \right) \tilde{K}_0(s).
\]
This can be simplified considerably using the fact that to order \( \psi^4 \),
\[
N(s)^2 = 0, \quad [\tilde{X}, N(s)] = 0, \quad N(s)Y(s) = 0, \quad \text{and} \quad N(s)[\tilde{X}, Y(s)] = 0,
\]
as is easily established from (22). Substituting the simplified expression into (17) results in the following differential equation for \( \tilde{K}_0(s) \):
\[
\frac{d\ln \tilde{K}_0(s)}{ds} = -(v^2 \tau^2 + b^2) 1 + F.A + Y + \frac{\tau^2 v^2 (\cosh 2sv - 1)^2}{\sinh^2 2sv} \left( 1 - \frac{4v}{\sinh 2sv} N(s) \right) + \frac{4}{\tau(\cosh 2sv - 1)} Y(s) + \frac{4}{\tau^2(\cosh 2sv - 1)^2} Y(s)^2 - v \cosh 2sv \left( 1 - \frac{2v}{\sinh 2sv} N(s) + \frac{2}{\cosh 2sv} [\tilde{X}, Y(s)] \right). \]
If $\tilde{K}_0(s)$ is factored in the form $\tilde{K}_0(s) = e^{-s b^2} K_0(s) M(s)$, then using the property (12), it follows that

$$\frac{d \ln M(s)}{ds} = (F.\Lambda + Y) + 4\tau v^2 \left( \frac{\cosh 2sv - 1}{\sinh^2 2sv} \right) Y(s)$$

$$+ \frac{4v^2}{\sinh^2 2sv} Y(s)^2 - \frac{2v}{\sinh 2sv} [\tilde{X}, Y(s)]$$

$$+ \left( -4\tau^2 v^3 \frac{(2 \cosh 2sv - 1)^2}{\sinh^3 2sv} + 2v^2 \frac{\cosh 2sv}{\sinh^2 2sv} \right) N(s).$$

(24)

Thus, to order $\psi^4$, the functional supertrace appearing in the effective action (6) has the following expression:

$$\text{Str} \, e^s \Delta_2 = \int d\tau e^{-s b^2} K_0(s) \text{str} M(s),$$

where $K_0(s)$ is given in (13) and, to order $\psi^4$, $M(s)$ is the solution to the differential equation (24).

5 The One-loop Effective Action to Order $\psi^4$

Using the above result together with (3), (8) and (9), the one-loop effective action for the matrix model in the presence of background fields relevant to the description of D0-brane scattering is

$$- \ln Z_1 = \int d\tau \int_0^\infty ds \frac{ds}{s} e^{-s b^2} \left( 2 - 8 \cosh s v - \frac{1}{2} \text{str} M(s) \right) K_0(s).$$

In the absence of a fermionic background, all terms except the first on the right hand side of (24) vanish, so that $M(s) = e^{s F.\Lambda}$. Thus $\text{str} M(s) = (4 \cosh 2sv + 16) - 32 \cosh sv$, where the +16 is the contribution from the piece of the identity matrix $1_{10} \otimes 1_2$ which is not in the $2 \times 2$ block in which $F_{\mu\nu}$ is nonvanishing. With $K_0(s)$ as in (13), this reproduces the well known results of [7] for the effective action. The leading term in the expansion (1) of the long-range potential between two D0-branes comes from the order $s^4$ term in the expansion of $\cosh sv$ and $\cosh 2sv$, the the order $s^0$ terms vanishing because of the equality of the number of bosonic and fermionic degrees of freedom, and the order $s^2$ terms vanishing because of the vanishing of the supertrace of the mass squared matrix as a result of supersymmetry.
In the presence of a fermionic background, the matrix $M(s)$ is more complicated, giving rise to the expansion of the effective action in the form (1). In particular, to order $\psi^4$ in the background fermions, $M(s)$ is obtained from equation (24). The supertrace of $M(s)$ is considered in Appendix B order by order in its expansion in powers of $s$, where it is found (using spinor identities established in Appendix A) that there is a nonvanishing $\psi^2$ term at order $s^5$, and the leading $\psi^4$ term emerges at order $s^6$. As will be explained below, once the leading $\psi^2$ and $\psi^4$ terms have been found, it is not necessary to consider such terms at higher order in the power series expansion in $s$. The $\psi^2$ term is

$$s^5 \text{str} M_5 = \frac{s^5}{120} \text{tr} \left( 40i(F)_{\mu\nu}^3 G_{\nu\mu} - 10i(F)_{\mu\nu}^2 H_{\nu\mu} ight) - 10i v^2 F_{\mu\nu} G_{\nu\mu} + \frac{5i}{2} v^2 H_{\mu\nu} A_i,$$

where the trace is over gauge indices and

$$G_{\mu\nu\rho} = (\bar{\psi} \Gamma_{\mu\nu\rho} \psi), \quad H_{\mu\nu\rho} = (\bar{\psi} \Gamma_{\mu} \tilde{F} \Gamma_{\rho} \psi).$$

Substituting $A_1 = v \tau, A_2 = b$ yields the result $-is^5 v^3 b \text{ tr} (\bar{\psi} \Gamma_{012} \psi)$. However, it is also possible to evaluate (25) in a SO(9) invariant manner using $A_i = x_i T_3$ and $F_{0k} = v_k T_3$, where $x_i$ and $v_i$ ($i = 1, \ldots, 9$) are the relative coordinate and speed for the two D0-branes. In this case

$$s^5 \text{str} M_5 = -is^5 v^2 v_i x_j \text{ tr} G_{0ij}.$$ 

Recalling the $\psi = \psi_3 T_3$ with $T_3$ given by (7), this becomes

$$s^5 \text{str} M_5 = -2is^5 v^2 v_i x_j (\bar{\psi}_3 \Gamma_{0ij} \psi_3).$$

The first nonvanishing $\psi^4$ term occurs at order $s^6$ in the power series expansion of $M(s)$, and is

$$s^6 \text{str} M_6 = \frac{s^6}{720} \text{ tr} \left( -12F_{\mu\nu} G_{\nu\rho} F_{\rho\sigma} G_{\sigma\mu} - 24(F^2)_{\mu\nu} G_{\nu\rho} G_{\rho\mu} ight) + 6 F_{\mu\nu} G_{\nu\rho} H_{\rho\mu} + 6 G_{\mu\nu} F_{\nu\rho} H_{\rho\mu} - \frac{3}{4} H_{\mu\nu} H_{\nu\mu} A_i A_j.$$

Again, it is convenient to evaluate this in a SO(9) invariant form. Using (31) in Appendix A, and then applying the identity (30),

$$s^6 \text{str} M_6 = \frac{s^6}{15} v_k v_l v_i x_j \left( -15(\bar{\psi}_3 \Gamma_{0kl} \psi_3) (\bar{\psi}_3 \Gamma_{0ij} \psi_3) ight) + 2\delta_{ij} (\bar{\psi}_3 \Gamma_{0i\mu} \psi_3) (\bar{\psi}_3 \Gamma_{0k\mu} \psi_3) - 3\delta_{il} (\bar{\psi}_3 \Gamma_{0j\mu} \psi_3) (\bar{\psi}_3 \Gamma_{0k\mu} \psi_3).$$
Collecting the above results, the effective action is

\[-\ln Z_1 = \int d\tau \int_0^\infty \frac{ds}{s} e^{-st^2} \left( -s^4 v^4 \frac{s^5}{2} \text{str } M_5 - \frac{s^6}{2} \text{str } M_6 \right) K_0(s).\]

From (13), the term of leading order in $v$ in $K_0(s)$ is

\[K_0(s) \approx (4\pi s)^{-1/2} e^{-v^2 r^2 s}.\]

So, after a rescaling $s \to \frac{s}{r^2}$ with $r^2 = b^2 + v^2 r^2$,

\[-\ln Z_1 = \frac{1}{\sqrt{4\pi}} \int d\tau \int_0^\infty ds e^{-s} \left( -s^{5/2} \frac{v^4}{r^7} - \frac{s^{7/2}}{2r^9} \text{str } M_5 - \frac{s^{9/2}}{2r^{11}} \text{str } M_6 \right)\]

\[= \int d\tau \left( -\frac{15}{16} \frac{v^4}{r^7} + \frac{105i}{32} \frac{v^2 x_i x_j}{r^9} (\bar{\psi}_3 \Gamma_{0ij} \psi_3) \right.\]

\[+ \left. \frac{63}{128} \frac{1}{r^{11}} v_k v_l x_i x_j \left( 15(\bar{\psi}_3 \Gamma_{0kl} \psi_3) (\bar{\psi}_3 \Gamma_{0lj} \psi_3) \right. \right.\]

\[- \left. \left. 2 \delta_{ij} (\bar{\psi}_3 \Gamma_{0ij} \psi_3) (\bar{\psi}_3 \Gamma_{0kl} \psi_3) + 3 \delta_{il} (\bar{\psi}_3 \Gamma_{0il} \psi_3) (\bar{\psi}_3 \Gamma_{0jk} \psi_3) \right) \right). (26)\]

The order $\psi^4$ terms are of the form $\frac{v^2 x^4}{r^{11}}$, and correspond to the $\frac{v^2 x^4}{r^9}$ terms in (1).

In principle, it is possible to obtain contributions of the form $\frac{v^3 x^2 \psi^2}{r^5}$ from the order $s^6$ term in the expansion of $M(s)$, namely through terms with the structure $v^3 x^3 \psi^2$. However, in practice, this is not possible. The coordinate $x_i$ in the form of $A_i$, is accompanied by a factor of at least $\psi$ in all the supermatrices used in the computation, the lowest order case being in the off-diagonal entries in the supermatrix $Y$ in (16). To get it into the diagonal to appear in supertraces, it picks up at least another factor of $\psi$. So a term with a factor of $x^3$ must be of at least order $\psi^6$. This argument shows that once a nonvanishing $\psi^2$ or $\psi^4$ term is found in the expansion of $M(s)$ in powers of $s$, higher order terms in this expansion need not be considered.

6 Interpretation of the Effective Action

In this section, we relate the matrix model effective action (24) to the spin dependence of the long-range potential between the moving D0-branes. After rotation back to the Minkowski metric, the order $\psi^2$ term is the same as that
obtained by Kraus [10]. By studying the action for a D0-brane probe moving in the linearized metric of a D0-brane with angular momentum, Kraus showed that this term correctly reproduces the spin-orbit interaction provided \( \frac{1}{2} (\bar{\psi}_3 \Gamma_{0ij} \psi_3) \) is identified with the angular momentum \( J_{ij} \) of the target D0-brane. This term in the effective action can be written in the form

\[
\int d\tau V_{\text{spin-orbit}}(r) = - \int d\tau \frac{15}{16} v^2 J_{ij} v^j \partial_i \frac{1}{r^7},
\]

which exhibits it as a derivative of the 9-dimensional Green’s function.

According to Harvey [9], the order \( \psi^4 \) terms in (26) should correspond to higher order spin-orbit couplings or possibly spin-spin couplings. To obtain an expression of this form, it is necessary to integrate the last term in (26) by parts with respect to \( \tau \) by writing

\[
\frac{(x_i v_i) v_j}{r^{11}} = - \frac{1}{9} \frac{\partial}{\partial \tau} \left( \frac{x_j}{r^9} \right) + \frac{1}{9} \frac{v_j}{r^9}.
\]

The last three terms in (26) can then be expressed as

\[
\frac{15}{128} \int d\tau v_k v_l (\bar{\psi}_3 \Gamma_{0ik} \psi_3) (\bar{\psi}_3 \Gamma_{0lj} \psi_3) \left( 63 \frac{x_i x_j}{r^{11}} - \frac{7 \delta_{ij}}{r^9} \right) = \frac{15}{128} \int d\tau v_k v_l (\bar{\psi}_3 \Gamma_{0ik} \psi_3) (\bar{\psi}_3 \Gamma_{0lj} \psi_3) \partial_i \partial_j \frac{1}{r^7}.
\]

Using the identification

\[
\frac{1}{2} (\bar{\psi}_3 \Gamma_{0ij} \psi_3) = J_{ij}
\]

from the spin-orbit interaction, this is clearly a higher order spin-orbit interaction.

This result can be compared with that of Morales et al [15], who have calculated spin dependent effects in scattering amplitudes for D-branes using string theoretic methods. In the notation of the present paper, they find that the contribution to the scattering amplitude for two D0-branes which is quadratic in the angular momentum of the form (in the Minkowski metric, and after restoring the trace over gauge indices)

\[
\frac{1}{192} V_1 T_0^2 \frac{1}{128} \text{tr} \left( 2 v^2 G^{i0} G^j_{0\mu} - v^2 G^{mn} G^j_{mn} + 4v^k v^l G^{i0k} G^{jl}_{0\mu} \right) \partial_i \partial_j G_9(r),
\]

\[\text{Comparison of the normalization of the spin-orbit terms shows that the quantity } J_{\mu\nu\rho} \text{ in the paper of Morales et al [15] is equivalent to } \frac{1}{9} (\bar{\psi}_3 \Gamma_{\mu\nu\rho} \psi_3) \text{ in the present paper.}\]
where $V_1$ is the D0-brane volume, $T_0$ is the tension (mass) and $G_9(r)$ is the nine-dimensional Green’s function. Using one of the identities (28),

$$2 G^{0\mu} G^i_{0\mu} - G^{mn} G^j_{mn} = 4 G^{0\mu} G^i_{0\mu}.$$ 

Noting the fact that there is an implicit $\delta_{00}$ factor on the $G_{ik\mu} G_{j\mu}$ term in the identity (30) which changes sign on rotation to the Minkowski metric, the result can be rewritten

$$\frac{1}{48} V_1 T_0^2 v^k v^l \frac{1}{128} \text{tr} \left( 6 G^{0l} G^{j}_{0l} - \delta^{ij} G^{0\mu} G_{0\mu} + 2 \delta^{ij} G^{0\mu} G_{0\mu} \right) \partial_i \partial_j G_9(r).$$

The last term is a total $\tau$ derivative and so doesn’t contribute to the effective action. The second term vanishes using the fact that the Green’s function is annihilated by the Laplacian. The remaining term is a higher order spin-orbit interaction of the form obtained in the matrix model calculation in this paper. Comparison with the spin-independent term in the potential in [15] shows that $V_1 T_0^2 G_9(r) = \frac{60}{r^7}$, which then produces agreement between the matrix model result and the string theoretic result.

7 Conclusion

The $v^2 \psi^4$ term in the effective action (1) has been calculated using the matrix model, and shown to be consistent with the corresponding term in the scattering amplitude for a pair of D0-branes using string theoretic methods. The technique employed in this paper is well adapted to the underlying supersymmetry of the matrix model by recognizing that the one-loop effective action is a superdeterminant. The procedure for evaluating the superdeterminant is straightforward (but laborious), in that it only requires the computation of commutators of supermatrices. In contrast, the method used in [10] and [11] to calculate the $\frac{v^2 \psi^4}{r^3}$ and $\frac{v^4}{r^7}$ terms respectively in the effective action involves many factors of the full bosonic propagator which are acted on by derivatives coming from the fermionic propagators. Each of the vertices also introduces a $\tau$ integral. Although tractable for the two point function computed in [10], and the static case in [11] (where the derivatives from the fermionic propagators play no role), it is likely to be cumbersome for higher order amplitudes such as the one computed in this paper. The technique used here will also extend to the treatment of more general Dp-brane scattering amplitudes, involving the computation of the one-loop effective action in the presence of a
Fermionic background for 10D supersymmetric Yang-Mills theory reduced to (p+1)-dimensions.

**Appendix A**

Here, spinor identities used in computing the supertrace of the matrix $M(s)$ determined by the differential equation (24) will be established. We use the spinor conventions of [12]. The ten-dimensional gamma matrices obey the anticommutation relations $\{\Gamma_\mu, \Gamma_\nu\} = -2\eta_{\mu\nu}1_{32}$, with metric $(-, +, +, \cdots, +)$. The spinor field $\psi$ of ten-dimensional supersymmetric Yang-Mills theory satisfies the Weyl constraint $(1 - \Gamma_{11})\psi = 0$, as well as the Majorana condition $\bar{\psi} = \psi^T C$, where $CT\mu C^{-1} = -\Gamma^T_\mu$. Although the gamma matrices in ten-dimensions are 32 dimensional, the Weyl constraint means that attention can be restricted to the appropriate 16 dimensional chiral projection. Throughout the paper, greek letters $\mu, \nu, \cdots$ take the values $0, \cdots, 9$ and latin letters $i, j, \cdots$ take the values $1, \cdots, 9$.

With $\Gamma_{\mu_1 \cdots \mu_n}$ denoting the totally antisymmetric product of the gamma matrices $\Gamma_{\mu_1}, \cdots, \Gamma_{\mu_n}$, the fermion bilinears $(\bar{\psi}\Gamma_{\mu_1 \cdots \mu_n}\psi)$ in the background fermions $\psi = \psi_3 T_3$ vanish for $n$ even as a consequence of the Weyl constraint. For $n$ odd, $(CT_{\mu_1 \cdots \mu_n})_{\alpha\beta}$ is antisymmetric in the spinor indices $\alpha$ and $\beta$ in the cases $n = 3$ and $n = 7$ and otherwise symmetric, so $\bar{\psi}^\alpha (CT_{\mu_1 \cdots \mu_n})_{\alpha\beta}\psi^\beta$ vanishes except in these two cases. Since $(\bar{\psi}\Gamma_{\mu_1 \cdots \mu_7}\psi)$ is proportional to $\epsilon_{\mu_1 \cdots \mu_7 \mu_8 \mu_9 \mu_{10}} (\bar{\psi}\Gamma_{\mu_8 \mu_9 \mu_{10}}\psi)$, the only independent nonvanishing bilinear in the background fermions is

$$(\bar{\psi}\Gamma_{\mu_1 \mu_2 \mu_3}\psi) \equiv G_{\mu_1 \mu_2 \mu_3}.$$

Consequently, the Fierz identity

$$\psi \bar{\psi} = -\frac{1}{96} (\bar{\psi}\Gamma_{\mu_1 \mu_2 \mu_3}\psi) P_- \Gamma_{\mu_1 \mu_2 \mu_3} P_+$$

applies, where $P_\pm = \frac{1}{2}(1 \pm \Gamma_{11})$. This is proved by multiplying both sides by $\Gamma_{\mu_1 \mu_2 \mu_3}$ and tracing, not forgetting that Weyl projectors make the Gamma matrices effectively 16 dimensional.

In computing the one-loop effective action, a rotation to the Euclidean metric is made, and the gamma matrices then obey $\{\Gamma_\mu, \Gamma_\nu\} = -2\delta_{\mu\nu}1_{16}$, where Weyl projectors are implicit on the gamma matrices. The only potentially nonvanishing bilinears are $(\bar{\psi}\Gamma_\mu \Gamma_\nu \Gamma_\rho \psi)$, for which the totally antisymmetric piece is $G_{\mu\nu\rho}$ defined above, and for which any symmetric piece...
vanishes. The contraction of $G_{\mu\nu\rho}$ with an expression symmetric in a pair of indices (such as $\delta_{\mu\nu}$ and $(F^2)_{\mu\nu}$) is thus zero. The following identities are also true:

$$G_{\mu\nu\rho} G_{\mu\nu\rho} = 0, \quad G_{\rho\mu\nu} G_{\sigma\mu\nu} = 0.$$  \hfill (28)

The first follows as a consequence of the second. The second is proved by contracting the Fierz identity on the left with $\bar{\psi} \Gamma_{\rho} \Gamma_{\mu} \Gamma_{\nu}$ and with $\Gamma_{\nu} \Gamma_{\mu} \Gamma_{\sigma}$ on the right, and using the identity

$$\Gamma_{\nu} \Gamma_{\mu_1} \cdots \Gamma_{\mu_n} \Gamma_{\nu} = (-1)^{(n+1)(10 - 2n)} \Gamma_{\mu_1} \cdots \Gamma_{\mu_n}$$  \hfill (29)

twice. By applying the Fierz identity “in reverse” on the result, one obtains an expression proportional $(\bar{\psi} \Gamma_{\rho} \psi) (\bar{\psi} \Gamma_{\sigma} \psi)$, which vanishes. This argument only works because two gamma matrices are moved through each of the projectors in the Fierz identity in order to use the result (29); if one attempts to apply the same trick to $G_{\rho\sigma\mu} G_{\gamma\delta\mu}$, only one gamma matrix is moved through each of the projectors, thus interchanging them and preventing the use of the Fierz identity “in reverse.” So $G_{\rho\sigma\mu} G_{\gamma\delta\mu}$ is in general nonzero.

Another identity which will be used is

$$0 = v_k v_l A_i A_j \left( -6 G_{0ik} G_{0jl} + \delta_{kl} G_{0ij} G_{0jk} + \delta_{ij} G_{0ik} G_{0lj} + G_{ikl} G_{jlm} - 2 \delta_{jk} G_{0im} G_{0lm} \right).$$  \hfill (30)

This is proved by applying the Fierz identity to obtain

$$G_{0ik} G_{0jl} = -\frac{1}{96} G_{\mu_1 \mu_2 \mu_3} (\bar{\psi} \Gamma_{0} \Gamma_{i} \Gamma_{k} \Gamma_{\mu_1 \mu_2 \mu_3} \Gamma_{0} \Gamma_{j} \Gamma_{\mu}),$$

and then “shuffling” the matrices $\Gamma_0, \Gamma_i$ and $\Gamma_k$ through $\Gamma_{\mu_1 \mu_2 \mu_3}$.

With $F_{0k} = v_k$ and $H_{\mu i} = (\bar{\psi} \Gamma_{\mu} \Gamma_{i} \Gamma \nu \psi)$, it is also possible to prove by similar tricks that

$$F_{\mu\nu} G_{\nu\rho} G_{\rho j} = v^2 G_{0\mu j} G_{0\nu j} + v_k v_l G_{0lp} G_{jkl},$$

$$F_{\mu\nu} G_{\nu\rho} H_{\rho j} = -2 v_k v_l G_{ikl} G_{jkl} - 2 v^2 G_{0\mu j} G_{0\nu j},$$

$$G_{\mu i} F_{\rho j} = 4 v_j v_k G_{0jk} G_{0\mu i} - 2 v_k v_l G_{0lp} G_{jkl} + 4 v_j v_k G_{0\nu j} G_{0\mu k},$$

$$H_{\mu i} H_{\nu j} A_i A_j = 64 v_k v_l G_{0ik} G_{0jl} A_i A_j.$$  \hfill (31)

Only the last of these is difficult to prove, and involves “shuffling” gamma matrices so that $\Gamma_\mu$ and $\Gamma_\nu$ appear together, then applying the Fierz identity and using (29) twice.
Appendix B

In this Appendix, the supertrace of the matrix $M(s)$ defined by (24) is computed order by order in its power series expansion in $s$. Many terms in the expansion have a vanishing supertrace as a result of the following identities:

$$str N_n = 0, \quad str Y_n = 0, \quad str Z_n = 0,$$

where $Z_n = [\tilde{X}, Y_n]$. We will prove these before proceeding.

Beginning with $str N_n$, using the definition (20) and the fact that the supertrace of a commutator of matrices is zero as a result of the cyclic property of the supertrace, the only potential contribution is from the $\tau$ derivative in the commutator $2[\tilde{X}, Y_{n-1}]$. Using the fact that $N_n$ and $Y_1$ are independent of $\tau$, and that $Y_1 = -[\tilde{X}, F.\Lambda + Y]$ has a vanishing supertrace because $\partial_\tau(F.\Lambda + Y)$ is off-diagonal, it follows from (21) that $2\partial_\tau str Y_{n-1} = 4v^2 str N_{n-2}$. Thus the vanishing of $str N_n$ follows by induction if $N_2$ and $N_3$ have vanishing supertraces. The former is easily computed explicitly as

$$N_2 = \begin{pmatrix} 0 & -G.\gamma \rho \psi \Gamma. & \rho \beta \Gamma. \\ 0 & 0 & \rho \beta \Gamma. \end{pmatrix}.$$  

This obviously has vanishing supertrace, while from (20) and (21),

$$str N_3 = 2 \partial_\tau str Y_2 = 2 \partial_\tau str [F.\Lambda + Y, Y_1] = 0.$$  

The vanishing of $str Y_n$ follows from the definitions (21), the fact that $Y_1$ and $N_n$ both have vanishing supertrace, and that $N_n Y_1$ vanishes to order $\psi^4$. Similarly, $str Z_n = str [\tilde{X}, Y_n]$, as the only potential contribution is $str \partial_\tau Y_n = \partial_\tau str Y_n$, which vanishes.

We now proceed with the evaluation of the supertrace of $M(s)$. The right hand side of (24) is expanded as a power series in $s$, integrated and then exponentiated. Consider first the terms in $M(s)$ up to order $s^3$:

$$M(s) = 1 + sG + \frac{s^2}{12} \left(6G^2 - N_2\right) + \frac{s^3}{144} \left(24G^3 - 12GN_2 + 48v^2 \tau Y_1 + 12Y_1^2 - 8Z_2 + N_3\right).$$

Here, the notation $G = F.\Lambda + Y$ and $Z_n = [\tilde{X}, Y_n]$ (so $Z_1 = \frac{1}{2}N_2$ by (20)) has been introduced for convenience. The terms independent of $\psi$ have already been discussed earlier. The supertrace of the terms dependent on $\psi$
vanishes. This follows from the identities (32) as well as the specific results that \(\text{str} Y^2\), \(\text{str} (F.A)Y^2\) and \(\text{str} Y_1\) and \(\text{str} GN_2\) all vanish. The latter two are true because, using the explicit expression for \(N_2\) given earlier and

\[
Y_1 = \begin{pmatrix}
-\frac{i}{2}G_{\mu\rho} & F_{\mu\rho} \bar{\psi} \Gamma_\rho \Gamma_0 \\
0 & \frac{i}{2} \Gamma_\rho \psi \bar{\psi} \Gamma_\rho \Gamma_0
\end{pmatrix},
\]

both \(\text{str} Y_1^2\) and \(\text{str} GN_2\) are proportional to \(G_{\mu0\rho}G_{\rho\mu}\), which vanish by (28). In the case of \(\text{str} Y^2\), using the expression (10), this is proportional to \(v\tau G_{\mu\rho\mu} + b G_{\mu2\mu}\), which vanishes due to the antisymmetry of \(G_{\mu\nu\rho}\). The term \(\text{str} (F.A)Y^2\), is the only nontrivial one, in that it vanishes due to a cancellation of two potentially nonzero contributions proportional to \(G_{012}\). These are

\[
-2iF_{\mu\nu}G_{\nu\mu}A_i = -4i vbG_{012}\] and 
\[
\frac{i}{2}(\bar{\psi} \Gamma_\mu \Gamma_i F_{\nu\rho} \Gamma_{\nu\rho} \Gamma_\mu \psi)A_i = 4i vbG_{012}.
\]

At order \(s^4\), after eliminating the terms which vanish via the identities (32),

\[
\text{str} M(s) = \frac{s^4}{288} \left( 12G^4 + 96v^2\tau GY_1 - 12G^2N_2 + 2GN_3 - 16GZ_2 + 24GY_1^2 + N_2^2 + 12Y_1Y_2 \right).
\]

The last two terms don’t contribute because \(\text{str} N_2^2\) is of order \(\psi^6\), and \(\text{str} Y_1Y_2 = \text{str} Y_1[G,Y_1]\) is zero using the cyclic property of the supertrace. The terms involving \(\text{str} GN_3\) and \(\text{str} GZ_2\) can be combined since \(\text{str} GN_3 = 2\text{str} GZ_2\) using (21). At this point, it is necessary to explicitly evaluate supertraces. To order \(\psi^2\), only the first two terms contribute, yielding the bilinears \((F^2)_{\mu\nu}G_{\nu\mu}\) and \((\bar{\psi} \Gamma_i \psi)\), both of which vanish by the results of Appendix A. To obtain the \(\psi^4\) terms, an explicit expression for \(N_3\) must be computed:

\[
N_3 = \begin{pmatrix}
2G_{\mu0\rho}G_{\rho\mu} & 4iF_{\mu\rho}G_{\rho0\sigma} \bar{\psi} \Gamma_\sigma \Gamma_0 - 2i v G_{\mu0\rho} \bar{\psi} \Gamma_\rho \Gamma_1 \\
0 & -2\Gamma_\rho \psi G_{\rho0\sigma} \bar{\psi} \Gamma_\sigma \Gamma_0
\end{pmatrix}.
\]

The order \(\psi^4\) terms are proportional to \(G_{\mu\nu\rho}G_{\rho\mu\nu}\) and \(G_{00\mu}G_{01\mu}\), which vanish via (28) and the antisymmetry of \(G_{\mu\nu\rho}\) respectively.

At order \(s^5\), again eliminating the terms which vanish via the identities (32), as well as terms of the form \(N_nN_m\), \(N_nY_m\) and \(N_nZ_m\) (which are all of order \(\psi^5\)),

\[
\text{str} M(s) = \frac{s^5}{120} G^5 - \frac{G^3N_2}{72} + \frac{G^2}{288} (N_3 + 48v^2\tau Y_1 + 12Y_1^2 - 8Z_2)
\]
\[
G_{2880} (160v^2N_2 + 3N_4 + 240v^2\tau Y_2 + 60Y_1Y_2 + 60Y_2Y_1 - 30Z_3)
\]
\[
+ \frac{1}{7200} (-480v^2Y_1^2 + 40Y_2^2 + 60Y_1Y_3) \Bigg). 
\]

This expression can be simplified somewhat. Using \(Y_2 = [G,Y_1]\), it follows that \(\text{str}GY_2\) vanishes by the cyclic property of the supertrace. Similarly, \(\text{str}G^4N_4 = 2\text{str}GZ_3\) using (20). Using (21) and the Jacobi identity, this can be further simplified to \(\text{str}G^4N_4 = \text{str}G(8v^2N_2 + 2[Y_2,Y_1])\). By a similar procedure, \(\text{str}G^2N_3 = 2\text{str}G^2[Y_1,Y_1] = 0\). Also \(\text{str}Y_1Y_3 = 4v^2\text{str}Y_1^2 - \text{str}[Y_1,Y_2]\). To order \(\psi^2\), the only potential contributions are from \(\frac{1}{6}v^2\tau \text{str}G^2Y_1 + \frac{1}{120} \text{str}G^5\), yielding

\[
\frac{1}{120} \text{tr} \left( 40i(F)^3_{\mu\nu}G_{\nu\mu} - 10i(F)^2_{\mu\nu}H_{\nu\mu} - 10iv^2F_{\mu\nu}G_{\nu\mu} + \frac{5i}{2}v^2H_{\mu\nu} \right) A_i, 
\]

where the trace is over gauge indices and

\[
H_{\mu\nu\iota} = (\bar{\psi}\Gamma_{\iota}\bar{F}\Gamma_{\mu}\psi). 
\]

At order \(\psi^4\), there is a nontrivial cancellation between potentially nonzero terms proportional to \(v^2G_{01\mu}G_{01\mu}\) to give a net result of zero.

Having identified the leading \(\psi^2\) term at order \(s^5\) in the expansion of \(M(s)\), only \(\psi^4\) terms need be considered at order \(s^6\). Again eliminating terms which are obviously zero,

\[
\text{str}M(s) = s^6 \text{str} \left( \frac{G^6}{720} - \frac{G^4N_2}{288} + \frac{1}{864}G^3(N_3 + 48v^2\tau Y_1 + 12Y_1^2 - 8Z_2) 
\]
\[
+ \frac{1}{5760}G^2(160v^2N_2 + 3N_4 + 240v^2\tau Y_2 + 120Y_1Y_2 - 30Z_3) 
\]
\[
+ \frac{1}{7200}G(-480v^4\tau^2N_2 + 20v^2N_3 + N_5 - 1440v^4\tau Y_1 - 480v^2Y_1^2 
\]
\[
+ 40Y_2^2 + 120v^2\tau Y_3 + 60Y_1Y_3 + 160v^2Z_2 - 12Z_4) 
\]
\[
+ \frac{1}{2160}(-80v^2Y_1Y_2 + 5Y_2Y_3 + 3Y_1Y_4 + 120v^4\tau^2Y_1^2) \right). 
\]

Many of the terms can be simplified using the definitions (20) and (21) together with the Jacobi identity and the fact that \(\text{str}G^n[W,G] = 0\) for any supermatrix \(W\). One finds that \(\text{str}G^2Y_2\), \(\text{str}G^3Z_2\), \(\text{str}G^3N_3\), \(\text{str}Y_1Y_2\), \(\text{str}Y_2Y_3\),
and $\text{str} Y_1 Y_4$ all vanish. Also,

$$
\text{str} G(N_5 - 12Z_4) = -20 \text{str} GN_3 = 0,
\text{str} GY_3 = 2v^2 \tau \text{str} GN_2 + 4v^2 \text{str} GY_1,
\text{str} G^2(3N_4 - 30Z_3) = -96v^2 \text{str} G^2 N_2 - 24 \text{str} G^2[Y_2, Y_1].
$$

On computing the remaining supertraces, one finds a nonvanishing contribution only from $\frac{1}{720} \text{str} G^6$, which yields

$$
\frac{1}{720} \text{tr} \left( -12F_{\mu\nu}G_{\nu\rho\sigma}F_{\rho\sigma}G_{\rho\mu} - 24(F^2)_{\mu\rho}G_{\nu\rho\sigma}G_{\rho\mu} \\
+ 6F_{\mu\nu}G_{\nu\rho\sigma}H_{\rho\mu} + 6G_{\mu\rho}F_{\nu\rho}H_{\nu\mu} - \frac{3}{4}H_{\mu\rho\sigma}H_{\nu\mu} \right) A_i A_j,
$$

with $H_{\mu\rho\sigma} = (\bar{\psi} \Gamma_\mu \Gamma_i \bar{F} \Gamma_\rho \psi)$.

**Note Added:** [hep-th/9806081](https://arxiv.org/abs/hep-th/9806081) has just appeared, which extends the work of [15] and compares it with 11-dimensional supergravity.

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