On the $f$-Norm Ergodicity of Markov Processes in Continuous Time

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Abstract

Consider a Markov process $\Phi = \{\Phi(t) : t \geq 0\}$ evolving on a Polish space $X$. A version of the $f$-Norm Ergodic Theorem is obtained: Suppose that the process is $\psi$-irreducible and aperiodic. For a given function $f: X \to [1, \infty)$, under suitable conditions on the process the following are equivalent:

(i) There is a unique invariant probability measure $\pi$ satisfying $\int f \, d\pi < \infty$.

(ii) There is a closed set $C$ satisfying $\psi(C) > 0$ that is “self $f$-regular.”

(iii) There is a function $V: X \to (0, \infty]$ that is finite on at least one point in $X$, for which the following Lyapunov drift condition is satisfied,

$$\mathcal{D}V \leq -f + b_C,$$

where $C$ is a closed small set and $\mathcal{D}$ is the extended generator of the process.

For discrete-time chains the result is well-known. Moreover, in that case, the ergodicity of $\Phi$ under a suitable norm is also obtained: For each initial condition $x \in X$ satisfying $V(x) < \infty$, and any function $g: X \to \mathbb{R}$ for which $|g|$ is bounded by $f$,

$$\lim_{t \to \infty} \mathbb{E}_x[g(\Phi(t))] = \int g \, d\pi.$$

Possible approaches are explored for establishing appropriate versions of corresponding results in continuous time, under appropriate assumptions on the process $\Phi$ or on the function $g$.

Keywords: Markov process, continuous time, generator, stochastic Lyapunov function, ergodicity

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1 Introduction

Consider a Markov process \( \Phi = \{ \Phi(t) : t \geq 0 \} \) in continuous time, evolving on a Polish space \( X \), equipped with its Borel \( \sigma \)-field \( \mathcal{B} \). Assume it is a nonexplosive Borel right process: It satisfies the strong Markov property and has right-continuous sample paths \([1, 9]\).

The distribution of the process \( \Phi \) is described by the initial condition \( \Phi(0) = x \in X \) and the transition semigroup: For any \( t \geq 0 \), \( x \in X \), \( A \in \mathcal{B} \),

\[
P^t(x, A) := \mathbb{P}_x(\{ \Phi(t) \in A \}) := \Pr(\Phi(t) \in A | \Phi(0) = x).
\]

A set \( C \) is called small if there is probability measure \( \nu \) on \((X, \mathcal{B}) \), a time \( T > 0 \), and a constant \( \varepsilon > 0 \) such that,

\[
P^T(x, A) \geq \varepsilon \nu(A), \quad \text{for every } A \in \mathcal{B}.
\]

It is assumed that the process is \( \psi \)-irreducible and aperiodic, where \( \psi \) is a probability measure on \((X, \mathcal{B}) \). This means that for each set \( A \in \mathcal{B} \) satisfying \( \psi(A) > 0 \), and each \( x \in X \),

\[
P^t(x, A) > 0, \quad \text{for all } t \text{ sufficiently large.}
\]

It follows that there is a countable covering of the state space by small sets \([8, \text{Prop. 3.4}]\).

The Lyapunov theory considered in this paper and in our previous work \([5, 9]\) is based on the extended generator of \( \Phi \), denoted \( D \). A function \( h : X \to \mathbb{R} \) is in the domain of \( D \) if there exists a function \( g : X \to \mathbb{R} \) such that the stochastic process defined by,

\[
M(t) = h(\Phi(t)) - \int_0^t g(\Phi(s)) \, ds, \quad t \geq 0,
\]

is a local martingale, for each initial condition \( \Phi(0) \) \([1, 13]\). We then write \( g = Dh \).

For example, consider a diffusion on \( X = \mathbb{R}^d \), namely, the solution of the stochastic differential equation,

\[
d\Phi(t) = u(\Phi(t)) \, dt + M(\Phi(t)) \, dB(t), \quad t \geq 0, \ \Phi(0) = x,
\]

where \( u = (u_1, u_2, \ldots, u_d)^T : X \to \mathbb{R}^d \) and \( M : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^k \) are Lipschitz, and \( B = \{ B(t) : t \geq 0 \} \) is \( k \)-dimensional standard Brownian motion. If the function \( h : X \to \mathbb{R} \) is \( C^2 \) then we can write \([13]\),

\[
Dh(x) = \sum_i u_i(x) \frac{d}{dx_i} h(x) + \frac{1}{2} \sum_{ij} \Sigma_{ij}(x) \frac{d^2}{dx_i dx_j} h(x), \quad x \in X.
\]

The Lyapunov condition considered in this paper is Condition (V3) of \([9]\): For a function \( V : X \to (0, \infty) \) which is finite for at least one \( x \in X \), a function \( f : X \to [1, \infty) \), a constant \( b < \infty \), and a closed, small set \( C \in \mathcal{B} \),

\[
DV \leq -\delta f + b\mathbb{1}_C. \quad \text{(V3)}
\]

It is entirely analogous to its discrete-time counterpart \([11]\), in which the extended generator is replaced by a difference operator \( D = P - I \), where \( P \) is the transition kernel of the discrete-time chain and \( I \) is the identity operator.
The lower bound \( f \geq 1 \) is imposed in (V3) because this function is used to define two norms: One on measurable functions \( g : X \to \mathbb{R} \) via,

\[
\|g\|_f := \sup_{x \in X} \frac{|g(x)|}{f(x)},
\]

and a second norm on signed measures \( \mu \) on \((X, \mathcal{B})\):

\[
\|\mu\|_f = \sup_{g : |g| \leq f} |\mu(g)|.
\]

Our main goal is to establish the ergodicity of \( \Phi \) in terms of this norm: There is an invariant measure \( \pi \) for the semi-group \( \{P^t\} \) satisfying,

\[
\lim_{t \to \infty} \|P^t(x, \cdot) - \pi(\cdot)\|_f = 0.
\]

The following result is a partial extension of the \( f \)-Norm Ergodic Theorem of [11] to the continuous time setting.

**Theorem 1.1.** Suppose that the Markov process \( \Phi \) is \( \psi \)-irreducible and aperiodic, and let \( f \geq 1 \) be a function on \( X \). Then the following conditions are equivalent:

(i) The semi-group admits an invariant probability measure \( \pi \) satisfying:

\[
\pi(f) := \int \pi(dx)f(x) < \infty.
\]

(ii) There exists a closed, small set \( C \in \mathcal{B} \) such that,

\[
\sup_{x \in C} E_x \left[ \int_0^{\tau_C(1)} f(\Phi(t)) \, dt \right] < \infty,
\]

where \( \tau_C(1) := \inf\{t \geq 1 : \Phi(t) \in C\} \) and \( E_x \) denotes the expectation operator under \( X_0 = x \).

(iii) There exists a closed, small set \( C \) and an extended-valued non-negative function \( V \) satisfying \( V(x_0) < \infty \) for some \( x_0 \in X \), such that Condition (V3) holds.

Moreover, if (iii) holds then there exists a constant \( b_f \) such that,

\[
E_x \left[ \int_0^{\tau_C(1)} f(\Phi(t)) \, dt \right] \leq b_f(V(x) + 1), \quad x \in X
\]

where \( V \) and \( C \) satisfy the conditions of (iii). The set \( S_V = \{x : V(x) < \infty\} \) is absorbing \( (P^t(x, S_V) = 1 \) for each \( x \in S_V \) and all \( t \geq 0 \)), and also full \( (\pi(S_V) = 1) \).

**Proof.** Theorem 1.2 (b) of [10] gives the equivalence of (i) and (ii). Theorem 4.3 of [10] gives the implication (iii) \( \Rightarrow \) (ii), along with the bound (5).

Conversely, if (ii) holds then we can define,

\[
V(x) = \int_0^\infty E_x \left[ f(\Phi(t)) \exp \left( - \int_0^t 1\{\Phi(s) \in C\} \, ds \right) \right] \, dt.
\]

We show in Proposition 2.2 that this is a solution to (V3) and that it is uniformly bounded on \( C \).
The function $V$ in (6) has the following interpretation. Let $\tilde{T}$ denote an exponential random variable that is independent of $\Phi$, and denote,

$$\tilde{\tau}_C = \min \left\{ t : \int_0^t \mathbb{I}\{\Phi(s) \in C\} \, ds = \tilde{T} \right\}.$$ 

We then have,

$$V(x) = \mathbb{E}_x \left[ \int_0^{\tilde{\tau}_C} f(\Phi(t)) \, dt \right],$$

where now the expectation is over both $\Phi$ and $\tilde{T}$. Consequently, this construction is similar to the converse theorems found in [11] for discrete-time models.

Theorem 1.1 is almost identical to the $f$-Norm Ergodic Theorem of [11], except that it leaves out the implications to ergodicity of the process. This brings us to two open problems: Under the conditions of Theorem 1.1:

Q1 Can we conclude that (3) holds for any initial condition $x \in S_V$?

Q2 Assume in addition that $\pi(V) < \infty$. Can we conclude that there exists a finite constant $B_f$ such that, for all $x \in S_V$,

$$\int_0^\infty \| P^t(x, \cdot) - \pi \|_f dt \leq B_f (V(x) + 1).$$

In discrete time, questions Q1 and Q2 are answered in the affirmative by the $f$-Norm Ergodic Theorem of [11], with the integral replaced by a sum in (8).

Q2 is resolved in the affirmative in this paper by an application of the discrete-time counterpart:

**Theorem 1.2.** Suppose that the Markov process $\Phi$ is $\psi$-irreducible and aperiodic, and that there is a solution to (V3) with $V$ everywhere finite. Then there is a constant $B^0_f$ such that for each $x, y \in X$,

$$\int_0^\infty \| P^t(x, \cdot) - P^t(x, \cdot) \|_f dt \leq B^0_f (V(x) + V(y) + 1)$$

If in addition $\pi(V) < \infty$, then (8) also holds for some constant $B_f$ and all $x$.

Although the full resolution of Q1 remains open, in Section 3 we discuss how (3) can be established under additional conditions on the process $\Phi$.

We begin, in the following section, with the proof of the implication $(ii) \Rightarrow (iii)$, which is based on theory of generalized resolvents and $f$-regularity [8]. Following this result, it is shown in Proposition 2.3 that $f$-regularity of the process is equivalent to $f_{\Delta}$-regularity for the sampled process, where $\Delta$ is the sampling interval, and,

$$f_{\Delta}(x) = \int_0^\Delta \mathbb{E}_x[f(\Phi(t))] \, dt, \quad x \in X.$$
Acknowledgment. The work reported in this note was prompted by a question of Yuanyuan Liu who, in a private communication, pointed out to us that some results in our earlier work [3] were stated inaccurately. Specifically: (1.) The implication (ii) ⇒ (iii) in Theorem 2.2 of [4], which is the same as the corresponding result in our present Theorem 1.1, was stated there without proof; and (2.) The convergence in (iii) was stated as a consequence of any of the three equivalent conditions (i)–(iii), again without proof. This note attempts to address and correct these omissions, although the relevant statements in [3] were only discussed as background material and do not affect any of the subsequent results in that paper.

2 f-Regularity

Following [8], we denote for each $r \geq 0$ and $B \in \mathcal{B}$,

$$G_B(x, f; r) := \mathbb{E}_x \left[ \int_0^{\tau_B(r)} f(\Phi(t)) \, dt \right], \quad (11)$$

where $\tau_B(r) = \inf\{t \geq r : \Phi(t) \in B\}$, and we write $G_B(x, f) = G_B(x, f; 0)$. The Markov process is called $f$-regular if there exists $r_0 > 0$ such that $G_B(x, f; r_0) < \infty$ for every $x$ and every $B \in \mathcal{B}$ satisfying $\psi(B) > 0$.

The following result, given here without proof, is a simple consequence of Lemma 4.1 and Prop. 4.3 of [8]:

Proposition 2.1. Suppose that the set $C$ is closed and small, and that the following self-regularity property holds: There exists $r_0 > 0$ such that $\sup_{x \in C} G_C(x, f; r_0) < \infty$. Then:

(i) There is $b_C < \infty$ such that $G_C(x, f; r) < G_C(x, f; r_0) + b_C r$ for each $x$ and $r$.

(ii) For each $B \in \mathcal{B}$ satisfying $\psi(B) > 0$, for each $r \geq 0$, and for each $x \in X$,

$$G_C(x, f; r) < \infty \Rightarrow G_B(x, f; r) < \infty.$$ 

Consequently, the process is $f$-regular if $G_C(x, f; r_0) < \infty$ for each $x$.

We next show that the function $V$ in (6) is finite-valued on $\{x \in X : G_C(x, f; r_0) < \infty\}$. We show that $V$ is in the domain of the extended generator, and obtain an expression for $DV$.

Consider the generalized resolvent developed in [8, 12]: For a function $h : X \to \mathbb{R}_+$, $A \in \mathcal{B}$, and $x \in X$, denote,

$$R_h(x, A) = \int_0^\infty \mathbb{E}_x \left[ I_A(\Phi(t)) \exp \left( - \int_0^t h(\Phi(s)) \, ds \right) \right] \, dt.$$ 

With the usual interpretation of $P^t$, or any kernel $Q(x, dy)$, as a lineal operator, $g \mapsto Qg = \int g(y)Q(\cdot, dy)$, it is shown in [12] that the following resolvent equation holds: For any functions $g \geq h \geq 0$,

$$R_h = R_g + R_g I_{g-h} R_h, \quad (12)$$

where, for any function $g$, $I_g$ denotes the (operator induced by the) kernel $I_g(x, dy) = g(x)\delta_x(dy)$. 
When \( h \equiv \alpha \) is constant, we obtain the usual resolvent,

\[
R_{\alpha} := \int_0^\infty e^{-\alpha t} P^t \, dt, \quad \alpha > 0,
\]

(13)

In the case \( \alpha = 1 \) we write \( R := R_1 = \int_0^\infty e^{-t} P^t \, dt \), and call \( R \) “the” resolvent kernel. For any non-negative function \( g: X \to \mathbb{R}_+ \) for which \( Rg \) is finite valued, the function \( \gamma = Rg \) is in the domain of the extended generator, with,

\[
\mathcal{D}\gamma = Rg - g.
\]

Proposition 2.2. Suppose that the assumptions of Theorem 1.1 (ii) hold: There is a closed, small set \( C \in \mathcal{B} \) such that, \( \sup_{x \in C} G_C(x, f; r_0) < \infty \) with \( r_0 = 1 \). Then the function \( V \) defined in (7) is finite on the full set \( S_V \subset X \) and (V3) holds with this function \( V \) and this closed set \( C \).

Proof. Proposition 4.3 (ii) of [8] implies that the set of \( x \) for which \( G_C(x, f; 1) < \infty \) is a full set. This result combined with Proposition 4.4 (ii) of [8] implies that \( V \) is bounded on \( C \).

For arbitrary \( x \) we have \( \bar{\tau}_C > \tau_C = \min\{t \geq 0 : \Phi(t) \in C\} \). Consequently, by the strong Markov property and the representation (7),

\[
V(x) = E_x \left[ \int_0^{\bar{\tau}_C} f(\Phi(t)) \, dt \right] + E_x \left[ E_{\Phi(\tau_C)} \left[ \int_0^{\bar{\tau}_C} f(\Phi(t)) \, dt \right] \right]
\leq G_C(x, f; 1) + \sup_{x' \in C} V(x').
\]

Hence \( V(x) \) is finite whenever \( G_C(x, f; 1) \) is finite.

To establish (V3), first observe that the function \( V \) in (7) can be expressed,

\[
V = R_h f, \quad \text{with } h = 1_C.
\]

Taking \( g \equiv 1 \), the resolvent equation gives,

\[
R_h f = R + R I_{1 - h} R_h = R[I + 1_C \circ R_h],
\]

where, for any set \( B \) and kernel \( Q \), \( I_B Q \) denotes the kernel \( \mathbb{1}_B(x)Q(x, dy) \). Combining the representation of \( V \) above with (14) we obtain,

\[
V = R[I + 1_C \circ R_h] f
\]

and

\[
\mathcal{D}V = (R - I)[I + 1_C \circ R_h] f.
\]

The second equation can be decomposed as follows,

\[
\mathcal{D}V = D_1 - D_2 - f,
\]

with \( D_1 = R[I + 1_C \circ R_h] f = V \) and \( D_2 = I_C \circ R_h f = I_C \circ V \). Substitution then gives,

\[
\mathcal{D}V = -f + 1_C V.
\]

This establishes (V3) with \( b = \sup_{x \in C} V(x) \).\)
The final results in this section concern the $\Delta$-skeleton chain. This is the discrete-time Markov chain with transition kernel $P^\Delta$, where $\Delta \geq 1$ is given. It can be realized by sampling the Markov process with sampling interval $\Delta$. The sampled process is denoted,

$$X(i) = \Phi(i\Delta), \quad i \geq 0.$$  \hspace{1cm} (15)

In prior work, the skeleton chain is used to translate ergodicity results for discrete-time Markov chains to the continuous time setting. For example, Theorem 6.1 of [9] implies that a weak version of the ergodic convergence (3) holds for an $f$-regular Markov process:

$$\lim_{t \to \infty} \|P^t(x, \cdot) - \pi(\cdot)\|_1 = 0.$$  \hspace{1cm} (16)

The proof consists of two ingredients: (i) The corresponding ergodicity result holds for the $\Delta$-skeleton chain, and (ii) the error $\|P^t(x, \cdot) - \pi(\cdot)\|_1$ is non-increasing in $t$.

In the next section we use a similar approach to address question Q2. The $f_\Delta$ norm is considered, where the function $f_\Delta$ is defined in (10). Denote,

$$\sigma_\Delta^C = \min\{i \geq 0 : X(i) \in C\}, \quad \tau_\Delta^C = \min\{i \geq 1 : X(i) \in C\}.$$  

The $\Delta$-skeleton is called $f_\Delta$-regular if,

$$G^\Delta_B(x, f_\Delta) := \mathbb{E}_x \left[ \sum_{i=0}^{\tau_\Delta^C} f_\Delta(X(i)) \right] < \infty,$$

for every $x \in X$ and every $B \in \mathcal{B}$ satisfying $\psi(B) > 0$.

**Proposition 2.3.** If the process $\Phi$ is $f$-regular, then each $\Delta$-skeleton is $f_\Delta$-regular. Moreover, there is a closed $f$-regular set $C$ such that:

(i) For a finite-valued function $V_\Delta : X \to (0, \infty]$ and a finite constant $b$,

$$P^\Delta V_\Delta \leq V_\Delta - f_\Delta + b\mathbb{1}_C,$$

and $\sup_x |V_\Delta(x) - G_C(x, f)| < \infty$.

(ii) For every $x \in X$ and every $B \in \mathcal{B}$ satisfying $\psi(B) > 0$, there is a constant $c_B < \infty$ such that,

$$G^\Delta_B(x, f_\Delta) \leq G_C(x, f) + c_B.$$  \hspace{1cm} (18)

**Proof.** It is enough to establish (i). Theorem 14.2.3 of [11] then implies that for every $B \in \mathcal{B}$ satisfying $\psi(B) > 0$, there is a constant $c_B^B < \infty$ satisfying $G^\Delta_B(x, f_\Delta) \leq V_\Delta(x) + c_B^B$.

Let $C$ denote any closed $f$-regular set for the process, satisfying $\psi(C) > 0$. For $V_0(x) = G_C(x, f)$ we obtain a bound similar to (17) through the following steps. First write,

$$P^\Delta V_0(x) = \mathbb{E}_x \left[ \int_\Delta^{\tau_C} f(\Phi(t)) \, dt \right].$$
The integral can be expressed as a sum,
\[
\int_{\Delta}^{\tau_C(\Delta)} f(\Phi(t)) \, dt = \int_{\Delta}^{\tau_C(\Delta)} f(\Phi(t)) \, dt \mathbb{I}\{\tau_C \leq \Delta\} + \int_{\Delta}^{\tau_C} f(\Phi(t)) \, dt \mathbb{I}\{\tau_C > \Delta\}.
\]
By the strong Markov property,
\[
E_x\left[I\{\tau_C \leq \Delta\} \int_{\Delta}^{\tau_C(\Delta)} f(\Phi(t)) \, dt\right] \leq E_x\left[I\{\tau_C \leq \Delta\} \int_{\tau_C}^{\tau_C(\Delta)} f(\Phi(t)) \, dt\right] \leq P_x\{\tau_C \leq \Delta\} \sup_y G_C(y, f; \Delta).
\]
Consequently,
\[
P^\Delta V_0(x) \leq E_x\left[\int_{\Delta}^{\tau_C} f(\Phi(t)) \, dt\right] + b_0 s(x) = V_0(x) - f_\Delta(x) + b_0 s(x),
\]
where \(b_0 = \sup_y G_C(y, f; \Delta) < \infty\), and \(s(x) = P_x\{\tau_C \leq \Delta\}\).

To eliminate the function \(s\) in (19) we establish the following bound: For some \(\varepsilon_0 > 0\) and \(k_0 \geq 1\),
\[
P^{k_0\Delta}(x, C) \geq \varepsilon_0 s(x), \quad x \in X.
\]
This establishes (20).

The Lyapunov function can now be specified as,
\[
V_\Delta(x) = V_0(x) + b_0 G_\Delta^\Delta(x, s),
\]
where \(b_0\) is defined in (19). The required bound \(\sup_x |V_\Delta(x) - G_C(x, f)| < \infty\) holds because \(V_0(x) = G_C(x, f)\), and the second term is uniformly bounded:
\[
G_\Delta^\Delta(x, s) = E_x\left[\sum_{i=0}^{\tau_B^\Delta} s(X(i))\right] = \varepsilon_0^{-1} E_x\left[\sum_{i=0}^{\tau_B^\Delta} P^{k_0\Delta}(\Phi(i\Delta), C)\right] \leq \varepsilon_0^{-1} (k_0 + 1).
\]
Consequently, from familiar arguments,

\[ PV_\Delta(x) - V_\Delta(x) \leq -f_\Delta(x) + b_0 s(x) \]

\[ + b_0 \left\{ G_C^\Delta(x, s) - s(x) + \mathbb{1}_{\text{C}}(x) \varepsilon_0^{-1}(k_0 + 1) \right\}. \]

This establishes (17) with \( b = b_0 \varepsilon_0^{-1}(k_0 + 1). \)

\[ \square \]

3 \hspace{1em} f-Norm Ergodicity

In this section we consider the implications to the ergodicity of the process. We assume that (V3) holds for a finite-valued function \( V : X \to (0, \infty), \) so that the process is \( f \)-regular.

Q1. \hspace{1em} \textbf{f-norm ergodicity.} The ergodicity of \( \Phi \) in terms of the \( f \)-norm as in (3) has only been established under special conditions. Theorem 5.3 of [10] implies that (3) will hold if \( f \) is subject to this additional bound: For some \( \beta \geq 0, \)

\[ P^t f \leq \beta e^{\beta t} f, \hspace{1em} t \geq 0. \]

This holds for example if \( f \equiv 1 \) and \( \beta = 1. \)

It is likely that the application of coupling bounds will lead to a more general theory. Under stronger conditions on the process, such a coupling time was obtained in [6], and it was used in [7] to obtain rates of convergence in the law of large numbers. However, to construct the coupling time, it is assumed in this prior work that the semi-group \( \{P^t\} \) admits a density for each \( t. \) No such assumptions are required in the discrete-time setting, so the full answer to Q1 remains open.

Q2. \hspace{1em} \textbf{Proof of Theorem Theorem 1.2.} The complete resolution of Q2 is possible by applying Proposition 2.3, which implies that the skeleton chain \( \{X(i) = \Phi(i\Delta) : i \geq 0\} \) is \( f_\Delta \)-regular. The bound (18) is the main ingredient in the proof of Theorem 1.2, but we also require the following relationship between a norm for the process and a norm for the sampled chain.

\textbf{Lemma 3.1.} For any signed measure \( \mu, \)

\[ \|\mu\|_{f_\Delta} \geq \int_0^\Delta \|\mu P^t\|_f dt, \]

where, for any measure \( \nu \) and kernel \( Q, \) \( \nu Q \) denotes the measure \( \nu Q(\cdot) = \int \nu(dx)Q(x, \cdot). \)

\textbf{Proof.} We first consider the right-hand side. Consider the signed measure \( \Gamma \) on \([0, \Delta] \times X\) defined by:

\[ \Gamma(dt, dy) = \mu P^t(dy) dt. \]

Define \( f_\Delta : [0, \Delta] \times X \to [1, \infty) \) via \( f(t, y) = f(y) \) for each pair \( t, y, \) and the associated norm,

\[ \|\Gamma\|_{f_\Delta} = \sup \int \int g(t, y) \Gamma(dt, dy), \]
where the supremum is over all \( g \) satisfying \( |g(t,y)| \leq f_\Delta(t,y) \) for all \( t,y \). It is shown next that the norm can be expressed,

\[
\|\Gamma\|_{f_\Delta} = \int_0^\Delta \|\mu P_t\|_f \, dt. \tag{21}
\]

The Jordan decomposition theorem \([2]\) implies that there is a minimal decomposition,
\[
\Gamma = \Gamma_+ - \Gamma_-,
\]
in which the two measures on the right-hand side are non-negative, with disjoint supports denoted \( S_+, S_- \), respectively. Hence \( |\Gamma| := \Gamma_+ + \Gamma_- \) is a non-negative measure. In this notation the norm is expressed,

\[
\|\Gamma\|_{f_\Delta} = \int f_\Delta(t,y) |\Gamma| \, (dt, dy)
\]

\[
= \int f(y) \left( \mathbb{1}_{S_+}(t,y) - \mathbb{1}_{S_-}(t,y) \right) \Gamma \, (dt, dy)
\]

\[
= \int_0^\Delta \left[ \int_{y \in \mathcal{X}} f(y) \left( \mathbb{1}_{S_+}(t,y) - \mathbb{1}_{S_-}(t,y) \right) \mu P_t(dy) \right] \, dt.
\]

For each \( t \), the measure on \((\mathcal{X}, \mathcal{B})\) defined by
\[
\mathbb{1}_{S_+}(t,y) - \mathbb{1}_{S_-}(t,y) \mu P_t(dy)
\]
is the marginal of \( |\Gamma| \), and is hence a non-negative measure for a.e. \( t \). It follows that for such \( t \),

\[
\int_{y \in \mathcal{X}} f(y) \left( \mathbb{1}_{S_+}(t,y) - \mathbb{1}_{S_-}(t,y) \right) \mu P_t(dy) = \|\mu P_t\|_f,
\]

which gives (21).

Consider next the left-hand side of the inequality in the lemma. Letting \( \mu = \mu_+ - \mu_- \) denote the Jordan decomposition for the signed measure \( \mu \), and \( |\mu| = \mu_+ + \mu_- \), we have,

\[
\|\mu\|_{f_\Delta} = \int f_\Delta(x) |\mu|(dx) = \int_{t=0}^\Delta \int_{x \in \mathcal{X}} |\mu|(dx) P_t(x,dy)f(y).
\]

The right-hand side can be expressed as,

\[
\int_0^\Delta \int |\mu|(dx) P_t(x,dy)f(y) = \int f_\Delta(t,y) \Lambda_+(dt,dy) + \int f_\Delta(t,y) \Lambda_-(dt,dy),
\]

where \( \Lambda_\pm(dt,dy) = \mu_\pm P_t(dy)dt \) defines a decomposition:

\[
\Gamma = \Lambda_+ - \Lambda_-.
\]

It follows that \( \|\mu\|_{f_\Delta} \geq \|\Gamma\|_{f_\Delta} \), by the minimality of the Jordan decomposition. This bound combined with (21) completes the proof. \( \square \)

**Proof of Theorem 1.2.** Theorem 1.1 combined with Proposition 2.3 establishes \( f_\Delta \)-regularity of the skeleton chain under (V3): The skeleton chain satisfies (V3) with Lyapunov function \( V_\Delta \) that satisfies \( \sup_x |V_\Delta(x) - G_C(x,f)| < \infty \). The bound (5) in Theorem 1.1 implies that \( V_\Delta(x) \leq b_\Delta^\Delta(V(x) + 1) \) for some constant \( b_\Delta^\Delta \) and all \( x \).
Theorem 14.3.4 of [11] then gives the bound, for some finite constant $M^0_f < \infty$,

$$\sum_{k=0}^{\infty} \| P^\Delta_k(x, \cdot) - P^\Delta_k(y, \cdot) \|_{f, \Delta} \leq M^0_f (V(x) + V(y) + 1). \tag{22}$$

Next apply Lemma 3.1 with $\mu(\cdot) = P^\Delta_k(x, \cdot) - P^\Delta_k(y, \cdot)$ to obtain,

$$\| P^\Delta_k(x, \cdot) - P^\Delta_k(y, \cdot) \|_{f, \Delta} \geq \int_0^\Delta \| \mu P^t \|_f \, dt, \tag{23}$$

and recognize that $\mu P^t(\cdot) = P^{\Delta k + t}(x, \cdot) - P^{\Delta k + t}(y, \cdot)$. Substituting the resulting bound into (22) establishes (9).

The proof of (8) is similar: If in addition $\pi(V) < \infty$, then Theorem 14.3.5 of [11] gives, for some constant $M_f < \infty$,

$$\sum_{k=0}^{\infty} \| P^\Delta_k(x, \cdot) - \pi(\cdot) \|_{f, \Delta} \leq M_f (V(x) + 1). \tag{24}$$

This combined with (23) completes the proof. \qed

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