RELATIVISTIC INVARIANT LIE ALGEBRAS FOR KINEMATICAL OBSERVABLES IN QUANTUM SPACE-TIME

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A deformation of the canonical algebra for kinematical observables of the quantum field theory in Minkowski space-time has been considered under the condition of Lorentz invariance. A relativistic invariant algebra obtained depends on additional fundamental constants $M$, $L$ and $H$ with the dimensions of mass, length and action, respectively. In some limiting cases the algebra goes over into the well-known Snyder or Yang algebras. In general case the algebra represents a class of Lie algebras, that consists of simple algebras and semidirect sums of simple algebras and integrable ones. Some algebras belonging to this class are noninvariant under $T$ and $C$ transformations. Possible applications of obtained algebras for descriptions of states of matter under extreme conditions are briefly discussed.

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At present the concept of continuous Minkowski space-time together with the group of its motions, namely the Poincaré group, is basic for description of all established physical phenomena in the framework of relativistic theory. The space-time properties of any physical object according to this concept are rather simple and well understood. But it is not so, when one would proceed to the investigation of unexplored states of matter under extreme conditions, for instance, in early universe or in quark-gluon plasma \cite{1, 2}. The space-time properties of these systems may be more complicated and probably determined by a generalized group of space-time symmetries in extra dimensions \cite{3}. Moreover their space-time properties may depend on additional fundamental constants as compared with the light velocity $c$ and Planck constant of action $\hbar$.

In this paper we present the results of study of possible generalizations of conventional space-time symmetries. In order to minimize a number of generalized symmetries we take into account some restrictions such as a preservation, if it is possible, of all known physical  

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properties of observables and a fulfilment of the mathematical condition that observables should generate a Lie group. The Poincaré group is a semidirect product of the Lorentz group and the Abelian four-dimensional translation group in Minkowski space-time. In quantum case the Heisenberg commutation relations allow one to consider coordinates as physical operators and on the same footing as momenta. Moreover, it is known that elaboration of the measurement procedure for the observables within an atom made Born to formulate the reciprocity principle for coordinates and momenta [4]. In the framework of the canonical quantum relativistic theory all fundamental space-time properties can be represented by a system of commutation relations between the Hermitian operators of the following 15-dimensional algebra:

\[
[x_i, x_j] = [p_i, p_j] = 0, \quad [p_i, x_j] = i\hbar g_{ij} I,
\]

\[
[p_i, I] = [x_i, I] = [F_{ij}, I] = 0,
\]

\[
[F_{ij}, F_{kl}] = i\hbar (g_{jk} F_{il} - g_{ik} F_{jl} + g_{il} F_{jk} - g_{jl} F_{ik}),
\]

\[
[F_{ij}, p_k] = i\hbar (g_{jk} p_i - g_{ik} p_j),
\]

\[
[F_{ij}, x_k] = i\hbar (g_{jk} x_i - g_{ik} x_j),
\]

where \(x_i\) and \(p_i\) are the operators of 4-coordinates and 4-momenta, respectively, \(F_{ij}\) are the proper Lorentz group generators, \(I\) is the "identity" operator.

The space-time points may be discriminated as eigenvalues of the \(x_i\) generators for certain irreducible representation of the algebra (1). For instance, in the \(x\)-representation the basis vectors of the representation have the form \(\psi_{\alpha\beta}(x)\), where \(x = \{x_0, x_1, x_2, x_3\}\) are the eigenvalues of operators for 4-coordinates, \(\alpha, \beta\) are the discrete spin indices, which are subject to an action of \(S_{ij}\), where \(S_{ij} = F_{ij} - x_i p_j + p_i x_j\) are the spin operators of some finite dimensional representation of the Lorentz group.

The conventional quantum theory, which is based on the system of commutation relations (1), has a long standing problem concerning a removal of shot-distance singularities. It was the reason, which induced Heisenberg to suggest the idea that the configuration-space coordinates may not commute [5]. Then Snyder introduced the Lorentz invariant quantized space-time characterized by a fundamental length [6]. This theory is, however, not invariant under translations. Yang presented a generalized translation invariance and suggested a new fundamental unit of mass [7]. In the limiting case, when the fundamental length and mass are eliminated from the theory, the standard commutation relations (1) are valid between the kinematical observables. But the Snyder and Yang theories are not maximal generalizations of the quantum relativistic theory providing a strict Lorentz invariance. In the work [8] a more general algebra for the kinematical observables has been found and a new fundamental unit of action has been defined.

Below we clear up the presumptions used in a deduction of deformed relativistic invariant Lie algebras for observables of quantum theories, generalize to a certain extent the results obtained in Ref. [8], and point to some possible applications of the generalized algebras. The obtained class of Lie algebras consists of simple algebras as well as semidirect sums of simple algebras and integrable ones. When a new constant with the dimensions of action enter in generalized commutation relations, the corresponding algebras became noninvariant under the time reversal \(T\) or the charge conjugation \(C\).
We suppose that a generalization of the algebra (1) is performed under the following conditions:

1. The generalized algebra should be a Lie algebra.
2. The dimensionality of the algebra, which is subject to a generalization, and the physical dimensions of operators entered in it should be preserved.
3. The generalized algebra should contain the Lorentz algebra as its subalgebra, and commutation relations of the Lorentz algebra with other generators should be the same as for the initial algebra.

The procedure of the generalization of the algebra (1) described above may be named a relativistic or Lorentz invariant deformation of the algebra (1) because of the property of Lorentz symmetry is conserved as a fundamental law of nature. Note that in some cases the canonical Poincaré invariance may be violated.

Under these conditions algebra spanned on the Hermitian operators $F_{ij}$, $p_i$, $x_i$ and $I$, which is the maximal generalization of algebra (1), has the following form:

\[
[F_{ij}, F_{kl}] = \varphi (g_{jk}F_{il} - g_{ik}F_{jl} + g_{il}F_{jk} - g_{jl}F_{ik}),
\]
\[
[p_{i}, x_{j}] = Ag_{ij}I + BF_{ij} + C\epsilon_{ijkl}F_{kl},
\]
\[
[p_{i}, p_{j}] = aF_{ij} + b\epsilon_{ijkl}F_{kl},
\]
\[
[x_{i}, x_{j}] = cF_{ij} + d\epsilon_{ijkl}F_{kl},
\]
\[
[p_{i}, I] = \alpha x_{i} + \beta p_{i},
\]
\[
[x_{i}, I] = \gamma x_{i} + \delta p_{i},
\]
\[
[F_{ij}, x_{k}] = h(g_{jk}x_{i} - g_{ik}x_{j}),
\]
\[
[F_{ij}, p_{k}] = f(g_{jk}p_{i} - g_{ik}p_{j}),
\]
\[
[F_{ij}, I] = 0,
\]

where $\epsilon_{ijkl}$ is the Levi-Civita tensor.

The relations (2) contain fourteen arbitrary pure imaginary parameters. Taking into account the Jacobi identities and the dimensions of physical operators entered in the commutation relations (2), ten parameters must be excluded, and the following relativistic invariant algebra, that is a maximal deformation of algebra (1) under conditions 1-3, can be obtained:

\[
[F_{ij}, F_{kl}] = i\varphi (g_{jk}F_{il} - g_{ik}F_{jl} + g_{il}F_{jk} - g_{jl}F_{ik}),
\]
\[
[p_{i}, x_{j}] = i\varphi (g_{ij}I + F_{ij}^{H}),
\]
\[
[p_{i}, p_{j}] = \frac{i\varphi}{L^2}F_{ij},
\]
\[
[x_{i}, x_{j}] = \frac{i\varphi}{M^2}F_{ij},
\]
\[ [p_i, I] = i f \left( \frac{x_i}{L^2} - \frac{p_i}{H} \right), \]
\[ [x_i, I] = i f \left( \frac{x_i}{H} - \frac{p_i}{M^2} \right), \]
\[ [F_{ij}, p_k] = i f (g_{jk}p_i - g_{ik}p_j), \]
\[ [F_{ij}, x_k] = i f (g_{jk}x_i - g_{ik}x_j), \]
\[ [F_{ij}, I] = 0. \]

The algebra (3) depends on four dimensional parameters: \( L \) is a constant with the dimensions of length, \( M \) is a constant with the dimensions of mass, \( H \) and \( f \) are the constants with the dimensions of action (\( M \) and \( L \) take real values as well as pure imaginary ones, \( c = 1 \) in the system of units being used).

In general case the algebra (3) can be considered as the algebra of observables for the Lorentz invariant quantum theory with noncommutative coordinates and momenta. In the limiting case, when \( M, L \) and \( H \) values become infinitely large, the commutation relations (3) go over into the commutation relations of contracted algebra (1) providing \( f = \hbar \). A more complicated case is also possible, when \( f \) is some function versus three parameters \( L, M \) and \( H \). However, in order that an agreement with the conventional commutation relations should take place in the limiting case, as \( L \to \infty, M \to \infty \) and \( H \to \infty \), the value of \( f(L, M, H) \) must be equal to \( \hbar \).

In other limiting cases, when \( f = \hbar \), but \( M, L \) and \( H \) have different magnitudes, the following theories may be obtained:

- a) \( H \to \infty, L \to \infty \) - the relativistic quantum theory with noncommutative coordinates;
- b) \( H \to \infty, M \to \infty \) - the relativistic quantum theory with noncommutative momenta;
- c) \( H \to \infty \) - the relativistic quantum theory with noncommutative coordinates and momenta.

From mathematical point of view the system of commutation relations (3) specify some class of Lie algebras, which consist of semisimple algebras as well as general type algebras. After the calculation of the Killing-Cartan form the condition for semisimplicity may be written as

\[ \frac{f^2(M^2L^2 - H^2)}{H^2M^2L^2} \neq 0. \]

If one performs the linear transformation of \( p_i, x_i, I \) generators in the form:

\[ F_{i5} = Bx_i + Dp_i, \quad F_{i6} = Ex_i + Gp_i, \quad F_{56} = AI, \]

then under the condition (4) one may obtain the commutation relations for the algebras of pseudoorthogonal groups \( O(3, 3) \), \( O(2, 4) \) and \( O(1, 5) \). These algebras correspond to the definite values of \( M^2, L^2, H^2 \) parameters, which are shown in Table 1.
For $H^2 = M^2L^2$ and $M^2 > 0$, $L^2 > 0$ the $o(1, 5)$ algebra degenerates into a semidirect product of the $o(1, 4)$ algebra and the algebra of 5-translations, while for $H^2 = M^2L^2$ and $M^2 < 0$, $L^2 < 0$ the $o(3, 3)$ algebra degenerates into a semidirect product of the $o(2, 3)$ algebra and the algebra of 5-translations. Note that a transition to the limit $A_\alpha \to \infty$, where $A_\alpha$ is any term of the set $\{M^2, L^2, H^2, (M^2, L^2)\}$, do not exclude the algebras (3) from the class of simple algebras, as distinct from the transitions $B_\alpha \to \infty$, where $B_\alpha \in \{(M^2, H^2), (L^2, H^2), (M^2, L^2, H^2)\}$ or $M^2L^2 \to H^2$.

The irreducible representations of algebras (3) are determined with the help of eigenvalues of Casimir operators. For the real simple algebras shown in Table 1 the Casimir operators have the known forms in terms of the generators $F_{ij}$, $i, j = 0, 1, \ldots, 5$ of pseudoorthogonal groups in six-dimensional spaces:

$$
K_1 = \epsilon_{ijklmn}F^{ij}F^{kl}F^{mn}, \quad K_2 = F_{ij}F^{ij}, \quad K_3 = (\epsilon_{ijklmn}F^{kl}F^{mn})^2.
$$

These operators can also be expressed through $p_i, x_i, F_{ij}$, $i, j = 0, \ldots, 3$, and $I$ generators. For instance, the second-order invariant operator may be represented in the form:

$$
C_2 = \sum_{i<j}F_{ij}F^{ij}\left(\frac{1}{M^2L^2} - \frac{1}{H^2}\right) + I^2 + \frac{x_ip^i + p_ix^i}{H} - \frac{x_ix^i}{L^2} - \frac{p_ip^i}{M^2}.
$$

that in the limit case $M \to \infty$, $L \to \infty$, $H \to \infty$ go over into the canonical "identity" operator squared (1). The limit expressions of Casimir operators (6), as $L \to \infty$, $M \to \infty$, have been found in Ref. [9] in the case $H = \infty$, while the explicit form for the quadratic Casimir operator (7) in this case has been presented in Ref. [10]. The quadratic Casimir operator for an algebra in three space dimensions, which is a particular case of the algebra (3), has been obtained in Ref. [11].

It should be noted that the presence of a new constant $H$ with the dimensions of action leads to the noninvariance of system (3) under the $T$ and $C$ transformations [5]. Indeed, the time reversal results in sign changes for all dimensional quantities, which include the time variable in an odd degree. Evidently, if simultaneously with the time reversal the appropriate transformations corresponding to this reversal for the physical operators have been done, then the system of commutation relations will remain invariant. In the conventional theory behaviour of the commutation relations (1) with respect to the time reversal is determined by the Planck constant, so the $T$ transformation is equivalent to the sign change for $\hbar$. 

### Table 1

| $M^2$, $L^2$ and $H^2$ values | Algebra |
|-------------------------------|---------|
| $H^2 < M^2L^2$, $M^2 > 0$, $L^2 > 0$ | $o(2, 4)$ |
| $H^2 < M^2L^2$, $M^2 < 0$, $L^2 < 0$ | $o(2, 4)$ |
| $M^2 > 0$, $L^2 < 0$, or $M^2 < 0$, $L^2 > 0$ | $o(2, 4)$ |
| $H^2 > M^2L^2$, $M^2 > 0$, $L^2 > 0$ | $o(1, 5)$ |
| $H^2 > M^2L^2$, $M^2 < 0$, $L^2 < 0$ | $o(3, 3)$ |
However, the transition to conjugate or transposed operators simultaneously with the time reversal preserves the commutation relations (1).

For the algebra (3) there is the additional constant $H$, which is also odd with respect to the time reversal and enters in (3) so that it is impossible to restore the $T$ invariance of the system (3) for $H \neq \infty$. Along the same lines one may obtain the $C$ noninvariance of the system (3), since the quantities with dimensionality of mass change its signs after replacement of particles by antiparticles. $TC$ transformation do not change $\hbar$ and $H$, thus the system (3) is invariant under $TC$ and $P$ transformations.

Moreover the generalization of the algebra (1) with $L \neq \infty$ leads to a breakdown of Poincaré invariance, as follows from the commutation relations (3). The investigation of the quantum theory in momentum space with constant curvature (which is related to the algebra (3) as $H \rightarrow \infty$, $M \rightarrow \infty$), including modifications to Feynman rules and the locality principle, has been started in Refs. [12, 13]. A connection of thermal properties of quantum system with the generalized space-time geometry was investigated as well [see, e.g., 14]. If the values of the parameters entering in the system (3) are of cosmic scales under the condition that the usual physical phenomena take place at least at distances of the order of Solar system, then, for instance, the value of $M$ parameter should be of the order of the mass of the Universe and the associated elementary length $\hbar/M$ turns out much less than typical nuclear distances and time intervals. Additional constants with the dimensions of length $H/M$ and $\hbar L/H$ or of mass $H/L$ and $\hbar M/H$ may also have some meaning. Their numerical values by no means are limited by the correspondence principle with the conventional theory for macroscopical phenomena. Moreover, one may consider algebras of type (3) for the description of such objects as quarks or other colour particles, which never have been observed in the conventional space-time in free states. Therefore some stringent restrictions for violation of the standard quantum theory principles, which take place at present for usual elementary particles, cannot be applied. In this case the values of parameters $\hbar/L$ and $\hbar/M$ might be of the order of 1 GeV and 1 Fm respectively [10, 15]. It is known, that at present there is a possibility, that the problem of confinement of the colour objects cannot be resolved in the framework of quantum chromodynamics (QCD) itself and, perhaps, for its resolution some extra postulates are needed. The use of algebra (3) instead of algebra (1) makes it possible to transform the problem of confinement, which is a dynamical problem in the framework of QCD, to a kinematical one. Moreover, one can to generalize the class of deformed algebras for quarks or elementary particles by means of weakening of the conditions under which the algebra (3) was found. For instance, in Refs. [16, 17] the number of generators of generalized algebra has been enlarged and the semidirect products of $su(1,3)$ algebra and some integrable algebras, that contains the operators of coordinates and momenta, have been considered. If one preserve the number of generators, the $su(1,3)$ algebra by itself is the generalization of the commutation relations (1), which is performed by weakening of the third condition just to the requirement that the generalized algebra should contain the Lorentz subalgebra.

The Coulomb and harmonic oscillator problems were considered in the work [11] in the three-dimensional quantum space described by $O(5)$ algebra. It has been shown that the energy spectrum of the Coulomb problem with conserving the Runge-Lenz vector coincides with a part of the spectrum found by Schrödinger for the space with a constant curvature. Some methods for evaluating matrix elements of operators in quantum space with the dimensional parameters $L$, $M$ and $H$ were worked out as well. In the last years a number of results were obtained in the framework of approaches with noncommutative configuration-space variables.
which have been initiated by string theory, noncommutative geometry and quantum gravitation and operate, as a rule, with more general algebraic structures than Lie algebras.

References

[1] J. Silk, The Big Bang, 3rd ed., W.H. Friman & Co., 2001.
[2] Proc. of Quark Matter Conf., 18-24 July, Nantes, France, 2002.
[3] V.A. Rubakov and M.E. Shaposhnikov, Phys. Lett. B125, 136 (1983).
[4] M. Born, Proc. Roy. Soc. A165, 291 (1938).
[5] R. Jackiw, physics/0209108.
[6] H. Snyder, Phys. Rev. 71, 38 (1947).
[7] C.N. Yang, Phys. Rev. 72, 874 (1947).
[8] A.N. Leznov and V.V. Khruschev, preprint IHEP 73-38, Serpukhov, 1973.
[9] A.N. Leznov, JETP Lett. 45, 321 (1965).
[10] V.V. Khruschev, Grav. & Cosmol. 3, 197 (1997).
[11] A.N. Leznov, Nucl. Phys. B640, 469 (2002).
[12] Yu.A. Golfand, JETP. 37, 504 (1959).
[13] V.G. Kadyshevsky, JETP. 41, 1885 (1961); in: I. E. Tamm memorial vol.
"Problems of Theoretical Physics", eds, V.I. Ritus, E.L. Feinberg, V.L. Ginsburg et al. Nauka, M. 1972.
[14] B.S. Kay and R.M. Wald, Phys. Rep. 207, 49 (1991).
[15] V.V. Khruschev, V.I. Savrin and S.V. Semenov, Phys. Lett. B525, 283 (2002).
[16] V.V. Khruschev, J. Nucl. Phys. 46, 219 (1987).
[17] S.G. Low, J. Phys. A35, 5711 (2002).
[18] A. Seiberg and E. Witten, J. High Ener. Phys. 09-032, 1 (1999).
[19] R. Oeckl, Nucl. Phys. B581, 559 (2000).
[20] P. Kosiński, J. Lukierski and P. Maślanka, hep-th/0012056.
[21] S. Doplicher, K. Fredenhagen and J. Roberts, Phys. Lett. B331, 39 (1994).
[22] L.J. Garay, Int. J. Mod. Phys. A10, 145 (1995).