Thermodynamics as Combinatorics: A Toy Theory

Abstract—We discuss a simple toy model which allows, in a natural way, for deriving central facts from thermodynamics such as its fundamental laws, including Carnot’s version of the second principle. Our viewpoint represents thermodynamic systems as binary strings, and it links their temperature to their Hamming weight. From this, we can reproduce the possibility of negative temperatures, the notion of equilibrium as the coincidence of two notions of temperature — statistical versus structural —, as well as the zeroth law of thermodynamics (transitivity of the thermal-equilibrium relation), which we find to be redundant, as other authors, yet at the same time not to be universally valid.

Index Terms—Thermodynamics, information theory, combinatorics, logic, complexity

I. INTRODUCTION

In the present article, we attempt to understand and develop the links between the well-established physical theory of thermodynamics on the one hand, and information (as well as algorithmic-complexity) theory on the other. Let us note first that, given the variety of already known connections — most obviously given through the notion of entropy —, it is not surprising that close ties exist between these fields.

In line of previous work described below, we propose a toy model of thermodynamics featuring binary strings of infinite length (the “thermodynamic limit”) as heat baths, where the strings’ Hamming weight per length (“Hamming fraction”) is linked to their temperature. Our model is in the same spirit as a proposal by Spekkens [1], presenting a toy model able to capture key properties of quantum physics that is based on a hidden-variable model. With this, he defends an epistemic view of quantum states. Our combinatorial model of thermodynamics, in turn, can capture key features of the physical theory, such as its fundamental laws, suggesting that the essence of thermodynamics can be derived through combinatorics, i.e., counting arguments.

In our model, we reproduce the notions of heat reservoir, temperature, equilibrium, and entropy. We express in a natural and simple way the fundamental laws of thermodynamics from the zeroth to the second, including notably the first (Carnot’s) version of the second law [2], bounding the efficiency of a circular process between two heat baths as one minus the ratio between their absolute temperatures.

We also obtain in a straightforward way the concept of negative temperature. The phenomenon arises for finite strings the Hamming fraction of which exceeds 1/2 — when the increase of energy (1s) effectively decreases the number of degrees of freedom —, but also with respect to structural temperature, an alternative measure we define based on the notion of Kolmogorov complexity as entropy.

A. Previous results

A close connection between the second law of thermodynamics and information was expressed by Rolf Landauer [4], whose starting point was his famous slogan “Information is Physical” [5]: The erasure of $N$ bits of information costs at least $k_B T \ln 2 \cdot N$ of free energy, the dissipation of which as heat into the environment compensates for the entropy decrease of the erased memory. This principle was used by Bennett [3] to resolve the paradox around “Maxwell’s demon” (see Figure 1): The latter, in the course of her gas-sorting activity, accumulates information in her brain, and the corresponding growth of entropy compensates for the external entropy defect. The erasure of that information in the demon’s

Fig. 1. The resolution of the Maxwell-demon paradox as described by Bennett: The order that is created outside the demon is compensated by the disorder arising inside the demon [3].

1Here, $k_B$ stands for Boltzmann’s constant.
internal state would in the end require exactly the same amount of free energy that is gained by the sorting — the paradox disappears.

Motivated by Landauer’s principle, Bennett [3] as well as Fredkin and Toffoli [6] have developed a theory of reversible computing, i.e., a computing model which does not require the erasure of information. More specifically, Bennett has described a generic procedure to turn any computation into a reversible one with essentially the same computational efficiency, whereas Fredkin and Toffoli have presented a model of computation — the “Ballistic computer,” based on elastic collisions of balls on a billiard table — that allows for carrying out any logically-reversible computation (no loss of information) in a thermodynamically-reversible way (no heat dissipation into the environment).

Based on this, and in generalization of Landauer’s principle, the second law of thermodynamics has been speculated to take the form that time evolutions are logically reversible [7]. In a spirit related to the toy model considered in the present article, we have derived from this fact consequences of this second law resembling the formulations due to Clausius as well as Kelvin: “Heat only flows from a hotter to a colder reservoir, not the other way around” and “No free energy from one heat bath alone” (see Figures 2, 3).

It has been an open question how to reproduce Carnot’s version in a similar way. We fill this gap in the present article by presenting a toy model powerful enough to reproduce additional notions such as temperature, free energy, heat, as well as the efficiency of a process connecting two heat baths.

The converse of Landauer’s principle states that certain, namely redundant, pieces of information have a work value: They allow for transforming environmental heat into free energy. In [7] and [9], general bounds are given, in terms of algorithmic complexity, on the work value of a generic string as well as the thermodynamic cost of a computation.

In a recently developed, process-oriented view of thermodynamics [10], the conclusion has been drawn that the zeroth law be redundant. We reproduce this conclusion in our toy model presented in this article: The law is a direct consequence of the notion of thermal equilibrium — but it has an error probability: It fails to hold with exponentially small probability in the size of the (random) heat baths. The reason, as we will see, is that the thermal-equilibrium relation fails to be reflexive: Two

\[
\begin{align*}
\cdots \, 0100110101 \cdots & \quad \mapsto \quad \cdots \, 0000100000 \cdots \\
& \quad \mapsto \quad \cdots \, 0011010101 \cdots \\
& \quad \mapsto \quad \cdots \, 1111110111 \cdots \\
\end{align*}
\]

Fig. 2. Clausius’ second law: Logically reversible maps do not accentuate differences [8], i.e., the depicted transformation is impossible for Hamming fractions \( t_1 > t'_1 \) and \( t_2 < t'_2 \) (the total number of 1s is preserved; this is the first law).

\[
0110100101110001 \quad \mapsto \quad 0000100001111111
\]

Fig. 3. Kelvin’s version of the second law: The depicted transformation is impossible for \( t'_1 < t_1 \) (the total number of 1s is preserved).

copies of the same heat bath contain redundancy, thus free energy.

**II. PRELIMINARIES**

First, we introduce the relevant concepts for our toy theory described in Section III. The underlying set for our model consists of binary strings \( s \) of unbounded length,

\[
s = s_1 s_2 s_3 \ldots,
\]

where \( s^{(n)} = s_1 s_2 s_3 \ldots s_n \) is the finite string consisting of the first \( n \) bits of \( s \).

**Definition 1** (Hamming fraction). The Hamming fraction \( t(s) \) of a string \( s \) is defined in the asymptotic limit as

\[
t(s) := \lim_{n \to \infty} \frac{H_W(s^{(n)})}{n},
\]

where \( H_W(s^{(n)}) \) denotes the Hamming weight of string \( s^{(n)} \), and \( n \) is the length of \( s^{(n)} \).

We restrict our model to strings for which this limit exists.

**Definition 2** (Kolmogorov complexity). The Kolmogorov complexity \( K(s) \) [11] of a string \( s \) is the length of the shortest program \( p \) for a universal Turing machine \( U \) generating \( s \):

\[
K_U(s^{(n)}) := \min \{|p| : U(p) = s^{(n)}\}.
\]

**Lemma 1.** The Kolmogorov complexity \( K(s) \) is upper bounded by:

\[
\frac{K(s^{(n)})}{n} \leq h(t),
\]

where \( h \) is the binary entropy function.

**III. THE TOY MODEL: STRINGS AND THEIR TEMPERATURE**

The macrostate of a system \( s \) is characterized by its length \( n \) and its Hamming fraction \( t \). The number of possible strings (i.e., microstates for given macroscopic observables) with fixed \( n \) and \( t \) is given by

\[
\binom{n}{t \cdot n} \approx 2^n h(t) = 2^{n \cdot (1 - t \cdot \log_2(1 - t) - (1 - t \cdot \log_2(1 - t)))}.
\]

We define below the (statistical) temperature of a string depending on its Hamming fraction. We first review an element of classical Boltzmann theory from statistical mechanics in order to motivate our definition, showing that the latter is in

\[
K_U(s^{(n)}) := \min \{|p| : U(p) = s^{(n)}\}.
\]

The Kolmogorov complexity \( K_U \) depends on the universal Turing machine \( U \) only up to an additive constant. Since our considerations concern asymptotic limits, we ignore the Turing-machine dependence from now on.
accordance with the traditional view. The probability to find a particle with energy \( E_i \) in a system at temperature \( T \) with total particle number \( N = \sum_i N_i \) and total energy \( E = \sum_i n_i \cdot E_i \), in the case that \( N_i \) particles are occupying the energy state \( E_i \) for \( i \in \{1, \ldots, J\} \), is given by the Boltzmann equilibrium distribution [12]:
\[
\frac{N_i}{N} = \frac{1}{Z} e^{-E_i/k_B T}.
\]  
(6)

This expression includes the classical partition function \( Z = \sum_i e^{-E_i/k_B T} \). The formula (6) was derived by Boltzmann in a combinatorial approach [13], maximizing the number of ways of arranging \( N \) particles on energy states \( E_i \) with occupation numbers \( N_i \), while keeping \( E \) and \( N \) conserved. For the binary string \( s^{(n)} \in \{0, 1\}^n \), each bit can be merely in two different states, i.e., the energy state 0 or \( \Delta E \), and based on (6), one obtains the following probability for a bit \( s_i \) being equal to 1 at energy \( \Delta E \):
\[
P(s_i = 1) \approx \frac{H_W(s^{(n)})}{n} = \frac{e^{-\Delta E/k_B T}}{1 + e^{-\Delta E/k_B T}} \text{ for } i \in \{1, \ldots, n\}.
\]  
(7)

Thereby, for each of the two-level systems, solely the energy difference \( \Delta E \) of both states is relevant. If we consider \( \Delta E \) as a constant for a given string \( s^{(n)} \), we may redefine the temperature \( T' := k_B T/\Delta E \), and we set \( P(s_i = 1) = 1 \) equal to the Hamming fraction \( t \). If we solve (7) for \( T' \), then we obtain a term for defining the following temperature.

**Definition 3** (Statistical Temperature). We define the *(statistical)* temperature \( T_{\text{stat}} \) of a string \( s \) as
\[
\frac{1}{T_{\text{stat}}(s)} := c_1 \cdot \log_2 \left( \frac{1 - t(s)}{t(s)} \right),
\]  
(8)

with a constant prefactor \( c_1 = 1/\log_2(e) \) that we omit in the following. The dependence of the temperature \( T_{\text{stat}} \) on \( t \) is shown in Figure 4.

**IV. ENERGY AND THE FIRST LAW**

The binary string represents an encoding of energy fractions for each degree of freedom of a macroscopic system, i.e., each element of the binary string \( s_i \) represents a degree of freedom that either carries energy or not. Hence, we may think of the 1s as representations of units of energy: Each (binary) element is either in its ground state (0) or upper energy state (1). Thus, the total number of 1s, i.e., the total Hamming weight, encodes the total energy contained within the specific system.

**Definition 4** (First Law). A transformation applied to a set of finite strings is said to respect the first law if it preserves the total Hamming weight of the strings.

**V. CARNOT’S VERSION OF THE SECOND LAW**

In accordance with previous related models, we consider the second law of thermodynamics as the logical reversibility of a transformation.

**Definition 5** (Second Law). A physical transformation acting on a set of finite strings is said to satisfy the second law of thermodynamics if it is logically reversible, i.e., no information is lost through the transformation.

From this fact, it has already been observed that Clausius- and Kelvin-like versions of the second law follow. It was an open question whether the same holds for Carnot’s theorem — which we show here. In order to achieve this, we consider physical transformation respecting both the first and the second law: A physical transformation on a finite string follows the first and second laws if it preserves the number of 1s and is logically reversible. For such transformations operating on a pair of strings, we now derive an upper bound on the efficiency of a process extracting free work.

Clausius’ version forbids the flow of information from a colder to a hotter string. We now consider the inverse scenario: If heat (a certain number of 1s) flows from a hotter to a colder string, then not all the 1s leaving the hotter string are required to be transferred to the colder one in order to enable the transformation to be logically reversible. (A logically reversible map cannot go from a larger to a smaller set.) The fraction of 1s not required for this compensation can be extracted as free energy from the process. This fraction, when compared to the total number of 1s leaving the hot string, is defined as the efficiency of the process.

**Theorem 1** (Carnot’s Theorem). The efficiency \( \eta \) of the Carnot process is
\[
\eta = 1 - \frac{T_{\text{stat},2}}{T_{\text{stat},1}},
\]  
(9)

where \( T_{\text{stat},1} \) is the temperature of the hot and \( T_{\text{stat},2} \) of the cold string.

**Proof.** Let us consider a string of Hamming fraction \( t_1 \) and a colder one with \( t_2 \) (\( < t_1 \)) being transformed by a heat flow to a lower fraction \( t'_2 = t_1 - \Delta_1 \) (where \( \Delta_1 \) is the fraction transferred away from the string), and for the second string a higher one \( t'_2 = t_2 + \Delta_2 \), (see Figure 5). It will turn out that the number of 1s to be added to the colder string \( (\Delta_2 \cdot n) \) can be smaller than the number of 1s \( (\Delta_1 \cdot n) \) taken out of the hotter, and the product of possible strings after the process is...
Hence, the ratio \( \Delta \), where fulfills, we must have in \( (10) \) can be approximated linearly in \( \Delta_1 \) and after does not get smaller. Specifically, i.e., guaranteeing that the number of pairs of strings before and after does not get smaller. Specifically, again, is the number of \( 1 \)s to be transferred to the second string in order to allow for logical reversibility, i.e., guaranteeing that the number of pairs of strings before and after does not get smaller. Specifically, the term \( 2^{n-h(t+\Delta)} \) corresponding to the physical entropy in our toy model. This, the information-theoretic entropy coincides with the physical entropy.

**Observation 1 (Entropy).** The change of the information-theoretic entropy \( \Delta h(t) \cdot n \) equals the change of the physical entropy \( \Delta S \).

The derivative \( h'(t) \) is the limit of the ratio of the change \( \Delta h \) of \( h \) given a change \( \Delta t \) of \( t \), and therefore, the inverse statistical temperature is

\[
\frac{1}{T_{\text{stat}}} \approx \frac{\Delta h(t)}{\Delta t}.
\]

This is consistent with Clausius' entropy definition based on physical heat \( Q \) and temperature \( T \)

\[
\Delta S = \frac{\Delta Q}{T},
\]

or, equivalently,

\[
\frac{1}{T} = \frac{\Delta S}{\Delta Q}.
\]

Thus, if \( \Delta Q = \Delta t \cdot n \) (which is in line with viewing the \( 1 \)s as the energy), then \( \Delta h(t) \cdot n \) corresponds to the entropy change, making the information-theoretic entropy \( h(t) \cdot n \) of the string correspond to the physical entropy in our toy model.

However, the actual information content of the string may be smaller than this if it has additional structure, namely, if the Kolmogorov complexity of the string is smaller than the entropy as defined simply by the fractions of \( 0 \)s and \( 1 \)s of the string. This serves as a motivation to define a second temperature measure, namely the **structural temperature**:

**Definition 6 (Structural Temperature).** We define the structural temperature \( T_{\text{struc}} \) through

\[
\frac{1}{T_{\text{struc}}} (s) := \max \limits_{\text{position choices}} \left( \frac{\Delta K(s)}{\Delta t} \right) .
\]

In contrast to the Hamming fraction, the Kolmogorov-complexity difference depends on the particular transformation, i.e., not merely on the number of \( 1 \)s added to the string, but the specific positions. We maximize over all possible choices of these positions. The defined notion of structural temperature allows for assigning a temperature to any string, without ignoring its structure beyond the Hamming fraction. An extremal example of a gap between the two temperatures is the alternating string

\[0101010101010101\ldots,\]

which is statistically hot but structurally cold: Flipping a bit at position \( k \) induces an arbitrarily large change \( \Delta K(s) \).
of $\log_2(k)$. Such a gap indicates that the string is far from thermal equilibrium. At the other end of the scale, the two temperature notions coincide exactly for those strings which have no structure besides the one given by the Hamming fraction, i.e., for which the Kolmogorov complexity equals the information-theoretic entropy:

$$K(s^{(n)}) \approx n \cdot h(t). \quad (21)$$

In these latter strings, all the energy is contained in form of heat; and it is exactly those we understand as being in thermal equilibrium or, in other words, as heat baths. Whereas we have, generally

$$T_{\text{struc}}(s) \leq T_{\text{stat}}(s), \quad (22)$$

heat baths are defined by the equality of the two temperatures.

**Definition 7** (Thermal Equilibrium). A string $s$ is in thermal equilibrium if its statistical temperature equals its structural temperature, i.e.,

$$T_{\text{stat}}(s) = T_{\text{struc}}(s). \quad (23)$$

Equivalently, we call such a string a heat bath.

The notion of thermal equilibrium allows us to talk about the zeroth law of thermodynamics.

**VII. The Zeroth Law**

A string is in thermal equilibrium and, therefore, a heat bath, if its statistical and structural temperature coincide. This definition extends naturally to pairs of heat baths: By $s^{(n)} \equiv_{t.e.} s'^{(n)}$, we mean:

$$K(s^{(n)})|s'^{(n)} = h \left( \frac{t + t'}{2} \right) \cdot 2n. \quad (24)$$

In (24), $s^{(n)}|s'^{(n)}$ represents the concatenation of two strings.

**Definition 8** (Zeroth Law). The relation $\equiv_{t.e.}$ of thermal equilibrium between pairs of heat baths is transitive.

Clearly, heat baths of different temperatures are not in thermal equilibrium. On the other hand, and this is the more surprising part, heat baths of the same temperature can fail to be in thermal equilibrium if

$$K(s^{(n)}|s'^{(n)}) < K(s^{(n)}) + K(s'^{(n)}). \quad (25)$$

A consequence thereof is, first, that a heat bath is not in thermal equilibrium with an identical copy of itself (the relation is irreflexive) since:

$$K(s^{(n)}|s^{(n)}) \approx K(s^{(n)}). \quad (26)$$

Furthermore, this irreflexivity of the relation implies that the zeroth law (transitivity) can also be violated: Assume that $s$ and $s'$ are heat baths that are in thermal equilibrium, then,

$$s \equiv_{t.e.} s' \text{ and } s' \equiv_{t.e.} s, \text{ but } s \not\equiv_{t.e.} s \quad (27)$$

The zeroth law, which would imply that $s$ is also in equilibrium with an identical copy of itself, is violated. For random heat baths, however, the law holds except with probability exponentially small in their size; in particular, the failure probability vanishes for infinitely large baths. In summary, the zeroth law suffers in our model from exactly the same deficit as the second law when stated as “entropy always increases:” It fails to hold with exponentially small probability.

**VIII. Negative Temperature**

Several authors have already proposed the concept of negative absolute temperature (see, e.g., [14]). This is consistent with our model. The dependence we propose of the temperature of a string and its Hamming fraction suggests that negative absolute temperatures occur for a string with Hamming fraction $> 1/2$ (see Figure 4, note that in some sense, strings with negative temperatures are hotter than those with positive ones). Intuitively, this is due to the fact that increasing the energy reduces the degrees of freedom in terms of the number of possible strings. In our model, also negative structural temperatures can occur if the string’s structure is such that adding energy — in the form of 1s — unavoidably makes the string more structured, i.e., its Kolmogorov complexity smaller.

**IX. Conclusion**

We have proposed a toy model, based on binary strings and their Hamming weight linked to the strings’ “temperature”, that is rich enough to represent basic facts of thermodynamics, such as its fundamental laws. With this, we suggest that thermodynamic concepts such as free energy, heat, and thermal equilibrium are of combinatorial and computational nature. In our model, the notion of negative temperature emerges naturally. First, with respect to statistical temperature: When the Hamming fraction of a string exceeds $1/2$, then additional energy (1s) mean reduction of degrees of freedom. Secondly, also the structural temperature, based on algorithmic complexity, can be negative (if additional energy, i.e., 1s, reduces complexity). Note that equality of the two temperature measures coincides with our notion of thermal equilibrium, and is related to the zeroth law, which appears in our model as redundant, underlining earlier observations by other authors. It is, however, valid only except with an error probability exponentially small in the size of the heat baths. This observation, which relates the zeroth law to the second, allowing for similar exceptions, is new.

In contrast to the zeroth and the second law, the situation around the third law is less clear. We propose as an open question to put its different formulations in context with our model in order to get more insight on their relation and correctness.

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