Quantising one-dimensional electromagnetic fields in position space

Daniel Hodgson, 1 Jake Southall, 1 Robert Purdy, 1 and Almut Beige 1

1 The School of Physics and Astronomy, University of Leeds, Leeds LS2 9JT, United Kingdom
(Dated: September 9, 2021)

An intuitive and straightforward modelling of many locally-interacting quantum systems requires a local description of the quantised electromagnetic field. In a recent paper [Southall et al., J. Mod. Opt. 68, 647 (2021)], we showed that such a description, which must overcome several no-go theorems, is indeed possible. To better justify our approach, this paper presents a systematic quantisation of one-dimensional electromagnetic fields in position space. Starting from the assumption that its basic building blocks are local bosonic particles with a clear direction of propagation—so-called bosons localised in position (BLiPs)—we identify the relevant Schrödinger equation and construct Lorentz-covariant electric and magnetic field observables. In addition we show that our approach simplifies to the standard description of quantum electrodynamics when restricted to a subspace of monochromatic photon Fock states.

I. INTRODUCTION

One way of quantising the electromagnetic (EM) field in one dimension is to postulate that the most basic constituents of the field are monochromatic photons with positive and negative wave numbers \( k \), positive frequencies \( \omega = c|k| \) and polarisations \( \lambda \in \{H,V\} \). The states describing a system of such photons are members of a Fock space and are generated by applying a set of bosonic creation operators \( \hat{a}_k^\dagger \) to the vacuum state \( |0\rangle \). Usually the resulting photon states evolve according to the Schrödinger equation with the Hamiltonian being given by the total field energy. If this is the case, then, for wave packets propagating along the \( x \)-axis, the observable for the electric field amplitude will be given by the expression

\[
E(x) = i \sum_{\lambda=H,V} \int_{-\infty}^{\infty} dk \sqrt{\frac{\hbar}{2\pi \varepsilon_0 c}} e^{ikx} (\hat{a}_k e_{k\lambda} + \text{H.c.})\]

(1)

Here the \( e_{k\lambda} \) polarisation vectors are oriented in the \( y-z \) plane; \( \varepsilon, \varepsilon, \text{ and } c \) denote the permittivity of the medium, the area occupied by photons in the \( y-z \) plane, and the speed of light respectively. The above equation is consistent with classical electrodynamics, and the expectation values of \( E(x) \) can be shown to evolve as predicted by Maxwell’s equations.

This standard description of the quantised EM field forms part of what is perhaps the most accurate and successful physical theory ever devised. Nevertheless, as we emphasised in a recent paper [2], it is incomplete. Like previous authors recognised, the above description prevents us from constructing localised and causally propagating wave packets [3,4]. Although the standard theory of the quantised EM field can be used to construct state vectors representing wave packets of any shape, it does not allow us to evolve them in all the different ways permitted by classical electrodynamics. For example, in one dimension, classical electrodynamics allows for the existence of highly localised wave packets which travel at the speed of light in the direction of the positive \( x \)-axis. However, in quantum electrodynamics, an initially localised wave packet delocalises almost immediately.

To see this, suppose that \( \alpha \) is a complex number and that the quantised EM field is initially prepared in a state \( |\psi_{z=0}\rangle \) where

\[
a_{k\lambda} |\psi_{z=0}\rangle = \frac{\alpha}{\sqrt{|k|}} e^{-ikx_0} |\psi_{z=0}\rangle
\]

(2)

for all \( k \) and a given polarisation \( \lambda \). In this case, the electric field expectation value \( \langle E(x) \rangle \) is given by

\[
\langle E(x) \rangle \propto i\alpha \int_{-\infty}^{\infty} dk e^{ik(x-x_0)} e_{k\lambda} + \text{c.c.}
\]

(3)

at \( t = 0 \). This expression is non-zero only for \( x = x_0 \), which shows that the state vector \( |\psi_{z=0}\rangle \) at time \( t = 0 \) corresponds to a highly localised wave packet at \( x_0 \). However, after a time \( t = L/c \), \( \langle E(x) \rangle \) equals

\[
\langle E(x) \rangle \propto
\]

\[
i\alpha \left[ \int_{0}^{\infty} dk e^{ik(x-x_0-L)} + \int_{-\infty}^{0} dk e^{ik(x-x_0+L)} \right] e_{k\lambda}
\]

(4)

which no longer simplifies to a \( \delta \)-function; rather \( \langle E(x) \rangle \) is non-zero everywhere for any time \( t \neq 0 \). If photons with positive and negative \( k \) travel in the positive and negative \( x \) directions respectively, an initially localised wave packet cannot remain so.

Now someone could argue that the above state \( |\psi_{z=0}\rangle \) is not the only state which describes a localised wave packet at \( x = x_0 \). This is true. Suppose the state of the quantised EM field equals \( |\phi_{x=0}\rangle \) with

\[
a_{k\lambda} |\phi_{x=0}\rangle = \frac{\alpha}{\sqrt{|k|}} e^{-ikx_0} |\phi_{x=0}\rangle
\]

(5)
for all $k > 0$ and a given polarisation $\lambda$ and with $a_{k\lambda}|\phi_{x_0\lambda}\rangle = 0$ for $k \leq 0$. In this case, the electric field expectation values $\langle \mathbf{E}(x) \rangle$ equal

$$\langle \mathbf{E}(x) \rangle \propto i\alpha \int_0^\infty dk \, e^{i k (x-x_0)} \, e_{k\lambda} + c.c. \quad (6)$$

at $t = 0$. This equation too simplifies to a $\delta$-function when the complex conjugate is taken into account but, unfortunately, this only applies for imaginary values of $\alpha$. As shown in Ref. [2], the state $|\phi_{x_0\lambda}\rangle$ becomes non-local when the corresponding wave packet passes through a phase shifter. This should not be the case, since phase shifters do not have this effect in classical electrodynamics. A complete theory of the quantised EM must allow for local wave packets that remain local while traveling at the speed of light even after experiencing a phase shifter. It is also worthwhile noting that in the standard theory a quantisation of the EM field in terms of modes that are restricted within a series of sub-regions of space is unitarily inequivalent to a quantisation in the total space, as recently shown in Ref. [9].

The above described problem also lies at the heart of what is known as the Fermi problem [10], first posed by Enrico Fermi in 1932. Here Fermi aimed to calculate the minimum time for a ground state atom to transition into an excited state through the absorption of radiation emitted by a second nearby atom [11]. As one would intuitively expect from causality considerations, he found that there is a zero probability for this transition to happen until a time has elapsed in which a light signal may propagate from the second atom to the first. However, in a collection of later papers by Shirokov [12] and others [13] [14], it was claimed that a causal result is due to an approximation that may not always be justified. As the calculations in Eqs. [9] [10] illustrate, coupling two atoms to local field states and evolving the quantised EM field as appointed by its standard description is bound to result in non-causal dynamics.

Incidentally, negative-frequency photons have already attracted some interest in the literature [19, 20, 21]. For example, quantum opticians often extend the lower bound of frequency integrations from 0 to $-\infty$ which often appears to be a convenient and well-justified approximation. Moreover, Hawton and Debierre [21] recently introduced negative-frequency states by extending the well-known expressions for the gauge field and canonical momenta in the canonical formalism. By including both the positive- and the negative-frequency excitations of the Klein-Gordon and photon fields, they were able to construct real and causal field excitations with properties analogous to those of the classical Maxwell theory. In their work, they made use of an inner product that is positive definite for negative-frequency states which allowed them to consider such states as physical. Their inner product of field states can be expressed as the time derivative of the field commutator, thus satisfying the micro-causality condition.

For many years, quantum opticians have recognised the importance of local photon theories in the study of locally-interacting quantum systems. For example, highly localised wave packets—so-called ultra-broadband photons—are frequently used in linear optics experiments [22, 23]. Similarly, an accurate description of individually trapped atomic particles with spontaneous photon emission requires a local theory of the surrounding free radiation field [24, 25]. This has resulted in a great deal of research into single-photon wave-functions [26, 27, 28]. Unfortunately, localising the photon is a difficult challenge for a number of reasons and attempts to do so have been met with varying levels of success. In what follows we shall review the most important of these theories to provide more background information for the discussion that follows.
Following the earlier work of Pryce [46], Newton and Wigner published a paper on the localisability of elementary systems [17] in which they defined position operators for massive particles of arbitrary spin. These operators have commuting components, are conjugate to the momentum operator and transform as a vector under rotations. In their paper, however, they claimed that such an operator does not exist for massless particles with spin $s > 1 \frac{1}{2}$. Similar statements were made by Wightman [45] and Fleming [44, 50], and a proof can be found in Ref. [51]. (See also Refs. [52, 53] for more recent discussions on Newton-Wigner localisation, and Ref. [54] for an example of alternative localisation criteria.) More recently, however, Hawton has shown that it is possible to define a position operator with commuting components that has transversally and longitudinally polarised eigenstates, but does not rotate as a vector [55]. This operator differs from that of Pryce by a Berry connection term. The transverse components of this position operator, combined with the generator of rotations about the propagation axis, were also shown to form an algebra for the two-dimensional Euclidean group [55].

Other authors have taken the more pragmatic approach of constructing wave-functions whose square modulus represent some useful local quantity (cf. e.g. Ref. [20]). The Landau-Peierls (LP) wave-function is an early example of such a wave-function whose square modulus represents local photon number density [56]. The LP wave function is not commonly used as it is related to the physical fields in a highly non-local way [57]. More recently, however, the LP wave function has been rediscovered by Mandel [58] and later by Cook [57, 59], who both derived expressions for a local, second-quantised photon number density operator, which was further shown by Amrein [60] to provide an upper bound for the number operator defined in the “weak localisation” scheme of Jauch and Piron [61]. Some authors have removed this non-local relation by defining first [37, 38, 41] second [38, 41, 43, 62] quantised forms of the energy density wave-function, which transform in a local way. This energy field is usually represented by the Riemann-Silberstein vector. Systems quantised in this way do require a non-local inner product in their standard form [38, 41, 62, 64], although recently techniques in biorthogonal quantum physics have been utilised to produce a local inner product [21, 25, 24, 41, 44]. For more details cf. e.g. Refs. [62, 65].

Unlike many other descriptions, Cook’s photon field is not vector-valued and its creation and annihilation components possess local bosonic commutation relations under the standard inner product: a necessary condition for generating localised particles. In the following, we adopt Cook’s notion. Different from the approach taken in Ref. [21], the starting point of our derivation is the assumption that our Hilbert space is a Fock space of local particles with bosonic commutation relations under the standard inner product. These local particles will be characterised by two distinct polarisations $\lambda = H, V$, and two distinct directions of propagation $s = \pm 1$. Moreover, at a given time $t$, they can be located anywhere along the $x$ axis and are obtained by applying bosonic creation operators $a_{s\lambda}^\dagger(0, x, t)$ to the vacuum state $|0\rangle$. From now on we simply refer to the local particles of light as bosons localised in position (BLiPs).

By specifying the direction of propagation, we immediately obtain an equation of motion in the Heisenberg picture. Afterwards we construct electric and magnetic field observables $E(x, t)$ and $B(x, t)$ that are consistent with Maxwell’s equations. As we shall see below, Lorentz covariance demands the presence of a position-independent superoperator $\mathcal{R}$ in the expressions of these observables in position space. The main effect of the superoperator $\mathcal{R}$ is to add $\sqrt{|k|}$ factors to their momentum space representations and to introduce non-local contributions to their position representations [57]. Therefore, like Cook [57, 59], we find that localised BLiP states are responsible for the emergence of electric and magnetic fields that are spread across all space, which may cause difficulties with fixing boundary conditions. This should not cause any great problems by and of itself, however, as measurable quantities are frequently related in very non-local ways to other, more fundamental quantities that are viewed as responsible in some way for the generation of the former. For example, the total mass density of a system is seen as responsible for the generation of the gravitational force at a single point. The two are related in a highly non-local way, but in spite of this, the mass density is still a very useful quantity.

As mentioned already above, introducing BLiP states is equivalent to doubling the standard Hilbert space of the quantised EM field and to adding negative-frequency photons. In this paper we find that this has immediate physical consequences. For example, in addition to providing a more complete model of photonic wave packets, we find that the local components of their electric and magnetic field observables commute with one another everywhere. This is contrary to the standard theory [68, 69] and suggests that we can always make simultaneous exact measurements of EM field amplitudes. It is also what allows us to construct strictly localised photonic particles in the sense of Knight [70] and Licht [71]. Due to the simplicity of calculations involving bosonic particles, we hope that our formalism will find a wide range of applications—from an improved modelling of inhomogeneous and of locally-interacting quantum optical systems [2, 43] to the modelling of the quantised EM field on curved spacetimes [72].

There are five sections to this paper. In Section II we introduce the equation of motion in position space and derive expressions for electric and magnetic field observables, up to the overall re-scaling operator $\mathcal{R}$, in the Heisenberg picture. In Section III we repeat the quantisation of the EM field in momentum space. Afterwards, in Section IV we establish a connection between both representations which share the vacuum state. It is shown that position and momentum space annihil-
tion operators can be linked via a Fourier transform, very much in the same way as we link local and non-local electric and magnetic field amplitudes in classical electrodynamics. Moreover, we eventually identify $R$ by imposing Lorentz covariance. Finally, we summarise our findings in Section V.

II. FIELD QUANTISATION IN POSITION SPACE

Before we derive the position space representations of the main observables and the dynamical Hamiltonian of the quantised EM field, let us have a closer look at the possible solutions of Maxwell’s equations in classical electrodynamics.

A. Maxwell’s equations in free space

Classical electromagnetism describes electric and magnetic fields by three-dimensional vectors, whose components are functions of position and time. Here we are only interested in the one-dimensional free theory in which light propagates in a homogenous dielectric medium along the $x$ axis. In this case, Maxwell’s equations can be written as

$$\begin{align*}
\partial_t B_1 &= \partial_t E_1 = 0, \\
\partial_t E_j &= \varepsilon_{1ij} c^2 \partial_1 B_i, \\
\partial_t B_j &= -\varepsilon_{1ij} \partial_1 E_i,
\end{align*}$$

where we consider an orthogonal, right-handed system of coordinates $(x_0, x_1, x_2, x_3) = (ct, x, y, z)$ with $t$ denoting time and $x, y$ and $z$ being position coordinates. Moreover, $c$ denotes the speed of light, $\varepsilon_{ijk}$ is the totally antisymmetric Levi-Civita symbol, and $E_i$ and $B_i$ are the components of the electric and magnetic field vectors $\mathbf{E}$ and $\mathbf{B}$. The presence of the index 1 ensures that the $i$ and $j$ components of $\mathbf{E}$ and $\mathbf{B}$ lie in the $y$-$z$ plane only. For convenience, we describe the EM field in this paper by complex vectors with the actual EM field vectors given by their real parts.

Using Maxwell’s equations, we see that the electric and magnetic field vectors independently obey the wave equations

$$\left( \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{O}(x, t) = 0$$

with $\mathbf{O} = \mathbf{E, B}$. In one dimension, we may also write this equation in the form

$$\left( \frac{\partial}{\partial x} \pm \frac{1}{c} \frac{\partial}{\partial t} \right) \left( \frac{\partial}{\partial x} \mp \frac{1}{c} \frac{\partial}{\partial t} \right) \mathbf{O}(x, t) = 0.$$  

This shows that the basic solutions of the free EM field in one dimension are wave packets of any shape which propagate in either the left or the right direction. In the following, $s = 1$ describes light moving towards $x = \infty$, while $s = -1$ is used to label light moving towards $x = -\infty$. Moreover, wave packets can take two independent polarisations $\lambda$. In the following $\lambda = H, V$ refers to horizontally and to vertically polarised light. Hence the electric and magnetic field vectors are each decomposed into four components,

$$\mathbf{O}(x, t) = \sum_{s=\pm1, \lambda=H,V} \mathbf{O}_{s\lambda}(x, t) ,$$

that independently satisfy the relation

$$\left( \frac{\partial}{\partial x} + \frac{s}{c} \frac{\partial}{\partial t} \right) \mathbf{O}_{s\lambda}(x, t) = 0.$$  

Since all wave packets propagate at the speed of light, we find that

$$\mathbf{O}_{s\lambda}(x, t) = \mathbf{O}_{s\lambda}(x - sct, 0),$$

where $\mathbf{O} = \mathbf{E, B}$. Later we shall use this relation to obtain an equation of motion for the basic building blocks of the quantised EM field.

In the absence of currents and charges, the energy density of the EM field is a locally conserved quantity that we can express in terms of the local field observables $\mathbf{E}(x, t)$ and $\mathbf{B}(x, t)$. Since we consider complex EM field vectors in this paper, with the physical field vectors given by their real parts, the energy of the EM field at any time $t$ equals

$$H_{\text{energy}}(t) = \frac{A}{2} \int_{-\infty}^{\infty} dx \left[ \varepsilon \text{E}_{\text{real}}(x, t)^2 + \frac{1}{\mu} \text{B}_{\text{real}}(x, t)^2 \right].$$

Here $A$ is the area inhabited by the field in the $y$-$z$ plane,

$$\begin{align*}
\text{E}_{\text{real}}(x, t) &= (\mathbf{E}(x, t) + \mathbf{H}.c)/2, \\
\text{B}_{\text{real}}(x, t) &= (\mathbf{B}(x, t) + \mathbf{H}.c)/2,
\end{align*}$$

$\varepsilon$ is the electric permittivity, and $\mu$ is the magnetic permeability of the dielectric medium. The latter relates to the speed of light via $c = 1/\sqrt{\varepsilon\mu}$.

B. The Hilbert space of the quantised EM field and its basic equation of motion

One possible starting point for quantising the EM field in momentum space is to assume that its basic building blocks are monochromatic photons $\lambda$. In the following, we take an analogous approach and assume that the basic building blocks of the quantised EM field in position space are local particles with bosonic commutator relations, so-called bosons localised in position (BLiPs) $\lambda$. At any given time $t$, each BLiP is fully characterised by its position $x$, its direction of motion $s$ and its polarisation $\lambda$. We therefore denote their annihilation operators...
by $a_{s\lambda}(x,t)$ in the Heisenberg picture and by $a_{s\lambda}(x,0)$ in the Schrödinger picture. Once quantised, the complex electric and magnetic field operators $E(x,t)$ and $B(x,t)$ and the energy observable $H_{\text{energy}}(t)$ from the previous subsection become operators acting on the Hilbert space of the quantised EM field. To identify this Hilbert space, we proceed as usual and define the vacuum state $|0\rangle$ to be the state that is annihilated by all local annihilation operators,

$$a_{s\lambda}(x,t)|0\rangle = 0,$$  \hspace{1cm} (17)

and equip it with the normalised inner product,

$$\langle 0|0 \rangle = 1.$$  \hspace{1cm} (18)

We then obtain single BLiP state vectors by applying a creation operator $a_{s\lambda}^\dagger(x,t)$ to the vacuum state. For example,

$$|1_{s\lambda}(x,t)\rangle = a_{s\lambda}^\dagger(x,t)|0\rangle$$

(19)

describes a single BLiP at $(x,t)$ with direction $s$ and polarisation $\lambda$. Repeated application of creation operators $a_{s\lambda}^\dagger(x,t)$ to the vacuum state generates a complete set of multi-particle states that eventually span the full Hilbert space of the quantised EM field.

Representing particles, we would like the BLiP states of local particles that are separated in space and time to be pairwise orthogonal with respect to the standard inner product of quantum physics. Hence we demand in the following that

$$\langle 1_{s\lambda}(x,t)|1_{s'\lambda'}(x',t') \rangle = \delta_{ss'}\delta_{\lambda\lambda'} \langle 0|0 \rangle$$

(20)

with the function $g_{s\lambda}(x,x')$ given by

$$g_{s\lambda}(x,x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ik(x-x')}$$

$$= \delta_{s}(x-x').$$  \hspace{1cm} (21)

The above inner product can be interpreted as a probability density. For example, the above $\delta$ function is strictly positive, translation independent, and both symmetric and real. Furthermore $g_{s\lambda}(x,x')$ depends on $x$, $x'$ and $s$, but not on $\lambda$, as one would expect for a probability density, and, given the state $|1_{s\lambda}(x,t)\rangle$, the probability of finding a BLiP within $(-\infty, \infty)$ is unity. The reason for the chosen $s$-dependence of terms in Eq. (21) is to ensure that the $s = 1$ and the $s = -1$ cases do not become formally the same.

Given a single particle with state vector $|1_{s\lambda}(x,t)\rangle$, its $x$ and $t$ are perfectly well known. Hence Heisenberg’s uncertainty relation implies that the momentum and energy uncertainties of the particle must be infinitely large. This means we clearly cannot associate this state with momentum or energy eigenstates. Consequently, we cannot proceed as in Ref. [1] and simply guess the Hamiltonian of the quantised EM field in position space. An alternative approach is needed to obtain an equation of motion. Fortunately, this is easily done since all light with a clear direction of propagation travels at the same constant speed. As we can see from Eq. (14), any expectation value of the EM field taken at some position $x$ and some time $t$ is identical to the expectation value taken at time $t = 0$ at position $x - sc t$. Accordingly, we conclude that $(a_{s\lambda}(x,t)) = (a_{s\lambda}(x - sc t,0))$. Since this relation must hold for all states, we find that

$$a_{s\lambda}(x,t) = a_{s\lambda}(x - sc t,0).$$  \hspace{1cm} (22)

This equation is the fundamental equation of motion of the quantised EM field in position space.

C. Local field observables

Next, we obtain expressions for the (complex) operators $E(x,t)$ and $B(x,t)$, and the energy observable $H_{\text{energy}}(t)$. As shown in Section II.A, the basic solutions of Maxwell’s equations in a one-dimensional homogeneous medium are wave packets which travel at the speed of light along the $x$ axis. Hence, $E(x,t)$ and $B(x,t)$ and the $a_{s\lambda}(x,t)$ operators in Eq. (22) have exactly the same $x$ and $t$. Taking this into account, we postulate that the observables of the complex vectors $E(x,t)$ and $B(x,t)$ can be written as

$$E(x,t) = \sum_{s = \pm 1} c R \left(a_{sH}(x,t) \hat{y} + a_{sV}(x,t) \hat{z}\right),$$

$$B(x,t) = \sum_{s = \pm 1} s R \left(a_{sH}(x,t) \hat{z} - a_{sV}(x,t) \hat{y}\right).$$  \hspace{1cm} (23)

Here the factor $c$ has been added for convenience. Moreover, $\hat{y}$ and $\hat{z}$ are unit vectors which point in the direction of the positive $y$ and $z$ axes, respectively. Eq. (23) tells us how field vectors evolve in time. Hence, proceeding as described in App. A in Ref. [2], it is relatively straightforward to show that the expectation values of the above field observables evolve indeed as predicted by Maxwell’s equations. The reason for this is the $x-sc t$ dependence of the $a_{s\lambda}(x,t)$ operators and the care that has been taken to ensure that the different components of the above field vectors point in the right directions.

Later on in Section III.V we see that $R$ denotes a position-independent superoperator. The magnitude of this regularisation operator determines the energy expectation value of each individual BLiP. Indeed, when we calculate the observable for the energy of the quantised EM field by substituting Eq. (23) into Eq. (16), we obtain the expression

$$H_{\text{energy}}(t) = \sum_{s = \pm 1} \sum_{\lambda = H,V} \frac{\varepsilon A^2}{4} \int_{-\infty}^{\infty} dx \left[R \left(a_{s\lambda}(x,t)\right) + \text{H.c.}\right]^2.$$  \hspace{1cm} (24)

Most importantly, because of the quadratic form of this observable, energy expectation values are always positive, as they are in classical electrodynamics.
D. Commutation relations in position space

Before we can map the basic equation of motion in Eq. (22) onto Heisenberg’s equation with a dynamical Hamiltonian, we need to specify the commutator relations of the BLiP operators. As is usual for bosons in quantum field theory, we assume that a state can contain any number of identical particles and, therefore, it does not matter in what order local excitations are created or destroyed. Hence BLiP annihilation operators, as well as BLiP creation operators, must commute amongst themselves,

\[
[a_{s\lambda}(x,t), a_{s'\lambda'}(x', t')] = [a_{s\lambda}^\dagger(x,t), a_{s'\lambda'}^\dagger(x', t')] = 0.
\]

(25)

Moreover, the local creation and annihilation operators of the quantised EM field must possess a suitable set of commutation relations,

\[
[a_{s\lambda}(x,t), a_{s'\lambda'}^\dagger(x', t')] \\
= [a_{s\lambda}(x - sc t, 0), a_{s'\lambda'}^\dagger(x' - s'c t', 0)] \\
= \tilde{g}_{s\lambda}(x - sc t, x' - s'c t'),
\]

(26)

where \(\tilde{g}_{s\lambda}(x, x')\) is a non-vanishing function. Since the \(a_{s\lambda}(x,t)\) and the \(a_{s\lambda}^\dagger(x,t)\) are single BLiP annihilation and creation operators, there is a close relationship between \(\tilde{g}_{s\lambda}(x - sc t, x' - s'c t')\) and the inner product in Eq. (20). Indeed one can show that

\[
\langle 1_{s\lambda}(x,t)|1_{s'\lambda'}(x', t') \rangle = \langle 0|a_{s\lambda}(x,t)a_{s'\lambda'}^\dagger(x', t')|0 \rangle
\]

which implies

\[
\langle 1_{s\lambda}(x,t)|1_{s'\lambda'}(x', t') \rangle = [a_{s\lambda}(x,t), a_{s'\lambda'}^\dagger(x', t')].
\]

(28)

Because of this equivalence, \(\tilde{g}_{s\lambda}(x, x')\) in Eq. (26) must coincide with \(g_{s\lambda}(x, x')\) in Eq. (20). Thus

\[
[a_{s\lambda}(x,t), a_{s'\lambda'}(x', t')] = \delta_s(x - x') \delta_{s,s'} \delta_{\lambda,\lambda'}
\]

(29)

which is the expected bosonic commutator relation.

E. Single photon wave functions

Suppose we want to describe an experiment involving certain types of single particle wave packets which are not completely local. If these have polarisation \(\lambda\) and travel in the \(s\) direction, we can describe them with the help of non-local annihilation operators \(b_{s\lambda}(x_0, t)\) of the form

\[
b_{s\lambda}(x_0, t) = \int_{-\infty}^{\infty} dx \ h(x - x_0) a_{s\lambda}(x,t).
\]

(30)

Here \(x_0\) depends on the position of the wave packet along the \(x\) axis and \(h(x)\) is a complex function of \(x\). Calculating the corresponding same-time commutator relation using Eq. (29) yields

\[
[b_{s\lambda}(x_0, t), b_{s'\lambda'}^\dagger(x'_0, t)] = \int_{-\infty}^{\infty} dx \ h(x - x_0) h^*(x - x'_0) .
\]

(31)

When \(x_0 = x'_0\), and \(h(x - x_0)\) is normalised, this commutator equals one and reduces to the well-known bosonic commutation relation of wave packets. Wave packets of light too are therefore often referred to as photons, e.g. in the context of linear optics experiments [30–33]. This commutator is also closely related to the single particle inner product as we saw in Eq. (27). The inner product between the non-local \(b_{s\lambda}(x_0, t)\) states is given by

\[
\langle b_{s\lambda}(x_0, t)|b_{s'\lambda}^\dagger(x_0, t) \rangle = \int_{-\infty}^{\infty} dx \ |h(x - x_0)|^2.
\]

(32)

Thus the normalised expression \(h(x)\) is a position wave function that determines a probability density in accordance with the Born rule. Although non-locally related to the field observables, as was intended with the LP and the Cook wave functions, this provides an intuitive local description of light.

F. The dynamical Hamiltonian

In this final subsection, we show that the equation of motion in Eq. (22) can be written as a Schrödinger equation. More concretely, we show that field observables \(O(x,t)\) evolve in the Heisenberg picture according to Heisenberg’s equation of motion,

\[
\frac{d}{dt} O(x,t) = -\frac{i}{\hbar} [O(x,t), H_{\text{dyn}}(t)].
\]

(33)

In the following we deduce the dynamical Hamiltonian \(H_{\text{dyn}}(t)\) in this equation from symmetry arguments and from Eq. (22). Suppose \(O(x,t) = a_{s\lambda}(x,t)\). In this case, Eq. (33) leads us to the relation

\[
\frac{d}{dt} a_{s\lambda}(x,t) = -\frac{i}{\hbar} [a_{s\lambda}(x,t), H_{\text{dyn}}(t)].
\]

(34)

Moreover, we see from Eq. (22) that the time derivative and the space derivative of \(a_{s\lambda}(x,t)\) differ only by a constant factor \(sc\). We can use this to obtain the relation

\[
\frac{d}{dx} a_{s\lambda}(x,t) = \frac{is}{\hbar c} [a_{s\lambda}(x,t), H_{\text{dyn}}(t)].
\]

(35)

This suggests that the dynamical Hamiltonian only affects the position but not the time coordinate of \(a_{s\lambda}(x,t)\). This is not surprising, since the purpose of \(H_{\text{dyn}}(t)\) is to propagate wave packets at the speed of light along the \(x\) axis. As the generator of such dynamics, the Hamiltonian must continuously annihilate BLiPs at their respective positions \(x\), while replacing them immediately with excitations of equal amplitudes at nearby positions \(x'\) different from \(x\).
Combining all of the above observations and taking into account that time evolution maps BLiPs into BLiPs we find that the dynamical Hamiltonian takes the general form

\[ H_{\text{dyn}}(t) = \sum_{s=\pm1} \sum_{\lambda=H,V} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx'' \hbar sc f_{s\lambda}(x'',x') \]

\[ \times a_{s\lambda}^\dagger(x'', t) a_{s\lambda}(x', t), \]

where \( f_{s\lambda}(x'',x') \) is a complex function. Consider for a moment a BLiP at a position \( x' \) that propagates to \( x'' \). Another BLiP at \( x'' \), which has been assigned the same direction \( s \), may only propagate to \( x' \), if time is reversed. Therefore, \( f_{s\lambda}(x'',x') \) must be antisymmetric in its arguments,

\[ f_{s\lambda}(x'',x') = -f_{s\lambda}(x',x''). \]  

(37)

The above considerations also imply that \( f_{s\lambda}(x',x') = 0 \) which prevents BLiP states from remaining stationary. Moreover, Eq. (22) suggests that \( f_{s\lambda}(x,x') \) is invariant under translations in \( x \). Since, we assume in the following that

\[ f_{s\lambda}(x'',x') = f_{s\lambda}(x''-x'). \]  

(38)

By combining the above equations and using the commutator in Eq. (24), one can now show that

\[ \frac{d}{dx} a_{s\lambda}(x,t) = i \int_{-\infty}^{\infty} dx' f_{s\lambda}(x-x') a_{s\lambda}(x',t). \]  

(39)

This relation only holds when \( i f_{s\lambda}(x-x') \) coincides with the spatial derivative of a \( \delta \)-function with argument \( x-x' \), that is, when

\[ f_{s\lambda}(x-x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \ e^{ik(x-x')} \]

\[ = -i \frac{d}{dx} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \ e^{ik(x-x')} \right] \]

\[ = -i \delta_s''(x-x') \]

(40)

which is consistent with the above symmetries. Here \( \delta_s''(x-x') \) denotes the spatial derivative of \( \delta_s(x-x') \) in Eq. (21) with respect to \( x \).

Overall, the dynamical Hamiltonian \( H_{\text{dyn}}(t) \) in Eq. (30) equals

\[ H_{\text{dyn}}(t) = \sum_{s=\pm1} \sum_{\lambda=H,V} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx'' \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hbar sc \]

\[ \times e^{i(k(x''-x'))} a_{s\lambda}^\dagger(x'',t) a_{s\lambda}(x',t) \]

(41)

which one can check to be Hermitian, and therefore a generator of unitary dynamics. As pointed out in Eq. (7), \( H_{\text{dyn}}(t) \) no longer coincides with the energy observable \( H_{\text{energy}}(t) \) of the quantised EM field in Eq. (24). We must therefore check that energy is conserved, i.e. that

\[ [H_{\text{energy}}(t), H_{\text{dyn}}(t)] = 0. \]

(42)

Owing to the symmetry of the BLiP commutator relations and the asymmetry of \( f_{s\lambda}(x,x') \), one can show that this is indeed the case.

### III. FIELD QUANTISATION IN MOMENTUM SPACE

Section 11 contains a complete description of the quantised EM field in position space. In this section, we proceed analogously to obtain a complete EM field description in momentum space. Both descriptions complement each other and can be used interchangeably. For example, in classical electrodynamics, we often describe light scattering experiments using Maxwell’s equations which only involve local field amplitudes. In such situations, it might be best to use the position space representation when modelling the quantised EM field. In other situations, classical electrodynamics introduces optical Green’s functions and decomposes the EM field into monochromatic waves to predict general optical properties. In such cases, it might be more convenient to consider a momentum space representation when analysing quantum properties.

In order to derive such a representation, we now introduce the complex electric and magnetic field vectors \( \tilde{E}(k,t) \) and \( \tilde{B}(k,t) \). Here the \( k \) variables are real numbers which can assume any value between \( -\infty \) and \( +\infty \). In close analogy to Eq. (12), we write these vectors as superpositions of electric and magnetic field vector components \( \tilde{E}_{s\lambda}(k,t) \) and \( \tilde{B}_{s\lambda}(k,t) \) of wave packets with polarisation \( \lambda \) and direction of propagation \( s \), i.e.

\[ \tilde{O}(k,t) = \sum_{\lambda=H,V} \sum_{s=\pm1} \tilde{O}_{s\lambda}(k,t) \]

(43)

with \( \tilde{O} = \tilde{E}, \tilde{B} \). As in classical electrodynamics, we relate local and non-local field vectors via a Fourier transform and assume that

\[ O_{s\lambda}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{i(kx+\varphi(k))} \tilde{O}_{s\lambda}(k,t), \]

(44)

where \( \varphi(k) \) is a function of \( k \). Hence, considering the inverse of the Fourier transform, we find that

\[ \tilde{O}_{s\lambda}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-i(kx+\varphi(k))} O_{s\lambda}(x,t). \]

(45)

From the Fourier inversion theorem, we know that this inverse transformation only exists for absolutely integrable functions \( O_{s\lambda}(x,t) \) if the above integrals cover the whole \( x \) and the whole \( k \) axes respectively, which is why \( k \) must be able to assume any real value. Moreover, the phases \( \varphi(k) \) can depend on \( k \).

In Eq. (11), we have added the factor \( s \) to the exponent to ensure that changing \( s \) reverses the dynamics of wave packets. In this way, the parameter \( s \) has the same physical meaning as in Section 11 it labels left- and right-moving wave packets differently. Unlike \( s \), \( k \) does not change its sign when we change the direction in which the wave packet travels. This means the parameter \( k \) no longer has the same meaning as the wave number \( k \) in classical electrodynamics. We only adopt the same notation here for convenience.
In the following, we first have a closer look at the basic solutions of the Fourier-transformed versions of Maxwell’s equations. Afterwards, we derive the complex electric and magnetic field observables \( \vec{E}(k, t) \) and \( \vec{B}(k, t) \) in terms of momentum space annihilation operators \( \hat{a}_s(k, t) \). In addition, we obtain expressions for the energy observable and for the dynamical Hamiltonian, \( \mathcal{H}_{\text{energy}}(t) \) and \( \mathcal{H}_{\text{dyn}}(t) \). Since we allow photons—the basic energy quanta of the EM field—to have positive and negative frequencies \( \pm \Omega \) the momentum space representation we obtain in this section is not the same as the EM field’s standard description. However, we show how to recover the standard representation of the quantised EM field in Section IV.

A. Maxwell’s equations in free space

Substituting Eq. (14) into Eq. (9), we find that Maxwell’s equations apply if the vectors \( \vec{O}_{s\lambda}(k, t) \) point in the same direction as the vectors \( \vec{O}_{s\lambda}(x, t) \). In addition, the Cartesian coordinates of \( \vec{E}_{s\lambda}(k, t) \) and \( \vec{B}_{s\lambda}(k, t) \) must obey the relations

\[
\frac{\partial}{\partial t} \vec{E}_{s\lambda}(k, t)_j = i \varepsilon_{1ij} sc^2 k \vec{B}_{s\lambda}(k, t)_i, \\
\frac{\partial}{\partial t} \vec{B}_{s\lambda}(k, t)_j = -i \varepsilon_{1ij} sk \vec{E}_{s\lambda}(k, t)_i.
\]

(46)

The above equations show that Maxwell’s equations assume a much simpler form in momentum than in position space. For example, the wave equation in Eq. (10) simplifies to the second-order linear differential equation

\[
\frac{\partial^2}{\partial t^2} \vec{O}_{s\lambda}(k, t) = (ck)^2 \vec{O}_{s\lambda}(k, t),
\]

(47)

where again \( \vec{O} = \vec{E}, \vec{B} \). Solving this equation while taking into account that \( s = 1 \) and \( s = -1 \) refer to right-moving and to left-moving wave packets respectively, we find that the EM field vectors evolve such that

\[
\vec{O}_{s\lambda}(k, t) = e^{-i \kappa t} \vec{O}_{s\lambda}(k, 0),
\]

(48)

where \( \vec{O}_{s\lambda}(k, 0) \) denotes a complex amplitude. Moreover, Eqs. (46) and (47) imply that initial electric and magnetic field amplitudes must relate such that

\[
\vec{E}_{s\lambda}(k, 0)_j = -\varepsilon_{1ij} sc \vec{B}_{s\lambda}(k, 0)_i, \\
sc \vec{B}_{s\lambda}(k, 0)_j = \varepsilon_{1ij} \vec{E}_{s\lambda}(k, 0)_i.
\]

(49)

The momentum space field vectors which must be orthogonal to each other and to the \( x \) axis remain so as they evolve in time.

Next we derive the EM field energy \( \mathcal{H}_{\text{energy}}(t) \) as a function of \( \vec{E}_{s\lambda}(k, t) \) and \( \vec{B}_{s\lambda}(k, t) \). Starting from Eq. (15) and proceeding as described in App. A, we arrive at the relation

\[
\mathcal{H}_{\text{energy}}(t) = \frac{\varepsilon A}{4} \sum_{s=\pm1} \sum_{\lambda=H,V} \int_{-\infty}^{\infty} dk \left[ e^{i \varphi(k)} \vec{E}_{s\lambda}(k, t) + e^{-i \varphi(-k)} \vec{E}_{s\lambda}^*(k, t) \right]^2.
\]

Here, one immediately notices a problem. This expression depends on the choice of \( \varphi(k) \) which suggests that a change of the phase—a change only in our point of view—changes the system’s energy. The phase dependence lies within the mixed frequency terms. Fortunately, Eq. (40) is independent of \( \varphi(k) \) which means that we can assume in the following, without restrictions, that

\[
\varphi(k) = -\varphi(-k).
\]

(51)

Now the energy \( \mathcal{H}_{\text{energy}}(t) \) no longer depends on \( \varphi(k) \). Notice also that fixing \( \varphi(k) \) in this way does not restrict the physical situations which we can describe. For example, different kinds of interference effects, e.g. constructive or destructive, between monochromatic waves with positive and negative frequencies can still occur. However, this now only depends on the relative phases of complex field amplitudes. Moreover, we learn from Eqs. (48) and (49) that \( \mathcal{H}_{\text{energy}}(t) \) is always time-independent and positive, as it should be.

B. The Hilbert space of the quantised EM field and its basic equation of motion

As we have seen above, the basic solutions of Maxwell’s equations in one dimension and in momentum space are left- and right-moving travelling waves with two different directions of propagation \( s \), a real parameter \( k \) and two different polarisations \( \lambda \). In the following, we denote the corresponding annihilation operators by \( \hat{a}_{s\lambda}(k, t) \). As usual, we refer to these energy quanta as photons \[.\] From Eq. (48), we see that their frequencies are given by

\[
\omega = ck.
\]

(52)

Since \( k \) varies between \(-\infty \) and \( +\infty \), these frequencies now take positive and negative values. This is important since, without considering the full range of frequencies, the Fourier transform in Eq. (14) would not have an inverse transformation (cf. Eq. (15)). As we shall see below in Section IV, extending the standard description of the quantised EM field into the negative frequency range will allow us to establish a connection between the highly localised states introduced in the previous section and the momentum space representation which we consider in this section.

To construct the relevant Hilbert space of the quantised EM field, we proceed as in Section IV and start again with the vacuum state \( |0\rangle \). Applying any annihilation operator to the vacuum state yields zero, i.e.

\[
\hat{a}_{s\lambda}(k, t) |0\rangle = 0.
\]

(53)

However, when applying a creation operator \( \hat{a}_{s\lambda}^\dagger(k, t) \) to the vacuum state, we obtain non-trivial states

\[
|1_{s\lambda}(k, t)\rangle = \hat{a}_{s\lambda}^\dagger(k, t) |0\rangle
\]

(54)

which we refer to in the following as single-photon states. Each photon carries a certain amount of strictly positive
energy. Applying creation operators $\hat{a}^\dagger_{s\lambda}(k,t)$ multiple times to the vacuum creates a complete set of photon Fock states which eventually span the full Hilbert space of the quantised EM field.

C. Non-local field observables

In analogy to Eq. (23), we assume in the following that the complex electric and magnetic field observables $\tilde{E}(k,t)$ and $\tilde{B}(k,t)$ can be written as

$$\tilde{E}(k,t) = \sum_{s=\pm 1} \bar{\Omega}(k) \left[ \hat{a}_{s\lambda}(k,t) \hat{y} + \hat{a}_{s\lambda}^\dagger(k,t) \hat{z} \right],$$

$$\tilde{B}(k,t) = \sum_{s=\pm 1} s \bar{\Omega}(k) \left[ \hat{a}_{s\lambda}(k,t) \hat{z} - \hat{a}_{s\lambda}^\dagger(k,t) \hat{y} \right]$$

in momentum space. Here $\bar{\Omega}(k)$ is a complex function of $k$ which determines field amplitudes and therefore also the energy of a single photon in the $(s,k,\lambda)$ mode. Because of the EM field’s isotropy, this function cannot depend on $s$ or on $\lambda$. Moreover, the above operators are non-Hermitian, and their expectation values are complex by construction. As in Section IV, we assume here that the actual field expectation values are given by their real parts (cf. Eq. (10)).

Next we notice that the time dependence of the expectation values of the above non-local fields, which can be found in Eq. (58), implies that

$$\hat{a}_{s\lambda}(k,t) = e^{-i\omega t} \hat{a}_{s\lambda}(k,0).$$

(56)

This equation is the basic equation of motion of the quantised EM field in momentum space. Combining Eqs. (55) and (56), one can verify that the expectation values of the Fourier-transformed complex field vectors $\tilde{E}(k,t)$ and $\tilde{B}(k,t)$ evolve indeed as predicted by Maxwell’s equations. They obey Eq. (46).

For completeness we now also derive the energy observable of the quantised EM field $H_{\text{energy}}(t)$ in the momentum representation. Starting again from Eq. (34) and proceeding as described in App. B we now find that

$$H_{\text{energy}}(t) = \frac{\varepsilon A e^2}{4} \sum_{s=\pm 1} \sum_{\lambda \neq \lambda'} \int_{-\infty}^{\infty} dk \left[ \bar{\Omega}(k) \hat{a}^\dagger_{s\lambda}(k,t) + \bar{\Omega}^*(k) \hat{a}^\dagger_{s\lambda}(k,t) \right] \hat{a}_{s\lambda}(k,t)$$

$$\times \left[ \bar{\Omega}(k) \bar{\Omega}^*(k) + \bar{\Omega}(\bar{k}) \bar{\Omega}^*(\bar{k}) \right] \hat{a}^\dagger_{s\lambda}(k,t) \hat{a}_{s\lambda}(k,t) \right].$$

(57)

As in the classical case, the expectation values of this operator are always positive. However, unlike the usual Hamiltonian for a system of uncoupled harmonic oscillators, the above energy observable contains additional mixed-frequency terms in the form of pairs of annihilation only and creation only operators. This is not surprising, since the analogous expression for the classical energy of the EM field in free space in Eq. (50) contains terms of the same form. As in Section IV A these terms arise due to possible interference effects between field excitations with identical parameters $s$ and $\lambda$ but exactly opposing $k$ values. However, using Eqs. (55) and (57), one can easily check that the energy of the quantised EM field is always conserved and always positive.

D. Commutation relations in momentum space

In principle, we now have a complete description of the quantised EM field in momentum space. We identified an equation of motion and the basic field observables. However, for completeness, we now also have a closer look at the commutation relations of the $\hat{a}_{s\lambda}(k,t)$ operators and derive the momentum space representation of the dynamical Hamiltonian $H_{\text{dyn}}(t)$. To do so, we notice again that it should not matter in which order we create or annihilate excitations. Hence

$$\left[ \hat{a}_{s\lambda}(k,t), \hat{a}^\dagger_{s'\lambda'}(k',t') \right] = 0 = \left[ \hat{a}^\dagger_{s\lambda}(k,t), \hat{a}^\dagger_{s'\lambda'}(k',t') \right].$$

(58)

Moreover, we demand that the single photon states in Eq. (59) are pairwise orthogonal such that

$$\langle 1_{s\lambda}(k,t)|1_{s'\lambda'}(k',t') \rangle = \delta_{s,s'} \delta_{\lambda,\lambda'} \delta(k-k')$$

(59)

where we used Eq. (60). As one would expect, the photons that we consider in this section are bosonic particles.

E. The dynamical Hamiltonian

Eq. (59) shows that monochromatic field excitations are eigenstates of the dynamical equation of motion. Given the bosonic commutator relation in Eq. (59), this suggests that the dynamical Hamiltonian of the quantised EM field equals

$$H_{\text{dyn}}(t) = \sum_{s=\pm 1} \sum_{\lambda \neq \lambda'} \int_{-\infty}^{\infty} dk \ hck \hat{a}^\dagger_{s\lambda}(k,t) \hat{a}_{s\lambda}(k,t)$$

(60)

in momentum representation. As already discussed in the Section V A the eigenvalues of this Hamiltonian which are integer values of $hck$ can take any value between $-\infty$ and $\infty$. The above equation contains an odd function in $k$ which also applies to the position space representation of $H_{\text{dy}}(t)$ in Eq. (11). The above expression for the dynamical Hamiltonian has a much simpler form than the equivalent expression in position space (cf. Eq. 11). The opposite is true for the energy observable $H_{\text{energy}}(t)$ which is much simpler in position than in momentum space (cf. Eqs. (24) and (57)). As pointed out previously, both representations have their respective merits.


IV. THE RELATIONS BETWEEN DIFFERENT EM FIELD REPRESENTATIONS

In this section, we first ensure that the electric and magnetic field observables in the momentum representation are consistent with relativistic transformations between reference frames. Afterwards, we identify the connection between position space and momentum space representations. Finally, we have a closer look at the relationship between our approach and the standard description of the quantised EM field in momentum space which employs only a subset of the photon annihilation operators that we consider here.

A. Lorentz covariance in momentum space representation

As of yet, we have not considered what states the photon operators describe from the point of view of other inertial or rotated observers. Since the inner product of quantum physics has a probabilistic interpretation, it must be invariant under translations, rotations and boosts. From classical physics, we know that the electric and magnetic field amplitudes are the components of an antisymmetric rank-2 tensor that transform accordingly.

It is therefore necessary that the expectation values of the electric and magnetic field observables, and hence the observables themselves, must have these same properties. In what follows, we shall use the invariance of the inner product of quantum physics to determine the photon operator transformations under such changes of reference frame, and following that determine the value of $\tilde{\Omega}(k)$ in Eq. (55) such that electric and magnetic field observables transform correctly.

As noted in Section II D the single-photon states $|s_\lambda(k, t)⟩$ with distinct parameters $k$, $s$ and $\lambda$ are orthogonal to one another. The right hand side of the inner product in Eq. (59) is already invariant under translations in $x$ and $t$, and under rotations about the $x$ axis. Under such transformations, state vectors change at most by a unitary phase, and an orthogonal rotation of polarisation vectors in the $y$-$z$ plane. Hence, for any choice of $\tilde{\Omega}(k)$, these transformations are fully consistent with the tensor transformation properties of the field observables. To determine $\tilde{\Omega}(k)$ we need only examine the transformations of state vectors under boosts.

Using the Lorentz-invariant expression $|k|\delta(k - k')$, the inner product in Eq. (59) is invariant under a Lorentz boost $\Lambda$ along the $x$ axis for all $k$ when

$$U(\Lambda) \tilde{a}_{s\lambda}(k, t) U^{-1}(\Lambda) = \sqrt{|p|/k} \tilde{a}_{s\lambda}(p, t). \quad (61)$$

Here $U(\Lambda)$ is a unitary operator describing the boost $\Lambda$ such that $p^\mu = \Lambda^\mu_{\nu}k^\nu$. Using the tensor transformation law, the EM field observables ought to transform as the time derivative and the curl respectively of a gauge-invariant four-vector field with no time-like or longitudinal polarisations. The orientation of the $\mathbf{E}$ and $\mathbf{B}$ fields have been determined in Eq. (59), so we more simply require that the components of the electric and magnetic field observables, namely

$$\int_{-\infty}^{\infty}dk \tilde{\Omega}(k) e^{ik(x-ct)} \tilde{a}_{s\lambda}(k, t), \quad (62)$$

transform as a time derivative. Using Eq. (61), we see that this applies when

$$\tilde{\Omega}(k) = \sqrt{|k|/\tilde{\Omega}_0}, \quad (63)$$

where $\tilde{\Omega}_0$ denotes a constant complex number. Since additional phases have been accounted for in the Fourier transform of $\check{\mathbf{E}}(k, t)$ in Eq. (14), we can assume that $\tilde{\Omega}_0$ has zero phase and is consequently real. If we want the expectation values of the energy observable $\hat{H}_{\text{energy}}(t)$ in Eq. (57) and of the dynamical Hamiltonian $\hat{H}_{\text{dyn}}(t)$ in Eq. (60) to be the same, at least in some cases, we must choose

$$|\tilde{\Omega}_0|^2 = \frac{4\hbar}{\varepsilon Ac}. \quad (64)$$

The latter equivalence only holds for states with positive $k$ values. In general, the above choice of $\tilde{\Omega}_0$ implies that the energy of a single photon in the $(s, k, \lambda)$ mode equals $\hbar c|k|$.

B. The connection between position and momentum space representations

Now that we have identified the constants $\tilde{\Omega}(k)$, we can finally have a closer look at the connection between the position and the momentum space representations that were introduced in Sections II I I and III I I. Of course, both representations must be equivalent since both describe the same physical system. This means that:

1. The momentum and position Hilbert spaces have the same vacuum state $|0⟩$.
2. There is a linear, invertible transformation between position and momentum space annihilation operators, $a_{s\lambda}(x, t)$ and $\tilde{a}_{s\lambda}(k, t)$, which preserves their bosonic commutator relations in Eqs. (23) and (59).
3. All observables of the quantised EM field need to be identical in either representation.

The third condition automatically guarantees that the position representation which we obtained in this paper is Lorentz covariant too.

The above considerations suggest that the BLiP annihilation operators $a_{s\lambda}(x, t)$ are unique linear superpositions of same-time photon annihilation operators $\tilde{a}_{s\lambda}(k, t)$ with positive- and negative-frequency components. To preserve the bosonic commutator relations in...
both cases, we link them via a Fourier transform. A closer look at Eq. (44) suggests that this transform can be written as
\[
an_s(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, e^{i(\mathbf{k} \cdot \mathbf{x} + \varphi(k))} \tilde{a}_s(k,t)
\] (65)
with the phases \(\varphi(k)\) being the same as in Section III. The inverse of this transformation is
\[
\tilde{a}_s(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \, e^{-i(\mathbf{k} \cdot \mathbf{x} + \varphi(k))} a_s(x,t)
\] (66)
which shows that the photon annihilation operators \(\tilde{a}_s(k,t)\) are unique superpositions of same-time BLiP annihilation operators \(a_s(x,t)\). One can also verify relatively easily that if the \(a_s(x,t)\) are bosonic, then so are the \(\tilde{a}_s(k,t)\).

Substituting Eq. (65) into the local electric and magnetic field observables in Eq. (23) and assuming that \(\mathcal{R}\) is a linear superoperator, we find that
\[
E(x,t) = \frac{1}{2\pi} \sum_{s=\pm1} \int_{-\infty}^{\infty} dk \, c \, e^{i(\mathbf{k} \cdot \mathbf{x} + \varphi(k))} \times \mathcal{R} (\tilde{a}_sH(k,t) \hat{y} + \tilde{a}_s\mathbf{v}(k,t) \hat{z}),
\]
\[
B(x,t) = \frac{1}{2\pi} \sum_{s=\pm1} \int_{-\infty}^{\infty} dk \, s \, e^{i(\mathbf{k} \cdot \mathbf{x} + \varphi(k))} \times \mathcal{R} (\tilde{a}_sH(k,t) \hat{z} - \tilde{a}_s\mathbf{v}(k,t) \hat{y}).
\] (67)

Moreover, substituting \(\mathbf{E}(k,t)\) and \(\mathbf{B}(k,t)\) in Eq. (65) into the Fourier transform in Eq. (44) yields
\[
E(x,t) = \frac{1}{\sqrt{2\pi}} \sum_{s=\pm1} \int_{-\infty}^{\infty} dk \, \mathcal{E}(k) \, e^{i(\mathbf{k} \cdot \mathbf{x} + \varphi(k))} \times [\tilde{a}_sH(k,t) \hat{y} + \tilde{a}_s\mathbf{v}(k,t) \hat{z}],
\]
\[
B(x,t) = \frac{1}{\sqrt{2\pi}} \sum_{s=\pm1} \int_{-\infty}^{\infty} dk \, \mathcal{B}(k) \, e^{i(\mathbf{k} \cdot \mathbf{x} + \varphi(k))} \times [\tilde{a}_s\mathbf{v}(k,t) \hat{y} - \tilde{a}_sH(k,t) \hat{z}].
\] (68)

As mentioned at the beginning of this subsection, both sets of operators must be identical and must act on the same Hilbert space. Hence, taking into account Eqs. (63) and (64), we find that the regularisation operator \(\mathcal{R}\) implements a non-unitary but invertible operation,
\[
\mathcal{R} (\tilde{a}_s(k,t)) = \frac{4\hbar}{\varepsilon A_C} \sqrt{|k|} \tilde{a}_s(k,t).
\] (69)

In momentum space, it simply multiplies \(a_s(k,t)\) with a \(k\)-dependent factor. As expected from the symmetries of the quantised EM field, the superoperator \(\mathcal{R}\) depends neither on \(x, t, s\) nor \(\lambda\).

The above trick of introducing \(\mathcal{R}\) allows us to describe the quantised EM field in position space with the help of local bosonic BLiP operators without having to sacrifice the Lorentz covariance of the local electric and magnetic field observables \(E(x,t)\) an \(B(x,t)\). These field observables are no longer simple linear superpositions of local annihilation and creation operators but contain an extra operator. However, since it is easier to work with bosonic annihilation and creation operators, we can perform all calculations in the Hilbert space created by the local bosonic operators. Fortunately, the dynamical Hamiltonian does not change \(k\). Hence, it does not matter if we regularise the \(a_s(x,t)\) operators at the beginning or at the end of a computation. More concretely, one can show that
\[
\mathcal{R} \left( U_{\text{dyn}}(t,0) \tilde{a}_sH(k,t) U_{\text{dyn}}^\dagger(t,0) \right) = U_{\text{dyn}}(t,0) \mathcal{R} (\tilde{a}_sH(k,t)) U_{\text{dyn}}^\dagger(t,0),
\] (70)
where \(U_{\text{dyn}}(t,0)\) denotes the time evolution operator associated with the dynamical Hamiltonian \(H_{\text{dyn}}\). Given the linearity of the above operators, this equation implies
\[
\mathcal{R} \left( U_{\text{dyn}}(t,0) a_s(x,t) U_{\text{dyn}}^\dagger(t,0) \right) = U_{\text{dyn}}(t,0) \mathcal{R} (a_s(x,t)) U_{\text{dyn}}^\dagger(t,0).
\] (71)

This means we can ignore \(\mathcal{R}\) when analysing the dynamics of state vectors and expectation values in BLiP representation. The regularisation operator only needs to be taken into account when calculating field vector and energy expectation values. The same is likely to apply when we use the local theory of the quantised EM theory in Section III to construct locally acting interaction Hamiltonians for the modelling of light scattering experiments. In this case, we can build interactions between local wave packets moving in opposite directions out of the local BLiP annihilation and creation operators [2]. If we do this, we obtain Hermitian interaction Hamiltonians whose dynamics can be analysed relatively easily. Moreover, if the interaction preserves energy for any possible incoming state and eventually only changes \(k\) into \(+k\) or \(-k\), then the regularisation operator \(\mathcal{R}\) is only needed at the very end in the calculation of field expectation values.

Moreover, using the above commutators and the field observables in Eq. (23), one can show that all components of the EM field observables commute with one another at all positions and times, i.e.
\[
0 = \left[ O(x,t)_i + \text{H.c.}, P(x',t)_j + \text{H.c.} \right]
\] (72)
where \(O, P = E, B\) and \(i, j = 1, 2, 3\). Hence, within the framework which we present here, we can always measure all local EM field components simultaneously. This is not surprising, since the electric and magnetic field vectors of travelling waves with a well-defined direction of propagation are closely related. From classical electrodynamics, we know that the EM field amplitudes of travelling waves only differ in orientation and amplitude by a constant rotation and a constant factor. Hence, \(E(x,t)\) and \(B(x,t)\) contain the same information about the system, and their individual components must commute [1]. However, some quantisation schemes lead to non-trivial commutation relations between electric and
magnetic field vector components. The reason for this is that these studies consider an incomplete set of standing-wave photon modes.

Finally, for completeness, let us check that the position and momentum representations of the dynamical Hamiltonian and the energy observable of the quantised EM field are identical. The latter holds due to the equivalence of the local EM field vectors in Eqs. (67) and (68). Moreover, one can show that all state vectors evolve according to the same Schrödinger equation. When substituting the annihilation operators \( \tilde{a}_{s\lambda}(k, t) \) in Eq. (66) into Eq. (60), we indeed obtain the dynamical Hamiltonian in Eq. (11). In other words, we have established a clear connection between the position and momentum space representations of the quantised EM field.

C. Alternative ways of achieving consistency between position and momentum space representations

Having a closer look at the standard expressions for the electric and magnetic field observables, \( E(x, 0) \) and \( B(x, 0) \) suggests alternative definitions of position-dependent field annihilation operators \([36, 41, 43–45]\). However, without the negative-frequency photon subspace, the creation operators that are obtained in this way can only be used to generate highly non-local field excitations. Hawton \([21, 25, 44, 45]\) and others \([41, 43]\) replaced the standard inner product of quantum physics with bosonic \( \langle 0 \rvert \lambda \rangle \) defined as

\[
\langle 0 \rvert \lambda \rangle = R(a_{s\lambda}(x, t)) \tag{76}
\]

Given this new inner product, the above states can be made to form a local orthogonal basis for the single-excitation states in position space.

D. The connection to the standard description of the quantised EM field

In this final subsection, we compare the description in Section III with the standard description of the quantised EM field in momentum space and ask which additional assumptions have to be made for the latter to emerge. Looking at Eqs. (57), (60), (63) and (64), and demanding that the energy observable \( H_{\text{energy}}(t) \) and the dynamical Hamiltonian \( H_{\text{dyn}}(t) \) become the same, we see that this automatically applies when we ignore the negative-frequency photons and restrict ourselves to positive \( k \) values \([2]\). As one can check relatively easily, in this case, the real parts of the local field observables in Eqs. (67) and (68) become the same as the local field observables in the standard description of the quantised EM field \([1]\).

As we know, the positive-frequency photon states provide a complete description of the quantised EM field in the sense that they can be superposed to re-produce the right electric and magnetic field expectation values of wave packets of any shape. However, as we have seen in the Introduction, they are not sufficient to generate the quantum versions of all possible solutions of Maxwell’s equations, like highly localised wave packets that remain localised \([1]\).

V. CONCLUSIONS

The results in this paper are based on the following basic aspects of classical optics which must hold simultaneously:

1. In one dimension, we can localise photons to arbitrarily small length scales, i.e. at positions \( x \).

2. For a given direction of propagation \( s \), those local photon excitations remain local.

3. Measurements are constant on the light cones.

In Section III, we use these properties to construct a straightforward and natural description of one-dimensional quantised EM fields in position space. Our starting point is the assumption that we can generate
local particles of light—so-called bosons localised in position (BLiPs)—by applying bosonic creation operators $a_\lambda^{\dagger}(x,t)$ to the vacuum state $|0\rangle$. Using the above postulates, we then identified their Schrödinger equation and constructed electric and magnetic field observables $E(x,t)$ and $B(x,t)$ which are consistent with Maxwell’s equations. The Lorentz covariance of these field operators is achieved with the help of a regularisation superoperator $\mathcal{R}$ (cf. Eqs. (23) and (69)). Although the BLiPs themselves are local, they generate non-local fields which can be felt in a region of the space surrounding them. Similarly, proceeding as described in the Introduction, the BLIP states can be used to construct highly localised wave packets of light which remain local when travelling along the $x$ axis. These localised fields are generated by non-local BLiP states.

Writing the BLiP excitations as superpositions of monochromatic photons, we show that our approach is consistent with the standard theory of the quantised EM field with the addition of countable negative-frequency states. Previously, these states have been widely overlooked but the concept of adopting them to negate the consequences of Hegerfeldt’s theorem [6] is not new. The idea of negative-frequency excitations has long been realised as important in a local description of light. For example, various papers [21, 23, 25, 29] have given explicit formulations of second quantised theories with negative-frequency solutions. However, by introducing local particles of light with a given direction of propagation, we have clarified how these solutions arise naturally in a covariant quantised theory. In addition, we exposed and elucidated some consequences of a theory containing these states. These consequences include the interference of positive- and negative-frequency states and the dissimilarity between the energy observable and the generator for time translation $\mathcal{L}$.

In classical electrodynamics, a local description of the EM field is often preferable to a non-local description. Similarly, we expect that the modelling of the quantised EM field in terms of BLiP states is often preferable to the standard description in terms of monochromatic photon states. For example, the position space representation in Section 11 should provide an extremely useful tool for the modelling of the quantised EM field in inhomogeneous dielectric media and on curved spacetimes [72]. Moreover, a local description is advantageous when modelling local light-matter and local light-light interactions. For example, in Ref. [2], we used the BLiP annihilation operators $a_\lambda(x,t)$ to construct locally acting mirror Hamiltonians. Other potential applications of the results in this paper include providing novel insight into fundamental effects, like the Fermi problem, the Casimir effect and the Unruh effect, as well as the modelling of linear optics experiments with ultra-broadband photons [30, 33]. Finally, our work might inspire other areas of quantum physics, where there is a need for a local description of quantum fields, including the modelling of massive particles, e.g. electrons in quantum transport problems.

Acknowledgement. AB and DH would like to thank Prof. Basil Altaie for interesting and stimulating discussions. Moreover, we acknowledge financial support from the Oxford Quantum Technology Hub NQIT (grant number EP/M013243/1). Statement of compliance with EPSRC policy framework on research data: This publication is theoretical work that does not require supporting research data.

[1] R. Bennett, T. M. Barlow, and A. Beige, Eur. J. Phys. 37, 014001 (2016).
[2] J. Southall, D. Hodgson, R. Purdy, and A. Beige, J. Mod. Opt. 68, 647 (2021).
[3] G. C. Hegerfeldt, Phys. Rev. D 10, 3320 (1974).
[4] B. S. K. Skagerstam, Int. J. Theor. Phys. 15, 213 (1976).
[5] J. Fernando Perez and I. F. Wilde, Phys. Rev. D 16, 315 (1977).
[6] G. C. Hegerfeldt, Nucl. Phys. B Proc. Suppl. 6, 231 (1989).
[7] D. B. Malament, In Defense of Dogma: Why There Cannot be a Relativistic Quantum Mechanics of (Localizable) Particles. In: R. Clifton (eds) Perspectives on Quantum Reality, The University of Western Ontario Series in Philosophy of Science (A Series of Books in Philosophy of Science, Methodology, Epistemology, Logic, History of Science, and Related Fields), vol 57 Springer Verlag, (Dordrecht, 1996).
[8] H. Halvorsen and R. Clifton, Philos. Sci. 69, 1 (2002).
[9] M. R Vásquez, M. del Rey, H. Westman and J. León, Ann. Phys. (N. Y.) 351, 112 (2014).
[10] J. S. Ben-Benjamin and L. Cohen, Phys. Lett. A 384, 18 (2020).
[11] E. Fermi, Rev. Mod. Phys. 4, 87 (1932).
[12] M. I. Shirokov, Yad. Fiz. 35, 1033 (1987).
[13] R. F. Mirollo, Phys. Lett. A. 88, 104 (1982).
[14] G. C. Hegerfeldt, Phys. Rev. Lett. 88, 1525 (1995).
[15] M. Bozzi, C. Sabatini, G. Adesso, F. Plastina and S. Maniscalco, New J. Phys. 14, 103011 (2012).
[16] J. A. Vaccaro, Found. Phys. 41, 1569 (2011).
[17] K. W. Chan, C. K. Law, and J. H. Eberly, Phys. Rev. Lett. 88, 100402 (2002).
[18] M. Pawlowski, and F. Persico, Phys. Rev. A 44, 798(E) (1991).
[19] P. W. Milonni, D. F. V. James and H. Fearn, Phys. Rev. A 52, 1525 (1995).
[20] M. Borrelli, C. Sabatini, G. Adesso, F. Plastina and S. Maniscalco, New J. Phys. 14, 103010 (2012).
[21] J. A. Vaccaro, Found. Phys. 41, 1569 (2011).
[22] K. W. Chan, C. K. Law, and J. H. Eberly, Phys. Rev. Lett. 88, 100402 (2002).
[23] M. Hawton and V. Debierrre, Phys. Lett. A. 381, 1926 (2017).
[24] V. Debierrre, The photon wave function in principle and in practice, PhD thesis in Optics, Ecole Centrale Marseille (2015).
the above equation. Doing so, we see that

\[ E_{\lambda}^2 = \sigma \sum_{s=1}^{\infty} \int_{-\infty}^{\infty} dx \left( \mathbf{E}_{s\lambda}(x,t) + c.c. \right) \left( \mathbf{E}_{s\lambda}(x,t) + c.c. \right) + c^2 \left( \mathbf{B}_{s\lambda}(x,t) + c.c. \right) \left( \mathbf{B}_{s\lambda}(x,t) + c.c. \right). \]  

(A1)

Taking into account that the electric and magnetic field vectors of waves which travel in the same direction only differ by a constant factor, as shown in Eq. (40), the above equation simplifies to

\[ H_{\text{energy}}(t) = \frac{\varepsilon A}{8} \sum_{\lambda=H\text{V}} \sum_{s,s'=1}^{\infty} \int_{-\infty}^{\infty} dx \left( 1 + ss' \right) \left( \mathbf{E}_{s\lambda}(x,t) + c.c. \right) \left( \mathbf{E}_{s\lambda}(x,t) + c.c. \right). \]  

(A2)

Hence one can show that

\[ H_{\text{energy}}(t) = \frac{\varepsilon A}{4} \sum_{\lambda=H\text{V}} \sum_{s=1}^{\infty} \int_{-\infty}^{\infty} dx \left( \mathbf{E}_{s\lambda}(x,t) + c.c. \right)^2. \]  

(A3)

Since we are looking for the EM field energy as a function of its complex electric and magnetic field amplitudes \( \mathbf{E}_{s\lambda}(k,t) \) and \( \mathbf{B}_{s\lambda}(k,t) \), we now substitute Eq. (44) into the above equation. Doing so, we see that

\[ H_{\text{energy}}(t) = \frac{\varepsilon A}{8\pi} \sum_{\lambda=H\text{V}} \sum_{s=1}^{\infty} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \left( e^{i(kx + \varphi(k))} \mathbf{E}_{s\lambda}(k,t) + c.c. \right) \left( e^{i(k'x + \varphi(k'))} \mathbf{E}_{s\lambda}(k',t) + c.c. \right). \]  

(A4)
Performing the $x$ integration yields $\delta$-functions,

\[
H_{\text{energy}}(t) = \frac{\varepsilon A}{4} \sum \sum \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \\
\times \left[ e^{i(\varphi(k) + \varphi(k'))} \tilde{E}_{s\lambda}(k, t) \tilde{E}_{s\lambda}(k', t) \delta(k + k') \\
+ e^{i(\varphi(k) - \varphi(k'))} \tilde{E}_{s\lambda}(k, t) \tilde{E}_{s\lambda}^*(k', t) \delta(k - k') \\
+ \text{c.c.} \right].
\]

(A5)

Finally, after also performing the $k'$ integration, we arrive at Eq. (50) in the main text.

Appendix B: The energy observable of the quantised EM field in momentum space

Substituting Eq. (55) into Eq. (A4), one can show that the energy observable of the $H_{\text{energy}}(t)$ of the quantised EM field equals

\[
H_{\text{energy}}(t) = \frac{\varepsilon A c^2}{8\pi} \sum \sum \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \\
\times \left[ e^{i(\varphi(k) + \varphi(-k))} \Omega(k) \Omega(-k) \tilde{a}_{s\lambda}(k, t) \tilde{a}_{s\lambda}^*(k', t) \\
+ e^{-i(\varphi(k) + \varphi(-k))} \Omega^*(k) \Omega^*(k) \tilde{a}_{s\lambda}^*(k, t) \tilde{a}_{s\lambda}^*(k', t) \\
+ |\Omega(k)|^2 \tilde{a}_{s\lambda}(k, t) \tilde{a}_{s\lambda}^*(k, t) \\
+ |\Omega^*(k)|^2 \tilde{a}_{s\lambda}^*(k, t) \tilde{a}_{s\lambda}(k, t) \right].
\]

(B1)

This equation has many similarities with Eq. (A5) and simplifies to Eq. (57) in the main text, when the phase relation in Eq. (51) is taken into account.