Effective field theory for the bulk and edge states of quantum Hall states in unpolarized single layer and bilayer systems

Ana Lopez¹ and Eduardo Fradkin²

¹Centro Atómico Bariloche, 8400 S. C. de Bariloche, Río Negro, Argentina
²Department of Physics, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, IL 61801-3080

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We present an effective theory for the bulk Fractional Quantum Hall states in spin-polarized bilayer and spin-1/2 single layer two-dimensional electron gases (2DEG) in high magnetic fields consistent with the requirement of global gauge invariance on systems with periodic boundary conditions. We derive the theory for the edge states that follows naturally from this bulk theory. We find that the minimal effective theory contains two propagating edge modes that carry charge and energy, and two non-propagating topological modes responsible for the statistics of the excitations. We give a detailed description of the effective theory for the spin-singlet states, the symmetric bilayer states and for the (m,m,m) states. We calculate explicitly, for a number of cases of interest, the operators that create the elementary excitations, their bound states, and the electron. We also discuss the scaling behavior of the tunneling conductances in different situations: internal tunneling, tunneling between identical edges and tunneling into a FQH state from a Fermi liquid.

I. INTRODUCTION

Two-dimensional electron gases (2DEG) in bilayers in high magnetic fields display a rich spectrum of fractional quantum Hall (FQH) states, some of them exhibiting rather unusual properties not found in single layer systems. Likewise unpolarized and partially polarized single layer FQH states have properties related to the FQH states in fully polarized bilayer wave functions. For the (m,m,m) primary, Laughlin-like, FQH states in bilayers were proposed long ago by Halperin [1]. A description of these states in terms of the hierarchy was given by Wen and Zee [2], and a fermion Chern-Simons construction [3] of their Jain-like states were given by us in ref. [4] and [5]. The basic properties of the bulk (m,m,m) Halperin states are well understood, both experimentally and theoretically, for a review see ref. [6]. The bulk Halperin states are well described by the effective K-matrix theories [4] [7], and their associated edge states have a simple representation in terms of a theory of two chiral bosons [8].

Among the multitude of bilayer states, there are three representative cases of special interest:

1. The spin-singlet states in single layers. These states have an SU(2) spin symmetry that organizes their spectra [9]. The most interesting representatives are the (3,3,2) state which corresponds to a ν = 2/5 singlet state, and the related state at ν = 2/3, which belongs to the reversed sequence of the (3,3,1) state with p < 0. Both states have been seen experimentally [10,11]. More recently higher order descendents in the Jain-like family of spin singlet states at ν = 4/7 and ν = 4/9 have also been seen [12].

2. The symmetric bilayer states. These states have a U(1) × U(1) symmetry. The most interesting representative of this group is the (3,3,1) FQH state in bilayers at ν = 1/2. This state has a number of unusual properties and it has been seen experimentally [13]. It has been conjectured that the (3,3,1) state may have a phase transition to the non-Abelian Pfaffian FQH state [14,15] triggered by inter-layer tunneling [16]. Experiments on wide quantum wells [17] show clear evidence of other bilayer states, including the (3,3,1) descendants at 4/5 and 4/7. They also show strong evidence unbalanced states (at zero parallel field) at 11/15, 13/21 among others.

3. The (m,m,m) bilayer states. These states are special in that they are ground states of the 2DEG with spontaneous interlayer coherence [18] [19] [20]. Consequently, their spectra have a massless bosonic excitation, a Goldstone boson. Recent tunneling experiments into the bulk (1,1,1) state at filling factor 1 have provided evidence for the Goldstone boson [21]. There is also a related state at filling factor 1 in single layers which exhibits both the integer quantum hall effect and ferromagnetism [22] [23]. NMR experiments [24] have provided firm evidence for the existence of skyrmion excitations in these states [25].

Recent experiments on tunneling of electrons into the edge of (so far) single layer fully polarized FQH states by A. Chang and coworkers [24,25] have opened a new window to study these interesting systems. No experiments have been done yet on bilayer systems or in unpolarized single layer 2DEGs. The experiments have raised a number of questions that theory has not answered so far. It has been found that in clean junctions the tunneling current at low voltages does follow a scaling law, as predicted by theory [26] [27]. The exponent found in experiment obeys, to a good approximation, the law
\( \alpha = 1/\nu \) where \( \nu \) is the filling factor. There is an indication in the experiment of a small feature that may be interpreted as a plateau in the exponent near \( \nu = 1/3 \) \[27\]. However, this interpretation requires a number of assumptions about the charge distribution near the edge which are hard to justify in detail. While other interpretations are also possible, the clearest conclusion that can be drawn from the data is that the dependence of the exponent on the magnetic field is much simpler than predicted by theory. At the theoretical level there are two types of predictions. On the one hand, Kane and Fisher \[27\], using the standard hierarchy construction, predicted a highly complex number theoretic function of the filling factor in the absence of disorder, and a significantly simpler dependence when the effects of disorder are accounted for \[27,28\]. Shytov, Levitov and Halperin \[29\] have considered the effects of long range Coulomb interactions and charge redistribution on edge tunneling and found a result identical to the case of short range interactions with disorder. However both predictions disagree quite significantly with the experimental results. Not only the predicted structure of the dependence of the exponent with the filling factor is much more complex than what is observed, but even the average tendency of this dependence fails to predict a simple \( \alpha = 1/\nu \) law. In fact, for \( 1/4 \leq \nu \leq 1/2 \) the prediction is a constant value of \( \alpha = 3 \).

This puzzle has motivated the development of a number of alternative approaches to this problem. D. H. Lee and X. G. Wen \[8\] have shown that if the velocity of the neutral modes of the standard \( K \)-matrix description for some reason becomes much smaller than the velocity of the charge mode, the \( \alpha = 1/\nu \) law follows. Ziulicke and Macdonald have suggested that the experiment could be understood if it were a simple consequence of charge conservation \[31\]. (Note however that the electron operator used in \[31\] has the correct charge but the wrong statistics; however this shortcoming does not affect the tunneling density of states.)

Recently we have proposed a conceptually different construction of the edge states which also yields the \( \alpha = 1/\nu \) law \[32\]. Our approach is based on the observation that the complex structure of the edge states predicted by the \( K \)-matrix theory \[8\] is a reflection on the edge of the standard hierarchical construction of the bulk \( \text{FQH} \) state, and that a possible, and consistent, interpretation of the experiments is that they fail to show evidence for the hierarchy. The principal idea behind the work of ref. \[32\] is that, for all the states in the Jain sequence, there is a much simpler construction of their bulk effective theories which does not involve the full structure of the hierarchy-based \( K \)-matrix theory, and the (sometimes) large number of additional conservation laws that it requires. The resulting effective theory contains one charge field and at least one additional field needed to give the correct topological degeneracies on a torus, and the correct quantum numbers for the excitations. One advantage of this approach is that, unlike the standard hierarchical construction, this theory has only one fundamental quasiparticle from which the entire spectrum is built.

In ref. \[33\] we also showed that there is a natural edge associated to this picture, which holds under the assumption that the edge is sharp and that no edge reconstruction takes place. The assumption that the edge is sharp means that the fine structure of edge modes, predicted by the mean field approximation to the fermion Chern Simons theory of the bulk state \[33\], occurs only on small scales (i.e. the magnetic length). At long scales these modes cannot be resolved (even assuming that the mean field approximation can be trusted in regions where the gap collapses, something that is far from clear). In this regime, the effective theory of the edge acquires a minimal structure which contains one propagating charged mode and, (at least) one extra mode required by the topological invariance of the bulk. This latter mode, which in this paper we will call the topological mode, has no relation with the neutral modes that appear in Wen’s theory \[8\]. In fact this mode does not propagate simply because there is no physical mechanism to give the excitations of this field a non-zero velocity. Thus, this field remains topological at the edge: its Hamiltonian is zero. In practice, the only role of the topological field is to contribute to the operators that create the excitations in the form of effective Klein operators that set the correct statistics. In this way, the operators of this effective theory have the right quantum numbers. A direct consequence is that the predicted tunneling density of states for electrons has the correct exponent and naturally yields the \( \alpha = 1/\nu \) law. However, this construction can only describe the system exactly at the values of the filling fraction for the Jain sequence. In other words, it can not make any prediction about the tunneling exponents for filling fractions interpolating between two \( \text{FQHE} \) states.

The \( \alpha = 1/\nu \) law of single layer fully polarized systems suggests that this effect may be a simple consequence of charge conservation (as suggested in ref. \[31\]). If this assumption were to be correct, a similar prediction should also hold for more complex situations such as in the \( \text{FQH} \) states in bilayers, spin unresolved single layer systems, and other cases where more that one subband may be present \[14\], or for 2D hole gases where Kramers states are present \[15\]. A few predictions for tunneling exponents for the simplest Halperin states have been proposed in the literature \[36\]. However, a theory that accounts for the rich spectrum of excitations and the related tunneling processes in these \( \text{FQH} \) states is lacking. In particular, a general theory of the tunneling exponents has not yet been formulated. (see however ref. \[15\]).

In this paper we present a theory of the edge states for all Jain-like states of \( \text{FQH} \) states in bilayers and unpolarized single layer systems. We generalize the approach of our earlier work for single layer fully polarized systems \[32\]. We begin by constructing the simplest possible effective theory of the bulk states, compatible with
the requirements of global gauge invariance, topological degeneracy on a torus, and with the smallest possible number of fundamental quasiparticles. We then use this theory to determine the physics of its edge states. Here too, under the assumption that the edge is sharp, unconstructed and clean, we proceed to derive the simplest theory compatible with these requirements. The effective theory of the edge states thus derived contains two propagating fields, the charge modes for each layer, and two topological non-propagating fields. We then use this construction to determine the structure of the edge states for all Jain sequences whose primary states are the Halperin \((m_1, m_2, n)\) states. We discuss in detail the spectrum of operators for the symmetric states. In particular we discuss the theory of the edge states for the \(SU(2)\) states, based on the \((m, m, m - 1)\) states and the general \(U(1) \times U(1)\) states. In each case we give an explicit construction of the operators that create the quasiparticles (and quasiholes), charged and neutral bound states (including neutral fermionic states) and the electron operators. For the case of the \(SU(2)\) states we show how the symmetry is realized in the spectrum and how it is promoted to a local \(su(2)_1\) current algebra. We use these theories of the edges to calculate the tunneling exponents for all cases in three different situations: internal tunneling, tunneling of electrons between identical liquids and tunneling of electrons into a FQH fluid from an external Fermi liquid lead. We find that although the tunneling exponents are universal, in general they no longer follow the \(1/\nu\) law observed in single layer fully polarized systems.

Finally we present a theory of the edge states for the \((m, m, m)\) states. Here we discuss the role of the bulk Goldstone boson and its effects on edge physics. The effective theory of the edges of the \((m, m, m)\) states is a chiral boson for the charge mode (with the same radius as the Laughlin states), and a non local non-chiral theory for the neutral modes. The non-locality of the neutral sector is due to the massless Goldstone mode. The effective action for the neutral sector is a generalization of the Caldeira-Leggett action to a problem of a field interacting with an active medium.

### A. Summary of Results and Organization

This paper is organized as follows. In Section \[\text{I}\] we derive an effective theory for bilayer FQH states in the bulk, consistent with the requirement of global gauge invariance on a torus. We apply this construction for general Jain-like states on bilayers. We find that a generic state is labeled by five integers, \(n_1, n_2, p_1, p_2\) and \(n\). Here, \(2n_1\) and \(2n_2\) are the number of flux quanta attached to a fermion on layers 1 and 2 respectively, \(n\) is the number of flux quanta attached to a fermion on layer 1 due to a fermion on layer 2 (and viceversa), and \(p_1\) and \(p_2\) are the occupation numbers of the effective Landau levels.

The signs of \(p_1\) and \(p_2\) determine determinme both direct and reversed sequences of FQH states. The formulas for the total filling fraction \(\nu = \nu_1 + \nu_2\) and polarization \(M = (\nu_1 - \nu_2)/2\nu\) are given by Eqs. \[\text{1.6}\] and \[\text{1.7}\]. These sequences were derived by us in reference \[\text{3}\]. In general the states in these sequences describe polarized states, namely states with \(M \neq 0\). The symmetric states, with \(2n_1 = 2n_2 = m - 1\), with \(m\) an odd integer, play a special role. In particular for \(m = n + 1\) these states have an effective \(SU(2)\) spin symmetry. Otherwise the symmetry is \(U(1) \times U(1)\).

We derive an effective \(K\)-matrix theory for a general bilayer state, and discuss its implications for the symmetric states, both \(SU(2)\) (where the layer index is regarded as the spin of the electron) and \(U(1) \times U(1)\). In both cases we discuss the structure of the excitation spectrum. Unlike the conventional \(K\)-matrix, based on the Haldane-Halperin hierarchical construction, here the rank of the \(K\)-matrix is the same for all states in the sequences, and the entries of the matrix are parametrized by the five integers \(n_1, n_2, p_1, p_2\) and \(n\). The spectrum contains only two fundamental quasiparticles (or quasihole), and the remaining states are constructed as bound states. These results generalize our recent work on single layer fully polarized systems, \[\text{32}\]. We also derive an effective theory for the \((m, m, m)\) states, and discuss the nature of the excitations, including a construction of the electron and of the quasiparticles.

In Section \[\text{II}\] we derive the theory of the edge states for bilayers, under the assumptions that the edge is sharp, clean and unreconstructed. In Section \[\text{III}\] we construct the operators that create the quasiparticles, their bound states, and the electron at the edge. We derive explicit expressions of these operators for both the \(SU(2)\) and \(U(1) \times U(1)\) symmetric states. Here we compute their propagators and find the tunneling exponents for different physical settings of particular interest. In particular we use these results to derive scaling laws for internal tunneling of quasiparticles, and for tunneling of electrons both between identical FQH states (on these sequences) and from a Fermi liquid into a state on these sequences. We find that, unlike the case of a single layer with fully polarized electrons, in this case the exponent \(\alpha\) of the tunneling current versus voltage is not simply determined by the filling fraction \(\nu\).

$$\begin{array}{|c|c|c|}
\hline
\nu & \alpha_{qp} & \alpha_{\nu} \\hline
2p & p(2n - 2np - 1) + 2 & 2n - 1 + \frac{2}{p} \\hline
2np + 1 & \frac{(2n - 2np - 1)}{2np + 1} & n + \frac{1}{p} \\hline
2/5 & 1/5 & 5 \\hline
2/3 & 1/3 & 3 \\hline
4/9 & -4/9 & 4 \\hline
4/7 & 1/7 & 3 \\hline
\hline
\end{array}$$

TABLE I. Tunneling exponents for \(SU(2)\) bilayer states.
In Tables I and II we summarize these results and give a few examples for states that have been observed experimentally. Recall that for the SU(2) symmetric bilayer states the integer \( n \) is even, and \( p \) is an arbitrary integer. We have listed the exponents for internal tunneling of quasiparticles on the left column on Table I. On the center column we give the exponents for electron tunneling between identical states and on the right column we give the exponents for tunneling of electrons from a Fermi liquid into an SU(2) FQH state. In the text we also give tunneling exponents for bound states in several filling fractions. In Table II the states at 2/5 and 4/9 and the reversed sequence states at 2/3 and 4/7. As a general rule these filling factors normally appear in more than one sequence. For instance 4/7 can be realized either as a singlet state (as we have just seen) or as a member of a \( U(1) \times U(1) \) sequence, as we will see next. These are distinct states and both can be realized in a given bilayer system. By varying a parameter (such as pressure or density) it should be possible to induce phase transitions among these states.

| \( \nu \) | \( \alpha_{dp} \) | \( \alpha_{c} \) | \( \alpha_{t} \) |
|-----------|----------------|----------------|----------------|
| \( \frac{2m-1+1/p}{2m} \) | \(-1\) for \( s > 0 \) | \(2(m-1+1/p)-1\) | \(m-1+1/p\) |
| \( \frac{2m-1-(1/p)}{2m} \) | \(-1\) for \( s < 0 \) | | |
| \( \nu/\nu_{2} \) | | | |
| \( \nu/\nu_{2} \times \frac{2m}{n} \) | \(-1\) | | |
| \( \nu/\nu_{2} \times \frac{2m}{n} \) | | | |
| | 2m - 1 | | |
| | m | | |
| \( 1/2 \) | \(-1/4\) | 5 | 3 |
| \( 4/7 \) | \(-16/21\) | 4 | 5/2 |
| \( 4/5 \) | \(-2/5\) | 2 | 3/2 |

TABLE II. Tunneling exponents for \( U(1) \otimes U(1) \) bilayer states.

In the case of the \( U(1) \times U(1) \), \( m \) and \( n \) are odd integers, \( s = \text{sign}(m-n-1+1/p) \), and \( p \) is an arbitrary integer. Here, 4/7 is the first descendant in the series of the primary Halperin state (3,3,1). It has \( m = 3 \), \( n = 1 \), \( p = 2 \) and \( s > 0 \). The state at 4/5 is the reversed state with \( p = -2 \). Likewise, the unbalanced states observed in wide quantum wells at \( \nu = 11/15 \) (with \( \Delta \nu/\nu = \pm 1/11 \)) and at \( \nu = 13/21 \) (with \( \Delta \nu/\nu = \pm 1/13 \)) are also descendants of the (3,3,1) state with \((p_1,p_2) = (-5,-3)\) and \((-3,-5)\) and \((p_1,p_2) = (3,7)\) and \((7,3)\) respectively.

Finally, in Section V we discuss in detail the special case of the \((m,m,m)\) states. Here we show how the properties of their edge states are affected by the existence of a gapless (Goldstone) mode in the bulk. In particular we give an explicit derivation of the theory of the edge states. We find that the gapless bulk states yield an effective theory at the edge which is non-local. We discuss how these features affect the electron operator and the tunneling exponents. In Section VI we discuss our conclusions and open questions. In Appendix A we compute some operator product expansions needed to find the physical operators at the edges.

II. BILAYER SYSTEMS

In this section we construct an effective bulk theory for the 2DEG in strong magnetic fields in bilayer systems. We can follow here exactly the same point of view that we presented in references [24] for single-layer fully polarized 2DEG, with the difference that now there is an extra degree of freedom represented by the layer index. Although the basic philosophy is the same, the physics of these systems is different.

We begin with a first quantized Feynman path-integral of the 2DEG. In the case of bilayers the worldlines of the particles can be represented by a conserved 3-current \( J_{\alpha}^a \). The index \( \alpha = 1, 2 \) labels the layer where the particle moves. From now on, whenever the layer indices are between brackets it will be understood that the indices are not summed over. The weight of a particular history of the 2DEG in the Feynman path-integral is given by the action of the history. This action contains terms representing the kinetic and interaction energies of the 2DEG. In addition, since the particles are fermions there is a minus sign whenever any pair of particles are exchanged.
We will assume that the individual histories are such that the paths do not cross. In other words, there are short range interactions that will keep the particles apart. Hence, relative to a “parent configuration” in which there are no exchanges, configurations with exchanges can be viewed as a set of linked paths. The sign associated with a given configuration is then \((-1)^{|N_P|}\), where the number of permutations \(|N_P|\) is equal to the linking number of the configuration.

We now proceed with the flux-attachment transformation, which maps fermions to fermions. The linking number of the trajectories of the particles is given by

\[
\nu^\alpha_\mu [j^\alpha_\mu] = \int d^3 x \, j^\alpha_\mu(x) b^\alpha_\mu(x)
\]

with \(j^\alpha_\mu(x) = \epsilon_{\mu\lambda\nu} \partial^\nu b^\lambda_\alpha(x)\), where \(\alpha, \beta = 1, 2\) are layer indices. If the world lines of the particles do not cross, then \(\nu^\alpha_\mu [j^\alpha_\mu]\) is a topological invariant. Clearly, for a bilayer system, the linking number \(\nu^\alpha_\mu\) is a symmetric matrix. Naturally, the same observation holds for particles with spin.

Thus, if \(S[j^\alpha_\mu]\) is the action for a given history, the quantum mechanical amplitudes of all physical observables remain unchanged if the action is modified by

\[
S[j^\alpha_\mu] \rightarrow S[j^\alpha_\mu] - 2\pi T^{\alpha\beta} \nu^\beta_\mu[j^\alpha_\mu]
\]

with

\[
2T^{\alpha\beta} = \begin{pmatrix} 2n_1 & n \\ n & 2n_2 \end{pmatrix}
\]

where \(n_1, n_2\) and \(n\) are arbitrary integers. The condition on the off-diagonal elements comes from the fact that \(T^{\alpha\beta}\) is necessarily a symmetric matrix since the linking number is symmetric as well.

Following the same approach introduced in references \[23\] we find that, for a bilayer system, the action is given by

\[
S_{\text{eff}}[a, b, j] = \frac{1}{2\pi} a^\alpha_\mu \epsilon_{\mu\lambda\nu} \partial^\nu b^\lambda_\alpha - a^\alpha_\mu j^\alpha_\mu - \frac{2T^{\alpha\beta}}{4\pi}\epsilon_{\mu\lambda\nu} b^\lambda_\mu \partial^\nu b^\lambda_\beta
\]

(2.4)

The indices \(\alpha, \beta = 1, 2\) label the layer or spin polarization, while \(\mu, \nu, \lambda\) are the space-time indices. If the currents \(j^\alpha_\mu\) correspond to the electrons living in the layer \(\alpha\) of the system, and since we assume that there is no tunneling between layers, the effective action written in second quantization becomes, in the composite fermion picture

\[
S_{\text{eff}} = \int d^3 x \left( \psi^\dagger_\alpha (iD^\alpha_\mu + \mu^\alpha) \psi_\alpha - \frac{1}{2M} |D^\alpha_\mu \psi_\alpha|^2 \right)
- \frac{1}{2} \int d^3 x d^3 x' (|\psi^\dagger_\beta(x')|^2 - \rho^\beta) V_{\alpha\beta} (|\vec{x} - \vec{x}'|)|\psi_\beta(x')|^2 - \rho^\beta)
+ \int d^3 x \left[ \frac{1}{2\pi} a^\alpha_\mu \epsilon_{\mu\lambda\nu} b^\lambda_\alpha - \frac{2T^{\alpha\beta}}{4\pi}\epsilon_{\mu\lambda\nu} b^\lambda_\mu \partial^\nu b^\lambda_\beta \right]
\]

(2.5)

where \(D^\alpha_\mu = \partial_{\mu} - i(eA^\alpha_\mu + a^\alpha_\mu)\), \(A^\alpha_\mu\) is the external electromagnetic field acting on layer \(\alpha\), and \(\mu^\alpha\) is the chemical potential for layer \(\alpha\).

The mean field theory of this problem, i.e. the average field approximation, is found by requiring that the effective action \(S_{\text{eff}}\) of Eq. \[2.3\] be stationary,

\[
\frac{\delta S_{\text{eff}}}{\delta a^\alpha_\mu(x)} = 0, \quad \frac{\delta S_{\text{eff}}}{\delta b^\alpha_\mu(x)} = 0
\]

(2.6)

for \(\alpha = 1, 2\). For fluid states these equations become,

\[
\bar{\rho}_\alpha = -\frac{1}{2\pi} \langle \epsilon_{ij} \partial_i b^j_\alpha \rangle, \quad \langle \epsilon_{ij} \partial_i a^\alpha_\mu \rangle = 2T^{\alpha\beta} \langle \epsilon_{ij} \partial_i b^j_\beta \rangle
\]

(2.7)

where \(\bar{\rho}_\alpha\) is the charge density of the fermions.

It is straightforward to show that these equations are identical to the mean field equations of our earlier work on bilayer FQH states \[24\]. Hence, we find the same series of fractions for the bilayer system and, for special choices of interactions, for single layer systems with unpolarized or partially polarized spins. For the bilayer system the allowed fractions are

\[
\nu_1 = \frac{n - \left(\pm \frac{1}{2} + 2n_1\right)}{n_2 - \left(\pm \frac{1}{2} + 2n_1\right)} \left(\frac{\pm 1}{2} + 2n_2\right)
\]

\[
\nu_2 = \frac{n - \left(\pm \frac{1}{2} + 2n_2\right)}{n_2 - \left(\pm \frac{1}{2} + 2n_1\right)} \left(\frac{\pm 1}{2} + 2n_2\right)
\]

(2.8)

Following the methods of reference \[24\], we expand the effective action around the mean field solution with uniform density. We find an effective Lagrangian \(\mathcal{L}_{\text{eff}}\) at long distances and low energies, of the form (including the coupling to a weak external gauge field \(A^\alpha_\mu\))

\[
\mathcal{L}_{\text{eff}}[a, b, j] = \frac{p_\alpha}{4\pi} \epsilon_{\mu\nu\lambda} a^\alpha_\mu \partial_\nu a^\lambda_\nu + \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} a^\alpha_\mu \partial_\nu b^\lambda_\alpha
- \frac{2T^{\alpha\beta}}{4\pi} \epsilon_{\mu\nu\lambda} b^\lambda_\mu \partial_\nu b^\lambda_\beta
+ \frac{1}{4\pi} \epsilon_{\mu\lambda\nu} \epsilon_{\nu\mu\lambda} \partial_\nu e_\lambda - \epsilon_{\mu\lambda\nu} \partial_\nu a^\alpha_\mu - \epsilon_\mu \partial_{\mu,1} + j_{\mu,2}
\]

(2.9)

where \(p_\alpha (\alpha = 1, 2)\) are arbitrary integers (positive or negative). As usual, the sequences of allowed FQH states can have either sign of \(p_\alpha\). We will refer to the sequences with \(p_\alpha > 0\) as the direct sequence and those with \(p_\alpha < 0\) as the reversed sequence. Intuitively, the sign of \(p_\alpha\) is the sign of the effective magnetic field felt by the composite
fermions, relative to the sign of the external field that acts on the electrons. We describe now the states such that \( \det K \neq 0 \). The current \( j^\mu_{qp,\alpha} \) represents the quasiparticles. Since in the fermionic picture the bare quasiparticles are composite fermions, the statistics of all the excitations that we will compute is defined relative to fermions. Therefore we have introduced the field \( e_\mu \) to keep track of the underlying statistics, i.e., to transform the fermions into bosons. Notice that the quasiparticle current couples directly to the gauge fields \( a_\mu^0 \) and \( e_\mu \) but not to the hydrodynamic field \( b_\mu^0 \) or to the external electromagnetic field.

We can write the effective Lagrangian in a more compact form by defining, in the basis \((b_\mu^1, a_\mu^1, b_\mu^2, a_\mu^2, e_\mu)\), the \((2 \times 5)\) charge matrix

\[
t_{ab} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]

as well as a \( 2 \times 5 \) quasiparticle coupling matrix \( \ell_{ab} \). The set of allowed quasiparticle states is represented by matrices \( \ell_{ab} \) of the form

\[
\ell_{ab} = \begin{pmatrix}
k_1 & 0 & 0 & -k_1 & 0 \\
k_2 & 0 & 0 & k_2 & 0
\end{pmatrix}
\]

where \( k_1 \) and \( k_2 \) are arbitrary integers. In addition, we also define the \((5 \times 5)\) Chern-Simons coupling constant matrix \( K \) of the form

\[
K = \begin{pmatrix}
-2n_1 & 1 & -n & 0 & 0 \\
1 & p_1 & 0 & 0 & 0 \\
0 & 0 & -2n_2 & 1 & 0 \\
0 & 1 & 0 & p_2 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

whose determinant is given by

\[
\det K = (1 + 2n_1p_1)(1 + 2n_2p_2) - n^2 p_1p_2
\]

It is well known that the determinant of the \( K \) matrix is equal to the degeneracy of the ground state of the FQH fluid on a torus \([39, 42]\).

It is easy to show that the only possible solutions for \( \det K = 0 \) are given by \( p_1 = p_2 = 1 \) and \( 2n_1 + 1 = n = m \) which correspond to the \((m, m, m)\) states. We describe now the states such that \( \det K \neq 0 \) and come back to the \((m, m, m)\) states later.

With the above definitions, the Lagrangian of the effective theory of the FQHE in bilayers takes the form

\[
\mathcal{L} = \frac{1}{4\pi} K_{IJ} \epsilon_{\mu\nu\lambda} a^I_\mu \partial^\nu a^J_\lambda - \frac{1}{2\pi} t_{ab} A^I_\mu \epsilon_{\mu\nu\lambda} \partial^\nu a^J_\lambda + \ell_{ab} j_{qp,\alpha} a^I_\mu
\]

where \( I, J = 1, \ldots, 5 \) and \( \alpha = 1, 2 \). There are two types of quasiparticles, given by the two choices of the coupling vectors \( \ell_1 \) of Eq. 2.11. The coupling to external electromagnetic perturbations is given by the charge matrix \( t^I \) of Eq. 2.10. Notice however that, even though the structure of the effective Lagrangian has the form required by the general classification of Abelian FQHE states of Wen and Zee \([3, 4]\), the physical interpretation of the component fields is actually quite different from those of the standard hierarchy.

The \( 2 \times 2 \) Hall conductance matrix for these states is found to be equal to

\[
\sigma_{xy}^{\alpha\beta} = \frac{1}{2\pi} t_{\alpha I} K^{-1}_{IJ} t_{J\beta} = \frac{1}{2\pi \det K} \begin{pmatrix}
p_1(2np_2 + 1) & -np_1p_2 \\
-np_1p_2 & p_2(2np_1 + 1)
\end{pmatrix}
\]

The total conductance of the bilayer system is \( \sigma_{xy}^{\alpha\beta} = \frac{1}{2\pi} \) for total filling factors of the form

\[
\nu = \sum_{\alpha,\beta} \nu_{\alpha,\beta} = p_1(2np_2 + 1) + p_2(2np_1 + 1) - 2np_1p_2
\]

which agrees with our earlier work \([4]\). The filling factor of each layer is \( \nu_\alpha = \sum_\beta \nu_{\alpha,\beta} \). Let us define the polarization per particle \( M_{\text{total}} \) of the ground state of the system as

\[
M_{\text{total}} = \frac{1}{2} \left( \frac{N_1 - N_2}{N_1 + N_2} \right) = \frac{\nu_1 - \nu_2}{2\nu}
\]

For non-polarized and partially polarized systems, in which the layer index \( \alpha \) is the spin index of the electrons, this definition of the polarization coincides with the total \( z \)-component magnetization per electron of the ground state \( S_{z_{\text{total}}} \). From Eq. 2.10 we find that \( M_{\text{total}} \) is given by

\[
M_{\text{total}} = \frac{1}{2} \left( \frac{(p_1 - p_2) - 2(n_1 - n_2)p_1p_2}{p_1(2np_2 + 1) + p_2(2np_1 + 1) - 2np_1p_2} \right)
\]

The fundamental vortices (or quasiholes) are represented by the coupling matrices of Eq. 2.11 with the choices \((k_1, k_2) = (1, 0), (0, 1)\). Let us label by \( a = 1, 2 \) these quasiparticles. For the quasihole \( \alpha \), the charge matrix \( Q_{qp,\alpha}^{bc} \) in units of the electric charge \( e \), is given by \((b, c = 1, 2)\)

\[
Q_{qp,1}^{bc} = t_{bc} K_{IJ} j_{qp,\alpha}^{-1} j_{Jc} = \frac{1}{\det K} \begin{pmatrix}
2np_2 + 1 & 0 \\
0 & np_1
\end{pmatrix}
\]

\[
Q_{qp,2}^{bc} = t_{bc} K_{IJ} j_{qp,\alpha}^{-1} j_{Jc} = \frac{1}{\det K} \begin{pmatrix}
0 & -np_1 \\
0 & 2np_1 + 1
\end{pmatrix}
\]

Each quasiparticle can be characterized by a charge \( Q_{qp,\alpha} \) and a polarization \( M_{qp}^{\alpha} \). The charges and polarizations have the form
\[
Q_{qp,a} = \sum_{bc} Q_{qp,a}^{bc}
\]
\[
M_{qp,a} = -\frac{1}{2} \sum_{bc} (-1)^c Q_{qp,a}^{bc} \equiv \frac{1}{2} Q_{qp,a}'
\]

Notice that we have defined the quasiparticle polarization as measured in units of the quasiparticle charge, and with the factor of \(\frac{1}{2}\) it is equal to the \(z\)-component of the spin of the quasiparticle (or quasihole). Alternatively, we may use the “spin charge” \(Q_{qp,a}'\) as defined in the last line of Eq. (2.21), which agrees with the notation of Milovanovic and Read [44].

Therefore the quasihole charges and polarizations are given by

\[
Q_{qp,1} = \frac{2n_2 p_2 + 1 - n p_2}{\text{det} \ K} = \frac{\nu_1}{p_1}
\]
\[
M_{qp,1} = \frac{1}{2} \frac{2n_2 p_2 + 1 + n p_2}{\text{det} \ K}
\]
\[
Q_{qp,2} = \frac{2n_1 p_1 + 1 - n p_1}{\text{det} \ K} = \frac{\nu_2}{p_2}
\]
\[
M_{qp,2} = \frac{1}{2} \frac{2n_1 p_1 + 1 + n p_1}{\text{det} \ K}
\]

The statistics of the quasiholes, measured relative to bosons, is given by the matrix \(\theta_{qp}^{ab}\) which, in general, has the form

\[
\frac{\theta_{qp}^{ab}}{\pi} = \ell_{a1} K_{1j}^{-1} \ell_{jb}
\]

The statistics of the elementary quasiparticles are found to be

\[
\frac{\theta_{qp}^{11}}{\pi} = 1 + \frac{2n_1 + p_2 (4n_1 n_2 - n^2)}{\text{det} \ K}
\]
\[
\frac{\theta_{qp}^{22}}{\pi} = 1 + \frac{2n_2 + p_1 (4n_1 n_2 - n^2)}{\text{det} \ K}
\]
\[
\frac{\theta_{qp}^{12}}{\pi} = 1 + \frac{n}{\text{det} \ K}
\]

Let us consider now the charge and statistics of a composite object made of \(k_1\) quasiparticles of type 1 and \(k_2\) quasiparticles of type 2, with \(k_1\), \(k_2\) integers. The total charges and polarizations of these composite objects, in units of the electron charge \(e\), are

\[
Q = k_1 (2n_2 p_2 + 1 - n p_2) + k_2 (2n_1 p_1 + 1 - n p_1)
\]
\[
M = k_1 (2n_2 p_2 + 1 + n p_2) - k_2 (2n_1 p_1 + 1 + n p_1)
\]

and their statistics is given by

\[
\frac{\theta}{\pi} = \frac{\nu_1}{p_1} + \frac{\nu_2}{p_2} + 2 \left(2n_1 k_1 + 2n_2 k_2 + 2nk_1k_2 + (p_2 k_1^2 + p_1 k_2^2)(4n_1 n_2 - n^2)\right) \frac{1}{\text{det} \ K}
\]

It is easy to check that if we take \(k_1 = 2n_1 p_1 + 1\) quasiparticles of type 1 and \(k_2 = n p_2\) quasiparticles of type 2, the resulting operator has \(Q = -e, M = 1/2\) and fermionic statistics since, in this case

\[
\frac{\theta}{\pi} = (2n_1 p_1 + 1)(2n_1 (p_1 + 1) + 1) + n_2 p_2 (p_2 + 1) + 2n_2 (2n_1 p_1 + 1)
\]

is always an odd integer.

In the next section will discuss the implications of these general results in the context of several cases of interest. In section [13] we will see how to combine quasiparticle operators to get the correct electron operator.

### A. \(SU(2)\) symmetry and Spin Singlet States

We will now regard the two layers as a spin index with \(\alpha = 1\) being spin \(\uparrow\) and \(\alpha = 2\) being spin \(\downarrow\). For these special cases, in order to preserve the \(SU(2)\) spin rotation invariance, we must choose \(2 n_i = m - 1 = n\), \(i = e,\) we do the flux attachment in such a way that it does not distinguish between in-layer and inter-layer labels. Then we get, as expected, \(\theta_{qp}^{11} = \theta_{qp}^{22} = \theta_{qp}^{12}\). Thus this formalism can be used to describe the problem of electrons with spin. Notice that \(n\) is an even integer.

The allowed FQH states compatible with the choice \(2 n_i = m - 1 = n\), which requires \(n\) to be even, have filling fractions

\[
\nu = \frac{p_1 + p_2}{1 + n (p_1 + p_2)}
\]

Of these states, only those in which \(p_1 + p_2\) is an even integer are fully compatible with the \(SU(2)\) symmetry. In other words, only for \(p_1 + p_2\) even it is possible to have a spin singlet state. In contrast, for \(p_1 + p_2\) odd the ground states are always polarized. The total ground state polarization per particle \(M_{\text{total}}\) is the spin polarization of the ground state \(S_{\text{total}}\), which for these states is

\[
S_{\text{total}} = \frac{1 - \nu m}{2 \nu} (p_1 - p_2) = \frac{1}{2} \left(\frac{p_1 - p_2}{p_1 + p_2}\right)
\]

We will classify the excitations according to their charge \(Q_{qp}\), their spin polarization \(S_{qp} = M_{qp}\) and their statistics \(\theta_{qp}\). In all the \(SU(2)\) states both fundamental quasiholes, specified by the choices \((k_1, k_2) = (1, 0)\) and \((k_1, k_2) = (0, 1)\) respectively, have the same charge (in units where the electric charge \(e = 1\))
\[ Q_{qp}^l = Q_{qp}^d = 1 - \nu n \]  

and their spin polarizations \( S_{qp}^z \) are, using Eq. 2.23 with \( S_{qp,\uparrow,\downarrow} = M_{qp,1,2} \)

\[ S_{qp,\uparrow,\downarrow}^z = \pm \frac{1}{2} - \nu n S_{\text{total}}^z \]  

The quasihole states have the statistics

\[ \frac{\theta_\pi}{\pi} = \frac{\theta_{1,\pi}}{\pi} = 1 + n - \nu n^2 \]  

The spin singlet (or unpolarized) FQH states are given by the choice \( p_1 = p_2 = p \), and the filling factors of these states are \( \nu = 2p/(1+2np) \). For the spin singlet states \( S_{\text{total}}^z = 0 \), and the elementary excitations are arranged into \( SU(2) \) multiplets. All states in a given multiplet have the same charge and statistics but different spin polarization, as can be read of Eqs. 2.30, 2.31. In particular, the elementary quasiholes form the doubly degenerate (fundamental) representation of \( SU(2) \) and carry the same charge and statistics, and opposite spin polarization.

A general composite excitation is made out of \( k_1 \) quasiparticles of type 1 and \( k_2 \) quasiparticles of type 2, and it has the following quantum numbers

\[ Q_{qp} = (k_1 + k_2)(1 - \nu n) \]

\[ S_{qp}^z = \frac{1}{2}(k_1 - k_2) \]

\[ \frac{\theta_\pi}{\pi} = (k_1 + k_2)^2(1 + n - \nu n^2) \]  

Thus, the quantum numbers of the states are specified by two integer-valued labels, \( k_1 \) and \( k_2 \), which span a two-dimensional integer lattice.

It is interesting to construct the quasi-electron states for all the spin singlet FQH states. Since these excitations must have \( Q = 1 \), we must require that \( k_1 + k_2 = 1 + 2np \). The spin polarization of these states is \( S_{qp}^z[k_1,k_2] = \frac{1}{2}(k_1 - k_2) \). The statistics of these states is \( 2\pi = (1 + 2np)(1 + 2np + n) \), which is an odd integer. Hence, these states are fermions with charge \( e \). However, since as we can see, these states actually belong to a \( 2np + 2 \)-fold degenerate representation, with the same charge and statistics but different spin polarization. The total spin of these states is \( S = np + \frac{1}{2} \). Thus, these states are not the “elementary” electron which must be a fermion with \( Q = 1 \) and \( S = \frac{1}{2} \).

Actually, a set of states of \( N \) elementary quasiparticles, each with \( S = \frac{1}{2} \), spans a Hilbert space of dimension \( 2^N \). Clearly, the states produced by the construction indicated above seems to give fewer states, apparently only the states with the largest total spin. In order to construct all the states created by a set of sources labeled by the pairs of integers \( (k_1,k_2) \), it is necessary to compute the wave functions of the effective Chern-Simons theory in the presence of the sources labeled by \((k_1,k_2)\). It is well known from Chern-Simons theory [45-47] that, in order to define the states created by a set of sources, a set of observables which act as canonically conjugate pairs need to be chosen. In Chern-Simons theory this is called a choice of polarization. For a Chern-Simons theory on a disk, a standard choice is to select \( A_2 = A_1 + iA_2 \) as the “coordinate” and \( A_\pm \) as the conjugate momentum. This choice is called holomorphic polarization. Within the holomorphic polarization, the wave functions are holomorphic functions with a singularity structure determined by the location and quantum numbers of the sources. It is well known [45] that these wave functions coincide with the (holomorphic) conformal blocks of the correlation functions of a chiral conformal field theory in two-dimensional Euclidean space. This correspondence is equivalent to the statement that there is a one-to-one correspondence between the states of the bulk and the correlation functions at the boundary.

For the bilayer problem that we are interested in here, these considerations imply that two sets of complex variables, \( z \) and \( w \), one for each layer, are needed to label the states. The full \( 2^N \)-dimensional Hilbert space of \( N \) quasiparticles is obtained by proper symmetrization and antisymmetrization of the (product) wave functions, as required by the \( SU(2) \) symmetry. Since this construction of the bulk wave functions of the excitations coincides with the correlation functions at the edge, we will discuss it in detail in the next sections where the correlation functions at the edge will be constructed. In what follows will refer to the “elementary” electron as the state with unit charge, Fermi statistics and spin polarization \( \pm \frac{1}{2} \). The unique choice that satisfies all of these requirements is the properly antisymmetrized state obtained from a set of sources with quantum numbers \((k_1,k_2) = ((m - 1)p + 1, (m - 1)p)\) for an electron with spin up, and \((k_1,k_2) = ((m - 1)p, (m - 1)p + 1)\) for an electron with spin down respectively. This choice also works for all \( U(1) \times U(1) \) symmetric states.

Finally, the excitations of polarized \( SU(2) \) states, i.e. states with \( S_{\text{total}}^z \neq 0 \), can be constructed in a similar manner. The quantum numbers of their excitations are the same as in the unpolarized case except that the polarization \( S_z \) is shifted downwards by \( -\frac{nu}{1-n} Q_{qp} S_{\text{total}}^z \). Hence, these states also form representations of \( SU(2) \). The only change is that the spin projections of the excitations are shifted by a constant amount determined by the filling fraction, the charge of the excitation and the polarization of the ground state.

Let us look in particular at the spectrum of quasiholes and electrons for the spin singlet state \((3,3,2)\) with filling fraction \( \nu = \frac{2}{3} \). This state has \( p_1 = p_2 = 1 \) and \( n = 2 \). Notice that, whether or not all of the states discussed below actually occur in the spectrum of a 2DEG with a specific Hamiltonian depends on the details of the interactions. In general, if the Hamiltonian is invariant under the global \( SU(2) \) rotations of spin there will not
be any matrix elements that will mix these states. However, even if the $SU(2)$ symmetry were exact, in general bound states with all of these quantum numbers will not be present. Naturally, the excitation with lowest charge (the quasiholes) is realized.

1. Elementary Quasiholes:
   The elementary quasiholes are vortices which have $(k_1, k_2) = (1, 0), (0, 1)$. Their charge is $Q = 1/5$, and there are two polarization states with $S_z = \pm 1/2$. The statistics of the quasiholes is $\theta = 1 + \frac{2}{5}$. This excitation is always present in the spectrum.

2. Neutral Quasihole Bound States:
   The neutral bosonic bound states are the collective modes of the fluid. They are described by the fluctuations of the Chern-Simons gauge fields. We already discussed their spectrum in reference [4]. From the point of view of the construction that we are discussing here, the neutral $Q = 0$ collective (bosonic) modes have $k_1 + k_2 = 0$. There is a spectrum of in-phase (charge fluctuations) and out-of-phase (spin fluctuations) collective modes.

3. Charged Quasihole Bound States:
   The simplest charged quasiholes have $(k_1, k_2) = (2, 0), (1, 1), (0, 2)$. Their charge is $Q = 2/5$, the spin polarization is $S_z = 0, \pm 1$, and their statistics is $\frac{\theta}{2} = 5 + \frac{2}{5}$. This is the $S = 1$ spin triplet representation of $SU(2)$. It is obvious that there should be an $SU(2)$ spin singlet as well. This state is obtained by antisymmetrization of the two-charge states. We will construct this state in section [IV] when we discuss the realization of these states at the edge.

4. Electron States:
   The electron states are bound states of five elementary quasiholes. Thus, they are created by sources with quantum numbers $(5, 0), (4, 1), (3, 2), (2, 3), (1, 4), (0, 5)$. These states have charge $Q = 1$. A naively constructed bound state is fully symmetric and it forms an $SU(2)$ multiplet whose spin polarizations are $S_z = \pm \frac{5}{2}, \pm \frac{3}{2}, \pm \frac{1}{2}$. These electron states are fermions since their statistics is $\theta = 35$. Hence, it can be viewed as a spin 5/2 multiplet constructed with five quasiparticles of both types. This multiplet is thus a set of states with maximal total spin. Obviously, there are other multiplets of five $S_z = 1/2$ charge 1/5 quasiparticles, forming fermionic bound states with charge 1 and with total spin 3/2 and 1/2. In these states, pairs of quasiparticles are placed in spin singlets. In particular, the actual electron state, with $S = 1/2$, is a product of two singlets and one doublet of quasiparticle states. These features are generally present in all the $SU(2)$-symmetric states. These issues will play an important role in Section [IV] where we give details of the construction of the electron operator at the edge.

It is also instructive to look at the spectrum of excitations of a partially polarized $\nu = 2/5$ state. Now we choose $p_1 \neq p_2$ but keep $p_1 + p_2 = 2$ and $n = 2$. The spectrum is still classified formally by the representations of $SU(2)$. For this particular filling fraction the allowed values of $p_1$ and $p_2$ are 0 and 2. Hence, these states are maximally polarized. As far as the quantum numbers are concerned, the effect of the asymmetry is a shift in the value of $S_z$. By an amount determined by the total polarization of the ground state, the filling factor and the charge of the state. Of course, this shift is not consistent with $SU(2)$. Thus, these states have fractional $z$-projection of the spin. Notice that, although there are no representations of $SU(2)$ with fractional quantum numbers (i.e. not integers or half-integers), there are such representations for the $U(1)$ subgroup of $SU(2)$ generated by $S_z$ since this group is Abelian. Hence, once the $SU(2)$ symmetry is broken, fractional quantum numbers for $z$-component of the spin are allowed. This is the case for generic $U(1) \times U(1)$ FQH states. For instance for the quasihole $(k_1, k_2) = (1, 0)$ ($(k_1, k_2) = (0, 1)$), the spin polarization is $S_z = 1/10$ ($S_z = -9/10$) given that $S_z^{\text{total}} = 1/2$. In the presence of a non-zero Zeeman interaction the energy of these two quasiholes states are not the same, and the lowest energy quasiholes has $S_z = 1/10$.

States with the quantum numbers of electrons can be constructed in the usual way, i.e. by binding 5 quasi-particles. However, here the electron with maximal $S_z$ is simply the tensor product of five quasiparticles with $S_z = 1/2$, and it coincides with the usual electron of a fully polarized $\nu = 2/5$ Jain state.

Another state of interest is the singlet state at $\nu = 2/3$ which has been observed experimentally [1]. This state has $p_1 = p_2 = -1$ and $m = n + 1 = 3$. Thus it belongs to the reversed sequence of the (3, 2, 1) FQH state. Transitions between the singlet state at $\nu = 2/3$ and the fully polarized state at the same filling factor have also been observed [1]. The excitation spectrum of the $\nu = 2/3$ spin singlet state can be constructed analogously.

B. $U(1) \times U(1)$ FQH states

Earlier in this section we gave formulas that describe the excitation spectrum of generic $U(1) \times U(1)$ FQH states. Here we will discuss the excitation spectra of symmetric $U(1) \times U(1)$ FQH states.

These states are given by the choices $2n_1 = 2n_2 = 2s = m - 1$ and $n \neq m - 1$. For simplicity we will restrict our discussion to the symmetric states $p_1 = p_2 = p$. The special case of the primary states at $p = 1$ are the Halperin states $(m, m, n)$ where $m = 2s + 1$. In an earlier publication [1] we presented a theory of these states, in-
Including the spectrum of collective modes for the $\nu = 1/2$ (3, 3, 1) FQH state. The filling fraction of these states is

$$\nu = \frac{2p}{(m + n - 1)p + 1} \quad (2.34)$$

By construction, these ground states have zero polarization $S_z = 0$, i.e. both layers are equally populated.

A generic quasiparticle state, with quasiparticle numbers $(k_1, k_2)$, has charge $Q$,

$$Q = \frac{k_1 + k_2}{(m + n - 1)p + 1} \quad (2.35)$$

and polarization $S_z$,

$$S_z = \frac{k}{2((m - n - 1)p + 1)} \quad (2.36)$$

where we have introduced the integer $k = k_1 - k_2$. The excitations can be classified by their charge $Q$ and by the integer $k$. The statistics of these states is

$$\frac{\theta}{\pi} = Q^2((m + n - 1)p + 1)^2 + \frac{Q^2}{2}(m + n - 1)(m + n - 1) + \frac{k^2}{2}(m - n - 1) + \frac{1}{2}((m + n - 1)p + 1)(m + n - 1) \quad (2.37)$$

In particular, we must choose $k_1 + k_2 = 1$ for the quasihole states whose charge $Q$, polarization $S_z$ and statistics $\theta$ are

$$Q = \frac{1}{(m + n - 1)p + 1}$$

$$S_z = \frac{k}{2((m + n - 1)p + 1)}$$

$$\frac{\theta}{\pi} = 1 + \frac{m + n - 1}{2((m + n - 1)p + 1)} + \frac{m - n - 1}{2((m - n - 1)p + 1)} \quad (2.38)$$

The electron states must have charge $Q = 1$, hence for these states we require $k_1 + k_2 = (2s + n)p + 1$. However, the polarization $S_z$, is still given by Eq. $2.38$. The statistics of the $Q = 1$ states is given by

$$\frac{\theta}{\pi} = ((m + n - 1)p + 1)^2 + \frac{k^2}{2}((m - n - 1)p + 1) + \frac{1}{2}((m + n - 1)p + 1)(m + n - 1) \quad (2.39)$$

In general these states are not fermions. However, the states with $k_1 + k_2 = (m + n - 1)p + 1$ and $|k_1 - k_2| = (m - n - 1)p + 1$ have charge $Q = 1$ and $S_z = \pm 1/2$. These composite objects are the electron excitations of the $U(1) \times U(1)$ FQH states on bilayers.

Let us now discuss the special case of the (3, 3, 1) FQH state, with filling factor $\nu = 1/2$. Many of the properties of the excitation spectrum of more general states are already present in the case of the (3, 3, 1) state. This state has $2s = 2$, $n = 1$ and $p = 1$, and $\nu = 1/2$.

1. Charged Quasiparticles:

The excitation with smallest charge is a particle-hole doublet with charge $\pm 1/4$ and polarization $S_z = \pm 1/4$ ($Q^s = \pm 1/2$). Since in general there is no particle-hole symmetry, the apparent degeneracy of this doublet is always lifted by terms not present in this effective theory. The statistics of the elementary quasiparticles is $\theta/\pi = 1 + 5/8$. Likewise there are other quasiparticle bound states (also doublets) with charges $Q = \pm 1/2, \pm 3/4$, spin polarizations $S_z = 0, \pm 1/2$ and $S_z = \pm 3/4, \pm 1/4$ respectively. Their statistics are $\theta/\pi = 5 + 1/2$ (for $Q = \pm 1/2$, $S_z = 0$), and $\theta/\pi = 6 + 1/2$ (for $Q = \pm 1/2$, $S_z = \pm 1/2$; $\theta/\pi = 14 + 5/8$ (for $Q = \pm 3/4$, $S_z = \pm 3/4$), and $\theta/\pi = 12 + 5/8$ (for $Q = \pm 3/4, S_z = \pm 1/4$).

2. Neutral Fermions:

An interesting feature of the primary $(p = 1)$ (3, 3, 1) state is the existence of neutral fermionic excitations [14]. These states have zero charge, $Q = 0$, layer polarization $S_z = \pm 1/4$ ($Q^s = \pm 1$), and statistics $\theta/\pi = 1$. Neutral bound states exist for all $U(1) \times U(1)$ FQH bilayer states with $m - n$ even and $p$ odd, e.g. the state (3, 3, 1) itself and its $p$ odd descendants. These excitations have $S_z = \pm 1/2$ ($Q^s = \pm 1$), and $Q = 0$. These bound states are bosons if $m - n = 4s$, and fermions if $m - n = 2s$ (with $s$ odd). The sources that create the neutral spin 1/2 excitations have the form $(k_1, k_2) = \pm \frac{k}{2}(1, -1)$, where $k = p(m - n - 1) + 1$ must be even.

3. Neutral Bosons:

Similarly, the spectrum has neutral bosons with integer $S_z$. These bosonic states are just the collective modes of the bilayer system, i.e. excitations of the out-of-phase Chern-Simons gauge field (see reference [4]).

4. Electron States:

The electron states have charge $Q = -1$ and are bound states of four elementary quasiparticles. Likewise, there are also hole states with charge $Q = 1$, which are bound states of four quasiholes. In the (3, 3, 1) both sets form particle-hole doublets (naturally, this degeneracy is also lifted). The states with $Q = 1$ are created by sources with the quantum numbers $(4, 0), (3, 1), (2, 2), (1, 3), (0, 4)$. Of these five states with charge 1 only two, the states denoted by (3, 1) and (1, 3), i.e. with polarization $S_z = \pm 1/2$, are fermions with $\theta/\pi = 11$, and represent the actual electron states. The three remaining states have $\theta/\pi = 10$ (for $(k_1, k_2) = (2, 2)$),
\[ \theta / \pi = 14 \text{ for } (k_1, k_2) = (4, 0), (0, 4) \], and are bosons with charge 1. The states with \( Q = -1 \) are constructed by reversing the sign of \((k_1, k_2)\).

As far as the bulk states are concerned it is obvious that, for a 2D-ER with general pair interactions, many of these apparently degenerate states will not have the same energy, even if particle-hole symmetry was exact. However, at the edge the situation may be different. In fact, except for possible differences in the effective velocities of the edge modes, the spectrum may be degenerate, not only at the level of the dimensions of the operators but also in terms of their energies.

Recently it has become clear that the properties of the elementary quasiparticles (or vortices) of the (3,1) FQH state can be related to the existence of underlying pairing correlations present in this state. In fact, it has been shown that the (3,1) Halperin state can be regarded as an Abelian paired Hall state \([38, 49]\). It has also been shown that the (3, 1) Halperin state is closely related to the non-Abelian Pfaffian paired Hall state \([38, 49]\).

C. The \((m, m, m)\) States.

In this section we have derived an effective low energy theory for a generic FQH state in a bilayer system, and given an explicit form for the effective Lagrangian. However, in the case of the Halperin \((m, m, n)\) states the effective low energy theory is considerably simpler. These states have been described in detail in the literature \([38, 49]\), particularly by Milovanovic and Read \([44]\) who gave a detailed description of their excitation spectra using a standard hierarchy construction. In the construction that we are presenting here (see also ref. \([38, 49]\) the \((m, m, n)\) states have level \( p_1 = p_2 = 1 \). It is easy to see, by direct inspection of their effective \( K \) matrix Eq. \(2.13\) that for all of these states the fields \( a_\mu^\alpha \) and \( e_\mu \) do not contribute to the quantum numbers of the excitations. Hence these fields are redundant and can be integrated out. The effective Lagrangian of these states thus involves only the two fields \( b_\mu^\alpha \), and it has the much simpler form

\[\begin{align*}
\mathcal{L}_{\text{eff}}[b] &= -\frac{2\tilde{T}_{\alpha\beta}}{4\pi} e_{\mu\nu\lambda} b_\nu^\alpha \partial_\mu b_\lambda^\beta - \frac{1}{2}\epsilon_{\mu\nu\lambda} A_\mu^\alpha \partial_\nu b_\lambda^\beta + b_\mu^\alpha j_{\mu q, \alpha} + \mathcal{L}_{\text{Maxwell}}[b] \\
&= b_\mu^+ (b_\mu^- + b_\mu^0) + b_\mu^- (b_\mu^+ + b_\mu^0)
\end{align*}\]

(2.40)

where

\[2\tilde{T}_{\alpha\beta} = \begin{pmatrix} 2n_1 + 1 & n \\ n & 2n_2 + 1 \end{pmatrix} \equiv \begin{pmatrix} m_1 & n \\ n & m_2 \end{pmatrix}\]

(2.41)

Here \( \mathcal{L}_{\text{Maxwell}}[b] \) represents subleading Maxwell-like terms, which are important for the case of the \((m, m, m)\) states for which the matrix \( \tilde{T} \) is degenerate. In the basis the effective Lagrangian for the \((m, m, n)\) states separates into the Lagrangians for the two (independent) fields \( b_\mu^+ \) and \( b_\mu^- \), with Chern-Simons coupling constants \( m + n \) and \( m - n \) respectively \([44, 8]\).

\[\begin{align*}
\mathcal{L}_{\text{eff}}[b] &= -m + n \frac{\epsilon_{\mu\nu\lambda}}{4\pi} b_\nu^\alpha \partial_\mu b_\lambda^\beta - m - n \frac{\epsilon_{\mu\nu\lambda}}{4\pi} b_\nu^\alpha \partial_\mu b_\lambda^\beta \\
&\quad - \frac{1}{2}\epsilon_{\mu\nu\lambda} A_\mu^\alpha \partial_\nu b_\lambda^\beta + b_\mu^+ j_{\mu q, +} + \mathcal{L}_{\text{Maxwell}}[b] \\
&= b_\mu^+ (b_\mu^- + b_\mu^0) + b_\mu^- (b_\mu^+ + b_\mu^0)
\end{align*}\]

(2.42)

Milovanovic and Read have given an extensive description of the excitations of the primary (level \( p_1 = p_2 = 1 \)) Halperin states using this form of the effective Lagrangian. Naturally their results for these Halperin states are the same ones we discussed in the previous subsections.

For the special case of the \((m, m, m)\) states it is also convenient to introduce the basis

\[b_\mu^+ = b_\mu^1 + b_\mu^2, \quad b_\mu^- = \frac{1}{2} (-b_\mu^1 + b_\mu^2)\]

(2.44)

The effective Lagrangian for the \((m, m, m)\) states becomes

\[\begin{align*}
\mathcal{L}_{\text{eff}}[b] &= -\frac{m + n}{4\pi} \epsilon_{\mu\nu\lambda} b_\nu^\alpha \partial_\mu b_\lambda^\beta - \frac{1}{2}\epsilon_{\mu\nu\lambda} A_\mu^\alpha \partial_\nu b_\lambda^\beta + b_\mu^+ j_{\mu q, +} + b_\mu^- j_{\mu q, -} \\
&\quad + \frac{1}{2g^2} \left( 1 - v^2 - v B^2 \right) - \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} A_\mu^- \partial_\nu b_\lambda^\beta + b_\mu^- j_{\mu q, -}
\end{align*}\]

(2.45)

where

\[A_\mu^+ = \frac{1}{2} (A_\mu^1 + A_\mu^2), \quad j_{\mu q, +} = \frac{1}{2} (j_{\mu q, 1} + j_{\mu q, 2}) \]

\[A_\mu^- = -A_\mu^1 + A_\mu^2, \quad j_{\mu q, -} = -j_{\mu q, 1} + j_{\mu q, 2}\]

(2.46)

In the effective Maxwell term of Eq. \(2.45\) the approximate value of the effective coupling constant is \( g = 2\pi \sqrt{B_0 \over 2}, \) and the approximate effective velocity is \( v = \sqrt{\mu^2 \over 2M}, \) \( \nu = 1/m \) is the filling factor of the \((m, m, m)\) state, and \( M \) is the electron mass (see ref. \([4]\)). Eq. \(2.45\) shows that the finite energy excitations of the charge sector of the \((m, m, m)\) state coincides with the charge sector of the corresponding single layer Laughlin state, and that the neutral excitations include a gapless mode, the “photon” of the effective Maxwell theory.

Wen and Zee \([18]\) where the first to note that in this state there is a gapless mode with speed \( v \). This mode has
been identified with the Goldstone boson for the “spontaneous development of interlayer coherence” \[^{10,20}\]. The order parameter of this condensate can be described in terms of the dual picture. In 2 + 1 dimensions a gauge theory is always dual to a theory of a scalar field. We define a real antisymmetric tensor field \(\Gamma_{\mu\nu}\) such that \(\Gamma^{0i} = e^i_\perp\) and \(\Gamma_{ij} = e^i_\parallel e^j_\parallel\), with \(e^i_\perp\) and \(e^i_\parallel\) real functions of space and time. It is simple to check that given the following Lagrangian

\[
\mathcal{L} = -\frac{g}{2} \left( v e^2_\perp - \frac{1}{v} b^2_\parallel \right) - \frac{1}{2} \Gamma_{\mu\nu} f_{\mu\nu} \tag{2.47}
\]

upon the integration of \(\Gamma_{\mu\nu}\), one obtains the Maxwell term of Eq. 2.43 for \(b^-_\mu\), except for an irrelevant constant.

The integral over \(b^-_\mu\) in Eq. 2.47 gives the constraint \(\partial_\mu \Gamma^{\mu\nu} = 0\). This constraint can be solved by means of the phase field \(\theta\) by \(\Gamma^{\mu\nu} = \frac{1}{2\pi} \epsilon^{\mu\lambda\rho} \partial_\lambda \theta\). By substituting back into the effective Lagrangian, we get the effective action for the field \(\theta\), dual to the gauge field \(b^-_\mu\)

\[
\mathcal{L}[\theta] = \frac{g}{8\pi^2} \left( \frac{1}{v} (\partial_\mu \theta + 2A^\mu_\perp)^2 - v (\mathcal{V} \theta + 2A_\perp)^2 \right) \tag{2.48}
\]

Notice that in the presence of the sources (currents) \(j_\mu^{\parallel,\perp}\), the field \(\theta\) is multivalued. Conversely, the operator \(\exp(i\theta)\) creates a monopole configuration of the field \(b^-_\mu\). This is natural since in 2 + 1 dimensions the dual of the order parameter field \(\exp(i\theta)\) is a monopole creation operator \[^{30}\].

Eq. 2.48 has a number of important features: (i) as expected, duality has replaced the coupling constant by its reciprocal divided by \((2\pi)^2\), and (ii) the field \(\theta\) behaves like the phase field of an order parameter with neutral charge 2, a feature emphasized by Ezawa and coworkers \[^{10}\]. This form of the dual theory, Eq. 2.43, is familiar from the theories of anyon superfluidity (see for example ref. \[^{51}\]).

The correlation function of the order parameter is

\[
\langle e^{i\theta(x)} e^{-i\theta(x')} \rangle = e^{\frac{\pi}{g} \left( \frac{1}{R^2} - \frac{1}{a^2} \right)} \tag{2.49}
\]

where (in imaginary time \(x_0\))

\[
R^2 = v^2 |x_0 - x'_0|^2 + |\vec{x} - \vec{x}'|^2 + a^2 \tag{2.50}
\]

where \(a\) is a short distance cutoff.

At equal times the correlation function is (for large distances compared to the cyclotron length)

\[
\langle e^{i\theta(0, \vec{x})} e^{-i\theta(0, \vec{x}')} \rangle \approx e^{\frac{\hbar}{\hbar} \frac{\ell_0}{2v}\sqrt{2\pi}} \tag{2.51}
\]

where \(\ell_0 = 1/\sqrt{B}\) is the cyclotron length. This equal-time correlation function has the same form as the contribution of the neutral sector to the wave function of the \((m,m,m)\) state found in ref. \[^{2}\].

One may also suspect that there may be more excitations than the ones we discussed so far. For example, one possibility is a quasiparticle (or quasihole) of a single layer. However, these states carry a non-zero polarization (or “neutral charge”), and as such they couple to the Maxwell gauge field \(b^-_\mu\). In addition, precisely because there is a massless mode, the self energy of quasi-particles that carry a non-vanishing polarization (neutral charge) is logarithmically divergent. Such excitations are thus confined to neutral pairs with respect to the neutral charge. These “composite fermions” are actually the vortices of the condensed state \[^{3,2}\].

Finally it is useful to write the dual form of the Wilson loop operators for the gauge field \(b^-_\mu\). Consider a closed loop \(\Gamma\) and a quasiparticle current with charge \(k_\perp = k_1 - k_2\). The dual of this Wilson loop operator is

\[
\langle e^{i\Sigma \int \sigma_\mu \mathcal{D} \theta} \rangle = \langle e^{i\pi k_\perp \int \mathcal{D} \theta} \rangle \tag{2.52}
\]

where \(\Sigma\) is an open surface whose boundary is \(\Gamma\).

Let us summarize the excitation spectrum of the bulk \((m,m,m)\) states:

1. There are electrically charged \(e/(2m)\) vortices with logarithmically divergent energy, statistics \(\pi/(4m)\), and “neutral charge” \(\pm 1\), which simply says that there is one for each layer. These excitations are the “composite fermions” of the individual layers.

2. There are electrically charged \(e/m\) finite energy excitations and statistics \(\pi/m\); these excitations are vortex-antivortex bound states, and are the analogs of the Laughlin quasiholes of a single layer system.

3. There are two types of electron states:

   (a) a state with electric charge \(-e\), zero polarization and finite energy. This is the electron state of the Laughlin sector. It is a bound state with \(m\) negatively charged vortices of one layer and \(m\) positively charged vortices of the other layer. Here we refer to the charge coupled to the field \(b^-_\mu\).

   (b) two states with electric charge \(-e\), polarization \(\pm 1\) and logarithmically divergent energy; these states are electrons made entirely from \(2m\) negatively charged vortices all in the same layer.

4. There is also a gapless mode or Goldstone boson.

In section \[^{5}\] we will discuss the spectrum of edge states.
III. EDGE THEORY FOR FQH STATES ON BILAYERS AND PARTIALLY POLARIZED STATES

In this section we use the theory derived in the previous section, to extract the effective theory for the edge states. The effective Lagrangian of Eq. (2.14) is globally well defined (on closed surfaces), and has excitations with the correct fractional charge and statistics. This effective Lagrangian has the standard form first introduced by Wen and Zee [3]. Following the general arguments of Wen [1, 2], it is straightforward to extract a theory for the edge states which reflects the structure of the bulk presented in the previous section.

The Hilbert space of states of a Chern-Simons theory on a manifold $\Omega$ (which we take to be a disk) with a spatial boundary $\partial \Omega \cong S_1$ (where $S_1$ is a circle) has support at the boundary. This is a special case of a general result originally derived by Witten [13]. We now imagine that there is a (sharp) potential that confines the electrons on some simply connected region on the torus, isomorphic to a disk. The gauge fields on the region forbidden to the electrons can be integrated out since they decouple. Furthermore, we will assume that the confining potential is sharp enough that there is no edge reconstruction. Whether or not this assumption is justified depends on (important) microscopic details, including the form of the interactions, the vicinity of gates, etc.

It is well known [33] that the mean field theory approximating the form of the interactions, the vicinity of gates, etc. isomorphic to a disk. The gauge fields on the region for-}

bidden depends on (important) microscopic details, includ-}

ing the form of the interactions, the vicinity of gates, etc.

It is well known [3] that the mean field theory approxi-

mation to the fermion Chern-Simons approach to the

FQHE predicts a rather delicate structure of the edge

modes. However, provided the assumption that the edge

is sharp holds, we can assume that the spatial separation

of these edge branches is small, i.e., of the order of the

magnetic length. In this case, even if these modes were

to exist, something that in this regime is highly question-

able, it is clear that they cannot be resolved as separate

Hilbert spaces. Furthermore, these modes will couple to

the same gauge field(s). Thus, we will adopt the physis-

ical point of view that, in this regime, it is legitimate to

"glue" these modes together and to treat them as if they

drew a single mode with an effective velocity. Notice that,

under these assumptions and due to the renormalization

of the mode velocity found below, the contribution of $p$

gauge modes glued together to the specific heat is the same

as the contribution of $p$ independent modes. Thus, this

assumption does not change the physics. Moreover, be-

low we will find that the structure of the bulk theory on

a torus requires the existence of a set of non-propagating

modes at the boundary (which we will refer to as topo-

logical modes). These topological modes are unrelated to

a possible fine structure of the edge and play a very dif-

ferent role. In particular they do not carry energy and

only show up in the statistics if the physical states [32].

Thus, in this regime we can derive an effective theory of

the edge states directly from the effective action of Eq.

(2.14) in its present form without further assumptions. It

is straightforward to show that the effective action can

be expanded in the form

\[
S = \frac{1}{4\pi} \mathcal{K}_{IJ} \int_{\Omega} d^2 x \ \partial^I (a^0_j \epsilon_{ij} a^i_j) \\
+ \frac{1}{4\pi} \mathcal{K}_{IJ} \int_{\Omega} d^2 x \ a^0_j \epsilon_{ij} \left( \partial^i a^j - \partial^j a^i \right) \\
- \frac{1}{4\pi} \mathcal{K}_{IJ} \int_{\Omega} d^2 x \ \epsilon_{ij} a^i_j \partial^0 a^j \\
+ \frac{1}{2\pi} \int_{\Omega} d^2 x \left( a^0_i \mathcal{J}^0_i + a^i_i \mathcal{J}^i_0 \right)
\]  

(3.1)

where we have used the current $\mathcal{J}^I_i$ defined by

\[
\mathcal{J}^I_i \equiv t_{a1} \epsilon^I\nu\lambda \partial_\nu A^\alpha_\lambda + 2\pi t_{a1} J^\mu a_{qp}
\]

(3.2)

We will impose the gauge condition $a^0_j = 0$ at the boundary $\partial \Omega$. In this gauge the first term of Eq. (3.1) vanishes. In this form of the action it is also apparent that the field $a^0_j$ is a Lagrange multiplier that enforces the local constraint

\[
\mathcal{J}^0_i = -\mathcal{K}_{IJ} \epsilon_{ij} \partial^i a^j
\]  

(3.3)

which is just Gauss’ Law. Similarly, the third term of

Eq. (3.1) determines the commutation relations.

The solution of Gauss’ Law is

\[
a^0_i = \partial_i \phi^l
\]  

(3.4)

where $\phi^l$ are five multivalued scalar fields, i.e., singular gauge transformations. If the quasiparticles and the external fluxes are quasistatic bulk perturbations of the condensate, of quasiparticle number $N^a_{qp}$ and flux $\Phi^a = 2\pi N^a_{\phi}$ with $a = 1, 2$, the scalar fields $\phi^l$ at the boundary $\partial \Omega$ must satisfy the conditions

\[
\Delta \phi^l = 2\pi \left( K^{-1} \right)_{IJ} \left[ \begin{array}{c} N^{(1)}_\phi \\
N^{(1)}_{qp} \\
N^{(2)}_\phi \\
N^{(2)}_{qp} \\
-N^{(1)}_{qp} - N^{(2)}_{qp} \end{array} \right]_J
\]

(3.5)

where $\Delta \phi^l \equiv \int_{\partial \Omega} d x_i \partial_i \phi^l$ is the change of the field $\phi^l$ once taken around the boundary $\partial \Omega$.

Once the constraint Eq. (3.3) is solved, it is immediate to show that the content of this theory resides at the boundary $\partial \Omega$. Indeed, the remaining term in the action Eq. (3.1) takes the form

\[
S = -\frac{1}{4\pi} \mathcal{K}_{IJ} \int_{\Omega} d^2 x \ \epsilon_{ij} a^i_j \partial^0 a^j \\
= -\frac{1}{4\pi} \mathcal{K}_{IJ} \int_{\partial \Omega} dx_0 \int_{\partial \Omega} dx_i \partial^0 \phi^l \partial^i \phi^j
\]  

(3.6)
which is a theory of chiral bosons. However, as emphasized by Wen [31], as they stand these bosons do not propagate. The reason is that the Chern-Simons gauge theory is actually a topological field theory. Thus, in addition to being gauge invariant, it is independent of the metric of the manifold where the electrons reside and hence it is also invariant under arbitrary local diffeomorphisms. In particular this means that the Hamiltonian of the Chern-Simons theory is zero. Naturally, this is just the statement that this is an effective theory for the degrees of freedom below the gap of the incompressible fluid. There are no local degrees of freedom left in this regime. The degrees of freedom only “materialize” at the boundary which, in addition to breaking gauge invariance, also break the topological invariance. This is physically obvious since the edge states at the boundary carry energy and their Hamiltonian does not vanish.

There are several possible ways to represent this physics in the effective theory. The generalization of our approach in reference [32] to the bilayer system is straightforward. The edge term in the action due to the presence of a confining potential for a sharp edge will be given by

\[ S_{\text{edge}} = -\frac{p_1 v_1}{4\pi} \int dx_0 \oint_{\partial \Omega} d\gamma_1 (\partial_1 \phi^1)^2 - \frac{p_2 v_2}{4\pi} \int dx_0 \oint_{\partial \Omega} d\gamma_1 (\partial_1 \phi^3)^2 \]  

(3.7)

with \( v_\alpha = eE_\alpha/B \) the speed of the edge excitations for layer \( \alpha = 1, 2 \), and we have used the gauge condition \( a^1_0 = a^3_0 = 0 \) at the boundary. At this point we allow for the possibility of the two physically separated layers to have different velocities.

The electron-electron interaction term of the Hamiltonian becomes

\[ H_{\text{int}} = \int \int x' x'' \left( \frac{1}{2} (\rho_a(x) - \bar{\rho}_a) V_{ab}(x - x') (\rho_b(x'') - \bar{\rho}_b) \right) \]

\[ = \int \oint_{\partial \Omega} d\gamma dx_1 dx'_1 \frac{t^{ab}}{8\pi^2} \partial_1 \phi_1(x) V_{ab}(x - x') \partial_1 \phi_J(x') \]

(3.8)

where \( a \) and \( b \) label the layers. We have only retained the boundary contribution since the bulk excitations have a finite (and for present purposes large) energy gap. Notice that, given the form of \( t_{ab} \) as in Eq. 2.14, the term of the Hamiltonian of Eq. 3.8 only affects the modes \( \phi_1 \) and \( \phi_3 \). Likewise, the interaction with an external potential with support at the boundary becomes

\[ H_{\text{ext}} = -\int d^2x (\rho_a(x) - \bar{\rho}_a) A^0_a(x) \]

\[ = -\oint_{\partial \Omega} d\gamma dx_1 \frac{t^{ab}}{2\pi} \partial_1 \phi_1(x) A^0_a(x) \]

(3.9)

and it involves only \( \phi_1 \) and \( \phi_3 \) for the same reason.

In summary the effective action involves five bosons \( \phi_I \) (with \( I = 1, ..., 5 \)) and takes the form

\[ S = \frac{1}{4\pi} \int_{\partial \Omega \times \mathbb{R}} dx_0 dx_1 ( -K_{IJ} \partial_1 \phi_I \partial_1 \phi_J - U_{1J} \partial_1 \phi_1 \partial_1 \phi_J ) \]

(3.10)

where \( U_{1J} = t_{ab} t_{bJ} \left( (\delta_{ab} v_1 p_1 \delta_0 + \delta_{ab} v_2 p_2 \delta_0) + \frac{1}{2\pi} V_{ab} \right) \), and its only effect is to determine the velocity of the edge modes. Notice that, as it is well known, the actual velocity of the edge modes is the sum of two terms, one of which is determined by the interactions. The modes with a non vanishing velocity are \( \phi_1, \phi_3 \) which are the only ones that couple to perturbations due to an external electromagnetic field. Thus we identify \( \phi_1, \phi_3 \) as the charge modes. The three remaining modes do not propagate. Their effect is to fix the statistics of the states.

Finally, we need to relate these fields to the edge charge density for each layer. The local charge and current density \( J_\mu^a(x) \) in layer \( a \) is given by

\[ J_\mu^a(x) = \frac{\partial S}{\partial A^\mu_a} = \frac{1}{2\pi} t_{aI} \epsilon_{\mu \nu \lambda} \partial^\nu a^\lambda \]

(3.11)

Following the same steps described in reference [32] we can integrate the bulk currents across the edge to obtain the edge densities and currents. If the physical width of the edge \( \Lambda \) is infinitesimal relative to the linear size of the system, we find that the edge charge density and current density for the edge of layer \( a \), in the \( x_1 \) direction, is given by

\[ j_0^1 = \frac{1}{\pi} \int_\Lambda dx_2 J_1^1(x) = \frac{1}{2\pi} \partial_1 \phi_1 \]

\[ j_0^2 = \frac{1}{\pi} \int_\Lambda dx_2 J_0^2(x) = \frac{1}{2\pi} \partial_1 \phi_3 \]

(3.12)

It is straightforward to check that if \( N^\alpha_{ap} \) quasiparticles of type \( \alpha \) are added to the bulk at constant magnetic field \( (N_\phi = 0) \), the edge acquires a charge

\[ Q_{\text{edge}} = Q^{(1)}_{\text{edge}} + Q^{(2)}_{\text{edge}} = \int dx_1 (j_0^1(x_1) + j_0^2(x_1)) = N^\alpha_{ap} \frac{v_\alpha}{p_\alpha} \]

(3.13)

which is, as expected, equal to the extra charge added to the bulk.

To conclude this section we remark once again that the correspondence between the bulk theory described in section 2 and the edge theory described here is only correct for a sharp edge. In this case we found that the theory of the edge states of the FQH states in bilayers can be described in terms of five chiral fields. Two of them are propagating fields, and are associated with charge fluctuations. The other three non-propagating fields are associated with the global topological consistency of flux-attachment. As we will see in the next section the only effect of these non-propagating topological modes is to
give the correct statistics to the excitations. We will also find that it is possible to reduce the number of non-propagating topological fields to just two.

However, in many experimental situations, the confining potentials are smooth. This implies that the edge scale is of the order of many magnetic lengths and the density gradually drops to zero within this scale. In this case the edge will show the usually called edge reconstruction in which two or more (possibly) interacting edge branches are generated. Within our approach, this effect will appear at the level of the mean field solution for the bulk action (eq 2.9). In the presence of a smooth confining potential in the sample, the calculation of the fermionic determinant has to take into account the presence of different edges and their interactions for the mean field solution. We will not deal with this problem here.

IV. ELECTRON AND QUASIPARTICLE OPERATORS FOR THE EDGE STATES ON BILAYERS

In this Section we will construct the operators for electrons and quasiparticles of the edge states in bilayer FQH states. As we have shown in section II, all elementary physical excitations can be written as combinations of $k_1$ quasiparticles of type 1 and $k_2$ quasiparticles of type 2. Therefore a generic edge operator that creates excitations can be written as

$$\Psi(x) = e^{i[k_1\phi_2 + k_2\phi_4 - (k_1 + k_2)\phi_5]} \tag{4.1}$$

At this stage, it is convenient to rewrite the effective theory in a new basis where the charged and topological fields decouple completely. We have already identified $\phi_1$ and $\phi_3$ as the charge fields, i.e., the fields that couple to an external electromagnetic field. We further introduce the topological fields $\phi_{T\alpha}$, $\phi_{T0}$ and define the new basis

$$\phi_{C1} = \phi_1$$
$$\phi_{T1} = \text{sign}(p_1) \frac{1}{\sqrt{|p_1|}} \phi_1 + \sqrt{|p_1|} \phi_2$$
$$\phi_{C2} = \phi_3$$
$$\phi_{T2} = \text{sign}(p_2) \frac{1}{\sqrt{|p_2|}} \phi_3 + \sqrt{|p_2|} \phi_4$$
$$\phi_{T0} = \phi_5 \tag{4.2}$$

The edge effective Lagrangian of Eq 4.10 in terms of the charged and topological fields becomes

$$\mathcal{L} = \frac{1}{4\pi} \left( \kappa_{\alpha\beta} \partial_1 \phi_{\alpha\beta} \partial_0 \phi_{\alpha\beta} - \partial_1 \phi_{\alpha\beta} v_{\alpha\beta} \partial_1 \phi_{\alpha\beta} \right) - \frac{1}{4\pi} \left( G_{\alpha\beta} \partial_1 \phi_{\alpha\beta} \partial_0 \phi_{\alpha\beta} + \partial_1 \phi_{T0} \partial_0 \phi_{T0} \right) \tag{4.3}$$

where

$$\kappa = \begin{pmatrix} 2n_1 + \frac{1}{p_1} & n_2 + \frac{1}{p_2} \\ n_1 & 2n_2 + \frac{1}{p_1} \end{pmatrix} \tag{4.4}$$
$$G = \begin{pmatrix} \text{sign}(p_1) & 0 \\ 0 & \text{sign}(p_2) \end{pmatrix} \tag{4.5}$$

and the velocities are given by

$$v = \left( \frac{V_{11} + 2\pi p_1 v_1}{V_{12}}, \frac{V_{12} V_{22} + 2\pi p_2 v_2}{V_{22}} \right) \tag{4.6}$$

Notice that we have allowed for $p_1$ and/or $p_2$ to be either positive or negative.

We see that only the $\phi_{C\alpha}$ modes propagate. The role of the three remaining modes is to give the correct statistics to the quasiparticles. In the new basis of Eq. 4.2, the most general quasiparticle operator of Eq. 4.1 can be written as

$$\Psi(x) = e^{i(a_{C\beta} \phi_{C\beta} + a_{T\beta} \phi_{T\beta} + a_{T0} \phi_{T0})} \tag{4.7}$$

where

$$a_{C1} = -\frac{k_1}{p_1}, \quad a_{C2} = -\frac{k_2}{p_2},$$
$$a_{T1} = \frac{k_1}{\sqrt{|p_1|}} \quad a_{T2} = \frac{k_2}{\sqrt{|p_2|}} \quad a_{T0} = -(k_1 + k_2) \tag{4.8}$$

As expected, the quantum numbers of the states created by these operators are given by Eq. 2.25 and Eq. 2.26.

For the quasiparticle (and quasihole) operators we choose $(k_1, k_2) = (1,0)$ for the quasiparticle of type 1, and $(k_1, k_2) = (0,1)$ for the quasiparticle of type 2 respectively,

$$\Psi_\downarrow = e^{i\left(-\frac{1}{p_1} \phi_{C1} + \frac{1}{\sqrt{|p_1|}} \phi_{T1} - \phi_{T0}\right)}$$
$$\Psi_\uparrow = e^{i\left(-\frac{1}{p_2} \phi_{C2} + \frac{1}{\sqrt{|p_2|}} \phi_{T2} - \phi_{T0}\right)} \tag{4.9}$$

where we have renamed the quasiparticles as $\Psi_\downarrow$ for $(k_1, k_2) = (1,0)$, and $\Psi_\uparrow$ for $(k_1, k_2) = (0,1)$, depending on whether their spin projection $S_z$ is negative or positive respectively.

From now on we will discuss the special case in which the Lagrangian 4.3 takes a simple (diagonal) form. In particular we will set $2p_1 + 1 = 2n_2 + 1 \equiv m$, and $p_1 = p_2 \equiv p$. We will also assume that the velocities in both layers are equal. Let us define the rotated and rescaled fields $\phi_{C\pm}$ and $\phi_{T\pm}$.
\[ \phi_{C \pm} = \sqrt{\left|\frac{m - 1 + \frac{1}{p} \pm n}{2}\right|} (\phi_{C1} \pm \phi_{C2}) \]

\[ \phi_{T \pm} = \frac{1}{\sqrt{2}} (\phi_{T1} \pm \phi_{T2}) \]

which will simplify the description considerably. The effective velocities \( v_\pm \) are

\[ v_\pm = \frac{V_{11} + 2\pi pv \pm V_{12}}{\sqrt{|m - 1 + \frac{1}{p} \pm n|}} \]

(4.10)

where we have assumed \( v_1 = v_2 = v \). We will also need the definition

\[ s = \text{sign} \left( m - n - 1 + \frac{1}{p} \right) \]

(4.12)

The sign \( s \) of Eq. 4.12 determines the chirality of the neutral (or spin) edge state, relative to the chirality of the charge mode. Recently, Moore and Haldane have found in an exact diagonalization in small systems that in the 2/3 singlet state the charge and spin edge modes have opposite chirality [54]. This result is consistent with the general rule of Eq. 4.12.

Furthermore, it is possible to find still another basis in which one of the topological non-propagating fields decouples completely. The details of this procedure depend on the sign of \( p \). For \( p > 0 \) we rotate the fields \( \phi_{T+} \) and \( \phi_{T0} \) by the orthogonal transformation

\[ \phi_{T+} \rightarrow \cos \theta \phi_{T+} - \sin \theta \phi_{T0} \]
\[ \phi_{T0} \rightarrow \sin \theta \phi_{T+} + \cos \theta \phi_{T0} \]

(4.13)

where the angle \( \theta \) is given by

\[ \tan \theta = \sqrt{2p} \]

(4.14)

whereas for \( p < 0 \) the required transformation has instead the form of a Lorentz transformation,

\[ \phi_{T+} \rightarrow \cosh \theta \phi_{T+} + \sinh \theta \phi_{T0} \]
\[ \phi_{T0} \rightarrow \sinh \theta \phi_{T+} + \cosh \theta \phi_{T0} \]

(4.15)

where the “rapidity” \( \theta \) is now given by

\[ \tanh \theta = \frac{1}{\sqrt{2|p|}} \]

(4.16)

Upon a simple redefinition of the fields it is possible to describe both cases simultaneously. In this condensed notation, the entire excitation spectrum can be generated in terms of two propagating chiral bosons, the fields \( \phi_{C \pm} \), and two non-propagating topological fields that we will denote by \( \phi_{T \pm} \). In terms of these fields, the effective Lagrangian that results is

\[ \mathcal{L} = \frac{1}{4\pi} \partial_1 \phi_{C+} (\partial_0 \phi_{C+} - v_+ \partial_1 \phi_{C+}) \]
\[ + \frac{1}{4\pi} \partial_1 \phi_{C-} (s \partial_0 \phi_{C-} - v_- \partial_1 \phi_{C-}) \]
\[ - \frac{1}{4\pi} \partial_1 \phi_{T+} \partial_0 \phi_{T+} - \frac{\text{sign}(p)}{4\pi} \partial_1 \phi_{T-} \partial_0 \phi_{T-} \]

(4.17)

The charge and spin currents are given by

\[ j_C = \frac{1}{2\pi} \frac{1}{\sqrt{|m + n - 1 + \frac{1}{p}|}} \partial_z \phi_{C+} \]

(4.18)

\[ j_s = \frac{1}{2\pi} \frac{1}{\sqrt{|m - n - 1 + \frac{1}{p}|}} \partial_z \phi_{C-} \]

(4.19)

The most general edge excitations are created by an operator of the form

\[ \Psi(x) = e^{i (a_{C+} \phi_{C+} + a_{C-} \phi_{C-} + a_{T+} \phi_{T+} + a_{T-} \phi_{T-})} \]

(4.20)

where, for states with \( 2n_1 + 1 = 2n_2 + 1 = m \), the coefficients \( a_{C \pm} \) and \( a_{T-} \) now take the form

\[ a_{C+} = -\frac{k_1 + k_2}{p \sqrt{2|m + n - 1 + \frac{1}{p}|}} \]
\[ a_{C-} = -\frac{k_1 - k_2}{p \sqrt{2|m - n - 1 + \frac{1}{p}|}} \]
\[ a_{T+} = \text{sign}(p) \frac{(k_1 + k_2)}{\sqrt{1 + \frac{1}{2p}}} \]
\[ a_{T-} = \frac{k_1 - k_2}{\sqrt{2|p|}} \]

(4.21)

Notice that here \( p \) can have either sign.

The quantum numbers of the excitations created by the operators of Eq. 4.21 are

\[ \frac{Q}{e} = \sqrt{\nu} a_{C+} \]
\[ S_z = \frac{1}{\sqrt{2|m - n - 1 + \frac{1}{p}|}} \]
\[ \frac{\theta}{\pi} = -(a_{C+})^2 - s (a_{C-})^2 + (a_{T+})^2 + \text{sign}(p) (a_{T-})^2 \]

(4.22)

It is easy to check that the operators
\[ \Psi_{\phi^\dagger \phi} = e^{-i \sqrt{1 + \frac{1}{2p}} \phi_T^+ \pm \frac{i}{\sqrt{2|p|}} \phi_T^-} \times e^{-i \frac{p}{\sqrt{2|m+n-1+\frac{1}{p}|}} \phi_C^+ \pm \frac{i}{p\sqrt{2|m-n-1+\frac{1}{p}|}} \phi_C^-} \] (4.23)

create excitations with the correct quantum numbers for the elementary quasiholes.

Next we compute the propagators for an excitation created by an operator of the form of Eq. 4.20. Since the effective action is quadratic in the fields, this calculation is straightforward. The propagators of the fields \( \phi_{C\pm} \) are identical since their actions are the same. In imaginary time they become \( \langle \phi_{C\pm}(x,t)\phi_{C\pm}(0,0) \rangle = -\ln z \) (4.24)

where \( z = x + i\tau \).

For the fields \( \phi_T \), which do not propagate (i.e., their velocity is zero), their propagators in the limit \( \epsilon \to 0 \) are

\[ \langle \phi_{T\pm}(x,t)\phi_{T\pm}(0,0) \rangle = -i \frac{\pi}{2} \text{sign}(xt) \] (4.25)

Using the above results we find that the propagator for an operator \( \Psi \) of the form of Eq. 4.20 in the limit \( x \to 0^+ \), is

\[ \langle \Psi(0^+,t)\Psi(0,0) \rangle \propto \frac{1}{|t|^g} e^{-\theta/2 \text{sign}(t)} \] (4.26)

where the exponent \( g \) is

\[ g_t = a_{C+}^2 + a_{C-}^2 \] (4.27)

and \( \theta \) is the statistical angle of Eq. 4.22.

Below we will apply these results to the computation of tunneling exponents to a number of FQH states of special interest. It is worth to notice that the exponent \( g_t \) is determined by the coefficients of the fields \( \phi_{C\pm} \) alone, and that the topological fields \( \phi_{T\pm} \) only contribute to the statistics. In addition, for \( p = \pm 1 \) the contribution of the topological fields to the statistics of the excitations yields a trivial multiple of \( 2\pi \). Hence, as anticipated above, for \( p = \pm 1 \) the fields \( \phi_{T\pm} \) drop out altogether. Finally, by symmetry, the propagator for quasiparticles with spin down is also given by Eq. 4.26 with the same exponents. Also by symmetry, the crossed propagator, which mixes up and down quasiparticles, vanishes identically.

It is also an interesting question to ask which operator can measure the relative statistics of two quasiparticles. The simplest way to do that is to add a boundary perturbation that will allow for tunneling between different types of quasiparticles. For example consider a perturbation well localized at some point on the edge, with tunneling amplitude \( \Gamma \). It is easy to see that, to first order in \( \Gamma \), a crossed propagator is induced. This crossed propagator is just the square of two propagators like Eq. 4.26 multiplied by a phase factor of the relative statistics. In any event, just as in the case of fractional statistics \([50]\), in practice it will be necessary to consider an interferometric experiment (with two tunneling centers) in order to measure relative statistics.

We will compute the tunneling exponents in two cases of special interest: the symmetric \( SU(2) \) and \( U(1) \times U(1) \) states. In particular we will discuss different tunneling processes and give the tunneling exponents. There are three situations of physical interest: i) internal tunneling of quasiparticles, ii) tunneling of electrons between identical fluids, and iii) electron tunneling between distinct fluids. Because these states are symmetric, even though there are several tunneling channels leading to a conductance matrix, it turns out that the exponents associated with different tunneling channels are equal while the amplitudes in general are different. In what follows we will only discuss the scaling exponents. These can be calculated by a straightforward generalization of the arguments of ref. [24], based on the formalism of ref. [37], for the case of both internal tunneling and of tunneling of electrons between identical fluids. The tunneling current \( I(V) \) at bias voltage \( V \) for identical layers, has the scaling form

\[ I_{bc}(V) \propto M_{bc} V^\alpha \] (4.28)

as discussed by Wen [26] and by Kane and Fisher [27] for Laughlin fluids. The exponent \( \alpha \) is determined by the scaling dimension of the tunneling operator. Here we have denoted the amplitudes for the different channels by a \( 2 \times 2 \) matrix \( M_{bc} \) (which we will not calculate).

For internal tunneling of quasiparticles, the tunneling exponent \( \alpha_{\phi\phi} \) is given in terms of the exponent of the quasiparticle propagator \( g_{\phi\phi} \) by the formula

\[ \alpha_{\phi\phi} = 2g_{\phi\phi} - 1 \] (4.29)

Internal tunneling of quasiparticles is always a relevant perturbation. Consequently, the system always flows at low energies to a regime in which the fluid splits in two [27]. Hence, in this regime quasiparticle tunneling yields a reduction of the differential conductance from its value at zero tunneling, the Hall conductance of the fluid. In this regime Eq. 1.28 for the tunneling current holds at high frequencies. In the case of electron tunneling between identical fluids Eq. 1.28 holds at low frequencies, with an exponent \( \alpha_e \) given by

\[ \alpha_e = 2g_e - 1 \] (4.30)

where \( g_e \) is the exponent of the electron propagator. Finally, for tunneling of electrons between an external lead (a Fermi liquid) and a bilayer FQH state, the scaling form Eq. 1.28, also valid at low frequencies, has an exponent \( \alpha_t \) given by

\[ \alpha_t = g_e \] (4.31)
where $g_e$ is the exponent of the electron propagator of the FQH fluid $[S]$. For the case of tunneling of electrons into a single layer FQH state in the Jain sequence, this exponent is equal to $1/\nu$ (see ref. [32]).

### A. SU(2) states.

Here we give a description of the spectrum of the edge states of the SU(2) FQH states. Recall that for these states $m = n + 1$, and the filling fraction is $\nu = 2p/(2np + 1)$. A special feature of the SU(2) states is that the chirality of the spin field is $s = \text{sgn}(p)$. Hence, for the SU(2) states only the sign of $p$ matters.

#### 1. Quasiholes:

The operators that create quasiholes with spin up or down are given in Eq. 4.23. For the SU(2) states, the quantum numbers of these quasiholes are

$$Q = \frac{e}{2np + 1}$$

$$S_z = \frac{1}{2} \text{sgn}(p)$$

$$\frac{\theta}{\pi} = 1 + \frac{n}{2np + 1}$$

(4.32)

The exponents $g_{qp}$ for quasiparticles of either spin are

$$g_{qp} = \begin{cases} n + \frac{1}{2} & \text{for } p > 0 \\ \frac{n}{2n|p| - 1} & \text{for } p < 0 \end{cases}$$

(4.33)

Below we will use these results to compute the exponents for internal tunneling.

#### 2. Bound States:

Let us consider bound states of a quasiparticle and a quasihole. Such states are electrically neutral and have spin projection $\pm 1$. For all symmetric FQH states in bilayers, the operator that creates a bound state of two quasiparticles, with total $S_z = \pm 1$ and zero electric charge, up to singular normalization factors (see Appendix A), are

$$S^\pm(0) \equiv \lim_{z \to 0} : \Psi_{qp,\uparrow,\downarrow}(z) \Psi_{qp,\downarrow,\uparrow}(0) : = \pm \sqrt{\frac{2}{|p|}} \phi_{T^-}(0) \mp \sqrt{\frac{2}{|p|}} \phi_{C^-}(0)$$

(4.34)

For example, in the special case of the SU(2) states we find

$$S^\pm(0) \propto e^{\pm i \sqrt{\frac{2}{|p|}} \phi_{T^-}(0)} \mp \frac{2}{|p|} \phi_{C^-}(0)$$

(4.35)

In addition to these operators, which have $S_z = \pm 1$, we also have the spin current operator $j_z$, given by Eq. 4.19, which has zero electric charge and $S_z = 0$. For $p = 1$, the primary Halperin states, these three operators are the local generators of a $su(2)_1$ Kac-Moody algebra. This algebra, and its generators, can be used to construct the entire Hilbert space of these spin edge states. However, for general $p \neq 1$ this is not possible since the off-diagonal generators do not have scaling dimension 1. Thus, for general $p$, there is only a global SU(2) spin symmetry in the sense that the states are arranged into SU(2) multiplets. However, for $p = r^2$ (where $r$ is an integer) there exist complexes of $r$ bound states, of the type constructed above, which constitute the off-diagonal generators of a Kac-Moody algebra $su(2)_1$.

Apart from these electrically neutral bound states, there are charged bound states as well. For instance, let us consider bound states of two quasiparticles. There are four operators that create such charged bound states. In the Appendix A we show that it is possible to arrange these four operators into linear combinations that create spin singlet pairs and three that create spin triplet pairs. The spin singlet operator

$$\Psi_{\text{singlet}} \propto e^{-i \sqrt{\frac{2}{p}} C_1(z)} : \phi_{C_1}(z) : \times e^{i \sqrt{\frac{2}{2p}} \phi_{C_1}(z)}$$

(4.36)

plays a central role in the construction of the electron operator. Here too, the three spin triplet operators only have the same scaling dimension for the primary Halperin FQH states, which have $p = 1$. Once again we see the same pattern: states with the correct charge and spin quantum numbers fail to have the same scaling dimension except for $p = 1$.

#### 3. Electrons:

In order to construct an electron operator for these FQH states we use the fact that they have filling fraction $\nu = \frac{2p}{2np + 1}$ with $n$ even, and that both quasiparticles have the same charge $Q = \frac{1}{2np + 1}$. Therefore we can obtain an object with charge $Q = -e$ if we construct a composite object made of $2np + 1$ quasiparticles. In addition, in order to be an electron, this object must have total spin $S = 1/2$. Clearly, out of $2np + 1$ objects we can construct a number of states (operators) with different total spin. For instance with two quasiparticles it is
possible to construct spin singlet states with total spin \( S = 0 \), or triplet states with \( S = 1 \). Since, in order to make an electron we must construct an object with total spin \( S = 1/2 \), we take \( np \) singletons constructed as it is shown in Appendix A, and an extra quasiparticle whose spin projection will determine the spin projection of the electron operator. Following this prescription the electron operator results

\[
\Psi(x)_{e,\uparrow\downarrow} = e^{-i \sqrt{\nu} \phi_{C^+} \mp \frac{i}{\sqrt{2|p|}} \phi_{C^-}} i(2np + 1) \text{sign}(p) \sqrt{1 + \frac{1}{2p}} \phi_{T^+} \mp \frac{i}{\sqrt{2|p|}} \phi_{T^-}
\]

up to irrelevant operators, whose form is discussed in Appendix A. This operator creates states with charge \( Q = -e \), spin projection \( S_z = \pm \frac{1}{2} \), and statistics \( \theta = \pi(2np + 1)(2np + n + 1) \). The exponent \( g_e \) for the electron operator is

\[
g_e = \begin{cases} n + \frac{1}{p} & \text{for } p > 0 \\ n & \text{for } p > 0 \end{cases}
\]

(4.38)

This result holds for both signs of \( p \).

For the special case of the Halperin state \((3, 3, 2)\), with \( \nu = 2/5 \), the electron operator is

\[
\Psi_{e,\uparrow\downarrow}(x) = e^{-i \sqrt{\frac{5}{2}} \phi_{C^+} \mp \frac{i}{\sqrt{2}} \phi_{C^-}}
\]

(4.39)

where we have dropped the contributions of the topological fields since for the \( p = 1 \) states they act like the identity operator. The exponent for the electron propagator results \( g_e = 3 \). In Eq. 4.39 we have kept only the most relevant operators which contribute to the electron operator, and neglected subleading irrelevant operators involving \( \partial_x \exp \left(-i \frac{\sqrt{2}}{2} \phi_{C^-} \right) \). These irrelevant operators appear in the operator product expansion with well defined coefficients which are calculated in Appendix A.

Similarly, for the singlet state at \( \nu = 2/3 \), which has \( m = n + 1 = 3 \) and \( p = -1 \) (i.e., it belongs to the reversed sequence), the electron operator is

\[
\Psi_{e,\uparrow\downarrow}(x) = e^{-i \frac{3}{\sqrt{2}} \phi_{C^+} \mp \frac{i}{\sqrt{2}} \phi_{C^-}}
\]

(4.40)

Notice that in this case the electron can be viewed as a bound state of a of a right moving electron and a right moving semion of the charge sector, and a left moving semion of the spin sector. Also, although the total exponent of the electron is in this case \( g_e = \frac{2}{p} + \frac{2}{p} = 2 \), since the velocities of the charge and spin bosons are different, the electron propagator formally still has a branch cut.

We will now apply the results derived above to the computation of the tunneling exponents for the \( SU(2) \) states.

1. Internal quasiparticle tunneling:

For the \( SU(2) \) states the exponent of the quasiparticle propagator is calculated in Eq. 4.33. In particular, for the primary Halperin state \((3, 3, 2)\) we find the exponent \( g_{qp} = \frac{2}{5} \). In this regime, the two-terminal differential conductance is reduced from the quantized bulk Hall conductance due to tunneling of quasiparticles. This reduction has the scaling form \( V^{\alpha_{qp} - 1} \), with \( \alpha_{qp} \) given by Eq. 4.29. In particular, for the \((m, m, m - 1)\) \( SU(2) \) Halperin states we find the exponent \( \alpha_{qp} = 1/(2m - 1) \).

For the case of the reversed sequence states, with \( p = -1 \), the exponent is obtained upon substituting \( m \to m - 2 \). For the 2/5 spin singlet state the exponent is \( g_{qp} = 3/5 \), we get \( \alpha_{qp} = 1/5 \), and the two-terminal conductance follows the law \( G_{qp} = \frac{2 e^2}{h} \text{const.} \times (T_K/V)^{4/5} \). Once again, this law holds only at large voltages \( V \gg T_K \), where \( T_K \) is a non-trivial crossover energy scale qualitatively similar to the Kondo scale in quantum impurity systems [58]. By a similar calculation, for the \( \nu = 2/3 \) singlet state we find \( g_{qp} = 2/3 \) and an exponent for the tunneling current of \( \alpha_{qp} = 1/3 \).

2. Electron tunneling between identical \( SU(2) \) states:

From the exponent of the electron propagator of Eq. 4.38 we find that the exponent of the differential conductance for tunneling of electrons between identical \( SU(2) \) singlet FQH states is

\[
\alpha_e = \begin{cases} 2n - 1 + \frac{2}{p} & \text{for } p > 0 \\ 2n - 1 & \text{for } p > 0 \end{cases}
\]

(4.41)

In particular, for the \((m, m, m - 1)\) primary states we find \( \alpha_e = 2m - 1 \), for \( p > 0 \), and \( \alpha_e = 2m - 3 \) for \( p = -1 \). For the 2/5 spin singlet state we find that the tunneling current of electrons obeys the law \( I \sim V^3 \) and \( G_e \sim V^4 \). Instead, for the spin singlet state at \( \nu = 2/3 \) we get \( \alpha_e = 3 \), and the current tunneling obeys the law \( I \sim V^3 \). For the states in the second level of the hierarchy which have already been observed experimentally [12], i.e., for \( 4/7 \) and \( 4/9 \) the exponents are \( \alpha_e = 3 \) and 4 respectively.

3. Electron tunneling into an \( SU(2) \) state from a Fermi liquid:

Finally, the exponent for the differential conductance for tunneling into an \( SU(2) \) singlet state from an external lead is
\[ \alpha_t = g_e = \begin{cases} m - 1 + \frac{1}{2} & \text{for } p > 0 \\ m - 1 & \text{for } p > 0 \end{cases} \quad (4.42) \]

Hence, for the 2/5 state we get \( I \sim V^3 \) and \( G_1 \sim V^2 \). For \( \nu = 2/3 \) we find that the tunneling current of electrons from a Fermi liquid obeys the law \( I \sim V^2 \). For the states 4/9 and 4/7 the current will obey \( I \sim V^{5/2} \) and \( I \sim V^2 \) respectively.

Quite generally, at the edge of an \( SU(2) \) state, at filling factor \( \nu = 2p/(2np+1) \), the tunneling density of states for electrons obeys the law \( |\omega|^{p-1} \), where \( g_e \) is given in Eq. (4.42). In contrast, the corresponding spin polarized state with the same filling factor has a tunneling density of states for electrons with the law \( |\omega|^{\mp 1} \), see ref. [32].

**B. \( U(1) \times U(1) \) states.**

The \( U(1) \times U(1) \) symmetric states have filling fraction \( \nu = 2p/(m+n-1)p+1 \) with \( n \) odd. Here we will consider the states which are not in the \( SU(2) \) subclass and are incompressible. The special case of the \((m,m,m)\) states, i.e., states with \( m = n + 2 \) and \( p = -1 \), are actually the same as the \((m,m,m)\) states and are not a separate case.

Here, unlike the special case of the \( SU(2) \) states, there are two general types of states: (a) states with \( m > n + 1 \), and (b) states with \( m < n \). For states with \( m > n + 1 \), we have \( m - n - 1 + \frac{1}{p} > 0 \). Hence, in this case the charge field \( \phi_{C+} \) and the spin field \( \phi_{C-} \) have the same chirality, i.e., \( s = +1 \). In the other case, \( m < n \), the opposite inequality holds, \( m - n - 1 + \frac{1}{p} < 0 \). Hence \( s = -1 \), and the chirality of the spin field is opposite to the chirality of the charge field. The same is true for \( m = n \) and \( p \neq 1 \).

The spectrum of the edge states of the \( U(1) \times U(1) \) FQH states consists of the following,

1. **Quasiholes:**
   There are two quasiholes with the quantum numbers
   \[ Q = \frac{e}{(m+n-1)p+1} \]
   \[ S_z = \pm \frac{2|\sigma(n)|}{|m-n-1+p|} \]
   \[ \theta = 1 + \frac{1}{p} - \frac{m-1 + \frac{1}{p}}{((m-1)p+1)^2 - n^2 p^2} \]
   \[ \frac{1}{\pi} = 1 + \frac{1}{p} - \frac{m-1 + \frac{1}{p}}{((m-1)p+1)^2 - n^2 p^2} \quad (4.43) \]

   The general form of the operator that creates the quasiholes of a general \( U(1) \times U(1) \) state was given in Eq. (4.23). In particular, in the case of the \((3,3,1)\) state the operators that creates quasiholes with both polarizations are much simpler,

\[ \Psi_{qH} \propto e^{-i \sqrt{8} \phi_{C+}} e^{\pm i \frac{1}{2} \phi_{C-}} \quad (4.44) \]

The quasiparticle propagators at the edge of a \( U(1) \times U(1) \) state have exponents \( g_{qP} \) given by

\[ g_{qP} = \begin{cases} \frac{m - 1 + \frac{1}{p}}{((m-1)p+1)^2 - n^2 p^2} & \text{for } s > 0 \\ \frac{n}{n^2 p^2 - ((m-1)p+1)^2} & \text{for } s < 0 \end{cases} \quad (4.45) \]

2. **Neutral fermions:**

There are both charged and neutral bound states, and their statistics depends on which particular FQH state is being discussed. Here we will consider only bound states for the \((3,3,1)\) state which have a number of interesting features. An operator quasiparticle-quasihole bound state with \((k_1, k_2) = (1, -1)\). It has zero charge, spin \( S_z = 1/2 \), and it is a fermion. It is a neutral fermion operator created by

\[ \Psi_{NF,\uparrow,\downarrow}(0) \propto e^{\pm i \phi_{C-}} \quad (4.46) \]

This is the chiral Dirac fermion at the edge discussed by Milovanovic and Read [44]. The exponent for the neutral fermion is \( g_{NF} = 1 \). The neutral fermion has dimension 1/2 and the corresponding tunneling operator has dimension 1. Hence, the operator that tunnels neutral fermions at one point is marginal.

The operators that create charge neutral quasiparticle-bound states with \( S_z = \pm 1/2 \) in a general symmetric \( U(1) \times U(1) \) FQH state with \( m - n \) even and \( p \) odd, have the form

\[ \Psi_{neutral,\uparrow,\downarrow} = e^{\mp i \sqrt{\frac{1}{2} |m - n - 1 + \frac{1}{p}| \phi_{C-}}} \times e^{\pm i \sqrt{\frac{|p|}{2} |m - n - 1 + \frac{1}{p}| \phi_{T-}}} \quad (4.47) \]

In particular, all of the primary \((m,m,m)\) states have a Dirac fermion in their spectrum. The exponents for the neutral states are

\[ g_{neutral} = \frac{1}{2} |m - n - 1 + \frac{1}{p}| \quad (4.48) \]

Except for the case of the FQH states \((m,m,m)\), and their descendants with \( p \) odd, the processes of tunneling of neutral \( S_z = \pm 1/2 \) excitations will
turn out to be irrelevant. However, in all cases where they are allowed, operators that represent the leading processes with spin flip without tunneling of charge will necessarily involve the neutral operators of Eq. 4.43.

3. Spin singlet bound states: Here we will discuss only the (3, 3, 1) primary state. Consider first a bound state with two quasiholes. There are four such states: the singlet state, and the three triplet states. These operators have \((k_1, k_2) = (2, 0), (0, 2),\) and the symmetric and antisymmetric combinations of the \((1, 1)\) operator (see Appendix A):

\[
\Psi_{\pm 1/2} \propto e^{-i/\sqrt{2} \phi_{C+}} e^{\mp i \phi_{C-}}, \\
\Psi_0 \propto e^{-i/\sqrt{2} \phi_{C+}}, \\
\Psi_{\text{singlet}} \propto e^{-i/\sqrt{2} \phi_{C+}} i \partial_z \phi_{C-}
\]  

(4.49)

These operators have dimensions \(3/4\) (two states), \(1/4\) (one state) and \(5/4\) (one state) respectively. The states they create have charge 1/2, \(S_z = 1/2\), and are semions. Their exponents are \(g = 3/2, 1/2, 5/2\). Hence only the tunneling operator \(\Psi_{0,R} \Psi_{0,L}\) is relevant (it has dimension 1/2). (Here \(R, L\) here denote the edges of a FQH fluid.) Finally, since the \((3, 3, 1)\) state is particle-hole symmetric (or rather, it is compatible with it), the adjoint of the operators of Eq. (4.49) create states with charge \(-1/2\).

Next we consider bound states of four quasiparticles, \(i.e\). bound states of the bound states. In particular the operators

\[
j_{\pm} \propto (\Psi_0)^2 \propto e^{\pm i \sqrt{2} \phi_{C+}}, \\
j_0 \propto i \partial_z \phi_{C+}
\]  

(4.50)

create states with charge 1, 0, \(-1\), \(S_z = 0\) and are bosons. These operators span an \(su(2)_1\) algebra of charge, a consequence of particle-hole symmetry. All three states have dimension 1 and their exponents are \(g = 2\). Notice that in the \((3, 3, 2)\) we found an \(su(2)_1\) algebra for spin.

4. Electrons: To construct a fermionic operator with charge \(Q = -1\) and the lowest spin projection in a general \(U(1) \times U(1)\) state, we need to take \((m + n - 1)p + 1\) quasiparticles arranged in such a way that they have the required properties. It is simple to check, following the prescriptions given in Appendix A, that the electron operator that results is given by

\[
\Psi_{e, \uparrow, \downarrow} \propto e^{-i \sqrt{2} \phi_{C+}} e^{\mp i \phi_{C-}}
\]  

(4.51)

where we have dropped an overall singular coefficient. The exponent of the propagator for the electron operator is

\[
g_e = m - 1 + \frac{1}{p}
\]  

(4.52)

Notice that for the Halperin states, \(p = 1\) and \(g_e = m\). For the reversed sequence states with \(p = -1\), we get \(g_e = m - 2\). In particular, in the special case of the state \((3, 3, 1)\) the electron operators are

\[
\Psi_{e, \uparrow, \downarrow} \propto e^{-i \sqrt{2} \phi_{C+}} e^{\mp i \phi_{C-}}
\]  

(4.53)

with an exponent \(g_e = 3\). Once again, there is no contribution from the topological fields \(\phi_{T\pm}\) to the electron operator since here \(p = 1\).

Note that the electron operator of Eq. (4.53) can be regarded as bound state of a neutral fermion \(\exp(i \phi_{C-})\) and a charge 1 boson \(\exp(-i \sqrt{2} \phi_{C+})\).

We will now apply the results derived above to the computation of the tunneling exponents for the \(U(1) \times U(1)\) states.

1. Internal quasiparticle tunneling: For the \(U(1) \times U(1)\) states, the exponents for quasiparticle propagators were calculated in Eq. (4.43). In particular, for the primary \((m, m, n)\) states, with \(p = 1\) and \(m > n + 1\), we get

\[
g_{qp} = \frac{m}{m^2 - n^2}
\]  

(4.54)

Thus, for the \((3, 3, 1)\) state the exponent is \(g_{qp} = \frac{3}{8}\), the tunneling exponent is \(\alpha_{qp} = 2g_{qp} - 1 = -1/4\), and the reduction of the conductance follows the law \(G_{qp} = \frac{1}{4} e^2 / h \times \langle T_K / V \rangle^{1/4}\).

It is also possible to consider tunneling of composites of quasiparticles. In any given state there
are always several tunneling processes that are relevant, although quasiparticle tunneling is always the most relevant operator. In particular, in the $(3, 3, 1)$ state, other relevant internal tunneling processes involving the operator $W_Q$ (this process has dimension 1), as well as tunneling of neutral fermions, which is a marginal operator. Thus the effect of a weak perturbation, such as a weakly coupled gate, on an otherwise decoupled $(3, 3, 1)$ state is rather complex. Nevertheless, one still expects that as the tunneling matrix element grows bigger the system should flow to the weakly coupled $(3, 3, 1)$ states, with a rather non-trivial crossover in between.

2. Electron tunneling between identical $U(1)\times U(1)$ states:

The exponent for the electron propagator in all $U(1)\times U(1)$ states is $m - 1 + \frac{1}{p}$. Thus, the exponent for electron tunneling equals

$$\alpha_e = 2 \left( m - 1 + \frac{1}{p} \right) - 1 \quad (4.55)$$

In particular, for the $(3,3,1)$ state, the electron tunneling exponent is $g_e = 3$, the tunneling current scales like $I_c \propto V^5$, and the tunneling differential conductance follows the law $G_e \propto V^4$. For all primary $(m,m,n)$ states, $g_e = m$, the tunneling current behaves like $V^{2m-1}$ and the differential conductance scales like $V^{2(m-1)}$.

3. Electron tunneling into an $U(1)\times U(1)$ state from a Fermi liquid:

Finally, we consider the case of tunneling of electrons from a Fermi liquid into an edge state of an $U(1)\times U(1)$ state. The tunneling current now scales with an exponent $\alpha_t = g_e$. Thus, we find the general result

$$I_t \propto V^{m - 1 + \frac{1}{p}} \quad (4.56)$$

In particular, the tunneling current into a $(m,m,n)$ state scales like $V^m$, and the differential conductance scales like $V^{m-1}$. Finally, we note that the tunneling density of states for electrons into a general $U(1)\times U(1)$ edge as a function of frequency $\omega$ scales like $|\omega|^{m-2+\frac{1}{p}}$.

V. EDGE THEORY FOR $(M, M, M)$ STATES

In section II C we derived an effective theory for the bulk $(m, m, m)$ states. The effective Lagrangian of this theory, Eq. 2.43, is a sum of two decoupled terms: (i) a charge sector $b_\mu^\nu$ with an effective Lagrangian identical to that of a single layer Laughlin state at filling factor $\nu = 1/m$, and (ii) a $2 + 1$-dimensional Maxwell-like Lagrangian for the neutral sector, which is dual to the Lagrangian of a (massless) phase field $\theta$ Eq. 2.43. In that section we showed that the $(m, m, m)$ states have an $m$-fold topological degeneracy on the torus and constructed its excitation spectrum. In this section we will construct the theory of the edge states for the $(m, m, m)$ states by means of a line of reasoning analogous to what we did for the other FQH states in bilayers.

The decoupling of the effective low energy theory means that the correlation functions in the bulk are products of a factor for the charge sector and a factor for the neutral sector. In principle we expect the same factorization ("separation") to take place on the edges as well. The only caveat here is that, while the charge sector has a simple edge structure, identical to that of the Laughlin single layer states constructed by Wen [52], the neutral sector has gapless excitations in the bulk and as such its Hilbert space does not project to the edge. In other words, the effective theory of the edge, if it exists at all, is not a local chiral conformal field theory. That is not chiral is obvious since the Maxwell theory is not chiral. In the case of the Laughlin states, the conformal invariance of the edges is a consequence of the bulk being an incompressible chiral topological fluid. While the charge sector does satisfy these requirements, the neutral sector does not. We will see that nevertheless a theory of the edge can be constructed, but it is neither chiral nor local.

A. The charge sector of the edge $(m, m, m)$ states.

Since the charge sector is identical to that of a Laughlin state for a single layer system, the effective Lagrangian for the edge states for the charge sector of the $(m, m, m)$ states is just a theory of a chiral boson field $\phi_e$, i. e.

$$\mathcal{L}_{\text{charge}}[\phi_e] = \frac{m}{4\pi} \left[ \partial_x \phi_e \partial_t \phi_e - v_c (\partial_x \phi_e)^2 \right]$$

(5.1)

where we have taken the edge to be a straight line along the $x$ axis, and $v_c$ is the velocity of the edge of the charged sector. As usual $v_c$ depends on the confining electric field and on the interactions.

In a single layer FQH Laughlin state the chiral edge boson $\phi_e$ would obey the periodicity condition

$$\phi_e(x + 2\pi R, t) = \phi_e(x, t) + 2\pi R n$$

(5.2)

which follows from the single-valuedness of the electron state. Here $R = \sqrt{\nu} = 1/\sqrt{m}$ is the so-called compactification radius. It follows [52] that the only allowed states are the Laughlin quasiparticle $V_{qp} = \exp(i/\sqrt{m}\phi_e)$ and the electron operator $V_e = \exp(i\sqrt{m}\phi_e)$. However, in the case of the $(m, m, m)$ states the Hilbert space is larger. Indeed, in addition to the analogs of the Laughlin quasiparticle and electrons, it is possible to construct more states by gluing multivalued (twisted) operators of the charge sector with twisted states from the
neutral sector. These states will be the projection of the bulk vortices discussed in section 1C to the edge. The simplest of these states is created by the operator \( V_{1/2} \sim \exp(i/(2\sqrt{m})\phi_n) \) which has charge \( e/(2m) \) and statistics \( \pi/(4m) \). This state is made consistent (i. e. single valued) by a contribution from the neutral sector that we will discuss below.

B. The neutral sector of the edge \((m, m, m)\) states.

The effective edge theory of the neutral sector \( \theta \) on a region \( \Omega \) is constructed as follows. We begin by demanding that the total neutral current should vanish at the boundary \( \partial \Omega \). This condition implies that the field \( \theta \) must obey von Neumann boundary conditions at \( \partial \Omega \). Let us denote by \( \phi_n \) the restriction of the bulk neutral field \( \theta \) to the boundary \( \partial \Omega \), i.e. \( \theta|_{\partial \Omega} = \phi_n \), and enforce this condition in the path-integral of the neutral sector by writing

\[
1 = \int D\phi_n \, \delta(\theta|_{\partial \Omega} - \phi_n) = \int D\phi_n \, D\omega \, e^{i \int_{\partial \Omega} \omega (\theta - \phi_n)}
\]  

(5.3)

The partition function (in imaginary time) \( Z_{\text{neutral}} \) of the neutral sector on \( \Omega \) with Neumann boundary conditions on \( \partial \Omega \) takes the form

\[
Z_{\text{neutral}} = \int D\phi_n \, D\omega \, D\theta \left[ -\frac{g}{8\pi^2} \int d^3x \, (\partial_\mu \theta)^2 + i \int_{\partial \Omega} d^2x \omega (\theta - \phi_n) \right]
\]  

(5.4)

After some elementary algebra and by making use of the Neumann boundary condition the bulk neutral field can be integrated out to give an expression of the partition function in terms of the fields \( \phi_n \) and \( \omega \),

\[
Z_{\text{neutral}} = \int D\phi_n \, D\omega \, e^{-i \int_{\partial \Omega} d^2x \omega(x) \phi_n(x)} \left[ -\frac{2\pi^2}{g} \int_{\partial \Omega} d^2x \int_{\partial \Omega} d^2x' \, \omega(x) \, G(x - x')|_{\partial \Omega} \, \omega(x') \right]
\]  

(5.5)

where \( G(x - x') \) is the Green function on \( \Omega \) satisfying Neumann boundary conditions on \( \partial \Omega \), i.e.

\[
-\left[ \frac{1}{v_0^2} + v\nabla^2 \right] G(x - x') = \delta^3(x - x')
\]  

(5.6)

with the Neumann boundary condition,

\[
\partial_n G|_{\partial \Omega} = 0
\]  

(5.7)

where \( n \) is the direction normal to the boundary \( \partial \Omega \). It is straightforward to compute this Green function for a straight edge.

We can now integrate out the field \( \omega \) to find

\[
Z_n = \int D\phi_n \, e^{-\frac{g}{2\pi^2} \int_{\partial \Omega} \int_{\partial \Omega} \frac{\partial_\mu \phi_n(x) \partial_\mu \phi_n(x')}{4\pi|x - x'|}}
\]  

(5.8)

where \( \mu = 0, 1 \). To simplify the notation we have dropped the explicit dependence on the velocity \( v \). In deriving Eq. 5.8 we have also made use of the fact that, for a straight edge, the Neumann Green function restricted to the edge is twice the Green function in free space.

The effective action for the edge neutral field \( \phi_n \) is non-local and not chiral. This expression is a generalization of the familiar Caldeira-Leggett effective action or a point-like degree of quantum mechanical freedom coupled to an extended system [59]. More recently, Castro Neto, Chamon and Nayak considered a generalization of the Caldeira-Leggett problem to the case of an open Luttinger liquid, a Luttinger liquid coupled to a higher dimensional massless field [60]. Although similar in spirit the details and assumptions of ref. [60] lead to a form of the effective action different from Eq. 5.8. From a physical point of view, the non-locality of the effective action of Eq. 5.8 simply means that in a system with massless bulk excitations there is no separation between edge and bulk, and that edge excitations leak into the bulk.

The form of the effective action for the neutral edge field \( \phi_n \), Eq. 5.8, has a number of important consequences. It is easy to show that the vacuum expectation value of Wilson loop operators infinitesimally close to the boundary (by a small distance \( a \)) decay exponentially fast (in imaginary time). This results follows as a consequence of the non-locality of the action or, equivalently, from the existence of the massless bulk mode. Thus, any excitation that carries a non vanishing polarization (neutral charge) acquires an exponentially decaying factor from its neutral sector, even if the charge sector alone yields the familiar power law behavior of conformal field theory. The decay length depends on both the coupling constant \( g \propto 1/\ell_0 \), and the small distance \( a \). In any case, which is determined by the spatial extent of the bound state, for typical pair interactions also scales with the magnetic length \( \ell_0 \). Therefore, the characteristic decay length is of the order of the magnetic length itself (up to a non-universal numerical constant). This exponential decay takes place in both periodic and twisted sectors of the field \( \phi_n \). Thus, any bulk state with non-zero polarization becomes a massive excitation at the edge.

On the other hand, states with zero polarization in the bulk remain massless at the edge. This happens because the operator \( \exp(i\phi_n) \), which creates a unit of flux quanta at \( x \) by the boundary, has the correlation function

\[
\langle \exp(i\phi_n(x)) e^{-i\phi_n(x')} \rangle \propto e^{\sqrt{2v}\ell_0/|x - x'|}
\]  

(5.9)
where we have set \(|x - x'|^2 \equiv v^2|x_0 - x'_0|^2 + |x_1 - x'_1|^2\). This correlation function has no effect at long distances and/or long times (up to small corrections). Thus the states with zero polarization are the low energy Hilbert space.

We can now summarize the spectrum of edge states for the \((m, m, m)\) states. It contains the Hilbert space of edge states of the single layer Laughlin states at filling factor \(\nu = 1/m\). The only change is that the operator that creates and electron in a state with zero polarization now requires a factor that creates a flux quantum of the neutral gauge field \(b_\mu\). Thus there are two types of electron operators

\[
\psi_{e,\pm}^\dagger(x) \propto e^{i\sqrt{m}\phi_e(x)} e^{\pm i\phi_n(x)} \tag{5.10}
\]

where the sign \(\pm\) indicates the sign of the charge that couples to the gauge field \(b_\mu\). Their correlation functions are

\[
\langle \psi_{e,+(x_0, x_1)}\psi_{e,+(x'_0, x'_1)} \rangle \propto \frac{1}{(z - z')^m} x e^{\sqrt{2}\nu|x - x'|} \tag{5.11}
\]

where \(z = x_1 + iv_\epsilon x_0\). This result implies that the tunneling density of states to the edge of an \((m, m, m)\) state is the same as in the Laughlin state with the same filling factor, up to corrections which are analytic in the frequency, i.e. \(|\omega|^{m-1} + O(|\omega|^m)\). For all practical purposes this contribution is a negligibly small effect.

In contrast, the propagators for the naively defined single-layer electron operators decay exponentially fast with distance (and imaginary time). Consequently, the tunneling density of states into an edge electron of a given layer will show an energy gap for frequencies low compared with a crossover energy scale, of the order of the cyclotron frequency, and the usual power law at higher frequencies, assuming that at these frequencies these states are still well defined (which is unlikely). Bulk vortex states exhibit similar exponential decays at the boundary.

It is interesting to use this edge theory to reconstruct the wave function of the \((m, m, m)\) states. Following Read and Moore \([14]\) we can compute the ground state wave function of the \((m, m, m)\) states. One has to compute the correlation functions of \(N/2\) electron operators of the form \(\psi_{e,\pm} \propto e^{i\sqrt{m}\phi_e(z)} e^{i\phi_n(z)}\) for \(z = z_1, ..., z_{N/2}\), and \(N/2\) electron operators of the form \(\psi_{e,-} \propto e^{i\sqrt{m}\phi_e(z)} e^{-i\phi_n(z)}\) for \(z = w_1, ..., w_{N/2}\), times a neutralizing background for the charged sector. Here the \(z's\) and \(w's\) are the coordinates of the electrons on each layer in complex notation. It is simple to check that this procedure gives the same expression for the wave function as the one we derived in reference \([3]\).

\[
\Psi(z_1, ..., z_{N/2}, w_1, ..., w_{N/2}) = \prod_{i<j=1}^{N/2} (z_i - z_j)^m \prod_{i<j=1}^{N/2} (w_i - w_j)^m \prod_{i=1}^{N/2} \prod_{j=1}^{N/2} (z_i - w_j)^m
\]

\[
-\frac{B}{e} \left(\sum_{i=1}^{N/2} |z_i|^2 + \sum_{i=1}^{N/2} |w_i|^2\right) - \frac{\ell_0}{\sqrt{2}\nu} \left(\sum_{i<j=1}^{N/2} \frac{1}{|z_i - z_j|} + \sum_{i<j=1}^{N/2} \frac{1}{|w_i - w_j|} - \sum_{i,j=1}^{N/2} \frac{1}{|z_i - w_j|}\right)
\]  

This wave function differs from the conventional wave function for the \((m, m, m)\) state by the last factor. This extra factor has the same form as the Jastrow factor of the wave function for a superfluid. Here, as in the case of a superfluid, this factor is due to the contribution of the “phonons”, the linearly dispersing Goldstone boson \([6]\).

These results imply that tunneling experiments into the (unreconstructed) edge states of an \((m, m, m)\) bilayer state are likely to yield results analogous to the tunneling experiments into a single layer Laughlin state \([21]\) provided that there is a significant tunneling matrix element with the zero polarization electron state. Such tunneling experiments have not yet been done. However, recently I. B. Spielman and coworkers \([22]\) have reported experiments of uniform tunneling into the bulk of a \((1,1,1)\)
state which show a strongly resonant tunneling conduction. Early on Wen and Zee predicted that the gapless neutral mode would make resonant tunneling into the bulk possible.\cite{18}

VI. CONCLUSIONS

In this paper we have presented a theory for the edge states for spin polarized bilayer and spin-1/2 single layer FQHE systems. We assumed that the edge is sharp, un-reconstructed and clean.

We began by constructing the simplest possible theory for bulk states, compatible with the requirement of global gauge invariance, with the correct topological degeneracy on a torus, and with the smallest number of fundamental quasiparticles. Later on, we used this bulk theory to find the physics of its edge states. We found that the minimal edge theory thus derived has two propagating fields that couple to the external sources and represent the charge mode in each layer, and two topological non-propagating fields. These latter fields play the role of Klein factors, providing the right statistics for all the physical operators.

We studied in detail all the Jain-like states for these systems, whose primary states are the Halperin \((m_1, m_2, n)\) ones. In particular, we described the spectrum of operators for the symmetric states, i.e., the \(SU(2)\) states (the descendants of the \((m, m, m - 1)\) states), where the layer index is regarded as the spin index, and the general \(U(1) \times U(1)\) states. In all these cases we explicitly constructed the operators that create the quasiparticles (and quasiholes), charged and neutral bound states (including neutral fermionic states) and the electron operators. For the case of the \(SU(2)\) states we showed how the symmetry is realized in the spectrum and how it is promoted to a local \(su(2)_1\) spin current algebra.

We also calculated the propagators for the physical excitations. We showed that the charge operators determine the exponent of the time dependence of the propagators, and that the topological operators only contribute to the statistics. For the primary Halperin states, which have \(p = \pm 1\), the contribution of the topological operators to the statistics is a multiple of \(2\pi\), and thus can be ignored.

Afterwards we applied these results to compute the tunneling exponents for all cases in three different situations: internal tunneling of quasiparticles, tunneling of electrons between identical liquids and tunneling of electrons into a FQH fluid from an external Fermi liquid. As a general rule, we found that although the tunneling exponents are universal, in general they are not equal to the inverse of the filling factor.

In particular we computed the tunneling exponents for the spin singlet states that have been observed experimentally \cite{11}, whose filling fractions are \(\nu = 2/3, 2/5, 4/7, 4/9\). For instance, for electron tunneling between two \(2/3\) states we found that the tunneling current obeys Ohm’s law. The same result is obtained for tunneling of electrons into the \(2/3\) state from a Fermi liquid. In general, at the edge of an \(SU(2)\) state, at filling factor \(\nu = 2p/(2np + 1)\), the tunneling density of states for electrons obeys the law \(|\omega|^{|g_e - 1|}\), where \(g_e = n + \frac{1}{p}\). In contrast, the corresponding spin polarized state with the same filling factor has a tunneling density of states for electrons with the law \(|\omega|^{|\nu| - 1}\) (see ref. \cite{12}). Experiments of tunneling of electrons into the edge of spin singlet systems have not been done yet although they should be possible in higher density samples. Experiments of this type would be very useful to sort out the subtle correlations that give rise these FQH states.

Finally, we presented a theory of the edges for the \((m, m, m)\) states. In this case there is a bulk Goldstone boson which affects the edge physics. The effective theory of the edges of the \((m, m, m)\) states is a chiral boson for the charge mode (with the same compactification radius as the Laughlin states), and a non local (both in space and time) non-chiral theory for the neutral mode. The non-locality of the neutral sector is due to the existence of a massless Goldstone mode in the bulk. Physically, it means that in a system with massless bulk excitations there is no separation between edge and bulk, and that the edge excitations leak into the bulk. For the \((m, m, m)\) states the operator that creates an electron in a state with zero polarization is the same as the one for the Laughlin states at filling factor \(\nu = 1/m\), times a factor that creates a flux quantum of the neutral gauge field. This last factor will give a negligible contribution to the tunneling density of states. Therefore, tunneling experiments into the edges of an \((m, m, m)\) state should give the same results as for the corresponding single layer Laughlin state, provided there is a sufficiently large overlap with the unpolarized electron state.

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APPENDIX A: OPERATOR PRODUCT EXPANSIONS

In this Appendix we calculate the Operator Product Expansions (OPE’s) for the quasiparticle operators of the symmetric bilayer states. These OPE’s apply for the edge states of both the \(SU(2)\) and the \(U(1) \times U(1)\) FQH states.

In order to compute products of quasiparticle operators Eq. \ref{E2} of the form \(\Psi(x)_{qp,\uparrow} \Psi(y)_{qp,\downarrow}\) or \(\Psi(x)_{qp,\uparrow} \Psi(y)_{qp,\downarrow}\), we use the following result for the OPE’s of two vertex operators \ref{52}
exponentials in powers of $(0, therefore the operator : $e^{i\beta\phi(x)} : e^{i\beta'\phi(y)} := e^{i[\beta\phi(x)+\beta'\phi(y)]} : e^{-\beta\beta'\phi(x)\phi(y)}$

Hence, the product of two different quasiparticle operators is

$$
\Psi_{qp,\uparrow}(x)\Psi_{qp,\downarrow}(y) = \left(\frac{|x-y|}{a_0}\right)^{p^2\left(\left(m-1+\frac{1}{2}\right)^2 - n^2\right)^{n}} e^{(\phi_T(x)\phi_T(y))}
\times -i \sqrt{p} \left(\phi_{C+}(x) + \phi_{C+}(y)\right) i \frac{1}{p\sqrt{2(m-n-1+\frac{1}{2})}} (\phi_{C-}(x) + \phi_{C-}(y))
\times :e e^{1 + \frac{1}{2p} (\phi_T(x) + \phi_T(y))} : e \frac{1}{\sqrt{2p}} (\phi_T(x) - \phi_T(y)) : 
$$

and the product of two identical quasiparticle operators results in the following expansion

$$
\Psi_{qp,\uparrow}(x)\Psi_{qp,\uparrow}(y) = \left(\frac{|x-y|}{a_0}\right)^{p^2\left((m-1+\frac{1}{2})^2 - n^2\right)^{n}} e^{(\phi_T(x)\phi_T(y))}
\times -i \sqrt{p} \left(\phi_{C+}(x) + \phi_{C+}(y)\right) i \frac{1}{p\sqrt{2(m-n-1+\frac{1}{2})}} (\phi_{C-}(x) + \phi_{C-}(y))
\times :e e^{1 + \frac{1}{2p} (\phi_T(x) + \phi_T(y))} : e \frac{1}{\sqrt{2p}} (\phi_T(x) - \phi_T(y)) : 
$$

To construct the singlet and triplet states we need to compute the following products of operators, in the limit $y \to (x-a_0)$,

$$
\Psi_{qp,\uparrow}(x)\Psi_{qp,\downarrow}(y) + \Psi_{qp,\uparrow}(y)\Psi_{qp,\downarrow}(x) \approx :e -i \sqrt{p} \phi_{C+}(x) : e \sqrt{1 + \frac{1}{2p} \phi_T(x)} :
\times \left[ i \frac{1}{p\sqrt{2(m-n-1+\frac{1}{2})}} (y-x) \partial \phi_{C-}(x) \right.
\left. + i \frac{1}{\sqrt{2p}} (y-x) \partial \phi_T(x) \right] 
$$

where we have ignored the amplitude factors. Since the topological operators do not propagate : $\partial \phi_T e^{i\alpha \phi_T} = 0$, therefore the operator : $e \sqrt{2p} (y-x) \partial \phi_T (x)$ : acts like the identity operator. Hence, upon expanding the exponentials in powers of $(x-y)$, Eq. [1.4] becomes

$$
\Psi_{qp,\uparrow}(x)\Psi_{qp,\downarrow}(y) + \Psi_{qp,\uparrow}(y)\Psi_{qp,\downarrow}(x) \approx :e -i \sqrt{p} \phi_{C+}(x) : e \sqrt{1 + \frac{1}{2p} \phi_T(x)} :
\times \left[ i \frac{1}{p\sqrt{2(m-n-1+\frac{1}{2})}} (y-x) \partial \phi_{C-}(x) \right.
\left. + i \frac{1}{\sqrt{2p}} (y-x) \partial \phi_T(x) \right] 
$$
\[
\times \left[ \frac{1}{p \sqrt{2(m-n-1+1/p)}} (x-y) \partial \phi_{C-}(x) \pm \frac{1}{p \sqrt{2(m-n-1+1/p)}} (x-y) \partial \phi_{C-}(x) \right]
\]

(1.5)

Analogously, one can calculate

\[
\Psi_{qp,\uparrow}(x)\Psi_{qp,\uparrow}(y) \approx e^{i \sqrt{p} \phi_{C+}(x)} : e^{i \sqrt{p} \phi_{C-}(x)} : e^{i 2 \sqrt{1 + 1/(2p)} \phi_{T+}(x)} : e^{-i \sqrt{2p} \phi_{T-}(x)} :
\]

in the limit \( x \to y \).

We now apply the above results for the SU(2) states. In this case the singlet and triplet (with \( S_z = 0 \) operators, \( \Psi_{\text{singlet}} \) and \( \Psi_0 \) respectively, can be written as follows:

\[
\Psi_{\text{singlet}}(x) \approx e^{-i \sqrt{p} \phi_{C+}(x)} : e^{i \sqrt{p} \phi_{C-}(x)} : e^{i 2 \sqrt{1 + 1/(2p)} \phi_{T+}(x)} : e^{-i \sqrt{2p} \phi_{T-}(x)} : = 1
\]

(1.7)

respectively. The triplet operator, with \( S_z = 1 \), is given by Eq. (1.8).

Notice that the operator \( \Psi_0 \) and the one in Eq. (1.7) have the same dimension only if

\[
\Delta(: e^{-i \sqrt{2p} \phi_{C-}(x)} :) = 1
\]

(1.8)

Since \( : e^{-i \beta \phi(x)} : e^{-i \beta \phi(y)} : = \frac{1}{|x-y|^\beta} \) and \( \Delta(: e^{-i \beta \phi(x)} :) = \beta^2 / 2 \), therefore we obtain

\[
\Delta(: e^{-i \sqrt{2p} \phi_{C-}(x)} :) = \frac{1}{p}
\]

(1.9)

Therefore both operators have the same dimension only if \( p = 1 \). This is the only case where these three operators form a triplet. In particular, the operators

\[
E^\pm \equiv e^{\pm i \sqrt{2p} \phi_{C-}(x)} : , \quad H \equiv i \partial \phi_{C-}(x)
\]

(1.10)

generate the \( su_1(2) \) algebra only if \( p = 1 \), i.e. only for the state \( (m, m, m - 1) \) with \( m \) odd, of which the state \( (3, 3, 2) \) is a particular case.

We discuss now the leading irrelevant operators that contribute to the electron operators. For instance for the \( (3, 3, 2) \) state, we have to compute the product of two singlets \( \Psi_{\text{singlet}}(x) \) and one quasiparticle operator \( \Psi_{qp,\uparrow}(x) \). The part corresponding to \( \phi_{C-}(x) \) involves the OPE

\[
(\partial \phi_{C-}(z))^2 : e^{i \sqrt{2p} \phi_{C-}(w)} : \equiv \frac{\Delta}{(z-w)^2} : e^{i \sqrt{2p} \phi_{C-}(w)} : + \frac{i}{z-w} \partial_w e^{i \sqrt{2p} \phi_{C-}(w)}
\]

(1.11)

where

\[
\Delta = \frac{1}{4p}
\]

(1.12)
Finally we compute the spin flip operator,

\[ \Psi_{qp,\uparrow}(x) \Psi_{qp,\downarrow}(y) = \left( \frac{|x - y|}{a_0} \right)^{m-1} \left( \frac{p^2(2m - 1)}{2p^2} \right) e^{i \frac{\sqrt{p}}{2p} (\phi_C(x) - \phi_C(y))} e^{-i \frac{\sqrt{p}}{2p} (\phi_T(x) + \phi_T(y))} \]

In the limit \( y \to (x - a_0) \), the spin flip operator becomes

\[ \Psi_{qp,\uparrow} \Psi_{qp,\downarrow} \approx e^{-i \frac{\sqrt{p}}{2p} \phi_C(x)} e^{i \frac{\sqrt{p}}{2p} \phi_T(x)} \]

(1.13)

\[ (1.14) \]
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