Modeling the Decoherence of Spacetime

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The question of whether unobserved short-wavelength modes of the gravitational field can induce decoherence in the long-wavelength modes (“the decoherence of spacetime”) is addressed using a simplified model of perturbative general relativity, related to the Nordström-Einstein-Fokker theory, where the metric is assumed to be conformally flat. For some long-wavelength coarse grainings, the Feynman-Vernon influence phase is found to be effective at suppressing the off-diagonal elements of the decoherence functional. The requirement that the short-wavelength modes be in a sufficiently high-temperature state places limits on the applicability of this perturbative approach.

I. INTRODUCTION

Any theory of quantum cosmology, which treats the entire universe as a quantum-mechanical system, should predict classical behavior in the regimes where we know classical physics to be valid. In particular, a quantum theory of gravity should predict classical spacetime on macroscopic scales. One way of formulating the quantum mechanics of a closed system (e.g., the universe) is generalized quantum mechanics [1], in which probabilities are assigned to alternatives (outcomes of a series of observations) only if the quantum-mechanical interference between pairs of alternatives vanishes. This non-interference, known as decoherence, is a minimum condition for classical behavior.

As described in Sec. II, The physical process associated with decoherence [2] occurs by interaction of the system of interest with an environment about which no information is gained. It has been conjectured that long-wavelength features of the gravitational field may be made to decohere by their interactions with the short-wavelength modes of the field, thus allowing classical behavior of the gravitational field when observed on large scales.

This paper provides evidence for this phenomenon, the decoherence of spacetime. This differs from previous work [3,4] which used an additional field to obtain decoherence of the gravitational field in cosmological models, in that the decoherence examined here is induced in a field theory representing only gravity with no external matter field.

The theory considered in this work is a scalar field theory with a self-interaction similar to that of the metric in a perturbative expansion of the action for General Relativity (GR). As described in Sec. II, it is also the perturbative form of the Nordström-Einstein-Fokker theory [5] of a conformally flat metric, if the scalar field is defined proportional to the deviation from unity of some power of the conformal factor, such as the scaling of the volume element.

Section IV demonstrates the effects of splitting the scalar field into long-wavelength and short-wavelength parts, and classifies the terms in the action by the number of short-wavelength modes (SWMs). The trace over the SWMs is complicated by the presence of a cubic term in the action.

Temporarily removing the terms with one and three SWM factors leaves an action whose terms are all quadratic in the SWMs, or independent of them. Thus the trace over the SWMs can be performed explicitly, and this is done in Sec. V.

As described in Sec. V D, the perturbative corrections to the decoherence functional can cause elements (namely those representing quantum interference) which are finite in the non-interacting theory to be suppressed if the SWMs are in a thermal state whose temperature is sufficiently high. Then certain terms in the perturbation series can become large in the high-temperature limit, producing seemingly non-perturbative effects, including decoherence.

Section VI summarizes the results of Appendix D, that reinserting the terms with one and three SWM factors into the action has no substantial effect on the result of Sec. V. The terms linear in the SWMs can be removed by completion of the square to recover the original result. The terms cubic in the SWMs are examined in a perturbation
series, and each term is seen to be perturbatively finite, even in the high-temperature limit. So, according to the perturbative analysis, the effect of the cubic terms is to multiply the influence functional by a factor of order unity.

Section VII applies the properties of the decoherence functional found in Sec. V to a class of practical coarse grainings. First, in Sec. VII A, we show that the suppression factor enforcing decoherence involves Fourier components of the field whose temporal frequency is less than their spatial frequency. In Sec. VII B, we construct the coarse grainings of interest. Taking the temperature of the short-wavelength background to be that of the cosmic graviton of the field whose temporal frequency is less than their spatial frequency. In Sec. VII B, we construct the coarse grainings. First, in Sec. VII A, we show that the suppression factor enforcing decoherence involves Fourier components and not the long-wavelength nature of the system, by showing that coarse grainings referring only to short-wavelength features can be made to decohere by their interaction with the long-wavelength “environment”.

For reference, a summary of the important notation used in the body of the paper is provided in Appendix A.

II. ENVIRONMENT-INDUCED DECOHERENCE AND THE INFLUENCE PHASE

A. Generalized Quantum Mechanics

In a sum-over-histories quantum mechanics, the natural definition of the probability \( p(\alpha) \) that a certain alternative \( c_\alpha \) is realized is as the square of an amplitude, which is constructed via a sum of \( e^{i \text{action}} \) only over those histories which are in the class corresponding to that alternative. However, probabilities defined in this way will in general not obey the probability sum rule, i.e., the probability calculated for a class which is the union of two disjoint smaller classes \( \{c_\alpha = c_\alpha' \cup c_\alpha'' : c_\alpha \cap c_\alpha'' = \emptyset\} \) will in general not be the sum of the probabilities for those two classes \( \{p(\alpha) \neq p(\alpha') + p(\alpha'')\} \). Generalized quantum mechanics (GQM) addresses this problem by replacing the probability \( p(\alpha) \) of a single alternative with a decoherence functional \( D(\alpha, \alpha') \) defined on each pair of alternatives. When an exhaustive set of mutually exclusive classes \( \{c_\alpha\} \) has the property that \( D(\alpha, \alpha') = 0 \) for \( \alpha \neq \alpha' \), known as decoherence, one can then identify the diagonal elements of the decoherence functional (thought of as a matrix) as the probabilities \( p(\alpha) = D(\alpha, \alpha) \) for the alternatives in that set.

A sum-over-histories generalized quantum mechanics, as formulated by Hartle [1], requires three elements:

1. A definition of the fine-grained histories, \( \{h\} \), the most precise descriptions of the state of the system. For example, these may be particle paths \( q(t) \) or field histories \( \varphi(x) \) over spacetime,

2. A rule for partitioning those fine-grained histories into coarse-grained classes or alternatives \( \{c_\alpha\} \), and

3. A decoherence functional \( D[h, h'] \) defined on pairs of histories (fine- or coarse-grained).

The decoherence functional must obey the following four conditions:

"Hermiticity":

\[
D(\alpha', \alpha) = D(\alpha, \alpha')^*; \tag{2.1a}
\]

positivity of diagonal elements:

\[
D(\alpha, \alpha) \geq 0; \tag{2.1b}
\]

normalization:

\[
\sum_\alpha \sum_{\alpha'} D(\alpha, \alpha') = 1; \tag{2.1c}
\]

superposition: If \( \{\overline{\alpha}\} \) is a coarse graining constructed by combining classes in \( \{c_\alpha\} \) to form larger classes ("coarser graining"), i.e., \( \overline{\alpha} = \bigcup_{\alpha \in \alpha} c_\alpha \), the decoherence functional for \( \{\overline{\alpha}\} \) can be constructed from the one for \( \{c_\alpha\} \) by

\[
D(\overline{\alpha}, \overline{\alpha'}) = \sum_{\alpha \in \pi} \sum_{\alpha' \in \pi'} D(\alpha, \alpha'). \tag{2.1d}
\]

Note that the superposition law \( \text{2.1d} \) allows one to construct the coarse-grained decoherence functional from the fine-grained one, via
\[ D(\alpha, \alpha') = \sum_{h \in \alpha} \int_{h' \in \alpha'} D[h, h']. \]  

(2.2)

When the initial state is described by a normalized density matrix \( \rho \), and there is no specified final state, the fine-grained decoherence functional for a field \( \varphi \) with action \( S \) is given by

\[ D[\varphi, \varphi'] = \rho(\varphi_i, \varphi'_i) \delta(\varphi'_f - \varphi_f) e^{i(S[\varphi] - S[\varphi'])}. \]  

(2.3)

B. The Influence Phase

Decoherence in most physical systems is caused by a division into the “system” of interest, and an “environment” about which no information is gathered. In the language of generalized quantum mechanics, this means that the coarse graining is described by alternatives which refer only to the system variables. (See [6] for further details and a bibliography of prior work.)

If we make a division of \( \varphi \) into system variables \( \Phi \) and environment variables \( \phi \), split up the action into a \( \phi \)-independent piece \( S_\phi[\Phi] = S_{\Phi=0} \) and a piece \( S_E \) describing the environment and its interaction with the system:

\[ S[\varphi] = S_\phi[\Phi] + S_E[\phi, \Phi], \]  

(2.4)

and assume that the initial state is the product of uncorrelated states for the system and the environment:

\[ \rho(\varphi_i, \varphi'_i) = \rho_\Phi(\Phi_i, \Phi'_i) \rho_\phi(\phi_i, \phi'_i), \]  

(2.5)

then the decoherence functional for a coarse-graining which makes no reference to the environment variables (but is still fine-grained in the system variables) can be written

\[ D[\Phi, \Phi'] = \int D\Phi D\phi' D[\varphi, \varphi'] = \rho_\Phi(\Phi_i, \Phi'_i) \delta(\Phi'_f - \Phi_f) e^{i(S_\phi[\Phi] - S_\phi[\Phi'] + W[\Phi, \Phi'])} \]  

(2.6a)

where

\[ e^{iW[\Phi, \Phi']} = \int D\phi D\phi' \rho_\phi(\phi_i, \phi'_i) \delta(\phi'_f - \phi_f) e^{i(S_E[\phi, \Phi] - S_E[\phi', \Phi'])}. \]  

(2.6b)

\( W[\Phi, \Phi'] \) is called the Feynman-Vernon influence phase [3]; if the influence functional \( e^{iW} \) becomes small for \( \Phi \neq \Phi' \), the “off-diagonal” parts of \( D[\Phi, \Phi'] \) will be suppressed, causing alternatives defined in terms of \( \Phi \) to decohere [3].

III. A SCALAR FIELD THEORY MODELING THE GRAVITATIONAL INTERACTION

The goal of this work is to perform the division described in Sec. III on a theory modeling vacuum gravity, with the short-wavelength modes acting as the environment which induces decoherence in the long-wavelength system. The idea behind this is that for coarse grainings which deal only with averages over sufficiently large regions of spacetime, gravity should behave classically, and thus such coarse grainings should decohere.

In order to model the self-interaction of the gravitational field without dealing with the gauge-fixing and other complications arising from the tensor nature of the metric in General Relativity, we will consider a toy model of a single scalar field \( \varphi \) moving on \( D + 1 \)-dimensional Minkowski space [3] with the action

\[ S[\varphi] = -\frac{1}{2} \int d^{D+1}x [1 - (2\pi)^{D/2} \xi \varphi] (\nabla \varphi)^2. \]  

(3.1)

\(^1\) Throughout this paper, we will use units in which \( \hbar = 1 = c. \)

\(^2\) As discussed in Sec. IVB 1, this should be a reasonable assumption in a cosmological scenario if the length scales in the problem do not approach the Hubble scale \( cH_0^{-1}. \) Note also that an analogous scalar field model, with a different Ricci-flat background metric \( g_{ab} \) in place of the Minkowski metric \( \eta_{ab} \), can also be constructed. That action is obtained by a perturbative expansion of the Einstein-Hilbert action when the metric is required to be conformally related to the background: \( g_{ab} = \Omega^2 \eta_{ab}. \)
This scalar field theory is a promising toy model for perturbative general relativity for two reasons. First, consider the Einstein-Hilbert action

$$S = -\frac{1}{16\pi G} \int d^{D+1}x \sqrt{|g|} R$$

and define the difference $\gamma_{ab} = g_{ab} - \eta_{ab}$ between the actual metric and a flat background metric. If we perform a perturbative expansion of $(3.2)$ in powers of $\gamma_{ab}$, the lowest order terms have two powers of $\gamma_{ab}$ and two derivatives and describe free wave propagation, while the first self-interaction terms have three powers of $\gamma_{ab}$ and two derivatives. If we replace the tensor-valued $\gamma_{ab}$ by a scalar field $\phi$, the most general action which has this form is $(3.1)$.

Second, $(3.1)$ can also be obtained by perturbative expansion of the action for the Nordström–Einstein–Fokker theory of gravity, which is given by the Einstein-Hilbert action $(3.2)$ restricted to conformally flat metrics

$$g_{ab} = \Omega^2 \eta_{ab}.$$  

(3.3)

The classical equation for $\Omega$ is obtained by varying the action

$$S[\Omega] = -\frac{4D}{D-1} \int d^{D+1}x \left( \nabla \Omega^{D-1} \right)^2.$$  

(3.4)

In $3 + 1$ spacetime dimensions ($D = 3$), this corresponds to the statement that the conformal factor $\Omega$ behaves as a free massless scalar field.

However, it may be that a quantity defining a useful coarse graining is proportional to some power $\Omega^\nu$ of the conformal factor. For example, the volume of a spacetime region $S$ in the metric $(3.3)$ will be

$$V = \int_S d^{D+1}x \sqrt{|g|} = \int_S d^{D+1}x \Omega^{D+1},$$

(3.5)

since the metric determinant in this theory is given by $-(\Omega^2)^{D+1}$. It is thus useful to express the theory in terms of

$$\zeta = \Omega^\nu - 1$$

(3.6)

so that $\zeta = 0$ corresponds to no deviation from the flat background metric.

Then we obtain a self-interacting action, which when expanded perturbatively for $\zeta \ll 1$ becomes

$$S[\zeta] \approx -\frac{D(D-1)}{\nu^2} \int d^{D+1}x \left( 1 - 2\nu + 1 - \frac{D}{\nu}\zeta \right) (\nabla \zeta)^2.$$  

(3.7)

Rescaling to define

$$\varphi = \frac{\zeta}{2\nu \ell_p} \sqrt{\frac{D(D-1)}{2\pi}},$$

(3.8)

and

$$\ell = \ell_p \sqrt{\frac{2(2\nu + 1 - D)}{D(D-1)(2\pi)^{D-1}}}.$$  

(3.9)

where $\ell_p = G^{1/2}$ is the Planck length, we obtain the action $(3.1)$ for the scalar field $\varphi$. Note that if $\nu = (D-1)/2$, then the coupling constant $\ell$ vanishes, and we have a free theory as dictated by $(3.4)$.

---

3Because quantities like the inverse metric $g^{ab}$ cannot be expressed in closed form in terms of $\gamma_{ab}$, the action will become an infinite series of terms containing increasing powers of $\gamma_{ab}$. One can treat the theory perturbatively, but it is not clear that $\gamma_{ab}$ is the most physically relevant quantity in which to carry out that expansion.

4The fact that $\zeta$ runs from $-1$ to $\infty$ (for positive $\nu$) as $\Omega$ runs from 0 to $\infty$ should not be cause for alarm, as the choice of $\zeta$ rather than $1 + \zeta$ is tailored to expanding the action for small $\zeta$.

5It is a common phenomenon that a theory which is free in one set of variables may exhibit interactions and the resulting decoherence when described in terms of another set. The theories considered in $(3.10)$ are of this sort, as are the linear oscillator models of $(11)$ when expressed in terms of normal modes.
If we take \( \nu = D + 1 \) in our definition of \( \zeta \), the volume of our spacetime region, when offset and rescaled by a reference volume \( V_0 \) (which we take to be the volume \( \int_\mathcal{S} d^{D+1}x \) of the region in the background metric) will be

\[
\frac{V - V_0}{V_0} = \frac{\int_\mathcal{S} d^{D+1}x \left( \sqrt{|g|} - 1 \right)}{\int_\mathcal{S} d^{D+1}x} = \frac{\int_\mathcal{S} d^{D+1}x \zeta}{\int_\mathcal{S} d^{D+1}x} = \langle \zeta \rangle_\mathcal{S} = \frac{(D + 1)(2\pi)^{D/2}}{D + 3} \langle \ell \varphi \rangle_\mathcal{S},
\]

(3.10)
a field average of \( \varphi \) over the region \( \mathcal{S} \).

Note that the quantization of the action \( (4.1) \), obtained from the Einstein-Hilbert action restricted to conformally flat metrics, is not the same as quantization of the conformal modes of general relativity, since we have made the restriction at the level of the action, producing a theory which has no other degrees of freedom. This scalar theory of gravity is, however, a geometric theory similar to GR, with a similar self-interaction, but without many of the complications of the full theory.

**IV. DIVIDING THE MODES**

We want to make a division of the field \( \varphi \) appearing in the action

\[
S[\varphi] = \int_{-T/2}^{T/2} dt \, L(t)
\]

(4.1a)

\[
L(t) = \frac{1}{2} \int d^Dx \left[ 1 - (2\pi)^{D/2} \ell \varphi \right] [\varphi^2 - (\nabla \varphi)^2]
\]

(4.1b)

into long-wavelength modes (LWMs), labelled by \( \Phi \), to act as the “system” and short-wavelength modes (SWMs), labelled by \( \phi \), to act as the “environment”. For reasons of mathematical convenience, we first make this division only in the spatial directions. First we reexpress the Lagrangian in terms of the Fourier transform

\[
\varphi_k(t) = \int \frac{d^Dx}{(2\pi)^{D/2}} e^{-ik \cdot x} \varphi(x, t);
\]

(4.2a)

\[
\varphi(x, t) = \int \frac{d^Dk}{(2\pi)^{D/2}} e^{ik \cdot x} \varphi_k(t)
\]

(4.2b)
to get

\[
L(t) = \frac{1}{2} \int d^Dk \left( |\varphi|^2 - k^2 |\varphi|^2 \right) - \frac{\ell}{2} \int d^Dk_1 d^Dk_2 d^Dk_3 \delta^D(k_1 + k_2 + k_3)(\varphi_1 \varphi_2 \varphi_3 + k_2 \cdot k_3 \varphi_1 \varphi_2 \varphi_3),
\]

(4.3)

where we have streamlined the notation by writing \( \varphi = \varphi_k \), \( \varphi_1 = \varphi_{k_1} \), etc.

We define the long-wavelength sector \( \mathcal{L} = \{ q \mid q < k_c \} \) and the short-wavelength sector \( \mathcal{S} = \{ k \mid k > k_c \} \), and define the long- and short-wavelength modes by

\[
\Phi_q(t) = \varphi_q(t) \Theta(k_c - q) \quad \text{LWM}
\]

(4.4a)

\[
\phi_k(t) = \varphi_k(t) \Theta(k - k_c) \quad \text{SWM}
\]

(4.4b)

so that the part of the Lagrangian quadratic in \( \varphi \) becomes

\[
\frac{1}{2} \int d^Dx [\varphi^2 - (\nabla \varphi)^2] = \frac{1}{2} \int_\mathcal{L} d^Dq \left( |\Phi_q|^2 - q^2 |\Phi_q|^2 \right) + \frac{1}{2} \int_\mathcal{S} d^Dk \left( |\phi_k|^2 - k^2 |\phi_k|^2 \right).
\]

(4.5)

Taking into account the fact that \( \varphi(x) \) is real, which means \( \varphi_{-k} = \varphi_k^* \), or \( \Phi_{-q} = \Phi_q^* \) and \( \phi_{-k} = \phi_k^* \), we can write any expression using only half of the complex modes, which define the other half by complex conjugation. We define \( \mathcal{L}/2 \) and \( \mathcal{S}/2 \) as arbitrarily chosen halves of \( \mathcal{L} \) and \( \mathcal{S} \) so that \( \{ \Phi_q | q \in \mathcal{L}/2 \} \) and \( \{ \phi_k | k \in \mathcal{S}/2 \} \) between them define \( \varphi_k \). This makes the noninteracting (\( \ell = 0 \)) action

\[
\frac{1}{2} \int d^Dx [\varphi^2 - (\nabla \varphi)^2] = \int_{\mathcal{L}/2} d^Dq \left( |\Phi_q|^2 - q^2 |\Phi_q|^2 \right) + \int_{\mathcal{S}/2} d^Dk \left( |\phi_k|^2 - k^2 |\phi_k|^2 \right).
\]

(4.6)
which is the action of a set of uncoupled harmonic oscillators. The interaction terms can be classified by the number of factors of the “environment” field $\phi$ to give

$$L[\varphi] = L[\phi, \Phi] = L_0[\Phi] + L_0[\phi] + \ell L_0[\phi, \Phi] + \ell L_{\phi\phi}[\phi, \Phi] + \ell L_{\phi\Phi}[\phi],$$

where

$$L_0[\Phi] = \int d^Dq \left( \left| \dot{\Phi} \right|^2 - q^2 |\Phi|^2 \right),$$

$$L_0[\phi] = \int d^Dk \left( \left| \dot{\phi} \right|^2 - k^2 |\phi|^2 \right),$$

$$L_\phi[\phi, \Phi] = -\frac{1}{2} \int d^Dq d^Dq_1 d^Dq_2 \delta^D(k + q_1 + q_2) \left[ \phi \dot{\Phi}_1 \ddot{\Phi}_2 + 2 \dot{\phi} \dot{\Phi}_1 \dot{\Phi}_2 + (2k + q_1) \cdot q_2 \Phi \dot{\Phi}_1 \Phi \dot{\Phi}_2 \right]$$

$$L_{\phi\phi}[\phi, \Phi] = -\frac{1}{2} \int d^Dq d^Dk_1 d^Dk_2 \delta^D(k_1 + k_2 + q) \left[ \Phi \dot{\phi}_1 \dot{\phi}_2 + 2 \dot{\Phi} \dot{\phi}_1 \dot{\phi}_2 + (2q + k_1) \cdot k_2 \Phi \dot{\phi}_1 \Phi \dot{\phi}_2 \right]$$

$$L_{\phi\Phi}[\phi, \Phi] = -\frac{1}{2} \int d^Dk_1 d^Dk_2 d^Dk_3 \delta^D(k_1 + k_2 + k_3) \left( \phi_1 \dot{\phi}_2 \dot{\phi}_3 + k_2 \cdot k_3 \Phi \dot{\phi}_1 \Phi \dot{\phi}_3 \right).$$

Under this division of the field $\varphi$, we can perform the division of the decoherence functional described by (2.6), with

$$S_E[\phi, \Phi] = S_0[\phi] + \ell S_0[\phi, \Phi] + \ell S_{\phi\phi}[\phi, \Phi] + \ell S_{\phi\Phi}[\phi].$$

V. THE QUADRATIC TERMS

We need to evaluate the path integral (2.6) for the influence functional, but the presence of a term (4.8e) in $S_0$, which is cubic in $\phi$ prevents us from doing that in closed form. We might be led then to treat the problem perturbatively, with the terms from (4.9) first-order in $\ell$ providing a correction to the answer obtained using the action $S_0$. However, to zeroth order in $\ell$, the theory if free and thus $e^{iW} = 1$. But if we are to examine situations where $e^{iW} \ll 1$, this is only possible if the perturbative correction is also of order unity, which cannot happen in a fully perturbative calculation. We will see in Sec. V.D one set of circumstances where we can perform some but not all of the calculations perturbatively and still obtain $e^{iW} \ll 1$. In the meantime, we will consider those parts of the action for which the integral can be done non-perturbatively, and add in the effects of the cubic term later, using a perturbative treatment which pays careful attention to the issues to be raised in Sec. V.D.

Since the parts of the action defined in (4.8a) and (4.8d) are quadratic in $\phi$, it would be possible to do the path integrals in (2.6) explicitly if the action $S_E$ included only those terms. Thus we turn our attention for the time being to the modified influence functional

$$e^{iW_{0+\phi\phi}[\phi, \Phi']} = \int \mathcal{D}\phi \mathcal{D}\phi' \rho_0(\phi_i, \phi'_j) \delta(\phi'_j - \phi_f) e^{i(S_0[\phi, \Phi] - S_{0+\phi\phi}[\phi, \Phi'])}.$$

We will find an upper limit

$$\left| e^{iW_{0+\phi\phi}[\phi, \Phi']} \right| \leq (1 + E^2[\Delta \Phi])^{-1/4},$$

on the absolute value of the influence functional in the absence of the terms linear and cubic in the short-wavelength part $\phi$ of the field. We will show in Sec. V.D and Appendix D that restoring those terms does not qualitatively change the limit (5.52).

A. A vector expression

The Lagrangian $L_{0+\phi\phi}$ can be written, using $\hat{\phi}_k^* = \phi_{-k}$, in the suggestive form
\[
L_{0+\phi\phi}[\phi, \Phi] = L_0[\phi] + \ell L_{\phi\phi}[\phi, \Phi] = \frac{1}{2} \int d^Dk_1 d^Dk_2 \left\{ \phi_1^* [\delta^D(k_1 - k_2) - \ell \Phi_{1-2}] \phi_2 - \frac{d}{dt}(\ell \phi_1^* \Phi_{1-2} \phi_2) \right\}
\]

where

\[
k_{12}^2 = -(q_{1-2}) \cdot (-k_1) - (-k_1) \cdot k_2 - (q_{1-2}) = k_1^2 + k_2^2 - k_1 \cdot k_2,
\]

and we have defined

\[
q_{1\pm 2} = k_1 \pm k_2, \quad \Phi_{1\pm 2} = \Phi_{q_{1\pm 2}}.
\]

We would like to write (5.2) as a matrix expression in terms of a vector which describes the short-wavelength modes \(\{\phi_k\}\). However, the reality conditions \(\phi_{-k} = -\phi_k^*\), which cause the measure for the integral over independent modes to be

\[
d\phi \propto \prod_{k \in S/2} d\phi_k^R d\phi_k^A,
\]

necessitate some caution.

If we express the modes \(\{\phi\}\) as a complex vector \(\phi_{\text{ex}} = \{\phi_k| k \in S\}\), on a vector space we call, with a slight abuse of notation, \(C^S\), with inner product

\[
w_{\text{ex}}^i w_{\text{ex}}^j = \int_S d^Dk w_{\text{ex}}^i w_{\text{ex}}^j,
\]

the Lagrangian (5.2) can be written

\[
L_{0+\phi\phi} = \frac{1}{2} \left[ \phi_{\text{ex}}^i m_{\text{ex}} \phi_{\text{ex}}^i - \phi_{\text{ex}}^i \varpi_{\text{ex}} \phi_{\text{ex}}^i + \frac{d}{dt}(\phi_{\text{ex}}^i \dot{m}_{\text{ex}} \phi_{\text{ex}}^i) \right]
\]

where \(m_{\text{ex}}\) and \(\varpi_{\text{ex}}\) are hermitian matrices acting on \(C^S\) with the form

\[
(m_{\text{ex}})_{k_1 k_2} = \delta^D(k_1 - k_2) - \ell \Phi_{1-2}
\]

(5.8a)

\[
(\varpi_{\text{ex}})_{k_1 k_2} = \delta^D(k_1 - k_2) k_1^2 - \ell (k_2^2 \Phi_{1-2} + \Phi_{1-2}).
\]

(5.8b)

Unfortunately, the components of \(\phi_{\text{ex}}\) represent twice as many degrees of freedom as are integrated over in \(5.5\). This means that a path integral over all the components would have to include the factor

\[
\prod_{k \in S/2} \delta [\phi_{-k} - \phi_k].
\]

On the other hand, a complex vector \(\phi_+ = \{\phi_k| k \in S/2\}\) in a space \(C^{S/2}\) with inner product

\[
w_{\text{ex}}^i w_{\text{ex}}^j = \int_{S/2} d^Dk w_{\text{ex}}^i w_{\text{ex}}^j,
\]

would completely specify the unique modes of \(\phi\), but (5.2) is not conveniently expressed in terms of \(\phi_+\). To see this, consider the velocity term

\[
\frac{1}{2} \int_S d^Dk_1 d^Dk_2 \phi_1^* [\delta^D(k_1 - k_2) - \ell \Phi_{1-2}] \phi_2
\]

\[
= \frac{1}{2} \int_{S/2} d^Dk_1 d^Dk_2 \left\{ \phi_1^* [\delta^D(k_1 - k_2) - \ell \Phi_{1-2}] \phi_2 + \phi_1 [\delta^D(k_1 - k_2) - \ell \Phi_{2-1}] \phi_2^* - \phi_1^* \ell \phi_{1+2} \phi_2^* - \phi_1 \ell \phi_{1+2} \phi_2 \right\}
\]

\text{or, more precisely, integral operators with these kernels}
FIG. 1. The addition of momenta $k_1, k_2 \in S/2$ can produce $k_1 + k_2 \in \mathcal{L}$. The long-wavelength (low-momentum) region $\mathcal{L}$ is shaded vertically. The short-wavelength (high-momentum) region $S$ is shaded diagonally in one direction or the other. The right half of $S$, shaded diagonally up and to the right, is $S/2$. In $D > 1$, we see that it is possible to add two “large” momenta on the right side of the origin ($k_1, k_2 \in S/2$) to get a “small” momentum ($k_1 + k_2 \in \mathcal{L}$).

Although the first two terms can be written as

$$\int_{S/2} d^D k_1 d^D k_2 \dot{\phi}_1^* \left[ \delta^D(k_1 - k_2) - \ell \Phi_{1+2} \right] \dot{\phi}_2 = \dot{\phi}_+^* m_+ \dot{\phi}_+$$

(5.12)

(where $m_+$ is the restriction of $m_{\text{ex}}$ to $\mathbb{C}^{S/2}$), the last two give

$$\text{Re} \left( \int_{S/2} d^D k_1 d^D k_2 \dot{\phi}_1 \left[ \delta^D(k_1 - k_2) - \ell \Phi_{1+2}^* \right] \dot{\phi}_2 \right),$$

(5.13)

which cannot be written in terms of the complex vector $\phi_+$ and its adjoint $\dot{\phi}_+^*$ without using the transpose $\phi_+^\text{tr}$ or the complex conjugate $\phi_+^*$. If $D = 1$, this is not a problem, since $k_1, k_2 \in S/2$ implies $k_1 + k_2 \notin \mathcal{L}$ and hence $\Phi_{1+2} = 0$. However, it is possible in $D > 1$ to have $k_1 + k_2 \in \mathcal{L}$ even when $k_1, k_2 \in S/2$, as illustrated in Fig. 1.

The most useful approach is to define a real vector

$$\phi = \{ \sqrt{2} \phi_R^k, \sqrt{2} \phi_I^k | k \in S/2 \}$$

(5.14)

in the space $\mathbb{R}^{S/2} \otimes \mathbb{R}^{S/2} = \mathbb{R}^{S/2 \oplus S/2}$ with inner product

$$w^\text{tr} v = 2 \int_{S/2} d^D k (w_R^k v_R^k + w_I^k v_I^k).$$

(5.15)

A straightforward calculation shows that it is possible to write (5.7) as

$$L_{0+\phi\phi} = \frac{1}{2} \left[ \dot{\phi}^\text{tr} m \dot{\phi} - \dot{\phi}^\text{tr} \varpi \phi + \frac{d}{dt} (\dot{\phi}^\text{tr} \dot{m} \phi) \right],$$

(5.16)

where $m$ and $\varpi$ are real symmetric matrices on $\mathbb{R}^{S/2 \oplus S/2}$, given by

$$m = \begin{pmatrix}
\{ \delta^D(k_1 - k_2) - \ell (\Phi_{1-2}^R + \Phi_{1+2}^R) \} & \{- \ell (\Phi_{1-2}^I + \Phi_{1+2}^I) \} \\
\{- \ell (\Phi_{1-2}^R + \Phi_{1+2}^R) \} & \{ \delta^D(k_1 - k_2) - \ell (\Phi_{1-2}^I - \Phi_{1+2}^I) \}
\end{pmatrix}$$

(5.17)

$$\varpi = \begin{pmatrix}
\varpi_{UL}^{k_1 k_2} & \varpi_{UR}^{k_1 k_2} \\
\varpi_{UL}^{k_2 k_1} & \varpi_{UR}^{k_2 k_1}
\end{pmatrix}$$

(5.18)

and
\[ \begin{align*}
\omega^\text{ul}_{k_1 k_2} &= \delta^D(k_1 - k_2)k_1^2 - \ell(\Phi^-_{1-2}k_1^2 + \Phi^R_{1-2}k_{1-2}^2 + \Phi^-_{1+2}k_{1+2}^2) \\
\omega^\text{ur}_{k_1 k_2} &= -\ell(-\Phi^-_{1-2}k_1^2 + \Phi^R_{1+2}k_{1-2}^2 - \Phi^-_{1+2}k_{1+2}^2) \\
\omega^\text{ll}_{k_1 k_2} &= -\ell(\Phi^R_{1-2}k_1^2 + \Phi^R_{1+2}k_{1-2}^2 + \Phi^-_{1-2} + \Phi^-_{1+2}) \\
\omega^\text{lr}_{k_1 k_2} &= \delta^D(k_1 - k_2)k_1^2 - \ell(\Phi^R_{1-2}k_1^2 - \Phi^-_{1+2}k_{1-2}^2 + \Phi^-_{1-2}k_{1+2}^2); \\
\end{align*} \]

where \( \phi \) is written as

\[ \phi = \sqrt{2} \left( \frac{\phi^R_k}{\phi^-_k} = \sqrt{2} \left( \begin{pmatrix} \phi^R_k \\ \phi^-_k \end{pmatrix} \right) \right). \] (5.19)

### B. The Propagator

We now have a workable vector expression for the path integral (5.1):

\[ e^{iW_{0+\phi^2}[\Phi, \Phi']} = \int D\phi D\phi' \rho_\phi(\phi_i, \phi'_i)\delta(\phi_j - \phi_f) e^{i(S_{0+\phi^2}[\phi, \Phi] - S_{0+\phi^2}[\phi', \Phi'])} \]

\[ = \int d\phi_i d\phi'_i d\phi_f \rho_\phi(\phi_i, \phi'_i)K_{0+\phi^2}(\phi_f | \phi_i; \Phi)K^*_{0+\phi^2}(\phi_f | \phi'_i; \Phi') \] (5.20)

where

\[ K_{0+\phi^2}(\phi_f | \phi_i; \Phi) = \int_{\phi_f, \phi_i} D\phi e^{iS_{0+\phi^2}[\phi, \Phi]} \] (5.21)

is the propagator for the quadratic action. It is useful to write

\[ K_{0+\phi^2}(\phi_f | \phi_i; \Phi) = e^{i\mathcal{H}(\phi_f^T \mathcal{m}_f \phi_f - \phi_i^T \mathcal{m}_i \phi_i) K\left( \phi_f \frac{T}{2} \middle| \phi_i - \frac{T}{2} \right)} \] (5.22)

where

\[ K(\phi_f t_f | \phi_i t_i) = \int_{\phi_f, \phi_i} D\phi e^{ij \int_{t_i}^{t_f} dt |\phi(t)^T \mathcal{m}(t)\phi(t) - \phi(t)^T \mathcal{w}(t)\phi(t)|} \] (5.23)

is the propagator for a simple harmonic oscillator with time-dependent matrices \( \mathcal{m}(t) \) and \( \mathcal{w}(t) \) in place of \( m \) and \( m\omega^2 \). [The dependence on \( \Phi \) is now implicit in the time dependence of \( m(t) \) and \( \mathcal{w}(t) \), given by (5.17–5.18).]

This propagator can be found explicitly to be \[12\]

\[ K(\phi_f t_f | \phi_i t_i) = \frac{1}{\sqrt{\det(2\pi i \mathcal{C}(t_f | t_i))}} \exp \left[ \frac{i}{2} \left( \begin{array}{c} \phi_f \\ \phi_i \end{array} \right)^T \left( \begin{array}{cc} \mathcal{C}^{-1}(t_f | t_i) \mathcal{B}(t_f | t_i) & -\mathcal{C}^{-1}(t_f | t_i) \\
-\mathcal{C}^{-1}(t_f | t_i)^T & \mathcal{A}(t_f | t_i) \mathcal{C}^{-1}(t_f | t_i) \end{array} \right) \left( \begin{array}{c} \phi_f \\ \phi_i \end{array} \right) \right], \] (5.24)

where

\[ \mathcal{B}(t_f | t_i) = \sum_{n=0}^{\infty} \left( \prod_{k=1}^{n} \int_{t_i}^{t_{k-1}} dt_k \int_{t_{k-1}}^{t_k} dt_{k} \right) \prod_{k=n}^{\infty} [ -m^{-1}(\ell_k) \mathcal{w}(t_k) ] \] (5.25a)

\[ \mathcal{C}(t_f | t_i) = -\frac{d \mathcal{B}(t_f | t_i)}{dt_f} \mathcal{w}^{-1}(t_f) \] (5.25b)

\[ \mathcal{A}(t_f | t_i) = -m(t_i) \frac{d \mathcal{C}(t_f | t_i)}{dt_i} \] (5.25c)

are the solutions to
One simple choice is a thermal state with temperature 1 to the action (4.9) would couple the short- and long-wavelength modes, preventing the separation (2.5) of the initial harmonic oscillator of frequency $\Omega_0$.

This means that (5.20) becomes

$$
\frac{d\mathfrak{A}(t_f|t_i)}{dt_f} = \mathfrak{A}(t_f|t_i)\mathfrak{C}^{-1}(t_f|t_i)\mathfrak{B}(t_f|t_i) - \mathfrak{C}^{-1}(t_f|t_i)^\text{tr} m^{-1}(t_f),
$$

(5.26a)

$$
\frac{d\mathfrak{B}(t_f|t_i)}{dt_f} = -\mathfrak{C}(t_f|t_i)\varpi(t_f)
$$

(5.26b)

$$
\frac{d\mathfrak{C}(t_f|t_i)}{dt_f} = \mathfrak{B}(t_f|t_i)m^{-1}(t_f),
$$

(5.26c)

or

$$
\frac{d\mathfrak{A}(t_f|t_i)}{dt_i} = \varpi(t_i)\mathfrak{C}(t_f|t_i)
$$

(5.27a)

$$
\frac{d\mathfrak{B}(t_f|t_i)}{dt_i} = -m^{-1}(t_i)\left[\mathfrak{A}(t_f|t_i)\mathfrak{C}^{-1}(t_f|t_i)\mathfrak{B}(t_f|t_i) - \mathfrak{C}^{-1}(t_f|t_i)^\text{tr}\right]
$$

(5.27b)

$$
\frac{d\mathfrak{C}(t_f|t_i)}{dt_i} = -m^{-1}(t_i)\mathfrak{A}(t_f|t_i),
$$

(5.27c)

with the initial conditions $\mathfrak{A}(t|t) = 1 = \mathfrak{B}(t|t)$ and $\mathfrak{C}(t|t) = 0$.

These exact expressions are expanded to first order in $\ell$ in Sec. 3 of Appendix B using the values of $m(t)$ and $\varpi(t)$ given by (5.17) and (5.18), respectively.

Given the expression (5.24) for the time-dependent propagator, (5.22) becomes

$$
\kappa_{0+\phi}(\phi_f|\phi_i; \Phi) = \frac{1}{\sqrt{\text{det}(2\pi\mathcal{C}[\Phi])}} \exp \left[\frac{i}{2} \left(\begin{array}{c} \phi_f \\ \phi_i \end{array}\right)^\text{tr} \left(\begin{array}{cc} B[\Phi] & -C[\Phi] \\ -C[\Phi]^\text{tr} & A[\Phi] \end{array}\right) \left(\begin{array}{c} \phi_f \\ \phi_i \end{array}\right)\right],
$$

(5.28)

where

$$
A[\Phi] = \mathfrak{A}\left(\frac{T}{2} \mid -\frac{T}{2}\right) \mathfrak{C}^{-1}\left(\frac{T}{2} \mid -\frac{T}{2}\right) - \hat{m}\left(\frac{T}{2}\right)
$$

(5.29a)

$$
B[\Phi] = \mathfrak{C}^{-1}\left(\frac{T}{2} \mid -\frac{T}{2}\right) \mathfrak{B}\left(\frac{T}{2} \mid -\frac{T}{2}\right) + \hat{m}\left(\frac{T}{2}\right)
$$

(5.29b)

$$
C[\Phi] = \mathfrak{C}^{-1}\left(\frac{T}{2} \mid -\frac{T}{2}\right).
$$

(5.29c)

This means that (5.20) becomes

$$
e^{i\mathcal{W}_{0+\phi}(\Phi; \Phi')} = \int \frac{d\phi_i d\phi'_i d\phi_f \rho_0(\phi_i, \phi_f)}{\sqrt{\text{det}(2\pi\mathcal{C}[\Phi])} \text{det}(2\pi\mathcal{C}[\Phi'])} \exp \left[\frac{i}{2} \left(\begin{array}{c} \phi_f \\ \phi_i \end{array}\right)^\text{tr} \left(\begin{array}{cc} B[\Phi] - B'[\Phi'] & -C[\Phi] \\ -C'[\Phi'] & A[\Phi] \end{array}\right) \left(\begin{array}{c} \phi_f \\ \phi_i \end{array}\right)\right]
$$

(5.30)

C. The initial state

Before we can perform the integrals over the endpoints $\phi_i$, $\phi'_i$ and $\phi_f$ of the SWM paths which remain in the expression (5.30) for the influence functional, we need to specify the the initial state $\rho_0$ of the SWM environment. One simple choice is a thermal state with temperature $1/k_B\beta$. This is physically reasonable if we consider the main source of such an environment to be, for instance, the primordial graviton background.

The density matrix for this state is given as an operator by $\hat{\rho} \propto e^{-\beta\hat{H}}$. Using the full Hamiltonian corresponding to the action (5.29) would couple the short- and long-wavelength modes, preventing the separation (2.5) of the initial state. So instead we use the zero-order non-interacting action $S_0$, which gives the thermal density matrix for a simple harmonic oscillator of frequency $\Omega_0 = \left(\begin{array}{cc} \text{diag}\{k\} & 0 \\ 0 & \text{diag}\{k\} \end{array}\right)$ and unit mass:

$$
\rho_0(\phi_i, \phi_i') \propto \exp \left[\frac{-1}{2} \left(\begin{array}{c} \phi_i \\ \phi_i' \end{array}\right)^\text{tr} \left(\begin{array}{cc} \frac{\Omega_0}{\tanh 1/k_B\beta} & -\frac{\Omega_0}{\sinh 1/k_B\beta} \\ -\frac{\Omega_0}{\sinh 1/k_B\beta} & \frac{\Omega_0}{\tanh 1/k_B\beta} \end{array}\right) \left(\begin{array}{c} \phi_i \\ \phi_i' \end{array}\right)\right]
$$

(5.31)
Equation (5.28) is simplified if we express it in terms of $\overline{\phi}_i = \frac{\phi_i + \phi'_i}{2}$ and $\Delta \phi_i = \phi_i - \phi'_i$ using

$$
\begin{pmatrix}
\phi_i \\
\phi'_i
\end{pmatrix}
= 
\begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}
\begin{pmatrix}
\overline{\phi}_i \\
\Delta \phi_i/2
\end{pmatrix}.
$$

(5.32)

Both $\frac{\Omega_0}{\sinh \Omega_0 \beta}$ and $\frac{\Omega_0}{\tanh \Omega_0 \beta} + \frac{\Omega_0}{\sinh \Omega_0 \beta}$ can be expressed in terms of

$$
V(\Omega_0) = \frac{2}{\Omega_0} \frac{\cosh \Omega_0 \beta - 1}{\sinh \Omega_0 \beta} = \frac{2}{\Omega_0} \frac{\sinh \Omega_0 \beta}{\cosh \Omega_0 \beta + 1} = \frac{2}{\Omega_0} \sqrt{\frac{\cosh \Omega_0 \beta - 1}{\cosh \Omega_0 \beta + 1}} = \frac{2}{\Omega_0} \tanh \frac{\Omega_0 \beta}{2}
$$

(5.33)

to give

$$
\rho_\phi(\phi_i, \phi'_i) \propto \exp\left[-\frac{1}{2} \left(\begin{array}{c} \overline{\phi}_i \\ \Delta \phi_i/2 \end{array}\right)^\text{tr} \left(\begin{array}{cc} \Omega_0^2 V(\Omega_0) & 0 \\ 0 & \nu^{-1}(\Omega_0) \end{array}\right) \left(\begin{array}{c} \overline{\phi}_i \\ \Delta \phi_i/2 \end{array}\right)\right];
$$

(5.34)

if we also define

$$
A_\pm = A[\Phi] \pm A[\Phi'],
$$

(5.35a)

$$
B_\pm = B[\Phi] \pm B[\Phi'],
$$

(5.35b)

$$
C_\pm = C[\Phi] \pm C[\Phi'],
$$

(5.35c)

Equation (5.30) becomes

$$
e^{iW_{0+\omega}[\Phi, \Phi']}
\propto \int \frac{d\overline{\phi}_i d\phi_i d\phi'_i}{\sqrt{\det(2\pi \mathcal{C}[\phi]) \det(2\pi \mathcal{C}[\phi'])}} 
\exp\left[-\frac{1}{2} \left(\begin{array}{c} \overline{\phi}_i \\ \Delta \phi_i/2 \end{array}\right)^\text{tr} \mathcal{M} \left(\begin{array}{c} \overline{\phi}_i \\ \Delta \phi_i/2 \end{array}\right)\right]
$$

(5.36)

$$
= \left\{\det(2\pi \mathcal{C}[\phi]) \det(2\pi \mathcal{C}[\phi']) \det(2\mathcal{M})\right\}^{-1/2},
$$

so that the calculation of $e^{iW_{0+\omega}}$ is reduced to the evaluation of the determinants of $\mathcal{C}$ and

$$
\mathcal{M} = 
\begin{pmatrix}
-iB_- & iC_- \\
-iC_-^\text{tr} & \Omega_0^2 V(\Omega_0) - iA_- \\
iC_+^\text{tr} & -iA_+ \\
-iA_+ & 4\nu^{-1}(\Omega_0) - iA_-
\end{pmatrix}.
$$

(5.37)

D. Controlling the breakdown of perturbation theory

Up to this point, the treatment has been completely non-perturbative and we have successfully performed all of the integrations over $\phi$ contained in (5.1). However, the expression obtained depends on the functionals $A[\Phi]$, $B[\Phi]$ and $C[\Phi]$, which, while they can be written exactly in terms of (5.29) and (5.25), are best understood using the expansions in powers of $\ell$ from Appendix B. If we are going to begin to expand in powers of $\ell$, however, we need to address the issue of how to obtain decoherence via a partially perturbative calculation.

As alluded to before, if we try to expand the influence functional $e^{iW}$ defined by (2.61) in powers of $\ell$, we note that as the zero-order term in $S_E[\phi, \Phi]$ is just $S_0[\phi]$ [i.e., the “system” and “environment” are decoupled to zeroth order; cf. (4.9)],

$$
e^{iW[\Phi, \Phi']} = 
\int \mathcal{D}\phi \mathcal{D}\phi' \rho_\phi(\phi_i, \phi'_i) \delta(\phi'_i - \phi_i) e^{i(S_0[\phi] - S_0[\phi'])} e^{i(S_0[\phi] + S_0[\phi'])} = \text{Tr} \left[ e^{-iT \hat{H}} \rho_\phi e^{iT \hat{H}} \right] = 1.
$$

(5.38)

Perturbatively, then, we would conclude $e^{iW} = 1 + O(\ell)$. The problem is that for the influence phase to be effective at producing decoherence, we need $e^{iW[\Phi, \Phi']} \ll 1$ for $\Phi$ and $\Phi'$ sufficiently different. This can only be possible if the perturbative analysis breaks down somehow. In this sense, decoherence is an inherently nonperturbative phenomenon.

---

\^ Although the action (5.1) was of course obtained using perturbative considerations.
We will focus on one scenario in which perturbation theory cannot be applied universally, but it is easy to keep track of which seemingly negligible terms must be retained. The basic idea can be expressed simply. Consider a quantity

$$F(a, b) = I(a) + \frac{A(a)}{b}$$

which depends on two parameters $a$ and $b$, and suppose the functions $I(a)$ and $A(a)$ have expansions

$$I(a) = 1 + O(a)$$
$$A(a) = a + O(a^2).$$

If we consider only the behavior of $F$ as perturbative expansion in $a$, we would conclude that

$$F(a, b) = 1 + O(a).$$

If we were looking for cases where $F \gg 1$, we would conclude that they do not exist in the perturbative regime. However, if $b$ is also small, have to be more careful about

$$F(a, b) = 1 + \frac{a}{b} + O\left(a, \frac{a^2}{b}\right).$$

Unless $a \ll b$, we cannot neglect $a/b$ relative to 1. However, we can still neglect the $O(a)$ terms [from expansion of $I(a)$] relative to 1, and the $O(a^2/b)$ terms [from further expansion of $A(a)$] relative to $a/b$. Thus even when it is valid to use perturbative (in fact, lowest order) expressions for $I(a)$ and $A(a)$, it is still possible to have $F \gg 1$ (when $b \ll a$), since it is not valid to expand $F(a, b)$ simply in powers of $a$.

In the current problem, where the role of $F$ is played by $1/|e^W|$, and the small parameter corresponding to $a$ is $\ell$, the role of the additional parameter $b$ can be played by $\beta$. If the temperature $\beta^{-1}$ is high enough, there will be some modes in $S$ for which $\nu(k) = \frac{\ell}{k} \tanh \frac{\beta k}{2} \rightarrow \beta$, and the $O(\beta)$ terms like $C_\pm \alpha^{-1} C_\pm^{\text{tr}}$ may become smaller than $O(\ell)$ terms like $B_-$. At that point, if the $O(\ell)$ correction to $e^W$ is also $O(\beta^{-1})$, it can cause perturbation theory to break down. We keep a handle on this breakdown by neglecting $O(\ell)$ terms only when they are not compared to potentially $O(\beta)$ terms.

### E. Evaluation of the influence functional

Since the matrix $A[\Phi]$ can be expanded (see Sec. 2 of Appendix B) as $A[\Phi] = A_0 + \ell A_1[\Phi] + O(\ell^2)$, where $A_1[\Phi]$ is a linear functional of its argument, we have $A_+ = 2A_0 + O(\ell)$ and $A_- = \ell A_1[\Delta \Phi] + O(\ell^2)$ (where $\Delta \Phi = \Phi - \Phi'$), with similar expressions holding for $B_\pm$ and $C_\pm$. This means the sub-matrices of $M$ are of the following order:

$$M = \begin{pmatrix}
O(\ell) & O(\ell) \\
O(\ell) & \Omega_0^2 \nu(\Omega_0) + O(\ell) \\
O(1) & O(1) \\
4V^{-1}(\Omega_0) + O(\ell)
\end{pmatrix}.\quad (5.44)$$

Given the relation

$$4V^{-1}(\Omega_0) = 2\Omega_0 \sqrt{\frac{\cosh \Omega_0 \beta + 1}{\cosh \Omega_0 \beta - 1}} \geq 2\Omega_0 \geq 2\Omega_0 \sqrt{\frac{\cosh \Omega_0 \beta - 1}{\cosh \Omega_0 \beta + 1}} = \Omega_0^2 \nu(\Omega_0)\quad (5.45)$$

we see that $\alpha = 4V^{-1}(\Omega_0) - iA_-$ is the largest of the sub-matrices on the diagonal, and is no smaller than $O(1)$. Thus we partially diagonalize $M$ about it to get

$$\mathcal{M} = \begin{pmatrix}
1 & 0 & -iC_+ \alpha^{-1} \\
0 & 1 & iA_+ \alpha^{-1} \\
0 & 0 & 1
\end{pmatrix} M \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-\alpha^{-1} \Omega_0 C_+^{\text{tr}} & i \alpha^{-1} A_+ & 1
\end{pmatrix}
= \begin{pmatrix}
\mathcal{C}_+ \alpha^{-1} C_+^{\text{tr}} - iB_- & -\alpha C_+ \alpha^{-1} A_+ + iC_- & 0 \\
\alpha \alpha^{-1} A_+ + \Omega_0^2 \nu(\Omega_0) - iA_- & 0 & \alpha
\end{pmatrix}.\quad (5.46)$$
Using the approximation of Sec. V D, we have

\[ \widetilde{M} = \left( \begin{array}{cc} C_+ & \frac{\sqrt{\Omega_0}}{4} A_+ - i B_+ \\ -A_+ & \frac{\sqrt{\Omega_0}}{4} C_+ + i C_- \\ \end{array} \right) \oplus 4 \mathcal{V}^{-1}(\Omega_0) \]

\[ = \left( \begin{array}{cc} \frac{\sqrt{\Omega_0}}{\sin^2 \Omega_0 T} - i t B_1 [\Delta \Phi] & -\frac{\sqrt{\Omega_0}}{\sin^2 \Omega_0 T} C_1 + i C_1 [\Delta \Phi] \\ -\frac{\sqrt{\Omega_0}}{\sin^2 \Omega_0 T} C_1 - i t A_1 [\Delta \Phi] & \frac{\sqrt{\Omega_0}}{\sin^2 \Omega_0 T} + i t A_1 [\Delta \Phi] \end{array} \right) \oplus 4 \mathcal{V}^{-1}(\Omega_0). \quad (5.47) \]

Noting that the matrices used to perform the diagonalization in (5.46) have unit determinant, we have

\[ \det \mathcal{M} = \det \widetilde{M} = \det (4 \mathcal{V}^{-1}(\Omega_0)) \det (N_0 - i t N_1 [\Delta \Phi]) \propto \det \left( 1 - i t N_0^{-1/2} N_1 [\Delta \Phi] N_0^{-1/2} \right) \quad (5.48) \]

where

\[ N_0 = \frac{\Omega_0^2 \mathcal{V}(\Omega_0)}{\sin^2 \Omega_0 T} \left( \begin{array}{cc} 1 & -\cos \Omega_0 T \\ -\cos \Omega_0 T & 1 \end{array} \right) \]

\[ N_1 [\Delta \Phi] = \left( \begin{array}{cc} B_1 [\Delta \Phi] & -C_1 [\Delta \Phi] \\ -C_1 [\Delta \Phi]^\dagger & A_1 [\Delta \Phi] \end{array} \right). \quad (5.49a) \]

\[ N_1 [\Delta \Phi] = \left( \begin{array}{cc} B_1 [\Delta \Phi] & -C_1 [\Delta \Phi] \\ -C_1 [\Delta \Phi]^\dagger & A_1 [\Delta \Phi] \end{array} \right). \quad (5.49b) \]

Now, \( e^{i \text{Re} W} \) is simply a phase multiplying the decoherence functional (5.4); the part which can actually make the off-diagonal components of \( D [\Phi, \Phi'] \) small is \( e^{-i \text{Im} W} = \left| e^{i W} \right| \). Noting from (5.36) that the factors of \( \det \mathcal{C} \) in (5.36) give, to lowest order in \( \ell \), the \( \Phi \)-independent values \( \det \left( \begin{array}{cc} \sin \Omega_0 T \\ \Omega_0^2 \mathcal{V}(\Omega_0) \end{array} \right) \), we have \( \text{cf. (5.36)} \)

\[ \left| e^{i W_{0+0}[\Phi, \Phi']} \right| \propto \left\{ \det (\mathcal{M}^\dagger \mathcal{M}) \right\}^{-1/4} \propto \left\{ \det \left( 1 + \ell^2 N_0^{-1/2} N_1 [\Delta \Phi] N_0^{-1} N_1 [\Delta \Phi] N_0^{-1/2} \right) \right\}^{-1/4}. \quad (5.50) \]

The normalization is set by (5.38), and in fact

\[ \left| e^{i W_{0+0}[\Phi, \Phi']} \right| = \left\{ \det \left( 1 + \ell^2 N_0^{-1/2} N_1 [\Delta \Phi] N_0^{-1} N_1 [\Delta \Phi] N_0^{-1/2} \right) \right\}^{-1/4}. \quad (5.51) \]

For any positive matrix \( a^2 \), a straightforward analysis in the diagonal basis shows \( \det (1 + a^2) \geq 1 + \text{Tr} a^2 \), so

\[ \left| e^{i W_{0+0}[\Phi, \Phi']} \right| \leq \left( 1 + E^2 [\Delta \Phi] \right)^{-1/4}, \quad (5.52) \]

where

\[ E^2 [\Delta \Phi] = \text{Tr} (\ell N_0^{-1} N_1 [\Delta \Phi])^2 \quad (5.53) \]

The magnitude of the influence functional (5.54) will be small when \( E^2 [\Delta \Phi] \) is large. This is calculated in Appendix C and found to be

\[ E^2 [\Delta \Phi] = \int_{k_1, k_2 > k_c} \frac{d^3 k_1 d^3 k_2 \Theta(k_c - q)}{4 \mathcal{V}(k_1) \mathcal{V}(k_2)} \left\{ \left| \int_{-T/2}^{T/2} dt \Delta \Phi_q(t) e^{ik\cdot t} - i \frac{k}{k_1 k_2} \left[ e^{i2k\cdot t} \Delta \Phi_q(t) \right] \right|_{-T/2}^{T/2} \right| \left[ e^{i2k\cdot t} \Delta \Phi_q(t) \right]_{-T/2}^{T/2} \right|^2 \]

\[ + \left| \int_{-T/2}^{T/2} dt \ell \Delta \Phi_q(t) e^{-ik\cdot t} + i \frac{k}{k_1 k_2} \left[ e^{-i2k\cdot t} \Delta \Phi_q(t) \right] \right|_{-T/2}^{T/2} \right|^2 \]

\[ + \left| \int_{-T/2}^{T/2} dt \ell \Delta \Phi_q(t) e^{ik\cdot t} + i \frac{k}{k_1 k_2} \left[ e^{i2k\cdot t} \Delta \Phi_q(t) \right] \right|_{-T/2}^{T/2} \right|^2 \]

\[ + \left| \int_{-T/2}^{T/2} dt \ell \Delta \Phi_q(t) e^{-ik\cdot t} - i \frac{k}{k_1 k_2} \left[ e^{-i2k\cdot t} \Delta \Phi_q(t) \right] \right|_{-T/2}^{T/2} \right|^2 \quad (5.54) \]

where \( q = k_1 - k_2, k_\pm = k_1 \pm k_2 \), and \( \cos \theta_{12} = \frac{k_1 \cdot k_2}{k_1 k_2} \).
There is also a dependence on \( \cos \theta \) however, is expressed in terms of the three components of \( \mathbf{q} \) and the two amplitudes \( k_1 \) and \( k_2 \) (or equivalently, \( k_\pm \)). There is also a dependence on \( \cos \theta_{12} = \frac{k_2}{k_1k_2} \), but that can be expressed in terms of the other five variables by

\[
q^2 = k_1^2 + k_2^2 - 2k_1k_2 \cos \theta_{12}.
\]

Changing variables from \( \{\mathbf{k}_1, \mathbf{k}_2\} \) to \( \{\mathbf{q}, k_+, k_-\} \) converts (5.54) to

\[
E^2[\Delta \Phi] = \int_0^{k_c} dq \int \int q^2 d\Omega_\mathbf{q} \int_{-q}^{q} dk_- \int_{2k_+|k_-|}^{\infty} dk_+ \frac{2\pi \coth \beta k_+ k_- \coth \beta k_1 k_2}{512q} \left( \begin{array}{c} |(q^2 - k_-^2) \int_{-T/2}^{T/2} dt \ell \Delta \Phi_\mathbf{q}(t)e^{ik_-t} - i4k_- [e^{i2k_-t} \ell \Delta \Phi_\mathbf{q}(t)]_{-T/2}^{T/2} |^2 \\ + |(q^2 - k_+^2) \int_{-T/2}^{T/2} dt \ell \Delta \Phi_\mathbf{q}(t)e^{-ik_+t} + i4k_+ [e^{-i2k_+t} \ell \Delta \Phi_\mathbf{q}(t)]_{-T/2}^{T/2} |^2 \\ + |(k_+ - q^2) \int_{-T/2}^{T/2} dt \ell \Delta \Phi_\mathbf{q}(t)e^{ik_+t} + i4k_+ [e^{i2k_+t} \ell \Delta \Phi_\mathbf{q}(t)]_{-T/2}^{T/2} |^2 \\ + |(k_- - q^2) \int_{-T/2}^{T/2} dt \ell \Delta \Phi_\mathbf{q}(t)e^{-ik_-t} - i4k_- [e^{-i2k_-t} \ell \Delta \Phi_\mathbf{q}(t)]_{-T/2}^{T/2} |^2 \end{array} \right). \tag{5.57}
\]

The Jacobian is straightforward to calculate, and the limits of integration come from combining the restrictions \( q < k_c < k_1, k_2 \) on (5.54) with the inherent geometrical requirement that \( k_-^2 < q^2 < k_+^2 \) (from \( |\cos \theta_{12}| \leq 1 \)), as illustrated in Fig. 2.
VI. THE FULL ACTION

Returning from the modified action \( S_{0+\phi\phi} \) to the full action \( S \), one finds that the result of the previous section is not substantially changed, as demonstrated in Appendix D.

Including the terms linear in \( \phi \) to produce the action \( S_3 = S_{0+\phi\phi} + \ell S_\phi \) changes the influence functional only by a phase (as is shown by completing the square in Sec. D.1):

\[
\left| e^{iW_s[\Phi,\Phi']} \right| = \left| e^{iW_{0+\phi\phi}[\Phi,\Phi']} \right|. \tag{6.1}
\]

The addition of the terms cubic in \( \phi \) to restore the full action is handled via a perturbative expansion in powers of \( \ell \) Sec. D.2. The cubic corrections should of course be \( O(\ell) \) or higher, but could in principle become large as described in Sec. V D if there were enough factors of \( \beta \) in the denominator. The calculation shows that these corrections multiply the influence functional by a factor of order unity:

\[
e^{iW[\Phi,\Phi']} = O(1) \times e^{iW_3[\Phi,\Phi']}. \tag{6.2}
\]

Thus we are still left with the result that

\[
\left| e^{iW[\Phi,\Phi']} \right| \lesssim \left\{ 1 + E^2[\Delta \Phi] \right\}^{-1/4}, \tag{6.3}
\]

with the suppression factor \( E^2[\Delta \Phi] \) given by (5.54).

A. A word about the perturbative analysis

The conclusion that

\[
e^{iW[\Phi,\Phi']-iW_5[\Phi,\Phi']} = O(1) \tag{6.4}
\]

is based upon an upper limit on each term in the perturbation series (the first term is obviously unity). There are two ways this analysis could fail. First, there may be cancellation among the various \( O(1) \) terms causing the net expression to be a higher order in \( \ell \) or \( \beta \). Since this would only make \( |e^{iW}| \) smaller than our estimate, it would only improve the upper limit given by (6.3).

The second is more problematic. While each individual term is at most \( O(1) \), the entire infinite series could be quite large, counteracting the tendency of \( e^{iW_{0+\phi\phi}} \) to become small. This is a shortcoming of the perturbative analysis, and there’s not a lot to be done, other than to tackle the non-perturbative problem.8 Note, however, that we can say with confidence that \( |e^{iW-iW_5}| \) does not have terms which are \( O(\ell^2/\beta^2) \), which could directly cancel similar terms in the expansion of \( |e^{iW_{0+\phi\phi}}| \). So if \( |e^{iW-iW_5}| \) becomes large, it is not in the same way which \( |e^{iW_5}| = |e^{iW_{0+\phi\phi}}| \) becomes small.

VII. INTERPRETATION

Having placed limits on the influence phase via the suppression factor \( E^2[\Delta \Phi] \) given in (5.54), we now consider the question of which coarse grainings can be made to decohere for reasonable values of the parameters \( k_c, \beta \) and \( T \).

A. Which modes are suppressed?

Having determined that the influence functional is bounded from above by

\[
\left| e^{iW[\Phi,\Phi']} \right| \lesssim \left\{ 1 + E^2[\Delta \Phi] \right\}^{-1/4}, \tag{6.3}
\]

\[8 \text{For instance, we can’t use (5.38) to conclude that the } O(1) \text{ factor in (6.2) is unity, since that would involve an illegal interchange of the } \beta \to 0 \text{ and } \ell \to 0 \text{ limits.} \]
and hence becomes small when

\[
E^2[\Delta \Phi] = \int_0^{k_c} dq \int q^2 d^2q \int_{-q}^q dk_- \int_{2k_c+|k_-|}^{\infty} dk_+ 2\pi \coth \beta \frac{k_++k_-}{2} \coth \beta \frac{k_+-k_-}{4} \frac{1}{512q} \times \left\{ (q^2 - k_-^2) \int_{-T/2}^{T/2} dt \ell \Delta \Phi_q(t) e^{i k_- t} - i 4k_- \left[ e^{i 2k_- t} \ell \Delta \Phi_q(t) \right]_{-T/2}^{T/2} \right\}^2
+ (q^2 - k_-^2) \int_{-T/2}^{T/2} dt \ell \Delta \Phi_q(t) e^{-i k_- t} + i 4k_- \left[ e^{-i 2k_- t} \ell \Delta \Phi_q(t) \right]_{-T/2}^{T/2} \right\}^2
+ (k_+^2 - q^2) \int_{-T/2}^{T/2} dt \ell \Delta \Phi_q(t) e^{i k_+ t} + i 4k_+ \left[ e^{i 2k_+ t} \ell \Delta \Phi_q(t) \right]_{-T/2}^{T/2} \right\}^2
+ (k_+^2 - q^2) \int_{-T/2}^{T/2} dt \ell \Delta \Phi_q(t) e^{-i k_+ t} - i 4k_+ \left[ e^{-i 2k_+ t} \ell \Delta \Phi_q(t) \right]_{-T/2}^{T/2} \right\}^2 \right\} \tag{5.56}
\]

becomes large, we would like to consider when that happens. Looking at \((5.56)\), and disregarding the surface terms (which will be shown in Sec. VII B 1 to be irrelevant), we see that not all of the space/time modes appear. The first two terms include only modes where \(|\omega| = |k_-| \leq q\), while the last two are limited to modes where \(|\omega| = k_+ \geq 2k_c\). This is illustrated in Fig. 4. Just as our coarse graining considers only long-wavelength modes \((q \leq k_c)\), it is reasonable to focus on long-period modes \((|\omega| \leq k_c)\) as well. Thus the limit of interest comes from the first two terms, and we write

\[
\Delta \Phi_q(t) = \int_{-T/2}^{T/2} dt \sqrt{2\pi} \Delta \Phi_q(t) e^{i \omega t} \tag{7.1}
\]

The factor

\[
R = \int_{2k_c+|k_-|}^{\infty} dk_+ \frac{2\pi \coth \beta \frac{k_+}{2} \coth \beta \frac{k_-}{4}}{256q} \tag{7.3}
\]

can be evaluated, to leading order in \(\beta\), by noting that

\[
\coth \eta_1 \coth \eta_2 = \frac{\cosh \eta_1 \cosh \eta_2}{\sinh \eta_1 \sinh \eta_2} = \frac{\cosh \eta_+ \cosh \eta_-}{\cosh \eta_+ - \cosh \eta_-} = 1 + \frac{2 \cosh \eta_-}{\cosh \eta_+ - \cosh \eta_-}, \tag{7.4}
\]

so that

\[
R = \int_{2k_c+|k_-|}^{\infty} \frac{2\pi dk_+}{256q} \left( 1 + \frac{2 \cosh \frac{\beta k_+}{2}}{\cosh \frac{\beta k_-}{2} - \cosh \frac{\beta k_-}{2}} \right) = R_0 + \int_{2k_c+|k_-|}^{\infty} \frac{2\pi dk_+}{256q}. \tag{7.5}
\]

Now,

\[
R_0 = \int_{2k_c+|k_-|}^{\infty} \frac{2\pi dk_+}{256q} \frac{2 \cosh \frac{\beta k_+}{2}}{\cosh \frac{\beta k_-}{2} - \cosh \frac{\beta k_-}{2}} = 4(2\pi) \coth \beta \frac{k_-}{4} \frac{1}{256q^3} \ln \left( \frac{\sinh \beta \frac{k_+}{2} \frac{k_-}{2}}{\sinh \beta \frac{k_-}{2} e^{\beta |k_-|/2}} \right). \tag{7.6}
\]

Again, since we only expect a useful answer when small \(\beta\) causes perturbation theory to break down, we look at the leading terms in \(\beta\), working in the high-temperature limit \(\beta k_c \gg 1\). (See Sec. VII B 1 for the physical significance of this.) In this limit, \((7.4)\) becomes
FIG. 3. The modes represented in (5.56), plotted by their $\omega$ and $q$ values. The modes with $q \geq k_c$ are traced over, and so that region is shaded horizontally. The first two terms in (5.56) can suppress modes with $|\omega| \leq q$, which are shaded vertically, the third can suppress modes which have $\omega \geq k_c$ and the fourth, $\omega \leq -k_c$; these last two are shaded diagonally. Since we are concerned with coarse grainings of low temporal frequency $\omega$ as well as spatial frequency $q$, the first two terms are the ones of interest.
\[ R_0 = \frac{8(2\pi)}{256q^3} |k_c| \ln \left( 1 + \frac{|k_c|}{k_c} \right) ; \]

since \( R - R_0 \) is independent of \( \beta \), the leading term in \( R \) is\[ R = \frac{2\pi}{32q^3 |k_c|} \ln \left( 1 + \frac{|k_c|}{k_c} \right) , \]

so

\[ E^2 |\Delta \Phi| \gtrsim \int_0^{k_c} dq \int q^2 d^2 \Omega \int_{-q}^q \frac{2\pi d\omega}{32q^3 |\omega|} \ln \left( 1 + \frac{|\omega|}{k_c} \right) |(q^2 - \omega^2)^{1/2} \Delta \Phi_{q\omega} - i4\omega \left[ e^{i2\omega t} \ell \Delta \Phi_{q}\ell(T/2) \right]_{-T/2}^T|^2 . \]  

(7.9)

\[ \text{B. Practical coarse grainings} \]

1. The physical scales

The expression (7.9) has three parameters, \( k_c, \beta, \) and \( T \), which are not integrated over. The scale \( k_c \) for division into SWMs and LWMs can be tailored to the coarse graining to give the strongest possible results, while the other two are features of the model. As alluded to in Sec. III, the time scale \( T \) over which we expect the Minkowski space model to be valid should be slightly below the Hubble scale \( H_0^{-1} \). In suitable units, this gives

\[ T \lesssim H_0^{-1} \sim 10^{10} \text{yr} \sim 10^{29} \text{cm}. \]  

(7.10)

This is so large that it allows us to set \( T \) much larger than all the other scales in the problem. In particular, it means that the cross terms in

\[ \left| (q^2 - \omega^2)^{1/2} \Delta \Phi_{q\omega} - i4\omega \left[ e^{i2\omega t} \ell \Delta \Phi_{q}\ell(T/2) \right]_{-T/2}^T \right|^2 , \]

(7.11)

will oscillate rapidly and vanish when \( \omega \) is integrated over, leaving

\[ 2\pi |(q^2 - \omega^2)\ell \Delta \Phi_{q\omega} |^2 + 16\omega^2 |e^{i\omega T} \ell \Delta \Phi_{qf} - e^{-i\omega T} \ell \Delta \Phi_{qi} |^2 \gtrsim 2\pi |(q^2 - \omega^2)\ell \Delta \Phi_{q\omega} |^2 , \]

(7.12)

so

\[ E^2 |\Delta \Phi| \gtrsim \int_0^{k_c} dq \int q^2 d^2 \Omega \int_{-q}^q \frac{2\pi d\omega}{32q^3 |\omega|} \ln \left( 1 + \frac{|\omega|}{k_c} \right) |(q^2 - \omega^2)^{1/2} \Delta \Phi_{q\omega} |^2 . \]

(7.13)

Turning our attention to the inverse temperature \( \beta \), we might reasonably treat the high-temperature thermal state \( \rho_0 \) as corresponding to the cosmic graviton background radiation \[ \text{[13]}, \]

which has a temperature on the order of 1 K. This means that in suitable units,

\[ \beta \sim \frac{1}{1 \text{K}} \sim \frac{1}{10^{-3} \text{eV}} \sim 10^{-1} \text{cm}. \]

(7.14)

This is the most severe limit to the usefulness of the calculations in this work. It means that to be in the high-temperature limit \( \beta k_c \ll 1 \), we need to have the cutoff scale \( k_c^{-1} \) dividing “short” and “long” wavelengths be above

\[ ^9 \text{Of course, this is a dubious approximation, since } R - R_0, \text{ while down by a factor of } \beta^2 \text{ from } R_0, \text{ is ultraviolet divergent. However, any suitable well-behaved regulation of the result will give a result which agrees with } R_0 \text{ to } \mathcal{O}(\beta^2) \text{ when the } \beta \to 0 \text{ limit is taken before the cutoff limit. Note also that our perturbative analysis has ignored terms like } \ell^2 (R - R_0), \text{ which are perturbatively small in } \ell \text{ without having corresponding factors of } \beta. \text{ One might hope that such terms will cancel the divergence in } R - R_0. \text{ However, this turns out not to be the case, as can be seen by calculating all of the } \mathcal{O}(\ell^2) \text{ terms in } |e^{iW_0 + q\phi}|. \]


the millimeter scale. While we don’t expect to have laboratory data on millimeter-scale oscillations of vacuum gravity any time soon (contrast this scale to the length corresponding to a typical component of the curvature tensor at the surface of a 1\(M_\odot\) black hole, which is \(GM_\odot \sim 1\ km \sim 10^6\ \beta\)), it might be a bit surprising to learn that coarse grainings corresponding to micron-scale variations in the gravitational field do not decohere. At any rate, that is not the prediction of this work, even assuming that the analysis of the conformally flat toy model is an accurate indicator of the behavior of the full theory. First, a perturbative analysis of vacuum gravity simply cannot make fruitful predictions outside of the perturbative regime. It is quite possible that for lower temperatures, non-perturbative effects can cause the influence functional to become small for large \(\Delta \Phi\). And of course, this analysis only models the decoherence of the vacuum gravitational field induced by gravity itself. If the gravitational field is coupled to some form of matter, unobserved modes of the matter can also induce decoherence, as described in [3,4]. So this work suggests an encouraging lower limit on the effectiveness of the decoherence of spacetime itself without the assistance of additional matter fields.

2. Field averages

When constructing the sum-over-histories generalized quantum mechanics of a field theory, it is useful to coarse grain by values of a field average over a particular region [14]. By defining the average

\[
\langle \ell \varphi \rangle = \int d^3x \int_{-T/2}^{T/2} dt \ w(x,t) \ell \varphi(x,t)
\]  

with a weighting function \(w(x,t)\) it is possible to study the behavior of different Fourier modes of the field via the choice of \(w(x,t)\). For now we take the weighting function to be normalized,

\[
\int d^3x \int_{-T/2}^{T/2} dt \ w(x,t) = 1,
\]

but later we will relax that restriction to allow for averages in Fourier space which do not include the zero mode.

As an example of a field average, recall the connection of our scalar field theory to the theory of a conformally flat metric discussed in Sec. III. As described there, the fractional deviation of the volume of a spacetime region \(S\) from its volume in flat the background metric, \(\int d^{D+1}\sqrt{|g|}w_S(x) - 1\), is a field average (given by (3.10)) which in the \(D = 3\) case currently being considered is

\[
\frac{2}{3}(2\pi)^{3/2}\langle \ell \Phi \rangle_S,
\]

with the weighting function taken to be the characteristic function for \(S\)

\[
w_S(x) = \begin{cases} V_0^{-1} & x \in S \\ 0 & x \notin S. \end{cases}
\]  

(\(V_0\) is of course the background volume of \(S\).)

In terms of Fourier modes, a general field average becomes

\[
\langle \ell \varphi \rangle = \int d^3q \ d\omega \ w^{*}_{\varphi q} \ell \varphi_{q \omega},
\]

where we have approximated the sum over \(\omega\) values separated by \(\delta \omega = 2\pi/T\) by an integral, and assumed that \(w(x,t)\) vanishes as \(t \to \pm T/2\), so that it is acceptable to replace the field \(\varphi_q(t)\) by its periodic counterpart

\[
\varphi^p_q(t) = \int \frac{d\omega}{\sqrt{2\pi}} \varphi_{q \omega} e^{-i\omega t} = \begin{cases} \frac{1}{2}(\varphi_{q t} + \varphi_{q f}) & t = \pm \frac{T}{2} \\ \varphi_{q t} & -\frac{T}{2} < t < \frac{T}{2}. \end{cases}
\]

If \(w_q\) contains only modes with \(q \in \mathcal{L}\) [i.e. \(w_q = \Theta(k_c - q) w_q\)], then we can write this average as

\[
\langle \ell \Phi \rangle = \int_{\mathcal{L}} d^3q \ d\omega \ w^{*}_{\Phi q} \ell \Phi_{q \omega}.
\]
The normalization condition (7.15) becomes \( w_{q_0} = (2\pi)^{-2} \), so a useful field average might be

\[
\frac{2}{3}(2\pi)^{3/2} \langle \Phi \rangle = \frac{2}{3} \int_0^{\Delta q/2} dq \int q^2 d\Omega_q \int_{-\Delta \omega/2}^{\Delta \omega/2} \frac{d\omega}{\sqrt{2\pi}} \Phi_{q\omega},
\]

(7.21)

where the width of the smoothing function in Fourier space is

\[
\Delta q \sim \frac{1}{\Delta x}
\]

(7.22a)

\[
\Delta \omega \sim \frac{1}{\Delta t}
\]

(7.22b)

and the origin of the spatial coordinates has been chosen to correspond with the center of \( w(\mathbf{x},t) \). To consider a group of modes not centered about the constant mode, we shift the center of the group of Fourier modes by \( q_0 \) and \( \omega_0 \), while keeping the mode volume the same, giving another dimensionless quantity

\[
\frac{2}{3}(2\pi)^{3/2} \langle \Phi \rangle = \frac{2}{3} \int_{-\Delta q/2}^{\Delta q/2} dq \int \Omega_q \int_{-\omega_0-\Delta \omega/2}^{\omega_0+\Delta \omega/2} \frac{d\omega}{\sqrt{2\pi}} (\Phi_{q\omega} + \Phi_{-q,-\omega}),
\]

(7.23)

where the solid angle integrated over is centered about \( \hat{q}_\omega \) and is chosen to preserve the mode volume:

\[
\frac{4\pi(\Delta q/2)^3}{3} = 2\Omega \int_{-\Delta q/2}^{\Delta q/2} q^2 dq = 2\Omega (q_0 + \Delta q/2)^3 - (q_0 - \Delta q/2)^3,
\]

(7.24)

so

\[
\Omega = \frac{\pi(\Delta q)^2}{12q_0^3 + (\Delta q)^2}.
\]

(7.25)

3. The influence phase

Now we can cast (7.3) into a useful form, so long as \( q_0 - \Delta q/2 \geq |\omega_0| + \Delta \omega \):

\[
E^2[\Delta \Phi] \geq \int_0^{k_c} dq \int q^2 d\Omega_q \int_{-q}^{q} \frac{2\pi d\omega}{32q^2 |\omega|} \ln \left( 1 + \frac{|\omega|}{k_c} \right) \left| (q^2 - \omega^2)\sqrt{2\pi} \ell |\Delta \Phi_{q\omega}| \right|^2
\]

\[
\geq \int_{q_0-\Delta q/2}^{q_0+\Delta q/2} dq \int \Omega_q \int_{-\omega_0-\Delta \omega/2}^{\omega_0+\Delta \omega/2} d\omega \left( \frac{2\pi}{32q^2 |\omega|} \right) \ln \left( 1 + \frac{|\omega|}{k_c} \right) \left( |\ell \Delta \Phi_{q\omega}|^2 + |\ell \Delta \Phi_{-q,-\omega}|^2 \right)
\]

(7.26)

The strongest result will be obtained if we take \( k_c = q_0 + \Delta q/2 \). If \( \Delta \omega \) and \( \Delta q \) are small relative to \( \omega_0 \) and \( q_0 \) (which means large \( \Delta t \) and \( \Delta x \)), we can approximate

\[
E^2[\Delta \Phi] \geq \Theta(q_0 - |\omega_0|) \frac{(2\pi)^2(q_0^2 - \omega_0^2)^2}{32q_0^3 |\omega_0|} \ln \left( 1 + \frac{|\omega_0|}{q_0} \right) \int_{q_0-\Delta q/2}^{q_0+\Delta q/2} dq \int \Omega_q \int_{-\omega_0-\Delta \omega/2}^{\omega_0+\Delta \omega/2} d\omega \left( |\ell \Delta \Phi_{q\omega}|^2 + |\ell \Delta \Phi_{-q,-\omega}|^2 \right)
\]

\[
\approx \Theta(q_0 - |\omega_0|) \frac{(2\pi)^2(q_0^2 - \omega_0^2)^2}{32q_0^3 |\omega_0|} \ln \left( 1 + \frac{|\omega_0|}{q_0} \right) \frac{(2\pi)^2}{\pi \Delta \omega (\Delta q)^3/6} \]

(7.27)

so that the influence phase is bounded by

\[
|e^{iW[\Phi,\Phi']}| \lesssim \left( 1 + \frac{3\pi^4(q_0^2 - \omega_0^2)^2 |\ell \Delta \Phi|^2}{q_0^3 |\omega_0| \Delta \omega (\Delta q)^3} \ln \left( 1 + \frac{|\omega_0|}{q_0} \right) \right)^{-1/4}.
\]

(7.28)

This means that if
the decoherence functional $D[\Phi, \Phi']$ corresponding to $\langle \tilde{\Phi} \rangle$ and $\langle \tilde{\Phi}' \rangle$ separated by $\langle \tilde{\Delta} \Phi \rangle$ will be small. This limit corresponds to

$$ \frac{2}{3} (2\pi)^{3/2} \left| \langle \tilde{\Delta} \Phi \rangle \right| \gg \frac{4\beta \sqrt{2q_0 |\omega_0| \Delta \omega (\Delta q)^3}}{3q_0^2 - \omega_0^2 \sqrt{3\pi}} \left[ \frac{|\omega_0|}{q_0} \right]^{-1/2} ; $$

(7.30)

Considering the static limit $|\omega_0| \ll q_0$ for simplicity, (7.30) becomes

$$ \frac{2}{3} (2\pi)^{3/2} \left| \langle \tilde{\Delta} \Phi \rangle \right| \gg \frac{4\beta \sqrt{2q_0 |\omega_0| \Delta \omega (\Delta q)^3}}{3q_0^2 \sqrt{3\pi}} \left[ \frac{|\omega_0|}{q_0} \right]^{-1/2} = \frac{4\beta q_0}{3} \sqrt{\frac{2\Delta \omega (\Delta q)^3}{3\pi q_0^3}}. $$

(7.31)

For sufficiently small $\Delta \omega$ and $\Delta q$ (which corresponds to averaging over a large spacetime region), the right hand side of (7.31) becomes small, and thus (7.31) can hold even when the quantity $\frac{2}{3} (2\pi)^{3/2} \left| \langle \tilde{\Delta} \Phi \rangle \right|$ representing the perturbation due to the metric is small, justifying the use of perturbation theory.

So a coarse graining which should decohere is one consisting of a set of alternatives $\{c_n\}$ which correspond to $\frac{2}{3} (2\pi)^{3/2} \langle \tilde{\Phi} \rangle \in \Delta_n = [n \Delta, (n + 1) \Delta]$. The decoherence functional for such a coarse graining will be

$$ D(n, n') = \int_{\langle \tilde{\Phi} \rangle \in \Delta_n} \int_{\langle \tilde{\Phi}' \rangle \in \Delta_{n'}} D[\Phi, \Phi'] \delta(\Phi' - \Phi) \rho_\Phi(\Phi_i, \Phi'_i) \delta(\Phi'_i - \Phi_i) e^{i(S_\Phi[\Phi] - S_\Phi[\Phi'] + W[\Phi, \Phi'])} \delta\left( f - \frac{2}{3} (2\pi)^{3/2} \langle \tilde{\Phi} \rangle \right) \delta\left( f' - \frac{2}{3} (2\pi)^{3/2} \langle \tilde{\Phi}' \rangle \right), $$

(7.32)

where

$$ G(f, f') = \int D\Phi D\Phi' \rho_\Phi(\Phi_i, \Phi'_i) \delta(\Phi' - \Phi) e^{i(S_\Phi[\Phi] - S_\Phi[\Phi'] + W[\Phi, \Phi'])} \delta\left( f - \frac{2}{3} (2\pi)^{3/2} \langle \tilde{\Phi} \rangle \right) \delta\left( f' - \frac{2}{3} (2\pi)^{3/2} \langle \tilde{\Phi}' \rangle \right). $$

(7.32a)

Equation (7.28) shows that $G(f, f')$ should be suppressed by the influence functional when

$$ f - f' \gtrsim \delta = \frac{4\beta q_0}{3} \sqrt{\frac{2\Delta \omega (\Delta q)^3}{3\pi q_0^3}}. $$

(7.33)

As long as the size $\Delta$ of the bins is much larger than $\delta$, the off-diagonal elements of $D(n, n')$ with $|n - n'| \geq 2$ will involve integrals only over the suppressed region, while the elements with $|n - n'| = 1$ should be down from the diagonal elements by a factor of $\delta/\Delta$. (Fig. 3)

To express the result in terms of a familiar measure of decoherence, we can define a decoherence time $T_{\text{dec}} = 2\pi/\Delta \omega$, which is the temporal extent of a weighting function leading to a decohering coarse graining. Solving for $T_{\text{dec}}$ gives

$$ T_{\text{dec}} \sim \frac{64 \pi^3}{27} \frac{\beta^2 (\Delta q)^3}{\delta^2 q_0^5} \sim \frac{512 \pi^3}{27} \frac{\beta^2}{\varepsilon^2 \Delta^2 q_0^3 V}, $$

(7.34)

where $V = (2\pi/\Delta q)^3$ is the spatial volume over which the weighting function is non-negligible, $q_0$ is the spatial frequency at which it oscillates, $\Delta$ is the size of the bins in our coarse graining, and $\varepsilon$ is our standard for approximate decoherence (how small the off-diagonal elements of the decoherence functional must be).

C. Impractical coarse graining

A question one might like to ask is whether the decoherence exhibited in the previous sections relied upon the fact that the modes of interest were the long-wavelength ones, or if the mere fact that some sufficiently large group of modes is traced over is enough to produce decoherence. In this section, we show that we obtain a similar result if the identification of system and environment in (4.4) are now reversed:
we see that the lower limit on the suppression factor becomes

\[ F = \frac{F_{\text{out}}}{F_{\text{in}}} \]

Focussing on the modes shaded horizontally in Fig. 6 and neglecting the boundary terms as described in Sec. VII B 1, squares on the diagonal \( (n = n') \) have a region of area \( 2\Delta \delta - \delta^2 \) over which \( G(f, f') \) is appreciable. Squares one spot off the diagonal \( (|n - n'| = 1) \) include some non-negligible values of \( G(f, f') \), but only in a triangular region of area \( \delta^2/2 \). Thus \( D(n, n) \) should be suppressed by a factor of \( \delta/\Delta \) relative to \( D(n, n) \). Compare Fig. 1 of [8],

\begin{align*}
\Phi_q(t) &= \varphi_q(t) \Theta(q - k_c) \\
\phi_{k}(t) &= \varphi_{k}(t) \Theta(k_c - q),
\end{align*}

so that now \( q \in S \) and \( k \in L \). Most of the calculation carries through unchanged until it comes to determining the limits of integration in Sec. VII A. The terms in (7.35) are thus changed so that \( q \) runs from \( k_c \) to \( 2k_c \) and \( k \) from \( q \) to \( 2k_c \), and \( k \) from \( -(2k_c - k_c) \) to \( 2k_c - k_c \) (or, equivalently, \( k \) runs from \(-2k_c - q\) to \( 2k_c - q \) and \( k \) from \( 2k_c - q \) to \( 2k_c \), with the same limits on the \( q \) integration). Moving to Sec. VII A, we find the new regions of potentially suppressed frequencies, illustrated in Fig. 5. The terms in (5.56) with \( |\omega| = |k_-| \) will have \( |\omega| \leq 2k_c - q \), and are shaded vertically in Fig. 5 while those with \( |\omega| = k_+ \) will have \( |\omega| \) between \( q \) and \( 2k_c \) and are shaded diagonally.

Since \( k_- \leq k_c \leq q \leq k_+ \leq k_c \), we can express the suppression factor in the limit that \( 2/3k_c \ll 1 \) as

\[ E^2[\Delta \Phi] \approx \int_{k_c}^{2k_c} dq \int q^2 d^2\Omega \frac{2\pi}{64q^3} 2 \int_{-(2k_c - q)}^{2k_c - q} d\omega \ln \left( \frac{k_c - |\omega|}{q - |\omega|} \right) \]

\[ \times \left( \omega^2 - \omega^2 \right) \sqrt{2\pi \ell \Delta \Phi_{q\omega}} - i4\omega \left[ e^{i2\omega t \ell \Delta \Phi_{q\omega}} \right]^{T/2} \]

\[ + \int_{q}^{2k_c} \frac{d\omega}{\omega} \ln \left( \frac{\omega}{k_c} - 1 \right) \left( \omega^2 - q^2 \right) \sqrt{2\pi \ell \Delta \Phi_{q\omega}} - i4\omega \left[ e^{i2\omega t \ell \Delta \Phi_{q\omega}} \right]^{T/2} \]

\[ + \left| \omega^2 - q^2 \right| \sqrt{2\pi \ell \Delta \Phi_{q,-\omega}} - i4\omega \left[ e^{-i2\omega t \ell \Delta \Phi_{q\omega}} \right]^{T/2} \right]. \quad (7.36) \]

Focussing on the modes shaded horizontally in Fig. 5 and neglecting the boundary terms as described in Sec. VII B 1, we see that the lower limit on the suppression factor becomes
FIG. 5. The modified regions of integration for (5.54) when the system is made up of short-wavelength modes and the environment of long-wavelength modes. As in Fig. 2, the geometrical restrictions $k_+ = k_1 + k_2 \geq q$ and $|k_-| = |k_1 - k_2| \leq q$ limit us to the region shaded vertically. Now the requirement producing the horizontally-shaded region is $k_1, k_2 \geq k_c$. Note that in contrast with the regions of integration in Fig. 2, the ranges of both the $k_+$ and $k_-$ integrations are finite.

$$E^2[\Delta \Phi] \gtrsim \int_{k_c}^{2k_c} dq \int \int q^2 d^3q_1 q \int_{-(2k_c-q)}^{2k_c-q} \frac{2\pi d\omega}{32q^3 \omega|\omega|} \ln \left(1 + \frac{|\omega|}{k_c}\right) \left|q^2 - |\omega|^2\right| \sqrt{2 \pi \ell} \Delta \Phi_{q\omega}^2,$$  

which is, other than the limits of integration on $q$ and $\omega$, the same as that given in (7.26). Thus we conclude that, at least for these groups of modes with $q_0 > |\omega_0|$, the tendency of unobserved modes to induce decoherence is just as effective whether they are of shorter or longer wavelength.

There is some precedent for this result in, for example, [3], where the long-wavelength modes of an additional scalar field induced decoherence in the gravitational field.

VIII. CONCLUSIONS

This work has demonstrated that, in a scalar field theory obtained by perturbative expansion of the Nordström-Einstein-Fokker action (which is given by the Einstein-Hilbert action restricted to conformally flat metrics), some coarse grainings which restrict only the long wavelength modes of the field should decohere. Using the self-interaction of this theory, which has a form analogous to that obtained by a perturbative expansion of GR, the ignored short-wavelength degrees of freedom can destroy quantum coherence between different long-wavelength alternatives. This lack of quantum-mechanical interference is a prerequisite for classical behavior of spacetime on large scales. The present result demonstrates that in some cases the gravitational self-interaction, as represented by this toy model, is sufficient to induce decoherence without adding any matter fields.

As demonstrated in Sec. VII C, the central feature of this mechanism is the division into a system and an environment. The split can also induce decoherence when we coarse grain by the short-wavelength features and let the long-wavelength modes act as an environment.

The decoherence properties were studied by calculating the influence functional $e^{iW}$ between pairs of long-wavelength histories, which describes the effect of tracing out the short-wavelength modes. Decoherence is expected in terms of an appropriate set of variables; in any number of dimensions, there is a reparametrization of the theory which is non-interacting, but it is the self-interaction in terms of the variables defining the coarse-graining which is relevant for decoherence; see footnote 3.
FIG. 6. The modes represented in the suppression term $E^2[\Delta \Phi]$, plotted by their $\omega$ and $q$ values. The modes with $q \leq k_c$ are traced over, and so that region is shaded horizontally. The terms with $\omega = \pm k_-$ can suppress modes with $|\omega| \leq 2k_c - k_+$, which are shaded vertically; those with $\omega = k_+$ can suppress modes which have $q \leq \omega \leq 2k_c$ and those with $\omega = -k_+$ can suppress $-q \geq \omega \geq -2k_c$; these last two are shaded diagonally.
when its absolute value $|e^{iW[Φ,Φ']}|$ becomes small for sufficiently large differences between the long-wavelength configurations $Φ$ and $Φ'$.

Even though the influence functional is unity to lowest order in the coupling constant $ℓ$, and one might normally assume that perturbative corrections cannot make $|e^{iW}|$ much smaller than one, decoherence is still possible if there is a second small parameter. In our case, this was accomplished by working in the high-temperature regime where the inverse temperature $β$ of the thermal state describing the SWMs was small. Then, as described in Sec. VII, terms which were higher order in the coupling $ℓ$ could still become large for high temperature if they were proportional to, for example, $(ℓ/β)^2$. This made it possible to find $|e^{iW}| \ll 1$, while still allowing us to treat terms higher order in $ℓ$ as small if they did not have corresponding powers of $β^{-1}$.

The $ℓ/β$ terms in the influence functional were handled non-perturbatively for the terms in the action which are quadratic or linear in the SWMs, but the cubic terms in the action were analyzed using a perturbative expansion. That expansion showed that while there are corrections which go like $O(ℓ)$ or $O(β)$, those are at largest $O(1)$, and there are no $O(ℓ/β)$ terms to cancel out the effect from the quadratic action.

The reliance on perturbative analysis is one of the limitations of this result. It means that we can only analyze the question of decoherence in the high-temperature limit, defined by $βk_c \ll 1$, where $k_c$ is the momentum which divides SWMs from LWMs. If the temperature of the SWM thermal state is taken to be that of the present-day cosmic graviton background, the length scale corresponding to this limit is on the order of a millimeter.

Another problem comes from the non-renormalizability of our derivative action (a property it shares with GR itself). While the terms in the influence functional proportional to $(ℓ/β)^2$ are finite, there are terms proportional to $ℓ^2$ alone which are ultraviolet divergent. We were able to ignore those by working in the high-temperature limit, but they may provide another way in which perturbation theory breaks down, demanding a fully non-perturbative analysis.

Before moving to a possible non-perturbative analysis, perhaps using the Regge calculus [15] to skeletonize geometry, another improvement of this work would be to see if the scalar field result is modified by considering a model where the full tensor nature of the theory is exhibited. In a full quantum theory of gravity, it is not the conformal variations of the metric which are expected to be the dynamical degrees of freedom. However, since the interaction should have the same form, we may expect that features of the present result will survive. In particular, the quantity $k_cβ$ is likely to be important.

And finally, the focus of this model has not been on cosmological systems (as contrasted to the matter-induced decoherence of spacetime described in [3] and the recent minisuperspace work [16]). The background spacetime was taken as Minkowski space and the temperature of the short-wavelength graviton state was taken to be its present-day value. Different background spacetimes might also be studied once the tensor nature of perturbative GR is restored.

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APPENDIX A: GLOSSARY OF NOTATION

For reference, we list here some of the notational conventions and important symbols used throughout the body of the paper.

$D$ is the number of space dimensions, so the case of physical interest is $D = 3$.

In the general discussion of Sec. II, the field $φ$ is divided into a set of variables $Φ$ which describe the system and the remainder, $ϕ$, which describe the environment.

In the more concrete subsequent calculations, $φ(x,t)$ is the field and $ϕ_k(t)$ its Fourier transform, defined by $\varphi_k(t)$.\footnote{The arguments or subscripts are sometimes omitted, in cases where the meaning of $φ$ (or $Φ$ or $ϕ$) should be clear from context.}
The space of wave vectors $k$ is divided into short- and long-wavelength regions $\mathcal{L}$ and $\mathcal{S}$ illustrated in Fig. [1]. An arbitrarily-chosen half $S/2$ of the region $\mathcal{S}$ is also defined there. When a wave vector falls in the long-wavelength $\mathcal{L}$, it is conventional to call it $q$ rather than $k$.

The Fourier modes $\varphi_k(t)$ are divided into long-wavelength modes (LWMs) $\Phi_k(t)$ and short-wavelength modes (SWMs) $\phi_k(t)$ defined by (1.4), making concrete the formal system-environment split described before.

The action $S[\varphi]$ is divided formally in Sec. [I] into a system-only part $S_\Phi[\Phi]$ and an interaction part $S_E[\phi, \Phi]$. In the specific realization for the scalar field theory, $S_E[\phi, \Phi]$ is divided into pieces $S_0[\phi], S_\phi[\phi, \Phi], S_{\phi\phi}[\phi, \Phi]$, and $S_{\phi\phi\phi}[\phi, \Phi]$, as defined in [4,8]. Also of use are $S_{1+\phi\phi}[\phi, \Phi] = S_0[\phi] + S_{\phi\phi}[\phi, \Phi]$ and $S_3[\phi, \Phi] = S_{1+\phi\phi}[\phi, \Phi] + S_{\phi\phi\phi}[\phi, \Phi]$. Each of these actions is defined by a corresponding Lagrangian $L[\phi, \Phi]$, $L_\Phi[\Phi]$, etc.

The influence phase $W[\Phi, \Phi']$ and influence functional $e^{iW[\Phi, \Phi']}$ are defined by (2.61). In addition, corresponding phases $W_0[\phi, \Phi]$ and $W_3[\Phi, \Phi']$ are defined by replacing $S_E[\phi, \Phi]$ with $S_{1+\phi\phi}[\phi, \Phi]$ and $S_3[\phi, \Phi]$, respectively.

The endpoints of the path $\varphi(x, t)$ are indicated by $\varphi_i(x) = \varphi(x, -T/2)$ and $\varphi_f(x) = \varphi(x, T/2)$. Similarly, $\Phi_i$ and $\Phi_f$ are the endpoints of the path $\Phi$, and $\phi_i$ and $\phi_f$ are the endpoints of the path $\phi$.

A prime on a path or its endpoints is used to distinguish the two arguments of the influence phase $W[\Phi, \Phi']$, or the decoherence functional, as in $D[\varphi, \varphi']$. The difference between the two paths is written as in $\Delta \Phi = \Phi - \Phi'$ and their average as in $\overline{\phi_i} = \frac{\phi_i + \phi'_i}{2}$.

When an object is a functional of an entire path, its argument is written in square brackets as in $S[\varphi]$. When it depends only on the value of the field at specific time, it is written, to emphasize the distinction, as an ordinary function as in $\rho(\varphi_i, \varphi'_i)$, even though the argument is technically still a function of the spatial coordinate $x$. Similarly, $\mathcal{D}[\phi]$ is reserved for path integrals, with the functional integration over the space $\mathcal{S}$.

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When an object is a functional of an entire path, its argument is written in square brackets as in $S[\varphi]$. When it depends only on the value of the field at specific time, it is written, to emphasize the distinction, as an ordinary function as in $\rho(\varphi_i, \varphi'_i)$, even though the argument is technically still a function of the spatial coordinate $x$. Similarly, $\mathcal{D}[\phi]$ is reserved for path integrals, with the functional integration over the endpoints written as an ordinary integral as in $\int d\phi$. The mismatched parentheses in $K_{1+\phi\phi}(\phi_f | \phi_i; \Phi)$ indicate that it is a “function” of the arguments before the semicolon and a functional of the argument after it.

The short-wavelength modes $\phi_k(t)$ are combined, by (5.14), into a vector $\phi$ in the space $\mathbb{R}^{S/2 \otimes S/2}$ described in Sec. [V A].

The following are all matrices which take vectors in $\mathbb{R}^{S/2 \otimes S/2}$ to other vectors in $\mathbb{R}^{S/2 \otimes S/2}$:

- $m(t)$, defined in (5.17); $\mathbf{w}(t)$, defined in (5.18); $A(t), B(t)$, and $C(t)$, defined in (5.25); $A[\Phi]$, $B[\Phi]$, and $C[\Phi]$, defined in (5.29); $A_\Phi$, $B_\Phi$, $C_\Phi$, defined in (5.32); and $R_0$ and $R_1[\Delta \Phi]$, defined in (5.44).

$V$ is a function having the character of its argument, so $V(k)$ is a number, and $V(\Omega_0)$ is a matrix.

In addition, $\mathcal{M}$ defined by (5.37) and $\mathcal{M}$ defined by (5.40) are matrices which take vectors in the product space $(\mathbb{R}^{S/2 \otimes S/2})^3$ to other vectors in $(\mathbb{R}^{S/2 \otimes S/2})^3$.

In most cases, a subscript of 0 or 1 indicates the zeroth- or first-order contribution to the quantity in question, in an expansion in powers of the coupling constant $\ell$.

Finally, the field averages $\langle \rangle$ and $\langle \rangle$ are defined in (7.14) and (7.23), respectively.

**APPENDIX B: PERTURBATIVE EXPANSIONS**

1. $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$

To make practical use of the exact expressions (5.25), we need to expand them in powers of the coupling constant $\ell$. Expanding to zeroth order is trivial, since $m_0 = 1$ and $\varpi_0 = \Omega_0^2$, where

$$\Omega_0 = \begin{pmatrix} \{k_1 \delta^D(k_1 - k_2)\} & \{0\} \\ \{0\} & \{k_1 \delta^D(k_1 - k_2)\} \end{pmatrix}$$

and of course

$$1 = \begin{pmatrix} \{\delta^D(k_1 - k_2)\} & \{0\} \\ \{0\} & \{\delta^D(k_1 - k_2)\} \end{pmatrix}.$$  

(B1)

(B2)

The expansion is

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12 Section VII C, in which the roles of the system and environment are reversed, is an exception to this.

13 This notation is also justified in the case of, for example, $\phi_i$, which may be thought of as a vector and not a function.

14 Note that in Appendix III there are equivalent matrices, defined according to (B10), mapping $\mathbb{R}^S$ to $\mathbb{R}^S$. This allows us to define, for instance, the components of $\mathfrak{B}_1$ as $\mathfrak{B}_1 k_i k_2$. 

26
\[ \mathcal{A}_0 = \cos \Omega_0 T = \mathcal{B}_0; \quad (B3a) \]
\[ \mathcal{C}_0 = \frac{\sin \Omega_0 T}{\Omega_0}. \quad (B3b) \]

Proceeding to the first order terms, we can substitute the first order expression
\[ \prod_{k=n}^{1} [-m^{-1}(\tilde{t}_k) \varpi(t_k)] = 1 + \ell \sum_{k=1}^{n} (-\Omega_0^2)^{n-k}[\varpi_1(t_k)\Omega_0^{-2} - m_1(t_k)](-\Omega_0^2)^k + \mathcal{O}(\ell^2) \quad (B4) \]
into (5.25a) and find
\[ \mathcal{B}_1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} \int_{t_i}^{t_f} dt \sum_{k=1}^{n} \left\{ \frac{\varpi_1(t_k)}{k_1} \sum_{k=1}^{2n-1} \left( \frac{2n-1}{2k-2} \right) \frac{\theta_{A1}}{k_1(t-t_i)} \right. \]
\[ \left. \times \frac{\theta_{B2}}{k_2(t_f-t)} \right\} \]
\[ -k_2 m_1(t_k) k_2 \sum_{k=1}^{n} \left( \frac{2n-1}{2k-1} \right) \frac{\theta_{A1}}{k_1(t-t_i)} \frac{\theta_{B2}}{k_2(t_f-t)} \]
\[ = \int_{t_i}^{t_f} dt \left[ m_1(t_k) k_1 k_2 \cos \theta_{A1} \sin \theta_{B2} - \varpi_1(t_k) k_1 k_2^{-1} \sin \theta_{A1} \cos \theta_{B2} \right] \quad (B7) \]

Then we can use the first order term in (5.25b),
\[ \mathcal{C}_1 = -\frac{d\mathcal{B}_1}{dt_f} \Omega_0^{-2} - \Omega_0 \sin \Omega_0 T \Omega_0^{-2} \varpi_1(t_f) \Omega_0^{-2}, \quad (B8) \]
to calculate
\[ \mathcal{C}_{1k_1 k_2} = -\int_{t_i}^{t_f} dt \left[ m_1(t_k) k_1 k_2 \cos \theta_{A1} \cos \theta_{B2} + \varpi_1(t_k) k_1 k_2 \frac{\sin \theta_{A1} \sin \theta_{B2}}{k_1 k_2} \right]; \quad (B9) \]
likewise, the first order term in (5.25c),
\[ \mathcal{A}_1 = -\frac{d\mathcal{C}_1}{dt_i} + m_1(t_i) \cos \Omega_0 T, \quad (B10) \]
gives
\[ \mathcal{A}_{1k_1 k_2} = \int_{t_i}^{t_f} dt \left[ m_1(t_k) k_1 k_2 \sin \theta_{A1} \cos \theta_{B2} - \varpi_1(t_k) k_1 k_2^{-1} \cos \theta_{A1} \sin \theta_{B2} \right]. \quad (B11) \]
It is straightforward to check that (B7), (B9), and (B11) satisfy (5.26) and (5.27).
2. $A$, $B$ and $C$

We can use the expansions for $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ calculated in Section B.1 to find expansions for

$$A[\Phi] = \mathfrak{A} \left( \frac{T}{2} \right) \mathcal{C}^{-1} \left( \frac{T}{2} \right) - \dot{m} \left( \frac{T}{2} \right), \quad (5.29a)$$

$$B[\Phi] = \mathcal{C}^{-1} \left( \frac{T}{2} \right) \mathfrak{B} \left( \frac{T}{2} \right) + \dot{m} \left( \frac{T}{2} \right), \quad (5.29b)$$

$$C[\Phi] = \mathcal{C}^{-1} \left( \frac{T}{2} \right). \quad (5.29c)$$

From B.3, the zero order terms are

$$A_0 = \frac{\Omega_0}{\tan \Omega_0 T} = B_0 \quad (B12a)$$

$$C_0 = \frac{\Omega_0}{\sin \Omega_0 T}; \quad (B12b)$$

Proceeding to the first order terms, we have

$$A_1[\Phi] = \mathfrak{A}_1 \frac{\Omega_0}{\sin \Omega_0 T} - \frac{\Omega_0}{\tan \Omega_0 T} \mathfrak{C}_1 - \frac{\Omega_0}{\sin \Omega_0 T} - \dot{m}_1 \left( \frac{T}{2} \right). \quad (B13)$$

Using (B11) and (B9) and rewriting the $\dot{m}$ boundary term using

$$- \dot{m}_1 \left( \frac{T}{2} \right) \sin k_1 T \sin k_2 T = \dot{m}(t)_{k_1 k_2} \sin \theta B_1 \sin \theta B_2 \left| \frac{T}{2} \right., \quad (B14)$$

we can calculate, via integration by parts,

$$A_{1 k_1 k_2} = m_1 \left( \frac{T}{2} \right)_{k_1 k_2} \left( \frac{k_1}{\tan k_1 T} + \frac{k_2}{\tan k_2 T} \right) - \frac{k_1 k_2}{\sin k_1 T \sin k_2 T} \int_{-T/2}^{T/2} dt \left[ m_1 (t)_{k_1 k_2} \cos \theta B_1 \cos \theta B_2 + n_1 (t)_{k_1 k_2} \sin \theta B_1 \sin \theta B_2 \right], \quad (B15)$$

where

$$n_1 (t)_{k_1 k_2} = \Omega_1 (t)_{k_1 k_2} - \dot{m}_1 (t)_{k_1 k_2} - m_1 (t)_{k_1 k_2} k_1^2 + k_2^2 \quad (B16)$$

Analogously, we find

$$B_{1 k_1 k_2} = m_1 \left( \frac{T}{2} \right)_{k_1 k_2} \left( \frac{k_1}{\tan k_1 T} + \frac{k_2}{\tan k_2 T} \right) - \frac{k_1 k_2}{\sin k_1 T \sin k_2 T} \int_{-T/2}^{T/2} dt \left[ m_1 (t)_{k_1 k_2} \cos \theta A_1 \cos \theta A_2 + n_1 (t)_{k_1 k_2} \sin \theta A_1 \sin \theta A_2 \right], \quad (B17)$$

The first order term $C_1 = - \frac{\Omega_0}{\sin \Omega_0 T} \mathfrak{C}_1 \frac{\Omega_0}{\tan \Omega_0 T}$ can be cast into a similar form after integration by parts to give

$$C_{1 k_1 k_2} = m_1 \left( \frac{T}{2} \right)_{k_1 k_2} \frac{k_2}{\sin k_2 T} + m_1 \left( \frac{T}{2} \right)_{k_1 k_2} \frac{k_1}{\sin k_1 T} - \frac{k_1 k_2}{\sin k_1 T \sin k_2 T} \int_{-T/2}^{T/2} dt \left[ m_1 (t)_{k_1 k_2} \cos \theta A_1 \cos \theta B_2 - n_1 (t)_{k_1 k_2} \sin \theta A_1 \sin \theta B_2 \right]. \quad (B18)$$

Application of trigonometric identities to expressions such as $\cos \theta A_1 \cos \theta B_2 = \cos k_1 (t + T/2) \cos k_2 (T/2 - t)$ allows us to rewrite (B15), (B17) and (B18) in terms of $k_\pm = k_1 \pm k_2$ as

28
These seem to be taking on a nice form in terms of more basically defined modes

\[ A_{1k, k_2} = m_1 \left( \frac{T}{2} \right) k_{1, k_2} \left( \frac{k_1}{\tan k_1 T} + \frac{k_2}{\tan k_2 T} \right) \]

\[ - \frac{k_{1, k_2}}{2 \sin k_1 T \sin k_2 T} \int_{-T/2}^{T/2} dt \left\{ [m_1(t) + n_1(t)]_{k_1, k_2} \left( \cos k_- t \cos \frac{k_- T}{2} + \sin k_- t \sin \frac{k_- T}{2} \right) 
+ [m_1(t) - n_1(t)]_{k_1, k_2} \left( \cos k_+ t \cos \frac{k_+ T}{2} + \sin k_+ t \sin \frac{k_+ T}{2} \right) \right\} \] (B19a)

\[ B_{1k, k_2} = m_1 \left( \frac{T}{2} \right) k_{1, k_2} \left( \frac{k_1}{\tan k_1 T} + \frac{k_2}{\tan k_2 T} \right) \]

\[ - \frac{k_{1, k_2}}{2 \sin k_1 T \sin k_2 T} \int_{-T/2}^{T/2} dt \left\{ [m_1(t) + n_1(t)]_{k_1, k_2} \left( \cos k_- t \cos \frac{k_- T}{2} - \sin k_- t \sin \frac{k_- T}{2} \right) 
+ [m_1(t) - n_1(t)]_{k_1, k_2} \left( \cos k_+ t \cos \frac{k_+ T}{2} - \sin k_+ t \sin \frac{k_+ T}{2} \right) \right\} \] (B19b)

\[ C_{1k, k_2} = m_1 \left( \frac{T}{2} \right) k_{1, k_2} \sin k_2 T + m_1 \left( \frac{T}{2} \right) k_{1, k_2} \sin k_1 T \]

\[ - \frac{k_{1, k_2}}{2 \sin k_1 T \sin k_2 T} \int_{-T/2}^{T/2} dt \left\{ [m_1(t) + n_1(t)]_{k_1, k_2} \left( \cos k_- t \cos \frac{k_- T}{2} - \sin k_- t \sin \frac{k_- T}{2} \right) 
+ [m_1(t) - n_1(t)]_{k_1, k_2} \left( \cos k_+ t \cos \frac{k_+ T}{2} - \sin k_+ t \sin \frac{k_+ T}{2} \right) \right\} \] (B19c)

\[ \chi_{\pm} = \int_{-T/2}^{T/2} dt [m_1(t) \mp n_1(t)]_{k_1, k_2} \cos k_{\pm} t + \text{boundary terms}, \quad \sigma_{\pm} = \int_{-T/2}^{T/2} dt [m_1(t) \mp n_1(t)]_{k_1, k_2} \sin k_{\pm} t + \text{boundary terms}, \] (B20a, b)

\[ \left( \begin{array}{c} B_{1k, k_2} \\ A_{1k, k_2} \\ C_{1k, k_2} \\ C_{1, k_2} \end{array} \right) = \frac{k_{1, k_2}}{2 \sin k_1 T \sin k_2 T} \left( \begin{array}{cccc} \cos \frac{k_- T}{2} & - \sin \frac{k_- T}{2} & \cos \frac{k_+ T}{2} & - \sin \frac{k_+ T}{2} \\ \cos \frac{k_+ T}{2} & - \sin \frac{k_+ T}{2} & \cos \frac{k_- T}{2} & - \sin \frac{k_- T}{2} \\ \cos \frac{k_- T}{2} & - \sin \frac{k_- T}{2} & \cos \frac{k_+ T}{2} & - \sin \frac{k_+ T}{2} \\ \cos \frac{k_+ T}{2} & - \sin \frac{k_+ T}{2} & \cos \frac{k_- T}{2} & - \sin \frac{k_- T}{2} \end{array} \right) \left( \begin{array}{c} \chi_- \\ \sigma_- \\ \chi_+ \\ \sigma_+ \end{array} \right), \] (B21)

where of course \( C_{1, k_2} = C_{1, k_2} \). We can use this to define \( \chi_{\pm} \) and \( \sigma_{\pm} \), and inverting the matrix determines the boundary terms, giving

\[ \chi_{\pm} = \int_{-T/2}^{T/2} dt [m_1(t) \mp n_1(t)]_{k_1, k_2} \cos k_{\pm} t \mp m_1(t)_{k_1, k_2} \frac{k_{\pm}}{k_{1, k_2}} \sin 2k_{\pm} t \bigg|_{-T/2}^{T/2}, \] (B22a)

\[ \sigma_{\pm} = \int_{-T/2}^{T/2} dt [m_1(t) \mp n_1(t)]_{k_1, k_2} \sin k_{\pm} t \pm m_1(t)_{k_1, k_2} \frac{k_{\pm}}{k_{1, k_2}} \cos 2k_{\pm} t \bigg|_{-T/2}^{T/2}. \] (B22b)

**APPENDIX C: EVALUATION OF THE TRACE IN SEC. V E**

The purpose of this appendix is to insert the values for \( A_1, B_1 \) and \( C_1 \) from Sec. B.2 of Appendix 3 into the expression \( E^2 |\Delta \Phi| \) appearing in the limit (5.52) on the influence phase, where

\[ \text{Note that the elements in these matrices are numbers, rather than matrices, i.e., the expression holds for a particular value of } k_1 \text{ and } k_2 \text{ so that there is no integral over } k_2 \text{ included in the matrix multiplication.} \]
Since 

Now it’s time to take the result in terms of the real matrices

Inverting \( \mathcal{R}_0 \) gives

so

Using the result (B21) from Sec. B3 of Appendix B and performing the matrix multiplication gives

Now it’s time to take the result in terms of the real matrices \( m_1[\Phi] \) and \( \varpi_1[\Phi] \) on \( \mathbb{R}^{S/2 \oplus S/2} \) defined by (5.17) and (5.18), and reconstruct from them useful expressions in terms of \( \{ \Phi_q \} \) and the complex matrices on \( \mathbb{C}^S \) with elements

defined by (B8), as well as [cf. (B10)]

These are related to the real matrices \( m \) and \( \varpi \) on \( \mathbb{R}^S \) (or \( \mathbb{R}^{S/2 \oplus S/2} \)) as follows: for \( k_1, k_2 \in S/2 \),

Since

\( 16 \) For the conversion of the range of the indices of these real matrices from \( S/2 \oplus S/2 \) to \( S \), see (B6).
we can use (C4d) and (C4e), along with \( |\sum_i \alpha_i \beta_i|^2 = \frac{1}{2} \left( |\sum_i \alpha_i|^2 + |\sum_i \beta_i|^2 \right) \), to write

\[
E^2[\Delta \Phi] = \int \frac{d^2k_1 d^2k_2}{4V(k_1)V(k_2)} \left\{ \sin^2 \frac{\theta_{12}}{2} \int_{-T/2}^{T/2} dt \Delta \Phi(t) e^{i k \cdot \mathbf{q}} - i \frac{k_1 k_2}{k_1 k_2} [e^{2ik \cdot \mathbf{q}} \Delta \Phi(t)]_{-T/2}^{T/2} \right\}^2 + \sin^2 \frac{\theta_{12}}{2} \int_{-T/2}^{T/2} dt \Delta \Phi(t) e^{-i k \cdot \mathbf{q}} + i \frac{k_1 k_2}{k_1 k_2} [e^{-2ik \cdot \mathbf{q}} \Delta \Phi(t)]_{-T/2}^{T/2} \right\}^2 \]

APPENDIX D: THE EFFECTS OF THE LINEAR AND CUBIC TERMS

Here we add the terms \( S_\phi \) and \( S_{\phi\phi} \) back into the action, and determine what effect, if any, this has on the influence phase (5.52).

1. The linear terms

The effect of the linear term \( S_\phi \) can, as usual, be elucidated by completing the square, as shown in this section. We define the “all-but-cubic” Lagrangian

\[
L_3[\phi, \Phi] = L_0[\phi, \Phi] + \ell L_\phi[\phi, \Phi] \tag{D1}
\]

by adding to the quadratic action considered in Sec. V the linear terms

\[
\ell L_\phi[\phi, \Phi] = \int_s d^2k \left( -\frac{\phi^*}{2} \tilde{x}_k + \frac{\phi}{2} \tilde{y}_k \right) \tag{D2}
\]

where\[^{17}\] (cf. (4.8d)]

\[
\tilde{x}_k = -\frac{\ell}{2} \int_s d^2q \left( Q^2 \Phi_{k-q} \Phi_q - \dot{\Phi}_{k-q} \dot{\Phi}_q \right) \tag{D3a}
\]

\[
\tilde{y}_k = -\frac{\ell}{2} \int_s d^2q \left( \Phi_{k-q} \Phi_q - \Phi_{k-q} \Phi_q \right) \tag{D3b}
\]

with \( Q^2 \) bearing the same relation to \( -\mathbf{k} \) and \( \mathbf{q} \) that \( k_1^2 \) [cf. (4.3)] bore to \( k_1 \) and \( k_2 \):

\[
Q^2 = -(-\mathbf{k}) \cdot (\mathbf{k} - \mathbf{q}) - (\mathbf{k} - \mathbf{q}) \cdot \mathbf{q} - \mathbf{q} \cdot (-\mathbf{k}) = k^2 + q^2 - k \cdot q. \tag{D4}
\]

The reality condition \( \Phi_{k-q} = \Phi^*_q \) forces \( \tilde{x}_k = \tilde{x}_{k}^* \) and \( \tilde{y}_k = \tilde{y}_{k} \), so we can use the identity

\[
\tilde{v}^{ex} v^{ex} = \int_s d^2k \tilde{v}_k^* w_k = \int_s d^2k \left( v_k^R w_k^R + v_k^I w_k^I \right) = 2 \int_s d^2k \left( v_k^R w_k^R + v_k^I w_k^I \right) = v^{IR} w, \tag{D5}
\]

\[^{17}\] Recall that \( \Phi_{k-q} = 0 \) when \( k - q \not\in \mathcal{L} \)
where $v$ and $w$ are vectors in $\mathbb{R}^{S^2/\mathbb{B}_2}$ defined as in (5.19), to write $\ell L_\phi = -\phi^{tr} \vec{x} + \phi^{tr} \vec{y}$; by integrating by parts, we can also write this, for an arbitrary $z(t)$ as

$$
\ell L_\phi = -\phi^{tr} x + \phi^{tr} y + \frac{d}{dt} (\phi^{tr} z),
$$

(D6)

where

$$
x = \vec{x} + \dot{z},
$$

(D7a)

$$
y = \vec{y} - z,
$$

(D7b)

to give

$$
L_3[\phi, \Phi] = \frac{1}{2} \left[ \phi^{tr} m \dot{\phi} + 2 \phi^{tr} y - \phi^{tr} \bar{\omega} \phi - 2 \phi^{tr} x + \frac{d}{dt} (\phi^{tr} \dot{m} \phi + 2 \phi^{tr} z) \right].
$$

(D8)

If we can choose $z$ such that

$$
y = m \frac{d}{dt} (\bar{\omega}^{-1} x),
$$

(D9)

$L_3$ is related to $L_{0+\phi}$ by

$$
L_{0+\phi}[\phi + \bar{\omega}^{-1} x] = L_3[\phi, \Phi] + \frac{y^{tr} m^{tr} y - \frac{x^{tr} \bar{\omega}^{-1} x}{2}}{2} + \frac{d}{dt} \left[ \phi^{tr} (\dot{m} \bar{\omega}^{-1} x - z) \right] + \frac{d}{dt} \frac{d}{dt} \frac{d}{dt} \left[ \phi^{tr} \bar{m} \bar{\omega}^{-1} x \right].
$$

(D10)

The condition (D9) is equivalent to the second order inhomogeneous ODE

$$
\frac{d}{dt} (\bar{\omega}^{-1} \dot{z}) + m^{-1} z = m^{-1} \dot{y} - \frac{d}{dt} (\bar{\omega}^{-1} \dot{x}).
$$

(D11)

A particular $\Phi(t)$ generates $\vec{x}(t)$ and $\vec{y}(t)$ via (D3), and for that source term, we can solve (D11), with the freedom to fix two boundary conditions which are functions of $z_i$, $z_f$, $(\dot{z})_i$, and $(\dot{z})_f$.

We can use this expression for $L_3$ to express $K_3$, the propagator for $S_3$, in terms of $K_{0+\phi}$ as

$$
K_3(\phi_f | \phi_i; \Phi) = K_{0+\phi}(\phi_f | \phi_i; \Phi) \exp \left\{ i \left( \begin{array}{c} \phi_f \\ \phi_i \end{array} \right)^{tr} A' \left( \begin{array}{c} \phi_f \\ \phi_i \end{array} \right) \right\} e^{i \psi[\Phi]},
$$

(D12)

where

$$
\psi[\Phi] = \frac{1}{2} \int_{-T/2}^{T/2} dt \left( x^{tr} \bar{\omega}^{-1} x - y^{tr} m^{tr} y \right) + \frac{1}{2} \left( \begin{array}{c} \bar{\omega}^{-1} x_i \\ \bar{\omega}^{-1} x_i \end{array} \right) \left( \begin{array}{ccc} B[\Phi] & -m_i \\ -C[\Phi] & -A[\Phi] \end{array} \right) \left( \begin{array}{c} \bar{\omega}^{-1} x_i \\ \bar{\omega}^{-1} x_i \end{array} \right),
$$

(D13)

is a real phase, and

$$
A'[\Phi] = \left( \begin{array}{cc} z_f - (\dot{m} \bar{\omega}^{-1} x_i) & f \\ -z_i + (\dot{m} \bar{\omega}^{-1} x_i) & f \end{array} \right) + \left( \begin{array}{cc} B[\Phi] & -m_i \\ -C[\Phi] & -A[\Phi] \end{array} \right) \left( \begin{array}{cc} (\bar{\omega}^{-1} x_i) & f \\ (\bar{\omega}^{-1} x_i) & f \end{array} \right).
$$

(D14)

By substituting for $x$ using (D7a), we see that $A'[\Phi] = 0$ is just a pair of linear first order boundary conditions on $z(t)$, and so we can choose the solution to (D11) to obey them, leaving

$$
K_3(\phi_f | \phi_i; \Phi) = K_{0+\phi}(\phi_f | \phi_i; \Phi) e^{i \psi[\Phi]}.
$$

(D15)

Proceeding along the same lines as (5.20), we find that

$$
e^{i W_3[\Phi; \Phi']} = e^{i W_{0+\phi}[\Phi; \Phi']} e^{i (\psi[\Phi] - \psi[\Phi'])}.
$$

(D16)

Since $\psi[\Phi]$ is real, and it is the imaginary part of $W$ which imposes decoherence,

$$
|e^{i W_3[\Phi; \Phi']}| = |e^{i W_{0+\phi}[\Phi; \Phi']}|
$$

(6.1)

and adding in the linear terms does not change the result (5.52).
2. The cubic terms

Now we are ready to consider the full environmental action

\[ S_E[\phi, \Phi] = S_3[\phi, \Phi] + S_{\phi\Phi}[\phi] \]  
(D17)

including the cubic terms from

\[ L_{\phi\Phi}[\phi] = -\frac{1}{2} \int d^Dk_1d^Dk_2d^Dk_3 \delta^D(k_1+k_2+k_3) \left( \phi_1 \dot{\phi}_2 \phi_3 - \frac{k_1^2+k_2^2+k_3^2}{6} \phi_1 \phi_2 \phi_3 \right). \]  
(D18)

Here we need to resort to using a generating functional

\[ Z[J, J', \Phi, \Phi'] = \int \mathcal{D}\phi \mathcal{D}\phi' \rho_\phi(\phi, \phi') \delta(\phi' - \phi) \exp \left\{ i \left[ S_3[\phi, \Phi] - S_3[\phi', \Phi'] \right] + \int_{-T/2}^{T/2} dt [\phi'^{tr} J - \phi^{tr} J'] \right\} \]  
(D19a)

and expressing

\[ e^{iW[\Phi, \Phi']} = \exp \left( i\ell S_{\phi\Phi} \left[ \frac{1}{i} \frac{\partial}{\partial J} \right] - i\ell S_{\phi\Phi} \left[ -\frac{1}{i} \frac{\partial}{\partial J'} \right] \right) Z[J, J', \Phi, \Phi']. \]  
(D19b)

\[ a. \text{ The wrong way to complete the square} \]

The “propagators” involved in constructing the generating functional

\[ Z[J, J', \Phi, \Phi'] = \int d\phi_1 d\phi_2 d\phi_3 \rho_\phi(\phi_1, \phi_2, \phi_3) \mathcal{K}_Z(\phi_1, \Phi, J) \mathcal{K}_Z(\phi_2, \Phi', J) \mathcal{K}_Z(\phi_3, \Phi, J') \]  
(D20)

are of the form

\[ \mathcal{K}_Z(\phi_f | \phi_i ; \Phi, J) = \int \mathcal{D}\phi e^{i \left( S_3[\phi, \Phi] + \frac{\ell^2}{2} \int_{-T/2}^{T/2} dt \phi^{tr} J \right)} \]  
(D21)

and involve the modified “Lagrangian”

\[ L_Z[\phi, \Phi, J] = L_3[\phi, \Phi] + J^{tr} \phi = \frac{1}{2} \left[ \phi^{tr} m \dot{\phi} + 2 \phi^{tr} \ddot{y} - \phi^{tr} \omega \phi - 2 \phi^{tr} (\ddot{x} - J) \right] + \frac{d}{dt} \left( \phi^{tr} \dot{m} \phi \right). \]  
(D22)

We might try to carry out the same completion of the square as was done in Sec. D11, getting the form (D6), where now

\[ x = \ddot{x} + \ddot{z} - J. \]  
(D7a)

The ODE for \( z \) becomes

\[ \frac{d}{dt} (\omega^{-1} \dot{z}) + m^{-1} = m^{-1} \ddot{y} - \frac{d}{dt} (\omega^{-1} (\ddot{x} - J)); \]  
(D12)

once again, we would find an expression like

\[ \mathcal{K}_Z(\phi_f | \phi_i ; \Phi, J) = \mathcal{K}_0(\phi_f | \phi_i ; \Phi) \exp \left\{ i \left( \phi_f \phi_i \right)^{tr} \mathcal{X}[\Phi, J] \right\} \]  
(D12)

with the expressions (D13) and (D14) for \( \psi[\Phi, J] \) and \( \mathcal{X}[\Phi, J] \) in terms of \( x \) still holding. Again, the boundary conditions on (D12) would allow us to set \( \mathcal{X}[\Phi, J] = 0 \), leaving

\[ \mathcal{K}_Z(\phi_f | \phi_i ; \Phi, J) = \mathcal{K}_0(\phi_f | \phi_i ; \Phi) e^{i\psi[\Phi, J]}. \]  
(D13)

However, this form is not convenient, even if we insert \( x = \ddot{x} + \ddot{z} - J \), since the expression would depend on \( J \) not only explicitly, but also implicitly via the solution \( z[\Phi, J] \) to (D11).

\[ ^{18} \text{The choice of sign of } J' \text{ may seem unusual, but it allows us to write the argument of the exponential in (D19a) as } S_Z[\phi, \Phi, J] - S_Z[\phi', \Phi', J'] \text{ rather than } S_Z[\phi, \Phi, J] - S_Z[\phi', \Phi', -J']. \]

33
b. The correct way to complete the square

Since we cannot fruitfully complete the square for the $J$-terms in the way we did for $L_\phi$ in Sec. [D1], let us instead combine $L_\phi$ with the $J$-terms by integrating by parts until $y = 0$, i.e.,

\[
x = \bar{x} + \frac{\dot{y}}{m} \quad \text{and} \quad z = \frac{\ddot{y}}{m} \tag{D23a}
\]

so that

\[
L_Z[\phi, \Phi, J] = \frac{1}{2} \left[ \phi'^T m \phi' - \phi'^T \omega \phi + 2 \phi'^T \bar{J} + \frac{d}{dt} (\phi'^T \dot{m} \phi + 2 \phi'^T \bar{y}) \right] \tag{D24}
\]

where $\bar{J} = \bar{J}$. We can construct this perturbatively, with the lowest order term being

\[
(G \circ J)(t) = \int_{-T/2}^{T/2} dt' G(t, t') J(t') \tag{D26}
\]

obeying $(\partial_t m \partial_t + \omega) G \circ J = J$. We can construct this perturbatively, with the lowest order term being

\[
G_0(t, t') = \frac{\sin \Omega_0 |t - t'|}{2 \Omega_0} \tag{D27}
\]

Then we can complete the square to obtain

\[
K_Z(\phi_f | \phi_i; \Phi, J) = K_{0+\phi\phi} \left( \phi_f - (G \circ \bar{J})_f \big| \phi_i - (G \circ \bar{J})_i \right)
\times \exp \left( i \left\{ -S_{0+\phi\phi}[G \circ J, \Phi] + \phi'^T (\partial_t (mG \circ J) + \bar{y}) \right|_{-T/2}^{T/2} \right) \tag{D28}
\]

The generating functional can thus be expanded, using the form of $\rho_\phi$ from [5.31], and making the transformation (cf. [5.32])

\[
\begin{pmatrix} \phi_f \\ \phi_i' \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \Delta \phi_f/2 \\ \phi_f \\ \phi_i \\ \Delta \phi_i/2 \end{pmatrix} \tag{D29}
\]

as

\[
Z[J, J', \Phi, \Phi'] \propto \int \frac{d\phi_f d\phi_i d\bar{J}}{\sqrt{\det(2\pi \mathcal{C}[\Phi]) \det(2\pi \mathcal{C}[\Phi']})} \exp \left\{ -\frac{1}{2} \left( \begin{pmatrix} \phi_f \\ \phi_i \\ \Delta \phi_i/2 \end{pmatrix}^T \mathcal{M} \begin{pmatrix} \phi_f \\ \phi_i \\ \Delta \phi_i/2 \end{pmatrix} + \frac{\phi_f}{\phi_i} \bar{J} \right) \right\}, \tag{D30}
\]

where we have defined the matrix

\[
\mathcal{P} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \Omega_0^2 \mathcal{V}(\Omega_0) & 0 & 0 \\ 0 & 0 & \mathcal{V}^{-1}(\Omega_0) \end{pmatrix} \tag{D31}
\]

so that $\mathcal{M}$, defined in [5.37], can be written
\[ M = P - i \begin{pmatrix} B_- & -C_- & -C_+ \\ -C_- & A_- & A_+ \\ -C_+ & A_+ & A_- \end{pmatrix} \]  \hspace{1cm} \text{(D32)}

as well as

\[ U[\tilde{J}] = \begin{pmatrix} (G \circ \tilde{J})_f \\ (G \circ \tilde{J})_i \\ \Delta(G \circ \tilde{J})_i/2 \end{pmatrix} \hspace{1cm} \text{(D33a)} \]

\[ W[\tilde{J}] = \begin{pmatrix} -B_+ \Delta(G \circ \tilde{J})_f/2 + \Delta \left[ \partial_i(mG \circ \tilde{J})_f \right] \\ C_+^{\text{tr}} \Delta(G \circ \tilde{J})_f/2 - \Delta \left[ \partial_i(mG \circ \tilde{J})_f \right] \\ C_-^{\text{tr}} \Delta(G \circ \tilde{J})_f/2 - 2 \left[ \partial_i(mG \circ \tilde{J})_f \right] \end{pmatrix} \hspace{1cm} \text{(D33b)} \]

\[ Q[\tilde{J}, \tilde{J}'] = -S_{0+\phi}[G \circ \tilde{J}, \Phi] + S_{0+\phi} [G' \circ \tilde{J}', \Phi'] \]

\[ + \frac{1}{2} \begin{pmatrix} \Delta(G \circ \tilde{J})_f/2 \\ (G \circ \tilde{J})_i \\ \Delta(G \circ \tilde{J})_i/2 \end{pmatrix}^{\text{tr}} \begin{pmatrix} \Delta(G \circ \tilde{J})_f/2 \\ (G \circ \tilde{J})_i \\ \Delta(G \circ \tilde{J})_i/2 \end{pmatrix} \hspace{1cm} \text{(D33c)} \]

Completing the square in (D33) and integrating gives

\[ Z[J, J', \Phi, \Phi'] = e^{i\omega_{0+\phi}[\Phi, \Phi']} \exp \left\{ \frac{1}{2} \left[ U[\tilde{J}]ight]^{\text{tr}} (M - P) + iW[\tilde{J}]^{\text{tr}} \right\} M^{-1} \left\{ (M - P)U[\tilde{J}] + iW[\tilde{J}] \right\} + O(1) \hspace{1cm} \text{(D34)} \]

Expanding (D19b) in a perturbation series, we see that terms beyond the zeroth have at least one factor of \( \ell \), from the \( \ell S_{0+\phi} \). Again, the only way that a perturbative expression could affect the non-perturbative result \( e^{iW} \ll 1 \) is if some of the terms have a \( \ell/\beta \) behavior. Thus, we should look for the terms in the exponential of (D34) which are larger than \( O(1) \) to see if any \( O(\beta^{-1}) \) terms can produce significant contributions. The only object which can be larger than \( O(1) \) is \( M^{-1} \) [the matrices \( M \) and \( P \) individually have \( V^{-1} \) eigenvalues, but the combination \( M - P \) is \( O(1) \)]. Since the smallest eigenvalue of \( M \) is \( O(V) + O(\ell) \), \( \ell M^{-1} \) will also be no larger than \( O(1) \). And since the terms \( \tilde{y} \) and \( \tilde{J} - J = -\tilde{x} + \tilde{y} \) coming from \( S_\phi \) are \( O(\ell) \), this means that

\[ Z[J, J', \Phi, \Phi'] = e^{i\omega_{0+\phi}[\Phi, \Phi']} \exp \left\{ \frac{1}{2} \left[ U[\tilde{J}]ight]^{\text{tr}} (M - P) + iW[\tilde{J}]^{\text{tr}} \right\} M^{-1} \left\{ (M - P)U[\tilde{J}] + iW[\tilde{J}] \right\} + O(1) \hspace{1cm} \text{(D35)} \]

Now,

\[ \left[ U^{\text{tr}}(M - P) + iW^{\text{tr}} \right] M^{-1} \left[ (M - P)U + iW \right] = -U^{\text{tr}}PU + (U^{\text{tr}}P - iW^{\text{tr}})M^{-1}(PU - iW) + O(1); \hspace{1cm} \text{(D36)} \]

if we use (5.46) to write \( M^{-1} \) in terms of \( \tilde{M}^{-1} \) and manipulate \( M^{-1}P \) using \( 4V^{-1}(\Omega_0) = \alpha + iA_- \), we have

\[ -U^{\text{tr}}PU + (U^{\text{tr}}P - iW^{\text{tr}})M^{-1}(PU - iW) \]

\[ = -U^{\text{tr}}PU + \left[ U^{\text{tr}} \begin{pmatrix} 0 & 0 & \Omega_0^2V(\Omega_0) \\ -i(1 + iA_- \alpha^{-1})C_+^{\text{tr}} & i(1 + iA_- \alpha^{-1})A_+ & 4V^{-1}(\Omega_0) \end{pmatrix} \right. \]

\[ \left. -iW^{\text{tr}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right] \times \tilde{M}^{-1} \left[ \begin{pmatrix} 0 & -iC_+ (1 + iA_- \alpha^{-1}) \\ 0 & \Omega_0^2V(\Omega_0) \end{pmatrix} \right] U - \left( \begin{pmatrix} 0 & -iC_+ (1 + iA_- \alpha^{-1}) \\ 0 & \Omega_0^2V(\Omega_0) \end{pmatrix} \right) \right] \hspace{1cm} \text{(D37)} \]

Because the matrices \( P \) and \( (\Omega_0 - i\Omega_1[\Delta\Phi])^{-1} + \alpha^{-1} \) are in block diagonal form, we split up the expression (D37) into parts corresponding to each block. That corresponding to the lower block is
\[-\mathcal{U}_3^{tr} \mathcal{V}^{-1}(\Omega_0) \mathcal{U}_3 + \left[ \mathcal{U}_3^{tr} \mathcal{V}^{-1}(\Omega_0) - i \mathcal{W}_3^{tr} \right] \alpha^{-1} \left[ 4 \mathcal{V}^{-1}(\Omega_0) \mathcal{U}_3 - i \mathcal{W}_3 \right], \]

where $\mathcal{U}_3$ is the bottom third of $\mathcal{U}$,

\[
\mathcal{U}_3 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \mathcal{U}.
\]

Using $4 \mathcal{V}^{-1}(\Omega_0) = \alpha + i \mathcal{A}$, the $\mathcal{V}^{-1}(\Omega_0)$ pieces of (D38) cancel, leaving an expression which is $\mathcal{O}(1)$. This leaves us with the piece from the upper block, making the exponential in (D34)

\[
\frac{1}{2} \left[ \mathcal{U}^{tr} \left( \begin{array}{cc} 0 & 0 \\ 0 & \Omega_0^2 \mathcal{V}(\Omega_0) \end{array} \right) - i \mathcal{W}^{tr} \left( \begin{array}{cc} 1 & 0 \\ -i\alpha^{-1} \mathcal{C}_+^{tr} & i\alpha^{-1} \mathcal{A}_+ \end{array} \right) \right] (\mathcal{N}_0 - i\hbar \mathcal{N}_1 [\Delta \Phi])^{-1} \\
\times \left[ \left( \begin{array}{cc} 0 & 0 \\ \Omega_0^2 \mathcal{V}(\Omega_0) & i \mathcal{A}_+ \end{array} \right) \mathcal{U} - i \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \mathcal{U} \right] + \mathcal{O}(1);
\]

inserting (D33a) and (D33b) and discarding $\mathcal{O}(1)$ terms \{including $(\mathcal{N}_0 - i\hbar \mathcal{N}_1 [\Delta \Phi])^{-1} \mathcal{V}(\Omega_0)$\}, we end up with

\[
Z[J, J', \Phi, \Phi'] = e^{i \mathcal{W}_3^{tr} \mathcal{V}^{tr} [\Phi, \Phi']} \exp \left[ -\frac{1}{2} \left( -C_0 G_0 \circ (\Delta J)_i + B_0 G_0 \circ (\Delta J)_f - \dot{G}_0 \circ (\Delta J)_f \right) \mathcal{U} \right] \\
\times (\mathcal{N}_0 - i\hbar \mathcal{N}_1 [\Delta \Phi])^{-1} \left( -C_0 G_0 \circ (\Delta J)_i + B_0 G_0 \circ (\Delta J)_f - \dot{G}_0 \circ (\Delta J)_f \right) + \mathcal{O}(1).
\]

The fact that the leading term in the exponential in (D41) depends only upon $\Delta J = J - J'$ is crucial, because of the operator

\[
\ell S_{\phi \phi} \left[ \frac{1}{i} \frac{D}{D \mathcal{J}} \right] - \ell S_{\phi \phi} \left[ \frac{1}{i} \frac{D}{D \mathcal{J}'} \right]
\]

in (D19b), which annihilates any functional independent of $\phi$ and depending on $J^{(i)}$ only in the combination $J - J'$. If all of the terms in the exponential in $Z$ were functions of $\Delta J$ alone, that would mean that $e^{i\mathcal{W}} = e^{i\mathcal{W}_3}$; however, there are $\mathcal{O}(1)$ terms in the exponential which depend on $\mathcal{J}$. The situation can be written as

\[
Z[J, J', \Phi, \Phi'] = e^{i \mathcal{W}_3^{tr} \mathcal{V}^{tr} [\Phi, \Phi']} \exp \left( \frac{1}{2} \Delta J \circ \mathcal{F}_{-1} \circ \Delta J + \frac{1}{2} J^{(i)} \circ \mathcal{F}_0 \circ \Delta J + J^{(i)} \circ \mathcal{G}_0 \right),
\]

where $\frac{1}{2} \Delta J \circ \mathcal{F}_{-1} \circ \Delta J$ is the argument of the exponential in (D43), and $\frac{1}{2} J^{(i)} \circ \mathcal{F}_0 \circ \Delta J$ and $J^{(i)} \circ \mathcal{G}_0$ are the quadratic and linear terms of $\mathcal{O}(1)$. Thinking in terms of a diagrammatic expansion, this means that there are three kinds of “propagators” in $Z$:

\[
\begin{align*}
\mathcal{F}_{-1} & \quad (D44a) \\
\mathcal{F}_0 & \quad (D44b) \\
\mathcal{G}_0 & \quad (D44c)
\end{align*}
\]

(note that the last is not truly a propagator, since it accepts only one “input”). These are used to connect the vertices, which all have the form

\[
\ell
\]

(D45)
A term in the series which has more $\ell$ vertices than $F_{-1}$ propagators will be perturbatively small, one with the same number will be $O(1)$, and one with more $F_{-1}$ propagators than $\ell$ vertices will be able to disrupt the perturbative analysis and have an impact upon $e^{iW}$. We can make a list of the objects in the theory by their order in perturbation theory and number of legs (with the legs on propagators counted negative so that a closed diagram has zero net legs):

| Graph | Order | Legs |
|-------|-------|------|
| D44a  | -1    | -2   |
| D44b  | 0     | -2   |
| D44c  | 0     | -1   |
| D45   | 1     | 3    |

Since the vertex (D43) has three legs and the propagator (D44a) has minus two, we’d expect divergent graphs starting with

\[ \begin{array}{c}
\text{F}_{-1} \\
\text{F}_{-1} \\
\text{F}_{-1} \\
\end{array} \]

\[ \ell \]

\[ \ell \]

\[ \text{F}_{-1} \]

\[ \text{D46} \]

However, in this case we have just the situation described above: all of the propagators depend only on $\Delta J$, so the graph vanishes. This sort of identity places the restriction that at least one leg of a vertex must be coupled to an $F_0$ or $G_0$ propagator. This means that we must abandon (D43) by itself and use as our primitive vertices

\[ \begin{array}{c}
\text{F}_0 \\
\text{F}_0 \\
\end{array} \]

\[ \ell \]

\[ \text{D47a} \]

\[ \begin{array}{c}
\text{G}_0 \\
\end{array} \]

\[ \ell \]

\[ \text{D47b} \]

which makes the pieces out of which non-vanishing graphs can be constructed

| Graph | Order | Legs |
|-------|-------|------|
| D44a  | -1    | -2   |
| D44b  | 0     | -2   |
| D44c  | 0     | -1   |
| D47a  | 2     | 4    |
| D47b  | 1     | 2    |

Now the most divergent graph which can be constructed with zero net legs is $O(1)$. This means that, perturbatively, the influence functional is

\[ e^{iW[\Phi,\Phi']} = O(1) \times e^{iW_3[\Phi,\Phi']} \]

so, perturbatively at least,

\[ |e^{iW[\Phi,\Phi']}| \lesssim \{1 + E^2[\Delta \Phi]\}^{-1/4}. \]

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