Implementation QSGS iteration applied to fractional diffusion equation

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Abstract. In this study, we propose approximate solution of the time-fractional diffusion equation (TFDE’s) based on a quarter-sweep implicit finite difference approximation equation. To derive this approximation equation, the Caputo’s time fractional derivative has been used to discretize the proposed problems. By using the Caputo’s finite difference approximation equation, a linear system will be generated and solved iteratively. In addition to that, formulation and implementation the Quarter-Sweep Gauss-Seidel (QSGS) iterative method are also presented. Based on numerical results of the proposed iterative method, it can be concluded that the proposed iterative method is superior to the FSGS and HSGS iterative method.

1. Introduction
In the recent years, many studies use fractional partial differential equations (FPDE’s) [1]–[5] for solving fractional problems to derive numerical and/or analytical solutions. Based on iteration methods for solving FPDE’s model with constant coefficients and analytical solutions in a diffusion model [4].

Therefore, there are numerical iterations proposed for solving the TFDE’s, such as explicit finite difference [6], weighted average finite difference method [7]. Nevertheless the finite difference method are available in the literature [8].

To solve the TFDE’s model needs to be discretized. Based on the Caputo’s implicit finite difference scheme, the approximation equations can be arranged to develop a linear system at each time level. In order to solve a linear systems, many researchers also have discussed concept of the iterative methods such as Young [9], Hackbusch [10] and Saad [11]. Besides these iterative methods, block iteration has shown by Evans [12]. Furthermore, Ibrahim and Abdullah [13], and Yousif and Evans [14] have pointed out the efficiency of block iterative methods. For solving the large linear system, Abdullah [15] initiated Half-Sweep (HS) iteration for solving in any linear-systems equations. Differently from the HS iteration approach, Othman and Abdullah [16] have expanded this approach to initiate the Modified Explicit Group (MEG) method based on the Quarter-Sweep (QS) approach. It is proved that this method is one of most efficient block iterative method in solving any linear-system equations as compared with ED and EDG iteration methods. Also, other researcher has showed the capability of the QS iteration, see [17]
In this paper, we implement the performance of the QSGS iteration method for solving TFDE’s based on the Caputo’s implicit finite difference approximation equation. For demonstrate the capability of the QSGS iteration, we also show implementation the Full-Sweep Gauss-Seidel (FSGS) and Half-Sweep Gauss-Seidel (HSGS) iteration which is use as a control method.

The content of this research as: In Sections 2 and 3, approximate the equation of the Caputo’s operator and Caputo’s implicit finite difference. In Section 4, formulation of the QSGS iteration is shown. In Section 5 display evaluation example and its results and conclusion is shown in Section 6.

2. Preliminaries

To begin the derivation of the QSGS iteration, the TFDE’s can be defined as

\[
\frac{\partial^\alpha Y(z,M)}{\partial z^\alpha} = a(z)\frac{\partial^2 Y(z,M)}{\partial z^2} + b(z)\frac{\partial Y(z,M)}{\partial z} + c(z)Y(z,M)
\]

with \(a(z), b(z)\) and \(c(z)\) are functions or constants value and \(\alpha\) is a parameter for the TFDe’s.

Previous to discretizing problem (1), some basic definitions given for fractional derivative theory:

**Definition 1.** [8] The Riemann-Liouville operator

\[
L^\alpha f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z (z-M)^{\alpha-1} f(M) dM, \alpha > 0, z > 0
\]

**Definition 2.** [8] The Caputo’s fractional operator,

\[
C^\alpha f(z) = \frac{1}{\Gamma(H - \alpha)} \int_0^z (z-M)^{\alpha-H+1} f(M) dM, \alpha > 0
\]

with \(H - 1 < \alpha \leq H, H \in \mathbb{N}, x > 0\)

From Definitions 1 and 2, \(\Gamma(\alpha)\) is known as a gamma function, which is given by

\[
\Gamma(\alpha) = \int_0^\infty z^{H-1} e^{-z} dH
\]

For solution the numerical of (TFDE’s), to Eq. (1), we get numerical approximations by using the Caputo’s derivative definition with Dirichlet boundary conditions and the non-local fractional operator. This estimation in Eq.(1) can be categories as unconditionally stable scheme. In order to solution Eq.(1), use associated with the initial conditions

\[Y(z,0) = f(z),\]

and boundary conditions

\[Y(0,M) = g_0(M), Y(\ell, M) = g_1(M),\]

with \(g_0(M), g_1(M)\), and \(f(M)\), are known functions. Based on a discretization approximation to the TFDE’s in Eq. (1), we using Caputo’s fractional derivative operator of order \(\alpha\), defined by \([8], [9]\)

\[
\frac{\partial^\alpha Y(z,M)}{\partial M^\alpha} = \frac{1}{\Gamma(\alpha-1)} \int_0^z (z-s)^{\alpha-1} (M-s)^{\alpha-1} ds, \ M > 0, \ 0 < \alpha < 1
\]

(4)

3. Implicit Finite Difference For TFDE’s

Based on Eq.(4), the formulation of Caputo’s fractional derivative operator of the first approximation method is given as

\[
D^n Y_{i,n} = \sigma_{a,k} \sum_{j=1}^{n} \omega_j^{\alpha_a} \left( Y_{i,n+j} - Y_{i,n-j} \right)
\]

with \(\sigma_{a,k} = \frac{1}{\Gamma(1-\alpha)\Gamma(1-\alpha)k^\alpha}\)

and
\[ \alpha_j^{(\alpha)} = j^{1-\alpha} - (j-1)^{1-\alpha}. \]

Firstly, to discretize Eq. (1), for solving domain of the Eq.1 be partitioned homogeneously. Furthermore, we have integers \( H \) and \( n \) in which the grid measure in space and time directions are given as \( h = \Delta Y = \frac{\gamma - 0}{H} \) and \( k = \Delta M = \frac{T}{n} \). So, we develop the grid network for domain with the grid points in the space interval \([0, \gamma]\) are defined as the numbers \( z_i = ih, \ i = 0,1,2,...,H \) and for the grid points in the time interval \([0,T]\) are given as \( M_j = jk, \ j = 0,1,2,...,n \). Then the measure of the function \( Y(z, M) \) at the grid points are shown as \( Y_{i,j} = Y(z_i, M_j) \). According to equation five and the Capuo’s implicit finite difference discretization scheme of Eq. (1) to the grid point centered at \((z_i, M_j) = (ih, jk)\) are given as

\[
\begin{align*}
\sigma_{\alpha,k} \sum_{j=1}^{n} \alpha_j^{(\alpha)}(Y_{i,n-j+1} - Y_{i,n-j}) &= a_i \frac{1}{16h^2}(Y_{i-4,n} - 2Y_{i,n} + Y_{i+4,n}) + b_i \frac{1}{8h}(Y_{i+4,n} - Y_{i-4,n}) + c_i Y_{i,n},
\end{align*}
\]

for \( i=4,6,...,m-4 \).

Thus, based on Eq. (6), this approximation equation is namely implicit finite difference approximation equation. Immediately, for \( n \geq 2 \), are given as

\[
\begin{align*}
\sigma_{\alpha,k} \sum_{j=1}^{n} \alpha_j^{(\alpha)}(Y_{i,n-j+1} - Y_{i,n-j}) &= p_i Y_{i,n} + q_i Y_{i,n} + r_i Y_{i+4,n},
\end{align*}
\]

where

\[
p_i = a_i \frac{1}{16h^2} - b_i \frac{1}{8h}, \quad q_i = c_i \frac{1}{8h^2}, \quad r_i = a_i \frac{1}{16h^2} + b_i \frac{1}{8h}
\]

Also, we get for \( n = 1 \),

\[
- p_i U_{i+1} + q_i U_{i+1} - r_i U_{i+4} = f_{i,1}, \quad i = 4,6,...,H-4
\]

where

\[
\alpha_j^{(\alpha)} = 1, \quad q_i^* = \sigma_{\alpha,k} - q_i, \quad f_{i,1} = \sigma_{\alpha,k} U_{i,1}.
\]

Furthermore, from equation eight, the tridiagonal linear system can be developed in matrix form as

\[
AY = f
\]

with

\[
A = \begin{bmatrix}
q_4 & -r_4 & & & \\
-p_s & q_s & -r_5 & & \\
& -p_{s2} & q_{s2} & -r_{s3} & \\
& & & \ddots & \ddots \\
& & & -p_{sH-1} & q_{sH-1} & -r_{sH} \\
& & & & -p_{sH} & q_{sH} & -r_{sH-1} & \left(\frac{a}{4}\right) \left(\frac{n}{4}\right)
\end{bmatrix},
\]

\[
Y = \begin{bmatrix}
Y_{4,1} & Y_{8,1} & Y_{12,1} & \cdots & Y_{H-8,1} & Y_{H-4,1} \\
Y_{4,1} + p_s Y_{0,1} & Y_{8,1} & Y_{12,1} & \cdots & Y_{H-8,1} & U_{H-4,1} + p_{sH} U_{H,1}
\end{bmatrix}^T,
\]

\[
f = \begin{bmatrix}
f_{4,1} & f_{8,1} & f_{12,1} & \cdots & f_{H-8,1} & f_{H-4,1}
\end{bmatrix}^T.
\]
4. Formulation of QSGS

Actually from equation (9), it can be expressed, for the characteristic of its coefficient matrix has large scale and sparse. Furthermore, we applied QSGS iteration for solution equation (9). Basically, the QSGS iteration is substantially derived from Half Sweep Gauss-Seidel (HSGS) iterative method. The main objective of the Quarter-Sweep iteration is to reduce the computational complexities during iteration process. Due to the advantage of this concept and implementation QSGS iteration, from the Eq. (9) can be shown as addition of the three matrices

\[ A = G - W - V \]  

with \( G, W \) and \( V \) are diagonal, lower triangular and upper triangular matrices.

Based on Eq. (10), Quarter-Sweep Gauss-Seidel (QSGS) iteration can be shown as

\[ Y^{(k+)} = (G - W)^{-1} \left( VY^{(k)} + f \right) \]  

with \( Y \) for an unknown vector at \( k^b \). The QSGS iteration method can be defined in Algorithm.

**ALGORITHM : QSGS Iteration**

i. Initializing all the parameters. Set \( k = 0 \).

ii. Start

\[ \tilde{Y}^{(k+)} = \frac{1}{N_{i,i}} \left( f_i - \sum_{j=p,2,p} N_{i,j} \tilde{Y}^{(k)}_j - \sum_{j=i+3p} N_{i,j} \tilde{Y}^{(k)}_j \right) \]

iii. Convergence test. If \( \| Y^{(k+)} - Y^{(k)} \| \leq \varepsilon = 10^{-10} \) is satisfied, go to iv.

iv. If not go back to ii.

v. Compute the remaining point via second order Lagrange scheme.

vi. Show for this approximate solutions.

5. Evaluation Example

For this section, via one example of the TFDE’s being used for show performance properties of QSGS compare Full-Sweep Gauss-Seidel (FSGS) and Half Sweep Gauss-Seidel (HSGS) iterative methods. These three parameters were executed on the computer using a program written in C language. To do this three criteria have been considered such as \( K \) (Number of Iterations), \( T \) (Time Execution Time in seconds) and \( MAE \) (Maximum Absolute Error) at three different values of \( \alpha = 0.25, 0.50 \) and 0.75.

For application of this, the convergence test considered the tolerance error, which is defined as \( \varepsilon = 10^{-10} \).

To illustrate performance of QSGS iteration method, TFDE’s initial value and boundary value problem be shown as [18]

\[ \frac{\partial^\alpha Y(z, M)}{\partial M^\alpha} = \frac{\partial^2 Y(z, M)}{\partial z^2}, \quad 0 < \alpha \leq M, 0 \leq z \leq \gamma, \quad M > 0 \]  

with the boundary conditions value are stated from

\[ Y(0, M) = \frac{2kM^\alpha}{\Gamma(\alpha + 1)}, \quad Y(\ell, M) = \ell^2 + \frac{2kM^\alpha}{\Gamma(\alpha + 1)}, \]  

then the initial condition value is
\[ Y(z, 0) = z^2. \]  

Following of the Problem (12), if \( \alpha = 1 \), so equation (12) can be rewritten as

\[
\frac{\partial Y(z, M)}{\partial M} = \frac{\partial^2 Y(z, M)}{\partial z^2}, \quad 0 \leq z \leq \gamma, \quad M > 0, 
\]

subjected for the initial condition

\[ Y(z, 0) = z^2, \]

and boundary conditions

\[ Y(0, M) = 2kM, \quad U(\ell, M) = \ell^2 + 2kM, \]

so for the analytical solving to Problem (13) is shown as

\[ Y(z, M) = z^2 + 2kM. \]

then it is obtained as

\[
Y(z, M) = \frac{\sum_{n=0}^{N-1} \delta^n Y(z, 0) M^n}{n!} + \frac{\sum_{i=0}^{N-1} \delta^i M^{\alpha i} Y(z, 0)}{\Gamma(\alpha + 1)} 
\]

for \( Y(z, M) \) to \( 0 < \alpha \leq 1 \), furthermore the analytical solution of Problem (12) is shown as

\[ Y(z, M) = z^2 + 2kM^{\alpha} \frac{M^{\alpha}}{\Gamma(\alpha + 1)}. \]

Throught the numerical experiment, the analyses were carried out in different \( m \) (mesh sizes) = “128”, “256”, “512”, “1024”, and “2048”. Result of numerical simulations which were obtained from implementation of the QSGS, HSGS and FSGS iterative methods have been recorded in Table 1 respectively.

**Table 1.** Comparison of K (Number Iterations), T (Execution Time In Seconds) and MAE (Maximum Absolute Errors) for the Iteration via \( \alpha = 0.25, 0.50, 0.75 \)

| M    | Iteration (Method) | (\( \alpha = 0.25 \)) | (\( \alpha = 0.50 \)) | (\( \alpha = 0.75 \)) |
|------|--------------------|------------------------|------------------------|------------------------|
|      | K                  | T                      | MAE                    | K                      | T                      | MAE                    |
| 128  | FSGS               | 21017                  | 37.01                  | 9.97e-05               | 13601                  | 23.92                  | 9.85e-05               | 6695                   | 12.1                   | 1.30e-04               |
|      | HSGS               | 5682                   | 5.28                   | 9.96e-05               | 3671                   | 3.77                   | 9.84e-05               | 1805                   | 2.18                   | 1.29e-04               |
|      | QSGS               | 1528                   | 1.42                   | 9.96e-05               | 987                    | 1.17                   | 9.84e-05               | 487                    | 0.96                   | 1.29e-04               |
| 256  | FSGS               | 77231                  | 332.11                 | 1.00e-04               | 50095                  | 213.28                 | 9.90e-05               | 24732                  | 104.04                 | 1.30e-04               |
|      | HSGS               | 21017                  | 34.81                  | 9.97e-04               | 13601                  | 22.72                  | 9.85e-05               | 6695                   | 11.41                  | 1.30e-04               |
|      | QSGS               | 5682                   | 6.34                   | 9.96e-05               | 3671                   | 4.36                   | 9.84e-05               | 1805                   | 2.38                   | 1.29e-04               |
| 512  | FSGS               | 281598                 | 2522.20                | 1.02e-04               | 183181                 | 162.08                 | 1.01e-05               | 90783                  | 831.58                 | 1.32e-04               |
|      | HSGS               | 77231                  | 333.92                 | 1.00e-04               | 50095                  | 214.58                 | 9.90e-05               | 24732                  | 105.44                 | 1.30e-04               |
|      | QSGS               | 21017                  | 49.96                  | 9.97e-05               | 13601                  | 32.3                   | 9.85e-05               | 6695                   | 16.20                  | 1.30e-04               |
| 1024 | FSGS               | 1017140                | 18485.43               | 1.09e-04               | 663971                 | 2454.53                | 1.08e-04               | 330622                 | 5870.9                 | 1.40e-04               |
|      | HSGS               | 90783                  | 771.42                 | 1.32e-04               | 183181                 | 1568.23                | 1.00e-04               | 90783                  | 771.42                 | 1.32e-04               |
|      | QSGS               | 77231                  | 419.16                 | 1.00e-04               | 50095                  | 267.61                 | 9.90e-05               | 24732                  | 131.53                 | 1.30e-04               |
| 2048 | FSGS               | 3631638                | 158914.30              | 1.38e-04               | 2380946                | 17795.25               | 1.38e-04               | 1192528                | 8794.26                | 1.71e-04               |
|      | HSGS               | 1017140                | 17798.81               | 1.09e-04               | 663971                 | 11482.81               | 1.4e-04                | 330622                 | 5653.5                 | 1.40e-04               |
|      | QSGS               | 281598                 | 3064                   | 1.02e-04               | 183181                 | 1953.61                | 1.0e-04                | 90783                  | 960.310                | 1.32e-04               |
6. Conclusions
In order to obtain the numerical solution of the TFDE’s problems, the paper presents the derivation of the QS (Quarter-Sweep) Caputo’s implicit finite difference approximation equations in which this approximation equation leads a tridiagonal linear system.

Via evaluation example result Table 1 by imposing the effectiveness and accuracy between the FSGS, HSGS and QSGS iteration at $\alpha = 0.25, 0.50$ and 0.75, it can be shown that the reduction of K (Number of Iterations) for the QSGS iteration have declined approximately by 71.99-92.72% 50.71-95.53%, and 40.41-96.45% corresponds to the QSGS iteration compared with the FSGS and HSGS methods. Again in terms of T (Execution Time), implementations QSGS iteration are more rapidly about 69.78-98.09%, 30.64-88.35%, and 25.03-89.99% than the FSGS and HSGS iteration method. It means that the QSGS iteration requires the least amount for K (Number of Iterations) and T (Execution Time) as compared with FSGS and HSGS Iteration. Based on the accuracy of FSGS, HSGS and QSGS iterations, it can be express that their numerical solutions are in good agreement.

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