A concept of $\frac{2}{3}$PROP and deformation theory of (co)associative bialgebras

Boris Shoikhet

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To my teacher Borya Feigin in the occasion of his 50th birthday

Abstract

We introduce a concept of $\frac{2}{3}$PROP generalizing the Kontsevich concept of $\frac{1}{2}$PROP. We prove that some Stasheff-type compactification of the Kontsevich spaces $K(m,n)$ defines a topological $\frac{2}{3}$PROP structure. The corresponding chain complex is a minimal model for its cohomology (both are considered as $\frac{2}{3}$PROPs). We construct a $\frac{2}{3}$PROP $\text{End}(V)$ for a vector space $V$. Finally, we construct a dg Lie algebra controlling the deformations of a (co)associative bialgebra. Philosophically, this construction is a version of the Markl’s operadic construction from [M1] applied to minimal models of $\frac{2}{3}$PROPs.

Introduction

The goal of this paper is to construct a deformation theory for (co)associative bialgebras. According to general principles, it means that we are looking for a dg Lie algebra (or, more generally, for an $L_\infty$ algebra) controlling the deformation theory of a (co)associative bialgebra. In the case of the deformation theory of associative algebras, such a dg Lie algebra controlling the deformations of an associative algebra $A$, is the cohomological Hochschild complex of $A$ with the Gerstenhaber bracket. (More precisely, this Hochschild complex controls the deformations of the category of $A$-modules).

First of all, recall that a (co)associative bialgebra is a vector space $A$ equipped with the maps $*: A^{\otimes 2} \to A$ (the product) and $\Delta: A \to A^{\otimes 2}$ (the coproduct). The product is supposed to be associative and the coproduct is supposed to be coassociative. Moreover, we suppose the following compatibility of them:

$$\Delta(a * b) = \Delta(a) * \Delta(b)$$  \hfill (1)
for any $a, b \in A$. (Here in the r.h.s. the product is the component product in $A^{\otimes 2}$ defined as $(a \otimes b) \ast (a_1 \otimes b_1) = (a \ast a_1) \otimes (b \ast b_1)$). Notice that we do not suppose the existence of unit and counit in $A$.

Here we meet our first difficulty: the r.h.s. of (1) is of the 4th degree and not quadratic. Recall that (little bit roughly) we associate the deformation theory with a dg Lie algebra $g^\bullet$ as follows: we consider the solutions of the Maurer-Cartan equation

$$d\alpha + \frac{1}{2}[\alpha, \alpha] = 0$$

for $\alpha \in g^1$ modulo the action of the gauge group associated with $g^0$ on these solutions. (Because of possible divergences in the action of the gauge group, we say instead of this direct construction that the deformation functor is a functor from the category of the Artinian algebras to the category of sets).

It is known that the Gerstenhaber-Schack complex [GS] associated with a bialgebra $A$ is a deformation complex of the bialgebra structure on $A$. It means that the first cohomology of this complex are isomorphic to the infinitesimal deformations on $A$. To pass from the infinitesimal deformations to the global ones, one needs to have an appropriate dg Lie algebra structure on the Gerstenhaber-Schack complex (or, more generally, an $L_\infty$-structure). Recall here that as a vector space, the Gerstenhaber-Schack complex of $A$ is

$$K_{GS}^\bullet = \bigoplus_{m,n \geq 1} \text{Hom}(A^{\otimes m}, A^{\otimes n})[-m-n+2]$$

In particular, in degree 1 we have: $K_{GS}^1 = \text{Hom}(A^{\otimes 2}, A) \oplus \text{Hom}(A, A^{\otimes 2})$. We could expect that for some dg Lie algebra structure on $K_{GS}^\bullet$ the Maurer-Cartan equation (2) for the element $\ast_1 \oplus \Delta_1 \in \text{Hom}(A^{\otimes 2}, A) \oplus \text{Hom}(A, A^{\otimes 2})$ means exactly that $(\ast + \ast_1, \Delta + \Delta_1)$ defines a new (co)associative bialgebra structure on $A$.

But it is impossible: because the r.h.s. of the equation (1) is of the 4th degree in $\ast_1$ and $\Delta_1$, while the Maurer-Cartan equation (2) is quadratic. It means that the best we could expect is to have an $L_\infty$ algebra structure on $K_{GS}^\bullet$ (which looks quite complicated). This crucial observation was explained to the author by Boris Tsygan about 3 years ago.

Now remember that the $L_\infty$ algebras and the dg Lie algebras is more or less the same: if we have an $L_\infty$ algebra structure on a graded vector space $V$, we necessarily have an $L_\infty$ isomorphic structure of pure dg Lie algebra on a (bigger) space $V_1$. It means that the question of which structure we have, dg Lie algebra or $L_\infty$ algebra, is the question of the right choice of ”generators”. This means, in particular, that we could expect the existence of a complex quasi-isomorphic to the Gerstenhaber-Schack complex (”with another generators”) and a dg Lie algebra on it, which solves the deformation problem of a (co)associative bialgebra $A$.

This idea is one source of the theory developed here in this paper. Another source is the Kontsevich spaces $K(m, n)$. The reader can find the definition of them in Section 2 of the paper. The original Kontsevich motivation when he invented these spaces was the
following: the space $K(2, 2)$ is the configuration space on two independent lines, we have 2 points in each line modulo independent common shift on each line, and modulo the following action of $R_+^*$ on this space: for $\lambda \in R_+^*$, we dilate the first line with the scale $\lambda$ and the second with the coefficient $\lambda^{-1}$. Then $K(2, 2)$ is a 1-dimensional space: we have an interval on the first line, an interval on the second (we identify the intervals of the same length), and we identify such two configurations with the same product of the lengths of the two intervals. Therefore, the configuration has the only one module—the product of the lengths of the intervals. Before compactification, it is isomorphic to $R_+$.

Now we compactify the space $K(2, 2)$ to the closed interval. The two limit configurations are shown in the Figure 1: The Kontsevich’s insight was that the left picture

\[ \begin{array}{c}
| & \infty^{-1} & | \\
\hline
| & \bullet & \bullet & | \\
1 \text{ cm} & \hline
\end{array} \quad \begin{array}{c}
| & \infty & | \\
\hline
| & \bullet & \bullet & | \\
1 \text{ cm} & \hline
\end{array} \]

Figure 1: The two limit points in $\overline{K(2, 2)}$

should give the left-hand side of the compatibility equation (\ref{eq:compatibility}), while the right picture should give the right-hand side of (\ref{eq:compatibility}). We say "should give" having in mind the Markl's construction in [M1], or even further, a construction of the type of Kontsevich formality.

After these short remarks, we pass to our constructions.

If the reader is interested mostly in our construction of the deformation dg Lie algebra, he can begin to read the paper from Section 3 and to come back to the previous two Sections if it is necessary.

1 The concept of $^{2/3}$PROP

1.1 The definition

Here we define our main technical tool—$^{2/3}$PROPs. The name $^{2/3}$PROPs indicates that this concept is a further generalization (or simplification) of the concept of $^{1/2}$PROP due to Maxim Kontsevich (see [K3], [MV]).

We define a $^{2/3}$PROP of vector spaces, a $^{2/3}$PROP of dg vector spaces, of topological spaces,... can be defined analogously.

Definition. A pre-$^{2/3}$PROP of vector spaces consists of the following data:

(i) a collection of vector spaces $F(m, n)$ defined for $m \geq 1, n \geq 1, m + n \geq 3$, with an action of symmetric groups $\Sigma_m \times \Sigma_n$ on $F(m, n)$,
(ii) a collection of vector spaces \( F_{m}^{1,1,...,1} \) defined for \( n \geq 2, m \geq 2 \), with an action of the symmetric group \( \Sigma_{n} \) on \( F_{m}^{1,1,...,1} \).

(iii) a collection of vector spaces \( F_{n}^{1,1,...,1} \) defined for \( m \geq 2, n \geq 2 \), with an action of the symmetric group \( \Sigma_{n} \) on \( F_{n}^{1,1,...,1} \).

(iv) compositions \( \circ_{i} : F(m,n) \otimes F(1,n_{1}) \rightarrow F(m,n+n_{1}-1), 1 \leq i \leq n, \)

(v) compositions \( j \circ : F(m_{1},1) \otimes F(m,n) \rightarrow F(m+m_{1}-1,n), 1 \leq j \leq m, \)

(vi) compositions \( \otimes_{i} : F_{1,1,...,1}^{1,1,...,1} \otimes F_{1,1,...,1}^{n} \rightarrow F_{1,1,...,1}^{n+1-1}, 1 \leq i \leq m, \)

(vii) compositions \( j \otimes : F_{m_{1}}^{1,1,...,1} \otimes F_{m}^{1,1,...,1} \rightarrow F_{m_{1}+m-1}^{1,1,...,1}, 1 \leq j \leq m, \)

(viii) compositions \( \otimes : F_{m}^{1,1,...,1} \otimes F_{1,1,...,1}^{n} \rightarrow F(m,n), \)

(ix) all the compositions are equivariant with respect to the actions of the symmetric groups.

This data should obey the following properties:

1. the composition \( \circ_{j} \ast \circ_{i} : F(m,n) \otimes F(1,n_{1}) \otimes F(1,n_{2}) \rightarrow F(n,n+n_{1}+n_{2}-2) \) is associative for \( i \leq j \leq i+n_{1} \): in this case \( \circ_{j-i} \ast \circ_{i} : F(m,n) \otimes F(1,n_{1}) \otimes F(1,n_{2}) \rightarrow F(n,n+n_{1}+n_{2}-2) \) coincides with \( \circ_{j} \ast \circ_{i} : (F(m,n) \otimes F(1,n_{1})) \otimes F(1,n_{2}) \rightarrow F(n,n+n_{1}+n_{2}-2), \)

2. in the notations of (1), if \( j < i \) or \( j > i+n_{1} \), we have \( \circ_{j} \ast \circ_{i} : (F(m,n) \otimes F(1,n_{1})) \otimes F(1,n_{2}) \rightarrow F(n,n+n_{1}+n_{2}-2) \) is equal to \( \circ_{1} \ast \circ_{j} : (F(m,n) \otimes F(1,n_{2})) \otimes F(1,n_{1}) \rightarrow F(n,n+n_{1}+n_{2}-2) \) where \( j_{1} = j, i_{1} = i+n_{2} \) if \( j < i \), and \( j_{1} = j-n_{1}, i_{1} = i \) if \( j > i+n_{1} \) (the commutativity),

3. the property analogous to (1) for \( j \circ, \)

4. the property analogous to (2) for \( j \circ, \)

5)-(8) the analogous properties for \( \otimes_{i} \) and for \( j \otimes \).

Notice that this structure without \( F_{m}^{1,1,...,1} \) and \( F_{1,1,...,1}^{n} \) is exactly the Kontsevich’s \( \frac{1}{2} \)PROP structure.

**Definition.** A pre-\( \frac{1}{2} \)PROP is a \( \frac{3}{2} \)PROP if the operations of \( \circ \)-type are compatible with the operations of \( \otimes \)-type, as follows:

There are extra maps \( \chi_{m_{1} \rightarrow m_{2}}^{n} : F_{1,1,...,1}^{m_{1} \rightarrow m_{2}}^{n} \rightarrow F_{1,1,...,1}^{m_{2} \rightarrow m_{1}}^{n} \) \( (m_{1} \leq m_{2}) \) and map \( \gamma_{m_{1} \rightarrow m_{2}}^{n} : F_{m}^{1,1,...,1}^{n} \rightarrow F_{m}^{1,1,...,1}^{n} \) \( (m_{1} \leq m_{2}) \), which are supposed to be equivariant with respect to the actions of symmetric groups. We also suppose that these maps are isomorphisms. Then we have:
(A) $\circ_i \otimes : (F_{m,1,...,1}^{1,n \times}) \otimes F_{1,1,...,1}^{n} (m \times) \otimes F(1,n_1) \rightarrow F(m,n+n_1-1)$ is equal to $\otimes \circ_i : F_{m,1,...,1}^{1,n \times} (n+n_1-1 \times) \otimes F_{1,1,...,1}^{n} (m \times) \rightarrow F(m,n+n_1-1)$. More precisely, let $\alpha \in F_{m,1,...,1}^{1,n \times}$, $\beta \in F_{n,1,...,1}^{1,m \times}$, and $\gamma \in F(1,n_1)$. Then

\[
(\alpha \otimes \beta) \circ_i \gamma = \gamma_{m}^{n+n_1-1} (\alpha) \otimes_i (\beta \otimes \lambda_{1 \rightarrow m}^{n_1} (\gamma))
\] (4)

(B) the analogous compatibility with $j \circ$.

We can imagine what is a free $\frac{2}{3}$PROP. It consists from all "free" words of the following two forms:

\[
\cdots \circ F(m_k,1) \circ \cdots F(m_1,1) \circ F(m,n) \circ F(1,n_1) \circ \cdots \circ F(1,n_\ell) \circ \cdots
\] (5)

and

\[
\otimes F_{m_k,1,...,1}^{1,n_1+\cdots+k+1 \times} \otimes \cdots \otimes F_{m_1,1,...,1}^{1,n_1+\cdots+k+1 \times} \otimes F_{1,1,...,1}^{1,m_2+\cdots-l+1 \times} \otimes F_{1,1,...,1}^{1,m_2+\cdots-l+1 \times} \otimes \cdots \otimes F_{1,1,...,1}^{1,m_2+\cdots \times}
\] (6)

We can draw these free elements as "two-sided trees", see Figure 2: The valences of the vertices are greater or equal than 3. We have trees of "two different colors" according to the equations (5)-(6) above.

Figure 2: A typical element of a free $\frac{2}{3}$PROP
The definition of \((\text{pre-})\mathcal{P}_2\) is motivated by the geometry of the Kontsevich spaces \(K(m, n)\) (see the next Section). The reader who is interested in the origin of this definition can pass directly to Section 2. We tried to construct a compactification of these spaces and to formalize the operations among the strata. The advantage of \(\mathcal{P}_2\) is that the chain complex of the compactification \(K(m, n)\) is a free dg \(\mathcal{P}_2\). Its homology \(\mathcal{P}_2\) is exactly the \(\mathcal{P}_2\) Bialg controlling the (co)associative bialgebras. Finally, any (co)associative bialgebra structure on a vector space \(V\) gives a map of the \(\mathcal{P}_2\) Bialg \(\rightarrow\) End(\(V\)).

1.2 The \(\text{pre-}\mathcal{P}_2\) \(\text{End}(V)\)

Here we define the \(\mathcal{P}_2\) \(\text{End}(V)\) for a vector spaces \(V\). We use here the notations \(\text{End}(m, n)\), \(\text{End}_1(1, n_1)\), and \(\text{End}_m(1, \ldots, 1)\).

We set:
\[
\begin{align*}
\text{End}(m, n) &= \text{Hom}(V^\otimes m, V^\otimes n), \\
\text{End}_1(1, n_1) (m \text{ times}) &= (\text{Hom}(V, V^\otimes n))^\otimes m, \\
\text{End}_m(1, \ldots, 1) (n \text{ times}) &= (\text{Hom}(V^\otimes m, V))^\otimes n
\end{align*}
\]

We should now define the compositions \(\circ_i\), \(\circ_j\), \(\otimes_i\), \(\otimes_j\), and \(\otimes\).

The case of the composition \(\circ_i\): \(\text{End}(m, n) \otimes \text{End}(1, n_1) \rightarrow \text{End}(m, n + n_1 - 1)\) can be schematically shown as follows (see Figure 3): Let \(\Psi \in \text{Hom}(V^\otimes m, V^\otimes n)\).

\[
\begin{array}{c}
\text{the i’th place} \\
\text{n}_1 \text{ points} \\
\text{n-1 points} \\
\text{m points}
\end{array}
\]

Figure 3: The composition \(\circ_i\)

\(\Theta \in \text{Hom}(V, V^\otimes n_1)\). Their composition \(\Psi \circ_i \Theta \in \text{Hom}(V^\otimes m, V^\otimes n + n_1 - 1)\) is defined as
\[
\Psi \circ_i \Theta(v_1 \otimes \cdots \otimes v_m) = (\text{Id} \otimes \cdots \otimes \text{Id} \otimes \Theta \otimes \cdots \otimes \text{Id}) \circ \Psi(v_1 \otimes \cdots \otimes v_m)
\]

(Here \(\Theta\) stands at the \(i\)th place).

The picture for the composition \(\circ_j\): \(\text{End}(m_1, 1) \otimes \text{End}(m, n) \rightarrow \text{End}(m + m_1 - 1, n)\) is the following (see Figure 4): For \(\Theta \in \text{Hom}(V^\otimes m_1, V)\) and \(\Psi \in \text{Hom}(V^\otimes m, V^n)\), their
composition $\Theta(\circ)\Psi \in \text{Hom}(V^{\otimes m+m_1-1}, V^{\otimes n})$ is
\[
\Theta(\circ)\Psi(v_1 \otimes \cdots \otimes v_{m+m_1-1}) = \\
\Psi(v_1 \otimes \cdots \otimes v_{j-1} \otimes \Theta(v_j \otimes \cdots \otimes v_{j+m_1-1}) \otimes v_{j+m_1} \otimes \cdots \otimes v_{m+m_1-1}) \tag{9}
\]

We have the particular case of the product $\circ_i$ when $m = 1$. Denote it by $\circ_1^i$. By definition, the composition $\circ_i^1: \text{End}_{1,1,\ldots,1}^n (m \text{ times}) \otimes \text{End}_{1,1,\ldots,1}^{n_1} (m \text{ times}) \rightarrow \text{End}_{1,1,\ldots,1}^{n+n_1-1} (m \text{ times})$ is the $n$th tensor power of the composition $\circ_1^i$. Analogously we define $\circ_1^j$ and the composition $\circ_j^1: \text{End}_{1,1,\ldots,1}^m (n \text{ times}) \otimes \text{End}_{1,1,\ldots,1}^{m} (n \text{ times}) \rightarrow \text{End}_{1,1,\ldots,1}^{m_1+m-1} (n \text{ times})$ as the $n$th tensor power of the composition $\circ_1^j$.

It remains to define the composition $\circ$. We define the composition $\circ: (\text{Hom}(V^{\otimes m}, V))^{\otimes n} \otimes (\text{Hom}(V, V^{\otimes n}))^{\otimes m} \rightarrow \text{Hom}(V^{\otimes m}, V^{\otimes n})$. Suppose $\Psi_1 \otimes \cdots \otimes \Psi_n \in (\text{Hom}(V^{\otimes m}, V))^{\otimes n}$, and $\Theta_1 \otimes \cdots \otimes \Theta_m \in (\text{Hom}(V, V^{\otimes n}))^{\otimes m}$. We are going to define their composition $(\Psi_1 \otimes \cdots \otimes \Psi_n) \circ (\Theta_1 \otimes \cdots \otimes \Theta_m) \in \text{Hom}(V^{\otimes m}, V^{\otimes n})$. Denote
\[
F(v_1 \otimes \cdots \otimes v_m) = \Theta_1(v_1) \otimes \cdots \otimes \Theta_m(v_m) \in V^{\otimes mn} \tag{10}
\]

Next, we define a map $G \in \text{Hom}(V^{\otimes mn}, V^{\otimes n})$ as follows:
\[
G(w_1^1 \otimes \cdots \otimes w_1^n \otimes w_2^1 \otimes \cdots \otimes w_2^n \otimes \cdots \otimes w_m^1 \otimes \cdots \otimes w_m^n) = \\
\Psi_1(w_1^1 \otimes w_2^2 \otimes w_m^1) \otimes \Psi_2(w_1^2 \otimes \cdots \otimes w_m^2) \otimes \cdots \otimes \Psi_n(w_1^n \otimes \cdots \otimes w_m^n) \in V^{\otimes n} \tag{11}
\]

Now we set
\[
(\Psi_1 \otimes \cdots \otimes \Psi_n) \circ (\Theta_1 \otimes \cdots \otimes \Theta_m) = (G \circ F)(v_1 \otimes \cdots \otimes v_m) \tag{12}
\]

It is clear that these compositions define a pre-$\frac{1}{2}$PROP structure on $\text{End}(V)$.

**Remark.** M. Markl communicated to the author that our composition $\circ$ is a particular case of his "fractions" composition [M2].
1.3 The \( \frac{2}{3} \)PROP of (co)associative bialgebras Bialg

For a (pre-)\( \frac{2}{3} \)PROP \( F \) we define an \( F \)-algebra structure on a vector space \( V \) as a map of pre-\( \frac{2}{3} \)PROP's \( F \rightarrow \text{End}(V) \). We are going to construct now a \( \frac{2}{3} \)PROP Bialg such that a Bialg-algebra structure on \( V \) is exactly a (co)associative bialgebra structure on \( V \).

Let \( \Sigma_n \) be the symmetric group on \( n \) points, and for a group \( G \) denote by \( G^\psi \) the dual group.

We can consider Bialg as \( \frac{2}{3} \)PROP of sets, or, if we like, as the corresponding \( \frac{2}{3} \)PROP of vector spaces (generated by these sets). We here consider Bialg as a \( \frac{2}{3} \)PROP of sets. Later it will appear also as the homology \( \frac{2}{3} \)PROP of the topological \( \frac{2}{3} \)PROP \( K(m,n) \), then we consider it as the corresponding \( \frac{2}{3} \)PROP of vector spaces. This functor replaces the direct product \( \times \) to the tensor product \( \otimes \).

We set:

\[
\begin{align*}
\text{Bialg}(m,n) &= \Sigma^\psi_m \times \Sigma_n, \\
\text{Bialg}_{1,1,...,1}^{\text{(m times)}} &= \Sigma_n, \\
\text{Bialg}_{1,1,...,1}^{\text{(n times)}} &= \Sigma^\psi_m.
\end{align*}
\]

The \( \frac{2}{3} \)PROP maps \( \lambda^n_{m_1 \rightarrow m} \) and \( \gamma^{n_1 \rightarrow n} \) are the identity maps. We define the compositions \( a_i, j, \sigma, \otimes, j \) and \( \otimes \) as follows:

Consider any of these compositions for the pre-\( \frac{2}{3} \)PROP \( \text{End}(V) \), a composition \( \star \). Suppose that \( \Psi \in \text{End}_\alpha \) and \( \Theta \in \text{End}_\beta \) are its arguments. These compositions were defined in the previous Subsection. There is the \( \Sigma^\psi_i \times \Sigma_{j_1} \)-action on \( \text{End}_\alpha \) and the \( \Sigma^\psi_{j_2} \times \Sigma_{j_2} \)-action on \( \text{End}_\beta \). Suppose \( \sigma^\psi_1 \times \sigma_1 \in \Sigma^\psi_{i_1} \times \Sigma_{j_1} \) and \( \sigma^\psi_2 \times \sigma_2 \in \Sigma^\psi_{i_2} \times \Sigma_{j_2} \). We are going to define the composition \( (\sigma^\psi_1 \times \sigma_1) \star (\sigma^\psi_2 \times \sigma_2) \) in Bialg. For this, consider the composition \( ((\sigma^\psi_1 \times \sigma_1) \Psi) \star ((\sigma^\psi_2 \times \sigma_2) \Theta). \) It is clear that it is equal to the action of some \( \sigma \in \text{Bialg} \) on the product of \( \Psi \) and \( \Theta \) in \( \text{End} \):

\[
((\sigma^\psi_1 \times \sigma_1) \Psi) \star ((\sigma^\psi_2 \times \sigma_2) \Theta) = \sigma(\Psi \star \Theta)
\]

The last equation holds for any \( \Psi \) and \( \Theta \) in the corresponding components of \( \text{End} \), that is, \( \sigma \) does not depend on the choice of \( |\psi \) and \( \Theta \). We define the composition \( (\sigma^\psi_1 \times \sigma_1) \star (\sigma^\psi_2 \times \sigma_2) \) as \( \sigma \).

It clear that this definition is correct, and in this way we define a \( \frac{2}{3} \)PROP Bialg.

Lemma. A map \( \phi : \text{Bialg} \rightarrow \text{End}(V) \) of pre-\( \frac{2}{3} \)PROP's is the same that a (co)associative bialgebra structure on \( V \).

Proof. First, let \( V \) be a (co)associative bialgebra with the product \( \star \) and the coproduct \( \Delta \). We define a map of pre-\( \frac{2}{3} \)PROP's \( \phi_{\star, \Delta} : \text{Bialg} \rightarrow \text{End}(V) \). Let \( \sigma^\psi \times \sigma \in \text{Bialg}(m,n) \). We put \( \phi_{\star, \Delta}(\sigma^\psi \times \sigma)(v_1 \otimes \cdots \otimes v_m) = \Delta^{n-1} \circ \sigma^{m-1}(v_1 \otimes \cdots \otimes v_m) \). Here in the formula \( \Delta^n \) and \( \star^m \) are the composition powers of the coproduct and of the product, correspondingly. Because of the (co)associativity, these powers are well-defined. Next, for \( \sigma \in \text{Bialg}_{1,1,...,1}^{\text{m times}} \) we set \( \phi_{\star, \Delta}(\sigma) \in (\text{Hom}(V, V^\otimes m))^\otimes m \) is the \( m \)th tensor power of the
map \( v \mapsto \sigma(\Delta^{n-1}(v)) \). Analogously, using the product, we define \( \phi_{*, \Delta} \) on \( \text{Bialg}^{1,1,\ldots,1}_m \). Now we explain why without the compatibility \( \Box \) this definition would be incorrect.

Consider many identity permutations: \( \text{Id}_2 \in \Sigma_2 = \text{Bialg}(1,2), \text{Id}^2 \in \Sigma_2^2 = \text{Bialg}(2,1), \text{Id}^{2,1}_2 \in \text{Bialg}^{2,1}_1, \) and \( \text{Id}^{1,1}_2 \in \text{Bialg}^{1,1}_2 \). Then we have the following identity in \( \text{Bialg} \):

\[
\text{Id}^2 \circ \text{Id}_2 = \text{Id}^2_1 \circ \text{Id}^{1,1}_2 = \text{Id}^\vee \times \text{Id} \in \text{Bialg}(2,2)
\]

(15)

It is clear that this identity in \( \text{Bialg} \) follow some identity in the images of these elements \( \text{Id}_2, \text{Id}^2, \text{Id}^{2,1}_2, \text{Id}^{1,1}_2 \) by the map \( \phi_{*, \Delta} \) of \( \text{pre-2PROP} \). The reader can easily verify that this identity is exactly the compatibility \( \Box \) in a (co)associative bialgebra. One can prove also that if the compatibility holds, the definition of \( \phi_{*, \Delta} \) is correct.

Vice versa, suppose we have a map \( \phi : \text{Bialg} \to \text{End}(V) \) of \( \text{pre-2PROP} \). Denote \( a \star b := \phi(\text{Id}^2)(a \otimes b) \) and \( \Delta(a) := \phi(\text{Id}_2)(a) \). The compatibility follows from (15). The reader can easily find analogous identities in \( \text{Bialg} \) which imply the associativity of \( \star \) and the coassociativity of \( \Delta \).

\[\Box\]

### 2 The Kontsevich spaces \( K(m, n) \), their Stasheff-type compactification, and the corresponding \( \frac{2}{3} \)PROP

First of all, recall the definition of the spaces \( K(m, n) \) due to Maxim Kontsevich (see also [Sh]). We show in the sequel that these spaces and its compactification introduced below play a crucial role in the deformation theory of (co)associative bialgebras.

First define the space \( \text{Conf}(m, n) \). By definition, \( m, n \geq 1, m + n \geq 3 \), and

\[
\text{Conf}(m, n) = \{p_1, \ldots, p_m \in \mathbb{R}^{(1)}, p_i < p_j \text{ for } i < j; \quad q_1, \ldots, q_n \in \mathbb{R}^{(2)}, q_i < q_j \text{ for } i < j\}
\]

(16)

Here we denote by \( \mathbb{R}^{(1)} \) and by \( \mathbb{R}^{(2)} \) two different copies of a real line \( \mathbb{R} \).

Next, define a 3-dimensional group \( G^3 = \mathbb{R}^2 \rtimes \mathbb{R}_+ \) (here \( \mathbb{R}_+ = \{x \in \mathbb{R}, x > 0\} \)) with the following group law:

\[
(a, b, \lambda) \circ (a', b', \lambda') = (\lambda'a + a', (\lambda')^{-1}b + b', \lambda \lambda')
\]

(17)

where \( a, b, a', b' \in \mathbb{R}, \lambda, \lambda' \in \mathbb{R}_+ \). This group acts on the space \( \text{Conf}(m, n) \) as

\[
(a, b, \lambda) \cdot (p_1, \ldots, p_m; q_1, \ldots, q_n) = (\lambda p_1 + a, \ldots, \lambda p_m + a; \lambda^{-1}q_1 + b, \ldots, \lambda^{-1}q_n + b)
\]

(18)

In other words, we have two independent shifts on \( \mathbb{R}^{(1)} \) and \( \mathbb{R}^{(2)} \) (by \( a \) and \( b \), and \( \mathbb{R}_+ \) dilatates \( \mathbb{R}^{(1)} \) by \( \lambda \) and dilatates \( \mathbb{R}^{(2)} \) by \( \lambda^{-1} \).

In our conditions \( m, n \geq 1, m + n \geq 3 \), the group \( G^3 \) acts on \( \text{Conf}(m, n) \) freely. Denote by \( K(m, n) \) the quotient-space. It is a smooth manifold of dimension \( m + n - 3 \).
We will need also a very special case of the spaces $K_{m_1,\ldots,m_{\ell_1}}$ introduced below. Recall here our definition of the space $K_{m_1,\ldots,m_{\ell_1}}$ (generalizing the Kontsevich space $K(m,n)$) from [Sh]:

First define the space $Conf_{n_1,\ldots,n_{\ell_2}}$. By definition,

$$Conf_{n_1,\ldots,n_{\ell_2}} = \left\{ p_1, \ldots, p_{m_1} \in \mathbb{R}^{(1,1)}, p_1, \ldots, p_{m_2} \in \mathbb{R}^{(1,2)}, \ldots, p_1, \ldots, p_{m_{\ell_1}} \in \mathbb{R}^{(1,\ell_1)}; \right. $$

$$\left. q_1, \ldots, q_{n_1} \in \mathbb{R}^{(2,1)}, q_1, \ldots, q_{n_2} \in \mathbb{R}^{(2,2)}, \ldots, q_1, \ldots, q_{n_{\ell_2}} \in \mathbb{R}^{(2,\ell_2)} \right| $$

$$p_i^j < p_{i+1}^j \text{ for } i_1 < i_2; q_i^j < q_{i+1}^j \text{ for } i_1 < i_2 \right\} \quad (19)$$

Here $\mathbb{R}^{(i,j)}$ are copies of the real line $\mathbb{R}$. Now we have an $\ell_1 + \ell_2 + 1$-dimensional group $G^{\ell_1,\ell_2,1}$ acting on $Conf_{n_1,\ldots,n_{\ell_2}}$. It contains $\ell_1 + \ell_2$ independent shifts

$$p_i^j \mapsto p_i^j + a_j, i = 1, \ldots, m_j, a_j \in \mathbb{R}; q_i^j \mapsto q_i^j + b_j, i = 1, \ldots, n_j, b_j \in \mathbb{R}$$

and one dilatation

$$p_i^j \mapsto \lambda \cdot p_i^j \text{ for all } i, j; q_i^j \mapsto \lambda^{-1} \cdot q_i^j \text{ for all } i, j.$$ 

This group is isomorphic to $\mathbb{R}^{\ell_1 + \ell_2} \times \mathbb{R}_+$. We say that the lines $\mathbb{R}^{(1,1)}, \mathbb{R}^{(1,2)}, \ldots, \mathbb{R}^{(1,\ell_1)}$ (corresponding to the factor $\lambda$) are the lines of the first type, and the lines $\mathbb{R}^{(2,1)}, \mathbb{R}^{(2,2)}, \ldots, \mathbb{R}^{(2,\ell_2)}$ (corresponding to the factor $\lambda^{-1}$) are the lines of the second type.

Denote

$$K_{m_1,\ldots,m_{\ell_1}} = Conf_{n_1,\ldots,n_{\ell_2}} / G^{\ell_1,\ell_2,1} \quad (20)$$

We construct a compactification $\overline{K(m,n)}$ the boundary strata of which are products of the spaces $K(m_1,n_1)$, $K_{1,1,\ldots,1}$, and $K_{m_2,1,\ldots,1}$ (it allowed to be several spaces of each type).

Example

Let $m = n = 2$. Then the space $K(2,2)$ is 1-dimensional. It is easy to see that $(p_2 - p_1) \cdot (q_2 - q_1)$ is preserved by the action of $G^3$, and it is the only invariant of the $G^3$ action on $K(2,2)$. Therefore, $K(2,2) \simeq \mathbb{R}_+$. There are two "limit" configurations: $(p_2 - p_1) \cdot (q_2 - q_1) \to 0$ and $(p_2 - p_1) \cdot (q_2 - q_1) \to \infty$. Therefore, the compactification $\overline{K(2,2)} \simeq [0,1]$. See Figure 1. We will construct a compactification of the space $K(m,n)$ which is a topological $\frac{2}{3}$PROP. More precisely,

$$F(m,n) = \overline{K(m,n)}$$

$$F_{1,1,\ldots,1} = \overline{K_{1,1,\ldots,1}}$$

$$F_{m,1,\ldots,1} = \overline{K_{m,1,\ldots,1}} \quad (21)$$
The compactifications will be defined below.

First of all, let us describe all strata of codimension 1 in $\overline{K(m,n)}$. There are boundary strata of codimension 1 of two different types. The first two strata are shown in Figures 3 and 4. In the picture in Figure 3 $n_1$ points on the upper line move infinitely close to each other, and the "scale" of this infinitely small number is irrelevant (we have in mind here the CROC compactification from [Sh] where this scale is relevant), $2 \leq n_1 \leq n$. In Figure 4 $m_1$ points on the lower line move close to each other, $2 \leq m_1 \leq m$. The remaining stratum of codimension 1 in $\overline{K(m,n)}$ (there is the only one such stratum) is shown in Figure 5: Here in the Figure all points on the upper line move infinitely far from each other will a finite ratio of any two among these infinite distances. The all distances on the lower line are finite. Of course, the stratum when the points on the lower line are infinite and the points on the upper line are in finite distances from each other, is the same: one stratum can be obtained from another by the application of the element $(0, 0, \infty) \in G^3$.

The strata in Figures 3 and 4 are isomorphic to $K(m,n) \times K(1,n_1)$ and $K(m_1,1) \times K(m,n)$, correspondingly. The stratum in Figure 5 is isomorphic to $K_{m_1,1,...,1}^{n_1,...,1}$ (n times) $\times K_{1,1,...,1}^m$ (m times). Let us explain the last formula: the distances between the points at the upper line are infinite, but their ratios are finite. Therefore, we should count these ratios. For this, we apply to Figure 5 the transformation $(0, 0, \infty) \in G^3$. Then the infinite distances will become finite, and then we can count the ratios.

Now we claim that using the 3 operations shown in Figures 3,4,5 we can obtain any limit configuration (here by a limit configuration we mean a configuration where some distances are infinitely large or/and infinitely small). Moreover, we can apply the configuration in Figure 5 not more than 1 time. Let us explain it:

Apply a transfotmation from $G^3$ such that there are no infinite distances at the lower line, and the diameter of the configuration of points at the lower line is finite (not infinitely small). We distinguish the following two cases: in the first case in the obtained
configuration there are no infinitely large distances in the upper line, and in the second
some distances are infinitely large. It is clear that we can reach any limit configuration
of the first type by the applying several times the degenerations shown in Figure 3
and Figure 4. Analogously, in the second case, we first apply degenerations in Figure
3 (several times), then apply the transformation from Figure 5 (with the scale of the
infinity depending on the configuration), and then apply several times the degenerations
from Figure 4. It is clear that in this case we can get any limit configuration.

Denote the operations shown in Figures 3, 4, 5 by $\circ_i$, $j \circ$, and $\otimes$, correspondingly. It
remains to define our operations $\otimes_i$ and $j \otimes$ in the Definition in Section 1.1, that is, to
compactify the spaces $K_{1,1,\ldots,1}^n$ and $K_{m,1,\ldots,1}^1$.

Notice that the space $K_{1,1,\ldots,1}^n \ (n \ \text{times})$ is isomorphic to the Stasheff polyhedron $St_m$
for any $n$, as well the space $K_{1,1,\ldots,1}^n \ (m \ \text{times})$ is isomorphic to the Stasheff polyhedron $St_n$
for any $m$. In particular, we define our maps $\lambda_{m_1 \to m_2}$ and $\gamma_{m \to n_1 \to n_2}$ as the identity maps.
Furthermore, we compactify these spaces as usual in the Stasheff compactification, and
the compositions $\otimes_i$ and $j \otimes$ are defined in the natural way. Moreover, it is clear that
the formulas (A) and (B) in the Definition of $\frac{2}{3}$PROP hold.

Thus, we constructed a topological $\frac{2}{3}$PROP $K(m,n)$.

Lemma. (i) The corresponding chain dg $\frac{2}{3}$PROP (formed by the chain complexes of
the spaces in our stratification) is a free $\frac{2}{3}$PROP of graded vector spaces (when we
forget about the differential),

(ii) the corresponding homology $\frac{2}{3}$PROP is the $\frac{2}{3}$PROP of bialgebras $Bialg$

Proof. (i) follows from the fact that any limit configuration can be written as the com-
position of the degenerations in Figures 3, 4, 5 in a unique way, (ii) follows from the
fact that all spaces $K(m,n)$ are contractible (and, therefore, have only 0-th nontrivi-
al homology).

Example. In our compactification, the left picture in Figure 1 is $K(2,1) \times K(1,2)$, and
the right picture is $K_{3,1}^3 \times K_{2,1}^1$.

Now we are ready to introduce our main object—a dg Lie algebra.

3 The dg Lie algebra

Denote by $\mathbb{C}\{\text{Hom}(V^\otimes m, V)\}$ the vector space generated by the infinite series of the form

$$\Psi + \Psi \otimes \Psi + \Psi \otimes \Psi \otimes \Psi + \cdots \in \prod_{n \geq 1} (\text{Hom}(V^\otimes m, V))^{\otimes n}$$

where $\Psi \in \text{Hom}(V^\otimes m, V), \ m \geq 2$. We denote the sum above by $\overline{\Psi}$. As a vector space,
$\mathbb{C}\{\text{Hom}(V^\otimes m, V)\}$ is "a very huge" vector space generated by the set $\text{Hom}(V^\otimes m, V)$.
Analogously, introduce the notation \( \mathbb{C}\{\text{Hom}(V, V^\otimes n)\}, \; n \geq 2 \), and the notation \( \Theta \) for \( \Theta \in \text{Hom}(V, V^\otimes n) \).

Now introduce a dg Lie algebra \( \mathfrak{N} \). First we introduce a graded vector space \( \mathfrak{N}_0 \), and then \( \mathfrak{N} \) will be a quotient space. We set:

\[
\mathfrak{N}_0 = \bigoplus_{m \geq 2} \mathbb{C}\{\text{Hom}(V^\otimes m, V)\}[-m + 1] \oplus \bigoplus_{n \geq 2} \mathbb{C}\{\text{Hom}(V, V^\otimes n)\}[-n + 1] \oplus \bigoplus_{m,n \geq 2} \text{Hom}(V^\otimes m, V^\otimes n)[-m - n + 2] \quad (23)
\]

Now we introduce a prebracket on \( \mathfrak{N}_0 \). It means that it is skew-symmetric but does not obey the Jacobi identity. The idea goes back to the constructions in Section 2. We associate generators with the strata of codimension 0, and their compositions (the bracket) by the strata of codimension 1. Then the Jacobi identity follows (after factorization) from the equation \( \partial^2 = 0 \) where \( \partial \) is the chain differential. Philosophically, it is a kind of the Markl’s construction from [M1], but the author can not verbalize it at the moment.

The appearance of the infinite power series \( \Psi \) and \( \Theta \) are motivated by the identity (4). Thus, the reader can say that no infinite sums appear among the strata of codimension 0. Nevertheless, the Markl’s construction is an operadic construction, and when we deal with \( \frac{2}{3} \)PROPs we need some modifications.

Only the following brackets are nonzero:

(i) \( [\Psi_1, \Psi_2] := [\Psi_1, \Psi_2]_G \)
where \( \Psi_1 \in \text{Hom}(V^\otimes m_1, V) \), \( \Psi_2 \in \text{Hom}(V^\otimes m_2, V) \), and \( [\Psi_1, \Psi_2]_G \) is the Gerstenhaber bracket,

(ii) \( [\Theta_1, \Theta_2] := [\Theta_1, \Theta_2]_G \)
where \( \Theta_1 \in \text{Hom}(V, V^\otimes n_1) \), \( \Theta_2 \in \text{Hom}(V, V^\otimes n_2) \),
and \( [\Theta_1, \Theta_2]_G \) is the Gerstenhaber cobracket,

(iii) \( [\Theta, \Psi] := \Theta \circ \Psi \pm (\Theta)^\otimes m \otimes (\Psi)^\otimes n \)
where \( \Psi \in \text{Hom}(V^\otimes m, V) \), \( \Theta \in \text{Hom}(V, V^\otimes n) \), \( \circ \) and \( \otimes \)
are the compositions in the pre-\( \frac{2}{3} \)PROP \( \text{End}(V) \),

(iv) for \( \alpha \in \text{Hom}(V^\otimes m, V^\otimes n) \), \( m, n \geq 2 \), \( \Psi \in \text{Hom}(V^\otimes m_1, V) \)
\[
[\Psi, \alpha] := \sum_{j=1}^{m} \pm \Psi(j \circ) \alpha,
\]

(v) for \( \alpha \in \text{Hom}(V^\otimes m, V^\otimes n) \), \( m, n \geq 2 \), \( \Theta \in \text{Hom}(V, V^\otimes n_1) \)
\[
[\alpha, \Theta] := \sum_{i=1}^{n} \pm \alpha \circ_i \Theta
\]
We also suppose that the bracket $[\ ,\ ]$ is graded-skew-commutative.

This bracket does not obey the Jacobi identity. The Jacobi identity fails for the brackets $[\Psi_1, \Psi_2, \Psi_3]$, $[[\Theta_1, \Theta_2], \Theta_3]$, $[[\Theta_1, \Theta_2], \Theta_3]$, and $[[\Psi, \Theta_1], \Theta_2]$. Our solution is to factorize by the vector space spanned by the Jacobi identities, as follows:

First, factorize the space generated by $\Psi$ by the space $I_G$ generated by the Jacobi identities. It means that we first consider a graded vector space $I_G$ generated by the vectors $[\Psi_1, \Psi_2, \Psi_3]$, $[[\Psi_1, \Psi_2], \Psi_3]$, and $[[\Psi_1, \Psi_2], \Theta_3]$. The same is true of the bracket $[\Psi_1, \Psi_2, \Psi_3]$. We also suppose that the bracket $[\Psi_1, \Psi_2, \Psi_3]$ obey the Jacobi identity. The Jacobi identity fails for the bracket $[\Psi_1, \Psi_2, \Psi_3]$. The Jacobi identity fails for the bracket $[\Psi_1, \Psi_2, \Psi_3]$. The formula for the bracket (24) defines a gradedLie algebra structure on $\Psi$. Theorem.

Now we define the quotient-spaces $\operatorname{Hom}(V^\otimes m, V^\otimes n)$ for $m, n \geq 2$.

Notice, that the bracket of $I_G$ and $I_G$ with $\operatorname{Hom}(V^\otimes m, V^\otimes n)$, $m, n \geq 2$ is zero without any factorization because it depends only on the "linear part" of $\Psi, \Theta$. The same is true for the product $\Psi \circ \Theta$. Then, define $I_G$ as the graded vector space spanned by all elements of the form $[\Theta_1, \Theta_2] \circ \Theta \pm \Psi_1 \circ \Theta_2 \pm \Psi_2 \circ \Theta_1 \pm \Psi_3 \circ \Theta_2$ where $\Psi_i \in \operatorname{Hom}(V^\otimes m, V)$, $m_i \geq 2$, $\Theta \in \operatorname{Hom}(V^\otimes m, V^\otimes n)$, $n \geq 2$ and the analogous expressions with 2 $\Theta$'s and 1 $\Psi$. Here $A \circ B$ denotes the sum with the signs as in (iv),(v) in the definition [24]. Next, denote by $I$ the graded vector space generated by the elements of the form

$$\Psi k_i (\circ) \Psi k_{i-1} (\circ) \cdots (\circ) \Psi k_1 (\circ) \alpha \circ \Theta_{\ell_i} \circ \cdots \circ \Theta_{\ell_i},$$

where $\alpha \in \operatorname{Hom}(V^\otimes m, V^\otimes n)$, $m, n \geq 2$.

Finally, denote by $\operatorname{Hom}(V^\otimes m, V^\otimes n)[-m-n+2]$, $m, n \geq 2$, the graded component of the quotient of $(\oplus_{m,n \geq 2} \operatorname{Hom}(V^\otimes m, V^\otimes n)[-m-n+2])/I$.

Denote

$$N = \bigoplus_{m \geq 2} \operatorname{Hom}(V^\otimes m, V)[-m+1] \oplus \bigoplus_{n \geq 2} \operatorname{Hom}(V, V^\otimes n)[-n+1] \oplus \bigoplus_{m,n \geq 2} \operatorname{Hom}(V^\otimes m, V^\otimes n)[-m-n+2] \\ (25)$$

Theorem. The formulas for the bracket [24] define a graded Lie algebra structure on $N$ (that means that the Jacobi identity is satisfied).

Proof. It follows from the definitions. \qed
Remark. It would be very interesting to specify in which sense our construction is an analog of the Markl’s construction [M1] applied to the case of $\mathbb{Z}_2$PROPs.

The dg Lie algebra $\mathfrak{N}$ defined above is a “deformation Lie algebra” (see Section 4) for the bialgebra $V$ with 0 product and 0 coproduct. When we want to consider deformation theory for a bialgebra $V$ with non-zero (co)product, we localize $\mathfrak{N}$ by the corresponding solution of the Maurer-Cartan equation.

Lemma. Let $\Psi \in \text{Hom}(V \otimes^2 V)$ and $\Theta \in \text{Hom}(V, V \otimes^2 V)$ are the product and the coproduct for a (co)associative bialgebra structure on $V$. Then $\beta = \Psi + \Theta \in \mathfrak{N}^1$ satisfies the Maurer-Cartan equation with 0 differential:

$$[\beta, \beta] = 0 \quad (26)$$

Proof. It is clear. $\square$

Now for such $\Psi, \Theta$ as above, we consider the dg Lie algebra $\mathfrak{N}_{\Psi, \Theta}$ which is the same as $\mathfrak{N}$ but with the differential $\text{ad}(\beta)$.

4 From dg Lie algebra to $L_\infty$ algebra

Let $g_1, g_2$ be two $L_\infty$ algebras. Recall that it means that we have odd vector fields of degree +1 $Q_1$ on $g_1[1]$ and $Q_2$ on $g_2[1]$ such that $Q_1^2 = Q_2^2 = 0$. Suppose we have an $L_\infty$ map $U: g_1 \to g_2$. It means, by definition, that we have a non-linear map $U: g_1[1] \to g_2[1]$ which maps the field $Q_1$ to the field $Q_2$. Suppose that the map $U$ is a (non-linear) imbedding of topological spaces. Then we can say that the vector field $Q_2$ is tangent to the image $U(g_1[1])$ (because it coincides with the image of $Q_1$). Vice versa, suppose we have an $L_\infty$ algebra $g_2$, a graded vector space $g_1$, and an imbedding $U: g_1[1] \to g_2[1]$ such that the vector field $Q_2$ on $g_2$ is tangent to the image. Then we claim that there is a unique $L_\infty$ structure on $g_1$ which makes $U$ an $L_\infty$ map.

Apply it now to the case when $g_2 = \mathfrak{N}$. Consider the Gerstenhaber-Schack space $\bigoplus_{m,n \geq 1, m+n \geq 3} \text{Hom}(V \otimes^m V \otimes^n V)[-m-n+2]$ as $g_1$. Consider the following non-linear map $U: g_1 \to \mathfrak{N}$: the map $U$ maps $\text{Hom}(V \otimes^m V \otimes^n V)[-m-n+2]$ identically to $\mathfrak{N}$ when $m,n \geq 2$. When $n = 1$, $U$ maps $\Psi \in \text{Hom}(V \otimes^m V)$ to $\Psi \in \mathfrak{N}$, and for $m = 1$, $U$ maps $\Theta \in \text{Hom}(V, V \otimes^n V) \in g_1$ to $\Theta \in \mathfrak{N}$. It is clear that we are in the assumptions above (before the localization by the solution $\beta$ of the Maurer-Cartan equation). It means that the quadratic vector field on $\mathfrak{N}[1]$ defining the dg Lie algebra structure on $\mathfrak{N}$ is tangent to the image of $U$. It allows us to define an $L_\infty$ structure on the Gerstenhaber-Schack space $g_1$ which makes $U$ an $L_\infty$ map.

The author hopes to construct this $L_\infty$ structure explicitly in the next paper. He is not sure that the linear part of this $L_\infty$ structure will be the Gerstenhaber-Schack differential, but he is sure that in this way we obtain a more right object.
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Dept. of Math., ETH-Zentrum, 8092 Zurich, SWITZERLAND
e-mail: borya@mccme.ru, borya@math.ethz.ch