Playing (Almost-)Optimally in Concurrent Büchi and co-Büchi Games

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Two-player games on graphs as a tool for formal verification (e.g. controller synthesis)

Win/lose games: the objectives of the two players are opposite

Concurrent games, as opposed to turn-based games
Concurrent games

Matching-penny game
Concurrent games

- Player A chooses a row

$q_0, \begin{bmatrix} \bot & \top \end{bmatrix}$

« Matching-penny game »
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- Player A chooses a row
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Major difference with turn-based games
Concurrent games

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Concurrent games

- Need for *randomization*!
- Randomized strategy: choose rows/columns according to a distribution

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Concurrent games

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- Randomized strategy: choose rows/columns according to a distribution
- Given randomized strategies $\sigma_A$ and $\sigma_B$, the payoff (for A) is the probability $\mathbb{P}_{\sigma_A, \sigma_B}(W)$
- Optimal strategy for A: $\sigma_A$ that maximizes $\inf_{\sigma_B} \mathbb{P}_{\sigma_A, \sigma_B}(W)$

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Concurrent games

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- Optimal strategy for B:
  - $\sigma_B$ that minimizes $\sup_{\sigma_A} \mathbb{P}_{\sigma_A,\sigma_B}(W)$
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- There are optimal strategies for both players:
  - Player A: chooses uniformly at random a row
Concurrent games

- Need for **randomization**!
- Randomized strategy: choose rows/columns according to a distribution
- Given randomized strategies $\sigma_A$ and $\sigma_B$, the **payoff** (for A) is the probability $\mathbb{P}_{\sigma_A,\sigma_B}(W)$
- Optimal strategy for B $\sigma_B$ that minimizes $\sup_{\sigma_A} \mathbb{P}_{\sigma_A,\sigma_B}(W)$
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- There are optimal strategies for both players:
  - Player A: chooses uniformly at random a row
  - Player B: chooses uniformly at random a column
Need for randomization!

Randomized strategy: choose rows/columns according to a distribution

Given randomized strategies $\sigma_A$ and $\sigma_B$, the payoff (for A) is the probability $\mathbb{P}_{\sigma_A, \sigma_B}(W)$

Optimal strategy for B $\sigma_B$ that minimizes $\sup_{\sigma_A} \mathbb{P}_{\sigma_A, \sigma_B}(W)$

$\varepsilon$-optimal strategy for A: $\sigma_A$ that achieves $\sup_{\sigma'_A} \inf_{\sigma'_B} \mathbb{P}_{\sigma'_A, \sigma'_B}(W)$ up to $\varepsilon$

There are optimal strategies for both players:
- Player A: chooses uniformly at random a row
- Player B: chooses uniformly at random a column

Value of the game: $\frac{1}{2}$
Properties of concurrent games

Martin’s determinacy theorem for Blackwell games

Concurrent games with Borel objectives have values:

$$\nu(q) = \sup_{\sigma_A} \inf_{\sigma_B} \mathbb{P}_{\sigma_A,\sigma_B}(W) = \inf_{\sigma_B} \sup_{\sigma_A} \mathbb{P}_{\sigma_A,\sigma_B}(W)$$
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  - Parity games require infinite memory for both optimal and almost-optimal strategies

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- (Infinite) Memory is sometimes needed by optimal and almost-optimal strategies
  - Parity games require infinite memory for both optimal and almost-optimal strategies
- Note: this is specific to concurrent games! (as compared to turn-based)
An example of a Büchi game

$q_0$, $\begin{bmatrix} q_0 & \top & \bot \\ \top & \bot & \bot \end{bmatrix}$

« The snowball game »

[AH00] L. De Alfaro, T. Henzinger. Concurrent omega-regular games (LICS’00)
An example of a Büchi game

- Objective is to visit $T$ infinitely often

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- Objective is to visit $T$ infinitely often
- Value of the game is $1$

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An example of a Büchi game

- Objective is to visit $T$ infinitely often
- Value of the game is 1
- Player A (rows) has no optimal strat.

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- Objective is to visit $T$ infinitely often
- Value of the game is $1$
- Player A (rows) has no optimal strat.
- Every finite-memory strat. has value $0$

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An example of a Büchi game

Objective is to visit $\top$ infinitely often

Value of the game is $1$

Player A (rows) has no optimal strat.

Every finite-memory strat. has value $0$

Player A needs infinite memory to play $\varepsilon$-optimal for every $\varepsilon > 0$:
- Play first row with probability $1 - \varepsilon_k$ and second row with probability $\varepsilon_k$
- $k$ is the number of visits to $\top$
- $(\varepsilon_k)_k$ quickly decreases to $0$

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The approach of this work
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- We are interested in low memory requirements for optimal and almost-optimal strategies in concurrent games with parity objectives in general, and more specifically Büchi and co-Büchi objectives.
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- Low memory requirement = **positional** strategies
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- Low memory requirement = positional strategies
- \( \sigma_A \) is positional if it depends only on the last visited state
The approach of this work

- We are interested in **low memory requirements** for optimal and almost-optimal strategies in concurrent games with parity objectives in general, and more specifically Büchi and co-Büchi objectives.

- Low memory requirement = **positional** strategies

- $\sigma_A$ is positional if it depends only on the last visited state.

Our approach: focus on interactions, and characterize well-behaved interactions.
A tool to apprehend concurrent interactions: game forms
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Elementary brick
A tool to apprehend concurrent interactions: game forms

Nice constructions

Nice bricks

Elementary brick
A tool to apprehend concurrent interactions: game forms

Games on graphs with good properties

Game forms with good properties

Game form

\[
\begin{bmatrix}
q_0, \\
\top & \top & \bot
\end{bmatrix}
\]

\[
\begin{bmatrix}
x & y \\
y & z
\end{bmatrix}
\]
Approach in this work
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- $\mathcal{I}$ set of game forms
Approach in this work

- $\mathcal{I}$ set of game forms
- Identify properties of $\mathcal{I}$ so that all concurrent games built using game forms $\mathcal{I}$ behave well
Approach in this work

- $\mathcal{F}$ set of game forms
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Behave well = positional (almost-)optimal strategies are sufficient
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Behave well = positional (almost-)optimal strategies are sufficient

(Co-)Büchi conditions
Previous works with a similar methodology

[BBL21] Bordais, Bouyer, Le Roux. From local to global determinacy in concurrent graph games (FSTTCS’21)
[BBL22] Bordais, Bouyer, Le Roux. Optimal Strategies in Concurrent Reachability Games (CSL’22)
Previous works with a similar methodology

- Determinacy of deterministic games [BBL21]
  - The matching-penny is not a good game form
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- Reachability objectives \[ \text{[BBL22]} \]
  - Optimal and almost-optimal strategies can be chosen positional (when they exist)
  - Local condition (called RM) on game forms to ensure existence (and therefore positionality) of optimal strategies everywhere

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Game in normal form:

$$
\begin{bmatrix}
1/2 & 1/4 \\
1/2 & 3/4 \\
\end{bmatrix}
$$
What game theory tells us

- One can associate to each state $q$ of the game its value $v(q)$, and these values satisfy local optimality equations.

- Both players have (local) optimal strategies in this game in normal form.
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- Locally optimal strategies may not be globally optimal (in the graph).
Example

$q_0, \begin{bmatrix} q_0 & T & \perp \end{bmatrix}$
Example

$q_0, \begin{bmatrix} q_0 & T & \top \end{bmatrix}$

0

1

1

⊥
Example

Locally optimal strategy $\sigma_A$:
Example

- Locally optimal strategy $\sigma_A$:
  - Player A chooses the first row
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- What is wrong?
Locally optimal strategy $\sigma_A$:
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What is wrong?
- In the MDP generated by $\sigma_A$, there is an end-component which is losing
Our contributions
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Characterize positional (almost-)optimal strategies using locally (almost-)optimal strategies (applies to tail objectives)
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Büchi objectives

- Optimal strategies may not exist (known)
- When optimal strategies exist from all states, they can be chosen positional (inherited from reachability games)
- Almost-optimal strategies may require infinite memory (known)
- Characterization of nice game forms (aBM) for ensuring:
  - Positional almost-optimal strategies
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co-Büchi objectives
- Optimal strategies may not exist (known)
- When optimal strategies exist, they may require infinite memory
- Almost-optimal strategies can be chosen positional (known [CDAH06])
- Characterization of nice game forms (coBM) for ensuring:
  - Positional optimal strategies

[CDAH06] K. Chatterjee, L. De Alfaro, T. Henzinger. The complexity of quantitative concurrent parity games (SODA'06).
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How should we restrict interactions to avoid this phenomenon?
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How should we restrict interactions to avoid this phenomenon?

$$\mathcal{F} = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$$
Recall: there exist Büchi games where infinite memory is required to play $\varepsilon$-optimally.

- How should we restrict interactions to avoid this phenomenon?

$$F = \begin{bmatrix} 1 \end{bmatrix}$$
Focus on Büchi conditions

\[
\begin{bmatrix}
  x & y \\
  y & z
\end{bmatrix}
\]
Focus on Büchi conditions

\[
\begin{bmatrix}
x & y \\
y & z \\
\end{bmatrix}
\]

Several local « environments »
Focus on Büchi conditions

\[
\begin{bmatrix}
x & y \\
y & z
\end{bmatrix}
\]

Several local « environments »

\[
\begin{bmatrix}
\overline{T} & T \\
T & 0
\end{bmatrix}
\]

Target

Not target
Focus on Büchi conditions

\[
\begin{bmatrix}
x & y \\ y & z
\end{bmatrix}
\]

Several local « environments »

Payoff if the game proceeds to here

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Not target
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Not target

\[
\begin{bmatrix}
  \overline{T} & T \\
  T & \overline{T}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \overline{T} & 1 \\
  1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
  1 & 0 \\
  0 & 1
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Focus on Büchi conditions

\[
\begin{bmatrix}
x & y \\
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Several local « environments »

Local environment

- \( O \) set of variables (\( \{x, y, z\} \) in the example)
- One small game
  - for every \( E \subseteq O \),
  - for every \( p_T : E \rightarrow [0,1] \), and
  - for every \( \alpha : O \setminus E \rightarrow [0,1] \)
Characterization

Definition of aBM (almost-Büchi maximizable)

- A game form $\mathcal{F}$ is aBM whenever every embedding of $\mathcal{F}$ into a local environment admits a positional $\epsilon$-optimal strategy for every $\epsilon > 0$. 
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- An aBM game form can be characterized and decided (it can be encoded as a formula of the first-order theory of the reals)
Definition of aBM (almost-Büchi maximizable)

- A game form $\mathcal{F}$ is aBM whenever every embedding of $\mathcal{F}$ into a local environment admits a positional $\epsilon$-optimal strategy for every $\epsilon > 0$.

- An aBM game form can be characterized and decided (it can be encoded as a formula of the first-order theory of the reals).

Characterization

- If all game forms used in a concurrent game $\mathcal{G}$ are aBM, then $\mathcal{G}$ admits positional $\epsilon$-optimal strategies for every $\epsilon > 0$.
- If a game form is not aBM, then there is a concurrent game which does not admit a positional $\epsilon$-optimal strategy for some $\epsilon > 0$. 
How to ensure positional (almost-)optimal strategies?

Existence of positional optimal or $\varepsilon$-optimal strategies under the following restrictions on game forms:

|                  | Positional opt. strat. | Positional almost-opt. strat. |
|------------------|------------------------|-------------------------------|
|                  | Target  | Not target | Target  | Not target |
| Safety obj.      | No restr. | No restr. | No restr. | No restr. |
| Reach. obj.      | No restr. | RM         | No restr. | No restr. |
| Büchi obj.       | No restr. | RM         | No restr. | aBM       |
| co-Büchi obj.    | RM      | coBM       | No restr. | No restr. |
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If game forms satisfy the properties below, then positional strategies exist and can be chosen positional.
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If game forms satisfy the properties below, then positional strategies exist and can be chosen positional.

If game forms at states not in target are coBM and in targets are RM, then optimal strategies exist and can be chosen positional.
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|-------------------|------------------------|-------------------------------|
|                   | Target | Not target | Target | Not target |
| No restr.         | No restr. | No restr. | No restr. | No restr. |

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|-------------------|------------------------|-------------------------------|
|                   | Target | Not target | Target | Not target |
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| co-Büchi obj.     | Positional opt. strat. | Positional almost-opt. strat. |
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| RM                | coBM                   | No restr. | No restr. |

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If game forms at states not in target are aBM then $\varepsilon$-optimal strategies can be chosen positional.
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If game forms satisfy the properties below, then positional strategies exist and can be chosen positional:

- Positional optimal strategies exist and can be chosen positional.
- Positional almost-optimal strategies exist and can be chosen positional.
- $\epsilon$-optimal strategies can always be chosen positional.

If game forms at states not in target are aBM then $\epsilon$-optimal strategies can be chosen positional.
How to ensure positional (almost-)optimal strategies?

Existence of positional optimal or $\varepsilon$-optimal strategies under the following restrictions on game forms:

|                      | Positional opt. strat. | Positional almost-opt. strat. |
|----------------------|------------------------|-------------------------------|
|                      | Target | Not target | Target | Not target |
| Safety obj.          | No restr. | No restr. | No restr. | No restr. |
| Reach. obj.          | No restr. | RM | No restr. | No restr. |
| Büchi obj.           | No restr. | RM | No restr. | aBM |
| co-Büchi obj.        | RM | coBM | No restr. | No restr. |
Properties of game forms
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- $\text{coBM} \subseteq \text{RM} \subseteq \text{aBM}$
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- All these notions RM, coBM, aBM, ... can be decided (can be expressed in FO(\(\mathbb{R}\)))
- coBM \(\subseteq\) RM \(\subseteq\) aBM
- These game forms are coBM:
  - "Turn-based" game forms:
    \[
    \begin{bmatrix}
    x & y & z \\
    x & y & z \\
    \end{bmatrix}
    \]
  - Two-variable game forms:
    \[
    \begin{bmatrix}
    x & y & x \\
    y & x & x \\
    \end{bmatrix}
    \]
  - Permutation game forms:
    \[
    \begin{bmatrix}
    x & y & z \\
    z & x & y \\
    y & z & x \\
    \end{bmatrix}
    \]
What you can bring home
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  - Optimal strategies might not exist
  - (Almost-)Optimal strategies might require infinite memory
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  - Study interactions (**game forms**) as first-class citizens
  - Identify interactions (game forms) that are well-behaved (with a property in mind)
  - Show that, all games on graphs with interactions taken in the set of well-behaved game forms behave well; and that this set is maximal
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Going further:
  • Understand beyond (co-)Büchi conditions, e.g. parity conditions
  • (Ongoing work) A different approach, which should be able to deal with parity conditions