ROPELENGTH, CROSSING NUMBER AND FINITE TYPE INvariants OF LINKS

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Abstract. Ropelength and embedding thickness are related measures of geometric complexity of classical knots and links in Euclidean space. In their recent work, Freedman and Krushkal posed a question regarding lower bounds for embedding thickness of \( n \)-component links in terms of the Milnor linking numbers. The main goal of the current paper is to provide such estimates, and thus generalizing the known linking number bound. In the process, we collect several facts about finite type invariants and ropelength/crossing number of knots. We give examples of families of knots, where such estimates outperform the well known knot–genus estimate.

1. Introduction

Given an \( n \)-component link (we assume class \( C^1 \) embeddings) in 3–space

\[ L : S^1 \sqcup \ldots \sqcup S^1 \rightarrow \mathbb{R}^3, \quad L = (L_1, L_2, \ldots, L_n), \quad L_i = L|_{the \ i'th \ circle}, \quad (1.1) \]

its ropelength \( \text{rop}(L) \) is the ratio \( \text{rop}(L) = \frac{\ell(L)}{r(L)} \) of length \( \ell(L) \), which is a sum of lengths of individual components of \( L \), to reach or thickness: \( r(L) \), i.e. the largest radius of the tube embedded as a normal neighborhood of \( L \). The ropelength within the isotopy class \( [L] \) of \( L \) is defined as

\[ \text{Rop}(L) = \inf_{L' \in [L]} \text{rop}(L'), \quad \text{rop}(L') = \frac{\ell(L')}{r(L')}, \quad (1.2) \]

(in \cite{7} it is shown that the infimum is achieved within \([L]\) and the minimizer is of class \( C^{1,1} \)). A related measure of complexity, called embedding thickness was introduced recently in \cite{16}, in the general context of embeddings’ complexity. For links, the embedding thickness \( \tau(L) \) of \( L \) is given by a value of its reach \( r(L) \) assuming that \( L \) is a subset of the unit ball \( B_1 \) in \( \mathbb{R}^3 \) (note that any embedding can be scaled and translated to fit in \( B_1 \)). Again, the embedding thickness of the isotopy class \([L]\) is given by

\[ \mathcal{T}(L) = \sup_{L' \in [L]} \tau(L'). \quad (1.3) \]

For a link \( L \subset B_1 \), the volume of the embedded tube of radius \( \tau(L) \) is \( \pi \ell(L) \tau(L)^2 \), \cite{20} and the tube is contained in the ball of radius \( r = 2 \), yielding

\[ \text{rop}(L) = \frac{\pi \ell(L) \tau(L)^2}{\pi \tau(L)^3} \leq \frac{\frac{4}{3} \pi 2^3}{\pi \tau(L)^3}, \quad \tau(L) \leq \left( \frac{11}{\text{rop}(L)} \right)^{\frac{3}{2}}. \quad (1.4) \]

In turn a lower bound for \( \text{rop}(L) \) gives an upper bound for \( \tau(L) \) and vice versa. For other measures of complexity of embeddings such as distortion or Gromov-Guth thickness see e.g. \cite{21}, \cite{22}.

Bounds for the ropelength of knots, and in particular the lower bounds, have been studied by many researchers, we only list a small fraction of these works here \cite{4, 5, 7, 12, 11, 10, 14, 29, 37, 39, 30, 38}. Many of the results are applicable directly to links, but the case of links is treated in more detail by Cantarella, Kusner and Sullivan \cite{7} and in the earlier work of Diao, Ernst, and Janse

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Van Rensburg [13] concerning the estimates in terms of the pairwise linking number. In [7], the authors introduce a cone surface technique and show the following estimate, for a link $L$ (defined as in [1.12]) and a given component $L_i$ [7, Theorem 11]:

$$\text{rop}(L_i) \geq 2\pi + 2\pi\sqrt{\text{lk}(L_i, L)}, \quad (1.5)$$

where $\text{lk}(L_i, L)$ is the maximal total linking number between $L_i$ and the other components of $L$. A stronger estimate was obtained in [7] by combining the Freedman and He asymptotic crossing number bound for energy of divergence free fields and the cone surface technique as follows

$$\text{rop}(L_i) \geq 2\pi + 2\pi\sqrt{\text{Ac}(L_i, L)}, \quad \text{rop}(L_i) \geq 2\pi + 2\pi\sqrt{2g(L_i, L) - 1}, \quad (1.6)$$

where $\text{Ac}(L_i, L)$ is the asymptotic crossing number (c.f. [17]) and the second inequality is a consequence of the estimate $\text{Ac}(L_i, L) \geq 2g(L_i, L) - 1$, where $g(L_i, L)$ is a minimal genus among surfaces embedded in $\mathbb{R}^3 \setminus \{L_1 \cup \ldots \cup L_i \cup \ldots \cup L_n\}$, [17] p. 220 (in fact, Estimate (1.6) subsumes Estimate (1.5) since $\text{Ac}(L_i, L) \geq \text{lk}(L_i, L)$). A relation between $\text{Ac}(L_i, L)$ and the higher linking numbers of Milnor, [32, 33] is unknown and appears difficult. The following question, concerning the embedding thickness, is stated in [16] p. 1424:

**Question A.** Let $L$ be an $n$-component link which is Brunnian (i.e. almost trivial in the sense of Milnor [32]). Let $M$ be the maximum value among Milnor’s $\bar{\mu}$-invariants with distinct indices i.e. among $|\bar{\mu}_{1,j}(L)|$. Is there a bound

$$\tau(L) \leq c_n M^{-\frac{1}{n}}, \quad (1.7)$$

for some constant $c_n > 0$, independent of the link $L$? Is there a bound on the crossing number $\text{Cr}(L)$ in terms of $M$?

Recall that the Milnor $\bar{\mu}$-invariants $\{\bar{\mu}_{i_1, i_2, \ldots, i_k; j}\}$ of $L$, are indexed by a subset of component indexes $\mathbb{I}(i; j) = (i_1, i_2, \ldots, i_k; j)$. These are a well known link homotopy invariants (if all the indexes $(i; j)$ in are different in $I$) of $n$–component links are often referred to simply as Milnor linking numbers or higher linking numbers, [32, 33]. The $\bar{\mu}$-invariants are defined as certain residue classes

$$\bar{\mu}_{1,j}(L) \equiv \mu_{1,j}(L) \mod \Delta_{\mu}(I; j), \quad \Delta_{\mu}(I; j) = \gcd(\Gamma_{\mu}(I; j)). \quad (1.8)$$

where $\mu_{1,j}$ are coefficients of the Magnus expansion of the $j$th longitude of $L$ in $\pi_1(\mathbb{R}^3 - L)$, and $\Gamma_{\mu}(I; j)$ is a certain subset of lower order Milnor invariants, c.f. [33]. Regarding $\bar{\mu}_{1,j}(L)$ as an element of $\mathbb{Z}_d = \{0, 1, \ldots, d - 1\}$, $d = \Delta_{\mu}(I; j)$ (or $\mathbb{Z}$, whenever $d = 0$) let us define

$$[\bar{\mu}_{1,j}(L)] := \begin{cases} \min(\bar{\mu}_{1,j}, d - \bar{\mu}_{1,j}) & \text{for } d > 0, \\ |\bar{\mu}_{1,j}| & \text{for } d = 0. \end{cases} \quad (1.9)$$

Our main result addresses Question $\overline{A}$ for general $n$–component links (without the Brunnian assumption) as follows.

**Theorem A.** Let $L$ be an $n$-component link and $\bar{\mu}(L)$ one of its top Milnor linking numbers, then

$$\text{rop}(L) \geq \sqrt[n]{\frac{1}{\sqrt[n]{\text{rop}(L)}}}, \quad \text{Cr}(L) \geq \frac{1}{3} (n - 1) \frac{1}{\sqrt[n]{\text{rop}(L)}}. \quad (1.10)$$

In the context of Question $\overline{A}$ the estimate of Theorem $\overline{A}$ transforms, using (1.4), as follows

$$\tau(L) \leq \left(\frac{141}{n^4}\right)^{\frac{1}{2}} M^{-\frac{1}{2(n-1)}}.$$ 

$^1$where the last index $j$ plays a special role c.f. [33].
Naturally, Question A can be asked for knots and links and lower bounds in terms of finite type invariants in general. Such questions have been raised for instance in [36], where the Bott-Taubes integrals [3] [40] have been suggested as a tool for obtaining estimates.

**Question B.** Can we find estimates for ropelength of knots/links, in terms of their finite type invariants?

In the remaining part of this introduction let us sketch the basic idea behind our approach to Question B, which relies on the relation between the finite type invariants and the crossing number.

Note that since rop(K) is scale invariant, it suffices to consider unit thickness knots, i.e. K together with the unit radius tube neighborhood (i.e. \( r(K) = 1 \)). In this setting, rop(K) just equals the length \( \ell(K) \) of K. From now on we assume unit thickness, unless stated otherwise. In [4], Buck and Simon gave the following estimates for \( \ell(K) \), in terms of the crossing number \( Cr(K) \) of K:

\[
\ell(K) \geq \frac{4\pi}{11} Cr(K)^{\frac{3}{2}}, \quad \ell(K) \geq 4\sqrt{\pi Cr(K)}.
\] (1.11)

Clearly, the first estimate is better for knots with large crossing number, while the second one can be sharper for low crossing number knots (which manifests itself for instance in the case of the trefoil). Recall that \( Cr(K) \) is a minimal crossing number over all possible knot diagrams of K within the isotopy class of K. The estimates in (1.11) are a direct consequence of the ropelength bound for the average crossing number \( PCr(L) \) of \( K \) (proven in [4, Corollary 2.1]) i.e.

\[
\ell(K)^{\frac{4}{3}} \geq \frac{4\pi}{11} acr(K), \quad \ell(K)^2 \geq 16\pi acr(K).
\] (1.12)

In Section 4, we obtain an analog of (1.11) for \( n \)-component links (\( n \geq 2 \)) in terms of the pairwise crossing number \( PCr(L) \), as follows:

\[
\ell(L) \geq \frac{1}{\sqrt{n-1}} \left( \frac{3}{2} PCr(L) \right)^{\frac{3}{2}}, \quad \ell(L) \geq \frac{n\sqrt{16\pi}}{\sqrt{n^2 - 1}} \left( PCr(L) \right)^{\frac{3}{2}}.
\] (1.13)

For low crossing number knots, the Buck and Simon bound (1.11) was further improved by Diao [10] as follows:

\[
\ell(K) \geq \frac{1}{2} \left( d_0 + \sqrt{d_0^2 + 64\pi Cr(K)} \right), \quad d_0 = 10 - 6(\pi + \sqrt{2}) \approx 17.334.
\] (1.14)

On the other hand, there are well known estimates for \( Cr(K) \) in terms of finite type invariants of knots. For instance,

\[
\frac{1}{4} Cr(K)(Cr(K) - 1) + \frac{1}{24} \geq |c_2(K)|, \quad \frac{1}{8} (Cr(K))^2 \geq |c_2(K)|.
\] (1.15)

Lin and Wang [28] considered the second coefficient of the Conway polynomial \( c_2(K) \) (i.e. the first nontrivial type 2 invariant of knots) and proved the first bound in (1.15). The second estimate of (1.15) can be found in Polyak–Viro’s work [36]. Further, Willerton, in his thesis [41] obtained estimates for the “second”, after \( c_2(K) \), finite type invariant \( V_3(K) \) of type 3, as

\[
\frac{1}{4} Cr(K)(Cr(K) - 1)(Cr(K) - 2) \geq |V_3(K)|.
\] (1.16)

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2i.e. an average of the crossing numbers of diagrams of K over all projections of K, see Equation (4.2).

3see (1.10) and Corollary C generally PCr(L) \( \leq Cr(L) \), as the individual components can be knotted.

4More precisely: \( 16\pi Cr(K) \leq \ell(K)(\ell(K) - 17.334) \) [40].
In the general setting, Bar-Natan [2] shows that if \( V(K) \) is a type \( n \) invariant then \( |V(K)| = O(Cr(K)^n) \). All these results rely on the arrow diagrammatic formulas for Vassiliev invariants developed in the work of Goussarov, Polyak and Viro [19].

Clearly, combining (1.15) and (1.16) with (1.11) or (1.14), immediately yields lower bounds for ropelength in terms of the Vassiliev invariant. One may take these considerations one step further and extend the above estimates to the case of the \( 2n \)th coefficient of the Conway polynomial \( c_{2n}(K) \), with the help of arrow diagram formulas for \( c_{2n}(K) \), obtained recently in [8, 9]. In Section 3 we follow the Polyak–Viro’s argument of [36] to obtain

**Theorem B.** Given a knot \( K \), we have the following crossing number estimate

\[
Cr(K) \geq (2^{n+1}n!)|c_{2n}(K)| \frac{1}{\pi n} \geq \frac{2n}{3} |c_{2n}(K)| \frac{1}{\pi n}.
\]  

(1.17)

Combining (1.17) with Diao’s lower bound (1.14) one obtains

**Corollary C.** For a unit thickness knot \( K \),

\[
\ell(K) \geq \frac{1}{2} \left( d_0 + (d_0^2 + 64\pi^2 |c_{2n}(K)| \frac{1}{\pi n}) \frac{1}{2} \right).
\]  

(1.18)

A somewhat different approach to ropelength estimates is due to Cantarella, Kusner and Sullivan. In [7], they introduce a cone surface technique, which combined with the asymptotic crossing number, \( Ac(K) \), bound of Freedman and He, [17] gives

\[
\ell(K) \geq 2\pi + 2\pi \sqrt{Ac(K)}, \quad \ell(K) \geq 2\pi + 2\pi \sqrt{2g(K) - 1},
\]  

(1.19)

where the second bound follows from the knot genus estimate of [17]: \( Ac(K) \geq 2g(K) - 1 \).

When comparing Estimate (1.19) and (1.18), in favor of Estimate (1.18), we may consider a family of pretzel knots: \( P(a_1, \ldots, a_n) \) where \( a_i \) is the number of signed crossings in the \( i \)th tangle of the diagram, see Figure 1. Additionally, for a diagram \( P(a_1, \ldots, a_n) \) to represent a knot one needs to assume either both \( n \) and all \( a_i \) are odd or one of the \( a_i \) is even, [23].

![pretzel knots](image)

**Figure 1.** \( P(a_1, \ldots, a_n) \) pretzel knots.

Genera of selected subfamilies of pretzel knots are known, for instance [18] Theorem 13 implies

\[
g(P(a, b, c)) = 1,
\]

\[
c_2(P(a, b, c)) = \frac{1}{4}(ab + ac + bc + 1),
\]

where \( a, b, c \) are odd integers with the same sign (for the value of \( c_2(P(a, b, c)) \) see a table in [18] p. 390]). Concluding, the lower bound in (1.18) can be made arbitrary large by letting \( a, b, c \to +\infty \), while the lower bound in (1.19) stays constant for any values of \( a, b, c \), under consideration. Yet another example of a family of pretzel knots with constant genus one and arbitrarily large \( c_2 \)-coefficient is \( D(m, k) = P(m, \varepsilon, k^{-\text{times}}, \varepsilon) \), \( m > 0, k \), where \( \varepsilon = \frac{k}{|k|} \) is the sign of \( k \) (e.g. \( D(3, -2) = P(3, -1, -1) \)). For any such \( D(m, k) \), we have \( c_2(D(m, k)) = \frac{mk}{4} \).

The paper is structured as follows: Section 3 is devoted to a review of arrow polynomials for finite type invariants, and Kravchenko-Polyak tree invariants in particular, it also contains the proof

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5 out of a few such examples given in [18]
Theorem B. Section 4 contains information on the average overcrossing number for links and link ropelength estimates analogous to the ones obtained by Buck and Simon [4] (see Equation (1.12)). The proof of Theorem A is presented in Section 5 together with final comments and remarks.

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3. Arrow polynomials and finite type invariants

Recall from [8], the Gauss diagram of a knot $K$ is a way of representing signed overcrossings in a knot diagram, by arrows based on a circle (Wilson loop, [1]) with signs encoding the sign of the crossing; see Figure 2 showing the $5_2$ knot and its Gauss diagram. Given a Gauss diagram $G$ of a knot, the arrow diagrammatic formulas of [19, 35] are defined simply as a signed count of selected subdiagrams in $G$. For instance the second coefficient of the Conway polynomial $c_2(K)$ is given by

$$c_2(K) = \langle \bigotimes, G \rangle = \sum_{\phi: \bigotimes \to G} \text{sign}(\phi), \quad \text{sign}(\phi) = \prod_{\alpha \in \bigotimes} \text{sign}(\phi(\alpha)), \quad (3.1)$$

where the sum is over all graph embeddings of $\bigotimes$ into $G$, and the sign is a product of signs of corresponding arrows in $\phi(\bigotimes) \subset G$. For example in the Gauss diagram of $5_2$ knot on Figure 2, there are two possible embeddings of $\bigotimes$ into the diagram. One matches the pair of arrows $\{a, b\}$ and another the pair $\{c, d\}$, since all crossings are positive we obtain $c_2(5_2) = 2$.

![Figure 2. $5_2$ knot and its Gauss diagram (all crossings are positive).](image)

For other even coefficients of the Conway polynomial, [9] provides the following recipe for their arrow polynomials. Given $n > 0$, consider any chord diagram $D$, on a single circle component with $2n$ chords, such as $\bigotimes$, $\bigcirc$, $\bigcirc$. A chord diagram $D$ is said to be a $k$-component diagram,
if after parallel doubling of each chord according to \(\overset{\rightarrow}{\sim}\), the resulting curve will have \(k\) components. For instance \(\otimes \overset{\rightarrow}{\sim} \otimes\) is a 1-component diagram and \(\square \overset{\rightarrow}{\sim} \square\) is a 3-component diagram. For the coefficients \(c_{2n}\), only one component diagrams will be of interest and we turn a one-component chord diagram with a base point into an arrow diagram according to the following rule [9]:

Starting from the base point we move along the diagram with doubled chords. During this journey we pass both copies of each chord in opposite directions. Choose an arrow on each chord which corresponds to the direction of the first passage of the copies of the chord (see Figure 3 for the illustration).

We call, the arrow diagram obtained according to this method, the ascending arrow diagram and denote by \(C_{2n}\) the sum of all based one-component ascending arrow diagrams with \(2n\) arrows. For example \(C_2 = \bigotimes\) and \(C_4\) is shown below (c.f. [9] p. 777).

\[
C_4 = \bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes.
\]

In [9], the authors show for \(n \geq 1\), that the \(c_{2n}(K)\) coefficient of the Conway polynomial of \(K\) equals

\[
c_{2n}(K) = \langle C_{2n}, G \rangle. \tag{3.2}
\]

**Theorem B.** Given a knot \(K\), we have the following crossing number estimate

\[
\Cr(K) \geq (2^{n+1}n!|c_{2n}(K)|)^\frac{1}{2n} \geq \sqrt{\frac{2n}{3}}|c_{2n}(K)|\frac{1}{2n}. \tag{3.3}
\]

**Proof.** Given \(K\) and its Gauss diagram \(G_K\), let \(X = \{1, 2, \ldots, \Cr(K)\}\) index arrows of \(G_K\) (i.e. crossings of a diagram of \(K\) used to obtain \(G_K\)). For diagram term \(A_i\) in the sum \(C_{2n} = \sum_i A_i\), and embedding \(\phi : A_i \hookrightarrow G_K\) covers a certain \(2n\) element subset of crossings in \(X\) we denote by \(X_\phi(i)\). Denote by \(E(i; G_K)\) the set of all possible embeddings \(\phi : A_i \hookrightarrow G_K\), and

\[
E(G_K) = \bigsqcup_i E(i; G_K).
\]

Note that for \(X_\phi(i) \neq X_\xi(j)\) for \(i \neq j\) and \(X_\phi(i) \neq X_\xi(i)\) for \(\phi \neq \xi\), thus for each \(i\) we have an injective map

\[
F_i : E(i; G_K) \hookrightarrow \mathcal{P}_{2n}, \quad F_i(\phi) = X_\phi(i),
\]

where \(\mathcal{P}_{2n} = \{2n\text{-element subsets in } C\}\). \(F_i\) extends in an obvious way to the whole disjoint union \(E(G_K)\), as \(F : E(G_K) \rightarrow \mathcal{P}_{2n}, F = \sqcup_i F_i\) and remains injective. In turn, for every \(i\) we have

\[
|\langle A_i, G_K \rangle| \leq \#E(i; G_K), \quad \text{and} \quad |\langle C_{2n}, G_K \rangle| \leq \#E(G_K) < \#\mathcal{P}_{2n} = \left(\frac{\Cr(K)}{2n}\right).
\]

Further, each arrow in \(G_K\) indexed by \(X\), either agrees with the orientation of the Wilson loop (then we say it is a right arrow) or not (then it is a left arrow), thus \(X = L \cup R, L \cap R = \varnothing\) where \(L\) is the subset of left arrows and \(R\) the subset or right arrows with cardinalities \(r = \#R\) and \(l = \#L\), \(l + r = \Cr(K)\). Since each arrow diagram \(A_i\) has exactly \(n\) right arrows and \(n\) left arrows we must have

\[
\sum_i \#E(i; G_K) \leq \binom{\Cr(K) - l}{n} \binom{l}{n}, \quad n \leq l \leq \Cr(K) - n.
\]
The left hand side is maximized for \( l = \left\lceil \frac{c(K)}{2} \right\rceil \) and thus we obtain
\[
2 |c_{2n}(K)| \leq \left( \frac{l}{n} \right)^2 \leq \frac{n^n}{n!},
\]
which proves the first inequality in (3.3). The second inequality is a simple consequence of Stirling’s approximation: \( n! \geq \sqrt{2\pi n^{n+\frac{1}{2}}} e^{-n} \).

Next, we turn to arrow polynomials for Milnor linking numbers. In [26] Kravchenko and Polyak introduced tree invariants of string links and established their relation to Milnor linking numbers via the skein relation of Polyak [34]. In the recent paper, the authors [24] showed that the \( \bar{m}_{23...n;1}(L) \) denoted by \( Z_{n;1} \). Changing the convention, adopted for knots, we follow [26], [24] and use vertical segments (strings) oriented downwards in place of circles (Wilson loops) as components. The polynomial \( Z_{n;1} \) is obtained from \( Z_{n-1;1} \) by expanding each term of \( Z_{n;1} \) through stacking.

**Figure 4.** Elementary trees \( e \) and \( \bar{e} \) and the \( Z_{2;1} \) arrow polynomial.

Elementary tree diagrams \( e \) and \( \bar{e} \), shown on Figure 4. We begin with the initial tree \( Z_{2;1} \), shown on Figure 4(right), and expand by stacking \( e \) and \( \bar{e} \) on the strings of \( Z_{2;1} \), this is shown on Figure 5 we avoid stacking \( \bar{e} \) on the first component (called the trunk, [24]). Thus \( Z_{3;1} \) is obtained as \( A + B - C \), where \( A = Z_{2;1} \prec_2 e \), \( B = Z_{2;1} \prec_1 e \), and \( C = Z_{2;1} \prec_3 e \). The sign of each term is \( (-1)^q \), where \( q \) =number of arrows pointing to the left, and \( \prec_i \) denotes stacking onto the \( i \)th string. Also, when obtaining terms of \( Z_{n;1} \), during the stacking process, we must pay attention and eliminate isomorphic (duplicate) diagrams. Given \( Z_{n;1} \), the main result of [24] (see also [25] for a related result) yields the following formula
\[
\bar{m}_{n;1}(L) \equiv \langle Z_{n;1}, G_L \rangle \mod \Delta_{\mu}(n;1),
\]
where \( \bar{m}_{n;1}(L) \) denotes a Gauss diagram of an \( n \)-component link \( L \), and the indeterminacy \( \Delta_{\mu}(n;1) \) is defined in (1.8). For \( n = 2 \), we obtain the usual linking number
\[
\bar{m}_{2;1}(L) = \langle Z_{2;1}, G_L \rangle = \langle 1 \rangle, G_L \rangle.
\]
For $n = 3$ and $n = 4$ the arrow polynomials can be obtained following the stacking procedure as follows

\[ \bar{\mu}_{3,1}(L) = \langle Z_{3,1}, G_L \rangle \mod \gcd \{ \bar{\mu}_{2,1}(L), \bar{\mu}_{3,1}(L), \bar{\mu}_{3,2}(L) \}, \]

and

\[ \bar{\mu}_{4,1}(L) = \langle Z_{4,1}, G_L \rangle \mod \Delta_{\bar{\mu}}(4; 1), \]

\[ Z_{3,1} = \begin{array}{c}
\text{component sublink} \end{array}, \]

\[ Z_{4,1} = \begin{array}{c}
\text{component sublink} \end{array}. \]

Given a formula for $\bar{\mu}_{n,1}(L) = \bar{\mu}_{23...n,1}(L)$ all remaining $\bar{\mu}$–invariants with distinct indices can be obtained from the following permutation identity (for $\sigma \in \Sigma(1, \ldots, n)$)

\[ \bar{\mu}_{\sigma(2)\sigma(3)\ldots\sigma(n),\sigma(1)}(L) = \bar{\mu}_{23...n,1}((\sigma(L)), \quad \sigma(L) = (L_{\sigma(1)}, L_{\sigma(2)}, \ldots, L_{\sigma(n)}). \quad (3.6) \]

By (3.4), (3.6) and (1.8) we have

\[ \bar{\mu}_{\sigma(2)\sigma(3)\ldots\sigma(n),\sigma(1)}(L) = \langle \sigma(Z_{n,1}), G_L \rangle \mod \Delta_{\bar{\mu}}(\sigma(2)\sigma(3)\ldots\sigma(n); \sigma(1)), \quad (3.7) \]

where $\sigma(Z_{n,1})$ is the arrow polynomial obtained from $Z_{n,1}$ by permuting the strings according to $\sigma$.

**Remark D.** One of the properties of $\bar{\mu}$–invariants is their cyclic symmetry. [33, Equation (21)], i.e. given a cyclic permutation $\rho$, we have

\[ \bar{\mu}_{\rho(2)\rho(3)\ldots\rho(n),\rho(1)}(L) = \bar{\mu}_{23...n,1}(L). \]

4. **OVERCROSSING NUMBER OF LINKS**

We will denote by $D_L$ a regular diagram of a link $L$, and by $D_L(v)$, the diagram obtained by the projection of $L$ onto the plane normal to a vector $v \in S^2$. For a pair of components $L_i$ and $L_j$ in $L$, define the overcrossing number in the diagram and the pairwise crossing number of components $L_i$ and $L_j$ in $D_L$ i.e.

\[ ov_{i,j}(D_L) = \{ \text{number of times } L_i \text{ overpasses } L_j \text{ in } D_L \}. \]

\[ cr_{i,j}(D_L) = \{ \text{number of times } L_i \text{ overpasses and underpasses } L_j \text{ in } D_L \} \quad (4.1) \]

\[ = ov_{i,j}(D_L) + ov_{j,i}(D_L) = cr_{j,i}(D_L). \]

In the following, we also use the average overcrossing number and average pairwise crossing number of components $L_i$ and $L_j$ in $L$, defined as an average over all $D_L(v)$, $v \in S^2$, i.e.

\[ aov_{i,j}(L) = \frac{1}{4\pi} \int_{S^2} ov_{i,j}(v) \, dv, \quad acr_{i,j}(L) = \frac{1}{4\pi} \int_{S^2} cr_{i,j}(v) \, dv = 2 aov_{i,j}(L) \quad (4.2) \]

**Lemma E.** Given a unit thickness link $L$, and any 2–component sublink $(L_i, L_j)$:

\[ \min(\ell_i, \ell_j, \frac{1}{\ell_i}, \frac{1}{\ell_j}) \geq \frac{5}{3} aov_{i,j}(L), \quad \ell_i \ell_j \geq 16\pi aov_{i,j}(L), \quad (4.3) \]

for $\ell_i = \ell(L_i)$, $\ell_j = \ell(L_j)$.

\footnote{unless otherwise stated we assume that $v$ is generic and thus $D_L(v)$ a regular diagram}
Proof. Consider the Gauss map of \( L_i = L_i(s) \) and \( L_j = L_j(t) \): 

\[
F_{i,j} : S^1 \times S^1 \rightarrow \text{Conf}_2(\mathbb{R}^3) \rightarrow S^2, \quad F_{i,j}(s,t) = \frac{L_i(s) - L_j(t)}{||L_i(s) - L_j(t)||}.
\]

If \( v \in S^2 \) is a regular value of \( F_{i,j} \) (which happens for the set of full measure on \( S^2 \)) then

\[
\text{oav}_{i,j}(v) = \# \{ \text{points in } F_{i,j}^{-1}(v) \}.
\]

i.e. \( \text{oav}_{i,j}(v) \) stands for number of times the \( i \)-component of \( L \) passes over the \( j \)-component, in the projection of \( L \) onto the plane in \( \mathbb{R}^3 \) normal to \( v \). As a direct consequence of Federer’s coarea formula \([15]\) (see e.g. \([31]\) for a proof)

\[
\int_{L_i \times L_j} |F_{i,j}^* \omega| = \frac{1}{4\pi} \int_{S^1 \times S^1} \frac{|\langle L_i(s) - L_j(t), L_i'(s), L_j'(t) \rangle|}{\|L_i(s) - L_j(t)\|^3} ds \, dt = \frac{1}{4\pi} \int_{S^2} \text{oav}_{i,j}(v) \, dv, \quad (4.4)
\]

where \( \omega = \frac{1}{4\pi} (x \, dy \wedge dz - y \, dx \wedge dz + z \, dx \wedge dy) \) is the normalized area form on the unit sphere in \( \mathbb{R}^3 \). Assuming the arc–length parametrization by \( s \in [0, \ell_i] \) and \( t \in [0, \ell_j] \) of the components we have \( \|L_i'(s)\| = \|L_j'(t)\| = 1 \) and therefore:

\[
\left| \frac{(L_i(s) - L_j(t), L_i'(s), L_j'(t))}{\|L_i(s) - L_j(t)\|^3} \right| \leq \frac{1}{\|L_i(s) - L_j(t)\|^2} \quad (4.5)
\]

Combining Equations \((4.4)\) and \((4.5)\) yields

\[
\int_0^{\ell_j} \int_0^{\ell_i} \frac{1}{\|L_i(s) - L_j(t)\|^2} ds \, dt = \int_0^{\ell_j} I_i(L_j(t)) dt \quad (4.6)
\]

where \( I_i(p) = \int_0^{\ell_i} \frac{1}{\|L_i(s) - p\|^2} ds \) is often called illumination of \( L_i \) from the point \( p \in \mathbb{R}^3 \), \( [4] \). Following the approach of \([4]\) and \([3]\) we estimate \( I_i(t) \). Denote by \( B_a(q) \) the ball at \( p \) of radius \( a \), and \( s(r) \) the length of a portion of \( L_i \) within the spherical shell: \( Sh(r) = B_r(p) \setminus B_2(p) \), where \( p = L_j(t) \) of and the radius \( r > 2 \). Note that, because the distance between \( L_i \) and \( L_j \) is at least \( 2 \), the unit thickness tube about \( L_i \) is contained entirely in \( Sh(r) \) for big enough \( r \). Clearly, \( s(r) \) is nondecreasing. Since the volume of a unit thickness tube of length \( a \) is \( \pi a \), comparing the volumes we obtain

\[
\pi s(r) \leq \text{Vol}(Sh(r)) = \frac{4}{3} \pi \left( (r + 1)^3 - 2^3 \right), \quad \text{and}
\]

\[
s(r) \leq 3.2 \pi r^3, \quad \text{for } r \geq 2.
\]

For a given \( s; 2.17 s^{-\frac{3}{2}} \geq \frac{1}{r^2} \), using the monotone rearrangement \( \left( \frac{1}{r^2} \right)^* \) of \( \frac{1}{r^2} \), \([27]\), we further obtain

\[
I_i(p) \leq \int_0^{\ell_i} \left( \frac{1}{r^2} \right)^*(s) ds \leq \int_0^{\ell_i} 2.17 s^{-\frac{3}{2}} ds = 6.51 \ell_i^{\frac{1}{2}},
\]

and integrating with respect to the \( t \)-parameter, we obtain

\[
\text{oav}(L) \leq \frac{1}{4\pi} \int_0^{\ell_j} \int_0^{\ell_i} \frac{1}{\|L_i(s) - L_j(t)\|^2} ds \, dt \leq \frac{13}{25} \ell_j \ell_i^{\frac{3}{4}}.
\]

Since the argument works for any choice of \( i \) and \( j \) estimates in Equation \((4.3)\) are proven. The second estimate in \((4.3)\) follows immediately from the fact that \( \frac{1}{\|L_i(s) - L_j(t)\|^2} \leq \frac{1}{4} \). \( \square \)

From the Gauss linking integral

\[
|\text{Lk}(L_i, L_j)| \leq \text{oav}_{i,j}(L),
\]
therefore

\[
\sum_{i \neq j} \ell_i \ell_j \leq n^2 \text{rop}(L) \text{rop}(L)^{3/2} - n^{-3/2} \text{rop}(L)^{3/2} = \frac{n - \frac{1}{n}}{n^2} \text{rop}(L)^{3/2}.
\] (4.8)

Analogously, using the second estimate in (4.7) and Jensen’s Inequality, yields

\[
32\pi \sum_{i<j} |\text{Lk}(L_i, L_j)| = 16\pi \sum_{i<j} |\text{Lk}(L_i, L_j)| \leq \sum_i \ell_i \ell_j \leq (1 - \frac{1}{n^2}) (\sum_i \ell_i)^2.
\]

We obtain

**Corollary F.** Let \( L \) be an \( n \)-component link \((n \geq 2)\), then

\[
\text{rop}(L) \geq \frac{1}{\sqrt{n-1}} \left(3 \sum_{i<j} |\text{Lk}(L_i, L_j)|\right)^{3/4}, \quad \text{rop}(L) \geq \frac{n \sqrt{32 \pi}}{\sqrt{n^2 - 1}} \left(\sum_{i<j} |\text{Lk}(L_i, L_j)|\right)^{1/2}.
\] (4.9)

In terms of growth of the pairwise linking numbers \(|\text{Lk}(L_i, L_j)|\), for a fixed \( n \), the above estimate performs better than the one in (1.5). One may also replace \( \sum_{i<j} |\text{Lk}(L_i, L_j)| \) with the isotopy invariant\(^8\)

\[
\text{PCr}(L) = \min_{D_L} \left(\sum_{i \neq j} \text{cr}_{i,j}(D_L)\right),
\] (4.10)

we call the **pairwise crossing number** of \( L \). This conclusion can be considered as an analog of the Buck and Simon estimate (1.11) for knots, and is stated in the following

**Corollary G.** Let \( L \) be an \( n \)-component link, and \( \text{PCr}(L) \) its pairwise crossing number, then

\[
\text{rop}(L) \geq \frac{1}{\sqrt{n-1}} \left(\frac{3}{2} \text{PCr}(L)\right)^{3/4}, \quad \text{rop}(L) \geq \frac{n \sqrt{16 \pi}}{\sqrt{n^2 - 1}} \left(\text{PCr}(L)\right)^{1/2}.
\] (4.11)

5. **Proof of Theorem A**

The following auxiliary lemma will be useful.

**Lemma H.** Given nonnegative numbers: \( a_1, \ldots, a_N \) we have for \( k \geq 2 \):

\[
\sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq N} a_{i_1} a_{i_2} \ldots a_{i_k} \leq \frac{1}{N^k} \binom{N}{k} \left(\sum_{i=1}^{N} a_i\right)^k.
\] (5.1)

*Proof.* It suffices to observe that for \( a_i \geq 0 \) the ratio \( \left(\sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq N} a_{i_1} a_{i_2} \ldots a_{i_k}\right) / \left(\sum_{i=1}^{N} a_i\right)^k \) achieves its maximum for \( a_1 = a_2 = \ldots = a_N \).

\(^8\)satisfying \( \text{PCr}(L) \leq \text{Cr}(L) \).
Recall from [1,8] that $\bar{\mu}_{n;1} := \bar{\mu}_{23...n+1;1}$, and

$$
\left[ \bar{\mu}_{n;1}(L) \right] := \begin{cases} 
\min \left( \bar{\mu}_{n;1}(L), d - \bar{\mu}_{n;1}(L) \right) & \text{for } d > 0, \\
\bar{\mu}_{n;1}(L) & \text{for } d = 0
\end{cases} \quad d = \Delta \mu(n;1). \quad (5.2)
$$

For convenience, recall the statement of Theorem A

**Theorem A.** Let $L$ be an $n$-component link of unit thickness, and $\bar{\mu}(L)$ one of its top Milnor linking numbers, then

$$
\ell(L) \geq \sqrt[n]{\left( \bar{\mu}(L) \right)^{\frac{2}{3}}}, \quad Cr(L) \geq \frac{1}{3} (n-1)^{n-1}/\sqrt[3]{\bar{\mu}(L)}. \quad (5.3)
$$

**Proof.** Let $G_L$ be a Gauss diagram of $L$ obtained from a regular link diagram $D_L$. Consider, any term $A$ of the arrow polynomial: $Z_{n;1}$ and index the arrows of $A$ by $(i_k, j_k)$, $k = 1, \ldots, n-1$ in such a way that $i_k$ is the arrowhead and $j_k$ is the arrowtail, we have the following obvious estimate:

$$
|\langle A, G_L \rangle| \leq \prod_{k=1}^{n-1} \text{ov}_{i_k,j_k}(D_L) \leq \prod_{k=1}^{n-1} \text{cr}_{i_k,j_k}(D_L). \quad (5.4)
$$

Let $N = \binom{n}{2}$, since every term (a tree diagram) of $Z_{n;1}$ is uniquely determined by its arrows indexed by string components, $\binom{n-1}{n-1}$ gives an upper bound for the number of terms in $Z_{n;1}$. Using Lemma [4] with $k = n-1$, $N$ as above and $a_k = \text{cr}_{i_k,j_k}(D_L)$, $k = 1, \ldots, N$, one obtains from (5.4)

$$
|\langle Z_{n;1}, G_L \rangle| \leq \frac{1}{N^{n-1} \left( \begin{array}{c} N \\ n-1 \end{array} \right)} \left( \sum_{i<j} \text{cr}_{i,j}(D_L) \right)^{n-1}. \quad (5.5)
$$

**Remark I.** The estimate (5.5) is valid for any arrow polynomial, in place of $Z_{n;1}$, which has arrows based on different components and no parallel arrows on a given component.

By (3.4), we can find $k \in \mathbb{Z}$ such that $\langle Z_{n;1}, G_L \rangle = \bar{\mu}_{n;1} + k d$. Since

$$
\left[ \bar{\mu}_{n;1}(D_L) \right] \leq |\bar{\mu}_{n;1}(D_L) + k d|, \quad \text{for all } k \in \mathbb{Z},
$$

replacing $D_L$ with a diagram obtained by projection of $L$ in a generic direction $v \in S^2$, we rewrite the above estimate as

$$
c_n^{-n+1} \left[ \bar{\mu}_{n;1}(D_L(v)) \right] \leq \sum_{i<j} \text{cr}_{i,j}(v), \quad c_n = \left( \frac{1}{N^{n-1} \left( \begin{array}{c} N \\ n-1 \end{array} \right)} \right)^{n-1}. \quad (5.6)
$$

Integrating over the sphere of directions and using invariance of $\left[ \bar{\mu}_{n;1} \right]$ yields

$$
4\pi c_n^{-n+1} \left[ \bar{\mu}_{n;1}(L) \right] \leq \sum_{i<j} \int_{S^2} \text{cr}_{i,j}(v) dv.
$$

By Lemma [5] we obtain

$$
c_n^{-n+1} \left[ \bar{\mu}_{n;1}(L) \right] \leq \sum_{i<j} \text{acr}_{i,j}(L) \leq \frac{3}{10} \sum_{i \neq j} \ell_i \ell_j. \quad (5.7)
$$

As in derivation of (4.8), applying Jensen Inequality yields

$$
\text{rop}(L) \geq \frac{10 \sqrt[n]{c_n}}{3(n-1)^{n-1}} \left[ \bar{\mu}_{n;1}(L) \right]. \quad (5.7)
$$

$^9$both $\bar{\mu}_{n;1}$ and $d$ are isotopy invariants.
Let us estimate the constant $c_n$, note that

$$\frac{N^{n-1}}{(n-1)!} = \frac{N^{n-1}}{(N-1)! \cdots (N-(n-1)+1)} \geq (n-1)!$$

From Stirling’s approximation: $(n-1)! \geq \sqrt{2\pi(n-1)^{(n-1)/2}} e^{-(n-1)}$ we obtain for $n \geq 2$:

$$c_n \geq ((n-1)!)^{\frac{1}{n-1}} \geq (\sqrt{2\pi(n-1)})^{\frac{1}{n-1}}(n-1)e^{-1} \geq \frac{n-1}{3}. \tag{5.8}$$

Simplifying (5.7) to

$$\text{rop}(L) \geq \sqrt[n-1]{|\bar{\mu}_{n;1}(L)|}, \tag{5.9}$$

we obtain the first inequality in Equation (5.3). Since $\sum_{i<j} c_{r_{i,j}}(D_L) \leq \text{Cr}(L)$, the second inequality of (5.3) is an immediate consequence of (5.6) and (5.8). Using the permutation identity (3.6) and the fact that $\text{rop}(\sigma(L)) = \text{rop}(L)$ for any $\sigma \in \Sigma(1, \ldots, n)$, we may replace $\bar{\mu}_{n;1}(L)$ with any other top $\mu$–invariant of $L$. \hfill \Box

In the case of almost trivial (Brunnian) links $d = 0$, and we may slightly improve the estimate in (5.5) of the above proof, by using cyclic symmetry of $\bar{\mu}$–invariants pointed out in [11]. We have in particular

$$n \bar{\mu}_{23 \ldots n;1}(L) = \sum_{\rho, \tilde{\rho} \text{ is cyclic}} \bar{\mu}_{\rho(2)\rho(3) \ldots \rho(n) ; \tilde{\rho}(1)}(L) = \sum_{\rho, \tilde{\rho} \text{ is cyclic}} \langle \rho(Z_{n;1}), G_L \rangle. \tag{5.10}$$

Since cyclic permutations applied to the terms of $Z_{n;1}$ produce distinct arrow diagrams by Remark 1, we obtain the following bound

$$n |\bar{\mu}_{n;1}(L)| \leq \sum_{\rho, \tilde{\rho} \text{ is cyclic}} |\langle \rho(Z_{n;1}), G_L \rangle| \leq \frac{1}{N^{n-1}} \left( \frac{N}{n-1} \right) \left( \sum_{i<j} c_{r_{i,j}}(D_L) \right)^{n-1}, \tag{5.11}$$

and disregarding the Stirling’s approximation step we have

$$\text{rop}(L) \geq 10^{\frac{\sqrt{n} \bar{c}_n}{3(n-1)}} n^{-\frac{1}{2}} |\bar{\mu}_{n;1}(L)|, \quad \bar{c}_n = \left( \frac{1}{nN^{n-1}} \left( \frac{N}{n-1} \right) \right)^{\frac{1}{n-1}}, \tag{5.12}$$

or using the second bound in (4.3)

$$\text{rop}(L) \geq 4^{3 \pi n} \bar{c}_n \left( \frac{n^2}{n^2 - 1} \right)^{n-\frac{1}{2}} |\bar{\mu}_{n;1}(L)|. \tag{5.13}$$

In particular, for $n = 3$ we have $N = 3$ and $\bar{c}_3 = 3$ and these estimates yield

$$\text{rop}(L) \geq \left( 5^{\sqrt{3}} \sqrt{|\bar{\mu}_{23;1}(L)|} \right)^{\frac{1}{3}}, \quad \text{rop}(L) \geq 6^{\sqrt{6\pi}} \sqrt{|\bar{\mu}_{23;1}(L)|}. \tag{5.14}$$

Since $6\sqrt{6\pi} \approx 26.049$, the second estimate is better for Borromean rings ($\mu_{23;1} = 1$) and improves the linking number bound of (1.5): $6\tau \approx 18.85$, but fails short of the genus bound (1.6): $12\tau \approx 37.7$. Numerical simulations (c.f. [7]) suggest that ropelength of Borromean rings $\approx 58.05$.

**Remark J.** This methodology can be easily extended to other families of finite type invariants of knots and links. For illustration, let us consider $c_3(L)$, whose arrow polynomial $C_3$ is given as [11].

---

1. there are $(n-2)!$ different top Milnor linking numbers [32].
2. since the trunk of a tree diagram is unique, c.f. [26], [24]
obtain Similarly, as in (5.6), replacing $D_L$ with $D_{L(v)}$ and integrating over the sphere of directions we obtain
\[
|c_3(L)|^{\frac{1}{3}} \leq \text{acr}_1,2(L) + \text{acr}(L_1).
\]
For a unit thickness link $L$, (1.12) and (4.3) yield $\text{acr}_1,2(L) + \text{acr}(L_1) \leq \frac{\ell_1^2}{16\pi} + \frac{\ell_1^2}{8\pi}$. Thus, for some constants: $A, B > 0$, we have
\[
\ell(L)^2 \geq A |c_3(L)|^{\frac{1}{2}}, \quad \ell(L)^{\frac{3}{2}} \geq B |c_3(L)|^{\frac{3}{2}}.
\]
In general, given a finite type $n$ invariant $V_n(L)$ and a unit thickness $m$–link $L$, we may expect constants $A_{m,n}, B_{m,n}$; such that
\[
\ell(L)^2 \geq A_{m,n} |V_n(L)|^{\frac{1}{2}}, \quad \ell(L)^{\frac{3}{2}} \geq B_{m,n} |V_n(L)|^{\frac{3}{2}}.
\]
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