p-Adic description of Higgs mechanism
II: General Theory

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Abstract

This paper is second one in the series devoted to the calculation of particle masses in the framework of p-adic conformal field theory limit of Topological GeometroDynamics. The concept of topological condensate generalizes the concept of 3-space in radical manner. The various hierarchically ordered levels of the condensate obey effective p-adic topology and fractal considerations motivate the hypothesis that physically interesting values of $p$ correspond to primes near prime powers of two, in particular Mersenne primes. This hypothesis relates succesfully the fundamental elementary particle mass scales to Planck mass scale and what remains is to predict particle mass ratios correctly. The fundamental description of Higgs mechanism is in terms of p-adic thermodynamics for the Virasoro generator $L^0$ (mass squared). The more phenomenological description is based on the breaking of super conformal symmetry. In thermal approach massivation follows from the small thermal mixing massless ground state with Planck mass excitations. A purely p-adic feature is the quantization of temperature at low temperature limit, which gives earlier length scale hypothesis as a prediction. The generalization of $M_+^4 \times CP_2$ spinor fields to Kac moody spinors allowing representation of Super Virasoro algebra and describing both bosons (N-S sector) and fermions (Ramond sector) is carried out and involves crucially the special features of $M_+^4 \times CP_2$ geometry. In TGD elementary particles (partons) correspond to boundary components of particle like 3-surface and elementary particle vacuum functionals defined in modular degrees of freedom of boundary component are generalized to p-adic context. The dominating contribution to elementary fermion and boson masses comes from boundary degrees of freedom and the general form of the contribution can be derived by a few line argument. The calculation of elementary fermion and boson masses is performed in the third paper of the series.
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1 Introduction

This is a second paper in the series devoted to the p-adic description of Higgs mechanism in TGD \cite{Pitkänen}, \cite{Pitkänen} (for p-adic numbers see for instance \cite{Brekke and Freund}). Concerning the general background reader is suggested to read the introduction of the first paper, where general formulation of p-adic conformal field theory limit was proposed and predictions were described generally.

The contents of this paper are following.

a) The physical picture behind p-adic description of Higgs mechanism is described. The basic elements of description are the concept of topological condensate, the hypothesis that favoured p-adic primes $p$ correspond to primes near prime powers of 2, Super Virasoro invariance, thermodynamical and conformal symmetry breaking descriptions of Higgs mechanism and the hypothesis that boundary components of particle like 3-surface are carriers of elementary particle quantum numbers. For mass calculations only the thermal description is needed.

b) Super Virasoro symmetry is one of the cornerstones of the approach and the construction of called Kac Moody spinors as generalization of $M_4^+ \times CP_2$ spinors is carried out. Kac Moody spinors give rise to Super Virasoro representations and Ramond/N-S type representations describe fermions/bosons.

c) Generalization of massive Dirac operator is constructed for fermions and bosons and purely p-adic elimination mechanism for tachyons is found.

d) Description of Higgs mechanism as breaking of conformal symmetry is considered at quantitative level. Some yet poorly understood problems (What determines primary condensation level, What happens in secondary condensation) are considered and the reader primary interested in mass calculations can skip over these sections.

e) p-Adic counterparts for the so called elementary particle vacuum functionals, which depend on moduli of boundary component of 3-surface are constructed. The construction involves identification of p-adic version of moduli space as a discrete space and p-adic generalization of theta functions. The contribution of boundary component to the mass squared of particle is calculated.

The calculation of masses of leptons, quarks and gauge bosons is carried out in the third paper of the series. It turns out that the predictions for
lepton and gauge bosons masses agree surprisingly well with known experimental masses. Errors are below one per cent except for $Z^0$ boson for which mass is 10 per cent too large. The reason is too large value $\sin^2(\theta_W) = 3/8$ for Weinberg angle: in the third paper it is shown that inclusion of Coulombic corrections and topological mixing effects of leptons leads to a correct prediction for gauge boson masses. One can safely conclude that the results of the calculation verify the essential correctness of both TGD and its p-adic conformal field theory limit at quantitative level.

2 p-Adic description of Higgs mechanism

The analysis of the weak points of the previous attempts to describe Higgs mechanism p-adically [Pitkänen] suggests what the basic assumptions of the new 'quark level' description might be.

2.1 Super Virasoro invariance

Super Virasoro invariance is the corner stone of the whole approach and follows from the criticality of TGD:ish Universe at quantum level (the concept is described in the books [Pitkänen, Pitkänen]). In order to realize Super Virasoro invariance one must solve several purely mathematical problems. One should construct the representations of Super Virasoro in $M^4$, $u(1)$ (Kähler charge), $so(4)$ and color degrees of freedom. Since the spinors of $H = M^4 \times CP_2$ play fundamental role in Quantum TGD it seems sensible to try to generalized the concept of H-spinor to some kind of Kac Moody spinor, which allows the description of both bosons and fermions as N-S type and Ramond representations respectively. There are several problems to be solved.

a) The generalization of $CP_2$ spinors is obviously based on the use of $so(4)$ Super Virasoro representations. The earlier simple description is however not correct but one must use tensor product of two $so(4)$ Super Virasoro representations of Ramond type in order to obtain generalization of $CP_2$ spinors. The tensor product of two $N - S$ type representation describes bosons.

b) The treatment of $M^4$ degrees of freedom is very difficult due to the anomalies: in string models these anomalies lead to critical dimensions $D = 10$ and $D = 26$. In TGD context the solution of the problem reduces to the observa-
tion that $so(4)$ Super Virasoro representations can be used in the construction of $M^4$ Kac Moody spinors, too. The mathematical structures in $M^4$ and $\mathbb{CP}^2$ degrees of freedom are practically identical. It is essential that the product of two representations is used. The trick is possible due to the fact that $so(3,1)$ is complexification of $so(4)$ so that dimensions are again important: now $H = M^4_+ \times \mathbb{CP}^2$ is the unique alternative. At quark level all these Super Virasoro representations have $c = h = 0$.

c) The inclusion of $u(1)$ degrees of freedom is simple: one just adds $u(1)$ type Super Virasoro representation as a tensor factor to the generalized $H$-spinor. For this representation one has necessarily $c = 3/2$. The value of the vacuum weight $h$ gets contribution $h_K = Q^2_K/2$ from the Kähler charge of the particle.

d) The problem of associating Super Virasoro representation to $su(3)$ degrees of freedom is highly nontrivial. Miraculously it turns out that $su(3)$ is the only Lie-group allowing $c = h = 0$ (color invariance is not broken), which allows the existence of this kind of representation. Essential role is played by the symmetric structure constants $d_{ABC}$, which allow the definition of anticommutator structure in triplet representation of $su(3)$.

2.2 Thermal description of Higgs mechanism

The basic idea is that massless particles become massive via thermal mixing of the massless ground state with excitations having Planck mass. The excitations in question are associated with boundary components identified as partons and are labeled by electroweak and color quantum numbers. Mass squared contains also interior contribution but this need not allow thermal description. For instance, states of higher spin on hadronic Regge trajectories correspond to to interior excitations so that thermal description predicts only the ground state masses of hadrons. Thermal description of the mixing applies in principle for all values of prime $p$. Thermalization takes place in Virasoro degrees of freedom $L^0$ being analogous to energy. It turns out that thermalization cannot be significant effect for spin, electroweak and color quantum numbers. Virasoro and isospin/spin degrees of freedom separate nicely for coset representation used to construct Kac-Moody spinors. There are obvious constraints on quantum numbers of, say, hadron. One can require that the overall ground state spin of the hadron is fixed, that hadron is color singlet, etc.
The quantization of temperature \((T = 1/k)\) makes scenario predictive and for large values of \(p\) predicts correct mass scale \(O(p^k)\) \((k = 1\) seems to be the most interesting value of \(k)\). This prediction in turn implies that \(m\) differs from \(m = 2\) by order \(O(p^{k+1})\) contribution. For Ramond (N-S) representations \(P, Q, Q_K\) differ by order \(O(p^k)\) \((p^{k+1}\) contribution from their values for massless representations. For \(m = 2\) representations with \(h = 0\) mass squared is in extremely good approximation given by the contribution of \(n = 1\) excited states to thermal average and is proportional to some degeneracy factor multiplying \(p^{n/T}\) factor.

\[
\langle m^2 \rangle \simeq k \sum_{n=1,2} A(n) p^{n/T} \\
1/T = 1, 2, ..., \\
A(1) = \frac{D(1)}{D(0)} \\
A(2) = \frac{(2D(2) - D^2(1))}{D(0)}
\]

Here \(D(i)\) denotes the degeneracy of \(M^2 = i\) state. \(k\) is proportionality constant, which turns out to have value \(k = \frac{3}{2}\). For \(T = 1\) mass squared scale is \(O(1/p)\) and for \(T = 1/2\) mass squared scale is \(O(1/p^2)\). It turns out that \(T = 1\) is the value of temperature for fermions and \(T = 1/2\) for gauge bosons. The powers \(p^n, n > 2\) give completely negligible contribution to mass squared so that computational inaccuracies are under very tight control.

There is very delicate and fundamental p-adic effect involved. If the coefficient \(k A(1) = kD(1)/D(0)\) of order \(O(p)\) contribution is not integer the mass of the state is practically Planck mass. Therefore only those states for which the condition

\[
\frac{kD(1)}{D(0)} \text{ integer}
\]

holds true are light. The degeneracies of massless states turn out to be larger than one due to the fact that ground state is p-adic tachyon with \(M^2(ground)\) being negative integer or half odd integer. The same mechanism implies also
that second order contribution to mass squared cannot be neglected unless $kA(2)$ is integer. This contribution is present in massive N-S and Ramond representations assumed to provide phenomenological description of Higgs mechanism. Clearly the degeneracy of massless states is analogous to vacuum degeneracy related to the direction of Higgs vacuum expectation value. Of course, Higgs field before symmetry breaking is also tachyon.

A further delicate p-adic effect is the fact that p-adic tachyons $M^2 < 0$ do in fact correspond to real p-adic masses due to the delicacies of p-adic square root under certain conditions for $p$: Mersenne primes belong to those p:s. What happens is that would be tachyons are actually Planck mass particles.

At which instances thermal description makes sense? Certainly it looks natural to assume that the mass of the particle at the primary condensation level is determined thermally. Since in TGD elementary particle quantum numbers reside on boundary components, boundary cm degrees of freedom, which are carriers of standard model quantum numbers, should be thermalized. A suitable generalization of the thermal description to the conformally invariant degrees of freedom of the boundary component (characterized by Teichmüller parameters) is also expected to make sense: it turns out that modular degrees of freedom give contribution to the vacuum weight of the Super Virasoro representation associated with boundary cm degrees of freedom. As already noticed, interior contribution to mass squared should be identifiable as counterpart for the known contributions such as Coulombic and color Coulombic binding energy, spin-spin interaction for quarks, etc... to particle mass and need not have any thermal description.

Thermal description for the secondary condensation of massive particles (Super Virasoro representations with vacuum weight $h > 0$) is not useful. For purely number theoretic reasons one must assume $T = \frac{1}{8m(m+2)N}$, where $N$ and $m$ are integers and $m = 2 + O(p)$ characterizes the Super Virasoro representation. The extreme smallness of the temperature implies that the contribution of the excited states to mass squared is proportional to $p^{m(m+2)N}$ and extremely small for large values of $p$. This means that thermal effects cannot explain the condensation energy assumed to be associated with secondary condensation in Pitkänen. Probably condensation energy corresponds to the change in Coulomb self energy, etc. associated with the interior of the particle like 3-surface. Neither does the use of thermal descrip-
tion resolve the problem of what happens to the values of p-adic quantum numbers $P, Q; m, Q_K$ associated with massive Super Virasoro representation in the condensation of level $p_1$ to level $p_2 > p_1$.

2.3 Higgs mechanism as breaking of Super Conformal Invariance

The description of Higgs mechanism as breaking of conformal symmetry parametrizes the predictions of fundamental thermal description at primary condensation level. The new element is the inclusion of electroweak symmetry breaking by decomposing the representations to the representations of $u(1)_V \times u(1)$. Equivalent description is in terms of central charge $c$ and vacuum weight $h$, which are linear combinations of $I$ and $I_3$. Correct order of magnitude for mass scales result if the deviations for $so(4)$ parameters are of order $O(p)$ and satisfy certain additional constraints. Also the order $O(p^2)$ contribution turns out to be important. If exact conservation of electromagnetic charge is assumed $u(1)$ charge is not free variable.

Thermal description, when applied at the primary condensate level should predict elementary particle and hadron ground state masses and it should be possible to deduce the values of the parameters $P, Q, m$ associated with the corresponding symmetry broken Super Virasoro representations. The basic formula for mass squared

$$\langle M^2(h = 0) \rangle_{\text{thermal}} = M^2(h(P, Q, m) \neq 0)$$

indeed gives a constraint on the parameters $P, Q, m$ of the symmetry broken $h \neq 0$ representation. The interactions associated with the interior of the particle like 3-surface (Coulomb self interaction, etc.) give also contributions to the mass squared. The simplest possibility is that interior contributions affect only the vacuum weight $h$ and therefore the parameters $P, Q, m$ of the symmetry broken Super Virasoro representation associated with particle. For hadrons interior contributions are large and responsible for the masses for the higher spin states on hadronic Regge trajectories and parameters $P, Q, m$ of the symmetry broken Super Virasoro representation associated with hadron (tensor product of quark representations) should contain additional contribution, which depends on hadronic spin and is in lowest order linear in hadronic
spin so that standard mass formula for Regge trajectories is obtained. Contrary to the original beliefs [Pitkänen], hadronic string tension is not predictable from the thermal approach but necessitates a model for the interior degrees of freedom (perhaps TGD:eish counterpart [Pitkänen] of the old fashioned string model).

2.4 Electroweak mass splitting in boundary degrees of freedom

The description of electroweak splitting in bosonic case requires no special assumptions. In fermionic case the mass squared operator must contain a part with depends on the isospin of the fermion. The kind of term is indeed allowed since Super Virasoro representation can always be decomposed to Super Virasoro representations associated with the vectorial $u(1)$ subgroup of $SO(4)$ so that the addition of isospin dependent term just shifts the vacuum weights $h$ of different Virasoro representations in isospin dependent manner. The vacuum weight can be expressed in the form

$$h = h_0 + dI_3$$

(4)

d must be integer: otherwise Planck mass results. $d = 1$ turns out to be the correct value of $d$ and that it turns out that vacuum weights are are given by

$$h(\nu_L) = h(U) = -\frac{5}{2}$$

$$h(L) = h(D) = -\frac{3}{2}$$

(5)

Since $M^2 = n$ states are created by operators of weight $\Delta = n + h$ the thermal averages are different for charged lepton and neutrino and thermal mass splitting results. For bosons isospin dependent term in mass squared operator is excluded by CP invariance.

2.5 Higgs mechanism without Higgs

In the standard description of Higgs mechanism massive gauge bosons receive their longitudinal polarization from Higgs field. In the description of Higgs
mechanism as a breaking of super conformal symmetry there is no need for Higgs field. Kac Moody spinors contain as tensor product factor discrete series representations of Super Virasoro in $so(4)$ degrees of freedom. The corresponding vacuum weight $h(P, Q, m = 2 + \Delta m)$ differs from its value at massless limit $\Delta m \to 0$ so that the norm of the longitudinal polarization is proportional to $p^2 \propto \Delta m$ and nonvanishing.

In the thermal description of Higgs mechanism using $h = 0$ representations one can define the norm of the massless longitudinal polarization states as the limiting value of the norm for $\Delta m \neq 0$ representations (this limit makes sense only p-adically!). One cannot exclude the possibility that in actual physical situation secondary topological condensation implies a small deviation of $m$ from $m = 2$ so that limiting procedure is only a convenient mathematical idealization of the actual situation. The ordinary description of Higgs mechanism indeed suggests that the $h = 0$ representation is unstable against generation of small value $\Delta m$.

2.6 Universal string tension

String tension $M_0^2$ does not depend on $p$ and is apart from some numerical constant equal to $1/G$ as in string models. String tension itself is prediction of Quantum TGD and involves the theory in Planck length scale. It turns out that $T$ is of same order as the string tension associated with cosmic strings in GUTs. p-Adic thermodynamics at low temperature limit predicts the mass scale at condensate level $p$ to be given by $M^2(p) = M_0^2/p$. The description simplifies drastically the description of elementary particles. There are no problems with light excitations (as in previous scenario) since they are totally absent. Primary condensation levels can be assumed to correspond primes, which define hadronic and leptonic mass scales that is Mersenne primes and primes near prime powers of 2.

As already noticed the ordinary hadronic string tension emerges in the conformal symmetry breaking description naturally via the dependence of the parameters $P, Q, m$ on the spin of the hadron. Since interactions related with the interior of the hadron are involved thermodynamic description cannot predict its value.
2.7 Boundaries as carriers of elementary particle quantum numbers

In TGD fermion families correspond to different boundary topologies characterized by the genus \( g = 0, 1, 2, .. \) of the boundary component. The natural expectation is that boundaries give the dominating contribution to the thermal expectation of the mass squared operator. There are several questions to be answered? Does the entire contribution to mass come from boundaries? Can one separate boundary contributions to cm contribution and contribution associated with conformally invariant degrees of freedom (modular degrees of freedom characterized by Teichmuller parameters)? Are cm and modular contributions independent and additive? Can one estimate modular contribution using thermodynamic approach or should one estimate it as quantum mechanical expectation value? Hard work with various alternatives has taught that answer is very simple. Mass squared for elementary particles is in good approximation sum of two contributions

\[
M^2 = M^2(cm) + M^2(mod)
\]

The first contribution correspond to either cm of boundary component and depends on 'standard' quantum numbers. The second contribution is the contribution of modular degrees of freedom and is same for all fermions and depends only on the genus of boundary component.

The most pleasant surprise of calculations was that \( M^2(mod) \) can be deduced apart from overall integer valued proportionality constant from the general form of elementary particle vacuum functionals constructed in [Pitkanen]. Together with the knowledge of cm contribution and Mersenne prime hypothesis one can predict lepton and gauge bosons masses with accuracy better than one per cent.

The actual calculation of \( M^2(mod) \) necessitates the introduction of p-adic generalization for the space of moduli (Teichmuller parameters) as well as construction of modular invariant partition functions plus exact definition of the mass squared operator. It turns out that surprisingly simple p-adication of moduli space as discrete space exists and that that elementary particle vacuum functionals as such provide the solution to the problem! In particular, the earlier arguments explaining the absence of \( g > 2 \) generations
remain valid as such. Amusingly enough, at this stage it is not clear whether one should regard modular contribution as quantum mechanical or thermal expectation value of the mass squared operator.

The harder part of the calculation is the evaluation of the cm contribution $M^2(cm)$. Fortunately, this contribution is same for all charged leptons/neutrinos. The contribution of quarks and leptons are also almost identical. It turns out that there are many delicacies related to the concept of Kac-Moody spinor: in particular, the definition of the inner product requires special care. The degeneracy of massless states turns out to be generic phenomenon very analogous to the vacuum degeneracy of Higgs field and light fermions are miracles in the sense that the ratio $D(1)/D(0)$ is integer for them.

What is the size of the boundary components associated with elementary particles? The belief has been that boundary components have size of order Planck length. The fact that elementary particle vacuum functionals depend on the conformal equivalence class of the particle only suggests that there is no preferred size. The success of $p$-adic mass formula in turn suggests that boundary components in fact correspond to the outer boundary of the small approximately flat piece of Minkowski space associated with the elementary particle and that the average size of surface is of order $p$-adic length scale and hierarchy of boundary size scales results. Hadron physics considerations gives additional support to this belief.

The mixing of boundary topologies is also possible. It turns out that Cabibbo-Kobayashi-Maskawa mixing can be understood essentially as difference of mixings of $g = 0, 1, 2$ topologies for $U$ and $D$ type quarks.

2.8 Open problems

There poorly understood aspects of TGD:eish description of Higgs mechanism can be condensed on few basic questions. What determines the primary condensation level? Why primes near prime powers of two are favoured? What happens in secondary condensation?
2.8.1 What happens to symmetry breaking parameters in secondary topological condensation?

The problem what happens in the secondary condensation of condensate level $p_1$ to condensate level $p_2$ belongs outside the context of p-adic conformal field theory and one can make only guesses at this stage. It makes no sense to relate to each other directly the values of various p-adic quantum numbers: only their real counterparts are comparable. What looks physically plausible is that mass decrease is necessary for condensation to occur. Concerning the details of condensation one can make only guesses at this stage. An attractive possibility is that some kind of variational principle is involved.

a) An untested hypothesis of [Pitkanen] is that real mass decreases in condensation and the decrease is as small as possible. This principle is essentially equivalent with the p-adic version of the statement that entropy gradient $dS/dt$ is as small as possible in time evolution since mass squared expectation value is for discrete p-adic temperature essentially equal to entropy. This requirement in principle determines the renormalization of the quantities $P, Q, m$ characterizing the conformal breaking. The problem is that without any constraints on the change of parameters $P, Q, m$ the mass change can be made practically zero. Of course, one can consider the addition of various physical constraints.

b) One possibility is that the real counterparts of $\Delta P, \Delta Q$ and $\Delta m$ are 'small' but the problem is how small. An other extreme would be that the real counterparts $(P_R, Q_R, m_R)$ of $P, Q, m$ remain invariant in condensation: this could be interpreted as conservation law in some sense and might well minimize the mass change. This implies that mass change results completely from the fact that for an algebraic expression $f(P, Q, m)$

$$f_R(P, Q, m) \neq f(P_R, Q_R, m_R) \quad (7)$$

This alternative is formally appealing since mass change is guaranteed to be small and the mass change is analogous to quantum corrections resulting from noncommutativity of quantum counterparts of classical observables. Furthermore, the mass change might be also positive: in this case the condensation obviously is not stable. Could it be that favoured condensation levels are just those for which the condensation energy is negative?
There is a counterargument against both scenarios. In [Pitkänen] the concept of condensation energy was introduced. The identification of the condensation energy as the change induced by the condensation to the various interior contributions (Coulomb self energy, etc.) to the vacuum weight $h(P, Q, m)$ looks natural. This suggests that condensation affects the values of $P_R, Q_R, m_R$ on the new condensation level so that scenario b) could not be correct as such. In the last paper of the series it will be found that the real counterpart of the p-adic gauge coupling $g^2$ can be defined via canonical identification $g^2_R = (g^2_p)_R$ assuming $g^2$ is rational number. The value of the real counterpart is very sensitive to the value of the prime $p$. If the effective values of gauge couplings are indeed sensitive to the value $p$ then condensation can affect the mass of the particle in decisive manner and lead to a dynamical selection of condensation levels and explain why Mersenne primes and primes near prime powers of two are favoured physically.

2.8.2 What determines the primary condensation level?

The basic assumption is that there exists some critical prime $p_{cr}$ so that for $p < p_{cr}$ it does not make sense to speak of the condensation of massless particle but secondary condensation of massive particle is in question and the change of particle mass in condensation can be regarded as renormalization of mass. What principle determines the value of $p_{cr}$? What stabilizes the primary condensation level $p_{cr}$? Why some condensation levels seem to be in special position physically (the considerations of [Pitkänen] suggest strongly that primes near prime powers of 2 are such primes)? One can imagine several alternative answers to these questions.

p-Adic thermodynamics might provide the principle selecting the favoured primary condensation levels. The maximization for the real counterpart of p-adic entropy is essentially equivalent with the minimization of mass squared as a function of $p$ and since second order term in mass squared depends very sensitively on $p$, local minima of mass squared are expected to occur and principle leads to favoured primes. An open question is whether the principle is strong enough to fix primary condensation level sufficiently uniquely. If one requires absolute mass minimum then there is no upper bound for $p_{cr}$ since mass behaves as $M^2 \propto \frac{1}{p}$. Therefore the correct question to ask is what stabilizes certain primary condensation level so that primary condensation
levels with \( p < p_{cr} \) are not possible. Of course, one cannot exclude the possibility of several values of \( p_{cr} \) and in the fifth paper of the series the possibility of scaled up copy of ordinary hadron physics \((M_{107})\) at \( M_{89} \) condensate level is considered.

A possible stabilization mechanism is following.

a) Massless elementary particles can in principle condense on several primary condensate levels \( p \geq p_{cr} \). The condensation on levels with \( p > p_{cr} \) is not however stable since the mass is always larger than at \( p = p_{cr} \) level. What stabilizes \( p_{cr} \) is that real mass squared is smaller for the primary condensation on \( p_{cr} \) followed by secondary condensation on some level \( p_0 < p_{cr} \) near \( p_{cr} \) than for direct primary condensation on \( p_0 \). Same principle applies to further secondary condensations, too and might explain why certain primes are in preferred position physically. The mass calculations for hadrons give support for this mechanism: both the primary condensation level \( p_{cr} \) for \( u, d, s \) quarks and condensation level of hadron (secondary condensation level for quarks) correspond to primes near \( 2^{107} \). Note that this principle is in accordance with local entropy maximization principle.

b) A more quantitative description is obtained by adopting either the minimization of \( \Delta M^2 \) in secondary condensation or the hypothesis that \( P_R, Q_R, m_R \) remain invariant in condensation. Later the latter possibility will be studied in more detail.

Consider next the question why primes near prime powers of two are favoured physically.

a) Period doubling analogy suggests one partial explanation. The underlying \( 2\)-adic fractality suggests a second partial explanation: scales \( 2^n L_0 \) are fundamental and are essentially identical with scales \( p^n L_0 \) (synchronization) if \( p \) is near power of 2.

b) \( p\)-Adic entropy maximization principle: the real counterpart of mass squared as a function of \( p \) has local minima for \( p \) near prime power of 2. This conjecture is testable via the study of the second order term in mass squared.

c) The following argument suggests a partial answer to the question why primes near powers of 2, especially Mersenne primes are favoured. The inverse temperature associated with condensed particle can be any non-negative integer \( n \). For large values of \( n \) (small temperature and small mixing with Planck mass excitations) also particles of small \( p \) can condense with small mass. This suggests the possibility that the value of \( n \) depends on
so that the value of the mass doesn’t depend very much on \( p \). For primes \( p \) near powers of 2: \( p \simeq 2^k \) it is indeed possible to get essentially identical masses by suitable choice of \( n \): \( p^k \simeq 2^n k \). For instance electron mass scale would correspond to 2-adic temperature \( T = 1/127 \). The difference between 2-adic masses and p-adic masses is smallest when primes \( p \) is as near as possible to power of 2 and this condition selects Mersenne primes as especially favourable candidates for \( p \).

d) The real counterparts of the various coupling constants \( g_{i}^{2} \) (such as \( e_{2}^{2}, g_{ew}^{2}, \ldots \)) turn out to correspond to the real counterparts of \( g_{i}^{2} p \) in canonical correspondence. If \( g_{i}^{2} \) is finite superposition of powers of 2 and \( p \) is Mersenne prime, the real counterpart is what it should be: for other primes this is true only under rather restrictive conditions. This means that the effective values of gauge couplings and other parameters depend sensitively on condensate level and this probably gives dynamical reason for the special role of Mersenne primes. It turns out also, that Mersenne primes very probably correspond to fixed points of color coupling constant evolution: there is one QCD for each Mersenne prime as was suggested in [Pitkanen]. The critical value of \( p \) is analogous to critical temperature since the real counterpart of p-adic temperature is proportional to \( 1/\ln(p) \) so that primary condensation is phase transition like phenomenon.

3 Super Virasoro symmetry

The existence of the p-adic super Virasoro representations poses a stringent consistency test for TGD.

a) In \( M^4 \) degrees of freedom the dynamical degrees of freedom involve quantized \( M^4 \) coordinate and Kac Moody version for spin degrees of freedom. The fact that spacetime dimension \( D \) is four implies that \( M^4 \) coordinates are in fact redundant dynamical variables for space times surfaces representable as graphs for a map \( M^4 \rightarrow CP_3 \). This in turn suggests the vanishing of the conformal anomaly \( c \) now proportional to \( 4 - D \), \( D \) the dimension of spacetime surface. The breaking of Lorentz invariance in string models for noncritical imbedding space dimension results from the nonvanishing of conformal anomaly and is expected to be absent for the simple reason that \( M^4 \) and space time surface have same dimensions. The elimination of vibrational coordinate degrees of freedom leaves only the Kac Moody version for spin
degrees of freedom and the remaining task is to construct generalization of $M^4$-spinors.
b) In $CP^2$ degrees of freedom one has both coordinate and spinorial degrees of freedom. Spinorial degrees of freedom correspond to $CP^2$ part of generalized H-spinor. Coordinate degrees of freedom correspond to Super Virasoro representation for color group.
c) The existence of spinor structure in $CP^2$ forces a modification of spinor structure via gauge coupling to Kähler potential. The counterpart of $u(1)$ coupling is $u(1)$ Super Virasoro representation appearing as tensor factor of Kac Moody spinor.

The basic guideline in the search for realization of Super Virasoro algebra is clearly the need to generalize the concept spinor field of $H = M^4 \times CP^2$ to 'Kac Moody' spinor allowing realization of Super conformal symmetry for $so(3,1)$, $so(4)$, $u(1)$ and $su(3)$. Also the description of bosons should be possible using Kac Moody spinors: fermions and bosons would correspond to Ramond and N-S type representations.

Ordinary spinors are constructed as tensor products of 2-dimensional spinors and same construction should apply now. The analog of two-dimensional spinor is clearly single $(c,h) = (0,0)$ so(4) Kac Moody representation [Goddard et al., Itzykson et al.], which decomposes into sum over tensor products of unitary $su(2)_V$ ($V$ refers to vectorial) Kac Moody and Super Virasoro representations [Goddard et al., Itzykson et al.] and for which vacua form isospin doublet in Ramond case and isospin triplet and singlet in N-S case. Four-fold tensor product of so(4) representation should give rise to both $M^4 \times CP^2$ Kac Moody spinors. Tensor product can contain only Ramond or N-S type representations and this implies that only spin $(J,I) = (1/0,1/0)$ vacuum states are possible for N-S sector and $(J,I) = (1/2,1/2)$ vacuum states in Ramond sector. In particular spin 0 isospin doublet (Higgs doublet) is excluded.

The description of $u(1)$ and $su(3)$ quantum numbers is in principle straightforward. One must add appropriate $u(1)$ and $su(3)$ Super Virasoro representations to the Kac Moody version of H-spinor as a tensor factor. $u(1)$ representations should be of type Ramond/N-S for fermionic/bosonic Kac Moody spinors. A good guess is that color Super Virasoro is of Ramond type for color triplet vacuum state (quarks) and N-S type for color singlet vacuum (leptons). This raises a little technical problem. Leptons corresponds
to tensor products of Ramond type and N-S type representations: how can one realize Ramond type Super Virasoro for this kind of structure. It turns out that problem can be solved.

The existence of required Super Virasoro representations is not at all obvious.

a) The transition from $CP_2$ Kac Moody spinors to $M^4$ Kac Moody spinors need not be trivial although the tangent space groups $so(3, 1)$ and $so(4)$ are related by complexification. The dimensions of $M^4$ and $CP_2$ are in fact crucial for the concept of Kac Moody spinor to make sense.

b) It could happen that color group does not allow Super Virasoro representation of required type: color would be totally invisible in the conformal field theory limit of TGD. In the previous work only Super Virasoro representations describing color confined states were considered (fit of particle masses \cite{Pitkanenb}) or the color degree of freedom was not treated explicitly (2-adic description of Higgs mechanism \cite{Pitkanenb}). The fact that quark level description is empirically found to work so well suggests that color Super Virasoro with physically acceptable properties indeed exists. To my personal knowledge the existence of this kind of representation has not been documented in literature and the motivation for the construction to be described came from TGD. It came as a pleasant surprise to find that the very special properties of $su(3)$ (and therefore of $CP_2$) allow the construction of this representation.

In the following the construction of the required Super Virasoro representations is described.

a) The most relevant properties of $so(4)$ Super Virasoro representations are reviewed.

b) Realization of Super Virasoro invariance for $M^4 \times CP_2$ Kac Moody spinors is considered.

c) $u(1)$ Super Virasoro is described.

d) Color Super Virasoro representation is constructed: crucial role is played by the completely symmetric structure constants $d_{ABC}$ existing for color group only.

e) The identification of elementary particles as generalized spinors is summarized with comparison with more standard theories.
3.1 Relevant properties of $so(4)$ Super Virasoro representation

$SO(4)$ is quite unique among other Lie groups in that it admits all unitary discrete series representations of Super Virasoro algebra \cite{Goddard et al, Pitkanen}. $su(2)_V$ Super Virasoro representations are obtained using $su(2)_L \times su(2)_R/su(2)_V$ (V refers to vectorial subgroup) coset construction \cite{Goddard et al} and can be decomposed into a sum of tensor products of Super Virasoro and $su(2)_V$ Kac Moody representations characterized by the isospin (or highest weight $q$) of the vacuum representation and by central charge $k_V$. This decomposition is of central importance since it implies separation of Super Virasoro and $su(2)_V$ Kac Moody degrees of freedom and plays central role in the construction of physical states and mass calculations.

$su(2)_R$ Kac Moody representation has highest weight $p$ and central charge $k_R$. $su(2)_L$ Kac Moody representation has central charge $k_L = 2$ and decomposes corresponds to isospin $1/2$ (Ramond) or direct sum of isospin $I = 0$ and $I = 1$ (N-S). The basic formulas for the conformal central charge $c$ and the allowed vacuum weights $h$ of general discrete series Super Virasoro representation appearing in tensor product composition read as:

\[
\begin{align*}
  c &= \frac{3}{2} \left(1 - \frac{8}{m(m+2)}\right) \\
  h &= \frac{((m+2)p - mq)^2 - 4}{8m(m+2)} + \frac{\epsilon}{16} \\
  m &= k_R + 2 \\
  \epsilon &= 1(0) \text{ for Ramond(N-S)}
\end{align*}
\]

(8)

Here one has $p = 1, \ldots, m - 1$ and $q = 1, \ldots, m + 1$. $p - q$ is odd for Ramond representation and even for N-S representation. $p$ and $q$ are the highest weights for $su(2)_R$ and $su(2)_V$ Kac Moody representations.

As far as thermal description of Higgs mechanism are considered the most interesting representations are $(c, h) = (0, 0)$ representations with unbroken conformal symmetry. These should correspond to massless particles. There are three representations with $(c, h) = (0, 0)$. They have $m = 2$ and therefore
vanishing central charge $k_R = 0$ for the second $su(2)$ factor of $SO(4)$. In Ramond sector the representation decomposes to tensor product of Super Virasoro and $su(2)_V$ Kac Moody representation having $I = 1/2 ((p, q) = (1, 2))$. In Neveu-Schwartz sector the representation decomposes into a tensor product of Super Virasoro with sum of $I = 0 ((p, q) = (1, 1))$ and $I = 1 ((p, q) = (1, 3))$ representations.

The identification of Kac Moody spinors as 4-fold tensor product of $(c, h) = (0, 0) so(4)$ Super Virasoro representations allows only spin 1/2 isospin 1/2 elementary fermions and spin 0/1 isospin 0/1 elementary bosons in accordance with experimental facts. Higgs doublet would require the tensor product of N-S ($M^4$ and Ramond $CP_2$ type representations and this probably leads to difficulties with the realization of Super Virasoro symmetry (one should somehow transform the N-S type generators to Ramond type generators in $CP_2$ type tensor factors).

3.2 Super Virasoro invariance in $M^4_+ \times CP_2$ spinorial degrees of freedom

There are strong physical constraints on the realization of Super Virasoro invariance in $M^4$ degrees of freedom.

a) $(c, h) = (0, 0)$ representation is required by Lorentz invariance. In the standard realization of string models in terms of bosonic and fermionic fields one encounters the well known problems. In present case however the dimension of spacetime surface is $D = 4$ and identical to the dimension of $M^4$ and since p-adic field theory limit is expected to make sense at the flat spacetime limit only, $M^4$ coordinate degrees of freedom can be eliminated almost totally: only cm degrees of freedom remain and the generator $L_0$ gets only the $p \cdot p$ contribution from $M^4$ coordinates. Fermionic degrees of freedom are not lost however and are necessary for describing the spin degrees of freedom.

b) The realization should provide some kind of local version of Lorentz group and respect Lorentz invariance. String theories have demonstrated how difficult this requirement is to satisfy.

c) The representations should allow generalization of $H$-spinors to Kac Moody context and also the generalization of Dirac equation. The formal analogies between Ramond and N-S representations suggest that the spinor concept and Dirac equation should also generalize to the N-S case.
The need to obtain generalization of H-spinors allows clue to the construction. The crucial observation is that single $so(4)$ coset representation for fermions gives just one $J = 1/2$ doublet. What is needed is two doublets. The only manner to achieve this is to use tensor product of two $so(4)$ coset representations. Same applies in $so(4)$ degrees of freedom so that 4-fold tensor product of $so(4)$ coset representation is needed in order to generalize the concept of H-spinors. Lorentz generators cannot be constructed in terms of single $so(4)$ representation but it turns out that the tensor product of two representations allows to construct the local version of Lorentz algebra in terms of commutators of the gamma matrix like operators $F_{i}^{am}$, $i = 1, 2$ and the construction respects Super conformal invariance.

The result means that there is amazingly close mathematical relationship between $M^4$ and $CP_2$ degrees of freedom. The result is unique to the 4-dimensional Minkowski space so that construction provides one further argument in favour of 8-dimensional imbedding space and its decomposition into $M^4$ and $CP_2$ factors.

3.2.1 Consistency of Lorentz and Super conformal symmetries

The previous construction does not make manifest the consistency of Lorentz and super conformal invariance. The consistency can be however seen by detailed construction of Lorentz algebra in terms $so(4)$ coset representation. Consider first Ramond sector. The counterparts of Dirac gamma matrices can be identified in terms of super generators $F_{i}^{a0}$, $i = 1, 2$ of $so(3, 1)$ sector

$$
\begin{align*}
\gamma^0 &= iF_{1}^{1,0} \\
\gamma^1 &= iF_{2}^{1,0} \\
\gamma^2 &= F_{2}^{2,0} \\
\gamma^3 &= F_{1}^{2,0}
\end{align*}
$$

(9)

The corresponding identification for the counterparts of $CP_2$ gamma matrices in sectors $i = 3, 4$ is obviously possible and the only difference is the absence of imaginary unit. The construction generalizes to bosonic case with the difference that now gamma matrices are $\Delta = 1/2$ operators $\gamma_{1/2}^k$ constructible from $F_{i}^{a_{1/2}}$ and their conjugates $\gamma_{-1/2}^k$ can be defined in terms of $F_{i}^{0,-1/2}$.

Sigma matrix representation for Lorentz algebra for Ramond representation can be defined as commutators of gamma matrices in standard manner.
In bosonic case one must define sigma matrices as commutators \( \gamma^k_{1/2} \) and \( \gamma^l_{-1/2} \) in fact localization is in both cases possible matrices \( \Sigma_{kl} = \sum_m \text{Comm}(\gamma^k_{n-m}, \gamma^l_m) \).

The most essential element of the definition is the use of tensor products of two \( so(4) \) coset representations. This is necessary for interpretation as generalized spinor and for proper identification of \( so(3,1) \) Lorentz algebra.

To sum up, all what is needed to obtain Kac Moody counterpart of local \( so(3,1) \) allowing interpretation in terms of generalized spinors is the replacement

\[
F^{1,k}_1 \rightarrow iF^{1,k}_1
\]

in the tensor product of two \( so(4) \) representations and in the definition of Super Virasoro algebra. The task is to show that super conformal invariance is not lost or equivalently: it is possible to replace the generators \( F^{1,k}_1 \) with \( iF^{1,k}_1 \) in the representation of Super Virasoro generators without spoiling the algebra. The definition of super generators \( G^k \) allows this as the following argument shows.

a) Isospin generators in sector where \( F^{am} \) act are of form \( T^a \propto f_{abc} F^{bm} F^{cn} \). This means the following transformation rule for isospin generators

\[
(T^1, T^2, T^3)_1 \rightarrow (T^1, iT^2, iT^3)_1
\]

in replacement \( F^1 \rightarrow iF^1 \). Same replacement must be done also in second \( su(2) \) sector of \( so(4) \) associated with the sector \( i = 1 \).

b) Super generators involve terms of general form \( f_{abc} F^{ak} F^{bl} F^{cn} \) from the \( su(2) \) sector where \( F^{am} \) act. This implies that \( G^k \) transforms as \( G^k \rightarrow iG^k \) in the replacement \( F^1 \rightarrow iF^1 \). If one modifies the definition of \( G^k \) by \( G^k \rightarrow iG^k \) Super Virasoro algebra suffers no change.

The states in Super Virasoro representation can be labeled by discrete eigenvalue of the axial Lorentz generator \( \Sigma^{03} \) and the representation contains infinite number of different eigenstates of this generator. An interesting problem is the relationship of the coset representation to the known discrete series unitary representations of Lorentz group, which are also infinite dimensional [Gelfand et al].
3.2.2 Kac Moody counterparts of quarks leptons and gauge bosons

A nontrivial problem is related to the construction of the exact Kac Moody counterparts of the imbedding space spinors and gauge bosons. Consider first the construction in case of imbedding space spinor.

a) The ground states for an appropriate tensor product of Kac Moody representations associated with $M^4$ and $SO(4)$ plus $u(1)$ and $su(3)$ degrees of freedom should behave as imbedding space spinor and different chiralities of this spinor correspond to quarks and leptons of single generation.

b) $M^4$ and $so(4)$ degrees can be treated in identical manner using spin or isospin $1/2$ Ramond representations having $(c, h) = (0, 0)$. Single Ramond representation gives single spin or isospin doublet. This means that one must take tensor product of two Ramond representations in both $M^4$ and $CP_2$ degrees of freedom to obtain the Kac Moody counterpart of H-spinor having $4 \times 4$ components. These representations correspond in natural manner to the two possible representations of $su(2)_R \times su(2)_L$ obtained as tensor products $(k_R = 0, \lambda = 0) \times (2, R)$ and $(2, R) \times (k_R = 0, \lambda = 0)$

\begin{equation}
H - \text{spinor} \leftrightarrow D(M^4) \otimes D(CP_2) = ((k_R = 0, \lambda = 0) \otimes (2, R)) \times ((2, R) \otimes (k_R = 0, \lambda = 0))
\end{equation}

The number of the vacuum states is correct and one can define various chirality operators associated with $M^4$ and $CP_2$ degrees of freedom and identify leptons and quarks as states of different H-chirality. Super fields $\Gamma_i$ associated with the $so(4)$ and $so(3, 1)$ super Virasoro representations change leptonic to quark chirality and vice versa.

c) The treatment of $U(1)$ degrees of freedom is simple: one just adds the appropriate Ramond type $U(1)$ Kac Moody representation to lepton and quark like representations as a tensor factor and takes into account that Kähler charges have their correct values for quarks, leptons and their antiparticles. The treatment of color degrees of freedom is similar: the tensor factor associated with leptons and quarks is $N - S$ and Ramond representation of color super Virasoro respectively.

The treatment of bosonic sector proceeds in similar manner. Also in NS-sector one must use the tensor product of two N-S representations.
in both $M^4$ and $CP_2$ sectors.

$$ Boson \leftrightarrow D(M^4) \otimes D(CP_2) $$
$$ D(M^4) = D(CP_2) = ((k_R = 0, \lambda = 0) \otimes (2, N - S)_i) \otimes ((2, N - S)_i \otimes (k_R = 0, \lambda = 0)) $$ (13)

Dirac equation can be generalized so that it applies to both Ramond and N-S case.

### 3.3 Super Virasoro in $U(1)$ degrees of freedom

The treatment of the $U(1)$ group associated with Kähler charge (essentially electroweak hyper charge) is completely analogous to the representation associated with $M^4$ degrees of freedom in string models. Nontrivial problem is associated with the nonvanishing vacuum expectation value

$$ h(U(1)) = \frac{p^2}{2} = \frac{Q_K^2}{2k(U1)} $$ (14)

of $L_0$. $p$ can be identified as the 'anomalous' Kähler charge of the particle deriving from the additional $U(1)$ term appearing in $CP_2$ spinor connection and making $CP_2$ spinor structure possible. One has $Q_K = 1/3$ for quarks and $Q_K = -1$ for leptons.

The requirement that $su(2) \times u(1)$ acts as symmetry group means that one can mix the generators of $I_3$ and $Q_K$. This requires that the values of $u(1)$ central charge is and vectorial central charge are identical:

$$ 2k(U(1)) = m = k_R + 2 $$ (15)

The dependence of $k(U(1))$ on $m$ introduces additional contribution into the mass formulas. At the symmetry limit one has $h(U(1), L) = 1/2$ for leptons.

For quarks one has at symmetry limit $h(U(1), q) = 1/18$. Free quarks have necessarily mass of order Planck mass unless the vacuum weight $h_0$ for quark representations contains a compensating term. One might think
that here lies an explanation for the confinement of fractional charges. It however turns out that quarks must be massless: this of course is also the only physically acceptable alternative in view of successes of parton model.

In string model $c_{tot} = 0$ is important consistency condition and this might be the case also now although it is difficult to see why: Poincare and color invariances are not lost and $so(4) \times u(1)$ is not an actual symmetry. In present case $c = 0$ for $M^4$ Super Virasoro, for $m = 2$ $so(4)$ Super Virasoro and for $su(3)$ Super Virasoro (as will be found) but for $U(1)$ Super Virasoro $c = 3/2$. It seems however impossible to apply coset construction (by applying it to $so(4) \times u(1)$ and $su(2) \times u(1)$) in order to get vanishing value of $c(U(1))$ without eliminating $U(1)$ contribution to Virasoro generators totally and at same time loosing nice explanation for the experimental absence of fractionally charged states.

### 3.4 Super conformal invariance and color symmetry

The construction of a unitary representation of Super Virasoro algebra having $(c, h) = (0, 0)$ based on color super Kac Moody group proceeds along the following lines.

a) Color is unbroken gauge symmetry: this suggests that the representation in question corresponds to vanishing Kac Moody central charge $k_B$. $k_B = 0$ implies the vanishing of the conformal central charge $c$. The vacuum expectation value $h$ of $L_0$ should also vanish. $(c, h) = (0, 0)$ representation of Virasoro has extension to a unitary Super Virasoro representation so that there are good hopes in case of $su(3)$, too.

b) Super Virasoro representations are of Ramond or Neveu-Schwartz type and this raises a problem. Ordinary Ramond representation has spinorial vacuum state and in case of color group the vacuum would correspond to a spinor of 8-dimensional Euclidean space. This is certainly an unphysical property. The Kac Moody algebra and/or its super counterpart must differ from the ordinary Kac Moody algebra in some crucial manner.

c) Color Lie-algebra is unique among other Lie-algebras in that it can be extended to a super algebra. The point is that for triplet representation color generators allow also anticommutator structure
\[
Anti(T^a, T^b) \equiv T^a T^b + T^b T^a = k\delta(a,b)I + d_{abc}T^c
\]  

(16)

For color triplet representation one has \( k = 1/3 \) and the quantities \( d_{abc} \) are completely symmetric 'structure constants' appearing in the completely symmetric color singlet constructed from three color octets. This implies that color algebra can be extended to super algebra \( \{T^a, F^a\} \):

\[
Comm(T^a, T^b) \equiv T^a T^b - T^b T^a = f_{abc}T^c
\]

\[
Comm(T^a, F^b) = f_{abc}F^c
\]

\[
Anti(F^a, F^b) \equiv F^a F^b + F^b F^a = k\delta(a,b)I + d_{abc}T^c
\]  

(17)

Super Jacobi identities

\[
Comm(Anti(F^a, F^b), T^c) - Comm(Comm(F^b, T^c), F^a) + Anti(Comm(T^c, F^a), F^b) = 0
\]  

(18)

boil down to the identities

\[
d_{abc}f_{dce} - f_{bcd}d_{ae} + f_{cad}d_{be} = 0
\]  

(19)

These identities follow from the condition

\[
Comm(T^d, d_{abc}T^a T^b T^c) = 0
\]  

(20)

The simplest representations of super color algebra are representations for which the condition

\[
T^a = F^a
\]  

(21)

holds true. The unique feature of these representations is the absence of additional degrees of freedom characteristic to the ordinary super algebras.
It is just this property, which suggests that the central term of color Super Kac Moody should be modified to include the $d_{abc}$ term.

e) The next step is to construct Color Super Kac Moody algebra $\{T_n^a, F_n^a\}$. The definition of this algebra is quite uniquely fixed by dimensional requirements. The defining equations are the following ones:

\[
\begin{align*}
\text{Comm}(T_n^a, T_n^b) &= f_{abc} T_{m+n}^c + k_B (n - m) \delta(a, b) \delta(m + n) I \\
\text{Comm}(T_n^a, F_n^b) &= f_{abc} F_{m+n}^c \\
\text{Anti}(F_n^a, F_n^b) &= (k \delta(a, b) + d_{abc} T^c) \delta(m + n)
\end{align*}
\]  

(22)

For fermionic generators $n$ is either integer (Ramond representation) or half integer (N-S representation). The condition

\[ F_0^a = T^a \]  

(23)
implies that the physically interesting Ramond representations does not have the usual $d = 8$ spinor degeneracy. The value of the fermionic central charge is fixed by vacuum representation. For quark triplet one has $k = 1/3$.

f) Concerning the construction of Super Virasoro representation there are two clues. The first and probably wrong clue (as it turned out) is provided by the observation that coset construction based on groups $su(3), u(2)$ with central charge $k_B = 1/2$ gives $c = 0$ representation of Virasoro algebra as is clear from the general formula for $c$ in coset construction [Goddard et al., Itzykson et al.] given by

\[
\begin{align*}
c &= c(su(3)) - c(su(2)) - c(u(1)) \\
c(g) &= \frac{2kdimg}{2k + c_g} \\
c_g &= \frac{f_{abc} f_{abc}}{dimg}
\end{align*}
\]  

(24)

One has $c_g = 3, 2$ and $0$ for $g = su(3), su(2)$ and $u(1)$ respectively so that $c$ indeed vanishes for $k = 1/2$. The representation in question has $h = 0$ as
one finds easily. I have not however been able to find the extension of this representation to Super Virasoro representation. The nonvanishing value of Kac Moody central charge $k$ as well as symmetry breaking to $u(2)$ (Virasoro commutes with $u(2)$ Kac Moody) suggests that this representation is not physically interesting: color group should act as unbroken local gauge group.

\[ g \] The second clue comes from the observation that ordinary color Super Kac Moody allows purely 'fermionic' representation \[ \text{Goddard et al, Itzykson et al} \] with bosonic generators defined in terms of the fermionic generators as

\[ T^a_n = K : f_{abc} \sum_{m \neq 0, m \neq n} F^b_{n-m} F^c_m : \]  

(25)

Note the exclusion of zero modes in present case. The value of the constant $K$ is $K = 1/2$ for the ordinary Kac Moody algebra. The value of the bosonic central charge is $k_B = c_g/2 = 3/2 \neq 1/2$ so that this representation cannot be used in the previous coset construction although the specific form of the generators suggests the possibility to define super Virasoro generators $G$ as linear combinations of generators of form $G_g \propto \sum_{a \in g} T^a$, $g = su(3), su(2), u(1)$.

A suitable modification of this representation might make sense also in the present case since the general structure of the commutation relations is correct.

\[ i \] The value of constant $K$ can be fixed by studying the contribution of $F^a_0 = T^a$ to the c-number part of $T^a_0$ and by requiring that this reduces to the ordinary Lie-algebra generator $T^a$: this condition reads as $T^a = K f^{abc} T^b T^c$ and gives

\[ K = \frac{2}{3} \]  

(26)

\[ ii \] The commutator $Comm(T^a_n, T^b_m)$ involves besides the ordinary term given by $\frac{16}{3} f_{abc} T^c_{n+m}$ terms proportional to $d_{abc}$ and involving $T^c$ besides the two fermionic generators. $T^c$ can be eliminated using the commutation relations of the bosonic generators. Due to the identity

\[ d_{abc}(f_{ACc} f_{BDd} f_{Da} + f_{ACd} f_{BCf} f_{Da}) = \frac{5}{24} f_{ABe} f_{edCD} \]  

(27)
relating the structure constants $f$ and $d$ this contribution is just the needed compensating term $-\frac{7}{9} f_{abc} T^c$. The identity follows by considering the commutator $\text{Comm}(T^a, T^b)$ for super color algebra in cm degrees of freedom, using the representation $T^a = \frac{2}{3} f_{abc} F^b F^c$ and expressing the commutator in terms of anticommutors only.

iii) For the ordinary representation the value of the Kac Moody central charge would be equal to $k_B = \frac{3}{2}$ but in present case also the $d_{abc}$ terms give contribution to the central charge term and for $k = 1/3$ representation the terms cancel each other! This can be understood by noticing that central charge term comes from two anticommutators of the 'fermionic' generators in the commutator. For the ordinary representation the term is proportional to $C_{ad} = f_{abc} f_{dce} = \frac{3}{2} \delta(a, d)$. In present case one has more general formula

$$C_{ad} = f_{abc} D_{be} D_{cf} f^{def}$$

$$D_{ab} = k \delta_{ab} + d_{abc} T^c_0$$  \hspace{1cm} (28)

The terms proportional to $T^c$ must vanish by the $su(3)$ invariance of the central term. The terms quadratic in $d_{abc}$ give a contribution, which must be proportional to unit matrix and a straightforward calculation for $a = d = Y$ shows that the two contributions cancel each other exactly. The vanishing of $k_B$ implies that if Super Virasoro for the representation in question exists it has $c = 0$.

h) The simplest guess is that Virasoro generators $L_n$ are given by the standard Sugawara construction

$$L_n = \frac{1}{\beta} : \sum_{m} T^a_{n-m} T^a_m :$$

$$\beta = 2k_B + c_g = 3 \text{ for } g = su(3)$$  \hspace{1cm} (29)

Since $k_B = 0$ the normal ordering is not in fact necessary. The general form of the super generator $G_n$ should be $T_a F_a$: this is achieved takin $G_n$ to be of the following general form

$$G_n = g f_{abc} \sum_{k,l,m \neq 0} F^a_k F^b_l F^c_m \delta(n - k - l - m)$$  \hspace{1cm} (30)
Again the normal ordering is unnecessary. The terms proportional to $d_{abc}$ in the anticommutator

$$L_{n+m} = \frac{1}{2} \text{Anti}(G_n, G_m)$$

(31)

vanish by Super Jacobi identities and one gets the standard Sugawara form by choosing the constant $g$ to have the value given by the condition

$$9g^2 = \frac{1}{\beta} K^2 = \frac{4}{27}$$

(32)

giving $g = \frac{2}{9\sqrt{3}}$. The commutators $\text{Comm}(L_n, L_m)$ are certainly of correct form and since they are expressible using the commutators between Virasoro generators and super generators, it is very plausible that also the commutors $\text{Comm}(L_n, G_m)$ have correct form.

To sum up, the unique property of $su(3)$ Lie-algebra allowing anticommutator structure allows to construct Super Virasoro algebra associated with color group with physically acceptable properties in Ramond sector. This is an additional item in the long list of very special properties of $CP^2$ geometry making the choice $H = M^4_+ \times CP^2$ as imbedding space unique.

### 3.5 How to achieve Super Virasoro invariance in leptonic case?

In leptonic case generalized spinor is tensor product of Ramond type representations with N-S type representation in color degrees of freedom. This leads to problems with Super Virasoro invariance. One should somehow be able to transform the half odd integer Super Virasoro generators associated with $su(3)$ degrees of freedom to generators labeled by integer. It can be assumed that the super Virasoro generators $G^n, n = 0, 1, 2, \ldots$ associated with $so_{-} \times u(1)$ degrees of freedom commute with the generators $G^k(su(3)), k = \pm 1/2, \ldots$ associated with color degrees of freedom. It can be assumed that the super Virasoro generators $G^n, n = 0, 1, 2, \ldots$ associated with $so_{-} \times u(1)$ degrees of freedom commute with the generators $G^k(su(3)), k = \pm 1/2, \ldots$ associated with color degrees of freedom.
A possible resolution of the problem is based on the following observations:
a) There must be operator $B^{1/2}$ of conformal weight $\Delta = 1/2$ acting on $u(1)$ degrees of freedom since Kähler charge contributes to the vacuum weight by $\Delta h = 1/2$. Assume therefore the existence of operators $B^{1/2}$ and $B^{-1/2}$, which are hermitian conjugates and anticommute to unit matrix whereas their commutator vanishes:

$$\text{Anti}(B^{1/2}, B^{-1/2}) = 1$$
$$\text{Comm}(B^{1/2}, B^{-1/2}) = 0$$

The conditions imply $B^{1/2}B^{-1/2} = 1/2$. Any diagonal matrix, whose diagonal elements are of form $\exp(i\phi)/\sqrt{2}$ gives a realization of these relations so that there are good hopes of finding solution to the conditions. These operators anticommute with Ramond type super generators and commute with all color Super Virasoro generators.
b) The commutation relations between Virasoro generators and Super generators do not produce troubles if one multiplies color Super generators $G^{n+1/2}$, $n > 0$ with $B^{-1/2}$ and $G^{n-1/2}$, $n < 0$ with $B^{1/2}$ to obtain color contribution to new Ramond type generator $\hat{G}^n, n \neq 0$. For $n = 0$ special care is needed since both $B^{-1/2}G^{1/2}$ and $G^{1/2}G^{-1/2}$ contribute to $\hat{G}^0$. A linear combination of these terms is obviously needed so that one has:

$$\hat{G}^n = G^n (s o \times u(1) + B^{1/2}G^{n-1/2}(su(3)), n > 0$$
$$\hat{G}^n = G^n (s o \times u(1) + B^{-1/2}G^{n+1/2}(su(3)), n < 0$$
$$\hat{G}^0 = G^0 (s o \times u(1)) + k(B^{-1/2}G^{1/2}(su(3)) + B^{1/2}G^{-1/2}(su(3)))$$

where $k$ must be chosen so that $G^0G^0 = L^0$ condition holds true.
c) The only problems are related to the central term in the anticommutator

$$\text{Anti}(\hat{G}^m, \hat{G}^n) = 2L^{m+n} + \frac{c}{3}(n^2 - \frac{1}{4})\delta(m, n)$$

since color contribution to the anticommutator is of form $(n + 1/2)^2 - 1/4$ rather than $n^2 - 1/4$. No troubles are however encountered if color algebra has vanishing central charge as it indeed does!
4 Identification of elementary particles as representations of $p$-adic Super Virasoro

The concept of generalized H-spinor implies unique identification of elementary particles as ground states of Super Virasoro representations.

a) Massless elementary particles are identified as generalized H-spinors having $(c, h) = (0, 0)$. Only $(J = 1/2, I = 1/2)$ fermions are obtained and sfermions typical for super symmetric theories are absent. Bosons have $(J = 1/0, I = 1/0)$ and again exotics are absent. Absent is also Higgs doublet.

b) All elementary particles correspond to Super Virasoro representation of $U(1)$ group with $(c, h) = (3/2, Q^2_K/2)$. For fermions the representation is of Ramond type and for bosons of type N-S.

c) With respect to color group leptons and bosons are N-S representations and quarks correspond to Ramond representations.

Physical states are either annihilated by Super Virasoro generators $L_n, n \neq 0$ and $G_n, n \neq 0, 1/2$ or these operators create zero norm states. Same applies to the generators $T_n$ of the vectorial $su(2)$ subgroups associated with $so(3, 1)$ and $so(4)$: otherwise one would get enormous mass degeneracy since $SO(4)$ Super Virasoro commutes with vectorial $su(2)$ Kac Moody generators. The assumption is in accordance with conserved vector current hypothesis. Color Kac Moody acts as an ordinary local gauge group and one must require that same conditions apply to the generators of color Kac Moody. A representation of local Lorentz group can be constructed in terms of 'fermionic' generators of $M^4$ Virasoro. Since conformal anomaly vanishes in $M^4$ sector this Kac Moody algebra has vanishing central charge and one can say that both local $so(4)$ local $SO(3, 1)$ acts as a local gauge group and physical states satisfy appropriate conditions. Symmetry breaking implies that these groups reduces to local $su(2)_V$.

In TGD quarks and leptons correspond to spinors of opposite H-chirality defined as product of $M^4$ and $CP_2$ chiralities and $B$ and $L$ are exactly conserved quantum numbers. One can associate definite baryon and lepton numbers both to bosons and fermions also in present approach [Pitkanen]. Super Virasoro generators with half odd integer conformal weight act as operators transforming quarks into leptons and vice versa. Leptoquarks, that is bosons with lepton and baryon number, if they exist, turn out to have Planck
mass. Exact conservation of $B$ and $L$ implies that only $\Delta \in \mathbb{Z}$ excitations are allowed and thermal mixing of different G-parities is not possible.

Thermal mixing between different G-parities, if possible, would imply breaking of $H$-chirality analogous to the ordinary breaking of Minkowski space chirality associated with massivation and provides an obvious explanation for matter antimatter asymmetry: symmetry breaking effects would be certainly small, of order $O(p^2)$ at the level of matrix elements: naive order of magnitude estimate for lifetime of proton as $M_p/p^2$ for $p = M_{107}$ gives lifetime of order $10^{48}$ years. In TGD there is however a mechanism for generation of matter antimatter asymmetry based on different rates for topological evaporation of fermions and antifermions in (necessary) presence of vacuum Kähler electric fields.

A more stringent requirement is suggested by string models: allow only excitations with G-parity equal to $+1$ or $-1$. This condition eliminates leptoquarks from spectrum and the excitations of leptons to quarks with wrong color triality $t = 0$ instead of $t = 1$. In TGD the absence of actual tachyons in TGD forces G-parity rule.

The main differences between p-adic TGD and more standard theories are following:

- a) Higgs doublet is absent in TGD.
- b) Leptoquarks are absent.
- c) Superpartners of ordinary elementary particles are absent.
- d) Graviton is not obtained in the simplest scenario as a massless elementary particle. States with quantum numbers of graviton are obtained as tensor products of basic representations but these correspond to ordinary hadrons rather than actual graviton. The result is in accordance with the basic assumption of p-adic approach: p-adic super conformal invariance makes sense only in the approximation that space time is flat [Pitkänen]. There is analogy with string model: in string model graviton corresponds to closed string, which corresponds to closed 3-surface without boundary. In p-adic approach [Pitkänen] one however considers only surfaces, which are essentially pieces of Minkowski space and have outer boundary. Perhaps the absorption of graviton corresponds to topological sum of absorbing particle and closed 3-surface describing graviton. One can consider the possibility that gravitons correspond to two-sheeted surfaces obtained by gluing to essentially flat
c) Color excitations of leptons obtained using the generators of N-S type color algebra with triality \( t = 1 \) are possible. It turns out that massless excitations exist for leptons as well as quarks. Color bound states of colored excitations are therefore possible and a conformation for the prediction of new branch of physics is obtained [Pitkänen and Mähonen, Pitkänen]. This prediction is in accordance with the TGD inspired explanation [Pitkänen and Mähonen, Pitkänen] of anomalous \( e^+e^- \) events observed in heavy ion collisions in terms of leptopions, bound states of color excited leptons. In the fourth paper of the series the rather deep consequences of colored excited leptons and quarks are discussed.

4.1 Definition of fermionic Dirac operator and the problem of tachyonic ground states

One should generalize Dirac equation in such a manner that mass shell condition results as a consequence of this equation. The generalization should apply in both fermionic and bosonic case and in bosonic case it should give the usual orthogonality condition for polarization vector and four momentum vector. Both bosons and fermions correspond to Kac Moody counterparts of H-spinors and the only difference between bosons and fermions is that they correspond to different representations of Super Virasoro.

In order to construct Dirac operator one needs gamma matrices. For fermions one can define Minkowski space gamma matrices in terms of super generators \( F_{i,K}^{a}, \ i = 1, 2 \). One has \( K = 0 \) in fermionic case and \( K = 1/2 \) in bosonic case.

\[
\begin{align*}
\gamma^0 &= iF_{1,K}^1, \\
\gamma^1 &= F_{1,K}^2, \\
\gamma^2 &= F_{2,K}^1, \\
\gamma^3 &= F_{2,K}^2
\end{align*}
\]  

(36)

Here the representation associated with \( so(3,1) \) is regarded as \( so(4)/su(2) \) coset representation and this explains the imaginary unit. For bosonic gamma
matrices one can define conjugate matrices in terms of $F_{k}^{i,-K}$. It is also possible to define $CP_2$ type gamma matrices applying previous formulas in $so(4)$ degrees of freedom and dropping the imaginary unit. Generalized Sigma matrices can be constructed in standard manner as commutators of gamma matrices.

There are infinitely many gauge equivalent ways to define gamma matrices by performing $su(2)$ rotations in both $so(4)$ factors of the tensor product defining the representation. These rotations are pure gauge rotations and cannot affect physics. In particular, the additional degeneracy of states resulting from the gauge invariance has no physical consequences and introduces just an integer factor in partition function.

### 4.2 Fermionic Dirac equation

Ground state should have such mass that for both for neutrinos, D type quarks and bosons $L^0 = 5/2$ states are massless whereas for charged leptons and U type quarks $L_0 = 3/2$ states are massless. This requirement can be contrasted with the anticommutation relations for Ramond type Super Virasoro generators. To begin with, the representation of $G^{0}(tot)$ given by

$$ G^{0}(tot) = \sqrt{\frac{2}{k(F)}} p^k \gamma_k + G^{0} \tag{37} $$

where $k(F)$ scale factor, whose value turns out to be $k(F) = 3/2$ for purely physical reasons. Anticommutations for super generators read as

$$ \text{Anti}(G^m, G^{-n}) = -\frac{1}{k(F)} p^2 \delta(m, n) + L^{m-n} - \frac{c}{3}(m^2 - \frac{1}{4}) \delta(m, n) \tag{38} $$

$G^{0}G^{0} = 0$, which can be regarded as a generalization for the square of massless Dirac equation gives mass formula

$$ p^2 = k(F)(L^0 - \frac{c}{24}) \tag{39} $$
The should be ground state with $L^0 = 5/2$ is massive and possesses Planck mass so that something goes wrong with the generalization of the massless Dirac equation. For ground state ($L^0 = 0$) one has $m^2 = -3/32$ for $c = 3/2$. One might regard this state as tachyon but in p-adic case the square root of $-3/32$ exists as real p-adic number under rather general conditions: for instance for Mersenne primes different from $M_2$.

One can modify the generalized massless Dirac equation for fermions by introducing fermion mass so that correct massless states are obtained. Symmetry breaking actually requires that $M^4$ chiralities get mixed so that massless Dirac equation is not realistic and mass term is needed. The problem is that the addition of scalar mass term breaks chirality invariance by mixing different $H$-chiralities. The properties of induced spinors suggest a resolution of the problem. The $CP_2$ part for the trace of second fundamental for for a submanifold appears as mass term in Dirac equation for induced spinors. Originally the trace of of the second fundamental form was identified as Higgs field. This term mixes $M^4$ chiralities but respects $H$-chirality conservation. In present case the mass operator must be linear combination of gamma matrices in $so(4)$ degrees of freedom and must anticommute with the operator $G_0$. This is achieved if Higgs term is linear combination of gamma matrices belonging to $su(2)_V$ since fermionic Super Virasoro generators in coset representation indeed anticommute with corresponding operators for factor group. Only $\gamma$ matrices $\gamma_0$ and $\gamma_3$ commuting with the sigma-matrix $\Sigma_{12}$ appearing in electromagnetic charge matrix are allowed. Since the values of $M^2$ are different for different charge states of fermions the mass operator must be decomposed into a sum of parts operating in different charged states only.

\[
M_{op} = \left( a\gamma_0 + b\gamma_3 \right) P_+ + \left( c\gamma_0 + d\gamma_3 \right) P_-
\]

\[
P_\pm = \frac{1 \pm 2I_3}{2}
\]

(40)

The gamma matrices correspond to $CP_2$ tangent space gamma matrices identified previously as $so(4)$ super generators.

The generalization of Dirac equation reads as

\[
\left( \frac{\sqrt{2}}{k(F)} \left( p^k \gamma_k + M_{op} \right) + G^0 \right) |PHYS\rangle = 0
\]

(41)
The solution to this equation is obtained in standard manner

\[ |\text{PHY}S\rangle = \left( \frac{\sqrt{2}}{k(F)}(p^k \gamma_k - M_{op}) + G^0 \right) |\text{phys}\rangle \]  

(42)

where \( |\text{phys}\rangle \) satisfies Super Virasoro gauge conditions and leads to the mass shell condition

\[
\begin{align*}
p^2 &= k(F)(L_0 - \frac{1}{16}) + M_{op}^2 \\
M_{op}^2 &= (a^2 + b^2)P_+ + (c^2 + d^2)P_- 
\end{align*}
\]

(43)

The requirement that \( L_0 = 2 \) (\( L_0 = 1 \)) state is massless for neutrinos and D type quarks (charged leptons and U type quarks) gives constraints on the eigenvalues of the mass squared operator

\[
\begin{align*}
a^2 + b^2 &= -k(F)(2 - \frac{1}{16}) \equiv A_+ = -\frac{3}{32} \cdot 31 \\
c^2 + d^2 &= -k(F)(1 - \frac{1}{16}) \equiv A_- = -\frac{3}{32} \cdot 15 \\
k(F) &= \frac{3}{2}
\end{align*}
\]

(44)

From these equations one can solve \( a, b, c \) and \( d \).

The solution ansatz relies on the existence/nonexistence of square roots for \( A_+, A_- \). \( \sqrt{A_-} \) does not exist as p-adically real number (\( \sqrt{p} \) is regarded real in this context) for Mersenne primes except \( M_{31} \). For \( M_n \neq M_2 = 3 \) the existence of square root reduces to the existence of \( \sqrt{31} \) in case of \( A_+ \). The existence or nonexistence of square root is verified easily by applying repeatedly reciprocity lemma stating that for two primes \( p \mod 4 = 3 \) and \( q \mod 4 = 3 \) one has \( p = x^2 \mod q \) iff \( q \neq x^2 \mod p \). Otherwise one has \( p = x^2 \mod q \) iff \( q = x^2 \mod p \). A useful auxiliary result is the existence of \( \sqrt{-3} \) as p-adically real number for all Mersenne primes except \( M_2 = 3 \) since neither \(-1\) nor \( 3 \) allows p-adically real square root for Mersenne primes. This means that although the arguments of square roots are negative as rational numbers their square roots can exist p-adically and negative mass squared...
ground states are not actually tachyons. $\sqrt{5}$ and $\sqrt{2}$ exist for all Mersenne primes except $M_2 = 3$ ($2 = x^2 \mod p$ for all $p \mod 8 = \pm1$). The following table summarizes the situation for different physically interesting Mersenne primes.

| $n$ | 127 | 107 | 89 | 61 | 31 | 19 | 17 | 13 | 7 | 3 | 2 |
|-----|-----|-----|----|----|----|----|----|----|---|---|---|
| $A_+$ | Y   | Y   | Y  | N  | N  | Y  | N  | Y  | N | N | N |
| $A_-$ | N   | N   | N  | N  | N  | N  | N  | N  | N | Y | N |

Table. The existence of square root of $A_+$ and of $A_-$ for different Mersenne primes $M_n$ (Y=yes, N=no).

For $I_3 = -1/2$ states the vector $(c, d)$ cannot allow real unit vector except for $M_2 = 3$. Perhaps the simplest solution to the condition $c^2 + d^2 = A_-$ is

$$
\begin{align*}
c &= \sqrt{3A_-} \\
d &= \sqrt{-2A_-}
\end{align*}
$$

and makes sense for $M_n, n \neq 2$ ($M_2 = 3$). Both square roots exist since $3$ and $-1$ and $A_-$ do not allow real square root whereas $2$ allows square root for $M_n, n > 2$. For $n = 2$ the ansatz

$$
\begin{align*}
(c, d) &= \sqrt{A_-} (n_1, n_2) \\
n_k n^k &= 1
\end{align*}
$$

and makes sense for $M_n, n \neq 2$ ($M_2 = 3$). Both square roots exist since $3$ and $-1$ and $A_-$ do not allow real square root whereas $2$ allows square root for $M_n, n > 2$. For $n = 2$ the ansatz

$$
\begin{align*}
(c, d) &= \sqrt{A_-} (n_1, n_2) \\
n_k n^k &= 1
\end{align*}
$$

works. The simplest rational solution is $(n_1, n_2) = (1, 0)$ or $(0, 1)$.

For $I_3 = 1/2$ states similar ansatz works for $(a, b)$ in $(N, N)$ case. For $(Y, N)$ case the vector $(a, b)$ allows unit vector and is of the form

$$
\begin{align*}
(a, b) &= \sqrt{A_+} (n_1, n_2) \\
n_k n^k &= 1
\end{align*}
$$

Any Pythagorean triangle with sides smaller than $p$ defines rational unit vector. For $n = 2, 3$ the only rational solution is $n_1 = 1$ or $n_2 = 1$. 

40
4.3 Bosonic Dirac equation

Also for bosons vacuum weight is $h = -5/2$ and massless states correspond to $L^0 = 5/2$. This necessitates the introduction of mass parameter to bosonic Dirac equation. In bosonic case the candidate for massless Dirac equation is

$$G^{1/2}(\text{tot})|\text{PHYS}\rangle = 0$$

$$G^{1/2}(\text{tot}) = \frac{\sqrt{2}}{k(B)p_k \gamma_{1/2}^k} + G^{1/2}$$

(47)

where $k(B)$ is some parameter measuring the ratio of fermionic and bosonic string tensions. The equation leads to mass shell condition mass shell condition $p^2 = k(B)L^0$, which fails to give massless states for $L^0 = 5/2$. In analogy with fermionic case 'Higgs' term to define mass operator

$$G^{1/2}(\text{tot}) = \frac{\sqrt{2}}{k(B)p_k \gamma_{1/2}^k} + G^{1/2} + M_{op}$$

$$M_{op} = a\gamma_0^{1/2} + b\gamma_3^{1/2}$$

(48)

The mass operator $M_{op}$ must be chosen so that $L^0 = 5/2$ states are massless. The square of Dirac equation corresponds to $G^{-1/2}G^{1/2}$ and gives mass shell condition

$$p^2 = k(B)L^0 + M_{op}^2$$

(49)

The construction also leads to a unique value of $k(B) = 3/2$. In this case $M_{op}$ is determined from the condition

$$M_{op}^2 = a^2 + b^2 = \frac{3}{4} \cdot 5 \equiv A$$

(50)

$\sqrt{5}$ and $\sqrt{-3}$ exist for all $M_n$ except $M_2 = 3$. Thus the square root of the left hand side exists for all $n$ and $(a, b)$ possessing unit vector defines acceptable mass operator.
The study of the bosonic sector shows that the standard solution ansatz for Dirac equation is not useful in bosonic case but that Dirac equation must be regarded as one gauge condition allowing many other solutions besides Dirac ansatz. Bosonic Dirac equation gives rise to mass shell condition plus conditions stating that bosonic polarization and momentum vectors are orthogonal. This follows from the fact that spin 1 bosonic state is proportional to the operator

\[ \epsilon_k \gamma^k_{1/2} \]  

and multiplication with \( p_k \gamma^k_{-1/2} \) gives \( p_k \epsilon_k = 0 \). Highest possible bosonic spin and isospin is \( J = 1, I = 1 \) as is clear from the fact that the operators \( F^1_{i/2} \), \( i = 1, 2 \) or \( i = 3, 4 \) anticommute so that only \( J = 0, 1 \) and \( I = 0, 1 \) states result from the tensor product. The fact that graviton is absent is in accordance with the assumption about the approximate flatness of p-adic spacetime surface, which is basic assumption of p-adic conformal field theory limit.

To sum up, the construction of fermionic and bosonic Dirac equation has provided two rather miraculous results: purely p-adic mechanism for the transformation of ground state tachyons to Planck mass particles and a mild suggestion that \( M_n \) with \( n = 89, 61, 17 \) are the only possible condensation level for gauge bosons among Mersenne primes.

5 Description of Higgs mechanism as a breaking of super conformal symmetry

In the following the phenomenological description of Higgs mechanism as breaking of Super conformal symmetry is considered quantitatively. General mass formulas are derived and the question what happens for mass in condensation is considered. The considerations are somewhat out of date in the sense that it is assumed that second order contribution to mass square correspond to small quantum number limit in the sense that second order term is of form \( k p^2 / 64 \) (Ramond) or \( k p^2 / 16 \) (N-S), where \( k \) is small integer. The thermodynamical calculation of masses has shown that this is however not
the case unless one performs suitable approximation or unless the secondary condensation changes the situation.

## 5.1 General mass formulas

Mass squared for the discrete series Super Virasoro representations is sum of $u(1)$ and $so(4)$ contributions:

$$M^2 = M^2(U(1)) + M^2(so(4))$$

(52)

$so(4)$ contribution is given by the following general expression as vacuum weight $h$ for general discrete series Super Virasoro representation:

$$M^2(so(4)) = h = \frac{((m+2)P-mQ)^2-4}{8m(m+2)} + \frac{\epsilon}{16}$$

$$m = k_R + 2$$

$$\epsilon = 1(0) \text{ for Ramond}(N-S)$$

(53)

Here one has $p = 1, \ldots, m-1$ and $q = 1, \ldots, m+1$. $p-q$ is odd for Ramond representation and even for N-S representation. There are three representations with vanishing mass squared $M^2 = h = 0$. They have $m = 2$ and therefore vanishing central charge $k_R = 0$ for the second $su(2)$ factor of $SO(4)$. In Ramond sector one $I = 1/2$ representation with $(p,q) = (1,2)$ (leptons and quarks). In Neveu-Schwartz sector there are $I = 0$ representation with $(p,q) = (1,1)$ (photon and gluons) and $I = 1$ representation with $(p,q) = (1,3)$ (intermediate gauge bosons $W$ and $Z$). Higgs mechanism corresponds to breaking of super conformal symmetry, when Kac Moody algebra $su(2)_R$ and as a consequence also super super conformal algebra develops central charge ($k_R \neq 0$). $I = 1/2$ and $I = 1$ states (quarks, leptons, intermediate gauge bosons) become massive in the breaking of conformal symmetry $I = 0$ states (photon, gluons) remain massless in the breaking of conformal symmetry.

$u(1)$ contribution is present for fermions only and for $Q_K = 1$ states (leptons) it is given by the following expression
\[
\frac{M^2(U(1))}{m_0^2} = \frac{Q_K^2}{m} - \frac{1}{2}
\]  

(54)

Exact conservation of electromagnetic charge \( Q_{em} = Q_K/2 + I_3 \) correlates \( Q_K \) with \( Q \).

\( P, Q, \) and \( m \) deviate in order \( O(p) \) from their values for massless representations. One can expand the expressions for \( P, Q \) and \( m \) in power series with respect to \( p \) and \( O(p^3) = 0 \) approximation is extremely accurate:

\[
m = 2 + m_1 p + m_2 p^2 + ...
\]

\[
P = p_0 + p_1 p + p_2 p^2 + ...
\]

\[
Q = q_0 + q_1 p + q_2 p^2 + ...
\]

\[
Q_K = q_0^K + q_1^K p + q_2^K p^2 + ...
\]

(55)

Expanding \( M^2(so(4)) \) and \( M^2(u(1)) \) in power series one obtains the general mass formula

\[
M^2 \simeq M_1^2 + M_2^2
\]

(56)

For Ramond representation the formula gives

\[
M_1^2 = (3t + q_1^K)p
\]

\[
M_2^2 = \frac{X}{64} p^2
\]

\[
X = (8(2p_1^2 + q_1^2) - 2p_1 q_1) + 32(q_1^K)^2 - 13m_2) \mod 64
\]

\[
m_1 = 64t
\]

(57)

The requirement that lowest order contribution is not of order Planck mass gives the condition \( m_1 = 64t \). It should be noticed that second order contribution to mass depends only weakly on the values of the integers appearing in it.
N-S representations can be assumed to have vanishing Kähler charge at elementary particle level. For N-S representations with $I_3 = 0$ one obtains

$$M_2^2 = \frac{X}{16}p^2$$

$$X = (q_1^2 - 12km_1 + 2(2p_2 - q_2)) \mod 16$$

$$2p_1 - q_1 = 8k$$

The condition $2p_1 - q_1 = 8k$ follows from the requirement that lowest order contribution is small in Planck mass scale.

For $I_3 = 1$ N-S representation the mass formula gives

$$M_1^2 = kp$$

$$M_2^2 = \frac{X}{16}p^2$$

$$X = (q_1^2 - 12km_1 + 2(m_2 - 2p_2 + q_2)) \mod 16$$

$$2p_1 - q_1 - m_1 = 8k$$

Again the smallness of lowest order contribution gives constraint on symmetry breaking parameters.

Since Kähler charge is $Q_K^0 = 1/3$ for quarks the $U(1)$ contribution to mass squared is of order Planck mass

$$M^2(u(1)) = \frac{Q_K^2}{m} = M_0^2 + M_1^2 + M_2^2$$

$$M_0^2 = \frac{1}{18}$$

$$M_1^2 = \frac{q_1^K}{3}p$$

$$M_2^2 = \left(\frac{q_1^2}{6} - \frac{m_2^2}{36}\right)p^2$$

(60)

unless vacuum weight for quark representations contains a compensating contribution making massless states possible: this turns out to be the case. This
is physically as the only possibility since partons are known to be essentially massless. In thermal approach massless quarks are predicted to spend fraction $\frac{1}{p} \simeq 2^{-107}$ of time in Planck mass state.

The relationship between $\Delta Q^K$ and $\Delta Q$ is fixed from the renormalization invariance of $Q_{em}$:

$$q_1 = \epsilon(L)q^K_1$$

$$\epsilon(L) = 2I_3 = \pm 1$$

(61)

where the sign factor depends on the isospin of the lepton. Electroweak symmetry breaking results. The actual breaking involves however also different critical condensation level for lepton and its neutrino so that one cannot deduce the ratio of electron and neutrino masses from the formula. It will be found that the sign of $q^K_1$ is same for, say, proton and neutron.

For certain primes the real counter part of the $so(4)$ contribution (and also of $u(1)$ contribution given by the canonical correspondence between p-adic and real numbers is minimized. If $p$ satisfies the condition $p \mod 64 = 63$ so that $p$ can be written in the form

$$p = 2^n - 64k - 1$$

(62)

the inverse of 64 in the finite field $G(p)$ formed by p-adic numbers mod p is given by

$$\frac{1}{64} \mod p = 2^{n-6} + small \ terms$$

(63)

Otherwise the inverse is

$$\frac{1}{64} = 2^{n-1} + small \ terms$$

(64)

Mersenne primes provide an example of the primes satisfying the condition. In [Pitkänen] it was erroneously argued that $p \mod 4 = 3$ guarantees that inverse satisfies the equation above.
With above constraint on the allowed values of $p$ the contributions to mass are

\[
\begin{align*}
M_1^2 &= \frac{(3k + q_1^1)}{p} \\
M_2^2 &= \frac{X}{64p} \\
X &= (8(2p_1^2 + q_1^2 - 2p_1q_1) + 32q_1^2 - 13m_2) \mod 64 \\
m_1 &= 64k
\end{align*}
\]

for leptonic Ramond representations,

\[
\begin{align*}
M_1^2 &= \frac{k}{p} \\
M_2^2 &= \frac{X}{16p} \\
X &= (q_1^2 - 12km_1 + 2(2p_2 - q_2)) \mod 16 \\
2p_1 - q_1 &= 8k
\end{align*}
\]

For N-S $I = 0$ representation and

\[
\begin{align*}
M_1^2 &= \frac{k}{p} \\
M_2^2 &= \frac{X}{16p} \\
X &= (q_1^2 - 12km_1 + 2(m_2 - 2p_2 + q_2)) \mod 16 \\
2p_1 - q_1 - m_1 &= 8k
\end{align*}
\]

for N-S $I = 1$ representation.

It is relatively easy to fit masses of known elementary particle and hadrons by choosing suitably the condensation level (presumably $p$ is near prime power of 2).

a) The dominant contribution to mass formula comes from $M_1^2$ and the integer appearing in this term is uniquely fixed by the fitted mass. For bosons
u(1) charge gives no contribution to this terms and for lowest lying mesons and gauge bosons this contribution turns out to be vanishing. For higher excitations of mesons on Regge trajectories this contribution must be non-vanishing and the integer \( k \) be expressed in terms of particle spin. In lowest order the lowest order contribution is linear in spin and one obtains the mass formula of the old-fashioned string model and prediction for the hadronic string tension.

b) The second order term has natural relative accuracy \( 1/64M^2 \) if the integers appearing in it are small and with a proper choice of the parameters rather accurate fit is obtained: the accuracy is the better the larger the fitted mass is. If one poses no constraints on the values of integers one can in fact select the parameters so that \( O(1/p^2) \) accuracy is achieved! The point is that p-adic numbers modulo \( p \) form a finite field and one can always choose the integers appearing in the mass formula so that the number \( K \) in \( M^2 = KP^2 \) is any p-adic integer in the range \( 1, p-1 \): this means that second order can have any value in the range \( (0,1/p) \) with accuracy \( O(1/p^2) \). This result implies that the concept of simplicity is doomed to be somewhat subjective.

### 5.2 Are the real counterparts of \( P, Q, m \) invariant in secondary condensation?

The mass formulas provide a test for the hypothesis that the real counterparts of \( P, Q, m \) remain invariant in topological condensation and condensation occurs only provided the mass decreases in condensation. The lowest order contribution to mass squared is linear in the quantum numbers \( p_1, q_1, m_1 \) and does not change but already the second order contribution changes since the algebra of p-adic numbers is not isomorphic with the algebra of real numbers.

Consider condensation of level \( p_a < p_b \) on level \( p_b \). The first constraint comes from the requirement that first order contribution to the real counterpart of \( P, Q, m \) remains invariant in order \( O(1/p^2) \). Writing the first order contribution in the general form \( n/p \) the invariance gives

\[
n_b = \frac{p_b}{p_a} n_a + O\left(\frac{1}{p^2}\right)
\]

Since \( n_2 \) must satisfy a condition of form \( n_a \mod k = 0, k = 8 \) or \( k = 64 \) in
order to avoid Planck mass the only manner to satisfy the constraint is to assume that the ratio $p_b p_a$ is integer in accuracy of $O(1/p_1)$.

\[
\frac{p_2}{p_1} = r_1 + O\left(\frac{1}{p_1}\right)
\]  

(69)

This condition together with the condition $p_i \mod 64 = 63$ restricts the set of allowed primes considerably since condensation to wrong levels implies Planck mass.

In second order approximation the invariance for the real counter parts to second order terms $n_2 \in \{m_2, p_2, q_2\}$ implies the transformation property

\[
n^b_2 = r_1^2 n^a_2 + r_2 n^a_1 + O\left(\frac{1}{p^2_b}\right)
\]

\[
r_2 = \text{int}\left(p_b\left(\frac{p_b}{p_a} - r_1\right)\right)
\]  

(70)

Second term corrects the error associated with lowest order term. In principle these formulas allow to estimate the change in mass squared to order.

The contribution of $O(p)$ terms to mass formula are linear in quantum numbers $p_i, q_i, m_i, q_K$ and does not change in condensation. An important factor affecting the situation is the appearence of $k \in \{1/8, 1/64\}$ factor, assumed to be essentially invariant in condensation ($p_i \mod 64 = 63$). The coefficient $X$ of the second order contribution contains two parts. The part $X_1$ quadratic in quantum numbers $p_1, q_1, m_1$ gets multiplied with $r_1^2$. The part $X_2$ linear in $p_2, q_2, m_2$ is mixed with the first order term of similar form and therefore transforms as $X_2 \rightarrow r_1^2 X_2 + r_2 s$, where $s$ is the coefficient of the first order term. The inhomogeneous term in fact cancels the change of first order term if the condition $s \mod k = 0$ is preserved on condensation. Neglecting the complication coming from the modulo conditions this gives for the change of the mass the following formula

\[
M^2_1 \rightarrow M^2_1 \\
M^2_2 \rightarrow M^2_2 r_1 \\
r_1 = \text{int}\left(p_b/p_a\right)
\]  

(71)
Mass increases in condensation!

The picture is however complicated by the fact that the integer valued coefficient \( X \) is defined only modulo \( k \) equal 16, 32 or 64 for small values of integers appearing in it. Furthermore, \( X \) contains terms, which are determined only modulo 2, 4, 16, 32, \( k \) and can have either positive or negative sign so that if \( r_1^2 \) is multiple of 2, 4, 16, ... the corresponding terms disappear from the mass formula and mass either increases or decreases depending on the sign of the disappearing term. If the contribution becomes \( p \)-adically negative it has essentially the maximal value \( 1/p_b \). This phenomenon occurs for primes near powers of two and satisfying the condition \( p \mod 64 = 63 \). More generally, primes near powers of 2 satisfy this condition. In particular, for primes near prime powers of two \( r_1^2 \) is some power of 2 and the sign of the condensation energy for to nearby prime powers of 2 is sensitive to the values of integers \( p_1, q_1, m_1 \) and this sensitivity.

An especially interesting situation occurs if \( r_1^2 \mod k = 0 \). In this case the contribution of the second order term seems to disappear totally from the mass formula! The condition is satisfied for the condensation of Mersenne prime \( p_a = 2^{m_a} - 1 \) on Mersenne prime \( p_b = 2^{m_b} - 1 \) since one has \( r_1 = 2^{m_b - m_a} \) when the ratio of Mersenne primes is sufficiently large. The situation is essentially same for all primes \( p \mod 64 = 63 \) near powers of two. If the interpretation is correct it would mean that second order contribution to mass disappears in few condensations if one assumes that physically interesting primes correspond to primes near prime powers of two and the first order term in mass is the only stable contribution to particle mass. If first order term to mass squared vanishes particle can lose its entire mass via condensation! Pions and intermediate gauge bosons indeed turn out to be examples of particles for which first order contribution to mass vanishes. Perhaps the unstability against decay makes the loss of mass via secondary condensation in practice impossible for these particles. An interesting prediction of the scenario is that the value of renormalized mass after several secondary condensations approaches always to the value obtained by dropping the second order contribution from the mass after primary condensation. This prediction is true for any model of condensation based on assumption that first order contribution to mass squared remains invariant in condensation.
5.3 Why primes near prime powers of two are favoured as primary condensation levels?

The difficult questions considered already earlier are:

a) Why primes near prime powers of 2 seem to be favoured as primary condensation levels?
b) What determines the primary condensation level $p_{cr}$.

The hypothesis about the invariance for the real counterparts of $P, Q, m$ are invariant in condensation allows to reconsider these problems on more quantitative level than previously.

Let us reconsider first the problem a). Assume that the primary condensation level of the particle is not unique so that also some primes $p < p_{cr}$ are possible as primary condensation levels. Assume however that particle mass is more or less independent on condensation level. Given prime $p_{cr} \simeq 2^k$ with temperature $T_{p_{cr}} = 1$, then for all primes $p_1 \simeq 2^{k_1}$, with $k_1$ a factor of $k$, the thermal masses are in good approximation same if the temperature $T_{p_1}$ is chosen to be $1/T_{p_1} = k/k_1 \in \mathbb{Z}$. For $p_1 = 2$ this requirement can be satisfied always so that 2-adic condensation level is in some sense fundamental. For $p = 2^k$, $k$ prime, only $p_1 = 2$ satisfies the constraint so that mass depends strongly on condensation level irrespectively how one chooses the temperature: amusingly these are just the physically interesting condensation levels! The condition implies that for $p > p_{cr}$ the inverse temperature had to be fractional number and this is not possible since fractional powers of $p$ does not exist p-adically. Therefore there is maximal value $p = p_{cr}$ of $p$ for which thermal description makes sense and this prime characterizes particle’s mass scale.

At the level of moduli space the choice $T = 1/k$ means that the allowed points of p-adic moduli space correspond to $q_{ij} = p^{k_{ij}}$, $k_{ij} \mod k = 0$: the restriction is modular invariant. The assumption implies that 2-adic level is the fundamental level and the value of the inverse temperature at 2-adic level determines the mass of the particle. For primes near prime powers of two the 2-adic inverse temperatures are primes:

$$\frac{1}{T_2} = \text{prime}$$

This explanation for favoured condensate levels looks very nice mathemat-
ically. What remains to be understood is how the prime \( k \) depends on the quantum numbers of particle: why \( k = 127 \) electron, 107 for u and d quarks, 89 for intermediate bosons,...?

It is useful to recall the second argument related to the real counterparts of the coupling constants defined by canonical identification as \( g^2_R = (g^2p)_R \). Mersenne primes have the special property that if p-adic \( g^2 \) is finite superposition of negative powers of two then its real counterpart is numerically equal to p-adic counterpart and real and p-adic theories do not differ drastically. It remains to be seen whether there exists also other primes near prime powers of two with the same property. For arbitrary primes \( g^2_R \) can differ widely from \( g^2 \) numerically if \( g^2 \) is rational number, which does not depend on \( p \). If this is indeed the case then Mersenne primes and primes near prime powers of two might be a result of a ‘natural selection’.

Consider next a possible mechanism determining the value of \( p_{cr} \). The simplest possibility one can imagine is that for \( p_{cr} \) the secondary condensation on nearby condensate level \( p \) not much larger than \( p_{cr} \) is energetically more favoured than primary condensation on level \( p \) so that particle is replaced with new structure: particle condensed on small piece of \( p_{cr} \) condensate level. This leads to essentially constant mass squared \( \propto 1/p_{cr} \) independent of condensate level instead of thermal mass squared proportional to \( 1/p \).

a) Consider two primes \( p_a < p_b \), which are near to each other: \( p_b/p_a < 2 \). Both \( p_a \) and \( p_b \) can serve as primary condensate level for particle and particle masses at these level are related by scaling \( m_b^2/m_a^2 = p_a/p_b \) if the condition \( p \mod 64 = 63 \) is satisfied.

b) Particle at \( p_a \) level can also suffer secondary condensation to the level \( p_b \). In this case the changes of the integers \( p_1, q_1, .. \) are small. Writing the first order term in form \( ap/k \) the condition that particle doesn’t get Planck mass in condensation reads \( \text{int}((p_b/p_a)a \mod k) = 0 \) so that \( a \) changes by a multiple of \( k \).

c) This would be guaranteed if the relative size of change of \( p \) is of order \( O(1/p) \) so that \( q_1, p_1, m_1 \) remain invariant and only \( m_2, p_2, q_2 \) change, in which case the particle mass changes, if the coefficient \( X_2 \) linear in these integers becomes proportional to \( k = 64, 16 \) or 32 appearing in the denominator of this term so that \( X_2 \) contribution from mass disappears. This is however impossible: \( r_2 = \text{int}((p_b/p_a)(p_b/p_a)) \propto 64 \) in this case so that \( X_2 \) remains unchanged.
d) The only possibility is that the change of $p$ is so large that $p_1$ or $q_1$ and therefore $X_1$ can change. The conditions for this is are $\text{int}(n(p_b/p_a) - n \equiv d \geq 1$ for $n = p_1$ or $n = q_1$. The change in $M^2$ caused by secondary condensation on one hand and by the change of primary condensation level on the other hand satisfies the upper bound

$$|\Delta M^2(\text{sec})| \leq M^2 = \frac{X_2}{p_{cr}}$$

$$|\Delta M^2(\text{prim})| < \frac{d}{n + d} M^2$$  \hspace{1cm} (73)

Therefore under the following necessary conditions

$$\Delta M^2(\text{sec}) < 0$$

$$\frac{d}{n + d} < \frac{M^2}{M^2}$$  \hspace{1cm} (74)

energetics can favour secondary condensation. If $M^2$ and/or $n$ is small it is quite possible that secondary condensation is not possible at all and particle remains massless (or possesses only thermal mass).

e) $p_{cr}$ might correspond to the smallest prime for which this kind of process is possible and resulting particle is long lived enough. This mechanism implies sensitive dependence of $p_{cr}$ on the integers $p_1, q_1, \ldots$ and therefore on particle's quantum numbers. If mechanism is really correct it predicts that for stable elementary particles the value of the parameter $X$ should be as small as possible.

This kind of mechanism could be at work in cosmological context: the cooling of the Universe corresponds to the gradual increase of the typical $p \propto \sqrt{1/T}$ associated with background and massive particles would separate from thermal equilibrium as 'hot spots' and remain in their own internal temperature in the proposed manner.

This mechanism implies that particles would correspond to double sheeted structures with $p_a$ and $p_b$ near each other. The calculation of particle masses shows that the idea about two $p$-adic levels very near each other seems to be correct. For example, the condensation levels with $u, d, s$ quarks is $k = 107$
and same as for hadrons. An amusing coincidence is that Connes and Lott have proposed the description of Higgs mechanism based on noncommutative geometry and one consequence of mechanism is double sheeted structure of space time! Of course, the presence of many sheeted structures is basic characteristic of the topological condensate.

6 p-Adic description of modular degrees of freedom

The success of the mass calculations give convincing support for generation-genus correspondence. The basic physical picture is following.
a) Mass squared is dominated by boundary contribution, which is sum of cm and modular contributions: $M^2 = M^2(cm) + M^2(mod)$. Here ‘cm’ refers to the center of mass of the boundary component. Modular contribution can be assumed to depend on the genus of the boundary component only.
b) Modular contribution to mass squared can be estimated apart from overall proportionality constant. Elementary particle vacuum functionals are proportional to product of all even theta functions and their conjugates the number of even theta functions and their conjugates being $2N(g) = 2^g(2^g + 1)$. Also the thermal partition function must also be proportional to $2^N(g):$th power of some elementary partition function. This implies that thermal/quantum expectation $M^2(mod)$ must be proportional to $2N(g)$. Since single handle behaves effectively as particle the contribution must be proportional to genus $g$ also. The surprising success of the resulting mass formula shows that the argument is correct.

The challenge is to construct theoretical framework reproducing the modular contribution to mass squared. There are two alternative manners to understand modular contribution.
a) Modular contribution is regarded as quantum mechanical expectation value of mass squared operator for elementary particle vacuum functional. Quantum treatment would be very straightforward in principle: generalize the concept of moduli space and theta function to p-adic context and find an acceptable definition for mass squared operator.
b) Modular contribution is calculated using thermodynamical treatment.

Thermodynamical treatment might go along following lines.
a) The minimal requirement is that the thermal expectation value for mass squared for modular invariant partition function is well defined. Probably the expectation value should be defined as logarithmic derivative with respect to temperature type parameter appearing in the partition function.

b) The structure of elementary particle vacuum functionals suggest that partition function must be expressible as product of $2N(g)$ elementary partition functions. Since $cm$ part contains all information about standard quantum numbers it seems useless to assume any electroweak structure for $Z(g)$. The fact that theta functions are expressible in terms of exponentials suggests that $Z$ is more or less equal to square of the elementary particle vacuum functional so that quantal and thermal approach seem to lead to same end result. This ansatz guarantees automatically modular invariance provided one can somehow modify the concept of moduli so that theta function or at least modular invariant combinations of them become well defined concepts.

c) Modular contribution to mass squared ought to be small in Planck mass scale and this is possible provided the concept of low temperature phase makes sense for modular invariant partition functions. This constraint is strong. The first alternative which comes into mind is that the concept of low temperature phase generalizes and leads to the quantization of allowed values of modular parameters (analogous to temperature parameters) so that moduli space is effectively discretized. This picture seems to work and implies also that theta function becomes a well defined concept.

The realization of either of these scenarios necessitates the generalization of ordinary Riemannian geometry to p-adic context in some sense. This would mean that boundary component is described as p-adic complex manifold, whose coordinate space can be imbedded to 4-dimensional square root allowing extension of p-adic numbers. The naive guess is that all formulas of Riemann geometry generalize as such: in particular the moduli space of p-adic Riemann surfaces is obtained simply by replacing its complex coordinates with their p-adic counterparts. It will turn out that p-adic existence requirements make the generalization far less trivial. It however seems that the concept of low temperature phase allows modular invariant generalization.
6.1 p-Adic moduli space

It is not at all obvious that the concept of moduli space generalizes to p-adic context in any sensible manner. Whatever the generalization is it should allow the p-adic version of elementary particle vacuum functionals. A further constraint comes from the requirement that low temperature phase is defined in some sense, which presumably means discretization of moduli space.

6.1.1 Θ function for torus

It is instructive to first study the problem for torus first. The ordinary moduli space of torus is parametrized by single complex number \( \tau \). The points related by \( SL(2, Z) \) are equivalent, which means that the transformation \( \tau \rightarrow (A\tau + B)/(C\tau + D) \) produces a point equivalent with \( \tau \). These transformations are generated by the shift \( \tau \rightarrow \tau + 1 \) and \( \tau \rightarrow -1/\tau \). One can choose the fundamental domain of moduli space to be the intersection of the slice \( Re(\tau) \in [1/2, -1/2] \) with the exterior of unit circle \( |\tau| = 1 \). The idea is to start directly from physics and to look whether one might some define p-adic version of elementary particle vacuum functionals in the p-adic counterpart of this set or in some modular invariant subset of this set.

Elementary particle vacuum functionals are expressible in terms of theta functions and a good guess is that this holds true for partition function, too. The general expression for theta function reads as

\[
\Theta[a, b](\Omega) = \sum_n exp(i\pi(n + a) \cdot \tau \cdot (n + a) + 2i\pi(n + a) \cdot b) \quad (75)
\]

\( a \) and \( b \) are half odd integers for torus. The obvious problem is that \( \pi \) does not exists p-adically and one should somehow make sense of the theta function perhaps somehow modifying the concept of modular parameter. Also the concept of low temperature phase ought to make sense. The idea is to regard \( q \equiv exp(i\pi\tau) \) as a basic coordinate in the fundamental domain instead of \( \tau \) so that one gets rid of the problems related to the nonexistence of \( \pi \).

Consider first the contribution of the real part of \( \tau \) to exponent appearing in \( q \) assuming that \( Re(\tau) \) belongs to the fundamental domain. One can always right \( Re(\tau) \) in the form \((m/n)p^{-k}, n > 1, k \geq 0\) in the fundamental domain and it corresponds to p-adic number with norm not smaller than one.
so that the exponent does not converge as power series and the only possible interpretation for $q$ is as a complex root of unity that is as a $p$-adic complex number $x + iy$ satisfying the condition $x^2 + y^2 = 1$, from which one gets $y = \sqrt{1 - x^2}$. Series converges if the $p$-adic norm $x$ is smaller than one so that one has $x \mod p = 0$.

An additional discrete set of phases is obtained by requiring that $y$ and $x$ are also finite with respect to real topology. With this assumption phases of form $(k/l) + i(m/n)$ satisfying $(k/l)^2 + (m/n)^2 = 1$ are allowed for $q$. The condition is equivalent with the age old problem considered already by Babylonian mathematicians of finding solutions of $k^2 + l^2 = n^2$ that is finding integer sided rectangular triangles (Pythagorean triangles). Solutions can be regarded as complex numbers of $k + il$ and form monod with respect to multiplication. The condition $Re(\tau) \in [-1/2, 1/2]$ for fundamental domain of torus gives $-\pi \leq \Phi \leq \pi$ so that all possible triangles are allowed in the fundamental domain. The action of the modular transformations $\tau \to A\tau D^{-1} + BD^{-1}$ on these phase factors is trivial for torus. For higher genera the action is just $\exp(i\Phi) \to \exp(in\Phi)$, which belongs to the allowed set since the allowed phases form a monod with respect to multiplication. By a little calculation one verifies that the explicit form for the allowed $(k, l)$ and $(l, k)$ pairs is given by

\[
\begin{align*}
k & = 2rs \\
l & = r^2 - s^2 \\
n & = r^2 + s^2
\end{align*}
\]

where $r$ and $s$ are relatively prime integers, not both odd. Note that $(l, k)$ is also an allowed solution. An important point to be noticed is that the $p$-adic norm of $\exp(i\Phi)$ is not larger than one for physically interesting primes satisfying $p \mod 4 = 3$ since $n \mod 4 = 1$ holds true as a simple calculation shows.

A possible source of problems is the appearance of terms $\exp(i\Phi/4)$ in the definition of theta function, when $a$ is half odd integer. The phase $\exp(i\Phi/4)$ however separates into a multiplicative factor and since elementary particle vacuum functional is proportional to the product of thetas and their conjugates these ill defined phases cancel each other. Note that the the phase
factor $\exp(i2i\pi(n+a)\cdot b)$ is for even theta functions appearing in the definition of elementary particle vacuum functional always equal to $\pm 1$ and therefore well defined.

An additional constraint to the allowed phase factors comes from the requirent that the integral (or rather discrete sum) of the square of elementary particle vacuum functional over moduli space converges. The problem is that infinite number of phases contribute to the sum and it is not clear whether the sum converges. The problem is that the p-adic norm of phase factor defined by Pythagorean triangle has p-adic norm equal to one.

A certainly working scenario is based on the assumption that the phase of $q$ is completely trivial. This means that only diagonal metrics $ds^2 = dx^2 + Im(\tau)^2 dy^2$ are allowed. The restriction to trivial phase factors respects modular invariance.

### 6.1.2 Imaginary part of the period matrix

Consider next the contribution $\exp(-2\pi Im(\tau))$ of the imaginary part of $\tau$ to $q$. The only sensible manner to define this quantity is to require that the modulus of $q$ rather than $Im(\tau)$ is the fundamental quantity just like the phase of $q$ rather than $Re(\tau)$ is taken as fundamental quantity. The constraint that $Im(\tau)$ corresponds in ordinary complex case the constraint $|q| < \exp(-2\pi\sqrt{1 - (m/(np^k))^2})$ for $Im(\tau) > 0$ (standard convention) or $|q| > \exp(2\pi\sqrt{1 - (m/(np^k))^2})$ if one takes $Im(\tau)$ to be negative. The subset of allowed p-adic values of $q$ should satisfy this constraint in some sense: as such the constraint does not make sense p-adically. The $\pm$ sign is present since one can choose between two alternatives $Im(\tau) < 0$ or $Im(\tau) > 0$ in complex case.

The requirement that low temperature phase is in question gives an additional constraint. Temperature quantization means in ordinary thermodynamic context that the exponent $\exp(1/T)$ reduces to integer power of $p$. The essential requirement is that the exponents $q^n$ are of order $O(p^{kn})$, $k > 0$, p-adically so that the power expansion of eta function with respect to $q$ converges extremely rapidly. This is achieved by requiring

$$N_p(mod(q)) < 1$$

(77)
This condition means that one can write \( mod(q) = p^k(m/n), k > 0 \), where \( (m/n) \) is p-adic rational number with unit p-adic norm. Condition is modular invariant since all modular transformations lead outside the fundamental domain. It turns out that a stronger condition \( q = p^k \) is needed in order to obtained well defined partition functions.

6.1.3 Discretization of moduli space

Restrictions on the values of the allowed modular parameters are probably necessary since the integration over p-adic moduli parameters is not obviously well defined concept.

a) Discretization of moduli of \( q \)

\[
mod(q) = p^k, k > 0
\]  

replaces integrations over \( mod(q) \) with summations and leads directly to the thermodynamic picture with the difference that all temperatures \( T \propto 1/k \) are allowed.

b) The assumption \( q = p^k \) so that the phase of \( q \) is completely trivial leads to mathematically well defined expressions for partition functions and to strict thermal interpretation of the partition function in the sense that temperature parameters are real. If phases defined by Pythagorean triangles are allowed the integration gives for for each power of \( p \) an infinite sum over phases, which have p-adic norm equal to one and it is not clear how the convergence of the sum could be achieved.

6.1.4 Higher genera

The generalization of the low temperature phase moduli space to the case of higher genera seems rather obvious. The rule is simple: \( q_{ij} = \exp(i2\pi r_{ij}) \rightarrow p^{k_{ij}}\exp(\Phi_{ij}), \quad k_{ij} > 0 \), where either of two possible scenarios for the phases is assumed. This definition is modular invariant if all modular transformations not leading out of this set respect this condition. The transformations permuting different nonintersecting homology cycles permute the rows of Teichmuller matrix and are not problematic. Also the transformations \( \Omega \rightarrow A\Omega D^{-1} \) leave this set invariant. The guess is that these are the only modular transformations not leading out of the discrete moduli space.
Since $\text{Im}(\Omega)$ and $\text{Re}(\Omega)$ transform independently in modular transformations in question the determinant $\text{det}(\text{Im}(\Omega))$ or equivalently the integer valued determinant

$$D_1 \equiv \text{det}(k_{ij})$$  \hspace{1cm} (79)$$is p-adically well defined modular invariant. This implies that one can perform gauge fixing in moduli space in order to avoid multiple summations in the construction of partition function. All what is needed is to pick up one representative $(k_{ij}, \text{exp}(\Phi_{ij}))$ from each orbit of the modular group. The orbits are labeled by the value of determinant of $D_1$ and all allowed values of the phase factor $\text{exp}(\Phi_{ij})$. Later considerations show that one must assume

$$D_1 \geq 0$$  \hspace{1cm} (80)$$for the allowed modular parameters in order to achieve convergence of the partition function.

### 6.2 Elementary particle vacuum functionals in p-adic context

The general definition of theta function $\Theta[a, b](\Omega)$ for genus $g$, where $a$ and $b$ are vectors having $g$ components with values in $\{0, 1/2\}$ reads as

$$\Theta[a, b](\Omega) = \sum_n \text{exp}(i\pi(n + a) \cdot \Omega \cdot (n + a) + 2i\pi(n + a) \cdot b)$$  \hspace{1cm} (81)$$For discrete moduli space $\text{exp}(i\pi\Omega_{ij}) = p^{k_{ij}}$ this quantity is indeed well defined. The only source of troubles is the term $\text{exp}(2i\pi(n + a) \cdot b)$. The quantity $\text{exp}(i2\pi n \cdot b)$ can be defined as $\pm 1$ depending on whether the inner product $n \cdot b$ is integer or half odd integer. For even theta functions appearing in elementary particle vacuum functionals the inner product $a \cdot b$ is half odd integer so that one obtains $-1$ from this term. This means that even theta functions are real quantities in discrete moduli space and can be written in the form
\[
\Theta[a, b](\Omega) = \sum_n p^{(n+a)^i k_{ij}(n+a)^j} \epsilon(n)
\]
\[
\epsilon(n) = -exp(2i\pi n \cdot b)(= \pm 1)
\] (82)

As far as practical calculations are considered only few terms are important due to the extremely rapid convergence for large values of \( p \).

The modular invariant vacuum functional can be defined just as in \cite{Pitkänen}

\[
\Omega(vac) = \frac{\prod_{a,b} \Theta[a, b](\Omega) \bar{\Theta}[a, b](\Omega)}{(\sum \Theta[a, b](\Omega) \bar{\Theta}[a, b](\Omega))^{N(g)}}
\] (83)

The product in the numerator is over all even theta functions. The nice feature of the definition is that vacuum functionals vanish for moduli possessing conformal symmetries: this is essential for the argument about 3 light fermions generations.

### 6.3 Definition of mass squared expectation value

The general form of the theta function suggests a natural definition of the mass squared expectation value. Theta function is analogous to a discrete version of functional integral. Functional integration corresponding to summation over over multi-integers \( n \) and the counterpart of action is the quadratic form \( S(a, n, k_{ij}) = (n+a)^i k_{ij}(n+a)^j \) so that 'free' theory is in question. The action of the mass squared operator on \( \Theta[a,b] \) should correspond to functional expectation value of the action \( 2S(a,n,k_{ij}) \) over \( n \) (note the appearence of factor 2, which turns out to be necessary)

\[
M^2 \circ \Theta[a,b](k_{ij}) \equiv \sum_n 2S(n, a, k_{ij})p^{S(n,a,k_{ij})} \epsilon(n)
\]
\[
S(n, a) = (n+a)^i k_{ij}(n+a)^j
\] (84)

In the expectation value of mass squared operator for vacuum functional \( M^2 \) operator should and like differential operator on the vacuum functional/partition function. This can be formally achieved by introducing formal temperature
parameter by scaling the ‘action’ $S$ appearing in the definition of theta function by temperature and defining the action of mass square operator as a logarithmic derivative with respect to $T$

$$\Theta[a, b](k_{ij}, T) \equiv \Theta[a, b]\left(\frac{k_{ij}}{T}\right)$$

$$M^2 \circ F = -2\frac{T}{dT}F|_{T=1}$$  \hspace{1cm} (85)

This definition is in accordance with the idea that mass squared operator, or essentially Virasoro generator $L_0$, measures the action of an infinitesimal scaling: now the scaling acts on modular parameters rather than complex coordinates. Since the contribution of modular degrees of freedom changes only the vacuum weight of Super Virasoro representation there is no necessity to define the action of $L_n$, $n \neq 0$ on vacuum functional.

Quantum mechanical and thermal mass expectation values are identical provided one identifies thermal partition function $Z$ in suitable manner

$$\langle M^2 \rangle_{qu} = \frac{\int dV \Omega M^2 \circ \Omega}{\int dV \Omega^2} = -2\frac{T}{dT}Z|_{T=1}$$

$$Z(T) \equiv \int dV \Omega^2(k_{ij}, T)$$  \hspace{1cm} (86)

Integration measure $dV$ corresponds actually to the summation over the nonequivalent points of the moduli space, one point for each value of $\det(k_{ij})$.

It remains to be shown that the action of mass squared operator on theta function is modular invariant. A straightforward calculation shows that the lowest order contribution to mass squared expectation value is what it is expected to be. Consider torus as an example. Even theta functions correspond to characteristics $[a, b] = [1/2, 0], [0, 1/2]$ and $[0, 0]$ and their explicit expressions read as

$$\Theta[1/2, 0](p^k) = p^{k/4} \sum_n p^{k(n^2 + n/2)}$$

$$\Theta[0, 1/2] = p^{k/4} \sum_n p^{kn^2} (-1)^n$$

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\[ \Theta[0,0] = p^{k/4} \sum_n p^{kn^2} \]

(87)

Notice that one must have \( k > 0 \) for the allowed powers of \( p \). A little calculation shows that leading order term in vacuum functional reads as

\[ \Omega_{\text{vac}} \simeq K(1 + 3p^k) \]
\[ K = 27 \]

(88)

Clearly the \( p^k = p \) of moduli space gives the dominating contribution to mass squared expectation. In principle there is \( O(p^{k/2}) \) term present but the contributions coming from numerator and denominator cancel. The coefficient \( K \) doesn’t affect the expectation value and lowest order contribution to mass squared can be read directly from the coefficient of \( O(p) \) term and one has \( M^2 = 6 = 2N(g)g \). Factor 2 comes from the definition of \( M^2 \) operator.

Mass expectation is of order \( O(p) \) for \( T = 1 \) unless the point of moduli space giving dominating contribution happens to be point with conformal symmetry: in this case the contribution to mass squared operator becomes \( O(1) \) since the integral in denominator is of order \( O(p) \) in this case. There is extremely rapid converge with respect to powers of \( p \) and the calculation of thermal average is a simple task. One can even define what \( T = 1/n \) expectation value means by restricting the points of discrete moduli space so that \( k_{ij} \) is multiple of \( n \).

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