COASSOCIATIVE K3 FIBRATIONS OF COMPACT $G_2$-MANIFOLDS

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Abstract. A class of examples of Riemannian metrics with holonomy $G_2$ on compact 7-manifolds was constructed by the author in [Ko] using a certain ‘generalized connected sum’ of two asymptotically cylindrical manifolds with holonomy $SU(3)$. We consider, on each of the two initial $SU(3)$-manifolds, a fibration arising from a Lefschetz pencil of K3 surfaces. The gluing of two such K3 fibrations yields a coassociative fibration of the connected sum $G_2$-manifold over a 3-dimensional sphere. The singular fibres of this fibration are diffeomorphic to K3 orbifolds with ordinary double points and are parameterized by a Hopf-type link in $S^3$. We believe that these are the first examples of fibrations of compact manifolds of holonomy $G_2$ by coassociative minimal submanifolds.

Introduction

It is well-known that if a K3 surface contains an elliptic curve then this elliptic curve has self-intersection zero and is a fibre of a holomorphic fibration of the K3 surface over the Riemann sphere. The self-intersection number is calculated by the adjunction formula and the elliptic fibration is defined by a pencil containing the elliptic curve (see [Ha, Propn. 11.1]). All but finitely many fibres of this fibration are smooth and are minimal submanifolds with respect to a Kähler metric on the ambient K3 surface. Moreover, by the Wirtinger theorem the fibres are volume-minimizing in their homology class. Harvey and Lawson [HL] generalized the volume-minimizing property of complex submanifolds of Kähler manifolds by developing a concept of calibrated minimal submanifolds of Riemannian manifolds. One type of examples of calibrated submanifolds found in [HL] is the coassociative submanifolds. These are four-dimensional submanifolds which occur in the Euclidean $\mathbb{R}^7$ and, more generally, in 7-dimensional Riemannian manifolds with holonomy contained in the Lie group $G_2$ (see [H] for precise definitions).

McLean [McL] studied the deformation theory for calibrated submanifolds. In particular, he showed that local deformations of a smooth compact coassociative submanifold $X$ are always unobstructed and the corresponding ‘moduli space’ is a smooth manifold of dimension $b^+(X)$, the positive index of the cup-product on $H^2(X)$. (This corresponds to the negative index in the actual statement of [McL Theorem 4.5] as we use a different sign convention for the $G_2$-structures.) If $b^+(X) = 3$ then, under some reasonable additional assumptions, the local coassociative deformations of $X$ give a local foliation. This naturally leads to a conjecture that some compact 7-manifolds with holonomy $G_2$ are fibred by coassociative submanifolds, possibly with singular fibres. Joyce [Jo1] [Jo2] constructed first examples of compact 7-manifolds with holonomy $G_2$, or $G_2$-manifolds; his results include examples of compact coassociative calibrated submanifolds obtained by carefully chosen
orientation-reversing involutions. Later the author gave a series of topologically different examples of compact $G_2$-manifolds using a ‘generalized connected sum’ construction \[\text{Ko},\] see also \[\text{KLe}.\] It is also shown in \[\text{KN}.\] that one of the $G_2$-manifolds obtainable as in \[\text{KLe}.\] is a deformation of a $G_2$-manifold constructed in \[\text{Jo1}.\] The purpose of this paper is to construct coassociative fibrations of the $G_2$-manifolds obtained in \[\text{Ko, KLe}.\] A generic fibre of these fibrations is diffeomorphic to a K3 surface. To the author’s knowledge, these are the first examples of fibrations of compact smooth manifolds with holonomy $G_2$ by coassociative minimal submanifolds.

Further motivation for this paper comes from similarities between coassociative submanifolds of $G_2$-manifolds and special Lagrangian submanifolds of Calabi–Yau manifolds. Special Lagrangian submanifolds are another instance of calibrated submanifolds appearing in \[\text{HL}.\] with unobstructed deformation theory established in \[\text{McL}.\] There are further similarities. For both types of compact calibrated submanifolds, the ‘number of moduli’ of the local deformations is given by some Betti number of the submanifold. Special Lagrangian submanifolds attracted much interest in recent years in connection with the SYZ conjecture \[\text{SYZ}.\] which explains mirror symmetry between Calabi–Yau threefolds. The SYZ conjecture was motivated by studies in string theory and inspired works on the mirror symmetry for $G_2$-manifolds admitting coassociative fibrations \[\text{Ac, GYZ}.\] A part of the SYZ conjecture asserts that some Calabi–Yau threefolds whose complex structures are close to a certain degenerate limit (the so-called large complex structure limit) admit fibrations by special Lagrangian tori. This may be compared to the requirement that in order to construct coassociative fibrations in this paper we assume that the torsion-free $G_2$-structure on a compact 7-manifold $M$ corresponds to a point near the boundary of the moduli space for torsion-free $G_2$-structures. (This boundary point is represented by a pair of asymptotically cylindrical manifolds used in the construction of $M$; this is made precise in \[\text{N, \S\ 5}.\]) More explicitly, a connected sum $G_2$-manifold should have a ‘sufficiently long neck’ (as measured by the parameter $T$ in the Main Theorem).

On the other hand, there is a difference concerning properties of the singular fibres. It was shown in \[\text{Jo3}.\] that generic fibrations of Calabi–Yau threefolds by special Lagrangian tori are piece-wise smooth and only continuous along the loci of the singular fibres which occur in families of codimension one. The coassociative fibrations constructed here are smooth and are continuously differentiable (more precisely, $C^{1,\alpha}$) at the singular fibres. The fibres develop singularities modelled on ordinary double points in the holomorphic deformation families of K3 surfaces, and the singular fibres occur in families of codimension two, parameterized by a Hopf-type link in $S^3$. This type of coassociative fibration survives under small perturbations of the ambient $G_2$-structure, defined by closed 3-forms.

The latter property is crucial as our construction of the coassociative fibrations relies on perturbative analysis, in the form of implicit function theorem in Banach spaces. In fact, we build up on the gluing construction of $G_2$-metrics in \[\text{Ko}.\] and initially construct an ‘approximating fibration’ whose fibres are coassociative with respect to a $G_2$-structure with small torsion. Some familiarity with \[\text{Ko}.\] is therefore required and we include in \[\S2.\] a review of the relevant details before stating the main result and explaining the strategy of the proof in \[\S3.\] The strategy requires, as a preliminary step, an extension of McLean’s
deformation theory to coassociative K3 orbifolds which we carry out in §4. Appropriate
definition and stability result was recently proved by Lotay [Lo3]. With the deformation
set-up in place, we proceed to an implicit function argument for coassociative K3 orbifold
fibres in §6 and complete the construction in §7 which deals with the smooth fibres.

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1. $G_2$-Structures and Coassociative Submanifolds

We give a short summary of some standard results concerning the Riemannian geometry
associated with the group $G_2$ and coassociative submanifolds associated with the
$G_2$-structures. The results given in this section are mostly gathered from [Br, HL, Jo2,
McL, Sa] and the reader is referred there for further details.

The group $G_2$ may be defined as the group of automorphisms of the cross-product algebra
on $\mathbb{R}^7$ interpreted as the space of pure imaginary octonions. As the octonions form
a normed algebra, any automorphism in $G_2$ necessarily preserves the Euclidean metric and
orientation of $\mathbb{R}^7$. The cross-product can be encoded as an anti-symmetric 3-linear form $\varphi_0$
on $\mathbb{R}^7$ defined by $\varphi_0(a, b, c) = (a \times b, c)$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product.
The 3-form $\varphi_0$ and its Hodge dual $*\varphi_0$ are explicitly expressed as

$$
\varphi_0 = e_{123} + e_{145} + e_{167} + e_{246} - e_{257} - e_{347} - e_{356},
$$
$$
*\varphi_0 = e_{4567} + e_{2367} + e_{2345} + e_{1357} - e_{1346} - e_{1256} - e_{1247},
$$

(1.1)

where $e_{i_1\ldots i_k} = e_{i_1} \wedge \ldots \wedge e_{i_k}$ and $e_i \in (\mathbb{R}^7)^*$, $i = 1, \ldots, 7$ is the standard orthonormal
co-frame. The group $G_2$ may be equivalently defined as the subgroup of the orientation-
preserving linear isomorphisms $GL_+(7, \mathbb{R})$ fixing $\varphi_0$ or the subgroup of $GL_+(7, \mathbb{R})$ fixing
$*\varphi_0$, in the action on, respectively, 3- or 4-forms. The group $G_2$ is a compact Lie group of
dimension 14 and thus a proper subgroup of $SO(7)$.

Let $M$ be a smooth oriented 7-manifold. Denote by $\Omega^3_+(M)$ the subset of 3-forms $\varphi$
on $M$ whose value $\varphi(p)$ at any point $p \in M$ can be identified with $\varphi_0$ via some orientation-
preserving linear isomorphism $T_p M \to \mathbb{R}^7$. Any $\varphi \in \Omega^3_+(M)$ induces a $G_2$-structure on $M$;
we shall sometimes, slightly inaccurately, say that $\varphi$ is a $G_2$-structure. The $GL(7, \mathbb{R})$-orbit
of $\varphi_0$ is open in $\Lambda^3(\mathbb{R}^7)^*$, so the subset $\Omega^3_+(M) \subset \Omega^3(M)$ is open in the uniform norm
topology. As $G_2 \subset SO(7)$ any $\varphi \in \Omega^3_+(M)$ induces a metric $g(\varphi)$ and an orientation of $M$,
so that the image of $e_1, \ldots, e_7$ via the isomorphisms $T_p M \to \mathbb{R}^7$ identifying $\varphi_p$ with $\varphi_0$ is
an orthonormal positive co-frame at each $p \in M$. The 4-form $*_{\varphi} \varphi$ on $M$ defined using the
Hodge star of the metric $g(\varphi)$ is, in a similar sense, point-wise isomorphic to $*\varphi_0$ in (1.1).

The metric $g(\varphi)$ induced by a $G_2$ structure $\varphi \in \Omega^3_+(M)$ will have holonomy contained
in $G_2$ if and only if

$$
d\varphi = 0, \quad d*_{\varphi} \varphi = 0,
$$

(1.2)

the $G_2$-structure $\varphi$ then is said to be torsion-free. Conversely, if the holonomy of a metric
$g$ is contained in $G_2$ then there is a $\varphi \in \Omega^3_+(M)$ satisfying (1.2) so that $g = g(\varphi)$. [Sa,
Lemma 11.5] The pair $(M, \varphi)$ then is called a $G_2$-manifold and a metric $g(\varphi)$ induced by
a torsion-free $G_2$-structure on a 7-manifold $M$ is called a $G_2$-metric. On a compact $M$, the holonomy of a $G_2$-metric is exactly $G_2$ if and only if the fundamental group of $M$ is finite [Jo2, Propn. 10.2.2].

Now we recall the concept of coassociative submanifolds. Any 4-tuple of orthogonal unit vectors $v_1, v_2, v_3, v_4 \in \mathbb{R}^7$ satisfies the inequality

$$\ast \varphi_0(v_1, v_2, v_3, v_4) \leq 1.$$  

(1.3)

If the equality in attained in (1.3) then the oriented 4-dimensional subspace of $\mathbb{R}^7$ defined by $v_1, v_2, v_3, v_4$ is called a coassociative subspace. More generally, if $M$ is a 7-manifold and $\varphi \in \Omega^3_+(M)$ is a $G_2$-structure form then any oriented 4-dimensional submanifold $X \subset M$ satisfies

$$\ast \varphi \varphi|_X \leq \text{vol}_X,$$  

(1.4)

in the sense that $\ast \varphi \varphi|_X = \alpha \text{vol}_X$ for some real constant $\alpha \leq 1$, where $\text{vol}_X$ is the volume form of the metric on $X$ induced by the embedding.

**Proposition 1.5** (cf. [HL, Cor. IV.1.20]). For an orientable 4-dimensional submanifold $X$ of a 7-manifold $M$ with a $G_2$-structure $\varphi \in \Omega^3_+(M)$, the equality $\ast \varphi \varphi|_X = \text{vol}_X$ is attained in (1.4) for some (necessarily unique) orientation of $X$ if and only if $\varphi|_X = 0$.

If a $G_2$-structure on $M$ satisfies $d \ast \varphi \varphi = 0$ then the 4-form $\ast \varphi \varphi$ is an instance of a calibration, called coassociative calibration, and a 4-dimensional submanifold $X$ with $\ast \varphi \varphi|_X = \text{vol}_X$ is then said to be calibrated by $\ast \varphi \varphi$. Any compact calibrated submanifold minimizes the volume in its homology class, in particular a compact $X$ calibrated by $\ast \varphi \varphi$ is a minimal submanifold of the Riemannian manifold $(M, g(\varphi))$.

The $G_2$ structures that we deal with in this paper will always be defined by closed 3-forms. The following terminology will be in use. A 4-dimensional submanifold $X \subset M$ will be called coassociative, with respect to a $G_2$-structure $\varphi \in \Omega^3_+(M)$, if $\varphi|_X = 0$. If, in addition, this $G_2$-structure satisfies $d \ast \varphi \varphi = 0$ (and so $\ast \varphi \varphi$ is a calibration) then we say that $X$ is a coassociative calibrated submanifold.

A remarkable property of compact coassociative submanifolds is that their local deformations are unobstructed, a result due to McLean [McL]. It uses a ‘tubular neighbourhood theorem’ (e.g. [La, Ch. IV]), which may be stated as follows.

**Theorem 1.6.** For any embedded submanifold $X$ of a Riemannian manifold $(M, g_M)$, the exponential map of $g_M$,

$$v(p) \in (N_{X\subset M})_p \subset T_p M \mapsto \exp_{v(p)} \in M, \quad p \in X,$$  

induces a diffeomorphism of a neighbourhood

$$\{v(p) \in N_{X\subset M} : |v| < C_X \min \{\delta(p), |II(p)|\} \text{ if } v(p) \in T_p M, p \in X\}$$

of the zero section of the normal bundle $N_{X\subset M}$ onto a neighbourhood of $X$ in $M$. Here $\delta(p)$ denotes the injectivity radius of $g_M$ and $II(p)$ the second fundamental form at $p$, $C_X > 0$ is a constant depending only on $X$. 
As $\delta(p)$ depends continuously on $p \in X$ it attains a maximum $\delta > 0$ when $X$ is compact. In that case, the inequality in Theorem 1.6 defining a tubular neighbourhood can be simplified to $|v(p)| < \delta$.

Thus any local deformation of compact coassociative $X$ can be written as $X_v = \exp_v(X)$, for some $v \in \Gamma(N_{X\subset M})$ with small $||v||_{C^0}$ in the metric induced from $M$. Suppose that $\varphi|_X = 0$. A local deformation $X_v$ of $X$ will be coassociative if and only if $F(v) = 0$, where

$$F : v \in \Gamma(N_{X\subset M}) \rightarrow F(v) = (\exp_v)^*\varphi \in \Omega^3(X), \tag{1.7}$$

The normal bundle $N_{X\subset M}$ of a coassociative submanifold is canonically isomorphic and isometric to the bundle of self-dual 2-forms $\Lambda^+T^*X$ via

$$v \in \Gamma(N_{X/M}) \rightarrow v \downarrow \varphi|_X \in \Omega^+(X). \tag{1.8}$$

(The $G_2$ structure 3-forms $\varphi$ used in [McL] differ from (1.1) by the opposite sign, which leads to $\Omega^+(X)$ rather than $\Omega^+(X)$ in the right-hand side of (1.8).) Composing $F$ with the inverse of (1.8), we obtain a map

$$\tilde{F} : \Omega^+(X) \rightarrow \Omega^3(X).$$

McLean proves:

**Theorem 1.9** (cf. [McL], §4). Suppose that a $G_2$-structure on a 7-manifold $M$ is given by a closed 3-form and $X \subset M$ is a coassociative submanifold. Then:

(a) the derivative $(d\tilde{F})_0$ is given by the exterior derivative $d : \Omega^+(X) \rightarrow \Omega^3(X)$, and

(b) the image of $\tilde{F}$ consists of exact forms.

**Remarks.** The statement of Theorem 1.9 extracts a part of McLean’s results which does not require the compactness of $X$. The result is stated in [McL] for coassociative calibrated submanifolds but it was later observed in [Go] that the condition $d^*\varphi = 0$ is not used in the proof.

If $X$ is compact then any exact 3-form on a compact oriented Riemannian 4-manifold is the differential of a self-dual form and Theorem 1.9 sets the scene for an application of the implicit function theorem in Banach spaces. It follows that the local deformations of a compact coassociative $X$ form a smooth manifold of dimension equal to the dimension $b^+(X)$ of harmonic self-dual forms on $X$, see [McL] Theorem 4.5 or [JS, Theorem 2.5].

Bryant [Br] proved that any closed real-analytic oriented Riemannian 4-manifold $X$ with trivial $\Lambda^+T^*X$ arises as a coassociative calibrated submanifold in some manifold with torsion-free $G_2$-structure. If there are three harmonic self-dual forms on $X$ that are linearly independent at every point then Bryant’s result produces examples of non-compact (local) $G_2$-manifolds foliated by the coassociative deformations of $X$.

Finally, an application of McLean’s theory shows that compact coassociative submanifolds are ‘stable’ under small deformations of the $G_2$ structure. The property will be crucial for the main results of this paper, the following theorem provides an introduction.

**Theorem 1.10** (cf. [Jo7, Theorem 12.3.6]). Suppose that $\varphi(s) \in \Omega^3(M)$, $s \in \mathbb{R}$, is a smooth path of closed $G_2$-structure forms on $M$, and $X$ is a compact submanifold of $M$ such that $\varphi(0)|_X = 0$ and the form $\varphi(s)|_X$ is exact for any $s$. Then there there is an $\varepsilon > 0$
and for each \(|s| < \varepsilon\) a section \(v(s)\) of \(N_{X/M}\) smoothly depending on \(s\), such that \(v(0) = 0\) and \(\varphi(s)\) vanishes on \(\exp_{v(s)}(X)\).

It is worth to point out the different roles of the two equations in (1.2): the second equation ensures that compact coassociative submanifolds are minimal, whereas the first equation ensures that they have a good deformation theory (Theorems 1.9 and 1.10).

Moreover, it is noted in [Br, 0.3.3] that the generic \(G_2\)-structure satisfying \(d\ast\varphi = 0\) (but not necessarily \(d\varphi = 0\)) will not admit any coassociative submanifolds.

2. FROM PENCILS OF K3 SURFACES TO THE APPROXIMATING FIBRATIONS

The method of construction of compact irreducible \(G_2\) manifolds that we shall consider was developed in [Ko]. Because some technical details of this construction will be important in what follows we shall review these details here. We shall also deduce some immediate consequences concerning coassociative submanifolds which are not given in [Ko].

Let \(W\) be a Ricci-flat Kähler complex threefold and \(\omega\) the Kähler form on \(W\). If the holonomy of the Kähler metric is contained in \(SU(3)\) (which will be the case e.g. if \(W\) is simply-connected) then there is a nowhere-vanishing holomorphic \((3,0)\)-form \(\Omega\) on \(W\), sometimes called a holomorphic volume form. We shall call the pair \((\omega, \Omega)\) a Calabi–Yau structure on \(W\). A Calabi–Yau structure \((\omega, \Omega)\) induces a torsion-free \(G_2\)-structure on the 7-manifold \(W \times S^1\) given by the 3-form

\[
\varphi_{CY} = \omega \wedge d\theta + \text{Im} \Omega,
\]

(2.11)

where \(\theta\) is the standard ‘angle coordinate’ on \(S^1\) (cf. [Jo2, Propn. 11.1.2], our holomorphic volume form differs from the one used there by the factor \(i\)). The form \(\varphi_{CY}\) induces the product metric \(g_W + d\theta^2\) on \(W \times S^1\) corresponding to the Ricci-flat Kähler metric \(g_W\) on \(W\) and we shall sometimes refer to \(\varphi_{CY}\) as a product \(G_2\)-structure.

If \(X\) is a complex surface in \(W\) then, for any \(\theta_0 \in S^1\),

\[
\varphi_{CY}|_{X \times \{\theta_0\}} = \text{Im} \langle \Omega|_X \rangle = 0,
\]

so \(X \times \{\theta_0\}\) is a coassociative calibrated submanifold of \(W \times S^1\). [MCL, Jo2]

Let \(V\) be a (non-singular) compact complex threefold with \(c_1(V) > 0\), i.e. a Fano threefold, and \(D \in |-K_V|\) a K3 surface in the anticanonical linear system of \(V\). Let \(D' \in |-K_V|\), \(D' \neq D\), be another K3 surface in the anticanonical class, so that \(C = D' \cap D\) is a non-singular connected curve in \(V\). Blowing up \(C\) we obtain a new threefold \(\tilde{V}\) with a holomorphic map \(\tau: \tilde{V} \to \mathbb{CP}^1\) whose fibres are proper transforms of the surfaces in the pencil defined by \(D\) and \(D'\).

The proper transform \(\tilde{D}\) of \(D\) is an anticanonical divisor on \(\tilde{V}\) and the complement non-compact complex threefold \(V = \tilde{V} \setminus \tilde{D}\) has trivial canonical bundle. We can define a holomorphic coordinate, \(\zeta\) say, on \(\mathbb{CP}^1\), so that \(D = \tau^{-1}(0)\). The fibre \(\tau^{-1}(\zeta)\) is diffeomorphic to \(\tilde{D}\) as a real 4-manifold whenever \(|\zeta|\) is sufficiently small, thus the real 6-manifold underlying \(W\) has a cylindrical end \(\tau^{-1}(\{0 < |\zeta| < \varepsilon\})\) diffeomorphic to \(\mathbb{R}_{>0} \times S^1 \times D\). Denote the real coordinates on the first two factors by \(t, \theta\) and then \(\zeta = e^{-t-i\theta}\). The complex structure on the end of \(W\) is asymptotic, as \(t \to \infty\), to the product complex
structure on $\mathbb{R}_{>0} \times S^1 \times D$, where $D$ is considered with the complex structure and Kähler form induced by the embedding in $V$. Note that the latter condition on the Kähler form is not restrictive: every Kähler metric on $D$ is obtainable as a restriction of some Kähler metric on $V$. \cite{Ko1} \cite{KLe1}

Another class of complex threefolds with similar properties was recently constructed in \cite{KLe1}. It uses K3 surfaces with non-symplectic involution.

Recall that by Yau’s solution of the Calabi conjecture \cite{Y} the Kähler K3 surface $D \subset V$ has a unique Ricci-flat Kähler metric in its Kähler class. We shall write $\kappa_I$ for the Kähler form of this metric and $\kappa_J + i\kappa_K$ for a holomorphic volume form on $D$. The following ‘non-compact version of the Calabi conjecture’ for $W$ is proved in \cite{Ko1} using the results of \cite{TY}.

Theorem 2.12 (\cite{Ko1}, §§2–3, 6). The threefold $W$ is simply-connected and has a complete Ricci-flat Kähler metric $g_W$ of holonomy $SU(3)$. This metric is asymptotically cylindrical in the sense that on the cylindrical end $\tau^{-1}\{0 < |\zeta| < \varepsilon\} \subset W$ the Kähler form $\omega$ of $g_W$ has an asymptotic expression

$$\omega|_{\tau^{-1}\{0 < |\zeta| < \varepsilon\}} = dt \land d\theta + \kappa_I + d\psi,$$

and there is a holomorphic volume form $\Omega$ on $W$ with an asymptotic expression

$$\Omega|_{\tau^{-1}\{0 < |\zeta| < \varepsilon\}} = (dt + i\theta) \land (\kappa_J + i\kappa_K) \land d\Psi,$$

where $\zeta = e^{-t-i\theta}$ and the differential forms $\psi, \Psi$ and all their derivatives decay at the rate $O(e^{-\lambda_W t})$ along the end of $W$, with the exponent $\lambda_W > 0$ depending only on the Kähler metric $\kappa_I$ on $D$.

For any $W$ satisfying the assertion of Theorem 2.12, the 7-manifold $W \times S^1$ has a cylindrical end $\mathbb{R}_{>0} \times S^1 \times D \times S^1$ and the product $G_2$-structure $\varphi_{CY}$ on $W \times S^1$ has an asymptotic expression

$$\varphi_{CY}|_{\tau^{-1}\{0 < |\zeta| < \varepsilon\} \times S^1} = \varphi_D + d\tilde{\psi},$$

where

$$\varphi_D = dt \land d\theta \land d\theta' + \kappa_I \land d\theta' + \kappa_J \land d\theta + \kappa_K \land dt,$$

$$\tilde{\psi} = \psi \land d\theta' + \text{Im} \Psi,$$

and $d\theta'$ denotes the standard non-vanishing 1-form on the last $S^1$ factor. Note that the asymptotic model $\varphi_D$ is determined by the choice of the K3 divisor $D \in |-K_V|$ alone and does not depend on the choice of $D'$ and the resulting pencil on $V$.

Proposition 2.14. For a generic choice of Fano threefold $V$ in its deformation family and a generic choice of $D' \in |-K_V|$, the fibres of the map $\tau : \widetilde{V} \to \mathbb{C}P^1$ define a ‘generic Lefschetz fibration’ in the following sense:

(1) the critical points of $\tau$ are non-degenerate (Morse points): if $d\tau(w) = 0$ then the Hessian of $\tau$ at $w$ is non-singular, and

(2) any fibre $\tau^{-1}(\zeta)$ contains at most one critical point of $\tau$. 
Proof. This is an application of [Ko] and some standard results in algebraic geometry and we only give an outline of the proof. A generic anticanonical divisor on $V$ is a smooth surface of type K3 [Sh]. The K3 surfaces arising as smooth anticanonical divisors on the deformations of $V$ form a Zariski open subset in a moduli space $\mathcal{K}$ of K3 surfaces whose Picard lattice contains a copy of $H^2(V,\mathbb{Z})$ as a sublattice [Ko] §7. The moduli space $\mathcal{K}$ is a quasiprojective complex algebraic variety of dimension $20 - b^2(V)$. The degenerations of the K3 surfaces in $\mathcal{K}$ developing an ordinary double point are generic. The anticanonical linear system $| - K_V|$ is parameterized by $\mathbb{C}P^N$ and an application of the Riemann–Roch theorem gives the dimension $N = -K_V^3/2 + 2$, so $N \geq 3$ [Is] Propn. 1.3.

The singular anticanonical divisors on $V$ are therefore parameterized by an algebraic subvariety $S$ of $\mathbb{C}P^N$ of codimension at least 1 and one can show that for a generic Fano threefold in the deformation family of $V$ any connected component of $S$ contains a K3 orbifold with the only singularity an ordinary double point. Each of the conditions (1) and (2) in Proposition 2.14 is an open condition in the Zariski topology of $S$. Violation of (1) or (2) defines a further subvariety of positive codimension in $S$. This latter subvariety therefore has codimension at least 2 in $\mathbb{C}P^N$ and can be avoided in the pencil through $D$ and $D'$ with a generic choice of $D'$.

If $w_0 \in W$ is a critical point of $\tau$ and the Hessian of $\tau$ at $w_0$ is non-degenerate then by the Morse lemma there is a system of local holomorphic coordinates $z_j$ near $w_0$, such that $\tau = z_1^2 + z_2^2 + z_3^2$ in these coordinates. The fibre $X_0$ through $w_0$ is an orbifold with an isolated singularity $z_1^2 + z_2^2 + z_3^2 = 0$ at $w_0$. (For a general theory of orbifolds see [B].) A neighbourhood of $w_0$ in $X_0$ admits a local parameterization by uniformizing coordinates $u_1, u_2 \in \mathbb{C}^2/\pm 1$,

$$ (u_1, u_2) \mapsto (i(u_1^2 + u_2^2), u_1^2 - u_2^2, 2u_1u_2), $$

(2.15)

so this neighbourhood is homeomorphic to a cone on $\mathbb{R}P^3$.

A K3 fibration $\tau$ has only finitely many singular fibres. Every singular fibre $X_0$ of a generic $\tau$ (in the sense of Proposition 2.14) is an ‘orbifold K3 surface’ with unique singularity which is an ordinary double point and $\tau$ near $X_0$ defines a one-parameter family of non-singular deformations of $X_0$. On the other hand, it is well-known (e.g. [As] §2.6) that by blowing up $w_0$ one achieves a resolution of the ±1 orbifold singularity of $X_0$ which is again a K3 surface (in general, not isomorphic to any nearby fibre of $\tau$) and the exceptional divisor is a complex curve with self-intersection $-2$.

The number $\mu_V$ of singular fibres of a generic $\tau$ is the number of K3 orbifolds in the corresponding generic Lefschetz pencil on $V$. This number is calculated by an application of Lefschetz theory of hyperplane sections to the topology of algebraic varieties [AF] §5,

$$ \mu_V = 2\chi(D) - \chi(C) - \chi(V) = 48 + (-K_V^3) - \chi(V) $$

(2.16)

where $C$ is the blow-up locus (the axis of the pencil of K3 surfaces in $V$), so $-\chi(C) = -K_V^3$ for a Fano threefold $V$ [Is] Propn. 1.6.

We may assume, rescaling $\zeta$ by a non-zero constant factor if necessary, that there are no singular fibres of $\tau$ on the cylindrical end $\{ t > 0 \} \subset W$, where $t = -\log |z|$ as before. Extend $t$ to a smooth function, still denoted by $t$, defined on all of $W$ with $t < 0$ away from
the cylindrical end. Fix once and for all a smooth cut-off function \( \alpha(s) \) with \( \alpha(s) = 0 \), for \( s \leq 0 \), and \( \alpha(s) = 1 \), for \( s \geq 1 \). The 3-form
\[
\varphi_{W,T} = \varphi_{CY} - d(\alpha(t - T)\tilde{\psi})
\]  
(2.17)
defines a \( G_2 \)-structure on \( W_T \times S^1 \) for every sufficiently large \( T \). It interpolates between the product torsion-free \( G_2 \)-structure \( \varphi_{CY} \) on \( W \times S^1 \) induced by the Ricci-flat Kähler structure on \( W \) and the product torsion-free \( G_2 \)-structure \( \varphi_D \) on the half-cylinder \([T − 1, \infty) \times S^1 \times D \times S^1 \) induced by the Calabi–Yau (hyper-Kähler) structure forms \( \kappa_I, \kappa_J, \kappa_K \) on \( D \) as in (2.13).

**Proposition 2.18.** For any \( T > 0 \), the 3-form \( \varphi_T \) vanishes on each fibre of the map \( \tau \times \text{id}_{S^1} : W \times S^1 \to \mathbb{C}P^1 \times S^1 \).

**Proof.** It is clear that the claim is true away from the cut-off region \( R = [T − 1, T] \times S^1 \times D \times S^1 \) because the map \( \tau \) is holomorphic both in the complex structure of \( W \) and in the product complex structure of \( \mathbb{R} \times S^1 \times D \) on the end of \( W \).

On the cut-off region, we have
\[
\varphi_T|_R = (1 - \alpha)\varphi_{CY} + \alpha\varphi_D - \alpha' dt \wedge \tilde{\psi}
\]
and the claim follows as any fibre of \( \tau \) on the end of \( W \times S^1 \) is contained in a level set \( \{ t = \text{const} \} \).

Now let \( W_1 \) and \( W_2 \) be two asymptotically cylindrical Calabi–Yau threefolds given by Theorem 2.12 and define \( W_j(T) = W_j \times \{ t_j > 0 \} \), for any \( T > 2, j = 1, 2 \). Assume that the respective two hyper-Kähler K3 surfaces \( D_j \) are ‘hyper-Kähler rotations’ of each other which means that there is an isometry \( f : D_1 \to D_2 \) of the Riemannian 4-manifolds such that \( f^* \) always exists after some deformations of the Fano threefolds \( V_1, V_2 \) used in the construction of \( W_1, W_2 \) [Ko, Theorem 6.44]. See also [KL, Theorem 5.3]

Construct a compact 7-manifold \( M \) by joining \( W_1(T) \) and \( W_2(T) \)
\[
M = (W_1(T) \times S^1) \cup_T (W_2(T) \times S^1),
\]
(2.19a)
identifying collar neighbourhoods of the boundaries via the orientation-preserving diffeomorphism
\[
\Upsilon : (y, \theta_1, \theta_2, T + t) \in D_1 \times S^1 \times S^1 \times [T + 1, T + 2] \mapsto (f(y), \theta_2, \theta_1, T + 3 - t) \in D_2 \times S^1 \times S^1 \times [T + 1, T + 2]
\]  
(2.19b)
Then \( \Upsilon^* \varphi_{D_2} = \varphi_{D_1} \), so the two \( G_2 \)-structures \( \varphi_{j,T} \in \Omega^3_+(W_j(T) \times S^1) \) defined by (2.17) agree on the overlap and together give a well-defined \( G_2 \)-structure \( \varphi_T \in \Omega^3_+(M) \) on the compact 7-manifold. This latter \( G_2 \)-structure form satisfies
\[
d\varphi_T = 0, \quad ||d^* \varphi_T||_{L^p_k} < C_{p,k}e^{-\lambda T},
\]
(2.20)
for each \( T > 2 \), where \( \lambda < \lambda_{W_j}, j = 1, 2 \), and \( \lambda_{W_j} \) are defined in Theorem 2.12.
Theorem 2.21 ([Ko], §5). There exists $T_0$ and for each $T > T_0$ a 2-form $\eta = \eta_T$ on the 7-manifold $M$ satisfying $\|\eta_T\|_{L^p_k} < K_{p,k} e^{-\lambda T}$ and such that
$$d * (\varphi_T + d\eta_T) = 0,$$
with the Hodge star $*$ in the above formula defined by the metric $g(\varphi_T + d\eta_T)$. Thus the 3-form $\varphi_T + d\eta_T$ induces a torsion-free $G_2$-structure on $M$ and the holonomy group of the metric $g(\varphi_T + d\eta_T)$ is $G_2$.

The last claim is true because $M$ is simply-connected. The 3-form $\varphi_T$ may be thought of as an approximation, improving as $T \to \infty$, of a torsion-free $G_2$-structure on $M$.

Return, for the moment, to the $G_2$-structures $\varphi_{j,T}$ on the two pieces of $M$. It is clear from the construction (2.19) of $M$ that the fibrations $\tau^{(j)}$ of $W_j(T) \times S^1$ agree on the overlap and hence can be patched to define, for any $T > 1$ a fibration $\tau_T$ of $M$ as shown in the following commutative diagram
$$
\begin{array}{ccc}
W_{1,T} \times S^1 \cup W_{2,T} \times S^1 & \longrightarrow & M \\
\tau^{(1)} & \downarrow \tau^{(2)} & \downarrow \tau_T \\
\Delta \times S^1 \cup \Delta \times S^1 & \longrightarrow & S^3,
\end{array}
$$

where the bottom row is a well-known splitting of the 3-sphere into two solid tori ($\Delta$ denotes a disc in $\mathbb{C}P^1$). We obtain from (2.22) and Proposition 2.18 the main result of this section.

Theorem 2.23. Let $(M, \varphi_T)$ be a compact 7-manifold with a $G_2$-structure constructed from a pair of Fano threefolds, as defined above. Then $\tau_T : M \to S^3$ is a coassociative fibration defined by (2.22), with respect to $\varphi_T$. The singular fibres of $\tau_T$ form a subset of codimension 2 in $M$ and are projected by $\tau$ onto a link in $S^3$.

This link consists of $\mu_{V_1}$ (disjoint) circles in $S^3 \setminus \tau_T(W_1(T) \times S^1)$ and $\mu_{V_2}$ (disjoint) circles in $S^3 \setminus \tau_T(W_2(T) \times S^1)$, where $\mu_{V_j}$ is defined in (2.16). Each circle in $S^3 \setminus \tau_T(W_j(T) \times S^1)$ is linked, with linking number 1, with each circle in the other subset $S^3 \setminus \tau_T(W_{3-j}(T) \times S^1)$ and is not linked with any circle in $S^3 \setminus \tau_T(W_j(T) \times S^1)$.

Of course, the 4-form $*_{\varphi_T} \varphi_T$ is not in general a calibration on $M$ and the fibres of $\tau_T$ are not necessarily calibrated by $*_{\varphi_T + d\eta_T} (\varphi_T + d\eta_T)$. The estimate on the 2-form $\eta_T$ given in Theorem 2.21 yields an upper bound on $d\eta_T|_X$, for each non-singular fibre $X$ of $\tau_T$,
$$
\|d\eta_T|_X\|_{L^p_k} < \tilde{K}_{p,k}(X)e^{-\lambda T},
$$
for each $T > T_0$. The constant $\tilde{K}_{p,k}(X)$ in (2.20) depends on a particular choice of norm (more precisely, on the value of $k - 4/p$) as well as on the choice of $X$. We shall return to this later, in Proposition 7.64.

3. The gluing theorem for coassociative K3 fibrations

The estimate (2.20) on the failure of the fibres of $\tau_T$ to be calibrated by the 4-form $*_{\varphi_T + d\eta_T} (\varphi_T + d\eta_T)$ also suggests that, the map $\tau_T$ might be in some sense an approximation
of a fibration of the holonomy-$G_2$ manifold $(M, g(\varphi_T + d\eta_T))$ with coassociative calibrated fibres. The next theorem—which is the main result of this paper—asserts that this is indeed the case.

**Main Theorem.** Let $(M, g(\varphi_T + d\eta_T))$ be a compact 7-manifold with holonomy $G_2$ constructed from a pair of Fano threefolds as defined in the previous section and let $\tau_T$ be the fibration of $M$ given by Theorem 2.23. There exists, for every sufficiently large $T$, a diffeomorphism $h_T$ of $M$ onto itself, exponentially close to $\text{id}_M$,

$$\sup_{x \in M} \text{dist}_{g(\varphi_T)}(h_T(x), x) < \text{const} \cdot e^{-\lambda T}$$

with $\lambda > 0$ as determined in Theorem 2.21, and such that the fibres of $\tau_T \circ h_T : M \to S^3$ are coassociative calibrated by $\ast_{\varphi_T + d\eta_T}(\varphi_T + d\eta_T)$. In particular, smooth fibres of $\tau_T \circ h_T$ are minimal submanifolds of $(M, g(\varphi_T + d\eta_T))$. The map $h_T^{-1}$ can be taken to be $C^1$ on the locus of the singular fibres of $\tau$ and $C^\infty$ elsewhere on $M$.

A generic fibre of $\tau_T \circ h_T$ is diffeomorphic to the real 4-manifold underlying a K3 surface. Each singular fibre is an orbifold diffeomorphic to a ‘K3 orbifold surface’ with one ordinary double point and no other singularities. The discriminant locus (image of the singular fibres) of $\tau_T \circ h_T$ is a link in $S^3$, as described in Theorem 2.23.

**Remarks.** (1) The smooth K3 fibres of $\tau_T \circ h_T$ have $b^+ = 3$ and thus form a maximal deformation family, by McLean’s results [McL]. We shall see, after some additional work below, that the singular fibres of $\tau_T \circ h_T$ also form a maximal deformation family.

Another well-known compact 4-manifold with $b^+ = 3$ is a 4-torus. A fibration by coassociative 4-tori was obtained by Goldstein [Go] for a compact 7-manifold constructed in [Jo1]. However, the $G_2$-structure used in [Go] only has a closed 3-form but does not define a coassociative calibration (it is close to a calibration and one can obtain an estimate similar to (2.20)), so the fibres need not be minimal submanifolds. The singular coassociative fibres in [Go] have non-isolated singularities and it appears that the problem of perturbing into a map with calibrated fibres would require a different analytic technique than that developed for the fibration (2.22). We hope to return to this problem in a future paper.

(2) A K3 fibre $X$ of $\tau$ has trivial normal bundle in $M$ and hence trivial bundle of self-dual forms $\Lambda^+T^*X$. A trivialization of $\Lambda^+T^*X$ induces an $Sp(1)$-structure on $X$ and hence a triple of orthogonal almost complex structures $I, J, K$ (relative to the metric on $X$) satisfying the quaternionic relations $IJ = -JI = K$. From the construction of the holonomy-$G_2$ metric on $M$ one can see that one of $I, J, K$ is a deformation of the complex structure induced by embedding of $X$ in the Calabi–Yau threefold $W_j$. As $T$ tends to infinity, the metric induced by $g(\varphi_T)$ on each fibre of $\tau_T$ converges uniformly with all derivatives to a Kähler metric. For fibres near the middle of the neck of $M$, the limit metric is, moreover, hyper-Kähler. However, there is no general reason for a holonomy reduction for the induced metric on the fibres, for any finite $T$.

In the rest of the paper we prove the Main Theorem.

We shall construct $h_T$ in the form $h_T = \exp_{v_T}$, for a smooth vector field $v_T$ on $M$ (more precisely, $h_T$ will be obtained as a composition of two exponential maps). A map $\exp_v$ of
the compact Riemannian manifold \((M, g(\varphi_T))\) is well-defined whenever the uniform norm of \(v\) is less than the injectivity radius of \(M\). Furthermore, \(\exp_v\) defines a \textit{diffeomorphism} of \(M\) whenever the uniform norm of both \(v\) and its first derivatives is sufficiently small, so that \(\exp_v\) is a local diffeomorphism near every point and a bijection of \(M\).

**Proposition 3.25.** Let \(M\) be a compact 7-manifold with a one-parameter family of Riemannian metrics induced by the \(G_2\)-structures \(\varphi_T \in \Omega^3(M)\) defined by the connected sum construction, as in \(\S\).

Then there exists \(\varepsilon > 0\) so that for any metric \(g(\varphi_T), T \geq 1\), the map \(\exp_v\) is a diffeomorphism of \(M\) onto itself whenever \(\|v\|_{C^1} < \varepsilon\).

**Proof.** The upper bound \(\varepsilon\) can be determined by considering restrictions of the metric \(g(\varphi_T)\) to small neighbourhoods and taking the supremum. The asymptotically cylindrical properties of the metrics constructed on \(W_j \times S^1\) imply that the \(\varepsilon\) can be taken positive on these manifolds and hence also on \(M\) as \(g(\varphi_T)\) is exponentially close to the asymptotically cylindrical metrics as \(T \to \infty\). \(\square\)

We require a \(C^1\)-small vector field \(v = v_T\) on \(M\) satisfying
\[
\exp^*_v(\varphi_T + d\eta_T)|_X = 0
\]
for each fibre \(X\) of \(\tau_T\), given that \(\varphi_T|_X = 0\) and \(\eta_T\) can be taken as small as we like by choosing a large \(T\). The construction of such \(v\) can be thought of as an infinite-dimensional version of the implicit function problem \(F_X(v, s) = 0\) on each fibre \(X\), for a family of vector fields \(v = v(s), 0 \leq s \leq 1\), with \(v(0) = 0\), where
\[
F : (v, s) \in \Gamma(TM|_X) \times [0, 1] \to \exp^*_v(\varphi + s d\eta)|_X \in \Omega^3(X)
\]
is a variant of the map appearing in McLean’s theory (cf. \(\S\)). Here we temporarily dropped the dependence on \(T\) from the notation.

If all the coassociative fibres of \(\tau\) were smooth, then the desired vector field \(v\) would be easily obtained by a slight modification of Theorem 1.10 for local deformation families of coassociative submanifolds and then patching together finitely many of these local families, using the compactness of \(M\). However, the fibres of \(\tau\) develop singularities. Recall also that the deformation problem for a coassociative submanifold is expressed as an equation for sections of vector bundles on the actual submanifold. In light of this, the proof of the Main Theorem naturally falls into two parts concerned, respectively, with the singular fibres of \(\tau_T\) and the nearby smooth fibres with ‘large’ curvature.

In order to implement the implicit function strategy we require, in the first place, an extension of the deformation theory to the coassociative \(K3\) orbifolds arising as the singular fibres. More precisely, the required property concerns the linearization \((D_1 F)_0\) of the deformation map in \(v\) at \(v = 0\). This map should be a surjective Fredholm map between appropriate Banach spaces (weighted Sobolev spaces in Theorem 4.35), so the deformations of coassociative \(K3\) orbifold fibres of \(\tau_T\) are unobstructed.

The perturbation \(h_T\) of \(\tau_T\) will be obtained as a composition of exponential maps via the following two results. The proof of Theorem A requires a ‘stability’ result for coassociative cones defined by complex 2-dimensional cones in \(\mathbb{C}^3\). Appropriate stability result for the
tangent cones at the singular points of K3 orbifold fibres of $\tau_T$ indeed holds and has been proved by Lotay in $[Lo3]$.

**Theorem A.** (compare $[Lo3]$) Let $M$ be a compact 7-manifold with a smooth one-parameter family of $G_2$-structures given by closed 3-forms $\varphi_T \in \Omega^3_c(M)$, $T > T_0$, defined by the generalized connected sum construction in $[2]$. Let $\tau_T : M \to S^3$, be a coassociative K3 fibration, with respect to $\varphi_T$, defined in Theorem 2.23. Suppose that $\varphi_T + d\eta_T$, is a smooth family of torsion-free $G_2$-structures on $M$, such that $\|\eta_T\|_{L^p_k} < K p, k e^{-\lambda T}$ for each $p > 1$, $k \geq 0$.

Then there exists $T_1$ and for any $T > T_1$ and $0 \leq s \leq 1$ a $C^{1,\alpha}$ vector field $v_{T,s}$ on $M$, smooth away from the singular fibres of $\tau_T$ and satisfying $\|v_{T,s}\|_{L^p_k} < K p, k e^{-\lambda T}$ and $\|v_{T,s}\|_{C^1} < K s e^{-\lambda T}$, with support of $v$ contained in a neighbourhood $U$ of the singular fibres of $\tau_T$, and such that $\varphi + s d\eta_T$ vanishes on every singular fibre of the perturbed fibration $\tau_T \circ \exp^{-1}_{v_{T,s}} : M \to S^3$. The neighbourhood $U$ may be chosen not to meet the neck of $M$, i.e. $U \subset (W_1(0) \times S^1 \sqcup W_2(0) \times S^1)$.

We explain in §4 a framework of appropriate weighted Sobolev spaces making $(D_1F)_0$ into a Fredholm map and in §5 show how the surjectivity of $(D_1F)_0$ then follows from Lotay’s stability result, in the case of conical singularities of K3 orbifold fibres. The surjectivity implies that the deformations of coassociative K3 orbifold fibres of $\tau_T$ are unobstructed.

The hypothesis of the next result assumes the assertion of Theorem A. We may now deal with an approximating fibration whose singular fibres are precisely coassociative with respect to a $G_2$-structure $\varphi_T$, for each $T$.

**Theorem B.** Let $M$ be a compact 7-manifold with a smooth one-parameter family of $G_2$-structures given by closed 3-forms $\varphi_T \in \Omega^3_c(M)$, $T > T_1$, defined by the generalized connected sum construction in $[2]$. Let $\tau_T : M \to S^3$ for $T > T_1$ be a coassociative K3 fibration, with respect to $\varphi_T$, defined in Theorem 2.23.

Then there exists $\varepsilon > 0$ so that if $T > T_1$ and an exact form $d\eta \in \Omega^3(M)$ vanishes on every singular fibre of $\tau_T$ and $\|d\eta\|_{C^0(M)} < \varepsilon$, relative to the metric $g(\varphi_T)$, then there is a unique smooth vector field $\tilde{v}_T(\eta)$ on $M$ such that:

(i) $\tilde{v}_T$ vanishes on the singular fibres of $\tau_T$ and is point-wise orthogonal to each smooth fibre $X$ of $\tau_T$ and $\tilde{v}_T \cdot \varphi_T|_X$ is $L^2$-orthogonal to the harmonic self-dual forms on $X$ relative to the metric $g(\varphi_T)|_X$;

(ii) $\tilde{v}_T(\eta)$ depends smoothly on $T$ and $d\eta$ and $\|\tilde{v}_T\|_{C^1} = O(\|d\eta\|_{C^1})$;

(iii) $\varphi_T + d\eta$ vanishes on the fibres of $\tau_T \circ \exp^{-1}_{\tilde{v}_T(\eta)}$.

Theorem B is proved in §7.

The estimates of the vector fields $v_{T,s}$ in Theorem A ensure that $\exp v_{T,s}$ for any large $T$ is a well-defined diffeomorphism of $M$ isotopic to $\id_M$. Denote $\tilde{\varphi}_T = \exp_{v_{T,s}}^*(\varphi_T + d\eta_T)$; then, for $T > T_1$, the form $\tilde{\varphi}_T$ vanishes on the singular fibres of $\tau_T$. The form $\tilde{\varphi}_T$ is in the cohomology class of $\varphi_T$ and $d\eta_T = \tilde{\varphi}_T - \varphi_T$ tends to zero in $C^\infty$ as $T \to \infty$. For any large $T$, Theorem B applies to $d\eta_T$ and gives a second vector field $\tilde{v}(\eta_T)$ on $M$, so that the diffeomorphism $\exp_{\tilde{v}(\eta_T)}$ of $M$ is well-defined and fixes the singular fibres of $\tau_T$. Then
\( \varphi_T + d\eta_T \) vanishes on the fibres of \( \tau_T \circ h_T \), where

\[
h_T = \exp_{\tau_T, 1}^{-1} \circ \exp_{\tau_T, a}^{-1}.
\]

Thus the fibres of \( \tau_T \circ h_T \) are coassociative calibrated by the 4-form \( *_{\varphi_T + d\eta_T} \varphi_T + d\eta_T \) and are minimal submanifolds of the holonomy-\( G_2 \) manifold \((M, \varphi_T + d\eta_T)\) as required in the Main Theorem.

4. Linear analysis on coassociative K3 orbifolds

Before going to prove Theorem A we need to deal with the analytic issues arising in the deformation theory of the singular, K3 orbifold coassociative fibres of the map (2.22). We begin, in this and the next section, by showing that the infinitesimal coassociative deformations of these K3 orbifold fibres are unobstructed.

Our treatment is similar in spirit to one previously used by Joyce in the series of papers on special Lagrangian submanifolds with conical singularities, including [Jo4, Jo6], in that we apply elliptic theory on non-compact manifolds using weighted Sobolev spaces and extend these by certain finite-dimensional spaces to eliminate the obstruction space. Lotay [Lo1] applied the method of [Jo4, Jo6] and other papers in the same series to show that deformations of coassociative submanifolds with conical singularities may in general be obstructed. Recently Lotay developed a rather general deformation and stability theory for coassociative submanifolds whose conical singularities arise from complex cones in \( \mathbb{C}^3 \).

The ordinary double point singularities of complex surfaces in a threefold \( W \) are an instance of conical singularities. On the other hand, complex surfaces in \( W \) define coassociative submanifolds in \( W \times S^1 \) with respect to the product \( G_2 \)-structure corresponding to a Calabi–Yau structure on \( W \). As we explain below, using the stability result of [Lo3], in the case of ordinary double point singularities it is possible to set up an unobstructed theory.

Throughout this section, we work on a neighbourhood \( M_{\text{loc}} = W_{\text{loc}} \times S^1 \subset M \) of a singular fibre, \( X_0 \) say, of the map \( \tau = \tau_T \) defined in (2.22). Here \( W_{\text{loc}} = (\tau_j^{-1}(U_0)) \subset W_j \), \( j = 1 \) or \( 2 \), and \( U_0 \subset \mathbb{C}P^1 \) is an open disc, such that \( X_0 \subset W_{\text{loc}} \) and \( W_{\text{loc}} \) contains no other singular fibres of \( \tau_j \). Respectively, \( M_{\text{loc}} = \tau^{-1}(U \times S^1) \); mark a point \( 0 \in S^1 \) and identify \( W_{\text{loc}} \) with \( W_{\text{loc}} \times \{0\} \subset M_{\text{loc}} \). (Note that writing \( \tau \) rather than \( \tau_T \) is justified here as there are no singular fibres on the neck of \( M \) and the restriction of the fibration map to a neighbourhood away from the neck does not depend on \( T \).) We consider the 7-manifold \( M_{\text{loc}} \) with the product \( G_2 \)-structure

\[
\varphi_{\text{CY}} = \omega \wedge d\theta + \text{Im} \Omega
\]

induced by a Calabi–Yau structure \((\omega, \Omega)\) on \( W \) (as before, \( \theta \) is an ‘angle coordinate’ on \( S^1 \), so \( \varphi_{\text{CY}} \) vanishes on the smooth part of \( X_0 \). We shall set up a technical framework to deal with the local coassociative deformations of \( X_0 \), extending McLean’s approach.

Recall that \( X_0 \) is an orbifold degeneration of K3 surface with one ordinary double point and no other singularities. Denote by \( w_0 \) the ordinary double point point of \( X_0 \) and by \( X_0' = X_0 \setminus \{w_0\} \) the complement smooth non-compact complex surface. We shall always use on a neighbourhood of \( w_0 \) in \( W \) the ‘Morse’ local complex coordinates \((z_1, z_2, z_3)\) discussed
in \([\ref{2}]\) recall that the local expression of \(\tau^{(j)}\) in these coordinates is \(z_1^2 + z_2^2 + z_3^2\) and a neighbourhood of \(w_0\) in \(X_0\) corresponds to a neighbourhood of 0 in the cone on \(\mathbb{R}P^3\)

\[
    C_0 = \{z_1^2 + z_2^2 + z_3^2 = 0\} \subset \mathbb{C}^3. \tag{4.26}
\]

We shall sometimes use on a neighbourhood of \(w_0\) in \(X_0\) the uniformizing coordinates \((u_1, u_2) \in \mathbb{C}^2\) defined in \([\ref{2}]\), so that

\[
    (z_1, z_2, z_3) = (i(u_1^2 + u_2^2), u_1^2 - u_2^2, 2u_1u_2). \tag{4.27}
\]

The Ricci-flat Kähler metric \(\omega\) on \(W_{\text{loc}}\) defines, by restriction, an incomplete Kähler metric on \(X'_0\) and which can be written near \(w_0\) in the form

\[
    dr^2 + r^2g_3 + O(r^3), \quad \text{as } r \to 0. \tag{4.28}
\]

where \(r\) is the geodesic polar radius at \(w_0\) in \(W\), \(g_3\) is some smooth metric on \(\mathbb{R}P^3\), and \(O(r^3)\) is understood in the sense of the uniform convergence on \(\mathbb{R}P^3\) with all derivatives.

**Remark.** More explicitly, the metric \(g_3\) can be computed by restricting the inner product on the real tangent space \(T_wW\) induced by \(\omega\) to the intersection of the unit sphere and the tangent cone of \(X_0\) at \(w_0\). In the uniformizing coordinates \(u_1, u_2, 3\), it is obtained by substituting the expressions \((4.27)\) into (the real part of) \(h_{ij}dz_idz_j\), where \(h_{ij}\) is the Hermitian inner product defined by \(\omega\) at \(w_0\). Note that the Kähler form \(\omega\) need not be ‘compatible’ with \(z_j\)’s in any special way. In particular, \(g_3\) need not in general be induced in the standard way from the round metric on \(S^3\), nor the ‘obvious’ metric on the link of \(C_0\) induced by the Euclidean metric on \(\mathbb{C}^3\).

Note also that the expression for the metric \((4.28)\) in the uniformizing coordinates degenerates at \(w_0\). In particular, the local 2-form \(du_1 \wedge du_2\) on \(X_0\) is smooth in the orbifold sense but its point-wise norm relative to the metric \(g(\varphi_{CY})\) blows up at \(w_0\), \(|du_1 \wedge du_2| = O(r^{-1})\) as \(r \to 0\).

In what follows, we extend the local coordinate \(r\) to a positive smooth function, still denoted by \(r\), defined on all of \(X'_0\) and such that \(r > 1\) away from a coordinate neighbourhood of \(w_0\).

A local deformation of \(X_0\) in \(M_{\text{loc}}\) is defined as \(\exp_v(X_0)\), where \(v\) is a vector field on a neighbourhood \(U\) of \(X_0\) in \(M_{\text{loc}}\) with a small \(C^1\)-norm, so that \(\exp_v : U \to M_{\text{loc}}\) is a diffeomorphism of \(U\) onto its image. We regard two local deformations \(\exp_{v_1}(X_0)\) and \(\exp_{v_2}(X_0)\) as equivalent if \(\exp_{v_1}(X_0)|_{x_0} = \exp_{v_2}(X_0)|_{x_0} \circ \Phi_0\), for some diffeomorphism \(\Phi_0\) of \(X_0\) onto itself. (Note that any diffeomorphism of the orbifold \(X_0\) necessarily fixes \(w_0\).)

The follows is a direct corollary of the tubular neighbourhood Theorem \([\ref{16}]\).

**Proposition 4.29.** Let \(B_{\varepsilon}(\rho)\) denote a ball of radius \(\rho\) about zero in the fibre of the normal bundle \(N_{X'_0/M_{\text{loc}}}\) over \(x\) with the inner product induced by the metric on \(M_{\text{loc}}\). There exists \(\varepsilon > 0\) such that the Riemannian exponential map \(\exp_{v} : U_{\varepsilon} \to M_{\text{loc}}\) defines a diffeomorphism of an open neighbourhood \(U_{\varepsilon} = \bigcup_{x \in X'_0} B_{\varepsilon}(\varepsilon) \subset N_{X'_0/M_{\text{loc}}}\) onto a neighbourhood of \(X_0'\) in \(M_{\text{loc}}\), where \(r\) is the ‘polar radius-function’ on \(X_0\) defined above.

It follows that the bundle isometry \(N_{X'_0/M_{\text{loc}}} \cong \Lambda^+ T^*X_0'\) (cf. \([\ref{18}]\)) bijectively identifies local deformations of \(X_0\) fixing \(w_0\) defined by vector fields \(v\) point-wise orthogonal to \(X_0\).
and the forms $\psi \in \Omega^+(X'_0)$ if the uniform norms of $r^{-1}v$ and $r^{-1}\psi$ are less than $\varepsilon$ given by Proposition 4.29. For any such $\psi = v \omega_{\text{CY}}|_{X'_0}$, an orbifold $\exp_v(X_0)$ will be coassociative if and only if $\omega$ is a zero of ‘McLean’s map’

$$F : \psi = v \omega_{\text{CY}}|_{X'_0} \in \Omega^+(X'_0) \rightarrow \exp_v^* \omega_{\text{CY}}|_{X'_0} \in \Omega^3(X'_0). \quad (4.30)$$

Recall from Proposition 1.9 that the linearization of $F$ at $\psi = 0$ is an overdetermined-elliptic differential operator $(dF)_0 = d : \Omega^+(X_0) \rightarrow \Omega^3(X_0)$ and that the image of $F$ consists of exact 3-forms on $X'_0$. In order to apply the implicit function theorem to $F$ we require a choice of Banach space completions for the space of self-dual forms $v \omega_{\text{CY}}$ on $X'_0$ arising from local deformations of $X_0$ and also for the space of bounded exact 3-forms on $X'_0$, so that the exterior derivative extends to a surjective operator between the two Banach spaces.

The Kähler metric induced on $X'_0$ from $W_{\text{loc}}$ does not extend to a smooth orbifold metric on $X_0$, so trying to work with orbifold versions of e.g. the usual Sobolev spaces on a compact $X_0$ and is not a very promising way. Instead, we use a ‘conformal blow-up’ of $X'_0$ at the singular point and apply the elliptic theory for manifolds with asymptotically cylindrical ends from [LM] MP Me.

The non-compact submanifold $X'_0$ is diffeomorphic to a smooth 4-manifold with cylindrical end $\mathbb{R}_{>0} \times \mathbb{R}^3$ which corresponds to a neighbourhood of $w_0$ via $r = e^{-t}$, where $t$ is the coordinate on the $\mathbb{R}_{>0}$ factor. Restricting to the end of $X'_0$ we can write

$$dr^2 + r^2 g_3 + o(r^3) = e^{-2t} g_{\text{cyl}} = e^{-2t} (dt^2 + g_3 + o(e^{-t})), \quad \text{as } t \rightarrow \infty, \quad (4.31)$$

which shows that the metric (4.28) induced on $X'_0$ from $M_{\text{loc}}$ is conformally equivalent to an asymptotically cylindrical metric $g_{\text{cyl}}$. In particular, the self-dual forms defined by the metrics $g_{\text{cyl}}$ and $\omega|_{X'_0}$ are the same.

We shall need exponentially weighted Sobolev spaces on $X'_0$. By definition, $e^{-\delta t} L^p(X'_0)_{\text{cyl}}$ is the space of functions $e^{-\delta t} f$ such that $f \in L^p(X'_0)$ and the norm is defined by $\|e^{-\delta t} f\|_{e^{-\delta t} L^p_k} = \|f\|_{L^p_k}$. Here we used the subscript ‘cyl’ to indicate that the $L^p_k$ norm in the previous sentence is calculated using the metric $g_{\text{cyl}}$. This will be important when we consider the differential forms on $X'_0$.

There is a preferred choice of weight $\delta = k - 4/p$. Denote the corresponding weighted spaces by $W^p_k(X_0)$. In terms of the radial parameter $r$ on $X'_0$ the $W^p_k$-norm is expressed as

$$\|f\|_{W^p_k(X_0)} = \|f\|_{r^k} + \|\nabla f\|_{r^{k-1}} + \ldots + \|\nabla^k f\|_{r^0}.$$ 

The above expression for $W^p_k$ norm extends to the differential forms of any degree $m$ on $X'_0$ but note that the point-wise norms of the $m$-forms are rescaled by the ‘conformal weight’ factor $e^{-mt}$ when passing to the metric $g_{\text{cyl}}$. In view of this, we define

$$W^p_k \Omega^m(X_0) = e^{-(k-4/p+m)t} L^p_k \Omega^m(X'_0)_{\text{cyl}}.$$ 

Then the exterior derivative extends to a bounded linear map $W^p_k \Omega^m(X'_0) \rightarrow W^p_{k-1} \Omega^{m+1}(X'_0)$.

There is a simple relation between the $W^p_k$ spaces and the usual, unweighted Sobolev spaces $L^p_k(X_0)$. The following result is proved in [Bi] in the case of a flat Euclidean ball (with $r$ the Euclidean distance to the origin). However, the argument of the proof works, with
only a change of notation, for a punctured ball endowed with a metric having the ‘conical’ form \( (1.28) \). Considering a punctured neighbourhood of \( w_0 \) in \( X_0 \) as the quotient of a 4-dimensional punctured ball with respective \( \pm 1 \)-invariant metric and restricting attention to \( \pm 1 \)-invariant functions we obtain.

**Proposition 4.32** (cf. [13] Theorem 1.3). Suppose that \( \ell \) is a non-negative integer such that \( \ell - 1 < k - 4/p < \ell \) and let \( \delta = k - 4/p \). Then one has

\[
W^p_k(X_0) = \{ f \in L^p_k(X_0) \mid \lim_{w \to w_0} \nabla^m f(w) = 0, \text{ for all } 0 \leq m \leq \ell - 1 \} \quad (4.33)
\]

and the \( W^p_k \)-norm on the left-hand side is equivalent to the \( L^p_k \)-norm on the right-hand side.

The vanishing condition in the right-hand side of (4.33) makes sense as \( L^p_k \) embeds in \( C^{\ell-1} \). Proposition 4.32 extends in the usual way to sections of vector bundles over \( X_0 \) by considering a \( \pm 1 \)-equivariant local trivialization near \( w_0 \).

In view of the local regularity results for coassociative calibrated manifolds [11] § IV.2.7 we require a Banach space consisting of the \( C^1 \) self-dual forms on \( X_0' \). We shall use the completion of the space of self-dual forms in a \( W^p_k \)-norm fixing a choice of \( p > 1 \), \( k \in \mathbb{Z} \) such that

\[
1 < k - 4/p < 2 \quad (4.34)
\]

With this choice, every self-dual form \( \psi \in W^p_k \Omega^+(X_0') \) vanishes to order two at \( w_0 \), in particular \( \psi \) is Lipschitz continuous at \( w_0 \) with any Lipschitz constant \( \varepsilon > 0 \). Whenever the \( W^p_k \)-norm of \( \psi \) is small, the corresponding section \( v \) of \( N_{X_0'/M_{\text{loc}}} \) defines a local deformation \( \exp_v \) of \( X_0 \) which fixes the singular point \( w_0 \) and the tangent cone at \( w_0 \).

In order to include local deformations of \( X_0 \) which move the tangent cone and the singular point, we extend the weighted Sobolev space of self-dual forms by adding a finite-dimensional space

\[
E_0 = \{ v_{e,L} \cdot \varphi_{\text{CY}}|_{X_0'} : e \in T_{w_0} M_{\text{loc}}, L \in \text{End}(T_{w_0} M_{\text{loc}}) \}.
\]

Here \( v_{e,L} \) denotes a choice of a smooth vector field on \( M_{\text{loc}} \) smoothly depending on the parameters \( e, L \), such that \( v_{e,L}(w_0) = e \) and \( dv_{e,L}(w_0) = L \). In the last condition we define \( dv_{e,L} \) by using the coordinates \( z_i, \theta \) near \( w_0 \) in \( M_{\text{loc}} \) to express \( v_{e,L} \) locally as a smooth map from a neighbourhood of zero in \( T_{w_0} M_{\text{loc}} \) to \( T_{w_0} M_{\text{loc}} \). The Banach space \( E_0 + W^p_k \Omega^+(X_0') \), for \( 1 < k - 4/p < 2 \), does not depend on the choice of vector fields \( v_{e,L} \) given above as the ambiguity is \( O(r^2) \), \( r \to 0 \), which is contained in \( W^p_k \) by Proposition 4.32. It is not difficult to check, using Proposition 4.32 that for every smooth local deformation \( \exp_v \) of \( X_0 \), the self-dual form \( v \cdot \varphi_{\text{CY}}|_{X_0'} \) is in \( E_0 + \varepsilon^{-\delta} L^p_k \Omega^+(X_0')_{\text{cyl}.} \). On the other hand, straightforward calculation in local coordinates on \( M_{\text{loc}} \) near \( w_0 \) shows that any smooth exact 3-form on \( M_{\text{loc}} \) restricts to a form in \( d(E_0) + W^p_{k-1} \Omega^3(X_0') \). (As \( k - 4/p > 0 \), the space \( W^p_{k-1} \Omega^3(X_0') \) alone only contains forms vanishing at \( w_0 \).)

We are now ready to state the main result of this section.

**Theorem 4.35.** If the Sobolev space parameters satisfy (4.34) then the exterior derivative defines a bounded linear map between Banach spaces

\[
d : E_0 + W^p_k \Omega^+(X_0') \to d(E_0) + \{ \eta \in W^p_{k-1} \Omega^3(X_0') : d\eta = 0 \} \quad (4.36)
\]
which is surjective and has a one-dimensional kernel spanned by the Kähler form $\omega|_{X'_0}$.

Remark. The 1-dimensional kernel of (4.36) arises from the $S^1$-symmetry of the torsion free $G_2$-structure $\varphi_{CY}$ on $W \times S^1$. Thus Theorem 4.35 shows that the $S^1$-families of coassociative K3 orbifolds arising in the approximating fibration (2.22) are maximal deformation families (the smooth fibres, of course, have the same property by Theorem 1.9 and McL as $b^+ = 3$ for a K3 surface).

5. Proof of Theorem 4.35

Remark. Putting $x = e^{-t}$ we can think of the manifold $X'_0$ as the interior of a compact manifold with boundary $\overline{X'_0} = X'_0 \cup \{x = 0\} \times \mathbb{R}P^3$ obtained by adding a copy of $\mathbb{R}P^3 = \{x = 0\}$ ‘at infinity’. (The added $\mathbb{R}P^3$ may also be canonically identified with the unit spherical space form $S^3/\pm 1$ in the tangent cone of $X_0 \subset W_{loc}$ at $w_0$.) The asymptotically cylindrical metric $g_{cyl}$ on $\overline{X'_0}$ can be written in the form

$$g_{cyl} = \frac{dx^2}{x^2} + \tilde{g},$$

where $\tilde{g}$ is a symmetric semi-positive definite form smooth up to the boundary: at any point in $\{x = 0\}$, $\tilde{g}$ is smooth in the $\mathbb{R}P^3$ directions and has one-sided derivatives in $x$ of any order. Then (5.37) gives an instance of a ‘smooth exact $b$-metric’, as defined by Melrose [Mc Ch. 2]. The results proved in [Mc] for manifolds with smooth exact $b$-metrics can therefore be applied to $(X'_0, g_{cyl})$.

In will be convenient to reduce Theorem 4.35 to a result concerning a linear operator between the $W^p_k$-spaces alone. We look at the kernel first. (The subscript ‘cyl’ at the exponentially weighted spaces on $X'_0$ will now be dropped from the notation.)

Proposition 5.38. Suppose that $k - 4/p > 1$. The map $d : e^{-\delta t}L^p_k(\Omega^+(X'_0)) \to e^{-\delta t}L^p_{k-1}(\Omega^3(X'_0))$ has a 3-dimensional kernel for any $0 \leq \delta < 1$, a 1-dimensional kernel for any $1 \leq \delta < 2$, and is injective for any $\delta \geq 2$.

Proof. This is an application of the Hodge theory on asymptotically cylindrical manifolds. The kernel of $d$ is contained in the kernel of the Laplacian acting on $e^{-\delta t}L^p_k(\Omega^+(X'_0))$. Furthermore, the standard integration by parts argument is valid when $\delta > 0$ and shows that the two kernels coincide. By [APS] Propn. 4.9 or [Mc] Propn. 6.14], the $L^2$-kernel of the Laplacian on the $m$-forms on an asymptotically cylindrical manifold $X'_0$ is isomorphic to the image of the natural inclusion homomorphism $H^m_c(X'_0) \to H^m(X'_0)$ of the de Rham cohomology groups, where the subscript ‘c’ indicates the cohomology with compact support.

Considering the exact sequence of the de Rham cohomology groups

$$\ldots \to H^{m-1}(\mathbb{R}P^3) \to H^m_c(X'_0) \to H^m(X'_0) \to H^m(\mathbb{R}P^3) \to \ldots$$

we find that $H^0(X'_0) = 0$, $H^1_c(X'_0) \cong \mathbb{R}$ and the inclusion homomorphism $H^m_c(X'_0) \to H^m(X'_0)$ is an isomorphism for $1 \leq m \leq 3$. Recall from [2] that $X'_0$ is isomorphic to the complement of a $(-2)$-curve in a K3 surface, $X$ say; this $(-2)$-curve is topologically a
sphere. Considering the Maier–Vitrih exact sequence for the union of $X'_0$ and a tubular neighbourhood of the ‘missing’ $(−2)$-curve we find that $H^2(X'_0)$ is a complement in $H^2(X)$ of the one-dimensional subspace generated by the Poincaré dual of the $(−2)$-curve. The cup-product on $H^2_0(X'_0)$ has maximal positive subspace $H^+(X'_0)$ of dimension $3$ and so the $L^2$-kernel of the Laplacian on $Ω^+(X'_0)$ is $3$-dimensional.

It is easy to identify, for $0 < δ < 1$, the $3$-dimensional space of $O(e^{−δt})$ exponentially decaying closed self-dual forms on $X'_0$. This space is spanned by the restriction $ω|_{X_0}$ of the Kähler form on $W_{loc}$ and the restrictions of the real and imaginary parts of the holomorphic $(2,0)$-form $((δτ)^2,ω)|_{X_0}$, defined using is the K3 fibration map $τ$ on $W_{loc}$. The Kähler form $ω|_{X_0}$, measured with the metric $g_{cyl}$, is $O(e^{−2t})$, as $t → ∞$, but not $O(e^{−(2+ε)t})$ for any $ε > 0$. As the $(1,0)$-form $δτ$ has a zero of order $1$ at $w_0$, the vector field $(δτ)^2$ is $O(1/r)$, as $r → 0$, when measured in to the Kähler metric $ω|_{X_0}$. Hence the real and imaginary parts of $((δτ)^2,ω)|_{X_0}$ are $O(e^{−t})$ but not $O(e^{−(1+ε)t})$in the metric $g_{cyl}$.

Neither of the three self-dual forms spanning the $L^2$-kernel of the Laplacian (and the $L^2$-kernel of $A$) is in $e^{−δt}L^p_k$ for $δ > 2$.

Corollary 5.39. If $1 < k − 4/p < 2$ then the map $\delta_{30}$ has a one-dimensional kernel spanned by the Kähler form $ω|_{X'_0}$.

Proof. It is not difficult to check that $ω|_{X'_0} ∈ E_0 + W_k^pΩ^+(X'_0) ⊂ e^{−δt}L^p_kΩ^+(X'_0)$, for any $1 < δ < 2$ but $E_0 + W_k^pΩ^+(X'_0)$.

The next two lemmas determine the codimensions of the relevant $W_k^p$ spaces.

Lemma 5.40. If $1 < k − 4/p < 2$ then the codimension of $W_k^pΩ^+(X'_0)$ in $E_0 + W_k^pΩ^+(X'_0)$ is $41$.

Proof. If $ε ≠ 0$ then $|e^{2t}v_{e,L}ω_{CY}|_{g_{cyl}}$ has a non-zero lower bound on an open subset of the cylindrical end, of the form $\mathbb{R}_+ × U'$ where $U'$ is open in $\mathbb{R}^3$. So $v_{e,L}ω_{CY}$ with $ε ≠ 0$ is never in $W_k^pΩ^+(X'_0)$ if $k − 4/p > 0$.

The form $v_{0,L}ω_{CY}$ will be in $W_k^pΩ^+(X'_0)$ with $1 < k − 4/p < 2$ precisely if $L$ leaves invariant the tangent cone of $X_0$ at $w_0$. This, in turn, will be the case if and only if the Zariski tangent space $T_{w_0}W_{loc}$ of $X_0$ is an invariant subspace of $L$ and the restriction of $L$ to $T_{w_0}W_{loc}$ is up to a complex factor an element of $SO(3, \mathbb{C})$ (so that $L$ preserves the tangent cone $(\Omega^3)^3$). We find that the subspace of endomorphisms in $End T_{w_0}M_{loc}$ preserving the tangent cone has dimension $15$. As the dimension of $E_0$ is $7 + 49$ the result follows.

Lemma 5.41. If $1 < k − 4/p < 2$ then the codimension of $W_k^pΩ^3(X'_0) ∩ Ker d$ in $(d(E_0) + (W_k^pΩ^3(X'_0) ∩ Ker d)$ is $18$.

Proof. Recall from $[34]$ that the space $d(E_0) + (W_k^pΩ^3(X'_0) ∩ Ker d)$ contains the restrictions to $X'_0$ of all the smooth exact $3$-forms on $M_{loc}$. On the other hand, $W_k^pΩ^3(X'_0)$ is precisely the space of $L^p_k$ $3$-forms on $X'_0$ with zero limit at $w_0$. Since $X_0 ⊂ W_{loc}$ and the Zariski tangent space of $X_0$ at $w_0$ is $T_{w_0}W_{loc}$ the codimension of interest may be computed as $\dim Λ^3T_{w_0}W_{loc} − \dim (d(E_0) ∩ W_k^pΩ^3(X'_0))$. 
A calculation in the local complex coordinates on $W_{\text{loc}}$ near $w_0$ shows that $d(v_{e,L} \varphi_{\text{CY}})|_{X_0'} \in W^p_{k-1} \Omega^3(X_0')$ holds precisely if $d(v_{e,L} \varphi_{\text{CY}})|_{W_{\text{loc}}(w_0)}$ is the real or imaginary part of a $(3,0)$-form on $T_{w_0} W_{\text{loc}}$. □

From Corollary 5.39 and Lemmas 5.40, 5.41 and some straightforward linear algebra we find that Theorem 4.35 is equivalent to the following technical result on an asymptotically cylindrical 4-manifold $(X'_0, g_{\text{cy}})$. Theorem 4.35. If $1 < k - 4/p < 2$ then the injective linear map

$$A : \psi \in W^p_k \Omega^+ (X_0') \rightarrow d\psi \in W^p_{k-1} \Omega^3(X_0') \cap \text{Ker} \, d,$$

(5.42)

has a 22-dimensional cokernel.

In the remainder of this section we prove Theorem 4.35. Note that the injectivity of $A$ follows from Proposition 5.38. Therefore, the dimension of $\text{Coker} \, A$ is minus the Fredholm index of $A$. For the index computation, it is convenient to observe that the map $A$ is equivalent to a component of an elliptic operator

$$D : (f, \psi) \in e^{-\delta t} L^p_k (\Omega^0 \oplus \Omega^+)(X_0') \rightarrow df - *d\psi \in e^{-\delta t} L^p_{k-1} \Omega^1(X_0').$$

(5.43)

For any real $\delta$, we write $\text{index}_{-\delta} \, D$ to indicate that $D$ is considered on the $e^{-\delta t}$-weighted Sobolev spaces. A similar notation will be used for $A$ and for the kernels and cokernels.

Proposition 5.44. $\text{index}_{-\delta} \, A = \text{index}_{-\delta} \, D$ for any $\delta > 0$. Proposition 5.44 will be deduced from the following.

Lemma 5.45. If $\varepsilon > 0$ is sufficiently small then $\text{Coker}_{-\varepsilon} \, D = \{0\}$.

Proof of Proposition 5.44 assuming Lemma 5.45. For any $\delta > 0$, the cohomology of the weighted de Rham complex on $X'_0$

$$e^{-\delta t} L^p_k \Omega^0(X_0') \xrightarrow{d} e^{-\delta t} L^p_k \Omega^1(X_0') \xrightarrow{d} e^{-\delta t} L^p_{k-1} \Omega^2(X_0') \xrightarrow{d} \ldots$$

(5.46)

is isomorphic to the de Rham cohomology with compact support $H^*_c(X'_0)$ [Mc Propn. 6.13]. The formal $L^2$-adjoint of (5.46) is a complex of the $e^{\delta t}$-weighted spaces. The cohomology of the latter complex at the $\Omega^{m-1}$ term is isomorphic to the de Rham cohomology $H^{4-m}(X'_0)$.

Recall from the proof of Proposition 5.38 that $H^0_k(X'_0) = H^1_k(X'_0) = 0$. So the exterior derivative maps $e^{-\delta t} L^p_k \Omega^0(X'_0)$, for each $\delta > 0$, isomorphically onto the subspace of closed forms in $e^{-\delta t} L^p_{k-1} \Omega^1(X'_0)$. Standard integration by parts argument of the Hodge theory for compact manifolds is valid for the exponentially decaying forms on $X'_0$ and shows that the spaces of 1-forms $d(e^{-\delta t} L^p_k \Omega^0(X'_0))$ and $d^*(e^{-\delta t} L^p_k \Omega^+(X'_0))$ are $L^2$-orthogonal if $\delta > 0$. By Lemma 5.45, $D$ is a surjective operator between $e^{-\varepsilon t}$-weighted spaces for any small $\varepsilon > 0$, so we obtain a decomposition

$$e^{-\varepsilon t} L^p_{k-1} \Omega^1(X'_0) = d(e^{-\varepsilon t} L^p_k \Omega^0(X'_0)) \oplus d^* (e^{-\varepsilon t} L^p_k \Omega^+(X'_0)).$$

For any arbitrary $\delta > 0$ we can write any $\xi \in e^{-\delta t} L^p_{k-1} \Omega^1(X'_0)$ as $\xi = d\xi_0 + d\xi_+$ where $\xi_+ \in e^{-\varepsilon t} L^p_k \Omega^+(X'_0)$ for some small $\varepsilon > 0$ but $\xi_0 \in e^{-\delta t} L^p_k \Omega^0(X'_0)$ (as explained above). Therefore, the image of $D$ on the $e^{-\delta t}$-weighted spaces is complemented by $d^*$-closed forms and the proposition follows. □
For the proof of Lemma 5.45 we need to recall from [LM] or [Me] some Fredholm theory for elliptic operators on asymptotically cylindrical Riemannian manifolds. Let \( \Lambda^\pm T^* (\mathbb{R}_{>0} \times \mathbb{R}P^3) \) denote the bundle of self-dual forms with respect to the metric \( dt^3 + g_3 \), the asymptotic model of \( g_{cyl} \). A point-wise orthogonal projection, relative to \( dt^3 + g_3 \), \( \sigma : \Lambda^+ T^* (\mathbb{R}_{>0} \times \mathbb{R}P^3) \rightarrow \Lambda^+_{\infty} T^* (\mathbb{R}_{>0} \times \mathbb{R}P^3) \) defines a bundle isomorphism asymptotic to the identity as \( t \rightarrow \infty \). The coefficients of \( D \) are determined by the metric \( g_{cyl} \) and \( D \) is asymptotic, on the end of \( X'_0 \), to an operator \( D_\infty \circ (1 \oplus \sigma) \), where \( D_\infty \) is given by the same formula as in (5.43) but using the product cylindrical metric \( dt^3 + g_3 \) rather than \( g_{cyl} \). The coefficients of the operator \( D_\infty \) on \( \mathbb{R} \times \mathbb{R}P^3 \) are independent of \( t \).

**Proposition 5.47.** (i) The elliptic operator \( D \) between \( e^{-\lambda t} \)-weighted Sobolev spaces, is Fredholm if and only if \( d(\lambda) = 0 \), where

\[
d(\lambda) = \dim \{ e^{-\lambda t} p(t,y) \mid p(t,y) \text{ is polynomial in } t \text{ and } D_\infty (e^{-\lambda t} p(y,t)) = 0 \}. \tag{5.48}
\]

The set \( \{ \delta \in \mathbb{R} : d(\delta) \neq 0 \} \) is discrete in \( \mathbb{R} \).

(ii) The index of \( D \) is independent of \( p, k \) but depends on the weight parameter \( \lambda \) according to the formula

\[
\text{index}_{\delta'} D - \text{index}_{\delta''} D = \sum_{\delta' < \lambda < \delta''} d(\lambda) \tag{5.49}
\]

for any \( \delta' < \delta'' \) such that \( d(\delta') \neq 0 \), \( d(\delta'') \neq 0 \).

(iii) The kernel of \( D \) consists of smooth forms and is independent of \( p, k \). The cokernel of \( D \) can be identified with the kernel of the formal \( L^2 \)-adjoint of \( D \) with respect to \( g_{cyl} \),

\[
D^* = d^* \oplus d^+ : e^{\delta t} L^p_{k-1}(\Omega^1_+ (X'_0)) \rightarrow e^{\delta t} L^p_{k}(\Omega^0_+ (X'_0)).
\]

In particular, \( \dim \text{Coker}_{-\lambda} D = \dim \text{Ker}_{-\lambda} D^* \) if \( d(\lambda) = 0 \).

Proposition 5.47 is a direct application of [LM] or [Me]. In the case of \( D \) we don’t need to worry about the possibility of \( \lambda \in \mathbb{C} \smallsetminus \mathbb{R} \) with \( d(\lambda) \neq 0 \), as we shall see in a moment.

**Proof of Lemma 5.45.** The operator \( D_\infty \) can be expressed, using some vector bundle isomorphisms, as \( \partial_t + D^{(3)} \) where \( D^{(3)} \) is a formally self-adjoint operator on \( \mathbb{R}P^3 \) whose square is the Laplacian on \( (\Omega^p_+ \Omega^1_+)(\mathbb{R}P^3) \) (cf. e.g. [MMR] pp. 132–134). This yields \( d(0) = 1 \). Therefore, \( D \) is a Fredholm operator on \( e^{-\delta t} \) and on \( e^{\delta t} \)-weighted spaces whenever \( \varepsilon > 0 \) is sufficiently small. Furthermore, for a small \( \varepsilon > 0 \), we obtain

\[
\dim \text{Ker}_{-\varepsilon} D = 3, \quad \dim \text{Ker}_{-\varepsilon} D^* = 0
\]

by the Hodge theory arguments as in the proof of Proposition 5.38 using also the relation \( DD^* = dd^* + \frac{1}{2}d^*d \). On the other hand,

\[
\text{index}_{\varepsilon} D - \text{index}_{-\varepsilon} D = 1,
\]

by (5.49), whence

\[
\dim_{-\varepsilon} \text{Coker} D + \dim_{-\varepsilon} \text{Ker} D = 4.
\]

by Proposition 5.47(iii). But \( \dim_{-\varepsilon} \text{Ker} D \geq 1 + \dim_{-\varepsilon} \text{Ker} D = 4 \) as the \( e^{\varepsilon t} L^p_{k} \) kernel contains the constants, so \( \text{Coker}_{-\varepsilon} D = \{0 \}. \) □
The next result is proved by Lotay \cite{Lo3} and is essentially the stability property for the coassociative cones defined by complex cones biholomorphic to $C_0$. Clause (i) will be needed for the $C^{1,\alpha}$-regularity claim in the next section.

**Proposition 5.50** (J. D. Lotay). Let $D$ be the elliptic operator defined in (5.43) over a $K3$ orbifold $X_0$ and let $d(\lambda)$ be as defined in (5.48). Then

(i) any $0 < \lambda \leq 0$ such that $d(\lambda) \neq 0$ is an integer;
(ii) $\sum_{0 < \lambda \leq \delta} d(\lambda) = 25$, for some $3 < \delta < 4$.

**Theorem 4.35'** follows from Proposition 5.50 by dimension counting. For if $3 < \delta < 4$ then using (5.49), Propositions 5.38 and 5.44, and Proposition 5.50(ii) we obtain $d(\delta) \neq 0$ and

$$\dim \text{Coker}_{-\delta} A = -\text{index}_{-\delta} A = -\text{index}_{-\delta} D = \sum_{0 < \lambda < \delta} d(\lambda) - 3 = 22$$

as required.

**Remarks on Proposition 5.50.** The conformal factor at the Hodge star in (5.43) is precisely the inverse of the conformal weight of 1-forms. Therefore, the operator $D$ and its asymptotic model $D_\infty$ are conformally invariant and can be interchangeably considered on $X_0'$ with the Kähler metric $\omega|_{X_0'}$.

The asymptotic model for the cylindrical end of $X_0'$ corresponds to the cone $C_0$ in $T_{w_0}W_{\text{loc}} \cong \mathbb{C}^3$ defined by (4.26) and the asymptotic model for $g_{\text{cyl}}$ is conformally equivalent to the Kähler metric $dr^2 + r^2 g_3$ on $C_0$ induced by restricting from $T_{w_0}W_{\text{loc}}$ a Hermitian inner product defined by $\omega(w_0)$. The kernel elements of $D_\infty$ contributing to the dimensions $d(\lambda)$ are expressed on $C_0$ as pairs

$$(r^\lambda p_0(\log r, y), r^{\lambda-2} p_+ (\log r, y)) \in (\Omega^0 \oplus \Omega^+)(C_0 \setminus \{0\}),$$

where a 0-form $p_0$ and a self-dual form $p_+$ are polynomial in $\log r$ and smooth in $y \in \mathbb{R}P^3$. As noted earlier, the conformal rescaling produces an extra factor $r^{-2}$ for the self-dual forms in (5.51).

For a Kähler surface $C_0 \setminus \{0\}$, there is a commutative diagram, cf. \cite[§4.3.3]{FM},

$$\begin{array}{ccc}
\Omega^0 \oplus \Omega^+ & \xrightarrow{D_\infty} & \Omega^1 \\
\iota, j : \pi^{0,2} & \cong & \cong \pi^{0,1} \\
\Omega^0 \oplus \Omega^{0,2} & \xrightarrow{2\bar{\partial} + \bar{\partial}^*} & \Omega^{0,1},
\end{array}$$

(5.52)

where $\Omega^0$ in the bottom row denotes the complex-valued 0-forms, $\iota$ is the inclusion map, $j(\alpha) = -i\alpha.\omega_{C_0}$ is the contraction with the Kähler form and the inverses of $\pi^{p,q}$ map the respective complex vector space to its underlying real vector space.

Suppose that

$$2\bar{\partial} f + \bar{\partial}^* \psi = 0$$

(5.53)

and that $(f, \psi) \in \Omega^0 \oplus \Omega^{0,2}$ is obtained from (5.51) for some $\lambda > 0$ via the left column of (5.52). Then since $\text{Re} f$ has degree $r^\lambda$, whereas $\text{Im} f$ and $\psi$ both have degree $r^{\lambda-2}$ we deduce that $\text{Re} f = 0$. 
If \((f, \psi)\) is a solution of (5.53) then, writing \(\psi = adu_1 \wedge du_2\), applying, respectively, \(\partial\) and \(\partial^*\) to the equation we deduce that \(\partial^* \partial f = 0\) and \(\partial^* \partial a = 0\), that is, \(f\) and \(a\) are harmonic functions on \(C_0\). Therefore, \(f\) and \(a\) each factorize as \(r^\lambda G(y)\) (we omit the details but cf. [Jo4, Propn. 2.4]). A function \(r^{\lambda-2}G(y)\) is harmonic if and only if \(G\) if an eigenfunction of the Laplacian on the link \(\mathbb{R}P^3 = \{r = 1\}\) of the singularity of \(C_0\). The eigenvalue of \(G\) is then \(\lambda (\lambda - 2)\) and must be non-negative, therefore there are no solutions \(f, \psi\) to (5.53) of the form (5.51) with \(0 < \lambda < 2\).

The harmonic \((0, 2)\)-form \(\psi\) may be written in the uniformizing coordinates as \(\psi = a\, du_1 \wedge d\bar{u}_2\), for some harmonic complex function \(a\). Note that \(d\bar{u}_1 \wedge d\bar{u}_2\) is homogeneous of degree \(-1\) in \(r\) and so \(a\) factorizes as \(r^{\lambda-1}G(y)\).

If we assume that both \(f\) and \(a\) are homogeneous polynomials of even degree in \(u_1, u_2, \bar{u}_1, \bar{u}_2\) then solving (5.53) with \(2 \leq \lambda < 4\) in this case becomes an elementary calculation. We obtain that \(\text{Im} f\) must be a constant if \(\lambda = 2\) and a harmonic homogeneous quadratic polynomial if \(\lambda = 3\). This contributes one real dimension to \(d(2)\) and 6 real dimensions to \(d(3)\) as a polynomial \(h\) can be found to satisfy (5.53). But \(h\) then is determined up to an anti-holomorphic homogeneous polynomial of \(\bar{u}_1, \bar{u}_2\) of degree \(2\lambda - 2\). This contributes further \(4\lambda - 2\) real dimensions to \(d(\lambda)\). We thus find that \(d(1) \geq 2\), \(d(2) \geq 7\), and \(d(3) \geq 16\). Proposition 5.50 in this context, asserts that these are all the solutions of (5.53) for \(0 < \lambda \leq 3\) and the above estimates for \(d(\lambda)\)'s are in fact equalities.

6. Perturbing the singular fibres

We can now deduce Theorem A.

**Theorem A.** Let \(M\) be a compact 7-manifold with a smooth one-parameter family of \(G_2\)-structures given by closed 3-forms \(\varphi_T \in \Omega^3_+(M), T > T_0\), defined by the generalized connected sum construction in [2]. Let \(\tau_T : M \to S^3\), be a coassociative K3 fibration, with respect to \(\varphi_T\), defined in Theorem 2.23. Suppose that \(\varphi_T + d\eta_T\), is a smooth family of torsion-free \(G_2\)-structures on \(M\), such that \(\|\eta_T\|_{L^p_k} < K_{p,k} e^{-\lambda T}\) for each \(p > 1, k \geq 0\).

Then there exists \(T_1\) and for any \(T > T_1\) and \(0 \leq s \leq 1\) a \(C^{1,\alpha}\) vector field \(v_{T,s}\) on \(M\), smooth away from the singular fibres of \(\tau_T\) and satisfying \(\|v_{T,s}\|_{C^{1}_k} < K_{p,k} e^{-\lambda T}\) and \(\|v_{T,s}\|_{C^{2}} < K_s e^{-\lambda T}\), with support of \(v\) contained in a neighbourhood \(U\) of the singular fibres of \(\tau_T\), such that \(\varphi + s\, d\eta_T\) vanishes on every singular fibre of the perturbed fibration \(\tau_T \circ \exp_{v_{T,s}}^{-1} : M \to S^3\). The neighbourhood \(U\) may be chosen not to meet the neck of \(M\), i.e. \(U \subset (W_1(0) \times S^1) \sqcup W_2(0) \times S^1)\).

Theorem A will be deduced by building up on the analysis of the previous two sections. The orbifold fibres of \(\tau_T\) are parameterized by finitely many disjoint circles. It will suffice to restrict attention to one such \(S^1\)-family. We assume the notation \(M_{\text{loc}} = W_{\text{loc}} \times S^1\), \(\varphi_{CY} \in \Omega^3_+(M_{\text{loc}}), \tau = \tau_T, X_0 \subset W_{\text{loc}}, X_0' = X_0 \setminus \{w_0\}\), and the radial coordinate \(r\) on \(X_0'\), as in [4]. For each \(\theta \in S^1\), define a finite-dimensional space of smooth vector fields

\[\tilde{E}_\theta = \{v_{e,L} | x'_0 \times \{\theta\} : e \in T_{(w_0, \theta)}(W_{\text{loc}} \times \{\theta\})\},\]

with \(v_{e,L}\) as defined in Theorem [4.35].
**Theorem 6.54.** Suppose that $1 < k - 4/p < 2$. Then there exists $\varepsilon_0 > 0$ such that for each $\eta \in \Omega^2(M_{\text{loc}})$ with $\|d\eta\|_{C^1(M_{\text{loc}})} < \varepsilon_0$ and $\theta \in S^1$ there is a unique vector field along $X'_0 \times \{\theta\}$, $v(\eta, \theta) \in \tilde{E}_0 + W^p_k N_{X'_0/M_{\text{loc}}}$, with the following properties:

(i) $v(0, \theta) = 0$ and $v$ depends smoothly on $\eta$ and $\theta$,

(ii) $\varphi_{CY} + d\eta$ vanishes on $\exp_{v(\eta, \theta)}(X'_0 \times \{\theta\})$.

(iii) the vector field $v(\eta, \cdot)$ on $X'_0 \times S^1$ is the restriction of some $C^{1,\alpha}$ vector field $v(\eta)$ on $M_{\text{loc}}$ smooth away from $X_0 \times S^1$, such that $\|v(\eta)\|_{L^p_k} < C_{p,k}\|d\eta\|_{L^p_k}$

Theorem A is a consequence of Theorem 6.54. Choose $T_1$ so that the $C^1$ norm of $d\eta_{T_1}$, for $T > T_1$, is smaller than $\varepsilon_0$ given by Theorem 6.54, for each $S^1$-family of the singular fibres of $\tau_T$. Further increasing $T_1$ if necessary, we can ensure that the vector field $v(\eta)$ can be chosen with compact support contained in $M_{\text{loc}}$ and with a small $C^1$-norm so that $\exp_{v(\eta)}$ is a diffeomorphism. Let $U$ be the union of the neighbourhoods $M_{\text{loc}}$ for all $S^1$-families of the singular fibres of $\tau$. The vector field $v_{T,s}^\tau$ required in Theorem A is then obtained by putting $\eta = s\, d\eta_T$ and extending the $v(\eta)$ by zero on $M \setminus U$.

In the rest of this section we prove Theorem 6.54.

The proof is based on an application of the Implicit Function Theorem in Banach spaces to a ‘parametric version’ of McLean’s map for the family $X_0 \times S^1$ of coassociative K3 orbifolds

$$F : (v, \eta, \theta) \in (\tilde{E}_0 + W^p_k N_{X'_0/M_{\text{loc}}}) \times \Omega^2(M_{\text{loc}}) \times S^1 \rightarrow$$

$$\exp^*_{v(\eta)}(\varphi_{CY} + R^\theta_{\text{loc}} d\eta)|_{X'_0} \in d(\tilde{E}_0, \varphi_{CY}) + (W^p_k \Omega^3(X'_0) \cap \text{Ker } d), \quad (6.55)$$

where $R^\theta : W_{\text{loc}} \times S^1 \rightarrow W_{\text{loc}} \times S^1$ denotes an isometry given by the translations by $\theta$ on the $S^1$ factor (here we exploited the $S^1$-symmetry to make the Banach space of self-dual forms independent of $\theta$). A choice of Banach space for $\eta$ is not particularly important here as long as it controls e.g. the $C^3$ norm. We need the following.

**Proposition 6.56.** The map $F$ defined in (6.55) is a smooth map of Banach spaces at any $(v, \eta, \theta)$ with sufficiently small $\|v\|$ in $\tilde{E}_0 + W^p_k N_{X'_0/M_{\text{loc}}}$.

**Proof.** The map $F$ is linear in $\eta$ and has an equivariant property

$$F(v, \eta, \theta) = F(v, R^\theta \eta, 0). \quad (6.57)$$

Therefore, $F$ is smooth in $\eta, \theta$ and it remains to show that $F$ is smooth in $v$.

A related argument was carried out by Baier in \cite[Theorem 2.2.15]{Ba}, see also \cite[Theorem 2.5]{JS}. We may disregard a finite-dimensional component $\tilde{E}_0$ and pretend that $v \in W^p_k N_{X'_0/M_{\text{loc}}} \subset C^1 N_{X'_0/M_{\text{loc}}}$, so $v$ has a point-wise estimate $|v| < \varepsilon r$ near $w_0$ for any $\varepsilon > 0$ (in the metric $g(\varphi_{CY})$). By Proposition 4.129 the Riemannian exponential map is a diffeomorphism between a neighbourhood of the zero section of $N_{X'_0/M_{\text{loc}}}$ and a neighbourhood of $X'_0$ in $M_{\text{loc}}$ of the form $U_\varepsilon = \bigcup_{x \in X'_0} B_\varepsilon(x) \varepsilon r$ for some $\varepsilon > 0$. Denote by $\tilde{\varphi} \in \Omega^3(U_\varepsilon)$ the pull-back of $\varphi_{CY} + R^\theta_{\text{loc}} d\eta$ via this diffeomorphism. Then $F$, as a function of $v$, becomes equivalent to

$$v \in W^p_k N_{X'_0/M_{\text{loc}}} \rightarrow v^* \tilde{\varphi} \in W^p_{k-1} \Omega^3(X'_0),$$
for $v$ with sufficiently small in the $W^p_k$ norm. Use a uniformizing coordinate neighbourhood of $w_0$ in $X'_0$ and a finite open cover of the compact complement by coordinate neighbourhoods and the respective local trivializations of $N_{X'_0/M_{loc}}$ to obtain local expressions for $v$. The pull-back $v^*\tilde{\varphi}$ of a given smooth 3-form is then expressed a cubic polynomial in the partial derivatives of $v$ with coefficients smoothly depending on $v$. The reader now should have no difficulty to check the smoothness of $F$ using standard results on Sobolev spaces. □

The first partial derivative $D_\theta F(0,0,\theta)$ is a composition of the bundle isometry $v \mapsto v\,\varphi_{CY}$ and the restriction of the exterior derivative operator (4.36) to a complement of its image. By Theorem 4.35, $D_\theta F(0,0,\theta)$ maps $\tilde{E}_0 \oplus W^p_k N_{X'_0/M_{loc}}$ isomorphically onto the image of $F$. Therefore, the implicit function theorem in Banach spaces applies to $F$ and gives for each $\theta_0 \in S^1$ a unique smooth family of fields $v(\eta,\theta)$ in $\tilde{E}_0 \oplus W^p_k N_{X'_0/M_{loc}}$, for small $\|d\eta\|$ and $|\theta - \theta_0|$, satisfying $v(0,\theta) = 0$ and $F(v(\eta,\theta), \eta, \theta) = 0$. By the compactness of $S^1$ and the uniqueness of the local solutions $v(\eta,\theta)$, a finite number of these solutions can be patched together to define a vector field $(dR_\theta)_0 v(\eta,\theta)$ on $X'_0 \times S^1$, for all $\|d\eta\| < \varepsilon$, where $\varepsilon$ is the smallest of the finitely many upper bounds coming from the local constructions.

This completes the proof of clauses (i) and (ii) of the theorem and it remains to establish the regularity clause (iii).

The interior regularity. We show that the vector fields $v(\eta,\theta)$ defined above are smooth on $X'_0$. In general, a $(\varphi_{CY} + d\eta)$-coassociative submanifold need not be calibrated, so it is not quite sufficient to fall back on the traditional argument for minimal submanifolds. If $v = v_0 + v_1 \in \tilde{E}_0 + W^p_k N_{X'_0/M_{loc}}$ is a solution of $F(v, \eta, \theta) = 0$ for some $\eta, \theta$ then $v_0$ is smooth and the equation satisfied by $v_1$ can be written in the form

$$F_0 + Gv_1 + Q(v_1) = 0. \quad (6.58)$$

Here $Gv_1 = d(v_1 \varphi_{CY} + d\eta))$ and $F_0 = R_0^* d\eta|_{X'_0} + Gv_0$ is a smooth 3-form on $X'_0$. It follows from the proof of Proposition 5.56 that the remainder $Q$ is a cubic polynomial in the first derivatives of $v_1$ with coefficients smoothly depending on $v_1$. To deal with this non-linearity note first that $v_1 \in W^p_k N_{X'_0/M_{loc}}$, with $k - 4/p > 1$, thus in $C^{1,\alpha}$, $\alpha > 0$. Then the first factor in each term $(\nabla v_1)^{\otimes n} \otimes \nabla v_1$ can be considered as a first order linear differential operator with Hölder continuous coefficients acting on the second term. If $\|v_1\|_{C^{1,\alpha}}$ is sufficiently small then the sum of the latter linear operator and $G$ is again an overdetermined-elliptic operator, so we still have the standard local regularity estimates [ADN]. As $v_1$ is already in $C^{1,\alpha}$ we can apply the usual bootstrapping to show that $v_1$, and hence also $v$, is smooth. Then $(dR_\theta)_0 (v(\eta,\theta))$ defines a $C^\infty$ vector field on $X'_0 \times S^1$.

Remark. If the 4-form $*_{\varphi_{CY} + d\eta}(\varphi_{CY} + d\eta)$ is closed then $v(\eta,\theta)$ defines a calibrated submanifold and is real analytic. This is a consequence of a general property of minimal submanifolds of a Ricci-flat (hence real analytic [DK]) Riemannian manifold.

$C^{1,\alpha}$-regularity at $X_0 \times S^1$. The normal bundle $N_{X'_0/M_{loc}}$ is trivial and $W^p_k$ sections are $C^{1,\alpha}$ and vanish at $w_0$ together with first derivatives when $k - 4/p > 1$. Multiplying by a smooth cut-off function which is equal to 1 on $X_0 \times S^1$ we obtain an extension of a $L^p_k$ field of normal vectors to a vector field supported on a compact neighbourhood of $X_0 \times S^1$. 

COASSOCIATIVE K3 FIBRATIONS 25
on $M_{\text{loc}}$. It is clear that such an extension is smooth except at points of $X_0 \times S^1$ where it is $C^{1,\alpha}$. It is easy to see that a $W^p_k N_{X_0/M_{\text{loc}}}$ section $v(\eta, \theta)$ has an extension to $M_{\text{loc}}$ with the same properties as an $E_0$ component is, by construction, the restriction to $X_0 \times S^1$ of some smooth vector field on $M_{\text{loc}}$.

If $v(\eta)$ has a sufficiently small $L^p_k$-norm on $X_0' \times \text{pt}$, then we can achieve estimates $\|v(\eta)\|_{L^p_k(M_{\text{loc}})} < \text{const} \|v(\eta, \cdot)\|_{L^p_k(X_0' \times S^1)}$, with constants independent of $v$. The estimates $\|v(\eta, \theta)\|_{p,k} < \text{const} \|d\eta\|_{p,k}$ follow by differentiating the identity $F(\psi(\eta, \theta), \eta, \theta) = 0$ in $\eta, \theta$ and taking account of the symmetry of $F$ in $\theta \in S^1$. If $\|d\eta\|_{C^1}$ is small then $\exp_v(\eta)$ is a well-defined diffeomorphism of $M_{\text{loc}}$ satisfying the assertions of Theorem 6.54.

7. A NEIGHBOURHood OF THE SINGULAR FIBRES

In this section we prove Theorem B and thus complete the proof of the Main Theorem.

Theorem B. Let $M$ be a compact 7-manifold with a smooth one-parameter family of $G_2$-structures given by closed 3-forms $\varphi_T \in \Omega^3_+ (M), T > T_1$, defined by the generalized connected sum construction in [2]. Let $\tau_T : M \to S^3$ for $T > T_1$ be a coassociative $K3$ fibration map defined in Theorem 2.23.

Then there exists $\varepsilon > 0$ so that if $T > T_1$ and an exact form $d\eta \in \Omega^3(M)$ vanishes on every singular fibre of $\tau_T$ and $\|d\eta\|_{C^0(M)} < \varepsilon$, relative to the metric $g(\varphi_T)$, then there is a unique smooth vector field $\tilde{\nu}_T(\eta)$ on $M$ such that:

(i) $\tilde{\nu}_T$ vanishes on the singular fibres of $\tau_T$ and is point-wise orthogonal to each smooth fibre $X$ of $\tau_T$ and $\tilde{\nu}_T \cdot \varphi_T|_X$ is $L^2$-orthogonal to the harmonic self-dual forms on $X$ relative to the metric $g(\varphi_T)|_X$;

(ii) $\tilde{\nu}_T(\eta)$ depends smoothly on $T$ and $d\eta$ and $\|\tilde{\nu}_T\|_{C^1} = O(\|d\eta\|_{C^1})$;

(iii) $\varphi_T + d\eta$ vanishes on the fibres of $\tau_T \circ \exp^{-1}_{\tilde{\nu}_T(\eta)}$.

The main technical issue in the proof of Theorem B is to establish certain uniform estimates for families of smooth fibres of $\tau_T$ in a neighbourhood of the singular fibres. We use the notation of [4].

$M_{\text{loc}} = W_{\text{loc}} \times S^1 \subset M$, $\varphi_{CY} \in \Omega^3_+(M_{\text{loc}})$, $X_0 \subset W_{\text{loc}}$, $X_0' = X_0 \setminus \{w_0\}$, and ‘holomorphic Morse coordinates’ $(z_1, z_2, z_3)$ near $w_0$. For $a \in \mathbb{C}$, let $X_{a^2} \subset W_{\text{loc}}$ denote the fibre of $\tau_T$ which is expressed near $w_0$ by the equation $z_1^2 + z_2^2 + z_3^2 = a^2$. Let $X_{a^2, \theta} = X_{a^2} \times \{\theta\} \subset W_{\text{loc}} \times S^1$ and, as before, identify $X_{a^2} = X_{a^2, 0}$, for a marked point $0 \in S^1$. The fibre $X_{a^2, \theta}$ is well-defined for any small $|a|$, say $|a| < \rho$, and is non-singular for $a \neq 0$. Throughout this section any $X_{a^2, \theta}$ is considered with the metric $g_{a^2}$ induced by restriction of $g(\varphi_{CY})$ unless stated otherwise. For each compact smooth coassociative submanifold $X \subset M$, define a decomposition

$L^p_k(N_{X/M}) = \Gamma^h(N_{X/M}) \oplus \Gamma^\perp(N_{X/M})_{p,k},$

where

$$\Gamma^h(N_{X/M}) = \{v \in \Gamma(N_{X/M}) \mid (v \cdot \varphi_T) \in \mathcal{H}^+(X)\},$$

$$\Gamma^\perp(N_{X/M}) = \{v \in \Gamma(N_{X/M}) \mid (v \cdot \varphi_T) \perp_{L^2} \mathcal{H}^+(X)\},$$
and $\Gamma^\perp(N_{X/M})_{p,k}$ means the completion of $\Gamma^\perp(N_{X/M})$ in the $L^p_k$ norm (as before, we require $1 < k - 4/p < 2$). Denote
\[
Y = \{ \eta \in \Omega^2(M_{\text{loc}}) \mid d\eta|_{X_{a,\theta}} = 0 \text{ for each } \theta \in S^1 \}.
\]

The proof of Theorem B builds up on the following.

**Proposition 7.59.** There is an $\epsilon_0 > 0$ so that whenever $\eta \in Y$ and $\|d\eta\|_{C^1(M_{\text{loc}})} < \epsilon_0$ the following holds. For each $0 < |a| < \rho$ and $\theta \in S^1$ there is a unique smooth $v = v_{a^2,\theta}(\eta) \in \Gamma^\perp(N_{X_{a^2,\theta}/M_{\text{loc}}})$ such that $v_{a^2,\theta}(0) = 0$, $v_{a^2,\theta}(\eta)$ depends smoothly on $\eta$, and $\varphi_{\text{CY}} + d\eta$ vanishes on the submanifold $\exp_{v_{a^2,\theta}(\eta)}(X_{a^2,\theta})$.

For a fixed $\eta \in Y$ with $\|d\eta\|_{C^1(M_{\text{loc}})} < \epsilon_0$, the family $v_{a^2,\theta}(\eta)$, for $0 < |a| < \rho$, $\theta \in S^1$ extended by zero over $X_{0,\theta}$, defines a vector field $v(\eta)$ which is $C^1$ on a neighbourhood $M_{\text{loc}}$ of $X_0 \times S^1$ and smooth on $M_{\text{loc}} \setminus X_0 \times S^1$. Furthermore, $\|v(\eta)\|_{C^1(M_{\text{loc}})} < K\|d\eta\|_{C^1(M_{\text{loc}})}$, for some constant $K$ independent of $\eta$.

**Proof.** For each $a \neq 0$, $\theta \in S^1$ we want to obtain $v_{a^2,\theta}(\eta)$ as $v(0, \eta)$, where $v(\beta, \eta)$ is the solution to implicit function problem $F(v(\beta, \eta), \beta, \eta) = 0$ for the extended McLean’s map for $X_{a^2,\theta}$

\[
F : (v, \beta, \eta) \in \Gamma^\perp(N_{X_{a^2,\theta}/M_{\text{loc}}})_{p,k} \oplus \Gamma^h(N_{X_{a^2,\theta}/M_{\text{loc}}}) \oplus Y \to 
exp_{v+\beta}(\varphi_{\text{CY}} + d\eta)|_{X_{a^2,\theta}} \in L^p_{k-1}\Omega^3(X_{a^2,\theta}) \cap \text{Ker } d,
\]

where $k - 4/p > 1$ we use the completion in the $C^3$ norm for $\eta \in Y$. That $F$ is a smooth map between the indicated Banach spaces follows by the argument of Proposition 7.56 up to a change of notation. We have $F(0, 0, 0) = 0$ and the derivative $(DvF)_{0,0,0}v = d(v_a\varphi_{\text{CY}})|_{X_{a^2,\theta}}$ is an isomorphism of Banach spaces by standard Hodge theory as $H^3(X_{a^2,\theta}, \mathbb{R}) = 0$ for a K3 surface $X_{a^2,\theta}$.

The implicit function theorem in Banach spaces gives a unique family of sections $v(\beta, \eta)$, with $v(0, 0) = 0$, defined for $\max\{|\beta|, \|d\eta\|_{C^1(X_{a^2,\theta})}\} < \delta$ say, but this $\delta$ may in general depend on $a$. (We can choose $\delta$ independent of $\theta$ by the $S^1$-symmetry of $\varphi_{\text{CY}}$.) To keep track of the relation between constants appearing in the estimates we use.

**Proposition 7.61** (Implicit function theorem in Banach spaces). Suppose that a smooth map $f : E = E_1 \oplus E_2 \to F$ between Banach spaces has an expansion
\[
f(\xi_1, \xi_2) = (D_1f(0, 0))\xi_1 + (D_2f(0, 0))\xi_2 + Q(\xi_1, \xi_2),
\]
so that $A = D_1f(0, 0) : E_1 \to F$ is an isomorphism of Banach spaces and for $\xi, \zeta \in E$,
\[
\|A^{-1}Q(\xi) - A^{-1}Q(\zeta)\| < C(\|\xi\| + \|\xi\|\|\xi - \zeta\|),
\]
for some constant $C$. Then there exists a uniquely determined smooth function $\varphi : B_{2,\delta} \to B_{1,\delta}$, $\varphi(0) = 0$, where $B_{i,\delta} = \{\xi_i \in E_i \mid \|\xi_i\| < \delta\}$, with $\delta = (4C)^{-1}$, so that all zeros of $f$ in $B_{1,\delta} \times B_{2,\delta}$ are of the form $(\varphi(\xi_2), \xi_2)$.

The proof of Proposition 7.61 is a standard application of the contraction mapping principle.
We require a lower bound on the linearization $(D_1 F)_0$ in $v$ and an upper bound on the quadratic remainder for $F$ so that the constant $(4C)^{-1}$ in Proposition 7.61 for the map $F$ on a manifold $X_{a^2}$ is greater than the norm of $d\eta|_{X_{a^2, \theta}}$ for any sufficiently small $|a| \neq 0$. As noted above, the estimates on $X_{a^2, \theta}$, can be taken independent of $\theta \in S^1$.

We begin with the linear part.

**Proposition 7.62.** There exists a $\rho > 0$ such that for any $v \in \Gamma^1(N_{X_{a^2, \theta}}/M_{\text{loc}})_{p,k}$ with $0 < |a| < \rho$,

$$\|v\|_{p,k} < C_{p,k}\|d(v \cdot \phi_{\text{CY}})|_{X_{a^2, \theta}}\|_{p,k-1}$$

with a constant $C_{p,k}$ independent of $v$ or $a$.

**Proof.** Since the linear map and the norms are symmetric in $\theta$ we drop $\theta$ from the notation. Consider a sphere $|z| = |a|^{1/2}$ in the coordinate neighbourhood of $u_0$ in $W_{\text{loc}}$, where $|z|^2 = \sum_{j=1}^3 |z_j|^2$. If $|a|$ is sufficiently small then this sphere intersects $X_{a^2}$ and we can write $X_{a^2} = X_{a^2}^- \cup X_{a^2}^+$, where $X_{a^2}^- = X_{a^2} \cap \{|z| \leq |a|^{1/2}\}$ and $X_{a^2}^+ = X_{a^2} \cap \{|z| < 2|a|^{1/2}\}$ are two open submanifolds of $X_{a^2}$. As $a \to 0$ the metric on $X_{a^2}^-$ is asymptotic in $C^\infty$ to the metric on $X_0 \setminus \{|z| \leq |a|^{1/2}\}$. The other piece $X_{a^2}^+$ with metric rescaled by a constant factor $|a|$ is asymptotic in $C^\infty$ to a complex surface $\Sigma = \{(\sum_{j=1}^3 z_j^2 = 1, \ |z| < |a|\}$ in $\mathbb{C}^3$ with the metric on $\Sigma$ induced by a Hermitian inner product on $T_{u_0} W_{\text{loc}} \cong \mathbb{C}^3$ defined by the Kähler form $\omega(u_0)$.

The decomposition $X_{a^2} = X_{a^2}^- \cup X_{a^2}^+$ can be thought of as a generalized connected sum $X_0' \#_{\mathbb{R}P^3} \Sigma$ of two manifolds taken at their ends $\mathbb{R}_{\geq 0} \times \mathbb{R}P^3$. The metric $g_{a^2}$ on the connected sum $X_a$ is $C^\infty$ asymptotic to a metric smoothly interpolating between metrics on compact subsets of $X_0'$ and $\Sigma$.

A family of Riemannian 4-manifolds $(X_{a^2}, g_{a^2})$ is an instance of the gluing construction studied in [KS, §2]. The estimate that we are interested in is equivalent to a lower bound on the operator

$$d : L^p_k \Omega^+(X_{a^2}) \to L^p_{k-1} \Omega^3(X_{a^2}).$$

(7.63)

The 'main estimate' proved in [KS, §4.1] deals with the invertibility of an elliptic differential operator on a generalized connected sum, as above, when the coefficients of this operator are obtained by gluing the coefficients of Fredholm elliptic operators on the two manifolds with cylindrical ends. The result, in particular, asserts a uniform lower bound on a subspace of finite codimension in the $L^p_k$ domain on a connected sum. Inspection of the proof of this uniform lower bound shows that it remains valid for the overdetermined elliptic operator (7.63) on $(X_{a^2}, g_{a^2})$ provided that the $L^2$ kernels of the respective exterior derivatives on self-dual forms on $X_0'$ and $\Sigma$ are finite-dimensional. Then the operator (7.63) admits a lower bound, independent of $a$ as $a \to 0$, on the complement of a finite-dimensional subspace in $L^p_k \Omega^+(X_{a^2})$. The dimension of this space is the sum of dimensions of the spaces of closed self-dual $L^2$ forms on $X_0'$ and $\Sigma$.

These latter spaces are contained in the $L^2$ kernels of the Laplacian on self-dual forms on $X_0'$ and $\Sigma$. The dimensions can be computed in the asymptotically cylindrical metrics in the conformal class of $X_0'$ and $\Sigma$ as the $L^2$ norm of 2-forms is conformally invariant.
in dimension 4. Recall from Proposition 5.38 that the $L^2$ kernel is 3-dimensional for $\Omega^+(X_0')$. The 4-manifold $\Sigma$ is diffeomorphic to the total space of the tangent bundle $TS^2$, so $b^+(\Sigma) = 0$ and the same argument as in Proposition 5.38 shows that the Laplacian on $L^p_k\Omega^+(\Sigma)$ is injective. Thus the uniform lower bound for (7.63) holds on the complement of a 3-dimensional subspace in $\Omega^+(X_{a^2})$. It is a posteriori clear that this 3-dimensional space can be taken to be $H^+(X_{a^2})$.

The argument of Proposition 5.38 shows that the non-linear remainder $Q$ of the map $F$ consists of polynomial terms in $\nabla v$ with coefficients smooth in $v$. This easily gives an upper bound on $Q$ independent of $a, \theta$ for small $|a|$. Therefore, by Propositions 7.61 and 7.62 there is an $\varepsilon > 0$ independent of $a, \theta$ so that if $\eta \in \Omega^2(M_{loc})$ satisfies $\|d\eta\|_{X_{a^2},\theta}^{C^1(X_{a^2},\theta)} < \varepsilon$ then the implicit function theorem defines sections $v(0, \eta)$ of $N_{X_{a^2,\theta}/M_{loc}}$ for every small $|a|$.

The metric on the submanifold $X_{a^2,\theta}$ is independent of $\theta \in S^1$ and in the next result we once again drop $\theta$ from the notation. It will be convenient to use Hölder norms commensurable with the $L^p_k$ norms that we required.

**Proposition 7.64.** For each $X_{a^2}$ with $a \neq 0$, we have

$$\|d\eta\|_{X_{a^2}}^{C^{1,\alpha}(X_{a^2})} < \lambda_{k,\alpha}\|d\eta\|_{C^{1,\alpha}(M_{loc})}|a|^{1-\alpha}, \quad 0 \leq \alpha < 1,$$

with a constant $\lambda_{k,\alpha}$ independent of $\eta$ or $a$.

Estimates similar to that in Proposition 7.64 are proved in [Jo6] and [Lo2]. In the present situation, we have an explicit algebraic local model for the submanifolds $X_{a^2}$ near the singular point of $X_0$ and some details are simplified.

**Proof.** Recall from the proof of Proposition 7.62 the decomposition $X_{a^2} = X_{a^2}^- \cup X_{a^2}^+$ as a generalized connected sum and denote by $(d\eta)^{\pm}$ the restrictions of $d\eta$ to the respective pieces $X_{a^2}^{\pm} \subset X_{a^2}$. Recall also that $X_{a^2}^-$ is diffeomorphic to $X_0 \setminus \{|z| \leq |a|^{1/2}\}$. As $X_{a^2}^+ = X_0 \setminus \{|z| \leq |a|^{1/2}\}$ is compact and non-singular, its injectivity radius of is bounded away from zero, furthermore, it is not difficult to check that the injectivity radius is $O(|a|)$, as $|a| \to 0$. We deduce that for $|a|$ sufficiently small, each $X_{a^2}^-$ is a well-defined deformation of $X_{a^2}'$ defined by a section $\nu_{a^2}$ of the normal bundle of the latter submanifold, using the exponential map for $g(\varphi_{CY})$. The $\nu_{a^2} \to 0$ depends smoothly on $a$ and vanishes when $a = 0$. As $d\eta$ is smooth and vanishes on $X_0'$ and the metric on $X_{a^2}'$ converges in $C^\infty$ to the metric on $X_0'$, we find that the $C^k$ norm of $(d\eta)^-$ is a smooth function of $a$. This function vanishes at $a = 0$, hence is bounded by a constant multiple of $\|d\eta\|_{C^k(M_{loc})}|a|$.

The form $(d\eta)^+$ is defined on a coordinate neighbourhood $\{|z| < |a|^{1/2}\} \times S^1$ of $w_0$ in $M_{loc}$. We may assume, by rescaling $z \in C^3$ if necessary, that the matrix of the inner product defined by $g(\varphi_{CY})$ on $T_{w_0}W_{loc}$ has determinant 1. The metric $g(\varphi_{CY})$ on the coordinate neighbourhood may be written as $g_0 + O(|z|)$, where $g_0$ denotes the value of $g(\varphi_{CY})$ at $w_0$ and $O(|z|)$ is estimated independently of $\theta$. A similar local expansion is valid for the derivatives of $g(\varphi_{CY})$.

The parameterizations of local submanifolds $X_{a^2}^+$ may be taken to be homogeneous of order 1 in $a$. The coefficients of the $k$-th derivatives of induced metric $g(X_{a^2}^+)$ on $X_{a^2}^+$
then have a local extension with the leading term homogeneous of order \(2 - k\) in \(a\) and
a remainder \(O(|a|^{\beta - k})\) as \(a \to 0\) (\(d\theta\) vanishes on each \(X_{a^2}\)). The difference between
the restriction of the Levi–Civita connection of \(g(\varphi_{CY})\) on the ambient \(M_{\text{loc}}\) to \(X_{a^2}^+\) and
the Levi–Civita connection of the metric \(g(X_{a^2}^+)\) induced by \(g(\varphi_{CY})\) on \(X_{a^2}\) is
determined by second fundamental form of \(g(X_{a^2}^+)\). Recall that for a submanifold defined by
submersion \(\tau_T\) the second fundamental form is the quotient of the Hessian of \(\tau_T\) and
the gradient of \(\tau_T\). Thus its point-wise norm has leading term homogeneous of order
\(a\) at \(w_0\), so the coefficients of \(d\eta|_{W_{\text{loc}}}\) have zeros of order two at \(w_0\) and the covariant
derivative has zero of order one. It follows that for \(0 < |a| < \rho\) with sufficiently small \(\rho\),
the \(C^{1,\alpha}\) norm of \(d\eta|_{X_{a^2}^+}\) measured using the Levi–Civita of \(g(X_{a^2}^+)\) is
\(O(||d\eta||_{C^k(M_{\text{loc}})}|a|^{1-\alpha})\).

As \(||d\eta||_{X_{a^2}^+} \leq \max\{||d\eta||_{X_{a^2}^-} ||C^k(X_{a^2}^-), ||d\eta||_{X_{a^2}^+} ||C^k(X_{a^2}^+)\}\) the proposition is proved.

The local deformation \(\exp_{v(\beta, \eta)}(X_{a^2, \theta})\) is well-defined if \(v(\beta, \eta)\) is small in the uniform
norm so that the tubular neighbourhood theorem is applicable to \(X_{a^2, \theta}\). It is not difficult
to see that, for any small \(a\), a suitable neighbourhood of the zero section of \(N_{X_{a^2, \theta}/M_{\text{loc}}}\) is
\(\|v\|_{C^0} < \varepsilon_1 |a|\), for a sufficiently small constant \(\varepsilon_1 > 0\) independent of \(a\).

For a small \(d\eta\) we can estimate on \(X_{a^2, \theta}\), \(\|v(0, \eta)\| = O(||(D_2v)_{0,0}|| ||d\eta||_{X_{a^2, \theta}})\). As
\((D_2v)_{0,0} = (D_1F)_{0,0,0} (D_3F)_{0,0,0}\) this gives
\(\|v(0, \eta)\|_{C^1(X_{a^2, \theta})} = O(||a||), \quad a \to 0, \quad (7.65)\)
using Proposition \ref{7.62} and Lemma \ref{7.64}. In particular, for a sufficiently small \(d\eta\) the
deformations \(\exp_{v(0, \eta)}(X_{a^2, \theta})\) are well-defined for every small \(|a|\).

The regularity results for the coassociative submanifolds \cite{HL} imply that \(v(\beta, \eta)\) is
a smooth section over \(X_{a^2, \theta}\), as \(v(\beta, \eta)\) is already in \(C^1\). By construction, \(\exp_{v(\beta, \eta)}(X_{a^2, \theta})\)
coincides with \(\exp_{v(0,\eta)}(\exp_{v(\beta, \eta)} (X_{a^2, \theta}))\). Therefore, the local families \(v(\beta, \eta)\) can be patched
together to define a smooth vector field \(v(\eta)\) on \((U_{0}|\rho|X_{a^2}) \times S^1\), for some \(\rho > 0\).
This vector field has a Lipschitz continuous extension by zero over \(U_{\|\beta\|S^1}X_{0, \theta}\).

Differentiating the identity \(F(v(\beta, \eta), \beta, \eta) = 0\) in \(\beta\) at \(\beta = 0\) we find that the derivative
of \(v(\eta)\) in the directions orthogonal to the fibres \(X_{a^2, \theta}\) satisfies
\((D_1F)_{v(0, \eta), 0, \eta} (D_2v)_{0,0,\eta} + (D_2F)_{v(0, \eta)} 0, \eta) = 0.

Using explicit expressions for the derivatives of \(F\) and the vanishing of \(\varphi_{CY}\) on \(X_{a^2, \theta}\) we obtain
\(d((D_1v)_{0, \eta, \beta}) \varphi_{CY} |_{X_{a^2, \theta}} = -d(\beta, d\eta) |_{X_{a^2, \theta}}\).

The operator \(d(\cdot, \varphi_{CY})|_{X_{a^2, \theta}}\) is bounded below independent of \(a\) by Proposition \ref{7.62}
and we deduce that \(||D_1v||_{C^0(X_{a^2, \theta})} < K||d\eta||_{C^1(M_{\text{loc}})}\), \(a \neq 0\).

We adapt the same method to show that \(v(\eta)\) is continuously differentiable at any point
in \(X_0 \times S^1\). Differentiating the identity \(F(v(\beta, \eta), \beta, \eta) = 0\) in \(\beta\) at \(\beta = 0\) we find that the
derivative of $v(\eta)$ in the directions orthogonal to the fibres $X_{a^2, \theta}$ satisfies
\[(D_1 F)_{v(0, \eta), 0, \eta} (D_1 v)_{0, \eta} + (D_2 F)_{v(0, \eta), 0, \eta} = 0.\]
The self-dual harmonic forms on $X_{a^2, \theta}$ are spanned by the Kähler form $\omega|_{X_{a^2, \theta}}$ and the real and imaginary parts of the $(2, 0)$-form $(\partial \tau)^2 \omega$ induced by the Calabi–Yau structure $(\omega, \Omega)$ on $W_{\text{loc}} \times \{\theta\}$. The definition of $\Gamma^h(N_{X_{a^2, \theta}/M})$ therefore can be extended to the smooth subset $X'_{0, \theta} \subset X_{0, \theta}$. The resulting $\Gamma^h(N_{X'_{0, \theta}})$ can be thought of as a limit of $\Gamma^h(N_{X_{a^2, \theta}/M})$ as $a \to 0$, in the sense that the 4-manifold $X'_{0, \theta}$ is diffeomorphic to an open subset of $X_{a^2, \theta}$ and the metric on $X'_{0, \theta}$ is a $C^\infty$ limit of the metrics on this subset. Using explicit expressions for the derivatives of $F$ and taking the limit as $a \to 0$ we obtain
\[d\left(\left(D_1 v\right)_{0, \eta} \beta \varphi_{\text{CY}}\right)|_{X'_{0, \theta}} = -d(\beta \varphi_{\text{CY}})|_{X'_{0, \theta}}\]
The operator $d(\cdot \varphi_{\text{CY}})|_{X'_{0, \theta}}$ is injective on $\Gamma^+(N_{X'_{0, \theta}/M_{\text{loc}}})$. Applying a left inverse we determine $(D_1 v)_{0, \eta}$ at any point in $X'_{0, \theta}$. This shows the continuity of $(D_1 v)_{0, \eta}$ away from critical point of $\tau_T$. The estimate (7.65) shows that the derivative of $v(\eta)$ is Lipschitz continuous at each point in $w_0 \times S^1$, thus $v(\eta)$ is $C^1$-regular on all of $M_{\text{loc}}$. The higher order derivatives of $v(\eta)$ can be handled by a similar method but the expressions become cumbersome; we omit the details.

We estimated the ‘intrinsic’ first derivatives of $v(\eta)$ on $X_{a^2, \theta}$ and in the transverse directions by $\|d\eta\|_{C^1(M_{\text{loc}})}$. To obtain the desired estimate of the $C^1$ norm of $v(\eta)$ on $M_{\text{loc}}$ we recall that the second fundamental form of $X_{a^2, \theta}$ is $O(|a|^{-1})$, as $a \to 0$, near the critical points of $\tau_T$ and bounded on the complement of a neighbourhood of the critical points. It follows that $\|v\|_{C^1(M_{\text{loc}})} = O(\|d\eta\|_{C^1(M_{\text{loc}})})$. This completes the proof of Proposition 7.59.

The singular fibres of $\tau_T$ occur in finitely many $S^1$-families. Applying Proposition 7.59 to each $S^1$-family we obtain for any $\eta \in \Omega^2(M_{\text{loc}})$ with small $\|d\eta\|_{C^1}$ a vector field $\tilde{v}_T(\eta)$ satisfying the assertions of Theorem B except that $\tilde{v}_T(\eta)$ is only defined on a neighbourhood $U$ of the singular fibres of $\tau_T$.

The implicit function argument for the extended McLean’s map $F(v, \beta, \eta)$ as in (7.60) for a fibre $X$ of $\tau_T$ in $M \setminus U$ gives a uniquely determined smooth local family $v(\beta, \eta)$ of sections in $\Gamma^+(N_{X'}/M)$, so that $v(0, 0) = 0$ and $\varphi_T + d\eta$ vanishes on $\exp(v(\beta, \eta))(X)$.

The estimates on $v(\beta, \eta)$ are similar to those in Proposition 7.59 but easier as the curvature of the fibres in $M \setminus U$ is bounded independent of the fibre. The complement $M \setminus U$ is a family of non-singular fibres of $\tau_T$. Recall from §2 that the metric on each of these fibres is up to a small deformation a metric in a compact closure of the finite-dimensional family of metrics on the fibres of holomorphic fibrations $\tau^{(j)}$, $j = 1, 2$, restricted to the asymptotically cylindrical ends of $W_j$. The deformation is bounded, independent of $T$ or the fibre, in $C^k$ norm for each $k$. It follows that the lower bound on $(D_1 F)_{0,0,0}$ and the upper bound on the quadratic remainder can be taken independent of $X \subset M \setminus U$. Then the sections $v(\beta, \eta)$ are defined for $\|d\eta\|_{C^1(M)} < \varepsilon$ where $\varepsilon$ can be taken independent of $T$ or the fibre $X$. 


Using once again the property that for a fixed $\beta$ the sections $v(\beta, \eta)$ define deformations of $\exp_{v(\beta,0)}(X)$, a fibre near $X$, we obtain that $v(\beta, \eta)$ induces a vector field satisfying the assertions of Theorem B except that it is only defined on an open neighbourhood of $X$ in $M$. The vector fields obtained from $v(\beta, \eta)$ for different $X$ agree on the overlaps of their domains by the uniqueness part of the implicit function theorem. As $M \setminus U$ is compact an extension of $v(\eta)$ from $U$ to $M$ is obtained by patching with a finite number of these local vector fields.

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