Quantum chromodynamics (QCD) is the gauge theory describing one of the fundamental forces of Nature, i.e. the one responsible for holding together the quarks inside the proton. For over four decades scientists have tried to understand its intricate dynamics using analytical methods as well as first principle computer simulations. Despite the many successes a complete understanding is still missing. The goal of this paper is to shed light on such a complicated dynamics in an innovative way by using a modern version of the Dirac’s time-honored idea of electro-magnetic duality to analytically compute nonperturbative quantities relevant to QCD. Our method is general and can be extended to determine novel nonperturbative quantities for different strongly coupled gauge theories also at nonzero matter density and temperature.

One of the most fascinating possibilities is that generic asymptotically free gauge theories have magnetic duals. Arguably the existence of a possible dual of a generic nonsupersymmetric asymptotically free gauge theory able to reproduce its infrared dynamics must match the ’t Hooft anomaly conditions [1]. We have exhibited several solutions of these conditions for QCD and gauge theories with higher dimensional representations respectively in [2] and [3].

In this work we suggest a direct test of the possible existence of gauge duals using the conformal S-parameter [4] i.e. the one associated to gauge theories within the conformal window. This parameter is calculable, using the electric theory, near the upper limit of the conformal window [4] since there the electric theory is in a perturbative regime. The results are relevant to shed light on the conformal dynamics and are directly applicable to unparticle extensions of the standard model (SM) [5, 6]. Near the lower boundary of the conformal window we cannot compute S analytically but we expect the magnetic dual to be weakly coupled and hence derive a closed form expression for it via the gauge dual. We will refer to it as the magnetic S parameter ($S_m$).

The S-parameter [7–10] is [11]:

\[
S = -16\pi \frac{\Pi_{YY}(m_Z^2) - \Pi_{YY}(0)}{m_Z^2},
\]

where $\Pi_{YY}$ is the vacuum polarization of one isospin and one hypercharge current. In the following we use as reference point, instead of the $Z_0$ mass $m_Z$, the external momentum $q^2$. We couple to the SM a generic gauge theory with sufficient fermionic matter to develop an infrared fixed point (IRFP) with $N_f$ Dirac flavors. The associated quantum global symmetries are $SU_L(N_f) \times SU_R(N_f) \times U_Y(1)$ if the fermion representation is complex or $SU(2N_f)$ if real or pseudoreal. We weakly gauge $N_D = N_f/2$ doublets. To probe the large scale conformal dynamics via $S$, which is UV and IR finite, we add to the underlying gauge theory a relevant mass operator. This is a standard procedure when trying to investigate the physics of fixed points. We give the up and down type fermions, with respect to the electroweak interactions an equal mass $m$. The language of the electroweak precision parameters is borrowed to connect more easily to the phenomenological world.

Having replaced $m_Z^2$ with the momentum $q^2$ the dimensionless S-parameter can only be a function of the ratio of $q^2/m^2$. This is so since we assumed the underlying massless gauge theory to be conformal at large distances. Of course, a dynamical scale is generated when endowing the fermions with masses, however it must be directly proportional to this fermion mass and parametrically smaller. If this were not the case one could never recover the conformal limit when sending the fermion masses to zero. We are henceforth entitled to consider at least two limits with respect to the $q^2/m^2$ ratio [4]: The one in which the fermion masses go to zero, at finite external momentum and the associated S-parameter vanishes and the other one in which the external momentum vanishes first and the S-parameter

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assumes a nonzero numerical value \[3\]. We have also argued that the latter is the limit which smoothly connects to the $S$-parameter in the chirally broken phase relevant for technicolor. We will therefore concentrate on the limit for $S$ for which $q^2/m^2 \to 0$.

The electric $S$-parameter ($S_e$) is defined here as the one computed using the electrical variables. Of course, if the magnetic and the electric theory are gauge duals of each other then $S_m = S_e$. Near the electric (or magnetic) Banks-Zaks IRFP this parameter can be computed reliably by means of perturbation theory \[4\]. We found that for an electric $SU(N)$ gauge theory with $N_f$ Dirac fermions transforming according to the representation $r$ of the $SU(N)$ gauge group, and a sufficiently large number of flavors to be near the upper line of the conformal window, the leading terms in the $q^2/m^2$ expansion and at the leading perturbative order in the gauge coupling constant:

$$
\lim_{q^2/m^2 \to 0} S_e = \frac{\pi}{6\pi} \left[ 1 + \frac{1}{10x^2} + \frac{1}{70x^2} + O(x^{-3}) \right],
$$

with $x = \frac{m^2}{\Lambda^2}$. Here $N = N_D d[r]$ counts the number of doublets times the dimension of the representation $d[r]$ under which the fermions transform. For example for the fundamental representation $d[F] = N$, for an $SU(N)$ gauge group and $d[S] = N(N + 1)/2$ for the two-index symmetric representation of the gauge group. Note that given that we are in the conformal window the mass to the fermions is given via the standard Higgs mechanism.

Consider the case of an underlying gauge group $SU(3)$. The quantum flavor group of the massless theory is $SU_l(N_f) \times SU_R(N_f) \times U_V(1)$. The classical $U_A(1)$ symmetry is destroyed at the quantum level by the Adler-Bell-Jackiw anomaly. We indicate with $Q_{\alpha\xi}^i$ the two component left spinor where $\alpha = 1, 2$ is the spin index, $c = 1, ..., 3$ is the color index while $i = 1, ..., N_f$ represents the flavor. $Q_{\eta}^{\alpha\xi}$ is the two component conjugated right spinor. We summarize the transformation properties in the following table. The global anomalies are associated to the triangle diagrams featuring at the vertices three $SU(N_f)$ generators (either all right or all left), or two $SU(N_f)$ generators (all right or all left) and one $U_V(1)$ charge. We indicate these anomalies for short with:

$$
SU_{L/R}(N_f)^3, \quad SU_{L/R}(N_f)^2 U_V(1).
$$

For a vector like theory there are no further global anomalies. The cubic anomaly factor, for fermions in fundamental representations, is 1 for $Q$ and $-1$ for $\bar{Q}$ while the quadratic anomaly factor is 1 for both leading to $SU_{L/R}(N_f)^3 \propto \pm 3$, and $SU_{L/R}(N_f)^2 U_V(1) \propto \pm 3$.

We have computed the $S$-parameter in the perturbative regime of the conformal window, however we would like now to determine this parameter near the lower boundary of the conformal window. Here perturbation theory fails, in the electric variables, and one has to resort to other methods. However, if a magnetic gauge dual exists one expects it to be weakly coupled near the critical number of flavors below which one breaks large distance conformality in the electric variables. We can then determine $S$ near the lower boundary of the conformal window using perturbation theory in the magnetic variables. Determining a possible unique dual theory for QCD is, however, not simple given the few mathematical constraints at our disposal. The saturation of the global anomalies is an important tool but is not able to select out a unique solution. The goal is to find the explicit expression for $S_m$ in terms of the magnetic variables by means of the most general expectation for the structure of the gauge dual.

As argued in \[2\] \[3\] \[13\] a candidate gauge dual theory within the conformal window, saturating the ‘t Hooft anomaly conditions, would be constituted by an $SU(X)$ gauge group with global symmetry group $SU_l(N_f) \times SU_R(N_f) \times U_V(1)$ featuring magnetic quarks $q$ and $\bar{q}$ together with $SU(X)$ gauge singlet fermions identifiable as baryons built out of the electric quarks $Q$. Since mesons do not affect directly global anomaly matching conditions we can add them to the spectrum of the dual theory. In particular they are needed to let the magnetic quarks and the gauge singlet fermions interact with each others. The new mesons will be massless and have no-self potential to respect the conformal invariance of the model at large distances. We add to the magnetic quarks gauge singlet Weyl fermions which can be identified with the baryons of QCD but are, in fact, massless. The generic dual spectrum is summarized in table \[\text{I}\]. The wave functions for the gauge singlet fields $A, C$ and $S$ are obtained by projecting the flavor indices of the following operator

$$
\epsilon^{i(123)} Q_{i\xi}^1 \bar{Q}^{i\eta} \bar{Q}^{i\zeta},
$$

over the three irreducible representations of $SU_l(N_f)$ as indicated in the table \[\text{I}\]. These states are all singlets under the $SU_R(N_f)$ flavor group. Similarly one can construct the only right-transforming baryons $\tilde{A}, \tilde{C}$ and $\tilde{S}$ via $\tilde{Q}$. The $B$ states are made by two $Q$ fields and one right field $\tilde{Q}$ while the $D$ fields are made by one $Q$ and two $\tilde{Q}$ fermions. $y$ is the, yet to be determined, baryon charge of the magnetic quarks while the baryon charge of com-
TABLE II: Massless spectrum of magnetic quarks and baryons and their transformation properties under the global symmetry group. The last column represents the multiplicity of each state and each state is a Weyl fermion.

| Fields | SU\(_3\) | SU\(_L(N_f)\) | SU\(_R(N_f)\) | U\(_V(1)\) | # of copies |
|--------|---------|-------------|-------------|-------------|------------|
| q \(q\) | 1       | 1           | y           | -y          | 1          |
| A      | 1       | 3           | \(\ell_A\)  |             |            |
| S      | 1       | 3           | \(\ell_S\)  |             |            |
| C      | 1       | 3           | \(\ell_C\)  |             |            |
| \(\bar{A}\) | 1     | 3           | \(\ell_{\bar{A}}\) |             |            |
| \(\bar{S}\) | 1  | 3           | \(\ell_{\bar{S}}\) |             |            |
| \(\bar{C}\) | 1   | 3           | \(\ell_{\bar{C}}\) |             |            |
| \(M_f\) | 1    | 0           | 1           |             |            |

The decomposition of the charged conjugated baryons is obtained from the one above by exchanging left with right.

Since we are gauging with respect to the electroweak theory the first two flavors we provide a mass term to them as done in \[14\], i.e. via the introduction of a SM Higgs-type interaction. Since we are operating within the conformal window this is the direct way to provide a mass to the fermions. By symmetry arguments we can pair only the states which do not transform with respect to \(SU_3(N_f - 2) \times SU_R(N_f - 2)\) but still transform nontrivially under \(SU_3(N_f - 2) \times SU_R(2)\). These states are \((1, 1, 1, 1)\) for the baryon \(S\); \((1, 1, 1, 1)\) for \(C\); \((1, 1, 1, 1)\) for \(B_A\) and for \(B_S\) the state \((1, 1, 1, 1)\). We need to consider the charge conjugated states as well. In terms of the spinorial representations of \(SU_3(2) \times SU_R(2)\) the states above are \(\ell_5(\frac{1}{2}, 0) \oplus \ell_C(\frac{1}{2}, 0) \oplus \ell_{B_A}(0, \frac{1}{2}) \oplus \ell_{B_S}(1, \frac{1}{2})\) with the \(\ell\) prefactor taking into account the multiplicity of each state. They will pair with their charged conjugated fermion via the mass term operator of the type \(\psi H \bar{\psi}\) with \(H\) the standard model Higgs field which transforms according to the \((\frac{1}{2}, \frac{1}{2})\) representation. Note that we can only pair states with \(j_Z = j_1 \pm \frac{1}{2}\).

Each pair of conjugated fermions transforming according to \((j_1, j_2)\) under \(SU_3(2) \times SU_R(2) \times U_V(1)\) leads to the following contribution to the \(S_m\) parameter \[14\]:

\[
S_b = \frac{2d_b}{3\pi} \sum_{jj'} X_{jj'} \left[ 2f \left( m_{j_f}^2, m_{j'_f}^2 \right) + g \left( m_{j_f}^2, m_{j'_f}^2 \right) \right] + \\
+ \left[ j (j + 1) \right]^2 \sum_{j} \frac{2j + 1}{9\pi} \left( f(j + 1) \right),
\] (9)
with the index $b$ indicating the specific baryon and $d_b$ its
degeneracy. We also have $j^- = |j_1 - j_2|$, $j^+ = j_1 + j_2$ and
$j^- \leq j \leq j^+$ the total spin for each baryon contribution.
If more than one spinorial representation belongs to the
same baryon $b$ the contributions of all the states must
be taken into account. The nonvanishing components of
the group theoretical factor $X_{\ell j \ell' j'}$ are:

$$X_{\ell j \ell' j'} = \left[ 1 - \frac{(j^+ + 1)^2}{(J(j+1))} \right] \frac{(J+1)(2J+1)}{12},$$

$$X_{\ell j-1 \ell' j'} = \frac{-1}{12} \left( (j^+ + 1)^2 - j^2 \right) \left( j^2 - j^2 \right).$$

(10)

The functions $f$ and $g$ read [14]:

$$f(m_f^2, m_r^2) = -6 \int_0^1 dx (1-x) \log \left( \frac{\sqrt{x m_f^2 + (1-x)m_r^2}}{\mu^2} \right),$$

$$g(m_f^2, m_r^2) = 6 \int_0^1 dx \frac{x(1-x)m_fm_r}{xm_r^2 + (1-x)m_f^2}. \quad (11)$$

The mass of each fermion is directly proportional to the
electric fermion mass $m$ and depends on the representa-
tion according to the formula $m_f = -m_f \frac{j^+ + j^-}{b(j^+ + j^-)}$. We have
chosen as a reference energy scale $\mu = m$. The contribu-
tion of the baryon sector is then:

$$S_B = \sum_b S_b. \quad (12)$$

The complex scalar meson $M$ decomposes as:

$$M \rightarrow \left[ (\Box 1, \overline{\Box} 1) \oplus (\Box 1, 1, \overline{\Box}) \oplus (1, \overline{\Box}, \overline{\Box} 1) \oplus (1, \Box 1, 1, \overline{\Box}) \right].$$

(13)

Only the first state, $(\frac{1}{2}, \frac{1}{2})$, contributes to $S_M$ and leads to:

$$S_M = \frac{1}{3^n} \sum_{j,j'} f(m_f^2, m_r^2). \quad (14)$$

with $j, j' = 1, 0$, $m_f^2 = m_r^2(1 + J(J + 1))$. This is a different
mass parameterization than the one given in [14]. We
also have $m_f^2 \propto m^2$. All factors of order unity have been
set to unity and finally set the scale $\mu = m_0$ in the function
$f$ for the scalars. The contribution to $S_M$ vanishes unless
there is a mass splitting between the different multiplets
of the unbroken $SU(2)_V$ symmetry.

Putting together the various terms we have for the
normalized $S_m$:

$$\frac{67}{3} S_m = \frac{X}{3} + \frac{\ell_e + \ell_B}{3} + \frac{25}{729} \ell_B (32 \log 2 - 39) - 0.14. \quad (15)$$

The explicit dependence on the quark masses disappear
for the $S_m$ parameter in agreement with the expectation
from the leading contribution in $q^2/m^2$ to the $S_e$
parameter. The above is the general expression for $S_m$ near
the lower end of the conformal window corresponding to
the nonperturbative regime in the electric variables.

From this expression is evident that the present definition
of the normalized $S$-parameter counts the relevant
degrees of freedom as function of the number of flavors.
We estimate $S_m$ using the possible dual provided in [2]
for which $X = 2N_f - 15$, $\ell_e = 2$, $\ell_B = -2$ (we take +2
since we are simply counting the states) with the other
$\ell$s vanishing. Asymptotic freedom for the magnetic dual
requires at least $N_f = 9$ for which $6\pi S_m/3 = 1.523$ while
if the lower bound of the conformal window occurs for
$N_f = 10$ we obtain $6\pi S_m/3 = 2.19$. Of course, only one
of these two values should be considered as the actual
value of the normalized magnetic $S$ parameter near the
lower end of the electric conformal window. Both values
are such that the normalized $S_m$ is always larger than the
electrical one near the upper end of the conformal win-
dow and are close to the one for two flavors QCD which
is around two [15].

The central result [15] rely on the existence of a gauge
dual to QCD built extending the famous suggestion
of 't Hooft. The form of the dual is general and can be ex-
tended to other strongly coupled gauge theories also at
nonzero temperature and matter density. Furthermore
the existence of a gauge dual can now be finally estab-
lished by comparing [15] with lattice computations of
the same two-point function using the electric variables,
i.e. ordinary QCD.
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