NONLINEAR MCKEAN-VLASOV DIFFUSIONS UNDER THE WEAK HÖRMANDER CONDITION WITH QUANTILE-DEPENDENT COEFFICIENTS

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ABSTRACT. In this paper, the strong existence and uniqueness for a degenerate finite system of quantile-dependent McKean-Vlasov stochastic differential equations are obtained under a weak Hörmander condition. The approach relies on the apriori bounds for the density of the solution to time inhomogeneous diffusions. The time inhomogeneous Feynman-Kac formula is used to construct a contraction map for this degenerate system.

Keywords: McKean-Vlasov equation, quantile, Langevin equation, weak Hörmander condition, Feynman-Kac formula, two-sided Gaussian estimates, quantile-dependent McKean-Vlasov equation, quantile-dependent PDE.

1. INTRODUCTION

Stochastic differential equations (SDEs) with coefficients depending on the probability distribution of the unknown have become a hot research area in recent years. One particular topic is the so-called mean field $d$-dimensional stochastic differential equations (see e.g. [2–4] and references therein): $dX_t = F(t, L(X_t), X_t)dt + \sigma(t, L(X_t), X_t)dW_t$, where $(W_t)_{t \geq 0}$ is a $d$-dimensional Brownian motion, and where $L(X_t) \in \mathcal{P}(\mathbb{R}^d)$ is the probability law of the unknown $X_t$. To guarantee the existence and uniqueness of the solution, researchers often assume that $F$ and $\sigma$ are Lipschitzian on $\mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d$ with respect to a Wasserstein metric on the space $\mathcal{P}(\mathbb{R}^d)$ of probability measures. However, this condition is sometimes hard to verify. For example, in finance and other applications (e.g. [5]), the following quantile-dependent equation is considered:

$$dX_t = F(t, Q_\alpha(X_t), X_t)dt + \sigma(t, Q_\alpha(X_t), X_t)dW_t,$$

where $F : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ are continuous functions, $\alpha = (\alpha_1, \ldots, \alpha_d) \in (0, 1)^d$, and $Q_\alpha(X_t)$ is the $\alpha$-quantile (vector) of the...
probability measure $\mathcal{L}(X_t)$ of $X_t$, namely,

$$(Q_\alpha(X_t))_j = (Q_\alpha(\mathcal{L}(X_t)))_j = \inf \left\{ y_j \in \mathbb{R}, \int_{x \in \mathbb{R}^d, x_j \leq y_j} \mathcal{L}(X_t)(dx) \geq \alpha_j \right\}, \quad j = 1, \ldots, d.$$  

It is well-known that for any two real valued random variables $X$ and $Y$ with cumulative distributions $F_X$ and $F_Y$, the $p$-Wasserstein distance is given by

$$I_p(X, Y) = \left( \int_0^1 |F_X^{-1}(\alpha) - F_Y^{-1}(\alpha)|^p d\alpha \right)^{1/p}.$$  

From the above expression it is obvious that the coefficients in (1.1) are not continuous with respect to the Wasserstein distance for any finite $p \geq 1$. Hence, we need a completely different approach to study the quantile-dependent equations.

The works [5] and [13] are among the first to study this type of equations. [5] suggested such a model from financial aspect, proved the existence of solution and posed the well-posedness problem as an open one. Under differentiable and Lipschitzian conditions on $\sigma$ and $F$, and under the uniform ellipticity condition on $a := \sigma \sigma^*$, namely, there exists constant $\Lambda > 0$, such that

$$\Lambda^{-1} |\xi|^2 \leq |\xi| a(t, y, x) \xi| \leq \Lambda |\xi|^2, \quad \forall (t, y, x) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d, \xi \in \mathbb{R}^d,$$

the well-posedness of equation (1.1) was obtained in [13].

The aim of this paper is to remove the uniform ellipticity condition (1.2). As in [2–5,13], we follow the main-stream approach of the fixed point theorem. However, to control the quantile, we require that the solution $X_t$ (as random vector) has density (with respect to Lebesgue measure) and this density is strictly positive with a certain decay property. This problem of existence of density is an important topic in probability theory and partial differential equations. To ensure the existence of the density of the solution of the McKean-Vlasov equation, we impose some weak Hörmander condition (see e.g. [11, Page 355]). Let us recall one such result on the following $nd$-dimensional Langevin-type stochastic differential equation:

$$\begin{align*}
    dX_t^1 &= F_1 \left( t, Q_\alpha(X_t), X_t^1, \cdots, X_t^n \right) dt + \sigma \left( t, Q_\alpha(X_t), X_t^{(1)}, \cdots, X_t^{(n)} \right) dW_t, \\
    dX_t^2 &= F_2 \left( t, Q_\alpha(X_t), X_t^1, \cdots, X_t^n \right) dt, \\
    dX_t^3 &= F_3 \left( t, Q_\alpha(X_t), X_t^2, \cdots, X_t^n \right) dt, \\
    &\vdots \\
    dX_t^n &= F_n \left( t, Q_\alpha(X_t), X_t^{n-1}, X_t^n \right) dt,
\end{align*}$$

(1.3)

where $d$ and $n$ are positive integers; $(W_t)_{t \geq 0}$ is a standard $d$-dimensional Brownian motion; $X^{(i)}, 1 \leq i \leq n$, are all $d$-dimensional processes, and $(X_t)_{t \geq 0} = (X_t^1, \ldots, X_t^n)_{t \geq 0}$; $F_i : \mathbb{R}_+ \times \mathbb{R}^{nd} \times \mathbb{R}^{nd} \to \mathbb{R}^d$; $F_i : \mathbb{R}_+ \times \mathbb{R}^{nd} \times \mathbb{R}^{(n-i+2)d} \to \mathbb{R}^d$ for $i = 2, \ldots, n$; and $\sigma : \mathbb{R}_+ \times \mathbb{R}^{nd} \times \mathbb{R}^{nd} \to \mathbb{R}^d \otimes \mathbb{R}^d$ are continuous functions.
Denote by $I_d$ and $0_d$ the $d \times d$ identity and zero matrices respectively. Introducing $D = (I_d, 0_d, \cdots, 0_d)^T \in \mathbb{R}^{nd \times d}$, and letting $F = (F_1, \cdots, F_n)^T$, we can rewrite (1.3) in the following abbreviated form

$$dX_t = F(t, Q_\alpha(X_t), X_t)dt + D\sigma(t, Q_\alpha(X_t), X_t)dW_t. \quad (1.4)$$

The system of equations (1.3) (or (1.4)) is highly degenerate if $n \geq 2$ and the elliptic condition (1.2) is obviously not satisfied. Still, in the special case that $F$ and $\sigma$ in (1.4) are independent of the quantile, namely, when (1.4) is reduced to

$$dX_t = \tilde{F}(t, X_t)dt + D\tilde{\sigma}(t, X_t)dW_t, \quad (1.5)$$

the existence of the density, its derivatives and its two-sided Gaussian bounds have been obtained in [6, 7, 14, 15], which are critical to this work.

The degenerate stochastic differential equations of the form (1.5) have been attracted more and more attention in the past years (see e.g. [16, 20, 21]). When a Newton equation $\ddot{x}(t) = F(t, x(t), \dot{x}(t))$ is under influence of some uncertainty, the corresponding stochastic differential equation could be $\ddot{x}(t) = F(t, x(t), \dot{x}(t)) + G(t, x(t), \dot{x}(t))\dot{W}(t)$. This equation is of the form (1.5) if we let $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$, namely, $dx_1(t) = F(t, x_1(t), x_2(t)) + G(t, x_1(t), x_2(t))dW(t)$ and $dx_2(t) = x_1(t)dt$. There are also many other examples. Equation (1.5) also corresponds to the dynamics of a finite-dimensional non-linear Hamiltonian system (a chain of anharmonic oscillators) coupled with two heat reservoirs at different temperatures, which was used by Eckmann et al. [8] (see also [10, 18, 19]) to study the statistical mechanics of such system. Rey-Bellet and Thomas [17] considered the low temperature asymptotic behavior of the invariant measure in the framework of (1.5). Additionally, there are some applications of the Langevin-type equation in pricing Asian options (see e.g. [1]).

To obtain the existence and uniqueness of equation (1.4), we use the fixed point theorem. But, to apply the fixed point theorem, we need to bound a certain distance between $Q_\alpha(h_1)$ and $Q_\alpha(h_2)$ by a certain distance of $h_1$ and $h_2$ (see (4.1)). This was already done in [13]. We also need to bound the distance between $u^{(1)}$ and $u^{(2)}$ by the distance between $\omega^{(1)}$ and $\omega^{(2)}$, where each $u^{(i)}$ ($i = 1, 2$) is the density of $X_t^{(i)}$ in (1.3) when $Q_\alpha(X_t^{(i)})$ is replaced by $\omega^{(i)}$ (see Proposition 4.2). This is relatively complicate and requires the fact that the density $u$ of the solution $X_t$ of (1.4) is characterized by the corresponding Fokker-Planck equation. Thus, the above problem of controlling the distance between $u^{(1)}$ and $u^{(2)}$ by the distance between $\omega^{(1)}$ and $\omega^{(2)}$ is reduced to how the solution of the corresponding Fokker-Planck equation depends on the coefficients. However, because of the degeneracy, it is hard to use the PDE approach as in [13]. Instead, we shall use the time-dependent Feynman-Kac formula.

The paper is organized as follows. In Section 2, we present the main hypotheses and main results of this paper as well as some notation. Some useful apriori estimates on the density including the tail estimates and lower bounds are given in Section
3. These two-sided bounds and the Feynman-Kac formula play a central role in the whole article. We present in Section 4 the proof of our main results.

2. MAIN RESULTS

For any \( x \in \mathbb{R}^n \), we write \( x = (x_1, \ldots, x_n) = (x_1^1, \ldots, x_1^d, \ldots, x_n^1, \ldots, x_n^d) \), where for \( i = 1, \ldots, n, j = 1, \ldots, d, x_i \in \mathbb{R}^d, x_i^j \in \mathbb{R} \). Let \( |x_i| \) denote the Euclidean norm of \( x_i \), that is, \( |x_i| = \left( \sum_{j=1}^d |x_i^j|^2 \right)^{1/2} \). Let \( F_i : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{(n-i+2)\wedge n}d \to \mathbb{R}^d \) be continuous mappings. For notational simplicity, we may consider \( F_i \) as a continuous mapping from \( \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \) to \( \mathbb{R}^d \) and write \( F = (F_1, \ldots, F_n) = (F_1^1, \ldots, F_1^d; \ldots; F_n^1, \ldots, F_n^d) \) as well.

For any \( d \times d \) matrix \( a = (a_{ij})_{i,j=1}^d \), denote by \( |a| = (\sum_{i,j=1}^d |a_{ij}|^2)^{1/2} \) its Hilbert-Schmidt norm. In what follows, we use \( \| \cdot \|_{p} \) for the \( L^p \) norm on the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). For any measurable function \( f \) on a Euclidean space, \( |f|_{L^p} \) denotes the \( L^p \) norm of \( f \) with respect to the Lebesgue measure.

The notation \( \nabla \) stands for the gradient with respect to all space variables. Let \( f \in C(\mathbb{R}_+^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}_+^n) \), \( k = 1, \ldots, n \). \( \nabla_x f(t, y, x) \) denotes the gradient operator w.r.t. the \( i \)th space variable \( x_i \in \mathbb{R}^d \), which is a \( d \times d \) Jacobian matrix.

Fix a time horizon \([0, T]\). We will need the following hypotheses for coefficients \( F \) and \( \sigma \) and the initial condition \( X_0 \).

\textbf{(H1)} \( F \) is uniformly bounded at the origin of the third argument. That is, there exists a positive constant \( \kappa \) such that

\[
\sup_{t \in [0, T], y \in \mathbb{R}^n} |F(t, y, 0)| \leq \kappa < \infty.
\]

\textbf{(H2)} The function \( a := \sigma \sigma^* \) is uniformly elliptic, namely, (1.2) is satisfied.

\textbf{(H3)} \( F \) and \( \sigma \) are uniformly Lipschitz continuous in space variables with constant \( \kappa > 0 \), i.e., for all \( y, \bar{y}, x, \bar{x} \in \mathbb{R}^n \),

\[
\sup_{t \in [0, T]} \left( |F(t, y, x) - F(t, \bar{y}, \bar{x})| + |\sigma(t, y, x) - \sigma(t, \bar{y}, \bar{x})| \right) \leq \kappa \left( |x - \bar{x}| + |y - \bar{y}| \right).
\]

(2.1)

\textbf{(H4)} The function \( x \mapsto F(t, y, x) \) is twice differentiable and function \( x_1 \mapsto a(t, y, x_1, \ldots, x_n) \) is three times differentiable. Moreover, the following inequalities hold

\[
\sup_{(t, y, x) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n} \left| \sum_{j,k=1}^d \frac{\partial^2}{\partial x_1^j \partial x_1^k} a_{jk}(t, y, x) \right| \leq \kappa.
\]
\[
\sup_{t \in [0,T]} \sum_{j,k=1}^{d} \left| \frac{\partial^2}{\partial x_1^j \partial x_1^k} a_{kj}(t, y, x) - \frac{\partial^2}{\partial x_1^j \partial x_1^k} a_{kj}(t, \bar{y}, \bar{x}) \right| \leq \kappa \left( |x - \bar{x}| + |y - \bar{y}| \right),
\]

and
\[
\sup_{t \in [0,T]} \sum_{i=1}^{d} \sum_{j=1}^{d} \left| \frac{\partial}{\partial x_i^j} F_i(t, y, x) - \frac{\partial}{\partial x_i^j} F_i(t, \bar{y}, \bar{x}) \right| \leq \kappa \left( |x - \bar{x}| + |y - \bar{y}| \right),
\]

for all \( x, \bar{x}, y, \bar{y} \in \mathbb{R}^{nd} \).

(H5) For any integer \( i = 2, \ldots, n \), the derivative \( \nabla_{x_{i-1}} F_i(t, y, x) \) is \( \eta \)-Hölder continuous in the first spatial variable \( x_{i-1} \) with constant \( \kappa \), and there exists a closed convex subset \( \varepsilon_{i-1} \) contained in the set of invertible \( d \times d \) matrices, such that for all \( t \in [0,T] \) and \( (x_{i-1}, \ldots, x_n) \in \mathbb{R}^{(n-i+2)d} \), the matrix \( \nabla_{x_{i-1}} F_i(t, y, x_{i-1}, \ldots, x_n) \) belongs to \( \varepsilon_{i-1} \).

(I) \( X_0 \) is a random variable independent of \( W \). The probability law of \( X_0 \) has a continuously differentiable density \( f > 0 \) satisfying the following integrability condition
\[
U = \int_0^\infty \sup_{|z| \geq r} |f(z)|^2 (r^{4n-1+\varepsilon} + r^{n-1}) dr
+ \int_0^\infty \left[ \sup_{|z| \geq \lambda} |\nabla f(z)|^4 \right] (\lambda^{4nd-1+\varepsilon} + \lambda^{nd-1}) d\lambda < \infty, \quad (2.2)
\]

for some constant \( \varepsilon > 0 \).

**Remark 2.1.** The most important hypotheses in this work are the hypotheses (H2) and (H5): the matrices \( (\nabla_{x_{i-1}} F_i)_{2 \leq i \leq n} \) have full rank, which imply a version of the (weak) Hörmander condition. It ensures the existence of the probability density of the solution to (1.4). Let us point out that in (H2) we assume that \( a = \sigma \sigma^* \) is uniformly elliptic. However, the diffusion coefficient \( D \sigma \sigma^* D^* \) of the whole system (1.4) is highly degenerate.

**Remark 2.2.** Hypotheses (H3), (H4) are to guarantee the Lipschitz continuity of the function \( c \) defined in (2.7) below. In addition, they imply that \( c \) is bounded, which is needed in the application of the Feynman-Kac formula (see Theorem 3.10).

**Remark 2.3.** At the first look, hypothesis (I) seems a little complicated. However, Gaussian densities and many other functions satisfy this condition. Furthermore, it is worth mentioning that to prove Proposition 4.2 (i.e. the local existence and uniqueness), hypothesis (I) can be weakened to the following form:
\[
\int_{\mathbb{R}^{nd}} f(y)^2 (|y|^{nd+\varepsilon} + 1) dy + \int_0^\infty \left[ \sup_{|z| \geq \lambda} |\nabla f(z)|^4 \right] (\lambda^{4nd-1+\varepsilon} + \lambda^{nd-1}) d\lambda < \infty.
\]

The condition (2.2) is used to guarantee the global existence and uniqueness of (1.4).
In the next theorem, we provide the existence and uniqueness result for equation (1.4), which is the main result of this paper.

**Theorem 2.4.** Assume hypotheses (H1)-(H5) and hypothesis (I). Then, there exists a unique strong solution to SDE (1.4) on \([0, T]\).

The idea to prove Theorem 2.4 follows the spirit of [13] to construct a contraction mapping associated to equation (1.4). To this end, we need to introduce an auxiliary equation. Given a continuous function \(\omega\) on \([0, T]\) with values in \(\mathbb{R}^{nd}\), we consider the following stochastic differential equation

\[
dX^\omega_t = F(t, \omega_t, X^\omega_t) dt + D\sigma(t, \omega_t, X^\omega_t) dW_t,
\]

with initial condition \(X_0\) satisfying hypothesis (I). Under hypotheses (H1)-(H5), equation (2.3) has a unique solution, whose density exists and satisfies the following Fokker-Planck equation

\[
\frac{\partial}{\partial t} u^\omega_t(x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(t, \omega_t, x) \frac{\partial^2}{\partial x_i \partial x_j} u^\omega_t(x) + \langle b(t, \omega_t, x), \nabla u^\omega_t(x) \rangle + c(t, \omega_t, x) u^\omega_t(x),
\]

with initial condition \(u_0^\omega = f\), where

\[
a = (a_{ij})_{i,j=1}^d = \sigma \sigma^*,
\]

\[b = (b_1, \ldots, b_n)\) with \(b_i = (b_1^i, \ldots, b_d^i), i = 1, \ldots, n\) and

\[
b^j_i(t, y, x) = -F^j_i(t, y, x) + 1_{\{i=1\}} \sum_{k=1}^d \frac{\partial}{\partial x_k} a_{kj}(t, y, x),
\]

and

\[
c(t, y, x) = -\sum_{i=1}^n \sum_{j=1}^d \frac{\partial}{\partial x_i} F^j_i(t, y, x) + \frac{1}{2} \sum_{j,k=1}^d \frac{\partial^2}{\partial x_j \partial x_k} a_{jk}(t, y, x),
\]

for any \((t, y, x) \in [0, T] \times \mathbb{R}^{nd} \times \mathbb{R}^{nd}\). Similarly, if (1.4) has a solution \(X_t\) with quantile \(Q_\alpha(X_t)\) being continuous in time, then the law of \(X_t\) has a density \(u\) that is the solution to the following equation

\[
\frac{\partial}{\partial t} u_t(x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(t, Q_\alpha(u_t), x) \frac{\partial^2}{\partial x_i \partial x_j} u_t(x) + \langle b(t, Q_\alpha(u_t), x), \nabla u_t(x) \rangle + c(t, Q_\alpha(u_t), x) u_t(x).
\]

Thus, the proof of Theorem 2.4 is reduced to prove that (2.8) admits a unique solution. However, it is not easy to deal with such PDE whose coefficients depend on quantiles. We shall find an appropriate Banach space \(\mathcal{B}\) and construct a mapping...
The well-posedness of \((2.8)\) is shown by proving that \(\mathcal{M}\) is a contraction map on \(\mathcal{B}\) in Proposition 4.4 (below).

In the next theorem, we prove the well-posedness of \((2.8)\).

**Theorem 2.5.** Let \(f\) be a continuous differentiable function on \(\mathbb{R}^{nd}\) satisfying hypothesis (I). Assume hypotheses (H1)-(H5). Then, there exists a function \(u\) on \([0, T] \times \mathbb{R}^{nd}\), which is the unique solution to PDE \((2.8)\) with initial condition \(f\).

In [13], to obtain the stability result the author uses the two-sided bounds of the density and its first order derivatives under uniform ellipticity condition. In our hypoellipticity case, we encounter the following difficulties.

1. For \((1.5)\), Pigato [15] obtained upper bounds for the derivatives of transition density of any order. The first derivative with respect to the variable \(x_j^i\) \((i = 1, \ldots, n; j = 1, \ldots, d)\) is given by
   \[
   |\partial_{x_j^i}p(t, x; 0, y)| \leq \frac{C}{t^{(2\frac{d-1}{d}+1+n^2d)/2}} \exp\left(\frac{|T_t^{-1}(x - \theta_t(y))|^2}{C}\right),
   \]
   where \(y\) is the initial position, \(C\) is a constant, \([\cdot]\) denote the integer part function, \(T\) and \(\theta_t\) are given by \((3.1)\) and \((3.2)\) below. As we see, this bound is more singular near \(t = 0\) than that in the elliptic case.

2. To overcome this problem, we assume that the initial condition \(f\) satisfies certain differentiability and integrability (over the whole \(\mathbb{R}^{nd}\)) conditions, in hope that the singularity difficulty can be absorbed in the initial condition. However, proceeding with this effort, we immediately encounter the difficulty that we don’t know how to pass the gradient \(\nabla_x p(t, x; s, y)\) to \(\nabla f\) in the following integral:
   \[
   \int_{\mathbb{R}^{nd}} f(y)\nabla_x p(t, x; s, y)dy
   \]
   since \(p(t, x; s, y)\) is not the form of \(p(t, s, x - y)\). To get around this difficulty, we apply the time inhomogeneous Feynman-Kac formula. This enables us to finish the stability analysis of the solution to \((2.4)\) with respect to \(\omega\).

3. **A priori estimates of the density**

In the remaining part of the paper, we assume that \(d = 1\) to simplify the presentation. The case \(d > 1\) can be treated analogously with only additional notational complexity. We use \(C > 0\) to denote a generic constant which may vary from occurrence to occurrence.

First, let us turn to \((1.5)\). We need a result from [6]. To state this result, we need to introduce the following conditions on the coefficients \(\bar{F}\) and \(\bar{\sigma}\).
(C1): $\bar{F}(t, 0)$ is bounded for all $t \in [0, T]$ and $\bar{a} = \bar{a}_0$ is uniformly elliptic with the positive constant $\Lambda$.

(C2): $\bar{a}$ is globally Lipschitz in the space variable uniformly in time variable. For all $j = 1, 2, \ldots, n$ the functions $\bar{F}_i, i = 1, \ldots, j$, are uniformly $\eta_j$-Hölder continuous in the $j$th spatial variable with $\eta_j \in (\frac{2j-2}{2j-1}, 1]$, uniformly in time and other spatial variables.

(C2’): The functions $\bar{F}_1, \ldots, \bar{F}_n$ and $\bar{a}$ are uniformly Lipschitz and $\eta$-Hölder continuous ($\eta \in (0, 1]$) with respect to the underlying space variables respectively.

(C3): For each integer $2 \leq i \leq n$, $(t, x_i, \ldots, x_n) \in \mathbb{R}_+ \times \mathbb{R}^{(n-i+1)d}$, the function $x_{i-1} \in \mathbb{R}^d \mapsto \bar{F}_i(t, x_{i-1}, \ldots, x_n)$ is continuously differentiable and its derivative, denoted by $(t, x_{i-1}, \ldots, x_n) \in \mathbb{R}_+ \times \mathbb{R}^{(n-i+2)d} \mapsto \nabla_{x_{i-1}} \bar{F}_i(t, x_{i-1}, \ldots, x_n)$, is $\eta$-Hölder continuous in the first space variable $x_{i-1}$ with constant $\kappa$. Moreover, there exists a closed convex subset $\varepsilon_{i-1}$ contained in the set of invertible $d \times d$ matrices, such that for all $t \geq 0$, $i = 2, \ldots, n$ and $(x_{i-1}, \ldots, x_n) \in \mathbb{R}^{(n-i+2)d}$, the matrix $\nabla_{x_{i-1}} \bar{F}_i(t, x_{i-1}, \ldots, x_n)$ belongs to $\varepsilon_{i-1}$.

Remark 3.1. Conditions (C1), (C2), (C2’) and (C3) can be easily verified by hypotheses (H1)-(H5). In fact, (C1) and (C3) are the same as (H1), (H2) and (H5). Additionally, hypotheses (H3) and (H4) imply (C2) and (C2’).

Theorem 3.2 (see [6]). Assume (C1), (C2) and (C3). There exists a unique strong solution to SDE (1.5).

We also need a result about the Gaussian estimate for the density of the solution to (1.5). To state this result we introduce the scale matrix $T$ and shift vector $\theta$ as follows. Fix $t \geq 0$ and $x \in \mathbb{R}^d$. Let $T$ denote the following $nd \times nd$ diagonal matrix:

$$
T_t = \begin{pmatrix}
T_t^1 & 0 & \cdots & 0 \\
0 & T_t^2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & T_t^n
\end{pmatrix} = \begin{pmatrix}
t^\frac{1}{2}I_d & 0 & \cdots & 0 \\
0 & t^\frac{1}{2}I_d & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & t^{n-\frac{1}{2}}I_d
\end{pmatrix}.
$$

(3.1)

Let $\theta_t(x) : [0, T] \mapsto \mathbb{R}^d$ be the solution to following (deterministic) ODE,

$$
\begin{aligned}
\frac{d}{dt} \theta_t(x) &= \bar{F}(t, \theta_t(x)), \\
\theta_0(x) &= x.
\end{aligned}
$$

(3.2)

Theorem 3.3 (see [7]). Assume (C1), (C2’) and (C3). Let $X$ be the solution to (1.5) with initial condition $X_0 = x$, where $x \in \mathbb{R}^d$. Then, for any $t \in [0, T]$, the law of $X_t$ admits a probability density, denoted by $p_t(\cdot, x)$. Moreover, there exists a constant $C_T \geq 1$, depending on $T, n, d, \Lambda, \eta$, the Lipschitz constants in (C1)-(C3),
and \( \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n-1} \), such that for any \( y \in \mathbb{R}^n \),

\[
\frac{1}{C_T t^{n^2/2}} \exp \left(-C_T |T^{-1}(\theta_t(x) - y)|^2\right) \leq p_t(y, x) \leq \frac{C_T}{t^{n^2/2}} \exp \left(-C_T^{-1} |T^{-1}(\theta_t(x) - y)|^2\right),
\]

(3.3)

where \( T_t \) and \( \theta_t(x) \) are given by (3.1) and (3.2) respectively.

**Remark 3.4.** We still cite the theorem for general dimension \( d \). However, we will continue to work on the case \( d = 1 \).

Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be defined as in Section 1 (with \( d = 1 \)). For any continuous function \( \omega : [0, T] \to \mathbb{R}^n \), and \( x \in \mathbb{R}^n \), we define, analogously to (3.2), a function \( \theta^\omega = (\theta^\omega_1, \ldots, \theta^\omega_n) \) on \([0, T]\) with values in \( \mathbb{R}^n \) by the following ODE

\[
\begin{aligned}
\frac{d}{dt} \theta^\omega(t, x) &= F(t, \omega_t, \theta^\omega(t, x)), t \in [0, T], \\
\theta^\omega(0, x) &= x.
\end{aligned}
\]

(3.4)

We have the following lemma about \( \theta^\omega \).

**Lemma 3.5.** Assume hypotheses (H1)-(H5) and assume that \( \omega \) is a continuous function of \( t \in [0, T] \). Let \( \theta^\omega \) satisfy (3.4). Then, for \( 0 \leq t \leq T \) and \( x \in \mathbb{R}^n \), the following inequalities hold

\[
\exp(-\kappa t)|x| - \kappa t \leq |\theta^\omega(t, x)| \leq (|x| + \kappa t) e^{\kappa t}
\]

(3.5)

and

\[
\exp(-n\kappa t) \leq \det(\nabla \theta^\omega(t, x)) \leq e^{n\kappa t},
\]

(3.6)

where \( \kappa \) is the positive constant that appeared in hypotheses (H1)-(H5).

**Proof.** By the Lipschitz property and uniformly boundedness (at the origin) of \( F \), we see that

\[
|\theta^\omega(t, x)| = |x + \int_0^t F(r, \omega_r, \theta^\omega(r, x)) dr| \\
\leq |x| + \int_0^t (|F(r, \omega_r, 0)| + \kappa |\theta^\omega(r, x)|) dr \\
\leq |x| + \kappa t + \int_0^t \kappa |\theta^\omega(r, x)| dr.
\]

An application of Gronwall’s inequality yields

\[
|\theta^\omega(t, x)| \leq (|x| + \kappa t) e^{\kappa t}.
\]

(3.7)
This proves the second inequality in (3.5). To prove the first inequality, we consider the following backward ODE:

\[
\frac{d}{ds}\hat{\theta}_s = -F(s, \omega_s, \hat{\theta}), \quad 0 \leq s < t, \\
\hat{\theta}_t = \xi \in \mathbb{R}^n.
\]

(3.8)

Similar to (3.7), we can show that

\[
|\hat{\theta}_s| \leq (|\xi| + \kappa(t-s)) e^{\kappa(t-s)},
\]

for all $s \in [0,t]$. Notice that $\hat{\theta} = \{\theta^\omega(s, x) ; s \in (0,t)\}$ is the solution to (3.8) with terminal condition $\hat{\theta}_t = \theta^\omega(t, x)$. Then, we have

\[
|x| = \hat{\theta}_0 \leq (|\theta^\omega(t, x)| + \kappa t) e^{\kappa t}.
\]

The proof of inequality (3.5) is then completed.

Taking the derivative of the following equation with respect to $x$,

\[
\theta^\omega(t, x) = x + \int_s^t F(r, \omega_r, \theta^\omega(r, x))dr,
\]

we have

\[
\nabla \theta^\omega(t, x) = I_d + \int_s^t \nabla F(r, \omega_r, \theta^\omega(r, x))\nabla \theta^\omega(r, x) dr.
\]

By Liouville’s formula, we can write

\[
\det(\nabla \theta^\omega(t, x)) = \exp\left(\int_s^t \text{tr}[\nabla F(r, \omega_r, \theta^\omega(r, x))]dr\right).
\]

(3.9)

Now the hypothesis (H3) can be applied to obtain (3.6). The lemma is then proved.

\[
\square
\]

By Lemma 3.5 and the implicit function theorem, we have the following corollary.

**Corollary 3.6.** Assume hypotheses (H1)-(H5) and assume $\omega(t)$, $0 \leq t \leq T$ is a continuous function. Let $\theta^\omega$ satisfy (3.4). Then, there exist a function $(\theta^\omega)^{-1}(t, \cdot)$ such that

\[
\theta^\omega(t, (\theta^\omega)^{-1}(t, x)) = (\theta^\omega)^{-1}(t, \theta^\omega(t, x)) = x,
\]

for all $x \in \mathbb{R}^n$. Moreover, the gradient of $(\theta^\omega)^{-1}$ with respect to the space variable satisfies the following inequality:

\[
e^{-n\kappa t} \leq \det(\nabla (\theta^\omega)^{-1}(t, x)) \leq e^{n\kappa t}.
\]

(3.10)

In the next proposition, we provide a tail estimate for the solution to (2.4).
Proposition 3.7. Assume hypotheses (H1)-(H5). Let $f$ be a continuous integrable function on $\mathbb{R}^n$. Then for any $\varepsilon > 0$, there exist $K > 0$ such that

$$\int_{G_K} |u^\omega_t(x)| dx \leq \varepsilon,$$

(3.11)

for any $t \in [0, T]$ and for any continuous function $\omega : [0, T] \to \mathbb{R}^n$, where

$$G_K = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \max_{1 \leq i \leq n} |x_i| \geq K \right\}.$$

Proof. For any $\varepsilon > 0$, due to integrability of $f$, we can choose $\bar{K}$ such that

$$\int_{|x| \geq \bar{K}} |f(x)| dx \leq \varepsilon. \quad (3.12)$$

Denote by $p^\omega_t(x, y)$ the transition density of $X^\omega$ from $y$ at time 0 to $x$ at time $t$. Then, it is well-known

$$u^\omega_t(x) = \int_{\mathbb{R}^n} p^\omega_t(x, \xi) f(\xi) d\xi.$$

Notice that hypotheses (H1)-(H5) ensures that functions $\bar{\sigma}$ and $\bar{F}$ given by

$$\bar{\sigma}(t, x) = \sigma(t, \omega_t, x), \quad \text{and} \quad \bar{F}(t, x) = F(t, \omega_t, x),$$

for all $(t, x) \in [0, T] \times \mathbb{R}^n$, satisfy conditions (C1)-(C3) and (C2'). Additionally, the independence of $t$ and $x$ of the constant of $\kappa$ in hypotheses (H1)-(H5) and Remark 3.1 imply that the constant $C_T$ appearing in Theorem 3.3 is independent of the choice of $\omega$. This allows us to apply Theorem 3.3 to obtain

$$\int_{G_K} |u^\omega_t(x)| dx = \int_{G_K} \left| \int_{\mathbb{R}^n} p^\omega_t(x, \xi) f(\xi) d\xi \right| dx$$

$$\leq \int_{G_K} \int_{\mathbb{R}^n} C_T t^{-n/2} \exp \left( -C_T^{-1} |T_t^{-1}(x - \theta^\omega_t(t, \xi))|^2 \right) |f(\xi)| d\xi dx$$

$$= \int_{G_K} \int_{\mathbb{R}^n} C_T t^{-n/2} \exp \left( -C_T^{-1} |T_t^{-1}(y)|^2 \right) |f(\xi)| d\xi dy,$$

where $\theta^\omega$ is defined by (3.4) and

$$G^1_K = \{ y \in \mathbb{R}^n : \max_{1 \leq i \leq n} |y_i + \theta^\omega_t(t, \xi)| > K \}.$$

Performing a change of variable $z = T_t^{-1}(y)$, we have

$$\int_{G_K} |u^\omega_t(x)| dx \leq \int_{G_K} \int_{\mathbb{R}^n} C_T \exp \left( -\frac{|z|^2}{C_T} \right) f(\xi) d\xi dz$$

$$\leq \int_{\mathbb{R}^n} \int_{|\xi| > K} C_T \exp \left( -\frac{|z|^2}{C_T} \right) f(\xi) d\xi dz + \int_{G_K} \int_{|\xi| \leq K} C_T \exp \left( -\frac{|z|^2}{C_T} \right) f(\xi) d\xi dz,$$
where
\[ G^2_K = \{ z \in \mathbb{R}^n : \max_{1 \leq i \leq n} \left| t_i^{\frac{1}{2}}z_i + \theta_i^\omega(t, \xi) \right| > K \}. \]

As a consequence of (3.12), we have
\[ \int_{\mathbb{R}^n} \int_{\{ |\xi| > \bar{K} \}} C_T \exp \left( -\frac{|z|^2}{C_T} \right) f(\xi) d\xi dz \leq (\pi C_T)^{\frac{n}{2}} C_T \epsilon. \]

On the other hand, by Lemma 3.5, we know that
\[ |\theta^\omega(t, \xi)| \leq (|\xi| + \kappa t) e^{\kappa t} \leq \bar{K} e^{\kappa T} + \kappa T e^{\kappa T}, \]
for all $|\xi| \leq \bar{K}$ and $t \in [0, T]$. Then, we have
\[ G^2_K \subseteq \{ z \in \mathbb{R}^n : \max_{1 \leq i \leq n} \left| t_i^{\frac{1}{2}}z_i \right| > K - \bar{K} e^{\kappa T} - \kappa T e^{\kappa T} \} =: G^3_K. \]

Therefore, for any $\epsilon > 0$, there exists $K$ sufficiently large such that
\[ \int_{G^2_K} \int_{\{ |\xi| \leq \bar{K} \}} C_T \exp \left( -\frac{|z|^2}{C_T} \right) f(\xi) d\xi dz \leq \int_{G^3_K} C_T f(\xi) d\xi \int_{G^2_K} \exp \left( -\frac{|z|^2}{C_T} \right) dz \leq \epsilon. \]

The proof of this proposition is complete. □

**Proposition 3.8.** Assume hypotheses (H1)-(H5). Let $f$ be a positive, continuously integrable function on $\mathbb{R}^n$. Then for any $K > 0$ there exists $\delta > 0$ such that
\[ \inf \left\{ u^\omega_t(x) : \max_{1 \leq j \leq n} |x_j| \leq K \right\} \geq \delta, \quad (3.13) \]
for all $t \in [0, T]$ and for all continuous functions $\omega$ on $[0, T]$ with values in $\mathbb{R}^n$.

**Proof.** Fix $K > 0$. For any $x \in \mathbb{R}$ with $|x| \leq K$. By the lower bound of (3.3) we get
\[ u^\omega_t(x) = \int_{\mathbb{R}^n} p^\omega_t(x, \xi) f(\xi) d\xi \]
\[ \geq \int_{\mathcal{R}} p^\omega_t(x, \xi) f(\xi) d\xi \]
\[ \geq \frac{1}{C_T t^{n^2/2}} \int_{\mathcal{R}} \exp \left( -C_T |\mathcal{T}_t^{-1}(x - \theta^\omega(t, \xi))|^2 \right) f(\xi) d\xi, \]
where
\[ \mathcal{R} = \{ \xi \in \mathbb{R}^n : |x_1 - \theta^\omega_1(t, \xi)| \leq \sqrt{t}, \ldots, |x_n - \theta^\omega_n(t, \xi)| \leq t^{(2n-1)/2}, \max_{1 \leq j \leq n} |x_j| \leq K \}. \]
Due to Lemma 3.5, we know that
\[ e^{-\kappa t} |\xi| - \kappa t \leq |\theta^\omega(t, \xi)|, \]
for all $t \in [0, T]$. Thus, we have
\[ |\xi| \leq |\xi| \leq (|\theta^\omega(t, \xi)| + \kappa t) e^{\kappa t}. \]
For any $\xi = (\xi_1, \ldots, \xi_n) \in \mathcal{R}$, it is easy to see

$$|\theta^\omega(t, \xi)| \leq \left[ \sum_{i=1}^{n} \left( t^{2i-1} + |x_i| \right)^2 \right]^{\frac{1}{2}} \leq \sqrt{2}|x| + \sqrt{2n(T^{\frac{n-1}{2}} + 1)}.$$

This means that

$$\mathcal{R} \subseteq \mathcal{R}^1 := \{ \xi \in \mathbb{R}^n : |\xi| \leq (\sqrt{2nK} + \sqrt{2n(T^{\frac{n-1}{2}} + 1)} + \kappa T)e^{\kappa T} \}. \quad (3.14)$$

Recall that $f$ is a continuous positive integrable function. Hence, there exists $\delta > 0$ such that $f(\xi) \geq \delta$ on the set $\mathcal{R}^1 \supseteq \mathcal{R}$. As a consequence, for any $x \in \mathbb{R}^n$ with $|x| \leq K$, we have

$$u^\omega_T(x) \geq \frac{\delta}{C_T} \int_{\mathcal{R}} \exp \left( -C_T|T_t^{-1}(x - \theta^\omega(t, \xi))|^2 \right) d\xi.$$

By change of variable $T_t^{-1}(x - \theta^\omega(t, \xi)) = y$ and then by Corollary 3.6, we have

$$u^\omega_t(x) \geq \frac{\delta}{C_T} \int_{\{y \in \mathbb{R}^n : |y| \leq 1, i=1,\ldots,n\}} \exp \left( -C_T|y|^2 \right) \det \left( (\theta^\omega)^{-1}(t, x - T_t(y)) \right) dy \geq \frac{\delta}{C_T} e^{-(nC_T^{n} + n\kappa T)} \int_{\{y \in \mathbb{R}^n : |y| \leq 1, i=1,\ldots,n\}} dy = \frac{\delta}{C_T} 2^n e^{-(nC_T^{n} + n\kappa T)},$$

which completes the proof of the proposition. \qed

Combining the Propositions 3.7 and 3.8, we arrive at the following result.

**Proposition 3.9.** Assume hypotheses (H1)-(H5) and that $\omega : [0, T] \to \mathbb{R}^n$ is a continuous function. Let $u^\omega$ be the solution to (2.4) with initial condition $f \in C(\mathbb{R}^n) \cup L^1(\mathbb{R}^n)$. For any $\alpha \in (0, 1)^n$ and $t \in [0, T]$, let $\hat{\omega}^\alpha = (\hat{\omega}_1, \cdots, \hat{\omega}_n) = Q_\alpha(u^\omega_t)$ be the $\alpha$-quantile of $u^\omega_t$. Then, there exist $K, \delta, \varepsilon > 0$, independent of $t$ and $\omega$, such that

$$\int_{\{x \in \mathbb{R}^n : \max_{1 \leq j \leq n} |x_j| \geq K\}} |u^\omega_t(x)| dx \leq \varepsilon, \quad (3.15)$$

$$\max_{1 \leq j \leq n} |\hat{\omega}_j| \leq K, \quad (3.16)$$

and

$$\inf \left\{ u^\omega_t(x) : \max_{1 \leq j \leq n} |x_j| \leq K \right\} \geq \delta. \quad (3.17)$$
Fix \( \alpha \in (0,1)^n \) and \( K, \delta, \varepsilon > 0 \). Denote by \( S = S_{\alpha,K,\delta,\varepsilon} \) the collection of density functions \( h \) on \( \mathbb{R}^n \) such that

\[
\int_{\left\{ x \in \mathbb{R}^n : \max_{1 \leq j \leq n} |x_j| \geq K \right\}} |h| \, dx \leq \varepsilon, \quad Q_{\alpha}(h) \leq K, \quad \text{and} \quad \inf \left\{ h(x) : \max_{1 \leq j \leq n} |x_j| \leq K \right\} \geq \delta.
\]

(3.18)

Then, \( S \) is a convex set.

**Proof.** Choose

\[
0 < \varepsilon < \min (\alpha_1, \ldots, \alpha_n, 1 - \alpha_1, \ldots, 1 - \alpha_n).
\]

Then, by Proposition 3.7, there exists \( K > 0 \) such that (3.15) is true. Inequality (3.16) also holds true, due to the fact that

\[
\int_{-\infty}^{-K} \, dx_j \int_{\mathbb{R}^{n-1}} u_i^\omega(x) \prod_{i \neq j, 1 \leq i \leq n} dx_i \leq \int_{\left\{ x \in \mathbb{R}^n : \max_{1 \leq j \leq n} |x_j| \geq K \right\}} |u_i^\omega(x)| \, dx \leq \varepsilon \leq \alpha_j,
\]

and

\[
\int_{K}^{\infty} \, dx_j \int_{\mathbb{R}^{n-1}} u_i^\omega(x) \prod_{i \neq j, 1 \leq i \leq n} dx_i \leq 1 - \alpha_j,
\]

for all \( j = 1, \ldots, n \). (3.17) is a straightforward consequence of Proposition 3.8.

In the next step, we prove the convexity of set \( S \). Let \( h_1, h_2 \in S \). For any \( \beta \in [0,1] \), \( h = \beta h_1 + (1 - \beta) h_2 \) is still a density function, and the first and the last properties in (3.18) are trivial for \( h \). It suffices to show that for any \( \beta \in [0,1] \),

\[
Q_{\alpha}(\beta h_1 + (1 - \beta) h_2) \leq K,
\]

which is true, because

\[
\begin{align*}
\int_{-\infty}^{-K} \, dx_j \int_{\mathbb{R}^{n-1}} (\beta h_1(x) + (1 - \beta) h_2(x)) \prod_{i \neq j, 1 \leq i \leq n} dx_i \\
= \beta \int_{-\infty}^{-K} \, dx_j \int_{\mathbb{R}^{n-1}} h_1(x) \prod_{i \neq j, 1 \leq i \leq n} dx_i + (1 - \beta) \int_{-\infty}^{-K} \, dx_j \int_{\mathbb{R}^{n-1}} h_2(x) \prod_{i \neq j, 1 \leq i \leq n} dx_i \\
\leq \beta \alpha_j + (1 - \beta) \alpha_j = \alpha_j,
\end{align*}
\]

and

\[
\begin{align*}
\int_{K}^{\infty} \, dx_j \int_{\mathbb{R}^{n-1}} (\beta h_1(x) + (1 - \beta) h_2(x)) \prod_{i \neq j, 1 \leq i \leq n} dx_i \leq 1 - \alpha_j,
\end{align*}
\]

for all \( j = 1, \ldots, n \). The proof of this proposition is completed. \( \Box \)
The next Feynman-Kac formula for time-inhomogeneous PDE is cited from [9, page 131-132]. Consider the following PDE,
\[
\frac{\partial}{\partial t} u(t) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} u(t) + \langle b(t, x), \nabla u(t) \rangle + c(t, x) u(t),
\]
\[
\quad u_0(x) = f(x).
\]
(3.19)
Let \( t > 0 \) and \( x \in \mathbb{R}^n \) and let the process \( (X^{t,x}_s)_{0 \leq s \leq t} \) be the solution to the following stochastic differential equation:
\[
\begin{aligned}
&dX^{t,x}_s = \sigma(t, X^{t,x}_s) \, dW_s + b(t, X^{t,x}_s) \, ds, \quad 0 \leq s \leq t, \\
&X^{t,x}_0 = x.
\end{aligned}
\]
(3.20)
Then, we have the following Feynman-Kac formula for the solution to (3.19).

**Theorem 3.10** (see [9]). Assume that the entries of the matrix \( \sigma(t, x) \) are continuous and bounded on the set \([0, \infty) \times \mathbb{R}^n\) and Lipschitz continuous in \( x \) with a Lipschitz constant which does not depend on \( t \). The vector \( b(t, x) \) are also assumed to be continuous, and Lipschitz continuous in \( x \) with a Lipschitz constant which does not depend on \( t \). Let \( u \) be the solution to (3.19). Suppose that \( u \) is continuous and bounded on \([0, T] \times \mathbb{R}^n\). Suppose further that the time derivative of \( u \) and its spatial derivatives up to order two are bounded and continuous in the region \((h < t < T, x \in \mathbb{R}^n)\) for every \( h \in (0, T) \). Then, for any \( t \in [0, T] \) and \( x \in \mathbb{R} \), \( u_t(x) \) can be represented in the form
\[
u_t(x) = \mathbb{E} f(X^{t,x}_t) \exp \left( \int_0^t c(t - s, X^{t,x}_s) \, ds \right),
\]
(3.21)
where \( X^{t,x}_s \) is given by (3.20).

**Remark 3.11.** Note that the Hypotheses (H1)-(H5) imposed on coefficients \( F \) and \( \sigma \) imply all the conditions required in Theorem 3.10. In other words, the density of \( X^\omega \) of SDE (2.3) can be represented by (3.21) under Hypotheses (H1)-(H5).

**Proposition 3.12.** Assume hypotheses (H1)-(H5). Let \( u^\omega \) be the solution to (2.4) with initial condition \( f \) satisfying hypothesis (I). Then, we have for any \( t_0 > 0 \),
\[
U^\prime := \sup_{t \in [t_0, T]} \sup_{\omega \in \mathcal{C}_b([0, T] : \mathbb{R}^n)} \left( \int_{\mathbb{R}^n} \sup_{\|y\| \geq r} u^\omega_t(y)^2 \left( |r|^{4n+\varepsilon-1} + r^{n-1} \right) dr \right.
\]
\[
\left. + \int_0^\infty \sup_{\|z\| \geq \lambda} |\nabla u^\omega_t(z)|^4 \left( \lambda^{4n-1+\varepsilon} + \lambda^{n-1} \right) d\lambda \right) < \infty.
\]
(3.22)

**Proof.** We shall show that the second term in (3.22) is uniformly bounded. The uniform boundedness of the first term can be proved in a similar way. Using Theorem 3.10, for any \( t \in [0, T] \) and \( x \in \mathbb{R} \), we can write
\[
u_t^\omega(x) = \mathbb{E} \left( f(X^{\omega_t,t,x}_t) \exp \left( \int_0^t c(t - s, \omega_{t-s}, X^{\omega_t,t,x}_s) \, ds \right) \right),
\]
where $X_{\omega,t,x}$ is the solution to (3.20), where $\sigma(t, x)$ and $b(t, x)$ are replaced by $\sigma(t, \omega(t), x)$ and $F(t, \omega(t), x)$. Differentiating this expression with respect to $x$, we have

$$\nabla u_t^\omega(x) = \mathbb{E} \left[ \exp \left( \int_0^t c(t-s, \omega, X_s^\omega,t,x) \, ds \right) \nabla f(X_t^\omega,t,x) \nabla X_t^\omega,t,x \
+ f(X_t^\omega,t,x) \exp \left( \int_0^t c(t-s, \omega_t-s, X_s^\omega,t,x) \, ds \right) \right] \nabla c \left( t-s, \omega_t-s, X_s^\omega,t,x \right) \nabla X_s^\omega,t,x \, ds.$$

Due to hypotheses (H4), we know that $c, \nabla c$ are both bounded functions. By Cauchy-Schwarz's and Minkowski’s inequalities, we can show that

$$|\nabla u_t^\omega(x)| \leq \mathbb{E} \left[ \left( \left\| \nabla f(X_t^\omega,t,x) \right\| \left\| \nabla X_t^\omega,t,x \right\| \right)^{1/2} + \left( \left\| f(X_t^\omega,t,x) \right\| \left\| \nabla X_t^\omega,t,x \right\| \right)^{1/2} \right] \int_0^t \left( \left\| \nabla X_t^\omega,t,x \right\| ^2 \right) \, ds.$$

Note that for any $r \in (0, t)$, $\nabla X_r^\omega,t,x$ satisfies the following equation

$$\nabla X_r^\omega,t,x = I_n + \int_0^r \nabla (D\sigma)(t-s, \omega_t-s, X_s^\omega,t,x) \nabla X_s^\omega,t,x \, dW_s$$

$$+ \int_0^r \nabla F(t-s, \omega_t-s, X_s^\omega,t,x) \nabla X_s^\omega,t,x \, ds.$$

From Burkholder-Davis-Gundy’s and Jensen’s and Minkowski’s inequalities it follows that

$$\left\| \nabla X_r^\omega,t,x \right\| ^2 \leq n + T^2 \int_0^r \left( \left\| \nabla F(t-s, \omega_t-s, X_s^\omega,t,x) \nabla X_s^\omega,t,x \right\| ^2 \right) \, ds$$

$$+ \int_0^r \left\| \nabla (D\sigma)(t-s, \omega_t-s, X_s^\omega,t,x) \nabla X_s^\omega,t,x \right\| ^2 \, ds$$

$$\leq n + C(\kappa, T) \int_0^r \left\| \nabla X_s^\omega,t,x \right\| ^2 \, ds.$$

By Gronwall’s inequality, we obtain

$$\left\| \nabla X_r^\omega,t,x \right\| _2 \leq \sqrt{n e^{C(\kappa, T)r}} \leq \sqrt{n e^{C(\kappa, T)T}}.$$

Inserting this inequality into (3.23), we obtain

$$|\nabla u_t^\omega(x)| \leq C(\kappa, T) \left( \left\| \nabla f(X_t^\omega,t,x) \right\| _2 + \left\| f(X_t^\omega,t,x) \right\| _2 \right).$$
This implies
\[
\int_0^\infty \left[ \sup_{|z| \geq \lambda} |\nabla u_t^\omega(z)|^4 \right] (\lambda^{4n-1+\varepsilon} + \lambda^{n-1}) \, d\lambda \\
\leq C \int_0^\infty \left[ \sup_{|z| \geq \lambda} \left( \|\nabla f(X_t^{\omega,x,z})\|_2 + \|f(X_t^{\omega,x,z})\|_2 \right)^4 \right] (\lambda^{4n-1+\varepsilon} + \lambda^{n-1}) \, d\lambda \\
\leq C \left( \int_0^\infty \left[ \sup_{|z| \geq \lambda} \left( \int_{\mathbb{R}^n} |\nabla f(x)|^2 p_t^\omega(x,z) \, dx \right)^2 \right] (\lambda^{4n-1+\varepsilon} + \lambda^{n-1}) \, d\lambda \\
+ \int_0^\infty \left[ \sup_{|z| \geq \lambda} \left( \int_{\mathbb{R}^n} |f(x)|^2 p_t^\omega(x,z) \, dx \right)^2 \right] (\lambda^{4n-1+\varepsilon} + \lambda^{n-1}) \, d\lambda \right) \\
:= C(D_1 + D_2),
\] (3.24)

where \( p_t^\omega(\cdot, \xi) \) is the probability density of the solution to (2.3) with initial condition \( X_0^\omega = \xi \in \mathbb{R}^n \). Applying Theorem 3.3 and Jensen’s inequality, we can show that
\[
D_1 \leq C \int_0^\infty \left[ \sup_{|z| \geq \lambda} \int_{\mathbb{R}^n} |\nabla f(x)|^4 \frac{C_T}{t^{n^2/2}} \exp \left( -\frac{|T_t^{-1}(\theta^\omega(t,z) - x)|^2}{C_T} \right) \, dx \right] (\lambda^{4n-1+\varepsilon} + \lambda^{n-1}) \, d\lambda.
\] (3.25)

Now that hypothesis (I) implies that
\[
\int_{\mathbb{R}^n} |\nabla f(x)|^4 \, dx \leq \int_0^\infty \sup_{|z| \geq \lambda} |\nabla f(x)|^4 \lambda^{n-1} \, d\lambda \leq U
\] (3.26)

and
\[
\sup_{|z| \geq \delta} |\nabla f(x)| \leq \delta^{-1} \int_0^\delta \sup_{|z| \geq \lambda} |\nabla f(x)| \, d\lambda \leq \delta^{-1} U,
\] (3.27)

for any \( \delta > 0 \). Using (3.26) we obtain
\[
\int_{\mathbb{R}^n} |\nabla f(x)|^4 \frac{C_T}{t^{n^2/2}} \exp \left( -\frac{|T_t^{-1}(\theta^\omega(t,z) - x)|^2}{C_T} \right) \, dx \\
= \int_{|x| \leq \delta} |\nabla f(x)|^4 \frac{C_T}{t^{n^2/2}} \exp \left( -\frac{|T_t^{-1}(\theta^\omega(t,z) - x)|^2}{C_T} \right) \, dx \\
+ \int_{|x| > \delta} |\nabla f(x)|^4 \frac{C_T}{t^{n^2/2}} \exp \left( -\frac{|T_t^{-1}(\theta^\omega(t,z) - x)|^2}{C_T} \right) \, dx \\
\leq \frac{C_T}{t^{n^2/2}} e^{\frac{(2n-1)\lambda^{-1}\lambda^2}{C_T}} U \exp \left( -\frac{|T_t^{-1}(\theta^\omega(t,z)|^2}{C_T} \right) \\
+ C_T \int_{\mathbb{R}^n} 1_{\{|\theta^\omega(t,z) - T_t(y)| > \delta\}} |\nabla f(\theta^\omega(t,z) - T_t(y))|^4 \exp \left( -\frac{|y|^2}{C_T} \right) \, dy,
\]
in the second part of the last inequality we perform change of variable \( x \rightarrow y = T_t^{-1}(\theta^*(t, z) - x) \).

By Lemma 3.5, we have that

\[ |\theta^*(t, z)| \geq (e^{-\kappa T}|z| - \kappa T)1_{\{|z|\geq e^{\kappa T}\kappa T\}}. \]

This implies that

\[
\int_0^\infty C_T \frac{e^{-(2n-1)\varepsilon - 1}t^2}{t^{n/2}} U \left[ \sup_{|z|\geq \lambda} \exp \left( -\frac{\left| T_t^{-1}(\theta^*(t, z)) \right|^2}{C_T} \right) \right] (\lambda^{4n-1+\varepsilon} + \lambda^{n-1}) d\lambda \leq C_T \frac{e^{-(2n-1)\varepsilon - 1}t^2}{t^{n/2}} U \int_0^\infty \exp \left( \frac{(t-2n+1\wedge t-1)(e^{-\kappa T} - e^{-T})^2}{C_T} \right) (\lambda^{4n-1+\varepsilon} + \lambda^{n-1}) d\lambda \leq C, \tag{3.28}
\]

for some \( C \) depending on \( n, C_T, t_0, T, \epsilon, \kappa \) and \( U \). Similarly, on the set

\[ \{|z| \geq \lambda \} \cap \{ |\theta^*(t, z) - T_t(y)| > \delta \}, \]

we can deduce that

\[ |\theta^*(t, z) - T_t(y)| \geq \delta \vee (e^{-\kappa T} - \kappa T - |T_t(y)|). \]

Therefore, it follows from (3.27) that

\[
\int_0^\infty \left[ \sup_{|z|\geq \lambda} \int_{\mathbb{R}^n} 1_{\{|\theta^*(t, z) - T_t(y)| > \delta\}} |\nabla f(\theta^*(t, z) - T_t(y))|^4 \exp \left( -\frac{|y|^2}{C_T} \right) dy \right] (\lambda^{4n-1+\varepsilon} + \lambda^{n-1}) d\lambda \leq \int_{\mathbb{R}^n} \int_0^\infty \left[ \sup_{|z|\geq \delta \vee (e^{-\kappa T} - \kappa T - |T_t(y)|)} |\nabla f(\tilde{z})|^4 \right] (\lambda^{4n-1+\varepsilon} + \lambda^{n-1}) e^{-\frac{|y|^2}{C_T}} d\lambda dy \leq \int_{\mathbb{R}^n} \int_0^\infty \left[ \sup_{|\tilde{z}|\geq \delta} |\nabla f(\tilde{z})|^4 \right] (\lambda^{4n-1+\varepsilon} + \lambda^{n-1}) e^{-\frac{|y|^2}{C_T}} d\lambda dy \leq \delta^{-1} U \int_{\mathbb{R}^n} e^{\epsilon T} (\delta + \kappa T + |T_t(y)|) (\lambda^{4n-1+\varepsilon} + \lambda^{n-1}) d\lambda e^{-\frac{|y|^2}{C_T}} dy + e^{(4n+\varepsilon) T} \int_{\mathbb{R}^n} \int_0^\infty e^{-\frac{|y|^2}{C_T}} \times \left[ \sup_{|\tilde{z}|\geq \tau} |\nabla f(\tilde{z})|^4 \right] (|\tau + \kappa T + |T_t(y)|)^{4n-1+\varepsilon} + |\tau + \kappa T + |T_t(y)| |^{n-1}) d\tau dy \leq C, \tag{3.29}
\]
where $C > 0$ depends on $n, C_T, t_0, T, \epsilon, \kappa, U$ and $\delta$. Combining (3.25), (3.28) and (3.29) we see that $D_1$ is bounded. Using a similar argument, we can prove that $D_2$ is bounded uniformly in $t \in [t_0, T]$ and $\omega \in C_b([0, T]; \mathbb{R}^n)$. The proof of this proposition is then completed.

**Remark 3.13.** The main difficulty in the above proof is to show the integrability over an unbounded domain with respect to $\lambda$. After (3.27) we divided the integral domain into $|x| \leq \delta$ and $|x| > \delta$ is for simplicity because even if we use $|x| \leq \delta \sqrt{t}$ and $|x| > \delta \sqrt{t}$, we cannot get rid of the $t_0$ in the statement (3.22).

4. **Proof of the main results**

Now, we are ready to prove Theorems 2.4 and 2.5. In the first subsection we shall prove the existence and uniqueness of the local solution to PDE (2.8) up to a small time $t_0$.

4.1. **Local solution.** In this subsection, we prove a local version of Theorem 2.5 (see Proposition 4.4). We shall use the fixed point theorem. First we need to bound the distance of quantiles by the distance of distributions. The following lemma is known (see e.g. [13]). We rewrite a short proof for the sake of completeness.

**Lemma 4.1.** Let $\alpha \in (0, 1)^n$ and let $K, \delta, \varepsilon$ be positive constants. Denote by $S$ the collection of density functions satisfying (3.18). Then, for any $h_1, h_2 \in S$,

$$|Q_\alpha(h_1) - Q_\alpha(h_2)| \leq \sqrt{n}(2K)^{-(n-1)}\delta^{-1}|h_1 - h_2|_{L^1}. \quad (4.1)$$

**Proof.** Since $h_1, h_2 \in S$ where $S$ is a convex set, we know that for any $\beta \in (0, 1)$,

$$h^\beta := \beta h_1 + (1 - \beta)h_2 \in S$$

as well. Write $\hat{\omega}(\beta) = (\hat{\omega}_1(\beta), \ldots, \hat{\omega}_n(\beta)) = Q_\alpha(h^\beta)$.

By definition of the quantile, for any $j = 1, \ldots, n$, we have

$$\int_{-\infty}^{\hat{\omega}_j(\beta)} dx_j \int_{\mathbb{R}^{n-1}} h^\beta(x) \prod_{k \neq j} dx_k = \alpha_j. \quad (4.2)$$

Differentiating both sides of (4.2) with respect to $\beta$ yields

$$\hat{\omega}_j'(\beta) \int_{\mathbb{R}^{n-1}} h^\beta(x) \prod_{k \neq j} dx_k \bigg|_{x_j = \hat{\omega}_j(\beta)} + \int_{-\infty}^{\hat{\omega}_j(\beta)} dx_j \int_{\mathbb{R}^{n-1}} (h_1(x) - h_2(x)) \prod_{k \neq j} dx_k = 0.$$

Thus

$$\hat{\omega}_j'(\beta) = -\left[ \int_{\mathbb{R}^{n-1}} h^\beta(x) \prod_{k \neq j} dx_k \bigg|_{x_j = \hat{\omega}_j(\beta)} \right]^{-1} \int_{-\infty}^{\hat{\omega}_j(\beta)} dx_j \int_{\mathbb{R}^{n-1}} (h_1(x) - h_2(x)) \prod_{k \neq j} dx_k.$$
It follows that

$$\left| Q_\alpha^j(h_1) - Q_\alpha^j(h_2) \right| = |\hat{\omega}_j(1) - \hat{\omega}_j(0)| = \left| \int_0^1 \hat{\omega}_j'(\beta) d\beta \right|$$

$$= \left| \int_0^1 \int_{-\infty}^{\hat{\omega}_j(\beta)} \int_{\mathbb{R}^{n-1}} (h_1(x) - h_2(x)) \prod_{k \neq j} dx_k \right|$$

$$\leq |h_1 - h_2|_{L^1} \left| \int_0^1 \left[ \int_{\mathbb{R}^{n-1}} h^\beta(x) \prod_{k \neq j} dx_k \right]_{x_j = \hat{\omega}_j(\beta)}^{-1} d\beta \right|,$$ \hspace{1cm} (4.3)

Recall that $h^\beta \in S$. This implies that $\max_{1 \leq j \leq n} |\hat{\omega}_j(\beta)| \leq K$, and thus by (3.18) we have

$$\int_{\mathbb{R}^{n-1}} h^\beta(x) \prod_{k \neq j} dx_k \bigg|_{x_j = \hat{\omega}_j(\beta)} \geq \int_{[-K,K]^{n-1}} h^\beta(x) \prod_{k \neq j} dx_k \bigg|_{x_j = \hat{\omega}_j(\beta)}$$

$$\geq \int_{[-K,K]^{n-1}} \delta \prod_{k \neq j} dx_k \geq (2K)^{n-1}\delta.$$

As a consequence, we have

$$\left| Q_\alpha^j(h_1) - Q_\alpha^j(h_2) \right| \leq (2K)^{-(n-1)}\delta^{-1}|h_1 - h_2|_{L^1},$$

for all $j = 1, \ldots, n$, which yields the lemma. \hfill \square

The next proposition describes the dependence of the solution of (2.4) with respect to $\omega$. It will be used to bound the distance of distributions of the solutions to (1.4) by the quantiles.

**Proposition 4.2.** Let the hypotheses (H1)-(H5) be satisfied. Let $u^{(1)} = u^{(1)}(\omega)$ and $u^{(2)} = u^{(2)}(\omega)$ be the solutions to equation (2.4) corresponding to the continuous functions $\omega = \omega^{(1)}$ and $\omega = \omega^{(2)}$ respectively and with the same initial condition $f$ satisfying hypothesis (I). Then, the following inequality holds true

$$\sup_{s \in [0,t]} |u^{(1)}_s - u^{(2)}_s|_{L^1} \leq C_0 \left( t + \sqrt{t} \right) \sup_{s \in [0,t]} |\omega^{(1)}_s - \omega^{(2)}_s|, \quad \forall \ t \in [0,T],$$ \hspace{1cm} (4.4)

where $C_0$ is a positive constant independent of $\omega^{(1)}$, $\omega^{(2)}$ and $t$.

**Proof.** By the Feynman-Kac formula (Theorem 3.10), for $i = 1$ and 2, we can write

$$u_t^{(i)}(x) = \mathbb{E} \left( f(X_t^{(i)}, t, x) \exp \left( \int_0^t c^{(i)}(t - s, X_s^{(i)}, t, x) \, ds \right) \right),$$
where $X^{(i), t, x} = X^{(i), t, x}$ is the solution to (3.20) with initial condition $X^{(i), t, x}_0 = x$ and coefficients

$$a^{(i)}(t - s, x) = a(t - s, \omega^{(i)}_{t-s}, x), \quad b^{(i)}(t - s, x) = b(t - s, \omega^{(i)}_{t-s}, x),$$

$$c^{(i)}(t - s, x) = c(t - s, \omega^{(i)}_{t-s}, x),$$

for all $t \in [0, T]$ and $x \in \mathbb{R}^n$ with $a, b$ and $c$ being defined by (2.5)-(2.7) respectively. Thus, we have

$$I = \int_{\mathbb{R}^n} |u^{(1)}_t(x) - u^{(2)}_t(x)| dx$$

$$= \int_{\mathbb{R}^n} \mathbb{E} \left[ f(X^{(1), t, x}_t) \exp \left( \int_0^t c^{(1)}(s, X^{(1), t, x}_s) ds \right) - f(X^{(2), t, x}_t) \exp \left( \int_0^t c^{(2)}(s, X^{(2), t, x}_s) ds \right) \right] dx$$

$$= \int_{\mathbb{R}^n} \mathbb{E} \left\{ f(X^{(1), t, x}_t) \left[ \exp \left( \int_0^t c^{(1)}(s, X^{(1), t, x}_s) ds \right) - \exp \left( \int_0^t c^{(2)}(s, X^{(2), t, x}_s) ds \right) \right] \right\} dx$$

$$= I_1 + I_2. \quad (4.5)$$

Due to hypothesis (H4), we know that the function $c$ is uniformly bounded on $[0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ by $2\kappa$, and Lipschitz continuous. Then, the first term of (4.5) is bounded by using the mean value theorem as follows:

$$I_1 = \int_{\mathbb{R}^n} \mathbb{E} \left[ f(X^{(1), t, x}_t) \left( \exp \left( \int_0^t c^{(1)}(s, X^{(1), t, x}_s) ds \right) - \exp \left( \int_0^t c^{(2)}(s, X^{(2), t, x}_s) ds \right) \right) \right] dx$$

$$\leq e^{2\kappa T} \int_{\mathbb{R}^n} \mathbb{E} \left[ f(X^{(1), t, x}_t) \left( \int_0^t c^{(1)}(s, X^{(1), t, x}_s) ds - \int_0^t c^{(2)}(s, X^{(2), t, x}_s) ds \right) \right] dx$$

$$= e^{2\kappa T} \int_{\mathbb{R}^n} \mathbb{E} \left[ f(X^{(1), t, x}_t) \left( \int_0^t c(s, \omega^{(1)}_{t-s}, X^{(1), t, x}_s) - c(s, \omega^{(1)}_{t-s}, X^{(2), t, x}_s) ds \right) \right] dx$$

$$+ e^{2\kappa T} \int_{\mathbb{R}^n} \mathbb{E} \left[ f(X^{(1), t, x}_t) \left( \int_0^t c(s, \omega^{(1)}_{t-s}, X^{(1), t, x}_s) - c(s, \omega^{(2)}_{t-s}, X^{(2), t, x}_s) ds \right) \right] dx$$

$$\leq e^{\kappa T} \int_0^t |\omega^{(1)}_{t-s} - \omega^{(2)}_{t-s}| ds \int_{\mathbb{R}^n} \mathbb{E}[f(X^{(1), t, x}_t)] dx$$

$$+ e^{\kappa T} \int_{\mathbb{R}^n} \mathbb{E} \left[ f(X^{(1), t, x}_t) \int_0^t |X^{(1), t, x}_s - X^{(2), t, x}_s| ds \right] dx$$

$$= e^{\kappa T} (I_{11} + I_{12}), \quad (4.6)$$
where $c_{\kappa,T}$ is a positive constant depending on $\kappa$ and $T$. For $i = 1, 2$, denote by $p_t^{(i)}(\cdot, x)$ the probability density of $X_t^{(i)}$ and write $\theta^{(i)} = \theta^{\omega^{(i)}}$ the solution to (3.4) with $\omega = \omega^{(i)}$. Then, by Theorem 3.3 and Corollary 3.6, we have

$$
\int_{\mathbb{R}^n} \mathbb{E}[f(X_t^{(1),t,x})] dx = \int_{\mathbb{R}^{2n}} f(y)p_t^{(1)}(y, x) dy dx
$$

$$
\leq C_T \int_{\mathbb{R}^{2n}} f(y)t^{-n^2/2} \exp \left( -C_T^{-1} \sum_{i=1}^n \left( \frac{\theta^{(1)}_i(t, x) - y_i}{t^{-i/2}} \right)^2 \right) dy dx
$$

$$
\leq C_T \int_{\mathbb{R}^{2n}} f(y) \exp \left( -C_T^{-1} |z|^2 \right) \det \left( \nabla \left( \theta^{(1)} \right)^{-1} (t, y - T_t(z)) \right) dz dy
$$

$$
\leq C_T e^{n\kappa T} \int_{\mathbb{R}^n} f(y) dy \int_{\mathbb{R}^n} \exp \left( -C_T^{-1} |z|^2 \right) dz
$$

$$
\leq C_T e^{n\kappa T} (C_T \pi)^{\frac{3}{2}}. \tag{4.7}
$$

Hence,

$$
I_{11} \leq C_1 t \sup_{s \in [0,t]} |\omega_s^{(1)} - \omega_s^{(2)}|, \tag{4.8}
$$

for some positive constant $C_1$ independent of $\omega^{(1)}$, $\omega^{(2)}$ and $t$. On the other hand, for any $p \geq 1$, we can deduce that, for some constant $c_{n,p} > 0$ depending on $n$ and $p$,

$$
\mathbb{E} \left| X_t^{(1),t,x} - X_t^{(2),t,x} \right|^{2p}
$$

$$
\leq c_{n,p} \left[ \sum_{i=1}^n \mathbb{E} \left( \int_0^t \left( F_i(t-s, \omega_{t-s}^{(1)}, X_s^{(1),t,x}) - F_i(t-s, \omega_{t-s}^{(2)}, X_s^{(2),t,x}) \right)^2 ds \right)^{2p} \right]
$$

$$
+ \mathbb{E} \left( \int_0^t \left( \sigma(t-s, \omega_{t-s}^{(1)}, X_s^{(1),t,x}) - \sigma(t-s, \omega_{t-s}^{(2)}, X_s^{(2),t,x}) \right) dW_s \right)^{2p}. \tag{4.9}
$$
By hypothesis (H1) and the Burkholder-Davis-Gundy and Jensen’s inequalities, we have
\[
\mathbb{E}\left| X_t^{(1),t,x} - X_t^{(2),t,x} \right|^{2p} \\
\leq c_{n,p} \mathbb{E}\left[ \int_0^t |\omega_{t-s}^{(1)} - \omega_{t-s}^{(2)}|^{2p} ds + \int_0^t \mathbb{E}\left| X_s^{(1),t,x} - X_s^{(2),t,x} \right|^{2p} ds \right] \\
+ \mathbb{E}\left( \int_0^t \left( |\omega_{t-s}^{(1)} - \omega_{t-s}^{(2)}| + |X_s^{(1),t,x} - X_s^{(2),t,x}| \right)^2 ds \right)^p \\
\leq c_{n,p,\kappa} \mathbb{E}\left[ \int_0^t |\omega_{t-s}^{(1)} - \omega_{t-s}^{(2)}|^{2p} ds + \int_0^t \mathbb{E}\left| X_s^{(1),t,x} - X_s^{(2),t,x} \right|^{2p} ds \right] \\
+ t^{p-1} \mathbb{E}\left[ \int_0^t |\omega_{t-s}^{(1)} - \omega_{t-s}^{(2)}|^{2p} ds + t^{p-1} \int_0^t \mathbb{E}\left| X_s^{(1),t,x} - X_s^{(2),t,x} \right|^{2p} ds \right] \\
\leq c_{n,p,\kappa}(t^p + t^{2p}) \sup_{s \in [0,t]} |\omega_{t-s}^{(1)} - \omega_{t-s}^{(2)}|^{2p} + c_{n,p,\kappa}(t^{2p-1} + t^{p-1}) \int_0^t \mathbb{E}\left| X_s^{(1),t,x} - X_s^{(2),t,x} \right|^{2p} ds.
\]

An application of Gronwall’s inequality yields that
\[
\mathbb{E}\left| X_t^{(1),t,x} - X_t^{(2),t,x} \right|^{2p} \leq c_{n,p,\kappa}(t^p + t^{2p}) e^{c_{n,p,\kappa}(T^{2p} + T^p)} \sup_{s \in [0,t]} |\omega_{s}^{(1)} - \omega_{s}^{(2)}|^{2p}. 
\tag{4.10}
\]

By Fubini’s theorem, Hölder’s inequality and (4.10), we get that
\[
I_{12} = \int_0^t \int_{\mathbb{R}^n} \mathbb{E}\left[ f(X_t^{(1),t,x}) | X_s^{(1),t,x} - X_s^{(2),t,x} \right] dx ds \\
\leq \int_0^t \int_{\mathbb{R}^n} \| f(X_t^{(1),t,x}) \|_2 \| X_s^{(1),t,x} - X_s^{(2),t,x} \|_2 dx ds \\
\leq c_{n,p,T} \sup_{s \in [0,t]} |\omega_{s}^{(1)} - \omega_{s}^{(2)}| \int_0^t \int_{\mathbb{R}^n} \| f(X_t^{(1),t,x}) \|_2 dx ds, 
\tag{4.11}
\]
for some positive constant $c_{n,p,T}$ depending on $n$, $p$ and $T$. Notice that by Theorem 3.3, Corollary 3.6 and Cauchy-Schwarz’s inequality, we can deduce that
\[
\int_{\mathbb{R}^n} \| f(X_t^{(1),t,x}) \|_2 dx = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} f(y)^2 p_t^{(1)}(y,x) dy \right)^{\frac{1}{2}} dx \\
\leq \frac{\sqrt{\mathcal{C}_T}}{t^{n/4}} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |f(y)|^2 \exp \left( C_T^{-1} \|T_t^{-1}(\theta^{(1)}(t,x) - \hat{y})\|_n \right) dy \right)^{\frac{1}{2}} dx \\
\leq \frac{\sqrt{\mathcal{C}_T}}{t^{n/4}} \left( \int_{\mathbb{R}^{2n}} |f(y)|^2 \exp \left( C_T^{-1} \|T_t^{-1}(\theta^{(1)}(t,x) - \hat{y})\|_n \right) \left( |\theta^{(1)}(t,x)|^{\frac{n+\kappa}{2}} \vee 1 \right)^2 dy dx \right)^{\frac{1}{2}} \\
\times \left( \int_{\mathbb{R}^n} \left( |\theta^{(1)}(t,h)|^{\frac{n+\kappa}{2}} \vee 1 \right)^{-2} dh \right)^{\frac{1}{2}}.
\]
By changing of variables \( x \rightarrow z = T^{-1}(\theta(1)(t, x) - y) \) and \( h \rightarrow l = \theta(1)(t, h) \), we can write

\[
\int_{\mathbb{R}^n} \| f(X^{(1)}(t, x)) \|_2^2 \, dx \\
\leq \sqrt{C_T} \left[ \int_{\mathbb{R}^{2n}} \det \left( \nabla \left( \theta(1)^{-1} \left( t, y + T(z) \right) \right) \right) |f(y)|^2 e^{-\frac{|l|^2}{C_T}} \left( |Tz + y|^{n+\epsilon} \vee 1 \right) \, dxdy \right]^{\frac{1}{2}} \\
\times \left[ \int_{\mathbb{R}^n} \det \left( \nabla \left( \theta(1)^{-1} \left( t, l \right) \right) \right) \left( |l|^{-(n+\epsilon)} \vee 1 \right) \, dl \right]^{\frac{1}{2}} \\
\leq c_{n, \epsilon, T} \sqrt{C_T} e^{n\kappa T} \left( \int_{\mathbb{R}^{2n}} f(y)^2 \exp \left( -C_T^{-1} |z|^2 \right) \left( |z|^{n+\epsilon} + |y|^{n+\epsilon} + 1 \right) \, dxdy \right)^{\frac{1}{2}} \\
\times \left[ \int_{\mathbb{R}^n} \left( |l|^{-(n+\epsilon)} \vee 1 \right) \, dl \right]^{\frac{1}{2}}. \quad (4.12)
\]

Recall that \( f > 0 \) is a probability density satisfying hypothesis (I). (4.12) tells us that

\[
\int_{\mathbb{R}^n} \| f(X^{(1)}(t, x)) \|_2^2 \, dx \leq C, \quad \forall t \in [0, T], \quad (4.13)
\]

where \( C > 0 \) depends on \( C_T, n, p, \kappa, \epsilon, T \) and \( U \). Combining inequalities (4.11) and (4.13), we finally obtain

\[
I_{12} \leq C_1 t \sup_{s \in [0, t]} |\omega_s^{(1)} - \omega_s^{(2)}|, \quad (4.14)
\]

for some \( C_1 \) independent of \( \omega^{(1)}, \omega^{(2)} \) and \( t \).

In the next step, we estimate the term \( I_2 \) in (4.5). By Cauchy-Schwarz’s inequality and the fact that \( c \) is uniformly bounded, we can write

\[
I_2 \leq \int_{\mathbb{R}^n} \| f(X^{(1)}(t, x)) - f(X^{(2)}(t, x)) \|_2 \left\| \exp \left( \int_0^t c(s) \left( t - s, X^{(2)}(s, t, x) \right) \, ds \right) \right\|_2 \, dx \\
\leq e^{2\kappa T} \int_{\mathbb{R}^n} \| f(X^{(1)}(t, x)) - f(X^{(2)}(t, x)) \|_2 \, dx. \quad (4.15)
\]

To bound the above integral, we first claim the following version of mean value theorem. For any \( x, y \in \mathbb{R}^n \), the following inequality holds true:

\[
|f(x) - f(y)| \leq 2 \sup_{|x| \leq |y| \leq |x| + |y|} |\nabla f(z)| |x - y|. \quad (4.16)
\]

In fact, consider a plane \( P \) such that 0, \( x, y \in P \). Without loss of generality, suppose that \( |x| \leq |y| \). Let \( x' \) be the intersection of the straight line connecting 0 and \( y \), and the circle \( O \) centered at 0 with radius \( |x| \). Applying the fundamental theorem of
calculus to the path integral of $\nabla f$ along the (shorter) arc $x \to x'$ on $O$, and then along the straight line $x' \to y$, we obtain immediately,

$$\left| f(x) - f(y) \right| \leq \sup_{|x| \leq |z| \leq |y|} |\nabla f(z)| (|xx'| + |y - x'|) \quad (4.17)$$

where $\overrightarrow{xx'}$ denotes the arc length. Since the angle between the ray $x'y$ and the line $x'x$ is greater than or equal to $\pi/2$, we see that both $\overrightarrow{xx'}$ and $|y - x|$ are less than or equal to $|y - x|$. Thus, inequality (4.16) follows immediately from (4.17). It is worth noticing that we do not apply the mean value theorem on the straight line $x \to y$. Since if so, we have $|f(x) - f(y)| \leq |\nabla f(\xi)||x-y|$, where the point $\xi = t_0 x + (1-t_0)y$ for some $t_0 \in [0, 1]$. We can have $|\xi| \leq |x| \vee |y|$. However, we cannot guarantee $|\xi| \geq |x| \wedge |y|$, which is critical in the following immediate application.

Using (4.16) and Cauchy-Schwarz’s inequality, we can write

$$\|f(X_t^{(1),t,x}) - f(X_t^{(2),t,x})\|_2 \leq \|g(|X_t^{(1),t,x}| \wedge |X_t^{(2),t,x}|)\|_4 \|X_t^{(1),t,x} - X_t^{(2),t,x}\|_4,$$

(4.18)

where $g : \mathbb{R}_+ \to \mathbb{R}$ is given by

$$g(\lambda) := \sup\{|\nabla f(z)| : |z| \geq \lambda\}, \quad \forall \lambda \geq 0.$$

Notice that $g(\lambda_1 \wedge \lambda_2) \leq g(\lambda_1) + g(\lambda_2)$ for all $\lambda_1, \lambda_2 \geq 0$. It follows that

$$\int_{\mathbb{R}^n} \|g(X_t^{(1),t,x} \wedge X_t^{(2),t,x})\|_4 dx \leq \int_{\mathbb{R}^n} \|g(X_t^{(1),t,x})\|_4 dx + \int_{\mathbb{R}^n} \|g(X_t^{(2),t,x})\|_4 dx.$$  

(4.19)

Therefore, proceeding with a similar argument to that in (4.12) and (4.13) and recalling hypothesis (I), we deduce that

$$\int_{\mathbb{R}^n} \|g(X_t^{(1),t,x})\|_4 dx \leq \sqrt{\frac{CT}{t^{n/4}}} \left( \int_{\mathbb{R}^n} |g(|y|)|^4 \exp \left( CT^{-1} |T^{-1}_t \theta^{(1)}(t, x) - y|^2 \right) \left( |\theta^{(1)}(t, x)|^{\frac{3(n+\epsilon)}{4}} \vee 1 \right)^4 dy dx \right) \frac{1}{4}$$

$$\times \left[ \int_{\mathbb{R}^n} \left( |\theta^{(1)}(t, x)|^{\frac{3(n+\epsilon)}{4}} \vee 1 \right)^{-\frac{3}{4}} dx \right] \frac{3}{4}$$

$$\leq c_{n,\epsilon} \sqrt{CT} e^{nT} \left( \int_{\mathbb{R}^{2n}} |g(|y|)|^4 \exp \left( C^{-1}_T |z|^2 \right) \left( |z|^{3(n+\epsilon)} + |y|^{3(n+\epsilon)} + 1 \right) dz dy \right)^\frac{1}{2}$$

$$\times \left[ \int_{\mathbb{R}^{2n}} \left( |z|^{-(n+\epsilon)} \wedge 1 \right) dz \right] \frac{1}{2} \leq C,$$  

(4.20)
for some constant $C > 0$ depends on $C_T, n, p, \kappa, \epsilon$ and $U$. Combining inequalities (4.10), (4.15), (4.18) - (4.20), we get

$$I_2 \leq C_2 \sqrt{t + t^2} \sup_{s \in [0, t]} |\omega_s^{(1)} - \omega_s^{(2)}|. \tag{4.21}$$

Therefore, inequality (4.4) follows by inserting inequalities (4.8), (4.14) and (4.21) into (4.5). □

**Remark 4.3.** In formulation (4.15), the function $c^{(2)}(t - s, X_s^{(2), t, x})$ is bounded because of the hypothesis (H4). This means that the integrability in $x$ has to be guaranteed by that of the term $\|f(X_s^{(1), t, x}) - f(X_s^{(2), t, x})\|_2$. This is the reason that we assume the integrability hypothesis (I) on $\nabla f$.

**Proposition 4.4.** Assume the conditions in Theorem 2.5. Then, there exists $t_0 > 0$ such that (2.8) with initial condition $f$ has a unique solution on the interval $[0, t_0]$.

**Proof.** For any continuous function $\omega \in C([0, T], \mathbb{R}^n)$ by a similar argument to that in Proposition 4.2, we have that

$$\lim_{s \to t} |u_s^\omega - u_s^\nu|_{L_1} = 0.$$ 

Then, it follow from (4.1) that

$$\lim_{s \to t} |Q_\alpha(u_s^\omega) - Q_\alpha(u_s^\nu)| \leq \lim_{s \to t} \sqrt{n} K^{1-n} \delta^{-1} |u_s^\omega - u_s^\nu|_{L_1} = 0.$$ 

In other words, $Q_\alpha(u_s^\omega)$ is a continuous function in $t$.

We shall use the fix point theorem to prove the proposition. Fix a $t_0 > 0$ satisfying the condition given by (4.27) below. Let $C([0, t_0], \mathbb{R}^n)$ be the Banach space of all continuous functions with the sup norm. For any $\omega \in C([0, t_0], \mathbb{R}^n)$, let $u^\omega : [0, t_0] \times \mathbb{R}^n$ be the (unique) solution to (2.4) associated with $\omega$. Define

$$\mathcal{B} = \{(\omega, u^\omega), \omega \in C([0, t_0], \mathbb{R}^n)\} \subseteq C([0, t_0], \mathbb{R}^n) \oplus C([0, t_0], L_1(\mathbb{R}^d)) \tag{4.22}$$

with the norm

$$\|\omega, u^\omega\|_{\mathcal{B}} = \sup_{0 \leq t \leq t_0} |\omega(t)| + \sup_{0 \leq t \leq t_0} |u_t^\omega|_{L_1}. \tag{4.23}$$

We claim that $\mathcal{B}$ is a closed set of $C([0, t_0], \mathbb{R}^n) \oplus C([0, t_0], L_1(\mathbb{R}^d))$. In fact, if $(\omega^{(n)}, u^{(n)}) \in \mathcal{B}$ converges to $(\omega, v) \in C([0, t_0], \mathbb{R}^n) \oplus C([0, t_0], L_1(\mathbb{R}^d))$, then $\omega^{(n)} \to \omega$ in $C([0, t_0], \mathbb{R}^n)$ and $u^{(n)} \to v$ in $C([0, t_0], L_1(\mathbb{R}^d))$. Thus, $\omega \in C([0, t_0], \mathbb{R}^n)$. Solving (2.4) associated with $\omega$, we obtain $u^\omega \in C([0, t_0], L_1(\mathbb{R}^d))$. By (4.4), we know that $u^{(n)} \to u^\omega$ in $C([0, t_0], L_1(\mathbb{R}^d))$. This implies that $v = u^\omega$. In other word, $\mathcal{B}$ is closed and hence it is also a Banach space.

Fix $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$. Let $K, \delta, \epsilon > 0$ be defined in (3.15)-(3.17). Now, we define a mapping $\mathcal{M} : \mathcal{B} \to \mathcal{B}$ as follows

$$\mathcal{M}(\omega, u^\omega) = (\mathcal{M}_1(\omega, u^\omega), \mathcal{M}_2(\omega, u^\omega)). \tag{4.24}$$
where \((\omega, u^\omega) \in \mathbb{B}\) and
\[
\begin{align*}
\mathcal{M}_1(\omega, u^\omega) &= Q_\alpha(u^\omega), \\
\mathcal{M}_2(\omega, u^\omega) &= u^{Q_\alpha(u^\omega)}.
\end{align*}
\]
Let \(\omega^{(1)}\) and \(\omega^{(2)}\) be continuous functions on \([0, t_0]\) with values in \(\mathbb{R}^n\), and let \(u^{(1)}\) and \(u^{(2)}\) be the solutions to equation (1.4) associated with \(\omega = \omega^{(1)}\) and \(\omega = \omega^{(2)}\) respectively, and with the same initial condition \(f\). Lemma 4.1 and Proposition 4.2 imply that
\[
\sup_{0 \leq t \leq t_0} |Q_\alpha(u_t^{(1)}) - Q_\alpha(u_t^{(2)})| \leq C_0 \sqrt{n} (2K)^{1-n} \delta^{-1} \left(t_0 + \sqrt{t_0}\right) \sup_{t \in [0,t_0]} |\omega_t^{(1)} - \omega_t^{(2)}| \quad (4.25)
\]
and
\[
\begin{align*}
\sup_{0 \leq t \leq t_0} |u_t^{Q_\alpha(\omega^{(1)})} - u_t^{Q_\alpha(\omega^{(2)})}|_{L_1} & \leq C_0 \left(t_0 + \sqrt{t_0}\right) \sup_{s \in [0,t_0]} |Q_\alpha(u_s^{(1)}) - Q_\alpha(u_s^{(2)})| \\
& \leq C_0 \sqrt{n} (2K)^{1-n} \delta^{-1} \left(t_0 + \sqrt{t_0}\right) \sup_{t \in [0,t_0]} |u_t^{(1)} - u_t^{(2)}|_{L_1}.
\end{align*}
\]
Choose \(t_0 > 0\) such that
\[
C_0 \sqrt{n} (2K)^{1-n} \delta^{-1} \left(t_0 + \sqrt{t_0}\right) = L < 1. \quad (4.26)
\]

Then, from (4.25)-(4.26) it follows that the mapping \(\mathcal{M}\) defined by (4.24) is a contraction map on \(\mathbb{B}\). It has then a fixed point \((\omega, u^\omega) \in \mathbb{B}\). By our construction, we see that \(u^\omega\) satisfies (2.4) with \(\omega = Q_\alpha(u^\omega)\). This means that \(u = u^\omega\) satisfies (2.8).

To show the uniqueness, we assume \(v\) is another solution to (2.8). Letting \(\omega' = Q_\alpha(v) = \{Q_\alpha(v_s)\}_{s \in [0,t_0]}\), replacing \(Q_\alpha(\omega)\) by \(\omega'\) in (2.8), we see that \(v\) is also a solution of (2.4) with \(\omega'\). Thus, \((\omega', v)\) is a fixed point of \(\mathcal{M}\). By the uniqueness of the fixed point of map \(\mathcal{M}\), we complete the proof of the proposition. \(\square\)

4.2. **Global solution and proof of main result.** In the previous subsection, we proved that (2.8) has a unique solution \(u\) on \([0, t_0]\) when \(t_0\) is small enough. A natural question is whether this solution can be uniquely extended to any time interval. A positive answer is given in this subsection by using Proposition 3.12.

**Proof of Theorem 2.5.** By Proposition 4.4, there exists \(t_0\), such that (2.8) has a unique solution on \([0, t_0]\). Consider (2.8) with \(t \geq t_0\) and with initial condition \(f = u_{t_0}\). Proposition 3.12 can be applied to obtain that there exists \(t_1 > 0\) depending on the initial condition \(f = u_{t_0}\) only through \(U'\) given by (3.22) such that equation (2.8) has a unique solution on \([t_0, t_0 + t_1]\). Notice that \(U'\) is independent of \(t \in [t_0, T]\). This allows us to extend the solution of (2.8) repeatedly to the interval \([0, t_0 + nt_1]\).
until time \( t_0 + nt_1 \geq T \). In other words, (2.8) has a unique solution on the whole time interval \([0, T]\).

**Proof of Theorem 2.4.** Under the hypotheses (H1)-(H5) and (I), Theorem 2.5 implies the weak existence and uniqueness to SDE (1.4). Because of the weak uniqueness, the \( \alpha \)-quantile of any weak solution to SDE (1.4) is the same function on \([0, T]\). Therefore, the strong existence and uniqueness is a straightforward result of Theorem 3.2.

**Remark 4.5.** Here we point out that our method can be applied to the regular degenerate model developed by Kolokoltsov in [12, page 72]:

\[
\frac{h}{\partial t} \frac{\partial u}{\partial t} = \frac{1}{2} h^2 \text{Tr} \left( G(x) \frac{\partial^2 u}{\partial x^2} \right) + h \left( A(x), \frac{\partial u}{\partial x} \right) - V(x)
\]

(4.28)

since its transition probability density also possesses a two sided Gaussian type bounds. However, to study its extension to

\[
\frac{h}{\partial t} \frac{\partial u}{\partial t} = \frac{1}{2} h^2 \text{tr} \left( G(Q_\alpha(u), x) \frac{\partial^2 u}{\partial x^2} \right) + h \left( A(Q_\alpha(u), x), \frac{\partial u}{\partial x} \right) - V(Q_\alpha(u), x)
\]

we also need the Feynman-Kac formula and, for this reason, we did not pursue this direction in this work further.

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