Quantum information is incompressible without errors

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A classical random variable can be faithfully compressed into a sequence of bits with its expected length, which lies within one bit of Shannon entropy. We generalize this variable-length and faithful scenario to the general quantum source producing mixed states \( \rho_i \) with probability \( p_i \). In contrast to the classical case, the optimal compression rate in the limit of large block length differs from the one in the fixed-length and asymptotically faithful scenario. The amount of this gap is interpreted as the genuine quantum part being incompressible in the former scenario.

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One of the fundamental questions in the information theory is about the data compression, namely, what is the shortest description of a data. This question is important not only for quantification of the amount of information, but also for understanding how well we can manipulate information stored in physical systems, which is a central topic in the field of quantum information. In the case of compressing a classical random variable \( X \) which takes letter \( x \) with probability \( p(x) \), the answer is given by the quantity called the Shannon entropy, which is defined as
\[
H(X) = -\sum_x p(x) \log_2 p(x)
\]
More precisely, a compression scheme is regarded as an assignment of each letter \( x \) with a codeword of length \( l(x) \) bits, which is not fixed and dependent on \( x \). Under the requirement that the codewords are uniquely decodable (faithful) even when they are concatenated to describe a sequence of letters, the minimum \( L_{\text{min}} \) of the expected length \( L = \sum_x p(x) l(x) \) satisfies
\[
H(X) \leq L_{\text{min}} \leq H(X) + 1.
\]
When a sequence of \( n \) letters independently drawn according to \( p(x) \) is collectively compressed, the minimum expected length lies between \( nH(X) \) and \( nH(X) + 1 \) because of the additivity of the entropy function. Hence the optimum compression rate \( R_{\text{opt}}^{(n)} \) for block length \( n \), defined as the minimum expected length per letter, satisfies
\[
H(X) \leq R_{\text{opt}}^{(n)} \leq H(X) + 1/n,
\]
and thus
\[
R_{\text{opt}}^{(\infty)} = \lim_{n \to \infty} R_{\text{opt}}^{(n)} = H(X).
\]
In addition to this ‘variable-length and faithful’ (VLF) scenario, we encounter the Shannon entropy also in a slightly different scenario of compression, in which all codewords have a common fixed length, and small errors in decoding are allowed as long as they vanish in the limit of large block-length. In this ‘fixed-length and asymptotically faithful’ (FLAF) scenario, we consider a sequence of compression schemes labeled by the block length \( n \), which have code length \( L_i^{(n)} \) and error probability \( p_{\text{err}}^{(n)} \). The sequence is asymptotically faithful when \( \lim_{n \to \infty} p_{\text{err}}^{(n)} = 0 \). The asymptotic rate of compression for the sequence is characterized by
\[
R_i = \lim_{n \to \infty} L_i^{(n)}/n.
\]
The optimal compression rate \( R_{\text{opt}}^{\text{AF}} \) in this scenario is defined as the infimum of \( R \) among all asymptotically faithful sequences. Requirement for this scenario is weaker than the VLF scenario in spite of the constraint on the codeword length. This is seen by composing a FLAF sequence with asymptotic rate \( R_{\text{opt}}^{\text{AF}} + \delta \) by just repeating \( m \) times a VLF scheme with rate \( R_{\text{opt}}^{\text{AF}} \) and treat the concatenated codewords longer than \( mn(R_{\text{opt}}^{\text{AF}} + \delta) \) as errors. With \( n \) fixed and in the limit of \( m \to \infty \), this sequence is asymptotically faithful for any \( \delta > 0 \) due to the law of large numbers. Hence we have
\[
R_{\text{opt}}^{(n)} \geq R_{\text{opt}}^{\text{AF}}, \tag{2}
\]
and
\[
R_{\text{opt}}^{(\infty)} \geq R_{\text{opt}}^{\text{AF}}.
\]
In the case of compressing a classical random variable \( X \), it is known that \( R_{\text{opt}}^{\text{AF}} \) is also equal to the Shannon entropy, namely,
\[
R_{\text{opt}}^{(\infty)} = R_{\text{opt}}^{\text{AF}} = H(X).
\]
In the quantum case of compressing an ensemble \( \mathcal{E} = \{ \rho_i \} \), we consider a source that emits a quantum system in state (letter) \( \hat{\rho}_i \) with probability \( p_i \), and \( n \) systems drawn from this source are compressed into qubits. The discussions were first focused on the FLAF scenario, and the result when \( \hat{\rho}_i \) are all pure states \( \mathbb{I} \) is
\[
R_{\text{opt}}^{\text{AF}} = S(\hat{\rho}),
\]
where \( S(\rho) = -\text{Tr}[\rho \log_2 \rho] \) is the von Neumann entropy of the average state \( \hat{\rho} = \sum_i p_i \hat{\rho}_i \). When \( \{ \hat{\rho}_i \} \) includes mixed states, \( R_{\text{opt}}^{\text{AF}} \) is still given by the von Neumann entropy after removing hidden redundancy \( \mathbb{I} \). This striking similarity to the classical case implies that there is no big difference in the compressibility between classical and quantum information, at least in the FLAF scenario. Recently, the investigations on different scenarios \( \mathbb{I} \) have been started to reveal how the optimal rates vary depending on small differences in the constraints. A notable example is the information defect, the difference between the rate \( R_{\text{opt}}^{\text{AF}} \) defined above (which is referred to as the blind scenario in this context) and the rate in an easier scenario (the visible scenario) in which the identity of the index \( i \) is available.
in the compression stage. The information defect has turned out to be nonzero even in the classical cases where all \{\hat{\rho}_i\} commute, implying that the origin of this gap does not necessarily lie in the nature of quantum information.

In this Letter, we discuss the VLF scenario for general quantum ensemble \(\mathcal{E} = \{p_i, \hat{\rho}_i\}\), and derive inequalities corresponding to Eq. (1), which identify \(R_{\text{opt}}^{(\infty)}\). We show that the gap \(\Delta_{\text{F}} - \Delta_{\text{F}}^{(\infty)} = R_{\text{opt}}^{(\infty)} - R_{\text{opt}}^{\Delta_F}\) is generally nonzero, and nonzero only if \{\hat{\rho}_i\} do not commute. When we further separate the information in \(\mathcal{E}\) into the classical part and the quantum part, the origin of the gap becomes transparent, namely, the genuinely quantum part of the information turns out to be incompressible in the VLF scenario.

The keystone in our derivation of \(R_{\text{opt}}^{(\infty)}\) is the observation that the length of the qubits used in a single round of the compression operation can be regarded as an outcome of a measurement on the system to be compressed. This is justified in an operational sense as follows. Suppose that initially there is a resource of \(N\) qubits, and an input state from the source \(\mathcal{E} = \{p_i, \hat{\rho}_i\}\) acting on Hilbert space \(\mathcal{H}'_A\) are compressed into a number of qubits \(L\) via a VLF scheme. The notion that \(L\) qubits has been used to compress the input state means that there should remain \(N - L\) qubits in the resource which can be utilized in other independent tasks. This means that after the state is decompressed back in \(\mathcal{H}_A\), the value \(L\) can be determined without accessing \(\mathcal{H}'_A\). In other words, if we generally write the whole quantum operation of the compression and decompression as a unitary operation \(\hat{U}\) on the combined system of \(\mathcal{H}_A\) and an auxiliary system \(\mathcal{H}_E\) initially prepared in a standard pure state \(\hat{\Sigma}_E\), we should be able to define an observable \(\hat{L}\) acting on \(\mathcal{H}_E\), which corresponds to the length of qubits used to compress. The expected length \(\langle \hat{L} \rangle\) of the compression scheme \(\hat{U}\) applied to the source \(\mathcal{E}\) is then written as

\[
\langle \hat{L} \rangle = \text{Tr}_E\{\hat{U}(\rho \otimes \hat{\Sigma}_E)\hat{U}^\dagger\},
\]

(3)

where \(\hat{\rho} = \sum_i p_i \hat{\rho}_i\). At the same time, since the compression scheme is faithful (no errors), \(\hat{U}\) obeys

\[
\text{Tr}_E[\hat{U}(\hat{\rho} \otimes \hat{\Sigma}_E)\hat{U}^\dagger] = \hat{\rho}.
\]

(4)

The length of qubits is thus regarded as the outcome of a generalized measurement on \(\mathcal{H}_A\) that induces no disturbance on the initial states \{\hat{\rho}_i\}.

The property of the operations that preserves the initial states \{\hat{\rho}_i\} was analyzed in detail recently [2], and it was shown that, given \{\hat{\rho}_i\}, we can find a unique decomposition of \(\mathcal{H}_A\), the support of \(\hat{\rho}\), written as

\[
\mathcal{H}_A = \bigoplus_l \mathcal{H}_j^{(l)} \otimes \mathcal{H}_K^{(l)}.
\]

(5)

Under this decomposition, \(\hat{\rho}_i\) is written as

\[
\hat{\rho}_i = \bigoplus_l p^{(i,l)} \hat{\rho}_j^{(i,l)} \otimes \hat{\rho}_K^{(l)},
\]

(6)

where \(\hat{\rho}_j^{(i,l)}\) and \(\hat{\rho}_K^{(l)}\) are normalized density operators acting on \(\mathcal{H}_j^{(l)}\) and \(\mathcal{H}_K^{(l)}\), respectively, and \(p^{(i,l)}\) is the probability for the state to be in the subspace \(\mathcal{H}_j^{(l)} \otimes \mathcal{H}_K^{(l)}\). \(\hat{\rho}_j^{(i,l)}\) is independent of \(i\), and \{\hat{\rho}_j^{(1,l)}, \hat{\rho}_j^{(2,l)}, \ldots\} cannot be expressed in a simultaneously block-diagonalized form. Any \(\hat{U}\) that satisfies Eq. (3) is then expressed in the following form

\[
\hat{U}(\hat{1}_A \otimes \hat{\Sigma}_E) = \bigoplus_l \hat{U}_l^{(l)}(\hat{1}_A \otimes \hat{\Sigma}_E),
\]

(7)

where \(\hat{U}_l^{(l)}\) are unitary operators acting on the combined space \(\mathcal{H}_j^{(l)} \otimes \mathcal{H}_E\), satisfying \(\text{Tr}_E[\hat{U}_l^{(l)}(\hat{\rho}_j^{(l)} \otimes \hat{\Sigma}_E)\hat{U}_l^{(l)}] = \rho_K^{(l)}\). An explicit procedure to obtain this particular decomposition is also given in [2].

The total density operator \(\hat{\rho} = \sum_i p_i \hat{\rho}_i\) for the ensemble \(\mathcal{E}\) is also decomposed as

\[
\hat{\rho} = \bigoplus_l p^{(l)} \hat{\rho}_j^{(l)} \otimes \hat{\rho}_K^{(l)},
\]

(8)

where \(p^{(l)} \equiv \sum_i p_i p^{(i,l)}\) and \(\hat{\rho}_j^{(l)} \equiv (\sum_i p_i p^{(i,l)} \hat{\rho}_j^{(i,l)})/p^{(l)}\). Now, let us take a basis \{\ket{a_j^{(l)}}\} \(j = 1, \ldots, \dim \mathcal{H}_j^{(l)}\) for each \(\mathcal{H}_j^{(l)}\), and consider another source \(\mathcal{E}' = \{q_{\lambda}, \hat{\sigma}_\lambda\}\) with double index \(\lambda \equiv (l, j)\), which is defined through the decomposition [2] such that \(q_{\lambda} = (\dim \mathcal{H}_j^{(l)})^{-1} p^{(l)}\) and

\[
\hat{\sigma}_\lambda = |a_j^{(l)}\rangle \langle a_j^{(l)}| \otimes \hat{\rho}_K^{(l)}.
\]

(9)
The total density operator $\hat{\sigma} \equiv \sum_\lambda q_\lambda \hat{\sigma}_\lambda$ for this source is

$$\hat{\sigma} = \bigoplus_l \rho^{(l)} (\dim H^{(l)}_J)^{-1} \hat{1}^{(l)} \otimes \hat{\rho}_K^{(l)}.$$  \hfill (10)

Suppose that this source is used as an input to the same compression scheme. The form (10) assures that the operation $\hat{U}$ also preserves $\hat{\sigma}_\lambda$, namely, the scheme $\hat{U}$ works as a VLF scheme for $\mathcal{E}'$. We further note that, since the difference between $\hat{\sigma}$ and $\hat{\sigma}$ lies in the internal states of $H^{(l)}_J$, and the form (10) ensures that the state of the auxiliary system ($\mathcal{H}_E$) after the operation $\hat{U}$ is insensitive to them, we obtain $\text{Tr}_A[\hat{U}(\hat{\rho} \otimes \hat{\Sigma}_E)\hat{U}^\dag] = \text{Tr}_A[\hat{U}(\hat{\sigma} \otimes \hat{\Sigma}_E)\hat{U}^\dag]$. Then, according to Eq. (3), the same expected length $\langle \hat{L} \rangle$ should be observed even when the source is replaced with $\mathcal{E}'$. Hence we have

$$L_{\text{min}}(\mathcal{E}) \geq L_{\text{min}}(\mathcal{E}').$$  \hfill (11)

In order to find a lower bound of $L_{\text{min}}(\mathcal{E}')$, we use Eq. (2), which also holds for general quantum ensembles. Since the letter states $\hat{\sigma}_\lambda$ of the ensemble $\mathcal{E}'$ are all orthogonal, the optimum rate $R_{\text{opt}}^{AF}(\mathcal{E}')$ should be equal to that of an orthogonal pure-state ensemble $\{q_\lambda, |\lambda\rangle\}$, and hence $R_{\text{opt}}^{AF}(\mathcal{E}') = -\sum_\lambda q_\lambda \log_2 q_\lambda$. Using this with $q_\lambda = (\dim H^{(l)}_J)^{-1} \rho^{(l)}$, Eq. (3) with $n = 1$, and $L_{\text{min}}(\mathcal{E}') = R_{\text{opt}}^{(1)}(\mathcal{E}')$ (by definition), we have

$$L_{\text{min}}(\mathcal{E}') \geq R_{\text{opt}}^{AF}(\mathcal{E}') = I_C(\mathcal{E}) + D_{\text{NC}}(\mathcal{E}),$$  \hfill (12)

where $I_C(\mathcal{E}) \equiv -\sum_l p^{(l)} \log_2 p^{(l)}$ and $D_{\text{NC}}(\mathcal{E}) \equiv \sum_l p^{(l)} \log_2 \dim H^{(l)}_J$.

For an upper bound for $L_{\text{min}}(\mathcal{E})$, we consider a specific example of the VLF scheme as follows. The compressor first makes a projection measurement to determine the index $l$, which appears in the decomposition (3). The projected state should be supported by subspace $H^{(l)}_J \otimes H^{(l)}_K$. Since the outcome $l$ occurs with probability $p^{(l)}$, there exists an instantaneous code for encoding $l$ with an expected length not larger than $H\{p^{(l)}\} + 1$ bits (3). The compressor write down this classical codeword onto qubits using the standard basis $\{|0\rangle, |1\rangle\}$. It then discards $H^{(l)}_K$, and transfer the state of $H^{(l)}_J$ (which should be $\rho^{(l)}_J$) into $\log_2 \dim H^{(l)}_J$ qubits and concatenate them after the qubits holding the classical codeword for $l$. The expected total length of the qubits is then not larger than $I_C(\mathcal{E}) + 1 + D_{\text{NC}}(\mathcal{E}) + 1$. The decompression is done by measuring qubits in the standard basis one by one, until the end of the codeword is reached. Note that the decompressor knows this end point since the code for $l$ is instantaneous. Learning $l$, it then transfers the contents of the next $\log_2 \dim H^{(l)}_J$ qubits into $H^{(l)}_J$, and prepare $H^{(l)}_K$ in the known state $\hat{\rho}_K^{(l)}$. When the input to the compressor was $\hat{\rho}_i$, the conditional probability of $l$ is $p^{(l)}(i)$. Hence the whole process faithfully reproduces $\hat{\rho}_i$ [see Eq. (3)]. We thus obtain an upper bound $I_C(\mathcal{E}) + D_{\text{NC}}(\mathcal{E}) + 2$ for $L_{\text{min}}(\mathcal{E})$. Together with Eqs. (11) and (12), we have

$$I_C(\mathcal{E}) + D_{\text{NC}}(\mathcal{E}) \leq L_{\text{min}}(\mathcal{E}) \leq I_C(\mathcal{E}) + D_{\text{NC}}(\mathcal{E}) + 2.$$  \hfill (13)

In order to extend this result to the case of $n$-block coding, we need to know the behavior of the functions $I_C(\mathcal{E})$ and $D_{\text{NC}}(\mathcal{E})$ for the concatenation of independently drawn letters (states). Suppose that $\mathcal{H}_A$ is prepared by a source $\mathcal{E}_A = \{p^A_l, \hat{\rho}^A_l\}$, and let $\mathcal{H}_A = \bigoplus_l \mathcal{H}^A_{l, J} \otimes \mathcal{H}^A_{l, K}$ be the decomposition determined by $\mathcal{E}_A$. Suppose further that another system $\mathcal{H}_B$ is independently prepared by a source $\mathcal{E}_B = \{p^B_j, \hat{\rho}^B_j\}$, and let $\mathcal{H}_B = \bigoplus_m \mathcal{H}^B_{m, J} \otimes \mathcal{H}^B_{m, K}$ be the decomposition determined by $\mathcal{E}_B$. This preparation can be also considered as the combined system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ being prepared by $\mathcal{E}_{AB} = \{p^A_l p^B_j, \hat{\rho}^A_l \hat{\rho}^B_j\}$. It was shown [2] that the decomposition determined by $\mathcal{E}_{AB}$ is simply given by the direct product $\mathcal{H}_{AB} = \bigoplus_{l,m} (\mathcal{H}^A_{l, J} \otimes \mathcal{H}^B_{m, J}) \otimes (\mathcal{H}^A_{l, K} \otimes \mathcal{H}^B_{m, K})$. This implies that $I_C(\mathcal{E})$ and $D_{\text{NC}}(\mathcal{E})$ are additive, namely, $I_C(\mathcal{E}_{AB}) = I_C(\mathcal{E}_A) + I_C(\mathcal{E}_B)$ and $D_{\text{NC}}(\mathcal{E}_{AB}) = D_{\text{NC}}(\mathcal{E}_A) + D_{\text{NC}}(\mathcal{E}_B)$. Just as in the classical case, this additivity leads to our main results,

$$I_C(\mathcal{E}) + D_{\text{NC}}(\mathcal{E}) \leq R_{\text{opt}}^{(n)}(\mathcal{E}) \leq I_C(\mathcal{E}) + D_{\text{NC}}(\mathcal{E}) + 2/n,$$  \hfill (14)

and

$$R_{\text{opt}}^{(\infty)}(\mathcal{E}) = I_C(\mathcal{E}) + D_{\text{NC}}(\mathcal{E}).$$  \hfill (15)

This optimal compression rate $R_{\text{opt}}^{(\infty)}(\mathcal{E})$ for the VLF scenario is generally larger than the one in the FLAF scenario, which was derived in [3] for general mixed-state cases as
\[ R_{\text{opt}}^{\text{AF}}(\mathcal{E}) = I_C(\mathcal{E}) + I_{NC}(\mathcal{E}), \]

where \( I_{NC}(\mathcal{E}) \equiv \sum_i p(i) S(\rho_i^{(i)}) \), and \( I_C(\mathcal{E}) = -\sum_i p(i) \log_2 p(i) \) is the same function as defined above. The difference \( \Delta_{F-\text{AF}} \equiv R_{\text{opt}}^{(\infty)} - R_{\text{opt}}^{\text{AF}} \) is

\[ \Delta_{F-\text{AF}} = D_{NC}(\mathcal{E}) - I_{NC}(\mathcal{E}) = \sum_i p(i) [\log_2 \dim \mathcal{H}_i^{(i)} - S(\hat{\rho}_i^{(i)})], \]

which is nonzero when there exists \( l \) such that \( \hat{\rho}_i^{(l)} \neq (\dim \mathcal{H}_i^{(l)})^{-1} \mathbb{1}_{\mathcal{H}_i^{(l)}} \). The gap \( \Delta_{F-\text{AF}} \) is zero when \( \hat{\rho}_i^{(l)} = (\dim \mathcal{H}_i^{(l)})^{-1} \mathbb{1}_{\mathcal{H}_i^{(l)}} \) for all \( l \). In particular, for the classical cases in which all \( \hat{\rho}_i \) commutes, \( (\dim \mathcal{H}_i^{(i)}) = 1 \) for all \( i \) \cite{12} and \( \Delta_{F-\text{AF}} = 0 \).

The gap \( \Delta_{F-\text{AF}} \) stems from the internal state of each \( \mathcal{H}_i^{(l)} \), which is regarded as the genuinely quantum (nonclassical) part of the ensemble \( \mathcal{E} \) in the following sense: (i) This part is inaccessible without introducing disturbance [Eq. (6)] and (ii) this part can be compressed into qubits, not into classical bits \cite{13,14}. Then, the present results are summarized in a simple statement, ‘the genuinely quantum part of the information is incompressible if no errors are allowed.’ In contrast, the classical part of the information, which is represented by \( \{ p(i;l) \} \), can be compressed into \( I_C \) bits in either one of the scenarios. The difference between the classical and the quantum part in the nature of compressibility may be understood as follows. The classical information can be effectively compressed by changing the description length adaptively depending on the input states, namely, using shorter descriptions for frequent inputs and longer descriptions for rare inputs. The quantum information, if it is to be compressed faithfully, does not allow such adaptation because learning the input state inevitably introduces irreversible disturbance.

It may be quite instructive to contrast the derived gap \( \Delta_{F-\text{AF}} \) with the information defect, the gap between the blind and the visible scenario. In the visible scenario, the classical index \( i \) is given to the compressor, rather than the state \( \rho_i \), to be reproduced in the decompression. The optimal compression rate \( I_{\text{eff}} \) for asymptotically faithful schemes in the visible scenario is called the effective information. By definition, \( R_{\text{opt}}^{\text{AF}} \geq I_{\text{eff}} \), and the gap \( \Delta_{b-v} \equiv R_{\text{opt}}^{\text{AF}} - I_{\text{eff}} \geq 0 \) is called the information defect. While the explicit form of the effective information is still open, it is bounded from below by the Levitin-Holevo function \cite{13} \( I_{\text{LH}} \equiv S(\hat{\rho}) - \sum_i p_i S(\hat{\rho}_i) \), namely, \( I_{\text{eff}} \geq I_{\text{LH}} \). An upper bound for \( \Delta_{b-v} \) is hence \( R_{\text{opt}}^{\text{AF}} - I_{\text{LH}} = I_C + I_{NC} - I_{\text{LH}} \), and using Eqs. (6) and (8), we obtain

\[ \Delta_{b-v} \leq \sum_i p_i S(\hat{\rho}_i) - \sum_i p(i) S(\hat{\rho}_i^{(l)}) = \sum_i p_i S(\hat{\rho}_i^{(l)}), \]

where the state \( \hat{\rho}_i^{(l)} = \bigoplus_l p(i;l) \hat{\rho}_i^{(l)} \) is the one obtained from \( \hat{\rho}_i \) by removing the redundant part \( \hat{\rho}_i^{(l)} \) [see Eq. (6)]. Let us call an ensemble \( \mathcal{E} \) ‘pure’ when all the states \( \hat{\rho}_i^{(l)} \) are pure states, and call it ‘mixed’ otherwise. Then, Eq. (13) implies that \( \Delta_{b-v} = 0 \) for pure ensembles. It should be noted that the pure/mixed classification is independent of the one by classical (all \( \rho_i \) commutes) and quantum. It has been shown \cite{13,14} that there exist classical ensembles with \( \Delta_{b-v} > 0 \). The properties of the two gaps \( \Delta_{F-\text{AF}} \) and \( \Delta_{b-v} \) are summarized in Table I. When the letter states are distinct, namely, \( \{ \hat{\rho}_i \} \) are all orthogonal, the optimal compression rates for the three scenarios are equal. When the letter states overlap with each other and are not completely distinguishable, there can be two types of overlaps. One is the cases when each letter state is noisy. In such cases, whether or not the identity of each letter is given as side information affect the compressibility. The other way of overlapping is quantum one, where the letter states are pure and suffers no noises, but are not completely distinguishable from each other due to nonorthogonality. In such cases, the compressibility depends on whether small errors (even an asymptotically vanishing one) are allowed or not.

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| Information | classical \( (\hat{\rho}_i, \rho_i) = 0 \) | quantum |
|------------|------------------|--------|
| pure \((S(\hat{\rho}_i^{(l)}) = 0)\) | \( R_{\text{opt}}^{(\infty)} = R_{\text{opt}}^{\text{AF}} = I_{\text{eff}} \) | \( R_{\text{opt}}^{(\infty)} \geq R_{\text{opt}}^{\text{AF}} = I_{\text{eff}} \) |
| mixed | \( R_{\text{opt}}^{(\infty)} = R_{\text{opt}}^{\text{AF}} \geq I_{\text{eff}} \) | \( R_{\text{opt}}^{(\infty)} \geq R_{\text{opt}}^{\text{AF}} \geq I_{\text{eff}} \) |

**TABLE I.** Nature of information and gaps between optimal compression rates for various scenarios — the variable-length, faithful and blind scenario \( (R_{\text{opt}}^{(\infty)}) \), the fixed-length, asymptotically faithful and blind scenario \( (R_{\text{opt}}^{\text{AF}}) \), and the fixed-length, asymptotically faithful and visible scenario \( (I_{\text{eff}}) \).
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