Recurrent Equilibrium Networks: Flexible Dynamic Models With Guaranteed Stability and Robustness

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Abstract—This article introduces recurrent equilibrium networks (RENs), a new class of nonlinear dynamical models for applications in machine learning, system identification, and control. The new model class admits “built-in” behavioral guarantees of stability and robustness. All models in the proposed class are contracting—a strong form of nonlinear stability—and can satisfy prescribed incremental integral quadratic constraints, including Lipschitz bounds and incremental passivity. RENs are otherwise very flexible: they can represent all stable linear systems, all previously known sets of contracting recurrent neural networks and echo state networks, all deep feedforward neural networks, and all stable Wiener/Hammerstein models, and can approximate all fading memory and contracting nonlinear systems. RENs are parameterized directly by a vector in $\mathbb{R}^N$, i.e., stability and robustness are ensured without parameter constraints, which simplifies learning since generic methods for unconstrained optimization such as stochastic gradient descent and its variants can be used. The performance and robustness of the new model set are evaluated on benchmark nonlinear system identification problems. This article also presents applications in data-driven nonlinear observer design and control with stability guarantees.

Index Terms—Deep learning, recurrent neural networks, robust stability, system identification, state estimation.

I. INTRODUCTION

DEEP neural networks (DNNs), recurrent neural networks (RNNs), and related models have revolutionized many fields of engineering and computer science [1]. Their remarkable flexibility, accuracy, and scalability have led to renewed interest in neural networks in many domains, including learning-based/data-driven methods in control, identification, and related areas (see, e.g., [2], [3], and [4] and references therein).

However, it has been observed that neural networks can be very sensitive to small changes in inputs [5], and this sensitivity can extend to control policies [6]. Furthermore, their scale and complexity makes them difficult to certify for use in safety-critical systems, and it can be difficult to incorporate prior physical knowledge into a neural network model, e.g., that a model should be stable. The most accurate current methods for certifying stability and robustness of DNNs and RNNs are based on mixed-integer programming [7] and semidefinite programming [8], [9], both of which face challenges when scaling to large networks.

In this article, we introduce a new model structure: the recurrent equilibrium network (REN), which has the following advantages.

1) RENs are highly flexible and include many established models as special cases, including DNNs, RNNs, echo-state networks, and stable linear dynamical systems.

2) RENs admit built-in behavioral guarantees such as stability, incremental gain, passivity, or other properties that are relevant to safety critical systems, and are compatible with most existing frameworks for nonlinear/robust stability analysis.

3) RENs are easy to use as they permit a direct (smooth, unconstrained) parameterization enabling learning of large-scale models via generic unconstrained optimization algorithms and off-the-shelf automatic-differentiation tools.

An REN is a dynamical model incorporating an equilibrium network [10], [11], [12], also known as implicit network [13]. Equilibrium networks are “implicit depth” neural networks, in which the output is generated as the zero set of an equation relating inputs and outputs, which can be viewed as the equilibrium of a “fast” dynamical system. This implicit structure brings the remarkable flexibility alluded to aforementioned, but also raises the question of existence and uniqueness of solutions, i.e., well-posedness. A benefit of our parameterization approach is that the resulting RENs are always well-posed.

RENS can be constructed to be contracting [14], a strong form of nonlinear stability, and/or to satisfy robustness guarantees in the form of incremental integral quadratic constraints (IQCs) [15]. This class of constraints includes user-definable bounds on the network’s Lipschitz constant (incremental gain), which can be used to tradeoff performance versus sensitivity.
to adversarial perturbations. The IQC framework also encompasses many commonly used tools for certifying stability and performance of system interconnections, including passivity methods in robotics [16], networked-system analysis via dissipation inequalities [17], $\mu$ analysis [18], and standard tools for analysis of nonlinear control systems [19].

A. Learning and Identification of Stable Models

The problem of learning dynamical systems with stability guarantees appears frequently in system identification. When learning models with feedback it is not uncommon for the model to be unstable even if the data-generating system is stable. For linear models, various methods have been proposed to guarantee stability via regularization and constrained optimization [20], [21], [22], [23], [24]. For nonlinear models, there has also been a substantial volume of research on stability guarantees, e.g., for polynomial models [25], [26], [27], [28], Gaussian mixture models [29], and RNNs [30], [31], [32], [33], [34]. However, the problem is substantially more complex than the linear case due to the many possible nonlinear model structures and differing definitions of nonlinear stability. Contraction is a strong form of nonlinear stability [14], which is particularly well-suited to problems in learning and system identification since it guarantees stability of all solutions of the model, irrespective of inputs or initial conditions. This is important in learning since the purpose of a model is usually to simulate responses to previously unseen inputs. The authors in [25], [26], [27], [28], [30], [33], and [34] guarantee to find contracting models.

B. Lipschitz Bounds for Neural Network Robustness

Model robustness can be characterized in terms of sensitivity to small perturbations in the input. It has recently been shown that RNN models can be extremely fragile [35], i.e., small changes to the input produce dramatic changes in the output.

Formally, sensitivity and robustness can be quantified via Lipschitz bounds on the input–output mapping associated with the model. In machine learning, Lipschitz constants are used in the proofs of generalization bounds [36], analysis of expressiveness [37], and guarantees of robustness to adversarial attacks [38], [39]. There is also ample empirical evidence to suggest that Lipschitz regularity (and model stability, where applicable) improves generalization in machine learning [40] and system identification [33]. In reinforcement learning [41], it has recently been found that the Lipschitz constant of policies has a strong effect on their robustness to adversarial attack [42]. In [43], it was shown that privacy preservation in dynamic feedback policies can be represented as an $\ell^2$ Lipschitz bound.

Unfortunately, even calculation of the Lipschitz constant of feedforward (static) neural networks is NP-hard [44]. The tightest tractable bounds known to date use incremental quadratic constraints to construct a behavioral description of the neural network activation functions [45], but using these results in training is complicated by the fact that the constraints are not jointly convex in model parameters and constraint multipliers. In [46], Lipschitz bounded feedforward models were trained using the alternating direction method of multipliers, and in [33], a custom interior point solver was used. However, the requirements to satisfy linear matrix inequalities (LMIs) at each iteration make these methods difficult to scale. Wang and Manchester [47] introduced a direct parameterization of feedforward neural networks satisfying the bounds of [45], using techniques related to this article.

C. Applications of Contracting and Robust Models in Data-Driven Control and Estimation

An ability to learn flexible dynamical models with contraction, robustness, and other behavioral constraints has many potential applications in control and related fields, some of which we explore in this article.

In robotics, passivity constraints are widely used to ensure stable interactions, e.g., in teleoperation, vision-based control, and multirobot control [16] and interaction with physical environments (e.g., [48] and [49]). More generally, methods based on quadratic dissipativity and IQCs are a powerful tool for the design of complex interconnected cyber-physical systems [15], [17]. Within these frameworks, the proposed REN architecture can be used to learn subsystems that specify prescribed or parameterized IQCs, and which, therefore, cannot destabilize the system when interconnected with other components.

A classical problem in the control theory is observer design: to construct a dynamical system that estimates the internal (latent) state of another system from partial measurements. A recent approach is to search for a contracting dynamical system that can reproduce true system trajectories [50], [51]. In Section VIII, we formulate the observer design problem as a supervised learning problem over a set of contracting nonlinear systems, and demonstrate the approach on an unstable nonlinear reaction diffusion partial differential equation (PDE).

In optimization of linear feedback controllers, the classical Youla-Kucera (or $Q$) parameterization provides a convex formulation for searching over all stabilizing controllers via a “free” stable linear system parameter [18], [52], [53]. This approach can be extended to nonlinear systems [19], [54] in which the “free parameter” is a stable nonlinear model. In Section IX, we apply this idea to optimize nonlinear feedback policies for constrained linear control.

D. Convex and Direct Parameterizations

The central contributions of this article are new model parameterizations that have behavioral constraints, and which are amenable to optimization. The first set of parameterizations we introduce includes (convex) LMI constraints, building upon [25] and [34]. LMI constraints can be incorporated into a learning process either through introduction of barrier functions or projections. However, they are computationally challenging for large-scale models. For example, a path-following interior point method, as proposed in [34], generally requires computing gradients of barrier functions, line search procedures, and a combination of “inner” and “outer” iterations as the barrier parameter changes.
To address this challenge, in this article, we also introduce direct parameterizations of contracting and robust RENs. That is, we construct a smooth mapping from $\mathbb{R}^N$ to the model weights such that every model in the image of this mapping satisfies the desired behavioral constraints. This can be thought of as constructing a (redundant) intrinsic coordinate system on the constraint manifold. The construction is related to the method of [55] for semidefinite programming, in which a positive-semidefinite matrix is parameterized by square-root factors. Our parameterization differs in that it avoids introducing any nonlinear equality constraints.

As mentioned previously, direct parameterization allows generic optimization methods such as stochastic gradient descent (SGD) and ADAM [56] to be applied. Another advantage is that it allows easy random sampling of nonlinear models with the required stability and robustness constraints by simply sampling a random vector in $\mathbb{R}^N$. This allows straightforward generation of echo state networks with prescribed behavioral properties, i.e., large-scale recurrent networks with fixed dynamics and learnable output maps (see, e.g., [57] and [58] and references therein).

E. Structure of This Article

The rest of this article is structured as follows.

1) Sections II–VI discuss the proposed model class and its properties. Section II formulates the problem of learning stable and robust dynamical models; in Section III, we present the REN model class; in Section IV, we present convex parameterizations of stable and robust RENs; in Section V, we present direct (unconstrained) parameterizations of RENs; and in Section VI, we discuss the expressivity of the REN model class, showing it includes many commonly used models as special cases.

2) Sections VII–IX present applications of learning stable/robust nonlinear models. Section VII presents applications to system identification; Section VIII presents applications to nonlinear observer design; Section IX presents applications to nonlinear feedback design for linear systems. Associated Julia code is available in the package RobustNeuralNetworks.jl [59].

3) Finally, Section X concludes this article.

A preliminary conference version was presented in [60]. This article expands the class of robustness properties to more general dissipativity conditions, removes the restriction that the model has zero direct-feedthrough, introduces the acyclic REN, adds proofs of all theoretical results, adds new material on echo state networks, and includes novel approaches to nonlinear observer design and optimization of feedback controllers enabled by the REN.

F. Notation

The set of sequences $x : \mathbb{N} \to \mathbb{R}^n$ is denoted by $\ell^0_{2\infty}_n$. Subscript $\infty$ is omitted when it is clear from the context. For $x \in \ell^0_{2\infty}_n$, $x_t \in \mathbb{R}^n$ is the value of the sequence $x$ at time $t \in \mathbb{N}$. The subset $\ell_2 \subset \ell_2$ consists of all square-summable sequences, i.e., $x \in \ell_2$ if and only if the $\ell_2$ norm $\|x\| := \sqrt{\sum_{t=0}^{\infty} |x_t|^2}$ is finite, where $|\cdot|$ denotes Euclidean norm. Given a sequence $x \in \ell_{2\infty}$, the $\ell_2$ norm of its truncation over $[0, T]$ is $\|x\|_T := \sqrt{\sum_{t=0}^{T} |x_t|^2}$. For two sequences $x, y \in \ell^0_{2\infty}$, the inner product over $[0, T]$ is $(x, y)_T := \sum_{t=0}^{T} x_t y_t$. We use $A > 0$ and $A \geq 0$ to denote a positive definite and positive semidefinite matrix, respectively. We denote the set of positive-definite diagonal matrices by $\mathbb{D}_+$. Given a positive-definite matrix $P$, we use $|x|_P$ to denote the weighted Euclidean norm, i.e., $|x|_P = \sqrt{x^\top P x}$.

II. Learning Stable and Robust Models

This article is concerned with learning of nonlinear dynamical models, i.e., finding a particular model within a set of candidates using some data relevant to the problem at hand. The central aim of this article is to construct model classes that are flexible enough to make full use of available data, and yet guaranteed to be well-behaved in some sense.

Given a dataset $\tilde{z}$, we consider the problem of learning a nonlinear state-space dynamical model of the form

$$x_{t+1} = f(x_t, u_t, \theta), \quad y_t = g(x_t, u_t, \theta)$$

that minimizes some loss or cost function depending (in part) on the data, i.e., to solve a problem of the form

$$\min_{\theta \in \Theta} \mathcal{L}(\tilde{z}, \theta).$$

In the abovementioned, $x_t \in \mathbb{R}^n, u_t \in \mathbb{R}^m, y_t \in \mathbb{R}^p$, and $\theta \in \Theta \subseteq \mathbb{R}^N$ are the model state, input, output, and parameters, respectively. Here, $f : \mathbb{R}^n \times \mathbb{R}^m \times \Theta \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^m \times \Theta \to \mathbb{R}^p$ are piecewise continuously differentiable functions.

Example 1: In the context of system identification, we may have $\tilde{z} = (\tilde{y}, \tilde{u})$ consisting of finite sequences of input–output measurements, and aim to minimize simulation error

$$\mathcal{L}(\tilde{z}, \theta) = \|y - \tilde{y}\|_T^2$$

where $y = \mathcal{A}_a(\tilde{u})$ is the output sequence generated by the nonlinear dynamical model (1) with initial condition $x_0 = a$ and inputs $u_t = \tilde{u}_t$. Here, the initial condition $a$ may be part of the data $\tilde{z}$, or considered a learnable parameter in $\theta$.

The main contributions of this article are model parameterizations, and we make the following definitions.

Definition 1: A model parameterization (1) is called a convex parameterization if $\Theta \subseteq \mathbb{R}^N$ is a convex set. Furthermore, it is called a direct parameterization if $\Theta = \mathbb{R}^N$.

Direct parameterizations are useful for learning large-scale models since many scalable unconstrained optimization methods (e.g., SGD) can be applied to solve (2). We will parameterize stable nonlinear models, and the particular form of stability we use is the following.

Definition 2: A model (1) is said to be contracting with rate $\alpha \in (0, 1)$ if for any two initial conditions $a, b \in \mathbb{R}^n$, given the same input sequence $u \in \ell_{2\infty}$, the state sequences $x^a$ and $x^b$ satisfy

$$|x^a_t - x^b_t| \leq K \alpha^t |a - b|$$

for some $K > 0$. 
Roughly speaking, contracting models forget their initial conditions exponentially. Beyond stability, we will also consider robustness constraints of the following form.

Definition 3: A model (1) is said to satisfy the incremental (IQC) defined by \((Q, S, R)\), where \(0 \geq Q \in \mathbb{R}^p \times p, S \in \mathbb{R}^{m \times p}\), and \(R = R^T \in \mathbb{R}^{m \times m}\), if for all pairs of solutions with initial conditions \(a, b \in \mathbb{R}^n\) and input sequences \(u, v \in \ell^2_{2e}\), the output sequences \(y^a = \mathcal{R}_a(u)\) and \(y^b = \mathcal{R}_b(v)\) satisfy

\[
\sum_{t=0}^{T} \begin{bmatrix} y^a_t - y^b_t \\ u_t - v_t \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y^a_t - y^b_t \\ u_t - v_t \end{bmatrix} \geq -d(a, b) \quad \forall T
\]

for some function \(d(a, b) \geq 0\) with \(d(a, a) = 0\).

Important special cases of incremental IQCs include the following.

1) \(Q = -\frac{1}{2} I, R = \gamma I, S = 0\): The model satisfies an \(L^2\) Lipschitz bound, also known as incremental \(L^2\)-gain bound, of \(\gamma\) as

\[
||\mathcal{R}_a(u) - \mathcal{R}_a(v)||_T \leq \gamma||u - v||_T
\]

for all \(u, v \in \ell^2_{2e}\), \(T \in \mathbb{N}\).

2) \(Q = 0, R = -\gamma_0 I, S = I\), where \(\gamma_0 \geq 0\): The model is monotone on \(L^2\) (strongly if \(\gamma_0 > 0\)), also known as incrementally passive (incrementally strictly input passive, respectively)

\[
(\mathcal{R}_a(u) - \mathcal{R}_a(v), u - v)_T \geq \nu ||u - v||^2_T
\]

for all \(u, v \in \ell^2_{2e}\) and \(T \in \mathbb{N}\).

3) \(Q = -\rho I, R = 0, S = I\), where \(\rho > 0\): The model is incrementally strictly output passive

\[
(\mathcal{R}_a(u) - \mathcal{R}_a(v), u - v)_T \geq \rho ||\mathcal{R}_a(u) - \mathcal{R}_a(v)||^2_T
\]

for all \(u, v \in \ell^2_{2e}\) and \(T \in \mathbb{N}\). If \(\rho = 1\), the model is firmly nonexpansive on \(L^2\).

In other contexts, \(Q, S, \text{ and } R\) may themselves be decision variables in a separate optimization problem to ensure the stability of interconnected systems (see, e.g., [15] and [17]).

Remark 1: Given a model class guaranteeing incremental IQC defined by constant matrices \(Q, S, \text{ and } R\), it is straightforward to construct models satisfying frequency-weighted IQCs. For example, by constructing a model \(R\) that is contracting and satisfies an \(L^2\) Lipschitz bound, and choosing stable linear filters \(W_1, W_2\), with \(W_1\) having a stable inverse, the new model

\[
y = \mathcal{M}_a(u) = W_1^{-1}(\mathcal{R}_a(W_2 u))
\]

is contracting and satisfies the frequency-weighted bound

\[
||W_1(\mathcal{M}_a(u) - \mathcal{M}_a(v))||_T \leq \gamma ||W_2(u - v)||_T.
\]

III. RECURRENT EQUILIBRIUM NETWORKS (RENS)

The model structure we propose—the REN—is a state-space model of the form (1) with

\[
x_{t+1} = Ax_t + B_1w_t + B_2u_t + b_x
\]

\[
y_t = C_2x_t + D_{21}w_t + D_{22}u_t + b_y
\]

in which \(w_t\) is the solution of an equilibrium network, also known as implicit network [10], [11], [12], [13]

\[
w_t = \sigma(D_{11}w_t + C_1x_t + D_{12}u_t + b_v)
\]

where \(A, B, C, \text{ and } D\) are matrices of appropriate dimension, \(b_x \in \mathbb{R}^n, b_y \in \mathbb{R}^p, \text{ and } b_y \in \mathbb{R}^q\) are “bias” vectors, and \(\sigma\) is a scalar nonlinearity applied elementwise, referred to as an “activation function.” We will show later how to ensure that a unique solution \(w_t^*\) to (8) exists and can be computed efficiently.

Remark 2: The term “equilibrium” comes from the fact that any solution of the aforementioned implicit equation is also an equilibrium point of the difference equation \(w_t^{k+1} = \sigma(Dw_t^k + b_w)\) or the ordinary differential equation \(\dot{w}_t(s) = -w_t(s) + \sigma(Dw_t(s) + b_w)\), where \(b_w = C_1x_t + D_{12}u_t\) is considered “frozen” for each \(t\). One interpretation of the REN model is that it represents a two-timescale or singular perturbation model, in which the “fast” dynamics in \(w\) are assumed to reach the equilibrium (8) well within each time step of the “slow” dynamics in \(x\) (6).

It will be convenient to represent the REN model as a feedback interconnection of a linear system \(G\) and a nonlinear activation \(\sigma\).

Fig. 1. REN as a feedback interconnection of a linear system \(G\) and a nonlinear activation \(\sigma\).
architectures included as special cases. For example, consider a standard $L$-layer DNN as
\[
\begin{align*}
z_0 &= u \\
z_{l+1} &= \sigma(W_l z_l + b_l), \quad l = 0, \ldots, L - 1 \\
y &= W_L z_L + b_L
\end{align*}
\] where $z_l$ is the output of the $l$th hidden layer. This can be written as an equilibrium network with
\[
w = \text{col}(z_1, \ldots, z_L), \quad b_w = \text{col}(b_0, \ldots, b_{L-1}), \quad b_y = b_L
\]
\[
C_1 = 0, \quad C_2 = 0, \quad D_{21} = \begin{bmatrix} 0 & \cdots & 0 \\ W_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & W_{L-1} \end{bmatrix}, \quad D_{22} = 0
\]
\[
D_{11} = \begin{bmatrix} 0 & \cdots & 0 \\ W_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} W_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]
Equilibrium networks can represent many other interesting structures including residual, convolution, and other feedforward networks. The reader is referred to [10], [11], [12], and [13] for further discussion of equilibrium networks and their properties.

Allowing $D_{11}$ to be nonzero is also key to our construction of direct parameterizations of contracting and robust RENs (see Section V). As discussed in Section I-D, this enables model learning via simple and generic first-order optimization methods, whereas [34] required a specialized interior-point method to deal with model behavioral constraints. Direct parameterization also enables easy random sampling of contracting models, so-called echo state networks (see Section V-C) and this enables convex learning of nonlinear feedback controllers (see Section IX).

B. Well-Posedness of Equilibrium Networks and Acyclic RENs

The added flexibility of equilibrium networks comes at a price: depending on the value of $D_{11}$, the implicit equation (8) may or may not admit a unique solution $w_t$ for a given $x_t, u_t$. An equilibrium network or REN is well-posed if a unique solution is guaranteed. In [12], it was shown that if there exists a $\Lambda \in \mathbb{D}_+$ such that
\[
2\Lambda - \Lambda D_{11} - D_{11}^\top \Lambda > 0
\]
then the equilibrium network is well-posed. We will show in Theorem 1 that this is always satisfied for our proposed model parameterizations.

A useful subclass of the REN that is trivially well-posed is the acyclic REN, where the weight $D_{11}$ is constrained to be strictly lower triangular. In this case, the elements of $w_t$ can be explicitly computed row-by-row from (8). We can interpret $D_{11}$ as the adjacency matrix of a directed graph defining interconnections between the neurons in the equilibrium network and if $D_{11}$ is strictly lower triangular, then this graph is guaranteed to be acyclic. Compared to the general REN, the acyclic REN is simpler to implement, and in our experience, often provides models of similar quality, as will be discussed in Section VII-B.

C. Evaluating RENs and Their Gradients

For a well-posed REN with full $D_{11}$, solutions can be computed by formulating an equivalent monotone operator splitting problem [61]. In the authors’ experience, the Peaceman–Rachford algorithm is reliable and efficient [12].

When training an equilibrium network via gradient descent, we need to compute the Jacobian $\partial w_t^\top / \partial (\cdot)$ where $w_t^\top$ is the solution of the implicit equation (8), and $(\cdot)$ denotes the input to the network or model parameters. By using the implicit function theorem, $\partial w_t^\top / \partial (\cdot)$ can be computed via
\[
\frac{\partial w_t^\top}{\partial (\cdot)} = (I - JD)^{-1} J \frac{\partial (Dw_t^\top + b_w)}{\partial (\cdot)}
\]
where $J$ is the Clarke generalized Jacobian of $\sigma$ at $Dw_t^\top + b_w$. From Assumption 1 in Section III-D, we have that $J$ is a singleton almost everywhere. It was shown in [12] that the condition (12) implies matrix $I - JD$ is invertible.

D. Contracting and Robust RENs

We call the model of (9) and (10) a contracting REN (C-REN) if it is contracting and a robust REN (R-REN) if it satisfies the incremental IQC. We make the following assumption on $\sigma$, which holds for commonly used activation functions [62].

\textbf{Assumption 1:} The activation function $\sigma$ is piecewise differentiable and slope restricted in $[0,1]$, i.e.,
\[
0 \leq \frac{\sigma(y) - \sigma(x)}{y - x} \leq 1 \quad \forall x, y \in \mathbb{R}, \quad x \neq y.
\]

The following theorem gives conditions for contracting and robust RENs.

\textbf{Theorem 1:} Consider the REN model (9), (10) satisfying Assumption 1, and a given $\alpha \in (0,1]$.

1) \textbf{Contracting REN:} Suppose there exists $P = P^\top > 0$ and $\Lambda \in \mathbb{D}_+$ such that
\[
\begin{bmatrix}
\alpha^2 P & -C_1^\top \Lambda \\
-\Lambda C_1 & W
\end{bmatrix} -
\begin{bmatrix}
A^\top \\
B_1^\top
\end{bmatrix} P
\begin{bmatrix}
A^\top \\
B_1^\top
\end{bmatrix}^\top > 0
\]

2) \textbf{Robust REN:} Consider the incremental defined in IQC (5) with given $(Q, S, R)$, where $Q \preceq 0$. Suppose there exist $P = P^\top > 0$ and $\Lambda \in \mathbb{D}_+$ such that
\[
\begin{bmatrix}
\alpha^2 P & -C_1^\top \Lambda \\
-\Lambda C_1 & W
\end{bmatrix} -
\begin{bmatrix}
A^\top \\
B_1^\top
\end{bmatrix} P
\begin{bmatrix}
A^\top \\
B_1^\top
\end{bmatrix}^\top -
\begin{bmatrix}
SC_2 & SD_{21} - D_{11}^\top A \\
SD_{21} & D_{12}^\top S^\top - \Lambda D_{12}
\end{bmatrix}
\begin{bmatrix}
C_2^\top S^\top \\
D_{12}^\top S^\top - \Lambda D_{12}
\end{bmatrix}
\begin{bmatrix}
Q \\
R + SD_{22} + D_{22}^\top S^\top
\end{bmatrix}
\begin{bmatrix}
C_2^\top S^\top \\
D_{12}^\top S^\top - \Lambda D_{12}
\end{bmatrix}^\top > 0.
\]

Then, the REN is well-posed, satisfies (5) and is contracting with a rate $\alpha < \alpha$.

The proof can be found in Appendix A. The main idea behind the LMI for the contracting REN is to use an incremental Lyapunov function $V(\Delta x) = \Delta x^2/2$, where $\Delta x$ denotes the
difference between a pair of solutions, and show that
\[ V(\Delta x_{t+1}) \leq \alpha^2 V(\Delta x_t) - \Gamma(\Delta v_t, \Delta w_t) \]  
and that \( \Gamma(\Delta v_t, \Delta w_t) \geq 0 \) for the activation function \( \sigma \), where \( \Gamma \) is an incremental quadratic constraint as in [9] and [34] with a multiplier matrix \( \Lambda \). The construction for the robust REN is similar, but uses an incremental dissipation inequality.

**Remark 3:** Note that (15) and (16) immediately imply that \( W \succ 0 \), which is precisely the equilibrium network well-posedness condition (12).

**Remark 4:** For a fixed REN model, conditions \( \{ E \} \) are well-posed, contracting with \( \Delta \). The proof is based on IQC characterization of unconstrained \( \ell^2 \) Lipschitz bound [65]. All models in \( Q \) and \( R \) can be chosen so that a robust REN verifies a particular Lipschitz bound \( \gamma \), the following weaker property is true of contracting RENs.

**Theorem 2:** Every contracting REN—i.e., a model (9), (10) satisfying Assumption 1 and (15)—satisfies the \( \ell^2 \) Lipschitz condition for some bound \( \gamma < \infty \).

The proof is in Appendix B.

**IV. CONVEX PARAMETERIZATIONS OF RENs**

In this section, we propose convex parameterizations for C-RENs/R-RENs, which are based on the following implicit representation of the linear component \( G \):

\[
\begin{bmatrix}
E x_{t+1} \\
\Delta v_t \\
y_t
\end{bmatrix} = \begin{bmatrix}
F & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix} \begin{bmatrix}
x_t \\
u_1 \\
u_2
\end{bmatrix} + \hat{b}
\]

where \( E \) is an invertible matrix and \( \Lambda \) is a positive-definite diagonal matrix. The model parameters are \( \theta_{cvx} := \{ E, \Lambda, \tilde{W}, \tilde{b} \} \).

Note that \( \theta_{cvx} \) can easily be mapped to \( \theta \) by multiplying the first and second rows of (18) by \( E^{-1} \) and \( \Lambda^{-1} \), respectively. Therefore, the parameters \( E \) and \( \Lambda \) do not expand the model set, however the extra degrees of freedom will allow us to formulate sets of C-RENs and R-RENs that are jointly convex in the model parameters, stability certificate, and multipliers.

**Definition 4:** A model of the form (18), (10) is said to be well-posed if it yields a unique \( (u_1, x_{t+1}) \) for any \( x_t, u_t \), and \( \tilde{b} \), and hence, a unique response to any initial conditions and input.

To construct a convex parameterization of C-RENs, we introduce the following LMI constraint:

\[
H(\theta_{cvx}) := \begin{bmatrix}
E + E^T - \frac{1}{\alpha} P & -C_1^T \\
-C_1 & W & B_1^T \\
F & B_1 & \mathcal{P}
\end{bmatrix} > 0
\]

where \( W = 2\Lambda - D_{11} - D_{11}^T \). The convex parameterization of C-RENs is then given by

\[
\Theta_C := \{ \theta_{cvx} \mid \exists \mathcal{P} = \mathcal{P}^T > 0 \text{ s.t. } H(\theta_{cvx}) > 0 \}.
\]

To construct convex parameterization of R-RENs, we propose the following convex constraint:

\[
\begin{bmatrix}
E + E^T - \frac{1}{\alpha} P & -C_1^T \\
-C_1 & W & B_1^T \\
F & B_1 & \mathcal{P}
\end{bmatrix} \begin{bmatrix}
D_{11}^T S^T \\
SC_2 \\
SD_{21} - D_{12}^T R + SD_{22} + D_{22}^T S^T
\end{bmatrix} + \begin{bmatrix}
C_2^T \\
D_{21}^T S^T \\
D_{22}^T S^T
\end{bmatrix} \begin{bmatrix}
Q \end{bmatrix} \begin{bmatrix}
C_2^T \\
D_{21}^T S^T \\
D_{22}^T S^T
\end{bmatrix} > 0
\]

where \( Q \preceq 0, S, \) and \( R \) are given. The convex parameterization of R-RENs is then defined as

\[
\Theta_R := \{ \theta_{cvx} \mid \exists \mathcal{P} = \mathcal{P}^T > 0 \text{ s.t. (20)} \}.
\]

The following results relates the aforementioned parameterizations to the desired model behavioral properties.

**Theorem 3:** All models in \( \Theta_C \) are well-posed and contracting with rate \( \alpha < \tilde{\alpha} \). All models in \( \Theta_R \) are well-posed, contracting with rate \( \alpha < \tilde{\alpha} \), and satisfy the IQC defined by \( (Q, S, R) \).

The proof can be found in Appendix C.

**Remark 6:** With the convex parameterizations, it is straightforward to enforce any desired sparsity structure on \( D_{11} \), e.g., corresponding to a multilayer neural network as per Section III-A. Since \( \Lambda \) is diagonal, the sparsity structures of \( D_{11} \) and \( D_{11} = \Lambda^{-1} D_{11} \) are identical, and so the desired structure can be added as a linear constraint on \( D_{11} \).

**V. DIRECT PARAMETERIZATIONS OF RENs**

In the previous section, we gave convex parameterizations of contracting and robust RENs in terms of LMIs, i.e., intersections of the cone of positive semidefinite matrices with affine constraints. While convexity of a model set is useful, LMIs are challenging to verify for large-scale models, and especially to enforce during training.

In this section, we provide direct parameterizations, i.e., smooth mappings from \( \mathbb{R}^N \) to the weights and biases of an REN, enabling unconstrained optimization methods to be applied. We do so by first constructing representations of RENs directly in terms of the positive semidefinite cone without affine constraints, and then, parameterize this cone in terms of its square-root factors.

**A. Direct Parameterizations of Contracting RENs**

The key observation leading to our construction is that the mapping from contracting REN parameters \( \theta_{cvx} \) to \( H \) in (21) is surjective, i.e., it maps onto the entire cone of positive-definite matrices. Furthermore, as we will show later it is straightforward to construct a (nonunique) inverse that maps from any positive-definite matrix back to \( \theta_{cvx} \) defining a well-posed and contracting REN.
1) **Free Parameters:** Of the parameters in $\theta_{\text{cvx}}$, the following have no effect on the stability and can be freely parameterized in terms of their elements: $B_2 \in \mathbb{R}^{n \times m}$, $C_2 \in \mathbb{R}^{p \times n}$, $D_{12} \in \mathbb{R}^{q \times m}$, $D_{21} \in \mathbb{R}^{p \times q}$, $D_{22} \in \mathbb{R}^{p \times m}$, and $\bar{b} \in \mathbb{R}^{(2n+q)}$.

2) **Constrained Parameters, Acyclic Case:** The parameters $E$, $F$, $\Lambda$, $B_1$, and $C_1$ relate to internal dynamics, and therefore, affect the stability properties of an REN. Here, we construct them from two free matrix variables $X \in \mathbb{R}^{(2n+q) \times (2n+q)}$ and $Y_1 \in \mathbb{R}^{n \times n}$.

We first construct $H$ from $X$ as

$$H = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} = X^\top X + \epsilon I \succ 0 \quad (21)$$

where $\epsilon$ is a small positive scalar, and we have partitioned $H$ into blocks of size $n$, $n$, and $q$. Comparing (21) with (19), we can immediately construct

$$F = H_{31}, \quad B_1 = H_{32}, \quad P = H_{33}, \quad C_1 = -H_{21}. \quad (22)$$

Further, it is straightforward to verify that the construction

$$E = \frac{1}{2} \left( H_{11} + \frac{1}{\epsilon} P + Y_1 - Y_1^\top \right) \quad (23)$$

results in $H_{11} = E + E^\top - \frac{2}{\epsilon} P$ for any $Y_1$.

We then construct a strictly lower triangular $D_{11}$ satisfying

$$H_{22} = \Phi - L - L^\top \quad (24)$$

by partitioning $H_{22}$ into its diagonal and strictly upper/lower triangular components

$$H_{22} = \Phi - L - L^\top \quad (25)$$

where $\Phi$ is a diagonal matrix and $L$ is a strictly lower triangular matrix, from which we construct the remaining parameters in $\theta_{\text{cvx}}$ as

$$\Lambda = \frac{1}{2} \Phi, \quad D_{11} = L. \quad (26)$$

3) **Constrained Parameters, Full Case:** The construction of a C-REN with full (not acyclic) $D_{11}$ is the same except that we introduce two additional free variables: $g \in \mathbb{R}^3$ and $Y_2 \in \mathbb{R}^{p \times q}$, and then, construct a positive diagonal matrix $\Lambda = \epsilon \text{diag}(g)$ and

$$D_{11} = \Lambda - \frac{1}{2} \left( H_{22} + Y_2 - Y_2^\top \right) \quad (27)$$

which also results in parameters satisfying (24).

### B. Direct Parameterizations of Robust RENs

We now provide a direct parameterization of RENs satisfying the robustness condition (20). The first step is to rearrange (20) into an equivalent form, which will turn out to be useful in the construction since it makes explicit the connection between the R-REN and C-REN conditions

$$\mathcal{R} := R + SD_{22} + D_{22}^\top S^T + D_{22}^\top Q D_{22} \succ 0 \quad (28a)$$

$$H(\theta_{\text{cvx}}) \succ \begin{bmatrix} C_2^T \\ D_{21}^T \\ B_2 \end{bmatrix} \mathcal{R}^{-1} \begin{bmatrix} C_2^T \\ D_{21}^T \\ B_2 \end{bmatrix}^\top - \begin{bmatrix} C_2^T \\ D_{21}^T \\ B_2 \end{bmatrix}^\top Q \begin{bmatrix} C_2^T \\ D_{21}^T \\ B_2 \end{bmatrix} \succ 0 \quad (28b)$$

where $H(\theta_{\text{cvx}})$ is the C-REN condition defined in (19), $C_2 = (D_{22}^\top Q + S)C_2$, and $D_{21} = (D_{22}^\top Q + S)D_{22} - D_{12}^\top$.

The first construction we give is for the simplest case without direct-feedthrough, i.e., $D_{22} = 0$. However, some practically useful constraints require $D_{22} \neq 0$, e.g., incremental passivity requires $D_{22} + D_{22}^\top \succ 0$. We consider this more general case as follows.

1) **Models With $D_{22} = 0$:** For models with no direct feedthrough, we have the following direct parameterization.

a) **Free variables:** The following matrix variables can be freely parameterized in terms of their elements: $B_2 \in \mathbb{R}^{n \times m}$, $C_2 \in \mathbb{R}^{p \times n}$, $D_{12} \in \mathbb{R}^{q \times m}$, $D_{21} \in \mathbb{R}^{p \times q}$, $\bar{b} \in \mathbb{R}^{(2n+q)}$.

b) **Constrained parameters:** The construction is similar to the contracting case in Section V-A.

Since $D_{22} = 0$, Condition (28a) reduces to $R \succ 0$, which is independent of model parameters. Now Condition (28b) can be satisfied if we construct $H$ as

$$H = X^\top X + \epsilon I$$

$$+ \begin{bmatrix} C_2^T \\ D_{21}^T \\ B_2 \end{bmatrix} \mathcal{R}^{-1} \begin{bmatrix} C_2^T \\ D_{21}^T \\ B_2 \end{bmatrix}^\top - \begin{bmatrix} C_2^T \\ D_{21}^T \\ B_2 \end{bmatrix}^\top Q \begin{bmatrix} C_2^T \\ D_{21}^T \\ B_2 \end{bmatrix} \succ 0 \quad (29)$$

with $X$ a free matrix variable, and then, recover the remaining model parameters from $H$ as per Section V-A. Note that $H \succ 0$, since $R \succ 0$ and $Q \preceq 0$.

2) **Models With $D_{22} \neq 0$:** In this case, we need to construct a $D_{22}$ satisfying (28a). In what follows it will be useful to have $Q$ invertible but we have only assumed that $Q \preceq 0$. If $Q$ is not negative definite, we introduce $Q = Q - \epsilon I \preceq 0$ and note that (28a) is equivalent to

$$R + SD_{22} + D_{22}^\top S^T + D_{22}^\top Q D_{22} \succ 0 \quad (30)$$

for sufficiently small $\epsilon \succ 0$. If $Q \prec 0$, we simply set $\epsilon = 0$, i.e., $Q = Q$.

We factor $Q = -L_R^\top L_R$, and we will show (see Proposition 1) that $R - SQ^{-1}S^T \succ 0$, hence, there is an invertible $L_R \in \mathbb{R}^{m \times m}$ such that $L_R^\top L_R = R - SQ^{-1}S^T$.

The direct parameterization of $D_{22}$ is

$$D_{22} = -Q^{-1}S^T + L_R^\top NL_R \quad (31)$$

where construction of $N$ depends on the input and output dimensions. If $p \geq m$, we take

$$M = X_3^\top X_3 + Y_3^\top - Z_3^\top Z_3 + \epsilon I$$

$$N = \begin{bmatrix} (I - M)(I + M)^{-1} \\ -2Z_3^\top (I + M)^{-1} \end{bmatrix} \quad (32)$$

with $X_3, Y_3 \in \mathbb{R}^{m \times m}$, and $Z_3 \in \mathbb{R}^{(p-m) \times m}$ as free variables. Note that $M + M^\top \succ 0$ so $I + M$ is invertible.

If $p < m$, $M$ is the same but we take

$$N = \begin{bmatrix} (I + M)^{-1} & -2(I + M)^{-1}Z_3^\top \end{bmatrix} \quad (33)$$

with $X_3, Y_3 \in \mathbb{R}^{p \times p}$ and $Z_3 \in \mathbb{R}^{(m-p) \times p}$ as free variables.

**Proposition 1:** The construction of $D_{22}$ in (31), (32), or (33) is well-defined and satisfies Condition (30).

The proof is in Appendix D.
a) Special cases: The following are direct parameterizations of $D_{22}$ for some commonly used robustness conditions.

1) Incrementally strictly output passive RENs (i.e., $Q = -2\rho I, R = 0, S = I$): We have $D_{22}$ given in (31) with $L_Q = I$ and $L_R = \gamma I$.

2) Incrementally strictly output passive RENs (i.e., $Q = -2\rho I, R = 0, S = I$): We have $D_{22} = \frac{1}{\rho}(I + M)^{-1}$.

3) Incrementally input passive RENs (i.e., $Q = 0, R = -2\nu I, S = I$): In this case, Condition (28a) becomes an LMI of the form $D_{22} + D_{22}^T - 2\nu I > 0$, which yields a simple parameterization with $D_{22} = \nu I + M$.

C. Random Sampling of Nonlinear Systems and Echo State Networks

One benefit of the direct parameterizations of RENs is that it is straightforward to randomly sample systems with the desired behavioral properties. Since contracting and robust RENs are constructed as the image of $\mathbb{R}^N$ under a smooth mapping (Sections V-A and V-B), one can sample random vectors in $\mathbb{R}^N$ and map them to random stable/robust nonlinear dynamical systems.

An “echo state network” is a model in which the state-space dynamics are randomly sampled but thereafter fixed, and with a learnable output map (see, e.g., [57] and [58])

$$x_{t+1} = f(x_t, u_t)$$
$$y_{t+1} = g(x_t, u_t, \theta)$$

where $f$ is fixed and $g$ is affine parameterized by $\theta$, i.e.,

$$g(x_t, u_t, \theta) = g_0(x_t, u_t) + \sum_i \theta_i g_i(x_t, u_t).$$

Then, system identification with a simulation-error criteria can be solved as a basic least-squares problem. This approach is reminiscent of system identification via a basis of stable linear responses (see, e.g., [66]).

For this approach to work over long horizons, it is essential that the random dynamics are stable. In [57] and [58] and references therein, contraction of (34) is referred to as the “echo state property,” and simple parameterizations are given for which contraction is guaranteed. The direct parameterizations of the REN can be used to randomly sample from a rich class of contracting models by sampling $X, Y_1, Y_2, B_2$, and $D_{12}$ to construct the state-space dynamics and equilibrium network. Such a model can be used, e.g., for system identification by simulating its response to inputs to generate data $\tilde{u}_t, \tilde{x}_t, \tilde{w}_t$, and then, the output mapping

$$y_t = C_2 \tilde{x}_t + D_{21} \tilde{w}_t + D_{22} \tilde{u}_t + b_y$$

can be fit to $\tilde{y}_t$, minimizing (3) via least squares to obtain the parameters $C_2, D_{21}, D_{22}$, and $b_y$. We will also see in Section IX how this approach can be applied in the data-driven feedback control design.

VI. Expressivity of REN Model Class

The set of RENs contain many widely used model structures as special cases, some of which we briefly describe here.

A. Deep, Residual, and Equilibrium Networks

As a special case with $A, C_1, C_2, B_1$, and $B_2$ all zero, RENs include (static) equilibrium networks, which as discussed in Section III-A and [11], [12], and [13] include standard DNNs (multilayer perceptrons), residual networks, and others.

B. Previously Proposed Stable RNNs

If we set $D_{11} = 0$, then the nonlinearity is not an equilibrium network but a single-hidden-layer neural network, and our model set $\Theta_C$ reduces to the model set proposed in [34]. Therefore, the RNN model class also includes all other models that were proven to be in that model set in [34, Th. 5], including prior sets of contracting RNNs including the ciRNN [33] and s-RNN [30].

C. Stable Linear Systems

Setting $B_1, C_1, D_{11}, D_{12}, D_{21}$, and $b$ to zero, RENs include all stable finite-dimensional linear time-invariant (LTI) systems (see [34, Th. 4]).

D. Previously Proposed Stable Echo State Networks

The stability condition for the ciRNN is the same as that proposed for echo state networks in [57] and [58], hence by randomly sampling RENs as in Section V-C, we sample from a strictly larger set of echo state networks than previously known.

E. Nonlinear Finite Impulse Response (NFIR) Models

An NFIR model a nonlinear mapping of a fixed history of inputs

$$y_t = f(u_t, u_{t-1}, \ldots, u_{t-h})$$

for some fixed $h$. Setting

$$A = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad B_1 = 0.$$  

The output $y$ is then a nonlinear function (an equilibrium network) of such truncated history of inputs.

F. Block Structured Models

These are constructed from series interconnections of LTI systems and static nonlinearities [67], [68], and are included within the REN model set. For example, the following.

1) Wiener systems consist of an LTI block followed by a static nonlinearity. This structure is replicated in (9) and (10) when $B_1 = 0$ and $C_2 = 0$. In this case, the linear dynamical system evolves independently of the nonlinearities and feeds into an equilibrium network.

2) Hammerstein systems consist of a static nonlinearity connected to an LTI system. This is represented in the REN when $B_2 = 0$ and $C_1 = 0$. In this case, the input passes through a static equilibrium network and into an LTI system.
More generally, arbitrary series and parallel interconnections of LTI systems and static nonlinearities can also be constructed.

G. Universal Approximation Properties

It is well known that even single-hidden-layer neural networks have universal approximation properties, i.e., as the number of neurons goes to infinity, they can approximate any continuous function over a bounded domain with arbitrary accuracy. RENs immediately inherit this property for universal approximation of static maps, NFIR models, and other block-structured models.

Furthermore, it was shown in [69] that as the number of states and activation functions grows, the REN structure is a universal approximator of fading-memory nonlinear systems as defined in [70], as well as all nonlinear dynamical systems that are contracting and have finite Lipschitz bounds.

VII. USE CASE: STABLE AND ROBUST NONLINEAR SYSTEM IDENTIFICATION

In this section, we demonstrate the proposed models on the F16 ground vibration [71] and Wiener Hammerstein with process noise [72] system identification benchmarks. We will compare the acyclic C-REN and Lipschitz-bounded R-REN with the prescribed Lipschitz bound of $\gamma$ with the widely used long-short-term memory (LSTM) [73] and standard RNN models with a similar number of parameters. We will also compare to the robust RNN proposed in [34] using the code from$^2$.

We fit models by minimizing simulation error as

$$L_{se}(\hat{z}, \theta) = \frac{||\hat{y} - R_{a}(\hat{u})||_T^2}{||y||_T}$$

using minibatch gradient descent with the Adam optimizer [56]. Model performance is measured by the normalized root mean square error (NRMSE) on the test sets, calculated as

$$\text{NRMSE} = \frac{||\hat{y} - R_{a}(\hat{u})||_T}{||y||_T}.$$  \hspace{1cm} (38)

Model robustness is measured in terms of the maximum observed sensitivity as

$$\gamma = \max_{u, v, a} \frac{||R_{a}(u) - R_{a}(v)||_T}{||u - v||_T}.$$ \hspace{1cm} (39)

We find a local solution to (39) using gradient ascent with the Adam optimizer. Consequently $\gamma$ is a lower bound on the true Lipschitz constant of the sequence-to-sequence map.

A. Benchmark Datasets and Training Details

1) F16 System Identification Benchmark: The F16 ground vibration benchmark dataset [71] consists of accelerations measured by three accelerometers, induced in the structure of an F16 fighter jet by a wing mounted shaker. We use the multisine excitation dataset with full frequency grid. This dataset consists of seven multisine experiments with 73728 samples and varying amplitude. We use datasets 1, 3, 5, and 7 for training and datasets 2, 4, and 6 for testing.

All models in our comparison have approximately 118000 parameters: the RNN has 340 neurons, the LSTM has 170 neurons and the RENs have width $n = 75$ and $q = 150$. Models were trained for 70 epochs with a sequence length of 1024. The learning rate was initialized at $10^{-3}$ and was reduced by a factor of 10 every 20 Epochs.

2) Wiener–Hammerstein With Process Noise Benchmark: The Wiener Hammerstein with process noise benchmark dataset [72] involves the estimation of the output voltage from two input voltage measurements from a Wiener–Hammerstein system with large process noise. We have used the multisine fade-out dataset consisting of two realizations of a multisine input signal with 8192 samples each. The test set consists of two experiments, a random phase multisine and a sine sweep, conducted without the added process noise.

All models in our comparison have approximately 42000 parameters: the RNN has 200 neurons, the LSTM has 100 neurons, and the RENs have $n = 40$ and $q = 100$. Models were trained for 60 epochs with a sequence length of 512. The initial learning rate was $10^{-4}$ and was reduced to $10^{-5}$ after 40 epochs.

B. Results and Discussion

In Figs. 2 and 3, we have plotted the test-set NRMSE (38) versus the observed sensitivity (39) for each of the models trained on the F16 and Wiener–Hammerstein benchmarks, respectively. The dashed vertical lines show the guaranteed Lipschitz bounds for the REN and robust RNN models.
We observe that the REN offers the best tradeoff between nominal performance and robustness, with the REN slightly outperforming the LSTM in terms of nominal test error for large $\gamma$. By tuning $\gamma$, nominal test performance can be tradedoff for robustness, signified by the consistent trend moving diagonally up and left with decreasing $\gamma$. In all cases, we found that the REN was significantly more robust than the RNN, typically having about 10% of the sensitivity for the F16 benchmark and 1% on the Wiener–Hammerstein benchmark. Also note that for small $\gamma$, the observed lower bound on the Lipschitz constant is very close to the guaranteed upper bound, showing that the real Lipschitz constant of the models is close to the upper bound.

Compared to the robust RNN proposed in [34], the REN has similar bounds on the incremental $\ell_2$ gain, however, the added flexibility from the term $D_{11}$ significantly improves the nominal model performance for a given gain bound. Additionally, while both the C-REN and Robust RNN $\gamma=\infty$ are contracting models, we note that the C-REN is significantly more expressive with an NRMSE of 0.16 versus 0.24.

It is well known that many neural networks are very sensitive to adversarial perturbations. This is shown, for instance, in Figs. 4 and 5, where we have plotted the change in output for a small adversarial perturbation $||\Delta u|| < 0.05$, for a selection of models trained on the F16 benchmark dataset. Here, we can see that both the RNN and LSTM are very sensitive to the input perturbation. The R-REN and R-RNN, on the hand, have guaranteed bounds on the effect of the perturbation and are significantly more robust.

We have also trained cyclic RENs (i.e., $D_{11}$ is a full matrix) for the F16 Benchmark dataset. The resulting nominal performance and sensitivities for the acyclic and cyclic RENs are shown in Table I. We do not observe a significant difference in performance between the cyclic and acyclic model classes. Finally, we have plotted the training loss (37) versus the number of epochs in Fig. 6 for some of the models on the F16 dataset. Compared to the LSTM, the REN takes a similar number of steps and achieves a slightly lower training loss.

### Table I: Nominal Performance (NRMSE) and Upper and Lower Bounds on Lipschitz Constant for Acyclic and Cyclic RENs on F16 Benchmark Dataset

|          | $\gamma$ | 10    | 20    | 40    | 60    | 100   | $\infty$ |
|----------|----------|-------|-------|-------|-------|-------|----------|
| Acyclic  | NRMSE (%)| 30.0  | 25.7  | 20.1  | 18.5  | 17.2  | 16.2     |
| Cyclic   | NRMSE (%)| 30.3  | 26.8  | 21.8  | 19.9  | 19.3  | 16.8     |

**Fig. 6.** Training loss versus epochs for models trained on the F16 ground vibration benchmark dataset.

**VIII. USE CASE: LEARNING NONLINEAR OBSERVERS**

Estimation of system states from incomplete and/or noisy measurements is an important problem in many practical applications. For linear systems with Gaussian noise, a simple and optimal solution exists in the form of the Kalman filter, but for nonlinear systems even finding a stable estimator (also known as observer) is nontrivial and many approaches have been investigated, e.g., [74], [75], and [76]. Observer design was one of the original motivations for contraction analysis [14], and in this section, we show how a flexible set of contracting models can be used to learn state observers via snapshots of a nonlinear system model.

The aim is to estimate the state of a nonlinear system of the form

$$x_{t+1} = f_m(x_t, u_t, w_t), \quad y_t = g_m(x_t, u_t, w_t)$$

where $x_t \in \mathbb{X}$ is an internal state to be estimated, $y_t$ is an available measurement, $u_t \in \mathbb{U}$ is a known (e.g., control) input, and $w_t \in \mathbb{W}$ comprises unknown disturbances and sensor noise.

A standard structure, pioneered by Luenberger, is an observer of the form

$$\hat{x}_{t+1} = f_m(\hat{x}_t, u_t, 0) + l(\hat{x}_t, u_t, y_t)$$
i.e., a combination of a model prediction $f_m$ and a measurement correction function $l$. A common special case is $l(\hat{x}_t, u_t, y_t) = L(\hat{x})(y_t - g_m(\hat{x}_t, u_t, 0))$ for some gain $L(\hat{x})$.

In many practical cases, the best available model $f_m, g_m$ is highly complex, e.g., based on finite-element methods or algorithmic mechanics [77]. This poses two major challenges to the standard paradigm, which are as follows.

1) How to design the function $l$ such that the observer (41) is stable (preferably globally) and exhibits good noise/disturbance rejection.

2) The model itself may be so complex that evaluating $f_m(\hat{x}_t, u_t, 0)$ in real time is infeasible, e.g., for stiff systems where short sample times are required.

Our parameterization of contracting models enables an alternative paradigm, first suggested for the restricted case of polynomial models in [50] and [51].

**Proposition 2:** If we construct an observer of the form

$$\hat{x}_{t+1} = f_o(\hat{x}_t, u_t, y_t)$$

such that the following two conditions hold:

1) the system (42) is contracting with rate $\alpha \in (0, 1)$ for some constant metric $P > 0$;

2) the following "correctness" condition holds:

$$f_m(x, u, 0) = f_o(x, u, g_m(x, u, 0)) \forall (x, u) \in \mathbb{R}^n \times U \tag{43}$$

then when $w = 0$, we have $\hat{x}_t \to x_t$ as $t \to \infty$. Suppose instead Condition 2) does not hold but that the observer (42) satisfies Conditions 1) and

3) the following error bound holds $\forall (x, u, w) \in \mathbb{R}^n \times U \times \mathbb{R}^n$:

$$|f_o(x, u, g_m(x, u, w)) - f_m(x, u, w)| \leq \rho. \tag{44}$$

then the estimation error satisfies, with exponential convergence

$$\lim_{t \to \infty} \|\hat{x}_t - x_t\| \leq \frac{\rho}{1 - \alpha} \sqrt{\frac{\sigma}{\sigma}} \tag{45}$$

where $\sigma$ and $\sigma$ denote the maximum and minimum singular values of the contraction metric $P$, respectively.

**Remark 7:** Note that the error term (44) may result from bounded disturbances $w_t$, modeling errors, or interpolation errors arising from fitting the correctness condition to finite data (see Section VIII-A), or some combination of such factors.

The reasoning for nominal convergence of the observer is simple: (43) implies that if $\hat{x}_0 = x_0$, then $\hat{x}_t \to x_t$ for all $t \geq 0$, i.e., the true state is a particular solution of the observer. But contraction implies that all solutions of the observer converge to each other. Hence, all solutions of the observer converge to the true state. The proof of the estimation error bound can be found in Appendix E.

Motivated by Proposition (2), we pose the observer design problem as a supervised learning problem over our class of contracting models.

1) Construct the dataset: Sample a set of points $\tilde{z} = \{(x^i, u^i)\}_{i=1}^N$ where $(x^i, u^i) \in \mathbb{R}^n \times U$, and for each compute $g_m^i = g_m(x^i, u^i, 0)$ and $f_m^i = f_m(x^i, u^i, 0)$.

2) Learn a contracting system $f_o$ minimizing the loss

$$L_o(\tilde{z}, \theta) = \sum_{i=1}^N \|f_o(x^i, u^i, y^i) - f_m(x^i, u^i, 0)\|^2. \tag{46}$$

**Remark 8:** An observer of the traditional form (41) with $l(\hat{x}_t, u_t, y_t) = L(\hat{x})(y_t - g_m(\hat{x}_t, u_t, 0))$ will always satisfy the correctness condition, but designing $L(\hat{x})$ to achieve global convergence may be difficult. In contrast, an observer design using the proposed procedure will always achieve global convergence, but may not achieve correctness exactly.

### A. Example: Reaction-Diffusion PDE

We illustrate this approach by designing an observer for the following semilinear reaction–diffusion PDE:

$$\frac{\partial \xi(z, t)}{\partial t} = \frac{\partial^2 \xi(z, t)}{\partial z^2} + R(\xi, z, t) \tag{47}$$

$$\xi(z, 0) = 1, \quad \xi(1, t) = \xi(0, t) = b(t)$$

$$y = g(\xi, z, t) \tag{49}$$

where the state $\xi(z, t)$ is a function of both the spatial coordinate $z \in [0, 1]$ and time $t \in \mathbb{R}_+$. Models of the form (47) model processes such as combustion [78], bioreactors [79], or neural spiking dynamics [78]. The observer design problem for such systems has been considered using complex back-stepping methods that guarantee only local stability [79].

We consider the case where the local reaction dynamics have the following form, which appears in models of combustion processes [78] as

$$R(\xi, z, t) = \frac{1}{2}(\xi - 1) (\xi - \frac{1}{2}).$$

We consider the boundary condition $b(t)$ as a known input and assume that there is a single measurement taken from the center of the spatial domain so $y(t) = \xi(0.5, t)$.

We discretize $z$ into $N$ intervals with points $z^0, ..., z^N$, where $z_i = i\Delta z$. The state at spatial coordinate $z^i$ and time $t$ is then described by $\xi_i = (\xi_{i-1}, \xi_i, ..., \xi_{i+1})$ where $\xi_i = \xi(z^i, t)$. The dynamics over a time period $\Delta t$ can then be approximated using the following finite differences:

$$\frac{\partial \xi(z, t)}{\partial t} \approx \frac{\xi_{i+1} - \xi_i}{\Delta t}, \quad \frac{\partial^2 \xi(z, t)}{\partial z^2} \approx \frac{\xi_{i+1} + \xi_{i-1} - 2\xi_i}{\Delta z^2 \Delta t};$$

Substituting them into (47) and rearranging for $\xi_{i+1}$ leads to an $(N+1)$-dimensional state-space model of the form

$$\xi_{i+1} = a_{i} \xi_i + b_{i}, \quad y_t = c_{i} \xi_i.$$

We generate training data by simulating the system (50) with $N = 50$ for $10^5$ time steps with the stochastic input $b_{i+1} = b_{i} + 0.05\omega_{i}$, where $\omega_{i} \sim N(0, 1)$. We denote this training data by $\tilde{z} = (\xi_i, y_i, b_i)$ for $t = 0, ..., 10^5 \Delta t$.

To train an observer for this system, we construct a C-REN with $n = 51$ and $q = 200$. We optimize the one step ahead prediction error as

$$\mathcal{L}(\tilde{z}, \theta) = \frac{1}{T} \sum_{t=0}^{T-1} |M(\xi_i, y_i) - f_o(\xi_i, b_i, y_i)|^2$$

using SGD with the Adam optimizer [56]. Here, $f_o(\xi, b, y)$ is a C-REN described by (9) and (10) using direct parameterization.
discussed in Section V-A. Note that we have taken the output mapping in (9) to be \([C_2, D_{21}, D_{22}] = [I, 0, 0]\).

We have plotted results of the PDE simulation and the observer state estimates in Fig. 7. The simulation starts with an initial state of \(\xi(z, 0) = 1\) and the observer has an initial state estimate of \(\bar{\xi}_0 = 0\). The error between the state estimate and the PDE simulation’s state quickly decays to zero and the observer state continues to track the PDE’s state.

We have also provided a comparison to a free run simulation of the PDE with initial condition \(\xi(z, 0) = 0\) in Fig. 8. Here, we can see that simulated trajectories with different initial conditions do not converge. This suggests that the system is not contracting and the state cannot be estimated by simply running a parallel simulation. The state estimates of the observer, however, quickly converge on the true state.

Fig. 7. Simulation of a semilinear reaction diffusion equation and the observer’s state estimate, with a measurement in the center of the spatial domain. The \(y\)-axis corresponds to the spatial dimension and the \(x\)-axis corresponds to the time dimension.

Fig. 8. True state and state estimates from the designed observer and a free run simulation of the PDE. (a) True and estimated states for \(\xi_1\), located at PDE boundary. (b) True and estimated states for \(\xi_{10}\).

IX. USE CASE: DATA-DRIVEN FEEDBACK CONTROL DESIGN

In this section, we show how a rich class of contracting nonlinear models can be useful for nonlinear feedback design for linear dynamical systems with stability guarantees. Even if the dynamics are linear, the presence of constraints, uncertain parameters, nonquadratic costs, and non-Gaussian disturbances can mean that nonlinear policies are superior to linear policies. Indeed, in the presence of constraints, model predictive control (a nonlinear policy) is a common approach.

The basic idea we illustrate in this section is to build on a standard method for linear feedback optimization: the Youla–Kucera parameterization, also known as Q-augmentation [18], [52], [53], [80]. For a discrete-time linear system model

\[
\begin{align*}
    x_{t+1} &= \dot{A} x_t + B_1 w_t + B_2 u_t \\
    \zeta_t &= C_1 x_t + D_{11} w_t + D_{12} u_t \\
    y_t &= C_2 x_t + D_{21} w_t
\end{align*}
\]

with \(x\) the state, \(u\) the controlled input, \(w\) external inputs (reference, disturbance, measurement noise), \(y\) a measured output, and \(\zeta\) comprises the “performance” outputs to kept small (e.g., tracking error, control signal). We assume the system is detectable and stabilizable, i.e., there exist \(L\) and \(K\) such that \(A - LC\) and \(A - BK\) are Schur stable. Note that if \(A\) is stable, we can take \(L = 0, K = 0\). Consider a feedback controller of the form

\[
\begin{align*}
    \dot{x}_{t+1} &= A \dot{x}_t + B_2 u_t + L \tilde{y} \\
    \tilde{y}_t &= y_t - C_2 \dot{x}_t \\
    u_t &= -K \dot{x}_t + \bar{u}_t
\end{align*}
\]
i.e., a standard output-feedback structure with $v_t$ an additional control augmentation. The closed-loop input–output dynamics can be written as the transfer matrix

$$
\begin{bmatrix}
\zeta \\
\tilde{y}
\end{bmatrix} = \begin{bmatrix}
T_0 & T_1 \\
T_2 & 0
\end{bmatrix}
\begin{bmatrix}
w \\
\tilde{u}
\end{bmatrix}
$$

(57)

where we have used the fact that $\tilde{u}$ maps to $x$ and $\dot{x}$ equally, hence the mapping from $\tilde{u}$ to $\tilde{y}$ is zero.

It is well-known that the set of all stabilizing linear feedback controllers can be parameterized by stable linear systems $Q : \tilde{y} \mapsto \tilde{u}$, and moreover, this convexifies the closed-loop dynamics. A standard approach (e.g., [53] and [80]) is to construct an affine parameterization for $Q$ via a finite-dimensional truncation of a complete basis of stable linear systems, and optimize to meet various criteria on frequency response, impulse response, and response to application-dependent test inputs. However, if the control augmentation $\tilde{u}$ is instead generated by contracting nonlinear system $\tilde{u} = Q(\tilde{y})$, then the closed-loop dynamics $w \mapsto \zeta$ are nonlinear but contracting and have the representation

$$
\zeta = T_0 w + T_1 Q(T_2 w).
$$

(58)

This presents opportunities for learning stabilizing controllers via parameterizations of stable nonlinear models.

A. Echo State Network and Convex Optimization

Here, we describe a particular setting in which the data-driven optimization of nonlinear policies can be posed as a convex problem. Suppose we wish to design a controller solving

$$
\min_\theta J(\zeta) \quad \text{s.t. } \ c(\zeta) \leq 0
$$

(59)

where $\zeta$ is the response of the performance outputs to a particular class of disturbances $w$, $J$ is a convex objective function, and $c$ is a set of convex constraints, e.g., state and control signal bounds.

If we take $Q$ as an echo state network, c.f., Section V-C

$$
q_{t+1} = f_q(q_t, \tilde{y}_t), \quad \tilde{u}_t = g_q(q_t, \tilde{y}_t, \theta)
$$

where $f_q$ is fixed and $g_q$ is linearly parameterized by $\theta$, i.e.,

$$
g_q(q_t, \tilde{y}_t, \theta) = \sum_i \theta_i g_q^i(q_t, \tilde{y}_t).
$$

Then, $Q$ has the representation

$$
Q(\tilde{y}) = \sum_i \theta_i Q^i(\tilde{y})
$$

where $Q^i$ is a state-space model with dynamics $f_q$ and output $g_q^i$. Then, we can perform data-driven controller optimization in the following way.

1) Construct (e.g., via random sampling, experiment) a finite set of test signals $w^j$.
2) Compute $\tilde{y}^j_t = T_2 w^j$ for each $j$.
3) For each $j$, compute the response to $\tilde{y}^j$ as

$$
q_{t+1} = f_q(q_t, \tilde{y}^j_t), \quad \tilde{u}^{ij}_t = g_q(q_t, \tilde{y}^j_t).
$$

4) Construct the affine representation

$$
\zeta^{ij} = T_0 w^j + \sum_i \theta_i T_1 \tilde{u}^{ij}.
$$

5) Solve the convex optimization problem as

$$
\theta^* = \arg \min_\theta J(\zeta) + R(\theta) \quad \text{s.t. } c(\zeta^{ij}) \leq 0
$$

where $R(\theta)$ is an optional regularization term.

The result will of course only be approximately optimal, since $w^j$ are but a representative sample and the echo state network provides only a finite-dimensional span of policies. However, it will be guaranteed to be stabilizing.

Remark 9: This framework can be extended to include learning over all REN parameters, however the optimization problem is no longer convex. We have recently shown that this amounts to learning over all stabilizing nonlinear controllers for a linear system [69] and extended the framework to learn robustly stabilizing controllers for uncertain systems [81].

B. Example

We illustrate the approach on a simple discrete-time linear system with transfer function

$$
T_0 = T_1 = -T_2 = \frac{0.3}{q^2 - 2\rho \cos(\phi)q + \rho^2}
$$

with $q$ the shift operator, $\rho = 0.8$, and $\phi = 0.2\pi$. We consider the task of minimizing the $\ell^1$ norm of the output in response to step disturbances, while keeping the control signal $u$ bounded: $|u_t| \leq 5$ for all $t$. This can be considered a data-driven approach to an explicit model predictive control [82] with stability guarantees.

Training data are generated by a 25,000 sample piece-wise constant disturbance that has a hold time of 50 samples and a magnitude uniformly distributed in the interval $[-10, 10]$.

We construct a contracting model $Q$ with $n = 50$ states and $q = 500$ neurons by randomly sampling a matrix $X \in \mathbb{R}^{(2n+q) \times (2n+q)}$ with $X_{ij} \sim \mathcal{N}(0, \frac{1}{2n+q})$ and constructing a C-REN via the method outline in Section V-A. The remaining parameters are sampled from the Glorot normal distribution [83]. For comparison, we construct a linear $Q$ parameter of the form

$$
q_{t+1} = A_q q_t + B_q \tilde{y}_t, \quad v_{t+1} = C_q q_t + D_q \tilde{y}_t
$$

where $A_q = \lambda \frac{\tilde{A}}{\mu(\tilde{A})}$ with $\lambda \in (0, 1)$ and $\tilde{A}_{ij} \sim \mathcal{N}(0, \frac{1}{2n+q})$. Note that $A_q$ is a stable matrix with a contraction rate of $\lambda$. We sample $B_q$ from the Glorot normal distribution [83].

The responses to test inputs are shown in Fig. 9. The benefits of learning a nonlinear $Q$ parameter are that the control can respond aggressively to small disturbances, driving the output quickly to zero, but respond less aggressively to large disturbances to stay within the control bounds. In contrast, the linear control policy must respond proportionally to disturbances of all sizes. Since the control constraints require a less aggressive response to large disturbances, the linear controller must also respond less aggressively to small disturbances, which does not drive the output to zero.

X. Conclusion

In this article, we have introduced RENs as a new model class for learning nonlinear dynamical systems with built-in stability and robustness constraints. The model set is flexible and admits a
direct parameterization, allowing learning of large-scale models via generic unconstrained optimization methods such as SGD.

We have illustrated the benefits of the new model class on problems in system identification, observer design, and control. On system identification benchmarks, the REN structure outperformed the widely used RNN and LSTM models in terms of model fit while achieving far lower sensitivity to input perturbations. We further showed that the REN model architecture enables new approaches to nonlinear observer design and optimization of nonlinear feedback controllers.

APPENDIX

A. Proof of Theorem 1

First, well-posedness follows directly from (15), since it implies $W > 0$, which is precisely (12).

To prove contraction and incremental IQCs, we consider the following dynamics, i.e., differences between two sequences $(x^a, w^a, u^a, y^a, w^a)$ and $(x^b, w^b, u^b, y^b, w^b)$, which we denote $\Delta x_t = x^b_t - x^a_t$, and similarly, for other variables. The incremental dynamics generated by (1) are

$$
\begin{bmatrix}
\Delta x_{t+1} \\
\Delta v_t \\
\Delta y_t \\
\Delta w_t
\end{bmatrix} =
\begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix}
\begin{bmatrix}
\Delta x_t \\
\Delta v_t \\
\Delta y_t \\
\Delta w_t
\end{bmatrix}
\tag{60}
$$

For $\Delta w_t = \sigma (v^b_t + \Delta v_t) - \sigma (v^a_t)$.

To deal with the nonlinear element (61), we note that the constraint (14) can be rewritten as $(\sigma(x) - \sigma(y))(y - x) \geq (\sigma(x) - \sigma(y))^2$, and by taking a conic combinations of this inequality for each channel with multipliers $\lambda_i > 0$, we obtain the following incremental quadratic constraint:

$$
\Gamma(\Delta v, \Delta w) =
\begin{bmatrix}
\alpha \Delta v \\
\Delta w
\end{bmatrix}^T
\begin{bmatrix}
\Lambda & -\alpha \Lambda \\
\alpha \Lambda & -2\alpha \Lambda
\end{bmatrix}
\begin{bmatrix}
\alpha \Delta v \\
\Delta w
\end{bmatrix} \geq 0
\tag{62}
$$

which is valid for any $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_q) \in \mathbb{R}_{++}^q$.

To prove contraction, we first note that if (15) holds, then

$$
\begin{bmatrix}
\alpha^2 P - C_i^T \Lambda A^T \\
-\Lambda C_i
\end{bmatrix} +
\begin{bmatrix}
A^T & 0 \\
W & B_i^T
\end{bmatrix}
\begin{bmatrix}
P & 0 \\
A & B_i
\end{bmatrix}^\top \geq 0
\tag{63}
$$

for some $\alpha < \bar{\alpha}$. Left-multiplying by $[\Delta x_t^T \Delta w_t^T]^\top$ and right-multiplying by $[\Delta x_t^T \Delta w_t^T]^\top$, we obtain the following incremental Lyapunov inequality:

$$
|\Delta x_{t+1}|^2_p \leq \alpha^2 |\Delta x_t|^2_p - \Gamma(\Delta v_t, \Delta w_t) \leq \alpha^2 |\Delta x_t|^2_p
\tag{64}
$$

where the second inequality follows by the incremental quadratic constraint (62). Iterating over $t$ gives (4) with $K = \sqrt{\frac{\bar{\alpha}}{\sigma^2}}$, where $\bar{\sigma}$ is the maximum singular value of $P$, and $\sigma$ the minimum singular value.

The proof for the incremental IQC is similar: from (16), we obtain a nonstrict version with $\alpha < \bar{\alpha}$. Left-multiplying by $[\Delta x_t^T \Delta w_t^T \Delta u_t^T]$ and right-multiplying by its transpose results in

$$
|\Delta x_{t+1}|^2_p \leq \alpha^2 |\Delta x_t|^2_p - \Gamma(\Delta v_t, \Delta w_t, \Delta u_t)
\tag{65}
$$

Since $\Gamma(\Delta v_t, \Delta w_t, \Delta u_t) \geq 0$ from (62), and $\alpha < 1$, we have

$$
|\Delta x_{t+1}|^2_p - |\Delta x_t|^2_p \leq
\begin{bmatrix}
\Delta y_t \\
\Delta u_t
\end{bmatrix}^T
\begin{bmatrix}
Q & S \\
S^T & R
\end{bmatrix}
\begin{bmatrix}
\Delta y_t \\
\Delta u_t
\end{bmatrix}.
$$

Telescoping sum of the aforementioned inequality yields the IQC (5) with $(a, b) = (b - a)^T P (b - a)$. Moreover, since $Q \geq 0$, taking $\Delta u_t = 0$ in (65) reduces to (64) proving contraction.

B. Proof of Theorem 2

We note that an REN has Lipschitz bound of $\gamma$ if (28) holds with $Q = -\frac{1}{2} I, R = \gamma I$, and $S = 0$. By taking Schur complements and permuting the third and fourth columns and rows, the condition to be verified can be rewritten as

$$
\begin{bmatrix}
\alpha^2 P - C_i^T \Lambda A^T \\
-\Lambda C_i
\end{bmatrix} +
\begin{bmatrix}
A^T & 0 \\
W & B_i^T
\end{bmatrix}
\begin{bmatrix}
P & 0 \\
A & B_i
\end{bmatrix}^\top \geq 0.
\tag{66}
$$

Now, the upper left quadrant is positive-definite via Schur complement of (15). Hence, by taking $\gamma$ sufficiently large, the condition (66) will be verified.

C. Proof of Theorem 3

To show well-posedness, from (19), we have $E + E^\top \succ P > 0$ and $W = 2I - \Lambda D_{11} - D_{11}^T \Lambda > 0$, where $D_{11} = \Lambda^{-1} D_{11}$. The first inequality implies that $E$ is invertible, and thus, (9) is well-posed. The second one ensures that the equilibrium network (8) is well-posed by the main result of [12].

To prove contraction, applying the inequality $\bar{\alpha}^2 E^\top P^{-1} E \succeq E + E^\top - \frac{1}{\bar{\alpha}^2} P$ [26, Sec. II] and a Schur complement to (19) gives

$$
\begin{bmatrix}
\bar{\alpha}^2 E^\top P^{-1} E - C_i^T \\
-\Lambda C_i
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
-\Lambda C_i
\end{bmatrix} \begin{bmatrix}
F\pi \\
B_i^T
\end{bmatrix}^\top \succeq 0.
$$

By substituting $F = EA, B_1 = EB_1, B_2 = EB_2, C_1 = \Lambda C_1, \text{and } D_{11} = \Lambda D_{11}$ into the aforementioned inequality, we obtain (15) with $P = E^\top P^{-1} E$. Thus, $\Theta_C$ is a set of C-RENs. Similarly, we can show that (20) implies (16) for R-RENs.

D. Proof of Proposition 1

With the factorization $Q = -L_Q^T L_Q$, (30) is equivalent to

$$
R - SQ^{-1} S^\top \succ \begin{bmatrix}
L_Q D_{22} - L_Q^T \bar{S}^T \\
L_Q D_{22} - L_Q^T \bar{S}^T
\end{bmatrix}^\top
$$

which implies that $R - SQ^{-1} S^\top > 0$, hence, $L_R$ is well-defined.

If $p \geq m$, from (32), we have $N^T N \prec I$ since

$$
(I + M)^T (I + M) - (I + M)^T N^T N (I + M)
$$

$$
= 2(M^T + M) - 4Z_3^T Z_3 = 4(X_3^T X_3 + \epsilon I) > 0.
$$

Similarly, for the case $p < m$, we can obtain $NN^T \prec I$ from (33), which also implies $NN^T \prec I$. Finally, by substituting (31) into (30), we have

$$
R + SD_{22}^T + D_{22}^T Q D_{22} = L_R^T (I - N^T N) L_R > 0.
$$
E. Proof of Proposition 2

When the correctness condition (43) holds, we have that \( \hat{x}_t = x_t \) for all \( t \geq 0 \) if \( \hat{x}_0 = x_0 \), i.e., the true state trajectory is a particular solution of the observer. But contradiction implies that all solutions of the observer converge to each other. Hence, when \( w = 0 \), we have \( \hat{x}_t \to x_t \) as \( t \to \infty \).

Now we consider the case where the correctness condition does not hold, but its error is bounded by (44). The dynamics of \( \Delta x := \hat{x} - x \) can be written as

\[
\Delta x_{t+1} = f_\theta(\hat{x}_t, u_t, y_t) - f_\theta(x_t, u_t) + \epsilon_t
\]

where \( \epsilon_t = f_\theta(x_t, u_t, y_t) - f_\theta(x_t, u_t) \). By the mean-value theorem, \( \Delta x_{t+1} = F(x_t, u_t) \Delta x_t + \epsilon_t \) where \( F_t = \frac{\partial f_\theta}{\partial z}(z, u_t) \) for some \( z \). By the triangle inequality \( |\Delta x_{t+1}|p \leq |F_t \Delta x_t|^p + |\epsilon_t|^p \), and by contraction, \( |F_t \Delta x_t|^p \leq \alpha |\Delta x_t|^p \). So, we have \( |\Delta x_{t+1}|p - |\Delta x_t|^p \leq (\alpha - 1) |\Delta x_t|^p + |\epsilon_t|^p \leq (\alpha - 1) |\Delta x_t|^p + \sqrt{\sigma p} \).

From which it follows that the set \( |\Delta x_t|^p \leq \frac{\sqrt{\sigma p}}{\alpha} \) is forward-invariant and exponentially attractive, since \( \alpha - 1 < 1 \). The claimed result then follows from \( \sqrt{\sigma} |\Delta x_t| \leq |\Delta x_t| \).

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