Driven diffusive systems have been at the focus of extensive theoretical and experimental studies in recent years. Many studies have been devoted to systems of particles evolving under local biased exchange processes in the bulk which may or may not be coupled to non-conserving reservoirs at the boundaries. Examples include the asymmetric exclusion process (ASEP), the zero-range process (ZRP) and many others [1–4]. Bulk-nonconserving processes are rather common in many physical systems such as molecular motors, traffic flow problems with road intersections, chemical reactions in an open environment and others (see for example [3]). The corresponding driven models of such systems are often less tractable analytically [3–12]. Their steady-state properties are usually very different from those of the corresponding conserving models.

In this Letter we study the effect of bulk nonconserving processes by considering the limit where they occur on a much longer time scale than that of the conserving dynamics. This separation of time scales allows the system to relax to its conserving steady state with a fixed number of particles in between nonconserving dynamical moves. As a result, the steady state of the system can be expressed in terms of an appropriate ‘ensemble’ of the conserving steady states which may be regarded as a kind of nonequilibrium grand-canonical ensemble. One may thus use the steady state properties of the conserving system to analyze the corresponding properties of the nonconserving one.

This approach is demonstrated on the ABC model [13, 14]. This is a one dimensional three species driven exclusion model which exhibits a phase separation transition, and which has previously been generalized to include particle-nonconserving processes. By considering the limit of slow nonconserving dynamics and following the approach described above, we compute explicitly the large deviation function (LDF) of the overall particle density, despite the fact that the LDF of the density profile is not known. The LDF yields a definition for a ‘chemical potential’ which unlike the equilibrium one, depends on the details of the nonconserving dynamical process. Based on this derivation, we draw the exact phase diagram of the model. The first order transition line exhibited by the model may be computed via Maxwell’s construction, even though detailed balance is not obeyed. As discussed at the end of the Letter, the method presented below is readily applicable to other driven models that are coupled slowly to an external reservoir.

The ABC model is defined on a one-dimensional periodic lattice of length $L$, where each site is occupied by one of the three species of particles, labeled $A$, $B$ and $C$. The model evolves by random sequential updates in the bulk which may or may not be coupled to non-conserving reservoirs at the boundaries. Examples include the asymmetric exclusion process (ASEP), the zero-range process (ZRP) and many others [1–4]. Bulk-nonconserving processes are rather common in many physical systems such as molecular motors, traffic flow problems with road intersections, chemical reactions in an open environment and others (see for example [3]). The corresponding driven models of such systems are often less tractable analytically [3–12]. Their steady-state properties are usually very different from those of the corresponding conserving models.

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asymmetric ABC model on a ring with particle-nonconserving dynamics [20, 21]. In this model particle sites can be occupied by inert vacancies, denoted by 0, whose dynamics is defined as

\[
A0 \xrightarrow{1} 0A, \quad B0 \xrightarrow{1} 0B, \quad C0 \xrightarrow{1} 0C. \tag{2}
\]

The total number of particles, \( N = N_A + N_B + N_C \leq L \), fluctuates through evaporation and deposition of triplets of neighbouring particles given by

\[
ABC \xrightarrow{p e^{-33\mu}} 000. \tag{3}
\]

Here \( p \) is the overall rate of the nonconserving dynamics and \( \mu \) is a parameter which as shown below, can be regarded as a chemical potential. This type of nonconserving process has been chosen because it maintains detailed balance with respect to an effective Hamiltonian if initially the densities are equal, \( N_A = N_B = N_C \).

In the equal-densities case, the equilibrium canonical and grand-canonical ensembles of this model correspond respectively to a \textit{conserving model}, defined by rules \( \ref{eq:ABC} \) and \( \ref{eq:ABC} \) and a \textit{nonconserving model}, which includes also rule \( \ref{eq:ABC} \). It has been found that for this case both models exhibit the same second order transition line, which turns into a first order line at a tricritical point only in the nonconserving model [20, 21]. Such inequivalence between the phase diagrams of the two ensembles is often observed in long-range interacting systems (for recent reviews see [22, 23]).

In this Letter we study the phase diagram of the generalized weakly-asymmetric ABC model with nonequal densities. In this case no free energy exists. This study can be accomplished by analyzing the steady state of its hydrodynamic equations, given by

\[
\partial_t \rho_\alpha = \beta \lambda^{-2} \partial_x \left[ \rho_\alpha \left( \rho_{\alpha+1} - \rho_{\alpha+2} \right) \right] + L^{-2} \partial_x^2 \rho_\alpha + p \left( \rho_0^{3} - e^{-33\mu} \rho_A \rho_B \rho_C \right). \tag{4}
\]

Here \( \rho_\alpha(x) \) is the coarse-grained density profile whose index \( \alpha \) denotes the species and runs cyclicly over \( A, B \) and \( C \). We denote the average density of each species by \( r_\alpha = N_\alpha / L = \int_0^1 dx \rho_\alpha(x) \) and the overall density by \( r = r_A + r_B + r_C \). In the original ABC model \(( r = 1, p = 0)\), Eq. \( \ref{eq:ABC} \) has been shown to be exact in the limit of \( L \to \infty \) for equal densities [13, 17], and has been argued to remain so even for arbitrary average densities [17].

The conserving steady state \(( r \leq 1, p = 0)\) of Eq. \( \ref{eq:ABC} \), denoted by \( \rho_\alpha^c(x, r) \), can be readily extracted from the known steady state of the original ABC model, \( \rho_\alpha^c(x, 1) \), via a scaling transformation discussed below. In the nonconserving model, we are able to derive the steady-state of Eq. \( \ref{eq:ABC} \) in the limit of slow nonconserving dynamics,

\[
p \sim L^{-\gamma}, \quad \gamma > 2, \tag{5}
\]

where the \( p \)-dependent term becomes subdominant. Dynamically this limit implies that on time scales of order \( L^2 \) the nonconserving model relaxes to the conserving steady state with a fixed overall particle density, \( r \), whereas on longer time scales of order \( L^\gamma \), \( r \) fluctuates around its steady-state value.

This separation of time scales suggests that Eq. \( \ref{eq:ABC} \) remains valid within the limit considered in Eq. \( \ref{eq:ABC} \) and that the steady state measure of the nonconserving model can be written in the limit of large \( L \) as

\[
P_{nc}(\zeta, N) \simeq P_c(\zeta ; N) P(N). \tag{6}
\]

Here \( \zeta = \{ \zeta_i \} \) denotes a microstate of the model with \( \zeta_i = A, B, C \) for \( i \in [1, L] \) and \( P_c(\zeta ; N) \) is the conserving steady-state measure. Although \( P_c(\zeta ; N) \) is not known, the knowledge of its extremizing profile in the hydrodynamic limit, \( \rho_\alpha^c(x, r) \), is sufficient for deriving the probability density of \( N \) in the nonconserving model, \( P(N) \).

We now derive \( P(N) \) by writing its master equation, which evolves on the slow time scale of the bath by

\[
\partial_t P(N) = \sum_{N' = N \pm 3} P(N') Q(N' \to N) - P(N) Q(N \to N'). \tag{7}
\]

The average transition rates, \( Q \), are given by

\[
Q(N \to N') = \sum_\zeta P_c(\zeta ; N) \sum_{\zeta' , N'} W(\zeta, N | \zeta', N'), \tag{8}
\]

where \( N' = N \pm 3 \) and \( W \) is the transition rate between two microstates. The latter is computed by counting the number of \( ABC \) and 000 triplets in each microstate, denoted by \( n_{ABC}(\zeta) \) and \( n_{000}(\zeta) \), respectively. In the limit of \( L \to \infty \) the evaporation rate can be computed using a saddle-point approximation, yielding

\[
Q(N \to N - 3) = p e^{-33\mu} \sum_\zeta P_c(\zeta ; N) n_{ABC}(\zeta) \approx p e^{-33\mu} L \int_0^1 dx p_A^c(x, r) \rho_B^c(x, r) \rho_C^c(x, r). \tag{9}
\]

The deposition rate is obtained similarly as

\[
Q(N \to N + 3) \simeq p L \int_0^1 dx |p_A^c(x, r)|^3, \tag{10}
\]

which can be simplified by noting that the inert vacancies have a flat steady-state profile, \( p_0^c(x, r) = 1 - r \).

Equation \( \ref{eq:ABC} \) corresponds to a one-dimensional random walk in \( N \) in the presence of a local potential whose steady state is

\[
P(N) \propto \prod_{N' = N_{\text{min}}}^{N-3} \frac{Q(N' \to N' + 3)}{Q(N' + 3 \to N')} \sim e^{-L G(\mu, r)}, \tag{11}
\]

where \( N_{\text{min}} = N - 3 \min_\alpha (N_\alpha) \) is the minimal number of particles. The LDF of the overall density, \( r \), is

\[
G(\mu, r) = -\mu r + \int_{r_0}^r dr' \mu(r'), \tag{12}
\]
phase transition when \( r = 0 \), for \( \beta = 50 \), \( r_A = r_B = r/3 = 0.025 \) and \( \mu = -0.053 \). For these parameters the LDF has two local minima, at \( r_{\text{hom}} \) and \( r_{\text{ord}} \), corresponding to the homogenous and ordered phases, respectively. The nonconserving model undergoes a first order phase transition when \( G(\mu, r_{\text{hom}}) = G(\mu, r_{\text{ord}}) \).

where \( r_0 \) is an arbitrary parameter and

\[
\mu(r) = \frac{1}{3\beta} \left[ \log \left( \int_0^1 dx \rho_A^* \rho_B^* \rho_C^* \right) - 3 \log (1 - r) \right]. \tag{13}
\]

The LDF, which is in fact proportional to the potential felt by the random walker, is plotted in Fig. 1 for some point in parameter-space. In the large \( L \) limit, the average value of \( r \) is given by the global minimum of \( G(\mu, r) \) which corresponds to \( \partial G/\partial r = \mu - \mu(r) = 0 \).

We now discuss briefly the role of \( \mu(r) \) as the chemical potential of the conserving model. This will enable us to compare the phase diagrams of the two models in the \((\beta, \mu)\)-plane. In the absence of a Hamiltonian which states the energy cost of adding or removing particles, the chemical potential of a conserving nonequilibrium system can be measured by coupling it to a microscopic gauge, customarily defined using Creutz method \[24\]. Here, this is done by allowing triplets of \( ABC \) particles to evaporate from the lattice into a \textit{demon} with the slow rate \( p \) defined in Eq. (15). If the demon contains triplets, it may depose them back into the lattice at the same rate \( p \). Following the lines of the derivation above, the probability density of the number particles in the demon, \( N_d \geq 0 \), can be shown to obey in the limit \( L \rightarrow \infty \),

\[
P(N_d + 3)/P(N_d) \approx \int_0^1 dx \rho_A^* \rho_B^* \rho_C^* \int_0^1 dx [\rho_0^3] = e^{\beta\mu(r)}, \tag{14}
\]

and hence \( P(N_d) \propto \exp(\beta\mu(r)) \) \[25\]. The function \( \mu(r) \) is therefore the chemical potential of the conserving model, as measured by the demon. In contrast to equilibrium, here \( \mu(r) \) is derived from \( W(\zeta, N|\zeta', N') \) and thus depends on the choice of nonconserving dynamics.

We now proceed to compute the phase diagrams of the model under conserving dynamics and slow nonconserving dynamics. The conserving phase diagram \((p = 0, r \leq 1)\) can be derived from that of the original ABC model \((p = 0, r = 1)\) using a mapping where the vacant sites are removed from each microstate of the conserving model.

The master equation of the resulting system corresponds to that of the original ABC model with \( N \) sites. By observing that \( q = \exp(\beta f/L) = \exp(\beta f/N) \equiv \exp(\beta N/N) = 1 \) we conclude that the \( N \)-size system has an effective bias of \( \beta' = \beta r \). Similarly the average densities of the \( N \)-size system can be shown to be given by \( r_{\alpha}' = r_{\alpha}/r \) for \( \alpha = A, B, C \) and \( r_0' = 0 \). The steady-state profile of the conserving model can thus be expressed as

\[
\rho_{\alpha}'(x, \beta, r_{\alpha}, r) = r \rho_{\alpha}(x, \beta r, r_{\alpha}/r, 1), \tag{15}
\]

where \( \rho_{\alpha} \) in the r.h.s. has been derived for arbitrary \( r_{\alpha} \) in \[14\]. Equation (15) maps the phase diagram of the original ABC model \[13, 16\] onto the conserving model \((r \leq 1, p = 0)\). The resulting conserving phase diagram consists of a second order transition line at

\[
\beta = 2\pi\sqrt{3}/\sqrt{r^2 - 36\Delta^2}, \tag{16}
\]

where \( \Delta^2 = \frac{1}{3} \sum_{\alpha=A,B,C} (r_{\alpha} - r/3)^2 \) is a measure for the deviation from equal densities. The transition becomes first order for \((r_A^2 + r_B^3 + r_C^3)/r > 2 (r_A^3 + r_B^3 + r_C^3)\). The phase diagram is shown in Fig. 2 for two equal densities, \( r_A = r_B = r = 3 - \Delta, \quad r_C = r/3 + 2\Delta \). \[17\]

Since the LDF of \( \rho_{\alpha}(x) \) in not known, one cannot compute the first order transition line of the conserving model. It is possible, however, to draw the stability limits (dashed lines) in between which both phases are stable with respect to small perturbations.

The phase diagram of the nonconserving model can be derived by studying the extrema of \( G(\mu, r) \), given by the equation \( \mu = \mu(r) \). The function \( \mu(r) \), defined

![Figure 1](image1.png)

**Figure 1.** The large deviation function of the overall density, \( r \), for \( \beta = 50 \), \( r_A = r_B = r/3 = 0.025 \) and \( \mu = -0.053 \). For these parameters the LDF has two local minima, at \( r_{\text{hom}} \) and \( r_{\text{ord}} \), corresponding to the homogenous and ordered phases, respectively. The nonconserving model undergoes a first order phase transition when \( G(\mu, r_{\text{hom}}) = G(\mu, r_{\text{ord}}) \).

![Figure 2](image2.png)

**Figure 2.** The \((1/\beta, \mu)\) phase diagrams of the conserving (a) and nonconserving (b) models for two equal densities with \( \Delta = 0.025 \). In the conserving model \( \mu \) is computed from Eq. (15). The thick and thin solid lines represent the first and second order phase transitions, respectively. They join at the conserving (*) and nonconserving (x) tricritical points. The dashed lines denote the stability limits of the two phases. The inset in (a) depicts schematically the area near the conserving tricritical point. The conserving tricritical point is irrelevant in (b), as it is located within the ordered phase of the nonconserving model.
The derivation above can be readily applied to other driven models that are coupled slowly to an external bath. The explicit expression of the LDF of the nonconserved parameter may, however, depend on the steady state properties of the conserving model which are known analytically only in a handful of models. We demonstrate this approach and the significance of the choice of the nonconserving process by considering the generalized ABC model with the usual grand-canonical dynamics, where the nonconserving process 3 is replaced by
\[
A \xrightarrow{\beta} p, \quad B \xrightarrow{\beta} p, \quad C \xrightarrow{\beta} 0.
\]  
Assuming slow nonconserving dynamics 5 and following the derivation above yields the same LDF as in Eq. 12 but with \( \mu(r) = [log(r) - log(1-r)]/\beta \). As expected, different nonconserving dynamics lead to different definitions of the chemical potential. Here, since \( \mu(r) \) is single-valued for any value of \( \beta \), the conserving and nonconserving model display the same phase diagram. It consists of a second order transition line given by Eq. 16 for \( \Delta = 0 \).

It would be interesting to study the borderline case of \( \gamma = 2 \) and investigate how the picture presented above changes when the ‘adiabatic’ approximation 4 breaks down. A similar limit has recently been studied in a boundary-driven diffusive model with nonconserving dynamics in the bulk, for which an implicit expression for the LDF of the profile was derived 12. In a different study, an approximate chemical potential has been de-
fined for several driven models where the nonconserving dynamics is not slow [32–34].

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