A quantum-copying machine for equatorial qubits

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Abstract

Bužek and Hillery proposed a universal quantum-copying machine (UQCM) (i.e., transformation) to analyze the possibility of cloning arbitrary states. The UQCM copies quantum-mechanical states with the quality of its output does not depend on the input. We propose a slightly different transformation to analyze a restricted set of input states. We impose the conditions (I) the density matrices of the two output states are the same, and that (II) the distance between input density operator and the output density operators is input state independent. Using Hilbert-Schmidt norm and Bures fidelity, we show that our transformation can achieve the bound of the fidelity.

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1 Introduction

Quantum computing and quantum information have been attracting a great deal of interests. They differ in many aspects from the classical theories. One of the most fundamental differences between classical and quantum information is the no-cloning theorem\cite{1}. It tells us that arbitrary quantum information can not be copied exactly. The no-cloning theorem for pure states is also extended to the case that a general mixed state can not be broadcast\cite{2}. However, no-cloning theorem does not forbid imperfect cloning. And several kinds of quantum copying (cloning) machines (QCM) are proposed\cite{3, 4, 5}. Some authors also try to find the optimal QCMs\cite{6, 7, 8}. In the proof of the no-cloning theorem, Wootters and Zurek introduced a QCM which has the property that the quality of the copy it makes depends on the input states\cite{1}. To diminish or cancel this disadvantage, Bužek and Hillery proposed a UQCM, the copying process is input-state independent. They use Hilbert-Schmidt norm to quantify distance between input density operator and the output density operators. Bruß et al discussed the performance of a UQCM by analyzing the role of the symmetry and isotropy conditions imposed on the system and found the optimal UQCM and the optimal state-dependent quantum cloning\cite{4}. Optimal quantum cloning of general $N \rightarrow M$ case is discussed in Ref\cite{6, 7, 8}. The relation between quantum cloning and superluminal signalling is proposed and discussed in Ref\cite{9, 10}.

In this paper, we propose a QCM for a restricted set of input states. The Bloch vector is restricted to the intersection of $x-z$ ($x-y$ and $y-z$) plane with the Bloch sphere, this kind of qubits are the so-called equatorial qubits\cite{11}. Applying the method by Bužek and Hillery, we propose a possible extension of the original transformation. We demand that (I) the density matrices of the two output states are the same, and that (II) the distance between input density operator and the output density operators is input state independent. To evaluate the distance of two states, we use both Hilbert-Schmidt norm and Bures fidelity. There are a family of transformations which satisfy the above two conditions. In a special point, we can obtain an optimal fidelity and the correspondent transformation agrees with the results of Bruß et al \cite{11} who study the optimal quantum cloning for equatorial qubits. The fidelity of quantum cloning for the equatorial qubits is higher than the original Bužek and Hillery UQCM. This is expected as that the more information about the input is given, the better one can clone each of its states. We also obtain the quantum cloning transformations for equatorial qubits in $x-y$ and $y-z$ planes.

The paper is organized as follows: In Section 2, we introduce the transformation for equator in $x-z$ plane. In Section 3, we use Hilbert-Schmidt norm to evaluate the distance between input state and output states, and the minimal distance is found. In section 4, we use Bures metric to define the fidelity, and the condition of orientation invariance of the Bloch vector is studied. In section 5, the cloning transformations for equator in $x-y$ and $y-z$ planes are obtained. Section 6 includes summary and discussions.

2 Transformation

We propose the following transformation

\[
|0\>_a \ Q \>_x \rightarrow (|0\>_a \ |0\>_b + \lambda |1\>_a |1\>_b) \ |Q_0\>_x + (|1\>_a \ |0\>_b + |0\>_a |1\>_b) \ |Y_0\>_x, \\
|1\>_a \ Q \>_x \rightarrow (|1\>_a \ |1\>_b + \lambda |0\>_a |0\>_b) \ |Q_1\>_x + (|1\>_a \ |0\>_b + |0\>_a |1\>_b) \ |Y_1\>_x, \quad (1)
\]
where the states $|Q_j>_{x},|Y_j>_{x},j=0,1$ are not necessarily orthonormal. Hereafter, we will drop the subscript $x$ for convenience. Explicitly, this transformation is a generalization of the original one proposed by Bužek and Hillery\cite{Bužek and Hillery}. When $\lambda = 0$, this transformation is reduced to the original transformation. For convenience, we restrict $\lambda$ to be real and $\lambda \neq \pm 1$. We also assume

$$<Q_0|Q_1>=<Q_1|Q_0>=0.$$  \hspace{1cm} (2)

Considering the unitarity of the transformation, we have the following relations:

$$(1 + \lambda^2) <Q_j|Q_j> + 2 <Y_j|Y_j> = 1, \ j = 0,1,$$  \hspace{1cm} (3)

$$<Y_0|Y_1>=<Y_1|Y_0>=0.$$  \hspace{1cm} (4)

As proposed by Bužek and Hillery, we further assume the following relations to reduce the free parameters

$$<Q_j|Y_j>=0, \ j = 0,1.$$  \hspace{1cm} (5)

$$<Y_0|Q_0>=<Y_1|Q_1>=\xi,$$  \hspace{1cm} (6)

$$<Q_0|Y_1>=<Q_1|Y_0>=<Q_0|Y_0>=<Q_1|Q_1>=1.$$  \hspace{1cm} (7)

For simplicity, we will also use the following standard notations

$$|j_k>=|j>_a|k>_b, \ j,k=0,1.$$  \hspace{1cm} (8)

and

$$|+>=\frac{1}{\sqrt{2}}(|10> + |01>), \ |->=\frac{1}{\sqrt{2}}(|10> - |01>).$$  \hspace{1cm} (9)

Obviously, $|+>$ and $|00>$, $|11>$ constitute an orthonormal basis. The input state is a pure superposition state (SU(2) coherent state)

$$|\psi>_a=\alpha|0>_a+\beta|1>_a$$  \hspace{1cm} (10)

with $\alpha^2 + \beta^2 = 1$. Here, we use an assumption that $\alpha$ and $\beta$ are real that means the $y$ component of the Bloch vector of the qubits is zero. And we just deal with a restricted set of input states.

The output density operator $\rho^{(out)}_{ab}$ describing output state after the copying procedure reads

\begin{align*}
\rho^{(out)}_{ab} &= |00><00|[1-2\xi(1+\lambda^2)]\lambda^2 + \alpha^2(1-\lambda^2)]} \\
&+ |00><10| + |00><01| + |11><10| + |11><01| + |01><11| + |01><00| + |10><10| + |10><00| \\
&+ |00><11| + |10><11|\frac{\eta}{2}\alpha\beta(\lambda + 1]) \\
&+ |01><11| + |11><00|\frac{1-2\xi}{1+\lambda^2}\lambda) \\
&+ \xi(|01><10| + |01><01| + |10><10| + |10><01|) \\
&+ |11><11|\frac{1-2\xi}{1+\lambda^2}\alpha^2(1+\lambda^2) + 1)],\end{align*}  \hspace{1cm} (11)
where \( \rho_{ab}^{(out)} = Tr_x[\rho_{abx}^{(out)}] \) with \( \rho_{abx}^{(out)} \equiv |\Psi_{abx}^{(out)} \rangle \langle \Psi_{abx}^{(out)}| \). Taking trace on mode \( b \) or mode \( a \), we can get reduced density operator for mode \( a \) or mode \( b \), \( \rho_{a}^{(out)} \) or \( \rho_{b}^{(out)} \),
\[
\rho_{a}^{(out)} = \rho_{b}^{(out)} = |0><0|_{a} \quad \text{or} \quad |0><0|_{b} \quad \text{with} \quad \rho_{a}^{(out)} = \rho_{b}^{(out)} = |0><0|_{a} \quad \text{or} \quad |0><0|_{b}.
\]

The density operators \( \rho_{a}^{(out)} \) and \( \rho_{b}^{(out)} \) are exactly the same. We see that the output density operators are identical to each other. However, it is well known that they are not equal to the original input density operator. Next, we first use Hilbert-Schmidt norm to evaluate the distance between input density operator and output density operators.

## 3 Hilbert-Schmidt norm

For two-dimensional space, the Hilbert-Schmidt norm is believed to give a reasonable result in comparing density matrices though it becomes less good for finite-dimensional spaces as the dimension increases. The Hilbert-Schmidt norm define the distance between input density operator and output density operator as
\[
D_{a} = Tr[\rho_{a}^{(out)} - \rho_{a}^{(in)}]^2,
\]
where \( \rho_{a}^{(in)} \) is the input density operator. The distance between the two-mode density operators \( \rho_{ab}^{(out)} \) and \( \rho_{ab}^{(in)} = \rho_{a}^{(in)} \otimes \rho_{a}^{(in)} \) is defined as:
\[
D_{ab}^{(2)} = Tr[\rho_{ab}^{(out)} - \rho_{ab}^{(in)}]^2.
\]

With the help of relation (12), we find
\[
D_{a} = \{ \xi - \frac{1}{1 + \lambda^2} [1 + \lambda^2 (1 - \lambda^2) - \alpha^2] + 2 \alpha^2 (1 - \alpha^2) (\lambda \eta + \eta - 1)^2 + \{ \xi - 1 + \frac{1}{1 + \lambda^2} [1 + \alpha^2 (\lambda^2 - 1)] + \alpha^2 \}^2. \tag{15}
\]

We demand that this distance is independent of the parameter \( \alpha^2 \). That means the quality of the copies it makes is independent of the input state.
\[
\frac{\partial}{\partial \alpha^2} D_{a} = 0 \tag{16}
\]

We can choose the following solution
\[
\eta = \frac{1 - \lambda}{1 + \lambda^2} (1 - 2 \xi). \tag{17}
\]

Thus we get
\[
D_{a} = 2 \left( \xi \frac{1 - \lambda^2}{1 + \lambda^2} + \frac{\lambda^2}{1 + \lambda^2} \right)^2. \tag{18}
\]

In case \( \lambda = 0 \), we find \( \eta = 1 - 2 \xi \) and \( D_{a} = 2 \xi^2 \). These are exactly the original results obtained by Bužek and Hillery [3].
To find the result of $D_{ab}^{(2)}$, we can rewrite the output density operator $\rho_{ab}^{(\text{out})}$ by choose basis in (9). Substituting the relation (17) into the two-mode output density operator, we can obtain

$$
\rho_{ab}^{(\text{out})} = |00><00|\{\frac{1-2\xi}{1+\lambda^2}\lambda^2 + \alpha^2(1-\lambda^2)\}
$$

$$
+ (|00><+| + |+><00| + |11><+| + |+><11|) \{\sqrt{2}\alpha\beta\frac{1-\lambda^2}{2(1+\lambda^2)}(1-2\xi)\}
$$

$$
+ (|00><11| + |11><00|) \{\frac{1-2\xi}{1+\lambda^2}\lambda\}
$$

$$
+ 2\xi|><+| + |11><11|\{\frac{1-2\xi}{1+\lambda^2}\lambda^2(\lambda^2-1) + 1\}. \quad (19)
$$

By straightforward calculations, we can write

$$
\rho_{ab}^{(\text{in})} = \alpha^4|00><00| + \sqrt{2}\alpha^3\beta(|00><+| + |+><00|) + \alpha^2\beta^2(|00><11| + |11><00|)
$$

$$
+ 2\alpha^2\beta^2|><+| + \sqrt{2}\alpha\beta^3(|+><11| + |11><+|) + \beta^4|11><11|. \quad (20)
$$

And with the definition (14), we have

$$
D_{ab}^{(2)} = (U_{11})^2 + (U_{22})^2 + (U_{33})^2 + 2(U_{12})^2 + 2(U_{13})^2 + 2(U_{23})^2, \quad (21)
$$

where

$$
U_{11} = \alpha^4 - \frac{1-2\xi}{1+\lambda^2}\lambda^2 + \alpha^2(1-\lambda^2), \quad U_{22} = 2\xi - 2\alpha^2 + 2\alpha^4,
$$

$$
U_{33} = \alpha^4 - 2\alpha^2 + 1 - \frac{1-2\xi}{1+\lambda^2}\lambda^2(\lambda^2-1) + 1, \quad U_{12} = \sqrt{2}\alpha\beta[\alpha^2 - \frac{1-\lambda^2}{1+\lambda^2}(\frac{1}{2} - \xi)],
$$

$$
U_{13} = \alpha^2\beta^2 - \frac{1-2\xi}{1+\lambda^2}\lambda, \quad U_{23} = \sqrt{2}\alpha\beta[\beta^2 - \frac{1-\lambda^2}{1+\lambda^2}(\frac{1}{2} - \xi)]. \quad (22)
$$

We still impose the condition

$$
\frac{\partial}{\partial \alpha} D_{ab}^{(2)} = 0. \quad (23)
$$

We find the result

$$
\xi = \frac{(1-\lambda)^2}{2(3-2\lambda + 3\lambda^2)}. \quad (24)
$$

Substitute these results into $D_a$ and $D_{ab}^{(2)}$, we have

$$
D_a = \frac{(1-2\lambda + 5\lambda^2)^2}{2(3-2\lambda + 3\lambda^2)^2},
$$

$$
D_{ab}^{(2)} = \frac{2(1-4\lambda + 12\lambda^2 - 8\lambda^3 + 7\lambda^4)}{(3-2\lambda + 3\lambda^2)^2}. \quad (25)
$$

So, we actually can have a family of transformations to satisfy the two conditions (I) and (II). In case $\lambda = 0$, we have Bužek and Hillery’s result

$$
D_a = \frac{1}{18} \approx 0.056, \quad D_{ab}^{(2)} = \frac{2}{9} \approx 0.22. \quad (26)
$$
Our aim is to find smaller $D_a$ and $D_{ab}^{(2)}$ for equatorial qubits. We can calculate that in the region $0 < \lambda < 1/3$, both $D_a$ and $D_{ab}^{(2)}$ take smaller values than the case $\lambda = 0$. When we choose

$$\lambda = 3 - 2\sqrt{2},$$

both $D_a$ and $D_{ab}^{(2)}$ take their minimal values,

$$D_a = \frac{99 - 70\sqrt{2}}{68 - 48\sqrt{2}} \approx 0.043, \quad D_{ab}^{(2)} = \frac{215 - 152\sqrt{2}}{8(3 - 2\sqrt{2})^2} \approx 0.17.$$ (28)

Thus for equatorial qubits, we can find smaller $D_a$ and $D_{ab}^{(2)}$, that means this QCM (1) has a higher fidelity than the original UQCM [3] by using the Hilbert-Schmidt norm. Actually, because that we assume $\alpha$ and $\beta$ are real, only a single unknown parameter is copied instead of two unknown parameters for the case of a general pure state. Thus a higher fidelity of quantum cloning can be achieved. The case of spin flip has a similar phenomenon [12, 13, 14].

Under the condition (27), we have

$$\xi = \frac{1}{8}, \quad \eta = \frac{\sqrt{2} - 1}{12 - 8\sqrt{2}}.$$ (29)

We can realize vectors $|Q_j\rangle$, $|Y_j\rangle$, $j = 0, 1$ in two-dimensional space

$$|Q_0\rangle = (0, \frac{1}{4 - 2\sqrt{2}}), \quad |Q_1\rangle = (\frac{1}{4 - 2\sqrt{2}}, 0),$$

$$|Y_0\rangle = (\frac{1}{2\sqrt{2}}, 0), \quad |Y_1\rangle = (0, \frac{1}{2\sqrt{2}}).$$ (30)

The transformation $|Q\rangle_x \rightarrow$ can be rewritten as

$$|0\rangle_a |Q\rangle_x \rightarrow \frac{1}{4 - 2\sqrt{2}} (|00\rangle + (3 - 2\sqrt{2})|11\rangle) \uparrow + \frac{1}{2} |\uparrow\rangle,$$

$$|1\rangle_a |Q\rangle_x \rightarrow \frac{1}{4 - 2\sqrt{2}} (|11\rangle + (3 - 2\sqrt{2})|00\rangle) \downarrow + \frac{1}{2} |\downarrow\rangle.$$ (31) (32)

This transformations agree with the results obtained by Bruß et al [11].

For an arbitrary $\lambda$ with the condition (17) and (24) satisfied, we can still realize vectors $|Q_j\rangle$, $|Y_j\rangle$, $j = 0, 1$ in two-dimensional space,

$$|Q_0\rangle = q |\uparrow\rangle, \quad |Q_1\rangle = q |\downarrow\rangle,$$

$$|Y_0\rangle = y |\downarrow\rangle, \quad |Y_1\rangle = y |\uparrow\rangle,$$ (33)

where we use notations

$$q = \sqrt{\frac{2}{3 - 2\lambda + 3\lambda^2}}, \quad y = \frac{1 - \lambda}{\sqrt{6 - 4\lambda + 6\lambda^2}}.$$ (34)

Thus all transformations satisfy the condition (I) and (II). Explicitly, the quantum cloning transformation for pure input states (10) can be written as

$$|0\rangle |Q\rangle_x \rightarrow (|00\rangle + \lambda |11\rangle) q |\uparrow\rangle_x + (|10\rangle + |01\rangle) y |\downarrow\rangle_x,$$

$$|1\rangle |Q\rangle_x \rightarrow (|11\rangle + \lambda |00\rangle) q |\downarrow\rangle_x + (|10\rangle + |01\rangle) y |\uparrow\rangle_x.$$ (35)

The distances defined by Hilbert-Schmidt norm take the form (25).
4 Bures fidelity

For finite-dimensional spaces, Hilbert-Schmidt norm becomes less good when the dimension increases. Bures fidelity provides a more exact measurement of the distinguishability of two density matrices. In this section, we will use Bures fidelity to check the result in the previous section. The fidelity is defined as

\[ F(\rho_1, \rho_2) = \text{Tr}(\rho_1^{1/2} \rho_2 \rho_1^{1/2})^{1/2}. \] (36)

The values of \( F \) range from 0 to 1, a larger \( F \) corresponds to a higher fidelity. \( F = 1 \) means two density matrices are equal.

We have a matrix

\[ U = \begin{pmatrix} -\frac{\beta}{\alpha} & \frac{\alpha}{\beta} \\ 1 & 1 \end{pmatrix} \] (37)

to diagonalize \( \rho^{(\text{in})}_a \)

\[ \rho^{(\text{in})}_a = U \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} U^{-1}. \] (38)

We thus have

\[ F(\rho^{(\text{in})}_a, \rho^{(\text{out})}_a) = \left\{ \xi + \frac{(1 - 2\xi)[2\alpha^4(1 - \lambda^2) + 2\alpha^2(\lambda^2 - 1) + 1]}{1 + \lambda^2} + 2\alpha^2(1 - \alpha^2)\eta(\lambda + 1) \right\}^{1/2}. \] (39)

We demand that the fidelity be independent of the input state

\[ \frac{\partial}{\partial \alpha^2} F(\rho^{(\text{in})}_a, \rho^{(\text{out})}_a) = 0, \] (40)

we can find

\[ \eta = \frac{1 - \lambda}{1 + \lambda^2}(1 - 2\xi), \] (41)

with

\[ F(\rho^{(\text{in})}_a, \rho^{(\text{out})}_a) = \left( \frac{1 - \xi + \lambda^2\xi}{1 + \lambda^2} \right)^{1/2}. \] (42)

Next, we use Bures fidelity to evaluate the distinguishability of density operators \( \rho^{(\text{out})}_{ab} \) and \( \rho^{(\text{in})}_{ab} = \rho^{(\text{in})}_a \otimes \rho^{(\text{in})}_a \). We have

\[ F(\rho^{(\text{in})}_{ab}, \rho^{(\text{out})}_{ab}) = \left\{ \frac{1 - 2\xi}{1 + \lambda^2}[\lambda^2 + \alpha^2(1 - \lambda^2)]\alpha^4 + 2\alpha^2(1 - \alpha^2)\lambda \frac{1 - 2\xi}{1 + \lambda^2} + 2\alpha^2(1 - \alpha^2)(1 - 2\xi) \right\} \frac{1 - \lambda^2}{1 + \lambda^2} + 4\alpha^2(1 - \alpha^2)\xi + \frac{1 - 2\xi}{1 + \lambda^2}[\alpha^2(\lambda^2 - 1) + 1](1 - \alpha^2)^2 \right\}^{1/2}. \] (43)

We still impose the condition

\[ \frac{\partial}{\partial \alpha^2} F(\rho^{(\text{in})}_{ab}, \rho^{(\text{out})}_{ab}) = 0. \] (44)
We have
\[ \xi = \frac{(1 - \lambda)^2}{2(3 - 2\lambda + 3\lambda^2)}. \] (45)

Thus, we finally have two Bures fidelities for one and two-mode density operators,
\[ F(\rho_a^{(\text{in})}, \rho_a^{(\text{out})}) = \left[ \frac{5 - 2\lambda + \lambda^2}{2(3 - 2\lambda + 3\lambda^2)} \right]^{1/2}, \] (46)
\[ F(\rho_{ab}^{(\text{in})}, \rho_{ab}^{(\text{out})}) = \left[ \frac{2}{3 - 2\lambda + 3\lambda^2} \right]^{1/2}. \] (47)

We can find that for both Hilbert-Schmidt norm and Bures fidelity, we have the same relations (17, 41) and (24, 45). However, the fidelity (46) and (47) do not take the maximums simultaneously which is different from the case of Hilbert-Schmidt norm. In the region 0 < \lambda < 1/3, for both \( F(\rho_a^{(\text{in})}, \rho_a^{(\text{out})}) \) and \( F(\rho_{ab}^{(\text{in})}, \rho_{ab}^{(\text{out})}) \), we can have a higher fidelity than the original UQCM which corresponds to \( \lambda = 0 \), this result agree with the previous result by Hilbert-Schmidt norm in last section. Here we remark that we just deal with the equatorial qubits.

When \( \lambda = 3 - 2\sqrt{2} \), \( F(\rho_a^{(\text{in})}, \rho_a^{(\text{out})}) \) takes its maximum
\[ F(\rho_a^{(\text{in})}, \rho_a^{(\text{out})})|_{\lambda=3-2\sqrt{2}} \approx 0.92388 \] (48)
which is larger than the original UQCM
\[ F(\rho_a^{(\text{in})}, \rho_a^{(\text{out})})|_{\lambda=0} \approx 0.912871. \] (49)

And we also have
\[ F(\rho_{ab}^{(\text{in})}, \rho_{ab}^{(\text{out})})|_{\lambda=3-2\sqrt{2}} \approx 0.853553 > F(\rho_{ab}^{(\text{in})}, \rho_{ab}^{(\text{out})})|_{\lambda=0} \approx 0.816497 \] (50)
Here the optimal fidelity (48) also agrees with the result obtained by Bruß et al [11].

In studying the optimal UQCM, the condition of orientation invariance of Bloch vector is generally imposed. Under the symmetry condition (I), the condition of orientation invariance of Bloch vector is equivalent to the condition (II) that the distance between input density operator and the output density operators is input state independent. We can check that for the case under consideration in this paper, the orientation invariance of Bloch vector means the relation (17) or (11) which is the subsequence of condition (II).
5 Quantum copying-machine for $x - y$ and $y - z$ planes equatorial qubits

In this section, instead of $x - z$ plane equator, we first study the transformations for equatorial qubits in $x - y$ plane. The input pure states take the form

$$|\psi > = \frac{1}{\sqrt{2}}(|0 > + e^{i\phi}|1 >),$$

where $\phi \in [0, 2\pi)$. We can check that the $z$ component of the Bloch vector is zero. Actually, the optimality of the fidelity should be independent from the choice of a particular basis. So, the optimal fidelity (48) for $x - z$ plane equator remains the same for $x - y$ equator. However, for different input states, the optimal transformation generally should be different. For $x - z$ plane equator, we have already found that a family of transformations (1) with restrictions (33) satisfy the conditions of quantum copying-machine (I) and (II), and also with the property of orientation invariance of Bloch vector. We expect the same result for the $x - y$ plane equator.

We can rewrite the input states as

$$|\psi > = e^{i\phi} \frac{1}{\sqrt{2}} (|0 > + e^{i\phi}|1 >)$$

$$= e^{i\phi} \frac{1}{\sqrt{2}} [\cos \frac{\phi}{2}(|0 > + |1 >) + i \sin \frac{\phi}{2}(|1 > - |0 >)] ,$$

(52)

It is obvious that two vectors ($|1 > + |0 >)/\sqrt{2}$ and ($|1 > - |0 >)/\sqrt{2}$ constitute an orthonormal basis. And the input states of $x - y$ (51) equator can be redefined as the input states of $x - z$ plane equator (11). With the help of results for $x - z$ plane equator, we can calculate the quantum cloning transformation for input states (51) as

$$|0 > |Q >_x \to |00 > \frac{2(1 - \lambda)}{\sqrt{6 - 4\lambda + 6\lambda^2}} |\uparrow_x > + (|01 > + |10 >) \frac{1 + \lambda}{\sqrt{6 - 4\lambda + 6\lambda^2}} |\downarrow_x > ,$$

(53a)

$$|1 > |Q >_x \to |11 > \frac{2(1 - \lambda)}{\sqrt{6 - 4\lambda + 6\lambda^2}} |\downarrow_x > + (|01 > + |10 >) \frac{1 + \lambda}{\sqrt{6 - 4\lambda + 6\lambda^2}} |\uparrow_x > .$$

(53b)

The Bures fidelity takes the same value as the case of $x - z$ equator (46). When $\lambda = 0$, we still obtain the results of UQCM[3]. When $\lambda = 3 - 2\sqrt{2}$, we obtain the optimal fidelity (48). And the optimal quantum cloning transformation for (51) becomes as

$$|0 > |Q >_x \to \frac{1}{\sqrt{2}} |00 > |\uparrow_x > + \frac{1}{2} \ (|01 > + |10 >) |\downarrow_x > ,$$

(54a)

$$|1 > |Q >_x \to \frac{1}{\sqrt{2}} |11 > |\downarrow_x > + \frac{1}{2} \ (|01 > + |10 >) |\uparrow_x > .$$

(54b)

For the case of $y - z$ equator, the results are similar as the case of $x - z$ plane. We can actually obtain the results by rename some vectors. We consider the input equatorial states as

$$|\psi > = \cos \theta |0 > + i \sin \theta |1 > .$$

(55)

The general and the optimal quantum cloning transformations can be written as follows:

$$|0 > |Q >_x \to \ (|00 > - \lambda |11 >) q |\uparrow_x > + (|10 > + |01 >) y |\downarrow_x > ,$$

(56a)

$$|1 > |Q >_x \to (|11 > - \lambda |00 >) q |\downarrow_x > + (|10 > + |01 >) y |\uparrow_x > ,$$

(56b)
where $q, y$ is defined in (34), and

$$
|0 >_{a} |Q >_{x} \rightarrow \frac{1}{4 - 2\sqrt{2}}|00 > - (3 - 2\sqrt{2})|11 >| \uparrow > + \frac{1}{2}|+ > | \downarrow >, \tag{59}
$$

$$
|1 >_{a} |Q >_{x} \rightarrow \frac{1}{4 - 2\sqrt{2}}|11 > - (3 - 2\sqrt{2})|00 >| \downarrow > + \frac{1}{2}|+ > | \uparrow >. \tag{60}
$$

6 Summary and discussions

We propose QCMs for equatorial qubits with the equator in the $x - z$, $x - y$ and $y - z$ planes respectively. We use both Hilbert-Schmidt norm and Bures fidelity to define the distinguishability of the density operator matrices. We can have a family of transformations, using Hilbert-Schmidt norm, the distances achieve the minimal values simultaneously for both one and two-mode operators in a special point $\lambda = 3 - 2\sqrt{2}$. Using Bures fidelity, the fidelity for one-mode operators also achieves the bound of fidelity in the case $\lambda = 3 - 2\sqrt{2}$.

We use only two conditions, (I) two output density matrices are identical and (II) the distance defined by Hilbert-Schmidt norm and the Bures fidelity for input density operator matrix and output density operator matrices are independent of the input state. We checked that the system also has the property of orientation invariance of the Bloch vector which is generally imposed to UQCM. For the case of arbitrary input state, $\lambda \neq 0$ generally breaks the condition of orientation invariance of the Bloch vector.

When we use Bures fidelity, the fidelity for one and two-mode does not reach their maximal points simultaneously which is different from the case of Hilbert-Schmidt norm, it is still need to clarify which measurement is more reasonable for two-dimensional space. We only consider the $1 \rightarrow 2$ cloning transformation in this paper, it is interesting to study the general $N \rightarrow M$ case. And the case of mixed states is also worth studying.

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