Entangling ability of a beam splitter in the presence of temporal which-path information

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We calculate the amount of polarization-entanglement induced by two-photon interference at a lossless beam splitter. Entanglement and its witness are quantified respectively by concurrence and the Bell-CHSH parameter. In the presence of a Mandel dip, the interplay of two kinds of which-path information — temporal and polarization — gives rise to the existence of entangled polarization-states that cannot violate the Bell-CHSH inequality.

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I. INTRODUCTION

Entanglement, the nonclassical correlations between spatially separated particles, is typically a signature of interactions in the past or emergence from a common source. However, it can also arise as the interference of identical particles \(1\). By postselecting experimental data based on the “click” of detectors \(2\), \(3\), photons scattered at a beam splitter have violated a Bell inequality, even if they originated from independent sources \(2\), \(3\). In reverse, triggered by an interferometric Bell-state measurement, entanglement has been swapped \(6\) to initially uncorrelated photons of different Bell pairs \(1\), \(7\), \(8\), \(9\). The observation of these nonclassical interference effects is an important step on the road towards an optical approach of quantum information processing \(11\), \(12\).

Being furnished by interference, the ability of a beam splitter to entangle the polarizations of two independent photons depends on their indistinguishability \(12\). One of the incident photons is horizontally polarized in state \(|H;\psi\rangle\), the other vertically polarized in \(|V;\phi\rangle\). The photons are partially distinguishable by their temporal degrees of freedom captured in the kets \(|\psi\rangle\) and \(|\phi\rangle\). Besides temporal which-path information inherited from incident photons, a scattered two-photon state possibly holds polarization which-path information. We make no assumptions about the scattering amplitudes connecting polarizations at the beam splitter, except that they constitute a unitary scattering matrix. Translated to a polarization-conserving beam splitter, this corresponds to incident photons in states \(|\sigma;\psi\rangle\) and \(|\sigma';\phi\rangle\) where \(\sigma\), \(\sigma'\) are arbitrary superpositions of \(H\), \(V\). Our analysis generalizes existing work on a polarization-conserving beam splitter \(V\) \(13\), \(15\).

The polarization-state \(\rho\) of a scattered photon pair is established from the scattering amplitudes of the beam splitter, the shape and timing of photonic wavepackets \(|\psi\rangle\), \(|\phi\rangle\) and the time-window of coincidence detection. If not erased by ultra-coincidence detection, an amount of temporal distinguishability of \(1 - |\langle\psi|\phi\rangle|^2\) pertains corresponding to a mixed state \(\rho\). We calculate both its concurrence and the Bell-CHSH parameter. The ability of the latter to witness entanglement can disappear in the presence of a Mandel dip. In terms of a polarization-conserving beam splitter, this corresponds to a deviation of \(\sigma\), \(\sigma'\) from \(\sigma = H\) and \(\sigma' = V\).

II. FORMULATION OF THE PROBLEM

In a second-quantized notation, the incident two-photon state \(|H;\psi\rangle_{L}|V;\phi\rangle_{R}\) takes the form

\[
|\Psi_{in}\rangle = \Psi^T_{H,L} \Phi^r_{V,R} |0\rangle,
\]

with field creation operators given by (see Fig. \(11\))

\[
\Psi^T_{H,L} = \int d\omega a^\dagger_H(\omega)\psi^\ast(\omega), \quad \Phi^r_{V,R} = \int d\omega b^\dagger_V(\omega)\phi^\ast(\omega).
\]

(The subscripts \(RL\) indicate the two sides of the beam splitter.) The operators \(a_i(\omega)\) with \(i = H, V\) satisfy commutation rules

\[
[a_i(\omega), a_j(\omega')] = 0, \quad [a_i(\omega), a^\dagger_j(\omega')] = \delta_{ij}\delta(\omega - \omega').
\]

The same commutation rules hold for the operators \(b_i(\omega)\), with commutation among \(a\) and \(b\).

The outgoing operators \(c_i(\omega), d_i(\omega)\) are related to the incoming ones \(a_i(\omega), b_i(\omega)\) by a \(4 \times 4\) unitary scattering matrix \(S\), decomposed in \(2 \times 2\) reflection and transmission matrices \(r,t,t',r'\):

\[
\begin{pmatrix}
c(\omega) \\
d(\omega)
\end{pmatrix} =
\begin{pmatrix}
r & t' \\
t & r'
\end{pmatrix}
\begin{pmatrix}
a(\omega) \\
b(\omega)
\end{pmatrix}, \quad a(\omega) \equiv \begin{pmatrix} a_H(\omega) \\ a_V(\omega) \end{pmatrix},
\]

and vectors \(b(\omega), c(\omega), d(\omega)\) defined similarly. The scattering amplitudes are frequency-independent.
The outgoing state $|\Psi_{\text{out}}\rangle$ can be conveniently written in a matrix notation

$$
|\Psi_{\text{out}}\rangle = \int d\omega \int d\omega' \psi^*(\omega)\phi^*(\omega') \left( c^\dagger(\omega) \right)^T \begin{pmatrix} r\sigma_{\text{in}}r'^T & r\sigma_{\text{in}}r'^T \\ t\sigma_{\text{in}}r'^T & t\sigma_{\text{in}}r'^T \end{pmatrix} \left( d^\dagger(\omega') \right) |0\rangle.
$$

(5)

FIG. 1: Schematic drawing of generation and detection of polarization-entanglement at a beam splitter. The independent sources SL and SR each create a photon in modes \{a\} and \{b\} cf. Eq. 1. The beam splitter with unitary $4 \times 4$ scattering matrix $S$ couples the polarization of incoming modes to the polarization of outgoing modes \{c\} and \{d\}. Polarizations are mixed by $R_L$, $R_R$ and absorbed by photodetectors DL,DR. A coincidence circuit C registers simultaneous detection of photons.

Here we used the unitarity of $S$ and $\sigma_{\text{in}} = (\sigma_x + i\sigma_y)/2$, with $\sigma_x$ and $\sigma_y$ Pauli matrices, corresponds to the polarizations of the incoming photons cf. Eq. 1. The matrix $\sigma_{\text{in}}$ has rank 1 reflecting the fact that polarizations are not entangled prior to scattering. Since we make no assumptions about the scattering amplitudes (apart from the unitarity of $S$), the choice of $\sigma_{\text{in}}$ is without loss of generality (see Appendix A).

The joint probability per unit (time)$^2$ of absorbing a photon with polarization $i$ at detector DL and a photon with polarization $j$ at detector DR at times $t$ and $t'$ respectively is given by

$$
w_{ij}(t, t') \propto \langle \Psi_{\text{out}} | E_{\text{IL}}(t) E_{\text{IR}}^{(-)}(t') E_{\text{IL}}^{(+)}(t') E_{\text{IR}}^{(+)}(t) | \Psi_{\text{out}} \rangle,
$$

(6)

where $E_{\text{IL}}(t)$ and $E_{\text{IR}}(t)$ are the positive frequency field operators of polarization $i$ at detectors DL and DR. The probability $C_{ij}(t)$ of a coincidence event within time-windows $\tau$ around $t$ is given by

$$
C_{ij}(t) = \int_{t-\tau}^{t+\tau} \int_{t-\tau}^{t+\tau} dt'' w_{ij}(t', t'').
$$

(7)

Experimentally, the time-window $\tau$ has typically a lower bound determined by the random rise time of an avalanche of charge carriers in response to a photon absorption event.

The polarization-entanglement is detected by violation of the Bell-CHSH inequality $\Box$. This requires two local polarization mixers $R_L$ and $R_R$. The Bell-CHSH parameter $\mathcal{E}$ is

$$
\mathcal{E} = |E(R_L, R_R) + E(R'_L, R_R) + E(R_L, R'_R) - E(R'_L, R'_R)|,
$$

(8)

where $E(R_L, R_R)$ is related to the correlators $C_{ij}(R_L, R_R)$ by

$$
E = \frac{C_{HH} + C_{VV} - C_{HV} - C_{VH}}{C_{HH} + C_{VV} + C_{HV} + C_{VH}}.
$$

(9)

Substituting the correlators of Eq. (7) into Eq. (10), we see that

$$
E(R_L, R_R) = \text{Tr} \rho (R_L^\dagger \sigma_z R_L) \otimes (R_R^\dagger \sigma_z R_R),
$$

(10)

where $\sigma_z$ is a Pauli matrix and $\rho$ a $4 \times 4$ polarization density matrix with elements

$$
\rho_{ij, mn} = \frac{1}{N} \left( (1 + |\alpha|^2)(\gamma_1)_{ij}(\gamma_1)^*_{mn} + (1 - |\alpha|^2)(\gamma_2)_{ij}(\gamma_2)^*_{mn} \right).
$$

(11)

The parameter $\alpha$ is given by

$$
\alpha = \frac{\int_{t-\tau}^{t+\tau} \int_{t-\tau}^{t+\tau} dt' \int d\omega \int d\omega' \phi(\omega)\phi^*(\omega')e^{i(t'-t')} \left( \int_{t-\tau}^{t+\tau} \int_{t-\tau}^{t+\tau} dt' \int d\omega \int d\omega' \phi(\omega)\phi^*(\omega')e^{i(t'-t')} \right)^{1/2}}{\left( \int_{t-\tau}^{t+\tau} \int_{t-\tau}^{t+\tau} dt' \int d\omega \int d\omega' \phi(\omega)\phi^*(\omega')e^{i(t'-t')} \right) \left( \int_{t-\tau}^{t+\tau} \int_{t-\tau}^{t+\tau} dt' \int d\omega \int d\omega' \phi(\omega)\phi^*(\omega')e^{i(t'-t')} \right)^{1/2}}.
$$

(12)

and $\gamma_1, \gamma_2$ are $2 \times 2$ matrices related to the scattering amplitudes by

$$
\gamma_1 = r\sigma_{\text{in}}r'^T + t'\sigma_{\text{in}}r'^T, \quad \gamma_2 = r\sigma_{\text{in}}r'^T - t'\sigma_{\text{in}}r'^T.
$$

(13)
The normalization factor $\mathcal{N}$ takes the form
\[ \mathcal{N} = (1 + |\alpha|^2) \text{Tr} \gamma_1 \gamma_1 + (1 - |\alpha|^2) \text{Tr} \gamma_2 \gamma_2. \]
(14)

The parameter $1 - |\alpha|^2 \in (0, 1)$ represents the amount of temporal which-path information. Generally, the time-window $\tau$ is much larger than the coherence times or temporal difference of the wavepackets. We may then take the limit $\tau \to \infty$ and $\alpha$ reduces to the overlap of wavepackets
\[ \alpha = \int d\omega \phi(\omega) \psi^*(\omega). \]
(15)

In the opposite limit of ultra-coincidence detection where $\tau \to 0$, temporal which-path information is completely erased corresponding to $|\alpha|^2 = 1$.

## III. ENTANGLEMENT OF FORMATION

The entanglement of formation of the mixed state $\rho$ is quantified by the concurrence $C$ given by
\[ C = \max (0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}). \]
(16)

The non-Hermitian matrix $\tilde{\rho}$ has eigenvalue-eigenvector decomposition
\[ \tilde{\rho} = \frac{\text{Tr} \gamma \gamma_1}{\mathcal{N}^2} \left( \sum_{i=1,2} \gamma_i s_i \right) \left( (1 + |\alpha|^2)^2 s_1 s_1 + (1 - |\alpha|^2)^2 s_2 s_2 \right) \left( \sum_{i=1,2} \gamma_i s_i \right)^{-1}, \]
(20)

where we have defined orthonormal states $s_1 = (1/2)(1 + \sigma_x)$ and $s_2 = (1/2)(\sigma_x + i\sigma_y)$. The pseudo-inverse is easily seen to be
\[ \left( \sum_{i=1,2} \gamma_i s_i \right)^{-1} = \frac{1}{\text{Tr} \gamma_1 \gamma_1^*} (s_1 \gamma_1^* - s_2 \gamma_2). \]
(21)

It follows that
\[ C = \frac{2|\alpha|^2 |\text{Tr} \gamma_1 \gamma_1^*|}{\mathcal{N}}. \]
(22)

The trace that appears in the numerator of Eq. (22) is given by
\[ |\text{Tr} \gamma_1 \gamma_1^*| = 2 \sqrt{\text{Det} X \text{Det}(\mathbb{I} - X^\dagger X)}, \]
(23)

where we have defined a “hybrid” $2 \times 2$ matrix $X$ as
\[ X = \begin{pmatrix} r_{HH} & t_{HV}^r \\ r_{VH} & t_{VV}^r \end{pmatrix}. \]
(24)

The normalization factor $\mathcal{N}$ given by Eq. (14) can be expressed in terms of $X$ using
\[ \text{Tr} \gamma_1 \gamma_1 = \text{Tr} X^\dagger X - 2 \text{Per} X^\dagger X, \]
(25)

The $\lambda_i$’s are the eigenvalues of the matrix product $\tilde{\rho}$, where $\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^*(\sigma_y \otimes \sigma_y)$, in the order $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$. The concurrence ranges from 0 (no entanglement) to 1 (maximal entanglement). For simplicity of notation it is convenient to define (xy)ij,mn $\equiv x_i y_m^*$. The matrix $\tilde{\rho}$ can be written as
\[ \tilde{\rho} = \frac{1}{\mathcal{N}} \left( (1 + |\alpha|^2)^2 \gamma_1 \gamma_2^* + (1 - |\alpha|^2)^2 \gamma_2 \gamma_2^* \right), \]
(17)

with $\gamma \equiv \sigma_y \gamma^* \sigma_y$. The product $\tilde{\rho} \tilde{\rho}$ takes the simple form
\[ \tilde{\rho} \tilde{\rho} = \frac{\text{Tr} \gamma_1 \gamma_1^*}{\mathcal{N}^2} \left( (1 + |\alpha|^2)^2 \gamma_1 \gamma_1^* - (1 - |\alpha|^2)^2 \gamma_2 \gamma_2^* \right), \]
(18)

where we have used the multiplication rule $xy^* = (\text{Tr} y^v \lambda \tilde{w})$ and
\[ \text{Tr} \gamma_1 \gamma_1^* = -\text{Tr} \gamma_2 \gamma_2^*, \quad \text{Tr} \gamma_2 \gamma_1^* = \text{Tr} \gamma_2 \gamma_1^* = 0. \]
(19)

The results for the tilde inner products of Eq. (19) hold since the photons are not polarization-entangled prior to scattering ($\text{Det} \sigma_m = 0$).

\[ \text{Tr} \gamma_1 \gamma_1^* = \text{Tr} X^\dagger X - 2 \text{Det} X^\dagger X, \]
(26)

(“Per” denotes the permanent of a matrix.) In the derivation of Eqs. (24-25) we have made use of the unitarity of $S$. The concurrence becomes
\[ C = \frac{2|\alpha|^2 \sqrt{\text{Det} X \text{Det}(\mathbb{I} - X^\dagger X)}}{\text{Tr} X \text{Tr}(\mathbb{I} - X^\dagger X) - (1 + |\alpha|^2) \text{Per} X \text{Tr}(\mathbb{I} - X^\dagger X) - (1 - |\alpha|^2) \text{Det} X^\dagger X}. \]
(27)

Entanglement depends on the amount of temporal indistinguishability $|\alpha|^2$ and the Hermitian matrix
\[ X^\dagger X = \begin{pmatrix} |r_H|^2 & r_H \cdot t_V^r \\ r_H \cdot t_V^r & |t_V|^2 \end{pmatrix}, \]
(28)

containing the states $r_H = (r_{HH}, r_{VH})$ and $t_V = (t_{HV}, t_{VV})$ of a reflected and transmitted photon to the left of the beam splitter. The determinant of $X^\dagger X$ measures the size of the span of $r_H$ and $t_V$ as
\[ \text{Det} X^\dagger X = |r_H|^2 |t_V|^2 \left( 1 - \frac{|r_H \cdot t_V^r|^2}{|r_H|^2 |t_V|^2} \right). \]
(29)
If $r_H$ and $r_V'$ are parallel ($\det X^\dagger X = 0$), a scattered photon to the left of the beam splitter is in a definite state, giving rise to an unentangled two-photon state ($C = 0$). Similarly,

$$\det(1 - X^\dagger X) = |t_H|^2 |r_V'|^2 \left(1 - \frac{|t_H \cdot r_V'|^2}{|t_H|^2 |r_V'|^2}\right)$$

involves scattered states $t_H = (t_{HH}, t_{HV})$ and $r_V' = (r_{HV}', r_{VV}')$ to the right of the beam splitter. The denominator of Eq. (27) is the probability of finding a scattered state with one photon on either side of the beam splitter. It deviates from its classical value $(X^\dagger X)_{HH} + (X^\dagger X)_{VV} - 2(X^\dagger X)_{HH}(X^\dagger X)_{VV}$ by an amount $-2|\alpha|^2|X^\dagger X|_{HV}|^2$ due to photon bunching. This reduction of coincidence count probability is the Mandel dip [17]. It measures the indistinguishability of a reflected and transmitted photon as the product of temporal indistinguishability $|\alpha|^2$ and polarization indistinguishability $|(X^\dagger X)_{HV}|^2$.

IV. VIOLATION OF THE BELL-CHSH INEQUALITY

The maximal value $\varepsilon_{\text{max}}$ of the Bell-CHSH parameter [8] for an arbitrary mixed state was analyzed in Refs. [18, 19]. For a pure state with concurrence $C$ one has simply $\varepsilon_{\text{max}} = 2\sqrt{1 + C^2}$ [20]. For a mixed state there is no one-to-one relation between $C$ and $\varepsilon_{\text{max}}$. Depending on the density matrix, $\varepsilon_{\text{max}}$ can take on values between $2\sqrt{C}$ and $2\sqrt{1 + C^2}$. The dependence of $\varepsilon_{\text{max}}$ on $\rho$ involves the two largest eigenvalues of the real symmetric $3 \times 3$ matrix $R^T R$ constructed from $R_{kl} = \Tr \rho \sigma_k \otimes \sigma_l$, where $\sigma_1 = \sigma_x, \sigma_2 = \sigma_y$, and $\sigma_3 = \sigma_z$. In terms of $\gamma_1$ and $\gamma_2$, the elements $R_{kl}$ take the form

$$R_{kl} = \frac{(1 + |\alpha|^2)}{N}\Tr \gamma_1^* \sigma_k \sigma_l^T + \frac{(1 - |\alpha|^2)}{N}\Tr \gamma_2^* \sigma_k \sigma_l^T.$$  

The matrix $\gamma_2$ has a polar decomposition $\gamma_2 = U \sqrt{\xi} V$ where $U$ and $V$ are unitary matrices and $\xi$ is a diagonal matrix holding the eigenvalues of $\gamma_2^* \gamma_2$. The real positive $\xi$’s are determined by

$$\xi_1 + \xi_2 = \Tr \gamma_1^* \gamma_2, \quad 2\sqrt{\xi_1 \xi_2} = |\Tr \gamma_2^* \gamma_2^*|.$$  

The matrix $\gamma_1$ can be conveniently expressed as (see Appendix B)

$$\gamma_1 = U Q \sqrt{\xi} V, \quad \text{where} \quad Q = \begin{pmatrix} c_1 & c_2 \\ c_3 & -c_1 \end{pmatrix}.$$  

The parameters $c_1, c_2, c_3$ are real numbers. The matrix $Q$ is traceless due to the orthogonality of $\gamma_1$ and $\gamma_2$. The number $c_1 \in (-1, 1)$ on the diagonal is related to the inner product of $\gamma_1$ and $\gamma_2$ and takes the form

$$c_1 = \frac{\Tr \gamma_1^* \gamma_2}{\xi_1 - \xi_2}, \quad \text{with} \quad \Tr \gamma_1^* \gamma_2 = \Tr \sigma_2 X^\dagger X.$$  

The numbers $c_2, c_3$ are determined by the norm and tilde inner product of $\gamma_1$ and satisfy the relations

$$c_2^2 + 2c_3 = 1, \quad c_2^2 (\xi_1 + \xi_2) + c_3^2 \xi_1 + c_3^2 \xi_2 = \Tr \gamma_1^* \gamma_1.$$  

We substitute $\gamma_1$ of Eq. (33) and the polar decomposition of $\gamma_2$ in Eq. (31) and parameterize

$$U^\dagger \sigma_k U = \sum_{i=1}^{3} N_{ki} \sigma_i, \quad V \sigma_i^T V^\dagger = \sum_{i=1}^{3} M_{ki} \sigma_i^T,$$

in terms of two $3 \times 3$ orthogonal matrices $N$ and $M$. The matrix $R$ takes the form

$$R = NR^T M^T,$$

where $R'$ is given by Eq. (31) with substitutions $R \rightarrow R'$, $\gamma_2 \rightarrow \sqrt{\xi} U$ and $\gamma_1 \rightarrow Q \sqrt{\xi}$. With the help of Eqs. (29, 34, 35), the eigenvalues $u_i$ of $R^T R$ can now be expressed as (see Appendix C)

$$u_1 = \frac{1}{2N^2} \left( T + \sqrt{T^2 - 4D} \right),$$

$$u_2 = \frac{1}{2N^2} \left( T - \sqrt{T^2 - 4D} \right),$$

$$u_3 = \frac{4|\alpha|^4 \Tr \gamma_1^* \gamma_1^2}{N^2},$$

where

$$T = N^2 + 4|\Tr \gamma_1^* \gamma_1^2| - 4(1 - |\alpha|^4) \left( \Tr \gamma_1^* \gamma_1 \Tr \gamma_2^* \gamma_2 - \Tr \gamma_1^* \gamma_2 \gamma_2^* \gamma_1 \right),$$

$$D = 4|\Tr \gamma_1^* \gamma_1^2| \left( N^2 - 4(1 - |\alpha|^4) \Tr \gamma_1^* \gamma_1 \Tr \gamma_2^* \gamma_2 \right).$$

We can relate the $u_i$’s to $X^\dagger X$ and $|\alpha|^2$ using Eqs. (31, 34, 35). The parameter $\varepsilon_{\text{max}}$ depends on the
we consider the special case \((X^\dagger X)_{ii} = 1/2\). The concurrence of Eq. (27) reduces to

\[ C = \frac{|\alpha|^2 (1 - 4|X^\dagger X_{HV}|^2)}{1 - 4|\alpha|^2|X^\dagger X_{HV}|^2}. \]  

(46)

To find \(E_{\text{max}}\) we have to consider the ordering of \(u_2\) and \(u_3\) which depends on \(|(X^\dagger X_{HV}|^2\) and \(|\alpha|^2\). The function

\[ f(|\alpha|^2) = \frac{|\alpha|^2}{2(1 + |\alpha|^2)} \]  

(47)

divides parameter space in the region \(|(X^\dagger X_{HV}|^2 \leq f\) where \(E_{\text{max}} = 2\sqrt{u_1 + u_3}\) and the region \(|(X^\dagger X_{HV}|^2 > f\) where \(E_{\text{max}} = 2\sqrt{u_1 + u_2}\). The equation \(E_{\text{max}} = 2\) has a solution \(g(|\alpha|^2)\) for \(|(X^\dagger X_{HV}|^2\) that lies in the region \(|(X^\dagger X_{HV}|^2 \leq f\). The function \(g\) takes the form

\[ g(|\alpha|^2) = \frac{1}{4} \left( 1 - |\alpha|^2 + |\alpha|^4 - (1 - |\alpha|^2)\sqrt{1 + |\alpha|^4} \right) \]  

(48)

and breaks parameter space in two fundamental regions: a region \(|(X^\dagger X_{HV}|^2 < f\) where \(E_{\text{max}} > 2\) and a region \(|(X^\dagger X_{HV}|^2 > g\) where \(E_{\text{max}} < 2\). We have drawn these regions in Fig. 2. The maximal value of the Bell-CHSH parameter is given by

\[ E_{\text{max}} = 2C|\alpha|^{-2} \sqrt{1 + |\alpha|^4} \]  

(49)

in the region \(|(X^\dagger X_{HV}|^2 \leq f\).

VI. CONCLUSIONS

In summary, we have calculated the amount of polarization-entanglement (concurrence \(C\)) and its witness (maximal value of the Bell-CHSH parameter \(E\)) induced by two-photon interference at a lossless beam splitter. The ability of \(E\) to witness entanglement \((E_{\text{max}} > 2)\) depends on the Mandel dip \(-2|\alpha|^2|X^\dagger X_{HV}|^2\). In the absence of a Mandel dip, \(C > 0\) implies \(E_{\text{max}} > 2\) cf. Eq. [45], whereas in its presence this is not necessarily true. In the latter case, as we have demonstrated in Sec. V with \((X^\dagger X)_{ii} = 1/2\), the witnessing ability of \(E\) depends on the individual contributions of temporal \(|\alpha|^2\) and polarization indistinguishability \(|(X^\dagger X_{HV}|^2\).

Our results can be applied to interference of other kinds of particles, getting entangled in some \(2 \otimes 2\) Hilbert space and being “marked” by an additional degree of freedom. However, determining the indistinguishability parameter \(|\alpha|^2\) requires careful analysis of the detection scheme. In case of fermions, the matrices \(\gamma_1\) and \(\gamma_2\) of Eq. [13] are to be interchanged. Systems without a time-reversal symmetry are captured by the analysis, as we did not make use of the symmetry of the scattering matrix.
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APPENDIX A: ARBITRARINESS OF TWO-PHOTON INPUT STATE

The unitary scattering matrix has a polar decomposition

\[ S = \left( \begin{array}{cc} K' & 0 \\ 0 & L' \end{array} \right) \left( \begin{array}{cc} \sqrt{T} & i\sqrt{T} \\ i\sqrt{\bar{T}} & \sqrt{\bar{T}} \end{array} \right) \left( \begin{array}{cc} K & 0 \\ 0 & L \end{array} \right), \]  

(A1)

where \( K', L', K, L \) are 2 \times 2 unitary matrices and \( T = \text{diag}(T_H, T_V) \) is a matrix of transmission eigenvalues \( T_H, T_V \in (0, 1) \). The outgoing state \( |\Psi_{\text{out}}\rangle \) is related to the 4 \times 4 matrix

\[ S \left( \begin{array}{cc} 0 & \sigma_{\text{in}} \\ 0 & 0 \end{array} \right) S^T \]  

(A2)

cf. Eq. (M). By group decomposition \( K = K_1 K_2 \) and \( L = L_1 L_2 \), \( |\Psi_{\text{out}}\rangle \) is easily seen to correspond to \( K_2 \sigma_{\text{in}} L_2^T \) scattered by \( S \) of Eq. (A1) with substitutions \( K \rightarrow K_1 \) and \( L \rightarrow L_1 \).

APPENDIX B: JOINT SEMI-POLAR DECOMPOSITION

The matrices \( \gamma_1 \) and \( \gamma_2 \) have a decomposition

\[ \gamma_1 = UAV, \quad \gamma_2 = U'\sqrt{\xi}V, \]  

(B1)

where \( U, V \) are unitary matrices and \( \xi \) is a diagonal matrix of eigenvalues of \( \gamma_2^\dagger \gamma_2 \). As we do not yet specify \( A \), such a joint decomposition always exists. In our case, the matrices \( \gamma_1 \) and \( \gamma_2 \) have the special properties

\[ \text{Tr} \gamma_1^\dagger \gamma_2 = -\text{Tr} \gamma_2^\dagger \gamma_1, \quad \text{Tr} \gamma_1^\dagger \gamma_2 = 0, \]  

(B2)

\[ |\text{Tr} \gamma_1^\dagger \gamma_1| = 2\sqrt{\text{Det} X^\dagger X \text{Det}(I - X^\dagger X)}, \]  

(B3)

\[ \text{Tr} \gamma_1^\dagger \gamma_1 = \text{Tr} X^\dagger X - 2\text{Per} \left( X^\dagger X \right), \]  

(B4)

\[ \text{Tr} \gamma_2^\dagger \gamma_2 = \text{Tr} X^\dagger X - 2\text{Det} \left( X^\dagger X \right), \]  

(B5)

\[ \text{Tr} \gamma_1^\dagger \gamma_2 = \text{Tr} \sigma_z X^\dagger X. \]  

(B6)

It is the purpose of this appendix to demonstrate that \( A = Q\sqrt{\xi} \) where \( Q \) is a real traceless matrix of Eq. (K) with \( c_1 \) given by Eq. (N) and \( c_2, c_3 \) satisfying Eq. (M).

The inner and tilde inner product of \( \gamma_1 \) and \( \gamma_2 \) take the form

\[ \text{Tr} \gamma_1^\dagger \gamma_2 = \text{Tr} A^\dagger \sqrt{\xi}, \]  

(B7)

\[ \text{Tr} \gamma_1^\dagger \gamma_2 = (\text{Det} UV)^{-2} \text{Tr} A^\dagger \sigma_y \sqrt{\xi} \sigma_y = 0. \]  

(B8)

(Here we have used the identity \( U \sigma_y U^T = \text{Det}^2 U \sigma_y \), valid for any 2 \times 2 unitary matrix \( U \).) The conditions of Eqs. (B7,B8) involve the diagonal elements of \( A \) as respectively

\[ \text{Tr} \gamma_1^\dagger \gamma_2 = (\text{Det} UV)^{-2} \left( \sqrt{\xi_1 A_{11}^*} + \sqrt{\xi_2} A_{22}^* \right) = 0. \]  

(B9)

\[ \text{Tr} \gamma_1^\dagger \gamma_2 = (\text{Det} UV)^{-2} \left( \sqrt{\xi_1} A_{11}^* + \sqrt{\xi_2} \right) = 0. \]  

(B10)

It follows that \( A_{11} = c_1^2 \sqrt{\xi_1} \) and \( A_{22} = -c_1 \sqrt{\xi_2} \) where \( c_1 \) is given by

\[ c_1 = \frac{\text{Tr} \gamma_1^\dagger \gamma_2}{\xi_1 - \xi_2}. \]  

(B11)

The number \( c_1 \) is real since \( \text{Tr} \gamma_1^\dagger \gamma_2 = \text{Tr} \sigma_z X^\dagger X \in \mathbb{R} \).

The determinant of \( A \) is fixed by \( \text{Tr} \gamma_1^\dagger \gamma_1 = -\text{Tr} \gamma_2^\dagger \gamma_2 \) implying

\[ \text{Det} A = -\sqrt{\xi_1 \xi_2}. \]  

(B12)

It follows that \( A_{12} = A_{12}' e^{i\phi} \) and \( A_{21} = A_{21}' e^{-i\phi} \) with real \( A_{12}', A_{21}' \). The numbers \( A_{12}', A_{21}' \) satisfy

\[ c_2^2 \sqrt{\xi_1 \xi_2} + A_{12}' A_{21}' = \sqrt{\xi_1 \xi_2}. \]  

(B13)

\[ c_2^2 (\xi_1 + \xi_2) + A_{12}^2 + A_{21}^2 = \text{Tr} \gamma_1^\dagger \gamma_1, \]  

(B14)

where Eq. (B11) comes from \( \text{Tr} A^\dagger A = \text{Tr} \gamma_1^\dagger \gamma_1 \). The undetermined phase \( \phi \) can be taken out,

\[ A = \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12}' \\ A_{21}' & A_{22} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix}, \]  

(B15)

and absorbed in the unitary matrices \( U \) and \( V \) by the transformations

\[ U \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix} \rightarrow U, \quad \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix} \rightarrow V. \]  

(B16)

(Note that these transformations also hold for \( \gamma_2 \) since \( \sqrt{\xi} \) commutes with a diagonal matrix of phase factors.)
The matrix $\mathcal{A}$ is related to $Q$ by $\mathcal{A} = Q \sqrt{\xi}$. It is now easily seen that the matrix $Q$ is real and traceless and takes the form of Eq. (33), with $c_1$ given by Eq. (34) and $c_2, c_3$ satisfying Eq. (35).

As a last step we perform a consistency check to demonstrate that Eqs. (B13, B14) have solutions for $\mathcal{A}_{12}$ and $\mathcal{A}_{21}$. The Hermitian matrix $X^\dagger X$ has an eigenvector-eigenvalue decomposition

$$X^\dagger X = W^\dagger \Lambda W. \quad \text{(B17)}$$

In terms of the eigenvalues $\Lambda_i \in (0, 1)$ and the unitary matrix $W$, the inner product of $\gamma_1$ and $\gamma_2$ and the $\xi_i$'s take the form

$$\text{Tr} \gamma_i^\dagger \gamma_2 = \Lambda_i (|W_{11}|^2 - |W_{12}|^2) + \Lambda_2 (|W_{21}|^2 - |W_{22}|^2), \quad \text{(B18)}$$

$$\xi_1 = \Lambda_1 (1 - \Lambda_2), \quad \xi_2 = \Lambda_2 (1 - \Lambda_1). \quad \text{(B19)}$$

It follows that $c_1 = \cos 2\eta$, where we have set $|W_{11}| = |W_{22}| = \cos \eta$ and $|W_{12}| = |W_{21}| = \sin \eta$. Eqs. (B13, B14) can be expressed as respectively

$$\mathcal{A}_{12}' \mathcal{A}_{21}' = \sin^2 2\eta \sqrt{\Lambda_1 \Lambda_2 (1 - \Lambda_1)(1 - \Lambda_2)}, \quad \text{(B20)}$$

$$\mathcal{A}_{12}^2 + \mathcal{A}_{21}^2 = \sin^2 2\eta (\Lambda_1 (1 - \Lambda_1) + \Lambda_2 (1 - \Lambda_2)) \quad \text{(B21)}$$

Since

$$2 \sqrt{\Lambda_1 \Lambda_2 (1 - \Lambda_1)(1 - \Lambda_2)} \leq \Lambda_1 (1 - \Lambda_1) + \Lambda_2 (1 - \Lambda_2) \quad \text{(B22)}$$

a family of solutions exists.

**APPENDIX C: EIGENVALUES OF $R^T R$**

The non-vanishing elements of $R'$ are given by

$$R'_{11} = \frac{2}{N} (1 - |\alpha|^2 - (1 + |\alpha|^2)(c_1^2 - c_2 c_3)) \sqrt{\xi_1 \xi_2}, \quad \text{(C1)}$$

$$R'_{13} = \frac{2}{N} (1 + |\alpha|^2) c_1 (c_2 \xi_2 + c_3 \xi_1), \quad \text{(C2)}$$

$$R'_{22} = \frac{2}{N} (-1 + |\alpha|^2 + (1 + |\alpha|^2)(c_1^2 + c_2 c_3)) \sqrt{\xi_1 \xi_2}, \quad \text{(C3)}$$

$$R'_{31} = \frac{2}{N} (1 + |\alpha|^2) c_1 (c_2 + c_3) \sqrt{\xi_1 \xi_2}, \quad \text{(C4)}$$

$$R'_{33} = \frac{2}{N} ((1 - |\alpha|^2) + (1 + |\alpha|^2) c_1^2) (\xi_1 + \xi_2) \quad \text{(C5)}$$

$$- \frac{1}{N} (1 + |\alpha|^2) (c_2^2 \xi_2 + c_3^2 \xi_1). \quad \text{(C6)}$$

The matrix $R'^T R'$ has eigenvalues

$$u_1 = \frac{1}{2N^2} (T + \sqrt{T^2 - 4D}), \quad \text{(C7)}$$

$$u_2 = \frac{1}{2N^2} (T - \sqrt{T^2 - 4D}), \quad \text{(C8)}$$

$$u_3 = R_{22}^2, \quad \text{(C9)}$$

where $T, D$ are the trace, determinant respectively of the $2 \times 2$ real symmetric matrix

$$N^2 \begin{pmatrix} R_{11}^2 + R_{31}^2 & R_{11} R_{13} + R_{31} R_{33} \\ R_{11} R_{13} + R_{31} R_{33} & R_{13}^2 + R_{33}^2 \end{pmatrix}. \quad \text{(C10)}$$

By making use of Eqs. (41, 46) $u_3, T, D$ can be simplified to yield the results of Eqs. (10, 11, 12) respectively.

[1] B. Yurke and D. Stoler, Phys. Rev. Lett. 68, 1251 (1992); Phys. Rev. A 46, 2229 (1992).
[2] Y. H. Shih and C. O. Alley, Phys. Rev. Lett. 61, 2921 (1988).
[3] Z. Y. Ou and L. Mandel, Phys. Rev. Lett. 61, 50 (1988).
[4] T. B. Pittman and J. D. Franson, Phys. Rev. Lett. 90, 240401 (2003).
[5] D. Fattal, K. Inoue, J. Vučković, C. Santori, G. S. Solomon, and Y. Yamamoto, Phys. Rev. Lett. 92, 037903 (2004).
[6] M. Žukowski, A. Zeilinger, M. A. Horne, and A. K. Ekert, Phys. Rev. Lett. 71, 4287 (1993).
[7] J. -W. Pan, D. Bouwmeester, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. 80, 3891 (1998).
[8] J. -W. Pan, M. Daniell, S. Gasparoni, G. Weihs, and A. Zeilinger, Phys. Rev. Lett. 86, 4435 (2001).
[9] T. Jennewein, G. Weihs, J. -W. Pan, and A. Zeilinger, Phys. Rev. Lett. 88, 047903 (2002).
[10] E. Knill, R. Laflamme, and G. J. Milburn, Nature 409, 46 (2001).
[11] J. D. Franson, M. M. Donegan, M. J. Fitch, B. C. Jacobs, and T. B. Pittman, Phys. Rev. Lett. 89, 137901 (2002).
[12] R. P. Feynman, *Lectures on Physics* (Addison-Wesley, 1969), Vol. 3.
[13] S. Bose and D. Home, Phys. Rev. Lett. 88, 050401 (2002).
[14] R. J. Glauber, Phys. Rev. 130, 2529 (1963); R. J. Glauber, Phys. Rev. 131, 2766 (1963).
[15] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. 23, 880 (1969).
[16] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[17] C. K. Hong, Z. Y. Ou, and L. Mandel, Phys. Rev. Lett. 59, 2044 (1987).
[18] R. Horodecki, P. Horodecki, and M. Horodecki, Phys. Lett. A 200, 340 (1995).
[19] F. Verstraete and M. M. Wolf, Phys. Rev. Lett. 89, 170401 (2002).
[20] N. Gisin, Phys. Lett. A 154, 201 (1991).