A dynamical analysis of the Kepler-11 planetary system

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ABSTRACT

The Kepler-11 planetary system hosts at least six transiting super-Earth planets detected through the precise photometric observations of the Kepler mission (Lissauer et al.). In this paper, we re-analyse the available Kepler data, using the direct N-body approach rather than an indirect transit timing variation method as employed in the discovery paper. The orbital modelling in the realm of the direct approach relies on the whole data set, not only on the mid-transits times. Most of the results in the original paper are confirmed and extended. We constrained the mass of the outermost planet g to less than 30 M⊕. The mutual inclinations between orbits b and c as well as between orbits d and e are determined with a good precision, in the range of [1°, 5°]. Having several solutions to the four qualitative orbital models of the Kepler-11 system, we analyse its global dynamics with the help of dynamical maps. They reveal a sophisticated structure of the phase space, with narrow regions of regular motion. The dynamics are governed by a dense net of three- and four-body mean motion resonances, forming the Arnold web. Overlapping of these resonances is a main source of instability. We found that the Kepler-11 system may be long-term stable only in particular multiple resonant configurations with small relative inclinations. The mass–radius data derived for all companions reveal a clear anticorrelation between the mean density of the planets and their distance from the star. This may reflect the formation and early evolution history of the system.

Key words: methods: data analysis – methods: numerical – celestial mechanics – planets and satellites: dynamical evolution and stability – stars: individual: Kepler-11.

1 INTRODUCTION

The Kepler space mission is a breakthrough in the field of searches for the Earth-like extrasolar planets (Borucki et al. 2010; Caldwell et al. 2010; Jenkins et al. 2010; Koch et al. 2010). About 150 000 solar dwarfs are monitored by the 0.95-metre Kepler telescope. The photometric data are publicly available from the MAST archive.†

To date, the mission identified more that 2200 planetary candidates (Batalha et al. 2012). Among them, many multoplanet systems were found. For instance, planets were confirmed in two-planet configurations, for example, Kepler-10 (Batalha et al. 2011; Fressin et al. 2011a), Kepler-25, Kepler-26, Kepler-27 and Kepler-28 (Steffen et al. 2012), Kepler-29, Kepler-31 and Kepler-32 (Fabrycky et al. 2012), and Kepler-23 and Kepler-24 (Ford et al. 2012); in three-planet systems, for example, Kepler-9 (Holman et al. 2010), Kepler-30 (Fabrycky et al. 2012) and Kepler-18 (Cochran et al. 2011); in four-planet systems (Borucki et al. 2011); in five-planet configurations, for example, Kepler-20 (Fressin et al. 2011b; Gautier et al. 2011) and Kepler-33 (Lissauer et al. 2012). Kepler-11 hosts six planetary companions (Lissauer et al. 2011). The transiting planet candidates can be confirmed through determining their masses with the help of the so-called transit timing variation (TTV, Agol et al. 2005; Holman & Murray 2005) method. In this approach, the (O–C) variations between observed mid-transit times and their ephemeris are the observables, which can be fitted by an appropriate orbital model. In recent papers, additional observables were also analysed, like the so-called transit duration variations (see e.g. Nesvorný et al. 2012).

In this paper, we re-analyse the photometric data of Kepler-11 with a modified, direct approach providing an alternative method for the estimation of masses and orbital elements. Before discussing this method in detail, we recall the main conclusions in Lissauer et al. (2011). Using the TTV method and an assumption of a strictly coplanar model of the system, they determined the masses of five inner planets in the range of a few M⊕. The outermost planet interacts weakly with the inner companions, and its mass could be roughly constrained as smaller than that of Jupiter. It has not been confirmed as a planet, although the probability of blending is very small, ~0.001. Orbital eccentricities in the Kepler-11 system were roughly constrained as smaller than that of Jupiter. It has not been confirmed as a planet, although the probability of blending is very small, ~0.001. Orbital eccentricities in the Kepler-11 system were determined only for the five inner objects. Due to the assumption of coplanarity, the determination of mutual inclinations between the orbits was not possible. Lissauer et al. (2011) argue that these inclinations should remain in the range of [0°, 2°]. The dynamical
analysis has revealed that the system is not involved in mean motion resonances (MMRs); however, a pair of planet b and planet c is close to a 5:4 MMR.

The determined masses and radii of the planets imply constraints on their chemical composition. Planets d, e and f might have similar internal compositions to that of Uranus or Neptune, while planets b and c are rather ice-rich, with a smaller amount of H₂/He mixture than present in Uranus or Neptune.

In this paper, we focus mostly on the global dynamics of the system and a few aspects which were not addressed in the discovery paper.

First of all, we model the available Kepler data through a direct algorithm that relies on the self-consistent N-body fitting of the light curves, instead of the TTV method applied in the discovery work. The TTV algorithm makes use of the transit times a posteriori, after they are determined from the light curves. Through extensive numerical experiments, we found that the direct approach brings more information than the TTV method. For instance, we could constrain the mass of the outermost planet to less than ~30 M⊕. We also found significant bounds for the mutual inclinations to less than 5° for planets b and c as well as for planets d and e.

The direct model, also called the dynamical photometric model, was already used in a few papers. For instance, it was applied to analyse the light curve of the triple-star system KOI-126 (Carter et al. 2011), and to estimate the masses of the two planets transiting Kepler-36 (Carter et al. 2012). This algorithm also verified the Kepler-9 model, which was first found with the help of the TTV algorithm (Holman et al. 2010).

A number of initial conditions found with the direct approach make it possible to investigate the dynamics of the system. We focus on the short-term time-scale, governed by the MMRs. We study the multidimensional structure of the phase space with the help of dynamical maps. In the vicinity of a few qualitative transit models considered in this work, the dynamics are governed by a dense net of three- and four-body MMRs. This net may be identified with the Arnold web, which is a feature of close-to-integrable Hamiltonian systems. Kepler-11 appears as a strongly resonant extrasolar system, and this feature may reflect its trapping into MMRs at the early stages of the formation and evolution.

Using a new method for the determination of the masses and radii, we found a curious mass–radius relation implying a clear anticorrelation between the mean density of the planets and their distances from the star. Their densities exhibit a sequence of planets: planet b, which is denser than Neptune, through Neptune-like planet c, Uranus-like planet d, Jupiter-like planets e and f, to planet g, which is likely Jupiter/Saturn like.

This paper is structured as follows. In Section 2, we shortly describe the photometric data of Kepler-11 available in the MAST archive. We also refine the observational TTV model. In Section 3, we present the results derived through intensive computations with the bootstrap algorithm. Furthermore, we discuss a possible composition of the planets (Section 4). Section 5 is devoted to the dynamical analysis of the Kepler-11 system. Conclusions and prospects for a future work are given at the end of this paper.

2 TRANSITS IN A MULTIPLANET SYSTEM

The photometric data of Kepler-11 were taken from the MAST archive. At the time of writing this paper, the publicly available light curves span about 500 d in six parts. These data were binned on ~30 minute intervals. We analysed a ‘de-trended’ data set derived through a smoothing procedure. At first, we isolate all transits from the light curve. Then the moving average with a time-step of 0.5 d provides the mean level of the flux. Next, we construct an interpolated, reference light curve with the cubic spline on these nodes. Finally, we divide the raw flux, with all transit data, by its values of the reference, mean level flux curve.

The de-trended data available in the MAST data base exhibit a growth of the flux shortly before and after a particular transit. In some parts of the available light curves, spanning approximately 300 d, the measurements appear in a raw form. We did not use these data, aiming to analyse a possibly uniform set of observations.

2.1 Modelling the stellar flux

A common model of photometric observations of a star transited by planetary companions consists of two major parts. The first part concerns the flux deficit due to small, dark objects passing in front of the star. At first, the average orbital periods are determined. Then transit depths and duration times are parametrized on the basis of phase-folded light curves. Single mid-transit times are also determined. At the second level, we can estimate the planetary masses and orbital elements fitting a model of motion of mutually interacting planets.

We focus on the first level of the photometric analysis. To compute the flux deficit, we use the quadratic limb-darkening model (Mandel & Agol 2002), recalling that the Kepler-11 light curves are relatively noisy and sampled with a low frequency,

\[ \Delta F(r) = 1 - \gamma_1 (1 - \cos \theta) - \gamma_2 (1 - \cos \theta)^2, \]

where \( r \) is the normalized radial coordinate with respect to the centre of the stellar disc and \( \theta \) is the angle between the direction to the observer and the normal to the stellar surface. The two limb-darkening coefficients \( \gamma_1 \) and \( \gamma_2 \) must be positive and \( \gamma_1 + \gamma_2 < 1 \) [see Howarth (2011) for a study of the limb-darkening coefficients for a few target stars of the Kepler mission]. For a small ratio of the planet radius \( R_p \) to the stellar radius \( R_\ast \), \( p = R_p/R_\ast \), Mandel & Agol (2002) found an analytic approximation of the flux deficit, \( \Delta F = \Delta F(\zeta; p, \gamma_1, \gamma_2) \), which depends on the normalized distance \( \zeta \) between the centres of stellar and planetary discs, projected on to the sky plane (see equation 8 in the cited paper), as well as on \( p \) and \( \gamma_1, \gamma_2 \).

If more than one planet transits the star at the same time, the total flux deficit can be computed as the sum of the deficits caused by particular planets. Obviously, \( \gamma_1 \) and \( \gamma_2 \) are the same for all planets, while \( p \) and \( \zeta \) are different for each object. If transiting planets are small, we can use a simple model of independent transits rather than a more general treatment (e.g. Pal 2011). Because we model the photometric measurements directly, by reconstructing the whole light curve, we are not restricted to single transits and mid-transit times. Also multiple transits can be covered. In light of a relatively narrow observational window, multiple transits are very helpful to constrain orbital elements of the transit model.

Fig. 1 displays a few selected fragments of the data set marked with the red dots and error bars which are overplotted on the synthetic curve best fitting the data (blue curve). The fitting procedure will be described in more detail in Section 2.3. The last panel shows transits of three planets (b, d and e).

2.2 The model of orbital motion

The orbital motion of multiple planetary systems is described in terms of the full N-body problem in the Poincaré reference frame (e.g. Morbidelli 2002). In this frame, the Cartesian coordinates of the
planets are astrocentric, while their velocities are barycentric. The equations of motion are integrated with the second-order symplectic integrator SABA2 (Wisdom & Holman 1991; Laskar & Robutel 2001). It provides two to three times better CPU performance than other algorithms, which we tested (like the Bulirsh–Stoer–Gragg scheme, hereafter BGS scheme) constrained with the same time-step accuracy. To speed up the computations even more, we did not integrate the system at all measurement moments. This would force an ~30 minute step-size of the integrator. Instead, we fixed this step-size to \( \Delta t \sim 1/20 \) of the innermost period of planet b, that is, \( \Delta t \approx 0.5 \) d. Furthermore, the flux function \( F(t) \) is computed only close to the mid-transits. Ingress and egress times of particular events are tabulated. When a transit takes place, the coordinates of a particular planet at time \( t \) required to evaluate the flux deficit are determined through the polynomial interpolation on five nodes around \( t \). Through a comparison with the direct, full-accuracy integrations with the BGS algorithm, we found that the selected time-step and the number of interpolation nodes provide a sufficient precision and an acceptable CPU overhead. We examined this method by changing the number of nodes in the polynomial interpolation, as well as the time-step-size. The flux level, interpolated on five nodes and with \( \Delta t \approx 0.5 \) d, differs from its exact value by less than \( 10^{-9} \).

### 2.3 Optimization algorithm and error estimation

We searched for the best-fitting model of the transits by a common minimization of the \( \chi^2 \) function. This function is defined as follows:

\[
\chi^2 = \frac{1}{N_{\text{obs}} - N_p - 1} \sum_{j=1}^{N_{\text{obs}}} \frac{1}{\sigma_j^2} [F_j - F(t_j)]^2,
\]

where \( N_{\text{obs}} \) is the number of observations, \( N_p \) is the number of free parameters, \( \nu = N_{\text{obs}} - N_p - 1 \) is the number of the degrees of freedom, \( \sigma_j \) is the error of the \( j \)th observation \( F_j \), and \( F(t_j) \) is a model function evaluated at time \( t_j \). This form of the \( \chi^2 \) function is correct if the uncertainties are uncorrelated (see e.g. Baluciev 2009). To verify whether the available photometric data fulfill this assumption, one has to use a more general statistical model incorporating the red-noise effect. However, under particular settings of our \( N \)-body photometric model, this would require an enormous CPU overhead. Hence, we use equation (2) as a reasonable first-order approximation.

The best-fitting parameters of the transit model are searched through a two-step optimization method. In the first step, we apply a robust and well-tested quasi-global genetic algorithm (GA, see Charbonneau 1995; Deb 2004) which makes it possible to find promising solutions. A local, fast gradient method (the Levenberg–Marquardt algorithm) is then used to refine the solutions found in the GA step. Such an approach is called hybrid optimization (see Gożdziewski, Migaszewski & Musieliski 2008b, and references therein). Let us note that the parameter space is huge as it has a dimension of 50. Some of these parameters can be determined very well, like the orbital periods of the transiting planets. Unfortunately, due to the relatively short observational time-window, many parameters that are critical for the stability (relative inclinations, masses, nodal lines) cannot be well constrained. It makes the fitting process a challenging problem.

The parameter errors are estimated through the bootstrap algorithm (see e.g. Press et al. 1992). The bootstrap method is CPU demanding, but it is a straightforward method to estimate standard errors in high-dimensional problems and for a large number of data. The light curves that we analysed have \( \sim 22,000 \) points. The bootstrap algorithm requires to find the best-fitting solutions to a large number of synthetic sets derived through random sampling with replacement from the original measurements. To obtain reliable error estimates of the best-fitting parameters, one needs at least \( \sim 10^3 \)–\( 10^4 \) synthetic solutions. When such a large set of best-fitting models were gathered, we constructed normalized histograms for each free parameter. These histograms reflect the parameter distribution in response to the errors of the measurements, and may be smoothly approximated by an asymmetric Gaussian function. This makes it possible to determine the standard uncertainties. To perform the bootstrap procedure, at first one needs to find reliable best-fitting parameters for the nominal data set. This step was performed through an intensive quasi-global search with the help of the hybrid algorithm. The bootstrap computations are CPU time-consuming and were performed on the ree CPU cluster of the Poznań Supercomputing Centre.

### 2.4 Numerical setup of the dynamical analysis

In spite of small eccentricities and apparently coplanar orbits, the Kepler-11 system is orbitally very active. It appears as a dynamically packed planetary system (the definition is given in Barnes, 2009). To verify whether the available photometric data fulfill this assumption, one has to use a more general statistical model incorporating the red-noise effect. However, under particular settings of our \( N \)-body photometric model, this would require an enormous CPU overhead. Hence, we use equation (2) as a reasonable first-order approximation.

2 We use publicly available implementation by Kalyanmoy Deb, see http://www.iitk.ac.in/kangal/pub.htm

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Goździewski & Raymond 2008), with only narrow stable zones in the phase space. For this reason, we used the best-fitting model solutions gathered in the bootstrap search as the input data to an extensive dynamical study of this system. As we will discuss later, studying the stability is a challenging task. Due to relatively short observational window ($\sim$500 d), weak transits having depths comparable with the measurements errors and a small number of data points covering particular transits (typically 10–15), the derived initial conditions may be shifted away from the real configurations.

To investigate the dynamics of the Kepler-11 system in a global manner, we applied an approach in our previous papers which is well established in the literature. It relies on reconstructing the structure of the phase space with the fast indicator the mean exponential growth factor of nearby orbits (MEGNO; Cincotta & Simó 2000; Cincotta, Giordano & Simó 2003). This dynamical characteristic makes it possible to distinguish between regular (stable) and irregular (chaotic, unstable) trajectories in the phase space by computing relatively short numerical orbits. Having representative solutions selected in the bootstrap statistics, we study their neighbourhood on the dynamical maps. Constructing a dynamical map relies on two model parameters, for example, the semi-major axes of a pair of planets. The selected parameters are varied in the given range at a discrete grid. The remaining components of the initial parameter vector are fixed at their nominal values. If it is necessary, they are altered to preserve the observational constraints. Then we calculate the MEGNO at each point of the grid. Dynamical maps are informative and become a standard numerical tool helpful to understand the global dynamics of multiple systems.

To compute the MEGNO indicator, we must solve the variational equations to the equations of motion of the planetary $N$-body problem. The Kepler-11 system architecture with low-eccentricity orbits and small masses is an ideal target for an efficient symplectic algorithm described in Goździewski (2003) and Goździewski et al. (2008a). The general-purpose integrators, like the Runge–Kutta or Bulirsh–Stoer–Gragg schemes, are neither efficient nor accurate enough in this case. These methods introduce a systematic drift of the energy and other integrals. To avoid such errors, and to solve the variational equations, we apply the tangent map introduced by Mikkola & Innanen (1999). As the very basic step, it requires to differentiate the 'drift' and 'kick' maps of the standard leapfrog algorithm. The variations may then be propagated within the same symplectic scheme, as the equations of motion. Having the variational vector $\delta$ computed at discrete times, we find the temporal $\gamma$ and mean $Y$ of the MEGNO at the $j$th integrator step, $j = 1, 2, \ldots$ (Cincotta et al. 2003; Goździewski et al. 2008a):

$$Y(j) = \frac{(j-1)Y(j-1) + y(j)}{j},$$

$$\gamma(j) = \frac{j-1}{j} \gamma(j-1) + 2 \ln \left( \frac{\delta_j}{\delta_{j-1}} \right)$$

with initial conditions $\gamma(0) = 0$, $Y(0) = 0$, $\delta = |\delta|$. The MEGNO maps tend asymptotically to

$$Y(j) = \alpha h j + b,$$

where $\alpha = 0$, $b \sim 2$ for quasi-periodic orbits, $\alpha = b = 0$ for stable, periodic orbits, and $\alpha = (1/2)\sigma$, $b = 0$ for chaotic orbits with the maximal Lyapunov exponent $\sigma$. The tangent MEGNO map is linear; hence, the variational vector can be normalized, if its value grows too large for chaotic orbits. In practice, we stop the integration if the MEGNO indicator reaches a given limit (usually, $Y = 5$).

The symplectic maps were propagated with the fourth-order SABA4 scheme in Laskar & Robutel (2001). A choice of the fixed step-size must be carefully controlled. We did this, checking whether the relative energy error is ‘flat’ across the dynamical map (Goździewski et al. 2008a) and sufficiently small. Indeed, the step-size $\sim 0.5$ d preserved this error at a level of $10^{-11}$ over the total integration times up to $T \sim 40{,}000$ yr ($\sim 100{,}000$ periods of the outermost planet). This time-scale is long enough to detect the most significant two- and three-body MMRs though even such an integration period may be insufficient to detect all ‘dangerous’ unstable resonances. Weakly chaotic motions due to multibody MMRs still may lead to catastrophic events after much longer time (Goździewski et al. 2008a).

The dynamical maps in this paper have a typical resolution up to $512 \times 512$ pixels. This requires an enormous CPU time. It is basically not possible to perform such intensive computations on a single workstation. Therefore, we used our new Message Passing Interface based environment MECHANIC (Slonina, Goździewski & Migaszewski 2012) to perform the computations in a reasonable time in CPU clusters. They were performed on the reef cluster at the Poznań Supercomputing Centre. A MECHANIC run of a typical dynamical map occupied up to 1200 CPU cores for $\sim 16$ h.

### 2.5 Free parameters of the transit model

The free parameters of the transit model are the stellar radius $R_\ast$, limb-darkening coefficients $y_1$, and $y_2$, mass $m_1$, radius $R_1$, and orbital elements of each planet in the system, where $i = b, c, d, e, f, g$. Planetary orbits are described through the Poincaré geometric, osculating elements at the epoch of the first observation JD 245 5964.511 28: a tuple $(a_1, e_1, I_1, \Omega_1, \omega_1, M_1)$ is for the semi-major axis, eccentricity, inclination to the plane of the sky, longitude of ascending nodes, argument of pericentre and mean anomaly, respectively. The orbital node of the first planet $\Omega_1 = 0^\circ$ due to the invariance of the model with respect to a rotation of the whole system. The inclinations are obviously close to $90^\circ$. A deviation from $90^\circ$ is irrelevant only for single-planet systems.

In a multi-planet system, some orbits may be inclined to the sky plane by an angle $\neq 90^\circ$, which implies different relative inclinations between orbits of particular planets, even for the same longitudes of nodes. Due to the invariance of transits with respect to the direction of the total angular momentum of the system, a combination of $(I_i \leq 90^\circ, \Omega_i)$ means the same geometry as $(|180^\circ - I_i | \geq 90^\circ, -\Omega_i)$. Thus, when necessary, for a given planet $i$ we can fix the range of $I_i \leq 90^\circ$ and $(I_i, \Omega_i)$ are corrected for the remaining companions, in accordance with the invariance relation.

Orbital elements $(a_i, e_i, \omega_i, M_i)$ are not fully suitable for transiting systems with small relative inclinations and small eccentricities. To avoid singularities and weakly constrained elements, like $\omega_i$ when $e_i = 0$ (circular or weakly eccentric orbits), we use the Poincaré modified elements $(X_i \equiv e_i \cos \omega_i, Y_i \equiv e_i \sin \omega_i)$ instead of $(e_i, \omega_i)$.

Similarly, the orbital period $P_i$ is more suitable for model fitting than $a_i$ since the semi-major axis depends on the planetary mass $m_i$ (a free parameter) and on the stellar mass $m_\ast$, which is fixed to 0.95 $M_\odot$, but it can be also fitted. Hence, we define $P_i$ as one of the osculating elements related to $a_i$ through the third Keplerian law.

The mean anomaly $M_i$ strongly depends on $\omega_i$. It determines the relative orbital phase (the mean longitude) but is also related to $e_i$. This may be avoided by choosing the time of the first transit $T_1$ as a

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free parameter instead of $\mathcal{M}_i$, because it is one of the directly determined observables from the light curves. Simple relations between $T_i$ and Poincaré canonical elements may be derived easily.

2.6 Direct and indirect transit model parameters

The direct parameters of the transit model are determined from the basic observables: it is the mean period of transits $P'_i$, depths, duration times of the transits, and shapes of the light curves (through the limb-darkening coefficients). These data are usually derived from the period-phased light curves of particular planets. The depth and duration of transits determine the ratio of planetary to stellar radii, $R_{p}/R_{\ast}$. If the stellar mass $M_{\ast}$ is fixed, then $R_{\ast}$ and $R_{p}$ may be resolved. We can also determine $I_\ast$ up to the angular momentum direction invariance, and $T_i$. In general, the mean period of transit events $P'_i$ is different from the osculating orbital period at the epoch of the first observation, $P_i$. The shape of the event-period-phased light curves makes it possible to fit the limb-darkening coefficients, $\gamma_1$ and $\gamma_2$.

These parameters of the transit model are independent of the $N$-planet dynamics. Hence, the remaining parameters are indirect parameters. To resolve them, a dynamical model of the orbital evolution is required. The indirect parameters consist of planetary masses $m_i$ as well as orbital elements, $e_i$, $\Omega_i$, $\omega_i$ and $P_i$ (instead of $P'_i$). Knowing $m_i$ and $P_i$, we may fix the osculating semi-major axis $a_i$ at the date of the first (or prescribed) observation.

We would like to note that the above distinction for two types of model parameters is somehow arbitrary in our photometric model. In our algorithm, both the direct and indirect parameters are fitted simultaneously, unlike, for instance, the TTV algorithm, in which the direct parameters are fitted at the first stage, and the indirect parameters are fitted in the next step.

Usually, the direct parameters can be estimated much more reliably than the indirect parameters. Even a potential derivation of the indirect parameters depends on the particular model of motion, that is, kinematic – Keplerian – or dynamic – Newtonian, and on the used method of modelling the observations. In the Keplerian (kinematic) model (see e.g. Agol et al. 2005), mid-transits of a given planet are governed by the geometric reflex motion of the star around the centre of mass in a subsystem composed of the star and all inner planets. For instance, transits of planet d are affected by planets b and c, but any outer planet does not affect transits of its inner companions. Hence, in accordance with the Keplerian model, the indirect parameters $(m_i, a_i, e_i, \Omega_i, \omega_i)$ of outermost planet g in the Kepler-11 system cannot be determined at all.

In a given pair of planets, the outer companion affects the transits times of the inner planet only through gravitational mutual perturbations which lead to changes of osculating orbital elements. To account for the mutual interactions, one has to apply the self-consistent N-body model of motion of the system.

Usually, to resolve the indirect parameters from photometric observations, the well-known TTV method is used (Agol et al. 2005). It consists of two steps. At first, we determine the mean periods, mid-transits, and then the $(O-C)$ residua, that is, differences between the measured and ephemeris transit times. The $(O-C)$ variations are observables in the second step during which we search for masses, eccentricities and arguments of pericentres of planetary companions. The TTV method in this form has a limitation, because it does not make any use of transit depths nor their duration times. If the individual inclinations of planets are different, the planets transit the parent star usually at different attitudes. Hence, the transit depths as well as duration times may vary, like the $(O-C)$ of mid-transits. This information can be used to better constrain the transit model. The mutual inclinations depend on the longitudes of ascending nodes in accordance with

$$\cos \Delta I_{i,j} = \cos I_i \cos I_j - \sin I_i \sin I_j \cos(\Omega_i - \Omega_j).$$

Because the inclinations of transiting planets, $(I_i, I_j)$, must be close to 90°, $\Delta I_{i,j} \approx |\Omega_i - \Omega_j|$. Within this approximation, the TTV method is apparently not sensitive for individual $\Omega_i$. In fact, different values of $\Omega_i$ imply different mutual inclinations affecting the dynamics and $(O-C)$. However, the dynamical variability of $(O-C)$ due to mutual interactions is weaker than the geometric variability due to changes of transit depths and duration times reflecting the motion of the star around the centre of mass of the system.

Overall, by direct modelling of the light curves (photometric measurements), rather than the mid-transit times, we can resolve $(O-C)$ with an improved precision. Modelling the light curves in terms of the $N$-body model is CPU demanding, but it makes it possible to estimate individual longitudes of nodes and mutual inclinations. In particular, as will be shown later, the direct method helped us to derive accurate relative inclinations between planets b and c, as well as between planets d and e of $\sim 2^\circ \pm 2^\circ$.

3 RESULTS OF THE BOOTSTRAP ANALYSIS

We performed the direct bootstrap TTV analysis of a few different orbital models of the Kepler-11 system. In the most general case (I), all parameters discussed in the previous section are the free parameters of the best-fitting model. Some of them are poorly constrained by the observations, in particular, the eccentricity of planet g and particular longitudes of nodes. Therefore, we also studied less general models, in which some of weakly constrained parameters are fixed. In the second model (II), $X_g = 0$, $Y_g = 0$, that is, $e_g = 0$. In the third model (III), also $\Omega_g = 0^\circ$, while in the last model (IV), $\Omega_{g}, \Omega_{d}, \Omega_{e}, \Omega_{c}$, $\Omega_{b}$ are all fixed at $0^\circ$. Because inclinations $I_i$ are not exactly equal to 90°, also $\Delta I_{i,j} \geq 0^\circ$.

For each of these four transit models, we applied the bootstrap algorithm and we gathered sets of $\sim 1500$ solutions for each instance of the transit model.

3.1 Model I: systems with $e_g \neq 0$

Fig. 2 shows an outcome of the bootstrap algorithm in the form of normalized histograms constructed for $X_g$, $Y_g$ and $\Omega_g$, and depicted from the left-hand to the right-hand panel, respectively. The red solid curves illustrate the best-fitting asymmetric Gaussian function to the histogram bins. The formal 1σ errors are marked with the red bars displayed above the histograms. The best-fitting parameters corresponding to the maximum of the Gaussian distribution are written in the respective panels, and they may be compared with the nominal solutions given in Table 1. The uncertainties of the eccentricity and longitude of nodes of planet g are relatively large. Because the nominal system is dynamically unstable, we examined the whole set of $\sim 1500$ bootstrap solutions by calculating their MEGNO indicator $(Y)$ on the time interval of $\sim 8000$ yr. It corresponds to $\sim 25000$ periods of the most distant companion. Such a characteristic time-scale should be long enough to detect unstable solutions due to low-order two- and three-body MMRs (Goździewski et al. 2008a, and references therein). Unfortunately, all initial configurations exhibit large values of $(Y)$, indicating that the system is strongly chaotic. The main source of instability are crossing orbits in the system that lead to disruptive events, that is,
one or more of the planets were ejected from the system or collided with the parent star. None of the tested solutions passed the direct integration over 10 Myr.

The parameter space of the Kepler-11 system is ~50 dimensional, and the dimension of the phase space of the N-body model is 36 dimensional.

The (Y) experiments indicate that this system can be locally chaotic and its phase space is filled with mostly unstable solutions. Then only small regions of stable MMRs may be present. In light of a large dimension of the phase space, the gathered statistics of best-fitting configurations are still very poor. We conclude that due to a short data span of only ~500 d, and unconstrained elements of the most general model, we cannot find reliably stable solutions assuming the most general transit model I. Unfortunately, in this high-dimensional problem, an alternative GAMP algorithm that relies on the optimization with imposed stability constraints (Goździewski et al. 2008b) would be CPU-time expensive.

3.2 Transit model II: systems with $e_g = 0$

In the next model, we narrow the most unconstrained parameters of the transit model. We fix the eccentricity $e_g = 0$; hence, $X_g$ and $Y_g$ are equal to 0. The results of the bootstrap algorithm are illustrated in Figs. 3–9. All panels in these figures are constructed in the same manner as Fig. 2. We tested whether the best-fitting parameters encompass at least marginally stable solutions with ($Y' \approx 2$ after $T = 16000$ yr. Fig. 3 shows the normalized histograms for masses of particular planets expressed in Earth masses. Besides formal uncertainties obtained through the bootstrap method (filled red circles), the best-fitting parameters derived in Lissauer et al. (2011) are plotted (blue filled circles). Clearly, these estimates coincide very well in both cases. There is one exception though, since the mass of planet g is not resolved in Lissauer et al. (2011). The direct code helps to resolve also this mass. It is constrained surprisingly well, in spite of a narrow observational window. This result confirms our predictions. Because the orbital model is constrained by all measurements, not the TTVs only, the direct algorithm makes use of dynamical information contained in the transit depths and widths.

For reference, the black and green asterisks in Fig. 3 mark the masses of Uranus and Neptune, respectively. The masses of planets b and f appear in a range specific for the super-Earths. They are significantly smaller than the masses of the two most distant planets b and f in our Solar system, but, as we will show in the next section, their chemical composition is likely similar to that of the ice giants in the Solar system. Fig. 4 shows histograms constructed for planetary radii expressed in units of Earth radii. These results confirm data in the discovery paper. Similarly to the previous plots, the radii of Uranus and Neptune are marked with asterisks. They are also labelled with $R_U$ and $R_N$, respectively. The derived radius of planet g confirms a hypothesis that it may belong to the Uranus/Neptune class. We note that most of the planets have radii smaller than $R_{U,N}$, and only planet e has a larger radius. Histograms of the mean densities are presented in Fig. 5. The x-axis is for the density expressed with respect to the Earth density. The black and green asterisks mark the values characteristic of Uranus and Neptune, respectively. The mean densities of Saturn and water are also marked with the red and blue symbols, respectively. According to this plot, the less dense planet e has the density of Saturn. The most dense planet b

Table 1. Bootstrap results for transit model I. The mass of the star is 0.95 $M_\odot$ (fixed). The best-fitting stellar parameters of this model are $R_\odot = 1.14^{+0.037}_{-0.037}$, $\gamma_1 = 0.32^{+0.30}_{-0.27}$, $\gamma_2 = 0.41^{+0.36}_{-0.24}$, and $\gamma_3 + \gamma_2 = 0.74^{+0.26}_{-0.23}$. The osculating Poincaré elements are given at the epoch of the first observation JD 245 5964.511 28.

| Parameter/planet | b | c | d | e | f | g |
|------------------|---|---|---|---|---|---|
| $m (M_\odot)$    | 4.2^{+2.4}_{-3.0} | 9.5^{+3.8}_{-2.9} | 8.9^{+4.5}_{-2.7} | 10.7^{+2.4}_{-1.1} | 3.6^{+5.4}_{-2.6} | 18^{+24}_{-15} |
| $R (R_\odot)$    | 2.04^{+0.18}_{-0.10} | 3.25^{+0.13}_{-0.14} | 3.58^{+0.17}_{-0.14} | 4.71^{+0.20}_{-0.18} | 2.82^{+4.19}_{-0.14} | 3.80^{+0.15}_{-0.14} |
| $\rho (\rho_\odot)$ | 0.50^{+0.18}_{-0.29} | 0.27^{+0.14}_{-0.17} | 0.19^{+0.07}_{-0.07} | 0.10^{+0.04}_{-0.02} | 0.16^{+0.04}_{-0.02} | 0.33^{+0.07}_{-0.23} |
| $a (au)$         | 0.091 089(13) | 0.106 522(17) | 0.154 241(19) | 0.193 937(15) | 0.249 489(20) | 0.463 918(59) |
| $\cos \omega$   | 0.010^{+0.017}_{-0.021} | 0.005^{+0.017}_{-0.018} | -0.013^{+0.008}_{-0.002} | -0.020^{+0.008}_{-0.011} | -0.026^{+0.008}_{-0.018} | -0.24^{+0.16}_{-0.08} |
| $\sin \omega$   | -0.011^{+0.031}_{-0.025} | -0.004^{+0.028}_{-0.020} | -0.009^{+0.006}_{-0.005} | -0.016^{+0.007}_{-0.011} | -0.017^{+0.016}_{-0.021} | -0.008^{+0.058}_{-0.085} |
| $I'$ (°)         | 88.40^{+0.76}_{-0.42} | 91.17^{+0.40}_{-0.20} | 89.18^{+0.22}_{-0.17} | 88.74^{+0.42}_{-0.60} | 89.30^{+0.12}_{-0.09} | 90.23^{+0.16}_{-0.11} |
| $\Omega$ (°)    | 0 (fixed) | 3.2^{+2.9}_{-2.9} | -3.2^{+2.9}_{-1.1} | -31^{+12}_{-11} | -32^{+12}_{-27} | -62^{+5}_{-46} |
| $M + \omega$ (°) | 205^{+7.3}_{-2.5} | 265^{+2.0}_{-2.0} | 182^{+2.0}_{-2.3} | 197^{+4.8}_{-1.4} | 89^{+7.5}_{-1.8} | 67^{+7}_{-19} |
| $P$ (d)          | 10.323^{+1.2}_{-0.18} | 13.028^{+1.2}_{-0.20} | 22.700^{+1.2}_{-0.24} | 32.005^{+4.2}_{-2.4} | 46.700^{+11}_{-6} | 118.410^{+16}_{-10} |
| $T_0$ (JD)       | 471.505^{+20}_{-7.7} | 471.175^{+20}_{-4.4} | 481.455^{+16}_{-10} | 487.178^{+22}_{-8} | 464.673^{+13}_{-8} | 501.916^{+42}_{-21} |
Figure 3. Bootstrap histograms for planetary masses (transit model II).

Figure 4. Bootstrap histograms for the planetary radii (transit model II).

Figure 5. Bootstrap histograms for the mean densities (transit model II).

may be almost as dense as the Earth. The densities of the other planets span a range characteristic of Saturn and Neptune, from $\rho_S$ to $\rho_N$.

Fig. 6 is for the bootstrap histograms constructed of the semi-major axes. These parameters are the best determined among all the transit models, with uncertainties of the order of $10^{-5}$ au only. We do not compare these results with data in Lissauer et al. (2011) because they accounted for the formal error of the stellar mass. Note that we fixed $m_0 = 0.95 M_\odot$, because we found that this parameter is unconstrained by the photometric data. Yet it seems
that the $\chi^2(m_0)$ function monotonically increases in the range of $m_0 \in (0.7, 1.2) M_\odot$.

The first five panels of Fig. 7 are for the eccentricities, and the bottom right-hand panel is for $\Delta\omega_{bc} \equiv \omega_b - \omega_c$. These histograms confirm that the eccentricities of planets b–f are small, typically less than 0.05, and the arguments of pericentres are not well constrained. The last panel assures us that $\Delta\omega_{bc}$ is determined with an error of only $\sim 10^\circ$, recalling a narrow time-window of the photometric data. The best-fitting parameters of model II are given in Table 2.

Inclination $I_b$ was constrained to the $\leq 90^\circ$ range, and due to the invariance rule implied by the direction of the total angular momentum, the remaining inclination $I_i$ may be smaller and larger than $90^\circ$. We tested whether there is a correlation of the transit events with a given half-disc of the star. We found that both cases are equally possible. Because the orbits are inclined to the plane of the sky at angles close to $90^\circ$, the relative inclinations with the same longitudes of nodes may be $\sim 2^\circ$–$3^\circ$. As expected, the indirect parameters $\Omega_i$ are unconstrained (see Table 2). Therefore, the main contribution to the uncertainties of the relative inclinations comes from ambiguous estimates of $\Omega_i$ rather than of $I_i$.

Curiously, there appears a clear correlation between mutual inclinations in particular pairs of orbits, namely c and e, f and e, as well as d and f. This can be seen in normalized histograms constructed for the inclinations (see Fig. 8). For a chosen planet, we transform $I_i$ to the $\leq 90^\circ$ range (in accordance with the inclination invariance rule), and we compute the bootstrap histogram for $I_j$. The panels of Fig. 8,
from the left-hand to right-hand side, are for pairs (i, j) = (e, c), (e, f), (f, d). If \( I_e \leq 90^\circ \), then it is much more likely that \( I_e, I_f \leq 90^\circ \) than \( I_e, I_f \geq 90^\circ \). Similarly, if \( I_f \leq 90^\circ \), then \( I_d \leq 90^\circ \) appears more likely than \( I_d \geq 90^\circ \).

For particular pairs of planets, the relative inclinations can be determined surprisingly well. Fig. 9 shows the bootstrap histograms \( \Delta I_{ij} \) for such pairs that exhibit well-constrained values. These histograms reveal that orbits of planets b and c are almost coplanar. Similarly, the pair of planets d and e form an almost coplanar subsystem. The mutual inclinations of orbits in these pairs are less than \( \sim 3^\circ \), with most likely values of \( 2^\circ \) – \( 3^\circ \). The remaining panels indicate that the mutual inclinations between the five inner orbits remain within a few degree range. Their upper limits are not so small as in the first two subsystems. The outermost orbit of planet g may by highly inclined to the rest of the system (see errors of \( \Omega _i \) in Table 2).

These results confirm a hypothesis in the discovery paper. In accordance with this work, planetary orbits in the Kepler-11 system should be mutually inclined by no more than a few degrees. It flows from estimating a probability that for a given orientation of the orbits, all six planets transit the star. This reasoning assumes that all inclinations are independent. However, we found that the Kepler-11 system is composed of two or three subsystems, which exhibit small mutual inclinations of orbits. Although a probability that the mutual inclinations between these subsystems is significant seems a bit larger than for fully independent orbits, it still remains very small. We estimate that a randomly located observer can detect transits of all six planets with a probability as small as \( \sim 0.05 \) per cent, for both models I and II.

### 3.3 Models III \( (e_g = 0, \Omega _g = 0) \) and IV \( (e_g = 0, \Omega _g = 0) \)

The results for models III and IV are given in Tables 3 and 4. Most of these results are common for all transit models I–IV. There are some differences regarding the determination of the mass of planet g. In the realm of models III and IV (note that both have fixed \( e_g = 0 \) and \( \Omega _g = 0 \)), only an upper limit of \( m_g \) smaller than 20–30 \( M_\odot \) may be found. The low limits of \( m_g \) in model I are likely due to weakly constrained \( e_g \) and \( \Omega _g \).

Let us recall that in the bootstrap set derived for model II, we found only two solutions with \( (Y) \approx 2 \) after 16 000 yr. However, this integration time-scale is too short to detect weak instability which actually leads to catastrophic disruption of these configurations. It was verified by the direct, long-term integrations. Hence, we did not detect any long-term stable configuration in the bootstrap set of model II. Similarly, stability tests performed for configurations of model III did not reveal any stable models. As compared to
model II, fixed $\Omega_g = 0$ does not seem to change the general view of the stability of the system.

For model IV we found many stable configurations which confirm the stability analysis in Lissauer et al. (2011). They found some stable solutions assuming that the Kepler-11 system is strictly coplanar. We may conclude that a factor of small relative inclinations is more important for maintaining the long-term stability than small eccentricities. This will be discussed in more detail later in this work.

We examined a probability that a randomly located observer could detect transits of all planets in the system. This is basically unlikely for model III (~0.09 per cent), while for model IV there is a probability of such an event to be larger, and we estimate it ~3.4 per cent.

4 DISCUSSION ON THE PLANET INTERIORS

Fig. 10 shows bootstrap diagrams of a few selected pairs of parameters. These results are for model II. The top row is for the semimajor axes and the planetary masses, the radii and mean densities, respectively. The red and green curves mark the data for Uranus and Neptune. The bottom row is for the mass–radius, mass–density and radius–density relations, respectively. Similarly, the red and green filled circles are for Uranus and Neptune, respectively. As we concluded above, the orbital solutions in set II are only marginally stable; however, it is a matter of unconstrained orbital angles. Note that a discussion in this section concerns semi-major axes (known with an excellent precision) as well as planetary masses and radii.

This figure reveals that the most inner four planets in the Kepler-11 system exhibit a clear and curious anti/correlation of masses, radii and densities with the semi-major axes. Masses and radii increase with $a_i$, while densities decrease. The last panel constructed for $(R, \rho)$ shows a weak anticorrelation between the radii and densities: the smaller the radius, the larger the density. The determined masses and radii of the planets provide some insight into their chemical composition. We use a simple analytic relation between the radius and the mass of a cold body (Lynden-Bell & O’Dwyer 2001; Lynden-Bell & Tout 2001) to estimate the characteristic density $\rho_0$ of planetary matter. This density can be compared with $\rho_0$ calculated for a given number of nucleons per electron of a chemical mixture forming the planet, $\mu_E$. The value of $\mu_E$ is a simple mean over the elements in each chemical substance or component. For instance, the H–He mixture has $\mu_E = 87/8$ for the mass proportions 3 to 1, and ice or rock has $\mu_E \approx 2$. In this way, we can
obtain some insight into the likely chemical composition of the planets.

Our results for Kepler-11 are presented in Fig. 11. The black curves with grey areas are for $\rho_0$ and its uncertainty $\Delta \rho_0$. Each panel is for one planet of the Kepler-11 system. Data for planets b–g are displayed from the left-hand to right-hand side, respectively. The coloured curves are for the Solar system, that is, for Uranus (red), Neptune (green), Jupiter (blue), Saturn (violet) and the Earth (light blue). The density $\rho_0$ was computed in a wide range of $\mu_e \in [1, 2]$. These values are known relatively well for the Sun companions. Following Helled et al. (2011), for Uranus and Neptune one finds $\mu_e \approx 1.1$ (for the icy model) and $\mu_e \approx 1.35$ (for the rocky model). The density $\rho_0$ in these particular cases is plotted with the filled circles. Similarly, for Jupiter and Saturn, $\mu_e$ may also be estimated to be $\approx 1.08–1.09$ (Guillot 1999). Values of $\rho_0$ for these particular $\mu_e$ are marked with circles. Let us note that the densities $\rho_0$ of Jupiter and Saturn are almost identical.

Lynden-Bell & Tout (2001) argue that $\rho_0$ is the zero-pressure density $\rho_{0,p}$ of the chemical mixture of the planets. Because their model has many simplifications, $\rho_0$ is usually two to five times larger than $\rho_{0,p}$. Keeping this in mind, the densities $\rho_0$ calculated for Kepler-11 planets can be compared with those of the Solar system planets. The value of $\rho_0$ is best determined for planet e. It is very close to the Jupiter/Saturn value $\approx 0.45 \text{ g cm}^{-3}$. This suggests that planet e is built mainly of an H/He mixture with mass proportions of the elements close to 3/1 with a portion of heavier elements contained in ices or rocks. This makes the planet classified as a smaller ‘cousin’ of Jupiter and Saturn rather than of Neptune and Uranus, as suggested in Lissauer et al. (2011).

The density $\rho_0$ of planet b is determined much less precisely than that for planet e. However, it is rather unlikely that it belongs to the same class as planet e. $\rho_0$ is larger than that for Jupiter and Saturn, even taking into account a large uncertainty. It is also larger than $\rho_0$ for Uranus and Neptune like planets but is smaller than for the Earth. We can conclude that planet b is a small planet containing a large percentage of heavy elements in its interior, which is likely larger than in the ice giants. It is reasonable to classify this planet in the super-Earth family, although it may be also a small Neptune-like planet.

Planet f has mass similar to planet b. However, its composition is likely between the Jupiter–Saturn and Uranus–Neptune classes. Planet d has a likely similar composition to planet f. The best-fitting estimate of $\rho_0$ for planet e is very close to the Uranus/Neptune value. For planet g there is only the upper limit of $\rho_0$. However, it is probably close to the Jupiter/Saturn value or, less likely, to the Neptune/Uranus value.

These conclusions are reinforced after inspecting the bottom left-hand panel of Fig. 10 illustrating the mass–radius diagram for the Kepler-11 system. The mass–radius relation computed for $\rho_0$ and $\mu_e$ of the Solar system planets is plotted with different colours. Data are shown for Uranus (red), Neptune (green), Jupiter (blue), Saturn (violet) and the Earth (light blue), respectively. Representations of this relation are plotted in the mass–density diagram (middle panel) and the radius–density diagram (right-hand panel).
For small masses, the characteristic density \( \rho_0 \) is very close to \( \rho \) (see equation 34 in Lynden-Bell & O’Dwyer 2001). This derived decay of \( \rho \) with the mean distance from the star suggests that the inner planets may contain a larger amount of heavy elements than more distant companions. If this correlation can be confirmed, it may provide an observational constraint for the planet formation theory. Allowing for some speculations here, let us note that all Kepler-11 planets exhibit masses in the same range of a few \( M_\oplus \). Hence, they likely have formed in a similar way and physical environment (Rogers et al. 2011). Small eccentricities and small relative inclinations suggest that the system evolved orbitally smoothly towards the current state, conserving the ordering of initial distances from the star. The observed relation between \( \rho_0 \) and \( \alpha_i \) may then indicate the chemical composition and mass density distribution in the primordial protoplanetary disc.

We underline that the results in this section must be considered as preliminary. Due to a relatively narrow time-window of the photometric data, masses of the planets are determined with large uncertainties.

### Table 5. Orbital parameters of marginally stable configuration IIa. The mass of the star is \( 0.95 M_\odot \). The osculating Poincaré elements are given at the initial epoch JD 245 5964.512 28.

| Parameter/planet | b       | c       | d       | e       | f       | g       |
|------------------|---------|---------|---------|---------|---------|---------|
| \( m (M_\oplus) \) | 4.550   | 1.542   | 7.224   | 15.698  | 4.340   | 18.530  |
| \( a (au) \)     | 0.091 097 | 0.106 515 | 0.154 241 | 0.193 939 | 0.249 532 | 0.463 815 |
| \( e \)          | 0.008 00  | 0.006 98  | 0.009 18  | 0.009 24  | 0.01827 0 | 0 (fixed) |
| \( I (\circ) \)  | 89.048   | 88.938   | 89.023   | 88.803   | 89.339   | 89.470   |
| \( \Omega (\circ) \) | 0 (fixed) | 2.119    | 15.216   | 14.804   | 20.626   | 52.574   |
| \( \omega (\circ) \) | 158.534  | 138.150  | 61.831   | 183.853  | 296.192  | 0 (fixed) |
| \( \lambda (\circ) \) | 47.285 91 | 128.232 48 | 118.949 29 | 12.278 74 | 151.624 81 | 336.224 07 |

### Table 6. Orbital parameters of solution IIb. The mass of the star is \( 0.95 M_\odot \). The osculating Poincaré elements are given at the initial epoch JD 245 5964.512 28.

| Parameter/planet | b       | c       | d       | e       | f       | g       |
|------------------|---------|---------|---------|---------|---------|---------|
| \( m (M_\oplus) \) | 6.227   | 4.422   | 8.467   | 11.293  | 6.866   | 32.651  |
| \( a (au) \)     | 0.091 102 | 0.106 525 | 0.154 254 | 0.193 939 | 0.249 501 | 0.463 815 |
| \( e \)          | 0.023 14  | 0.017 80  | 0.011 48  | 0.004 01  | 0.018 59  | 0 (fixed) |
| \( I (\circ) \)  | 88.000   | 90.849   | 89.296   | 91.206   | 90.677   | 89.733   |
| \( \Omega (\circ) \) | 0 (fixed) | −1.748   | −5.944   | −2.902   | −2.393   | 92.907   |
| \( \omega (\circ) \) | 178.847  | 175.740  | 35.793   | 336.443  | 350.572  | 0 (fixed) |
| \( \lambda (\circ) \) | 29.837 04 | 92.040 62 | 144.325 68 | 218.200 59 | 96.062 46 | 336.171 21 |

### 5. RESULTS OF THE DYNAMICAL ANALYSIS

The best-fitting solutions gathered with the help of the bootstrap algorithm provide us primary information required to perform an extensive study of the dynamical stability of the system. Because the orbits of Kepler-11 super-Earth planets are confined within the mean distance of Mercury in the Solar system, we could expect that the dynamics of this system are extremely complex. Indeed, preliminary integrations demonstrated that the Kepler-11 system is dynamically packed, in accordance with the definition and PPS hypothesis in Barnes et al. (2008). In spite of apparently ordered motions centred at their nominal \( n_{i,0} \), we try to resolve this paradox and we try to detect sources of this seemingly odd and strong instability. To illustrate this paradox, we try to detect such solutions, we choose a few representative solutions and we construct the MEGNO maps in their vicinity.

#### 5.1 Quasi-stable solutions in transit model II

For model II, among \( \sim 1500 \) initial conditions, we found only two configurations exhibiting the MEGNO close to 2 after \( T = 16000 \) yr. Parameters of these solutions are listed in Tables 5 and 6. The first stable solution (hereafter referred to as IIa, see Table 5) has a relatively low mass of planet c, \( \sim 1.5 M_\oplus \). Two innermost planets b and c have almost coplanar orbits. The next three planets, d, e and f, also form a nearly coplanar subsystem (d--e--f), which is inclined to the first two orbits by a large angle \( \sim 15^\circ \). The outermost orbit is inclined even more, by \( \sim 50^\circ \) to the inner subsystem of b--c, and by \( \sim 30^\circ \) to the triple-planet subsystem of d--e--f. The second stable solution IIb (see Table 6) has all masses close to the nominal best-fitting values. The mutual inclinations of five inner orbits are close to 0°, while the outermost orbit of planet g is highly inclined to the inner orbits by \( \sim 90^\circ \), similarly to solution IIa. Because the relative inclinations between particular pairs of orbits are large in these best-fitting solutions, such systems might be unlikely observed by a randomly located observer. We estimate a probability of such an event as \( \sim 0.07 \) and \( \sim 0.05 \) percent for solutions IIa and IIb, respectively.

#### 5.2 Triple-planet resonances as the main source of instability

Let us now study the vicinity of these particular solutions through the dynamical maps. For each initial condition of the discrete grid with \( 512 \times 512 \) resolution, we compute \( \langle \gamma \rangle \) over \( T = 8000 \) yr. Fig. 12 shows the MEGNO maps for solution IIa. Each panel is for a different pair of planets. The coordinate axes are rescaled mean motions centred at their nominal \( n_{i,0} \):

\[
x_i = \frac{n_i - n_{i,0}}{n_{i,0}} \times 10^3.
\]
The $x_i$-axes span 1σ uncertainties of the semi-major axes $a_i$, in accordance with Table 2. The semi-major axes are determined very precisely; hence, the 1σ interval spans a range of $10^{-5}$ to $10^{-4}$ au. The rescaled $x_i$ are confined to the order of 10.

The MEGNO is colour-coded in the dynamical maps. The blue regions mean regular solutions with $\langle Y \rangle \approx 2$, while the yellow colour is for chaotic (unstable) initial conditions, $\langle Y \rangle \gtrsim 5$. The resolution is $512 \times 512$ pixels, total integration time is $T = 8000$ yr pixel$^{-1}$.
and SABA$_3$ integrator step-size is 0.5 d. A single computation of each map took $\sim 16$ h of 1200 CPU cores. The integrations of each pixel were performed up to the end time $T$, regardless of the value of the MEGNO.

Still, although the maps cover tiny regions of the phase space, close to the fixed initial condition, they reveal a sophisticated structure. Because we consider the dynamics in terms of conservative, close-to-integrable Hamiltonian system, this structure is governed by the resonant motions. A relatively short integration time, $\sim 10^3$–$10^5$ characteristic periods, equivalent to the orbital period of the outermost planet makes it possible to detect unstable MMRs. They appear as yellow straight bands of different widths and slopes. Inspecting the dynamical maps, we can identify particular resonances.

A condition for the MMR in the $N$-planet system may be written in the following form:

$$\sum_{i=1}^{N} p_i \frac{d\lambda_{i}}{d\tau} = \mathcal{O}[f_{i+1}, f_{i+2}], \quad \mathcal{O} = \sum_{i=1}^{N} p_i n_i = \mathcal{O}[f_{i+1}, f_{i+2}], \quad (3)$$

where $n_i$ is the mean motion of the ith planet, $f_{i+1}$ and $f_{i+2}$ are the fundamental frequencies associated with the pericentre arguments $\omega_i$ and the longitudes of nodes $\Omega_i$ (for all orbits), and $p_i$ are relatively prime integers. The linear relations must obey the d’Alambert rule.

The two-planet MMR takes place when two values of $p_i$ are non-zero. If three coefficients in this linear relation are non-zero, it means that the system exhibits three-body MMR. In the Kepler-11 system, the fundamental frequencies associated with $\omega_i$ and $\Omega_i$ are much smaller than $n_i$ (these are the secular frequencies); hence, the right-hand sides of equation (3) are close to 0. This makes it possible to skip the secular terms, as the first-order approximation, and to identify the MMRs through approximate resonance conditions involving the mean motions only.

To identify the MMRs in the MEGNO maps, we apply a simple method described in Guzzo (2005). In the $(n_j, n_k)$ plane, the slope

$$\alpha_{j,k} \equiv \frac{d \lambda_k}{d \lambda_j}$$

of a particular resonance line determines the ratio of coefficients $p_j$, that is,

$$\alpha_{j,k} = - \frac{p_j}{p_k} \quad (4)$$

If $\alpha_{j,k} = 0$, then the MMR forms a horizontal line, planet $k$ is involved in the MMR, while planet $j$ is not. If $\alpha_{j,k} = 1$ then the MMR forms a vertical line, planet $j$ is involved in the resonance, while planet $k$ is not. In these cases, other planets may be involved in this particular resonance. If $\alpha_{j,k}$ is finite and non-zero, then both considered planets are involved in the MMR. To identify this particular resonance, one has to compute slopes corresponding to this resonance in all $(n_j, n_k)$ planes. It may be not possible, if the map ranges do not cover the resonance band. If $\alpha_{j,k}$ is non-zero and finite only for one pair of planets $(j, k)$, it means that a two-planet MMR is present. It should be verified whether $p_j n_k + p_k n_j \approx 0$. The coefficients $p_j$ and $p_k$ of the MMR condition can be computed from the slopes $\alpha_{j,k}$. Similarly, the three-body MMR takes place, if there exist relatively prime integers $i, j \neq i$ and $k \neq i, j$ such that $\alpha_{i,j}, \alpha_{i,k}$ and $\alpha_{j,k}$ are all finite and non-zero. The three-body MMR may be identified by computing the slope coefficients in appropriate planes of the mean motions. An identification of four-body and $N$-body MMRs can be derived as well.

Let us note that Fig. 12 shows the $(x_i, x_j)$ planes but $(n_i, n_j)$ may be easily computed from these data.

Using this simple MMR identification algorithm, we found most significant MMRs close to solution IIa. The identified three-body MMRs were labelled at the panels, and listed in Table 7. The mean motions of solution IIa permit a few low-order two-body MMRs in the vicinity of this solution, for example, $4n_b - 5n_c, 1n_b - 3n_c$, $2n_c - 5n_x, 1n_d - 2n_i$ and $2n_x - 3n_i$. There is no two-planet MMR in the range of $n_i$ implied by $1\sigma$ uncertainty. All straight bands with finite and non-zero $\alpha_{i,j}$ have at least two images in the planes constructed for other planets. All features seen in the dynamical maps then correspond to three- and four-body MMRs. Possibly, even more complex $N$-body resonances may be found.

The labels in Fig. 12 correspond to data in Table 7. The labels with asterisks are written in open circles in the dynamical maps and denote MMRs in the neighbourhood of solution IIa. Resonances labelled with numbers without asterisks and written in the filled circles in the dynamical map are also present in the neighbourhood of solution IIb.

The MMRs $(7n_c - 11n_1 + 2n_g)$ labelled as ‘2’ and $(5n_c - 8n_d - 1n_g)$ labelled as ‘1’ are the most close to solution IIa. Solution IIa passed the 16 000 yr MEGNO test as a stable solution. However, because it is close to two three-body MMRs, it is found in a dense web of low-order three- and four-body MMRs, its long-term stability cannot be guaranteed by this test. The integration time of 16 000 yr corresponds to $\sim 50 000$ orbital periods of the outermost planet g. This time is usually too short to detect a chaotic nature of the orbit when the three-body resonances are present. Unfortunately, the CPU overhead does not permit to derive the dynamical map with the integration time per single initial condition which should be 10–10$^2$ times longer. Indeed, a test run over $T = 40 000$ yr illustrated in Fig. 13 reveals that solution IIa is chaotic and unstable. Besides, almost the whole $(x_b, x_c)$ plane corresponds to chaotic motions with $\langle Y \rangle > 5$.

We analysed the second solution IIb in the same manner. The MEGNO maps computed for 8000 yr are presented in Fig. 14. Also these maps reveal a dense net of three- and four-body resonances. Some of these resonances may be identified as close to solution IIa as well as to solution IIb. They are labelled with the same numbers written within the filled circles, as in Fig. 12. We also found a few new MMRs, labelled within the open circles by ‘3’, ‘4’ and ‘5’. The remaining four-body MMRs which are visible in this figure are not labelled. All identified MMRs are listed in Table 8.

Similarly to configuration IIa, the integration over a longer time of $T = 40 000$ yr reveals that solution IIb is unstable. Almost the whole plane of the dynamical map corresponds to $\langle Y \rangle > 5$. The MEGNO map is similar to Fig. 13, and is not shown here.

| Label | $p_b$ | $p_c$ | $p_d$ | $p_e$ | $p_t$ | $p_g$ |
|-------|-------|-------|-------|-------|-------|-------|
| 1     | 7     | -10   | 2     | 0     | 0     | 0     |
| 2     | 0     | 0     | 0     | 7     | -11   | 2     |
| 3     | 0     | 0     | 5     | -5    | -3    | 0     |
| 4     | 1     | 0     | -10   | 11    | 0     | 0     |
| 5     | 9     | -13   | 0     | 4     | 0     | 0     |
| 6     | 0     | 0     | 6     | -9    | 0     | 2     |
| 7     | 0     | 0     | 5     | -16   | 11    | 0     |
| 1*    | 0     | 5     | -8    | -1    | 0     | 0     |
| 2*    | 0     | 6     | -14   | 5     | 0     | 0     |

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Table 7. Three-planet resonances near the best-fitting model IIa (see Fig. 12).

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5.3 Dynamical maps in the \((\omega_i, \omega_j)\) and \((e_i, \Omega_i)\) planes

Fig. 15 illustrates the MEGNO maps in planes of the arguments of pericentres. The MEGNO is calculated over \(T = 8000\) yr. The \((\omega_i, \omega_j)\) maps are constructed a bit differently from the mean motion maps.
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Figure 15. Dynamical maps in the \((\omega_i, \omega_j)\) plane for solution IIa. The MEGNO range is [1, 5] and colour-coded: blue is for stable solutions and yellow is for unstable systems. Each map has elements of a different pair of planets; these planets are labelled at the bottom right-hand corner of each panel. The arguments of the pericentre are expressed in degrees. The nominal solution is marked with the asterisk.

dynamical maps. Because we intend to analyse configurations coherent with the observations, \(\omega_i\) can be freely varied over the grid, if also the mean anomalies are modified to preserve the time of the first transit, \(T_i\). For instance, for a circular orbit, when the argument of pericentre is shifted from the nominal value by \(\Delta \omega_i\), the mean anomaly should be shifted by \(-\Delta \omega_i\). For eccentric orbits, such a correction flows from the first Keplerian law.

The MMRs form an even more sophisticated web in the \((\omega_i, \omega_j)\) planes than in the mean motion planes. These dynamical maps reveal that the stability depends on the initial arguments of pericentres. It is not obvious a priori, because eccentricities are very small. The initial phases of the system are preserved across the maps, and each point corresponds to the same \(T_i\). Keeping in mind that the photometric data spanning only \(\sim 500\) d imply weak constraints on angles \(\omega_i\), we realize how difficult it is to find a stable initial condition in the huge, 50-dimensional parameter space of the system. Fig. 16 illustrates MEGNO maps in the \((e, \Omega_i)\) planes calculated over \(T = 8000\) yr. Each panel is for one planet. An identification of particular MMRs is much more complex than in the mean motion planes. It would require the frequency analysis of these solutions. Still, regions of stable, quasi-periodic motions usually form only small islands in the phase space. For the four innermost planets, the regular motions are confined to an \(\sim 5^\circ\) range of \(\Omega_i\) around the nominal value and also to a small range of eccentricities \(\sim 0.01\). For the two outermost planets f and planet g the maps look different. A zone of stable motion of planet g extends towards large \(\Omega_g\). It implies a large mutual inclination of its orbit to the rest of the system. Small \(\Omega_g\) provoke unstable motions. The nominal solution is found at the very edge between the regular and chaotic zones.

5.4 Stable solution of model IV

Let us recall that in transit model IV all \(\Omega_i = 0^\circ\). Hence, the mutual inclinations between all pairs of orbits are close to 0° but the system remains non-coplanar because \(I\) is still different from 90° (hence, the transits of all planets in this system can be detected by a randomly located observer with a significant probability of \(\sim 3.4\) per cent). In this case, we found several solutions with the MEGNO converging to 2 after \(T = 16000\) yr. This indicates a possibility of quasi-regular orbits. We chose one of such solutions. Its
parameters are listed in Table 9, and we compute dynamical maps in its vicinity. Fig. 17 shows the results of this experiment in the mean motions planes. The integration time is $T = 8000 \text{ yr pixel}^{-1}$. The tested configurations are found in a narrow region of regular motions. The most prominent dynamical feature visible in the maps is associated with stable three-planet MMRs between planets b, c and d. We identified it as a $(7, -10, 2, 0, 0, 0)$ MMR. It has also been found in dynamical maps associated with solutions IIa and IIb as the MMR labelled as '1'. Due to altered parameters of the model, the unstable region of this resonance is visible in these maps. The solid black lines plotted across panels shown in the top row of Fig. 17 have slope equal to $7/10$, $-7/2$ and 5, from the left-hand to right-hand side, respectively. Other three-body resonances visible in the dynamical maps form a dense web, which may be better seen in the $(x_d, x_e)$ and $(x_d, x_f)$ planes displayed in the bottom panels.

To examine the stability of this solution over a longer time-scale, we calculated a single dynamical map in the $(x_b, x_c)$ plane for a much longer integration time $T = 40 000 \text{ yr}$. It is shown in Fig. 18. This solution is located in a tiny region of stable motions. In this case, the three-body MMRs have a protective role for the system, saving it from disruption. Actually, this solution is chaotic. To demonstrate this, we computed the critical argument of the three-body MMR and we plot over two intervals of time, at the beginning of the 1 Myr integration period (left-hand panel of Fig. 19) and at the end of this period (right-hand panel of Fig. 19). The critical arguments exhibit librations alternated with circulations. Such behaviour of the critical argument indicates a crossing of the separatrix of the resonance, and chaotic dynamics. In such a case, the configuration may be geometrically stable over a very long time but it may be suddenly disrupted due to a slow diffusion along the resonance.

5.5 The Arnold web structure in the phase space

The results of experiments described in the previous sections may be interpreted on the grounds of the dynamical system theory. The dynamical stability of planetary orbits in systems with strong perturbations is influenced by even small errors and the resulting diffusion due to resonances overlapping. This dynamical phenomenon has been investigated in the outer Solar system. Murray & Holman (2001) identified the chaos among the Jovian planets as resulting from the overlap of the components of three-body MMRs among Jupiter, Saturn and Uranus. In spite of short Lyapunov time

Table 9. Orbital parameters of solution IVa. The mass of the star is 0.95 $M_\odot$. The osculating elements of Poincaré are given at the epoch JD 245 5964.511 28.

| Parameter/plane | b   | c   | d   | e   | f   | g   |
|-----------------|-----|-----|-----|-----|-----|-----|
| $m$ ($M_\oplus$) | 2.359 | 3.386 | 5.630 | 10.841 | 7.524 | 25.161 |
| $a$ (au)        | 0.091 113 | 0.106 505 | 0.154 243 | 0.193 940 | 0.249 511 | 0.463 991 |
| $e$             | 0.044 23   | 0.017 19   | 0.006 33   | 0.002 58   | 0.010 73   | 0 (fixed)  |
| $I$ (°)         | 89.141     | 91.215     | 89.332     | 88.837     | 89.394     | 89.770     |
| $\omega$ (°)    | 20.651     | 55.728     | 140.753    | 236.761    | 355.845    | 0 (fixed)  |
| $\lambda$ (°)   | 178.881 74 | 209.600 77 | 40.792 59  | 318.518 31 | 91.575 69  | 336.265 02 |
(10^7 yr), they found the dynamical lifetime of Uranus ~10^{18} yr. In this way, the analytic theory of Murray & Holman (2001) explained an apparent paradox of long-term stable systems which is actually chaotic.

The structure of the three- and four-body MMRs in the outer Solar system was further investigated numerically by Guzzo (2005, 2006). Very recently, it was also studied by Quillen (2011) for strongly interacting extrasolar systems. Through the dynamical map technique, Guzzo (2005) found that if the chaotic motions appear in a regular net, they may be practically stable over very long times.

Such a state of chaotic systems is called the Nekhoroshev regime. In contrast, if the chaotic resonances do not constitute a regular web, and a minority of orbits form a global chaotic zone, the stability of the system is influenced by strong chaotic diffusion. Such a regime is related to the resonance overlap, and is called the Chirikov regime of the dynamics (Froeschlé, Guzzo & Lega 2000; Guzzo 2005).

These results may be applied to interpret the dynamical maps of the Kepler-11 system. The dense net of the multiple-body MMRs forms the Arnold web in the neighbourhood of the best-fitting model configurations. Our experiments reveal that this system may undergo the Chirikov regime rather than a long-term stable Nekhoroshev regime. We have no strong evidence of this phenomenon, because it would require non-trivial and intensive computations of the chaotic diffusion. Conclusions regarding the real state of the Kepler-11 system require more precise determination of the initial conditions than obtained in this work.

We also note that solutions IIa and Ib investigated in detail are found at the very border of the chaotic and regular zones. This can be well seen in the (ω_i, ω_j) and (e_i, Ω_i) planes. A change of these angles constrained by the best-fitting errors could ‘move’ the system into larger zones of stable motions. Because the MEGNO quantifies the stability of the system as a whole, such a change would imply a shift of the initial condition in all parameter maps (Goździewski & Maciejewski 2001). The altered initial conditions would be also more ‘distant’ from the unstable strips of three- and four-body MMRs in the mean motions planes. The dynamical maps would be more similar to those computed for solution IV revealing most extended zones of stable motions. Overall, the dynamical state of the system is very fragile and depends on subtle changes of the initial conditions.

Still most of the best-fitting model configurations obtained with the bootstrap algorithm, which appear as regular over the short-term dynamics time-scale, are self-disrupting. This behaviour is...
similar to unstable evolution in three-body MMRs observed in the HD 37124 system (Goździewski et al. 2008a). In this sense, the dynamical state of the Kepler-11 system is still puzzling. The results of long-term integrations of the obtained sets of initial conditions indicate that the system resides on the edge of dynamical stability. These conclusions might be changed, if more data are gathered and analysed.

6 CONCLUSIONS

In this paper, we derived an improved method for the dynamical analysis of photometric light curves of stars with multiple transiting planets. Its main purpose is to determine planetary masses, as well as a number of indirect parameters affecting the dynamical stability of the system. This algorithm improved the well-known TTV algorithm. The crucial point of this method is to model the whole photometric curve directly with the help of an efficient symplectic N-body integration. Such a direct approach makes it possible to account for multiple transits, as well for the transit depths and their widths. This in turn makes it possible to impose dynamical constraints on parameters which cannot be determined in terms of the TTV, like the longitude of nodes and mutual inclinations of orbits.

With the help of this new method, we re-analysed available photometric data for Kepler-11. The direct algorithm imposes constraints on the mass of the outermost planet g and helps us to determine the mutual inclinations between orbits of planets b and c as well as between planets d and e with a good accuracy of 2°. These results extend the analysis performed in the discovery paper (Lissauer et al. 2011). Overall, the conclusions in the discovery paper and in our work coincide very well, in spite of quite different transit models, optimization algorithms and uncertainty estimation methods applied.

Thanks to the in-depth analysis of the Kepler-11 light curves, we investigated a possible chemical composition of the planets detected in this intriguing system. The most curious finding is a clear anticorrelation of the mean densities of the planets with their mean distances from the star. The inner planets exhibit a larger abundance of heavy elements than the outer companions. Because all eccentricities as well as the mutual inclinations of stable systems remain small, the system unlikely suffered violent scattering processes in the past. It follows that the ordering of planets has been preserved since their formation. A dynamical relaxation should imply large $e_i$ and $\Delta i_j$ (see e.g. Adams & Laughlin 2003; Chatterjee et al. 2008). These factors indicate that the primordial protoplanetary disc might have a significant gradient of metallicity and the present Kepler-11 system is a real fossil of this disc. If this suggestion is confirmed, it can constrain the planet formation theories, in particular concerning multiple systems of super-Earth planets.

This conclusion is reinforced by the dynamical analysis of the system. We found, in accordance with the discovery paper, that the system is basically free from two-planet MMRs. However, its global dynamics are governed by three- and four-body MMRs. Overlapping of these resonances near the best-fitting solutions implies an extended zone of dynamical chaos and very unstable configurations. Particular multibody resonances may stabilize the system. We identified such a three-body resonance. However, the observation window spanning only $\sim500$ d does not make it possible to pick up this MMR as the only possible. Besides, the MMRs form the structure of the Arnold web characteristic for the Chirikov regime. In this dynamical state, the phase-space orbits are strongly unstable due to overlapping of the MMRs. In such a case, the chaotic diffusion in the action (semi-major axes) space is significant and easily leads to strong geometric changes of orbits. Actually, our numerical experiments indicate this in the Kepler-11 system. It remains an open question though how such an apparently ordered configuration of planets could survive the formation phase in an extremely complex and fragile dynamical environment.

The discovery team argue that the system is non-resonant. This factor would prevent a scenario of trapping the planets into MMRs during the inward migration at the early stages of the evolution. As we showed here, the system is in fact extremely resonant, but in quite a different sense. Combining this fact with small values of the eccentricities and anticorrelation of the chemical composition with the distance to the star, we may conclude quite an opposite: the migration/trapping scenario could be the only way of preserving the primary architecture in the present form. However, these suggestions might be verified only if more photometric data are gathered and are available.

Most likely, many other Kepler-discovered multiple extrasolar systems with transiting planets exhibit qualitatively similar behaviours to that we found in Kepler-11. Hence, the approach in this paper is general and may be applied in the studies of other compact systems composed of low-massive, super-Earth or Neptune-like planets.

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Figure 19. Temporal evolution of the critical argument of the three-body MMR for the initial condition IVa. The left-hand panel is for the beginning of the integration period and the right-hand panel is for the end of the integration period spanning 1 Myr.
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