A Short Proof of a Conjecture of Erdős Proved by Moreira, Richter and Robertson

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Abstract: We give a short proof of a sumset conjecture of Erdős, recently proved by Moreira, Richter and Robertson: every subset of the integers of positive density contains the sum of two infinite sets. The proof is written in the framework of classical ergodic theory.

Key words and phrases: Sumset, Erdős Conjecture, Positive density, Følner sequence

1 Introduction

In this note we give a short proof of a result of Moreira, Richter and Robertson, conjectured by Erdős:

Theorem 1 ([MRR]). If A ⊂ ℤ has positive density then there exist infinite subsets B and C of ℤ with A ⊃ B + C.

Here and below, by the density of a subset A of ℤ we mean its upper uniform density d∗(A) defined by

$$d^*(A) := \lim_{N \to \infty} \sup_{M \in \mathbb{N}} \frac{|A \cap [M, M+N)|}{N}.$$  

The proof extends to the case of countable abelian groups up to notational changes. We believe that a generalization of the method to countable amenable groups is possible, by using the maximal isometric factor in place of the Kronecker factor.

The combinatorial part of our proof is closely related to the corresponding part of [MRR]: Theorem 3 is the dynamical counterpart of [MRR, Theorem 2.2]. But the whole proof of Theorem 1 presented here is very different from [MRR], as it is written inside the framework of classical ergodic theory.

In [MRR] the authors ask whether any set of positive density contains the sum of three infinite sets. A possible strategy would be to show that it contains the sum of a set of positive density and an infinite set, and then to use Theorem 1. However, this method does not work:
Proposition 2. There exists a set of positive density not containing any sum of a set of positive density and an infinite set.

Section 2 contains some preliminaries. Theorem 1 is proved in Section 3 and Proposition 2 in Section 4.

2 Preliminaries

In this paper we use the convention that \( \mathbb{N} = \{0, 1, 2, \ldots \} \).

2.1 Følner sequences

We make a constant use of Følner sequences but, since we are dealing only with subsets of the integers, we never need general Følner sequences and we can assume that a Følner sequence is a sequence \( \Phi = (\Phi_N)_{N \geq 1} \) of intervals of \( \mathbb{N} \) whose lengths tend to \( \infty \). If \( A \subset \mathbb{N} \) we write
\[
\overline{d}_\Phi(A) = \limsup_{N \to \infty} \frac{|A \cap \Phi_N|}{|\Phi_N|};
\]

\[
d_\Phi(A) = \lim_{N \to \infty} \frac{|A \cap \Phi_N|}{|\Phi_N|} \text{ if this limit exists.}
\]

Therefore,
\[
d^*(A) = \sup_{\Phi} d_\Phi(A)
\]

where the supremum is taken for all Følner sequences \( \Phi \) such that \( d_\Phi(A) \) exists, and the supremum is attained.

2.2 Topological dynamical systems

A topological dynamical system \((X, T)\) is a compact metric space endowed with a map \( T : X \to X \), continuous and onto. We write \( d_X \) for the distance on \( X \).

Let \( x_0 \in X \), \( \Phi \) a Følner sequence and \( \mu \) a probability measure on \( X \). We say that \( x_0 \) is generic for \( \mu \) along \( \Phi \) if
\[
\frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \delta_{T^n x_0} \to \mu \text{ weakly* as } N \to \infty
\]

where \( \delta_x \) is the Dirac mass at \( x \). This is equivalent to
\[
\text{for every } f \in C(X), \quad \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} f(T^n x_0) \to \int f \, d\mu \text{ as } N \to \infty.
\]

In this case, \( \mu \) is supported on the closed orbit of \( x_0 \) under \( T \) and is invariant under \( T \). For \( U \subset X \) we have
\[
\overline{d}_\Phi(\{n : T^n x_0 \in U\}) \geq \mu(U) \text{ if } U \text{ is open};
\]

\[
d_\Phi(\{n : T^n x_0 \in U\}) = \mu(U) \text{ if } U \text{ is clopen (that is, open and closed).}
\]
We remark that for every $x_0 \in X$, every Følner sequence admits a subsequence along which $x_0$ is generic for some measure $\mu$.

We say that $x_0$ is generic for $\mu$ without mention of a Følner sequence if it is generic for the usual Følner sequence $([0,N])_{N \geq 1}$.

### 2.3 The key combinatorial result

The next theorem is the dynamical counterpart of [MRR, Theorem 2.2].

**Theorem 3.** Let $(X,T)$ be a topological dynamical system, $x_0 \in X$, $E$ a clopen subset of $X$ and

$$A = \{ n \in \mathbb{N} : T^n x_0 \in E \}.$$  

Let $X \times X$ be endowed with the transformation $T \times T$. Let $x_1 \in X$ and let $\nu$ be a measure on $X \times X$, for which the point $(x_0,x_1)$ is generic along some Følner sequence $\Phi = (\Phi_N)_{N \in \mathbb{N}}$. Assume that there exists $\epsilon > 0$ and a sequence $(m_i)_{i \geq 1}$ of integers tending to $\infty$ with

$$T^{m_i} x_0 \to x_1 \quad \text{and} \quad \nu(T^{-m_i} E \times E) \geq \epsilon \quad \text{for every } i.$$  

Then there exist infinite subsets $B, C$ of $\mathbb{N}$ with $A \supset B + C$.

**Proof.** We use the following result of Bergelson:

**Lemma 4 ([1]).** Let $(Y, \mathcal{F}, \lambda)$ be a probability space, $\epsilon > 0$, and $(B_n)_{n \geq 1}$ a sequence in $\mathcal{F}$ with $\lambda(B_n) \geq \epsilon$ for every $n$. Then this sequence admits a subsequence $(B_{n_j})_{j \geq 1}$ such that

$$\lambda\left( \bigcap_{j=1}^k B_{n_j} \right) > 0.$$

Applying this lemma to the probability $\nu$ on $X \times X$ and to the sets $T^{-m_i} E \times E$, we obtain that the sequence $(m_i)_{i \geq 1}$ admits a subsequence, still written $(m_i)$, such that

$$\nu\left( \bigcap_{i=1}^r (T^{-m_i} E \times E) \right) > 0. \quad (2)$$

We write

$$L = \{ \ell \in \mathbb{N} : T^\ell x_1 \in E \}.$$  

For $n \in \mathbb{N}$, we have $(T \times T)^n(x_0,x_1) \in T^{-m_i} E \times E$ if and only if $n \in (A - m_i) \cap L$. By definition of $\nu$ and (2),

$$\text{for every } r \geq 1, \quad d_{\Phi}\left( \bigcap_{i=1}^r (A - m_i) \cap L \right) > 0. \quad (3)$$
and in particular this set is infinite.

We build by induction strictly increasing sequences \((b_n)_{n \geq 1}\) and \((c_n)_{n \geq 1}\) with

\[
b_n \in \{m_i : i \geq 1\} \text{ and } c_n \in L \text{ for every } n; \ b_i + c_j \in A \text{ for all } i, j \geq 1.
\]

Let \(n \geq 0\). If \(n = 0\) we chose an arbitrary \(c_1 \in L\). Assume that \(n \geq 1\) and that \(b_1, \ldots, b_n\) and \(c_1, \ldots, c_n\) are built with \(b_i + c_j \in A\) for all \(i, j \leq n\). By (3) applied with \(r = b_n\) we can chose \(c_{n+1} > c_n\) with

\[
c_{n+1} \in \bigcap_{i=1}^{n} (A - b_i) \cap L
\]

and we have \(b_i + c_{n+1} \in A\) for \(i \leq n\).

In both cases, for \(j \leq n + 1\) we have \(c_j \in L\) that is \(T^j x_1 \in E\). Since \(E\) is open and since \(T^m x_0 \to x_1\), for every sufficiently large \(i\) we have \(T^{m_i + c_j} x_0 \in E\) for every \(j \leq n + 1\), that is \(m_i + c_j \in A\). We chose \(i\) with this property and moreover \(m_i > b_n\) if \(n \geq 1\), and we set \(b_{n+1} = m_i\). We have \(b_{n+1} + c_j \in A\) for \(1 \leq j \leq n + 1\) and we are done. \(\square\)

### 3 Proof of Theorem 1

In this Section, \(A\) is a subset of \(\mathbb{N}\) with \(\delta = d^\ast(A) > 0\), and we build the different objects appearing in the statement of Theorem 3 and check that they satisfy the required conditions. Theorem 1 then follows from Theorem 3.

#### 3.1 Construction of a first system

Let \(\{0, 1\}^\mathbb{N}\) be endowed with the product topology and with the shift \(T\). Elements of \(\{0, 1\}^\mathbb{N}\) are written as \(x = (x(n))_{n \in \mathbb{N}}\). We consider \(I_A\) as an element of \(\{0, 1\}^\mathbb{Z}\) that we write \(x_0\), and define

(i) \(X\) is the closed orbit of \(x_0\) under \(T\).

Let \(E\) be the cylinder set \(E = \{x \in X : x(0) = 1\}\). We have:

(ii) \(E\) is a clopen subset of \(X\) and \(A = \{n \in \mathbb{N} : T^n x_0 \in E\}\).

Let \(\Phi\) be a Følner sequence with \(d_\Phi(A) = \delta\). Replacing \(\Phi\) by a subsequence we can assume that

(iii) \(x_0\) is generic along \(\Phi\) for some measure \(\mu\) on \(X\)

and since \(E\) is clopen we have

(iv) \(\mu(E) \geq \delta\).
3.2 Construction of an ergodic measure

Let \( \mu = \int_\Omega \mu_\omega dP(\omega) \) be the ergodic decomposition of \( \mu \) under \( T \). Since \( \int_\Omega \mu_\omega(E) dP(\omega) = \delta \), we have \( P(\{ \omega \in \Omega: \mu_\omega(E) \geq \delta \}) > 0 \). Therefore there exists an ergodic measure \( \mu' \) with \( \mu'(E) \geq \delta \).

By the pointwise ergodic theorem, the measure \( \mu' \) admits a generic point \( x_1 \in X \) in the usual sense, meaning that for every \( f \in \mathcal{C}(X) \) we have
\[
\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x_1) \to \int f d\mu \text{ as } N \to \infty.
\]

Since the orbit of \( x_0 \) is dense and \( T \) is continuous there exists a sequence \( (m_N)_{N \geq 1} \) of integers such that
\[
\text{for every } N \text{ and } 0 \leq n < N, \ d_X(T^{m_N+n} x_0, T^n x_1) < 1/N.
\]

For every \( f \in \mathcal{C}(X) \), by uniform continuity of \( f \) we have
\[
\frac{1}{N} \sum_{n=0}^{N-1} f(T^{m_N+n} x_0) \to \int f d\mu \text{ as } N \to \infty.
\]

This means that \( x_0 \) is generic of \( \mu' \) along the Følner sequence \( \Phi' = (\Phi'_N)_{N \geq 1} \) where \( \Phi'_N = [m_N, m_N + N) \).

Substituting \( \Phi' \) for \( \Phi \) and \( \mu' \) for \( \mu \), the properties (i)-(iv) remain valid and moreover

(v) \( \mu \) is ergodic under \( T \).

3.3 Using the Kronecker factor

We recall that the Kronecker factor \((Z, m_Z, S)\) of an ergodic system \((X, \mu, T)\) is a compact abelian group \( Z \) endowed with its Haar measure \( m_Z \), with a translation \( S \) and with a factor map \( \pi: X \to Z \), characterized by

for all \( \phi, \psi \in L^2(\mu) \) with \( E_\mu(\psi | Z) = 0 \)
\[
\text{for every } \varepsilon > 0, \quad d^*(\left\{ n \in \mathbb{N}: \left| \int T^n \psi \cdot \phi d\mu \right| \geq \varepsilon \right\}) = 0. \quad (4)
\]

The transformation \( S \) has the form
\[
\text{for every } z \in Z, \quad Sz = z + \alpha
\]
where \( \alpha \) is a fixed element of \( Z \).

It should be noticed that, even when \( \mu \) is an invariant ergodic measure on a topological dynamical system \((X, T)\), the factor map \( \pi: X \to Z \) is not continuous in general. The next Proposition explains how to modify the given system and obtain continuity. It is stated in [2, 3] for a distal system, and thus applies in particular to the isometric system \((Z, S)\).
Proposition 5 ([2, Proposition 6.1], [3, Proposition 24.3]). Let \((X, T)\) be a topological dynamical system, \(x_0 \in X\), and \(\mu\) an ergodic invariant probability measure supported on the closed orbit of \(x_0\) under \(T\). Let \((Z, m_Z, S)\) be the Kronecker factor of \((X, \mu, T)\), with factor map \(\pi : X \to Z\). Let \(X \times Z\) be endowed with the transformation \(T \times S\). Let \(\bar{\mu}\) be the measure on \(X \times Z\), image of \(\mu\) under the map \(x \mapsto (x, \pi(x)) : X \to X \times Z\). Then there exists a Følner sequence \(\Phi\) and a point \(z_0\) of \(Z\) such that \((x_0, z_0)\) is generic for \(\bar{\mu}\) along \(\Phi\).

The conclusion means that, for every \(f \in \mathcal{C}(X)\) and every \(h \in \mathcal{C}(Z)\) we have

\[
\int_X f \cdot h \circ \pi \, d\mu = \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} f(T^nx_0) h(S^nz_0).
\]

The first projection \(p : (X \times Z, T \times T) \to (X, T)\) is a factor map in the topological sense. Moreover, \(p : (X \times Z, \bar{\mu}, T \times T) \to (X, \mu, T)\) is an isomorphism in the ergodic sense. In particular, \((X \times Z, \bar{\mu}, T \times T)\) is ergodic, its Kronecker factor is \((Z, m_Z, T)\), the factor map is the second projection and thus is continuous. The key point in the Proposition is the genericity of \((x_0, z_0)\).

We apply this Proposition to the system \((X, \mu, T)\) and the point \(x_0\) introduced above and let \((Z, m_Z, S), \pi, \bar{\mu}\) and \(z_0\) and \(\Phi\) be given by this Proposition. Let \(\tilde{x}_0 = (x_0, z_0)\) and let \(\tilde{X}\) be the closed orbit of \(\tilde{x}_0\) in \(X \times Z\). Since the point \(\tilde{x}_0\) is generic for \(\bar{\mu}\) along \(\Phi\), \(\bar{\mu}\) is supported on \(\tilde{X}\). Let \(\tilde{\rho} : \tilde{X} \to X\) and \(\tilde{\pi} : \tilde{X} \to Z\) be the restriction to \(\tilde{X}\) of the two natural projections \(X \times Z \to X\) and \(X \times Z \to Z\) respectively, and let \(\tilde{E} = \tilde{\rho}^{-1}(E)\). Then \(\tilde{E}\) is a clopen subset of \(\tilde{X}\), \(A = \{n : T^n\tilde{x}_0 \in \tilde{E}\}\) and \(d_\Phi(A) = \bar{\mu}(\tilde{E}) = \mu(E) \geq \delta\).

Substituting \(\tilde{x}_0\) for \(x_0\), \(\tilde{X}\) for \(X\), \(\tilde{\Phi}\) for \(\Phi\), \(\tilde{\mu}\) for \(\mu\) and \(\tilde{\pi}\) for \(\pi\), the properties (i)-(v) above remain valid and moreover,

(vi) The factor map \(\pi : X \to Z\) from \(X\) to the Kronecker factor \((Z, m_Z, S)\) of \((X, \mu, T)\) is continuous.

We define \(z_0 = \pi(x_0)\).

3.4 Choosing a point \(x_1\)

We copy the next definition and results from [4].

**Definition ([4])**. A Følner sequence \(\Phi = (\Phi_N)\) is tempered if there exists a constant \(C > 0\) such that

\[
\left| \bigcup_{k < N} (\Phi_N - \Phi_k) \right| \leq C|\Phi_N|
\]

for every \(N\). (Here \(\Phi_N - \Phi_k = \{a - b : a \in \Phi_N, b \in \Phi_k\}\).)

**Lemma 6 ([4])**. Every Følner sequence admits a tempered subsequence.

**Theorem 7 ([4])**. If \((X, \mu, T)\) is a system and \(\Phi\) is a tempered Følner sequence in \(\mathbb{N}\) then for every \(f \in L^1(\mu)\) the limit

\[
\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} f(T^nx)
\]

exists of \(\mu\)-almost every \(x\). If \(\mu\) is ergodic, then the limit is the constant \(\int f \, d\mu\).
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**Corollary 8.** Let \((X, T)\) be a topological dynamical system, \(\mu\) an ergodic measure on \(X\) and \(\Phi\) a tempered Følner sequence. Then \(\mu\)-almost every \(x \in X\) is generic for \(\mu\) along \(\Phi\).

**Proof.** It suffices to apply Theorem 7 to all functions in a countable subset of \(\mathcal{C}(X)\), dense in \(\mathcal{C}(X)\) under the uniform norm. □

Returning to our construction, we use Lemma 6 and replace the Følner sequence with a tempered subsequence, still written \(\Phi\). We write

\[ X_1 = \{ x \in X : x \text{ is generic for } \mu \text{ along } \Phi \}. \]

By Corollary 8 we have \(\mu(X_1) = 1\). We define also

\[ X_2 \text{ is the topological support of } \mu. \]

We have \(\mu(X_2) = 1\). Since the image of \(\mu\) under \(\pi\) is equal to \(m_Z\) we have \(\pi(X_2) = Z\).

We write

\[ \phi = \mathbb{E}_\mu (1_E | Z) \text{ and } \psi = 1_E - \phi \circ \pi. \]

We chose an open neighborhood \(V\) of 0 in \(Z\) such that

\[ \int_Z |\phi(z) - \phi(z + \xi)| \, dm_Z(z) < \delta^2/4 \text{ for every } \xi \in V \] (6)

and define

\[ X_3 = \pi^{-1}(V + z_0). \]

Note that for every \(n\) we have \(T^n x_0 \in X_3\) if and only if \(n x \in V\).

\(X_3\) is an open neighborhood of \(x_0\) and, since \(m_Z(V + z_0) > 0\), we have \(\mu(X_3) > 0\) and we can chose

\[ x_1 \in X_1 \cap X_2 \cap X_3. \]

We write

\[ z_1 = \pi(x_1) \text{ and } \beta = z_1 - z_0. \]

Since \(x_1 \in X_3\) we have \(z_1 \in V + z_0\) and

\[ \beta \in V. \]

### 3.5 The joining \(\nu\)

Replacing the Følner sequence \(\Phi\) by a subsequence we can assume that the point \((x_0, x_1)\) is generic along \(\Phi\) for a measure \(\nu\) on \(X \times X\). This measure is invariant under \(T \times T\); since \(x_0\) is generic for \(\mu\), the image of \(\nu\) under the first projection is equal to \(\mu\); since \(x_1 \in X_1\), \(x_1\) is generic for \(\mu\) too, and the second projection of \(\nu\) is equal to \(\mu\). We have that \((X \times X, \nu, T \times T)\) is a joining of \((X, \mu, T)\) with itself.

Let \(\lambda\) be the image of \(\nu\) under the projection \(\pi \times \pi : X \times X \to Z \times Z\). Then \((Z \times Z, \lambda, S \times S)\) is a joining of \((Z, m_Z, S)\) with itself. Since \((x_0, x_1)\) is generic for \(\nu\) along \(\Phi\), \((z_0, z_1)\) is generic for \(\lambda\) along the same Følner sequence. Therefore, \(\lambda\) is supported on a single closed orbit under \(S \times S\) and, since \(S \times S\) is a...
translation on the compact abelian group \( Z \times Z \), the measure \( \lambda \) is ergodic under \( S \times S \), and for \( \lambda \)-almost every \((z,z')\) we have \( z' - z = z_1 - z_0 = \beta \). It follows that for all \( h,h' \in \mathcal{C}(Z) \) we have
\[
\int_{Z \times Z} h(z)h'(z') \, d\lambda(z,z') = \int_Z h(z)h'(z+\beta) \, dm_Z(z). \tag{7}
\]
By density the same relation holds for \( h,h' \in L^2(m_Z) \).

Let \( \theta \in L^2(\mu) \) be the image of \( 1_E \) under the Markov operator associated to the joining \( \nu \). This function is characterized by
\[
\text{for every } h \in L^2(\mu), \int_{X \times X} h \otimes 1_E \, d\nu = \int_X h \cdot \theta \, d\mu.
\]

For every \( n \) we have
\[
\nu(T^{-n}E \times E) - \int_{X \times X} (S^n \phi \circ \pi) \otimes 1_E \, d\nu = \int_{X \times X} T^n \psi \otimes 1_E \, d\nu = \int_X T^n \psi \cdot \theta \, d\mu.
\]
Since \( E \mu(\psi | Z) = 0 \), by the characterization (4) of the Kronecker factor we have
\[
d^\ast\left( \left\{ n \in \mathbb{N} : \left| \nu(T^{-n}E \times E) - \int_{X \times X} (S^n \phi \circ \pi) \otimes 1_E \, d\nu \right| \geq \delta^2/8 \right\} \right) = 0. \tag{8}
\]

For every \( n \), by definition of the measure \( \lambda \) we have
\[
\int_{X \times X} (S^n \phi \circ \pi) \otimes 1_E \, d\nu - \int_{X \times X} (S^n \phi) \otimes 1_E \, d\nu = \int_{X \times X} (S^n \phi \circ \pi) \otimes \psi \, d\nu = \int_{X \times X} (\phi \circ \pi) \otimes (T^{-n} \psi) \, d\nu.
\]

Exchanging the role of the coordinates and proceeding as above we obtain
\[
d^\ast\left( \left\{ n \in \mathbb{N} : \left| \int_{X \times X} (S^n \phi \circ \pi) \otimes 1_E \, d\nu - \int_{Z \times Z} (S^n \phi) \otimes \phi \, d\lambda \right| \geq \delta^2/8 \right\} \right) = 0. \tag{9}
\]

Combining (8) and (9) and using Formula (7) for \( \lambda \) and formula (5) for \( S \) we obtain that
\[
d^\ast\left( \left\{ n \in \mathbb{N} : \left| \nu(T^{-n}E \times E) - \int_Z \phi(z+n\alpha) \, \phi(z+\beta) \, dm_Z(z) \right| \geq \delta^2/4 \right\} \right) = 0.
\]

We use now the fact that \( \beta \in V \), the definition of this set and the bound \( \int \phi^2 \, dm_Z \geq \delta^2 \) and we obtain that the set
\[
\Lambda = \{ n \in \mathbb{N} : n\alpha \in V \text{ and } \nu(T^{-n}E \times E) \leq \delta/4 \}
\]
satisfies
\[
d^\ast(\Lambda) = 0.
\]
3.6 Conclusion

We check now that the point $x_1$ and the measure $\nu$ satisfy all the hypothesis needed to apply Theorem 3. It remains only to build a good sequence $(m_i)$.

For every integer $i > 0$ the set

$$X_3 \cap \{ x \in X : d_X(x, x_1) < 1/i \}$$

is an open neighborhood of $x_1$, and since $x_1$ belongs to the topological support $X_2$ of $\mu$, the measure of this set is $> 0$ and thus the set

$$F_i = \{ n \in \mathbb{N} : T^n x_0 \in X_3 \text{ and } d_X(T^n x_0, x_1) < 1/i \}$$

satisfies $\delta\Phi(F_i) > 0$. Since $\delta\Phi(\Lambda) = 0$, $\delta\Phi(F_i \setminus \Lambda) > 0$ and in particular $F_i \setminus \Lambda$ is infinite. We thus can chose a sequence of integers $(m_i)$, tending to $\infty$, with $m_i \in F_i \setminus \Lambda$ for every $i$. We have that

$$T^{m_i} x_0 \to x_1 \text{ as } i \to \infty$$

and, for every $i$,

$$T^{m_i} x_0 \in X_3 \text{ and thus } m_i \alpha \in V;$$

$$m_i \notin \Lambda \text{ and thus } \nu(T^{-m_i} E \times E) > \delta^2/4.$$  

All the hypothesis of Theorem 3 are satisfied with $\epsilon = \delta^2/4$ and we are done.

4 Proof of Proposition 2

Let $(X, T)$ be an uniquely ergodic topological dynamical system with invariant measure $\mu$ and assume that the measure preserving system $(X, \mu, T)$ is mixing. Let $U \subset X$ be an open subset with $0 < \mu(U) \leq \mu(\overline{U}) < 1$, $x_0 \in X$ and

$$A = \{ n \in \mathbb{N} : T^n x_0 \in U \}. $$

Since $\mu(U) > 0$ and by unique ergodicity, $d^*(A) > 0$. We show that it does not contain a sum $B + C$ with $d^*(B) > 0$ and $C$ infinite.

Assume by contradiction that $B$ and $C$ exist with these properties and let

$$K = \{ T^n x_0 : n \in B \}. $$

By unique ergodicity,

$$\mu(K) \geq d^*(\{ n \in \mathbb{N} : T^n x_0 \in K \}) \geq d^*(B) > 0.$$  

On the other hand, for every $c \in C$ we have $T^c \{ T^b x_0 : b \in B \} \subset U$ and thus

for every $c \in C$, $T^c K \subset \overline{U}.$

Since $(X, \mu, T)$ is mixing and since $K$ and $X \setminus \overline{U}$ have positive measure, for every sufficiently large $n$ we have $\mu(T^n K \cap (X \setminus \overline{U})) > 0$ and thus $T^n K \not\subset \overline{U}$. We have a contradiction. \qed
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