SHARP HARDY-SOBOLEV-MAZ’YA, ADAMS AND HARDY-ADAMS
INEQUALITIES ON THE SIEGEL DOMAINS AND COMPLEX
HYPERBOLIC SPACES

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ABSTRACT. This paper continues the program initiated in the works by the authors [60], [61]
and [62] and by the authors with Li [51] and [52] to establish higher order Poincaré-Sobolev,
Hardy-Sobolev-Maz’ya, Adams and Hardy-Adams inequalities on real hyperbolic spaces
using the method of Helgason-Fourier analysis on the hyperbolic spaces. The aim of this
paper is to establish such inequalities on the Siegel domains and complex hyperbolic spaces.
Firstly, we give a factorization theorem for the operators on the complex hyperbolic space
which is closely related to Geller’ operator, as well as the CR invariant differential operators
on the Heisenberg group and CR sphere. Secondly, by using, among other things, the
Kunze-Stein phenomenon on a closed linear group SU(1, n) and Helgason-Fourier analysis
techniques on the complex hyperbolic spaces, we establish the Poincaré-Sobolev, Hardy-
Sobolev-Maz’ya inequality on the Siegel domain $U^n$ and the unit ball $B^n_C$. Finally, we
establish the sharp Hardy-Adams inequalities and sharp Adams type inequalities on Sobolev
spaces of any positive fractional order on the complex hyperbolic spaces. The factorization
theorem we proved is of its independent interest in the Heisenberg group and CR sphere
and CR invariant differential operators therein.

1. INTRODUCTION

Let $B^n_C$ denote the unit ball in $C^n$. In [25], Geller introduced a family of second order
degenerate elliptic operators arising in several variables

$$\Delta_{\alpha,\beta} = 4(1-|z|^2) \left\{ \sum_{j=1}^{n} \sum_{k=1}^{n} (\delta_{j,k} - z_j \bar{z}_k) \frac{\partial^2}{\partial z_j \partial \bar{z}_k} + \alpha R + \beta \bar{R} - \alpha \beta \right\},$$

where $\delta_{j,k}$ denotes the Kronecker symbol and

$$R = \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}, \quad \bar{R} = \sum_{j=1}^{n} \bar{z}_j \frac{\partial}{\partial \bar{z}_j}.$$
If $\alpha = \beta = 0$, $\Delta_{0,0}$ is the invariant Laplacian or Laplace-Beltrami operator for the Bergman metric on $\mathbb{B}^n_C$. Set

$$\Delta'_{\alpha,\beta} = \frac{1}{4(1 - |z|^2)} \Delta_{\alpha,\beta} = \sum_{j=1}^{n} \sum_{k=1}^{n} (\delta_{j,k} - z_j z_k) \frac{\partial^2}{\partial z_j \partial \overline{z}_k} + \alpha R + \beta \overline{R} - \alpha \beta.$$

The above family of operators $\Delta_{\alpha,\beta}$ and $\Delta'_{\alpha,\beta}$ have been considered by many authors. For example, Geller [25] introduced such operators to discuss the $H^p$ theory of Hardy spaces on the Heisenberg group and Graham [28], [29] has given a precise description of the associated Dirichlet problem. We remark that such operators are closely related to the invariant differential operator on the line bundle of $SU(1,n)/S(U(n) \times U(1))$ associated with the one-dimensional representation of $S(U(n) \times U(1))$ (see [73]).

There are analogous operators on the Siegel domain $U^n := \{ z \in \mathbb{C}^n : \varrho(z) = \text{Im} z_n - \sum_{j=1}^{n-1} |z_j|^2 > 0 \}$.

Let $t = \text{Re} z_n$ and $L_0$ be the Folland-Stein operator on the Heisenberg group (see Section 2 for details). The analogous operators are then

$$P_{\alpha,\beta} = 4 \varrho[\varrho \varrho_{\varrho} + \varrho^2] - L_0 - (n-1+\alpha+\beta) \varrho - i(\alpha-\beta) \partial_t].$$

If $\alpha = \beta = 0$, $P_{0,0}$ is also the Laplace-Beltrami operator for the Bergman metric on $U^n$.

The purpose of this paper is to establish Hardy-Sobolev-Maz’ya, Adams and Hardy-Adams type inequalities for $\Delta_{0,0}$ on $\mathbb{B}^n_C$ with the Bergman metric, that is, on the complex hyperbolic space. Moreover, we should mention that our result are closely related to operator $\Delta'_{\alpha,\alpha}$ as well as the CR invariant differential operators on the Heisenberg group and the CR sphere.

1.1. Higher order Hardy-Sobolev-Maz’ya inequalities on hyperbolic spaces. Maz’ya ([65], Section 2.1.6) gave a refinement of both the first order Sobolev and the Hardy inequalities in half spaces which are known as the Hardy-Sobolev-Maz’ya inequality. It reads as follows

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 dx - \frac{1}{4} \int_{\mathbb{R}^n_+} \frac{u^2}{x_1} dx \geq C_n \left( \int_{\mathbb{R}^n_+} x_1^\gamma |u|^p dx \right)^{\frac{2}{p}}, \quad u \in C^\infty_0(\mathbb{R}^n_+),$$

where $2 < p \leq \frac{2n}{n-2}$, $\gamma = \frac{(n-2)p}{n-2} - n$, $\mathbb{R}^n_+ = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_1 > 0\}$ and $C_n$ is a positive constant which is independent of $u$.

We recall that the classical Sobolev inequalities and sharp constants play an essential role in analysis and geometry. If $k$ is a positive integer, then the $k$–th order Sobolev inequality states as follows (see [53, 13, 79])

$$\int_{\mathbb{R}^n} |(-\Delta)^{k/2} u|^2 dx \geq S_{n,k} \left( \int_{\mathbb{R}^n} |u|^\frac{2n}{n-2k} dx \right)^\frac{n-2k}{n}, \quad u \in C^\infty_0(\mathbb{R}^n), \quad 1 \leq k < \frac{n}{2},$$
where
\[ S_{n,k} = \frac{\Gamma\left(\frac{n+2k}{2}\right)}{\Gamma\left(\frac{n-2k}{2}\right)} \omega_n^{2k/n} \]
is the best Sobolev constant and \( \omega_n = \frac{2^n \pi^{n+1}}{\Gamma\left(n+\frac{3}{2}\right)} \) is the surface measure of \( S^n = \{ x \in \mathbb{R}^{n+1} : |x| = 1 \} \).

It is known that the upper half space can be regarded as a hyperbolic space. Then the Hardy-Sobolev-Maz’ya inequality on the upper half space is equivalent to the corresponding inequality on the hyperbolic ball. By using the Helgason-Fourier analysis on hyperbolic spaces, the authors established in [60] the sharp Hardy-Sobolev-Maz’ya inequalities for higher order derivatives. To state the results, we now recall the ball model as a hyperbolic space. The unit ball
\[ B^n = \{ x = (x_1, \cdots, x_n) \in \mathbb{R}^n ||x|| < 1 \} \]
equipped with the usual Poincaré metric
\[ ds^2 = \frac{4(dx_1^2 + \cdots + dx_n^2)}{(1-|x|^2)^2} \]
is known as the hyperbolic space of ball model. The hyperbolic gradient is \( \nabla_H = \frac{1-|x|^2}{2} \nabla \) and the Laplace-Beltrami operator is given by
\[ \Delta_H = \frac{1-|x|^2}{4} \left\{ (1-|x|^2) \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} + 2(n-2) \sum_{j=1}^n x_j \frac{\partial}{\partial x_j} \right\}. \]
The hyperbolic volume is \( dV = \left(\frac{2}{1-|x|^2}\right)^n dx. \)

Then the GJMS operators on \( B^n \) is defined as follows (see [18], [31], [26], [44])
\[ P_k = P_1(P_1 + 2) \cdots \cdots (P_1 + k(k - 1)), \quad k \in \mathbb{N}, \]
where \( P_1 = -\Delta_{\mathbb{H}} - \frac{n(n-2)}{4} \) is the conformal Laplacian on \( B^n \) and \( P_2 \) is the well-known Paneitz operator (see [69]). Then, the authors have proved in [60] the following higher order Hardy-Sobolev-Maz’ya inequalities on the real hyperbolic spaces.

**Theorem 1.1.** Let \( 2 \leq k < \frac{n}{2} \) and \( 2 < p \leq \frac{2n}{n-2k} \). There exists a positive constant \( C = C(n, k, p) \) such that for each \( u \in C_0^\infty (B^n) \),
\[ \int_{B^n} (P_k u) u dV - \prod_{i=1}^k \frac{(2i - 1)^2}{4} \int_{B^n} u^2 dV \geq C \left( \int_{B^n} |u|^p dV \right)^{\frac{2}{p}}. \]

This improves substantially the Poincaré-Sobolev inequalities (1.1) for higher order derivatives on the hyperbolic spaces \( B^n \) by subtracting a Hardy term on the left hand side. If \( p = \frac{2n}{n-2k} \), then by (1.11), the sharp constant in (1.5) is less than or equal to the best \( k \)-th order Sobolev constant \( S_{n,k} \).

As an application of Theorem 1.1, we have the following Hardy-Sobolev-Maz’ya inequalities for higher order derivatives (see [60]):
Theorem 1.2. Let \( 2 \leq k < \frac{n}{2} \) and \( 2 < p \leq \frac{2n}{n-2k} \). There exists a positive constant \( C \) such that for each \( u \in C_0^\infty(\mathbb{R}^n_+) \),

\[
\int_{\mathbb{R}^n_+} |\nabla^k u|^2 \, dx - \prod_{i=1}^{k} \frac{(2i-1)^2}{4} \int_{\mathbb{R}^n_+} \frac{u^2}{x_1^2} \, dx \geq C \left( \int_{\mathbb{R}^n_+} x_1^\gamma |u|^p \, dx \right)^\frac{2}{p},
\]

where \( \gamma = \frac{(n-2k)p}{2} - n \).

In terms of the Poincaré ball model \( \mathbb{B}^n \), inequality (1.5) can be written as follows:

\[
\int_{\mathbb{B}^n} |\nabla^k u|^2 \, dx - \prod_{i=1}^{k} \frac{(2i-1)^2}{4} \int_{\mathbb{B}^n} \frac{u^2}{(1-|x|^2)^{2k}} \, dx \geq C \left( \int_{\mathbb{B}^n} (1-|x|^2)^\gamma |u|^p \, dx \right)^\frac{2}{p}.
\]

We remark that the best constant in the above Hardy-Sobolev-Maz’ya inequalities when \( k = 1 \) and \( n = 3 \) is the same as the Sobolev constant (see [8]) and is strictly smaller than the Sobolev constant (see [33]). In the higher order derivative cases, it was proved in all the cases of \( n = 2k + 1 \), the best constants are the same as the Sobolev constants [61] (see also [35]).

To state our results on the complex hyperbolic spaces, let us introduce some conventions. We denote by \( U^n \) the Siegel domain model and \( \mathbb{B}^n_C \) the ball model of complex hyperbolic space (see Section 2 for details). We may assume \( n \geq 2 \) since in the case \( n = 1 \), the complex hyperbolic space is isomorphic to the real hyperbolic space of dimension two. Let \( \Delta_B \) be the Laplace-Beltrami operator on the complex hyperbolic space. The spectral gap of \( -\Delta_B \) is \( n^2 \).

In fact, we have

\[
\text{spec}(-\Delta_B) = [n^2, +\infty).
\]

As one of our main theorems in this paper, firstly, we have the following factorization theorem for the operators on the complex hyperbolic space. This factorization theorem is important in establishing our Hardy-Sobolev-Maz’ya inequalities on complex hyperbolic spaces. We refer the reader to Section 2 for the notations.

Theorem 1.3. Let \( a \in \mathbb{R} \) and \( k \in \mathbb{N} \setminus \{0\} \). In terms of the Siegel domain model, we have, for \( u \in C^\infty(U^n) \),

\[
\prod_{j=1}^{k} \left[ \delta_{\partial^2} + a \delta_{\partial} + \delta T^2 - \mathcal{L}_0 - i(k+1-2j)T \right] (\phi^{-\frac{k-a}{2}} u)
= 4^{-k} \phi^{-\frac{k+a}{2}} \prod_{j=1}^{k} \left[ \Delta_B + n^2 - (a - k + 2j - 2)^2 \right] u.
\]

In terms of the ball model, we have, for \( f \in C^\infty(\mathbb{B}_C^n) \),

\[
\prod_{j=1}^{k} \left[ \Delta_{\frac{k-a}{2}, \frac{k-a}{4}} + \frac{(k+1-2j)^2}{4} - \frac{k+1-2j}{2} (R - \bar{R}) \right] [(1-|z|^2)^{-\frac{k-a}{2}} f]
= 4^{-k} (1-|z|^2)^{-\frac{k+a}{2}} \prod_{j=1}^{k} \left[ \Delta_B + n^2 - (a - k + 2j - 2)^2 \right] f.
\]
We remark that the operators on the left side of (1.8) and (1.9) are closely related to Geller’s operator, as well as the CR invariant differential operators on the Heisenberg group and CR sphere, respectively (see Section 2) and can have important applications on the Heisenberg group and the CR sphere.

Secondly, we establish in this paper the following Poincaré-Sobolev inequalities on $\mathbb{B}_n^C$. For simplicity, we denote by

$$
\|u\|_p = \left( \int_{\mathbb{B}_n^C} |u|^p dV \right)^{\frac{1}{p}}.
$$

**Theorem 1.4.** Let $0 < \alpha < 3$ and $\zeta > 0$. If $0 < \beta < 2n - \alpha$, then for $2 < p \leq \frac{4n}{2n-(\alpha+\beta)}$ there exists $C > 0$ such that for all $u \in C_0^\infty(\mathbb{B}_n^C)$,

$$
\|u\|_p \leq C \|(-\Delta_{\mathbb{B}} - n^2 + \zeta^2)^{\beta/4}(-\Delta_{\mathbb{B}} - n^2)^{\alpha/4} u\|_2.
$$

If $\beta = 2n - \alpha$, then for $p > 2$, we have

$$
\|u\|_p \leq C \|(-\Delta_{\mathbb{B}} - n^2 + \zeta^2)^{\beta/4}(-\Delta_{\mathbb{B}} - n^2)^{\alpha/4} u\|_2.
$$

As the application of Theorems 1.3 and 1.4, we have the following Hardy-Sobolev-Maz’ya inequalities on the complex hyperbolic space, which is another main result of this paper.

**Theorem 1.5.** Let $a \in \mathbb{R}$, $1 \leq k < n$ and $2 < p \leq \frac{2n}{n-k}$ and $\mathbb{H}^{n-1}$ is the Heisenberg group. In terms of the Siegel domain model, there exists a positive constant $C$ such that for each $u \in C_0^\infty(U^n)$ we have

$$
\int_{\mathbb{H}^{2n-1}} \int_0^\infty \prod_{j=1}^k \left[ -\partial_{\varphi} \varphi - a \partial_\rho - \varphi T^2 + \mathcal{L}_0 + i(k+1-2j)T \right] u \frac{dz dt d\varphi}{\rho^{1-a}}
$$

$$
- \prod_{j=1}^k \frac{(a-k+2j-2)^2}{4} \int_{\mathbb{H}^{2n-1}} \int_0^\infty \frac{u^2}{\rho^{k+1-a}} dz dt d\varphi
$$

$$
\geq C \left( \int_{\mathbb{H}^{2n-1}} \int_0^\infty |u|^p \rho^\gamma dz dt d\varphi \right)^{\frac{1}{p}},
$$

where $\gamma = \frac{(n-k+a)p}{2} - n - 1$. In terms of the ball model, we have for $f \in C_0^\infty(\mathbb{B}_n^C)$,

$$
\int_{\mathbb{B}_n^C} f \prod_{j=1}^k \left[ \Delta'_{1-a,n,1-a-n} + \frac{(k+1-2j)^2}{4} - \frac{k+1-2j}{2} (R - \bar{R}) \right] f \frac{dz}{(1-|z|^2)^{1-a}}
$$

$$
- \prod_{j=1}^k \frac{(a-k+2j-2)^2}{4} \int_{\mathbb{B}_n^C} \frac{f^2}{(1-|z|^2)^{k+1-a}} dz
$$

$$
\geq C \left( \int_{\mathbb{B}_n^C} |f|^p (1-|z|^2)^\gamma dz \right)^{\frac{1}{p}}.
$$
1.2. Adams and Hardy-Adams inequalities on hyperbolic spaces. We note the Trudinger-Moser inequality is a borderline case of the Sobolev embedding $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ where $p < N$ when $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with $1 \leq q \leq \frac{Np}{N-p}$, Trudinger [77] proved that $W_0^{1,N}(\Omega) \subset L^{\varphi_N}(\Omega)$, where $L^{\varphi_N}(\Omega)$ is the Orlicz space associated with the Young function $\varphi_N(t) = \exp(\beta |t|^{N/N-1}) - 1$ for some $\beta > 0$ (see also Yudovich [80], Pohozaev [70]). The sharp form of this inequality was established by J. Moser [66] by identifying the best constant for $\beta$.

In 1988, D. Adams extended the Trudinger-Moser inequality to high order Sobolev spaces and proved the following

**Theorem 1.6.** Let $\Omega$ be a domain in $\mathbb{R}^n$ with finite $n$-measure and $m$ be a positive integer less than $n$. There is a constant $c_0 = c_0(m, n)$ such that for all $u \in C^m(\mathbb{R}^n)$ with support contained in $\Omega$ and $\|\nabla^m u\|_{n/m} \leq 1$, the following uniform inequality holds

$$
\frac{1}{|\Omega|} \int_{\Omega} \exp(\beta_0(m, n)|u|^{n/(n-m)}) \, dx \leq c_0,
$$

where

$$
\beta_0(m, n) = \begin{cases} 
\frac{n}{\omega_{n-1}} \left[ \frac{\pi^{n/2} m^{((m+1)/2)}}{\Gamma((n-m+1)/2)} \right]^{n/(n-m)}, & m = \text{odd}; \\
\frac{n}{\omega_{n-1}} \left[ \frac{\pi^{n/2} m^{(m)/2}}{\Gamma((n-m)/2)} \right]^{n/(n-m)}, & m = \text{even}
\end{cases}
$$

and $\omega_{n-1}$ is the surface measure of the unit sphere in $\mathbb{R}^n$. Furthermore, the constant $\beta_0(m, n)$ in (1.12) is sharp in the sense that if $\beta_0(m, n)$ is replaced by any larger number, then the integral in (1.12) cannot be bounded uniformly by any constant.

Such inequalities have been generalized in many directions by many authors. For other Trudinger-Moser-Adams inequalities inequalities on $\mathbb{R}^n$, the Heisenberg group and CR sphere, hyperbolic spaces and Riemannian manifolds, we refer to, to just name a few, [5], [6], [7], [9], [10], [11], [12], [45], [49], [57], [58], [54], [59], [63], [72], [81], and many references therein.

More recently, Hardy-Adams inequalities have been established for all the case $n \geq 3$ (see [62, 51] and [82], Corollary 1.4). We state it here as follows.

**Theorem 1.7.** Let $n \geq 3$, $\zeta > 0$ and $0 < s < 3/2$. Then there exists a constant $C_{\zeta, n} > 0$ such that for all $u \in C_0^\infty(\mathbb{B}^n)$ with

$$
\int_{\mathbb{B}^n} |(-\Delta_H - (n-1)^2/4)^{s/2}(-\Delta_H - (n-1)^2/4 + \zeta^2)^{n-s/4} u|^2 \, dV \leq 1,
$$

there holds

$$
\int_{\mathbb{B}^n} (e^{\beta_0(n/2, n) u^2} - 1 - \beta_0(n/2, n) u^2) \, dV \leq C_{\zeta, n},
$$

where $dV = \left(\frac{2}{1-|x|^2}\right)^n \, dx$ is the hyperbolic volume element and $\Delta_H$ is the Laplace-Beltrami operator on hyperbolic space $\mathbb{B}^n$. 

We remark that the proof of Theorem 1.7 relies on hard analysis of Green's functions estimates and Fourier analysis on hyperbolic spaces. By the following factorization theorem for the operators on real hyperbolic space (see [55] for the ball model and [60] for the half space model)

\begin{equation}
\left( \frac{1 - |x|^2}{2} \right)^{k + \frac{\eta}{2}} \Delta^k \left[ \left( \frac{1 - |x|^2}{2} \right)^{k + \frac{\eta}{2}} f \right] = \prod_{i=1}^{k} \left[ \Delta_{\mathbb{H}} + \frac{(n - 2i)(n + 2i - 2)}{4} \right] f,
\end{equation}

we have, by Theorem 1.7, the following Hardy-Adams inequalities

**Theorem 1.8.** There exists a constant $C > 0$ such that for all $u \in C_0^\infty(\mathbb{B}_n)$ with

\[
\int_{\mathbb{B}_n} |\nabla^2 u|^2 dx - \frac{n}{2} \int_{\mathbb{R}^n} \frac{u^2}{(1 - |x|^2)^n} dx \leq 1,
\]

there holds

\[
\int_{\mathbb{B}_n} e^{\beta_0(n/2,n)u^2} - 1 - \beta_0(n/2,n)u^2 (1 - |x|^2)^n dx \leq C.
\]

We should mention that the Hardy-Trudinger-Moser inequality on the unit ball $\mathbb{B}_n \subset \mathbb{R}^n$ when $n = 2$ was first established by Wang and Ye [78]. In a recent paper [59], the authors give a rearrangement-free argument of the result in [78] and confirm that the result in [78] holds for any bounded and convex domain in $\mathbb{R}^2$ as conjectured in [78]. The method in [59] is via an argument from local inequalities to global ones together with the Riemann mapping theorem.

In this paper, we will establish the following Adams inequalities on the complex hyperbolic space.

**Theorem 1.9.** Let $0 < \alpha < 3$ and $\zeta > 0$. Then there exists a constant $C > 0$ such that for all $u \in C_0^\infty(\mathbb{B}_n)$ with

\[
\|(-\Delta_{\mathbb{B}} - n^2 + \zeta^2)^{(2n-\alpha)/4}(-\Delta_{\mathbb{B}} - n^2)^{\alpha/4} u\|_2 \leq 1,
\]

there holds

\[
\int_{\mathbb{B}_n} (e^{\beta_0(n,2n)u^2} - 1 - \beta_0(n,2n)u^2) dV \leq C.
\]

As an application of Theorem 1.9, we have the following Hardy-Adams inequalities on the complex hyperbolic space: both for the complex ball model and the Siegel domain model, which is another main theorem in this paper.

**Theorem 1.10.** Let $a \in \mathbb{R}$. There exists a constant $C > 0$ such that for all $u \in C_0^\infty(\mathbb{B}_n)$ with

\[
4^{n-1} \int_{\mathbb{B}_n^C} f \prod_{j=1}^{n} \left[ \Delta_{1 - a} - \frac{(n + 1 - 2j)^2}{4} - \frac{n + 1 - 2j}{2} (R - \bar{R}) \right] f \frac{dz}{(1 - |z|^2)^{1-a}} - \frac{1}{4} \prod_{j=1}^{k} (a - k + 2j - 2) \int_{\mathbb{B}_n^C} \frac{u^2}{(1 - |z|^2)^{n+1-a}} dz \leq 1,
\]
there holds
\[ \int_{\mathbb{B}_C^n} e^{\beta_0(n, 2n)(1-|z|^2)u^2} - 1 - \beta_0(n, 2n)(1-|z|^2)u^2 \frac{dz}{(1-|z|^2)^{n+1-a}} \leq C \]
and
\[ \int_{\mathbb{B}_C^n} e^{\beta_0(n, 2n)(1-|z|^2)u^2} dz \leq C, \quad \text{if } a \leq n + 1. \]

In terms of the Siegel domain model, we have for all \( u \in C_0^\infty(U^n) \) with
\[
4^{n-1} \int_{\mathbb{H}^{2n-1}} \int_0^\infty u \prod_{j=1}^n \left[ -\varrho \partial_{\varrho} - a \partial_\theta + \varrho T^2 + L_0 + i(k+1-2j)T \right] u^{\frac{dzd\varrho}{\varrho^{1-a}}} - \frac{1}{4} \prod_{j=1}^n (a-n+2j-2) \int_0^\infty \frac{u^2}{\varrho^{n+1-a}} d\varrho d\varrho \leq 1,
\]
there holds
\[ \int_{\mathbb{H}^{2n-1}} \int_0^\infty e^{\beta_0(n, 2n)\varrho^a u^2} - 1 - \beta_0(n, 2n)\varrho^a u^2 \frac{dzd\varrho}{\varrho^{n+1-a}} \leq C. \]

Finally, we also set up some Adams type inequalities on Sobolev spaces \( W^{\alpha, \frac{2n}{n}}(\mathbb{B}_C^n) \) for arbitrary positive fractional order \( \alpha < 2n \). More precisely, we will prove the following local result:

**Theorem 1.11.** Let \( n \geq 2, 0 < \alpha < 2n \) be an arbitrary real positive number, \( p = \frac{2n}{\alpha} \) and \( \zeta \) satisfies \( \zeta > 0 \) if \( 1 < p < 2 \) and \( \zeta > 2n(\frac{1}{2} - \frac{1}{p}) \) if \( p \geq 2 \). Then for measurable \( E \) with finite \( 2n \)-dimensional complex hyperbolic volume measure in \( \mathbb{B}_C^n \), there exists \( C = C(\zeta, \alpha, n, |E|) \) such that
\[
\frac{1}{|E|} \int_E \exp(\beta(n, \alpha)|u|^p) dV \leq C
\]
for any \( u \in W^{\alpha, p}(\mathbb{B}_C^n) \) with \( \int_{\mathbb{B}_C^n} |(-\Delta_\mathbb{B} - n^2 + \zeta^2)^\frac{3}{2} u|^p dV \leq 1 \). Here \( p' = \frac{p}{p-1} \) and
\[
\beta(2n, \alpha) = \frac{2n}{\omega_{2n-1}} \left[ \frac{\pi^{n2}\alpha\Gamma(\alpha/2)}{\Gamma(\frac{2n-\alpha}{2})} \right]^{p'}. \]

Furthermore, this inequality is sharp in the sense that if \( \beta(n, \alpha) \) is replaced by any \( \beta > \beta(n, \alpha) \), then the above inequality can no longer hold with some \( C \) independent of \( u \).

It is fairly standard by now, we can establish the following global Hardy-Adams inequalities on the complex hyperbolic spaces using the symmetrization-free argument of the local inequalities to global ones developed earlier by Lam and the first author [48, 49].

**Theorem 1.12.** Let \( n \geq 2, 0 < \alpha < 2n \) be an arbitrary real positive number, \( p = \frac{2n}{\alpha} \) and \( \zeta \) satisfies \( \zeta > 2n \left| \frac{1}{2} - \frac{1}{p} \right| \). Then there exists \( C = C(\zeta, \gamma, n) \) such that
\[
(1.15) \quad \int_{\mathbb{B}_C^n} \Phi_p(\beta(2n, \alpha)|u|^p) dV \leq C
\]
and

\[(1.16) \quad \int_{B_n^C} e^{\beta(2n,\alpha)|u|^p'} \, dz \leq C\]

hold simultaneously for any \( u \in W^{\alpha,p}(B_n^C) \) with \( \int_{B_n^C} |(-\Delta_{B} - n^2 + \zeta^2)^{\frac{\alpha}{2}} u|^p \, dV \leq 1 \). Here

\[\Phi_p(t) = e^t - \sum_{j=0}^{j_p-2} \frac{t^j}{j!}, \quad j_p = \min\{j \in \mathbb{N} : j \geq p\} \]

Furthermore, this inequality is sharp in the sense that if \( \beta(2n,\alpha) \) is replaced by any \( \beta > \beta(2n,\alpha) \), then the above inequality can no longer hold with some \( C \) independent of \( u \).

Notice that \( 2n|\frac{1}{2} - \frac{1}{p}| < n \) provided \( p > 1 \). Choosing \( \zeta = n \) in Theorem 1.5, we have the following

**Corollary 1.13.** Let \( n \geq 2, \, 0 < \alpha < 2n \) be an arbitrary real positive number and \( p = n/\alpha \). There exists \( C = C(\alpha, n) \) such that

\[\int_{B_n^C} \Phi_p(\beta(2n,\alpha)|u|^p') \, dV \leq C\]

and

\[\int_{B_n^C} e^{\beta(2n,\alpha)|u|^p'} \, dz \leq C\]

hold simultaneously for any \( u \in W^{\alpha,p}(B_n^C) \) with \( \int_{B_n^C} |(-\Delta_{B})^{\frac{\alpha}{2}} u|^p \, dV \leq 1 \).

The following remarks are in order.

First, as we have shown in the works on higher order Poincaré-Sobolev and Hardy-Sobolev-Maz’ya inequalities by the authors ([60], [61]), factorization theorems play an important role in establishing these sharp inequalities for higher order. Proving such a factorization theorem in the complex hyperbolic spaces is significantly more difficult than in the real hyperbolic space. This is one of our main results in this paper. Its connection between the Heisenberg group and CR sphere and the complex hyperbolic space makes this particularly interesting.

Second, as we have demonstrated in the works by the authors with Li [62], [51] and [52], as long as the Green’s kernel estimates are established, we have developed a general scheme in [62], [51] and [52] to combine D. Adams’ method of using O’Neil’s lemma and the method by Lam and the first author [48, 49] passing from the local inequalities to the global inequalities using the level set approach to carry out the proofs of the Adams and Hardy-Adams inequalities on the hyperbolic spaces. This method is fairly standard by now and it is easy to see that these general schemes work well for more general settings almost identically, especially for Riemannian symmetric spaces of noncompact type since the sharp estimates of Bessel-Green-Riesz kernels is well known (see [3, 4]). In fact, if we denote by \( \Delta_M \) the Laplace-Beltrami operaor on a Riemannian symmetric space of noncompact type and let

\[k_\alpha = (-\Delta_M - q^2)^{-\frac{\alpha}{2}} \quad \text{and} \quad k_{\zeta,\alpha} = (-\Delta_M - q^2 + \zeta^2)^{-\frac{\alpha}{2}},\]
where $\rho$ is the half-sum of the positive roots of $M$, then for $|x| > 0$,

$$k_\alpha(x) \sim |x|^{\alpha - l - 2|\Sigma^+|} \varphi_0(|x|), \quad 0 < \alpha < \min\{l + |\Sigma_0^+|, N\};$$

$$k_{\zeta,\alpha}(x) \sim |x|^{\alpha - l - 1 - \frac{1}{2}|\Sigma^+|} \varphi_0(|x|), \quad 0 < \alpha < N,$$

where $l$ is the rank of $M$, $\Sigma^+$ is the set of positive simple roots, $N = \dim M$ and $\varphi_0$ is spherical function (see [4], Theorem 4.2.2). The $L^p - L^q$ estimates for $f * k_\alpha$ and $f * k_{\zeta,\alpha}$ are well known (see [3] and [15]). For example, there holds the Poincaré-Sobolev inequality for $f \in C^\infty_0(M)$ (see [15], Theorem 6.1):

$$\|f\|_{L^p(M)} \leq \int_M |\sqrt{\Delta_M - \rho^2}f|^2 dV, \quad 2 < p \leq \frac{2N}{N - 2}.$$  

The Adams’ inequality on bounded domain then follows by combining such estimates and Adams’ method of using O’Neil’s lemma. To obtain Hardy-Adams inequality, one needs to get the estimates for $k_\alpha * k_{\zeta,\alpha}$. Though the estimates of $k_\alpha * k_{\zeta,\alpha}$ for $|x| \to 0$ is trivial (the method is similar to that in Euclidean space), it is nontrivial for $|x| \to \infty$, as well as the $L^p - L^q$ estimates for $f * (k_\alpha * k_{\zeta,\alpha})$. In fact, in this paper, we show some of estimates of $f * (k_\alpha * k_{\zeta,\alpha})$ through Funk-Hecke formula and Kunze-Stein phenomenon for $SU(n,1)$. To overcome this problem, one can use the the Plancherel formula on $M$ and the $L^2$ estimates of $k_\alpha * k_{\zeta,\alpha}$ outside of the origin, as we do for the real and complex hyperbolic space (see Lemma [3.2] or [52], Lemma 4.1). The Hardy-Adams inequality then follows by combining this fact, Adams’ method and the method by Lam and the first author [48, 49] passing from the local inequalities to the global inequalities. Therefore, in this paper, we will not give all the details when we prove the Adams and Hardy-Adams inequalities on the complex hyperbolic space (Theorems 1.11, 1.12) and simply refer the interested reader to [62], [51] and [52] to fill out all the details.

Third, in a recent paper [19], Joshua Flynn and the authors have shown analogous factorization theorems on the quaternionic hyperbolic space and the Cayley hyperbolic plane, and use these to establish the similar Hardy-Sobolev-Maz’ya inequalities and Hardy-Adams inequalities. Therefore, such inequalities are established for all rank one symmetric spaces of non-compact type (i.e. the real, complex and quaternionic hyperbolic spaces, and the Cayley hyperbolic plane). Though, as we have illustrated above, showing the Adams and Hardy-Adams inequalities can be standard by now following the general scheme in [62], [51] and [52] and this paper, it remains a nontrivial problem to establish the Hardy-Sobolev-Maz’ya inequalities on general symmetric spaces of higher ranks.

The organization of the paper is as follows: In Section 2, we recall some necessary preliminary facts of complex hyperbolic spaces. We shall prove the factorization theorem, namely Theorem 1.3 in Section 3. Sharp estimates of Bessel-Green-Riesz kernels of certain fractional Laplacians and their rearrangement estimates on complex hyperbolic spaces are given in Section 4 and Section 5, respectively. The higher order Hardy-Sobolev-Maz’ya inequalities, namely Theorem 1.4 and 1.5 are proved in Section 6. In Section 7, we prove the Hardy-Adams inequality on complex hyperbolic spaces, namely Theorem 1.9 and 1.10. In Section 8, we show the Adams type inequality on complex hyperbolic spaces, namely Theorem 1.11 and 1.12.
2. Notations and preliminaries

We firstly recall some geometric properties of complex hyperbolic space. Consider $\mathbb{C}^{n+1}$ equipped with the Hermitian product

$$F(z,w) = -z_0 \bar{w}_0 + \sum_{j=1}^{n} z_j \bar{w}_j.$$ 

Let $M = \{z \in \mathbb{C}^{n+1} : F(z,z) = -1\}$ be a real hypersurface in $\mathbb{C}^{n+1}$ and $U(1,n)$ be a closed linear group defined as

$$U(1,n) = \{A \in GL(n+1,\mathbb{C}) : F(Az,Aw) = F(z,w), z,w \in \mathbb{C}\}.$$ 

It is known that $U(1,n)$ is connected and acts transitively on $M$. Furthermore, the group $U(1) = \{e^{i\theta} : 0 \leq \theta < 2\pi\}$ acts freely on $M$ by $z \rightarrow e^{i\theta}z$. Let $M' = M/U(1)$. Then $M'$ is the base space of the fibration

$$U(1) \rightarrow M \rightarrow M'.$$

We note $M'$ is simply connected. Denote by $P_{\pi} : M \rightarrow M'$ the natural projection. The metric induced by $\operatorname{Re}F$ on $M$ has signature $(1,2n)$. However, its restriction to the orthogonal of the fiber is positive definite. In fact, if we define an inner product $g$ in each tangent space $T_pM'$, $p \in M'$ by

$$g(X,Y) = \operatorname{Re}F(X,Y), \quad X,Y \in T_pM',$$

then $g$ is a Riemannian metric $M'$. The complex structure $w \rightarrow iw$ on $T'_z = \{w \in \mathbb{C}^{n+1} : F(z,w) = 0\}$ are compatible with the action of $U(1)$. It induce an almost complex structure $J$ on $M'$ such that the metric $g$ is Hermitian with respect to $J$. Moreover, $(M',g)$ is a complete Kaehler manifold and has constant holomorphic section curvature $-4$ (see [46], Chapter XI, Example 10.7).

We remark that the complex hyperbolic space is also holomorphically isometric to the Hermitian symmetric space $SU(1,n)/S(U(1) \times U(n))$, where

$$SU(1,n) = \{A \in U(1,n) : \det A = 1\}.$$ 

In fact, the closed linear group $SU(1,n)$ acts transitively on $M'$ and the isotropy group at $P_{\pi}(e_0)$ is $S(U(1) \times U(n))$, where $e_0 = (1,0,\cdots,0)$.

There are also other models of complex hyperbolic space, for example, the Siegel domain model $\mathcal{U}^n$ and the ball model $B^n_{\mathbb{C}}$. We note these models are holomorphically isometric to each other because any two simply connected complete Kaehler manifolds of constant holomorphic sectional curvature $-4$ are holomorphically isometric to each other (see [46], Chapter IX, Theorem 7.9).

2.1. The Siegel domain model $\mathcal{U}^n$. The Siegel domain $\mathcal{U}^n \subset \mathbb{C}^n$ is defined as

$$\mathcal{U}^n := \{z \in \mathbb{C}^n : g(z) > 0\},$$

where

$$g(z) = \operatorname{Im}z_n - \sum_{j=1}^{n-1} |z_j|^2.$$
Its boundary $\partial U := \{ z \in \mathbb{C}^n : \varrho(z) = 0 \}$ can be identified with the Heisenberg group. Recall that the Heisenberg group $\mathbb{H}^{2n-1} = (\mathbb{C}^{n-1} \times \mathbb{R}, \circ)$ is a nilpotent group of step two with the group law

$$(z, t) \circ (z', t') = (z + z', t + t' + 2 \text{Im}(z, z')),$$

where $z, z' \in \mathbb{C}^{n-1}$ and $(z, z')$ is the Hermite inner product

$$(z, z') = \sum_{j=1}^{n} z_j \overline{z}_j'.$$

Set $z_j = x_j + iy_j (1 \leq j \leq n - 1)$ and define

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t} \quad \text{for } j = 1, \ldots, n - 1, \quad T = \frac{\partial}{\partial t}.$$

The $2n - 1$ vector fields $X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1}, T$ are left-invariant and form a basis for Lie algebra of $\mathbb{H}^{2n-1}$. Furthermore, we have the commutation relations

$$[X_j, Y_j] = -4T, \quad j = 1, \ldots, n - 1,$$

and that all other commutations vanish. The sub-Laplacian on $\mathbb{H}^{2n-1}$ is given by

$$\Delta_b = \frac{1}{4} \sum_{j=1}^{n-1} (X_j^2 + Y_j^2).$$

Set $L_0 = -\Delta_b$. The Bergman metric on $U^n$ is the metric with Kaehler form $\omega = \frac{i}{2} \partial \bar{\partial} \log \frac{1}{\varrho}$. Through the coordinates $(z, t, \varrho) \in \mathbb{H}^{2n-1} \times (0, +\infty) \cong U^n$, we can give the volume form in Bergman metric as follows (see [28])

$$dV = \frac{1}{4\varrho^{n+1}} dz dtd\varrho.$$

The Laplace-Beltrami operator is given by

$$\Delta_b = 4\varrho(\varrho \partial_{\varrho} + T^2) - L_0 - (n - 1) \partial_{\varrho}.$$

2.2. **The ball model** $\mathbb{B}_n^\mathbb{C}$. It is given by the unit ball

$$\mathbb{B}_n^\mathbb{C} = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z| < 1 \}$$

equipped with the Kaehler metric

$$ds^2 = -\partial \bar{\partial} \log(1 - |z|^2).$$

The volume form in this Bergman metric is

$$dV = \frac{dz}{(1 - |z|^2)^{n+1}},$$

where $dz$ is the usual Euclidean volume form. The Laplace-Beltrami operator is given by

$$\Delta_b = 4(1 - |z|^2) \sum_{j,k=1}^{n} (\delta_{jk} - z_j \bar{z}_k) \frac{\partial^2}{\partial z_j \partial \bar{z}_k},$$
where
\[ \delta_{j,k} = \begin{cases} 1, & j = k; \\ 0, & j \neq k. \end{cases} \]

The Cayley transform \( C : \mathbb{B}_C^n \to \mathcal{U}^n \), defined by
\[ (2.3) \quad C(z) = \left( \frac{z_1}{1 + z_n}, \ldots, \frac{z_{n-1}}{1 + z_n}, 1 - z_n \right), \]
is an isometry of \( \mathbb{B}_C^n \) onto the space \( \mathcal{U}^n \). The geodesic distance between 0 and \( z \in \mathbb{B}_C^n \) is
\[ (2.4) \quad \rho(0, z) = \frac{1}{2} \log \frac{1 + |z|}{1 - |z|}. \]

For simplicity, we write \( \rho(z) = \rho(0, z) \). A function \( f \) is said to be radial if \( f(z) = \tilde{f}(\rho(z)) \) for some function \( \tilde{f} \). In terms of polar coordinates, we can rewrite the volume form as following
\[ (2.5) \quad dV = (\sinh \rho)^{2n-1} \cosh \rho d\rho d\sigma, \]
where \( d\sigma \) is the Lebesgue measure on the sphere \( S^{2n-1} = \{ z = (z_1, \cdots, z_n) \in \mathbb{C}^n : |z| = 1 \} \).

Recall that Geller’s operator \( \Delta_{\alpha, \beta} \) is given by
\[ (2.6) \quad \Delta_{\alpha, \beta} = 4(1 - |z|^2) \left[ \sum_{j,k=1}^n (\delta_{jk} - z_j \bar{z}_k) \frac{\partial^2}{\partial z_j \partial \bar{z}_k} + \alpha \sum_{j=1}^n z_j \frac{\partial}{\partial z_j} + \beta \sum_{j=1}^n \bar{z}_j \frac{\partial}{\partial \bar{z}_j} - \alpha \beta \right]. \]

There is a radial-tangential expression for \( \Delta_{\alpha, \beta} \) obtained in [25] and we state it as follows:
\[ \Delta_{\alpha, \beta} = 4(1 - |z|^2) \left[ \frac{1 - |z|^2}{|z|^2} R \bar{R} - \frac{1}{|z|^2} \mathcal{L}'_0 + \frac{n-1}{2} \frac{1}{|z|^2} (R + \bar{R}) + \alpha R + \beta \bar{R} - \alpha \beta \right], \]
where
\[ \mathcal{L}'_0 \]
and \( \mathcal{L}'_0 \) is the Folland-Stein operator [20] defined as follows:
\[ \mathcal{L}'_0 = -\frac{1}{2} \sum_{j<k} (M_{jk} \bar{M}_{jk} + \bar{M}_{jk} M_{jk}) , \quad M_{j,k} = z_j \partial \bar{z}_k - \bar{z}_k \partial z_j. \]

### 2.3. CR invariant differential operators on the Heisenberg group and CR sphere.

It is known that the CR invariant sub-Laplacian of Jerison and Lee ([39]-[42]) in CR geometry plays a role analogous to that of the conformal Laplacian in Riemannian geometry. In the case of CR sphere \( S^{2n-1} \), the CR invariant sub-Laplacian is defined as \( \mathcal{L}'_0 + \frac{(n-1)^2}{4} \). Let \( \partial C \) be the restriction of Cayley transform \( C \) defined in (2.3) to the boundary. The relationship between \( \mathcal{L}_0 = -\Delta_b \), the sub-Laplacian on the Heisenberg group, and the \( \mathcal{L}'_0 + \frac{(n-1)^2}{4} \) is
\[ \left( \mathcal{L}'_0 + \frac{(n-1)^2}{4} \right) \left( |J_{\partial C}| \frac{Q-2}{2} (F \circ \partial C) \right) = |J_{\partial C}| \frac{Q-2}{2} \mathcal{L}_0 F, \quad F \in C^\infty(\mathbb{H}^{2n-1}), \]
where \( J_{\partial C} \) is the Jacobian determinant of \( \partial C \) and \( Q = 2n \) is the homogeneous dimension.
The CR invariant differential operators for high order are the product of the Folland-Stein operators. In fact, if we denote the Folland-Stein operators $L_\alpha$ by

$$ L_\alpha = L_0 + i\alpha T, $$

then the CR invariant differential operator of $2k$ is, up to a constant,

$$ (2.7) 
P_{2k} = \prod_{j=1}^{k} L_{k+1-2j}. $$

A simple calculation shows

$$ P_1 = L_0 = -\Delta_b, \quad P_2 = L_0^2 + T^2 = \Delta_b^2 + T^2. $$

We remark that the operator $P_{2k}$ has been firstly obtained by Cowling [14] via the computation of the group Fourier transform of $(|z|^4 + i^2)^{2k-2}Q$.

Similarly, there is an analogous operator on the CR sphere. Let $T = i \frac{1}{2}(R - \bar{R})$ be the transversal direction. The CR invariant differential operator of order $2k$ on $\mathbb{S}^{2n-1}$ is given by, up to a constant,

$$ P'_{2k} = \prod_{j=0}^{k} \left( L'_0 + \frac{(n-1)^2}{4} - \frac{(k+1-2j)^2}{4} - (k+1-2j)iT \right). $$

The relationship between $P_{2k}$ and the $P'_{2k}$ is (see [30])

$$ P'_{2k} \left( |J_{\partial C}|^{\frac{2-2k}{2q}} (F \circ \partial C) \right) = |J_{\partial C}|^{\frac{2-2k}{2q}} P_{2k} F, \quad F \in C^\infty(\mathbb{H}^{2n-1}). $$

For the conformally invariant sharp Sobolev inequality on the Heisenberg group and CR sphere, we refer to Jerison-Lee [41] and Frank-Lieb [23].

2.4. The automorphisms. For each $a \in \mathbb{B}^n_C$, we define the automorphisms $\varphi_a$ of $\mathbb{B}^n_C$ by (see e.g. [71])

$$ \varphi_a(z) = \frac{a - P_a(z) - \sqrt{1 - |a|^2} Q_a(z)}{1 - (z,a)}, $$

where

$$ P_a(z) = \left\{ \begin{array}{ll} \frac{(z,a)}{|a|^2}a, & a \neq 0; \\ 0, & a = 0, \end{array} \right. $$
and \( Q_a(z) = z - P_a(z) \). It is easy to check \( \varphi_a(0) = a \) and \( \varphi_a(a) = 0 \). Furthermore, \( \varphi_a \) has the following properties:

\[
1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - (z, a)|^2}
\]

(2.8)

\[
\sinh \varphi_a(z) = \frac{|\varphi_a(z)|}{\sqrt{1 - |\varphi_a(z)|^2}} = \frac{\sqrt{|z - a|^2 + |(z, a)|^2 - |z|^2|a|^2}}{\sqrt{(1 - |a|^2)(1 - |z|^2)}}
\]

\[
\cosh \varphi_a(z) = \frac{1}{\sqrt{1 - |\varphi_a(z)|^2}} = \frac{\sqrt{(1 - |a|^2)(1 - |z|^2)}}{|1 - (z, a)|}
\]

Since \( \varphi_a \) is an isometry, the distance from \( z \) to \( a \) is

\[
\rho(x, a) = \rho(\varphi_a(z)) = \frac{1}{2} \log \frac{1 + |\varphi_a(z)|}{1 - |\varphi_a(z)|}
\]

The convolution of measurable functions \( f \) and \( g \) on \( \mathbb{B}_C^n \) is

(2.9)

\[
(f \ast g)(z) = \int_{\mathbb{B}_C^n} f(\varphi_1(w))g(w)dV(w)
\]

provided this integral exists. It is easy to check that if \( f \) is radial, then

(2.10)

\[
f \ast g = g \ast f.
\]

2.5. Helgason-Fourier transform on the complex hyperbolic spaces. In this subsection we recall some basics of Helgason-Fourier analysis on complex hyperbolic spaces in term of ball model and refer the reader to [24, 34, 36, 75, 76, 68] for more information about this subject.

It is known that the Poisson kernel on \( \mathbb{B}_C^n \) is given by (see [43], Lemma 2.1)

\[
\left( \frac{1 - |z|^2}{|1 - (z, \zeta)|^2} \right)^n, \quad x \in \mathbb{B}_C^n, \quad \zeta \in \mathbb{S}^{2n-1}.
\]

Therefore, if we set

\[
e_{\lambda, \zeta}(z) = \left( \frac{1 - |z|^2}{|1 - (z, \zeta)|^2} \right)^{\frac{n+1}{2}}, \quad x \in \mathbb{B}_C^n, \quad \lambda \in \mathbb{R}, \quad \zeta \in \mathbb{S}^{2n-1},
\]

then the Helgason-Fourier transform of a function \( f \) on \( \mathbb{B}_C^n \) can be defined as

\[
\hat{f}(\lambda, \zeta) = \int_{\mathbb{B}_C^n} f(z)e^{-\lambda, \zeta}(z)dV
\]

provided this integral exists. It is easy to check that if \( f, g \in C_0^\infty(\mathbb{B}_C^n) \) and \( g \) is radial, then

\[
(\hat{f} \ast \hat{g}) = \hat{f} \cdot \hat{g}.
\]

The following inversion formula holds for \( f \in C_0^\infty(\mathbb{B}_C^n) \):

\[
f(z) = C_n \int_{-\infty}^{+\infty} \int_{\mathbb{S}^{2n-1}} \hat{f}(\lambda, \zeta)e_{\lambda, \zeta}(z)|\zeta(\lambda)|^{-2}d\lambda d\sigma(\zeta),
\]
where $C_n$ is a positive constant and $c(\lambda)$ is the Harish-Chandra $c$-function given by

$$c(\lambda) = \frac{2^{n/2-i\lambda} \Gamma(n/2) \Gamma(i\lambda)}{\Gamma((n/2+i\lambda)/2) \Gamma((n/2+1+i\lambda)/2)}.$$ 

Similarly, there holds the Plancherel formula

$$ \int_{\mathbb{B}_C^n} |f(z)|^2 dV = C_n \int_{-\infty}^{+\infty} \int_{\mathbb{S}^{2n-1}} |\hat{f}(\lambda, \zeta)|^2 |\lambda|^{-2} d\lambda d\sigma(\zeta). $$

(2.11)

Since $e_{\lambda,\zeta}(z)$ is an eigenfunction of $\Delta_{\mathbb{B}}$ with eigenvalue $-n^2 - \lambda^2$, we have, for $f \in C_0^\infty(\mathbb{B}_C^n)$,

$$ \widehat{\Delta_{\mathbb{B}}} f(\lambda, \zeta) = -(n^2 + \lambda^2) \hat{f}(\lambda, \zeta). $$

Therefore, in analogy with the Euclidean setting, we define the fractional Laplacian on hyperbolic space as follows:

$$ \Delta_{\mathbb{B}}^{\gamma} f(\lambda, \zeta) = (n^2 + \lambda^2)^\gamma \hat{f}(\lambda, \zeta), \quad \gamma \in \mathbb{R}. $$

The fractional Sobolev spaces on $\mathbb{B}_C^n$ are defined by

$$ W^{\alpha,p}(\mathbb{B}_C^n) = (1 - \Delta_{\mathbb{B}})^{-\frac{\alpha}{2}} L^p \quad (1 < p < \infty, \alpha \in \mathbb{R}). $$

In [2], J.-P. Anker proved a Hörmander-Mikhlin type multiplier theorem in the context of Riemannian symmetric spaces of the noncompact type. In the case of complex hyperbolic space, the Hörmander-Mikhlin type multiplier theorem reads as follows:

**Theorem 2.1.** Let $1 < p < \infty$, $\kappa$ be a tempered radial distribution on $\mathbb{B}_C^n$ and $m$ be its Fourier transformation. Then $f \ast \kappa$ is a bounded operator on $L^p(\mathbb{B}_C^n)$, provided that

(i) $m$ extends to a holomorphic function inside the tube $\Xi = \{ \lambda \in \mathbb{C} : |\text{Im}\lambda| \leq 2 n^2 \frac{1}{p} - \frac{1}{2} \}$;

(ii) $\partial_\lambda^i m(\lambda)$ $(i = 0, 1, \cdots, \left[2n\frac{1}{p} - \frac{1}{2}\right] + 1)$ extends continuously to the whole of $\Xi$, with

$$ \sup_{\lambda \in \Xi} (1 + |\lambda|)^i |\partial_\lambda^i m(\lambda)| < \infty $$

Observe that $(-\Delta_{\mathbb{B}} - n^2 + \zeta^2)^{-\frac{\alpha}{2}}$ has the symbol $(\lambda^2 + \zeta^2)^{-\frac{\alpha}{2}}$ which can be extended to a holomorphic function inside the tube $\{ \lambda \in \mathbb{C} : |\text{Im}\lambda| \leq \zeta \}$. Therefore, by Theorem 2.1 there exists a constant $C > 0$ such that for $f \in L^p(\mathbb{B}_C^n)$

$$ \|(-\Delta_{\mathbb{B}} - n^2 + \zeta^2)^{-\frac{\alpha}{2}} f\|_p \leq C\|f\|_p $$

provided $\zeta > 2n\frac{1}{2} - \frac{1}{p}$ and $\alpha > 0$. This leads to the following Sobolev type inequality immediately.

**Corollary 2.2.** Let $1 < p < \infty$, $\alpha > 0$ and $\zeta > 2n\frac{1}{2} - \frac{1}{p}$. Then there exists a positive constant $C$ such that for $f \in W^{\alpha,p}(\mathbb{B}_C^n)$,

$$ C\|f\|_p \leq \|(-\Delta_{\mathbb{B}} - n^2 + \zeta^2)^{\frac{\alpha}{2}} f\|_p. $$

For more information about the fractional Laplacian on noncompact symmetric spaces, we refer to [2], [3], [16].
2.6. **Funk-Hecke formula.** It is known that \( L^2(\mathbb{S}^{2n-1}) \) can be decomposed as follows

\[
L^2(\mathbb{S}^{2n-1}) = \bigoplus_{j,k=0}^{\infty} \mathcal{H}_{j,k},
\]

where \( \mathcal{H}_{j,k} \) is the space of restrictions to \( \mathbb{S}^{2n-1} \) of harmonic polynomials \( p(z, \bar{z}) \) which are homogeneous of degree \( j \) in \( z \) and degree \( k \) in \( \bar{z} \). The Funk-Hecke formula of Frank and Lieb reads as follow (see [23]):

\[
\int_{\mathbb{S}^{2n-1}} K((\xi, \eta)) Y_{j,k}(\eta) d\sigma(\eta) = \lambda_{j,k} Y_{j,k}, \quad Y_{j,k} \in \mathcal{H}_{j,k},
\]

(2.13)

\[
\lambda_{j,k} = \frac{\pi^m m!}{2^{n-1+j-k}(m+n-2)!} \int_{-1}^{1} \left( \int_{-\pi}^{\pi} K(e^{i\theta} \sqrt{(1+t)/2}) e^{i(j-k)\theta} d\theta \right) \cdot (1-t)^{n-2}(1+t)^{|j-k|/2} P_{m}^{(n-2,|j-k|)}(t), \quad m = \min\{j, k\},
\]

where \( K \) is an integrable function on the unit ball in \( \mathbb{C} \) and \( P_{m}^{(\alpha, \beta)} \) is the Jacobi polynomials.

Next, we will prove the following

**Proposition 2.3.** Let \( 0 < \alpha < 2n \). There holds, for \( 0 < r < 1 \) and \( \xi \in \mathbb{S}^{2n-1} \),

(2.14)

\[
\int_{\mathbb{S}^{2n-1}} \frac{1}{|1 - (r\xi, \eta)|^\alpha} d\sigma(\eta) = \frac{2\pi}{\Gamma(\alpha/2)} F(\alpha/2, \alpha/2; n; r^2),
\]

where \( F(a, b; c; z) \) is the hypergeometric function defined by

(2.15)

\[
F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(c)_k k!}, \quad c \neq 0, -1, \ldots, -n, \ldots,
\]

and \( (a)_k \) is the rising Pochhammer symbol defined by

\[
(a)_0 = 1, \quad (a)_k = a(a+1) \cdots (a+k-1), \quad k \geq 1.
\]

**Proof.** By Funk-Hecke formula, we have

(2.16)

\[
\int_{\mathbb{S}^{2n-1}} \frac{1}{|1 - (r\xi, \eta)|^\alpha} d\sigma(\eta) = \frac{1}{2^{n-1}(n-2)!} \int_{-1}^{1} \left( \int_{-\pi}^{\pi} \frac{1}{1 - r \sqrt{\frac{1+t}{2} e^{-i\theta} |^\alpha} d\theta \right) (1-t)^{n-2} P_0^{n-2,0}(t) dt.
\]

We compute

(2.17)

\[
\int_{-\pi}^{\pi} \frac{1}{1 - r \sqrt{\frac{1+t}{2} e^{-i\theta} |^\alpha} d\theta = \int_{-\pi}^{\pi} \left( 1 - 2r \sqrt{\frac{1+t}{2} \cos \theta + \frac{1+t}{2} r^2 \right)^{-\frac{\alpha}{2}} d\theta \right.
\]

\[
= \frac{2\pi}{\Gamma^2(\alpha/2)} \sum_{\mu=0}^{\infty} \frac{\Gamma^{2}(\alpha/2 + \mu)}{(\mu)!^2} r^{2\mu} \left( \frac{1+t}{2} \right)^{\mu}.
\]
Here we use the fact (see [23], (5.11))
\[ \int_{-\pi}^{\pi} (1 - 2r \cos \theta + r^2)^{-\frac{\alpha}{2}} d\theta = \frac{2\pi}{\Gamma^2(\alpha/2)} \sum_{\mu=0}^{\infty} \frac{\Gamma^2(\alpha/2 + \mu)}{(\mu!)^2} r^{2\mu}. \]

Therefore, we have
\[ \int_{S^{2n-1}} \frac{1}{|1 - (r\xi, \eta)|^\alpha} d\sigma(\eta) = \frac{2\pi}{\Gamma^2(\alpha/2)} \sum_{\mu=0}^{\infty} \frac{\Gamma^2(\alpha/2 + \mu)}{(\mu!)^2} r^{2\mu} \int_{-1}^{1} \left( \frac{1 + t}{2} \right)^\mu (1 - t)^{n-2} P^{n-2,0}_\mu(t) dt \]

To get the last equation, we use the fact (see [23], (5.12))
\[ \int_{-1}^{t} (1 - t)^{n-1}(1 + t)^{|j-k|+\mu} P_m^{(n-1,|j-k|)}(t) dt = \begin{cases} 0, & \text{if } \mu < m; \\ \frac{1}{2^{j-k}+n+\mu} \frac{\mu!}{m!(\mu-m)!} \frac{(|j-k|+\mu)!}{(|j-k|+n+\mu)!}, & \text{if } \mu \geq m. \end{cases} \]

Therefore, we have, by (2.18),
\[ \int_{S^{2n-1}} \frac{1}{|1 - (r\xi, \eta)|^\alpha} d\sigma(\eta) = \frac{2\pi}{\Gamma(n)} F(\alpha/2, \alpha/2; n; r^2). \]

This completes the proof.

3. A factorization theorem for the Operators on complex hyperbolic space: Proof of Theorem 1.3

Factorization theorem plays an important role in establishing the Hardy-Sobolev-Maz’ya inequalities. The main purpose of this section is to prove Theorem 1.3.

Firstly, we consider the Siegel domain model. The proof depends on the following three lemmas.

**Lemma 3.1.** Let \( a \in \mathbb{R} \) and \( f \in C^\infty(U^n) \). There holds
\[ \left[ \rho \partial \rho + a \partial \theta + \rho T^2 + \Delta_b \right] \rho^{1-n+a} f \]
\[ = \rho^{1-n+a} \left\{ \rho \left[ \rho \partial \rho + T^2 \right] + \Delta_b - (n-1) \partial \theta \right\} + \frac{n^2}{4} - \frac{(a-1)^2}{4} \]

**Proof.** A simple calculation shows, for each \( \beta \in \mathbb{R} \), there holds
\[ \rho^{\beta+1} \left[ \rho \partial \rho + a \partial \theta + \rho T^2 + \Delta_b \right] \rho^{-\beta} f \]
\[ = \rho \left[ \rho \partial \rho + T^2 + \Delta_b - (2\beta - a) \partial \theta \right] f + \beta(\beta + 1 - a) f. \]
Choosing $\beta = \frac{n-1+a}{2}$ in (3.2), we have

$$\varrho^{1+\frac{n+a}{2}} \left[ \varrho \partial_{\varrho \varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b \right] \left( \varrho^{1-\frac{n-a}{2}} f \right) = \left\{ \varrho \left[ \varrho (\partial_{\varrho \varrho} + T^2) + \Delta_b - (n-1)\partial_{\varrho} \right] + \frac{n^2}{4} - \frac{(a-1)^2}{4} \right\} f.$$ 

The desired result follows. \hfill \Box

**Lemma 3.2.** Let $\beta \in \mathbb{R}$. There holds

\begin{equation}
[\varrho \partial_{\varrho \varrho} + (a+\beta)\partial_{\varrho} + \varrho T^2 + \Delta_b] \left\{ \left[ \varrho \partial_{\varrho \varrho} + (a-1)\partial_{\varrho} + \varrho T^2 + \Delta_b \right]^2 + (\beta - 1)^2 T^2 \right\} = \left\{ \varrho \partial_{\varrho \varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b \right\} \left[ \varrho \partial_{\varrho \varrho} + (a+\beta-2)\partial_{\varrho} + \varrho T^2 + \Delta_b \right]. \tag{3.3}\end{equation}

**Proof.** Since

\begin{equation}
\partial_{\varrho} \left[ \varrho \partial_{\varrho \varrho} + (a-1)\partial_{\varrho} + \varrho T^2 + \Delta_b \right] = \left[ \varrho \partial_{\varrho \varrho} + (a-1)\partial_{\varrho} + \varrho T^2 + \Delta_b \right] \partial_{\varrho} + T^2, \tag{3.4}\end{equation}

we have

\begin{align}
\partial_{\varrho} \left[ \varrho \partial_{\varrho \varrho} + (a-1)\partial_{\varrho} + \varrho T^2 + \Delta_b \right]^2 & = \left[ \varrho \partial_{\varrho \varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b \right] \partial_{\varrho} \left[ \varrho \partial_{\varrho \varrho} + (a-1)\partial_{\varrho} + \varrho T^2 + \Delta_b \right] \\
& + \left[ \varrho \partial_{\varrho \varrho} + (a-1)\partial_{\varrho} + \varrho T^2 + \Delta_b \right] T^2 \\
& = \left[ \varrho \partial_{\varrho \varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b \right]^2 \partial_{\varrho} + \left[ \varrho \partial_{\varrho \varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b \right] T^2 \\
& + \left[ \varrho \partial_{\varrho \varrho} + (a-1)\partial_{\varrho} + \varrho T^2 + \Delta_b \right] T^2 \\
& = \left[ \varrho \partial_{\varrho \varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b \right]^2 \partial_{\varrho} + 2 \left[ \varrho \partial_{\varrho \varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b \right] T^2 - T^2 \partial_{\varrho}. \tag{3.5}\end{align}

Similarly,

\begin{align}
\partial_{\varrho} \left[ \varrho \partial_{\varrho \varrho} + (a-1)\partial_{\varrho} + \varrho T^2 + \Delta_b \right]^2 & = \left[ \varrho \partial_{\varrho \varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b \right] \left[ \varrho \partial_{\varrho \varrho} + (a-1)\partial_{\varrho} + \varrho T^2 + \Delta_b \right] - \\
& - \partial_{\varrho} \left[ \varrho \partial_{\varrho \varrho} + (a-1)\partial_{\varrho} + \varrho T^2 + \Delta_b \right] \\
& = \left[ \varrho \partial_{\varrho \varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b \right]^2 - 2 \left[ \varrho \partial_{\varrho \varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b \right] \partial_{\varrho} - T^2. \tag{3.6}\end{align}
Therefore, we obtain, by using (3.5) and (3.6),

\[
\begin{align*}
&\left[\varrho \partial_{\varrho\varrho} + (a + \beta) \partial_{\varrho} + \varrho T^2 + \Delta_b\right] \left\{\left[\varrho \partial_{\varrho\varrho} + (a - 1) \partial_{\varrho} + \varrho T^2 + \Delta_b\right]^2 + (\beta - 1)^2 T^2\right\} \\
&= \left[\varrho \partial_{\varrho\varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b\right] \left\{\left[\varrho \partial_{\varrho\varrho} + (a - 1) \partial_{\varrho} + \varrho T^2 + \Delta_b\right]^2 + (\beta - 1)^2 T^2\right\} + \\
&\beta \partial_{\varrho} \left\{\left[\varrho \partial_{\varrho\varrho} + (a - 1) \partial_{\varrho} + \varrho T^2 + \Delta_b\right]^2 + (\beta - 1)^2 T^2\right\} \\
&= \left[\varrho \partial_{\varrho\varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b\right] \cdot \\
&\left\{\left[\varrho \partial_{\varrho\varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b\right]^2 - 2 \left[\varrho \partial_{\varrho\varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b\right] \partial_{\varrho} + \beta(\beta - 2) T^2\right\} \\
&+ \beta \left\{\left[\varrho \partial_{\varrho\varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b\right]^2 \partial_{\varrho} + 2 \left[\varrho \partial_{\varrho\varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b\right] T^2 + \beta(\beta - 2) T^2 \partial_{\varrho}\right\} \\
&= \left\{\left[\varrho \partial_{\varrho\varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b\right]^2 + \beta^2 T^2\right\} \left[\varrho \partial_{\varrho\varrho} + (a + \beta - 2) \partial_{\varrho} + \varrho T^2 + \Delta_b\right].
\end{align*}
\]

This completes the proof of Lemma 3.2.

Lemma 3.3. There holds, for \(k \in \mathbb{N} \setminus \{0\}\),

\[
\begin{align*}
&\left[\varrho \partial_{\varrho\varrho} + (a + 2k) \partial_{\varrho} + \varrho T^2 + \Delta_b\right] \prod_{j=1}^{k} \left\{\left[\varrho \partial_{\varrho\varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b\right]^2 + (2j - 1)^2 T^2\right\} \\
&= \left(\varrho \partial_{\varrho\varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b\right) \prod_{j=1}^{k} \left\{\left[\varrho \partial_{\varrho\varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b\right]^2 + 4j^2 T^2\right\}
\end{align*}
\]

(3.7)

and

\[
\begin{align*}
&\left[\varrho \partial_{\varrho\varrho} + (a + 2k) \partial_{\varrho} + \varrho T^2 + \Delta_b\right] \\
&\left\{\left(\varrho \partial_{\varrho\varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b\right) \prod_{j=1}^{k-1} \left[\left(\varrho \partial_{\varrho\varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b\right)^2 + 4j^2 T^2\right]\right\} \\
&= \prod_{j=1}^{k} \left\{\left[\varrho \partial_{\varrho\varrho} + a \partial_{\varrho} + \varrho T^2 + \Delta_b\right]^2 + (2j - 1)^2 T^2\right\}.
\end{align*}
\]

(3.8)
Proof. By Lemma 3.2, we have
\[
\left(\phi \partial_{\phi} + (a + 2k) \partial_{\theta} + \phi T^2 + \Delta_b \right) \prod_{j=1}^{k} \left\{ \left[ \phi \partial_{\phi} + a \partial_{\theta} + \phi T^2 + \Delta_b \right]^2 + (2j - 1)^2 T^2 \right\} 
\]
\[
= \left(\phi \partial_{\phi} + (a + 2k) \partial_{\theta} + \phi T^2 + \Delta_b \right) \prod_{j=1}^{k} \left\{ \left[ \phi \partial_{\phi} + a \partial_{\theta} + \phi T^2 + \Delta_b \right]^2 \right\} 
\]
\[
= \left\{ \left[ \phi \partial_{\phi} + a \partial_{\theta} + \phi T^2 + \Delta_b \right]^2 + 4k^2 T^2 \right\} \prod_{j=1}^{k-1} \left\{ \left[ \phi \partial_{\phi} + a \partial_{\theta} + \phi T^2 + \Delta_b \right]^2 + (2j - 1)^2 T^2 \right\} . 
\]
Repeating this process in a suitable manner we get (3.7). The proof of (3.8) is similar and we omit it. The proof of Lemma 3.3 is thereby completed. \( \square \)

Proposition 3.4. Let \( a \in \mathbb{R} \) and \( f \in C^\infty(U^n) \). We have, for \( k \in \mathbb{N} \setminus \{0\} \),
\[
4^k \phi^{k+n+1-2} \prod_{j=1}^{k} \left[ \phi \partial_{\phi} + a \partial_{\theta} + \phi T^2 + \Delta_b - i(k + 1 - 2j)T \right] \left( \phi^{k-n-a} \frac{2}{4} f \right) 
\]
\[= \prod_{j=1}^{k} \left[ \Delta_b + n^2 - (a - k + 2j - 2)^2 \right] f. \tag{3.9} \]

Proof. It is enough to show
\[
\prod_{j=1}^{k} \left[ \phi \partial_{\phi} + a \partial_{\theta} + \phi T^2 + \Delta_b - i(k + 1 - 2j)T \right] \left( \phi^{k-n-a} \frac{2}{4} f \right) 
\]
\[= \phi^{k+n+1-a} \prod_{j=1}^{k} \left\{ \phi [\phi \partial_{\phi} + T^2 + \Delta_b - (n - 1) \partial_{\theta}] + \frac{n^2}{4} - \frac{(a - k + 2j - 2)^2}{4} \right\} f, \tag{3.10} \]
We shall prove (3.10) by induction. By Lemma 3.1 (3.10) is valid for \( k = 1 \). Now assume (3.10) is valid for \( k = l \), i.e.,
\[
\prod_{j=1}^{l} \left[ \phi \partial_{\phi} + a \partial_{\theta} + \phi T^2 + \Delta_b - i(l + 1 - 2j)T \right] \left( \phi^{l-n-a} \frac{2}{4} f \right) 
\]
\[= \phi^{l+n+1-a} \prod_{j=1}^{l} \left\{ \phi [\phi \partial_{\phi} + T^2 + \Delta_b - (n - 1) \partial_{\theta}] + \frac{n^2}{4} - \frac{(a - l + 2j - 2)^2}{4} \right\} f, \tag{3.11} \]
Replacing $a$ by $a - 1$ in (3.11), we get

\[
\prod_{j=1}^{l} \left[ \phi \partial_{\phi} + (a - 1) \partial_{\phi} + \phi T^2 + \Delta_b - i(l + 1 - 2j)T \right] \left( \phi^{\frac{l+1-n-a}{2}} f \right)
\]

(3.12)

\[
= e^{-\frac{l+n+a-1}{2}} \prod_{j=1}^{l} \left\{ \phi [\phi(\partial_{\phi} + T^2) + \Delta_b - (n - 1) \partial_{\phi}] + \frac{n^2}{4} - \frac{(a - l + 2j - 3)^2}{4} \right\} f.
\]

If $l$ is even, then by (3.7), we have

\[
\left[ \phi \partial_{\phi} + (a + l) \partial_{\phi} + \phi T^2 + \Delta_b \right] \prod_{j=1}^{l/2} \left\{ \left[ \phi \partial_{\phi} + a \partial_{\phi} + \phi T^2 + \Delta_b \right]^2 + (2j - 1)^2 T^2 \right\}
\]

\[
= \prod_{j=1}^{l+1} \left[ \phi \partial_{\phi} + a \partial_{\phi} + \phi T^2 + \Delta_b - i(l + 2 - 2j)T \right]
\]

and by Lemma 3.1

\[
\left[ \phi \partial_{\phi} + (a + l) \partial_{\phi} + \phi T^2 + \Delta_b \right] \prod_{j=1}^{l/2} \left\{ \phi [\phi(\partial_{\phi} + T^2) + \Delta_b - (n - 1) \partial_{\phi}] + \frac{n^2}{4} - \frac{(a - l + 2j - 3)^2}{4} \right\} f
\]

Similarly, if $l$ is odd, then by (3.8) and Lemma 3.1 we get that (3.11) is also valid for $l + 1$. These complete the proof of Proposition 3.4.

Next we consider the ball model.

**Lemma 3.5.** Let $s \in \mathbb{R}$ and $u \in C^\infty(\mathbb{B}^n_C)$. There holds

\[
\Delta_{s-n,s-n} \left[(1 - |z|^2)^{s-n}u\right] = 4^{-1}(1 - |z|^2)^{s-n-1} \left[ \Delta_{\mathbb{B}} + 4s(n - s) \right] u.
\]
Proof. A simple calculation shows

\[
R [(1 - |z|^2)^{n-s}]u = -(n-s)(1 - |z|^2)^{n-1-s} |z|^2 u + (1 - |z|^2)^{n-s} R u;
\]
\[
\overline{R} [(1 - |z|^2)^{n-s}]u = -(n-s)(1 - |z|^2)^{n-1-s} |z|^2 u + (1 - |z|^2)^{n-s} \overline{R} u;
\]
\[
RR [(1 - |z|^2)^{n-s}]u = -(n-s) \left[ -(n-1-s)(1 - |z|^2)^{n-s-2} |z|^4 + (1 - |z|^2)^{n-1-s} |z|^2 \right] u
- (n-s)(1 - |z|^2)^{n-1-s} |z|^2 Ru + (1 - |z|^2)^{n-s} RR u
- (n-s)(1 - |z|^2)^{n-1-s} |z|^2 \overline{R} u
= (1 - |z|^2)^{n-s} RR u - (n-s)(1 - |z|^2)^{n-1-s} |z|^2 (\overline{R} + R) u
- (n-s) \left[ -(n-1-s)(1 - |z|^2)^{n-s-2} |z|^4 + (1 - |z|^2)^{n-1-s} |z|^2 \right] u.
\]

Therefore,

\[
\left\{ \frac{1 - |z|^2}{|z|^2} RR - \frac{1}{|z|^2} L_0' + \frac{n-1}{2} \cdot \frac{1}{|z|^2} (R + \overline{R}) - (n-s)(\overline{R} + R) - (n-s)^2 \right\} [(1 - |z|^2)^{s-n} u]
= (1 - |z|^2)^{s-n} \left[ \frac{(1 - |z|^2)^2}{|z|^2} RR - \frac{1 - |z|^2}{|z|^2} L_0' + \frac{n-1}{2} \cdot \frac{1}{|z|^2} (R + \overline{R}) + s(n-s) \right] u.
\]

That is

\[
\Delta'_{s-n,s-n} [(1 - |z|^2)^{s-n} u] = 4^{-1} (1 - |z|^2)^{s-n-1} [\Delta_{\overline{R}} + 4s(n-s)] u.
\]

This prove Lemma 3.5 \( \square \)

Lemma 3.6. Let \( a \in \mathbb{R} \) and \( l \in \mathbb{R} \). There holds, for \( u \in C^\infty(\mathbb{B}_C^n) \),

\[
\Delta'_{\frac{1-a-n-l}{2}, \frac{1-a-n-l}{2}} \left[ \left( \Delta'_{\frac{1-a-n}{2}, \frac{1-a-n}{2}} + \frac{(l-1)^2}{4} \right)^2 - \frac{(l-1)^2}{4} (R - \overline{R})^2 \right]
= \left[ \left( \Delta'_{\frac{1-a-n}{2}, \frac{1-a-n}{2}} + \frac{l^2}{4} \right)^2 - \frac{l^2}{4} (R - \overline{R})^2 \right] \Delta'_{\frac{3-a-n-l}{2}, \frac{3-a-n-l}{2}}.
\]

Proof. since

\[
\Delta'_{\frac{1-a-n}{2}, \frac{1-a-n}{2}} + \frac{(l-1)^2}{4} = \Delta'_{\frac{1-a-n}{2}, \frac{1-a-n}{2}} + \frac{l^2 + 2n + 2a - 2l - 2}{4} + \frac{1}{2} (R + \overline{R});
\]

\[
\Delta'_{\frac{1-a-n-l}{2}, \frac{1-a-n-l}{2}} = \Delta'_{\frac{1-a-n}{2}, \frac{1-a-n}{2}} - \frac{l}{2} (R + \overline{R}) - \frac{l^2 - 2(1-a-n)l}{4},
\]

(3.13)
we have

\[
\begin{align*}
&\left(\Delta'_{1, a, n-1, 1-a, n-1} + \Delta'_{1-a, n-1, 1, a-1} + \frac{(l-1)^2}{4}\right) \\
&= \left(\Delta'_{1, a, n-1, 1-a, n-1} + \frac{l^2}{4} - \frac{l}{2}(R + \bar{R}) - \frac{l^2 - (1 - a - n)}{2}\right) \\
&\quad \left(\Delta'_{1-a, n-1, 1, a-1} + \frac{l^2}{4} + \frac{1}{2}(R + \bar{R}) + \frac{n + a - l - 1}{2}\right) \\
&= \left(\Delta'_{1, a, n-1, 1-a, n-1} + \frac{l^2}{4}\right)^2 - \frac{l}{2}(R + \bar{R}) \left(\Delta'_{1, a, n-1, 1-a, n-1} + \frac{l^2}{4}\right) + \frac{1}{2} \left(\Delta'_{1, a, n-1, 1-a, n-1} + \frac{l^2}{4}\right) (R + \bar{R}) \\
&\quad - \frac{l}{4}(R + \bar{R})^2 - \frac{l(n + a - 1)}{2}(R + \bar{R}) - \frac{l}{4} \left[(n + a - 1)^2 - l^2\right].
\end{align*}
\]

Using the identity

\[
\Delta'_{\alpha, \alpha} \Delta'_{\beta, \beta} - \Delta'_{\beta, \beta} \Delta'_{\alpha, \alpha} = (\alpha - \beta) [(R + \bar{R}) \Delta'_{\beta, \beta} - \Delta'_{\beta, \beta} (R + \bar{R})] = -2(\alpha - \beta)(\Delta'_{0, 0} + R\bar{R}),
\]

we obtain

\[
\begin{align*}
&\left(\Delta'_{1, a, n-1, 1-a, n-1} + \Delta'_{1-a, n-1, 1, a-1} + \frac{(l-1)^2}{4}\right) \\
&= \left(\Delta'_{1, a, n-1, 1-a, n-1} + \frac{l^2}{4}\right)^2 - \frac{l - 1}{2} \left(\Delta'_{1, a, n-1, 1-a, n-1} + \frac{l^2}{4}\right) (R + \bar{R}) + l\Delta'_{0, 0} + lR\bar{R} \\
&\quad - \frac{l}{4}(R + \bar{R})^2 - \frac{l(n + a - 1)}{2}(R + \bar{R}) - \frac{l}{4} \left[(n + a - 1)^2 - l^2\right] \\
&= \left(\Delta'_{1, a, n-1, 1-a, n-1} + \frac{l^2}{4}\right)^2 - \frac{l - 1}{2} \left(\Delta'_{1, a, n-1, 1-a, n-1} + \frac{l^2}{4}\right) (R + \bar{R}) + l\Delta'_{0, 0} \\
&\quad - \frac{l}{4}(R - \bar{R})^2 - \frac{l(n + a - 1)}{2}(R + \bar{R}) - \frac{l}{4} \left[(n + a - 1)^2 - l^2\right] \\
&= \left(\Delta'_{1, a, n-1, 1-a, n-1} + \frac{l^2}{4}\right)^2 - \frac{l - 1}{2} \left(\Delta'_{1, a, n-1, 1-a, n-1} + \frac{l^2}{4}\right) (R + \bar{R} + a + n + l - 1) - \frac{l}{4}(R - \bar{R})^2.
\end{align*}
\]
Therefore,
\[
\begin{align*}
\Delta'_{1,a-n-1,1,a-n-1} & \left( \Delta'_{1,a-n,1,a-n} \frac{3}{2} + \frac{(l - 1)^2}{4} \right) - \left( \Delta'_{1,a-n,1,a-n} \frac{3}{2} + \frac{l^2}{4} \right) \\
& = \left( \Delta'_{1,a-n,1,a-n} \frac{3}{2} + \frac{l^2}{4} \right) \left( \Delta'_{2,a-n,1,a-n} \frac{3}{2} + \frac{(l - 1)^2}{4} \right) - \left( \Delta'_{2,a-n,1,a-n} \frac{3}{2} + \frac{l^2}{4} \right) \\
& - \frac{l - 1}{2} \left( \Delta'_{1,a-n,1,a-n} \frac{3}{2} + \frac{l^2}{4} \right) \left( R + \bar{R} + a + n + l - 1 \right) \left( \Delta'_{2,a-n,1,a-n} \frac{3}{2} + \frac{(l - 1)^2}{4} \right) \\
& - \frac{l}{4} (R - \bar{R})^2 \left( \Delta'_{2,a-n,1,a-n} \frac{3}{2} + \frac{(l - 1)^2}{4} \right) \\
& = \frac{l - 1}{2} \left( \Delta'_{1,a-n,1,a-n} \frac{3}{2} + \frac{l^2}{4} \right) \left( \frac{R + \bar{R}}{2} + \frac{l + 1 - a - n}{2} \right) \left( R + \bar{R} + a + n + l - 1 \right) \\
& - \frac{l + 1}{2} \left( \Delta'_{1,a-n,1,a-n} \frac{3}{2} + \frac{l^2}{4} \right) \left( \frac{R + \bar{R}}{2} + \frac{l + 1 - a - n}{2} \right) \left( \Delta'_{2,a-n,1,a-n} \frac{3}{2} + \frac{(l - 1)^2}{4} \right) \\
& - \frac{l}{4} (R - \bar{R})^2 \left( \Delta'_{2,a-n,1,a-n} \frac{3}{2} + \frac{(l - 1)^2}{4} \right) \\
& = - \frac{l}{4} (R - \bar{R})^2 \left( \frac{l - 1}{2} \left( \Delta'_{1,a-n,1,a-n} \frac{3}{2} + \frac{l^2}{4} \right) - \frac{l}{4} (R - \bar{R})^2 \left( \Delta'_{2,a-n,1,a-n} \frac{3}{2} + \frac{(l - 1)^2}{4} \right) \\
& = \frac{(R - \bar{R})^2}{4} \left( \frac{1}{2} \Delta'_{1,a-n,1,a-n} \frac{3}{2} + \frac{l^2}{4} \right) - \frac{l}{4} (R - \bar{R})^2 \left( \Delta'_{2,a-n,1,a-n} \frac{3}{2} + \frac{(l - 1)^2}{4} \right) \\
& = \frac{(R - \bar{R})^2}{4} \left[ (l - 1)^2 \Delta'_{1,a-n,1,a-n} \frac{3}{2} - l^2 \Delta'_{2,a-n,1,a-n} \frac{3}{2} \right].
\end{align*}
\]
This proves Lemma 3.6.

**Proposition 3.7.** Let \( a \in \mathbb{R} \) and \( u \in C^\infty (\mathbb{B}^n_\mathbb{C}) \). We have, for \( k \in \mathbb{N} \setminus \{0\} \),
\[
\begin{align*}
\prod_{j=1}^{k} \left[ \Delta'_{1,a-n,1,a-n} + \frac{(k + 1 - 2j)^2}{4} - \frac{k + 1 - 2j}{2} (R - \bar{R}) \right] \left( 1 - |z|^2 \right)^{\frac{k+n-2}{2}} f
\end{align*}
\]
(3.14)
\[
= 4^{-k} \left( 1 - |z|^2 \right)^{-\frac{k+n-2}{4}} \prod_{j=1}^{k} \left[ \Delta_{\mathbb{B}} + n^2 - (a - k + 2j)^2 \right] f.
\]
Proof. By letting $s = \frac{n+1-a}{2}$ in Lemma 3.3, we obtain that (3.14) is valid for $k = 1$. Assume it is valid for $k = l$.

\[
\prod_{j=1}^{l} \left[ \Delta'_{\frac{1-a-n}{2}, \frac{1-a-n}{2}} + \frac{(l + 1 - 2j)^2}{4} - \frac{l + 1 - 2j}{2} (R - R) \right] [(1 - |z|^2)^{\frac{l-n-a}{2}} f] \]

(3.15)

Replacing $a$ by $a - 1$ in (3.15), we get

\[
\prod_{j=1}^{l} \left[ \Delta'_{\frac{2-a-n}{2}, \frac{2-a-n}{2}} + \frac{(l + 1 - 2j)^2}{4} - \frac{l + 1 - 2j}{2} (R - R) \right] [(1 - |z|^2)^{\frac{l+n-a}{2}} f] \]

\[
= 4^{-l} (1 - |z|^2)^{-\frac{l+n-a}{2}} \prod_{j=1}^{l} \left[ \Delta_B + n^2 - (a - l + 2j - 2)^2 \right] f. \]

Therefore, by Lemma 3.5, we get

\[
\Delta'_{\frac{1-a-n-l}{2}, \frac{1-a-n-l}{2}} \prod_{j=1}^{l} \left[ \Delta'_{\frac{2-a-n}{2}, \frac{2-a-n}{2}} + \frac{(l + 1 - 2j)^2}{4} - \frac{l + 1 - 2j}{2} (R - R) \right] [(1 - |z|^2)^{\frac{l+n-a}{2}} f] \]

\[
= \Delta'_{\frac{1-a-n-l}{2}, \frac{1-a-n-l}{2}} \left[ 4^{-l} (1 - |z|^2)^{-\frac{l+n-a}{2}} \prod_{j=1}^{l} \left[ \Delta_B + n^2 - (a - 1 - l + 2j - 2)^2 \right] f \right] \]

\[
= 4^{-l-1} (1 - |z|^2)^{-\frac{l+n-a+1}{2}} \prod_{j=1}^{l+1} \left[ \Delta_B + n^2 - (a - 1 - l + 2j - 2)^2 \right] f. \]

On the other hand, if $l$ is even, then by Lemma 3.6, we have

\[
\Delta'_{\frac{1-a-n-l}{2}, \frac{1-a-n-l}{2}} \prod_{j=1}^{l/2} \left[ \Delta'_{\frac{2-a-n}{2}, \frac{2-a-n}{2}} + \frac{(l + 1 - 2j)^2}{4} - \frac{l + 1 - 2j}{2} (R - R) \right] \]

\[
= \Delta'_{\frac{1-a-n-l}{2}, \frac{1-a-n-l}{2}} \left[ \left( \Delta'_{\frac{2-a-n}{2}, \frac{2-a-n}{2}} + \frac{(l + 1 - 2j)^2}{4} \right)^2 - \frac{(l + 1 - 2j)^2}{4} (R - R)^2 \right] \]

\[
= \left[ \left( \Delta'_{\frac{1-a-n-l}{2}, \frac{1-a-n-l}{2}} + \frac{l^2}{4} \right)^2 - \frac{l^2}{4} (R - R)^2 \right] \Delta'_{\frac{1-a-n-l}{2}, \frac{1-a-n-l}{2}} \]

\[
\prod_{j=2}^{l/2} \left[ \left( \Delta'_{\frac{2-a-n}{2}, \frac{2-a-n}{2}} + \frac{(l + 1 - 2j)^2}{4} \right)^2 - \frac{(l + 1 - 2j)^2}{4} (R - R)^2 \right]. \]
Repeating this process in a suitable manner we get

\[
\Delta'_{1-a,n-l,1-a,n-l} \prod_{j=1}^{l+1} \left[ \frac{(l+1-2j)^2}{4} - \frac{l+2-2j}{2}(R - \bar{R}) \right]
\]

Therefore, we have proved that if \( l \) is even, then

\[
\prod_{j=1}^{l+1} \left[ \frac{(l+2-2j)^2}{4} - \frac{l+2-2j}{2}(R - \bar{R}) \right] [(1 - |x|^{\frac{4}{l+2-a-n}} f] \]

(3.16)

\[4^{l-1}(1 - |z|^2)^{\frac{l+1+n-a}{2}} \prod_{j=1}^{l+1} \left[ \Delta_3 + n^2 - (a - l + 2j - 3)^2 \right] f.\]

The proof of (3.16) for \( l \) odd is similar and we omit it. These complete the proof of Proposition 3.7.

\[\square\]

**Proof of Theorem 1.3** Combining Proposition 3.4 and 3.7

4. Asymptotic Estimates of Bessel-Green-Riesz Kernels

In what follows, \( a \lesssim b \) or \( a = O(b) \) will stand for \( a \leq Cb \) with a positive constant \( C \) and \( a \sim b \) stand for \( C^{-1}b \leq a \leq Cb \). We shall frequently use the following

(4.1) \( \cosh r - 1 \sim r^2, \sinh r - r \sim r^3 \), \( 0 < r < 1 \), and \( \sinh r \sim \cosh r \sim e^r, \ r \geq 1. \)

We also need the following identity of convolution on Euclidean space (see [74])

(4.2)

\[
\int_{\mathbb{R}^n} |x|^{\alpha-n} |y-x|^{\beta-n} dx = \frac{\gamma_n(\alpha) \gamma_n(\beta)}{\gamma_n(\alpha + \beta)} |y|^{\alpha + \beta - n}, \ 0 < \alpha < n, \ 0 < \beta < n, \ 0 < \alpha + \beta < n.
\]

where

(4.3) \( \gamma_n(\alpha) = \pi^{n/2} 2^n \Gamma(\alpha/2)/\Gamma \left( \frac{n}{2} - \frac{\alpha}{2} \right), \ 0 < \alpha < n. \)

4.1. Heat kernel and Bessel-Green-Riesz kernels on real hyperbolic space. In this subsection we recall some fact about asymptotic estimates of Bessel-Green-Riesz kernels on real hyperbolic space. Denote by \( h_t(\rho; n) \) the heat kernel on real hyperbolic space of dimension \( n \) which is given explicitly by the following formulas (see e.g. [17] [32]):

- If \( n = 2m + 1 \), then

(4.4) \( h_t(\rho; 2m + 1) = 2^{-m-1} \pi^{-m-1/2} t^{-\frac{1}{2}} e^{-m^2 t} \left( -\frac{1}{\sinh \rho} \frac{\partial}{\partial \rho} \right)^m e^{-\frac{t}{4}}. \)

- If \( n = 2m \), then

(4.5) \( h_t(\rho; 2m) = (2\pi)^{-m-\frac{1}{2}} t^{-\frac{1}{2}} e^{-\frac{(2m-1)^2}{4}} \int_\rho^{+\infty} \frac{\sinh r}{\cosh r - \cosh \rho} \left( -\frac{1}{\sinh \rho} \frac{\partial}{\partial r} \right)^m e^{-\frac{t}{4}} dr. \)
The asymptotic estimates of Bessel-Green-Riesz kernels on real hyperbolic space satisfies (see e.g. [52]):

- \( \zeta > 0 \):
  \[
  (-\Delta_{\mathbb{H}} - (n-1)^2/4 + \zeta)^{-\alpha/2} \leq \frac{1}{\gamma_n(\alpha)} \cdot \frac{1}{\rho^{n-\alpha}} + O\left(\frac{1}{\rho^{n-\alpha-\epsilon}}\right), \quad 0 < \alpha < n, \quad 0 < \rho < 1
  \]
  \[
  (-\Delta_{\mathbb{H}} - (n-1)^2/4 + \zeta)^{-\alpha/2} \sim \rho^{-\alpha/2} e^{-\zeta \rho - \frac{n-\alpha}{2} \rho}, \quad \alpha > 0, \quad \rho \geq 1;
  \]

- \( \zeta = 0 \):
  \[
  (-\Delta_{\mathbb{H}} - (n-1)^2/4)^{-\alpha/2} \leq \frac{1}{\gamma_n(\alpha)} \cdot \frac{1}{\rho^{n-\alpha}} + O\left(\frac{1}{\rho^{n-\alpha-2}}\right), \quad 0 < \alpha < 3, \quad 0 < \rho < 1
  \]
  \[
  (-\Delta_{\mathbb{H}} - (n-1)^2/4)^{-\alpha/2} \sim \rho^{\alpha-2} e^{-\frac{n-3}{2} \rho}, \quad 0 < \alpha < 3, \quad \rho \geq 1;
  \]

where \( 0 < \zeta' < \zeta \) and \( 0 < \epsilon < \min\{1, n - \alpha\} \).

In particular, in case \( \alpha = 2 \), we have the following explicit formula (see [64] for \( \nu \geq \frac{n-1}{2} \), [50] for \( \nu > 0 \) and [61] for \( \nu = 0 \))

\[
(\nu^2 - (n-1)^2/4 - \Delta_{\mathbb{H}})^{-1}
\]

\[
= (2\pi)^{-\frac{n-1}{2}} \Gamma(\frac{n-1}{2} + \nu) (\sinh \rho)^{2-n} \int_0^\pi (\cosh \rho + \cos t)^{\frac{n-3}{2} - \nu} (\sin t)^{2\nu} dt.
\]

4.2. Heat kernel and Bessel-Green-Riesz kernels on complex hyperbolic space. Denote by \( e^{t\Delta_{\mathbb{H}}} \) the heat kernel on \( \mathbb{H}^n_C \). It is known that \( e^{t\Delta_{\mathbb{H}}} \) is given explicitly by the following formulas (see [56])

\[
e^{t\Delta_{\mathbb{H}}} = 2^{-n+\frac{1}{2}n-\frac{1}{2}} e^{-\frac{n-2}{2} t} \int_\rho^{+\infty} \frac{\sinh r}{\sqrt{\cosh 2r - \cosh 2\rho}} \left( -\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^n e^{-\frac{t}{2r}} dr.
\]

For simplicity, we denote by

\[
k_{\zeta, \alpha} = (-\Delta_{\mathbb{H}} - n^2 + \zeta^2)^{-\alpha/2}, \quad 0 < \alpha < 2n, \quad \zeta > 0;
\]

\[
k_{\alpha} = (-\Delta_{\mathbb{H}} - n^2)^{-\alpha/2}, \quad 0 < \alpha < 3.
\]

The following asymptotic estimates of Bessel-Green-Riesz kernels for large \( \rho \) is given by Anker and Ji ([41], page 1083, (iii)):

- \( \zeta > 0 \):
  \[
  k_{\zeta, \alpha} \sim \rho^{-\alpha/2} e^{-\zeta \rho - \rho}, \quad \alpha > 0, \quad \rho \geq 1
  \]

- \( \zeta = 0 \):
  \[
  k_{\alpha} \sim \rho^{\alpha-2} e^{-\rho}, \quad 0 < \alpha < 3, \quad \rho \geq 1;
  \]

In order to obtain the sharp constant of fractional Hardy-Adams type inequalities, we need to give more refined and optimal estimates of such kernels for both small and large \( \rho \).

In particular, by (4.1), and (4.1), we have

\[
e^{t\Delta_{\mathbb{H}}} = 2^3 \int_\rho^{+\infty} \frac{\sinh r}{\sqrt{\cosh 2r - \cosh 2\rho}} h_t(r; 2n+1) dr.
\]
Combing (4.8) and (4.12) yields the following explicit formula of Green’s function:

\[
(\nu^2 - n^2 - \Delta_B)^{-1} = \frac{(2\pi)^{\frac{2n+1}{2}} \Gamma(n+\nu)}{2^{n-1}\Gamma(\nu + \frac{1}{2})} \int_{\rho}^{+\infty} \frac{(\sinh r)^{2-2n}}{\sqrt{\cosh 2r - \cosh 2\rho}} \left( \int_0^{\pi} (\cosh r + \cos t)^{n-1-\nu} (\sin t)^{2\nu} dt \right) dr.
\]

4.3. Estimate of \( k_{\alpha} \), \( 0 < \alpha < 3 \). We firstly prove the following lemma:

**Lemma 4.1.** Let \( \beta > 0 \). Then for \( \rho > 0 \), we have

\[
\int_{\rho}^{+\infty} \frac{\cosh r}{(\sinh r)^\beta \sqrt{\cosh 2r - \cosh 2\rho}} dr = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{\beta}{2}\right)}{2\sqrt{2} \Gamma\left(\frac{1+\beta}{2}\right) (\sinh \rho)^\beta}.
\]

**Proof.** Substituting \( t = \cosh 2r - \cosh 2\rho \), we have

\[
\int_{\rho}^{+\infty} \frac{\cosh r}{(\sinh r)^\beta \sqrt{\cosh 2r - \cosh 2\rho}} dr = \frac{1}{2} \int_0^{+\infty} \frac{1}{\sqrt{t}} \cdot \frac{1}{(t/2 + \sinh^2 \rho)^{1+\beta/2}} dt = \frac{1}{2\sqrt{2}(\sinh \rho)^\beta} \int_0^{+\infty} s^{-\frac{1}{2}} (1 + s)^{-\frac{1+\beta}{2}} ds.
\]

To get last equation, we use the substitution \( t = 2s \sinh^2 \rho \). Therefore, by using the fact

\[
\int_0^{+\infty} s^{-x-1} (1 + s)^{-x-y} ds = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}, \quad x > 0, \quad y > 0,
\]

we have

\[
\int_{\rho}^{+\infty} \frac{\cosh r}{(\sinh r)^\beta \sqrt{\cosh 2r - \cosh 2\rho}} dr = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{\beta}{2}\right)}{2\sqrt{2} \Gamma\left(\frac{1+\beta}{2}\right) (\sinh \rho)^\beta}.
\]

This completes the proof of Lemma 4.1.

Now we give the asymptotic estimates of \( k_{\alpha} \) as \( \rho \to 0 \). The main result is the following:

**Lemma 4.2.** Let \( 0 < \alpha < 3 \) and \( n \geq 2 \). There holds

\[
k_{\alpha} \leq \frac{1}{\gamma_{2n}(\alpha) \rho^{2n-\alpha}} + O\left( \frac{1}{\rho^{2n-\alpha-1}} \right), \quad 0 < \rho < 1.
\]

**Proof.** By the Mellin type expression and (4.12), we have, for \( \rho \in (0, 1) \),

\[
k_{\alpha}(\rho) = \frac{1}{\Gamma(\alpha/2)} \int_0^{+\infty} t^{\frac{\alpha}{2} - 1} e^{t(\Delta_B + n^2)} dt = 2^{\frac{\alpha}{2}} \int_{\rho}^{+\infty} \frac{\sinh r}{\sqrt{\cosh 2r - \cosh 2\rho}} \left( \frac{1}{\Gamma(\alpha/2)} \int_0^{+\infty} t^{\frac{\alpha}{2} - 1} e^{nt} h_t(r; 2n + 1) dt \right) dr = A_1 + A_2,
\]
where

\[
A_1 = 2^{\frac{3}{2}} \int_0^2 \frac{\sinh r}{\sqrt{\cosh 2r - \cosh 2\rho}} \left( \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha-1} h_t(r; 2n + 1) dt \right) dr;
\]

\[
A_2 = 2^{\frac{3}{2}} \int_2^{+\infty} \frac{\sinh r}{\sqrt{\cosh 2r - \cosh 2\rho}} \left( \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha-1} h_t(r; 2n + 1) dt \right) dr.
\]

By (4.7), we have

\[
A_1 \leq 2^{\frac{3}{2}} \int_0^2 \frac{\sinh r}{\sqrt{\cosh 2r - \cosh 2\rho}} \left( \frac{1}{\gamma_{2n+1}(\alpha)} \cdot \frac{1}{r^{2n+1-\alpha}} + O \left( \frac{1}{r^{2n-\alpha-1}} \right) \right) dr
\]

\[
\leq 2^{\frac{3}{2}} \int_0^2 \frac{\sinh r}{\sqrt{\cosh 2r - \cosh 2\rho}} \left( \frac{1}{\gamma_{2n+1}(\alpha)(\sinh r)^{2n+1-\alpha}} + O \left( \frac{1}{(\sinh r)^{2n-\alpha-1}} \right) \right) dr.
\]

By Lemma 4.1, we obtain

\[
2^{\frac{3}{2}} \int_0^2 \frac{\sinh r}{\sqrt{\cosh 2r - \cosh 2\rho}} \cdot \frac{1}{\gamma_{2n+1}(\alpha)(\sinh r)^{2n+1-\alpha}} dr
\]

\[
= \frac{2^{\frac{3}{2}}}{\gamma_{2n+1}(\alpha)} \int_0^2 \frac{1}{\sqrt{\cosh 2r - \cosh 2\rho}} \cdot \frac{1}{(\sinh r)^{2n-\alpha}} dr
\]

\[
\leq \frac{2^{\frac{3}{2}}}{\gamma_{2n+1}(\alpha)} \int_0^{+\infty} \frac{\cosh r}{\sqrt{\cosh 2r - \cosh 2\rho}} \cdot \frac{1}{(\sinh r)^{2n-\alpha}} dr
\]

\[
= \frac{2^{\frac{3}{2}}}{\gamma_{2n+1}(\alpha)} \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2n-\alpha}{2}\right)}{2\sqrt{\Gamma\left(\frac{1+2n-\alpha}{2}\right)}} (\sinh \rho)^{2n-\alpha} = \frac{1}{\gamma_{2n}(\alpha)(\sinh \rho)^{2n-\alpha}}.
\]

Similarly,

\[
\int_0^2 \frac{\sinh r}{\sqrt{\cosh 2r - \cosh 2\rho}} \cdot \frac{1}{(\sinh r)^{2n-\alpha-1}} dr
\]

\[
\leq \int_0^2 \frac{\cosh r}{\sqrt{\cosh 2r - \cosh 2\rho}} \cdot \frac{1}{(\sinh r)^{2n-\alpha-1}} dr \lesssim \frac{1}{(\sinh \rho)^{2n-\alpha-1}}.
\]

Therefore,

\[
A_1 \leq \frac{1}{\gamma_{2n}(\alpha)(\sinh \rho)^{2n-\alpha}} + O \left( \frac{1}{(\sinh \rho)^{2n-\alpha-1}} \right).
\]

We claim \( A_2 \) is bounded for \( 0 < \rho < 1 \). In fact, we have, for \( 0 < \rho < 1 \),

\[
A_2 \sim \int_2^{+\infty} \frac{\sinh r}{\sqrt{\cosh 2r - \cosh 2\rho}} r^{\alpha-2} e^{-nr} dr
\]

\[
\sim \int_2^{+\infty} r^{\alpha-2} e^{-nr} dr \sim 1.
\]

Therefore, we have

\[
k_\alpha = A_1 + A_2 \leq \frac{1}{\gamma_{2n}(\alpha)(\sinh \rho)^{2n-\alpha}} + O \left( \frac{1}{(\sinh \rho)^{2n-\alpha-1}} \right), \quad 0 < \rho < 1.
\]
The desired result follows. \hfill ∎

4.4. Estimate of $k_{ζ,α}$.

**Lemma 4.3.** Let $α > 0$ and $0 < ε < \min\{1, 2n - α\}$. There holds

\begin{equation}
(4.15) \quad k_α \leq \frac{1}{γ_2n(α)ρ^{2n-α}} + O\left(\frac{1}{ρ^{2n-α-ε}}\right), \quad 0 < ρ < 1.
\end{equation}

**Proof.** Also by the Mellin type expression and (4.12), we have, for $ρ \in (0, 1)$,

\[ k_{ζ,α} = \frac{1}{Γ(α/2)} \int_0^∞ t^{2α-1} e^{t(Δ_2n+2-ζ^2)} dt \]
\[ = 2^{α/2} \int_0^ρ \frac{\sinh r}{\sqrt{\cosh 2r - \cosh 2ρ}} \left( \frac{1}{Γ(α/2)} \int_0^∞ t^{2α-1} e^{n^2t-ζ^2} h_t(r; 2n + 1) dt \right) dr \]
\[ =: A_3 + A_4, \]

where

\[ A_3 = 2^{α/2} \int_0^2 \frac{\sinh r}{\sqrt{\cosh 2r - \cosh 2ρ}} \left( \frac{1}{Γ(α/2)} \int_0^∞ t^{2α-1} e^{n^2t-ζ^2} h_t(r; 2n + 1) dt \right) dr; \]
\[ A_4 = 2^{α/2} \int_2^{+∞} \frac{\sinh r}{\sqrt{\cosh 2r - \cosh 2ρ}} \left( \frac{1}{Γ(α/2)} \int_0^∞ t^{2α-1} e^{n^2t-ζ^2} h_t(r; 2n + 1) dt \right) dr. \]

By (4.6), we have

\[ A_3 = 2^{α/2} \int_0^2 \frac{\sinh r}{\sqrt{\cosh 2r - \cosh 2ρ}} \left( \frac{1}{Γ(α/2)} \cdot \frac{1}{r^{2n+1-α}} + O\left(\frac{1}{r^{2n+1-α-ε}}\right) \right) dr \]
\[ \leq \frac{2^{α/2}}{γ_2n+1(α)} \int_0^2 \frac{\sinh r}{\sqrt{\cosh 2r - \cosh 2ρ}} \left( \frac{1}{(\sinh r)^{2n+1-α}} + O\left(\frac{1}{(\sinh r)^{2n+1-α-ε}}\right) \right) dr. \]

Since

\[ \frac{2^{α/2}}{γ_2n+1(α)} \int_0^2 \frac{\sinh r}{\sqrt{\cosh 2r - \cosh 2ρ}} \cdot \frac{1}{(\sinh r)^{2n+1-α}} dr \]
\[ \leq \frac{2^{α/2}}{γ_2n+1(α)} \int_0^{+∞} \frac{\cosh r}{\sqrt{\cosh 2r - \cosh 2ρ}} \cdot \frac{1}{(\sinh r)^{2n-α}} dr \]
\[ = \frac{2^{α/2}}{γ_2n+1(α)} \cdot \frac{Γ\left(\frac{1}{2}\right)Γ\left(\frac{2n-α}{2}\right)}{2√2Γ\left(\frac{1+2n-α}{2}\right)(\sinh r)^{2n-α}} = \frac{1}{γ_2n(α)(\sinh r)^{2n-α}} \]

and

\[ \int_0^2 \frac{\sinh r}{\sqrt{\cosh 2r - \cosh 2ρ}} \cdot \frac{1}{(\sinh r)^{2n+1-α-ε}} dr \]
\[ \leq \frac{2^{α/2}}{γ_2n+1(α)} \int_0^{+∞} \frac{\cosh r}{\sqrt{\cosh 2r - \cosh 2ρ}} \cdot \frac{1}{(\sinh r)^{2n-α-ε}} dr \]
\[ \approx \frac{1}{(\sinh r)^{2n-α-ε}}. \]
we get
\[ A_3 \leq \frac{1}{\gamma_{2n}(\alpha)(\sinh \rho)^{2n-\alpha}} + O \left( \frac{1}{(\sinh \rho)^{2n-\alpha-\varepsilon}} \right). \]

On the other hand,
\[ A_4 \sim \int_2^{+\infty} \frac{\sinh r}{\sqrt{\cosh 2r - \cosh 2\rho}} r^{\frac{\alpha-2}{2}} e^{-\zeta r - nr} dr \sim \int_2^{+\infty} r^{\frac{\alpha-2}{2}} e^{-\zeta r - nr} dr \sim 1. \]

Therefore, we have
\[ k_{\zeta,\alpha} = A_3 + A_4 \leq \frac{1}{\gamma_{2n}(\alpha)(\sinh \rho)^{2n-\alpha}} + O \left( \frac{1}{(\sinh \rho)^{2n-\alpha-\varepsilon}} \right), \quad 0 < \rho < 1. \]
The desired result follows. \( \square \)

4.5. **Estimate of** \( k_{\alpha} k_{\zeta,\beta} \), \( 0 < \alpha < 3, \zeta > 0, 0 < \beta < 2n - \alpha \). Before we give the proof of the main result in this subsection, we need the following two lemmas.

**Lemma 4.4.** Let \( 0 < \alpha < 2n \), \( 0 < \beta < 2n \) and \( \lambda_1 + \lambda_2 > \alpha + \beta - 2n \). If \( 0 < \alpha + \beta < 2n - 1 \), then for \( 0 < \rho < 1 \),
\[ \frac{1}{(\sinh \rho)^{2n-\alpha}(\cosh \rho)^{\lambda_1}} \cdot \frac{1}{(\sinh \rho)^{2n-\beta}(\cosh \rho)^{\lambda_2}} \leq \frac{1}{\gamma_{2n}(\alpha)\gamma_{2n}(\beta)} \frac{1}{\gamma_{2n}(\alpha + \beta)} \frac{1}{\rho^{n-\alpha-\beta}} + O \left( \frac{1}{\rho^{n-\alpha-\beta-1}} \right). \]

If \( 2n - 1 \leq \alpha + \beta < 2n \), then for \( 0 < \rho < 1 \) and \( 0 < \epsilon < 2n - \alpha - \beta \),
\[ \frac{1}{(\sinh \rho)^{2n-\alpha}(\cosh \rho)^{\lambda_1}} \cdot \frac{1}{(\sinh \rho)^{2n-\beta}(\cosh \rho)^{\lambda_2}} \leq \frac{1}{\gamma_{2n}(\alpha)\gamma_{2n}(\beta)} \frac{1}{\gamma_{2n}(\alpha + \beta)} \frac{1}{\rho^{n-\alpha-\beta}} + O \left( \frac{1}{\rho^{n-\alpha-\beta-\epsilon}} \right). \]

**Proof.** We compute
\[
\frac{1}{(\sinh \rho)^{2n-\alpha}(\cosh \rho)^{\lambda_1}} \cdot \frac{1}{(\sinh \rho)^{2n-\beta}(\cosh \rho)^{\lambda_2}} = \int_{B_{\rho}^\circ} \left( \frac{\sqrt{1 - |z|^2}}{|z|} \right)^{2n-\alpha} (1 - |z|^2)^{\lambda_1} \left( \frac{(1 - |z|^2)(1 - |a|^2)}{|z - a|^2 + |(z, a)|^2 - |z|^2|a|^2} \right)^{\frac{2n-\beta}{2}}.
\]

(4.16)
\[
= (1 - |a|^2)^{2n-\beta + \lambda_2} \int_{B_{\rho}^\circ} \frac{1}{|z|^{2n-\alpha}} \left( \frac{1}{|z - a|^2 + |(z, a)|^2 - |z|^2|a|^2} \right)^{\frac{2n-\beta}{2}} d\rho.
\]

Thus,
\[
= (\cosh \rho(a))\left( 1 - |z|^2 \right)^{\frac{\alpha + \beta - 2n - \lambda_1 - \lambda_2}{2}} \cdot (A_5 + A_6),
\]
where

\[ A_5 = \int_{\{|z| \leq \frac{1}{2}\}} \frac{1}{|z|^{2n-\alpha} \left| \frac{1}{|z-a|^2 + |(z,a)|^2 - |z|^2 a|^2 \right|^{2n-\beta}} dz \frac{1}{(1 - (z,a)^{\lambda_2} (1 - |z|^2)^{1+\frac{a+\beta-2n-\lambda_1-\lambda_2}{2}}}; \]

\[ A_6 = \int_{\{|z| > \frac{1}{2}\}} \frac{1}{|z|^{2n-\alpha} \left| \frac{1}{|z-a|^2 + |(z,a)|^2 - |z|^2 a|^2 \right|^{2n-\beta}} dz \frac{1}{(1 - (z,a)^{\lambda_2} (1 - |z|^2)^{1+\frac{a+\beta-2n-\lambda_1-\lambda_2}{2}}}; \]

We note if \( \rho(a) < 1 \) and \( |z| \leq \frac{1}{2} \), then

\[ |1 - (z,a)|^{\lambda_2} (1 - |z|^2)^{1+\frac{a+\beta-2n-\lambda_1-\lambda_2}{2}} = 1 + O(|z|). \]

On the other hand,

\[ |(z,a)|^2 - |z|^2 a|^2 = |(z,z) + (z,a - z)|^2 - |z|^2 a - z|^2 \]

\[ = |z|^4 + |(z,a - z)|^2 + 2|z|^2 \text{Re}(z,a - z) - |z|^2 |a - z|^2 + |z|^2 + 2 \text{Re}(z,a - z) \]

\[ = |z|^2 |z - a|^2 \left[ \left( \frac{z - a}{|z| \cdot |z - a|} \right)^2 - 1 \right]. \]

Therefore, we have

\[ |z - a|^2 + |(z,a)|^2 - |z|^2 a|^2 = |z - a|^2 \left\{ 1 + |z|^2 \left[ \left( \frac{z - a}{|z| \cdot |z - a|} \right)^2 - 1 \right] \right\} \]

\[ = |z - a|^2 \left[ 1 + O(|z|^2) \right]. \]

Thus, if \( 0 < \alpha + \beta < 2n - 1 \), then by using (4.2), we get

\[ A_5 = \int_{\{|z| \leq \frac{1}{2}\}} \frac{1}{|z|^{2n-\alpha} \left| \frac{1}{|z-a|^2 - |z|^2 a|^2} \right|^{2n-\beta}} \left[ 1 + O(|z|) \right] dz \]

\[ \leq \int_{C^n} \frac{1}{|z|^{2n-\alpha} \left| \frac{1}{|z-a|^2 - |z|^2 a|^2} \right|^{2n-\beta}} dz + O \left( \int_{C^n} \frac{1}{|z|^{2n-\alpha-1} \left| \frac{1}{|z-a|^2 - |z|^2 a|^2} \right|^{2n-\beta}} dz \right) \]

\[ = \frac{\gamma_{2n}(\alpha) \gamma_{2n}(\beta)}{\gamma_{2n}(\alpha + \beta)} \frac{1}{|a|^{2n-\alpha-\beta}} + O \left( \frac{1}{|a|^{2n-\alpha-\beta-1}} \right). \]

Similarly, if \( 2n - 1 < \alpha + \beta < 2n \), then

\[ A_5 = \int_{\{|z| \leq \frac{1}{2}\}} \frac{1}{|z|^{2n-\alpha} \left| \frac{1}{|z-a|^2 - |z|^2 a|^2} \right|^{2n-\beta}} \left[ 1 + O(|z|^\varepsilon) \right] dz \]

\[ \leq \int_{C^n} \frac{1}{|z|^{2n-\alpha} \left| \frac{1}{|z-a|^2 - |z|^2 a|^2} \right|^{2n-\beta}} dz + O \left( \int_{C^n} \frac{1}{|z|^{2n-\alpha-\varepsilon} \left| \frac{1}{|z-a|^2 - |z|^2 a|^2} \right|^{2n-\beta}} dz \right) \]

\[ = \frac{\gamma_{2n}(\alpha) \gamma_{2n}(\beta)}{\gamma_{2n}(\alpha + \beta)} \frac{1}{|a|^{2n-\alpha-\beta}} + O \left( \frac{1}{|a|^{2n-\alpha-\beta-\varepsilon}} \right). \]

Now we give the estimate of \( A_6 \). In fact,

\[ A_6 \sim \int_{\{|z| > \frac{1}{2}\}} \frac{dz}{(1 - |z|^2)^{1+\frac{a+\beta-2n-\lambda_1-\lambda_2}{2}}} < \infty. \]
The result follows by combining (4.17)-(4.19).

**Lemma 4.5.** Let $0 < \alpha < 2n$, $0 < \beta < 2n$ and $\lambda_1 + \lambda_2 > \alpha + \beta - 2n$. If $\lambda_2 - \beta < \lambda_1 - \alpha$, then for $\rho > 1$,

$$\frac{1}{(\sinh \rho)^{2n-\alpha}(\cosh \rho)^{\lambda_1}} \ast \frac{1}{(\sinh \rho)^{2n-\beta}(\cosh \rho)^{\lambda_2}} \sim e^{-(2n-\beta+\lambda_2)\rho}.$$

**Proof.** By , we have

$$\frac{1}{(\sinh \rho)^{2n-\alpha}(\cosh \rho)^{\lambda_1}} \ast \frac{1}{(\sinh \rho)^{2n-\beta}(\cosh \rho)^{\lambda_2}} = (\cosh \rho)^{-(2n-\beta+\lambda_2)\rho} \int_{B_\rho} \frac{1}{|z|^{2n-\alpha} \left( |z-a|^2 + |(z,a)|^2 - |z|^2|a|^2 \right)} \frac{1}{|1 - (z,a)|^{\lambda_2}} dz.$$

Set

$$F(a) = \int_{S^{2n-1}} \left( \frac{1}{|z-a|^2 + |(z,a)|^2 - |z|^2|a|^2} \right)^{2n-\beta \over 2} |1 - (z,a)|^{-\lambda_2} d\sigma.$$

It is easy to check $F(a)$ depends only on $|a|$. Furthermore, by Proposition 2.3

$$\lim_{|a| \to 1^-} F(a) = \int_{S^{2n-1}} |1 - (z,a)|^{-(2n-\beta+\lambda_2)} d\sigma$$

$$= F((2n - \beta + \lambda_2)/2, (2n - \beta + \lambda_2)/2; n; |z|^2).$$

Therefore,

$$\lim_{|a| \to 1^-} \int_{B_\rho} \frac{1}{|z|^{2n-\alpha}} \left( \frac{1}{|z-a|^2 + |(z,a)|^2 - |z|^2|a|^2} \right)^{2n-\beta \over 2} \frac{dz}{|1 - (z,a)|^{\lambda_2}(1 - |z|^2)^{1+\alpha+\beta-2n-\lambda_1-\lambda_2}}$$

$$= \int_0^1 r^{\alpha-1}(1 - r^2)^{-1-\alpha+\beta-2n-\lambda_1-\lambda_2} F((2n - \beta + \lambda_2)/2, (2n - \beta + \lambda_2)/2; n; r^2) dr$$

$$= {1 \over 2} \int_0^1 t^{\alpha-1}(1 - t)^{-1-\alpha+\beta-2n-\lambda_1-\lambda_2} F((2n - \beta + \lambda_2)/2, (2n - \beta + \lambda_2)/2; n; t) dr$$

$$= \frac{\Gamma(\alpha)\Gamma(2n+\lambda_1+\lambda_2-\alpha-\beta)}{2\Gamma(2n+\lambda_1+\lambda_2-\beta)} {}_3F_2((2n - \beta + \lambda_2)/2, (2n - \beta + \lambda_2)/2; \alpha \over 2; n, 2n + \lambda_1 + \lambda_2 - \alpha - \beta \over 2; 1),$$

where ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ is the generalized hypergeometric series defined by

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}.$$
To get the last equation, we use the following (see [27], Page 813, 7.5 12(4))

\[
\int_0^1 x^{\mu-1}(1-x)^{\nu-1} F(a, b; c; x) dx = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} _3F_2(a, b, \mu; c, \nu; 1),
\]

\[\text{Re} \mu > 0, \text{Re} \nu > 0, \text{Re}(c+\nu-a-b) > 0.\]

The desired result follows. \(\Box\)

Now we can give the estimates of \(k_\alpha * k_{\zeta,\beta}\). The main results are the following two lemmas.

**Lemma 4.6.** Let \(n \geq 2, \zeta > 0, 0 < \alpha < 3\) and \(0 < \beta < 2n - \alpha\). Then for \(0 < \epsilon_2 < \min\{1, 2n - \alpha - \beta, \zeta\}\), we have

\[
k_\alpha * k_{\zeta,\beta} \leq \frac{1}{\gamma_{2n}(\alpha + \beta)} \cdot \frac{1}{\rho^{2n-\alpha-\beta}} + O\left(\frac{1}{\rho^{2n-\alpha-\beta-\epsilon_2}}\right), \quad 0 < \rho < 1.
\]

and

\[
k_\alpha * k_{\zeta,\beta} \lesssim e^{(\epsilon_2-n)\rho}, \quad 0 < \rho < 1.
\]

**Proof.** Recall that

\[
k_\alpha \lesssim \frac{1}{\gamma_{2n}(\alpha) (\sinh \rho)^{2n-\alpha}} + O\left(\frac{1}{(\sinh \rho)^{2n-\alpha-1}}\right), \quad 0 < \rho < 1,
\]

\[
k_\alpha \sim \rho^{\alpha-2} e^{-n\rho} \lesssim e^{\epsilon_1\rho-n\rho}, \quad 0 < \alpha < 3, \quad \rho \geq 1,
\]

where \(\epsilon_1 > 0\). Therefore, we have the following global estimate of \(k_\alpha\): for \(\rho > 0\),

\[
k_\alpha \leq \frac{(\cosh \rho)^{n-\alpha+\epsilon_1}}{\gamma_{2n}(\alpha) (\sinh \rho)^{2n-\alpha}} + O\left(\frac{(\cosh \rho)^{n-\alpha+\epsilon_1-1}}{(\sinh \rho)^{2n-\alpha-1}}\right).
\]

Similarly, for \(\rho > 0\),

\[
k_{\zeta,\beta} \leq \frac{(\cosh \rho)^{n-\beta-\zeta'}}{\gamma_{2n}(\beta) (\sinh \rho)^{2n-\beta}} + O\left(\frac{(\cosh \rho)^{n-\beta-\zeta'-\epsilon_2}}{(\sinh \rho)^{2n-\beta-\epsilon_2}}\right).
\]

If we choose \(\epsilon_1 < \zeta'\), then by Lemma 4.4 we have, for \(0 < \rho < 1\),

\[
k_\alpha * k_{\zeta,\beta} \leq \frac{1}{\gamma_{2n}(\alpha + \beta)} \cdot \frac{1}{\rho^{2n-\alpha-\beta}} + O\left(\frac{1}{\rho^{2n-\alpha-\beta-\epsilon_2}}\right).
\]

Similarly, by Lemma 4.5 we get (1.23). These complete the proof of Lemma 4.6 \(\Box\)

**Lemma 4.7.** Let \(n \geq 2, \zeta > 0, 0 < \alpha < 3\) and \(0 < \beta < 2n - \alpha\). For each \(\zeta' \in (0, \zeta)\), we have

\[
k_\alpha * k_{\zeta,\beta} \lesssim e^{-\zeta' \rho-n\rho} + \rho^{\alpha-2} e^{-n\rho} * k_{\zeta,\beta}, \quad \rho > 1.
\]
Proof. We have, by (4.6),

\[ k_{\alpha} \ast k_{\zeta,\beta} = \int_{\mathbb{B}^n} k_{\alpha}(\rho(z))k_{\zeta,\beta}(\rho(z),\alpha)dV(z) \]

\[ = \int_{\rho(z)<1/2} k_{\alpha}(\rho(z))k_{\zeta,\beta}(\rho(z),\alpha)dV(z) + \int_{\rho(z)\geq1/2} k_{\alpha}(\rho(z))k_{\zeta,\beta}(\rho(z),\alpha)dV(z) \]

\[ \lesssim \int_{\rho(z)<1/2} k_{\alpha}(\rho(z))k_{\zeta,\beta}(\rho(z),\alpha)dV(z) + \int_{\rho(z)\geq1/2} \rho(\alpha)^{-2} e^{-n\rho(z)}k_{\zeta,\beta}(\rho(z),\alpha)dV(z) \]

\[ \leq \int_{\rho(z)<1/2} k_{\alpha}(\rho(z))k_{\zeta,\beta}(\rho(z),\alpha)dV(z) + \rho^{-2}e^{-z\rho} \ast k_{\zeta,\beta}. \]

Next we shall show that

\[ \int_{\rho(z)<1/2} k_{\alpha}(\rho(z))k_{\zeta,\beta}(\rho(z),\alpha)dV(z) \lesssim e^{-\zeta(\rho(\alpha) - n\rho(\alpha))}, \quad \rho(\alpha) \geq 1. \]

Notice that, if \( \rho(z) < 1/2 \), then \( \rho(z,\alpha) \geq \rho(z) - \rho(\alpha) \geq 1/2 \) since \( \rho(y) \geq 1 \). We have, by ,

\[ k_{\alpha}(\rho(z)) \lesssim \frac{1}{\rho(z)^{2n-\alpha}} \sim \frac{1}{|z|^{2n-\alpha}}, \quad \rho(z) < 1/2; \]

\[ k_{\zeta,\beta}(\rho(z,\alpha)) \lesssim e^{-\zeta(\rho(z,\alpha) - n\rho(z,a))} \sim (\cosh \rho(z,\alpha))^{-(n+\zeta')}, \quad \rho(z,\alpha) \geq 1/2. \]

Therefore, we have

\[ \int_{\rho(z)<1/2} k_{\alpha}(\rho(z))k_{\zeta,\beta}(\rho(z,\alpha))dV(z) \]

\[ \lesssim \int_{\rho(z)<1/2} \frac{1}{|z|^{2n-\alpha}} (\cosh \rho(z,\alpha))^{-(n+\zeta')} \left( \frac{1}{1 - |z|^2} \right)^{n+1} dz \]

\[ = \int_{\rho(z)<1/2} \frac{1}{|z|^{2n-\alpha}} \left( \frac{\sqrt{1 - |a|^2}(1 - |z|^2)}{|1 - (z, a)|} \right)^{n+\zeta'} \left( \frac{1}{1 - |z|^2} \right)^{n+1} dz \]

\[ \sim (\sqrt{1 - |a|^2})^{n+\zeta'} \int_{\rho(z)<1/2} \frac{1}{|z|^{2n-\alpha}} dz \]

\[ \sim e^{-\zeta'\rho - n\rho}, \quad \rho(y) \geq 1. \]

This completes the proof of Lemma 4.7.

\[ \square \]

5. REARRANGEMENT OF REAL FUNCTIONS ON \( \mathbb{B}^n_C \)

We now recall the rearrangement of a real functions on \( \mathbb{B}^n_C \). Suppose \( f \) is a real function on \( \mathbb{B}^n_C \). The non-increasing rearrangement of \( f \) is defined by

\[ f^*(t) = \inf \{ s > 0 : \lambda_f(s) \leq t \}, \]

where

\[ \lambda_f(s) = |\{ x \in \mathbb{B}^n_C : |f(z)| > s \}| = \int_{\{x \in \mathbb{B}^n_C : |f(z)| > s\}} dV. \]
Here we use the notation $|\Sigma|$ for the measure of a measurable set $\Sigma \subset \mathbb{B}_C^n$. Set
\[
 f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds.
\]
Denote by $L^{p,q}(\Omega)$ the Lorentz space of those function $f$ on $\Omega$ satisfying
\[
\|f\|_{L^{p,q}(\Omega)} = \begin{cases} \left( \frac{1}{t^{p-1} q} \right)^{\frac{1}{q}} \left\| f^*(t) \right\|_{L^q(0,|\Omega|)}, & 1 \leq q < \infty; \\ \sup_{t>0} t^{1/p} f^*(t)^{1/q}, & q = \infty \end{cases}
\]
is finite. For simplicity, we denote by $\|f\|_{p,q} = \|f\|_{L^{p,q}(B^n_C)}$. Similarly, denote by
\[
\|f\|_{p,q}^{**} = \begin{cases} \left( \frac{1}{t^{p-1} q} \right)^{\frac{1}{q}} \left\| f^{**}(t) \right\|_{L^q(0,|\Omega|)}, & 1 \leq q < \infty; \\ \sup_{t>0} t^{1/p} f^{**}(t)^{1/q}, & q = \infty \end{cases}
\]
We have the following generalization of Young's inequality for convolution (see [67], Theorem 2.6):
\[
\|f * g\|_{L^{r,s}} \leq C \|f\|_{L^{p_1,q_1}}^{*} \|g\|_{L^{p_2,q_2}}^{*}, \quad f \in L(p_1, q_1), \quad g \in L(p_2, q_2),
\]
where $C > 0$ and $r, s, p_1, p_2, q_1$ and $q_2$ satisfy
\[
\frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{r} > 0, \quad s \geq 1 \quad \text{and} \quad \frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s}.
\]
Notice that for $1 < q < +\infty$ and $0 < r \leq +\infty$, we have
\[
\|f * g\|_{L^{r,s}} \leq \|f * g\|_{L^q} \leq \frac{q}{q-1} \|f * g\|_{L^q}.
\]
The proof for $1 \leq r < +\infty$ can be found in [67] while the rest case has been proved by Yap (see [83], Theorem 3.4), is an exercise in calculus (see also [37], equation (2.2) on page 258). Combining (5.1) and (5.2) yields the following:

**Proposition 5.1.** Let $1 < r, p_1, p_2 < +\infty$ and $1 \leq s, q_1, q_2 \leq \infty$. If
\[
\frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{r} > 0 \quad \text{and} \quad \frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s},
\]
then there exists $C > 0$ such that
\[
\|f * g\|_{L^{r,s}} \leq C \|f\|_{L^{p_1,q_1}}^{*} \|g\|_{L^{p_2,q_2}}^{*}, \quad f \in L(p_1, q_1), \quad g \in L(p_2, q_2).
\]
In the previous sections, we obtain the following asymptotic estimates of $k_\alpha$ and $k_{\zeta, \alpha}$:

- $\zeta > 0$:
\[
\begin{align*}
 k_{\zeta, \alpha} & \leq \frac{1}{\gamma_{2n}(\alpha)} \cdot \frac{1}{\rho^{2n-\alpha}} + O \left( \frac{1}{\rho^{2n-\alpha-\epsilon}} \right), \quad 0 < \alpha < n, \quad 0 < \rho < 1 \\
 k_{\zeta, \alpha} & \sim \rho^{\frac{\alpha-2}{2}} e^{-\zeta \rho^{-\mu}}, \quad \alpha > 0, \quad \rho \geq 1;
\end{align*}
\]
• $\zeta = 0$:

$$k_\alpha \leq \frac{1}{\gamma_{2n}(\alpha)} \cdot \frac{1}{\rho^{2n-\alpha}} + O\left(\frac{1}{\rho^{2n-\alpha-1}}\right), \quad 0 < \alpha < 3, \quad 0 < \rho < 1$$

$$k_\alpha \sim \rho^{\alpha - 2} e^{-n\rho} , \quad 0 < \alpha < 3, \quad \rho \geq 1;$$

• If $\zeta > 0$, $0 < \alpha < 3$ and $0 < \beta < 2n - \alpha$, then

$$k_\alpha \cdot k_{\zeta,\beta} \leq \frac{1}{\gamma_{2n}(\alpha + \beta)} \cdot \frac{1}{\rho^{2n-\alpha-\beta}} + O\left(\frac{1}{\rho^{2n-\alpha-\beta-1}}\right), \quad 0 < \rho < 1,$$

$$k_\alpha \cdot k_{\zeta,\beta} \lesssim e^{-\zeta'\rho - n\rho} + \rho^{\alpha - 2} e^{-n\rho} \cdot k_{\zeta,\beta}, \quad \rho > 1,$$

where $0 < \zeta' < \zeta$ and $\epsilon > 0$ is small enough. If we denote by $B_\rho$ the ball centered at the origin with radius $\rho$, then the volume of $B_\rho$ satisfies the estimates

$$|B_\rho| \sim \frac{\omega_{2n-1}}{2n} \rho^{2n} + O(\rho^{2n-1}), \quad 0 < \rho < 1;$$

$$|B_\rho| \sim e^{2n\rho}, \quad \rho \geq 1.$$

Therefore, we have, by (5.1)-(5.3), the non-increasing rearrangement of such kernels satisfies

• $\zeta > 0$ and $0 < \alpha < 2n$:

$$(5.8) \quad [k_{\zeta,\alpha}]^*(t) \leq \frac{1}{\gamma_{2n}(\alpha)} \cdot \left(\frac{2n}{\omega_{2n-1}}\right)^{\frac{\alpha - 2n}{2n}} + O\left(t^{\frac{\alpha + 2n - 2n}{2n}}\right), \quad 0 < \alpha < n, \quad 0 < t < 2,$$

$$[k_{\zeta,\alpha}]^*(t) \sim t^{-\frac{\alpha}{2}} - \frac{1}{2n} \zeta (\ln t)^{\frac{\alpha - 2n}{2}}, \quad \alpha > 0, \quad t \geq 2;$$

• $\zeta = 0$ and $0 < \alpha < 3$:

$$(5.9) \quad [k_\alpha]^*(t) \leq \frac{1}{\gamma_{2n}(\alpha)} \cdot \left(\frac{2n}{\omega_{2n-1}}\right)^{\frac{\alpha - 2n}{2n}} + O\left(t^{\frac{\alpha + 2n - 2n}{2n}}\right), \quad 0 < \alpha < 3, \quad 0 < t < 2,$$

$$[k_\alpha]^*(t) \sim t^{-\frac{\alpha}{2}} (\ln t)^{\alpha - 2}, \quad 0 < \alpha < 3, \quad t \geq 2;$$

• $\zeta > 0$, $0 < \alpha < 3$ and $0 < \beta < 2n - \alpha$,

$$(5.10) \quad [k_\alpha \cdot k_{\zeta,\beta}]^*(t) \leq \frac{1}{\gamma_{2n}(\alpha + \beta)} \cdot \left(\frac{2n}{\omega_{2n-1}}\right)^{\frac{\alpha + \beta - 2n}{2n}} + O\left(t^{\frac{\alpha + \beta + 2n - 2n}{2n}}\right), \quad 0 < t < 2.$$

Next, we will show the following

**Lemma 5.2.** Let $n \geq 2$, $0 < \alpha < 3/2$, $\zeta > 0$ and $0 < \beta < 2n - \alpha$. Then for each each $c > 0$, we have $\int_c^\infty |[k_\alpha \cdot k_{\zeta,\beta}]^*(t)|^2 dt < \infty$.

**Proof.** It is enough to show that for some $c_0 > 0$, $\int_{c_0}^\infty |[k_\alpha \cdot k_{\zeta,\beta}]^*(t)|^2 dt < \infty$. By (5.6), there exists a constant $A > 0$ such that

$$k_\alpha \cdot k_{\zeta,\beta} \leq A(e^{-\zeta'\rho - n\rho} + \rho^{\alpha - 2} e^{-n\rho} \cdot k_{\zeta,\beta}), \quad 0 < \zeta' < \zeta, \quad \rho > 1.$$
Therefore, for some $c_0 > 0$, we have
\[
\int_{c_0}^{\infty} |[k_{\alpha} * k_{\zeta, \beta}](t)|^2 dt \leq \int_{c_0}^{\infty} |[A(e^{-c't - np} + \rho^{\alpha/2 - np} * k_{\zeta, \beta})](t)|^2 dt \\
\leq \int_{0}^{\infty} |[A(e^{-c't - np} + \rho^{\alpha/2 - np} * k_{\zeta, \beta})](t)|^2 dt \\
= A^2 \int_{\mathbb{C}} |e^{-c't - np} + \rho^{\alpha/2 - np} * k_{\zeta, \beta}|^2 dV \\
\leq \int_{\mathbb{C}} e^{-2c't - 2np} dV + \int_{\mathbb{C}} |\rho^{\alpha/2 - np} * k_{\zeta, \beta}|^2 dV.
\]
By (2.5), we have
\[
\int_{\mathbb{C}} e^{-2c't - 2np} dV = \omega_{2n-1} \int_{0}^{\infty} e^{-2c't - 2np} \sinh^{2n-1} \rho \cosh \rho d\rho < \infty,
\]
where $\omega_{2n-1}$ is the volume of $S^{2n-1}$. On the other hand, by the Plancherel formula, we have, for $0 < \alpha < 3/2$,
\[
\int_{\mathbb{C}} |\rho^{\alpha/2 - np} * k_{\zeta, \beta}|^2 dV \\
= C_n \int_{0}^{\infty} \int_{S^{2n-1}} |\hat{k}_{\zeta, \beta}|^2 \cdot |\rho^{\alpha/2 - np}|^2 |c(\lambda)|^{-2} d\lambda d\sigma(\zeta) \\
= C_n \int_{0}^{\infty} \int_{S^{2n-1}} \left(\lambda + \frac{\zeta}{2}\right)^{-\frac{\alpha}{2}} \cdot |\rho^{\alpha/2 - np}|^2 |c(\lambda)|^{-2} d\lambda d\sigma(\zeta) \\
\leq C_n \int_{0}^{\infty} \int_{S^{2n-1}} |\rho^{\alpha/2 - np}|^2 |c(\lambda)|^{-2} d\lambda d\sigma(\zeta) \\
= \int_{\mathbb{C}} |\rho^{\alpha/2 - np}|^2 dV = \omega_{2n-1} \int_{0}^{\infty} \frac{\sinh^{2n-1} \rho \cosh \rho}{\rho^{4 \alpha/2 - 2np}} d\rho < \infty.
\]
Thus $\int_{c_0}^{\infty} |\phi^*(t)|^2 dt < \infty$ and the desired result follows.

6. Proofs of Theorems 1.4 and 1.5

Before the proof of main result in this section, we recall the Kunze-Stein phenomenon on the closed linear group $SU(1, n)$. For simplicity, we denote by $G = SU(1, n)$ and $K = S(U(1) \times U(n))$. By $L^p(G)$ and $L^{p,q}(G)$, we denote the usual Lebesgue space and Lorentz space, respectively. We define $L^{p,q}(G/K)$, $L^{p,q}(K \setminus G)$ and $L^{p,q}(K \setminus G/K)$ to be the closed subspaces of $L^{p,q}(G)$ of the right-$K$-invariant, left-$K$-invariant and $K$-bi-invariant functions, respectively. Cowling, Meda and Setti (16) gave the following sharp version of the Kunze-Stein phenomenon for Lorentz space on $G$

\[
(6.1) \\
L^{p,q_1}(G) \ast L^{p,q_2}(G) \subseteq L^{p,q_3}(G),
\]
where $1 < p < 2, 1 \leq q_k \leq \infty (k = 1, 2, 3)$ and $1 + 1/q_3 \leq 1/q_1 + 1/q_2$ (see 38 for an endpoint estimate of (6.1)).
Lemma 6.1. There holds, for \( p \in (1, 2) \),
\[
L^p(K \setminus G) \ast L^p(G/K) \subset L^{p, \infty}(K \setminus G/K).
\]

**Proof.** By (6.1), it is enough to show that if \( f \in L^p(K \setminus G) \) and \( h \in L^p(G/K) \), then \( f \ast h \) is a \( K \)-bi-invariant function. For simplicity, we denote by \( \mu(dg) \) a left Harr measure on \( G \). Since \( G \) is semi-simple, \( G \) is unimodular and \( \mu(dg) \) is also a right Harr measure on \( G \). Therefore we have, for \( k \in K \) and \( g \in G \),
\[
f \ast h(kg) = \int_G f(kgg_1^{-1})h(g_1)\mu(dg_1) = \int_G f(gg_1^{-1})h(g_1)\mu(dg_1) = f \ast h(g)
\]
and by substituting \( g_2 = g_1k^{-1} \),
\[
f \ast h(gk) = \int_G f(gkg_1^{-1})h(g_1)\mu(dg_1) = \int_G f(g(g_1k^{-1})^{-1})h(g_1)\mu(dg_1) = \int_G f(gg_2^{-1})h(g_2k)\mu(dg_2) = \int_G f(gg_2^{-1})h(g_2)\mu(dg_2) = f \ast h(g).
\]
The proof of Lemma 6.1 is thereby completed. \( \square \)

**Lemma 6.2.** There holds, for \( p \in (1, 2) \) and \( p' = p/(p-1) \),
\[
L^{p', -1}(K \setminus G/K) \ast L^p(G/K) \subset L^{p'}(G/K).
\]

**Proof.** Since \( G \) is unimodular, we have
\[
\int_G v(g^{-1})\mu(dg) = \int_G v(g)\mu(dg), \quad v \in L^1(G).
\]

Let \( f \in L^{p', -1}(K \setminus G/K) \) and \( h \in L^p(G/K) \). By (2.10), we have
\[
f \ast h(g) = h \ast f(g) = \int_G h(gg_1^{-1})f(g_1)\mu(dg_1).
\]
Set \( h_1(g) = h(g^{-1}) \). Then \( h_1 \) is left-\( K \)-invariant since \( h \) is right-\( K \)-invariant. Furthermore, by (6.3), we have
\[
\|h_1\|_{L^p(K \setminus G)} = \|h\|_{L^p(G/K)}.
\]

By duality, we have
\[
\|f \ast h\|_{L^{p'}(G/K)} = \sup_{\|u\|_{L^p(G/K)} \leq 1} \int_G u(g)f \ast h(g)\mu(dg) = \sup_{\|u\|_{L^p(G/K)} \leq 1} \int_G u(g)h \ast f(g)\mu(dg)
\]
\[
= \sup_{\|u\|_{L^p(G/K)} \leq 1} \int_G \int_G u(g)h_1(g_1g^{-1})f(g_1)\mu(dg_1)\mu(dg_1)
\]
\[
= \sup_{\|u\|_{L^p(G/K)} \leq 1} \int_G f(g_1)h_1 \ast u(g_1)\mu(dg_1)
\]
\[
\lesssim \sup_{\|u\|_{L^p(G/K)} \leq 1} \|f\|_{L^{p', -1}(K \setminus G/K)}\|h_1 \ast u\|_{L^{p, \infty}(K \setminus G/K)}.
\]
To get the last inequality above, we use the Hölder’s inequality for Lorentz spaces. On the other hand, by Lemma 6.1, we have
\[ \|h_1 \ast u\|_{L^p,\infty(K\setminus G/K)} \lesssim \|h_1\|_{L^p(G/K)} \|u\|_{L^p(G/K)}. \]
Therefore,
\[ \|f \ast h\|_{L^{p'}(G/K)} \lesssim \sup_{\|u\|_{L^p(G/K)} \leq 1} \|f\|_{L^{p',1}(K\setminus G/K)} \|h\|_{L^p(G/K)} \|u\|_{L^p(G/K)} \]
\[ \lesssim \|f\|_{L^{p',1}(K\setminus G/K)} \|h\|_{L^p(G/K)}. \]
The desired result follows. □

Lemma 6.3. Let \( 0 < \alpha < 3, \ 0 < \beta < 2n - \alpha, \ \zeta > 0 \) and \( \frac{4n}{2n+4n+\beta} \leq p < 2 \). Denote by \( p' = p/(p-1) \). Then there exists \( C > 0 \) such that for all \( f \in C^\infty_0(\Bbb H^n_C) \),
\[ (6.4) \]
\[ \|k_\alpha * k_{\zeta, \beta} * f\|_{p'} \leq C\|f\|_p. \]
Proof. Set
\[ \eta_1(\rho) = \begin{cases} \ k_\alpha * k_{\zeta, \beta}, & 0 < \rho < 1; \\ 0, & \rho \geq 1. \end{cases} \]
and \( \eta_2(\rho) = k_\alpha * k_{\zeta, \beta} - \eta_1(\rho) \). By (5.10), there exists \( t_0 > 0 \) such that
\[ \eta_1^*(t) \lesssim t^{-\frac{\alpha + \zeta - 2\alpha}{2n}}, \ t \leq t_0, \text{ and } \eta_1^*(t) = 0, \ t > t_0. \]
Therefore, we have, by (5.3),
\[ (6.5) \]
\[ \|\eta_1 * f\|_{p'} = \|\eta_1 * f\|_{L^{p',p'}} \leq C\|\eta_1\|_{L^{p',\infty}} \|f\|_p \lesssim \|f\|_p. \]
Here we use the fact
\[ \|\eta_1\|_{L^{p',\infty}} = \sup_{0 < t < \infty} t^{\frac{2}{p'}} \eta_1^*(t) \lesssim \sup_{0 < t \leq t_0} t^{\frac{2}{p'} + \frac{\alpha + \zeta - 2\alpha}{2n}} < \infty. \]
On the other hand, by Lemma 4.6
\[ \eta_2(\rho) = 0 \text{ for } 0 < \rho < 1 \text{ and } \eta_2(\rho) \leq e^{(e-n)\rho}, \rho \geq 1. \]
We have, by (5.10),
\[ \eta_2^*(t) \lesssim 1, \ 0 < t < 1 \text{ and } \eta_2^*(t) \leq t^{-\frac{\alpha + \zeta}{2n}}, \ t \geq 1. \]
Therefore,
\[ \|\eta_2\|_{p',1} = \int_0^\infty t^{\frac{1}{p'} - 1} \eta_2^*(t) dt < \infty \]
if we choose \( 0 < \varepsilon < 2n(1/p - 1/2) \). Therefore, by Lemma 6.2 we have
\[ (6.6) \]
\[ \|\eta_2 * f\|_{p'} \leq C\|f\|_p. \]
Combining (6.5) and (6.6) yields
\[ (6.7) \]
\[ \|k_\alpha * k_{\zeta, \beta} * f\|_{p'} \leq \|\eta_1 * f\|_{p'} + \|\eta_2 * f\|_{p'} \leq C\|f\|_p. \]
This completes the proof of Lemma 6.3 □
Proof of Theorem 1.4. By Lemma 6.3 we have

\[ \|(-\Delta_B - n^2 + \zeta^2)^{-\beta/4}(-\Delta_B - n^2)^{-\alpha/4}f\|_{p'} \leq \|f\|_p \]

which is equivalent to (1.10) (see e.g. [6], Appendix). Now, we prove inequality (1.11). For each \( p > 2 \), we choose \( \beta' < \beta = 2n - \alpha \) be such that \( p \leq \frac{4n}{2n - (\alpha + \beta')} \). Then by (1.10), we have

\[ \|f\|_p \leq C\|(-\Delta_B - n^2 + \zeta^2)^{\beta'/4}(-\Delta_B - n^2)^{\alpha/4}f\|_2. \]

On the other hand, by the Plancherel formula, we have

\[ \|(-\Delta_B - n^2 + \zeta^2)^{\beta'/4}(-\Delta_B - n^2)^{\alpha/4}f\|_2 \]

\[ = C_n \int_{\Sigma^{2n-1}} \int_0^{+\infty} (\lambda^2 + \zeta^2)^{\beta'/2} |\lambda|^\alpha |\hat{f}(\lambda, \zeta)|^2 |\lambda(\lambda)|^{-2} d\lambda d\sigma(\zeta) \]

\[ \geq \zeta^{\beta - \beta'} C_n \int_{\Sigma^{2n-1}} \int_{-\infty}^{+\infty} (\lambda^2 + \zeta^2)^{\beta'/2} |\lambda|^\alpha |\hat{f}(\lambda, \zeta)|^2 |\lambda(\lambda)|^{-2} d\lambda d\sigma(\zeta) \]

\[ = \zeta^{\beta - \beta'} \|(-\Delta_B - n^2 + \zeta^2)^{\beta'/4}(-\Delta_B - n^2)^{\alpha/4}f\|_2 \geq C\|f\|_p. \]

The desired result follows.

Proof of Theorem 1.5. Set \( u = \varrho^{\frac{k-n-a}{2}} f \), we have, by Theorem 1.3

\[ 4^k \int_{\Sigma^{2n-1}} \int_0^{+\infty} \prod_{j=1}^k \left[ -\varrho \partial_{\varrho^2} - a \partial_{\varrho^2} - \varrho T^2 - \Delta_b + i(k + 1 - 2j)T \right] u \frac{dz dt d\varrho}{\varrho^{1-a}} \]

\[ = \int_{\Sigma^{2n-1}} \int_0^{+\infty} \prod_{j=1}^k \left[ -\Delta_B - n^2 + (a - k + 2j - 2)^2 \right] f \frac{dz dt d\varrho}{\varrho^{a+1}} \]

\[ = 4 \int_{U^n} \prod_{j=1}^k \left[ -\Delta_B - n^2 + (a - k + 2j - 2)^2 \right] f dV. \]

Since \( \text{spec}(-\Delta_B) = [n^2, +\infty) \), we have

\[ \int_{U^n} \prod_{j=1}^k \left[ -\Delta_B - n^2 + (a - k + 2j - 2)^2 \right] f dV \]

\[ \geq \prod_{j=1}^k (a - k + 2j - 2) \int_{U^n} f^2 dV \]
and the constant $\prod_{j=1}^{k} (a - k + 2j - 2)^2$ is sharp. Furthermore, by Plancherel formula, we have
\[
\int_{\mathbb{U}^n} f \prod_{j=1}^{k} \left[ -\Delta_B - n^2 + (a - k + 2j - 2)^2 \right] f dV - \prod_{j=1}^{k} (a - k + 2j - 2)^2 \int_{\mathbb{U}^n} f^2 dV
\]
\[
= C_n \int_{-\infty}^{\infty} \int_{\mathbb{S}^{2n-1}} \left[ \prod_{j=1}^{k} \left( \lambda^2 + (a - k + 2j - 2)^2 \right) - \prod_{j=1}^{k} (a - k + 2j - 2)^2 \right] |\hat{f}(\lambda, \zeta)|^2 |c(\lambda)|^{-2} d\lambda d\sigma(\zeta).
\]
Let $\delta > 0$ be such that
\[
\prod_{j=1}^{k} \left( \lambda^2 + (a - k + 2j - 2)^2 \right) - \prod_{j=1}^{k} (a - k + 2j - 2)^2 \geq \lambda^2(\lambda^2 + \delta)^{k-1}, \forall \lambda \in \mathbb{R}.
\]
Then by Theorem 1.4, we have
\[
\int_{\mathbb{U}^n} f \prod_{j=1}^{k} \left[ -\Delta_B - n^2 + \left(1/2 - k + 2j - 2\right)^2 \right] f dV - \prod_{j=1}^{k} (a - k + 2j - 2)^2 \int_{\mathbb{U}^n} f^2 dV
\]
\[
\geq C_n \int_{-\infty}^{\infty} \int_{\mathbb{S}^{2n-1}} \lambda^2(\lambda^2 + \delta)^{k-1} |\hat{f}(\lambda, \zeta)|^2 |c(\lambda)|^{-2} d\lambda d\sigma(\zeta)
\]
\[
= \int_{\mathbb{U}^n} f(-\Delta_B - n^2) \left( -\Delta_B - n^2 + \delta \right)^{k-1} f dV \geq C\|f\|^2_p.
\]
Combining (6.8) and (6.9) yields
\[
4^k \int_{\mathbb{H}^{2n-1}} \int_{0}^{\infty} u \prod_{j=1}^{k} \left[ -\varrho \partial_{\varrho} - a \partial_{\varrho} - \varrho T^2 - \Delta_b + i(k + 1 - 2j)T \right] u \frac{dz dtd\varrho}{\varrho^{k-a}}
\]
\[
- \prod_{j=1}^{k} (a - k + 2j - 2)^2 \int_{\mathbb{H}^{2n-1}} \int_{0}^{\infty} \frac{u^2}{\varrho^{k+1-a}} dz dtd\varrho
\]
\[
= 4 \int_{\mathbb{U}^n} f \prod_{j=1}^{k} \left[ -\Delta_B - n^2 + \left(1/2 - k + 2j - 2\right)^2 \right] f dV - 4 \prod_{j=1}^{n} \frac{(2j - 1)^2}{4} \int_{\mathbb{U}^n} f^2 dV
\]
\[
\geq C\|f\|^2_p = C \left( \int_{\mathbb{S}^{2n-1}} \int_{0}^{\infty} |u|^p \varrho^q dz dtd\varrho \right)^\frac{2}{p}.
\]
Similarly, we can also obtain the Hardy-Sobolev-Maz'ya inequality on $\mathbb{B}^n_C$. Since the proof is similar, we omit it. This completes proof of Theorem 1.5.

7. Proofs of Theorems 1.9 and 1.10

Firstly, we shall prove Theorem 1.9. The main idea is to adapt the level set argument developed by Lam and the first author to derive a global Moser-Trudinger inequality from a local one (see [48, 49]).
Proof of Theorem 1.9 Let \( u \in C_0^\infty(\mathbb{R}^n_C) \) be such that
\[
\|(-\Delta_B - n^2 + \zeta^2)(2n-\alpha)\sqrt{-\Delta_B - n^2}u\|_2 \leq 1.
\]
Set \( \Omega(u) = \{ x \in \mathbb{B}^n_C : |u(x)| \geq 1 \} \). By Theorem 1.5 we have, for \( p > 2 \),
\[
|\Omega(u)| = \int_{\Omega(u)} dV \leq \int_{\mathbb{B}^n_C} |u|^p dV \lesssim 1.
\]
Therefore, we may assume
\[
(7.1) \quad |\Omega(u)| \leq \Omega_0
\]
for some constant \( \Omega_0 \) which is independent of \( u \).

We write
\[
\int_{\mathbb{B}^n_C} (e^{\beta_0(n,2n)u^2} - 1 - \beta_0(n,2n)u^2)dV = \int_{\Omega(u)} (e^{\beta_0(n,2n)u^2} - 1 - \beta_0(n,2n)u^2)dV + \int_{\mathbb{B}^n_C \setminus \Omega(u)} (e^{\beta_0(n,2n)u^2} - 1 - \beta_0(n,2n)u^2)dV \leq \int_{\Omega(u)} e^{\beta_0(n,2n)u^2}dV + \int_{\mathbb{B}^n_C \setminus \Omega(u)} (e^{\beta_0(n,2n)u^2} - 1 - \beta_0(n,2n)u^2)dV.
\]
Notice that on the domain \( \mathbb{B}^n_C \setminus \Omega(u) \), we have \( |u(z)| < 1 \). Thus,
\[
\int_{\mathbb{B}^n_C \setminus \Omega(u)} (e^{\beta_0(n,2n)u^2} - 1 - \beta_0(n,2n)u^2)dV = \int_{\mathbb{B}^n_C \setminus \Omega(u)} \sum_{n=2}^{\infty} \frac{(\beta_0(n,2n)u^2)^n}{n!}dV \leq \int_{\mathbb{B}^n_C \setminus \Omega(u)} \sum_{n=2}^{\infty} \frac{(\beta_0(n,2n))u^4}{n!}dV \leq \sum_{n=2}^{\infty} \frac{(\beta_0(n,2n))}{n!} \int_{\mathbb{B}^n_C} |u(x)|^4 dV.
\]
Therefore, by Theorem 1.5, \( \int_{\mathbb{B}^n_C \setminus \Omega(u)} (e^{\beta_0(n,2n)u^2} - 1 - \beta_0(n,2n)u^2)dV \) is bounded by some constant which is independent of \( u \).

To finish the proof, it is enough to show \( \int_{\Omega(u)} e^{\beta_0(n,2n)u^2}dV \) is also bounded by some universal constant. Set
\[
v = (-\Delta_B - n^2 + \zeta^2)(2n-\alpha)\sqrt{-\Delta_B - n^2}u.
\]
Then
\[
(7.2) \quad \int_{\mathbb{B}^n_C} |v|^2 dV \leq 1
\]
and we can write \( u \) as a potential
\[
(7.3) \quad u = (-\Delta_B - n^2 + \zeta^2)^{-\alpha}(-\Delta_B - n^2)^{-\alpha/4}v = v \ast \phi,
\]
where \( \phi = (-\Delta_B - n^2 + \zeta^2)^{-\frac{(2n-\alpha)}{4}}(-\Delta_B - n^2)^{-\frac{\alpha}{4}} = k_{\zeta,(2n-\alpha)/2}*k_{\alpha/2} \). By (5.10) and Lemma 5.2,
\[
\phi^*(t) \leq \frac{1}{\gamma_{2n}(n)} \left( \frac{2nt}{\omega_{2n-1}} \right)^{\frac{\alpha}{4}} + O \left( \frac{t^{\frac{\alpha}{2}}}{\omega_{2n-1}} \right), \quad 0 < t < 2 \quad \text{and} \quad \int_0^\infty |\phi^*(t)|^2 dt < \infty, \forall c > 0.
\]
Closely following the proof of Theorem 1.7 in [51], we have that there exists a constant \( C \) which is independent of \( u \) and \( \Omega(u) \) such that
\[
\int_{\Omega(u)} e^{\beta_0(n,2n)u^2} dV = \int_0^{[\Omega(u)]} \exp(\beta_0(n,2n)|u^*(t)|^2) dt \leq \int_0^{\Omega_0} \exp(\beta_0(n,2n)|u^*(t)|^2) dt \leq C.
\]
The proof of Theorem 1.10 is thereby completed.

**Proof of Theorem 1.10** It is enough to show that for some \( \zeta > 0 \),
\[
\|(-\Delta_B - n^2 + \zeta^2)^{(n-1)/2}(-\Delta_B - n^2)^{1/2} u\|_2
\leq 4^{n-1} \int_{\mathbb{H}^{2n-1}} \int_0^\infty \frac{\varphi^{-a/2}u}{\prod_{j=1}^n [-\varphi \partial_{\varphi^j} - a \partial_{\varphi^j} + \varphi T^2 - \Delta_B + i(k + 1 - 2j)T]} \left( \varphi^{-a/2}u \right) \frac{dzdt\varphi}{\varphi^{n+1-a}}
\]
\[
- \frac{1}{4} \prod_{j=1}^n (a - n + 2j - 2)^2 \int_{\mathbb{H}^{2n-1}} \int_0^\infty \frac{\varphi^{-a/2}u}{\varphi^{n+1-a}} dzdt\varphi.
\]
In fact, by Theorem 1.4 and the Plancherel formula,
\[
4^{n-1} \int_{\mathbb{H}^{2n-1}} \int_0^\infty \frac{\varphi^{-a/2}u}{\prod_{j=1}^n [-\varphi \partial_{\varphi^j} - a \partial_{\varphi^j} + \varphi T^2 - \Delta_B + i(k + 1 - 2j)T]} \left( \varphi^{-a/2}u \right) \frac{dzdt\varphi}{\varphi^{n+1-a}}
\]
\[
- \frac{1}{4} \prod_{j=1}^n (a - n + 2j - 2)^2 \int_{\mathbb{H}^{2n-1}} \int_0^\infty \frac{\varphi^{-a/2}u}{\varphi^{n+1-a}} dzdt\varphi
\]
\[
= \int_{\mathcal{U}^n} u \prod_{j=1}^n [-\Delta_B - n^2 + (a - k + 2j - 2)^2] u dV - \prod_{j=1}^n (a - n + 2j - 2)^2 \int_{\mathcal{U}^n} u^2 dV
\]
\[
= C_n \int_{-\infty}^{+\infty} \int_{\mathbb{S}^{2n-1}} \left[ \prod_{j=1}^n \left( \lambda^2 + (a - k + 2j - 2)^2 \right) - \prod_{j=1}^n (a - k + 2j - 2)^2 \right] |\hat{u}(\lambda, \zeta)|^2 |c(\lambda)|^{-2} d\lambda d\sigma(\zeta)
\]
\[
\geq C_n \int_{-\infty}^{+\infty} \int_{\mathbb{S}^{2n-1}} \lambda^2 (\lambda^2 + \delta)^{k-1} |\hat{f}(\lambda, \zeta)|^2 |c(\lambda)|^{-2} d\lambda d\sigma(\zeta)
\]
\[
\geq \|(-\Delta_B - n^2 + \delta)^{(n-1)/2}(-\Delta_B - n^2)^{1/2} u\|_2^2,
\]
where \( \delta > 0 \) is such that
\[
\prod_{j=1}^n (\lambda^2 + (a - k + 2j - 2)^2) - \prod_{j=1}^n (a - k + 2j - 2)^2 \geq \lambda^2 (\lambda^2 + \delta)^{n-1}, \forall \lambda \in \mathbb{R}.
\]
This completes the proof of Theorem 1.10.
8. Proofs of Theorems 1.11 and 1.12

**Proof of Theorem 1.11** Since \( u \in W^{\alpha,p}(\mathbb{R}^{n}) \), we write \( f = (-\Delta_B - n^2 + \zeta^2)^{\frac{\alpha}{2}} u \). Then \( \|f\|_p \leq 1 \) and

\[
 u = (-\Delta_B - n^2 + \zeta^2)^{-\frac{\alpha}{2}} f = f * k_{\zeta,\alpha}.
\]

Applying O’Neil’s lemma (67), we have for \( t > 0 \),

\[
 u^*(t) \leq \frac{1}{t} \int_0^t f^*(s)ds \int_0^t k_{\zeta,\alpha}^*(s)ds + \int_t^\infty f^*(s)k_{\zeta,\alpha}^*(s)ds.
\]

By (5.8), we have

\[
[k_{\zeta,\alpha}]^*(t) \leq \frac{1}{\gamma_{2n}(\alpha)} \left( \frac{2nt}{\omega_{2n-1}} \right)^{\frac{n}{2}} + O \left( \frac{t^{\frac{n+2m}{2n}}}{} \right), 0 < t < 2, \quad \text{and} \quad \int_t^\infty \| [k_{\zeta,\alpha}]^*(t) \|^p dt < \infty, \quad \forall c > 0.
\]

Using O’Neil’s lemma and closely following the proof of Theorem 1.13 in [52], we have that there exists a constant \( C \) which is independent of \( u \) such that

\[
\frac{1}{|E|} \int_E \exp(\beta(2n,\alpha)|u|^p) dV \leq \frac{1}{|E|} \int_0^{[E]} \exp(\beta(2n,\alpha)|u^*(t)|^p) dt \\
\leq \frac{1}{|E|} \int_0^{[E]} \exp \left( \beta(2n,\alpha) \frac{1}{t} \int_0^t f^*(s)ds \int_0^t k_{\zeta,\alpha}^*(s)ds + \int_t^\infty f^*(s)k_{\zeta,\alpha}^*(s)ds \right)^p dt \leq C.
\]

The sharpness of the constant \( \beta(2n,\alpha) \) can be verified by the process similar to that in [1, 47] and thus the proof of Theorem 1.11 is completed.

Next we will prove Theorem 1.12. The main idea is also apply the method of the level set argument for functions under consideration and then derive a global inequality from a local one.

**Proof of Theorem 1.12** Let \( u \in W^{\alpha,p}(\mathbb{B}^{n}_0) \) with \( \int_{\mathbb{B}^{n}_0} \|(-\Delta_B - n^2 + \zeta^2)^{\frac{\alpha}{2}} u \|^p dV \leq 1 \). By Corollary 2.2 we have

\[
\int_{\mathbb{B}^{n}_0} |u|^p dV \leq \int_{\mathbb{B}^{n}} \|(-\Delta_B - n^2 + \zeta^2)^{\frac{\alpha}{2}} u \|^p dV \leq 1
\]

provided \( \zeta > 2n|\frac{1}{2} - \frac{1}{p}| \). Set \( \Omega(u) = \{ z \in \mathbb{B}^{n}_0 : |u(z)| \geq 1 \} \). Then we have

\[
|\Omega(u)| = \int_{\Omega(u)} dV \leq \int_{\mathbb{B}^{n}_0} |u|^p dV \leq \Omega_0,
\]

where \( \Omega_0 \) is a constant independent of \( u \). We write

\[
\int_{\mathbb{B}^{n}_0} \Phi_p(\beta(2n,\alpha)|u|^p) dV = \int_{\Omega(u)} \Phi_p(\beta(2n,\alpha)|u|^p) dV + \int_{\mathbb{B}^{n}_0 \setminus \Omega(u)} \Phi_p(\beta(2n,\alpha)|u|^p) dV.
\]
Notice that on the domain $\mathbb{B}_C \setminus \Omega(u)$, we have $|u(z)| < 1$. Thus,

$$
\int_{\mathbb{B}_C \setminus \Omega(u)} \Phi_p(\beta(2n, \alpha)|u|^p) dV \leq \sum_{k=j_p-1}^{\infty} \frac{\beta(2n, \alpha)^k}{k!} \int_{\mathbb{B}_C \setminus \Omega(u)} |u|^{p'} dV
$$

$$
\leq \sum_{k=j_p-1}^{\infty} \frac{\beta(2n, \alpha)^k}{k!} \int_{\mathbb{B}_C \setminus \Omega(u)} |u|^p dV
$$

$$
\leq \sum_{k=j_p-1}^{\infty} \frac{\beta(2n, \alpha)^k}{k!} \|u\|^p \leq C. \tag{8.1}
$$

Moreover, by Theorem 1.11 if $\zeta$ satisfies $\zeta > 0$ if $1 < p < 2$ and $\zeta > 2n \left| \frac{1}{p} - \frac{1}{2} \right|$ if $p \geq 2$, then

$$
\int_{\Omega(u)} \Phi_p(\beta(2n, \alpha)|u|^p) dV \leq \int_{\Omega(u)} \exp(\beta(2n, \alpha)|u|^p) dV \leq C. \tag{8.2}
$$

Combining (8.1) and (8.2) yields

$$
\int_{\mathbb{B}_C} \Phi_p(\beta(2n, \alpha)|u|^p) dV = \int_{\Omega(u)} \Phi_p(\beta(2n, \alpha)|u|^p) dV + \int_{\mathbb{B}_C \setminus \Omega(u)} \Phi_p(\beta(2n, \alpha)|u|^p) dV \leq C
$$

provided that $\zeta$ satisfies $\zeta > 2n \left| \frac{1}{p} - \frac{1}{2} \right|$.

On the other hand,

$$
\int_{\mathbb{B}_C} e^{\beta(2n, \alpha)|u|^p} dx = \int_{\mathbb{B}_C} \Phi_p(\beta(2n, \alpha)|u|^p) dz + \sum_{j=0}^{j_p-2} \frac{\beta(2n, \alpha)^j}{j!} \int_{\mathbb{B}_C} |u|^{jp'} dz
$$

$$
\leq \int_{\mathbb{B}_C} \Phi_p(\beta(2n, \alpha)|u|^p) dV + \sum_{j=0}^{j_p-2} \frac{\beta(2n, \alpha)^j}{j!} \int_{\mathbb{B}_C} |u|^{jp'} dz
$$

$$
\leq C + \sum_{j=0}^{j_p-2} \frac{\beta(2n, \alpha)^j}{j!} \int_{\mathbb{B}_C} |u|^{jp'} dz \leq C'.
$$

To get the last inequation, we use the fact

$$
\left( \int_{\mathbb{B}_C} |u|^q dz \right)^{\frac{p}{q}} \leq \left( \int_{\mathbb{B}_C} dz \right)^{\frac{p-q}{q}} \int_{\mathbb{B}_C} |u|^p dz \leq \int_{\mathbb{B}_C} |u|^p dV \lesssim 1, \text{ } \forall 1 < q \leq p.
$$

The sharpness of the constant $\beta(2n, \alpha)$ can be verified by the process similar to that in the proof of Theorem 1.11.

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