Integrable discretization of hodograph-type systems, hyperelliptic integrals and Whitham equations

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Based on the well-established theory of discrete conjugate nets in discrete differential geometry, we propose and examine discrete analogues of important objects and notions in the theory of semi-Hamiltonian systems of hydrodynamic type. In particular, we present discrete counterparts of (generalized) hodograph equations, hyperelliptic integrals and associated cycles, characteristic speeds of Whitham-type and (implicitly) the corresponding Whitham equations. By construction, the intimate relationship with integrable system theory is maintained in the discrete setting.

1. Introduction

Systems of quasi-linear first-order differential equations of the form

\[ u_i^t = \lambda_i(u)u_i^x, \quad i = 1, \ldots, N + 1, \]  

(1.1)

where subscripts denote partial derivatives, represent an important subclass of partial differential equations that admit special properties and a variety of applications [1]. In physics, such systems arise, in particular, as limits of nonlinear partial differential equations without dissipation or dispersion and as Whitham equations for slow modulations [2,3]. The theory of (more general) Hamiltonian quasi-linear systems of hydrodynamic type has been developed by Dubrovin & Novikov [4–6]. It has been established in references [7–12] that these systems are intimately related to important notions in...
classical differential geometry. In particular, it has been demonstrated by Tsarev [7–9] that a semi-Hamiltonian system of the type (1.1) possesses an infinite set of integrals of motion with densities $\psi$ obeying the system of linear hyperbolic equations

$$
\psi_{it} = A_{ik} \psi_{iu} + A_{ki} \psi_{it}, \quad i \neq k,
$$

(1.2)

where the coefficients $A_{ik}$ are defined by the system

$$
\lambda^i_{ik} = A_{ik} (\lambda^k - \lambda^i),
$$

(1.3)

and the fluxes $\psi_\ast$ in the corresponding conservation laws are (uniquely) determined via integration of the compatible system

$$
\psi_\ast_{it} = \lambda^i \psi_{iu}.
$$

(1.4)

These semi-Hamiltonian systems admit an infinite number of symmetries [7–9]

$$
u^i_{\alpha} = \lambda^i_{\alpha} (u), \quad \alpha = 2, 3, 4, \ldots,
$$

(1.5)

where each set of characteristic speeds $\{\lambda^i_{\alpha}\}$ constitutes a solution of (1.3) regarded as a linear system. The compatibility of the latter (in the sense of a natural Cauchy problem) is equivalent to the existence of the semi-Hamiltonian structure. In the differential–geometrical context, the constituent equations of (1.2) are known as conjugate net equations because these constitute the governing linear equations in the classical theory of conjugate nets [13]. Moreover, the connection between the density $\psi$ and the flux $\psi_\ast$ is of Combescure type [14]. In modern integrable system terminology, the densities $\psi$ constitute eigenfunctions, whereas the characteristic speeds $\lambda^i_{\alpha}$ represent associated adjoint eigenfunctions.

Remarkably, Tsarev [7,8] has proved that, locally, all solutions of a semi-Hamiltonian system of the form (1.1) are given implicitly by the algebraic system

$$
x + \lambda^i (u) t - \omega^i (u) = 0, \quad i = 1, \ldots, N + 1
$$

(1.6)

with $\{\omega^i\}$ denoting the general set of adjoint eigenfunctions obeying the linear system (1.3). This linearization technique has come to be known as the generalized hodograph method because, in the case $N = 1$, the quantities $x$ and $t$ regarded as the unknowns of the system (1.6) obey the classical hodograph equations [15]

$$
x_{ui} + \lambda^2 (u) t_{ui} = 0, \quad x_{u^2} + \lambda^1 (u) t_{u^2} = 0.
$$

(1.7)

In fact, in this paper, it is shown that such (generalized) hodograph equations exist for arbitrary $N$. In summary, the properties of semi-Hamiltonian systems of the form (1.1) and their solutions are completely encoded in the classical surface theory of conjugate nets. This highlights the privileged nature of semi-Hamiltonian systems of hydrodynamic type.

The particular class of conjugate nets governed by the compatible hyperbolic equations

$$
\phi_{x'i x'} = \frac{1}{\lambda^i - \lambda^j} (\epsilon^j \phi_{x'i} - \epsilon^i \phi_{x'j}),
$$

(1.8)

where the $\epsilon^i$'s constitute constants, plays a distinguished role in the theory of semi-Hamiltonian hydrodynamic-type systems (with the identification $u = x$). However, it is important to note that, in general, these special conjugate net equations are, a priori, unrelated to the conjugate net equations (1.2). In particular, the eigenfunction $\phi$ does not necessarily play the role of a density. The linear equations (1.8) are known as Euler–Poisson–Darboux equations and have been the subject of extensive investigation in classical differential geometry [16]. Their importance in the one-phase Whitham equations for the Korteweg–de Vries (KdV) and nonlinear Schrödinger (NLS) equations has been observed in references [17–19] and, in the multi-phase case, in reference [20]. In fact, the explicit expressions for the characteristic speeds in the multi-phase Whitham equations
derived in the pioneering paper [21] and also in reference [20] contain, as elementary building blocks, particular solutions of the Euler–Poisson–Darboux equations with parameters $\epsilon_i = 1/2$.

Recently, it has been demonstrated [22] that the characteristic speeds for multi-phase Whitham equations may be obtained by means of iterated Darboux transformations generated by contour integrals of separable solutions of (extended) Euler–Poisson–Darboux systems. Euler–Poisson–Darboux systems for different values of $\epsilon_i$ also play a central role in the treatment of various dispersionless soliton equations and $\epsilon$-systems [23–25].

The connection between the Euler–Poisson–Darboux system and the characteristic speeds of multi-phase Whitham equations (and therefore the associated conjugate net system (1.2)) is provided by the observation that the hyperelliptic integrals

$$\oint_{b_k} \frac{\zeta^k}{\prod_{i=1}^{2g+1} (\zeta - x^i)} d\zeta,$$  (1.9)

where the contours $b_k$, $k = 1, \ldots, g$ are appropriately chosen cycles [20,21,26] and $k \leq g - 1$, may be regarded as superpositions of separable solutions of the Euler–Poisson–Darboux system (1.8) for $\epsilon^i = 1/2$. The methods recorded in [20–22] are then used in the (algebraic) construction of the characteristic speeds $\lambda^i$. For instance, in the case $g = 1$, the elliptic integral (1.9) may be calculated to be essentially

$$\phi^O = \frac{2}{\pi \sqrt{x^3 - x^1}} K \left( \sqrt{\frac{x^3 - x^2}{x^3 - x^1}} \right),$$  (1.10)

where $K$ denotes the complete elliptic integral of the first kind. Each of the three coordinates $x^i$ gives rise to a classical Levy transform [27]

$$\lambda^i = \phi^0 - \frac{\phi^O}{\phi^0} \phi^0_x$$  (1.11)

of the simplest non-constant solution $\phi^0 = 1/2 (x^1 + x^2 + x^3)$ of the Euler–Poisson–Darboux system (1.8) generated by $\phi^O$ and these coincide with the characteristic speeds for the one-phase Whitham equations [2]. The avatar (1.11) of these characteristic speeds may be found in references [19,28]. It is remarked in passing that the action of the Levy transformation on semi-Hamiltonian systems of hydrodynamic type has been discussed in detail in reference [29].

As indicated above, many systems of hydrodynamic type (1.1) admit dispersive counterparts which are integrable by means of the inverse spectral transform (IST) method [2,6]. One of the remarkable properties of IST integrable equations is that they admit integrable discretizations which reveal their fundamental properties [30]. Such discretizations are usually constructed via invariances of the integrable equations under Bäcklund, Darboux or similar discrete transformations [14]. This method simultaneously leads to a discretization of the underlying linear representation (Lax pair [31]).

Based on the standard discretization [32] of the conjugate net equations (1.2) and associated adjoint equations (1.5), we here propose a canonical integrability-preserving way of discretizing the theory outlined in the preceding. In particular, this is shown to lead to integrable discretizations of generalized hodograph equations, canonical cycles and associated hyperelliptic integrals and characteristic speeds of commuting flows of hydrodynamic type such as those corresponding to the multi-phase Whitham equations. It is noted that large classes of solutions of the standard discrete conjugate net equations may be obtained by means of, for instance, the $\delta$-bar-dressing method [33], Darboux-type transformations [34,35] or the algebro-geometric approach employed in reference [36]. Our approach exploits the existence of a canonical discretization of the classical Euler–Poisson–Darboux system (1.8) and associated separable solutions.
2. Generalized hodograph equations

We are concerned with commuting flows of diagonal systems of hydrodynamic type, that is, compatible systems of first-order equations of the type

\[ u_i^\alpha_t = \lambda_i^\alpha(u)u_i^\alpha_x, \quad i = 1, \ldots, N + 1, \quad \alpha = 1, \ldots, N, \quad (2.1) \]

where the subscripts on the functions \( u_i^\alpha \) denote derivatives with respect to the independent variables \( x \) and \( t^\alpha \). It is emphasized that even though the above systems constitute the point of departure, many of the mathematical notions presented in this paper go beyond these systems and turn out to be of interest in their own right. It is known [8] that diagonal systems of hydrodynamic type commute if and only if the \( N \) sets of characteristic speeds \( \{\lambda_i^\alpha\} \) labelled by \( \alpha \) obey the same linear system

\[ \lambda_i^\alpha u_k = A_{ik}(\lambda_k^\alpha - \lambda_i^\alpha), \quad (2.2) \]

where the coefficients \( A_{ik} \) may be regarded as being defined by the equations for, say, \( \alpha = 1 \). Here, \( \lambda_i^\alpha u_k = \partial \lambda_i^\alpha / \partial u_k \). Furthermore, it is readily verified that the above linear equations may also be regarded as the compatibility conditions for the existence of some functions \( \psi\alpha \) defined (up to constants of integration) by

\[ \psi_{\alpha u_t} = \lambda_i^\alpha \psi_{u^\alpha}, \quad (2.3) \]

where \( \psi \) is a solution of the linear hyperbolic equations

\[ \psi_{u_t} = A_{ik}\psi_{u^i} + A_{ki}\psi_{u^k}. \quad (2.4) \]

It is important to note that the coefficients \( A_{ik} \) cannot be arbitrary as these are constrained by the compatibility conditions for the hyperbolic equations (2.4) or, equivalently, the first-order equations (2.2). In fact, the coefficients \( A_{ik} \) must be solutions of an integrable system of nonlinear partial differential equations known as the Darboux system. Indeed, in the context of the geometric theory of integrable systems (see, e.g. [14] and references therein), the function \( \psi \) constitutes an eigenfunction of the conjugate net equations (2.4) and the sets \( \{\lambda_i^\alpha\} \) represent adjoint eigenfunctions. The functions \( \psi\alpha \) are Combescure transforms of the eigenfunction \( \psi \) and, for reasons of symmetry, it is evident that each Combescure transform is a solution of another system of conjugate net equations with different coefficients.

(a) The generalized hodograph method

In order to motivate the approach adopted in this paper, we here recall the generalized hodograph method developed by Tsarev in [8] for a single system of hydrodynamic-type equations

\[ u_i^\alpha = \lambda_i^\alpha(u)u_i^\alpha_x, \quad i = 1, \ldots, N + 1 \quad (2.5) \]

with associated functions \( A_{ik} \) defined by (2.2), that is,

\[ \lambda_i^\alpha u_k = A_{ik}(\lambda_k^\alpha - \lambda_i^\alpha). \quad (2.6) \]

Thus, Tsarev’s theorem states that if \( \{\omega^i\} \) is another set of adjoint eigenfunctions obeying the above linear system then any local solution \( u(x, t) \) of the nonlinear system

\[ \omega^i(u) = \lambda_i^\alpha(u)t + x \quad (2.7) \]

constitutes a solution of the hydrodynamic-type system (2.5). Conversely, any solution of the hydrodynamic-type system may locally be represented in this manner. As indicated in the Introduction, in the case, \( N = 1 \), (2.7) may be regarded as a linear system for \( x(u, t) \) rather than a nonlinear system for \( u^1 \) and \( u^2 \) and differentiation of \( x(u^1, u^2) \) and \( t(u^1, u^2) \) leads to the classical hodograph system [15]

\[ x_{u^1} + \lambda_1^2 t_{u^1} = 0, \quad x_{u^2} + \lambda_1^1 t_{u^2} = 0. \quad (2.8) \]

Here, the coefficients \( \lambda_i^\alpha \) are regarded as known functions of the independent variables \( u^k \). In the original context, this linear system is obtained from the nonlinear two-component system
(2.5)_{N=1} by merely interchanging dependent and independent variables, whereby the Jacobian determinant drops out.

(b) Generalized hodograph equations

Even though the generalized hodograph method encapsulated in the algebraic system (2.7) is applicable for all \( N \), an associated system of hodograph-type equations is not available for \( N > 1 \), because the number of independent variables does not coincide with the number of dependent variables. However, because any flow which commutes with the hydrodynamic-type equations (2.5) does not impose any constraint on the space of solutions, it is natural to supplement (2.5) by \( N-1 \) commuting flows, leading to the larger system (2.1). Thus, if \( \{ \mu^i \} \) constitutes another set of adjoint eigenfunctions, then we may locally define a coordinate transformation

\[
u = \nu(x, t)
\]

via the system

\[
\mu^i(u) = \sum_{\alpha=1}^{N} \lambda^i_{\alpha} (u) t^\alpha + x
\]

which coincides with the system (2.7) in the case \( N = 1 \).

It is now easy to see that Tsarev’s generalized hodograph method is still valid in this more general setting so that, locally, the general solution of the hydrodynamic-type system (2.1) is encapsulated in the algebraic system (2.10) regarded as a definition of \( u \). In fact, this observation may be interpreted as a corollary of Tsarev’s theorem because if we select a ‘time’ \( t^{a_0} \) and regard all other \( t^\alpha \)s as parameters then system (2.10) may be formulated as

\[
\omega^i(u) = \lambda^i_{a_0} (u) t^{a_0} + x,
\]

where the quantities

\[
\omega^i = \mu^i - \sum_{\alpha \neq a_0} \lambda^i_{\alpha} t^\alpha
\]

represent linear superpositions of adjoint eigenfunctions, so that, according to the generalized hodograph method, (2.1) holds for \( \alpha = a_0 \).

As in the classical case (\( N = 1 \)), the algebraic system (2.10) turns out to be equivalent to a system of first-order differential equations. Indeed, if we regard (2.10) as a definition of some functions \( \mu^i \) then, on substitution into the adjoint eigenfunction equations (2.2), it is readily verified that these functions constitute adjoint eigenfunctions if and only if the generalized hodograph equations

\[
x^{i}_{\mu^i} + \sum_{\alpha=1}^{N} \lambda^i_{\alpha} \mu^i_{\alpha} = 0, \quad i \neq k
\]

are satisfied. By construction, this system of hodograph type is equivalent to the original hydrodynamic-type system (2.1).

(c) Iterated adjoint Darboux transformations

It turns out that, just like the characteristic speeds \( \lambda^i_{\alpha} \) and the quantities \( \mu^i \), the remaining ingredients \( x \) and \( t^\alpha \) of the algebraic system (2.10) have distinct soliton-theoretic meaning. Thus, we first consider two sets \( \{ \mu^i \} \) and \( \{ \lambda^i \} \) of adjoint eigenfunctions obeying

\[
\mu^i_{\mu^k} = A_{ik}(\mu^k - \mu^i), \quad \lambda^i_{\mu^k} = A_{ik}(\lambda^k - \lambda^i)
\]

for some solution \( \{ A_{ik} \} \) of the underlying Darboux system. This system is known to be invariant under adjoint Darboux transformations \([27,37]\). Specifically, for fixed \( i \), the adjoint Darboux
transformation $\mathbf{D}^i$ generated by $\lambda^i$ transforms the adjoint eigenfunctions $\mu^i$ according to

$$
\mathbf{D}^i(\mu^i) = \mu^i - \frac{\lambda^i}{\lambda^i_{\alpha}} \mu_{\alpha},
$$

and

$$
\mathbf{D}^i(\mu^k) = \frac{\lambda^i \mu^k - \lambda^k \mu^i}{\lambda^i - \lambda^k}, \ k \neq i.
$$

(2.15)

By construction, the above Darboux transforms obey a linear system of the type (2.14) with coefficients depending on $A_{ik}$ and the adjoint eigenfunctions $\lambda^i$ only. The latter property guarantees that adjoint Darboux transformations may be iterated in the following purely algebraic manner. Given any $N$ sets of eigenfunctions $\{\lambda^i_{\alpha}\}$, we begin with the adjoint Darboux transformation $\mathbf{D}^i_{\alpha}$ generated by $\lambda^i_{\alpha}$. The quantities $\mathbf{D}^1_{\alpha}(\mu^i)$ then constitute new adjoint eigenfunctions. In particular, if we focus on the new adjoint eigenfunctions $\lambda^1_{\alpha}$, we begin with the adjoint eigenfunctions $\mathbf{D}^i_{\alpha}$, then we may use the adjoint eigenfunction $\mathbf{D}^1_{\alpha}(\lambda^2_{\alpha})$ to define an adjoint Darboux transformation acting on the new adjoint eigenfunctions which we denote by $\mathbf{D}^2_{\alpha}$. This procedure may be repeated to construct $N$ adjoint Darboux transformations $\mathbf{D}^1_{\alpha}, \ldots, \mathbf{D}^N_{\alpha}$ generated by the adjoint eigenfunctions $\lambda^1_{\alpha}, \mathbf{D}^1_{\alpha}(\lambda^2_{\alpha}), \mathbf{D}^2(\mathbf{D}^1_{\alpha}(\lambda^3_{\alpha})), \ldots$. On use of Jacobi’s identity for determinants [38], it is then straightforward to verify by induction that the $N$th Darboux transform of the adjoint eigenfunction $\mu^{N+1}$ is given by

$$
(\mathbf{D}^N_{\alpha} \circ \cdots \circ \mathbf{D}^1_{\alpha})(\mu^{N+1}) = \left| \begin{array}{ccccc}
\lambda^1_{\alpha} & \cdots & \lambda^1_{\alpha} & \mu^1 \\
\vdots & & \vdots & \vdots \\
\lambda^{N+1}_{\alpha} & \cdots & \lambda^{N+1}_{\alpha} & \mu^{N+1} \\
\lambda^1_{\alpha} & \cdots & \lambda^1_{\alpha} & 1 \\
\vdots & & \vdots & \vdots \\
\lambda^{N+1}_{\alpha} & \cdots & \lambda^{N+1}_{\alpha} & 1
\end{array} \right|.
$$

However, the right-hand side of the expression (2.16) is completely symmetric in both the upper and lower indices. Hence, the $N$th Darboux transform depends neither on the order of application of the adjoint Darboux transformations nor on the components of the sets of adjoint eigenfunctions $\{\lambda^i_{\alpha}\}$ which are chosen to generate the corresponding adjoint Darboux transformations. More precisely, for any permutations $(\alpha_1, \ldots, \alpha_N)$ and $(i_1, \ldots, i_{N+1})$ of $(1, \ldots, N)$ and $(1, \ldots, N + 1)$, respectively, the iterated Darboux transform

$$
(\mathbf{D}^N_{\alpha_1} \circ \cdots \circ \mathbf{D}^1_{\alpha_1})(\mu^{i_{N+1}}) = \mu^{(N)}
$$

is the same. In fact, application of Cramer’s rule shows that

$$
x = \mu^{(N)}
$$

(2.17)

corresponds to the unique solution of the algebraic system (2.10) regarded as a linear system for $x$ and $t^a$. Thus, remarkably, by virtue of the commutativity of the flows (2.1), the ‘spatial’ independent variable $x$ may be interpreted as the unique $N$-fold Darboux transform constructed from the characteristic speeds $\lambda^i_{\alpha}$.

The interpretation of the ‘times’ $t^a$ is now based on the observation that the system (2.10) is implicitly symmetrical in $x$ and $t^a$. Indeed, for any fixed $\alpha$, the system (2.10) may be reformulated as

$$
\frac{\mu^i}{\lambda^i_{\alpha}} = \left( \sum_{\beta \neq \alpha} \frac{\lambda^i_{\beta}}{\lambda^i_{\alpha}} t^\beta + \frac{1}{\lambda^i_{\alpha}} x \right) + t^\alpha,
$$

(2.19)

so that the roles of $x$ and $t^\alpha$ have been interchanged. In fact, the a priori formal symmetry obtained in this manner may indeed be exploited by rewriting the linear system (2.3) as

$$
\psi_{\alpha i} = \frac{1}{\lambda^i_{\alpha}} \psi_{\alpha i}, \quad \psi_{\beta i} = \frac{\lambda^i_{\beta}}{\lambda^i_{\alpha}} \psi_{\alpha i}, \quad \beta \neq \alpha.
$$

(2.20)
The latter implies that the quantities \( \tilde{\lambda}_i^\alpha = 1/\lambda_i^\alpha, \tilde{\lambda}_i^\beta = \lambda_i^\beta/\lambda_i^\alpha \) and, in fact, \( \tilde{\mu}_i^\alpha = \mu_i^\alpha/\lambda_i^\alpha \) constitute adjoint eigenfunctions of the Darboux system associated with the eigenfunction \( \psi_\alpha \). Hence, for reasons of symmetry, the time \( t^\alpha \) obtained by means of Cramer’s rule from (2.19) or, equivalently, the original system (2.10) coincides with the iterated Darboux transform

\[
t^\alpha = \mu_{(N,\alpha)}, \quad \mu_{(N,\alpha)} = (\tilde{D}_{iN}^{\alpha} \circ \cdots \circ \tilde{D}_{i1}^{\alpha})(\tilde{\mu}_{iN}^\alpha),
\]

where the Darboux transformations \( \tilde{D}_{ik}^{\alpha} \) are now generated by the adjoint eigenfunctions \( \tilde{\lambda}_{ik}^\alpha \). It is also observed that the generalized hodograph system (2.13) may be solved for the derivatives of \( t^\alpha \) to deduce that

\[
t^\alpha_{it} = A_k^{\alpha}(\tilde{\lambda}_i^\beta)x_{it} \tag{2.22}
\]

for some functions \( A_k^\alpha \) and, hence, the \( N + 1 \) variables \( x \) and \( t^\alpha \) may also be regarded as Combescure transforms of each other.

### 3. Discrete generalized hodograph equations

The formulation of the classical hodograph equations and their generalization in the language of (adjoint) eigenfunctions may instantly be used to derive their canonical integrable discrete counterparts. Indeed, the standard integrable discretization of the conjugate net equations (2.4) turns out to be the fundamental structure on which this discretization technique is based. Thus, if

\[
\psi : \mathbb{Z}^{N+1} \to \mathbb{R} \quad \left\{(n_1, \ldots, n_{N+1}) \mapsto \psi(n_1, \ldots, n_{N+1})\right\} \tag{3.1}
\]

is an eigenfunction obeying the discrete conjugate net equations [32]

\[
\Delta_{ik} \psi = A_{ik} \Delta_i \psi + A_{ki} \Delta_k \psi, \tag{3.2}
\]

where the forward difference operators \( \Delta_i \) are defined by \( \Delta_i \psi = \psi[i] - \psi \) and \( \Delta_{ik} = \Delta_i \Delta_k = \Delta_k \Delta_i \), then a discrete Combescure transform \( \psi^* \) of \( \psi \) defined by

\[
\Delta_i \psi^* = \tilde{\lambda}_i^i \Delta_i \psi \tag{3.3}
\]

exists if the associated discrete adjoint eigenfunctions \( \tilde{\lambda}_i^i \) constitute solutions of the linear system [33,39]

\[
\Delta_{ik} \tilde{\lambda}_i^i = A_{ik}(\tilde{\lambda}_i^k - \tilde{\lambda}_k^i). \tag{3.4}
\]

Here, a subscript \([i]\) denotes the relative unit increment \( n_i \to n_i + 1 \), so that the mixed difference operator \( \Delta_{ik} \) acts according to \( \Delta_{ik} \psi = \psi[i][k] - \psi[k][i] - \psi + \psi \). It is noted that (3.2) and (3.3) represent the discrete analogues of the linear equations (1.2) and (1.4) defining the densities \( \psi \) and fluxes \( \psi^* \) associated with the conservation laws for semi-Hamiltonian systems of hydrodynamic type. In connection with an appropriate Cauchy problem, it is convenient to reformulate the adjoint linear system (3.4) as

\[
\Delta_{ik} \tilde{\lambda}_i^i = \frac{A_{ik}}{A_{ik} + A_{ki} + 1}(\lambda_k^i - \tilde{\lambda}_i^i). \tag{3.5}
\]

As in the continuous case, the compatibility conditions for the (adjoint) eigenfunction equations (3.2) and (3.4) (or (3.5)) give rise to the same nonlinear system of discrete equations for the coefficients \( A_{ik} \) which constitutes the standard integrable discretization of the aforementioned Darboux system [33,40].

The discrete analogues of the classical adjoint Darboux transformations may be obtained by formally replacing derivatives by differences in the transformation laws (2.15). Indeed, the
Darboux transforms of another set of adjoint eigenfunctions \( \{ \mu^j \} \) are given by
\[
D_i^{j}(\mu^j) = \mu^j - \frac{\lambda^j_i}{\Delta^i \lambda^j_i} \Delta_i \mu^j = \frac{\lambda^j_i \mu^j - \lambda^j_i \mu^j}{\lambda^j_i - \lambda^j_i} \tag{3.6}
\]
and
\[
D_i^{j}(\mu^k) = \frac{\lambda^j_i \mu^k - \lambda^j_k \mu^i}{\lambda^j_i - \lambda^j_k}, \quad k \neq i
\]
for any fixed \( i \) corresponding to the adjoint eigenfunction \( \lambda^j_i \) which generates the adjoint Darboux transformation \( D_i^{j} \). Because iteration of the discrete adjoint Darboux transformations only involves the algebraic transformation law (3.6), which coincides with the transformation law (2.15), the expressions (2.16), (2.17) and (2.21) for the iterated Darboux transforms are also valid in the discrete case. Moreover, the quantities \( x \) and \( t^\alpha \) defined by
\[
\begin{align*}
x &= \mu(N), & t^\alpha &= \mu(N, \alpha) \\
\mu^i &= \sum_{\alpha=1}^{N} \lambda^i_\alpha t^\alpha + x,
\end{align*}
\]
still constitute the unique solution of the linear system (2.10), that is,
\[
\Delta_k x + \sum_{\alpha=1}^{N} \lambda^i_\alpha \Delta_k t^\alpha = 0, \quad i \neq k. \tag{3.9}
\]
Conversely, any solution of this integrable discretization of the generalized hodograph equations (2.13) provides via (3.8) a set of adjoint eigenfunctions \( \{ \mu^j \} \). Finally, the discrete generalized hodograph equations adopt the form
\[
\Delta_k t^\alpha = \Lambda^\alpha_k (\lambda^j_i)^{\beta_{[k]}} \Delta_k x, \tag{3.10}
\]
which demonstrates that, as in the continuous case, the \( N+1 \) variables \( x \) and \( t^\alpha \) may be interpreted as discrete Combescure transforms of each other.

As pointed out in the previous section, there exists complete equivalence between the hydrodynamic-type system (2.1) and the generalized hodograph equations (2.13). In fact, this is verified directly by employing a formulation in terms of differential forms (cf. [41]). Indeed, it is seen that the system
\[
\text{d}u^j \wedge \text{d}x + \sum_{\alpha=1}^{N} \lambda^i_\alpha \text{d}u^j \wedge \text{d}t^\alpha = 0 \tag{3.11}
\]
reduces to the hydrodynamic-type system (2.1) if \( x \) and \( t^\alpha \) are chosen as the independent variables. Alternatively, one may select the \( u^i \)'s as the independent variables, so that the generalized hodograph equations (2.13) result. Accordingly, the algebraic system (2.10) encodes the \( N+1 \)-dimensional integral manifolds \( \mathcal{M} \) of the differential system (3.11). Thus, if we interpret a solution \((x, t)(u)\) of the generalized hodograph equations (2.13) as an \( N+1 \)-dimensional submanifold \( \mathcal{M} \) of the space of (in)dependent variables \( \mathbb{R}^{2N+2} \), then this submanifold \( \mathcal{M} \) admits the parametrization
\[
u \mapsto (x(u), t(u), u). \tag{3.12}
\]
However, locally, we may also use the parametrization
\[
(x, t) \mapsto (x, t, u(x, t)), \tag{3.13}
\]
where \( u(x, t) \) represents a corresponding solution of the hydrodynamic-type system (2.1). In the discrete case, the algebraic system (3.8) encapsulates ‘discrete integral manifolds’ \( \mathcal{M}^\Delta \) in the
following sense. Any solution
\[ (x,t) : \mathbb{Z}^{N+1} \to \mathbb{R}^{N+1}, \quad n \mapsto (x,t)(n) \]
(3.14)
of the discrete generalized hodograph equations (3.9) may be used to parametrize a ‘discrete
submanifold’ \( M^A \) of \( \mathbb{R}^{N+1} \times \delta^1 \mathbb{Z} \times \cdots \times \delta^{N+1} \mathbb{Z} \) according to
\[ n \mapsto (x(n), t(n), u^\Lambda), \]
(3.15)
where \( u^\Lambda = (\delta^1 n^1, \ldots, \delta^{N+1} n^{N+1}) \) for prescribed lattice parameters \( \delta^i \). Hence, the variable \( u^\Lambda \) may
be regarded as a discretization of either the independent variables of the generalized hodograph
equations (2.13) or the dependent variables of the system of hydrodynamic type (2.1). The latter
corresponds to an ‘implicit discretization’ of the hydrodynamic-type system with variable spacing
between the lattice points on the \( N + 1 \)-dimensional submanifold \( \mathbb{R}^{N+1} \) of the independent
variables \( x \) and \( t \).

4. Discrete Euler–Poisson–Darboux systems

It is well known [17–20] that the characteristic speeds \( \lambda^i \) associated with the multi-phase-averaged
KdV equations are related to linear hyperbolic equations of Euler–Poisson–Darboux-type. In fact,
recently, it has been demonstrated [22] that these characteristics speeds may be generated by
means of iterated Darboux transformations applied to separable solutions of (extended) Euler–
Poisson–Darboux-type systems. It turns out that one may construct canonical discretizations
of the multi-phase characteristic speeds if one carefully defines analogues of the hyperelliptic
integrals associated with the underlying Riemann surfaces of genus \( g \geq 1 \). Here, we demonstrate
how one may derive particular classes of discrete characteristic speeds from the discrete Euler–
Poisson–Darboux-type system
\[ \frac{\delta^i (n^i + v^i)}{\delta^i} \Delta_{ik} \phi = \delta^k \epsilon^k \Delta_i \phi - \delta^i \epsilon^i \Delta_k \phi, \]
(4.1)
where \( i \neq k \in \{1, \ldots, 2g + 1\} \), which include those of ‘averaged KdV’ type. Here, the constants
\( \delta^i \) are lattice parameters and the constants \( \epsilon^i \) determine the nature of the contour integrals to
be defined in §5. For \( \epsilon^i = 1/2 \), this leads to analogues of the above-mentioned hyperelliptic
integrals. The parameters \( v^i \) reflect the fact that it is crucial to maintain the freedom of placing
the discretization points not necessarily on the vertices of the \( \mathbb{Z}^{2g+1} \) lattice but, possibly, on the
edges, faces, etc. Thus, we regard the hyperbolic system (4.1) as a discretization of the classical
Euler–Poisson–Darboux system
\[ (x^i - x^k) \phi_{x^i x^k} = \epsilon^k \phi_{x^k} - \epsilon^i \phi_{x^i}, \]
(4.2)
obtained in the limit \( x^i = \delta^i (n^i + v^i) \), \( \delta^i \to 0 \). In the following, the key idea is to introduce an
auxiliary continuous variable \( y \) and supplement the discrete Euler–Poisson–Darboux system by
the differential–difference equations
\[ [y - \delta^i (n^i + v^i)] \Delta_i \phi_y = \delta^i \epsilon^i \phi_y + (g - 1) \Delta_i \phi. \]
(4.3)
The function \( \phi = \phi(n,y) \) is well-defined, because the semi-discrete Euler–Poisson–Darboux
system (4.1) and (4.3) remains compatible.

(a) Separable solutions

As in the continuous case [22], we now focus on separable solutions of the semi-discrete Euler–
Poisson–Darboux system (4.1) and (4.3). Thus, it is readily verified that the ansatz
\[ \phi_{\text{sep}} = \rho(\zeta) \prod_{i=1}^{2g+1} \rho^i(\zeta), \]
(4.4)
where we have suppressed the dependence of $\rho$ and $\rho^i$ on $y$ and $n^i$, respectively, leads to the first-order differential/difference equations

$$\Delta_i \rho^i = \frac{\delta^i \epsilon^i}{\zeta - \delta^i(n^i + v^i)} \rho^i, \quad \rho_y = \frac{(1 - g)}{\zeta - y} \rho$$

(4.5)

with $\zeta$ being a (complex) constant of separation. The latter may be solved to obtain

$$\rho = (\zeta - y)^{\delta^{1-1}}$$

(4.6)

without loss of generality and, in the continuum limit $\delta^i \to 0$, the difference equations (4.5) reduce to

$$\rho^i_{\chi} = \frac{\epsilon^i}{\zeta - x^i} \rho^i.$$  

(4.7)

Hence, up to a multiplicative constant, $\rho^i$ represents a canonical discretization of $(\zeta - x^i)^{-\epsilon^i}$ so that

$$\phi_{\text{sep}} = \rho(\zeta) \prod_{i=1}^{2g+1} \rho^i(\zeta) \to (\zeta - y)^{\delta^{1-1}} \prod_{i=1}^{2g+1} (\zeta - x^i)^{-\epsilon^i}$$

(4.8)

in the continuum limit.

(b) Superposition and iterated Darboux transformations

The separable solutions derived in the preceding may be superimposed to obtain large classes of solutions of the semi-discrete Euler–Poisson–Darboux system (4.1) and (4.3). Here, we consider the contour integrals

$$\phi_\kappa = \oint_{b_\kappa} (\zeta - y)^{\delta^{1-1}} \prod_{i=1}^{2g+1} \rho^i(\zeta) \, d\zeta, \quad \kappa = 1, \ldots, g,$$

(4.9)

where the contours $b_\kappa$ on the complex $\zeta$-plane are assumed to be independent of $y$ and 'locally' independent of $n$, that is, we demand that

$$\Delta_i \oint_{b_\kappa} f(n, y; \zeta) \, d\zeta = \oint_{b_\kappa} \Delta_i f(n, y; \zeta) \, d\zeta$$

(4.10)

for any relevant functions $f$. Accordingly, the semi-discrete Euler–Poisson–Darboux system (4.1) and (4.3) admits vector-valued solutions of the form

$$\phi = \begin{pmatrix} \phi^1 \\ \vdots \\ \phi^g \end{pmatrix}, \quad \hat{\phi} = \begin{pmatrix} \hat{\phi}^1 \\ \vdots \\ \hat{\phi}^g \end{pmatrix}.$$  

(4.11)

The components of these solutions may be used to generate iteratively solutions of semi-discrete conjugate net equations with increasingly complex coefficients. Thus, the $(g - 1)$-fold Darboux transform [42] of any solution $\phi$ of the semi-discrete Euler–Poisson–Darboux system (4.1) and (4.3) with respect to the independent variable $y$ is given by the compact expression

$$\phi^{g-1} = \begin{vmatrix} \phi & \phi_y & \cdots & \phi_{(g-1)y} \\ \hat{\phi} & \hat{\phi}_y & \cdots & \hat{\phi}_{(g-1)y} \end{vmatrix}.$$  

(4.12)

In the context of classical differential geometry, this Darboux transform is known as the $(g - 1)$-fold Levy transform [27] with respect to $y$. The Levy transforms of the particular solutions

$$\phi^0 = \sum_{i=1}^{2g+1} \epsilon^i \delta^i n^i - (g - 1)y, \quad \phi^1$$

(4.13)
of the semi-discrete Euler–Poisson–Darboux system therefore read

\[
\phi_{g-1}^0 = \frac{\sum_{i=1}^{2g+1} e^{i\delta^i n^i} |\hat{\phi}_y \cdots \hat{\phi}_{(g-1)y}| + (g-1)|\hat{\phi} - y \hat{\phi}_y \cdots \hat{\phi}_{(g-1)y}|}{|\hat{\phi}_y \cdots \hat{\phi}_{(g-1)y}|}
\]

and

\[
\phi_{g-1}^1 = \frac{|\phi \cdots \phi_{(g-1)y}|}{|\hat{\phi}_y \cdots \hat{\phi}_{(g-1)y}|}
\]

so that it is readily verified that

\[
\phi_{g-1}^0 = \frac{\sum_{i=1}^{2g+1} e^{i\delta^i n^i} |\hat{I}_{g-2} \cdots \hat{I}_0| - |\hat{I}_{g-1} \hat{I}_{g-3} \cdots \hat{I}_0|}{|\hat{I}_{g-2} \cdots \hat{I}_0|}
\]

and

\[
\phi_{g-1}^1 = \frac{|I_{g-1} \cdots I_0|}{|I_{g-2} \cdots I_0|}
\]

with the contour integrals

\[
I_k^e = \oint_{b_k} \zeta^k \prod_{i=1}^{2g+1} \rho^i(\zeta) d\zeta.
\]

It is observed that, remarkably, the Levy transforms \(\phi_{g-1}^0\) and \(\phi_{g-1}^1\) are independent of \(y\) and that, by definition, \(\phi_{g-1}^0 = \phi^0\) and \(\phi_{g-1}^1 = \phi^1\) in the case \(g = 1\).

The action of another Levy transformation with respect to the variable \(n^i\) now produces the \(g\)-fold Levy transform

\[
\lambda^i = \phi_{g-1}^i[\phi^0] = \phi_{g-1}^0 - \Delta_i \phi_{g-1}^1 \Delta_i \phi_{g-1}^0
\]

of \(\phi^0\). Here, the symbol \(\lambda^i\) has been chosen to indicate that the set \(\{\lambda^i\}\) will indeed be shown to constitute a set of adjoint eigenfunctions. Because the Levy transform \(\lambda^i\) may be formulated as

\[
\lambda^i = \frac{\Delta_i (\phi_{g-1}^0 / \phi_{g-1}^1)}{\Delta_i (1 / \phi_{g-1}^1)},
\]

we may set

\[
H_1 = \frac{\hat{I}_0 \cdots \hat{I}_{g-2}}{|I_0 \cdots I_{g-1}|}, \quad H_2 = -\frac{\hat{I}_0 \cdots \hat{I}_{g-3} \hat{I}_{g-1}}{|I_0 \cdots I_{g-1}|}
\]

and

\[
\tilde{I}_0 = 1, \quad \tilde{I}_1 = \sum_{i=1}^{2g+1} e^{i\delta^i n^i}
\]

to obtain the final expression

\[
\lambda^i = \frac{\Delta_i (H_1 \tilde{I}_1 + H_2 \tilde{I}_0)}{\Delta_i H_1}.
\]

This constitutes a natural discretization of a particular case of the characteristic speeds obtained in an entirely different manner by Tian in the continuous context (cf. [20, p. 218] for \(\alpha_1 = 1, \alpha_k = 0\) otherwise and \(H_{1/2} \sim K_{1/2}^1\) in Tian’s notation). Once again, in the case \(g = 1\), the interpretation \(H_1 = 1/I_0\) and \(H_2 = 0\) is to be adopted.

(c) Discrete characteristic speeds

The connection with discrete characteristic speeds and the associated discrete generalized hodograph equations (3.9) is now made as follows. By construction, \(\phi_{g-1}^0\) and \(\phi_{g-1}^1\) are solutions
of the same system of discrete conjugate net equations

\[ \Delta_{ik} \phi_{g-1} = B_{ik} \Delta_i \phi_{g-1} + B_{ki} \Delta_k \phi_{g-1}. \]  

(4.21)

On the other hand, the classical Levy transforms of an eigenfunction corresponding to different ‘directions’ \( x^i \) may also be regarded as adjoint eigenfunctions of another system of conjugate net equations [37]. The analogous statement is true in the discrete case and, accordingly, the quantities \( \lambda^i \) constitute adjoint eigenfunctions associated with the discrete conjugate net equations

\[ \Delta_{ik} \psi = A_{ik} \Delta_i \psi + A_{ki} \Delta_k \psi, \]  

(4.22)

wherein the coefficients \( A_{ik} \) are related to the coefficients \( B_{ik} \) by

\[ C_{ik} = \frac{A_{ik}}{A_{ik} + A_{ki} + 1} = \frac{B_{ki}\phi^1_{g-1[k]} \Delta_k \phi^1_{g-1}}{B_{ki}\phi^1_{g-1} \Delta_i \phi^1_{g-1} - \phi^1_{g-1}[k]}. \]  

(4.23)

The above observation allows us to identify a particular set of discrete characteristic speeds which may be used in the discrete generalized hodograph equations (3.9). However, the definition of the latter requires \( 2g + 1 \) sets of adjoint eigenfunctions \( \{ \lambda^i \} \), each of which represents a solution of the adjoint eigenfunction equations

\[ \Delta_{ik} \lambda^i = C_{ik}(\lambda^k - \lambda^i) \]  

(4.24)

satisfied by the Levy transforms \( \lambda^i \).

In order to construct canonical sets of adjoint eigenfunctions satisfying (4.24), it is required to introduce an explicit parametrization of the functions \( \rho^i \) in the base separable solution (4.4) of the associated semi-discrete Euler–Poisson–Darboux system. Thus, in terms of Gamma functions [43], the general solution of the difference equation (4.5) formulated as

\[ \rho^i_{[i]} = \frac{\xi - \delta^i(n^i + \nu^i - \epsilon^i)}{\xi - \delta^i(n^i + \nu^i)} \rho^i \]  

(4.25)

is given by

\[ \rho^i = \delta^i - \epsilon^i \frac{\Gamma(\xi^i - n^i - \nu^i + 1)}{\Gamma(\xi^i - n^i - \nu^i + \epsilon^i + 1)}, \quad \xi^i = \frac{\zeta}{\delta^i} \]  

(4.26)

up to a constant of ‘integration’ which may depend on \( \zeta \). In fact, the multiplicative factor has been chosen in such a manner that \( \rho^i \to (\zeta - x^i)^{-\epsilon^i} \) in the continuum limit \( \delta^i \to 0 \). This is a consequence of the well-known asymptotic behaviour

\[ \lim_{|z| \to \infty} z^{b-a} \frac{\Gamma(z + a)}{\Gamma(z + b)} = 1, \quad |\arg(z)| < \pi \]  

(4.27)

of ratios of Gamma functions. In fact, the first two terms of the associated classical asymptotic expansion [44] read

\[ \frac{\Gamma(z + a)}{\Gamma(z + b)} = z^{a-b} \left[ 1 + \frac{(a-b)(a+b-1)}{2z} + O(|z|^{-2}) \right]. \]  

(4.28)

It is noted that (4.26) regarded as a discretization of a ‘power function’ essentially coincides with that considered in reference [45].

It has been pointed out that the Levy transforms \( \lambda^i = \phi^i_{g}[\phi^0] \) are independent of the auxiliary variable \( y \). This is due to the fact that the seed solution \( \phi^0 \) of the semi-discrete Euler–Poisson–Darboux system is a polynomial in \( y \) of degree at most \( g - 1 \). A canonical way of generating an
infinite number of seed solutions which admit this property is to expand the separable solution
\[
\phi = \zeta^{\alpha} \phi_{\text{sep}} = \zeta^{\alpha} (\zeta - y)^{g-1} \prod_{i=1}^{2g+1} \rho^i(\zeta), \quad \sigma = \sum_{i=1}^{2g+1} \epsilon^i - (g-1)
\]  
(4.29)
about \(\zeta = \infty\) to obtain
\[
\phi = (1 - y\zeta^{-1})^{g-1} \sum_{m=0}^{\infty} \Gamma_m(n)\zeta^{-m}. \tag{4.30}
\]
The existence of this formal power series in \(\zeta^{-1}\) is readily established by applying the asymptotic expansion (4.28) to the function \(\rho^i\) as given by (4.26) and reformulating it as an asymptotic series in \(\zeta^{-1}\), namely
\[
\rho^i(\zeta) = \zeta^{-\epsilon^i} \left[ 1 + \epsilon^i \delta^i \left(n^i + v^i - \frac{\epsilon^i + 1}{2}\right) \zeta^{-1} + O(|\zeta|^{-2}) \right]. \tag{4.31}
\]
Thus, for instance, the first two coefficients \(\Gamma_0\) and \(\Gamma_1\) are seen to be
\[
\Gamma_0 = 1, \quad \Gamma_1 = \sum_{i=1}^{2g+1} \epsilon^i \delta^i \left(n^i + v^i - \frac{\epsilon^i + 1}{2}\right). \tag{4.32}
\]
It is evident that the expansion (4.30) is of the form
\[
\phi = \sum_{\alpha=0}^{\infty} \mathcal{E}_\alpha(n, y)\zeta^{-\alpha}, \tag{4.33}
\]
where the coefficients \(\mathcal{E}_\alpha(n, y)\) are polynomials in \(y\) of degree \(\alpha\) if \(\alpha \leq g - 1\) and of degree \(g - 1\) if \(\alpha > g - 1\). In fact,
\[
\mathcal{E}_\alpha(n, y) = \sum_{k=0}^{g-1} (-y)^k \binom{g-1}{k} \Gamma_{\alpha,k}(n), \tag{4.34}
\]
where
\[
\Gamma_{\alpha,k} = \Gamma_{\alpha-k} \quad \text{if } 0 \leq k \leq \min(\alpha, g-1) \tag{4.35}
\]
and \(\Gamma_{\alpha,k} = 0\) otherwise. By construction, each coefficient \(\mathcal{E}_\alpha\) constitutes a solution of the semi-discrete Euler–Poisson–Darboux system (4.1) and (4.3). For instance, \(\mathcal{E}_0 = \Gamma_0 = 1\) represents the trivial constant solution, whereas
\[
\mathcal{E}_1 = (g-1)y\Gamma_0 = \phi^0 + c^0 \tag{4.36}
\]
turns out to be a linear superposition of the trivial solution, and the seed solution \(\phi^0\) which has been used to construct the discrete characteristic speeds \(\lambda^i\) given by (4.20). The constant \(c^0\) may be read off (4.32).

The general expression (4.12) for the iterated Darboux transform \(\phi_{g-1}\) may be used to generate the \((g-1)\)-fold Levy transform \(\phi_{g-1}[\mathcal{E}_\alpha]\) of any seed solution \(\mathcal{E}_\alpha\). Because the degree of \(\mathcal{E}_\alpha\) in \(y\) is less than \(g\), the Levy transform \(\phi_{g-1}[\mathcal{E}_\alpha]\) is independent of \(y\). Hence, the procedure outlined in §4b may be simplified by evaluating the analogue of (4.14) at \(y = 0\). As a result, one is immediately led to the compact expression
\[
\phi_{g-1}[\mathcal{E}_\alpha] = \begin{vmatrix}
\Gamma_{\alpha,0} & \Gamma_{\alpha,1} & \cdots & \Gamma_{\alpha,g-1} \\
\hat{I}_{g-1} & \hat{I}_{g-2} & \cdots & \hat{I}_0 \\
\end{vmatrix}. \tag{4.37}
\]
In particular, by virtue of (4.36), it may be concluded that
\[
\phi_{g-1}[\mathcal{E}_1] = \phi_{g-1}^0 + c^0 \tag{4.38}
\]
so that, essentially, the \((g-1)\)-fold Levy transform associated with the discrete characteristic speeds \(\lambda^i\) is retrieved. We may now employ the eigenfunctions \(\phi_{g-1}[\mathcal{E}_\alpha]\) and \(\phi_{g-1}^1\) to generate
additional discrete characteristic speeds in the manner described in § 4b by replacing \( \phi_{g-1}^0 [\Xi_a] \) in (4.17) and (4.18). In terms of the coefficients \( \Gamma_{a,k} \) and the ratios of determinants

\[
H_k = (-1)^{k+1} \frac{|I_0 \cdots \tilde{I}_{g-k-1} \tilde{I}_{g-k+1} \cdots \tilde{I}_{g-1}|}{|I_0 \cdots I_{g-1}|},
\]

these turn out to be

\[
\lambda_i^j = \phi_i^j [\Xi_a] = \frac{\Delta_j (H_1 \Gamma_{a,0} + \cdots + H_g \Gamma_{a,g-1})}{\Delta_i H_1},
\]

and encode the discrete characteristic speeds \( \lambda_i^j \) via

\[
\lambda_i^j = \lambda_i^j + c^0.
\]

Accordingly, any choice of \( 2g + 1 \) sets of adjoint eigenfunctions \( \{ \lambda_i^j \} \) such as \( \alpha = 1, \ldots, 2g + 1 \) gives rise to a discrete system of generalized hodograph equations (3.9) with associated ‘implicitly defined’ discrete commuting flows of hydrodynamic type. Once again, it is observed that (4.40) represents a natural discretization of the compact formulation of the corresponding characteristic speeds recorded in reference [20].

5. ‘Discrete’Hyperelliptic Integrals and Characteristic Speeds of Whitham Type

It has been demonstrated that the characteristic speeds \( \lambda_i^j \) are independent of the auxiliary variable \( y \) and, accordingly, the contour integrals

\[
I_k^\kappa = \oint_{b_k} \xi^k \prod_{i=1}^{2g+1} \rho_i(\xi) \, d\xi
\]

constitute the main ingredients in their construction. Here, we are concerned with contours \( b_k \) which mimic canonical cycles associated with classical hyperelliptic integrals [26]. To this end, we make the choice

\[
\epsilon^i = \frac{1}{2}, \quad \delta^i = \delta,
\]

so that the underlying discrete Euler–Poisson–Darboux system (4.1) reduces to

\[
2(n^i + \nu^j - n^k - \nu^k) \Delta_{ik} \phi = \Delta_i \phi - \Delta_k \phi
\]

and the continuum limit is represented by \( \delta \to 0 \) with \( \delta n^i \) in \( x^i = \delta (n^i + \nu^i) \) held constant as before. Up to the factor \( \xi^k \), the integrand of the contour integrals (5.1) may be formulated as

\[
\varphi = \prod_{i=1}^{2g+1} p(\xi, n^i, \nu^i), \quad \xi = \frac{\xi}{\delta},
\]

where the function \( p \) representing all functions \( \rho^i \) is defined by

\[
p(\xi, n, \nu) = \frac{\Gamma(\xi - n - \nu + 1)}{\sqrt{\delta} \Gamma(\xi - n - \nu + 3/2)}
\]

in agreement with the choice (4.26).
**Figure 1.** The distribution of zeros (circles) and poles (crosses) of the function $\varphi$ on the $\xi$-plane and associated $b$-cycles for $g = 2$.

(a) ‘Discrete’ cycles and hyperelliptic integrals

We now introduce the ordering

$$n_1 < n_2 < \ldots < n_{2g} < n_{2g+1}$$

(5.6)

and choose

$$\nu^{2k} = \frac{1}{2}, \quad \nu^{2k+1} = 0$$

(5.7)

corresponding to the discretization points $x^{2k} = \delta(n^{2k} + \frac{1}{2})$ and $x^{2k+1} = \delta n^{2k+1}$. Hence, the separable solution (5.4) of the discrete Euler–Poisson–Darboux system (5.3) becomes

$$\varphi = \frac{1}{\delta^{3/2}} \frac{\Gamma(\xi - n^{2k+1} + 1)}{\Gamma(\xi - n^{2k+1} + 3/2)} \prod_{k=1}^{g} \frac{\Gamma(\xi - n^{2k} + 1/2)}{\Gamma(\xi - n^{2k} + 1)}.$$  

(5.8)

For instance, in the case $g = 1$, we obtain

$$\varphi = \frac{1}{\delta^{3/2}} \frac{\Gamma(\xi - n^1 + 1)}{\Gamma(\xi - n^1 + 3/2)} \frac{\Gamma(\xi - n^2 + 1/2)}{\Gamma(\xi - n^2 + 1)}.$$  

(5.9)

Because the Gamma function is non-zero but has simple poles at non-positive integers, the distribution of zeros and poles of the function $p(\xi, n, 0)$ is given by

$$p(\xi, n, 0) = 0, \quad \xi = \ldots, n - \frac{5}{2}, n - \frac{3}{2}$$

and

$$p(\xi, n, 0) = \pm \infty, \quad \xi = \ldots, n - 2, n - 1,$$

(5.10)

whereas

$$p\left(\xi, n, \frac{1}{2}\right) = 0, \quad \xi = \ldots, n - 2, n - 1$$

and

$$p\left(\xi, n, \frac{1}{2}\right) = \pm \infty, \quad \xi = \ldots, n - \frac{3}{2}, n - \frac{1}{2}.$$  

(5.11)

Accordingly, the zeros and poles of the functions $p$ which make up $\varphi$ partially cancel each other in such a manner that, as a function of $\xi$, $\varphi$ has no zeros or poles in the region

$$\bigcup_{k=1}^{g+1} (n^{2k-1} - 1, n^{2k}), \quad n^{2g+2} = \infty.$$  

(5.12)

It is therefore natural to define the $g$ contours $b_\kappa$ (on the $\xi$-plane) as closed paths of anticlockwise orientation which pass through the pairs of intervals $(n^{2k-1} - 1, n^{2k})$ and $(n^{2k+1} - 1, \infty)$ for $\kappa = 1, \ldots, g$ as indicated in figure 1. The contour integrals

$$\oint_{b_\kappa} \xi^k \varphi(\xi) \, d\xi, \quad \kappa = 1, \ldots, g, \quad k = 0, \ldots, g - 1$$

(5.13)
constitute ‘discrete’ analogues of the hyperelliptic integrals

\[
\oint_{b_n} \frac{\zeta^k}{\sqrt{\prod_{i=1}^{g+1} (\zeta - x_i)}} \, d\zeta
\]

(5.14)

with the contours \( b_n \) essentially becoming the \( b \)-cycles employed in references [20,21] in the limit \( \delta \to 0 \). The intervals \( (n^{2k-1} - 1, n^{2k}) \) and \( (n^{2k+1} - 1, \infty) \), \( k = 1, \ldots, g \) correspond to the cuts \( (x^{2k-1}, x^{2k}) \) and \( (x^{2k+1}, \infty) \) along which the upper and lower sheets of the underlying Riemann surface of genus \( g \) are joined. In fact, the \( \zeta \)-plane represents the union of the half of the upper sheet, and the half of the lower sheet which contains the \( b \)-cycles. This union is discontinuous between the cuts and, in the discrete case, this is reflected by the presence of poles and zeros between the intervals \( (n^{2k-1} - 1, n^{2k}) \) and \( (n^{2k+1} - 1, \infty) \). As in the continuous case, the ‘discrete’ \( b \)-cycles are ‘locally’ independent of \( n \) in the sense of (4.10), so that the ‘discrete’ hyperelliptic integrals (5.13) regarded as functions of \( n \) are indeed solutions of the discrete Euler–Poisson–Darboux system (4.1).

The discrete hyperelliptic integrals (5.13) may be evaluated explicitly in terms of the residues of the meromorphic integrand \( \varphi \), because one only requires the known relationships

\[
\Gamma \left( \frac{1}{2} \pm i \right) = \left[ \frac{(2l)!}{(±4)^l l!} \right]^{±1} \sqrt{\pi}, \quad \text{res}(\Gamma(z), z = -l) = \frac{(-1)^l}{l!}, \quad l \in \mathbb{N}. \tag{5.15}
\]

Specifically, in the case \( g = 1 \), the contour integral

\[
\phi^\Delta = \frac{1}{2\pi i} \oint_{b_1} \varphi(\xi) \, d\xi = \frac{\delta}{2\pi} \oint_{b_1} \varphi(\xi) \, d\xi = \delta \sum_{k=n^2}^{n^3-1} \text{res}(\varphi(\xi), \xi = k) \tag{5.16}
\]

is given by

\[
\phi^\Delta = \frac{1}{\sqrt{\delta}} \sum_{k=n^2}^{n^3-1} \frac{\Gamma(k - n^1 + 1)}{\Gamma(k - n^2 + 1/2)} \frac{\Gamma(k - n^2 + 1/2)}{\Gamma(k - n^3 + 1/2)} \frac{\Gamma(k - n^3 + 1)}{\Gamma(k - n^2 + 3/2) \Gamma(k - n^1 + 1/2)} \tag{5.17}
\]

This is to be compared with the corresponding elliptic integral (5.14) (divided by \( 2\pi i \)) evaluated at the points

\[
x^1 = \delta n^1, \quad x^2 = \delta(n^2 + \frac{1}{2}), \quad x^3 = \delta n^3. \tag{5.18}
\]

In terms of the complete elliptic integral \( K \) of the first kind [43], this elliptic integral may be expressed as

\[
\phi^O = \frac{2}{\pi \sqrt{x^3 - x^1}} K \left( \sqrt{\frac{x^3 - x^2}{x^3 - x^1}} \right). \tag{5.19}
\]

By construction, the latter constitutes an eigenfunction of the continuous Euler–Poisson–Darboux system (4.2) and may be used to generate three adjoint eigenfunctions \( \lambda^1, \lambda^2, \lambda^3 \) by means of the continuous analogue of the Levy transformation (4.18) for \( g = 1 \). These turn out to be the characteristic speeds in the one-phase averaged KdV equations derived by Whitham [2,8]. Thus, the discrete elliptic integral \( \phi^\Delta \) gives rise to discrete characteristic speeds of Whitham type.

It is observed that the elliptic integral \( \phi^O \) only depends on the differences of the \( x^i \). In fact, the same applies, \emph{mutatis mutandis}, to the discrete elliptic integral (5.16), because the function \( p(\xi, n, \nu) \) is invariant under a shift of \( \xi \) and \( n \) by the same amount. Accordingly, it is natural to regard the (discrete) elliptic integrals \( \phi^\Delta \) and \( \phi^O \) as functions of the differences \( n^2 - n^1 \) and \( n^3 - n^2 \). Their graphs are displayed in figure 2 and it its seen that there exists virtually no difference between the discrete and continuous elliptic integrals represented by points and a mesh, respectively. It is noted that this statement is independent of the lattice parameter \( \delta \) in the sense that, as a function of \( n \), the ratio \( \phi^\Delta / \phi^O \) does not depend on \( \delta \). Thus, remarkably, there exists a unique relationship between the discrete and continuous elliptic integrals. We conclude with the remark that the summation involved in the determination of the discrete hyperelliptic integrals may
Figure 2. The (discrete) elliptic integrals $\phi^D$ (points) and $\phi^O$ (mesh) plotted as functions of the differences $n^2 - n^1$ and $n^3 - n^2$.

be reformulated, so that it becomes transparent that the discrete hyperelliptic integrals may be expressed in terms of generalized hypergeometric functions [43].

(b) Even number of branch points

It is natural to inquire as to the existence of contour integrals of the type (5.13) which may be regarded as the analogues of hyperelliptic integrals associated with an even number $2g + 2$ of branch points. These hyperelliptic integrals arise in connection with the multi-phase-averaged NLS equations [46]. In principle, the analogue of the solution (5.8) of the discrete Euler–Poisson–Darboux system, that is,

$$\tilde{\phi} = \prod_{k=0}^{g} p(\xi, n^{2k+1}, 0) \prod_{k=1}^{g+1} p \left( \xi, n^{2k}, \frac{1}{2} \right),$$

(5.20)

is still valid but it is seen that this ansatz does not lead to the distribution of poles and zeros in the case of odd ‘genus’ by formally letting $n^{2g+2} \to \infty$. However, this situation may be rectified by annihilating the poles of $\tilde{\phi}$ and introducing new poles in the ‘non-singular’ regions by multiplication of $\tilde{\phi}$ by an appropriate function of $\xi$ which has zeros and poles at half-integers and integers, respectively. By virtue of the symmetries of the Gamma function, it turns out natural to introduce the ‘complementary’ function

$$\varphi = -\tilde{\phi} \cot \pi \xi.$$  

(5.21)

Indeed, in terms of the ‘complementary’ solution

$$q(\xi, n, \nu) = \frac{\Gamma(n + \nu - \frac{1}{2} - \xi)}{\sqrt{\pi} \Gamma(n + \nu - \xi)}$$

(5.22)
of the difference equation (4.25) which is related to $p(\xi, n, \nu)$ by

$$q(\xi, n, \nu) = p(\xi, n, \nu) \tan \pi (\xi - \nu), \quad (5.23)$$

it is readily verified that

$$\varphi = q(\xi, n^{2k+2}, 1/2) \prod_{k=0}^{q} p(\xi, n^{2k+1}, 0) \prod_{k=1}^{q} p(\xi, n^{2k}, 1/2). \quad (5.24)$$

Hence, the poles and zeros are distributed as required, that is, there are no zeros or poles in the intervals $(n^{2k-1}, n^{2k})$. It is noted that, for convenience, the scaling of $q$ has been chosen in such a manner that it approximates the function $(x - \zeta)^{-1/2}$ rather than $(\zeta - x)^{-1/2}$ in the sense of (4.27). Furthermore, up to a sign, $\varphi$ is symmetric in $p$ and $q$ due to the identity

$$p(\xi, n, 0)p(\xi, m, 1/2) = -q(\xi, n, 0)q(\xi, m, 1/2) \quad (5.25)$$

for any integers $m$ and $n$. Once again, in the simplest case $g = 1$, the contour integral (5.16), where the contour $b_1$ passes anticlockwise through the intervals $(n^1 - 1, n^2)$ and $(n^3 - 1, n^4)$, turns out to be a very good approximation of the corresponding elliptic integral

$$\phi^0 = \frac{2}{\pi \sqrt{(x^3 - x^1)(x^4 - x^2)}} \kappa \left\{ \sqrt{(x^3 - x^1)(x^4 - x^2)} \right\} \quad (5.26)$$

and

$$x^1 = \delta n^1, \quad x^2 = \delta \left( n^2 + 1/2 \right), \quad x^3 = \delta n^3, \quad x^4 = \delta \left( n^4 + 1/2 \right)$$

valid in the classical continuous case. In general, discrete $b$-cycles are defined as closed paths of anticlockwise orientation passing through the pairs of intervals $(n^{2k-1}, n^{2k})$ and $(n^{2k+1} - 1, n^{2k+2})$ for $k = 1, \ldots, g$.

6. Perspectives

We conclude with a selection of open problems which naturally arise in connection with the theory presented in this paper. For instance, it has been pointed out in references [2,22,24,47] that the theory of semi-Hamiltonian systems of hydrodynamic type is closely related to the analysis of the critical points of appropriately chosen functions. In the current context, if $\psi$ is an eigenfunction satisfying the discrete conjugate net equations (4.22) and $\{\mu_i\}$ are associated sets of adjoint eigenfunctions, then one may introduce the corresponding Combescure transforms $\psi_\alpha$ and $\tilde{\Theta}$ according to

$$\Delta_i \psi_\alpha = \lambda_\alpha^i \Delta_i \psi, \quad \Delta_i \tilde{\Theta} = \mu_i^i \Delta_i \psi. \quad (6.1)$$

The key function $\Theta$ is now defined by

$$\Theta = x \psi + \sum_{\alpha=1}^{N} t^\alpha \psi_\alpha - \tilde{\Theta}, \quad (6.2)$$

where, $a$ priori, $x$ and $t^\alpha$ are merely parameters. In analogy with the continuous case, critical points $n_c$ of the function $\Theta$ are defined as points $n$, where the ‘discrete derivatives’ of $\Theta$ vanish, that is, $\Delta_i \Theta|_{n=n_c} = 0$. Accordingly, we obtain

$$x \Delta_i \psi(n_c) + \sum_{\alpha=1}^{N} t^\alpha \Delta_i \psi_\alpha(n_c) - \Delta_i \tilde{\Theta}(n_c) = 0, \quad (6.3)$$

so that the definitions (6.1) imply that

$$x + \sum_{\alpha=1}^{N} t^\alpha \lambda_\alpha^i (n_c) - \mu_i^i (n_c) = 0. \quad (6.4)$$
The latter relate $x$ and $t^\alpha$ to $n_c$ in the same manner (with the index on $n_c$ being dropped) as the algebraic system (3.8) which gives rise to the discrete generalized hodograph equations (3.9). The implications of this observation are currently being investigated.

In the preceding, we have regarded ‘complete’ hyperelliptic integrals as functions of their branch points $x_i$ and, in this context, put forward a canonical definition of their discrete analogues. It is natural to inquire as to the existence of similar analogues of ‘incomplete’ hyperelliptic integrals and their associated differential equations. For instance, in the classical case, elliptic integrals are related by inversion to the differential equation

$$\frac{d\xi}{ds} = \sqrt{(\xi - x_1)(\xi - x_2)(\xi - x_3)}$$

(6.5)

which essentially defines the elliptic Weierstrass $\wp$ function [43]. It is evident that the approach pursued in this paper suggests that one should examine in detail the properties of the differential equation

$$\frac{d\xi}{ds} = \delta^{3/2} \frac{\Gamma(\xi - n^1 + \frac{3}{2})}{\Gamma(\xi - n^1 + 1)} \frac{\Gamma(\xi - n^2 + \frac{3}{2})}{\Gamma(\xi - n^2 + 1)} \frac{\Gamma(\xi - n^3 + \frac{3}{2})}{\Gamma(\xi - n^3 + 1)}, \quad \xi = \xi(s)$$

(6.6)

which may be regarded as a one-parameter deformation of the classical differential equation (6.5). The latter is retrieved in the usual limit $\delta \to 0$.

In §5, we have confined ourselves to a detailed discussion of the relevance of the discrete Euler–Poisson–Darboux-type system (4.1) for $\epsilon_i = \frac{1}{2}$. It is easy to see that, in the classical case, separable solutions of the Euler–Poisson–Darboux system (4.2) for $\epsilon_i = 1/M$, where $M$ is a positive integer, are obtained in terms of the superelliptic $(M, N)$-curves

$$y^M = \prod_{i=1}^{N}(\xi - x^i).$$

(6.7)

As in the hyperelliptic case $M = 2$, the corresponding superelliptic integrals are relevant in the theory of Whitham-type equations. For instance, trigonal curves $(3, N)$ appear in connection with the Benney equations and the dispersionless Boussinesq hierarchy (see, e.g. [48,49] and references therein). It is therefore desirable to investigate whether it is possible to extend the theory developed in this paper to define canonical discrete analogues of superelliptic integrals and associated discrete characteristic speeds of Whitham type.

Acknowledgements. B.G.K. acknowledges support by the PRIN 2010/2011 grant no. 2010JJ4KBA_003. W.K.S. expresses his gratitude to the DFG Collaborative Research Centre SFB/TRR 109 Discretization in Geometry and Dynamics for its support and hospitality.

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