PAPER

Equivalence and superposition of real and imaginary quasiperiodicities

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Abstract

We take non-Hermitian Aubry–André–Harper models and quasiperiodic Kitaev chains as examples to demonstrate the equivalence and superposition of real and imaginary quasiperiodic potentials (QPs) on inducing localization of single-particle states. We prove this equivalence by analytically computing Lyapunov exponents (or inverse of localization lengths) for systems with purely real and purely imaginary QPs. Moreover, when superposed and with the same frequency, real and imaginary QPs are coherent on inducing the localization, in a way which is determined by the relative phase between them. The localization induced by a coherent superposition can be simulated by the Hermitian model with an effective strength of QP, implying that models are in the same universality class. When their frequencies are different and relatively incommensurate, they are incoherent and their superposition leads to less correlation effects. Numerical results show that the localization happens earlier and there is an intermediate mixed phase lacking of mobility edge.

1. Introduction

Ever since the seminal work by Anderson in 1958, quantum localization has been a central topic in condensed matter physics [1, 2]. According to scaling theory, an infinitesimal random disorder localizes all single-particle states in one dimension (1D) [3]. However, as an intermediate case between disordered and periodic systems, quasiperiodic ones exhibit distinct behaviors and may support localization phase transitions. This is due to intrinsic spatial correlations of quasiperiodicities, which are absent for the true disorder. A well-known example is the Aubry–André–Harper (AAH) model [4], which undergoes a transition from metal to insulator phases at a finite strength of the quasiperiodic potential (QP), guaranteed by a self-duality. The AAH model and its various extensions have been theoretically studied extensively [5–12], and experimentally realized in a variety of systems, such as ultracold atoms [13–19] and photonic crystals [20, 21].

On the other hand, there has been growing interest in non-Hermitian physics recently [22, 23]. Non-Hermiticity originates from exchanges of energy and/or particles with environment, and is embodied in the Hamiltonian as nonreciprocal hoppings, complex potentials, etc. It leads to various unique phenomena, such as parity-time (PT) symmetry breaking [24, 25], non-Hermitian topology [26, 27], and skin effect [28–35]. Quantum localization also has been studied in non-Hermitian disordered [36–43] and quasiperiodic [44–63] systems. By extending to the non-Hermitian realm, systems gain extra degrees of freedom and may host novel localization phenomena, such as nonreciprocal hopping induced delocalization [36–38], and new universality classes of Anderson transitions [42]. However, the relation and difference between localizations of Hermitian and non-Hermitian systems are unrevealed yet. In particular, the extension of QP from real to complex results in various AAH models with both real and imaginary QPs. Although several special cases have been studied before [44, 57, 61, 64], showing similar localization
behaviors, the extension naturally raises following fundamental questions. First, what are the roles on inducing localization of states, played by purely real and corresponding imaginary QPs? Are they simply equivalent? Note that they have distinct physical origins, and spectra of systems with them can be quite different. Second, when superposed, whether real and imaginary parts of complex QPs are coherent on inducing the localization? How does the superposition affect the spatial correlations? Furthermore, is it possible to simulate in the Hermitian system, the localization induced by a superposition?

In the paper, we attempt to address above questions by studying the localization of 1D non-Hermitian AAH models with both real and imaginary QPs. Applying Avila’s global theory of one-frequency Schrödinger operators [65] and Thouless’s result [66], we analytically prove that purely real and imaginary QPs result in the same localization length and phase transition point, which implies that they are equivalent on inducing localization of states. When superposed, superposition principles for real and imaginary QPs with the same frequency are analytically established. Meanwhile, the localization induced by their coherent superposition is examined. Furthermore, the incoherent superposition of them with different and relatively incommensurate frequencies is numerically studied, which shows less correlation effects.

2. Model and Hamiltonian

We consider the 1D non-Hermitian AAH model with and without $p$-wave pairing, described by the following Hamiltonian

$$H = \sum_j (-t c_j^\dagger c_{j+1} + \Delta c_j c_{j+1} + \text{h.c.}) + \sum_j V_j c_j^\dagger c_j,$$

where $c_j^\dagger$ ($c_j$) is the creation (annihilation) operator of a spinless fermion at site $j$. $t$ is the hopping amplitude and sets the unit of energy ($t = 1$). $\Delta$ is the $p$-wave pairing amplitude, which can be made positive real [67].

We choose a general form of QP, which is given by

$$V_j = 2VR \cos(2\pi\beta j + \theta) + 2iVR \cos(2\pi\beta j + \theta + \delta),$$

with $VR$ and $V_I$ strengths of the real and imaginary QPs, respectively. $\delta$ is the relative phase between the real and imaginary QPs. It characterizes the relative displacement between these two QPs along the spatial dimension, which determines how they superpose. It affects the spatial correlation of the resulting complex QP and the $\mathcal{PT}$ symmetry of the model. Thus, spectrum and localization properties depend on it crucially.

$\theta$ is a global phase accounting for the global shift of two QPs along the spatial dimension, which is trivial on the localization, and we will set $\theta = 0$ if not specified. In the presence of phases $\theta$ and $\delta$, we can set both $VR$ and $V_I$ positive real. $\beta_R$ and $\beta_I$ are irrational numbers characterizing quasiperiodicities of the real and imaginary QPs, respectively. In this paper, we choose the metallic mean family [68] of irrational numbers. Considering a generalized $k$-Fibonacci sequence given by $F_m = kF_{m-1} + F_{m-2}$ with $F_0 = 0$ and $F_1 = 1$, the limit $\beta = \lim_{m \to \infty} F_{m+1}/F_m$ with $k = 1, 2, 3, \ldots$ yields the metallic mean family. The first three members of the family, which are used in this paper, are the well-known golden mean $\beta_k = (\sqrt{5} - 1)/2$, the silver mean $\beta_s = \sqrt{2} - 1$, and the bronze mean $\beta_b = (\sqrt{13} - 3)/2$, respectively. Each number of the family satisfies the relation $k\beta + \beta^2 = 1$.

When $V_I = 0$, the model reduces to the Hermitian quasiperiodic Kitaev chain [6], which further reduces to the classic AAH model [4] when $\Delta = 0$. The classic AAH model undergoes a delocalization–localization phase transition at the critical point $VR = t$. When $VR < t$ the system is in the extended phase where all single-particle states in spectrum are extended, whereas it is in the localized phase and all states are localized when $VR > t$. With a finite $\Delta$, the critical point expands to a critical phase, ranging from $|t - \Delta|$ to $t + \Delta$, and the model turns into the Hermitian quasiperiodic Kitaev chain [69]. As for the non-Hermitian extensions, localization phenomena in a few models with specific QPs, which can be rewritten as the superpositions in equation (2) with $\beta_R = \beta_I = \beta$, have been studied before. The potential $2V \cos(2\pi\beta j + ih)$ studied in [44, 61, 70] corresponds to $VR = V \cosh(h)$, $V_I = V \sinh(h)$, and $\delta = -\pi/2$; the potential $2V e^{2i\phi} \cos(2\pi\beta j)$ studied in [57, 64] has $VR = V$ and $\delta = -\pi/2$; and the potential $2\lambda \cos(2\pi\beta j) + 2i\lambda \sin(2\pi\beta j)$ numerically studied in [59] corresponds to $\delta = \pm\pi/2$, $VR = |\lambda|$,

3. Equivalence and superposition in AAH models

We first examine the case without $p$-wave pairing ($\Delta = 0$), which is a typical non-Hermitian AAH model. Before embarking on the main study, we briefly explore the $\mathcal{PT}$ symmetry of the model and the
real–complex transition of spectra. The model is $PT$-symmetric only when $\beta_R = \beta_I$ and $\delta = (2m + 1)\pi/2$, $m \in \mathbb{Z}$ or when $V_R = 0$. Otherwise, it is not $PT$-symmetric and its spectrum is complex. The case of $V_R = 0$ can be seen as a special case of $\delta = (2m + 1)\pi/2$, given that $\delta$ does not affect the $PT$ symmetry when $V_R = 0$. When $\delta = (2m + 1)\pi/2$, the potential reduces to $2V_R \cos(2\pi j \beta_I) \pm 2V_I \sin(2\pi j \beta_I)$. Spectra of the AAH model with it were studied in [59]. Spectrum is real only when the system is in the extended phase and $V_R \geq V_I$. The same things happen when $\Delta \neq 0$.

To prove the equivalence between purely real and imaginary QPs on inducing the localization, we analytically compute Lyapunov exponents (LEs) (or inverse of localization lengths) of single-particle states. Given an eigenstate $|\Phi\rangle = \sum_j \phi_j |j\rangle$, the Schrödinger equation of amplitudes $\phi_j$ in transfer matrix form is written as

$$T_j \left[ \begin{array}{c} \phi_{j+1} \\ \phi_j \end{array} \right] = \left[ \begin{array}{cc} (V_j - E)/t & -1 \\ 1 & 0 \end{array} \right] \left[ \begin{array}{c} \phi_{j} \\ \phi_{j-1} \end{array} \right],$$

with $E$ the eigenenergy. The LE of state is computed by

$$\gamma(\theta) = \lim_{L \to \infty} \frac{1}{L} \ln \| T_j(\theta + \varepsilon t) \|,$$

where $\| \cdot \|$ denotes the norm of a matrix and $L$ is the number of lattice sites. Since our computation relies on Avila’s global theory of one-frequency analytical $SL(2, \mathbb{C})$ cocycle [65], complexification of the global phase $\theta$ in $T_j$ has been performed. Let $\varepsilon$ go to infinity, then a direct calculation yields transfer matrices

$$T_j^{R(I)}(\varepsilon) = e^{-i2\pi \beta_R j \varepsilon - i\theta} \left[ \begin{array}{cc} V_R (iV_I e^{-i\varepsilon})/t & 0 \\ 0 & 0 \end{array} \right] + o(1),$$

for purely real and imaginary QPs, respectively. Thus we obtain $\gamma^{R(I)} = \varepsilon + \ln |V_R(t)|$. According to Avila’s global theory, as a function of $\varepsilon$, $\gamma_\varepsilon$ is a convex, piecewise linear function with integer slopes. Moreover, the energy $E$ does not belong to the spectrum, if and only if $\gamma_{\varepsilon=0}(E) > 0$ and $\gamma_\varepsilon$ is an affine function in the neighborhood of $\varepsilon = 0$. It follows that

$$\gamma^{R(I)} = \max \{ \ln |V_R(t)|, 0 \},$$

for AAH models with purely real and purely imaginary QPs, respectively. The LEs do not depend on the energy $E$, frequencies $\beta_R$ and $\beta_I$, and phases $\theta$ and $\delta$. More importantly, purely real and imaginary QPs with the same strength have exactly the same LE, implying that they are equivalent on inducing localization of states. When $V_R(t)/t < 1$, the LE $\gamma^{R(I)} = 0$ and states are extended, whereas states are localized when $V_R(t)/t > 1$. The localization phase transition happens at $V_R(t)/t = 1$. The equivalence and parameter-(in)dependence are further verified numerically. We adopt exponentially decaying wave functions $\phi_j^{\pm} = \exp(-\gamma_n(j - j_0))$ with $j_0$ the localization center, $n$ the index of states, and $\gamma_n$ the LE. Extracted by fitting numerical single-particle states with the above wave functions, the mean LEs $\gamma = \sum_n \gamma_n/L$ of systems with purely real and purely imaginary QPs are shown in figure 1. All curves coincide with the theoretical prediction in equation (6).
Having established the equivalence, we study the superposition of real and imaginary QPs with the same frequency ($\beta_R = \beta_I = \beta$). By applying Avila’s global theory, we obtain the LE in the presence of both QPs

$$\gamma = \max \left\{ \frac{1}{2} \ln \left( \frac{V_R^2 + V_I^2 + 2V_R V_I |\sin \delta|}{t^2} \right), 0 \right\},$$

(7)

(see appendix A for analytical computation). Thus, localization phase transition points are determined by

$$V_R^2 + V_I^2 + 2V_R V_I |\sin \delta| = t^2.$$  

(8)

These clearly reflect the coherent superposition of real and imaginary QPs, where their relative phase $\delta$ plays a decisive role. The LE and condition for phase transition are periodic functions of the phase $\delta$ with a period $\pi$. Moreover, they are symmetric with respect to $\delta = m\pi/2, m \in \mathbb{Z}$. When $\delta = 0$, we have LE

$$\gamma = \ln \left( \sqrt{V_R^2 + V_I^2} / t \right)$$

in the localized phase, and phase transition points are determined by $V_R^2 + V_I^2 = t^2$, which is part of the unit circle (red dash dot line in figure 2(b)). When $\delta = \pi/2$, the LE

$$\gamma = \max \{ \ln(V_R + V_I) / t, 0 \}$$

and phase transitions occur at $V_R + V_I = t$ (blue dash dot straight line in figure 2(b)). With substituting $V_R = |\lambda_r|$ and $V_I = |\lambda_i|$, this equation of the straight phase transition line is exactly the one numerically obtained in [59]. With a general $\delta$, the phase transition line varies between the above two cases back and forth. The coherent superposition induces the localization transition earlier [at smaller magnitudes of the vector $(V_R, V_I)$] than purely real or imaginary QP solely, implying that the superposition leads to less correlation effects. Effects of real and imaginary QPs are equal in the superposition. Additionally, near a phase transition point the LE in localized phase scales linearly with the distance from the phase transition point on the $(V_R, V_I)$ plane. Thus the non-Hermitian AAH model considered here is in the same universality class as the Hermitian one, with a localization exponent $\alpha = 4, 5$. Furthermore, note that equations (6) and (7) are in the same form, and the localization induced by a superposition of real and imaginary QPs can be simulated by the Hermitian AAH model with an effective strength of QP $|V_{loc}| = \sqrt{V_R^2 + V_I^2 + 2V_R V_I |\sin \delta|}$. The above analytical results are consistent with numerical simulations. In figure 2(a) we show examples of mean LEs $V_R$ for systems with different $V_I$ and $\delta$, which agree with equation (7). To explore the localization in detail, we further compute inverse of the participation ratios (IPRs) and fractal dimensions of single-particle states. For a normalized state the IPR is defined by $P = \sum_j |\phi_j|^4$. In general, the IPR $P \propto L^{-\alpha}$ with the fractal dimension $0 \leq \alpha \leq 1$. For an extended state, $P \propto 1/L$ and $\alpha = 1$, whereas the IPR approaches 1 and $\alpha = 0$ for a localized state. States with $0 < \alpha < 1$ are critical, and have multi-fractal properties. Extracted by the box-counting method [71], a typical mean fractal dimension $MFD = \sum_n \alpha_n / L$ in $(V_R, V_I)$ plane is shown in figure 2(b). When both $V_R$ and $V_I$ are small, the system is in the extended phase with $MFD \simeq 1$, whereas it is in the localized phase with $MFD \simeq 0$ when $V_R$ and/or $V_I$ are large. Critical states only exist at the boundary between two phases, which agrees with the theoretical prediction in equation (8) (black dash line in figure 2(b)).

![Figure 2](image_url)

**Figure 2.** Superposition in AAH models: same frequency case. (a) Mean LEs vs $V_R$ for systems with different $V_I$ and $\delta$. (b) Mean fractal dimension in the $(V_R, V_I)$ plane for the system with $\delta = 0.2\pi$. Black dash line represents equation (8) with $\delta = 0.2\pi$, red dash dot line is described by equation $V_R^2 + V_I^2 = t^2$, and blue dash dot straight line corresponds to equation $V_R + V_I = t$. Other parameters: $\beta_R = \beta_I = \beta_\delta = 610/987$, and $L = 987$. The superposition of real and imaginary QPs with different frequencies only can be numerically studied. We focus on the case where two frequencies are incommensurate to each other, chosen from the metallic mean family. The localization is basically independent of frequencies $\beta_R/\beta_I$ and phases $\theta$ and $\delta$, showing the incoherent nature of the superposition (see supplemental material [71] for details). Quantities
characterizing the localization, such as the mean IPR, MFD, and mean LE, are functions of \((V_R, V_I)\) only, and a typical one is shown in figure 3(a), which is defined by [12]

\[
\kappa = \log_{10}(\text{MIPR} \times \text{MNPR}),
\]

(9)

where MIPR = \(\sum_n P_n/L\) is the mean IPR. The normalized participation ratio is defined as the ‘inverse’ of IPR for a single-particle state, and the mean one is MNPR = \(\sum_n 1/(P_nL^2)\). The quantity \(\kappa\) is introduced to distinguish the mixed phase (the intermediate region bounded by dash lines, where \(\kappa\) is finite) from the fully extended and localized phases [blue regions, where \(\kappa \propto \log_{10}(1/L)\)]. In the mixed phase, the system consists of extended, localized, and even critical single-particle states. The boundary between mixed and localized phases is well described by the empirical equation \(V_R + V_I = t\) (black dash straight line in figure 3(a)), while the lower left quarter of circle \((V_R - t)^2 + (V_I - t)^2 = t^2\) (red dash dot line in figure 3(a)) approximately describes the boundary between extended and mixed phases. Compared to the same frequency case (figure 2(b)), the superposition of real and imaginary QPs with different frequencies induces the localization earlier, thus has weaker correlation effects. To further explore details of the localization, we present distributions of IPRs in figure 3(b) for systems with different strengths of QPs. States are arranged in ascending order of the real parts of energies. When both \(V_R\) and \(V_I\) are small, the system is in the extended phase with \(P \propto 1/L\). However, numerically there are some localized defect states caused by the mismatch of rational approximations of \(\beta_R\) and \(\beta_I\), and cusps exist in the upper panel of figure 3(b). In the mixed phase, the IPR varies from a few tenths to \(1/L\), and the system contains all three kinds of states. There is no mobility edge, a critical energy separating extended and localized states in spectrum, since no large sudden change exists in distributions of IPRs (see the third panel of figure 3(b)). The sudden change, which indicates a dramatic change in the localization of states, is the characteristic feature of the presence of a mobility edge [11, 12].

4. Equivalence and superposition in quasiperiodic Kitaev chains

Turning on the \(p\)-wave pairing (\(\Delta > 0\)), the model can be thought of as the Kitaev chain subjected to both real and imaginary QPs. It can be diagonalized by the Bogoliubov–de Gennes (BdG) transformation \(\eta^\dagger = \sum_j (\phi_j \chi_A^j + i\psi_j \chi_B^j)\), where \(\chi_A^j(B)\) are Majorana operators, and \(\phi (\psi)\) play the role of single-particle states (see appendix B). By generalizing to the non-Hermitian realm Thouless’s result relating LE to the density of state [66], the analytical LEs are

\[
\gamma^{R(I)}_K = \max \left\{ \ln \frac{V^{R(I)}}{t + \Delta}, 0 \right\},
\]

(10)

for quasiperiodic Kitaev chains with purely real and purely imaginary QPs, respectively (see appendix B for analytical computation). Purely real and imaginary QPs are still equivalent on inducing the localization, and a numerical verification is presented in figure 4. The LEs are independent of the energy \(E\), frequencies \(\beta_R/\beta_I\) and phases \(\theta\) and \(\delta\). When \(V^{R(I)} > t + \Delta\), the system is in the localized phase with \(\gamma_K > 0\). However, it
Figure 4. Equivalence in quasiperiodic Kitaev chains. Mean LEs of systems with purely real and purely imaginary QPs. Parameters: $\Delta = 0.5$, $\delta = 0$, $\beta_g \simeq 610/987$, $\beta_l \simeq 408/985$, $\beta_b \simeq 360/1189$, and $L = 987, 985$, or $1189$.

Figure 5. Superposition in quasiperiodic Kitaev chains: same frequency case. (a) Mean LEs vs $V_R$ for systems with $\Delta = 0.5$ and different $V_I$ and $\delta$. (b) Mean fractal dimension in the $(V_R, V_I)$ plane for the system with $\Delta = 0.2$ and $\delta = \pi/4$. Black and blue dash lines correspond to equations (12) and (13), respectively. Other parameters: $\beta_R = \beta_I = \beta_g \simeq 610/987$, and $L = 987$.

is not necessarily in the extended phase when $V_{R(I)} < t + \Delta$, despite $\gamma_K = 0$. There is an extra transition point separating extended and critical phases at $V_{R(I)} = |t - \Delta|$. In the intermediate critical phase, states are critical and have fractal dimensions $0 < \alpha < 1$ (see the vertical and horizontal axes of figure 5(b)).

When real and imaginary QPs are superposed and have the same frequency, the non-Hermitian extension of Thouless’s result leads to the LE of quasiperiodic Kitaev chain

$$\gamma_K = \max\left\{ \ln \frac{V_R^2 + V_I^2 + 2V_R V_I |\sin \delta|}{t + \Delta}, 0 \right\}$$

(11)

(see appendix B for analytical computation). Localization phase transition points are determined by condition

$$V_R^2 + V_I^2 + 2V_R V_I |\sin \delta| = (t + \Delta)^2.$$  

(12)

Comparing equations (7) and (8) with above two equations, respectively, the difference is that $t$ is replaced by $t + \Delta$. Thus, conclusions drawn in AAH models still hold for the quasiperiodic Kitaev chain. The LE and condition for phase transition are verified by numerical simulations, which are presented in figure 5. An example of mean fractal dimension is shown in figure 5(b). Different from AAH models, there is a critical phase between extended and localized phases. The location of extended-critical phase transition is well described by equation

$$V_R^2 + V_I^2 + 2V_R V_I |\sin \delta| = (t - \Delta)^2,$$  

(13)

which is similar to equations (8) or (12).

Finally, we study the superposition of QPs with two relatively incommensurate frequencies. The localization is still independent of $\theta$, $\delta$, and $\beta_{R/I}$, and thus QPs are incoherent. In figure 6 we present the quantity $\kappa$ in $(V_R, V_I)$ plane. Two lines are added, corresponding to empirical equations that are similar to
the ones for AAH models and obtained by substituting $t$ with $t \pm \Delta$. The intermediate region corresponds to the mixed phase, and the localization happens before the extended-critical phase transition shown in the same frequency case. The superposition of real and imaginary QPs with different frequencies leads to weaker correlations. As strengths of QPs increase, more and more states become localized in the mixed phase (see supplemental material [71] for more localization details).

5. Conclusion and discussion

Through our work proves the equivalence between purely real and imaginary QPs, and studies the superposition of them on inducing localization of states. Specifically, LEs are analytically computed for AAH models by applying Avila’s global theory, and for quasiperiodic Kitaev chains by generalizing to non-Hermitian realm Thouless’s result relating LE to the density of state. Purely real and imaginary QPs induce the same localization. A sign of the equivalence also has been shown in a study of the localization at energy $E = 0$ [43]. In the presence of both QPs and with the same frequency, they are coherent on inducing the localization, in a way which is determined by the relative phase between them. Localization phase diagrams are symmetric with respect to $V_R$ and $V_I$. The localization induced by a coherent superposition can be simulated by the Hermitian model with an effective strength of QP, and both Hermitian and non-Hermitian models are in the same universality classes. The symmetry in phase diagrams was also reported in a three dimensional non-Hermitian Anderson model, but the universality class changes as the superposition alters [43]. With different and relatively incommensurate frequencies, QPs are incoherent and less correlated, and induce the localization earlier. Compared to previous studies on non-Hermitian AAH models with specific QPs, our study on the superposition of independent real and imaginary QPs gives more flexibility to the experimental design of complex QPs and observation of the localization in non-Hermitian systems.

The physics of equivalence and superposition can be experimentally tested in photonic waveguides and electric circuits. Photonic waveguides have been routinely used to demonstrate the localization of light [72, 73]. In the tight-binding limit, propagation of classical light is governed by $i \partial \phi_j/\partial z = \kappa_j \phi_j + \sum_{l \neq j} t_{jl} \phi_l$, which resembles the Schrödinger equation. $\kappa_j$ is the refractive index of $j$th waveguide, which plays the role of complex potential. $t_{jl}$ is the hopping between different waveguides. In photonics, the equation specified for the single-particle physics of Kitaev chain is even termed ‘photonic Kitaev chain’ [74]. In electric circuits the single-particle eigenvalue problem is simulated by the Kirchhoff’s current law $I_a = \sum_{b=1}^L J_{ab} V_b$, where the Laplacian of circuit $J$ acts as the effective Hamiltonian, and $I_a$ and $V_a$ are the current and voltage at node $a$ [23], respectively. On-site complex potentials are realized by grounding nodes with proper resistors.
It would be interesting to extend the study of equivalence and superposition to other systems, such as in the presence of disorders, mobility edges, and even interactions, where non-Hermitian many-body localization occurs.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Computation of equations (6)–(8)

Given a single-particle eigenstate $|\Phi\rangle = \sum_{j=1}^{L} \phi_j |\psi\rangle$ of the non-Hermitian AAH model, the amplitude $\phi$ satisfies eigenequation

$$-t\phi_{j-1} - t\phi_{j+1} + V_j \phi_j = E \phi_j,$$

where $E$ is the energy of state, and the complex QP $V_j$ was given in equation (2). In the language of transfer matrix, the eigenequation can be rewritten as

$$\begin{pmatrix} \phi_{j+1} \\ \phi_j \end{pmatrix} = T_j \begin{pmatrix} \phi_j \\ \phi_{j-1} \end{pmatrix},$$

with the transfer matrix

$$T_j = \begin{pmatrix} V_j - E & -1 \\ t & 0 \end{pmatrix}.$$  

Then, the transfer matrix of the whole system is $T_L = \prod_{j=1}^{L} T_j$. LE (or inverse of localization length) of the single-particle state is defined by

$$\gamma(E) = \lim_{L \to \infty} \frac{1}{L} \ln \| T_L \|,$$

where $\| A \|$ denotes the norm of matrix $A$, which is defined by the largest absolute value of its eigenvalues.

In the case where real and imaginary QPs have the same frequency, i.e. $\beta_R = \beta_I = \beta$, we apply Avila’s global theory of one-frequency analytical SL(2, $\mathbb{C}$) cocycle [65]. We first carry out an analytical continuation of the global phase, i.e., $\theta \to \theta + i \varepsilon$, in the computation of LE. Thus, the complex QP becomes

$$V_j(\varepsilon) = 2V_R \cos(2\pi \beta j + \theta + i \varepsilon) + 2iV_I \cos(2\pi \beta j + \delta + \theta + i \varepsilon).$$  

(A5)

In the absence of ambiguity, we use the same symbol for a quantity and its analytical continuation. In the limit $\varepsilon \to +\infty$, a direct computation yields

$$T_j(\varepsilon \to +\infty) = e^{-i2\pi \beta j + \theta + \varepsilon} \begin{pmatrix} V_R + iV_I e^{-i\delta} & 0 \\ t & 0 \end{pmatrix} + o(1),$$

(A6)

which leads to

$$\gamma(E, \varepsilon \to +\infty) = \varepsilon + \ln \left| \frac{V_R + iV_I e^{-i\delta}}{t} \right|. $$

(A7)

On the other side, in the limit $\varepsilon \to -\infty$, we obtain

$$T_j(\varepsilon \to -\infty) = e^{i2\pi \beta j + \theta - \varepsilon} \begin{pmatrix} V_R + iV_I e^{i\delta} & 0 \\ t & 0 \end{pmatrix} + o(1),$$

(A8)

and the corresponding LE is

$$\gamma(E, \varepsilon \to -\infty) = -\varepsilon + \ln \left| \frac{V_R + iV_I e^{i\delta}}{t} \right|.$$  

(A9)
Avila's global theory shows that, as a function of $\varepsilon$, $\gamma(E, \varepsilon)$ is a convex, piecewise linear function with integer slopes. This implies $\gamma(E, \varepsilon) \geq \max\{|\ln(V_R + iV_I e^{-i\theta})/t|, \ln(V_R + iV_I e^{i\theta})/t|\}$, where $\gamma(E, \varepsilon) \geq 0$ has been used. Moreover, according to Avila's global theory, the energy $E$ does not belong to the spectrum, if and only if $\gamma(E, \varepsilon) > 0$ and $\gamma(E, \varepsilon)$ is an affine function in the neighborhood of $\varepsilon = 0$. Consequently, we obtain that if the energy $E$ lies in the spectrum, LE of the single-particle state is

$$\gamma = \max\{|\ln(V_R + iV_I e^{-i\theta})/t|, \ln(V_R + iV_I e^{i\theta})/t|\} \geq \max\left\{\frac{1}{2} \ln \frac{V_R^2 + V_I^2 + 2V_R V_I |\sin \delta|}{t^2}, 0\right\}.$$ (A10)

$$\gamma > 0$$ indicates that the state is localized, while it is extended or critical when $\gamma = 0$. Localization phase transition points are determined by condition

$$\frac{V_R^2 + V_I^2 + 2V_R V_I |\sin \delta|}{t^2} = 1.$$ (A12)

The LE and phase transition points are energy-independent. Thus there is no mobility edge.

To prove the equivalence between purely real and imaginary QPs, we just need to set $V_I = 0 = V_R$ in equations (A11) and (A12). Thus, the LEs are

$$\gamma^{R(I)} = \max\{|\ln |V_{R(I)}|/t|, 0\},$$ (A13)

and localization phase transitions happen at $V_{R(I)}/t = 1$, for systems with purely real and purely imaginary QPs, respectively.

**Appendix B. Computation of equations (10)–(12)**

With a finite amplitude of $p$-wave pairing ($\Delta > 0$), the model can be thought of as the Kitaev chain subjected to both real and imaginary QPs. It can be diagonalized by the BdG transformation

$$\eta_n^i = \sum_{j=1}^L \left[ c_{ij}^\dagger \chi_j^A + i\psi_j^H \chi_j^B \right],$$ (B1)

with $n = 1, \ldots, L$ the index of states. $\chi_j^A \equiv c_j^A + c_j^A$ and $\chi_j^B \equiv i(c_j - c_j^\dagger)$ are operators of two Majorana fermions belonging to physical site $j$. They satisfy relations $(\chi_j^A)^\dagger = \chi_j^\dagger$ and $\{\chi_j^A, \chi_j^\dagger\} = 2\delta_{jk}\delta_{\kappa\lambda}$ ($\kappa, \lambda = A, B$). Under the BdG transformation, the eigenvalue problem turns into

$$(M - N)(M + N)\phi_n = E_n^2 \phi_n,$$ (B2)

$$(M + N)(M - N)\psi_n = E_n^2 \psi_n,$$ (B3)

with vectors $\phi_n = [\phi_1^n, \phi_2^n, \ldots, \phi_L^n]^T$ and $\psi_n = [\psi_1^n, \psi_2^n, \ldots, \psi_L^n]^T$, and $E_n$ the eigenenergy. The symmetric and antisymmetric tridiagonal matrices $M$ and $N$ are

$$M = \begin{pmatrix} V_1 & -t & & -t \\ -t & V_2 & \ddots & \\ & \ddots & \ddots & -t \\ -t & & -t & V_L \end{pmatrix}, \quad N = \begin{pmatrix} 0 & -\Delta & & \\ \Delta & 0 & \ddots & \\ & \ddots & \ddots & -\Delta \\ -\Delta & & \Delta & 0 \end{pmatrix}.$$ (B4)

Solving above equations, we obtain the spectrum and all single-particle states $(\phi_n, \psi_n)$.

In order to compute the LE of quasiperiodic Kitaev chain, we generate to the non-Hermitian realm Thouless's result relating LE to the density of state [66]. According to Thouless, the LE of an eigenstate with energy in the neighborhood of $E_R$ is given by

$$\gamma_K = \int d\varepsilon \rho(\varepsilon) \ln |\varepsilon - E_R| - \ln(t'),$$ (B5)

where $\rho(\varepsilon)$ is the density of state. Due to the presence of $p$-wave pairing, we introduced a parameter $t'$ to reset the energy scale, which will be determined later. Moreover, numerical results show that LEs are energy-independent, and we will set $E_R = 0$ for the sake of simplicity. On the other hand, we define

$$g \equiv \ln |\det(H)| = \frac{1}{2} \ln |\det(H^2)| = \ln |\det(M + N)|.$$ (B6)
Equation (B3) and the fact \((M - N)^T = (M + N)\) have been used above. The tridiagonal matrix \(M + N\) is

\[
M + N = \begin{pmatrix}
-(t + \Delta) & -(t - \Delta) & 0 & \cdots & 0 \\
-(t - \Delta) & V_2 & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-(t + \Delta) & -(t - \Delta) & \cdots & V_L \\
0 & \cdots & \cdots & \cdots & -(t + \Delta)
\end{pmatrix}.
\] (B7)

Indicating by \(\lambda_1, \ldots, \lambda_L\) the eigenvalues of \(M + N\), we can rewrite it as

\[
g = \sum_{n=1}^{L} \ln |\lambda_n|.
\] (B8)

In the large-\(L\) limit, we replace the summation by integration and

\[
g = L \int \mathrm{d}\varepsilon \rho(\varepsilon) \ln |\varepsilon|.
\] (B9)

Now we obtain

\[
\int \mathrm{d}\varepsilon \rho(\varepsilon) \ln |\varepsilon| = \frac{g}{L} = \frac{\ln |\det(M + N)|}{L},
\] (B10)

and the relation between LE and the determinant of Hamiltonian

\[
\gamma_K = \frac{\ln |\det(M + N)|}{L} - \ln(t').
\] (B11)

To compute \(\det(M + N)\), we redefine parameters

\[
t_1 e^{\eta} \equiv t + \Delta, \quad t_1 e^{-\eta} \equiv |t - \Delta|.
\] (B12)

Then the tridiagonal matrix \(M + N\) turns into

\[
M + N = \begin{pmatrix}
-t_1 e^{\eta} & \xi t_1 e^{-\eta} & \cdots & 0 \\
\xi t_1 e^{-\eta} & V_2 & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-t_1 e^{\eta} & \cdots & \cdots & -t_1 & \xi t_1 e^{-\eta} \\
0 & \cdots & \cdots & \cdots & -(t_1 e^{\eta})
\end{pmatrix},
\] (B13)

with \(\xi\) a possible sign. Now we perform a similarity transformation

\[
S(M + N)S^{-1} = \begin{pmatrix}
-t_1 & \xi t_1 e^{-\eta} \\
\xi t_1 & V_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-t_1 e^{\eta} & 0 & \cdots & \cdots & -t_1 \\
0 & \cdots & \cdots & \cdots & \xi t_1 e^{\eta}
\end{pmatrix},
\] (B14)

with diagonal matrix \(S = \text{diag}(e^{\eta}, e^{2\eta}, \ldots, e^{L\eta})\). Then in the large-\(L\) limit,

\[
\det(M + N) = -t_1 e^{L\eta} + \det(H_1),
\] (B15)

where

\[
H_1 = \begin{pmatrix}
-t_1 \\
\xi t_1 & V_2 & \cdots \\
\vdots & \vdots & \ddots & \cdots & 0 \\
-t_1 e^{\eta} & 0 & \cdots & \cdots & -t_1 \\
0 & \cdots & \cdots & \cdots & \xi t_1 e^{\eta}
\end{pmatrix}.
\] (B16)

The matrix \(H_1\) describes the single-particle physics of the non-Hermitian AAH model with a possible sign difference between left and right hoppings. Following the same procedure shown in appendix A, the LE corresponding to matrix \(H_1\) is the same as for the non-Hermitian AAH model studied in the paper, i.e.

\[
\gamma_1 = \max \left\{ \ln \frac{\sqrt{V_R^2 + V_I^2} + 2V_R V_I |\sin \delta|}{t_1}, 0 \right\},
\] (B17)

when both real and imaginary QPs have the same frequency. To simplify the presentation, we will use \(V_{\text{eff}} = \sqrt{V_R^2 + V_I^2} + 2V_R V_I |\sin \delta|\) for short.
Now we apply Thouless’s result, inversely, to get \( \det(H_1) \). For the non-Hermitian AAH model, whose single-particle physics is described by the matrix \( H_1 \), the LE is

\[
\gamma_1 = \int \text{d}\varepsilon \rho(\varepsilon) \ln |\varepsilon| - \ln(t_1).
\]

(B18)

Following the same procedure shown above, we can get

\[
\gamma_1 = \frac{\ln |\det(H_1)|}{L} - \ln(t_1). 
\]

(B19)

Using equation (B17), we obtain

\[
|\det(H_1)|_{L \to \infty} = \max \left[ t_1^L, V_{\text{eff}}^L \right].
\]

(B20)

Substituting it back into equation (B15), we have

\[
\det(M + N)|_{L \to \infty} = -t_1^L e^{i\theta} + \zeta \max \left[ t_1^L, V_{\text{eff}}^L \right] 
\]

\[
= \zeta' \max \left[ (t_1 e^{i\theta})^L, V_{\text{eff}}^L \right] = \zeta' \max \left[ (t + \Delta)^L, V_{\text{eff}}^L \right],
\]

(B21)

with \( \zeta \) and \( \zeta' \) possible signs. Then from equation (B11), we obtain

\[
\gamma_K = \frac{\ln \left[ \max \left[ (t + \Delta)^L, V_{\text{eff}}^L \right] / t_1^L \right]}{L} 
\]

\[
= \begin{cases} 
\ln \left[ (t + \Delta) / t_1 \right], & t + \Delta > V_{\text{eff}}, \\
\ln \left[ V_{\text{eff}} / t_1 \right], & t + \Delta < V_{\text{eff}}.
\end{cases}
\]

(B22)

Considering that the LE must be zero when \( V_{\text{eff}} = 0 \) or at the phase transition point, we obtain \( t' = t + \Delta \). Finally, the LE of bulk single-particle states of the non-Hermitian quasiperiodic Kitaev chain is

\[
\gamma_K = \max \left\{ \ln \frac{V_R^2 + V_I^2 + 2V_R V_I \sin \delta}{(t + \Delta)^2}, 0 \right\}.
\]

(B23)

Localization phase transition points are determined by condition

\[
\frac{V_R^2 + V_I^2 + 2V_R V_I \sin \delta}{(t + \Delta)^2} = 1.
\]

(B24)

Compared with equations (A11) and (A12), the LE and condition for phase transition are in the same forms as for the AAH model, but with a replacement of \( t \) by \( t + \Delta \).

To prove the equivalence between purely real and imaginary QPs, we set \( V_I = 0 \) or \( V_R = 0 \) in equations (B25) and (B26). Consequently, the LEs are

\[
\gamma_K^{R(I)} = \max \left\{ \ln \left[ V_{R(I)} / (t + \Delta) \right], 0 \right\},
\]

and conditions for phase transition are \( V_{R(I)} / (t + \Delta) = 1 \), for quasiperiodic Kitaev chains with purely real and purely imaginary QPs, respectively.

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