The Baryon Isgur-Wise Function 
in the Large $N_c$ Limit

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Abstract

In the large $N_c$ limit, the $\Lambda_b$ and $\Lambda_c$ can be treated as bound states of chiral solitons and mesons containing a heavy quark. We show that the soliton and heavy meson are bound in an attractive harmonic oscillator potential. The Isgur-Wise function for $\Lambda_b \rightarrow \Lambda_c e^{-\nu_e}$ decay is computed in the large $N_c$ limit. Corrections to the form factor which depend on $m_N/m_Q$ can be summed exactly ($m_N$ and $m_Q$ are the nucleon and heavy quark masses). We find that this symmetry breaking correction at zero recoil is only 1%.
1. Introduction

In the heavy quark limit, the form factors for semileptonic $\Lambda_b \to \Lambda_c e^- \bar{\nu}_e$ decay \[1\] are characterized by a single universal function $\eta(v \cdot v')$

$$\langle \Lambda_c(v',s')|\bar{\psi}\gamma^\mu(1-\gamma_5)b|\Lambda_b(v,s)\rangle = \eta(v \cdot v') \overline{\psi}(v',s')\gamma^\mu(1-\gamma_5)u(v,s), \quad (1.1)$$

where $v^\mu$ and $v'^\mu$ are the four-velocities of the $\Lambda_b$ and $\Lambda_c$, respectively. The Isgur-Wise function $\eta(v \cdot v')$ \[2\] has logarithmic dependence on the heavy $b$ and $c$ quark masses which is calculable using perturbative QCD methods. The quark mass dependence can be put into a multiplicative factor \[3\]

$$\eta(v \cdot v') = C_{cb}(v \cdot v') \eta_0(v \cdot v'), \quad (1.2)$$

where

$$C_{cb}(v \cdot v') = \left[ \frac{\alpha_s(m_b)}{\alpha_s(m_c)} \right]^{-6/25} \left[ \frac{\alpha_s(m_c)}{\alpha_s(\mu)} \right]^{a_L(v \cdot v')}, \quad (1.3)$$

and

$$a_L(v \cdot v') = \frac{8}{27} [v \cdot v' r(v \cdot v') - 1], \quad (1.4)$$

$$r(v \cdot v') = \frac{1}{\sqrt{(v \cdot v')^2 - 1}} \ln \left( v \cdot v' + \sqrt{(v \cdot v')^2 - 1} \right). \quad (1.5)$$

For very large heavy quark masses (and $\mu$ of order the QCD scale), $C_{cb}(v \cdot v')$ has a rapid dependence on $v \cdot v'$. The function $\eta_0(v \cdot v')$ is determined by low-momentum strong interaction physics. It depends on the subtraction point $\mu$ in a way that cancels the subtraction point dependence of $C_{cb}(v \cdot v')$. At zero recoil, i.e. $v \cdot v' = 1$, $\eta_0$ is independent of $\mu$ and is normalized to unity \[2\] \[4\] \[5\] by heavy quark flavor symmetry,

$$\eta_0(1) = 1. \quad (1.6)$$

Some of the low momentum properties of QCD are determined by its symmetries (e.g. chiral symmetry and heavy quark symmetry). Those nonperturbative aspects of the theory which are not determined by symmetries cannot be treated using perturbation theory in the strong coupling constant. QCD, however, does have an expansion parameter which can be used to study low momentum features of the strong interactions analytically. In the limit that the number of colors $N_c$ is large, the theory simplifies and many predictions
are possible. The main purpose of this paper is to examine $\eta_0(v \cdot v')$ in the large $N_c$ limit.

In the large $N_c$ limit, baryons containing light $u$ and $d$ quarks can be viewed as solitons of the nonlinear chiral Lagrangian for pion self-interactions. Baryons containing a single heavy $c$ (or $b$) quark and light $u$ and $d$ quarks are then described as bound states of these solitons with $D$ and $D^*$ mesons (or $\bar{B}$ and $\bar{B}^*$ mesons). The large $N_c$ behavior of $\eta_0(v \cdot v')$ can be determined using the bound state wavefunctions of the $\Lambda_b$ and $\Lambda_c$. For large $N_c$, the function $\eta_0(v \cdot v')$ is strongly peaked about zero recoil since any velocity change must be transferred to $\sim N_c$ light quarks. Independent of the details of the bound state approach, we find that

$$
\eta_0(v \cdot v') = \exp\left[-\lambda N_c^{3/2} (v \cdot v' - 1)\right],
$$

where $\lambda$ is a constant of order unity. This equation is valid for $v \cdot v' - 1$ of order $N_c^{-3/2}$. In this kinematic region, $\eta_0$ falls from unity to a very small quantity. The derivation of this result is the main purpose of this paper. In addition, the effect on $\eta_0$ of corrections to the heavy quark limit that depend on $m_N/m_Q$ is examined. Some features of the soliton picture of heavy baryons not discussed in previous work on this subject will also be derived here. In Refs. it was shown that the leading term in the chiral Lagrangian for heavy-meson–pion interactions gives rise to a heavy-meson–soliton potential that is attractive at the origin in the $\Lambda_Q$, $\Sigma_Q$ and $\Sigma_Q^*$ channels. We show in this paper that the curvature of the soliton–heavy-meson potential is positive, indicating that the origin is a stable minimum of the potential energy for these channels. The curvature is negative for the exotic channels.

2. $\Lambda_Q$ as a Heavy Meson-Soliton Bound State

The starting point for discussing soliton–heavy-meson bound states is the chiral Lagrangian for the interactions of mesons containing a heavy quark $Q$ with pions. In the limit $m_Q \to \infty$, the total angular momentum of the light degrees of freedom, $\vec{S}_\ell$, is a symmetry generator. The lowest mass mesons with $Q\bar{Q}_a$ ($q_1 = u, q_2 = d$) flavor quantum numbers have $s_\ell = 1/2$ and form a degenerate doublet consisting of pseudoscalar and vector mesons. In the case $Q = c$, these are the $D$ and $D^*$ mesons, and in the case $Q = b$, these are the $\bar{B}$ and $\bar{B}^*$ mesons.
It is convenient to combine the fields $P_\alpha$ and $P_{a\mu}^\ast$ for the ground state $s_\ell = 1/2$ mesons into the bispinor matrix

$$H_\alpha = \frac{(1 + \not{v})}{2} \left[ P_{a\mu}^\ast \gamma^\mu - P_\alpha \gamma^5 \right], \quad (2.1)$$

where $v^\mu$ is the heavy quark four velocity, and $v^2 = 1$. The vector meson field is constrained to satisfy $v^\mu P_{a\mu}^\ast = 0$. In this section, we work in the rest frame of the heavy meson, $v^\mu = (1, \vec{0})$. Under the heavy quark spin symmetry,

$$H_\alpha \rightarrow S H_\alpha, \quad (2.2)$$

where $S \in SU(2)_v$ is the heavy quark spin transformation. The transformation property of $H$ under $SU(2)_L \times SU(2)_R$ chiral symmetry has an arbitrariness associated with field redefinitions. We will use the basis chosen in Ref. [10],* with the transformation rule

$$H_\alpha \rightarrow (HR^\dagger)_\alpha, \quad (2.3)$$

under $SU(2)_L \times SU(2)_R$, where $R \in SU(2)_R$. It is also convenient to introduce the field

$$\overline{H}^a = \gamma^0 H_a^\dagger \gamma_0 = \left[ P_{a\mu}^\ast \gamma^\mu + P_\alpha^\dagger \gamma^5 \right] \frac{(1 + \not{v})}{2}. \quad (2.4)$$

The Goldstone bosons occur in the field

$$\Sigma = \exp \left( \frac{2iM}{f} \right), \quad (2.5)$$

where

$$M = \begin{bmatrix} \pi^0 / \sqrt{2} & \pi^+ \\ \pi^- & -\pi^0 / \sqrt{2} \end{bmatrix}, \quad (2.6)$$

and $f \approx 132$ MeV is the pion decay constant. Under $SU(2)_L \times SU(2)_R$

$$\Sigma \rightarrow L\Sigma R^\dagger, \quad (2.7)$$

with $L \in SU(2)_L$ and $R \in SU(2)_R$. Under parity,

$$\Sigma(x^0, \vec{x}) \rightarrow \Sigma^\dagger(x^0, -\vec{x}), \quad (2.8)$$

since $M(x^0, \vec{x}) \rightarrow -M(x^0, -\vec{x})$. If $H_\alpha$ transforms under chiral symmetry as in Eq. (2.3), then the parity transform of $H$ must transform under chiral symmetry with a factor of $L^\dagger$.

* The computations are repeated for the $\xi$ basis in the appendix.
Consequently, in the basis we are using, the action of parity on the $H$ field is somewhat unusual,

$$H_a(x^0, \vec{x}) \rightarrow \gamma^0 H_b(x^0, -\vec{x}) \gamma^0 \Sigma^{tb}_a(x^0, -\vec{x}).$$  \hfill (2.9)

The chiral Lagrangian density for heavy meson-pion strong interactions is

$$L = -i \text{Tr} \overline{H} v \cdot \partial H + \frac{i}{2} \text{Tr} \overline{H} v^\mu \Sigma^{tb} \partial_\mu \Sigma + \frac{ig}{2} \text{Tr} \overline{H} \gamma^\mu \gamma_5 \Sigma^{tb} \partial_\mu \Sigma + \ldots,$$ \hfill (2.10)

where the ellipsis denotes the contribution of terms containing more derivatives or factors of $1/m_Q$. The coefficient of the second term in the Lagrangian density Eq. (2.10) is fixed (relative to the first) by parity invariance. The coupling $g$ determines the $D^* \rightarrow D\pi$ decay rate. Present experimental information on the $D^*$ width and the $D^* \rightarrow D\pi$ branching ratio implies that $g^2 < 0.4$ \cite{13}. The constituent quark model predicts that $g$ is positive.

The soliton solution of the $SU(2)_L \times SU(2)_R$ chiral Lagrangian for baryons containing $u$ and $d$ quarks is

$$\Sigma = A(t) \Sigma_0(\vec{x}) A^{-1}(t),$$ \hfill (2.11)

where

$$\Sigma_0 = \exp (iF(r) \hat{x} \cdot \vec{\tau}),$$ \hfill (2.12)

and $r = |\vec{x}|$. $A(t)$ contains the dependence on the collective coordinates associated with rotations and isospin transformations of the soliton solution. For solitons with baryon number one, $F(0) = -\pi$ and $F(\infty) = 0$. The detailed shape of $F(r)$ depends on the chiral Lagrangian for pion self interactions including terms with more than two derivatives. We expect that $\Sigma_0(\vec{x})$ has a power series expansion in $\vec{x}$. Consequently, the even powers of $r$ must vanish when $F(r)$ is expanded in a power series in $r$, e.g. $F''(0) = 0$. The chiral Lagrangian for pion self interactions is of order $N_c$. However, the chiral Lagrangian for heavy-meson–pion interactions is only of order one. Thus, to leading order in $N_c$ the shape of the soliton $F(r)$ is not altered by the presence of the heavy meson.

In the large $N_c$ limit, baryons containing light $u$ and $d$ quarks are very heavy and time derivatives on the $\Sigma$ field can be neglected. Consequently, it is the interaction Hamiltonian

$$H_I = -\frac{ig}{2} \int d^3 \vec{x} \; \text{Tr} \overline{H} H \gamma^{tb} \gamma_5 \Sigma^{tb} \partial_j \Sigma + \ldots,$$ \hfill (2.13)

with $\Sigma$ given by Eqs. (2.11) and (2.12) that determines the potential energy of a configuration with a heavy meson at the origin and a baryon at position $\vec{x}$. Neglecting operators
with more than one derivative (the ellipsis in Eq. (2.13)), and expanding the interaction potential operator in \( \vec{x} \) gives

\[
\hat{V}_I(\vec{x}) = g S^j_{I_H} I^k_H \text{ Tr } A^{j\ell} A^{-1} \tau^k \left\{ \delta^{ij} \left[ F'(0) - \frac{2}{3} r^2 [F'(0)]^3 + \frac{1}{6} r^2 F'''(0) \right] \right.
\]
\[
+ x^i x^j \left[ \frac{2}{3} [F'(0)]^3 + \frac{1}{3} F'''(0) \right] + \epsilon^{ijm} x^m [F'(0)]^2 \} + \mathcal{O}(x^3),
\]

(2.14)

where \( S^j_{I_H} \) denotes the angular momentum of the light degrees of freedom of the heavy meson, and \( I^k_H \) denotes the isospin of the heavy meson. The interaction potential has terms which superficially have the wrong parity, e.g. the term involving the \( \epsilon \) symbol. However, these terms are required because of the \( \Sigma^\dagger \) factor in the parity transformation of \( H \) in Eq. (2.9).

The \( \Lambda_Q \) baryon has isospin zero and total angular momentum of the light degrees of freedom equal to zero. In the large \( N_c \) limit, it arises from a bound state of nucleons with \( P \) and \( P^* \) mesons. Baryons with \( I > 1/2 \) such as the \( \Delta \) cannot produce a heavy baryon bound state with \( I = 0 \). On nucleon states, \( \text{Tr } A^{j\ell} A^{-1} \tau^k \) is equal to \(-8 S^i_N I^k_N / 3 \) where \( S^i_N \) is the spin of the nucleon, and \( I^k_N \) is the isospin of the nucleon \([14]\). Using this simplification, the potential operator becomes

\[
\hat{V}_I(\vec{x}) = \hat{V}_I^{(0)} + \hat{V}_I^{(1)} + \hat{V}_I^{(2)} + \mathcal{O}(x^3),
\]

(2.15)

where \( \hat{V}_I^{(n)} \) denotes the term of order \( r^n \) in the potential

\[
\hat{V}_I^{(0)} = -\frac{8}{3} g F'(0) \vec{I}_H \cdot \vec{I}_N \vec{S}_{I_H} \cdot \vec{S}_N,
\]
\[
\hat{V}_I^{(1)} = \frac{8}{3} g \left[ F'(0) \right]^2 \vec{I}_H \cdot \vec{I}_N \vec{x} \cdot (\vec{S}_{I_H} \times \vec{S}_N),
\]
\[
\hat{V}_I^{(2)} = -\frac{8}{3} g \vec{I}_H \cdot \vec{I}_N \left\{ \vec{S}_{I_H} \cdot \vec{S}_N \left[ -\frac{2}{5} r^2 [F'(0)]^3 + \frac{1}{5} r^2 F'''(0) \right] \right.
\]
\[
\left. + (\vec{S}_{I_H} \cdot \vec{x})(\vec{S}_N \cdot \vec{x}) \left[ \frac{2}{3} [F'(0)]^3 + \frac{1}{3} F'''(0) \right] \right\}.
\]

(2.16)

The potential \( \hat{V}_I(\vec{x}) \) commutes with the total angular momentum of the light degrees of freedom, \( \vec{L} + \vec{S}_{I_H} + \vec{S}_N \), where \( \vec{L} \) is the orbital angular momentum.

To find the \( \Lambda_Q \) wavefunction \( \Psi_\Lambda(\vec{x}) \) and its potential energy \( V_\Lambda(\vec{x}) \), the potential energy operator must be diagonalized at each point \( \vec{x} \) on the product space of nucleon heavy meson states (e.g. \( |p, \uparrow\rangle |P^*, \downarrow\rangle \)). It is convenient to consider linear combinations of these product states that have definite isospin \( \vec{I} = \vec{I}_H + \vec{I}_N \), spin of the light degrees of freedom.
freedom, $\vec{S}_\ell = \vec{S}_{\ell H} + \vec{S}_N$, and total spin $\vec{S} = \vec{S}_Q + \vec{S}_\ell$. These states are labeled $| I, s, s_\ell \rangle$. It is straightforward to diagonalize $\hat{V}_I(\vec{x})$ in this basis, and we find that

$$\Psi_\Lambda(\vec{x}) = \left[ 1 - F'(0) \vec{x} \cdot (\vec{S}_{\ell H} \times \vec{S}_N) \right] | 0, \frac{1}{2}, 0 \rangle \phi(\vec{x}),$$

(2.17)
is an eigenstate of $\hat{V}_I(\vec{x})$ with eigenvalue

$$V_\Lambda(\vec{x}) = -\frac{3}{2} g F'(0) + g r^2 \left[ \frac{1}{6} | F'(0) |^3 - \frac{5}{12} F''''(0) \right]$$

$$= -\frac{3}{2} g F'(0) + \frac{1}{2} \kappa r^2,$$

(2.18)

where $\kappa$ is defined by

$$\kappa = g \left[ \frac{1}{3} | F'(0) |^3 - \frac{5}{6} F''''(0) \right].$$

(2.19)
The form of the wavefunction in Eq. (2.17) can also be found using Eq. (2.9) and demanding that it has definite parity. The factor in square brackets in Eq. (2.17) compensates for the factor of $\Sigma^\dagger$ in Eq. (2.9). In Eqs. (2.17) and (2.18), $r \Lambda_{\text{QCD}}$ is treated as a small quantity* and the wavefunction is given to linear order in $r \Lambda_{\text{QCD}}$, while the potential energy is given to quadratic order in $r \Lambda_{\text{QCD}}$. The spatial part of the wavefunction $\phi(\vec{x})$ has a more rapid dependence on $r$ which will be computed later in this article. As $r$ goes from zero to infinity, $F(r)$ goes from $-\pi$ to 0. Consequently, we expect that $F'(0)$ is positive and $F''''(0)$ is negative. This is true for example for the solution given in Ref. [14] where a particular four derivative term in the chiral Lagrangian for pion self-interactions is used to stabilize the soliton. Furthermore, the constituent quark model suggests that $g$ is positive. Thus, Eq. (2.18) implies that the $\Lambda_Q$ is bound by a harmonic oscillator potential with $\kappa > 0$.

In the limit $N_c \to \infty$, the nucleon is infinitely heavy and terms in the Hamiltonian involving the nucleon momentum are neglected. The lowest energy state then has the spatial wavefunction $\phi(\vec{x}) = \delta^3(\vec{x})$ corresponding to the minimum energy classical configuration where the nucleon is located at $\vec{x} = 0$. The $1/N_c$ terms in the Hamiltonian involving the nucleon momentum give the spatial wavefunction a finite spread and it is this finite extent which is responsible for the Isgur-Wise function $\eta_0(v \cdot v')$. Terms of order $1/N_c$ involving the momentum are found by writing $\Sigma = \Sigma(\vec{x} - \vec{r}(t))$ and quantizing the collective coordinate $\vec{r}(t)$. This procedure yields

$$H_{\text{kin}} = \frac{\vec{p}^2}{2 m_N} - \frac{4}{3} F'(0) \left( \vec{I}_H \cdot \vec{I}_N \right) \frac{\vec{S}_N \cdot \vec{p}}{m_N},$$

(2.20)

* $\Lambda_{\text{QCD}}$ denotes a nonperturbative strong interaction scale that is finite as $N_c \to \infty$. 

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where $m_N$ is the mass of the nucleon. The first term is the usual kinetic energy of the soliton and the second term in Eq. (2.20) results from the second term in Eq. (2.10). The Schrödinger equation is

$$[H_{\text{kin}} + V_\Lambda(\vec{x})] \Psi_\Lambda(\vec{x}) = E \Psi_\Lambda(\vec{x}),$$

which implies that $\phi(\vec{x})$ obeys the differential equation

$$\left[ -\frac{\vec{\nabla}^2}{2m_N} + V_\Lambda(\vec{x}) \right] \phi(\vec{x}) = E \phi(\vec{x}),$$

in the rest frame of the bound state. Note that the term linear in $\vec{p}$ in Eq. (2.20) is necessary for the wavefunction in Eq. (2.17) to obey the Schrödinger equation (2.21). This result is not surprising because both the term linear in $\vec{p}$ and the part of the wavefunction linear in $\vec{x}$ arise from the peculiar definition of parity in Eq. (2.9). In deriving Eqs. (2.21) and (2.22) we have treated the term linear in $\vec{p}$ in $H_{\text{kin}}$ as a perturbation and neglected its action on the “small” part of the wavefunction $\Psi_\Lambda(\vec{x})$ (i.e. the piece proportional to $\vec{x}$). As we shall see shortly, for large $N_c$ the term proportional to $\vec{x}$ in $\Psi_\Lambda(\vec{x})$ and the term linear in $\vec{p}$ in $H_{\text{kin}}$ are subdominant and can be neglected in the calculation of $\eta_0(v \cdot v')$.

In general, for large $N_c$ we expect the potential $V_\Lambda(\vec{x})$ to have the harmonic oscillator form

$$V_\Lambda(\vec{x}) = V_0 + \frac{1}{2}\kappa \vec{x}^2.$$  

(2.23)

The absence of a term linear in $r$ requires $F''(0) = 0$ which is a consequence of the derivative expansion of the chiral Lagrangian for pion self-interactions. Quantum corrections can induce nonanalytic behavior in $\vec{x}$, but because of an explicit factor of $1/N_c$ such terms are less important than those we have kept. The particular expression for $\kappa$ in Eq. (2.19) is, however, model dependent and arises from keeping only terms with one derivative in the chiral Lagrangian for heavy meson-pion interactions.

The model independence of the results of this paper follows from an analysis of large $N_c$ power counting. In the limit $m_Q \to \infty$, the typical size (or momentum) of the bound state wavefunction occurs when the kinetic energy $\vec{p}^2/2m_N$ and potential energy $\kappa \vec{x}^2/2$ of the bound state contribute equally to the total energy $E - V_0$,

$$r \sim (\kappa m_N)^{-1/4}, \quad p \sim (\kappa m_N)^{1/4}.$$  

(2.24)

Since $\kappa$ is of order $\Lambda_{\text{QCD}}^3$ and $m_N$ is of order $\Lambda_{\text{QCD}} N_c$, the $N_c$ dependence of the typical binding energy is given by

$$E - V_0 \sim \Lambda_{\text{QCD}} N_c^{-1/2}.$$  

(2.25)
It is now straightforward to see that higher order terms in the effective Lagrangian can be neglected. Any term in the effective Lagrangian is a function of $\vec{r} = \vec{r}_N - \vec{r}_H$, $\dot{\vec{r}}_N$, and $\dot{\vec{r}}_H$. Terms involving only the soliton field carry an overall factor of $N_c$. These terms are independent of the relative coordinate $\vec{r}$ and depend only on $\dot{\vec{r}}_N$. In the effective Hamiltonian, the dependence of these terms on $\dot{\vec{r}}_N$ enters through powers of the nucleon momentum. A term with $n$ powers of the nucleon momentum has the following large $N_c$ behavior,

$$\Lambda_{\text{QCD}} N_c \left( \frac{\hat{p}}{M_N} \right)^n \sim \Lambda_{\text{QCD}} N_c N_{c}^{-3n/4}. \quad (2.26)$$

Thus, all terms with higher powers of the nucleon momentum than the leading order kinetic energy term $\hat{p}^2/2m_N$ are suppressed by more powers of $1/N_c$ and can be neglected in the large $N_c$ limit. Terms in the Lagrangian which involve the interaction between the soliton and the heavy meson can depend on $\vec{r}$, but they are at most of order one in the large $N_c$ limit. The typical scale of the $r$ dependent interaction is $\Lambda_{\text{QCD}}$, so higher order interaction terms have the form

$$\Lambda_{\text{QCD}} (\Lambda_{\text{QCD}} r)^m \left( \frac{\hat{p}}{M_N} \right)^n \sim \Lambda_{\text{QCD}} N_c^{-m/4} N_{c}^{-3n/4}. \quad (2.27)$$

Hence, all the higher order interaction terms in the effective Lagrangian other than the harmonic potential are higher order in $1/N_c$ and can be neglected (including the term linear in $\hat{p}$ in $H_{\text{kin}}$ Eq. (2.20)).

There is also a class of $1/m_Q$ corrections which can be summed exactly. So far, we have concentrated on the order of limits $m_Q \to \infty$ followed by $N_c \to \infty$. We now switch to the situation in which $m_Q, N_c \to \infty$ simultaneously, with the ratio $\Lambda_{\text{QCD}} N_c/m_Q$ held fixed. The only additional term in the effective Lagrangian which must be included in this double scaling limit is the kinetic energy of the heavy meson,

$$\frac{\hat{p}^2}{m_H} \sim \frac{m_N}{m_Q} \frac{p^2}{m_N} \sim \left( \frac{\Lambda_{\text{QCD}} N_c}{m_Q} \right) \frac{p^2}{m_N}, \quad (2.28)$$

which is of the same order as a term we have included, since $\Lambda_{\text{QCD}} N_c/m_Q$ is of order one. Higher order terms in $1/m_Q$ such as

$$\frac{p^4}{m_Q^3} \sim \left( \frac{m_N}{m_Q} \right)^3 \frac{p^4}{m_N^3}, \quad (2.29)$$

are of order one times terms which can be neglected by the power counting arguments of the previous paragraph, so they too can be neglected. Additional $1/m_Q$ effects arise from
1/$m_Q$ operators in the heavy quark effective theory. The $P^* - P$ mass difference is of order $\Lambda_{QCD}^2 / m_Q \sim (N_c \Lambda_{QCD} / m_Q) \Lambda_{QCD} / N_c$ which is subleading in $N_c$. Higher derivative operators in the current are also suppressed. For example, the leading correction

$$\frac{1}{m_c} \text{Tr} \ H^{(c)}(\gamma^\mu (1 - \gamma_5)) H^{(b)} \sim \frac{\Lambda_{QCD} N_c^{1/4}}{m_c} \sim \left( \frac{\Lambda_{QCD} N_c}{m_Q} \right) N_c^{-3/4},$$

since the typical momentum of the heavy quark in the baryon is of order $\Lambda_{QCD} N_c^{1/4}$. Thus, no additional terms other than the kinetic energy of the heavy meson are relevant.

3. $\Lambda_b \rightarrow \Lambda_c e^- \bar{\nu}_e$ Decay

For non-relativistic $\Lambda_c$ recoil, the matrix element of the weak current Eq. (1.1) in the $\Lambda_b$ rest frame is

$$\langle \Lambda_c(v', s') | \overline{c} \gamma^\mu (1 - \gamma_5) b | \Lambda_b(v, s) \rangle = \int \frac{d^3 \vec{p}'}{m_N} \int \frac{d^3 \vec{p}}{m_D} \phi_c^*(\vec{p}') \phi_b(\vec{p})$$

$$\frac{1}{4} \langle N(-\vec{p}' + m_N \vec{v}', s') | N(-\vec{p}, s) \rangle \langle D(\vec{p}' + m_D \vec{v}') | c \gamma^\mu (1 - \gamma_5) b | B(\vec{p}) \rangle + \ldots$$

where the ellipsis represents terms involving at least one vector meson. $\phi_c$ and $\phi_b$ are the Fourier transform of the ground state wave function

$$\phi_{c,b}(\vec{p}) = \frac{1}{(\pi \mu_{c,b} \kappa)^{3/8}} \text{exp} \left( -\frac{\vec{p}^2}{2 \sqrt{\mu_{c,b} \kappa}} \right),$$

where $\kappa$ is defined in Eq. (2.19), and $\mu_{c,b}$ is the reduced mass $\mu = m_N m_H / (m_N + m_H)$ of the bound state, with $m_H = m_D, m_B$ for the $c$ and $b$ subscripts, respectively. The $\Lambda_Q$ state is a superposition of products of $N$ with $P$ and $P^*$ states. However, we do not need the details of the Clebsch-Gordan structure* of the $\Lambda_Q$ state, as will become clear soon.

The nucleon matrix element vanishes unless

$$\vec{p}' = \vec{p} + m_N \vec{v}',$$

at which point, for the term explicitly displayed in Eq. (3.1), the required heavy meson matrix element is

$$\langle D(\vec{p} + (m_N + m_D) \vec{v}') | c \gamma^\mu (1 - \gamma_5) b | B(\vec{p}) \rangle.$$

* The factor of 1/4 is the square of the appropriate Clebsch-Gordan coefficient for the term displayed explicitly in Eq. (3.1).
This matrix element can be evaluated in terms of heavy meson form factors. In the $\Lambda_Q$ heavy meson-nucleon bound state, the typical momentum is of order $\langle p \rangle \sim (m_N \kappa)^{1/4}$ and so form factors for semileptonic $\Lambda_b \to \Lambda_c e^- \bar{\nu}_e$ decay are smooth functions of $m_N \bar{v}'/(m_N \kappa)^{1/4}$ which is of order $N_c^{3/4} \bar{v}'$. Consequently, we are interested in the kinematic region $v'$ of order $N_c^{-3/4}$, and so for large $N_c$ we can replace the heavy meson form factors by their rapidly varying part, $C_{cb}(v \cdot v')$, and neglect the slow variation in the meson Isgur-Wise function. This is why we do not need the details of the Clebsch-Gordan structure of the $\Lambda_Q$ bound state. The $\Lambda_b \to \Lambda_c e^- \bar{\nu}_e$ form factor in the large $N_c$ limit can be written in terms of an Isgur-Wise function, as in Eq. (1.1), with

$$
\eta_0 = \int d^3 \vec{p} \quad \phi_c^*(\vec{p} + m_N \vec{v}') \phi_b(\vec{p}) \\
= \left[ \frac{2 (\mu_c \mu_b)^{1/4}}{\sqrt{\mu_b + \sqrt{\mu_c}}} \right]^{3/2} \exp \left[ -m_N^2 \bar{v}'^2 / 2 \sqrt{\kappa} \left( \sqrt{\mu_b + \sqrt{\mu_c}} \right) \right].
$$

(3.5)

Since for non-relativistic recoils $\bar{v}'^2 \approx 2(v \cdot v' - 1)$, the expression for $\eta_0$ in a general frame has the form

$$
\eta_0(v \cdot v') = \left[ \frac{2 (\mu_c \mu_b)^{1/4}}{\sqrt{\mu_b + \sqrt{\mu_c}}} \right]^{3/2} \exp \left[ -m_N^2 (v \cdot v' - 1) / \sqrt{\kappa} \left( \sqrt{\mu_b + \sqrt{\mu_c}} \right) \right].
$$

(3.6)

The result we have obtained is valid in the heavy quark and large $N_c$ limits, where we have included all terms of order $(m_N/m_Q)^n \sim (N_c \Lambda_{QCD}/m_Q)^n$. The baryon form factors for $\Lambda_b \to \Lambda_c e^- \bar{\nu}_e$ are still written in terms of a single function in this limit, even though we have included a class of $1/m_Q$ corrections. In the large $N_c$ limit, the function $\eta_0(v \cdot v')$ falls off rapidly away from zero recoil. Derivatives of $\eta_0$ at zero recoil diverge as $N_c \to \infty$. Eq. (3.6) indicates that the $m^{th}$ derivative is of order $N_c^{3m/2}$, and includes all contributions of this order neglecting less divergent pieces. For example, there could be corrections to $\eta_0$ of the form $N_c^{1/2} (v \cdot v' - 1)$ in the exponent. This term is small compared with the leading term $N_c^{3/2} (v \cdot v' - 1)$, but is significant when $v \cdot v' - 1$ is of order $N_c^{-1/2}$. Thus Eq. (3.6) is valid in the large $N_c$ limit in the region where $v \cdot v' - 1 \lesssim O(N_c^{-3/2})$, i.e. in the region where $\eta_0$ falls from about unity to a very small quantity.

At zero recoil in the limit that $m_b \to \infty$, $\eta_0$ has the form

$$
\eta_0(1) = \left[ \frac{2 (\mu_c m_N)^{1/4}}{\sqrt{m_N + \sqrt{\mu_c}}} \right]^{3/2} = 1 - \frac{3}{64} \left( \frac{m_N}{m_D} \right)^2 + \ldots,
$$

(3.7)
where we have expanded the exact expression in a power series in $m_N/m_D$. The term linear in $m_N/m_D$ vanishes, which is consistent with Luke’s theorem \cite{Luke}. For physical values of $m_N/m_D$, the correction to the symmetry limit prediction $\eta_0(1) = 1$ is only 1%.

The parameter $\kappa$ can be determined to be $(530 \text{ MeV})^3$ in the Skyrme model using the shape function used in Ref. \cite{shape}, and the value of $g$ obtained in Ref. \cite{g}. With this value of $\kappa$, we find that the orbitally excited $\Lambda_Q$ state should be about 400 MeV above the ground state, and that the form factor $\eta_0$ of Eq. (3.6) is

$$\eta_0(v \cdot v') \sim 0.99 \exp \left[-1.3(v \cdot v' - 1)\right],$$

(3.8)

using the known values of $m_N$, $m_D$ and $m_B$. The Skyrme model prediction for $\kappa$ is sensitive to the precise shape of the soliton solution. A better way to determine $\kappa$ is to use the experimentally measured excitation energy of the orbitally excited $\Lambda_Q$, which is $\sqrt{\kappa/\mu_Q}$.

The large $N_c$ predictions of this paper rely on the number of light quarks in the heavy baryon being large. For $N_c = 3$ there are only two light quarks in $\Lambda_Q$, so we expect our results to only be qualitatively correct. Nevertheless, it is interesting that the baryon form factors are calculable in the large $N_c$ limit of QCD. There are other results that can be computed using the methods developed here. For example, the Isgur-Wise functions for transitions to excited states are also calculable. It should also be possible to derive our results using the methods of Witten \cite{Witten}. It would be interesting to explore that approach.

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Appendix A. The $\xi$ Basis

In this appendix, we briefly discuss the computation of the binding potential in the $\xi$ basis discussed in Ref. \cite{xi}. The notation is the same as found in Ref. \cite{xi}. The $\xi$ basis is singular at the origin, but has a simple transformation rule for the $H$ field under parity,

$$H(x^0, \vec{x}) \rightarrow \gamma^0 H(x^0, -\vec{x}) \gamma^0.$$
The interaction Hamiltonian in this basis is

\[ H_I = -\frac{ig}{2} \int d^3x \text{Tr} \, HH\gamma^j \gamma_5 \left( \xi^\dagger \partial_j \xi - \xi \partial_j \xi^\dagger \right). \]  

(A.1)

Expanding the Goldstone boson field

\[ \xi_0 = \exp \left( iF(r) \hat{x} \cdot \vec{\tau}/2 \right), \]  

(A.2)
in a power series about \( \vec{x} = 0 \), and using \( \xi = A \xi_0 A^{-1} \), we get the interaction potential

\[ \hat{V}_I(\vec{x}) = 2g \, \text{Tr} \, A^j A^{-1} r^k \left\{ F'(0) \left( S_{\ell H} \cdot \hat{x} \, I_H^k \right) \hat{x}^j \right. - \left. \frac{1}{2} S_{\ell H} \cdot I_H^k \right\} \\
+ r^2 \left( \frac{1}{12} [F'(0)]^3 - \frac{1}{12} F''''(0) \right) S_{\ell H} \cdot I_H^k + S_{\ell H} \cdot \vec{x} \, I_H^k \, x^j \left( -\frac{1}{12} [F'(0)]^3 + \frac{1}{3} F''''(0) \right) \}. \]  

(A.3)

Note that only even powers of \( x \) occur in Eq. (A.3) because the parity transformation is trivial in the \( \xi \) basis. To leading order in \( N_c \), the kinetic term of the soliton can be neglected, so that states with a definite value of \( A \) are eigenstates of the Hamiltonian. It is convenient to consider the soliton state that is an eigenstate of \( A \) with \( A = 1 \). States with other values of \( A \) can be obtained by an isospin transformation, and so have the same energy. On states with \( A = 1 \), the interaction potential reduces to

\[ \hat{V}_I(\vec{x}) = 4g \, \left\{ F'(0) \left( S_{\ell H} \cdot \hat{x} \, I_H \right) \right. \\
+ r^2 \left( \frac{1}{12} [F'(0)]^3 - \frac{1}{12} F''''(0) \right) S_{\ell H} \cdot I_H + S_{\ell H} \cdot \vec{x} \, I_H \left. \cdot \vec{x} \left( -\frac{1}{12} [F'(0)]^3 + \frac{1}{3} F''''(0) \right) \right\} \]. \]  

(A.4)

This Hamiltonian is singular at the origin because of the coordinate singularity in the \( \xi \) basis, and the eigenstates of the Hamiltonian will also be singular at the origin. We know that the singularity at the origin has the form \( \vec{\tau} \cdot \hat{x} \), because that is the transformation function from the singular \( \xi \) basis to the non-singular basis used earlier in this article. We therefore write the eigenstates of the interaction potential Eq. (A.4), \( |\psi\rangle \) in terms of new eigenstates \( |\phi\rangle \) which are related by the unitary transformation

\[ |\psi\rangle = \left( 2 \, \vec{I}_H \cdot \hat{x} \right) |\phi\rangle. \]  

(A.5)

The interaction potential in the \( |\phi\rangle \) basis is

\[ \hat{V}'_I(\vec{x}) = \left( 2 \, \vec{I}_H \cdot \hat{x} \right) \hat{V}_I(\vec{x}) \left( 2 \, \vec{I}_H \cdot \hat{x} \right) = 2g \left\{ F'(0) \, \vec{S}_{\ell H} \cdot \vec{I}_H \right. \\
+ r^2 \left( -\frac{1}{3} [F'(0)]^3 + \frac{2}{15} F''''(0) \right) \vec{S}_{\ell H} \cdot \vec{I}_H \left. \right\} \\
+ \left( \frac{1}{6} [F'(0)]^3 + \frac{1}{3} F''''(0) \right) \left( \vec{S}_{\ell H} \cdot \vec{x} \vec{I}_H \cdot \vec{x} - \frac{1}{3} r^2 \vec{S}_{\ell H} \cdot \vec{I}_H \right) \}. \]  

(A.6)
where we have used the relations
\[
\left( \vec{I}_H \cdot \hat{x} \right)^2 = \frac{1}{4},
\]
\[
\left( \vec{I}_H \cdot \hat{x} \right) \left( \vec{S}_{\ell H} \cdot \vec{I}_H \right) \left( \vec{I}_H \cdot \hat{x} \right) = \frac{1}{2} \left( \vec{S}_{\ell H} \cdot \hat{x} \right) \left( \vec{I}_H \cdot \hat{x} \right) - \frac{1}{4} \left( \vec{S}_{\ell H} \cdot \vec{I}_H \right),
\]
that follow from the anticommutation relation
\[
I^i_H I^k_H + I^k_H I^i_H = \frac{1}{2} \delta^{ij},
\]
for the isospin-1/2 operators \(I_H\). It is convenient to define the operator \(\vec{K}_H = \vec{I}_H + \vec{S}_{\ell H}\), in terms of which Eq. (A.6) can be rewritten as
\[
\hat{V}'_I(\vec{x}) = g \left\{ F'(0) \left( \vec{K}_H^2 - \frac{3}{2} \right) + r^2 \left( -\frac{1}{9} [F'(0)]^3 + \frac{5}{18} F''(0) \right) \left( \vec{K}_H^2 - \frac{3}{2} \right) \right.
\]
\[
+ \left. \left( \frac{1}{6} [F'(0)]^3 + \frac{1}{3} F''(0) \right) \left( \vec{K}_H \cdot \vec{x} \vec{K}_H \cdot \vec{x} - \frac{1}{3} r^2 \vec{K}_H^2 \right) \right\}.
\]

The allowed values of \(K_H\) for the \(H\) field are \(K_H = 0\) and \(K_H = 1\) obtained by combining \(I_H = 1/2\) with \(S_{\ell H} = 1/2\). The states with \(K_H = 0\) are the bound physical baryon states containing a heavy quark. On the \(K_H = 0\) states, Eq. (A.9) reduces to the potential
\[
\hat{V}^K_{I=0} = \frac{3}{2} g \left\{ F'(0) + r^2 \left( -\frac{1}{9} [F'(0)]^3 + \frac{5}{18} F''(0) \right) \right\}
\]
\[
= -\frac{3}{2} g F'(0) + \frac{1}{2} \kappa r^2,
\]
where \(\kappa\) is defined in Eq. (2.19). Thus the states with \(K_H = 0\) are bound in an attractive harmonic oscillator potential, and the potential at the origin has the value \(-3gF'(0)/2\). The potential is more complicated for exotic states which have \(K_H = 1\). The last term in Eq. (A.9) is an irreducible tensor operator with \(K_H = 2\) and contributes a spin-orbit term to the interaction potential (but note that it does not cause any mixing between \(K_H = 0\) and \(K_H = 1\) states). The spherically averaged potential for \(K_H = 1\) states is simple to compute,
\[
\hat{V}^K_{I=1} = \frac{1}{2} g \left\{ F'(0) + r^2 \left( -\frac{1}{9} [F'(0)]^3 + \frac{5}{18} F''(0) \right) \right\}
\]
\[
= \frac{1}{2} g F'(0) - \frac{1}{6} \kappa r^2.
\]
Thus the \(K_H = 1\) states are unbound since the potential at the origin is positive, and the interaction potential is a repulsive inverted harmonic oscillator potential. The \(\Lambda_Q\) state is obtained from the \(K_H = 0\) state by applying a projection operator \(\Pi\), so the interaction potential for the \(\Lambda_Q\) state is Eq. (A.10). This is precisely the potential given in Eq. (2.18) of the text.
The kinetic term in the $\xi$ basis on the $A = 1$ soliton states has the form

$$\mathcal{L} = \frac{1}{2} M_N \dddot{x}^2 - \frac{2}{r^2} \dddot{x} \cdot (\ddot{x} \times \vec{I}_H).$$  \hspace{1cm} (A.12)$$

The first term is the usual soliton kinetic energy, and the second term is from the expansion of

$$\mathcal{L} = \frac{i}{2} \text{Tr} \vec{H} \dot{H} v^\mu (\xi^\dagger \partial_\mu \xi + \xi \partial_\mu \xi^\dagger).$$  \hspace{1cm} (A.13)$$

The kinetic Hamiltonian in the $\xi$ basis obtained from the Lagrangian Eq. (A.12) is

$$H_{\text{kin}} = \frac{1}{2M_N} \left( \vec{p}_N + \frac{2}{r^2} \dddot{x} \times \vec{I}_H \right)^2.$$  \hspace{1cm} (A.14)$$

Transforming from the singular basis using Eq. (A.5) gives the kinetic energy

$$H'_{\text{kin}} = \left( 2 \vec{I}_H \cdot \dddot{x} \right) H_{\text{kin}} \left( 2 \vec{I}_H \cdot \dddot{x} \right) = \frac{\vec{p}^2}{2M_N}.$$  \hspace{1cm} (A.15)$$

The bound state problem in the $\xi$ basis reduces to that of a three-dimensional harmonic oscillator with a conventional kinetic term.
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