A UNIFIED FRAMEWORK FOR REGULARIZED REINFORCEMENT LEARNING

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ABSTRACT

We propose and study a general framework for regularized Markov decision processes (MDPs) where the goal is to find an optimal policy that maximizes the expected discounted total reward plus a policy regularization term. The extant entropy-regularized MDPs can be cast into our framework. Moreover, under our framework there are many regularization terms that bring multi-modality and sparsity which are potentially useful in reinforcement learning. In particular, we present sufficient and necessary conditions that induce a sparse optimal policy. We also conduct a full mathematical analysis of the proposed regularized MDPs, including the optimality condition, performance error and sparseness control. We provide a generic method to devise regularization forms and propose off-policy actor-critic algorithms in complex environment settings. We empirically analyze the numerical properties of optimal policies and compare the performance of different sparse regularization forms in discrete and continuous environments.

1 Introduction

Reinforcement learning (RL) aims to find an optimal policy that maximizes the expected discounted total reward in an MDP [4, 36]. Since the Bellman equation for the optimal policy is nonlinear [2], greedily solving the Bellman equation might result in less efficient and far poor policies. Even the optimal policy is accurately obtained, its ability to cope with unexpected situations is limited because it is deterministic so that it does not have the knowledge of multiple optimal actions. For example, an autonomous vehicle has multiple optimal routes to reach its destination but the optimal policy only selects one of them. If a traffic accident occurs at the currently selected optimal route, it is impossible to avoid the accident unless a new optimal route is computed. However, this could be avoided if the optimal policy is multi-modal rather deterministic. Therefore, in a real-life application we hope the optimal policy to possess some properties, such as computational easiness and multi-modality.

Entropy-regularized RL methods have been proposed to handle these issues. More specifically, an entropy is added to expected long-term returns. As a result, it not only softens the non-linearity of the original Bellman equation, making it more easily solvable but also forces the optimal policy to be stochastic, which is desirable in problems where dealing with unexpected situations is crucial. In prior work, the Shannon entropy is usually used. The optimal policy is of the form of softmax which has been shown can encourage exploration [8, 40]. However, a softmax policy assigns a non-negligible probability mass to non-optimal actions, even if these actions are suboptimal and dismissable, which may result in an unsafe policy. For RL problems with high dimensional action spaces, a sparse distribution is preferred in modeling a policy function, because it implicitly does action filtration, i.e., weeds out suboptimal actions and maintains near optimal actions. Thus, [19] proposed to use Tsallis entropy [39] instead, giving rise to a sparse MDP where only few actions have non-zero probability at each state in the optimal policy. What’s more, the Tsallis regularized RL has a

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lower performance error, i.e., the optimal value of the Tsallis regularized RL is closer to the original optimal value than that of the Shannon regularized RL.

Above discussions manifest that an entropy regularization characterizes the solution to the corresponding regularized RL. From [23], any entropy-regularized MDP can be viewed as a regularized convex optimization problem where the entropy serves as the regularizer and decision variable is a stationary policy. For each specific regularized formulation, algorithms have been proposed. On the other hand, a sparse optimal policy distribution is more favored in large action space RL problems. However, prior works [19][25] obtain a sparse optimal policy by the Tsallis entropy regularization. Considering the diversity and generality of regularization forms in convex optimization, it is natural to ask whether there are other regularizations can lead to sparseness. A positive answer to that question could be given.

In this paper, we propose a framework for regularized MDPs where a general form of regularizers is imposed on the expected discounted total reward. This framework includes the entropy regularized MDP as a special case, implying certain regularizers can induce sparseness. We first give the optimality condition in regularized MDPs under the framework and then give necessary and sufficient conditions to decide which kind of regularization can lead to a sparse optimal policy. Interestingly, there are a lot of regularization can lead to sparseness and the degree of sparseness can be controlled by the regularization coefficient. What’s more, we show that regularized MDPs have a regularization-dependent performance error caused by the introduced regularization term, which could guide us to choose less-error regularization when it comes to solving problems with a continuous action space. To solve regularized MDPs, we adopt the idea of generalized policy iteration and propose an off-policy actor-critic algorithm to figure out the performance of different regularizers.

2 Notation and Preliminaries

Markov Decision Process In reinforcement learning (RL) problems, the agent’s interaction with the environment is often modeled as an Markov decision process (MDP). An MDP is defined by a tuple \((S, A, P, r, P_0, \gamma)\), where \(S\) is the state space and \(A\) the action space with \(|A|\) actions. We use \(\Delta_X\) to denote the simplex on any set \(X\), which is defined as the set of distributions over \(X\), i.e., \(\Delta_X = \{P : \sum_{x \in X} P(x) = 1, P(x) \geq 0\}\). The vertex set of \(\Delta_X\) is defined as \(V_X = \{P \in \Delta_X : \exists x \in X, \text{s.t. } P(x) = 1\}\). \(P : S \times A \rightarrow \Delta_S\) is the unknown state transition probability distribution and \(r : S \times A \rightarrow [0, R_{\text{max}}]\) is the bounded reward on each transition. \(P_0\) is the distribution of initial state and \(\gamma \in (0, 1)\) is the discount factor.

Optimality Condition of MDP The goal of RL is to find a stationary policy which maps from state space to a simplex over the actions \(\pi : S \rightarrow \Delta_A\) that maximizes the expected discounted total reward, i.e.,

\[
\max_{\pi} \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \mid \pi, P_0 \right],
\]

where \(s_0 \sim P_0, a_t \sim \pi(|s_t), s_{t+1} \sim P(|s_t, a_t)\). Given any policy \(\pi\), its states value and Q-value functions are defined respectively as

\[
V^\pi(s) = \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \mid s = s, \pi \right],
\]

\[
Q^\pi(s, a) = \mathbb{E}_{a' \sim \pi(|s)} \left[ r(s, a) + \gamma \mathbb{E}_{s' \sim P(s') \mid s, a} V^\pi(s') \right].
\]

Any solution of the problem (1) is called an \textit{optimal} policy and denoted by \(\pi^*\). Optimal policies may not be unique in an MDP, but the optimal states value is unique (denoted \(V^*\)). Actually, \(V^*\) is the unique fixed point of the Bellman operator \(T\), i.e., \(V^*(s) = TV^*(s)\) and

\[
TV(s) \triangleq \max_{\pi} \mathbb{E}_{a \sim \pi(|s)} \left[ r(s, a) + \gamma \mathbb{E}_{s' \sim P(s') \mid s, a} V(s') \right].
\]

Any \(\pi^*\) must be a \textit{deterministic} policy which puts all probability mass on one action [31]. Actually, it can be obtained as the greedy action w.r.t. the optimal Q-value function, i.e., \(\pi^*(s) \in \arg\max_a Q^*(s, a)\). The optimal Q-value can be obtained from the state value \(V^*(s)\) by definition.

As a summary, any optimal policy \(\pi^*\) and its optimal states value \(V^*\) and Q-value \(Q^*\) satisfy the following \textit{optimality condition} for all states and actions,

\[
Q^*(s, a) = r(s, a) + \gamma \mathbb{E}_{s' \sim P(s') \mid s, a} V^*(s),
\]

\[V^*(s) = \max_a Q^*(s, a), \quad \pi^*(s) \in \arg\max_a Q^*(s, a).\]
3 Regularized MDPs

To obtain a sparse but multi-modal optimal policy, we impose a general regularization term to the objective (1) and solve the following regularized MDP problem

$$\max_{\pi} E \left[ \sum_{t=0}^{\infty} \gamma^t (r(s_t, a_t) + \lambda \phi(\pi(a_t|s_t))) \bigg| \pi, \pi_0 \right],$$  \hspace{1cm} (2)

where $\phi(\cdot)$ is a regularizer function. Problem (2) can be seen as a RL problem in which the reward function is the sum of the original reward function $r(s, a)$ and a term $\phi(\pi(a|s))$ that provides regularization. If we take expectation to the regularization term $\phi(\pi(a|s))$, it can be found that the quantity

$$H_\phi(\pi) = E_{a \sim \pi(a|s)} \phi(\pi(a|s)),$$

is entropy-like but not necessarily an entropy in our work. However Problem (2) is not well-defined since arbitrary regularizers would be more of a hindrance than a help. In the following, we make some assumptions about $\phi(\cdot)$.

3.1 Assumption for Regularizers

A regularizer $\phi(\cdot)$ characterizes solutions to Problem (2). In order to make Problems (2) analyzable, a basic assumption (Assumption 1) is prerequisite. Explanation and examples will be provided to show that such an assumption is reasonable and not rigorous.

**Assumption 1** The regularizer $\phi(x)$ is assumed to satisfy the following conditions on the interval $(0, 1)$: (1) **Monotonicity**: $\phi(x)$ is non-increasing; (2) **Non-negativity**: $\phi(1) = 0$; (3) **Differentiability**: $\phi(x)$ is differentiable; (4) **Expected Concavity**: $x \phi(x)$ is strictly concave.

The assumption of monotonicity and non-negativity makes the regularizer be an positive exploration bonus. This consideration follows from the optimism in the face of uncertainty principle [15], which could encourage exploration. The bonus for choosing an action of high probability is larger than that of choosing an action of low probability. When the policy becomes deterministic, the bonus is forced to zero. The assumption of differentiability facilitates theoretic analysis and benefits practical implementation due to the widely used automatic derivation in deep learning platforms. The last assumption of expected concavity makes $H_\phi(\pi)$ a concave function w.r.t. $\pi$. Thus any solution to (2) hardly lies in the vertex set of the action simplex $V_\Delta$ (where the policy is deterministic). As a byproduct, let $f_\phi(x) = x \phi(x)$, then its derivative $f_\phi'(x) = \phi(x) + x \phi'(x)$ is a strictly decreasing function on $(0, 1)$ and thus $(f_\phi')^{-1}(x)$ exists. For simplicity, we denote $g_\phi(x) = (f_\phi')^{-1}(x)$.

There are plenty of options for the regularizer $\phi(\cdot)$ that satisfy Assumption 1. First, entropy can be recovered by $H_\phi(\pi)$ with specific $\phi(\cdot)$. For example, when $\phi(x) = -\log x$, the Shannon entropy is recovered; when $\phi(x) = \frac{x}{q-1} (1-x)^{q-1}$ with $k > 0$, the Tsallis entropy is recovered. Second, there are many instances that are not viewed as an entropy but can serve as a regularizer. We find two families of such functions, namely, the exponential function family $q - x^k q^x$ with $k \geq 0, q \geq 1$ and the trigonometric function family $\cos(\theta x) - \cos(\theta)$ and $\sin(\theta) - \sin(\theta x)$ both with hyper-parameter $\theta \in (0, \frac{\pi}{2})$. Since these functions are simple, we term them basic functions.

Apart from basic functions mentioned above, we come up with a generic method to combine different basic functions. Let $F$ be the set of all functions satisfying Assumption 1. By Proposition 1, the operations of positive addition and minimum can preserve the properties shared among $F$. Therefore, finite-time application of such operation still leads to an available regularizer.

**Proposition 1** Given $\phi_1, \phi_2 \in F$, we have that $\alpha \phi_1 + \beta \phi_2 \in F$ for all $\alpha, \beta \geq 0$ and $\min\{\phi_1, \phi_2\} \in F$.

Here we only consider those differentiable $\min\{\phi_1, \phi_2\}$, because the minimum of any two functions in $F$ may be non-differentiable on some points. For instance, given any $q > 1$, the minimum of $-\log(x)$ and $q(1-x)$ has a unique undifferentiable point on $(0, 1)$.

3.2 Optimality and Sparsity

Once the regularizer $\phi(\cdot)$ is given, similar to non-regularized case, the (regularized) states value and Q-value functions of any given policy $\pi$ in a regularized MDP are defined as the following

$$V_\pi^\phi(s) = E \left[ \sum_{t=0}^{\infty} \gamma^t (r(s_t, a_t) + \lambda \phi(\pi(a_t|s_t))) \bigg| s_0 = s, \pi \right],$$

$$Q_\pi^\phi(s, a) = r(s, a) + \gamma E_{s' \sim \pi'(s|a)} E_{\pi'}[s, a] V_\pi^\phi(s'),$$ \hspace{1cm} (4)
Any solution to Problem (2) is called the regularized optimal policy (denoted \(\pi^*_\lambda\)). The corresponding regularized optimal state value and Q-value are also optimal and denoted by \(V^*_\lambda\) and \(Q^*_\lambda\) respectively. If the context is clear, we will omit the word regularized for simplicity. Similar to non-regularized RL where the optimal value and Q-value are unique, we can prove not only the regularized value and Q-value are unique but the optimal policy are also unique. We postpone the discussion about their uniqueness in next section. In this part, we aim to show the optimality condition for the regularized MDPs (Theorem 2). The proof of Theorem 2 is given in Appendix A.

**Theorem 2** Any optimal policy \(\pi^*_\lambda\) and its optimal state value \(V^*_\lambda\) and Q-value \(Q^*_\lambda\) satisfy the following optimality condition for all states and actions:

\[
Q^*_\lambda(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \mid s,a} V^*_\lambda(s'),
\]

\[
\pi^*_\lambda(a \mid s) = \max \left\{ \frac{\mu^*_\lambda(s)}{V^*_\lambda(s)} Q^*_\lambda(s,a), 0 \right\},
\]

\[
V^*_\lambda(s) = \mu^*_\lambda(s) - \lambda \sum_a \pi^*_\lambda(a \mid s)^2 \phi'(\pi^*_\lambda(a \mid s)).
\]

where \(\phi(x) = (f'_\phi)^{-1}(x)\) is strictly decreasing and \(\mu^*_\lambda(s)\) is a normalization term so that \(\sum_a \pi^*_\lambda(a \mid s) = 1\).

Theorem 2 shows how the regularization influences the optimality condition. Let \(f'_\phi(0) = \lim_{x \to 0^+} f'_\phi(x)\) for short. From (5), it can be shown that the optimal policy \(\pi^*_\lambda\) assigns zero probability to the actions whose \(Q^*_\lambda\) values are below the threshold \(\mu^*_\lambda(s) - \lambda f'_\phi(0)\) and assigns positive probability to near optimal actions in proportion to their \(Q\)-values (since \(g_\phi(x)\) is decreasing). The threshold involves \(\mu^*_\lambda(s)\) and \(f'_\phi(0)\). \(\mu^*_\lambda(s)\) can be uniquely solved from the equation obtained by plugging (5) into \(\sum_a \pi^*_\lambda(a \mid s) = 1\). Note that the resulting equation only involves \(\{Q^*_\lambda(s,a)\}\) for each \(s,a\). However, the value \(f'_\phi(0)\) can be infinity, making the threshold out of function. To see this, if \(f'_\phi(0) = \infty\), the threshold will be \(-\infty\) and all actions will be assigned positive probability in any optimal policy. Therefore, a regularized MDP has the property of a sparse optimal policy only when \(f'_\phi(0)\) is finite. Theorem 3 gives a necessary and sufficient condition of regularization that leads to a sparse optimal policy.

**Theorem 3** The following statements are equivalent: (1) A regularized MDP has the property of a sparse optimal policy; (2) \(f'_\phi(0) < \infty\); (3) \(0 \in \text{dom } f'_\phi\).

Theorem 3 provides us with criteria to determine whether a regularization could render its corresponding regularized optimal policy the property of sparseness. To facilitate understanding, let us see two examples. When \(\phi(x) = -\log(x)\), we have that \(\lim_{x \to 0^+} f'_\phi(x) = \lim_{x \to 0^-} -\log(x) - 1 = \infty\), which implies that the Shannon entropy-regularized MDP does not have sparseness. When \(\phi(x) = \frac{k}{q-1} (1 - x^{q-1})\) for \(q > 1\), the corresponding optimal policy is sparse because \(\lim_{x \to 0^+} f'_\phi(x) = \frac{k}{q-1}\). What’s more, Theorem 3 shows that the sparseness property of Tsallis entropy still keeps for \(1 < q < \infty\). Interestingly, there exists a phase transition phenomenon in sparseness as \(q\) approaches 1, because when \(q = 1\), the Tsallis entropy degenerates to the Shannon entropy due to \(\lim_{q \to 1^+} \frac{1}{q-1} (1 - x^{q-1}) = -\log(x)\). Additionally, when \(0 < q < 1\), the Tsallis entropy could no longer lead to sparseness due to \(\lim_{x \to 0^+} f'_\phi(x) = \lim_{x \to 0^+} \frac{k}{q-1} (x^{q-1} - 1) = \infty\).

The sparseness property is first discussed in [19] which shows the Tsallis entropy with \(k = \frac{1}{2}\) and \(q = 2\) can devise a sparse MDP. However, we point out that any \(\phi(x)\) such that \(f'_\phi(0) < \infty\) leads to a sparse MDP. Let \(G \subseteq F\) be the set of all sparse regularizers. The positive sum of any two sparse regularizers is still a sparse regularizer.

**Proposition 4** Given \(\phi_1, \phi_2 \in G\), we have that \(\alpha \phi_1 + \beta \phi_2 \in G\) for all \(\alpha, \beta \geq 0\). However, if \(\phi_1 \in G\) but \(\phi_2 \notin G\), \(\alpha \phi_1 + \beta \phi_2 \notin F\) for any positive \(\beta\).

It is easily checked that the two families (i.e., exponential functions and trigonometric functions) given in Section 3.1 can also induce a sparse MDP. For convenience, we prefer to term the function \(\phi(x) = \frac{k}{q-1} (1 - x^{q-1})\) that defines the Tsallis entropy as a polynomial function, because when \(q > 1\) it is a polynomial function of degree \(q - 1\). From the experiments in Section 6, we find that the trigonometric functions have stronger sparseness power than the rest, while the polynomial functions work better in deep RL continuous tasks. Additionally, these three basic families of functions which could be combined to construct more complex regularizers by addition (Proposition 4).
3.3 Properties of Regularized MDPs

In this section we present some properties of regularized MDPs. We first prove the uniqueness of the optimal policy and values. Next we give the bound of the performance error between $\pi_\lambda^*$ (the optimal policy obtained by a regularized MDP) and $\pi^*$ (the policy obtained by the original MDP). Finally, we show that the hyperparameter $\lambda$ controls the sparseness of the optimal policy when $0 \leq f_\phi'(0) < \infty$. In the proofs of this section, we need an additional assumption for regularizers. Assumption 2 is quite weak. All the functions introduced in Section 3.1 satisfy it.

**Assumption 2** The regularizer $\phi(\cdot)$ satisfies $f_\phi(0) \triangleq \lim_{x \to 0^+} x \phi(x) = 0$.

**Generic Bellman Operator** $T_\lambda$ We define a new operator $T_\lambda$ for regularized MDPs, which defines a smoothed maximum. Given one state $s \in S$ and current value function $V_\lambda$, $T_\lambda$ is defined as

$$T_\lambda V_\lambda(s) \triangleq \max_a \sum_s \pi(a|s) \left[ Q_\lambda(s,a) + \lambda \phi(\pi(a|s)) \right],$$

where $Q_\lambda(s,a) = r(s,a) + \gamma E_{s',a} V_\lambda(s')$ is Q-value function derived from one-step foreseeing according to $V_\lambda$. By definition, $T_\lambda$ maps $V_\lambda(s)$ to its possible highest value which considers both future discounted rewards and regularization term. We provide simple upper and lower bounds of $T_\lambda$ w.r.t. $T$, i.e.,

**Theorem 5** Under Assumption 7 and 2 for any value function $V$ and $s \in S$,

$$TV(s) \leq T_\lambda V(s) \leq TV(s) + \lambda \phi \left( \frac{1}{|A|} \right).$$

The bound (7) shows that $T_\lambda$ is a bounded and smooth approximation of $T$. When $\lambda = 0$, $T_\lambda$ reduces to the Bellman operator $T$. What’s more, it can be proved that $T_\lambda$ is a $\gamma$-contraction. By the Banach fixed point theorem, $V_\lambda^*$, the fixed point of $T_\lambda$, is unique. As a result of Theorem 6, $Q_\lambda^*$ and $\pi_\lambda^*$ are both unique. Above conclusion is formally stated and proved in Appendix D.

**Performance Error Between $V_\lambda^*$ and $V^*$** In general, $V^* \neq V_\lambda^*$. But their difference is controlled by both $\lambda$ and $\phi(\cdot)$. The behavior of $\phi(x)$ around the origin represents the regularization ability of $\phi(x)$. Theorem 7 shows that when $|A|$ is quite large (which means $\phi \left( \frac{1}{|A|} \right)$ is close to $\phi(0)$ due to its continuity), the closeness of $\phi(0)$ to 0 also determines their difference. As a result, the Tsallis entropy regularized MDPs have always tighter error bounds than the Shannon entropy regularized MDPs, because the value at the origin of the concave function $\frac{k}{x^{t-1}} (1 - x^{t-1}) (q > 1)$ is much lower than that of $-\log x$, both function satisfying in Assumption 2. Our theory incorporates the result of [19] which show a similar performance error for Tsallis entropy RL. The proof of Theorem 6 is detailed in Appendix E.

**Theorem 6** Under Assumption 7 and 2 the error between $V_\lambda^*$ and $V^*$ can be bounded, $\|V_\lambda^* - V^*\|_\infty \leq \frac{\lambda}{1-\gamma} \phi \left( \frac{1}{|A|} \right)$.

**Support Set of Optimal Policy** Let $S(s,\lambda)$ be the set of actions with nonzero probabilities in the optimal policy. When $f_\phi'(0) = \infty$, the support set of the optimal policy is always the same as the entire action space. When $f_\phi'(0) < \infty$, the optimal policy has a sparse distribution. Theorem 7 shows the cardinality of $S(s,\lambda)$ can be controlled by regularization coefficient $\lambda$ when $0 \leq f_\phi'(0) < \infty$. When $\lambda$ increases, $|S(s,\lambda)|$ also increases. In extreme cases, if $\lambda$ goes zero, the action with largest Q-value will be included in $|S(s,\lambda)|$ and if $\lambda$ goes infinity, the entire actions will be included in $|S(s,\lambda)|$. This property enables us to give a zero probability to non-optimal actions by controlling $\lambda$. The proof is detailed in Appendix F.

**Theorem 7** Let $Q_\lambda^*(s,a)$ and $\mu_\lambda^*(s)$ defined in Theorem 2 When $0 \leq f_\phi'(0) < \infty$, the cardinality of the set

$$S(s,\lambda) = \{ a : Q_\lambda^*(s,a) > \mu_\lambda^*(s) - \lambda f_\phi'(0) \}$$

is controlled by $\lambda$. Specifically, $|S(s,\lambda)|$ is a non-decreasing function of $\lambda$ for all $s \in S$.

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2It is worth to note that if $f_\phi'(0) < \infty$, it is not necessary $f_\phi'(0) \geq 0$. To see this, consider the example, $\phi(x) = \frac{4 - (x + 1)^2}{x}$ which satisfies Assumption 2 but $f_\phi'(0) = \lim_{x \to 0^+} -2(x + 1) = -2 < 0$. 

[ARCH 2019]
4 Regularized Actor-Critic

To solve problem (2), we introduce Regularized Policy Iteration (RPI), an algorithm that alternates between policy evaluation and policy improvement in the maximum regularized MDP framework. We first derive RPI on a tabular setting and show it provably converges to an optimal policy. Then we approximate RPI into a more practical algorithm which is an actor-critic method and thus named as regularized actor-critic (RAC).

4.1 Derivation of Regularized Policy Iteration

The derivation of RPI stems from generalized policy iteration [36] that alternates between policy evaluation and policy improvement. In the policy evaluation step, we wish to compute the Q-value $Q^\pi$ of a given policy $\pi$. When $\pi$ is fixed, $Q^\pi$ can be computed iteratively by initializing any Q-value function and repeatedly applying the modified Bellman backup operator $T^\pi_\lambda$ defined by

$$T^\pi_\lambda(s, a) \triangleq r(s, a) + \gamma \mathbb{E}_{s'|s, a} V^\lambda(s'),$$

where $V^\lambda$ is the state value function derived from $Q^\pi$,

$$V^\lambda(s') = \mathbb{E}_{a' \sim \pi(\cdot|s')} [Q^\lambda(s', a') + \phi(a')].$$

One can show that by repeatedly applying $T^\pi_\lambda$ to any initialized value function, the regularized Q-value $Q^\pi_\lambda$ of the policy $\pi$ will be obtained.

In the policy improvement step, we wish to update the evaluated policy $\pi_{old}$ to an improved policy $\pi_{new}$ in terms of its regularized Q-values. As a result, for each state $s$ we update the policy according to

$$\pi_{new}(a|s) = \arg \max \limits_\pi \mathbb{E}_{a' \sim \pi(\cdot|s)} [Q^\lambda_{old}(s, a) + \lambda \phi(\pi(a|s))].$$

If $\phi$ is good enough, we can find a closed form of $\pi_{new}$ for problem (10). For example, for Shannon entropy [26] with $\phi(x) = -\log(x)$, $\pi_{new}(a|s) \propto \exp(Q^\lambda_{old}(s, a))/\sum_{a \in A} \exp(Q^\lambda_{old}(s, a))$ is the normalization term and $S(s, \lambda)$ is given in Theorem 4. However, for a general $\phi$, it is unlikely to find a closed form of $\pi_{new}$. Thus in that case the solution can be obtained through a numerical optimization method, since the maximization problem (10) is a convex optimization whose domain is the probability simplex $\Delta_A$ and traditional convex solvers could solve it efficiently. Actually, in the experiments of RPI, for the regularization forms introduced in Section 3.1 except the two examples mentioned above, there is no closed form and we use numerical optimization to improve the old policy.

Once evaluating and improving the current policy $\pi_{old}$, we can prove the resulting policy $\pi_{new}$ has a higher regularized Q-value than that of the old one. Therefore, by alternating the policy evaluation and the policy improvement, any initializing policy will provably converge to the optimal policy $\pi^*_\lambda$ under the framework of regularized MDPs (Theorem 8).

However, there are some limitations on RPI. First, it records all Q-values in a tabular form, which is impracticable in an RL problem with large state space or actions space. Second, it takes a long time to wait for convergence of the policy evaluation and improvement. Especially problem (10) involves a possibly intricate convex optimization, which makes the policy improvement step more computationally expensive. To fix these weaknesses, we approximate RPI into a more practical algorithm. For the first problem, we apply neural networks to parameterize the Q-value and policy to increase expressive power. For the second, we truncate the two steps before convergence and do not solve (10) exactly. The approximation gives rise to a more practical algorithm RAC.

**Theorem 8** For any policy $\pi_0$, by repeatedly applying policy evaluation and regularized policy improvement, $\pi_0$ will converge to the optimal policy $\pi^*_\lambda$ in the sense that $Q^\lambda_\pi(s, a) \geq Q^\lambda_\pi(s, a)$ for all $\pi$ and $s \in S, a \in A$.

4.2 Regularized Actor Critic

As discussed in previous section, we aim to approximate RPI into a practical off-policy algorithm RAC. Actually, RAC is created by consulting the previous work SAC [12] [13] and making some necessary changes so that it is able to be agnostic to the form of regularization. To validate RAC, we will test it in continuous problems. Now we begin to describe the algorithm.

We use function approximators for both the Q-value function and policy to enhance their expressive power. We consider a parameterized regularized Q-value function $Q_\theta(s, a)$ and a tractable policy $\pi_\psi(a|s)$. Specifically, $Q_\theta(s, a)$ is a neural network. $\pi_\psi(a|s)$ is modeled as a factorized Gaussian distribution with the mean and variance modeled as neural...
We now approximate the gradient of $J$, where $a$ is an input noise vector, sampled from some fixed distribution (denoted $N$), such as the standard Gaussian. Therefore we can rewrite (10) as a minimization form:

$$J_Q(\theta) = \frac{1}{2} \mathbb{E}_D(Q_\theta(s_t, a_t) - y)^2,$$

(11)

where $D$ is the replay buffer used to eliminate the correlation of sampled trajectory data and $y$ is the target function defined as follows

$$y = r(s_t, a_t) + \gamma \left[ Q_\theta(s_{t+1}, a_{t+1}) + \lambda \phi(\pi_\psi(a_{t+1}|s_{t+1})) \right].$$

The target involves a target regularized Q-value function with parameters $\theta$ that are updated in a moving average fashion, which can stabilize the training process [24][12]. Thus the stochastic gradient of $J_Q(\theta)$ w.r.t. $\theta$ can be estimated by

$$\nabla J_Q(\theta) = \mathbb{E}_D \nabla_\phi Q_\theta(s_t, a_t) (Q_\theta(s_t, a_t) - y).$$

Note that the target Q-value function is modeled as a differentiable neural network. Rather then using the likelihood ratio gradient estimator [42] which does not require backpropagating the gradient through the policy and the target density networks, we apply the reparameterization trick to update the policy. The reparameterization trick can result in lower variance, so it is commonly used in prior work such as DDPG [20] and SAC [12]. Specifically, the policy is reparameterized as an factorized gaussian with tanh output, i.e.,

$$\pi_\psi(s, \epsilon) = \tanh(\text{mean}_\psi(s) + \epsilon \cdot \text{std}_\psi(s)),$$

where $\epsilon$ is an input noise vector, sampled from some fixed distribution (denoted $N$), such as the standard Gaussian. Therefore we can rewrite (10) as a minimization form:

$$J_\pi(\psi) = \mathbb{E}_D [\phi(\pi_\psi(s_t, \epsilon_t)) - Q_\theta(s_t, \phi(\pi_\psi(s_t, \epsilon_t)))].$$

We now approximate the gradient of $J_\pi(\psi)$ with

$$\hat{\nabla} J_\pi(\psi) = \nabla_\psi [\phi(\pi_\psi(a_t|s_t)) + (\nabla_{a_t} \phi(\pi_\psi(a_t|s_t)) - \nabla_{a_t} Q(s_t, a_t)) \nabla_a \pi_\psi(s_t, \epsilon_t),$$

where $a_t$ is evaluated at $\pi_\psi(s_t, \epsilon_t)$.

Our formal RAC is described in Algorithm[1] The method alternates between data collection and parameter updating. Trajectory data is collected by executing the current policy in the environment and then is stored in a replay buffer. Parameters of the function approximators are updated by descending along the stochastic gradients calculated from the batch sampled from that replay buffer. The method makes use of two Q-functions to overcome the positive bias incurred by overestimations of Q-value, which is known to yield a poor performance [14][9]. Specifically, these two Q-function are parameterized by parameters $\theta_t$ and are independently trained to minimize $J_Q(\theta_t)$. The minimum of the Q-functions is used to calculate the value gradient in $\hat{\nabla} J_Q(\theta)$ and the policy gradient $\hat{\nabla} J_\pi(\psi)$.
5 Related Work

Regularization in RL Recent deep reinforcement learning algorithms deploy the technique of regularization to enhance their theoretical or practical performance. The regularization function often is picked from $L_1$, $L_2$, entropy, Kullback-Leibler divergence, and relative entropy. These regularizations could be classified into three classes on their usage.

The first class aims to control the complexity of value function approximation. The use of function approximation makes it possible to model value (or Q-value) function when the state space is large or even infinite. The main regularization form is $L_2$ or $L_1$ regularization. For example, [23, 7] uses $L_2$ regularization to control the complexity of fitting value (or Q-value) functions. [18, 16] uses $L_1$ regularization for sparse feature selection.

The second class aims to better capture the geometry of parameter spaces and confine the information loss of policy search [50]. A lot of works propose to constraint the updated policy $\pi_{\text{new}}$ so that it is close to the old one $\pi_{\text{old}}$ in some sense. [50, 25, 22] use the Kullback-Leibler (KL) divergence as the measure for closeness and [3] considers a more general class of $f$-divergences.

The third class aims to modify the original MDP to a more tractable one. One considers the case the transition probabilities can be rescaled [37]. Others add a policy-related regularization term to the rewards, where entropy-regularized RL belongs. [29, 26, 11, 12] consider using the Shannon entropy, which is shown to improve both exploration and robustness. An MDP with Shannon entropy maximization is termed as soft MDP where the hard max operator is replaced by a softmax [11]. However, the optimal policy in soft MDPs put probability mass on all actions, implying some significantly unimportant actions would be executed. To fix this problem, [10] proposes to dynamically learn a prior that weights the importance of actions by using the mutual information. Alternatively, [25, 19] replace Shannon entropy with Tsallis entropy, since a special case ($q = 2$ in our notation) of Tsallis entropy can devise a sparse optimal policy [19].

In order to address the issues discussed in the introduction (i.e., to obtain a sparse but multi-modal optimal policy), only the regularization in the third class could work. However, they either focus on entropy regularization or consider too large function, the former ignoring various regularization forms in convex optimization and the latter having no implications for the choice of regularization. Thus we are motivated to propose a unified framework for regularized RL which extends current entropy-regularized RL and provides enough practical guidance.

Optimization for Entropy-regularized MDPs In the literature, there are many algorithms to solve entropy-regularized MDP problems. Similarly, these methods can be modified to solve regularized MDPs since the regularization we proposed is an extension of the traditional entropy.

[11, 19] consider the general modified value iteration approach. They repeatedly solve greedily the target regularized Q-values and updates the Q-value function in a Q-learning-like pattern. [52] discussed the equivalence between policy gradients and Q-learning where the entropy regularizer is Shannon entropy. [12] adopted actor-critic methods to solve the Shannon regularized MDP in an off-policy fashion and achieves the state-of-the-art performance in continuous control tasks in RL. [26] point out there exists a path consistency equation which only the (near) optimal value and policy satisfy and propose to minimize the residual of that equation by simultaneously updating value and policy functions. This method is called as Path Consistency Learning (PCL). [27, 25, 6] share the same methodology with PCL for Shannon entropy. [28] provides a unified view of entropy-regularized MDPs which enables us to formalize most entropy-regularized RL algorithms as approximate variants of Mirror Descent or Dual Averaging.

6 Experiments

We investigate the performance of different regularizers in discrete problems first in two numerical environments and then in continuous problems in Mujoco control environments. We use RPI to compute optimal policies in discrete environments and use RAC to tackle continuous problems.

6.1 Discrete Control

Discrete environments We test with four different regularizers including $-\log x$, $\frac{1}{2}(1 - x)$, $\cos(\frac{2}{3}x)$ and $\exp(1) - \exp(x)$ (Figure 1(a) plots those regularizers) in two following discrete environments. The first environment is a simple random generated MDP with state space $|S| = 50$ and action space $|A| = 10$. The rewards and transition probability matrix are sampled from a uniform distribution on $[0, 1]$. In order to make a sparse environment, those entries of the transition probability matrix which have the probability 0.95 are clipped as 0 and then each row of the clipped matrix is scaled to a probability distribution. The second environment is a Gridworld with size $(2N - 1) \times (2N - 1)$. A
A preprint - March 5, 2019

reward of 1 would be obtained at four corners, i.e., $\pm (N - 1) \times \pm (N - 1)$. The agent is initialized at the origin $(0, 0)$. At each state, the agent chooses one of the four directions (left, right, up, down) and then moves along that direction deterministically. We focus on the Gridworld at $N = 10$.

![Graphs showing support action ratio and optimal policy probability](image)

(a) Different regularizers $\phi$  
(b) $f_\phi'$ for different $\phi$  
(c) $\mathcal{R}$ for Random MDP  
(d) $\mathcal{R}$ for Gridworld

Figure 1: (a) plots four different regularization forms we will investigate. (b) shows the plot of $f_\phi'(0)$ since it implies the optimal policy has a sparse distribution. (c) and (d) shows the results of the support action ratio $\mathcal{R}$ on two environments (Random MDP and Gridworld). The support action ratio is the ratio of the number of total support actions and the total action number $|\mathcal{A}| |\mathcal{S}|$, which is a measure of the degree of sparseness for regularized optimal policies.

**Support action ratio** To measure the sparseness, we define the support action ratio $\mathcal{R}$ as the ratio of the number of total non-zero actions in the optimal policy and total action number $|\mathcal{A}| |\mathcal{S}|$. Specifically, the support action ratio is $\mathcal{R} = \frac{\sum_{s \in \mathcal{S}} |\mathcal{A}(s, \lambda)|}{|\mathcal{A}| |\mathcal{S}|}$ for a given regularization coefficient $\lambda$. (b) shows the plot of $f_\phi'(0) = \phi + x \phi'$ for four regularizers. From Theorem 3, all except $-\log$ should have a low $\mathcal{R}$. This is validated by (c) and (d) since no matter what value the regularization coefficient $\lambda$ is, the support action ratio of $-\log x$ is always 1, and, when the regularization coefficient value $\lambda$ is low, $\mathcal{R}$ for the rest regularizers is less than 1. As $\lambda$ increases, the support action ratios $\mathcal{R}$ of the four regularizers approximate 1 monotonously, implying that the regularization coefficient $\lambda$ can control the degree of sparseness of optimal policies as Theorem 7 states.

![Graphs showing changing process of probability mass on each action in optimal policy](image)

(a) $\cos: \cos(\frac{\pi}{2} x)$  
(b) $\exp: \exp(1) - \exp(x)$  
(c) tsallis: $\frac{1}{2}(1 - x)$  
(d) shannon: $- \log x$

Figure 2: (a)-(d) shows the changing process of the probability mass on each action in the optimal policy in a random MDP where $|\mathcal{A}| = 10$. There are totally ten colored curves in each figure with one color representing one action.

**Sparseness power** It can be seen from Figure 1 that when $\lambda$ is extremely large, $\mathcal{R} = 1$ holds for all regularizers. This is because large $\lambda$ reduces the importance of discounted reward sum and makes $H_\phi(\pi)$ dominate the loss, which forces the optimal policy to put probability mass evenly on all actions in order to maximize $H_\phi(\pi)$. We regard the ability to defend the tendency towards converging to a uniform distribution as sparseness power. To determine the power of sparseness for these regularizers, we gradually increase the regularization coefficient $\lambda$ and monitor the changing process of the probability mass on each action in the optimal policy at a fixed state. Limited by space, we only show the result in the random MDP in Figure 2. The results in the Gridworld can be looked up in Appendix E. When $\lambda$ is small, all regularizers except $-\log x$ have some zero-probability actions. When $\lambda$ is just over 2, $\exp$ and tsallis already have a full action support set. By contrast, $\cos$ is still sparse enough, implying the trigonometric function $\cos$ has a stronger ability in modeling sparseness. In the extreme case where $\lambda$ is sufficiently large, the optimal policy will converge to a uniform distribution on the action space as we expect.

6.2 Continuous Control

**Regularizers** We test RAC across four continuous control tasks from OpenAI Gym benchmark [5] with the MuJoCo simulator [38], namely Hopper-v2, Walker-v2, HalfCheetah-v2 and Ant-v2. Here we only choose two regularization
form: \text{shannon} \left(- \log x \right) \text{ and } \text{tsallis} \left( \frac{1}{2} \left(1 - x \right) \right). \) We don’t include \text{exp} and \text{cos} since they are quite unstable and prone to gradient exploding problems in deep RL training process. We speculate their unstableness results from numerical issues where the probability density function often diverges into infinity. All the details of the following experiments are given in Appendix F.

![Graphs showing performance comparison between Shannon and Tsallis regularization forms](image)

**Figure 3**: Evaluation curves on continuous control benchmarks. Each curve is the average of four experiments with different seeds. Each entry in the legend is named with the rule the regularization form \(+\lambda\). The score is smoothed with 30 windows while the shaded area is the 0.475 standard deviation.

**Performance compare** Figure 3 shows the total average return of evaluation rollouts during training for RAC with two regularization forms and different regularization coefficients \([0.01, 0.1, 1]\). For each curve, we train four different instances with different random seeds. The solid curves correspond to the mean and the shaded region to the 95% confidence intervals. \text{Tsallis} performs steadily better than \text{Shannon} given the same regularization coefficient \(\lambda\). \text{Tsallis} is also more stable since its shaded area is thinner than \text{Shannon}.

**Parameter sensitivity** As reported by [12], \text{Shannon} is very sensitive to the regularization coefficient \(\lambda\) (which is also referred as the temperature parameter). As an extreme example, when \(\lambda = 1\), \text{Shannon} fails to converge in Walker-v2 and Ant-v2. By contrast, \text{Tsallis} is less sensitive to \(\lambda\). As \(\lambda\) varies from 0.01 to 1, the performance of \text{Tsallis} doesn’t degrade to much.

7 Conclusion

In this paper, we have proposed a unified framework for regularized reinforcement learning, which includes entropy-regularized RL as a special case. Under this framework, the regularization characterizes the optimal policy and value of the corresponding regularized MDPs. We have shown there are many regularizers can lead to a sparse but multi-modal optimal policy. We have specified a necessary and sufficient condition for the sparse optimal policy. We have given three families of building block functions, namely the polynomial, trigonometric and exponential functions, and provided a generic method to combine them into more complicated regularizers. In numerical experiments, we have observed that the trigonometric function has stronger power in sparseness while the polynomial function performs stable and better in continuous control tasks. Moreover, we have studied other properties of our regularized MDPs. The performance error between the regularized optimal policy and the original optimal policy can be controlled by the regularization. The sparseness of the optimal policy can be controlled by the regularization coefficient. We have presented the logical and mathematical foundations of these properties and also conducted the experimental evaluation.
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Appendix

A Proof for Optimality Condition of Regularized MDPs

In this section, we give the detail proof for Theorem 2 which states the optimality condition of regularized MDPs. The proof follows from the Karush-Kuhn-Tucker (KKT) conditions where the derivative of a Lagrangian objective function with respect to policy \( \pi(a|s) \) is set zero.

Proof [for Theorem 2] The Lagrangian function of (2) obtained by the optimal policy is written as follows

\[
L(\pi, \beta, \mu) = \sum_s d_\pi(s) \left( \sum_a (Q_\lambda(s, a) + \lambda \phi(\pi(a|s))) \right) - \sum_s d_\pi(s) \left( \mu(s)(\sum_a \pi(a|s) - 1) + \sum a \beta(a|s) \pi(a|s) \right)
\]

where \( d_\pi \) is the stationary state distribution of the policy \( \pi, \mu \) and \( \beta \) are Lagrangian multipliers for the equality and inequality constraints respectively. Let \( f_\phi(x) = x\phi(x) \). Then the KKT condition of (2) are as follows, for all states and actions

\[
0 \leq \pi(a|s) \leq 1 \quad \text{and} \quad \sum_a \pi(a|s) = 1 \tag{12}
\]

\[
0 \leq \beta(a|s) \tag{13}
\]

\[
\beta(a|s) \pi(a|s) = 0 \tag{14}
\]

\[
Q_\lambda(s, a) + \lambda f_\phi(\pi(a|s)) - \mu(s) + \beta(a|s) = 0 \tag{15}
\]

where (12) is the feasibility of the primal problem, (13) is the feasibility of the dual problem, (14) results from the complementary slackness and (15) is the stationarity condition. We eliminate \( d_\pi(s) \) since we assume all policies induce an irreducible Markov chain.

Since \( f_\phi(x) = x\phi(x) \) is a strictly decreasing function due to (4) in Assumption 1 its inverse function \( g_\phi(x) = (f_\phi')^{-1}(x) \) is also strictly decreasing. From (15), we can resolve \( \pi(a|s) \) as

\[
\pi(a|s) = g_\phi \left( \frac{1}{\lambda}(\mu(s) - Q_\lambda(s, a)) - \beta(a|s) \right) \tag{16}
\]

Fix a state \( s \). For any positive action, its corresponding Lagrangian multiplier \( \beta(a|s) \) is zero due to the complementary slackness and \( Q_\lambda(s, a) > \mu(s) - \lambda f_\phi'(0) \) must hold. For any zero-probability action, its Lagrangian multiplier \( \beta(a|s) \) will be set such that \( \pi(a|s) = 0 \). Note that \( \beta(a|s) \geq 0 \), thus \( Q_\lambda(s, a) \leq \mu(s) - \lambda f_\phi'(0) \) must hold in this case. From these observations, \( \pi(a|s) \) can be reformulated as

\[
\pi(a|s) = \max \left\{ g_\phi \left( \frac{1}{\lambda}(\mu(s) - Q_\lambda(s, a)) \right), 0 \right\} \tag{17}
\]

By plugging (17) into (12), we obtain a new equation

\[
\sum_a \max \left\{ g_\phi \left( \frac{1}{\lambda}(\mu(s) - Q_\lambda(s, a)) \right), 0 \right\} = 1 \tag{18}
\]

Lemma 9 states that (18) has and only has one solution denoted as \( \mu^*_\lambda \). Therefore, \( \mu^*_\lambda \) can be solved uniquely. We defer the proof of Lemma 9 later in this section.

Next we aim to obtain the optimal state value \( V_\lambda^* \). It follows that

\[
V_\lambda^* (s) = \mathcal{T}_\lambda V_\lambda^*(s) \]

\[
= \sum_a \pi^*_\lambda(a|s) (Q_\lambda^*(s, a) + \lambda \phi(\pi^*_\lambda(a|s))) \]

\[
= \sum_a \pi^*_\lambda(a|s) (\mu^*_\lambda(s) - \lambda \pi^*_\lambda(a|s) \phi(\pi^*_\lambda(a|s))) \]

\[
= \mu^*_\lambda(s) - \lambda \sum_a \pi^*_\lambda(a|s)^2 \phi'(\pi^*_\lambda(a|s)).
\]

The first equality follows from the definition of the optimal state value. The second equality holds because \( \pi^*_\lambda \) maximizes \( \mathcal{T}_\lambda V_\lambda^*(s) \). The third equality results from plugging (15).
To summarize, we obtain the optimality condition of regularized MDPs as follows

\[ Q^\lambda(s,a) = r(s,a) + \gamma \mathbb{E}_{s'|s,a} V^\lambda(s'), \]
\[ \pi^\lambda(a|s) = \max \left\{ g_\phi \left( \frac{1}{\lambda} (\mu^\lambda(s) - Q^\lambda(s,a)) \right), 0 \right\}, \]
\[ V^\lambda(s) = \mu^\lambda(s) - \lambda \sum_a \pi^\lambda(a|s)^2 \phi'(\pi^\lambda(a|s)), \]

where \( g_\phi(x) = (f_\phi')^{-1}(x) \) is strictly decreasing and \( \mu^\lambda(s) \) is a normalization term so that \( \sum_{a \in A} \pi^\lambda(a|s) = 1. \]

\[ \text{Lemma 9} \quad \text{For any Q-value function } Q(s,a), \text{ the equation} \]
\[ \sum_a \max \left\{ g_\phi \left( \frac{1}{\lambda} (\mu(s) - Q(s,a)) \right), 0 \right\} = 1 \tag{19} \]

\[ \text{has and only has one } \mu^* \text{ satisfying it.} \]

**Proof**  Denote the left hand side of (18) which is a continuous function of \( \mu \) as \( h(\mu) \). We first prove that \( h(\mu) \) is a strictly decreasing function on \((-\infty, \mu_{\text{max}})\), where \( \mu_{\text{max}} = \max_a Q(s,a) + \lambda f_\phi'(0) \). Let \( \Lambda(s,\mu) \) the set of actions such that their maximum term in (19) is not obtained at 0, i.e., \( \Lambda(s,\mu) = \{a : Q(s,a) > \mu(s) - \lambda f_\phi'(0)\} \). Then for \( \mu_1 < \mu_2 < \mu_{\text{max}} \) it follows that \( \Lambda(s,\mu_2) \subseteq \Lambda(s,\mu_1) \) and
\[ h(\mu_1) - h(\mu_2) = \sum_{\alpha \in \Lambda(s,\mu_2)} \Delta(\mu_1,\mu_2) + \sum_{\alpha \in \Lambda(s,\mu_1) - \Lambda(s,\mu_2)} g_\phi \left( \frac{1}{\lambda} (\mu_1(s) - Q^\lambda(s,a)) \right), \]

where
\[ \Delta(\mu_1,\mu_2) = g_\phi \left( \frac{1}{\lambda} (\mu_1(s) - Q^\lambda(s,a)) \right) - g_\phi \left( \frac{1}{\lambda} (\mu_2(s) - Q^\lambda(s,a)) \right) \]
is positive for all actions in \( \Lambda(s,\mu_2) \). Since there must be at least one action in \( \Lambda(s,\mu_2) \), \( h(\mu_1) - h(\mu_2) > 0 \). Therefore, we have proved that \( h(\mu) \) decreases strictly on \((-\infty, \mu_{\text{max}})\). Note that \( h(\mu_{\text{max}}) = 0 < 1 \) and \( h(\mu_{\text{min}}) > 1 \) where \( \mu_{\text{min}} = \min_a Q(s,a) + \lambda f_\phi'(1) \). This result implies there exist a unique \( \mu^* \in (\mu_{\text{min}}, \mu_{\text{max}}) \) satisfying (19) as the result of the intermediate value theorem.

\[ \text{B \quad Proof for General Bellman Operator} \]

In [6], we define a general Bellman operator \( T_\lambda \) for regularized MDPs. Given one state \( s \in S \) and current value function \( V^\lambda \),
\[ (T_\lambda V^\lambda)(s) := \max \pi \sum_a \pi(a|s) [Q(s,a) + \lambda \phi(\pi(a|s))], \]

where \( Q(s,a) = r(s,a) + \gamma \mathbb{E}_{s'|s,a} V^\lambda(s') \) is Q-value function deriving from one-step foreseeing according to \( V^\lambda \). In Lemma 10, we prove \( T_\lambda \) is a \( \gamma \)-contraction. In Lemma 11, we prove the simple lower and upper bound for \( T_\lambda \) under Assumption 2.

\[ \text{Lemma 10} \quad T_\lambda \text{ is a } \gamma \text{-contraction.} \]

**Proof**  For any two state value functions \( V_1 \) and \( V_2 \), let \( \pi_i \) be the policy that maximize \( T_\lambda V_i, i \in \{1, 2\} \). Then it follows that for any state \( s \in S \),
\[ (T_\lambda V_1)(s) - (T_\lambda V_2)(s) \]
\[ = \sum_a \pi_1(a|s) \left[ r(s,a) + \gamma \mathbb{E}_{s'|s,a} V_1(s') + \lambda \phi(\pi_1(a|s)) \right] - \sum_a \pi_2(a|s) \left[ r(s,a) + \gamma \mathbb{E}_{s'|s,a} V_2(s') + \lambda \phi(\pi_2(a|s)) \right] \]
\[ \leq \sum_a \pi_1(a|s) \left[ r(s,a) + \gamma \mathbb{E}_{s'|s,a} V_1(s') + \lambda \phi(\pi_1(a|s)) \right] - \sum_a \pi_2(a|s) \left[ r(s,a) + \gamma \mathbb{E}_{s'|s,a} V_2(s') + \lambda \phi(\pi_1(a|s)) \right] \]
\[ = \gamma \sum_a \pi_1(a|s) \mathbb{E}_{s'|s,a} (V_1(s') - V_2(s')) \leq \gamma \|V_1 - V_2\|_\infty. \]
By symmetry, it follows that for any state $s$ in $S$,

$$(T_{\lambda}V_2)(s) - (T_{\lambda}V_1)(s) \leq \gamma \|V_1 - V_2\|_{\infty}$$

Therefore, it follows that

$$\|T_{\lambda}V_2 - T_{\lambda}V_1\|_{\infty} \leq \gamma \|V_1 - V_2\|_{\infty}$$

Lemma 11  Under Assumption[1] and [2] for any value function $V$ and $s \in S$, it follows that

$$TV(s) \leq T_{\lambda}V(s) \leq TV(s) + \lambda \phi\left(\frac{1}{|A|}\right).$$

Proof  Fix any value function $V$ and $s \in S$. Note that $\phi(\pi(a|s))$ is non-negative due to (1) and (2) in Assumption[1]. Therefore, by definition the left inequality follows from

$$T_{\lambda}V(s) = \max_\pi \sum_a \pi(a|s) \left[ r(s, a) + \gamma \mathbb{E}_{s'|s,a}V(s') + \lambda \phi(\pi(a|s)) \right]$$

$$\geq \max_\pi \sum_a \pi(a|s) \left[ r(s, a) + \gamma \mathbb{E}_{s'|s,a}V(s') \right] = TV(s).$$

For the right inequality, note that

$$T_{\lambda}V(s) = \max_\pi \sum_a \pi(a|s) \left[ r(s, a) + \gamma \mathbb{E}_{s'|s,a}V(s') + \lambda \phi(\pi(a|s)) \right]$$

$$\leq \max_\pi \sum_a \pi(a|s) \left[ r(s, a) + \gamma \mathbb{E}_{s'|s,a}V(s') \right] + \lambda \max_\pi H_\phi(\pi)$$

$$= TV(s) + \lambda \max_\pi H_\phi(\pi).$$

where $H_\phi(\pi) = \sum_a \pi(a|s)\phi(\pi(a|s))$ defined in (3) is what we next aim to bound.

The Lagrangian of solving $\max_\pi H_\phi(\pi)$ is

$$L(\pi, \beta, \mu) = H_\phi(\pi) + \mu(\sum_a \pi(a|s) - 1) + \beta_a \pi(a|s).$$

Its stationary condition is

$$\frac{\partial L}{\partial \pi(a|s)} = f_\phi(\pi(a|s)) + \mu + \beta_a = 0.$$
Then it follows that for any \( s \in S \),

\[
T\lambda V_1(s) = \max_{a} \sum_{a} \pi(a|s) \left[ r(s, a) + \gamma E_{a'|s,a} V_1(a') + \lambda \phi(\pi(a|s)) \right] \\
\leq \max_{a} \sum_{a} \pi(a|s) \left[ r(s, a) + \gamma E_{a'|s,a} V_2(a') + \lambda \phi(\pi(a|s)) \right] = T\lambda V_2(s)
\]

\[\blacksquare\]

**Lemma 13 (Translation)** Let \( c \) denote any constant. Define \((V + c)(s) \triangleq V(s) + c\) as the value function shifted by \( c \). Then it follows that for any \( s \in S \),

\[(T\lambda(V + c))(s) = (T\lambda V)(s) + \gamma c\]

**Proof** By definition, it directly follows from

\[(T\lambda(V + c))(s) = \max_{a} \sum_{a} \pi(a|s) \left[ r + \gamma E_{a'|s,a} (V + c)(a') + \lambda \phi(\pi(a|s)) \right] \\
= \max_{a} \sum_{a} \pi(a|s) \left[ r + \gamma E_{a'|s,a} V(a') + \gamma c + \lambda \phi(\pi(a|s)) \right] = (T\lambda V)(s) + \gamma c\]

\[\blacksquare\]

**Lemma 14 (Convergence of Repeated Applications)** For any initial value function \( V_0 \), define \( V_\lambda = \bigotimes_{n} T\lambda V_0 \) as the value function resulting from \( n \) times application of \( T\lambda \) to \( V_0 \). Then

\[
\lim_{n \to \infty} \| V_n - V_\lambda^* \|_\infty = 0.
\]

**Proof** Note that \( V_n^\lambda = \bigotimes_{n} T\lambda V_\lambda^* \). It follows that

\[
\| V_n - V_\lambda^* \|_\infty = \| \bigotimes_{n} T\lambda V_{n-1} - \bigotimes_{n} T\lambda V_\lambda^* \|_\infty \leq \gamma \| V_{n-1} - V_\lambda^* \|_\infty \leq \cdots \leq \gamma^n \| V_0 - V_\lambda^* \|_\infty.
\]
The first equality follows from definition. The first inequality results from Lemma[10] The last inequality is due to \( n \)-times applications of the first inequality.

**Proof [for Theorem 6]** Fix any initial value function \( V_0 \). We aim to use mathematical induction to prove the statement that for any \( n \geq 1 \), it follows for any \( s \in S \)

\[
T^n V_0(s) \leq T^\lambda V_0(s) \leq T^n V_0(s) + \lambda \phi \left( \frac{1}{|A|} \right) \sum_{t=0}^{n-1} \gamma^t.
\]  

(20)

When \( n = 1 \), (20) results from Lemma[11].

Suppose the statement holds when \( n = k(k \geq 1) \). Consider the case where \( n = k + 1 \). First it follows that

\[
T^{k+1} V_0(s) \leq T T^k V_0(s) \leq T^\lambda V_0(s).
\]

The first inequality follows from the hypothesis and the monotonicity of \( T \) (which is a special case of \( T\lambda \) when \( \lambda = 0 \)) from Lemma[12] The second inequality results from letting \( V = T^\lambda V_0 \) in Lemma[11].

Second, it follows that

\[
T^\lambda V_0(s) = T\lambda T^k V_0(s) \\
\leq T\lambda (T^k V_0(s) + \lambda \phi \left( \frac{1}{|A|} \right) \sum_{t=0}^{k-1} \gamma^t) \\
= T\lambda T^k V_0(s) + \lambda \phi \left( \frac{1}{|A|} \right) \sum_{t=0}^{k} \gamma^t \\
\leq T^{k+1} V_0(s) + \lambda \phi \left( \frac{1}{|A|} \right) \sum_{t=0}^{k} \gamma^t,
\]

16
where the first inequality follows from the induction where \( n = k \) and the monotonicity of \( T_n \) from Lemma 12. The second equality holds by setting \( V = T^n V_0 \) and \( c = \lambda \phi(\frac{1}{|A|}) \sum_{i=0}^{k-1} \gamma^i \) in Lemma 13. The last inequality results from letting \( V = T_n V_0 \) in Lemma 11.

Putting above results together, we prove that (20) holds when \( n = k + 1 \). Therefore by mathematical induction, (20) holds for any positive integer \( n \). From Lemma 14 we have \( V^*(s) = \lim_{n \to \infty} T^n V_0(s) \) and \( V^*_\lambda(s) = \lim_{n \to \infty} T_n^\lambda V_0(s) \). Now let \( n \) go infinity in both sides of (20), we obtain

\[
V^*(s) \leq V^*_\lambda(s) \leq V^*(s) + \frac{\lambda}{1 - \gamma} \phi(\frac{1}{|A|}),
\]

which proves the theorem.

\[
\text{D Proof for Support Set of Optimal Policy}
\]

In this section, we show that the number of positive actions can be controlled by regularization coefficient \( \lambda \). Similar results about Tsallis entropy regularized MDPs can be found in [19]. However their proof focuses on a specific regularization. The proof we provide is suitable for any regularizers satisfying Assumption 1.

Proof [for Theorem 7] Fix any \( s \in S \). A sufficient condition for the conclusion is \( S(s, \lambda_1) \subseteq S(s, \lambda_2) \) for all \( 0 < \lambda_1 < \lambda_2 \). We use proof by contradiction to prove it. If there exist \( a \in A \) and \( 0 < \lambda_1 < \lambda_2 \) such that \( a \in S(s, \lambda_1) \) but \( a \notin S(s, \lambda_2) \), by definition it follows that for that action \( a \),

\[
\frac{\mu^*_\lambda(s) - Q^*_\lambda(s, a)}{\lambda_1} < f'_\phi(0) \leq \frac{\mu^*_\lambda(s) - Q^*_\lambda(s, a)}{\lambda_2} \quad (21)
\]

Next we show that \( Q^*_\lambda \) is an increasing function of \( \lambda \) for all states and actions. To see this, by definition of \( Q^*_\lambda \) and non-negativity of \( \phi(\cdot) \), we have

\[
Q^*_\lambda(s, a) = \max_{\pi} \mathbb{E} \left[ \sum_{t=0}^{\infty} (r(s_t, a_t) + \lambda \phi(\pi(a_t|s_t))) | s_0 = s, a_0 = a \right]
\]

\[
\leq \max_{\pi} \mathbb{E} \left[ \sum_{t=0}^{\infty} (r(s_t, a_t) + \lambda \phi(\pi(a_t|s_t))) | s_0 = s, a_0 = a \right] = Q^*_{\lambda_2}(s, a).
\]

Since

\[
0 \leq \lambda_2 f'_\phi(0) \leq \mu^*_\lambda(s) - Q^*_\lambda(s, a) \quad (22)
\]

and \( Q^*_\lambda(s, a) \leq Q^*_{\lambda_2}(s, a) \), it follows that

\[
\frac{\mu^*_\lambda(s) - Q^*_\lambda(s, a)}{\lambda_2} \leq \frac{\mu^*_\lambda(s) - Q^*_\lambda(s, a)}{\lambda_1} \quad (23)
\]

(21) and (23) together imply that \( \mu^*_\lambda(s) < \mu^*_\lambda(s) \) and thus for all actions,

\[
\frac{\mu^*_\lambda(s) - Q^*_\lambda(s, a)}{\lambda_1} < \frac{\mu^*_\lambda(s) - Q^*_\lambda(s, a)}{\lambda_2} \quad (24)
\]

Therefore it follows that

\[
1 = \sum_{a \in S(s, \lambda_2)} \pi^*_\lambda(a|s) = \sum_{a \in S(s, \lambda_1) \cap S(s, \lambda_2)} \pi^*_\lambda(a|s) \quad (25)
\]

\[
< \sum_{a \in S(s, \lambda_1)} \pi^*_\lambda(a|s) = 1,
\]

17
which is a contradiction. The first and last equalities results from the constrain that the sum of positive probabilities in any optimal policy is one, i.e., \( \sum_{(a,s)} \pi_\lambda \tau(a|s) = 1 \). The first strict inequality uses (24) and the fact that \( g_0 \) is strictly decreasing. The second inequality uses the definition of the optimal policy (5). The last strict inequality uses the hypothesis that there exist \( a \) such that \( a \in S(s, \lambda_1) \) but \( a \notin S(s, \lambda_2) \), implying there is always a positive probability mass lost in the right hand side of (25).

\[ \vdots \]

**E Proof for Regularized Policy Iteration**

In this section, we give the proof of convergence of RPI. We first that repeatedly applying \( T_\pi^\lambda \) to any initialized policy leads to the Q-value of a given policy \( \pi_{\text{old}} \). Then we prove the policy improvement step will lead to a new policy \( \pi_{\text{new}} \) which has higher Q-value than \( \pi_{\text{old}} \).

**Lemma 15 (Policy Evaluation)** Fix any policy \( \pi \). Consider the Bellman backup operator \( T_\pi^\lambda \) in (8), for any initial Q-value \( Q_0 \), let \( Q_n = T_\pi^\lambda Q_{n-1} \) \((n \geq 1)\). Then \( \lim_{n \to \infty} \|Q_n - Q_\infty\|_\infty = 0 \).

**Proof** Similar to Lemma 10, we can prove \( T_\pi^\lambda \) is a \( \gamma \) contraction. Note that \( Q_n = T_\pi^\lambda Q_{n-1} \). Therefore we have that

\[ \|Q_n - Q_\infty\|_\infty = \|T_\pi^\lambda Q_{n-1} - T_\pi^\lambda Q_\infty\|_\infty \leq \gamma \|Q_{n-1} - Q_\infty\|_\infty \leq \cdots \leq \gamma^n \|Q_0 - Q_\infty\|_\infty. \]

When \( n \) goes infinity, \( Q_n \) will converge to the regularized Q-value of \( \pi \).

**Lemma 16 (Policy Improvement)** Let \( \pi_{\text{old}} \) be the evaluated policy with \( Q^\tau_{\lambda_{\text{old}}} \) its regularized Q-value and \( \pi_{\text{new}} \) be the optimizer of the maximization problem defined in (10). Then \( Q^\tau_{\lambda_{\text{old}}}(s, a) \leq Q^\tau_{\lambda_{\text{new}}}(s, a) \) for all \( s \in S \) and \( a \in A \).

**Proof** Since \( \pi_{\text{new}} \) is the maximizer of the problem defined in (10), it follows for all states and actions

\[ \mathbb{E}_{a \sim \pi_{\text{old}}} [Q^\tau_{\lambda_{\text{old}}}(s, a) + \lambda \phi(\pi_{\text{old}}(a|s))] \leq \mathbb{E}_{a \sim \pi_{\text{new}}} [Q^\tau_{\lambda_{\text{new}}}(s, a) + \lambda \phi(\pi_{\text{new}}(a|s))]. \]

Let \( \tau_i = (s_0, a_0, \cdots, s_i, a_i) \) denotes the trajectory and \( \tau \) is the whole trajectory (with infinite horizon). \( \tau \sim \pi_{\text{old}} \) means the trajectory is generated by \( \pi_{\text{old}} \). It follows that

\[ Q^\tau_{\lambda_{\text{old}}}(s_0, a_0) \]
\[ = \mathbb{E}[r(s_0, a_0) + \gamma \mathbb{E}_{s_1|s_0} V^\tau_{\lambda_{\text{old}}}(s_1)] \]
\[ = \mathbb{E}[r(s_0, a_0) + \gamma \mathbb{E}_{s_1|s_0} \mathbb{E}_{a_1 \sim \pi_{\text{old}}} [Q^\tau_{\lambda_{\text{old}}}(s_1, a_1) + \lambda \phi(\pi_{\text{old}}(a_1|s_1))]] \]
\[ \leq \mathbb{E}[r(s_0, a_0) + \gamma \mathbb{E}_{s_1|s_0} \mathbb{E}_{a_1 \sim \pi_{\text{new}}} [Q^\tau_{\lambda_{\text{old}}}(s_1, a_1) + \lambda \phi(\pi_{\text{new}}(a_1|s_1))]] \]
\[ = \mathbb{E}_{\tau_i \sim \pi_{\text{old}}} [r(s_0, a_0) + \gamma (r(s_1, a_1) + \lambda \phi(\pi_{\text{old}}(a_1|s_1))) + \gamma^2 \mathbb{E}_{s_2|s_1} V^\tau_{\lambda_{\text{old}}}(s_2)] \]
\[ \leq \mathbb{E}_{\tau_i \sim \pi_{\text{new}}} [r(s_0, a_0) + \sum_{t=1}^{\infty} \gamma^t (r(s_t, a_t) + \lambda \phi(\pi_{\text{new}}(a_t|s_t))) + \gamma^{n+1} \mathbb{E}_{s_{n+1}|s_n} V^\tau_{\lambda_{\text{old}}}(s_{n+1})] \]
\[ \leq \mathbb{E}_{\tau_i \sim \pi_{\text{new}}} [r(s_0, a_0) + \sum_{t=1}^{\infty} \gamma^t (r(s_t, a_t) + \lambda \phi(\pi_{\text{new}}(a_t|s_t)))] \]
\[ = Q^\tau_{\lambda_{\text{new}}}(s_0, a_0), \]

where the last inequality is because we repeatedly expanded \( Q^\tau_{\lambda_{\text{old}}} \) on the RHS by applying (9) and \( Q^\tau_{\lambda_{\text{old}}} \) is bounded by

\[ \frac{R_{\text{max}}}{1 - \gamma}. \]

**Proof [for Theorem 5]** Let \( \pi_i \) be the policy at iteration \( i \) of RPI. By Lemma 16, the sequence \( Q^\tau_{\pi_i} \) is monotonically increasing. Since \( Q^\tau_{\lambda} \) is bounded by \( \frac{R_{\text{max}}}{1 - \gamma} \) for any policy \( \pi_i \), therefore \( Q^\tau_{\lambda} \) will converge to a limit, denoted by \( Q^\tau_{\lambda_{\text{lim}}} \). Let \( \pi_{\text{lim}} = \arg\max_{\pi} \mathbb{E}_{a \sim \pi} [Q^\tau_{\lambda_{\text{lim}}}(s, a) + \lambda \phi(\pi(a|s))]. \) It is obvious that \( Q^\tau_{\lambda_{\text{lim}}} = Q^\tau_{\lambda_{\text{lim}}}. \) We aim to prove \( \pi_{\text{lim}} = \pi_{\lambda}. \)

To that end, we only need to prove \( Q^\tau_{\lambda} = Q^\tau_{\lambda_{\text{lim}}}. \) For one hand, \( Q^\tau_{\lambda_{\text{lim}}}(s, a) = \lim_{n \to \infty} Q^\tau_{\pi_i}(s, a) \leq Q^\tau_{\lambda}(s, a) = Q^\tau_{\lambda_{\text{lim}}}(s, a). \)

For another hand, at convergence, it must be the case that for all policy \( \pi \),

\[ \mathbb{E}_{a \sim \pi} [Q^\tau_{\lambda_{\text{lim}}}(s, a) + \lambda \phi(\pi(a|s))] \leq \mathbb{E}_{a \sim \pi_{\text{lim}}} [Q^\tau_{\lambda_{\text{lim}}}(s, a) + \lambda \phi(\pi_{\text{lim}}(a|s))]. \]
Using the same iterative argument as in the proof of Lemma 16, we get $Q^\pi_\lambda(s, a) \leq Q^{\text{lim}}_\lambda(s, a)$ for all states and actions. Putting above results together, it follows that $Q^\pi_\lambda = Q^{\text{lim}}_\lambda$ therefore $\pi_{\text{lim}} = \pi^*_\lambda$.

**F Experiment Details**

**F.1 Discrete Environments**

**F.1.1 Environment setup**

For the random MDP model, we choose $|A| = 50$, $|S| = 10$ and $\gamma = 0.99$. Each state is assigned an index ranging from 0 to 49. The transition probabilities are generated by uniform distribution $[0, 1]$ and each entry of transition is clipped as zero with probability 0.95. Then each row of the clipped matrix is scaled to a probability distribution. The state we monitored is the state with index zero. The rewards are generated by uniform distribution $[0, 1]$. The initial Q-value is generated by uniform distribution $[0, 10]$ and policies are calculated explicitly or implicitly from Q-values.

For $(2N - 1) \times (2N - 1)$ GridWorld model, we choose $N = 10$ and $\gamma = 0.99$. The action space includes four actions (left, right, up, down). Each grid is indexed by an Cartesian coordinates $(x, y)$ with $x$ the row index and $y$ the column index. $x$ and $y$ are all range from $-(N - 1)$ to $N - 1$. The agent is initialized at the origin $(0, 0)$. Once it achieves four corners (i.e., $\pm(N - 1) \times \pm(N - 1)$), a reward with value 1 will be obtained. Otherwise, no reward will be given. Due to the symmetry of GridWorld, we are interesting on the three states $(0, 0)$, $(0, N/2)$, $(N/2, 0/N)$. In the origin $(0, 0)$, all actions should be equal. While the agent locates at $(0, N/2)$ or $(N/2, 0/N)$, the optimal policy should put more probability mass on the action which could lead to

**F.1.2 Optimization**

**Policy evaluation** Given a regularization term, we run 500 iterations of RPI that alternates between policy evaluation and policy improvement. Since in our experiments the transition probability is known, the evaluation of a given policy is conducted by DP. Specifically, let $P^\pi = \mathbb{P} \in \mathbb{R}^{|S| \times |S|}$ denote the transition matrix deduced from $\pi$, i.e., $P^\pi(s, s') = \sum_a \pi(a'|s)P(s'|a', s)$ and $r^\pi = \mathbb{R}^{|S|}$ the reward vector deduced from $\pi$, i.e., $r^\pi(s) = \sum_a r(s, a')\pi(a'|s)$. Then the regularized state value function $V^\pi_\lambda$ is solved from

$$V^\pi_\lambda = r^\pi + \gamma P^\pi V^\pi_\lambda \Rightarrow V^\pi_\lambda = (1 - \gamma P^\pi)^{-1}r^\pi$$

where by slightly notation abuse, $V^\pi_\lambda \in \mathbb{R}^{|S|}$ is the vector with each coordinate $V^\pi_\lambda(s)$. Then $Q^\pi_\lambda$ can be computed from $V^\pi_\lambda$ by definition 4.

**Policy improvement** The policy improvement step involves an possibly intricate convex optimization. Here we detail how we solve the involved convex optimization.

Let $Q^\pi_{\text{old}}(s, a)$ denote the already evaluated Q-value function of $\pi_{\text{old}}$. For $\phi(x) = \frac{1}{2}(1 - x)$, since the improved policy has an explicit form $\pi_{\text{new}}(a|s) = \max \left( \frac{Q^\pi_{\text{old}}(s, a)}{\lambda} - \tau \left( \frac{Q^\pi_{\text{old}}(s, \cdot)}{\lambda} \right) \right)$, where $\tau(Q^\pi_{\text{old}}(s, \cdot)) = \sum_{s' \in S(s, \lambda)} \frac{Q^\pi_{\text{old}}(s', a)}{|S(s, \lambda)|}$ is the normalization term and $S(s, \lambda)$ is given in Theorem 7. We only need to figure out $S(s, \lambda)$. Note that $S(s, \lambda)$ only contains those actions whose Q-values are among the $|S(s, \lambda)|$ largest Q-values. We then use binary search method to compute the cardinality of support action of $\pi_{\text{new}}$, i.e., $|S(s, \lambda)|$. For $\phi(x) = \cos(\frac{x}{2})$ and $\phi(x) = \exp(1) - \exp(x)$ which do not have a closed form and their corresponding $g_\phi$ are hard to formulate, thus we solve the convex optimization problem 10 directly. Specifically, for each $s \in S$, we solve

$$\max_a \sum_a \pi(a|s)Q^\pi_{\text{old}}(s, a) + \lambda \sum_a \pi(a|s)\phi(\pi(a|s))$$

In practice, we use CVXOPT package to compute the improved policy.

**F.1.3 Results for Gridworld**

Figure 4 shows the probability mass on four actions in the optimal policy at selected three states. When $\lambda$ is large, the optimal policies tend to uniform distribution. In the regime of low $\lambda$, the optimal policy at different states show different preference. As shown in Random MDP, $\cos$ still has the strongest sparseness power.
We use OpenAI Gym benchmark with the MuJoCo simulator, including Hopper-v2, Walker-v2, HalfCheetah-v2 and Ant-v2. We test the same four regularizers in discrete environments. However, the regularization form +\( \lambda \) to suffer from gradients explosion. Thus we only show two regularizers, i.e., -\( \log x \) and \( \frac{1}{2} (1 - x) \) for their stable performance in deep RL training process. Since RAC is very similar to SAC except RAC is agnostic to regularization forms. We build our code on the work of SAC [13]. For comparison’s purpose, we use the same network structure and hyper-parameter settings. We provide the total average return of training rollouts in Figure 5.

Figure 4: The probability mass on four actions in the optimal policy at selected three states. (a)-(d) shows the results for the origin \((0,0)\). (e)-(h) shows the results for the state \((0,N/2)\) and (i)-(l) shows the results for the state \((N/2,N/2)\).

F.2 Continuous Environments

We use OpenAI Gym benchmark with the MuJoCo simulator, including Hopper-v2, Walker-v2, HalfCheetah-v2 and Ant-v2. We test the same four regularizers in discrete environments. However, \( \exp \) and \( \cos \) are quite unstable and prone to suffer from gradients explosion. Thus we only show two regularizers, i.e., -\( \log x \) and \( \frac{1}{2} (1 - x) \) for their stable performance in deep RL training process. Since RAC is very similar to SAC except RAC is agnostic to regularization forms. We build our code on the work of SAC [13]. For comparison’s purpose, we use the same network structure and hyper-parameter settings. We provide the total average return of training rollouts in Figure 5.

Figure 5: Training curves on continuous control benchmarks. Each curve is the average result of four experiments with different seeds. Each entry in the legend is named as the regularization form + \( \lambda \).