Exploring the Steiner-Soddy Porism

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Abstract. We explore properties and loci of a Poncelet family of polygons – called here Steiner-Soddy – whose vertices are centers of circles in the Steiner porism, including conserved quantities, loci, and its relationship to other Poncelet families.

Keywords: triangle · inversive · pedal polygon · porism · Poncelet · invariant

1 Introduction

A Steiner chain is a set of pairwise-tangent circles, all of whom are tangent to a pair of disjoint circles, called the inner and outer “Soddy” circles, see Figure 1(left). The chain is poristic since it is the inversive image of a set of identical, mutually-tangent circles, centered at the vertices of a regular polygon, see Figure 1(right). In this article, we explore the family of “Steiner-Soddy” polygons whose vertices are the centers of circles in a Steiner porism.

Main Results. Let $\mathcal{P}$ denote polygons in the Steiner-Soddy family. Let $N$ denote the number of its polygonal sides.

- The $\mathcal{P}$ are conic-inscribed and circumscribe a circle, and are therefore a Poncelet porism [7].
- The outer conic of the $\mathcal{P}$ has foci on the centers of the inner and Soddy circles of the Steiner chain.
- When the center of the caustic is on the circumference of the inner Soddy circle, the family becomes parabola-inscribed and the outer Soddy circle degenerates to a line.
- As a corollary to a result proved in [19], the family conserves the sum of powers of half-angle tangents, up to power $N - 1$.
- The locus of the perimeter centroid⁴ is a conic. Recall this is not guaranteed for a generic Poncelet family [18].

For the case when the $\mathcal{P}$ are triangles ($N = 3$) we obtain:

- The half-tangents of the family are equal to the cotangents of its intouch triangles.

⁴ This is the weighted average of the midpoints of sides, where the weights are side-lengths.
– The sum of tangents of half angles is less than, equal, or greater than two if the family is ellipse, parabola, or hyperbola inscribed, respectively.
– When the family parabola-inscribed, the locus of the orthocenter is a line.
– In the spirit of [14], we tour loci of some triangle centers over the \( \mathcal{P} \), showing some to be stationary, circles, lines, conics, and non-conics.

![Diagram](image)

**Fig. 1:** **Left:** Consider a set of pairwise tangent circles (blue) centered on the vertices of a regular polygon (gray). By symmetry, an inner and an outer “Soddy” circle (light blue) can be drawn tangent to all circles. Let the shaded red circle be an inversion circle. **Right:** a Steiner chain is the inversive image of the setup in the left with respect to said inversion circle. Also shown is the Steiner-Soddy polygon \( \mathcal{P} \) (orange) whose vertices are the centers of circles in the chain. A Poncelet porism of \( \mathcal{P} \) exists inscribed in a conic (gray) whose foci are the centers of the inversive images of the two original Soddy circles (light blue). The caustic is a circle (dashed gray) centered at \( I \). Also shown is the pedal polygon of \( \mathcal{P} \) (magenta) with respect to \( I \), also Ponceletian.

**Related Work.** In [19] it is proved that the first \( k - 1 \) moments of curvatures are invariant over Steiner’s porism. Loci of vertex, area, and perimeter centroids of a generic Poncelet family are studied in [18], where it is shown that the first two are always conics. In [11] a condition is derived which guarantees that a certain center of a triangle exists iff the sum of half-tangents is less than two; this is equivalent to whether the center of the outer Soddy circle is everted or not [9]. The Poncelet family which is the polar image of bicentrics was studied in [3], and that of harmonic polygons was studied in [16]. Seminal studies of loci of triangle centers over families of Poncelet triangles include [14, 20, 23, 24]. A theory for the type of locus swept by triangle centers over the confocal Poncelet family is presented in [12].

**Article organization.** In Section 2 we prove the main properties of the Steiner-Soddy family for all \( N \). In Section 3 we prove certain results specialized to the
N = 3 case. Loci of triangle centers in the N = 3 are toured Section 4. We finish with a discussion in Section 5 comparing invariants of various Poncelet families. Explicit formulas for some of the objects mentioned herein appear in Section 5.

2 The Steiner-Soddy Porism

Consider polygons $\mathcal{P}$ with vertices at the centers of circles in a Steiner porism. We call these “Steiner-Soddy” polygons (or family). Referring to Figure 1, the points of contact between consecutive circles in the chain are concyclic, since these lie on the inversive image of a regular polygons’ incircle. Call this circle $\mathcal{C}$ and its center $I$. Let $\mathcal{H}$ denote the circle-inscribed polygon whose vertices are said contact points. $\mathcal{H}$ is known to be harmonic, i.e., it is circle-inscribed and contains a special point $K$ whose distance to the sides are a constant proportion of the sidelengths [5].

It is known that the tangent lines between consecutive pairs of circles in a Steiner chain meet at $I$ [22, pp. 120,244–245]. Therefore:

Remark 1. $\mathcal{P}$ is the pedal polygon of $\mathcal{H}$ with respect to $I$, and therefore the sides of $\mathcal{P}$ are tangent to $\mathcal{C}$.

The Steiner porism has two fixed “Soddy” circles $S$ and $S'$ which are tangent to all $N$ circles in chain. Let $S_{\text{reg}}$ and $S'_{\text{reg}}$ denote their inversive pre-images in the regular setup, Figure 1(left).

Proposition 1. $\mathcal{P}$ is inscribed in a conic $\mathcal{E}$ which is an (i) ellipse, (ii) hyperbola, or (iii) parabola if the inversion center is (i) interior to $S_{\text{reg}}$ or exterior to $S'_{\text{reg}}$, (ii) in the annulus between $S_{\text{reg}}$ and $S'_{\text{reg}}$, or (iii) on either circle. Furthermore, the foci of $\mathcal{E}$ are the centers of the inner and outer Soddy circles of $S$ and $S'$. In the case of a parabola, the center of $S'$ is at infinity and $S'$ becomes a line parallel to the directrix.

Proof. It is well-known that locus of the center of a circle internally tangent to two fixed circles pair of circles is an ellipse (resp. hyperbola) if one circle is contained in another (resp. disjoint). Likewise, said locus when one circle has infinite radius is a parabola [13].

Since $\mathcal{P}$ is inscribed to a conic $\mathcal{E}$ and circumscribed a circle $\mathcal{C}$:

Corollary 1. The family $\mathcal{P}$ is Ponceletian.

In [10] it is shown that the pedal family $\mathcal{H}$ is also Ponceletian. Its envelope or caustic is known as the Brocard inellipse $\mathcal{B}$, see Section 5 for explicit expressions.

Let $\theta_i$ denote the internal angle of $\mathcal{P}$ at its $i$-th vertex $P_i$.

Theorem 1. The $\mathcal{P}$ family conserves $\sum_{i=1}^{N} \tan(\theta_i/2)$ is conserved.

Proof. Referring to Figure 1, let $P_i$ (resp. $P'_i$) be the centers (resp. the contact points between) consecutive circles in the Steiner chain. Let $r$ (resp. $r_i$) denote the radius of $\mathcal{C}$ (resp. a circle in chain centered at $P_i$). Since $|P_iP'_i| = |P_iP'_{i+1}|$ =
$r_i$ and $|OP_i'| = |OP_i'| = r$, the line $P_iO$ bisects $P_iP_i'P_{i+1} = \theta_i$. Since $\mathcal{H}$ is the $I$-pedal of $P$, $\mathcal{C}$ is tangent to $P_iP_{i+1}$ at $P_i'$, so $\Delta P_iP_i'O$ is right-angled, i.e., $P_iP_i'O = 90^\circ$. Hence, $\tan(P_iP_i'O) = \frac{OP_i'}{P_iP_i'}$. Thus:

$$\tan \frac{\theta_i}{2} = \frac{r}{r_i} \text{ hence } \sum_{n=1}^{n} \tan^k(\theta_i/2) = \sum_{n=1}^{n} \left( \frac{r}{r_i} \right)^k,$$

for any $k$. (1)

The claim is obtained by invoking a result from [19, Theorem 1], namely that over Steiner’s porism, the sum of the $k^{th}$ powers of curvatures of circles in chain, up to $N - 1$, is invariant.

**Remark 2.** When $\mathcal{E}$ is a hyperbola, the sign of $\tan(\theta_i/2)$ must be flipped for the two angles whose neighboring vertices lie on different branches of the hyperbola.

**Remark 3.** Richard Schwartz [17] has suggested an elegant interpretation of conservations of the above type: All homogeneous polynomials in 2-variables of degree less than $N$ have the same average on the unit circle as they do on a regular $N$-gon inscribed in the circle.

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**Fig. 2:** Left: If the inversion center is on $S_{reg}'$, the Soddy polygon $P$ (orange) is inscribed to a parabola (gray), and $S'$ degenerates to a line (vertical cyan), parallel to the directrix; in this case, $I$, the center of the caustic $\mathcal{C}$ (dashed grey), is on $S$ (cyan circle). Right: When $S$ and $S'$ are disjoint, the family is hyperbola-inscribed. The family oscillates between two states: (i) as shown, all but one vertex lie on a single branch, and (ii) not shown, all vertices lie on a single branch.

**Proposition 2.** Over a hyperbola-inscribed Steiner-Soddy family, the vertices of $\mathcal{P}$ oscillate between two states: (i) all on a single branch of the hyperbola or
(ii) \( N - 1 \) contiguous vertices on one branch and one on the other branch. This corresponds to whether \( I \) is interior (resp. exterior) to \( \mathcal{H} \), the \( I \)-pedal polygon.

**Proof.** When \( \mathcal{S} \) and \( \mathcal{S}' \) are disjoint (hyperbola case), the circles in the chain are either (all) externally tangent to both, or there exists a circle \( \Gamma \) which is internally tangent to one of the Soddy circles. In this case, \( \Gamma \) is centered at a point which lies on the distal branch of the hyperbola, and is also the unique solution to an Apollonius problem for the two Soddy circles and one neighbor in the chain, which is externally tangent to all of them. So there can be at most one center on the distal branch. In this case, \( \Gamma \) is internally tangent to two of the circles in the chain and the perpendiculars at the tangency points meet at \( I \), which lies outside the pedal polygon.

**Relationship with the Homothetic family** Let the homothetic family refer to a Poncelet family interscribed between two concentric, homothetic conics. Let their foci be called “inner” and “outer” ones. In [16] it was shown that the harmonic family is the polar image of the homothetic family with respect to an inversion circle centered on one of the outer foci. Referring to Figure 3:

**Proposition 3.** The Steiner-Soddy family is the polar image of the homothetic family with respect to an inversion circle centered on one of its inner foci.

![Fig. 3: The Steiner-Soddy family, orange (resp. harmonic family, magenta), is the polar image of the “homothetic” family, with respect to a circle centered on an inner (resp. outer) focus.](image)
Centroid Loci In [18] it was proved that the locus of the vertex and area centroids $C_0, C_2$ are conics over any Poncelet family whereas the locus of $C_1$ is not always a conic. Arseniy Akopyan reminded us that if a polygon circumscribes a circle (of center $O$), $C_1, C_2, O$ are collinear and $(C_1 - O) = (3/2)(C_2 - O)$ [1]. Referring to Figure 4:

Corollary 2. Over the Steiner-Soddy porism of any number $N$ of sides, the locus of the perimeter centroid $C_1$ is a conic.

By symmetry, it follows that the major axes of the loci of $C_0, C_1$ are coaxial with the outer ellipse of the porism.

![Fig. 4: Over the Soddy-Steiner porism (orange), the loci of the vertex, area, and perimeter centroids $C_0, C_1, C_2$ are conics (dashed red, green, blue) coaxial with the outer ellipse (gray). Notice $I, C_2, C_1$ are collinear (green line) and $|IC_1|/|IC_2| = 3/2$ [1].](image)

3 The Special case of N=3

Consider an $N = 3$ Steiner-Soddy family. We present a direct proof of Theorem 1 based on Descartes’ theorem.

Proof. Let $1/\rho = 1/r_1 + 1/r_2 + 1/r_3$, i.e., the sum of the curvatures of the (three) circles in the chain. By Descartes’ theorem [21]:
where $r_4$ and $r_5$ denotes the radii of the inner and outer Soddy circles. Subtracting and factoring, we obtain:

$$\left(\frac{1}{r_4} - \frac{1}{r_5}\right)\left(\frac{2}{\rho} + \frac{1}{r_4} + \frac{1}{r_5}\right) = 2 \left(\frac{1}{r_4} - \frac{1}{r_5}\right)\left(\frac{1}{r_4} + \frac{1}{r_5}\right)$$

Hence $1/\rho = (1/r_4 + 1/r_5)/2$. Referring to Figure 5(left), it can be seen that:

$$\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} = \frac{r}{\rho} \quad (2)$$

Finally, since $r_4$, $r_5$, and $r$ are fixed, $\rho$ is constant, and so is the sum of half tangents.

*Remark 4.* The outer Soddy circle is a line when $r_5 = +\infty$. By Equation (2) this gives $\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} = 2$. Note the latter was derived in [11] in the context of showing if a certain condition in a triangle is satisfiable, which turns out to be equivalent to ours.

**Proposition 4.** Let $\triangle ABC$ be in the Steiner-Soddy family and $\triangle A'B'C'$ be its I-pedal. Over the porism, the left and right hand side of each equation are conserved and identical:

$$\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} = \cot(A') + \cot(B') + \cot(C')$$

$$\tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} = \cot^2(A') + \cot^2(B') + \cot^2(C')$$

**Proof.** Referring to Figure 5(left), let the circles in the Steiner chain be centered at $A, B, C$, the vertices of $P$. The perpendiculars in $A', B', C'$ meet at $O$, the incenter of $\triangle ABC$ as well as the circumcenter of $\triangle A'B'C'$. From properties of tangents, $AO \perp BC$. On the other hand $OC' \perp AB$. This proves that $OAC'$ and $OC'B'$ are either equal, or supplementary. Since both angles are acute, the latter is impossible. Hence $OAC' = OC'B'$. Similarly, $OBC' = OC'A'$. Thus:

$$\cot(C') = \cot\left(\frac{\hat{A} + \hat{B}}{2}\right) = \cot\left(\frac{\pi - \hat{C}}{2}\right) = \tan \frac{C}{2}$$

Equation (1) finishes the proof.

**Proposition 5.** In the $N = 3$ case, the outer Soddy circle degenerates to a line when any one of the following equivalent conditions is fulfilled: (i) $\tau = 2$; (ii) the outer conic is a parabola; (iii) the circumcenter of $\mathcal{B}_3$ is at a co-vertex of the Brocard inellipse, whose aspect ratio is $\sqrt{5}/2 = \phi - 1/2 \approx 1.118$, where $\phi = (1 + \sqrt{5})/2$ is the golden ratio.
Fig. 5: **Left:** Consider a $\triangle ABC$ and 3 mutually-tangent circles (blue) centered on $A, B, C$. These are tangent at the contact points $A', B', C'$ of the incircle with the sides of the triangle, which are the vertices of the pedal triangle with respect to the incenter $I$. Same-color angles have the same measure. **Right:** Let $ABC$ be a parabola-inscribed Poncelet family of triangles whose caustic is a circle centered on the parabola axis. Over the family, the locus of the orthocenter $X_4$ is a line (purple) parallel to the directrix (dashed red).

### 4 Loci in the N=3 case

Referring to Figure 5(right):

**Proposition 6.** For any parabola-inscribed Poncelet family whose caustic is an axis-centered circle, the locus of the orthocenter $X_4$ is a line parallel to the directrix.

The following proof was kindly contributed by Alexey Zaslavsky [25]:

**Proof.** Consider the unit parabola $y = x^2$, and let $y_0$, $r$ be the center and the radius of the incircle. Then the points $(r, y_0 - r)$ and $(-r, y_0 - r)$ lie on the parabola, thus $y_0 = r^2 + r$. Now let $A(a, a^2)$, $B(b, b^2)$, $C(c, c^2)$ be the vertices of the triangle. Then the distances from the incenter $(0, r^2)$ to lines AB and AC equal $r$ and we obtain $b+c$ and $bc$ as functions of $a$. After this we have that the ordinate of orthocenter $-(1 + ab + ac + bc) = r^3 + 2r - 1$ do not depend on $a$.

**Remark 5.** Consider the parabola $4cy = x^2$ and the circle centered at $(0, y_0)$. For the pair to admit Poncelet triangles, it can be shown $r = 4y_0c/\sqrt{16c^2 + x_0^2}$, where $x_0 = 2\sqrt{-2c^2 + y_0c + 2\sqrt{c^3(c + y_0)}}$. In this case, the locus of $X_4$ is the line $y = -4c + x_0^2/(4c) = (-6c^2 + y_0c + 2\sqrt{c^3(c + y_0)})/c$.

**Corollary 3.** Over a parabola-inscribed Soddy-Steiner family, the locus of its orthocenter $X_4$ is a line parallel to the directrix.
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For example, assume the parabola is $4cy = x^2$. Then the locus of $X_4$ is the line $y = -7c/4$.

It is known that if the intouch triangle is acute, its circumcenter $X_3$ (resp. symmedian $X_6$) coincides with the incenter $X_1$ (resp. Gergonne point $X_7$) of the reference.

While the intouch of $S_3$ is acute (its harmonic $O$-pedal), $X_1$ and $X_7$ are stationary. Otherwise, while $S_3$ is everted (it is inscribed in a hyperbola), $X_1$ and $X_7$ follow arcs curves of degree at least four. Here are some experimental observations about the loci of $X_k$, some of which are explicitly derived in Section 5:

1. $X_k, k = 2, 3, 4, 5$ are conics, as predicted in [12], see Figure 6.
2. $X_k, k = 6, 8, 9, 10$ are also conics, though not in general [12].
3. $X_k, k = 13, 14, 15, 16, 80$ are circles.
4. $X_k, k = 20, 77, 170$ are segments along the major axis of $E$.
5. $X_{105}$ is a circle centered on the major axis of $E$ and tangent to it at two points.

5 Comparing Poncelet Conservations

Conservations of various Poncelet families studied so far as well as their polar images with respect to certain significant points is shown in Table 1. As we study more of these canonical families, one of our goals (still elusive) is to identify a functional pattern in the conserved quantities.

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Appendix: Explicit Formulas

Referring to Figure 1, let the inversion circle be centered at \( \text{inv} = (x_0, 0) \) and have radius \( \lambda \) and \( \alpha = \pi / N \). Let \( \mathcal{E} \) denote the conic the Steiner-Soddy family is inscribed to and \( \mathcal{C} \) its circular caustic.

**Proposition 7.** The \( x \) coordinates of foci \( f_1 \) and \( f_2 \) and vertex \( V \) of \( \mathcal{E} \) are given by:

\[
\begin{align*}
    f_{1,2} &= x_0 \left( R^4 \cos^4 \alpha - R^2 \cos^2 \alpha (\lambda^2 - 2x_0^2) \pm 2R^2 \sin \alpha \lambda^2 + 2R^2 (\lambda^2 - 2x_0^2) - x_0^2 (\lambda^2 - x_0^2) \right) \\
    V_z &= \cos(2\alpha) R^2 x_0 + R (Rx_0 + 2\lambda^2 - 4x_0^2) - 2x_0 (\lambda^2 - x_0^2) \\
        &= \frac{R^2 \cos(2\alpha) + R^2 - 2Rx_0 + 2x_0^2}{R^2 \cos(2\alpha) + R^2 - 2Rx_0 + 2x_0^2}
\end{align*}
\]

**Proposition 8.** The parameters of the caustic \( \mathcal{C} = (I, r) \) are given by:

\[
I = \left[ x_0 + \frac{2x_0 \lambda^2}{2(R^2 \cos^2 \alpha - x_0^2)}, 0 \right], \quad r = \frac{\lambda^2 R \cos \alpha}{R^2 \cos^2 \alpha - x_0^2}
\]

**Proposition 9.** Let Brocard inellipse (caustic of the pedal family with respect to \( \mathcal{I} \)) be centered at \( O' \) with semi-axes \( a', b' \). These are given by:

\[
\begin{align*}
    O' &= \left[ x_0 + \frac{\lambda^2 x_0 (R^2 \cos^2 \alpha \cos(2\alpha) - x_0^2)}{R^2 (R^2 - 4x_0^2) \cos^2 \alpha + 2R^2 x_0^2 \cos^2 \alpha + x_0^2}, 0 \right] \\
    a' &= \frac{\lambda^2 R (x_0^2 - R^2 \cos^2 \alpha \cos^2 \alpha)}{R^2 (R^2 - 4x_0^2) \cos^2 \alpha + 2R^2 x_0^2 \cos^2 \alpha + x_0^2} \\
    b' &= \sqrt{R^2 (R^2 - 4x_0^2) \cos^2 \alpha + 2R^2 x_0^2 \cos^2 \alpha + x_0^2}
\end{align*}
\]
Proposition 10. The loci of $X_2$, $X_3$ and $X_4$ are the conics given by:

$$X_2: (R^2 - 4x_0^2)(R^6 - 56R^2x_0^6 + 16x_0^8)x^2 + (R^2 - 4x_0^2)^3y^2$$

$$\quad - 2x_0(R^6 - 60R^2x_0^6 + 12R^4x_0^4 + 240R^2x_0^4 - 192R^2x_0^4\lambda^2 - 64x_0^6 + 64x_0^8\lambda^2)x$$

$$\quad + x_0^2(R^6 - 60R^4x_0^4 + 240R^2x_0^4 - 384R^2x_0^4\lambda^2 + 144R^2\lambda^4 - 64x_0^6 + 128x_0^4\lambda^2 - 64x_0^4\lambda^4) = 0$$

$$X_3: (R^4 - 56R^2x_0^2 + 16x_0^4)(R^2 - 4x_0^2)^3x^2 + (R^2 - 4x_0^2)^2(R^4 + 40R^2x_0^2 + 16x_0^4)y^2$$

$$\quad - 2x_0(R^2 - 4x_0^2)(R^6 - 68R^2x_0^6 + 736R^2x_0^4 - 928R^2x_0^4\lambda^2 - 2944R^2x_0^4\lambda^4 - 768R^4x_0^4\lambda^2$$

$$\quad + 4352R^2x_0^6 + 2560R^2x_0^4\lambda^2 - 1024x_0^4 + 1024x_0^4\lambda^2)x$$

$$\quad + x_0^2(R^2 + (-72x_0^2 + 56\lambda^2)R^2 + (1008x_0^4 - 2080x_0^2\lambda^2 + 784\lambda^4)R^6 - 256x_0^2(23x_0^4 - 23x_0^4\lambda^2$$

$$\quad + 16\lambda^4)R^8 + 256x_0^4(63x_0^4 + 4x_0^2\lambda^2 - 18\lambda^4)R^4 - 2048x_0^2(x_0^2 - \lambda^2)(9x_0^2 - 2\lambda^2)R^2 + 4096x_0^2(x_0^2 - \lambda^2)^2) = 0$$

$$X_4: (R^2 - 4x_0^2)^3x^2 + (R^2 - 56R^2x_0^2 + 16x_0^4)(R^2 - 4x_0^2)^2y^2$$

$$\quad x_0(-2R^2 + 32R^2x_0^2 + 40R^2\lambda^2 - 192R^2x_0^4 + 96R^2x_0^4\lambda^2 + 512R^2x_0^4\lambda^2 - 1152R^2x_0^4\lambda^2 - 512x_0^6$$

$$\quad + 512x_0^4\lambda^2)x$$

$$\quad + x_0^2(R^2 - 16x_0^4 - 40R^2\lambda^2 + 96R^2x_0^4 - 96R^2x_0^4\lambda^2 + 400R^4\lambda^4 - 256R^2\lambda^4 + 1152R^2x_0^4\lambda^2$$

$$\quad - 896R^2x_0^4\lambda^4 + 256x_0^8 - 512x_0^4\lambda^2 + 256x_0^4\lambda^4) = 0$$

Proposition 11. The locus of $X_6$ is the ellipse given by:

$$X_6: (R^2 - 4x_0^2)(R^4 + 544R^4x_0^4 + 256x_0^8)(R^2 + 4x_0^2)^2x^2 + (R^2 + 16x_0^2)^2(R^2 - 4x_0^2)^2y^2$$

$$\quad - 2x_0(R^2 + 4x_0^2)^3(R^6 - 4R^4\lambda^2 - 4R^2x_0^2 - 160R^2\lambda^2x_0^2 + 544R^4x_0^4$$

$$\quad + 1536R^4x_0^4\lambda^2 - 2176R^4x_0^4\lambda^2 - 512R^2\lambda^2x_0^4 + 256R^4x_0^4\lambda^2 + 1024x_0^4\lambda^2 - 1024x_0^4\lambda^4$$

$$\quad + x_0^2(R^2 + 4x_0^2)^2(R^6 - 8\lambda^2 + 4x_0^2)R^4 + (16x_0^4 - 320\lambda^2x_0^2 + 544x_0^4)R^8 - 64x_0^4(15x_4$$

$$\quad - 48x_0^4 + 34x_0^4)R^4 + 256x_0^4(\lambda^4 - x_0^4)(3\lambda^2 + x_0^4)R^2 - 1024x_0^4(\lambda^2 - x_0^4)^2) = 0$$

Proposition 12. When the center of inversion is internal to the Soddy circle, $S_{\text{reg}}$ the locus $X_{15}$ is the circle given by:

$$X_{15} : \left( x - \frac{x_0(-256x_0^4 + 144R^2x_0^2 - 24(R^4 + 6R^2\lambda^2)x_0^2 + R^6 + 12R^2\lambda^2)}{R^6 - 24R^2x_0^2 + 144R^2x_0^2 - 256x_0^6} \right)^2 + y^2$$

$$= \frac{36864R^2x_0^6\lambda^4}{(R^6 - 24R^2x_0^2 + 144R^2x_0^2 - 256x_0^6)^2}$$

Proposition 13. The locus of $X_{20}$ is a segment $[X_{20}^-, X_{20}^+]$ contained in axis $x$ of length $L_{20}$. These are given by:

$$X_{20}^+ = \frac{x_0(\zeta + 4x_0R^6 + 304R^2x_0^2\lambda^2 - 64x_0^2(x_0^2 - \lambda^2)(R + x_0))}{(R^2 + 8Rx_0 + 4x_0^2)(R + 2x_0)}$$

$$= \frac{x_0(\zeta - 4x_0R^6 - 304R^2x_0^2\lambda^2 + 64x_0^2(x_0^2 - \lambda^2)(R - x_0))}{(R^2 - 8Rx_0 + 4x_0^2)(R + 2x_0)}$$

$$L_{20} = \frac{18432x_0^4R^6\lambda^2}{(R^2 - 4x_0^2)^3(R^3 - 56R^2x_0^2 + 16x_0^4)}$$

where $\zeta = R^6 + (76x_0^2 - 28x_0^2)R^4 + 16x_0^2(12x_0^2 + 7x_0^4)$. 

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References

1. Akopyan, A.: Private communication (2022)
2. Akopyan, A., Schwartz, R., Tabachnikov, S.: Billiards in ellipses revisited. Eur. J. Math. (9 2020), doi:10.1007/s40879-020-00426-9
3. Belio, F., Garcia, R., Reznik, D.: Properties of parabola-inscribed Poncelet polygons. arXiv:2111.00979 (10 2021)
4. Bialy, M., Tabachnikov, S.: Dan Reznik’s identities and more. Eur. J. Math. (9 2020), doi:10.1007/s40879-020-00428-7
5. Casey, J.: A sequel to the first six books of the Elements of Euclid. Hodges, Figgis & Co., Dublin, 5th edn. (1888)
6. Chavez-Caliz, A.: More about areas and centers of Poncelet polygons. Arnold Math J. (8 2020), doi:10.1007/s40598-020-00154-8
7. Dragović, V., Radnović, M.: Poncelet Porisms and Beyond: Integrable Billiards, Hyperelliptic Jacobians and Pencils of Quadrics. Frontiers in Mathematics, Springer, Basel (2011)
8. Galkin, S., Garcia, R., Reznik, D.: On affine images of regular polygons (2022), in preparation
9. Garcia, R., Gheorghe, L., Moses, P., Reznik, D.: Triads of conics associated with a triangle (12 2021), arXiv:TBD
10. Garcia, R., Reznik, D., Roitman, P.: New properties and invariants of harmonic polygons (12 2021), arXiv:2112.02545
11. Hajja, M., Yff, P.: The isoperimetric point and the point of equal detour in a triangle. J. of Geom. 87(7), 76–82 (2007)
12. Helman, M., Laurain, D., Reznik, D., Garcia, R.: Poncelet triangles: a theory for locus ellipticity. Beitr. Algebra Geom. (2022), to appear
13. Lucca, G.: Circle chains inside a circular segment. Forum Geometricorum 9, 173–179 (2009)
14. Odehnal, B.: Poristic loci of triangle centers. J. Geom. Graph. 15(1), 45–67 (2011)
15. Reznik, D., Garcia, R., Koiller, J.: Fifty new invariants of n-periodics in the elliptic billiard. Arnold Math J. (2 2021)
16. Roitman, P., Garcia, R., Reznik, D.: New invariants of Poncelet-Jacobi bicentric polygons. arXiv:2103.11260 (3 2021)
17. Schwartz, R.: Private communication (2021)
18. Schwartz, R., Tabachnikov, S.: Centers of mass of Poncelet polygons, 200 years after. Math. Intelligencer 38(2), 29–34 (2016). https://doi.org/10.1007/s00283-016-9622-9
19. Schwartz, R.E., Tabachnikov, S.: Descartes circle theorem, Steiner porism, and spherical designs. The American Mathematical Monthly 127(3), 238–248 (2020). https://doi.org/10.1080/00029890.2020.1690909
20. Skutin, A.: On rotation of an isogonal point. J. of Classical Geom. 2, 66–67 (2013)
21. Weisstein, E.: Mathworld. MathWorld–A Wolfram Web Resource (2019), mathworld.wolfram.com
22. Wells, D.: The Penguin Dictionary of Curious and Interesting Geometry. Penguin Books, London (1991)
23. Zaslavsky, A., Chehnokov, G.: The Poncelet theorem in euclidean and algebraic geometry. Mathematical Education 4(19), 49–64 (2001), in Russian
24. Zaslavsky, A., Kosov, D., Muzafarov, M.: Trajectories of remarkable points of the Poncelet triangle. Kvant 2, 22–25 (2003)
25. Zaslavsky, A.: Private communication (2021)