**Conservation laws of the Haldane-Shastry type spin chains**

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A systematic method to construct the complete set of conserved quantities of the Haldane-Shastry type spin chains is proposed. The hidden relationship between the Yang-Baxter relation and the conservation laws of the long-range interacting integrable models is exposed explicitly. An integrable anisotropic Haldane-Shastry model is also constructed.

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A typical characteristic of the integrable models is that each of them possesses a complete set of conservation laws. For an integrable system with a finite number of degrees of freedom, the number of linearly independent conserved quantities is exactly the same as that of the degrees of freedom, while for a continuous integrable model, there is an infinite number of conserved quantities. It is well known that the conservation laws of the integrable models with short-range interactions are tightly related to the Yang-Baxter relation. The conserved quantities can be obtained from the derivatives of the transfer matrix of the system [1, 2]. On the other hand, there is another class of integrable models with inverse square type interaction potentials which are called as the Calogero-Sutherland (CS) model [3] in the continuous case and the Haldane-Shastry (HS) model [4, 5] in the lattice case. These models belong to the long-range interacting ones and have many applications in the fields of two-dimensional fractional quantum Hall effect and fractional statistics [6]. Because of the importance, these models have been studied extensively [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Several people have tried to construct the conservation laws of these models. For example, by using the Dunkl operators [18], Polychronakos [19] constructed the invariants of motion of the CS continuous model. Nevertheless, the demonstration of the integrability of the HS Lattice model is still a challenge problem. Borrowed Polychronakos’s idea, Fowler and Minahan [20] proposed a set of conserved quantities for the HS model. However, there are still some debates [21, 22] on this construction because the HS Hamiltonian appears in the third level of invariants, and should act on some magnon states to erase the unwanted terms.

The main difficulty of the demonstration of the complete integrability of HS type models is lack of a systematic method to construct the complete set of conservation laws. Recalling the case of nearest neighbor interacting integrable models, an interesting question may arise: Is there any intrinsic relationship between the HS model and the Yang-Baxter relation [21]? In this Letter, we expose this intrinsic relationship exactly and develop a systematic method to construct the complete set of conservation laws of HS model. This provides us not only a deep understanding to this important model but also a general method to construct other integrable models with long-range couplings.

Our starting point is the following Lax operator

$$L_0(u) = 1 + \frac{\eta}{u} \sigma_0 \cdot S_j,$$

where $u$ is the spectral parameter; $\eta$ is the crossing parameter; $\sigma_0$ is the auxiliary Pauli matrix and $S_j$ is the spin-$1/2$ operator on site $j$. It is well known that the integrability of the Heisenberg spin chain model is related to the monodromy matrix $T_0(u) = L_0(u) \cdots L_0(u)$ which satisfies the Yang-Baxter relation

$$L_{12}(u_1 - u_2)T_1(u_1)T_2(u_2) = T_2(u_2)T_1(u_1)L_{12}(u_1 - u_2).$$

Define the transfer matrix $t(u) = tr_0 T_0(u)$. From Eq. (2), one can prove that the transfer matrices with different spectral parameters are mutually commutative, i.e., $[t(u), t(v)] = 0$. Therefore, $t(u)$ serves as the generating function of the conserved quantities of the corresponding system. The first order derivative of logarithm of the transfer matrix gives the Hamiltonian of the Heisenberg spin chain

$$H_H = \frac{1}{2\eta} \frac{\partial}{\partial u} \ln t(u) \bigg|_{u=0} + \left( \frac{1}{u^2} - \frac{1}{4} \right) N = \sum_{j=1}^{N} S_j \cdot S_{j+1}. \quad (3)$$

In fact, the inhomogeneous Lax operator $L_0(u - \delta_j)$ with a site-dependent shift $\delta_j$ to the spectral parameter $u$ and the inhomogeneous transfer matrix $T_0(u) \equiv L_0(u - \delta_1) \cdots L_0(u - \delta_N)$ also satisfy the Yang-Baxter relation Eq. (2). Consider the classical expansion of the Lax operator $L_0(u - \delta_j) = 1 + \eta L_{0j}(u - \delta_j)$ and define the classical monodromy matrix $T_0(u) = \sum_{j=1}^{N} L_{0j}(u - \delta_j)$. We find that they satisfy the following classical Yang-Baxter relation [23]

$$[T_1(u_1), T_2(u_2)] = [T_1(u_1) + T_2(u_2), L_{12}(u_1 - u_2)]. \quad (4)$$

The Eq. (4) ensures that the functional $\tau(u) \equiv tr_0 T_0^2(u)/4$ with different spectral parameters are mutually commutative, i.e., $[\tau(u), \tau(v)] = 0$. Therefore, $\tau(u)$ can be treated as the generating functional of a series of conserved quantities which can be written out explicitly as

$$\tau(u) = \sum_{j=1}^{N} \frac{3}{8(u - \delta_j)} + \sum_{j=1}^{N} \frac{h_j}{u - \delta_j}. \quad (5)$$
where
\[ h_j = \sum_{l=1, \neq j}^{N} \frac{1}{\delta_j - \delta_l} S_j^z S_l^z. \] (6)

The operators \( h_j \) are nothing but the Gaudin operators [24] associated with the Heisenberg spin chain. It is easy to prove that the Gaudin operators commute with each other, i.e., \([h_j, h_k] = 0\). This allows us to construct the mutually commutative operators \( L_n = \sum_{j=1}^{N} h_j^n \) for arbitrary \( n \). If we choose one of \( L_n \) as the Hamiltonian, \( \{L_n\} \) form a set of conserved quantities and the Hamiltonian describes an integrable system. For a translational invariant lattice \( \delta_j = j \) and \( N \to \infty \), we find
\[ H_{ISE} = \lim_{N \to \infty} \left[ -\sum_{j=1}^{N} h_j^2 + \frac{\pi^2}{16} N \right] = \sum_{l \neq j}^{N} \frac{1}{(j-l)^2} S_j^z S_l^z. \] (7)

\( H_{ISE} \) is just the Hamiltonian of inverse square exchanging (ISE) model. The prime in the summation means that \( j \) and \( l \) in the summation take values from negative to positive infinity. Obviously, operators \( h_j = \sum_{l=1}^{N} \delta_j S_j^z S_l^z \) and their arbitrary combinations commute with the Hamiltonian \((7)\) and form a set of the conserved quantities. In addition, the model \((7)\) has another set of conserved quantities \( h_j' = \sum_{l=1}^{N} S_j^z S_l^z \) by simply checking \( [H_{ISE}, h_j'] = 0 \). We note that \((7)\) with arbitrary \( \delta_j \) gives a disordered integrable system. Now it is clear that there is an intrinsic relationship between the short-range coupling Heisenberg model and the long-range coupling ISE model, i.e., they share the common Yang-Baxter equation. This provides us a powerful method to construct new integrable models with long-range interactions from the known solutions of the Yang-Baxter equation or to obtain the conservation laws of the predicted integrable models with \( r^{-2} \) type potentials. For example, from the Lax operator of the anisotropic XXZ Heisenberg spin chain [1] we have the following mutually commutative Gaudin operators
\[ h_j = \sum_{l=1, \neq j}^{N} \left[ S_j^z S_l^z + S_l^z S_j^z \right] \frac{1}{\sin(\delta_j - \delta_l) + \cot(\delta_j - \delta_l) S_j^z S_l^z}. \] (8)

For the equally spaced \( \delta_j = \pi j/N \), we obtain
\[ H_{AHS} = -\sum_{j=1}^{N} h_j^2 + \frac{1}{4} \left[ \sum_{j=1}^{N} S_j^z \right]^2 + \frac{1}{16} N(N^2 - N + 1) = \sum_{l \neq j}^{N} \cos \frac{\pi}{N}(j-l) \left[ S_j^z S_l^z + S_l^z S_j^z \right] \frac{1}{\sin^2 \frac{\pi}{N}(j-l)}. \] (9)

Notice that in the anisotropic XXZ spin chain, the total spin is no longer a good quantum number but \( \sum_{j=1}^{N} S_j^z \) is indeed a conserved quantity which commutes with the \( h_j \) in Eq. \((8)\) and the \( H_{AHS} \) in Eq. \((9)\). Therefore, \( H_{AHS} \) can be treated as the Hamiltonian of an anisotropic HS model. In fact, the ISE Hamiltonian \((7)\) is the limiting case of the anisotropic HS model \((9)\), i.e., \( H_{ISE} = \lim_{N \to \infty} \frac{\pi^2}{N^2} H_{AHS} \). By putting \( N \to \infty \) and \( \delta_m = \pi m \), where \( i \) is the imaginary unit, we readily obtain the hyperbolic version of this integrable Hamiltonian
\[ H_{AHS} = \sum_{l \neq j}^{N} \frac{\cosh(j-l)(S_j^z S_l^z + S_l^z S_j^z) + S_j^z S_l^z}{\sinh^2(j-l)}. \] (10)

Motivated by these findings, we introduce the following local operators
\[ h_j = \sum_{l=1, \neq j}^{N} f(\delta_j - \delta_l) S_j^z S_l^z, \] (11)

and look for the solutions of \([h_j, h_k] = 0\), where \( f(\delta_j - \delta_l) \) is a function to be determined. For simplicity, we denote \( f(\delta_j - \delta_l) \equiv f_{jk} \). With the relation \([S_j^z S_l^z, S_k^z S_l^z] = i S_j^z (S_l^z \times S_k^z)\) for \( l \neq j \neq k \), we obtain
\[ [h_j, h_k] = i \sum_{l=1, \neq j, k}^{N} (f_{jk} f_{kl} - f_{jl} f_{kj}) S_j^z (S_l^z \times S_k^z). \] (12)

Therefore, the constraint for \([h_j, h_k] = 0\) is the solution of \( f_{jk} f_{kl} = f_{jl} f_{kj} \). After some simple algebra, we find three sets of solutions:

(i) The first solution is \( f(x) = x^{-1} \). This solution just gives Eq. \((6)\) and thus the ISE Hamiltonian \((7)\).

(ii) The second solution is \( f(x) = \cot(x) \pm i \). The operators \( h_j \) and \( h_j' \) take the forms
\[ h_j = \sum_{l=1, \neq j}^{N} \left[ \cot(\delta_j - \delta_l) \pm i \right] S_j^z S_l^z, \]
\[ h_j' = \sum_{l=1, \neq j}^{N} \left[ \cot(\delta_j - \delta_l) - i \right] S_j^z S_l^z. \] (13)

Both \([h_j]\) and \([h_j']\) are not hermitian for real \( \delta_j \) but each of them form a set of mutually commutative operators, though \( h_j \) and \( h_j' \) do not commute with each other, \([h_j, h_j'] = 2 \sum_{l=1, \neq j, k}^{N} \left[ \cot(\delta_j - \delta_k) \pm i \right] S_j^z (S_l^z \times S_k^z)\). A key problem is to construct a hermitian Hamiltonian from those non-hermitian operators. Fortunately, we find that \( l_2 \equiv \sum_{j=1}^{N} h_j^2 = \sum_{j=1}^{N} h_j'^2 \) is a hermitian operator. For \( \delta_j = \pi j/N \), the HS Hamiltonian \([4, 5] \) can be derived as
\[ H_{HS} = -\sum_{j=1}^{N} h_j^2 + \frac{1}{2} (N-4) \sum_{j=1}^{N} h_j + \frac{1}{16} N(3N^2 - 6N - 5). \]
\[ = \sum_{j=1, \neq j}^{N} \frac{1}{\sin^2 \frac{\pi}{N}(j-l)} S_j^z S_l^z. \] (14)

An obvious fact is that \([H_{HS}, h_j] = [H_{HS}, h_j'] = 0\). We readily have two independent sets of conserved hermitian quantities for \( H_{HS} \):
\[ l_j^+ = \frac{1}{2} (h_j + h_j') = \sum_{l=1, \neq j}^{N} \cot \frac{\pi}{N}(j-l) S_j^z S_l^z, \]
\[ I_j = \frac{1}{2i} (h_j - h_j^*) = \sum_{l=1,\pi_j}^{N} S_j \cdot S_l. \] (15)

For the spin half system, we know that the number of degrees of freedom of each site is two. The two linearly independent conserved quantities \( I_j \) clearly show that the HS Hamiltonian is completely integrable.

Actually, we can also define the classical operators \( L_{\delta j}(u - \delta j) = [\cot(u - \delta j) + i]S_0 \cdot S_j \) and \( T_\delta (u) = \sum_{\delta j=1}^{N} L_{\delta j}(u - \delta j) \) for this solution. They satisfy the following deformed classical Yang-Baxter relation

\[ [T_1(u), T_2(v)] = [L_{21}(v - u), T_1(u)] + [T_2(v), L_{12}(u - v)]. \] (16)

Define a functional \( \tau(u) = \tr_0 T_0^2(u) \), which can be written out explicitly as

\[
\tau(u) = \sum_{j=1}^{N} \left[ \frac{3 \cos 2(u - \delta j)}{8 \sin^2(u - \delta j)} + \frac{3}{4} i \cot(u - \delta j) \right]
+ \sum_{j=1}^{N} \cot(u - \delta j) + i \hbar_j.
\] (17)

From the deformed Yang-Baxter relation (16), we can prove that the functional \( \tau(u) \) with different spectral parameters are mutually commutative and thus can be treated as the generation functional of a integrable system. Obviously, for \( \delta_j = j\pi/N \), \([H_{HS}, \tau(u)] = 0 \), implying that the conserved quantities of the HS model can also be generated by \( \tau(u) \).

(iii) The third solution is \( f(x) = \cot(x) \pm 1 \) and \( h_j^\pm = \sum_{l=1,\pi_j}^{N} \coth(\delta_j - \delta_l) \pm 1 \) for the Inozemtsev Hamiltonian \([7]\)

\[ H_I = \sum_{l<j}^{N} \frac{1}{\sinh^2(j-l)} S_j \cdot S_l \] (18)

can be easily derived from \( \sum_j h_j^{+2} \) or \( \sum_j h_j^{-2} \) by taking \( \delta_j = j \) and \( N \rightarrow \infty \). The operators \( h_j^\pm = \sum_{l\neq j}^{N} \coth(j-l) \pm 1 \) span the complete space of the conserved hermitian quantities and the Inozemtsev model (18) is also completely integrable.

Similarly, we can construct the conservation laws of the \( SU(M) \)-invariant HS and Inozemtsev models. In these cases, the generating operators \( h_j \) take the form:

\[ h_j = \sum_{l=1,\pi_j}^{N} f(\delta_j - \delta_l)(P_{\beta} - c), \] (19)

where \( P_{ij} \) is the \( SU(M) \) spin permutation operator and \( c \) is a constant. One can easily check that

\[ [h_j, h_k] = \sum_{l=1,\pi_j,\pi_k}^{N} (f_{\beta f_{k\delta}} - f_{\beta f_{k\delta}} + f_{\delta f_{k\beta}}) P_{\beta}(P_{k} - P_{\delta}). \] (20)

The constraint \( [h_j, h_k] = 0 \) gives the same solutions of \( f(x) \) as those in the (i) - (iii). The \( SU(M) \) HS Hamiltonian can be constructed from the linear combination of \( \sum_{j=1}^{N} I_j^{\pm 2} \) and \( \sum_{j=1}^{N} I_j^\pm \), whose explicit form reads

\[ H_{SU(M)} = \sum_{l<j}^{N} \frac{P_{\beta}}{\sin^2(j-l)}. \] (21)

In conclusion, we establish the intrinsic relationship between the inverse square potential spin chain models and the Yang-Baxter relation. This provides us a powerful method to construct new integrable models with long-range couplings from the solutions of the Yang-Baxter relation obtained from the systems with short-range interactions. As an example, an integrable anisotropic HS spin chain model is derived. The complete sets of conservation laws of the ISE, HS and Inozemtsev models are constructed in quite simple forms.

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