Article

Fractional Hermite–Hadamard–Fejer Inequalities for a Convex Function with Respect to an Increasing Function Involving a Positive Weighted Symmetric Function

Pshtiwan Othman Mohammed, Thabet Abdeljawad and Artion Kashuri

1 Department of Mathematics, College of Education, University of Sulaimani, Sulaimani 46001, Kurdistan Region, Iraq
2 Department of Mathematics and General Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia
3 Department of Medical Research, China Medical University, Taichung 40402, Taiwan
4 Department of Computer Science and Information Engineering, Asia University, Taichung 41354, Taiwan
5 Department of Mathematics, Faculty of Technical Science, University Ismail Qemali, 9401 Vlora, Albania; artion.kashuri@univlora.edu.al

Abstract: There have been many different definitions of fractional calculus presented in the literature, especially in recent years. These definitions can be classified into groups with similar properties. An important direction of research has involved proving inequalities for fractional integrals of particular types of functions, such as Hermite–Hadamard–Fejer (HHF) inequalities and related results. Here we consider some HHF fractional integral inequalities and related results for a class of fractional operators (namely, the weighted fractional operators), which apply to function of convex type with respect to an increasing function involving a positive weighted symmetric function. We can conclude that all derived inequalities in our study generalize numerous well-known inequalities involving both classical and Riemann–Liouville fractional integral inequalities.

Keywords: symmetric; weighted fractional operators; convex functions; Hermite–Hadamard–Fejer inequality

1. Introduction

First of all, we recall the basic notation in convex analysis. A set $\mathcal{V} \subset \mathbb{R}$ is said to be convex if

$$\varepsilon \vartheta_1 + (1 - \varepsilon) \vartheta_2 \in \mathcal{V}$$

for each $\vartheta_1, \vartheta_2 \in \mathcal{V}$ and $\varepsilon \in [0, 1]$. Based on a convex set $\mathcal{V}$, we say that a function $h: \mathcal{V} \rightarrow \mathbb{R}$ is convex, if the inequality

$$h(\varepsilon \vartheta_1 + (1 - \varepsilon) \vartheta_2) \leq \varepsilon h(\vartheta_1) + (1 - \varepsilon) h(\vartheta_2), \quad \forall \vartheta_1, \vartheta_2 \in \mathcal{V}, \varepsilon \in [0, 1]$$

holds. We say that $h$ is concave if $-h$ is convex.

Theory and application of convexity play an important role in the field of fractional integral inequalities due to the behavior of its properties and definition, especially in the past few years. There is a strong relationship between theories of convexity and symmetry. Whichever one we study, we can apply it to the other one; see, e.g., [1]. There are plenty of well-known integral inequalities that have been established for the convex functions (1) in the literature; for example, Ostrowski type

Symmetry 2020, 12, 1503; doi:10.3390/sym12091503 www.mdpi.com/journal/symmetry
where \( \ell > \alpha \psi \), \(-\psi\)-convex functions [20], \(-\psi\)-convex functions [21] and many other types can be found in [18]. Besides, the HH type inequality (2) has been applied to a huge number of convex functions, such as \( F\)-convex functions [19], \( \lambda \)-\( g\)-convex functions [20], MT-convex functions [21] and (\( m \), \( m \))-convex functions [22], a new class of convex functions [23], and many other types can be found in the literature. Meanwhile, it has been applied to other models of fractional calculus, such as standard RL-fractional operators [24], conformable fractional operators [25,26], generalized fractional operators [27], \( \psi \)-RL-fractional operators [28,29], tempered fractional operators [30] and AB and Prabhakar fractional operators [31].

After growing the field of Hermite–Hadamard type inequalities (2) and (3), many classical and fractional integral inequalities have been established by many authors; for more details, see [24–31].

**Definition 1** ([32]). Let \( g : [\theta_1, \theta_2] \rightarrow [0, \infty) \) be an integrable function; then we say \( g \) is symmetric with respect to \((\theta_1 + \theta_2)/2\), if

\[
g(\theta_1 + \theta_2 - x) = g(x),
\]

holds for each \( x \in [\theta_1, \theta_2] \).

Based on this definition, the authors in [33,34] extended the HH-type inequalities (2) and (3) and they could deduce the so-called Hermite–Hadamard–Fejér (HHF) type inequalities, and their results were, respectively, as follows:

\[
h \left( \frac{\theta_1 + \theta_2}{2} \right) \int_{\theta_1}^{\theta_2} g(x)dx \leq \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} h(x)g(x)dx \leq \frac{h(\theta_1) + h(\theta_2)}{2} \int_{\theta_1}^{\theta_2} g(x)dx,
\]

\[
\left( \frac{\theta_1 + \theta_2}{2} \right) \left[ RL J_{\theta_1}^{\ell} g(\theta_2) + RL J_{\theta_2}^{\ell} g(\theta_1) \right] \leq \left[ RL J_{\theta_1}^{\ell} h(\theta_2) + RL J_{\theta_2}^{\ell} h(\theta_1) \right] \leq \frac{h(\theta_1) + h(\theta_2)}{2} \left[ RL J_{\theta_1}^{\ell} g(\theta_2) + RL J_{\theta_2}^{\ell} g(\theta_1) \right],
\]

where \( h : V \rightarrow \mathbb{R} \) is supposed to be a positive convex function, \( h \in L^1(\theta_1, \theta_2) \) with \( \theta_1 < \theta_2 \), and \( RL J_{\theta_1}^{\ell} \) and \( RL J_{\theta_2}^{\ell} \) stand for the left-sided and right-sided Riemann–Liouville fractional integrals of order \( \ell > 0 \), respectively, and these are defined by [12,13]:

\[
RL J_{\theta_1}^{\ell} h(x) = \frac{1}{\Gamma(\ell)} \int_{\theta_1}^{x} (x - \varepsilon)^{\ell - 1} h(\varepsilon) d\varepsilon, \quad x > \theta_1;
\]

\[
RL J_{\theta_2}^{\ell} h(x) = \frac{1}{\Gamma(\ell)} \int_{x}^{\theta_2} (\varepsilon - x)^{\ell - 1} h(\varepsilon) d\varepsilon, \quad x < \theta_2.
\]
where $h$ is as before and $g$ is as defined in Definition 1.

**Definition 2.** Let $(\bar{\theta}_1, \bar{\theta}_2) \subseteq \mathbb{R}$ and $\sigma(x)$ be an increasing positive and monotonic function on the interval $(\bar{\theta}_1, \bar{\theta}_2)$ with a continuous derivative $\sigma'(x)$ on the interval $(\bar{\theta}_1, \bar{\theta}_2)$ with $\sigma(0) = 0$, $0 \in [\bar{\theta}_1, \bar{\theta}_2]$. Then, the left-side and right-side of the weighted fractional integrals of a function $h$ with respect to another function $\sigma(x)$ on $[\bar{\theta}_1, \bar{\theta}_2]$ are defined by [35]:

\[
\begin{align*}
\left( \bar{\theta}_1 + J^\ell_{\bar{\theta}_1} \sigma h \right)(x) &= \frac{w^{-1}(x)}{\Gamma(\ell)} \int_{\bar{\theta}_1}^{x} \sigma'(\epsilon)(\sigma(x) - \sigma(\epsilon))^{\ell-1} h(\epsilon) w(\epsilon) d\epsilon, \\
\left( w J^\ell_{\bar{\theta}_2} \sigma h \right)(x) &= \frac{w^{-1}(x)}{\Gamma(\ell)} \int_{x}^{\bar{\theta}_2} \sigma'(\epsilon)(\sigma(x) - \sigma(\epsilon))^{\ell-1} h(\epsilon) w(\epsilon) d\epsilon, \quad \ell > 0,
\end{align*}
\]

where $w^{-1}(x) = \frac{1}{w(x)}$, $w(x) \neq 0$.

**Remark 1.** From the Definition 2, one can observe that

- If $\sigma$ is specialized by $\sigma(x) = x$ and $w(x) = 1$, then the weighted fractional integral operators (8) reduce to the classical Riemann–Liouville fractional integral operators (4).
- If $w(x) = 1$, we get the fractional integral operators of a function $h$ with respect to another function $\sigma(x)$, which is defined in [36,37] as follows:

\[
\begin{align*}
\left( \bar{\theta}_1 + J^\ell_{\bar{\theta}_1} \sigma h \right)(x) &= \frac{1}{\Gamma(\ell)} \int_{\bar{\theta}_1}^{x} \sigma'(\epsilon)(\sigma(x) - \sigma(\epsilon))^{\ell-1} h(\epsilon) d\epsilon, \\
\left( J^\ell_{\bar{\theta}_2} \sigma h \right)(x) &= \frac{1}{\Gamma(\ell)} \int_{x}^{\bar{\theta}_2} \sigma'(\epsilon)(\sigma(x) - \sigma(\epsilon))^{\ell-1} h(\epsilon) d\epsilon, \quad \ell > 0.
\end{align*}
\]

This study investigates several inequalities of HHF type via the weighted fractional operators (8) with positive weighted symmetric functions in the kernel.

The rest of the study is structured in the following way: In Section 2, we prove the necessary and auxiliary lemmas that are useful in the next section. Section 3 contains our main results which consist of proving several HHF fractional integral inequalities and some related results. In Section 4, we discuss our results and give the comparison between our results and the existing results, and we point out the future work. Section 5 is for the conclusions.

### 2. Auxiliary Results

Here, we shall prove analogues of the fractional HH inequalities (2) and (3) and HHF inequalities (6) and (7) for weighted fractional integrals with positive weighted symmetric function kernels. The main results here are Theorem 1 (a generalization of HH inequalities (2) and (3) and HHF inequality (6), and a reformulation of HHF inequality (7)) and Lemma 2 (a consequence of Theorem 1). First, we need the following fact.

**Lemma 1.** (i) Let $w : [\bar{\theta}_1, \bar{\theta}_2] \to [0, \infty)$ be an integrable function and symmetric with respect to $(\bar{\theta}_1 + \bar{\theta}_2)/2$, $\bar{\theta}_1 < \bar{\theta}_2$; then we have

\[ w(\epsilon \bar{\theta}_1 + (1 - \epsilon) \bar{\theta}_2) = w((1 - \epsilon) \bar{\theta}_1 + \epsilon \bar{\theta}_2), \quad (10) \]

for each $\epsilon \in [0, 1]$.

(ii) Let $w : [\bar{\theta}_1, \bar{\theta}_2] \to [0, \infty)$ be an integrable and symmetric function with respect to $(\bar{\theta}_1 + \bar{\theta}_2)/2$, $\bar{\theta}_1 < \bar{\theta}_2$; then we have for $\ell > 0$: 
Thus, let

\begin{align}
\left(\sigma^{-1}(\theta_1) + J_{\sigma^{-1}(\theta_2)}(w \circ \sigma)\right)\left(\sigma^{-1}(\theta_2)\right) &= \left(\sigma^{-1}(\theta_2)\right) \\
\cdot \left(\sigma^{-1}(\theta_1)\right)
\end{align}

\begin{align}
&= \frac{1}{2} \left[\left(\sigma^{-1}(\theta_1) + J_{\sigma^{-1}(\theta_2)}(w \circ \sigma)\right)\left(\sigma^{-1}(\theta_2)\right) + \left(\sigma^{-1}(\theta_1)\right)\right].
\end{align}

(11)

**Proof.** (i) Let \( x = \varepsilon \theta_1 + (1 - \varepsilon) \theta_2 \). It is clear that \( x \in [\theta_1, \theta_2] \) for each \( \varepsilon \in [0, 1] \) and then \( \theta_1 + \theta_2 - x = (1 - \varepsilon) \theta_1 + \varepsilon \theta_2 \). Then, by using the assumptions and Definition 1, we get (10).

(ii) By using the symmetric property of \( w \), we have

\((w \circ \sigma(\varepsilon)) = w(\sigma(\varepsilon)) = w(\theta_1 + \theta_2 - \sigma(\varepsilon)), \quad \forall \varepsilon \in \left[\sigma^{-1}(\theta_1), \sigma^{-1}(\theta_2)\right].\)

From this and by setting \( \sigma(x) = \theta_1 + \theta_2 - \sigma(\varepsilon) \), it follows that

\begin{align}
\left(\sigma^{-1}(\theta_1) + J_{\sigma^{-1}(\theta_2)}(w \circ \sigma)\right)\left(\sigma^{-1}(\theta_2)\right) &= \frac{1}{\Gamma(\ell)} \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} (\theta_2 - \sigma(x))^{\ell-1}(w \circ \sigma(x))\sigma'(x)dx \\
&= \frac{1}{\Gamma(\ell)} \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} (\sigma(x) - \theta_1)^{\ell-1}w(\theta_1 + \theta_2 - \sigma(x))\sigma'(x)dx \\
&= \frac{1}{\Gamma(\ell)} \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} (\sigma(x) - \theta_1)^{\ell-1}(w \circ \sigma)(x)\sigma'(x)dx \\
&= \left( J_{\sigma^{-1}(\theta_2)}(w \circ \sigma)\right)\left(\sigma^{-1}(\theta_1)\right).
\end{align}

This rearranges to the required (11). \( \Box \)

**Remark 2.** Throughout this study \( w^{-1}(x) = \frac{1}{w(x)} \) and \( \sigma^{-1}(x) \) is the inverse of the function \( \sigma(x) \).

**Example 1.** Consider the following integrable and positive weighted function

\[
w(x) = \begin{cases} 
2x + \frac{1}{2}, & 0 \leq x \leq \frac{1}{2}, \\
-2x + \frac{5}{2}, & \frac{1}{2} \leq x \leq 1.
\end{cases}
\]

One can easily show that

\[
w(1 - x) = \begin{cases} 
2x + \frac{1}{2}, & 0 \leq x \leq \frac{1}{2}, \\
-2x + \frac{5}{2}, & \frac{1}{2} \leq x \leq 1.
\end{cases}
\]

Thus, \( w(x) = w(1 - x) \) and hence the given weighted function is symmetric on \([0, 1]\) with respect to \( \frac{1}{2} \).

**Theorem 1.** Let \( h : [\theta_1, \theta_2] \subseteq [0, \infty) \to \mathbb{R} \) be an \( L^1 \) convex function with \( 0 \leq \theta_1 < \theta_2 \) and \( w : [\theta_1, \theta_2] \to \mathbb{R} \) be an integrable, positive and weighted symmetric function with respect to \( \frac{\theta_1 + \theta_2}{2} \). If \( \sigma \) is an increasing and positive function on \([\theta_1, \theta_2]\) and \( \sigma'(x) \) is continuous on \((\theta_1, \theta_2)\), then, we have for \( \ell > 0 \):

\[
h\left(\frac{\theta_1 + \theta_2}{2}\right) \left[\left(\sigma^{-1}(\theta_1) + J_{\sigma^{-1}(\theta_2)}(w \circ \sigma)\right)\left(\sigma^{-1}(\theta_2)\right) + \left( J_{\sigma^{-1}(\theta_2)}(w \circ \sigma)\right)\left(\sigma^{-1}(\theta_1)\right)\right]
\]

\[
\leq w(\theta_2) \left(\sigma^{-1}(\theta_1) + J_{\sigma^{-1}(\theta_2)}(h \circ \sigma)\right)\left(\sigma^{-1}(\theta_2)\right) + w(\theta_1) \left( J_{\sigma^{-1}(\theta_2)}(w \circ \sigma)\right)\left(\sigma^{-1}(\theta_1)\right)
\]

\[
\leq \frac{h(\theta_1) + h(\theta_2)}{2} \left[\left(\sigma^{-1}(\theta_1) + J_{\sigma^{-1}(\theta_2)}(w \circ \sigma)\right)\left(\sigma^{-1}(\theta_2)\right) + \left( J_{\sigma^{-1}(\theta_2)}(w \circ \sigma)\right)\left(\sigma^{-1}(\theta_1)\right)\right].
\]

(12)

**Proof.** Since \( h \) is a convex function on \([\theta_1, \theta_2]\), we have

\[
h\left(\frac{x + y}{2}\right) \leq \frac{h(x) + h(y)}{2}, \quad \forall x, y \in [\theta_1, \theta_2].
\]
Thus, for $x = \epsilon \theta_1 + (1 - \epsilon) \theta_2$ and $y = (1 - \epsilon) \theta_1 + \epsilon \theta_2$, $\epsilon \in [0, 1]$, it follows that

$$2h\left(\frac{\theta_1 + \theta_2}{2}\right) \leq h(\epsilon \theta_1 + (1 - \epsilon) \theta_2) + h((1 - \epsilon) \theta_1 + \epsilon \theta_2). \tag{13}$$

By multiplying both sides of (13) by $\epsilon^{t-1}w(\epsilon \theta_1 + (1 - \epsilon) \theta_2)$, and then, by integrating the resulting inequality with respect to $\epsilon$ over $[0, 1]$, we get

$$2h\left(\frac{\theta_1 + \theta_2}{2}\right) \int_0^1 \epsilon^{t-1}w(\epsilon \theta_1 + (1 - \epsilon) \theta_2)d\epsilon \leq \int_0^1 \epsilon^{t-1}h(\epsilon \theta_1 + (1 - \epsilon) \theta_2)w(\epsilon \theta_1 + (1 - \epsilon) \theta_2)d\epsilon + \int_0^1 \epsilon^{t-1}h((1 - \epsilon) \theta_1 + \epsilon \theta_2)w(\epsilon \theta_1 + (1 - \epsilon) \theta_2)d\epsilon. \tag{14}$$

For the left hand side inequality, we make use of (11) to get

$$\frac{\Gamma(t)}{2(\theta_2 - \theta_1)^r} \left[ (\sigma^{-1}(\theta_1) + J_{\theta_2}^{\epsilon \sigma}(w \circ \sigma)) (\sigma^{-1}(\theta_2)) + (J_{\theta_1}^{\epsilon \sigma}(w \circ \sigma)) (\sigma^{-1}(\theta_1)) \right]$$

$$= \frac{\Gamma(t)}{2(\theta_2 - \theta_1)^r} \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} (\theta_2 - \sigma(x))^{t-1}(w \circ \sigma)(x)dx$$

$$= \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} (\theta_2 - \sigma(x))^{t-1}(w \circ \sigma)(x)dx.$$

Now, we evaluate the weighted fractional operators as follows:

$$w(\theta_2) \left(\sigma^{-1}(\theta_1) + J_{\theta_2}^{\epsilon \sigma}(w \circ \sigma)\right) (\sigma^{-1}(\theta_2)) + w(\theta_1) \left(\sigma^{-1}(\theta_1) + J_{\theta_1}^{\epsilon \sigma}(w \circ \sigma)\right) (\sigma^{-1}(\theta_1))$$

$$= w(\theta_2) \left(\sigma^{-1}(\theta_1) + J_{\theta_2}^{\epsilon \sigma}(w \circ \sigma)\right) (\sigma^{-1}(\theta_2)) + w(\theta_1) \left(\sigma^{-1}(\theta_1) + J_{\theta_1}^{\epsilon \sigma}(w \circ \sigma)\right) (\sigma^{-1}(\theta_1))$$

$$= \frac{(\theta_2 - \theta_1)^t}{\Gamma(t)} \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \left(\frac{\theta_2 - \sigma(x)}{\theta_2 - \theta_1}\right)^{t-1}(w \circ \sigma)(x)dx$$

$$+ \frac{(\theta_2 - \theta_1)^t}{\Gamma(t)} \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \left(\frac{\sigma(x) - \theta_1}{\theta_2 - \theta_1}\right)^{t-1}(w \circ \sigma)(x)dx,$$

where

$$\frac{1}{(w \circ \sigma)^{-1}(\sigma^{-1}(z))} = \frac{1}{(w \circ \sigma)(\sigma^{-1}(z))} = \frac{1}{w(z)}t, \quad z = \theta_1, \theta_2. \tag{16}$$

Setting $t_1 = \frac{\theta_2 - \sigma(x)}{\theta_2 - \theta_1}$ and $t_2 = \frac{\sigma(x) - \theta_1}{\theta_2 - \theta_1}$, it follows that
We multiply both sides of (19) by \( \sigma^{-1}(\theta_1) \)
\[
\left( \sigma^{-1}(\theta_1) + \mathcal{J}_{w0}^{\ell}(h \circ \sigma) \right) \left( \sigma^{-1}(\theta_1) \right)
\]
\[
= \left( \frac{\theta_2 - \theta_1}{\Gamma(\ell)} \right) \int_0^1 t_1^{\ell-1} h(t_1\theta_1 + (1 - t_1)\theta_2) w(t_1\theta_1 + (1 - t_1)\theta_2) dt_1
\]
\[
+ \int_0^1 t_2^{\ell-1} h((1 - t_2)\theta_1 + t_2\theta_2) w((1 - t_2)\theta_1 + t_2\theta_2) dt_2
\]
\[
= \left( \frac{\theta_2 - \theta_1}{\Gamma(\ell)} \right) \int_0^1 e^{\ell-1} h(\epsilon\theta_1 + (1 - \epsilon)\theta_2) w(\epsilon\theta_1 + (1 - \epsilon)\theta_2) d\epsilon
\]
\[
+ \int_0^1 e^{\ell-1} h((1 - \epsilon)\theta_1 + \epsilon\theta_2) w((1 - \epsilon)\theta_1 + \epsilon\theta_2) d\epsilon
\].

By making use of (15) and (17) in (14), we get
\[
\int \left( \frac{\theta_1 + \theta_2}{2} \right) \left[ RL^{\ell} \mathcal{J} w(\theta_2) + RL^{\ell} J_{\theta_2} w(\theta_1) \right] \leq w(\theta_2) \left( \sigma^{-1}(\theta_1) + \mathcal{J}_{w0}^{\ell}(h \circ \sigma) \right) \left( \sigma^{-1}(\theta_1) \right)
\]
\[
+ w(\theta_1) \left( w_{w0} \mathcal{J}_{\sigma^{-1}(\theta_2)}^{\ell}(h \circ \sigma) \right) \left( \sigma^{-1}(\theta_1) \right).
\]

The first inequality of (12) is proved.

On the other hand, we will prove the second inequality of (12). By making use of the convexity of \( h \), we get
\[
\int e^{\ell-1} h(\epsilon\theta_1 + (1 - \epsilon)\theta_2) w(\epsilon\theta_1 + (1 - \epsilon)\theta_2) d\epsilon
\]
\[
+ \int e^{\ell-1} h((1 - \epsilon)\theta_1 + \epsilon\theta_2) w((1 - \epsilon)\theta_1 + \epsilon\theta_2) d\epsilon \leq (h(\theta_1) + h(\theta_2)) \int e^{\ell-1} w(\epsilon\theta_1 + (1 - \epsilon)\theta_2) d\epsilon.
\]

Then, by using (10) and (17) in (20), we get
\[
\left( \sigma^{-1}(\theta_1) + \mathcal{J}_{w0}^{\ell}(h \circ \sigma) \right) \left( \sigma^{-1}(\theta_2) \right)
\]
\[
+ w(\theta_1) \left( \sigma^{-1}(\theta_2) \right)
\]
\[
\leq \frac{h(\theta_1) + h(\theta_2)}{2} \left[ RL^{\ell} \mathcal{J} w(\theta_2) + RL^{\ell} J_{\theta_2} w(\theta_1) \right].
\]

This completes the proof of our theorem. \( \square \)

Remark 3. Particularly, in Theorem 1, if we take

(i) \( \sigma(x) = x \), then inequality (12) becomes

\[
\int e^{\ell-1} h(\epsilon\theta_1 + (1 - \epsilon)\theta_2) w(\epsilon\theta_1 + (1 - \epsilon)\theta_2) d\epsilon
\]
\[
+ \int e^{\ell-1} h((1 - \epsilon)\theta_1 + \epsilon\theta_2) w((1 - \epsilon)\theta_1 + \epsilon\theta_2) d\epsilon \leq (h(\theta_1) + h(\theta_2)) \int e^{\ell-1} w(\epsilon\theta_1 + (1 - \epsilon)\theta_2) d\epsilon.
\]

This completes the proof of our theorem. \( \square \)

Remark 3. Particularly, in Theorem 1, if we take

(i) \( \sigma(x) = x \), then inequality (12) becomes

\[
\left( \frac{\theta_1 + \theta_2}{2} \right) \left[ RL^{\ell} \mathcal{J} w(\theta_2) + RL^{\ell} J_{\theta_2} w(\theta_1) \right] \leq w(\theta_2) \left( RL^{\ell} \mathcal{J} w \right) (\theta_2) + w(\theta_1) \left( RL^{\ell} J_{\theta_2} w \right) (\theta_1)
\]
\[
\leq \frac{h(\theta_1) + h(\theta_2)}{2} \left[ RL^{\ell} \mathcal{J} w(\theta_2) + RL^{\ell} J_{\theta_2} w(\theta_1) \right].
\]

where \( RL^{\ell} \mathcal{J} w \) and \( RL^{\ell} J_{\theta_2} w \) are the left and right weighted Riemann–Liouville fractional operators, defined by

\[
\frac{RL^{\ell} \mathcal{J} w}{RL^{\ell} J_{\theta_2} w} = \frac{RL^{\ell} \mathcal{J} w}{RL^{\ell} J_{\theta_2} w}.
\]
\[
\left( RL_{\theta_1} J_{w}^\ell h \right) (x) = \frac{w^{-1}(x)}{\Gamma(\ell)} \int_{\theta_1}^{x} (x - \varepsilon)^{\ell - 1} h(\varepsilon) d\varepsilon,
\]
\[
\left( RL_{w} J_{\theta_2}^\ell h \right) (x) = \frac{w^{-1}(x)}{\Gamma(\ell)} \int_{x}^{\theta_2} (\varepsilon - x)^{\ell - 1} h(\varepsilon) d\varepsilon, \quad \ell > 0,
\]
respectively.

(ii) \( \sigma'(x) = x \) and \( \ell = 1 \); then inequality (12) reduces to inequality (6).

(iii) \( \sigma'(x) = x \) and \( w(x) = 1 \); then inequality (12) reduces to inequality (3).

(iv) \( \sigma(x) = x, w(x) = 1 \) and \( \ell = 1 \); then inequality (12) reduces to inequality (2).

Remark 4. From Remark 3, we can observe that the HH inequality (3) and the HHF inequality (6) are essentially particular cases of our HHF inequality (12). Additionally, the HHF inequality (21) can be seen as a reformulation of HHF inequality (12), even though it is about weighted fractional and RL-fractional integrals rather than RL-fractional integrals explicitly.

Lemma 2. Let \( h : [\theta_1, \theta_2] \subseteq [0, \infty) \rightarrow \mathbb{R} \) be an \( L^1 \) function with \( h' \in L^1 \) and \( 0 \leq \theta_1 < \theta_2 \), and \( w : [\theta_1, \theta_2] \rightarrow \mathbb{R} \) be an integrable, positive and weighted symmetric function with respect to \( \frac{\theta_1 + \theta_2}{2} \). If \( \sigma \) is an increasing and positive function on \( [\theta_1, \theta_2] \) and \( \sigma'(x) \) is continuous on \( (\theta_1, \theta_2) \), then, we have for \( \ell > 0 \):

\[
\frac{h(\theta_1) + h(\theta_2)}{2} \left[ \left( \sigma^{-1}(\theta_1) + J_{w}^{\ell \sigma}(w \circ \sigma) \right) \left( \sigma^{-1}(\theta_2) \right) + \left( J_{w}^{\ell \sigma}(w \circ \sigma) \right) \circ \sigma^{-1}(\theta_1) \right]
- \left[ w(\theta_2) \left( \sigma^{-1}(\theta_2) + J_{w}^{\ell \sigma}(w \circ \sigma) \right) \circ \sigma^{-1}(\theta_2) + w(\theta_1) \left( J_{w}^{\ell \sigma}(w \circ \sigma) \right) \circ \sigma^{-1}(\theta_1) \right]
= \frac{1}{\Gamma(\ell)} \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \sigma'(x) (\theta_2 - \sigma(x))^{\ell - 1} (w \circ \sigma)(x) dx
- \frac{1}{\Gamma(\ell)} \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \sigma'(x) (\sigma(x) - \theta_1)^{\ell - 1} (w \circ \sigma)(x) dx \right) (h' \circ \sigma)(\varepsilon) d\varepsilon. \tag{22}
\]

Proof. Setting

\[
\frac{1}{\Gamma(\ell)} \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \left[ \int_{\sigma^{-1}(\theta_1)}^{\sigma(x)} \sigma'(x) (\theta_2 - \sigma(x))^{\ell - 1} (w \circ \sigma)(x) dx \right. 
- \left. \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \sigma'(x) (\sigma(x) - \theta_1)^{\ell - 1} (w \circ \sigma)(x) dx \right] (h' \circ \sigma)(\varepsilon) d\varepsilon
= \frac{1}{\Gamma(\ell)} \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \left[ \int_{\sigma^{-1}(\theta_1)}^{\sigma(x)} \sigma'(x) (\theta_2 - \sigma(x))^{\ell - 1} (w \circ \sigma)(x) dx \right. 
+ \left. \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \sigma'(x) (\sigma(x) - \theta_1)^{\ell - 1} (w \circ \sigma)(x) dx \right] (h' \circ \sigma)(\varepsilon) d\varepsilon
:= E_1 + E_2.
\]

By integration by parts, making use of Lemma 1, and definitions (8) and (9), we obtain
\[ \Xi_1 = \frac{1}{\Gamma(\ell)} \left( \int_{\sigma^{-1}(\theta_1)}^{\ell} \sigma'(x)(\theta_2 - \sigma(x))f^{-1}(w \circ \sigma)(x)dx \right) (h \circ \sigma)(\epsilon)de \bigg|_{t=\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \]

\[ - \frac{1}{\Gamma(\ell)} \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \sigma'(\epsilon)(\theta_2 - \sigma(\epsilon))f^{-1}(w \circ \sigma)(\epsilon)(h \circ \sigma)(\epsilon)de \]

\[ = \left( \frac{1}{\Gamma(\ell)} \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \sigma'(x)(\theta_2 - \sigma(x))f^{-1}(w \circ \sigma)(x)dx \right) h(\theta_2) \]

\[ - w(\theta_2) \frac{(w \circ \sigma)^{-1}(\sigma^{-1}(\theta_2))}{\Gamma(\ell)} \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \sigma'(\epsilon)(\theta_2 - \sigma(\epsilon))f^{-1}(w \circ \sigma)(\epsilon)(h \circ \sigma)(\epsilon)de \]

by using (16)

\[ = h(\theta_2) \left( \sigma^{-1}(\theta_2) + J_{\sigma^{-1}(\theta_2)}^{\sigma^{-1}(\theta_1)} (w \circ \sigma) \right) \left( \sigma^{-1}(\theta_2) \right) - w(\theta_2) \left( \sigma^{-1}(\theta_1) + J_{\sigma^{-1}(\theta_2)}^{\sigma^{-1}(\theta_1)} (w \circ \sigma) \right) \left( \sigma^{-1}(\theta_2) \right) \]

\[ = \frac{h(\theta_2)}{2} \left[ \left( \sigma^{-1}(\theta_1) + J_{\sigma^{-1}(\theta_2)}^{\sigma^{-1}(\theta_1)} (w \circ \sigma) \right) \left( \sigma^{-1}(\theta_2) \right) + \left( J_{\sigma^{-1}(\theta_2)}^{\sigma^{-1}(\theta_1)} (w \circ \sigma) \right) \left( \sigma^{-1}(\theta_1) \right) \right] \]

- \[ w(\theta_2) \left( \sigma^{-1}(\theta_1) + J_{\sigma^{-1}(\theta_2)}^{\sigma^{-1}(\theta_1)} (w \circ \sigma) \right) \left( \sigma^{-1}(\theta_2) \right) . \]

Analogously, one can get

\[ \Xi_2 = \frac{-1}{\Gamma(\ell)} \left( \int_{\ell}^{\sigma^{-1}(\theta_2)} \sigma'(x)(\sigma(x) - \theta_1)f^{-1}(w \circ \sigma)(x)dx \right) (h \circ \sigma)(\epsilon)de \bigg|_{t=\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \]

\[ - \frac{1}{\Gamma(\ell)} \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \sigma'(\epsilon)(\sigma(\epsilon) - \theta_1)f^{-1}(w \circ \sigma)(\epsilon)(h \circ \sigma)(\epsilon)de \]

\[ = \left( \frac{1}{\Gamma(\ell)} \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \sigma'(x)(\sigma(x) - \theta_1)f^{-1}(w \circ \sigma)(x)dx \right) h(\theta_1) \]

\[ - w(\theta_1) \frac{(w \circ \sigma)^{-1}(\sigma^{-1}(\theta_1))}{\Gamma(\ell)} \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \sigma'(\epsilon)(\sigma(\epsilon) - \theta_1)f^{-1}(w \circ \sigma)(\epsilon)(h \circ \sigma)(\epsilon)de \]

by using (16)

\[ = \frac{h(\theta_1)}{2} \left[ \left( \sigma^{-1}(\theta_1) + J_{\sigma^{-1}(\theta_2)}^{\sigma^{-1}(\theta_1)} (w \circ \sigma) \right) \left( \sigma^{-1}(\theta_2) \right) + \left( J_{\sigma^{-1}(\theta_2)}^{\sigma^{-1}(\theta_1)} (w \circ \sigma) \right) \left( \sigma^{-1}(\theta_1) \right) \right] \]

- \[ w(\theta_1) \left( \sigma^{-1}(\theta_1) + J_{\sigma^{-1}(\theta_2)}^{\sigma^{-1}(\theta_1)} (w \circ \sigma) \right) \left( \sigma^{-1}(\theta_2) \right) . \]

Thus, we can deduce

\[ \Xi_1 + \Xi_2 = \frac{h(\theta_1) + h(\theta_2)}{2} \left[ \left( \sigma^{-1}(\theta_1) + J_{\sigma^{-1}(\theta_2)}^{\sigma^{-1}(\theta_1)} (w \circ \sigma) \right) \left( \sigma^{-1}(\theta_2) \right) + \left( J_{\sigma^{-1}(\theta_2)}^{\sigma^{-1}(\theta_1)} (w \circ \sigma) \right) \left( \sigma^{-1}(\theta_1) \right) \right] \]

\[ - \left[ w(\theta_2) \left( \sigma^{-1}(\theta_1) + J_{\sigma^{-1}(\theta_2)}^{\sigma^{-1}(\theta_1)} (w \circ \sigma) \right) \left( \sigma^{-1}(\theta_2) \right) + w(\theta_1) \left( \sigma^{-1}(\theta_1) + J_{\sigma^{-1}(\theta_2)}^{\sigma^{-1}(\theta_1)} (w \circ \sigma) \right) \left( \sigma^{-1}(\theta_2) \right) \right] , \]

which completes the proof. \( \square \)
Remark 5. Particularly, in Lemma 2, if we take:

(i) \( \sigma(x) = x \), then equality (12) becomes

\[
\frac{h(\theta_1) + h(\theta_2)}{2} - \left[ \frac{RL_t^\ell \mathcal{J}_w^{\ell}(\theta_2) + RL^{\ell}_{w} \mathcal{J}_{\theta_2} w(\theta_1)}{\theta_2 - \theta_1} \right]_{\theta_1}^{\theta_2} h(x) dx - \int_{\epsilon}^{\theta_2}(x - \theta_1)^{\ell-1} w(x) dx = h'(\epsilon \theta_1 + (1-\epsilon) \theta_2) \epsilon, \tag{23}
\]

where \( RL_t^\ell \) and \( RL^{\ell}_{w} \mathcal{J}_{\theta_2} w \) are as defined in Remark 3.

(ii) \( \sigma(x) = x \) and \( w(x) = 1 \), then equality (12) becomes

\[
\frac{h(\theta_1) + h(\theta_2)}{2} - \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} h(x) dx = \frac{\theta_2 - \theta_1}{2} \int_0^1 [\epsilon^{\ell} - (1-\epsilon)^{\ell}] h'(\epsilon \theta_1 + (1-\epsilon) \theta_2) d\epsilon, \tag{24}
\]

which is already established in ([11], lemma 2).

(iii) \( \sigma(x) = x \), \( w(x) = 1 \) and \( \ell = 1 \), we obtain

\[
\frac{h(\theta_1) + h(\theta_2)}{2} - \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} h(x) dx = \frac{\theta_2 - \theta_1}{2} \int_0^1 [1 - 2\epsilon] h'(\epsilon \theta_1 + (1-\epsilon) \theta_2) d\epsilon,
\]

which is already established in ([38] lemma 2.1).

Remark 6. From Remark 5 (i), we can observe that our result Lemma 2 is essentially a reformulation of the result of ([34], lemma 2.4), even though it is about weighted fractional and RL-fractional integrals rather than RL-fractional integrals explicitly. Additionally, from Remark 5 (ii) and (iii), we can observe that the results of ([11], lemma 2) and ([38], lemma 2.1) are basically particular cases of our result Lemma 2.

3. Main Results

In view of Lemma 2, we can obtain the following HHF inequalities.

Theorem 2. Let \( h : [\theta_1, \theta_2] \subseteq [0, \infty) \to \mathbb{R} \) be an \( L^1 \) function with \( h' \in L^1 \) and \( 0 \leq \theta_1 < \theta_2 \), and \( w : [\theta_1, \theta_2] \to \mathbb{R} \) be an integrable, positive and weighted symmetric function with respect to \( \theta_1, \theta_2 \). If \( |h'| \) is convex on \([\theta_1, \theta_2]\), \( \sigma \) is an increasing and positive function on \([\theta_1, \theta_2]\), and \( \sigma'(x) \) is continuous on \((\theta_1, \theta_2)\). Then, we have for \( \ell > 0 \):

\[
|\Xi_1 + \Xi_2| \leq \frac{\|w \circ \sigma\|_{\infty}}{(\theta_2 - \theta_1)\Gamma(\ell + 1)} \left[ A_{\ell}(\theta_1, \theta_2)|h'(\theta_1)| + B_{\ell}(\theta_1, \theta_2)|h'(\theta_2)| \right], \tag{25}
\]

where \( \Xi_1 \) and \( \Xi_2 \) are defined as in the proof of Lemma 2, and

\[
A_{\ell}(\theta_1, \theta_2) := \frac{1}{\ell + 1} \left[ (\theta_2 - \theta_1)^{\ell+1} - (\theta_2 - \sigma \left( \frac{\sigma^{-1}(\theta_2)}{\ell + 1} \right))^{\ell+1} - \sigma \left( \frac{\sigma^{-1}(\theta_1)}{\ell + 1} \right)^{\ell+1} \right] - \frac{1}{\ell + 2} \left[ (\theta_2 - \theta_1)^{\ell+2} - (\theta_2 - \sigma \left( \frac{\sigma^{-1}(\theta_2)}{\ell + 2} \right))^{\ell+2} - \sigma \left( \frac{\sigma^{-1}(\theta_1)}{\ell + 2} \right)^{\ell+2} \right] + \frac{\theta_1}{\ell + 1} \left( \sigma \left( \frac{\sigma^{-1}(\theta_1)}{2} \right) \right)^{\ell+1} + \frac{1}{\ell + 2} \left( \sigma \left( \frac{\sigma^{-1}(\theta_1)}{2} \right) \right)^{\ell+2} - \sigma \left( \frac{\sigma^{-1}(\theta_1)}{2} \right)^{\ell+2}
\]
Additionally, since \( w \) can write

\[
\begin{align*}
\text{Symmetry} \, 2020 & : \quad \text{can write} \\
|\mathcal{E}_1 + \mathcal{E}_2| & \leq \frac{1}{\Gamma(\ell)} \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \int_{\sigma^{-1}(\theta_1)}^{\epsilon} \sigma'(x)(\theta_2 - \sigma(x))^{\epsilon-1}(w \circ \sigma')(x) dx \\
& \quad - \int_{\epsilon}^{\sigma^{-1}(\theta_2)} \sigma'(x)(\sigma(x) - \theta_1)^{\epsilon-1}(w \circ \sigma')(x) dx \\
& \quad \left( (h' \circ \sigma')(\epsilon) \right) |\sigma'(\epsilon)| de. \quad (26)
\end{align*}
\]

Since \(|h'|\) is convex on \([\theta_1, \theta_2]\), we get for \(\epsilon \in [\sigma^{-1}(\theta_1), \sigma^{-1}(\theta_2)]\):

\[
|h'(\circ \sigma')(\epsilon)| = |h'\left(\frac{\theta_2 - \sigma(\epsilon)}{\theta_2 - \theta_1} \theta_1 + \frac{\sigma(\epsilon) - \theta_1}{\theta_2 - \theta_1} \theta_2\right)| \leq \frac{\theta_2 - \sigma(\epsilon)}{\theta_2 - \theta_1} |h'\left(\theta_1\right)| + \frac{\sigma(\epsilon) - \theta_1}{\theta_2 - \theta_1} |h'\left(\theta_2\right)|. \quad (27)
\]

Additionally, since \(w : [\theta_1, \theta_2] \to \mathbb{R}\) is symmetric weighted function with respect to \(\frac{\theta_1 + \theta_2}{2}\), so we can write
After simple calculations of integrals arising from inequality (29), we can obtain the desired result (25).

Then, we obtain

\[ \int_{\varepsilon}^{\theta_1} \sigma'(x)(\sigma(x) - \theta_1)^{\ell-1}(w \circ \sigma)(x)dx \]
\[ = \int_{\varepsilon}^{\theta_1} \sigma'(x)(\theta_2 - \sigma(x))^{\ell-1}(w \circ \sigma)(\sigma^{-1}(\theta_1) + \sigma^{-1}(\theta_2) - x)dx \]
\[ = \int_{\varepsilon}^{\theta_1} \sigma'(x)(\theta_2 - \sigma(x))^{\ell-1}(w \circ \sigma)(x)dx. \]

Then, we obtain

\[ \int_{\varepsilon}^{\theta_1} \sigma'(x)(\theta_2 - \sigma(x))^{\ell-1}(w \circ \sigma)(x)dx - \int_{\varepsilon}^{\theta_1} \sigma'(x)(\sigma(x) - \theta_1)^{\ell-1}(w \circ \sigma)(x)dx \]
\[ = \int_{\varepsilon}^{\theta_1} \sigma'(x)(\theta_2 - \sigma(x))^{\ell-1}(w \circ \sigma)(x)dx \]
\[ \leq \left\{ \begin{array}{ll}
\int_{\varepsilon}^{\theta_1} \sigma'(x)(\theta_2 - \sigma(x))^{\ell-1}(w \circ \sigma)(x)dx, & \varepsilon \in \left[\sigma^{-1}(\theta_1), \frac{\sigma^{-1}(\theta_1) + \sigma^{-1}(\theta_2)}{2}\right] \\
\int_{\varepsilon}^{\theta_1} \sigma'(x)(\theta_2 - \sigma(x))^{\ell-1}(w \circ \sigma)(x)dx, & \varepsilon \in \left[\frac{\sigma^{-1}(\theta_1) + \sigma^{-1}(\theta_2)}{2}, \sigma^{-1}(\theta_2)\right].
\end{array} \right. \]

By applying the inequalities (26)–(28), we have

\[ \|\Xi_1 + \Xi_2\| \leq \frac{1}{\Gamma(\ell)} \int_{\varepsilon}^{\theta_1} \sigma'(x)|\theta_2 - \sigma(x)|^{\ell-1}(w \circ \sigma)(x)dx \]
\[ \times \left( \frac{\theta_2 - \sigma(x)}{\theta_2 - \theta_1} |h'(x)| + \frac{\sigma(x) - \theta_1}{\theta_2 - \theta_1} |h'(\theta_2)| \right) \sigma'(x)dx \]
\[ \leq \frac{1}{\Gamma(\ell)} \int_{\varepsilon}^{\theta_1} \sigma'(x)|\theta_2 - \sigma(x)|^{\ell-1}(w \circ \sigma)(x)dx \]
\[ \times \left( \frac{\theta_2 - \sigma(x)}{\theta_2 - \theta_1} |h'(x)| + \frac{\sigma(x) - \theta_1}{\theta_2 - \theta_1} |h'(\theta_2)| \right) \sigma'(x)dx. \]

After simple calculations of integrals arising from inequality (29), we can obtain the desired result (25). □

Remark 7. Particularly, in Theorem 2, if we take

(i) \( \sigma(x) = x \), we have

\[ \left| \frac{h(\theta_1) + h(\theta_2)}{2} - \int_{\theta_1}^{\theta_2} w(\theta_2) + \int_{\theta_1}^{\theta_2} w(\sigma(\theta_2)) \right| \]
\[ \leq \left\| w \right\|_{\infty} (\theta_2 - \theta_1)^{\ell+1} \left( 1 - \frac{1}{2^\ell} \right) \left[ |h'(\theta_1)| + |h'(\theta_2)| \right]. \]

(ii) \( \sigma(x) = x \) and \( w(x) = 1 \), we get

\[ \left| \frac{h(\theta_1) + h(\theta_2)}{2} - \int_{\theta_1}^{\theta_2} w(\theta_2) + \int_{\theta_1}^{\theta_2} w(\sigma(\theta_2)) \right| \]
\[ \leq \left( 1 - \frac{1}{2^\ell} \right) \left[ |h'(\theta_1)| + |h'(\theta_2)| \right].
\]

which is already established in ([11] Theorem 3).
(iii) \( \sigma(x) = x, w(x) = 1 \) and \( \ell = 1 \), we obtain

\[
\left| \frac{h(\theta_1) + h(\theta_2)}{2} - \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} h(x) \, dx \right| \leq \frac{\theta_2 - \theta_1}{8} \left[ |h'(\theta_1)| + |h'(\theta_2)| \right],
\]

which is already established in ([38], Theorem 2.2).

**Remark 8.** Again, from Remark 7 (i), we can observe that our result Lemma 2 is essentially a reformulation of the result of ([11], Theorem 3) and ([38], Theorem 2.2), even though it is about weighted fractional and RL-fractional integrals rather than RL-fractional integrals explicitly. In addition, from Remark 7 (ii) and (iii), we can observe that the results of ([11], Theorem 3) and ([38], Theorem 2.2) are basically particular cases of our result Lemma 2.

**Theorem 3.** Let \( h : [\theta_1, \theta_2] \subseteq [0, \infty) \to \mathbb{R} \) be an \( L^1 \) function with \( h' \in L^1 \) and \( 0 \leq \theta_1 < \theta_2 \), and \( w : [\theta_1, \theta_2] \to \mathbb{R} \) be an integrable, positive and weighted symmetric function with respect to \( \frac{1}{\theta_1} + \frac{1}{\theta_2} \).

If \( |h'|^q, q \geq 1 \) is convex on \( [\theta_1, \theta_2], \sigma \) is an increasing and positive function on \( [\theta_1, \theta_2] \) and \( \sigma'(x) \) is continuous on \( [\theta_1, \theta_2] \). Then, we have for \( \ell > 0 \):

\[
|\Xi_1 + \Xi_2| \leq \left( \frac{\|w \circ \sigma\|_{\infty}}{(\theta_2 - \theta_1)^{1/2} \Gamma(\ell + 1)} (C_{\sigma}(\ell; \theta_1, \theta_2))^{1-\frac{1}{\ell}} \right) \times \left( D_{\sigma}(\ell; \theta_1, \theta_2) |h'(\theta_1)|^q + E_{\sigma}(\ell; \theta_1, \theta_2) |h'(\theta_2)|^q \right)^{\frac{1}{\ell}},
\]

where

\[
C_{\sigma}(\ell; \theta_1, \theta_2) := \frac{1}{\ell} \left[ \left( \frac{\theta_2 - \theta_1}{\ell + 1} - \left( \frac{\sigma^{-1}(\theta_1) + \sigma^{-1}(\theta_2)}{2} \right) \right)^{\ell + 1} - C^{(1)}_{\sigma}(\ell; \theta_1, \theta_2)
\]

\[
+ C^{(2)}_{\sigma}(\ell; \theta_1, \theta_2) - \left( \frac{\theta_2 - \sigma^{-1}(\theta_1) + \sigma^{-1}(\theta_2)}{\ell + 1} \right)^{\ell + 1} \right];
\]

\[
C^{(1)}_{\sigma}(\ell; \theta_1, \theta_2) := \int_{\theta_1}^{\theta_2} \left( \frac{\sigma^{-1}(\theta_1) + \sigma^{-1}(\theta_2)}{2} \right)^{\ell + 1} \left[ \theta_2 - \sigma^{-1} \left( \sigma^{-1}(\theta_1) + \sigma^{-1}(\theta_2) - \sigma^{-1}(x) \right) \right] \, dx;
\]

\[
C^{(2)}_{\sigma}(\ell; \theta_1, \theta_2) := \int_{\theta_1}^{\theta_2} \left( \frac{\sigma^{-1}(\theta_1) + \sigma^{-1}(\theta_2)}{2} \right)^{\ell + 1} \left[ \theta_2 - \sigma^{-1} \left( \sigma^{-1}(\theta_1) + \sigma^{-1}(\theta_2) - \sigma^{-1}(x) \right) \right] \, dx;
\]

\[
D_{\sigma}(\ell; \theta_1, \theta_2) := \frac{1}{\ell} \left[ \left( \frac{\theta_2 - \theta_1}{\ell + 2} - \left( \frac{\sigma^{-1}(\theta_1) + \sigma^{-1}(\theta_2)}{2} \right) \right)^{\ell + 2} - \theta_2 C^{(1)}_{\sigma}(\ell; \theta_1, \theta_2) - D^{(1)}_{\sigma}(\ell; \theta_1, \theta_2)
\]

\[
+ \theta_2 C^{(2)}_{\sigma}(\ell; \theta_1, \theta_2) - \left( \frac{\theta_2 - \sigma^{-1}(\theta_1) + \sigma^{-1}(\theta_2)}{\ell + 2} \right)^{\ell + 2} - D^{(2)}_{\sigma}(\ell; \theta_1, \theta_2) \right];
\]

\[
D^{(1)}_{\sigma}(\ell; \theta_1, \theta_2) := \int_{\theta_1}^{\theta_2} \left( \frac{\sigma^{-1}(\theta_1) + \sigma^{-1}(\theta_2)}{2} \right)^{\ell + 2} \left[ \theta_2 - \sigma^{-1} \left( \sigma^{-1}(\theta_1) + \sigma^{-1}(\theta_2) - \sigma^{-1}(x) \right) \right] \, dx;
\]

\[
D^{(2)}_{\sigma}(\ell; \theta_1, \theta_2) := \int_{\theta_1}^{\theta_2} \left( \frac{\sigma^{-1}(\theta_1) + \sigma^{-1}(\theta_2)}{2} \right)^{\ell + 2} \left[ \theta_2 - \sigma^{-1} \left( \sigma^{-1}(\theta_1) + \sigma^{-1}(\theta_2) - \sigma^{-1}(x) \right) \right] \, dx,
\]
and

$$E_\phi(\ell; \theta_1, \theta_2) := \frac{1}{\ell} \left\{ \frac{\theta_2}{\ell + 1} \left[ (\theta_2 - \theta_1)^{\ell+1} - \left( \theta_2 - \sigma \left( \frac{\sigma^{-1}(\theta_1) + \sigma^{-1}(\theta_2)}{2} \right) \right)^{\ell+1} \right] \right. $$

$$- \frac{1}{\ell + 2} \left[ (\theta_2 - \theta_1)^{\ell+2} - \left( \theta_2 - \sigma \left( \frac{\sigma^{-1}(\theta_1) + \sigma^{-1}(\theta_2)}{2} \right) \right)^{\ell+2} \right] $$

$$- \frac{\theta_1}{\ell + 1} \left[ (\theta_2 - \theta_1)^{\ell+1} - \left( \theta_2 - \sigma \left( \frac{\sigma^{-1}(\theta_1) + \sigma^{-1}(\theta_2)}{2} \right) \right)^{\ell+1} \right] - D_\phi^{(1)}(\ell; \theta_1, \theta_2) + \theta_1 C_\phi^{(1)}(\ell; \theta_1, \theta_2) $$

$$+ D_\phi^{(2)}(\ell; \theta_1, \theta_2) + \theta_1 \left[ \frac{1}{\ell + 1} \left( \theta_2 - \sigma \left( \frac{\sigma^{-1}(\theta_1) + \sigma^{-1}(\theta_2)}{2} \right) \right)^{\ell+1} - C_\phi^{(2)}(\ell; \theta_1, \theta_2) \right] $$

Proof. By using Lemma 2, the well–known power mean inequality, inequality (28), convexity of $|h'|^q$ and properties of modulus, we can deduce

$$|\Xi_1 + \Xi_2| \leq \frac{1}{\Gamma(\ell)} \left[ \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \left( \int_{\ell \epsilon}^{\epsilon} \left| \sigma'(x)(\theta_2 - \sigma(x))^{\ell-1}(w \circ \sigma)(x) \right| dx \right) \sigma'(\epsilon) d\epsilon \right]^{1-\frac{1}{q}} $$

$$\times \left[ \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \left( \int_{\epsilon}^{\ell \epsilon} \left| \sigma'(x)(\theta_2 - \sigma(x))^{\ell-1}(w \circ \sigma)(x) \right| dx \right) \left| (h' \circ \sigma)(\epsilon) \right|^q \sigma'(\epsilon) d\epsilon \right]^{\frac{1}{q}} $$

$$\leq \frac{\|w \circ \sigma\|_\infty}{\Gamma(\ell)} \left[ \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \left( \int_{\epsilon}^{\ell \epsilon} \left| \sigma'(x)(\theta_2 - \sigma(x))^{\ell-1} dx \right) \right) \sigma'(\epsilon) d\epsilon \right]^{1-\frac{1}{q}} $$

$$\times \left[ \int_{\sigma^{-1}(\theta_1)}^{\sigma^{-1}(\theta_2)} \left( \int_{\epsilon}^{\ell \epsilon} \left| \sigma'(x)(\theta_2 - \sigma(x))^{\ell-1} dx \right) \right) \left| (h' \circ \sigma)(\epsilon) \right|^q \sigma'(\epsilon) d\epsilon \right]^{\frac{1}{q}} $$
After simple calculations of integrals arising from inequality (34), one can obtain the desired result (33). □

**Remark 9.** Particularly, in Theorem 3, if we take:

(i) \( \sigma(x) = x \), we get

\[
\left| \frac{h(\theta_1) + h(\theta_2)}{2} \right| \leq \frac{||w \circ \sigma||_{\infty}}{(\theta_2 - \theta_1)^{\frac{1}{2}} \Gamma(\ell)} \left[ \int_{c^{-1}(\theta_1)}^{c^{-1}(\theta_2)} \left( \int_{c^{-1}(\theta_1)}^{c^{-1}(\theta_2)} |\sigma'(x)| |\theta_2 - \sigma(x)|^{\ell+1} dx \right) \times \left( \frac{\theta_2 - \sigma(x)}{\theta_2 - \theta_1} |h'(\theta_1)|^q + \frac{\sigma(x) - \theta_1}{\theta_2 - \theta_1} |h'(\theta_2)|^q \right) \sigma'(x) dx \right]^{\frac{1}{q}}.
\]

(ii) \( \sigma(x) = x \) and \( w(x) = 1 \), we get

\[
\left| \frac{h(\theta_1) + h(\theta_2)}{2} \right| \leq \frac{||w||_{\infty}}{(\theta_2 - \theta_1)^{\frac{1}{2}} \Gamma(\ell + 1)} \left[ (C(\ell; \theta_1, \theta_2))^{\frac{1}{2}} \left[ D(\ell; \theta_1, \theta_2) |h'(\theta_1)|^q + E(\ell; \theta_1, \theta_2) |h'(\theta_2)|^q \right] \right]^{\frac{1}{2}}.
\]

where

\[
C(\ell; \theta_1, \theta_2) := \frac{2(\theta_2 - \theta_1)^{\ell+1}}{\ell(\ell + 1)} \left( 1 - \frac{1}{2^\ell} \right),
\]

\[
D(\ell; \theta_1, \theta_2) := \frac{1}{\ell + 1} \left[ \left( \frac{2}{\ell + 1} \right)^{\ell+2} - 2 \theta_2 \left( \frac{\theta_2 - \theta_1}{\theta_2 - \theta_1} \right)^{\ell+1} \right] - \frac{2}{\ell + 2} \left( \frac{\theta_2 - \theta_1}{\theta_2 - \theta_1} \right)^{\ell+2},
\]

and

\[
E(\ell; \theta_1, \theta_2) := \frac{(\theta_2 - \theta_1)^{\ell+2}}{\ell(\ell + 1)} \left( 1 - \frac{1}{2^\ell} \right).
\]
Symmetry 2020, 12, 1503

(iii) \( \sigma(x) = x, w(x) = 1 \) and \( \ell = 1 \), we obtain

\[
\left| \frac{h(\vartheta_1) + h(\vartheta_2)}{2} - \frac{1}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} h(x) dx \right| \leq \frac{1}{(\vartheta_2 - \vartheta_1)^{\frac{3}{2}}} \left( \frac{(\vartheta_2 - \vartheta_1)^2}{2} \right)^{1 - \frac{1}{\eta}} \times \left[ \frac{(\vartheta_2 - \vartheta_1)^2}{12} (2\vartheta_2 - 5\vartheta_1) |h'(\vartheta_1)|^\eta + \frac{3(\vartheta_2 - \vartheta_1)^3}{4} |h'(\vartheta_2)|^\eta \right]^{\frac{1}{\eta}}. \tag{37}
\]

**Remark 10.** The specific results are different from those obtained in [11,34,38] according to Remark 9.

4. Discussion

We have considered the weighted fractional operators. In our present investigation, we have established new fractional HHF integral inequalities involving the weighted fractional operators associated with positive symmetric functions. The HHF fractional integral inequality (7) has been applied to other class of convex functions, such as \( p \)-convex functions [39], generalized convex functions [40], \((\eta_1, \eta_2)\)-convex functions [41] and many others that can be found in the literature. Thus, the results obtained here can be also be applied to the above class of convex functions.

It is worthwhile to mention that there are three well-known versions of fractional Hermite–Hadamard integral inequalities. The first version was established by Sarikaya et al. in [11] and their result is given in (3). The other versions consist of

\[
\frac{h\left( \frac{\vartheta_1 + \vartheta_2}{2} \right)}{2} \leq 2^\ell \frac{\Gamma(\ell + 1)}{(\vartheta_2 - \vartheta_1)^\ell} \left[ RL\mathcal{J}^\ell_{\vartheta_1, \vartheta_2} h(\vartheta_1) + RL\mathcal{J}^\ell_{\vartheta_1, \vartheta_2} h(\vartheta_2) \right] \leq \frac{h(\vartheta_1) + h(\vartheta_2)}{2}, \tag{38}
\]

and

\[
\frac{h\left( \frac{\vartheta_1 + \vartheta_2}{2} \right)}{2} \leq 2^\ell \frac{\Gamma(\ell + 1)}{(\vartheta_2 - \vartheta_1)^\ell} \left[ RL\mathcal{J}^\ell_{\vartheta_1, \vartheta_2} h(\vartheta_1) + RL\mathcal{J}^\ell_{\vartheta_1, \vartheta_2} h(\vartheta_2) \right] \leq \frac{h(\vartheta_1) + h(\vartheta_2)}{2}; \tag{39}
\]

these were already established by Sarikaya and Yaldiz [42], and Mohammed and Brevik [1], respectively. We believe that the results in this study are very generic and can be extended to give further potentially interesting and useful integral inequalities involving other versions of fractional integral inequalities (38) and (39).

5. Conclusions

Integral inequality forms a significant branch of mathematical analysis, which has been combined with all models of fractional calculus but never before with weighted fractional calculus models. For this reason, in this study we have considered the Hermite–Hadamard–Fejer integral inequalities in the context of fractional calculus with positive weighted symmetric function kernels.

**Author Contributions:** Conceptualization, P.O.M. and T.A.; methodology, A.K.; software, P.O.M.; validation, P.O.M., A.K.; formal analysis, P.O.M.; investigation, P.O.M.; resources, T.A.; data curation, P.O.M.; writing—original draft preparation, P.O.M.; writing—review and editing, A.K.; visualization, A.K.; supervision, T.A.; project administration, T.A.; funding acquisition, T.A. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Acknowledgments:** The authors express their special thanks to the associate editor and the referees.

**Conflicts of Interest:** The authors declare no conflict of interest.
Symmetry 2020, 12, 1503

References
1. Mohammed, P.O.; Brevik, I. A New Version of the Hermite–Hadamard Inequality for Riemann-Liouville Fractional Integrals. Symmetry 2020, 12, 610. [CrossRef]
2. Gavrea, B.; Gavrea, I. On some Ostrowski type inequalities. Gen. Math. 2010, 18, 33–44.
3. Vivas-Cortez, M.; Abdeljawad, T.; Mohammed, P.O.; Rangel-Oliveros, Y. Simpson’s Integral Inequalities for Twice Differentiable Convex Functions. Math. Probl. Eng. 2020, 2020, 1936461. [CrossRef]
4. Kaibser, S.; Nikolova, L.; Persson, L.-E.; Wedestig, A. Hardy type inequalities via convexity. Math. Inequal. Appl. 2005, 8, 403–417. [CrossRef]
5. Gunawan, H. Eridani, Fractional integrals and generalized Olsen inequalities. Kyungpook Math. J. 2009, 49, 31–39. [CrossRef]
6. Sawano, Y.; Wadade, H. On the Gagliardo-Nirenberg type inequality in the critical Sobolev-Morrey space. J. Fourier Anal. Appl. 2013, 19, 20–47. [CrossRef]
7. Mohammed, P.O.; Abdeljawad, T. Opial integral inequalities for generalized fractional operators with nonsingular kernel. J. Inequal. Appl. 2020, 2020, 148. [CrossRef]
8. Sarikaya, M.Z.; Bilisik, C.C.; Mohammed, P.O. Some generalizations of Opial type inequalities. Appl. Math. Inf. Sci. 2014, 14, 809–816.
9. Zhao, C.-J.; Cheung, W.-S. On improvements of the Rozanova’s inequality. J. Inequal. Appl. 2011, 2020, 33. [CrossRef]
10. Hadamard, J. Étude sur les propriétés des fonctions entières en particulier d’une fonction considérée par Riemann. J. Math. Pures Appl. 1893, 58, 171–215.
11. Sarikaya, M.Z.; Set, E.; Yıldız, H.; Başak, N. Hermite-Hadamard’s inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model. 2013, 57, 2403–2407. [CrossRef]
12. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. Theory and Applications of Fractional Differential Equations; North-Holland Mathematics Studies; Elsevier Sci. B.V.: Amsterdam, The Netherlands, 2006; Volume 204.
13. Bardaro, C.; Butzer, P.L.; Mantellini, I. The foundations of fractional calculus in the Mellin transform setting with applications. J. Fourier Anal. Appl. 2015, 21, 961–1017. [CrossRef]
14. Zhang, T.-Y.; Ji, A.-P.; Qi, F. On Integral Inequalities of Hermite-Hadamard Type for s-Geometrically Convex Functions. Abstr. Appl. Anal. 2012, 2012, 560586. [CrossRef]
15. Zhang, T.-Y.; Ji, A.-P.; Qi, F. Some inequalities of Hermite-Hadamard type for GA-convex functions with applications to means. Le Mat. 2013, 68, 229–239.
16. Mohammed, P.O. Some new Hermite-Hadamard type inequalities for MT-convex functions on differentiable coordinates. J. King Saud Univ. Sci. 2018, 30, 258–262. [CrossRef]
17. Shi, D.-P.; Xi, B.-Y.; Qi, F. Hermite–Hadamard type inequalities for Riemann–Liouville fractional integrals of \((a, m)\)-convex functions. Fract. Differ. Calc. 2014, 4, 31–43. [CrossRef]
18. Dragomir, S.S.; Pearce, C.E.M. Selected Topics on Hermite-Hadamard Inequalities and Applications; RGMIA Monographs; Victoria University: Footscray, Australia, 2000.
19. Mohammed, P.O.; Sarıkaya, M.Z. Hermite-Hadamard type inequalities for F-convex function involving fractional integrals. J. Inequal. Appl. 2018, 2018, 359. [CrossRef] [PubMed]
20. Baleanu, D.; Mohammed, P.O.; Zeng, S. Inequalities of trapezoidal type involving generalized fractional integrals. Alex. Eng. J. 2020. [CrossRef]
21. Han, J.; Mohammed, P.O.; Zeng, H. Generalized fractional integral inequalities of Hermite-Hadamard-type for a convex function. Open Math. 2020, 18, 794–806. [CrossRef]
22. Qi, F.; Mohammed P.O.; Yao, J.C.; Yao, Y.H. Generalized fractional integral inequalities of Hermite–Hadamard type for \((a, m)\)-convex functions. J. Inequal. Appl. 2019, 2019, 135. [CrossRef]
23. Mohammed, P.O.; Abdeljawad, T.; Zeng, S.; Kashuri, A. Fractional Hermite-Hadamard Integral Inequalities for a New Class of Convex Functions. Symmetry 2020, 12, 1485. [CrossRef]
24. Mohammed, P.O.; Abdeljawad, T. Modification of certain fractional integral inequalities for convex functions. Adv. Differ. Equ. 2020, 2020, 69. [CrossRef]
25. Baleanu, D.; Mohammed, P.O.; Vivas-Cortez, M.; Rangel-Oliveros, Y. Some modifications in conformable fractional integral inequalities. Adv. Differ. Equ. 2020, 2020, 374. [CrossRef]
26. Abdeljawad, T.; Mohammed, P.O.; Kashuri, A. New Modified Conformable Fractional Integral Inequalities of Hermite-Hadamard Type with Applications. J. Funct. Space 2020, 2020, 4352357. [CrossRef]
27. Mohammed, P.O.; Sarikaya, M.Z. On generalized fractional integral inequalities for twice differentiable convex functions. J. Comput. Appl. Math. 2020, 372, 112740. [CrossRef]
28. Mohammed, P.O.; Abdeljawad, T. Integral inequalities for a fractional operator of a function with respect to another function with nonsingular kernel. Adv. Differ. Equ. 2020, 2020, 363. [CrossRef]
29. Mohammed, P.O. Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals of a convex function with respect to a monotone function. Math. Methods Appl. Sci. 2019, 1–11. [CrossRef]
30. Mohammed, P.O.; Sarikaya, M.Z.; Baleanu, D. On the Generalized Hermite-Hadamard Inequalities via the Tempered Fractional Integrals. Symmetry 2020, 12, 595. [CrossRef]
31. Fernández, A.; Mohammed, P. Hermite-Hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels. Math. Methods Appl. Sci. 2020, 1–18. [CrossRef]
32. Macdonald, I.G. Symmetric Functions and Orthogonal Polynomials; Providence, RI; American Mathematical Soc.: New York, NY, USA, 1997
33. Fejér, L. Überdie Fourierreihen, II, Math. Naturweiz Anz Ung. Akad. Wiss. 1906, 24, 369–390.
34. İşcan, İ. Hermite-Hadamard-Fejér type inequalities for convex functions via fractional integrals. Stud. Univ. Babeş Bolyai Math. 2015, 60, 355–366.
35. Jarad, F.; Abdeljawad, T.; Shah, K. On the Weighted Fractional operators of a function with respect to another function. Fractals 2020. [CrossRef]
36. Osler, T.J. The Fractional Derivative of a Composite Function. SIAM J. Math. Anal. 1970, 1, 288–293. [CrossRef]
37. Almeida, R. A Caputo fractional derivative of a function with respect to another function. Commun. Nonlinear Sci. Numer. Simul. 2017, 44, 460–481. [CrossRef]
38. Dragomir, S.S.; Agarwal, R.P. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Appl. Math. Lett. 1998, 11, 91–95. [CrossRef]
39. Kunt, M.; İşcan, İ. On new Hermite-Hadamard-Fejer type inequalities for p-convex functions via fractional integrals. CMMA 2017, 2, 1–15. [CrossRef]
40. Delavar, M.R.; Aslani, M.; De La Sen, M. Hermite-Hadamard-Fejér Inequality Related to Generalized Convex Functions via Fractional Integrals. ScienceAsia 2018, 2018, 5864091.
41. Mehmood, S.; Zafar, F.; Asmin, N. New Hermite-Hadamard-Fejér type inequalities for \( (\eta_1, \eta_2) \)-convex functions via fractional calculus. ScienceAsia 2020, 46, 102–108. [CrossRef]
42. Sarikaya, M.Z.; Yaldiz, H. On generalization integral inequalities for fractional integrals. Nihonkai Math. J. 2014, 25, 93–104.

© 2020 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).