Time-dependent rationally extended Pöschl–Teller potential and some of its properties

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Abstract We examine time-dependent Schrödinger equation with oscillating boundary condition. More specifically, we use separation of variable technique to construct time-dependent rationally extended Pöschl–Teller potential (whose solutions are given by in terms of $X_1$ Jacobi exceptional orthogonal polynomials) and its supersymmetric partner, namely the Pöschl–Teller potential. We have obtained exact solutions of the Schrödinger equation with the above-mentioned potentials subjected to some boundary conditions of the oscillating type. A number of physical quantities like the average energy, probability density, expectation values, etc., have also been computed for both the systems and compared with each other.

1 Introduction

Schrödinger equation with time-dependent potentials is of interest for various reasons, and they have been studied over the years by many authors [1–9]. Among the various time-dependent systems, there is a class of problems for which the boundary moves with time [10–29]. During the last few years, moving boundary problems have been studied by various methods like the method of invariants [30], separation of variables [31–34], supersymmetry [6,25], etc. It may be noted that in most of these works the initial potential at zero time was chosen as one of the standard solvable quantum mechanical potentials.

The solutions of the standard quantum mechanical potentials are usually given in terms of classical orthogonal polynomials. On the other hand, some years back a class of orthogonal polynomials, called the exceptional orthogonal polynomials, have been found and they are distinct from the classical ones [35,36]. Subsequently, through Darboux transformation or supersymmetry, a large number of exactly solvable potentials have been found whose solutions are given in terms of exceptional orthogonal polynomials [37,38]. One of the main features of these new potentials is that they share the same or nearly the same spectrum as their classical counterparts although their wavefunctions are quite different. Recently, a method

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has also been proposed to generate rational extensions of time-dependent potentials \[39\]. In the present work, we shall consider the Pöschl–Teller potential and its supersymmetric partner, namely the rationally extended Pöschl–Teller potential \[40\] and apply the separation of variable technique \[34\] to construct time-dependent versions of both the potentials subjected to oscillating boundary condition \[21\]. One of the reasons for studying this problem stems from the fact that barring an exception \[39\] time-dependent versions of potentials whose solutions are given in terms of exceptional orthogonal polynomials have been not been studied before and secondly, it is of interest to examine how far physical quantities like average energy or expectation values, etc., for the time-dependent rationally extended Pöschl–Teller potential differ from their time-dependent classical counterpart, namely the time-dependent Pöschl–Teller potential. In particular, we shall obtain exact solutions and examine various features like behavior of the probability density, time-dependent localization property, average energy, root mean square values, etc., and also compare these properties with those of the time-dependent Pöschl–Teller potential.

The paper is organized as follows: In Sect. 2, we present briefly the separation of variable approach to the time-dependent Schrödinger equation; in Sect. 3, we shall use the method of Sect. 2 and supersymmetry to construct time-dependent rationally extended Pöschl–Teller potential. We shall also examine various features of the time-dependent Pöschl–Teller potential and the rationally extended Pöschl–Teller potential and compare the differences, if any, at different points of time; finally, Sect. 4 is devoted to a conclusion.

2 Separation of variables

To begin with, let us consider the time-dependent Schrödinger equation (in units of $\hbar = 2m = 1$)

$$
\left[ -\frac{\partial^2}{\partial x^2} + V(x, t) \right] \psi(x, t) = i \frac{\partial \psi(x, t)}{\partial t},
$$

subjected to the boundary conditions

$$
\psi(0, t) = 0 = \psi(L(t), t).
$$

The above boundary condition describes a moving boundary \[12\] problem in quantum mechanics. Depending on the choice of $L(t)$, the boundary can be either expanding, contracting or oscillating. We shall now use the separation of variable technique \[31\] to transform Eq. (1) to a problem with fixed boundary. To do this, let us transform the variable $x \mapsto q$ as

$$
q = \frac{\pi (x - \alpha(t))}{L(t)}, \quad \alpha(t) = \frac{1}{2} L(t), \quad -\frac{\pi}{2} \leq q \leq \frac{\pi}{2}.
$$

We now consider the potential and the wavefunction to be of the form

$$
V(x(q, t), t) = g(t)\tilde{V}(q) + U(q, t) + g_0(t),
$$

$$
\psi(q, t) = e^{\Phi(q, t)} Q(q)T(t).
$$

Then, substituting (5) in Eq. (1), we obtain

$$
-\frac{1}{Q} \frac{d^2 Q}{dq^2} + L_1^2(t) \left[ g(t)\tilde{V}(q) \right] = i L_1^2(t) \frac{1}{T} \frac{dT}{dt} = \epsilon.
$$
where $\epsilon$ is a separation constant and

$$\Phi(q, t) = a(t) \frac{q^2}{2} + b(t)q + c(t), \quad a(t) = \frac{i}{2} L_1(t) \dot{L}_1(t),$$

$$b(t) = \frac{i}{2} L_1(t) \dot{a}(t), \quad L_1(t) = \frac{1}{\pi} L(t),$$

and $c(t)$ is a constant of integration. Now without any loss of generality, we can choose [34]

$$c(t) = -\frac{1}{2} \ln L_1(t) + \frac{i}{4} \int_0^t \dot{a}^2(s) ds - i \int_0^t g_0(s) ds,$$

$$U(q, t) = -\frac{1}{4}(\pi q + q^2)L_1(t) \bar{L}_1(t),$$

and

$$g(t)L_1^2(t) = \text{constant} = 1, \text{ say}. \quad (10)$$

Then, separating Eq. (6) into the space-dependent and the time-dependent part, we obtain

$$- \frac{d^2 Q}{dq^2} + \tilde{V}(q) Q = \epsilon Q. \quad (11)$$

and

$$T(t) = e^{-i\epsilon t(t)}, \quad \tau(t) = \int_0^t \frac{1}{L_1^2(s)} ds, \quad (12)$$

where $\tilde{V}(q)$ is the effective potential. Finally, from Eqs. (4) and (9), the time-dependent potential can be found to be

$$V(x, t) = \frac{\pi^2}{L^2(t)} \tilde{V} \left[ \frac{\pi (x - \frac{1}{2}L(t))}{L(t)} \right] + \frac{1}{16} L(t) \bar{L}(t) - \frac{1}{4} \frac{\bar{L}(t)}{L(t)} x^2 + g_0(t). \quad (13)$$

### 2.1 Time-dependent rationally extended Pöschl–Teller potential via supersymmetry

Here, we shall construct a potential appropriate for the moving boundary problem using supersymmetry formalism [41, 42]. As mentioned earlier, here we are interested in obtaining time-dependent potentials whose time-independent counterparts are solvable in terms of exceptional orthogonal polynomials. Thus, we consider the following superpotential [40]:

$$\tilde{W}(q, A, B) = \left(-B - \frac{1}{2}\right) \tan q + \left(A - \frac{1}{2}\right) \sec q + \frac{2B \cos q}{2A - 1 - 2B \sin q}. \quad (14)$$

Then, the supersymmetric partner potentials $\tilde{V}^{\pm}(q, A, B) = \tilde{W}^2 \pm \tilde{W}'$ can be found to be

$$\tilde{V}^{(-)}(q, A, B) = \left[A(A - 1) + (B + 1)^2\right] \sec^2 q - (B + 1)(2A - 1) \sec q \tan q - \left(B - \frac{1}{2}\right)^2,$$

$$\tilde{V}^{(+)}(q, A, B) = \left[A(A - 1) + B^2\right] \sec^2 q - B(2A - 1) \sec q \tan q + 2 \left[2(2A - 1)^2 - 4B^2\right] \left(B - \frac{1}{2}\right)^2.$$  

The eigenvalues and the corresponding wavefunctions are given by [40]

$$E_n^{(-)} = (n + A)^2 - \left(B - \frac{1}{2}\right)^2 = E_n^{(+)}, \quad (16)$$

$$Q_n^{(-)}(q) = N_n^{(\alpha - 1, \beta + 1)}(1 - z)^{\frac{1}{2}(\alpha - \frac{1}{2})}(1 + z)^{\frac{1}{2}(\beta + \frac{1}{2})}P_n^{(\alpha - 1, \beta + 1)}(z), \quad (17)$$
where 
\[
\hat{\phi}_n^{(\alpha, \beta)}(z) = \frac{1}{2} \left( \frac{\beta + \alpha}{\beta - \alpha} - z \right) P_n^{(\alpha, \beta)}(z) + (\beta + \alpha + 2n)^{-1} \left( \frac{\beta + \alpha}{\beta - \alpha} P_n^{(\alpha, \beta)}(z) - P_{n-1}^{(\alpha, \beta)}(z) \right),
\]
\[
V^{(-)}(x, t) = \frac{\pi^2}{L^2(t)} \left[ A(A - 1) + (B + 1)^2 \right] \sec^2 \left( \frac{\pi (x - \frac{1}{2}L(t))}{L(t)} \right)
- \frac{\pi^2}{L^2(t)}(B + 1)(2A - 1) \sec \left( \frac{\pi (x - \frac{1}{2}L(t))}{L(t)} \right) \tan \left( \frac{\pi (x - \frac{1}{2}L(t))}{L(t)} \right)
+ \frac{1}{16} L(t) \ddot{L}(t) - \frac{1}{4} \frac{\dot{L}(t)}{L(t)} x^2 - (B - 1)^2.
\]
Now, using Eqs. (11), (12), we obtain the time-dependent wavefunction as
\[
\psi^{(-)}_n(x, t) = \sqrt{\Omega^{(-)}(x, t)} \Phi^{(-)}_n(x, t) \ e^{i F^{(-)}_n(x, t)},
\]
where 
\[
\Omega^{(-)}(x, t) = \frac{\pi}{L(t)} \cos^2 A \left[ \frac{\pi (x - \frac{1}{2}L(t))}{L(t)} \right] \left( \frac{1 + \sin \left[ \frac{\pi (x - \frac{1}{2}L(t))}{L(t)} \right]}{1 - \sin \left[ \frac{\pi (x - \frac{1}{2}L(t))}{L(t)} \right]} \right)^{B+1},
\]
\[
\Phi^{(-)}_n(x, t) = P_n^{(A-B-\frac{3}{2}, A+B+\frac{1}{2})} \left( \sin \left[ \frac{\pi (x - \frac{1}{2}L(t))}{L(t)} \right] \right),
\]
and \(e^{i F^{(-)}_n(x, t)}\) is the position- and time-dependent phase factor:
\[
F^{(-)}_n(x, t) = \frac{\ddot{L}(t)}{4L(t)} x^2 - \frac{1}{16} L(t) \ddot{L}(t) + \frac{1}{16} \int_0^t L^2(s) ds - E^{(-)}(t).
\]
It is not difficult to show that the normalization condition is satisfied:
\[
\int_0^L \Omega^{(-)}(x, t) \Phi^{(-)}_n(x, t) \Phi^{(-)}_m(x, t) dx = \frac{\delta_{n,m}}{N_n^{(A,B+1)^2}}.
\]
Now we come to the time-dependent rational extension of $\tilde{V}^{(-)}(q)$. From Eq. (4), the time-dependent potential $V^{(+)}(x, t)$ is found to be

$$V^{(+)}(x, t) = \frac{\pi^2}{L^2(t)} \left[ A(A - 1) + B^2 \right] \sec^2 \left[ \frac{\pi \left( x - \frac{1}{2} L(t) \right)}{L(t)} \right] - \frac{\pi^2}{L^2(t)} B(2A - 1) \sec \left[ \frac{\pi \left( x - \frac{1}{2} L(t) \right)}{L(t)} \right] \tan \left[ \frac{\pi \left( x - \frac{1}{2} L(t) \right)}{L(t)} \right]
\begin{align*}
&+ \frac{2(2A - 1)}{2A - 1 - 2B} \left[ \frac{\pi \left( x - \frac{1}{2} L(t) \right)}{L(t)} \right] - \frac{2[(2A - 1)^2 - 4B^2]}{2A - 1 - 2B \sin \left[ \frac{\pi \left( x - \frac{1}{2} L(t) \right)}{L(t)} \right]^2} \\
&+ \frac{1}{16} L(t) \dot{L}(t) - \frac{1}{4} \dot{L}(t)^2 - \left( B - \frac{1}{2} \right)^2. 
\end{align*}
(27)

The corresponding wave function can be found using Eqs. (11) and (12):

$$\psi_n^{(+)}(x, t) = \sqrt{\Omega^{(+)}(x, t)} \Phi_n^{(+)}(x, t) e^{i F_n^{(+)}(x, t)},$$
(28)

where

$$\Omega^{(+)}(x, t) = \frac{1 - \sin \left[ \frac{\pi \left( x - \frac{1}{2} L(t) \right)}{L(t)} \right]}{1 + \sin \left[ \frac{\pi \left( x - \frac{1}{2} L(t) \right)}{L(t)} \right]} \times \frac{\Omega^{(-)}(x, t)}{(2A - 1 - 2B \sin \left[ \frac{\pi \left( x - \frac{1}{2} L(t) \right)}{L(t)} \right])^2},$$
(29)

$$\Phi_n^{(+)}(x, t) = \frac{1}{2} \left( \frac{\beta + \alpha}{\beta - \alpha} - \sin \left[ \frac{\pi \left( x - \frac{1}{2} L(t) \right)}{L(t)} \right] \right) P_n^{(\alpha, \beta)} \left( \sin \left[ \frac{\pi \left( x - \frac{1}{2} L(t) \right)}{L(t)} \right] \right)
\begin{align*}
&+ (\beta + \alpha + 2n)^{-1} \frac{\beta + \alpha}{\beta - \alpha} P_n^{(\alpha, \beta)} \left( \sin \left[ \frac{\pi \left( x - \frac{1}{2} L(t) \right)}{L(t)} \right] \right) \\
&- (\beta + \alpha + 2n)^{-1} P_{n-1}^{(\alpha, \beta)} \left( \sin \left[ \frac{\pi \left( x - \frac{1}{2} L(t) \right)}{L(t)} \right] \right),
\end{align*}
(30)

are the time-dependent exceptional orthogonal polynomials and $e^{i F_n^{(+)}(x, t)}$ is a position and time-dependent phase factor:

$$F_n^{(+)}(x, t) = \frac{\dot{L}(t)}{4L(t)} x^2 - \frac{1}{16} L(t) \dot{L}(t) + \frac{1}{16} \int_0^t \dot{L}^2(s) ds - E_n^{(+)} \int_0^t \frac{\pi^2}{L^2(s)} ds. \quad (31)$$

The normalization condition reads

$$\int_0^L \Omega^{(+)}(x, t) \Phi_n^{(+)}(x, t) \Phi_m^{(+)}(x, t) dx = \frac{E_n^{(+)}}{4 \left( \beta - \alpha \right) \left( \beta + n \right) N_n^{(A,B+1)}} \delta_{n,m}. \quad (32)$$

### 3 Expectation values

In this section, we shall evaluate several expectation values of interest for both the time-dependent potentials (21) and (27). To this end, we first note that the root mean square
(RMS) or standard deviation is defined as:
\[
(\Delta x) = \sqrt{\langle x^2 \rangle - \langle x \rangle^2},
\]
where
\[
\langle x^k \rangle = \int \psi^*(x, t) x^k \psi(x, t) \, dx, \quad k = 1, 2,
\]
is the \(k\)th order moment of \(x\) and \(\psi^*(x, t)\) is the complex conjugate of \(\psi(x, t)\). The RMS is the direct spreading measure of \(x\), and it has unit of length \(x\). It has a number of interesting properties of which we shall use the following ones: It is invariant under (i) translations and reflections, (ii) linear scaling [43,44]. The solutions \(\psi_n(\pm) (x, t)\) of the time dependent potentials \(V(\pm) (x, t)\) are invariant neither under translations nor scaling, that is, under the transformation (3). On the other hand, the probability densities \(|\psi_n(\pm) (x, t)|^2 = \frac{\pi}{L(t)} |Q_n(\pm) (q)|^2\) are invariant under the above-mentioned transformation. Moreover, it is seen that the time-dependent potential is shifted by two additional terms \(U(q, t)\) and \(g_0(t)\) which are also linearly scaled by the factor \(\frac{\pi^2}{L^2(t)}\). Then, using the second property mentioned above, we obtain
\[
(\Delta x)^2 = \frac{L^2(t)}{\pi^2} \left( I_2 - I_1^2 \right),
\]
where \(I_k = \int Q(q)^2 q^k \, dq, \quad k = 1, 2.\) (36)

From (35), it may be seen that the RMS for any state of the time-dependent potential can be obtained by linearly scaling by \(\frac{L(t)}{\pi}\) the RMS of corresponding state of the time independent potential. Next, the RMS in the momentum space is defined by
\[
(\Delta p) = \sqrt{\langle p^2 \rangle - \langle p \rangle^2},
\]
where
\[
\langle p^k \rangle = \int \psi^* \left( -i \frac{\partial}{\partial x} \right)^k \psi \, dx = (-i)^k \left( \frac{\pi}{L(t)} \right)^{k-1} \int \psi^* \frac{\partial^k \psi}{\partial q^k} \, dq.
\]
After some calculations, we finally obtain
\[
(\Delta p)^2 = \frac{\pi^2}{L^2(t)} \left[ I_3 - a^2(t) \left( I_2 - I_1^2 \right) \right],
\]
where \(I_3 = \int \left( \frac{\partial Q}{\partial q} \right)^2 \, dq\). Therefore, the Heisenberg uncertainty relation is
\[
(\Delta x)(\Delta p) = \left\{ \left( I_2 - I_1^2 \right) \left[ I_3 - a^2(t) \left( I_2 - I_1^2 \right) \right] \right\}^{1/2} \geq \frac{1}{2}.
\]
Now from Eqs. (35), (39), we may write
\[
(\Delta x)^{\pm}_n = \frac{L(t)}{\pi} \sqrt{I_{2,n}^{\pm} - \left[ I_{1,n}^{\pm} \right]^2},
\]
\[
(\Delta p)^{\pm}_n = \frac{\pi}{L(t)} \sqrt{I_{3,n}^{\pm} - a^2(t) \left( I_{2,n}^{\pm} - \left[ I_{1,n}^{\pm} \right]^2 \right)}.
\]
Thus, we have
\[ b \] and
\[ \] where
\[ \]
Therefore, the Heisenberg uncertainty relation can be expressed as
\[
(\Delta x)_n^{(\pm)}(\Delta p)_n^{(\pm)} = \left\{ \left( I_{2,n}^{(\pm)} - I_{1,n}^{(\pm)} \right)^2 \left[ I_{3,n}^{(\pm)} - a^2(t) \left( I_{2,n}^{(\pm)} - I_{1,n}^{(\pm)} \right)^2 \right] \right\}^{1/2} \geq \frac{1}{2}.
\] (43)

Next, we note that the average energy of a normalized time-dependent state \( \psi(x, t) \) is defined by [45]
\[
\bar{E} = \langle H \rangle = i \int \psi^*(x, t) \frac{\partial \psi(x, t)}{\partial t} \, dx
\]
\[
= i \int Q(q) \left\{ \left( i \dot{F}(t) - \frac{\dot{L}_1(t)}{2L_1(t)} \right) Q(q) - \frac{\dot{\alpha}(t) + q \dot{L}_1(t)}{L_1(t)} \left( \frac{\partial Q}{\partial q} + i Q(q) \frac{\partial F}{\partial q} \right) \right\} dq,
\] (44)

where
\[
F(q, t) = \frac{1}{4} L_1(t) \dot{L}_1(t) q^2 + \frac{1}{2} L_1(t) \dot{\alpha}(t) q + \int_0^t \left[ \frac{(\dot{\alpha}^2(s) - 4g_0(s)) L_1^2(s) - 4\epsilon}{4L_1^2(s)} \right] ds.
\] (45)

Thus, we have
\[
\bar{E}_n^{(\pm)} = h_0^{(\pm)}(t) + h_1(t) I_{1,n}^{(\pm)} + h_2(t) I_{2,n}^{(\pm)},
\] (46)

where
\[
h_0^{(\pm)}(t) = -\frac{(\dot{\alpha}^2(t) - 4g_0(t)) L_1^2(t) - 4E_n^{(\pm)}}{4L_1^2(t)} - i \left( \frac{\dot{L}_1(t)}{2L_1(t)} + \dot{\alpha}^2(t) \right),
\]
\[
h_1(t) = -\frac{1}{2} \left( \dot{L}_1(t) \dot{\alpha}(t) + L_1(t) \ddot{\alpha}(t) \right) + i \left( \frac{\dot{L}_1(t)}{2L_1(t)} - \dot{\alpha}(t) \dot{L}_1(t) \right),
\]
\[
h_2(t) = -\frac{1}{4} \left( L_1(t) \dot{L}_1(t) + L_1^2(t) \dddot{\alpha}(t) \right) - \frac{i}{2} \dot{L}_1^2(t),
\]
\[
I_{k,n}^{(\pm)} = \int Q_n^{(\pm)}(q)^2 q^k \, dq, \quad k = 1, 2,
\]
\[
I_{3,n}^{(\pm)} = \int \left( \frac{\partial Q_n^{(\pm)}}{\partial q} \right)^2 dq.
\] (47)
From Eq. (46), we can see that the average energy of the stationary state of a time-dependent potential is complex in general, whereas for time-independent potential, it is real.

4 Results and discussion

So far our analysis has been quite general in the sense that no specific form of the quantity $L(t)$ was necessary. However, to obtain quantitative understanding of the results, it now becomes necessary to prescribe specific form(s) of $L(t)$. Here, we shall consider the following forms of $L(t)$:
Fig. 3 Comparison of the average energy: (1) a, b real parts of $(\bar{E}_0^-, \bar{E}_0^+)$ (red), $(\bar{E}_1^-, \bar{E}_1^+)$ (blue) and $(\bar{E}_2^-, \bar{E}_2^+)$ (black), (2) c, d imaginary part of $(\bar{E}_0^-, \bar{E}_0^+)$ (red), $(\bar{E}_1^-, \bar{E}_1^+)$ (blue) and $(\bar{E}_2^-, \bar{E}_2^+)$ (black). The solid curves denote $\psi_{0,1,2}^-$, and the dashed curves denote $\psi_{0,1,2}^+$. In the left column, we have taken $L(t) = \pi(2 + \sin t)$, while in the right column, $L(t) = \frac{A_1 \pi}{\sqrt{1 + B_1 \cos \omega t}}$. The other parameters are $A = 5$, $B = 3.4$, $A_1 = 1$; $B_1 = 0.5$, $\omega = 1$

\[ L(t) = L^{(1)}(t) = \pi(2 + \sin t), \]
\[ L(t) = L^{(2)}(t) = \frac{A_1 \pi}{\sqrt{1 + B_1 \cos \omega t}}, \] (48)

where $A_1 = \frac{ab\sqrt{2}}{\sqrt{a^2 + b^2}}$, $B_1 = \frac{b^2 - a^2}{b^2 + a^2}$ are free parameters. To get a visual understanding of how the time-dependent potential look at different points of time, we present in Fig. 1 plots of the potentials $V^{\pm}(x, t)$ for $L(t) = \pi(1 + \sin t)$, $L(t) = \frac{A_1 \pi}{\sqrt{1 + B_1 \cos \omega t}}$ for different values of $t$. It is seen from Fig. 1 that both the potentials show well structure although the wells for $V^+(x, t)$ are deeper than those of $V^-(x, t)$ for each value of $t$. This is true for both the choices of $L(t)$.

To avoid the singularity, we consider $A > \text{Max}\{B + 1.5, |B| + 0.5\}$. Next, we examine the instantaneous probability densities $\rho_n^\pm = |\psi_n^\pm(x, t)|^2$ at different times. A plot of the instantaneous densities is given in Fig. 2. From the figure, we see that particles in both the potentials have more or less similar localization behavior at different values of time $t$ although the wells are narrower in the $(+)$ sector. It may also be observed that localization is periodic in nature. For example, starting from $t = 10$, localization increases as $t$ increases to $t = 20$ and thereafter, it decreases as $t$ increases to $t = 30$. This is generally true for all the levels.

We now examine the behavior of the average energy $\langle \bar{E} \rangle$. It can be seen that the average energy depends on the first- and second-order moments of the variable $q$ of the time-independent Schrödinger equation. It is, however, difficult to obtain analytically the first-order moment $I_{1,n}^{(\pm)}$ and second-order moment $I_{2,n}^{(\pm)}$ of $q$ for the states $Q_0^{(\pm)}$. Therefore, we have
Fig. 4 Comparison of the RMS in position space \(a, b\) \((\Delta x)_0 (\pm)\) (red), \((\Delta x)_1 (\pm)\) (blue) and \((\Delta x)_2 (\pm)\) (black), in momentum space. \(c, d\) \((\Delta p)_0 (\pm)\) (red), \((\Delta p)_1 (\pm)\) (blue) and \((\Delta p)_2 (\pm)\) (black). The solid curves represent the \((\sim)\) sector, while dashed curves represent \((\sim)\) sector. In the left panel, \(L(t) = \pi(2 + \sin t)\) and in the right panel \(L(t) = \frac{A_1 \pi}{\sqrt{1 + B_1 \cos \omega t}}\). The parameter values are \(A = 5, B = 3.4, A_1 = 1; B_1 = 0.5, \omega = 1\)

It may be pointed out that for the uncertainty relation to be satisfied, the behavior of the RMS \((\Delta x)_n^{(\pm)}\) and \((\Delta p)_n^{(\pm)}\) should be opposite for all values of \(t\). In other words, if one of

\begin{align*}
\text{evaluated numerically the moments } I_{1,n}^{(\pm)} \text{ and } I_{2,n}^{(\pm)} \text{ for } n = 0, 1, 2. \text{ In Fig. 3, we have presented plots of the real and imaginary parts of the average energy corresponding to both the potentials } V^{\pm}(x, t). \text{ It can be seen from (a), (b) of Fig. 3 that the real parts of the average energy exhibit similar behavior and are almost identical at all times, while from (c), (d) we find that the imaginary parts of the average energy coincide at particular instants of time and for other values of } t, \text{ they are different for both the } (\pm) \text{ sectors. In addition, it is to be noted that the average energies } \bar{E}_n^{(\pm)} \text{ are equivalent to energies } \frac{2\pi^2}{L^2} E_n^{(\pm)} \text{ of the time-independent Schrödinger equation when the boundary conditions are fixed, where } E_n^{(\pm)} \text{ are the energies of the effective potentials } \tilde{V}^{(\pm)}. \end{align*}
Fig. 5 Variation of the uncertainty in position space (solid curves) and momentum space (dashed curves) against time. The left column is defined for 
\[ L(t) = \pi (2 + \sin t) \]
and the right column is defined for 
\[ L(t) = \frac{A_1 \pi}{\sqrt{1 + B_1 \cos \omega t}} \]
with red \((n = 0)\), blue \((n = 1)\) and black \((n = 2)\). The parameter values are \(A = 5, B = 3.4, A_1 = 1, B_1 = 0.5, \omega = 1\).

Fig. 6 Comparison of the uncertainty product for 
\[ a \] \( L(t) = \pi (2 + \sin t) \)
\[ b \] \( L(t) = \frac{A_1 \pi}{\sqrt{1 + B_1 \cos \omega t}} \). The solid curves represent the \((-)\) sector, while the dashed curves represent the \((+\) sector with red \((n = 0)\), blue \((n = 1)\) and black \((n = 2)\). The parameter values are \(A = 5, B = 3.4, A_1 = 1, B_1 = 0.5, \omega = 1\).

the quantities is increasing, the other must be decreasing and vice versa. In Fig. 5, we have presented plots of \((\Delta x)_{n}^{\pm}\) and \((\Delta p)_{n}^{\pm}\) against \(t\) for both the choices of \(L(t)\). The figure shows the desired features, thus confirming the above assertion.

Finally, we consider the uncertainty product and present plots of the uncertainty product at different times in Fig. 6. From Fig. 6, we see that the uncertainty relation (43) always holds.
good for all values of $t$. Secondly, it may be observed that the uncertainty product of the $(+)$ sector is always lower than that of the $(-)$ sector for all values of $n$ considered here and this may again be attributed to the nature of the solutions of the $(+)$ sector. These observations are true for both profiles of the moving boundaries.

5 Conclusion

In this paper, we have constructed the time-dependent versions of the Pöschl–Teller potential and its supersymmetric partner, namely the rationally extended Pöschl–Teller potential and obtained exact solutions using separation of variable technique. Using these solutions, one of which is expressed in terms of $X_1$ Jacobi exceptional orthogonal polynomials, we have computed several quantities, for example, the average energy, expectation values, etc., and examined their similarity/differences. It may be noted that a host of potentials whose solutions are given in terms of exceptional orthogonal polynomials of different types have been constructed in recent years [46,47]. We feel it would be interesting to treat the potentials considered here as well as others of different types [46,47] using other methods like the method of invariants and examine the differences, if any, with the present approach. In addition, other types of time-dependent boundary conditions, for example, when both boundaries are moving, may also be used to study the above-mentioned problems.

Compliance with ethical standards

Conflict of interest The authors do not have any conflict of interest.

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