SAT-UNSAT transitions of classical, quantum, and random field spherical antiferromagnetic Hopfield model

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Here we study the fully connected spherical antiferromagnetic Hopfield model as a toy model of the convex continuous satisfaction problem. The model exhibits a sharp SAT-UNSAT transition at a certain point, where the system becomes isostatic: the number of degrees of freedom agrees with that of the constraints. The model is simple enough to investigate the scaling behavior near the SAT-UNSAT transition point without relying on the replica method. Furthermore, the simplicity of the model allows us to take into account the effects of the quantum fluctuation and random field. We found that the pure classical model, quantum model, and model with the random field have different critical exponents and thus belong to different universal classes.

I. INTRODUCTION

The purpose of the constraint satisfaction problem is to find solutions that satisfy given constraints. There are many such solutions for a sufficiently small number of constraints. On increasing the number of constraints, the number of solutions decreases, and eventually, the solution ceases to exit at a certain point. This is the so-called satisfaction (SAT) unsatisfaction (UNSAT) transition [1,3].

A promising way to study phase transitions is first to consider solvable mean-field models [4]. There are already several well-investigated mean-field models for satisfaction problems for discrete variables [5,9]. On the contrary, for continuous variables, the complete scaling analysis near the SAT-UNSAT transition has been performed only recently [3,10,21].

A prototypical mean-field model of the continuous satisfaction problem is the single-layer perceptron [2,22]. However, due to the complexity of the model, the previous investigations are highly relied on the uncontrollable approximations such as the replica method even in the convex phase [3,10,19,21]. In this work, we propose a simpler model to study the SAT-UNSAT transition of the convex continuous satisfaction problem. The model can be analyzed without relying on the replica method [23]. In particular, the minimal eigenvalue of the Hessian, which may control the relaxation time, can be calculated easily. Furthermore, the simplicity of the model allows us to investigate the effects of the quantum fluctuation and random field.

This paper is organized as follows. In Sec. II, we investigate the classical model. We characterize the criticality in terms of the condensation transition as previously done for $p=2$-spin spherical model [24]. In Sec. III, we investigate the quantum model. In Sec. IV, we investigate the model with the random field. Finally, in Sec. V, we summarize the work.

II. CLASSICAL MODEL

We consider the following continuous constraint satisfaction problem. Let $\mathbf{x} = \{x_1, \ldots, x_N\}$ be the state vector of norm $\mathbf{x} \cdot \mathbf{x} = N$. The problem is to find out $\mathbf{x}$ such that

$$\mathbf{x} \cdot \mathbf{\xi}^\nu = 0 \text{ for } \nu = 1, \ldots, M.$$  \hspace{1cm} (1)

where $\mathbf{\xi}^\nu = \{\xi_1^\nu, \ldots, \xi_N^\nu\}$ denotes a $N$ dimensional random vector whose components $\xi_i^\nu$ are i.i.d Gaussian of zero mean and unit variance. The inequality version of the problem is referred to as the perceptron and has already been well investigated [2,3,10,22]. It is known that the perceptron exhibits a sharp phase transition from the satisfiable (SAT) phase, where all constraints are satisfied, to the unsatisfiable (UNSAT) phase, where some constraints are violated [2]. Later, we show that our model also exhibits a similar (convex) SAT-UNSAT transition. To solve this problem, we consider the following convex cost function:

$$V_N = \frac{1}{N} \sum_{\nu=1}^{M} (\mathbf{x} \cdot \mathbf{\xi}^\nu)^2 + \frac{\mu}{2} (N - \mathbf{x} \cdot \mathbf{x}),$$  \hspace{1cm} (2)

where $\mu$ denotes the Lagrange multiplier to impose the spherical constraint $\mathbf{x} \cdot \mathbf{x} = N$. When the conditions Eqs. (1) are satisfied, one obtains $V_N = 0$ and vice versa. After some manipulations, Eq. (2) is rewritten as

$$V_N = \frac{1}{2} \mathbf{x} \cdot W \cdot \mathbf{x} + \frac{\mu}{2} (N - \mathbf{x} \cdot \mathbf{x}),$$  \hspace{1cm} (3)

where $W$ is a $N \times N$ symmetric matrix whose $ij$ component is given by

$$W_{ij} = \frac{1}{N} \sum_{\nu=1}^{M} \xi_i^\nu \xi_j^\nu.$$  \hspace{1cm} (4)

The potential Eq. (3) with the opposite sign is that of the spherical Hopfield model [20,27]. So hereafter we call the model Eq. (3) as the spherical antiferromagnetic Hopfield model.
To solve the model, we diagonalize the matrix $W$ and expand the potential by the normal modes:

$$V_N = \sum_{i=1}^{N} \frac{\lambda_i - \mu}{2} u_i^2 + \frac{\mu}{2} N,$$

where $\lambda_i$ denotes the $i$-th eigenvalue of $W$. We will order $\lambda_i$ such that

$$\lambda_1 < \lambda_2 < \cdots < \lambda_N.$$  

(6)

Since $W$ is a Wishart matrix, in the thermodynamic limit $N \to \infty$, its distribution is given by the Marcenko-Pastur law [25, 28]:

$$\rho(\lambda) = \frac{1}{2\pi} \frac{\sqrt{(\lambda - \lambda_+)(\lambda - \lambda_-)}}{\lambda},$$

(7)

where $\alpha = M/N$ and

$$\lambda_{\pm} = (\sqrt{\alpha} \pm 1)^2.$$  

(8)

Now it is easy to show that the ground state energy of Eq. [5] is given by

$$\frac{V_{GS}}{N} = \frac{\lambda_{\text{min}}}{2} = \begin{cases} \frac{1}{(\alpha-1)^2} & \alpha \leq 1 \\ \alpha > 1 \end{cases},$$

(9)

where $\lambda_{\text{min}}$ denotes the minimal eigenvalue of $W$. When $\alpha \leq 1$, $V_{GS} = 0$, implying that all constraints Eq. [1] are satisfied. On the contrary, when $\alpha > 1$, $V_{GS} > 0$, implying that some constraints are unsatisfied. Therefore, the model exhibits the SAT-UNSAT transition at $\alpha_c = 1$. Also, at the transition point, the system is isostatic: the number of degrees of freedom is the same that of the constraints $N = M$ [3].

Now we characterize the criticality around $\alpha_c$ in terms of the condensation transition [24]. For this purpose, we first consider the model in equilibrium at temperature $T$ and take the limit $T \to 0$ at the end of the calculation. At temperature $T$, the equipartition theorem predicts [29]

$$\langle u_i^2 \rangle = \frac{k_B T}{\lambda_i - \mu},$$

(10)

where $k_B$ denotes the Boltzmann constant. Hereafter, we set $k_B = 1$ to simplify the notation. Since $\langle u_i^2 \rangle \geq 0$, $\mu$ should satisfy

$$\mu \leq \lambda_{\text{min}}.$$  

(11)

The spherical constraint is now written as [30]

$$1 = \frac{1}{N} \sum_{i=1}^{N} \langle u_i^2 \rangle = \frac{1}{N} \sum_{i=1}^{N} \langle u_i^2 \rangle = \int d\lambda \rho_N(\lambda) \frac{T}{\lambda - \mu},$$

(12)

where

$$\rho_N(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i).$$

(13)

In the limit $N \to \infty$, $\rho_N(\lambda)$ converges to Eq. [7]. Therefore, we get

$$1 = -(1 - \alpha) \frac{T}{\mu} + \int_{\lambda_-}^{\lambda_+} d\lambda g(\lambda) \frac{T}{\lambda - \mu}.$$  

(14)

For $\alpha < 1$ in the limit $T \ll 0$, the dominant contribution comes from the first term, thus we get

$$1 = -(1 - \alpha) \frac{T}{\mu} \to \mu = -T(1 - \alpha),$$

(15)

and

$$\langle u_i^2 \rangle = \begin{cases} (1 - \alpha)^{-1} & i = 1, \ldots, (1 - \alpha)N \\ 0 & \text{otherwise} \end{cases}.$$  

(16)

As we approach the transition point $\alpha = 1$, the relaxation time $\tau$ would diverge. $\tau$ is controlled by the slowest mode, which has the smallest curvature $\kappa = \min_i \partial^2 u_i V_N = \partial^2 u_1 V_N$ along that direction. Assuming the exponential decay $\dot{u}_1(t) \propto -\kappa u_1(t)$, $\tau$ is estimated as

$$\tau \sim \left( \frac{\partial^2 V_N}{\partial u_1^2} \right)^{-1} = (1 - \alpha)^{-1} = T^{-1}(1 - \alpha)^{-1}.$$  

(17)

Therefore, we get the dynamical critical exponent $z = -1$.

For $\alpha > 1$, $\mu$ is determined by

$$1 = F(\mu),$$

(18)

where

$$F(\mu) = T \int_{\lambda_-}^{\lambda_+} d\lambda \frac{g(\lambda)}{\lambda - \mu}. $$

(19)

![FIG. 1. Phase diagrams for (a) classical model, (b) quantum model, and (c) model with random field. The filled region represents the condensed phase, and the red point represents the SAT-UNSAT transition point. For the model with the random field, the condensed phase appears only at $\Delta = 0$.](image)
One can show that $F(\mu)$ takes the maximal value when $\mu = \lambda$:

$$F(\lambda) = \frac{T}{\sqrt{\alpha - 1}}.$$  \hfill (20)

Below $T_c = \sqrt{\alpha} - 1$ however, Eq. 18 has no solution, which is the signature of the condensation to the lowest eigenmode $\lambda_{c}$ [24, 31, 32]. We plot $T_c$ in Fig. 1 (a). For $T < T_c$, one should separate the first term and others in Eq. 12 as in the case of the Bose-Einstein condensation [29]:

$$1 = \frac{\langle u_1^2 \rangle}{N} + \frac{1}{N} \sum_{i=2}^{N} \langle u_i^2 \rangle = \frac{\langle u_1^2 \rangle}{N} + F(\lambda_{c}).$$  \hfill (21)

From the above equation, we get

$$\frac{\langle u_1^2 \rangle}{N} = 1 - \frac{T}{\sqrt{\alpha - 1}}.$$  \hfill (22)

In particular, at $T = 0$, $x$ is completely localized to the ground state $\langle u_1^2 \rangle = N$. To characterize the criticality for $T \ll 0$, following [33], we define the susceptibility:

$$\chi = \frac{1}{T} \left[ 1 - \frac{\langle u_1^2 \rangle}{N} \right] = \frac{1}{\sqrt{\alpha - 1}}.$$  \hfill (23)

The susceptibility diverges as $\chi \propto (\alpha - 1)^{-1}$ on approaching the SAT-UNSAT transition point. The same critical exponent has been previously reported for the perceptron in the convex phase [3].

III. QUANTUM MODEL

We consider the quantum version of the model:

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i=1}^{N} \left[ \lambda_i - \frac{\mu}{2} u_i^2 \right],$$  \hfill (24)

where we removed the constant term $\mu N/2$ to simplify the notation. We require the standard canonical commutation relation [33, 34]:

$$[u_i, u_j] = 0, \quad [p_i, p_j] = 0, \quad [u_i, p_j] = \hbar \delta_{ij},$$  \hfill (25)

where $\hbar$ corresponds to the plank constant, but here we treat it as a control parameter to control the strength of the quantum fluctuation. Following the standard operation of the quantum statistical mechanics [29], one can calculate the partition function for the $i$-th harmonic oscillator as

$$Z_i = \text{Tr} e^{-\beta H} = \sum_{n=0}^{\infty} e^{-\beta \hbar \sqrt{\lambda_i - \mu}} (n + 1/2) \left[ 2 \sinh \left( \frac{\beta \hbar \sqrt{\lambda_i - \mu}}{2} \right) \right]^{-1},$$  \hfill (26)

where $\beta = 1/T$ denotes the inverse temperature. Then, the second moment is

$$\langle u_i^2 \rangle = \frac{2 d \log Z_i}{d \mu} = \left[ \frac{2 \sqrt{\lambda_i - \mu} \tanh \left( \frac{h \sqrt{\lambda_i - \mu}}{2T} \right)}{h} \right]^{-1}.$$  \hfill (27)

In the high temperature limit, we get

$$\frac{2 \sqrt{\lambda_i - \mu} \tanh \left( \frac{h \sqrt{\lambda_i - \mu}}{2T} \right)}{h} \sim \frac{\lambda_i - \mu}{T}.$$  \hfill (28)

Substituting it back into Eq. (27), we recover the classical result Eq. (10). Instead, here we first take the $T \rightarrow 0$ limit and then observe the asymptotic behavior for $h \ll 1$. At $T = 0$, Eq. (27) reduces to

$$\langle u_i^2 \rangle = \frac{h}{2 \sqrt{\lambda_i - \mu}}.$$  \hfill (29)

As before, $\mu$ is determined by the spherical constraint:

$$1 = \frac{1}{N} \sum_{i=1}^{N} \langle u_i^2 \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{h}{2 \sqrt{\lambda_i - \mu}}.$$  \hfill (30)

Repeating the same analysis of that of the classical model, one can see that for $h \ll 1$, Eq. (30) reduces to

$$1 = \frac{1}{N} \sum_{i=1}^{N} \langle u_i^2 \rangle = (1 - \alpha) \frac{h}{2 \sqrt{\mu}},$$  \hfill (31)

which leads to

$$\mu = -\frac{h^2(1 - \alpha)^2}{4},$$

$$\langle u_i^2 \rangle = \begin{cases} \frac{(1 - \alpha)^{-1}}{1}, & i = 1, \ldots, (\alpha - 1)N, \\ 0, & \text{otherwise} \end{cases}.$$  \hfill (32)

We find the same exponent for $\langle u_i^2 \rangle$ and the different exponent for $\mu$ from those of the classical model, see Eqs. (15) and (16). This is consistent with the previous result for the perceptron [21], where the authors mentioned the differences in the critical exponent between the classical and quantum models. For $\alpha > 1$, the condensation occurs at a finite $h = h_c$. As before, the transition point is calculated as

$$1 = \frac{h}{2} \int_{\lambda_-}^{\lambda_+} d\lambda \frac{g(\lambda)}{\sqrt{\lambda - \lambda_-}} \rightarrow h_c = \frac{2}{\int d\lambda g(\lambda)(\lambda - \lambda_-)^{-1/2}},$$  \hfill (33)

see Fig. 1 (b). In the limit $\alpha \rightarrow 1$, $h_c$ vanishes as

$$h_c \propto -\frac{1}{\log(\alpha - 1)}. $$  \hfill (34)
For \( h < h_c \), we define the order parameter

\[
\frac{\langle u_i^2 \rangle}{N} = 1 - \frac{h}{2} \sum_{i=2}^{N} \frac{1}{\sqrt{\lambda_i}},
\]

(35)

and susceptibility

\[
\chi = \lim_{h \to 0} \frac{1}{N} \left( 1 - \frac{\langle u_i^2 \rangle}{N} \right) = \frac{1}{2} \int_{\lambda}^{\lambda_c} d\lambda \frac{g(\lambda)}{\sqrt{\lambda - \lambda_-}}.
\]

(36)

In the limit \( \alpha \to 1 \), \( \chi \) diverges logarithmically

\[
\chi \sim -\log(\alpha - 1),
\]

(37)

instead of the power-law found in the classical model Eq. (23).

\section{IV. MODEL WITH RANDOM FIELD}

Finally, we consider the model with the random field:

\[
V_N = \sum_{i=1}^{N} \frac{\lambda_i - \mu}{2} u_i^2 + \sum_{i=1}^{N} h_i u_i,
\]

(38)

where \( h_i \) is an i.i.d random variable of zero mean and variance \( \Delta \). In equilibrium at temperature \( T \), we get

\[
\langle u_i^2 \rangle = \frac{T}{\lambda_i - \mu} + \frac{\Delta}{(\lambda_i - \mu)^2},
\]

(39)

where the overline denotes the average for \( h_i \), and \( \Delta = \overline{h_i^2} \). Hereafter we consider the model at \( T = 0 \), and observe the asymptotic behavior for \( \Delta \ll 1 \). As before, \( \mu \) is determined by the spherical constraint:

\[
1 = \frac{1}{N} \sum_{i=1}^{N} \frac{\Delta}{(\lambda_i - \mu)^2}.
\]

(40)

Repeating the same analysis of that of the classical model, in the limit \( \Delta \ll 1 \), we get for \( \alpha > 1 \)

\[
\mu = -(1 - \alpha)^{1/2} \Delta^{1/2},
\]

\[
\langle u_i^2 \rangle = \begin{cases} 
(1 - \alpha)^{-1} & i = 1, \ldots, (\alpha - 1)N \\
0 & \text{otherwise}
\end{cases}
\]

(41)

We find the same exponent for \( \langle u_i^2 \rangle \) and the different exponent for \( \mu \) from those of the classical and quantum models. As in the case of the pure classical system, the relaxation time \( \tau \) would be inversely proportional to curvature along the minimal eigenmode:

\[
\tau \sim \left( \frac{\partial^2 V_N}{\partial u_i^2} \right)^{-1} = (-\mu)^{-1} = \Delta^{-1/2}(1 - \alpha)^{-1/2}.
\]

(42)

We get a larger critical exponent \(-1/2\) than that of the pure system \(-1\), see Eq. (17), suggesting that the external random field accelerates the relaxation.

The condensation transition point for \( \alpha > 1 \) is estimated as

\[
\Delta_c = \frac{1}{\int d\lambda g(\lambda)(\lambda - \lambda_-)^{-2}} = 0,
\]

(43)

meaning that the condensation does not occur at finite \( \Delta \), see Fig. (c). In a previous work, we investigated a similar equation as Eq. (17), and found that the localization transition at finite \( \Delta \) occurs only for \( \rho(\lambda) \sim (\lambda - \lambda_-)^n \) with \( n > 1 \).

\section{V. SUMMARY}

In this work, we investigated the convex continuous SAT-UNSAT transition of the fully connected spherical antiferromagnetic Hopfield model. Since the interaction potential has a quadratic form, the model can be easily analyzed by decomposing the potential into the normal vibration modes. We successfully characterized the criticality near the SAT-UNSAT transition point in terms of the condensation transition. The simplicity of the model also allowed us to investigate the quantum effects. We found different critical behaviors from those of the classical model. In particular, the susceptibility of the order parameter diverges logarithmically when approaching the transition point from the UNSAT side. This is qualitatively different behavior from that of the classical model where the power-law divergence is observed. Finally, we investigated the model with the random field. We found different values of the critical exponent from those of classical and quantum models. Interestingly, we found a smaller critical exponent of the minimal eigenvalue than that of the classical model, suggesting that the random field accelerates the relaxation. Further studies about this point would be beneficial.

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