Lepton mixing and discrete symmetries

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Abstract

The pattern of lepton mixing can emerge from breaking a flavor symmetry in different ways in the neutrino and charged lepton Yukawa sectors. In this framework, we derive the model-independent conditions imposed on the mixing matrix by the structure of discrete groups of the von Dyck type which include $A_4$, $S_4$ and $A_5$. We show that, in general, these conditions lead to at least two equations for the mixing parameters (angles and CP phase $\delta$). These constraints, that correspond to unbroken residual symmetries, are consistent with non-zero 13 mixing and deviations from maximal 2-3 mixing. For the simplest case, which leads to an $S_4$ model and reproduces the allowed values of the mixing angles, we predict $\delta = (90^\circ - 120^\circ)$.

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1 Introduction

The fact that the leptonic mixing angles seem to take “special” values [1] has long been presented as an argument for the existence of an approximate underlying symmetry. Based on this, the following program for model building has been devised [2]. One starts with a Lagrangian symmetric under a flavor group $G$ and breaks this symmetry down to two different symmetries described by subgroups, $G_\nu$ and $G_\ell$, that are preserved in the neutrino and the charged lepton sector respectively. These different residual symmetries of the Yukawa sectors are responsible for the mismatch of the rotations which diagonalize the neutrino and charged lepton matrices. In particular, they can give rise to the tribimaximal (TBM) [3] and bimaximal (BM) [4] mixing patterns. Inversely, the flavor group $G$ can be reconstructed from the residual symmetries of the mass matrices. This constructive approach was pioneered in [5] where it was shown that the minimal $G$ that included all the symmetries of the neutrino mass matrix and leads to TBM mixing is $S_4$. In [8, 9, 10, 11] consequences of the postulated residual symmetries of the neutrino mass matrix for the lepton mixing have been explored.

In this paper we reformulate and generalize the described approach. We deviate from the road pursued in [5] by recognizing that not all the symmetries of the neutrino mass term necessarily belong to $G$ since they appear accidentally anyway [7]. We will assume in this paper that the subgroup of $G$ conserved by the neutrino mass term is only a $Z_2$. The consequence of this milder assumption is that the mixing matrix is not completely fixed thus broadening greatly the possibilities.

It is known that in specific models different symmetry groups $G$ can lead to the same mixing. Inversely, the same symmetry group $G$ can lead to different mixing patterns, e.g. TBM or BM. Such an ambiguity makes rather unclear the connection between symmetry groups and specific mixing patterns. In this paper we develop a model-independent formalism which clarifies a correspondence between symmetries and mixing pattern and establishes to which extent non-abelian discrete symmetries can fix the mixing.

The paper is organized as follows. In Sec. 2 we reformulate the program of “symmetry building”. We conclude that the relevant symmetries are in general of the von Dyck type. We obtain the condition which the mixing matrix should satisfy in terms of symmetry generating elements. In Sec. 3, using this condition we derive the relations between the mixing angles that appear from different symmetries. Sec. 4 contains discussion and conclusions.

2 Mixing and symmetry building

We assume throughout this paper that the neutrino masses are of Majorana type (the Dirac case allows for a broader set of symmetries and will be considered separately).
We begin by writing the corresponding Lagrangian in the mass basis

\[ \mathcal{L} = \frac{g}{\sqrt{2}} \bar{\ell}_L U_{PMNS} \gamma^\mu \nu_L W^\mu_L + \bar{E}_R m_\nu \nu_L + \frac{1}{2} \bar{\nu^c}_L m_\nu \nu_L + \ldots + \text{h.c.} , \]

where \( U \) is the mixing matrix, \( \ell = (e, \mu, \tau)^T \), \( \nu = (\nu_1, \nu_2, \nu_3)^T \) are vectors of mass states and \( m_\ell, m_\nu \) are the diagonal mass matrices of the charged leptons and neutrinos.

Taken by itself, each mass term in Eq. (1) has certain symmetries. The neutrino mass term has two independent and non-trivial \( \mathbb{Z}_2 \) symmetries with respect to which \( \nu \) transforms as

\[ \nu \rightarrow S_i \nu, \quad m_\nu \rightarrow S_i^T m_\nu S_i = m_\nu, \quad k = 1, 2 \]

with

\[ S_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \]

The mass matrix is also invariant under an additional transformation \( S_3 \) described by

\[ S_3 = S_1 S_2 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}. \]

For simplicity, the matrices \( S_i \) are taken to be elements of \( SU(3) \), so they satisfy \( \text{Det}[S_i] = 1 \).

The charged lepton mass term has a full \( U(1)^3 \) symmetry. Restricting to discrete symmetries implies that the values of the phases must be rational divisors of \( 2\pi i \) and the symmetry that remains is \( \mathbb{Z}_m \). The transformation can be written explicitly as

\[ \ell \rightarrow T \ell_L, \quad \ell_R \rightarrow T \ell_R, \]

where

\[ T = \begin{pmatrix} e^{2\pi i k_1 / m} & e^{2\pi i k_2 / m} & e^{2\pi i k_3 / m} \\ e^{2\pi i k_2 / m} & e^{2\pi i k_3 / m} & e^{2\pi i k_1 / m} \end{pmatrix} \]

satisfies \( T^m = I \).

Once again, we assume that \( T \) belongs to \( SU(3) \) so that \( \text{Det}[T] = 1 \) which implies

\[ k_1 + k_2 + k_3 = 0, m, 2m. \]

We take one of the \( k_i \) to be zero, which fixes the \( \mathbb{Z}_m \) charges of the charged leptons, and consider in the rest of the paper the three \( T \) matrices

\[ T_e = \text{diag}\{1, e^{2\pi i k / m}, e^{-2\pi i k / m}\}, \]

\[ T_\mu = \text{diag}\{e^{2\pi i k / m}, 1, e^{-2\pi i k / m}\}, \]

\[ T_\tau = \text{diag}\{e^{2\pi i k / m}, e^{-2\pi i k / m}, 1\}. \]
The assumption that both \( S_i \) and \( T \) are elements of \( SU(3) \) simplifies our considerations, e.g., it reduces the number of independent phases of \( T \). A more general form for the \( T \) generator will be discussed elsewhere [6].

Notice that the charged current interactions in Eq. (1) are not invariant under the transformations \( S_i \) or \( T \) in this basis.

We now examine these symmetries in an arbitrary basis, related to the mass basis by the rotations

\[
U_\nu \nu_L = \nu'_L, \quad U_\ell \ell_L = \ell'_L, \quad V_\ell \ell_R = \ell'_R
\]

The mass matrices of neutrinos and charged leptons become

\[
m_\nu U = U^*_\nu m_\nu U^\dagger, \quad m_U = V m_\ell U^\dagger.
\]

It is easy to see that \( m_\nu U \) is invariant under the transformed matrix \( S U \)

\[
S U = U_\nu S U^\dagger,
\]

while \( m_U \) is invariant under the transformation

\[
m_U = T^\dagger \alpha m_U T \alpha
\]

where

\[
T^\alpha V = V^\dagger_\ell T^\alpha V_\ell, \quad T^\alpha U = U^\dagger_\ell T^\alpha U_\ell.
\]

These satisfy the same \( Z_2 \) and \( Z_m \) relations as in the mass basis:

\[
S^2 U = T^m_\alpha V = T^m_\alpha U = 1.
\]

The transformations of the RH components of leptons are irrelevant for the discussion of the mixing matrix \( U_{PMNS} \), since only the left-handed lepton basis enters in the physical mixing. The neutrino and charged lepton mass terms in an arbitrary basis are invariant under the \( Z_2 \) and \( Z_m \) symmetries irrespective of what the mixing matrix is. In the new basis the weak interaction term reads

\[
\bar{\ell}_L U_\ell U_{PMNS} U^\dagger_\nu \nu_L.
\]

It becomes diagonal, and therefore invariant under the flavor symmetry transformations in Eq. (13), if

\[
U^\dagger_\ell U_\nu = U_{PMNS}.
\]
The first relation in Eq. (13) can be rewritten as
\[ S_i U_\nu = U_\nu S_i. \] (19)
It means that the columns of the matrix \( U_\nu \), which coincides with \( U_{PMNS} \) in the flavor basis \( (U_\ell = \mathbb{I}) \), are the eigenstates of the symmetry matrix \( S_i \) with eigenvalues given by the diagonal elements of \( S_i \). If \( S_i \) is known (postulated), Eq. (19) can be used to put conditions on the elements of mixing matrix.

We now make the key assumption that the residual symmetries in both the charged lepton and neutrino sectors are one-generator groups. That is, we assume that \{ \( S_i \), \( T_\alpha \) \} form a set of generators for the flavor group \( G \) for given \( i \) and \( \alpha \) (this excludes in particular, the case in which the residual symmetry in the charged lepton sector is a Klein group). Then the definition of \( G \) requires a relation linking \( S_i \) and \( T_\alpha \). In this paper we will study
\[ (S_i T_\alpha)^p = (U_{PMNS} S_i U_{PMNS}^\dagger T_\alpha)^p = \mathbb{I} \] (20)
which, for small values of \( m \) and \( p \), leads to well known groups. Other relations with the structure \( W^p = \mathbb{I} \), where \( W \) is a product of \( S_i \) and \( T_\alpha \) matrices are possible. We stress that all these relations, including Eq. (20), do not depend on \( U_\nu \) or \( U_\ell \) as such but only in the combination \( U_{PMNS} \) in Eq. (18). Thus, the mixing pattern, which follows from symmetry, is independent of the basis, as it should be.

Notice that Eq. (5) doesn’t imply that \( \ell_L \) and \( \ell_R \) must transform under the same representation of \( G \). The requirement that a cyclic group is preserved in the charged lepton sector is enough to guarantee that Eq. (5) holds.

Since we use only one of \( S_i \) as a generator of the flavor group \( G \), the others \( S_j \) that do not belong to \( G \) should be treated as generators of accidental symmetries of the mass matrix (recall that all three symmetries \( S_i \) exist simply because the Majorana mass matrix is symmetric). This point is the departure from [5] where it was assumed that all \( S_i \) belong to \( G \). In our case, as we will see later, the choice of \( S_i \) leads to different relations between the mixing parameters. The accidental symmetries play no role in our model-independent considerations. In specific models, the situation with a single \( Z_2 \) in the neutrino sector can be realized in two different ways: (a) from the beginning the flavor group contains only one generator \( Z_2 \) or (b) the flavor group contains both generators, but one is broken in the lowest order. The inclusion of the second \( Z_2 \) into \( G \) enhances the symmetry and therefore leads to additional constraints on the mixing parameters. In this sense, the framework explored in this paper is more general than the one in [5].

The relation in Eq. (20) can be presented in a different form. We consider the combination \( W_U = T_{aU}^{-1} S_i U \), so that
\[ S_i T_{aU} W_U = \mathbb{I}. \] (21)
Then, noticing that \( T_{\alpha U}^{-1} S_{\alpha U} = (S_{\alpha U}^{-1} T_{\alpha U})^{-1} \), one finds from Eq. (20)

\[
W_{\alpha U}^p = (T_{\alpha U}^{-1} S_{\alpha U})^p = [(S_{\alpha U}^{-1} T_{\alpha U})^{-1}]^p = \mathbb{I}.
\]

The three relations

\[
S_{\alpha U}^2 = T_{\alpha U}^m = W_{\alpha U}^p = S_{\alpha U} T_{\alpha U} W_{\alpha U} = \mathbb{I}
\]

(23)
define the von Dyck group \( D(2, m, p) \) \[12\]. Some well known cases are

\[
D(2, 2, 3) = S_3, \\
D(2, 3, 3) = A_4, \\
D(2, 3, 4) = S_4, \\
D(2, 3, 5) = A_5.
\]

(24)

We will take from now on the relations in Eq. (20) and (16) as the defining relation for the von Dyck group \( D(2, m, p) \).

One ambiguity appears at Eq. (24). An \( m \leftrightarrow p \) exchange defines the same von Dyck group despite requiring different \( T_{\alpha U} \) matrices. Starting from \( G \), different \( T_{\alpha U} \) matrices may appear as a result of the choice of the representations for the fields and of the ways of breaking the symmetry. In such cases the same group may lead to different patterns of mixing. We stress again that the fact that we consider Majorana masses amounts to the condition that \( n = 2 \) in \( S_{1, i}^n = 1 \).

Let us mention in passing that the von Dyck groups \( D(n, m, p) \) with

\[
\frac{1}{n} + \frac{1}{m} + \frac{1}{p} \leq 1
\]

(25)
have infinite order. Besides those listed in Eq. (24), this inequality leaves only the dihedral groups \( D_p = D(2, 2, p) \) as finite von Dyck groups. We will not study the dihedral groups in this paper. They have been less used for model building because they do not have irreducible representations of dimension 3. On the other hand, finite subgroups of the infinite von Dyck groups can be generated if one imposes additional relations to those in Eq. (23). The formalism developed in this paper will apply to those as well. The possibility of using infinite discrete groups to generate patterns of mixing will be explored elsewhere \[6\].

**3 Symmetries and constraints on mixing**

For a given set \( \{S_i, T_\alpha\} \) and a given value of \( p \), the group relation, Eq. (20), provides the constraints on the mixing matrix \( U_{PMNS} \). These constraints can be found in the following way. From Eq. (20), the eigenvalues \( \lambda_i \) of the unitary matrix

\[
W_{\alpha U}^{-1} = U_{PMNS} S_i U_{PMNS}^\dagger T_\alpha
\]

(26)
must be such that \( \lambda^p_i = 1 \). That is, Eq. (20) holds if and only if all solutions of the equation

\[
\text{Det}[W_{ioU} - \lambda \mathbb{I}] = 0
\]  

(27)
satisfy

\[
\lambda^p_i = 1.
\]  

(28)

For any unitary matrix \( U_{P M N S} \) with determinant 1, Eq. (27) is a cubic equation of the form

\[
\lambda^3 + a\lambda^2 - a^*\lambda - 1 = 0,
\]  

(29)

where \( a \) is a complex function of the angles and phases of \( U_{P M N S} \).

From Eq. (27), \( a \) is given by

\[
a = -\text{Tr}[W_{ioU}].
\]  

(30)

If \( a \) is known, this equation can be considered as a condition on \( W_{ioU} \), and consequently, \( U_{P M N S} \). This gives two constraints on the parameters of \( U_{P M N S} \) which correspond to the real and imaginary part of \( a \).

We find the conditions on \( U_{P M N S} \) assuming for simplicity that one of the eigenvalues of \( W_{ioU} \) is equal to the identity, e.g. \( \lambda_1 = 1 \), which implies that \( a \) is real according to Eq. (29). Then, substituting \( S_i \) and \( T_\alpha \) in Eq. (30) we find the general conditions on the values of the mixing matrix elements

\[
|U_{\beta i}|^2 = |U_{\gamma i}|^2
\]  

(31)

\[
|U_{\alpha i}|^2 = \eta, \quad \beta, \gamma \neq \alpha,
\]  

(32)

where

\[
\eta \equiv \frac{1 - a}{4 \sin^2 \left( \frac{\pi k}{m} \right)}.
\]  

(33)

The parameter \( a \) can be found as the sum of the three eigenvalues \( \lambda_i \) (see footnote) which, along with the previous assumption, reduces to

\[
a = -1 - \lambda_2 - \lambda_3.
\]  

(34)

\( W_{ioU} \) is a matrix of \( SU(3) \). If its characteristic equation is

\[
\lambda^3 + a\lambda^2 + b\lambda + c = 0.
\]

Then

\[
a = -(-\lambda_1 + \lambda_2 + \lambda_3), \quad b = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \quad c = -\lambda_1\lambda_2\lambda_3 = -\text{Det}[W_i] = -1
\]

and therefore

\[
b^* = b^*\lambda_1\lambda_2\lambda_3 = \lambda_1 + \lambda_2 + \lambda_3 = -a,
\]

where we have used that the eigenvalues of a unitary matrix are phase factors.
Since $W_{\alpha U}$ is a matrix of $SU(N)$, we must have $\lambda_2 \lambda_3 = 1$. This determines that $\text{Im}[a] = 0$ and, supplemented with $\lambda_2^p = \lambda_3^p = 1$, it implies that $a$ is uniquely fixed for $p \leq 4$. For larger $p$ two or more possibilities appear which should be taken into account (see below).

The parameter $\eta$ contains information about the flavor symmetry group. Eqs. (31) and (32) have solutions for $\eta \leq 1$, or explicitly

$$\frac{1 - a}{4} \leq \sin^2 \left( \frac{\pi k}{m} \right),$$

and describe the constraints imposed on the mixing matrix by a flavor group $G$ defined by the residual symmetries $S_i$ and $T_\alpha$ of the neutrinos and charged leptons. Notice that Eq. (31) does not depend on $m$ or $k$.

Notice that $S_i$ has a diagonal submatrix that is proportional to the identity. This implies that the mixing matrix that satisfies Eq. (30) is fixed at most up to a $2 \times 2$ rotation. On the other hand, the symmetry imposes conditions for the elements of a single column of the mixing matrix (determined by $i$), Eq. (31). It leads to the equality of the elements that correspond to the charged leptons that transform non-trivially under $T$. From experimental data it follows that Eq. (31) can be satisfied either for $\alpha = e$, $i = 1, 2, 3$ or for $i = 2$, $\alpha = e, \mu, \tau$. That is, experimental data implies that $T_e$ or/and $S_2$ are generators of $G$. We consider first the case in which $T_e$ is a generator.

We will write Eqs. (31) and (32) explicitly in terms of the mixing angles and the CP phase using the standard parametrization

$$U_{\text{PMNS}} = \begin{pmatrix}
  c_{12} c_{13} & -s_{12} c_{13} & e^{-i \delta} s_{13} \\
  s_{12} c_{23} + e^{i \delta} c_{12} s_{13} s_{23} & c_{12} c_{23} - e^{i \delta} s_{12} s_{13} s_{23} & -c_{13} s_{23} \\
  s_{12} s_{23} - e^{i \delta} c_{12} s_{13} c_{23} & c_{12} s_{23} + e^{i \delta} s_{12} s_{13} c_{23} & c_{13} c_{23}
\end{pmatrix}. \quad (36)$$

Taking $S_1$ as a generator of $G$ and setting $\alpha = e$ we obtain

$$\tan 2\theta_{23} = \frac{\sin^2 \theta_{12} - \cos^2 \theta_{12} \sin^2 \theta_{13}}{\sin 2\theta_{12} \sin \theta_{13} \cos \delta}, \quad (37)$$

$$\cos^2 \theta_{12} = \frac{\eta}{\cos^2 \theta_{13}}. \quad (38)$$

For $S_2$ we have

$$\tan 2\theta_{23} = \frac{\cos^2 \theta_{12} - \sin^2 \theta_{12} \sin^2 \theta_{13}}{\sin 2\theta_{12} \sin \theta_{13} \cos \delta}, \quad (39)$$

$$\sin^2 \theta_{12} = \frac{\eta}{\cos^2 \theta_{13}}. \quad (40)$$

The relations between $\theta_{12}$ and $\theta_{13}$, Eqs. (38) and (40), do not depend on $\theta_{23}$ or $\delta$ while Eqs. (37) and (39) yield the relation $\tan 2\theta_{23} \propto (\cos \delta \sin \theta_{13})^{-1}$. The latter implies, in
particular, that 2-3 mixing becomes maximal, $\sin^2 \theta_{23} = 0.5$, in the limits $\delta \to \pi/2$ and $\theta_{13} \to 0$. For $\delta = 0, \pi$ one obtains maximal deviation from maximal mixing. Notice that all these limits are independent of the value of $k, m$ and $p$.

We can substitute $\theta_{12}$ in Eqs. (37) and (39) using Eqs. (38) and (40). This gives a relation between $\theta_{23}$ and $\theta_{13}$

$$\tan 2\theta_{23} = \mp \frac{1}{2 \sin \theta_{13} \cos \delta} \frac{\cos^2 \theta_{13} - \eta(1 + \sin^2 \theta_{13})}{\sqrt{\eta \cos^2 \theta_{13} - \eta^2}},$$

(41)

where the minus sign (plus) refers to the case $S_1 (S_2)$. Thus, the expressions for $\tan 2\theta_{23}$ differ by the sign which amounts to a substitution $\delta \to \pi - \delta$.

Eqs. (37) and (39) coincide with relations obtained in [8, 9, 10]. In those papers, the invariance of the neutrino mass matrix in the flavor basis with respect to transformations $G_i$ was assumed. These $G_i$ in [8, 9, 10] correspond to our $S_i U = U_{PMNS} S_i U_{PMNS}$ with the equality of Eq. (31) imposed on $U_{PMNS}$ for $\alpha = e$. Then, the relations between mixing angles were found using Eq. (19). Our relations have been obtained from a general group theoretical consideration without assuming a certain form for the transformations. Furthermore, symmetries of the charged lepton mass matrices are included and, in fact, we obtain two equations which should be satisfied simultaneously to have a consistent group embedding. Our analysis is more general and can be applied to wide class of symmetries.

Choosing $S_3$ as a generating element yields

$$\sin^2 \theta_{23} = \cos^2 \theta_{23}, \quad \sin^2 \theta_{13} = \eta,$$

(42)

which reproduces the well known fact that $S_3$ leads to maximal 2-3 mixing. For $p = 2$ one has $a = 1$ which gives $\sin^2 \theta_{13} = 0$. For $p = 3, 4$ one obtains the value $\sin^2 \theta_{13} \geq 0.25$.

On the other hand, the cases of $p = 3, 4$ are relevant, for both $S_1$ and $S_2$, in connection to models that have been explored extensively and we analyze them in turn. The models below are characterized by the choice of the neutrino generator $S_i$ and by numbers $p$ and $m$ which fix the charged lepton generator and the relation linking the two generators. We restrict to the cases with $k = 1$. For $m > 4$, larger values of $k$ will lead to different models.

Model 2T\footnote{We will denote the model by $iM$, where the number $i = 1, 2$ corresponds to selected generator $S_i$ and the letter $M = B, T$ indicates the type of mixing one obtains in the limit $\theta_{13} = 0$: $BM(B)$ or $TBM(T)$.}: This model uses the generator $S_2$ and $(p, m) = (3, 3)$ which corresponds to the group $G = D(2, 3, 3) = A_4$ [1]. For value $p = 3$ we have $\lambda^3 = 1$. This leads, according to Eq. (23), to $a = 0$ and therefore $\eta = 1/3$. Then from Eq. (40) we have

$$\sin^2 \theta_{12} = \frac{1}{3 \cos^2 \theta_{13}},$$

(43)
TBM is a point on the curve determined by this equation with $\theta_{13} = 0$ and $\sin^2 \theta_{12} = 1/3$. A larger absolute value of $\theta_{13}$ pulls $\theta_{12}$ up which is disfavored by data. The second relation is as in Eq. (39). From Eq. (41) we obtain a relation between the 2-3 and 1-3 mixings:

$$\tan 2\theta_{23} = \frac{1 - 2s_{13}^2}{\cos \delta s_{13} \sqrt{2 - 3s_{13}^2}}.$$  \hspace{1cm} (44)

Choosing $S_1$ leads, according to Eq. (38), to the experimentally excluded value $\sin^2 \theta_{12} = 2/3$.

**Model 2B:** Uses $S_2$; $(p, m) = (3, 4)$, and consequently, the group is $S_4$. We obtain from Eq. (40)

$$\sin^2 \theta_{12} = \frac{1}{2 \cos^2 \theta_{13}}.$$ \hspace{1cm} (45)

It corresponds to BM mixing for $\theta_{13} = 0$. Hence, $\sin^2 \theta_{12} \geq 0.5$ which is excluded by experimental data unless there are large corrections from residual symmetry breaking. The second relation is given by Eq. (37).

**Model 1B:** $S_1$, $(p, m) = (3, 4)$, i.e. the group $S_4$. Here we have the relation

$$\cos^2 \theta_{12} = \frac{1}{2 \cos^2 \theta_{13}}.$$ \hspace{1cm} (46)

which contains the BM prediction \[\text{[14]}\]. For allowed values of 1-3 mixing, we have from this equation $\sin^2 \theta_{12} \sim 0.5$ so that large corrections are required to make this case experimentally viable. Now the second relation is given by Eq. (37).

**Model 1T:** $S_1$, $(p, m) = (4, 3)$, the group $G = D(2, 3, 4) = S_4$ \[\text{[10]}\]. The value $p = 4$ leads to $a = -1$ so that $\eta = 2/3$, and consequently,

$$\cos^2 \theta_{12} = \frac{2}{3 \cos^2 \theta_{13}}.$$ \hspace{1cm} (47)

TBM corresponds to $\theta_{13} = 0$, $\cos \theta_{12} = \sqrt{2/3}$. From Eq. (41) we obtain

$$\tan 2\theta_{23} = -\frac{1 - 5s_{13}^2}{2 \cos \delta s_{13} \sqrt{2(1 - 3s_{13}^2)}}.$$ \hspace{1cm} (48)

**Model GR:** $S_1$, $(p, m) = (3, 5)$, the group is $A_5$. For $\theta_{13} = 0$, Eq. \[\text{[38]}\] becomes

$$\sin^2 \theta_{12} = \frac{2}{5 + \sqrt{5}}.$$ \hspace{1cm} (49)
Figure 1: Symmetry relations between mixing parameters. Shown are $\sin^2 \theta_{12}$ and $\sin^2 \theta_{23}$ as functions of $\sin^2 \theta_{13}$ in models 1B (left, solid), 1T (left, dashed) for $\delta = \pi$, 2B (right, solid), 2T (right, dashed) for $\delta = 0$. The dotted line in the lower panels represents $\delta = \pi/2$. For $\pi/2 < \delta < \pi$ ($0 < \delta < \pi/2$) the curve is inside the region delimited by these two lines for the left (right) panels. Values of $\delta \leq (\geq) \pi/2$ correspond to $\theta_{23} > \pi/4$. The crosses show the $1\sigma$ experimentally allowed regions for the mixing angles.

which reproduces the value of the Golden Ratio model [13].

In Fig. 1 we plot $\theta_{12}$ and $\theta_{23}$ as functions of $\theta_{13}$ for the models 1T, 1B, 2T and 2B. The solid lines in the bottom panels delimit the predicted regions from below. They correspond to $\delta = \pi$ for models 1T and 1B and to $\delta = 0$ for models 2T and 2B. The panels depict regions that correspond to $\sin^2 \theta_{23} \leq 0.5$. The predicted regions are extended to $\sin^2 \theta_{23} > 0.5$ symmetrically with respect to $\sin^2 \theta_{23} = 0.5$ (dotted horizontal lines). These upper parts (not shown) would correspond to the ranges $0 < \delta < \pi/2$ (1T and 1B) and $\pi/2 < \delta < \pi$ (2T and 2B). The leftmost points in these curves correspond to the TBM and BM values. We emphasize that in the lower panels of Fig. 1 all the points above the curves can be reached by allowing for values of $\delta \neq 0$. We show also best fit points and $1\sigma$ ranges for the mixing parameters obtained from the global fit [1] and result of the Daya Bay experiment [17] which are in agreement with other
measurements of 1-3 mixing \[18, 19, 20, 21\]. We notice that model 1T fits well all the measured values for the mixing angles. For this model, the values of \(\sin^2 \theta_{23}\) and \(\sin^2 \theta_{13}\) correspond to \(90^\circ < \delta < 119^\circ\) with a central value of \(\delta = 103^\circ\).

Since the flavor symmetry \(G\) is assumed not to be an exact symmetry of the whole Lagrangian the residual symmetries are broken at the sub-leading order. The effect of this breaking is, however, model-dependent and therefore can not be traced in our approach. The breaking can modify the predictions presented above. Still there is certain sense to confront them with experimental results: (i) corrections can be small; (ii) they can improve agreement with data; (iii) requirement that corrections are small can further restrict models.

In [5] it was shown that the minimal group leading to TBM that includes all symmetries of the neutrino mass matrix is \(S_4\). This fact is consistent with our formalism. Indeed, if all \(S_iU\) belong to the \(G\), then

\[
(S_iU)T_U^{p_i} = I, \tag{50}
\]

so we are looking for a simultaneous solution to all the Eqs. (37) - (42) with not necessarily equal \(a_i\) such that all mixing angles have the TBM values. This is easily realized for \(a_1 = -1, a_2 = 0, a_3 = 1\) corresponding to \((p_1, p_2, p_3) = (4, 3, 2), k = 1, m = 3,\) that is, \(S_4\). That is, for TBM and the values of \(a_i, p_i, k\) and \(m\) above, \(S_2\) and \(S_3\) belong to the group generated by \(S_1\) and \(T\).

Let us now consider the cases when the group is generated by \(S_2\) and \(T_\beta\), with \(\beta = \mu\) or \(\tau\). For \(T_\mu\) the relations read

\[
|U_{e2}|^2 = |U_{\mu(\tau)2}|^2, \quad |U_{\tau(\mu)2}|^2 = \eta. \tag{51}
\]

Eliminating terms which contain the phase \(\delta\) we obtain from these equations in both cases to the same relation

\[
\sin^2 \theta_{12} = \frac{1 - \eta}{2 \cos^2 \theta_{13}}. \tag{52}
\]

As before, the second relation does not depend on \(k\) or \(m\) but involves the CP phase \(\delta\). For \(T_\mu\) we have

\[
\cos \delta = -2 \frac{\sin^2 \theta_{12}(\cos^2 \theta_{23} \sin^2 \theta_{13} - \cos^2 \theta_{13}) + \cos^2 \theta_{12} \sin^2 \theta_{23}}{\sin 2\theta_{12} \sin 2\theta_{23} \sin \theta_{13}}, \tag{53}
\]

while for \(T_\tau\) the corresponding expression becomes

\[
\cos \delta = 2 \frac{\sin^2 \theta_{12}(\sin^2 \theta_{23} \sin^2 \theta_{13} - \cos^2 \theta_{13}) + \cos^2 \theta_{12} \cos^2 \theta_{23}}{\sin 2\theta_{12} \sin 2\theta_{23} \sin \theta_{13}}. \tag{54}
\]

In the limit \(\sin^2 \theta_{13} \to 0\) the 2-3 mixing converges as \(\sin^2 \theta_{23} \to \tan^2 \theta_{12}\) in the \(T_\mu\)-case and \(\sin^2 \theta_{23} \to 1 - \tan^2 \theta_{12}\) in the \(T_\tau\)-case.
Figure 2: Symmetry relations between mixing parameters. Shown are $\sin^2 \theta_{12}$ (left panel) and $\sin^2 \theta_{23}$ (right panel) as functions of $\sin^2 \theta_{13}$ for the model with $\{T_\mu; a = 0, k = 2, m = 5\}$. The curves in the right panel correspond to $\delta = 0$ (thick), $\pi/4$ (dashed), $\pi/2$ (dotted).

Notice that for $a = 0$ we can have $|U_e2|^2 = |U_\mu2|^2 = |U_\tau2|^2$ ($k = 1$ and $m = 3$), both for $T_\mu$ and $T_\tau$, which leads to the model $2T$ analyzed previously. An excellent agreement with data can be obtained, for instance, for $p = 5$, $m = 6$ and $k = 1$, if $a = -1 + 2 \cos \pi/5 \approx 0.618$ is selected among different possibilities. This leads to $\eta = 0.382$. However such a choice of $p$ and $m$ corresponds to an infinite group, see Eq. (25).

In Fig. 2 we show the symmetry relations for the model with $T_\mu$ and $\{a, k, m\} = \{0, 2, 5\}$ which leads to an $A_5$ group ($p = 3$). The lines in the right panel correspond to different values of the CP phase $\delta$. They move up with increasing $\delta$ and the upper line is for $\delta = \pi/2$. In this case, the experimental measurements select small values of the CP phase: $\delta < 30^\circ$.

4 Discussion and conclusions

In this paper we considered the framework in which lepton mixing appears as a result of different ways of breaking the underlying flavor symmetry in the neutrino and charged lepton Yukawa sectors. Focusing on the case of Majorana neutrinos, we developed a formalism that can be applied to various group presentations and opens up a way of "symmetry building" and explanation of the data with discrete symmetries.

We assumed that the flavor group is determined by the residual symmetries of the mass matrices and considered a minimal, and at the same time general, case in which only a cyclic group from each sector (the neutrino and charged lepton) is a subgroup of the full flavor group $G$. A relation between the generators of these two groups that completes the presentation is imposed. The remaining symmetries of the mass matrices
that do not belong to $G$ are treated as accidental ones. We showed that in general, the flavor group that emerges from this construction is of the von Dyck type.

In this framework, we explored to which extent the symmetry determines the mixing. We derived a non-trivial matrix equation which constrains the mixing matrix in terms of the symmetry transformations of the neutrino and charged leptons in the mass basis. These relations depend on the charges of neutrinos and charged leptons under the residual symmetries and are independent of the basis in which the fermion fields are written. The solution of the matrix equation is reduced to 2 equations for the mixing parameters. Therefore, in the minimal scenario in which the residual symmetries in each sector are cyclic groups, the symmetry group imposes 2 relations between the 4 parameters (mixing angles and CP phase). This result is compatible with the well known fact that if only one $Z_2$ symmetry is imposed in the neutrino sector then at least one parameter of the mixing matrix must remain undetermined.

We showed that the symmetry group allows for non-zero 1-3 mixing and deviation from 2-3 maximal mixing even in the limit of exact residual symmetries. With respect to TBM and BM, this paper clarifies what is actually “special” about them. They correspond to limiting cases ($\theta_{13} \to 0$) in the continuous set of values allowed by the symmetry. We showed that, in general, discrete symmetries impose maximal 2-3 mixing, if either $\theta_{13} = 0$ or $\delta = \pi/2$.

The mixing can be fixed completely in models in which the full Klein symmetry of the neutrino mass matrix belongs to the flavor group. These models are particular cases of the framework developed here in the sense that the larger symmetry leads to new relations in addition to the ones derived in this paper. In the case of the flavor group $D(2,3,4) = S_4$, TBM is indeed compatible with the four relations that need to hold if the Klein symmetry of the neutrino mass matrix is a subgroup of $G$.

We applied our formalism to the finite von Dyck groups. We made a number of assumptions that simplify our analysis and reduce the number of possibilities. Namely, we assumed that: (i) generators of the group belong to $SU(3)$, (ii) one of the eigenvalues of $T$ is equal to one, (iii) one of the eigenvalues of the matrix $W$ is equal to 1. Furthermore, we took the group relation in the simple form, Eq. (20) which leads to the von Dyck groups. Nevertheless we find that many models discussed in the literature are included in our analysis. Removing these assumptions can lead to different flavor groups and, in general, to new relations between the mixing parameters.

We found that $S_4$ can lead to a mixing matrix, as in model $2T$ discussed in Sec. 3, which is in a good agreement with the data including the latest value for the 1-3 mixing. In the simplest cases of finite groups according to the symmetry relations and the experimental data on mixing angles, the CP-violating phase should be in the interval $\delta = (90^\circ - 120^\circ)$. 

14
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