Multiplicity of singularities is not a bi-Lipschitz invariant

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Abstract. It was conjectured that multiplicity of a singularity is bi-Lipschitz invariant. We disprove this conjecture, constructing examples of bi-Lipschitz equivalent complex algebraic singularities with different values of multiplicity.

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1 Introduction

The famous multiplicity conjecture, stated by Zariski in 1971 (see \cite{Z2}), is formulated as follows: if two germs of complex analytic hypersurfaces are ambient topological equivalent, then they have the same multiplicity. It was proved by Zariski \cite{Z1} for germs of plane analytic curves. The results of Pham-Teisser in \cite{PT} show that this result can be extended in the following “metric” way: if the two germs of complex analytic curves in n-dimensional space are bi-Lipschitz equivalent with respect to the outer metric, then the germs of the space curves have the same multiplicity. Comte in \cite{Co} proves that the multiplicity of complex analytic germs (not necessarily codimension 1 sets) is invariant under bi-Lipschitz homeomorphism with Lipschitz constants close enough to 1 (this is a severe assumption). These results motivated the following question, closely related to the multiplicity conjecture:

\textsuperscript{1}Partially supported by CNPq grant 302655/2014-0.
\textsuperscript{2}Partially supported by CNPq grant 302764/2014-7.
\textsuperscript{3}Partially supported by the Russian Academic Excellence Project "5-100".
is the multiplicity of a germ of analytic singularity a bi-Lipschitz invariant? This question was stated as a conjecture in [BFS].

The Lipschitz Regularity Theorem in [BFLS] shows that if the multiplicity of a complex analytic germ is equal to one, then it is a bi-Lipschitz invariant. Namely, if a germs of an analytic set is bi-Lipschitz equivalent to a smooth germ, then it is smooth itself. Later, Fernandes and Sampaio in [FS] give a positive answer to this question for surfaces in 3-dimensional space with respect to the ambient bi-Lipschitz equivalence. More recently, Neumann and Pichon ([NP]) showed that the multiplicity is an invariant under bi-Lipschitz equivalence, for germs of normal surface singularities. Another important result in [FS] is the following: in order to prove (or disprove) the bi-Lipschitz invariance of the multiplicity, it is enough to prove it for the algebraic cones, i.e. for the algebraic sets, defined by homogeneous polynomials. In [BFS] the authors show that the conjecture has a positive answer for 1 or 2 dimensional complex analytic sets.

The present paper shows that the multiplicity of complex algebraic sets is not a bi-Lipschitz invariant for the sets of dimension bigger or equal to three. Moreover, we show that there exists an infinite family of 3-dimensional germs, such that all of them are bi-Lipschitz equivalent, but they have different multiplicities. The idea of the construction is to consider the complex cones over different embeddings of $\mathbb{C}P^1 \times \mathbb{C}P^1$ to complex projective spaces. Using the topology of Smale-Barden manifolds, we show that all the links of such singularities are diffeomorphic. That is why the germs of the corresponding cones are bi-Lipschitz equivalent. From the other hand, the multiplicities of the cones at the origin may be explicitly calculated in terms of the embeddings.

2 Smale-Barden manifolds

The classification of 5-manifolds is due to S. Smale ([S]) and D. Barden ([B]).

Definition 2.1: A simply connected, compact, oriented 5-manifold is called Smale-Barden manifold.

The Smale-Barden manifolds are uniquely determined by their second Stiefel-Whitney class and the linking form.
Theorem 2.2: ([B]) Let $X, X'$ be two Smale-Barden manifolds. Assume that $H^2(X) = H^2(X')$ and this isomorphism is compatible with the linking form and preserves the second Stiefel-Whitney class. Then $X$ is diffeomorphic to $X'$. ■

Corollary 2.3: There exists only two Smale-Barden manifolds $M$ with $H^2(M) = \mathbb{Z}$, the product $S^2 \times S^3$ and the total space of a non-trivial $S^3$-bundle over $S^2$ (see [C] for an introduction to Barden theory, where this manifold is formally introduced).

Proof: Indeed, the linking form on $\mathbb{Z}$ vanishes, therefore the manifold is uniquely determined by the Stiefel-Whitney class $w_2(M)$. Hence we have only two possibilities: $w_2(M) = 0$ and $w_2(M) \neq 0$. ■

In early 2000-ies, the classification of 5-manifolds attracted interest coming from algebraic geometry, in the context of Sasakian geometry and geometry of generalized Seifert manifolds ([K1], [K2]). In the present paper we are interested in $S^1$-bundles over $\mathbb{C}P^1 \times \mathbb{C}P^1$.

Proposition 2.4: Let $\pi : M \rightarrow B$ be a simply connected 5-manifold obtained as a total space of an $S^1$-bundle $L$ over $B = \mathbb{C}P^1 \times \mathbb{C}P^1$. Then $H^2(M)$ is torsion-free, and $M$ is diffeomorphic to $S^2 \times S^3$.

Proof. Step 1: Universal coefficients formula gives an exact sequence

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(H_1(M; \mathbb{Z}), \mathbb{Z}) \rightarrow H^2(M; \mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}}(H_2(M; \mathbb{Z}), \mathbb{Z}) \rightarrow 0.$$ 

This implies that $H^2(M; \mathbb{Z})$ is torsion-free.

Step 2: Consider the following exact sequence of homotopy groups

$$0 \rightarrow \pi_2(M) \rightarrow \pi_2(B) \xrightarrow{\phi} \pi_1(S^1) \rightarrow \pi_1(M) \rightarrow 0$$

Since $\pi_1(M) = 0$, the map $\phi$, representing the first Chern class of $L$, is surjective. This exact sequence becomes

$$0 \rightarrow \pi_2(M) \rightarrow \mathbb{Z}^2 \rightarrow \mathbb{Z} \rightarrow 0$$

giving $\pi_2(M) = \mathbb{Z}$, and $H^2(M) = \mathbb{Z}$ because $H^2(M)$ is torsion-free.
Step 3: To deduce Proposition 2.4 from the Smale-Barden classification, it remains to show that $w_2(M) = 0$. However, $w_2(M) = \pi^*(\omega_2(B))$ ([K2, Lemma 36]), and the latter clearly vanishes, because $w_2(S^2) = 0$. ■

3 Multiplicity of homogeneous singularities

Given a projective variety $X \subset \mathbb{C}P^n$, the projective cone of $X$ is the union of all 1-dimensional subspaces $l \subset \mathbb{C}^{n+1}$ such that $l$, interpreted as a point in $\mathbb{C}P^n$, belongs to $X$. The link of $C(X)$ is an intersection of $C(X)$ with a unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$.

Let $A$ be complex algebraic set of $\mathbb{C}^{n+1}$ and $x \in A$. The multiplicity of $A$ at $x$, denoted by $\text{mult}(A, x)$, is defined to be the multiplicity of the maximal ideal of the local ring $O_{A,x}$. Given a projective variety $X \subset \mathbb{C}P^n$, we see that multiplicity of the projective cone $C(X)$ at the origin $0 \in \mathbb{C}^{n+1}$ coincides with degree of $X$ (see [Ch], subsection 11.3).

Next, we shall be interested in the following geometric situation. Let $X \subset \mathbb{C}P^n$ be a variety isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$. Then the Picard group of $X = \mathbb{C}P^1 \times \mathbb{C}P^1$ is isomorphic to $\mathbb{Z}^2$, and a line bundle is determined by its bidegree. We shall denote a line bundle of bidegree $a, b$ by $O(a,b)$.

Proposition 3.1: Let $X \subset \mathbb{C}P^n$ be a variety isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$, and $S \subset C(X)$ be its link. Assume that and $O(1)|_X = O(a,b)$. Then $X$ has degree $2ab$. If, in addition, $a$ and $b$ are relatively prime, the link of $C(X)$ is diffeomorphic to $S^2 \times S^3$.

Proof. Step 1: Since $c_1(O(a,b))^2 = 2ab$, and degree of a subvariety $X \subset \mathbb{C}P^n$ is its intersection with the top power of $O(1)|_X$, one has $\deg X = 2ab$.\l

Step 2: Consider the homotopy exact sequence

$$0 \rightarrow \pi_2(S) \rightarrow \pi_2(X) \xrightarrow{\phi} \pi_1(S^1) \rightarrow \pi_1(S) \rightarrow \pi_1(X) \rightarrow 0$$

for the circle bundle $\pi : S \rightarrow X$. Since the map $\phi$ represents the first Chern class of $O(1)|_X$, it is obtained as a quotient of $\mathbb{Z}^2$ by a subgroup generated by $(a,b)$, and this map is surjective because $a$ and $b$ are relatively prime. Then
\[ \pi_1(S) = \pi_1(X) = 0, \] and Proposition 2.4 implies that \( S \) is diffeomorphic to \( S^2 \times S^3 \).

4 Lipschitz invariance of singularities

Let \( X \subset \mathbb{C}^n \) be a complex variety. The induced metric from the Euclidean distance on \( \mathbb{C}^n \) gives a distance on \( X \); it is called the \textbf{outer metric} on \( X \).

**Definition 4.1:** Let \( X \subset \mathbb{C}^n \) and \( X' \subset \mathbb{C}^{n'} \) be complex varieties equipped with the outer metrics, \( x \in X, x' \in X' \) marked points. We say that \( (X, x) \) is \textbf{bi-Lipschitz equivalent} to \( (X', x') \) if there exist neighborhoods \( U \) of \( x \) in \( \mathbb{C}^n \) and \( U' \) of \( x' \) in \( \mathbb{C}^{n'} \), and a bi-Lipschitz homeomorphism of \( X \cap U \) to \( X' \cap U' \) mapping \( x \) to \( x' \).

**Definition 4.2:** Let \( X, X' \subset \mathbb{C}^n \) be complex varieties equipped with the outer metrics, \( x \in X, x' \in X' \) marked points. We say that \( (X, x) \) is \textbf{ambient bi-Lipschitz equivalent} to \( (X', x') \) if there exists a bi-Lipschitz equivalence of a neighborhood \( U \) of \( x \) in \( \mathbb{C}^n \) and a neighborhood \( U' \) of \( x' \) in \( \mathbb{C}^{n'} \) mapping \( X \cap U \) to \( X' \cap U' \) and \( x \) to \( x' \).

Actually, the two definitions above do not coincide. The ambient bi-Lipschitz equivalence implies bi-Lipschitz equivalence, but the examples presented in [BG] show that the converse does not hold true in general.

As it was already mentioned in Introduction, it was conjectured in [BFS] that the multiplicity is a bi-Lipschitz invariant. We prove that this is false. Here is the main result of this paper.

**Theorem 4.3:** For each \( n \geq 3 \), there exists a family \( \{Y_i\}_{i \in \mathbb{Z}} \) of \( n \)-dimensional complex algebraic varieties \( Y_i \subset \mathbb{C}^{n_i+1} \) such that:

(a) for each pair \( i \neq j \), the germs at the origin of \( Y_i \subset \mathbb{C}^{n_i+1} \) and \( Y_j \subset \mathbb{C}^{n_j+1} \) are bi-Lipschitz equivalent, but \( (Y_i, 0) \) and \( (Y_j, 0) \) have different multiplicity.

(b) for each pair \( i \neq j \), there are \( n \)-dimensional complex algebraic varieties \( Z_{ij}, \tilde{Z}_{ij} \subset \mathbb{C}^{n_i+n_j+2} \) such that \( (Z_{ij}, 0) \) and \( (\tilde{Z}_{ij}, 0) \) are ambient bi-Lipschitz equivalent, but \( \text{mult}(Z_{ij}, 0) = \text{mult}(Y_i, 0) \) and \( \text{mult}(\tilde{Z}_{ij}, 0) = \text{mult}(Y_j, 0) \) and, in particular, they have different multiplicity.
Proof. Let \( \{p_i\}_{i \in \mathbb{Z}} \) be the family of odd prime numbers. For each \( i \in \mathbb{Z} \), let \( L_i \) be a very ample bundle on \( X = \mathbb{C}P^1 \times \mathbb{C}P^1 \) of bidegree \((2, p_i)\). Let \( X_i \) be projective variety obtained by the embedding of the very ample bundle \( L_i \). Consider the link of the singularity \( S_i := C(X_i) \cap S^{2m_i+1} \), where \( S^{2m_i+1} \) is the unit sphere centered in \( 0 \in \mathbb{C}^{m_i+1} \). Then, for each pair \( i \neq j \) the links \( S_i, S_j \) are diffeomorphic to \( S^2 \times S^3 \) (Proposition 3.1). In particular, \( S_i \) to \( S_j \) are bi-Lipschitz homeomorphic. Since a bi-Lipschitz map from \( S_i \) to \( S_j \) induces a bi-Lipschitz map of their cones, then the affine cones \( (C(X_i), 0) \) and \( (C(X_j), 0) \) are bi-Lipschitz equivalent, but \( \text{mult}(C(X_i), 0) = 4p_i \) and \( \text{mult}(C(X_j), 0) = 4p_j \) (Proposition 3.1). Thus, if for each \( i \in \mathbb{Z} \) we define \( Y_i := C(X_i) \times \mathbb{C}^{n-3} \), then we have that the family \( \{Y_i\}_{i \in \mathbb{Z}} \) satisfies the item (a), since \( \text{mult}(Y_i, 0) = \text{mult}(C(X_i), 0) = 4p_i \), for all \( i \in \mathbb{Z} \).

Concerning to the item (b), let \( \phi_{ij} : Y_i \rightarrow Y_j \) be a bi-Lipschitz homeomorphism such that \( \phi_{ij}(0) = 0 \). Let \( \tilde{\phi}_{ij} : \mathbb{C}^{n_i+1} \rightarrow \mathbb{C}^{n_j+1} \) (resp. \( \tilde{\psi}_{ij} : \mathbb{C}^{n_j+1} \rightarrow \mathbb{C}^{n_i+1} \)) be a Lipschitz extension of \( \phi_{ij} \) (resp. \( \psi_{ij} = \phi_{ij}^{-1} \)) (see [Ki], [M] and [W]). Let us define \( \varphi, \psi : \mathbb{C}^{n_i+1} \times \mathbb{C}^{n_j+1} \rightarrow \mathbb{C}^{n_i+1} \times \mathbb{C}^{n_j+1} \) as follows:

\[
\varphi(x, y) = (x - \tilde{\psi}_{ij}(y + \tilde{\phi}_{ij}(x)), y + \tilde{\phi}_{ij}(x))
\]

and

\[
\psi(z, w) = (z + \tilde{\psi}_{ij}(w), w - \tilde{\phi}_{ij}(z + \tilde{\psi}_{ij}(w))).
\]

It easy to verify that \( \psi = \varphi^{-1} \) and since \( \varphi \) and \( \psi \) are composition of Lipschitz maps, they are also Lipschitz maps. Moreover, if \( Z_{ij} = Y_i \times \{0\} \) and \( \bar{Z}_{ij} = \{0\} \times Y_j \), we obtain that \( \varphi(Z_{ij}) = \bar{Z}_{ij} \) (see [Sa]). Therefore, \( (Z_{ij}, 0) \) and \( (\bar{Z}_{ij}, 0) \) are bi-Lipschitz equivalent, but \( \text{mult}(Z_{ij}, 0) = \text{mult}(Y_i, 0) \) and \( \text{mult}(\bar{Z}_{ij}, 0) = \text{mult}(Y_j, 0) \) and, in particular, they have different multiplicity.

References

[BFLS] L. Birbrair, A. Fernandes, Lê D. T. and J. E. Sampaio. *Lipschitz regular complex algebraic sets are smooth*. Proceedings of the American Mathematical Society 144 (2016), no. 3, 983–987.
[BG] Lev Birbrair and Andrei Gabrielov. Ambient Lipschitz equivalence of real surface singularities. arXiv:1707.04951v2 [math.AG], (2017), to appear in IMRN.

[B] D. Barden, Simply connected five-manifolds. Ann. of Math. 82 (1965), no. 2, 365–385.

[BFS] J. de Bobodilla, A. Fernandes and E. Sampaio. Multiplicity and degree as bi-Lipschitz invariants for complex sets. arXiv:1706.06614v2 [math.AG], (2017).

[Ch] E. M. Chirka. Complex analytic sets Kluwer Academic Publishers, 1989.

[C] Diarmuid Crowley. 5-manifolds: 1-connected. Bulletin of the Manifold Atlas (2011), 49–55.

[Co] Georges Comte. Multiplicity of complex analytic sets and bi-Lipschitz maps. Real analytic and algebraic singularities (Nagoya/Sapporo/Hachioji, 1996) Pitman Res. Notes Math. Ser., 381 (1998), 182–188.

[F] A. Fernandes Topological equivalence of complex curves and bi-Lipschitz maps. Michigan Math. J. 51 (2003), 593–606.

[FS] A. Fernandes and J. Edson Sampaio. Multiplicity of analytic hypersurface singularities under bi-Lipschitz homeomorphisms. Journal of Topology 9 (2016), 927–933.

[Ki] M. Kirszbraun. Über die zusammenziehende und Lipschitzsche Transformationen. Fundamenta Math. 22 (1934), no. 1, 77–108.

[K1] J. Kollár. Seifert Gm-bundles. arXiv:math/0404386v2 [math.AG], (2004).

[K2] J. Kollár. Circle actions on simply connected 5-manifolds, Topology 45 (2006), no. 3, 643–671.

[M] E. J. Mcshane. Extension of range of functions. Bull. Amer. Math. Soc. 40 (1934), 837–842.

[NP] W. Neumann and A. Pichon Lipschitz geometry of complex surfaces: analytic invariants and equisingularity. arXiv:1211.4897v3 [math.AG], (2016).

[Sa] J. Edson Sampaio Bi-Lipschitz homeomorphic subanalytic sets have bi-Lipschitz homeomorphic tangent cones. Selecta Math. (N.S.), 22 (2016), no. 2, 553–559.
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