General formulas for the central and non-central moments of the multinomial distribution

Frédéric Ouimet\textsuperscript{a,1,*}

\textsuperscript{a}California Institute of Technology, Pasadena, USA.

Abstract

We present the first general formulas for the central and non-central moments of the multinomial distribution, using a combinatorial argument and the factorial moments previously obtained in Mosimann (1962). We use the formulas to give explicit expressions for all the non-central moments up to order 8 and all the central moments up to order 4. These results expand significantly on those in Newcomer (2008; Newcomer \textit{et al.} (2008), where the non-central moments were calculated up to order 4.

**Keywords:** multinomial distribution, higher moments, central moments, non-central moments

**2010 MSC:** Primary : 62E15 Secondary : 60E05

1. The multinomial distribution

For any $d \in \mathbb{N}$, the $d$-dimensional (unit) simplex is defined by $S := \{x \in [0,1]^d : \sum_{i=1}^d x_i \leq 1\}$, and the probability mass function $k \mapsto P_{k,m}(x)$ for $\xi := (\xi_1, \xi_2, \ldots, \xi_d) \sim \text{Multinomial}(m, x)$ is defined by

$$P_{k,m}(x) := \frac{m!}{(m - \sum_{i=1}^d k_i)! \prod_{i=1}^d k_i !} \cdot (1 - \sum_{i=1}^d x_i)^{m - \sum_{i=1}^d k_i} \prod_{i=1}^d x_i^{k_i}, \quad k \in \mathbb{N}_0^d \cap mS, \quad (1.1)$$

where $m \in \mathbb{N}$ and $x \in S$. If $x_{d+1} := 1 - \sum_{i=1}^d x_i$, then (1.1) is just a reparametrization of $(\xi, 1 - \sum_{i=1}^d \xi_i) \sim \text{Multinomial}(m, (x, x_{d+1}))$ where $\sum_{i=1}^{d+1} x_i = 1$. In this paper, our main goal is to give general formulas for the non-central and central moments of (1.1), namely

$$E \left[ \prod_{i=1}^d \xi_i^{p_i} \right] \quad \text{and} \quad E \left[ \prod_{i=1}^d (\xi_i - E[\xi_i])^{p_i} \right], \quad p_1, p_2, \ldots, p_d \in \mathbb{N}_0. \quad (1.2)$$

We obtain the formulas using a combinatorial argument and the general expression for the factorial moments found in Mosimann (1962), which we register in the lemma below.

**Lemma 1** (Factorial moments). Let $\xi \sim \text{Multinomial}(m, x)$. Then, for all $k_1, k_2, \ldots, k_d \in \mathbb{N}_0$,

$$E \left[ \prod_{i=1}^d \xi_i^{(k_i)} \right] = m^{(\sum_{i=1}^d k_i)} \prod_{i=1}^d x_i^{k_i}, \quad (1.3)$$

where $m^{(k)} := m(m-1)\ldots(m-k+1)$ denotes the $k$-th order falling factorial of $m$.

The formulas that we develop for the expectations in (1.2) will be used to compute explicitly all the non-central moments up to order 8 and all the central moments up to order 4, which expands on the third and fourth order non-central moments that were previously calculated in (Newcomer, 2008, Appendix A.1). We should also mention that explicit formulas for several lower-order (mixed) cumulants were presented in Wishart (1949) (see also (Johnson \textit{et al.}, 1997, p.37)), but not for the moments.

\textsuperscript{*}Corresponding author

\textit{Email address: ouimetfr@caltech.edu} (Frédéric Ouimet)
2. Motivation

To the best of our knowledge, general formulas for the central and non-central moments of the multinomial distribution have never been derived in the literature. The central moments can arise naturally, for example, when studying asymptotic properties, via Taylor expansions, of statistical estimators involving the multinomial distribution. For a given sequence of i.i.d. observations $X_1, X_2, \ldots, X_n$, two examples of such estimators are the Bernstein estimator for the cumulative distribution function

$$F_{n,m}^*(x) := \sum_{k \in \mathbb{N}^d} \frac{1}{n} \sum_{i=1}^{n} I_{(\infty, \frac{k}{m})}(X_i) P_{k,m}(x), \quad x \in S, \quad m, n \in \mathbb{N},$$

(2.1)

and the Bernstein estimator for the density function (also called smoothed histogram)

$$f_{n,m}(x) := \sum_{k \in \mathbb{N}^d} \frac{m}{k} \sum_{i=1}^{n} I_{(\infty, \frac{k}{m})}(X_i) P_{k,m-1}(x), \quad x \in S, \quad m, n \in \mathbb{N},$$

(2.2)

over the $d$-dimensional simplex. Some of their asymptotic properties were investigated by Vitale (1975); Gawronski & Stadtmüller (1981); Stadtmüller (1983); Gawronski (1985); Stadtmüller (1986); Tenbusch (1997); Petrone (1999a,b); Ghosal (2001); Petrone & Wasserman (2002); Babu et al. (2002); Kakizawa (2004); Bouezmarni & Rolin (2007); Bouezmarni et al. (2007); Leblanc (2009, 2010); Curtis & Ghosh (2011); Leblanc (2012a,b); Igarashi & Kakizawa (2014); Turnbull & Ghosh (2014); Lu (2015); Guan (2016, 2017); Belalia et al. (2017, 2019) when $d = 1$, by Tenbusch (1994) when $d = 2$, and by Ouimet (2020a,b) for all $d \geq 1$, using a local limit theorem from Ouimet (2020c) for the multinomial distribution (see also Arenbaev (1976)).

The estimator (2.2) is a discrete analogue of the Dirichlet kernel estimator introduced by Aitchison & Lauder (1985) and studied theoretically in Brown & Chen (1999); Chen (1999, 2000); Bouezmarni & Rolin (2003) when $d = 1$ (among others), and in Ouimet (2020c) for all $d \geq 1$.

3. Results

First, we give a general formula of the non-central moments of the multinomial distribution in (1.1).

**Theorem 1 (Non-central moments).** Let $\xi \sim \text{Multinomial}(m, x)$. Then, for all $p_1, p_2, \ldots, p_d \in \mathbb{N}_0$,

$$\mathbb{E}\left[\prod_{i=1}^{d} \xi_i^{p_i}\right] = \sum_{k_1=0}^{p_1} \cdot \sum_{k_d=0}^{p_d} m^{(\sum_{i=1}^{d} k_i)} \prod_{i=1}^{d} \binom{p_i}{k_i} x_{i}^{k_i},$$

(3.1)

where $\binom{p}{k}$ denotes a Stirling number of the second kind (i.e., the number of ways to partition a set of $p$ objects into $k$ non-empty subsets).

**Proof.** We have the following well-known relation between the power $p \in \mathbb{N}_0$ of a number $x \in \mathbb{R}$ and the falling factorials of $x$:

$$x^p = \sum_{k=0}^{p} \binom{p}{k} x^{(k)}.$$

(3.2)

See, e.g., (Graham et al., 1994, p.262). Apply this relation to every $\xi_i^{p_i}$ and use the linearity of the expectation to get

$$\mathbb{E}\left[\prod_{i=1}^{d} \xi_i^{p_i}\right] = \sum_{k_1=0}^{p_1} \cdot \sum_{k_d=0}^{p_d} \prod_{i=1}^{d} \binom{p_i}{k_i} \mathbb{E}\left[\prod_{i=1}^{d} \xi_i^{(k_i)}\right],$$

(3.3)

The conclusion follows from Lemma 1.

We can now deduce a general formula for the central moments of the multinomial distribution.

**Theorem 2 (Central moments).** Let $\xi \sim \text{Multinomial}(m, x)$. Then, for all $p_1, p_2, \ldots, p_d \in \mathbb{N}_0$,

$$\mathbb{E}\left[\prod_{i=1}^{d} (\xi_i - \mathbb{E}[\xi_i])^{p_i}\right] = \sum_{\ell_1=0}^{p_1} \cdot \sum_{\ell_d=0}^{p_d} \prod_{i=1}^{d} m^{(\sum_{i=1}^{d} k_i)} \{-m\}^{\sum_{i=1}^{d} (p_i - \ell_i) \prod_{i=1}^{d} \binom{p_i}{k_i} \ell_i^{p_i - \ell_i + k_i},$$

(3.4)

where $\binom{p}{\ell}$ denotes the binomial coefficient $\frac{p!}{\ell!(p-\ell)!}$.
Proof. By applying the binomial formula to each factor \((\xi_i - \mathbb{E}[\xi_i])^{p_i}\) and using the fact that \(\mathbb{E}[\xi_i] = mx_i\) for all \(i \in \{1, 2, \ldots, d\}\), note that

\[
\mathbb{E} \left[ \prod_{i=1}^{d} (\xi_i - \mathbb{E}[\xi_i])^{p_i} \right] = \sum_{\ell_1=0}^{p_1} \cdots \sum_{\ell_d=0}^{p_d} \mathbb{E} \left[ \prod_{i=1}^{d} \xi_i^{\ell_i} \right] \cdot \prod_{i=1}^{d} \binom{p_i}{\ell_i} (-mx_i)^{p_i - \ell_i}.
\]  

(3.5)

The conclusion follows from Theorem 1.

\(\square\)

4. Numerical implementation

The formulas in Theorem 1 and Theorem 2 can be implemented in Mathematica as follows:

```mathematica
NonCentral[a_, x_, p_, d_] :=
  Sum[FactorialPower[m, Sum[k[i], {i, 1, d}]] * 
  Product[StirlingS2[p[[i]], k[i]] * x[[i]] - k[i], {i, 1, d}], ##] & @@ 
  ((k[#], 0, p[[#]]) & /@ Range[d]);

Central[a_, x_, p_, d_] :=
  Sum[Sum[FactorialPower[m, Sum[k[i], {i, 1, d}]] * 
  Product[Binomial[p[[i]], ell[i]] * StirlingS2[ell[i], k[i]] * 
  x[[i]] - (p[[i]] - ell[i] + k[i]), {i, 1, d}], ##] & @@ 
  ((ell[#], 0, p[[#]]) & /@ Range[d]), ##] & @@ 
  ((ell[#], 0, p[[#]]) & /@ Range[d]);
```

5. Explicit formulas

In Newcomer (2008), explicit expressions for the non-central moments of order 3 and 4 where obtained for the multinomial distribution, see also Newcomer et al. (2008); Ouimet (2020d). To expand on those results, we use the formula from Theorem 1 in the two subsections below to calculate (explicitly) all the non-central moments up to order 8 and all the central moments up to order 4.

Here is a table of the Stirling numbers of the second kind that we will use in our calculations:

| \(k\) | \(\ell\) |
|-------|--------|
| 0     | 0      |
| 0     | 1      |
| 1     | 0      |
| 2     | 0      |
| 3     | 0      |
| 4     | 0      |
| 5     | 0      |
| 6     | 0      |
| 7     | 0      |
| 8     | 0      |

\[
\begin{align*}
\binom{0}{0} &= 1, \\
\binom{0}{1} &= 1, \\
\binom{1}{0} &= 0, \\
\binom{2}{0} &= 1, \\
\binom{3}{0} &= 1, \\
\binom{4}{0} &= 1, \\
\binom{5}{0} &= 1, \\
\binom{6}{0} &= 1, \\
\binom{7}{0} &= 1, \\
\binom{8}{0} &= 1, \\
\binom{1}{1} &= 1, \\
\binom{2}{1} &= 1, \\
\binom{3}{1} &= 1, \\
\binom{4}{1} &= 1, \\
\binom{5}{1} &= 1, \\
\binom{6}{1} &= 1, \\
\binom{7}{1} &= 1, \\
\binom{8}{1} &= 1, \\
\binom{1}{2} &= 2, \\
\binom{2}{2} &= 3, \\
\binom{3}{2} &= 3, \\
\binom{4}{2} &= 6, \\
\binom{5}{2} &= 15, \\
\binom{6}{2} &= 31, \\
\binom{7}{2} &= 63, \\
\binom{8}{2} &= 127, \\
\binom{1}{3} &= 3, \\
\binom{2}{3} &= 6, \\
\binom{3}{3} &= 7, \\
\binom{4}{3} &= 11, \\
\binom{5}{3} &= 25, \\
\binom{6}{3} &= 50, \\
\binom{7}{3} &= 101, \\
\binom{8}{3} &= 203, \\
\binom{1}{4} &= 4, \\
\binom{2}{4} &= 6, \\
\binom{3}{4} &= 10, \\
\binom{4}{4} &= 22, \\
\binom{5}{4} &= 46, \\
\binom{6}{4} &= 96, \\
\binom{7}{4} &= 196, \\
\binom{8}{4} &= 403, \\
\binom{1}{5} &= 5, \\
\binom{2}{5} &= 8, \\
\binom{3}{5} &= 15, \\
\binom{4}{5} &= 30, \\
\binom{5}{5} &= 60, \\
\binom{6}{5} &= 126, \\
\binom{7}{5} &= 253, \\
\binom{8}{5} &= 512, \\
\binom{1}{6} &= 6, \\
\binom{2}{6} &= 10, \\
\binom{3}{6} &= 21, \\
\binom{4}{6} &= 46, \\
\binom{5}{6} &= 108, \\
\binom{6}{6} &= 254, \\
\binom{7}{6} &= 547, \\
\binom{8}{6} &= 1112, \\
\binom{1}{7} &= 7, \\
\binom{2}{7} &= 12, \\
\binom{3}{7} &= 28, \\
\binom{4}{7} &= 70, \\
\binom{5}{7} &= 168, \\
\binom{6}{7} &= 432, \\
\binom{7}{7} &= 1046, \\
\binom{8}{7} &= 2322, \\
\binom{1}{8} &= 8, \\
\binom{2}{8} &= 16, \\
\binom{3}{8} &= 44, \\
\binom{4}{8} &= 120, \\
\binom{5}{8} &= 336, \\
\binom{6}{8} &= 936, \\
\binom{7}{8} &= 2464, \\
\binom{8}{8} &= 5868.
\end{align*}
\]
5.1. Computation of the non-central moments up to order 8

By applying the general expression in Theorem 1 and by removing the Stirling numbers \( \{p_k\} \) that are equal to 0, we get the following results directly.

**Order 1:** For any \( j_1 \in \{1, 2, \ldots, d\} \),

\[
E[\xi_{j_1}] = x_{j_1} m. \tag{5.1}
\]

**Order 2:** For any distinct \( j_1, j_2 \in \{1, 2, \ldots, d\} \),

\[
E[\xi_{j_1}^2] = x_{j_1} [m + m^{(2)} x_{j_1}], \tag{5.2}
E[\xi_{j_1} \xi_{j_2}] = x_{j_1} x_{j_2} m^{(2)}. \tag{5.3}
\]

**Order 3:** For any distinct \( j_1, j_2, j_3 \in \{1, 2, \ldots, d\} \),

\[
E[\xi_{j_1}^3] = x_{j_1} [m + 3m^{(2)} x_{j_1} + m^{(3)} x_{j_1}^2], \tag{5.4}
E[\xi_{j_1}^2 \xi_{j_2}] = x_{j_1} x_{j_2} [m^{(2)} + m^{(3)} x_{j_1}], \tag{5.5}
E[\xi_{j_1} \xi_{j_2} \xi_{j_3}] = x_{j_1} x_{j_2} x_{j_3} m^{(3)}. \tag{5.6}
\]

**Order 4:** For any distinct \( j_1, j_2, j_3, j_4 \in \{1, 2, \ldots, d\} \),

\[
E[\xi_{j_1}^4] = x_{j_1} [m + 7m^{(2)} x_{j_1} + 6m^{(3)} x_{j_1}^2 + m^{(4)} x_{j_1}^3], \tag{5.7}
E[\xi_{j_1}^3 \xi_{j_2}] = x_{j_1} x_{j_2} [m^{(2)} + 3m^{(3)} x_{j_1} + m^{(4)} x_{j_1}^2], \tag{5.8}
E[\xi_{j_1}^2 \xi_{j_2}^2] = x_{j_1} x_{j_2} [m^{(2)} + m^{(3)} (x_{j_1} + x_{j_2}) + m^{(4)} x_{j_1} x_{j_2}], \tag{5.9}
E[\xi_{j_1}^3 \xi_{j_2} \xi_{j_3}] = x_{j_1} x_{j_2} x_{j_3} [m^{(3)} + m^{(4)} x_{j_1}], \tag{5.10}
E[\xi_{j_1} \xi_{j_2} \xi_{j_3} \xi_{j_4}] = x_{j_1} x_{j_2} x_{j_3} x_{j_4} m^{(4)}. \tag{5.11}
\]

**Order 5:** For any distinct \( j_1, j_2, j_3, j_4, j_5 \in \{1, 2, \ldots, d\} \),

\[
E[\xi_{j_1}^5] = x_{j_1} [m + 15m^{(2)} x_{j_1} + 25m^{(3)} x_{j_1}^2 + 10m^{(4)} x_{j_1}^3 + m^{(5)} x_{j_1}^4], \tag{5.12}
E[\xi_{j_1}^4 \xi_{j_2}] = x_{j_1} x_{j_2} [m^{(2)} + 7m^{(3)} x_{j_1} + 6m^{(4)} x_{j_1}^2 + m^{(5)} x_{j_1}^3], \tag{5.13}
E[\xi_{j_1}^3 \xi_{j_2}^2] = x_{j_1} x_{j_2} [m^{(2)} + 3m^{(3)} (3x_{j_1} + x_{j_2}) + m^{(4)} (2x_{j_1}^2 + 3x_{j_1} x_{j_2}) + m^{(5)} x_{j_1}^2 x_{j_2}], \tag{5.14}
E[\xi_{j_1}^4 \xi_{j_2} \xi_{j_3}] = x_{j_1} x_{j_2} x_{j_3} [m^{(3)} + 3m^{(4)} x_{j_1} + m^{(5)} x_{j_1}^2], \tag{5.15}
E[\xi_{j_1}^3 \xi_{j_2}^2 \xi_{j_3}] = x_{j_1} x_{j_2} x_{j_3} [m^{(3)} + m^{(4)} (x_{j_1} + x_{j_2}) + m^{(5)} x_{j_1} x_{j_2}], \tag{5.16}
E[\xi_{j_1}^4 \xi_{j_2} \xi_{j_3} \xi_{j_4}] = x_{j_1} x_{j_2} x_{j_3} x_{j_4} [m^{(4)} + m^{(5)} x_{j_1}], \tag{5.17}
E[\xi_{j_1} \xi_{j_2} \xi_{j_3} \xi_{j_4} \xi_{j_5}] = x_{j_1} x_{j_2} x_{j_3} x_{j_4} x_{j_5} m^{(5)}. \tag{5.18}
\]

**Order 6:** For any distinct \( j_1, j_2, j_3, j_4, j_5, j_6 \in \{1, 2, \ldots, d\} \),

\[
E[\xi_{j_1}^6] = x_{j_1} [m + 31m^{(2)} x_{j_1} + 90m^{(3)} x_{j_1}^2 + 65m^{(4)} x_{j_1}^3 + 15m^{(5)} x_{j_1}^4 + m^{(6)} x_{j_1}^5], \tag{5.19}
E[\xi_{j_1}^5 \xi_{j_2}] = x_{j_1} x_{j_2} [m^{(2)} + 15m^{(3)} x_{j_1} + 25m^{(4)} x_{j_1}^2 + 10m^{(5)} x_{j_1}^3 + m^{(6)} x_{j_1}^4], \tag{5.20}
E[\xi_{j_1}^4 \xi_{j_2}^2] = x_{j_1} x_{j_2} \begin{bmatrix} m^{(2)} + m^{(3)} (7x_{j_1} + x_{j_2}) + m^{(4)} (6x_{j_1}^2 + 7x_{j_1} x_{j_2}) + m^{(5)} x_{j_1} x_{j_2} \end{bmatrix}, \tag{5.21}
E[\xi_{j_1}^5 \xi_{j_2} \xi_{j_3}] = x_{j_1} x_{j_2} x_{j_3} [m^{(3)} + 7m^{(4)} x_{j_1} + 6m^{(5)} x_{j_1}^2 + m^{(6)} x_{j_1}^3], \tag{5.22}
E[\xi_{j_1}^4 \xi_{j_2}^2 \xi_{j_3}] = x_{j_1} x_{j_2} \begin{bmatrix} m^{(2)} + m^{(3)} (3x_{j_1} + 3x_{j_2}) + m^{(4)} (2x_{j_1}^2 + 9x_{j_1} x_{j_2} + x_{j_2}^2) + m^{(5)} (3x_{j_1}^2 + 3x_{j_1} x_{j_2} + m^{(6)} x_{j_1} x_{j_2}) \end{bmatrix}, \tag{5.23}
E[\xi_{j_1}^3 \xi_{j_2}^2 \xi_{j_3}] = x_{j_1} x_{j_2} x_{j_3} \begin{bmatrix} m^{(3)} + m^{(4)} (3x_{j_1} + x_{j_2}) + m^{(5)} (x_{j_1}^2 + 3x_{j_1} x_{j_2} + m^{(6)} x_{j_1} x_{j_2}) \end{bmatrix}, \tag{5.24}
E[\xi_{j_1}^3 \xi_{j_2} \xi_{j_3} \xi_{j_4}] = x_{j_1} x_{j_2} x_{j_3} x_{j_4} [m^{(4)} + 3m^{(5)} x_{j_1} + m^{(6)} x_{j_1}^2], \tag{5.25}
\]
\[ \mathbb{E}[\xi_{1j_s}^2 \xi_{2j_s}^2 \xi_{3j_s}^2] = x_{j_1} x_{j_2} x_{j_3} \left[ (m^3 + m^4 \xi_{j_1} + x_{j_2} + x_{j_3}) + m^5 \xi_{j_1} x_{j_2} + x_{j_3} x_{j_3} x_{j_3} x_{j_3} + m^6 \xi_{j_1} x_{j_2} x_{j_3} x_{j_3} \right] \quad (5.26) \]

\[ \mathbb{E}[\xi_{1j_s}^2 \xi_{2j_s} \xi_{3j_s} \xi_{4j_s}] = x_{j_1} x_{j_2} x_{j_3} x_{j_4} \left[ m^4 + m^5 (\xi_{j_1} + x_{j_2}) + m^6 x_{j_1} x_{j_2} \right] \quad (5.27) \]

\[ \mathbb{E}[\xi_{1j_s} \xi_{2j_s}^2 \xi_{3j_s} \xi_{5j_s}] = x_{j_1} x_{j_2} x_{j_3} x_{j_4} x_{j_5} \left[ m^5 + m^6 \xi_{j_4} \right] \quad (5.28) \]

\[ \mathbb{E}[\xi_{1j_s} \xi_{2j_s} \xi_{3j_s} \xi_{4j_s} \xi_{5j_s} \xi_{6j_s}] = x_{j_1} x_{j_2} x_{j_3} x_{j_4} x_{j_5} x_{j_6} m^6 \quad (5.29) \]

**Order 7:** For any distinct \( j_1, j_2, j_3, j_4, j_5, j_6, j_7 \in \{1, 2, \ldots, d\},

\[ \mathbb{E}[\xi_{1j_s}^3] = x_{j_1} \left[ m + 63m^2 \xi_{j_1} + 301m^3 \xi_{j_1}^2 + 350m^4 \xi_{j_1}^3 \right] + 140m^5 \xi_{j_1}^4 + 21m^6 \xi_{j_1}^5 + m^7 \xi_{j_1}^6 \quad (5.30) \]

\[ \mathbb{E}[\xi_{1j_s}^2 \xi_{2j_s}] = x_{j_1} x_{j_2} \left[ m^2 + 31m^3 \xi_{j_1} + 90m^4 \xi_{j_1}^2 + 65m^5 \xi_{j_1}^3 + 15m^6 \xi_{j_1}^4 + m^7 \xi_{j_1}^5 \right] \quad (5.31) \]

\[ \mathbb{E}[\xi_{1j_s} \xi_{2j_s}^2] = x_{j_1} x_{j_2} \left[ m^2 + m^3 (15 \xi_{j_1} + x_{j_2}) + m^4 (25 \xi_{j_1} x_{j_2}) + m^5 (10 \xi_{j_1}^2 + 25 \xi_{j_1} x_{j_2}) + m^6 (\xi_{j_1}^3 + 10 \xi_{j_1} x_{j_2}) + m^7 \xi_{j_1} x_{j_2} \right] \quad (5.32) \]

\[ \mathbb{E}[\xi_{1j_s} \xi_{2j_s} \xi_{3j_s}] = x_{j_1} x_{j_2} x_{j_3} \left[ m^3 + 15m^4 \xi_{j_1} + 25m^5 \xi_{j_1}^2 + 10m^6 \xi_{j_1}^3 + m^7 \xi_{j_1}^4 \right] \quad (5.33) \]

\[ \mathbb{E}[\xi_{1j_s} \xi_{2j_s} \xi_{3j_s}^2] = x_{j_1} x_{j_2} \left[ m^3 + m^4 (7 \xi_{j_1} + x_{j_2}) + m^5 (6 \xi_{j_1} x_{j_2}) + m^6 (\xi_{j_1}^2 + 2 \xi_{j_1} x_{j_2}) + m^7 \xi_{j_1} x_{j_2} \right] \quad (5.34) \]

\[ \mathbb{E}[\xi_{1j_s} \xi_{2j_s} \xi_{3j_s} \xi_{4j_s}] = x_{j_1} x_{j_2} x_{j_3} x_{j_4} \left[ m^3 + m^4 (7 \xi_{j_1} + x_{j_2}) + m^5 (6 \xi_{j_1} x_{j_2}) + m^6 (\xi_{j_1}^2 + 7 \xi_{j_1} x_{j_2}) + m^7 \xi_{j_1} x_{j_2} \right] \quad (5.35) \]

\[ \mathbb{E}[\xi_{1j_s} \xi_{2j_s} \xi_{3j_s} \xi_{4j_s} \xi_{5j_s}] = x_{j_1} x_{j_2} x_{j_3} x_{j_4} x_{j_5} \left[ m^4 + 7m^5 \xi_{j_1} + 6m^6 \xi_{j_1}^2 + m^7 \xi_{j_1}^3 \right] \quad (5.36) \]

\[ \mathbb{E}[\xi_{1j_s} \xi_{2j_s} \xi_{3j_s}^2] = x_{j_1} x_{j_2} x_{j_3} \left[ m^3 + m^4 (3 \xi_{j_1} + 3 \xi_{j_2}) + m^5 (\xi_{j_1} + 9 \xi_{j_1} x_{j_2} + x_{j_2}^2) + m^6 (3 \xi_{j_1} x_{j_2} + 3 \xi_{j_1} x_{j_2} + m^7 \xi_{j_1} x_{j_2} \right] \quad (5.37) \]

\[ \mathbb{E}[\xi_{1j_s}^2 \xi_{2j_s} \xi_{3j_s}] = x_{j_1} x_{j_2} x_{j_3} \left[ m^3 + m^4 (3 \xi_{j_1} + x_{j_2} + x_{j_3}) + m^5 (\xi_{j_1} + 3 \xi_{j_1} x_{j_2} + 3 \xi_{j_1} x_{j_3} + 3 \xi_{j_1} x_{j_3} x_{j_3}) + m^6 \xi_{j_1} x_{j_2} x_{j_3} x_{j_3} \right] \quad (5.38) \]

**Order 8:** For any distinct \( j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8 \in \{1, 2, \ldots, d\},

\[ \mathbb{E}[\xi_{1j_s}^8] = x_{j_1} \left[ m + 127m^2 \xi_{j_1} + 960m^3 \xi_{j_1}^2 + 1701m^4 \xi_{j_1}^3 + 1050m^5 \xi_{j_1}^4 + 2660m^6 \xi_{j_1}^5 + 28m^7 \xi_{j_1}^6 + m^8 \xi_{j_1}^7 \right] \quad (5.45) \]

\[ \mathbb{E}[\xi_{1j_s}^7 \xi_{2j_s}] = x_{j_1} x_{j_2} \left[ m + 63m^3 \xi_{j_1} + 301m^4 \xi_{j_1}^2 + 350m^5 \xi_{j_1}^3 + 140m^6 \xi_{j_1}^4 + 21m^7 \xi_{j_1}^5 + m^8 \xi_{j_1}^6 \right] \quad (5.46) \]

\[ \mathbb{E}[\xi_{1j_s}^6 \xi_{2j_s}^2] = x_{j_1} x_{j_2} \left[ m^2 + 31m^3 \xi_{j_1} + 90m^4 \xi_{j_1}^2 + 65m^5 \xi_{j_1}^3 + 15m^6 \xi_{j_1}^4 + m^7 \xi_{j_1}^5 \right] \quad (5.47) \]
\begin{align*}
E[\xi_j^4 \xi_j^2 \xi_j]\, &= \, x_{j1}x_{j2}x_{j3} \left[ m^{(3)} + \frac{1}{2}m^{(4)}(15x_{j1} + 3x_{j2}) + m^{(4)}(25x_{j1}^2 + 15x_{j1}x_{j2} + x_{j2}^2) \right], \\
E[\xi_j^4 \xi_j^2 \xi_j^2]\, &= \, x_{j1}x_{j2} \left[ m^{(2)} + m^{(3)}(15x_{j1} + 3x_{j2}) + m^{(4)}(25x_{j1}^2 + 15x_{j1}x_{j2} + x_{j2}^2) \right], \\
E[\xi_j^4 \xi_j^2 \xi_j^3]\, &= \, x_{j1}x_{j2}x_{j3} \left[ m^{(3)} + m^{(4)}(15x_{j1} + 3x_{j2}) + m^{(5)}(25x_{j1}^2 + 15x_{j1}x_{j2} + x_{j2}^2) \right].
\end{align*}

\begin{align*}
E[\xi_j^4 \xi_j^2 \xi_j^3 \xi_j]\, &= \, x_{j1}x_{j2}x_{j3} \left[ m^{(2)} + m^{(3)}(15x_{j1} + 3x_{j2}) + m^{(4)}(25x_{j1}^2 + 15x_{j1}x_{j2} + x_{j2}^2) \right], \\
E[\xi_j^4 \xi_j^2 \xi_j^3 \xi_j^2]\, &= \, x_{j1}x_{j2} \left[ m^{(2)} + m^{(3)}(7x_{j1} + 7x_{j2}) + m^{(4)}(6x_{j1}^2 + 99x_{j1}x_{j2} + 6x_{j2}^2) \right], \\
E[\xi_j^4 \xi_j^2 \xi_j^3 \xi_j^3]\, &= \, x_{j1}x_{j2}x_{j3} \left[ m^{(3)} + m^{(4)}(7x_{j1} + 3x_{j2}) + m^{(5)}(6x_{j1}^2 + 21x_{j1}x_{j2} + x_{j2}^2) \right].
\end{align*}
5.2. Computation of the central moments up to order 4

With the results of the previous subsection and some algebraic manipulations (or the formula in Theorem 2), we can now calculate the central moments explicitly. We could calculate them up to order 8, but it would be very tedious. Instead, we write them up to order 4 for the sake of brevity. The simplifications we make to obtain the boxed expressions below are done with Mathematica.

Order 2: For any distinct $j_1, j_2 \in \{1, 2, \ldots, d\}$,

$$
E[(\xi_{j_1} - E[\xi_{j_1}])^2] = E[\xi_{j_1}^2] - (E[\xi_{j_1}])^2 = x_{j_1} [m + m^{(2)} x_{j_1}] - m^2 x_{j_1}^2 = m x_{j_1} (1 - x_{j_1}) \tag{5.67}
$$

$$
E[(\xi_{j_1} - E[\xi_{j_1}])(\xi_{j_2} - E[\xi_{j_2}])] = E[\xi_{j_1} \xi_{j_2}] - E[\xi_{j_1}] E[\xi_{j_2}] = m^{(2)} x_{j_1} x_{j_2} - m x_{j_1} m x_{j_2} = -m x_{j_1} x_{j_2} \tag{5.68}
$$

Order 3: For any distinct $j_1, j_2, j_3 \in \{1, 2, \ldots, d\}$,

$$
E[(\xi_{j_1} - E[\xi_{j_1}])^3] = E[\xi_{j_1}^3] - 3 E[\xi_{j_1}^2] E[\xi_{j_1}] + 2 (E[\xi_{j_1}])^3
= x_{j_1} [m + 3 m^{(2)} x_{j_1} + m^{(3)} x_{j_1}^2] - 3 x_{j_1} [m + m^{(2)} x_{j_1}] m x_{j_1} + 2 m^3 x_{j_1}^3
= m x_{j_1} (x_{j_1} - 1)(2 x_{j_1} - 1) \tag{5.69}
$$

$$
E[(\xi_{j_1} - E[\xi_{j_1}])^2(\xi_{j_2} - E[\xi_{j_2}])] = E[\xi_{j_1}^2] E[\xi_{j_2}] - E[\xi_{j_1}^2] E[\xi_{j_2}] - 2 E[\xi_{j_1} \xi_{j_2}] E[\xi_{j_1}] + 2 (E[\xi_{j_1}])^2 E[\xi_{j_2}]
= x_{j_1} x_{j_2} [m^{(2)} + m^{(3)} x_{j_1}] - x_{j_1} [m + m^{(2)} x_{j_1}] m x_{j_2} - 2 m^{(2)} x_{j_1} x_{j_2} m x_{j_1} + 2 m^3 x_{j_1}^3 m x_{j_2}
= m x_{j_1} x_{j_2} (2 x_{j_1} - 1) \tag{5.70}
$$

$$
E[(\xi_{j_1} - E[\xi_{j_1}])(\xi_{j_2} - E[\xi_{j_2}])(\xi_{j_3} - E[\xi_{j_3}])] = E[\xi_{j_1} \xi_{j_2} \xi_{j_3}] - E[\xi_{j_1} \xi_{j_2}] E[\xi_{j_3}] - E[\xi_{j_1} \xi_{j_3}] E[\xi_{j_2}] - E[\xi_{j_2} \xi_{j_3}] E[\xi_{j_1}] + 2 E[\xi_{j_1}] E[\xi_{j_2}] E[\xi_{j_3}]
= m^{(3)} x_{j_1} x_{j_2} x_{j_3} - m^{(2)} x_{j_1} x_{j_2} m x_{j_3} - m^{(2)} x_{j_1} x_{j_3} m x_{j_2} - m^{(2)} x_{j_2} x_{j_3} m x_{j_1} + 2 m^3 x_{j_1} x_{j_2} x_{j_3}
= 2 m x_{j_1} x_{j_2} x_{j_3} \tag{5.71}
$$

Order 4: For any distinct $j_1, j_2, j_3, j_4 \in \{1, 2, \ldots, d\}$,

$$
E[(\xi_{j_1} - E[\xi_{j_1}])^4] = E[\xi_{j_1}^4] - 4 E[\xi_{j_1}^3] E[\xi_{j_1}] + 6 E[\xi_{j_1}^2] (E[\xi_{j_1}])^2 - 3 (E[\xi_{j_1}])^4
= x_{j_1} [m + 7 m^{(2)} x_{j_1} + 6 m^{(3)} x_{j_1}^2 + m^{(4)} x_{j_1}^3] - 4 x_{j_1} [m + 3 m^{(2)} x_{j_1} + m^{(3)} x_{j_1}^2] m x_{j_1}
+ 6 x_{j_1} [m + m^{(2)} x_{j_1}] (m x_{j_1})^2 - 3 m^4 x_{j_1}^4
= 3 m^2 x_{j_1}^2 (x_{j_1} - 1)^2 + m x_{j_1} (1 - x_{j_1})(6 x_{j_1}^2 - 6 x_{j_1} + 1) \tag{5.72}
$$

$$
E[(\xi_{j_1} - E[\xi_{j_1}])^3(\xi_{j_2} - E[\xi_{j_2}])] = E[\xi_{j_1}^3] E[\xi_{j_2}] - E[\xi_{j_1}^3] E[\xi_{j_2}] - E[\xi_{j_1}^2] E[\xi_{j_2}] + 3 E[\xi_{j_1}^2] (E[\xi_{j_1}])^2 - 3 (E[\xi_{j_1}])^3 E[\xi_{j_2}]
= x_{j_1} x_{j_2} [m^{(2)} + 3 m^{(3)} x_{j_1} + m^{(4)} x_{j_1}^2] - x_{j_1} [m + 3 m^{(2)} x_{j_1} + m^{(3)} x_{j_1}^2] m x_{j_2}
- 3 x_{j_1} x_{j_2} [m^{(2)} + m^{(3)} x_{j_1}] m x_{j_1} + 3 x_{j_1} [m + m^{(2)} x_{j_1}] m x_{j_1} m x_{j_2} + 3 m^{(2)} x_{j_1} x_{j_2} m^2 x_{j_1}^2 - 3 m^3 x_{j_1}^3 m x_{j_2}
= m x_{j_1} x_{j_2} (3 m - 2) x_{j_1} (x_{j_1} - 1) \tag{5.73}
$$

$$
E[(\xi_{j_1} - E[\xi_{j_1}])^2(\xi_{j_2} - E[\xi_{j_2}]^2)] = E[\xi_{j_1}^2] E[\xi_{j_2}]^2 - 2 E[\xi_{j_1}^2] (E[\xi_{j_2}])^2 - 2 E[\xi_{j_1}] (E[\xi_{j_2}])^3 + E[\xi_{j_1}^2] (E[\xi_{j_2}])^2 + E[\xi_{j_2}]^2 (E[\xi_{j_1}])^2
= \ldots \tag{5.74}
$$
\begin{align*}
&+ 4 E(\xi_1, \xi_2) E(\xi_1) E(\xi_2) \left(3 - (E(\xi_1))^2 - (E(\xi_2))^2 \right)
&= x_1 x_2 \left[ m(2) + m(3) x_1 x_2 + m(4) x_1 x_2 \right] - 2 x_1 x_2 \left[ m(2) + m(3) x_1 \right] m x_2
&\quad + 2 x_1 x_2 \left[ m(2) + m(3) x_2 \right] m x_1 + x_1 \left[ m + m(2) x_1 \right] m^2 x_1 + x_2 \left[ m + m(2) x_2 \right] m^2 x_2
&\quad + 4 m(2) x_1 x_2 m x_1 m x_2 - 3 m^2 x_1^2 m^2 x_2^2
&= \frac{(m-2) x_1 x_2 (3x_1 x_2 - (x_1 + x_2) + 1) + m x_1 x_2}{(5.74)}
\end{align*}

\begin{align*}
E(\xi_3) &= E(\xi_1)^2 (\xi_3 - E(\xi_2))(\xi_3 - E(\xi_1))
\end{align*}

\begin{align*}
&= E(\xi_1, \xi_2, \xi_3, E(\xi_1) - E(\xi_2)) \left(2 - 2 E(\xi_1, \xi_2, \xi_3, \xi_4, E(\xi_3) - E(\xi_3) \right) - 3 (E(\xi_1))^2 E(\xi_2) E(\xi_3)
&= x_1 x_2 x_3 \left[ m(3) + m(4) x_1 - x_1 x_2 \right] \left[ m(2) + m(3) x_2 \right] m x_3 + x_2 \left[ m + m(2) x_3 \right] m^2 x_3
&\quad + 2 m(2) x_1 x_3 m^2 x_1 m^2 x_3
&= \frac{(m-2) x_1 x_2 x_3 (3x_1 x_2 x_3 - 1)}{(5.75)}
\end{align*}

\begin{align*}
E(\xi_3) &= E(\xi_1)^2 (\xi_3 - E(\xi_2))(\xi_3 - E(\xi_1))
\end{align*}

\begin{align*}
&= E(\xi_1, \xi_2, \xi_3, E(\xi_1) - E(\xi_2)) \left(2 - 2 E(\xi_1, \xi_2, \xi_3, \xi_4, E(\xi_3) - E(\xi_3) \right) - 3 (E(\xi_1))^2 E(\xi_2) E(\xi_3)
&= x_1 x_2 x_3 \left[ m(3) + m(4) x_1 - x_1 x_2 \right] \left[ m(2) + m(3) x_2 \right] m x_3 + x_2 \left[ m + m(2) x_3 \right] m^2 x_3
&\quad + 2 m(2) x_1 x_3 m^2 x_1 m^2 x_3
&= \frac{(m-2) x_1 x_2 x_3 (3x_1 x_2 x_3 - 1)}{(5.76)}
\end{align*}

6. Conclusion

In this short paper, we found general formulas for the central and non-central moments of the multinomial distribution as well as explicit formulas for all the non-central moments up to order 8 and all the central moments up to order 4. Our work expands on the results in Newcomer (2008), where the central moments were calculated up to order 4. It also complements the general formula for the (joint) factorial moments from Mosimann (1962) and the explicit formulas for some of the lower-order (mixed) cumulants that were presented in Wishart (1949).

References

Aitchison, J., & Lauter, I. J. 1985. Kernel Density Estimation for Compositional Data. *Journal of the Royal Statistical Society. Series C (Applied Statistics)*, 34(2), 129–137. doi:10.2307/2347365.

Arenhaev, N. K. 1976. Asymptotic behavior of the multinomial distribution. *Teor. Verojatnost. i Primenen.*, 21(4), 826–831. MR0478288.

Babu, G. J., Canty, A. J., & Chaubey, Y. P. 2002. Application of Bernstein polynomials for smooth estimation of a distribution and density function. *J. Statist. Plann. Inference*, 105(2), 377–392. MR1910059.

Belalia, M., Bouzamani, T., & Leblanc, A. 2017. Smooth conditional distribution estimators using Bernstein polynomials. *Comput. Statist. Data Anal.*, 111, 166–182. MR3630225.

Belalia, M., Bouzamani, T., & Leblanc, A. 2019. Bernstein conditional density estimation with application to conditional distribution and regression functions. *J. Korean Statist. Soc.*, 48(3), 356–383. MR3983257.

Bouzamani, T., & Rolin, J.-M. 2003. Consistency of the beta kernel density function estimator. *Canad. J. Statist.*, 31(1), 89–98. MR1985506.

Bouzamani, T., & Rolin, J.-M. 2007. Bernstein estimator for unbounded density function. *J. Nonparametr. Stat.*, 19(3), 145–161. MR2351744.

Bouzamani, T. M., Mesfioui, & Rolin, J. M. 2007. L1-rate of convergence of smoothed histogram. *Statist. Probab. Lett.*, 77(14), 1497–1504. MR2395509.

Brown, B. M., & Chen, S. X. 1999. Beta-Bernstein smoothing for regression curves with compact support. *Scand. J. Statist.*, 26(1), 47–59. MR1685301.

Chen, S. X. 1999. Beta kernel estimators for density functions. *Comput. Statist. Data Anal.*, 31(2), 131–145. MR1718494.

Chen, S. X. 2000. Beta kernel smoothers for regression curves. *Statist. Sinica*, 10(1), 73–91. MR1742101.

Currie, S. M., & Ghoosh, S. K. 2011. A variable selection approach to monotonic regression with Bernstein polynomials. *J. Appl. Stat.*, 38(5), 961–976. MR2782409.
Gawronski, W. 1985. Strong laws for density estimators of Bernstein type. *Period. Math. Hungar.*, **16**(1), 23–43. MR0791719.

Gawronski, W., & Stadtmüller, U. 1981. Smoothing histograms by means of lattice and continuous distributions. *Metrika*, **28**(3), 155–164. MR0638651.

Ghosal, S. 2001. Convergence rates for density estimation with Bernstein polynomials. *Ann. Statist.*, **29**(5), 1264–1280. MR1873330.

Graham, R. L., Knuth, D. E., & Patashnik, O. 1994. *Concrete mathematics*. Second edn. Addison-Wesley Publishing Company, Reading, MA. MR1397498.

Guan, Z. 2016. Efficient and robust density estimation using Bernstein type polynomials. *J. Nonparametr. Stat.*, **28**(2), 250–271. MR3488598.

Guan, Z. 2017. Bernstein polynomial model for grouped continuous data. *J. Nonparametr. Stat.*, **29**(4), 831–848. MR3740722.

Igarashi, G., & Kakiwawa, Y. 2014. On improving convergence rate of Bernstein polynomial density estimator. *J. Nonparametr. Stat.*, **26**(1), 61–84. MR3174309.

Johnson, N. L., Kotz, S., & Balakrishnan, N. 1997. *Discrete multivariate distributions*. Wiley Series in Probability and Statistics: Applied Probability and Statistics. John Wiley & Sons, Inc., New York. MR1429617.

Kakiwawa, Y. 2004. Bernstein polynomial probability density estimation. *J. Nonparametr. Stat.*, **16**(5), 709–729. MR2068610.

Leblanc, A. 2009. Chung-Smirnov property for Bernstein estimators of distribution functions. *J. Nonparametr. Stat.*, **21**(2), 133–142. MR2488150.

Leblanc, A. 2010. A bias-reduced approach to density estimation using Bernstein polynomials. *J. Nonparametr. Stat.*, **22**(3-4), 459–475. MR2662607.

Leblanc, A. 2012a. On estimating distribution functions using Bernstein polynomials. *Ann. Inst. Statist. Math.*, **64**(5), 919–943. MR2960052.

Leblanc, A. 2012b. On the boundary properties of Bernstein polynomial estimators of density and distribution functions. *J. Statist. Plann. Inference*, **142**(10), 2762–2778. MR2925964.

Lu, L. 2015. On the uniform consistency of the Bernstein density estimator. *Statist. Probab. Lett.*, **107**, 52–61. MR3412755.

Mosimann, J. E. 1962. On the compound multinomial distribution, the multivariate hypergeometric law, and the study of proportions. *Biometrika*, **49**, 65–82. MR143299.

Newcomer, J. T. 2008. *Estimation procedures for multinomial models with overdispersion*. PhD thesis, University of Maryland.

Newcomer, J. T., Neerchal, N. K., & Morel, J. G. 2008. Computation of higher order moments from two multinomial overdispersion likelihood models. Preprint, 1–11. [URL] http://www.math.umbc.edu/~kogan/technical_papers/2008/Newcomer_Nagaraj_Morel.pdf.

Ouimet, F. 2019. Extreme values of log-correlated random fields and the Riemann–zeta function, and some asymptotic results for variational estimators in statistics. PhD thesis, Université de Montréal. http://hdl.handle.net/1866/22667.

Ouimet, F. 2020a. Asymptotic properties of Bernstein estimators on the simplex. Preprint, 1–27. arXiv:2002.07758.

Ouimet, F. 2020b. Asymptotic properties of Bernstein estimators on the simplex. Part 2: the boundary case. *Preprint*, 1–23. arXiv:2006.11756.

Ouimet, F. 2020c. Density estimation using Dirichlet kernels. Preprint, 1–39. arXiv:2002.06956.

Ouimet, F. 2020d. Explicit formulas for the joint third and fourth central moments of the multinomial distribution. Preprint, 1–7. arXiv:2006.09059.

Ouimet, F. 2020e. A precise local limit theorem for the multinomial distribution and some applications. *Preprint*, 1–20. arXiv:2001.08512.

Petrone, S. 1999a. Bayesian density estimation using Bernstein polynomials. *Canad. J. Statist.*, **27**(1), 105–126. MR1703623.

Petrone, S. 1999b. Random Bernstein polynomials. *Scand. J. Statist.*, **26**(3), 373–393. MR1712051.

Petrone, S., & Wasserman, L. 2002. Consistency of Bernstein polynomial posteriors. *J. Roy. Statist. Soc. Ser. B*, **64**(1), 79–100. MR1881846.

Stadtmüller, U. 1983. Asymptotic distributions of smoothed histograms. *Metrika*, **30**(3), 145–158. MR0726014.

Stadtmüller, U. 1986. Asymptotic properties of nonparametric curve estimates. *Period. Math. Hungar.*, **17**(2), 83–108. MR0858310.

Tenbusch, A. 1994. Two-dimensional Bernstein polynomial density estimators. *Metrika*, **41**(3-4), 233–253. MR1293514.

Tenbusch, A. 1997. Nonparametric curve estimation with Bernstein estimates. *Metrika*, **45**(1), 1–30. MR1437794.

Turnbull, B. C., & Ghosh, S. K. 2014. Unimodal density estimation using Bernstein polynomials. *Comput. Statist. Data Anal.*, **72**, 13–29. MR3139345.

Vitale, R. A. 1975. Bernstein polynomial approach to density function estimation. Pages 87–99 of: *Statistical Inference and Related Topics*. Academic Press, New York. MR0397097.

Wishart, J. 1949. Cumulants of multivariate multinomial distributions. *Biometrika*, **36**, 47–58. MR33996.