On y-closed Dual Rickart Modules

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Abstract
In this paper, we develop the work of Ghawi on close dual Rickart modules and discuss y-closed dual Rickart modules with some properties. Then, we prove that, if $M_1$ and $M_2$ are y-closed simple $R$-modules and if $M_1$ is $M_2$-y-closed is a dual Rickart module, then either $\text{Hom}(M_1, M_2) = 0$ or $M_1 \cong M_2$. Also, we study the direct sum of y-closed dual Rickart modules.

Keywords: Endomorphism ring, y-closed submodule, Image of endomorphism, y-closed simple, y-closed dual Rickart modules.

1. INTRODUCTION
A module $M$ is called a dual Rickart module if for every $\varphi \in \text{End}(M)$, then $\text{Im} \varphi = eM$ for some $e^2 = e \in S$. Equivalently, a module $M$ is a dual Rickart module if and only if for every $\varphi \in \text{End}(M)$, then $\text{Im} \varphi$ is a direct summand of $M$ [1]. A module $M$ is called a closed dual Rickart module, if for any $f \in \text{End}(M)$, $\text{Im} f$ is a closed submodule in $M$ [1]. Recall that a submodule $A$ of an $R$-module $M$ is called a y-closed submodule of $M$ if $A$ is nonsingular [2]. It is known that every y-closed submodule is closed.

In this paper, we give some results on the y-closed dual Rickart modules.

In section 2, we give the definition of the y-closed dual Rickart modules with some examples and basic properties. Moreover, we prove that for two $R$-modules $M$ and $N$, and let $B$ be a submodule of $N$ if $M$ is $N$-y-closed dual Rickart module, then $M$ is $B$-y-closed dual Rickart module, see proposition (2.4).

In section 3, we study the direct sum of y-closed dual Rickart modules. Furthermore, we prove that, let $M$ and $N$ be two $R$-modules, such that $M = A \oplus B$ if $M$ is $N$-y-closed dual Rickart module, then $A$ is $L$-y-closed dual Rickart module, for every submodule $L$ of $N$, see proposition (3.1).

Throughout this article, $R$ is a ring with identity and $M$ is a unital left $R$-module. For a left module $M$, $S = \text{End}_R (M)$ will denote the endomorphism ring of $M$.

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§2: Y-CLOSED DUAL RICKART MODULES

In this section, we introduce the concept of the y-closed dual Rickart modules and we illustrate it by some examples. Also, we give some basic properties. We start by the definition.

Definition 2.1: Let $M$ and $N$ be two $R$-modules. We say that $M$ is $N$-y-closed dual Rickart module if for every homomorphism $0 \neq f: M \rightarrow N$, $Im f$ is a y-closed submodule of $N$.

For a module $M$. If $M$ is $M$-y-closed dual Rickart module, then we say that $M$ is a y-closed dual Rickart module.

Examples 2.2

1- The module $Z_2 \oplus Z_2$ as $Z_2$-module is $Z_2$-y-closed dual Rickart module. To show that, let $0 \neq f: Z_2 \oplus Z_2 \rightarrow Z_2$ be any $R$-homomorphism. Then $Im f = Z_2$ is a y-closed submodule of $Z_2$.

2- Consider the module $Z$ as $Z$-module and let $f: Z \rightarrow Z$ be a map defined by $f(n) = 4n$, $\forall n \in Z$. It is clear that $f$ is an $R$-homomorphism and $Im f = 4Z$. But $\frac{Z}{4Z} = Z_4$ and $Z_4$ as $Z$-module is singular, therefore $Im f = 4Z$ is not a y-closed submodule of $Z$. Thus $Z$ is not y-closed dual Rickart module.

3- Consider the modules $Z_p$ and $Z_{p\infty}$ as $Z$-modules. The module $Z_p$ is not $Z_{p\infty}$-y-closed dual Rickart module. To show that, let $i: \frac{1}{p}Z \rightarrow Z_{p\infty}$ be the inclusion map. Since $Z_{p\infty}$ is singular, then $\frac{Z_{p\infty}}{\frac{1}{p}Z}$ is singular, by [2]. Therefore $\frac{1}{p}Z$ is not a y-closed submodule of $Z_{p\infty}$. But $Z_p \simeq \frac{1}{p}Z$ as $Z$-module, therefore, is not $Z_{p\infty}$-y-closed dual Rickart module.

Remark 2.3: A dual Rickart module needs not to be a y-closed dual Rickart module. For example, the module $Z_6$ as $Z$-module is a dual Rickart module, where $Z_6$ as $Z$-module is semisimple. Claim that $Z_6$ as $Z$-module is not y-closed dual Rickart module. To show that, let $f: Z_6 \rightarrow Z_6$ be a map defined by $f(x) = 2x$, $\forall x \in Z_6$. It is clear that $f$ is a homomorphism and $Im f = \{0, 2, 4\}$. But $\frac{Z_6}{Im f} \simeq Z_2$ and $Z_2$ as $Z$-module is singular, therefore $Im f$ is not a y-closed submodule of $Z_6$. Thus $Z_6$ is not y-closed dual Rickart module.

Proposition 2.4: Let $M$ and $N$ be two $R$-modules and let $B$ be a submodule of $N$. If $M$ is $N$-y-closed dual Rickart module, then $M$ is $B$-y-closed dual Rickart module.

Proof. Let $f: M \rightarrow B$ be an $R$-homomorphism and let $i: B \rightarrow N$ be the inclusion map. Consider the map $i \circ f: M \rightarrow N$. Since $M$ is $N$-y-closed dual Rickart module, then $Im f = Im i \circ f$ is a y-closed submodule of $N$ and hence $\frac{N}{Im f}$ is nonsingular. But $\frac{N}{Im f}$ is a submodule of $\frac{N}{Im f}$, therefore $\frac{B}{Im f}$ is nonsingular and hence $Im f$ is a y-closed submodule of $B$. Thus $M$ is $B$-y-closed dual Rickart module.

Definition 2.5: Let $M$ be an $R$-module, then $M$ is called a y-closed simple if $M$ and $0$ are the only y-closed submodules of $M$.

Proposition 2.6: Let $M$ be an $R$-module and let $N$ be a y-closed simple $R$-module. If $M$ is $N$-y-closed dual Rickart module, then either

1. $\text{Hom}(M,N)=0$ or
2. Every nonzero $R$-homomorphism from $M$ to $N$ is an epimorphism.

Proof. Assume that $\text{Hom}(M,N) \neq 0$ and let $f: M \rightarrow N$ be a non-zero $R$-homomorphism. Since $M$ is $N$-y-closed dual Rickart, then $Im f$ is y-closed submodule of $N$. But $N$ is y-closed simple, therefore $Im f = N$ and $f$ is an epimorphism.

Recall that an $R$-module $M$ is called a Co-Quasi-Dedekind $R$-module if every nonzero endomorphism of $M$ is an epimorphism, see[3, p2].

Proposition 2.7: Let $M_1$ and $M_2$ be $R$-modules such that $M_2$ is y-closed simple and $M_1$ is $M_2$-y-closed dual Rickart module. If $\text{Hom}(M_1,M_2) \neq 0$, then $M_2$ is Co-Quasi-Dedekind $R$-module.

Proof. Assume that there is an $R$-homomorphism $0 \neq f: M_1 \rightarrow M_2$. Then by proposition (2.6), $Im f = M_2$. Now let $0 \neq g: M_2 \rightarrow M_2$ be an $R$-homomorphism. Consider the map $g \circ f: M_1 \rightarrow M_2$. Since $M_1$ is $M_2$-y-closed dual Rickart module, then $Im g \circ f$ is a y-closed submodule of $M_2$. But $f$ is an epimorphism, therefore $Im g \circ f = Im g$ is a y-closed submodule of $M_2$. Since $M_2$ is y-closed simple, then $Im g = M_2$. Thus $M_2$ is Co-Quasi-Dedekind $R$-module.

Proposition 2.8: Let $M_1$ and $M_2$ be y-closed simple $R$-modules. If $M_1$ is $M_2$ y-closed dual Rickart module, then either $\text{Hom}(M_1,M_2)=0$ or $M_1 \cong M_2$.

Proof. Assume that $\text{Hom}(M_1,M_2) \neq 0$ and let $0 \neq f: M_1 \rightarrow M_2$ be an $R$-homomorphism. Since $M_1$ is $M_2$ y-closed dual Rickart module, then by Proposition (2.7), $f$ is an epimorphism. Now consider
the following short exact sequence
\[ 0 \rightarrow \ker f \rightarrow M_1 \rightarrow f \rightarrow M_2 \rightarrow 0 \]
where \( i \) is the inclusion map. Since \( M_2 \) is nonsingular, then \( M_2 \cong \frac{M_1}{\ker f} \) is nonsingular. Hence \( \ker f \) is y-closed submodule of \( M_1 \). But \( M_1 \) is y-closed simple and \( M_1 \neq \ker f \), therefore \( \ker f = 0 \). Thus \( M_1 \cong M_2 \).

§3 DIRECT SUM OF Y-CLOSED DUAL RICKART MODULES
In this section, we study the direct sum of the y-closed dual Rickart modules. We begin with the following theorem.

**Theorem 3.1:** Let \( M \) and \( N \) be two \( R \)-modules such that \( M = A \oplus B \). If \( M \) is \( N \)-y-closed dual Rickart module, then \( A \) is \( L \)-y-closed dual Rickart module for every submodule \( L \) of \( N \).

**Proof.** Let \( M \) be \( N \)-y-closed dual Rickart module and \( f : A \rightarrow L \) be an \( R \)-homomorphism. Let \( p : M \rightarrow A \) be the projection map and \( i : L \rightarrow N \) be the inclusion map. Consider the map \( (i \circ f \circ p) : M \rightarrow N \). Since \( M \) is \( N \)-y-closed dual Rickart, then \( \text{Im}(i \circ f \circ p) \) is a y-closed submodule of \( N \). But
\[ \text{Im}(i \circ f \circ p) = \{ i \circ f \circ p(x), \ x \in M \} \]
\[ = \{ i(f(p(a + b))), \ a \in A, b \in B \} \]
\[ = \{ f(a), \ a \in A \} = \text{Im} f \]
Therefore \( \text{Im}(i \circ f \circ p) = \text{Im} f \) is a y-closed submodule of \( N \). Hence \( \text{Im} f \) is a y-closed submodule of \( L \). Thus \( A \) is \( L \)-y-closed dual Rickart module.

**Proposition 3.2:** Let \( M = \bigoplus_{i \in I} M_i \) and \( N = \bigoplus_{i \in I} N_i \) be two \( R \)-modules, such that \( f(M_i) \subseteq N_i, \forall i \in I \). Then \( M \) is \( N \)-y-closed dual Rickart module if and only if each \( M_i \) is \( N_i \)-y-closed dual Rickart module.

**Proof.** \( \Rightarrow \) Clear by Propositions (2.4) and (3.1)

For the converse, let \( f : M \rightarrow N \) be an \( R \)-homomorphism. We want to show that \( \text{Im} f \) is a y-closed submodule of \( N \). Since \( f(M_i) \subseteq N_i, \forall i \in I \), then we can consider \( f\big|_{M_i} : M_i \rightarrow N_i \). First, claim that \( \text{Im}(f\big|_{M_i}) = \text{Im} f \cap N_i, \forall i \in I \). To show that, let \( f(x_i) \in \text{Im}(f\big|_{M_i}), x_i \in M_i \), then \( f(x_i) \in (\text{Im} f \cap N_i) \). Now let \( f(x) \in (\text{Im} f \cap N_i) \). Then \( x = \sum_{j \in I} x_j \), where \( x_j \in M_j \), for each \( j \in I \) and \( x_j \neq 0 \) for at most a finite number of \( j \in I \). Now \( f(x) = f(\sum_{j \in I} x_j) = \sum_{j \in I} f(x_j) \in \bigoplus_{i \in I} N_i \). But \( f(x) \in N_i \). Therefore \( f(x_j) = 0, \forall j \neq i \) and \( f(x) = f(x_i) \). Hence \( f(x) \in \text{Im}(f\big|_{M_i}) \). Thus \( \text{Im}(f\big|_{M_i}) = \text{Im} f \cap N_i, \forall i \in I \). Claim that \( \text{Im} f = \bigoplus_{i \in I} (\text{Im} f \cap N_i) \). To show that, let \( f(x) \in \text{Im} f, x \in M \). Then \( x = \sum_{i \in I} x_i \), where \( x_i \in M_i, \forall i \in I \) and \( x_i \neq 0 \), for at most a finite number of \( i \in I \). Hence \( f(x) = f(\sum_{i \in I} x_i) = \sum_{i \in I} f(x_i) \), for all \( i \in I \). By our assumption \( f(x_i) \in \text{Im} f \cap N_i, \forall i \in I \). Hence \( f(x) \in \bigoplus_{i \in I} (\text{Im} f \cap N_i) \). Thus \( \text{Im} f = \bigoplus_{i \in I} (\text{Im} f \cap N_i) = \bigoplus_{i \in I} \text{Im}(f\big|_{M_i}) \). Since \( M_i \) is \( N_i \)-y-closed dual Rickart module, for each \( i \in I \), then \( \text{Im}(f\big|_{M_i}) \) is a y-closed submodule of \( N_i \) and hence \( \bigoplus_{i \in I} \text{Im}(f\big|_{M_i}) \) is a y-closed submodule of \( N \). By [4, proposition (2.1.20), p29]. So, \( \text{Im} f \) is a y-closed submodule of \( N \). Thus \( M \) is \( N \)-y-closed dual Rickart module.

**Proposition 3.3:** Let \( M, N \) be two \( R \)-modules with the property that the sum of any two y-closed submodule of \( N \) is a y-closed submodule of \( N \). The following statements are equivalent
(a) \( M \) is a y-closed dual Rickart module,
(b) \( \sum_{f \in I} f(M) \) is a y-closed submodule of \( M \), where \( I \) is a finitely generated left ideal of \( \text{End}_R(M) \).

**Proof.** (a) \( \Rightarrow \) (b). Let \( I = (f_1, \ldots, f_n) \) be a finitely generated left ideal of \( \text{End}_R(M) \). Since \( M \) is a y-closed dual Rickart module, then \( \text{Im}(f_j) \) is a y-closed submodule of \( N, \forall 1 \leq j \leq n \). But \( \text{Im}(f_j) = \text{Im}(f_1 + \cdots + f_n) \). Hence \( \sum_{j=1}^n f_j(M) \) is a y-closed submodule of \( N \).

(b) \( \Rightarrow \) (a). Clear.
Recall that an $R$-module $M$ is called a faithful module if $ann(M) = 0$, where $ann(M) = \{ r \in R \mid rx = 0, \forall x \in M \}$, see [5, p206].

Before we give our next result, let us recall that an $R$-module $M$ is called dualizable if $\text{Hom}(M, R) \neq 0$, see [6, p10].

**Proposition 3.4:** Let $M$ be a $y$-closed simple, faithful $R$-module. If $M$ is $y$-closed dual Rickart module, then $M$ is divisible.

**Proof.** Suppose that $M$ is $y$-closed simple, faithful and $y$-closed dual Rickart module. Let $R$ must be commutative. $f(m) = rm$, $\forall m \in M$. It is clear that $f$ is an epimorphism. Since $M$ is a $y$-closed dual Rickart module, then $Imf = rM$ is a $y$-closed submodule of $M$. Since $M$ is a faithful module, then $rM \neq 0$. But $M$ is $y$-closed simple, therefore $rM = M$. Thus $M$ is divisible.

Recall that an $R$-module $M$ is called 1/2 cancellation module if it is faithful and for any ideal $A$ of $R$ such that $AM = M$ implies $A = R$, see [7].

**Proposition 3.5:** Let $M$ be a faithful, finitely generated and $y$-closed simple $R$-module, where $R$ is not a field. Then $M$ is not $y$-closed dual Rickart module.

**Proof.** Assume that $M$ is a $y$-closed dual Rickart module and let $0 \neq r \in R$ such that $R = (r)$. Define $f:M \rightarrow M$ by $f(m) = rm$, $\forall m \in M$. It is clear that $f$ is an epimorphism, then $Imf = rM$ is a $y$-closed submodule of $M$. Since $M$ is a faithful module, then $rM \neq 0$. But $M$ is an $y$-closed simple module, therefore $rM = M$. Since $M$ is finitely generated and faithful, then $M$ is 1/2 cancellation, by [7]. So, $R = (r)$, which is a contradiction. Thus $M$ is not $y$-closed dual Rickart module.

**Proposition 3.6:** Let $M$ be an $R$-module such that $R$ is $M$-$y$-closed dual Rickart module. Then every cyclic submodule of $M$ is a $y$-closed submodule.

**Proof.** Suppose that $M$ is an $R$-module such that $R$ is $M$-$y$-closed dual Rickart module and let $0 \neq m \in M$. Define $f : R \rightarrow Rm$ by $f(r) = rm$, $r \in R$. Let $i : Rm \rightarrow M$ be the inclusion map. Consider the map $i \circ f : R \rightarrow M$. It is clear that $Im(i \circ f) = Rm$. Since $R$ is $M$-$y$-closed dual Rickart, then $Im i \circ f$ is a $y$-closed submodule of $M$. Thus $Rm$ is a $y$-closed submodule of $M$.

Recall that an $R$-module $M$ is called $y$-extending if for any submodule $A$ of $M$ there exists a direct summand $K$ of $M$ such that $A \cap K$ is essential in $A$ and $A \cap K$ is essential in $K$, see [8].

**Proposition 3.7:** Let $M$ be a $y$-extending $R$-module. If $\bigoplus_i R$ is $M$-$y$-closed dual Rickart module, for every index set $I$, then $M$ is a semisimple module.

**Proof.** Let $N$ be a submodule of $M$ and let $\{n_{\alpha}; \alpha \in \Lambda \}$ be a set of generators of $N$. For each $\alpha \in \Lambda$, define $f_\alpha : R \rightarrow Rn_\alpha$ by $f_\alpha(r) = rn_\alpha$, $\forall r \in R$. Now define $f : \bigoplus_i R \rightarrow N$ by $(f_\alpha)_{\alpha \in \Lambda}(r) = \sum_{\alpha \in \Lambda} r_\alpha n_\alpha$. Its is clear that $f$ is an epimorphism. Let $i : N \rightarrow M$ be the inclusion map. Consider the map $i \circ f : \bigoplus_i R \rightarrow M$. Since $\bigoplus_i R$ is $M$-$y$-closed dual Rickart module, then $Im i \circ f = N$ is a $y$-closed submodule of $M$. But $M$ is a $y$-extending module, therefore $N$ is a direct summand of $M$. Thus $M$ is semisimple.

**Proposition 3.8:** Let $M$ be $y$-extending and a self-generator $R$-module. If $\bigoplus_i M$ is a $y$-closed dual Rickart module for every index set $I$, then $M$ is semisimple.

**Proof.** Let $N$ be a submodule of $M$. Since $M$ is a self-generator, then there exists a family $\{f_\alpha\}_{\alpha \in \Lambda}$ where $f_\alpha : M \rightarrow N$ is an $R$-homomorphism such that $\sum_{\alpha \in \Lambda} Imf_\alpha = N$. Define $f : \bigoplus_{\alpha \in \Lambda} M_\alpha \rightarrow N$ by $f((m_\alpha)_{\alpha \in \Lambda}) = \sum_{\alpha \in \Lambda} f_\alpha(m_\alpha)$. It is clear that $f$ is an epimorphism. Let $i : N \rightarrow M$ be the inclusion map. Consider the map $i \circ f : \bigoplus_{\alpha \in \Lambda} M_\alpha \rightarrow M$. Since $\bigoplus_{\alpha \in \Lambda} M_\alpha M$ is a $y$-closed dual Rickart module, then $Im i \circ f = Imf = N$ is a $y$-closed submodule of $M$. But $M$ is a $y$-extending module, therefore $N$ is a direct summand of $M$. Thus $M$ is semisimple.

Now, we give the following characterization.

**Theorem 3.9:** Let $M_1$ and $M_2$ be two $R$-modules. Then the following statements are equivalent.

1. $M_1$ is $M_2$-$y$-closed dual Rickart module;
2. For every submodule $N$ of $M_2$, every direct summand $K$ of $M_1$ is $N$-$y$-closed dual Rickart module;
3. For every direct summand $K$ of $M_1$, every $y$-closed submodule $L$ of $M_2$, and for every $f \in \text{Hom}_R(M_1, L)$, the Image of the restricted map $f|_K$ is a $y$-closed submodule of $K$.

**Proof.** (1) $\Rightarrow$ (2) Let $K$ be a direct summand of $M_1$, $N$ be a submodule of $M_2$, and $f : K \rightarrow N$ be an $R$-homomorphism. Let $M = K \bigoplus K_1$ for some submodule $K_1$ of $M$. Define $g: M_1 \rightarrow M_2$ by

$$
g(x) = \begin{cases} f(x), & \text{if } x \in K \\ 0, & \text{if } x \in K_1 \end{cases}
$$
Clearly, \( g \) is an \( R \)-homomorphism. Since \( M_1 \) is \( M_2 \)-\( y \)-closed dual Rickart module, then \( \text{Im} g \) is a \( y \)-closed submodule of \( M_2 \) and hence \( \frac{M_2}{\text{Im} g} \) is nonsingular. But, 
\[
\text{Im} g = \{ g(a + b), \quad a \in K, \quad b \in K_1 \} = \{ f(a), \quad a \in K \} = \text{Im} f. \quad \text{So} \quad \text{Im} f \text{ is a } y\text{-closed submodule of } M_2. \quad \text{Hence} \quad \frac{M_2}{\text{Im} f} \text{ is nonsingular. But } \frac{N}{\text{Im} f} \text{ is a submodule of } \frac{M_2}{\text{Im} f}, \text{ therefore } \frac{N}{\text{Im} f} \text{ is nonsingular. Thus } \text{Im} f \text{ is a } y\text{-closed submodule of } N.
\]

(2) \( \Rightarrow \) (3). Let \( K \) be a direct summand of \( M_3 \) and \( L \) is \( y \)-closed submodule of \( M_2 \). Let \( f: M_1 \to L \) be \( R \)-homomorphism. Since \( f|_K: K \to L \) and \( K \) is \( L \)-\( y \)-closed dual Rickart module, then \( \text{Im}(f|_K) \) is a \( y \)-closed submodule of \( L \).

(3) \( \Rightarrow \) (1) Let \( f: M_1 \to M_2 \) be an \( R \)-homomorphism. Take \( K = M_1 \) and \( L = M_2 \). Since \( f|_K: K \to L \) and \( L \) is a \( y \)-closed submodule of \( M_2 \), therefore \( \text{Im} f \) is a \( y \)-closed submodule of \( M_2 \). Thus \( M_2 \) is \( M_2 \)-\( y \)-closed dual Rickart module.

**Remark 3.10:** Let \( M \) and \( N \) be two \( R \)-modules and \( f: M \to N \) be an \( R \)-homomorphism. Let \( A_M = M \oplus 0, B_N = 0 \oplus N \). \( \tilde{f}: A_M \to B_N \) be a map defined by \( \tilde{f}(m,0) = (0,f(m)) \), for every \( m \in M \) and \( T_f = \{ x + \tilde{f}(x), x \in A_M \} \). Then,

1- \( M \oplus N = A_M \oplus B_N \)
2- \( \tilde{f} \) is an \( R \)-homomorphism
3- \( \ker \tilde{f} = \ker f \oplus 0 \)
4- \( T_f \) is a submodule of \( M \oplus N \)
5- \( A_M + T_f = A_M \oplus \text{Im} \tilde{f} \).

In this paper, by \( A_M, B_M, \tilde{f}, T_f \), we mean the same concepts in the previous above remark.

Now, we will give characterization for the notion that \( M \) is \( N \)-\( y \)-closed dual Rickart module.

**Theorem 3.11:** Let \( M \) and \( N \) be two \( R \)-modules. Then \( M \) is \( N \)-\( y \)-closed dual Rickart module if and only if, for every homomorphism \( f: M \to N \), \( A_M + T_f \) is a \( y \)-closed of \( M \oplus N \).

**Proof.** Let \( f: M \to N \) be an \( R \)-homomorphism. Since \( M \) is \( N \)-\( y \)-closed dual Rickart, then \( \text{Im} f \) is a \( y \)-closed submodule of \( N \) and so \( 0 \oplus \text{Im} f \) is \( y \)-closed submodule of \( 0 \oplus N \). Therefore \( \text{Im} \tilde{f} \) is \( y \)-closed submodule of \( \text{Im} f \). Hence \( A_M \oplus \text{Im} \tilde{f} \) is \( y \)-closed submodule of \( A_M \oplus B_N \). So \( A_M \oplus \text{Im} \tilde{f} \) is \( y \)-closed submodule of \( M \oplus N \). By the same argument of the proof of the theorem in [9, Theorem(2.2)], \( A_M \oplus \text{Im} \tilde{f} = A_M + T_f \). Thus \( A_M + T_f \) is a \( y \)-closed sumodule of \( M \oplus N \).

For the converse, Let \( f: M \to N \) be an \( R \)-homomorphism. Since \( A_M + T_f \) is \( y \)-closed submodule of \( M \oplus N \), and \( A_M + T_f = A_M \oplus \text{Im} \tilde{f} \), therefore \( \frac{M \oplus N}{A_M + T_f} = \frac{A_M \oplus B_N}{A_M \oplus \text{Im} \tilde{f}} \simeq \frac{B_N}{\text{Im} \tilde{f}} \) is nonsingular. Therefore \( \text{Im} \tilde{f} \) is \( y \)-closed submodule of \( 0 \oplus N \). Hence \( \frac{0 \oplus N}{\text{Im} \tilde{f}} \simeq \frac{\text{Im} \tilde{f}}{\text{Im} \tilde{f}} \simeq \frac{\frac{N}{\text{Im} \tilde{f}}}{\text{Im} \tilde{f}} \) is nonsingular. So \( \text{Im} f \) is \( y \)-closed submodule of \( M \). Thus \( M \) is \( N \)-\( y \)-closed dual Rickart module.

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