Perturbed inertial primal-dual dynamics with damping and scaling terms for linearly constrained convex optimization problems *

Xin He\textsuperscript{a}, Rong Hu\textsuperscript{b}, Ya-Ping Fang\textsuperscript{a,*}

\textsuperscript{a}Department of Mathematics, Sichuan University, Chengdu, Sichuan, P.R. China
\textsuperscript{b}Department of Applied Mathematics, Chengdu University of Information Technology, Chengdu, Sichuan, P.R. China

Abstract

We propose a perturbed inertial primal-dual dynamic with damping and scaling coefficients, which involves inertial terms both for primal and dual variables, for a linearly constrained convex optimization problem in a Hilbert setting. With different choices of damping and scaling coefficients, by a Lyapunov analysis approach we discuss the asymptotic properties of the dynamic and prove its fast convergence properties. Our results can be viewed extensions of the existing ones on inertial dynamical systems for the unconstrained convex optimization problem to the linearly constrained convex optimization problem.

Keywords: Perturbed inertial primal-dual dynamic, linearly constrained convex optimization problem, damping and scaling, Lyapunov analysis approach, convergence rate

1. Introduction

1.1. Problem statement

Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be two real Hilbert spaces with inner \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). Let \( f : \mathcal{H}_1 \to \mathbb{R} \) be a differentiable convex function and \( A : \mathcal{H}_1 \to \mathcal{H}_2 \) be a continuous linear operator with its adjoint operator \( A^T \). Consider the perturbed inertial primal-dual dynamical system

\[
\begin{align*}
\ddot{x}(t) + \alpha(t) \dot{x}(t) &= -\beta(t) (\nabla f(x(t)) + A^T (\lambda(t) + \delta(t) \dot{\lambda}(t)) + \sigma A^T (Ax(t) - b)) + \epsilon(t), \\
\ddot{\lambda}(t) + \alpha(t) \dot{\lambda}(t) &= \beta(t) (A(x(t) + \delta(t) \dot{x}(t)) - b),
\end{align*}
\]

where \( t \in [t_0, +\infty) \) with \( t_0 \geq 0, \sigma \geq 0, \alpha : [t_0, +\infty) \to (0, +\infty) \) is a viscous damping coefficient, \( \beta : [t_0, +\infty) \to (0, +\infty) \) is a scaling coefficient, \( \delta : [t_0, +\infty) \to (0, +\infty) \) is an extrapolation coefficient, and \( \epsilon : [t_0, +\infty) \to \mathcal{H}_1 \) is an integrable source term that can be interpreted as a small external perturbation. In terms of the dynamic (1), in this paper, we shall develop a fast primal-dual dynamic approach to solve the linearly constrained convex optimization problem

\[
\min_x f(x), \quad \text{s.t. } Ax = b.
\]

\*This work was supported by the National Science Foundation of China (11471230) and the Scientific Research Foundation of the Education Department of Sichuan Province (16ZA0213).

*Corresponding author

Email addresses: hexinuser@163.com (Xin He), ronghumath@aliyun.com (Rong Hu), ypfang@aliyun.com (Ya-Ping Fang)
The primal-dual dynamic (1) involves three important parameters: the damping coefficient \( \alpha(t) \), the extrapolation coefficient \( \delta(t) \), and the scaling coefficient \( \beta(t) \), which play crucial roles in deriving the fast convergence properties. The importance of the damping coefficient and the scaling coefficient has been widely recognized in inertial dynamical approaches [4, 19, 40] as well as fast algorithms [35, 40, 9, 39, 44] for unstrained optimization problems. Recently, the damping technique and the scaling technique were also used to develop inertial primal-dual dynamic approaches and inertial primal-dual algorithms for linearly constrained optimization problems, see [47, 27, 28, 26]. Extrapolation coefficients were also considered in [47, 27].

Let \( L(x, \lambda) \) and \( L^\sigma(x, \lambda) \) be the Lagrangian function and the augmented Lagrangian function of the problem (2) respectively, i.e.,

\[
L(x, \lambda) = f(x) + \langle \lambda, Ax - b \rangle
\]
and

\[
L^\sigma(x, \lambda) = L(x, \lambda) + \frac{\sigma}{2} \|Ax - b\|^2 = f(x) + \langle \lambda, Ax - b \rangle + \frac{\sigma}{2} \|Ax - b\|^2,
\]

where \( \sigma \geq 0 \) is the penalty parameter and \( \lambda \) is the Lagrangian multiplier. Let \( \Omega \subset H_1 \times H_2 \) be the saddle point set of \( L(L^\sigma) \). It is known that \( (x^*, \lambda^*) \in \Omega \) if and only if

\[
\begin{align*}
-A^T \lambda^* &= \nabla f(x^*), \\
Ax^* - b &= 0.
\end{align*}
\]

Throughout this paper, we always assume that \( f \) is a convex continuously differentiable function and \( \Omega \neq \emptyset \). We will investigate the asymptotical behavior of the dynamic (1) with the damping coefficient \( \alpha(t) = \alpha \) and the extrapolation coefficient \( \delta(t) = \delta t^s \), where \( \alpha > 0 \), \( \delta > 0 \), and \( 0 \leq r \leq s \leq 1 \).

1.2. Related works

1.2.1. Inertial dynamical systems with damping coefficients

Let’s recall some important inertial dynamical systems with damping coefficients for the unstrained optimization problem

\[
\min \Phi(x),
\]

where \( \Phi(x) \) is a smooth convex function. The following inertial gradient system:

\[
(\text{IGS}_\alpha) \quad \ddot{x}(t) + \alpha(t)\dot{x}(t) + \nabla \Phi(x(t)) = 0,
\]

and its perturbed version

\[
(\text{IGS}_{\alpha, \epsilon}) \quad \ddot{x}(t) + \alpha(t)\dot{x}(t) + \nabla \Phi(x(t)) = \epsilon(t),
\]

have been intensively studied in the literature. When damping coefficient \( \alpha(t) = \alpha \) with \( \alpha > 0 \): \( \text{IGS}_\alpha \) becomes the heavy ball with friction system, which was introduced by Polyak [36], and the asymptotic behavior has been investigated in [1,13]; under the assumption \( \int_{t_0}^{+\infty} \|\epsilon(t)\|dt < +\infty \), Haraux and Jendoubi [25] studied the
asymptotic behavior of solutions of \((IGS_{\alpha,\epsilon})\). When \(\alpha(t) = \frac{\alpha}{t^r}\) with \(\alpha > 0, r \in (0, 1)\): Cabot and Frankel \cite{20} and May \cite{33} investigated the asymptotic behavior of \((IGS_\alpha)\) as \(t\) goes to infinity; Jendoubi and May \cite{29} generalized the results of \cite{20} to \((IGS_{\alpha,\epsilon})\) with \(\int_0^t \|\epsilon(t)\|\,dt < +\infty\) and \(\int_0^t \|\epsilon(t)\|\,dt < +\infty\) respectively; Balti and May \cite{13} obtained the \(O(1/t^2)\) convergence rate with \(\int_0^{+\infty} t^r \|\epsilon(t)\|\,dt < +\infty\) and the \(o(1/t^{1+r})\) convergence rate with \(\int_0^{+\infty} t^{(1+r)/2} \|\epsilon(t)\|\,dt < +\infty\) for \((IGS_{\alpha,\epsilon})\); Sebbouh et al. \cite{37} investigated the convergence rate of the values along the trajectory of \((IGS_{\alpha,\epsilon})\) under some additional geometrical conditions on \(\Phi(x)\). When \(\alpha(t) = \frac{\alpha}{t^2}\) Su et al. \cite{40} pointed out that \((IGS_\alpha)\) with \(\alpha = 3\) can be viewed as a continuous version of the Nesterov’s accelerated gradient algorithm \((\ref{14}, \ref{34})\), and obtained the convergence rate \(\Phi(x(t)) - \min \Phi = O(1/t^2)\) as \(\alpha \geq 3\); Attouch et al. \cite{6} investigated the asymptotic behavior of \((IGS_{\alpha,\epsilon})\) as \(\alpha \geq 3\) under the assumption \(\int_0^t \|\epsilon(t)\|\,dt < +\infty\); May \cite{32} proved an improved convergence rate \(\Phi(x(t)) - \min \Phi = o(1/t^2)\) with \(\alpha > 3\); in the case \(\alpha \leq 3\) of \((IGS_\alpha)\) and \((IGS_{\alpha,\epsilon})\), the \(O(1/t^\alpha/3)\) rate of convergence can be found in \cite{7, 13}; the optimal convergence rates under some additional geometrical conditions was studied by \cite{11} for \((IGS_\alpha)\) with \(\alpha > 0\). For general damping coefficient \(\alpha(t)\), it has been investigated by \cite{4, 8, 19}. 

\subsection{1.2.2. Inertial dynamical systems with scaling coefficients}

Balhag el al. \cite{12} considered following inertial gradient system with time scaling and constant damping coefficient:

\[ \ddot{x}(t) + \alpha \dot{x}(t) + \beta(t) \nabla \Phi(x(t)) = 0, \quad (6) \]

for solving problem \((5)\), under the assumption \(\beta(t) = e^{\beta t}\) with \(\beta \leq \alpha\), they can obtain the linear convergence without strong convexity of \(\Phi\). From the calculus of variations, Wibisono et al. \cite{14} proposed the following dynamic

\[ \ddot{x}(t) + \frac{\alpha}{t} \dot{x}(t) + C(\alpha - 1)^2 t^{\alpha-3} \nabla \Phi(x(t)) = 0, \quad (7) \]

with time scaling \(\beta(t) = C(\alpha - 1)^2 t^{\alpha-3}\) for problem \((5)\) where \(\alpha > 1\) and \(C > 0\), and obtained the \(O(1/t^{\alpha-1})\) rate of convergence. Fazlyab et al. \cite{23} extended the dynamic \((7)\) to following dual dynamic for solving problem \((2)\):

\[ \dot{\lambda}(t) + \frac{\alpha}{t} \lambda(t) + C(\alpha - 1)^2 t^{\alpha-3} \nabla G(\lambda(t)) = 0, \quad \] (8)

where \(G(\lambda) = \min_x L(x, \lambda), \alpha > 1, C > 0\), the convergence rate \(G(\lambda^*) - G(\lambda(t)) = O(1/t^{\alpha-1})\) also obtained. In \cite{9}, they consider following dynamic:

\[ \ddot{x}(t) + \frac{\alpha}{t} \dot{x}(t) + \beta(t) \nabla \Phi(x(t)) = 0 \]

for problem \((5)\), and showed \(O(1/t^2 \beta(t))\) rate of convergence under assumption \(t \dot{\beta}(t) \leq (\alpha - 3) \beta(t)\). The general damped inertial gradient system with time scaling can be found in \cite{3, 10, 17}.

\subsection{1.2.3. Inertial primal-dual dynamics}

For the affine constrained convex optimization problem \((2)\), the most popular numerical methods and dynamics are based on the primal-dual framework. In recent years, many first-order dynamical systems were proposed for
In this case, the damping coefficients $\alpha, \beta$. This leads to the time scaling coefficient $\sigma$. The problem (2) was investigated by [26, 28]. Recently, Zeng et al. [47] proposed the following damped primal-dual dynamical system for solving the problem (2):

$$
\begin{align*}
\dot{x}(t) + \frac{\alpha}{\upsilon} \dot{x}(t) &= -\nabla f(x(t)) - A^T (\lambda(t) + \delta t \dot{\lambda}(t)) - \sigma A^T (Ax(t) - b), \\
\dot{\lambda}(t) + \frac{\beta}{\upsilon} \dot{\lambda}(t) &= A(x(t) + \delta t \dot{x}(t)) - b,
\end{align*}
$$

(8)

In this dynamic, the damping coefficients $\alpha(t) = \frac{\upsilon}{t}, \delta(t) = \frac{\delta}{t}$. When $\alpha > 3$ and $\delta = \frac{1}{2}$, they showed that the trajectory satisfies the following asymptotic convergence rate

$$L(x(t), \lambda^*) - L(x^*, \lambda^*) = O(1/t^2), \quad \|Ax(t) - b\| = O(1/t),$$

(9)

they also obtained $L(x(t), \lambda^*) - L(x^*, \lambda^*) = O(1/t^{2\alpha/3})$ with $\alpha \leq 3$, $\delta = \frac{3}{2\alpha}$. He et al. [27] and Attouch et al. [3] extended dynamic (8) to solve separable convex optimization problems with general conditions. The “second-order” + “first-order” primal-dual dynamics with time scaling was investigated by [26, 28].

In the next, by the substitution of variables in dynamic (8), let’s illustrate the role of time scaling $\beta(t)$ in dynamic (1). Suppose that $\alpha > 3$ and $\delta = \frac{1}{2}$ in (8), $(x^*, \lambda^*) \in \Omega$. Let’s make the change of time variable $t = v(p)$, where $v : \mathbb{R} \to \mathbb{R}$ and $\lim_{p \to +\infty} v(p) = +\infty$. Set $\dot{x}(p) = x(v(p))$ and $\dot{\lambda}(p) = \lambda(v(p))$. By the chain rule, we have

$$\dot{x}(p) = \dot{x}(v(p))\dot{v}(p), \quad \ddot{x}(p) = \dot{x}(v(p))\ddot{v}(p) + \ddot{x}(v(p))\dot{v}(p)^2,$$

and

$$\dot{\lambda}(p) = \dot{\lambda}(v(p))\dot{v}(p), \quad \ddot{\lambda}(p) = \dot{\lambda}(v(p))\ddot{v}(p) + \ddot{\lambda}(v(p))\dot{v}(p)^2.$$

Then rewritten (8) in terms of $\ddot{x}(\cdot), \ddot{\lambda}(\cdot)$ and its derivatives, we obtain

$$
\begin{align*}
\ddot{x}(p) + \left( \frac{\alpha(v(p))}{v(p)} - \frac{\upsilon(p)}{v(p)} \right) \dot{x}(p) &= -\upsilon(p)^2(\nabla f(\ddot{x}(p)) + A^T (\ddot{\lambda}(p) + \frac{\upsilon(p)}{2v(p)} \dddot{\lambda}(p))) + \sigma A^T (A\ddot{x}(p) - b), \\
\ddot{\lambda}(p) + \left( \frac{\alpha(v(p))}{v(p)} - \frac{\upsilon(p)}{v(p)} \right) \dot{\lambda}(p) &= \upsilon(p)^2(A(\ddot{x}(p) + \frac{\upsilon(p)}{2v(p)} \dddot{\lambda}(p))) - b.
\end{align*}
$$

(10)

This leads to the time scaling coefficient $\beta(p) = \upsilon(p)^2$ and the damping coefficients $\alpha(p) = \alpha - 1, \delta(p) = \frac{\upsilon(p)}{2v(p)}$. The convergence rate (9) becomes

$$L(\ddot{x}(p), \lambda^*) - L(x^*, \lambda^*) = O(\frac{1}{v(p)^2}), \quad \|A\ddot{x}(p) - b\| = O(\frac{1}{v(p)}).$$

In the next, we investigate two model examples. First, taking $v(p) = e^p$, then (10) reads

$$
\begin{align*}
\ddot{x}(p) + (\alpha - 1) \dot{x}(p) &= -e^{2p}(\nabla f(\ddot{x}(p)) + A^T (\ddot{\lambda}(p) + \frac{1}{2} \dddot{\lambda}(p))) + \sigma A^T (A\ddot{x}(p) - b), \\
\ddot{\lambda}(p) + (\alpha - 1) \dot{\lambda}(p) &= e^{2p}(A(\ddot{x}(p) + \frac{1}{2} \dddot{\lambda}(p))) - b.
\end{align*}
$$

(11)

In this case, the damping coefficients $\alpha(p) = \alpha - 1$ are constants, the time scaling coefficient is $\beta(p) = e^{2p}$, and the convergence rate becomes

$$L(\ddot{x}(p), \lambda^*) - L(x^*, \lambda^*) = O(\frac{1}{e^{2p}}), \quad \|A\ddot{x}(p) - b\| = O(\frac{1}{e^{2p}}).$$
Taking $v(p) = p^\kappa$ with $\kappa > 0$, then (11) reads

\begin{align}
\begin{cases}
\ddot{x}(t) + \frac{1+(\alpha-1)\kappa}{p} \dot{x}(t) &= -\kappa^2 p^{2(\kappa-1)} (\nabla f(\bar{x}(p))) + A^T(\bar{\lambda}(p) + \frac{\kappa}{\alpha} \dot{\lambda}(p)) + \sigma A^T(A\bar{x}(p) - b)), \\
\dot{\lambda}(p) + \frac{1+(\alpha-1)\kappa}{p} \lambda(p) &= \kappa^2 p^{2(\kappa-1)} (A(\bar{x}(p) + \frac{\kappa}{\alpha} \dot{x}(p)) - b).
\end{cases}
\end{align}

the convergence rate becomes

$$\mathcal{L}(\bar{x}(p), \lambda^*) - \mathcal{L}(x^*, \lambda^*) = O\left(\frac{1}{p^{2\kappa}}\right), \quad \|A\bar{x}(p) - b\| = O\left(\frac{1}{p^\kappa}\right),$$

the damping coefficient $\alpha(p) = \frac{1+(\alpha-1)\kappa}{p}$. For $\kappa \geq 1$, we have $1 + (\alpha - 1)\kappa \geq \alpha$, so damping coefficient similar to (8), where $\alpha(t) = \frac{\alpha}{\sigma}$.

1.3. Organisation

In Section 2, we present the rate of convergence in the different choice of damping coefficient and extrapolation coefficient under the suitable assumptions on time scaling coefficient and external perturbation. Section 3 concludes the paper. Some technical proofs and lemmas are postponed to Appendix.

2. Main results

In this paper, we will investigate the dynamic (11) with damping coefficient $\alpha(t) = \frac{\alpha}{\sigma}$ and extrapolation coefficient $\delta(t) = \delta t^s$, where $\alpha > 0$, $\delta > 0$, $0 \leq r \leq s \leq 1$. The the dynamic (11) becomes:

\begin{align}
\begin{cases}
\ddot{x}(t) + \frac{\alpha}{\sigma} \dot{x}(t) &= -\beta(t) (\nabla f(x(t)) + A^T(\lambda(t) + \delta t^s \dot{\lambda}(t)) + \sigma A^T(Ax(t) - b)) + \epsilon(t), \\
\dot{\lambda}(t) + \frac{\alpha}{\sigma} \dot{\lambda}(t) &= \beta(t) (A(x(t) + \delta t^s \dot{x}(t)) - b).
\end{cases}
\end{align}

Before investigating the rate of convergence, we first discuss the existence and uniqueness of solutions for dynamical system (13).

When $\nabla f(x)$ is Lipschitz continuous on $\mathcal{H}_1$, from [3, Theorem 4.2], for any $(x_0, \lambda_0, u_0, v_0)$, the dynamic (13) has a unique strong global solution $(x(t), \lambda(t))$, in which (i): $x(t) \in C^2([t_0, +\infty), \mathcal{H}_1)$, $\lambda(t) \in C^2([t_0, +\infty), \mathcal{H}_2)$; (2): $(x(t), \lambda(t))$ and $(\dot{x}(t), \dot{\lambda}(t))$ are locally absolutely continuous; (3): for almost every $t \in [0, +\infty)$, (13) holds, and $(x(t_0), \lambda(t_0)) = (x_0, \lambda_0)$ and $(\dot{x}(t_0), \dot{\lambda}(t_0)) = (u_0, v_0)$.

When $\nabla f(x)$ is locally Lipschitz continuous on $\mathcal{H}_1$, following from the Picard-Lindelof Theorem (see [42, Theorem 2.2]), we can establish the local existence and uniqueness solution of dynamic (13) as follows:

**Proposition 2.1.** Let $f$ be continuously differentiable function such that $\nabla f$ is locally Lipschitz continuous, $\beta : [t_0, +\infty) \to (0, +\infty)$ be a continuous function, $\epsilon : [t_0, +\infty) \to \mathcal{H}_1$ be locally integrable. Then for any $(x_0, \lambda_0, u_0, v_0)$, there exists a unique solution $(x(t), \lambda(t))$ with $x(t) \in C^2([t_0, T), \mathcal{H}_1)$, $\lambda(t) \in C^2([t_0, T), \mathcal{H}_2)$ of the dynamic (13) satisfying $(x(t_0), \lambda(t_0)) = (x_0, \lambda_0)$ and $(\dot{x}(t_0), \dot{\lambda}(t_0)) = (u_0, v_0)$ on a maximal interval $[t_0, T) \subseteq [t_0, +\infty)$. 
So under the assumptions in Proposition 2.1, we obtain that there exists a unique solution \((x(t), \lambda(t))\) defined on maximal interval \([t_0, T) \subseteq [t_0, +\infty)\). If we can prove that the derivative of trajectory \((\dot{x}(t), \dot{\lambda}(t))\) is bounded on \([t_0, T)\), it follows from assumptions that \((\ddot{x}(t), \ddot{\lambda}(t))\) is also bounded on \([t_0, T)\). This implies that \((x(t), \lambda(t))\) and its derivative \((\dot{x}(t), \dot{\lambda}(t))\) have a limit at \(t = T\), and therefore can be continued, a contradiction. Thus \(T = +\infty\), we obtain the existence and uniqueness of global solution of dynamic (13). To simplify the proof process, we assume that the global solution of dynamic (13) exists. We will discuss the existence and uniqueness of global solution of dynamics (13) in the case \(r = 0, s \in [0, 1]\) later, and it can be proved similarly for other cases.

In order to investigate the convergence rates of dynamic (13) under different choices of \(r, s\). We construct the different energy functions, fixed \((x^*, \lambda^*) \in \Omega\), for any \(\lambda \in \mathcal{H}_2\), define the energy function \(E_{\epsilon}^{\lambda, \rho} : [t_0, +\infty) \rightarrow \mathbb{R}\) as

\[
E_{\epsilon}^{\lambda, \rho}(t) = E_{0}(t) + E_{1}(t) + E_{2}(t),
\]

where

\[
E_{0}(t) = t^{2\rho}\beta(t)(L\sigma(x(t), \lambda) - L\sigma(x^*, \lambda)),
E_{1}(t) = \frac{1}{2}\|\theta(t)(x(t) - x^*) + t^{\rho}\dot{x}(t)\|^2 + \frac{\eta(t)}{2}\|x(t) - x^*\|^2,
E_{2}(t) = \frac{1}{2}\|\theta(t)(\lambda(t) - \lambda) + t^{\rho}\dot{\lambda}(t)\|^2 + \frac{\eta(t)}{2}\|\lambda(t) - \lambda\|^2.
\]

\(\theta, \eta : [t_0, +\infty) \rightarrow \mathbb{R}\) are two smooth functions, and \(\rho \geq 0\).

The key point of our proof is to find the appropriate \(\theta(t), \eta(t)\) to ensure that the energy function \(E_{\epsilon}^{\lambda, \rho}(t)\) is decreasing. To avoid repeated calculations, we list the main calculation procedures in Appendix A.1.

2.1. Case \(r = 0, s \in [0, 1]\)

Let us first consider the case when \(r = 0, s \in [0, 1]\), i.e., the dynamic (13):

\[
\begin{align*}
\ddot{x}(t) + \alpha \dot{x}(t) &= -\beta(t)(\nabla f(x(t))) + A^T(\lambda(t) + \delta t^s \dot{\lambda}(t)) + \sigma A^T(Ax(t) - b)) + \epsilon(t), \\
\ddot{\lambda}(t) + \alpha \dot{\lambda}(t) &= \beta(t)(A(x(t) + \delta t^s \dot{x}(t)) - b),
\end{align*}
\]

with \(\alpha > 0, \delta > 0, \sigma \geq 0, \ t \geq t_0 > 0\).

Theorem 2.1. Assume that \(\beta : [t_0, +\infty) \rightarrow (0, +\infty)\) is continuous differentiable function with

\[
t^s \beta(t) \leq \left(\frac{1}{\delta} - st^{s-1}\right) \beta(t),
\]

and \(\epsilon : [t_0, +\infty) \rightarrow \mathcal{H}_1\) is an integrable function with

\[
\int_{t_0}^{+\infty} t^{\frac{s}{2}} \|\epsilon(t)\| \, dt < +\infty.
\]

Suppose \(\alpha \delta > 1\) when \(s = 0; \ \delta \leq 1\) when \(s = 1\), and \(\sigma > 0\). Let \((x(t), \lambda(t))\) be a global solution of the dynamic (13) and \((x^*, \lambda^*) \in \Omega\). Then \((x(t), \lambda(t))\) is bounded, and the following conclusions hold:
\( (i) \int_{t_0}^{+\infty} (1 - st^{-s-1}) \beta(t) - t^s \dot{\beta}(t) (L^\sigma (x(t), \lambda^*) - L^\sigma (x^*, \lambda^*)) dt < +\infty. \)

\( (ii) \int_{t_0}^{+\infty} t^s (\| \dot{x}(t) \|^2 + \| \dot{\lambda}(t) \|^2) dt < +\infty, \int_{t_0}^{+\infty} \beta(t) \| Ax(t) - b \|^2 dt < +\infty. \)

\( (iii) \| \dot{x}(t) \| + \| \dot{\lambda}(t) \| = O(\frac{1}{t^{s/2}}). \)

\( (iv) \text{When} \lim_{t \to +\infty} t^s \beta(t) = +\infty: \)

\[ \mathcal{L}(x(t), \lambda^*) - \mathcal{L}(x^*, \lambda^*) = O(\frac{1}{t^s \beta(t)}), \quad \| Ax(t) - b \| = O(\frac{1}{t^{s/2} \beta(t)}). \]

**Proof.** Given \( \lambda \in \mathcal{H}_2 \), define energy functions \( E^{\lambda, \rho}(t) \) and \( E^{\lambda, \rho}_e(t) \) same as [15], [14] with \( r = 0, s \in [0, 1], \rho = \frac{1}{t} \), and

\[ \begin{align*}
\theta(t) &= \frac{1}{t} t^{-s/2}, \\
\eta(t) &= \frac{1}{t} (\alpha - \frac{1}{t} t^{-s}).
\end{align*} \]  

By computations, we can verify that \( (A.2) \) and \( (A.4) \) hold.

**Case** \( s = 0 \): \( \theta(t) = \frac{1}{t} \) and \( \eta(t) = \frac{a_0 - 1}{2s} \). Since \( a_0 > 1 \), we obtain that \( (A.1), (A.3) \) hold, and then \( (A.5) \) holds,

\[ \theta(t) + \rho t^{\rho - 1} - \alpha t^{\rho - r} = \frac{1}{\delta} \alpha < 0. \]  

It follows from [17] that

\[ t^\rho \dot{\beta}(t) + (2\rho t^{\rho - 1} - \theta(t)) \beta(t) = \dot{\beta}(t) - \frac{1}{\delta} \beta(t) \leq 0 \]  

for all \( t \geq t_0 \). Taking \( \lambda = \lambda^* \), then \( \mathcal{L}^\sigma (x(t), \lambda^*) - \mathcal{L}^\sigma (x^*, \lambda^*) \geq 0 \), it follows from [19], [20] and \( (A.5) \) that

\[ \begin{align*}
\dot{E}^{\lambda, \rho}_e(t) &\leq \frac{1}{\delta} (\| \dot{x}(t) \|^2 + \| \dot{\lambda}(t) \|^2) - \frac{\sigma}{2\delta} \| Ax(t) - b \|^2 \\
&\quad + (\dot{\beta}(t) - \frac{1}{\delta} \beta(t)) (\mathcal{L}^\sigma (x(t), \lambda^*) - \mathcal{L}^\sigma (x^*, \lambda^*)) \\
&\leq 0.
\end{align*} \]  

So \( \dot{E}^{\lambda, \rho}_e(\cdot) \) is nonincreasing on \([t_0, +\infty)\), and then

\[ \dot{E}^{\lambda, \rho}_e(t) \leq \dot{E}^{\lambda, \rho}_e(t_0), \quad \forall t \geq t_0. \]  

By the definition of \( \dot{E}^{\lambda, \rho}_e(\cdot) \) and \( \dot{E}^{\lambda, \rho}(\cdot) \), we have

\[ \frac{1}{2} \left\| \frac{1}{\delta} (x(t) - x^*) + \dot{x}(t) \right\|^2 \leq \dot{E}^{\lambda, \rho}_e(t_0) + \int_{t_0}^{t} \frac{1}{\delta} (x(w) - x^*) + \dot{x}(w), \epsilon(w) dw. \]

By Cauchy-Schwarz inequality, we get

\[ \frac{1}{2} \left\| \frac{1}{\delta} (x(t) - x^*) + \dot{x}(t) \right\|^2 \leq |\dot{E}^{\lambda, \rho}_e(t_0)| + \int_{t_0}^{t} \left\| \frac{1}{\delta} (x(w) - x^*) + \dot{x}(w) \right\| \| \epsilon(w) \| dw, \]

then applying Lemma \( \text{Appendix A.1} \) with \( \mu(t) = \| \frac{1}{\delta} (x(t) - x^*) + \dot{x}(t) \| \), we obtain

\[ \sup_{t \geq t_0} \left\| \frac{1}{\delta} (x(t) - x^*) + \dot{x}(t) \right\| \leq \sqrt{2 |\dot{E}^{\lambda, \rho}_e(t_0)|} + \int_{t_0}^{+\infty} \| \epsilon(t) \| dt < +\infty. \]  

7
It is easy to verify $\mathcal{E}^{x^*}(t) \geq 0$ for all $t \geq t_0$, then we have
\[
\inf_{t \geq t_0} \mathcal{E}^{x^*}(t) \geq - \sup_{t \geq t_0} \| \frac{1}{\delta}(x(t) - x^*) + \dot{x}(t) \| \times \int_{t_0}^{+\infty} \| \epsilon(s) \| ds > -\infty
\]
and
\[
\sup_{t \geq t_0} \mathcal{E}^{x^*}(t) \leq \mathcal{E}^{x^*}(t_0) + \sup_{t \geq t_0} \| \frac{1}{\delta}(x(t) - x^*) + \dot{x}(t) \| \times \int_{t_0}^{+\infty} \| \epsilon(s) \| ds < +\infty.
\]
This together with (22) and the definition of $\mathcal{E}^{x^*}(\cdot)$ yields the boundedness of $\mathcal{E}^{x^*}(\cdot)$ and $\mathcal{E}^{x^*}(\cdot)$. By integrating inequality (21) on $[t_0, +\infty)$, it follows the boundedness of $\mathcal{E}^{x^*}(\cdot)$ that
\[
(\alpha - \frac{1}{\delta}) \int_{t_0}^{+\infty} \| \dot{x}(t) \|^2 + \| \dot{\lambda}(t) \|^2 dt + \int_{t_0}^{+\infty} \left(\frac{1}{\delta} \beta(t) - \dot{\beta}(t)\right)(\mathcal{L}(x(t), \lambda^*) - \mathcal{L}(x^*, \lambda^*))dt
\]
\[
+ \frac{\sigma}{28} \int_{t_0}^{+\infty} \beta(t) \| Ax(t) - b \|^2 dt < +\infty.
\]
This together with $\frac{1}{\delta} < \alpha$ yields (i) - (ii).

Since $\eta(t) = \frac{\alpha - 1}{\beta} > 0$, By the boundedness of $\mathcal{E}^{x^*}(\cdot)$, we obtain that $\| x(t) - x^* \|^2, \| \lambda(t) - \lambda^* \|^2, \| \frac{1}{\delta}(x(t) - x^*) + \dot{x}(t) \|$ and $\| \frac{1}{\delta}(\lambda(t) - \lambda^*) + \dot{\lambda}(t) \|$ are bounded, and then the trajectory $(x(t), \lambda(t))$ is bounded,
\[
\sup_{t \in [t_0, +\infty)} \| \dot{x}(t) \| \leq \frac{1}{\delta} \sup_{t \in [t_0, +\infty)} \| x(t) - x^* \| + \sup_{t \in [t_0, +\infty)} \| \frac{1}{\delta}(x(t) - x^*) + \dot{x}(t) \| < +\infty,
\]
similarly, $\sup_{t \in [t_0, +\infty)} \| \dot{\lambda}(t) \| < +\infty$, this is (iii). When $\lim_{t \to +\infty} \beta(t) = +\infty$, following from the boundedness of $\mathcal{E}^{x^*}(\cdot)$, we get
\[
\mathcal{L}(x(t), \lambda^*) - \mathcal{L}(x^*, \lambda^*) = O\left(\frac{1}{\beta(t)}\right).
\]
Since $\mathcal{L}(x(t), \lambda^*) - \mathcal{L}(x^*, \lambda^*) = \mathcal{L}(x(t), \lambda^*) - \mathcal{L}(x^*, \lambda^*) + \frac{\sigma}{2} \| Ax(t) - b \|^2$, then we obtain (iv).

**Case** $s \in (0, 1)$: There exists $t_1 \geq t_0$ such that
\[
\frac{1}{\delta} t^{-s} + \frac{s}{2} t^{-1} \leq \frac{\alpha}{2}, \quad \forall \ t \geq t_1,
\]
this together with (18) yields
\[
\eta(t) \geq \frac{\alpha}{2\delta} > 0, \quad \forall \ t \geq t_1.
\]
We can compute that
\[
\theta(t) \dot{\theta}(t) + \frac{\dot{\eta}(t)}{2} = 0.
\]
Then (21)-(24) are satisfied for any $t \geq t_1$. It follows from (24) that
\[
\theta(t) + \alpha t^{\rho - 1} - \alpha t^{\rho - r} = t^{s/2} \left(\frac{1}{\delta} t^{-s} + \frac{s}{2} t^{-1} - \alpha\right) \leq -\frac{\alpha}{2} t^{s/2}, \quad \forall \ t \geq t_1.
\]
By computation,
\[
t^{s}(t^{\rho} \dot{\beta}(t) + \theta(t) \beta(t)) = t^{s} \dot{\beta}(t) - \left(\frac{1}{\delta} - s \delta^{s-1}\right) \beta(t) \leq 0, \quad \forall \ t \geq t_0.
\]
Let $\lambda = \lambda^*, \mathcal{L}^\sigma(x(t), \lambda^*) - \mathcal{L}^\sigma(x^*, \lambda^*) \geq 0$. Combining (28), (27) and (A.5), we get

$$
\dot{\mathcal{E}}^\lambda_{\sigma}(t) \leq - \frac{\alpha}{2} t^s \|\dot{x}(t)\|^2 + \|\dot{\lambda}(t)\|^2 \frac{\sigma^2(t)}{2\tilde{e}} \|Ax(t) - b\|^2 + (t^s \tilde{\beta}(t) - (\frac{1}{\delta} - st^{s-1})\beta(t)) (\mathcal{L}^\sigma(x(t), \lambda^*) - \mathcal{L}^\sigma(x^*, \lambda^*))
$$

(28)

for all $t \geq t_1$. $\mathcal{E}^\lambda_{\sigma}(\cdot)$ is nonincreasing on $[t_1, +\infty)$,

$$
\mathcal{E}^\lambda_{\sigma}(t) \leq \mathcal{E}^\lambda_{\sigma}(t_1), \quad \forall \ t \geq t_1.
$$

By the definition of $\mathcal{E}^\lambda_{\sigma}(\cdot)$ and $\mathcal{E}^\lambda_{\sigma}(\cdot)$, for all $t \geq t_1$ we have

$$
\frac{1}{2} \|t^{-s/2}(x(t) - x^*) + t^{s/2}\dot{x}(t)\|^2 \leq \mathcal{E}^\lambda_{\sigma}(t_1) + \int_{t_1}^t \left(\frac{1}{2} t^{-s/2}(x(w) - x^*) + s^{s/2}\dot{x}(w), w^{s/2}c(w)\right) dw.
$$

By similar arguments in Case s=0, we obtain the boundedness of $\mathcal{E}^\lambda_{\sigma}(\cdot)$ and $\mathcal{E}^\lambda_{\sigma}(\cdot)$. Integrating inequality (28) on $[t_1, +\infty)$, we get the results (i) - (ii).

Since $\mathcal{E}^\lambda_{\sigma}(\cdot)$ is bounded, following from the definition of $\mathcal{E}^\lambda_{\sigma}(\cdot)$, we obtain (iv),

$$
\sup_{t \geq t_0} \mathcal{E}^\lambda_{\sigma}(t) \leq \mathcal{E}^\lambda_{\sigma}(t_0) < +\infty
$$

and

$$
\sup_{t \geq t_0} \|t^{-s/2}(x(t) - x^*) + t^{s/2}\dot{x}(t)\| < +\infty.
$$

This together with (25) and $s \in (0, 1]$ implies

$$
\sup_{t \geq t_0} \|x(t) - x^*\| < +\infty
$$

and

$$
\sup_{t \geq t_0} t^{s/2} \|\dot{x}(t)\| \leq \frac{1}{\delta} \sup_{t \geq t_0} t^{-s/2} \|x(t) - x^*\| \leq \frac{1}{\delta} \sup_{t \geq t_0} \left(\frac{1}{\delta} t^{-s/2}(x(t) - x^*) + t^{s/2}\dot{x}(t)\right) \|x(t) - x^*\| + \|t^{-s/2}(x(t) - x^*) + t^{s/2}\dot{x}(t)\| < +\infty.
$$

Similarly, we have $\sup_{t \geq t_0} \|\lambda(t) - \lambda^*\| < +\infty$, $\sup_{t \geq t_0} t^{s/2} \|\dot{\lambda}(t)\| < +\infty$. Then we obtain the boundedness of $(x(t), \lambda(t))$ and (iii).

\[\square\]

**Remark 2.1.** From Proposition 2.1, there exists a unique local solution $(x(t), \lambda(t))$ of the dynamic (10) defined on a maximal interval $[t_0, T]$ with $T \leq +\infty$. If we pick an appropriate $t_0 > 0$, following from the proof process in Theorem 2.4 and (iii), we can obtain $\sup_{t \in [t_0, T]} \|\dot{x}(t)\| + \|\dot{\lambda}(t)\| < +\infty$, and then $T = +\infty$, the existence and uniqueness of global solution of the dynamic (10) is established.
Remark 2.2. From Theorem 2.1, we can see that for same damping $\alpha(t) = \alpha$, choosing another damping $\delta(t) = \frac{\delta}{t}$ different, the different rates of convergence can be obtained. Taking $A = 0$, $b = 0$, we can obtain the $O(1/t^s\beta(t))$ convergence rate for dynamic (6) under the assumption $t^s\dot{\beta}(t) \leq (\frac{1}{s} - st^{-1})\beta(t)$ with $\delta > 0$, so Theorem 2.1 complements the results in [12]. The assumption $\int_{t_0}^{+\infty} \|\epsilon(t)\|dt < +\infty$ for perturbation $\epsilon(t)$ has been used in [24] for asymptotic analysis of heavy ball dynamic.

Remark 2.3. When $s = 0$, choosing $\beta(t) \equiv 1$, then (17) is automatically satisfied. Then from (i), we have
$$\int_{t_0}^{+\infty} L^\sigma(x(t), \lambda^*) - L^\sigma(x^*, \lambda^*)dt < +\infty.$$ Since $L^\sigma(\cdot, \lambda^*)$ is a convex function with respect to first variable, taking
$$\bar{x}(t) = \frac{\int_{t_0}^{t} x(s)ds}{t - t_0},$$ we have
$$L^\sigma(\bar{x}(t), \lambda^*) - L^\sigma(x^*, \lambda^*) \leq \frac{1}{t - t_0} \int_{t_0}^{t} L^\sigma(x(s), \lambda^*) - L^\sigma(x^*, \lambda^*)ds \leq \frac{1}{t - t_0} \int_{t_0}^{+\infty} L^\sigma(x(s), \lambda^*) - L^\sigma(x^*, \lambda^*)ds.$$ Following from the definition of $L^\sigma(\bar{x}(t), \lambda^*)$, we obtain $L(\bar{x}(t), \lambda^*) - L(x^*, \lambda^*) = O(1/t) \text{ and } \|A\bar{x}(t) - b\| = O(1/rt^s)$, the $O(1/t)$ ergodic convergence rate corresponds to the convergence rate of the discrete heavy ball algorithm in [24]; for general $\beta(t)$ with $s = 0$, the similar convergence rate results can be found in [28]. When $s = 1$, choosing $\beta(t) \equiv 1$ and $\delta \leq 1$, the $O(1/t)$ rate of convergence also was investigated in [24]. Theorem 4.4 with $r = 0$ for problem (2), and it is consistent with results of heavy ball dynamic and algorithm in [41] for problem (2).

In Theorem 2.1, when $\lim_{t \to +\infty} t^s\beta(t) = +\infty$, we show the $O(1/t^s\beta(t))$ convergence rate of Lagrangian function and $O(1/t^{s/2}\sqrt{\beta(t)})$ convergence rate of constraint, then
$$|f(x(t)) - f(x^*)| \leq L(x(t), \lambda^*) - L^\sigma(x^*, \lambda^*) + \|\lambda^*\|\|Ax(t) - b\| = O\left(\frac{1}{t^{s/2}\sqrt{\beta(t)}}\right).$$
We only can obtain the $O(1/t^{s/2}\sqrt{\beta(t)})$ convergence rate of objection function.

In the next, we will investigate the best convergence rates of objection function and constrain for suitable $\beta(t)$. When $s = 0$, let $\hat{\beta}(t) = \frac{1}{3}\beta(t)$. then $\frac{\hat{\beta}(t)}{\beta(t)} = \frac{1}{3}$; integrating it on $[t_0, t]$, we have
$$\beta(t) = \frac{\beta(t_0)}{e^{\frac{1}{3}\int_{t_0}^{t} \frac{1}{\beta(t)}dt}}.$$ 
In this case, from Theorem 2.1, we can obtain the $O\left(\frac{1}{e^{\frac{1}{3}\int_{t_0}^{t} \frac{1}{\beta(t)}dt}}\right)$ convergence rate of objective function and constraint. Let $\beta(t) = \mu e^{t/\delta}$ with $\mu > 0$, we list the following improved convergence rate results, which also can be found in [37, Proposition 6.2] with $\epsilon(t) = 0$.

**Theorem 2.2.** Let $\beta(t) = \mu e^{t/\delta}$ with $\mu > 0$, $\alpha \delta > 1$, $s = 0$, $\sigma \geq 0$. Assume $\int_{t_0}^{+\infty} \|\epsilon(t)\|dt < +\infty$. Let $(x(t), \lambda(t))$ be a solution of dynamic (16) and $(x^*, \lambda^*) \in \Omega$. Then:
$$|f(x(t)) - f(x^*)| = O\left(\frac{1}{e^{t/\delta}}\right), \quad \|Ax(t) - b\| = O\left(\frac{1}{e^{t/\delta}}\right).$$
Proof. Given \( \lambda \in \mathcal{H}_2 \), recall the energy functions \( \mathcal{E}^{\lambda, \rho}(t) \) and \( \mathcal{E}^{\lambda, \rho}_e(t) \) from Theorem 2.1 with \( \beta(t) = \mu e^{t/8} \), \( s = \rho = 0 \). Then

\[
t^\rho \dot{\beta}(t) + (2 \rho t^{\rho-1} - \theta(t)) \beta(t) = 0,
\]

this together with \([19]\) and \([A.5]\) yields

\[
\dot{\mathcal{E}}^{\lambda, \rho}_e(t) \leq \left( \frac{1}{\delta} - \alpha \right)(\| \dot{x}(t) \|^2 + \| \dot{\lambda}(t) \|^2) - \frac{\sigma \beta(t)}{2\delta} \| Ax(t) - b \|^2 \leq 0, \quad \forall \ t \geq t_0, \lambda \in \mathcal{H}_2.
\]

So for any \( \lambda \in \mathcal{H}_2 \), \( \mathcal{E}^{\lambda, \rho}_e(\cdot) \) is nonincreasing on \([t_0, +\infty)\) such that

\[
\mathcal{E}^{\lambda, \rho}_e(t) \leq \mathcal{E}^{\lambda, \rho}_e(t_0), \quad \forall \ t \geq t_0.
\]

By the definition of \( \mathcal{E}^{\lambda, \rho}_e(\cdot) \) and \( \sigma \geq 0 \), we have

\[
f(x(t)) - f(x^*) + \langle \lambda, Ax(t) - b \rangle \leq \frac{1}{\mu e^{t/8}} \left( \sup_{\| \lambda \| \leq \varrho} \mathcal{E}^{\lambda, \epsilon}(t_0) + \sup_{t \geq t_0} \frac{1}{\delta} \| x(t) - x^* \| + \| \dot{x}(t) \| \right) \int_{t_0}^{+\infty} \| \epsilon(t) \| dt
\]

for any \( \lambda \in \mathcal{H}_1 \) and \( t \geq t_0 \). Taking \( \varrho > \| \lambda^* \| \), it follows from Lemma Appendix A.2 that

\[
f(x(t)) - f(x^*) + \varrho \| Ax(t) - b \| \leq \frac{1}{\mu e^{t/8}} \left( \sup_{\| \lambda \| \leq \varrho} \mathcal{E}^{\lambda, \epsilon}(t_0) + \sup_{t \geq t_0} \frac{1}{\delta} \| x(t) - x^* \| + \| \dot{x}(t) \| \right) \int_{t_0}^{+\infty} \| \epsilon(t) \| dt.
\]

Denote \( C = \sup_{\| \lambda \| \leq \varrho} \mathcal{E}^{\lambda, \epsilon}(t_0) + \sup_{t \geq t_0} \| x(t) - x^* \| + \| \dot{x}(t) \| \int_{t_0}^{+\infty} \| \epsilon(t) \| dt \). Since \( \varrho > \| \lambda^* \| \), \( \sup_{\| \lambda \| \leq \varrho} \mathcal{E}^{\lambda, \epsilon}(t_0) \geq \mathcal{E}^{\lambda^*, \epsilon}(t_0) \geq 0 \), this together with \([28]\) yields \( 0 \leq C < +\infty \). Following from \([41]\), we have

\[
f(x(t)) - f(x^*) \geq -\| \lambda^* \| \| Ax(t) - b \|,
\]

this together with \([30]\) implies

\[
\| Ax(t) - b \| \leq \frac{C}{\mu (\varrho - \lambda^*) e^{t/8}}
\]

and then

\[
\frac{-\| \lambda^* \| C}{\mu (\varrho - \lambda^*) e^{t/8}} \leq f(x(t)) - f(x^*) \leq \frac{C}{\mu e^{t/8}}.
\]

We obtain results from above inequalities. \( \square \)

**Remark 2.4.** When \( s = 0 \) and \( \beta(t) = \mu e^{t/8} \), Theorem 2.1 obtains \( O(\frac{1}{\epsilon^{1/2}}) \) convergence rate of objective function and constraint, it is consistent with convergence rates of dynamic \([11]\), which is derived from dynamic \([8]\). Theorem 2.2 shows that the rate of convergence is actually \( O(\frac{1}{\epsilon^{1/2}}) \). Then we can obtain the linear convergence rate of dynamic \([16]\) merely under the convexity assumption of \( f \), and in this case we also allow the penalty parameter \( \sigma \) of augmented Lagrangian function to be zero, which is different in Theorem 2.1.

When \( s \in (0, 1) \), let \( t^s \dot{\beta}(t) = (\frac{1}{s} - st^{s-1}) \beta(t) \). It leads

\[
\beta(t) = \frac{t_0 \beta(t_0)}{e^{\frac{1}{e^{1-\gamma}-1} t^{1-s}}} e^{rac{1}{e^{1-\gamma}-1} t^{1-s}}.
\]

Take \( \beta(t) = \mu e^{\frac{1}{e^{1-\gamma}-1} t^{1-s}} \) with \( \mu > 0 \). We investigate the following optimal results.
Theorem 2.3. Let $\beta(t) = \mu t^{1-s}$ with $\mu > 0$, $s \in (0, 1)$. Suppose $\int_{t_0}^{+\infty} t^{s/2} \| \epsilon(t) \| dt < +\infty$. Let $(x(t), \lambda(t))$ be a solution of dynamic (16) and $(x^*, \lambda^*) \in \Omega$. Then

$$|f(x(t)) - f(x^*)| = \mathcal{O}\left(\frac{1}{e^{\frac{1}{r_1-s}}t}\right), \quad \|Ax(t) - b\| = \mathcal{O}\left(\frac{1}{e^{\frac{1}{r_1-s}}t}\right).$$

Proof. Given $\lambda \in \mathcal{H}_2$, recall the energy functions $\mathcal{E}^{\lambda, \rho}(t)$ and $\mathcal{E}_c^{\lambda, \rho}(t)$ from Theorem 2.1 with $\beta(t) = \mu t^{1-s}$, $s \in (0, 1)$, $\rho = \frac{1}{2}$. Then

$$(2\rho t^{\rho-1} - \theta(t))\beta(t) = 0,$$

this together with (19) and (A.5) yields

$$\dot{\mathcal{E}}^{\lambda, \rho}(t) \leq \frac{\alpha}{2} t^{s} \| \dot{x}(t) \|^2 + \| \dot{\lambda}(t) \|^2 - \frac{\sigma \beta(t)}{2\delta} \|Ax(t) - b\|^2 \leq 0, \quad \forall t \geq t_1, \lambda \in \mathcal{H}_2,$$

for some $t_1 \geq t_0$. By similar arguments in Theorem 2.1, we obtain the results.

When $s = 1$, let $t\dot{\beta}(t) = (\frac{1}{\delta} - 1)\beta(t)$. It leads

$$\beta(t) = \frac{\beta(t_0)}{t^{\frac{1}{\delta} - 1}} t^{\frac{1}{\delta} - 1}.$$ 

Taking $\beta(t) = \mu t^{\frac{1}{\delta} - 1}$ with $\mu > 0$. By similar arguments in Theorem 2.2 and Theorem 2.3 we obtain the following results.

Theorem 2.4. Let $\beta(t) = \mu t^{\frac{1}{\delta} - 1}$ with $\mu > 0$, $\delta \leq 1$, $s = 1$, $\sigma \geq 0$. Suppose $\int_{t_0}^{+\infty} t^{1/2} \| \epsilon(t) \| dt < +\infty$. Let $(x(t), \lambda(t))$ be a solution of dynamic (16) and $(x^*, \lambda^*) \in \Omega$. We have

$$|f(x(t)) - f(x^*)| = \mathcal{O}\left(\frac{1}{t^{1/\delta}}\right), \quad \|Ax(t) - b\| = \mathcal{O}\left(\frac{1}{t^{1/\delta}}\right).$$

Remark 2.5. When $s = 1$, taking $\delta = 1$ and $\beta(t) = 1$, from Theorem 2.4, we obtain $\mathcal{O}(\frac{1}{t})$ convergence rates of objective function and constraint, which improves results in Theorem 2.1 with time scaling $\beta(t) = 1$.

Remark 2.6. For damping $\alpha(t) = \alpha$, $\delta = \frac{1}{r}$ with $s \in [0, 1]$, the $\mathcal{O}(1/t^s \beta(t))$ convergence rate in Theorem 2.4 shows that convergence results is better as $s$ larger in $[0, 1]$. Conversely, following from Theorem 2.2 Theorem 2.4 when $s$ is smaller in $[0, 1]$, we can obtain better optimal convergence rates with suitable $\beta(t)$.

2.2. Case $r \in (0, 1)$, $s \in [r, 1]$

In the case $r \in (0, 1)$, $s \in [r, 1]$, the dynamic (11) reads:

$$(32)\begin{cases}
\dot{x}(t) + \phi \dot{x}(t) = -\beta(t)(\nabla f(x(t)) + A^T \lambda(t) + \delta t^s \dot{\lambda}(t)) + \sigma A^T (Ax(t) - b) + \epsilon(t), \\
\dot{\lambda}(t) + \phi \dot{\lambda}(t) = \beta(t)(A(x(t) + \delta t^s \dot{x}(t)) - b).
\end{cases}$$

with $\alpha > 0$, $\delta > 0$, $\sigma \geq 0$, $t \geq t_0 > 0$. We will investigate the convergence properties of dynamic (32).
Theorem 2.5. Assume that $\beta : [t_0, +\infty) \rightarrow (0, +\infty)$ is continuous differentiable function with

$$t^s \beta(t) \leq \left( \frac{1}{\delta} - \tau t^{s-1} \right) \beta(t)$$

and $\epsilon : [t_0, +\infty) \rightarrow \mathcal{H}_1$ satisfies

$$\int_{t_0}^{+\infty} t^{-s/2} \|\epsilon(t)\| dt < +\infty,$$

where $\tau \in (0, r + s)$. Assume $\alpha \delta > 1$ when $s = r$; $\tau \delta \leq 1$ when $s = 1$. Let $(x(t), \lambda(t))$ be a global solution of the dynamic \(32\) and $(x^*, \lambda^*) \in \Omega$. The following results hold:

(i) $\int_{t_0}^{+\infty} \left( \left( \frac{1}{\delta} t^{-s} - \tau t^{r-1} \right) \beta(t) - t^s \beta(t) \right) (L^\delta (x(t), \lambda^*) - L^\delta (x^*, \lambda^*)) dt < +\infty$.

(ii) $\int_{t_0}^{+\infty} t^{-s} \beta(t) \|Ax(t) - b\|^2 dt < +\infty$, $\int_{t_0}^{+\infty} t^{r-s} (\|\dot{x}(t)\|^2 + \|\dot{\lambda}(t)\|^2) dt < +\infty$.

(iii) $\|\dot{x}(t)\| + \|\dot{\lambda}(t)\| = O\left( \frac{1}{t^\beta(t)} \right)$.

(iv) When $\lim_{t \rightarrow +\infty} t^s \beta(t) = +\infty$,

$$L(x(t), \lambda^*) - L(x^*, \lambda^*) = O\left( \frac{1}{t^\beta(t)} \right), \quad \|Ax(t) - b\| = O\left( \frac{1}{t^\beta(t)} \right).$$

Proof. Given $\lambda \in \mathcal{H}_2$, recall energy functions $E^\lambda(t)$ and $E^\lambda_\ast(t)$ from (13), (14) with $r \in (0, 1), s \in [r, 1], \rho = \frac{r}{s}$ and

$$\theta(t) = \frac{1}{\delta} t^{-s/2}, \quad \eta(t) = -\frac{1}{\delta} t^{-s-r} \left( \frac{1}{\delta} t^{-s-r} + (\tau - s) t^{r-1} - \alpha \right).$$

Then the equations (A.2) and (A.4) are automatically satisfied.

We claim that there exists $C_1 < 0$ and $t_1 \geq t_0$ such that

$$\frac{1}{\delta} t^{-r-s} + \frac{r}{2} t^{r-1} - \alpha \leq C_1, \quad \forall \ t \geq t_1. \quad (35)$$

Indeed, when $s = r$, since $\alpha \delta > 1$ and $r \in (0, 1)$, there exists $t_1 \geq t_0$ such that $\frac{1}{\delta} t^{-r-s} + \frac{r}{2} t^{r-1} - \alpha = \frac{1}{\delta} - \alpha + \frac{r}{2} t^{r-1} \leq \frac{1}{\delta} (1 - \alpha) < 0$; when $s \in (r, 1)$, since $r \in (0, 1)$, there exists $t_1 \geq t_0$ such that $\frac{1}{\delta} t^{-r-s} + \frac{r}{2} t^{r-1} - \alpha \leq -\frac{r}{2} < 0$. Since $\frac{r}{2} < \frac{r}{s} = s$, it follows from (35) that

$$\frac{1}{\delta} t^{-r-s} + (\tau - s) t^{r-1} - \alpha \leq C_1, \quad \forall \ t \geq t_1,$$

and it yields

$$\eta(t) \geq \frac{-C_1}{\delta} t^{-r-s-\tau} \geq 0, \quad \forall \ t \geq t_1. \quad (36)$$

Since $\tau \in (0, s + r)$, then there exist $t_2 \geq t_1$ such that

$$\alpha (\tau - s - r) - (\tau - s - 1)(\tau - s) t^{r-1} < 0, \quad \forall \ t \geq t_2,$$

so we can compute

$$\theta(t) \dot{\theta}(t) + \frac{\dot{\eta}(t)}{2} = \frac{1}{2\delta} t^{-r-s-r-1} \left( \alpha (\tau - s - r) - (\tau - s - 1)(\tau - s) t^{r-1} \right) < 0$$

13
for all \( t \geq t_2 \). Then (A.1) and (A.3) hold for any \( t \geq t_2 \).

It follows from (35) that
\[
\theta(t) + \rho t^{\tau - r} - \alpha t^{\tau - r} = t^{\tau - r} \left( \frac{1}{\delta} t^r - s + \frac{\tau}{2} t^{\tau - 1} - \alpha \right) \leq C_1 t^{\tau - r} < 0, \quad \forall t \geq t_1.
\] (37)

By computation, and from (33), we have
\[
t^\tau \dot{\beta}(t) + (2\rho t^{\tau - 1} - \theta(t))\beta(t) = t^{\tau - s}(t^\tau \dot{\beta}(t) - (\frac{1}{\delta} - \tau t^{\tau - 1})\beta(t)) \leq 0.
\]
for all \( t \geq t_0 \). Let \( \lambda = \lambda^* \), then \( \mathcal{L}^\sigma(x(t), \lambda^*) - \mathcal{L}^\sigma(x^*, \lambda^*) \geq 0 \), this together with (34), (37) and (A.5) yields
\[
\mathcal{E}^{\lambda^*, \rho}(t) \leq C_1 t^{\tau - r} \left( \| \dot{x}(t) \|^2 + \| \dot{\lambda}(t) \|^2 \right) - \frac{\sigma t^{\tau - s} \beta(t)}{2\delta} \| A\lambda(t) - b \|^2
\] (38)
\[+ (t^\tau \dot{\beta}(t) - (\frac{1}{\delta} t^r - \tau t^{\tau - 1})\beta(t)) (\mathcal{L}^\sigma(x(t), \lambda^*) - \mathcal{L}^\sigma(x^*, \lambda^*)) \leq 0
\]
for all \( t \geq t_2 \). Then \( \mathcal{E}^{\lambda^*, \rho}(\cdot) \) is nonincreasing on \([t_2, +\infty)\),
\[
\mathcal{E}^{\lambda^*, \rho}(t) \leq \mathcal{E}^{\lambda^*, \rho}(t_2), \quad \forall t \geq t_2.
\]

Since \( \int_{t_0}^{+\infty} t^{\tau/2} \| \epsilon(t) \| dt < +\infty \), by similar arguments in proof of Theorem 2.1 and using the fact \( C_1 < 0 \), we obtain that \( \mathcal{E}^{\lambda^*, \rho}(\cdot) \) and \( \mathcal{E}^{\lambda^*, \rho}(\cdot) \) are bounded on \([t_0, +\infty)\), and then (i), (ii), (iv) hold. It follows from (34), (36) and the definition of \( \mathcal{E}^{\lambda^*, \rho}(\cdot) \) that
\[
\sup_{t \geq t_0} t^{(\tau - s - r)/2} \| x(t) - x^* \| < +\infty, \quad \sup_{t \geq t_0} \frac{1}{\delta} t^{\tau/2 - s} (x(t) - x^*) + t^{\tau/2} \dot{x}(t) \| < +\infty.
\]

Since \( s \in [r, 1] \), then
\[
\sup_{t \geq t_0} t^{\tau/2} \| \dot{x}(t) \| \leq \frac{1}{\delta} \sup_{t \geq t_0} t^{\tau/2 - s} \| x(t) - x^* \| + \sup_{t \geq t_0} \frac{1}{\delta} t^{\tau/2 - s} (x(t) - x^*) + t^{\tau/2} \dot{x}(t) \| \leq \frac{1}{\delta} \sup_{t \geq t_0} t^{(\tau - s - r)/2} \| x(t) - x^* \| + \sup_{t \geq t_0} \frac{1}{\delta} t^{\tau/2 - s} (x(t) - x^*) + t^{\tau/2} \dot{x}(t) \| < +\infty.
\]

Similarly, \( \sup_{t \geq t_0} t^{\tau/2} \| \dot{\lambda}(t) \| < +\infty \), the result (iii) holds. \( \square \)

If we take \( \beta(t) \) satisfying
\[
t^\tau \dot{\beta}(t) \leq (\frac{1}{\delta} - (r + s) t^{\tau - 1}) \beta(t),
\]
then for any \( \tau \in (0, r + s) \), (33) is satisfied, and then we obtain the following results from Theorem 2.5

**Corollary 2.1.** Assume that
\[
\int_{t_0}^{+\infty} t^{(r+s)/2} \| \epsilon(t) \| dt < +\infty.
\] (39)
Suppose \( \alpha \delta > 1 \) when \( s = r; \ \delta (r + s) \geq 1 \) when \( s = 1 \). Let \( (x(t), \lambda(t)) \) be a global solution of the dynamic (32). Then for any \( (x^*, \lambda^*) \in \Omega \) and \( \tau \in (0, r + s) \):

\[14\]
(i) \( \| \dot{x}(t) \| + \| \dot{\lambda}(t) \| = O \left( \frac{1}{t^{1/2}} \right) \).

(ii) When \( \lim_{t \to +\infty} t^r \beta(t) = +\infty \),

\[
\mathcal{L}(x(t), \lambda^*) - \mathcal{L}(x^*, \lambda^*) = O \left( \frac{1}{t^{r/2} \beta(t)} \right), \quad \| Ax(t) - b \| = O \left( \frac{1}{t^{r/2} \sqrt{\beta(t)}} \right).
\]

**Remark 2.7.** In proof process of Theorem 2.4 we can note that the boundedness of trajectory \((x(t), \lambda(t))\) is not guaranteed. If (39) holds, we can obtain

\[
\sup_{t \geq t_0} t^{(r-s-\tau)/2}(\|x(t) - x^*\| + \|\lambda(t) - \lambda^*\|) < +\infty
\]

is satisfied for any \((x^*, \lambda^*) \in \Omega\) and \(\tau \in (0, r + s)\), then we get that \(t^p(\|x(t) - x^*\| + \|\lambda(t) - \lambda^*\|)\) is bounded for any \(p < 0\). When objective function \(f\) satisfying the following coercive condition:

\[
\lim_{\|x\| \to +\infty} f(x) = +\infty,
\]

we also can obtain the boundedness of \(x(t)\) of dynamic (32) from (iv) of Theorem 2.5.

**Remark 2.8.** Taking \(\beta(t) \equiv 1, s = 1\), and letting \(\int_{t_0}^{+\infty} t^{(r+1)/2}\|f(t)\| dt < +\infty\) and \(\delta(r+1) \geq 1\). We obtain the \(O(1/t^r)\) rate of convergence for any \(\tau \in (0, r+1)\), since \(r \in (0, 1)\), \(r + 1 > 2r\), so the results in Corollary 2.7 improve the corresponding results in [24, Theorem 3.4] which only obtain the \(O(1/t^{2r})\) convergence rate. In the case \(\alpha(t) = \frac{1}{\tau} \) with \(\alpha > 0\), \(r \in (0, 1)\), the \(o(1/t^{r+1})\) convergence of \((IGS_{\alpha})\) and \((IGS_{\alpha, \epsilon})\) for problem (i) have been obtained in [24, Corollary 4.5] and [13, Theorem 1.2] respectively, which have subtle differences of dynamic (32) for problem (2). The assumption \(\int_{t_0}^{+\infty} t^{(r+1)/2}\|f(t)\| dt < +\infty\) also can find in [13].

By similar discussions in Section 2.1, we obtain the following optimal convergence rates of Theorem 2.5 and the proof is similar to Theorem 2.2 so we omit it.

**Theorem 2.6.** Let \(\beta(t) = \mu \frac{t^{r-s}}{t^{r+s}}\) with \(\mu > 0\), \(r \in (0, r+s)\), \(r \in (0, 1)\), \(s \in [r, 1)\), \(\sigma \geq 0\). Assume \(\alpha \delta > 1\) when \(s = r\). Suppose \(\int_{t_0}^{+\infty} t^{r/2}\|f(t)\| dt < +\infty\). Let \((x(t), \lambda(t))\) be a solution of dynamic (16) and \((x^*, \lambda^*) \in \Omega\). Then

\[
|f(x(t)) - f(x^*)| = O \left( \frac{1}{e^{\pi (r-s)r / 4}} \right), \quad \| Ax(t) - b \| = O \left( \frac{1}{e^{\pi (r-s)r / 4}} \right).
\]

**Theorem 2.7.** Let \(\beta(t) = \mu t^{1-r}\) with \(\mu > 0\), \(r \in (0, r+1)\), \(r \in (0, 1)\), \(s = 1\), \(\delta r \leq 1, \sigma \geq 0\). Suppose \(\int_{t_0}^{+\infty} t^{r/2}\|f(t)\| dt < +\infty\). Let \((x(t), \lambda(t))\) be a solution of dynamic (16) and \((x^*, \lambda^*) \in \Omega\). Then

\[
|f(x(t)) - f(x^*)| = O \left( \frac{1}{t^{1/8}} \right), \quad \| Ax(t) - b \| = O \left( \frac{1}{t^{1/8}} \right).
\]

**Remark 2.9.** When \(s = 1\), taking \(\tau = \frac{1}{\sigma}\), then \(\beta(t) = \mu > 0\) is a positive constant time scaling. For any \(\frac{1}{\sigma} < r+1\), we can obtain the \(O(1/t^{1/2})\) convergence rates of objective function and constraint.
2.3. Case $r = 1, s = 1$

Consider the case when $r = 1, s = 1$, i.e., the dynamic (41) becomes:

$$\begin{cases}
\ddot{x}(t) + \frac{\dot{\theta}}{\tau} \dot{x}(t) = -\beta(t)(\nabla f(x(t)) + AT(\lambda(t) + \delta t \dot{\lambda}(t)) + \sigma A^T(Ax(t) - b)) + \epsilon(t), \\
\dot{\lambda}(t) + \frac{\dot{\theta}}{\tau} \dot{\lambda}(t) = \beta(t)(Ax(t) + \delta t \dot{x}(t)) - b).
\end{cases}$$

We will discuss dynamic (41) with $\alpha \leq 3$ and $\alpha > 3$ respectively.

**Theorem 2.8.** Assume that $\beta : [t_0, +\infty) \to (0, +\infty)$ is continuous differentiable function with

$$t \beta(t) \leq \tau \beta(t),$$

and $\epsilon : [t_0, +\infty) \to \mathcal{H}_1$ satisfies

$$\int_{t_0}^{+\infty} t^{\frac{\alpha - 3}{\alpha}} \|\epsilon(t)\| dt < +\infty.$$

Let $0 \leq \tau \leq \alpha \leq 3$, $\delta = \frac{3}{2\alpha + \tau}$ and $(x(t), \lambda(t))$ be a global solution of the dynamic (41). Then for any $(x^*, \lambda^*) \in \Omega$, the following conclusions hold:

(i) $\int_{t_0}^{+\infty} t^{\frac{2(\alpha - 3)}{3}} \beta(t) \|Ax(t) - b\|^2 dt < +\infty$.

(ii) When $\tau \in (0, \alpha)$: for any $\rho \in [0, \frac{\alpha + \tau}{3})$,

$$\int_{t_0}^{+\infty} t^{2\rho - 1} \beta(t) (\mathcal{L}(x(t), \lambda^*) - \mathcal{L}(x^*, \lambda^*)) dt < +\infty, \quad \int_{t_0}^{+\infty} t^{2\rho - 1} \|\dot{x}(t)\|^2 + \|\dot{\lambda}(t)\|^2 dt < +\infty.$$

(iii) When $\lim_{t \to +\infty} t^{\frac{2(\alpha - 3)}{3}} \beta(t) = +\infty$:

$$\mathcal{L}(x(t), \lambda^*) - \mathcal{L}(x^*, \lambda^*) = \mathcal{O}(\frac{1}{t^{\frac{2(\alpha - 3)}{3}} \beta(t)}), \quad \|Ax(t) - b\| = \mathcal{O}(\frac{1}{t^{\frac{\alpha + \tau}{3}} \sqrt{\beta(t)}}).$$

(iv) When $\tau = 0$ and $\alpha = 3$:

$$\|\dot{x}(t)\| + \|\dot{\lambda}(t)\| = \mathcal{O}(\frac{1}{t^\rho}), \quad \forall \rho \in (0, 1).$$

Otherwise:

$$\|\dot{x}(t)\| + \|\dot{\lambda}(t)\| = \mathcal{O}(\frac{1}{t^\rho}).$$

**Proof.** Given $\lambda \in \mathcal{H}_2$, define $\mathcal{E}_{\lambda,\rho}(t)$ and $\mathcal{E}_{\beta,\rho}(t)$ as (11) with $r = s = 1, \rho \in [0, \frac{\alpha + \tau}{3}]$ and

$$\theta(t) = \frac{2\alpha + \tau}{3} t^{\rho - 1}, \quad \eta(t) = \frac{2\alpha + \tau}{3} \left(1 + \frac{\alpha - \tau}{3} - 2\rho\right) t^{2\rho - 2}.$$  

By computation, we have

$$\eta(t) \geq \frac{2\alpha + \tau}{3} \left(1 - \frac{\alpha - \tau}{3}\right) t^{2\rho - 2} \geq 0,$$

and (A.2), (A.4) are satisfied. Since $0 \leq \tau \leq \alpha \leq 3$ and $\rho \in [0, \frac{\alpha + \tau}{3}]$, we also can verify that

$$\theta(t) \dot{\theta}(t) + \frac{\dot{\eta}(t)}{2} = \frac{2\alpha + \tau}{3} (\alpha + 1 - 2\rho)(\rho - 1) t^{2\rho - 3} \leq 0.$$
Then (A.1) - (A.4) hold for any $t \geq t_0$.

It is easy to verify that

$$
\theta(t) + \rho t^{\rho-1} - \alpha t^{\rho-r} = (\rho - \frac{\alpha - \tau}{3}) t^{\rho-1} \leq 0
$$

and

$$
t^\rho \dot{\beta}(t) + (2\rho t^{\rho-1} - \theta(t)) \beta(t) = t^{\rho-1} (t \dot{\beta}(t) - \tau \beta(t) + (\tau + 2\rho - \frac{2\alpha + \tau}{3}) \beta(t))
\leq 2(\rho - \frac{\alpha - \tau}{3}) t^{\rho-1} \beta(t)
\leq 0
$$

for all $t \geq t_0$. This together with (A.5) in case $\lambda = \lambda^*$ implies

$$
\mathcal{E}_\epsilon^{\lambda^*, \rho}(t) \leq (\rho - \frac{\alpha - \tau}{3}) t^{2\rho-1} (\|\dot{x}(t)\|^2 + \|\dot{\lambda}(t)\|^2) + 2(\rho - \frac{\alpha - \tau}{3}) t^{\rho-1} \beta(t) (\mathcal{L}^\rho(x(t), \lambda^*) - \mathcal{L}^\rho(x^*, \lambda^*))
- \frac{\sigma t^{2\rho-1} \beta(t)}{2\delta} \|Ax(t) - b\|^2
\leq 0.
$$

Then $\mathcal{E}_\epsilon^{\lambda^*, \rho}(\cdot)$ is nonincreasing on $[t_0, +\infty)$,

$$
\mathcal{E}_\epsilon^{\lambda^*, \rho}(t) \leq \mathcal{E}_\epsilon^{\lambda^*, \rho}(t_0), \quad \forall t \geq t_0.
$$

Since $\int_{t_0}^{+\infty} \epsilon(t) dt < +\infty$, for any $\rho \in [0, \frac{\alpha - \tau}{3}]$, we have

$$
\int_{t_0}^{+\infty} t^\rho \epsilon(t) dt < +\infty.
$$

By similar arguments in proof of Theorem 2.1 we obtain the boundedness of $\mathcal{E}_\epsilon^{\lambda^*, \rho}(\cdot)$ and $\mathcal{E}_\epsilon^{\lambda^*, \rho}(\cdot)$. Since (47) holds for any $\rho \in [0, \frac{\alpha - \tau}{3}]$, integrating it on $[t_0, +\infty)$, and following from the boundedness of $\mathcal{E}_\epsilon^{\lambda^*, \rho}(\cdot)$, we get the results (i) - (ii).

Since $\mathcal{E}_\epsilon^{\lambda^*, \rho}(\cdot)$ is bounded for any $\rho \in [0, \frac{\alpha - \tau}{3}]$, by the definition of $\mathcal{E}_\epsilon^{\lambda^*, \rho}(\cdot)$ and (3), (43), we obtain (iii),

$$
\sup_{t \geq t_0} \sqrt{1 + \frac{\alpha - \tau}{3} - 2\rho \times t^{\rho-1} \|x(t) - x^*\|} < +\infty
$$

and

$$
\sup_{t \geq t_0} \frac{2\alpha + \tau}{3} t^{\rho-1} (x(t) - x^*) + t^\rho \dot{x}(t) \| < +\infty
$$

for any $\rho \in [0, \frac{\alpha - \tau}{3}]$.

When $\tau = 0$ and $\alpha = 3$: for any $\rho \in (0, 1)$, $1 + \frac{2\alpha - \tau}{3} - 2\rho = 2(1 - \rho) > 0$, it follows from (48) and (49) that

$$
\sup_{t \geq t_0} t^{\rho-1} \|x(t) - x^*\| < +\infty
$$

and then

$$
\sup_{t \geq t_0} t^\rho \|\dot{x}(t)\| \leq 2 \sup_{t \geq t_0} t^{\rho-1} \|x(t) - x^*\| + \sup_{t \geq t_0} \|2 t^{\rho-1} (x(t) - x^*) + t^\rho \dot{x}(t)\| < +\infty,
$$

17
for any $\rho \in (0, 1)$. Similarly $\sup_{t \geq t_0} t^\rho \|\dot{\lambda}(t)\| < +\infty$.

Otherwise $\alpha - \tau < 3$, taking $\rho = \frac{\alpha - \tau}{3}$, then $1 + \frac{\alpha - \tau}{3} - 2\rho = 1 - \frac{\alpha - \tau}{3} > 0$, by similar discussions in above, we get (iv).

\[ \square \]

**Remark 2.10.** Following from above proof process, when $\tau = 0$ and $\alpha = 3$:

\[
\sup_{t \geq t_0} t^{\rho-1}(\|x(t) - x^*\| + \|\lambda(t) - \lambda^*\|) < +\infty, \quad \forall \rho \in (0, 1),
\]

Otherwise:

\[
\sup_{t \geq t_0} t^{\frac{\alpha - \tau - 3}{3}}(\|x(t) - x^*\| + \|\lambda(t) - \lambda^*\|) < +\infty.
\]

When the coercive condition \[40\] satisfied, we also can obtain the boundedness of $x(t)$ of dynamic (42) with $\alpha \leq 3$.

**Remark 2.11.** Theorem 2.8 extends the results in \[27\], Corollary 2.9] and \[47, Theorem 3.2] to general case. Taking $A = 0$, $b = 0$, the dynamic (41) reduces to

\[
\dot{x}(t) + \frac{\alpha}{t} x(t) + \beta(t) \nabla f(x(t)) = \epsilon(t),
\]

with $\alpha \leq 3$ for solving unconstrained optimization problem, then Theorem 2.8 also complements the results in \[47, Theorem A.1\], which considered the case $\alpha \geq 3$.

Taking $t\beta(t) = \tau \beta(t)$, in which $\beta(t) = \mu t^\tau$ with $\mu > 0$, we investigate the improved rate of convergence.

**Theorem 2.9.** Let $\beta(t) = \mu t^\tau$ with $\mu > 0$, $0 \leq \tau \leq \alpha \leq 3$, $\delta = \frac{3}{2\alpha + 3}$, $\sigma \geq 0$. Suppose $\int_{t_0}^{+\infty} t^{(\alpha - \tau)/3} |\epsilon(t)| dt < +\infty$. Let $(x(t), \lambda(t))$ be a solution of dynamic (41). For any $(x^*, \lambda^*) \in \Omega$:

\[
|f(x(t)) - f(x^*)| = O\left(\frac{1}{t^{(2\alpha + 3)/3}}\right), \quad \|Ax(t) - b\| = O\left(\frac{1}{t^{(2\alpha + 3)/3}}\right).
\]

**Proof.** Recall the energy functions $E^\lambda(t)$ and $E^\epsilon(t)$ from Theorem 2.8 with $\beta(t) = \mu t^\tau$, $\rho = \frac{\alpha - \tau}{3}$. Then

\[
t^\rho \beta(t) + (2\rho t^{\rho-1} - \theta(t))\beta(t) = 0,
\]

this together with (41) and (A.5) yields

\[
E^\lambda(t) \leq -\frac{\sigma \beta(t)}{2\delta} \|Ax(t) - b\|^2 \leq 0, \quad \forall t \geq t_0, \lambda \in H_2.
\]

By similar arguments in Theorem 2.2 we obtain the results. \[\square\]

From Theorem 2.9 we obtain the following results in the case $\tau = 0$ and $\tau = \alpha$, respectively.

**Corollary 2.2.** Let $\beta(t) = \beta > 0$, $\alpha \leq 3$, $\delta = \frac{3}{2\alpha}$, $\sigma \geq 0$. Suppose $\int_{t_0}^{+\infty} t^{\alpha/3} |\epsilon(t)| dt < +\infty$. Let $(x(t), \lambda(t))$ be a solution of dynamic (41). For any $(x^*, \lambda^*) \in \Omega$:

\[
|f(x(t)) - f(x^*)| = O\left(\frac{1}{t^{2\alpha + 3}}\right), \quad \|Ax(t) - b\| = O\left(\frac{1}{t^{2\alpha + 3}}\right).
\]
Corollary 2.3. Let \( \beta(t) = \mu_t^{\alpha} \) with \( \mu > 0, \alpha \leq 3, \delta = \frac{1}{\alpha}, \sigma \geq 0. \) Suppose \( \int_{t_0}^{+\infty} \|\varepsilon(t)\|dt < +\infty. \) Let \( (x(t), \lambda(t)) \) be a solution of dynamic (11). For any \( (x^*, \lambda^*) \in \Omega: \)

\[
|f(x(t)) - f(x^*)| = O\left(\frac{1}{\|x(t)\|}\right), \quad \|Ax(t) - b\| = O\left(\frac{1}{\|x(t)\|}\right).
\]

Remark 2.12. Taking \( \beta = 1, \) the dynamic (11) has been investigate in [27] and [47] for \( \alpha \leq 3. \) Corollary 2.2 improves the convergence rates of [27, Corollary 2.9] and [47, Theorem 3.2], which only obtain \( O\left(\frac{1}{\|x(t)\|}\right) \) convergence rate of \( |f(x(t)) - f(x^*)| \) and \( \|Ax(t) - b\|, \) and it also can be viewed as analogs of the results in [27, [47], where the convergence rate analysis of (IGS) with \( \alpha(t) = \frac{\alpha}{\tau}, \alpha \leq 3 \) for unconstrained optimization problem [5]. Corollary 2.3 shows the optimal convergence rate we can expect of dynamic (11) with \( \alpha \leq 3. \)

Next, we investigate the convergence rate of dynamic (11) with \( \alpha > 3. \) The similar results can be found in [20].

Theorem 2.10. Assume that \( \beta : [t_0, +\infty) \to (0, +\infty) \) is continuous differentiable function with

\[
t\beta'(t) \leq \left(\frac{1}{\delta} - 2\right)\beta(t),
\]

and \( 2 \leq \frac{1}{\delta} < \alpha - 1. \) Let \( \varepsilon : [t_0, +\infty) \to \mathcal{H}_1 \) with

\[
\int_{t_0}^{+\infty} t\|\varepsilon(t)\|dt < +\infty.
\]

Let \( (x(t), \lambda(t)) \) be a global solution of the dynamic (11) and \( y(x^*, \lambda^*) \in \Omega. \) Then \( (x(t), \lambda(t)) \) is bounded and the following conclusions hold:

(i) \( \int_{t_0}^{+\infty} t((\frac{1}{\delta} - 2)\beta(t) - t\beta'(t))(\mathcal{L}^\sigma(x(t), \lambda^*) - \mathcal{L}^\sigma(x^*, \lambda^*))dt < +\infty. \)

(ii) \( \int_{t_0}^{+\infty} t\beta'(t)\|Ax(t) - b\|^2dt < +\infty, \int_{t_0}^{+\infty} t\|\dot{x}(t)\|^2 + \|\dot{\lambda}(t)\|^2 dt < +\infty. \)

(iii) \( \|\dot{x}(t)\|^2 + \|\dot{\lambda}(t)\|^2 = O\left(\frac{1}{t}\right). \)

(iv) When \( \lim_{t \to +\infty} t^2\beta(t) = +\infty: \)

\[
\mathcal{L}(x(t), \lambda^*) - \mathcal{L}(x^*, \lambda^*) = O\left(\frac{1}{t^2\beta(t)}\right), \quad \|Ax(t) - b\| = O\left(\frac{1}{t\sqrt{\beta(t)}}\right).
\]

Proof. Given \( \lambda \in \mathcal{H}_2, \) define \( \mathcal{E}^{\lambda^\rho}(t) \) and \( \mathcal{E}^{\lambda^\rho}(t) \) as [15], [14] with \( r = s = \rho = 1 \) and

\[
\theta(t) = \frac{1}{\delta}, \quad \eta(t) = \frac{\alpha\delta - \delta - 1}{\delta^2}.
\]

Since \( \alpha - 1 > \frac{1}{\delta} \geq 2, \) by simple computations we can verify (A.1)-(A.4). It follows from assumptions that

\[
t^\alpha \dot{\beta}(t) + (2\rho t^{\rho-1} - \theta(t))\beta(t) = t\dot{\beta}(t) + (2 - \frac{1}{\delta})\beta(t) \leq 0.
\]

Taking \( \lambda = \lambda^*, \) this together with (A.5) implies

\[
\mathcal{E}^{\lambda^\rho}(t) \leq \left(\frac{1}{\delta} + 1 - \alpha\right)t(\|\dot{x}(t)\|^2 + \|\dot{\lambda}(t)\|^2) + t(t\dot{\beta}(t) + (2 - \frac{1}{\delta})\beta(t))(\mathcal{L}^\sigma(x(t), \lambda^*) - \mathcal{L}^\sigma(x^*, \lambda^*))
\]
\[-\frac{\sigma t \beta(t)}{2\delta} \|Ax(t) - b\|^2 \leq 0.\]

By similarly arguments in proof of Theorem 2.1, we obtain the boundedness of $\mathcal{E}^{\lambda^*} \cdot \rho(\cdot)$ and $\mathcal{E}_\epsilon^{\lambda^*} \cdot \rho(\cdot)$. This yields (i), (ii), (iv). Since $\eta(t) = \frac{\alpha \delta - \beta - 1}{\sigma} > 0$, we get that $(x(t), \lambda(t))$ is bounded and

$$\sup_{t \geq t_0} (\|\dot{x}(t)\|^2 + \|\lambda(t)\|^2) < +\infty.$$ 

This implies (iii).

\textbf{Remark 2.13.} Theorem 2.10 extends the results in [3, Theorem A.1] and [16, Section 3.2] from $(IGS_{\alpha, \epsilon})$ with $\alpha(t) = \frac{\alpha}{t}$, $\alpha > 3$ for problem (13) to primal-dual dynamic for problem (2). Taking $\beta(t) \equiv 1$, we recover the convergence rate of [27, Corollary 2.9] and [47, Theorem 3.1], moreover when $A = 0$, $b = 0$, we get the classical results for $(IGS_{\alpha})$ and $(IGS_{\alpha, \epsilon})$ with $\alpha(t) = \frac{\alpha}{t}$ with $\alpha > 3$, which can be seen as a continuous version of the Nesterov method, see [6, 17, 33, 40].

Let $t \dot{\beta}(t) = (\frac{1}{\alpha} - 2) \beta(t)$. We have $\beta(t) = \mu t^{\frac{1}{\alpha} - 2}$ with $\mu > 0$. By similar proof of Theorem 2.2, we obtained following results, and the corresponding results of unperturbed case can be found in [3, Proposition 6.3].

\textbf{Theorem 2.11.} Let $\beta(t) = \mu t^{1/\delta - 2}$ with $\mu > 0$, $2 \leq \frac{1}{\alpha} < \alpha - 1$, $\sigma \geq 0$. Suppose $\int_{t_0}^{+\infty} t \|\epsilon(t)\| dt < +\infty$. Let $(x(t), \lambda(t))$ be a solution of dynamic (11). For any $(x^*, \lambda^*) \in \Omega$:

$$|f(x(t)) - f(x^*)| = \mathcal{O}(\frac{1}{t^{1/\delta}}), \quad \|Ax(t) - b\| = \mathcal{O}(\frac{1}{t^{1/\delta}}).$$

From Theorem 2.11, we have following result.

\textbf{Corollary 2.4.} Let $\beta(t) = \beta > 0$, $\delta = \frac{1}{2}$, $\alpha > 3$, $\sigma \geq 0$. Suppose $\int_{t_0}^{+\infty} t \|\epsilon(t)\| dt < +\infty$. Let $(x(t), \lambda(t))$ be a solution of dynamic (11). For any $(x^*, \lambda^*) \in \Omega$:

$$|f(x(t)) - f(x^*)| = \mathcal{O}(\frac{1}{t^{1/2}}), \quad \|Ax(t) - b\| = \mathcal{O}(\frac{1}{t^{1/2}}).$$

\textbf{Remark 2.14.} Theorem 2.11 shows the optimal convergence rates of dynamic (11) in the case $\alpha > 3$. The $O(1/t^{\alpha/2 + 2})$ convergence rate results associated with the time scaling $\beta(t) = \mu t^\alpha$ for unconstrained optimization problem (6) can be found in [3, 40]; it also can be found in [24] with Euclidean setting of Bregman distance for problem (25). Corollary 2.4 shows that the convergence rate of objective function and constraint of dynamical system (8) is $\mathcal{O}(\frac{1}{t^2})$ instead of $\mathcal{O}(\frac{1}{t^3})$.

2.4. Summary of results

In the subsection, we complete the tables giving a synthetic view of convergence results in before.

For dynamic (13) with different $r$ and $s$, chose suitable parameters $\alpha$, $\delta$. Table 1 lists the convergences rates for $L(x(t), \lambda^*) - L(x^*, \lambda^*)$ of dynamic (11) under different assumptions of $\beta(t)$ and $\epsilon(t)$. Table 2 summarizes
the properties of trajectory \((x(t), \lambda(t))\) and its derivates \((\dot{x}(t), \dot{\lambda}(t))\). (See Theorem 2.3, Corollary 2.4, Theorem 2.7, Corollary 2.8, Theorem 2.10, Remark 2.9, Remark 2.10). The results extend the inertial dynamic with time scaling in \([9, 10, 12, 44]\) for problem \((3)\) to primal-dual dynamic \((1)\) for problem \((2)\). Taking \(A = 0, b = 0\), our results also can complement the existing results of the inertial dynamic with time scaling.

\[
\begin{array}{|c|c|c|c|}
\hline
r, s & \beta(t) \leq (t) & \epsilon(t) & L(x(t), \lambda(t)) - L(x^*, \lambda^*) \\
\hline
r = 0, s \in [0, 1] & t^{r}\dot{\beta}(t) \leq \frac{1}{\beta} - st^{s-1}\beta(t) & \int_{t_0}^{\infty} t^{r}\epsilon(t) dt < +\infty & O\left(\frac{1}{r^{\min(\beta(t))}}\right) \\
\hline
r \in (0, 1), s \in [r, 1] & t^{r}\dot{\beta}(t) \leq \frac{1}{\beta} - (r + s)t^{s-1}\beta(t) & \int_{t_0}^{\infty} t^{r}\epsilon(t) dt < +\infty & O\left(\frac{1}{r^{\min(\beta(t))}}\right), \forall \rho \in (0, r + s) \\
\hline
r = s = 1 & \alpha \leq 3 & \int_{t_0}^{\infty} t^{r}\epsilon(t) dt < +\infty & O\left(\frac{1}{r^{\min(\beta(t))}}\right) \\
\hline
& \alpha > 3 & \int_{t_0}^{\infty} t^{r}\epsilon(t) dt < +\infty & O\left(\frac{1}{r^{\min(\beta(t))}}\right) \\
\hline
\end{array}
\]

Table 1: Convergence rates for \(L(x(t), \lambda(t)) - L(x^*, \lambda^*)\) of dynamic \((1)\)

Table 2: Summary of trajectory properties

| \(r, s\) | \(||\dot{x}(t)|| + ||\dot{\lambda}(t)||\) | \(I = ||x(t) - x^*|| + ||\lambda(t) - \lambda^*||\) |
|---|---|---|
| \(r = s = 0\) | bounded | \(I\) bounded |
| \(r = 0, s \in (0, 1)\) | \(O\left(\frac{1}{r^{\min(\beta(t))}}\right)\) | \(I\) bounded |
| \(r \in (0, 1), s \in [r, 1]\) | \(O\left(\frac{1}{r^{\min(\beta(t))}}\right), \forall \rho \in (0, r + s)\) | \(t^rI\) bounded, \(\forall \rho \in \left(-\frac{r + s}{2}, 0\right)\) |
| \(r = s = 1\) | \(\alpha = 3, \tau = 0\) | \(O\left(\frac{1}{r^{\min(\beta(t))}}\right), \forall \rho \in (0, 1)\) | \(t^rI\) bounded, \(\forall \rho \in (-1, 0)\) |
| & \(\alpha \leq 3, 0 \leq \alpha - \tau < 3\) | \(O\left(\frac{1}{r^{\min(\beta(t))}}\right)\) | \(t^{\frac{\alpha + 3}{4} - 1}I\) bounded |
| & \(\alpha > 3\) | \(O\left(\frac{1}{r^{\min(\beta(t))}}\right)\) | \(I\) bounded |

Select a specific time scaling \(\beta(t)\) with suitable parameters \(\alpha, \delta\). Table 3 shows optimal convergence rates we can expect for different choices of coefficients. (See Theorem 2.2, Theorem 2.3, Theorem 2.4, Theorem 2.6, Theorem 2.7, Corollary 2.8, Theorem 2.11).

Taking time scaling \(\beta(t) \equiv 1\), Table 4 lists the corresponding convergence rates (See Remark 2.3, Theorem 2.4, Theorem 2.5, Theorem 2.6, Corollary 2.7, Corollary 2.8), it extends the convergence rates of \((IGS_\alpha)\) and \((IGS_{\alpha, \epsilon})\) in \([9, 10, 12, 44]\) for unconstrained optimization problems to primal-dual dynamic \((1)\) for linear equality constrained optimization problems. It also extend and complements the existing results of inertial primal-dual dynamic in \([1, 26, 27, 28, 47]\).
Table 3: Optimal convergence rates of $|f(x(t) - f(x^*))|$ and $\|Ax(t) - b\|$  

| $r, s$ | $\beta(t)$ | $|f(x(t) - f(x^*))|$ and $\|Ax(t) - b\|$ |
|-------|------------|---------------------------------
| $r = 0, s \in [0, 1)$ | $t^{\frac{1-\alpha}{1-s}}$ | $O\left(\frac{1}{t^{\frac{1-\alpha}{1-s}}}ight)$ |
| $r = 0, s = 1$ | $\mu t^{\frac{1}{\alpha}}$ | $O\left(\frac{1}{t^{\frac{1}{\alpha}}}ight)$ |
| $r \in (0, 1), s \in [r, 1)$ | $t^{\frac{1-\alpha}{1-s}}$, $\forall \tau \in (0, r+s)$ | $O\left(\frac{1}{t^{\frac{1-\alpha}{1-s}}}ight)$ |
| $r \in (0, 1), s = 1$ | $\mu t^{\frac{1}{\alpha}}$, $\forall \tau \in (0, r+1)$ | $O\left(\frac{1}{t^{\frac{1}{\alpha}}}ight)$ |
| $r = s = 1$ | $\alpha \leq 3$ | $O\left(\frac{1}{t^{\frac{1}{\alpha}}}ight)$ |
| $r = s = 1$ | $\alpha > 3$ | $O\left(\frac{1}{t^{\frac{1}{\alpha}-2}}\right)$ |

Table 4: Convergence rates of dynamic $\left(\frac{1}{t^{\frac{1}{\alpha}}}ight)$ with $\beta(t) \equiv 1$  

| $r, s$ | $|f(x(t)) - f(x^*)|$ and $\|Ax(t) - b\|$ | $\mathcal{L}(x(t), \lambda^*) - \mathcal{L}(x^*, \lambda^*)$ |
|-------|---------------------------------|---------------------------------|
| $r = 0, s = 0$ | $O\left(\frac{1}{t^{\frac{1}{\alpha}}}ight)$ | $O\left(\frac{1}{t^{\frac{1}{\alpha}}}ight)$ |
| $r = 0, s \in (0, 1)$ | $O\left(\frac{1}{t^{\frac{1}{\alpha}}}ight)$ | $O\left(\frac{1}{t^{\frac{1}{\alpha}}}ight)$ |
| $r = 0, s = 1$ | $O\left(\frac{1}{t^{\frac{1}{\alpha}}}ight)$ | $O\left(\frac{1}{t^{\frac{1}{\alpha}}}ight)$ |
| $r \in (0, 1), s \in [r, 1)$ | $O\left(\frac{1}{t^{\frac{1}{\alpha}}}ight)$, $\forall \tau \in (0, r+s)$ | $O\left(\frac{1}{t^{\frac{1}{\alpha}}}ight)$, $\forall \tau \in (0, r+s)$ |
| $r \in (0, 1), s = 1$ | $O\left(\frac{1}{t^{\frac{1}{\alpha}}}ight)$, $\forall \tau \in (0, r+1)$ | $O\left(\frac{1}{t^{\frac{1}{\alpha}}}ight)$, $\forall \tau \in (0, r+1)$ |
| $r = s = 1$ | $\alpha \leq 3$ | $O\left(\frac{1}{t^{\frac{1}{\alpha}}}ight)$ |
| $r = s = 1$ | $\alpha > 3$ | $O\left(\frac{1}{t^{\frac{1}{\alpha}}}ight)$ |

3. Conclusion

In this paper, we propose a family of damped inertial primal-dual dynamical systems with time scaling for solving problem (2) in Hilbert space. We extend the inertial dynamic in [6, 7, 9, 12, 13, 40, 41, 43, 44] for solving unconstrained optimization problems to primal-dual dynamic (1) for solving linear equality constrained convex optimization problems. Our results also extend and complement the existing results of inertial primal-dual dynamics in [5, 26, 27, 28, 47]. Taking $A = 0, b = 0$, our results also complement the convergence rate results of existing inertial dynamic for solving unconstrained convex optimization problems. By discretization of primal-dual dynamic [13], it may lead to new primal-dual algorithms for solving problem (2), how to chose suitable discretization scheme of [13] to get rate-matching algorithms is an interesting direction of research. From
Similarly, we have terms, we get This yields Differentiating of $E$ of $(t)$ satisfy $t$ $0_{\dot{\rho}}(\rho_{\dot{\rho}} - \rho_0)$ $\dot{\rho}_1^*(t)$ $t\rho\delta\theta(\delta\theta) = \alpha t + \rho t^{\rho-1} \dot{x}(t) + t^\rho \dot{\rho}(t)$ $t^\rho \dot{\rho}(t) = -\alpha t^{\rho-1} \dot{x}(t) - t^\rho \beta(t)(\nabla f(x(t)) + A^T(\lambda(t) + \delta t^s \dot{\lambda}(t)) + \sigma A^T(Ax(t) - b)) + t^\rho \epsilon(t)$.

This yields

\[
\begin{align*}
\dot{\rho}_1(t) &= \langle \theta(t)(x(t) - x^*) + \dot{\theta}(t)(x(t) - x^*) + \theta(t)\dot{x}(t) + \rho t^{\rho-1} \dot{x}(t) + t^\rho \dot{\rho}(t) \rangle \\
+ \frac{\dot{\rho}_1(t)}{2} \|x(t) - x^*\|^2 + \eta(t)\langle x(t) - x^*, \dot{x}(t) \rangle \\
= \langle \theta(t)(x(t) - x^*) + \dot{\theta}(t)(x(t) - x^*) + \theta(t) + \rho t^{\rho-1} - \alpha t^{\rho-1} \dot{x}(t) \\
- t^\rho \beta(t)(\nabla f(x(t)) + A^T(\lambda(t) + \delta t^s \dot{\lambda}(t)) + \sigma A^T(Ax(t) - b)) + t^\rho \epsilon(t) \rangle \\
+ \frac{\dot{\rho}_1(t)}{2} \|x(t) - x^*\|^2 + \eta(t)\langle x(t) - x^*, \dot{x}(t) \rangle \\
= \langle \theta(t)(\dot{x}(t) + \dot{\theta}(t))\|x(t) - x^*\|^2 + t^\rho(\theta(t) + \rho t^{\rho-1} - \alpha t^{\rho-1})\|\dot{x}(t)\|^2 \\
+ \theta(t)(\theta(t) + \rho t^{\rho-1} - \alpha t^{\rho-1}) + t^\rho \dot{\theta}(t) + \eta(t)\langle x(t) - x^*, \dot{x}(t) \rangle \\
- \delta t^{\rho+s} \beta(t)\langle x(t) - x^*, A^T\dot{\lambda}(t) \rangle - \delta t^{2\rho+s} \beta(t)\langle A\dot{x}(t), \dot{\lambda}(t) \rangle \\
- \theta(t)t^\rho \beta(t)(\nabla f(x(t)) + A^T\lambda(t) + \sigma A^T(Ax(t) - b)) \\
- t^{2\rho} \beta(t)(\dot{x}(t), \nabla f(x(t)) + A^T\lambda(t) + \sigma A^T(Ax(t) - b)) \\
+ \langle \theta(t)(x(t) - x^*) + t^\rho \dot{x}(t), t^\rho \epsilon(t) \rangle.
\end{align*}
\]

Similarly, we have

\[
\begin{align*}
\dot{\rho}_2(t) &= \langle \theta(t)(\dot{x}(t) + \dot{\theta}(t))\|x(t) - \lambda\|^2 + t^\rho(\theta(t) + \rho t^{\rho-1} - \alpha t^{\rho-1})\|\dot{\lambda}(t)\|^2 \\
+ \theta(t)(\theta(t) + \rho t^{\rho-1} - \alpha t^{\rho-1}) + t^\rho \dot{\theta}(t) + \eta(t)\langle \lambda(t) - \lambda, \dot{\lambda}(t) \rangle \\
+ \theta(t)t^\rho \beta(t)(\lambda(t) - \lambda, Ax(t) - b) + \delta \theta(t)t^{\rho+s} \beta(t)(\lambda(t) - \lambda, A\dot{x}(t)) \\
+ t^{2\rho} \beta(t)(\lambda(t), Ax(t) - b) + \delta t^{2\rho+s} \beta(t)(\dot{\lambda}(t), A\dot{x}(t)).
\end{align*}
\]

Differentiating of $E_0(t)$ to get

\[
\begin{align*}
\dot{E}_0(t) &= t^{2\rho} \beta(t)(\nabla f(x(t)) + A^T\lambda + \sigma A^T(Ax(t) - b), \dot{x}(t)) \\
+ (2\rho t^{\rho-1} \beta(t) + \dot{t}^{2\rho} \dot{\beta}(t))\langle \mathcal{L}^\sigma(x(t), \lambda) - \mathcal{L}^\sigma(x^*, \lambda) \rangle.
\end{align*}
\]

Let $\theta(t)$ satisfy $t^{2\rho} \beta(t) = \delta \theta(t)t^{\rho+s} \beta(t)$. Adding $\dot{E}_0(t)$, $\dot{E}_1(t)$, $\dot{E}_2(t)$ together, using $Ax^* = b$ and rearranging the terms, we get

\[
\begin{align*}
\dot{E}^{\lambda-p}(t) &= \dot{E}_0(t) + \dot{E}_1(t) + \dot{E}_2(t) = \sum_{i=1}^5 V_i(t),
\end{align*}
\]
where

\[
\begin{align*}
V_1(t) &= \left(\dot{\theta}(t) + \frac{\dot{\lambda}(t)}{2}\right) (\|x(t) - x^*\|^2 + \|\lambda(t) - \lambda\|^2), \\
V_2(t) &= (\theta(t)(\theta(t) + pt^{\nu-1} - \alpha t^{\rho - r}) + t^\rho \dot{\theta}(t) + \eta(t))(\langle x(t) - x^*, \dot{x}(t) \rangle + \langle \lambda(t) - \lambda, \dot{\lambda}(t) \rangle), \\
V_3(t) &= t^\nu(\theta(t) + pt^{\nu-1} - \alpha t^{\rho - r})(\|\dot{x}(t)\|^2 + \|\lambda(t)\|^2), \\
V_4(t) &= t^\nu(t^\rho \dot{\beta}(t) + (2pt^{\rho-1} - \theta(t))\beta(t))(\mathcal{L}^\sigma(x(t), \lambda) - \mathcal{L}^\sigma(x^*, \lambda)) \\
&\quad + \theta(t)t^\rho \beta(t) f(x(t)) - f(x^*) - \langle x(t) - x^*, \nabla f(x(t)) \rangle - \frac{\sigma t^\rho \beta(t)}{2} \|Ax(t) - b\|^2, \\
V_5(t) &= \langle \theta(t)(x(t) - x^*) + t^\rho \dot{x}(t), t^\rho \epsilon(t) \rangle.
\end{align*}
\]

To investigate the rates of convergence of dynamical system \([\ref{13}]\), we need to find the appropriate \(\theta(t)\) and \(\eta(t)\) to satisfy the following conditions:

\[
\begin{align*}
\theta(t) &\geq 0, \quad \eta(t) \geq 0, \quad (A.1) \\
t^2 \beta(t) - \delta(t)t^{\rho+\beta'}(t) &= 0, \quad (A.2) \\
\theta(t)\dot{\theta}(t) + \frac{\dot{\lambda}(t)}{2} &\leq 0, \quad (A.3) \\
\theta(t)(\theta(t) + pt^{\nu-1} - \alpha t^{\rho - r}) + t^\rho \dot{\theta}(t) + \eta(t) &= 0, \quad (A.4)
\end{align*}
\]

Then \(V_1 \leq 0, V_2 = 0\), this together with the convexity of \(f\) yields

\[
\dot{\mathcal{E}}_c^\lambda(t) = \dot{\mathcal{E}}_c^\lambda(t) - \langle \theta(t)(x(t) - x^*) + t^\rho \dot{x}(t), t^\rho \epsilon(t) \rangle \\
\leq t^\rho(\theta(t) + pt^{\nu-1} - \alpha t^{\rho - r})(\|\dot{x}(t)\|^2 + \|\lambda(t)\|^2) - \frac{\sigma t^\rho \beta(t)}{2} \|Ax(t) - b\|^2 \\
+ t^\rho(t^\rho \dot{\beta}(t) + (2pt^{\rho-1} - \theta(t))\beta(t))(\mathcal{L}^\sigma(x(t), \lambda) - \mathcal{L}^\sigma(x^*, \lambda)) \tag{A.5}
\]

for any \(\lambda \in \mathcal{H}_2\).

Appendix A.2. Technical lemmas:

In convergence analysis for the dynamical system, we shall recall the following lemmas.

Lemma Appendix A.1. \([\ref{18}, \text{Lemma A.5}]\) Let \(\nu : [t_0, T] \to [0, +\infty)\) be integrable, and \(M \geq 0\). Suppose \(\mu : [t_0, T] \to \mathbb{R}\) is continuous and

\[
\frac{1}{2} \mu(t)^2 \leq \frac{1}{2} M^2 + \int_{t_0}^t \nu(s)\mu(s)ds
\]

for all \(t \in [t_0, T]\). Then \(\mu(t) \leq M + \int_{t_0}^t \nu(s)ds\) for all \(t \in [t_0, T]\).

Lemma Appendix A.2. \([\ref{16}, \text{Lemma 2.1}]\) For problem \([\ref{2}], x^\ast\) be a solution. Given a function \(\phi\) and a fix point \(x\), if for any \(\lambda\) it holds that

\[
f(x) - f(x^\ast) + \langle \lambda, Ax - b \rangle \leq \phi(\lambda),
\]

then for any \(\varrho > 0\), we have

\[
f(x) - f(x^\ast) + \varrho\|Ax - b\| \leq \sup_{\|\lambda\| \leq \varrho} \phi(\lambda),
\]

\(24\).
References

[1] Alvarez F. On the minimizing property of a second order dissipative system in Hilbert spaces. SIAM Journal on Control and Optimization. 2000;38(4):1102-1119.

[2] Attouch H. Fast inertial proximal ADMM algorithms for convex structured optimization with linear constraint. 2020;hal-02501604.

[3] Attouch H, Balhag A, Chbani Z, Riahi H. Fast convex optimization via inertial dynamics combining viscous and Hessian-driven damping with time rescaling. Evolution Equations & Control Theory, 2021; https://doi.org/10.3934/eect.2021010.

[4] Attouch H, Cabot A. Asymptotic stabilization of inertial gradient dynamics with time-dependent viscosity. Journal of Differential Equations. 2017;263(9):5412-5458.

[5] Attouch H, Chbani Z, Fadili J, Riahi H. Fast convergence of dynamical ADMM via time scaling of damped inertial dynamics. 2021; arXiv:2103.12675.

[6] Attouch H, Chbani Z, Peypourquet J, Redont P. Fast convergence of inertial dynamics and algorithms with asymptotic vanishing viscosity. Mathematical Programming. 2018;168(1-2):123-175.

[7] Attouch H, Chbani Z, Riahi H. Rate of convergence of the Nesterov accelerated gradient method in the subcritical case $\alpha \leq 3$. ESAIM: Control, Optimisation and Calculus of Variations. 2019:25:2.

[8] Attouch H, Cabot A, Chbani Z, Riahi H. Rate of convergence of inertial gradient dynamics with time-dependent viscous damping coefficient. Evolution Equations & Control Theory. 2018;7(3):353-371.

[9] Attouch H, Chbani Z, Riahi H. Fast proximal methods via time scaling of damped inertial dynamics. SIAM Journal on Optimization. 2019;29(3):2227-2256.

[10] Attouch H, Chbani Z, Riahi H. Fast convex optimization via time scaling of damped inertial gradient dynamics. 2019; hal-02138954.

[11] Aujol J.F, Dossal C, Rondepierre A. Optimal convergence rates for Nesterov acceleration. SIAM Journal on Optimization. 2019;29(4):3131-3153.

[12] Balhag A, Chbani Z, Riahi H. Linear convergence of inertial gradient dynamics with constant viscous damping coefficient and time-dependent rescaling parameter. 2020;hal-02610699

[13] Balti M, May R. Asymptotic for the perturbed heavy ball system with vanishing damping term. Evolution Equations & Control Theory. 2017;6(2):177-186.

[14] Beck A, Teboulle M. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. SIAM journal on imaging sciences. 2009;2(1):183-202.
[15] Bégout P, Bolte J, Jendoubi MA. On damped second-order gradient systems. Journal of Differential Equations. 2015;259(7):3115-3143.

[16] Boţ RI, Csetnek ER, László SC. A primal-dual dynamical approach to structured convex minimization problems. Journal of Differential Equations. 2020;269(12):10717-10757.

[17] Boţ RI, Csetnek ER. Second order forward-backward dynamical systems for monotone inclusion problems. SIAM Journal on Control and Optimization. 2016;54(3):1423-1443.

[18] Brezis H, Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, Elsevier, New York, 1973.

[19] Cabot A, Engler H, Gadat S. On the long time behavior of second order differential equations with asymptotically small dissipation. Transactions of the American Mathematical Society. 2009;361(11):5983-6017.

[20] Cabot A, Frankel P. Asymptotics for some semilinear hyperbolic equations with non-autonomous damping. Journal of Differential Equations. 2012;252(1):294-322.

[21] Cherukuri A, Mallada E, Cortés J. Asymptotic convergence of constrained primal–dual dynamics. Systems and Control Letters. 2016;87:10-15.

[22] Feijer D, Paganini F. Stability of primal-dual gradient dynamics and applications to network optimization. Automatica. 2010;46(12):1974-1981.

[23] Fazlyab M, Koppel A, Preciado VM, Ribeiro A. A variational approach to dual methods for constrained convex optimization. American Control Conference (ACC). 2017; 5269-5275.

[24] Ghadimi E, Feyzmahdavian HR, Johansson M. Global convergence of the heavy-ball method for convex optimization. European control conference (ECC). 2015;310-315.

[25] Haraux A, Jendoubi MA. On a second order dissipative ODE in Hilbert space with an integrable source term. Acta Mathematica Scientia. 2012;32(1):435-443.

[26] He X, Hu R, Fang YP. Convergence rate analysis of fast primal-dual methods with scalings for linearly constrained convex optimization problems. 2021; arXiv:2103.10118.

[27] He X, Hu R, Fang YP. Convergence rates of inertial primal-dual dynamical methods for separable convex optimization problems. 2020;arXiv:2007.12428.

[28] He X, Hu R, Fang YP. Fast convergence of primal-dual dynamics and algorithms with time scaling for linear equality constrained convex optimization problems. 2021; arXiv:2103.12931

[29] Jendoubi M.A, May R. Asymptotics for a second-order differential equation with nonautonomous damping and an integrable source term. Applicable Analysis. 2015;94(2):435-443.
[30] Lin Z, Li H, Fang C. Accelerated algorithms for constrained convex optimization. In Accelerated Optimization for Machine Learning 2020:57-108. Springer, Singapore.

[31] Luo H. A primal-dual flow for affine constrained convex optimization. 2021; arXiv:2103.06636.

[32] May R. Asymptotic for a second-order evolution equation with convex potential and vanishing damping term. Turkish Journal of Mathematics. 2017;41(3):681-685.

[33] May R. Long time behavior for a semilinear hyperbolic equation with asymptotically vanishing damping term and convex potential. Journal of Mathematical Analysis and Applications. 2015;430(1):410-416.

[34] Nesterov Y. A method of solving a convex programming problem with convergence rate $O(1/k^2)$. In Sov. Math. Dokl 1983;27(2):372-376.

[35] Nesterov Y. Introductory lectures on convex optimization: A basic course. Springer Science and Business Media; 2013.

[36] Polyak BT. Some methods of speeding up the convergence of iteration methods. USSR Computational Mathematics and Mathematical Physics. 1964;4(5):1-7.

[37] Sebbouh O, Dossal C, Rondepierre A. Convergence rates of damped inertial dynamics under geometric conditions and perturbations. SIAM Journal on Optimization. 2020;30(3):1850-1877.

[38] Qu G, Li N. On the exponential stability of primal-dual gradient dynamics. IEEE Control Systems Letters. 2018;3(1):43-48.

[39] Shi B, Du SS, Jordan MI, Su WJ. Understanding the acceleration phenomenon via high-resolution differential equations. 2018;arXiv:1810.08907.

[40] Su W, Boyd S, Candes E.J. A differential equation for modeling Nesterov's accelerated gradient method: theory and insights. The Journal of Machine Learning Research. 2016;17(1):5312-5354.

[41] Sun T, Yin P, Li D, Huang C, Guan L, Jiang H. Non-ergodic convergence analysis of heavy-ball algorithms. Proceedings of the AAAI Conference on Artificial Intelligence 2019;33:5033-5040.

[42] Teschl G. Ordinary differential equations and dynamical systems. American Mathematical Soc;2012.

[43] Vassilis A, Jean-François A, Charles D. The Differential Inclusion Modeling FISTA Algorithm and Optimality of Convergence Rate in the Case $b \leq 3$. SIAM Journal on Optimization. 2018;28(1):551-574.

[44] Wibisono A, Wilson AC, Jordan MI. A variational perspective on accelerated methods in optimization. Proceedings of the National Academy of Sciences. 2016;113(47):E7351-E7358.

[45] Wilson AC, Recht B, Jordan MI. A lyapunov analysis of momentum methods in optimization. 2016;arXiv:1611.02635.
[46] Xu Y. Accelerated first-order primal-dual proximal methods for linearly constrained composite convex programming. SIAM Journal on Optimization. 2017;27(3):1459-1484.

[47] Zeng X, Lei J, Chen J. Dynamical primal-dual accelerated method with applications to network optimization. 2019;arXiv:1912.03690.