A QUILLEN ADJUNCTION BETWEEN
GLOBULAR AND COMPLICIAL APPROACHES TO
$(\infty, n)$-CAT EGORIES

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Abstract. We prove the compatibility between the suspension construction and the complicial nerve of $\omega$-categories. As a motivating application, we produce a Quillen pair between the models of $(\infty, n)$-categories given by Rezk’s complete Segal $\Theta_n$-spaces and Verity’s $n$-complicial sets.

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Introduction

It is by now known that many mathematical phenomena of interest can only be properly formalized using the language of $(\infty, n)$-categories. Several mathematical objects have been identified to implement the notion of an $(\infty, n)$-category, each with its own advantages and disadvantages. Amongst those, there are Verity’s $n$-complicial sets [Ver08b, Ver17, Rie18, OR20b, RV22] and Rezk’s complete Segal $\Theta_n$-spaces [Rez10].

The homotopy theories of $n$-complicial sets and complete Segal $\Theta_n$-spaces are only known to be equivalent for $n \leq 2$, and this paper reports progress towards establishing the equivalence of these homotopy theories for general $n$, which was conjectured more than three decades ago (see e.g. [Str87, Ver17, BSP21]).

Theorem A. There is an adjunction of $\infty$-categories between the $\infty$-category of complete Segal $\Theta_n$-spaces and the $\infty$-category of $n$-complicial sets.

More precisely, we achieve this by constructing an adjunction between the model categories $sSet_{\Theta^n}$ and $mSSet_{(\infty, n)}$, which we show in Theorem 4.10 to be a Quillen pair.

2020 Mathematics Subject Classification. 18N65; 55U35; 18N50; 55U10.
Key words and phrases. $(\infty, n)$-categories, $n$-categories, complicial sets, complete Segal $\Theta_n$-spaces, suspension of $n$-categories, complicial nerve.
(and which is conjecturally a Quillen equivalence). This results (partially) generalizes joint work of the authors with Bergner [BOR21].

In order to prove Theorem 4.1.6 the crucial ingredient is to understand how the two-point suspension interacts with the complicial nerve of certain $n$-categories. We prove the following as Theorem 3.22.

**Theorem B.** If $C$ admits an algebraic model in an appropriate sense, then $N^{RS}C$ is equivalent to $\Sigma N^{RS}C$ in the model structure for $n$-complicial sets.

Here, the precise condition on $C$ requires it to be obtained from an (augmented directed) chain complex via Steiner’s functor $\nu: adCh \to \omega Cat$ (see [Ste04, AM20]), the functor $N^{RS}: nCat \to msSet$ is the Roberts–Street nerve and the functors $\Sigma: nCat \to (n+1)Cat$ and $\Sigma: msSet(\infty,n) \to msSet(\infty,n+1)$ implement the two-point suspension construction in a strict and weak context. The theorem is also used in work by Loubaton [Lou22], who gives a criterion to identify self-equivalences on the $\infty$-category of $n$-complicial sets.

**Acknowledgements.** It is hard to overestimate the role of Andrea Gagna for this paper, who has taught the authors the language of Steiner’s theory of augmented directed chain complexes, without which the current result would have been out of our reach. This work was completed while the authors visited the Instituto de Matemáticas de UNAM in Cuernavaca for the program Higher categories – Part 2, supported by the National Science Foundation under Grant No. DMS-1928930. The second author is grateful for support from the National Science Foundation under Grant No. DMS-2203915.

1. Steiner’s augmented directed chain complexes

We recall the basic definitions around Steiner’s augmented directed chain complexes, as well as some constructions based on augmented directed chain complexes: the suspension, tensor product, and the total dual, as well as the main properties that we use later in the paper and relevant examples. Most of the material is drawn from [Ste04] (see also [AM20]).

1.1. Augmented directed chain complexes. By a chain complex $C$ we will always mean an $\mathbb{N}$-graded chain complex of abelian groups with homological indexing, that is, a family $(C_q)_{q\geq 0}$ of abelian groups, together with maps $\partial_q: C_{q+1} \to C_q$ satisfying $\partial_q\partial_{q+1} = 0$. We also assume that, whenever occurring, $C_{-1} = 0$, and $\partial_0 = 0$.

Given chain complexes $C$ and $\overline{C}$, a chain map or morphism of chain complexes $\phi: C \to \overline{C}$ consists of a family of homomorphisms $(\phi_q: C_q \to \overline{C}_q)_{q\geq 0}$ that commutes with the differentials in the sense that $\overline{\partial_q\phi_{q+1}} = \phi_q\partial_q$ for every $q \geq 0$.

An augmented chain complex is a pair $(C, \varepsilon)$ of a chain complex $C$ and an augmentation, namely a map $\varepsilon: C_0 \to \mathbb{Z}$ such that $\varepsilon\partial_0 = 0$.

An augmented chain map $\phi: (C, \varepsilon) \to (\overline{C}, \overline{\varepsilon})$ between augmented chain complexes $(C, \varepsilon)$ and $(\overline{C}, \overline{\varepsilon})$ consists of a chain map $\phi: C \to \overline{C}$ that is moreover compatible with the augmentations, namely such that $\overline{\varepsilon}\partial_0 = \varepsilon$.

We recall the enhancement of augmented chain complexes developed by Steiner [Ste14, §2].

**Definition 1.1** ([Ste14, Def. 2.2]). An augmented directed complex is a triple $(C, C^+, \varepsilon)$ where $(C, \varepsilon)$ is an augmented chain complex and $C^+ = (C^+_q)_{q\geq 0}$ is a collection of commutative monoids, where $C^+_q$ is a submonoid of $C_q$ called the positivity submonoid of $C_q$. 

A morphism of augmented directed chain complexes, or an augmented directed chain map \( \phi: (C, C^+, \varepsilon) \to (C, C^+, \varepsilon) \) between augmented directed chain complexes \((C, C^+, \varepsilon)\) and \((C, C^+, \varepsilon)\) is an augmented chain map \( \phi: (C, \varepsilon) \to (C, \varepsilon) \) that moreover preserves the positivity submonoids, namely such that

\[
\phi_q(C^+_q) \subseteq C^+_q
\]

for all \( q \geq 0 \).

We denote by \( \text{ad}Ch \) the category of augmented directed chain complexes and maps of chain complexes that preserve the augmentation and the positivity submonoids.

Remark 1.2. The category \( \text{ad}Ch \) is cocomplete, colimits are computed degreewise, and epimorphisms are detected pointwise in the category \( \text{Ab} \) of abelian groups and the category \( \text{cMon} \) of commutative monoids, which are both cocomplete. That is, the forgetful functor

\[
\text{ad}Ch \to \prod_{q \geq 0} (\text{Ab} \times \text{cMon})
\]

given by \( C \mapsto (C_q, C^+_q)_{q \geq 0} \) creates colimits (and in particular epimorphisms).

Remark 1.3. Consider the following left adjoint functors.

1. The free abelian group functor on a set and the free commutative monoid functor on a set,

\[
\mathbb{Z}[-]: \text{Set} \to \text{Ab} \quad \text{and} \quad \mathbb{N}[-]: \text{Set} \to \text{cMon},
\]

given by \( X \mapsto \mathbb{Z}[X] \) and \( X \mapsto \mathbb{N}[X] \). The right adjoint functors are the forgetful functors.

2. The free abelian group functor on a pointed set and the free commutative monoid functor on a pointed set,

\[
\mathbb{Z}[-]: \text{Set}_* \to \text{Ab} \quad \text{and} \quad \mathbb{N}[-]: \text{Set}_* \to \text{cMon},
\]

given by \( (X, x_0) \mapsto \mathbb{Z}[X \setminus \{x_0\}] \) and \( (X, x_0) \mapsto \mathbb{N}[X \setminus \{x_0\}] \). The right adjoint functors are the forgetful functors that retain the identity as a base point.

3. The functor that freely adds a base point to a set,

\[
(-)_*: \text{Set} \to \text{Set}_*,
\]

given by \( X \mapsto (X \amalg \{\ast\}, \ast) \). The right adjoint functor is the functor that forgets the base point.

Being left adjoint functors, they all preserve colimits (and in particular epimorphisms).

Notation 1.4. Let \( m \geq -1 \) and \( q \geq -1 \). We denote

- by \( \Delta[m]_q = \text{Cat}([q], [m]) \) the set of \( q \)-simplices of the standard simplex\footnote{We follow the convention that \([-1]\) is the empty category, and \(\Delta[-1]\) is the initial simplicial set, which is levelwise empty.} \( \Delta[m] \). A generic \( q \)-simplex in \( \Delta[m] \) is of the form

\[
[a] = [a_0, a_1, \ldots, a_q]
\]

with \( 0 \leq a_0 \leq a_1 \leq \ldots \leq a_q \leq m \). We say that \( q \) is the length \( |a| \) of \([a]\).

- by \( B[m]_q \subseteq \Delta[m]_q \) the set of non-degenerate \( q \)-simplices of \( \Delta[m] \), namely those simplices for which \( 0 \leq a_0 < a_1 < \ldots < a_q \leq m \).
• by $O[m]_q = \mathbb{Z}[B[m]_q] \cong \mathbb{Z}^{(m)}_{\leq q}$ the abelian group freely generated by non-degenerate $q$-simplices of $\Delta[m]$. The generic element of $O[m]_q$ is a formal sum

$$c = \sum_{[a] \in B[m]_q} c_{[a]} \cdot [a]$$

where $c_{[a]} \in \mathbb{Z}$.

• by $O[m]_q^+ = \mathbb{N}[B[m]_q] \cong \mathbb{N}^{(m)}_{\leq q}$ the abelian monoid freely generated by non-degenerate $q$-simplices of $\Delta[m]$. The generic element of $O[m]_q^+$ is one for which $c_{[a]} \in \mathbb{N}$.

There are canonical inclusions $\Delta[m]_q \supseteq B[m]_q \subseteq O[m]_q^+ \subseteq O[m]_q$.

The augmented directed chain complex $O[m]$ is the algebraic model of the $m$-oriental $O[m]$, in a sense that will be made precise in Example 2.10.

**Example 1.5 ([Ste04, Ex. 3.8]).** For $m \geq -1$, we consider the augmented directed chain complex $O[m]$ with the following structure.

• For $q \geq 0$ the abelian group of $q$-chains is given by $O[m]_q$.

• For $q \geq 0$ the commutative monoid of positive $q$-chains is given by $O[m]_q^+$.

• For $q \geq -1$ the differential $\partial_q: O[m]_{q+1} \to O[m]_q$ is given by

$$\partial_q[a] = \partial_q(a_0, \ldots, a_q, a_{q+1}) = \sum_{i=0}^{q+1} (-1)^i \cdot [a_0, \ldots, \hat{a}_i, \ldots, a_{q+1}] \in O[m]_q$$

• The augmentation map $\varepsilon: O[m]_0 \to \mathbb{Z}$ is given by

$$\varepsilon[a] = 1 \in \mathbb{Z}.$$ 

Later in the paper, we will make use of the following dual construction for an augmented directed chain complex.

**Definition 1.6 ([AM20, §2.18]).** Let $C$ be an augmented directed complex. The total dual $C^\circ$ of $C$ is the augmented directed complex with the following structure

• For $q \geq 1$ the abelian group of $q$-chains is given by $C^\circ_q = C_q$;

• For $q \geq 1$ the commutative monoid of positive chains is given by $(C^\circ)_q^+ = C_q^+$;

• For $q \geq 1$, the differential $\partial_q^\circ: C^\circ_q \to C^\circ_{q-1}$ is given by $\partial_q^\circ(c) = -\partial_q(C)(c)$.

• The augmentation $\varepsilon^\circ: C^\circ_0 \to \mathbb{Z}$ is given by $\varepsilon^\circ(a) = \varepsilon(C)(a)$.

This construction defines an involution $(-)^\circ: \text{adCh} \to \text{adCh}$.

1.2. **Suspension of augmented directed chain complexes.** We define a two-point suspension for augmented directed chain complexes. This is the construction that Steiner denotes $V(1,C)$ in [Ste07b, §5.2].

**Definition 1.7.** Let $C$ be an augmented directed chain complex. The suspension of $C$ is the augmented directed chain complex $\Sigma C$ with the following structure:

• For $q \geq 0$, the abelian group $(\Sigma C)_q$ of $q$-chains is given by

$$\begin{cases}
\mathbb{Z}^{[\bot, \top]} & \text{if } q = 0, \\
C_{q-1} & \text{if } q \geq 1.
\end{cases}$$

\[\text{This is different from the one-point suspension considered by Ara–Martsiniotis in AM20 §6.3.}\]
For $q \geq 0$, the commutative monoid $(\Sigma C)_q^+$ of positive $q$-chains is given by
\[(\Sigma C)_q^+ = \begin{cases} \mathbb{N}[\perp, \top] & \text{if } q = 0, \\ C_{q-1}^+ & \text{if } q \geq 1. \end{cases} \]

For $q \geq 0$, the differential $\partial_q : (\Sigma C)_q \to (\Sigma C)_{q-1}$ is given by
\[\partial_q^{\Sigma C}(a) := \begin{cases} \varepsilon^C(c)(\top - \perp) = -\varepsilon^C(c) \cdot \perp + \varepsilon^C(c) \cdot \top & \text{if } q = 0, \\ \partial_{q-1}(c) & \text{if } q \geq 1. \end{cases} \]

The augmentation $\varepsilon^{\Sigma C} : (\Sigma C)_0 \to \mathbb{Z}$ is given by
\[\varepsilon^{\Sigma C} \perp = 1 = \varepsilon^{\Sigma C} \top.\]

The augmented directed chain complex $\Sigma C$ comes with a map $\Sigma \mathbb{O}[-1] \to \Sigma C$ so it can be naturally regarded as an object of $\Sigma \mathbb{O}[-1]/\text{adCh}$. The following is a consequence of [Ste07] Theorem 5.6.

**Proposition 1.8.** The suspension functor $\Sigma : \text{adCh} \to \Sigma \mathbb{O}[-1]/\text{adCh}$ is fully faithful.

1.3. Tensor product of augmented directed chain complexes. We consider the tensor product of abelian groups $\otimes : \text{Ab} \times \text{Ab} \to \text{Ab}$, as well as the (less known) tensor product of commutative monoids $\otimes : \text{cMon} \times \text{cMon} \to \text{cMon}$. See e.g. [Gol99] Chapter 16 for more details on this construction. We will mostly use instances of the tensor product of free abelian groups and free commutative monoids, which is described by the following remark.

**Remark 1.9.** Recall the functors from Remark 1.3.

1. The free abelian group functor $\mathbb{Z}[-] : (\text{Set}, \times) \to (\text{Ab}, \otimes)$ and the free commutative monoid functor $\mathbb{N}[-] : (\text{Set}, \times) \to (\text{cMon}, \otimes)$ is strong monoidal, namely, there are natural bijections
\[\mathbb{Z}[X] \otimes \mathbb{Z}[Y] \cong \mathbb{Z}[X \times Y] \quad \text{and} \quad \mathbb{N}[X] \otimes \mathbb{N}[Y] \cong \mathbb{N}[X \times Y],\]
for any $X$ and $Y$ sets. In particular, the tensor product of free abelian groups, resp. commutative monoids, is a free abelian group, resp. free commutative monoid. It follows that also the free abelian group functor $\mathbb{Z}[-] : (\text{Set}_*, \times) \to (\text{Ab}, \otimes)$ and the free commutative monoid functor $\mathbb{N}[-] : (\text{Set}_*, \times) \to (\text{cMon}, \otimes)$ is strong monoidal

2. The free abelian group functor $\mathbb{Z}[-] : (\text{Set}, I) \to (\text{Ab}, \oplus)$ and the free commutative monoid functor $\mathbb{N}[-] : (\text{Set}, I) \to (\text{cMon}, \oplus)$ is strong monoidal.

**Definition 1.10** ([Ste04] Example 3.10). Let $C$ and $D$ be augmented directed chain complexes. The tensor product of $C$ and $D$ is the augmented directed chain complex $C \otimes D$ with the following structure:

- For $q \geq 0$, the abelian group $(C \otimes D)_q$ of $q$-chains is given by
\[(C \otimes D)_q = \bigoplus_{k+\ell=q} C_k \otimes D_\ell.\]

- For $q \geq 0$, the commutative monoid $(C \otimes D)_q^+$ of positive $q$-chains is given by
\[(C \otimes D)_q^+ = \bigoplus_{k+\ell=q} C_k^+ \otimes D_\ell^+.\]

- For $q \geq 0$, the differential $\partial_{q}^{C \otimes D} : (C \otimes D)_q \to (C \otimes D)_{q-1}$ is given by
\[\partial_{q}^{C \otimes D}(c \otimes d) := \partial^C c \otimes D + (-1)^{|c|} c \otimes \partial^D d\]
• The augmentation $\varepsilon^{C \otimes D} : (C \otimes D)_0 \cong C_0 \otimes D_0 \to \mathbb{Z}$ is given by

$$\varepsilon^{C \otimes D}(c \otimes d) = \varepsilon^C \cdot \varepsilon^D \cdot d.$$ 

The construction defines a functor $\otimes : adCh \times adCh \to adCh$.

We now unpack tensor product of orientals.

**Example 1.11.** Let $k, \ell \geq 0$.

• For $q \geq 0$, the abelian group $(O[k] \otimes O[\ell])_q$ of $q$-chains is given by

$$(O[k] \otimes O[\ell])_q = \bigoplus_{i=0}^q O[k]_i \otimes O[\ell]_{q-i} \cong \bigoplus_{i=0}^q \mathbb{Z}[B[k]_i] \otimes \mathbb{Z}[B[\ell]_{q-i}] \cong \bigoplus_{i=0}^q \mathbb{Z}^{(\binom{k}{i})} \otimes \mathbb{Z}^{(\binom{\ell}{q-i})}$$

• For $q \geq 0$, the commutative monoid $(O[k] \otimes O[\ell])_q^+$ of positive $q$-chains is given by

$$(O[k] \otimes O[\ell])_q^+ = \bigoplus_{i=0}^q O[k]_i^+ \otimes O[\ell]_{q-i}^+ \cong \bigoplus_{i=0}^q N[B[k]_i] \otimes N[B[\ell]_{q-i}] \cong \bigoplus_{i=0}^q N^{(\binom{k}{i})} \otimes N^{(\binom{\ell}{q-i})}$$

• For $q > 0$, the differential $\partial_q^{O[k] \otimes O[\ell]} : (O[k] \otimes O[\ell])_{q+1} \to (O[k] \otimes O[\ell])_q$ is given by

$$\partial_q^{O[k] \otimes O[\ell]}([a] \otimes [b]) := \partial^{O[k]}[a] \otimes [b] + (-1)^{|a|}[a] \otimes \partial^{O[\ell]}[b]$$

• The augmentation $\varepsilon^{O[k] \otimes O[\ell]} : (O[k] \otimes O[\ell])_0 \cong O[k]_0 \otimes O[\ell]_0 \to \mathbb{Z}$ is given by

$$\varepsilon^{O[k] \otimes O[\ell]}([a] \otimes [b]) := \varepsilon^{O[k]}([a]) \cdot \varepsilon^{O[\ell]}([b]) = 1 \cdot 1 = 1$$

Recall the functor $(-)_+ : Set \to Set_*$, which is the left adjoint to the forgetful functor.

**Remark 1.12.** For $q \geq 0$, there is a canonical map of pointed sets

$$\left(\left[\frac{k+1+\ell}{q}\right]\right)_+ \to \bigoplus_{r=0}^q \left(\left[\frac{k}{r}\right]\right) \times \left(\left[\frac{\ell}{q-r}\right]\right)_+$$

that splits a subset of $[k+1+\ell]$ into its $[k]$-part and $[\ell]$-part, using the base point whenever any of them is empty. This induces a map of abelian groups

$$\phi_q : O[k+1+\ell]_q \cong \mathbb{Z}^{(\binom{k+1+\ell}{q})} \to \bigoplus_{r=0}^q \mathbb{Z}^{(\binom{k}{r})} \otimes \mathbb{Z}^{(\binom{\ell}{q-r})} \cong \Sigma(O[k] \otimes O[\ell])_q$$

and commutative monoids

$$\phi_q : O[k+1+\ell]_q^+ \cong N^{(\binom{k+1+\ell}{q})} \to \bigoplus_{r=0}^q N^{(\binom{k}{r})} \otimes N^{(\binom{\ell}{q-r})} \cong \Sigma(O[k] \otimes O[\ell])_q^+.$$ 

Explicitly, $\phi_0 : O[k+1+\ell]_0 \to \Sigma(O[k] \otimes O[\ell])_0$ is given by

$$\phi_0([a'']) := \begin{cases} \perp & \text{if } 0 \leq a'' \leq k; \\
\top & \text{if } k + 1 \leq a'' \leq k + 1 + \ell. \end{cases}$$

For $q > 0$, the map $\phi_q : O[k+1+\ell]_q \to \Sigma(O[k] \otimes O[\ell])_q$ is given by

$$\phi_q([a, a']) := \begin{cases} [a] \otimes (s^0)^{k+1}[a'], & \text{with } a \in [0, k], a' \in [k+1, m], |a| \geq 0, |a'| \geq 0; \\
0 & \text{else.} \end{cases}$$

**Proposition 1.13.** Let $k, \ell \geq 0$. There is a map of augmented directed chain complexes

$$(1.14) \quad \phi : O[k+1+\ell] \to \Sigma(O[k] \otimes O[\ell]).$$

given degreewise by the maps $\phi_q$ described in Remark 1.12.
Proof. Given \( a \subseteq [0, k] \), and \( a' \subseteq [k + 1, n] \), with \(|a| \geq 0\) and \(|a'| \geq 0\), which is the only case of interest, we obtain

\[
\partial \phi([a, a']) = \partial \left( [a] \otimes (s^0)^{k+1}[a'] \right) = \partial[a] \otimes (s^0)^{k+1}[a'] + (-1)^{|a|+1}[a] \otimes \partial(s^0)^{k+1}[a'] = \phi(\partial([a, a']))
\]
as desired. \( \square \)

For \( k, \ell \geq 0 \), the maps from Proposition 1.13 together with other canonical maps, can be used to build a commutative diagram

\[
\begin{array}{ccc}
O[k] \oplus O[\ell]^0 & \longrightarrow & O[k+1+\ell] \\
\downarrow & & \downarrow \\
O[0] \oplus O[0]^0 & \longrightarrow & \Sigma(O[k] \otimes O[\ell]^0).
\end{array}
\]

Proposition 1.16. Let \( k, \ell \geq 0 \). The diagram (1.15) induces a natural isomorphism of augmented directed chain complexes

\[\Sigma(O[k] \otimes O[\ell]^0) \cong (O[0] \oplus O[0]^0) \otimes_{O[k] \oplus O[\ell]^0} O[k+1+\ell].\]

Proof. There are pushout squares of abelian groups and of commutative monoids:

\[
\begin{array}{ccc}
Z[k] \oplus Z[\ell] & \xrightarrow{=} & Z[k+1+\ell] \\
\downarrow & & \downarrow \\
Z \oplus Z & \xrightarrow{=} & Z[\bot, T].
\end{array} \quad \text{and} \quad \begin{array}{ccc}
N[k] \oplus N[\ell] & \xrightarrow{=} & N[k+1+\ell] \\
\downarrow & & \downarrow \\
N \oplus N & \xrightarrow{=} & N[\bot, T].
\end{array}
\]

Here, the left vertical maps are the sums of the canonical map that folds the first \( k+1 \) copies of \( Z \) and the one that folds the last \( \ell+1 \) copies of \( Z \). This means that (1.15) induces a pushout of abelian groups (resp., commutative monoids):

\[
\begin{array}{ccc}
(O[k] \oplus O[\ell]^0)^{(+)0} & \longrightarrow & O[k+1+\ell]^{(+)0} \\
\downarrow & & \downarrow \\
O[0]^{(+)0} \oplus (O[0]^0)^{(+)0} & \longrightarrow & \Sigma(O[k] \otimes O[\ell]^0)^{(+)0}.
\end{array}
\]

Let \( q > 0 \). Vandermonde’s identity

\[
\binom{k+1+\ell}{q} = \sum_{i=-1}^{q} \binom{k}{i} \cdot \binom{\ell}{q-1-i} = \binom{k}{q} + \binom{\ell}{q} + \sum_{i=0}^{q-1} \binom{k}{[i]} \cdot \binom{\ell}{[q-1-i]},
\]
can be equivalently expressed as a pushout of pointed sets

\[
\begin{array}{ccc}
\binom{k}{q} + \vee \binom{\ell}{q} & \longrightarrow & \binom{k+1+\ell}{[q]} \\
\downarrow & \downarrow \\
\{*\} & \longrightarrow & \bigvee_{i=0}^{q-1} \binom{k}{[i]} \times \binom{\ell}{[q-1-i]},
\end{array}
\]
Here, the components of the top horizontal map are the canonical, and the right vertical map is the one from Remark 1.12. By Remark 1.2(2), we then obtain a pushout of abelian groups and of commutative monoids:

\[
\begin{array}{ccc}
\mathbb{Z}^{(k)} \oplus \mathbb{Z}^{(\ell)} & \rightarrow & \mathbb{Z}^{(k+1+\ell)} \\
0 \oplus 0 & \rightarrow & \bigoplus_{r=0}^{q} \mathbb{Z}^{(k)} \otimes \mathbb{Z}^{(q-1-r)} \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathbb{N}^{(k)} \oplus \mathbb{N}^{(\ell)} & \rightarrow & \mathbb{N}^{(k+1+\ell)} \\
0 \oplus 0 & \rightarrow & \bigoplus_{r=0}^{q} \mathbb{N}^{(k)} \otimes \mathbb{N}^{(q-1-r)}. \\
\end{array}
\]

This means that (1.15) induces a pushout of abelian groups (resp. commutative monoids):

\[
\begin{array}{ccc}
(O[k] \oplus O[\ell])^{(+)} & \rightarrow & O[k+1+\ell]^{(+)} \\
O[0]^{(+)} \oplus (O[0])^{(+)} & \rightarrow & \Sigma(O[k] \otimes O[\ell])^{(+)}. \\
\end{array}
\]

Combining (1.17) and (1.18), by Remark 1.2 we obtain the desired pushout of augmented directed chain complexes (1.15). □

**Proposition 1.19.** Let \( k, \ell \geq 0 \). The map

\[
O[k] + \ell \rightarrow \Sigma \left( O[k] \otimes O[\ell] \right)
\]

from Proposition 1.13 is an epimorphism of augmented directed chain complexes.

**Proof.** Using the explicit computations from Proposition 1.16 we see that for every \( q \geq 0 \) the canonical map

\[
(O[k] \oplus O[\ell])^{(+)} \rightarrow O[k+1+\ell]^{(+)}
\]

is an epimorphism of abelian groups (resp. commutative monoids). By Remark 1.3 the canonical map

\[
O[k] \oplus O[\ell]^{(+)} \rightarrow O[0] \oplus O[0]^{(+)}
\]

is then an epimorphism of augmented directed chain complexes. Given that epimorphisms are closed under pushout, by Proposition 1.16 the map

\[
O[k+1+\ell] \rightarrow \Sigma \left( O[k] \otimes O[\ell] \right)
\]

from Proposition 1.13 is then an epimorphism of augmented directed chain complexes, too. □

We can understand how to map orientals into suspensions:

**Proposition 1.20.** For \( m \geq 1 \) and \( C \) an augmented directed chain complex, the map from Proposition 1.13 induces a natural bijection

\[
\prod_{k+1+\ell=m, \ k, \ell \geq 0} \text{adCh}(O[k] \otimes O[\ell], C) \xrightarrow{\cong} \text{adCh}(O[m], \Sigma C).
\]

**Proof.** For any \( x: O[m] \rightarrow \Sigma C \) we set

\[
k := \# \{ 0 \leq i \leq m \mid x(i) = \perp \} - 1 \quad \text{and} \quad \ell := m - 1 - k.
\]

and construct a corresponding preimage \( \tilde{x}: O[k] \otimes O[\ell] \rightarrow C \).
Let $k = -1$ (resp. $k = m$). Then we take
\[
\hat{x}: O[-1] \otimes O[m] \cong O[-1] \to C, \text{ resp. } \hat{x}: O[m] \otimes O[-1] \cong O[-1] \to C
\]
to be the trivial map.

Let $0 \leq k \leq m - 1$. By Proposition 1.19 the function
\[
\Sigma: \prod_{k+1+ℓ=m}^{\geq k}\text{adCh}(O[k] \otimes O[ℓ]^o, C) \to \prod_{k+1+ℓ=m}^{\leq k,ℓ\geq 0}\Sigma O[-1]/\text{adCh}(\Sigma(O[k] \otimes O[ℓ]^o), \Sigma C)
\]
is bijective, so $x$ can be uniquely identified with the suspension map
\[
\Sigma x: \Sigma(O[k] \otimes O[ℓ]^o) \to \Sigma C
\]
under $\Sigma O[-1]$. By composing with the map
\[
\prod_{k+1+ℓ=m}^{\geq k,ℓ\geq 0}\Sigma O[-1]/\text{adCh}(\Sigma(O[k] \otimes O[ℓ]^o), \Sigma C) \to \text{adCh}(O[m], \Sigma C)
\]
induced by Proposition 1.13, which is is injective by Proposition 1.8. $\Sigma x$ can be uniquely identified with a map of the form
\[
\hat{x}: O[k] \otimes O[ℓ]^o \to C.
\]
It is a straightforward verification that the assignment $x \mapsto \hat{x}$ defines the inverse for the desired bijection. \qed

2. $\omega$-CATEGORIES AND ALGEBRAIC MODELS

We recall the basic definitions around $\omega$-categories, as well as some constructions based on $\omega$-categories: the $\omega$-categorical suspension, the tensor product, the total dual, and Steiner’s linearization, as well as the main properties that we use later in the paper, and relevant examples.

2.1. $\omega$-CATEGORIES. While we refer the reader to e.g. [Str87] for a traditional approach to the definition of an $\omega$-category, we briefly recall the main features here.

The data of an $\omega$-category $\mathcal{C}$ consists of a collection of sets $\mathcal{C}_q$, for $q \geq 0$, where $\mathcal{C}_0$ is called the set of objects of $\mathcal{C}$ and $\mathcal{C}_q$ for $q > 0$ is the set of $q$-cells or cells of dimension $q$ of $\mathcal{C}$, together with:

- **source** and **target** operators $s_q, t_q: \mathcal{C}_p \to \mathcal{C}_q$ for all $p > q \geq 0$;
- **identity** operators $id_q: \mathcal{C}_p \to \mathcal{C}_q$ for all $q \geq p \geq 0$;
- **composition** operators $*_{p, q}: \mathcal{C}_q \times_{C_p} \mathcal{C}_p \to \mathcal{C}_q$ defined for all $q > p \geq 0$ and all pairs of $q$-cells $(g, f)$ for which $s_p(g) = t_p f$.

We say that $\mathcal{C}$ is an $\omega$-category if for all $r > q > p \geq 0$ the triple $(\mathcal{C}_p, \mathcal{C}_q, \mathcal{C}_r)$ together with all the relevant source, target, identity and composition operators is a 2-category. In particular,
\[
s_p s_q f = s_p f \quad \text{and} \quad t_p t_q f = t_p f
\]
for any $r$-cell $f$ of $\mathcal{C}$ and $r > q > p$.

An $\omega$-functor $F: \mathcal{C} \to \mathcal{D}$ between $\omega$-categories $\mathcal{C}$ and $\mathcal{D}$ is a collection of maps $F_q: \mathcal{C}_q \to \mathcal{D}_q$ for $q \geq 0$ that preserves source, target, identity, and composition operators. We denote by $\omega \text{Cat}$ the category of (small) $\omega$-categories and $\omega$-functors.

A cell in an $\omega$-category $\mathcal{C}$ is said to be **trivial** if it is the identity of a cell of lower dimension. For $m \geq 0$, an $m$-**category** is an $\omega$-category in which all $q$-cells are trivial for
\( q > n \), and an \( n \)-functor is an \( \omega \)-functor between \( n \)-categories. We denote by \( n\text{Cat} \) the (full) subcategory of \( \omega\text{Cat} \) given by \( n \)-categories and \( n \)-functors.

**Example 2.2.** For \( m \geq -1 \), the \( m \)-oriental \( \mathcal{O}[m] \) from [Str87], [Ste07a, Theorem 3.2] or [AM20, §7.2] is an \( m \)-category, and in particular an \( \omega \)-category. The reader who is not familiar with the original definition can also take the formula from Example 2.10 as the definition of \( \mathcal{O}[m] \).

**Definition 2.3** ([AM20 §1.8]). Let \( C \) be an \( \omega \)-category. The total dual of \( C \) is the \( \omega \)-category \( C^\circ \) with the following structure.

- The set of \( q \)-cells \( C^q_q \) is \( C^q_q := C_q \)
- The source map \( s_q : C^q_p \to C^q_q \) is given by \( s^C_q f = t^C_q f \) for all \( p > q \geq 0 \);
- The target map \( t_q : C^q_p \to C^q_q \) is given by \( t^C_q f = s^C_q f \) for all \( p > q \geq 0 \);
- The composition map \( *_p : C^q_q \times C^p_q \to C^q_q \) is given by \( f *^C_p g = g *^C_p f \) for all \( p > q \geq 0 \);
- The identity map \( \text{id}_q : C^q_p \to C^q_q \) is given by \( \text{id}^C_q f = \text{id}^C_q f \) for all \( p \geq q \geq 0 \);

The construction defines a functor \((-)^\circ : \text{Cat} \to \text{Cat}\).

### 2.2. Suspension of \( \omega \)-categories

The following is a variant\(^3\) of the construction treated in [AM20, §B.6.5]. When the input is an ordinary 1-category \( C \), the suspension agrees with the one that we previously considered in [OR20a].

**Definition 2.4.** Let \( C \) be an \( \omega \)-category. The suspension of \( C \) is the \( \omega \)-category \( \Sigma C \) with the following structure.

- The set of \( q \)-cells \( (\Sigma C)_q \) is

\[
(\Sigma C)_q := \{\bot, \top\} \cup C_{q-1}, \quad \text{with} \quad (\Sigma C)_0 := \{\bot, \top\}
\]

- The source map \( s_q : \Sigma C_p \to (\Sigma C)_q \) for \( q > 1 \) is given by

\[
s^\Sigma_q f = s^C_{q-1} f, \quad s^\Sigma_q \bot = \bot, \quad s^\Sigma_q \top = \top, \quad \text{with} \quad s^\Sigma_0 f = \bot.
\]

- The target map \( t_q : \Sigma C_p \to (\Sigma C)_q \) for \( q > 1 \) is given by

\[
t^\Sigma_q f = t^C_{q-1} f, \quad t^\Sigma_q \bot = \bot, \quad t^\Sigma_q \top = \top, \quad \text{with} \quad t^\Sigma_0 f = \top.
\]

- The identity map \( \text{id}_q : \Sigma C_p \to (\Sigma C)_q \) is given by

\[
\text{id}^\Sigma_q f = \text{id}^C_q f, \quad \text{id}^\Sigma_q \bot = \bot, \quad \text{id}^\Sigma_q \top = \top.
\]

- The composition map \( *_p : \Sigma C_q \times (\Sigma C)_p \to (\Sigma C)_q \) is given by

\[
g *^\Sigma_p f = g *^C_{p-1} f
\]

Regarding \( \Sigma C \) as an \( \omega \)-category bipointed on \( \bot \) and \( \top \), the construction defines a functor \( \Sigma : \text{Cat} \to \text{Cat}^\ast_\ast \).

\(^3\)Precisely, what we present in Definition 2.3 is the composite of the one used in [AM20, §B.6.5] with the total dual from [AM20 §1.8].
2.3. **Steiner’s functors.** We briefly recall Steiner’s adjoint pair that relates $\omega$-categories and augmented directed chain complexes. For a more detailed treatment, see [Ste04 Definition 2.8] or [AM20 §2.4].

**Definition 2.5.** Let $C$ be an augmented directed chain complex. A **Steiner table** in $C$ is a matrix

$$x = \begin{pmatrix} x_{0}^- & \cdots & x_{q-1}^- & x_{q}^- \\ x_{0}^+ & \cdots & x_{q-1}^+ & x_{q}^+ \end{pmatrix}$$

such that, for $\alpha = +, -$ and $0 \leq p \leq q$, the following hold:

1. $x_{p}^\alpha$ belongs to $C_{p}^\alpha$;
2. $\partial(x_{p}^\alpha) = x_{p-1}^\alpha - x_{p-1}^\alpha$ for $0 < p \leq q$;
3. $\varepsilon(x_{0}^\alpha) = 1$;
4. $x_{q}^- = x_{q}^+.$

**Definition 2.6 ([Ste04 Definition 2.8], [AM20 §2]).** Let $C$ be an augmented directed chain complex. The $\omega$-categorical realization of $C$ is the $\omega$-category $\nu C$ is defined as follows.

- The set $(\nu C)_q$ of $q$-cells is given by
  $$(\nu C)_q := \left\{ x = \begin{pmatrix} x_{0}^- & \cdots & x_{q-1}^- & x_{q}^- \\ x_{0}^+ & \cdots & x_{q-1}^+ & x_{q}^+ \end{pmatrix} \right\}.$$

- The source map $s_q: (\nu C)_p \to (\nu C)_q$ is given by
  $$s_q \begin{pmatrix} x_{0}^- & \cdots & x_{p-1}^- & x_{p}^- \\ x_{0}^+ & \cdots & x_{p-1}^+ & x_{p}^+ \end{pmatrix} := \begin{pmatrix} x_{0}^- & \cdots & x_{q-1}^- & x_{q}^- \\ x_{0}^+ & \cdots & x_{q-1}^+ & x_{q}^+ \end{pmatrix}.$$

- The target map $t_q: (\nu C)_p \to (\nu C)_q$ is given by
  $$t_q \begin{pmatrix} x_{0}^- & \cdots & x_{p-1}^- & x_{p}^- \\ x_{0}^+ & \cdots & x_{p-1}^+ & x_{p}^+ \end{pmatrix} := \begin{pmatrix} x_{0}^- & \cdots & x_{q-1}^- & x_{q}^- \\ x_{0}^+ & \cdots & x_{q-1}^+ & x_{q}^+ \end{pmatrix}.$$

- The composition map $*_{p}: (\nu C)_q \times (\nu C)_p \to (\nu C)_q$ is given by
  $$\begin{pmatrix} x_{0}^- & \cdots & x_{q-1}^- & x_{q}^- \\ x_{0}^+ & \cdots & x_{q-1}^+ & x_{q}^+ \end{pmatrix} *_{p} \begin{pmatrix} y_{0}^- & \cdots & y_{q-1}^- & y_{q}^- \\ y_{0}^+ & \cdots & y_{q-1}^+ & y_{q}^+ \end{pmatrix} := \begin{pmatrix} x_{0}^- & \cdots & x_{p-1}^- & x_{p}^+ & y_{p}^- & x_{p+1}^+ & \cdots & x_{q}^- & y_{q}^+ \\ x_{0}^+ & \cdots & x_{p-1}^+ & x_{p}^- & y_{p}^+ & x_{p+1}^- & \cdots & x_{q}^+ & y_{q}^- \end{pmatrix}.$$

- The identity map $id_q: (\nu C)_p \to (\nu C)_q$ is given by
  $$id_q \begin{pmatrix} x_{0}^- & \cdots & x_{p-1}^- & x_{p}^- \\ x_{0}^+ & \cdots & x_{p-1}^+ & x_{p}^+ \end{pmatrix} := \begin{pmatrix} x_{0}^- & \cdots & x_{p-1}^- & x_{p}^- & 0 & 0 & \cdots & 0 \\ x_{0}^+ & \cdots & x_{p-1}^+ & x_{p}^+ & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The construction extends to a functor $\nu: adCh \to \omega Cat$.

**Definition 2.7.** Let $\mathcal{C}$ be an $\omega$-category. The **linearization** of $\mathcal{C}$ is the augmented directed chain complex $\lambda \mathcal{C}$ defined as follows.
The abelian group \((\lambda C)_q\) of \(q\)-chains of \(\lambda C\) is the quotient of \(\mathbb{Z}[C_q]\) given by
\begin{equation}
(\lambda C)_q := \mathbb{Z}[C_q] \left\langle [x*p, y]_q - [x]_q - [y]_q \mid x, y \in C_q; p < q \right\rangle.
\end{equation}

The positivity submonoid \((\lambda C)_q^+\) is the submonoid of \((\lambda C)_q\) generated by the collection of elements \([f]_q\) for \(f\) a \(q\)-cell of \(C\).

The differential map \(\partial_q: (\lambda C)_{q+1} \rightarrow (\lambda C)_q\) is determined by the condition on generators \(f \in C_q\) given by
\[\partial_q([f]_{q+1}) := [t_q f]_q - [s_q f]_q,\]

The augmentation map \(\varepsilon: (\lambda C)_0 \rightarrow \mathbb{Z}\) is determined by the condition on generators \(x \in C_0\) given by
\[\varepsilon([x]_0) := 1.\]

The construction extends to a functor \(\lambda: \omega Cat \rightarrow \text{adCh}\).

**Theorem 2.9** ([Ste04, §2]). The functors \(\nu\) and \(\lambda\) form an adjoint pair
\[
\lambda: \omega Cat \rightleftarrows \text{adCh}: \nu.
\]
In other words, for any \(\omega\)-category \(C\) and any augmented directed chain complex \(\overline{C}\) there is a natural bijection
\[
\text{adCh}(\lambda C, \overline{C}) \cong \omega Cat(C, \nu \overline{C}).
\]

**Example 2.10** ([Ste07a, Theorem 3.2]). For \(m \geq 0\), there is an isomorphism of augmented directed chain complexes
\[
\lambda O[m] \cong O[m]
\]
and an isomorphism of \(\omega\)-categories
\[
O[m] \cong \nu O[m].
\]

**Lemma 2.11** ([AM20, Proposition 2.19]). Let \(C\) be an augmented directed chain complex and \(\mathcal{C}\) an \(\omega\)-category.

1. There is a natural isomorphism of \(\omega\)-categories
\[
\nu(C^\circ) \cong (\nu C)^\circ.
\]
2. There is a natural isomorphism of augmented directed chain complexes
\[
\lambda(C^\circ) \cong (\lambda C)^\circ.
\]

### 2.4. Steiner’s functors and suspension.

**Lemma 2.12.** For an augmented chain complex \(C\), there is a natural isomorphism
\[
\nu \Sigma C \cong \Sigma \nu C.
\]

**Proof.** The prototypical element of both \((\nu \Sigma C)_q\) and \((\Sigma \nu C)_q\) can be expressed as a table of the form
\[
x = \begin{pmatrix}
\bot & x_0^- & \cdots & x_{q-2}^- & x_{q-1}^- \\
\top & x_0^+ & \cdots & x_{q-2}^+ & x_{q-1}^+
\end{pmatrix},
\]
where
1. \(x_p^+\) belongs to \(C_p^+\);  
2. \(\partial(x_p^+) = x_{p-1}^- - x_{p-1}^-\) for \(0 < p < q\);  
3. \(\varepsilon(x_0^+) = 1\);  
4. \(x_{q-1}^- = x_{q-1}^-\).
One can check that this identification is compatible with source, target, identity and composition operations, and the desired isomorphism of \(\omega\)-categories follows.

\[\square\]

2.5. **Tensor product of \(\omega\)-categories.** The statement of the following theorem relies on the notion of a **strong Steiner \(\omega\)-category**, a.k.a. \(\omega\)-category that admits a strongly loop-free atomic basis. Those are particularly nice \(\omega\)-categories that are in a sense “free” and “loop-free” and we refer the reader to [Ste04] [AM20] [AGOR22] for an account on strong Steiner \(\omega\)-categories. There is also a notion of a **strong Steiner complex**, a.k.a. augmented directed chain complex that admits a strongly loop-free and unital basis, which correspond in a precise sense to Strong Steiner \(\omega\)-categories under the adjunction \((\lambda, \nu)\). For the purpose of this paper, it is sufficient to know the following.

- For every \(m \geq 0\) the \(m\)-oriental \(O[m]\) is a strong Steiner \(\omega\)-category (as shown in [Ste04 Example 3.8]), and so its total dual \(O[m]^\circ\) (which can be verified directly).
- For every \(m \geq 0\) the \(m\)-cell is a strong Steiner \(\omega\)-category (as shown in [Ste04 Example 3.9]).
- For any strong Steiner \(\omega\)-category \(C\), the unit of the adjunction from Theorem 2.9 is an isomorphism of \(\omega\)-categories \(\eta_C : C \cong \nu\lambda C\) (as shown in [Ste04 Theorem 5.11]).
- For any strong Steiner complex \(C\), the counit of the adjunction from Theorem 2.9 is an isomorphism of augmented directed chain complexes \(\epsilon_C : \lambda\nu C \cong C\) (as shown in [Ste04 Theorem 5.11]).
- For any strong Steiner \(\omega\)-category \(C\), the augmented directed chain complex \(\lambda C\) is a strong Steiner complex (as shown in [Ste04 Theorem 5.11]).
- For any strong Steiner complex \(C\), the \(\omega\)-category \(\nu C\) is a strong Steiner \(\omega\)-category (as shown in [Ste04 Theorem 5.11]).
- For any strong Steiner complexes \(C\) and \(\overline{C}\) there is a natural isomorphism of augmented directed chain complexes \(\nu C \otimes \nu \overline{C} \cong \nu(C \otimes \overline{C})\) (as shown in [AM20 Theorem A.15]).

**Theorem 2.13** ([AM20 Theorem A.15]). There exists a unique – up to unique monoidal isomorphism – monoidal structure \(\otimes : \omega\text{Cat} \times \omega\text{Cat} \to \omega\text{Cat}\) on \(\omega\text{Cat}\), called the tensor product of \(\omega\)-categories, such that

- for any strong Steiner \(\omega\)-categories \(C\) and \(\overline{C}\) the tensor product \(C \otimes \overline{C}\) is the \(\omega\)-category \(C \otimes \overline{C} := \nu(\lambda C \otimes \lambda \overline{C})\);
- the functor \(- \otimes -\) commutes with colimits in each variable.

**Proposition 2.14.** The linearization functor defines a strong monoidal functor \(\lambda : (\omega\text{Cat}, \otimes) \to (\text{adCh}, \otimes)\). That is, for any \(\omega\)-categories \(C\) and \(\overline{C}\) there is a natural isomorphism of augmented directed chain complexes

\[\lambda(C \otimes \overline{C}) \cong \lambda C \otimes \lambda \overline{C}.\]

**Proof.** First, we observe that for any strong Steiner \(\omega\)-categories \(C\) and \(\overline{C}\) we have

\[\lambda(C \otimes \overline{C}) \cong \lambda(\nu\lambda C \otimes \nu\lambda \overline{C}) \cong \lambda\nu(\lambda C \otimes \lambda \overline{C}) \cong \lambda C \otimes \lambda \overline{C},\]

so the desired isomorphism holds for strong Steiner \(\omega\)-categories. Since any \(\omega\)-category is a colimit of strong Steiner \(\omega\)-categories (as cells are in particular strong Steiner \(\omega\)-categories),
and the functors \( \lambda (\otimes - ), (\lambda - ) \otimes (\lambda - ) : \omega \text{Cat} \times \omega \text{Cat} \to \omega \text{Cat} \) commute with colimits in both variables, the desired isomorphism follows.

We can understand tensor product of orientals:

**Example 2.15.** Let \( k, \ell \geq -1 \). Have isomorphism of \( \omega \)-categories

\[
O[k] \otimes O[\ell]^\circ \cong \nu O[k] \otimes (\nu O[\ell])^\circ \quad \text{Example 2.10}
\]
\[
\cong \nu O[k] \otimes \nu (O[\ell]^\circ) \quad \text{Lemma 2.11(1)}
\]
\[
\cong \nu (O[k] \otimes O[\ell]^\circ) \quad \text{Theorem 2.13}
\]

and of augmented directed chain complexes

\[
\lambda(O[k] \otimes O[\ell]^\circ) \cong \lambda O[k] \otimes \lambda (O[\ell]^\circ) \quad \text{Proposition 2.14}
\]
\[
\cong \lambda O[k] \otimes (\lambda O[\ell]^\circ) \quad \text{Lemma 2.11(2)}
\]
\[
\cong O[k] \otimes O[\ell]^\circ \quad \text{Example 2.10}
\]

Can understand the suspension of tensor product of orientals:

**Remark 2.16.** Let \( k, \ell \geq 0 \). Applying \( \nu \) to the square (1.15) – and evoking Examples 2.10 and 2.15 and Lemma 2.12 – we obtain the diagram of \( \omega \)-categories

\[
\begin{align*}
O[k] \oplus O[\ell] & \to O[k] + \ell \\
\downarrow & \downarrow \\
O[0] \oplus O[0] & \to \Sigma(O[k] \otimes O[\ell]^\circ).
\end{align*}
\]

(2.17)

In particular, the map

\[
O[k] + \ell \to \Sigma(O[k] \otimes O[\ell]^\circ)
\]

(2.18)

is induced by (1.13).

**Proposition 2.19.** Let \( k, \ell \geq -1 \). The diagram (2.17) induces a natural isomorphism of \( \omega \)-categories

\[
\Sigma(O[k] \otimes O[\ell]^\circ) \cong \nu C
\]

\[
\cong (O[0] \oplus O[0]^\circ) \bigg|_{O[k] \otimes O[\ell]^\circ} \bigg|_{O[k] + \ell},
\]

\[
\cong \big(\bigwedge_{O[k] \otimes O[\ell]^\circ} \bigg|_{O[k] + \ell} \bigg)
\]

\[
\cong \big(\bigwedge_{O[k] \otimes O[\ell]^\circ} \bigg|_{O[k] + \ell} \bigg).
\]

Proof. Consider the commutative diagram of augmented directed chain complexes on the left, and the induced commutative diagram of \( \omega \)-categories on the right:

\[
\begin{align*}
O[k] \oplus O[\ell]^\circ & \to O[k] + \ell \\
\downarrow & \downarrow \\
O[0] \oplus O[0]^\circ & \to \Sigma(O[k] \otimes O[\ell]^\circ)
\end{align*}
\]

\[
\begin{align*}
O[k] \oplus O[\ell]^\circ & \to O[k] + \ell \\
\downarrow & \downarrow \\
O[0] \oplus O[0]^\circ & \to \Sigma(O[k] \otimes O[\ell]^\circ).
\end{align*}
\]

(2.18)

The square on the left is a pushout by (1.15) and, as an application of [Lou21, Théorème 3.1.5], so is the pushout on the right.

Can understand how to map orientals into suspension \( \omega \)-categories:

**Proposition 2.20.** For \( m \geq 1 \) and \( C \) an \( \omega \)-category of the form \( C \cong \nu C \), there is a natural bijection

\[
\bigwedge_{k, \ell \geq -1, k + \ell = m} \omega \text{Cat}(O[k] \otimes O[\ell]^\circ, C) \xrightarrow{\cong} \omega \text{Cat}(O[m], \Sigma C).
\]
Proof. There’s a natural bijection
\[
\prod_{k,\ell} \omega \text{Cat}(O[k] \otimes O[\ell]^\circ, C) \cong \prod_{k,\ell} \omega \text{Cat}(O[k] \otimes O[\ell]^\circ, \nu C)
\]
\[
\cong \prod_{k,\ell} \text{adCh}(\lambda(O[k] \otimes O[\ell]^\circ), C) \quad \text{Theorem 2.9}
\]
\[
\cong \prod_{k,\ell} \text{adCh}(O[k] \otimes O[\ell]^\circ, C) \quad \text{Example 2.10}
\]
and a natural bijection
\[
\omega \text{Cat}(O[m], \Sigma C) \cong \omega \text{Cat}(O[m], \Sigma \nu C)
\]
\[
\cong \omega \text{Cat}(O[m], \nu \Sigma C) \quad \text{Lemma 2.12}
\]
\[
\cong \text{adCh}(\lambda O[m], \Sigma C) \quad \text{Theorem 2.9}
\]
\[
\cong \text{adCh}(O[m], \Sigma C) \quad \text{Example 2.10}
\]
They fit into a commutative diagram of sets
\[
\omega \text{Cat}(O[m], \Sigma C) \leftarrow \prod_{k,\ell} \omega \text{Cat}(\Sigma(O[k] \otimes O[\ell]^\circ), \Sigma C) \leftarrow \prod_{k,\ell} \omega \text{Cat}(O[k] \otimes O[\ell]^\circ, C)
\]
\[
\text{adCh}(O[m], \Sigma C) \leftarrow \prod_{k,\ell} \text{adCh}(\Sigma(O[k] \otimes O[\ell]^\circ), \Sigma C) \leftarrow \prod_{k,\ell} \text{adCh}(O[k] \otimes O[\ell]^\circ, C)
\]
so we obtain two equal bijections of the desired form. □

3. Complicial sets and complicial nerve of \(\omega\)-categories

We recall the basic definitions around simplicial sets with marking and \(n\)-complicial sets, as well as some constructions based on simplicial sets with marking: the suspension and the complicial nerve, as well as the main properties that we use later in the paper, and relevant examples. The study of the homotopy theory of complicial sets originated with [Ver08b, Ver17], and continued with [Rie18, OR20a, OR20b, RV22].

3.1. Complicial sets. We recall the main facts about complicial sets that will be used in this paper.

Definition 3.1. A simplicial set with marking is a pair \((X, tX)\) where \(X\) is a simplicial set, and \(tX = \bigsqcup_{m \geq 1} tX_m \subseteq \bigsqcup_{m \geq 1} X_m\) is a collection of subsets of simplices of \(X\), called marked simplices, which have positive dimension that contain all degenerate simplices of \(X\).

We denote by \(msSet\) the category of simplicial sets with marking and marking-preserving simplicial maps.

Remark 3.2. The category \(msSet\) is cocomplete, and colimits are computed degreewise (a simplex is marked in a colimit if it admits a marked representative).

Definition 3.3. A sub-simplicial set with marking \(X\) of a simplicial set with marking \(Y\) is regular if a simplex of \(X\) is marked in \(X\) precisely when it is marked in \(Y\).

Notation 3.4. We denote
• by $\Delta^k[m]$, for $0 \leq k \leq m$, the standard $m$-simplex in which a non-degenerate simplex is marked if and only if it contains the vertices $\{k - 1, k, k + 1\} \cap [m]$;
• by $\Delta^k[m]'$, for $0 \leq k \leq m$, the standard $m$-simplex with marking obtained from $\Delta^k[m]$ by additionally marking the $(k - 1)$-st and $(k + 1)$-st face of $\Delta[m]$;
• by $\Delta^k[m]''$, for $0 \leq k \leq m$, the standard $m$-simplex with marking obtained from $\Delta^k[m]'$ by additionally marking the $k$-th face of $\Delta[m]$;
• by $\Lambda^k[m]$, for $0 \leq k \leq m$, the regular sub-simplicial set of $\Delta^k[m]$ with marking whose simplicial set is the $k$-horn $\Lambda^k[m]$.
• by $\Delta[3]_\sharp$ the standard 3-simplex with the maximal marking.
• by $\Delta[3]_\equiv$ the standard 3-simplex in which the 1-simplices $[0, 2]$ and $[1, 3]$ are marked, as well as all simplices in dimension 2 or higher.

The following class of maps plays a role in the model structure on $\mathsf{msSet}$ for $(\infty, n)$-categories, with $n \in \mathbb{N} \cup \{\infty\}$.

**Definition 3.5.** Let $n \in \mathbb{N} \cup \{\infty\}$.

1. For $m > 1$ and $0 < k < m$, the **complicial inner horn extension** is the inclusion
   \[ \Lambda^k[m] \to \Delta^k[m]. \]
2. For $m \geq 2$ and $0 < k < m$, the **complicial thinness extension** is the inclusion
   \[ \Delta^k[m]' \to \Delta^k[m]''. \]
3. For $m > n$, the **triviality extension** is the inclusion
   \[ \Delta[m] \to \Delta[m]_t. \]
4. For $m \geq -1$, the **complicial saturation extension** is the inclusion
   \[ \Delta[3]_\equiv \star \Delta[m] \to \Delta[3]_\sharp \star \Delta[m]. \]

We fix the following terminology (cf. [Ver08b, Def. 15]).

**Definition 3.6.** A map of simplicial sets with marking $X \to Y$ is a **complicial inner anodyne extension** if it can be written as a retract of a transfinite composition of pushouts of maps of type (1) and (2) from Definition 3.5.

**Remark 3.7.** One can prove with standard model categorical techniques the following formal properties of complicial inner anodyne extensions.

1. The underlying simplicial map of a complicial inner anodyne extension is an inner anodyne extension of simplicial sets.
2. The class of complicial inner anodyne extensions is closed under transfinite composition and pushouts.

**Lemma 3.8** ([OR20a, Lemma 1.12]). For $m \geq 2$ and $0 < k < m$, let $\Lambda^k[m]'$ denote the regular subset of $\Delta^k[m]'$ whose underlying simplicial set is given by the $k$-horn $\Lambda^k[m]$. The inclusion
   \[ \Lambda^k[m]' \to \Delta^k[m]'' \]
   is a complicial inner anodyne extension.

The category $\mathsf{msSet}$ hosts a model for $(\infty, n)$-categories.

---

4The reader can find the join of simplicial sets with marking $\star$: $\mathsf{msSet} \times \mathsf{msSet} \to \mathsf{msSet}$ in [Ver08b, §3.1], but it will not be needed explicitly in this paper.
Definition 3.9. An $n$-complicial set is a simplicial set that has the right lifting property with respect to all maps of type (1)-(4) from Definition 3.5.

Theorem 3.10 ([OR20b Theorem 1.25]). Let $n \in \mathbb{N} \cup \{\infty\}$. The category $\text{msSet}$ supports a cartesian closed model structure $\text{msSet}_{(\infty,n)}$, that we call the model structure for $(\infty,n)$-categories, where

- the fibrant objects are precisely the saturated $n$-complicial sets,
- the cofibrations are precisely the monomorphisms (of underlying simplicial sets),
- all complicial anodyne extensions are weak equivalences.

Remark 3.11. Other model structures are sometimes considered on $\text{msSet}$, for instance those from [Ver08b Theorem 100] and [Rie18 Examples 3.33–3.36]. But in all the aforementioned model structures complicial inner anodyne extensions are weak equivalences.

3.2. Suspension of complicial sets. We now define the suspension of simplicial sets with marking.

Definition 3.12 ([OR20a Definition 2.6]). Let $X$ be a simplicial set with marking. The suspension $\Sigma X$ of $X$ is the simplicial set with marking defined as follows.

- The set $(\Sigma X)_m$ of $m$-simplices is given by
  
  $$(\Sigma X)_m = \begin{cases} 
  \{\bot, \top\} & \text{if } m = 0, \\
  \{\bot, \top\} \cup \prod_{k=0}^{m-1} X_k & \text{if } m > 0.
  \end{cases}$$

- The face map $d_i : (\Sigma X)_m \to (\Sigma X)_{m-1}$ satisfies
  $$d_i \bot = \bot \text{ and } d_i \top = \top,$$
  and restricts to the map $X_k \subseteq (\Sigma X)_m \to (\Sigma X)_{m-1}$ given by
  $$d_i(x) = \begin{cases} 
  d_i x \in X_{k-1} \subseteq (\Sigma X)_{m-1} & \text{if } 0 \leq i \leq k, \\
  x \in X_k \subseteq (\Sigma X)_m & \text{if } k+1 \leq i \leq m.
  \end{cases}$$

- The degeneracy map $s_i : (\Sigma X)_m \to (\Sigma X)_{m+1}$ satisfies
  $$s_i \bot = \bot \text{ and } s_i \top = \top,$$
  and restricts to the map $X_k \subseteq (\Sigma X)_m \to (\Sigma X)_{m+1}$ given by
  $$s_i(x) = \begin{cases} 
  s_i x \in X_{k+1} \subseteq (\Sigma X)_{m+1} & \text{if } 0 \leq i \leq k, \\
  x \in X_k \subseteq (\Sigma X)_m & \text{if } k+1 \leq i \leq m.
  \end{cases}$$

- The set $t(\Sigma X)_m$ of marked $k$-simplices is given by
  $$t(\Sigma X)_m = \prod_{k=0}^{m} tX_k.$$

Regarding $\Sigma X$ as a simplicial set with marking bipointed on $\bot$ and $\top$, the construction defines a functor $\Sigma : \text{msSet} \to \text{msSet}_{\ast,\ast}$.

Remark 3.13. The set $(\Sigma X)_m^{\text{nd}}$ of non-degenerate $m$-simplices of $\Sigma X$ for $m > 0$ is contained in the set of the non-degenerate $(m-1)$-simplices of $X$, namely

$$(\Sigma X)_m^{\text{nd}} \subseteq X_{m-1}.$$
As a special case of the slice model structures, constructed e.g. in [Hir21], we also obtain model structure $\langle m\text{Set}_{(\infty,n+1)},\circ,\ast,\ast\rangle$ on the category $m\text{Set}_{*,*}$ of bi-pointed simplicial sets with marking.

**Lemma 3.14** ([OR20a, Lemma 2.7]). The marked suspension defines a left Quillen functor $\Sigma : m\text{Set}_{(\infty,n)} \to (m\text{Set}_{(\infty,n+1)},\circ,\ast,\ast)$.

### 3.3. Complicial nerve of $\omega$-categories.

The geometry of orientals is such that the construction $m \mapsto O[m]$ defines a cosimplicial object $O[\bullet]$ in $\omega\text{Cat}$, and in particular it makes sense to define the nerve construction $N : \omega\text{Cat} \to s\text{Set}$ originally due to Street [Str87]. The Street nerve can be endowed with the following marking, originally considered by Roberts in unpublished work and Street in [Str87], further studied by Verity in [Ver08a], and later discussed by Riehl in [Rie18], obtaining a functor $N_{RS} : \omega\text{Cat} \to m\text{Set}$.

**Definition 3.15.** Let $C$ be an $\omega$-category. The Roberts–Street nerve of $C$ is the simplicial set with marking defined as follows:

- The set of $m$-simplices is the set of $\omega$-functors $O[m] \to C$, namely $N_{m}C = \omega\text{Cat}(O[m],C)$,

- the simplicial structure is induced by the geometry of orientals,

- an $m$-simplex of $NC$ is marked in $N_{RS}NC$ if and only if the corresponding $\omega$-functor $O[m] \to C$ sends the unique non-trivial $m$-cell $\langle[0,1,\ldots,m]\rangle$ of $O[m]$ to a trivial $m$-cell of $C$, namely

  $$x \in t(N_{m}C) \iff x(\langle[0,1,\ldots,m]\rangle) = \text{id},$$

  where $\langle[0,1,\ldots,m]\rangle$ denotes the top non-identity $m$-cell of $O[m]$.

The construction extends to a functor $N_{RS} : \omega\text{Cat} \to m\text{Set}$.

In particular, in the Street nerve of an $n$-category $C$ all simplices in dimension at least $n+1$ are marked.

### 3.4. Complicial nerve and suspension.

We describe a comparison map between $\Sigma N_{RS}C$ and $N_{RS}\Sigma C$, which we will show to be furthermore a weak equivalence for an $\omega$-category of the form $C \cong \nu C$.

The simplicial sets have the same sets of 0-simplices, namely

$$(\Sigma N_{RS}C)_0 = \{\bot, \top\} = (N_{RS}\Sigma C)_0,$$

and we now analyze the set of $m$-simplices for $m > 0$.

**Remark 3.16.** We have the following description for the set $(N\Sigma C)_m$ of $m$-simplices of $N\Sigma C$:

$$(N\Sigma C)_m = \omega\text{Cat}(O[m],\Sigma C).$$

Moreover,

$$x \in t(N\Sigma C)_m \iff x(\langle[0,1,\ldots,m]\rangle) = \text{id}.$$
If furthermore $C \cong \nu C$, by Theorem 2.9 and Propositions 1.13 and 2.20 we also obtain the “algebraic” descriptions of $(N\Sigma C)_m$:

\[
\begin{align*}
(N\Sigma C)_m & \cong \omega \text{Cat}(O[m], \Sigma C) \cong \prod_{k+1+\ell=m, \, k, \ell \geq -1} \omega \text{Cat}(O[k] \otimes O[\ell]^\circ, C) \\
adCh(O[m], \Sigma C) & \cong \prod_{k+1+\ell=m, \, k, \ell \geq -1} adCh(O[k] \otimes O[\ell]^\circ, C)
\end{align*}
\]

Moreover, 
\[x \in t(N\Sigma C)_m \iff x([0, \ldots, k] \otimes [0, \ldots, \ell]) = 0.\]

Recall the bijection from Proposition 2.20 which we use in the following definition.

**Definition 3.17.** Let $C$ be an $\omega$-category and $x \in (N\Sigma C)_m$. We say that the $m$-simplex $x$ if of type $k$ if

\[x \in \omega \text{Cat}(O[k] \otimes O[\ell]^\circ, C) \subseteq (N\Sigma C)_m.\]

**Remark 3.18.** Let $C$ be an $\omega$-category and $x \in (N\Sigma C)_m$. The following are equivalent.

1. The simplex $x$ is the degeneracy of a 0-simplex, namely $x = s_0^m \bot$ or $x = s_0^m \top$.
2. The simplex $x$ has type $-1$ or $m$.

If the equivalent conditions are met, we say that $x$ is *totally degenerate.*

**Remark 3.19.** Combining results from previous sections, we have the following equivalent descriptions for the set of $m$-simplices of $\Sigma NC$:

\[(\Sigma NC)_m \cong \{s_0^m \bot, s_0^m \top\} \coprod \prod_{k=0}^{m-1} (NC)_k \cong \{s_0^m \bot, s_0^m \top\} \coprod \prod_{k=0}^{m-1} \omega \text{Cat}(O[k], C).
\]

Moreover, for a non-totally degenerate simplex $x$, we have

\[x \in t(\Sigma N_m C) \iff k < m - 1 \text{ or } x([0,1, \ldots, k]) = \text{id}.
\]

If furthermore $C \cong \nu C$, by Theorem 2.9 also get the “algebraic” descriptions:

\[(\Sigma NC)_m \cong \{s_0^m \bot, s_0^m \top\} \coprod \prod_{k=0}^{m-1} (NC)_k \cong \{s_0^m \bot, s_0^m \top\} \coprod \prod_{k=0}^{m-1} \omega \text{Cat}(O[k], C)
\cong \{s_0^m \bot, s_0^m \top\} \coprod \prod_{k=0}^{m-1} adCh(O[k], C).
\]

Moreover,
\[x \in t(\Sigma NC)_m \iff k < m - 1 \text{ or } x([0, \ldots, k]) = 0.
\]

The canonical map(s) from either (1.14) or (2.17) then induce a canonical natural map $(\Sigma NC)_m \to (N\Sigma C)_m$ which assembles into a map $\Sigma NC \to N\Sigma C$.

**Proposition 3.20.** For any $\omega$-category of the form $C \cong \nu C$, either of the maps Proposition 1.13 or Proposition 2.20 induces:

1. a natural inclusion of simplicial sets
\[\Sigma NC \to N\Sigma C;\]
2. a natural regular inclusion of simplicial sets with marking
\[\Sigma N^{RS} C \to N^{RS} \Sigma C.\]
**Proof.** At the level of simplicial sets, both inclusions act as follows:

- they are identities on 0-simplices, namely they send the 0-simplex $\bot$ to $\bot$ and the 0-simplex $\top$ to $\top$, and
- referring to the identifications Remarks 3.16 and 3.19, they act on an $(m+1)$-simplex $y: O[k] \to C$ of $\Sigma(NC)$ as

$$[y: O[k] \to C] \mapsto \Sigma(O[k] \otimes O[\ell]) \to \Sigma(O[k] \otimes O[0]) \cong \Sigma(O[k]) \xrightarrow{\Sigma y} \Sigma C].$$

From this explicit description, we see that both maps are inclusions, and that the second one is regular, namely, a non-degenerate $(m+1)$-simplex of $\Sigma NC$ is marked in $\Sigma N^{RS}C$ if and only if the corresponding $m$-simplex of $NC$ is marked in $N^{RS}C$. □

**Remark 3.21.** Given an $\omega$-category of the form $C \cong \nu C$, the map from Proposition 3.20 can be seen as induced by the canonical map

$$N^{RS}C * \Delta[0] \to N^{RS}(C * \Delta[0])$$

from [GOR21] Theorem 5.2.

We now prove that the comparison map is a weak equivalence if $C \cong \nu C$.

**Theorem 3.22.** Let $C$ be an $\omega$-category of the form $C \cong \nu C$.

1. The inclusion from Proposition 3.20(1) is an inner anodyne extension, and in particular a weak equivalence in the Joyal model structure $sSet(\infty, 1)$

$$\Sigma NC \xrightarrow{\cong} N \Sigma C.$$

2. The inclusion from Proposition 3.20(1) is a complicial inner anodyne extension, and in particular a weak equivalence in the model structure $msSet(\infty, n)$.

$$\Sigma N^{RS}C \xrightarrow{\cong} N^{RS} \Sigma C.$$

The proof is given in the coming subsections.

**3.5. Proof of Theorem 3.22.** Given an $\omega$-category $C \cong \nu C$, by Remark 3.18 we know that the non-degenerate $m$-simplices of $N \Sigma C$ for $m \geq 1$ are all of the form

$$x: O[k] \otimes O[\ell]^0 \to C \leftrightarrow x: O[k] \otimes O[\ell]^0 \to C,$$

for some $k, \ell \geq 0$ with $m = k + 1 + \ell$. Each such simplex has the following features:

- the dimension $m = k + 1 + \ell$;
- the type $k$, which was defined in Definition 3.17;
- the suspect index $r$, which will be defined in Definition 3.23.

The goal is to filter the inclusion(s) from Proposition 3.20 by a sequence of anodyne extensions $\Sigma N^{RS}C =: X_0 \subseteq X_1 \subseteq \ldots \subseteq X_m \subseteq \ldots \subseteq N^{RS} \Sigma C$, where the inclusions are regular and the underlying simplicial set of $X_m$ contains $X_0$, all the simplices of $N^{RS} \Sigma C$.

---

5The same map is also a weak equivalences in the model structures mentioned in Remark 3.14
of dimension less than \(m\) as well as all \(m+1\)-simplices that are \textit{suspect}, a notion that we'll give in Lemma 3.25 and can be computed as

\[
k := \#\{0 \leq i \leq m \mid x([i]) = \bot\} - 1.
\]

To this end, we will filter the inclusion \(X_{m-1} \subseteq X_m\) again via a sequence of anodyne extensions, as \(X_{m-1} = Y_{m-1} \subseteq Y_{m-2} \subseteq \ldots \subseteq Y_1 \subseteq Y_0 = X_m\), where \(Y_0\) contains \(X_{m-1}\), and show that all the inclusions of \(N^{\text{RS}}\) of dimension \(m\) and type at least \(k\) as well as all suspect simplices of dimension \(m\) and type at least \(k-1\). Perhaps surprisingly, to show that the inclusions \(Y_k \subseteq Y_{k-1}\) are anodyne, we will construct a further filtration again via a sequence of anodyne extensions.

**Definition 3.23.** Let \(\ell > 0\), \(k \geq 0\) and \(m = k + 1 + \ell\). Let \(x : O[k] \otimes O[\ell] \to C\) be a non-totally degenerate \((k+1+\ell)\)-simplex of \(N\Sigma C\). The \textit{suspect index} of \(x\) is the minimal integer \(0 \leq r \leq k\) if existing so that the following two conditions hold.

\(\text{(SusInd 1)}\) Whenever \(a \subseteq [r, k], \ a' \subseteq [0, r-1], \ b \subseteq [0, \ell], \) with \(|a| \geq 0, \ |a'| \geq 0, \ |b| \geq 0,\) we have

\[
x([a', a] \otimes [0, b]) = 0
\]

\(\text{(SusInd 2)}\) Whenever \(a \subseteq [r, k], \ b \subseteq [0, \ell],\) with \(|b| \geq 1,\) we have

\[
x([a] \otimes [b]) = 0.
\]

If there is no integer \(r\) for which the conditions hold, we say that the \textit{suspect index} of \(x\) is \(k+1\).

The following are direct consequences of [Stel12 Proposition 3.4].

**Lemma 3.24.** Let \(x : O[k] \otimes O[\ell] \to C\) be a non-totally degenerate \((k+1+\ell)\)-simplex of \(N\Sigma C\). For \(0 \leq i \leq k-1\) the following are equivalent:

1. Whenever \(a' \subseteq [0, i-1], \ a \subseteq [i+2, k],\) with \(|a| \geq -1, \ |a'| \geq -1, \ |b| \geq 0\) we have

\[
x([a', i, i+1, a] \otimes [b]) = 0.
\]

2. The \((k+1+\ell)\)-simplex \(x : O[k] \otimes O[\ell] \to C\) of \((N\Sigma C)_{k+1+\ell}\) is degenerate at \(i\).

For \(k+1 \leq i \leq m-1 = k + \ell\) the following are equivalent:

1. Whenever \(a \subseteq [0, k], \ b' \subseteq [0, i-(k+1)-1], b \subseteq [i-(k+1)+2, \ell],\) with \(|a| \geq 0, \ |b'| \geq -1, \ |b| \geq -1,\) we have \(x([a] \otimes [b', i-(k+1), i-(k+1)+1, b]) = 0\).

2. The \((k+1+\ell)\)-simplex \(x : O[k] \otimes O[\ell] \to C\) of \((N\Sigma C)_{k+1+\ell}\) is degenerate at \(i\).

**Lemma 3.25.** Let \(y : O[k+1] \otimes O[\ell] \to C\) be a non-totally degenerate \((k+2+\ell)\)-simplex of \(N\Sigma C\) of suspect index \(r\). The following are equivalent:

1. Whenever \(a' \subseteq [0, r-2], \ a \subseteq [r+1, m], \) with \(|a| \geq -1, \ |a'| \geq -1,\) we have \(y([a', r-1, a] \otimes [0]) = 0\).

2. The \((k+1)\)-simplex \(y([-] \otimes [0]) : O[k+1] \to C\) of \((N\Sigma)_{k+1}\) is degenerate at \(r-1\).

If the equivalent conditions are satisfied, we say that \(y\) is a suspect simplex.

We analyze the faces of a suspect simplex.

**Lemma 3.26.** Let \(y : O[k+1] \otimes O[\ell] \to C\) be a suspect simplex of \(N\Sigma C\) suspect index \(r\). Then, if \(d_i y\) is not degenerate, we have that \(d_i y\) is

1. If \(\text{Face 1) of suspect index at most } r-1\) if \(0 \leq i \leq r-2\),

2. If \(\text{Face 2) of suspect index at most } r-1\) if \(i = r-1\),
We distinguish several cases, which correspond to the different cases appearing in the statement.

(Face 1) Let \(0 \leq i \leq r - 2\). Whenever \(a' \subseteq [0, r - 2], a \subseteq [r - 1, k]\), with \(|a| \geq 0, |a'| \geq 0, |b| \geq 0\), we have \(d'[a'] \subseteq [0, r - 1], d'[a] \subseteq [r, k + 1]\) so that

\[
(d_y)([a', a] \otimes [0, b]) = y(d'[a', a] \otimes [0, b]) = 0,
\]

yielding that (SuspInd 1) holds for \(r - 1\) and \(d_y\). Whenever \(a \subseteq [r - 1, k]\), with \(|a| \geq 0, |b| \geq 1\), we have \(d'[a'] \subseteq [r, k + 1]\) and so

\[
(d_y)([a] \otimes [b]) = y(d'[a] \otimes [b]) = 0,
\]

yielding that (SuspInd 2) holds for \(r - 1\) and \(d_y\). So the suspect index of \(d_y\) is at most \(r - 1\).

(Face 2) Let \(i = r - 1\). Whenever \(a' \subseteq [0, r - 2], a \subseteq [r - 1, k]\), with \(|a| \geq 0, |a'| \geq 0, |b| \geq 0\), we have \(d'^{-1}[a'] \subseteq [0, r - 1], d'^{-1}[a] \subseteq [r, k + 1]\) so that

\[
(d_y)([a', a] \otimes [0, b]) = y(d'[a', a] \otimes [0, b]) = 0,
\]

yielding that (SuspInd 1) holds for \(r - 1\) and \(d_y\). Whenever \(a \subseteq [r - 1, k]\), with \(|a| \geq 0, |b| \geq 1\), we have \(d'^{-1}a \subseteq [r, k + 1]\) and so

\[
(d_y)([a] \otimes [b]) = y(d'[a] \otimes [b]) = 0,
\]

yielding that (SuspInd 2) holds for \(r - 1\) and \(d_y\). So the suspect index of \(d_y\) is at most \(r - 1\).

(Face 3) Let \(r + 1 \leq i \leq k + 1\). Whenever \(a' \subseteq [0, r - 1], a \subseteq [r, k]\), with \(|a| \geq 0, |a'| \geq 0, |b| \geq 0\), we have

\[
(d_y)([a] \otimes [b]) = 0,
\]

yielding that (SuspInd 1) holds for \(r\) and \(d_y\).

Moreover, whenever \(|a| \geq 0, a \subseteq [r, k]\), with \(|b| \geq 1\), we have

\[
(d_y)([a] \otimes [b]) = 0,
\]

yielding that (SuspInd 2) holds for \(r\) and \(d_y\). So the suspect index of \(d_y\) is at most \(r\).

To see that the simplex is suspect in the case that the suspect index is exactly \(r\), we observe that for any \(a' \subseteq [0, r - 2], a \subseteq [r - 1, k]\), with \(|a| \geq -1, |a'| \geq -1\) we have

\[
(d_y)([a', r - 1, r, a] \otimes [0]) = y(d'[a', r - 1, r, a] \otimes [0]) = 0.
\]

(Face 4) The last case is clear since the face operator acts on the second coordinate. □

**Lemma 3.27.** Let \(x\) be a non-degenerate non-suspect simplex of suspect index \(r\). There is a simplex \(\tilde{x}: O[k + 1] \otimes O[l] \to C\) defined by the following formulas:

\begin{align*}
(P1) \quad \tilde{x}(d'[a] \otimes [b]) &= x([a] \otimes [b]) & \text{if } |a| \geq -1, |b| \geq -1, \\
(P2) \quad \tilde{x}([r, a] \otimes [b]) &= x(s^{-1}[r, a] \otimes [0]) & \text{if } |a| \geq -1, \\
(P3) \quad \tilde{x}([a', r] \otimes [b]) &= x([a'] \otimes [0, b]) & \text{if } |a'| \geq 0, |b| \geq 1, \\
(P4) \quad \tilde{x}([a', r] \otimes [0]) &= x([a'] \otimes [0, b]) + x([a', r - 1] \otimes [0]) & \text{if } |a'| \geq 0, \\
(P5) \quad \tilde{x}([a', r, a] \otimes [0]) &= x(s^{-1}[a', r, a] \otimes [0]) & \text{if } |a| \geq 0, |a'| \geq 0, \\
(P6) \quad \tilde{x}([a', r, a] \otimes [b]) &= x([a'] \otimes [0, b]) & \text{if } |a| \geq 0, |a'| \geq -1, |b| \geq 1, \\
(P7) \quad \tilde{x}([r] \otimes [b]) &= 0 & \text{if } |b| \geq 1.
\end{align*}
The proof consists of a careful (and tedious) analysis of all possible cases, and is postponed until Section 3.3.

Remark 3.28. Given a non-degenerate non-suspect simplex \( x \) of suspect index \( r \), by construction we have \( d_r(\bar{x}) = x \).

We record the following features of \( \bar{x} \).

**Lemma 3.29.** If \( x \) is a non-suspect simplex of \( \mathcal{N}\Sigma \mathcal{C} \) with dimension \( m = k + 1 + \ell \), type \( k \) and suspect index \( r \), then the simplex \( \bar{x} \) is a suspect simplex of dimension \( m + 1 \), type \( k + 1 \) and suspect index \( r \).

**Proof.** By construction, the simplex \( \bar{x} \) is a suspect simplex of dimension \( m \), type \( k \) and suspect index at most \( r \). We now prove the suspect index of \( \bar{x} \) is exactly \( r \).

Assume by contradiction that the suspect index of \( \bar{x} \) is at most \( r - 1 \). Then, whenever \( \bar{a}' \subseteq [0, r - 2] \), \( \bar{a} \subseteq [r - 1, k] \), we have \( d'(a) \subseteq [r - 1, k] \), so that

\[
x((\bar{a}', a) \otimes [0, b]) = \bar{x}(d'(\bar{a}', a) \otimes [0, b])) = 0,
\]
yielding that \([\text{SuspInd 1}]\) holds for \( r - 1 \) and \( x \). Also, whenever \( \bar{a} \subseteq [r - 1, k] \), with \( |a| \geq 0 \), and \( |b| \geq 1 \), we have

\[
x([a] \otimes [b]) = \bar{x}(d'(a) \otimes [b]) = 0,
\]
yielding that \([\text{SuspInd 2}]\) holds for \( r - 1 \) and \( x \). This would thus imply that \( x \) is also of suspect index at most \( r - 1 \), contrary to the assumption. \( \square \)

**Lemma 3.30.** Let \( x \) be a non-suspect simplex of \( \mathcal{N}\Sigma \mathcal{C} \). If \( x \) is non-degenerate, then \( \bar{x} \) is non-degenerate.

**Proof.** Let \( x \) be a simplex of dimension \( m \), type \( k \) and suspect index \( r \). Assume that \( \bar{x} = s_i\bar{x} \) is degenerate at some \( 0 \leq i \leq m \), and deduce a contradiction by distinguishing several cases.

- If \( i = r - 1 \), we can prove that \( x \) would be of suspect index at most \( r - 1 \), contradicting the assumption. Since \( \bar{x} \) is degenerate at \( r - 1 \), we have \( x = d_r\bar{x} = d_{r-1}\bar{x} \).

We check first that then the condition \([\text{SusInd 1}]\) for \( r - 1 \) and \( x \). Assume \( \bar{a}' \subseteq [0, r - 2] \), \( \bar{a} \subseteq [r - 1, k] \), with \( |a| \geq 0 \), \( |a'| \geq 0 \), \( |b| \geq 0 \). If \( \bar{a} \) does not contain \( r - 1 \), then we have

\[
x([a', a] \otimes [0, b]) = 0
\]
since \( x \) has suspect index \( r \). If \( a \) contains \( r - 1 \) and at least one further element, then we have

\[
x([a', r, a] \otimes [0, b]) = 0
\]
since \( x \) has suspect index \( r > r - 1 \). Finally, if \( \bar{a} = [r - 1] \), we have

\[
x([a', r - 1] \otimes [0, b]) = d_{r-1}\bar{x}([a', r - 1] \otimes [0, b]) = \bar{x}([a', r] \otimes [0, b]) = 0.
\]
This yields the condition \([\text{SuspInd 1}]\) for \( r - 1 \) and \( x \).

For \([\text{SuspInd 2}]\), we observe that, whenever \( \bar{a} \subseteq [r, k] \), we have

\[
x([r - 1, a] \otimes [b]) = \bar{x}([r, a] \otimes [b]) = 0.
\]
This yields the condition \([\text{SuspInd 2}]\) for \( r - 1 \) and \( x \).
• If \( i = r \), we can prove that \( x \) would be suspect, contradicting the assumption. Indeed, we observe that
\[
\begin{align*}
x([a', r-1, r, a] \otimes [0]) &= x(s^r \partial^r \cdot ([a', r-1, r, a] \otimes [0]) \\
&= \frac{\partial^r}{\partial^r} (d^r \cdot ([a', r-1, r, a] \otimes [0])) \\
&= (s_r z) (d^r \cdot ([a', r-1, r, a] \otimes [0])) \quad \text{[P5], [P2]} \\
&= 0
\end{align*}
\]
so that \( x \) would be indeed suspect. This yields the desired contradiction.

• If \( i \neq r - 1, r \), we can prove that \( x \) would be degenerate, contradicting the assumption. Indeed, we would obtain
\[
x = d_r \overline{x} = d_r s_i z = \begin{cases} 
  s_{i-1} d_r z & \text{if } r + 1 \leq i \leq k + 1 + l \\
  s_i d_r z & \text{if } 0 \leq i \leq r - 2
\end{cases}
\]
contrary to the assumption that \( x \) is non-degenerate.

This concludes the proof. \( \square \)

The following shows that the values of a suspect simplex \( y \) are enforced by those of its face \( d_r y \).

**Lemma 3.31.** Let \( y : O[k+1] \otimes O[l]^\circ \to C \) be a suspect simplex of \( N \Sigma \mathcal{C} \) with suspect index \( r \).

1. \( y([d^r a] \otimes [b]) = d_r y([a] \otimes [b]) \) if \( |a| \geq -1, |b| \geq -1 \),
2. \( y([r, a] \otimes [b]) = d_r y(s^{r-1}[r, a] \otimes [0]) \) if \( |a| \geq -1 \),
3. \( y([a', r] \otimes [b]) = d_r y([a'] \otimes [0, b]) \) if \( |a'| \geq 0, |b| \geq 1 \),
4. \( y([a', r, a] \otimes [b]) = d_r y(s^{r-1}[a', r, a] \otimes [0]) \) if \( |a'| \geq 0 \),
5. \( y([a', r, a] \otimes [b]) = d_r y(s^{r-1}[a', r, a] \otimes [0]) \) if \( |a'| \geq 0 \),
6. \( y([r, a] \otimes [b]) = 0 \) if \( |a| \geq 0, |a'| \geq -1, |b| \geq 1 \),
7. \( y([r] \otimes [b]) = 0 \) if \( |b| \geq 1 \).

**Proof.** We now show that a suspect \( m \)-simplex \( y : O[k+1] \otimes O[l]^\circ \to C \) of suspect index \( r \) is already completely specified by \( d_r y \), in the way described more precisely by the statement.

1. The value of \( y \) in this case is by definition of simplicial structure of \( N \Sigma \mathcal{C} \).
2. We prove the formula for \( y([r, a] \otimes [b]) \) in this case. We observe that
\[
0 = \partial y([r, a] \otimes [0, b]) \quad \text{[SuspInd 2]}
\]
\[
\begin{align*}
&= y([a] \otimes [0, b]) - y([r, \partial a] \otimes [0, b]) \\
&\quad + (-1)^{|a|+1} y([r, a] \otimes [0]) - (-1)^{|a|+1} y([r, a] \otimes [b]) \\
&= (1)^{|a|+1} y([r, a] \otimes [0]) - (1)^{|a|+1} y([r, a] \otimes [b]) \\
&= y([r, a] \otimes [0]) - y([r, a] \otimes [b]) \\
&= y(d^r s^{r-1}[a, r, a] \otimes [0]) - y([r, a] \otimes [b]) \quad \text{[SuspInd 2]}
\end{align*}
\]

The desired formula follows.

3. We prove the formula for \( y([a', r] \otimes [b]) \) in this case. If \( b \) contains 0, the term vanishes and the formula follows by ([SuspInd 1]). If \( b \) does not contain 0, we have
\[
0 = \partial y([a', r] \otimes [0, b]) \quad \text{[SuspInd 1]}
\]
\[
\begin{align*}
&= (-1)^{|a'|+1} y([a'] \otimes [0, b]) + y([\partial a', r] \otimes [0, b]) \\
&\quad + (-1)^{|a'|+1} y([a', r] \otimes [b]) + (-1)^{|a'|+1} y([a', r] \otimes [0, \partial^2 b]) \\
&= (1)^{|a'|+1} y([a'] \otimes [0, b]) + (1)^{|a'|+1} y([a', r] \otimes [b]) \\
&= y([a'] \otimes [0, b]) - y([a', r] \otimes [b]) \\
&= y([a'] \otimes [0, b]) - y([a', r] \otimes [b]) \quad \text{[SuspInd 1]([SuspInd 2])}
\end{align*}
\]

The desired formula follows.
We prove the formula for \( y([a',r] \otimes [b]) \) in this case. If \( b = 0 \), the formula follows from Lemma 3.25. If \( b > 0 \), we have

\[
0 = \partial y([a',r] \otimes [0,b]) = (−1)^{|a'|+1}y([a',r] \otimes [0,b]) + y((\partial a',r) \otimes [0,b]) + (−1)^{|a'|+2}y([a',r] \otimes [0,b]) + (−1)^{|a'|+3}y([a',r] \otimes [0,b])
\]

\[
= (−1)^{|a'|+1}y([a'] \otimes [0,b]) + (−1)^{|a'|+2}y([a',r] \otimes [0,b]) + (−1)^{|a'|+3}y([a',r] \otimes [0,b])
\]

\[
y([a'] \otimes [0,b]) = y([a',r] \otimes [0,b]) + y([a',r] \otimes [0,b])
\]

The desired formula follows.

(S5) We prove the formula for \( y([a',r,a] \otimes [b]) \) in this case. If \( b = 0 \), then the equality follows from Lemma 3.25. If \( b > 0 \), we have

\[
0 = \partial y([a',r,a] \otimes [0,b]) = (−1)^{|a'|+1}y([a',a] \otimes [0,b]) + y((\partial a',r,a) \otimes [0,b]) + (−1)^{|a'|+2}y([a',r,a] \otimes [0,b]) + (−1)^{|a'|+3}y([a',r,a] \otimes [0,b])
\]

\[
= (−1)^{|a'|+1}y([a'] \otimes [0,b]) + (−1)^{|a'|+2}y([a',r,a] \otimes [0,b]) + (−1)^{|a'|+3}y([a',r,a] \otimes [0,b])
\]

\[
y([a',r,a] \otimes [0,b]) = y([a',r,a] \otimes [0,b]) + y([a',r,a] \otimes [0,b])
\]

The desired formula follows.

(S6) We prove that \( y([a',r,a] \otimes [b]) \) necessarily vanishes. If \( b \) contains \( 0 \), this is a special case of (S5). If \( b \) does not contain \( 0 \), we have

\[
0 = \partial y([a',r,a] \otimes [0,b]) = (−1)^{|a'|+1}y([a',a] \otimes [0,b]) + y((\partial a',r,a) \otimes [0,b]) + (−1)^{|a'|+2}y([a',r,a] \otimes [0,b]) + (−1)^{|a'|+3}y([a',r,a] \otimes [0,b])
\]

\[
= (−1)^{|a'|+2}y([a',r,a] \otimes [0,b]) + (−1)^{|a'|+3}y([a',r,a] \otimes [0,b])
\]

If \( |a| > 0 \), the desired vanishing follows directly from (S5). If \( |a| = 0 \), the desired vanishing follows from (S3) which we have treated before.

(S7) The fact that \( y([r] \otimes [b]) \) vanishes in this case can be seen applying (S5) for \( y \) and \( r \).

This concludes the proof.

Lemma 3.32. Let \( y: O[k+1] \otimes O[\ell] \to C \) be a non-degenerate suspect simplex of \( N\Sigma \) with suspect index \( r \). Then the face \( d_r y \) is a simplex of dimension \( m \), type \( k \) and suspect index \( r \).

We record the following features of \( d_r y \).

Proof. The value of the dimension and type of \( d_r y \) are immediate from the definitions.

We now show that \( d_r y \) is of suspect index \( r \). It is straightforward from the construction that the suspect index of \( d_r y \) is at most \( r \). Assuming for contradiction the suspect index of \( d_r y \) to be at most \( r - 1 \), we show that the suspect index of \( y \) would be also at most \( r - 1 \), contrary to the assumptions.
One can show that the conditions (SusInd 1) and (SusInd 2) hold for \( y \) and \( r - 1 \). If \([a]\) contains \( r \), this follows from (SusInd 1) and (SusInd 2). If \([a]\) does not contain \( r \), this follows from (S3), (S6), (S7) of Lemma 3.31.

Finally, we show that \( d_r y \) is a non-suspect simplex. Assuming for contradiction that \( d_r y \) is suspect, we show using the characterization from Lemma 3.24 that then \( y \) is degenerate at \( r \), contrary to the assumptions. To this end, we need to show that \( y([a', r, r + 1, a] \otimes |b|) \) vanishes. If \(|b| \geq 1\), this term vanishes by (S6). If \(|b| = 0\) and \(|a'| = -1\), then this term vanishes by (S2) using the assumption that \( d_r y \) is suspect with suspect index \( r \). If \(|b| = 0\) and \(|a'| \geq 0\), this term vanishes by (S5) using the assumption that \( d_r y \) is suspect. □

**Lemma 3.33.** Let \( y \) be a suspect simplex of \( N\Sigma C \) with suspect index \( r \). If \( y \) is non-degenerate, then the face \( d_r y \) is a non-degenerate simplex.

**Proof.** Let \( k + 1 \) be the type of \( y \). Assuming for contradiction that \( d_r y \) is degenerate, we show that \( y \) itself has to be degenerate at some \( 0 \leq i \leq k + \ell \). Notice that the case \( i = k \) cannot happen, because it would violate the type property from Lemma 3.32.

- If \( 0 \leq i < r - 1 \), we use Lemma 3.24 to show that \( y \) is degenerate at \( i \). The fact that 
  
  \[ y([a', i, i + 1, a] \otimes |b|) \]
  
  vanishes is by definition when \( r \) does not occur in \([a]\). Otherwise, it can be deduced from the formulas (S3), (S4), (S5), (S6) together with the assumption that \( d_r y \) is degenerate at \( i \).

- If \( i = r - 1 \), we use Lemma 3.24 to show that \( y \) is degenerate at \( r \). The fact that
  
  \[ y([a', r, r + 1, a] \otimes |b|) \]
  
  vanishes can be deduced from the formulas (S2), (S5), (S6) together with the assumption that \( d_r y \) is degenerate at \( r - 1 \) in the formulation from Lemma 3.24.

- If \( r \leq i < k \), we use Lemma 3.24 to show that \( y \) is degenerate at \( i + 1 \). The fact that
  
  \[ y([a', i + 1, i + 2, a] \otimes |b|) \]
  
  vanishes can be deduced from the formulas (S2), (S5), (S6) together with the assumption that \( d_r y \) is degenerate at \( i \) in the formulation from Lemma 3.24.

- If \( k + 1 \leq i \leq k + \ell \), we use Lemma 3.24 to show that \( y \) is degenerate at \( i + 1 \). The fact that
  
  \[ y([a] \otimes |b|) \]
  
  vanishes follows from the formulas from Lemma 3.31 together with the fact that \( d_r y \) is degenerate at \( i \) in the formulation from Lemma 3.24.

This concludes the proof. □

We can now establish a correspondence between the suspect and non-suspect simplices of \( N\Sigma C \).

**Proposition 3.34.** Let \( C \) be a 1-category and \( m \geq 0 \). Recall the inclusion \( \Sigma\NC \hookrightarrow N\Sigma C \) from Proposition 3.20.

(i) The non-degenerate \((m + 1)\)-simplices in \( \Sigma\NC \), regarded as a simplicial subset of \( N\Sigma C \), are contained in the \((m + 1)\)-simplices of type \( m \).

(ii) The non-degenerate \((m + 1)\)-simplices in \( N\Sigma C \) that do not belong to \( \Sigma\NC \) are non-degenerate \((m + 1)\)-simplices of type \( 0 \leq k \leq m - 1 \) and suspect index \( 1 \leq r \leq k + 1 \).
(iii) The \( r \)-th face map gives a bijective correspondence between the non-degenerate suspect \((m+1)\)-simplices \( \tilde{x} \) in \( \Sigma NC \setminus \Sigma NC \) of type \( 1 \leq k \leq m-1 \) and suspect index \( 1 \leq r \leq k+1 \) and the non-degenerate non-suspect \( m \)-simplices \( x \) of type \( 0 \leq k-1 \leq m-2 \) and suspect index \( 1 \leq r \leq k+1 \).

Proof. The first two statements can be verified by direct inspection. For the third statement, we observe that the assignment \( (\tilde{\cdot}) \) from Lemma \( 3.24 \) is an inverse for the function \( d_r \) with the given domain and codomain. Indeed, Lemmas \( 3.29 \) and \( 3.30 \) show that \( y \mapsto d_r y \) is a well-defined function, Lemmas \( 3.29 \) and \( 3.30 \) show that \( x \mapsto \tilde{x} \) is a well-defined function, Remark \( 3.28 \) shows that \( d_r \tilde{x} = x \), and the formulas from Lemmas \( 3.27 \) and \( 3.31 \) together imply that \( d_r y = y \). \( \square \)

We now prove the theorem.

Proof of Theorem \( 3.22 \). We prove (2), and (1) follows then by applying the forgetful functor from marked simplicial sets to simplicial sets. In order to show that the inclusion \( \Sigma N^{RS}C \to N^{RS}N \Sigma C \) is a complicial inner anodyne extension, we will realize it as a transfinite composite of intermediate complicial inner anodyne extensions

\[
\Sigma N^{RS}C = X_1 \hookrightarrow X_2 \hookrightarrow \ldots \hookrightarrow X_{m-1} \hookrightarrow X_m \hookrightarrow \ldots \hookrightarrow N^{RS}N \Sigma C.
\]

For \( m \geq 2 \), we let \( X_m \) be the smallest regular subsimplicial set of \( N^{RS}N \Sigma C \) containing \( X_{m-1} \), all \( m \)-simplices of \( N \Sigma C \), as well as the suspect \((m+1)\)-simplices of \( N \Sigma C \). Note that \( X_1 \) already contains all non-degenerate 1-simplices of \( N \Sigma C \) and that there are no non-degenerate suspect 2-simplices. Moreover, by Proposition \( 3.34 \), the subsimplicial set \( X_1 \) contains all non-degenerate \((m+1)\)-simplices of type \( m \). We see that the difference between \( X_{m-1} \) and \( X_m \) are the non-degenerate non-suspect \( m \)-simplices of type at most \( m-2 \) and the non-degenerate suspect \((m+1)\)-simplices of type at most \( m-1 \).

In order to show that the inclusion \( X_{m-1} \hookrightarrow X_m \) is a complicial inner anodyne extension for all \( d \geq 2 \), we realize it as a composite of intermediate complicial inner anodyne extensions

\[
X_{m-1} =: Y_m \hookrightarrow Y_{m-1} \hookrightarrow \ldots \hookrightarrow Y_{k+1} \hookrightarrow Y_k \hookrightarrow \ldots \hookrightarrow Y_1 = X_m.
\]

For \( 1 \leq k < m \), let \( Y_k \) be the smallest regular subset of \( X_m \) containing \( Y_{k+1} \) as well as all non-degenerate suspect \((m+1)\)-simplices \( \bar{x} \) of \( N \Sigma C \) of type \( k \) and all non-degenerate non-suspect \( m \)-simplices of type \( k-1 \). Note that \( Y_m \) already contains all non-degenerate \( m \)-simplices of type \( m-1 \) and that any suspect \((m+1)\)-simplex of type \( m \) is necessarily a degeneracy of a \( m \)-simplex of type \( m-1 \) and thus can be checked to be already in \( Y_m \). We see using Lemma \( 3.26 \) and Proposition \( 3.34 \) that the difference between \( Y_k \) and \( Y_{k+1} \) are the non-degenerate suspect \((m+1)\)-simplices of type \( k \) and possibly some of their faces (precisely those that are neither suspect nor of type \( k \) or higher).

In order to show that the inclusion \( Y_{k+1} \hookrightarrow Y_k \) is a complicial inner anodyne extension for \( 1 \leq k \leq m-1 \), we realize it as a filtration made by intermediate complicial inner anodyne extensions

\[
Y_{k+1} =: W_0 \hookrightarrow W_1 \hookrightarrow \ldots \hookrightarrow W_{r-1} \hookrightarrow W_r \hookrightarrow \ldots \hookrightarrow W_k = Y_k.
\]

For \( 0 < r \leq k \), we let \( W_r \) be the smallest regular simplicial subset of \( Y_k \) containing \( W_{r-1} \) as well as all suspect \((m+1)\)-simplices of \( N^{RS}N \Sigma C \) of type \( k \) and suspect index \( r \), namely those \( \bar{x} \) for which each \( i \)-th row constant for \( r \leq i \leq k \). Note that any \( m \)-simplex of suspect index 0 is either degenerate or of type \( m-1 \) and thus can be checked to be already in...
We argue by induction and using Lemma 3.26 that the above, as shown in Proposition 3.34.

We argue that the \( r \)-horn of \( \bar{x} \) belongs to \( W_{r-1} \); in particular, the \( r \)-horn defines a map of (underlying) simplicial sets

\[
\Lambda^r[m+1] \to W_{r-1}.
\]

Indeed, using Lemma 3.26 we see that the \( a \)-th face of \( \bar{x} \) is already in \( W_{r-1} \) for \( a \neq r \) since it is either a degeneracy of a simplex of smaller dimension or:

\( \diamond \) if \( 0 \leq a \leq r-2 \), the face \( d_a\bar{x} \) is of suspect index at most \( r-1 \), and in particular it belongs to \( W_{r-1} \).

\( \diamond \) if \( a = r-1 \), the face \( d_a\bar{x} \) has suspect index at most \( r-1 \), and in particular it belongs to \( W_{r-1} \).

\( \diamond \) if \( r+1 \leq a \leq k \), the face \( d_a\bar{x} \) is either of suspect index at most \( r-1 \) or suspect of dimension \( m \) and suspect index \( r \); in either case, it belongs to \( W_{r-1} \).

\( \diamond \) if \( k+1 \leq a \leq m+1 \), the face \( d_a\bar{x} \) is of type \( k+1 \), and in particular it belongs to \( Y_{k+1} \subseteq W_{r-1} \).

We argue that the \( r \)-th horn of \( \bar{x} \) defines a map of simplicial sets

\[
\Lambda^r[m+1] \to W_{r-1}
\]

with marking.

Let \([a']\) be a marked \( p \)-simplex of \( \Lambda^r[m+1] \), namely \([a]\) contains the vertices \( \{r-1, r, r+1\} \cap [m+1] \). If the simplex \( \bar{x}([a']) \) is totally degenerate, it is in particular marked, so we will exclude this case for the rest of the discussion. By definition of the suspect index, we have \( 0 \leq r \leq k+1 \). Note that \( r = 0 \) would imply \( x = s_{k+1}d_{k+1} x \) using that \( \ell > 0 \), the characterization Lemma 3.24 and (SuspInd 1). Thus, we can assume \( 0 < r \leq k+1 < k+1 + \ell = m+1 \). In particular, \( \{r-1, r, r+1\} \subseteq [m+1] \).

If \( r \leq k \), using \([P6]\) or \([P5]\) we have

\[
\bar{x}([a']) = \bar{x}([a', r-1, r, r+1, a] \otimes [b]) = 0,
\]

so \( \bar{x}([a']) \) is marked by Remark 3.16. If instead \( r = k+1 \), then using \([\text{SuspInd 1}]\) we obtain

\[
\bar{x}([a', r-1, r] \otimes [0, b]) = 0,
\]

so \( \bar{x}([a']) \) is marked by Remark 3.16. In total, we see that such a face is necessarily marked. Since this holds for all faces as above, we indeed obtain a map of simplicial sets with marking

\[
\Lambda^r[m+1] \to W_{r-1}.
\]

If furthermore \( x \) is marked, we argue that the \( r \)-th horn of \( \bar{x} \) defines a map of simplicial sets with marking

\[
\Lambda^r[m+1]' \to W_{r-1},
\]

with the simplicial set with marking \( \Lambda^r[m+1]' \) defined in Lemma 3.8.

We show that the \( (r-1) \)-st face is marked using Remark 3.10. If \( r \leq k \), since \( \ell \geq 1 \) we can use \([P5]\) and obtain

\[
\bar{x}([0, \ldots, r-1, \ldots, k+1] \otimes [0, \ldots, \ell]) = 0,
\]
so the $(r - 1)$-st face is marked in this case. If $r = k + 1$, by \([P7]\) or \([P3]\) we obtain
\[
\bar{x}([0, \ldots, k, k + 1] \otimes [0, \ldots, \ell]) = 0
\]
so the $(r - 1)$-st face is marked in this case.

We show that the $(r + 1)$-st face is marked using Remark 3.16. If $r \leq k$, since $\ell \geq 1$, we can use \([P3]\) or \([P6]\) to obtain that
\[
\bar{x}([0, \ldots, \widehat{r}, \ldots, k + 1] \otimes [0, \ldots, \ell]) = 0.
\]
so the $(k + 1)$-st face is marked in this case. If $r = k + 1$, using \([P3]\) and the fact that $x$ is marked we obtain
\[
\bar{x}([0, \ldots, k + 1] \otimes [1, \ldots, \ell]) = x([0, \ldots, k] \otimes [0, 1, \ldots, \ell]) = 0.
\]
So the $(k + 1)$-st face is marked in this case.

By filling all $r$-horns of suspect $(m + 1)$-simplices $\bar{x}$ of $W_r$, we then obtain their $r$-th face $x$, which was missing in $W_{r-1}$, as well as the suspect $(m + 1)$-simplex $\bar{x}$ itself. This can be rephrased by saying that there is a pushout square
\[
\begin{array}{ccc}
\coprod_{x} \Lambda^r[m + 1] & \coprod_{x} \Lambda^r[m + 1] & \coprod_{x} \Lambda^r[m + 1] \\
\text{marked} & \text{marked} & \text{marked} \\
W_{r-1} & W_r & W_r.
\end{array}
\]

Since the involved horn inclusions are in fact inner horn inclusions, the inclusions of simplicial sets with marking $\Lambda^r[m + 1] \hookrightarrow \Delta^r[m + 1]$ and $\Lambda^r[m + 1] \hookrightarrow \Delta^r[m + 1]' \hookrightarrow \Delta^r[m + 1]''$ are complicial inner anodyne extensions by Lemma 3.8.

It follows that the inclusion $W_{r-1} \hookrightarrow W_r$ for any $1 \leq r \leq m - j$, the inclusion $Y_{j-1} \hookrightarrow Y_j$ for any $1 \leq j \leq m$, the inclusion $Y_{j-1} \hookrightarrow Y_j$ for any $1 \leq j \leq m$, the inclusion $X_{m-1} \hookrightarrow X_m$ for any $m \geq 1$, and the inclusion $\Sigma N^{RS}C \hookrightarrow N^{RS}\Sigma C$ are complicial inner anodyne extensions, as desired. \(\square\)

4. $\Theta_n$-spaces and Quillen pair with complicial sets

In this section we apply Theorem 3.22 to produce an explicit Quillen adjunction between the model structure of $n$-complicial sets, and the model structure for complete Segal $\Theta_n$-spaces, which we first recall.

4.1. $\Theta_n$-spaces. We recall the main facts about $\Theta_n$-spaces that will be used in this paper.

Remark 4.1. Let $n \geq 0$. The suspension functor $\Sigma: \omega Cat \rightarrow \omega Cat_{*, *}$ restricts and corestricts to a functor $\Sigma: (n - 1)Cat \rightarrow nCat_{*, *}$.

Let $\Theta_n$ denote Joyal’s cell category from [Joy97], which is by [Ber02, MZ01] a full subcategory of $nCat$. By definition, $\Theta_0$ is the terminal category, $\Theta_1$ is the ordinal category $\Delta$, and $\Theta_n$ is for $n > 0$ the full subcategory of $nCat$ whose generic object is obtained as a pushout of $n$-categories
\[
\theta = [k; \theta_1, \ldots, \theta_k] = \Sigma \theta_1 \Pi \Sigma \theta_2 \Pi \ldots \Pi \Sigma \theta_k
\]
for $k \geq 0$ and $\theta_1, \ldots, \theta_k \in \Theta_{n-1}$.

For $n > 0$, there is a full inclusion $\Theta_{n-1} \hookrightarrow \Theta_n$, and whenever needed we will regard any object of $\Theta_{n-1}$ as an object of $\Theta_n$ without further specification.
Definition 4.2. Let $\geq 0$. A $\Theta_n$-space (resp. $\Theta_n$-set) is a presheaf $W: \Theta_n^{op} \to sSet$ (resp. $W: \Theta_n^{op} \to Set$).

For $n \geq 0$, we denote by $sSet^{\Theta_n^{op}}$ (resp. $Set^{\Theta_n^{op}}$) the category of $\Theta_n$-spaces (resp. $\Theta_n$-sets).

Remark 4.3. The categories $sSet^{\Theta_n^{op}}$ and $Set^{\Theta_n^{op}}$ are cocomplete, and colimits are computed componentwise.

For $n \geq 0$ the canonical inclusion $Set \hookrightarrow sSet$ of sets as discrete simplicial sets induces a canonical inclusion $sSet^{\Theta_n^{op}} \hookrightarrow sSet^{\Theta_n^{op}}$, which preserves limits and colimits. In particular, we often regard $\Theta_n$-sets as discrete $\Theta_n$-spaces without further specification.

For any object $\theta$ in $\Theta_n$, we denote by $\Theta_n[\theta]$ the $\Theta_n$-set represented by $\theta$ via the Yoneda embedding $\Theta_n \hookrightarrow Set^{\Theta_n^{op}}$.

Remark 4.4. As a special case of [Ara14, §3.1], given any $\Theta_n$-set $A$ and any space $B$ one can consider the $\Theta_n$-space $A \boxtimes B$, which is defined levelwise as the simplicial set $(A \boxtimes B)_\theta := A_\theta \times B$.

The construction extends to a bifunctor
\[ \boxtimes: Set^{\Theta_n^{op}} \times sSet \to sSet^{\Theta_n^{op}} \]
that preserves colimits in each variable.

4.2. Suspension of $\Theta_n$-spaces.

Remark 4.5. Let $n \geq 0$. The suspension functor $\Sigma: \omega Cat \to \omega Cat$ restricts and corestricts to a functor $\Sigma: \Theta_{n-1} \to \Theta_n$.

As discussed in [Rez10, Remark 4.5] and in [Rez10, Lemma 11.10], the following functor agrees with the functor $V[1]$ from [Rez10, § 4.4].

Definition 4.6. Let $n > 0$, and $\theta \in \Theta_{n-1}$. The suspension of the representable presheaf $\Theta_{n-1}[\theta]$ is the (discrete) $\Theta_n$-space
\[ \Sigma \Theta_n[\theta] := \Theta_n[\Sigma \theta]. \]

The enriched left Kan extension of the functor $\Theta_{n-1} \to \Theta_n \hookrightarrow sSet^{\Theta_n^{op}}$ defines a functor $\Sigma: sSet^{\Theta_n^{op}} \subseteq sSet^{\Theta_n^{op}}$.

4.3. The adjunction. Let us begin by defining the functor that we use to make our comparison.

Construction 4.7. Let $n \geq 0$. The functor $\Theta_n \times \Delta \subseteq sSet^{\Theta_n^{op}} \to msSet$ given by
\[ (\theta, [\ell]) \mapsto (\Theta_n \times \Delta)[\theta, \ell] = \Theta_n[\theta] \boxtimes \Delta[\ell] \to N^{\text{its}} \theta \times \Delta[\ell]^2. \]

induces an adjunction
\[ L_n: sSet^{\Theta_n^{op}} \rightleftarrows msSet: R_n. \]
4.4. The Quillen pair before localizing.

**Proposition 4.8.** Let \( n \geq 0 \). The category \( s\Sigma^\Theta_{\infty}^{op} \) supports the projective model structure \( s\Sigma^\Theta_{\infty}^{\text{proj}} \) where the fibrant objects are precisely the projectively fibrant presheaves and the cofibrations are precisely the projective cofibrations.

**Remark 4.9.** Let \( n \geq 0 \). Combining [Hir03, Theorem 11.6.1, Definition 11.5.33, Definition 11.5.25], we know that

1. a set of generating cofibrations for the projective model structure on \( s\Sigma^\Theta_{\infty}^{\text{proj}} \) is given by all maps of the form
   \[
   \Theta_n[\theta] \Box \partial \Delta[\ell] \to \Theta_n[\theta] \Box \Delta[\ell]
   \]
   for \( \theta \in \Theta_n \) and \( \ell \geq 0 \);

2. a set of generating acyclic cofibrations for the projective model structure on \( s\Sigma^\Theta_{\infty}^{\text{proj}} \) is given by all maps of the form
   \[
   \Theta_n[\theta] \Box \Lambda^k[\ell] \to \Theta_n[\theta] \Box \Delta[\ell]
   \]
   for \( \theta \in \Theta_n \) and \( 0 \leq k \leq \ell \).

The following can be proven similarly to [BOR21, Lemma 1.27].

**Proposition 4.10.** Let \( n \geq 0 \). The functor
\[
(\cdot)^{\Theta}: s\Sigma_{\infty,0} \to m\Sigma_{\infty,0}
\]
is left Quillen.

**Proposition 4.11.** Let \( n \geq 0 \). The functor
\[
L_n: s\Sigma_{\infty,\Theta}^{\text{proj}} \to m\Sigma_{\infty,n}
\]
is left Quillen.

We include the proof for the reader’s convenience, but the argument is the evident generalization of the 2-dimensional case treated in [BOR21, Proposition 2.2].

**Proof.** We want to show that the functor \( L_n \) preserves cofibrations and acyclic cofibrations. Using the facts that \( (\cdot)^{\Theta} \) commutes with colimits, which is a consequence of Proposition 4.10 and that the box product \( \Box \) preserves colimits in each variable, which was recalled in Remark 4.4, we see that

1. the image of the generic generating cofibration via \( L_n \) is the map
   \[
   N^{RS}\theta \times \partial \Delta[\ell]^{\Theta} \to N^{RS}\theta \times \Delta[\ell]^{\Theta}
   \]
   for \( \theta \in \Theta_n \) and \( \ell \geq 0 \);

2. the image of the generic generating acyclic cofibration via \( L_n \) is the map
   \[
   N^{RS}\theta \times \Lambda^k[\ell]^{\Theta} \to N^{RS}\theta \times \Delta[\ell]^{\Theta}
   \]
   for \( \theta \in \Theta_n \) and \( 0 \leq k \leq \ell \).

Since the model structure \( m\Sigma_{\infty,n} \) is cartesian closed by Theorem 3.10 and \( (\cdot)^{\Theta} \) is a left Quillen functor by Proposition 4.10, we conclude that

1. the map \( N^{RS}\theta \times \partial \Delta[\ell]^{\Theta} \to N^{RS}\theta \times \Delta[\ell]^{\Theta} \) is a cofibration and
2. the map \( N^{RS}\theta \times \Lambda^k[\ell]^{\Theta} \to N^{RS}\theta \times \Delta[\ell]^{\Theta} \) is an acyclic cofibration

It follows that \( L_n \) preserves cofibrations and acyclic cofibrations, so it is a left Quillen functor, as desired. \( \square \)
4.5. **The Quillen pair before localizing.** We construct a variant of Rezk’s model structure from [Rez10, §11.4] (see Remark 4.15).

**Definition 4.12.** Let $n \geq 0$. A map of (discrete) $\Theta_n$-spaces is an **elementary acyclic cofibration** if it is of one of the following kinds:

1. For $2 \leq j \leq n$, $k \geq 1$ and objects $\theta_1, \ldots, \theta_k$ of $\Theta_{n-j}$, the $j$-fold $k$-Segality extension $\Sigma^j \Theta_n[\theta_1] \cup \cdots \cup \Sigma^j \Theta_n[\theta_k] \hookrightarrow \Sigma^{j-1} \Theta_n[k|\theta_1, \ldots, \theta_k]$

2. For $0 \leq j \leq n-1$, the $j$-fold completeness extension $\Sigma^j \Theta_n[0] \cup \cdots \cup \Sigma^j \Theta_n[3] \cup \cdots \cup \Sigma^j \Theta_n[0]$.  

**Definition 4.13.** A **complete Segal $\Theta_n$-space** is a $\Theta_n$-space that is local with respect to all maps of type (1) and (2) from Definition 4.12.

By localizing the projective model structure $\mathcal{S}et_{\Theta_n^{\mathcal{O}p} \text{proj}}$ at the class of maps from Definition 4.12, we obtain the following.

**Theorem 4.14.** Let $n \geq 0$. The category $\mathcal{sSet}_{\Theta_n^{\mathcal{O}p}}$ supports a cartesian closed model structure $\mathcal{sSet}_{\Theta_n^{\mathcal{O}p}((\infty,n))}$ where the fibrant objects are precisely the projectively fibrant complete Segal $\Theta_n$-spaces and the cofibrations are precisely the projective cofibrations.

**Remark 4.15.** The model structure from Theorem 4.14 differs from Rezk’s from [Rez10, §11.4] in the following aspects:

1. We work with localizations of the projective model structure, instead of the injective model structure.
2. To express the completeness extension, we use $\Theta_n[0] \cup \Theta_n[1] \cup \Theta_n[3] \cup \cdots \cup \Theta_n[0]$ instead of the $\Theta_n$-nerve of the walking isomorphism.

However, the two model structures are Quillen equivalent (see [Rez10, §2.5–2.13, §10]).

We now show that we still have a Quillen pair after localizing the projective model structure on $\mathcal{sSet}_{\Theta_n^{\mathcal{O}p}}$.

**Theorem 4.16.** Let $n \geq 0$. The functor $L_n: \mathcal{sSet}_{\Theta_n^{\mathcal{O}p}((\infty,n))} \to \mathcal{msSet}_{((\infty,n))}$ is left Quillen.

**Proof.** We prove this by induction on $n \geq 2$. The basis of the induction $n = 0, 1, 2$ are treated in Proposition 4.10, the combination of [Ver08b, §6.5] with [JT07, Theorem 2.4], respectively. We now assume $n > 2$ and that the statement is true for $n-1$. Since cofibrations are unchanged by localization, it suffices to prove that $L_n$ preserves acyclic cofibrations. We do so by proving that $L_n$ preserves all elementary acyclic cofibrations from Definition 4.12 which we do in Propositions 4.18 and 4.19. □

The following result by Steiner will allow us to apply Theorem 3.22 to the case $C = \theta$, for some $\theta \in \Theta_n$.

**Theorem 4.17.** ([Ste07b]). Let $n \geq 0$ and $\theta \in \Theta_n$. There is an isomorphism of $\omega$-categories $\theta \simeq \nu T$ for some augmented directed chain complex $T$. 

We analyze the action of $L_n$ on the Segality extensions.

**Proposition 4.18.** Let $n > 0$. If the functor $L_{n-1} : sSet^{\Theta_n^{op}}_{(\infty,n-1)} \to msSet_{(\infty,n-1)}$ is left Quillen, the functor $L_n : sSet^{\Theta_n^{op}}_{(\infty,n)} \to msSet_{(\infty,n)}$ sends the $j$-fold Segal acyclic cofibration from Definition 4.12:

\[
\Sigma^j \Theta_n[k_1, \ldots, k_n] \to \Sigma^j \Theta_n[k_1, \ldots, k_n] \quad \text{for } 1 \leq j \leq n, \quad k \geq 1 \quad \text{and objects } k_1, \ldots, k_n \text{ of } \Theta_{n-j}, \text{ to a weak equivalence in } msSet_{(\infty,n)}.
\]

**Proof.** We prove the statement by induction on $j \geq 1$, and fixed $k \geq 1$. The basis case(s) $j = 1$ is a direct consequence of [OR20a, Theorem 4.9], and we now assume $j > 1$ for the inductive step. By definition, the map

\[
\Sigma^{j-1} \Theta_n[k_1, \ldots, k_n] \to \Sigma^{j-1} \Theta_n[k_1, \ldots, k_n]
\]

is an acyclic cofibration in $sSet^{\Theta_n^{op}}_{(\infty,n-1)}$. This acyclic cofibration can be rewritten as

\[
\Theta_n \Sigma^{j-1} \Theta_n[k_1, \ldots, k_n] \to \Theta_n \Sigma^{j-2} \Theta_n[k_1, \ldots, k_n].
\]

By applying to it the left Quillen functor $L_{n-1} : sSet_{(\infty,n-1)}^{\Theta_n^{op}} \to msSet_{(\infty,n-1)}$ we obtain an acyclic cofibration in $msSet_{(\infty,n-1)}$

\[
N^{RS} \Theta_n \Sigma^{j-1} \Theta_n[k_1, \ldots, k_n] \to N^{RS} \Theta_n \Sigma^{j-2} \Theta_n[k_1, \ldots, k_n].
\]

By applying to it the left Quillen functor $\Sigma : msSet_{(\infty,n-1)} \to (msSet_{(\infty,n)})_{*,*}$ from Lemma 3.13 we obtain an acyclic cofibration in $msSet_{(\infty,n)}$

\[
\Sigma N^{RS} \Theta_n \Sigma^{j-1} \Theta_n[k_1, \ldots, k_n] \to \Sigma N^{RS} \Theta_n \Sigma^{j-2} \Theta_n[k_1, \ldots, k_n].
\]

Since $\Sigma$ commutes with nerve by Theorems 3.22 and 4.17 we also get an acyclic cofibration

\[
N^{RS} \Theta_n \Sigma^{j-1} \Theta_n[k_1, \ldots, k_n] \to N^{RS} \Theta_n \Sigma^{j-2} \Theta_n[k_1, \ldots, k_n],
\]

which is

\[
L_n \Theta_n \Sigma^{j-1} \Theta_n[k_1, \ldots, k_n] \to L_n \Theta_n \Sigma^{j-2} \Theta_n[k_1, \ldots, k_n].
\]

This concludes the proof. \qed

We analyze the action of $L_n$ on the completeness extensions.

**Proposition 4.19.** Let $n > 0$. If the functor $L_{n-1} : sSet^{\Theta_n^{op}}_{(\infty,n-1)} \to msSet_{(\infty,n-1)}$ is left Quillen, the functor $L_n : sSet^{\Theta_n^{op}}_{(\infty,n)} \to msSet_{(\infty,n)}$ sends the $j$-fold Segal acyclic cofibration from Definition 4.12:

\[
\Sigma^j \Theta_n[0] \to \Sigma^j \Theta_n[0] \quad \text{for } 0 \leq j \leq n-1, \text{ to a weak equivalence in } msSet_{(\infty,n)}.
\]
Proof. We prove the statement by induction on \( j \geq 0 \). The basis case(s) \( j = 0, 1 \) are proven in [BOR21, Propositions 2.7, 2.9], and we now assume \( j > 0 \) for the inductive step. By definition, the map

\[
\Sigma^{j-1} \Theta_n[0] \rightarrow \Sigma^{j-1} \Theta_n[0] \amalg \Sigma^{j-1} \Theta_n[3] \amalg \Sigma^{j-1} \Theta_n[0]
\]

is an acyclic cofibration in \( \text{Set}_{\infty}^{\Delta} \). This acyclic cofibration can be rewritten as

\[
\Theta_n \Sigma^{j-1}[0] \rightarrow \Theta_n \Sigma^{j-1}[0] \amalg \Theta_n \Sigma^{j-1}[3] \amalg \Theta_n \Sigma^{j-1}[0].
\]

By applying to it the left Quillen functor \( L_{n-1} : \text{Set}_{\infty}^{\Delta} \rightarrow \text{Set}_{(\infty)}^{\Delta} \) we obtain an acyclic cofibration in \( \text{Set}_{(\infty)}^{\Delta} \):

\[
N^R \Theta_n \Sigma^{j-1}[0] \rightarrow N^R \Theta_n \Sigma^{j-1}[0] \amalg N^R \Theta_n \Sigma^{j-1}[3] \amalg N^R \Theta_n \Sigma^{j-1}[0].
\]

By applying to it the left Quillen functor \( \Sigma : \text{Set}_{(\infty)}^{\Delta} \rightarrow (\text{Set}_{(\infty)}^{\Delta})_* \), from Lemma 3.14 we obtain an acyclic cofibration in \( \text{Set}_{(\infty)}^{\Delta} \)

\[
\Sigma N^R \Theta_n \Sigma^{j-1}[0] \rightarrow \Sigma N^R \Theta_n \Sigma^{j-1}[0] \amalg \Sigma N^R \Theta_n \Sigma^{j-1}[3] \amalg \Sigma N^R \Theta_n \Sigma^{j-1}[0].
\]

Since \( \Sigma \) commutes with nerve by Theorems 3.22 and 4.17 we also get an acyclic cofibration

\[
N^R \Theta_n \Sigma^{j}[0] \rightarrow N^R \Theta_n \Sigma^{j}[0] \amalg N^R \Theta_n \Sigma^{j}[3] \amalg N^R \Theta_n \Sigma^{j}[0],
\]

which is

\[
L_n \Theta_n \Sigma^{j}[0] \rightarrow L_n \Theta_n \Sigma^{j}[0] \amalg L_n \Theta_n \Sigma^{j}[3] \amalg L_n \Theta_n \Sigma^{j}[0].
\]

This concludes the proof. \( \square \)

With the establishment of Propositions 4.18 and 4.19 the proof of Theorem 4.10 is now complete.

**Appendix A. Proof of Lemma 3.27**

We now prove Lemma 3.27.

**Proof of Lemma 3.27.** Since these cases are mutually exclusive and cover all possibilities, this at least defines a map, and by construction we will have \( d_x x = x \).

It is immediate that the map is directed. Observe that neither the case (P4) nor the case (P5) can apply to a chain of dimension 0, proving that \( x \) is augmented since \( x \) is augmented. What we need to check is that \( x \) is a chain map.

(P1) This case is immediate since \( x \) is a chain map (and ‘not containing \( r \) in the first component’ is preserved by the differential).

(P2) For \( |a| = -1 \), there is nothing to check. For \( |a| \geq 0 \), on the one hand, we have

\[
\bar{x}(\partial [r, a] \otimes [b]) = \bar{x}([a] \otimes [b] - [r, \partial a] \otimes [b]) = x(s^{r-1}[a] \otimes [b]) - x(s^{r-1}[r, \partial a] \otimes [0]);
\]

on the other hand we have

\[
\partial x([r, a] \otimes [b]) = \partial x(s^{r-1}[r, a] \otimes [0]) = x(s^{r-1}[a] \otimes [0]) - x(s^{r-1}[r, \partial a] \otimes [0]).
\]
From \([\text{SuspInd 2}]\) we obtain

\[
0 = \partial x(s^{-1}[a] \otimes [0, b]) = x(s^{-1}[\partial a] \otimes [0, b]) + (-1)^{n_1} x(s^{-1}[a] \otimes [0]) - (-1)^{n_1} x(s^{-1}[a] \otimes [b]).
\]

Using \([\text{SuspInd 2}]\) again, the first summand vanishes, yielding the equality of the other two. This shows the desired equality.

(P3) For \([a'] \geq 1, [b] \geq 2\), on the one hand we have

\[
\tilde{x}(\partial([a', r] \otimes [b])) = \tilde{x}(\partial([a', r] \otimes [b]) + (-1)^{n_1}[a'] \otimes [b] + (-1)^{n_2}[a', r] \otimes \partial^a[b])
\]

\[
= x([\partial a'] \otimes [0, b]) + (-1)^{n_1}[a'] \otimes [b] + (-1)^{n_2}[x([\partial a'] \otimes [b]) +
\]

\[
+(-1)^{n_1}[a'] \otimes [0, \partial^a[b])
\]

on the other hand, we have

\[
\partial \tilde{x}([a', r] \otimes [b]) = \partial(x([a'] \otimes [0, b]))
\]

\[
= x([\partial a'] \otimes [0, b]) + (-1)^{n_1}[a'] \otimes [b] + (-1)^{n_2}[x([\partial a'] \otimes [b])+
\]

\[
+(-1)^{n_1}[a'] \otimes [0, \partial^a[b])
\]

so the two expressions coincide.

For \([a'] = 0, [b] \geq 2\), on the one hand we have

\[
\tilde{x}(\partial([a', r] \otimes [b])) = \tilde{x}([r] \otimes [b] - [a'] \otimes [b] - [a', r] \otimes \partial^a[b])
\]

\[
= -x([a'] \otimes [b]) - x([a'] \otimes [0, \partial^a[b])
\]

On the other hand we have

\[
\partial \tilde{x}([a', r] \otimes [b]) = \partial(x([a'] \otimes [0, b]))
\]

\[
= -x([a'] \otimes [b]) - x([a'] \otimes [0, \partial^a[b])
\]

so the two expressions coincide.

For \([a'] \geq 1, [b] = 1\), on the one hand we have

\[
\tilde{x}(\partial([a', r] \otimes [b_0, b_1])) = \tilde{x}(\partial([a', r] \otimes [b_0, b_1]) + (-1)^{n_1}[a'] \otimes [b_0, b_1] + (-1)^{n_2}[a', r] \otimes [b_0] - (-1)^{n_2}[a', r] \otimes [b_1])
\]

\[
= x([\partial a'] \otimes [0, b_0, b_1]) + (-1)^{n_2}[a'] \otimes [b_0, b_1] + (-1)^{n_1}[a'] \otimes [0, b_0] + x([a', r - 1] \otimes [0]) - (-1)^{n_2}[x([a'] \otimes [0, b_1]) - x([a', r - 1] \otimes [0])
\]

on the other hand we have

\[
\partial \tilde{x}([a', r] \otimes [b_0, b_1]) = \partial(x([a'] \otimes [0, b_0, b_1])
\]

\[
= x([\partial a'] \otimes [0, b_0, b_1]) + (-1)^{n_1}[a'] \otimes [b_0, b_1] + (-1)^{n_2}[x([a'] \otimes [0, b_1]) - (-1)^{n_2}[x([a'] \otimes [0, b_0])
\]

so the two expressions coincide.
For \(|a'| = 0, |b| = 1\), on the one hand we have
\[
\bar{x}(\partial([a', r] \otimes [b_0, b_1])) = \bar{x}([r] \otimes [b_0, b_1] - [a'] \otimes [b_0, b_1] - [a', r] \otimes [b_0] + [a', r] \otimes [b_1])
\]
\[
= -x[a'] \otimes [b_0, b_1] - x([a'] \otimes [0, b_0]) - x([a', r - 1] \otimes [0]) + x([a'] \otimes [0, b_1]) + x([a', r - 1] \otimes [0]);
\]
on the other hand we have
\[
\partial x([a', r] \otimes [b_0, b_1]) = \partial x([a'] \otimes [0, b_0, b_1])
\]
\[
= -x([a'] \otimes [b_0, b_1] - x([a'] \otimes [0, b_0]) + x([a'] \otimes [0, b_1]),
\]
so the two expressions coincide.

(P4) For \(|a'| \geq 1, |a'| \geq 0\), on the one hand we have
\[
\bar{x}(\partial([a', r] \otimes [b])) = \bar{x}((-1)^{|a'|+1} [a'] \otimes [b] + [(\partial a'), r] \otimes [b])
\]
\[
= (-1)^{|a'|+1} x([a'] \otimes [b]) + x((\partial a')) \otimes [0, b]) + x((\partial a'), r - 1] \otimes [0]);
\]
on the other hand we have
\[
\partial x([a', r] \otimes [b]) = \partial x([a'] \otimes [0, b]) + x([a', r - 1] \otimes [0])
\]
\[
= x((\partial a')) \otimes [0, b]) - (-1)^{|a'|} x([a'] \otimes [b]) + (-1)^{|a'|} x([a'] \otimes [0]) + x((\partial a'), r - 1] \otimes [0]);
\]
so the two expressions coincide.

For \(|a'| = 0, |a'| = 0\), on the one hand we have
\[
\bar{x}(\partial([a', r] \otimes [b])) = \bar{x}([r] \otimes [b] - [a'] \otimes [b])
\]
\[
= x([r - 1] \otimes [0]) - x([a'] \otimes [b]);
\]
on the other hand we have
\[
\partial x([a', r] \otimes [b]) = \partial x([a'] \otimes [0, b]) + F([a', r - 1] \otimes [0])
\]
\[
= x([a'] \otimes [0]) - x([a'] \otimes [b]) + x([r - 1] \otimes [0]) - x([a'] \otimes [0]);
\]
so the two expressions coincide.

(P5) For \(|a'| \geq 1, |a'| \geq 1\), on the one hand we have
\[
\bar{x}(\partial([a', r, a] \otimes [b])) = \bar{x}((\partial a', r, [a] \otimes [b])
\]
\[
= (-1)^{|a'|+1} x([a', a] \otimes [b]) + (-1)^{|a'|+2} x([a', r, \partial a] \otimes [b])
\]
\[
= x(s^{-1} (\partial a', r, a] \otimes [0]) + (-1)^{|a'|+1} x(s^{-1}[a', a] \otimes [b]) + (-1)^{|a'|+2} x(s^{-1}[a', r, \partial a] \otimes [0]);
\]
on the other hand we have
\[
\partial x([a', r, a] \otimes [b]) = \partial x(s^{-1}[a', r, a] \otimes [0])
\]
\[
= x(s^{-1} [\partial a', r, a] \otimes [0]) + (-1)^{|a'|+1} x(s^{-1}[a', a] \otimes [0]) + (-1)^{|a'|+2} x(s^{-1}[a', r, \partial a] \otimes [0]).
\]
Observe that from [Susplnd 1] we obtain

\[ 0 = \partial x(s^{r-1}[a', a] \otimes [0, b]) \]
\[ = x(s^{r-1}[\partial a', a] \otimes [0, b]) \]
\[ + (-1)^{|a|+1} x(s^{r-1}[a', \partial a] \otimes [0, b]) \]
\[ + (-1)^{|a|+1} x(s^{r-1}[a', a] \otimes [0]) \]
\[ - (-1)^{|a|+1} x(s^{r-1}[a', a] \otimes [b]). \]

The first two summands vanish again by [Susplnd 1] again, yielding the equality of the other two summands. This shows the desired equality.

For \(|a| = 0, |a'| \geq 1\), on the one hand we have

\[ \tilde{x}(\partial ([a', r, a] \otimes [b])) = \tilde{x}([\partial a', r, a] \otimes [b]) \]
\[ + (-1)^{|a|+1} \tilde{x}([a', a] \otimes [b]) \]
\[ + (-1)^{|a|+1} \tilde{x}([a', r] \otimes [b]) \]
\[ = x(s^{r-1}[\partial a', r, a] \otimes [0]) \]
\[ + (-1)^{|a|+1} x(s^{r-1}[a', a] \otimes [0]) \]
\[ + (-1)^{|a|+1} x(s^{r-1}[a', r] \otimes [0]); \]

on the other hand we have

\[ \partial \tilde{x}([a', r, a] \otimes [b]) = \partial x(s^{r-1}[a', r, a] \otimes [0]) \]
\[ = x(s^{r-1}[\partial a', r, a] \otimes [0]) \]
\[ + (-1)^{|a|+1} x(s^{r-1}[a', a] \otimes [0]) \]
\[ + (-1)^{|a|+1} x(s^{r-1}[a', r] \otimes [0]). \]

Observe that from [Susplnd 1] we obtain

\[ 0 = \partial x(s^{r-1}[a', a] \otimes [0, b]) \]
\[ = x(s^{r-1}[\partial a', a] \otimes [0, b]) \]
\[ + (-1)^{|a|+1} x(s^{r-1}[a'] \otimes [0, b]) \]
\[ + (-1)^{|a|+1} x(s^{r-1}[a', a] \otimes [0]) \]
\[ - (-1)^{|a|+1} x(s^{r-1}[a', a] \otimes [b]). \]

The first summand vanishes again by [Susplnd 1]. This implies the desired equality.

For \(|a| \geq 1, |a'| = 0\), on the one hand we have

\[ \tilde{x}(\partial ([a', r, a] \otimes [b])) = \tilde{x}([r, a] \otimes [b]) - \tilde{x}([a', a] \otimes [b]) \]
\[ + \tilde{x}([a', r, \partial a] \otimes [b]) \]
\[ = x(s^{r-1}[r, a] \otimes [0]) \]
\[ - x(s^{r-1}[a', a] \otimes [b]) \]
\[ + x(s^{r-1}[a', r, \partial a] \otimes [0]); \]

on the other hand we have

\[ \partial \tilde{x}([a', r, a] \otimes [b]) = \partial x(s^{r-1}[a', r, a] \otimes [0]) \]
\[ = x(s^{r-1}[r, a] \otimes [0]) - x(s^{r-1}[a', a] \otimes [0]) \]
\[ + x(s^{r-1}[a', r, \partial a] \otimes [0]). \]
Observe that from \([\text{SuspInd 1}]\) we obtain
\[
0 = \partial x(s^{r-1}[a', a] \otimes [0, b])
\]
\[
= x(s^{r-1}[a] \otimes [0, b])
\]
\[
- x(s^{r-1}[a', \partial a] \otimes [0, b])
\]
\[
+ (-1)^{|a'|+1} x(s^{r-1}[a', a] \otimes [0])
\]
\[
- (-1)^{|a'|+1} x(s^{r-1}[a', a] \otimes [b]).
\]

The first summand vanishes again by \([\text{SuspInd 2}]\) and the second by \([\text{SuspInd 1}]\) so that the other two summands are equal. This implies the desired equality.

For \(|a| = |a'| = 0\), on the one hand we have
\[
\bar{x}(\partial([a', r, a] \otimes [b])) = \bar{x}([r, a] \otimes [b])
\]
\[
- \bar{x}([a', a] \otimes [b])
\]
\[
+ \bar{x}([a', r] \otimes [b])
\]
\[
= x(s^{r-1}[r, a] \otimes [0])
\]
\[
- x(s^{r-1}[a', a] \otimes [b])
\]
\[
+ x(s^{r-1}[a', r] \otimes [0])
\]
\[
+ x([a'] \otimes [0, b]);
\]
on the other hand we have
\[
\partial \bar{x}([a', r, a] \otimes [b]) = \partial x(s^{r-1}[a', r, a] \otimes [0])
\]
\[
= x(s^{r-1}[r, a] \otimes [0])
\]
\[
- x(s^{r-1}[a', a] \otimes [0])
\]
\[
+ x(s^{r-1}[a', r] \otimes [0]).
\]

Using \([\text{SuspInd 1}]\) we obtain
\[
0 = \partial x([a', a - 1] \otimes [0, b])
\]
\[
= x([a - 1] \otimes [0, b])
\]
\[
- x([a'] \otimes [0, b])
\]
\[
+ x([a', a - 1] \otimes [b]) - x([a', a - 1] \otimes [0]).
\]

Now the first summand vanishes by \([\text{SuspInd 2}]\). This yields the desired equality.

\((P6)\) For \(|a| \geq 1, |a'| \geq 0\) and \(|b| \geq 2\), on the one hand we have
\[
\bar{x}(\partial([a', r, a] \otimes [b])) = \bar{x}([\partial a', r, a] \otimes [b])
\]
\[
+ (-1)^{|a'|+1} \bar{x}([a', a] \otimes [b])
\]
\[
+ (-1)^{|a'|+2} \bar{x}([a', r, \partial a] \otimes [b])
\]
\[
+ (-1)^{|a'|+|b|+1} \bar{x}([a', r, a] \otimes \partial b]
\]
\[
= (-1)^{|a'|+1} x(s^{r-1}[a', a] \otimes [0, b]);
\]
on the other hand we have
\[
\partial \bar{x}([a', r, a] \otimes [b]) = 0.
\]

Observe that from \([\text{SuspInd 1}]\) we obtain
\[
0 = \partial x(s^{r-1}[a', a] \otimes [0, b])
\]
\[
= x(s^{r-1}[\partial a', a] \otimes [0, b])
\]
\[
+ (-1)^{|a'|+1} x(s^{r-1}[a', \partial a] \otimes [0, b])
\]
\[
- (-1)^{|a'|+|b|+1} x(s^{r-1}[a', a] \otimes [b])
\]
\[
+ (-1)^{|a'|+|b|+1} x(s^{r-1}[a', a] \otimes [0, \partial b]).\]
The first two summands as well as the last one vanish again by \([\text{SuspInd 1}]\) so the third summand also needs to vanish. This yields the desired equality.

For \(|a| \geq 1, |a'| \geq 0\) and \(|b| = 1\), on the one hand

\[
\partial \bar{x}(\partial[a', r, a] \otimes [b_0, b_1]) = \bar{x}(\partial[a', r, a] \otimes [b_0, b_1]) \\
= (-1)^{|a'|+1} \bar{x}(\partial[a', r, a] \otimes [b_0, b_1]) \\
= (-1)^{|a'|+2} \bar{x}(\partial[a', r, a] \otimes [b_0, b_1]) \\
\]

on the other hand, we have

\[
\partial \bar{x}(\partial[a', r, a] \otimes [b_0, b_1]) = 0.
\]

Observe that from \([\text{SuspInd 1}]\) we obtain

\[
0 = \partial x(s^{n-1}[a', a] \otimes [0, b_0, b_1]) \\
= x(s^{n-1}[\partial[a', a] \otimes [0, b_0, b_1]) \\
= x(s^{n-1}[\partial[a', a] \otimes [0, b_0, b_1]) \\
= (-1)^{|a'|+1} x(s^{n-1}[a', a] \otimes [0, b_0, b_1]) \\
= (-1)^{|a'|+2} x(s^{n-1}[a', a] \otimes [0, b_0, b_1]) \\
\]

The first two summands as well as the last two vanish again by \([\text{SuspInd 1}]\) so the third summand also needs to vanish. This yields the desired equality.

For \(|a| = 0, |a'| \geq 0\) and \(|b| \geq 2\); on the one hand we have

\[
\partial \bar{x}(\partial[a', r, a] \otimes [b_0, b_1]) = \bar{x}(\partial[a', r, a] \otimes [b_0, b_1]) \\
= (-1)^{|a'|+1} \bar{x}(\partial[a', r, a] \otimes [b_0, b_1]) \\
= (-1)^{|a'|+2} \bar{x}(\partial[a', r, a] \otimes [b_0, b_1]) \\
\]

on the other hand we have

\[
\partial \bar{x}(\partial[a', r, a] \otimes [b_0, b_1]) = 0.
\]

Observe that by \([\text{SuspInd 1}]\) we have

\[
0 = \partial x(s^{n-1}[a', a] \otimes [0, b_0]) \\
= x(s^{n-1}[\partial[a', a] \otimes [0, b_0]) \\
= x(s^{n-1}[\partial[a', a] \otimes [0, b_0]) \\
= (-1)^{|a'|+1} x(s^{n-1}[a', a] \otimes [0, b_0]) \\
= (-1)^{|a'|+2} x(s^{n-1}[a', a] \otimes [0, b_0]) \\
\]

The first and the last summands vanish using \([\text{SuspInd 1}]\) once again. This yields the desired equality.
For \(|a| = 0, |a'| \geq 0, |b| = 1\), on the one hand we have
\[
\tilde{x}(\partial([a', r, a] \otimes [b_0, b_1])) = \tilde{x}([a', a] \otimes [b_0, b_1]) + (-1)^{|a'|+1} \tilde{x}([a', r] \otimes [b_0, b_1]) + (-1)^{|a'|+2} \tilde{x}([a', a] \otimes [b_0 - b_1]) = (-1)^{|a'|+1} x(s^{-1}[a', a] \otimes [b_0, b_1]) + (-1)^{|a'|+2} x([a'] \otimes [0, b_0, b_1]) + x(s^{-1}[a', r, a] \otimes [0]) - x(s^{-1}[a', r, a] \otimes [0]);
\]
on the other hand we have
\[
\partial \tilde{x}([a', r, a] \otimes [b_0, b_1]) = 0.
\]
Observe that by \([\text{SuspInd} 1]\) we have
\[
0 = x(s^{-1}[a', a] \otimes [0, b_0, b_1]) + (-1)^{|a'|+1} x(s^{-1}[a'] \otimes [0, b_0, b_1]) - (-1)^{|a'|+1} x(s^{-1}[a', a] \otimes [0, b_0]) + (-1)^{|a'|+1} x(s^{-1}[a', a] \otimes [0, b_1]).
\]
The first and the last two summands vanish using \([\text{SuspInd} 1]\) once again. This yields the desired equality.

For \(|a| \geq 1, |a'| = -1, |b| \geq 2\), on the one hand we have
\[
\tilde{x}(\partial([r, a] \otimes [b])) = \tilde{x}([a] \otimes [b]) - \tilde{x}([r, \partial a] \otimes [b]) + (-1)^{|a'+|a|+1} \tilde{x}([r, a] \otimes \partial^o[b]) = x(s^{-1}[a'] \otimes [b]);
\]
on the other hand we have
\[
\partial \tilde{x}([r, a] \otimes [b]) = 0.
\]
Observe that by \([\text{SuspInd} 2]\) the two results coincide.

For \(|a| \geq 1, |a'| = -1, |b| = 1\), on the one hand we have
\[
\tilde{x}(\partial([r, a] \otimes [b_0, b_1])) = \tilde{x}([a] \otimes [b_0, b_1]) - \tilde{x}([r, \partial a] \otimes [b_0, b_1]) + (-1)^{|a|+1} \tilde{x}([r, a] \otimes [b_0]) = x(s^{-1}[a] \otimes [b_0, b_1]) + (-1)^{|a|+1} x(s^{-1}[r, a] \otimes [0]) - (-1)^{|a|+1} x(s^{-1}[r, a] \otimes [0]);
\]
on the other hand we have
\[
\partial \tilde{x}([r, a] \otimes [b_0, b_1]) = 0.
\]
Then \([\text{ SuspInd } 2]\) yields the desired equality. For \(\lvert a \rvert = 0, \lvert a' \rvert = -1, \lvert b \rvert \geq 2\), on the one hand we have
\[
\bar{x}(\partial([r, a] \otimes [b])) = \bar{x}([a] \otimes [b]) - \bar{x}(r \otimes [b]) + (-1)^1 \bar{x}([r, a] \otimes \partial^0[b]) = x(s^{-1}[a] \otimes [b]);
\]
on the other hand we have
\[
\partial \bar{x}([r, a] \otimes [b]) = 0.
\]
Using \([\text{ SuspInd } 2]\) once again, we obtain the desired equality.

For \(\lvert a \rvert = 0, \lvert a' \rvert = -1, \lvert b \rvert = 1\); on the one hand
\[
\bar{x}(\partial([r, a] \otimes [b_0, b_1])) = \bar{x}([a] \otimes [b_0, b_1]) - \bar{x}([r] \otimes [b_0, b_1]) + \bar{x}([r, a] \otimes [b_1]) - \bar{x}([r, a] \otimes [b_0]) = x(s^{-1}[a] \otimes [b_0, b_1]) + x(s^{-1}[r, a] \otimes [0]) - x(s^{-1}[r, a] \otimes [0]);
\]
on the other hand we have
\[
\partial \bar{x}([r, a] \otimes [b_0, b_1]) = 0.
\]
Using \([\text{ SuspInd } 2]\) once again yields the desired equality.

(P7) In this case, all constituents can be seen to be 0 in a straightforward manner.

Since we treated all possible cases, we conclude that \(\bar{x}\) is indeed a chain map. \(\square\)

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