Riemann-Hilbert approach and $N$-soliton formula for the $N$-component Fokas-Lenells equations

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Abstract

In this work, the generalized $N$-component Fokas-Lenells(FL) equations, which have been studied by Guo and Ling (2012 J. Math. Phys. 53 (7) 073506) for $N=2$, are first investigated via Riemann-Hilbert(RH) approach. The main purpose of this is to study the soliton solutions of the coupled Fokas-Lenells(FL) equations for any positive integer $N$, which have more complex linear relationship than the analogues reported before. We first analyze the spectral analysis of the Lax pair associated with a $(N+1) \times (N+1)$ matrix spectral problem for the $N$-component FL equations. Then, a kind of RH problem is successfully formulated. By introducing the special conditions of irregularity and reflectionless case, the $N$-soliton solution formula of the equations are derived through solving the corresponding RH problem. Furthermore, take $N=2, 3$ and 4 for examples, the localized structures and dynamic propagation behavior of their soliton solutions and their interactions are discussed by some graphical analysis.

Keywords: $N$-component Fokas-Lenells equations, Riemann-Hilbert approach, Multi-soliton solutions.

1. Introduction

As we all known, the nonlinear Schrödinger equation (NLS) is an important integrable system, which plays an important role in nonlinear optics, water waves and plasma physics. Furthermore, the Fokas-Lenells(FL) equation is closely related to the NLS equation in the same way as the Camassa-Holm equation associated with the KdV equation. Since the FL system is one of the important models from both mathematical and physical considerations, obtaining a series solutions of FL system has
been always a focusing subject for many scholars, and much of the research has been carried out on the coupled FL system. The single-component FL equation was first constructed by Fokas \[1\]. After that, the bi-hamiltonian structure, the Lax pair and conservation laws were constructed by Fokas and Lenells \[2\]. Besides, many other scholars have obtained a series of solutions of single-component FL equation, such as dark soliton \[3\], algebraic geometry solution \[4\] and long-time asymptotic behavior of solutions \[5\]. As for multi-component FL equations, Yang has constructed the generalized Darboux transformation(DT) method for the generalized two-component FL equations to get the high-order rogue wave solutions \[6\]. Zhang et al have obtained the soliton, breather and rogue waves solutions for a special two-component FL equations via DT method \[7\]. With the aid of Riemann-Hilbert(RH) approach, Kang et al have solve the two-component FL equations to get multi-soliton solutions formul\[8\]. Hu et al have considered the initial boundary value problem for the two-component FL(FL) equations on the half-line via RH approach \[9\]. In addition, the multi-soliton solutions of a \(m\)-component FL equations with vanishing boundary conditions and nonvanishing boundary conditions are also given by bilinear transformation method \[10\].

Of particular concern in the field of nonlinear science is to find multi-soliton solutions for nonlinear partial differential equations, and a number of effective methods have been produced to solve this problem, such as Hirota bilinear method \[11\], Darboux and Bäcklund transformation \[12\], inverse scattering transformation \[13–15\] and RH approach. Moreover, many scholars show an increasing interest in using the RH approach as a powerful tool to solve certain important problem. For instance, applying the RH approach can slove the soliton solutions of a series of nonlinear evolution equations \[16–30\], study integrable systems with non-zero boundaries \[31–35\], discuss the asymptoticity of integrable system solutions \[36–39\] and so on. The main purpose of our work is to use RH approach which is a powerful tool to solve the multi-soliton solutions of a new class of multi-component FL equations.

In this work, we mainly consider a generalized \(N\)-component FL equations

\[
\begin{align*}
    u_{x,t} - 3u + i(u_1 u^\dagger A u + uu^\dagger A u_1) = 0, \quad (1.1)
\end{align*}
\]

where \(u(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_\mathcal{N}(x,t))^T\) is a \(N\)-component vector function and matrix \(A\) is Hermitian. In \[40\], the \(N\)-component FL equations are given as a by-product of the coupled derivative Schrödinger equations(cDNLS). In addition, the \(N\)-component FL equations \[40\] is equivalent to the coupled FL system in \[7\] by a gauge transformation. Since the authors only considered the multi-soliton solutions under the simplest non-trivial case, i.e.,

\[
\begin{align*}
    u_{x,t} - 3u + i(u_1 u^\dagger A u + uu^\dagger A u_1) = 0, \quad u(x,t) = (u_1, u_2)^T, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}, \sigma = \pm 1, \quad (1.2)
\end{align*}
\]
we decided to generalize the author’s result to obtain the $N$-soliton solutions in more complex case. In our work, we consider the more complex case that the FL system is a $N$-component equations and the relationship matrix $A$ is promoted to the general Hermitian matrix, i.e.,

\[
\begin{align*}
  u_{x,t} - 3u + i(u, u^t Au + uu^t Au_x) &= 0, \\
  u(x, t) &= (u_1, u_2, \ldots, u_{N-1}, u_N)^T, 
\end{align*}
\]  

(1.3)

and

\[
A = \begin{pmatrix}
  a_{1,1} & a_{1,2}^* & \cdots & a_{1,N}^* \\
  a_{2,1} & a_{2,2}^* & \cdots & a_{2,N}^* \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{N,1} & a_{N,2} & a_{N,3} & \cdots & a_{N,N} \\
\end{pmatrix},
\]

(1.4)

with $a_{i,j}(1 \leq i \leq N + 1)$ are real constants and $a_{i,j}(i \neq j)$ are complex constants. Via carrying out the RH approach, we successfully obtain the multi-soliton solutions of the new generalized FL equations and get some certain interesting phenomena about the solutions.

The outline of this work is as follows. In Section 2, starting from analyzing the spectral problem of the Lax pair and analyticity of scattering matrix, a RH problem for the $N$-component FL equations is formulated. Next, via solving the RH problem, we obtain the explicit multi-soliton solutions of the $N$-component FL equations. In Section 3, we consider the solutions under the special case that $N$ and the elements of $A$ are taken as fixed values. Moreover, the localized structures and dynamic propagation behavior of these solutions are presented vividly by some graphics. Finally, the conclusions are given in the last section.

2. Riemann-Hilbert problem

We begin our discussion by considering the Lax pair representation of the $N$-component FL equations

\[
\begin{align*}
  \Phi_x &= U \Phi, & U &= \frac{1}{\xi^2} \sigma_0 + i \frac{1}{\xi} U_{1,x}, \\
  \Phi_t &= V \Phi, & V &= -\frac{1}{3} \xi^2 i \sigma_0 + \xi \sigma_1 U_1 - i \sigma_1 U_1^2,
\end{align*}
\]

(2.1)

where

\[
\begin{align*}
  \sigma_0 &= \begin{pmatrix}
    -2 & 0_{1 \times N} \\
    0_{N \times 1} & I_{N \times N}
  \end{pmatrix}, & \sigma_1 &= \begin{pmatrix}
    1 & 0_{1 \times N} \\
    0_{N \times 1} & -I_{N \times N}
  \end{pmatrix}, & U_1 &= \begin{pmatrix}
    0 & v^T \\
    u & 0_{N \times N}
  \end{pmatrix},
\end{align*}
\]

(2.2)
with \( u = (u_1, u_2, \ldots, u_N)^T \) and \( v = (v_1, v_2, \ldots, v_N)^T \). If taking \( v = A^T u^* \), Eq. (1.1) can be derived by the compatibility condition of Eq. (2.1). The Lax pair (2.1) can be rewritten as the equivalent form

\[
\begin{align*}
\Phi_+ &= \frac{1}{\zeta} \phi_0 \Phi + P \Phi, \\
\Phi_- &= -\frac{1}{3} i \zeta^2 \phi_0 \Phi + Q \Phi,
\end{align*}
\]

(2.3)

where \( P = i \zeta U_{1,1}, Q = \zeta \sigma_1 U_1 - i \sigma_1 U_1^2 \). From Eq. (2.3), when \(|x| \to \infty\), we have

\[
\Phi \propto \exp\left(\frac{i}{\zeta^2} \phi_0 x - \frac{1}{3} i \zeta^2 \sigma_0 t\right).
\]

(2.4)

Let \( J = \Phi \exp(-i \frac{i}{\zeta^2} \phi_0 x + \frac{1}{3} i \zeta^2 \sigma_0 t) \), then we obtain

\[
\begin{align*}
J_+ &= i \frac{1}{\zeta} [\phi_0, J] + PJ, \\
J_- &= -\frac{1}{3} i \zeta^2 [\phi_0, J] + QJ,
\end{align*}
\]

(2.5)

where \([\phi_0, J] = \sigma_0 J - J \sigma_0\) is the commutator. Moreover, we can get the following formula

\[
d(e^{-i (\frac{i}{\zeta^2} \phi_0 x + \frac{1}{3} i \zeta^2 \sigma_0 t)} J) = e^{-i (\frac{i}{\zeta^2} \phi_0 x + \frac{1}{3} i \zeta^2 \sigma_0 t)} (P dx + Q dt) J.
\]

(2.6)

Now, let us construct two matrix solutions of Eq. (2.5)

\[
\begin{align*}
J_- &= ([J_-]_1, [J_-]_2, \ldots, [J_-]_{N+1}), \\
J_+ &= ([J_+]_1, [J_+]_2, \ldots, [J_+]_{N+1}),
\end{align*}
\]

(2.7)

where each \([J_s]_l\) denotes the \( l \)-th column of \( J_s \). In addition, \( J_+ \) are determined by

\[
\begin{align*}
J_- &= I + \int_{-\infty}^{\infty} e^{i \frac{1}{\zeta^2} \phi_0 (s-\xi)} P(\xi) J_- (\xi, \zeta) e^{-i \frac{1}{\zeta^2} \sigma_0 (s-\xi)} d\xi, \\
J_+ &= I - \int_{-\infty}^{\infty} e^{i \frac{1}{\zeta^2} \phi_0 (s-\xi)} P(\xi) J_+ (\xi, \zeta) e^{-i \frac{1}{\zeta^2} \sigma_0 (s-\xi)} d\xi,
\end{align*}
\]

(2.8)

which satisfy the asymptotic conditions

\[
\begin{align*}
J_- \to I, \quad \zeta \to -\infty, \\
J_+ \to I, \quad \zeta \to +\infty.
\end{align*}
\]

(2.9)

It is easy to find that \([J_-]_1, [J_+]_2, [J_+]_3, \ldots, [J_+]_{N+1}\) are analytic for \( \zeta \in \mathbb{D}^+ \), and \([J_+]_1, [J_-]_2, [J_-]_3, \ldots, [J_-]_{N+1}\) are analytic for \( \zeta \in \mathbb{D}^- \), where

\[
\mathbb{D}^+ = \left\{ \zeta \mid \arg \zeta \in (0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, \frac{5\pi}{2}) \right\},
\]

\[
\mathbb{D}^- = \left\{ \zeta \mid \arg \zeta \in (\frac{\pi}{2}, \pi) \cup (\frac{3\pi}{2}, 2\pi) \right\}.
\]

(2.10)
Next, we pay attention to the properties of $J_\pm$. According to the Abel’s identity and $\text{tr}(U) = 0$, we obtain that $\det(J_\pm)$ are independent of $x$ and $\det(J_\pm) = 1$. Moreover, since $J_-E$ and $J_+E$ are the matrix solutions of the spectral problem, where $E = e^{i\frac{k}{\mu}x}$, they are linearly related by a $(N + 1) \times (N + 1)$ scattering matrix $S(\zeta) = (s_{kj})_{(N+1)\times(N+1)}$, namely

$$J_-E = J_+ES(\zeta), \quad \zeta \in \mathbb{R} \cup i\mathbb{R},$$

where

$$S(\zeta) = \begin{pmatrix}
    s_{1,1} & \ldots & s_{1,N} & s_{1,N+1} \\
    \vdots & \ddots & \vdots & \vdots \\
    s_{N,1} & \ldots & s_{N,N} & s_{N,N+1} \\
    s_{N+1,1} & \ldots & s_{N+1,N} & s_{N+1,N+1}
\end{pmatrix}$$

represents the inverse scattering matrix, $\zeta \in \mathbb{R} \cup i\mathbb{R}$. Furthermore, it is easy to know $\det(S(\zeta)) = 1$.

A Riemann–Hilbert problem to be formulated for the $N$-component FL equations need two matrix functions: one is analytic in $\mathbb{D}^+$ and the other is analytic in $\mathbb{D}^-$. Let $\Gamma_1 = \Gamma_1(x, \zeta)$ be an analytic function of $\zeta$

$$\Gamma_1(x, \zeta) = ([J_-]_1, [J_-]_2, [J_+]_3, \ldots, [J_+]_{N+1})(x, \zeta),$$

defining in $\mathbb{D}^+$, with the asymptotic behavior $\Gamma_1 \rightarrow \mathbb{I}, \zeta \in \mathbb{D}^+ \rightarrow \infty$.

To formulate a Riemann–Hilbert problem for the $N$-component FL equations, we also need to consider the inverse matrices of $J_\pm$. We write the matrices $J_\pm^{-1}$ as a collection of rows

$$J_\pm^{-1} = \begin{pmatrix}
    [J_\pm^{-1}]_1 \\
    [J_\pm^{-1}]_2 \\
    \vdots \\
    [J_\pm^{-1}]_N \\
    [J_\pm^{-1}]_{N+1}
\end{pmatrix}$$

It can be seen that $J_\pm^{-1}$ satisfy

$$K_\pm = i\frac{1}{\zeta^2}[\sigma_0, K] - KP,$$

and meet the boundary condition $J_\pm^{-1} \rightarrow \mathbb{I}, x \rightarrow \pm\infty$. The matrix function $\Gamma_2(x, \zeta)$ which is analytic in $\mathbb{D}^-$, can be defined as follows:

$$\Gamma_2(x, \zeta) = \begin{pmatrix}
    [J_\pm^{-1}]_1 \\
    [J_\pm^{-1}]_2 \\
    [J_\pm^{-1}]_3 \\
    \vdots \\
    [J_\pm^{-1}]_{N+1}
\end{pmatrix}(x, \zeta).$$
Similar to $\Gamma_1(x, \zeta)$, we obtain that $\Gamma_2(x, \zeta) \to \mathbb{I}$ as $\zeta \in \mathbb{D}_- \to \infty$. Besides, we can get the linear relationship
\begin{equation}
 e^{-i \phi_0(\zeta)} J_0^{-1} = R(\zeta) e^{-i \phi_0(\zeta)} J_0^{-1}, \quad \zeta \in \mathbb{R} \cup i\mathbb{R},
\end{equation}
where
\begin{equation}
 R(\zeta) = \begin{pmatrix}
 r_{1,1} & \ldots & r_{1,N} & r_{1,N+1} \\
 \vdots & \ddots & \vdots & \vdots \\
 r_{N,1} & \ldots & r_{N,N} & r_{N,N+1} \\
 r_{N+1,1} & \ldots & r_{N+1,N} & r_{N+1,N+1}
\end{pmatrix}
\end{equation}
represents the inverse scattering matrix, $\zeta \in \mathbb{R} \cup i\mathbb{R}$. As for the analyticity of the scattering matrix and the inverse scattering matrix, we have the following theorem.

**Theorem 2.1.** The spectral function $\Gamma_1(x, \zeta)$, the element $s_{k,j}(2 \leq k, j \leq N + 1)$ are analytic in $\mathbb{D}^+$; The spectral function $\Gamma_2(x, \zeta)$, the element $r_{1,1}$ and $s_{k,j}(2 \leq k, j \leq N + 1)$ are analytic in $\mathbb{D}^-$. In addition, $s_{1,k}(2 \leq k \leq N + 1)$ and $s_{j,1}(2 \leq j \leq N + 1)$ are not analytic in $\mathbb{D}^+$ or $\mathbb{D}^-$ but continuous to the real axis and image axis, and $r_{1,1}(2 \leq k \leq N + 1)$ and $r_{j,1}(2 \leq j \leq N + 1)$ are not analytic in $\mathbb{D}^+$ or $\mathbb{D}^-$ and not continuous to the real axis and image axis.

Based on the above analysis, the Riemann-Hilbert problem of the $N$-component FL equations can be formulated.

**Theorem 2.2.** Let’s make the convention that the limit of $\Gamma_1(x, \zeta)$ when $\zeta \in \mathbb{D}^+$ approaches $\mathbb{R} \cup i\mathbb{R}$ is $\Gamma^+$, and the limit of $\Gamma_2(x, \zeta)$ when $\zeta \in \mathbb{D}^-$ approaches $\mathbb{R} \cup i\mathbb{R}$ is $\Gamma^-$, then the Riemann-Hilbert problem can be set up as follows:

\begin{equation}
\begin{array}{c}
\Gamma^+ \text{ are analytic in } \mathbb{D}^+,
\Gamma^- \Gamma^+ = G(x, \zeta), \quad \zeta \in \mathbb{R},
\Gamma^+ \to \mathbb{I}, \quad \zeta \to \infty,
\end{array}
\end{equation}

where
\begin{equation}
 G = \begin{pmatrix}
 1 & r_{1,2} e^{-\frac{i \phi_0}{2}} & \ldots & r_{1,N} e^{-\frac{i \phi_0}{2}} & r_{1,N+1} e^{-\frac{i \phi_0}{2}} \\
 s_{2,1} e^{\frac{i \phi_0}{2}} & 1 & 0 & \ldots & 0 & 0 \\
 s_{3,1} e^{\frac{i \phi_0}{2}} & 0 & 1 & \ldots & 0 & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 s_{N,1} e^{\frac{i \phi_0}{2}} & 0 & 0 & \ldots & 1 & 0 \\
 s_{N+1,1} e^{\frac{i \phi_0}{2}} & 0 & 0 & \ldots & 0 & 1
\end{pmatrix}
\end{equation}

and the canonical normalization condition of the RH problem is
\begin{equation}
\begin{array}{c}
\Gamma_1(x, \zeta) \to \mathbb{I}, \quad \zeta \in \mathbb{D}^+ \to \infty,
\Gamma_2(x, \zeta) \to \mathbb{I}, \quad \zeta \in \mathbb{D}^- \to \infty.
\end{array}
\end{equation}
To solve the RH problem, we suppose that it is irregular, which means that \( \det(\Gamma_1) \) and \( \det(\Gamma_2) \) have certain zeros in the analytic domains, respectively. According to the definition of them, we can obtain
\[
\begin{align*}
\det(\Gamma_1(x, \zeta)) &= s_{1,1}(\zeta), \quad \zeta \in \mathbb{D}^+, \\
\det(\Gamma_2(x, \zeta)) &= r_{1,1}(\zeta), \quad \zeta \in \mathbb{D}^-.
\end{align*}
\] (2.21)

Because of the above important results, it is necessary to consider the characteristics of zeros by the symmetry of \( U(x, \zeta) \), which is helpful for classifying the soliton solutions of the \( N \)-component FL equations. At first, we can see that the matrix \( U_1 \) satisfy
\[
U_1^\dagger = -BU_1B^{-1},
\] (2.22)
where \( B = \begin{pmatrix} -1 & 0 \\ 0 & A \end{pmatrix} \), and the symbol \( \dagger \) represents the Hermitian transpose of one matrix. According to Eqs. (2.5) and (2.22), we obtain
\[
J_\pm^1(\zeta^*) = BJ_\mp^{-1}(\zeta)B^{-1},
\] (2.23)
furthermore, the scattering matrix \( S(\zeta) \) meet the condition
\[
B^{-1}S^\dagger(\zeta^*)B = S^{-1}(\zeta) = R(\zeta),
\] (2.24)
moreover,
\[
\Gamma_1^\dagger(\zeta^*) = B\Gamma_2(\zeta)B^{-1}.
\] (2.25)

Besides, the matrix \( U_1 \) meets the relation \( U_1 = -\sigma_1U_1\sigma_1 \), based on which we can conclude that
\[
J_\pm(\zeta) = \sigma_1J_\pm(\zeta)\sigma_1,
\] (2.26)
and
\[
\Gamma_1(\zeta) = \sigma_1\Gamma_1(\zeta)\sigma_1.
\] (2.27)

At this point, we suppose that \( \det(\Gamma_1) \) has \( 2N_1 \) simple zeros \( \{\zeta_j \}(1 \leq j \leq 2N_1) \) in \( \mathbb{D}^+ \) which satisfy \( \dot{\zeta}_{N+1} = -\zeta_j \), \( 1 \leq l \leq N_1 \). At the same time, \( \det(\Gamma_2) \) has \( 2N_1 \) simple zeros \( \{\hat{\zeta}_j \}(1 \leq j \leq 2N_1) \) in \( \mathbb{D}^- \), where \( \hat{\zeta}_j = \zeta_j^*, 1 \leq j \leq 2N_1 \).

In fact, the scattering data needed to solve the RH problem include the continuous scattering data \( \{s_{2,1}, s_{3,1}, \ldots, s_{N+1,1}\} \) as well as the discrete data \( \{\zeta_j, \dot{\zeta}_j, \theta_j, \dot{\theta}_j \}(1 \leq j \leq 2N_1) \), where \( \theta_j \) and \( \dot{\theta}_j \) are nonzero column vectors and row vectors, respectively, satisfying
\[
\Gamma_1(\zeta_j)\theta_j = 0, \\
\dot{\theta}_j\Gamma_2(\hat{\zeta}_j) = 0.
\] (2.28)
According to Eqs. (2.25) and (2.28), we can reveal the relation
\[ \hat{\theta}_j = \theta_j^T B, \quad 1 \leq j \leq 2N_1, \] (2.29)

Similarly, from Eqs. (2.27) and (2.28), we can obtain
\[ \theta_{N_1+j} = \sigma_1 \theta_j, \quad 1 \leq j \leq N_1. \] (2.30)

To obtain the explicit form of \( \theta_j (1 \leq j \leq 2N_1) \), we take the derivatives of the first expression of Eq. (2.28) with respect to \( x \) and \( t \) and get
\[
\begin{align*}
\frac{\partial \theta_j}{\partial x} &= i \frac{1}{\xi_j} \sigma_0 \theta_j, \\
\frac{\partial \theta_j}{\partial t} &= -i \frac{1}{3} \xi_j^2 \sigma_0 \theta_j,
\end{align*}
\] (2.31)
then the \( \theta_j \) and \( \hat{\theta}_j \) are determined by
\[
\theta_j = \begin{cases} 
\frac{i}{\xi_j} \xi_j^0 e^{i \xi_j^0 \sigma_0} \theta_j, & 1 \leq j \leq N_1, \\
\sigma_3 e^{-i \xi_j^0 \tau_0} \theta_{-N_1,0}, & N_1 + 1 \leq j \leq 2N_1,
\end{cases}
\] (2.32)
and
\[
\hat{\theta}_j = \begin{cases} 
\frac{i}{\xi_j} \xi_j^0 e^{i \xi_j^0 \sigma_0} \hat{\theta}_j, & 1 \leq j \leq N_1, \\
\sigma_3 e^{-i \xi_j^0 \tau_0} \hat{\theta}_{-N_1,0} B, & N_1 + 1 \leq j \leq 2N_1,
\end{cases}
\] (2.33)
where \( \theta_{j,0} \) are the complex constant vectors.

It is pointed out that the Riemann-Hilbert problem examined corresponds to the reflectionless case, namely, \( s_{2,1} = s_{3,1} = \cdots = s_{N+1,1} = 0 \). We introduce a \( 2N_1 \times 2N_1 \) matrix \( M \) with elements
\[ m_{k,j} = \frac{\hat{\theta}_k \theta_j}{\xi_j - \xi_k}, \quad 1 \leq k, j \leq 2N_1, \] (2.34)
and suppose that the inverse matrix \( M^{-1} \) exists, then the solutions of the RH problem can be given by
\[
\begin{align*}
\Gamma_1(x, \zeta) &= \mathbb{I} - \sum_{k=1}^{2N_1} \sum_{j=1}^{2N_1} \frac{\theta_k \hat{\theta}_j (M^{-1})_{k,j}}{\xi_j - \xi_k}, \\
\Gamma_2(x, \zeta) &= \mathbb{I} + \sum_{k=1}^{2N_1} \sum_{j=1}^{2N_1} \frac{\theta_k \hat{\theta}_j (M^{-1})_{k,j}}{\xi_j - \xi_k}.
\end{align*}
\] (2.35)

Furthermore, we take the expansion for \( \Gamma_1(x, \zeta) \)
\[ \Gamma_1(x, \zeta) = \mathbb{I} + \frac{\Gamma_1^{(1)}}{s} + \frac{\Gamma_1^{(2)}}{s^2} + \frac{\Gamma_1^{(3)}}{s^3} + O\left(\frac{1}{s^4}\right). \] (2.36)

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Comparing Eq. (2.35) and Eq. (2.36), we obtain

$$\Gamma^{(1)}_1 = - \frac{2N_{1}}{3} \sum_{k=1}^{2N_{1}} \sum_{j=1}^{2N_{1}} \theta_k \theta_j (M^{-1})_{k,j},$$

(2.37)

substituting the above expression into Eq. (2.33), the following relationship can be obtained

$$U_1 = \frac{i}{3} \sigma_1 [\sigma_0, \Gamma^{(1)}_1],$$

(2.38)

more explicitly,

$$\begin{align*}
\left\{ \begin{array}{l}
u_1(x, t) = -i(\Gamma^{(1)}_1)_{2,1}, \\
u_2(x, t) = -i(\Gamma^{(1)}_1)_{3,1}, \\
\ldots, \\
u_N(x, t) = -i(\Gamma^{(1)}_1)_{N+1,1},
\end{array} \right.
\end{align*}$$

(2.39)

where $(\Gamma^{(1)}_1)_{k,j}$ denotes the $(k, j)$-entry of matrix $\Gamma^{(1)}_1$. Besides, from Eq. (2.37), the potential functions $u_j(x, t) (1 \leq j \leq N)$ can be recovered as follows

$$\begin{align*}
\left\{ \begin{array}{l}
u_1(x, t) = \frac{i}{3} \sum_{k=1}^{2N_{1}} \sum_{j=1}^{2N_{1}} \theta_k \theta_j (M^{-1})_{k,j} \end{array} \right., \\
\nu_2(x, t) = \frac{i}{3} \sum_{k=1}^{2N_{1}} \sum_{j=1}^{2N_{1}} \theta_k \theta_j (M^{-1})_{k,j} \end{align*}$$

(2.40)

where $(M^{-1})_{k,j}$ denotes the $(k, j)$-entry of inverse matrix of $M$.

3. Multi-soliton solutions

To obtain the explicit expression of the multi-soliton solutions, we should make more efforts. At first, we set $\theta_{j0} = (\mu_{j1}, \mu_{j2}, \ldots, \mu_{jN+1})^T$, $\theta_j = i \left( x - \frac{\zeta_j}{\epsilon} t \right)$ and $\zeta_j = \alpha_j + i \beta_j$. When $1 \leq j \leq N_1,$

$$\theta_j = e^{\theta_j \sigma_1} \theta_{j0} = \begin{pmatrix} e^{-2\beta_j} & 0 & 0 & \cdots & 0 \\
0 & e^{\beta_j} & 0 & \cdots & 0 \\
0 & 0 & e^{\beta_j} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & e^{\beta_j} \end{pmatrix} \begin{pmatrix} \mu_{j1} \\
\mu_{j2} \\
\vdots \\
\mu_{jN} \\
\mu_{jN+1} \end{pmatrix} = \begin{pmatrix} \mu_{j1} e^{-2\beta_j} \\
\mu_{j2} e^{\beta_j} \\
\vdots \\
\mu_{jN} e^{\beta_j} \\
\mu_{jN+1} e^{\beta_j} \end{pmatrix},$$

(3.1)
According to Eqs. (3.1), (3.2) and (2.33), we can obtain the explicit expressions of $\vartheta$ when

$$
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & -1 & 0 & \cdots & 0 \\
0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -1
\end{pmatrix}
\begin{pmatrix}
\mu_{j-N_1,1}e^{-2\theta_{j-N_1}} \\
\mu_{j-N_1,2}e^{\theta_{j-N_1}} \\
\vdots \\
\mu_{j-N_1,N}e^{\theta_{j-N_1}} \\
\mu_{j-N_1,N+1}e^{\theta_{j-N_1}}
\end{pmatrix}
= 
\begin{pmatrix}
\mu_{j-N_1,1}e^{-2\theta_{j-N_1}} \\
-\mu_{j-N_1,2}e^{\theta_{j-N_1}} \\
\vdots \\
-\mu_{j-N_1,N}e^{\theta_{j-N_1}} \\
-\mu_{j-N_1,N+1}e^{\theta_{j-N_1}}
\end{pmatrix}.
$$

(3.2)

According to Eqs. (3.1), (3.2) and (2.33), we can obtain the explicit expressions of $\vartheta_j$ and $\dot{\vartheta}_j$ ($1 \leq j \leq 2N_1$) needed to solve the RH problem. Inserting these data into Eq. (2.40), we have the explicit expressions of multi-soliton solutions.

3.1. Case 1: multi-soliton solutions of two-component FL equations

To observe the propagation behavior of the solutions, we take $N = 2$, $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, i.e. $v_1 = u_1^*$ and $v_2 = -u_2^*$, Eq. (1.1) is reduced into the following form

$$
u_1 = 3u_1 - i(2|u_1|^2u_{1,x} - u_2^*u_1u_{2,x} - |u_2|^2u_{1,x}),$$

$$
u_2 = 3u_2 - i(-2|u_2|^2u_{2,x} - u_1^*u_2u_{1,x} - |u_1|^2u_{2,x}),$$

(3.3)

when $N_1 = 1$, we can express the solution to Eq. (2.40) explicitly

$$
\begin{cases}
\nu_1(x,t) = i(-\mu_{1,1}\mu_1^*e^{\theta_1}(M^{-1})_{1,1} - \mu_{1,2}\mu_1^*e^{\theta_1}(M^{-1})_{1,2} \\
\quad + \mu_{1,2}\mu_1^*e^{\theta_1}(M^{-1})_{2,1} + \mu_{1,1}\mu_1^*e^{\theta_1}(M^{-1})_{2,2}), \\
\nu_2(x,t) = i(-\mu_{1,1}\mu_1^*e^{\theta_1}(M^{-1})_{1,1} - \mu_{1,2}\mu_1^*e^{\theta_1}(M^{-1})_{1,2} \\
\quad + \mu_{1,2}\mu_1^*e^{\theta_1}(M^{-1})_{2,1} + \mu_{1,1}\mu_1^*e^{\theta_1}(M^{-1})_{2,2}),
\end{cases}
$$

(3.4)

where

$$
\begin{align}
m_{1,1} &= -|\mu_{1,1}|^2e^{-2\theta_1} + |\mu_{1,2}|^2e^{\theta_1} - |\mu_{1,3}|^2e^{\theta_1}, \\
m_{1,2} &= -|\mu_{1,1}|^2e^{-2\theta_1} - |\mu_{1,2}|^2e^{\theta_1} + |\mu_{1,3}|^2e^{\theta_1}, \\
m_{2,1} &= -|\mu_{1,1}|^2e^{-2\theta_1} - |\mu_{1,2}|^2e^{\theta_1} + |\mu_{1,3}|^2e^{\theta_1}, \\
m_{2,2} &= -|\mu_{1,1}|^2e^{-2\theta_1} + |\mu_{1,2}|^2e^{\theta_1} - |\mu_{1,3}|^2e^{\theta_1}.
\end{align}
$$

(3.5)
Figure 1. One-hump solutions to Eq. (3.4) with parameters $\zeta_1 = 1.2 + 0.5i$, $\mu_{1,1} = 0.97$, $\mu_{1,2} = 1.02$, and $\mu_{1,3} = 1.21$. (a)(b)(c): the local structure, density and wave propagation of the one-hump solution $|u_1(x,t)|$. (d)(e)(f): the local structure, density and wave propagation of the one-hump solution $|u_2(x,t)|$.

Figure 2. One-soliton solutions to Eq. (3.4) with parameters $\zeta_1 = 1.2 + 0.5i$, $\mu_{1,1} = 0.97 + 0.5i$, $\mu_{1,2} = 1.02 + 1.2i$, and $\mu_{1,3} = 1.21 + 0.5i$. (a)(b)(c): the local structure, density and wave propagation of the one-soliton solution $|u_1(x,t)|$. (d)(e)(f): the local structure, density and wave propagation of the one-soliton solution $|u_2(x,t)|$. 
The above results for the two-component FL equations is consistent with the results in [40]. Moreover, the localized structures and dynamic behavior of one-hump solutions and one-soliton solutions are shown in Fig. 1 and Fig. 2, respectively. From the following two figures, we can see that the single solitons all travel in an exchanging direction, but the amplitude of the single-hump solution in Fig. 1 is larger than the single soliton in Fig. 2, and the single soliton in Fig. 2 is much wider than the single hump in Fig. 1.

Next, if we take \( N_1 = 2 \), the solutions to Eq. (2,40) can be expressed explicitly by

\[
\begin{align*}
\{ &u_1(x, t) = i(\mu_1 \mu_1 e^{\theta_{1,1}} (M^{-1})_{1,1} - \mu_1 \mu_2 e^{\theta_{1,2}} (M^{-1})_{1,2} \\
&- \mu_2 \mu_1 e^{\theta_{2,1}} (M^{-1})_{2,1} - \mu_2 \mu_2 e^{\theta_{2,2}} (M^{-1})_{2,2} \\
&- \mu_2 \mu_2 e^{\theta_{2,3}} (M^{-1})_{2,3} - \mu_2 \mu_2 e^{\theta_{2,4}} (M^{-1})_{2,4} \\
&+ \mu_1 \mu_1 e^{\theta_{1,3}} (M^{-1})_{3,1} + \mu_1 \mu_2 e^{\theta_{1,3}} (M^{-1})_{3,2} \\
&+ \mu_1 \mu_2 e^{\theta_{1,4}} (M^{-1})_{3,3} + \mu_1 \mu_2 e^{\theta_{1,4}} (M^{-1})_{3,4} \\
&+ \mu_2 \mu_2 e^{\theta_{2,5}} (M^{-1})_{4,4} + \mu_2 \mu_2 e^{\theta_{2,5}} (M^{-1})_{4,4}, \tag{3.6} \}
\end{align*}
\]

\[
\begin{align*}
\{ &u_2(x, t) = i(\mu_1 \mu_1 e^{\theta_{1,1}} (M^{-1})_{1,1} - \mu_1 \mu_2 e^{\theta_{1,2}} (M^{-1})_{1,2} \\
&- \mu_2 \mu_1 e^{\theta_{2,1}} (M^{-1})_{2,1} - \mu_2 \mu_2 e^{\theta_{2,2}} (M^{-1})_{2,2} \\
&- \mu_2 \mu_2 e^{\theta_{2,3}} (M^{-1})_{2,3} - \mu_2 \mu_2 e^{\theta_{2,4}} (M^{-1})_{2,4} \\
&+ \mu_1 \mu_1 e^{\theta_{1,3}} (M^{-1})_{3,1} + \mu_1 \mu_2 e^{\theta_{1,3}} (M^{-1})_{3,2} \\
&+ \mu_1 \mu_2 e^{\theta_{1,4}} (M^{-1})_{3,3} + \mu_1 \mu_2 e^{\theta_{1,4}} (M^{-1})_{3,4} \\
&+ \mu_2 \mu_2 e^{\theta_{2,5}} (M^{-1})_{4,4} + \mu_2 \mu_2 e^{\theta_{2,5}} (M^{-1})_{4,4}, \tag{3.6} \}
\end{align*}
\]

where

\[
m_{k,j} = \frac{\hat{\theta}_k \theta_j}{\xi_j - \xi_k}, \quad 1 \leq k, j \leq 4, \tag{3.7}
\]

with

\[
\xi_3 = -\xi_1, \quad \xi_4 = -\xi_2, \quad \xi_{j'} = \xi_j, \quad 1 \leq j \leq 4. \tag{3.8}
\]
Figure 3. Two-soliton solutions to Eq. (3.6) with parameters $\zeta_1 = 1.2 + 0.5i$, $\zeta_2 = 0.9 + 1.3i$, $\mu_{1,1} = 0.7$, $\mu_{1,2} = 1.5$, $\mu_{1,3} = 0.9$ $\mu_{2,1} = 0.8$, $\mu_{2,2} = 1.2$, and $\mu_{2,3} = 1.4$. (a)(b)(c): the local structure, density and wave propagation of the two-soliton solution $|u_1(x,t)|$, (d)(e)(f): the local structure, density and wave propagation of the two-soliton solution $|u_2(x,t)|$.

The localized structures and dynamic propagation behavior of the two soliton solutions are displayed in Fig. 3. From Fig. 3, it can be seen that before two solitons collide each other, they spread forward in directions that cross each other. After they collide each other, the directions of two solitons are not exchanged, but the positions of them has been shifted and the energy of them has been swapped.
When $N_1 = 3$, we can rewrite Eq. (2.40) as the following form

\[
\mu_1(x, t) = i(-\mu_{1,1}^* e^{i \theta_1 - 2\theta_1} (M^{-1})_{1,1} - \mu_{1,2}^* e^{i \theta_2 - 2\theta_2} (M^{-1})_{1,2} - \mu_{1,3}^* e^{i \theta_3 - 2\theta_3} (M^{-1})_{1,3} - \mu_{1,4}^* e^{i \theta_4 - 2\theta_4} (M^{-1})_{1,4} - \mu_{1,5}^* e^{i \theta_5 - 2\theta_5} (M^{-1})_{1,5} - \mu_{1,6}^* e^{i \theta_6 - 2\theta_6} (M^{-1})_{1,6} - \mu_{1,7}^* e^{i \theta_7 - 2\theta_7} (M^{-1})_{1,7} - \mu_{1,8}^* e^{i \theta_8 - 2\theta_8} (M^{-1})_{1,8} - \mu_{1,9}^* e^{i \theta_9 - 2\theta_9} (M^{-1})_{1,9})
\]

\[
\mu_2(x, t) = i(-\mu_{2,1}^* e^{i \theta_1 - 2\theta_1} (M^{-1})_{2,1} - \mu_{2,2}^* e^{i \theta_2 - 2\theta_2} (M^{-1})_{2,2} - \mu_{2,3}^* e^{i \theta_3 - 2\theta_3} (M^{-1})_{2,3} - \mu_{2,4}^* e^{i \theta_4 - 2\theta_4} (M^{-1})_{2,4} - \mu_{2,5}^* e^{i \theta_5 - 2\theta_5} (M^{-1})_{2,5} - \mu_{2,6}^* e^{i \theta_6 - 2\theta_6} (M^{-1})_{2,6} - \mu_{2,7}^* e^{i \theta_7 - 2\theta_7} (M^{-1})_{2,7} - \mu_{2,8}^* e^{i \theta_8 - 2\theta_8} (M^{-1})_{2,8} - \mu_{2,9}^* e^{i \theta_9 - 2\theta_9} (M^{-1})_{2,9})
\]

where

\[
m_{k,j} = \frac{\hat{\theta}_k \hat{\theta}_j}{\zeta_j - \zeta_k}, \quad 1 \leq k, j \leq 6, \quad (3.10)
\]

with

\[
\zeta_1 = -\zeta_1, \quad \zeta_5 = -\zeta_5, \quad \zeta_6 = -\zeta_3, \quad \zeta_j = \zeta_j^*, \quad 1 \leq j \leq 6. \quad (3.11)
\]
3.2. Case 2: multi-soliton solutions of three-component FL equations

If we take $N=3$ and $v = A^T u^*$, where $A = \begin{pmatrix} a_{11} & a_{21}^* & a_{31}^* \\ a_{21} & a_{22}^* & a_{32}^* \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$, when $N_1 = 1$, we can express the solution to Eq. (2.40) explicitly

$$
\begin{align*}
 u_1(x, t) &= \ii \left( -\mu_{1,2} \mu_{1,1} e^{\theta_{1,2} - 2\theta_{1}} (M^{-1})_{1,1} - \mu_{1,2} \mu_{1,1} e^{\theta_{1,2} - 2\theta_{1}} (M^{-1})_{1,2} \\
 &\quad + \mu_{1,2} \mu_{1,1} e^{\theta_{1,2} - 2\theta_{1}} (M^{-1})_{2,1} + \mu_{1,2} \mu_{1,1} e^{\theta_{1,2} - 2\theta_{1}} (M^{-1})_{2,2} \right), \\
 u_2(x, t) &= \ii \left( -\mu_{1,3} \mu_{1,1} e^{\theta_{1,3} - 2\theta_{1}} (M^{-1})_{1,1} + \mu_{1,3} \mu_{1,1} e^{\theta_{1,3} - 2\theta_{1}} (M^{-1})_{1,2} \\
 &\quad + \mu_{1,3} \mu_{1,1} e^{\theta_{1,3} - 2\theta_{1}} (M^{-1})_{2,1} + \mu_{1,3} \mu_{1,1} e^{\theta_{1,3} - 2\theta_{1}} (M^{-1})_{2,2} \right), \\
 u_3(x, t) &= \ii \left( -\mu_{1,4} \mu_{1,1} e^{\theta_{1,4} - 2\theta_{1}} (M^{-1})_{1,1} - \mu_{1,4} \mu_{1,1} e^{\theta_{1,4} - 2\theta_{1}} (M^{-1})_{1,2} \\
 &\quad + \mu_{1,4} \mu_{1,1} e^{\theta_{1,4} - 2\theta_{1}} (M^{-1})_{2,1} + \mu_{1,4} \mu_{1,1} e^{\theta_{1,4} - 2\theta_{1}} (M^{-1})_{2,2} \right),
\end{align*}
$$

(3.12)

where

$$
\begin{align*}
 m_{11} &= -|\mu_{1,1}|^2 e^{-2\theta_{1} - 2\theta_{2} + \ii \theta_{1,2}} (\mu_{1,2})^2 + 2 \mu_{1,2} \mu_{1,3} + 3 \mu_{1,3} \mu_{1,4} + 2 \mu_{1,4} \mu_{1,1} + |\mu_{1,2}|^2 + 3 |\mu_{1,1}|^2 + |\mu_{1,3}|^2 + |\mu_{1,4}|^2, \\
 m_{12} &= -|\mu_{1,1}|^2 e^{-2\theta_{1} - 2\theta_{2}} - |\mu_{1,1}|^2 (\mu_{1,2})^2 + 2 \mu_{1,2} \mu_{1,3} + 3 \mu_{1,3} \mu_{1,4} + 2 \mu_{1,4} \mu_{1,1} + |\mu_{1,2}|^2 + 3 |\mu_{1,1}|^2 + |\mu_{1,3}|^2 + |\mu_{1,4}|^2, \\
 m_{13} &= -|\mu_{1,1}|^2 e^{-2\theta_{1} - 2\theta_{2}} - |\mu_{1,1}|^2 (\mu_{1,2})^2 + 2 \mu_{1,2} \mu_{1,3} + 3 \mu_{1,3} \mu_{1,4} + 2 \mu_{1,4} \mu_{1,1} + |\mu_{1,2}|^2 + 3 |\mu_{1,1}|^2 + |\mu_{1,3}|^2 + |\mu_{1,4}|^2, \\
 m_{14} &= -|\mu_{1,1}|^2 e^{-2\theta_{1} - 2\theta_{2}} - |\mu_{1,1}|^2 (\mu_{1,2})^2 + 2 \mu_{1,2} \mu_{1,3} + 3 \mu_{1,3} \mu_{1,4} + 2 \mu_{1,4} \mu_{1,1} + |\mu_{1,2}|^2 + 3 |\mu_{1,1}|^2 + |\mu_{1,3}|^2 + |\mu_{1,4}|^2, \\
 m_{21} &= -|\mu_{1,1}|^2 e^{-2\theta_{1} - 2\theta_{2}} - |\mu_{1,1}|^2 (\mu_{1,2})^2 + 2 \mu_{1,2} \mu_{1,3} + 3 \mu_{1,3} \mu_{1,4} + 2 \mu_{1,4} \mu_{1,1} + |\mu_{1,2}|^2 + 3 |\mu_{1,1}|^2 + |\mu_{1,3}|^2 + |\mu_{1,4}|^2, \\
 m_{22} &= -|\mu_{1,1}|^2 e^{-2\theta_{1} - 2\theta_{2}} - |\mu_{1,1}|^2 (\mu_{1,2})^2 + 2 \mu_{1,2} \mu_{1,3} + 3 \mu_{1,3} \mu_{1,4} + 2 \mu_{1,4} \mu_{1,1} + |\mu_{1,2}|^2 + 3 |\mu_{1,1}|^2 + |\mu_{1,3}|^2 + |\mu_{1,4}|^2.
\end{align*}
$$

(3.13)
Figure 5. One-hump solutions to Eq. (3.12) with parameters $\zeta_1 = 1.5 + 0.5i$, $\zeta_2 = -\zeta_1$, $\mu_{1,1} = 1.1 + 0i$, $\mu_{1,2} = 0.9 + 1.5i$, $\mu_{1,3} = 0.2 + 0.5i$ and $\mu_{1,4} = 1.2 + 1.6i$. (a)(d)(g): the structures of $|u_1(x,t)|$, Re($u_1$) and Im($u_1$), (b)(e)(h): the structures of $|u_2(x,t)|$, Re($u_2$) and Im($u_2$), (c)(f)(i): the structures of $|u_3(x,t)|$, Re($u_3$) and Im($u_3$).

When $N_1 = 2$, we can express the solutions to Eq. (2.40) explicitly

$$
\begin{align*}
\left\{ \begin{array}{l}
\quad u_1(x,t) = & i(\mu_{1,2}\mu_{1,1}^{*}e^{\frac{i}{2}(M^{-1})_{1,1}} + \mu_{1,2}\mu_{2,1}^{*}e^{\frac{i}{2}(M^{-1})_{1,2}} - \mu_{1,2}\mu_{1,1}^{*}e^{\frac{i}{2}(M^{-1})_{1,3}} - \mu_{1,2}\mu_{2,1}^{*}e^{\frac{i}{2}(M^{-1})_{1,4}} - \mu_{2,2}\mu_{1,1}^{*}e^{\frac{i}{2}(M^{-1})_{2,1}} - \mu_{2,2}\mu_{2,1}^{*}e^{\frac{i}{2}(M^{-1})_{2,2}} - \mu_{2,2}\mu_{1,1}^{*}e^{\frac{i}{2}(M^{-1})_{2,3}} - \mu_{2,2}\mu_{2,1}^{*}e^{\frac{i}{2}(M^{-1})_{2,4}} + \mu_{1,2}\mu_{1,1}^{*}e^{\frac{i}{2}(M^{-1})_{3,1}} + \mu_{1,2}\mu_{2,1}^{*}e^{\frac{i}{2}(M^{-1})_{3,2}} + \mu_{1,2}\mu_{1,1}^{*}e^{\frac{i}{2}(M^{-1})_{3,3}} + \mu_{1,2}\mu_{2,1}^{*}e^{\frac{i}{2}(M^{-1})_{3,4}} + \mu_{2,2}\mu_{1,1}^{*}e^{\frac{i}{2}(M^{-1})_{4,1}} + \mu_{2,2}\mu_{2,1}^{*}e^{\frac{i}{2}(M^{-1})_{4,2}} + \mu_{2,2}\mu_{1,1}^{*}e^{\frac{i}{2}(M^{-1})_{4,3}} + \mu_{2,2}\mu_{2,1}^{*}e^{\frac{i}{2}(M^{-1})_{4,4}})
\end{array} \right.
\end{align*}
\tag{3.14}$$
\[
\begin{align*}
\mathbf{u}(x,t) &= \mathbf{u}_2(x,t) + \mathbf{u}_3(x,t), \\
\mathbf{u}_2(x,t) &= \mathbf{u}^{\theta_1} - \mathbf{u}^{\theta_2} + \mathbf{u}^{\theta_3} + \mathbf{u}^{\theta_4}, \\
\mathbf{u}_3(x,t) &= \mathbf{u}^{\theta_5} - \mathbf{u}^{\theta_6} + \mathbf{u}^{\theta_7} + \mathbf{u}^{\theta_8},
\end{align*}
\]

where

\[
m_{k,j} = \frac{\hat{\theta}_k \theta_j}{\eta_j - \xi_k}, \quad 1 \leq k, j \leq 4, (3.15)
\]

with

\[
\xi_3 = -\xi_1, \quad \xi_4 = -\xi_2, \quad \xi_j = \xi_j, \quad 1 \leq j \leq 4. (3.16)
\]
Figure 6. Two-soliton solutions to Eq. (3.14) with parameters \( \zeta_1 = 1 + 0.9i, \zeta_2 = 0.9 + 1.3i, \mu_{1,1} = 1 + i, \mu_{1,2} = 2 + i, \mu_{1,3} = 3 + i, \mu_{1,4} = 1 + 2i, \mu_{2,1} = 1 - i, \mu_{2,2} = 2 - i, \mu_{2,3} = 3 - i \) and \( \mu_{2,4} = 1 + 2i \). (a)(b)(c): the local structure, density and wave propagation of \( |u_1(x, t)| \), (d)(e)(f): the local structure, density and wave propagation of \( \Re(u_1) \), (g)(h)(i): the local structure, density and wave propagation of \( \Im(u_1) \).

The localized structures, density plot and the dynamic propagation behavior of the two soliton solutions are presented in Fig. 6. From Fig. 6, we can learn the propagation process and interaction mechanism of the two solitons. To be more concrete, the energy of two solitons changes significantly before and after collision, and the direction and the position of them has also changed to some extent.
Next, if we take \( N_1 = 3 \), the solutions to Eq. (2.40) can be expressed explicitly by

\[
u(x, t) = i(-\mu_{1, v+1} \mu_{1, 1} e^{\theta_1 - \theta_2} (M^{-1})_{1, 1, 1} - \mu_{1, v+1} \mu_{2, 1} e^{\theta_1 - \theta_2} (M^{-1})_{1, 2} - \mu_{1, v+1} \mu_{3, 1} e^{\theta_1 - \theta_2} (M^{-1})_{1, 3} - \mu_{2, v+1} \mu_{1, 1} e^{\theta_1 - \theta_2} (M^{-1})_{2, 1} - \mu_{2, v+1} \mu_{2, 1} e^{\theta_1 - \theta_2} (M^{-1})_{2, 2} - \mu_{2, v+1} \mu_{3, 1} e^{\theta_1 - \theta_2} (M^{-1})_{2, 3} - \mu_{2, v+1} \mu_{1, 1} e^{\theta_1 - \theta_2} (M^{-1})_{3, 1} - \mu_{2, v+1} \mu_{2, 1} e^{\theta_1 - \theta_2} (M^{-1})_{3, 2} - \mu_{2, v+1} \mu_{3, 1} e^{\theta_1 - \theta_2} (M^{-1})_{3, 3} - \mu_{3, v+1} \mu_{1, 1} e^{\theta_1 - \theta_2} (M^{-1})_{3, 4} - \mu_{3, v+1} \mu_{2, 1} e^{\theta_1 - \theta_2} (M^{-1})_{3, 5} - \mu_{3, v+1} \mu_{3, 1} e^{\theta_1 - \theta_2} (M^{-1})_{3, 6} + \mu_{1, v+1} \mu_{1, 1} e^{\theta_1 - \theta_2} (M^{-1})_{4, 1} + \mu_{1, v+1} \mu_{2, 1} e^{\theta_1 - \theta_2} (M^{-1})_{4, 2} + \mu_{1, v+1} \mu_{3, 1} e^{\theta_1 - \theta_2} (M^{-1})_{4, 3} + \mu_{2, v+1} \mu_{1, 1} e^{\theta_1 - \theta_2} (M^{-1})_{4, 4} + \mu_{2, v+1} \mu_{2, 1} e^{\theta_1 - \theta_2} (M^{-1})_{4, 5} + \mu_{2, v+1} \mu_{3, 1} e^{\theta_1 - \theta_2} (M^{-1})_{4, 6} + \mu_{3, v+1} \mu_{1, 1} e^{\theta_1 - \theta_2} (M^{-1})_{5, 1} + \mu_{3, v+1} \mu_{2, 1} e^{\theta_1 - \theta_2} (M^{-1})_{5, 2} + \mu_{3, v+1} \mu_{3, 1} e^{\theta_1 - \theta_2} (M^{-1})_{5, 3} + \mu_{2, v+1} \mu_{1, 1} e^{\theta_1 - \theta_2} (M^{-1})_{5, 4} + \mu_{2, v+1} \mu_{2, 1} e^{\theta_1 - \theta_2} (M^{-1})_{5, 5} + \mu_{2, v+1} \mu_{3, 1} e^{\theta_1 - \theta_2} (M^{-1})_{5, 6} + \mu_{3, v+1} \mu_{1, 1} e^{\theta_1 - \theta_2} (M^{-1})_{6, 1} + \mu_{3, v+1} \mu_{2, 1} e^{\theta_1 - \theta_2} (M^{-1})_{6, 2} + \mu_{3, v+1} \mu_{3, 1} e^{\theta_1 - \theta_2} (M^{-1})_{6, 3} + \mu_{3, v+1} \mu_{1, 1} e^{\theta_1 - \theta_2} (M^{-1})_{6, 4} + \mu_{3, v+1} \mu_{2, 1} e^{\theta_1 - \theta_2} (M^{-1})_{6, 5} + \mu_{3, v+1} \mu_{3, 1} e^{\theta_1 - \theta_2} (M^{-1})_{6, 6})
\]

where

\[
\nu = 1, 2, 3, \quad m_{k,j} = \frac{\partial_{k} x_{j}}{\xi_j - \xi_k}, \quad 1 \leq k, j \leq 6,
\]

with

\[
\xi_1 = -\xi_2, \quad \xi_3 = -\xi_2, \quad \xi_6 = -\xi_3, \quad \xi_j = \xi_j^*, \quad 1 \leq j \leq 6.
\]

Figure 7. Three-soliton solution to Eq. (3.27) with parameters \( \xi_1 = 1.2 + 0.8i, \xi_2 = 1.5 + 0.7i, \xi_3 = 1.1 + 0.9i, \mu_{1, 1} = 0.1, \mu_{1, 2} = \mu_{2, 1} = \mu_{3, 1} = 0.2, \mu_{1, 3} = \mu_{3, 3} = 0.3, \mu_{1, 4} = \mu_{2, 2} = \mu_{3, 2} = 0.4, \mu_{1, 5} = \mu_{2, 5} = \mu_{3, 5} = 0.5, \mu_{2, 3} = \mu_{3, 4} = 0.6, \mu_{2, 4} = 0.8, \mu_{3, 3} = 0.9 \) and \( \mu_{3, 4} = 1.2 \). (a): the local structures of the three soliton solutions \( u_1(x, t) \), (b): the density plot of \( u_1(x, t) \), (c): the wave propagation of the three soliton solutions \( u_1(x, t) \).
3.3. Case 3: multi-soliton solutions of four-component FL equations

If we take $N = 4$ and $v = A^T u^*$, where

$$
A = \begin{pmatrix}
    a_{11} & a_{21}^* & a_{31}^* & a_{41}^* \\
    a_{21} & a_{22}^* & a_{32}^* & a_{42}^* \\
    a_{31} & a_{32} & a_{33}^* & a_{43}^* \\
    a_{41} & a_{42} & a_{43} & a_{44}
\end{pmatrix} = \begin{pmatrix}
    1 & 1 - i & 1 - 2i & 1 - 3i \\
    1 + i & 2 & 2 - i & 2 - 2i \\
    1 + 2i & 2 + i & 3 & 3 - i \\
    1 + 3i & 2 + 2i & 3 + i & 4
\end{pmatrix}, \quad (3.21)
$$

when $N_1 = 1$, we can express the solution to Eq. 3.20 explicitly

$$
\begin{align*}
    u_1(x, t) &= i(-\mu_{1,2}\mu_{1,1}^* e^{\theta_{1,2} - 2\theta_{1,1}}(M^{-1})_{1,1} - \mu_{1,2}\mu_{1,1}^* e^{\theta_{1,2} - 2\theta_{1,1}}(M^{-1})_{1,2} \\
                 &\quad + \mu_{1,2}\mu_{1,1}^* e^{\theta_{1,2} - 2\theta_{1,1}}(M^{-1})_{2,1} + \mu_{1,2}\mu_{1,1}^* e^{\theta_{1,2} - 2\theta_{1,1}}(M^{-1})_{2,2}), \\
    u_2(x, t) &= i(-\mu_{1,3}\mu_{1,1}^* e^{\theta_{1,3} - 2\theta_{1,1}}(M^{-1})_{1,1} - \mu_{1,3}\mu_{1,1}^* e^{\theta_{1,3} - 2\theta_{1,1}}(M^{-1})_{1,2} \\
                 &\quad + \mu_{1,3}\mu_{1,1}^* e^{\theta_{1,3} - 2\theta_{1,1}}(M^{-1})_{2,1} + \mu_{1,3}\mu_{1,1}^* e^{\theta_{1,3} - 2\theta_{1,1}}(M^{-1})_{2,2}), \\
    u_3(x, t) &= i(-\mu_{1,4}\mu_{1,1}^* e^{\theta_{1,4} - 2\theta_{1,1}}(M^{-1})_{1,1} - \mu_{1,4}\mu_{1,1}^* e^{\theta_{1,4} - 2\theta_{1,1}}(M^{-1})_{1,2} \\
                 &\quad + \mu_{1,4}\mu_{1,1}^* e^{\theta_{1,4} - 2\theta_{1,1}}(M^{-1})_{2,1} + \mu_{1,4}\mu_{1,1}^* e^{\theta_{1,4} - 2\theta_{1,1}}(M^{-1})_{2,2}), \\
    u_4(x, t) &= i(-\mu_{1,5}\mu_{1,1}^* e^{\theta_{1,5} - 2\theta_{1,1}}(M^{-1})_{1,1} - \mu_{1,5}\mu_{1,1}^* e^{\theta_{1,5} - 2\theta_{1,1}}(M^{-1})_{1,2} \\
                 &\quad + \mu_{1,5}\mu_{1,1}^* e^{\theta_{1,5} - 2\theta_{1,1}}(M^{-1})_{2,1} + \mu_{1,5}\mu_{1,1}^* e^{\theta_{1,5} - 2\theta_{1,1}}(M^{-1})_{2,2}).
\end{align*} \quad (3.22)
$$
Figure 8. One-hump solutions to Eq. (3.22) with parameters $\zeta_1 = 1.5 + 0.5i$, $\zeta_2 = -\zeta_1$, $\mu_{1,1} = 1.1 + 0.7i$, $\mu_{1,2} = 0.9 + 1.5i$, $\mu_{1,3} = 0.2 + 0.5i$, $\mu_{1,4} = 1.2 + 1.6i$, and $\mu_{1,5} = 0.7 + 0.8i$. (a)(b)(c): the local structure, density and wave propagation of $|u_1(x,t)|$. (d)(e)(f): the local structure, density and wave propagation of Re($u_1$). (g)(h)(i): the local structure, density and wave propagation of Im($u_1$).

Figure 9. Two-soliton solutions to Eq. (3.24) with parameters $\zeta_1 = 1.2 + 0.5i$, $\zeta_2 = 0.9 + 1.3i$, $\mu_{1,1} = 0.7$, $\mu_{1,2} = 1.5$, $\mu_{1,3} = 0.9$, $\mu_{1,4} = 1.2$, $\mu_{1,5} = 0.8$, $\mu_{2,2} = 1.2$, $\mu_{2,3} = 1.4$, $\mu_{2,4} = 0.9$ and $\mu_{2,5} = 0.5$. (a)(b)(c): the local structure, density and wave propagation of $|u_1(x,t)|$. (d)(e)(f): the local structure, density and wave propagation of Re($u_1$). (g)(h)(i): the local structure, density and wave propagation of Im($u_1$).
where

\[
\begin{align*}
m_{1,1} &= -\mu_{1,1}^2 e^{-2\theta_1} - (a_{11}\mu_{1,2} e^{\theta_1} + a_{21}\mu_{1,3} e^{\theta_1} + a_{31}\mu_{1,4} e^{\theta_1} + a_{41}\mu_{1,5} e^{\theta_1})\mu_{1,2} e^{\theta_1} \\
&
\quad \left(\frac{\xi_1 - \xi_1}{\xi_1 - \xi_1} + \frac{\xi_1 - \xi_1}{\xi_1 - \xi_1}\right) + \left(\frac{\xi_1 - \xi_1}{\xi_1 - \xi_1} + \frac{\xi_1 - \xi_1}{\xi_1 - \xi_1}\right) + \left(\frac{\xi_1 - \xi_1}{\xi_1 - \xi_1} + \frac{\xi_1 - \xi_1}{\xi_1 - \xi_1}\right),
\end{align*}
\]

\[
\begin{align*}
m_{1,2} &= -\mu_{1,1}^2 e^{-2\theta_1} - (a_{11}\mu_{1,2} e^{\theta_1} + a_{21}\mu_{1,3} e^{\theta_1} + a_{31}\mu_{1,4} e^{\theta_1} + a_{41}\mu_{1,5} e^{\theta_1})\mu_{1,2} e^{\theta_1} \\
&
\quad \left(\frac{\xi_1 - \xi_1}{\xi_1 - \xi_1} + \frac{\xi_1 - \xi_1}{\xi_1 - \xi_1}\right) + \left(\frac{\xi_1 - \xi_1}{\xi_1 - \xi_1} + \frac{\xi_1 - \xi_1}{\xi_1 - \xi_1}\right) + \left(\frac{\xi_1 - \xi_1}{\xi_1 - \xi_1} + \frac{\xi_1 - \xi_1}{\xi_1 - \xi_1}\right),
\end{align*}
\]

\[
\begin{align*}
m_{2,1} &= -\mu_{1,1}^2 e^{-2\theta_1} - (a_{11}\mu_{1,2} e^{\theta_1} + a_{21}\mu_{1,3} e^{\theta_1} + a_{31}\mu_{1,4} e^{\theta_1} + a_{41}\mu_{1,5} e^{\theta_1})\mu_{1,2} e^{\theta_1} \\
&
\quad \left(\frac{\xi_1 + \xi_1}{\xi_1 + \xi_1} + \frac{\xi_1 + \xi_1}{\xi_1 + \xi_1}\right) + \left(\frac{\xi_1 + \xi_1}{\xi_1 + \xi_1} + \frac{\xi_1 + \xi_1}{\xi_1 + \xi_1}\right) + \left(\frac{\xi_1 + \xi_1}{\xi_1 + \xi_1} + \frac{\xi_1 + \xi_1}{\xi_1 + \xi_1}\right),
\end{align*}
\]

\[
\begin{align*}
m_{2,2} &= -\mu_{1,1}^2 e^{-2\theta_1} - (a_{11}\mu_{1,2} e^{\theta_1} + a_{21}\mu_{1,3} e^{\theta_1} + a_{31}\mu_{1,4} e^{\theta_1} + a_{41}\mu_{1,5} e^{\theta_1})\mu_{1,2} e^{\theta_1} \\
&
\quad \left(\frac{\xi_1 + \xi_1}{\xi_1 + \xi_1} + \frac{\xi_1 + \xi_1}{\xi_1 + \xi_1}\right) + \left(\frac{\xi_1 + \xi_1}{\xi_1 + \xi_1} + \frac{\xi_1 + \xi_1}{\xi_1 + \xi_1}\right) + \left(\frac{\xi_1 + \xi_1}{\xi_1 + \xi_1} + \frac{\xi_1 + \xi_1}{\xi_1 + \xi_1}\right). 
\end{align*}
\]

(3.23)
When $N_1 = 2$, we can express the solutions to Eq. (2.40) explicitly

\[
\begin{align*}
 u_1(x,t) &= i(-\mu_1 \mu_{1,1} e^{\mu_1 t}(M^{-1})_{1,1} - \mu_1 \mu_{2,1} e^{\mu_1 t}(M^{-1})_{1,2} \\
& \quad - \mu_2 \mu_{1,1} e^{\mu_2 t}(M^{-1})_{1,3} - \mu_2 \mu_{2,1} e^{\mu_2 t}(M^{-1})_{1,4} \\
& \quad - \mu_2 \mu_{1,1} e^{\mu_2 t}(M^{-1})_{2,1} - \mu_2 \mu_{2,1} e^{\mu_2 t}(M^{-1})_{2,2} \\
& \quad - \mu_2 \mu_{1,1} e^{\mu_2 t}(M^{-1})_{3,3} + \mu_2 \mu_{2,1} e^{\mu_2 t}(M^{-1})_{3,4} \\
& \quad + \mu_2 \mu_{1,1} e^{\mu_2 t}(M^{-1})_{3,1} + \mu_2 \mu_{2,1} e^{\mu_2 t}(M^{-1})_{3,2} \\
& \quad + \mu_2 \mu_{1,1} e^{\mu_2 t}(M^{-1})_{4,1} + \mu_2 \mu_{2,1} e^{\mu_2 t}(M^{-1})_{4,2} \\
& \quad + \mu_2 \mu_{1,1} e^{\mu_2 t}(M^{-1})_{4,3} + \mu_2 \mu_{2,1} e^{\mu_2 t}(M^{-1})_{4,4}, \\
 u_2(x,t) &= i(-\mu_3 \mu_{1,1} e^{\mu_3 t}(M^{-1})_{1,1} - \mu_3 \mu_{2,1} e^{\mu_3 t}(M^{-1})_{1,2} \\
& \quad - \mu_3 \mu_{1,1} e^{\mu_3 t}(M^{-1})_{1,3} - \mu_3 \mu_{2,1} e^{\mu_3 t}(M^{-1})_{1,4} \\
& \quad - \mu_3 \mu_{1,1} e^{\mu_3 t}(M^{-1})_{2,1} - \mu_3 \mu_{2,1} e^{\mu_3 t}(M^{-1})_{2,2} \\
& \quad - \mu_3 \mu_{1,1} e^{\mu_3 t}(M^{-1})_{3,3} + \mu_3 \mu_{2,1} e^{\mu_3 t}(M^{-1})_{3,4} \\
& \quad + \mu_3 \mu_{1,1} e^{\mu_3 t}(M^{-1})_{3,1} + \mu_3 \mu_{2,1} e^{\mu_3 t}(M^{-1})_{3,2} \\
& \quad + \mu_3 \mu_{1,1} e^{\mu_3 t}(M^{-1})_{4,1} + \mu_3 \mu_{2,1} e^{\mu_3 t}(M^{-1})_{4,2} \\
& \quad + \mu_3 \mu_{1,1} e^{\mu_3 t}(M^{-1})_{4,3} + \mu_3 \mu_{2,1} e^{\mu_3 t}(M^{-1})_{4,4}, \\
 u_3(x,t) &= i(-\mu_4 \mu_{1,1} e^{\mu_4 t}(M^{-1})_{1,1} - \mu_4 \mu_{2,1} e^{\mu_4 t}(M^{-1})_{1,2} \\
& \quad - \mu_4 \mu_{1,1} e^{\mu_4 t}(M^{-1})_{1,3} - \mu_4 \mu_{2,1} e^{\mu_4 t}(M^{-1})_{1,4} \\
& \quad - \mu_4 \mu_{1,1} e^{\mu_4 t}(M^{-1})_{2,1} - \mu_4 \mu_{2,1} e^{\mu_4 t}(M^{-1})_{2,2} \\
& \quad - \mu_4 \mu_{1,1} e^{\mu_4 t}(M^{-1})_{3,3} + \mu_4 \mu_{2,1} e^{\mu_4 t}(M^{-1})_{3,4} \\
& \quad + \mu_4 \mu_{1,1} e^{\mu_4 t}(M^{-1})_{3,1} + \mu_4 \mu_{2,1} e^{\mu_4 t}(M^{-1})_{3,2} \\
& \quad + \mu_4 \mu_{1,1} e^{\mu_4 t}(M^{-1})_{4,1} + \mu_4 \mu_{2,1} e^{\mu_4 t}(M^{-1})_{4,2} \\
& \quad + \mu_4 \mu_{1,1} e^{\mu_4 t}(M^{-1})_{4,3} + \mu_4 \mu_{2,1} e^{\mu_4 t}(M^{-1})_{4,4}, \\
 u_4(x,t) &= i(-\mu_5 \mu_{1,1} e^{\mu_5 t}(M^{-1})_{1,1} - \mu_5 \mu_{2,1} e^{\mu_5 t}(M^{-1})_{1,2} \\
& \quad - \mu_5 \mu_{1,1} e^{\mu_5 t}(M^{-1})_{1,3} - \mu_5 \mu_{2,1} e^{\mu_5 t}(M^{-1})_{1,4} \\
& \quad - \mu_5 \mu_{1,1} e^{\mu_5 t}(M^{-1})_{2,1} - \mu_5 \mu_{2,1} e^{\mu_5 t}(M^{-1})_{2,2} \\
& \quad - \mu_5 \mu_{1,1} e^{\mu_5 t}(M^{-1})_{3,3} + \mu_5 \mu_{2,1} e^{\mu_5 t}(M^{-1})_{3,4} \\
& \quad + \mu_5 \mu_{1,1} e^{\mu_5 t}(M^{-1})_{3,1} + \mu_5 \mu_{2,1} e^{\mu_5 t}(M^{-1})_{3,2} \\
& \quad + \mu_5 \mu_{1,1} e^{\mu_5 t}(M^{-1})_{4,1} + \mu_5 \mu_{2,1} e^{\mu_5 t}(M^{-1})_{4,2} \\
& \quad + \mu_5 \mu_{1,1} e^{\mu_5 t}(M^{-1})_{4,3} + \mu_5 \mu_{2,1} e^{\mu_5 t}(M^{-1})_{4,4}, \\
\end{align*}
\]
where
\[ m_{k,j} = \frac{\hat{\theta}_k \theta_j}{\zeta_j - \zeta_k}, \quad 1 \leq k, j \leq 4, \quad (3.25) \]

with
\[ \zeta_3 = -\zeta_1, \quad \zeta_4 = -\zeta_2, \quad \hat{\zeta}_j = \zeta_j', \quad 1 \leq j \leq 4. \quad (3.26) \]

In Figs. 8 and 9, we display the localized structures, density plot and the wave propagation of the one-soliton and two-soliton solutions. It is interesting that whatever the solutions are single-soliton or two-soliton solutions, their real part and image part are all breather-like solitons.

Similar to Case 1 and Case 2, the three soliton solutions are given by
\[
u = 1, 2, 3, 4, \quad m_{k,j} = \frac{\hat{\theta}_k \theta_j}{\zeta_j - \zeta_k}, \quad 1 \leq k, j \leq 6, \quad (3.28)\]

with
\[ \zeta_4 = -\zeta_1, \quad \zeta_5 = -\zeta_2, \quad \zeta_6 = -\zeta_3, \quad \hat{\zeta}_j = \zeta_j', \quad 1 \leq j \leq 6. \quad (3.29) \]
Figure 10. Three-soliton solution to Eq. (3.27) with parameters $\zeta_1 = 1.2 + 0.8i$, $\zeta_3 = 1.5 + 0.7i$, $\zeta_5 = 1.1 + 0.9i$, $\mu_{1,1} = 0.1$, $\mu_{1,2} = \mu_{2,1} = \mu_{3,1} = 0.2$, $\mu_{1,3} = \mu_{3,3} = 0.3$, $\mu_{1,4} = \mu_{2,2} = \mu_{3,5} = 0.4$, $\mu_{1,5} = \mu_{2,5} = 0.5$, $\mu_{3,2} = 0.6$, $\mu_{3,4} = 0.8$ and $\mu_{5,3} = 0.9$. (a): the local structures of the three soliton solutions $u_1(x,t)$, (b): the density plot of $u_1(x,t)$, (c): the wave propagation of the three soliton solutions $u_1(x,t)$.

4. Conclusions

Based on the previous work [40], the main purpose of our work is to investigate a generalized $N$-component FL equations via the Riemann-Hilbert approach, practically speaking, which is to greatly promote the results of previous work. In this work, the spectral analysis of the associated Lax pair is first carried out and a Riemann-Hilbert problem is established. After that, via solving the presented Riemann-Hilbert problem with reflectionless case, the $N$-soliton solution to the $N$-component FL equations are obtained at last. Furthermore, by selecting specific values of the involved parameters, a few plots of one-, two- and three- soliton solutions are made to display the localized structures and dynamic propagation behaviors.

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