ROTA-BAXTER OPERATORS ON INVOLUTIVE ASSOCIATIVE ALGEBRAS

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ABSTRACT. In this paper, we consider Rota-Baxter operators on involutive associative algebras. We define cohomology for Rota-Baxter operators on involutive algebras that governs the formal deformation of the operator. This cohomology can be seen as the Hochschild cohomology of a certain involutive associative algebra with coefficients in a suitable involutive bimodule. We also relate this cohomology with the cohomology of involutive dendriform algebras. Finally, we show that the standard Fard-Guo construction of the functor from the category of dendriform algebras to the category of Rota-Baxter algebras restricts to the involutive case.

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1. INTRODUCTION

Rota-Baxter operators are an algebraic abstraction of the integral operator that was first introduced by Baxter in his study of the fluctuation theory in probability [2]. The study of Rota-Baxter operators was further developed by Rota [16] and Cartier [4] in relationship with combinatorics. They were found important applications in the Connes-Kreimer’s algebraic approach of the renormalization of quantum field theory [5]. Rota-Baxter operators are also useful to study splitting of algebras. Namely, Rota-Baxter operators give rise to dendriform algebras which are splitting of associative algebras [1,14]. In [10] Ebrahimi-Fard and Guo constructs the universal enveloping Rota-Baxter algebra of a dendriform algebra in view of the standard universal enveloping algebra of a Lie algebra. The cohomology and deformation problem of associative Rota-Baxter operators (more generally of relative Rota-Baxter operators [17]) has been recently studied by the author in [7].

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On the other hand, classical algebras such as associative algebras, $A_\infty$-algebras and $L_\infty$-algebras equipped with involutions are studied in the last few years. An involutive associative algebra is an associative algebra $A$ together with a linear map $*: A \to A$, $a \mapsto a^*$ satisfying $a^{**} = a$ and $(ab)^* = b^*a^*$, for $a, b \in A$. Such involutive algebras first appeared in mathematical physics in the context of an unoriented version of topological field theory [6]. Involutive algebras often appear in the standard constructions of algebras arising in geometric contexts, when the underlying geometric object has an involution [3,6]. For example, the de Rham cohomology of a manifold with an involution carries an involutive $A_\infty$-algebra structure [15].

Involutive algebras often appear in the standard constructions of algebras arising in geometric contexts, when the underlying geometric object has an involution [3,6]. For example, the de Rham cohomology of a manifold with an involution carries an involutive $A_\infty$-algebra structure [15]. An interpretation of Braun’s Hochschild cohomology is given by the authors in [11] using involutive Bar complex which led them to also introduce Hochschild homology of involutive associative algebras. Recently, with Saha, the present author gave a more explicit description of Hochschild cohomology of involutive associative algebras [9]. More precisely, they defined involutive dendriform algebras, their cohomology and find relations with the Hochschild cohomology of involutive associative algebras.

Our aim in this paper is to study (relative) Rota-Baxter operators on involutive associative algebras. Let $(A, *)$ be an involutive associative algebra and $(M, *)$ be an involutive $A$-bimodule. A linear map $T: M \to A$ is said to be a relative Rota-Baxter operator on $A$ with respect to the involutive $A$-bimodule $M$ if $T$ satisfies $T(u^*) = T(u)^*$ and the following identity

$$T(u)T(v) = T(uT(v) + T(u)v), \text{ for } u, v \in M.$$ 

From the last identity, it follows that $T$ is a relative Rota-Baxter operator on the ordinary associative algebra $A$ with respect to the ordinary $A$-bimodule $M$. Here the word ‘ordinary’ means that we are not considering the involution. By definition, a Rota-Baxter operator on an involutive associative algebra $A$ is a relative Rota-Baxter operator on the involutive algebra $A$ with respect to itself. A (relative) Rota-Baxter operator on an involutive algebra induces an involutive dendriform algebra structure on the domain of the operator. Using Gerstenhaber’s bracket on involutive Hochschild cochains and Voronov’s derived bracket [18], in Section 2, we construct a graded Lie algebra whose Maurer-Cartan elements are relative Rota-Baxter operators. Thus, a relative Rota-Baxter operator $T$ on an involutive algebra $A$ with respect to an involutive $A$-bimodule $M$ induces cohomology, called the cohomology of $T$.

In Section 3, we show that the cohomology of $T$ introduced in the previous section can be seen as the Hochschild cohomology of an involutive associative algebra with coefficients in a suitable involutive bimodule. For a relative Rota-Baxter operator $T$ on an involutive associative algebra $A$ with respect to an involutive bimodule $M$, we show that the ordinary cohomology of $T$ (viewed as a relative Rota-Baxter operator on the ordinary algebra $A$ with respect to the ordinary bimodule $M$) has a direct sum decomposition of the involutive cohomology of $T$ and a skew-factor. Finally, we obtain a morphism from the cohomology of a relative Rota-Baxter operator $T$ and the cohomology of the induced involutive dendriform algebra.

The classical deformation theory of Gerstenhaber [13] has been extended to associative Rota-Baxter operators in [7]. In Section 4, we study deformations of a relative Rota-Baxter operator $T$ on an involutive associative algebra with respect to an involutive bimodule. Our main results in this section are similar to the results of [7]. We show that the linear term in a formal deformation of $T$ is a 1-cocycle in the cohomology of $T$, called the infinitesimal of the deformation. Moreover, equivalent deformations have cohomologous infinitesimals. Given a finite order deformation of $T$, we associate a 2-cocycle in the cohomology complex of $T$, called the obstruction 2-cocycle. When the corresponding cohomology class vanishes, the given deformation extends to deformation of next order.

Finally, in Section 5, we first recall the construction of the universal enveloping Rota-Baxter algebra of a dendriform algebra. Then we show that this construction restricts to the corresponding algebras equipped with involutions.
All vector spaces, linear maps and tensor products are over a field $\mathbb{K}$ of characteristic 0.

2. (Relative) Rota-Baxter operators on involutive associative algebras

In this section, we introduce relative Rota-Baxter operators on involutive associative algebra with respect to an involutive bimodule. A particular case is given by Rota-Baxter operators on involutive algebra. We construct a graded Lie algebra whose Maurer-Cartan elements are relative Rota-Baxter operators.

2.1. Involutive associative algebras and Hochschild cohomology. An involution on a vector space $V$ is a linear map $*: V \to V$, $v \mapsto v^*$ satisfying $v^{**} = v$, for all $v \in V$. Thus, an involution on $V$ is an invertible linear map on $V$ that equals to its inverse.

2.1. Definition. An involutive associative algebra is an associative algebra $A$ together with an involution $*: A \to A$ that satisfies $(ab)^* = b^*a^*$, for all $a, b \in A$.

A morphism between involutive associative algebras is a morphism between underlying algebras preserving the involutions. Let $A$ be an involutive associative algebra. An involutive $A$-bimodule is an ordinary $A$-bimodule $M$ together with an involution $*: M \to M$ that satisfies $(au)^* = u^*a^*$ and $(ua)^* = a^*u^*$, for $a \in A, u \in M$.

In this case, the direct sum $A \oplus M$ carries an involutive associative algebra structure (called the semi-direct product) with the involution $(a, u)^* = (a^*, u^*)$ and the product

$$(a, u) \cdot (b, v) = (ab, av + ub).$$

In the following, we recall the Hochschild cohomology of an involutive associative algebra $A$ with coefficients in an involutive $A$-bimodule $M$. First consider the ordinary Hochschild cochain complex $\{C^n_{Hoch}(A, M), \delta_{Hoch}\}$, where $C^n_{Hoch}(A, M) = \text{Hom}(A \otimes^n, M)$ for $n \geq 0$ and the differential $\delta_{Hoch}: C^n_{Hoch}(A, M) \to C^{n+1}_{Hoch}(A, M)$ given by

$$\delta_{Hoch}(f)(a_1, \ldots, a_{n+1}) = a_1f(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n}(-1)^i f(a_1, \ldots, a_{i-1}, a_ia_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1}f(a_1, \ldots, a_n)a_{n+1}.$$ 

For $n \geq 0$, consider the collection of subspaces $iC^n_{Hoch}(A, M) \subset C^n_{Hoch}(A, M)$ given by $iC^n_{Hoch}(A, M) = \{m \in C^n_{Hoch}(A, M) | m^{**} = -m \}$ and

$$iC^n_{Hoch}(A, M) = \{ f \in C^n_{Hoch}(A, M) | f(a_1, \ldots, a_n)^* = (-1)^{\frac{(n-1)(n-2)}{2}} f(a_1^*, \ldots, a_n^*) \}, \quad \text{for } n \geq 1.$$ 

It has been shown in [9] that $\{iC^n_{Hoch}(A, M), \delta_{Hoch}\}$ is a subcomplex of the ordinary Hochschild complex and the cohomology of this subcomplex is called the Hochschild cohomology of the involutive algebra $A$ with coefficients in the involutive bimodule $M$.

Next we show that the classical Gerstenhaber bracket on ordinary Hochschild cochains passes onto the involutive Hochschild cochains. Let us first recall the classical Gerstenhaber bracket [12]. For $f \in C^n_{Hoch}(A, A)$ and $g \in C^n_{Hoch}(A, A)$, the Gerstenhaber bracket $[f, g] \in C^{n+1}_{Hoch}(A, A)$ is given by

$$[f, g] = \sum_{i=1}^{n}(1-)^{(i-1)(n-1)} f \circ_i g - (-1)^{(i-1)(n-1)} \sum_{i=1}^{n}(-1)^{(i-1)(n-1)} g \circ_i f,$$

where

$$(f \circ_i g)(a_1, \ldots, a_{m+n-1}) = f(a_1, \ldots, a_{i-1}, g(a_i, \ldots, a_{i+n-1}), a_{i+n, \ldots, a_{m+n-1}}).$$

With this notation, we have the following.

2.2. Proposition. If $f \in iC^n_{Hoch}(A, A)$ and $g \in iC^n_{Hoch}(A, A)$, then $[f, g] \in iC^{n+1}_{Hoch}(A, A)$.

Proof. First observe that

$$(f \circ_i g)(a_1, \ldots, a_{m+n-1})^* = (-1)^{\frac{(n-1)(m-2)+(n-1)(n-2)}{2}} (f \circ_{m-i+1} g)(a_{m+n-1}^*, \ldots, a_i^*).$$
Hence
\[
\left( \sum_{i=1}^{m} (-1)^{(i-1)(n-1)} f \circ g \right)(a_1, \ldots, a_{m+n-1})^* = (-1)^{\frac{(n-1)(m-2)+(n-1)(n-2)}{2}} \left( \sum_{i=1}^{m} (-1)^{(i-1)(n-1)} (f \circ g)(a_{m+n-1}^*, \ldots, a_i^*) \right)
\]
\[
= (-1)^{\frac{(n-1)(m-2)+(n-1)(n-2)}{2}} \sum_{i=1}^{m} (-1)^{(m-i)(n-1)} (f \circ g)(a_{m+n-1}^*, \ldots, a_i^*)
\]

Therefore,
\[
[f, g](a_1, \ldots, a_{m+n-1})^* = (-1)^{\frac{(n-1)(m-2)+(n-1)(n-2)}{2}} \sum_{i=1}^{m} (-1)^{(m-i)(n-1)} [f, g](a_{m+n-1}^*, \ldots, a_i^*)
\]

\[
= (-1)^{\frac{(m+n-2)(m+n-3)}{2}} [f, g](a_{m+n-1}^*, \ldots, a_i^*)
\]

This shows that \([f, g] \in \mathfrak{C}_{\text{Roch}}^{m+n-1}(A, A)\).

\[\square\]

2.2. Relative Rota-Baxter operators.

2.3. Definition. Let \(A\) be an associative algebra. A linear map \(R : A \to A\) is a Rota-Baxter operator on \(A\) if \(R\) satisfies
\[
R(a)R(b) = R(aR(b) + R(a)b), \text{ for } a, b \in A.
\]

If \(A\) is an involutive associative algebra, then a linear map \(R : A \to A\) is said to be a Rota-Baxter operator on \(A\) if \(R(a^*) = R(a)^*\) and satisfies (2).

2.4. Definition. Let \(A\) be an involutive associative algebra and \(M\) be an involutive \(A\)-bimodule. A linear map \(T : M \to A\) is called a relative Rota-Baxter operator on \(A\) with respect to the involutive \(A\)-bimodule \(M\) if \(T\) satisfies \(T(u^*) = T(u)^*\) and
\[
T(u)v = T(uT(v) + T(u)v), \text{ for } u, v \in M.
\]

They are also called involutive relative Rota-Baxter operators. Thus, it follows that a Rota-Baxter operator on an involutive associative algebra \(A\) is a relative Rota-Baxter operator on \(A\) with respect to the involutive bimodule \(A\) itself.

2.5. Proposition. Let \(A\) be an involutive associative algebra and \(M\) be an involutive \(A\)-bimodule. A linear map \(T : M \to A\) is a relative Rota-Baxter operator on \(A\) with respect to the bimodule \(M\) if and only if the graph of \(T\),
\[
\text{Gr}(T) = \{(Tu, u) \mid u \in M\}
\]
is an involutive subalgebra of the semi-direct product \(A \oplus M\).

Let \(T\) (resp. \(T'\)) be a relative Rota-Baxter operator on an involutive associative algebra \(A\) with respect to an involutive \(A\)-bimodule \(M\) (resp. on an involutive associative algebra \(A'\) with respect to an involutive \(A'\)-bimodule \(M'\)).

2.6. Definition. A morphism from \(T\) to \(T'\) consists of a pair \((\phi, \psi)\) in which \(\phi : A \to A'\) is an involutive algebra morphism and \(\psi : M \to M'\) is a linear map satisfying \(\psi(u^*) = \psi(u)^*\) and
\[
T' \circ \psi = \phi \circ T, \quad \psi(au) = \phi(a)\psi(u) \quad \text{and} \quad \psi(ua) = \psi(u)\phi(a),
\]
for all \(a \in A\) and \(u \in M\). A morphism \((\phi, \psi)\) is called an isomorphism if \(\phi\) and \(\psi\) are both linear isomorphisms.

In [1] Aguiar showed that a (relative) Rota-Baxter operator induces a dendriform structure. Here we observe the corresponding result in the involutive case.
2.7. **Definition.** A dendriform algebra is a vector space $D$ together with bilinear operations $\prec, \succ : D \otimes D \to D$ satisfying the following three identities

$$(a \prec b) \prec c = a \prec (b \prec c + b \succ c), \quad (a \succ b) \prec c = a \prec (b \prec c), \quad \text{for all } a, b, c \in D.$$ 

2.8. **Proposition.** Let $T$ be a relative Rota-Baxter operator on an involutive associative algebra $A$ with respect to an involutive $A$-module $M$. Then $M$ carries an involutive dendriform algebra structure with products

$$u \prec v = uT(v) \quad \text{and} \quad u \succ v = T(u)v, \quad \text{for } u, v \in M.$$ 

2.3. **Gauge transformations.** Let $A$ be an involutive associative algebra and $M$ be an involutive $A$-bimodule. Let $T : M \to A$ be a relative Rota-Baxter operator. Consider the involutive subalgebra $Gr(T) \subset A \oplus M$ of the semi-direct product.

For any involutive 1-cocycle $B \in iC^1_{Hoch}(A, M)$, we consider the deformed subspace

$$\tau_B(Gr(T)) = \{(Tu_u + B(Tu_u)) | u \in M\} \subset A \oplus M.$$ 

2.9. **Lemma.** If $B \in iC^1_{Hoch}(A, M)$ is an involutive Hochschild 1-cocycle then the subspace $\tau_B(Gr(T)) \subset A \oplus M$ is an involutive subalgebra of the semi-direct product $A \oplus M$.

**Proof.** For any $u, v \in M$, we have

$$(Tu_u + B(Tu_u)) \cdot (Tv_v + B(Tv_v))$$

$$= (Tu_uT(v_v), Tu_uv + uTv_v + Tu_u(B(Tv_v)) + B(Tu_u)Tv_v)$$

$$= (Tu_uT(v_v), Tu_uv + uTv_v + B(Tu_u)Tv_v) \quad \text{(since } B \text{ is a 1-cocycle)}.$$ 

This is in $\tau_B(Gr(T))$ as $T$ is a relative Rota-Baxter operator. Finally, this is an involutive subspace as $B$ is an involutive 1-cocycle. \hfill \Box

We now ask the question whether the involutive subalgebra $\tau_B(Gr(T))$ is the graph of a new involutive relative Rota-Baxter operator. We observe that if the linear map $id_M + B \circ T : M \to M$ is invertible, then $\tau_B(Gr(T))$ is the graph of a linear map $T \circ (id_M + B \circ T)^{-1} : M \to A$. In such a case, by Proposition 2.5, the linear map $T \circ (id_M + B \circ T)^{-1}$ is a relative Rota-Baxter operator on the involutive algebra $A$ with respect to the involutive bimodule $M$. The relative Rota-Baxter operator $T \circ (id_M + B \circ T)^{-1}$ is called the gauge transformation of $T$ associated with $B$.

2.4. **Maurer-Cartan characterization and cohomology.** In this subsection, we first recall from [7] that ordinary relative Rota-Baxter operators are Maurer-Cartan elements in a suitable graded Lie algebra $\mathfrak{g}$. Then we will show that involutive relative Rota-Baxter operators are Maurer-Cartan elements in a suitable graded Lie subalgebra of $\mathfrak{g}$.

Let $A$ be an ordinary associative algebra with product $\mu$ and $M$ be an $A$-bimodule with left and right $A$ actions $l, r$. Then the graded space $\oplus_{n \geq 0} \text{Hom}(M^\otimes n, A)$ carries a graded Lie bracket defined by Voronov’s derived bracket

$$[P, Q] := (-1)^m[[\mu + l + r, P], Q],$$

for $P \in \text{Hom}(M^\otimes n, A), Q \in \text{Hom}(M^\otimes n, A)$. Here $\mu + l + r$ can be considered as an element in $\text{Hom}((A \oplus M)^{\otimes 2}, A \oplus M)$. Similarly, $P$ can be considered as element in $\text{Hom}((A \oplus M)^{\otimes m}, A \oplus M)$ and same for $Q$. Finally, the bracket $[,]$ on the right hand side of (3) is the Gerstenhaber’s bracket (1) on multilinear maps
on the vector space \( A \oplus M \). Explicitly, the bracket \((3)\) is given by

\[
(4) \quad [P, Q](u_1, \ldots, u_{m+n}) = \\
= \sum_{i=1}^{m} (-1)^{(i-1)n} P(u_1, \ldots, u_{i-1}, Q(u_{i+1}, \ldots, u_{i+n-1})u_{i+n}, \ldots, u_{m+n}) \\
- \sum_{i=1}^{n} (-1)^{in} P(u_1, \ldots, u_{i-1}, u_iQ(u_{i+1}, \ldots, u_{i+n}), u_{i+n+1}, \ldots, u_{m+n}) \\
- (-1)^{mn} \left\{ \sum_{i=1}^{n} (-1)^{(i-1)m} Q(u_1, \ldots, u_{i-1}, P(u_{i+1}, \ldots, u_{i+m-1})u_{i+m}, \ldots, u_{m+n}) \right\} \\
+ (-1)^{mn} [P(u_1, \ldots, u_m)Q(u_{m+1}, \ldots, u_{m+n}) - (-1)^{mn} Q(u_1, \ldots, u_n)P(u_{n+1}, \ldots, u_{m+n})],
\]

for \( P \in \text{Hom}(M^{\otimes m}, A), \ Q \in \text{Hom}(M^{\otimes n}, A), \ a, b \in A \) and \( u_1, \ldots, u_{m+n} \in M \).

It is easy from the above bracket that a linear map \( T \in \text{Hom}(M, A) \) is an ordinary relative Rota-Baxter operator on \( A \) with respect to the \( A \)-bimodule \( M \) if and only if \( T \) is a Maurer-Cartan element in the above-graded Lie algebra. The cohomology induced from the Maurer-Cartan element \( T \) is called the cohomology of the relative Rota-Baxter operator \( T \), and they are denoted by \( H_{T}^{\#}(M, A) \).

Next, let \( A \) be an involutive associative algebra and \( M \) be an involutive \( A \)-bimodule. Consider the graded space of involutive multilinear maps \( \otimes_{n \geq 0} \text{Hom}(M^{\otimes n}, A) \), where \( \text{Hom}(M^{\otimes n}, A) = iA = \{ a \in A \mid a^* = -a \} \) and for \( n \geq 1 \),

\[
i\text{Hom}(M^{\otimes n}, A) = \{ f \in \text{Hom}(M^{\otimes n}, A) \mid (u_1, \ldots, u_n)^* = (-1)^{(n-1)(n-2)} f(u_{n+1}^*, \ldots, u_1^*) \}.
\]

Since involutive multilinear maps are closed under the Gerstenhaber’s bracket, it follows that the bracket \((3)\) restricts to the graded subspace \( \otimes_{n \geq 0} i\text{Hom}(M^{\otimes n}, A) \) by the same formula as \((4)\). It follows that a linear map \( T : M \rightarrow A \) is an involutive relative Rota-Baxter operator if and only if \( T \in i\text{Hom}(M, A) \) is a Maurer-Cartan element in the graded Lie algebra \((\otimes_{n \geq 0} i\text{Hom}(M^{\otimes n}, A), [\ , \ ]_t)\).

Thus, an involutive relative Rota-Baxter operator \( T \) induces a degree 1 differential \( d_T = [T, \ ]_t \) on the graded space \( \otimes_{n \geq 0} i\text{Hom}(M^{\otimes n}, A) \). The corresponding cohomology groups are called the cohomology of the involutive relative Rota-Baxter operator \( T \), and they are denoted by \( iH_{T}^{\#}(M, A) \).

### 3. Some properties of the cohomology

In this section, we first show that the cohomology of an involutive relative Rota-Baxter operator can be seen as the Hochschild cohomology of an involutive associative algebra. We also obtain a splitting theorem of the ordinary cohomology of a relative Rota-Baxter operator on an involutive associative algebra. Finally, we relate the cohomology of an involutive relative Rota-Baxter operator to the cohomology of the corresponding involutive dendriform algebra.

#### 3.1. Cohomology as involutive Hochschild cohomology

Let \( T : M \rightarrow A \) be a relative Rota-Baxter operator on an involutive associative algebra \( A \) with respect to the involutive \( A \)-bimodule \( M \). Then by Proposition 2.8, \( M \) carries an involutive dendriform algebra structure. Hence \( M \) has an involutive associative algebra structure with product

\[
u \odot v = uT(v) + T(u)v, \text{ for } u, v \in M.
\]
The following lemma is a generalization of [17] in the involutive context.

3.1. **Lemma.** Let $T : M \to A$ be a relative Rota-Baxter operator on an involutive associative algebra $A$ with respect to the involutive $A$-bimodule $M$. Then the maps

$$l_T : M \otimes A \to A, \quad (u, a) \mapsto T(u)a - T(ua),$$

$$r_T : A \otimes M \to A, \quad (a, u) \mapsto aT(u) - T(au)$$

defines an involutive $M$-bimodule structure on $A$.

**Proof.** In [17] it has been proved that the maps $l_T$ and $r_T$ define an $M$-bimodule structure on $A$. Thus we need to verify the compatibility of involution and the maps $l_T, r_T$. We have

$$l_T(u, a)^* = a^*T(u)^* - T((ua)^*) = a^*T(u^*) - T(a^*u^*) = r_T(a^*, u^*).$$

Similarly, $r_T(a, u)^* = l_T(u^*, a^*)$. Hence the proof.

It follows from the above lemma that we may consider the Hochschild cochain complex of the involutive associative algebra $M$ with coefficients in the involutive $M$-bimodule $A$. More precisely, consider the cochain complex $\{iC^n_{\text{Hoch}}(M, A), \delta^n_{\text{Hoch}}\}$, where $iC^n_{\text{Hoch}}(M, A) = \{a \in A| a^* = -a\}$ and

$$iC^n_{\text{Hoch}}(M, A) = \{ f : M^{\otimes n} \to A | f(u_1, \ldots, u_n)^* = (-1)^{(n-1)(n-2)/2} f(u_n^*, \ldots, u_1^*) \}, \quad \text{for } n \geq 1$$

and the differential $\delta^n_{\text{Hoch}} : iC^n_{\text{Hoch}}(M, A) \to iC^{n+1}_{\text{Hoch}}(M, A)$ given by

$$(\delta^n_{\text{Hoch}} f)(u_1, \ldots, u_{n+1}) = l_T(u_1, f(u_2, \ldots, u_{n+1})) + \sum_{i=1}^{n} (-1)^i f(u_1, \ldots, u_{i-1}, u_i \otimes u_{i+1}, \ldots, u_{n+1}) + (-1)^{n+1} r_T(f(u_1, \ldots, u_n), u_{n+1}).$$

It has been shown in [7] that the coboundary operator $d_T$ induced from the Maurer-Cartan element $T$ and the coboundary operator (5) are related by

$$d_T f = (-1)^{n} \delta^n_{\text{Hoch}} f, \quad \text{for } f \in iC^n_{\text{Hoch}}(M, A) = i\text{Hom}(M^{\otimes n}, A).$$

Hence we get that the cohomology of the involutive relative Rota-Baxter operator $T$ is isomorphic to the Hochschild cohomology of the involutive associative algebra $M$ with coefficients in the involutive $M$-bimodule $A$.

3.2. **Splitting theorem.** In [3] Braun has shown that for involutive associative algebras, the ordinary Hochschild cohomology splits as a direct sum of involutive Hochschild cohomology and a skew-factor. This splitting theorem has been explicitly described in a recent paper by the present author [9] and further extended it to the dendriform context. Here we conclude a similar result for relative Rota-Baxter operators.

Let $T : M \to A$ be a relative Rota-Baxter operator on an involutive associative algebra $A$ with respect to an involutive $A$-bimodule $M$. For each $n \geq 0$, consider a linear map $S_n : \text{Hom}(M^{\otimes n}, A) \to \text{Hom}(M^{\otimes n}, A)$ by

$$S_0(a) = -a^* \quad \text{and} \quad (S_n P)(a_1, \ldots, a_n) = (-1)^{(n-1)(n-2)/2} P(a_n^*, \ldots, a_1^*)^*, \quad \text{for } n \geq 1.$$ 

Then we have $(S_n)^2 = \text{id}$. Therefore, the map $S_n$ has eigenvalues $\pm 1$. Observe that the eigenspace corresponding to the eigenvalue $+1$ is precisely given by $i\text{Hom}(M^{\otimes n}, A)$. Denote the eigenspace corresponding to the eigenvalue $-1$ by $i_-\text{Hom}(M^{\otimes n}, A)$. Then we have

$$\text{Hom}(M^{\otimes n}, A) \cong i\text{Hom}(M^{\otimes n}, A) \oplus i_-\text{Hom}(M^{\otimes n}, A), \quad f \mapsto \left(\frac{f + S_n f}{2}, \frac{f - S_n f}{2}\right).$$

It is easy to verify that $\{i_-\text{Hom}(M^{\otimes n}, A), d_T\}$ is a subcomplex of the complex $\{\text{Hom}(M^{\otimes n}, A), d_T\}$. We denote the corresponding cohomology groups by $i_-H^n(M, A)$. Note that the isomorphisms (6) preserve the corresponding differentials on both sides. Hence we get the following.
3.2. **Proposition.** For an involutive relative Rota-Baxter operator $T$, the ordinary cohomology of $T$ splits as a direct sum $H^*_T(M,A) \cong iH^*_T(M,A) \oplus i-H^*_T(M,A)$.

3.3. **Relation with the cohomology of involutive dendriform algebras.** The cohomology of dendriform algebras was first defined by Loday [14] with trivial coefficients and the operadic approach was given in [15]. An explicit description of the cohomology was given in [8]. Here we require the cohomology of involutive dendriform algebras given in [9].

Let $C_n$ be the set of first $n$ natural numbers. For convenience, we denote the elements of $C_n$ by $\{1, [2], \ldots, [n]\}$. It has been shown in [8] that for any vector space $D$, the collection of spaces

$$O(n) = \text{Hom}(K[C_n] \otimes D^\otimes n, D), \quad \text{for } n \geq 1$$

forms a non-symmetric operad with partial compositions

$$(f \circ_i g)([r]; a_1, \ldots, a_{m+1}) =
\begin{cases}
  f([r]; a_1, \ldots, a_{i-1}, g([1] + \cdots + [n]; a_i, \ldots, a_{i+n-1}), \ldots, a_{m+n-1}) & \text{if } 1 \leq r \leq i - 1 \\
  f([r]; a_1, \ldots, a_{i-1}, g([r - i + 1]; a_i, \ldots, a_{i+n-1}), \ldots, a_{m+n-1}) & \text{if } i \leq r \leq i + n - 1 \\
  f([r - n + 1]; a_1, \ldots, a_{i-1}, g([1] + \cdots + [n]; a_i, \ldots, a_{i+n-1}), \ldots, a_{m+n-1}) & \text{if } i + n \leq r \leq m + n - 1,
\end{cases}$$

for $f \in O(m)$, $g \in O(n)$, $1 \leq i \leq m$ and $[r] \in C_{m+n-1}$. Therefore, there is a graded Lie bracket on the graded vector space $O(\bullet + 1) = \oplus_{n \geq 0} O(n + 1)$ given by

$$\|f, g\| = \sum_{i=1}^{n+1} (-1)^{(i-1)n} f \circ_i g - (-1)^{mn} \sum_{i=1}^{n+1} (-1)^{(i-1)m} g \circ_i f,$$

for $f \in O(m + 1)$ and $g \in O(n + 1)$. More generally, if $(D, \prec, \succ)$ is a dendriform algebra, then the element $\pi \in O(2)$ defined by

$$\pi([1]; a, b) = a \prec b \quad \text{and} \quad \pi([2]; a, b) = a \succ b$$

satisfies $\|\pi, \pi\| = 0$, i.e. \( \pi \) defines a Maurer-Cartan element in the above graded Lie algebra. Hence $\pi$ induces a differential $\delta_\pi : O(n) \to O(n + 1)$ given by $\delta_\pi(f) := (-1)^{n-1}\|\pi, f\|$, for $f \in O(n)$.

Let $(D, \prec, \succ, \ast)$ be an involutive dendriform algebra. We define

$$iC^n_{\text{dend}}(D, D) = \{ f \in O(n) \mid f([r]; a_1, \ldots, a_n) = (-1)^{(n-1)(n-2)/2} f([n-r+1]; a_n^\ast, \ldots, a_1^\ast) \}, \quad \text{for } n \geq 1.$$

Then it has been shown in [9] that \{ $iC_{\text{dend}}^*(D, D), \delta_\pi$ \} is a subcomplex of the cochain complex \{ $O(\bullet), \delta_\pi$ \}. The cohomology groups of this subcomplex are called the cohomology of the involutive dendriform algebra $(D, \prec, \succ, \ast)$ and they are denoted by $iH^*_{\text{dend}}(D, D)$.

Let $T$ be a relative Rota-Baxter operator on an involutive associative algebra $A$ with respect to an involutive $A$-bimodule $M$. Consider the involutive dendriform algebra structure on $M$. We denote by $\pi_T \in iC^n_{\text{dend}}(M, M)$ the corresponding Maurer-Cartan element. Define a collection of maps $\Theta_n : i\text{Hom}(M^\otimes n, A) \to iC^{n+1}_{\text{dend}}(M, M)$ by

$$\Theta_n(P)([r]; u_1, u_2, \ldots, u_{n+1}) = \begin{cases}
  (-1)^{n+1} u_1 P(u_2, \ldots, u_{n+1}) & \text{if } r = 1 \\
  0 & \text{if } 2 \leq r \leq n \\
  P(u_1, \ldots, u_n) u_{n+1} & \text{if } r = n + 1.
\end{cases}$$

Note that $\Theta_n(P) \in iC^{n+1}_{\text{dend}}(M, M)$ as

$$\Theta_n(P)([1]; u_1, \ldots, u_{n+1})^\ast = (-1)^{n+1} P(u_2, \ldots, u_{n+1})^\ast u_1^\ast = (-1)^{n+1} (-1)^{\frac{n(n-1)}{2}} P(u_{n+1}^\ast, \ldots, u_2^\ast) u_1^\ast$$

$$= (-1)^{\frac{n(n-1)}{2}} P(u_{n+1}^\ast, \ldots, u_2^\ast) u_1^\ast$$

$$= (-1)^{\frac{n(n-1)}{2}} \Theta_n(P)([n+1]; u_{n+1}^\ast, \ldots, u_1^\ast).$$
For $2 \leq r \leq n$, we have $\Theta_n(P)([r]; u_1, \ldots, u_{n+1})^* = 0 = \Theta_n(P)([n - r + 2]; u_{n+1}^*, \ldots, u_1^*)$.

With these notations, we have the following [7, Lemma 3.4].

3.3. Lemma. The collection $\{\Theta_n\}$ of maps preserve the corresponding graded Lie brackets, i.e.

$$\|[\Theta_m(P), \Theta_n(Q)]\| = \Theta_{m+n}(\{P, Q\}).$$

Hence as a consequence, we get the following.

3.4. Proposition. Let $T$ be a relative Rota-Baxter operator on an involutive associative algebra $A$ with respect to the involutive $A$-bimodule $M$. Then the collection $\{\Theta_n\}$ of maps induces a morphism $\Theta_n : iH^*_T(M, A) \to iH^*_{T\text{dend}}(M, M)$ from the cohomology of $T$ to the cohomology of the involutive dendriform algebra structure on $M$.

4. Deformations

In this section, we study formal deformations of relative Rota-Baxter operators on involutive associative algebras from cohomological perspectives.

Let $A$ be an involutive associative algebra and $M$ be an involutive $A$-bimodule. Consider the space $A[[t]]$ of formal power series in $t$ with coefficients from $A$. The involution on $A$ induces an involution on $A[[t]]$ and the associative multiplication on $A$ induces an associative multiplication on $A[[t]]$ by $\mathbb{K}[[t]]$-bilinearity. With these structures, $A[[t]]$ is an involutive associative algebra. Moreover, the space $M[[t]]$ can be given the structure of an involutive $A[[t]]$-bimodule with the obvious left and right actions.

4.1. Definition. Let $T : M \to A$ be a relative Rota-Baxter operator on the involutive algebra $A$ with respect to the involutive $A$-bimodule $M$. A formal one-parameter deformation of $T$ consists of a formal sum

$$T_t = T_0 + tT_1 + t^2T_2 + \cdots \in \text{Hom}(M, A)[[t]]$$

in which $T_0 = T$ such that as a $\mathbb{K}[[t]]$-linear map $T_1 : M[[t]] \to A[[t]]$ is a relative Rota-Baxter operator on the involutive algebra $A[[t]]$ with respect to the involutive $A[[t]]$-bimodule $M[[t]]$.

Thus, the followings are hold: $T_t(u^*) = T_t(u)^*$ and

$$T_t(u)T_t(v) = T_t(uT_t(v) + T_t(u)v), \quad \text{for } u, v \in M.$$ 

These conditions are equivalent to the followings: for each $k \geq 0$, we have $T_k(u^*) = T_k(u)^*$ and

$$\sum_{i+j=k} T_i(u)T_j(v) = T_i(uT_j(v) + T_j(u)v), \quad \text{for } u, v \in M.$$ 

For $k = 1$, we get $T_1(u^*) = T_1(u)^*$ and

$$T(u)T_1(v) + T_1(u)T(v) = T(uT_1(v) + T_1(u)v) + T_1(uT(v) + T(v)u).$$

This says that $T_1 \in i\text{Hom}(M, A)$ is a 1-cocycle in the cohomology of the involutive relative Rota-Baxter operator $T$.

4.2. Definition. Two deformations $T_t = \sum_{i \geq 0} t^i T_i$ and $T'_t = \sum_{i \geq 0} t^i T'_i$ of an involutive relative Rota-Baxter operator $T$ are said to be equivalent if there is an element $a \in A$ with $a^* = -a$ and linear maps $\phi_j \in i\text{Hom}(A, A)$, $\psi_j \in i\text{Hom}(M, M)$, for $j \geq 2$ such that

$$\phi_t = \text{id}_A + t(ad^*_a - ad^*_a) + \sum_{j \geq 2} t^j \phi_j, \quad \psi_t = \text{id}_M + t(l_a - r_a) + \sum_{j \geq 2} t^j \psi_j$$

defines a morphism of relative Rota-Baxter operators from $T_t$ to $T'_t$.

Hence by Definition 2.6, the following conditions must hold: for all $a, b \in A$ and $u \in M$,

$$\phi_t(a)\phi_t(b) = \phi_t(ab), \quad T'_t \circ \psi_t(u) = \phi_t \circ T_t(u), \quad \psi_t(au) = \phi_t(a)\psi_t(u)$$

and

$$\psi_t(ua) = \psi_t(u)\phi_t(a).$$
In the second equality, by equating coefficients of $t$ from both sides, we get

$$T_i(u) - T'_i(u) = T(au - ua) - (aT(u) - T(u)a) = \delta_{Roch}(a)(u).$$

Summarizing the above discussions, we get the following.

4.3. **Theorem.** Let $T_i = \sum_{t \geq 0} t^iT_i$ be a formal one-parameter deformation of an involutive relative Rota-Baxter operator $T$. Then the linear term $T_1$ is a 1-cocycle in the cohomology of $T$ whose cohomology class depends only on the equivalence class of the deformation $T_i$.

4.1. **Extensions of finite order deformations.** In this subsection, we consider extensions of a finite order deformation of an involutive relative Rota-Baxter operator $T$. Given a finite order deformation of $T$, we associate a second cohomology class in the cohomology of $T$. When the class is trivial, the deformation extends to next order.

Let $T : M \to A$ be a relative Rota-Baxter operator on an involutive associative algebra $A$ with respect to the involutive $A$-bimodule $M$.

4.4. **Definition.** An order $N$ deformation of $T$ consists of a finite sum $T_i = \sum_{i=0}^{N} t^iT_i \in \text{Hom}(M,A)[[t]]/(t^{N+1})$ such that $T_0 = T$ and as a $\mathbb{K}[t]/(t^{N+1})$-linear map $T_i : M[[t]]/(t^{N+1}) \to A[[t]]/(t^{N+1})$ is an involutive relative Rota-Baxter operator on $A[[t]]/(t^{N+1})$ with respect to the involutive $A[[t]]/(t^{N+1})$-bimodule $M[[t]]/(t^{N+1})$.

Therefore, we must have $T_k(u^*) = T_k(u)^*$ and

$$\sum_{i+j=k} T_i(u)T_j(v) = T_i(uT_j(v) + T_j(u)v), \text{ for } u, v \in M \text{ and } k = 0, 1, \ldots, N.$$  

The last condition is equivalent to the fact that

$$d_T(T_k) = -\frac{1}{2} \sum_{i+j=k, i,j \geq 1} [T_i, T_j], \text{ for } k = 0, 1, \ldots, N.$$  

4.5. **Definition.** A deformation $T_i = \sum_{i=0}^{N} t^iT_i$ of order $N$ is said to be extensible if there exists an element $T_{N+1} \in i\text{Hom}(M,A)$ such that $\tilde{T}_i = T_i + t^{N+1}T_{N+1}$ is a deformation of order $N + 1$.

In such a case, one more deformation equation needs to be satisfied, namely,

$$d_T(T_{N+1}) = -\frac{1}{2} \sum_{i+j=N+1, i,j \geq 1} [T_i, T_j].$$

Note that the right hand side of (7) depends only on $\{T_1, \ldots, T_N\}$ and doesn’t involve $T_{N+1}$. Hence it depends on the deformation $T_i$. This is called the obstruction to the extend the deformation $T_i$, denoted by $\text{Ob}_{T_i}$.

4.6. **Proposition.** $\text{Ob}_{T_i}$ is a 2-cocycle in the cohomology complex of $T$.

**Proof.** See [7, Proposition 4.17].

The above proposition shows that a finite order deformation $T_i$ gives rise to a second cohomology class $[\text{Ob}_{T_i}] \in iH^2_F(M,A)$, called the obstruction class.

Hence from (7) and Proposition 4.6, we get the following.

4.7. **Theorem.** A finite order deformation $T_i$ of an involutive relative Rota-Baxter operator $T$ extends to a deformation of next order if and only if the corresponding obstruction class $[\text{Ob}_{T_i}] \in iH^2_F(M,A)$ is trivial.

4.8. **Corollary.** If $iH^2_F(M,A) = 0$ then every finite order deformation of $T$ extends to a deformation of next order.
5. Fard-Guo functor for involutive algebras

In [10] Ebrahimi-Fard and Guo constructs the universal enveloping Rota-Baxter algebra of a dendriform algebra. Here we recall their construction and observe that it passes to the involutive case.

Let $B$ be a nonunitary associative algebra. Let $X$ be a basis for $B$, and let $X' = X \cup \{[,]\}$. Here $[$ and $]$ are two symbols, called brackets. Let $M(X')$ be the free semigroup generated by $X'$.

There is a sequence $\{X_n\}$ of subsets of $M(X')$ defined by the following recursive formula: $X_0 = X$ and for $n \geq 0$,

$$X_{n+1} = \left( \bigcup_{r \geq 1} (X[X_n])^r \right) \cup \left( \bigcup_{r \geq 0} (X[X_n])^r X \right) \cup \left( \bigcup_{r \geq 1} ([X_n]X)^r \right) \cup \left( \bigcup_{r \geq 0} ([X_n]X)^r [X_n] \right).$$

Then $X_{n+1} \supset X_n$, for $n \geq 0$. Define $X_\infty = \cup_{n \geq 0} X_n = \lim_{n} X_n$. The words of $X_\infty$ are called Rota-Baxter words. Every Rota-Baxter word $x \neq 1$ has a unique decomposition (called standard decomposition) $x = x_1 \cdots x_b$, where $x_i$, $1 \leq i \leq b$, is alternatively in $X$ or in $[X_\infty]$. The number $b$ is called the breadth of $x$, denoted by $b(x)$. We define the head $h(x)$ of $x$ to be 0 (resp. 1) if $x_1$ is in $X$ (resp. in $[X_\infty]$). Similarly, the tail $t(x)$ to be defined as 0 (resp. 1) if $x_0$ is in $X$ (resp. in $[X_\infty]$). Finally, the depth of $x$ is defined as $d(x) = \min \{n | x \in X_n\}$.

Define $\wp^{NC,0}_r(B) = \bigoplus_{x \in X_\infty} \kappa x$. For $x, x' \in X_\infty$ with $t(x) \neq h(x')$, we define a product $x \diamond x'$ by the concatenation. For $x, x' \in X_\infty$ with $t(x) = h(x')$, we define $x \diamond x'$ using the induction on $n = d(x) + d(x')$.

If $n = 0$, then $x, x'$ is in $X$, hence in $B$, and the product $x \diamond x' := x \cdot x'$ (the product in $B$). Suppose the product is defined for $n = k \geq 0$ and we want to define for $n = k + 1$. If $b(x) = b(x') = 1$, then

$$(8) \quad x \diamond x' = \begin{cases} x \cdot x' \text{ (the product in } B) & \text{if } x, x' \in X \\ xx' \text{ (concatenation)} & \text{if } x \in X, x' \in [X_\infty] \text{ or } x \in [X_\infty], x' \in X \\ [|[x] \diamond [x']| + [x] \diamond [x']] & \text{if } x = [x], x' \in [X_\infty]. \end{cases}$$

Finally, if $b(x) > 1$ or $b(x') > 1$, take $x = x_1 \cdots x_b$ and $x' = x'_1 \cdots x'_b$ be standard decompositions of $x$ and $x'$. In this case, we define

$$x \diamond x' = x_1 \cdots x_{b-1}(x_b \diamond x'_1) x'_2 \cdots x'_b,$$

where $x_b \diamond x'_1$ is defined by (8). Then $(\wp^{NC,0}_r(B), \diamond)$ is a nonunimodular associative algebra and $R_B : \wp^{NC,0}_r(B) \to \wp^{NC,0}_r(B)$ defined by $R_B(x) = [x]$, for $x \in X_\infty$ is a Rota-Baxter operator on $(\wp^{NC,0}_r(B), \diamond)$. We also consider the natural inclusion $j_X : X \to X_\infty \to \wp^{NC,0}_r(B)$ which extends to an injective algebra map $j_B : B \to \wp^{NC,0}_r(B)$.

For any vector space $V$, consider the tensor algebra $T(V) = \oplus_{n \geq 1} V^\otimes n$. Then $(\wp^{NC,0}_r(T(V)), \diamond, R_{T(V)})$ is a ‘free’ nonunimodular Rota-Baxter algebra over $V$ [10]. Let $(D, \langle , \rangle)$ be a dendriform algebra. Consider the free nonunimodular Rota-Baxter algebra $\wp^{NC,0}_r(T(D))$ over the vector space $D$. Let $R_B$ be the Rota-Baxter ideal of $\wp^{NC,0}_r(T(D))$ generated by the set

$$\langle x \prec y - x[y], \ x \succ y - [x]y \mid x, y \in D \rangle.$$

Then the quotient Rota-Baxter algebra $\wp^{NC,0}_r(T(D))/JR_B$ is the universal enveloping Rota-Baxter algebra of $D$.

Note that, if we start with a nonunimodular involutive associative algebra $B$, then $(\wp^{NC,0}_r(B), \diamond, R_B)$ can be given an involutive Rota-Baxter algebra with the involution given on basis elements by the involution on $B$ (when $x \in X \subset B$),

$$[x]^* = [x^*] \quad \text{and} \quad (x_1 \cdots x_b)^* = x_b^* \cdots x_1^*.$$

If $V$ is an involutive vector space, then $T(V)$ is an involutive algebra with involution $(v_1 \otimes \cdots \otimes v_n)^* = v_n^* \otimes \cdots \otimes v_1^*$. Hence $(\wp^{NC,0}_r(T(V)), \diamond, R_{T(V)})$ is an involutive Rota-Baxter algebra. This is free in the following sense [10].
5.1. Proposition. Let $V$ be an involutive vector space. Then for any nonunitary involutive Rota-Baxter algebra $A$ and a linear map $f : V \to A$ preserving involutions, there exists a unique nonunitary involutive Rota-Baxter algebra morphism $\tilde{f} : \varpi_{NC,0}(T(V)) \to A$ such that $\tilde{f} \circ (j_{T(V)} \circ i) = f$, where $i : V \to T(V)$ is the inclusion.

Finally, for an involutive dendriform algebra $(D, \prec, \succ, \ast)$, the ideal $J_R$ of the nonunitary involutive Rota-Baxter algebra $\varpi_{NC,0}(T(D))$ preserves under the involution as

$$(x \prec y - x[y])^* = y^* \succ x^* - [y^*]x^* \in J_R \quad \text{and} \quad (x \succ y - [x]y)^* = y^* \prec x^* - y^*[x^*] \in J_R.$$

Hence we get the following.

5.2. Proposition. If $(D, \prec, \succ, \ast)$ is an involutive dendriform algebra then the universal enveloping Rota-Baxter algebra $\varpi_{NC,0}(T(D))/J_R$ is involutive.

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