SINGULAR INTEGRALS OF SUBORDINATORS WITH APPLICATIONS TO STRUCTURAL PROPERTIES OF SPDES

CHANG-SONG DENG, RENÉ L. SCHILLING, AND LIHU XU

ABSTRACT. We study stochastic integrals driven by a general subordinator and establish a zero-one law for the finiteness of the resulting integral as well as moment estimates. As an application, we use these results to obtain structural properties of SPDEs driven by multiplicative pure jump noise, which include (1) a maximal inequality for a multiplicative stochastic convolution $Z_t$, (2) a small ball probability of $Z_t$, (3) the existence of invariant measures and accessibility to zero of SPDEs, and (4) a Galerkin approximation of solutions to SPDEs.

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1. INTRODUCTION

A subordinator $(S_t)_{t \geq 0}$ is an increasing Lévy process on $[0, \infty)$ starting at $S_0 = 0$. As usual, we use a càdlàg (finite left limits, right-continuous) modification of $S_t$. The law of a subordinator is determined by the Laplace transform of the random variables $S_t$. Because of the independent and stationary increments property of a subordinator, its Laplace transform is of the form

$$E \left[ e^{-\tau S_t} \right] = e^{-t \phi(r)}, \quad r > 0, \ t \geq 0,$$

where the exponent $\phi : (0, \infty) \to (0, \infty)$ is a Bernstein function with $\phi(0+) = 0$, i.e. a $C^\infty$-function such that $\phi \geq 0$ and with alternating derivatives $(-1)^{n+1}\phi^{(n)} \geq 0, \ n \in \mathbb{N}$. Every such $\phi$ has a unique Lévy–Khintchine representation

$$\phi(r) = br + \int_{(0,\infty)} \left(1 - e^{-rs}\right) \nu(ds)$$

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with a drift parameter $b \geq 0$ and a Lévy measure $\nu$, i.e. a Radon measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (1 + s) \nu(ds) < \infty$. We use [20] as a standard reference for Bernstein functions.

The parameters $b$ and $\nu$ also determine the structure of $S_t$ via the Lévy–Itô representation

$$S_t = bt + \sum_{0 < r \leq t} \Delta S_r,$$

where $\Delta S_r = S_r - S_r^-$ is the jump of $(S_t)_{t \geq 0}$ at time $t = r$. The jumps form a Poisson point process with intensity measure $dt \times \nu(ds)$; note that $S_0 = S_{0+}$, i.e. there is a.s. no instantaneous jump at time $t = 0$. It is well known that $t \mapsto S_t$ is a.s. strictly increasing if $b > 0$ or $\nu(0, \infty) = \infty$; this is also equivalent to saying that $\phi$ is an unbounded function.

Among the most important subordinators are the $\alpha$-stable subordinators $(0 < \alpha < 1)$ whose Bernstein functions are of the form $\phi(r) = r^\alpha$, i.e. $b = 0$ and $\nu(ds) = \alpha \Gamma(1 - \alpha)^{-1} s^{-\alpha-1} ds$. We refer the reader to [2, 3, 4, 7, 12, 15, 21] for results on $\alpha$-stable subordinated Brownian motion.

Since subordinators are a.s. increasing, we may use them as random time-changes of other stochastic processes. This procedure is called subordination (in the sense of Bochner) and it allows us to represent many Lévy processes as time-changed (“subordinated”) Brownian motions; in this way we can get, for example, all symmetric stable Lévy processes. Our standard reference for Lévy processes and subordinators is [1].

We are interested in (stochastic) integrals of the following form

$$\int_0^\infty f(t) \, dS_t,$$

where $f$ is a non-random integrand which may be a singular function, and we are going to establish a zero-one law for the finiteness and various moment formulas. A related zero-one law for integral functionals of spectrally positive Lévy processes can be found in [13].

As an application of these results we shall use them to study structural properties of SPDEs driven by multiplicative pure jump noise as follows. Let $(H, |\cdot|)$ be a separable Hilbert space, and $W = (W_t)_{t \geq 0}$ a cylindrical Wiener process on $H$ with filtration $(\mathcal{F}_t)_{t \geq 0}$, see e.g. [5]. We consider the following SPDE:

$$dX_t = [-AX_t + F(X_t)] \, dt + Q(X_t-) \, dW_t, \quad X_0 = x \in H,$$

where $S = (S_t)_{t \geq 0}$ is a subordinator with Bernstein function $\phi$. We assume that $S$ is independent of $W$; moreover we need:

(A1) $Q : H \to \mathcal{L}_{HS}(H)$ is a bounded, Lipschitz-continuous function, taking values in the set $\mathcal{L}_{HS}(H)$ of Hilbert–Schmidt operators on $H$, such that

$$\|Q\|_{HS, \infty} := \sup_{x \in H} \|Q(x)\|_{HS} < \infty,$$

$$\|Q(x) - Q(y)\|_{HS} \leq C|x - y| \quad \forall x, y \in H.$$

(A2) $A$ is a self-adjoint operator such that there exists an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of eigenvectors $A e_k = \gamma_k e_k$, $k \in \mathbb{N}$, and the eigenvalues satisfy

$$0 < \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n \leq \ldots, \quad \lim_{n \to \infty} \gamma_n = \infty.$$

(A3) $F : H \to H$ is a bounded, Lipschitz-continuous function.

If $(\mathcal{F})_{t \geq 0}$ is the filtration generated by $(S_t)_{t \geq 0}$ and $(\mathcal{G}_t)_{t \geq 0}$, then $(W_{S_t})_{t \geq 0}$ is adapted to the filtration $\mathcal{F}_{S_t} = \{ F \in \mathcal{F}_{\infty} : F \cap \{ S_t \leq r \} \in \mathcal{F}_r \text{ for all } r \geq 0 \}$. By a standard Picard
iteration argument, see e.g. [16, Theorems 9.29, Theorem 9.34], the conditions (A1)–(A3) ensure that there is a unique $H$-valued càdlàg process $(X_t)_{t \geq 0}$ which is adapted to the filtration $\mathcal{F}_t$ of the driving noise $(W_{S_t})_{t \geq 0}$, and satisfies the SPDE

$$
X_t = e^{-tA}x + \int_0^t e^{-(t-r)A}F(X_r) \, dr + \int_0^t e^{-(t-r)A}Q(X_r) \, dW_{S_r}.
$$

(1.3)

In order to study the SPDE (1.2) we need to understand the following stochastic convolution

$$
Z_t = \int_0^t e^{-(t-r)A}Q(X_{r-}) \, dW_{S_r}.
$$

(1.4)

Using our results on (1.1), we will prove a maximal inequality and a small ball probability estimate of $Z_t$ in Section 5, which is crucial for the proof of structural properties of $X_t$ and approximation results. The structural properties include (1) a maximal inequality of multiplicative stochastic convolution $Z_t$ (Theorem 5.3), (2) a small ball probability for $Z_t$ (Theorem 5.5), (3) the existence of invariant measures and accessibility to zero of the SPDE (Theorem 6.1, Theorem 7.3), and (4) a Galerkin approximation for the solution of the SPDE (Theorem 8.1). For the study of structural properties of SPDEs driven by a pure jump noise, we refer the reader to [17, 11, 16, 25] and the references therein.

For the readers’ convenience, let us briefly recall the following standard estimates which will be frequently used in the sequel. Denote by $\|A\| = \sup_{\|x\| = 1} |Ax|$ the operator norm induced by the Hilbert norm $\| \cdot \|$.

$$
\left\| A^\theta e^{-tA} \right\| \leq C e^{-\theta t} \quad \text{for all } \theta > 0,
$$

(1.5)

$$
|A^\theta x| \geq \gamma_1^\theta |x| \quad \text{for all } x \in H,
$$

(1.6)

$$
\left\| e^{-tA} \right\| \leq e^{-\gamma_1 t}.
$$

(1.7)

The first inequality is from [23, (3.2)] or [18, Lemma 2.3], the second and third inequalities are both due to the spectral gap of $A$. In fact, if $(e_n)_{n \in \mathbb{N}}$ is an orthonormal basis of $H$, we have $x = \sum_{k \in \mathbb{N}} a_k e_k$ with $a_k \in \mathbb{R}$ for each $k$ and

$$
|A^\theta x|^2 = \left| \sum_{k \in \mathbb{N}} a_k A^\theta e_k \right|^2 = \sum_{k \in \mathbb{N}} \gamma_k^\theta |a_k e_k|^2 = \sum_{k \in \mathbb{N}} \gamma_k^\theta |a_k|^2 \geq \gamma_1^\theta \sum_{k \in \mathbb{N}} |a_k|^2 = \gamma_1^\theta |x|^2.
$$

The other relations can be obtained by very similar arguments. As usual, we write $X \overset{d}{=} Y$ if the random variables $X$ and $Y$ have the same distribution.

2. A ZERO-ONE LAW FOR INTEGRALS DRIVEN BY SUBORDINATORS

Since the integrator $S_t$ is a.s. increasing, the integral (1.1) has a classical pathwise meaning as Lebesgue–Stieltjes integral and we can consider any measurable positive $f : (0, \infty) \to [0, \infty]$. The following simple but useful lemma on the characteristic functional of a subordinator will be crucial for our study.

Lemma 2.1 (characteristic functional). Let $(S_t)_{t \geq 0}$ be a subordinator with Bernstein function $\phi$. For any measurable and positive function $f : (0, \infty) \to [0, \infty)$ the following equality holds:

$$
\mathbb{E} \left( \exp \left[ - \int_0^\infty f(t) \, dS_t \right] \right) = \exp \left[ - \int_0^\infty \phi(f(t)) \, dt \right].
$$
Proof. Assume first that \( f(t) \) is a step function of the form \( \sum_{i=1}^{n} f_{i-1} \mathbf{1}_{[t_{i-1}, t_i)}(t) \), \( f_i \geq 0 \), \( 0 \leq t_0 < t_1 < \cdots < t_n < \infty \). Using the fact that a subordinator has stationary and independent increments gives

\[
E \left( \exp \left[ -\int_{0}^{\infty} f(t) \, dS_t \right] \right) = E \left( \exp \left[ -\sum_{i=1}^{n} f_{i-1} (S_{t_i} - S_{t_{i-1}}) \right] \right)
\]

\[
= \prod_{i=1}^{n} E \left[ e^{-f_{i-1}S_{t_{i-1}}} \right] = \prod_{i=1}^{n} e^{-(t_{i-1}-t_i)\phi(f_{i-1})}
\]

\[
= \exp \left[ -\sum_{i=1}^{n} \phi(f_{i-1}) (t_i - t_{i-1}) \right] = \exp \left[ -\int_{0}^{\infty} \phi(f(t)) \, dt \right].
\]

If \( f \) is a general positive measurable function such that \( \int_{0}^{\infty} \phi(f(t)) \, dt < \infty \), we can approximate \( f \), hence \( \phi \circ f \), in \( L^1((0, \infty); dt) \)-sense by step functions as above, and the claim follows by a standard density argument. If \( \int_{0}^{\infty} \phi(f(t)) \, dt = \infty \), we approximate \( \phi \) by the increasing sequence \( \phi_n(t) := \min\{\phi(t), n\} \mathbf{1}_{[0,n]}(t) \). Since \( \int_{0}^{\infty} \phi_n(f(t)) \, dt \leq n^2 \), we approximate this, as before, by step functions and use a monotone convergence theorem. \(\square\)

The first application of the characteristic functional is the following result on time reversals. Throughout the paper we will need the following elementary identities

\[
(2.1) \quad s^p = \frac{p}{\Gamma(1-p)} \int_{0}^{\infty} \left(1 - e^{-sr}\right) \frac{dr}{r^{p+1}}, \quad s \geq 0, \quad p \in (0, 1),
\]

\[
(2.2) \quad s^p = \frac{1}{\Gamma(-p)} \int_{0}^{\infty} e^{-sr} \frac{dr}{r^{p+1}}, \quad s > 0, \quad p < 0.
\]

**Corollary 2.2** (time reversal). Let \((S_t)_{t \geq 0}\) be a subordinator with Bernstein function \(\phi\), \(T > 0\), and \(f : (0, \infty) \to [0, \infty)\) a measurable function. For any \(-\infty < p < 1\) it holds that

\[
E \left[ \left( \int_{0}^{T} f(T-t) \, dS_t \right)^p \right] = E \left[ \left( \int_{0}^{T} f(t) \, dS_t \right)^p \right] \in [0, +\infty].
\]

**Proof.** The case \( p = 0 \) is trivial. Since

\[
\int_{0}^{T} \phi(f(T-t)) \, dt = \int_{0}^{T} \phi(f(t)) \, dt,
\]

the assertion follows immediately from Lemma 2.1, Tonelli’s theorem, and the identities (2.1), (2.2). \(\square\)

In the following two sections we will obtain conditions ensuring the finiteness of the moments appearing in Corollary 2.2.

If \( f : (0, \infty) \to [0, \infty) \) is a bounded measurable function, the integral \( \int_{0}^{\infty} f(t) \, dS_t \) is finite if, and only if, the tail integrals \( \int_{n}^{\infty} f(t) \, dS_t \), \( n \in \mathbb{N} \), are finite. This means that the set \( \{ \omega : \int_{0}^{\infty} f(t) \, dS_t(\omega) < \infty \} \) is a terminal event for the natural filtration of \((S_t)_{t \geq 0}\), hence it has probability either 0 or 1 by Kolmogorov’s zero-one law. The following result contains both a generalization (to all positive \( f \)) and a criterion to decide whether the probability is 1.

**Proposition 2.3** (zero-one law). Let \((S_t)_{t \geq 0}\) be a subordinator with Bernstein function \(\phi\) and \(f : (0, \infty) \to [0, \infty)\) a measurable function. The following assertions are equivalent:
i) $P\left(\int_0^\infty f(t) \, dS_t < \infty \right) > 0.$

ii) $P\left(\int_0^\infty f(t) \, dS_t < \infty \right) = 1.$

iii) $\int_0^\infty \phi(f(t)) \, dt < \infty.$

Proof. iii) $\Rightarrow$ ii): If we use Lemma 2.1 with $f$ replaced by $\lambda f$ for some $\lambda > 0$ and combine it with the monotone convergence theorem we get

$$P\left(\int_0^\infty f(t) \, dS_t < \infty \right) = \lim_{\lambda \to 0} E\left[\exp\left[-\lambda \int_0^\infty f(t) \, dS_t \right] \mathbf{1}_{\{\int_0^\infty f(t) \, dS_t < \infty \}} \right] = \lim_{\lambda \to 0} E\left[\exp\left[-\int_0^\infty \phi(\lambda f(t)) \, dt \right] \right] = 1.$$ 

The direction ii) $\Rightarrow$ i) is obvious, and i) $\Rightarrow$ iii) follows thus: Suppose that $\int_0^\infty \phi(f(t)) \, dt = \infty$. By Lemma 2.1,

$$E\left[\exp\left[-\int_0^\infty f(t) \, dS_t \right] \right] = 0,$$

hence $P\left(\int_0^\infty f(t) \, dS_t = \infty \right) = 1$, which contradicts i). This completes the proof. □

3. Moment Formulas for Singular Integrals Driven by a Stable Subordinator

Throughout this section $(S_t)_{t \geq 0}$ is an $\alpha$-stable subordinator; the corresponding Bernstein function is of the form $\phi(r) = r^\alpha$, $\alpha \in (0, 1)$. For the special case with $f(t) = t^{-\theta}$, $\theta \in (0, \infty)$ in Proposition 3.1 below, moment estimates have been established in [24].

**Proposition 3.1.** Let $S_t$ be an $\alpha$-stable subordinator, $0 < \alpha < 1$. If $f : (0, \infty) \to [0, \infty)$ is a measurable function such that $\text{Leb}\{f > 0\} > 0$, then

$$E\left[\left(\int_0^\infty f(t) \, dS_t \right)^p \right] = \begin{cases} \frac{\Gamma \left(1 - \frac{p}{\alpha} \right)}{\Gamma(1 - p)} \left(\int_0^\infty f(t)^\alpha \, dt \right)^{\frac{p}{\alpha}}, & \text{if } -\infty < p < \alpha, \\ \infty, & \text{if } p \geq \alpha. \end{cases}$$

Proof: Without loss of generality we may assume that $0 < \int_0^\infty f(t)^\alpha \, dt < \infty$ and $p \neq 0$. We distinguish between three cases.

**Case 1:** $0 < p < 1$. Combining the elementary identity (2.1) with Tonelli’s theorem and Lemma 2.1, yields

$$E\left[\left(\int_0^\infty f(t) \, dS_t \right)^p \right] = \frac{p}{\Gamma(1 - p)} \int_0^\infty \left(1 - e^{-r f(t)} \right)^\frac{d}{\Gamma(1 - p)} \frac{\, dr}{r^{p+1}} = \frac{p}{\Gamma(1 - p)} \int_0^\infty \left(1 - e^{-r} f(t)^\alpha \right)^\frac{d}{\Gamma(1 - p)} \frac{\, dr}{r^{p+1}}.$$
If we change variables according to $s = r^\alpha \int_0^\infty f(t)^\alpha \, dt$ and use (2.1) once again, we obtain

$$E \left[ \left( \int_0^\infty f(t) \, dS_t \right)^p \right] = \frac{\Gamma \left( \frac{1 - \frac{p}{\alpha}}{1 - \alpha \theta} \right) \Gamma(1 - p) \Gamma(1 - \alpha \theta)}{\alpha \Gamma(1 - \alpha \theta) \Gamma(1 - p)} \left( \int_0^\infty f(t)^\alpha \, dt \right)^{\frac{p}{\alpha}} \Gamma(1 - p) \frac{\Gamma(1 - p)}{\Gamma(1 - p)} \left( \int_0^\infty f(t)^\alpha \, dt \right)^{\frac{p}{\alpha}} \frac{1}{\Gamma(1 - p)},$$

if $p \in (0, \alpha)$,

$$\infty,$$

if $p \in [\alpha, 1)$.

Case 2: $p \geq 1$. It follows from Jensen’s inequality and the first case that

$$E \left[ \left( \int_0^\infty f(t) \, dS_t \right)^p \right] \geq \left( E \left[ \left( \int_0^\infty f(t) \, dS_t \right)^\alpha \right] \right)^{\frac{p}{\alpha}} \Gamma(1 - \alpha \theta) \Gamma(1 - p) \frac{\Gamma(1 - p)}{\Gamma(1 - p)} \left( \int_0^\infty f(t)^\alpha \, dt \right)^{\frac{p}{\alpha}} \frac{1}{\Gamma(1 - p)};$$

in the last equality we use the functional equation $\Gamma(1 + r) = r\Gamma(r)$ of the Gamma-function.

**Corollary 3.2.** Let $S_t$ be an $\alpha$-stable subordinator, $0 < \alpha < 1$, $p, \theta \in \mathbb{R}$ and $T > 0$.

i) According to $\theta < \frac{1}{\alpha}$ or $\theta \geq \frac{1}{\alpha}$ one has with probability one

$$\int_0^T t^{-\theta} \, dS_t < \infty, \quad \text{resp.}, \quad = \infty,$$

and

$$E \left[ \left( \int_0^T t^{-\theta} \, dS_t \right)^p \right] = \begin{cases} \frac{\Gamma \left( \frac{1 - \frac{p}{\alpha}}{1 - \alpha \theta} \right) T^{p \left( \frac{1}{\alpha} - \theta \right)}}{(1 - \alpha \theta)^{\frac{p}{\alpha}} \Gamma(1 - \alpha \theta) \Gamma(1 - p)}, & \text{if } \theta < \frac{1}{\alpha} \text{ and } p < \alpha, \\ 0, & \text{if } \theta \geq \frac{1}{\alpha} \text{ and } p < 0, \\ 1, & \text{if } \theta \geq \frac{1}{\alpha} \text{ and } p = 0 \\ \infty, & \text{if } \theta \geq \frac{1}{\alpha} \text{ and } p > 0. \end{cases}$$

ii) According to $\theta > \frac{1}{\alpha}$ or $\theta \leq \frac{1}{\alpha}$ one has with probability one

$$\int_T^\infty t^{-\theta} \, dS_t < \infty, \quad \text{resp.}, \quad = \infty,$$
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and

\[
E \left[ \left( \int_{0}^{T} t^{-\theta} \, dS_t \right)^p \right] = \begin{cases} 
\frac{\Gamma \left( 1 - \frac{p}{\alpha} \right)}{(\alpha\theta - 1)^{\frac{p}{\alpha}} \Gamma(1 - p)} T^{p(1 - \theta)}, & \text{if } \theta > \frac{1}{\alpha} \& p < \alpha, \\
0, & \text{if } \theta \leq \frac{1}{\alpha} \& p < 0, \\
1, & \text{if } \theta \leq \frac{1}{\alpha} \& p = 0, \\
\infty, & \text{if } \theta \leq \frac{1}{\alpha} \& p > 0.
\end{cases}
\]

iii) For all \( \lambda > 0 \) one has

\[
E \left[ \left( \int_{0}^{T} e^{-\lambda t} \, dS_t \right)^p \right] = \begin{cases} 
\frac{\Gamma \left( 1 - \frac{p}{\alpha} \right)}{\Gamma(1 - p)} \left( \frac{1 - e^{-\alpha\lambda T}}{\alpha\lambda} \right)^{\frac{p}{\alpha}}, & \text{if } p < \alpha, \\
\infty, & \text{if } p \geq \alpha.
\end{cases}
\]

Proof. The assertions i) and ii) follow from Lemma 2.1 and Proposition 3.1 with \( f(t) = t^{-\theta} \mathbb{1}_{(0,T]}(t) \) and \( f(t) = t^{-\theta} \mathbb{1}_{(T,\infty)}(t) \). In a similar way iii) can be obtained from Proposition 3.1 if we use \( f(t) = e^{-\lambda t} \mathbb{1}_{(0,T]}(t) \).

\[\square\]

4. MOMENT ESTIMATES FOR SINGULAR INTEGRALS DRIVEN BY A GENERAL SUBORDINATOR

We will now consider a subordinator \((S_t)_{t \geq 0}\) with Bernstein function \( \phi \). Other than in the stable case, we cannot hope for exact moment formulae. Therefore we aim for estimates of the following type:

\[(4.1)\]

\[
E \left[ \left( \int_{0}^{T} t^{-\theta} \, dS_t \right)^p \right] \leq C T^{-p\theta} \left[ \phi^{-1} \left( \frac{1}{T} \right) \right]^{-p},
\]

\[(4.2)\]

\[
E [S_T^p] \leq C \left[ \phi^{-1} \left( \frac{1}{T} \right) \right]^{-p},
\]

\[(4.3)\]

\[
E \left[ \left( \int_{0}^{T} e^{-\lambda t} \, dS_t \right)^p \right] \leq C \left[ \phi^{-1} \left( \frac{1}{T \wedge 1} \right) \right]^{-p},
\]

with constants \( C \) depending on \( p \in \mathbb{R}, \theta \geq 0 \) and \( \lambda > 0 \).

Proposition 4.1. Let \( S_t \) be a subordinator with Bernstein function \( \phi \).

i) The estimate (4.1) holds for \( p \leq 0, \theta \geq 0 \) and all \( T \in [1, \infty) \), if

\[
\liminf_{s \to \infty} \frac{\phi(s)}{\log s} > 0 \quad \text{and} \quad \liminf_{s \to 0} \frac{\phi(2s)}{\phi(s)} > 1.
\]

ii) The estimate (4.1) holds for \( p \leq 0, \theta \geq 0 \) and all \( T \in (0,1] \), if

\[
\liminf_{s \to \infty} \frac{\phi(2s)}{\phi(s)} > 1.
\]

iii) The estimate (4.2) holds for all \( T > 0 \) [resp. \( T \geq 1 \)] if

\[(4.4)\]

\[
0 \leq p < \log_2 \left( \inf_{s > 0} \frac{\phi(2s)}{\phi(s)} \right) \quad \text{[resp.} \quad 0 \leq p < \log_2 \left( \liminf_{s \to 0} \frac{\phi(2s)}{\phi(s)} \right)\].
\]

iv) The estimate (4.1) holds for all \( T \geq 1 \) if

\[(4.5)\]

\[
0 \leq p < \log_2 \left( \liminf_{s \to 0} \frac{\phi(2s)}{\phi(s)} \right) \quad \text{and} \quad 0 \leq \theta < \left[ \log_2 \left( \sup_{s > 0} \frac{\phi(2s)}{\phi(s)} \right) \right]^{-1}.
\]
v) The estimate (4.1) holds for all $T \in (0, 1]$ if
\[
0 \leq p < \log_2 \left( \inf_{s > 0} \frac{\phi(2s)}{\phi(s)} \right) \quad \text{and} \quad 0 < \theta < \left[ \log_2 \left( \limsup_{s \to \infty} \frac{\phi(2s)}{\phi(s)} \right) \right]^{-1}.
\]

vi) The estimate (4.3) holds for all $T > 0$, $\lambda > 0$ and $p < 0$, if
\[
\liminf_{s \to \infty} \frac{\phi(2s)}{\phi(s)} > 1.
\]

vii) If $p > 0$ and $\lambda > 0$, then
\[
\int_0^1 \frac{\phi(s)}{s^{p+1}} ds < \infty \iff E \left[ \left( \int_0^\infty e^{-\lambda t} dS_t \right)^p \right] < \infty.
\]

Before we are going to prove Proposition 4.1 we will add a few remarks on the assumptions made in this proposition and give some examples.

**Remark 4.2.**

i) Corollary 3.2 shows that all assertions of Proposition 4.1 are sharp for $\alpha$-stable subordinators.

ii) Since Bernstein functions are subadditive, we have $\phi(2s) \leq 2\phi(s)$ for all $s > 0$. This means that both (4.4) and the first condition in (4.5) imply $p \in [0, 1)$.

iii) Since Bernstein functions are concave, we get
\[
\phi'(s) \leq \frac{\phi(s)}{s}, \quad s > 0,
\]
and, therefore,
\[
\int_0^1 \frac{\phi(s)}{s^{p+1}} ds \geq \int_0^1 \frac{\phi'(s)}{s^p} ds \geq \phi'(1) \int_0^1 \frac{ds}{s^p}.
\]
This means that (4.7) can only happen if $p < 1$.

iv) The condition (4.4) implies that there is some $\tilde{p} > p$ such that for all $s > 0$
\[
\frac{\phi(2s)}{\phi(s)} > 2^{\tilde{p}} \quad \text{and} \quad \text{for all } k \in \mathbb{N} \cup \{0\}, \quad \phi \left( 2^{-k}s \right) \leq 2^{-k\tilde{p}} \phi(s).
\]
A routine monotonicity argument shows that this implies
\[
\phi \left( 2^{-x}s \right) \leq 2^{\tilde{p}} 2^{-x \tilde{p}} \phi(s) \quad \text{for all } x \geq 0 \text{ and } s > 0.
\]
Under the alternative condition, this estimate is still valid for small values $0 < s < s_0$.

v) The second condition in (4.5) implies that there is some $0 < \tilde{\theta} < 1/\theta$ such that
\[
\phi \left( 2^k s \right) \leq 2^{\theta k} \phi(s) \quad \text{for all } k \in \mathbb{N} \text{ and } s > 0.
\]
A routine monotonicity argument shows that this implies
\[
\phi \left( 2^{x} s \right) \leq 2^{1/\theta \tilde{\theta} x} \phi(s) \quad \text{for all } x \geq 0 \text{ and } s > 0.
\]
If we assume, instead, the weaker second condition in (4.6), the estimate (4.9) is still valid for large values $s > s_0$.

**Example 4.3.** From [20, Proposition 7.16(ii)] and [20, table entry 16.2.6] we know that the functions
\[
\phi(s) = s^\alpha \log^\beta (1 + s), \quad 0 < \alpha < 1, \quad 0 \leq \beta \leq 1 - \alpha
\]
\[
\psi(s) = s^\alpha \log^{-\beta} (1 + s), \quad 0 \leq \beta \leq \alpha < 1
\]
\[ \omega(s) = s(1 + s)^{-\alpha}, \quad 0 < \alpha < 1 \]

are (complete) Bernstein functions. The results of Proposition 4.1 are summarized for these functions in Table 1.

**Table 1.** Overview of the results of Proposition 4.1 for some concrete examples.

| Estimate | \( s^\alpha \log^2(1 + s) \) | \( s^\alpha \log^{-\beta}(1 + s) \) | \( s(1 + s)^{-\alpha} \) |
|----------|-----------------|-----------------|-----------------|
| (4.1), \( T > 0 \) | \( p \leq 0, \theta \geq 0 \) | \( p \leq 0, \theta \geq 0 \) | \( p \leq 0, \theta \geq 0 \) |
| (4.1), \( T \geq 1 \) | \( 0 \leq p < \alpha + \beta \) | \( 0 \leq p < \alpha - \beta \) | \( 0 \leq p < 1 \) |
| \( 0 \leq \theta < (\alpha + \beta)^{-1} \) | \( 0 \leq \theta < \alpha^{-1} \) | \( 0 \leq \theta < 1 \) |
| (4.1), \( T \leq 1 \) | \( 0 \leq p < \alpha \) | \( 0 \leq p < \alpha - \beta \) | \( 0 \leq p < 1 - \alpha \) |
| \( 0 \leq \theta < \alpha^{-1} \) | \( 0 \leq \theta < \alpha^{-1} \) | \( 0 \leq \theta < (1 - \alpha)^{-1} \) |
| (4.2), \( T > 0 \) | \( 0 \leq p < \alpha \) | \( 0 \leq p < \alpha - \beta \) | \( 0 \leq p < 1 - \alpha \) |
| (4.3), \( T > 0 \) | \( p \leq 0, \lambda > 0 \) | \( p \leq 0, \lambda > 0 \) | \( p \leq 0, \lambda > 0 \) |
| (4.7) applies | \( 0 \leq p < \alpha + \beta \) | \( 0 \leq p < \alpha - \beta \) | \( 0 \leq p < 1 \) |
| \( \lambda > 0 \) | \( \lambda > 0 \) | \( \lambda > 0 \) |

**Proof of Proposition 4.1.** i) & ii): Since \( p < 0 \), the monotonicity of the integral gives

\[ E \left[ \left( \int_0^T t^{-\theta} \, dS_t \right)^p \right] \leq E \left[ \left( \int_0^T T^{-\theta} \, dS_t \right)^p \right] = T^{-\theta p} E \left[ S_T^p \right], \]

so i) and ii) follow from the moment estimates in [8, Theorem 2.1(ii)].

iii) By (2.1), Tonelli's theorem, Lemma 2.1, and the inequality \( 1 - e^{-r} \leq 1 \wedge r, r \geq 0 \), we get for any \( p \in (0, 1) \) and \( T > 0 \)

\[
\Gamma(1 - p) E \left[ S_T^p \right] = p \int_0^{\infty} \left( 1 - e^{-r} S_r \right) \frac{dr}{r^{p+1}} \cdot
\]

\[
= p \int_0^{\infty} \left( 1 - e^{-r} \phi(r) \right) \frac{dr}{r^{p+1}} \leq p T \int_0^{\phi^{-1}(\frac{1}{T})} \phi(r) \frac{dr}{r^{p+1}} + p \int_{\phi^{-1}(\frac{1}{T})}^{\infty} \frac{dr}{r^{p+1}} \cdot
\]

\[
= p T \int_0^{\phi^{-1}(\frac{1}{T})} \phi(r) \frac{dr}{r^{p+1}} + \left[ \phi^{-1} \left( \frac{1}{T} \right) \right]^{-p} \cdot
\]

Since \( \phi(0+) = 0 \), we get using integration by parts,

\[
p \int_0^{\phi^{-1}(\frac{1}{T})} \phi(r) \frac{dr}{r^{p+1}} = \int_0^{\phi^{-1}(\frac{1}{T})} s^{-p} \, ds \, \phi(s) - \frac{1}{T} \left[ \phi^{-1} \left( \frac{1}{T} \right) \right]^{-p}.\]
Note that
\[
\int_0^{\phi^{-1}(\frac{1}{T})} s^{-p} \, d\phi(s) = \sum_{k=0}^{\infty} \int_{2^{-(k+1)} \phi^{-1}(\frac{1}{T})}^{2^{-k} \phi^{-1}(\frac{1}{T})} s^{-p} \, d\phi(s)
\]
(4.12)
\[
\leq \sum_{k=0}^{\infty} \left( \frac{1}{2^{k+1}} \phi^{-1}(\frac{1}{T}) \right)^{-p} \phi\left(\frac{1}{2^k} \phi^{-1}(\frac{1}{T})\right) = 2^p \left[ \phi^{-1}(\frac{1}{T}) \right]^{-p} \sum_{k=0}^{\infty} 2^{pk} \phi\left(\frac{1}{2^k} \phi^{-1}(\frac{1}{T})\right).
\]

Combining (4.10), (4.11) and (4.12), we get for any \( p \in (0, 1) \) and \( T > 0 \),
\[
(4.13) \quad \Gamma(1 - p) \mathbb{E}\left[ S_T^p \right] \leq 2^p T \left[ \phi^{-1}(\frac{1}{T}) \right]^{-p} \sum_{k=0}^{\infty} 2^{pk} \phi\left(\frac{1}{2^k} \phi^{-1}(\frac{1}{T})\right).
\]
Since we assume (4.4), we may use the estimate from Remark 4.2.iv) in (4.13), and this gives for all \( T > 0 \) [resp. \( T \geq 1 \)]
\[
(4.14) \quad \Gamma(1 - p) \mathbb{E}\left[ S_T^p \right] \leq \left( 2^p \sum_{k=0}^{\infty} 2^{-(\frac{p}{2} + 1)k} \right) \left[ \phi^{-1}(\frac{1}{T}) \right]^{-p}.
\]
iv) If \( \theta = 0 \), we are in the situation of part iii) with \( T \geq 1 \).
Assume that \( 0 < \theta < 1/p \). As in (4.10), we have for any \( p \in (0, 1) \) and \( T > 0 \)
\[
\begin{align*}
\Gamma(1 - p) \mathbb{E}\left[ \left( \int_0^{r} t^{-\theta} \, dS_t \right)^p \right] & = \frac{1}{\theta} \int_0^{T^\theta \phi^{-1}(\frac{1}{T})} \left( \int_0^T \phi(r t^{-\theta}) \, dt \right) \frac{dr}{r^{p+1}} \\
& \leq \frac{1}{\theta} \int_0^{T^\theta \phi^{-1}(\frac{1}{T})} \left( \int_0^T \phi(r t^{-\theta}) \, dt \right) \frac{dr}{r^{p+1}} + p \int_{T^\theta \phi^{-1}(\frac{1}{T})}^{\infty} \frac{dr}{r^{p+1}} \\
& = \frac{1}{\theta} \int_0^{T^\theta \phi^{-1}(\frac{1}{T})} \left( \int_0^T \phi(r t^{-\theta}) \, dt \right) \frac{dr}{r^{p+1}} + p \int_{T^\theta \phi^{-1}(\frac{1}{T})}^{\infty} \frac{dr}{r^{p+1}} + T^{-p\theta} \left[ \phi^{-1}(\frac{1}{T}) \right]^{-p} \\
& = \frac{T^{1-p\theta}}{1 - p\theta} \int_0^{\phi^{-1}(\frac{1}{T})} \phi(s) \frac{ds}{s^{p+1}} + T^{-p\theta} \left[ \phi^{-1}(\frac{1}{T}) \right]^{-p}.
\end{align*}
\]
In order to estimate the middle term in the above expression, we use integration by parts and get

\[
\int_{\phi^{-1}(\frac{1}{T})}^{\infty} \phi(s) \frac{ds}{s^{\frac{1}{p}+1}} = \theta \int_{\phi^{-1}(\frac{1}{T})}^{\infty} r^{-\frac{1}{p}} d\phi(r) + \frac{\theta}{T} \left[ \phi^{-1} \left( \frac{1}{T} \right) \right]^{-\frac{1}{p}}
\]

\[
= \theta \sum_{k=0}^{\infty} \int_{2^k \phi^{-1}(\frac{1}{T})}^{2^{k+1} \phi^{-1}(\frac{1}{T})} r^{-\frac{1}{p}} d\phi(r) + \frac{\theta}{T} \left[ \phi^{-1} \left( \frac{1}{T} \right) \right]^{-\frac{1}{p}}
\]

\[
\leq \theta \left[ \phi^{-1} \left( \frac{1}{T} \right) \right]^{-\frac{1}{p}} \sum_{k=0}^{\infty} 2^{-\frac{k}{p}} \phi \left( 2^{k+1} \phi^{-1} \left( \frac{1}{T} \right) \right) + \frac{\theta}{T} \left[ \phi^{-1} \left( \frac{1}{T} \right) \right]^{-\frac{1}{p}}.
\]

Using (4.11) and (4.12) for the first integral, we obtain for any \( p \in (0, 1), \theta \in (0, 1/p) \) and \( T > 0 \),

(4.15)

\[
(1 - p\theta) \Gamma(1 - p) E \left[ \left( \int_{0}^{T} t^{-\theta} dS_t \right)^p \right] \leq T^{1-p\theta} \left[ \phi^{-1} \left( \frac{1}{T} \right) \right]^p \left[ 2^p \sum_{k=0}^{\infty} 2^{p(k-1)} \phi \left( \frac{1}{2^k} \phi^{-1} \left( \frac{1}{T} \right) \right) + p\theta \sum_{k=0}^{\infty} 2^{-\frac{k}{p}} \phi \left( 2^{k+1} \phi^{-1} \left( \frac{1}{T} \right) \right) \right].
\]

The conditions (4.5) allow us (cf. Remark 4.2.iv, v)) to estimate the terms under the sum for some \( \tilde{p} > p \) and \( \tilde{\theta} < 1/\theta \) for all large values of \( T \), say \( T \geq 1 \). Therefore,

(4.16)

\[
(1 - p\theta) \Gamma(1 - p) E \left[ \left( \int_{0}^{T} t^{-\theta} dS_t \right)^p \right] \leq \left( 2^p \sum_{k=0}^{\infty} 2^{-(\tilde{p}-p)k} + p\theta \sum_{k=0}^{\infty} 2^{-\left(\frac{1}{p} - \tilde{\theta}\right)k} \right) T^{-p\theta} \left[ \phi^{-1} \left( \frac{1}{T} \right) \right]^{-p},
\]

and iv) follows.

v) If we replace in the proof of iii) the conditions (4.5) by (4.6), we get from (4.15) that (4.16) holds for small \( T \), say \( T \leq 1 \), and v) follows.

vi) Since \( p < 0 \), we get by monotonicity

\[
E \left[ \left( \int_{0}^{T} e^{-\lambda t} dS_t \right)^p \right] \leq E \left[ \left( \int_{0}^{T \land 1} e^{-\lambda t} dS_t \right)^p \right] \leq e^{-p\lambda(T \land 1)} E \left[ S_{T \land 1}^p \right] \leq e^{-p\lambda} E \left[ S_{T \land 1}^p \right] .
\]

Under the condition \( \lim \inf_{s \to \infty} \phi(2s)/\phi(s) > 1 \) there is some constant \( C_p \) such that

\[
E \left[ S_{T \land 1}^p \right] \leq C_p \left[ \phi^{-1} \left( \frac{1}{T \land 1} \right) \right]^{-p}, \quad T > 0,
\]

see [8, Theorem 2.1 (ii) (c)], and we get vi).

vii) In view of Remark 4.2.iii) we may assume that \( 0 < p < 1 \). We see with (2.1), Tonelli’s theorem and Lemma 2.1

(4.17)

\[
\Gamma(1-p) E \left[ \left( \int_{0}^{\infty} e^{-\lambda t} dS_t \right)^p \right] = pE \left[ \int_{0}^{\infty} \left( 1 - \exp \left[ -r \int_{0}^{\infty} e^{-\lambda t} dS_t \right] \right) dr \right]_{r=p+1} \]

\[
= p \int_{0}^{\infty} \left( 1 - \exp \left[ - \int_{0}^{\infty} \phi \left( re^{-\lambda t} \right) dt \right] \right) dr \right]_{r=p+1}.
\]
Note the following elementary inequalities

\[ (4.18) \quad \frac{1}{2} (1 \wedge x) \leq (1 - e^{-1})(1 \wedge x) \leq 1 - e^{-x} \leq 1 \wedge x, \quad x \geq 0. \]

Assume that \( \int_0^1 \phi(s)s^{-1-p} \, ds < \infty \). Using in (4.17) the upper estimate from (4.18) we get

\[
\begin{aligned}
\Gamma(1-p)E\left[\left(\int_0^\infty e^{-\lambda t} \, dS_t\right)^{p}\right] &\leq p \int_0^1 \left( \int_0^\infty \phi \left( r e^{-\lambda t} \right) \, dt \right) \frac{dr}{p^{p+1}} + p \int_1^\infty \frac{dr}{p^{p+1}} \\
&= \frac{p}{\lambda} \int_0^1 \left( \int_s^\infty \frac{dr}{p^{p+1}} \right) \frac{\phi(s)}{s} \, ds + 1 \\
&\leq \frac{p}{\lambda} \int_0^1 \left( \int_s^\infty \frac{dr}{p^{p+1}} \right) \frac{\phi(s)}{s} \, ds + 1 \\
&= \frac{1}{\lambda} \int_0^1 \frac{\phi(s)}{s^{p+1}} \, ds + 1.
\end{aligned}
\]

This proves the direction “⇒” in (4.7).

In order to see the other implication, we assume that \( E\left[\left(\int_0^\infty e^{-\lambda t} \, dS_t\right)^{p}\right] < \infty \). Because of Proposition 2.3 (applied with \( f(t) = e^{-\lambda t} \)) this means that \( \int_0^1 \phi(s)s^{-1} \, ds = \lambda \int_0^\infty \phi \left( e^{-\lambda t} \right) \, dt < \infty \). Applying the lower estimate from (4.18) to (4.17) gives

\[
\begin{aligned}
\Gamma(1-p)E\left[\left(\int_0^\infty e^{-\lambda t} \, dS_t\right)^{p}\right] &\geq \frac{p}{2} \int_0^\infty 1 \wedge \left( \int_0^\infty \phi \left( r e^{-\lambda t} \right) \, dt \right) \frac{dr}{r^{p+1}} \\
&= \frac{p}{2} \int_0^\infty 1 \wedge \left( \frac{1}{\lambda} \int_s^r \frac{\phi(s)}{s} \, ds \right) \frac{dr}{r^{p+1}} \\
&\geq \frac{p}{2\lambda} \int_0^{r_0} \left( \int_0^r \frac{\phi(s)}{s} \, ds \right) \frac{dr}{r^{p+1}},
\end{aligned}
\]

where \( r_0 = r_0(\lambda) > 0 \) is so small that \( \int_0^{r_0} \phi(s) s^{-1} \, ds \leq \lambda \). Thus, by Tonelli’s theorem,

\[
\begin{aligned}
\Gamma(1-p)E\left[\left(\int_0^\infty e^{-\lambda t} \, dS_t\right)^{p}\right] &\geq \frac{p}{2\lambda} \int_0^{r_0} \left( \int_s^{r_0} \frac{dr}{r^{p+1}} \right) \frac{\phi(s)}{s} \, ds \\
&= \frac{1}{2\lambda} \int_0^{r_0} \frac{\phi(s)}{s^{p+1}} \, ds - \frac{1}{2\lambda r_0^p} \int_0^{r_0} \phi(s) \, ds.
\end{aligned}
\]

Since the second summand is finite, the proof is complete. \( \square \)

5. STOCHASTIC CONVOLUTIONS

Recall that \( W = (W_t)_{t \geq 0} \) is a cylindrical Brownian motion with values in a Hilbert space \( H \), and that \( S = (S_t)_{t \geq 0} \) is a subordinator which is independent of \( W \). We assume that \( t \mapsto S_t \) is a.s. strictly increasing; this is equivalent to assuming that \( \lim_{\xi \to \infty} \phi(\xi) = \infty \). In order to justify the method of conditioning on the subordinator \( S \), we introduce in this section a product construction which will allow us to freeze the subordinator, see e.g. [26, 22].

Let \( \Omega^W \) be the space of all continuous functions from \( \omega : [0, \infty) \to H, t \mapsto \omega_t \), which vanish at \( t = 0 \); we endow \( \Omega^W \) with the topology of locally uniform convergence and the
Wiener measure $\mathbb{P}^W$; under $\mathbb{P}^W$, the canonical process $(\omega_t)_{t \geq 0}$ is a cylindrical Brownian motion valued on $H$, that is

$$W_t(\omega) = \omega_t, \quad t \geq 0.$$  

Similarly, we construct a canonical realization of the subordinator $(S_t)_{t \geq 0}$ on the space $\Omega^S$ of all strictly increasing càdlàg functions $t : [0, \infty) \to [0, \infty)$, $t \mapsto \ell_t$, such that $\ell_0 = 0$; we endow $S$ with the Skorohod topology and an probability measure $\mathbb{P}^S$, such that $(S_t)_{t \geq 0}$ is the canonical coordinate process

$$S_t(\ell) = \ell_t, \quad t \geq 0.$$  

We consider the SPDE (1.2) on the product space

$$(\Omega, \mathcal{F}, \mathbb{P}) := (\Omega^W \times \Omega^S, \mathcal{B}(\Omega^W \times \Omega^S), \mathbb{P}^W \otimes \mathbb{P}^S).$$

Recall the unique solution $(X_t)_{t \geq 0}$ of the SPDE (1.2) is defined as (1.3), and the stochastic convolution $(Z_t)_{t \geq 0}$ is defined as (1.4). For a $\ell \in S$, we have $S_t(\ell) = \ell_t$ for $t \geq 0$, i.e., the subordinator takes a sample path $(\ell_t)_{t \geq 0}$, let us consider the following SDE in the probability space $(\Omega^W, \mathcal{B}(\Omega^W), \mathbb{P}^W)$:

$$(5.1) \quad dX_t^\ell = [-AX_t^\ell + F(X_t^\ell)] \, dt + Q(X_t^{\ell -}) \, dW_{\ell_t}, \quad X_0^\ell = x.$$  

Note that $(W_{\ell_t})_{t \geq 0}$ is a càdlàg martingale (with deterministic jump-times) for the filtration $\mathcal{G}_{\ell_t}$ where $(\mathcal{G}_{\ell_t})_{t \geq 0}$ is the filtration of the cylindrical Brownian motion $\mathbb{W}$. Identifying cylindrical Brownian motion with a Hilbert space valued Wiener process (in general, a larger Hilbert space $U \supset H$, cf. [16, Theorem 7.13]) we can use the results in [16, §8.1] to see that the bracket of the time-changed process satisfies $\langle W_{t}\ell, W_{t}\ell \rangle = \langle W_{t}, W_{t} \rangle_{\ell_t}$, and the same relation holds for the operator (or tensor) bracket.

By [16, p.142], the SPDE (5.1) has a unique mild solution given by

$$X_t^\ell(x) = e^{-tA}x + \int_0^t e^{-(t-s)A}F(X_s^\ell(x)) \, ds + Z_t^\ell$$

where $Z_t^\ell$ is the following stochastic convolution:

$$(5.2) \quad Z_t^\ell = \int_0^t e^{-(t-s)A}Q(X_{s-}^\ell(x)) \, dW_{\ell_s}.$$  

To keep notation simple, we will suppress the initial condition $x$ and write $X_t^\ell_{s-} = X_{s-}^\ell(x)$. Moreover, we use $X = (X_t)_{t \geq 0}$, $Z = (Z_t)_{t \geq 0}$, $X^\ell = (X^\ell_t)_{t \geq 0}$ and $Z^\ell = (Z^\ell_t)_{t \geq 0}$.

**Lemma 5.1.** Let $F$ and $G$ be measurable functionals of $X$ and $Z$ respectively. We have

$$\mathbb{E}[F(X)] = \mathbb{E}^S \left[ \mathbb{E}^W(F(X_t^\ell)) \right]_{t \geq S},$$

$$\mathbb{E}[G(Z)] = \mathbb{E}^S \left[ \mathbb{E}^W(F(Z_t^\ell)) \right]_{t \geq S}. $$

**Proof.** Because of (1.3), $X$ is a measurable map of $x$ and $W_S$ which we will denote by $X := X(x, W_S)$. We have

$$\mathbb{E}[F(X)] = \mathbb{E}^{\mathbb{P}^W \times \mathbb{P}^S} \left[ F(X(x, W_S)) \right] = \int_S \mathbb{E}^W \left[ F(\tilde{X}(x, W_t)) \right] \mathbb{P}^S(d\ell) = \int_S \mathbb{E}^W \left[ F(X^\ell_t(x)) \right] \mathbb{P}^S(d\ell)$$
where the last equality follows from the fact that $\bar{X}(x, W_t)$ is the solution to Equation (5.1).

It is easy to see that $Z = (Z_t)_{t \geq 0}$ is a measurable functional of $X$ and $W_S$; we denote it by $Z = \bar{Z}(X, W_S)$, so for any measurable function $G$ of $Z$, we have

$$E[G(Z)] = E_{P^W \times P^S}[G(\bar{Z}(X, W_S))]$$

where the last equality follows from the fact that $Z = \bar{Z}(X, W_S)$.

5.1. Estimate of the $p$-th moment of $Z_t$. From Lemma 5.1, in order to estimate $X_t$ and $Z_t$, we can first estimate $X^\ell_t$ and $Z^\ell_t$ for a given $\ell \in S$ and then integrate the estimations over $S$.

**Theorem 5.2.** Let $(W_t)_{t \geq 0}$, $(S_t)_{t \geq 0}$ and $(Z_t)_{t \geq 0}$ be as above and assume (A1) and (A2).

i) Assume that $\inf_{s > 0} \phi(2s)/\phi(s) > 1$ and let $p, \theta > 0$ be such that

$$\frac{p}{2} < \log_2 \left( \frac{\phi(2s)}{\phi(s)} \right) \leq \log_2 \left( \limsup_{s \to \infty} \frac{\phi(2s)}{\phi(s)} \right) < \frac{1}{2\theta}.$$

There exists a constant $C = C(p, \theta) > 0$ such that

$$E[|A^\theta Z_t|^p] \leq Ct^{p\theta} \left[ \phi^{-1} \left( \frac{1}{t} \right) \right]^{-\frac{p}{2\theta}} \quad \text{for all } t \in (0, 1].$$

ii) Assume that $\liminf_{s \to 0} \phi(2s)/\phi(s) > 1$ and let $p, \theta > 0$ be such that

$$\frac{p}{2} < \log_2 \left( \liminf_{s \to 0} \frac{\phi(2s)}{\phi(s)} \right) \leq \log_2 \left( \limsup_{s \to 0} \frac{\phi(2s)}{\phi(s)} \right) < \frac{1}{2\theta},$$

then one has

$$\sup_{t > 0} E[|A^\theta Z_t|^p] < \infty.$$

**Proof.** Since $\phi(2s) \leq 2\phi(s)$, our assumptions guarantee that $p \leq 2$. An application of Jensen’s inequality and Itô’s isometry (e.g. [16, Theorem 8.7]) shows

$$E^W[|A^\theta Z_t^\ell|^p] = E^W\left[ \left| \int_0^t A^\theta e^{-(t-s)A} Q(X^\ell_{s-}) dW_s \right|^p \right]$$

$$\leq \left( E^W\left[ \left| \int_0^t A^\theta e^{-(t-s)A} Q(X^\ell_{s-}) dW_s \right|^2 \right] \right)^{\frac{p}{2}}$$

$$\leq \left( \int_0^t \|A^\theta e^{-(t-s)A}\|^2 \cdot \|Q\|_{HS, \infty}^2 \, ds \right)^{\frac{p}{2}}.$$
By (1.5) and (1.7) we get

\[ E^W \left[ |A^\theta Z_t|^p \right] \leq \|Q\|_{HS,\infty}^p \left( \int_0^t \|A^\theta e^{-\frac{t}{2}A(t-s)}\|^2 \|e^{-\frac{1}{2}A(t-s)}\|_2^2 \, d\ell_s \right)^{\frac{p}{2}} \]

\[ \leq C_\theta \|Q\|_{HS,\infty}^p \left( \int_0^t (t-s)^{-2\theta} e^{-\gamma_1(t-s)} \, d\ell_s \right)^{\frac{p}{2}}. \]

This estimate, together with Lemma 2.2, yields

\[ E \left[ |A^\theta Z_t|^p \right] = E^S \left( E^W \left[ |A^\theta Z_t|^p \right] \bigg|_{\ell=S} \right) \]

\[ \leq C_\theta \|Q\|_{HS,\infty}^p E \left[ \left( \int_0^t (t-r)^{-2\theta} e^{-\gamma_1(t-r)} \, dS_r \right)^{\frac{p}{2}} \right] \]

\[ = C_\theta \|Q\|_{HS,\infty}^p E \left[ \left( \int_0^t r^{-2\theta} e^{-\gamma_1 r} \, dS_r \right)^{\frac{p}{2}} \right]. \]

The assertions i) and ii) follow directly from this estimate:

i) Since we have

\[ E \left[ \left( \int_0^t r^{-2\theta} e^{-\gamma_1 r} \, dS_r \right)^{\frac{p}{2}} \right] \leq E \left[ \left( \int_0^t r^{-2\theta} \, dS_r \right)^{\frac{p}{2}} \right], \]

we get i) directly from Proposition 4.1.v).

ii) Observe that

\[ \sup_{t>0} \left( \int_0^t r^{-2\theta} e^{-\gamma_1 r} \, dS_r \right)^{\frac{p}{2}} \leq \left( \int_0^1 r^{-2\theta} \, dS_r + \int_1^{\infty} e^{-\gamma_1 r} \, dS_r \right)^{\frac{p}{2}} \]

\[ \leq \left( \int_0^1 r^{-2\theta} \, dS_r \right)^{\frac{p}{2}} + \left( \int_1^{\infty} e^{-\gamma_1 r} \, dS_r \right)^{\frac{p}{2}}. \]

From Proposition 4.1.iv) we know that \( E \left[ \left( \int_0^1 r^{-2\theta} \, dS_r \right)^{p/2} \right] < \infty. \) On the other hand, \( \frac{p}{2} < \log_2 (\liminf_{s \to 0} \phi(2s)/\phi(s)) \) allows us to use Proposition 4.1.vii) with \( p \) replaced by \( p/2 \) (see Lemma 5.7 below), and so \( E \left[ \left( \int_0^{\infty} e^{-\gamma_1 r} \, dS_r \right)^{p/2} \right] < \infty. \) This completes the proof. \( \square \)

5.2. A maximal inequality and a small ball probability of \( Z_t. \)

**Theorem 5.3.** Let \((W_t)_{t \geq 0}, (S_t)_{t \geq 0} \) and \((Z_t)_{t \geq 0} \) be as above and assume (A1), (A2). If

\[ 0 < p < 2 \log_2 \left( \liminf_{s \to 0} \frac{\phi(2s)}{\phi(s)} \right), \quad \text{resp.} \quad 0 < p < 2 \log_2 \left( \inf_{s > 0} \frac{\phi(2s)}{\phi(s)} \right), \]

then there exists a constant \( C = C(p, \|Q\|_{HS,\infty}) > 0 \) such that

\[ E \left[ \sup_{0 \leq t \leq T} |Z_t|^p \right] \leq C \left[ \phi^{-1} \left( \frac{1}{T} \right) \right]^{\frac{p}{2}} \quad \text{holds for all } T \geq 1, \text{ resp. } T > 0. \]

Let us note an immediate consequence of Theorem 5.3.
Corollary 5.4. Assume that (A1), (A2) and \( \lim_{s \to 0} \phi(2s)/\phi(s) > 1 \) hold. Then \( Z \in L^\infty([0, T], H) \) a.s.

A further consequence of the maximal inequality is the following small ball probability estimate.

Theorem 5.5. Let \( (W_t)_{t \geq 0}, (S_t)_{t \geq 0} \) and \( (Z_t)_{t \geq 0} \) be as above, and assume (A1), (A2).

i) If \( \phi \) has zero drift, then for any \( \delta \in (0, 1) \) and any \( T > 0 \), the following small ball estimate holds

\[
P \left( \sup_{0 \leq t \leq T} |Z_t| < \delta \right) > 0.
\]

ii) If \( \inf_{s > 0} \frac{\phi'(2s)}{\phi(s)} > 1 \), then for any \( \delta \in (0, 1) \), any \( \kappa \in (0, 1) \), and any \( 0 < p < \log_2 \left( \inf_{s > 0} \frac{\phi'(2s)}{\phi(s)} \right) \), there exists some \( C = C(p, \kappa, \|Q\|_{HS, \infty}) > 0 \) such that for sufficiently small \( T > 0 \)

\[
P \left( \sup_{0 \leq t \leq T} |Z_t| < \delta \right) \geq \kappa \left( 1 - C \left( \delta^4 \phi^{-1} \left( \frac{1}{T} \right) \right)^{-p} \right) > 0.
\]

The proofs of Theorems 5.3 and 5.5 rely on the following auxiliary result.

Lemma 5.6. Let \( (W_t)_{t \geq 0}, (S_t)_{t \geq 0} \) and \( (Z_t^\ell)_{t \geq 0} \) be as above and assume (A1), (A2). The following maximal inequality holds:

\[
E^W \left[ \sup_{0 \leq t \leq T} |Z_t^\ell|^2 \right] \leq 9\|Q\|_{HS, \infty}^2 \ell T.
\]

Proof. Note that \( dZ_t^\ell = Q(X_{t-}^\ell) dW_t - AZ_t^\ell dt \). By Itô’s formula [14, Theorem 27.2], we have

\[
|Z_t^\ell|^2 + 2 \int_0^t |A^\frac{1}{2} Z_s^\ell|^2 ds = 2 \int_0^t \langle Z_{s-}^\ell, Q(X_{s-}^\ell) dW_s \rangle + \sum_{0 < s \leq t} [ |Z_s^\ell|^2 - |Z_s^\ell - 2 (Z_{s-}^\ell, Q(Z_{s-}^\ell) \Delta W_s)|^2 ].
\]

A direct calculation shows that

\[
|Z_t^\ell|^2 - |Z_{t-}^\ell|^2 - 2 (Z_{t-}^\ell, Q(Z_{t-}^\ell) \Delta W_t) = |Z_{t-}^\ell + Q(Z_{t-}^\ell) \Delta W_t|^2 - |Z_{t-}^\ell|^2 - 2 (Z_{t-}^\ell, Q(Z_{t-}^\ell) \Delta W_t) = |Q(X_{t-}^\ell) \Delta W_t|^2,
\]

and, therefore,

\[
|Z_t^\ell|^2 + 2 \int_0^t |A^\frac{1}{2} Z_s^\ell|^2 ds = 2 \int_0^t \langle Z_{s-}^\ell, Q(X_{s-}^\ell) dW_s \rangle + \sum_{0 < s \leq t} |Q(X_{s-}^\ell) \Delta W_s|^2.
\]

If we take expectations on both sides of the above equality and apply Itô’s isometry, we see

\[
E^W [ |Z_t^\ell|^2 ] + 2 \int_0^t E^W [ |A^\frac{1}{2} Z_s^\ell|^2 ] ds = \sum_{0 < s \leq t} E^W [ |Q(X_{s-}^\ell) \Delta W_s|^2 ] = \int_0^t E^W [ ||Q(X_{s-}^\ell)||_{HS}^2 ] d\ell_s.
\]
This implies, in particular,

\[(5.6)\quad E^W \left[ |Z_t^\ell|^2 \right] \leq \int_0^t E^W \left[ \|Q(X_{s-}^\ell)\|_{HS}^2 \right] \, d\ell_s \leq \|Q\|_{HS,\infty}^2 \ell_t.\]

Since $Z_t^\ell - Z_{t-}^\ell = Q(X_{t-}^\ell)\Delta W_t$, we see using (5.6) and, again, Itô’s isometry,

\[(5.7)\quad E^W \left[ |Z_t^\ell|^2 \right] = E^W \left[ |Z_t^\ell - Q(X_{t-}^\ell)\Delta W_t|^2 \right] \leq 2E^W \left[ |Z_t^\ell|^2 \right] + 2E^W \left[ \|Q(X_{t-}^\ell)\|_{HS}^2 \right] \Delta \ell_t \leq 2\|Q\|_{HS,\infty}^2 \ell_t + 2\|Q\|_{HS,\infty}^2 \Delta \ell_t \leq 4\|Q\|_{HS,\infty}^2 \ell_t.

Combining (5.5), the Cauchy-Schwarz inequality, Doob’s martingale maximal $L^2$-inequality and Itô’s isometry, we get

\[
E^W \left[ \sup_{0 \leq t \leq T} |Z_t^\ell|^2 \right] \leq 2 \left\{ E^W \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \langle Z_{s-}^\ell, Q(X_{s-}^\ell) \rangle \, dW_s \right| \right] \right\}^{\frac{1}{2}} + E^W \left[ \sum_{0 < s \leq t} |Q(X_{s}^\ell)\Delta W_s|^2 \right]^\frac{1}{2} \leq 4 \left( E^W \left[ \left( \int_0^T \langle Z_{s-}^\ell, Q(X_{s}^\ell) \rangle \, dW_s \right)^2 \right] \right)^\frac{1}{2} + \int_0^T \|Q(X_{s}^\ell)\|_{HS}^2 \, d\ell_s \leq 4 \|Q\|_{HS,\infty} \left( \int_0^T E^W \left[ |Z_{s-}^\ell|^2 \right] \, d\ell_s \right)^\frac{1}{2} + \|Q\|_{HS,\infty}^2 \ell_T.
\]

This estimate together with (5.7) finally yields

\[
E^W \left[ \sup_{0 \leq t \leq T} |Z_t^\ell|^2 \right] \leq 4\|Q\|_{HS,\infty} \left( \int_0^T 4\|Q\|_{HS,\infty}^2 \ell_t \, d\ell_t \right)^\frac{1}{2} + \|Q\|_{HS,\infty}^2 \ell_T \leq 8\|Q\|_{HS,\infty}^2 \left( \int_0^T \ell_t \, d\ell_t \right)^\frac{1}{2} + \|Q\|_{HS,\infty}^2 \ell_T = 9\|Q\|_{HS,\infty}^2 \ell_T. \tag*{\Box}
\]

**Proof of Theorem 5.3.** Since $W$ and $S$ are independent, we get with Jensen’s inequality

\[
E \left[ \sup_{0 \leq t \leq T} |Z_t^p|^p \right] = E^S \left( E^W \left[ \sup_{0 \leq t \leq T} |Z_t^p|^p \right] \right)_{t=S} \leq E^S \left( E^W \left[ \sup_{0 \leq t \leq T} |Z_t^p|^p \right] \right)_{t=S}^{\frac{p}{2}} \leq 3^p \|Q\|_{HS,\infty}^p E^S \left[ S_T^p/2 \right].
\]

The claim follows now from Proposition 4.1.iii). \(\Box\)
Proof of Theorem 5.5. By Chebyshev’s inequality and Lemma 5.6, we have for all $T > 0$
\[
\mathbb{P} \left( \sup_{0 \leq t \leq T} |Z_t| \geq \delta, S_T < \delta^4 \right) = \mathbb{E}^\varphi \left[ \mathbb{I}_{\{\delta^4 \leq Z_t \leq \delta \}} \mathbb{P}^W \left( \sup_{0 \leq t \leq T} |Z_t| \geq \delta \right) \right]_{t = S} \\
\leq \delta^{-2} \mathbb{E}^\varphi \left[ \mathbb{I}_{\{\delta^4 \leq Z_t \leq \delta \}} \mathbb{E}^W \left( \sup_{0 \leq t \leq T} |Z_t|^2 \right) \right]_{t = S} \\
\leq 9 \|Q\|_{H, \infty}^2 \delta^{-2} \mathbb{E}^\varphi \left( \mathbb{I}_{\{S_T < \delta^4 \}} S_T \right) \\
\leq 9 \|Q\|_{H, \infty}^2 \delta^2 \mathbb{P} \left( S_T < \delta^4 \right).
\]

Let $\kappa \in (0, 1)$. If $0 < \delta < \sqrt{1 - \kappa/(3\|Q\|_{H, \infty})}$, we get
\[
\mathbb{P} \left( \sup_{0 \leq t \leq T} |Z_t| < \delta \right) \geq \mathbb{P} \left( \sup_{0 \leq t \leq T} |Z_t| < \delta, S_T < \delta^4 \right) \\
= \mathbb{P} \left( S_T < \delta^4 \right) - \mathbb{P} \left( \sup_{0 \leq t \leq T} |Z_t| \geq \delta, S_T < \delta^4 \right) \\
\geq (1 - 9 \|Q\|_{H, \infty}^2 \delta^2) \mathbb{P} \left( S_T < \delta^4 \right) \\
\geq \kappa \mathbb{P} \left( S_T < \delta^4 \right).
\]

i) If $\phi$ has zero drift, then $\mathbb{P} \left( S_T < \delta^4 \right) > 0$ for any $\delta > 0$ and $T > 0$, see e.g. [19, Corollary 24.8] and observe that the support of $S_T$ contains zero. This, with (5.8), gives the first assertion.

ii) By Chebyshev’s inequality and Proposition 4.1.iii), there exists $C_1 = C_1(p) > 0$ such that
\[
\mathbb{P} \left( S_T < \delta^4 \right) = 1 - \mathbb{P} \left( S_T^p \geq \delta^{4p} \right) \\
\geq 1 - \delta^{-4p} \mathbb{E} \left[ S_T^p \right] \\
\geq 1 - C_1 \left[ \delta^4 \phi^{-1} \left( \frac{1}{T} \right) \right]^{-p}.
\]

Combining this with (5.8), we get for all $T > 0$ and $0 < \delta < \sqrt{1 - \kappa/(3\|Q\|_{H, \infty})}$
\[
\mathbb{P} \left( \sup_{0 \leq t \leq T} |Z_t| < \delta \right) \geq \kappa \left( 1 - C_1 \left[ \delta^4 \phi^{-1} \left( \frac{1}{T} \right) \right]^{-p} \right).
\]

Fix $\delta \in (0, 1)$ and pick $C_2 = C_2(\kappa, \|Q\|_{H, \infty}) > 1$ such that
\[
\delta := \frac{\delta}{C_2} \leq \delta \wedge \frac{\sqrt{1 - \kappa}}{3\|Q\|_{H, \infty}}.
\]

Then, for all $\delta \in (0, 1)$ and $T > 0$
\[
\mathbb{P} \left( \sup_{0 \leq t \leq T} |Z_t| < \delta \right) \geq \mathbb{P} \left( \sup_{0 \leq t \leq T} |Z_t| < \delta \right) \\
\geq \kappa \left( 1 - C_1 \left[ \delta^4 \phi^{-1} \left( \frac{1}{T} \right) \right]^{-p} \right) \\
= \kappa \left( 1 - C_1 C_2^{4p} \left[ \delta^4 \phi^{-1} \left( \frac{1}{T} \right) \right]^{-p} \right).
\]
Since \( \inf_{s>0} \frac{\phi(2s)}{\phi(s)} > 1 \) implies \( \lim_{s \to \infty} \phi(s) = \infty \), we know that the last term is positive if \( T > 0 \) is small enough. This completes the proof.

In the proof of Theorem 5.2, we need the following auxiliary result.

**Lemma 5.7.** Let \( g : (0, 1) \to (0, \infty) \) be an increasing function. Then

\[
\kappa := \log_2 \left( \lim_{s \to 0} \frac{g(2s)}{g(s)} \right) > 0 \implies \forall \epsilon > 0 : \lim_{s \to 0} \frac{g(s)}{s^{\kappa+\epsilon}} < \infty.
\]

**Proof.** It follows from the assumption that for any \( \epsilon \in (0, \kappa) \) there exists some sufficiently small \( \delta = \delta(\epsilon) > 0 \) such that

\[
2^{\kappa-\epsilon} g(s) \leq g(2s), \quad 0 < s \leq \delta.
\]

By iteration, we get for any \( n \in \mathbb{N} \),

\[
g(s) \leq 2^{-n(\kappa-\epsilon)} g(2^n s), \quad 0 < s \leq \delta 2^{-n+1}.
\]

For \( s \in (0, \delta) \) there is a unique \( n = n_s \in \mathbb{N} \) such that \( 2^{-n} \delta < s \leq 2^{-n+1} \delta \). Since \( g \) is increasing,

\[
g(s) \leq g(\delta 2^{-n+1}) \leq 2^{-n(\kappa-\epsilon)} g(2^n \delta 2^{1-n}) \leq \left( \frac{s}{\delta} \right)^{\kappa-\epsilon} g(2\delta) = \frac{g(2\delta)}{\delta^{\kappa-\epsilon}} s^{\kappa-\epsilon},
\]

which implies that

\[
\lim_{s \to 0} \frac{g(s)}{s^{\kappa-\epsilon}} \leq \frac{g(2\delta)}{\delta^{\kappa-\epsilon}} < \infty.
\]

6. INVARIANT MEASURES

Recall that the solution to Eq. (1.2) has the following form:

\[
X_t = e^{-tA} x + \int_0^t e^{-(t-s)A} F(X_s) \, ds + Z_t,
\]

where

\[
Z_t = \int_0^t e^{-(t-s)A} Q(X_{s-}) \, dW_s.
\]

**Theorem 6.1.** Assume (A1)–(A3), and \( \lim_{s \to 0} \phi(2s)/\phi(s) > 1 \). Then the system (1.2) admits at least one invariant measure.

**Proof.** Pick \( p > 0 \) and \( \theta \in (0, 1) \) such that (5.3) holds. Note that

\[
E \left[ \|A^\theta X_t\|^p \right] \leq 3^p \left( \|A^\theta e^{-tA} x\|^p + \int_0^t \|A^\theta e^{-(t-s)A}\| \, ds \right)^p F_\infty^p + E \left[ \|A^\theta Z_t\|^p \right].
\]

By (1.5), we have for \( 0 \leq s \leq t \)

\[
\|A^\theta e^{-(t-s)A}\| \leq \|A^\theta e^{-\frac{1}{2}(t-s)A}\| \|e^{-\frac{1}{2}(t-s)A}\| \leq C_\theta (t-s)^{-\theta} e^{-\gamma_1(t-s)/2}
\]

which implies that

\[
\sup_{t>0} \int_0^t \|A^\theta e^{-(t-s)A}\| \, ds \leq C_\theta \sup_{t>0} \int_0^t (t-s)^{-\theta} e^{-\gamma_1(t-s)/2} \, ds
\]

\[
= C_\theta \int_0^\infty s^{-\theta} e^{-\gamma_1 s/2} \, ds
\]

\[
= C_{\theta, \gamma_1}.
\]
Then by Theorem 5.2.ii),
\[ E \left[ |A^\theta X_t|^p \right] \leq 3^p \left[ C_{\theta}^p t^{-p}\epsilon^{-p\gamma t/2}|x|^p + C_{\theta,\gamma t}^p \|F\|_\infty^p + C_{\theta,p} \right]. \]
Hence, we obtain that for any \( T > 0 \)
\[ \frac{1}{T} \int_1^{T+1} E \left[ |A^\theta X_t|^p \right] \, dt \leq C_{\theta,p,\gamma t,\|F\|_\infty,x}. \]
Because of (A2), the inverse \( A^{-1} \) is a compact operator and, therefore, the set
\[ C_K := \{ x \in H ; |A^\theta x| \leq K \} \]
is compact in \( H \). By the Chebyshev inequality,
\[ \frac{1}{T} \int_1^{T+1} P_t(x, H \setminus C_K) \, dt = \frac{1}{T} \int_1^{T+1} \mathbb{P} (|A^\theta X_t| > K) \, dt \]
\[ \leq \frac{1}{T} \int_1^{T+1} E \left[ |A^\theta X_t|^p \right] \frac{1}{K^p} \, dt \]
\[ \leq C_{\theta,p,\gamma t,\|F\|_\infty,x} K^{-p}. \]
This yields that \( \left( \frac{1}{T} \int_1^{T+1} P_t(x, \cdot) \, dt \right)_{T>0} \) is tight and thus admits a subsequence which converges to an invariant measure, as long as the transition probability of \( X_t \) has the Feller property [6, Theorem 3.1.1].

It remains to show that \( (X_t)_{t \geq 0} \) has the Feller property. By [16, Theorem 9.29 (ii)], we have
\[ E \left[ |X_t(x) - X_t(y)|^2 \right] \leq C|x - y|^2, \quad t > 0. \]
Fix \( \delta > 0 \) and set \( D_\delta = \{ |X_t(x) - X_t(y)| \leq \delta \}. \) Since \( f \) is continuous, we can assume that \( \delta = \delta(\epsilon) \) is so small that \( |f(X_t(x)) - f(X_t(y))| \leq \epsilon \) on \( D \). For every bounded continuous function \( f \),
\[ |P_t f(y) - P_t f(x)| = |E f(X_t^y) - E f(X_t^x)| \]
\[ \leq |E [(f(X_t(x)) - f(X_t(y)))1_D]| + |E [(f(X_t(x)) - f(X_t(y)))1_{D^c}]| \]
\[ \leq \epsilon + 2\|f\|_\infty \mathbb{P} (|X_t(x) - X_t(y)| > \delta) \]
\[ \leq \epsilon + \frac{2C\|f\|_\infty}{\delta^2} |x - y|^2, \]
where we use Chebyshev’s inequality in the last step. Since \( \epsilon > 0 \) is arbitrary, we see that \( P_t f(y) \to P_t f(x) \) as \( y \to x \), i.e. we have the Feller property. \( \square \)

7. ACCESSIBILITY AND AN ASSOCIATED CONTROL PROBLEM

As before, we use the independence of \( S = (S_t)_{t \geq 0} \) and \( W = (W_t)_{t \geq 0} \) to represent \( \mathbb{P} \)
as \( \mathbb{P} = \mathbb{P}^S \otimes \mathbb{P}^W \). This means that we can condition on the event \( S_{t_0} = \ell_t \), which is an increasing càdlàg function \( [0, \infty) \ni t \mapsto \ell_t \) such that \( \ell_0 = 0 \), and consider the following auxiliary equation:
\[ dX_t^\ell = [-AX_t^\ell + F(X_t^\ell)] \, dt + Q(X_t^\ell) \, dW_t, \quad X_0^\ell = x \in H, \]
whose (unique, mild) solution [5, Theorem 7.4] is
\[ X_t^\ell = e^{-tA}x + \int_0^t e^{-(t-s)A} F(X_s^\ell) \, ds + \int_0^t e^{-(t-s)A} Q(X_s^\ell) \, dW_s. \]
For any real-valued, bounded and measurable function on $H$, $f \in \mathcal{B}_b(H, \mathbb{R})$, we have
\begin{equation}
E(f(X_t)) = E^S \left( E^W \left[ f(X^*_t) \right] \big| \ell = S \right).
\end{equation}

From now on we need the following additional condition:

**(A4)** There exist some $\delta > 0$ with $\int_0^\infty \phi(s^{-2\delta}) \, ds < \infty$ and $C > 0$ such that for all $x \in H$
\begin{equation*}
\left\| Q(x)^{-1} e^{-tA} \right\| \leq Ct^{-\delta}.
\end{equation*}

**Remark 7.1.** (A4) means that the noise is not too weak; this is necessary to guarantee accessibility of the solution to Eq. (1.2). Let us illustrate this for the $\frac{\alpha}{2}$-stable subordinator case, i.e. for $\phi(s) = s^{\alpha/2}$, $0 < \alpha < 2$. Assume, for simplicity, $Q(x) = A^\beta$ with $\beta \in \mathbb{R}$. From (A4) we know that $\delta < 1/\alpha$. By (1.5), we need $\beta > -1/\alpha$, which means that the strength of the noise is bounded from below. The requirement $\delta < 1/\alpha$ is also consistent with [17, Assumption 2.2 (A4)] for SPDEs driven by additive $\alpha$-stable noises.

As before, we write $\ell : [0, \infty) \to [0, \infty)$ for a fixed trajectory of the subordinator $(S_t)_{t \geq 0}$. Since $P^S$-almost all $\ell$ are strictly increasing, we can define the (generalized, right-continuous) inverse of $\ell$:
\begin{equation*}
\ell_t^{-1} := \inf \{ s \geq 0 : \ell_s > t \}, \quad t \geq 0.
\end{equation*}
It is easy to check that we have for any measurable function $f : [0, \infty) \to [0, \infty)$
\begin{equation*}
\int_0^t f(\ell_s^{-1}) \, ds = \int_0^{\ell_t^{-1}} f(s) \, d\ell_s \quad \text{for all } t > 0.
\end{equation*}
See also [26] for more applications of this transform.

The following proposition is crucial in order to prove the accessibility to zero of (1.2).

**Proposition 7.2.** Assume that (A4) holds and fix $t > 0$ and $m > 0$. For $P^S$ almost every trajectory $\ell$ of $S$ satisfying $\ell_t^{-1} \leq m$, we have
\begin{equation}
\left| \int_0^t Q(z)^{-1} e^{-\ell_s^{-1} A} x \, ds \right| \leq C_1 |x|,
\end{equation}
\begin{equation}
\left| \int_0^t Q(z)^{-1} e^{-\ell_s^{-1} A} x \, ds \right|^2 \leq C_2 |x|^2
\end{equation}
for all $z \in H$, where $C_1, C_2 > 0$ are constants which may depend on $t$ and $m$.

**Proof.** Using the Cauchy–Schwarz inequality along with (A4) we get
\begin{align*}
\left| \int_0^t Q(z)^{-1} e^{-\ell_s^{-1} A} x \, ds \right|^2 &\leq t \int_0^t \left| Q(z)^{-1} e^{-\ell_s^{-1} A} x \right|^2 \, ds \\
&\leq tC^2 |x|^2 \int_0^t (\ell_s^{-1})^{-2\delta} \, ds \\
&= tC^2 |x|^2 \int_0^t s^\delta \, ds \\
&\leq tC^2 |x|^2 \int_0^m s^\delta \, ds.
\end{align*}
Since $\int_0^\infty \phi(s^{-2\delta}) \, ds < \infty$, we know from Proposition 2.3 that $\int_0^m s^{-2\delta} \, ds$ is a.s. finite.

This proves both (7.4) and (7.5) for $P^S$-almost all trajectories $\ell$ of $S$ satisfying $\ell_t^{-1} \leq m$. Considering $t, m \in \mathbb{N}$, we see that the $P^S$-null can be chosen independently of $t$ and $m$. \qed
Let \( \nu((0, \infty)) = \infty \), then the system (1.2) is accessible to zero, i.e., for all \( x \in H \), \( \epsilon > 0 \), \( T > 0 \), we have

\[
P\left( |X_T(x)| < \epsilon \right) > 0.
\]

Proof. Observe that for every \( m \in \mathbb{N} \)

\[
(7.6) \quad P\left( |X_T(x)| < \epsilon \right) \geq E^S \left[ P^W \left( |X_T^\ell(x)| < \epsilon \right) \right] \bigg| \ell = S \mathbb{1}_{\{S_T \leq m\}}.
\]

Since \( \lim_{m \to \infty} P^S(S_T \leq m) = 1 \), we can choose \( m \) in such a way that \( P^S(S_T \leq m) > 0 \). Because of the assumption \( \nu((0, \infty)) = \infty \), almost all trajectories of \( S \) are strictly increasing. This observation and Lemma 7.4 below give

\[
E^S \left[ P^W \left( |X_T^\ell(x)| < \epsilon \right) \right] > 0. \quad \square
\]

Lemma 7.4. Let \( \ell \) be a strictly increasing trajectory which is right continuous with left limits. If the assumptions (A1)–(A4) hold, then the system (7.1) is accessible to zero, i.e., for all \( x \in H \), \( \epsilon > 0 \), \( T > 0 \), we have for all \( m \in \mathbb{N} \) and \( P^S \)-almost all \( \ell \) with \( \ell_T \leq m \)

\[
P^W \left( |X_T^\ell(x)| < \epsilon \right) > 0.
\]

For the proof of Lemma 7.4 we need to study some auxiliary control problems. Consider the following problem:

\[
dY_t^\ell = [-A Y_t^\ell + F(Y_t^\ell)] dt + Q(Y_{t-}^\ell) \, d\ell_t,
\]

where \( u \) is some controller which is to be determined. We say that (7.7) is

- exactly controllable: if for all \( x, y \in H \) and \( T > 0 \) there exists some \( u \in C([0, \ell_T]; H) \) such that

\[
(7.8) \quad Y_0^\ell = x, \quad Y_T^\ell = y.
\]

- approximately controllable: if for all \( x, y \in H \), \( T > 0 \) and \( \epsilon > 0 \), there exists some \( u \in C([0, \ell_T]; H) \) such that

\[
(7.9) \quad Y_0^\ell = x, \quad |Y_T^\ell - y| < \epsilon,
\]

- approximately controllable to 0: if for all \( x \in H \), \( T > 0 \) and \( \epsilon > 0 \), there exists some \( u \in C([0, \ell_T]; H) \) such that

\[
(7.10) \quad Y_0^\ell = x, \quad |Y_T^\ell| < \epsilon,
\]

If the target point 0 is replaced by some fixed \( y_0 \in H \), then the problem (7.7) is said to be approximately controllable to \( y_0 \).

Proof of Lemma 7.4. Choose some \( \tilde{T} \in (0, 1/\|F\|_{\text{lip}}) \), we shall use an iteration procedure to show that there exists some \( u \in C([0, \ell_{\tilde{T}}], H) \) with bounded total variation such that

\[
(7.11) \quad \begin{cases}
    d\Phi_t^\ell = -A\Phi_t^\ell \, dt + Q(Y_{t-}^\ell) \, d\ell_t, \\
    \Phi_0^\ell = x, \\
    \Phi_{\tilde{T}}^\ell = 0,
\end{cases}
\]
where $Y_t^\ell$ is the solution to Eq. (7.7). In order to keep notations simple, we drop in this proof the superscripts "$\ell$" of $\Phi^\ell$ and $Y^\ell$. Because $Q$ depends on $Y$, we need to use an iteration procedure to find the controller $u$.

Define $Y_t^{(0)} \equiv x$ for all $t \in [0, \tilde{T}]$. We consider the following control problem:

$$
\begin{cases}
\frac{d\Phi_t^{(1)}}{dt} = -A\Phi_t^{(1)} + Q(Y_t^{(0)})u_t^{(1)}, \\
\Phi_0^{(1)} = x, \\
\Phi_T^{(1)} = 0.
\end{cases}
$$

Choose

$$u_t^{(1)} = -\frac{1}{\ell_T} \int_0^t Q(Y_s^{(0)})^{-1} e^{-sA}x \, ds.$$

It is easy to check that the assumption (A4) ensures $u_t^{(1)} \in H$ for all $t \geq 0$. We have

$$\Phi_t^{(1)} = e^{-tA}x - \frac{1}{\ell_T} \int_0^t e^{-(t-s)A}Q(Y_s^{(0)})Q(Y_s^{(0)})^{-1}e^{-sA}x \, ds = \frac{\ell_T - \ell_t}{\ell_T} e^{-tA}x.$$

Further, define

$$Y_t^{(1)} = e^{-tA}x + \int_0^t e^{-(t-s)A}F(Y_s^{(0)}) \, ds + \int_0^t e^{-(t-s)A}Q(Y_s^{(0)})u_s^{(1)};$$

using Proposition 7.2 it is easy to see that

$$Y_t^{(1)} = \int_0^t e^{-(t-s)A}F(Y_s^{(0)}) \, ds + \Phi_t^{(1)}.$$

For $n \in \mathbb{N}$, we define recursively

$$u_t^{(n+1)} = -\frac{1}{\ell_T} \int_0^t Q(Y_s^{(n)})^{-1} e^{-sA}x \, ds,$$

$$\Phi_t^{(n+1)} = e^{-tA}x + \int_0^t e^{-(t-s)A}Q(Y_s^{(n)})u_s^{(n+1)},$$

$$Y_t^{(n+1)} = \int_0^t e^{-(t-s)A}F(Y_s^{(n)}) \, ds + \Phi_t^{(n+1)}.$$

The first two equalities yield

$$\Phi_t^{(n+1)} = \frac{\ell_T - \ell_t}{\ell_T} e^{-tA}x.$$

Therefore, we have

$$|Y_t^{(n+1)} - Y_t^{(n)}| \leq \int_0^t |e^{-(t-s)A}F(Y_s^{(n)}) - e^{-(t-s)A}F(Y_s^{(n-1)})| \, ds$$

$$\leq \int_0^t \|F\|_{\text{Lip}} |Y_s^{(n)} - Y_s^{(n-1)}| \, ds.$$

Since $\tilde{T} < 1/\|F\|_{\text{Lip}}$, we see that

$$\sup_{0 \leq t \leq \tilde{T}} |Y_t^{(n+1)} - Y_t^{(n)}| \leq \tilde{T} \|F\|_{\text{Lip}} \sup_{0 \leq t \leq \tilde{T}} |Y_t^{(n)} - Y_t^{(n-1)}|$$

$$\leq \ldots \leq \left(\tilde{T} \|F\|_{\text{Lip}}\right)^n \sup_{0 \leq t \leq \tilde{T}} |Y_t^{(1)} - Y_t^{(0)}|.$$
So there exists some uniformly bounded \((Y_t)_{0 \leq t \leq T}\), which is right continuous and has left limits, such that
\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} |Y_t^{(n)} - Y_t| = 0.
\]

Letting \(n \to \infty\) in (7.14), we obtain
\[
(7.16) \quad u_{t_\ell} = -\frac{1}{\ell} \int_{0}^{t} Q(Y_{t-})^{-1} e^{-A_s} x \, d\ell_s,
\]
\[
(7.17) \quad \Phi_t = e^{-A_s} x + \int_{0}^{t} e^{-(t-s)A} Q(Y_{s-}) \, du_s,
\]
\[
(7.18) \quad Y_t = \int_{0}^{t} e^{-(t-s)A} F(Y_s) \, ds + \Phi_t.
\]

From the first two equalities we see that \(\Phi_T = 0\). For \(\epsilon > 0\), we get
\[
|Y_T| = \left| \int_{0}^{T} e^{-(t-s)A} F(Y_s) \, ds \right| \leq \|F\| \leq \epsilon, \quad \tilde{T} < \min \left\{ \frac{\epsilon}{2\|F\|}, \frac{1}{\|F\|_{\text{Lip}}} \right\}.
\]

Recall that
\[
(7.19) \quad X_t = \int_{0}^{t} e^{-(t-s)A} F(X_s) \, dt + Z_t,
\]
where \(Z_t = e^{-A_s} x + \int_{0}^{t} e^{-(t-s)A} Q(X_{s-}) \, dW_s\), then
\[
\begin{aligned}
|X_t - Y_t| &\leq \int_{0}^{t} \left| e^{-(t-s)A} [F(X_s) - F(Y_s)] \right| \, ds + |Z_t - \Phi_t| \\
&\leq \int_{0}^{t} \|F\|_{\text{Lip}} |X_s - Y_s| \, ds + |Z_t - \Phi_t|.
\end{aligned}
\]

By (7.16), \((u_{t_\ell})_{0 \leq t \leq T}\), as a function which is right continuous with left limit, can be embedded into a continuous function \(u \in C([0, \ell_T]; H)\) defined by
\[
u t = \frac{1}{\ell} \int_{0}^{t} Q(Y_{t-1})^{-1} e^{-A_s} x \, ds, \quad \forall t \in [0, \ell_T].
\]

Because of (7.4), the function \(u_t\) is well-defined. Moreover, because of (7.5), we have \(\dot{u} \in L^2([0, \ell_T]; H)\).

Since \(Q\) is Lipschitz by the assumption (A1),
\[
\begin{aligned}
|Z_t - \Phi_t| &\leq \left| \int_{0}^{t} Q(X_s) \, du_s - \int_{0}^{t} Q(Y_s) \, du_s \right| + D(t, Q, X, W, u) \\
&\leq \|Q\|_{\text{Lip}} \int_{0}^{t} |X_s - Y_s| \, |du_s| + D(t, Q, X, W, u),
\end{aligned}
\]

where
\[
D(t, Q, X, W, u) = \left| \int_{0}^{t} Q(X_{s-}) \, dW_s - \int_{0}^{t} Q(X_{s-}) \, du_s \right| = \left| \int_{0}^{t} Q(X_{0-}) \, dW_s - \int_{0}^{t} Q(X_{0-}) \, du_s \right|,
\]
with $\theta_s = \ell_s^{-1}$ for all $s > 0$. Hence,

$$|X_t - Y_t| \leq \int_0^t |e^{-(t-s)A}[F(X_s) - F(Y_s)]| \, ds + |Z_t - \Phi_t| \leq \|F\|_{\text{Lip}} \int_0^t |X_s - Y_s| \, ds + \|Q\|_{\text{Lip}} \int_0^t |X_s - Y_s| \, du_x + D^*(t, Q, X, W, u),$$

where $D^*(t, Q, X, W, u) = \sup_{0 \leq s \leq t} D(s, Q, X, W, u)$. As we have seen earlier, $\hat{u} \in L^2([0, \ell_T]; H)$, and so [6, Theorem 7.4.1] yields

$$P (D^*(t, Q, X, W, u) \leq \gamma) > 0 \quad \forall \gamma > 0.$$  
(7.22)

By Gronwall’s inequality (recall that $u$ has finite total variation), we have

$$|X_t - Y_t| \leq D^*(t, Q, X, W, u) \exp \left[ \|F\|_{\text{Lip}} t + \|Q\|_{\text{Lip}} \|u\|_{\text{TV}(0, \ell_t)} \right].$$

In view of the previous two inequalities, we obtain

$$P(|X_{\hat{T}} - Y_{\hat{T}}| < \epsilon/2) > 0.$$  

Since $|Y_{\hat{T}}| < \epsilon/2$, this further implies

$$P(|X_{\hat{T}}| < \epsilon) > 0.$$  
(7.23)

Recall that the initial datum of $X$ is $x$, which can be an arbitrary point in $H$; (7.23) implies that the transition probability $P_{\hat{T}}(x, B(0, \epsilon))$ satisfies

$$P_{\hat{T}}(x, B(0, \epsilon)) > 0 \quad \forall x \in H.$$  

Combining the Chapman–Kolmogorov equations and the above inequality, we get

$$P_{2\hat{T}}(x, B(0, \epsilon)) = \int_H P_{\hat{T}}(y, B(0, \epsilon)) P_{\hat{T}}(x, dy) > 0.$$  

Using the above argument repeatedly, we see that for all $n \in \mathbb{N}$

$$P_{n\hat{T}}(x, B(0, \epsilon)) > 0.$$  

Since $\hat{T} \in (0, t_0)$ and $\epsilon > 0$ are both arbitrary, $X$ is accessible to zero. \hfill \Box

8. GALERKIN APPROXIMATION

For every $n \in \mathbb{N}$, we define an orthogonal projection $\Pi_n : H \rightarrow H_n$, where $H_n$ is the subspace of $H$ generated by $\{e_1, \ldots, e_n\}$; that is, for any $x \in H$ with the orthogonal expansion $x = \sum_{k=1}^\infty x_k e_k$, we have $\Pi_n x = \sum_{k=1}^n x_k e_k \in H_n$.

The Galerkin approximations of Eq. (1.2) and Eq. (7.1) in $H_n$ are, respectively, as follows:

$$dX^n_t = [-AX^n_t + F^n(X^n_t)] \, dt + Q^n(X^n_{t-}) \, dL^n_t, \quad X^n_0 = x^n,$$

and

$$dX^{n,\ell}_t = [-AX^{n,\ell}_t + F^n(X^{n,\ell}_t)] \, dt + Q^n(X^{n,\ell}_{t-}) \, dW^n_{\ell_t}, \quad X^{n,\ell}_0 = x^n,$$

where $x^n = \Pi_n x$, $F^n = \Pi_n F$, and $Q^n = \Pi_n Q\Pi_n$.

The main result of this section is the following.

**Theorem 8.1.** Assume that (A1) and (A3) hold. For any $T > 0$ and $\delta > 0$,

$$\lim_{n \rightarrow \infty} P \left( \sup_{0 \leq t \leq T} |X^n_t - X_t| > \delta \right) = 0.$$  
(8.3)

The proof of Theorem 8.1 relies on the following lemma.
Lemma 8.2. Assume that (A1) and (A3) hold. For all $T > 0$, we have

\begin{equation}
\lim_{n \to \infty} \mathbb{E}_W \left[ \sup_{0 \leq t \leq T} |X_t^n - X_t| \right] = 0,
\end{equation}

Proof. Throughout this proof, $C$ denotes some generic constant which may change its value from line to line. Observe that

First, we have

\begin{align*}
d(X_t^n - X_t^\ell) &= -A(X_t^n - X_t^\ell) \, dt + \left[ F(X_t^n) - F^n(X_t^\ell) \right] \, dt \\
&\quad + \left[ Q(X_t^n) - Q(X_t^\ell) + Q(X_t^n - X_t^\ell) - Q^n(X_t^n - X_t^\ell) \right] \, dW_t,
\end{align*}

and

\begin{align*}
\langle X_t^n - X_t^\ell, F(X_t^n) - F^n(X_t^\ell) \rangle &= \langle X_t^n - X_t^\ell, F(X_t^n) \rangle - \langle X_t^n - X_t^\ell, F^n(X_t^\ell) \rangle \\
&\quad + \langle X_t^n - X_t^\ell, F^n(X_t^n) - F^n(X_t^\ell) \rangle \\
&= \langle X_t^n - X_t^\ell, F(X_t^n) - F^n(X_t^\ell) \rangle,
\end{align*}

where the last equality uses the fact that $X_t^n - X_t^\ell$ and $F(X_t^n) - F^n(X_t^\ell)$ are orthogonal. By Itô’s formula, we have

\begin{align}
|X_t^n - X_t^\ell|^2 &= |x - x^n|^2 - 2 \int_0^t |A^\frac{1}{2}(X_s^n - X_s^\ell)|^2 \, ds \\
&\quad + 2 \int_0^t \langle X_s^n - X_s^\ell, F(X_s^n) - F^n(X_s^\ell) \rangle \, ds + 2 \mathcal{M}_t + [\mathcal{M}, \mathcal{M}]_t,
\end{align}

where

\begin{align*}
\mathcal{M}_t := \int_0^t \left\langle X_s^n - X_s^\ell, \left[ Q(X_s^n) - Q(X_s^\ell) + Q(X_s^\ell) - Q^n(X_s^n) \right] \right\rangle \, dW_s, \\
[\mathcal{M}, \mathcal{M}]_t := \sum_{0 < s \leq t} \left| \left[ Q(X_s^n) - Q(X_s^\ell) + Q(X_s^\ell) - Q^n(X_s^n) \right] \Delta W_s \right|^2.
\end{align*}

For $t \geq 0$, set

\begin{align*}
\Lambda_{n,t}^\ell := \mathbb{E}_W \left[ \sup_{0 \leq s \leq t} |X_s^n - X_s^\ell|^2 \right].
\end{align*}

First, we have

\begin{align*}
\mathbb{E}_W \left[ \sup_{0 \leq s \leq t} \int_0^s \langle X_t^n - X_t^\ell, F(X_s^n) - F^n(X_s^\ell) \rangle \, dr \right] \\
&\leq \frac{1}{2} \mathbb{E}_W \left[ \sup_{0 \leq s \leq t} \int_0^s |X_t^n - X_t^\ell|^2 \, dr \right] + \frac{1}{2} \mathbb{E}_W \left[ \sup_{0 \leq s \leq t} \int_0^s |F(X_s^n) - F^n(X_s^\ell)|^2 \, dr \right] \\
&\leq \frac{1}{2} \int_0^t \mathbb{E}_W [ |X_r^n - X_r^\ell|^2 ] \, dr + \frac{1}{2} \int_0^t \mathbb{E}_W [ |F(X_r^n) - F^n(X_r^\ell)|^2 ] \, dr \\
&\leq \frac{1}{2} \int_0^t \Lambda_{n,s}^\ell \, ds + \frac{1}{2} \int_0^t \mathbb{E}_W [ |F(X_r^n) - F^n(X_r^\ell)|^2 ] \, ds.
\end{align*}

By the Burkholder-Davis-Gundy inequality with $p = 1$ and (A1), we obtain

\begin{align*}
\mathbb{E}_W \left[ \sup_{0 \leq s \leq t} |\mathcal{M}_s| \right]
\end{align*}
By the dominated convergence theorem and the previous two relations, we get

\[
\frac{1}{2} \Lambda_{n,t} + \frac{1}{2} C^2 \int_0^t E^W \left[ \|Q(X_{s-}^\ell) - Q^n(X_{s-}^\ell)\|_{HS}^2 \right] \, ds \leq C E^W \left[ \sup_{0 \leq s \leq t} |X_{s-}^\ell - X_{s-}^{n,\ell}| \left( \int_0^t \|Q(X_{s-}^\ell) - Q^n(X_{s-}^\ell)\|_{HS}^2 \, ds \right)^{\frac{1}{2}} \right]
\]

By the Itô isometry and (A1),

\[
E^W \left[ \sup_{0 \leq s \leq t} |\mathcal{M}_s| \right] = \sum_{0 \leq r \leq t} E^W \left[ \left| \left[ Q(X_{r-}^\ell) - Q^n(X_{r-}^\ell) \right] \mathcal{W}_r \right|^2 \right] \leq 2 \int_0^t \left( E^W \left[ \|Q(X_{s-}^\ell) - Q^n(X_{s-}^\ell)\|_{HS}^2 \right] + E^W \left[ \|Q(X_{s-}^n) - Q^n(X_{s-}^n)\|_{HS}^2 \right] \right) \, ds \leq C \int_0^t \left( \Lambda_{n,s}^\ell + E^W \left[ \|Q(X_{s-}^n) - Q^n(X_{s-}^n)\|_{HS}^2 \right] \right) \, ds.
\]

Combining the previous three inequalities with (8.5) and moving the term \( \frac{1}{2} \Lambda_{n,t}^\ell \) to the left hand side, we get

\[
\frac{1}{2} \Lambda_{n,t}^\ell \leq |x - x^n|^2 + C \int_0^t \Lambda_{n,s}^\ell (ds + d\ell_s) + C \int_0^t E^W \left[ \|Q(X_{s-}^\ell) - Q^n(X_{s-}^\ell)\|_{HS}^2 \right] \, ds + C \int_0^t E^W \left[ \|F(X_s^\ell) - F^n(X_s^\ell)\|^2 \right] \, ds.
\]

This, together with Gronwall’s inequality, implies that for all \( t > 0 \)

\[
\Lambda_{n,t}^\ell \leq C e^{C(t + \ell_t)} \left( |x - x^n|^2 + \int_0^t E^W \left[ \|Q(X_{s-}^\ell) - Q^n(X_{s-}^\ell)\|_{HS}^2 \right] \, ds \right. \\
\left. + \int_0^t E^W \left[ \|F(X_s^\ell) - F^n(X_s^\ell)\|^2 \right] \, ds \right).
\]

By (A1), as \( n \to \infty \),

\[
\|Q(x^n) - Q^n(x^n)\|_{HS}^2 \leq 4 \|Q(x^n) - Q(x)\|_{HS}^2 + 4 \|\Pi_n(Q(x^n) - Q(x))\|_{HS}^2 + 4 \|Q^n(x) - Q(x)\|_{HS}^2 \to 0.
\]

By the dominated convergence theorem and the previous two relations, we get

\[
\lim_{n \to \infty} \Lambda_{n,t}^\ell = 0,
\]
which completes the proof. □

Proof of Theorem 8.1. By Chebyshev’s inequality, for \( m \in \mathbb{N} \),

\[
P \left( \sup_{0 \leq t \leq T} |X^n_t - X^\ell_t| > \delta \right) = E^S \left[ P^W \left( \sup_{0 \leq t \leq T} |X^n_t - X^\ell_t| > \delta \right) \right] \]

\[
= E^S \left[ \mathbb{1}_{\{\ell_T \leq m\}} P^W \left( \sup_{0 \leq t \leq T} |X^n_t - X^\ell_t| > \delta \right) \right] + E^S \left[ \mathbb{1}_{\{\ell_T > m\}} P^W \left( \sup_{0 \leq t \leq T} |X^n_t - X^\ell_t| > \delta \right) \right] \]

\[
\leq \delta^{-2} E^S \left[ \mathbb{1}_{\{\ell_T \leq m\}} \Lambda^\ell_{n,T} \right] + P^S (S_T > m) .
\]

Because of the bound (8.6) we can use dominated convergence. From Lemma 8.2 we get

\[
\lim_{n \to \infty} E^S \left[ \mathbb{1}_{\{\ell_T \leq m\}} \Lambda^\ell_{n,T} \right] = 0 \quad \text{for all } m \in \mathbb{N}.
\]

Since \( P^S (S_T > m) \to 0 \) as \( m \to \infty \), we can finish the proof by letting first \( n \to \infty \) and then \( m \to \infty \) in (8.7).

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(C.-S. Deng) School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China

Email address: dengcs@whu.edu.cn

(R.L. Schilling) TU Dresden, Fakultät Mathematik, Institut für Mathematische Stochastik, 01062 Dresden, Germany

Email address: rene.schilling@tu-dresden.de

(L. Xu) Department of Mathematics, Faculty of Science and Technology, University of Macau, Macau S.A.R., China

Email address: lihuxu@um.edu.mo