GEOMETRY OF $A_g$ AND ITS COMPACTIFICATIONS

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Abstract. In this survey we give a brief introduction to, and re-
view the progress made in the last decade in understanding the
geometry of the moduli spaces $A_g$ of principally polarized abelian
varieties and its compactifications, concentrating on results ob-
tained over $\mathbb{C}$.

This is an expanded and updated version of the talk given at
the 2005 Summer Institute for Algebraic Geometry.

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1. Introduction

In this survey we review the progress made in the last decade, the
current state of knowledge, and the open problems and possible direc-
tions in the study of the geometry of the moduli spaces of principally
polarized abelian varieties and their compactifications, primarily over
the field of complex numbers.

We discuss the results on the geometric interpretation and construc-
tion of compactifications; the study of the birational geometry of $A_g$,
including nef and effective cones, and the canonical model; the work
on homology and Chow rings of $A_g$; constructions of special loci within
$A_g$ by using the geometry of the theta divisor. Since the moduli space
of curves $M_g$ is perhaps the best-studied moduli space, and is natu-
really a subvariety of $A_g$ via the Torelli map, we also draw analogies
with the study of $\mathcal{M}_g$ when appropriate. We mostly give references to
the original papers instead of complete proofs, but try to explain the
motivation for study, and some ideas leading to the proofs.

In this survey we focus primarily on the geometry of $\mathcal{A}_g$ rather than
that of individual abelian varieties, or of loci in $\mathcal{A}_g$ arising from special
dynamic constructions. In particular we do not cover the exciting
recent developments in understanding the geometry of linear systems
on one abelian variety (surveyed, for example, in [PaPo05]). The more
modular-theoretic aspects of the theory, including a detailed study of
subgroups of $\text{Sp}(2g, \mathbb{Z})$ and the associated moduli spaces, are also not
covered. Neither do we survey the extensive literature on the problems
of characterizing Jacobians of Riemann surfaces within $\mathcal{A}_g$ (known as
the Schottky problem), including Krichever’s recent proof [Kr06] of
Welters’ trisecant conjecture, of characterizing Prym varieties — charac-
terized by the existence of a pair of quadrisection planes in [GrKr07],
immediate Jacobians of cubic threefolds — characterized by the exis-
tence of a triple point on the theta divisor [C-MFr04], [C-M06], etc. The
history of the first two of these characterization problems are surveyed,
for example, in [Ta97], from a more analytic viewpoint.

An earlier introduction and survey, with much more details on the
cycles on $\mathcal{A}_g$ and characteristic $p$, is [vdGOo99]. A survey giving more
details on the work on birational geometry of $\mathcal{A}_g$, including the study
of the non-principal polarizations, is [HuSa02]. The study of complex
tori that are not necessarily algebraic is also surveyed in [De05]. The
book [BiLa04] contains a wealth of information about complex abelian
varieties, special loci, theta functions, and moduli. The survey [vdG06]
is focused more on the theory of Siegel modular forms and related
questions in number theory.

**Acknowledgements.** Those of the results surveyed in which I par-
ticipated have been obtained in collaboration with Cord Erdenberger,
Klaus Hulek, David Lehavi, and Riccardo Salvati Manni, to all of whom
I am grateful for the chance to investigate the subject together. I am
indebted to Klaus Hulek for detailed discussions on the geometry of
compactifications, and to Riccardo Salvati Manni for discussions on
the intricacies of the rings of theta constants, and especially for ex-
plaining how Tai’s technique could be improved to yield small slopes
(theorem 5.19). I would like to thank Izzet Coskun, Gerard van der
Geer, Klaus Hulek, Nicholas Shepherd-Barron, and especially Riccardo
Salvati Manni for reading drafts of this text very carefully, and for many
useful suggestions and advice on content and presentation.
2. Notations

We start by defining the object of our discussion — the moduli space of principally polarized abelian varieties. Throughout the text we will work over the base field \( \mathbb{C} \), though many of the results, especially the purely algebraic ones, carry over to arbitrary characteristic. We make a few comments about the situation in positive characteristic in section \([S]\).

**Definition 2.1.** Algebraically, an *abelian variety* is a projective algebraic variety \( A \), with the structure of an abelian group on the set of its points, such that the group operations are morphisms \( + : A \times A \to A \) and \( -1 : A \to A \).

A *polarization* on an abelian variety is an ample line bundle \( L \) on \( A \).

A polarization \( L \) on an abelian variety \( A \) is called *principal* if its space of sections is one-dimensional, i.e. if \( h^0(A, L) = 1 \). We often think of any non-zero section of \( L \) as the polarization.

**Definition 2.2.** We denote by \( \mathcal{A}_g \) the moduli space of principally polarized abelian varieties (or ppavs for short) of dimension \( g \), up to isomorphisms preserving the principal polarization.

**Remark 2.3.** \( \mathcal{A}_g \), and its compactifications, to be defined below, are properly to be thought of as stacks. However, for many considerations thinking of \( \mathcal{A}_g \) naïvely as if it were a variety, or, more carefully, an orbifold, suffices. To formally justify some of the work done on \( \mathcal{A}_g \) one needs to either work properly with a stack, or use the fact that \( \mathcal{A}_g \) admits finite covers (see below) that are actually manifolds; often the stackiness does not present a problem. Note, however, that any abelian variety has an involution \( x \mapsto -x \), and thus a general point of \( \mathcal{A}_g \) in fact parametrizes an object with an automorphism, so should be counted with multiplicity 1/2 as a stacky point.

**Definition 2.4.** We would now like to say that there exists a *universal family of principally polarized abelian varieties* \( \pi : \mathcal{X}_g \to \mathcal{A}_g \), with the fiber over the point \([A] \in \mathcal{A}_g\) being the variety \( A \) itself. The existence of the universal family, even as a stack, is not a trivial fact, and has to be proven and discussed in more detail. However, for the base field being \( \mathbb{C} \), the universal family can be constructed as an explicit quotient (see below), and this will allow us to think of it very explicitly. Note that the \( \pm 1 \) involution is no longer trivial on \( \mathcal{X}_g \) (and thus a generic point of \( \mathcal{X}_g \) is in fact smooth). See \([FaCh90]\) for a complete discussion of the moduli stack and the universal family and of the compactifications.
Definition 2.5. There is a very important Hodge vector bundle $H := \pi_*(\Omega^1_{A_g/A_g})$ on $A_g$. This is just to say that the fiber of the Hodge vector bundle over a point $[A] \in A_g$ is the $g$-dimensional space of holomorphic 1-forms on $A$. We denote by $L := \det H$ the corresponding determinant Hodge line bundle.

$A_g$ can be thought of algebraically, over any field. Let us now give the analytic picture of it over $\mathbb{C}$. If the base field is $\mathbb{C}$, the universal cover of any abelian variety is $\mathbb{C}^g$, and $A$ is given as a quotient of $\mathbb{C}^g$ by some action of $\pi_1(A)$. This is to say that an abelian variety is a quotient of $\mathbb{C}^g$ by the translations by elements of a full-rank lattice $\Lambda$, where by a lattice we mean a subgroup of $\mathbb{C}^g$ (under addition) isomorphic to $\mathbb{Z}^{2g}$, and a lattice is said to be of full rank if $\Lambda \otimes \mathbb{R} = \mathbb{C}^g$. If we act on $\Lambda \subset \mathbb{C}^g$ by an element of $GL(g, \mathbb{C})$, the quotient is going to be biholomorphic to the original one. Thus, up to biholomorphisms, any abelian variety is a quotient of $\mathbb{C}^g$ by a lattice the first $g$ generators of which are the unit vectors in all the directions. It turns out that (this is known as Riemann’s bilinear relations) the other $g$ vectors constitute a $g \times g$ matrix $\tau$ with a positive-definite imaginary part (otherwise the quotient $\mathbb{C}^g/\Lambda$ is a complex torus, but not a projective algebraic variety). Such a complex matrix is called a period matrix.

Definition 2.6. We denote by $\mathcal{H}_g$ the Siegel upper half-space — the set of all period matrices — and for a period matrix $\tau \in \mathcal{H}_g$ denote by $A_\tau := \mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g)$ the corresponding abelian variety. Notice that $\mathcal{H}_g$ is contractible.

Given a point $\tau \in \mathcal{H}_g$, there is a canonical choice of the principal polarization on $A_\tau$.

Definition 2.7. We define the theta function to be the holomorphic function of $\tau \in \mathcal{H}_g$ and $z \in \mathbb{C}^g$, given by the following formula:

$$\theta(\tau, z) := \sum_{n \in \mathbb{Z}^g} \exp(\pi i (n^t \tau n + 2n^t z)).$$

The theta function is even in $z$, and automorphic in $z$ w.r.t. to the lattice $\mathbb{Z}^g + \tau\mathbb{Z}^g$: for any $n, m \in \mathbb{Z}^g$ we have the transformation law

$$\theta(\tau, z + \tau n + m) = \exp(-\pi i n^t \tau n - 2\pi i n^t z)\theta(\tau, z).$$

Thus for a fixed $\tau$ the zero locus in $\mathbb{C}^g$ of the theta function, as a function in $z$, descends to a subvariety of $A_\tau$, which is called the theta divisor $\Theta_\tau$. This divisor then gives a principal polarization on $A_\tau$, for which the theta function generates the space of sections.
The theta function satisfies the very important heat equation
\[
\frac{\partial \theta(\tau, z)}{\partial \tau_{jk}} = 2\pi i (1 + \delta_{j,k}) \frac{\partial^2 \theta(\tau, z)}{\partial z_j \partial z_k}.
\]

The map \( \tau \mapsto A_\tau \) exhibits \( \mathcal{H}_g \) as the universal cover of \( \mathcal{A}_g \), and it is natural to ask what is the deck group of this cover, i.e. if the ppav \((A_\tau, \Theta_\tau)\) is isomorphic to \((A_{\tau'}, \Theta_{\tau'})\), how are \( \tau \) and \( \tau' \) related?

**Definition 2.8.** It turns out that there is an action of \( \text{Sp}(2g, \mathbb{R}) \) on \( \mathcal{H}_g \). If we think of \( \text{Sp}(2g, \mathbb{R}) \) as the group of \( 2g \times 2g \) matrices written in the form of four \( g \times g \) blocks such that the symplectic condition is
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
then the action is given by
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \circ \tau := (A\tau + B)(C\tau + D)^{-1}.
\]
A general element of \( \text{Sp}(2g, \mathbb{R}) \) does not map ppavs to isomorphic ppavs; however, \( \text{Sp}(2g, \mathbb{Z}) \) does: if \( \tau' = \gamma \circ \tau \) for some \( \gamma \in \text{Sp}(2g, \mathbb{Z}) \), then the ppav \( A_\tau \) is isomorphic to \( A_{\tau'} \) (the map is \( z \mapsto (C\tau + D)z \)), and it turns out that this is the only way \( A_\tau \) and \( A_{\tau'} \) can be isomorphic as ppavs, i.e. that \( \mathcal{A}_g = \mathcal{H}_g / \text{Sp}(2g, \mathbb{Z}) \).

We observe that \( \dim \mathcal{H}_g = \dim \mathcal{A}_g = \frac{g(g+1)}{2} \). The universal family \( \mathcal{X}_g \) is then the quotient of \( \mathcal{H}_g \times \mathbb{C}^g \) by the semidirect product action of \( \text{Sp}(2g, \mathbb{Z}) \times \mathbb{Z}^{2g} \) (where \( \mathbb{Z}^{2g} \) acts on \( \mathbb{C}^g \) by adding a lattice vector), and the fiber of the Hodge bundle over \( \tau \) is \( H|_\tau = H^{1,0}(A_\tau) = \mathbb{C}dz_1 \oplus \ldots \oplus \mathbb{C}dz_g \). Notice that \( H \) is of course a trivial bundle on the contractible space \( \mathcal{H}_g \), but not trivial on the quotient \( \mathcal{A}_g \).

**Remark 2.9.** To be able to talk of \( \mathcal{A}_g \) and \( \mathcal{X}_g \) constructed as quotients of \( \mathcal{H}_g \) and \( \mathcal{H}_g \times \mathbb{C}^g \), respectively, as moduli spaces or fine moduli stacks, one needs to verify that the stabilizer of any point in \( \mathcal{A}_g \) under the action of \( \text{Sp}(2g, \mathbb{Z}) \) (respectively, of any point in \( \mathcal{X}_g \) under \( \text{Sp}(2g, \mathbb{Z}) \times \mathbb{Z}^{2g} \)) is finite. To show that this is the case for \( \mathcal{A}_g \), note that any automorphism of an abelian variety can be lifted to a holomorphic map \( \mathbb{C}^g \to \mathbb{C}^g \) of the universal covers fixing \( 0 \), which is of linear growth and thus linear. Then such a linear map must map the lattice to itself, and have determinant one (to be an isomorphism, and not finite-to-one), and then there can only be finitely many such maps. The proof for \( \mathcal{X}_g \) is similar.
3. Modular forms and projective embeddings of $A_g$

The moduli space $A_g$ is not compact. There are various compactifications that one can define by studying what happens in degenerating families of ppavs, and we devote the next section to discussing these. Another approach to compactifying an algebraic variety, however, is to construct an explicit embedding of it into a projective space, and then compactify the image. For $A_g$ this is done by considering Siegel modular forms, which can be also thought of as functions on $H_g$ with certain automorphy properties, or as some representations of $\text{Sp}(2g, \mathbb{Z})$, or as sections of certain bundles on $A_g$. The study of modular forms is a vast subject, of which we barely touch the tip here — it is exposed, for example, in the books [Ig72], [Fr83]. A comprehensive recent survey of Siegel modular forms and of the questions arising already in dimension 2 is [vdG06].

Perhaps the simplest way to embed a variety into a projective space is by sections of a very ample linear system. Luckily, the Hodge line bundle $L$ is actually ample on $A_g$, though not very ample, but for full generality it pays to consider the more general situation.

In general any vector bundle $V$ on a variety $X$ can be lifted to its universal cover $\tilde{X}$. If $\tilde{X}$ is contractible (and thus any bundle on $\tilde{X}$ is trivial), then a section of $V$ lifts to a global vector-valued function on $\tilde{X}$, which transforms appropriately under the action of $\pi_1(X)$ on $\tilde{X}$. This is the concept of automorphic forms: studying sections of bundles on a variety as functions on the universal cover, subject to certain transformation rules.

**Definition 3.1.** Given a subgroup $\Gamma \subset \text{Sp}(2g, \mathbb{Z})$ and a rational representation $\rho : \text{GL}(g, \mathbb{C}) \to \text{GL}(W)$ for some vector space $W$, a $\rho$-valued modular form is a map $F : H_g \to W$ such that

$$F(\gamma \circ \tau) = \rho(C\tau + D) \circ F(\tau) \quad \forall \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma, \forall \tau \in H_g$$

(where, as always, we write $\gamma$ as four $g \times g$ blocks), such that moreover for $g = 1$ we require $F$ to be regular at the cusps of $H_1/\Gamma$.

If $W = \mathbb{C}$, and the representation is $\rho(\gamma) = \det(C\tau + D)^k$, then the modular form is called a (scalar) weight $k$ modular form for $\Gamma$. It can be shown, by writing down the transformation law for holomorphic 1-forms on $A_1$ under the action of $\text{Sp}(2g, \mathbb{Z})$ on $H_g$, that the Hodge vector bundle $H$ is in fact the bundle of modular forms for the standard (identity) representation, and thus $L$ is the bundle of (scalar) modular forms of weight 1.
It is hard to construct scalar modular forms of small weight for the entire group $\text{Sp}(2g, \mathbb{Z})$. However, one can use the theta function to construct modular forms for subgroups.

**Definition 3.2.** For any $m \geq 2$ and any $\varepsilon, \delta \in (\frac{1}{m}\mathbb{Z}/\mathbb{Z})^g$ the *level $m$ theta function with characteristics* $[\varepsilon, \delta]$ is defined as

$$\theta_{[\varepsilon, \delta]}(\tau, z) := \sum_{n \in \mathbb{Z}^g} \exp(\pi i (n + \varepsilon)^t \tau (n + \varepsilon) + 2(n + \varepsilon)^t (z + \delta))) = \exp(\pi i (\varepsilon^t \tau \varepsilon + 2\varepsilon^t (z + \delta))) \theta(\tau, z + \tau \varepsilon + \delta).$$

As a function of $z$, the level $m$ theta function is a section of the theta bundle translated by the corresponding point of order $m$, and thus for all $\varepsilon, \delta$, $
abla_{[\varepsilon, \delta]}(\tau, z)^m$ is a section of the bundle $m\Theta_\tau$ on $A_\tau$. The space $H^0(A_\tau, m\Theta_\tau)$ is $m^g$-dimensional, with the basis given by *theta functions of order $m$*: for $\varepsilon \in (\frac{1}{m}\mathbb{Z}/\mathbb{Z})^g$ these are defined as

$$\Theta_{\varepsilon}(\tau, z) := \theta_{[\varepsilon, 0]}(m \tau, mz).$$

**Remark 3.3.** To see that $\Theta_{\varepsilon}$ is a section of the bundle $m\Theta_\tau$ on $A_\tau$, note that for a general ppav we have $H^2(A_\tau) = \mathbb{C}\Theta_\tau$. Now compute the top power of the divisor of $\Theta_{\varepsilon}$ on $A_\tau$, using $\Theta_\varepsilon^g = g!$. Indeed, the multiplication by $m$ map has degree $m^2g$ on $A_\tau = \mathbb{C}^g/(m\mathbb{Z}^g + m\tau\mathbb{Z}^g) \equiv \mathbb{C}^g/(\mathbb{Z}^g + m\tau\mathbb{Z}^g)$, which is a degree $m^g$ cover of $A_{m\tau} = \mathbb{C}^g/(\mathbb{Z}^g + m\tau\mathbb{Z}^g)$, and thus the top power of the divisor of $\Theta_{\varepsilon}$ on $A_\tau$ is $m^g g!$.

The value of the (level or order) theta function at $z = 0$ is called the associated (level or order) *theta constant*. As a function of $\tau$ for fixed $\varepsilon, \delta$, the *order $m$ theta constant* is a modular form of weight $1/2$ wrt the finite index subgroup $\Gamma(m, 2m) \subset \text{Sp}(2g, \mathbb{Z})$ (normal for $m$ even), defined as follows in two steps:

$$\Gamma(m) := \left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod m \right\}$$

$$\Gamma(m, 2m) := \{ \gamma \in \Gamma(m) \mid \text{diag}(A^t B) \equiv \text{diag}(C^t D) \equiv 0 \mod 2m \}$$

The *level $m$ theta constant* is also a modular form, also of weight $1/2$, with respect to the (smaller) group $\Gamma(m^2, 2m^2)$.

**Remark 3.4.** Notice a peculiar feature of the theta functions here: as functions of $z$, $m$'th powers of the level theta functions are sections of the same bundle, $m\Theta$, on a fixed abelian variety, as the theta functions of order $m$. However, theta *constants* of any order or level are all of weight $1/2$, with respect to the appropriate level subgroups.
Definition 3.5. We call the quotient $A_g(m, 2m) := \mathcal{H}_g/\Gamma(m, 2m)$ the level moduli space of ppavs — this is a finite cover of $A_g$. The subgroup $\Gamma(m, 2m) \subset \text{Sp}(2g, \mathbb{Z})$ is normal if and only if $m$ is even, and in this case the cover is Galois. Since all theta constants of order $m$ are sections of $\frac{1}{2}L$ on $A_g(m, 2m)$, we can use them to define the theta constant map

$$Th_m : A_g(m, 2m) \longrightarrow \mathbb{P}^{m^g - 1}$$

$$[\tau] \mapsto \{\Theta[\epsilon](\tau, 0)\}_{\epsilon \in \left(\frac{1}{m}\mathbb{Z}/\mathbb{Z}\right)^g}$$

A priori this is just a rational map, but the main result about it is

Theorem 3.6 (Igusa for $m = 4r^2$, Mumford for $m \geq 4$, Salvati Manni for $m \geq 3$; see [Ig72], [BiLa04]). $Th_m$ is an embedding for all $m \geq 3$.

Algebraically this theorem says that the bundle $\frac{1}{2}L$ is very ample on $A_g(m, 2m)$, which implies that a sufficiently high power of $L$ is very ample on $A_g$, and so $L$ is ample on $A_g$. This can be also checked directly by computing the natural metric on $L$ and checking that it is positive.

The map $Th_2$ is known to be generically injective, and believed to be in fact an embedding — see [SM94b]. It can in fact be shown that for $m = 2k > 2$ the level moduli space $A_g(m, 2m)$ (or in fact $A_g(m)$ for any $m \geq 3$) is a smooth variety, i.e. that the group $\Gamma_g(m, 2m)$ acts freely on $\mathcal{H}_g$. Thus the orbifold $A_g$ has a global manifold cover of a finite degree, which often allows one to work rigorously on the orbifold $A_g$ by passing to the level cover.

Remark 3.7. Taking the closure of the image $Th_m(A_g(m, 2m))$ in $\mathbb{P}^{m^g - 1}$ defines a compactification of the moduli space. It turns out that modular forms extend to the Satake compactification (which we define in the next section). Igusa used theta functions to study the fiber of $Th_m$ over the boundary, and showed that for $m > 4$ it consists of more than one point (he computed the number of points for $m = 4r^2$, and bounded it below for other $m$), while the map $Th_4$ is injective on the boundary of the Satake compactification as well. However, for $g \geq 6$ the map $Th_4$ is not an embedding of the Satake compactification — the inverse is not regular near the boundary. The fact that there exist modular forms that are not polynomial in theta constants, and the relation of the analytic structure near the boundary of $Th_m(A_g(m, 2m))$ with the analytic structure of the Satake boundary are considered in [Ig64], [Ig81], [SM90], [SM94a].

3.8 (Vector-valued modular forms). The above discussion tells us that the line bundle of scalar modular forms is ample on $A_g$. What about vector-valued modular forms? This is some kind of ampleness.
question for a vector bundle. Let us see what happens if \( \rho : GL(g, \mathbb{C}) \rightarrow GL(\mathbb{C}^g) \) is the standard representation \( std \) tensored with a power of the det (i.e. a power of \( L \)).

It can be shown that the \( z \)-gradients at zero of order \( m \) theta functions
\[
\text{grad}_z \Theta[\varepsilon](\tau, z)|_{z=0},
\]
are \( std \otimes \text{det}^{1/2} \)-valued modular forms for \( \Gamma(m, 2m) \). Varying \( \varepsilon \) one gets different modular forms, and thus for \( m > 2 \) we can define the map
\[
\Phi_m : \mathcal{A}_g(m, 2m) \rightarrow G(g, m^g)
\]
\[
\tau \mapsto \{\text{grad}_z \Theta[\varepsilon](\tau, z)|_{z=0}\}_{\varepsilon \in \left(\frac{1}{m} \mathbb{Z}/\mathbb{Z}\right)^g},
\]
where \( G(g, m^g) \) denotes the Grassmannian of \( g \)-dimensional subspaces of \( \mathbb{C}^{m^g} \) (a priori it is a map to \( \text{Mat}_{g \times m^g}(\mathbb{C}) \), but it turns out [SM96] that the rank of the image matrix is always \( g \)). Notice that all theta functions of order 2 are even in \( z \), and thus the map \( \Phi_2 \) is undefined.

**Theorem 3.9** ([GrSM04] for \( m = 4 \), [GrSM06] for \( m = 4k > 4 \)). If the level \( m \) is divisible by 4, then the map \( \Phi_m \) is an embedding for all \( m > 4 \), and is generically injective for \( m = 4 \) (though we actually believe \( \Phi_4 \) is also an embedding).

The condition that \( m \) is divisible by 4 is likely technical, but our proof, which deduces the injectivity of \( \Phi_m \) from the injectivity of \( Th_{m/2} \) and \( Th_m \), uses it. Note also that one can consider the gradients at zero of theta functions of level \( m \), but this does not give any new information.

**Remark 3.10.** This implies that the vector bundle of \( std \otimes \text{det}^{1/2} \)-valued modular forms is very ample on \( \mathcal{A}_g(m, 2m) \) in some sense (it can be shown that the space of such modular forms is generated by gradients of theta functions). This theorem has a geometric interpretation, and is related to classical algebraic geometry. Indeed, on any ppav \( A_\tau \) the line bundle \( \Theta_\tau \) is ample, and \( m\Theta_\tau \) is very ample for \( m \geq 3 \) (this fact is known as the Lefschetz theorem). For any characteristic of level \( m \) the function \( \theta \left[ \frac{\varepsilon}{\delta} \right] (z)^m \) is a section of \( |m\Theta_\tau| \) (a theta function with characteristics is a section of \( \Theta_\tau \) shifted by \( \varepsilon \tau + \delta \), and thus the shift for the \( m' \)-th power is \( m(\varepsilon \tau + \delta) = 0 \in A_\tau \)). It can be shown that the space of sections of \( |m\Theta_\tau| \) is generated by these \( m' \)-th powers. \( Th_m(\tau) \) is then the image of the origin in the corresponding embedding \( F : A_\tau \hookrightarrow \mathbb{P}^{m^2 - 1} \). Instead of taking \( F(0) \), one can take the the differential \( dF(0) \), which is exactly \( \Phi_m(\tau) \).
Given a plane quartic, its bitangent lines are sections of one half of the canonical system, i.e. are level 2 theta constants, and this is what the map $\Phi_2$ is for the corresponding Jacobian in $\mathcal{M}_3 \subset \mathcal{A}_3$. In [Caporaso03a] Caporaso and Sernesi show that a plane quartic is generically determined by its bitangents, in [Caporaso03b] they generalize this to higher genus curves, and in [Lehavi05] Lehavi explicitly reconstructs quartics from their bitangents. Our result is almost a generalization of all these from curves to ppavs (though not quite: there are some issues with symmetrizing and projectivizing that we cannot deal with for $\mathcal{A}_g$), and it is also a step towards better understanding the rings of vector-valued modular forms and to perhaps answering an old question of Weil, essentially on the relation of the maps $\det \Phi_m$ and $\Theta_m$. We refer to [Fay79], [Igus80] and [SM83] for more details on the problem and past results; we used the above framework to further investigate this with Salvati Manni in [GrSM05].

4. Degeneration: compactifications of $\mathcal{A}_g$

In the previous section we constructed explicit projective embeddings of level covers of $\mathcal{A}_g$, which thus naturally induce some compactifications. We will now proceed to construct abstractly compactifications of $\mathcal{A}_g$ and understand their geometry — their relation to the ones obtained from projective embeddings is still not entirely clear. The discussion we present is necessarily greatly simplified — we refer to [FarkasCh90] for the complete details in full generality, and also to [AMRT75], [NaY96], [An00], [AlNa99], [Hu00b], [Ol06] for more comprehensive explanations and the intuition about toroidal compactifications. A more detailed discussion of the explicit boundary geometry, especially for $g = 2$, can also be found in the book [HKW93] and the survey [HuSa02], while the original constructions are given in [Mu72].

The Siegel space $\mathcal{H}_g$ is not compact — the entries of a period matrix $\tau$ can tend to infinity, or $\text{Im}\, \tau$ can become degenerate instead of being positive definite. It can be shown that the action of $\text{Sp}(2g, \mathbb{Z})$ can conjugate the second kind of degeneration into the first kind of degeneration — so the only degeneration one needs to consider in working with $\mathcal{A}_g$ is when the entries of the period matrix grow unboundedly.

To compactify $\mathcal{A}_g$ we need to attach some boundary points as limits of degenerating families; it would also be nice to have some geometric objects that are degenerations of abelian varieties correspond to the extra points we add as the boundary. There are two possible approaches.
**Approach 1:** we take \([\tau] \in A_{g-1}\) as the limit of the degenerating family \(\lim_{t \to \infty} \begin{pmatrix} it & w \\ w^t & \tau \end{pmatrix}\) (where \(w \in \mathbb{C}^{g-1}\) and \(\tau \in A_{g-1}\) are fixed), i.e. we add \(A_{g-1}\) as a boundary component. This means that the boundary is going to be high codimension and very singular. However, the good thing is that when we consider more complicated degenerations, the choice of what to do is natural. Indeed, we can set for example
\[
\lim_{t_1, t_2 \to \infty} \begin{pmatrix} it_1 & x & w_1 \\ x & it_2 & w_2 \\ w_1^t & w_2^t & \tau \end{pmatrix} = [\tau] \in A_{g-2}
\]
(recall that the imaginary part of a period matrix is positive-definite, so this is the way the degeneration has to look).

**Definition 4.1.** The object we get as the result is called the Satake, or Baily-Borel, or minimal, compactification of \(A_g\). As a set, it is
\[
A^S_g := A_g \sqcup A_{g-1} \sqcup \ldots \sqcup A_1 \sqcup A_0,
\]
and much more work is necessary to properly describe the analytic and algebraic structure near the boundary. It can be seen that modular forms extend to \(A^S_g\), i.e. that the bundle \(L\) extends to \(A^S_g\) as a line bundle. The extension of theta constants to the level Satake compactification can be computed directly:
\[
\lim_{t \to \infty} \Theta[\varepsilon_1 \varepsilon_2] \begin{pmatrix} it & w \\ w^t & \tau \end{pmatrix} = \left( \lim_{t \to \infty} \Theta[\varepsilon_1](it) \right) \Theta[\varepsilon_2](\tau) = \delta_{\varepsilon_1,0} \Theta[\varepsilon_2](\tau),
\]
where \(\delta_{\varepsilon_1,0}\) is the Kronecker symbol, i.e. the extension is zero if \(\varepsilon_1 \neq 0\). Thus the map \(Th_m\) extends to \(A^S_g(m, 2m)\).

The space \(A^S_g\) is highly singular, and the boundary points represent lower-dimensional ppavs, which of course are not degenerations of \(g\)-dimensional ppavs, so let us try to get a different compactification.

**Approach 2:** We say that \(\lim_{t \to \infty} \begin{pmatrix} it & w \\ w^t & \tau \end{pmatrix}\) is the pair \((\tau, w)\). The vector \(w\) is only defined up to \(\tau \mathbb{Z}^{g-1} + \mathbb{Z}^{g-1}\) (we can act by the symplectic group, preserving the one infinity in the period matrix), i.e. we have \(w \in A_\tau\), and so can think of the pair \((\tau, w)\) as a point in the universal family. Note, however, that if \(A_\tau\) has an automorphism \(\sigma\) (and all ppavs have involution \(\pm 1\)), then the points \(\tau, w\) and \(\tau, \sigma(w)\) would define the same semiabelian object.

**Definition 4.2.** The object we get by adding all of these boundary points is called the partial compactification of \(A_g\). Set-theoretically it
is
\[ \mathcal{A}_g^* := \mathcal{A}_g \sqcup \mathcal{X}_{g-1}/ \pm 1. \]
\( \mathcal{A}_g^* \) is the blowup of the partial Satake compactification \( \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \) along the boundary.

### 4.3 (Rank one semiabelian varieties)

The boundary of \( \mathcal{A}_g^* \) is codimension one; its points represent (torus rank one) semiabelian varieties, which are defined as follows: given \( (\tau, w) \in \mathcal{X}_{g-1} \) compactify the \( \mathbb{C}^* \)-extension

\[ 1 \to \mathbb{C}^* \to G \to A_\tau \to 0 \]

to a \( \mathbb{P}^1 \)-bundle \( \tilde{G} \), by adding 0 and \( \infty \) sections, and then identify the 0 and \( \infty \) sections with a shift by \( w \in A_\tau \), getting a non-normal variety \( \bar{G} := \tilde{G}/(0, \infty) \sim (x + w, \infty) \). The principal polarization on such a semiabelian variety is a codimension one subvariety of \( \bar{G} \), which intersects the zero section of the \( \mathbb{P}^1 \)-bundle \( \tilde{G} \) in the theta divisor of \( A_\tau \), and is globally a blowup of a section of \( \tilde{G} \) with center \( \Theta_{A_\tau} \cap t_w \Theta_{A_\tau} \) (\( t_w \) denotes the translation by \( w \)). The existence of such a subvariety determines the extension in (1) uniquely — it depends on \( \tau \) and \( w \).

No choice is involved in the construction of \( \mathcal{A}_g^* \), but it is still not compact. How can we extend it to an actual compactification, i.e. what should for example be the limit

\[ \lim_{t_1, t_2 \to \infty} \begin{pmatrix} it_1 & x & w_1 \\ x & it_2 & w_2 \\ w_1^t & w_2^t & \tau \end{pmatrix} ? \]

We can certainly keep track of \( (\tau, w_1, w_2) \in \mathcal{X}_{g-2}^{\times 2 (\text{fiberwise})} \), but if we want this type of degenerations to form a codimension two stratum in the compactification — after all, we have two entries of the period matrix degenerating — we need one more piece of data, and that is \( x \). The problem is that \( x \) may also go to \( i \infty \), and may change when we conjugate the period matrix by elements of \( \text{Sp}(2g, \mathbb{Z}) \) while leaving the two infinities intact. Thus to keep track of this extra coordinate properly (and to do this in general for higher codimension generations) we need to make a choice of a so-called cone decomposition. We now give an idea of what this entails, and encourage the reader to learn the theory properly by looking at [AMRT75], [Na76], [FaCh90], [HKW93], [Hu00b], [Al02], [Ol06] and references therein.

### 4.4

Instead of making the entries of the period matrix go to infinity, we would now rather think of the imaginary part becoming positive semidefinite. Fix generators \( x_1, \ldots, x_g \) of \( \mathbb{Z}^g \), and think of the space
Sym$_2$(\(\mathbb{Z}^g\)) of integer-valued bilinear forms on \(\mathbb{Z}^g\). Identifying this with the space of quadratic forms, it is a finite-dimensional free \(\mathbb{Z}\)-module generated by \(x_i^2\) and \(2x_ix_j\) for \(i \leq j\). Denote by \(\mathcal{C}(\mathbb{Z}^g)\) the \(\mathbb{R}_{\geq 0}\)-span of the positive semidefinite quadratic rational forms on \(\mathbb{Z}^g\), i.e. \(\mathcal{C}(\mathbb{Z}^g)\) is the cone generated by positive semidefinite \(g \times g\) rational matrices. All of this is the data used to understand the orbits of the \(\text{Sp}(2g, \mathbb{Z})\) action on the boundary of \(\mathcal{H}_g\), i.e. on the set of symmetric matrices with positive semidefinite imaginary part.

What we would now need is to somehow have local “coordinates” on \(\mathcal{C}(\mathbb{Z}^g)\), in which we would be able to keep track of the degeneration happening. Doing so globally is impossible since \(\mathcal{C}(\mathbb{Z}^g)\) is not finitely generated. Thus what we need to do is decompose it into infinitely many finitely-generated polyhedral cones, i.e. each cone should be a finite span \(\mathbb{R}_{\geq 0}q_1 + \ldots + \mathbb{R}_{\geq 0}q_k\), where \(q_i \in \text{Sym}_2(\mathbb{Z}^g)\) are semipositive definite, and when two cones intersect, they should intersect along a face. Moreover, note that the natural action \(\text{GL}(g, \mathbb{Z}) : \mathbb{Z}^g\) extends to an action on \(\mathcal{C}(\mathbb{Z}^g)\), and thus it is natural to ask for our cone decomposition to be invariant under this \(\text{GL}(g, \mathbb{Z})\) action. There may of course exist different cone decompositions (each encoded by a finite amount of data, though, as the cones in it would fall into finitely many \(\text{GL}(g, \mathbb{Z})\)-orbits), and choosing different ones yields different toroidal compactification.

**Definition 4.5.** The names for some common choices of the cone decompositions and the corresponding toroidal compactifications are the following (unfortunately it seems that defining and discussing the precise construction of each of these would be quite long — the readers interested in this are advised to read more comprehensive sources listed above):

- The **perfect cone**, also called **first Voronoi compactification** \(\overline{\mathcal{A}}_g^P\).
- The **second Voronoi compactification** \(\overline{\mathcal{A}}_g^V\).

It was shown by Namikawa [Na76] that the Torelli embedding \(\mathcal{M}_g \hookrightarrow \mathcal{A}_g\) extends to a map (no longer an embedding) \(\overline{\mathcal{M}}_g \rightarrow \overline{\mathcal{A}}_g^V\) of the Deligne-Mumford compactification.

The **Igusa compactification** \(\overline{\mathcal{A}}_g^\text{Igusa}\), which is the monoidal blowup of the Satake compactification along the boundary, corresponding to the central cone decomposition.

**Example:** For genus 2 all the toroidal compactifications we mentioned above coincide. They are defined by considering the polyhedral
cone

\[
\sigma := \mathbb{R}_{\geq 0} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathbb{R}_{\geq 0} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \mathbb{R}_{\geq 0} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \subset \mathcal{C}(\mathbb{Z}^2),
\]

(notice that all generators are indeed degenerate), and the cone decomposition of \(\mathcal{C}(\mathbb{Z}^2)\) is obtained by taking the \(GL(2, \mathbb{Z})\) orbits of \(\sigma\) and of its faces.

**Remark 4.6.** All toroidal compactifications of \(\mathcal{A}_g\) admit a contracting morphism to \(\mathcal{A}_g^S\). We remark, however, that the stratum over \(\mathcal{A}_{g-i} \subset \mathcal{A}_g^S\) is in general very complicated. Even the dimension of the preimage of \(\mathcal{A}_{g-i}\) in \(\overline{\mathcal{A}}_g\) depends on the choice of the compactification: for example the stratum of \(\overline{\mathcal{A}}_g^P\) lying over \(\mathcal{A}_{g-i}\) always has codimension \(i\), while already the preimage of \(\mathcal{A}_0\) under the map \(\overline{\mathcal{A}}_4^V \to \mathcal{A}_4^S\) is a divisor.

It is natural to ask if boundary points of a compactification of \(\mathcal{A}_g\) have a geometric interpretation; do they parameterize some degenerate objects that live in a universal family? For the case of \(\overline{\mathcal{A}}_g^V\), the answer to these questions was recently shown to be positive:

**Theorem 4.7** (Alexeev [Al02]). The second Voronoi compactification \(\overline{\mathcal{A}}_g^V\) is an irreducible component of a functorial compactification of \(\mathcal{A}_g\), i.e. of some “natural” compactification from the point of view of moduli theory, over which the universal family exists. Thus the boundary points of \(\overline{\mathcal{A}}_g^V\) represent geometric objects, and \(\overline{\mathcal{A}}_g^V\) is projective.

**Theorem 4.8** (Olsson [Ol06]). Within the functorial compactification \(\overline{\mathcal{A}}_g^V\) is distinguished as the component parameterizing log smooth objects.

In view of this theorem, and especially since it is still not even known whether there is a universal family over \(\overline{\mathcal{A}}_g^P\) or any other toroidal compactification, one may ask whether \(\overline{\mathcal{A}}_g^V\) is then the “natural” choice of a toroidal compactification, or whether any other toroidal compactifications are singled out by some geometric constructions? In the next section we will discuss why \(\overline{\mathcal{A}}_g^P\) is also very important. Meanwhile, there is another naturally singled out compactification, though it may be one of those that we have defined above.

**Open Problem 1.** Which compactification does the map \(\Phi_m\) from theorem 3.9 induce, i.e. what is the structure of the closure of the image \(\Phi_m(\mathcal{A}_g(m, 2m))\), for \(m = 4k\)?
Remark 4.9. It can be shown by studying the degenerations of theta functions directly that \( \Phi_m \) extends to an embedding of \( \mathcal{A}_g^*(m, 2m) \) for \( m = 4k \). Since the Hodge vector bundle and its determinant line bundle extend as bundles to any toroidal compactification \([\text{Mu77}]\), the gradients of theta functions extend to the boundary of any toroidal compactification. However, the map \( \Phi_m \) may not be defined on the boundary if the gradients no longer span a \( g \)-dimensional space, and injectivity seems very hard to deal with. We certainly get some blowup of \( \mathcal{A}_g^S \), since essentially we are somehow resolving the singularities of \( \mathcal{A}_g^S \) by taking derivatives of modular forms, but it is not even clear if the induced compactification is toroidal.

One can also ask what happens for maps induced by vector-valued modular forms for representations of \( \text{GL}(g, \mathbb{C}) \) other than \( \text{std} \otimes \det^{1/2} \), but this currently seems to be entirely out of reach: while we can hope to understand the degeneration of the polarization and thus of theta functions, it is not clear how to understand the extensions of general modular forms.

5. Birational geometry: divisors on \( \mathcal{A}_g \)

In this section we discuss the recent progress and the open questions in the study of the birational geometry of \( \mathcal{A}_g \) and its compactifications. We give the description of the nef cone of \( \mathcal{A}_g^* \) (and of \( \mathcal{A}_g^{g'} \)), due to Hulek and Sankaran; of the nef cone of \( \mathcal{A}_g^{P} \), due to Shepherd-Barron, and the possible approaches and known results about the effective cone. We also draw comparisons with moduli of curves.

It is a by now classical result of Borel in group cohomology saying that \( h^2(\text{Sp}(2g, \mathbb{Z})) = 1 \) for \( g \geq 3 \). Since \( \text{Sp}(2g, \mathbb{Z}) \) is the universal covering group for \( \mathcal{A}_g \), and \( \mathcal{H}_g \) is contractible, this yields \( \text{Pic}_Q(\mathcal{A}_g) = QL \) (in fact for all \( g \)). Since the boundary divisor of \( \mathcal{A}_g^* \) is irreducible, it follows that \( \text{Pic}_Q(\mathcal{A}_g^*) = QL \oplus QD \). It can in fact be shown that the boundary divisor is also irreducible on \( \mathcal{A}_g^{P} \), so that it follows that \( \text{Pic}_Q(\mathcal{A}_g^{P}) = \text{Pic}_Q(\mathcal{A}_g^*) = QL \oplus QD \), while in general \( \text{Pic}_Q(\mathcal{A}_g^{g'}) \) is higher-dimensional.

Definition 5.1. Recall that a divisor (we always talk about \( \mathbb{Q} \)-divisors, since we are on an orbifold/stack) is called ample if on any subvariety (including the variety itself) its top power is positive; a divisor is called nef (numerically effective) if it intersects all curves non-negatively; and a divisor is called effective if it is a positive linear combination of codimension one subvarieties.
For a divisor \( E = aL - bD \in \text{Pic}(\mathcal{A}_g) = \text{Pic}(\overline{\mathcal{A}}_g^p) \) we call the ratio \( s(E) := a/b \) the slope of \( E \); if \( E \) is (the closure in \( \mathcal{A}_g^* \) or \( \overline{\mathcal{A}}_g^p \) of) the zero locus in \( \mathcal{A}_g \) of a modular form, then the slope is the weight of the modular form divided by the generic vanishing order on the boundary.

The sets of effective/nef/ample divisors form respectively the cones \( \text{Eff}/\text{Nef}/\text{Amp} \), which are important invariants. Since \( \text{Pic}(\overline{\mathcal{A}}_g^p) = \text{Pic}(\mathcal{A}_g^*) \) is two-dimensional for any genus, the slopes of the boundaries of the cone (which we then call the slope of the cone, denoted \( s(\text{Eff}(\mathcal{A}_g^*)) \), etc.) determine the cone, and computing these cones may be more amenable than, say, for \( \overline{\mathcal{M}}_g \), where the Picard group is higher dimensional, and though there has been significant progress in understanding the nef cone [GKM02] and the minimal slope of the effective cone (reviewed in [Fa06b]) the nef and effective cones of \( \overline{\mathcal{M}}_g \) are still unknown.

**Definition 5.2.** For birational geometry it is especially important to know whether the canonical class is ample, effective, or neither. The **Kodaira dimension** of a variety \( X \) is a number \( \kappa \) such that \( h^0(X, mK_X) \) grows as \( m^{\kappa} \) for \( m \) large (more precisely, \( \kappa(X) := \limsup_{m \to \infty} \frac{\ln h^0(X, mK_X)}{\ln m} \)).

In general we have \( \kappa(X) \in \{-\infty, 0, \ldots, \dim X\} \), and a variety is said to be of **general type** if \( \kappa = \dim X \).

The Kodaira dimension of a variety is a birational invariant. The minimal model conjecture/program states that any variety of general type is birational to a **canonical model**, i.e. a variety with only canonical singularities, and such that on it the canonical divisor is ample. Thus if the canonical class is ample and the singularities are canonical, the variety is its own canonical model.

To compute the canonical class of \( \mathcal{A}_g \) and \( \mathcal{A}_g^* \), one writes down the explicit volume form \( \omega(\tau) := \bigwedge_{i \leq j} \tau_{ij} \) on \( \mathcal{H}_g \). To get the class \( K_{\mathcal{A}_g} \in \text{Pic}(\mathcal{A}_g) \) one needs to determine the transformation properties of the form \( \omega \) under the action of \( \text{Sp}(2g, \mathbb{Z}) \). It turns out that \( \omega(\gamma \tau) = \det(C\tau + D)^{-g-1} \omega(\tau) \), which means that \( K_{\mathcal{A}_g} = (g+1)L \). Now determining the class of \( K_{\mathcal{A}_g^*} \) is very easy — we just need to see how fast \( \omega(\tau) \) degenerates as \( \tau \) goes to the boundary of \( \mathcal{A}_g^* \), i.e. as say \( \tau_{11} \to i\infty \). Clearly in this case there is one factor in \( \omega \), precisely \( d\tau_{11} \), which degenerates, and thus we get

\[
K_{\mathcal{A}_g^*} = (g+1)L - D.
\]

The same expression is true for \( K_{\overline{\mathcal{A}}_g^p} \).
5.3 (The nef cone of \( \mathcal{A}^*_g \)). Determining the nef cone is equivalent to determining the cone of effective curve classes, as these are dual. From our review of modular forms we know that \( L \) is ample on \( \mathcal{A}^*_g \), and thus it is nef on \( \mathcal{A}^*_g \), which admits a contracting map to \( \mathcal{A}^*_g \). Moreover, on the fiber of the map \( \partial \mathcal{A}^*_g = \mathcal{X}_{g-1} \rightarrow \mathcal{A}_{g-1} \) over some point \([B] \in \mathcal{A}_{g-1}\) the restriction \( D|_D = -2\Theta_B \) (see [Mu83]), and thus \(-D\) is relatively ample with respect to this contraction map.

There are two easy to construct curve classes in \( \mathcal{A}^*_g \). Let \( C_1 \subset \mathcal{A}^*_g \) be any curve in the boundary projecting to a point in \( \mathcal{A}^*_g \), i.e. \( C_1 \subset B \), where \( B \) is the fiber of \( \partial \mathcal{A}_g = \mathcal{X}_{g-1} \rightarrow \mathcal{A}_{g-1} \) over \([B] \in \mathcal{A}_{g-1}\). Since \( L \) is ample on \( \mathcal{A}_g^* \) and \( L.C_1 = 0 \), the curve \( C_1 \) must lie in the boundary of the cone of effective curves. Dually, \( L \) must lie in the boundary of the nef cone.

Another curve class in \( \mathcal{A}^*_g \) one can consider is \( C_2 := \overline{\mathcal{A}}_1 \times [B] \), where \([B] \in \mathcal{A}_{g-1}\) is fixed, i.e. this is the family of elliptic tails. The intersection \( L.C_2 = 1/24 \) — this is the (stacky) degree of \( L \) on \( \mathcal{A}_1 \), which can be computed by computing the appropriate orbifold structure on \( \mathbb{P}^1 = \mathcal{H}/\text{SL}(2,\mathbb{Z}) \) or by integrating the volume form over this fundamental domain. The intersection \( D.C_2 \) is equal to \( 1/2 \) — there is exactly one point in the boundary of \( \overline{\mathcal{A}}_1 \), and the corresponding semialbelian object \( C^* \) has an involution. Thus we have \( (12L - D).C_2 = 0 \). If we had a map of \( \mathcal{A}^*_g \) contracting \( C_2 \), we would conclude that \( 12L - D \) is the other boundary of the nef cone. Unfortunately, such a map is not known, but the result still holds.

**Theorem 5.4.** a) (Hulek and Sankaran, [HuSa04]) The cone of effective curves on \( \mathcal{A}^*_g \) is generated by \( C_1 \) and \( C_2 \), i.e. the nef cone is

\[
\text{Nef}(\mathcal{A}^*_g) = \{aL - bD \mid a \geq 12b \geq 0\},
\]

so the minimal slope of nef divisors is 12.

b) (Hulek [Hu00a]) For the genus 2 and 3 toroidal compactifications the same result holds (for \( g \leq 3 \) the perfect cone, central cone, and the second Voronoi compactifications coincide).

Genus 3 is the highest in which the first and second Voronoi compactifications coincide. In general \( \overline{\mathcal{A}}_g^P \neq \overline{\mathcal{A}}_g^V \), and the birational map from one to the other is regular in neither direction.

However, in dimension 4 there exists a contracting morphism \( \overline{\mathcal{A}}_4^V \rightarrow \overline{\mathcal{A}}_4^P \), with an irreducible exceptional divisor that we denote by \( E \), over the stratum \( \mathcal{A}_0 \subset \mathcal{A}_4^S \). The explicit geometric and combinatorial description of the toroidal compactifications in dimension 4, though very hard, gives an approach to the nef cone of \( \overline{\mathcal{A}}_4^V \) — here is the result.
Theorem 5.5 (Hulek and Sankaran [HuSa04]). The nef cone of $\overline{A}_4^P$ is the same as for the partial compactification, i.e.

$$\text{Nef}(\overline{A}_4^P) = \{aL - bD \mid a \geq 12b \geq 0\}.$$ 

For the second Voronoi compactification, we have

$$\text{Pic}(\overline{A}_4^V) = \mathbb{Q}L \oplus \mathbb{Q}D \oplus \mathbb{Q}E;$$

$$\text{Nef}(\overline{A}_4^V) = \{aL - bD - cE \mid a \geq 12b \geq 24c \geq 0\}.$$ 

5.6 (Canonical model of $A_g$). In [HuSa02] the question of determining the cone $\text{Nef}(\overline{A}_g^V)$ for arbitrary $g$ is posed, but, as explained in [HuSa04], it seems that the dimensions of $\text{Pic}(\overline{A}_g^V)$ grow fast with $g$, and thus this question, though very interesting especially because of Alexeev’s interpretation of $\overline{A}_g^V$ as the functorial compactification, currently seems beyond reach.

However, $\text{Pic}(\overline{A}_g^P)$ is always two-dimensional, and in view of the above $g \leq 4$ results it is tempting to conjecture that $\text{Nef}(\overline{A}_g^P) = \text{Nef}(\overline{A}_g^*)$. This is indeed the case, as was recently proven:

**Theorem 5.7** (Shepherd-Barron [S-B06]). In any genus the nef cone of $\overline{A}_g^P$ is the same as that of $\overline{A}_g^*$, i.e. has minimal slope 12:

$$\text{Nef}(\overline{A}_g^P) = \{aL - bD \mid a \geq 12b \geq 0\}.$$ 

Proving this requires a very detailed study of the structure of $\partial \overline{A}_g^P$ and the torus action on it. One describes the strata of $\overline{A}_g^P \to \overline{A}_g^S$ over each $\mathcal{A}_i \subset \partial \overline{A}_g^S$ explicitly, as torus fibrations over the fiberwise $(g-i)\text{th}$ power of the universal family $\mathcal{A}_i$ of ppavs over $\mathcal{A}_i$ — this uses the specific geometry and combinatorics of the perfect cone decomposition. One then uses the torus action along the fibers in each stratum to “average” any effective curve — for the perfect cone compactification we get then a curve on the “zero-section” of the torsor, i.e. on $\mathcal{A}_i \times (g-i)(\text{fiberwise})$.

One then uses the fact that the stratum of $\overline{A}_g^P$ lying over $\mathcal{A}_{i-1} \subset \overline{A}_g^S$ is up to codimension two essentially the partial compactification of the power of the universal family over $\mathcal{A}_i^*$. After more hard work one eventually deduces that if there exists a curve $C \subset \overline{A}_g^P$ projecting to a point of $\mathcal{A}_i$ such that $(12L - D).C < 0$, then there exists such a curve over $\mathcal{A}_{i-1}$, and then induction yields a contradiction. In doing this, the explicit understanding of the geometry of the perfect cone enters in many places and plays a crucial role.
Corollary 5.8 (Shepherd-Barron [S-B06]). \(\overline{A}_g^P\) is the canonical model of \(A_g\) for \(g \geq 12\), since \(K_{\overline{A}_g^P} = (g + 1)D - L\) is then ample.

The corollary follows from the theorem once it is established that all the singularities of \(\overline{A}_g^P\) are terminal, which is done, building upon the local ideas of the computations from [Ta82], in [S-B06].

Thus for \(g \geq 12\) the minimal model program for \(A_g\) is complete — we know that \(\overline{A}_g^P\) is the canonical model.

**Open Problem 2.** Determine the canonical model of \(A_g\) for \(g < 12\).

5.9 (Kodaira dimension of \(A_g\)). Comparing the Kodaira dimension of a variety and its compactification is a bit tricky — a priori it is not clear that pluricanonical forms on a variety would extend to a compactification. However, for \(A_g^*\) there is no problem by the following result.

**Theorem 5.10 (Tai [Ta82]).** Any section of \(mK_{A_g^*}\) extends to a section of \(mK_{\overline{A}_g^P}\).

The study of Kodaira dimension of \(A_g\) was pioneered by Freitag, who in [Fr77b] showed that \(A_g^*\) is of general type for \(g\) divisible by 24, by explicitly constructing many pluricanonical forms in this case. In [Ta82] Tai studied the spaces of modular forms and obtained estimates for the dimension of the space of pluricanonical forms (see theorem 5.19 and proof for more details), which allowed him to prove directly from the definition of Kodaira dimension

**Theorem 5.11 (Tai [Ta82]).** For \(g \geq 9\) the space \(A_g\) is of general type.

5.12 (Effective divisors). For any variety \(X\) if we have \(K_X = E + A\), where \(E\) is an effective divisor, and \(A\) is a big \(\mathbb{Q}\)-divisor\(^1\) and the singularities are canonical, then \(X\) is of general type. Since we know that \(L\) is big and nef on \(A_g^*\), it follows that \(A_g^*\), or, properly speaking, \(\overline{A}_g^P\) is of general type if we can find an effective \(\mathbb{Q}\)-divisor \(E\) such that \(K_X = E + \varepsilon L\), for some \(\varepsilon > 0\), i.e. if there exists an effective divisor of slope \(s(E) < s(K_{A_g^*}) = g + 1\).

A direct way to construct effective divisors is to consider the zero loci of explicit modular forms. As observed by Freitag [Fr83], one can consider the modular form

\[
\theta_{\text{null}} := \prod_{\varepsilon, \delta \in (\frac{1}{2}\mathbb{Z})^g \text{ even}} \theta\left[\frac{\varepsilon}{\delta}\right](\tau),
\]

\(^1\) A divisor \(D\) on \(X\) is called big if \(h^0(X, mD)\) grows as \(m^{\dim X}\).
(where even means that the scalar product $4\varepsilon \cdot \delta = 0 \mod 2$), for which the weight and the vanishing order can be easily computed. This gives the slope $s(\theta_{\text{null}}) = 8 + \frac{1}{2g-3}$, which is less than $g + 1$ for $g \geq 8$, so this implies that $A_g$ is of general type for $g \geq 8$.

Constructing other explicit modular forms of small slope is quite hard, and if one writes down a random modular form, chances are it would be of very high slope — indeed, if a modular form belongs to a family that has no base locus, then its zero locus must intersect any curve non-negatively, and thus the modular form defines a nef divisor, which is thus of slope at least 12.

Alternatively one can construct effective divisors on $A_g$ by considering loci of abelian varieties satisfying some special geometric property. This approach has been very successful for moduli of curves (see [FaPo05], [Fa06a], [Fa07] for recent results and [Fa06b] for a survey), but is harder to pursue for $A_g$ than for $M_g$, as there are fewer geometric constructions known that are associated to a ppav than to an algebraic curve.

**Definition 5.13.** The Andreotti-Mayer divisor $N_0 \subset A_g^*$ is the closure in $A_g^*$ of the locus in $A_g$ of those ppavs for which the theta divisor is a singular $(g-1)$-dimensional variety.

Mumford used Grothendieck-Riemann-Roch for universal families and studied the geometry of the boundary to compute the class of $N_0$.

**Theorem 5.14** (Mumford [Mu83]). The slope of the Andreotti-Mayer divisor is $s(N_0) = 6 + \frac{12}{g+1}$; by comparison with $s(K_{A_g^*}) = g+1$ it follows that $A_g$ is of general type for $g \geq 7$.

**Remark 5.15.** The class of the divisor $N_0$ was later also computed by Yoshikawa [Y090] by more analytic methods. Since $N_0$ is an effective geometric divisor in $A_g^*$, one can ask whether it is given as the zero locus of a modular form. Work in this direction was done, and an integral expression for $N_0$ was obtained by Kramer and Salvati Manni in [KrSM02], but there is still more to be understood about the relationship of the geometry and modular forms here.

**Open Problem 3.** Write down an explicit modular form for which $N_0$ is the zero locus.

This of course does not mean that all $A_g$ are of general type. It was known classically that $A_1 = M_1$ and $A_2 = M_2$ are rational, and thus of Kodaira dimension $-\infty$. 

Theorem 5.16. (Katsylo [Ka96]) $\mathcal{M}_3$, and thus also $\mathcal{A}_3$, is rational. (Clemens [Cl83]) $\mathcal{A}_4$ is unirational. (Donagi [Do84], Mori and Mukai [MoMu83], Verra [Ve84]) $\mathcal{A}_5$ is unirational.

Thus since the 1980s only the Kodaira dimension of $\mathcal{A}_6$ remained unknown.

Since $\text{Pic}_Q(\mathcal{A}_g^*) = \mathbb{Q}^2$, to compute the class of any divisor in it all that is needed is to compute the intersection numbers of this divisor with two numerically non-equivalent test curves. Since $\text{Pic}_Q(\mathcal{A}_g) = \mathbb{Q}$, only one test curve can be taken to be an arbitrary curve lying completely in $\mathcal{A}_g$ (these exist for $g \geq 3$, see [KeSa03] or section 7 below for a discussion of related questions). Since $\text{Pic}_Q(\mathcal{A}_g^S) = \mathbb{Q}$, for the other test curve we can take a curve in $\mathcal{A}_g^S$ contracted to a point in $\mathcal{A}_g^S$ — this means that we can choose a ppav $[B] \in \mathcal{A}_{g-1} \subset \partial \mathcal{A}_g^S$ general, and take a general curve $C \in B \subset \partial \mathcal{A}_g^S$.

Then to compute the class of $N_0$ one can do the following: restrict the universal theta divisor and the universal family $\Theta_g \subset \mathcal{X}_g$ to a test curve $C$ (and denote the restrictions $\Theta \subset \mathcal{X}$), and then use the ramification formula for the map $\Theta \to C$, which would thus give the intersection number $N_0.C$ in terms of some intersection numbers of classes $\Theta$ and $c_1(T_{\mathcal{X}/C})$ on $\mathcal{X}$. Mumford performed this computation for a test curve $C \subset \mathcal{A}_g$, but over the boundary relied on the geometric description of $N_0$ to compute the corresponding coefficient. If one were to try to compute the class of any other geometrically defined divisor, such a geometric approach might not work.

However, the intersection theoretic computation can also be carried out over the boundary. Indeed, in this case $\mathcal{X}$ should be the universal semiabelian family over a curve $C \subset \partial \mathcal{A}_g^*$ (that is contracted to $[B] \in \mathcal{A}_{g-1} \subset \partial \mathcal{A}_g^S$). This universal family is in fact the total space of the projectivized Poincaré bundle on $B \times B$ restricted to $B \times C$ — see, for example, [Al02], [Hu00b]. Once the intersection numbers on this family are computed, the class of $N_0$ (and thus potentially of other divisors) can be computed directly, without appealing to the specific geometry of the situation. This was recently accomplished, and the computation results are as follows.

**Proposition 5.17** (Mumford [Mu83]). For $p: \mathcal{X} \to C$ being the universal family over a test curve $C \subset \mathcal{A}_g$, the pushforwards are

$$p(\Theta^{g+1}) = \frac{(g+1)!}{2}L; \quad p(\Theta^gL) = g!L.$$
Proposition 5.18 (- and Lehavi [GrLe08]). For \( p : \mathcal{X} \to C \) being the universal semiabelian family over a test curve \( C \subset \partial \mathcal{A}_g^* \), contracted to a point \( [(B, \Theta_B)] \in \mathcal{A}_{g-1} \), the pushforwards are

\[
p(\Theta^{g+1}) = \frac{(g+1)!}{6} \Theta_B, \quad p(\Theta^g c_1(T_{\mathcal{X}/C}) = 0.
\]

These results should allow computation of the classes in \( \text{Pic}_Q(\mathcal{A}_g^*) \) of many geometrically defined divisors — unfortunately the ones we have already tried did not give low slope.

The table of slopes of various effective divisors is as follows; here \( N_0^* := N_0 - 2\theta_{\text{null}} \) has slope slightly less than \( N_0 \), and was thus used by Mumford:

| \( g \) | \( s(K_{\mathcal{A}_g^*}) \) | \( s(\theta_{\text{null}}) \) | \( s(N_0^*) \) |
|-------|----------------|----------------|----------|
| 4     | 5              | 8.5            | 8        |
| 5     | 6              | 8.25           | 7.71     |
| 6     | 7              | 8.13           | 7.53     |
| 7     | 8              | 8.06           | 7.40     |
| \underline{\ldots} | \underline{\ldots} | \underline{\ldots} | \underline{\ldots} |
| \( \infty \) | \( \infty \) | 8              | 6        |

Notice that for all genera \( g \geq 5 \) we in fact have \( s(\theta_{\text{null}}) > s(N_0^*) > 6 \), and it seems very natural to wonder whether the minimal slope of \( \text{Eff}(\mathcal{A}_g^*) \) is always at least 6. This is absolutely not the case.

Theorem 5.19 (Riccardo Salvati Manni explained to us how this is obtained by improving the bounds in Tai [Ta82]). There exists an effective divisor on \( \mathcal{A}_g^* \) of slope at most

\[
\frac{(2\pi)^2}{\sqrt{g!} \sqrt[3]{2}}.
\]

Corollary 5.20. The slope of the effective cone goes to zero as \( g \) increases:

\[
\lim_{g \to \infty} s(\text{Eff}(\mathcal{A}_g^*)) = 0.
\]

Proof of the corollary. Indeed, we have \( \lim_{g \to \infty} \zeta(2g) = 1 \), and \( \sqrt[3]{g!} \sim g/e \), so for large \( g \) the asymptotics of the expression in the theorem is \( \frac{(2\pi)^2 e}{g} \), which tends to 0 as \( g \) increases.

Proof of the theorem. The improvement of Tai’s result is obtained by looking more carefully at his dimension estimates. For convenience, we recall Tai’s notations and results.

Denote by \( \mathcal{A}_{g,k} \) the vector space of scalar modular forms on \( \mathcal{A}_g \) of weight \( k(g+1) \). The reason for this notation is that \( \mathcal{A}_{g,k} \) are forms in
$kK_{\mathcal{A}_g}$, i.e. $k$-pluricanonical forms. Tai computes the asymptotics of the dimension for $g$ fixed and $k$ large ([Ta82], Proposition 2.1):

$$
\dim A_{g,k} \sim 2^{(g-1)(g-2)} k(g+1)^{g(g+1)/2} \prod_{j=1}^{g} \frac{(j-1)!}{(2j)!} B_j
$$

The slope of a modular form is its weight divided by the vanishing order at the boundary. Thus Tai defines (page 429) $\Theta_{g-1,m}^k(\ell)$ to be essentially the space of all possible expansions of weight $k(g+1)$ modular forms on $\mathcal{A}_g(\ell)$ near the boundary $\partial \mathcal{A}_g^*(\ell)$, vanishing to order $m$ along the boundary (this is somewhat confusing in [Ta82] — he does not have the upper index $k$ in notations, which is important for the computation). Such a boundary expansion determines the modular form uniquely; more precisely $\dim \Theta_{g-1,m}^k(1) = \dim \mathcal{H}^0(\mathcal{A}_g^*(\ell), k(g+1)L - mD)$, where “even” means that we are taking the even expansions, which are roughly one half of all expansions.

Thus if for some $M$ we have $\dim A_{g,k} > \sum_{m \leq M} \dim \Theta_{g-1,m}^k(1)$, it follows that there must exist a form in $A_{g,k}$ with boundary vanishing order at least $M$ and thus slope at most $\frac{k(g+1)}{M}$. One then estimates ([Ta82], Corollary 2.6)

$$
\dim \Theta_{g-1,m}^k(1) \sim (2m)^{g-1} \dim A_{g-1,k}.
$$

Combining this with the formula for $\dim A_{g,k}$ and taking the sum, we get (this is the last formula on page 431 of [Ta82] — be warned that there $M = k$, and one needs to carefully retrace Tai’s computations to verify that on the right-hand-side one of the two places $k$ appears it should now be $M$, while in the other it is still $k$):

$$
\sum_{m \leq M} \dim \Theta_{g-1,m}^k(1) \sim 2^{(g-2)(g-1)} \frac{M^g}{g} k(g+1)^{g(g+1)/2} \prod_{j=1}^{g-1} \frac{(j-1)!}{(2j)!} B_j,
$$

for $k$ and $M$ large enough, where $B_j$ are the even Bernoulli numbers.

Finally, to show the existence of a modular form of slope $s$, we need to have a modular form of weight $N := k(g+1)$ (for $k$ very large), vanishing at the boundary to order $M := N/s$. Such a modular form must exist if

$$
1 < \frac{\dim A_{N,s+1}}{\sum_{m \leq M} \dim \Theta_{g-1,m}^k(1)} \sim \frac{2^{(g-1)(g-2)} \frac{N^{g(g+1)} N \prod_{j=1}^{g-1} \frac{(j-1)!}{(2j)!} B_j}{g^{(g-2)(g-1)} \left(\frac{N}{s}\right)^{g/2} \prod_{j=1}^{g-1} \frac{(j-1)!}{(2j)!} B_j}}
$$
= s^g B_g \frac{g!}{(2g)!} = s^g g! (2g) \zeta(2g) \frac{(2\pi)^{2g}}{(2\pi)^{2g}}

where we used the explicit formula for $B_g$ in terms of the zeta function.

This inequality holds for

\[ s > \frac{(2\pi)^2}{\sqrt{g!} \sqrt{2\zeta(2g)}} \]

and thus there exist modular forms of this slope. \hfill \Box

Comparing the slope bound from theorem 5.19 to $s(N_0^*)$, we see that at least for $g \geq 13$ (and likely for smaller $g$ as well) $N_0^*$ cannot be the effective divisor of the smallest slope. This leaves the following important question wide open.

**Open Problem 4.** What is the slope of the cone $Eфф(A_g^*)$?

Since the slope of $Eфф(M_g) = 8.4$ is known, and $M_4 \subset A_4$ is codimension one, given by the Schottky modular form of slope 8, it follows that $Eфф(A_4) = \{aL - bD \mid a \geq 8b \geq 0\}$ has slope 8. However, already $Eфф(A_g^*)$ is not known. Oura, Poor and Yuen [OPY08] have been studying this question from the point of view of code polynomials etc., but a complete answer still seems beyond reach.

**Remark 5.21.** It is very interesting to compare what we know about the slopes of the effective cones of $M_g$ and $A_g^*$. A long-standing slope conjecture for $M_g$ predicted that the Brill-Noether divisor had minimal slope, and these slopes tended to 6 as $g$ went to infinity. The slope conjecture was disproven by Farkas and Popa [FaPo05], and divisors of smaller slopes have been constructed by them and by Farkas [Fa06a, Fa07]. However, all of these have slope at least 6, while it is not even clear whether there exists a genus-independent lower bound for slopes of effective divisors on $M_g$.

By the above theorem, there is no such bound for $A_g$, and it is tempting to try to apply techniques similar to Tai’s to $M_g \subset A_g$ to prove that there is no such bound for $M_g$, either. Since the dimension count in theorem 5.19 produces effective divisors on $A_g$ of slope smaller than $6.5 = s(K_{M_g})$ for $g \geq 14$, and since in this range (except for $M_{14}$, which is known to be unirational [Ve04]) the Kodaira dimension $\kappa(M_g)$ is not always known, it would also be very interesting to try to use Tai’s dimension-counting techniques to approach this computation, but this also seems hard.

**Remark 5.22.** There is also a very curious coincidence: the slope of the Brill-Noether divisor on $M_g$ is equal to $6 + \frac{12}{g+1}$, the same as the
slope of $N_0$ on $\mathcal{A}_g$. For $g \geq 4$ under the Torelli map we have $\mathcal{M}_g \subset N_0$, but since $\mathcal{M}_g \subset \mathcal{A}_g$ is of high codimension for $g$ large, so far this equality of slopes seems to be just a numerical coincidence, though a very strange one. Finding a reason for it, if there is one, could shed more light on the relationship of $Eff(\mathcal{M}_g)$ and $Eff(\mathcal{A}_g^*)$, and perhaps on the geometry of the Schottky problem.

6. Homology and Chow rings: intersection theory on $\mathcal{A}_g$

Having discussed the birational geometry, i.e. divisors, in the previous section, we now review the progress made in understanding the higher-dimensional cohomology and Chow rings of $\mathcal{A}_g$ and compactifications, and the intersection theory.

**Definition 6.1.** In $\text{Pic}_q(\mathcal{A}_g)$ we had one natural class — the Hodge line bundle $L = \det H$. Similarly, the most natural homology or Chow classes on $\mathcal{A}_g$ are the Hodge classes, i.e. the Chern classes of the Hodge bundle 

$$\lambda_i := c_i(H).$$

The cohomology of the open space $\mathcal{A}_g$ is the same as the group cohomology of $\text{Sp}(2g, \mathbb{Z})$, and a lot is known about it. Notice that choosing $[A] \in \mathcal{A}_h$ gives a natural embedding $\mathcal{A}_g \hookrightarrow \mathcal{A}_{g+h}$, by taking the Cartesian product with $A$. All of these embeddings are homotopic, and thus one can talk of the stable cohomology of $\mathcal{A}_g$. In comparison, for $\mathcal{M}_g$ there is no natural map $\mathcal{M}_g \hookrightarrow \mathcal{M}_{g+h}$, as taking the product with a fixed curve of genus $h$ gives reducible stable curves, which lie in $\partial \mathcal{M}_{g+h}$ — so for $\mathcal{A}_g$ we get an analog of Harer’s stability for free. The stable cohomology of $\mathcal{A}_g$, the same as that of $\text{Sp}(2g, \mathbb{Z})$, has been computed much before the recent proof by Madsen and Weiss [MaWe02] of Mumford’s conjecture on the stable cohomology of $\mathcal{M}_g$.

**Theorem 6.2** (Borel, see [Kn01] for an exposition). The stable cohomology ring of $\mathcal{A}_g$ is freely generated by a class in each dimension $4k + 2$, i.e. for any fixed $n$ there exists a $G(n)$ (some explicit formula for $G(n)$ is actually known) such that for all $g > G(n)$ the cohomology ring $H^*_S(\mathcal{A}_g)$ in dimensions $\leq n$ is the free algebra generated by the odd Hodge classes $\lambda_1, \lambda_3, \lambda_5, \ldots$.

In comparison, the stable cohomology of $\mathcal{M}_g$ is generated by a class in every even dimension, i.e. while the classes $\lambda_{2k}$ on $\mathcal{A}_g$ are expressible algebraically in terms of $\lambda_{2k+1}$ (and this relation of course also holds over $\mathcal{M}_g$), on $\mathcal{M}_g$ there are also stably algebraically independent from $\lambda$’s Miller-Morita-Mumford’s classes $\kappa_{2k}$. 


Moreover, for $\mathcal{A}_g$ there exist also product maps for compactifications: $\mathcal{A}_g^S \times \mathcal{A}_h^S \to \mathcal{A}_{g+h}^S$, $\mathcal{A}_g^P \times \mathcal{A}_h^P \to \mathcal{A}_{g+h}^P$, and $\mathcal{A}_g^V \times \mathcal{A}_h^V \to \mathcal{A}_{g+h}^V$. Thus we are naturally led to ask whether the stable homology can be computed for various compactifications of $\mathcal{A}_g$. The answer is in fact known for the Satake compactification.

**Theorem 6.3** (Charney and Lee [ChLe83]). The stable homology ring of $\mathcal{A}_g^S$ is freely generated by the odd Hodge classes $\lambda_{2k+1}$, for $k \geq 0$, and some other classes $\alpha_{2k+1}$, for $k \geq 1$.

It appears that the classes $\alpha$ may not be algebraic, but the algebraic geometry interpretation of this result is still now known. The stable homology of toroidal compactifications is completely unknown.

**Open Problem 5.** What are the stable homology rings, or maybe Chow rings, if this makes sense, of $\mathcal{A}_g^P$ and $\mathcal{A}_g^V$?

We thank Nicholas Shepherd-Barron for discussions relating to this question, drawing our attention to [ChLe83], and telling us about the following considerations.

These cohomology rings could be understood as the cohomology rings of the corresponding inductive limits $\mathcal{A}_\infty^P$ and $\mathcal{A}_\infty^V$ — these actually exist in the appropriate monoid category, but their topology may depend on the choice of the base point for the embedding $\mathcal{A}_g \hookrightarrow \mathcal{A}_{g+1}$. Moreover, the cohomology of ind-limits is naturally a graded Hopf algebra, and thus by a theorem of Milnor and Moore [MiMo65] is a product of a polynomial ring and an exterior algebra. We note that the number of irreducible boundary components of $\mathcal{A}_g^V$ grows unboundedly as $g$ grows, as thus the stable homology of $\mathcal{A}_g^V$ may only exist in some sense in which similarly the stable homology of the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ could exist.

The topic of stable modular forms (i.e. the structure of the limit $\mathcal{A}_\infty^S$) has been studied at least since the work of Freitag [Fr77a]. The space $\mathcal{A}_\infty^P$ is of interest in particular due to the work of Shepherd-Barron: one can try to think of it as the universal canonical model for $\mathcal{A}_g$ in some sense.

We also note that there do not exist any “stable compactly supported” cohomology classes, i.e. there cannot exist families of complete subvarieties of $\mathcal{A}_g$ of the same codimension for all $g$ — by theorem 7.2 below the codimension of a complete subvariety of $\mathcal{A}_g$ must be more than $g$.

**6.4 (Tautological ring).** Analogously to the case of $\mathcal{M}_g$ (see [To05] for the case of $\mathcal{M}_4$ and [GrPa03] for the case of $\mathcal{M}_{1,11}$), there may
exist cohomology classes in $A_g$ not lying the algebra generated by the Hodge classes. There has been much progress for $M_g$ in studying the tautological ring — the subring of the Chow generated by the naturally defined classes; one major goal being proving Faber’s conjecture [Fa99b]. The tautological ring can also be studied for $A_g$ — one simply considers the subring of the Chow generated by the Hodge classes $\lambda_i$. This has been determined entirely.

**Theorem 6.5** (van der Geer [vdG99] for $A_g$, Esnault and Viehweg [EsVi02] for a compactification). For an appropriate toroidal compactification the tautological subring of $CH^*_Q(A_g)$ generated by the Hodge classes has only one relation:

$$ (1 + \lambda_1 + \lambda_2 + \ldots + \lambda_g)(1 - \lambda_1 + \lambda_2 - \ldots + (-1)^g\lambda_g) = 1. $$

The tautological subring of $CH^*_Q(A_g)$ has one more relation: $\lambda_g = 0$.

Writing out all the terms of relation (2), we can immediately see that the even Hodge classes are expressible in terms of the odd Hodge classes. For example equating to zero the $CH^2$ term gives $2\lambda_2 = \lambda_1^2$, the $CH^4$ term gives $\lambda_4 = 2\lambda_3\lambda_1 - \lambda_2^2 = 2\lambda_1\lambda_3 - \frac{1}{4}\lambda_1^4$, etc.

Note that the above equalities are in $CH^*_Q(A_g)$, and thus one can wonder what happens in $CH^*_Z$. The torsion of $\lambda_g \in CH^*_Z(A_g)$, and subvarieties representing it on the compactification (since $\lambda_g$ is zero on $A_g$, it defines some subvariety of the boundary) were studied by Ekedahl and van der Geer [EkvdG04], [EkvdG05]. It is interesting to compare this to the recent work of Galatius, Madsen and Tillmann [GMT05] on the divisibility of the tautological classes on $M_g$.

The full homology and Chow rings (as opposed to just the tautological subring) were computed for $A_g$ for $g \leq 3$. The results for genera 1 and 2 are classical, and the same as for the moduli space of curves. For genus 3 we have the following two computations.

**Theorem 6.6** (Hain [Ha02]). The dimensions of the rational cohomology groups for $A_3$ and its Satake compactification $A_3^S$ are

| $n$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
|-----|---|---|---|---|---|----|----|
| $\dim H^0_Q(A_3)$ | 1 | 1 | 1 | 2 | 0 | 0 | 0 |
| $\dim H^0_Q(A_3^S)$ | 1 | 1 | 1 | 2 | 1 | 1 | 1 |

while the homology in all odd dimensions is zero. Moreover, the space $H^0(A_3)$ is described explicitly as a mixed Hodge structure.

**Theorem 6.7** (van der Geer [vdG98]). The Chow groups of $A_3$ (which are actually equal to the cohomology, though this is not a priori clear) have the following dimensions
In fact van der Geer describes the generators of all the Chow groups and the entire ring structure. While it seems very hard to describe the entire Chow ring in higher genera, one result that could potentially be generalized is the intersection theory of divisors, as we know that $\text{Pic}_Q(\overline{\mathcal{A}}_g^P)$ is always two-dimensional. For genus 3 the numbers are

**Theorem 6.8** (van der Geer [vdG98]). The intersection numbers of divisors on $\overline{\mathcal{A}}_3$ are

| $L^6$ | $L^5D$ | $L^4D^2$ | $L^3D^3$ | $L^2D^4$ | $LD^5$ | $D^6$ |
|-------|--------|----------|----------|----------|--------|-------|
| $\frac{1}{181440}$ | $0$ | $0$ | $\frac{1}{720}$ | $0$ | $-\frac{203}{240}$ | $-\frac{4103}{144}$ |

Compared to $\overline{\mathcal{A}}_g^P$, the intersection theory on $\overline{\mathcal{M}}_g$ has been extensively studied. Using Faber’s intersection computations program [Fa99a] for $\overline{\mathcal{M}}_4$ and the computation of the class of $\overline{\mathcal{M}}_4$ as a divisor in $\overline{\mathcal{A}}_4^P$ and $\overline{\mathcal{A}}_4^V$ by Harris and Hulek [HaHu04], in a recent work we have determined the intersection theory of divisors in genus 4.

**Theorem 6.9** (Erdenberger, –, Hulek [EGH06]). a) The intersection numbers of divisors on $\overline{\mathcal{A}}_4^P$ (recall $\text{Pic}_Q(\overline{\mathcal{A}}_g^P) = QL \oplus QD$) are

| $L^{10}$ | $L^6D^4$ | $L^3D^7$ | $LD^9$ | $D^{10}$ |
|----------|----------|----------|--------|--------|
| $\frac{1}{907200}$ | $-\frac{1}{3780}$ | $-\frac{1759}{1680}$ | $1636249$ | $101449217$ |

while all others are zero.

b) For $\overline{\mathcal{A}}_4^V$, recall from theorem 5.5 that there is a contracting morphism $\pi : \overline{\mathcal{A}}_4^V \rightarrow \overline{\mathcal{A}}_4^P$, with exceptional divisor $E$, and $\text{Pic}_Q(\overline{\mathcal{A}}_4^V) = QL \oplus QD \oplus QE$. For our purposes use $F := D + 4E$ instead of $D$ in the basis: $F \subset \overline{\mathcal{A}}_4^V$ is the pullback of $D \subset \overline{\mathcal{A}}_4^P$ under $\pi$. Then we have

$$\langle L^{i}F^j \rangle_{\overline{\mathcal{A}}_4^V} = \langle L^{i}D^j \rangle_{\overline{\mathcal{A}}_4^P} ; \quad \langle E^{10} \rangle_{\overline{\mathcal{A}}_4^V} = -\frac{35}{24} ;\quad \langle E^{10} \rangle_{\overline{\mathcal{A}}_4^V} = -\frac{35}{24} ,$$

while all other intersection numbers $\langle L^{i}F^jE^k \rangle_{\overline{\mathcal{A}}_4^V}$ with $k(i+j) \neq 0$ are zero.

We now remark that in the above results many of the intersection numbers turn out to be zero, and thus the following conjecture is plausible.
Conjecture 6.10 (Erdenberger, –, Hulek [EGH07]). An intersection number $\langle L^{i}D^{(g+1)/2-i} \rangle_{A_g}$ is zero unless $i = \frac{k(k+1)}{2} = \dim A_k$ for some $k \leq g$.

This is indeed true by inspection of the above numbers for $g \leq 4$, and by explicitly studying the geometry of the boundary strata of $A_g^{p}$ and the intersection numbers on them, the following result was also obtained.

Theorem 6.11 (Erdenberger, –, Hulek [EGH07]). The above conjecture is true for $i > \frac{(g-3)(g-2)}{2}$; explicit formulae for the non-zero intersection numbers in this range are also obtained.

The above considerations suggest that the homology and intersection homology of $A_g^{p}$ and $A_g^{s}$ could be related; it is natural to look more generally at the full Chow and cohomology rings instead of just the top intersection numbers of divisors. Since the class of $M_4 \subset A_4^p$ is known and much is known about the Chow ring and cohomology of $M_4$, there is the following natural

Open Problem 6. Determine the cohomology and Chow rings for $A_4^p$, $A_4^v$, or at least for $A_4$.

Tommasi [To05] recently computed the cohomology of $M_4$, which turns out to have an odd class.

Open Problem 7. Is some odd cohomology $H^{2k+1}(A_g)$ ever non-zero? In particular, do $A_4$ or its compactifications have any odd cohomology?

7. Special loci: subvarieties of $A_g$

In section 5 we discussed the question of constructing geometric divisors on $A_g^*$. In the previous section we discussed the Chow and homology rings of $A_g$ and its compactifications. We will now consider the question of constructing and studying subvarieties of $A_g$ of any dimension. One possible motivation for researching this would be to try to see if perhaps the cohomology is supported on a closed subvariety. On the other hand, stratifying $A_g$ in a geometrically meaningful way could shed more light on the geometry of individual abelian varieties, depending on which stratum they lie in, and yield results related to characterizing geometrically constructible loci. Many of the constructions and problems we survey are discussed in more detail in [BiLa04].

7.1 (Complete subvarieties). In [vdG99] it is shown that $\lambda_1^{(g-1)/2+1} = 0 \in CH_0^*(A_g)$. Since $\lambda_1$ is ample on $A_g$, it follows that there cannot
exist a closed subvariety $\mathcal{A}_g$ of dimension larger than \( \frac{g(g-1)}{2} \) (i.e. of codimension less than \( g \)), since otherwise the top power of $\lambda_1$ on it would have to be non-zero, contradicting the above equality. However, it is known $\lambda_1^{\frac{g(g-1)}{2}} \neq 0 \in CH^*_Q(\mathcal{A}_g)$, so it natural to ask if there exists a codimension $g$ closed subvariety $X \subset \mathcal{A}_g$ which could then perhaps carry all the cohomology (i.e. such that $H^*(X) = H^*(\mathcal{A}_g)$)? We discuss in section 8 that in characteristic $p$ there exists a complete codimension $g$ subvariety of $\mathcal{A}_g$, but over $\mathbb{C}$ this was conjectured by Oort (stated in [vdGOo99]) not to be the case. This was recently proven:

**Theorem 7.2** (Keel and Sadun [KeSa03]). Over $\mathbb{C}$, there does not exist a complete subvariety of $\mathcal{A}_g$ of codimension $g$.

This leads to the following

**Open Problem 8.** What is (over $\mathbb{C}$) the maximal dimension of a complete subvariety of $\mathcal{A}_g$?

Since $\partial \mathcal{A}_g$ is codimension $g$, if we start intersecting general hypersurfaces in $\mathcal{A}_g$, then once the dimension of the intersection drops down to $g - 1$, we know that it generally should not intersect the boundary — thus there exist complete subvarieties of $\mathcal{A}_g$ of dimension $g - 1$. The theorem above says that the maximal dimension of a complete subvariety of $\mathcal{A}_g$ cannot be greater than $\frac{g(g-1)}{2} - 1$. We do not have any reasons to believe that either the lower or upper bound are close to the actual maximal dimension of subvarieties. Instead of studying the maximal dimension of a closed subvariety of $\mathcal{A}_g$, one can also ask for the maximal dimension of a closed subvariety of $\mathcal{A}_g$ passing through a general point, etc. — some questions in this direction, for both $\mathcal{M}_g$ and $\mathcal{A}_g$, are discussed by Izadi in [Iz98].

One can also consider the following related

**Open Problem 9.** What is the homological dimension of $\mathcal{A}_g$, i.e. what is the smallest $n$ such that for any coherent sheaf $\mathcal{F}$ on $\mathcal{A}_g$ we have $H^k(\mathcal{A}_g, \mathcal{F}) = 0$ $\forall k > n$?

It is clear that if the cohomological dimension is $n$, then the maximal possible dimension of a complete subvariety is at most $n$, but we are not aware of a bound going the other way. For $\mathcal{M}_g$ it is conjectured by Looijenga that the homological dimension is equal to $g - 2$, and in fact that $\mathcal{M}_g$ can be covered by $g - 1$ affine open sets, while for $\mathcal{A}_g$ we do not even have a conjecture. The issue of homological dimension and affine covers was recently studied by Roth and Vakil [RoVa04].

### 7.3 (Stratifications of $\mathcal{A}_g$). As we saw above, constructing (over $\mathbb{C}$) explicit complete subvarieties of $\mathcal{A}_g$ is very hard. Maybe it is easier to
construct some non-complete subvarieties? One can consider the loci of ppavs given by various geometric constructions: Jacobians, Pryms, intermediate Jacobians, etc., but all of these seem to be, for $g$ large enough, of exceedingly high codimension in $\mathcal{A}_g$, and thus probably do not capture much of the geometry of $\mathcal{A}_g$. Thus it is natural to wonder whether one can define stratifications of $\mathcal{A}_g$ and obtain some geometric information about each of the strata.

**Definition 7.4.** We define the *Andreotti-Mayer locus* $N_k \subset \mathcal{A}_g$ to be the locus of ppavs for which $\dim \text{Sing} \Theta \geq k$. Clearly we then have

$$\emptyset = N_{g-1} \subseteq N_{g-2} \subseteq \ldots \subseteq N_1 \subset N_0 \subset N_{-1} = \mathcal{A}_g.$$ 

(In [Mu83] Mumford proved $N_1 \subset N_0$.)

These loci were originally introduced as an approach to the Schottky problem:

**Theorem 7.5** (Andreotti and Mayer [AnMa67]). $N_{g-4}$ contains the Jacobian locus as an irreducible component; $N_{g-3}$ contains the hyperelliptic locus.

The locus $N_{g-4}$ in low genera was studied by Beauville [Be77] and Debarre [De92], [De88] who described the extra components in it, other than the Jacobian locus, explicitly. One can also ask what are the dimensions of other Andreotti-Mayer loci.

**Theorem 7.6** (Ciliberto and van der Geer [CivdG00], [CivdG07]). For all $k \leq g - 3$ we have $\text{codim } N_k \geq k + 2$ (for $k \geq g/3$ this bound can be improved to $k + 3$).

Is this a reasonable bound for codimension? The codimension in $\mathcal{A}_g$ of the Jacobian locus, which is a component of $N_{g-4}$. A naïve, and thus completely unjustified, dimension count for the number of conditions for a point to be in $N_k$ seems to indicate that the codimension should indeed be quadratic in $k$. This motivates the following:

**Conjecture 7.7** (Ciliberto and van der Geer [CivdG00]). Within the locus of simple abelian varieties, $\text{codim } N_k \geq \frac{(k+1)(k+2)}{2}$.

Notice that this conjectural bound is exact for the Jacobian locus and for the hyperelliptic locus. This conjecture, however, seems very hard, as even the answer to the following question is unknown

**Open Problem 10** (Ciliberto and van der Geer [CivdG00]). Is it possible that there exists some $k < g - 4$ such that $N_k = N_{k+1}$?

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2We remind that a ppav is called simple if it does not have an abelian subvariety. A very general ppav is simple.
We know that $\mathcal{N}_{g-3}$ contains the hyperelliptic locus. What can we say about $\mathcal{N}_{g-2}$? Consider a decomposable\(^3\) ppav $A = A_1 \times A_2$. We then have

$$\Theta_A = (A_1 \times \Theta_{A_2}) \cup (\Theta_{A_1} \times A_2)$$

and thus

$$\text{Sing } \Theta_A \supset \Theta_{A_1} \times \Theta_{A_2},$$

so for decomposable ppavs $\dim \text{Sing } \Theta = g - 2$, i.e. $\bigcup_i \mathcal{A}_i \times \mathcal{A}_{g-i} \subset \mathcal{N}_{g-2}$. Since the codimension in $\mathcal{A}_g$ of the locus of decomposable abelian varieties is only $g - 1$, the condition of abelian variety being simple was needed in conjecture \[7.7\] Arbarello and De Concini conjecture in \[ArDC87\] that $\mathcal{N}_{g-2}$ is in fact equal to the locus of decomposable abelian varieties. This was proven to be true.

**Theorem 7.8** (Ein and Lazarsfeld \[EiLa97\]). $\mathcal{N}_{g-2}$ is equal to the locus of decomposable abelian varieties.

The loci $\mathcal{N}_k$ are of great interest, but very hard to study, as even their dimensions are still not known. From the analytical point of view it is very hard to determine the dimension of a solution set of a certain system of equations (singular points are where $\theta(\tau, z) = \text{grad}_z \theta(\tau, z) = 0$). Thus one wonders if it could be easier to look at some local singularity conditions instead.

**Definition 7.9.** We denote by $\text{Sing}_k \Theta := \{x \in A \mid \text{mult}_x \Theta \geq k\}$ the multiplicity $k$ locus of the theta divisor, i.e. the locus of points $z$ where the theta function, as a function of $z$, has multiplicity at least $k$, i.e. such that the theta function and its partial $z$-derivatives up to order $k - 1$ vanish. By the heat equation this means that all partial $\tau$-derivatives of the theta function up to order $\lfloor \frac{k-1}{2} \rfloor$ vanish at $(\tau, z)$.

Since multiplicity is a local condition, it is natural to study it from the point of view of singularity theory and multiplier ideals. This was done quite successfully.

**Theorem 7.10** (Kollár \[Ko95\]). The pair $(A, \Theta)$ is log canonical; thus $\text{codim}_A(\text{Sing}_k \Theta) \geq k$. In particular the multiplicity of the theta function at any point is at most $g$.

**Open Problem 11.** Give a direct analytic proof of this theorem, or at least of the fact that the theta function cannot vanish at any point to order higher than $g$.

\(^3\)We remind that a ppav is called decomposable if it is isomorphic (with polarization) to a product of two lower-dimensional ppavs. The term “reducible” is often used instead of “decomposable”.
Though the statement is entirely elementary, we have no idea on how to approach this problem.

**Definition 7.11.** We define the multiplicity locus $S_k \subset A_g$ to be the locus of abelian varieties for which $\text{Sing}_k \Theta$ is non-empty. We then have

$$\emptyset = S_{g+1} \subsetneq S_g \subsetneq \ldots \subsetneq S_2 = N_0 \subsetneq S_1 = A_g.$$ 

Similarly to the discussion above for $N_{g-2}$, one can see that for a $k$-fold product of abelian varieties we have $\text{Sing}_k \neq \emptyset$, thus in particular products of $g$ elliptic curves lie in $S_g$.

**Theorem 7.12** (Smith and Varley [SmVa96]). $S_g = \{\text{products of } g \text{ elliptic curves}\}$.

This is a special case of a more general theorem

**Theorem 7.13** (Ein and Lazarsfeld [EiLa97]). If for some $k > 1$ we have $\text{codim}_A(\text{Sing}_k \Theta) = k$, then $A$ is decomposable.

This result allows one to say something about ppavs for which $\text{Sing}_k \Theta$ has the maximal possible dimension. What happens if the dimension is one less — are these ppavs special in any way? Another question is

**Open Problem 12.** What is the maximal $k$ for which $S_k$ contains indecomposable abelian varieties?

One can also try to ask the same question for sections of multiples of the theta bundle on an abelian variety, rather than only for the theta function. This has been investigated by Hacon [Ha99], and Debarre and Hacon [DeHa05], with results generalizing theorem 7.10. However, we note that by Riemann’s theta singularity theorem for Jacobians, and by its generalizations for Prym varieties — see, for example, [C-M04], the maximal multiplicity of the theta function for Jacobians and Pryms is $\left\lfloor \frac{g+1}{2} \right\rfloor$. Since the dimension of $\text{Sing} \Theta$ for hyperelliptic Jacobians is largest possible for indecomposable ppavs, and Pryms lie in $N_{g-6}$ (see [De90]) it is natural to make the following

**Conjecture 7.14.** The maximal multiplicity of the theta function for indecomposable ppavs is equal to $\left\lfloor \frac{g+1}{2} \right\rfloor$, i.e. $S_{\left\lfloor \frac{g+3}{2} \right\rfloor}$ is a subvariety of the locus of decomposable abelian varieties.

We do not know of an approach to this conjecture short of trying to define a Prym-like construction for arbitrary ppavs, which would be very hard, and likely not possible. Another obvious question to ask is

**Open Problem 13.** What is the dimension of $S_k$? Is it possible to have $S_k = S_{k+1}$?
These are also entirely open. Some attempts to study these conditions by degeneration techniques were made in [CivdG07], [GrSM07].

**7.15 (Seshadri constants).** The above stratifications of $A_g$ encode some geometric information about the theta divisor. The multiplicity is a local invariant of the theta divisor, but from the point of view of the modern study of singularities, the multiplicity may not be the best invariant. Something perhaps more intrinsic is the following.

**Definition 7.16.** Given a variety $X$ with a divisor $D$ the Seshadri constant is defined to be

$$\varepsilon(x, D) := \inf_{x \in C \subset X} \frac{C.D}{\text{mult}_x(C)},$$

where the infimum is taken over all points $x \in X$, and all curves $C \subset X$ passing through the point $x$.

This is a very important invariant of a pair $(X, D)$ — for example the Seshadri constant is positive if and only if $D$ is ample.

One can study the Seshadri constants of general ppavs and then of special loci in $A_g$, and see whether the Seshadri constants in fact capture some geometric information.

**Theorem 7.17** (Lazarsfeld [La96]). There exists a constant $c$ independent of $g$ such that for the Jacobian $(J, \Theta)$ of any curve of genus $g$ the Seshadri constant $s(J, \Theta) \leq c\sqrt{g}$.

In comparison, for general ppavs we have

**Theorem 7.18** (Lazarsfeld [La96], see also Bauer [Ba98]). For a general ppav the Seshadri constant is at least of the order of a constant times $\sqrt{g}$.

There is also the following conjecture.

**Conjecture 7.19** (Debarre [De04], following Lazarsfeld). For $g \geq 4$, if $\varepsilon(A, \Theta) < 2$, then either $A$ is decomposable, or it is a hyperelliptic Jacobian.

**Remark 7.20.** It appears that recent results of Krichever [Kr05], [Kr06] provide techniques that could potentially be applied in an attempt to prove the so-called $\Gamma_{00}$ conjecture of van Geemen and van der Geer [vGvdG86], which is closely related to the half-degenerate case of the trisecant conjecture. As pointed out in [De04], the $\Gamma_{00}$ conjecture would imply this characterization of hyperelliptic Jacobians.
This leads one to hope that perhaps a characterization of Jacobians by Seshadri constants could be possible, or that one could better understand the stratification of $A_g$ by the value of the Seshadri constant. However, this is not so simple:

**Theorem 7.21** (Debarre [De04], see also Lazarsfeld [La96] for Jacobians). There exist Jacobians with Seshadri constants at least constant times $\ln g$. However, in each genus $g \geq 4$ there exist ppavs that are not Jacobians, but with Seshadri constant equal to 2.

**Open Problem 14.** What is the actual order of growth of the Seshadri constants for generic Jacobians? We know it is between $\ln g$ and $\sqrt{g}$, but it seems not much more is known.

Thus the stratification by the value of the Seshadri constant is also quite complicated. We believe that, if possible, giving a meaningful answer to the following loosely-phrased question would be extremely useful in understanding the geometry of $A_g$.

**Open Problem 15.** Define a stratification of $A_g$ with geometrically tractable strata, i.e. such that the number of the strata, and at least their dimensions are computable. Try to also say something about the special properties of the geometry of ppavs in each strata, perhaps inductively in the stratification.

8. A Glimpse of $A_g$ in Finite Characteristic

In this section we very briefly list the differences between the results over $\mathbb{C}$ that we discussed so far, and the case of the base field of finite characteristic. There is vast literature, and lots of other interesting questions on $A_g$ in finite characteristic — we refer to [vdGO099], [Oo99], [Oo01], [vdGMo] for more details, reviews, and further references. Here we just list what happens — from now on we are always talking about characteristic $p$.

The concept of a ppav is still defined, and the algebraic definition of the moduli space $A_g$ still makes sense. However, the universal cover of a ppav is no longer $\mathbb{C}^g$, and thus the discussion about period matrices, lattices, the Siegel upper half-space and the symplectic group action no longer applies. There is, however, a way to define theta functions algebraically over any base field, though not all the techniques used in working with holomorphic theta functions are still applicable.

The Satake and toroidal compactifications are defined over arbitrary base fields; the theory of Siegel modular forms and induced embeddings as we gave it is specific to the base field $\mathbb{C}$, but there is a concept of modular forms in finite characteristic.
The results on the nef cones of $A_g^*$ and $\overline{A}_g^P$ hold in any characteristic. However, the resolution of singularities in finite characteristic is not known, and the minimal model program is not established, so we cannot speak of the canonical models anymore. Neither does the discussion of effective divisors and Kodaira dimension/general type issues carry over to the case of finite characteristic.

The study of subvarieties of $A_g$ in finite characteristic is entirely different. Recall that over $\mathbb{C}$ by theorem 7.2 $A_g$ does not have a closed subvariety of codimension $g$, in stark contrast to the following.

**Theorem 8.1** (Koblitz [Ko75]). In finite characteristic the moduli space $A_g$ has a complete subvariety of codimension $g$ — the locus of ppavs that do not have points of order $p$.

**Definition 8.2.** This observation is in a sense a byproduct of the study of the powerful Ekedahl-Oort stratification [Oo99] of $A_g$. What one does is consider the group scheme $A[p]$ of points of order $p$ on a ppav, with the symplectic pairing on it induced by the principal polarization on $A$. One then defines the *Ekedahl-Oort stratum* as the locus of ppavs $A$ for which the group scheme $A[p]$ is of a given type. It can be shown with a lot of work that there are finitely many strata, each of which is quasi-affine, so that this stratification gives a cell decomposition of $A_g$.

Let $k$ be an algebraically closed field with $\text{char } k = p$. We define the $p$-rank of a ppav $A$ to be $f := \log_p \sharp A[p](k)$. Let $V_f$ be the locus of abelian varieties of $p$-rank at most $f$.

**Theorem 8.3** (van der Geer [vdG99]). The cycle class of the locus of ppavs of $p$-rank $\leq f$ is

$$[V_f] = (p - 1)(p^2 - 1)\cdots(p^{g-f} - 1)\lambda_{g-f},$$

so in finite characteristic the Hodge classes are effectively represented by subvarieties (not complete for $f > 0$) of $A_g$.

It turns out that in fact all cycle classes of the Ekedahl-Oort stratification lie in the tautological ring and can be computed explicitly.

There also exists another stratification of $A_g$ in finite characteristic, by Newton polygon — see [Oo04] for recent work on it. There is a multitude of other constructions, results, and questions concerning $A_g$ in finite characteristic, which we do not discuss here. The forthcoming book [vdGMo] will be a great source of information on moduli spaces of abelian varieties in finite characteristics.
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