Considerations on the thermal equilibrium between matter and the cosmic horizon

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A common feature in the thermodynamic analysis of homogeneous and isotropic world models is the assumption that the temperature of the fluids inside the cosmic horizon (including dark energy) coincides with the temperature of the latter, whether it be either the event or the apparent horizon. We examine up to what extent this assumption may be justified, given that these temperatures evolve under different time-temperature laws. We argue that while radiation cannot reach thermal equilibrium with the horizon, nonrelativistic matter may, and dark energy might though only approximately.

I. INTRODUCTION

Spatially homogeneous and isotropic universe models are usually adopted as starting point in cosmological studies, both mathematical and observational. This class of models is characterized by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric,

\[ \text{ds}^2 = -c^2 dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \quad (1) \]

which relies on the cosmological principle \cite{1} whose validity, at large scales, has not been contradicted \cite{4} and thus far looks rather robust \cite{3, 6}. In the latter expression the parameter \( k = 0, \pm 1 \) discriminates the three types of spatial curvatures and \( \Omega \) is the unit two-sphere.

These spaces entail for each observer an apparent horizon \( \tilde{r} \) of radius

\[ \tilde{r}_h = \left[ (H/c)^2 + ka^{-2} \right]^{-1/2} \quad (2) \]

with \( H = \dot{a}/a \) the Hubble factor. Since the observer has no information about what is going on beyond the horizon, the latter has an entropy, namely: \( S_h = \frac{k_B \pi \tilde{r}_h^2}{\ell_P^2} \) and a temperature \( T_h = \hbar H/(2\pi k_B) \). Clearly for the spatially flat FLRW metric, \( k = 0 \), the apparent horizon coincides with the Hubble horizon whose radius simplifies to \( cH^{-1} \).

Before going any further it is expedient to recall the meaning of the apparent horizon. We begin deriving its radius —see e.g. \cite{8, 9} for details. To this end we recast the FLRW metric as

\[ \text{ds}^2 = h_{ab} dx^a dx^b + \tilde{r}^2(x) d\Omega^2, \quad (3) \]

where \( x^0 = ct \), \( x^1 = r \) and \( h_{ab} = \text{diag} \left[ -1, \frac{a^2}{1 - kr^2} \right] \). The radius of the dynamical apparent horizon is set by the condition \( h^{ab}\partial_a \tilde{r} \partial_b \tilde{r} = 0 \), where \( \tilde{r} = a(t)r \). A straightforward calculation produces Eq. \( (2) \). Note that the expansion of the ingoing and outgoing null geodesic congruences is given by

\[ \theta_{IN} = H - \frac{1}{\tilde{r}} \sqrt{1 - \frac{k\tilde{r}^2}{a^2}} \quad \text{and} \quad \theta_{OUT} = H + \frac{1}{\tilde{r}} \sqrt{1 - \frac{k\tilde{r}^2}{a^2}}, \quad (4) \]

respectively. A spherically symmetric spacetime region will be called “trapped” if the expansion of ingoing and outgoing null geodesics, normal to the spatial two-sphere of radius \( \tilde{r} \) centered at the origin, is negative. By contrast, the region will be called “antitrapped” if the expansion of the geodesics is positive. In normal regions outgoing null rays have positive expansion and ingoing null rays, negative expansion. Thus, the antitrapped region is given by the condition \( \tilde{r} > (H^2 + ka^{-2})^{-1/2} \). Clearly, the surface of the apparent horizon is nothing but the boundary hypersurface of the spacetime antitrapped region. Obviously, in the case of an exact de Sitter expansion, the apparent and event horizons coincide.

When analyzing the thermodynamics of FLRW models the hypothesis that the temperature of either matter or dark fluids (or both) inside the horizon equals the temperature of the latter or, at least, is proportional to it is often made — see e.g. \cite{10, 11}. This is more frequently seen in the case of dark energy \cite{9}, for there is no clue about which expression its temperature should have. In the absence of further information, such a viewpoint provides the most economical and simplifying assumption, although no clear physical argument supports it.

The purpose of this note is to discuss and clarify the validity of the above hypothesis. While the issue of the validity of the first and second laws of thermodynamics in cosmology has attracted a great deal of interest \cite{1, 19, 22}, the present question has somewhat been overlooked. As it turns out, neither radiation nor dark energy can be in thermal equilibrium (equality of temperatures)

\footnote{Assuming it differs from the cosmological constant, as the latter has neither entropy nor temperature.}
with the horizon for a long period of time. In the case of nonrelativistic matter, the equilibrium is in principle attainable, and, after being reached, it can be stable forever. But the latter does not hold in the earliest stages of cosmic expansion. For simplicity, we restrict ourselves to spatially flat universes.

II. RELATIVISTIC MATTER

A natural and necessary condition for the equilibrium between the horizon (say, Hubble horizon, or what amounts to the same, the apparent horizon in a spatially flat universe) and thermal radiation (e.g., black body photons) at temperature $T$, is that the wavelength of the photons at which the Planck’s spectrum peaks given by Wien’s law,

$$2.82 = \frac{c h}{k_B \lambda_m T},$$  \hspace{1cm} (5)

be no larger than the horizon radius,

$$\lambda_m \leq c H^{-1}.$$  \hspace{1cm} (6)

Using $T_h = h H/(2\pi k_B)$ alongside (5) in (6), it follows that the ratio between temperatures must fulfill

$$\frac{T_h}{T} \leq \frac{2.82}{4\pi^2} < 1.$$  \hspace{1cm} (7)

Thus, condition (6) implies $T_h < T$; no thermodynamic equilibrium can prevail between the horizon and a bath of black body photons inside it. Actually, nowadays the temperature of the cosmic microwave radiation, $T_{\text{CMB}}$, exceeds by about 31 one orders of magnitude $T_h$ \cite{23, 24}.

$$\left| \frac{T_h}{T_{\text{CMB}}} \right| \leq 10^{-31}.$$  \hspace{1cm} (8)

Nevertheless given the relationship

$$\left( \frac{T_h}{T} \right) = -\frac{T_h}{T} q H,$$  \hspace{1cm} (9)

and the fact that the deceleration parameter $q = -(1 + \ddot{H} H^{-2})$ is negative at present, the gap between both temperatures is slowly narrowing (because the current value of $H$ is very small) though it will never vanish.

III. NONRELATIVISTIC MATTER

To study whether thermal equilibrium between nonrelativistic matter, of mass $m$ and temperature $T_m$, and the horizon is feasible, we proceed as follows. First, we bear in mind that the energy density and pressure of this fluid are given by $\rho = n mc^2 + (3/2)nk_BT_m$ and $P = nk_BT_m$, respectively, where $n$ is the number density of massive particles. Then we explore whether the de Broglie wavelength, $\lambda = h/p$, satisfies the reasonable condition $\lambda < cH^{-1}$.

To do this recall that $p = mv \approx \sqrt{mT_m}$, thus the heavier the particle species, the earlier the said condition is satisfied. Nevertheless, a simple estimate reveals that in general this will occur after the decoupling of the corresponding particles species.

On the other hand, within a very good approximation $p \propto a^{-1}$; then if the universe is dominated by thermal matter, the Hubble factor will be nearly given by $H \propto a^{-3/2}$, and, consequently, $(1 + z)^{1/2} < c$, where $z$ is the redshift. Thus, at very early times the condition involving the de Broglie wavelength will not be fulfilled, but eventually it will be and it will remain so forever.

If the universe is dominated by thermal matter and the cosmological constant $\Lambda$, then, sooner or later, the Hubble factor will very approximately obey $H \propto \sqrt{\Lambda}$; whereby $1 + z < \Lambda^{-1/2}$. Accordingly, the de Broglie condition will be met as soon as the cosmological constant starts dominating the expansion, or even earlier.

After having seen that, in principle, the equality $T_m = T_h$ can be reached at some point during the cosmic expansion, it is worthwhile to study whether the equilibrium, when it happens, will be stable. To do so we must consider the total heat capacity of the system, horizon plus matter. The heat capacity of the horizon is negative. Indeed keeping in mind that $\partial H/\partial T_h > 0$ and that we have chosen $k = 0$, it follows

$$C_h = T_h \frac{\partial S_h}{\partial T_h} = T_h \frac{\partial}{\partial H} \left( \frac{\pi k_B n c^2}{\ell_{\text{pl}}^3} \right) \frac{\partial H}{\partial T_h} < 0.$$  \hspace{1cm} (10)

On the other hand, the heat capacity at constant volume of thermal matter inside the horizon is

$$C_m = \frac{4\pi c}{3} \frac{\partial}{\partial T_m} \left( nmc^2 + \frac{3}{2} nk_BT_m \right) = 2\pi k_B n \frac{c^3}{H^3}.$$  \hspace{1cm} (11)

Stable thermal equilibrium between two systems whose respective heat capacities are of opposite signs is possible only if their combined heat capacity is negative \cite{23, 24}. In the case at hand, the inequality $C_h + C_m < 0$ yields the lower bound on the expansion rate

$$H > n c \ell_{\text{pl}}^2,$$  \hspace{1cm} (12)

i.e., $T_h > \hbar^2 Gn/(2\pi k_B c^2)$. Recalling that the de Broglie condition $\lambda = h/p < cH^{-1}$ is satisfied at least at late times and that $n \propto a^{-3}$, the right-hand side of the above inequality involving the Planck’s length decreases faster than $H$ in most reasonable cosmological models. Therefore, there is a long period (possibly lasting forever) in which the equilibrium—if it occurs—between thermal
matter and the apparent horizon is stable.

A somewhat related question is the current value of $T_m$. This can be evaluated recalling that the latter was in equilibrium with the cosmic microwave background radiation until the redshift was about $10^4$. Using this, alongside the relations $T_r = T_{r0}(1 + z)$ and $T_m = T_{m0}(1 + z)^2$ (although the latter expression is strictly valid for pressureless matter only), yields $T_{m0} \approx 3 \times 10^{-4}$ K. Consequently, $T_m$ is at all times 4 orders of magnitude closer to the temperature of the horizon than the cosmic photon gas.

In the case at hand, the rate of the ratio $T_h/T_m$ is readily found after realizing that $\dot{T}_m/T_m = -2H$. It results in

$$
\left( \frac{T_h}{T_m} \right)' = -\frac{T_h}{T_m} (q - 1) H.
$$

(13)

Because of $q < 0$ nowadays, the gap $T_m - T_h$ is narrowing, and this happens faster than in the case of black body radiation, as given by Eq. (9).

IV. DARK ENERGY

We now explore whether dark energy, with an equation of state $P = w \rho$, where $w$ is lower than $-1/3$ and not necessarily constant, can be in stable thermal equilibrium with the apparent horizon. First notice that, if dark energy is a scalar field, it is unclear whether it may have an entropy and a temperature. In principle, this would be possible if the scalar field is not in a pure quantum state, but rather in a mixed state. However, we do not know which case we are dealing with. In the latter one, we can use the expression [27]

$$
\frac{\dot{T}_{de}}{T_{de}} = -3H \left( \frac{\partial P}{\partial \rho} \right)_n
$$

(14)

to derive that $T_{de}$ grows with expansion, as the last factor on the right-hand side is negative. Since $T_h$ decreases with expansion, it is hard to see how thermal equilibrium between the dark energy and the apparent horizon can prevail.

From the above equation it is seen that the larger $H$, the higher the rate of variation of $T_{de}$. However, away from the primeval inflationary era, in any sensible cosmological model (e.g. no phantom dominated) $H$ always decreases implying that the grow of $T_{de}$ goes steadily down —modulo $w$ does not increase.

Also, in spite of that a state of stable thermal equilibrium between dark energy and the apparent horizon is not achievable, since both $H$ and its current time variation are extremely small, the rate of variation of their ratio $T_h/T_{de}$ will also be tiny. Indeed,

$$
\left( \frac{T_h}{T_{de}} \right) = -\frac{T_h}{T_{de}} [1 + q - 3w] H.
$$

(15)

Then, given the present smallness of $H$, we may say that the ratio between both temperatures stays practically constant during most of the history of the universe. Furthermore, the rate at which both temperatures depart from each other currently is decreasing.

On the other hand, nowadays the size of the apparent horizon changes little in a Hubble time (at most of order unity), i.e.,

$$
t_H \frac{\dot{r}_h}{r_h} = 1 + q,
$$

(16)

where $t_H = H^{-1}$ and for simplicity we have assumed $k = 0$. Therefore if both temperatures are similar at some not extremely remote early time, they will also remain so today, and this will last for another Hubble time at least.

V. CONCLUDING REMARKS

In summary, thermal equilibrium between radiation and the cosmic horizon cannot be achieved because Wien’s law yields a wavelength larger than the horizon radius, Eq. (7), at all times. Nonrelativistic particles can attain equilibrium at some point in the expansion that depends on the particle mass. The stability of this equilibrium is submitted to a condition that sets a lower bound for the Hubble factor, Eq. (12), but it is easily fulfilled. As for dark energy (assuming the corresponding scalar field is not in a pure quantum state), a stable equilibrium can never be accomplished; but if at some point in the expansion $T_h$ and $T_{de}$ happen to be equal or close to each other, then they will stay practically so for most of the history of the universe. Thus the hypothesis, made by several authors, of thermal equilibrium between dark energy and the horizon is, in this regard, not unjustified.

ACKNOWLEDGMENTS

The authors are indebted to Alberto Rozas for comments on an earlier version of this manuscript. D.P. is deeply grateful to the “Instituto de Astrofísica e Ciências do Espaço (IA)”, where part of this work was done, for warm hospitality and financial support. J.P.M. acknowledges the kind hospitality of the Universitat Autònoma de Barcelona during his visit. J.P.M. and D.P. were supported by Fundaçao para a Ciência e a Tecnologia (FCT) through the research grant UID/FIS/04434/2013. The authors also acknowledge the COST Action CA15117, supported by COST (European Cooperation in Science and Technology). And J.P.M. further acknowledges the Project 6818 FCT/DAAD/2016-17.
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