Space time symmetry in quantum mechanics

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New prescription to treat position and time equally in quantum mechanics is presented. Using this prescription, we could successfully derive some interesting formulae such as time-of-arrival for a free particle and quantum tunneling formula. The physical interpretation will be discussed.

I. INTRODUCTION

Though we can treat time and space symmetric way in relativity, in quantum mechanics the time seems different to other observables: It seems we don’t have proper operator for time. A particle detected at one position can be detected at the same position at later time, namely, we encounter the difficulty on orthogonality and normalization and these two measurements do not commute each other. This non commuting property leads us to think about time-of-arrival which means the time that a particle first arrives to a specific position.

Allcock [2] tried to build time-of-arrival eigenstates which are orthogonal each other for different time but could not define a consistent time-of-arrival. His study says that because we cannot absorb the particle in an arbitrarily short time, we cannot measure the time-of-arrival at any accuracy. Oppenheim et al [3] insist that using two level detector absorbing a particle in arbitrarily short period time, we can overcome this restrictions. However they found that the limitation on measuring the time-of-arrival with arbitrarily accuracy comes from the clock coupled to the trigger. They show that if we couple the system to a clock to measure the time-of-arrival at which the particle arrives at specific position, then the accuracy of measurement is limited by

$$\delta t_A > h/E_k$$

where $\delta t_A$ is the minimum uncertainty of measuring time of arrival and $E_k$ is the energy of clock.

One of interesting approaches to find the time-of-arrival operator has been studied by Grot et al [1]. They tried to develop the time-of-arrival operator for free non-relativistic particle by proper ordering of space time operator in Heisenberg picture, analogous to classical picture. But the eigenstates of time-of-arrival they calculated satisfying eigenvalue equation, did not satisfy the orthogonal condition for different times. They bypassed this difficulty by modifying the time-of-arrival operator so that in the classical limit would not reproduced the time-of-arrival exactly, but would reproduce a quantity arbitrary close to the time-of-arrival.

In this article, I will not attempt to develop the time-of-arrival operator nor discuss about dynamical limitation on treating position and time of quantum mechanics in equal manner. Rather it will be focused on how we can put an equal footing on position and time in quantum mechanical evolution. Contrast to other approaches, I assumed that we cannot put an equal footing on both position and time simultaneously. That is, when we treat the position as a quantum operator, we have to treat the time as an evolution parameter. And when we treat the time as a quantum operator, we have to treat the position as an evolution parameter. We will discuss how we can apply this prescription on quantum tunneling process.

II. PRESCRIPTION

In this section notational conventions will be defined in symmetrical way for both position and time. When the time is used as an evolution parameter (TEP), the position is used as a usual quantum observable. When the position is used as an evolution parameter (PEP), the time is used as a usual quantum observable. We can specify any quantum states with one state vector and one evolution parameter.

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A. TEP

We denote the quantum state $\psi$ at time $t_1$ by
\[ |\psi, t_1\rangle \] (1)
where the underline on $t_1$ means that time is the evolution parameter. That is, the first one represents the quantum state and the second one stands for the evolution parameter. We can express (1) it shorter by
\[ |\psi_1\rangle \equiv |\psi, t_1\rangle \] (2)
where the subscript $1$ on $\psi$ means the state is of time $t_1$. The probability amplitude to find the position $x$ is then
\[ \langle x | \psi, t_1 \rangle = \psi(x, t_1) \] (3)
where the position $x$ is the usual quantum observable and the time $t_1$ is the evolution parameter. Thus while the operation $\langle x | x_1 \rangle$ is possible, the operation $\langle t | t_1 \rangle$ is not possible because both $t$ and $t_1$ are just evolution parameters.

By the same reason $\langle x | \psi, t_1 \rangle = \psi(t, t_1)$ does not make sense. Since this represents the pure evolution process up to $t$, it must be denoted by
\[ \langle x | \psi, t_1 \rangle = |\psi, t\rangle \] (4)

B. PEP

We denote the quantum state $\varphi$ at position $x_1$ by
\[ |\varphi, x_1\rangle \] (5)
where the underline on $x_1$ means that position is the evolution parameter. That is, the first one represents the quantum state and the second one stands for the evolution parameter. We can express (5) it shorter by
\[ |\varphi_1\rangle \equiv |\varphi, x_1\rangle \] (6)
where the subscript $1$ on $\varphi$ means the state is of position $x_1$. The probability amplitude to find the time $t$ is then
\[ \langle t | \varphi, x_1 \rangle = \varphi(t, x_1) \] (7)
where the time $t$ is the usual quantum observable and the position $x_1$ is the evolution parameter. Thus while the operation $\langle t | t_1 \rangle$ is possible, the operation $\langle x_1 | x \rangle$ is not possible because both $x$ and $x_1$ are just evolution parameters.

By the same reason $\langle x | \varphi, x_1 \rangle = \varphi(x, x_1)$ does not make sense. Since this represents the pure evolution process up to $x$, it must be denoted by
\[ \langle x | \varphi, x_1 \rangle = |\varphi, x\rangle \] (8)

As we have seen, any quantum state is expressed by one state vector and one evolution parameter as $|\psi, t\rangle$ or $|\varphi, x\rangle$. We cannot specify a quantum state by $|\psi, x, t\rangle$ (two state vectors) or by $|\psi, x, t\rangle$ (two evolution parameters).

The rule is simple: The state vector $\psi$ does not operate to the evolution parameter $t$. So $\langle t | \psi \rangle$ does not work. Only two exceptions are $\langle t | E_n \rangle = e^{-iE_nt}$ and $\langle \varphi | p_n \rangle = e^{ip_nx}$ where $E_n, p_n$ are the components of identity operator $I = \sum_n |E_n\rangle\langle E_n|$. If $E$ is not a component of identity operator and $\langle t | E \rangle = e^{-iEt}$ contributes only overall phase, then it is physically meaningless.

And two state vectors with different evolution parameters do not directly operate each other. For example, two state vectors $\varphi$ and $\psi$ in $\langle \varphi, t_2 | \psi, t_1 \rangle$ do not directly operate each other. In this case, if needed, we may sandwich
\[ I = \sum_n |E_n\rangle\langle E_n| \] (9)
between $t_2$ and $t_1$ or we may sandwich
\[ I = \sum_n |p_n\rangle\langle p_n| \] (10)
between $x_2$ and $x_1$. [10]
III. EXPRESSION IN $E$ AND $p$ BASIS

A. TEP

We can express the quantum state $\psi$ at time $t_1$ by
\[
| \psi_{1,t_1} \rangle = |E_n \rangle \langle E_n | \psi_{1,t_1} \rangle
\]  
where the summation is assumed for repeated index $n$. Thus the probability amplitude to find the state $\psi_2$ at $t_2$ is
\[
\langle \psi_2, t_2 | \psi_{1,t_1} \rangle = \langle \psi_2, t_2 | E_n \rangle \langle E_n | \psi_{1,t_1} \rangle
\]  
For example, $\psi_2 = x_2$,
\[
\langle x_2, t_2 | \psi_{1,t_1} \rangle = \langle x_2, t_2 | E_n \rangle \langle E_n | \psi_{1,t_1} \rangle
\]
\[
\langle x_2 | \psi_{2,t_2} \rangle = \langle x_2 | E_n \rangle \langle t_2 | E_n \rangle \langle E_n | \psi_{1,t_1} \rangle
\]
\[
\rightarrow \psi_2(x_2, t_2) = \int \langle x_2 | E \rangle \psi_1(E) e^{-iE(t_2-t_1)} dE
\]
where the subscripts 1 and 2 in $\psi$ mean the states $\psi$ is of $t_1$ and $t_2$, we set $\hbar = 1$.

B. PEP

We can express the quantum state $\varphi$ at position $x_1$ by
\[
| \varphi_{1,x_1} \rangle = |p_n \rangle \langle p_n | \varphi_{1,x_1} \rangle
\]
The probability amplitude to find the state $\varphi_2$ at $x_2$ is
\[
\langle \varphi_2, x_2 | \varphi_{1,x_1} \rangle = \langle \varphi_2, x_2 | p_n \rangle \langle p_n | \varphi_{1,x_1} \rangle
\]
For example, $\varphi_2 = x_2$,
\[
\langle x_2, x_2 | \varphi_{1,x_1} \rangle = \langle x_2, x_2 | p_n \rangle \langle p_n | \varphi_{1,x_1} \rangle
\]
\[
\langle x_2 | \varphi_{2,x_2} \rangle = \langle x_2 | p_n \rangle \langle x_2 | p_n \rangle \langle p_n | \varphi_{1,x_1} \rangle
\]
\[
\rightarrow \varphi_2(x_2, x_2) = \int \langle x_2 | p \rangle \varphi_1(p) e^{ip(x_2-x_1)} dp
\]

IV. EXPRESSION IN SPACE TIME BASIS

A. TEP

In (11) and (12), let’s use $| x_i, t_i \rangle \langle x_i, t_i |$ instead of $| E_n \rangle \langle E_n |$.
\[
| \psi_{1,t_1} \rangle = | x_i, t_i \rangle \langle x_i, t_i | \psi_{1,t_1} \rangle = \int | x, t_1 \rangle \langle x | \psi_{1} \rangle dx
\]
\[
= \int | x, t_1 \rangle \langle x | p_n \rangle \langle p_n | \psi_{1} \rangle dx = \int | x, t_1 \rangle \frac{e^{ip_n x}}{\sqrt{2\pi}} \langle p_n | \psi_{1} \rangle dx
\]
For the momentum eigenstate $\psi_1 = p_1$,
\[
| p_1, t_1 \rangle = \int | x, t_1 \rangle \frac{e^{ip_1 x}}{\sqrt{2\pi}} dx
\]
From (18), we can see that a particle in momentum eigenstate evolving from time $t_1$ starts its motion at all equally different positions. (17) is illustrated in figure (I) (a). We can check that if we are more certain about the momentum of a particle, we are less certain about the position of departure at time $t_1$. This is the fundamental meaning of position-momentum uncertainty relation.
In (13) and (14), let’s use $| t_i, x_1 \rangle \langle t_i, x_1 |$ instead of $| p_n \rangle \langle p_n |$.

\[ | \varphi_1, x_1 \rangle = | t_i, x_1 \rangle \langle t_i, x_1 | \varphi_1, x_1 \rangle = \int | t, x_1 \rangle \langle t | \varphi_1 \rangle dt \]

\[ = \int | t, x_1 \rangle \langle t | E_n \rangle \langle E_n | \varphi_1 \rangle dt = \int | t, x_1 \rangle \frac{e^{-iE_n t}}{\sqrt{2\pi}} \langle E_n | \varphi_1 \rangle dt \]

For the energy eigenstate $\varphi_1 = E_1$,

\[ | E_1, x_1 \rangle = \int | t, x_1 \rangle \frac{e^{-iE_1 t}}{\sqrt{2\pi}} dt \]

From (21), we can see that the particle in energy eigenstate evolving from the position $x_1$, starts its motion at all equally different times. (20) is illustrated in figure 1(b). We can check that if we are more certain about the energy of a particle, we are less certain about the time of departure at position $x_1$. This is the fundamental meaning of time-energy uncertainty relation.

**V. QUANTUM EVOLUTION IN SPACE TIME BASIS**

**A. TEP**

Let’s find the expression for the probability amplitude of quantum state $\psi_1$ at $t_1$ to be measured $\psi_2$ at $t_2$.

\[ \langle \psi_2, t_2 | \psi_1, t_1 \rangle = \iint \langle \psi_2 | x_2 \rangle \langle x_2, t_2 | x_1, t_1 \rangle \langle x_1 | \psi_1 \rangle dx_1 dx_2 \]

\[ = \iint \langle \psi_2 | x_2 \rangle \langle x_2, t_2 | x_1, t_1 \rangle \langle x_1 | \psi_1 \rangle dx_1 dx_2 \]

Equation (22) is illustrated in figure 2(a).

For example

\[ \langle x_2, t_2 | \psi, t_1 \rangle = \iint \langle x_2 | x_2' \rangle \langle x_2', t_2 | x_1, t_1 \rangle \langle x_1 | \psi_1 \rangle dx_1 dx_2 \]

\[ \langle x_2 | \psi, t_2 \rangle = \iint \langle x_2, t_2 | x_1, t_1 \rangle \langle x_1 | \psi_1 \rangle dx_1 \]

\[ \rightarrow \psi(x_2, t_2) = \int \langle x_2, t_2 | x_1, t_1 \rangle \psi(x_1, t_1) dx_1 \]
FIG. 2: (a) Illustration of (22); (b) Illustration of (27)

Take an another example

\[ \langle p_2, t_2 | p_1, t_1 \rangle = \int \int \langle p_2 | x_2 \rangle \langle x_2, t_2 | x_1, t_1 \rangle \langle x_1 | p_1 \rangle dx_1 dx_2 \]
\[ = \int \int \frac{e^{-i p_2 x_2}}{\sqrt{2\pi}} \langle x_2, t_2 | x_1, t_1 \rangle \frac{e^{i p_1 x_1}}{\sqrt{2\pi}} dx_1 dx_2 \]  

(26)

B. PEP

Let’s find the expression for the probability amplitude of quantum state \( \varphi_1 \) at \( x_1 \) to be measured \( \varphi_2 \) at \( x_2 \).

\[ \langle \varphi_2, x_2 | \varphi_1, x_1 \rangle = \int \langle \varphi_2, x_2 | t_2, x_2 \rangle \langle t_2, x_2 | t_1, x_1 \rangle \langle t_1, x_1 | \varphi_1, x_1 \rangle dt_1 dt_2 \]
\[ = \int \langle \varphi_2 | t_2 \rangle \langle t_2, x_2 | t_1, x_1 \rangle \langle t_1 | \varphi_1 \rangle dt_1 dt_2 \]  

(27)

Equation (27) is illustrated in figure 2(b).

For example

\[ \langle t_2, x_2 | \varphi, x_1 \rangle = \int \langle t_2, x_2 | t_2' \rangle \langle t_2', x_2 | t_1, x_1 \rangle \langle t_1 | \varphi, x_1 \rangle dt_1 dt_2' \]  

(28)

\[ \langle t_2 | \varphi, x_2 \rangle = \int \langle t_2, x_2 | t_1, x_1 \rangle \langle t_1 | \varphi, x_1 \rangle dt_1 \]  

(29)

\[ \rightarrow \varphi(t_2, x_2) = \int \langle t_2, x_2 | t_1, x_1 \rangle \varphi(t_1, x_1) dt_1 \]  

(30)

Take an another example

\[ \langle E_2, x_2 | E_1, x_1 \rangle = \int \int \langle E_2 | t_2 \rangle \langle t_2, x_2 | t_1, x_1 \rangle \langle t_1 | E_1 \rangle dt_1 dt_2 \]
\[ = \int \int \frac{e^{i E_2 t_2}}{\sqrt{2\pi}} \langle t_2, x_2 | t_1, x_1 \rangle \frac{e^{-i E_1 t_1}}{\sqrt{2\pi}} dt_1 dt_2 \]  

(31)
VI. ORTHOGONALITY OF POSITION AND TIME

In this section it will be shown how to achieve $\langle x \mid x' \rangle = \delta(x-x')$ in TEP and $\langle t \mid t' \rangle = \delta(t-t')$ in PEP. In doing so, we will find the following expression for $\langle E \mid p \rangle$.

$$\langle E \mid p \rangle = \sqrt{\frac{E}{p_E}} \delta(\pm p_E - p) \quad \text{in TEP} \tag{32}$$

The range of energy is either $[m, \infty]$ or $[-m, -\infty]$. The range of momentum is $[-\infty, \infty]$.

$$\langle E \mid p \rangle = \sqrt{\frac{p}{E_p}} \delta(\pm E_p - E) \quad \text{in PEP} \tag{33}$$

The range of momentum is either $[0, \infty]$ or $[0, -\infty]$. The range of energy is $[-\infty, \infty]$ except $[-m, m]$. Where $p_E = \sqrt{p^2 - m^2}$ and $E_p = \sqrt{p^2 + m^2}$ for a free particle. Let’s check this out.

A. TEP

In order to find out the expression of $\langle E \mid p \rangle$, first verify the orthogonality of momentum, $\langle p' \mid p \rangle = \int \langle p' \mid E \rangle \langle E \mid p \rangle dE = \delta(p' - p)$. Since $EdE = p_E dp_E$, try $\langle E \mid p \rangle = \sqrt{E/p_E} \delta(\pm p_E - p)$, where $\pm$ indicates that when we integrate over $p_E$, we have to do it for both $+p_E$ and $-p_E$.

$$\langle p' \mid p \rangle = \int \sqrt{\frac{E}{p_E}} \delta(\pm p_E - p') \sqrt{\frac{E}{p_E}} \delta(\pm p_E - p) dE \tag{34}$$

This is an odd function of $E$, thus if we integrate over $[-\infty, \infty]$, it turns out to be zero. We can fix it by restricting $E$ to either $[m, \infty]$ or $[-m, -\infty]$. Then,

$$\langle p' \mid p \rangle = \int_m^\infty \frac{E}{p_E} \delta(\pm p_E - p') \delta(\pm p_E - p) dE \tag{35}$$

The sign of $E$ does not specify the sign of $p_E$. Thus we have to count both positive and negative momentum cases.

$$\langle p' \mid p \rangle = \int_0^\infty \delta(p_E - p') \delta(p_E - p) dp_E + \int_0^\infty \delta(p_E + p') \delta(p_E + p) dp_E \tag{36}$$

$$= \int_0^\infty \delta(p_E - p') \delta(p_E - p) dp_E + \int_0^\infty \delta(-u + p') \delta(-u + p) d(-u) \tag{37}$$

$$= \int_{-\infty}^{\infty} \delta(p_E - p') \delta(p_E - p) dp_E = \delta(p' - p) \tag{38}$$

with $E$ either $[m, \infty]$ or $[-m, -\infty]$ ensures the orthogonality of position, $\langle x \mid x' \rangle = \delta(x-x')$:

$$\langle x \mid E \rangle = \int \langle x \mid p \rangle \langle p \mid E \rangle dp = \frac{1}{\sqrt{2\pi}} \int e^{ipx} \sqrt{\frac{E}{p_E}} \delta(\pm p_E - p) dp = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{E}{p_E}} e^{\pm ip_E x} \tag{39}$$

$$\langle x \mid x' \rangle = \int_m^\infty \langle x \mid E \rangle \langle E \mid x' \rangle dE = \frac{1}{2\pi} \int_m^\infty \frac{E}{p_E} e^{\pm ip_E(x-x')} dE \tag{40}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ip_E(x-x')} dp_E = \delta(x-x') \tag{41}$$

If we did not restrict $E$ to either positive or negative values, we couldn’t have $\langle x \mid x' \rangle = \delta(x-x')$. This is expected because the negative energy particle comes backward in time to be detected at another position at the same time it has already been detected.
B. PEP

In order to find out the expression of \( \langle E \mid p \rangle \), first verify the orthogonality of energy, \( \langle E' \mid E \rangle = \int \langle E' \mid p \rangle \langle p \mid E \rangle dp = \delta(E' - E) \). Since \( EdE = pEdp_E \), try \( \langle E \mid p \rangle = \frac{1}{\sqrt{p/Ep}} \delta(\pm Ep - E) \).

\[
\langle E' \mid E \rangle = \int \sqrt{\frac{p}{Ep}} \delta(\pm Ep - E') \sqrt{\frac{p}{Ep}} \delta(\pm Ep - E) dp
\]

This is an odd function of \( p \), thus if we integrate over \([ -\infty, \infty ]\), it turns out to be zero. We can fix it by restricting \( p \) to either \([0, \infty]\) or \([0, -\infty]\). Then,

\[
\langle E' \mid E \rangle = \int_0^{\infty} \frac{p}{Ep} \delta(\pm Ep - E') \delta(\pm Ep - E) dp
\]

The sign of \( p \) does not specify the sign of \( Ep \). Thus we have to count both positive and negative energy cases.

\[
\langle E' \mid E \rangle = \int_0^{\infty} \delta(Ep - E') \delta(Ep - E) dEp + \int_0^{\infty} \delta(Ep + E') \delta(Ep + E) dEp
\]

\( = \int_0^{\infty} \delta(Ep - E') \delta(Ep - E) dEp + \int_{-\infty}^{0} \delta(-u + E') \delta(-u + E) d(-u)
\]

\( = \int_{-\infty}^{\infty} \delta(Ep - E') \delta(Ep - E) dEp = \delta(E' - E)
\]

\( \text{[33]} \) with \( p \) either \([0, \infty]\) or \([0, -\infty]\) ensures the orthogonality of time, \( \langle t \mid t' \rangle = \delta(t - t') \):

\[
\langle p \mid t \rangle = \int \langle p \mid E \rangle \langle E \mid t \rangle dE = \frac{1}{\sqrt{2\pi}} \int \sqrt{\frac{p}{Ep}} \delta(\pm Ep - E) e^{iEt} dEp = \frac{1}{\sqrt{2\pi}} \int \sqrt{\frac{p}{Ep}} e^{iEp(t-t)} dp
\]

\[
\langle t \mid t' \rangle = \int_0^{\infty} \langle t \mid p \rangle \langle p \mid t' \rangle dp = \frac{1}{2\pi} \int_0^{\infty} \frac{p}{Ep} e^{iEp(t-t')} dp
\]

\( = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iEp(t-t')} dEp = \delta(t - t')
\]

If we did not restrict \( p \) to either positive or negative values, we couldn’t have \( \langle t \mid t' \rangle = \delta(t - t') \). This is expected because the negative momentum particle comes backward in space to be detected at another time at the same position it has already been detected.

VII. APPLICATION

We have seen how to treat the position and time equally in quantum mechanics especially in evolution process. Let’s consider some application of our prescription.

A. Time-of-arrival

We may apply new prescription to the time-of-arrival introduced earlier. By putting \( \langle 47 \rangle \) into \( \langle 15 \rangle \), we can derive the expression of time-of-arrival for a free particle. Then \( \langle 15 \rangle \) turns out

\[
\varphi_2(t) = \frac{1}{\sqrt{2\pi}} \int \sqrt{\frac{p}{Ep}} e^{iEp(t+i\pi)(x_2-x_1)} \varphi_1(p) dp
\]

where the subscript 1 and 2 in \( \varphi \) stands for the position \( x_1 \) and \( x_2 \). The range of \( p \) goes either \([0, \infty]\) or \([0, -\infty]\); \( \mp Ep \) correspond positive and negative energy particle respectively. The negative energy particle evolves in opposite direction to the positive energy particle in time. \( \text{[50]} \) is well consistent with the final result Grot et al\( \text{[1]} \) derived for
a free particle.

We have derived (50) from $\langle t_2, x_2 \mid p_n \rangle \langle p_n \mid \varphi_1, x_1 \rangle$. We could also derive (50) from $\langle t_2, x_2 \mid t_1, x_1 \rangle \langle t_1, x_1 \mid \varphi_1, x_1 \rangle$.

$$\langle t_2, x_2 \mid \varphi_1, x_1 \rangle = \int dt_1 \langle t_2, x_2 \mid t_1, x_1 \rangle \langle t_1, x_1 \mid \varphi_1, x_1 \rangle$$  \hspace{1cm} (51)

$$\langle t_2 \mid \varphi_2, x_2 \rangle = \int dt_1 \langle t_2, x_2 \mid p_m \rangle \langle p_m \mid t_1, x_1 \rangle \langle t_1, x_1 \mid p_m \rangle \langle p_m \mid \varphi_1, x_1 \rangle$$  \hspace{1cm} (52)

$$\varphi_2(t_2) = \int dt_1 \langle t_2 \mid p_m \rangle \langle p_m \mid t_1 \rangle \langle t_1 \mid x_2 \rangle \langle x_2 \mid p_m \rangle \langle p_m \mid t_1 \rangle \langle t_1 \mid p_m \rangle \langle p_m \mid \varphi_1 \rangle$$  \hspace{1cm} (53)

$$\varphi_2(t_2) = \int \int \int \int dt_1 \left( \frac{p_m}{2\pi E_{p_m}} \right)^{\frac{1}{2}} e^{i(E_{p_m} - E_{p_n})t_1} e^{\pm i E_{p_m} t_2} e^{i p_m (x_2 - x_1)} \varphi_1(p_n) dp_m dp_n$$  \hspace{1cm} (54)

$$= \int \int \frac{p_m}{E_{p_m}} \sqrt{\frac{p_n}{2\pi E_{p_n}}} \delta(E_{p_m} - E_{p_n}) e^{\mp i E_{p_m} t_2 + i p_m (x_2 - x_1)} \varphi_1(p_n) \frac{E_{p_m} dE_{p_m}}{p_m} dp_n$$  \hspace{1cm} (55)

$$= \int_0^\infty \sqrt{\frac{p_n}{2\pi E_{p_n}}} e^{\mp i E_{p_m} t_2 + i p_n (x_2 - x_1)} \varphi_1(p_n) dp_n$$  \hspace{1cm} (56)

which reduce to (50). Note that if we did not restrict the momentum to either $[0, \infty]$ or $[0, -\infty]$, we could not have $e^{i p_n (x_2 - x_1)}$ in (56) from (55).

**B. Quantum tunneling**

Another application is the region of quantum tunneling. (Or inside event horizon.) Thus let’s apply the prescription to derive quantum tunneling formula. For $E_1 = E_2$ (51) becomes

$$\langle E_1, x_2 \mid E_1, x_1 \rangle = \int \int \langle t_2, x_2 \mid t_1, x_1 \rangle \frac{1}{2\pi} e^{i E_1 (t_2 - t_1)} dt_1 dt_2$$

$$= \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty \sum_{[x(t)]} \exp \left( i \int_{t_1}^{t_2} (L + E_1) dt \right) dt_1 dt_2$$  \hspace{1cm} (57)

where we have used Feynman kernel. And we can make it simpler by

$$\int_{t_1}^{t_2} (L + E_1) dt = \int_{t_1}^{t_2} p dt = \int_{x_1}^{x_2} p dx = W(x) \bigg|_{x_1}^{x_2}$$

For a classical object ($W \gg h$) or for WKB approximation \[5\],

$$\sum_{[x(t)]} \exp \left( \frac{i W(x)}{h} \bigg|_{x_1}^{x_2} \right) \sim \exp \left( \frac{i W(x_\ell)}{h} \bigg|_{x_1}^{x_2} \right) F(t_2, t_1)$$  \hspace{1cm} (59)

where $F(t_2, t_1)$ is some function of only $t_2$ and $t_1$. $x_\ell$ stands for the least action path satisfying Euler-Lagrange equation

$$\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0$$  \hspace{1cm} (60)

Thus

$$\langle E_1, x_2 \mid E_1, x_1 \rangle \sim \exp \left( \frac{i}{h} W(x_\ell) \bigg|_{x_1}^{x_2} \right)$$  \hspace{1cm} (61)
For tunneling particle, $p^2 < 0$, $W = i(\text{Im}W)$,

$$
\langle E_1, x_2 \mid E_1, x_1 \rangle \sim e^{-\frac{1}{\hbar} \text{Im}W(x_1)} \left|_{x_2}^{x_1} \right.
$$

(62)

The tunneling probability is

$$
P(x_1 \rightarrow x_2, E) \sim e^{-\frac{2}{\hbar} \text{Im}W(x_1)} \left|_{x_2}^{x_1} \right.
$$

(63)

In (21), we have seen that a particle in energy eigenstate departs the initial position $x_1$ at all different times equally. This applies also to the final position $x_2$ in (31). Thus it is meaningless to talk about tunneling time of a particle in energy eigenstate; (50) reveals this property clearly. We can consider (50) as tunneling time for a zero potential, $V = 0$. For an energy eigenstate $\varphi_1 = p_1$, $|\varphi_2(t)|^2$ of (50) has no time dependence. It makes sense, because the particle in energy eigenstate departs $x_1$ and arrives $x_2$ at all different times equally. We may discuss time-of-arrival or tunneling time only for particles which are not in energy eigenstate.

**VIII. CONCLUSION**

We have seen how to put an equal footing on position and time in quantum mechanics. Unlike other approaches, I proposed that we cannot take both position and time as evolution parameters or both as observables. We have to take one as an observable and the other as an evolution parameter; With set of simple prescriptions, we could formulate quantum mechanics in space time symmetric manner. Combining with Feynman path integral, we could understand the fundamental meaning of time-energy uncertainty principle. We could derive the time-of-arrival for a free particle. We could also develop quantum tunneling formula expressed in Jacobi action for classical or WKB limit. This approach may contribute to the development of quantum gravity.

One drawback of this prescription is that, for example figure 2(b) suggest that the particle can travel faster than the speed of light or even backward in time. We can fix this problem by just assuming it cannot do it and modifying the integral range in formula for final position. But this is not an elegant way to bypass the problem and it ruins the spirit of space time symmetry we are trying to achieve. Does that mean the prescription for the position as an evolution parameter applies only to stationary case where there is no measurable distinction between past and future? Further study is needed to answer it.

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[10] Note in order to use new prescription, it is not good idea to sandwich (9) between $x_2$ and $x_1$; / to sandwich (10) between $t_2$ and $t_1$, because it can destroy the information about the Lagrangian of the system.

[11] We may fix this problem by making the integrand to an even function of $p$ (i.e., $p \rightarrow |p|$). But as explained at the appendix A, this cause another problem.
Appendix A: Inconsistency of making $\langle E \mid p \rangle$ even function

We saw that (42) is an odd function of $p$. In order to achieve $\langle E' \mid E \rangle = \delta(E' - E)$ we had to restrict $p$ to either $[0, \infty)$ or $[0, -\infty)$. However, we may achieve it by choosing $\langle E \mid p \rangle = \sqrt{\frac{|p|}{2E_p}}\delta(\pm E_p - E)$ (A1)

$$\langle p \mid t \rangle = \int \langle p \mid E \rangle \langle E \mid t \rangle dE = \int_{-\infty}^{\infty} \sqrt{\frac{|p|}{2E_p}}\delta(\pm E_p - E) e^{iEt} \sqrt{\frac{2}{2\pi}}dE = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{|p|}{2E_p}} e^{\pm iE_p t}$$ (A2)

Then, we can show that (A1) satisfies $\langle E' \mid E \rangle = \delta(E' - E)$ and $\langle t' \mid t \rangle = \delta(t' - t)$ without constraining allowed range of momentum for the same evolution parameter $x_1$. However, we can also show that (A1) does not work consistently for $\langle t_2, x_2 \mid \varphi_1, x_1 \rangle$. Using (A2) to (15), we have

$$\varphi_2(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{|p|}{2E_p}} e^{\mp iE_p t + ip(x_2 - x_1)} \varphi_1(p) dp$$ (A3)

and (55) changes to

$$\varphi_2(t_2) = \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{1}{2} \sqrt{\frac{|p_n|}{2\pi 2E_{p_n}}} \delta(E_{p_m} - E_{p_n}) e^{\mp iE_{p_m} t_2 + ip_m(x_2 - x_1)} \varphi_1(p_n) dE_{p_m} dp_n$$ (A4)

$$+ \int_{-\infty}^{\infty} \int_{0}^{\infty} -\frac{1}{2} \sqrt{\frac{|p_n|}{2\pi 2E_{p_n}}} \delta(E_{p_m} - E_{p_n}) e^{\mp iE_{p_m} t_2 + ip_m(x_2 - x_1)} \varphi_1(p_n) dE_{p_m} dp_n$$ (A5)

$$= \int_{-\infty}^{\infty} \int_{0}^{\infty} \sqrt{\frac{|p_n|}{2\pi 2E_{p_n}}} \delta(E_{p_m} - E_{p_n}) e^{\mp iE_{p_m} t_2 + ip_m(x_2 - x_1)} \varphi_1(p_n) dE_{p_m} dp_n$$ (A6)

(A6) is not consistent with (A3) unless $x_2 = x_1$ or $p_m$ and $p_n$ have the same sign. We can also check this inconsistency in classical limit.