The unicity of real Picard–Vessiot fields

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Abstract

Using Deligne’s work on Tannakian categories, the unicity of real Picard-Vessiot fields for differential modules over a real differential field is derived. The inverse problem for real forms of a semi-simple group is treated. Some examples illustrate the relations between differential modules, Picard–Vessiot fields and real forms of a group.

1 Introduction

$K$ denotes a real differential field with field of constants $k$. We suppose that $k \neq K$ and that $k$ is a real closed field. Let $M$ denote a differential module over $K$ of dimension $d$, represented by a matrix differential equation $y' = Ay$ where $A$ is a $d \times d$-matrix with entries in $K$. A Picard–Vessiot field $L$ for $M/K$ is a field extension of $K$ such that:
(a) $L$ is equipped with a differentiation extending the one of $K$,
(b) $M$ has a full space of solutions over $L$, i.e., there exists an invertible $d \times d$-matrix $F$ (called a fundamental matrix) with entries in $L$ satisfying $F' = AF$,
(c) $L$ is (as a field) generated over $K$ by the entries of $F$,
(d) the field of constants of $L$ is again $k$.

A real Picard–Vessiot field $L$ for $M/K$ is a Picard–Vessiot field which is also a real field. In [CHS1] and [CHS2] the existence of a real Picard–Vessiot field is proved using results of Kolchin.

The main result of this paper is:

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Theorem 1.1 Let $L_1, L_2$ denote two real Picard–Vessiot extensions for $M/K$. Suppose that $L_1$ and $L_2$ have total orderings which induce the same total ordering on $K$. Then there exists a $K$-linear isomorphism $\phi : L_1 \to L_2$ of differential fields.

Remarks 1.2 Suppose that $\phi$ exists. Choose a total ordering of $L_1$ and define the total ordering of $L_2$ to be induced by $\phi$. Then $L_1$ and $L_2$ induce the same total ordering on $K$. Therefore the condition of the Theorem 1.1 is necessary.

If $K$ happens to be real closed, then the assumption in the theorem is superfluous since $K$ has a unique total ordering. On the other hand, consider the example $K = k(z)$ with differentiation $' = \frac{d}{dz}$ and the equation $y' = \frac{1}{2}y$. Let $L_1 = K(t_1)$ with $t_1^2 = z$ and $L_2 = K(t_2)$ with $t_2^2 = -z$. Both fields are real Picard–Vessiot fields for this equation. They are not isomorphic as differential field extensions of $K$, since $z$ is positive for any total ordering of $L_1$ and $z$ is negative for any total ordering of $L_2$. □

The proof of Theorem 1.1 uses Tannakian categories as presented in [DM] and P. Deligne’s fundamental paper [De]. We adopt much of the notation of [De]. Let $< M >$ denote the Tannakian category generated by the differential module $M$. The forgetful functor $\rho : < M > \to \text{vect}(K)$ associates to any differential module $N \in < M >$ the finite dimensional $K$-vector space $N$. Let $\omega : < M > \to \text{vect}(k)$ be a fibre functor with values in the category $\text{vect}(k)$ of the finite dimensional vector spaces over $k$.

Now we recall some results of [De], §9. The functor $\text{Aut}(\omega)$ is represented by a linear algebraic group $G$ over $k$. By Proposition 9.3, the functor $\text{Isom}_K^G(K \otimes \omega, \rho)$ is represented by a torsor $P$ over $G_K := K \times_k G$. This torsor is affine, irreducible and its coordinate ring $O(P)$ has a natural differentiation extending the differentiation of $K$. Moreover, the field of fractions $K(P)$ of $O(P)$ is a Picard–Vessiot field for $M/K$.

On the other hand, let $L$ be a Picard–Vessiot field for $M/K$. Define the fibre functor $\omega_L : < M > \to \text{vect}(k)$ by $\omega_L(N) = \ker(\partial : L \otimes_K N \to L \otimes_K N)$. Then $\omega_L$ produces a Picard–Vessiot field $L'$ which is isomorphic to $L$ as differential field extension of $K$. The conclusion is:
Proposition 1.3 ([De], §9) The above constructions yield a bijection between
the (isomorphism classes of) fibre functors \( \omega : \langle M \rangle \otimes \rightarrow \text{vect}(k) \) and the
(isomorphism classes of) Picard–Vessiot fields \( L \) for \( M/K \).

The following result will also be useful.

Proposition 1.4 ([DM], Thm. 3.2) Let \( \omega : \langle M \rangle \otimes \rightarrow \text{vect}(k) \) be a fibre
functor and \( G = \text{Aut}^\otimes(\omega) \).

(a) For any field \( F \supset k \) and any fibre functor \( \eta : \langle M \rangle \otimes \rightarrow \text{vect}(F) \), the
functor \( \text{Isom}^\otimes_F(F \otimes \omega, \eta) \) is representable by a torsor over \( G_F = F \times_k G \).

(b) The map \( \eta \mapsto \text{Isom}^\otimes_F(F \otimes \omega, \eta) \) is a bijection between the (isomorphy
classes of) fibre functors \( \eta : \langle M \rangle \otimes \rightarrow \text{vect}(F) \) and the (isomorphy classes
of) \( G_F \)-torsors.

The main ingredient in the proof of Theorem 1.1, given in §2, is:

Theorem 1.5 Suppose that \( K \) is real closed. Let \( L \) be a Picard–Vessiot field
for \( M/K \). Then \( L \) is a real field if and only if the torsor \( \text{Isom}_K^\otimes(K \otimes \omega_L, \rho) \)
is trivial.

2 The proof of Theorem 1.1

2.1 Reduction to \( K \) is a real closed differential field

For notational convenience, the differential module \( M/K \) is represented by
a scalar homogeneous linear differential equation \( \mathcal{L}(y) := y^{(d)} + a_{d-1}y^{(d-1)} + 
\cdots + a_1y^{(1)} + a_0y = 0 \). A Picard–Vessiot field \( L \) for \( \mathcal{L} \) has the properties:
k is the field of constants of \( L \), the solution space \( V = \{ v \in L \mid \mathcal{L}(v) = 0 \} \)
is a \( k \)-linear space of dimension \( d \) and \( L \) is generated over the field \( K \) by \( V \) and all the derivatives of the elements in \( V \). One writes \( L = K < V > \) for
this last property.

Lemma 2.1 Let \( L_1, L_2 \) be two real Picard–Vessiot fields for \( M \) over \( K \). Suppose
that \( L_1 \) and \( L_2 \) have total orderings extending a total ordering \( \tau \) on \( K \). Let \( K^r \supset K \) be the real closure of \( K \) inducing the total ordering \( \tau \). Then:

The fields \( L_1, L_2 \) induce Picard–Vessiot fields \( \tilde{L}_1, \tilde{L}_2 \) for \( K^r \otimes M \) over \( K^r \).
These fields are isomorphic as differential field extensions of \( K^r \) if and only if \( L_1 \) and \( L_2 \) are isomorphic as differential field extensions of \( K \).
Proof. Let, for \( j = 1, 2 \), \( \tau_j \) be a total ordering on \( L_j \) inducing \( \tau \) on \( K \) and let \( L_j^r \) be the real closure of \( L_j \) which induces the ordering \( \tau_j \). The algebraic closure \( K_j \) of \( K \) in \( L_j^r \) is real closed. Since \( \tau_j \) induces \( \tau \), there exists a \( K \)-linear isomorphism \( \phi_j : K^r \to K_j \). This isomorphism is unique since the only \( K \)-linear automorphism of \( K^r \) is the identity. We will identify \( K_j \) with \( K^r \).

Let \( V_j \subset L_j \) denote the solution space of \( M \). Then, for \( j = 1, 2 \), the field \( \tilde{L}_j := K^r < V_j > \subset L_j^r \) is a real Picard–Vessiot field for \( K^r \otimes M \).

Assume the existence of a \( K^r \)-linear differential isomorphism \( \psi : K^r < V_1 > \to K^r < V_2 > \). Clearly \( \psi(V_1) = V_2 \) and \( \psi \) induces therefore a \( K \)-linear differential isomorphism \( L_1 = K < V_1 > \to L_2 = K < V_2 > \).

On the other hand, an isomorphism \( \phi : L_1 \to L_2 \) (of differential field extensions of \( K \)) extends to an isomorphism \( \tilde{\phi} : L_1^r \to L_2^r \). Clearly \( \tilde{\phi} \) maps \( \tilde{L}_1 \) to \( \tilde{L}_2 \). \( \square \)

\[ \sum \]

2.2 Real algebras and connected linear groups

An algebra \( R \) (commutative with 1 and without zero divisors) is called real if \( x_1, \ldots, x_n \in R \) and \( \sum_{j=1}^n x_j^2 = 0 \) implies \( x_1 = \cdots = x_n = 0 \). By lack of a reference we give a proof of the following statement.

Lemma 2.2 Let \( F \) be a real closed field and let \( G \) be a linear algebraic group over \( F \) such that \( G_{F(i)} = F(i) \times_F G \) is connected. Then the coordinate ring \( F[G] \) of \( G \) over \( F \) is a real algebra.

Proof. We consider the case \( F \) is equal to \( \mathbb{R} \), the field of real numbers. Consider \( x_1, \ldots, x_n \in \mathbb{R}[G] \) with \( \sum_{j=1}^n x_j^2 = 0 \). We regard \( G(\mathbb{R}) \) as a real analytic group. There is an exponential map \( \text{Lie}(G)(\mathbb{R}) \to G(\mathbb{R}) \), where \( \text{Lie}(G) \) is the Lie algebra of \( G \). Define the real analytic map \( \tilde{x}_j : \text{Lie}(G)(\mathbb{R}) \xrightarrow{\text{exp}} G(\mathbb{R}) \xrightarrow{x_j} \mathbb{R} \).

Now \( \sum \tilde{x}_j^2 = 0 \) and hence all \( \tilde{x}_j = 0 \). The complex analytic morphism \( X_j : \text{Lie}(G)(\mathbb{C}) \xrightarrow{\text{exp}} G(\mathbb{C}) \xrightarrow{x_j} \mathbb{C} \) is the complex extension of \( \tilde{x}_j \). It is zero since it is zero on the subset \( \text{Lie}(G)(\mathbb{R}) \) of \( \text{Lie}(G)(\mathbb{C}) \). The image of the complex exponential map generates the component of the identity of \( G(\mathbb{C}) \) and \( x_j \) is zero on this set. By assumption \( G_{\mathbb{C}} \) is connected and thus \( x_j = 0 \) is zero for all \( j \). Hence \( \mathbb{R}[G] \) is a real algebra.

For any real field \( F \) which has an embedding in \( \mathbb{R} \), one has \( F[G] \subset \mathbb{R}[G] \) and \( F[G] \) is a real algebra. Further, a real field \( k \) which is finitely generated over \( \mathbb{Q} \) has an embedding in \( \mathbb{R} \) ([Si] Proposition 3).
We consider the general case: $G$ is a linear algebraic group defined over a real closed field $F$. Now $G$ is defined over a subfield $F_0$ of $F$ which is finitely generated over $\mathbb{Q}$. Then $F[G]$ is the union of the subrings $k[G]$, where $k$ runs in the set of the subfields of $F$ which are finitely generated over $\mathbb{Q}$ and contain $F_0$. Thus $F[G]$ is a real algebra since every $k[G]$ is a real algebra.

Remark. The condition that $G_{F(i)}$ is connected (or equivalently $G$ is connected) is necessary. Indeed, consider the example of the group $\mu_3$ over $\mathbb{R}$ with coordinate algebra $\mathbb{R}[X]/(X^3 - 1)$. This algebra is isomorphic to the direct sum $\mathbb{R} \oplus \mathbb{R}[X]/(X^2 + X + 1)$ and therefore is not real.

Corollary 2.3 ([La] Corollary 6.8) Let $G$ be a linear algebraic group over the real closed field $F$. Suppose that $G_{F(i)}$ is connected. Then the group $G(F)$ is Zariski dense in $G(F(i))$.

Theorem 2.4 (1.5) Suppose that $K$ is real closed. Let $L$ be a Picard–Vessiot field for a differential module $M/K$. Then $L$ is a real field if and only if the torsor $\text{Isom}^\otimes_K(K \otimes \omega_L, \rho)$ is trivial.

Proof. $G := \text{Aut}^\otimes_K(\omega_L)$ coincides with the group of the $K$-linear differential automorphisms of $L$. Let $R$ denote the coordinate ring of the torsor $\text{Isom}^\otimes_K(K \otimes \omega_L, \rho)$. Then $L$ is the field of fractions of $R$.

If $L$ is a real Picard–Vessiot field, then $R \subset L$ is a finitely generated real $K$-algebra. From the real Nullstellensatz and the assumption that $K$ is real closed it follows that there exists a $K$-linear homomorphism $\phi : R \rightarrow K$ with $\phi(1) = 1$. The torsor $\text{Spec}(R)$ has a $K$-valued point and is therefore trivial.

We observe that $L(i)$ is a Picard–Vessiot field for the differential module $K(i) \otimes M$ over $K(i)$. Further $G_{K(i)}$ is the group of the $K(i)$-linear differential automorphisms of $L(i)$ and is the ‘usual’ differential Galois group of $K(i) \otimes M$ over $K(i)$. This group is connected since $K(i)$ is algebraically closed.

Suppose that the torsor $\text{Spec}(R)$ is trivial. Then $R \cong K \otimes_k k[G] \cong K[G]$. According to Lemma 2.2, $K[G]$ is a real $K$-algebra and therefore its field of fractions $L$ is a real field. \qed
2.3 The final step

By Lemma 2.1, we may suppose that $K$ is real closed. Let $L_1, L_2$ denote two real Picard-Vessiot fields for a differential module $M/K$.

Write $\omega_j = \omega_{L_j} : < M > \to \text{vect}(k)$ for the corresponding fibre functors. Let $G = \text{Aut}_k^\otimes(\omega_j)$. Then $\text{Isom}_k^\otimes(\omega_1, \omega_2)$ is a $G$-torsor over $k$ corresponding to an element $\xi \in H^1(\{1, \sigma\}, G(k(i)))$, where $\{1, \sigma\}$ is $\text{Gal}(k(i)/k)$, represented by a 1-cocycle $c$ with $c(1) = 1$, $c(\sigma) \in G(k(i))$ and $c(\sigma) \cdot \sigma c(\sigma) = 1$.

The $G_K$-torsor $\text{Isom}_K^\otimes(K \otimes \omega_1, K \otimes \omega_2)$ corresponds to an element $\eta \in H^1(\{1, \sigma\}, G(K(i)))$. This element is the image of $\xi$ under the map, induced by the inclusion $G(k(i)) \subset G(K(i))$, from $H^1(\{1, \sigma\}, G(k(i)))$ to $H^1(\{1, \sigma\}, G(K(i)))$ (we note that $\text{Gal}(K(i)/k) = \text{Gal}(k(i)/k)$). Since $L_j$ is real, the torsor $\text{Isom}_K^\otimes(K \otimes \omega_j, \rho)$ is trivial for $j = 1, 2$, by Theorem 1.5. Thus there exists isomorphisms $\alpha_j : K \otimes \omega_j \to \rho$ for $j = 1, 2$. The isomorphism $\alpha_2^{-1} \circ \alpha_1 : K \otimes \omega_1 \to K \otimes \omega_2$ implies that $\eta$ is trivial. In particular, there is an element $h \in G(K(i))$ such that $c(\sigma) = h^{-1} \sigma(h)$.

There exists a finitely generated $k$-algebra $B \subset K$ with $h \in G(B(i))$. Since $B$ is real and $k$ is real closed, there exists, by the real Nullstellensatz, a $k$-linear homomorphism $\phi : B \to k$ with $\phi(1) = 1$. Applying $\phi$ to the identity $c(\sigma) = h^{-1} \sigma(h)$ one obtains $c(\sigma) = \phi(h)^{-1} \sigma(\phi(h))$. Thus $c$ is a trivial 1-cocycle and there is an isomorphism $\omega_1 \to \omega_2$. Hence $L_1$ and $L_2$ are isomorphic as differential field extensions of $K$.

Remark. The natural map $H^1(\{1, \sigma\}, G(k(i))) \to H^1(\{1, \sigma\}, G(K(i)))$ is injective, by the above argument.

\[\square\]

3 Comments and Examples

The proof of the unicity of a real Picard–Vessiot field uses almost exclusively properties of Tannakian categories. This implies that the proof remains valid for other types of equations, such as:

(a). linear partial differential equations, like $\frac{\partial}{\partial z_j} Y = A_j Y$ for $j = 1, \ldots, n$,
(b). linear ordinary difference equations, like $Y(z + 1) = AY(z),$
(c). linear q-difference equations with $q \in \mathbb{R}^*$, like $Y(qz) = AY(z)$.

For case (a), the existence of a real Picard-Vessiot field has been proved in [CH]. The proof of the uniqueness result (Theorem 1.1) for real differential
fields with real closed field of constants can probably be rephrased for the case of differential modules over a formally p-adic differential field with a p-adically closed field of constants of the same rank.

**Observations 3.1**

Let $K$ be a real closed differential field with field of constants $k$, $M/K$ a differential module and $\omega :< M >_{\otimes} \rightarrow vect(k)$ a fibre functor. Let $L$ be the Picard–Vessiot field corresponding to $\omega$ and $G$ the group of the differential automorphisms of $L/K$. Let $H$ be the differential Galois group of $K(i) \otimes M$ over $K(i)$. We recall that $G$ is a form of $H$ over the field $k(i)$. Using the identification $k(i) \times_k G = H$, one obtains on $H$ and on $Aut(H)$ a structure of algebraic group over $k$. Let $\{1, \sigma\}$ be the Galois group of $k(i)/k$. Then $H^1(\{1, \sigma\}, Aut(H))$ has a natural bijection to the set of forms of $H$ over $k$. Although the action of $\sigma$ on $Aut(H)$ depends on $G$, this set does not depend on the choice of $G$.

Let $\eta :< M >_{\otimes} \rightarrow vect(k)$ be another fibre functor. Then $\eta$ is mapped, according to Proposition 1.4, to an element in $\xi(\eta) \in H^1(\{1, \sigma\}, G(k(i)))$ (and this induces a bijection between $\eta$’s and elements in this cohomology set). A 1-cocycle $c$ for the group $\{1, \sigma\}$ has the form $c(1) = 1$, $c(\sigma) = a$ and $a$ should satisfy $a \cdot \sigma(a) = 1$ (and is thus determined by $a$).

A 1-cocycle for $\xi(\eta)$ can be made as follows. The fibre functor $\eta$ corresponds to a Picard–Vessiot field $L_\eta$. Both $L(i)$ and $L_\eta(i)$ are Picard–Vessiot fields for $K(i) \otimes M$ over $K(i)$. Thus there exists a $K(i)$-linear differential isomorphism $\phi : L(i) \rightarrow L_\eta(i)$. On the field $L(i)$ we write $\tau$ for the conjugation given by $\tau(i) = -i$ and $\tau$ is the identity on $L$. The similar conjugation on $L_\eta(i)$ is denoted by $\tau_\eta$. Now $\tau_\eta \circ \phi \circ \tau : L(i) \rightarrow L_\eta(i)$ is another $K(i)$-linear differential isomorphism. A 1-cocycle $c$ for $\xi(\eta)$ is now $c(\sigma) = \phi^{-1} \circ \tau_\eta \circ \phi \circ \tau$.

Let $G_\eta$ denote the group of the $K$-linear differential automorphism of $L_\eta$. The group $G_\eta$ is a form of $G$ and produces an element in $H^1(\{1, \sigma\}, G(k(i)))$ with $H = k(i) \times G$. We want to compute a 1-cocycle $C$ for this element. Define the isomorphism $\psi : k(i) \times G \rightarrow k(i) \times G_\eta$ of algebraic groups over $k(i)$, by $\psi(g) = \phi \circ g \circ \phi^{-1}$. Define $\tau_G$, the ‘conjugation’ on $k(i) \times G$, by the formula $\tau_G(g) = \tau \circ g \circ \tau$ for the elements $g \in G(k(i))$. Let $\tau_{G_\eta}$ be the similar conjugation on $k(i) \times G_\eta$. Now $\tau_{G_\eta} \circ \psi \circ \tau_G : k(i) \times G \rightarrow k(i) \times G_\eta$ is another isomorphism between the algebraic groups over $k(i)$. The 1-cocycle $C$ is given by $C(\sigma) = \psi^{-1} \circ \tau_{G_\eta} \circ \psi \circ \tau_G$. One observes that $C(\sigma)(g) = c(\sigma)gc(\sigma)^{-1}$. 

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The map, which associates to \( h \in G(k(i)) \), the automorphism \( g \mapsto hgh^{-1} \) of \( G \), induces a map \( H^1(\{1, \sigma\}, G(k(i))) \to H^1(\{1, \sigma\}, G/Z(G)(k(i))) \to H^1(\{1, \sigma\}, \text{Aut}(H)) \), denoted by \( \xi(\eta) \mapsto \tilde{\xi}(\eta) \). The forms corresponding to elements in the image of \( H^1(\{1, \sigma\}, G/Z(G)(k(i))) \to H^1(\{1, \sigma\}, \text{Aut}(H)) \) are called ‘inner forms of \( G \).’ By §1, \( \eta \) induces a Picard–Vessiot field and a form \( G(\eta) \) of \( H \). Above we have verified (see \([B]\) for a similar computation) that \( G(\eta) \) is the inner form of \( G \) corresponding to the element \( \tilde{\xi}(\eta) \). For the delicate theory of forms we refer to the informal manuscript \([B]\) and the standard text \([Sp]\). \( \square \)

Examples.

We continue with the notation and assumptions of Observations (3.1).

(1). Let \( M/K, \omega, L, G \) be such that \( G = \text{SL}_{n,k} \). Since \( H^1(\{1, \sigma\}, \text{SL}_{n,k}(k(i))) \) is trivial, \( L \) is the unique Picard–Vessiot field and is a real field (because a real Picard–Vessiot field exists).

The group \( \text{SL}_n \) has non trivial forms. For instance, \( \text{SU}(2) \) is an inner form of \( \text{SL}_{2,\mathbb{R}} \). There are examples, according to Proposition 3.2 below, of differential modules \( M/K \) having a real Picard–Vessiot field \( L \) with group of differential automorphisms of \( L/K \) equal to \( \text{SU}(2) \).

From \([Sp]\), 12.3.7 and 12.3.9 one concludes that \( H^1(\{1, \sigma\}, \text{SU}(2)(\mathbb{C})) \) is trivial. Again \( L \) is the only Picard–Vessiot field.

(2). If \( G \) is the symplectic group \( \text{Sp}_{2n,k} \), then there are no forms and \( H^1(\{1, \sigma\}, G(k(i))) \) is trivial. Therefore there is only one Picard–Vessiot field \( L \) and this is a real field.

(3). Consider a \( k \)-form \( G \) of \( \text{SO}(n)_k \) with odd \( n \geq 3 \). The center \( Z \) of \( G \) consists of the scalar matrices of order \( n \), thus \( Z \) is the group \( \mu_{n,k} \) of the \( n \)th roots of unity. Since \( n \) is odd, one has \( Z(k) = \{1\} \). Further, again since \( n \) is odd, the automorphisms of \( H = G_k(i) \) are interior and \( \text{Aut}(H)(k(i)) = G/Z(k(i)) \).

We claim the following.

The natural map \( H^1(\{1, \sigma\}, G(k(i))) \to H^1(\{1, \sigma\}, G/Z(k(i))) \) is a bijection.

Proof. A 1-cocycle \( c \) for \( G/Z(k(i)) \) is given by \( c(1) = 1 \) and \( c(\sigma) = a \in G/Z(k(i)) \) with \( a\sigma(a) = 1 \). Choose an \( A \in G(k(i)) \) which maps to \( a \). Thus \( A\sigma(A) \in Z(k(i)) \) and \( A \) commutes with \( \sigma A \). Further \( \sigma(A\sigma(A)) = \sigma(A)A = A\sigma(A) \) and thus \( A\sigma(A) \in \mu_{n}(k) = \{1\} \). Therefore \( C \) defined by \( C(1) = 1 \), \( C(\sigma) = A \) is a 1-cocycle for \( G(k(i)) \) and maps to \( c \). Hence the map is surjective.
Consider for \( j = 1, 2 \) the 1-cocycle \( C_j \) for \( G \) given by \( C_j(\sigma) = A_j \). Suppose that the images of \( C_j \) as 1-cocycles for \( G/Z(k(i)) \) are equivalent. Then there exists \( B \in G(k(i)) \) such that \( B^{-1}A_1\sigma(B) = xA_2 \) for some element \( x \in Z(k(i)) \). We may replace \( B \) by \( yB \) with \( y \in Z(k(i)) \). Then \( x \) is changed into \( xy^{-1}\sigma(y) \). And the latter is equal to 1 for a suitable \( y \). This proves the injectivity of the map.

We conclude from the above result that there exists a (unique up to isomorphism) fibre functor \( \eta : M > \otimes \rightarrow \text{vect}(k) \) (or, equivalently, a Picard–Vessiot field) for every form of \( H = SO(n)_{k(i)} \) over \( k \). Moreover, only one of these fibre functors corresponds to a real Picard–Vessiot field.

Let \( \omega : M > \otimes \rightarrow \text{vect}(k) \) denote the fibre functor corresponding to a real Picard–Vessiot field \( L_\omega \) and \( G_\omega \) the group of the differential automorphisms of \( L_\omega/K \). We want to identify this form \( G_\omega \) of \( H := SO(n)_{k(i)} \).

Since the differential Galois group of \( K(i) \otimes M \) is \( SO(n)_{k(i)} \), there exists an element \( F \in \text{sym}^2(K(i) \otimes M^*) \) with \( \partial F = 0 \). Further \( F \) is unique up to multiplication by a scalar and \( F \) is a non degenerate bilinear symmetric form. The non trivial automorphism \( \sigma \) of \( K(i)/K \) and of \( k(i)/k \) acts in an obvious way on \( K(i) \otimes M \) and on constructions by linear algebra of \( K(i) \otimes M \). Now \( \sigma(F) \) has the same properties as \( F \) and thus \( \sigma(F) = cF \) for some \( c \in K(i) \). After changing \( F \) into \( aF \) for a suitable \( a \in K(i) \), we may suppose that \( \sigma(F) = F \). Then \( F \) belongs to \( \text{sym}^2(M^*) \) and is a non degenerate form of degree \( n \) over the field \( K \). Further \( F \) is determined by its signature because \( K \) is real closed. Moreover \(KF\) is the unique 1-dimensional submodule of \( \text{sym}^2(M^*) \). We claim the following:

\( G_\omega \) is the special orthogonal group over \( k \) corresponding to a form \( f \) over \( k \) which has the same signature as \( F \).

Let \( V = \omega(M) \). The group \( G_\omega \) is the special orthogonal group of some non degenerate bilinear symmetric form \( f \in \text{sym}^2(V^*) \). Since \( L_\omega \) is real, there exists a isomorphism \( m : K \otimes_k \omega \rightarrow \rho \) of functors. Applying \( m \) to the modules \( M \) and \( \text{sym}^2(M^*) \) one finds an isomorphism \( m_1 : K \otimes_k V \rightarrow M \) of \( K \)-vector spaces which induces an isomorphism of \( K \)-vector spaces \( m_2 : K \otimes_k \text{sym}^2(V^*) \rightarrow \text{sym}^2(M^*) \). The latter maps the subobject \( K \otimes k f \) to \( K F \) by the uniqueness of \( K F \). One concludes that the forms \( f \) and \( F \) have the same signature.
Proposition 3.2 Suppose that $K$ is real closed. Given is a connected semi-simple group $H$ over $k(i)$ and a form $G$ of $H$ over $k$. Then there exists a differential module $M$ over $K$ and a real Picard–Vessiot field for $M/K$ such that the group of the differential automorphisms of $L/K$ is $G$.

Proof. Let $G$ be given as a subgroup of some $GL_{n,k}$, defined by a radical ideal $I$. Then $k[G] = k[\{X_{k,l}\}_{k,l=1}^n, \frac{1}{\det}] / I$. The tangent space of $G$ at $1 \in G$ can be identified with the $k$-linear derivations $D$ of this algebra, commuting with the action of $G$. These derivations $D$ have the form $(DX_{k,l}) = B \cdot (X_{k,l})$ for some matrix $B \in \text{Lie}(G)(k)$ (where $\text{Lie}(G) \subset \text{Matr}(n,k)$ is the Lie algebra of $G$).

The same holds for $K[G] = K \otimes_k k[\{X_{k,l}\}_{k,l=1}^n, \frac{1}{\det}] / I$. Any $K$-linear derivation $D$ on the algebra, commuting with the action of $G$, has the form $(DX_{k,l}) = A \cdot (X_{k,l})$ with $A \in \text{Lie}(G)(K)$. We choose $A$ as general as possible.

The differential module $M/K$ is defined by the matrix equation $y' = Ay$. It follows from [PS], Proposition 1.3.1 that the differential Galois group of $K(i) \otimes M$ is contained in $H = G_{k(i)}$. Now one has to choose $A$ such that the differential Galois group (which is connected because $K(i)$ is algebraically closed) is not a proper subgroup of $H$. Since $H$ is semi-simple, there exists a Chevalley module for $H$. Using this Chevalley module one can produce a general choice of $A$ such that differential Galois group of $y' = Ay$ over $K(i)$ is in fact $G_{k(i)}$ (compare [PS], §11.7 for the details which remain valid in the present situation).

The usual way to produce a Picard–Vessiot ring for the equation $y' = Ay$ is to consider the differential algebra $R_0 := K[\{X_{k,l}\}_{k,l=1}^n, \frac{1}{\det}]$, with differentiation defined by $(X'_{k,l}) = A \cdot (X_{k,l})$, and to produce a maximal differential ideal in $R_0$. Since $A \in \text{Lie}(G)(K) \subset \text{Lie}(H)(K(i))$, the ideal $J \subset R_0[i]$, generated by $I$ is a differential ideal. It is in fact a maximal differential ideal of $R_0[i]$, since the differential Galois group is precisely $H$. Then $J \cap R_0 = IR_0$ is a maximal differential ideal of $R_0$ and $K[G] = R = R_0/IR_0$ is a Picard–Vessiot ring for $M$ over $K$. The field of fractions $L$ of $R$ is real because the $G$-torsor $\text{Spec}(K[G])$ is trivial.

It seems that, imitating the proofs in [MS], one can show that Proposition 3.2 remains valid under the weaker conditions: $K$ is a real differential field and a $C_1$-field and $H$ is connected.
References

[B] K. Buzzard, *Forms of reductive algebraic groups*, On the web, February 7, 2012.

[BCR] J. Bochnak, M. Coste, M.-F. Roy, *Real Algebraic Geometry*, Springer Verlag, Berlin, 1998.

[CH] T. Crespo, Z. Hajto, *Picard-Vessiot theory for real partial differential fields*, submitted.

[CHS1] T. Crespo, Z. Hajto, E. Sowa, *Constrained extensions of real type*, C. R. Acad. Sci. Paris, Ser. I 350 (2012), 235-237.

[CHS2] T. Crespo, Z. Hajto, E. Sowa, *Picard-Vessiot theory for real fields*, Israel J. Math., to appear.

[De] P. Deligne, *Catégories galoisiennes*, in: “The Grothendieck Festschrift”, vol. 2, Progr. Math. Birkhäuser 87, 1990, pp. 111-195.

[DM] P. Deligne, J.S. Milne, *Tannakian Categories*, in: “Hodge cycles, motives, and Shimura varieties”, Lect. Notes Math. 900, Springer, 1982, pp.101-228.

[La] T.Y. Lam, *An introduction to real algebra*, Rocky Mt. J. Math. 14 (1984), 767-814.

[MS] C. Mitschi, M.F. Singer, *Connected linear algebraic groups as differential Galois groups*, J.Algebra, 184 (1996), 333-361.

[PS] M. van der Put, M.F. Singer, *Galois Theory of Linear Differential Equations*, Grundlehren, Volume 328, Springer Verlag 2003

[Si] M.F. Singer *The model theory of ordered differential fields*, J. Symbolic Logic 43 (1978), 82-91.

[Sp] T.A. Springer, *Linear Algebraic Groups, Second Edition*, Progress in Mathematics, Volume 9, 1998.
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