STANDARD ELEMENTS OF THE LATTICE OF MONOID VARIETIES

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ABSTRACT. We completely classify all standard elements in the lattice of all monoid varieties. In particular, we prove that an element of this lattice is standard if and only if it is neutral.

The article is devoted to investigation of the lattice of varieties of monoids which will be denoted by \textit{MON} (when referring to monoid varieties, we consider monoids as semigroups equipped by an additional 0-ary operation that fixes the identity element). Until recently, this lattice has been studied very little. However, recently, the articles [2–6] which are devoted to this subject appeared. In particular, the study of the special elements of the lattice \textit{MON} lattice was begun in [3]. In this paper, we continue to study them.

Let us recall definitions of special elements which will be used below. An element \(x\) of a lattice \(L\) is called

- **neutral** if \(\forall y, z \in L: (x \lor y) \land (y \lor z) \land (z \lor x) = (x \land y) \lor (y \land z) \lor (z \land x);\)
- **standard** if \(\forall y, z \in L: (x \lor y) \land z = (x \land z) \lor (y \land z);\)
- **modular** if \(\forall y, z \in L: y \leq z \rightarrow (x \lor y) \land z = (x \land z) \lor y;\)
- **lower-modular** if \(\forall y, z \in L: x \leq y \rightarrow x \lor (y \land z) = y \land (x \lor z).\)

Costandard and upper-modular elements are defined dually to standard and lower-modular elements respectively. It is evident that a neutral element is both standard and costandard; a standard element is both modular and lower modular; a costandard element is both modular and upper-modular. Some information about special elements in arbitrary lattices can be found in [1, Section III.2].

The neutral and costandard elements of the lattice \textit{MON} were completely described in [3]. In this paper, we classify the standard elements of this lattice.

The trivial variety of monoids is denoted by \textit{T}, while \textit{MON} denotes the variety of all monoids. We denote by \textit{SL} the variety of all semilattice monoids. The main result of the paper is the following

**Theorem 1.** For a monoid variety \(V\), the following are equivalent:

(i) \(V\) is a modular and lower-modular element of the lattice \textit{MON};
(ii) \(V\) is a standard element of the lattice \textit{MON};
(iii) \(V\) is a neutral element of the lattice \textit{MON};

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(iv) \( V \) is one of the varieties \( T, \ SL \) or \( \text{MON} \).

Theorem 1 implies

**Corollary 2.** Every standard element is costandard one in \( \text{MON} \).

We note that the equivalence of the claims (iii) and (iv) of Theorem 1 was proved in [3, Theorem 1.1]. It is natural to compare Theorem 1 and the results of [3] with the results concerning the special elements of lattice of semigroup varieties denoted by \( \text{SEM} \) (the survey of these results can be found in [13]). The properties of being neutral and standard elements are not equivalent in the lattice \( \text{SEM} \) (this fact follows from Theorems 3.3 and 3.4 of [13]), while the properties of being neutral and costandard elements are equivalent in this lattice (see [13, Theorem 3.4]). In contrary with the semigroup case, the properties of being neutral and standard elements are not equivalent (see Theorems 1.1 and 1.2 of [3]), while the properties of being neutral and standard elements are equivalent in \( \text{MON} \) (Theorem 1 of this work). Theorem 1 implies that an element of \( \text{MON} \) is neutral if and only if it is both modular and lower-modular in \( \text{MON} \). However, this is not true for the lattice \( \text{SEM} \) (this fact follows from Theorems 3.2 and 3.4 and Corollary 3.9 of [13]). At the same time, an element of \( \text{SEM} \) is neutral if and only if it is both upper-modular and lower-modular (see [13, Theorem 3.4]). The question about whether the same result holds in the lattices \( \text{MON} \) remains open. Finally, every standard element is costandard one but a standard element doesn’t have to be costandard one in \( \text{MON} \) (Theorem 1 and [3, Theorem 1.2]). At the same time, every costANDARD element is standard one but the properties of being standard and costandard elements are not equivalent in \( \text{SEM} \) (Theorems 3.3 and 3.4 of [13]).

To prove the main result, we need several auxiliary statements. We start with the fact that is a part of the semigroup folklore (it is noted in [6, Section 1.1] and [3, Proposition 2.1], for instance).

**Proposition 3.** The map from \( \text{MON} \) into \( \text{SEM} \) that maps a monoid variety generated by a monoid \( M \) to the semigroup variety generated by \( M \) is an embedding of the lattice \( \text{MON} \) into the lattice \( \text{SEM} \). □

Recall that the variety \( V \) is said to be periodic if all its monoids are periodic and aperiodic if it does not contain any non-trivial group. A monoid variety \( V \) is called proper if \( V \neq \text{MON} \). The proof of the following statement is similar to the arguments from the second paragraph of Section 2.1 of [11].

**Lemma 4.** Let \( V \) be a proper monoid variety. If \( V \) is a lower-modular element of \( \text{MON} \) then \( V \) is periodic.

**Proof.** Suppose that \( V \) is non-periodic. Then \( V \) contains the variety of all commutative monoids. It is proved [12, Lemma 2.16] that the variety of all semigroups is generated by all minimal non-Abelian varieties of groups. This fact and Proposition 3 imply that there exists a minimal non-Abelian group variety \( G \) such that \( G \not\subseteq V \). Put \( W = V \lor G \). Clearly, \( V \subseteq W \). As is well known, every monoid variety that contains the variety of all commutative monoids is generated by all its aperiodic members. Hence there exists an aperiodic variety \( K \) such that \( K \subseteq W \) but \( K \not\subseteq V \). Put \( Y = V \lor K \). Clearly, \( V \subseteq Y \subseteq W \). It is proved in [11, Lemma 1.4] that if \( U \) is a semigroup variety and \( X \) is an aperiodic semigroup variety then every group from the variety \( U \lor X \) belongs to \( U \). Since \( G \not\subseteq V \), this fact and Proposition 3 imply that \( G \not\subseteq V \lor K = Y \). Therefore, the variety \( G \lor Y \) is commutative,
the other hand, $S$ satisfies the identity

$$x^{(1)}$$

such a form holds in $x^{(3)}$. Thus, $Y$ contains $C$. Therefore, this identity does not hold in $x^{(3)}$. We have proved that the variety $V$ is periodic. \qed

We need a series of notation and definitions. The free monoid over a countably infinite alphabet is denoted by $F^1$. As usual, elements of $F^1$ are called words, while elements of $A$ are said to be letters. The words unlike letters are written in bold. Two parts of an identity are connected by the symbol $\approx$, while the symbol $=$ denotes, among other things, the equality relation on the free monoid. The following notion was introduced by Perkins [10] and often appeared in the literature. For any word $w$, let $S(w)$ denote the Rees quotient monoid of $F^1$ over the ideal of all words that are not subwords of $w$. A non-empty word $w$ is an isoterm for a variety $V$ if $V$ does not satisfy any non-trivial identity of the form $w \approx w'$. Monoid variety given by an identity system $\Sigma$ is denoted by $\text{var } \Sigma$. Put

$$C_n = \text{var}\{x^n \approx x^{n+1}, xy \approx yx\} \text{ where } n \geq 2,$$

$$E = \text{var}\{x^2 \approx x^3, x^2y \approx xyx, x^2y^2 \approx y^2x^2\}.$$

**Lemma 5.** Let $V$ be a monoid variety that contains the variety $E$. Suppose that there exists $n \geq 2$ such that $V$ does not contain the variety $C_{n+1}$. Put

$$X = (V \lor C_{n+1}) \land \text{var}\{x^ky \approx yx^k \mid k > n\},$$

$$Y = (V \lor C_{n+1}) \land \text{var}\{x^ky \approx yx^k \mid k \geq n\}.$$ 

Then $Y \subset X$ and the varieties $X$, $Y$ and $V$ generate the 5-element non-modular sublattice in $\text{MON}$. In particular, $V$ is not a modular element of the lattice $\text{MON}$.

**Proof.** Evidently, $Y \subseteq X$. We are going to verify that this inclusion is strict. In view of [5, Proposition 4.2], if $E$ satisfies an identity $yx^n \approx w$ then $w = xy^t$ for some $t \geq 2$. If the identity $yx^n \approx w$ holds in $C_{n+1}$ then it follows from commutative law. Taking into account the inclusion $E \subseteq V$ we have that the word $yx^n$ is an isoterm for $V \lor C_{n+1}$. Then $S(yx^n) \in V \lor C_{n+1}$ by [7, Lemma 5.3]. Evidently, $S(yx^n)$ satisfies the identity $x^ky \approx yx^k$ whenever $k > n$. It follows that $S(yx^n) \notin X$. On the other hand, $S(yx^n) \notin Y$ because $S(yx^n)$ violates the identity

$$x^ny \approx yx^n. \tag{1}$$

Thus, $Y \subset X$.

It is well known and can be easily verified that if a monoid variety does not contain $C_{n+1}$ then this variety satisfies the identity

$$x^n \approx x^{n+m} \tag{2}$$

for some natural $m$ (see [5, Lemma 2.5], for instance). In particular, an identity of such a form holds in $V$. Then $V$ violates the identity

$$x^{n+m}y \approx yx^{n+m}. \tag{3}$$

Therefore, this identity does not hold in $V \lor C_{n+1}$, whence $X \subseteq V \lor C_{n+1}$.

Evidently, $V \lor X = V \lor C_{n+1} = V \lor Y$. To complete the proof it remains to note that $V \land X = V \land Y$. Indeed, the variety $V \land X$ satisfies the identity (1) because this identity follows from the identities (2) and (3). This implies the required conclusion. \qed
A variety of monoids is called \emph{completely regular} if it consists of completely regular monoids (i.e., unions of groups).

\textit{Proof of Theorem 1.} The claims (iii) and (iv) are equivalent by [3, Theorem 1.1]. The implications (iii) \(\Rightarrow\) (ii) \(\Rightarrow\) (i) are obvious. It remains to prove the implication (i) \(\Rightarrow\) (iv). Let \(\mathbf{V}\) be a proper monoid variety that is a modular and lower-modular element of the lattice \(\text{MON}\). Suppose that \(\mathbf{V}\) is completely regular. In view of Lemmas 3.1 and 3.2 of [3], every completely regular monoid variety that is a modular element of \(\text{MON}\) is commutative and aperiodic. Since every completely regular aperiodic variety is a variety of idempotent monoids, \(\mathbf{V} \subseteq \text{SL}\). Therefore, \(\mathbf{V} \in \{\text{T}, \text{SL}\}\).

Suppose now that \(\mathbf{V}\) is a non-completely regular monoid variety. Lemma 4 implies that \(\mathbf{V}\) is periodic. It is well known that \(\mathbf{V}\) satisfies the identity (2) for some \(n \geq 2\) and \(m \geq 1\). The identity (2) does not hold in \(\mathbf{C}_{n+1}\), whence \(\mathbf{C}_{n+1} \not\subseteq \mathbf{V}\). Then \(\mathbf{E} \not\subseteq \mathbf{V}\) by Lemma 5. Put \(\mathbf{W} = \mathbf{V} \lor \mathbf{E}\). Clearly, \(\mathbf{V} \subseteq \mathbf{W}\).

Put \(\text{LRB} = \text{var}\{xy \approx xyx\}\). We are going to verify that \(\text{LRB} \not\subseteq \mathbf{W}\). If \(\mathbf{V}\) is non-commutative then Lemmas 2.14 and 4.1 and Proposition 4.2 of [5] imply that \(\mathbf{V}\) satisfies the identity
\[
yx^r \approx x^sy^tx^t
\]
for some \(s \geq 1\), \(t \geq 0\), \(s + t \geq 2\) and \(r \geq 2\). If \(\mathbf{V}\) is commutative then \(\mathbf{V}\) satisfies the identity (4) with \(s = t = 1\) and \(r = 2\). The variety \(\mathbf{E}\) satisfies the identity
\[
y^2x^r \approx x^sy^2x^t.
\]
The identity (5) follows from the identity (4). Therefore, the identity (5) holds in \(\mathbf{W}\). On the other hand, \(\text{LRB}\) satisfies the identities \(y^2x^r \approx yx\) and \(x^sy^2x^t \approx xy\). Therefore, \(\text{LRB}\) violates the identity (5), whence \(\text{LRB} \not\subseteq \mathbf{W}\).

In view of [14, Proposition 4.7], the subvariety lattice of \(\text{LRB}\) is the chain \(\text{T} \subset \text{SL} \subset \text{LRB}\). It is well known that every non-group monoid variety contains the variety \(\text{SL}\) (see [5, Lemma 2.1], for instance). In particular, \(\text{SL} \subseteq \mathbf{V}\) and, therefore, \(\text{SL} \subseteq \mathbf{W}\). It follows that
\[
\mathbf{V} \lor (\mathbf{W} \land \text{LRB}) = \mathbf{V} \lor \text{SL} = \mathbf{V}.
\]
On the other hand, in view of [5, Corollary 2.6], every non-completely regular monoid variety contains the variety \(\mathbf{C}_2\). It is proved in [9, Proposition 4.1] that \(\text{E} \subseteq \mathbf{C}_2 \lor \text{LRB}\). This implies that \(\mathbf{W} \subseteq \mathbf{V} \lor \text{LRB}\). Thus,
\[
\mathbf{W} \land (\mathbf{V} \lor \text{LRB}) = \mathbf{W}.
\]
Then, since \(\mathbf{V} \subseteq \mathbf{W}\), we have that \(\mathbf{V}\) is not a lower-modular element in \(\text{MON}\). A contradiction.

\(\square\)

It is proved in [3, Theorem 1.1] that an element is neutral in \(\text{MON}\) if and only if this element is both modular, lower-modular and upper-modular in \(\text{MON}\). Theorem 1 establishes more stronger result. Namely, the property of being upper-modular element can be omitted. In view of Theorems 1.1 and 1.2 of [3], the variety \(\mathbf{C}_2\) is a costandard (and, therefore, modular) element but is not a neutral one in \(\text{MON}\). Thus, the property of being lower-modular element cannot be omitted. The following question is still open

\textbf{Question 6.} Is it true that an arbitrary lower-modular element of the lattice \(\text{MON}\) is a neutral element of this lattice?
In conclusion, we note that the properties of being modular and costandard elements are not equivalent in \( \text{MON} \). Indeed, put

\[ D = \text{var}\{ x^2 \approx x^3, x^2 y \approx xyx \approx yx^2 \}. \]

In view of [3, Theorem 1.2], \( D \) is not a costandard element of \( \text{MON} \). At the same time, the following statement is true.

**Fact 7.** The variety \( D \) is a modular element of the lattice \( \text{MON} \).

**Proof.** Suppose that \( D \) is not a modular element of the lattice \( \text{MON} \). Then [8, Proposition 2.1] implies that there exist varieties \( U \) and \( W \) such that \( U \subset W \) and the varieties \( D, U \) and \( W \) generate the 5-element non-modular sublattice in \( \text{MON} \). Clearly, \( D \not\subset U \) and \( D \not\subset W \). In view of [5, Lemma 2.12], \( W \) is either completely regular or commutative. If \( W \) is completely regular then \( U \) is completely regular too. The arguments from the latest paragraph of the proof of Theorem 1.2 of [3] imply a contradiction. Therefore, we can assume that \( W \) is commutative. In view of [3, Proposition 1.4], each commutative variety of monoids is an upper-modular element of the lattice \( \text{MON} \). This contradicts the fact that the varieties \( D, U \) and \( W \) generate the 5-element non-modular sublattice in \( \text{MON} \). \( \square \)

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**References**

[1] G. Grätzer, Lattice Theory: Foundation, Springer Basel AG, 2011.
[2] S.V. Gusev, On the lattice of overcommutative varieties of monoids, Izv. VUZ. Matem. No. 5 (2018), 28–32 [Russian; Engl. translation: Russ. Math. Izv. VUZ, 62, No. 5 (2018), 23–26].
[3] S.V. Gusev, Special elements of the lattice of monoid varieties, Algebra Universalis, 79 (2018), Article 29, 1–12.
[4] S.V. Gusev, On the ascending and descending chain conditions in the lattice of monoid varieties, Siberian Electronic Math. Reports, 16 (2019), 983–997.
[5] S.V. Gusev and B.M. Vernikov, Chain varieties of monoids, Dissertationes Math., 534 (2018), 1–73.
[6] M. Jackson and E.W.H. Lee, Monoid varieties with extreme properties, Trans. Amer. Math. Soc., 370 (2018), 4785–4812.
[7] M. Jackson and O. Sapir, Finitely based, finite sets of words, Int. J. Algebra and Comput., 10 (2000), 683–708.
[8] J. Ječek, The lattice of equational theories. Part I: modular elements, Czechosl. Math. J., 31 (1981), 127–152.
[9] E.W.H. Lee, Varieties generated by 2-testable monoids, Studia Sci. Math. Hungar, 49 (2012), 366–389.
[10] P. Perkins, Bases for equational theories of semigroups, J. Algebra, 11 (1969), 298–314.
[11] B.M. Vernikov, Lower-modular elements of the lattice of semigroup varieties, Semigroup Forum, 75 (2007), 554–566.
[12] B.M. Vernikov, Upper-modular elements of the lattice of semigroup varieties, Algebra Universalis, 59 (2008), 405–428.
[13] B.M. Vernikov, Special elements in lattices of semigroup varieties, Acta Sci. Math. (Szeged), 81 (2015), 79–109.
[14] S.L. Wismath, The lattice of varieties and pseudovarieties of band monoids, Semigroup Forum, 33 (1986), 187–198.

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