Abstract

The main purpose of this paper is to study under what condition compressible modules are critically compressible. A sufficient condition for the injective hull of a critically compressible module to be critically compressible is also provided. Furthermore we prove sufficient conditions for a critically compressible module to be continuous. In addition, some characterization of critically compressible modules in terms of CS modules, nonsingular modules and cyclic modules are also provided.

Key Words: - CS modules, continuous modules, uniform modules, compressible modules, self-similar modules, injective and quasi injective modules.

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Introduction

Throughout all rings have non-zero identity elements and all modules are unital right $R$ – modules. Any terminology not defined hare may be found in McConnell and Robson [19] and Gooderl and field [18]. J. Zelmanowitz considered compressible modules in detail in a series of papers [8, 9, 10, 11 and 12]. Zelmanowitz introduced the concept of a weakly primitive ring as a ring having a faithful compressible module. Following [8] the right $R$-module $M$ is called compressible if for each non-zero submodule $N$ of $M$, there exists a monomorphism $\alpha: M \to N$. In [13], author consider a problem due to Zelmanowitz and study under what condition a uniform
compressible module whose non-zero endomorphisms are monomorphisms is critically compressible and they have provided positive answer to this problem for the class of nonsingular modules, quasi-projective modules. In [13], it was proved that the concepts of compressible and critically compressible modules are equivalent over commutative rings or duo rings.

Zelmanowitz [8] claimed that a “compressible uniform module whose nonzero endomorphisms are monomorphisms would be critically compressible”. In [2] it was proved that for a commutative ring compressible and critically compressible modules are equivalent and the author called the above statement the “Zelmanowitz Conjecture”. Zelmanowitz also raises a question that under what conditions a compressible uniform module whose non-zero endomorphisms are monomorphisms is critically compressible.

Khuri [14] introduced the notion of retractable modules (slightly compressible modules) and gave some results for non-singular retractable modules when the endomorphism ring is (quasi-) continuous. Khuri considered retractable modules in details in series of papers [14, 15, and 16].

This paper is organized in two sections. In section 1 we give preliminary definitions and some results that help me to characterize critically compressible modules.

1. Preliminaries

Throughout this paper, it is assumed that R is an associative ring with an identity element. Unless otherwise indicated modules are unitary left modules and homomorphisms are written as right operators. If N is a submodule of M, we write N ≤ M and if N is an essential submodule of M then we write N ≪ M.

Now we will list some definitions and results about critically compressible modules used in this paper.

Definition 1.1. Let M be nonzero R-module. Then:
(1) M is called compressible if it can be embedded in each of its nonzero submodules.

(2) M is called critically compressible if it is compressible and, additionally, it cannot be embedded in any of its proper factor modules.

**Definition 1.2.** A partial endomorphism of M is a homomorphism from a submodule of M into M.

**Definition 1.3.** An R-module M is called uniform if for every pair of non zero submodule N₁ and N₂, N₁ \( \cap \) N₂ \( \neq \) 0. Clearly M will be uniform if every non zero submodule of M is essential submodule.

**Example 1.1.** If every nonzero partial endomorphism on M is monomorphism then M will be uniform but converse is not true. For exp, \( Z(p^\infty) \) is uniform Z-module. If we define \( f \in \text{End}_z(Z(p^\infty)) \) by \( f([x]) = [px] \). Then f is nonzero but not monomorphism so, in \( Z(p^\infty) \) every nonzero partial endomorphism is not monomorphism.

**Definition 1.4.** Given two R-module M and N, M is called N-injective if for every submodule N₁ of N, any homomorphism \( \alpha : N₁ \rightarrow M \) can be extended to a homomorphism \( \beta : N \rightarrow M \). M is called quasi injective if M is M – injective.

Consider the following condition for a module M.

\( (c₁) \) Every submodule of M is essential in a direct summand of M.

\( (c₂) \) If a submodule of M is isomorphic to a direct summand of M, then it is itself a direct summand of M.

\( (c₃) \) If A & B are direct summand of M with A \( \cap \) B = 0 then A + B is also a direct summand of M.

M is called CS module (extending module) if it satisfies \( (C₁) \). M is called continuous module if it satisfies \( C₁ \ & C₂ \) and quasi continuous if it satisfies \( C₁ \ & C₃ \).

**Definition 1.5.** A nonzero R-module M is called self similar if every non zero submodule of M is isomorphic to M.
Definition 1.6. An R-module M is said to be rational extension of $M_1$, in case for each submodule $B$, $M \supseteq B \supseteq M_1$, $f \in \text{Hom}(B, M)$ satisfies $f(M_1) = 0$ if and only if $f = 0$. The injective hull of M denoted by $E(M)$.

Proposition 1.1. [7, Prop.1.1] The following conditions are equivalent for a compressible module M:

(1) M is critically compressible;

(2) Every non-zero partial endomorphism on M is monomorphism.

An R-module M is called polyform if every essential submodule of M is dense in M and M is called if every submodule of M is essential. An R-module M is slightly compressible if $\text{Hom}_R(M, X) \neq 0$ for every non-zero submodule X of M. Clearly every compressible module is slightly compressible module

Proposition 1.2. If every non-zero endomorphism of a slightly compressible module M is monomorphism, then M is compressible.

Since every endomorphism of M is also a partial endomorphism of M, then Proposition 1.1 can be re-written as;

Proposition 1.3. Let M is slightly compressible R-module. The following conditions are equivalent:

(i) M is critically compressible;

(ii) Every non-zero partial endomorphism of M is monomorphism.

2. Compressible and Critically compressible modules

In this section, we are giving some useful results concerning critically compressible and compressible modules.

Proposition 2.1. Every compressible module over a commutative ring R is torsion free.

Proof: Since every compressible module over commutative ring is critically compressible, then every non zero partial endomorphism on M is monomorphism.
Take any nonzero partial endomorphism on $M$. $f_{\alpha} : M \to M$. Suppose $M$ is not torsion free. Take a torsion element $x \in M$. $\alpha \neq 0 \in R$ then $\alpha x = 0$ Now define $f_{\alpha}(x) = \alpha x$ is homomorphism then $f_{\alpha}(x) = 0 = \alpha x \Rightarrow x \in \text{Ker } f$, but due to critically Compressibility of $M$, $f_{\alpha}$ will be monomorphism, then $x = 0$ which is a contradiction. Therefore $M$ is torsion free.

**Proposition 2.2.** Let $M$ be compressible module. If $M$ is cyclic and torsion free then $M$ will be critically compressible.

**Proof:** Let $N$ be any nonzero submodule of $M$ and $f$ be any non zero partial endomorphism from $N$ to $M$. Then $f(n) \neq 0$ for some $o \neq n \in N$. Since otherwise $f(N) = f(Rn) = 0$, a contradiction. Now let $\alpha n \in \text{Ker } f \Rightarrow f(\alpha n) = 0 \Rightarrow \alpha f(n) = 0$ since $M$ is torsion free and $f(n) \neq 0 \Rightarrow \text{Ker } f = 0$ Hence every nonzero partial endomorphism on $M$ is monomorphism. Then $M$ is critically compressible.

**Proposition 2.3.** If $R$ be a torsion free ring with unity then $R^R$ is compressible module if and only if $R^R$ it is critically compressible module.

**Proof:** Let $R^R$ is compressible module. Now $R = R.1$, then $R$ is cyclic. Since $R$ is also Torsion free. Then from Prop. 2.2, $R^R$ is critically compressible, converse is obvious.

**Proposition 2.4.** Every critically compressible module $M$ is indecomposabale.

**Proof:** Suppose $M = M_1 \oplus M_2$ with $0 \neq M_1, M_2 \neq M$. Let $p_i$ be the projection map from $M$ to $M_i$ for $i = 1, 2$. Due to compressibility of $M$, $p_i$ is monomorphism then $\text{Ker } p_i = m_j = 0$ for $i \neq j$ which is a contradiction Hence $M$ is indecomposable.

**Proposition 2.5.** Let $M$ be a compressible module. Then every indecomposable injective module $M$ whose nonzero partial endomorphism have uniform non singular kernels then $M$ will be critically compressible.

**Proof:** Let $N$ be a nonzero submodule of $M$. Now Let $f : N \to M$ be any non zero partial endomorphism on $M$. suppose $\text{Ker } f \neq 0$ since $\text{Ker } f$ is uniform, $N = \text{Ker } f$ [by 2, Prop. 2.2] then $N$ is non singular. Therefore $N$ is rational extension of $\text{Ker } f$ [by 3,
Prop 5, p. 59]. Hence \( f(\text{Ker}f) = 0 \implies f = 0 \) which is a contradiction. Hence \( \text{Ker} f = 0 \) then every non zero partial endomorphism on \( M \) is monomorphism. Then \( M \) is critically compressible.

**Proposition 2.6.** Every critically compressible module is CS (extending module) and it is continuous if and only if every partial monomorphism is isomorphism.

**Proof:** Let \( M \) be a critically compressible module. Then every non zero partial endomorphism on \( M \) is monomorphism. So \( M \) is uniform. Hence every critically compressible module is CS module (extending module).

Let \( f : M \rightarrow M \) be any non zero partial endomorphism on \( M \). Due to critically compressibility of \( M \), \( f \) will be monomorphism. Suppose \( f \) is not surjective then \( f(M) \subseteq M \) and \( f(M) \) will be proper essential submodule of \( M \). Now define \( g : M \rightarrow f(M) \) by \( g(x) = f(x) \), then \( g \) is isomorphism. Since \( M \) is continuous, \( f(M) \) being on isomorphic copy of \( M \) cannot be a proper essential submodule of \( M \), a contradiction. Hence \( f \) is surjective. Converse is obvious.

We note that, every non-zero submodule of compressible or critically compressible module is also compressible or critically compressible module respectively.

It is clear that if \( E(M) \) is compressible or critically compressible then \( M \) is also compressible or critically compressible module respectively. Now we are giving sufficient condition for an injective hull of compressible or critically compressible module to be compressible or critically compressible.

**Proposition 2.7.** Let \( M \) is a critically compressible then \( E(M) \) will be critically compressible if

1. \( M \) is stable under partial endomorphism of \( E(M) \)
2. \( E(M) \) is rational extension of \( M \).

**Proof:** Let \( M \) is critically compressible. Now suppose \( M^1 \) be any non zero submodule of \( E(M) \) and \( h : M^1 \rightarrow E(M) \) be any zero partial endomorphism on \( E(M) \). Suppose
Ker h \neq 0. Then Ker h will be submodule of E (M) \Rightarrow M \cap Ker h \neq 0. Now f = h/N is any non zero partial endomorphism on M. Since M is stable under endomorphism of E (M). Also f \neq 0 since E (M) is rational extension of M. Thus Ker f = Ker h \cap M \neq 0 which is a contradiction. Hence E (M) is critically compressible.

Now if M is non-singular then rational extension of M and injective hull of M coincides and this leads immediately to the following.

**Corollary2.8.** Let M and E (M) be compressible module and M be nonsingular module stable under partial endomorphism of E (M). Then E(M) is critically compressible module if and only if M is compressible module.

A ring with unit 1 is said to be an S.P. ring if Z (M) is a direct summand of M for every R-module M, where Z (M) = \{x \in M \mid R \Delta 0 (x)\}, 0 (x) denotes the annihilator ideal of x. In [7] author claimed that if M is compressible module then M is singular or nonsingular. Now we are giving here the result that for a S.P. ring every critically compressible module is either singular or nonsingular.

**Proposition2.9.** Let R be a S.P. ring. Then any critically compressible module over R is either singular or non singular.

**Proof.** Let M is neither singular nor non singular, then Z (M) is a non zero direct summand and proper submodule of M. Now by prop. 2.4 M is indecomposable Hence Z (M) = M, a contradiction. Hence M is either singular or non singular.

**Proposition2.10.** A ring R with unit 1 \neq 0 is self similar and in which every non zero endomorphism is monomorphism if and only if it is a principal ideal ring without zero divisor.

**Proof.** Assume R is self similar and in which every non zero endomorphism is monomorphism. Then R has no zero divisor. To prove R is a principal ideal ring, Let S be any non zero ideal of R. Take \alpha \in R, r \neq 0 then there exist a isomorphism f : R \rightarrow
Rr and g : Rr \to S. Let \ f(1) = xr and g(xr) = b, Now if y \in S then \ y = g \ of \ (a) \ for \ some \ a \in R
\ y = g \circ f(a.1)
\ = g (a f(1))
\ = a g(xr) = ab \in R
Hence, S = Rb is a principal ideal.

The converse is obvious.

**Corollary 2.11.** If R is a principal ideal ring without zero divisor then \( R^R \) is critically compressible.

**Proof.** If R is Principal ideal ring without zero divisor then R is self similar in which every non zero endomorphism is monomorphism (By prop. 2.10). Then from [1, thereon 4.1], \( R^R \) is critically compressible.

Hence the following implications hold:

\[ R \text{ is self similar in which every non zero endomorphism is monomorphism} \Rightarrow R \text{ is a principal ideal ring without zero divisor} \Rightarrow R^R \text{ is critically compressible.} \]

**Proposition 2.12.** Let M be compressible module and every non zero \( f \in \text{Hom}_R (M, M') \) is monomorphism, where \( M' \) is quasi injective hull of M. Then M is critically compressible module.

**Proof.** Let \( g : N \to M \) be any non zero partial endomorphism on M. Such that \( f|_N = g \) by assumption \( f \) is monomorphism and so \( g \) is monomorphism. Hence every non zero partial endomorphism on M is monomorphism. Then M is critically compressible.

**Proposition 2.13.** Let R be a noetherian ring and R be a module over ring R. Then following conditions are equivalent:

(i) \( R^R \) is compressible module

(ii) \( R^R \) is critically compressible

(iii) Monomorphism\( _R (R, U) \) is non zero for every uniform submodule U of R.
(iv) Monomorphism$_R (R, U)$ is non zero for every cyclic uniform submodule $U$ of $R$.

**Proof.**

$(1) \Rightarrow (2)$ from [1, theorem 3.2]

$(2) \Rightarrow (3) \Rightarrow (4)$ is clear.

Now $(4) \Rightarrow (1)$

Let $N$ be any non zero submodule of $R$. Let $m$ be any non zero element of $N$.

By hypothesis $mR$ is noetherian and hence $mR$ contains a uniform submodule $U$ of $R$.

Let $u$ be any non zero element of $U$. Then $uR$ is cyclic uniform submodule of $R$. By

$(4)$ mono$_R (R, UR)$ is non zero. Hence mono$_R (R, N)$ is non zero. Hence $R^R$ is
compressible module.

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