The fourth moment of Dirichlet L-functions

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Abstract. Extending a result of Heath-Brown, we prove an asymptotic formula for the fourth moment of $L(1/2, \chi)$ where $\chi$ ranges over the primitive Dirichlet characters $(\text{mod } q)$.

1. Introduction

In [HB81], D.R. Heath-Brown showed that

$$\sum_{\chi \pmod{q}}^* |L(1/2, \chi)|^4 = \frac{\varphi^*(q)}{2\pi^2} \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})(\log q)^4} + O(2^{\omega(q)} q (\log q)^3).$$

Here $\sum^*$ denotes summation over primitive characters $\chi \pmod{q}$, $\varphi^*(q)$ denotes the number of primitive characters $(\text{mod } q)$, and $\omega(q)$ denotes the number of distinct prime factors of $q$. Note that $\varphi^*(q)$ is a multiplicative function given by $\varphi^*(p) = p-2$ for primes $p$, and $\varphi^*(p^k) = p^k (1 - 1/p)^2$ for $k \geq 2$ (see Lemma 1 below). Also note that when $q \equiv 2 \pmod{4}$ there are no primitive characters $(\text{mod } q)$, and so below we will assume that $q \not\equiv 2 \pmod{4}$. For $q \equiv 2 \pmod{4}$ it is useful to keep in mind that the main term in (1.1) is $\asymp q (\varphi(q)/q)^6 (\log q)^4$.

Heath-Brown’s result represents a $q$-analog of Ingham’s fourth moment for $\zeta(s)$:

$$\int_0^T |\zeta(1/2 + it)|^4 dt \sim \frac{T}{2\pi^2} (\log T)^4.$$

When $\omega(q) \leq (1/\log 2 - \epsilon) \log \log q$ (which holds for almost all $q$) the error term in (1.1) is dominated by the main term and (1.1) gives the $q$-analog of Ingham’s result. However if $q$ is even a little more than ‘ordinarily composite’, with $\omega(q) \geq (\log \log q)/\log 2$, then the error term in (1.1) dominates the main term. In this note we remedy this, and obtain an asymptotic formula valid for all large $q$.

Theorem. For all large $q$ we have

$$\sum_{\chi \pmod{q}}^* |L(1/2, \chi)|^4 = \frac{\varphi^*(q)}{2\pi^2} \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})(\log q)^4} \left(1 + O\left(\frac{\omega(q)}{\log q (\sqrt{\varphi(q)})} \sqrt{\varphi(q)}\right)\right) + O(q (\log q)^7).$$

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Since $\omega(q) \ll \log q / \log \log q$, and $q/\varphi(q) \ll \log \log q$, we see that

$$(\omega(q)/\log q) \sqrt{q/\varphi(q)} \ll 1/\sqrt{\log \log q}.$$ 

Thus our Theorem gives a genuine asymptotic formula for all large $q$.

For any character $\chi \pmod q$ (not necessarily primitive) let $a = 0$ or $1$ be given by $\chi(-1) = (-1)^a$. For $x > 0$ we define

$$(1.2) \quad W_a(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( \frac{\Gamma(s + \frac{1}{2})}{\Gamma(\frac{3}{2})} \right)^2 x^{-s} ds,$$

for any positive $c$. By moving the line of integration to $c = -\frac{1}{2} + \epsilon$ we may see that

$$(1.3a) \quad W(x) = 1 + O(x^{\frac{1}{2} - \epsilon}),$$

and from the definition (1.2) we also get that

$$(1.3b) \quad W(x) = O_e(x^{-c}).$$

We define

$$(1.4) \quad A(\chi) := \sum_{a,b=1}^{\infty} \frac{\chi(a) \chi(b)}{\sqrt{ab}} W_a(\frac{\pi ab}{q}),$$

If $\chi$ is primitive then $|L(\frac{1}{2}, \chi)|^2 = 2A(\chi)$ (see Lemma 2 below). Let $Z = q/2^{\omega(q)}$ and decompose $A(\chi)$ as $B(\chi) + C(\chi)$ where

$$B(\chi) = \sum_{a,b \geq 1 \atop ab \leq Z} \frac{\chi(a) \chi(b)}{\sqrt{ab}} W_a(\frac{\pi ab}{q}),$$

and

$$C(\chi) = \sum_{a,b > Z \atop ab \leq Z} \frac{\chi(a) \chi(b)}{\sqrt{ab}} W_a(\frac{\pi ab}{q}).$$

Our main theorem will follow from the following two Propositions.

**PROPOSITION 1.** We have

$$\sum_{\chi \pmod q} |B(\chi)|^2 = \frac{\varphi^*(q)}{8\pi^2} \prod_{p|q} \frac{(1 - 1/p)^3}{(1 + 1/p)^3} (\log q)^4 \left(1 + O\left(\frac{\omega(q)}{\log q}\right)\right).$$

**PROPOSITION 2.** We have

$$\sum_{\chi \pmod q} |C(\chi)|^2 \ll q \left(\frac{\varphi(q)}{q}\right)^5 (\omega(q) \log q)^2 + q(\log q)^3.$$

**PROOF OF THE THEOREM.** Since $|L(\frac{1}{2}, \chi)|^2 = 2A(\chi) = 2(B(\chi) + C(\chi))$ for primitive characters $\chi$ we have

$$\sum_{\chi \pmod q} |L(\frac{1}{2}, \chi)|^4 = 4 \sum_{\chi \pmod q} \left(|B(\chi)|^2 + 2B(\chi)C(\chi) + |C(\chi)|^2\right).$$

The first and third terms on the right hand side are handled directly by Propositions 1 and 2. By Cauchy’s inequality

$$\sum_{\chi \pmod q} |B(\chi)C(\chi)| \leq \left( \sum_{\chi \pmod q} |B(\chi)|^2 \right)^{\frac{1}{2}} \left( \sum_{\chi \pmod q} |C(\chi)|^2 \right)^{\frac{1}{2}},$$
and thus Propositions 1 and 2 furnish an estimate for the second term also. Combining these results gives the Theorem.

In [HB79], Heath-Brown refined Ingham’s fourth moment for \( \zeta(s) \), and obtained an asymptotic formula with a remainder term \( O(T^{1/2 + \epsilon}) \). It remains a challenging open problem to obtain an asymptotic formula for \( \sum_{\chi} |L(\frac{1}{2}, \chi)|^4 \) where the error term is \( O(q^{1-\delta}) \) for some positive \( \delta \).

This note arose from a conversation with Roger Heath-Brown at the Gauss-Dirichlet conference where he reminded me of this problem. It is a pleasure to thank him for this and other stimulating discussions.

2. Lemmas

**Lemma 1.** If \( (r, q) = 1 \) then

\[
\sum_{\chi \pmod{q}} \chi(r) = \sum_{\varphi(k) \mu(q/k)} \varphi(k) \mu(q/k).
\]

**Proof.** If we write \( h_r(k) = \sum_{\chi \pmod{q}} \chi(r) \) then for \( (r, q) = 1 \) we have

\[
\sum_{k|q} h_r(k) = \sum_{\chi \pmod{q}} \chi(r) = \begin{cases} \varphi(q) & \text{if } q \mid r-1 \\ 0 & \text{otherwise.} \end{cases}
\]

The Lemma now follows by Möbius inversion.

Note that taking \( r = 1 \) gives the formula for \( \varphi^*(q) \) given in the introduction. If we restrict attention to characters of a given sign \( a \) then we have, for \( (mn, q) = 1 \),

\[
\sum_{\chi \pmod{q} \chi(-1) = (-1)^a} \chi(m)\chi(n) = \frac{1}{2} \sum_{k|q, m-n} \varphi(k) \mu(q/k) + \frac{(-1)^a}{2} \sum_{k|q, m+n} \varphi(k) \mu(q/k).
\]

**Lemma 2.** If \( \chi \) is a primitive character \( \pmod{q} \) with \( \chi(-1) = (-1)^a \) then

\[
|L(\frac{1}{2}, \chi)|^2 = 2A(\chi),
\]

where \( A(\chi) \) is defined in (1.4).

**Proof.** We recall the functional equation (see Chapter 9 of [Dav00])

\[
\Lambda(\frac{1}{2} + s, \chi) = \left( \frac{q}{\pi} \right)^s 2^{1/2} \Gamma\left( \frac{s + \frac{1}{2} + a}{2} \right) L(\frac{1}{2} + s, \chi) = \frac{\tau(\chi)}{s^a \sqrt{q}} \Lambda(\frac{1}{2} - s, \chi),
\]

which yields

\[
\Lambda(\frac{1}{2} + s, \chi)\Lambda(\frac{1}{2} + s, \chi) = \Lambda(\frac{1}{2} - s, \chi)\Lambda(\frac{1}{2} - s, \chi).
\]

For \( c > \frac{1}{2} \) we consider

\[
I := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Lambda(\frac{1}{2} + s, \chi)\Lambda(\frac{1}{2} + s, \chi) ds.
\]

We move the line of integration to \( \Re(s) = -c \), and use the functional equation (2.2). This readily gives that \( I = |L(\frac{1}{2}, \chi)|^2 - I \), so that \( |L(\frac{1}{2}, \chi)|^2 = 2I \). On the other hand, expanding \( L(\frac{1}{2} + s, \chi)L(\frac{1}{2} + s, \chi) \) into its Dirichlet series and integrating termwise, we get that \( I = A(\chi) \). This proves the Lemma.
We shall require the following bounds for divisor sums. If \( k \) and \( \ell \) are positive integers with \( \ell k \ll x^{2/3} \) then
\[
\sum_{\substack{n \leq x \\ (n, k) = 1}} d(n)d(\ell k \pm n) \ll x \log x \sum_{d \mid \ell} d^{-1},
\]
provided that \( x \leq \ell k \) if the negative sign holds. This is given in (17) of Heath-Brown [HB81]. Secondly, we record a result of P. Shiu [Shi80] which gives that
\[
\sum_{\substack{n \leq x \\ n \equiv r \pmod{k}}} d(n) \ll \frac{\varphi(k)}{k^2} x \log x,
\]
where \( (r, k) = 1 \) and \( x \geq k^{1+\delta} \) for some fixed \( \delta > 0 \).

**Lemma 3.** Let \( k \) be a positive integer, and let \( Z_1 \) and \( Z_2 \) be real numbers \( \geq 2 \). If \( Z_1 Z_2 > k \frac{209}{20} \) then
\[
\sum_{\substack{Z_1 \leq ab < 2Z_1 \\ Z_2 \leq cd < 2Z_2 \\ (abcd, k) = 1 \\ ac \equiv \pm bd \pmod{k} \not\equiv \pm ab \pmod{k}}} 1 \ll \frac{Z_1 Z_2}{k} (\log(Z_1 Z_2))^3.
\]
If \( Z_1 Z_2 \leq k \frac{209}{20} \) the quantity estimated above is \( \ll (Z_1 Z_2)^{1+\epsilon}/k \).

**Proof.** By symmetry we may just focus on the terms with \( ac > bd \). Write \( n = bd \) and \( ac = k\ell \pm bd \). Note that \( k\ell \leq 2ac \) and so \( 1 \leq \ell \leq 8Z_1 Z_2/k \). Moreover since \( ac \geq k\ell/2 \) we have that \( bd \leq 4Z_1 Z_2/(ac) \leq 8Z_1 Z_2/(k\ell) \). Thus the sum we desire to estimate is
\[
\ll \sum_{1 \leq \ell \leq 8Z_1 Z_2/k} \sum_{\substack{n \leq 8Z_1 Z_2/(k\ell) \\ n \equiv \ell k \pm n \pmod{k} \\ (n, k) = 1}} d(n)d(\ell k \pm n).
\]
Since \( d(n)d(\ell k \pm n) \ll (Z_1 Z_2)^2 \) the second assertion of the Lemma follows.

Now suppose that \( Z_1 Z_2 > k \frac{209}{20} \). We distinguish the cases \( k\ell \leq (Z_1 Z_2) \frac{209}{20} \) and \( k\ell > (Z_1 Z_2) \frac{209}{20} \). In the first case we estimate the sum over \( n \) using (2.3). Thus such terms contribute to (2.5)
\[
\ll \sum_{k\ell \leq (Z_1 Z_2) \frac{209}{20}} \frac{Z_1 Z_2}{k\ell} (\log Z_1 Z_2)^2 \sum_{d \mid \ell} d^{-1} \ll \frac{Z_1 Z_2}{k} (\log Z_1 Z_2)^3.
\]
Now consider the second case. Here we sum over \( \ell \) first. Writing \( m = k\ell \pm n (= ac) \) we see that such terms contribute
\[
\ll \sum_{n \leq 8Z_1 Z_2/k} d(n) \sum_{\substack{(Z_1 Z_2)^{1/2} \leq m \leq 4Z_1 Z_2/n \\ m \equiv \pm n \pmod{k} \not\equiv \pm ab \pmod{k}}} d(m),
\]
and by (2.4) (which applies as \( (Z_1 Z_2)^{1/2} > k \frac{209}{20} \)) this is
\[
\ll \sum_{n \leq 8Z_1 Z_2/k} d(n) \frac{Z_1 Z_2}{kn} \log Z_1 Z_2 \ll \frac{Z_1 Z_2}{k} (\log Z_1 Z_2)^3.
\]
The proof is complete. \( \square \)
The next two Lemmas are standard; we have provided brief proofs for completeness.

**Lemma 4.** Let \( q \) be a positive integer and \( x \geq 2 \) be a real number. Then
\[
\sum_{\substack{n \leq x \\ (n, q) = 1}} \frac{1}{n} = \frac{\varphi(q)}{q} \left( \log x + \gamma + \sum_{p|q} \frac{\log p}{p-1} \right) + O\left( \frac{2^{\omega(q)} \log x}{x} \right).
\]

Further \( \sum_{p|q} \log p/(p-1) \ll 1 + \log \omega(q) \).

**Proof.** We have
\[
\sum_{\substack{n \leq x \\ (n, q) = 1}} \frac{1}{n} = \sum_{d|q} \mu(d) \sum_{\substack{n \leq x \\ d|n \cap d \leq x}} \frac{1}{n} = \sum_{d|q} \mu(d) \left( \frac{\log x}{d} + \gamma + O\left( \frac{d}{x} \right) \right)
= \sum_{d|q} \mu(d) \left( \frac{\log x}{d} + \gamma \right) + O\left( \frac{2^{\omega(q)} \log x}{x} \right).
\]

Since \(-\sum_{d|q} \mu(d)/d \log d = \varphi(q)/q \sum_{p|q} (\log p)/(p-1)\) the first statement of the Lemma follows. Since \( \sum_{p|q} \log p/(p-1) \) is largest when the primes dividing \( q \) are the first \( \omega(q) \) primes, the second assertion of the Lemma holds. \( \square \)

**Lemma 5.** We have
\[
\sum_{\substack{n \leq q \\ (n, q) = 1}} 2^{\omega(n)} n \ll \left( \frac{\varphi(q)}{q} \right)^2 (\log q)^2.
\]

For \( x \geq \sqrt{q} \) we have
\[
\sum_{\substack{n \leq x \\ (n, q) = 1}} 2^{\omega(n)} \left( \frac{\log x}{n} \right)^2 = \frac{(\log x)^4}{12 \zeta(2)} \prod_{p|q} \left( \frac{1}{1+1/p} \right) \left( 1 + O\left( \frac{1 + \log \omega(q)}{\log q} \right) \right).
\]

**Proof.** Consider for \( \Re(s) > 1 \)
\[
F(s) = \sum_{n = 1}^{\infty} \frac{2^{\omega(n)}}{n} \frac{\zeta(s)}{\zeta(2s)} \prod_{p|q} \frac{1-p^{-s}}{1+1/p}.
\]

Since
\[
\sum_{\substack{n \leq q \\ (n, q) = 1}} 2^{\omega(n)} n \leq e \sum_{\substack{n = 1 \\ (n, q) = 1}}^{\infty} \frac{2^{\omega(n)}}{n^{1+1/\log q}} = eF(1+1/\log q),
\]

the first statement of the Lemma follows. To prove the second statement we note that, for \( c > 0 \),
\[
\sum_{\substack{n \leq x \\ (n, q) = 1}} 2^{\omega(n)} \left( \frac{\log x}{n} \right)^2 = \frac{2}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(1+s) x^s \frac{ds}{s^3}.
\]

We move the line of integration to \( c = -\frac{1}{2} + \epsilon \) and obtain that the above is
\[
2 \sum_{k=0}^{\infty} F(1+s) x^{s} + O(x^{-1/2+\epsilon q'}).
\]

A simple residue calculation then gives the Lemma. \( \square \)
3. Proof of Proposition 1

Applying (2.1) we easily obtain that

\[ \sum_{\chi \mod q} |B(\chi)|^2 = M + E, \]

where

\[ (3.1) \]

\[ M := \frac{\varphi^*(q)}{2} \sum_{\substack{a, b, c, d \geq 1 \\ ab \leq Z, cd \leq Z \\ ac \neq bd \atop (abcd, q) = 1}} \frac{1}{abcd} \left( W_0 \left( \frac{\pi ab}{q} \right) W_0 \left( \frac{\pi cd}{q} \right) + W_1 \left( \frac{\pi ab}{q} \right) W_1 \left( \frac{\pi cd}{q} \right) \right) \]

and

\[ E = \sum_{k \mid q} \varphi(k) \mu^2(q/k) E(k), \]

with

\[ E(k) \ll \sum_{\substack{(abcd, q) = 1 \\ k \mid (ac \pm bd) \\ ac \neq bd \atop ab, cd \leq Z}} \frac{1}{\sqrt{abcd}} \]

To estimate \( E(k) \) we divide the terms \( ab, cd \leq Z \) into dyadic blocks. Consider the block \( Z_1 \leq ab < 2Z_1 \) and \( Z_2 \leq cd < 2Z_2 \). By Lemma 3 the contribution of this block to \( E(k) \) is, if \( Z_1 Z_2 > k \frac{19}{10} \),

\[ \ll \frac{1}{\sqrt{Z_1 Z_2}} \left( \log Z_1 Z_2 \right)^3 \ll \frac{\sqrt{Z_1 Z_2}}{k} (\log q)^3, \]

and is \( \ll (Z_1 Z_2)^{\frac{3}{2} + \epsilon} / k \) if \( Z_1 Z_2 \leq k \frac{19}{10} \). Summing over all such dyadic blocks we obtain that \( E(k) \ll (Z/k)(\log q)^3 + k^{-\frac{1}{2} + \epsilon} \), and so

\[ E \ll Z^{2-(q/4)} (\log q)^3 \ll q (\log q)^3. \]

We now turn to the main term (3.1). If \( ac = bd \) then we may write \( a = gr \), \( b = gs \), \( c = hs \), \( d = hr \), where \( r \) and \( s \) are coprime. We put \( n = rs \), and note that given \( n \) there are \( 2^{\omega(n)} \) ways of writing it as \( rs \) with \( r \) and \( s \) coprime. Note also that \( ab = g^2 rs = g^2 n \), and \( cd = h^2 rs = h^2 n \). Thus the main term (3.1) may be written as

\[ M = \frac{\varphi^*(q)}{2} \sum_{n=0,1} \sum_{\substack{n \leq Z \\ (n, q) = 1}} \frac{2^{\omega(n)}}{n} \left( \sum_{\substack{g \leq \sqrt{Z/n} \\ (g, q) = 1}} \frac{1}{g} W_0 \left( \frac{\pi g^2 n}{q} \right) \right)^2. \]

By (1.3a) we have that \( W_0(\pi g^2 n/Z) = 1 + O(\sqrt{g n^{1/4} / q^{1/4}}) \), and using this above we see that

\[ M = \varphi^*(q) \sum_{n \leq Z} \frac{2^{\omega(n)}}{n} \left( \sum_{\substack{g \leq \sqrt{Z/n} \\ (g, q) = 1}} \frac{1}{g} + O(2^{-\omega(q)/4}) \right)^2. \]

We split the terms \( n \leq Z \) into the cases \( n \leq Z_0 \) and \( Z_0 < n \leq Z \), where we set \( Z_0 = Z/q^{\omega(q)} = q/18^{\omega(q)}. \) In the first case, Lemma 4 gives that the sum over \( g \) is
Since these terms contribute to \( M \), the orthogonality relation for characters gives that

\[
\varphi^*(q) \sum_{n \leq Z_0} \frac{2^{\omega(n)}}{n} \left( \frac{\varphi(q)}{2q} \log \frac{Z}{n} + O(1 + \log \omega(q)) \right)^2
\]

\[
= \varphi^*(q) \left( \frac{\varphi(q)}{2q} \right)^2 \sum_{n \leq Z_0} \frac{2^{\omega(n)}}{n} \left( \left( \log \frac{Z_0}{n} \right)^2 + O(\omega(q) \log q) \right).
\]

Using Lemma 5 we conclude that the terms \( n \leq Z_0 \) contribute to \( M \) an amount

\[
(3.2) \quad \frac{\varphi^*(q)}{8\pi^2} \prod_{p|q} \left( \frac{1}{1 + \log q} \right)^3 \left( \log q \right)^4 \left( 1 + O\left( \frac{\omega(q)}{\log q} \right) \right).
\]

In the second case when \( Z_0 \leq n \leq Z \), we extend the sum over \( g \) to all \( g \leq 3\omega(q) \) that are coprime to \( q \), and so by Lemma 4 the sum over \( g \) is \( \ll \omega(q) \varphi(q)/q \). Thus these terms contribute to \( M \) an amount

\[
\ll \varphi^*(q) \left( \frac{\varphi(q)}{q} \right)^2 \sum_{Z_0 < n \leq Z} \frac{2^{\omega(n)}}{n} \ll \varphi^*(q) \left( \frac{\varphi(q)}{q} \right)^2 (\omega(q))^3 \log q.
\]

Since \( q \omega(q)/\varphi(q) \ll \log q \), combining this with (3.2) we conclude that

\[
M = \frac{\varphi^*(q)}{8\pi^2} \prod_{p|q} \left( \frac{1}{1 + \log q} \right)^3 \left( \log q \right)^4 \left( 1 + O\left( \frac{\omega(q)}{\log q} \right) \right).
\]

Together with our bound for \( E \), this proves Proposition 1.

4. Proof of Proposition 2

The orthogonality relation for characters gives that

\[
\sum_{\chi \pmod{q}} |C(\chi)|^2 \ll \varphi(q) \sum_{\chi \pmod{q}} \frac{1}{\sqrt{abcd}} \sum_{a=0,1} \left| W_a \left( \frac{\pi ab}{q} \right) W_a \left( \frac{\pi cd}{q} \right) \right|
\]

\[
\ll \varphi(q) \sum_{\chi \pmod{q}} \frac{1}{\sqrt{abcd}} \left( 1 + \frac{ab}{q} \right)^{-2} \left( 1 + \frac{cd}{q} \right)^{-2},
\]

using (1.3a,b). We write the last expression above as \( R_1 + R_2 \), where \( R_1 \) contains the terms with \( ac = bd \), and \( R_2 \) contains the rest.

We first get an estimate for \( R_2 \). We break up the terms into dyadic blocks; a typical one counts \( Z_1 \leq ab < 2Z_1 \) and \( Z_2 \leq cd < 2Z_2 \) (both \( Z_1 \) and \( Z_2 \) being larger than \( Z \)). The contribution of such a dyadic block is, using Lemma 3 (note that \( Z_1 Z_2 > Z^2 > q^{10} \))

\[
\ll \frac{\varphi(q)}{\sqrt{Z_1 Z_2}} \left( 1 + \frac{Z_1}{q} \right)^{-2} \left( 1 + \frac{Z_2}{q} \right)^{-2} \frac{Z_1 Z_2}{q} (\log Z_1 Z_2)^3.
\]

Summing this estimate over all the dyadic blocks we obtain that

\[
R_2 \ll q (\log q)^3.
\]
We now turn to the terms $ac = bd$ counted in $R_1$. As in our treatment of $M$, we write $a = gr$, $b = gs$, $c = hs$, $d = hr$, with $(r, s) = 1$, and group terms according to $n = rs$. We see easily that

$$R_1 \ll \varphi(q) \sum_{(n, q) = 1} \frac{2^{\omega(n)}}{n} \left( \sum_{g > \sqrt{Z/n}} \frac{1}{g} \left( 1 + \frac{2n}{q} \right)^{-2} \right)^2.$$  

First consider the terms $n > q$ in (4.1). Here the sum over $g$ gives an amount $\ll q^2/n^2$ and so the contribution of these terms to (4.1) is

$$\ll \varphi(q) \sum_{n > q} \frac{2^{\omega(n)}}{n} \ll \varphi(q) \log q.$$  

For the terms $n < q$ the sum over $g$ in (4.1) is easily seen to be

$$\ll 1 + \sum_{\sqrt{Z/n} \leq g \leq \sqrt{q/n}} \frac{1}{g} \ll 1 + \frac{\varphi(q)}{q} \omega(q).$$  

The last estimate follows from Lemma 4 when $n < Z/9 \omega(q)$, while if $n > Z/9 \omega(q)$ we extend the sum over $g$ to all $g \leq 6 \omega(q)$ with $(g, q) = 1$ and then use Lemma 4. Thus the contribution of terms $n < q$ to (4.1) is, using Lemma 5,

$$\ll \varphi(q) \left( 1 + \frac{\varphi(q)}{q} \omega(q) \right)^2 \sum_{(n, q) = 1} \frac{2^{\omega(n)}}{n} \ll q \log^2 q \left( \frac{\varphi(q)}{q} \right)^5 \omega(q)^2.$$  

Combining these bounds with our estimate for $R_2$ we obtain Proposition 2.

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