MINIMAL HILBERT-KUNZ MULTIPLICITY

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ABSTRACT. In this paper, we ask the following question: what is the minimal value of the
difference \( e_{HK}(I) - e_{HK}(I') \) for ideals \( I' \supseteq I \) with \( l_A(I'/I) = 1 \)?

In order to answer to this question, we define the notion of minimal Hilbert-Kunz
multiplicity for strongly F-regular rings. Moreover, we calculate this invariant for quotient
singularities and for the coordinate ring of the Segre embedding:
\( P^{r-1} \times P^{s-1} \hookrightarrow P^{rs-1} \),
respectively.

Introduction

Throughout this paper, let \( A \) be a Noetherian ring of positive prime characteristic \( p \). The purpose of this paper is to introduce the notion of minimal Hilbert-Kunz
multiplicity which is a new invariant of local rings in positive characteristic.

The notion of Hilbert-Kunz multiplicity has been introduced by Kunz [Ku1] in 1969,
and has been studied in detail by Monsky [Mo]; see also e.g. [BC], [BCP], [Co], [HM],
[Se1], or [WY1,WY2,WY3].

Further, Hochster and Huneke [Hu2] have pointed out that the tight closure \( I^* \) of
\( I \) is the largest ideal containing \( I \) having the same Hilbert-Kunz multiplicity as \( I \); see
Lemma 1.3. Thus it seems to be important to understand the Hilbert-Kunz multiplicity
well. For example, the authors [WY1] have proved that an unmixed local ring whose
Hilbert-Kunz multiplicity one is regular. Also, they [WY3] have given a formula of
\( e_{HK}(I) \) for any integrally closed ideal \( I \) in a two-dimensional F-rational double point
using McKay correspondence, Riemann–Roch formula.
One of the most important conjectures about Hilbert-Kunz multiplicities is that it is always a rational number. Let \( A \) be a local ring and \( I, J \) be \( \mathfrak{m} \)-primary ideals in \( A \). Also, suppose that \( J \) is a parameter ideal. Then it is known that \( e_{HK}(J) = e(J) \), the usual multiplicity (and hence \( e_{HK}(J) \) is an integer). In order to investigate the value of \( e_{HK}(I) \), we study the difference “\( e_{HK}(J) - e_{HK}(I) \)”. So it is natural to ask the following question.

**Question.** What is the minimal value of the difference \( e_{HK}(I) - e_{HK}(I') \) for ideals \( I' \supseteq I \) with \( l_A(I'/I) = 1 \).

To give an answer to this question, we introduce the notion of relative minimal Hilbert-Kunz multiplicity \( \text{rel.} m_{HK}(A) \) and that of minimal Hilbert-Kunz multiplicity \( m_{HK}(A) \) as follows:

\[
\text{rel.} m_{HK}(A) := \inf \{ e_{HK}(I) - e_{HK}(I') : I \subseteq I' \text{ with } l_A(I'/I) = 1 \}
\]

\[
m_{HK}(A) := \lim \inf_{e \to \infty} \frac{l_A(A/\text{ann}_A F_e(z))}{p^{ed}},
\]

where \( z \) is a generator of the socle of the injective hull \( E_A(A/\mathfrak{m}) \).

Note that \( \text{rel.} m_{HK}(A) \geq m_{HK}(A) \); see Proposition 1.10.

In Section 2, we prove that if \( A \) is a Gorenstein local ring then

\[
e_{HK}(J) - e_{HK}(J : \mathfrak{m}) = \text{rel.} m_{HK}(A) = m_{HK}(A)
\]

for any parameter ideal \( J \) of \( A \); see Theorem 2.1 for details. Note that a similar result is independently proved by Huneke and Leuschke [HuL].

In general, if \( A \) is not weakly F-regular (see Definition 1.1), then \( \text{rel.} m_{HK}(A) = 0 \) (and hence \( m_{HK}(A) = 0 \)). Thus it suffices to consider weakly F-regular local rings in our context.

In Section 3, we will give a formula for minimal Hilbert-Kunz multiplicities of the canonical cover of \( \mathbb{Q} \)-Gorenstein F-regular local rings as follows:

**Theorem 3.1.** Let \( A \) be a \( \mathbb{Q} \)-Gorenstein strongly F-regular local ring. Also, let \( B = A \oplus K_A t \oplus K_A^{(2)} t^2 \oplus \cdots \oplus K_A^{(r-1)} t^{r-1} \), the canonical cover of \( A \), where \( r = \text{ord}(cl(K_A)) \), \( K_A^{(r)} = fA \) and \( ft^r = 1 \). Also, suppose that \( (r, p) = 1 \). Then we have

\[
m_{HK}(B) = r \cdot m_{HK}(A).
\]

In Section 4, as an application of Theorem 3.1, we will give a formula for the minimal Hilbert-Kunz multiplicities of quotient singularities.

**Theorem 4.2.** Let \( k \) be a field of characteristic \( p > 0 \), and let \( A = k[x_1, \ldots, x_d]^G \) be the invariant subring by a finite subgroup \( G \) of \( GL(d, k) \) with \( (p, |G|) = 1 \). Also, assume that \( G \) contains no pseudo-reflections. Then \( m_{HK}(A) = 1/|G| \).

In Section 5, we will give a formula for the minimal Hilbert-Kunz multiplicities of Segre products.
Theorem 5.6. Let $A = k[x_1, \ldots, x_r] \# k[y_1, \ldots, y_s]$, where $2 \leq r \leq s$, and put $d = r + s - 1$. Then

$$m_{HK}(A) = \frac{r!}{d!} S(d, r) + \frac{1}{d!} \sum_{k=1}^{r-1} \sum_{j=1}^{r-k} \binom{r}{k} \binom{r}{j} (-1)^{r+k} k^d,$$

where $S(n, k)$ denotes Stirling number of the second kind; see Section 5.

In particular,

$$e_{HK}(A) + m_{HK}(A) = \frac{r! \cdot S(d, r) + s! \cdot S(d, s)}{d!}.$$

1. Definition of the minimal Hilbert-Kunz multiplicity

In this section, we define the notion of minimal Hilbert-Kunz multiplicity and give its fundamental properties. In the following, let $A$ be a Noetherian excellent reduced local ring of positive characteristic $p > 0$ with perfect residue field $k = A/m$, unless specified.

1A. Peskine-Szpiro functor.

First, let us recall the definition of Peskine–Szpiro functor. Let $^e A$ denote the ring $A$ viewed as an $A$-algebra via $F^e : A \to A (a \mapsto a^p)$. Then $F^e_A(-) = ^e A \otimes_A -$ is a covariant functor from $A$-modules to $^e A$-modules. Since $^e A$ is isomorphic to $A$ as rings (via $F^e$), we can regard $F^e_A$ as a covariant functor from $A$-modules to themselves. We call this functor $F^e_A$ the Peskine–Szpiro functor of $A$. The $A$-module structure on $F^e_A(M)$ is such that $a'(a \otimes m) = a'a \otimes m$. On the other hand, $a' \otimes am = a'a^q \otimes m$. See e.g. [PS], [Ro] or [Hu]. Suppose that an $A$-module $M$ has a finite presentation $A^m \xrightarrow{\phi} A^n \to M \to 0$ where the map $\phi$ is defined by a matrix $(a_{ij})$. Then $F^e_A(M)$ has a finite presentation $A^m \xrightarrow{\phi_q} A^n \to F^e_A(M) \to 0$ where the map $\phi_q$ is defined by the matrix $(a_{ij}^q)$. For example, $F^e_A(A/I) = A/I[^p e]$, where $I[^p e]$ is the ideal generated by $\{a^p : a \in I\}$.

1B. Tight closure, Hilbert-Kunz multiplicity.

Using the Peskine–Szpiro functor, we define the notion of tight closure.

Definition 1.1. ([HH1, HH2, Hu]) (i) Let $M$ be an $A$-module, and let $N$ be an $A$-submodule of $M$. Put $N^\omega_M = \ker(F^e_A(M) \to F^e_A(M/N))$. Also, we denote by $F^e(x)$ the image of $x$ the Frobenius map $M \to F^e_A(M) (x \mapsto 1 \otimes x)$. Then the tight closure $N^*_M$ of $N$ (in $M$) is the submodule generated by elements for which there exists an element $c \in A^0 := A \setminus \bigcup_{P \in \Min(A)} P$ such that for all sufficiently large $q = p^e$, $cF^e(x) \in N^q_M$. By definition, we put $I^*_A = I^*_A$. Also, we say that $N$ is tightly closed (in $M$) if $N^*_M = N$.

(ii) A local ring $A$ in which every ideal is tightly closed is called weakly $F$-regular. Also, the ring whose localization is always weakly $F$-regular is called $F$-regular.

(iv) A reduced ring $A$ is said to be strongly $F$-regular if for any element $c \in A^0$ there exists $q = p^e$ such that the $A$-linear map $A \to A^{1/q}$ defined by $a \mapsto c^{1/q}a$ is split injective.
A Noetherian ring $R$ is (resp. weakly, strongly) F-regular if and only if so is $R_m$ for every maximal ideal $m$.

**Remark.** Strongly F-regular rings are F-regular. In general, it is not known whether the converse is true or not. But it is known that F-finite $\mathbb{Q}$-Gorenstein weakly F-regular rings are always strongly F-regular; see [AM, Mc, Wi].

The notion of Hilbert-Kunz multiplicity plays the central role in this paper.

**Definition 1.2.** ([Ku2, Mo, Se1]) Let $I$ be an $m$-primary ideal in $A$ and $M$ a finite $A$-module. Then we define the Hilbert-Kunz multiplicity $e_{HK}(I, M)$ of $M$ with respect to $I$ as

$$e_{HK}(I, M) := \lim_{e \to \infty} \frac{l_A(M/I^{[p^e]}M)}{p^{de}}.$$ 

By definition, we put $e_{HK}(I) := e_{HK}(I, A)$ and $e_{HK}(A) := e_{HK}(m)$.

Also, the multiplicity of $I$ is defined as

$$e(I) = \lim_{n \to \infty} \frac{d! \cdot l_A(A/I^n)}{n^d}.$$ 

Suppose that $\widehat{A}$ (the $m$-adic completion of $A$) is reduced. Let $I \subseteq I'$ be $m$-primary ideals in $A$. Then it is known that $I'$ and $I$ have the same integral closure (i.e. $\overline{I'} = \overline{I}$) if and only if $e(I) = e(I')$. The similar result holds for tight closures and the Hilbert-Kunz multiplicities.

**Lemma 1.3.** (cf. [HH2, Theorem 8.17]) Let $I \subseteq I'$ be $m$-primary ideals in $A$.

1. If $I' \subseteq I^*$, then $e_{HK}(I) = e_{HK}(I')$.
2. Further assume that $\widehat{A}$ is excellent, equidimensional, and reduced. Then the converse of (1) is also true.

1C. Minimal Hilbert-Kunz multiplicity.

Our work is motivated by the following question.

**Question 1.4.** What is the minimal value of the difference $e_{HK}(I) - e_{HK}(I')$ for ideals $I' \supseteq I$ with $l_A(I'/I) = 1$?

In order to represent the “difference”, we define the following notion.

**Definition 1.5 (Relative Hilbert-Kunz multiplicity).** Let $L$ be an $A$-module, and let $N \subseteq M$ be finite $A$-submodules of $L$ with $\text{Supp}_A(M/N) \subseteq \{m\}$. Then we set

$$e_{HK}(N, M; L) := \liminf_{e \to \infty} \frac{l_A(M^{[p^e]}L/N^{[p^e]}L)}{p^{de}}.$$ 

We call $e_{HK}(N, M; L)$ the relative Hilbert-Kunz multiplicity with respect to $N \subseteq M$ of $L$. In particular, $e_{HK}(I, I'; A) = e_{HK}(I) - e_{HK}(I')$ for $m$-primary ideals $I \subseteq I'$ in $A$. 

We further define the notion of the relative minimal Hilbert-Kunz multiplicity of $A$ as follows:

\[(1.5.2) \quad \text{rel. } m_{HK}(A) := \inf\{e_{HK}(I, I'; A) : I \subseteq I' \text{ such that } l_A(I'/I) = 1\}.\]

The following properties of the “relative minimal Hilbert-Kunz multiplicity” essentially follows from [WY1, Theorem 1.5].

**Proposition 1.6.** Let $A$ be an excellent local ring. Then

1. $0 \leq \text{rel. } m_{HK}(A) \leq 1$.
2. $\text{rel. } m_{HK}(A) = 1$ if and only if $A$ is regular.
3. If $\text{rel. } m_{HK}(A) > 0$, then $A$ is weakly F-regular.

**Proof.** We first prove (3). Suppose that $A$ is not weakly F-regular. Then there exists an $m$-primary ideal $I$ such that $I \neq I^*$. Taking an ideal $I'$ such that $I \subseteq I' \subseteq I^*$ and $l_A(I'/I) = 1$, we have $e_{HK}(I) = e_{HK}(I')$ by Lemma 1.3(1). This implies that $\text{rel. } m_{HK}(A) = 0$; this is a contradiction.

To see (1),(2), it is enough to show that if $\text{rel. } m_{HK}(A) \geq 1$ then $A$ is regular. Actually, if $A$ is regular, then $e_{HK}(I) = l_A(A/I)$ for every ideal $I$ of $A$. Thus $\text{rel. } m_{HK}(A) = 1$.

Now suppose that $\text{rel. } m_{HK}(A) \geq 1$. Then $A$ is weakly F-regular, and thus is Cohen-Macaulay (cf. [HH3]). Let $J$ be a parameter ideal of $A$. Then $e_{HK}(J) = e(J) = l_A(A/J)$.

By the assumption that $\text{rel. } m_{HK}(A) \geq 1$, we get

$$e_{HK}(m) \leq e_{HK}(J) - e_{HK}(m/J) = l_A(A/J) - l_A(m/J) = 1.$$  

Hence $A$ is regular by [WY1, Theorem 1.5].

**Question 1.7.** Is the converse of Proposition 1.6(3) true?

**Remark.** In Section 3, we will give an affirmative answer to this question in case of $\mathbb{Q}$-Gorenstein F-regular local rings.

In the following, to study $\text{rel. } m_{HK}(A)$ in detail, we introduce the notion of minimal Hilbert-Kunz multiplicity.

**Definition 1.8 (Minimal Hilbert-Kunz multiplicity).** Let $E_A := E_A(A/m)$ denote the injective hull of the residue field $k = A/m$. Let $z$ be a generator of the socle $\text{Soc}(E_A) := \{x \in E_A | mx = 0\}$ of $E_A$. Then we put

$$m_{HK}(A) := e_{HK}(0, \text{Soc}(E_A); E_A) = \liminf_{e \to \infty} \frac{l_A(A/\text{ann}_A(F_A^e(z)))}{p^{ed}},$$

where $F_A^e : E_A \to F_A^e(E_A)$ ($u \mapsto 1 \otimes u$). We call $m_{HK}(A)$ the minimal Hilbert-Kunz multiplicity of $A$. 
Example 1.9. Let $A = k[[x^e, x^{e-1} y, \ldots, x y^{e-1}, y^e]]$, the $e$th Veronese subring of $k[[x, y]]$. Then $\rel m_{HK}(A) = m_{HK}(A) = 1/e$.

In the following proposition, we will prove $\rel m_{HK}(A) \geq m_{HK}(A)$. We expect that $\rel m_{HK}(A) = m_{HK}(A)$ always holds, but have no proof yet in general. This will be proved for Gorenstein local rings in the next section.

Proposition 1.10. $\rel m_{HK}(A) \geq m_{HK}(A)$.

Proof. Since $m_{HK}(A) = m_{HK}(\hat{A})$, we may assume that $A$ is complete. For given $m$-primary ideals $I \subseteq I'$ with $l_A(I'/I) = 1$, we must show that $e_{HK}(I) - e_{HK}(I') \geq m_{HK}(A)$. To see this, we need the following lemma.

Lemma 1.11. Let $I \subseteq I'$ be $m$-primary ideals in $A$ such that $l_A(I'/I) = 1$. Also, let $z$ be a generator of $\soc(E_A)$. Then we can take elements $a \in I'$ and $u \in [0 :_E I]$ which satisfy the following conditions:

(i) $I' = I + aA$.
(ii) $[0 :_E I] = [0 :_E I'] + A u$.
(iii) $z = au$.
(iv) $ma \subseteq I$.
(v) $mu \subseteq [0 :_E I']$.

Proof. Since $l_A(I'/I) = 1$, we can write $I' = I + aA$ for some $a \in I' \setminus I$ with $ma \subseteq I$. Applying Matlis duality to the natural surjective $A$-homomorphism $A/I \rightarrow A/I'$, we get $[0 :_E I'] \subseteq [0 :_E I]$ with $l_A([0 :_E I]/[0 :_E I']) = l_A(A/I) - l_A(A/I') = 1$. Thus we can write $[0 :_E I] = Au + [0 :_E I']$ for some $u \in [0 :_E I] \setminus [0 :_E I']$ with $mu \subseteq [0 :_E I']$. Then $A z = \soc(E_A)$ is contained in $I'[0 :_E I]$ since $I'[0 :_E I] = a[0 :_E I] \neq 0$. Hence we can write $z = rau$ for some $r \in A$. Then $r$ is a unit of $A$. Otherwise, since $ru \in [0 :_E I']$, we have $z = a(ru) = 0$; this is a contradiction. Hence we may assume that $r = 1$ without loss of generality.

We now return to the proof of Proposition 1.10. Let $a \in I'$ and $u \in [0 :_E I]$ be the elements described as in Lemma 1.11. We want to show the following claim.

Claim. $I^{[q]} : a^q \subseteq \ann_A(F^e_A(z))$ for all $q = p^e$, $e \geq 1$.

Let $c \in I^{[q]} : a^q$. Then $c F^e_A(z) = c F^e_A(au) = ca^q F^e_A(u) = 0$, where the last equality follows from $I^{[q]} F^e_A([0 :_E I]) = 0$. Hence $c \in \ann_A(F^e_A(z))$, as required.

On the other hand, since $I^{[q]} = I^{[q]} + A a^q$, we have

$$I^{[q]} / I^{[q]} = \frac{I^{[q]} + a^q A}{I^{[q]} / I^{[q]}} \cong A / I^{[q]} : a^q.$$ 

By the above claim, we get

$$\frac{l_A(I^{[q]} / I^{[q]})}{q^d} = \frac{l_A(A / I^{[q]} : a^q)}{q^d} \geq \frac{l_A(A / \ann_A(F^e_A(z)))}{q^d}.$$
This yields the required inequality. □

Discussion 1.12. Let \( I \subseteq I' \) be \( m \)-primary ideals in \( A \) with \( l_A(I'/I) = 1 \). Then an exact sequence \( F_A(A/m) = A/m[q] \to A/I[q] \to A/I'[q] \to 0 \) implies that

\[
(1.12.1) \quad e_{HK}(I) - e_{HK}(I') \leq e_{HK}(A).
\]

Thus

\[
\max\{e_{HK}(I, I', A) : I \subseteq I' \text{ with } l_A(I'/I) = 1\} = e_{HK}(A).
\]

Remark. One can prove the similar result as in Proposition 1.10 and Discussion 1.12 for any pair \((N, M)\) of \( A \)-submodules of an \( A \)-module \( L \) with \( N \subseteq M \) and \( l_A(M/N) = 1 \).

2. Gorenstein local rings

In this section, we prove that if \((A, m)\) is a Gorenstein local ring then \( e_{HK}(J) - e_{HK}(J : m)\) is independent on the choice of parameter ideal \( J \) of \( A \). In fact, this invariant is equal to \( m_{HK}(A) \) defined in the previous section.

In the following, let \((A, m, k)\) be a \( d \)-dimensional local ring of characteristic \( p > 0 \) and assume that \( k \) is an infinite field.

Now suppose that \( A \) is Cohen-Macaulay. Then the highest local cohomology \( H^d_m(A) \) may be identified with \( \varinjlim A/(a_1^n, \ldots, a_d^n)A \), where \( a_1, a_2, \ldots, a_d \) is a system of parameters for \( A \) and the maps in the direct limit system are given by multiplication by \( a = \prod_{i=1}^d a_i \). Put \((a)[n] = (a_1^n, a_2^n, \ldots, a_d^n)A\) for every integer \( n \geq 1 \). Then any element \( \eta \in H^d_m(A) \) can be represented as the equivalence class \( [x + (a)[n]] \) for some \( x \in A \) and some integer \( n \geq 1 \). Also, note that we can write \((a)[q] = J[q]\) for all \( q = p^e \).

Considering the Frobenius action to \( H^d_m(A) \), we can regard as

\[
F^e_A(H^d_m(A)) \cong \varinjlim A/(a)[nq] = H^d_m(A),
\]

where \( q = p^e \). Then \( F^e(\eta) = [x^q \mod (a)[nq]] \in H^d_m(A) \) for \( \eta = [x + (a)[n]] \in H^d_m(A) \). See [Sm] for more details. Using this fact, we can prove the following theorem.

Theorem 2.1. Let \((A, m)\) be a Gorenstein local ring of characteristic \( p > 0 \). Then for any \( m \)-primary ideal \( J \) of \( A \) such that \( \text{pd}_A A/J < \infty \) and \( A/J \) is Gorenstein, we have

\[
e_{HK}(J) - e_{HK}(J : m) = m_{HK}(A).
\]

In particular, \( \text{rel.} m_{HK}(A) = m_{HK}(A) \).

Proof. First, we consider the case of parameter ideals. Since \( A \) is Gorenstein, \( E_A \cong H^d_m(A) \). In the above notation, the generator \( z \) of \( \text{Soc}(E_A) \) can be written as \( z = [b + J] \), where \( b \) is a generator of \( \text{Soc}(A/J) \). For any element \( c \in A \) and for all \( q = p^e \),

\[
cF^e_A(z) = cF^e_A([b + J]) = [cb^q + J[q]] = 0 \in H^d_m(A)
\]
if and only if there exists an integer \( n \geq 1 \) such that

\[
ch^q \in (a_1^{nq}, \ldots, a_d^{nq}) : (a_1^{n-1} \cdots a_d^{n-1})^q = J^q.
\]

It follows that \( \text{ann}_A F^e_A(z) = J^q : b^q \). Hence we get

\[
m_{HK}(A) = \lim_{e \to \infty} \frac{l_A(A/J^q : b^q)}{q^d} = \lim_{e \to \infty} \frac{l_A((J : m)^q/J^q)}{q^d} = e_{HK}(J) - e_{HK}(J : m),
\]

as required.

Next we consider the general case. Let \( J \) be an \( \mathfrak{m} \)-primary ideal such that \( \text{pd}_A A/J < \infty \) and \( A/J \) is Gorenstein. Take a parameter ideal \( \mathfrak{q} \) which is contained in \( J \). Then it is enough to show that

\[
e_{HK}(J) - e_{HK}(J : \mathfrak{m}) = e_{HK}(\mathfrak{q}) - e_{HK}(\mathfrak{q} : \mathfrak{m}).
\]

This follows from the argument as in the proof of [Vr, Proposition 3.5], but we put a sketch here for sake of completeness. As \( \mathfrak{q} \subseteq J \), there exists a natural surjective map \( A/\mathfrak{q} \to A/J \). Also, since both \( A/\mathfrak{q} \) and \( A/J \) are Gorenstein rings of dimension \( d \), we have the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \to & A = F'_d & \to & \cdots & \to A^d & \to & A & \to & A/\mathfrak{q} & \to 0 \\
& & \delta & \downarrow & & \downarrow & & \downarrow & & \downarrow_{\text{nat}} \\
0 & \to & A = F_d & \to & \cdots & \to A^d & \to & A & \to & A/J & \to 0
\end{array}
\]

(2.1.1)

where the horizontal sequences are minimal free resolutions of \( A/\mathfrak{q} \) and \( A/J \), respectively. In particular, the map \( F'_d \to F_d \) is given by the multiplication of an element (say \( \delta \)). Then we have \( J = \mathfrak{q} : \delta \). Let \( L = J : \mathfrak{m} \) and \( L' = \mathfrak{q} : \mathfrak{m} \). Since \( \mathfrak{q} : (\mathfrak{q}, \delta L) = \mathfrak{q} : \delta L = (\mathfrak{q} : \delta) : L = J : L = \mathfrak{m} \), we get

\[
(\mathfrak{q}, \delta L) = \mathfrak{q} : (\mathfrak{q}, \delta L) = \mathfrak{q} : \mathfrak{m} = L'.
\]

Taking a Frobenius power \( (\mathfrak{q}^q)_{[q]} \), we have \( (\mathfrak{q}^q, \delta^q L^q) = L^q_{[q]} \).

On the other hand, \( J^q = \mathfrak{q}^q : \delta^q \) because if one can apply the Peskine–Szpiro functor to the diagram (2.1.1) then one can obtain the minimal free resolutions of \( A/J^q \) and \( A/\mathfrak{q}^q \), respectively. Hence

\[
J^q : L^q = (\mathfrak{q}^q : \delta^q) : L^q = \mathfrak{q}^q : (\mathfrak{q}^q, \delta^q L^q) = \mathfrak{q}^q : L^q_{[q]}.
\]

If we write \( L = J + aA \) and \( L' = \mathfrak{q} + bA \), then \( J^q = a^q = \mathfrak{q}^q : b^q \). Thus

\[
l_A(L^q/J^q) = l_A(A/J^q : a^q) = l_A(A/\mathfrak{q}^q : b^q) = l_A(L^q_{[q]}/\mathfrak{q}^q)
\]

for all \( q = p^s \). The required assertion easily follows from this.
Corollary 2.2. Under the same notation as in Theorem 2.1, we have

1. \(0 \leq m_{HK}(A) \leq 1\).
2. \(m_{HK}(A) = 1\) if and only if \(A\) is regular.
3. \(m_{HK}(A) > 0\) if and only if \(A\) is F-regular.

If, in addition, \(e(A) \geq 2\), then

\[
m_{HK}(A) \leq \frac{e(A) - e_{HK}(A)}{e(A) - 1},
\]

where \(e(A) = e(m)\) denotes the usual multiplicity of \(A\).

Proof. The first two statement immediately follows from Proposition 1.6 and the previous theorem.

To see (3), it is enough to see “if” part. Actually, since Gorenstein weakly F-regular local ring is always F-regular, “only if” part follows from Proposition 1.6 and Theorem 2.1. Now suppose that \(A\) is weakly F-regular, that is, every ideal of \(A\) is tightly closed. Thus \(e_{HK}(J) \neq e_{HK}(J : m)\) for every parameter ideal \(J\) of \(A\) by Lemma 1.3(2). Therefore we have \(m_{HK}(A) = e_{HK}(J) - e_{HK}(J : m) > 0\).

To see the last inequality, taking a minimal reduction \(J\) of \(m\), we have

\[
e_{HK}(J) - e_{HK}(m) \geq l_A(m/J) \cdot m_{HK}(A).
\]

This yields a required inequality since \(e_{HK}(J) = e(J) = e(A)\).

Remark. In [HuL], Huneke and Leuschke independently proved the similar result. In fact, in case of Gorenstein local rings, the notion of minimal Hilbert-Kunz multiplicity coincides with that of “rational signature” which was defined in [HuL]. Also, see [AL] for more details.

Example 2.3. Assume that \(A\) is a hypersurface local ring of multiplicity 2. Then we have \(m_{HK}(A) = 2 - e_{HK}(A)\).

Let \(A\) be a two-dimensional Gorenstein F-regular local ring which is not regular. Then \(e(A) = 2\) since \(A\) has minimal multiplicity. Moreover, suppose that \(k\) is an algebraically closed field. Then it is known that the \(m\)-adic completion \(\hat{A}\) of \(A\) is isomorphic to the completion of the invariant subring by a finite subgroup \(G \subseteq SL(2, k)\) which acts on the polynomial ring \(k[x, y]\). Furthermore, we have \(e_{HK}(A) = 2 - 1/|G|\); see [WY1,Theorem 5.1]. Hence \(m_{HK}(A) = 1/|G|\) by Example 2.3. This result will be generalized in Section 4.

By the above observation, we have an inequality \(m_{HK}(A) \leq \frac{1}{2}\) for hypersurface local rings with \(\dim A = e(A) = 2\). We can extend this result for hypersurface local rings of higher dimension.

Proposition 2.4. Let \((A, m)\) be a hypersurface local ring of characteristic \(p > 0\). Suppose that \(e(A) = \dim A = d \geq 1\). Then

\[
m_{HK}(A) \leq \frac{1}{2^{d-1} \cdot (d-1)!}.
\]
Proof. We may assume that $A$ is a complete F-rational local domain with infinite residue field. Let $J$ be a minimal reduction of $m$. Take an element $x \in m$ such that $m = xA + J$. Then since $x^{d-1}$ is a generator of $\text{Soc}(A/J)$, we have

$$m_{HK}(A) = \lim_{q \to \infty} \frac{l_A(Ax^{(d-1)q} + J^q/J^q)}{q^d}$$

by Theorem 2.1. For any $q = p^e$, we have the following claim.

Claim.

$$l_A(Ax^{(d-1)q} + J^q/J^q) \leq 2 \cdot l_A(A/m^{\frac{d+1}{2}}).$$

To prove the claim, we put $B = A/J^q$, $y = x^{(d-1)q}$ and $a = m^{\frac{d+1}{2}}$. Then since $ya^2 \subseteq x^{(d-1)q}m^q \subseteq m^{dq} \subseteq J^q$, we have $yaB \subseteq 0 : aB = K_A/a$. By Matlis duality, we get

$$l_A(yB) \leq l_A(yB/yaB) + l_A(yaB) \leq l_A(A/a) + l_A(A/a),$$

as required. Since $l_A(A/m^n) = \frac{e(A)}{d} n^d + O(n^{d-1})$ for all large enough $n$, the assertion easily follows from the claim.

Discussion 2.5. Let $(A, m)$ be a three-dimensional F-regular hypersurface local ring. Then $e_{HK}(A) \geq \frac{2}{3} e(A)$ by the following formula:

$$e_{HK}(A) \geq \frac{e(A)}{\pi} \int_{-\infty}^{\infty} \left( \frac{\sin \theta}{\theta} \right)^{d+1} d\theta = \frac{e(A)}{2^d d!} \sum_{i=0}^{\frac{d}{2}} (-1)^i (d+1-2i)^d \binom{d+1}{i}.$$ 

In particular, if furthermore $e(A) = 3$, then $e_{HK}(A) \geq \frac{2}{3} \cdot 3 = 2$. Thus $m_{HK}(A) \leq \frac{3}{3-1} = \frac{1}{2}$ by Corollary 2.2. On the other hand, Proposition 2.4 implies that $m_{HK}(A) \leq \frac{1}{3}$.

Question 2.6. Let $d$ be an integer with $d \geq 2$, and let

$$A = k[[x_0, x_1, \ldots, x_d]]/(x_0^d + x_1^d + \cdots + x_d^d),$$

where $k$ is a field of characteristic $p > 0$. Does $m_{HK}(A) = \frac{1}{2^{d-1}(d-1)!}$ hold if $p > d$?

3. Canonical covers

In the previous section, we have showed how to compute $m_{HK}(A)$ in the case of Gorenstein local rings. In this section, we study the minimal Hilbert-Kunz multiplicity in case of $\mathbb{Q}$-Gorenstein local rings using “canonical cover”.

Now let us recall the notion of canonical cover. Let $A$ be a normal local ring and let $I$ be a divisorial ideal (i.e. an ideal of pure height one) of $A$. Also, let $\text{Cl}(A)$ denote the divisor class group of $A$. Now suppose that $\text{cl}(I)$ is a torsion element in $\text{Cl}(A)$, that is, $I^{(r)} := \cap_{P \in \text{Ass}_A(A/I)} I^r A_P \cap A$ is a principal ideal for some integer $r \geq 1$. Putting
\( r = \text{ord}(\text{cl}(I)) \), one can write as \( I^{(r)} = fA \) for some element \( f \in A \). Then a \( \mathbb{Z}^r \)-graded \( A \)-algebra

\[
B(I, r, f) := A \oplus I \oplus I^{(2)} \oplus \cdots \oplus I^{(r-1)} = \sum_{i=0}^{r-1} I^{(i)} t^{i}, \quad \text{where} \quad t^r f = 1
\]

is called the \( r \)-cyclic cover of \( A \) with respect to \( I \). Also, suppose that \( r \) is relatively prime to \( p = \text{char}(A) > 0 \). Then \( B(I, r, f) \) is a local ring with the unique maximal ideal \( n := m \oplus I \oplus \cdots \oplus I^{(r-1)} \), and the natural inclusion \( A \hookrightarrow B(I, r, f) \) is étale in codimension one; thus \( B(I, r, f) \) is also normal.

Also, we further assume that \( A \) admits a canonical module \( K_A \). Note that one can regard \( K_A \) as an ideal of pure height one. The ring \( A \) is called \( \mathbb{Q} \)-Gorenstein if \( \text{cl}(K_A) \) is a torsion element in \( \text{Cl}(A) \). Put \( r := \text{ord}(\text{cl}(K_A)) < \infty \). Then the \( r \)-cyclic cover with respect to \( K_A \)

\[
B := A \oplus K_A \oplus K_A^{(2)} \oplus \cdots \oplus K_A^{(r-1)}
\]

is called the canonical cover of \( A \).

Roughly speaking, the notion of \( \mathbb{Q} \)-Gorenstein F-regular local rings (resp. Gorenstein F-rational local rings) is “a positive characteristic analogy” of that of log terminal singularities (resp. 1-Gorenstein rational singularity) in characteristic zero. Thus “canonical cover trick” is one of the important tools not only in the theory of singularities but also in the theory of tight closures. Actually, if \( A \) is a \( \mathbb{Q} \)-Gorenstein F-regular local ring then the canonical cover is not only F-regular but also Gorenstein. Further, \( A \) is a direct summand of \( B \) as an \( A \)-module. Thus one can reduce several problems of \( \mathbb{Q} \)-Gorenstein F-regular local rings to those of Gorenstein F-regular local rings (this trick is called “canonical cover trick”). See e.g. [Wa3], [TW] for details.

In fact, in our context, we can prove the following theorem.

**Theorem 3.1.** Let \((A, m, k)\) be a \( \mathbb{Q} \)-Gorenstein F-regular local ring and let \( B = \bigoplus_{i=0}^{r-1} K_A^{(i)} t^i \) be the canonical cover of \( A \), where \( r \) is the order of \( \text{cl}(K_A) \) in \( \text{Cl}(A) \) and \( t^r f = 1 \). Also, suppose that \( r \) is relatively prime to \( p = \text{char}(A) \). Then we have

\[
m_{\text{HK}}(B) = r \cdot m_{\text{HK}}(A).
\]

The following corollary gives a partial answer to Question 1.7.

**Corollary 3.2.** Let \( A \) be a \( \mathbb{Q} \)-Gorenstein F-regular local ring of characteristic \( p > 0 \) such that \( (\text{ord}(\text{cl}(K_A)), p) = 1 \). Then \( m_{\text{HK}}(A) > 0 \).

In the following, we shall prove Theorem 3.1. We begin with recalling several properties of canonical covers for convenience of the readers.
Lemma 3.3. Let \((A, m, k)\) be a Cohen-Macaulay normal local ring, and suppose that \(A\) is \(\mathbb{Q}\)-Gorenstein. Let \(B = \oplus_{i=0}^{r-1} K_A^{(i)} t^i\) be the canonical cover of \(A\), where \(t^r f = 1_A\). Then the following statements hold.

1. \(B\) is quasi-Gorenstein, that is, \(B \cong K_B\) as \(B\)-modules. In particular, \(B\) is Gorenstein if it is Cohen-Macaulay.
2. \(A\) is strongly F-regular if and only if so is \(B\).

In the following, we further assume that \(B\) is Cohen-Macaulay.

3. The injective hull \(E_B := E_B(B/n)\) of \(B/n\) is given as follows:

\[
E_B = \bigoplus_{i=0}^{r-1} H^d_m(K_A^{(i)}) t^i.
\]

4. \(\text{Soc}_B(E_B) = \text{Hom}_B(B/n, E_B)\) is generated by \(zt\), where \(z\) is a generator of the socle of \(E_A \cong H^d_m(K_A)\).

Proof. The assertion (1) follows from [TW, Sect.3]. Also, the assertion (2) follows from [Wa3,Theorem 2.7].

In the following, assume that \(B\) is Cohen-Macaulay. Then since \(B\) is Gorenstein by (1) and \(mB\) is \(n\)-primary, we have

\[
E_B \cong H^d_n(B) \cong H^d_m(B) = \bigoplus_{i=0}^{r-1} H^d_m(K_A^{(i)}) t^i.
\]

Thus we get the assertion (3). To see (4), it is enough to show that \(zt \in \text{Soc}_B(E_B)\) since \(\text{dim}_k \text{Soc}_B(E_B) = 1\). Namely, we must show that \(az = 0\) in \(H^d_m(K_A^{(i+1)})\) for all \(i\) with \(1 \leq i \leq r - 1\) and for all \(a \in K_A^{(i)}\).

Fix an integer \(i\) with \(1 \leq i \leq r - 1\) and suppose that \(0 \neq a \in K_A^{(i)}\). Applying the local cohomology functor to the short exact sequence

\[
0 \to K_A \xrightarrow{a} K_A^{(i+1)} \to K_A^{(i+1)}/aK_A \to 0
\]

implies that

\[
0 = H^{d-1}_m(K_A^{(i+1)}) \to H^{d-1}_m(K_A^{(i+1)}/aK_A) \to H^d_m(K_A) \xrightarrow{a} H^d_m(K_A^{(i+1)}),
\]

where the first vanishing follows from the fact that \(K_A^{(i+1)}\) is a direct summand of a maximal Cohen-Macaulay \(A\)-module \(B\). To get the lemma, it is enough to show the following claim:

Claim. \(H^{d-1}_m(K_A^{(i+1)}/aK_A) \neq 0\).

Since \(A\) is Cohen-Macaulay, \(aK_A \cong K_A\) is a maximal Cohen-Macaulay \(A\)-module. Hence \(aK_A\) is a divisorial ideal of \(A\). Thus it is enough to show that \(K_A^{(i+1)}/aK_A \neq 0\).
Suppose not. Then we have \((i + 1) \text{div}(K_A) = \text{div}(K_A) + \text{div}(a)\), and thus \(i \cdot \text{cl}(K_A) = 0\). This contradicts the assumption that \(r = \text{ord}(\text{cl}(K_A))\). Hence we get the claim, as required.

**Proof of Theorem 3.1.** Let \(x_1, \ldots, x_d\) be a system of parameters of \(A\) and fix it. Since \(A\) is Cohen–Macaulay, we have \(E_A = H^d_{\text{inj}}(K_A) = \lim_{q \to \infty} K_A / \overline{z}^q K_A\). Also, \(A\) is \(\mathbb{Q}\)-Gorenstein, one can regard the Frobenius map \(F_A^e\) in \(E_A\) as

\[
F_A^e : E_A \to F_A^e(E_A) \equiv H^d_{\text{inj}}(K_A) = \lim_{q \to \infty} K_A / \overline{z}^q K_A = \lim_{q \to \infty} ([b + \overline{x} K_A] \to [b + \overline{z}^q K_A]);
\]

see [Wa3] for details. Thus we have

\[
(3.1.1) \quad m_{HK}(A) = \lim_{q \to \infty} \inf l_A \left( \frac{z^q A + \overline{x}^q K_A}{\overline{z}^q K_A} \right) / q^d,
\]

where we denote the inverse image of the generator \(z\) of the socle of \(K_A / \overline{x} K_A\) by the same symbol as \(z\).

On the other hand, since \(zt \in K_A t\) generates the socle of \(E_B\) by Lemma 3.3, we get

\[
(3.1.2) \quad m_{HK}(B) = \lim_{q \to \infty} l_A \left( \frac{z^q t^q B + \overline{x}^q B}{\overline{z}^q B} \right) / q^d
\]

by Theorem 2.1. Also, as \(B\) is a \(\mathbb{Z}/r\mathbb{Z}\)-graded ring (especially, \(K_A^{(i+r)} t^{i+r} = K_A^{(i)} t^i\)), (3.1.2) is reformulated as follows:

\[
(3.1.3) \quad m_{HK}(B) = \sum_{i=0}^{r-1} \lim_{q \to \infty} l_A \left( \frac{z^q K_A^{(i)} + \overline{x}^q K_A^{(i+q)}}{\overline{z}^q K_A^{(i+q)}} \right) / q^d
\]

If necessary, we may assume that \(q \equiv 1 \pmod{r}\). Taking a nonzero element \(a_i \in K_A^{(i)}\) for each \(i\) with \(0 \leq i \leq r - 1\), we consider the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \to & K_q & \longrightarrow & z^q A + \overline{x}^q K_A^{(q)} & \longrightarrow & a_i \to z^q K_A^{(i)} + \overline{x}^q K_A^{(i+q)} & \longrightarrow & C_q & \to & 0 \\
\downarrow & & \downarrow \text{inj.} & & \downarrow \text{inj.} & & \downarrow & & \downarrow & \\
0 & \to & X_q & \longrightarrow & K_A^{(q)} / \overline{x}^q K_A^{(q)} & \longrightarrow & a_i \to K_A^{(i+q)} / \overline{x}^q K_A^{(i+q)} & \longrightarrow & Y_q & \to & 0.
\end{array}
\]

In order to complete the proof of the theorem, it suffices to prove the following claim.

**Claim.** \(\lim_{q \to \infty} \frac{l_A(K_q)}{q^d} = \lim_{q \to \infty} \frac{l_A(C_q)}{q^d} = 0\).
First, note that if \( N \) is a finite \( A \)-module with \( \dim N \leq d-1 \) then \( l_A(N/\mathbb{x}[q]N)/q^d = 0 \). By definition of \( Y_q \), we have \( Y_q = K_A^{(i+q)}/(a_i K_A^{(q)} + \mathbb{x}[q] K_A^{(i+q)}) \cong (K_A^{(i+1)}/a_i K_A) \otimes_A A/\mathbb{x}[q] \). Since \( \dim K_A^{(i+1)}/a_i K_A \leq d - 1 \), we get \( \lim_{q \to \infty} l_A(Y_q)/q^d = 0 \). On the other hand, as \( q \equiv 1 \pmod{r} \), we have

\[
\lim_{q \to \infty} \frac{l_A(K_A^{(q)}/\mathbb{x}[q] K_A^{(q)})}{q^d} = e_{\HK}(y) \cdot \rank K_A = e_{\HK}(y),
\]

\[
\lim_{q \to \infty} \frac{l_A(K_A^{(i+q)}/\mathbb{x}[q] K_A^{(i+q)})}{q^d} = e_{\HK}(y) \cdot \rank K_A^{(i+1)} = e_{\HK}(y).
\]

That is, \( \lim_{q \to \infty} l_A(X_q)/q^d = \lim_{q \to \infty} l_A(Y_q)/q^d = 0 \) and thus \( \lim_{q \to \infty} l_A(K_q)/q^d = 0 \). On the other hand,

\[
C_q = \frac{z^q K_A^{(i)} + \mathbb{x}[q] K_A^{(i+q)}}{a_i z^q A + \mathbb{x}[q] K_A^{(i+q)}} \cong \frac{z^q K_A^{(i)}}{a_i z^q A + z^q K_A^{(i)} \cap \mathbb{x}[q] K_A^{(i+q)}}
\]

\[
= \frac{z^q K_A^{(i)}}{a_i z^q A + z^q [K_A^{(i)} \cap (\mathbb{x}[q] K_A^{(i+q)} : z^q)]}
\]

Since \( \mathfrak{m}[q] K_A^{(i)} \subseteq K_A^{(i)} \cap (\mathbb{x}[q] K_A^{(i+q)} : z^q) \) by the choice of \( z \in K_A \), we get

\[
l_A(C_q) \leq l_A(K_A^{(i)}/a_i A + \mathfrak{m}[q] K_A^{(i)}) = l_A(K_A^{(i)}/a_i A \otimes_A A/\mathfrak{m}[q]).
\]

By the similar argument as above, we can prove \( \lim_{e \to \infty} l(C_q)/q^d = 0 \), as required. \( \blacksquare \)

**Question 3.4.** Let \( A \) be a weakly F-regular local ring and let \( I \) be a divisorial ideal of \( A \) such that \( \text{cl}(I) \) has a finite order (say \( r \)). If \( B = A \oplus I t \oplus I(2)t^2 \oplus \cdots \oplus I(r-1)t^{r-1} \), the \( r \)-cyclic cover, then does \( m_{\HK}(B) = r \cdot m_{\HK}(A) \) hold?

**Question 3.5.** Under the same notation as in Theorem 3.1, does \( \text{rel.} m_{\HK}(B) = r \cdot \text{rel.} m_{\HK}(A) \) hold? Equivalently, does \( \text{rel.} m_{\HK}(A) = m_{\HK}(A) \) hold?

**Discussion 3.6.** Let \( (A, \mathfrak{m}) \hookrightarrow (B, \mathfrak{n}) \) be a module-finite extension of local domains of characteristic \( p > 0 \). Put \( r = [Q(B) : Q(A)] \). If \( A \hookrightarrow B \) is pure, then \( \text{rel.} m_{\HK}(B) \leq r \cdot \text{rel.} m_{\HK}(A) \).

In fact, let \( I \subseteq I' \) be \( \mathfrak{m} \)-primary ideals of \( A \) with \( l_A(I'/I) = 1 \). Since \( A \subseteq B \) is pure, we have \( L = LB \cap A \) for all \( L \subseteq A \). In particular, \( IB \neq I'B \). By definition of \( \text{rel.} m_{\HK}(B) \), we get

\[
\text{rel.} m_{\HK}(B) \leq e_{\HK}(IB) - e_{\HK}(I'B) = r \cdot \{e_{\HK}(I) - e_{\HK}(I')\}.
\]

Hence \( \text{rel.} m_{\HK}(B) \leq r \cdot \text{rel.} m_{\HK}(A) \).
4. Quotient singularities

In this section, as an application of Theorem 3.1, we study the minimal Hilbert-Kunz multiplicities for quotient singularities (i.e. the invariant subrings by a finite group; see below in detail). In general, quotient singularities are not necessarily Gorenstein but \( \mathbb{Q} \)-Gorenstein normal domains. So using the notion of canonical cover trick, we can reduce our problem to the case of Gorenstein rings.

Let \( k \) be a field and \( V \) a \( k \)-vector space of finite dimension (say \( d = \dim_k V \)). Assume that a finite subgroup \( G \) of \( GL(V) \cong GL(d, k) \) acts linearly on \( S := \text{Sym}_k(V) \cong k[x_1, \ldots, x_d] \), a polynomial ring with \( d \)-variables over \( k \). Then

\[
S^G := \{ f \in S : g(f) = f \quad \text{for all } g \in G \}
\]

is said to be an invariant subring of \( S \) by \( G \).

In this section, we consider positive characteristic (say \( p = \text{char}(k) \)) only, and assume that the order \( |G| \) is non-zero in \( k \), that is, \( |G| \) is not divided by \( p \). Then the existence of the Reynolds operator

\[
\rho : S \to S^G \quad \left(a \mapsto \frac{1}{|G|} \sum_{g \in G} g(a)\right),
\]

claims that \( S^G \) is a direct summand of \( S \). Put \( n = (x_1, \ldots, x_d)S \) and \( m = n \cap S^G \). Then the ring \( A = (S^G)_m \) is said to be a quotient singularity (by a finite group \( G \)). Such a quotient singularity as above is a \( \mathbb{Q} \)-Gorenstein strongly F-regular domain, but it is not always Gorenstein; see e.g. [Wa1, Wa2] for details.

In [WY1], we gave a formula for Hilbert-Kunz multiplicity \( e_{HK}(A) \) of quotient singularities as follows.

**Theorem 4.1.** (cf. [WY1, Theorem 2.7], [BCP]) Under the same notation as above, we have

\[
e_{HK}(I) = \frac{1}{|G|} l_A(S_n/IS_n),
\]

for every \( m \)-primary ideal \( I \) in \( A \). In particular, \( e_{HK}(A) = \frac{1}{|G|} \mu_A(S_n) \), where \( \mu_A(M) \) denotes the number of minimal system of generators of a finite \( A \)-module \( M \).

The main purpose of this section is to prove the following theorem.

**Theorem 4.2.** Let \( A = (S^G)_m \) be a quotient singularity by a finite group \( G \) described as above. Also, assume that \( G \) contains no pseudo-reflections. Then we have

\[
m_{HK}(A) = \frac{1}{|G|}.
\]

**Proof.** First, suppose that \( G \subseteq SL(d, k) \). Then \( S^G \) is Gorenstein by [Wa1, Theorem 1a]. Since \( G \) acts linearly on \( S \), \( S^G \) is a graded subring of \( S \). Thus one can take a
homogeneous system of parameters $a_1, \ldots, a_d$ of $S^G$ with the same degree $m$. Also, we may assume that $m$ is a multiple of $|G|$. Put $J = (a_1, \ldots, a_d)S^G$. Then since $S/JS$ is a homogeneous Artinian Gorenstein ring having the same Hilbert function as that of $S/(x_1^m, \ldots, x_d^m)S$, there exists an element $z \in S_{d(m-1)}$ which generates $S/JS$. Then we have $z \in S^G$, which follows from the proof of [Wa1, Theorem 1a]. But since this fact is essential point in the proof in this case, we put a sketch here.

To see $z \in S^G$, it is enough to show that $z \in S^{(g)}$ for any element $g \in G$. The property $z \in S^{(g)}$ does not change if we consider $S \otimes_k \overline{k}$ instead of $S$, where $\overline{k}$ is the algebraic closure of $k$. Then we may assume $k = \overline{k}$ and assume $g$ is diagonal. Then $x_1 \cdots x_d \in S^{(g)}$ and $x_i^m \in S^{(g)}$ since $\det(g) = 1$. If we put $(x)^m = (x_1^m, \ldots, x_d^m)$, then

$$\dim_k [S^{(g)}/JS^{(g)}]_{d(m-1)} = \dim_k [S^{(g)}/(x)^m S^{(g)}]_{d(m-1)} \geq 1.$$  

On the other hand, since $JS^{(g)} = JS \cap S^{(g)}$, we have

$$\dim_k [S^{(g)}/JS^{(g)}]_{d(m-1)} \leq \dim_k [S/JS]_{d(m-1)} = 1.$$  

It follows that $z \in S^G$, as required.

Now let $J, z$ be as above. Then $JA : mA = (J, z)A$ and $JS : n = (J, z)S$. Hence we get

$$e_{HK}(JA) - e_{HK}(JA : mA) = \frac{1}{|G|} l_A(S_n/JS_n) - \frac{1}{|G|} l_A(S_n/(J : m)S_n)$$

$$= \frac{1}{|G|} l_S(n/S_n) = \frac{1}{|G|}.$$  

The required assertion follows from Theorem 2.1.

Next, we consider the general case. If we put $H = G \cap SL(n, k)$, then $S^H$ is Gorenstein by [Wa2, Theorem 1]. Further, since $H$ is a normal subgroup of $G$ and $G/H$ is a finite subgroup of $k^\times$, $G/H$ is a cyclic group. Say $G/H = \langle \sigma H \rangle$ and $r = |G/H|$. Also, $S^G = (S^H)^{(\sigma)}$.

Then $B = (S^H)_{n \cap S^H}$ is a cyclic $r$-cover of $A = (S^G)_{m}$. In fact, it is known that $B$ is isomorphic to the canonical cover of $A$.

$$B \cong A \oplus K_A t \oplus K_A^{(2)} t^2 \oplus \cdots \oplus K_A^{(r-1)} t^{r-1},$$

where $K_A^{(r)} = fA$, $t^r f = 1$. See [TW] in detail.

Since $m_{HK}(B) = \frac{1}{|H|}$, by Theorem 3.1, we get

$$m_{HK}(A) = \frac{1}{r} m_{HK}(B) = \frac{1}{(G : H)|H|} = \frac{1}{|G|},$$

as required.  

**Conjecture 4.3.** Under the same notation as in Theorem 4.2, rel. $m_{HK}(A) = 1/|G|$.
5. Segre products

Throughout this section, let $k$ be a perfect field of characteristic $p > 0$, and let $R = k[x_1, \ldots, x_r]$ (resp. $S = k[y_1, \ldots, y_s]$) be a polynomial ring with $r$-variables (resp. $s$-variables) over $k$. Also, we regard these rings as homogeneous $k$-algebras with $\deg(x_i) = \deg(y_j) = 1$ as usual. Then we define the graded subring $A = R \# S$ of $R \otimes_k S$ by putting $A_n := R_n \otimes_k S_n$ for all integer $n \geq 0$. Then $A = R \# S$ is said to be the Segre product of $R$ and $S$. Actually, the ring $A$ is the coordinate ring of the Segre Embedding $\mathbb{P}^{r-1} \times \mathbb{P}^{s-1} \hookrightarrow \mathbb{P}^{rs-1}$.

As the Segre product $A$ is a direct summand of $R \otimes_k S$ (which is isomorphic to a polynomial ring with $r+s$-variables), it is a strongly $F$-regular domain. Further, it is known that $\dim A = r + s - 1$ and $e(A) = \binom{r + s - 2}{r-1}$. See also [GW, Chapter 4] for more details.

The main purpose of this section is to compute the minimal Hilbert-Kunz multiplicity $m_{HK}(A)$ for Segre products. Before stating our result, we recall related results.

In [BCP], Buchweitz, Chen and Purdue have given the Hilbert-Kunz multiplicity $e_{HK}(A)$ of $A$. Also, Eto and the second-named author [Et] simplified their result in terms of “Stirling numbers of the second kind” as follows.

**Theorem 5.1.** (cf. [BCP, 2.2.3], [EY, Theorem 3.3], [Et]) Suppose that $2 \leq r \leq s$ and put $d = r + s - 1$. Let $A = k[x_1, \ldots, x_r] \# k[y_1, \ldots, y_s]$. Then

$$e_{HK}(A) = \frac{s!}{d!} S(d, s) - \frac{1}{d!} \sum_{k=1}^{r-1} \sum_{j=1}^{r-k} \binom{r}{k+j} \binom{s}{j} (-1)^{r+k} k^d,$$

where $S(n, k)$ denotes the Stirling number of the second kind; see below.

Stirling numbers of the second kind also plays an important role in the study of the minimal Hilbert-Kunz multiplicity of the Segre product. So we recall the notion of Stirling numbers.

**Definition 5.2.** ([St,Chapter 1, §1.4]) We denote by $S(n, k)$ the number of partitions of the set $[n] := \{1, \ldots, n\}$ into $k$ blocks. Then $S(n, k)$ is called the Stirling number of the second kind.

The following properties are well-known. See [St].

**Fact 5.3.** If we denote by $S(n, k)$ the Stirling number of the second kind, then

1. The following identity holds:

$$x^n = \sum_{k=0}^{\infty} S(n, k) x(x-1) \cdots (x-k+1).$$

2. $S(n, k) = kS(n-1, k) + S(n-1, k-1)$ and $S(0, 0) = 1$. 

(3) \( S(n, k) \) admits the following exponential generating function:

\[
\sum_{n \geq k} S(n, k) \frac{x^n}{n!} = \frac{1}{k!} (e^x - 1)^k.
\]

In particular,

\[
S(n, k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n.
\]

For example, since \( S(s + 1, s) = \binom{s+1}{2} \), we have the following example.

**Example 5.4.** Let \( A = R \# S = k[x_1, x_2] \# k[y_1, \ldots, y_s] \), which is isomorphic to the Rees algebra \( S[nt] \) over \( S \). Then

\[
e_{HK}(A) = s \left( \frac{1}{2} + \frac{1}{(s + 1)!} \right).
\]

In the following, we will give a formula for the minimal Hilbert-Kunz multiplicities of the Segre product. Now let \( A \) be the Segre product of \( R \) and \( S \) described as above: \( A = R \# S = k[x_1, \ldots, x_r] \# k[y_1, \ldots, y_s] \), and suppose that \( 2 \leq r \leq s \). Put \( d = r + s - 1(= \dim A) \) and set

\[
m = (x_1, \ldots, x_r)R, \quad n = (y_1, \ldots, y_s)S, \quad \text{and} \quad \mathfrak{M} = m \# n = \bigoplus_{n=1}^{\infty} R_n \otimes S_n.
\]

Then the (graded) canonical module \( K_A \) of \( A \) is isomorphic to \( K_R \# K_S \) by [GW, Theorem 4.3.1] (In particular, \( A \) is Gorenstein if and only if \( r = s \)). Thus by virtue of [GW, Theorem 4.1.5], we get

\[
E_A = H^d_{\mathfrak{M}}(K_A) = H^d_{\mathfrak{M}}(K_R \# K_S) = H^r_m(K_R) \# H^s_n(K_S) = E_R \# E_S.
\]

Further, since \( E_R \) (resp. \( E_S \)) can be represented as a graded module \( k[x_1^{-1}, \ldots, x_r^{-1}] \) (resp. \( k[y_1^{-1}, \ldots, y_s^{-1}] \)) which is called the inverse system of Macaulay, we get

\[
E_A \cong k[x_1^{-1}, \ldots, x_r^{-1}] \# k[y_1^{-1}, \ldots, y_s^{-1}] = k[x_1^{-1}, \ldots, x_r^{-1}] \otimes k[y_1^{-1}, \ldots, y_s^{-1}].
\]

Then \( z = 1 \# 1 \in E_A \) generates the socle of \( E_A \).

Using this, we have
Proposition 5.5. Let $A = R\#S$ and $z = 1\#1$ be as above. Then
\[
l_A(A/\text{ann}_A(F^e_A(z))) = \# \left\{ (a_1, \ldots, a_r, b_1, \ldots, b_s) \in \mathbb{Z}^{r+s} \left| \begin{array}{c}
0 \leq a_1, \ldots, a_r \leq q - 1 \\
0 \leq b_1, \ldots, b_s \leq q - 1 \\
a_1 + \cdots + a_r = b_1 + \cdots + b_s
\end{array} \right. \right\}.
\]

Proof. We use the same notation as in the above argument. Now we shall investigate the Frobenius action of $z$ in $E_A$. First note that $F^e_A(E_A) \cong F^e_R(E_R)\#F^e_S(E_S)$. Thus it is enough to investigate the Frobenius action of $z_1 = 1$ in $E_R$. Since $E_R = H_m^r(R)(-r)$, that is, $H_m^r(R) \cong (x_1 \cdots x_r)^{-r}E_R$, the generator $z_1$ of $E_R$ corresponds to the element $w_1 = (x_1 \cdots x_r)^{-1}$ via this isomorphism. Then we have $F^e_R(w_1) = (x_1 \cdots x_r)^{-q}$ since there exists an isomorphism
\[
(x_1 \cdots x_r)^{-1}k[x_1, \ldots, x_r] \to H_m^r(R) = \lim_{n \to \infty} R/(x_1^n, \ldots, x_d^n).
\]
\[
(x_1^{-a_1} \cdots x_d^{-a_d} \mapsto [x_1^{a_1} \cdots x_d^{a_d} + (\mathbb{Z})], \text{where } a := \max\{a_1, \ldots, a_d\})
\]
If we identify $F^e_R(E_R)$ with $E_R$, then
\[
F^e_R(z_1) = (x_1 \cdots x_r) \cdot F^e(w_1) = (x_1 \cdots x_r)^{-(q-1)}.
\]
Therefore
\[
(5.5.1) \quad F^e_A(z) = F^e_R(z_1)\#F^e_S(z_2) = (x_1 \cdots x_r)^{-(q-1)}\#(y_1 \cdots y_s)^{-(q-1)} \text{ in } E_A.
\]

For any element $c = x_1^{a_1} \cdots x_r^{a_r} \# y_1^{b_1} \cdots y_s^{b_s}$ in $R$, we have
\[
e^c F^e(z) \neq 0 \text{ in } E_A \iff \left\{ \begin{array}{c}
0 \leq a_1, \ldots, a_r \leq q - 1,
0 \leq b_1, \ldots, b_s \leq q - 1,
a_1 + \cdots + a_r = b_1 + \cdots + b_s.
\end{array} \right.
\]

Thus we get the required assertion. \(\square\)

We are now ready to state our main theorem in this section.

Theorem 5.6. Let $A = k[x_1, \ldots, x_r] \# k[y_1, \ldots, y_s]$, where $2 \leq r \leq s$, and put $d = r + s - 1$. Then
\[
m_{HK}(A) = \frac{r!}{d!} S(d, r) + \frac{1}{d!} \sum_{k=1}^{r-1} \sum_{j=1}^{r-k} \binom{r}{k+j} \binom{s}{j} (-1)^{r+k} k^d,
\]
where $S(n, k)$ denotes the Stirling number of the second kind; see below.

In particular,
\[
e_{HK}(A) + m_{HK}(A) = \frac{r! \cdot S(d, r) + s! \cdot S(d, s)}{d!}.
\]

The following two corollaries easily follow from Theorem 5.1 and Theorem 5.6.
Corollary 5.7. Let $A = R\#S = k[x_1, x_2] \# k[y_1, \ldots, y_s]$, which is isomorphic to the Rees algebra $S[nt]$ over $S$. Then

$$m_{HK}(A) = \frac{2^{s+1} - s - 2}{(s + 1)!}.$$ 

Corollary 5.8. Under the same notation as in Theorem 5.6, further, assume that $A$ is Gorenstein, that is, $r = s$. Then

$$e_{HK}(A) + m_{HK}(A) = \frac{2 \cdot r!}{(2r - 1)!} S(2r - 1, r).$$

Proof of Theorem 5.6. If we put $\alpha_{r,n} := l_R(m^n/m^{n+1}) = \binom{n + r - 1}{r - 1}$ and $\alpha_{r,n,q} := l_R(m^n/m^{n-q}m^q + m^{n+1})$, then

$$e_{HK}(A) = \lim_{q \to \infty} \frac{1}{q^d} \sum_{n=0}^{r(q-1)} \alpha_{r,n} \alpha_{s,n,q}$$

$$+ \lim_{q \to \infty} \frac{1}{q^d} \sum_{n=0}^{s(q-1)} \alpha_{r,n,q} \alpha_{s,n} - \lim_{q \to \infty} \frac{1}{q^d} \sum_{n=0}^{r(q-1)} \alpha_{r,n,q} \alpha_{s,n,q}.$$

Also, by virtue of Proposition 5.5, we get

$$m_{HK}(A) = \lim_{q \to \infty} \frac{1}{q^d} \sum_{n=0}^{r(q-1)} \alpha_{r,n,q} \alpha_{s,n,q}.$$

Hence the required assertion follows from the following lemma. □

Lemma 5.9. (cf. [EY, Lemma 3.8, Lemma 3.9]) Under the same notation as above, we have

1. $$\lim_{q \to \infty} \frac{1}{q^d} \sum_{n=0}^{r(q-1)} \alpha_{r,n,q} \alpha_{s,n} = \frac{r!}{d!} S(d, r).$$

2. $$\lim_{q \to \infty} \frac{1}{q^d} \sum_{n=0}^{r(q-1)} \alpha_{r,n,q} \alpha_{s,n,q} = \frac{r!}{d!} S(d, r) + \frac{1}{d!} \sum_{0 < j < i \leq r} \binom{r}{i} \left( \binom{s}{j} \right) (-1)^{r-i+j} (i - j)^d.$$
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