Equivariant formality of corank-one isotropy actions and products of rational spheres

Jeffrey D. Carlson and Chen He

November 28, 2023

Abstract

We completely characterize the pairs of connected Lie groups $G > K$ such that $\text{rk } G - \text{rk } K = 1$ and the isotropy action of $K$ on $G/K$ is equivariantly formal. The analysis requires us to correct and extend an existing partial classification of homogeneous quotients $G/K$ with the rational homotopy type of a product of an odd- and an even-dimensional sphere.

1. Introduction

Among the most consequential algebraic conditions on a continuous group action $G \times X \rightarrow X$ is surjectivity of the map $H^*(X_G; \mathbb{Q}) \rightarrow H^*(X; \mathbb{Q})$ induced by the fiber inclusion of the Borel fibration $X \rightarrow X_G \rightarrow BG$, known by the trade name of equivariant formality. This notion was already considered by Borel [Bo60, Ch. XII] long before being brought to its present prominence in well-known work of Goresky–Kottwitz–MacPherson [GorKM], and makes cohomology computations tractable and powerful integral localization theorems applicable [BeV, AtB, JK].

The most fundamental type of action neither guaranteed to be equivariantly formal nor equivariantly informal is that of the translation action $k \cdot gK = kgK$ of a subgroup $K$ of a Lie group $G$ on the homogeneous space $G/K$, the (global) isotropy action whose tangent action at $1K$ is so important for local study of smooth actions. One always assumes $G$ and $K$ are connected, and then a reduction due to one of the authors [C, Prop. 3.1] allows one to assume they are compact as well. Then, despite a number of equivalent characterizations [CF], and a number of special classes of cases known to be equivariantly formal [Bri98, Sh, GoN], one only has a classification of all equivariantly formal isotropy actions when $\text{rk } G = \text{rk } K$ or $\text{rk } K = 1$ [Bri98, C].

In this work we add $\text{rk } G - \text{rk } K = 1$ to the fully classified range, and find to our surprise that we require the classification of effective transitive actions of compact Lie groups on spaces rationally homotopy equivalent to a product $S^\text{odd} \times S^\text{even}$ of spheres. Such homogeneous rational sphere products have received extended consideration as the subject of at least three Ph.D. dissertations, and an AMS Memoirs volume on isoparametric hypersurfaces admitting a transitive isometry group on one focal manifold [Kam, Kr, BiSo2, Wfm], but these invaluable and generally very thorough analyses unfortunately do not consider all cases we need and suffer from minor errors and omissions (see Discussions 3.1, 3.23, and 3.24), so that we must revise and complete the classification ourselves. This classification appears as part of Table 1.6 and occupies Section 3.

The classification appears at the end of a sequence of reductions, which begins with a generic compact, connected pair $(G, K)$ with $\text{rk } G - \text{rk } K = 1$ and ends with an item in Table 1.6. The reduction involving rational sphere products requires a standard setup we maintain throughout.
Notation 1.1. If $G$ is a compact, connected Lie group with closed, connected subgroup $K$ whose maximal torus $S$ is of codimension one in a maximal torus $T$ of $G$, we write $W_G$ for the Weyl group, $w_0 = w^G_0 \in W_G$ for the longest word, $\mathfrak{g}$ for the Lie algebra, $N_G(S)$ and $Z_G(S)$ respectively for the normalizer and the centralizer, $N$ for the component group $\pi_0N_G(S)$, and $H_S$ for the largest closed, connected subgroup of $G$ containing $S$ as a maximal torus and centralizing the orthogonal complement to $s$ in $t$ under a fixed $\text{Ad}(G)$-invariant inner product. (See Definition 2.9 and the following material for much more on this group.)

Definition 1.2. We call a pair of topological groups $(G, K)$ with $G > K$ isotropy-formal if the isotropy action of $K$ on $G/K$ is equivariantly formal.

Theorem 1.3. Let $(G, K)$ be a pair of compact, connected Lie groups such that a maximal torus $S$ of $K$ is of codimension one in a maximal torus $T$ of $G$.

1. If $G/K$ has the rational cohomology of an odd-dimensional sphere, then $(G, K)$ is isotropy-formal.

2. If $G/K$ has the rational cohomology of a product $S^n \times S^m$, with $n$ odd and $m$ even, then $(G, K)$ is isotropy-formal if and only if $|N| \neq |W_K|$.

3. If $K = H_S$, then $(G, K)$ is isotropy-formal if and only if one of the two conditions above holds.

4. We have $N \neq W_{H_S}$ if and only if $w_0$ stabilizes $s$ but $w_0|_s$ is not in $W_{H_S}$.

The entire procedure is as follows; irreducible pairs are $(G, H)$ such that no proper normal subgroup of $G$ acts transitively on $G/H$, and (virtually) effective pairs those such that $\ker(G \to \text{Homeo}(G/H))$ is trivial (resp., finite). Other vocabulary should be intuitive and will be elaborated over the course of Section 2.

Theorem 1.4. Let $(G, K)$ be a corank-one pair of compact, connected Lie groups, with maximal tori $(T, S)$ as in Notation 1.1. To determine whether it is isotropy-formal, we may do the following.

Step 1: Construct the associated maximal regular pair $(G, H_S)$ per Proposition 2.11.

Step 2: If $\pi_1(G/H_S)$ is infinite, then $(G, K)$ is isotropy-formal.

- If $\pi_1(G/H_S)$ is finite, compute the (finite, even) number $d = \dim Q H^a(G/H_S)$.

Step 3: If $d = 2$, then $(G, K)$ is isotropy-formal.

- If $d \geq 6$, then $(G, K)$ is not isotropy-formal.

- If $d = 4$, then $(G, H_S)$ is a rational sphere product pair.

Take the effective quotient $(\overline{G}, \overline{H})$ of $(G, H_S)$, which is isotropy-formal if and only if $(G, K)$ is.

Step 4: The pair $(\overline{G}, \overline{H})$ is irreducible if and only if it is in Table 1.6 up to a finite covering (in the sense of Definition 2.24).

- If so, read from Table 1.6 whether it is isotropy-formal.

- If not, then it is not isotropy-formal.

Evidently Theorem 1.3 is responsible for Step 3. The classification in the irreducible case in Section 3 can be extended to the virtually effective case with the material of Section 2.4, and isotropy-formality in this case is determined with some additional information involving $\overline{H}$ in Theorem 1.4 and the quotient $Z_{\overline{H}}/\overline{H}/\overline{H}$. The considerations in Step 4 may be summarized as follows; notation is explained in Remark 3.19 and Notation 3.22.
Table 1.6: Irreducible compact, connected pairs $(G, H)$ with $\pi_1(G/H) = 0$ and $H^*(G/H) \cong H^*(S^n \times S^m)$ for $n \geq 3$ odd and $m \geq 2$ even

| $G$     | $H$                  | $Z_G(H)^0$                | $H_6$                     | $H_6 \cong H^7$ | $(m, n)$ | $G/H$ |
|---------|----------------------|---------------------------|----------------------------|-----------------|----------|-------|
| SU(3)   | $i_{p,q} U(1)$, $p, q$ coprime | If and only if $\{p, q\} = \{0, \pm 1\}$ or $[1, -1]$; | $i_{p,q} U(1) \cdot SU(2)_p \cdot SU(2)_q \cdot SU(2)_{-p}$ | $SU(2)_p \cdot SU(2)_q \cdot SU(2)_{-p}$ | >        | (2, 5) |
|         |                      | $U(1)^2$ if $p \cdot q \in \{1, -2\}$;                   | $U(1)^2 \cdot SU(2)_p \cdot SU(2)_q \cdot SU(2)_{-p}$ | $U(1)^2 \cdot SU(2)_p \cdot SU(2)_q \cdot SU(2)_{-p}$ | =        |       |
|         |                      | otherwise                | $H$ otherwise             | $H$ otherwise | =        |       |
| Sp(2)   | $i_{p,q} U(1)$, $p, q$ coprime | ✓                         | $i_{p,q} U(1) \cdot Sp(1)_p \cdot Sp(1)_q \cdot Sp(1)_{-p}$ | $Sp(1)_p \cdot Sp(1)_q \cdot Sp(1)_{-p}$ | >        | (2, 7) |
|         |                      | $U(1)^2$ if $p \cdot q \in \{0, \pm 1\}$;                   | $U(1)^2 \cdot Sp(1)_p \cdot Sp(1)_q \cdot Sp(1)_{-p}$ | $U(1)^2 \cdot Sp(1)_p \cdot Sp(1)_q \cdot Sp(1)_{-p}$ | =        |       |
|         |                      | otherwise                | $H$ otherwise             | $H$ otherwise | =        |       |
| $G_2$   | $i_{p,q} U(1)$, $p, q$ coprime | ✓                         | $i_{p,q} U(1) \cdot SU(2)_p \cdot SU(2)_q \cdot SU(2)_{-p}$ | $SU(2)_p \cdot SU(2)_q \cdot SU(2)_{-p}$ | >        | (2, 11) |
|         |                      | $U(1)^2$ if $p \cdot q = 0$;                                     | $U(1)^2 \cdot SU(2)_p \cdot SU(2)_q \cdot SU(2)_{-p}$ | $U(1)^2 \cdot SU(2)_p \cdot SU(2)_q \cdot SU(2)_{-p}$ | =       |       |
|         |                      | otherwise                | $H$ otherwise             | $H$ otherwise | =        |       |
| $G' \times Sp(1)$ | $H', i_{p,q} U(1)$, $p, q$ coprime | Unless $(G', H') = (SU(k+1), SU(k))$ for $k \geq 2$ | $U(1)^2$ if $p \cdot q = 0$; | $U(1)^2 \cdot SU(2)_p \cdot SU(2)_q \cdot SU(2)_{-p}$ | >        | (2, dim $G'/H'$) |
|         |                      | $Z_G(H')^0 \neq 1$                  | $H'$ otherwise            | $H'$ otherwise | =        |       |
| $K_1 \times K_2$ | $H_1 \times H_2$ | ✓                         | $S^1 \times Z_{K_1}(H_2)^0$ if $K_1 = A_1$ | $K_1 \times H_2$ | $H_2 \cong H_2^3$ | (dim $K_1/H_1$, dim $K_2/H_2$) | $K_1/H_1 \times K_2/H_2$ |
| SU(5)   |                      |                           | $SU(2) \oplus SU(3)$             | SU(2) \oplus SU(2) | $SU(2) \oplus SU(2)$ | (4, 9) | $G_2(C^3)$ |
| SO(4)   |                      |                           | $SU(2) \oplus SU(2)$             | SU(2) \oplus SU(2) | $SU(2) \oplus SU(2)$ | (4, 11) | $G_2(C^3) \cong V_2(R^6)$ |
| SO(2k+1), $k \geq 3$ | SO(2k-2) | ✓                         | $SO(3)$ if $k = 1$             | $SO(3)$ if $k = 1$ | $SO(3)$ if $k = 1$ | (2k-2, 4k-1) | $V_3(R^{2k+1})$ |
| Spin(9) |                      |                           | $U(1)$                      | $U(1)$             | $U(1)$           | (6, 12) | $V_2(R^9) \cong S^6 \times S^7$ |
| Spin(7) |                      |                           | $U(1)$                      | $U(1)$             | $U(1)$           | (6, 11) |       |
| SO(7)   |                      |                           | $U(1)$                      | $U(1)$             | $U(1)$           | (6, 11) |       |
| Sp(4)   |                      |                           | $SO(3)$                     | $SO(3)$            | $SO(3)$          | (4, 11) |       |
| Sp(3)   |                      |                           | $Sp(1) \cdot Sp(1) \oplus [1]$ if $Sp(1) \cdot Sp(1) \oplus [1]$ | $Sp(1) \cdot Sp(1) \oplus [1]$ if $Sp(1) \cdot Sp(1) \oplus [1]$ | $Sp(1) \cdot Sp(1) \oplus [1]$ if $Sp(1) \cdot Sp(1) \oplus [1]$ | (4, 11) |       |
|         |                      |                           | $[1] \oplus [1] \oplus Sp(1)$ if $Sp(1) \cdot Sp(1) \oplus [1]$ | $[1] \oplus [1] \oplus Sp(1)$ if $Sp(1) \cdot Sp(1) \oplus [1]$ | $[1] \oplus [1] \oplus Sp(1)$ if $Sp(1) \cdot Sp(1) \oplus [1]$ | (4, 11) |       |
| SO(2k), $k \geq 5$ | SO(2k-2) | ✓                         | $SO(2)$                     | $SO(2)$            | $SO(2)$          | (2k-2, 2k-1) | $V_2(R^{2k})$ |
| Spin(10) |                      |                           | $U(1)$                      | $U(1)$             | $U(1)$           | (6, 15) | $Spin(9)/SU(4)$ |
| Spin(8) |                      |                           | $U(1)$                      | $U(1)$             | $U(1)$           | (6, 7)  | $Spin(7)/SU(3)$ |
| SO(8)   |                      |                           | $U(1)$                      | $U(1)$             | $U(1)$           | (6, 7)  | $Spin(7)/SU(3)$ |
| SO(6)   |                      |                           | $U(1)$                      | $U(1)$             | $U(1)$           | (4, 5)  | $Spin(7)/SU(3)$ |
| F_4     | Spin(7)             | ✓                         | $SU(2)$                     | $SU(2)$            | $SU(2)$          | (8, 23) |       |
|         | Spin(3)             |                           | $Sp(3)$                     | $Sp(3)$            | $Sp(3)$          | (8, 23) |       |
|         |                      |                           | $H$                         | $H$                | $H$              | (8, 23) |       |
| Sp(k) x Sp(2), $k \geq 2$ | Sp(k-1) \times Sp(1) \times Sp(1) | ✓                         | $Sp(k-1) \times U(1) \times U(1)$ | $Sp(k-1) \times U(1) \times U(1)$ | $Sp(k-1) \times U(1) \times U(1)$ | (4k-4, 1) |       |
| G_2 x Sp(2) |                       |                           | $SU(2)_1 \cdot U(1) \cdot Sp(1)$ | $SU(2)_1 \cdot U(1) \cdot Sp(1)$ | $SU(2)_1 \cdot U(1) \cdot Sp(1)$ | (4, 11) |       |
|         |                       |                           | $SU(2)_1 \cdot U(1) \cdot Sp(1)$ | $SU(2)_1 \cdot U(1) \cdot Sp(1)$ | $SU(2)_1 \cdot U(1) \cdot Sp(1)$ | (4, 11) |       |
**Theorem 1.5.** Let \((G, H)\) be a pair of compact, connected Lie groups such that \(G/H\) has the rational homotopy type of a product \(S^n \times S^n\) with \(n \geq 3\) odd and in even. All irreducible such pairs (considered up to covers in the sense of Definition 2.24) are listed in Table 1.6, and the isotropy-formal pairs among them are noted. All reducible but virtually effective such pairs (up to covers) are obtained as

\[
\left( G \times P, (H \times 1) \cdot \{(p, p) : p \in P\} \right)
\]

for \((G, H)\) in Table 1.6 and \(P\) a rank-one subgroup of \(Z_G(H)^0\) not contained in \(H\). Existence of such subgroups can be determined from the \(Z_G(H)^0\) column of the table. When such a \(P\) exists, it is isomorphic to one of \(U(1), SO(3),\) and \(SU(2),\) and the equivalence class of the pair depends only on the isomorphism type of \(P\). Such a pair is isotropy-formal if and only if \(H_S > H\), as also noted in the table.

**Remark 1.7.** As the construction of the associated maximal pair, the computation of the fundamental group and cohomology ring, and the construction of the effective quotient and simply-connected cover of a pair are all effectively computable tasks and the table one consults is finite as well, Theorem 1.4 in principle gives an effective algorithm for determining whether a compact, connected pair \((G, K)\) is isotropy-formal. Thanks go to the referee for asking for this clarification.

**Acknowledgments.** The first author thanks Federico Pasini for help with a program evaluating degree multisets for rational sphere products \(G/H,\) Linus Kramer for verifying his analysis did not consider the cases \(S^{\text{even}} \times S^{\text{odd}}\) with even \(\geq 3\) odd and discussing the cases in which our results differed, Oliver Goertsches for comments and suggestions on an early draft, and Jason DeVito for useful conversations and literature references on homogeneous spaces, especially the dissertation of Kamerich [Kam] and the argument in Discussion 3.26(f). The second author thanks Mychelle Parker for letting us use her Maple code to study various subalgebras of simple Lie algebras, and thanks the Fundamental Research Funds for the Central Universities of China (2020MS040, 2023MS078) for their support. Both authors would additionally like to thank the anonymous referee for a careful and insightful reading resulting in many corrections and clarifications.

## 2. Reductions and characterization

We have set ourselves the task of describing all the isotropy-formal pairs \((G, K)\) of compact, connected groups with \(\text{rk } K = \text{rk } G - 1\). We say \(K\) and the pair \((G, K)\) are of \textit{corank one}.\(^4\) We now embark on the voyage of reduction described in Theorem 1.4. As a first step, we may replace \(K\) by its maximal torus \(S\).

**Theorem 2.1** ([C, Thm. 1.1][CF, Thm. 1.4, Prop. 3.12]). Let \((G, K)\) be a pair of compact, connected Lie groups and \(S\) a maximal torus of \(K\). Then \((G, K)\) is isotropy-formal if and only if \((G, S)\) is isotropy-formal, and if and only if either of the following hold:

- \(G/S\) (equivalently, \(G/K\)) is formal in the sense of rational homotopy theory and \(H^\ast (BG; \mathbb{Q}) \to H^\ast (BS; \mathbb{Q})^{N_C(S)}\) is surjective;

- \(H^\ast (BG; \mathbb{Q}) \to H^\ast (BS; \mathbb{Q})^{N_C(S)}\) is surjective and the action of the component group \(\pi_0 N_G(S)\) on the Lie algebra \(S\) of \(S\) induced from the conjugation action of \(N_G(S)\) on \(S\) displays \(\pi_0 N_G(S)\) as a reflection group.

\(^4\) N.B. This terminology differs from Onishchik’s usage [On94, p. 207], in which the \textit{corank} is the rational dimension of the cokernel of the map \(H^{\geq 1} G \to H^{\geq 1} K \to H^{\geq 1}(K) / H^{\geq 1}(K)^2\).
Of course, applying this result twice, we may replace \((G, K)\) in our considerations with any other pair \((G, H)\) such that \(H\) shares a maximal torus with \(K\). There is a natural notion of equivalence of pairs:

**Definition 2.2.** Two pairs of Lie groups \((G_1, K_1)\) and \((G_2, K_2)\) are equivalent if there is an isomorphism \(f: G_1 \to G_2\) such that \(f(K_1) = K_2\).

An equivalence induces a diffeomorphism \(G_1/K_1 \to G_2/K_2\) taking the isotropy \(K_1\)-action to the isotropy \(K_2\)-action, so isotropy-formality depends only on equivalence classes of pairs.

Since conjugation by an element of \(G\) is a Lie group automorphism, all tori in \(G\) lie in a maximal torus, and all maximal tori are conjugate, we may assume the maximal torus \(S\) of \(K\) lies in some fixed maximal torus \(T\) of \(G\). Then we may relate the Weyl group \(W_G\) with respect to \(T\) and the component group \(N = \pi_0 N_G(S)\) of Notation 1.1.

**Definition 2.3.** Let \(G\) be a compact, connected Lie group with maximal torus \(T\) containing a subtorus \(S\). We write \(\tilde{N}\) for the subgroup \((N_G(S) \cap N_G(T))/T\) of \(W_G\).

**Lemma 2.4** ([C, Lem. 3.10, 4.3]). With the notation set in 1.1 and 2.3, the adjoint action of \(\tilde{N}\) on \(s\) factors as \(\tilde{N} \to N \to \text{Aut} s\).

Thus \(N\) can be identified as a subquotient of \(W_G\). If \(S\) is a maximal torus of some closed, connected subgroup \(K\) of \(G\), then \(N_K(S)\) is contained in \(N_G(S)\), so \(W_K\), viewed as a subgroup of \(\text{Aut} s\), is a subgroup of \(N\). A well-known characterization [Bo⁺60, IV.5.5, XII.3.4] of equivariant formality of the action of a torus \(T\) on a compact space \(X\) in terms of the sums of Betti numbers of \(X\) and of \(X^T\) reduces in our case \(T = S\) and \(X = G/K\) to the following:

**Theorem 2.5** (Shiga–Takahashi [ShT, Sec. 2]; Goetsches–Noshari [GoN, Prop. 2.3]; Carlson [C, Prop. 3.11] for this phrasing). With the notation set in 1.1, we have

\[
\dim_Q H^* (G/K) \geq 2^{\text{rk } G - \text{rk } N} [N : W_K],
\]

and the left translation action of \(S\) (hence \(K\)) on \(G/K\) is equivariantly formal if and only if equality holds.

## 2.1. Reduction to the regular case

We now justify the group \(H_S\) occurring in Notation 1.1 and Theorems 1.3 and 1.4.

The essential special characteristic of the case \(\dim S = \dim T - 1\) is that there exists a character \(T \to S^1\) whose kernel is \(S\), and the Lie algebra \(s\) of \(S\) can be identified as the kernel of the derivative of this character, a functional \(\alpha: t \to \mathbb{R}\) on the Lie algebra \(t\) of \(T\). We can normalize \(\alpha\) to lie in a fundamental domain for the Weyl group action as follows. Conjugating \(S\) by an element \(w \in N_G(T)\) replaces the functional \(\alpha\) by \(w \cdot \alpha\), so that fixing a basis \(\Delta\) of simple roots for \(G\), we may assume \(\alpha\) lies in the closed fundamental dual Weyl chamber in \(t^*\) consisting of functionals whose invariant inner product with each \(\gamma \in \Delta\) is nonnegative. Equivalently, we may consider the unique vector \(v \in t^*\) such that \(-B(v, -) = \alpha\), where \(B\) is an \(\text{Ad}(G)\)-invariant negative-definite bilinear form, and then our normalization constrains \(v\) to lie in the closed fundamental Weyl chamber \(C\) of vectors on which each \(\gamma \in \Delta\) is nonnegative.

Thus, by Theorem 2.1, isotropy-formality of corank-one pairs \((G, K)\) can be determined wholly in terms of the group \(G\) and a nonzero vector \(v\) in the closed fundamental Weyl chamber with respect to some maximal torus and some basis of simple roots for \(G\). We fix this notation for the rest of this subsection.
Notation 2.6. Let $G$ be a compact, connected Lie group and $T$ a fixed maximal torus as in Notation 1.1. We write

- $\Delta$ for a basis of simple roots for $G$ in the dual $\mathfrak{t}^*$ to the Lie algebra $\mathfrak{t}$ of $T$,
- $\overline{\mathcal{C}} \subseteq \mathfrak{t}$ for the closed fundamental Weyl chamber $\{u \in \mathfrak{t} : \gamma(u) \geq 0 \text{ for all } \gamma \in \Delta\},$
- $v \in \mathfrak{t}$ for a fixed nonzero vector in $\overline{\mathcal{C}},$
- $\Delta_v \subseteq \Delta$ for the set of simple roots annihilating $v$,
- $W_v$ for the stabilizer of $v$ in $W_G$, generated by reflections with respect to $\gamma \in \Delta_v$,
- $B$ for a negative-definite $\text{Ad}(G)$-invariant bilinear form on $\mathfrak{t}$,
- $\alpha \in \mathfrak{t}^*$ for the functional $-B(v, -) : \mathfrak{t} \rightarrow \mathbb{R},$
- $s = (\mathbb{R}v)^\perp < \mathfrak{t}$ for the $B$-orthogonal complement of $\mathbb{R}v$ in $\mathfrak{t}$,
- $S = \exp s$ for the connected subgroup of $T$ whose Lie algebra is $s$.

Although the logical dependency between the defined symbols is different than in Notation 1.1, the notations are compatible when $v$ is chosen such that $\exp \mathbb{R}v$ is closed (hence a circle) in $T$, in which case $S$ is a complementary codimension-one subtorus of $T$.

The stabilizer $W_v$ is closely related to the group $\tilde{N}$ of Definition 2.3 and Lemma 2.4. As the tangent space $s < \mathfrak{t}$ is defined to be $\ker \alpha = (\mathbb{R}v)^\perp$, and since $-B$ is $W$-invariant, it follows an element $\bar{w} \in \mathcal{W}$ stabilizes $s$, and hence lies in $\tilde{N}$, if and only if it stabilizes $\{\pm v\}$. In symbols, $\tilde{N} = W_{|v|}.$

Corollary 2.7. With the notation set in 1.1, 2.3, and 2.6, we have

$$\tilde{N} = \begin{cases} \langle W_v, w_0 \rangle & \text{if } w_0 \cdot v = -v, \\ W_v & \text{otherwise,} \end{cases} \quad \text{and hence} \quad N = \begin{cases} \langle W_v|_{s}, w_0|_{s} \rangle & \text{if } w_0 \cdot v = -v, \\ W_v|_{s} & \text{otherwise.} \end{cases}$$

Proof. Recall [A69, Thm. 5.13, Cor. 5.16] that $W_G \cdot v$ meets each closed Weyl chamber in precisely one point. Since $-v$ lies in $-\overline{\mathcal{C}} = w_0 \cdot \overline{\mathcal{C}}$ and $w_0 \cdot v$ is the unique point of the orbit $W_G \cdot v$ lying in $w_0 \cdot \overline{\mathcal{C}}$, it follows that if there exist any elements of $W_G$ sending $v$ to $-v$, then $w_0$ is among them. If so, the index $[\tilde{N} : W_v]$ is $|\tilde{N} \cdot v| = |\{v\}| = 2$, and so $\tilde{N} = \langle W_v, w_0 \rangle$; otherwise, $[\tilde{N} : W_v] = |\{v\}| = 1.$

Remark 2.8. If we have $[\tilde{N} : W_v] = 2$, it does not necessarily follow that $[N : W_v|_{s}] = 2$; it can be the case that $w_0 \cdot v = -v$ but $w_0|_{s}$ lies in $W_v|_{s}$.

We follow Onishchik [On94, pp. 61–5, 218–9] in associating another Lie group containing $S$.

Definition 2.9. With notation as in Notation 1.1 and 2.6, let $S^\perp := \exp \mathbb{R}v$ be the one-parameter subgroup of $T$ tangent to $\mathbb{R}v$ at $1 \in T$, let $L_S := Z_G(S^\perp)$ be its centralizer in $G$, write $[L_S, L_S]$ for the commutator subgroup of $L_S$, and set $H_S := [L_S, L_S] \cdot S$. We call $(G, H_S)$ the maximal regular pair associated to $S$.

\footnotesize
\begin{itemize}
  \item See, e.g., Adams [A69, Thm. 5.13(vii)], notation fromDefs. 4.13, 4.38).
  \item This can be taken to be the Killing form if $G$ is semisimple and otherwise may be taken as the direct sum of the Killing form and the negative of an arbitrary inner product on the center $\mathfrak{z}(g)$.
\end{itemize}
The nomenclature is explained by the following definition and result.

**Definition 2.10** (Dynkin [D]). A Lie subalgebra \( \mathfrak{t} \) of \( \mathfrak{g} \) is **regular with respect to** a maximal abelian subalgebra \( \mathfrak{a} \) of \( \mathfrak{g} \) if \([\mathfrak{t}, \mathfrak{a}]\) lies in \( \mathfrak{a} \) (or in other words if \( \mathfrak{t} \) lies in the normalizer \( \mathfrak{n}_\mathfrak{g}(\mathfrak{a}) \)) and **regular** if \( \mathfrak{n}_\mathfrak{g}(\mathfrak{t}) \) has the same rank as \( \mathfrak{g} \). Equivalently, a closed, connected subgroup \( K \) of \( G \) is **regular with respect to** a maximal torus \( T \) of \( G \) if \( T \) lies in the normalizer \( N_G(K) \), and **regular** if \( N_G(K) \) is of full rank in \( G \).

**Proposition 2.11.** With the notation set in 2.6 and 2.9, we have the following.

1. The Weyl group \( W_{L_S} \) of \( L_S \) is the stabilizer \( W_0 \).

2. The group \( H_S \) is the largest closed, connected subgroup of \( G \) containing \( S \) as a maximal torus and centralizing \( S^\perp \). Equivalently, \( H_S \) is the largest closed, connected subgroup of \( G \) regular with respect to \( T \) and containing \( S \) as a maximal torus.

3. The inclusion \( H_S \hookrightarrow L_S \) induces an isomorphism \( W_{H_S} \cong \) \( W_{L_S} \); i.e., \( W_{H_S} = W_{L_S}|_{S} = W_{V}|_{S} \equiv W_{0} \).

4. With the notation set in 1.1, we have

\[
N = \begin{cases} 
\langle W_{H_S}, w_0|_{S} \rangle & \text{if } w_0 \cdot v = -v, \\
W_{H_S} & \text{otherwise.}
\end{cases}
\]

Thus \( N \neq W_{H_S} \) holds if and only if both \( w_0 \cdot v = -v \) and \( w_0|_{S} \notin W_{H_S} \).

**Proof.** Writing \( \Phi \) for the set of nonzero roots of \( G \), the adjoint action of \( t \) on \( g_C = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \) has the root space decomposition \( g_C = t_C \oplus \bigoplus_{\beta \in \Phi} C e_{\beta} \) with coroot vectors \( h_\beta = [e_\beta, e_{-\beta}] \in \) it implicitly defined by \( \beta = 2 \frac{B(h_\beta, -)}{B(\beta, \beta)} \). The associated real root space decomposition is thus

\[
\mathfrak{g} = t \oplus \bigoplus_{\beta \in \Phi^+} \mathbb{R}\{e_{\beta} + e_{-\beta}, i(e_{\beta} - e_{-\beta})\}.
\]

1. Since \( T \) is abelian and contains \( S^\perp \), it centralizes \( S^\perp \) and so lies in \( L_S \). If we write \( \Phi_{L_S} \subseteq \Phi \) for the set of nonzero roots of \( L_S \), then the complexified Lie algebra \( \mathfrak{t}_C \) of \( L_S \), as a \( \mathfrak{t} \)-representation, decomposes as \( t_C \oplus \bigoplus_{\beta \in \Phi_{L_S}} C e_{\beta} \). To determine \( \Phi_{L_S} \), note that since \( \mathfrak{t} \) centralizes \( V \), for each \( \beta \in \Phi_{L_S} \) we have \( \beta(v) = 0 \). The positive roots \( \Phi^+ \) are \( \mathbb{N} \)-linear combinations of the simple roots \( \delta \in \Delta \), and \( \delta(v) \geq 0 \) for each simple root because \( v \in \overline{C} \), so \( \beta \in \Phi^+ \) lies in \( \Phi_{L_S} \) if and only if it is an \( \mathbb{N} \)-linear combination of roots \( \delta \in \Delta \) vanishing on \( v \). Thus we have \( \Phi_{L_S} = \Phi \cap \mathbb{Z}\Delta \). But then \( W_{L_S} \) is the subgroup of \( W_G \) generated by reflections through the hyperplanes ker \( \gamma \) for \( \gamma \in \Delta \), namely \( W_\gamma \).

2. Since \( L_S \) and \( T \) centralize \( S^\perp \), so evidently does \( H_S = [L_S, L_S] \cdot S \leq L_S \). On the Lie algebra level, we have \( \mathfrak{t} = \mathfrak{t} \oplus \bigoplus_{\beta \in \Phi_{L_S}^+} \mathbb{R}\{e_{\beta} + e_{-\beta}, i(e_{\beta} - e_{-\beta})\} \), so since \( [i(e_{\beta} - e_{-\beta}), e_{\beta} + e_{-\beta}] = 2i\beta_{H_S} \) and \( [h_{\beta}, e_{\beta}] = \beta(h_\beta)e_{\beta} = 2e_\beta \), the derived subalgebra of \( \mathfrak{t} \) decomposes as

\[
[\mathfrak{t}, \mathfrak{t}] = \mathfrak{t}_{[\mathfrak{t}, \mathfrak{t}]} \oplus \bigoplus_{\beta \in \Phi_{L_S}^+} \mathbb{R}\{e_{\beta} + e_{-\beta}, i(e_{\beta} - e_{-\beta})\},
\]

where the Cartan algebra \( \mathfrak{t}_{[\mathfrak{t}, \mathfrak{t}]} \) is spanned by \( i[e_{\beta}, e_{-\beta}] = ih_\beta \in \mathfrak{t} \) for \( \beta \in \Phi_{L_S}^+ \).

For \( \beta \in \Phi_{L_S} \), since \( \beta(v) = 0 \), we have \( h_\beta \in \mathfrak{v}^\perp = s \), so that \( \mathfrak{t}_{[\mathfrak{t}, \mathfrak{t}]} \leq s \). Since \( [s, \mathfrak{t}_{[\mathfrak{t}, \mathfrak{t}]}] \leq [s, \mathfrak{t}] \leq 0 \) and \( [s, \mathfrak{R}_{e_{\beta}}] \leq [\mathfrak{t}, \mathfrak{R}_{e_{\beta}}] = \mathfrak{R}_{e_{\beta}} \) for each \( \beta \in \Phi_{L_S} \), the root space decomposition of \( [\mathfrak{t}, \mathfrak{t}] \) shows
\[ s, [l, l] \leq [l, l], \text{ meaning } [l, l] + s \text{ is a Lie subalgebra of } l \text{ with Cartan algebra } s. \text{ So } H_S = [L_S, L_S] \cdot S \text{ is a Lie group with maximal torus } S \text{ and hence is a proper, closed, connected Lie subgroup of } L_S. \]

To see \( H_S \) is the largest subgroup containing \( S \) as a maximal torus and centralizing \( S^\perp \), suppose \( H' \) is another such subgroup. By definition, \( H' \) lies in the centralizer \( L_S \) of \( S^\perp \), so \([H', H'] \) lies in \([L_S, L_S]\). As \( S \) is a maximal torus of \( H' \), it contains the identity component \( Z(H')^0 \) of the center \( Z(H') \), giving a containment \( H' = [H', H']Z(H')^0 \leq H_S = [L_S, L_S]S \).

To see \( H_S \) can be equivalently characterized as the largest closed, connected subgroup of \( G \) regular with respect to \( T \) and with maximal torus \( S \), let \( H' \) be any closed, connected subgroup of \( G \) containing \( S \) as a maximal torus. If \( H' \) centralizes \( S^\perp \), then both \( S \) and \( S^\perp \) normalize \( H' \), and hence so does \( T = SS^\perp \). Conversely, if \( T \) normalizes \( H' \), then the adjoint representation of \( g \) restricts to \( T \)-representation on \( h' \). The complexification of this representation decomposes as the direct sum of \( \mathfrak{g}_C \) and weight spaces \( \mathbb{C}e_\beta \). Taking \( h_\beta \in \mathfrak{g} \) as above, \([e_\beta, v]\) is a multiple of \( B(h_\beta, v)e_\beta = 0 \), and since \( h'_C \) is spanned by \( s \) and the \( e_\beta \), it follows that \( H' \) centralizes \( S^\perp \).

3. As \( L_S = Z_G(S^\perp) \) and \( T = SS^\perp \), we have \( N_{L_S}(S) = N_{L_S}(T) \). As \( T \) is a maximal torus in \( L_S \), it in particular contains the identity component \( Z(L_S)^0 \) of the center of \( L_S \), and hence \( H_S \leq L_S = [L_S, L_S]Z(L_S)^0 \leq [L_S, L_S]T = H_ST \). Since \( T \) evidently normalizes \( S \), we get \( N_{H_S}(S) = N_{L_S}(S) \leq N_{L_S}(T) \). Thus \( N_{H_S}(T) = N_{L_S}(T) = N_{L_S}(T) \). It follows

\[
W_{H_S} = \frac{N_{H_S}(S)}{S} \to \frac{N_{H_S}(S)T}{T} = \frac{N_{L_S}(T)}{T} = W_{L_S}
\]

is surjective, and since \( T \) acts trivially by conjugation on \( S \), the injection \( W_{H_S} \to \text{Aut } S \) factors through \( W_{L_S} \), showing \( W_{H_S} \to W_{L_S} \) is injective.

4. The displayed equations follow from Corollary 2.7 and Items 1 and 3. The new inequality comes from the fact \( w_0|_S \) may lie in \( W_{H_S} \) even if \( w_0 \) does not lie in \( W_{L_S} \).

Recall that in the statement of Theorem 1.3, we needed to understand the cohomology of \( G/K \). We set yet more notation to facilitate a cohomological characterization of formality for homogeneous spaces.

**Notation 2.12.** Given a map \( f : B \to A \) of graded \( \mathbb{Q} \)-algebras, we write

\[
A \bowtie_B B := A \otimes \mathbb{Q} = A / f(B^{\geq 1}) \cdot A.
\]

Given a group \( W \leq \text{Aut } A \) of algebra automorphisms, we let \( A^W \leq A \) denote the subalgebra of \( W \)-fixed elements. We write \( H^*_A := H^*(BT; \mathbb{Q}) \) for \( \Gamma \) a topological group.

**Theorem 2.13 ([On94, p. 211]).** If \((G, H)\) is a pair of compact, connected groups, then \( G/H \) is formal if and only if

\[
H^*(G/H) \cong H^*_H \bowtie_H H^*_G \otimes \Lambda \hat{P},
\]

where \( \Lambda \hat{P} \cong \text{im}(H^*(G/H) \to H^*G) \) is an exterior algebra on \( \text{rk } G - \text{rk } H \) generators of odd degree and \( H^*_H \bowtie_H H^*_G \) is a complete intersection ring.

Thus, by Theorem 2.1, to assess isotropy-formality of \((G, H)\), it will help to understand \( H^*_H \bowtie_H H^*_G \).

**Proposition 2.14.** For a maximal regular pair \((G, H)\) with maximal tori \((T, S)\), we have isomorphisms

\[
H^\text{even}(G/H) \cong H^*_H \bowtie_H H^*_G \cong (H^*_G)^\text{Wit} \bowtie \text{im}(H^*_G \to H^*_S).
\]
Proof. First note that there is a finite covering \(T/S \to L/H\) where \(L = L_S\) and \(H = [L,L] \cdot S\) are as in Definition 2.9. This map arises because, as noted in the proof of Proposition 2.11.3, we have \(L = [L,L] \cdot T = HT\), and hence \(T \to L \to L/H\) is surjective with kernel \(T \cap H\). This kernel is compact and properly contained in \(T\), so that its rank equals that of \(S\), and hence by compactness the index \(k\) of \(S\) is finite, so that \(T/S \to L/H\) is a degree-\(k\) map of circles.

Consider the Serre spectral sequence \((E_r,d_r)\) associated to the principal circle bundle

\[
L/H \to G/H \to G/L.
\]

Since this spectral sequence’s \(E_2\) page has only its 0th and 1st rows nonzero, it collapses at \(E_3\). Moreover since \(L_S\) is of full rank in \(G\), the cohomology groups of \(G/L_S\) are concentrated in even degree, so the spectral sequence can be equivalently understood as the exact (Gysin) sequence of graded vector spaces

\[
0 \to H^\text{odd}(G/H)[-1] \to H^*(G/L) \xrightarrow{e} H(G/L) \to H^\text{even}(G/H) \to 0,
\]

where the class \(e\) is \(d_2(1 \otimes [L/H])\). Particularly, the ring \(H^\text{even}(G/H)\) is isomorphic to the quotient of \(H^*(G/L)\) by the ideal \((e)\).

The covering spaces \(W_L \to BT \to BN_L(T)\) and \(W_H \to BS \to BN_H(S)\) induce isomorphisms \(H^*_L \xrightarrow{\sim} (H^*_T)^{W_L}\) and \(H^*_H \xrightarrow{\sim} (H^*_S)^{W_H}\) [H, Prop. 3G.1]. We claim the class \(e\) is the Euler class of the bundle \(G/H \to G/L\), and also, up to a scalar multiple, the image under \((H^*_T)^{W_L} \xrightarrow{\sim} H^*_T \to H^*(G/L)\) of the Euler class \(\tilde{e}\) of the circle bundle \(BS \to BT\). This follows from the commutative diagram

\[
\begin{array}{ccc}
BS & \to & E(T/S) \\
\downarrow & & \downarrow \\
G/H & \to & E(L/H) \\
\downarrow & & \downarrow \\
BT & \to & B(T/S) \\
\downarrow & & \downarrow \\
G/L & \to & B(L) \\
\end{array}
\]

of principal bundles. Here the vertical arrows are the bundle projections of \(L/H\)-bundles in front, and of \(T/S\)-bundles in back, the front horizontal maps in the left square can be seen as \(gH \to xgH\) and \(gL \to xgL\) for a fixed \(x \in EL\), the maps of classifying spaces are up to homotopy those functorially induced from inclusions and quotient projections of groups, the bottom edge of the back square is also the classifying map \(BT \to B(T/S)\) of the principal bundle \(T/S \to BS \to BT\), and similarly for the edge \(BL \to B(L/H)\) of the front square. The maps \(BS \to BH\) and \(E(T/S) \to E(L/H)\) induce maps of principal bundles, equivariant with respect to the covering map \(T/S \to L/H\).

The map of spectral sequences induced by the front left square shows that under \(G/L \to BL\), the class \(e = d_2(1 \otimes [L/H])\) (name reused) indeed does pull back to \(e = d_2(1 \otimes [L/H])\). The spectral sequence map induced by the front right square identifies \(e\) as the transgression of the class \(1 \otimes [L/H]\) in the \(E_2\) page \(H^*_T \otimes H^*(L/H)\). The map of spectral sequences induced by the back square identifies \(\tilde{e}\) as the transgression of the class \(1 \otimes [T/S]\) in the \(E_2\) page \(H^*_T \otimes H^*(T/S)\). Finally, the map of spectral sequences induced by the right face sends \(1 \otimes [L/H]\) to \(k \otimes [T/S]\), and hence the transgression of the former is sent to \(k\) times the transgression of the latter. Since the bundle square commutes, this means \(H^*_L \to H^*_T\) sends \(e\) to \(ke\), so the identification \(H^*_L \xrightarrow{\sim} (H^*_T)^{W_L}\) shows \(\tilde{e}\) lies in \((H^*_T)^{W_L}\).
The ideal $(\bar{e})$ is the kernel of the surjection $H^*_{\mathbb{F}_2} \to H^*_{\mathbb{S}}$, which can be seen either directly by observing it generates the kernel of $H^2_T \to H^2_S$, or else from the Serre spectral sequence of $T/S \to BS \to BT$. As the inclusion $H \hookrightarrow L$ induces a restriction isomorphism $W_L \sim W_H$ by Proposition 2.11.3, the identifications yield $(H^*_{\mathbb{F}_2})^{W_L} / (\bar{e}) \cong (H^*_{\mathbb{S}})^{W_H}$, for any $x + (\bar{e}) \in H^*_{\mathbb{S}} = H^*_{\mathbb{F}_2} / (\bar{e})$ invariant under the action of $W_L = W_H$ has $x = w \cdot x \pmod{\bar{e}}$ for all $w \in W_L$, and hence $\frac{1}{|W_L|} \sum w \cdot x \in (H^*_{\mathbb{F}_2})^{W_L}$ represents $x + (\bar{e})$. As $L$ is of full rank in $G$, the Serre spectral sequence of $G/L \to BL \to BG$ collapses (the $E_2$ page is concentrated in even degree), yielding the standard isomorphisms $H^*(G/L) \cong H^*_L / H^*_G = (H^*_{\mathbb{F}_2})^{W_L} / (H^*_{\mathbb{S}})^{W_C}$. Thus, as hoped,

$$H^{\text{even}}(G/H) \cong \frac{(H^*_{\mathbb{F}_2})^{W_L}}{(\hat{e}, (H^*_{\mathbb{S}})^{W_C})} \cong \frac{(H^*_{\mathbb{F}_2})^{W_L} / (\bar{e})}{\ker((H^*_{\mathbb{F}_2})^{W_C} \to H^*_T \to H^*_S)} \cong (H^*_S)^{W_H} / \ker(H^*_G \to H^*_S).$$

Now we can prove the target result of this subsection.

**Proof of Theorem 1.3.** The final statement follows immediately from Proposition 2.11.4. If $G/K$ is a rational cohomology sphere of odd dimension, then $\dim \mathbb{Q} H^*(G/K) = 2$; in that case, we see from Theorem 2.5 that $|N| / |W_K| = 1$ and $(G, K)$ is isotropy-formal. On the other hand, if $G/K$ has the rational cohomology of a product of spheres, one even- and one odd-dimensional, then evidently $\dim \mathbb{Q} H^*(G/K) = 4$. Thus in this case, by Theorem 2.5, the pair $(G, K)$ is isotropy-formal if and only if $|N : W_K| = 2$, and in particular $N$ is not isomorphic to $W_K$. 

For the converse implication of clause 3, write $H = H_S$ and assume $(G, H)$ is isotropy-formal; we must show $G/H$ has the cohomology of an odd-dimensional sphere or the product of an odd- and an even-dimensional sphere. By Theorem 2.1, it is formal, so by Theorem 2.13, its cohomology is of the form $(H^*_H / H^*_G) \otimes \Lambda \hat{P}$ for a one-dimensional vector space graded in odd degree (the cohomology of an odd-dimensional sphere) and $H^*_H / H^*_G$ a complete intersection ring. Thus $\dim \mathbb{Q} H^*(G/H) = 2 \cdot \dim \mathbb{Q} (H^*_H / H^*_G)$. By Theorem 2.5, we also know $\dim \mathbb{Q} H^*(G/H) = 2|N : W_H|$, so $\dim \mathbb{Q} (H^*_H / H^*_G) = |N : W_H|$, which by Proposition 2.11.4 is either 1 or 2. Thus $H^*_H / H^*_G$ has the cohomology either of a point or of an even-dimensional sphere.

**Remark 2.16.** We have the anonymous referee to thank for pointing us to the current proof of the converse direction, which at one point we had found independently but failed to implement. An older version runs as follows. Note $H^*_H / H^*_G$ is $H^{\text{even}}(G/H)$, which by Proposition 2.14 is isomorphic to $(H^*_S)^{W_H} / \ker(H^*_G \to H^*_S)$. From the assumption $(G, H)$ is isotropy-formal and Theorem 2.1 again, the image of $H^*_G \to H^*_S$ is $(H^*_S)^N$ and $N$ acts on $s$ as a reflection group, so the Chevalley–Shephard–Todd theorem [NS, Thm. 7.1.4] implies $(H^*_S)^N$ is a polynomial ring as well. By Theorem 2.13, then, $(H^*_S)^{W_H} / (H^*_S)^N = H^*_H / H^*_G$ is a complete intersection ring, meaning $(H^*_S)^{W_H}$ too is free over $(H^*_S)^N$. Since by a result of Chevalley [NS, Thm. 7.2.1] $H^*_S$ is of rank $|N|$ over $(H^*_S)^N$ and of rank $|W_H|$ over $(H^*_S)^{W_H}$, it follows that

$$\dim \mathbb{Q} H^{\text{even}}(G/H) = \text{rk}_{(H^*_S)^N}(H^*_S)^{W_H} = |N| / |W_H|.$$ 

But by Proposition 2.11.4, this is 1 or 2.

---

4 If we make the identifications $H^2(BT; \mathbb{R}) \cong \mathbb{R}^r$ and $H^2(BS; \mathbb{R}) \cong \mathbb{R}^s$, then $\bar{e}$ can be identified with $a$, which is another way of showing it is $W_l$-invariant.
This theorem distinguishes an especially interesting class of transitive actions.

**Definition 2.17.** We call a Lie group pair \((G, H)\) a **rational sphere product pair** if \(G/H\) has the rational cohomology of a product \(S^\text{even} \times S^\text{odd}\) and a **rational sphere pair** if \(G/H\) has the rational cohomology of \(S^\text{odd}\). In both cases, we call \((G, H)\) a **rational sphere (product) pair**.

**Remark 2.18.** It is well known that when \(G/H\) is simply-connected, any homogeneous space having the rational cohomology ring of a product of spheres (and more generally, any for which the pure Sullivan model \(H^*(BH) \otimes H^*(G)\) is formal) is rationally homotopy equivalent to one. Because the vocabulary differs when \(G/H\) is not simply-connected, we have opted not to speak in terms of rational homotopy type in the general case (but see Remark 2.29).

In our new terminology, Theorem 1.3 says that to understand isotropy-formality in the corank-one case, we need only examine rational sphere (product) pairs which are also maximal regular pairs. The two notions are closely related.

**Lemma 2.19.** Let \((G, H)\) be a rational sphere product pair. If there exists a closed, connected, regular subgroup \(H^- \leq H\) sharing the maximal torus \(S\) with \(H\), and \(H/H^-\) is an even-dimensional rational homotopy sphere, then either \((G, H_S)\) is a rational sphere (product) pair or \(H_S = H^-\).

**Proof.** Consider the bundle \(H/H^- \to G/H^- \to G/H\), which admits a bundle map to \(H/H^- \to BH^- \to BH\). As the Serre spectral sequence of the latter bundle is concentrated in even degree, it collapses at \(E_2\), so the map of spectral sequences implies that of the former bundle does as well; we conclude \(\dim_Q H^*(G/H^-) = \dim_Q(H/H^-) \cdot \dim_Q(G/H) = 8\). Considering instead the bundle \(H_S/H^- \to G/H^- \to G/H_S\), we see that if \(H_S > H^-\), then \(\dim_Q H^*(G/H_S)\) is 2 or 4. Since \(\dim G/T\) and \(\dim H_S/S\) are even and \(\dim T/S = 1\), it follows \(\dim G/H_S\) is odd, and hence by Poincaré duality the fundamental class \([G/H_S]\) is the product of an odd-dimensional class and its even-dimensional Poincaré dual. It follows \((G, H_S)\) is a rational sphere (product) pair.

**Lemma 2.20.** Any rational sphere pair \((G, H)\) is isotropy-formal. The associated maximal regular pair \((G, H_S)\) is a rational sphere (product) pair, and \(H = H_S\) if and only if \(H\) is regular.

**Proof.** Isotropy-formality of an rational sphere pair is well-known and already shown in the proof of Theorem 1.3. In that case Theorem 1.3 implies \((G, H_S)\) is a rational sphere (product) pair. If \(H\) is irregular, then by definition \(H \neq H_S\). On the other hand, if \(H\) is regular, it is contained in \(H_S\), and we have a fiber bundle \(H_S/H \to G/H \to G/H_S\) which by the argument of Lemma 2.19 gives

\[
2 = \dim_Q H^*(G/H) = \dim_Q H^*(H_S/H) \cdot \dim_Q H^*(G/H_S).
\]

As \(\dim_Q H^*(G/H_S) \geq 2\), we must have \(\dim_Q H^*(H_S/H) = 1\). But then as \(H_S\) and \(H\) are of equal rank, we have \(\chi(H_S/H) = |W_{H_S}|/|W_H| = 1\), implying \(H_S = H\).

We can use this to show that if the isotropy group of a corank-one pair properly contains that of a rational sphere product pair, it is isotropy-formal.

**Lemma 2.21.** Suppose \((G, H)\) is a rational sphere product pair. If there exists a closed, connected subgroup \(K > H\) of \(G\) sharing the same maximal torus \(S\), then \(K/H\) is an even-dimensional sphere, \((G, K)\) is a rational sphere pair, and \((G, H)\) is isotropy-formal.
Proof. Consider the bundle $K/H \to G/H \to G/K$; as in the proof of Lemma 2.20, we have $4 = \dim_Q H^*(G/H) = \dim_Q H^*(K/H) \cdot \dim_Q H^*(G/K)$. Since $K > H$ is of equal rank, we have $\dim_Q H^*(K/H) \equiv 2$, and since $\text{rk} G - \text{rk} K = 1$, we have $\dim_Q H^*(G/K) \equiv 2$, so we must actually have equality in both cases. Thus $K/H$ has the rational cohomology an even-dimensional sphere, and hence by a result of Borel [Bo49, Thm. IV] is an even-dimensional sphere, and $(G, K)$ is a rational sphere pair. Hence by Lemma 2.20, $(G, K)$ and $(G, H)$ are isotropy-formal. \qed

In fact, we do not need the quotient rational homotopy sphere $G/H$ to be odd-dimensional.

Lemma 2.22. If $G/H$ is a rational cohomology sphere, then $(G, H)$ is isotropy-formal.

Proof. We have addressed the odd-dimensional case in Lemma 2.20. The even-dimensional case is well-known and follows from Theorem 2.5 since $\chi(G/H) = 2$ implies $\text{rk} G = \text{rk} H$, so that $T = S$ and $N = W_G$, and then $2 = \chi(G/H) = |W_G|/|W_H| = |N|/|W_H|$. \qed

We will need one more lemma of this type for our proof of Theorem 1.5.

Corollary 2.23. Let $(G, H)$ be a regular rational sphere product pair such that the longest word $w^G_0$ of $W_G$ acts as $-\text{id}$ on $t$. Then $(G, H)$ fails to be isotropy-formal if and only if the longest word $w^H_0$ of $H$ acts as $-\text{id}$ on $s$ and $H = H_S$ in the notation of Definition 2.9.

Proof. If $H_S > H$, then $(G, H)$ is isotropy-formal by Lemma 2.21. Thus we assume $H_S = H$. Since $w^G_0$ acts on $t$ as $-\text{id}$, in particular $w^G_0 \cdot v = -v$ and by Corollary 2.7, we have $N = \langle W_0|_s, w^G_0|_s \rangle = \langle W_0|_s, -\text{id}_s \rangle$. If $-\text{id}_s \notin W_H$, then we must have $N > W_H$, so $(G, H)$ is isotropy-formal by Theorem 1.3. On the other hand, by Proposition 2.11.3, we have $W_H = W_0|_s$, so if $-\text{id}_s \in W_H$, then $N = W_0|_s = W_H$, and $(G, H_S)$ is by Theorem 1.3 not isotropy-formal, so $(G, H)$ is not isotropy-formal. But by the argument of Corollary 2.7, applied to an interior point $v'$ of $\mathcal{C}$, the only element of $W_H$ that might act as $-\text{id}_s$ is $w^H_0$. \qed

2.2. Covers

It will be convenient to be able to make replacements of pairs ensuring $\pi_1(G/H)$ is free abelian. Recall that the fundamental group of a Lie group is abelian, so that $\pi_1 G$ is abelian and if $H$ is connected, then so is $\pi_1(G/H)$. In this section we will frequently reason in terms of the rational vector spaces $\pi_*(\cdot; \mathbb{Q}) := \pi_*(\cdot) \otimes \mathbb{Q}$.

Definition 2.24. We call a pair $(\tilde{G}, \tilde{H})$ of compact, connected Lie groups a cover of another such pair $(G, H)$ if there exists a finite covering map $q: \tilde{G} \longrightarrow G$ such that $\tilde{H}$ is the identity component $q^{-1}(H)^0$ of the preimage $q^{-1}(H)$. We call a cover equal-sheeted if $\tilde{H} = q^{-1}(H)$.

This definition is rigged to induce a covering $\tilde{G}/\tilde{H} \longrightarrow G/H$, which in the equal-sheeted case is a diffeomorphism. Coverings preserve and reflect all properties we are interested in.

Proposition 2.25. A compact, connected Lie group pair $(\tilde{G}, \tilde{H})$ covering a compact, connected pair $(G, H)$ is isotropy-formal if and only if $(G, H)$ is, is (maximal) regular if and only if $(G, H)$ is, and is a rational sphere (product) pair (respectively) if and only if $(G, H)$ is. The projection $\tilde{G}/\tilde{H} \longrightarrow G/H$ induces isomorphisms of all rational homotopy groups.
Proof. The statement about isotropy-formality appears in earlier work of one of the authors [C, Thm. 1.2]. For the statement about rational sphere (product) pairs, another lemma in that work [C, Prop. 3.12] observes that one has \( H^*(\tilde{G}/\tilde{H}) \cong H^*(G/H) \). The statement about regularity follows because it can be expressed in terms of the common Lie algebras \( g, h, t \) of \( G, H, T \) and \( \tilde{G}, \tilde{H}, q^{-1}(T)^0 \), and that about maximality because closed, connected subgroups are in bijection with Lie subalgebras. The isomorphism of rational homotopy groups follows from the long exact homotopy sequences of the bundles \( \tilde{H} \to H, \tilde{G} \to G, \tilde{G}/\tilde{H} \to G/H \), and \( G \to G/H \).

We will therefore only consider pairs up to covers. For this, it will be useful to recall some fundamental results in the structure theory of compact Lie groups we will use throughout the rest of the work, so we now make a brief digression to rehearse a sufficiency of this theory.

**Definition 2.26.** We say a compact Lie group is simple if it contains no nontrivial connected, proper, normal subgroups. If \( H \) embeds as a normal subgroup of each of two groups \( K_1 \) and \( K_2 \), we write \( K_1 \otimes H K_2 \) for the balanced product, the quotient of \( K_1 \times K_2 \) by the equivalence relation setting \( (k_1 h, k_2) \sim (k_1 h k_2) \) for \( (k_1, k_2) \in K_1 \times K_2 \) and \( h \in H \). A virtual direct product of two Lie groups \( K_1 \) and \( K_2 \), written \( G = K_1 \cdot K_2 \), is a Lie group \( G \) of the form \( K_1 \otimes F K_2 \) for some finite subgroup \( F \) central in \( K_1 \) and \( K_2 \).

More generally a virtual direct product of two topological spaces \( X_1 \) and \( X_2 \) is any space finitely and centrally covered by the direct product \( X_1 \times X_2 \). The definition generalizes to virtual direct products of any finite number of factors.

**Theorem 2.27.** Every compact, connected Lie group \( G \) is a virtual direct product of the central torus \( Z(G)^0 \) and the commutator subgroup \([G, G]\), which is in turn the virtual direct product of finitely many simple groups. If \( H \leq G \) is a connected normal subgroup, then \( G \) is the virtual direct product of \( H \) and a complement \( P \), meaning there is a compact, connected subgroup \( P \leq G \) such that \( G = H P \cong H \otimes_{H \wedge P} P \). Particularly, \( P \) centralizes \( H \) and \( H \wedge P \) is a finite central subgroup of both \( H \) and \( P \).

In the case \( \pi_1 \) is infinite, we have the following.

**Theorem 2.28.** Let a compact, connected pair \((G, H)\) of Lie groups be such that \( \pi_1(G/H) \) is infinite and \( H^* (G/H) \cong H^* (S^1 \times S^1) \) for some \( \ell \geq 2 \). Then \((G, H)\) finitely covers a pair of the form \((G', H') \times (S^1, 1)\) where \( G'/H' \) is a rational cohomology \( S^1 \), and hence is isotropy-formal.

**Proof.** Note from the cohomology that we must have \( \pi_1(G; \mathbb{Q}) \cong \mathbb{Q} \) one-dimensional, and that the maps \( Z(G)^0 \hookrightarrow G \) and \( G \twoheadrightarrow G_{ab} := G/[G, G] \) induce isomorphisms on \( \pi_1(-; \mathbb{Q}) \). Hence \( \dim_{\mathbb{Q}} \pi_1(G; \mathbb{Q}) \) is equal to the dimension of the torus \( G_{ab} \), and the (toral) image \( \overline{S} \) of \( H \to G \to G_{ab} \) is of dimension one less. Select a circle \( C \leq Z(G)^0 \) whose image \( \overline{C} \) under \( Z(G)^0 \to G \to G_{ab} \) is complementary to \( \overline{S} \), so that \( \overline{C} \cdot \overline{S} = G_{ab} \) and \( \overline{C} \cap \overline{S} \) is finite and \( \overline{C} \to G_{ab} \to G_{ab}/\overline{S} \) is a finite covering. The map \( C \longrightarrow \overline{C} \) is also a finite covering since \( Z(G)^0 \longrightarrow G_{ab} \) is. Write \( \hat{S} \) for the identity component of the preimage of \( \overline{S} \) under \( Z(G)^0 \to G \to G_{ab} \) and set \( G' := \hat{S} \cdot [G, G] \). Then the image of \( G' \to G \to G_{ab} \) is \( \overline{S} \), so the image of \( G' \cdot C \to G \to G_{ab} \) is all of \( \overline{S} \cdot \overline{C} = G_{ab} \), and hence \( G' \cdot C = G \). The intersection \( F = G' \cap C \) of the two is also the kernel of the surjection \( C \to Z(G)^0 \to G_{ab} \to G_{ab}/\overline{S} \to \overline{C}/(\overline{C} \cap \overline{S}) \) and hence is finite since \( \overline{C} \cap \overline{S} \) and the kernel of \( Z(G)^0 \longrightarrow G_{ab} \) are. Of course, since \( H = Z(H)^0 \cdot [H, H] \leq \overline{S} \cdot [G, G] = G' \), the intersection \( F_H := H \cap C \leq F \) is also finite.

---

\(^5\) In this case the traditional terminology is that \( G \) is locally isomorphic to \( K_1 \times K_2 \), a sensible wording since this means the Lie algebras are isomorphic.
Since $G$ is isomorphic to $G' \otimes_F C$, we see $(G, H)$ covers $(G'/F \times C/F, H/F \times 1)$. The latter has homogeneous quotient $G'/FH \times C/F$. Evidently $C/F$ is a circle, and since covers are rational cohomology isomorphisms, we see $G'/FH$ is a rational cohomology $S^I$. The old pair is isotropy-formal if and only if the new pair is by Proposition 2.25. For the new pair, the homotopy orbit space $(G'/F)_{H \cap FH} \times C/F$, so the new pair is isotropy-formal if and only if $(G'/F, H/FH)$ is, but this is always the case by Lemma 2.22.

\[ \square \]

Remark 2.29. This implies the converse to Bletz-Siebert’s Corollary 6.3.2 [BlSo2], which states that if $(G, H)$ is a compact, connected pair of Lie groups with $\pi_*(G/H; \mathbb{Q}) \cong \pi_*(S^I \times S^I; \mathbb{Q})$, then $H^*(G/H) \cong H^*(S^I \times S^I)$. With Remark 2.18, this shows that for such a pair, $G/H$ has the rational cohomology of the product of (one or) two spheres if and only if it has the rational homotopy of such a product.

If $\pi_1$ is finite, we may assume for our purposes it is zero by applying the following:

**Proposition 2.30.** Let $(G, H)$ be a pair of compact, connected Lie groups. Then all connected finite covers of $G/H$ are realized as $\tilde{G}/\tilde{H}$ for some compact, connected pair $(\tilde{G}, \tilde{H})$ such that $\tilde{H} \cong H$.

**Proof.** Considering the long exact sequence of the fibration $H \rightarrow G \rightarrow G/H$, since $H$ is path-connected, we see $\pi_1(G/H) = \text{coker } \pi_1(i)$. Since $G$ is a topological group, $\pi_1(G/H)$ is then abelian, so connected finite covers of $G/H$ correspond bijectively to finite-index subgroups $\Gamma \leq \pi_1(G)$ containing $\text{im } \pi_1(i)$, which also correspond to connected finite covers $\tilde{G} \rightarrow G$ along which $i$ lifts. If we write $\tilde{H} \cong H$ for a lift, then $(\tilde{G}, \tilde{H})$ covers $(G, H)$ and the image of $\pi_1(\tilde{G}/\tilde{H}) \rightarrow \pi_1(G/H)$ is isomorphic to $\Gamma/\text{im } \pi_1(i)$.

If $\pi_1(G/H_5)$ is finite, then taking $\Gamma = \text{im } \pi_1(i)$, we get a maximal regular pair $(\tilde{G}, \tilde{H})$ such that $\pi_1(\tilde{G}/\tilde{H}) = 0$ which is isotropy-formal (respectively a rational sphere (product) pair) if and only if $(G, H)$ is.

### 2.3 Reduction to the effective quotient

It is in a naive sense impossible to fully write out rational sphere (product) pairs $(G, H)$, because if $G$ and another group $K$ are closed normal subgroups in some larger group and $G \cap K = 1$, we may take the pair $(GK, HK)$ and obtain the same quotient. While no one is defending this as a sound use of time, it does obstruct a full classification. To get rid of boring components of an action, note that the non-identity elements of the image of the group homomorphism $\rho: G \rightarrow \text{Homeo}(G/H)$ act nontrivially by definition, and the kernel of $\rho$, the subgroup of elements fixing each $gH$, is a normal subgroup of $G$, namely $\bigcap_{g \in G} gHg^{-1}$, the largest normal subgroup of $G$ contained in (and hence also normal in) $H$.

**Definition 2.31.** We call a pair $(G, H)$ of Lie groups (virtually) effective if the action of $G$ on $G/H$ is, which is to say the kernel of $\rho: G \rightarrow \text{Homeo}(G/H)$ is trivial (respectively, finite).\(^\text{6}\) We call a pair $(\tilde{G}, \tilde{H})$ equivalent to $(\rho(G), \rho(H))$ an effective quotient of $(G, H)$.

Evidently the effective quotient of a rational sphere (product) pair $(G, H)$ is again a rational sphere (product) pair. Slightly less obviously, effective quotient also reflects maximal regular pairs and isotropy-formality.

---

\(^6\) Almost effective is standard, but we prefer to make the analogy with geometric group theory, where $G$ is said to be virtually of some class $C$ if it admits a finite-index subgroup of class $C$. 
**Proposition 2.32.** Let $(G,H)$ be a pair of compact, connected Lie groups, $T \leq G$ a maximal torus, $K \leq H$ a closed subgroup normal in $G$, and $S \leq H \cap T$ a maximal torus of $H$ such that the identity component $(S \cap K)^0$ is a maximal torus of $K$. Then $(G,H)$ is a maximal regular pair with respect to $T$ if and only if $(G/K,H/K)$ is a maximal regular pair with respect to $TK/K$.

**Proof.** One checks easily that the bijection from (closed) subgroups $H' \supseteq K$ of $G$ to (closed) subgroups $H'/K \leq G/K$ restricts to a bijection between groups $H' \supseteq K$ satisfying $T \leq N_G(H')$ and groups $H'/K$ with $TK/K \leq N_{G/K}(H'/K)$, so maximality is preserved and reflected.

Evidently, if $S$ is maximal in $H$, then $SK/K$ is maximal in $H/K$. On the other hand, suppose tori $S \leq S' \leq H$ are given such that $S \cap K$ contains a maximal torus $T_K$ of $K$ and $SK/K$ is a maximal torus of $H/K$. As $SK/K \leq H/K$ is a maximal torus, we have $S'K/K \leq SK/K$ and hence may write any $s' \in S'$ as $sk$ for some $s \in S$ and $k \in K$. Then $k = s^{-1}s'$ lies in $S' \cap K$, so $S' \leq S(S' \cap K)$. As $S' \cap K$ is a compact abelian group with identity component $K$, this means $S'$ is contained in the union of finitely many translates of $ST_K$, which is only possible if $\dim S' \leq \dim ST_K$. But $T_K \leq S \cap K \leq S \leq S'$, so $S' = S$.

**Proposition 2.33.** (Equivariant formality and effective quotients). Let a compact, connected Lie group $H$ act on a compact space $X$ in such a way that the restricted action of a closed normal subgroup $K$ is trivial. Then the $H$-action on $X$ is equivariantly formal if and only if the induced action of $T = H/K$ is.

**Proof.** Let $S$ be a maximal torus of $H$ and $\overline{S}$ its image in $\overline{T}$. By Borel’s well-known characterization [Bo+60, IV.5.5, XII.3.4] of equivariant formality of the action of a torus $T$ on a compact space $X$ (also leading to Theorem 2.5), we have $\dim_{\mathbb{Q}} H^*(X) \geq \dim_{\mathbb{Q}} H^*(X^S)$, with equality if and only if the $S$-action on $X$ (or equivalently the $H$-action) is equivariantly formal, and similarly for the $\overline{S}$- and $\overline{T}$-actions. Of course, since we have assumed $K$ acts trivially, the $S$-action on $X$ factors through an $\overline{S}$-action and the fixed point sets $X^S$ and $X^{\overline{S}}$ are thus equal.

Now per Theorem 1.3, it remains only to examine effective maximal regular pairs, determine which are rational sphere (product) pairs, and examine the action of $\omega_0$. We will see in Section 3 there are up to equivalence only finitely many classes of cases to be dealt with.

### 2.4. Reduction to the irreducible

Even an effective action can sometimes be simplified.

**Definition 2.34.** We call a pair $(G,H)$ **(normally) irreducible** if the action of $G$ on $G/H$ is irreducible in the sense that the restricted action of each proper normal subgroup of $G$ is intransitive. Given a pair $(\tilde{G},\tilde{H})$ and closed subgroups $G \leq \tilde{G}$ and $H = G \cap \tilde{H}$ such that the induced map $G/H \rightarrow \tilde{G}/\tilde{H}$ is a homeomorphism and $(G,H)$ is irreducible, we call the latter a **(normally) irreducible subpair** of $(\tilde{G},\tilde{H})$. We also call the embedding of pairs $(G,H) \rightarrow (\tilde{G},\tilde{H})$ an **enlargement** in this case, and $(\tilde{G},\tilde{H})$ an **enlargement of** $(G,H)$.

---

7 At least one direction survives even when $H$ is only be a topological group acting continuously on a topological space $X$, and in fact holds with arbitrary coefficients. As $EH \times E\overline{T}$ is contractible and a principal $H$-bundle under the diagonal action, we can use it in the Borel construction

$$X_H = \frac{EH \times E\overline{T} \times X}{H} = \frac{(EH \times E\overline{T} \times X)/K}{H/K} = \frac{BK \times E\overline{T} \times X}{\overline{T}}.$$

With this identification, the map $X_H \rightarrow X_{\overline{T}}$ induced by $H \rightarrow \overline{T}$ projects out the $BK$-coordinate, and $X \rightarrow X_{\overline{T}}$ factors through $X \rightarrow X_{H}$, so if the former induces a surjection in cohomology, then so does the latter.
Remark 2.35. Every pair contains an irreducible subpair, and an irreducible pair is always virtually effective, but not vice versa [Ons94, p. 76]. For example, the standard action of \( U(n) \) on \( S^{2n-1} \subset \mathbb{C}^n \) is effective but the action of its normal subgroup \( SU(n) \) is also transitive.

We mean to find irreducible subpairs of the effective quotient. In Definition 2.34, there is no guarantee that if \( \hat{G}, \hat{H}, \) and \( G \) are connected, then \( H \) will be, but if \( \pi_1(G/H) = 0 \), as we arranged in Section 2.2, then this is true by examination of the long exact homotopy sequence. Isotropy-formality is inherited by irreducible subpairs.

**Proposition 2.36** (Inheritance of equivariant formality). Let a topological group \( \hat{H} \) act on space \( X \) and let \( H \leq \hat{H} \) be a subgroup. Then if the \( \hat{H} \)-action on \( X \) is equivariantly formal, so is the restricted \( H \)-action.

This is very well known, but the proof is too simple to be worth excluding.

**Proof.** Taking \( EH = E\hat{H} \), the fiber inclusion \( i: X \rightarrow X_{\hat{H}} \) of the Borel construction factors as \( X \rightarrow X_H \rightarrow X_{\hat{H}} \). Evidently if \( i \) induces a surjection in cohomology, so must \( X \rightarrow X_H \). \( \Box \)

Irreducibility is also preserved by finite covers.

**Proposition 2.37** (Irreducibility and covers). A compact, connected pair \((G, H)\) with \( G/H \) positive-dimensional is irreducible if and only if any one of its finite connected covers \((\hat{G}, \hat{H})\) is irreducible.

**Proof.** If \( \hat{K} \leq \hat{G} \) acts transitively on \( \hat{G}/\hat{H} \), then the image \( K \) of \( \hat{K} \rightarrow \hat{G} \rightarrow G \) acts transitively on \( G/H \) and is again proper since \( \hat{K} \) cannot be contained in \( \text{ker}(\hat{G} \rightarrow G) \) lest one have cardinalities \( |\hat{G}/\hat{H}| = |\hat{K} \cdot \hat{H}| \leq |K| \) finite. On the other hand \( K \leq G \) acts transitively on \( G/H \) if and only if \( K \cdot 1_H = G/H \), in which case the identity component \( \hat{K} \leq \hat{G} \) of the preimage of \( K \) is such that \( \hat{K} \cdot 1_{\hat{H}} \leq \hat{G}/\hat{H} \) is connected and covers \( G/H \), so that it is all of \( \hat{G}/\hat{H} \).

Evidently we can create ineffective enlargements \((G, H) \rightarrow (G \times P, H \times P)\) of compact, connected Lie group pairs with wild abandon, and if \( G \) is normal in \( \hat{G} \), then \( \hat{G} \) must virtually be such a product, but effectiveness means the action of the new factor must commute with the given \( G \)-action, severely restricting our options. The interaction between normalizers and centralizers is thus crucially important.

**Lemma 2.38.** Let \((G, H)\) be a compact, connected pair. Then \( N_G(H)^0 \) is \( Z_G(H)^0 \cdot H \).

**Proof.** As \( H \) is normal in \( N_G(H)^0 \), it has a connected complement \( Q \) in \( N_G(H)^0 \), by Theorem 2.27. Since \( Q \) centralizes \( H \), we have \( N_G(H)^0 = Q \cdot H \leq Z_G(H)^0 \cdot H \leq N_G(H)^0 \cdot H = N_G(H)^0 \).

**Lemma 2.39.** Let \((G, H)\) be a pair of compact, connected Lie groups. Then a subgroup \( Q \leq Z_G(H)^0 \) can be found which is a complement to \( H \) in \( N_G(H)^0 \) and such that every closed, connected subgroup \( \bar{P} \leq N_G(H)^0/H \) is of the form \( PH/H \) for a unique closed, connected subgroup \( P \leq Q \).

**Proof.** Recall that \( g \) carries an \( \text{Ad}(G) \)-invariant inner product \(-B\). The orthogonal complement \( q = h^\perp \) in the normalizer \( n = n_G(h) \) is automatically an ideal of \( n \), since as \( h \) is an ideal, for \((h, n, q) \in h \times n \times q\), we have \( B([h, n], q) = B([h, n], q) = 0 \). Let \( Q = \exp q \). Writing \( \pi: N_G(H)^0 \twoheadrightarrow N_G(H)^0/H \) for the projection and given a connected \( \bar{P} \leq N_G(H)^0/H \), we take \( P = (\pi^{-1}(\bar{P}) \cap Q)^0 \).

\[8\] On the other hand, if \( \hat{G}, G, \) and \( H \) are connected, \( \hat{H} \) is connected too.
\textbf{Theorem 2.40} (Cf. Kramer [Kr, §3.5, p. 21–22], Onishchik [On94, pp. 73–76]).\footnote{A result of this kind for $G/H = G^{\text{odd}}$ is already proved by Montgomery–Samelson [MS, §§4–7].} Let $(G, H)$ be an effective pair of compact, connected Lie groups. Then for any effective enlargement $(\hat{G}, \hat{H})$ there exists a closed, connected subgroup $P$ of $Z_G(H)^0$, meeting $H$ in a finite subgroup, such that $(G \times P, (H \times 1) \cdot \Delta P)$ is a cover of $(\hat{G}, \hat{H})$. This is again an enlargement of $(G, H)$ and is isotropy-formal if and only if $(\hat{G}, \hat{H})$ is.

The action of $G \times P$ on $G/H$ is given by $(g, p) \cdot xH := gxp^{-1}H$.

\textbf{Proof.} Let $\hat{P} \leq \hat{G}$ be a complement of $G$. By effectiveness, we may identify $\hat{G}$ with a subgroup of Homeo$(G/H)$ and $\hat{P}$ with a subgroup of the centralizer $C$ of $G$ in Homeo$(G/H)$. Any map $\phi \in C$ satisfies $\phi(1H) = \ell H$ for some $\ell \in G$, and since $hH = H$ for each $h \in H$, one has $h\ell H = h\phi(H) = \phi(hH) = \phi(H) = \ell H$, so $\ell$ lies in the normalizer $N_G(H)$. Now, $\phi(gH) = g\ell H = gH\ell H$ for $g \in G$, so that $\phi$ is the right translation by $\ell H$ on $G/H$. Thus $C$ is isomorphic to $N_G(H)/H$, acting on $G/H$ by the standard effective right action.

Since $\hat{G} = G\hat{P}$ acts on the left on $G/H$, to proceed we convert the action of $\hat{P} \leq N_G(H)/H$ to a left action. Using Lemma 2.39, we fix a centralizing complement $Q$ to $H$ in $N_G(H)^0$ and find a unique closed, connected subgroup $P \leq Q$ such that $\hat{P} = PH/H$. Then $pH \cdot xH = xp^{-1}H$ for $pH \in P = PH/H$ and $xH \in G/H$, so the action of $\hat{G}$ on $G/H$ descends from the ineffective action $(g, hp) \cdot xH := gxp^{-1}H$ of $G \times HP$. The stabilizer of $1H \in G/H$ under this action is the set $(H \times 1) \cdot \Delta(HP)$ of pairs $(h'hp, hp) \in G \times HP$ for $h' \in H$ and $hp \in HP$, so we have an ineffective enlargement $(G \times HP, (H \times 1) \cdot \Delta(HP))$.

Now, $P$ contains a full set of coset representatives of $\hat{P} = PH/H$, so the composite $G \times P \hookrightarrow G \times HP \twoheadrightarrow G \times \hat{P}$ is a surjection with kernel $1 \times (H \cap P)$. The stabilizer $(H \times 1) \cdot \Delta(HP)$ meets $G \times P$ in $(H \times 1) \cdot \Delta P$, which we will abbreviate $H\Delta P$. Thus $(G \times P, H\Delta P)$ is a virtually effective pair covering $(\hat{G}, \hat{H})$; by Proposition 2.33, it is isotropy-formal if and only if $(\hat{G}, \hat{H})$ is.

As $\hat{P}$ in Theorem 2.40 is a subgroup of $N_G(H)^0/H$, which equals $Z_G(H)^0H/H$ by Lemma 2.38 (and $QH/H$ by Lemma 2.39), we have the following.

\textbf{Proposition 2.41.} Let $(G, H)$ be an effective pair of compact, connected Lie groups. Then the existence of a proper virtually effective enlargement $(\hat{G}, \hat{H})$ of $(G, H)$ such that $G$ is normal in $\hat{G}$ is equivalent to each of the following conditions:

- $Z_C(H)^0$ is not contained in $H$;
- $H$ is strictly contained in $N_G(H)^0 = Z_G(H)^0H$;
- $\text{rk } N_G(H)^0 = \text{rk } Z_G(H)^0H > \text{rk } H$;
- $\text{rk } Z_G(H)^0H/H = \text{rk } Q > 0$.

In the case of a corank-one pair, the third condition of Proposition 2.41 implies that $N_G(H)^0$ is of full rank in $G$, or in other words that $H$ is regular.

\textbf{Proposition 2.42.} Let $(G, H)$ be an effective corank-one pair of compact, connected Lie groups. Then the existence of a proper virtually effective enlargement $(\hat{G}, \hat{H})$ of $(G, H)$ such that $G$ is normal in $\hat{G}$ is equivalent to each of the following conditions:

- $H$ is regular in $G$;
• \( \text{rk} Z_G(H)^0 H/H = \text{rk} Q = 1; \) i.e., \( Q \) is isomorphic to one of \( \text{U}(1), \text{Sp}(1), \) or \( \text{SO}(3). \)

In this case, letting \( P \leq Q \) be a closed rank-one subgroup with maximal torus \( T_P \) as in Theorem 2.40, \( ST_P \times T_P \) is a common maximal torus of \( G \times P \) and of \( HP \times P, \) and \( S \Delta T_P \) is a maximal torus of \( H \Delta P. \)

**Corollary 2.43.** If \( (G, H) \) is an effective rational sphere pair, there is a proper, virtually effective enlargement \( (\hat{G}, \hat{H}) \) of \( (G, H) \) such that \( G \) is normal in \( \hat{G} \) if and only if \( H = H_S. \)

**Proof.** By Lemma 2.20, in this case \( H \) is regular with respect to \( ST_P \) if and only if \( H = H_S. \)

When there does exist an extension, it is essentially unique:

**Lemma 2.44.** Two enlargements \( (G \times P_1, H \Delta P_1) \) and \( (G \times P_2, H \Delta P_2) \) of an irreducible corank-one pair \( (G, H) \) as constructed in Theorem 2.40 are equivalent if and only if \( P_1 \) and \( P_2 \) are isomorphic.

**Proof.** Necessity is immediate. For sufficiency, recall that in Theorem 2.40, the \( P_i \) can be taken as subgroups of a fixed rank-one \( Q \leq Z_G(H)^0. \) If \( P_1 \cong P_2 \leq Q \) are of type \( A_1, \) then all must be equal. Otherwise, \( P_1 \cong P_2 \cong S^1 \) are maximal tori of \( Q, \) so \( P_2 = qP_1q^{-1} \) for some \( q \in Q. \) Then \( (g, p) \mapsto (qgq^{-1}, qpq^{-1}) \) induces an equivalence of pairs \( (G \times P_1, H \Delta P_1) \to (G \times P_2, H \Delta P_2). \)

An effective quotient of a maximal regular pair gives another maximal regular pair, and a virtually effective covering of a maximal regular pair is again a maximal regular pair, so our reduction only requires us to handle the case when \( H \Delta P \) is regular.

**Corollary 2.45.** In the situation of Proposition 2.42, the stabilizer \( H \Delta P \) is a regular subgroup of \( G \times P \) with respect to \( ST_P \times T_P \) if and only if \( P \cong S^1. \)

**Proof.** Since \( H \) centralizes \( \Delta P, \) we see \( ST_P \times T_P \) normalizes \( H \Delta P \) if and only if \( T_P \times T_P \) normalizes \( \Delta P. \) This happens if \( P = T_P \) and not otherwise.

Now we finally can reduce isotropy-formality to the irreducible case.

**Theorem 2.46.** If \( (G, H) \) is a regular, irreducible rational sphere product pair, then there exists an isotropy-formal enlargement \( (G \times S^1, H \Delta S^1) \) if and only if \( H_S > H. \) In this case \( (G, H) \) is also isotropy-formal.

**Proof.** For the second clause, if any enlargement \( (\hat{G}, \hat{H}) \) is isotropy-formal, then \( (G, H) \) inherits isotropy-formality by Proposition 2.36.

For the equivalence, first suppose \( (G, H) \) is a rational sphere product pair and \( H_S > H; \) then Lemma 2.21 says \( (G, H_S) \) is a rational sphere pair. As \( H_S \) is regular, Proposition 2.42 implies \( \text{rk} Z_G(H_S)^0 H_S = 1 \) so there is a circle \( C \leq Z_G(H_S)^0 \) meeting \( H_S \) finitely and a fortiori also meeting \( H \) finitely. The enlargement \( (G \times C, H_S \Delta C) \) is again a rational sphere pair, hence isotropy-formal by Lemma 2.20, and hence so is the enlargement \( (G \times C, H \Delta C) \) as \( H \Delta C \) and \( H_S \Delta C \) share a maximal torus. By Lemma 2.44, any other enlargement \( (G \times S^1, H \Delta S^1) \) is also isotropy-formal.

On the other hand, suppose an isotropy-formal enlargement \( (G \times C, H \Delta C) \) exists for some circle \( C \leq G. \) By Proposition 2.42, \( (G, H) \) is regular, so \( H \leq H_S \) for some maximal torus \( S \) of \( H. \) By Lemma 2.44, \( (G \times S^1, H \Delta S^1) \) is isotropy-formal as well. We will prove that if \( H_S = H, \) this leads to a contradiction, so that in fact \( H_S > H. \) Now \( S^1 \) centralizes \( H = H_S \) by Proposition 2.11, and by definition \( T = SS^1 \) is a maximal torus of \( G, \) so \( (\hat{T}, \hat{S}) = (T \times S^1, S \Delta S^1) \) are maximal tori in the enlargement \( (G \times S^1, H \Delta S^1), \) and an orthogonal complement is given by \( \hat{S}^1 := \{(z, z^{-1}) \in \)
\(G \times S^\perp : z \in S^\perp\). Since \(N_{G \times S^\perp}(T) = N_G(T) \times S^\perp\) and \(N_{H \times S^\perp}(S \Delta S^\perp) = N_H(S) \Delta S^\perp\), the inclusion of \((G, H)\) induces isomorphisms of Weyl groups \(W_G \isom W_{G \times S^\perp}\) and \(W_H \isom W_{H \times S^\perp}\). In particular, if \(n_0 \in N_G(T)\) represents the longest word \(w_0^G\) of \(W_G\), then \((n_0, 1) \in N_G(T) \times S^\perp\) represents \(w_0^G \times 1\) in \(W_{G \times S^\perp}\). Moreover, since an element \((g, s^\perp) \in G \times S^\perp\) normalizing \(S \Delta S^\perp\) also normalizes \(G \cap S \Delta S^\perp = S\), and then all elements of \(\{g\} \times S^\perp\) induce the same automorphism of \(S \Delta S^\perp\), the projection \(G \times S^\perp \longrightarrow G\) induces an injection \(\tilde{N} := \pi_0 N_{G \times S^\perp}(S \Delta S^\perp) \longrightarrow \pi_0 N_G(S) = N\). By the second clause in this theorem, \((G, H)\) is isotropy-formal, so applying Theorem 1.3 and Proposition 2.11, the longest word \(w_0^G\) satisfies \(w_0^G \cdot v = -v\) and \(w_0^G|_S \notin W_H\). But the adjoint action \(w_0^G \times S^\perp\) of \((n_0, 1)\) sends \((v, v) \in \Delta S^\perp < s + \Delta s^\perp\) to \((-v, v) \notin s + \Delta s^\perp\), so the restriction \(w_0^G \times S^\perp|_{s + \Delta s^\perp}\) is not an element of \(\tilde{N}\) and hence by Proposition 2.11 and Theorem 1.3 again, the enlarged pair is not isotropy-formal.

Corollary 2.47. A maximal regular, effective but reducible rational sphere product pair is not isotropy-formal.

Proof. By Theorem 2.40, Proposition 2.42 and Corollary 2.45, such a pair \((G, H)\) is finitely covered by \((G' \times S^1, H' \Delta S^1)\), where \((G', H') < (G, H)\) is irreducible with \(G'/H' \longrightarrow G/H\) a homeomorphism and \(S^1 \leq Z_G(H')\) meets \(H'\) in a finite set. This cover is again maximal regular by Proposition 2.25 and virtually effective. If \((G, H)\) were isotropy-formal, then by Theorem 2.46 applied to \((G', H')\), we would have \(H_S > H'\) for \(S\) a maximal torus of \(H'\), but then we would have \(H_S \Delta S^1 > H' \Delta S^1\), so that \((G \times S^1, H_S \Delta S^1)\) would be a regular pair properly containing \((G' \times S^1, H' \Delta S^1)\), which we assumed was maximal regular.

Corollary 2.48. A maximal regular, effective rational sphere product pair \((G, H)\) with \(Z_G(H)^0 H/H \cong A_1\) is not isotropy-formal.

Proof. Due to maximal regularity, so \(H = H_S = [L_S, L_S]S\), which by Proposition 2.11.2 centralizes \(S^\perp\). Since \(S^\perp\) does not lie in \(H\) (lest \(H\) contain \(T = SS^\perp\)), it descends to a circle subgroup of \(Z_G(H)^0 H/H\). Notice that the nontrivial element \(\tilde{w}\) of the Weyl group of \(Z_G(H)^0 H/H = A_1\) reflects the tangent line to this circle. The quotient \(A_1\) corresponds to a normal complement \(Q\) to \(H\) in \(N_G(H)^0\) which lies in \(Z_G(H)^0\), and has \(S^\perp\) as a maximal torus, by Lemma 2.39, and \(\tilde{w}\) lifts to the nontrivial element of \(W_Q\), represented by some \(q \in Q\). Since \(Q\) centralizes \(H \leq S\) and normalizes \(S^\perp\), it normalizes \(T = SS^\perp\), and so induces an element \(w \in W_G\) reflecting the tangent line \(\mathbb{R}v\) to \(S^\perp\). Thus, in the notation of Definition 2.3 and Corollary 2.7, we have \(\tilde{N} = W_{[\perp v]} = \langle W_0, w\rangle \neq W_0\). But \(q\) centralizes \(S\), so \(w\) acts trivially on \(s\), and hence \(\tilde{N} = \langle W_0|_s, w|_s\rangle = W_0|_s\). By Theorem 1.3.2, then, \((G, H)\) cannot be isotropy-formal.

2.5. Recapitulation

We have now assembled a long list of reductions from the generic case to a highly constrained one, which we summarize as a proof of the algorithm.

Proof of Theorem 1.4.

1. By Theorem 2.1, isotropy-formality of \((G, K)\) is equivalent to that of \((G, H_S)\).

2. The evenness of \(d\) follows since exactness of \((2.15)\) implies the dimensions of \(H^{\text{even}}(G/H_S)\) and \(H^{\text{odd}}(G/H_S)\) are equal. By Theorem 2.28, if \(\pi_1(G/H_S)\) is infinite, then \((G, H_S)\) is isotropy-formal.
3. The case analysis is Theorem 1.3. By Proposition 2.33, isotropy-formality of \((G, H_S)\) is equivalent to that of the effective quotient \((\overline{G}, \overline{H})\). By Proposition 2.32, \((\overline{G}, \overline{H})\) is again a maximal regular pair, and by definition, \((\overline{G}, \overline{H})\) is a rational sphere product pair if and only if \((G, H_S)\) is.

4. This is a rational sphere product pair. By Proposition 2.30, we can find a pair \((\tilde{G}, \tilde{H})\) covering \((\overline{G}, \overline{H})\) with \(\pi_1(\tilde{G}/\tilde{H}) = 0\) and \(\tilde{H} = \overline{H}\). This is again a maximal regular pair and a rational sphere product pair by Proposition 2.25. In Section 3.4.2, we construct the list of irreducible rational sphere product pairs \((\tilde{G}, \tilde{H})\) with simply-connected \(\tilde{G}/\tilde{H}\) in Table 1.6. This list includes the irreducible maximal pairs which are rational sphere product pairs. If \((\tilde{G}, \tilde{H})\) is not in the table, then it is reducible, and by Corollary 2.47, it is not isotropy-formal. If \((\tilde{G}, \tilde{H})\) is in the table, it is irreducible; in Section 4 we determine which entries of the table are isotropy-formal through a case analysis hinging on Theorem 1.3 and Proposition 2.11.4.

3. Homogeneous products of rational spheres

The previous section should make it clear that we need to classify irreducible rational sphere product pairs, which we will accomplish in this section.

Discussion 3.1 (Classes of cases). Up to a bounded error (a handful of missing and spurious cases, which we will repair and exclude in detail in Discussion 3.24), this classification is already present in the work of Kramer and his students in the cases of

- \(S^n \times S^m\) for \(n\) odd, \(m\) even and \(n > m \geq 4\) (Kramer [Kr]),
- \(S^n \times S^{n'}\) for odd \(n, n' \geq 3\) (Kramer [Kr]),
- \(S^n \times S^2\) for odd \(n > 2\) (Wolfro [Wfm]), and
- \(S^1 \times S^\ell\) for any \(\ell\) (Bletz-Siebert [BlSo2]).

Beyond examining this work, it thus remains only for us to analyze the cases of

- \(S^n \times S^m\) for \(n\) odd, \(m\) even and \(m > n \geq 3\), which we need, and
- \(S^m \times S^{m'}\) for \(m, m'\) both even, which we include for completeness.

We will prove in Section 3.3 that the pairs in the two previously unexamined cases are all covered by products of two homogeneous rational cohomology spheres, so that we do not find any new classes of example.

Our analysis and repair of the classification of rational sphere product pairs is greatly aided by the doctoral dissertation of Kamerich [Kam], which classifies the irreducible pairs \((G, H)\) such that \(G/H\) is homeomorphic to the product of two spheres, but which the other authors named above seem not to use.\(^{10}\) Although Kamerich’s goals differ from ours, the rational classification can be extracted with some additional analysis from his preliminary results assembling a list of candidates, which includes a number of cases not considered by Kramer; see Discussion 3.23.

\(^{10}\) Kramer also classifies the irreducible pairs such that \(G/H\) has the integral cohomology of a product of spheres [Kr, pp. x–xi]—in fact, this is his main goal of the first part of his monograph—although he is missing some cases implied by Kamerich’s classification, which was not accessible to him at the time [personal communication].
In the first subsection to follow, we state what is known in the \( n = 1 \) case. In the next two subsections we will extend the classification to cover the new cases, such as they are, and in the subsequent, much longer, subsection, we will recapitulate, revise, and verify the existing classification.

### 3.1. The case of \( S^1 \times S^\ell \)

By Theorem 2.28, we do not need a full classification of rational sphere product pairs with \( n = 1 \), but for completeness we recount what is known. As we have seen in Theorem 2.28, in this case any such pair covers a pair of the form \((G', H') \times (S^1, 1)\) with \(G'/H'\) a rational cohomology sphere, so the classification of such (effective, resp. irreducible) pairs up to covers reduces to the classification of (effective, resp. irreducible) pairs \((G', H')\) with \(G'/H'\) a rational cohomology sphere up to covers. These are completely known, and tabulated in Tables 3.16 to 3.18.

Bletz-Siebert has also studied pairs \((G, H)\) with the rational homotopy of \( S^1 \times S^\ell \) and \( H \) not necessarily connected, producing something a bit finer-grained. We quote three such results to give the flavor.

**Theorem 3.2 (Bletz-Siebert [BlSz2, Thm. 2.5.11]).** Let \((G, H)\) be a pair of compact Lie groups, \( G \) connected, such that \( \pi_1(G/H) \cong \mathbb{Z} \) and \( \pi_*(G/H; Q) \cong \pi_*(S^1 \times S^\ell; Q) \) with \( \ell \geq 2 \). Then component group \( \pi_0(H) \) is cyclic and there is a circle subgroup \( C \leq \mathbb{Z}(G)^0 \) complementary to the commutator subgroup \([G, G] \) such that the restricted action of \( G' = [G, G] \) on \( G/H \) is transitive with stabilizer \( H' = [G, G] \cap H = (G' \cap H)^0 \), and \( G/H \) is finitely covered by \( S^1 \times [G, G]/H' \).

**Theorem 3.3 (Bletz-Siebert [BlSz2, Lem. 6.3.6]).** Let \((G, H)\) be a pair of compact, connected Lie groups such that \( \pi_*(G/H; Q) \cong \pi_*(S^1 \times S^\ell; Q) \) with \( \ell \geq 3 \). Then there are a circle subgroup \( C \leq \mathbb{Z}(G)^0 \) and a simple group \( G' \leq G \) such that the virtual direct product \( G' \cdot C \leq G \) acts transitively on \( G/H \), with stabilizer \( (G' \cap H)^0 \) either simple or trivial.

**Theorem 3.4 (Bletz-Siebert [BlSz2, Thm. 6.3.7]).** Let \((G, H)\) be an irreducible pair of compact Lie groups such that \( G \) is connected, \( \pi_1(G/H) \) is torsion-free, and \( \pi_*(G/H; Q) \cong \pi_*(S^1 \times S^\ell; Q) \) with \( \ell \geq 3 \). Then \( \pi_0(H) \) is cyclic, \([G, G] \) is simple, \( \mathbb{Z}(G)^0 \) is a circle, \( H^0 \) is \( H \cap [G, G] \), and \([G, G]/H^0 \) is a simply-connected rational cohomology \( S^\ell \).

### 3.2. Reduction to the case \( \text{rk} \pi_3 \leq 2 \)

For the general discussion of rational sphere product pairs \((G, H)\), we require a result comparing the ranks and degrees of \( G \) and \( H \).

**Definition 3.5.** By the *degrees* of a connected Lie group \( G \) we mean the multiset \( \deg G \) of degrees \( \{x\} \) of generators of the exterior algebra \( H^*(G; \mathbb{Q}) \cong \Lambda[\mathbb{Z}] \). These can be enumerated as the degrees of a homogeneous basis of the graded vector space \( PG \) of primitives of the Hopf algebra \( H^*(G; \mathbb{Q}) \).

**Proposition 3.6 ([Kam, §9], cf. Kramer [Kr, Thm. 3.11]).** Let \((G, H)\) be a rational sphere product pair. Then \( \text{rk} G = \text{rk} H + 1 \) and one of the following two possibilities holds:

a) \( n \neq m - 1 \) and \( \deg G \setminus \deg H = \{n, 2m - 1\} \), while \( \deg H \setminus \deg G = \{m - 1\} \);
b) \( n = m - 1 \in \text{deg } G \cap \text{deg } H \) and \( \text{deg } G = \text{deg } H \cap \{2m - 1\} \).

We can briefly reproduce the proof in Kamerich’s dissertation.

**Proof.** It is a result of Onishchik [On94, Rmk., p.206][On63, Thm. 1] following from uniqueness of the minimal model that if homogeneous spaces \( G/H \) and \( G'/H' \) with \( G, G', H, H' \) compact, connected have the same real homotopy type, the quotients \( p(G)/p(H) \) and \( p(G')/p(H') \in \mathbb{Q}(t) \) of Poincaré polynomials are equal. Taking \( G' = SO(n+1) \times SO(m+1) \) and \( H' = SO(n) \times SO(m) \), we have

\[
\frac{\prod_{j \in \text{deg } G} (1 + t^j)}{\prod_{k \in \text{deg } H} (1 + t^k)} = \frac{p(G)}{p(H)} = \frac{p(G')}{p(H')} = (1 + t^m)^{1 + t^{2m-1}},
\]

or simply \( 1 + t^{2m-1} \) if \( n = m - 1 \), giving the result. \( \square \)

**Remark 3.7.** While Proposition 3.6 rules out potential rational sphere product pairs very efficiently, it is not a sufficient condition. For example, Kramer’s lists [Kr, 5.12. p. 60] include the pair \( (G, H) = (SO(8), SO(3) \times SO(5)) \), which has

\[
\begin{align*}
\text{deg } G & = \{3, 7, 7, 11\}, \\
\text{deg } H & = \{3, 3, 7\}, \\
\text{deg } G \setminus \text{deg } H & = \{7, 11\}, \\
\text{deg } H \setminus \text{deg } G & = \{3\}.
\end{align*}
\]

This numerical data satisfies the conclusion of Proposition 3.6 with \( m = 4 \) and \( n = 11 \), but the cohomology ring of the oriented real Grassmannian \( G_3(\mathbb{R}^8) = G/H \) is actually \( \mathbb{Q}[p_1]/\langle p_1 \rangle^3 \otimes \Lambda[\eta] \) with \( |p_1| = 4 \) and \( |\eta| = 7 \); the relevant arithmetic of degrees is that \( 11 = 3m - 1 \), the differential \( d_{12} \) of the degree-11 generator of \( H^*SO(8) \) cancelling \( p_1^3 \in H^{12}(BH) \) in the Serre spectral sequence of \( G \to G/H \to BH \), while the fact \( n = 7 = 2m - 1 \) is a red herring.

We are thus forced to compute \( H^*(G/H) \) for all pairs \( (G, H) \) in the existing classification to verify that they truly belong. Fortunately, we will find we only need to exclude a few cases.

Proposition 3.6 enables us to make a useful reduction on \( \text{rk } G \).

**Proposition 3.8.** Suppose \( (G, H) \) is a rational sphere product pair with \( m, n \geq 2 \). Then there is a closed, connected, semisimple normal subgroup \( \bar{G} \leq G \) acting transitively on \( G/H \) and \( \text{rk } \pi_3 \bar{G} \leq 2 \). This \( \bar{G} \) can be taken to act irreducibly.

**Proof.** This is Kramer’s Proposition 3.14 [Kr], minus his hypothesis that \( n > m \). It turns out this hypothesis is unnecessary: examining his proof, he uses it only in his observation that \( \dim \ker(PG \to PH) = 2 \), a tacit reference to his Theorem 3.11; but this is subsumed in Kamerich’s Proposition 3.6 [Kam], which does not require the dimension hypothesis. \( \square \)

**Remark 3.9.** Since \( \bar{G} \) is semisimple, we have \( \pi_1 \bar{G} \) finite and can replace \( \bar{G} \) with its universal cover; \( \text{rk } \pi_3 \bar{G} \) is just the number of simple factors of this cover. If \( \text{rk } \pi_3 \bar{G} = 2 \), then \( \bar{G} \) will be a virtual direct product of two simple groups.
3.3. The case with \( m > n \geq 3 \) and the case with both \( m, m' \) even

In this subsection, we show that when a homogeneous space has the rational homotopy of \( S^n \times S^m \) with \( m \) even, \( n \) odd, and \( m > n \geq 3 \), or the rational homotopy of \( S^m \times S^{m'} \) with both \( m, m' \) even, it is a virtual product of homogeneous rational cohomology spheres.

We continue to write \( G^0 \) for the identity component of a topological group \( G \).

**Theorem 3.10.** Let \( (G, H) \) be an irreducible rational sphere product pair with \( m > n \geq 3 \). Then \( G = K_1 \cdot K_2 \) is a virtual direct product of two simple subgroups and \( H = (H \cap K_1)^0 \cdot (H \cap K_2)^0 \) is also a virtual direct product. Hence \( G/H \) is a virtual direct product of homogeneous rational cohomology spheres of dimensions \( n \) and \( m \).

This says, in essence, that this newly considered class of cases contains nothing morally new. The proof will be given in several steps.

**Lemma 3.11.** Under the assumptions of Theorem 3.10, both \( G \) and \( H \) are semisimple and \( \text{rk} \, \pi_3 H \leq \text{rk} \, \pi_3 G \leq 2 \).

**Proof.** By Proposition 3.8, we know \( G \) contains a connected, semisimple normal subgroup \( \overline{G} \trianglelefteq G \) with \( \text{rk} \, \pi_3 \overline{G} \leq 2 \) and acting transitively on \( G/H \). Since we have assumed \( G \) itself acts irreducibly, we must have \( G = \overline{G} \); hence \( G \) is semisimple with \( \text{rk} \, \pi_3 G \leq 2 \).

By assumption, \( \pi_1 (G/H) \otimes \mathbb{Q} \) is trivial, so \( \pi_1 (G/H) \) is finite. Let \( \widetilde{G/H} \) be its universal cover, \( \tilde{G} \) the universal cover of \( G \), and \( \tilde{H} < \tilde{G} \) the stabilizer of a point of \( G/H \) lying over \( 1H \in G/H \). Then \( \tilde{H} \) is a finite-sheeted cover of \( H \) as well. The long exact homotopy sequence of the fibration \( \tilde{H} \to \tilde{G} \to G/H \) contains the subsequence \( \pi_2 (G/H) \to \pi_1 \tilde{H} \to \pi_1 \tilde{G} = 0 \), implying \( \pi_2 (G/H) \to \pi_1 \tilde{H} \) is surjective. As we have assumed \( m > n \geq 3 > 2 \), we have \( \pi_2 (G/H) \otimes \mathbb{Q} \cong \pi_2 (G/H) \otimes \mathbb{Q} = 0 \), so \( \pi_2 (G/H) \) is finite. But then by surjectivity, \( \pi_1 \tilde{H} \) and hence \( \pi_1 H \) are finite as well, so that \( H \) too is semisimple.

We also have the exact fragment

\[
\pi_4 \overline{G} \to \pi_4 (G/H) \to \pi_3 H \to \pi_3 G \to \pi_3 (G/H) \to \pi_2 (G/H). \quad (3.12)
\]

Now we show by contradiction that \( \text{rk} \, \pi_3 H \leq \text{rk} \, \pi_3 G \). Assume instead that \( \text{rk} \, \pi_3 H > \text{rk} \, \pi_3 G \). Then by exactness of (3.12) we have \( \text{rk} \, \pi_4 (G/H) \geq 1 \) and hence \( m = 4 \). Since we have assumed \( m > n \geq 3 \), this forces \( n = 3 \) and hence \( \text{rk} \, \pi_3 (G/H) = 1 = \text{rk} \, \pi_4 (G/H) \). But then by exactness of (3.12) again, we have \( \text{rk} \, \pi_3 \tilde{H} = \text{rk} \, \pi_3 G \), contradicting our assumption.

**Lemma 3.13.** Under the assumptions of Theorem 3.10, we have \( \text{rk} \, \pi_3 G = 2 \).

**Proof.** By Lemma 3.11, we have \( \text{rk} \, \pi_3 G \leq 2 \). If we had \( \text{rk} \, \pi_3 G = 1 \), or in other words if \( G \) were simple, then \( H \) would also be simple by Lemma 3.11. But we are able to rule out all examples of pairs \( (G, H) \) of simple groups such that \( H^*(G/H) \) is of the requisite form \( H^*(S^n \times S^m) \) using the necessary condition of Proposition 3.6 and working through the Killing–Cartan classification. We have no particular insight as to why such examples cannot occur, so such a proof is just a verification, either manual or (better) automated, and not particularly rewarding. We list some simplifications that make the verification process finite:

- For each simple \( G \), there are only finitely many simple \( H \) with \( \ell := \text{rk} \, H = \text{rk} \, G - 1 \).
• For each $\ell \geq 9$ there are only 3 types of $G$ and $H$ to consider.

• The gaps in degrees for Killing–Cartan types $BC_\ell$ and $D_\ell$ are generically 4, while the gaps for type $A_\ell$ are 2. Since Proposition 3.6 implies the cardinality of $(\deg G \setminus \deg H) \cup (\deg H \setminus \deg G)$ is at most 3, examples with $G = A_{\ell+1}$ and $H \in \{BC_\ell, D_\ell\}$ or $G \in \{BC_{\ell+1}, D_{\ell+1}\}$ and $H = A_\ell$ are impossible for $\ell \geq 4$.

• The degrees for the cases when $G$ and $H$ both lie in the same one of the infinite families $A$, $BC$, and $D$ correspond to rational cohomology spheres.

• For $G = D_\ell$ and $H = BC_{\ell-1}$, one has $\deg G = \deg H \cup \{2\ell - 1\}$, corresponding to a rational homotopy sphere for $H = B_{\ell-1}$ and not actually possible for $H = C_{\ell-1}$.

• For $G = BC_\ell$ and $H = D_{\ell-1}$, one has $\deg G \setminus \deg H = \{4\ell - 1, 4\ell - 5\}$ and $\deg H \setminus \deg G = \{2\ell - 1\}$, unless some of these degrees are equal. If that is the case, then $2m - 1 = 4\ell - 1$ and $n = m - 1 = 4\ell - 5 = 2\ell - 1$, so that $\ell = 2$, but the rational cohomology calculations for $Sp(2)/SO(2)$ and $SO(5)/SO(2)$ yield the cohomology of $S^2 \times S^7$ after all, contradicting $m > n$. If the numbers are distinct, then $m = 2\ell$, so $n = 4\ell - 5 > m$ in all but finitely many cases.

• The finitely many cases where $\deg G = \deg H \cup \{2m - 1\}$ and $G$ and $H$ do not lie in the same infinite family all correspond to cases where $G/H$ is a rational cohomology $S^{2m-1}$.

Proof of Theorem 3.10. By Lemmas 3.11 and 3.13, we see $G$ and $H$ are both semisimple and that $\text{rk}\, \pi_3 H \leq \text{rk}\, \pi_3 G = 2$. Write $G = K_1 \cdot K_2$ with $K_j$ simple and normal in $G$, and let $H_j$ be the identity component of $H \cap K_j$. Each $H_j$ is normal in $H$, so there is a connected complement $H_0$ to $H_1 H_2$. As $\text{rk}\, \pi_3 H \leq 2$, this implies at least one of the three $H_j$ must be trivial.

We claim the trivial factor must be $H_0$. Otherwise, the trivial factor is one of $H_1$ or $H_2$, say the latter. The compositions of Lie algebra maps $\mathfrak{h}_0 \hookrightarrow \mathfrak{g} \rightarrow \mathfrak{k}_j$ must be injective for $j \in \{1, 2\}$. Then $H_0$ lies in $K_1$ or $K_2$. Writing the corresponding lowercase Roman letters for ranks of Lie groups, we get $k_1 \geq h_0 + h_1$ and $k_2 \geq h_0$. As

$$k_1 + k_2 = g = h + 1 = h_0 + h_1 + 1,$$

arithmetic then shows $k_2 = h_0 = 1$, meaning $\mathfrak{g}_2 \cong \mathfrak{h}_0$ and $K_2 \cong H_0$ owing to semisimplicity. But this would mean $K_1$ acted transitively on $G/H = K_1 h_0 / H_1 H_0$, violating our assumption that the $G$-action on $G/H$ was irreducible. Thus $H_0$ is trivial.

Since $H$ is assumed to be connected, we thus have $H = H_1 \cdot H_2$. Writing $G \cong (K_1 \times K_2)/F$ for $F$ a finite central subgroup, we see $H \cong (H_1 \times H_2)F/F$, so $K_1/H_1 \times K_2/H_2 \twoheadrightarrow G/H$ is a finite-sheeted cover with abelian fiber $F(H_1 \times H_2)/(H_1 \times H_2)$. Since $F$ is central in $K_1 \times K_2$, the covering transformations are given by $(x_1, x_2)(H_1 \times H_2) \cdot f(H_1 \times H_2) = f \cdot (x_1, x_2)(H_1 \times H_2)$, the left action of $f \in K_1 \times K_2$, and hence are homotopic to the identity map since $K_1 \times K_2$ is path-connected. Thus the covering induces an isomorphism

$$H^*(G/H) \cong H^*(\left(\begin{array}{c} K_1 \\ H_1 \end{array}\right) \times \left(\begin{array}{c} K_2 \\ H_2 \end{array}\right))^{\Gamma(H_1 \times H_2)} = H^*(\left(\begin{array}{c} K_1 \\ H_1 \end{array}\right) \times \left(\begin{array}{c} K_2 \\ H_2 \end{array}\right)).$$

By the Künneth formula, $K_1/H_1$ and $K_2/H_2$ are rational cohomology spheres.  

$\square$
Remark 3.14. Kamerich [Kam, §13] conducts a similar analysis of ranks but with the stronger hypotheses that \( G \) is a product of two simple groups and \( G/H \) is homeomorphic to a product of two spheres, and without the hypothesis \( m > n \).\textsuperscript{11} His analysis in the subsequent sections finds many examples, all of the form \( S^2 \times S^n \) for \( n \) odd or \( S^n \times S^m \) for both \( n \) and \( m \) odd.

Next we consider the case of \( m, m' \) both even.

Theorem 3.15. Let \( (G, H) \) be a pair of compact, connected Lie groups such that \( G \) acts effectively on \( G/H \) and \( G/H \) has the rational homotopy type of \( S^m \times S^{m'} \) with \( m, m' \) both even. Then \( G/H \) is homeomorphic to a direct product of two homogeneous spheres of dimensions \( m \) and \( m' \).

Proof. The Euler characteristic \( \chi(G/H) \) is 4, so by a theorem of Wang [W, Thm. I, p. 927] on homogeneous spaces of positive Euler characteristic, \( G/H \) is homeomorphic to the direct product of homogeneous spaces of compact simple groups. There can a priori be arbitrarily many such factors with \( \chi = 1 \) and must also be either exactly two factors with \( \chi = 2 \) or one with \( \chi = 4 \). We will rule out the former and latter possibilities.

Wang’s proof descends from \( G \) to its effective quotient \( \overline{G} \) in Homeo \( G/H \), which is a direct product of centerless simple groups \( K_j \), notes \( \overline{H} \) is connected because \( H \) is, and argues \( \overline{H} \) is the direct product of subgroups \( H_j < K_j \) which then are connected as well. We cannot have any factor \( K_j/H_j \) with \( \chi(K_j/H_j) = 1 \), because \( 1 = \chi(K_j/H_j) = |W_{K_j}|/|W_{H_j}| \) would imply \( K_j = H_j \).

Thus if \( K_j/H_j \) is a factor with \( \chi = 4 \), then \( K_j/H_j \) is actually all of \( \overline{G}/\overline{H} \), so \( \overline{G} \) is simple. We will show this too is impossible by examining \( H^*(\overline{G}/\overline{H}) \cong H^*(G/H) \) and showing it cannot be isomorphic to \( H^*(S^m \times S^{m'}) \).

Assume first \( H \) is a maximal closed, connected subgroup of \( K \). Then by Borel and de Siebenthal’s classification [BodS49, p. 219], since \( |W_{\overline{G}}|/|W_{\overline{H}}| = \chi(\overline{G}/\overline{H}) = 4 \), the type of \((\overline{G}, \overline{H})\) must be one of \((A_3, A_2 \times S^1), (C_4, C_3 \times C_1), \) and \((C_2, A_1 \times S^1)\), corresponding respectively to the homogeneous spaces \( \mathbb{C}P^3, \mathbb{H}P^3, \) and \( G_2(\mathbb{R}^5) \) or \( \text{Sp}(2)/(\text{Sp}(1) \oplus U(1)) \). But in each of these cases the cohomology ring is a truncated polynomial ring.

Now assume instead there is an intermediate closed, connected subgroup \( \overline{K} \) between \( \overline{G} \) and \( \overline{H} \). Then one must have \( |W_{\overline{G}}|/|W_{\overline{K}}| = 2 = |W_{\overline{K}}|/|W_{\overline{H}}| \). By Borel–de Siebenthal again, as Borel notes [Bog49, p. 586], the pairs \((\overline{G}, \overline{K})\) and \((\overline{K}, \overline{H})\) must be of the types \((B_2, D_r)\) or \((G_2, A_2)\), but \( \overline{K} \) cannot be simultaneously of types \( D_2 \) and \( G_2 \), nor simultaneously of types \( A_2 \) and \( B_2 \).

So we may conclude that \( \overline{G}/\overline{H} \) is the product of homogeneous spaces \( K_1/H_1 \) and \( K_2/H_2 \) of simple groups, both factors of Euler characteristic two. But Borel observes that pairs of this type give rise to honest spheres.

\[ \square \]

3.4. The classifications of irreducible actions

Having shown that the only irreducible rational sphere product pairs with even \( m > n \) odd are covered by products of homogeneous rational cohomology spheres, it remains to evaluate and update the existing classifications for \( n > m \).

\textsuperscript{11} He coins the charming term \textit{auletic} for a pair leading to a space of the form \((K_1/H_1) \times (K_2/H_2)\), from the ancient Greek \( \alphaυλός \) (\textit{aulos}), a wind instrument with two pipes, one keyed by each hand. An online search for Kamerich finds in later life he was a teacher of mathematics as well as a longtime bassoonist for the Arnhem Symphony Orchestra and a member of a local Renaissance music ensemble. A memorial for him in 2017 was titled “His life was music.”
Table 3.16: Irreducible pairs \((G,H)\) with \(\pi_1(G/H) = 0\) and \(H^*(G/H) \cong H^*(S^{\text{even}})\)

| \(G\)            | \(H\)          | \(\dim G/H\) | \(G/H\)       |
|-------------------|-----------------|---------------|----------------|
| \(\text{SO}(2k+1), \ k \geq 1\) | \(\text{SO}(2k)\) | 2\(k\)     | \(S^{2k}\)     |
| \(G_2\)          | \(\text{SU}(3)\) | 6            | \(S^6\)        |
| \(\text{Sp}(2)\) | \(\text{Sp}(1) \times \text{Sp}(1)\) | 4            | \(S^4 = \text{Spin}(5)/\text{Spin}(4)\) |

Table 3.17: Odd-dimensional irreducible rational sphere pairs \([\text{On}63, \text{Table 2}, \text{p. 457}]\) \([\text{On}94, \text{Table 10, p. 265}]\) \([\text{Bes}78, \text{Thm. F.7.50-7.54, p. 195-196}]\) \([\text{Bes}87, \text{Ex. B.7.13, p. 179}]\) \([\text{Kr}, \text{p. 64-66}]\) \([\text{KaZ}, \text{Table III, p. 154}]\)

| \(G\)            | \(H\)          | \(Z_G(H)^0\) | \(H_S\)       | \(\dim G/H\) | \(G/H\)       |
|-------------------|-----------------|---------------|---------------|---------------|---------------|
| \(\text{SU}(k+1), \ k \geq 2\) | \(\text{SU}(k)\) | \(\U(1)\)     | \(H\)         | 2\(k+1\)     | \(S^{2k+1}\)  |
| \(\text{SU}(4)\) | \(\text{Sp}(2)\) | 1             | \(\text{SU}(2) \oplus \text{SU}(2)\) | 5             | \(S^5 = \text{Spin}(6)/\text{Spin}(5)\) |
| \(\text{SU}(3)\) | \(\text{SO}(3)_4\) | 1             | \(\text{SU}(2)\) | 5             | \(\text{Wu manifold}\) |
| \(\text{SO}(2k+1), \ k \geq 3\) | \(\text{SO}(2k-1)\) | \(\text{SO}(2)\) | \(H\)         | 4\(k-1\)     | \(V_2(\mathbb{R}^{2k+1})\) |
| \(\text{Spin}(9)\) | \(\text{Spin}(7)\) | 1             | \(\text{SU}(3)\) | 7             | \(S^7\)       |
| \(\text{Spin}(5) \cong \text{Sp}(2)\) | \(\text{SU}(2) \cong \text{Sp}(1)\) | \(\text{SU}(2)^+\) | \(H\)         | 7             | \(S^7 = \text{Sp}(2)/\text{Sp}(1)\) |
| \(\text{Sp}(k), \ k \geq 1\) | \(\text{Sp}(k-1)\) | \(\text{Sp}(1)\) | \(H\)         | 4\(k-1\)     | \(S^{4k-1}\)  |
| \(\text{Sp}(2) \cong \text{Spin}(5)\) | \(\text{SU}(2)_1 \oplus \mathbb{R}^{p_{3\lambda_1}}\) | 1             | \(i_{3,1} \U(1)\) | 7             | \(\text{Berger 7-space}\) |
| \(\text{Sp}(2)\) | \(\Delta \text{Sp}(1)_2 \text{ or } \text{SU}(2)_2\) | \(\text{SO}(2)\) | \(H\)         | 7             | \(V_2(\mathbb{R}^7) \cong \text{SO}(5)/\text{SO}(3)\) |
| \(\text{SO}(2k), \ k \geq 3\) | \(\text{SO}(2k-1)\) | 1             | \(\text{SO}(2k-2)\) | 2\(k-1\)     | \(S^{2k-1}\)  |
| \(G_2\)          | \(\text{SU}(2)_1 \oplus \mathbb{R}^{p_{\lambda_1}}\) | \(\text{SU}(2)_3\) | \(H\)         | 11            | \(V_2(\mathbb{R}^7)\) |
| \(G_2\)          | \(\text{SU}(2)_3 \oplus \mathbb{R}^{p_{\lambda_1}} + \mathbb{R}^{p_{2\lambda_1}}\) | \(\text{SU}(2)_1\) | \(H\)         | 11            | \(\text{V}(2)\) |
| \(G_2\)          | \(\text{SO}(3)_4 \oplus \mathbb{R}^{p_{2\lambda_1}}\) | 1             | \(i_{3,1} \U(1)\) | 11            | \(\text{V}(2)\) |
| \(G_2\)          | \(\text{SO}(3)_{28} \oplus \mathbb{R}^{p_{6\lambda_1}}\) | 1             | \(i_{3,1} \U(1)\) | 11            | \(\text{V}(2)\) |

3.4.1. Sphere pairs

We first collect simply-connected irreducible rational sphere pairs in Tables 3.16 and 3.17; notation is explained in Remark 3.19. The computations of \(Z_G(H)^0\) and \(H_S\) (as defined in Definition 2.9) are not part of the standard table. We tabulate them here and will use them in Section 4. As we are only concerned with rational homotopy type, when \(F\) is a finite subgroup of \(H\) acting trivially on \(G/H\), we do not distinguish between \((G,H)\) and \((G/F,H/F)\) in our tabulation.

Remark 3.19 (Notation for Table 3.17).

- The block-diagonal product of two matrix groups is denoted with \(\oplus\), according with the standard notation \(h_1 \oplus h_2: V_1 \oplus V_2 \rightarrow V_1 \oplus V_2\), and the trivial \(1 \times 1\) block-diagonal subgroup of a matrix group is denoted \([1]\). Thus, for example, \(\text{Sp}(1) \oplus [1]\) is the subgroup of \(\text{Sp}(2)\) with elements \(\text{diag}(q,1)\) for \(q \in \text{Sp}(1)\).

- For rational cohomology spheres in the list which are not spheres, the symbols after the colon denote representations named under the conventions of Chapter 4 of Kramer’s Habilitationsschrift [Kr, Ch. 4], identifying specific subgroups \(H\) of \(G\) isomorphic to the group
Table 3.18: Odd-dimensional virtually effective but reducible rational sphere pairs

| \( G \)                  | \( H \)                  | \( \dim G/H \) | \( G/H \)                  |
|--------------------------|--------------------------|----------------|--------------------------|
| \( U(k+1), \ k \geq 2 \) | \( U(k) \)              | \( 2k+1 \)     | \( S^{2k+1} \)            |
| \( \text{SO}(2k+1) \times \text{SO}(2) \ k \geq 3 \) | \( \text{SO}(2) \Delta \text{SO}(2) \) | \( 4k-1 \)     | \( V_2(\mathbb{R}^{2k+1}) \) |
| \( \text{Sp}(k) \times \text{Sp}(1) \ k \geq 1 \) | \( \text{Sp}(k-1) \Delta \text{Sp}(1) \) | \( 4k-1 \)     | \( S^{4k-1} \)            |
| \( \text{Sp}(k) \times \text{U}(1) \ k \geq 1 \) | \( \text{Sp}(k-1) \Delta \text{U}(1) \) | \( 4k-1 \)     | \( S^{4k-1} \)            |
| \( \text{Sp}(2) \times \text{SO}(2) \) | \( \text{SU}(2) \Delta \text{SO}(2) \) | \( 7 \)         | \( V_2(\mathbb{R}^7) \cong \text{SO}(5)/\text{SO}(3) \) |
| \( G_2 \times \text{SU}(2) \) | \( \text{SU}(2) \Delta \text{SU}(2) \) | \( 11 \)        | \( V_2(\mathbb{R}^7) \)            |
| \( G_2 \times \text{U}(1) \) | \( \text{SU}(2) \Delta \text{U}(1) \) | \( 11 \)        | \( V_2(\mathbb{R}^7) \)            |
| \( G_2 \times \text{SU}(2) \) | \( \text{SU}(2) \Delta \text{SU}(2) \) | \( 11 \)        | \( V_2(\mathbb{R}^7) \)            |
| \( G_2 \times \text{U}(1) \) | \( \text{SU}(2) \Delta \text{U}(1) \) | \( 11 \)        | \( V_2(\mathbb{R}^7) \)            |

preceding the colon. We are able to mostly avoid engaging directly with these representations, but list them for completeness. The subscript attached to a subgroup \( H \) denotes the Dynkin index of \( H \) in \( G \), which can be identified with the order of the finite group \( \pi_3(G/H) \) [On94, p. 257]. The Dynkin indices in the above table are taken from Onishchik [On63, Table 2, p. 457] and date back to Dynkin’s work [D]. Those \( H \) without subscripts (except for \( \text{Sp}(0) \vartriangleleft \text{Sp}(1) \), which has no Dynkin index) have Dynkin index 1 in the ambient group. The subscript “2” of \( G_2 \), though, is just its rank; its Dynkin index in \( \text{Spin}(7) \) is 1.

- For the pair \((\text{Spin}(5), \text{SU}(2))\), the group denoted \( \text{SU}(2)^\perp \) is the copy of \( \text{SU}(2) \) such that \( \text{SU}(2) \cdot \text{SU}(2)^\perp = \text{Spin}(4) \vartriangleleft \text{Spin}(5) \), or equivalently \( \text{Sp}(1) \cdot \text{Sp}(1)^\perp = \text{Sp}(1) \oplus \text{Sp}(1) \vartriangleleft \text{Sp}(2) \).

- The map \( i_{p,q} : \text{U}(1) \rightarrow T^2 \), for \( p, q \in \mathbb{Z} \), denotes a parameterization by \( t \mapsto \sigma_1(t^p) \sigma_2(t^q) \) of a circle in a maximal torus \( T^2 = \sigma_1 \text{U}(1) \cdot \sigma_2 \text{U}(1) \) of \( G \). We may as well assume \( p \) and \( q \) are coprime. In \( \text{SU}(3) \), we take \( T^2 = S(\text{U}(1)^3) \) the diagonal matrices of determinant 1, and in \( \text{Sp}(2) \), we take \( T^2 = \text{U}(1) \oplus \text{U}(1) \vartriangleleft \text{Sp}(1) \oplus \text{Sp}(1) \vartriangleleft \text{Sp}(2) \), with the standard parameterizations. For \( G_2 \), we will explain our choice of \( \sigma_1 \) and \( \sigma_2 \) in Discussion 3.20; we will have \( \sigma_1 \text{U}(1) \cap \sigma_2 \text{U}(2) \approx \{ \pm 1 \} \), so that \( i_{p,q} \) is not injective, but this is fine as we are only after a systematic way of naming the image circles \( i_{p,q} \text{U}(1) \).

Note that depending on the Weyl group of \( G \), different pairs \((p, q)\) can correspond to conjugate subgroups and hence result in equivalent pairs \((G, H)\). For example, two circular groups \( i_{p,q} \text{U}(1) \) and \( i_{p',q'} \text{U}(1) \) in \( \text{Sp}(2) \) are conjugate if and only if their ordered pairs of weights \((p, q)\) and \((p', q')\) lie in the same orbit of the Weyl group action on the weight lattice, i.e., if the sets \( \{p', q'\} \) and \( \{\pm p, \pm q\} \) are equal.

**Discussion 3.20 (Discussion of examples).**

(a) The standard embedding \( \text{SO}(3) \vartriangleleft \text{SU}(3) \) has Dynkin index 4 because the Dynkin index of the composite \( \text{SO}(3) \hookrightarrow \text{SU}(3) \hookrightarrow G_2 \) is the product of the Dynkin indices of the two intermediate embeddings [On94, Prop. 9, p. 58], and the index of \( \text{SU}(3) \) in \( G_2 \) is 1 since \( \pi_3(S^6) = 0 \), while the index in \( G_2 \) of \( \text{SO}(3) \vartriangleleft \text{SU}(3) \) (which is not a maximal proper, closed,
connected subgroup of G2) must be 4 because the SO(3) subgroups of G2 all have index 4 or 28, and those of index 28 are maximal in G2 (see (e)).

(b) The embedding Spin(7) → Spin(9) is not the standard one covering SO(7) ⊕ \{1\} ⊕ 2 SO(9), but the composition of a standard embedding Spin(8) → Spin(9) with a lift \iota: Spin(7) → Spin(8) of one of the (faithful) half-spin representations of Spin(7) on \mathbb{R}^8.

We will encounter \iota again in Discussion 3.26(c).

(c) The standard embedding SO(3)_2 = SO(3) ⊕ \{1\} ⊕ 2 of SO(3) in SO(5) has Dynkin index 2 because the real Stiefel manifold V = V_2(\mathbb{R}^5) = SO(5)/SO(3) has \pi_3 SO(3) \cong \mathbb{Z}/2 [St, 25.6, p. 132]. This manifold is the total space of a bundle SO(4)/SO(3) → SO(5)/SO(3) → SO(5)/SO(4), which Kapovitch–Ziller [KaZ, p. 155] observe is diffeomorphic to Sp(1) → Sp(2)/ΔSp(1) → \mathbb{HP}^1, the unit tangent bundle of the quaternionic projective line. But ΔSp(1) is conjugate in Sp(2) to the standard subgroup SU(2)_2 [MiT, 2.14, p. 23], so Sp(2)/ΔSp(1) ≃ Sp(2)/SU(2) and both subgroups have Dynkin index 2 in Sp(2) just as SO(3) does in SO(5).

(d) The irreducible representation \mathbb{R}^{3\lambda_1} of SU(2) ≃ Sp(1) (in Kramer’s notation [Kr, Ch. 4]) is of complex dimension 4, hence of quaternionic dimension 2, and induces a nonstandard embedding SU(2)_10 of SU(2) in Sp(2). Berger [Ber, p. 237–239] studied the topological and geometric properties of the quotient Sp(2)/SU(2)_10, now called the Berger 7-space, which also appears in Onishchik’s work [On63, p. 457] as SO(5)/SO(3)_10, where the denominator subgroup is the nonstandard SO(3) subgroup of Dynkin index 10 in SO(5) arising from the representation of SO(3) on the 5-dimensional space \mathbb{R}\{xy, yz, xz, x^2 - y^2, y^2 - z^2\} of homogeneous quadratic polynomials on \mathbb{R}^3 = \mathbb{R}\{x, y, z\}. This representation is \mathbb{R}^{3\lambda_1} in Kramer’s notation. According to Berger [Ber, p. 237], one maximal torus of the embedded subgroup SU(2)_10 in Sp(2) is the diagonal circle \iota_{3,1} U(1) < U(1) \oplus U(1) < Sp(2).

(e) According to Dynkin [D, p. 411], there are four conjugacy classes of A_1 subgroups of G2, with SU(2)_1, SU(2)_3, SO(3)_4, and SO(3)_28, of which only the first two are regular. Let \alpha and \beta respectively be the short and the long simple roots of G2 with respect to a fixed maximal torus, so that ||\alpha||^2 = 1, ||\beta||^2 = 3, and -B(\alpha, \beta) = -3/2. Then \gamma = 3\alpha + 2\beta is another long root and is orthogonal to \alpha, so that \h_\alpha and \h_\gamma give an orthogonal basis of \mathbb{R}^2. According to Mayanskiy [May, p. 4], any coroot corresponding to a long root (such as \h_\gamma) is tangent to the maximal torus of an SU(2)_1 subgroup of G2 while any coroot corresponding to a short root (such as \h_\alpha) is tangent to that of an SU(2)_3 subgroup. Mayanskiy [May, p. 17] also notes that SO(3)_4 and SO(3)_28 subgroups of G2 can be found with Cartan algebras respectively spanned by 2h_{3\alpha + \beta} and 14h_{9\alpha + 5\beta}. By pairing with 2\beta + 3\alpha and \alpha, we see these vectors are respectively h_{2\beta + 3\alpha} + h_\alpha and 5h_{2\beta + 3\alpha} + h_\alpha. Dynkin shows these representative SU(2)_1 and SU(2)_3 subgroups centralize one another, yielding a virtual direct product SO(4) which is a maximal proper closed subgroup of G2 and can be chosen to have \alpha and 2\beta + 3\alpha as its simple roots. If we take \mathbb{T}^2 to be the product U(1) \cdot U(1) < SU(2)_1 \cdot SU(2)_3 = SO(4), then the maximal tori of SU(2)_1, SU(2)_3, SO(3)_4, SO(3)_28 are respectively realized as \iota_{1,0} U(1), \iota_{0,1} U(1), \iota_{1,1} U(1), \iota_{5,1} U(1).

---

\(\text{12} \) A proof not involving knowledge of G2 is possible by computing that the cohomology of the Wu manifold W = SU(3)/SO(3) is the sum of \mathbb{H}^4(W) \cong \mathbb{Z} \cong \mathbb{H}^5(W) and \mathbb{H}^2(W) \cong \mathbb{Z}/2 \cong \mathbb{H}^3(W; \mathbb{Z}/2) then taking the homotopy fiber F of a representing map W → K(\mathbb{Z}/2, 2) to kill \pi_2(W) = \mathbb{Z}/2 and finding the order of \mathbb{H}^4(F) \cong H_3(F) \cong \pi_3(F) \cong \pi_4(W) through the Serre spectral sequence of K(\mathbb{Z}/2, 1) → F → W.
**Remark 3.21 (History).** The list of pairs resulting in actual spheres is a classical result due to Montgomery–Samelson [MS] and Borel [Bo49, Thm. 3, p. 486][Bo50, Thm. 3]. Borel also showed in 1953 that a simply connected integral homology sphere is a sphere [Bo53, (4.61), p. 455], and Bredon [Bre61] showed in 1961 that the only integral homology sphere that is not a sphere is the Poincaré homology sphere SO(3)/(icosahedral group). Matsushima [Mat] showed that the infinite series in Table 3.17 were the only odd-dimensional rational homology spheres $G/H$ with (perforce simple) $G$ acting irreducibly and virtually effectively, up to possible low-dimensional exceptions, and that $G_2$ was the only possible exceptional $G$ for $H$ of type $A$ or $C$; Montgomery and Samelson had already done this for $B$ and $D$.

**Proof of centralizers and maximal regular subgroups for Table 3.17.** In most of these examples, $H$ is regular, which by Lemma 2.20 implies it is the maximal closed, connected subgroup $H_S$ containing $S$, as defined in Definition 2.9. We can usually verify this by checking $\text{rk } Z_G(H)H/H = 1$ using Proposition 2.42. Then $Z_G(H)^0H/H$ is either isomorphic to $S^1$ or a group of type $A_1$. When it is $S^1$, it is often simple to verify this directly, but it is simpler to note that were it of type $A_1$, then $G$ would admit a maximal-rank subgroup $H \times A_1$, which can be ruled out by consultation with Borel–de Siebenthal [BodS49, p. 219], and we do not spell out this verification. When the centralizer really is of type $A_1$, our approach is *ad hoc*.

- For $(\text{SU}(k+1), \text{SU}(k) \oplus [1])$, the circle $\{ \text{diag}(z, \ldots, z, z^{-k}) \}$ centralizes $H$, so $H = H_S$.
- For $(\text{SO}(2k+1), \text{SO}(2k-1) \oplus [1]_{\mathbb{G}_2})$, the circle $[1]^{\oplus 2k-1} \oplus \text{SO}(2)$ centralizes $H$, so $H = H_S$.
- For $(\text{Sp}(k), \text{Sp}(k-1) \oplus [1])$, the $A_1$ subgroup $[1]^{\oplus k} \oplus \text{Sp}(1)$ centralizes $H$, so $H = H_S$.
- For $(\text{Sp}(2), \text{SU}(2))$, the diagonal $\Delta U(1)$ centralizes $H$, so $H = H_S$.
- For $(G_2, \text{SU}(2))$, the $\text{SU}(2)$ can be of index 1 or 3, and each centralizes the other.

- For $(G_2, \text{SO}(3))$, where $H = \text{SO}(3)$ is either $\text{SO}(3)_4$ or $\text{SO}(3)_{28}$, it is known [May, Prop. 4.1] that the complement to $\mathfrak{so}(3)_C$ in the adjoint representation of $\text{SO}(3)$ on $(\mathfrak{g}_2)_C$ is a sum of nontrivial irreducible representations, so $Z_G(H)^0 = 1$, and hence $H$ is not regular. If $S$ is a maximal torus of $H$, then it is centralized by a maximal torus $T < G_2$ containing it, and so regular, and if $H_S > S$, then $H_S$ is an $A_1$ subgroup; but we have seen in Discussion 3.20(e) that up to conjugacy there are only the four such subgroups, that only the $\text{SU}(2)$ subgroups are regular, and that these $A_1$ groups have distinct maximal tori, so $H_S = S$.

- For $(\text{SU}(4), \text{Sp}(2))$, one sees manually the centralizer is $\Delta\{\pm 1\}$, so $H$ is not regular. A maximal torus of $\text{Sp}(2)$ is given by $\{ \text{diag}(z, z^{-1}, w, w^{-1}) : w, z \in U(1) \}$, which is also maximal in $H' = \text{SU}(2) \oplus \text{SU}(2)$, and $H'$ is centralized by the circle $\{ \text{diag}(z, z^{-1}, z, z^{-1}) : z \in U(1) \}$, so it is regular. In fact, $(G, H')$ is a rational sphere product pair. To see $H'$ is maximal regular, note that by Lemma 2.21, if we had $H_S > H'$, then $(G, H_S)$ would be a rational sphere pair, but consulting Table 3.17, there are no candidates.

- For $(\text{SU}(3), \text{SO}(3))$, one computes directly that the centralizer is trivial. The subgroup $S = \text{SO}(2) \oplus [1]$ is a maximal torus, which is contained in $H' = \text{SU}(2) \oplus [1]$. The circle $\{ \text{diag}(z, z, z^{-2}) \}$ centralizes $H'$, which thus is regular, and $(G, H')$ is a rational sphere pair, so by Lemma 2.20, $H' = H_S$. 

• In \((\text{Spin}(9), \text{Spin}(7))\), as we have discussed in Discussion 3.20(b), the embedding of \text{Spin}(7) in \text{Spin}(8) < \text{Spin}(9) is nonstandard, and hence is not centralized by the expected \text{Spin}(2). Consulting the (yet to come) Discussion 3.26(c), we see \text{Spin}(7) contains an \text{SU}(4) subgroup \text{H}' sharing its maximal torus and such that \((G,H')\) is a rational sphere product pair. The double-covering \text{Spin}(8) \to \text{SO}(8) takes \text{H}' isomorphically to the standard \text{SU}(4) < \text{U}(4) < \text{SO}(8), which is centralized by \Delta_4 \text{U}(1), and this circle is double-covered by a circle in \text{Spin}(8) centralizing \text{H}', which thus is regular. If we had \text{H}_5 > \text{H}', then by Lemma 2.21, \((G,H)\) would be a rational sphere pair, but consulting Table 3.17, there are no candidates.

• For \((\text{Spin}(7), G_2)\), the centralizer is \(Z(G)\) by Borel–de Siebenthal. But (see Discussion 3.26(c)), \(G_2\) shares a maximal torus with a standardly embedded \text{SU}(3) subgroup \text{H}' of the copy of \text{SU}(4) in \text{Spin}(7), and \text{SU}(3) is centralized within \text{SU}(4) by \(\{\text{diag}(z,z,z^{-3})\}\), and hence is regular. To see \text{H}' is regular, again note that \((G,H')\) is a rational sphere product pair but \((G,K)\) is not a rational sphere pair for any \(K > \text{H}'\).

• For \((\text{Sp}(2), \text{SU}(2)_{10})\), note that the maximal torus \(S = i_{3.1} \text{U}(1)\) of \text{SU}(2)_{10} is centralized by the diagonal maximal torus, hence is regular. Since \(H/S \approx S^2\), the Serre spectral sequence of \(H/S \to G/S \to G/H\) collapses, so \((G,S)\) is a rational sphere product pair. By Lemma 2.20, either \(H_5 = S\) or \((G,H_5)\) is a rational sphere pair, but there are no possibilities for \(H_5 > S\) in Table 3.17. Hence, we have \(H_5 = S\).

• For \((\text{SO}(2k), \text{SO}(2k - 1))\), the centralizer is \(\pm [1]^{\otimes 2k}\), so \text{H} is not regular, but \text{H} shares the maximal torus \(S = \text{SO}(2)^{\otimes k - 1}\) with \text{H}' = \text{SO}(2k - 2) and \((G,H')\) is a rational sphere product pair with \text{H}' centralized by \([1]^{\otimes 2k-2} \oplus \text{SO}(2)\). Again \(H' = H_5\) by lack of candidates \(K > H'\) with \((G,K)\) a rational sphere pair.

3.4.2. Sphere product pairs

Our Theorem 3.10 shows the existing classifications of Kramer [Kr, Thm. 3.15, Chs. 5–6] for \(n > m \geq 4\) and Wolfrom [Wfm, Thm. 2.1] for \(n > m = 2\) should contain all simply-connected irreducible cases.13 As mentioned earlier, this is nearly but not entirely true, since we must re-analyze the candidates from these lists to make sure they have the correct cohomology, as explained in Remark 3.7, and since there are a few omitted cases. Kamerich, in classifying homogeneous products of two spheres, first lists all the possible pairs of Lie algebras satisfying the condition of Proposition 3.6 as candidates [Kam, Table 1, p. 55–60]. His list includes all the examples in Kramer and Wolfrom’s lists, and after extensive checking, we are confident his list is complete. In this subsection we justify this conclusion, which leads to Table 1.6.

After establishing notation, we discuss how our table and the work of previous authors relate to each other in Discussion 3.23 and Discussion 3.24, elaborate on several cases in Discussion 3.26, and then finally prove that the candidates do have the correct cohomology in several representative cases, meant to convey the flavor of this somewhat lengthy verification process.

Although \(Z_G(H)\) and \(H_5\) appear in Table 1.6, we will only need them later in Section 4 to determine isotropy-formality, so we will defer their calculation until then.

---

13 Note that we must have \(n,m \geq 2\) due to the simple-connectivity hypothesis.
Notation 3.22 (for Table 1.6). This discussion will strictly cover notation per se; Discussion 3.26 will describe the less obvious embeddings of $H$ in $G$.

- In the first four lines of the table, if $Z_G(i_{p,q}U(1))^0$ contains an $A_1$ subgroup, then its complexified Lie algebra $a_1 \otimes \mathbb{C} = \mathbb{C}\{e_\beta, e_{-\beta}, h_\beta\}$ is centralized by $\mathfrak{h} = s = \mathbb{R}u$, so that $\beta(u) = 0$ and the hence the tangent line $\mathbb{R}h_\beta/i < t^2$ to its maximal torus is $B$-orthogonal to $s$. This maximal torus will be $i_{p*,q*}U(1)$, for a pair $(p^*,q^*) \in \mathbb{Z}^2$ we will determine in the proof of Theorem 1.5. We then write $(A_1)p^*,q^*$ for this $A_1$ subgroup. In the case, $p \cdot q = 1, -2$, we have written $SU(2)p+2q, -2p = SU(2)\pm3,\mp3 = SU(2)\pm1,\pm1$ and $SU(2)\pm3,0 = SU(2)\pm1,0$ and $SU(2)\pm3,0 = SU(2)\pm1,1$. In all these entries, this classification is only unique up to the assumption $s\setminus\{0\}$ meets the fundamental closed Weyl chamber. The conditions on $p$ and $q$ necessary to ensure this are left to the reader.

- In the fourth line of the table, the subpair $(K_1,H_1)$ can be any pair from Table 3.17 such that $Z_{K_1}(H_1)^0 \neq 1$ and the embedding $i_{p,q}: U(1) \to T^2$ is as defined in Remark 3.19 for $T^2 \cong U(1) \times U(1)$ the product of a maximal torus of $Z_{K_1}(H_1)$ and a maximal torus of $Sp(1)$.

- In the fifth line of the table, the pairs $(K_j,H_j)$ for $j \in \{1,2\}$ are respectively drawn from Table 3.17 and Table 3.16.

- The notation $\tilde{G}_k(\mathbb{C}^n)$ reflects that the space can be seen as the total space of a bundle over the complex Grassmannian $G_k(\mathbb{C}^n) = SU(n)/SU(k) \oplus SU(n-k)$ with fiber the circle $SU(k) \oplus SU(n-k)/SU(K) \oplus SU(n-k)$. We might think of this as an oriented complex Grassmannian by analogy with the oriented real Grassmannians.

Discussion 3.23 (Comparison with Kamerich). Recall that Kamerich wants to determine pairs $(G,H)$ with $G/H$ homeomorphic to a product of two spheres, considering all parities of dimensions for the two potential spheres. He rules out simple groups $G$ acting irreducibly on homogeneous spaces $G/H$ homeomorphic to $S^1 \times S^n$ for all $n$, to $S^2 \times S^m$ for $m \geq 2$ even [Kam, §10], through an analysis that shows there are no possibilities rationally, and to $S^2 \times S^n$ for $n \geq 3$ odd, through an analysis that leaves $G \in \{SU(3),Sp(2),G_2\}$ as the only possibilities rationally.

He applies Proposition 3.6 to determine corresponding Lie algebra pairs $(g,h)$ with $g$ simple and $h$ semisimple satisfying the degree criterion, excluding cases where the Dynkin index is not 1 and (1) $h$ is simple and $m,n \geq 5$ or (2) $g$ is exceptional, since $\pi_3(G/H) \neq 0$ is not consistent with a product of spheres unless one of the factors is $S^2$ or $S^3$. We briefly run through the cases from his resulting Table 1 [Kam, p. 55] that we exclude and our reasoning.

Case 11 is a typo, which should be $(B_3,A_1)$. The cases 13–16, all of type $(B_3,A_1 \oplus A_1)$, respectively correspond to the cases $(SO(7),SO(4))$ (which we have absorbed as $(B_3,D_2)$ in our analysis), $(SO(7),SO(3) \oplus SU(2))$, $(SO(7),SO(3)\oplus^2 \oplus [1])$, and $(Spin(7),SO(4))$, which is rationally equivalent to $\tilde{G}_3(\mathbb{R}^6)$, and which we noted in Remark 3.7 has the wrong cohomology. We write $(B_4,A_3)$, case 17, as $(SO(2k+1),SO(2k-2))$, whereas case 18, also $(B_4,A_3)$, is our $(Spin(9),SU(4))$. The cases of type $(C_3,A_1 \oplus A_1)$, comprising one unlabeled case and Kamerich’s cases 28–30, are respectively $(Sp(3),SO(3)\cdot\Delta_3Sp(1))$, $(Sp(3),Sp(1)\oplus^2 \oplus [1])$, $(Sp(3),Sp(1)\cdot SU(2)\,10)$, and $(Sp(3),Sp(1)\oplus\Delta_2Sp(1))$. Cases 36 and 37, both $(D_4,A_3)$, we have written as $(SO(8),SO(6))$.

---

14 This argument, as written, would incorrectly rule out $SU(3)$, on considering the long exact homotopy sequence of $H \to G \to G/H$, but this is due to the claim that $\pi_3(S^2) \cong \mathbb{Z}$ (probably a typo), which does not affect his argument since he only needs that this group is nonzero; of course $\pi_3(S^2) \cong \mathbb{Z}/2$. 
and \((\text{Spin}(8), \text{SU}(4))\) respectively. We do not include \((B_n, B_{n-1})\), case 19, since the Stiefel manifolds \(V_2(\mathbb{R}^{2n+1})\) are rationally \((4n - 1)\)-spheres; \((D_4, A_1 \oplus B_2)\), case 43, both embeddings of which represent \(\tilde{G}_3(\mathbb{R}^8)\); or \((G_2, A_1)\), case 48, all four subcases of which are rational spheres. Since all yield spheres, we omit the cases 20, 25, 31, 33, 39, and 40, respectively \((B_n, B_3) = (\text{Spin}(9), \text{Spin}(7)), (B_3, G_2) = (\text{Spin}(7), G_2), (C_3, B_2) = (\text{Sp}(3), \text{Sp}(2)), (C_n, C_{n-1}) = (\text{Sp}(n), \text{Sp}(n - 1)), (D_n, B_{n-2}) = (\text{SO}(4n), \text{SO}(4n - 1)),\) and \((D_4, B_3) = (\text{Spin}(8), \text{Spin}(7))\) either of the conjugacy classes of inclusions not covering \((\text{SO}(8), \text{SO}(7))\)—these two \((D_4, B_3)\) pairs are equivalent by triality.

Kamerich next deals with the possibility \(G\) is a simply-connected connected group times a torus (possibly trivial), but not simple. He is interested in virtually effective, irreducible actions, and finds the cases \((K_1, H_1) \times (K_2, H_2)\) and another case which he subjects to an analysis similar to that of our Theorem 3.10, but without the restriction on dimensions of spheres, to arrive at his Table 2 of candidates satisfying the Onishchik degree criterion. When we exclude cases with two odd dimensions, there remain only the examples \((K_1 \times \text{Sp}(1), H_1 : i_{p,q}U(1))\) for \(K_1 \in \{\text{SU}(3), \text{Sp}(2), G_2\}\) and \(H_1 \cong \text{SU}(2)\) in cases 1–4, and for \((K_1, H_1)\) equal to \((\text{SU}(k + 1), \text{SU}(k)), (\text{Spin}(2k + 1), \text{Spin}(2k - 1)),\) and \((\text{Spin}(k + 1), \text{Spin}(k))\) respectively in cases 5–7. We do not find any irreducible rational sphere product pairs not found in Kamerich’s preliminary lists.

As far as his real goal goes, pairs giving rise to genuine homogeneous spaces \(G/H = S^m \times S^n,\) for \(m\) even and \(n\) odd, he finds only product pairs, \(\text{Spin}(7)/\text{SU}(3) = \text{SO}(8)/\text{SO}(6) \cong S^5 \times S^7,\) and three classes of examples with \(G\) semisimple: the class \(\text{Sp}(2)/i_{p,q}U(1) \cong S^2 \times S^3\) with \(\text{gcd}(p, q) = 1,\) the class \((\text{SU}(2n + 1) \times \text{Sp}(1))/(\text{SU}(2n) \cdot i_{p,q}U(1)) \cong S^{4n+1} \times S^2\) with \(\text{gcd}(p, q) = 1\) and \(q|n,\) and the class \((\text{Sp}(n) \times \text{Sp}(1))/(\text{Sp}(n - 1) \cdot i_{p,q}U(1)) \cong S^{4n-1} \times S^2.\) Kamerich notes that when \(G/H\) is homeomorphic to a product of spheres, it is in fact diffeomorphic to the standard product, but also finds an example which is merely homotopy equivalent without being homeomorphic.

**Discussion 3.24** (Comparison with later work). Kramer and his students want to find pairs \((G, H)\) such that \(H^*(G/H; \mathbb{Z})\) is isomorphic to \(H^*(S^m \times S^n; \mathbb{Z})\) with \(m\) and \(n\) both odd or \(n > m\) with \(m\) even and \(n\) odd, but need to find candidates with the desired rational cohomology along the way. The union of their classifications contains most of what we need, with a few exceptions, as we have already noted. Briefly, these are as follows: the pair \((\text{SU}(4), \text{SO}(4))\) is missing, as are two of the four \((C_3, A_1 \oplus A_1)\) pairs, namely \((\text{Sp}(3), \text{Sp}(1) \oplus \Delta_2 \text{Sp}(1))\) and \((\text{Sp}(3), \text{SO}(3) \cdot \Delta_3 \text{Sp}(1))\). The case \((G_2 \times \text{Sp}(2), \text{Sp}(1) \cdot \Delta \text{Sp}(1) \cdot \text{Sp}(1))\) is mentioned in passing (p. 73) and shown to not have the correct integral cohomology, but not tabulated as having the desired rational cohomology. Also, Kramer’s pairs \((\text{Spin}(7), \text{SO}(4))\) and \((\text{SO}(8), \text{SO}(3) \times \text{SO}(5))\), as noted in Remark 3.7, must be excluded as they do not give the proper cohomology. Apart from this, our results agree.

When Kramer goes on to consider cohomology over \(\mathbb{Z}\), he finds the following examples with the integral cohomology of a product \(S^m \times S^n\) with \(m\) even less than \(n\) odd: \((\text{Spin}(9), \text{SU}(4))\) and \((\text{Spin}(10), \text{SU}(5))\), giving rise to the same quotient, two of the three cases \((\text{Sp}(3), \text{Sp}(1) \cdot \text{Sp}(1)),\) \(^{15}\) the pairs \((\text{SO}(2k), \text{SO}(2k - 2))\) giving rise to Stiefel manifolds \(V_2(\mathbb{R}^{2k})\), and \((\text{SU}(5), \text{SU}(2) \oplus \text{SU}(3))\).

The majority of our calculations of \(Z_G(H)^0\) can also be found in Kramer’s work. However, we have found that \(Z_G(H)^0\) for the pairs \((G, H) = (\text{SO}(7), \text{SO}(3) \oplus \text{SU}(2)), (\text{Sp}(4), \text{SU}(4)),\) and \((\text{Sp}(3), \text{SU}(3))\) is in fact \(U(1)\), rather than the trivial group, our computations of \(Z_G(H)^0\) for the first four lines of Table 1.6 seem to be original, and the excluded cases seem not to have been completed.

---

\(^{15}\) He does not consider the remaining case, which one can however show has \(H^8 \cong \mathbb{Z}/3.\) In fact, the pair \((\text{Sp}(3), \text{Sp}(1) \oplus \text{SU}(2)_{10})\) also appears to have torsion, namely \(H^8 \cong \mathbb{Z}/91.\)
Discussion 3.25 (General patterns).

- From Table 1.6, we see that whenever $Z_C(H)^0 \cong S^1$, we also have $H = H_5$. We can see this more directly by observing that if we could have $Z_C(H)^0 \cong S^1$ and $H < H_5$, then $(G, H_5)$ would be a rational sphere pair with $Z_C(H_5)^0 \leq Z_C(H)^0 = S^1$ a circle, and this does not occur in the classification Table 3.17, but it would be nice to see a conceptual reason for this that does not pass through some classification.

- The converse also seems to nearly hold: if $H \not\cong S^1$ and $H = H_5$, then in all but one case, we have $Z_C(H)^0 H/H \cong S^1$. Again, we do not have a direct argument why this should be.

Discussion 3.26 (Discussion of selected cases).

(a) The quotient $SU(4)/SO(4)$ is not included by Kramer [Kr], but we find it in Kamerich’s classification [Kam, Table 1, p. 55] as the unlabeled case before case 5. To compare $SU(4)/SO(4)$ with $SU(4)/((SU(2) \oplus SU(2)))$, we first note that the quotients are both simply-connected, but have nonisomorphic $\pi_2$ since $\pi_2(SU(4))$ and $\pi_2(SU(2) \oplus SU(2))$ are nonisomorphic, and different Dynkin indices, as Kamerich notes. In terms of more familiar spaces, we have

$$\frac{SU(4)}{SO(4)} \approx \frac{SU(4)/\mathbb{Z}_2}{SO(4)/\mathbb{Z}_2} \approx \frac{SO(6)}{SO(3) \oplus SO(3)} = \tilde{G}_3(\mathbb{R}^6),$$

$$\frac{SU(4)}{SU(2) \oplus SU(2)} \approx \frac{SU(4)/\mathbb{Z}_2}{(SU(2) \oplus SU(2))/\mathbb{Z}_2} \approx \frac{SO(6)}{SO(4)} = V_2(\mathbb{R}^6).$$

For explicit matrix expressions for the isomorphisms, see Yokota [Y].

(b) The quotients $Sp(3)/((Sp(1) \oplus \Delta_2 Sp(1)) \cong Sp(3)/((Sp(1) \oplus SU(2))$ and $Sp(3)/(SO(3) \cdot \Delta_3 Sp(1))$ are not included by Kramer [Kr], but are by Kamerich [Kam, Table 1, p. 58]. Here $\Delta_2 Sp(1)$ and $\Delta_3 Sp(1)$ denote the diagonally embedded copies of $Sp(1)$ in $Sp(2)$ and $Sp(3)$ respectively and $SO(3)$ is the subgroup of real matrices in $Sp(3)$. As noted in Discussion 3.20(c), $\Delta_2 Sp(1)$ is conjugate to $SU(2)$ in $Sp(2)$, explaining the above diffeomorphism. We will check both pairs have the cohomology of $S^4 \times S^{11}$ in the proof to follow.

(c) The quotients $Spin(7)/SU(3)$ and $SO(8)/SO(6)$ can be seen to both be $V_2(\mathbb{R}^8) \cong S^6 \times S^7$. For the latter, identifying $\mathbb{R}^8$ with the octonions $\mathbb{O}$, we see $SO(8)/SO(6) = V_2(\mathbb{R}^8)$ can be identified with the unit tangent bundle to the sphere $S^7$ of unit octonions. Although $S^7$ is not a group, it is a smooth H-space, and so left translation to $1 \in \mathbb{R} < \mathbb{O}$ shows this unit tangent bundle is parallelizable, hence diffeomorphic to $S^6 \times S^7$ (see also Kamerich [Kam, pp. 70, 77]). The quotient $V_2(\mathbb{R}^4) = SO(4)/SO(2)$ is the unit tangent bundle to $S^3 = Sp(1)$, hence diffeomorphic to $S^3 \times S^2$ by the same reasoning.

For the former, write $\mathbb{O} = \mathbb{H} \oplus \mathbb{H} \mathbb{E}$ with $\epsilon$ a new pure imaginary and fix the $\mathbb{R}$-basis $\{1, j, e, je, i, ij, ie, ije\}$ for $\mathbb{O}$ to endow it with a linear $SO(8)$-action. It is well known that $Spin(8)$ embeds in $SO(8)^3$ as the set of triples $(A, B, C)$ such that for all $o, o' \in \mathbb{O}$ we have $A(o)B(o') = C(oo')$, and it is a manifestation of triality that $A$ determines $\pm(B, C)$ and symmetrically, so that each of the three coordinate projections is a double-covering $Spin(8) \rightarrow SO(8)$. The conditions $A = B = C$ cut out the subgroup $G_2$ of $\mathbb{R}$-algebra automorphisms of $\mathbb{O}$. The equivalent conditions $A(1) = 1$ and $B = C$ cut out a group $Spin(7)^+$ isomorphic to $Spin(7)$, double-covering $[1] \oplus SO(7) < SO(8)$ under $(A, B, B) \rightarrow A$. The
further demand $A(i) = i$ gives $iB(i) = B(io)$, and so cuts out a subgroup $SU(4)^+$ taken injectively to $SU(4) < SO(8)$ by $(A, B, B) \mapsto B$. The subgroup $Spin(7)^- < Spin(8)$ cut out by $B(1) = 1$ and hence comprising triples $(A, B, A)$, meets $SU(4)^+$ in the group of triples $(A, A, A)$ with $A(i) = i$, namely the stabilizer of $i \in S^6 \subseteq \text{Im} \mathcal{O}$ under the action of $G_2 = \text{Aut}_R \mathcal{O}$. Hence this copy of $SU(3)$ is the stabilizer of $(i, 1)$ under the transitive action of $Spin(7)^+$ on $S^6 \times S^7 \subseteq \mathcal{O} \times \mathcal{O}$ by $(A, B, B) \cdot (o, o') := (A(o), B(o'))$, yielding a diffeomorphism $Spin(7)^+/SU(3) \cong S^6 \times S^7$.

This explanation is mostly drawn from Kerr [Kerr98, §6].

(d) In the vocabulary of (c), the group $SU(4)^-$ of triples $(A, B, C) \in SO(8)$ with $B(1) = 1$ and $B(i) = i$ meets $Spin(7)^+$ in $SU(3) = \text{Stab}_{G_2} i$ and they together generate $Spin(8)$, so the inclusion $Spin(7)^+ \hookrightarrow Spin(8)$ induces a diffeomorphism $Spin(7)^+/SU(3) \rightarrow Spin(8)/SU(4)^-.$

(e) The homogeneous space $Spin(9)/SU(4)$ is the total space of a bundle

$$S^6 \times S^7 = Spin(8)/SU(4)^- \rightarrow Spin(9)/SU(4)^- \rightarrow Spin(9)/Spin(8) = S^8,$$

where $Spin(8)/SU(4)^+$ is as in (c) and (d) and the inclusion $Spin(8) \hookrightarrow Spin(9)$ is the standard one covering $SO(8) \oplus [1] \rightarrow SO(9)$. As we have $SU(4)^- < Spin(7)^-$, there is a map from this bundle to

$$S^7 = Spin(8)/Spin(7)^- \rightarrow Spin(9)/Spin(7)^- \rightarrow Spin(9)/Spin(8) = S^8.$$

The fiber of $Spin(8)/SU(4)^- \rightarrow Spin(8)/Spin(7)^-$ is $Spin(7)^-/SU(4)^- \cong S^6$, as noted in (c), so we may view this map as the coordinate projection $S^6 \times S^7 \rightarrow S^7$, inducing an isomorphism on $H^7$. The projection $(A, B, A) \mapsto A$ from (c) is a double-covering from $Spin(7)^-$ to an SO(7) subgroup of SO(8) giving one of the half-spin representations, and it follows $Spin(7)^- \hookrightarrow Spin(8) \hookrightarrow Spin(9)$ is the map $\iota$ of Discussion 3.20(b). Thus the total space $Spin(9)/Spin(7)^-$ of the second displayed bundle is $S^{15}$. Accordingly, in the Serre spectral sequence, the fundamental class of the fiber $S^7$ transgresses to that of the base $S^8$, so the induced map of spectral sequences shows the generator of degree 7 in $H^7(S^6 \times S^7)$ transgresses to the fundamental class of $S^8$ as well. It follows by Poincaré duality that $H^*(Spin(9)/SU(4); \mathbb{Z})$ is isomorphic to $H^*(S^6 \times S^{15}; \mathbb{Z})$.

(f) The embedding of SU(5) in Spin(10) is obtained by lifting the standard embedding of SU(5) in SO(10). The subgroup $SU(4) \oplus [1]$ maps to $SO(8) \oplus [1][\oplus 2] < SO(9) \oplus [1]$, so the composite embedding $SU(4) \rightarrow SU(5) \rightarrow Spin(10)$ factors through the map $SU(4)^- \rightarrow Spin(9)$ of (e). To see the diffeomorphism $Spin(9)/SU(4) \rightarrow Spin(10)/SU(5)$ it remains to show Spin(9) and SU(5) meet in SU(4) and together generate Spin(10). For the former, it is clear SU(4) lies in the intersection $H$, and the converse follows by a dimension count and the exact sequence $\pi_1(Spin(10)/SU(5)) \rightarrow \pi_0 H \rightarrow \pi_0 Spin(9)$. For the latter, note the action $(A, B) \cdot C = ABC^{-1}$ of $SU(5) \times Spin(9)$ on Spin(10) is transitive since the induced action of SU(5) on Spin(10)/Spin(9) = SO(10)/SO(9) = $S^9$, which is the standard action, is transitive.

(g) It is interesting that the two quotients $Spin(7)/SO(4)$ and $SO(8)/(SO(3) \times SO(5)) = \tilde{G}_3(\mathbb{R}^8)$ cited by Kramer [Kr, pp. 59–60] are diffeomorphic. Kerr [Kerr96, §6] notes the projection $(A, B, B) \mapsto B$ is an embedding $Spin(7)^- \hookrightarrow SO(8)$ and that the action of the image on $\mathbb{R}^8$ induces a transitive action on $\tilde{G}_3(\mathbb{R}^8)$, and it can be shown that any element stabilizing
the 3-plane $\mathbb{R}\{i,j,k\}$ also fixes 1, so that the stabilizer is a subgroup of $G_2$ stabilizing $\mathbb{H}$. A dimension count shows that it is this entire subgroup, which is known to be isomorphic to $SO(4)$. We have seen in Remark 3.7 that this Grassmannian does not have the cohomology of $S^8 \times S^{11}$, so these two quotients are not included in Table 1.6.

(h) One of the descriptions of $F_4$ is as the group of $\mathbb{R}$-algebra automorphisms of the 27-dimensional Jordan algebra $h_3(\mathbb{O})$ and it follows directly from this description that $F_4$ acts transitively on the octonionic projective plane $\mathbb{O}P^2$ with stabilizer $Spin(9)$. In fact, this realizes $\mathbb{O}P^2$ as a rank-one symmetric space, and the adjoint representation of $Spin(9)$ on $f_4$ decomposes as the +1-eigenspace $spin(9)$ of the involutive automorphism of $F_4$ plus the $(-1)$-eigenspace, of dimension $52 - 36 = 16$, the isotropy representation of $Spin(9)$ on the tangent space to $\mathbb{O}P^2$ at $1F_4$. Because $\mathbb{O}P^2$ is the irreducible symmetric space of type $F_4$ and has positive Euler characteristic, this 16-dimensional representation is irreducible [Wf, p. 302, 8.13.4], and hence it can only be the spin representation, so $Spin(9)$ acts transitively on the fiber $S^15$ of the unit tangent bundle with stabilizer $Spin(7)^+$ as discussed in (e). Thus the unit tangent bundle can be written as $S^{15} \to F_4/Spin(7)^+ \to \mathbb{O}P^2$. Since the Euler characteristic $\chi(\mathbb{O}P^2)$ is $\dim(H^0 \oplus H^8 \oplus H^{16}) = 3$, the fundamental class $[S^{15}]$ transgresses to $3[\mathbb{O}P^2]$ in the Serre spectral sequence, so $E_2 \cong H^*(F_4/Spin(7); \mathbb{Q})$ is spanned by the four remaining classes, of degrees 0, 8, 23, and 31 and hence the rational cohomology ring is isomorphic to $H^*(S^8 \times S^{23})$ by Poincaré duality.

(i) For the pair $(F_4, Sp(3))$, recall that $Sp(3)$ is contained in a rank-4 subgroup $K = Sp(1) \cdot Sp(3)$ of $G = F_4$ such that $G/K$ is the symmetric space $F_1$ of $\mathbb{H}P^2$ subspaces of $\mathbb{O}P^2$. Ishitoya and Toda [IT, Cor. 4.6] determine that the torsion-free quotient of $H^*(G/H; \mathbb{Z})$ is isomorphic to $H^*(S^8 \times S^{23})$ on the way to computing the integral cohomology of $G/K$.

(j) For $G = Sp(k) \times Sp(2)$ and $H = Sp(k-1) \times \Delta Sp(1) \times Sp(1)$, if we take $K = Sp(k) \times Sp(1) \times Sp(1)$, then $K/H \to G/H \to G/K$ can be identified with the unit sphere bundle of the Whitney sum of $k$ copies of the tautological quaternionic line bundle over $\mathbb{H}P^1 = G_1(\mathbb{H}^2)$.

(k) For $G = G_2 \times Sp(2)$ and $H \cong Sp(1) \cdot \Delta Sp(1) \cdot Sp(1) = H_1 \cdot H_2 \cdot H_3$, there are two possibilities. In both, we take $H_3$ to be $[1] \oplus Sp(1) < Sp(2)$. As for $H_1$ and $H_2$, recall from Discussion 3.20(e) that $G_2$ contains a virtual product subgroup $SO(4) = SU(2)_1 \cdot SU(2)_3$. We may then take $H_1 = SU(2)_1$ and $H_2$ to be the diagonal $Sp(1)$ in $SU(2)_3 \times (Sp(1) \oplus [1]) < SO(4) \times Sp(2)$, or we may take $H_1 = SU(2)_3$ and $H_2$ the diagonal in $SU(2)_1 \times (Sp(1) \oplus [1])$.

**Sketch proofs for inclusions of $(G, H)$ in Table 1.6.** We noted that Proposition 3.6 gives a necessary condition for $(G, H)$ to be a rational sphere product pair, but saw in Remark 3.7 that it is possible a pair satisfying that criterion not have the correct cohomology, so we need to compute the cohomology of all candidates.

The tool of choice in most cases is the **Cartan model** $H^*(BH) \otimes H^*(G)$, a cdga whose cohomology is $H^*(G/H)$ and for which the spectral sequence associated to the filtration by degree in $H^*(BH)$ is the Serre spectral sequence of the Borel fibration $EH \otimes H G \to BH$. The differential of the Cartan model is a derivation vanishing on $H^*(BH)$; on homogeneous exterior generators of $H^*(G)$, it is the composite $H^{*-1}(G) \xrightarrow{\partial} H^*(BG) \to H^*(BH)$ of a lift of the universal transgression and the functorial map $H^*B(H \to G)$. See Onishchik [On94, §§8 & 12] for setup (with real coefficients) and Greub, Halperin, and Vanstone [GHV, Ch. XI.4] for examples of computations in this model.
In most of the cases in the table, computation of $H^*(G/H)$ via the Cartan model is not particularly challenging, so we will content ourselves to illustrate a few representative examples not already covered in Discussion 3.26.

- For $(G, H) = (G_2, i_{p,q}U(1))$, one has $H^*(G) = \Lambda[z_3, z_{11}]$ and $H^*(BH) = \mathbb{Q}[c_1]$, and $H^4(BG) \longrightarrow H^4(BH) = \mathbb{Q}c_2^2$ is nonzero whenever $G$ is semisimple, independent of $p$ and $q$, so the ideal $(dz_3)$ of $H^*(BH)$ is $(c_2^2)$. Irrespective of what $dz_{11} \in \mathbb{Q}c_1^6$ is, it will lie in this ideal, so $d_{12}z_{11} = 0$ in the Serre spectral sequence, showing $H^*(G/H) \cong \mathbb{Q}[t]/(t^2) \otimes \Lambda[z_{11}]$. The same argument applies in all cases with $H \cong U(1)$.

- The pairs $(G, H) = (\text{Sp}(3), (\text{Sp}(1) \oplus \Delta_2\text{Sp}(1)))$ and $(\text{Sp}(3), \text{SO}(3) \cdot \Delta_3\text{Sp}(1))$ both have the rational cohomology of $S^4 \times S^{11}$. For the former, we can factor the inclusion $\text{Sp}(1) \oplus \Delta_2\text{Sp}(1) \longrightarrow \text{Sp}(3)$ through $\text{Sp}(1)^{\otimes 3}$. The factor map
  \[
  H^*\text{Sp}(3) \longrightarrow H^*\text{BSp}(1)^{\otimes 3} \cong (H^*\text{BSp}(1))^{\otimes 3} \cong \mathbb{Q}[q, q', q'']
  \]
embeds the symplectic Pontrjagin classes $q_1, q_2, q_3$ as the elementary symmetric polynomials on the generators $q, q', q''$. Under $(\Delta_2)^* : H^*\text{BSp}(1)^{\otimes 2} \longrightarrow H^*\text{BSp}(1)$, both generators $q', q''$ are taken to the polynomial generator $r \in H^4\text{BSp}(1)$, so the map $H^*\text{BSp}(3) \longrightarrow H^*\text{B}(\text{Sp}(1) \oplus \Delta_2\text{Sp}(1))$ is given by
  \[
  q_1 \longrightarrow q + 2r, \quad q_2 \longrightarrow 2qr + r^2, \quad q_3 \longrightarrow qr^2.
  \]
Computing the cohomology of the Cartan model $\mathbb{Q}[q, r] \otimes \Lambda[z_3, z_7, z_{11}]$, one finds $H^*(G/K) \cong \mathbb{Q}[r]/(r^2) \otimes \Lambda[z_{11}]$ for an element $\bar{z}_{11} \in z_{11} + (\text{im} \bar{d})$. As noted earlier, integrally one has $H^8 \cong \mathbb{Z}/3$, as one has $q \sim -2r$ in $H^4$ and $0 \sim 2qr + r^2 \sim -4r^2 + r^2$ in $H^8$ using the Serre spectral sequence.

As for the inclusion $\text{SO}(3) \cdot \Delta_3\text{Sp}(1) \longrightarrow \text{Sp}(3)$, the maximal torus $\text{SO}(2) \oplus [1]$ of $\text{SO}(3)$ can be conjugated to the diagonal subgroup with elements $\text{diag}(\omega, \omega^{-1}, 1)$ for $\omega \in U(1)$ without moving the maximal torus $\{\text{diag}(\zeta, \zeta, \zeta)\}$ of $\Delta_3\text{Sp}(1)$. The two-dimensional maximal torus $S$ generated by these two subtori has generic element $(\omega \zeta, \omega^{-1} \zeta, \zeta)$. Taking $s \in H^2\text{BS}$ to correspond to $\omega$ and $t$ to $\zeta$, the restriction from $\mathbb{Q}[t_1, t_2, t_3] \cong H^*\text{BU}(1)^{\otimes 3}$ is
  \[
  t_1 \longrightarrow t + s, \quad t_2 \longrightarrow t - s, \quad t_3 \longrightarrow t.
  \]
Restricting the $q_j$, which are the elementary symmetric polynomials in $t_1^2, t_2^2, t_3^2$, one finds the map to $H^*(BH) \cong \mathbb{Q}[s_1^2, t^2]$ is given by
  \[
  q_1 \longrightarrow 2s^2 + 3t^2, \quad q_2 \longrightarrow s^4 + 3t^4, \quad q_3 \longrightarrow s^4 t^2 - 2s^2 t^4 + t^6.
  \]
One then can compute the cohomology of the model $\mathbb{Q}[s_2^2, t_2^2] \otimes \Lambda[z_3, z_7, z_{11}]$ and find it is isomorphic to $\mathbb{Q}[t^2]/(t^4) \otimes \Lambda[z_{11}]$ for an element $\bar{z}_{11} \in z_{11} + (\text{im} \bar{d})$.

The cases involving classical groups are mostly resolved by choosing a maximal torus $T_H$ of $H$ contained in a maximal torus $T_G$ of $G$ and identifying $H^*(BG) \longrightarrow H^*(BH)$ with the map of invariants $H^*(BT_G)^{W_G} \longrightarrow H^*(BT_H)^{W_H}$, using the fact that these invariants are generated by the symmetric polynomials $c_i$ in the generators $t_j$ of $H^*(BT) = \mathbb{Q}[t]$ for groups of type $A_k$, symmetric polynomials $p_j$ in the $t_j^2$ for groups of type $BC_k$ and $D_k$, and additionally the product $e = t_1 \cdots t_k = \sqrt{p_k}$ for groups of type $D_k$. We do two examples of this type to communicate the basic idea.
• For \((G, H) = (SU(5), SU(2) \oplus SU(3))\), the differential \(\Lambda[z_3, z_5, z_7, z_9] \to \mathbb{Q}[c_2, c_3, c_4, c_5] = H^*(BG) \to H^*(BH) = \mathbb{Q}[c_2'] \otimes \mathbb{Q}[c_3', c_3'']\) sends

\[
\begin{align*}
z_3 & \mapsto c_2' \mapsto c_2'' + c_2''', \\
z_5 & \mapsto c_3' \\
z_7 & \mapsto c_4' \mapsto c_4'' + c_4''', \\
z_9 & \mapsto c_5' \mapsto c_5'' + c_5'''.
\end{align*}
\]

In the Serre spectral sequence of \(G \to G/H \to BH\), then, we see \(c_2' = -c_2''\) after \(E_4\) and \(c_3'' = 0\) after \(E_6\), so \(d_5z_7 = -(c_2')^2\) and \(d_10z_9 = 0\). Thus \(H^*(G/H) \cong \mathbb{Q}[c_2']/(c_2')^2 \otimes \Lambda[z_9]\).

• For \(G = G_2 \times \text{Sp}(2)\) and \(H = \text{Sp}(1) \cdot \Delta \text{Sp}(1) \cdot \text{Sp}(1)\), the maximal torus \(T \cong \text{U}(1)^4\) of \(\text{Sp}(1) \cdot \text{Sp}(1) \times \text{Sp}(1) \oplus \text{Sp}(1) < G\) meets \(H\) in \(T' = \text{U}(1) \times \Delta \text{U}(1) \times \text{U}(1)\),\(^{16}\) so the induced map \(\mathbb{Q}[t_1, u_1, u_2, t_2] = H^*(BT) \to H^*(BT') = \mathbb{Q}[s_1, v, s_2]\) is given by \(t_1 \mapsto s_1\) and \(u_1 \mapsto v\).

Writing \(H^*(BG_2) \otimes H^*(B\text{Sp}(2)) = \mathbb{Q}[r_4, q_{12}] \otimes \mathbb{Q}[p_1, p_2]\) and \(H^*(BH) = \mathbb{Q}[p', p'', p''']\), we get the splittings of characteristic classes

\[
\begin{align*}
r & \mapsto p' + p'', \\
p_1 & \mapsto p'' + p''', \\
p_2 & \mapsto p'''p'''.
\end{align*}
\]

For our purposes, we do not need to work out what \(q_{12}\) goes to, but observe only that it will be some only polynomial in \(p'\) and \(p''\). In the spectral sequence of \(G \to G/H \to BH\), we have \(p' = -p'' = p'''\) from \(E_6\) on since \(p' + p''\) and \(p'' + p'''\) are images of \(d_5\) and \((p')^2 = 0\) from \(E_{10}\) on since \(p''p''' = (-p')p''\) is in the image of \(d_6\). Since the image of \(q_{12}\) is congruent modulo the previous differentials to a polynomial in \(p'\) alone and \((p')^2 = 0\), it follows \(d_{12}z_{11} = 0\), so we have \(H^*(G/H) \cong \mathbb{Q}[p']/(p')^2 \otimes \Lambda[z_{11}]\). \(\square\)

4. Isotropy-formal pairs

In Section 2 we reduced the study of isotropy-formality of corank-one pairs \((G, K)\) to the computation of \(H^*(G/H_S)\) and examination of the corresponding irreducible pair in Table 1.6. In this section we determine which of those pairs is isotropy-formal. The largest and simplest class of cases is that of products \((K_1 \times K_2, H_1 \times H_2)\).

**Lemma 4.1.** Suppose \(G = K_1 \cdot K_2\) is a virtual direct product of two compact, connected Lie groups, and \(H = H_1 \cdot H_2 \leq G\) is also a virtual direct product, with \(H_j \leq K_j\). The pair \((G, H)\) is isotropy-formal if and only if both \((K_1, H_1)\) and \((K_2, H_2)\) are.

**Proof.** Noting that finite coverings do not impact the question of isotropy-formality by Proposition 2.25, we may assume \((G, H)\) is an honest product of \((K_1, H_1)\) and \((K_2, H_2)\). Then the lemma follows from Theorem 2.1 or Theorem 2.5. \(\square\)

We now check isotropy-formality for the remaining pairs in Table 1.6. For all pairs, we will simultaneously determine \(Z_G(H)^0\) and \(H_S\), which play crucial roles in Theorem 2.46 and Proposition 2.42 respectively, determining isotropy-formality in the virtually effective but reducible

\(^{16}\) The two copies of \(\text{U}(1) < \text{Sp}(1)\) in \(G_2\) meet in \(\{\pm 1\}\), but the diagonal \(\Delta \text{U}(1)\) with one coordinate in \(\text{Sp}(1) \oplus [1]\) does not meet the \(\text{U}(1)\), so this really is a direct product.
case. The calculation of $Z_G(H)^0$ appears in Kramer [Kr] and requires only mild expansion and correction, but the calculation of $H_S$ in the cases we need does not seem to be in the literature.

**Discussion 4.2.** There are a few basic principles underlying these computations. For $G$ and $H$ presented as matrix groups or products thereof, it is elementary and usually uncomplicated to compute $Z_G(H)^0$ directly. When $H$ is a product, one can compute the centralizer of one factor and then within that subgroup the centralizer of the other factor. For the remaining pairs, in which $G$ is exceptional, we take recourse to the literature.

As for the maximal regular subgroup $H_S$ sharing the maximal torus $S$ of $H$ (Definition 2.9), it is usually not fastest to compute it in terms of the root space composition of Proposition 2.11. In many cases, $H$ is regular, which we usually verify by checking $\text{rk} Z_G(H)H/H = 1$ using Proposition 2.42, and then $Z_G(H)^0H/H$ is either isomorphic to $S^1$ or a group of type $A_1$. It is almost invariably $S^1$. This is usually simple to verify directly, but it is simpler still to note that if it is of type $A_1$, then $G$ admits a maximal rank subgroup $H \times A_1$, which can be ruled out by consultation with Borel–de Siebenthal [BodS49]. Then to check if $H$ agrees with $H_S$, we note that if $H_S > H$, then by Lemma 2.21, the pair $(G, H_S)$ appears in the Table 3.17 of rational sphere pairs, so if no pair $(G, K)$ with $K > H$ appears in this list, then $H = H_S$. If, on the other hand, there are any such $(G, K)$ on the list, then it remains to check if any of the $K$ are regular, which again can be accomplished by determining the centralizer. For brevity we call this the centralizer-and-sphere argument.

**Proof of Theorem 1.5.** Theorem 2.40 and Lemma 2.44 extend the classification of irreducible rational sphere product pairs in Table 1.6 to the stated classification of virtually effective rational sphere product pairs in terms of subgroups $P < Z_G(H)^0$ descending to rank-one subgroups of $N_G(H)^0/H = Z_G(H)^0H/H$. Theorem 2.46 shows such extensions are isotropy-formal if and only if $H_S > H$. Thus it remains only to complete Table 1.6 by determining which irreducible rational sphere product pairs are isotropy-formal and finding $Z_G(H)^0$ and $H_S$ in all cases.

By Lemmas 2.22 and 4.1, all the pairs $(G, H)$ whose resulting homogeneous spaces are rational cohomology spheres or virtual direct products of rational cohomology spheres are isotropy-formal. This particularly covers the case $S^1 \times S^m$ studied by Bletz-Siebert. It remains to check Kamerich, Kramer, and Wolfrom’s non-product cases from Table 1.6.

When there is obviously some larger closed, connected $K$ sharing the maximal torus $S$ of $H$, we apply Lemma 2.21 to show the following pairs $(G, H)$ are isotropy-formal in the following cases. We interleave a brief argument computing $Z_G(H)^0$ and $H_S$ in each case.

- $(\text{SU}(4), \text{SU}(2) \oplus \text{SU}(2))$ and $K = \text{Sp}(2) > \text{Sp}(1) \oplus \text{Sp}(1)$
  Here $Z_G(H)^0 = \{\text{diag}(z, z, z^{-1}, z^{-1})\}$ and $H = H_S$ by a centralizer-and-sphere argument.

- $(\text{SO}(2k + 1), \text{SO}(2k - 2))$ and $K = \text{SO}(2k - 1)$
  Here $Z_G(H)^0 = [1]^{\oplus 2k-2} \oplus \text{SO}(3)$ directly and $[1]^{\oplus 2k-1} \oplus \text{SO}(2)$ centralizes $K$, which is thus regular; hence $K = H_S$.

- $(\text{Spin}(9), \text{SU}(4))$ and $K = \text{Spin}(7) > \text{SU}(4)$
  To see regularity, project down to $\text{SO}(8)$, preserving $\text{SU}(4)$, note that $\text{SU}(4) < U(4) < \text{SO}(8)$ is normalized by the diagonal $\Delta_4 U(1)$, and lift to find its double cover in $\text{Spin}(8)$ centralizes $\text{SU}(4)$ as well. That $Z_G(H)^0 \cong U(1)$ and $H = H_S$ follow by a centralizer-and-sphere argument.
• \((\text{Spin}(7), \text{SU}(3))\) and \(K = G_2\)

Since \(\text{SU}(3) < \text{SU}(4)\) is normalized by \(\{\text{diag}(z, z, z^{-3})\}\), and \(\text{SU}(4)\) is contained in \(\text{Spin}(7)\) (see Discussion 3.26(c)), we may run a centralizer-and-sphere argument.

• \((\text{Sp}(3), \text{Sp}(1) \oplus \text{Sp}(1) \oplus [1])\) and \(K = \text{Sp}(2) \oplus [1]\)

One has \(Z_G(H)^0 = [1]^{\oplus 2} \oplus \text{Sp}(1)\) by direct computation and this group also centralizes \(K\), so \(K = H_S\) is regular.

We can also directly find a group \(K\) sharing a maximal torus \(S\) and such that \((G, K)\) is a rational sphere pair and use Lemma 2.20, to show the following \((G, H)\) are isotropy-formal.

• \((\text{SU}(4), \text{SO}(4))\) and \(K = \text{Sp}(2)\) and \(S = \text{SO}(2) \oplus \text{SO}(2)\)

The centralizer is \(\pm [1]^{\oplus 4}\) and \(H' = \text{SU}(2)^{\oplus 2}\) contains \(S\); we saw above that \(H' = H_S\).

• \((\text{SO}(7), \text{SO}(3) \oplus \text{SO}(3) \oplus [1])\) and \(K = \text{SO}(5) \oplus [1]^{\oplus 2}\) and \(S = \text{SO}(2) \oplus [1] \oplus \text{SO}(2) \oplus [1]^{\oplus 2}\)

The centralizer is \(\pm [1]^{\oplus 6} \oplus [1]\), but \(K\) is centralized by \([1]^{\oplus 5} \oplus \text{SO}(2)\), and by a centralizer-and-sphere argument, \(K = H_S\).

• \((\text{SO}(2k), \text{SO}(2k-2))\) with \(k \geq 3\) and \(K = \text{SO}(2k-1)\) and \(S = \text{SO}(2)^{\oplus k-1}\)

One sees the centralizer is \(\pm [1]^{\oplus k-2} \oplus \text{SO}(2)\) and applies a centralizer-and-sphere argument \((K\) is not regular, since its centralizer is \(\pm [1]^{\oplus k}\)).

Not quite of this class, but closely related, is the following:

• \((\text{Spin}(8), \text{SU}(4))\) is isotropy-formal.

Recall that in \(\text{Spin}(8)\), the triality automorphism takes the subgroup \(\text{SU}(4)\) to the standard \(\text{Spin}(6)\) [Kam, pp. 40–1] [Kr, p. 60], but \((\text{Spin}(8), \text{Spin}(6))\) double-covers the just discussed isotropy-formal pair \((\text{SO}(8), \text{SO}(6))\). We found \(H = H_S\) is regular and \(Z_G(H)^0 \cong S^1\) while discussing \((\text{Spin}(9), \text{SU}(4))\) above.

For a few examples, we are able to determine the maximal regular pair \((G, H_S)\) associated to \(S\) by observing \(H\) is not regular, but a subgroup \(H'\) is, and \(H/H'\) is rationally homotopy equivalent to an even-dimensional sphere, so that by Lemma 2.19 we have \(H' = H_S\). Then because \((G, H_S)\) is not a rational sphere product pair, we may conclude by Theorem 1.3 that \((G, H)\) is not isotropy-formal in the following cases. Moreover, since \(H > H_S\), we conclude \(H\) is not regular, and so \(Z_G(H)^0 H = H\) by Proposition 2.42; but then \(Z_G(H)^0 = Z(H)^0\) is trivial.

• \((\text{Sp}(3), \text{Sp}(1) \oplus \text{SU}(2)_{10})\), with \(H' = \text{Sp}(1) \cdot i_{3,1} \text{U}(1)\)

The standard maximal torus \(\text{U}(1)^{\oplus 3}\) normalizes \(H'\). According to Berger [Ber, p. 237], \(i_{3,1} \text{U}(1)\) is a maximal torus of the embedded subgroup \(\text{SU}(2)_{10}\) in \(\text{Sp}(2)\), and so \(H/H' \approx S^2\).

• \((\text{Sp}(k) \times \text{Sp}(2), \text{Sp}(k-1) \times \Delta \text{Sp}(1) \times \text{Sp}(1))\) for \(k \geq 2\), with \(H' = \text{Sp}(k-1) \times \Delta \text{U}(1) \times \text{Sp}(1)\)

• \((G_2 \times \text{Sp}(2), \text{Sp}(1) \times \Delta \text{Sp}(1) \times \text{Sp}(1))\), with \(H' = \text{Sp}(1) \times \Delta \text{U}(1) \times \text{Sp}(1)\)

Note that this actually comprises two cases, depending which of the subgroups \(\text{SU}(2)_1\) and \(\text{SU}(2)_3\) is chosen to serve as which \(\text{Sp}(1)\) in \(H\).
Again not quite in this class, but related, is the following:

- \((\text{Sp}(3), \text{SO}(3) \cdot \Delta_3 \text{Sp}(1))\) is not isotropy-formal.

For the centralizer, one has \(Z_G(\text{SO}(3)) = \Delta_3 \text{Sp}(1)\) but \(Z(\text{Sp}(1)) = \{\pm 1\}\), so \(H\) is not regular. A maximal torus \(S\) is \(\text{SO}(2) \cdot \Delta_3 \text{U}(1)\), and one has \(H/S \approx S^2 \times S^2\). This torus \(S\) is also contained in \(H' = \text{SU}(2) \cdot \Delta_3 \text{U}(1)\), which is normalized by the torus \(\text{SO}(2) \cdot \Delta \text{U}(1) \oplus \text{U}(1)\), hence regular. To see \(H' = H_S\), note that by two spectral sequence collapse arguments we have \(\dim Q H^*(G/S) = 16\) and \(\dim Q H^*(G/H') = 8\) by the collapse of the corresponding Serre spectral sequence. A larger \(H'' > H'\) sharing \(S\) as a maximal torus would have \(\dim Q H^*(G/H'') \in \{2, 4\}\) and \(H^{11}(G/H'') \cong \mathbb{Q}\) by comparing the Cartan algebras of \(G/H''\) and \(G/H'\) using Theorem 2.13 and the resulting observation that formality is determined by \((G,S)\) alone. But then \((G,H'')\) would appear in Table 3.17, and it does not, since \(H'\) is not contained in any conjugate of \(\text{Sp}(2)\).

For the remaining examples, we compare \(N, W_{H_H}\) and \(W_{H_H}\) per Theorem 1.3. In the easier of these, \(H\) is regular and \(w^G_0\) acts as \(-\text{id}\) on \(t\), so by Corollary 2.23, \((G,H)\) is isotropy-formal if and only if \(w^H_0\) does not act as \(-\text{id}\) on \(s\). In the following cases, both \(w^G_0\) and \(w^H_0\) act as \(-\text{id}\), so \((G,H)\) is not isotropy-formal.

- \((\text{SO}(7), \text{SO}(3) \oplus \text{SU}(2))\)
  The block \(\text{SU}(2) < \text{U}(2) < \text{SO}(4)\) is normalized by the standard maximal torus \(\text{U}(1)^{\oplus 2} < \text{U}(2)\), so \(H\) is regular. That \(Z_G(H)^0 \cong \text{U}(1)\) and \(H = H_S\) follow by a centralizer-and-sphere argument.

- \((\text{Sp}(3), \text{Sp}(1) \oplus \Delta_2 \text{Sp}(1))\) (or \((\text{Sp}(3), \text{Sp}(1) \oplus \text{SU}(2))\))
  Here \(W_H\) and \(N\) are the same as for \((\text{SO}(7), \text{SO}(3) \oplus \text{SU}(2))\). Only \([1] \oplus \text{Sp}(2)\) centralizes the first factor \(\text{Sp}(1) \oplus [1]^{\oplus 2}\) of \(H\). Within \(\text{Sp}(2)\), only \(\text{SO}(2)\) centralizes \(\Delta_2 \text{Sp}(1)\), so \(Z_G(H) = [1] \oplus \text{SO}(2)\). That \(H = H_S\) follows by a centralizer-and-sphere argument.

- \((F_4, \text{Spin}(7))\)
  Within \(\text{Spin}(9) < F_4\), the subgroup \(\text{Spin}(2)\) normalizes the standardly embedded \(\text{Spin}(7) = H'\) (as \(\text{SO}(7) \oplus [1]^{\oplus 2}\) and \([1]^{\oplus 7} \oplus \text{SO}(2)\) commute), so \(H'\) is regular. But \(H'\) is taken to \(H\) by an outer automorphism of \(\text{Spin}(8)\), and all automorphisms of \(\text{Spin}(8)\) are realized through conjugation by an element of \(N_{F_4}(\text{Spin}(8))\) [A96, Thm. 14.2], so \(H\) too is regular in \(F_4\). That \(Z_G(H)^0 \cong \text{U}(1)\) and \(H = H_S\) follow by a centralizer-and-sphere argument.

- \((F_4, \text{Sp}(3))\)
  The subgroup \(\text{Sp}(3)\) is contained in a copy of \(\text{Sp}(3) \otimes_{\mathbb{Z}/2} \text{SU}(2) < F_4\), within which \(\text{SO}(2)\) centralizes it. That \(Z_G(H)^0 = \text{SO}(2)\) and \(H = H_S\) follow by a centralizer-and-sphere argument.

In the following cases, \(H\) is regular and \(w^G_0\) acts as \(-\text{id}\) while \(w^H_0\) does not, so \((G,H)\) is isotropy-formal.

- \((\text{Sp}(4), \text{SU}(4))\)
  The central circle \(\Delta_4 \text{U}(1)\) of \(\text{U}(4) < \text{Sp}(4)\) centralizes \(\text{SU}(4)\). That \(Z_G(H)^0 \cong \text{U}(1)\) and \(H = H_S\) follow by a centralizer-and-sphere argument.
• (\(\text{Sp}(3), \text{SU}(3)\))

The central circle \(\Delta_3\text{U}(1)\) of \(\text{U}(3) < \text{Sp}(3)\) centralizes \(\text{SU}(3)\). That \(Z_G(H)^0 \cong \text{U}(1)\) and \(H = H_S\) follow by a centralizer-and-sphere argument.

• (\(\text{Sp}(2), i_{p,q}\text{U}(1)\)) for any coprime pair \((p, q)\)

To find \(H_S\) and \(Z_G(H)^0\), note that \(H\) is regular with centralizer at least the maximal torus since it is contained within it. To determine when it is contained in an \(A_1\) subgroup or centralized by one, recall from the representation theory of \(\text{Sp}(1)\) and \(\text{SO}(3)\) and Proposition 2.42 that the regular \(A_1\)-subgroups of \(\text{Sp}(2)\) up to conjugacy are \(\text{Sp}(1) \oplus [1]\) and \(\Delta\text{Sp}(1)\).

• (\(G_2, i_{p,q}\text{U}(1)\)) for any coprime pair \((p, q)\)

Here \(H\) lies within the maximal torus \(T\) of \(G_2\) from Discussion 3.20(e), and hence is at least centralized by \(T\), hence regular. If we have \(Z_G(H)^0 H / H\) an \(A_1\) group, then \(Z_G(H)\) contains an \(A_1\) subgroup virtually disjoint from \(S\), and following Discussion 3.20(e), these fall into four conjugacy classes, of which only those of \(\text{SU}(2)_1\) and \(\text{SU}(2)_3\) admit a centralizing circle. These are also the only regular \(A_1\) subgroups. Thus \(H = H_S\) and \(Z_G(H)^0 = T\) unless \(H\) is (conjugate to) the maximal circle of either \(\text{SU}(2)\), in which case \(H_S > H\) is that \(\text{SU}(2)\) and \(Z_G(H)^0\) is the product of \(H\) and the other \(\text{SU}(2)\).

In the remaining cases we apply Theorem 1.3 again, but require subcases or \textit{sui generis} arguments in determining whether \(N > W_H\).

• (\(\text{SU}(5), \text{SU}(2) \oplus \text{SU}(3)\)) is not isotropy-formal.

One has \(N = W_H = \Sigma_{(1,2)} \times \Sigma_{(3,4,5)}\) in \(W_G = \Sigma_{(1,2,3,4,5)}\) and sees \(H\) is centralized by elements \(\text{diag}(z, z^3, z^{-2}, z^{-2}, z^{-2})\). That \(Z_G(H)^0 = \text{U}(1)\) and \(H = H_S\) follow by a centralizer-and-sphere argument.

• (\(\text{Spin}(10), \text{SU}(5)\)) is not isotropy-formal.

In this example the embedding is defined by lifting the standard block embedding of \(\text{SU}(5)\) in \(\text{SO}(10)\), but we may work locally in \(\text{so}(10) \cong \text{spin}(10)\). Hence \(t = \text{so}(2) \otimes 5 \cong \mathbb{R}^5\) and \(W_{\text{Spin}(10)}\) is the semi-direct product of \(\Sigma_5\) with the kernel \(S(\pm 1)^5\) of the multiplication map \(\{\pm 1\}^5 \rightarrow \{\pm 1\}\). Then \(s = \{\vec{x} \in \mathbb{R}^5 : \sum x_j = 0\}\) is the nullspace of the weight \(\alpha = (1, 1, 1, 1, 1) \in \mathbb{Z}^5\). The stabilizer \(N\) of \(\{\pm \alpha\}\) is just \(\Sigma_5 = W_{\text{SU}(5)}\) since elements of \(W_{\text{Spin}(10)}\) can negate only an even number of coordinates and 5 is not even.

That the centralizer contains a nontrivial circle follows from lifting the standard circle \(\Delta_5\text{SO}(2) < \text{SO}(10)\) centralizing \(\text{SU}(5)\). That \(Z_G(H)^0 = \text{U}(1)\) and \(H = H_S\) follow by a centralizer-and-sphere argument.

• (\(\text{SU}(3), i_{p,q}\text{U}(1)\)) is isotropy-formal if and only if \((p, q)\) is one of \((\pm 1, 0), (0, \pm 1), \text{or} (\pm 1, \mp 1)\).

Since \(s \cong \mathbb{R}\), we have \(N > W_H\) if and only if some element \(w \in W_G\) acts as \(-\text{id}|_s\). This only happens for \(S\) conjugate to \(\text{SU}(1)^{\otimes 2} < \text{SU}(3)\) under the coordinate-permuting action of \(\Sigma_3 = W_{\text{SU}(3)}\) on \(T = S(\text{U}(1)^{\otimes 3})\). If we parameterize \(T\) by \((z, w) \mapsto \text{diag}(z, w, z^{-1}w^{-1})\) so that \(i_{p,q}(t) = \text{diag}(t^p, t^q, t^{-p-q})\), this is just what we have claimed.

\textsuperscript{17} Isotropy-formality in the case \(H\) is of rank one was already characterized in one of the authors’ earlier works \([C]\), where isotropy-formal pairs \((G, H)\) were classified.
To find \( H_S \) and \( Z_G(S)^0 \), we view \( t \) as \( \{ x \in \mathbb{R}^3 : \sum x_j = 0 \} \), where \( \mathbb{R}^3 \) carries the standard inner product, so that \( s = \mathbb{R}(q, -p, -q) \) and \( s^\perp = \mathbb{R}(p + 2q, q, -q - 2p, p - q) =: (p^*, q^*, -p^* - q^*) \). Because the only representations of SU(2) and SO(3) of dimension \( \leq 3 \) are the standard ones, all \( A_1 \)-subgroups of SU(3) are conjugate to the standard ones. By Proposition 2.42, SO(3) is not a regular subgroup of SU(3), so any \( H = H_S \) is isomorphic to \( S \) or SU(2), and the semisimple virtual factor of \( Z_G(H)^0 \) can only be, respectively, SU(2) or 1. Since any SU(2) is conjugate to the standard one, if \( H_S > S = i_{p,q}U(1) \), then \( p \cdot q \in \{ 0, -1 \} \). For \( Z_G(H)^0 \) to be one of the conjugates of the standard SU(2) on the other hand, we need \( (p^*, q^*) \) to be \( (\pm 1, 0), (0, \pm 1), \) or \( (\pm 1, \mp 1) \) up to scaling, and one checks this happens if \( (p, q) \) is respectively \( (\pm 1, \mp 2), (\mp 2, \pm 1), \) or \( (\pm 1, \pm 1) \).

• \((G', \text{Sp}(1), H' : i_{p,q}U(1))\), where \((G', H')\) is a rational cohomology sphere as in Table 3.17 with \( Z_{G'}(H')^0 \neq \{ 1 \} \) and \( p \) and \( q \) are coprime and nonzero, is isotropy-formal except when \((G', H') = (SU(k + 1), SU(k))\) for \( k \geq 2 \).

Let \( u_1 \) be the Lie algebra of a maximal torus of the rank-1 group \( Z_G(H') \). Since the rank of \( H' \) is one less than that of \( G' \), it follows the sum of \( u_1 \) with the Lie algebra \( s_1 \) of a maximal torus \( S_1 \) of \( H' \) spans the Lie algebra \( t_1 \) of a maximal torus of \( G' \), i.e., \( t_1 = s_1 \oplus u_1 \). We consider the Weyl group of \( G' \) with respect to this particular torus. The Lie algebra \( u_2 \) of a maximal torus of Sp(1) is one dimensional, so we may make the identifications

\[
t = t_1 \oplus u_2 = s_1 \oplus u_1 \oplus u_2 \cong s_1 \oplus \mathbb{R} \oplus \mathbb{R}
\]

in describing the Lie algebra \( t \) of the maximal torus of \( G = G' \times \text{Sp}(1) \). Under this identification, the image \( u \) of the tangent space \( \mathbb{R} \) of \( S^1 \) under the inclusion \( i_{p,q} \) of Notation 3.22 becomes \( \mathbb{R}((0, p), q) \) and the Lie algebra \( s \) of the maximal torus of \( H = H' \cdot i_{p,q}U(1) \) becomes \( s_1 \oplus u = s_1 \oplus \mathbb{R}((0, p), q) \).

Because we assume \( q \) is nonzero, \( H \) is abstractly isomorphic to \( H' \times U(1) \), so we have isomorphisms \( W_H \cong W_{H'} \times W_{U(1)} \cong W_H \). The longest word \( \tilde{w}_0 \) of \( W_{G'} \times \text{Sp}(1) \cong W_{G'} \times W_{\text{Sp}(1)} \) is the pair \((\tilde{w}_0, -\text{id}_{u_2})\), for \( \tilde{w}_0 \in \text{Aut}(s_1 \oplus \mathbb{R}) \) the longest word of \( W_{G'} \).

- For \( G' = \text{SO}(2k + 1), \text{Sp}(k) \) and \( G_2 \), we have \( w_0|_{u_1} = -\text{id}_{u_1} \), hence the word \( \tilde{w}_0 \) induces \(-\text{id}\) on all of \( t \). Since \( W_H = W_{H'} \) acts trivially on \( u \), we see \( \tilde{w}_0|_s = -\text{id}_s \notin W_H \). Thus, we have \( N \neq W_H \).
- For \( (G', H') = (SU(k + 1), SU(k) \oplus [1]) \) with \( k \geq 2 \), the Lie algebra of the maximal torus of \( G = SU(k + 1) \times \text{Sp}(1) \) can be written as

\[
t = \left\{ (x_1, x_2, \ldots, x_{k+1}; y) \in \mathbb{R}^{k+1} \oplus \mathbb{R} : \sum_{j=1}^{k+1} x_j = 0 \right\}.
\]

Under this identification, the Lie algebras of the maximal tori of \( H' = SU(k) \) and \( i_{p,q}U(1) \) become respectively

\[
s_1 = \left\{ (x_1, x_2, \ldots, x_k, 0; 0) \in \mathbb{R}^{k+1} \oplus \mathbb{R} : \sum_{j=1}^{k} x_j = 0 \right\}
\]

and

\[
u = \mathbb{R} \cdot (p, p, \ldots, p, -kp; q).
\]
Thus the Lie algebra \( s = s_1 \oplus u \) of the maximal torus of \( H \) is perpendicular to the vector \( v = (q, q, \ldots, q, -kq; -(k^2 + k)p) \). Since \( k \geq 2 \), we have \( kq \neq q \), so the stabilizer \( \tilde{N} = W(\pm v) \) of \( \{\pm v\} \) under the action of \( W_G = \Sigma_{k+1} \times \{\pm 1\} \) is exactly \( W_v = \Sigma_k \). By Lemma 2.4, we have \( N = W_v|_s \cong \Sigma_k \). However, since \( W_H = W_H|_s = \Sigma_k \), we see \( N = W_H \).

To find \( Z_{G^{'}}(H^{'}) \) for \( G^{'}, H^{'}, i_{p,q}U(1) \) in \( G^{'}, \text{Sp}(1) \) contains both this centralizer \( U(1) \times U(1) \) and \( S_1 \), and hence all of \( T \). Thus \( (G^{'}, \text{Sp}(1), H^{'}, i_{p,q}U(1)) \) is a rational sphere pair by Lemma 2.21. Since \( G^{'}, \text{Sp}(1) \) is not simple, the pair \( (G^{'}, \text{Sp}(1), H_S) \) is not irreducible. As \( G^{'}, H^{'}, i_{p,q}U(1) \) is an \( A_1 \) subgroup and \( (p, q) = (1, \pm 1) \) (up to conjugacy). Thus \( H_S \) is determined by consulting Table 3.17.

References

[A69] J. Frank Adams. Lectures on Lie groups. Univ. Chicago Press, 1969.
[A96] J. Frank Adams (edited by Z. Mahmud and M. Mimura). Lectures on Exceptional Lie Groups. Chicago Lectures in Mathematics. Univ. Chicago Press, 1996.
[AtB] Michael F. Atiyah and Raoul Bott. The moment map and equivariant cohomology. Topology, 23(1):1–28, 1984. doi:10.1016/0040-9383(84)90021-1.
[Ber] Marcel Berger. Les variétés riemanniennes homogènes normales simplement connexes à courbure strictement positive. Ann. Scuola Norm. Sup. Pisa Cl. Sci., 15(3):179–246, 1961. http://numdam.org/item/ASNSP_1961_3_15_3_179_0/.
[BeV] Nicole Berline and Michèle Vergne. Classes caractéristiques équivariantes. Formule de localisation en cohomologie équivariante. C. R. Acad. Sci. Paris, 295(2):539–541, Nov. 1982. http://gallica.bnf.fr/ark:/12148/bpt6k62356694/f777.
[Bes78] Arthur L. Besse. Manifolds all of whose geodesics are closed, volume 93 of Ergeb. Math. Grenzgeb. (2). Springer-Verlag, Berlin, 1978.
[Bes87] Arthur L. Besse. Einstein Manifolds, volume 10 of Ergeb. Math. Grenzgeb. (2). Springer, 1987.
[BIS02] Oliver Bltz-Siebert. Homogeneous spaces with the cohomology of sphere products and compact quadrangles. 2002 dissertation. http://opus.bibliothek.uni-wuerzburg.de/files/332/bltzzsis.pdf.
[B049] Armand Borel. Some remarks about Lie groups transitive on spheres and tori. Bull. Amer. Math. Soc., 55(6):580–587, 1949. doi:10.1090/S0002-9904-1949-09251-0.
[B050] Armand Borel. Le plan projectif des octaves et les spheres comme espaces homogenes. C. R. Acad. Sci. Paris, 230(178-138):22, 1950. https://gallica.bnf.fr/ark:/12148/bpt6k3182n/f1378.
[B053] Armand Borel. Les bouts des espaces homogènes de groupes de Lie. Ann. of Math. (2), 58:443–457, 1953. doi:10.2307/1969747.
[B060] Armand Borel, Glen Bredon, Edwin E. Floyd, Deane Montgomery, and Richard Palais. Seminar on transformation groups. Number 46 in Ann. of Math. Stud. Princeton Univ. Press, 1960. http://indiana.edu/~jf Stall/Seminar On Transformation Groups.pdf.
[Bos49] Armand Borel and Jean de Siebenthal. Les sous-groupes fermés de rang maximum des groupes de Lie clos. Comment. Math. Helv., 23(1):200–221, 1949. doi:10.1007/bf02565599.
[Bre61] Glen E. Bredon. On homogeneous cohomology spheres. *Ann. of Math.* (2), pages 556–565, 1961. doi:10.2307/1970317.

[Brig8] Michel Brion. Equivariant cohomology and equivariant intersection theory. In *Representation theories and algebraic geometry*, pages 1–37. Springer, 1998. http://link.springer.com/chapter/10.1007/978-94-015-9131-7_1, arXiv:9802063.

[C] Jeffrey D. Carlson. Equivariant formality of isotropic torus actions. *J. Homotopy Relat. Struct.*, 14(1):199–234, 2019. arXiv:1410.5740, doi:10.1007/s40062-018-0207-5.

[CF] Jeffrey D. Carlson and Chi-Kwong Fok. Equivariant formality of isotropy actions. *J. Lond. Math. Soc.*, Mar. 2018. arXiv:1511.06228, doi:10.1112/jlms.12116.

[D] Evgenii Borisovich Dynkin. Semisimple subalgebras of semisimple Lie algebras. *Mat. Sb. (N.S.)*, 30(72)(2):349–462, 1952. http://mi.mathnet.ru/msb5435.

[GGK] Viktor L. Ginzburg, Victor Guillemin, and Yael Karshon. *Moment maps, cobordisms, and Hamiltonian group actions*, volume 98 of *Math. Surveys Monogr.* Amer. Math. Soc., Providence, RI, 2002. http://utmt.utoronto.ca/~karshony/HUJI/monograph/index-pdf.html.

[GHV] Werner H. Greub, Stephen Halperin, and Ray Vanstone. *Connections, curvature, and cohomology*, vol. III: Cohomology of principal bundles and homogeneous spaces. Academic Press, 1976.

[GoN] Oliver Goertsches and Sam Haghshenas Noshari. Equivariant formality of isotropy actions on homogeneous spaces defined by lie group automorphisms. *J. Pure Appl. Algebra*, 220(5):2017–2028, 2016. arXiv:1405.2655, doi:10.1016/j.jpaa.2015.10.013.

[GorKM] Mark Goresky, Robert Kottwitz, and Robert MacPherson. Equivariant cohomology, Koszul duality, and the localization theorem. *Invent. Math.*, 131(1):25–83, 1998. http://math.ias.edu/~goresky/pdf/equivariant.jour.pdf, doi:10.1007/s002220050197.

[H] Allen Hatcher. *Algebraic topology*. Cambridge Univ. Press, 2002. http://math.cornell.edu/~hatcher/AT/ATpage.html.

[IT] Kiminao Ishitoya and Hirosi Toda. On the cohomology of irreducible symmetric spaces of exceptional type. *J. Math. Kyoto Univ.*, 17(2):229–243, 1977. doi:10.1215/kjm/125052765.

[JK] Lisa C. Jeffrey and Frances C. Kirwan. Localization for nonabelian group actions. *Topology*, 34(2):291–327, 1995. arXiv:alg-geom/9307001, doi:10.1016/0040-9383(94)90035-9.

[Kam] Berno Nico Plechelmus Kamerich. *Transitive transformation groups of products of two spheres*. PhD thesis, Katholieke Universiteit Nijmegen, 1997. https://hdl.handle.net/2066/147554.

[KaZ] Vitali Kapovitch and Wolfgang Ziller. Biquotients with singly generated rational cohomology. *Geom. Dedicata*, 104(1):149–160, 2004. doi:10.1023/B:GEDM.0000022860.89824.2f.

[Kerr96] Megan Kerr. Some new homogeneous Einstein metrics on symmetric spaces. *Trans. Amer. Math. Soc.*, 348(1):153–171, 1996. doi:10.1090/S0002-9947-96-01512-7.

[Kerr98] Megan M. Kerr. New examples of homogeneous Einstein metrics. *Michigan Math. J.*, 45(1):115–134, 04 1998. doi:10.1307/mmj/103120806.

[Kr] Linus Kramer. *Homogeneous spaces, Tits buildings, and isoparametric hypersurfaces*, volume 158 of *Mem. Amer. Math. Soc.* Amer. Math. Soc., 2002. arXiv:math:0109133.

[Mat] Yozô Matsushima. On a type of subgroups of a compact Lie group. *Nagoya Math. J.*, 21:1–15, 1951. doi:10.1017/S0027763000009995.

[May] Evgeny Mayanskiy. The subalgebras of $G_2$. 2016. arXiv:1611.04070v1.

[MiT] Mamoru Mimura and Hiroshi Toda. *Topology of Lie groups, I and II*, volume 91 of *Transl. Math. Monogr.* Amer. Math. Soc., Providence, RI, 2000.

[MS] Deane Montgomery and Hans Samelson. Transformation groups of spheres. *Ann. of Math.* (2), pages 454–470, 1943. doi:10.2307/1968975.

[NS] Mara D. Neusel and Larry Smith. *Invariant Theory of Finite Groups*, volume 94 of *Math. Surveys Monogr.* Amer. Math. Soc., 2002.

[On63] Arkadii L’vovich Onishchik. Transitive compact transformation groups. *Mat. Sb. (N.S.)*, 102(4):447–485, 1963. http://mi.mathnet.ru/msb4555.
 Arkadi L. Onishchik. *Topology of transitive transformation groups*. Johann Ambrosius Barth, 1994.

Hiroo Shiga. Equivariant de Rham cohomology of homogeneous spaces. *J. Pure Appl. Algebra*, 106(2):173–183, 1996. doi:10.1016/0022-4049(95)00018-6.

Hiroo Shiga and Hideo Takahashi. Remarks on equivariant cohomology of homogeneous spaces. Technical report 17, Tech. Univ. Nagaoka, May 1995. dl.ndl.go.jp/info:ndljp/pid/8760355.

Norman Earl Steenrod. *The topology of fibre bundles*, volume 14 of *Princeton Math. Ser*. Princeton Univ. Press, 1951.

Hsien-Chung Wang. Homogeneous spaces with non-vanishing Euler characteristics. *Ann. of Math.* (2), 50(4):925–953, 1949. doi:10.2307/1969588.

Joseph Albert Wolf. *Spaces of constant curvature*. AMS Chelsea Publishing, Providence, RI, 6th edition, 2011.

Martin Wolfrom. Isoparametric hypersurfaces with a homogeneous focal manifold. 2002 dissertation. http://opus.bibliothek.uni-wuerzburg.de/files/293/WOLFROM.PDF.

Ichiro Yokota. Explicit isomorphism between SU(4) and Spin(6). *J. Fac. Sci. Shinshu Univ*, 14(1):29–34, 1979.

jeffrey.carlson@tufts.edu

NORTH CHINA ELECTRIC POWER UNIVERSITY
che@ncepu.edu.cn