Wave-Style Token Machines and Quantum Lambda Calculi

Ugo Dal Lago∗ Margherita Zorzi†

Abstract

Particle-style token machines are a way to interpret proofs and programs, when the latter are written following the principles of linear logic. In this paper, we show that token machines also make sense when the programs at hand are those of a simple quantum λ-calculus. This, however, requires generalizing the concept of a token machine to one in which more than one particle travel around the term at the same time. The presence of multiple tokens is intimately related to entanglement and allows to give a simple operational semantics to the calculus, coherently with the principles of quantum computation.

1 Introduction

One of the strongest trends in computer science is the (relatively recent) interest in exploiting new computing paradigms which go beyond the usual, classical one. Among these paradigms, quantum computing plays an important role. In particular, the quantum paradigm is having a deep impact on the notion of a computationally (in)tractable problem. In this respect, two of the most surprising results are due to Peter Shor, who proved that prime factorization of integers and the discrete logarithm can be efficiently solved (i.e. in polynomial time) by a quantum computer [20].

Even if quantum computing has catalyzed the interest of a quite large scientific community, several theoretical aspects are still unexplored. As an example, the definition of a robust theoretical framework for quantum programming is nowadays still a challenge. A number of (paradigmatic) calculi for quantum computing have been introduced in the last ten years. Among them, some functional calculi, typed and untyped, have been proposed [3, 5, 4, 18, 21], but we are still at a stage where it is not clear whether one calculus could be considered canonical. Moreover, the meta-theory of most of these formalisms lack the simplicity of the one of their “classical” siblings.

It is clear that linear logic and quantum computing are strongly related: since quantum data have to undergo restrictions such as no-cloning and no-erasing, it is not surprising that in most of the cited quantum calculi the use of resources is controlled. Linear logic therefore provides an ideal framework where rooting quantum data treatment, but also offers another tool which has not been widely exploited in the quantum setting: its mathematical model in terms of operator algebras, i.e. the Geometry of Interaction (GoI in the following). Indeed, the latter provides a dynamical interpretation and a semantic account of the cut-elimination procedure as a flow of information circulating into a net structure. This idea can be formulated both as an algebra of bounded operators on a infinitely dimensional Hilbert space [10] or as a token-based machine (a rewriting automata model with local transition rules) [11, 14]. Both formulations seem to be promising in the quantum setting. On the one hand, the Hilbert space on top of which the first formulation of GoI is given is precisely the canonical
state space of a quantum Turing machine (see for example [1]). On the other hand, the definition of a token machine provides a mathematically simpler setting, which has already found a role in this context [2, 12].

In this paper, we show that token machines are also a model of a linear quantum λ-calculus $Q_\Lambda$ defined along the lines of van Tonder’s $\lambda_q$ [21]. This allows to give an operational semantics to $Q_\Lambda$ which renders the quantum nature of $Q_\Lambda$ explicit: type derivations become quantum circuits built on exactly the set of gates occurring in the underlying $\lambda$-term. This frees us from the burden of having to define the operational semantics of quantum calculi in reduction style, which is known to be technically challenging in a similar setting [21]. On the other hand, the power of $\beta$-style axioms is retained in the form of an equational theory for which our operational semantics can be proved sound.

Technically, the design of our token machine for $Q_\Lambda$, called $IAM_{Q_\Lambda}$ is arguably more challenging than the one of classical token machines. Indeed, the principles of quantum computing, and the so-called entanglement in particular, force us to go towards wave-style machines, i.e., to machines where more than one particle can travel inside the program at the same time. Moreover, the possibly many tokens at hand are subject to synchronization points, each one corresponding to unitary operators of arity greater than 1. This means that $IAM_{Q_\Lambda}$, in principle, could suffer from deadlocks, let alone the possibility of non-termination. We here prove that these pathological situations can not happen.

In Section 2 we recall the token machine for multiplicative linear logic. In Section 3 we propose a gentle introduction to quantum computing. The calculus $Q_\Lambda$ and its token machine $IAM_{Q_\Lambda}$ are introduced in Section 4 and Section 5 respectively. Main results about $IAM_{Q_\Lambda}$ are in Section 6. Sections 7 and 8 are respectively devoted to related works and conclusion/future plans.

2 Linear Logic and Token Machines

In this section, we give some ideas about the simplest token machine, namely the one for the propositional, multiplicative fragment of linear logic. This not only encourages the unfamiliar reader to understand the basic concepts underlying this concrete approach to the geometry of interaction, but will also be useful in the following, when proving basic results about quantum token machines. More details can be found in [7, 11].

Let $A = \{\alpha, \beta, \ldots\}$ be a countable set of propositional atoms. Formulas of Multiplicative Linear Logic (MLL) are given by the following grammar:

$$ A, B ::= \alpha \mid \alpha^\perp \mid A \otimes B \mid A \multimap B. $$

Linear negation can be extended to all formulas in the usual way:

$$(\alpha^\perp)^\perp = \alpha;$$

$$ A \otimes B^\perp = A^\perp \multimap B^\perp;$$

$$ A \multimap B^\perp = A^\perp \otimes B^\perp. $$

This way, $A^{\perp \perp}$ is just $A$. The one-sided sequent calculus for MLL is very simple:

$$ \vdash A, A^\perp \quad \vdash \Gamma, A, \Delta, A^\perp \quad \vdash \Gamma, \Delta \quad \vdash \Gamma, A \otimes B \quad \vdash \Gamma, A \multimap B \quad \vdash \Gamma, A \multimap B $$

The logic MLL enjoys cut-elimination: there is a terminating algorithm turning any MLL proofs into a cut-free proof of the same conclusion.
Consider the following MLL proof ξ (where different occurrences of the same propositional (co)atom have been numbered):

\[
\begin{array}{c}
\vdash \alpha_4^\perp, \alpha_4 & \text{ax} \\
\vdash \alpha_5^\perp, \alpha_5 & \text{ax} \\
\vdash \beta_2^\perp, \beta_2 & \text{cut} \\
\vdash \beta_3^\perp & \text{cut} \\
\vdash \beta_3^\perp & \text{ax} \\
\vdash \alpha_2^\perp, \alpha_2 \otimes \beta_2^\perp & \otimes \\
\vdash \alpha_1^\perp, \alpha_1 \otimes \beta_1^\perp & \otimes \\
\end{array}
\]

The token machine for ξ is a simple automaton whose internal state is nothing more than an occurrence of a propositional (co)atom in ξ. This state evolves by “following” this occurrence, keeping in mind that atoms go down, while coatoms go up. A run of the token machine of ξ is, as an example, the following one:

\[
\alpha_1^\perp \mapsto \alpha_2^\perp \mapsto \alpha_3^\perp \mapsto \alpha_4^\perp \mapsto \alpha_5^\perp \mapsto \alpha_3^\perp \mapsto \alpha_2^\perp \mapsto \alpha_1^\perp.
\]

This tells us that the occurrences \(\alpha_1^\perp\) and \(\alpha_1\) are somehow related. Similarly, one could find a run relating \(\beta_1\) to \(\beta_1^\perp\). Remarkably, these correspondences survive cut-elimination.

All this can be formalized through the notion of a context, which is an MLL formula with a hole:

\[
\mathcal{C} ::= [\cdot] \mid \mathcal{C} \otimes A \mid A \otimes \mathcal{C} \mid \mathcal{C} \mathcal{Y} A \mid A \mathcal{Y} \mathcal{C}.
\]

\(\mathcal{C}[A]\) is the formula obtained by replacing the unique occurrence of \([\cdot]\) in \(\mathcal{C}\) with \(A\). If \(A = \mathcal{C}[\alpha]\) (\(A = \mathcal{C}[\alpha^\perp]\), respectively), we say that \(\mathcal{C}\) is a positive (negative, respectively) context for \(A\). If \(\mathcal{C}\) is positive (negative, respectively) for \(A\), we sometime write it as \(P_A\) (as \(N_A\), respectively). An atom occurrence in an MLL proof ξ is a pair \((A, \mathcal{C})\) where \(A\) is an occurrence of an MLL formula in ξ and \(\mathcal{C}\) is a context for it. Linear negation can be easily extended to contexts:

\[
\begin{align*}
[\cdot]^\perp &= [\cdot]; \\
(C \otimes B)^\perp &= C^\perp \mathcal{Y} B^\perp; \\
(C \mathcal{Y} B)^\perp &= C^\perp \otimes B^\perp; \\
(A \otimes C)^\perp &= A^\perp \mathcal{Y} C^\perp; \\
(A \mathcal{Y} C)^\perp &= A^\perp \otimes C^\perp.
\end{align*}
\]

Please observe that \(\mathcal{C}\) is a negative context for \(A\) iff \(\mathcal{C}^\perp\) is a positive context for \(A^\perp\). To every proof ξ in MLL, we associate an automaton \(\mathcal{M}_\xi\) which consists of:

- The finite set \(\mathcal{S}_\xi\) of states of \(\mathcal{M}_\xi\), which are all the atom occurrences of ξ;
- a transition relation \(\rightarrow_\xi \subseteq \mathcal{S}_\xi \times \mathcal{S}_\xi\), which is described by the rules in Figure 1.

An atom occurrence in ξ is said to be initial (respectively, final) iff it is in the form \((A, N_A)\) (respectively, in the form \((A, P_A)\)), where \(A\) is one among the formulas among the conclusions of ξ. It is easy to verify that:

- for every non-final occurrence \(O\) there is exactly one occurrence \(P\) such that \(O \rightarrow_\xi P\);
- for every non-initial occurrence \(O\) there is exactly one occurrence \(P\) such that \(P \rightarrow_\xi O\).

As a consequence, every initial occurrence is put in correspondence with a final occurrence in a bijective way — the number of occurrences in ξ is anyway finite, and cycles cannot be reached from initial occurrences. It is this correspondence which is taken as the semantics of ξ, after being shown to be invariant by cut-elimination.

One last observation is now in order. Suppose \(O_1, \ldots, O_n\) are all the initial occurrences for ξ. Then, every occurrence in ξ is visited exactly once along one of the \(n\) maximal computations starting in \(O_1, \ldots, O_n\). This can be proved as follows:
Figure 1: Defining Rules for $\vdash_{\xi}$
• First, prove the statement for any cut-free proof $\xi$, by induction on the structure of $\xi$;
• Then show that if $\xi$ has the property and $\mu$ reduces to $\xi$ by cut-elimination, $\mu$ has the property, too. Incidentally, this shows that cyclic $\Rightarrow \xi$ is acyclic.

3 Quantum Computing in a Nutshell

Quantum computing principles are non-standard notions to the largest part of the “lambda community”. The aim of this section is to provide to the non-expert reader an overview of quantum computing basic concepts. This will guide her or him in understanding the “quantum content” of our calculus (in particular, the meaning of unitary steps and the linear management of quantum data, see Section 4).
Moreover, notions like quantum entanglement, a peculiar feature of quantum data, offers some intuitions about how and why the choice of a wave-style token machine as operational model is the right choice.

The simplest quantum system is a two-dimensional state space whose elements are called quantum bits or qubits for short. The qubit is the most basic unit of quantum information. The most direct way to represent a quantum bit is as a unitary vector in the 2-dimensional Hilbert space $\ell^2(\{0, 1\})$, which is isomorphic to $\mathbb{C}^2$. We will denote with $|0\rangle$ and $|1\rangle$ the elements of the computational basis of $\ell^2(\{0, 1\})$. The states $|0\rangle$ and $|1\rangle$ of a qubit correspond to the boolean constants 0 and 1, which are the only possible values of a classical bit. A qubit, however, can assume other values, different from $|0\rangle$ and $|1\rangle$. In fact, every linear combination $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where $\alpha, \beta \in \mathbb{C}$, and $|\alpha|^2 + |\beta|^2 = 1$, represents a possible qubit state. These states are said to be superposed, and the two values $\alpha$ and $\beta$ are called amplitudes. The amplitudes $\alpha$ and $\beta$ univocally represent the qubit with respect to the computational basis. Given a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, we commonly denote it by the vectorial notation $\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$.

In particular, the vectorial representation of the elements of the computational basis $|0\rangle$ and $|1\rangle$ is the following:

$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

While we can determine the state of a classical bit, for a qubit we can not establish with the same precision the values $\alpha$ and $\beta$: quantum mechanics says that a measurement of a qubit with state $\alpha|0\rangle + \beta|1\rangle$ has the effect of changing the state to $|0\rangle$ with probability $|\alpha|^2$ and to $|1\rangle$ with probability $|\beta|^2$. For example, if $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$, one can observe 0 or 1 with the same probability $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$. In this brief survey on quantum computing, we will not enter in the details about qubit measurement, since the syntax of the calculus $Q\Lambda$ does not include an explicit measurement operator (a constant whose — probabilistic — operational semantics mimics the observation of quantum data). This choice is sound from a theoretical viewpoint, since it is possible to assume to have a unique, final measurement, at the end of the computation. Notwithstanding, the measurement operator is a useful programming tool in order to encode quantum algorithms and the extension of the syntax with a measurement operator is one of our planned future works. For a complete overview about measurement of qubits and relationships between different kind of measurement, see [16].

In order to define arbitrary set of quantum data, we need a generalization of the notion of qubit, called quantum register or, more commonly, quantum state [21, 18, 17]. A quantum register can be viewed as a system of $n$ qubits and, mathematically, it is a normalized vector in the Hilbert space...
$\ell^2(\{0, 1\}^n)$ (\{0, 1\}^n is a compact notation to represent any binary sequence of length \(n\)). The standard computational basis for $\ell^2(\{0, 1\}^n)$ is $B = \{|i\} \mid i \text{ is a binary string of length } n\}$.

**Notation 1** We use the notation $|b_1 \ldots b_k\rangle$ ($b_i \in \{0, 1\}$) for $|b_1\rangle \otimes \ldots \otimes |b_k\rangle$, where $\otimes$ is the tensor product (see below).

With a little abuse of language, we say that the number of qubits \(n\) corresponds to the dimension of the space. Notice that if the dimension is \(n\), then the basis \(B\) contains $2^n$ elements, and each quantum state is a normalized linear combination of these elements:

\[
\alpha_1|00\ldots 0\rangle + \alpha_2|00\ldots 1\rangle + \ldots + \alpha_2^n|11\ldots 1\rangle
\]

**Example 1** Let us consider a 2-level quantum system, i.e. a system of two qubits. Each 2-qubit quantum register is a normalized vector in $\ell^2(\{0, 1\}^2)$ and the computational basis is $\{\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

For example, $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{4}}|01\rangle + \frac{1}{\sqrt{8}}|10\rangle + \frac{1}{\sqrt{8}}|11\rangle$ is a quantum register of two qubits and we can represent it as

\[
\psi = \begin{pmatrix}
\frac{\sqrt{2}}{2} \\
\frac{1}{4} \\
\frac{1}{\sqrt{8}} \\
\frac{1}{\sqrt{8}}
\end{pmatrix}.
\]

An Hilbert space of dimension \(n\) can be built from smaller Hilbert spaces by means of the tensor product $\otimes$. If \(H_1\) is an Hilbert space of dimension \(k\) and \(H_2\) is an Hilbert space of dimension \(m\), \(H_3 = H_1 \otimes H_2\) is an Hilbert space of dimension \(km\) (each element is a vector of \(km\) coordinates obtained by “hooking” a vector in \(H_2\) to a vector in \(H_1\)). In other words, an \(n\)-qubit quantum register with \(n \geq 2\) can be viewed as a composite system. It is possible to combine two (or more) distinct physical systems into a composite one. If the first system is in the state $|\phi_1\rangle$ (a vector in a Hilbert Space \(H_1\)) and the second system is in the state $|\phi_2\rangle$ (a vector in a Hilbert Space \(H_2\)), then the state of the combined system is $|\phi_1\rangle \otimes |\phi_2\rangle$ (a vector in a Hilbert Space \(H_1 \otimes H_2\)).

We will often omit the “$\otimes$” symbol, and will write the joint state as $|\psi_1\rangle|\psi_2\rangle$ or as $|\psi_1\psi_2\rangle$.

Not all quantum states can be viewed as composite systems: this case occurs in presence of entanglement phenomena (see below). Since normalized vectors of quantum data represent physical systems, the (discrete) evolution of systems can be viewed as a suitable transformation on Hilbert spaces. The evolution of a quantum register is linear and unitary. Giving an initial state $|\psi_1\rangle$, for each evolution to a state $|\psi_2\rangle$, there exists a unitary operator \(U\) such that $|\psi_2\rangle = U|\psi_1\rangle$. Informally, “unitary” referred to an algebraic operator on a suitable space means that the normalization constraint of the amplitudes ($\sum_i |\alpha_i|^2 = 1$) is preserved during the transformation. Thus, a quantum physical system, i.e. a normalized vector which represents our data, can be described in term of linear operators and in a deterministic way. In quantum computing we refer to a unitary operator \(U\) acting on a \(n\)-qubit quantum register as an \(n\)-qubit quantum gate. We can represent operators on the $2^n$-dimensional Hilbert space $\ell^2(\{0, 1\}^n)$ with respect to the standard basis of $\mathbb{C}^{2^n}$ as $2^n \times 2^n$ matrices, and it is possible to prove that to each unitary operator on a Hilbert Space it is possible to associate an algebraic representation. Matrices which represent unitary operators enjoy some important property: for example they are easily invertible (reversibility is one of the peculiar features of quantum computing). The application of quantum gates to quantum registers represents the pure quantum computational step and captures the internal evolution of quantum systems. The simplest quantum gates act on a single qubit: they are operators on the space $\ell^2(\{0, 1\})$, represented in $\mathbb{C}^2$ by $2 \times 2$ complex matrices. For
example, the quantum gate $X$ is the unitary operator which maps $|0\rangle$ to $|1\rangle$ and $|1\rangle$ to $|0\rangle$ and it is represented by the matrix

$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
$$

Being a linear operator, it maps a linear combination of inputs to the corresponding linear combination of outputs, and so $X$ maps the general qubit state $\alpha|0\rangle + \beta|1\rangle$ into the state $\alpha|1\rangle + \beta|0\rangle$ i.e

$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}
= 
\begin{pmatrix}
\beta \\
\alpha
\end{pmatrix}
$$

An interesting unitary gate is the *Hadamard gate* denoted by $H$ which acts on the computational basis in the following way:

$$
|0\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad |1\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)
$$

The Hadamard gate, which therefore is given by the matrix

$$
H = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}
$$

is useful when we want to create a superposition starting from a classical state. It also holds that $H(H(|c\rangle)) = |c\rangle$ for $c = \{0, 1\}$. 1-qubit quantum gates can be used in order to build gates acting on $n$-qubit quantum states. If we have a 2-qubit quantum system, we can apply a 1-qubit quantum gate only to one component of the system, and we implicitly apply the identity operator (the identity matrix) to the other one. For example suppose we want to apply $X$ to the first qubit. The 2-qubits input $|\psi_1\rangle \otimes |\psi_2\rangle$ gets mapped to $X|\psi_1\rangle \otimes I|\psi_2\rangle = (X \otimes I)|\psi_1\rangle \otimes |\psi_2\rangle$.

The **CNOT** is one of the most important quantum operators. It is mathematically described by the standard operator $CNOT : \ell^2(\{0, 1\})^2 \to \ell^2(\{0, 1\})^2$ defined by

$$
CNOT|00\rangle = |00\rangle \quad CNOT|10\rangle = |11\rangle \\
CNOT|01\rangle = |01\rangle \quad CNOT|11\rangle = |10\rangle
$$

Intuitively, cnot acts as follows: it takes two distinct quantum bits as inputs and complements the target bit (the second one) if the control bit (the first one) is 1; otherwise it does not perform any action. The control qubit is a “master” agent: its evolution is independent from the evolution of the target bit (if the first input of the cnot is $|\phi\rangle$ the output is the same); the target qubit is a “slave” agent: its evolution is controlled by the value of the first qubit. In some sense, a communication between the agents is required and the quantum circuit is a simple distributed system. By adopting this perspective, controlled operators like cnot acts as “synchronization points” between token (ground type occurrences) in our definition of quantum token machine: this is one of the main features of our semantics (see Section[5]).

Not all quantum states can be viewed as composite systems. In other words, if $|\psi\rangle$ is a state of a tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$, it is not generally true that there exists $|\psi_1\rangle \in \mathcal{H}_1$ and $|\psi_2\rangle \in \mathcal{H}_2$ such that $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$. Instead, it is not always possible to decompose an $n$-qubit register as the tensorial product of $n$ qubits.
These non-decomposable registers are called entangled and enjoy properties that we cannot find in any object of classical physics (and therefore in classical data). If \( n \) qubits are entangled, they behave as if connected, independently of the real physical distance. The strength of quantum computation is essentially based on the existence of entangled states (see, for example, the teleportation protocol [16]).

**Example 2** The 2-qubit states 
\[
|\psi\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle
\]
and 
\[
|\psi\rangle = \frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{2}} |10\rangle
\]
are entangled. The 2-qubit state 
\[
|\phi\rangle = \alpha |00\rangle + \beta |01\rangle
\]
is not entangled. Trivially, notice that it is possible to rewrite it in the mathematically equivalent form 
\[
\phi = |0\rangle \otimes (\alpha |0\rangle + \beta |1\rangle).
\]

A simple way to create an entangled state is to feed a CNOT gate with a target qubit \( |c\rangle \) and a particular control qubit, more precisely the output of the Hadamard gate applied to a base qubit, therefore a superposition \( \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \) or \( \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \). This composition of quantum gates is actually encoded by the terms defined in the Example 3.

We previously said that each \( n \)-ary unitary transformation (or composition of unitary transformations) can be represented by a suitable \( n \times n \) matrix. From a computer science viewpoint, it is common to reason about quantum states transformations in terms of *quantum circuits*. Through the paper, we frequently say that “a lambda term encodes a quantum circuit”. What does this mean? What is a quantum circuit? One more time, this is a long and complex subject and we refer to [16, 15] for a complete and exhaustive explanation. Since quantum circuits are invoked in the proof of Soundness Theorem 1, we give here some intuitions and a qualitative description (enough to understand the Soundness proof) of quantum circuits. We have introduced qubits to store quantum information, in analogy with the classical case. We have also introduced operations acting on them, i.e. quantum gates, and we can think about quantum gates in analogy with gates in classical logic circuits.

A quantum circuit on \( n \) qubits implements an unitary operator on a Hilbert space of dimension \( \mathbb{C}^{2^n} \). This can be views as a primitive collection of quantum gates, each implementing a unitary operator on \( k \) (small) qubits.

It is useful to graphically represent quantum circuit in terms of sequential and parallel composition of quantum gates and wires, as for boolean circuits (notwithstanding, in the quantum case the graphical representation does not reflect the physical realization of the circuit).

For example, the following diagram represents the quantum circuit implemented by the term in Example 3.

![Quantum Circuit Diagram](image)

The calculus QA is purely linear (see Section 4). Each (well typed) lambda terms encode a quantum transformation or, equivalently, a quantum circuit built on the set of (the constants representing)
quantum gates occurring in the lambda-term.

One of the primitive operations in information theory is the copy of a datum. When we deal with quantum data as qubits, quantum information suffers from lack of accessibility in comparison to classical one. In fact, a quantum bit can not be duplicated. This curious feature is well-known in literature as no-cloning property: it does not allow to make a copy of an unknown quantum state (it is only possible to duplicate “trivial” qubits, i.e. basis states $|0\rangle$ and $|1\rangle$). In other words, it is not possible to build a quantum transformation/a quantum circuit able to maps an arbitrary quantum state $|\psi\rangle \otimes |\psi\rangle$. No-cloning property is one of the main difference between classical and quantum data and any paradigmatic quantum language has deal with to this fact. Notwithstanding, even if no-cloning property made the design of quantum languages more challenging, quantum data enjoy some properties (which have no classical counterpart) which can be exploited in the design of quantum algorithms.

4 The Calculus $\mathcal{Q}\Lambda$

An essential property of quantum programs is that quantum data, i.e. quantum bits, should always be uniquely referenced. This restriction follows from the well-known no-cloning and no-erasing properties of quantum physics, which state that a quantum bit cannot be duplicated nor canceled [16].

Syntactically, one captures this restriction by means of linearity: if every abstraction $\lambda x. M$ is such that there is exactly one free occurrence of $x$ in $M$, then the substitution triggered by firing any redex is neither copying nor cancelling and, as a consequence, coherent with the just stated principles.

In this Section, we introduce a quantum linear $\lambda$-calculus in the style of van Tonder’s $\lambda_q$ [21] and give an equational theory for it. This is the main object of study of this paper, and is the calculus for which we will give a wave-style token machine in the coming sections.

4.1 The Language of Terms

Let us fix a finite set $\mathcal{U}$ of unitary operators, each on a finite-dimensional Hilbert space $\mathbb{C}^{2^n}$, where $n$ can be arbitrary. To each such $U \in \mathcal{U}$ we associate a symbol $U$ and call $n$ the arity of $U$. The syntactic categories of patterns, bits, constants and terms are defined by the following grammar:

\[
\begin{align*}
\pi & ::= x \mid \langle x, y \rangle; \\
B & ::= |0\rangle_n \mid |1\rangle_n; \\
C & ::= B \mid U; \\
M, N & ::= x \mid C \mid M \otimes N \mid MN \mid \lambda \pi.M; \\
\end{align*}
\]

where $n$ ranges over $\mathbb{N}$ and $x$ ranges over a denumerable, totally ordered set of variables $V$. We always assume that the natural numbers occurring next to bits in any term $M$ are pairwise distinct. This condition, by the way, is preserved by substitution when the substituted variable occurs (free) exactly once. Whenever this does not cause ambiguity, we elide labels and simply write $|b\rangle$ for a bit. Notice that pairs can be formed via the binary operator $\otimes$. We will sometime write $|b_1 b_2 \ldots b_k\rangle$ for $|b_1\rangle \otimes |b_2\rangle \otimes \ldots \otimes |b_k\rangle$ (where $b_1, \ldots, b_n \in \{0, 1\}$). In the following, capital letters such as $M, N, L, Q$ (possibly indexed), denote terms. We work modulo variable renaming; in other words, terms are equivalence classes modulo $\alpha$-conversion. Substitution up to $\alpha$-equivalence is defined in the usual way. Observe that the terms of $\mathcal{Q}\Lambda$ are the ones of a $\lambda$-calculus with pairs (which are accessed by pattern-matching) endowed with constants for bits and unitary operators. We don’t consider measurements here, and discuss the possibility of extending the language of terms in Section 8.
Since in QA all terms are assumed to be non-duplicable by default, we adopt a linear type-discipline. Formally, the set of types is defined as

\[ A := B | A \multimap B | A \otimes B, \]

where \( B \) is the ground type of qubits. We write \( B^n \) for the \( n \)-fold tensor product

\[ \underbrace{B \otimes \ldots \otimes B}_n. \]

Judgements are defined from a linear notion of environment.

- A linear environment \( \Gamma \) is a (possibly empty) finite set of assignments in the form \( x : A \). We impose that in a linear environment, each variable \( x \) occurs at most once.
- If \( \Gamma \) and \( \Delta \) are two linear environments assigning types to distinct sets of variables, \( \Gamma, \Delta \) is their union.
- A judgement is an expression \( \Gamma \vdash M : A \), where \( \Gamma \) is a linear environment, \( M \) is a term, and \( A \) is a type in QA.

Typing rules are in Figure 2. Observe that contexts are treated multiplicatively and, as a consequence, variables always appear exactly once in terms. In other words, a strictly linear type discipline is enforced.

**Example 3** Consider the following term:

\[ M_{EPR} = \lambda (x, y). \text{CNOT}(Hx \otimes y). \]

\( M_{EPR} \) encodes the quantum circuit which takes two input qubits and returns an entangled state (a quantum state that cannot in general be expressed as the tensor product of single qubits). It can be given the type \( B \otimes B \multimap B \otimes B \) in the empty context. Indeed, here is a type derivation \( \pi_{EPR} \) for it:

\[
\frac{\cdot \vdash H : B \multimap B \quad x : B \vdash x : B}{\cdot \vdash H : x : B} \quad \text{(E_\otimes)}
\]

\[
\frac{\cdot \vdash \text{CNOT} : B \otimes B \multimap B \otimes B}{\cdot \vdash \text{CNOT} : x : B, y : B \vdash Hx \otimes y : B \otimes B} \quad \text{(E_\otimes)}
\]

\[
\frac{\cdot \vdash M_{EPR} : B \otimes B \multimap B \otimes B}{\cdot \vdash M_{EPR} : x : B, y : B \vdash M_{EPR} : x : B \otimes B} \quad \text{(E_\otimes)}
\]
$M_{EPR}$ and $\pi_{EPR}$ will be used as running examples in the rest of this paper, together with the following type derivation $\rho_{EPR}$:

\[
\begin{align*}
\pi_{EPR} \triangleright & \vdash M_{EPR} : B \otimes B \rightarrow B \otimes B \\
& \vdash |0\rangle_1 \otimes |1\rangle_2 : B \otimes B \\
\cdot & \vdash M_{EPR}(|0\rangle_1 \otimes |1\rangle_2) : B \otimes B
\end{align*}
\]

If $\pi \triangleright \Gamma \vdash (\lambda x. M) N : A$, one can build a type derivation $\pi^\uparrow$ with conclusion $\Gamma \vdash M\{x/N\} : A$ in a canonical way, by going through a constructive substitution lemma. Similarly when $\pi \triangleright \Gamma \vdash (\lambda \langle x, y \rangle. M) (N \otimes L) : A$.

**Lemma 1** If $\pi \triangleright \Gamma, x_1 : A_1, \ldots, x_n : A_n \vdash M : B$ and for every $1 \leq i \leq n$ there is $\rho_i \triangleright \Delta_i \vdash N_i : A_i$, then there is a canonically defined derivation $\pi\{x_1, \ldots, x_n/\rho_1, \ldots, \rho_n\}$ of $\Gamma, \Delta_1, \ldots, \Delta_n \vdash M\{x_1, \ldots, x_n/N_1, \ldots, N_n\} : B$.

**Proof.** Just proceed by the usual, simple induction on $\pi$. \qed

The notion of type derivation $\pi$ of a term $M$ and the related definition of $\pi^\uparrow$, the type derivation of the reduct of $M$, will be generalized in the following section taking into account quantum superposition.

### 4.3 An Equational Theory

The $\lambda$-calculus is usually endowed with notions of reduction or equality, both centered around the $\beta$-rule, according to which a function $\lambda x. M$ applied to an argument $N$ reduces to (or can be considered equal to) the term $M\{N/x\}$ obtained by replacing all free occurrences of $x$ with $N$. A reduction relation implicitly provides the underlying calculus with a notion of computation, while an equational theory is more akin to a reasoning technique. Giving a reduction relation on $Q\Lambda$ terms directly, however, is problematic. What happens when a $n$-ary unitary operator $U$ is faced with an $n$-tuple of qubits $|b_1 \ldots b_n\rangle$? Superposition should somehow arise, but how can we capture it?

In this section, an equational theory for $Q\Lambda$ will be introduced. In the next sections, we will prove that the semantics induced by token machines is sound with respect to it. The equational theory we are going to introduce will be a binary relation on formal, weighted sums of type derivations for $Q\Lambda$ terms.

**Definition 1 (Superposed Type Derivation)** A superposed type derivation of type $(\Gamma, A)$ is a formal sum

\[
\mathcal{T} = \sum_{i=1}^{n} \kappa_i \pi_i
\]

where for every $1 \leq i \leq n$, $\kappa_i \in \mathbb{C}$ and it holds that $\pi_i \triangleright \Gamma \vdash M_i : A$. In this case, we write $\Gamma \vdash \mathcal{T} : A$. Superposed type derivations will be denoted by metavariables like $\mathcal{T}$ or $\mathcal{S}$.

Please, notice that:
- If $\pi \triangleright \cdot \vdash U|b_1 \ldots b_k\rangle$, then $\pi^\uparrow$ is a superposed type derivation in the form $\sum_{x \in B_k} \kappa_x \pi_x$, where $B_k$ is the set of all binary strings of length $k$, $\pi_x$ is the trivial type derivation for $|x\rangle$, and $\kappa_x$ is the complex number corresponding to $|x\rangle$ in the vector $U|b_1 \ldots b_k\rangle$. 


### Axioms

\[
\pi \vdash \Gamma \vdash (\lambda x.M)(N \otimes L) : A \\
\pi \approx \pi' \quad \text{beta}
\]

\[
\pi \vdash \Gamma \vdash (\lambda x.M)N : A \\
\pi \approx \pi' \quad \text{beta}
\]

\[
\pi \vdash \Gamma \vdash U|b_1 \ldots b_k : B_k \\
\pi \approx \pi' \quad \text{quant}
\]

### Context Closure

| Rule | Left | Right |
|------|------|-------|
| l.a  | \( T \approx S \) | \( T \pi \approx S \pi \) |
| r.a  | \( T \approx S \) | \( \pi T \approx \pi S \) |
| in.\( \lambda \) | \( T \approx S \) | \( \lambda x.T \approx \lambda x.S \) |
| in.\( \lambda.pair \) | \( T \approx S \) | \( \lambda(x,y).T \approx \lambda(x,y).S \) |
| l.in.tens | \( T \approx S \) | \( T \otimes \pi \approx S \otimes \pi \) |
| r.in.tens | \( T \approx S \) | \( \pi \otimes T \approx \pi \otimes S \) |
| sum | \( T \approx S \) | \( \alpha T + \nu \approx \alpha S + \nu \) |

### Reflexive, Symmetric and Transitive Closure

| Rule | Left | Right |
|------|------|-------|
| refl | \( T \approx T \) | \( S \approx \) |
| sym | \( T \approx S \) | \( S \approx T \) |
| trans | \( T \approx S \approx \nu \) | \( \\nu \approx T \) |

### Figure 3: Equational Theory

- If \( \pi \vdash \Gamma \vdash (\lambda x.M)N : A \), \( \pi' \) is the type derivation with conclusion \( \Gamma \vdash M\{x/N\} : A \) built in a canonical way, by going through a constructive substitution lemma. Similarly when \( \pi \vdash \Gamma \vdash (\lambda x.M)(N \otimes L) : A \).
- All the term constructs can be generalized to operators on superposed type derivations, with the proviso that the types match. As an example if \( T = \sum_i \alpha_i \pi_i \) where \( \pi_i \vdash \Gamma \vdash M_i : A \rightarrow B \) and \( \rho \vdash \Delta \vdash N : A \), \( T \rho \) denotes the superposed type derivation \( S = \sum_i \alpha_i \sigma_i \) where \( \sigma_i \vdash \Gamma, \Delta \vdash M_i N : B \) and each \( \sigma_i \) is obtained applying the rule \( (E_{\rightarrow}) \) to \( \pi_i \) and \( \rho \).

A binary relation \( \approx \) on superposed type derivations having the same type can be given by way of the rules in Figure 3 where we tacitly assume that the involved superposed type derivations have the appropriate type whenever needed. Notice that \( \approx \) is by construction an equivalence relation. When the underlying type derivation is clear from the context, we denote superposed derivations simply by superposed terms. As an example, consider the term \( M_{EPR}(|0\rangle_1 \otimes |1\rangle_2) \) from Example 3 and the corresponding type derivation \( \rho_{EPR} \) for it. It is convenient to be able to reason as follows, directly on the former:
\[ M_{EPR}(|0\rangle \otimes |1\rangle) \approx CNOT(H|0\rangle \otimes |1\rangle) \]
\[ \approx \frac{1}{\sqrt{2}} CNOT(|0\rangle \otimes |1\rangle) + \frac{1}{\sqrt{2}} CNOT(|1\rangle \otimes |1\rangle) \]
\[ \approx \frac{1}{\sqrt{2}} |0\rangle \otimes |1\rangle + \frac{1}{\sqrt{2}} |1\rangle \otimes |1\rangle \]

Please observe that the equational theory we have just defined can hardly be seen as an operational semantics for QA. Although equations can of course be oriented, it is the very nature of a superposed type derivation which is in principle problematic from the point of view of quantum computation: what is the mathematical nature of a superposed type derivation? Is it an element of an Hilbert Space? And if so, of which one? If we consider a simple language such as QA, the questions above may appear overly rhetorical, but we claim they are not. For example, what would be the quantum meaning of linear beta-reduction? If we want to design beta-reduction according to the principles of quantum computation, it has to be, at least, easily reversible (unless measurement is implicit in it). Moving towards more expressive languages, this non-trivial issue becomes more difficult and a number of constraints have to be imposed (for example, superposition of terms can be allowed, but only between “homogenous” terms, i.e. terms which have an identical skeleton \[21\]). This is the reason for which promising calculi \[21\] fail to be canonical models for quantum programming languages. This issue has been faced in literature without satisfactory answers, yielding a number of convincing arguments in favor of the (implicit or explicit) classical control of quantum data \[3, 18\].

4.3.1 Equational Theory Derivations in Normal Form

Sometime it is quite useful to assume that a derivation for \( T \approx S \) is in a peculiar form, defined by giving an order on the rules in Figure 3. More specifically, define the following two sets of rules:

\[ AX = \{ \text{beta, beta.pair, quant} \}; \]
\[ CC = \{ \text{l.a, r.a, in.\lambda, in.\lambda.pair, l.in.tens, r.in.tens} \}. \]

A derivation of \( T \approx S \) is said to be in normal form (and we write \( T \sim S \)) iff:
- either the derivation is obtained by applying rule refl;
- or any branch in the derivation consists in instances of rules from AX, possibly followed by instances of rules in CC, possibly followed by instances of sum, possibly followed by instances of sym, possibly followed by instances of trans.

In other words, a derivation of \( T \approx S \) is in normal form iff rules are applied in a certain order. As an example, we cannot apply transitivity or symmetry closure rules too early, i.e., before context closure rules. One may wonder whether this restricts the class of provable equivalences. Infact it does not:

**Proposition 1** \( T \approx S \) iff \( T \sim S \).

**Proof.** If \( T \sim S \), then of course \( T \approx S \). The converse can be showed by induction on the height \( n \) of a proof of \( T \approx S \), enriching the thesis by prescribing that the height of the obtained proof of \( T \approx S \) must be at most \( n \):
- If \( T \approx S \) is proved by rules in AX or by refl, then by definition \( T \sim S \).
• If \( T \approx S \) is derived by rules in CC from a proof \( \pi \), then:
  
  - If the rules in \( \pi \) are all from AX and CC, then there is nothing to do.
  - If the last rule in \( \pi \) is sum, then we can apply one of the following transformations, so as to be able to apply the induction hypothesis:
    
    \[
    \begin{align*}
    & V \approx X \\
    & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
\[\begin{align*}
V \approx X \quad X \approx Y & \quad \text{trans} \\
\quad V \approx Y & \quad \Rightarrow \quad \text{trans} \\
\alpha V + W \approx \alpha Y + W & \quad \text{sum} \\
\quad \alpha X + W \approx \alpha Y + W \quad & \quad \text{sum} \\
\end{align*}\]

- If the last rule in \(\pi\) is \(\text{refl}\), then we can derive \(T \approx S\) by a single application of \(\text{refl}\).
- If \(T \approx S\) is derived by \(\text{sym}\) from a proof \(\pi\), then:
  - If the rules in \(\pi\) are all from \(\text{AX}\) or \(\text{CC}\), or are \(\text{sum}\) or \(\text{sym}\), then there is nothing to do.
  - If the last rule in \(\pi\) is \(\text{trans}\), then we can apply the following transformation, so as to be able to apply the induction hypothesis:
    \[\begin{align*}
    V \approx X \quad X \approx Y & \quad \text{trans} \\
    \quad V \approx Y & \quad \Rightarrow \quad \text{sym} \\
    \alpha V + W \approx \alpha Y + W & \quad \Rightarrow \quad \text{trans} \\
    \end{align*}\]
- If the last rule in \(\pi\) is \(\text{refl}\), then we can derive \(T \approx S\) by a single application of \(\text{refl}\).
- If \(T \approx S\) is derived by \(\text{trans}\) from two proofs of \(\pi\) and \(\rho\), then if either \(\pi\) or \(\rho\) is derived by \(\text{refl}\), then the required proof is already in our hand. Otherwise, there is nothing to do.

This concludes the proof. \(\square\)

5 A Token Machine for \(Q\Lambda\)

In this section we describe an interpretation of \(Q\Lambda\) type derivations in terms of a specific token machine called IAM\(Q\Lambda\).

With a slight abuse of notation, a permutation \(\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}\) will be often applied to sequences of length \(n\) with the obvious meaning: \(\sigma(a_1, \ldots, a_n) = a_{\sigma(1)}, \ldots, a_{\sigma(n)}\). Similarly, such a permutation can be seen as the unique unitary operator on \(\mathbb{C}^2^n\) which sends \(|b_1 \ldots b_n\rangle\) to \(|b_{\sigma(1)} \ldots b_{\sigma(n)}\rangle\). Suppose given an operator \(U \in U\) of arity \(n \in \mathbb{N}\). Now, take a natural number \(m \geq n\) and \(n\) distinct natural numbers \(j_1, \ldots, j_n\), all of them smaller or equal to \(m\). With \(U_{j_1 \ldots j_n}^m\) (or simply with \(U_{j_1 \ldots j_n}\)) we indicate the operator of arity \(m\) which acts like \(U\) on the qubits indexed with \(j_1, \ldots, j_n\) and leave all the other qubits unchanged.

In the following, with a slight abuse of notation, occurrences of types in type derivations are confused with types themselves. On the other hand, occurrences of types inside other types will be defined quite precisely, as follows. \textit{Contexts} (types with an hole) are denoted by metavariables like \(C, D\). A context \(C\) is said to be a context for a type \(A\) if \(C[B] = A\). \textit{Negative contexts} (i.e., contexts where the hole is in negative position) are denoted by metavariables like \(N, M\). \textit{Positive} ones are denoted by metavariables like \(P, Q\). An occurrence of \(B\) in the type derivation \(\pi\) is a pair \((A, C)\), where \(A\) is an occurrence of a type in \(\pi\) and \(C\) is a context for \(A\). Sequences of occurrences are indicated with metavariables like \(\varphi, \psi\) (possibly indexed). All sequences of occurrences we will deal with do not contain duplicates. Type constructors \(\rightarrow\) and \(\otimes\) can be generalized to operators on occurrences and sequences of occurrences, e.g. \((A, C) \rightarrow B\) is just \((A \rightarrow B, C \rightarrow B)\).

Given (an occurrence of) a type \(A\), all positive and negative occurrences of \(B\) inside \(A\) can be put
in sequences called $\mathcal{P}(A)$ and $\mathcal{N}(A)$ as follows (where $\cdot$ is sequence concatenation):

\[
\mathcal{P}(B) = (B, \varepsilon);
\]

\[
\mathcal{N}(B) = \varepsilon;
\]

\[
\mathcal{P}(A \otimes B) = (\mathcal{P}(A) \otimes B) \cdot (A \otimes \mathcal{P}(B));
\]

\[
\mathcal{N}(A \otimes B) = (\mathcal{N}(A) \otimes B) \cdot (A \otimes \mathcal{N}(B));
\]

\[
\mathcal{P}(A \rightarrow B) = (\mathcal{N}(A) \rightarrow B) \cdot (A \rightarrow \mathcal{P}(B));
\]

\[
\mathcal{N}(A \rightarrow B) = (\mathcal{P}(A) \rightarrow B) \cdot (A \rightarrow \mathcal{N}(B)).
\]

As an example, the positive occurrences in the type $B \rightarrow B \otimes B$ should be the two rightmost ones. And indeed:

\[
\mathcal{P}(B \rightarrow \o B \otimes B) = (\mathcal{N}(B) \rightarrow \o B \otimes B) \cdot (B \rightarrow \mathcal{P}(B \otimes B))
\]

\[
= \varepsilon \cdot (B \rightarrow \mathcal{P}(B \otimes B)) = B \rightarrow \mathcal{P}(B \otimes B)
\]

\[
= (B \rightarrow (\mathcal{P}(B) \otimes B)) \cdot (B \rightarrow (B \otimes \mathcal{P}(B)))
\]

\[
= (B, B \rightarrow ([: \otimes B]), (B, B \rightarrow ([ \cdot [:]))).
\]

For every type derivation $\pi$, $B(\pi)$ is the sequence of all occurrences of $B$ in $\pi$ which are introduced by the rules $(\mathfrak{a}_q0)$ and $(\mathfrak{a}_q1)$ (from Figure 2). Similarly, $V(\pi)$ is the corresponding sequence of binary digits, seen as a vector in $\mathbb{C}^{2^{B(\pi)}}$. Both $B(\pi)$ and in $V(\pi)$, the order is the one induced by the natural number labeling the underlying bit in $\pi$. As an example, consider the following type derivation, and call it $\pi$:

\[
\begin{array}{c}
\frac{\cdot \vdash |0\rangle_2 : \mathbb{B}_1 \cdot \vdash |1\rangle_1 : \mathbb{B}_2}{\cdot \vdash |0\rangle_2 \otimes |1\rangle_1 : \mathbb{B}_3 \otimes \mathbb{B}_4}
\end{array}
\]

There are four occurrences of $B$ in it, and we have indexed it with the first four positive natural numbers, just to be able to point at them without being forced to use the formal, context machinery. Only two of them, namely the upper ones, are introduced by instances of the rules $(\mathfrak{a}_q0)$ and $(\mathfrak{a}_q1)$. Moreover, the rightmost one serves to type a bit having an index (namely 1) greater than the one in the other instance (namely 2). As a consequence, $B(\pi)$ is the sequence $\mathbb{B}_2, \mathbb{B}_1$. The two instances introduces bits 0 and 1; then $V(\pi) = |1\rangle \otimes |0\rangle$. As another example, one can easily compute $B(\pi_{EPR})$ and $V(\pi_{EPR})$ (where $\pi_{EPR}$ is from Example 3), finding out that both are the empty sequence.

Finally, we are able to define, for every $\pi$, the abstract machine $A_{\pi}$ interpreting it:

- The states of $A_{\pi}$ form a set $S_{\pi}$ and are in the form $(O_1, \ldots, O_n, Q)$ where:
  - $O_1, \ldots, O_n$ are occurrences of the type $B$ in $\pi$;
  - $Q$ is a quantum register on $n$ qubits, i.e. a normalized vector in $\mathbb{C}^2^n$ (see Section 3).
- The transition relation $\rightarrow_{\pi} \subseteq S_{\pi} \times S_{\pi}$ is defined based on $\pi$, following Figure 4 and Figure 5. In the latter, each of the $2n$ occurrences of $B$ in the type of $U$ is simply denoted through its index, and for every $1 \leq k \leq m$, $i_k$ is the position of $\mathbb{B}_k$ in the sequence $(\varphi_1, \mathbb{B}_{j_1}, \varphi_2, \ldots, \varphi_m, \mathbb{B}_{j_m}, \varphi_{m+1})$. The number of positive (negative, respectively) occurrences of $B$ in the conclusion of $\pi$ is said to be the output arity (the input arity, respectively) of $\pi$. Given a type derivation $\pi$, the relation $\rightarrow_{\pi}$ enjoys a strong form of confluence:

**Proposition 2 (One-step Confluence of $\rightarrow_{\pi}$)** Let $S, R, T \in S_{\pi}$ be such that $S \rightarrow_{\pi} R$ and $S \rightarrow_{\pi} T$. Then either $R = T$ or there exists a state $U$ such that $R \rightarrow_{\pi} U$ and $T \rightarrow_{\pi} U$.  

16
Figure 4: Quantum GoI Machine — Classical Rules

\[
\begin{array}{c}
x : A_1 \vdash x : A_2 \\
\Gamma_1, x : A_1 \vdash M : B_1 \\
\Gamma_2 \vdash \lambda x. M : A_2 \rightarrow B_2 \\
\Gamma_1, x : A_1, y : B_1 \vdash M : C_1 \\
\Gamma_2 \vdash x(y). M : (A_2 \otimes B_2) \rightarrow C_2 \\
\Gamma_1 \vdash M : A_1 \rightarrow B_1 \\
\Delta_1 \vdash N : A_2 \\
\Gamma_2, \Delta_2 \vdash MN : B_2 \\
\Gamma_1 \vdash M : A_1 \\
\Delta_1 \vdash N : B_1 \\
\Gamma_2, \Delta_2 \vdash M \otimes N : A_2 \otimes B_2 \\
\end{array}
\]

\[(\varphi, (A_1, P), \psi), Q) \rightarrow_\pi ((\varphi, (A_2, P), \psi), Q)\]
\[(\varphi, (A_2, N), \psi), Q) \rightarrow_\pi ((\varphi, (A_2 \rightarrow B_2, N \rightarrow B_2), \psi), Q)\]
\[(\varphi, (A_2 \rightarrow B_2, P \rightarrow B_2), \psi), Q) \rightarrow_\pi ((\varphi, (B_1, P), \psi), Q)\]
\[(\varphi, (A_2 \rightarrow B_2, \psi), Q) \rightarrow_\pi ((\varphi, (B_1, N), \psi), Q)\]
\[(\varphi, (A_2 \rightarrow B_2, \psi), Q) \rightarrow_\pi ((\varphi, (B_2, P), \psi), Q)\]
\[(\varphi, (A_1 \rightarrow B_1, A \rightarrow N), \psi), Q) \rightarrow_\pi ((\varphi, (\Gamma_1, P), \psi), Q)\]
\[(\varphi, (\Gamma_2, P), \psi), Q) \rightarrow_\pi ((\varphi, (\Gamma_1, N), \psi), Q)\]
\[(\varphi, (\Delta_2, P), \psi), Q) \rightarrow_\pi ((\varphi, (\Delta_1, P), \psi), Q)\]
\[(\varphi, (\Delta_1, N), \psi), Q) \rightarrow_\pi ((\varphi, (\Delta_1, P), \psi), Q)\]

\[(\varphi, (A_1, N), \psi), Q) \rightarrow_\pi ((\varphi, (A_1 \rightarrow B_1, P \rightarrow B_1), \psi), Q)\]

\[(\varphi, (A_2 \otimes B_2, N \otimes B_2), \psi), Q) \rightarrow_\pi ((\varphi, (A_1, N), \psi), Q)\]
\[(\varphi, (A_2 \otimes B_2, A \otimes N), \psi), Q) \rightarrow_\pi ((\varphi, (B_1, N), \psi), Q)\]
\[(\varphi, (A_2 \otimes B_2, P \otimes B_2), \psi), Q) \rightarrow_\pi ((\varphi, (B_1, P), \psi), Q)\]
\[(\varphi, (A_2 \otimes B_2, \psi), Q) \rightarrow_\pi ((\varphi, (\Gamma_1, N), \psi), Q)\]
\[(\varphi, (\Gamma_2, N), \psi), Q) \rightarrow_\pi ((\varphi, (\Gamma_2, N), \psi), Q)\]
\[(\varphi, (\Delta_1, N), \psi), Q) \rightarrow_\pi ((\varphi, (\Delta_1, P), \psi), Q)\]
\[(\varphi, (\Delta_2, N), \psi), Q) \rightarrow_\pi ((\varphi, (\Delta_2, P), \psi), Q)\]

Figure 5: Quantum GoI Machine — Quantum Rules

\[
\vdash U : B_1 \otimes \ldots \otimes B_m \rightarrow B_{m+1} \otimes \ldots \otimes B_{2m} \\
((\varphi_1, B_{j_1}, \varphi_2, \ldots, \varphi_m, B_{j_m}, \varphi_{m+1}), Q) \\
\rightarrow_\pi ((\varphi_1, B_{j_1+m}, \varphi_2, \ldots, \varphi_m, B_{j_m+m}, \varphi_{m+1}), U^{1 \rightarrow m}(Q))
\]
Proof. By simply inspecting the various rules. Notice that there are no critical pairs in $\rightarrow_\pi$. □

Suppose, for the sake of simplicity, that $\pi$ is a type derivation of $\cdot \vdash M : A$. An initial state for $Q$ is a state in the form $(N(A) \cdot B(\pi), Q \otimes \nu(\pi))$. Given a permutation $\sigma$ on $n$ elements, a final state for $Q$ is one in the form $(\phi, Q \otimes \nu(\pi))$, where $\phi = \sigma(P_\pi)$.

Definition 2 Given a type derivation $\pi$, the partial function computed by $\pi$ is $[\pi] : C^2_n \rightarrow C^2_m$ (where $n$ and $m$ are the input and output arity of $\pi$) and is defined by stipulating that $[\pi](Q) = R$ iff any initial state for $Q$ rewrites into a final state for $S$ and $\sigma$, where $S = \sigma^{-1}(R)$.

Given a type derivation $\pi$, $[\pi]$ is either always undefined or always defined. Indeed, the fact any initial configuration (for, say, $Q$) rewrites to a final configuration or not does not depend on $Q$ but only on $\pi$:

Lemma 2 (Uniformity) For every type derivation $\pi$ and for every occurrences $O_1, \ldots, O_n, P_1, \ldots, P_n$, there is a unitary operator $U$ such that whenever $(O_1, \ldots, O_n, Q) \rightarrow_\pi (P_1, \ldots, P_n, R)$ it holds that $R = U(Q)$.

Proof. Observe that for every $O_1, \ldots, O_n, P_1, \ldots, P_n$ there is at most one of the rules defining $\rightarrow_\pi$ which can be applied. Moreover, notice that each rule acts uniformly on the underlying quantum register. □

In the following section, we will prove that $[\pi]$ is always a total function, and that it makes perfect sense from a quantum point of view.

6 Main Properties of IAM$_{QA}$

In this section, we will prove some crucial results about IAM$_{QA}$. More specifically, we prove that runs of this token machine are indeed finite and end in final states. We proceed by putting QA in correspondence to MLL, thus inheriting the same kind of very elegant and powerful results enjoyed by MLL token machines.

6.1 A Correspondence Between MLL and QA

Any type derivation $\pi$ can be put in correspondence with some MLL proofs. We inductively define the map $(\cdot)^*$ from QA types to MLL formulas as follows:

$$(B)^* = \alpha;$$

$$(A \multimap B)^* = (A)^* \perp \forall (B)^*;$$

$$(A \otimes B)^* = (A)^* \otimes (B)^*.$$ 

Given a judgment $J = \Gamma \vdash M : A$ and a natural number $n \in \mathbb{N}$, the MLL sequent corresponding to $J$ and $n$ is the following one:

$\vdash \alpha^+, \ldots, \alpha^+, ((B_1)^*)^+, \ldots, ((B_m)^*)^+, (A)^*$,

where $\Gamma = x_1 : B_1, \ldots, x_m : B_m$. For every $\pi$, we define now a set of MLL proofs $\mathcal{F}(\pi)$. This way, every type derivation $\pi$ for $J = \Gamma \vdash M : A$ such that $n$ bits occur in $M$, is put in relation to
possibly many MLL proofs of the sequent corresponding to $J$ and $n$. One among them is called the canonical proof for $\pi$. The set $\mathcal{I}(\pi)$ and canonical proofs are defined by induction on the structure of the underlying type derivation $\pi$:

- If $\pi$ is the type derivation
  \[
  \vdash [0] : B^{(a_{q0})},
  \]
  then the only proof $\xi$ in $\mathcal{I}(\pi)$ is an atomic axiom. Similarly if the only rule in $\pi$ is $(\alpha U)$. Please notice that $\pi$ contains one bit, and as a consequence $\xi$ has the correct conclusion.

- If $\pi$ is
  \[
  \vdash U : B^n \rightarrow B^n^{(a_{q})},
  \]
  then $\pi$ is in correspondence to all of the $n!$ possible cut-free proofs of the sequent
  \[
  \vdash ((\alpha \otimes \ldots \otimes \alpha) \vdash \gamma) (\alpha \otimes \ldots \otimes \alpha)
  \]
  $n$ times $n$ times
  obtained by starting from $n$ instances of an atomic axiom, gluing them together by the rule $\otimes$, and finally choosing one of the $n!$ possible permutations before applying $n$ times rule $\gamma$. The canonical proof is the one corresponding to the identity permutation.

- If $\pi$ is the type derivation
  \[
  \vdash x : A \vdash x : (A_v)
  \]
  then the only proof corresponding to $\pi$ is the following
  \[
  \vdash (A)^\perp, (A)^\perp
  \]

- If $\pi$ is
  \[
  \rho \triangleright \Gamma, x : A \vdash M : B
  \]
  \[
  \Gamma \vdash \lambda x.M : A \rightarrow B^{(1_{\omega})}
  \]
  where $\Gamma = x_1 : A_1, \ldots, x_m : A_m$. Then for all possible MLL proof $\mu \in \mathcal{I}(\rho)$ of the MLL sequent
  \[
  J = \vdash \alpha^\perp, \ldots, \alpha^\perp, ((A_1)^\perp)^\perp, (A_1)^\perp \perp, \ldots, ((A_m)^\perp)^\perp, (A_m)^\perp \perp, (B)^\perp
  \]
  $n$ times
  the following MLL proof is in $\mathcal{I}(\pi)$:
  \[
  \mu \triangleright J
  \]
  \[
  \vdash \alpha^\perp, \ldots, \alpha^\perp, ((A_1)^\perp)^\perp, (A_1)^\perp \perp, \ldots, ((A_m)^\perp)^\perp, (A_m)^\perp \perp, (B)^\perp \perp \gamma
  \]
  $n$ times

- If $\pi$ is
  \[
  \rho \triangleright \Gamma, x : A, y : B \vdash M : C
  \]
  \[
  \Gamma \vdash \lambda(x, y).M : (A \otimes B) \rightarrow C^{(1_{\omega})}
  \]
  where $\Gamma = z_1 : D_1, \ldots, z_m : D_m, x : A, y : B$, then for all possible MLL proofs $\mu \in \mathcal{I}(\rho)$ of the MLL sequent
  \[
  J = \vdash \alpha^\perp, \ldots, \alpha^\perp, ((D_1)^\perp)^\perp, (A)^\perp \perp, (A^\perp)^\perp, \ldots, ((D_m)^\perp)^\perp, (A)^\perp \perp, (B)^\perp \perp, (C)^\perp
  \]
  $n$ times
the following MLL proof is in $\mathcal{J}(\pi)$:

$$\vdash \alpha^\perp,\ldots,\alpha^\perp,((D_1)^\perp)^\perp,\ldots,((D_m)^\perp)^\perp,((A)^\perp)^\perp \not\subseteq (B)^\perp, (C)^\perp$$

\[\text{n times}\]

$$\vdash \alpha^\perp,\ldots,\alpha^\perp,((D_1)^\perp)^\perp,\ldots,((D_m)^\perp)^\perp,((A)^\perp)^\perp \not\subseteq (B)^\perp \not\subseteq (C)^\perp$$

\[\text{n times}\]

- If $\pi$ is

$$\rho \triangleright \Gamma \vdash M : A \rightarrow \ B \quad \sigma \triangleright \Delta \vdash N : A \quad \Gamma, \Delta \vdash \langle E \rangle$$

where $\Gamma = x_1 : A_1, \ldots, x_m : A_m$ and $\Delta = y_1 : B_1, \ldots, y_k : B_k$ then for all possible MLL proofs $\xi \in \mathcal{J}(\rho)$ and $\mu \in \mathcal{J}(\sigma)$ of the MLL sequents

$H = \vdash \alpha^\perp,\ldots,\alpha^\perp,((A_1)^\perp)^\perp,\ldots,((A_m)^\perp)^\perp, (A)^\perp \not\subseteq (B)^\perp$

\[\text{n_1 times}\]

$G = \vdash \alpha^\perp,\ldots,\alpha^\perp,((B_1)^\perp)^\perp,\ldots,((B_k)^\perp)^\perp, (A)^\perp$

\[\text{n_2 times}\]

the following MLL proof is in $\mathcal{J}(\pi)$:

$$\mu \triangleright G \vdash (B)^\perp, (B)^\perp$$

$\xi \triangleright H \vdash \alpha^\perp,\ldots,\alpha^\perp,((B_1)^\perp)^\perp,\ldots,((B_k)^\perp)^\perp, (A)^\perp \not\subseteq (B)^\perp$

\[\text{n_2 times}\]

$$\vdash \alpha^\perp,\ldots,\alpha^\perp,((A_1)^\perp)^\perp,\ldots,((A_m)^\perp)^\perp,((B_1)^\perp)^\perp,\ldots,((B_k)^\perp)^\perp, (B)^\perp$$

\[\text{n_1 + n_2 times}\]

- If $\pi$ is

$$\rho \triangleright \Gamma \vdash M : A \quad \sigma \triangleright \Delta \vdash N : B \quad \Gamma, \Delta \vdash \langle \otimes \rangle$$

where $\Gamma = x_1 : A_1, \ldots, x_m : A_m$ and $\Delta = y_1 : B_1, \ldots, y_k : B_k$, then for all possible MLL proofs $\xi \in \mathcal{J}(\rho)$ and $\mu \in \mathcal{J}(\sigma)$ of the MLL sequents

$H = \vdash \alpha^\perp,\ldots,\alpha^\perp,((A_1)^\perp)^\perp,\ldots,((A_m)^\perp)^\perp, (A)^\perp$

\[\text{n_1 times}\]

$G = \vdash \alpha^\perp,\ldots,\alpha^\perp,((B_1)^\perp)^\perp,\ldots,((B_k)^\perp)^\perp, (B)^\perp$

\[\text{n_2 times}\]

$\pi$ is in correspondence to the MLL proof

$$\xi_1 \triangleright J_1 \quad \xi_2 \triangleright J_2$$

$$\vdash \alpha^\perp,\ldots,\alpha^\perp,((A_1)^\perp)^\perp,\ldots,((A_m)^\perp)^\perp,((B_1)^\perp)^\perp,\ldots,((B_k)^\perp)^\perp, (A)^\perp \not\subseteq (B)^\perp \not\subseteq \otimes$$

\[\text{n_1 + n_2 times}\]

20
Observe how \( \mathcal{I}(\pi) \) is a singleton whenever \( \pi \) does not contain any unitary operator of arity (strictly) greater than 1.

Given an MLL proof \( \xi \), let us denote as \( T_\xi \) the class of all finite sequences of atom occurrences in \( \xi \). The relation \( \rightarrow_\xi \) can be extended to a relation on \( T_\xi \) by stipulating that

\[
(O_1, \ldots, O_{n-1}, P, O_{n+1}, \ldots, O_m) \rightarrow_\xi (O_1, \ldots, O_{n-1}, R, O_{n+1}, \ldots, O_m)
\]

whenever \( P \rightarrow_\xi R \). As usual, \( \rightarrow^+_\xi \) is the transitive closure of \( \rightarrow_\xi \).

Let us now consider a type derivation \( \pi \) in \( \mathbb{Q}\Lambda \) and its quantum token machine \( A_\pi \) and any \( \xi \in \mathcal{I}(\pi) \). States of \( A_\pi \) can be mapped to \( T_\xi \) by simply forgetting the underlying quantum register and mapping any occurrence of \( \pi \) to the corresponding atom occurrence in \( \xi \). This way one gets a map \( \mathcal{R}_{\pi,\xi}(\cdot) : S_\pi \rightarrow T_\xi \) such that, given a state \( S = (O_1, \ldots, O_n, Q) \) in \( S_\pi \), \( |\mathcal{R}_{\pi,\xi}(S)| = n \), number of occurrences in \( S \) is the same as the length of \( \mathcal{R}_{\pi,\xi}(S) \). Each reduction step on the token machine \( A_\pi \) corresponds to at least one reduction step in the MLL machine \( M_\xi \), where \( \xi \in \mathcal{I}(\pi) \) is the canonical proof:

**Lemma 3** Let us consider a token machine \( A_\pi \) and two states \( S, R \in S_\pi \). If \( S \rightarrow_\pi R \) and \( \xi \in \mathcal{I}(\pi) \) is canonical, then \( \mathcal{R}_{\pi,\xi}(S) \rightarrow^+_\xi \mathcal{R}_{\pi,\xi}(R) \).

**Proof.** This goes by induction on the structure of \( \pi \). \( \square \)

Any (possible) pathological situation on the quantum token machine, then, can be brought back to a corresponding (absurd) pathological situation in the MLL token machine. This is the principle that will guide us in the rest of this section.

### 6.2 Termination

The first property we want to be sure about is that every computation of any token machine \( A_\pi \) always terminates. This is relatively simple to state and prove:

**Proposition 3 (Termination)** Given a quantum token machine \( A_\pi \), any sequence \( S \rightarrow_\pi R \rightarrow_\pi \ldots \) is finite.

**Proof.** Suppose, for the sake of contradiction, than there exists an infinite computation in \( A_\pi \). This implies by Lemma [3] that there exists an infinite path in the token machine \( M_\xi \) where \( \xi \in \mathcal{I}(\pi) \) is the canonical MLL proof for \( \pi \). Absurd. \( \square \)

### 6.3 Progress

Progress (i.e. deadlock-freedom) is more difficult to prove than termination. Again, however, we use in an essential way the correspondence between \( \mathbb{Q}\Lambda \) and MLL:

**Proposition 4 (Progress)** Suppose \( \pi \) is a type derivation in \( \mathbb{Q}\Lambda \) and \( S \in S_\pi \) is initial. Moreover, suppose that \( S \rightarrow^*_\pi R \). Then either \( R \) is final or \( R \rightarrow_\pi T \) for some \( T \in S_\pi \).

Given a type derivation \( \pi \), an argument occurrence is any negative occurrence \((A, N)\) of \( B \) in a \((\text{au})\) axiom. We extend this definition to the corresponding atom occurrence when \( \xi \in \mathcal{I}(\pi) \). A result occurrence is defined similarly, but the occurrence has to be positive.
Proof. Let us consider a computation \( S_1 \rightarrow_{\pi} \ldots \rightarrow_{\pi} S_k \) on a quantum token machine \( A_\pi \). Suppose that the state \( S_k \) is a deadlocked state, i.e. \( S_k \) is not a final state, and that there exists no \( S_m \) such that \( S_k \rightarrow_{\pi} S_m \). The fact \( S_k \) is a deadlocked state means that \( l \geq 1 \) occurrences in \( S_k \) are argument occurrences, since the latter are the only points of synchronization of the machine. Let us consider any maximal sequence

\[
\mathcal{R}_{\pi,\xi}(S_1) \rightarrow_{\xi} \ldots \rightarrow_{\xi} \mathcal{R}_{\pi,\xi}(S_k) \rightarrow_{\xi} Q_1 \rightarrow_{\xi} \ldots \rightarrow_{\xi} Q_n,
\]

where \( \xi \in \mathcal{I}(\pi) \) is the canonical proof corresponding to \( \pi \). Observe that in (1), all occurrences of atoms in \( \xi \) are visited exactly once, including those corresponding to argument and result occurrences from \( \pi \). Notice, however, that the argument and result occurrences of the unitary operators affected by \( S_k \) cannot have been visited along the subsequence \( \mathcal{R}_{\pi,\xi}(S_1) \rightarrow_{\xi} \ldots \rightarrow_{\xi} \mathcal{R}_{\pi,\xi}(S_k) \) (otherwise we would visit the occurrences in \( S_k \) at least twice, which is not possible). Now, form a directed graph whose nodes are the unitary constants \( U_1, \ldots, U_h \) which block \( S_k \), plus a node \( F \) (representing the conclusion of \( \pi \)), and whose edges are defined as follows:

- there is an edge from \( U_i \) to \( U_j \) iff along \( Q_1 \rightarrow_{\xi} \ldots \rightarrow_{\xi} Q_n \) one of the \( l \) independent computations corresponding to a blocked occurrence in \( S_k \) is such that a result occurrence of \( U_i \) is followed by an argument occurrence of \( U_j \) and the occurrences between them are neither argument nor result occurrences.
- there is an edge from \( U_i \) to \( F \) iff along \( Q_1 \rightarrow_{\xi} \ldots \rightarrow_{\xi} Q_n \) one of the \( l \) traces is such that a result occurrence of \( U_i \) is followed by a final occurrence of an atom and the occurrences between them are neither argument nor result occurrences.

The thus obtained graph has the following properties:

- Every node \( U_i \) has at least one incoming edge, because otherwise the configuration \( S_k \) would not be deadlocked.
- As a consequence, the graph must be cyclic, because otherwise we could topologically sort it and get a node with no incoming edges (meaning that some of the \( U_i \) would not be blocked!).

Moreover, the cycle does not include \( F \), because the latter only has incoming nodes. From any cycle involving the \( U_j \), one can induce the presence of a cycle in the token machine \( M_{\mu} \) for some \( \mu \in \mathcal{I}(\pi) \). Indeed, such a \( \mu \) can be formed by simply choosing, for each \( U_j \), the “good” permutation, namely the one linking the incoming edge and the outgoing edge which are part of the cycle. This way, we have reached the absurd starting from the existence of a deadlocked computation. \( \square \)

The token machine \( A_\pi \) can be built by following the structure of \( \pi \). However, the fact this gives rise to a well-behaved, unitary, function requires proving some properties of \( A_\pi \) (i.e. termination and progress) externally. One may wonder whether this could be avoided by taking a categorical approach and apply the so-called Int-Construction [13] to the underlying category. This is not going to work, however, because finite dimensional Hilbert spaces and unitary maps on them are not a traced category. Of course, one could switch to linear maps, which indeed turn Hilbert spaces into a traced category; one loses the strong link with quantum computation this way, however.

6.4 Discussion

The immediate consequence of the termination and progress results from Section 6 is that \( |\pi| \) is always a total function. The way \( A_\pi \) is defined ensures that \( |\pi| \) is obtained by feeding some of the input of a unitary operator \( U \) with some bits (namely those occurring in \( \pi \)). \( U \) is itself obtained by composing the unitary operators occurring in \( \pi \), which can thus be seen as a program computing a quantum
consider the term $\langle \pi \rangle$. Of course, $[\pi]$ is nothing more than the function computed by $\langle \pi \rangle$. In a way, then, token machines both show that $Q\Lambda$ is a true quantum calculus and can be seen as the right operational semantics for it.

**Example 4** Consider the term $M_{EPR} = \lambda(x,y).\text{CNOT}(Hx \otimes y)$ and a type derivation $\pi$ for it:

$$
\begin{align*}
\vdash H : B \rightarrow B & \quad x : B \vdash x : B \\
\vdash \text{CNOT} : B \otimes B \rightarrow B \otimes B & \quad x : B, y : B \vdash Hx \otimes y : B \otimes B \\
& \quad x : B, y : B \vdash \text{CNOT}(Hx \otimes y) : B \otimes B
\end{align*}
$$

Forgetting about terms and marking different occurrences of $B$ with distinct indices, we obtain:

$$
\begin{align*}
& \vdash : B_{21} \rightarrow B_{22} \quad B_{23} \vdash B_{24} \\
& \vdash B_{17} \vdash B_{18} \\
& \vdash B_{5,6} \vdash B_{7,8} \\
& \vdash B_{19} \vdash B_{20}
\end{align*}
$$

Now, consider the IAM$_{\Lambda}$ computation:

$$(B_1, B_2, Q) \rightarrow^* \pi \ (B_5, B_6, Q) \rightarrow^* \pi \ (B_{13}, B_{14}, Q)$$

$$(B_{17}, B_{19}, Q) \rightarrow^* \pi \ (B_{23}, B_{20}, Q)$$

$$(B_{24}, B_{10}, Q) \rightarrow^* \pi \ (B_{21}, B_{10}, Q)$$

$$(B_{22}, B_{10}, H^1(Q)) \rightarrow^* \pi \ (B_{18}, B_{10}, H^1(Q))$$

$$(B_{15}, B_{10}, H^1(Q)) \rightarrow^* \pi \ (B_{9}, B_{10}, H^1(Q))$$

$$(B_{11}, B_{12}, \text{CNOT}^{1,2}(H^1(Q))) \rightarrow^* \pi \ (B_{7}, B_{8}, \text{CNOT}^{1,2}(H^1(Q)))$$

$$(B_{3}, B_{4}, \text{CNOT}^{1,2}(H^1(Q)))$$

Notice that $\text{CNOT}$ acts as a synchronization operator: the second token is stuck in the occurrence $B_{10}$ until the first token arrives as a control input of the $\text{CNOT}$ and the corresponding reduction step actually occurs.

### 6.5 Soundness

What is the relation between token machines and the equational theory on superposed type derivations introduced in Section [4.3]? It is easy to extend the definition of $[\cdot]$ to superposed type derivations: if $T = \sum_{i=1}^{n} \alpha_i \pi_i$ then $[T]$ when fed with a vector $x$ returns $\sum_{i=1}^{n} \alpha_i [\pi_i](x)$. In the rest of this section, we will prove that token machines behave in accordance to the equational theory.

Suppose $\pi$ is a type derivation for $\Gamma, x_1 : A_1, \ldots, x_m : A_m \vdash M : B$ and that, for every $1 \leq i \leq m$ there is a type derivation $\rho_i$ for $\Delta_i \vdash N_i : A_i$. By induction on the structure of $\pi$, one can define a type derivation $\pi \{ \rho_1, \ldots, \rho_m / x_1, \ldots, x_m \}$ of $\Gamma, \Delta_1, \ldots, \Delta_m \vdash M \{ N_1, \ldots, N_m / x_1, \ldots, x_m \} : B$ (see Lemma [1]). Moreover, from $\pi, \rho_1, \ldots, \rho_m$ we can form a machine $A_{\pi,\rho_1,\ldots,\rho_m}$ as follows:
The states of $A_{\pi}^{\rho_1,\ldots,\rho_m}$ are in the form $(O_1,\ldots,O_n,Q)$ where:

- $O_1,\ldots,O_n$ are occurrences of the type $B$ in $\pi,\rho_1,\ldots,\rho_m$;
- $Q$ is a quantum register on $n$ qubits;

The transition function is itself obtained by taking the disjoint union of $\rightarrow_{\pi},\rightarrow_{\rho_1},\ldots,\rightarrow_{\rho_m}$, plus

- transitions of any positive occurrence of $B$ in $A_i$ (in the conclusion of $\rho_i$) to the corresponding occurrence of $B$ in $A_i$ (in the conclusion of $\pi$);
- transitions of any negative occurrence of $B$ in $A_i$ (in the conclusion of $\pi$) to the corresponding occurrence of $B$ in $A_i$ (in the conclusion of $\rho_i$).

Initial and final states are defined in the natural way, taking into account occurrences of $B$ in $\Delta_1,\ldots,\Delta_m,B$, but not those in $A_1,\ldots,A_m$.

The just defined machine is equivalent to the one built from the derivation $\pi\{\rho_1,\ldots,\rho_n/x_1,\ldots,x_m\}$. This is stated by the following substitution lemma:

**Lemma 4** Let $\pi\vdash\Gamma, x_1 : A_1,\ldots,x_m : A_n \vdash M : B$ and for every $1 \leq i \leq m$ let $\rho_i \vdash\Delta_i \vdash N_i : A_i$. Then the automaton $A_{\pi}\{x_1,\ldots,x_m/\rho_1,\ldots,\rho_n\}$ is equivalent to $A_{\pi}^{\rho_1,\ldots,\rho_m}$.

It is now possible to prove two key intermediate results towards soundness:

**Lemma 5** Let $\pi\vdash\Gamma \vdash (\lambda x.M)N : A$. Then $\langle \pi \rangle = \langle \pi^\downarrow \rangle$.

**Lemma 6** Let $\pi\vdash\Gamma \vdash (\lambda\langle x,y \rangle.M)(N\otimes L) : A$. Then $\langle \pi \rangle = \langle \pi^\downarrow \rangle$.

In order to prove Soundness Theorem, we need to introduce the following technical tool:

**Definition 3 (Superposed Quantum Circuits)** A superposed quantum circuits of arity $(n,m)$ (where $n \leq m$) is a formal sums in the form

$$\sum_{i=1}^{n} \alpha_i C_i$$

where $\alpha_i \in \mathbb{C}$ and $C_i$ is a quantum circuit on $m$ qubits of which $n$ are assigned a bit.

As an example, a superposed quantum circuit of arity $(2,4)$ looks as follows:

$$\alpha_1 \cdot \begin{pmatrix} |b_1^1\rangle \\ |b_2^1\rangle \end{pmatrix} C_1 + \alpha_2 \cdot \begin{pmatrix} |b_1^2\rangle \\ |b_2^2\rangle \end{pmatrix} C_2$$

Since every type derivation $\pi$ computes a quantum circuit $\langle \pi \rangle$, every superposed type derivation $T$ can be seen as a superposed quantum circuit $\langle T \rangle$. Moreover, the function $[\sum_{i=1}^{n} \alpha_i C_i]$ computed by a superposed quantum circuit $\sum_{i=1}^{n} \alpha_i C_i$ can be defined similarly to what we have done for superposed type derivations. Of course, $[\langle T \rangle] = [T]$.

We now define the set of admissible circuit transformations.

**Definition 4 (Admissible Transformations)** Assume $\langle T \rangle = \sum_{i=1}^{n} \alpha_i C_i$ is a superposed quantum circuit. The following transformation are called admissible:

1. One summand $\alpha C_i$ is replaced by $\beta C_i + \gamma C_i$, where $\alpha = \beta + \gamma$;
2. One summand $\alpha C_i$ where $C_i$ has the following form

$$\begin{vmatrix}
|b_1\rangle

|b_m\rangle
\end{vmatrix}$$

is replaced by a sum $\sum_{x \in B_m} \alpha \cdot \beta_x \cdot C_x$ where $B_m$ is the set of binary strings of length $m$, $\beta_x$ is the coefficient of $|x\rangle$ in $U|b_1 \ldots b_m\rangle$ and $C_x$ is the following circuit:

$$\begin{vmatrix}
|x_1\rangle

|x_m\rangle
\end{vmatrix}$$

Admissible transformations can be applied in both directions. It is easy to prove that admissible transformations, when applied to a superposed circuit $\langle T \rangle$, leave the underlying function unchanged.

We are now ready to prove our soundness result:

**Theorem 1 (Soundness)** If $T \approx S$, then $[T] = [S]$.

**Proof.** Since $[\langle T \rangle] = [T]$, it is sufficient, by Proposition [1], to show that, if $T \sim S$, then $\langle S \rangle$ can be obtained from $\langle T \rangle$ by iteratively applying one or more admissible transformations. This is an induction on the structure of a proof $d$ of $T \sim S$. Let $r$ be the last rule applied in $d$, where we enrich the thesis by stipulating that if the rules in $d$ are all from $AX \cup CC$, then $T$ is a single type derivation and that going from $\langle T \rangle$ to $\langle S \rangle$ can be done by performing at most one admissible transformation of the second kind. Some interesting cases:

- $r$ is (beta,pair). The result follows by means of Lemma [5].
- $r$ is (beta). The result follows by means of Lemma [6].
- $r$ is (quant). Then $d$ is simply

$$\begin{align*}
\pi & \triangleright \cdot \triangleright U|b_1 \ldots b_k\rangle : \mathbb{I}^k \\
\pi & \approx U|b_1 \ldots b_k\rangle \quad \text{quant}
\end{align*}$$

and $\langle \pi \rangle$ is simply the quantum circuit built on the unitary operator $U$, feeded with the input $|b_1 \ldots b_k\rangle$. We know that $U|b_1 \ldots b_k\rangle$ is a superposed type derivation in the form $S = \sum_{x \in B_k} \alpha_x \pi_x$, where $B_k$ is the set of all binary strings of length $k$ and $\pi_x$ is the type derivation for $|x\rangle$ ($k$ applications of the rule $(I \otimes I)$ starting from the axioms for $|b_1 \ldots b_k\rangle$). Such a derivation can be seen as the superposed quantum circuit of arity $(k, k)$ $\langle S \rangle = \sum_{x \in B_k} \alpha_x |x\rangle$ (where the binary string $|x\rangle$ can also seen as the trivial circuit that act on it as the identity) and the amplitudes $\alpha_x$ are exactly the coefficient of $|x\rangle$ in $U|b_1 \ldots b_k\rangle$. $\langle S \rangle$ can be plainly obtained from $\langle \pi \rangle$ by means of the admissible transformation of the second kind by replacing the only summand $1 \cdot C$ with the sum $\sum_{x \in B_k} 1 \cdot \alpha_x |x\rangle$.

- $r$ is a reflexive or a symmetric or a transitive closure. Trivial.

- $r \in CC$, then we know that $T \sim S$ is derived from $V \sim W$, where $V$ is a single type derivation and $\langle W \rangle$ is obtained by applying either zero or one admissible transformations of the second kind.
to $\langle V \rangle$. In other words, $V$ is
\[
|b_1\rangle \quad U \quad |b_m\rangle
\]
while $W$ is $\sum_{x \in B_m} \alpha \cdot \beta_x \cdot C_x$ where $B_m$ is the set of binary strings of length $m$, $\beta_x$ is the coefficient of $|x\rangle$ in $U|b_1 \ldots b_m\rangle$ and $C_x$ is the following circuit:
\[
|x_1\rangle \quad D \quad |x_m\rangle
\]
It is then clear that the effect of $r$ to $\langle V \rangle$ consists in modifying $D$, because $U$ cannot be affected. Moreover, the same modification is performed by $r$ uniformly on $D$ in any $C_x$. We can then conclude that there exists $E$ such that $T$ is
\[
|b_1\rangle \quad U \quad |b_m\rangle \quad E
\]
while $S$ is
\[
|x_1\rangle \quad E \quad |x_m\rangle
\]
This concludes the proof.

7 Related Works

The role of GoI in quantum computing has already been explored in at least two works. In [12] a geometry of interaction model for Selinger and Valiron’s quantum lambda calculus [18] is defined. The model is formulated in particle-style. In [2] QMLL, an extension of MLL with quantum modalities is studied. QMLL is sound and complete with respect to quantum circuits, and an interactive, particle-style token machine is defined. The computational meaning of QMLL proofs is given by means of the token machine: each cut-free QMLL proof corresponds to an unique quantum circuit. In both cases, adopting a particle-style approach has a bad consequence: the “quantum” tensor product does not coincide with the tensor product in the sense of linear logic. Here we show that adopting the wave-style approach solves the problem.

Quantum extensions of game semantics are partially connected to our subject. In [8] a game semantics for a simply-typed lambda calculus (similar to QA) is introduced. The language uses a notion of extended variable, able to deal with tensor products. The game semantics is built around
classical game semantics where, however, quantum operations are the questions and measurements are the answers. A soundness result for the semantics is given. A similar approach for a lambda calculus with quantum stores (i.e. in which quantum data are referred through pointers) has been explored in [9]. Again, two tensor products are needed, unless one wants to drop the possibility of entangling qubits.

Purely linear quantum lambda-calculi (with measurements) can be given a fully abstract denotational semantics, like the one proposed by Selinger and Valiron [19]. In their work, closure (necessary to interpret higher-order functions) is not obtained via traces and is not directly related in any way to the geometry of interaction. Moreover, morphisms are just linear maps, and so the model is far from being an quantum operational semantics like the IAMQΛ.

8 Conclusions

The definition of an elegant semantics is always a challenge in the case of quantum functional languages. This mainly holds for denotational models, but remains true also for operational, reduction-style semantics. In this paper we introduce QA, a linear quantum calculus with explicit qubits, where quantum circuits can be easily encoded. This simple calculus is a good framework to further investigate the (deep) relationships between quantum computing and Girard’s Geometry of Interaction. We describe IAMQA, an interactive abstract machine which provides a sound operational characterization of any QA’s type derivation. QA quantum features force to move from the (usual) particle-style token machine model to the wave-style one, where different tokens circulate around a net (a type derivation) at the same time. Constants for n-ary unitary operators act as synchronization points: every token trips independently since it arrives at a unitary operator constant. In this case, computation takes place only if all input qubits occurrence has reached the unitary operator. IAMQA is a sound model: critical behaviors potentially introduced by the synchronization mechanism, can not happen in IAMQA computations. Our contribution can be summarized as follows:

- The IAMQA provides an elegant model for quantum programs written in QA: each type derivation is interpreted as a quantum circuit built on the set of quantum gates occurring in the underlying lambda-term;
- we show that also wave-style token machines are sound with respect to an operational theory of superposed type derivations;
- we give evidence that wave-style provides an original account of the quantum data entanglement phenomenon, since the notion of synchronization we implicitly define is strongly connected to what happens to entangled data.

Our investigation is open to some possible future directions. A natural step will be to extend the syntax of terms and type grammar with an exponential modality. The generalization of the wave-style token machine to this more expressive language would be an interesting and technically challenging subject. Something we see as relatively easy is an extension of this framework to a calculus with measurements: token machines could cope with measurements by evolving probabilistically[6], while adapting the equational theory would probably be nontrivial. Finally, giving a formal status to the connection between wave-style and the presence of entanglement is a fascinating subject which we definitely aim to investigate further.

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28
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