A Simulation of Shallow Water Wave Equation Using Finite Volume Method: Lax-Friedrichs Scheme

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Abstract. Long wave propagation above a bottom topography such as tsunami waves can be modeled mathematically by applying shallow water wave equations. The finite volume method was developed to determine the numerical solution of shallow water wave equations. The method uses Lax-Friedrichs scheme for the determination of numerical fluxes at cell interfaces. Furthermore, the model and method was applied to the simulation of long wave propagation on a sloping beach. The simulation results show that the present model and method has a power of simulation of long wave propagation of the tsunami wave on the beach topography with a slope.

1. Introduction

Tsunamis are one of the disasters caused by air flow. According to Synolakis [9], tsunami waves are a long wave form. Long wave propagation can be mathematically modeled using the shallow water wave equation. Flouri [2] states that shallow water waves are waves that occur on shallow water surfaces where the wavelength is quite large compared to its depth.

The shallow water wave equation is the system of non-linear first order partial differential equations (PDE) [10]. The dynamics of the water wave phenomenon can be known through the solution of the PDE system [1]. The solutions obtained are useful for predicting where water will flow, water flow velocity, and wave height. However, it is sometimes difficult to obtain analytical solutions from non-linear first-order PDE systems. Therefore, numerical methods are used to obtain numerical solutions from the shallow water wave equation.

The finite volume method (FVM) can overcome this problem. Hieu [4] states that the volume method is derived based on an integral form of conservation law. According to Godlewski[3], the basic principle of finite volume method is to integrate the model of the volume control.

In this paper the authors are interested in examining shallow water wave equations involving topography using the volume up method. The volume method is chosen because the method can be used for the 1D shallow water wave equation. In addition, the method can also be applied to a structured or unstructured grid. Volume discretization up to the conservative form of the equation that connects the average integral of the conserved and flux inter cell, so that the solutions obtained are more accurate. The solution is useful for predicting where the water will flow, water flow velocity, and wave height that is influenced by the beach topography. Furthermore, these results are applied to a case related to coastal waves.
2. Shallow Water Wave Equation

Shallow Water Wave Equation is a system of partial differential equations that describe the problem of fluid flow. The shallow water wave equation applies to homogeneous fluids that have a constant density of $\rho$, inviscid, incompressible which flows by irrotational. In addition, the shallow water wave equation applies if the fluid depth is much smaller than its wavelength or the ratio of wave amplitude and wavelength is of small value. In addition, it is assumed that the pressure increases linearly with the fluid depth and the vertical velocity in the fluid is ignored.

The one-dimensional shallow water wave equation is the function of one variable space $x$ and the time variable $t$. The fluid domain for the 1D shallow water wave equation presented in Figure 1.

![Figure 1: Domain sketch of the 1D shallow water wave equation](image)

The Figure 1 describes the sketch of the domain at the $xOz$ cartesian coordinate. A surface free of water waves above the average water level is expressed as $\eta(x,t)$. Notation $H(x) = D - b(x)$ in Figure 1 states the water depth above the bottom topography and $b(x)$ denotes the bottom topographic function. The $D$ notation represents the average water depth. The total water depth at the position of $x$ and at time to $t$ is expressed in the following equation

$$h(x,t) = \eta(x,t) + H(x) = \eta(x,t) + D - b(x).$$

According to Khakimzyanov et al.[6], if the bottom topography is a flat topography then $b_x = 0$ so that $H(x) = D$.

The total water depth at the position of $x$ at the time of $t$ for the case of a flat topography is given

$$h(x,t) = \eta(x,t) + D.$$

The construction of 1D shallow water wave equation is derived from two physical conservation laws, namely mass conservation law and momentum conservation law. Based on these two laws, the 1D shallow water wave equation is presented in the following equation

$$h_t + (uh)_x = 0$$

$$u t + \left(u^2 h + \frac{1}{2} gh^2\right)_x = -ghb_x. \tag{1}$$

By substituting $h = \eta + D - b$ to the system (1) is obtained

$$\eta_t + (u(\eta + D - b))_x = 0$$

$$u_t + \left(\frac{1}{2} u^2 + g\eta\right)_x = 0. \tag{2}$$
3. Finite Volume Method

Finite volume method (FVM) is derived on the basis of an integral form of conservation law [7]. According to Godlewski [3], the basic principle of finite volume method is to integrate the model of the volume control.

Flouri [2] writes that the finite volume method can be used in any geometry, and can be used in structured or unstructured. The operational steps of the volume method consist of integrating, discretizing, and solving equations.

The differential form of conservation law referred to from Leveque [8] and Leveque [7] is expressed in Equation (2). It is known that a 1D shallow water wave equation depends only on the space variable and time variable, so that the System (2) can be rewritten in the form

\[ q_t(x, t) + f(q(x, t))_x = 0, \]  

(3)

with

\[ q = \begin{bmatrix} \eta \\ u \end{bmatrix} \] \quad \text{and} \quad f(q) = \begin{bmatrix} u(\eta + D - b) \\ \frac{1}{2} u^2 + g \eta \end{bmatrix}. \]

Also assumed that \( q \) and \( f(q) \) are smooth functions.

The basic principle of the volume method is to integrate the conservative form of the 1D shallow water wave equation over the volume control. The volume control in the \( x - t \) field for the shallow water wave equation is presented in Figure 2.

\[ \int_{C_i} q_t(x, t) \, dx = \int_{C_i} f(q(x, t))_x \, dx = f(q(x_{i-1/2}, t)) - f(q(x_{i+1/2}, t)). \]

(5)

Next, the average value at time \( t_{n+1} \) that is \( Q_{i,n}^{t_{n+1}} \) is approached by integrating equation (5) against the time variable from \( t_n \) to \( t_{n+1} \) and obtained

\[ \int_{t_n}^{t_{n+1}} \left( \int_{C_i} q_t(x, t) \, dx \right) dt = \int_{t_n}^{t_{n+1}} \left( f(q(x_{i-1/2}, t)) - f(q(x_{i+1/2}, t)) \right) dt \]

(6)
with a time step of length \( \Delta t = t_{n+1} - t_n \). Using Calculus Fundamental Theorem in equation (6), obtained

\[
\int_{C_i} q(x, t_{n+1}) \, dx - \int_{C_i} q(x, t_n) \, dx = \int_{t_n}^{t_{n+1}} \left( f(q(x_{i-1/2}, t)) - f(q(x_{i+1/2}, t)) \right) \, dt.
\]

(7)

Next, equation (7) multiplied by \( \frac{1}{\Delta x \Delta t} \) and based on equation (4) so that it is obtained

\[
\frac{1}{\Delta t} \left( Q_{i+1}^{n+1} - Q_i^n \right) = \frac{1}{\Delta x} \int_{t_n}^{t_{n+1}} \left( f(q(x_{i-1/2}, t)) - f(q(x_{i+1/2}, t)) \right) \, dt.
\]

(8)

The integral value on the right hand side is approximated by

\[
F_{i+1/2}^n \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(q(x_{i+1/2}, t)) \, dt.
\]

That is the approach to the average flux along \( x = x_{i+1/2} \), so that equation (8) becomes

\[
\frac{1}{\Delta t} \left( Q_{i+1}^{n+1} - Q_i^n \right) = \frac{1}{\Delta x} \left( F_{i+1/2}^n - F_{i-1/2}^n \right).
\]

(9)

The equations (9) can be written in the form

\[
Q_{i+1}^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} \left( F_{i+1/2}^n - F_{i-1/2}^n \right).
\]

(10)

Formula (10) is a finite volume formula for the 1D shallow water wave equation.

The value \( F_{i+1/2}^n \) is approached with the numerical flux \( F \) which is only based on the value of \( Q_{i-1}^n \) and \( Q_i^n \) with

\[
F_{i-1/2}^n = F(Q_{i-1}^n, Q_i^n)
\]

(11)

Based on equation (11), algorithm (10) becomes

\[
Q_{i+1}^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} \left[ F(Q_i^n, Q_{i+1}^n) - F(Q_{i-1}^n, Q_i^n) \right].
\]

(12)

Furthermore, the Lax-Friedrichs scheme is selected for the \( F \) formula, i.e.

\[
F(Q_{i-1}^n, Q_i^n) = \frac{1}{2} \left[ f(Q_{i-1}^n) + f(Q_i^n) \right] - \frac{\Delta x}{2 \Delta t} \left( Q_i^n - Q_{i-1}^n \right).
\]

(13)

Scheme (13) is substituted to the equation (12) so that it is obtained

\[
Q_{i+1}^{n+1} = \frac{1}{2} \left( Q_{i+1}^n + Q_{i-1}^n \right) - \frac{\Delta t}{2 \Delta x} \left( f(Q_{i+1}^n) - f(Q_{i-1}^n) \right).
\]

(14)

The equation (14) is a finite volume formula with Lax-Friedrichs scheme for the 1D shallow water wave equation.
4. Simulation

In this section, numerical experiments are taken from experiments carried out by McHale [5], namely long wave propagation on shallow beaches. The case is a depiction of the propagation of the tsunami wave on the beach topography with a slope of \( s = \frac{5040}{20000} \) which is connected with the bottom topography with a water depth of \( D = 5000 \) meters. The beach toes are located at a distance of \( x_1 = 20000 \) meters and the coastline is located at a distance of 158.73 meters with a height of 5000 meters. The experiment was discretized at \( x \in [0, 10000] \) meters and \( t = [0, 1000] \) seconds.

In this experiment, the incoming wave is assumed to be a single wave with an amplitude of 30 meters generated at a distance of \( x_2 = 50000 \) meters. The initial requirement for the wave used is a solitary wave known as soliton. The sketch for this experiment is given by Figure 3.

\[ \eta(x, 0) = A sech\left(\sqrt{\frac{3}{4D^2}}(x - x_2)\right). \] (15)

The velocity of fluid particles in the direction of \( x \) when \( t = 0 \) is given the initial condition

\[ u(x, 0) = 0. \] (16)

Based on the sketch in Figure 3, the beach topography function is given

\[ b(x) = \begin{cases} 0, & x \geq 20000; \\ \frac{5040}{20000}(20000 - x), & x < 20000. \end{cases} \] (17)

We know the average water depth in this case is as high as 5000 meters, so the total water depth at \( t = 0 \) is

\[ h(x, 0) = \eta(x, 0) + D - b(x). \]

Initial conditions of wave \( \eta(x, 0) \) and topography function \( b(x) \) are presented in the Figure 4.
Figure 4: Initial conditions $\eta(x,0)$ and topography function $b(x)$ for domain $x \in [0, 100000]$.

Numerical experiments are performed on domains $x \in [0, 100000]$ and $t \in [0, 1000]$. Discretization is done on a structured grid with the length of the cell $\Delta x = \frac{100000}{N}$ where $N$ is the number of grid cells. In this experiment, $N = 150$ is taken, which means that the domain $x \in [0.100000]$ is divided into 150 grid cells. In this experiment a Courant value of 0.68 was taken and an error tolerance of $h_{\text{min}} = 1 \times 10^{-3}$.

The numerical experimental results of shallow water wave equations for the case of ocean waves with initial conditions (15) and (16) are presented in Figure 5.

Based on Figure 5a it appears that at 55.1014 the water wave is split and moves left and right. The wave that moves to the left has a wave amplitude of 14.5 meters. At the time of $t = 104.0805$, the amplitude of the wave decreases to 13.1 meters. As time goes by, waves that move to the left will approach the coastline. Meanwhile, the wave that moves to the right will
go to the deep sea with a smaller amplitude wave. The movement of the tsunami wave for the next time step is presented in Figure 6.

![Figure 6](image1)

(a) $\max(\eta(x,t)) = 21.5 \text{ m}$

(b) $\max(\eta(x,t)) = 37.3 \text{ m}$

Figure 6: The results of numerical experiments for single wave from $t = 257.14$ to $t = 306.1191$

Figure 6a shows that the wave height above the mean sea level reaches 21.5 meters higher than before. The wave height above the average water level reaches a maximum height of 37.3 meters and occurs at the position of $x = 10,713$ and at 306.111 as shown in Figure 6b. This means that at maximum altitude, the wave propagates as far as 148.016 meters from the shoreline. The wave movement for the next time step is presented in Figure 7.

![Figure 7](image2)

(a) $\min(\eta(x,t)) = -26.9 \text{ m}$

(b) $\max(\eta(x,t)) = 1.8 \text{ m}$

Figure 7: The results of numerical experiments for single wave when $t = 453.0563$ and $t = 997.9482$

Based on Figure 7a it appears that when $t = 453.0563$ wave height reaches its lowest point at an altitude of $-26.9 \text{ m}$. In other words, the lowest point of the water level is 26.9 meters below the average sea level. Furthermore, Figure 7b shows that when $t = 997.9482$ the highest wave height is 1.8 meters above the average water level and the lowest wave height is 4.3 meters below the mean sea level. In other words, the wave height at the last time step is around the average sea level.
Next, this experiment is simulated with a different total water depth. This is done to
determine the relationship of the total water depth to the wave height above the average water
level. The simulation is given two initial conditions, namely the initial conditions with a total
water depth of 5025 meters with amplitude waves as high as 25 meters and a total water depth
of 5035 meters with wave amplitude as high as 25 meters. The numerical experimental results
for both cases are presented in Figure 8.

Based on the Figure 8 the highest wave height for the initial height of $A = 25$ meters reaches
31.2 meters above the average surface of sea air. Based on the figure, it can also be seen
that the highest wave height for initial requirements with $A = 35$ meters reaches 43.3 meters
above the average sea level. The maximum height of the approved waves at the same time
step is $t = 306.1191$. Next, fixing waves generated by waves with different waves is presented in
Table 1.

Based on Table 1, it appears that the higher the total water depth given, the higher the wave
height generated. This results in further waves from the shoreline.

### 5. Conclusion

The 1D shallow water wave equation with a flat topography is constructed based on two
physical conservation laws, namely mass conservation law and momentum conservation law.
Numerical solutions are determined by the finite volume method. The Lax-Friedrichs scheme
was applied to determine the cell interface flux function in the 1D shallow water wave equation.
Based on the experimental results it appears that the 1D shallow water wave equation and
the finite volume method to the Lax-Friedrichs scheme have the power to simulate long wave
propagation on the shallow coast. The simulation results also show that the higher the total
water depth given, the greater the wave height generated.
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