A minimal representation for continuous functions

Franz Brauße* and Florian Steinberg†

Abstract

This paper presents a variation of a celebrated result of Kawamura and Cook specifying the least set of information about a continuous function on the unit interval which is needed for fast function evaluation. To make the above description precise, one has to specify what is considered a ‘set of information’ about a function and what ‘fast’ means. Kawamura and Cook use second-order complexity to define what ‘fast’ means but do not use the most general notion of a ‘set of information’ this framework is able to handle. Instead they additionally require the codes to be ‘length-monotone’. This paper changes the setting in that it removes the additional premise of length-monotonicity, and instead imposes further conditions the speed of the evaluation and the translations. The sense in which the evaluation has to be fast is given the name ‘hyper-linear’, for the translations another technical condition is necessary. Hyper-linear time computation is very restrictive: The running time is not only forbidden to contain iterations of the length functions, but also the length function is only allowed to be applied to a shift by a constant of the argument instead of an arbitrary polynomial. The paper proves that there is a minimal set of information necessary for hyper-linear evaluation and that the set of information obtained by Kawamura and Cook is polynomial-time translatable, but not polynomial-time equivalent to it. Indeed, the encoding constructed in this paper is not polynomial-time equivalent to any encoding only using length-monotone names.

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*Universität Trier, 54286 Trier, room H 420; Email: brausse@informatik.uni-trier.de; supported by the German Research Foundation (DFG), project WERA, grant MU 1801/5-1
†Technische Universität Darmstadt, Schloßgartenstraße 7, 64289 Darmstadt, Building S215, room 203; Email: steinberg@mathematik.tu-darmstadt.de
1 Introduction

This paper discusses subjects that are from the field of real complexity theory. Real complexity theory is the resource sensitive refinement of computable analysis. Real complexity theory and computable analysis aim to carry the merits of classical computability and complexity theory, which are only applicable to compute on discrete structures, to apply to continuous structures. Computable analysis can be traced back to one of the foundational papers of computability theory itself [Tur36]. It branched off as a separate discipline in the 50’s [Grz55] and has since been extended and refined steadily.

Weihrauch’s framework of Cantor space representations [Wei00] provides a plausible and sufficiently general model of encodings of elements from a continuous structure. In particular it provides a canonical encoding of each metric space with a distinguished dense subsequence and a working function space construction. While it is possible to do complexity theory in this framework, the outcome coincides with the expectations of what should be feasible very rarely. For instance, it is provable that no encoding of the real valued continuous functions on the unit interval exists such that the evaluation operation is fast [Wei03].

Therefore, real complexity theory, while it works fairly well as long as the encoded objects are real numbers or elements of $\mathbb{R}^n$, has for a long time been restricted to point-wise considerations about objects of higher type, i.e. functionals, operators on function spaces etc. [Ko91]. (These results are important and interesting nonetheless.) Stepping up the type of objects one considers requires to also use complexity theory of higher type. This step was done very recently: In 2010 Kawamura and Cook defined a framework for complexity of operators in analysis [KC12] using a characterization of Kapron and Cook from 1996 [KC96] of the basic feasible functionals introduced by Mehlhorn 1976 [Meh76].

Kawamura and Cook’s framework of second-order representations is accepted as introducing the right notion of complexity for operators in analysis. For this reason the part of real complexity theory that considers complexity of operators and functionals has become very active in the past years [Kaw11, KO14, KP14, FGH14, KMRZ15, FZ15, Ste16, and many more]. One of the celebrated results that contributed to the popularity and acceptance of the framework of second-order representations is that Kawamura and Cook succeeded to provide a second-order representation of the set of continuous functions on the unit interval, that is minimal up to polynomial-time reductions with the property that the evaluation operator is polynomial-time computable. The present paper provides a variation of this very result.

While working very well for theoretical considerations, the framework of second-order representations imposes some assumptions on the encodings that lead to extensive padding that seems unnatural in practical applications. Furthermore, some of the theoretical predictions seem to be out of sync with the behavior of popular implementations of real complexity theory: iRRAM is a framework for and implementation of error-free real arithmetic based on the ideas of real complexity theory [Mü01, Mü]. In iRRAM it is possible to implement functions and, as long as the implementation of the function is reasonable, evaluation of the function is fast. Computing the modulus of continuity of a function, on the other hand, does not seem to be possible in a reasonable amount of time. In contrast to that, in the framework of second-order representations one can prove that polynomial-time computability of evaluation implies polynomial-time computability of a modulus. Thus, it seems reasonable to assume that the behavior of iRRAM on functions can not be modeled by second-order representations.
The content of this paper

This paper uses a more relaxed notion of encoding than that provided by Kawamura and Cook’s second-order representations (Definition 1.1). It proves that in this more general setting it is possible to partly recover Kawamura and Cook’s construction of a weakest representation of the continuous functions on the unit interval such that evaluation is fast: It provides a representation \( \xi \) (Definition 2.3) such that evaluation is fast (Theorem 2.4). Then it proves that for any other representation such that evaluation is fast, there is a fast translation to \( \xi \) (Theorem 2.5). Here, being ‘fast’ (Definition 2.1) is more restrictive than polynomial-time computability and to be able to prove minimality of the representation it is necessary to restrict the translations considered even more than that (Definition 2.2). The algorithm for evaluation with respect to \( \xi \) is strikingly similar to how \( \text{iRRAM} \) works internally.

In the end the paper proves that it is impossible to compute a modulus of a function in polynomial-time with respect to \( \xi \) (Theorem 2.6) and uses this to compare \( \xi \) to the minimal second-order representation constructed by Kawamura and Cook. It proves that these representations are not polynomial-time equivalent (Corollary 2.10). From the minimality results proven by Kawamura and Cook it follows that \( \xi \) is not polynomial-time equivalent to any second-order representation (Corollary 2.12).

The use of a different notion of ‘fastness’ is necessary for the proofs, but can also partially be justified by other means: In the past of real complexity theory there has been a lot of discussion about whether or not iteration of length function in the running time should be considered feasible. Thus, forbidding iterations is justifiable. The restriction, however, goes further to only allows a constant lookahead instead of the more usual polynomial one. This seems to be a real restriction, and is only done since it seems unavoidable for the proofs.

1.1 Notations

Fix the finite alphabet \( \Sigma := \{0, 1, \#\} \). Denote the set of finites words over \( \Sigma \) by \( \Sigma^* \). The empty string is denoted by \( \varepsilon \).

For convenience of notation, this paper considers some sets from mathematics as subsets of \( \Sigma^* \): Let \( \mathbb{N} \) denote the set \( \{1, 10, 11, 100, 101, \ldots\} \) of positive integers in binary notation. Let \( \omega = \{\varepsilon, 1, 11, \ldots\} \) denote the non-negative integers in unary notation. To avoid notational confusion this paper uses \( 2^n \) instead of \( n \) if an integer in unary notation is handed to a machine. Let \( \mathbb{Z} \) denote the set \( 00\mathbb{N} \cup 01\mathbb{N} \cup \{00\} \), where 00 is interpreted as 0, 01n is interpreted as \( n \) and 01n is interpreted as \( -n \). Finally, interpret a string \( c \) that has a single \# and starts in either 01, 11 or 00# as the binary expansion of a rational number. I.e. identify \( c \) with the rational number \((-1)^{\varepsilon(\#)}(\sum_{i=1}^{m-1} c_i2^{-(m-i)} + \sum_{i=m+1}^{n-1} c_i2^{-(m-n)})\), where \( m \) is the position of the \#. The set of numbers that have a code as above is called dyadic numbers and denoted by \( \mathbb{D} \). Note that this does not provide \( \mathbb{D} \subseteq \Sigma^* \) but only defines partial a surjective mapping from \( \Sigma^* \) to \( \mathbb{D} \), a so-called notation. Furthermore it holds that the \( m+n \) initial segment of a dyadic number is again a dyadic number and a \( 2^{-n} \)-approximation to the original number. The above sets \( \mathbb{N}, \mathbb{Z}, \mathbb{D} \subseteq \Sigma^* \) are pair-wise disjoint.

The Baire space \( \mathcal{B} \) is the space of all string functions \( \varphi : \Sigma^* \rightarrow \Sigma^* \). The reader is assumed to be familiar with the definitions of computability and complexity of string functions. The above can be used to talk about these concepts for functions between natural and dyadic numbers. Note that all string functions are required to be total, however, usually only the values of the functions on natural or rational inputs are required to fulfill some conditions. As a consequence it is possible to consider multivariate functions by just separating the arguments with \( \#\# \). This paper uses the following pairing
function on string functions:

$$\langle \varphi, \psi \rangle(a) := \begin{cases} 
\varphi(b) & \text{if } a = 0 \\
\psi(b) & \text{if } a = 1 \\
\varepsilon & \text{otherwise.}
\end{cases}$$

Throughout this paper $C([0, 1])$ denotes the set of continuous real-valued functions on the unit interval. The following short notation for intervals is used:

$$[x \pm \epsilon] := [x + \epsilon, x - \epsilon].$$

1.2 Representations

Computability theory encodes discrete structures by strings. Since the set of all strings $\Sigma^*$ is countable, this can only work for countable structures. To compute on structures of continuum cardinality one has to encode the elements by string functions instead of strings.

Definition 1.1

A representation $\xi$ of a space $X$ is a partial surjective mapping $\xi : \mathbb{B} \to X$ from the Baire space to $X$.

An element of $\xi^{-1}(x)$ is called an $\xi$-name or simply a name of $x$. An element of a space with a distinguished representation is called computable resp. polynomial-time computable if it has a name which is computable resp. polynomial-time computable.

Example 1.2

This paper always considers the following representation when computations on the real numbers are carried out: A string function $\varphi$ is a name of $x \in \mathbb{R}$ if and only if for all $n \in \omega$ it holds that $\varphi(2^n) \in D$ and $|\varphi(2^n) - x| \leq 2^{-n}$.

That is: a name of a real number encodes dyadic approximations of arbitrary precision. Instead of fixing the error to $2^{-n}$ one could have used $\frac{1}{n}$ or dyadic numbers. This paper, however, adopts the convention to encode precision requirements as integers in unary, which is standard in the field of real complexity theory.

Definition 1.3

Let $\xi_X$ and $\xi_Y$ be representations of spaces $X$ and $Y$. A realizer of a function $f : X \to Y$ is a function $F : \mathbb{B} \to \mathbb{B}$ such that for all $\varphi \in \mathbb{B}$

$$\varphi \in \text{dom}(\xi_X) \implies \xi_Y(F(\varphi)) = f(\xi_X(\varphi)).$$

That is: $F$ translates $\xi_X$-names of $x$ into $\xi_Y$-names of $f(x)$. Note that computability of functionals on the Baire space can be defined using oracle Turing machines. I.e. a functional $F : \mathbb{B} \to \mathbb{B}$ is called computable if there is an oracle Turing machine $M'$ such that the run of $M'$ on input $a$ and with oracle $\varphi \in \text{dom}(F)$ halts with output $M'(a) = F(\varphi)(a)$.

A function $f : X \to Y$ between spaces with distinguished representations is called computable if it has a computable realizer.

Finally, this paper needs the product construction. Recall that a pairing $\langle \cdot, \cdot \rangle$ of string functions was fixed in the introduction.

Definition 1.4

Let $\xi_X$ and $\xi_Y$ be representations of spaces $X$ and $Y$. Define a representation $\xi_{X \times Y}$ of the Cartesian product $X \times Y$ as follows: A string function $\varphi$ is a name of an element $(x, y) \in X \times Y$ if and only if there exist string functions $\psi \in \xi_X^{-1}(x)$ and $\psi' \in \xi_Y^{-1}(y)$ such that $\varphi = (\psi, \psi')$.

It is true that an element $(x, y)$ of the product is computable resp. polynomial-time computable if and only if both $x$ and $y$ are computable resp. polynomial-time computable.
Example 1.5 For a given representation $\xi$ of the continuous functions on the unit interval $C([0,1])$, the above definitions together with the standard representation of the reals from Example 1.2 allow to discuss computability and polynomial-time computability of the operator

$$\text{eval} : C([0,1]) \times [0,1] \rightarrow \mathbb{R}, \quad (f,x) \mapsto f(x).$$

(2)

1.3 Second-order complexity theory

For complexity considerations this paper uses the framework of second-order complexity theory introduced by Mehlhorn [Meh76]. Instead of the original definition, it uses a characterization of the class of functionals that are considered feasible by means of resource restricted oracle Turing machines that was proven by Kapron and Cook [KC96]. An oracle Turing machine $M'$ is considered to compute a functional $F : \mathcal{B} \rightarrow \mathcal{B}$, thus the oracle is considered input and the time granted should be allowed to depend on the ‘size’ of the oracle.

Definition 1.6 For a string function $\varphi \in \mathcal{B}$ define its length $|\varphi| : \omega \rightarrow \omega$ to be the function

$$|\varphi|(n) := \max\{|\varphi(a)| \mid |a| \leq n\}.$$ 

Therefore, a running time $T$ should be an object of the type $\omega \times \omega \rightarrow \omega$:

- Whenever $p$ is a polynomial whose coefficients are natural numbers, then $(l,n) \mapsto p(n)$ is a second-order polynomial.
- Whenever $P$ is a second-order polynomial, so is the function $(l,n) \mapsto l(P(l,n))$.
- Whenever $P$ and $Q$ are second-order polynomials, so are their point-wise sum and product.

To obtain the class of polynomial time computable functionals, some conventions have to be fixed: This paper considers oracle calls to take one time step. That is: writing the return value is part of the job of the oracle. The machine still has to invest time to read the return value.

Definition 1.7 An oracle Turing machine $M'$ is said to run in polynomial time if there is a second-order polynomial $P$ such that on oracle $\varphi$ with input $a$ it halts after at most $P(|\varphi|,|a|)$ computation steps.

A functional $F : \subseteq \mathcal{B} \rightarrow \mathcal{B}$ is called polynomial-time computable if there is an oracle Turing machine $M'$ that runs in polynomial time and such that for all $\varphi \in \text{dom}(F)$ and strings $a$ it holds that $M'(a) = F(\varphi)(a)$. A function between spaces with distinguished representations is called polynomial-time computable if it has a polynomial time computable realizer.

An important special case of where one is interested in computability or complexity of an operation is to compare different representations of the same space.

Definition 1.8 Let $\xi$ and $\xi'$ be representations of some space $X$. A translation from $\xi$ to $\xi'$ is a realizer of the identity, i.e. a mapping $F : \subseteq \mathcal{B} \rightarrow \mathcal{B}$ such that for all $\varphi \in \text{dom}(F)$ it holds that $\varphi \in \text{dom}(\xi) \Rightarrow \xi'(F(\varphi)) = \xi(\varphi)$.

The representation $\xi$ is called topologically, computably or polynomial-time translatable to $\xi'$ if there exists a continuous, computable or polynomial-time computable translation. The representations $\xi$ and $\xi'$ are called
topologically, computably or polynomial-time equivalent if there exist continuous, computable or polynomial-time computable translations in both directions.

In literature the corresponding relation is usually called reducibility and denoted by \( \leq \). This terminology is often confusing as intuitively ‘\( \xi \) is reducible to \( \xi' \)’ should mean that there is a reduction mapping from \( \xi' \) to \( \xi \). Therefore, it is avoided in this paper.

Example 1.9 The different versions of the definition presented in Example 1.2 lead to polynomial-time equivalent representations.

2 A minimal representation

Due to the use of general representations, this paper imposes the following more restrictive condition on the evaluation operator:

Definition 2.1 A second-order polynomial \( H \) is called hyper-linear, if there exists some integer polynomial \( p \) and a constant \( C \in \omega \) such that

\[
H(l, n) \leq p(l(n + C) + n)
\]

A polynomial-time computable function between represented spaces is called computable in hyper-linear time if there is a hyper-linear second-order polynomial that witnesses its polynomial-time computability.

One should keep in mind that this definition is tailored to fit the application at hand. No care about complexity theoretical well-behavedness was taken. Indeed, the class of hyper-linear time computable operators may change with subtle changes in the model of computation. To make the above definition meaningful, more details about the model of computation have to be fixed: From now on oracle Turing machines are assumed to have distinct oracle query and oracle answer tapes, furthermore the position of the reading resp. writing heads on the oracle tapes is not changed during an oracle query.

Hyper-linear-time computability is not preserved under composition. To guarantee the compositions this paper is interested in to still be hyper-linear-time computable, the following additional assumption is imposed. Recall that in this paper, pairs of strings are encoded by using ## as separator and that all encodings of input strings use at most one further #.

Definition 2.2 A machine \( M \) is said to run in 2-independent hyper-linear time, if there is a hyper-linear running time \( H \) such that the following additional assumption is fulfilled: For any oracle \( \varphi \) and input string of the form \( a##b \), where both \( a \) and \( b \) are strings containing at most one #, the run of \( M \) with oracle \( \varphi \) and input \( a##b \) takes at most \( H(|\varphi|, |a|) \) steps to terminate. An operator on the Baire-space is said to run in 2-independent hyper-linear time if it is computed by a machine that runs in 2-independent hyper-linear time.

This definition does only make sense if the output’s input data is known. I.e. if the returned string function does encode information about pairs. If the input is indeed a pair, the definition can be interpreted as ‘the size of the second element of the pair is irrelevant for the computation time’. This property is needed in the next chapter to provide a very specific kind of closure of the hyper-linear-time computable operators under composition with 2-independent hyper-linear-time computable operators.

2.1 The representation and evaluation

Recall that this paper simulates multivariate input and output from \( \mathbb{N} \) or \( \mathbb{D} \) by separating the different arguments by ##. Furthermore, recall that intervals
of the form \([r + \epsilon, r - \epsilon]\) are abbreviated as \([r \pm \epsilon]\). This chapter proves the following representation to be the minimal representation such that evaluation is hyper-linear-time computable:

**Definition 2.3** Define the **representation** \(\xi_C\) of \(C([0,1])\) as follows: A string function \(\varphi\) is a \(\xi_C\)-name of a function \(f \in C([0,1])\) if and only if

1. For all \(r \in \mathbb{D} \cap [0,1]\) and \(n \in \omega\) there are \(q \in \mathbb{D}\) and \(m \in \omega\) such that
   \[
   \varphi(2^n \# q) = 2^m \# q \quad \text{and} \quad f([r \pm 2^{-m}] \cap [0,1]) \subseteq [q \pm 2^{-n}].
   \]

2. For all \(r, q \in \mathbb{D} \cap [0,1]\) it holds that
   \[
   \varphi(2^{m} \# r) = 2^m \# q \Rightarrow m \leq |\varphi|(n).
   \]

Note that the length of a name can be increased arbitrarily without interfering with the other condition by changing the values of the string function on strings that do not contain any \(\#\). Using this it is quite easy to see that the above indeed defines a representation, i.e. that any continuous function has a name.

Also note that the second condition implies that \(|\varphi|\) is a modulus of continuity of \(\xi_C(\varphi)\) in the following sense: A function \(\mu : \omega \to \omega\) is called **modulus of continuity** of \(f \in C([0,1])\) if it fulfills

\[
\forall x,y \in [0,1] |x - y| \leq 2^{\mu(n)} \Rightarrow |f(x) - f(y)| \leq 2^{-n}. \quad \text{(mod)}
\]

The above is automatically fulfilled for \(\mu(n) : = |\varphi|(n + 1)\) and \(f := \xi_C(\varphi)\).

**Theorem 2.4** The evaluation operator

\[
\text{eval} : C([0,1]) \times [0,1] \to \mathbb{R}, \quad (f, x) \mapsto f(x)
\]

is hyper-linear-time computable with respect to \(\xi_C\).

**Proof** A machine computing the evaluation operator can be described as follows: When given a pair \((\varphi, \psi)\) of a \(\xi_C\)-name \(\varphi\) of a function \(f \in C([0,1])\) and a name \(\psi\) of a real number \(x \in [0,1]\) and an precision requirement \(2^n\) as input, the machine carries out the following loop for increasing \(i\): First it obtains an encoding of a dyadic \(2^{-i}\)-approximation \(x_i\) of \(x\) by evaluating \(\psi(2^n)\). Then it evaluates \(\varphi(2^n \# x_i)\) to obtain an encoding of a dyadic number \(q_i\) and an integer \(m_i\) such that \(f([x_i \pm 2^{m_i}]) \subseteq [q_i \pm 2^{-n}]\). It checks if \(m_i \leq i\). If this it not the case, it increases \(i\) and restarts the loop. If it is the case it exits the loop and returns \(q_i\).

It should be clear that if the machine exits the loop at some point, then the return value is a valid approximation to \(f(x)\). Therefore, it remains to prove that the machine always terminates and runs in polynomial time. Note that by the second condition of the definition of the representation \(\xi_C\), the length of the name is a modulus of continuity. Claim that whenever \(i \geq |\varphi|(n)\), then the machine exits the loop. Indeed, in this case by the second condition of the definition of the representation \(\xi_C\), it holds that \(m_i \leq |\varphi|(n) \leq i\). Thus, the loop is carried out at most \(|\varphi|(n)\) times and the machine runs in polynomial time.

Carrying out the loop takes time \(\mathcal{O}(|\varphi|(n))\): In each run through the loop the number \(i\) is smaller than \(|\varphi|(n)\). The loop also needs to copy \(2^n\), which takes \(\mathcal{O}(n)\) steps. To see that copying the second argument \(q_i\) of \(\varphi(2^n \# x_i)\) is possible within the time bound, it is necessary to extract a bound on the integer part of \(q_i\). This can be done as follows: The string 00\#1 encodes the dyadic number \(\frac{1}{2}\). Thus, by the first condition of the definition of \(\xi_C\) it holds that \(\varphi(1\#00\#1) = 2^{-m} \# q\) and \(q\) and \(m\) fulfill

\[
f([1/2 \pm 2^{-m}]) \subseteq [q \pm 1].
\]
In addition to this, \( \mu(n) := |\varphi| \cdot (n + 1) \) is a modulus of continuity of \( f \) and by dividing the distance to any \( x \in [0, 1] \) to \( \frac{1}{n} \) it follows that

\[
f([0, 1]) \subseteq \{ q \pm 1 + 2^{2^{(1) - 1}} \}.
\]

This finally implies that any of the integer part of the second argument of the return value of \( \varphi(2^n \# \# r) \) is smaller than \( 2^{2^{(1) + 2^{w(7)}}} \), where the second term is a bound on the integer part of \( q \). Since \( |\varphi| \cdot (1) \leq |\varphi| \cdot (7) \), such integers have codes that are of length less than \( |\varphi| \cdot (7) + 3 \).

Therefore, the loop can be carried out in \( O(\max\{|\varphi| \cdot (7), |\varphi| \cdot (n, n)\}) \subseteq O(n + |\varphi| \cdot (n + 7)) \) steps and all of the computation takes less than \( O((n + |\varphi| \cdot (n + 7))^3) \). This time bound is hyper-linear.

\[\square\]

### 2.2 Minimality of the representation

Note that the Definition 2.3 of the representation \( \xi_C \) only poses requirements for the values of a name on pairs, therefore, for an operator \( T \) translating an arbitrary representation to \( \xi_C \), the notion of 2-independent hyper-linear-time computability from Definition 2.2 is meaningful.

**Theorem 2.5** For a representation \( \xi \) of \( C([0, 1]) \) the following are equivalent:

1. The evaluation operator

\[
\text{eval} : C([0, 1]) \times [0, 1] \rightarrow \mathbb{R}, \quad (f, x) \mapsto f(x)
\]

is hyper-linear-time computable with respect to \( \xi \).

2. The representation \( \xi \) can be translated to \( \xi_C \) in 2-independent hyper-linear time.

**Proof**

1. \( \Rightarrow \) 2.: Assume the evaluation operator is computable in hyper-linear time.

To build a machine that translates \( \xi \) into \( \xi_C \) proceed as follows: Given input of the form \( 2^n \# \# r \) (i.e. input for a \( \xi_C \)-name such that the first condition of Definition 2.3 applies) and a \( \xi \)-name \( \varphi \) as oracle. The machine executes a modified version of the source code of the evaluation operator on \( 2^n \): Note that the evaluation operator expects to be handed a pair \( (\psi, \psi') \) of a \( \xi \)-name for the function and a name of a real number \( x \). Thus, whenever there is a leading 0 on the query tape and a query command is issued, the machine first removes the leading 0, and then queries the oracle. Whenever there is a leading 1 on the query tape, the oracle query command in the code of the evaluation are replaced with a code snippet that notes the maximum precision that was asked to the memory tape and then copies an appropriate initial segment of the encoding of the rational number \( r \) to the oracle answer band. The described procedure produces an encoding of a dyadic number \( q \) on the output tape. Finally the machine adds \( 2^m \# \# \) in front of the encoding, where \( m \) is the highest precision that was required of the oracle for the real number. Then it terminates.

This produces a valid output of a \( \xi_C \) name on \( 2^n \# \# r \): The output is valid, as any \( x \in [r \pm 2^{-m}] \) has a name that returns the exact same initial segments of \( r \) on queries less than \( 2^m \). The run of the evaluation operator on this oracle is identical to the run simulated above. Thus the return value is a valid approximation to \( f(x) \) for each of these \( x \). I.e. \( f([r \pm 2^{-m}]) \subseteq [q \pm 2^{-m}] \).

To guarantee that the second condition from Definition 2.3 recall that the evaluation operator being hyper-linear-time computable means that there is an integer polynomial \( p \) and a natural number \( C \) such that the run of the machine computing \( \text{eval} \) with oracle \( \varphi \) on input \( a \) takes at
most $p(|\varphi|(n + C) + n)$ steps. Let the machine proceed on inputs $a$ that are not of the form $2^n \# \# r$ as follows: For any of the $3^n$ strings $c$ of length $C$ it queries the oracle $\varphi$ on $ca$ and $ca'$, where $a'$ is the string where the first symbol after the first $\#$ is replaced by a $\#$ (and $a = a'$ if there is no $\#$ or the only one is the last symbol). It takes the maximum $m$ of the lengths of the oracle answers and returns the string consisting only of 1s and of length $p(m + n)$.

The above guarantees that the string function produced by the machine has length bigger than $p(|\varphi|(n + C) + n)$: Let $b$ be a string of length $n + C$ such that $|\varphi(b)| = |\varphi|(n + C)$. Let $a$ be the last $n$ bits of $b$ where in the first occurrence of $\# \#$ the second $\#$ is replaced by $0$.

Then the machine described above carries out the previous paragraph on input $a$. By the procedure described there it is guaranteed that the query $b$ is posed to the oracle and that the return value is longer than $p(|\varphi(b)| + n) = p(|\varphi|(n + C) + n)$.

The final thing to verify is that the second condition of the Definition 2.3 of $\xi_C$ is fulfilled by the function produced by the above procedure: Let $\psi$ be the string function produced by the machine above. By the previous it is clear that $|\psi|(n) \geq p(|\varphi|(n + C) + n)$. Since $(l, n) \mapsto p((n + C) + n)$ is a running time of the evaluation operator, which is simulated on an oracle of length $|\varphi|$ and input $2^n$, it is clear that the number $m$ produced in the second paragraph of the proof is smaller than $p(|\varphi|(n + C) + n)$ and therefore also as $|\psi|(n)$.

Finally argue that there is a machine that runs in 2-independent hyper-linear time. If the input is of the form $2^n \# \# r$, to guarantee that the procedure only takes time hyper-linear in $n$ the procedure must be modified to only copy a beginning segment of $r$ to the oracle answer band. This does not lead to changes in the run as the running time requirement for the evaluation operator bounds the number of digits that are accessible to the machine. If the input is not of this form, then the number of steps needed is polynomial in $3^n$, $|\varphi|(n + C)$, the degree of $p$ and the maximum of its coefficients. Since all but $|\varphi|(n + C)$ are constants, this leads to a hyper-linear time-bound.

2. $\Rightarrow$ 1:. It can be verified that the 2-independent hyper-linear-time computability of the translation is exactly what is needed to guarantee that the composition of the translation and the evaluation operator on $\xi_C$ (which is computable in hyper-linear time by Theorem 2.4) is hyper-linear-time computable.

2.3 Comparison to second-order representations

This final chapter presents a restriction of the representation $\xi_C$: It is impossible to compute a modulus of continuity of a function in polynomial time with respect to $\xi_C$. Computing a modulus of continuity is an inherently multivalued operation. Recall that a multivalued mapping $f : X \rightrightarrows Y$ is an assignment of elements of $x$ to non-empty sets $f(x) \subseteq Y$. The elements of $f(x)$ are interpreted as the ‘acceptable return values’. Definition 1.3 of a realizer can straight-forwardly be extended to apply to multivalued mappings and thus it makes sense to talk about computability and complexity of multivalued mappings.

**Theorem 2.6** The modulus function

$\text{mod} : C([0, 1]) \rightrightarrows \omega^\omega; f \mapsto \{\mu \mid \mu \text{ is mod. of cont. of } f \text{ (see eq. (mod))}\}$

is not polynomial-time computable with respect to $\xi_C$. 

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Definition 2.7 ([KC12]) A string function \( \varphi \) imposes an additional condition on the names: this paper, however, for a well-behaved second-order complexity theory they

\[ \text{if } \varphi \text{ is a modulus of continuity of } f \] Consider the following name \( \psi \) of the constant zero function:

\[ \psi(a) := \begin{cases} 2^n \# \# 00\# & \text{if } a = 2^{n+1} \# \# r \text{ for some } r \in \mathbb{D} \cap [0,1] \\ \varepsilon & \text{otherwise.} \end{cases} \]

Obviously \( |\psi|(n) = n + 1 \). The function \( P(n \mapsto n + 1, n) \) is a polynomial \( p \) and bounds the number of steps until the machine returns some value \( m \) of \( \mu(n) \). Choose some \( N \) such that \( 3p(N) < 2^N \). Think of \([0,1]\) as the union of \( 2^N \) closed intervals of equal length \( 2^{-N} \). Since the \( 2^{-N-1} \) neighborhood of a rational number can at most intersect three such intervals, and the machine can at most ask \( p(N) \) queries, at least one closed interval \( I \) is such that no rational number in its \( 2^{-N-1} \) neighborhood is queried. Let \( f' \) be the function that is zero everywhere but in \( I \), where it takes the value \( \frac{3}{2} 2^{-N} \) in the middle and then goes linearly to zero with slope \( 3 \cdot 2^{\mu(N) - N \delta} \). Note that any modulus of continuity of \( f' \) at \( N \) is strictly larger than \( \max\{\mu(N), N\} \).

To change the name \( \psi \) of the zero function to a name \( \psi' \) of \( f' \) without changing any of the values the machine looked at during the computation, first note that due to the choice of the interval \( I \) each query the machine makes either is a query with a precision such that the values of \( f' \) are valid approximations or the name only returns information about the values on an interval disjoint from \( I \). Therefore, it is possible to change the values of \( \psi \) at strings the machine does not query to obtain a string function \( \tilde{\psi} \) that fulfills the first condition of being a name of \( f' \). Where the corresponding return values can be chosen to be the exact values of \( f' \) and the intervals can be chosen to be optimal.

Furthermore, there are at least \( 2^M \) strings of length \( M \) that do not represent any pair of a natural number and a dyadic number, for instance the binary strings. Thus for any \( M \geq N \) there is at least one such string \( a_M \) the machine does not query. To obtain a valid name \( \psi' \) of \( f' \) change the values of \( \psi \) on strings \( a_M \) to have length according to a modulus of continuity of \( f' \).

Thus, the run of the machine on input \( N \) with oracle \( \psi' \) returns \( \mu(N) \). However, by construction, \( \mu \) is not a modulus of continuity of \( f' \). Therefore, no polynomial-time machine computing a modulus function exists. \( \blacksquare \)

Kawamura and Cook introduced a framework for complexity considerations in analysis. This framework uses the same kind of representation as this paper, however, for a well-behaved second-order complexity theory they impose an additional condition on the names:

**Definition 2.8 ([KC12])** A representation is a second-order representation if its domain is contained in \( \Sigma^{**} \).

It is noteworthy, that they also adjust the notion of polynomial-time computability by relaxing the time bound to only be valid for length-monotone
oracles. I.e. a polynomial-time machine in their sense may diverge on a not
length-monotone input. However, since there is a polynomial-time computable
retraction from the Baire space to $\Sigma^\omega$ [KS], this does not change the class of
polynomial-time computable functions and we may stick with the definitions
used in this paper.

**Definition 2.9 ([KC12])** Define a second-order representation $\delta$ of $C([0,1])$
as follows: A length-monotone string function $\varphi$ is a name of a function
$f \in C([0,1])$ if $\varphi = \langle \psi, \psi' \rangle$ for string functions $\psi$ and $\psi'$ that fulfill both
of the following:

1. $n \mapsto |\psi(2^n)|$ is a modulus of continuity of $f$.
2. for any encoding $r$ of a dyadic number in $[0,1]$ and $n \in \omega$ it holds that
$\psi'(2^n \# \# r)$ is an encoding of a dyadic number $q$ and

$$|f(r) - q| \leq 2^{-n}.$$ 

A polynomial-time translation of $\delta$ to $\xi_C$ is readily written down. The modulus
function as defined in Theorem 2.6 is obviously polynomial-time computable with respect to $\delta$. With respect to $\xi_C$ Theorem 2.6 proves it not to be polynomial-time computable. Therefore, the representations $\delta$ and $\xi_C$ are
not polynomial-time equivalent.

**Corollary 2.10** $\xi_C$ can not be translated to $\delta$ in polynomial time.

Kawamura and Cook succeeded to prove the following:

**Theorem 2.11 (Lemma 4.9 in [KC12])** For a second-order representation
$\delta$ of $C([0,1])$ the following are equivalent

- The evaluation operator from eq. (eval) is polynomial-time computable.
- $\delta$ is polynomial-time translatable to $\delta_C$.

Since the hyper-linear-time computability implies polynomial-time computability
this entails the following:

**Corollary 2.12** $\xi_C$ is not polynomial-time equivalent to any second-order
representation.

### 3 Conclusion

Before we go off into the technical details note the following: The representation $\xi_C$ was invented in an attempt to model the behavior of iRRAM within
the framework of second-order complexity theory. There is empirical evidence
that within iRRAM function evaluation is fast but computing a modulus of
continuity is slow. The representation $\xi_C$ indeed reflects this: It renders eval-
uation polynomial-time computable but does not allow to extract a modulus
of continuity in polynomial time. It is remarkable that it is possible to do this
within the framework of second-order complexity theory as previous results
seemed to indicate that this is not possible.

However, the correspondence between $\xi_C$ and iRRAM is imperfect: The
running time of the straight forward algorithm for computing a modulus of
continuity in iRRAM is still way worse than that with respect to the representation $\xi_C$. It is improbable that representations that feature polynomial-time
evaluation and an even worse running time of the modulus function can be
produced as polynomial-time evaluation seems to necessitate the length to be
comparable to a modulus of continuity, while the length function can always
be computed in exponential time by brute force search. Furthermore the rep-
resentation $\xi_C$ seems to have deficiencies that are not reflected in the behavior
of iRRAM. For instance the authors suspect that composition of functions is
not polynomial-time computable with respect to \( \xi_C \). However, so far they failed to provide a proof that this is so.

That hyper-linear-time computable operators are not closed under composition can be seen as follows:

**Example 3.1** Let \( T \) be the operator such that \( T(\varphi)(a) \) is \( \varphi(a) \) spelled from back to front. Let \( S \) be the operator such that \( S(\varphi)(a) \) is the first bit of \( \varphi(\varphi(a)) \). Both of these operators are hyper-linear time computable. However, their composition evaluates the last bit of \( \varphi(\varphi(a)) \). But this is impossible without doing at least \( |\varphi(\varphi(a))| \) steps to shift the reading head, therefore this composition is not hyper-linear-time computable.

One might argue that this indicates that writing the oracle answer should be counted towards the time consumption, and therefore \( S \) should not be hyper-linear-time computable. However, in Theorem 2.4 a situation is encountered where the oracle is evaluated on big input, but the output is guaranteed to be small. While this is intuitively still linear-time, it would not be possible in linear time when writing the oracle answer was considered to consume time:

The realizer has to be total and linear-time computable and for a general oracle there is no way to guarantee that writing the return value does not take more steps than the machine is allowed to take.

In the proof of the hyper-linear-time computability of the evaluation operator with respect to \( \xi_C \) in Theorem 2.4 the precision in each try is increased by one. This may lead to many useless queries. One could instead use the precision that the name requires the input approximation to have as next precision. However, this may lead to unnecessary high precision. Both approaches lead to the same worst case complexity. The later, however, seems to be empirically superior and is thus used in \texttt{iRRAM}.

Definition 2.1 of hyper-linear time could be relaxed: The construction in Theorem 2.5 still works if the constant \( C \) depends polynomially on the logarithm of \( n \). If \( C \) were allowed to depend on \( n \) polynomially, the class would coincide with a class that some authors argue should be used to define polynomial-time computability anyway. However, the proof-technique used in this paper does not generalize this far.

The introduction of 2-independent hyper-linear-time computability is due to principal restrictions of second-order complexity theory. If one insists on inserting the length function of a string function into the running time, one looses the possibility to use parameters. The notion takes a step away from second-order complexity theory and towards a parameterized complexity theory with a first-order parameter instead. From a programming point of view this is a reasonable step: If there exists some input to the string function that produces a long output but this argument is never queried in the current computation, the length of the return value should not count towards the time restriction imposed on the machine.

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