Examples of potentials with convergent Schwinger — DeWitt expansion

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Abstract

Convergence of the Schwinger — DeWitt expansion for the evolution operator kernel for special class of potentials is studied. It is shown, that this expansion, which is in general case asymptotic, converges for the potentials considered (widely used, in particular, in one-dimensional many-body problems), and besides, convergence takes place only for definite discrete values of the coupling constant. For other values of the charge divergent expansion determines the kernels having essential singularity at origin (beyond usual $\delta$-function). If one consider only this class of potentials then one can avoid many problems, connected with asymptotic expansions, and one get the theory with discrete values of the coupling constant that is in correspondence with discreteness of the charge in the nature. This approach can be transmitted into the quantum field theory.

Key words

Evolution operator kernel, Schrödinger equation, convergence, quantization of charge.

1 Introduction

In the quantum theory expansions in different parameters such as the coupling constant [Bender 1969, 1971, Lipatov 1977], the WKB—expansion, the short-time Schwinger — DeWitt expansion [Schwinger 1951, DeWitt 1965, 1975] the perturbation expansion in phase-space technique [Barvinsky 1995], $1/n$-expansion [Popov 1992], etc. are, as a rule, asymptotic. This circumstance imposes essential restrictions on possibilities of their using, makes the theory incomplete and compels one to look for the ways of overcoming these restrictions either by summation of divergent series with special methods (see, e.g., [Kazakov 1980]), or by constructing new convergent expansions [Holliday 1980, Ushveridze 1983, Sissakian 1992], or by creating different approximate methods taking into consideration the so-called nonperturbative effects.

Interesting information about possible way of overcoming the problem of divergencies in some cases may be obtained from investigation of the time
dependence of the evolution operator kernel with a help of the Schwinger—
DeWitt expansion. For example, such investigations for a class of potentials
which is the family of bounded and continuous functions that are formed from
the Fourier transforms of complex bounded measures were made in [Osborn
1983]. An important feature of this expansion is that after factorization of
the contribution of the free kernel ("free" case corresponds to $V \equiv 0$), having
at $t = 0$ the singularity in the form of $\delta$-function in space variables, one can
concentrate attention on rest part (denote it as $F$) which, according to the
initial condition, should be equal to 1 when $t = 0$. To understand behaviour
at $t = 0$ it is necessary to make analytical continuation into the complex
plain $t$. When this continuation is made for the kernel, then its analytical
properties are masked by the singularity which provides $\delta$-like behaviour in
space variables. But if the factorization of the free part of the kernel is made,
then the rest function can be continued into the entire complex plain $t$ and
one can accurately examine its properties in the neighborhood of origin.

If the Schwinger—DeWitt expansion is convergent, then the point $t = 0$
is regular and initial condition is fulfilled in rigorous sense. But if this
expansion is divergent (note, that usually it is treated as asymptotic [Osborn
1983, Slobodenyuk 1995, 1996 [IS]]) then the point $t = 0$ is essential singular
point for the function $F$. In this case the initial condition may be fulfilled only
in asymptotic sense. The function $F$ tends to 1 when $t \to 0$ along the real
positive semiaxis as continuous function, but it is not analytic at $t = 0$ and it
does not have any meaning at this point.

Nevertheless, it is enough to fulfill the initial condition even in asym-
ptotic sense that unambiguous solution of the evolution problem to exist. So,
divergence of the Schwinger—DeWitt expansion does not put any formal re-
strictions on choose of the potentials in the quantum theory. But using of the
potentials with divergent expansion (if exact solution is not known) is usually
connected with problems of different divergences (see, e.g., Sect. 4).

There exist possibility to avoid many of these problems in some cases. If
one consider the potentials for which the Schwinger—DeWitt expansion con-
verges, then one may get convergent representation for the kernel and other
physical values. Such nontrivial potentials really exist. This paper is devoted
to consideration of some examples of such potentials and to proving of conver-
gence of the expansion for them. These are the potentials being widely used
in one-dimensional many-body problems [Olshanetsky 1983, Calogero 1975,
Sutherland 1971, 1972]. For definite discrete values of the coupling constant
the expansions for them are convergent in the entire complex plain \( t \). For
other values of the charge the expansions are asymptotic. Phenomenon of
existence of such potentials is very interesting. Moreover, convergence of the
expansion only for discrete values of the coupling constant may be connected
with discreteness of the charge in the nature.

2 The method of research

The evolution operator kernel of the Schrödinger equation in one-dimensional
case is the solution of the problem

\[
\frac{i}{\partial t} \langle q', t \mid q, 0 \rangle = -\frac{1}{2} \frac{\partial^2}{\partial q'^2} \langle q', t \mid q, 0 \rangle + V(q') \langle q', t \mid q, 0 \rangle, \tag{1}
\]

\[
\langle q', t = 0 \mid q, 0 \rangle = \delta(q' - q). \tag{2}
\]

Here and everywhere below dimensionless values, which are derived from di-
mension ones in an obvious way, are used for the sake of convenience. The
variable \( t \) is treated as a complex one. If one means the proper Schrödinger
equation then \( t \) is real. We imply that \( V(q) \) does not apparently depend on
time.

As it is well known, in the free case \((V \equiv 0)\) the solution of the problem (1),
(2) is

\[
\langle q', t \mid q, 0 \rangle = \frac{1}{\sqrt{2\pi t}} \exp \left\{ \frac{i (q' - q)^2}{2t} \right\} \equiv \phi(t; q', q). \tag{3}
\]

The function \( \phi \) has essential singularity at \( t = 0 \), but this singularity is such,
that it provides the initial condition (2) to be fulfilled.

When interaction is present the kernel can be represented as

\[
\langle q', t \mid q, 0 \rangle = \frac{1}{\sqrt{2\pi it}} \exp \left\{ \frac{i (q' - q)^2}{2t} \right\} F(t; q', q), \tag{4}
\]

and besides, one can write for \( F \) the expansion (the short-time Schwinger–
DeWitt expansion)

\[
F(t; q', q) = \sum_{n=0}^{\infty} (it)^n a_n(q', q), \tag{5}
\]

which, as a rule, is asymptotic and is usually utilized only in that quality.
We shall use representation (4), (5) to test the analytical properties of the
evolution operator kernel in variable $t$ and, in particular, ascertain its behavior for $t \to 0$.

For this purpose let us derive from (1) the equation for $F$
\[ \frac{i}{t} \frac{\partial F}{\partial t} = -\frac{1}{2} \frac{\partial^2 F}{\partial q'^2} + \frac{q' - q}{it} \frac{\partial F}{\partial q'} + V(q') F. \] (6)

Because the factorized function $\phi$ yet fulfills the initial condition (2), then $F$ should satisfy the initial condition
\[ F(t = 0; q', q) = 1. \] (7)

It seems, at first sight, that it is possible to add to the right-hand side of (7) an arbitrary function of $q' - q$, which vanishes at $q' = q$. However, this is not true. The equation for the coefficient $a_0$
\[ \frac{q' - q}{\partial q'} \frac{\partial a_0(q', q)}{\partial q'} = 0, \]
taken from general recursion relations for $a_n(q', q)$, and condition $a_0(q, q) = 1$ determines unambiguously
\[ a_0(q', q) = F(0; q', q) = 1. \]

The problem (1), (2), from which we have started, has a physical sense only for the real positive $\tau$, where $\tau = it$ (if the heat equation and heat kernel are considered), or for real $t$ (if the quantum mechanical evolution equation is considered). The same restrictions are initially fair for equation (3) too. But we can analytically continue the function $F$ into complex plain of the variable $t$ using the differential equation (3) with condition (7). Now the variable $t$ may vary in hole complex plain $t$. There is no restriction $\text{Re} \ t > 0$, which takes place for the analytic semigroup.

If $q$ is regular point of the function $V(q)$ and at any domain expansion in powers of $\Delta q = q' - q$ is fair
\[ V(q') = \sum_{k=0}^{\infty} \Delta q^k \frac{V^{(k)}(q)}{k!}. \] (8)

(the notation
\[ V^{(k)}(q) \equiv \frac{d^k V(q)}{dq^k} \]
is used here and will be used further everywhere), then one can use concrete form of the coordinate dependence of the coefficients $a_n$

$$F(t; q', q) = 1 + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} (it)^n \Delta q^k b_{nk}(q). \quad (9)$$

It is obvious that

$$\sum_{k=0}^{\infty} \Delta q^k b_{nk}(q) = a_n(q', q) = -Y_n(q', q),$$

where $Y_n$ are the functions introduced in [Slobodenyuk 1993]. The behavior of $Y_n$ was studied in [Slobodenyuk 1995] using representation adduced in [Slobodenyuk 1993].

Substitution of (9), (8) into (6) leads to recurrent relations for the coefficients $b_{nk}$

$$b_{nk} = \frac{1}{n+k} \left[ \frac{(k+1)(k+2)}{2} b_{n-1,k+2} - \sum_{m=0}^{k} \frac{V^{(m)}(q)}{m!} b_{n-1,k-m} \right] \quad (10)$$

with condition $b_{0k} = \delta_{k0}$. Specifically,

$$b_{1k} = -\frac{V^{(k)}(q)}{(k+1)!}. \quad (11)$$

Expressions (9), (10) determine a formal solution of problem (3), (7). As to the expansion in powers of $\Delta q$ in (3), one may expect that its convergence range is equal to one for expansion (8) of the potential. The series in $t$ in (3) is usually treated as divergent one. At first sight, it is really always so. Let us estimate the convergence of the series in (3).

At the beginning let $n$ be fixed and $k \to \infty$. Expressing $b_{n-1,k+2}$ from (10) via the coefficients with smaller $k$ we shall come to some linear combination of the coefficients of type $b_{n_0,0}$ and $b_{n_1,1}$ with any indexes $n_0, n_1$ (for the sake of brevity we shall write further only the terms with $b_{n_0,0}$, implying that the same statements are concerned with the terms with $b_{n_1,1}$). The main growth for large $k$ takes place if the second index of $b_{nk}$ is diminished: a) with using the term $V^{(k)}(q)b_{n-1,0}/k!$, b) with using the expression on the left-hand side of (10).

In the case a) we get for $k \to \infty$

$$|b_{n-1,k+2}^{(a)}| \sim \frac{2}{(k+1)(k+2)} \frac{|V^{(k)}|}{k!} |b_{n-1,0}|. \quad (12)$$
Because series (8) converges at some circle with radius $R(q)$ the estimate

$$\frac{|V^{(k)}|}{k!} \sim \frac{1}{R^k(q)}$$

for $k \to \infty$ is fair. So, for every fixed $n$ and for $k \to \infty$ we have

$$|b^{(a)}_{nk}| \sim \frac{|b_{n0}|}{R^k(q)}.$$  \hspace{1cm} (13)

The contributions of type (13) correspond to the expansion in $\Delta q$, which is convergent for every fixed $n$ with convergence range $R(q)$.

In the case b) for $k \to \infty$ we get

$$|b^{(b)}_{n-1,k+2}| \sim \frac{2^{k/2+1}(n+k)!}{k!(n+k/2-1)!}|b_{n+k/2,0}|.$$  \hspace{1cm} (14)

Behavior of $b^{(b)}_{nk}$ for $k \to \infty$ depends on the behavior of $b_{n0}$ for $n \to \infty$. If $b_{n0}$ decreases when $n \to \infty$ or increases more slowly than $\Gamma(\alpha n)$ ($\alpha$ is any positive number), then

$$|b^{(b)}_{nk}| \sim \frac{|b_{n0}|}{\Gamma(k/2)}$$

for $k \to \infty$, i.e., these contributions will disappear at large $k$. If $b_{n0}$ increases as $\Gamma(\alpha n)$ (here $0 < \alpha \leq 1$, in [Slobodenyuk 1995] showed that $\alpha$ cannot be larger then 1), then for $k \to \infty$ and $\alpha < 1$

$$|b^{(b)}_{nk}| \sim \frac{|b_{n0}|}{\Gamma\left(\frac{1-\alpha}{2}k\right)},$$

so, these contributions will disappear too with the growth of $k$. If $\alpha = 1$, then the following estimate will take place ($n$ is fixed, $k \to \infty$)

$$|b^{(b)}_{nk}| \sim |b_{n0}|k^c \rho^k.$$  \hspace{1cm} (15)

In this case the expansion in $\Delta q$ in (3) will have the finite convergence range too, but it will be equal to minimum from two values $R(q)$ and $\rho$.

Now let us examine the behavior of $|b_{n0}|$ (the same will be also correct for $|b_{n1}|$) when $n \to \infty$. Consider the decreasing of $n$ till 1 by means of the first term on the right-hand side of (14)

$$|b_{n0}| \sim \frac{|b_{n-1,2}|}{n} \sim \ldots \sim \frac{(n-1)!}{2^{n-1}}|b_{1,2n-2}| = \frac{(n-1)!}{2^{n-1}} \frac{|V^{(2n-2)}|}{(2n-1)!}.$$  \hspace{1cm} (15)
Because $|V^{(k)}| \sim k!/R^k(q)$ for $k \to \infty$, then for $n \to \infty$ we get

$$|b_{n0}| \sim \frac{(n-1)!}{2^{n-1}(2n-1)} \sim n!.$$  \hspace{1cm} (16)

Really, the contributions taken into account in (15) provide the main growth only for the potentials, for which $R(q) < \infty$. If the potentials with $R(q) = \infty$ are considered (e.g., polynomial ones), then, at first sight, one can conclude from (15) that $|b_{n0}| \sim 1/n!$. But it is not so, in fact. As it was shown in [Slobodenyuk 1995], the combination of contributions of the first term and terms of sum over $m$ in (10) leads to the estimate of type $|b_{n0}| \sim \Gamma(\alpha n)$. So, for arbitrary potentials the series in $t$ in (9) is divergent. But in our estimates, in fact, absolute values of all contributions to every coefficient $b_{nk}$ were summed. Nevertheless, for some potentials the cancellation of different terms may occur. It can lead to convergence of the expansion in (3). For the potentials considered in Secs. 3–5 this cancellation takes place only for definite values of the coupling constant.

Note that we, really, test expansion (9) for the absolute convergence. So, it is enough for the convergence of double series that (9), in which instead of $b_{nk}$ absolute values $|b_{nk}|$ taken, would converge for any order of summation. Our consideration corresponds to the following order: at first the series over $k$ for every fixed $n$ are summed and then summation over $n$ is made. If one assumes that there is convergence of the series in index $n$ then, as it was shown before, the convergence in index $k$ will take place at every fixed $n$, and to establish the convergence of the series in index $n$ it is enough to determine the behavior of the coefficients $b_{n0}$, $b_{n1}$ only (but not all $b_{nk}$) at $n \to \infty$.

3 Modified Pöschl — Teller potential

Let us introduce standard notation for the coupling constant $g = \lambda(\lambda - 1)/2$ ($\lambda > 0$) and investigate modified Pöschl — Teller potential

$$V(q) = -\frac{\lambda(\lambda - 1)}{2} \frac{\beta^2}{\cosh^2(\beta q)}$$  \hspace{1cm} (17)

for the convergence of expansion (3).

Because the constant $\beta$ is connected with the choice of length scale one can put $\beta = 1$ without the restriction of generality. Further, for the sake of
brevity we shall denote
\[ f(q) = -\frac{1}{\cosh^2 q}. \] (18)

Then the potential reads briefly \( V(q) = gf(q) \).

The potential (17) has the expansion of type (8) about every real point \( q \). Its convergence range is equal to \( R(q) = \sqrt{\left(\frac{\pi}{2}/k\right)^2 + q^2} \) and is determined by the distance to the nearest singularities of the function \( 1/\cosh^2 q \) placed at the points \( q = \pm i\pi/2 \). The derivatives can be calculated as follows
\[ V^{(k)}(q) = gf^{(k)}(q), \] (19)
where \( f^{(k)} \) are represented as expansions in powers of \( f 
\]
\[ f^{(2n)}(q) = \sum_{l=1}^{n+1} a^{(2n)}_l f^l(q), \] (20)

\[ f^{(2n+1)}(q) = \sum_{l=1}^{n+1} l a^{(2n)}_l f^{l-1} f^{(1)} = \sum_{l=1}^{n+1} a^{(2n+1)}_l f^{l-1} f^{(1)}. \] (21)

To obtain all coefficients \( a^{(k)}_l \) it is enough to put \( a^{(0)}_l = \delta_{l1} \) and take into account
\[ (f^{(1)})^2 = 4f^3 + 4f^2. \]

For \( a^{(2n)}_l \) one has the recursion relations
\[ a^{(2n)}_l = 4l^2 a^{(2n-2)}_l + 4(l - 1)(l - 1/2) a^{(2n-2)}_{l-1}. \] (22)

So, every derivative of the function \( f(q) \) is represented as a polynomial in powers of this function.

From (10) one gets for potential (17)
\[ b_{nk} = \frac{1}{n+k} \left[ \frac{(k+1)(k+2)}{2} b_{n-1,k+2} - \frac{\lambda(\lambda - 1)}{2} \sum_{m=0}^{k} \frac{f^{(m)}}{m!} b_{n-1,k-m} \right], \] (23)

where the derivatives \( f^{(m)} \) are calculated via (21)–(22).

According to the note at the end of Sec. 2, it is enough for testing the convergence of series (9) to examine the behavior at \( n \to \infty \) of the coefficients \( b_{n0}, b_{n1} \) only. Introduce in this connection the functions
\[ B_k(t,q) = \sum_{n=0}^{\infty} t^n b_{nk}(q) \] (24)
and consider them for \( k = 0, 1 \).

The analysis of relations (23) with taking into account of (20)–(22) shows that \( B_0, B_1 \) can be represented in the form

\[
B_0(t, q) = 1 + \sum_{n=1}^{\infty} (it)^n \sum_{l=1}^{n} \frac{(-1)^l}{l!} f^l(q) \beta_{nl} \prod_{j=1}^{l} \left( \frac{\lambda(\lambda - 1)}{2} - \frac{j(j - 1)}{2} \right)
\]

\[
= 1 + \sum_{n=1}^{\infty} (it)^n \sum_{l=1}^{n} \frac{(-1)^l}{l!} f^l(q) \beta_{nl} \frac{\Gamma(\lambda + l)}{2^l \Gamma(\lambda - l)},
\]

(25)

\[
B_1(t, q) = \sum_{n=1}^{\infty} (it)^n \sum_{l=1}^{n} \frac{(-1)^l}{l!} \frac{l}{2} f^{l-1}(q) f^{(1)}(q) \beta_{nl} \prod_{j=1}^{l} \left( \frac{\lambda(\lambda - 1)}{2} - \frac{j(j - 1)}{2} \right)
\]

\[
= \sum_{n=1}^{\infty} (it)^n \sum_{l=1}^{n} \frac{(-1)^l}{l!} \frac{l}{2} f^{l-1}(q) f^{(1)}(q) \beta_{nl} \frac{\Gamma(\lambda + l)}{2^l \Gamma(\lambda - l)},
\]

(26)

where

\[
\beta_{nl} = \frac{1}{2^{n-l}} \frac{(n - 1)!}{(l - 1)! (2n - 1)!}.
\]

(27)

To estimate the behavior of \( \beta_{nl} \) when \( n \to \infty \) we probe the asymptotics of \( a_l^{(2n-2)} \). Let us take \( a_l^{(2n-2)} \) for sufficiently large \( n \) and begin to express \( a_l^{(2n-2)} \) with the help of (22) via the coefficients with smaller \( n \) and \( l \) so that to come to \( a_1^{(0)} = 1 \) at the end. Maximal contribution arises in this procedure when, at the beginning, \( n \) will be diminished at fixed \( l \) by means of the first term on the right-hand side of (22), and then, when the equality \( n = l - 1 \) becomes valid, \( n \) and \( l \) will start to be decreased simultaneously by unit at every step by means of the second term in (22). For large \( n \) this gives the estimate

\[
a_l^{(2n-2)} \sim 4^{n-l} l^{2(n-l)} (2l - 1)!
\]

Then \( \beta_{nl} \) behaves itself as

\[
\beta_{nl} \sim 2^{n-l} l^{2(n-l)} (n - 1)! (2l - 1)! \frac{(l - 1)! (2n - 1)!}{(l - 1)! (2n - 1)!}.
\]

(28)

Now one can evaluate the asymptotics at \( n \to \infty \) of the coefficients of series (23), (20). Taking in (23), (20) in the sum over \( l \) the term with \( l = n \) we shall obtain for noninteger \( \lambda \) that the coefficients in front of \( t^n \) growth in (23) as

\[
\frac{f^n}{2^n n! \Gamma(\lambda - n)} \sim n!,
\]
and in (26) as
\[ f^{n-1} f^{(1)} - 1 \begin{array}{c} n+1 \\ n+1 \end{array} \Gamma(\lambda + n) \sim (n-1)!\Gamma(\lambda - n) \sim n! , \]
So, for noninteger \( \lambda \) series (25), (26), and, hence, (9) are asymptotic ones.

Let now \( \lambda \) be integer (\( \lambda > 1 \)). Then in (25), (26) in the sums over \( l \) only the terms with \( l < \lambda \) are different from zero, and, in fact, one should take instead of \( \sum_{l=1}^{n} \) the sum \( \sum_{l=1}^{\min(n,\lambda-1)} \). For \( n \geq \lambda - 1 \) the sum over \( l \) will always contain \( \lambda - 1 \) terms, and its dependence on \( n \) will be determined only by the dependence on \( n \) of the coefficients \( \beta_{nl} \). And the dependence of the latter on \( n \), as it is clear from estimate (28), at fixed \( l \leq \lambda - 1 \) and at \( n \to \infty \) is determined by the factor
\[ \beta_{nl} \sim \left( 2(\lambda - 1)^2 \right)^n \frac{(n-1)!}{(2n-1)!} . \]
So, the coefficients in front of \( t^n \) in (25), (26) behave themselves at large \( n \) as
\[ C^n \frac{(n-1)!}{(2n-1)!} \]
with any positive \( C \), i.e., the series will be convergent at the circle of infinite range.

To obtain finally the function \( F(t; q', q) \) it is necessary either to take the coefficients \( b_{n0}, b_{n1} \) from (25), (26) to calculate other \( b_{nk} \) using (23), or starting from \( B_0, B_1 \) to calculate other functions \( B_k(t, q) \) from the equations
\[ B_{k+2} = \frac{2}{(k+1)(k+2)} \left( \frac{1}{i} \frac{\partial B_k}{\partial t} + \frac{k}{it} B_k + g \sum_{m=0}^{k} \frac{f^{(m)}}{m!} B_{k-m} \right) , \]
which in an obvious way are derived from (3) after substitution
\[ F(t; q', q) = \sum_{k=0}^{\infty} \Delta q^k B_k(t, q) , \]
and substitute them into (30).

Particularly, for \( \lambda = 2 \) (\( g = 1 \)) we have the potential \( V(q) = -1/\cosh^2 q \), for which
\[ B_0(t, q) = 1 - f(q) \sum_{n=1}^{\infty} \frac{(it)^n}{(2n-1)!!} = 1 - f(q) \sqrt{\frac{\pi it}{2}} e^{it/2} \text{erf}(\sqrt{it}/2) , \]
\[ B_1(t, q) = -\frac{1}{2} f^{(1)}(q) \sum_{n=1}^{\infty} \frac{(it)^n}{(2n-1)!!} = -\frac{1}{2} f^{(1)}(q) \sqrt{\frac{\pi it}{2}} e^{it/2} \text{erf}(\sqrt{it}/2) . \]
With the help of (29) one is able to determine all coefficient functions $B_k$ starting from (31), (32) and then to substitute them into (30). In this manner the function $F$ will be found.

We established, that for integer $\lambda$ expansion (9) was convergent if $|\Delta q| < R(q)$ and the representation (4), (9) for the evolution operator kernel was not asymptotic. The function $F$ is single-valued analytic in the entire complex plain of the variable $t$ function and it has an essential singularity at the infinite ($t = \infty$) point.

The potential (17) is the representative of class of potentials studied in [Osborn 1983]. It can be written as the Fourier transform

$$V(x) = \int_{-\infty}^{+\infty} e^{i\alpha x} d\mu(\alpha),$$

where

$$d\mu(\alpha) = -\frac{g\alpha d\alpha}{2 \sinh(\pi\alpha/2)}.$$  

In [Osborn 1983] showed that $|a_n| < n^{2n}/n! \sim n!$ when $n \to \infty$ for the potentials of that class. This does not mean that the Schwinger — DeWitt expansion should be divergent in every case, because this is only bound from up, but not from down. So, our result about convergence of the expansion for the potential (17) for integer $\lambda$ does not contradict to conclusions of paper [Osborn 1983].

4 Potential $V(q) = g/q^2$

Another example of convergent series (8) we shall get considering the potential $V(q) = \frac{\lambda(\lambda - 1)}{2} \frac{1}{q^2}$ on half-line $q > 0$. This potential is well studied and it is known analytic expression for the kernel for it. Our purpose is to show how the method described above can be applied to singular potentials.

Expansion (8) for the potential (33) has the finite convergence range $R(q) = q$, finiteness of which is connected with singularity of $V(q)$ at the point $q = 0$. The derivatives $V^{(k)}$ may be easily calculated

$$V^{(k)}(q) = (-1)^k \frac{\lambda(\lambda - 1)}{2} \frac{(k + 1)!}{q^{k+2}}. $$
But for this potential additional problem arises because of its singularity at the origin. To obtain the kernel that provides fulfilment of boundary condition for the wave function $\psi(q)$ at $q = 0$ ($\psi(q)$ should vanish at $q = 0$) one is to use initial condition of more general form as compared with (2). Namely, in this case

$$\langle q', t = 0 | q, 0 \rangle = \delta(q' - q) + A\delta(q' + q),$$

(35)

where constant $A$ is determined by demand that the kernel does not have singularity at $q = 0$ and/or $q' = 0$ ($t \neq 0$). In correspondence with (35) and analogously to (4) the kernel may be represented as

$$\langle q', t | q, 0 \rangle = \frac{1}{\sqrt{2\pi it}} \exp \left\{ \frac{i(q' - q)^2}{2t} \right\} F^{(-)}(t; q', q) + A \frac{1}{\sqrt{2\pi it}} \exp \left\{ \frac{i(q' + q)^2}{2t} \right\} F^{(+)}(t; q', q).$$

(36)

The equations for the functions $F^{(\pm)}$ are

$$i \frac{\partial F^{(\pm)}}{\partial t} = -\frac{1}{2} \frac{\partial^2 F^{(\pm)}}{\partial q'^2} + q' \pm q \frac{\partial F^{(\pm)}}{\partial q'} + V(q') F^{(\pm)},$$

(37)

and initial conditions are

$$F^{(\pm)}(t = 0; q', q) = 1.$$ 

(38)

Directly in (37), (38) $q', q > 0$. But it is possible to consider analytic continuation of $F^{(\pm)}$ into the complex plain $q$ and adopt negative values of $q$. Then one may write

$$F^{(+)}(t; q', q) = F^{(-)}(t; q', -q)$$

(39)

and study only one function $F(t; q', q) = F^{(-)}(t; q', q)$, where $q', q$ may be both positive and negative.

It is convenient to begin with positive $q', q$. In this case technique described above can be used without any changes. From (10) with account of (34) we take

$$b_{nk} = \frac{1}{n + k} \left[ \frac{(k + 1)(k + 2)}{2} b_{n-1,k+2} + \frac{\lambda(\lambda - 1)}{2} \sum_{m=0}^{k} (-1)^{m+1} \frac{m + 1}{q^{m+2}} b_{n-1,k-m} \right].$$

(40)

If we shall diminish $n$ times the first index of $b_{nk}$ by means of (44), then we get

$$b_{nk} = \frac{(-1)^{n+k} (k + n - 1)!}{q^{2n+k} n!(n - 1)!k!} \prod_{j=1}^{n} \left( \frac{\lambda(\lambda - 1)}{2} - \frac{j(j - 1)}{2} \right).$$

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\[ (-1)^{n+k} \frac{(k+n-1)!}{q^{2n+k}} \frac{\Gamma(\lambda+n)}{n!(n-1)!|k| \cdot 2^n \Gamma(\lambda-n)}. \] (41)

It is obvious that for noninteger \( \lambda \) \( |b_{nk}| \sim n! \) when \( n \to \infty \). So, for noninteger \( \lambda \) expansion (9) for potential (33) is divergent. But if \( \lambda \) is integer (\( \lambda > 1 \), because cases \( \lambda = 0, \lambda = 1 \) are trivial) then one can easy see from (41) that only the coefficients \( b_{nk} \) for \( n < \lambda \) are different from zero, and in (9) the series in powers of \( t \) is really the polynomial of finite degree \( \lambda - 1 \).

Let us substitute (41) into (9) and make summation over \( k \). Then we get finally

\[ F(t; q', q) = 1 + \sum_{n=1}^{\infty} \left( -\frac{i t}{2q'q} \right)^n \frac{\Gamma(\lambda+n)}{n! \Gamma(\lambda-n)}. \] (42)

Derivation of (42) is made in supposition that \( |\Delta q| < |q| \). If this condition is not satisfied, then calculations should be made with the expansion about the point \( q' \) in powers of \( q - q' \). But because \( F \) is symmetric in \( q', q \), then it is clear that the answer in this case would be the same as in (42). Note, that representation (42) for \( F \) does not suppose that \( q', q > 0 \). One can put \( q' \) and/or \( q \) negative and, hence, (42) gives expressions both for \( F(-) \) and for \( F(+) \).

Expansion (42) has singularity at \( q = 0 \) (\( q' = 0 \)). For noninteger \( \lambda \) this expression is asymptotic, so it cannot be applied to analysis the behaviour of \( F \) at \( q', q \to 0 \). It is correct only for sufficiently small values of variable \( t/q'q \).

But for integer \( \lambda \) (42) becomes finite series because in this case \( 1/\Gamma(\lambda-n) = 0 \) for \( n \geq \lambda \) and sum over \( n \) should be made only till \( \lambda - 1 \), but not till infinity. One can state, so, that \( F(\pm) \) is really singular at the point \( q = 0 \) (or \( q' = 0 \)). This feature, nevertheless, is not so dangerous, because only the kernel (38) should be finite at \( q = 0 \) (\( q' = 0 \)).

After substitution of (42) into (39) we get

\[ \langle q', t \mid q, 0 \rangle = \frac{1}{\sqrt{2\pi i t}} \exp \left\{ -\frac{i q'^2 + q^2}{2t} \right\} \left( -it \right)^n \frac{\Gamma(\lambda+n)}{n! \Gamma(\lambda-n)} + A e^{\frac{i q'^2 + q^2}{2t}} \left( it \right)^n \frac{\Gamma(\lambda+n)}{n! \Gamma(\lambda-n)} \] (43)

Expanding \( \exp\{\pm i q'q/t\} \) into series in \( q'q/t \) and considering terms with singularity in variable \( q'q \) we can see that if \( A = e^{-i\pi \lambda} \) then all these terms will be cancelled and the kernel will be equal to zero when \( q'q = 0 \) (this will be zero of order \( \lambda \)). So, the initial condition (35) with \( A = e^{-i\pi \lambda} \) provides fulfillment of the boundary condition for the kernel at origin.

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Finally the kernel may be represented as

\[
\langle q', t | q, 0 \rangle = \frac{1}{\sqrt{2\pi it}} \exp \left\{ i \frac{(q' - q)^2}{2t} \right\} \sum_{n=0}^{\infty} \left( \frac{-it}{2q'q} \right)^n \frac{\Gamma(\lambda + n)}{n!\Gamma(\lambda - n)} 
+ e^{-i\pi\lambda} \frac{1}{\sqrt{2\pi it}} \exp \left\{ i \frac{(q' + q)^2}{2t} \right\} \sum_{n=0}^{\infty} \left( \frac{it}{2q'q} \right)^n \frac{\Gamma(\lambda + n)}{n!\Gamma(\lambda - n)}.
\]

(44)

For integer \( \lambda \) sums are made only till \( \lambda - 1 \) and this expansion is finite, for noninteger \( \lambda \) it is asymptotic.

Naturally, this result exactly coincides with well known representation

\[
\langle q', t | q, 0 \rangle = e^{-i\frac{\pi}{2}(\lambda-1/2)} \sqrt{q'q} \frac{\pi q'q}{2it} e^{-i\frac{\pi q'^2}{2t}} H^{(2)}_{\lambda-1/2} \left( \frac{q'q}{t} \right) 
+ e^{-i\pi\lambda} \frac{1}{\sqrt{2\pi it}} \exp \left\{ i \frac{(q' + q)^2}{2t} \right\} \sqrt{\pi q'q} e^{-i\frac{\pi q'^2}{2t}} H^{(1)}_{\lambda-1/2} \left( \frac{q'q}{t} \right).
\]

(45)

which may be derived directly by reducing the Schrödinger equation to the equation for the cylindric functions (here \( J_\nu \) is Bessel function, \( H^{(1,2)}_\nu \) are Hankel functions of first and second kinds). Strictly speaking, validity of (44) for noninteger \( \lambda \) follows only from (45), but not from previous considerations.

Note, that essential feature of representation (44) for the kernel is following: the sums over index \( n \) are divergent if \( \lambda \) is noninteger and, contrary, they are finite if \( \lambda \) is integer. The kernel is well defined in both cases and apparent expression (45) allows us to study its behaviour in different variables: \( t, q', q \) or \( \lambda \). But if we do not know exact solution, as it usually takes place in more complicated problems with other potentials, and if we have only asymptotic expansion of the form (36), then we will have many problems for noninteger \( \lambda \) and much less problems for integer \( \lambda \).

For example, for the potential (33) we cannot study from the asymptotic expansion (44) behaviour of the kernel at \( q \to 0 \). Particularly, if we had not exact expression (45), but had only asymptotic one (44), we would not know that the kernel has the zero of order \( \lambda \) at \( q'q \to 0 \).

From the other side, if we try to get the expansion in powers of the coupling constant \( g = \lambda(\lambda - 1)/2 \) for \( F \) starting from (42), we would get after some transformations

\[
F(t; q', q) = 1 + \sum_{k=1}^{\infty} g^k \sum_{n=k}^{\infty} \left( \frac{-it}{q'q} \right)^n \frac{C_{nk}}{n!},
\]

(46)
where $C_{nk}$ are the coefficients of the polynomial
\[ \prod_{j=1}^{n} \left( g - \frac{j(j-1)}{2} \right) = \sum_{k=1}^{n} g^k C_{nk}. \] (47)

Equation (46) is the series of the conventional perturbation theory. This series is not simply divergent, but its coefficients are divergent too. Consider contribution of the first order in $g$:
\[ F^{(1)}(t; q', q) = g \sum_{n=1}^{\infty} \left( \frac{-it}{q'q} \right)^n \frac{C_{n1}}{n!}. \]
From (47) one can easily derive
\[ C_{n1} = (-1)^{n-1} \frac{n! (n-1)!}{2^{n-1}}. \]
Hence,
\[ F^{(1)}(t; q', q) = -2g \sum_{n=1}^{\infty} \left( \frac{-it}{2q'q} \right)^n (n-1)!, \] (48)
and the coefficient in front of $g$ is divergent series. Now we see, that without knowledge of exact solution for noninteger $\lambda$ we cannot build even conventional perturbation theory for $F$. Nevertheless, for integer $\lambda$ the Schwinger — DeWitt expansion is convergent and it can be used for further applications.

5 Other samples of potentials

Calculations made in Secs. 3, 4 may be easily repeated for some similar potentials which are often used in one-dimensional many-body problems [Olshanet-sky 1983, Calogero 1975, Sutherland 1971, 1972]. These are the potentials
\[ V(q) = \frac{\lambda(\lambda - 1)}{2} \frac{1}{\sinh^2 q}. \] (49)
and
\[ V(q) = \frac{\lambda(\lambda - 1)}{2} \frac{1}{\sin^2 q}. \] (50)
To prove convergence of the series for $F^{(-)}(t; q', q)(q', q > 0)$ it is enough to make a little modification of considerations of Sec. 3.

For the potential (49) denote
\[ f(q) = \frac{1}{\sinh^2 q}. \] (51)
and notice, that

\[(f^{(1)})^2 = 4f^3 + 4f^2,\]
i.e., it exactly coincides with the corresponding expression for the function \(f(q)\) defined by (18) in Sec. 3. So, all relations for the derivatives of \(f\) obtained there and, hence, expressions for \(b_{nk}, B_k,\) and \(F\) (now it is \(F^{(-)}\)) remain right in this case. There exist only two differences: function \(f\) is defined now by (51), but not by (18), and convergence range of the expansion (8) is \(R(q) = \sqrt{\pi^2 + q^2}.\)

Hence, convergence of (8) for the potential (49) when \(q^\prime, q > 0\) takes place for integer \(\lambda.\) The function \(F^{(-)}\) is single-valued and analytic function in the entire complex plain of the variable \(t\) for all \(q^\prime, q > 0.\)

The potential (50) in region \(0 < q < \pi\) can be also considered in similar way. Denote

\[f(q) = \frac{1}{\sin^2 q}\] (52)

and take into account, that

\[(f^{(1)})^2 = 4f^3 - 4f^2.\]

This expression differs from analogous ones for potentials (17), (49) only by the sign of the second term. So, we are able to reconstruct the expressions from Sec. 3 with small changes only: in (25), (26) will appear an additional multiplier \((-1)^{n+l},\) and the function \(f\) will be defined by (52).

Conclusion about convergence of the expansion (5) for \(F^{(-)}\) at region \(0 < q^\prime, q < \pi\) when \(\lambda\) is integer remains fair for the potential (50). But both these potentials are singular at \(q = 0.\) So, one is to consider initial condition (35) and additional function \(F^{(+)\prime}(t; q^\prime, q)\) as it was made in Sec. 4. Really it is enough to continue \(F^{(-)}\) into region \(q < 0\) and use (38). Expressions obtained in Sec. 3 do not allow us to make any conclusions about behaviour of \(F\) for \(q < 0.\)

Let us consider another representation for \(a_n(q^\prime, q).\) The potentials (49), (50) may be written as follows

\[V(q) = g \left( \frac{1}{q^2} + \sum_{k=0}^{\infty} s_k q^k \right),\] (53)

where \(s_k\) are known coefficients, \(g = \lambda(\lambda - 1)/2.\) The coefficient functions \(a_n(q^\prime, q)\) have a form of Loran series with finite number of pole terms

\[a_n(q^\prime, q) = \sum_{k=-n}^{\infty} \sum_{l=-n}^{\infty} q^{k+l} d_{nk}^{pl}.\] (54)
Substitution of (3), (53), (54) into (36) gives us algebraic recurrent relations for $d_{kl}^n$

$$(n + k)d_{kl}^n \pm (k + 1)d_{k+1,l}^n = \left(\frac{(k + 1)(k + 2)}{2} - g\right)d_{k+2,l}^{n-1} - g \sum_{m=k+n-1}^{\infty} s_md_{k-m,l}^{n-1}.$$  

(55)

If $k + n - 1 < 0$ then sum at last term of (55) should be equated to zero. One can find solutions of equation (55) and be convinced of validness of representation (54). It is obvious, besides, that the most singular at $q', q = 0$ terms in $a_n$ do not depend on $s_m$ and they exactly coincide with ones for the potential (33).

We know that $F^{(-)}(q', q > 0)$ for considered potentials is represented by convergent series of type (9). Then there exist representation of type (5), (54) and Loran series (54) is convergent at pierced in zero polycircle of finite radii. At this series sign of $q$ has no any meaning. One may consider both $q > 0$ and $q < 0$. Convergence will take place in any case.

One can state that convergence of expansion (9) for $F^{(+)}$ follows from convergence of it for $F^{(-)}$. The latter was proved earlier. So, we established that for the potentials (49), (50) the Schwinger — DeWitt expansion (36) is convergent for integer $\lambda$. Singularities of $F^{(\pm)}$ at $q', q = 0$ cancel each other in combination (36) if $A = e^{-i\pi\lambda}$ so as in the case of the potential (33).

6 Conclusion

Usually the Schwinger — DeWitt expansion is used as asymptotic. Its general property is rising of the coefficients $a_n(q', q)$ as $n \to \infty$ (or as $\Gamma\left(\frac{L-2}{L+2}n\right)$ if the potential is polynomial of order $L$) [Osborn 1983, Slobodenyuk 1995]. Such growth takes place always when no any cancellations of different contributions into $a_n$ occur. It is so for the most number of potentials. But there exist some potentials, for which cancellations really occur. Examples of such potentials were considered in present paper. It was proved convergence of the Schwinger — DeWitt expansion for them when constant $\lambda$ is integer and divergence when $\lambda$ is noninteger.

Besides the potentials mentioned above one more example is known [Slobodenyuk 1996 [19]], which has the property of convergence of the expansion (9).
It is following:

\[ V(q) = a^2q^2 + \frac{\lambda(\lambda - 1)}{2} \frac{1}{q^2}. \]  

(56)

The expansion (5) converges for it when \( \lambda \) is integer. But convergence range is finite contrary to examples of this paper. It is naturally, because expansion for the harmonic oscillator \( V(q) = a^2q^2 \) has finite convergence range. And it is so for the perturbed oscillator (56) too.

So, we discovered existence of the class of nontrivial potentials in the quantum mechanics, for which the Schwinger — DeWitt expansion is convergent and for which initial condition for the evolution problem is fulfilled in rigorous (analytic) sense, but not only asymptotically (when \( F \) is not analytic at \( t = 0 \) and its value at origin is determined from condition of continuity).

The potentials belonging to this class has at least two remarkable features: 1) the Schwinger — DeWitt expansion is convergent for them, hence, other expansions, which may be derived from it are convergent too, and many problems, connected with divergence of the expansions are absent for such potentials, 2) the potentials of this class have discrete coupling constants that corresponds to discreteness of the charge in the nature.

This is why the potentials from this class is to be well studied. But research of quantum mechanical models is only preparation for practical using of discovered phenomenon. One can expect to construct the fundamental theory of elementary particles as a result of transmission of this approach into the quantum field theory. It is possible to introduce interaction in the field theory which is analogous in any meaning to the quantum mechanical potentials studied in this article. One of such quantum field models is under consideration at present time and will be described in consequent paper. The model conserves essential features discussed above.
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