A unified theory of quantum Hall effect and high
temperature superconductivity

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Abstract. The quantum Hall effect (QHE) and high temperature superconductivity (HTSC)
have remarkable common features. They occur only in two-dimensional (2D) solids. The critical
temperature $T_c$ of some HTSC exceeds 160 K while the room temperature QHE is observed
in graphene. The cause of both QHE and HTSC is the phonon exchange attraction. We
develop a theoretical model for the QHE in terms of the composite bosons (fermions), each
containing an electron and an odd (even) number of fluxons (magnetic flux quanta). The
composite particles (boson, fermion) are bound by the phonon exchange attraction. If the Bose-
Einstein condensation (BEC) of the composite (c) - bosons occurs, then the system exhibits zero
resistivity and the associated Hall conductivity plateau. The Hall conductivity is calculated
rigorously without averaging. The mystery of the fractional charge carried by the c-bosons is
resolved in our model.

1. Introduction

In 1983 Laughlin introduced a Laughlin ground-state wave function for a system of $N$ electrons at
the Landau level occupation ratio (also called filling factor) $\nu = 1/3$, and studied the elementary
excitations (quasiparticles) over the ground-state [1]. He predicted that the quasiparticle has a
fractional charge:

\[ e^* = e/3, \quad e = \text{electron charge}. \tag{1} \]

This prediction appears to have been later confirmed in the magnetotransport experiments by
Clark et al. and others [2]. To interpret the experimental data it is convenient to introduce composite (c) - particles (bosons, fermions). The c-boson (fermion), each containing an electron and an odd (even) number of flux quanta (fluxons), were introduced by Zhang et al. [3] and others (Jain [4]) for the description of the fractional QHE (Fermi liquid). The fractional charge is also known in quantum chromodynamics. Quarks have charges 1/3 and 2/3 in unit of $e$ [5].

But quarks are fermions. Bosons can be constructed from two fermions while no fermions can
be constructed from bosons. Hence fermions are considered as more elementary building blocks
of material. Any c-fermion in solid state physics is thought to have the charge magnitude $e$. Hence the fractional charge carried by c-bosons is a little strange, which must be grounded in the full quantum statistical theory. Besides, an unlimited set of charges: 1/2, 1/5, $\cdots$ conceived for QHE is unnatural. The quarks’ charges are limited to 1/3 and 2/3 by the SU3 symmetry.

The prevalent theories [6] based on the Laughlin wave function in the Schrödinger picture
deal with the QHE at 0 K and immediately above. The system ground-state does not carry a
current, however. It is desirable to treat the QHE below and above the critical temperature $T_c$ in a unified manner. Beside, the prevalent theories have limitations:

- The high-field limit is taken at the outset. The integer QHE at $\nu \equiv P/Q$ are observed for small integer $P$ only. The question why the QHE is not observed for high $P$ (weak field) cannot be answered. We better describe the phenomena for all fields.

- The Hall resistivity $\rho_H$ value $(Q/P)(h/e^2)$ is obtained in a single stroke. To obtain $\rho_H$ we need two separate measurements of the Hall field $E_H$ and the current density $j$. We must calculate $(E_H, j)$ and take the ratio $E_H/j$ to find $\rho_H(= E_H/j)$.

The main purposes of the present work is to report that a unified theory can be developed for the QHE and HTSC, that the Hall resistivity $\rho_H$ can be calculated rigorously without averaging, and that the fractional QHE can be described within the frame-work of c-particles model without using Laughlin’s theory and results about the fractional charges.

2. The Hamiltonian

Fujita and Okamura developed a quantum statistical theory of the QHE [7]. We follow this theory. See this reference for more details.

There is a remarkable similarity between the QHE and the High-Temperature Superconductivity (HTSC), both occurring in two-dimensional (2D) systems as pointed out by Laughlin [8]. We regard the phonon exchange attraction as the causes of both QHE and HTSC. Starting with a reasonable Hamiltonian, we calculate everything, using quantum statistical method.

The countability concept of the fluxons, known as the flux quantization:

$$B = \frac{N_\phi}{A} \frac{h}{e} \equiv n_\phi \frac{\hbar}{e},$$

where $A =$ sample area, $N_\phi =$ fluxon number (integer) and $h =$ Planck constant, is originally due to Onsager [9]. The magnetic (electric) field is an axial (polar) vector and the associated fluxon (photon) is a half-spin fermion (full-spin boson). The magnetic (electric) flux line cannot (can) terminate at a sink, which supports the fermionic (bosonic) nature of the associated fluxon (photon). No half-spin fermion can annihilate itself because of angular momentum conservation. The electron spin originates in the relativistic quantum equation (Dirac’s theory of electron) [10]. The discrete (two) quantum numbers ($\sigma_z = \pm 1$) cannot change in the continuous limit, and hence the spin must be conserved. The countability and statistics of the fluxon are fundamental particle properties. We postulate that the fluxon is a half-spin fermion with zero mass and zero charge. The fluxons (neutrinos) occur in electron (nucleon) dynamics. Hence fluxon and neutrino are regarded as distinct from each other.

We assume that the magnetic field $B$ is applied perpendicular to the interface. The 2D Landau level energy,

$$\varepsilon = \hbar \omega_e (N_L + 1/2), \quad \omega_e \equiv eB/m^* \quad N_L = 0, 1, 2, \cdots,$$

with the states $(N_L, k_y)$, have a great degeneracy. The cyclotron frequency $\omega_e$ contains the electron effective mass $m^*$. The Center-of-Mass (CM) of any c-particle moves as a fermion (boson). The eigenvalues of the CM momentum are limited to 0 or 1 (unlimited) if it contains an odd (even) number of elementary fermions. This rule is known as the Ehrenfest-Oppenheimer-Bethe’s (EOB’s) rule [11]. Hence the CM motion of the composite containing an electron and $Q$ fluxons is bosonic (fermionic) if $Q$ is odd (even). The system of c-bosons condenses below some critical temperature $T_c$ and exhibits a superconducting state while the system of c-fermions shows a Fermi liquid behavior.
A longitudinal phonon, acoustic or optical, generates a density wave, which affects the electron (fluxon) motion through the charge displacement (current). The exchange of a phonon between electrons and fluxons can generate an attractive transition.

Bardeen, Cooper and Schrieffer (BCS) [12] assumed the existence of Cooper pairs [13] in a superconductor, and wrote down a Hamiltonian containing the “electron” and “hole” kinetic energies and the pairing interaction Hamiltonian with the phonon variables eliminated. We start with a BCS-like Hamiltonian $\mathcal{H}$ for the QHE [7]:

$$\mathcal{H} = \sum_{k,s} \epsilon_k n_{ks} + \sum_{q,k,k',s} v_0 \left( B_{k'q,s}^\dagger B_{kq,s} + B_{k'q,s}^\dagger B_{kq,s} - B_{k'q,s} B_{kq,s} \right) - \sum_{q,k,k',s} v_0 \left( B_{k'q,s}^\dagger B_{kq,s}^\dagger - B_{k'q,s} B_{kq,s} \right)$$

where $n_{ks} = c_k^\dagger c_k$ is the number operator for the “electron” (1) “hole” (2), fluxon (3) at momentum $k$ and spin $s$ with the energy $\epsilon_{(j)}^k$, with annihilation (creation) operators $c$ ($c^\dagger$) satisfying the Fermi anti-commutation rules:

$$\{c_{k,s}^\dagger, c_{k',s'}\} = c_{k,s}^\dagger c_{k,s} - c_{k,s} c_{k,s}^\dagger = \delta_{k,k'} \delta_{s,s'} \delta_{i,j}, \quad \{c_{k,s}^\dagger, c_{k,s}\} = 0.$$  

The fluxon number operator $n_{ks}^{(3)}$ is represented by $a_{k,s}^\dagger a_{k,s}$ with a $a^\dagger$ satisfying the anti-commutation rules:

$$\{a_{k,s}, a_{k',s'}^\dagger\} = \delta_{k,k'} \delta_{s,s'} \delta_{i,j}, \quad \{a_{k,s}, a_{k,s'}\} = 0.$$  

The phonon exchange attraction can create electron-fluxon composites. We call the conducting-electron composite with an odd (even) number of fluxons c-boson (c-fermion). The electron (hole)-type c-particles carry negative (positive) charge. The pair operators $B$ in Eq. (4) are defined by

$$B_{kq,s} = c_{k+q/2,s} a_{-k+q/2,-s}^\dagger, \quad B_{kq,s}^\dagger = a_{-k+q/2,-s} c_{k+q/2,s}^\dagger.$$  

The prime on the summation in Eq. (4) means the restriction:

$$0 < \epsilon_{(j)}^k < \hbar \omega_D \quad \omega_D = \text{the Debye frequency.}$$  

The pairing interaction terms in Eq. (4) conserve the charge. The term $-v_0 B_{k'q,s}^\dagger B_{kq,s}^\dagger$, where $v_0 = |V_q V_q'| (\hbar \omega_0 A)^{-1}$, $A$ = sample area, is the pairing strength, generates a transition in electron-type c-fermion states. Similarly, the exchange of a phonon generates a transition between hole-type c-fermion states, represented by $-v_0 B_{k'q,s}^\dagger B_{kq,s}^\dagger$. The phonon exchange can also pair-create (pair-annihilate) electron (hole)-type c-boson pairs, and the effects of these processes are represented by $-v_0 B_{k'q,s}^\dagger B_{kq,s}^\dagger \left(-v_0 B_{k'q,s} B_{kq,s}^\dagger\right)$.

The Cooper pair is formed from two “electrons” (or “holes”). Likewise the c-bosons may be formed by the phonon-exchange attraction from c-fermions and fluxons. If the density of the c-bosons is high enough, then the c-bosons will be condensed and exhibit a superconductivity.

To treat superconductivity we modify the pair operators in Eq. (7) as

$$B_{kq,s}^\dagger = c_{k+q/2,s}^\dagger c_{-k+q/2,-s}, \quad B_{kq,s} = c_{-k+q/2,-s} c_{k+q/2,s}.$$  

Then, the pairing interaction terms in Eq. (4) are formally identical with those in the generalized BCS Hamiltonian [14]. If we assume that only zero momentum Cooper pairs ($q = 0$) are
generated, then the Hamiltonian $\mathcal{H}$ in Eq. (4) is reduced to the original BCS Hamiltonian, ref. [12], Eq. (2.14).

We first consider integer QHE. We choose a conduction electron and a fluxon for the pair. The $c$-bosons, having the linear dispersion relation:

$$\varepsilon^{(j)} = w_0 + \frac{2}{\pi} v_F^{(j)} p, \quad (10)$$

can move in all directions in the plane with the constant speed $(2/\pi)v_F^{(j)}$. A brief derivation of Eq. (10) is given in Appendix. The supercurrent is generated by $\mp c$-bosons monochromatically condensed, running along the sample length. The supercurrent density (magnitude) $j$, calculated by the rule: $j = (\text{carrier charge} e^*) \times (\text{carrier density} n_0) \times (\text{drift velocity} v_d)$, is given by

$$j \equiv e^* n_0 v_d = e^* n_0 \frac{2}{\pi} |v_F^{(1)} - v_F^{(2)}|. \quad (11)$$

The Hall field (magnitude) $E_H$ equals $v_d B$. The magnetic flux is quantized:

$$B = n_0 \Phi_0, \quad \Phi_0 \equiv e/h, \quad (12)$$

where $n_0 \equiv N_\phi/A$ is the fluxon density. Hence the Hall resistivity $\rho_H$ is given by

$$\rho_H \equiv \frac{E_H}{j} = \frac{v_d B}{e^* n_0 v_d} = \frac{1}{e^* n_0} n_0 \left( \frac{h}{e} \right) = h/e. \quad (13)$$

We assume that the \textit{c-fermion has a charge magnitude} $e$. For the integer QHE, $e^* = e$, $n_0 = n_0$, thus we obtain $\rho_H = h/e^2$, the correct plateau value observed for the principal QHE at $\nu = 1$.

The supercurrent generated by equal numbers of $\mp c$-bosons condensed monochromatically is neutral. This is reflected in our calculations in Eq. (11). In the calculation of $\rho_H$ in Eq. (13), we used the \textit{unaveraged} drift velocity difference $(2/\pi)|v_F^{(1)} - v_F^{(2)}|$, which is significant. Only the unaveraged drift velocity $v_d$ cancels out exactly from numerator/denominator, leading to an exceedingly accurate plateau value.

We now extend our theory to include elementary fermions (electron, fluxon) as members of the \textit{c-fermion} set. We can then treat the QHE in a unified manner, using the same Hamiltonian $\mathcal{H}$.

\section{3. Fractional Quantum Hall Effect}

We assume that \textit{any} \textit{c-fermion} has the effective charge $e^*$ equal to the electron charge (magnitude) $e$:

$$e^* = e \quad \text{for any c-fermion.} \quad (14)$$

After studying the low-field QH states of \textit{c-fermions}, we obtain

$$n_\phi^{(Q)} = n_e/Q, \quad Q = 0, 2, 4, \ldots, \quad (15)$$

for the density of the \textit{c-fermions} with $Q$ fluxons, where $n_e$ is the electron density. All fermionic QH states (points) lie on the classical-Hall straight line passing the origin with a constant slope when $\sigma_H$ is plotted as a function of $B^{-1}$. The density $n_\phi^{(Q)}$ is proportional to the magnetic field $B$. As the magnetic field is raised, the separation between the LL becomes greater, and the higher-$Q$ \textit{c-fermion} is more difficult to form energetically. This condition is unlikely to depend on the statistics of the \textit{c-particles}. Hence Eq. (15) should be valid for all integers, odd or even.
We take the case of $Q = 3$. The c-boson containing an electron and three (3) fluxons can be formed from a c-fermion with two (2) fluxons and a fluxon. If the c-bosons are Bose-condensed, then the supercurrent density $j$ is given by Eq. (11). Hence we obtain

$$
\rho_H \equiv \frac{E_H}{j} = \frac{\nu_d B}{e^* n_0 \nu_d} = \frac{n_0^{(3)}}{e^* n_0} \left( \frac{h}{e^*} \right) = \frac{1}{3} \frac{h}{e^*} \sqrt{2},
$$

(16)

where we used $e^* = e$ from Eq. (14), and $n_0^{(3)} / n_0 = 1/3$, a bosonic extension of Eq. (15).

The principal fractional QHE occurs at $\nu = 1/3$, where the Hall resistivity value is $h/(3e^2)$ as shown in Eq. (16). A set of weaker QHE occur on the lower field side at

$$
\nu = \frac{1}{3}, \frac{2}{3}, \ldots.
$$

(17)

The QHE behavior at $\nu = P/Q$ for any odd $Q$ is similar. We illustrate it by taking integer QHE with $\nu = P$. The field magnitude becomes smaller with increasing $P$. The LL degeneracy is proportional to $B$, and hence $P$ LL’s must be considered. First consider the case $P = 2$. Without the phonon-exchange attraction the electrons occupy the lowest two LL’s with spin. See Fig. 1 (a). The electrons at each level form c-bosons. In the superconducting state

the supercondensate occupy the monochromatically condensed state, which is separated by the superconducting gap $\varepsilon_g$ from the continuum states (band) as shown in Fig. 1 (b). The temperature-dependent energy gap $\varepsilon_g(T)$ is defined in terms of the BCS energy parameter $\Delta$. Its temperature dependence is discussed in Appendix. The c-boson density $n_0$ at each LL is one-half the density at $\nu = 1$, which is equal to the electron density $n_e$ fixed for the sample. Extending the theory to a general integer, we have

$$
n_0 = n_e / P.
$$

(18)

The critical temperature $T_c (= 1.24h \nu B k_B^{-1} n_0^{1/2})$ and the gap energy $\varepsilon_g$ are smaller for higher $P$, making the plateau width (a measure of $\varepsilon_g$) smaller in agreement with experiments. The c-bosons have lower energies than the conduction electrons. Hence at the extreme low temperatures the

![Diagram of electron and Landau levels](image)

**Figure 1.** The electrons which fill up the lowest two LL’s, shown in (a) form the QH state at $\nu = 2$ in (b) after the phonon-exchange attraction and the BEC of the c-bosons.
The c-bosons, each with one fluxon, will be called the fundamental (f) c-bosons. Their energies are obtained from\cite{7, 15}:

$$w_0 = \frac{-2\hbar \omega_D}{\exp(1/(v_F^0 D_0)) - 1},$$  \hspace{1cm} (A.2)

where $$v_F^0 = (2\varepsilon_F/m_j)^{1/2}$$ is the Fermi velocity and $$D_0 \equiv D(\varepsilon_F)$$ the density of states per spin. Note that the energy $$w_0$$ depends linearly on the momentum $$q$$.

The system of fc-bosons undergoes a Bose-Einstein condensation (BEC) in 2D at the critical temperature\cite{7, 15}:

$$T_c = 1.24 \hbar v_F k_B^{-1} n_0^{1/2},$$  \hspace{1cm} (A.3)

The interboson distance $$R_0 \equiv n_0^{1/2}$$ calculated from this expression is $$1.24 \hbar v_F (k_B T_c)^{-1}$$. The boson size $$r_0$$ calculated from Eq. (A.3), using the uncertainty relation ($$q_{max} r_0 \sim \hbar$$) and $$|w_0| \sim k_B T_c$$, is $$(2/\pi)\hbar v_F (k_B T_c)^{-1}$$, which is a few times smaller than $$R_0$$.

Hence, the bosons do not overlap in space, and the model of free bosons is justified. For GaAs/AlGaAs, $$m^* = 0.067m_e$$, $$m_e$$ = electron mass. For the 2D electron density $$10^{11}$$ cm$$^{-2}$$, we have $$v_F = 1.36 \times 10^6$$ cm s$$^{-1}$$.

Not all electrons are bound with fluxons since the simultaneous generations of ± fc-bosons is required. The minority carrier (“hole”) density controls the fc-boson density. For $$n_0 = 10^{10}$$ cm$$^{-2}$$, $$T_c = 1.29$$ K, which is reasonable.

In the presence of the Bose condensate below $$T_c$$ the unfluxed electron carries the energy $$E_{j}^{(f)} = \sqrt{\varepsilon_k^2 + \Delta^2}$$, where the BCS energy gap $$\Delta$$ is the solution of

$$1 = v_0 D_0 \int_0^{\varepsilon_{max}} \frac{d\varepsilon}{(\varepsilon^2 + \Delta^2)^{1/2}} \left[ 1 + \exp[-\beta(\varepsilon^2 + \Delta^2)^{1/2}] \right]^{-1},$$  \hspace{1cm} (A.4)

Note that the gap $$\Delta$$ depends on $$T$$. At $$T_c$$, there is no condensate and hence $$\Delta$$ vanishes.
Now the moving fc-boson below \( T_c \) has the energy \( \tilde{\omega}_q \) obtained from

\[
\tilde{\omega}_q^{(j)} \Psi(k,q) = E^{(j)}_{k+q} \Psi(k,q) - \frac{v_0^2}{(2\pi \hbar)^2} \int d^2k' \Psi(k',q),
\]

where \( E^{(j)} \) replaced \( \varepsilon^{(j)} \) in Eq. (A.1). We obtain

\[
\tilde{\omega}_q^{(j)} = \tilde{\omega}_0 + \frac{2}{\pi} v_F^{(j)} q = \omega_0 + \varepsilon_g + \frac{2}{\pi} v_F^{(j)} q,
\]

where \( \tilde{\omega}_0(T) \) is determined from

\[
1 = D_0 v_0^2 \int_0^{k_{FD}} \frac{d\varepsilon}{|\tilde{\omega}_0| + (\varepsilon^2 + \Delta^2)^{1/2}}.
\]

The energy difference,

\[
\tilde{\omega}_0(T) - \omega_0 \equiv \varepsilon_g(T) > 0,
\]

represents the \( T \)-dependent energy gap. The energy \( \tilde{\omega}_q \) is negative. Otherwise, the fc-boson should break up. This limits \( \varepsilon_g(T) \) to be \( |\omega_0| \) at 0 K. The \( \varepsilon_g \) declines to zero as the temperature approaches \( T_c \) from below.

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[15] Fujita S and Suzuki A 2013 Electrical Conduction in Graphene and Nanotubes (Weinheim Germany: Wiley-VCH) Sec. 11.3 (also see Appendix A.6).