On the Equivalence of Bound State Solutions

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Abstract

In this paper we show the equivalence of various (non-threshold) bound state solutions of branes, or equivalently branes in background potentials, in ten- and eleven-dimensional supergravity. We compare solutions obtained in two very different ways. One method uses a zero mode analysis to make an Ansatz which makes it possible to solve the full non-linear supergravity equations. The other method utilises T-duality techniques to turn on the fields on the brane. To be specific, in eleven dimensions we show the equivalence for the (M2,M5) bound state, or equivalently an M5-brane in a $C$\textsubscript{3} field, where we also consider the (MW,M2,M2\textacuted, M5) solution, which can be obtained from the (M2,M5) bound state by a boost. In ten dimensions we show the equivalence for the ((F,D1),D3) bound states as well as the bound states of $(p,q)$ 5-branes with lower dimensional branes in type IIB, corresponding to D3-branes in $B$\textsubscript{2} and $C$\textsubscript{2} fields and $(p,q)$ 5-branes in $B$\textsubscript{2}, $C$\textsubscript{2} and $C$\textsubscript{4} fields. We also comment on the recently proposed V-duality related to infinitesimally boosted solutions.

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1 Introduction

A lot of papers have appeared on the subject of bound states of branes, and the supergravity solutions can be obtained in several different but equivalent ways. The purpose of this paper is to establish this equivalence explicitly.

The most commonly used method utilises T-duality techniques to turn on the fields on the brane, but alternatively the solutions can be obtained by performing a zero mode analysis, which enables one to make an Ansatz for the fields and then solve the supergravity equations. This was done by Cederwall et al.\ [1, 2, 3].

Furthermore, the solutions can be viewed either as a brane in background potentials or as a bound state of a brane and lower dimensional branes. For instance, an \((F,D_p)\) bound state corresponds to a \(D_p\)-brane in an electric (i.e., a component including the time direction) \(B\)-field background, and a \((D(p-2),D_p)\) bound state corresponds to a \(D_p\)-brane in a magnetic \(B\)-field background.

While the method of analysing the zero-modes generally gives a unique solution parametrised by the field strength on the brane, the method of using T-duality techniques generally gives a multitude of solutions for the same brane. We argue that the method of analysing the zero-modes always gives the most general half supersymmetric solution by construction, since there is a unique way of fitting the half supersymmetric zero-modes into the target space fields and since the subsequent full solution is uniquely obtained from the zero-mode solution. Therefore, all the solutions obtained using T-duality techniques are related to solutions obtained by analysing the zero-modes and thus do not generate a larger family of solutions despite the multitude of different looking solutions. As a consequence of this, the most general bound states in type IIB was first obtained in \[3\] for the \(D_3\)-brane and in \[3\] for the \((p,q)\) 5-branes.

The zero-mode method does not generate any waves in the solutions, but solutions including waves \[4, 5\] have been shown to be related to solutions without waves via finite boosts \[4, 6\]. Solutions with light-like fields have been shown to be obtainable by taking a limit involving an infinite boost \[6, 8\].

The solutions have been seen to be important as the supergravity duals \[1, 10, 11, 12\] of noncommutative theories on branes \[3, 13, 14, 15, 16\] and since some papers on this subject use the solutions of Cederwall et al.\ as supergravity duals \[14, 15\] it also seems important to get an understanding of the relation between the different formulations.

The following is a chronological description of some of the important papers\[3\].

To our knowledge, the first explicit ten or eleven-dimensional supergravity solution, involving a tensor field on the brane, was obtained by Izquierdo et al. in 1995 \[19\]. They construct an M5-brane in a background of a 3-form potential, by lifting a known eight dimensional solution to eleven dimensions. The solution corresponds to a bound state of M5- and M2-branes. In 1996 Russo and Tseytlin constructed the same solution, but for the first time the solution was obtained using the mentioned duality techniques \[14\]. In this paper, they also construct \((F,D_3), (D_1,D_3)\)

\[\text{3We will only consider the non-marginal half-supersymmetric bound states, where the lower dimensional branes lie within the higher dimensional brane.}\]

\[\text{4This is not intended to be a complete list.}\]
and (D0,D2) bound states for the first time. Almost at the same time Breckenridge et al. used the same method on D-branes, yielding (D(p − 2),Dp) bound states \[20\]. Shortly afterwards Costa and Papadopoulos constructed the (F,D6) bound state (and they discuss the other (F,Dp) bound states) \[21\]. In late 98, Cederwall et al. constructed the most general SL(2,Z)-covariant ((F,D1),D3) bound state for the first time, using a zero mode analysis \[2\], and the (M2,M5) solution was also constructed in this way. The following year Lu and Roy constructed (F,D(p − 2),Dp) bound states. Concerning the 5-branes in type IIB and the 2-form on them, they write down complete solutions (i.e., including all the supergravity fields) in the cases where there is an electric rank 2 field on the brane (in the rank 4 case, they only give the solution for the metric). The general rank D5- and NS5-brane solutions were first obtained in the fall of 99 \[25\], but these solutions were not complete (the D5 solution lacked the RR fields and the NS5 solution lacked the NS-NS 2-form as well as the RR 4-form). The solutions correspond to the (F,D3,D5), (D3,D3,D5), (F,D3,D3,D5) bound states (and similarly for the NS5-brane). The solutions for general Dp-branes with general rank of the B-field as well as the general NS5 solution in type IIA (corresponding to (D0,D2,NS5) and (D2,D4,NS5)) were also obtained (but again not with all the fields). In late 99 the most general (p,q) 5-brane solution was constructed in \[3\]. This includes the following bound states, (F,D5), (D3,D5), (F,D1,D3,D5), (D1,D3,D3,D5), (F,D1,D3,D3,D5) as well as the SL(2,Z) transformations of these (and the cases mentioned above are obtained as special cases). These states correspond to the following B-field configurations: electric rank 2, magnetic rank 2, electric and magnetic rank 4, magnetic rank 4 and rank 6. The complete NS5-branes solutions in type IIA appeared in \[26\].

In section 2, we describe the Goldstone mechanism, which yields an understanding of the zero modes on a brane and enables us to make an Ansatz for the fields and solve the full non-linear supergravity equations. This is done for the M5-brane and after that the D3- and (p,q) 5-branes are discussed. In section 3, we show the equivalence between various M5-brane solutions as well as for the D3- and (p,q) 5-branes. We end with a conclusion in section 4.

2 Zero modes

2.1 The M5-brane

It is well known that massless degrees of freedom, so called Goldstone modes, arise when a continuous symmetry is broken. Put, e.g., an M2-brane into an eleven dimensional space. This breaks half of the supersymmetry, another half is broken by the Dirac equation when going on-shell, resulting in eight fermionic zero-modes. The translational symmetry in the transverse directions is also broken, generating eight bosonic zero-modes. Since we get the same number of bosonic and fermionic degrees of freedom, there is a supersymmetric theory living on the M2-brane. The M2-brane case, however, contains only scalar modes, which have been understood for quite some time. The situation is a bit different if we instead look at the M5-
brane. Here we get eight fermionic degrees of freedom as above, but now we only get five bosonic degrees of freedom from the breaking of translational symmetries. The three extra bosonic degrees of freedom needed to get a supersymmetric theory come from an anti-self-dual three-form field strength on the brane. In [1], the Goldstone mechanism is generalised to tensor fields of arbitrary rank, providing the same level of understanding in terms of broken symmetries as for the scalar modes. Here we will just sketch the ideas.

The generalisation to tensor modes can be made by understanding how the scalar modes arise. Since we are studying a theory with gravity, \textit{i.e.}, a theory having local diffeomorphism invariance, we have to be more careful when we say that introducing a brane breaks translational symmetry in the transverse directions. We can always make a \textit{small} diffeomorphism, \textit{i.e.}, a diffeomorphism taking the same value in the two asymptotic regions of the brane solution, \( r \to 0 \) and \( r \to \infty \), without changing any conserved quantities like, \textit{e.g.}, the momentum. If we instead make a \textit{large} diffeomorphism, taking different values in the asymptotic regions, we change conserved quantities and it is therefore these symmetries that are broken in the presence of a brane. Since diffeomorphisms are the gauge symmetry associated with gravity, we come to the conclusion that Goldstone modes are associated with broken large gauge symmetries. In the case of the fermionic modes it is the large supersymmetry transformations that are broken and, \textit{e.g.}, the tensor modes on the M5-brane come from broken large gauge transformations of the background three-form potential.

By doing a “rigid” transformation, \textit{i.e.}, an \textit{x}-independent gauge transformation, where \( x \) denotes the longitudinal coordinates, we get information on how to introduce the zero-modes in the relevant field. By \textit{then} turning on the \textit{x}-dependence, to obtain a theory on the brane, we can get the equations of motion for the zero-modes by using the supergravity field equation, since after turning on the \textit{x}-dependence we are no longer considering just a gauge transformation. To illustrate this method we will now analyse the tensor modes on the M5-brane [1]. Using the notation of [1], we make a gauge transformation of the background three-form potential \( \delta C = d \Lambda \), where \( \Lambda = \Delta^k A \) and \( A \) is a \textit{constant} two-form which lies in the transverse directions since we want a theory on the brane and \( k \) is a constant which will be determined from the equations of motion. We first calculate \( \delta C = d \Delta^k \wedge A \) and \textit{then} turn on the \textit{x}-dependence of \( A \), after which the variation of \( C \) is no longer just a gauge transformation and we can therefore obtain equations of motion for the zero-modes by using the supergravity equations. We can now compute the variation of the four-form field strength \( H = dC \),

\[
h = \delta H = d \Delta^k \wedge F \tag{1}
\]

where \( F = dA \). The field equation for \( H \) is, to linear order in \( h \),

\[
d \ast h - H \wedge h = 0 \tag{2}
\]

By inserting the M5-brane solution for \( H \) we get

\[
\Delta d \ast_x F \wedge \ast_y d \Delta - (k \ast_x F - F) \wedge d \Delta \wedge \ast_y d \Delta = 0 \tag{3}
\]
where $*_{x}$ and $*_{y}$ denote dualisation in the longitudinal and transverse directions respectively. By considering the two duality components of $F$ separately (fulfilling $*_{x}F = \pm F$) we get that $k = -1$ for the anti-self-dual part and $k = 1$ for the self-dual part. We also get the equation of motion $d*_{x}F = 0$. Since each duality component of $F$ contributes with three bosonic degrees of freedom, we seem to have twice the number of extra degrees of freedom that we needed in order to get supersymmetry. However, since we want a theory on the brane, we must require that the zero modes are normalisable when integrating out the transverse directions. By doing this, we see that the self-dual part of $F$ has non-normalisable zero-modes, and must therefore be discarded. We have thus seen how the tensor modes on the M5-brane can be understood as arising from broken large gauge transformations of the background three-form potential.

### 2.2 Type IIB branes

Just as for the M5-brane, we can do a zero mode analysis for the 3- and 5-branes in type IIB [1, 3], but here we get complications due to the SL(2,$\mathbb{Z}$) symmetry. As discussed in detail in [1], the additional bosonic zero modes on D-branes, correspond to vector modes and the deformed supergravity solutions are then parametrised by the corresponding 2-form field strength on the brane. These zero modes arise when we break the large gauge transformations of the background 2-form potentials. In a manifestly SL(2,$\mathbb{Z}$)-covariant formulation, we have a doublet of 2-forms which can be combined into a complex 2-form, see appendix A for details. To be specific, the deformations are parametrised by a complex anti-selfdual 2-form $F_{(2)}$ on the D3-brane [1] and by a real 2-form $F$ on the $(p,q)$ 5-branes [3]. This also yields a matching of the number of fermionic and bosonic zero modes. In both cases we have 8 fermionic zero modes. On the D3-brane we have 6 scalar zero modes and the last two exactly correspond to half of the modes for a complex vector in 4 dimensions (and the half is due to the anti-selfduality of the field strength). On the 5-branes, we have 4 scalars, and the remaining 4 bosonic zero modes correspond to the number of degrees of freedom for a real vector in 6 dimensions.

In general, we can start from any brane, whose normalisable zero-modes we identify using the prescription described above. This gives us exact knowledge of how the zero-modes appear in all target space fields and enables us to make an Ansatz for the full solution. The non-linear supergravity equations are then solved for the unknown functions in the Ansatz. The results are presented in the following sections. Apart from the general M5-brane solution, only special cases of these solutions were known prior to [2, 3].

### 3 Equivalence of solutions

#### 3.1 The M5-brane

In this section we show the equivalence of various M5-brane solutions. We will start from the solution obtained in [2], representing an (M2,M5) bound state, and derive an explicit mapping to the (M2,M5) solution of Izquierdo et al. [19], who were the
first to obtain the (M2,M5) solution. We will then boost the solution of Izquierdo et al. leading to the (MW,M2,M2',M5) solution of Bergshoeff et al. [5], essentially following [6] but using a slightly generalised form of the mapping.

The (M2,M5) solution obtained in [2] is

\[ ds^2 = (\Delta_+ \Delta_-)^{1/3} \left[ \Delta_-^{-1} (-dt^2 + (dx^1)^2 + (dx^2)^2) + \Delta_+^{-1} ((dx^3)^2 + (dx^4)^2 + (dx^5)^2) + dr^2 + r^2 d\Omega_4^2 \right] \]

\[ \ell_p^3 C_3 = \sqrt{2\nu} \left[ \Delta_-^{-1} dt \wedge dx^1 \wedge dx^2 - \Delta_+^{-1} dx^3 \wedge dx^4 \wedge dx^5 \right] \]

\[ H_4 = dC_3 + 3\pi N \epsilon_4 \tag{4} \]

where \( \ell_p \) is the eleven dimensional Planck length, \( N \) is the number of M5-branes in the bound state, \( \epsilon_4 \) is the volume element on the unit 4-sphere and

\[ \Delta = k + \left( \frac{R}{r} \right)^3, \quad R = \pi N^{1/3} \ell_p \tag{5} \]

is the harmonic function and \( \Delta_\pm = \Delta \pm \nu \), where \( \nu \) is proportional to the square of the field strength on the brane, see [2] for details. We must have \( \nu \leq k \) in order to avoid naked singularities. The critical case \( \nu = k \), discussed at the end of this section, is related to the supergravity dual of OM theory. The (M2,M5) solution of Izquierdo et al. [19] is

\[ ds^2 = H^{-1/3} h^{-1/3} \left[ -dt^2 + (dx^1)^2 + (dx^2)^2 + h ((dx^3)^2 + (dx^4)^2 + (dx^5)^2) + H (dr^2 + r^2 d\Omega_4^2) \right] \]

\[ \ell_p^3 C_3 = H^{-1} \sin \alpha dt \wedge dx^1 \wedge dx^2 - H^{-1} h \tan \alpha dx^3 \wedge dx^4 \wedge dx^5 \]

\[ H_4 = dC_3 + 3\pi N \epsilon_4 \tag{6} \]

where the function \( h \) and the harmonic function \( H \) are defined as

\[ H = A + \frac{R^3}{\cos \alpha r^3}, \quad h^{-1} = H^{-1} \sin^2 \alpha + \cos^2 \alpha \tag{7} \]

where we have allowed for an arbitrary constant \( A \) in the harmonic function.

The equivalence can be seen by making the following substitutions

\[ \frac{\Delta_-}{2\nu} = H \frac{\tan^2 \alpha}{H} \]

\[ \frac{\Delta_+}{2\nu} = H \frac{h \sin^2 \alpha}{H} \]

\[ \frac{(k - \nu)}{2\nu} = A \frac{\tan^2 \alpha}{\tan^2 \alpha} \tag{8} \]

\[ ^5 \text{Note that we have chosen a sign such that } h_{+++} = -h_{---} = -\sqrt{2\nu} \text{ in the formulation of [2].} \]
keeping $R$ unchanged and rescaling the coordinates according to

$$
\begin{align*}
    r &\to \left( \frac{\tan^2 \alpha \cos \alpha}{2\nu} \right)^{1/3} r \\
x^{0,1,2} &\to \left( \frac{2\nu \cos^2 \alpha}{\tan^2 \alpha} \right)^{1/6} x^{0,1,2} \\
x^{3,4,5} &\to \left( \frac{2\nu}{\tan^2 \alpha \cos^4 \alpha} \right)^{1/6} x^{3,4,5}
\end{align*}
$$

where $x^0 \equiv t$. Note that we can not solve for $A$ and $\alpha$ in terms of $k$ and $\nu$, we just have the relation in (8). Usually one chooses $A = k = 1$ and then the relation determines $\alpha$ in terms of $\nu$. This, however, prevents us from being able to map the critical solutions in a non-singular manner, which will be important later on.

Having shown the equivalence of the (M2,M5) solution, we now turn to the boosted solutions. We perform the boost

$$
\begin{align*}
    t &\to t \cosh \gamma - x^5 \sinh \gamma, \\
x^5 &\to x^5 \cosh \gamma - t \sinh \gamma
\end{align*}
$$

on the solution of Izquierdo et al. (6), for details see [6]. Instead of the parameters $\alpha$ and $\gamma$, we can use the angles $\theta_1$ and $\theta_2$, with $\theta_1 \geq \theta_2$, defined by

$$
\begin{align*}
    \cos \alpha &= \frac{\cos \theta_2}{\cos \theta_1}, \\
    \cosh \gamma &= \frac{1}{\cos \theta_1 \sin \alpha}, \\
    \sinh \gamma &= \frac{\sin \theta_1}{\cos \theta_2 \tan \alpha}
\end{align*}
$$

The boosted solution becomes

$$
\begin{align*}
    ds^2 &= (H'h_1 h_2)^{-1/3} \left[ -dt^2 + h_1 ((dx^1)^2 + (dx^2)^2) + h_2 ((dx^3)^2 + (dx^4)^2) + h_1 h_2 \left( dx^5 + \sin \theta_1 \sin \theta_2 (H'^{-1} - 1)dt \right)^2 + H'(dr^2 + r^2 d\Omega_4^2) \right] \\
    \ell^3 p C_3 &= H'^{-1} \left( \frac{\sin \theta_1}{\cos \theta_1} h_1 dt \wedge dx^1 \wedge dx^2 + \frac{\sin \theta_1}{\cos \theta_2} h_2 dt \wedge dx^3 \wedge dx^4 \\
    &\quad - h_1 \tan \theta_1 dx^1 \wedge dx^2 \wedge dx^5 - h_2 \tan \theta_2 dx^3 \wedge dx^4 \wedge dx^5 \right)
\end{align*}
$$

$$
H_4 = dC_3 + 3\pi N \epsilon_4
$$

with the harmonic functions

$$
H' = B + \frac{R^3}{\cos \theta_1 \cos \theta_2 r^3}, \quad h_i^{-1} = H'^{-1} \sin^2 \theta_i + \cos^2 \theta_i
$$

which satisfy

$$
H = H'h_1^{-1}, \quad h^{-1} = h_1 h_2^{-1}
$$

and we have allowed for an arbitrary constant in the harmonic function which is related to $A$ as follows

$$
B = \frac{A - \sin^2 \theta_1}{\cos^2 \theta_1}
$$
Note that for $A = 1$ we get $B = 1$, which is the case considered in [6].

We now want to match this to the solution of Bergshoeff et al. [5].

\[ ds^2 = \left( E_1 E_2 \right)^{1/3} \left[ -\tilde{H}^{-1} \left[ 1 - (1 - \tilde{H})^2 \frac{s_1 s_2}{E_1 E_2} \right] dt^2 
\right. \\
\left. + \frac{2}{E_1 E_2} (1 - \tilde{H}) s_1 s_2 dt dx^5 + \frac{\tilde{H}}{E_1 E_2} (dx^5)^2 + \frac{1}{E_1} ((dx^1)^2 + (dx^2)^2) 
\right. \\
\left. + \frac{1}{E_2} ((dx^3)^2 + (dx^4)^2) + dr^2 + r^2 d\Omega_4^2 \right] \quad (16) \]

\[ dC_3 = d \left( \frac{1 - \tilde{H}}{E_1} \right) c_1 s_2 \wedge dt \wedge dx^1 \wedge dx^2 \\
+ d \left( \frac{1 - \tilde{H}}{E_2} \right) c_2 s_1 \wedge dt \wedge dx^3 \wedge dx^4 \\
- d \left( \frac{1 - \tilde{H}}{E_1} \right) c_1 s_1 \wedge dx^1 \wedge dx^2 \wedge dx^5 \\
- d \left( \frac{1 - \tilde{H}}{E_2} \right) c_2 s_2 \wedge dx^3 \wedge dx^4 \wedge dx^5 \\
- c_1 c_2 \star d\tilde{H} \]

where $s_i = \sin \theta_i$, $c_i = \cos \theta_i$ and

\[ E_i = s_i^2 + \tilde{H} c_i^2 \quad (17) \]

with the harmonic function

\[ \tilde{H} = a + \left( \frac{\tilde{R}}{r} \right)^3 \quad (18) \]

We have allowed for a constant $\tilde{R}$ in order not to have to rescale the radial coordinate in the mapping. The mapping is obtained by setting

\[ H' = \tilde{H}, \quad H' h_i^{-1} = E_i \quad (19) \]

without any coordinate rescalings. The first requirement implies that

\[ \tilde{R}^3 = \frac{R^3}{\cos \theta_1 \cos \theta_2} \quad (20) \]

and also that the constants in the harmonic functions must be related, i.e.,

\[ a = \frac{A - \sin^2 \theta_1}{\cos^2 \theta_1} \quad (21) \]

As in [3], we have obtained a complete mapping between the boosted solution of Izquierdo et al. and the solution of Bergshoeff et al. [5] but now with arbitrary constants in the harmonic functions.

We end this section with a discussion of the supergravity dual of OM theory where light membranes are obtained from a critical 3-form, here corresponding to
\( k = \nu \). Recently, possible decoupled theories of the boosted solution were discussed in [1]. The coordinate rescalings between the unboosted solutions can be written as

\[
x^{0,1,2} \rightarrow \left( \frac{(k - \nu)}{A} \left[ \frac{2\nu}{A} \left( \frac{A}{k - \nu} \right) + 1 \right]^{-1} \right)^{1/6} x^{0,1,2}
\]

\[
x^{3,4,5} \rightarrow \left( \frac{(k - \nu)}{A} \left[ \frac{2\nu}{A} \left( \frac{A}{k - \nu} \right) + 1 \right]^{2} \right)^{1/6} x^{3,4,5}
\]

\[
r \rightarrow \left( \frac{A}{(k - \nu)} \left[ \frac{2\nu}{A} \left( \frac{A}{k - \nu} \right) + 1 \right]^{-1/2} \right)^{1/3} r
\]

(22)

(23)

If we want to be able to relate critical solutions we must require that the above mapping is non-singular when \( \nu \rightarrow k \). We therefore let \( A \rightarrow 0 \) as \( \nu \rightarrow k \) in such a way that the quotient \( A/(k - \nu) \) is kept fixed. Note also that the critical case \( A = 0 \) corresponds to

\[
a = -\tan^{2} \theta_{1}
\]

(24)

and therefore, as soon as we consider a boosted solution (see (21)), the value of \( a \) which yields the critical solution is negative. Taking into account the change in \( a \) there is no problem associated with boosting a critical solution and in particular a Lorentz transformation does not change the critical or non-critical aspect of a solution, as expected. For the above value of \( a \), the asymptotic metric will be that of a smeared membrane, just as in the usual supergravity dual of OM-theory. We have seen that it is crucial to have arbitrary constants in the harmonic functions in order to be able to have a non-singular mapping, thereby being able to relate critical solutions, in particular when the solutions are boosted.

We end this section with some comments on the recently proposed V-duality [27, 28, 6]. In [1] it is argued that it is only possible to obtain a decoupled OM theory from an infinitesimally boosted (M2,M5) bound state, but not from a finitely boosted one. Important for this conclusion is that the decoupling limit is assumed to be the same after the boost as before. We see that the restriction to infinitesimal boosts, i.e., galilean transformations, follows directly from this assumption regarding the decoupling limit. The decoupling limit is obtained by scaling \( t \sim \epsilon^{0} \) and \( x^{5} \sim \epsilon^{3/2} \) and demanding this scaling both before and after the boost gives, when inserted into the Lorentz transformation (10), that \( \sinh \gamma \sim \epsilon^{3/2} \), i.e., \( \gamma \sim \epsilon^{3/2} \). The restriction to infinitesimal Lorentz transformations can therefore be seen to arise due to the different scalings of the coordinates when one tries to keep the decoupling limit fixed, which amounts to an extra constraint on the system. If we instead use the decoupling limit from [17], where all the coordinates on the M5-brane scale in the same way, we do not get any restriction to infinitesimal boosts and V-duality reduces to ordinary Lorentz transformations.
3.2 The D3-brane

In this section, we present the translation between two D3-brane solutions, the one by Cederwall et al. [2] and the one by Lu and Roy [23]. The first of these is an SL(2, \mathbb{R})-covariant description of all the bound states of a D3-brane and (F,D1)-strings, whereas the second is written as a specific but general ((F,D1),D3) bound state. When comparing the solutions, we can pick a certain bound state, and the general equivalence follows from SL(2, \mathbb{R})-covariance.

As mentioned in the previous section, the solution of [2] is described by a complex anti-selfdual 2-form \( F^{(2)} \). The radial dependence of the undeformed solution is described by the harmonic function
\[
\Delta = 1 + \frac{R^4}{r^4}
\]
(25)

We can then define the deformed harmonic functions \( \Delta_\pm = \Delta \pm \nu \), where \( \nu \) describes the deformation. More precisely, \( \nu = 2|\mu| \), where \( \mu = \frac{1}{4} \text{tr} \left( F^{(2)} \right)^2 \). Then the solution is
\[
ds^2 = \Delta_+^{1/4} \Delta_-^{-3/4} \left( - (dx^0)^2 + (dx^1)^2 \right) + \Delta_-^{3/4} \Delta_+^{3/4} \left( (dx^2)^2 + (dx^3)^2 \right) + \Delta_+^{1/4} \Delta_-^{1/4} dy^2
\]
\[
\mathcal{H}_{(3)} = (\Delta_+ \Delta_-)^{-1/4} d\Delta \wedge \left( \Delta_-^{1/2} \Pi_+ F^{(2)} + \Delta_+^{1/2} \Pi_- F^{(2)} \right)
\]
\[
P = i \mu (\Delta_+ \Delta_-)^{-1} d\Delta, \quad Q = 0
\]
where \( y^n, m = 1, 2, \ldots, 6 \) are the coordinates transverse to the D3-brane, \( P \) and \( Q \) are the Mauer-Cartan 1-forms, see appendix A for details, and the following projectors have been defined
\[
(\Pi_{\pm})_{\mu\nu} = \frac{1}{4} \left( 1 \pm \frac{1}{2} F^{(2)} \tilde{F}^{(2)} \right)_{\mu\nu}
\]
(27)

The solution is an SL(2, \mathbb{R})-covariant description of a bound state of a D3-brane and (F,D1)-strings. The charges of the strings are specified by the doublet of 3-forms. Choosing to get a specific bound state therefore corresponds to choosing the scalar doublet (from the set of possible solutions). Since \( F^{(2)} \) is antisymmetric and anti-selfdual, we can use a basis in which it takes the following form
\[
F^{(2)} = \sqrt{\mu} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}
\]
(28)

We can then find a certain scalar doublet, corresponding to a bound state of an F-string and a D3-brane, or equivalently a solution with an electric NS-NS 2-form. From the doublet of scalars we get the dilaton and the axion as well as the doublet of 3-forms which can be integrated to yield the doublet of 2-forms
\[
\mathcal{U}^1 = -\frac{1}{2} \eta \Delta_+^{1/2} \Delta_-^{1/2}, \quad \mathcal{U}^2 = i \eta \Delta_+^{1/2} \Delta_-^{-1/4} \\
C_{(2)1} = e^{-i \sqrt{2\nu} \Delta_-^{-1} dx^0 \wedge dx^1}, \quad C_{(2)2} = e^{-i \sqrt{2\nu} \Delta_+^{-1} dx^2 \wedge dx^3}
\]
\[
e^\phi = e^{\sqrt{2\mu} \Delta_-^{1/4}}, \quad \chi = 0
\]
(29)

\footnote{We will use the version given in [29].}
where \( c \) is an arbitrary real constant (actually \( c^2 \) is the undeformed asymptotic dilaton) and \( \eta = \mu/|\mu| \). This solution has string charges \((p^1, p^2) = (p, 0)\). Arbitrary string charges and background scalars can be obtained by performing an \( SL(2, \mathbb{R}) \)-transformation

\[
\begin{pmatrix}
p \\
q
\end{pmatrix} = \begin{pmatrix}
1 & p\tilde{q} \\
q/p & p\tilde{q}
\end{pmatrix} \begin{pmatrix}
p \\
0
\end{pmatrix}
\]

(30)

where \( \tilde{p}, \tilde{q} \) are real numbers fulfilling \( p\tilde{q} - q\tilde{p} = 1 \). Through such a transformation we can thus, e.g., get a non-vanishing axion.

It is the \((F,D3)\) bound state above we will use for comparison with the solution of Lu and Roy. Omitting the transverse 5-form, which is not important in this context, their solution is\(^7\)

\[
ds^2 = g_s^{-1/2} \left( H^{-\frac{n}{2}} H'^{\frac{n}{2}} (d\tilde{x}^2_0 + d\tilde{x}^2_1) + H^\frac{3}{2} H'^{-\frac{3}{2}} (d\tilde{x}^2_2 + d\tilde{x}^2_3) + (H H')^{\frac{1}{2}} d\gamma^2 \right)
\]

\[
e^\phi = g_s \frac{H''}{\sqrt{HH'}}, \quad \chi = \frac{pq(H - H') + g_s \chi_0 \Delta_{(p,q)} H'}{q^2 H + g_s^2 (p - \chi_0 q) H'}
\]

\[
2\pi \alpha^'B = g_s^{1/2} (p - \chi_0 q) \Delta_{(p,q,n)}^{1/2} H^{-1} d\tilde{x}^0 \wedge d\tilde{x}^1 - \frac{q}{n} H'^{-1} d\tilde{x}^2 \wedge d\tilde{x}^3
\]

\[
A_2 = g_s^{-1/2} (qg_s^{-1} - \chi_0 (p - \chi_0 q) g_s) \Delta_{(p,q,n)}^{1/2} H^{-1} d\tilde{x}^0 \wedge d\tilde{x}^1 + \frac{p}{n} H'^{-1} d\tilde{x}^2 \wedge d\tilde{x}^3
\]

where

\[
H = 1 + \frac{Q_3}{\tilde{r}^4}, \quad H' = 1 + \frac{n^2 g_s^{-1} Q_3}{\Delta_{(p,q,n)}^{-1} \tilde{r}^4}, \quad H'' = 1 + \frac{g_s^{-1} (q^2 + n^2) Q_3}{\Delta_{(p,q,n)}^{-1} \tilde{r}^4}
\]

(32)

and

\[
\Delta_{(p,q)} = g_s (p - \chi_0 q)^2 + g_s^{-1} q^2, \quad \Delta_{(p,q,n)} = \Delta_{(p,q)} + g_s^{-1} n^2
\]

\[
Q_3 = 4\pi \alpha'^2 \Delta_{(p,q,n)}^{1/2} g_s^{3/2}
\]

Here \( p \) and \( q \) are the charges of the strings and \( n \) is the number of D3-branes lying on top of each other. The solution can also be written in terms of two angles

\[
\cos \theta = \frac{n}{\sqrt{q^2 + n^2}}, \quad \cos \alpha = \frac{\sqrt{q^2 + n^2}}{\sqrt{g_s^2 (p - \chi_0 q)^2 + q^2 + n^2}}
\]

(34)

The cases \( p - \chi_0 q = 0 \) \((q = 0)\) correspond to \( \alpha = 0 \) \((\theta = 0)\) respectively, and these special cases are of the usual form used when constructing the solutions via T-duality and boosts/rotations. To see the equivalence between the two solutions, we only need to compare the \((F,D3)\) bound states with vanishing axion, since the general solution is just obtained by an \( SL(2,\mathbb{R}) \)-transformation. The form of the metric is the same independently of the charges, so this part of the translation can

\(^7\)There is a small typo in the solution in [29], a missing factor of \( g_s \) in \( A_2 \), as can be seen when checking the \( SL(2,\mathbb{R}) \)-covariance of the solution.
be done for the general configuration. From the metrics, the harmonic functions can be identified

\[ \Delta_+ = (1 - \nu)(1 + \frac{\nu^4}{(1+\nu)^4}) = (1 - \nu)H \]
\[ \Delta_- = (1 + \nu)(1 + \frac{\nu^4}{(1+\nu)^4}) = (1 + \nu)H' \]

(35)

We get exactly the same metric by doing the following coordinate transformation

\[ x^{0,1} = (1 - \nu)^\frac{3}{8}(1 + \nu)^{-\frac{1}{8}} g_s^{-\frac{1}{4}} \tilde{x}^{0,1} \]
\[ x^{2,3} = (1 - \nu)^{-\frac{1}{8}}(1 + \nu)^{\frac{3}{8}} g_s^{-\frac{1}{4}} \tilde{x}^{2,3} \]
\[ r = (1 - \nu)^{-\frac{1}{8}}(1 + \nu)^{-\frac{1}{8}} g_s^{-\frac{1}{4}} \tilde{r} \]

(36)

And we also get a relation between the parameters governing the deformation

\[ \cos^2 \theta \cos^2 \alpha = \frac{n^2}{g^2[p-\chi_0]q^2 + q^2 + n^2} = \frac{1-\nu}{1+\nu} \]

(37)

Using these relations in the special case of the (F,D3) bound state with vanishing axion, corresponding to \( q = \chi_0 = 0 \), we get exact agreement for all the fields, and as mentioned, the general case follows from the SL(2,R)-covariance. Thus it has been demonstrated that the two ((F,D1),D3) solutions indeed are equivalent.

A comment regarding the application of the above solution for supergravity duals of noncommutative theories on branes: Using our solution, critical electric field and infinite magnetic deformation parameter, \( \tan \theta \), are obtained in the limit \( \nu \to 1 \). The coordinate transformation (36) therefore becomes singular in this limit. This is not a problem in itself, it will just change the scaling of the coordinates with respect to \( \alpha' \). However, one has to be careful, if one wants to boost the critical solutions. In particular, one has to keep track of which coordinates should be fixed when \( \alpha' \) goes to zero. If one wants to relate critical solutions, in particular when the solutions are boosted, the coordinate transformation between the different formulations is required to be non-singular in the critical limit \( \nu = 1 \). Hence (36) has to be modified in this case, and the result will be similar to that obtained in the previous section for the M5-branes, where we had to allow for arbitrary integration constants in the harmonic functions in order to be able to map between critical solutions in a non-singular manner. The same remarks of course hold for the \((p,q)\) 5-branes in the next section.

### 3.3 \((p,q)\) 5-branes

In this section we will show the equivalence of various 5-brane solutions in type IIB supergravity. In the rank six case, our solution [3] is the only one containing \((p,q)\) 5-branes; other papers only consider the D5 and/or the NS5 solutions. We will therefore compare our D5-brane solution with that of [25]. The latter solution does not contain the RR fields, though. To be able to compare all the supergravity fields, we therefore also consider the rank two case \([3,3]\). In the rank 2 case, general \((p,q)\) 5-branes were first obtained in [24], and just as in the previous section, by relating two specific solutions the equivalence then follows from SL(2,R)-covariance.
Our solution in \( \mathbb{R} \) looks quite complicated, but it can actually be simplified a bit, by using the basis mentioned in section 5 of that paper. From the zero mode analysis we know that the solution is parametrised by a real 2-form \( F \) which then takes the form

\[
F = \begin{pmatrix}
0 & \tilde{\nu}_1 & 0 & 0 & 0 & 0 \\
-\tilde{\nu}_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \tilde{\nu}_2 & 0 & 0 & 0 \\
0 & 0 & -\tilde{\nu}_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \tilde{\nu}_3 & 0 \\
0 & 0 & 0 & 0 & -\tilde{\nu}_3 & 0 \\
\end{pmatrix}
\] (38)

Define the harmonic functions

\[
\Delta_{\pm \pm} = \Delta \pm \nu_1 \pm \nu_2 \pm \nu_3
\] (39)

where \( \nu_i \) corresponds to \( \nu^2_i \) in \( \mathbb{R} \). The solution contains expressions for the complex 3-form as well as the 5-form field strengths. From the scalar doublet we can get the solution in the Einstein frame takes the following form

\[
d s^2 = \Delta_{-\ldots-}^{-3/4} (\Delta_{++} \Delta_{+-})^{1/4} ( - dx_0^2 + dx_3^2 ) + \Delta_{-\ldots-}^{-3/4} (\Delta_{--} \Delta_{-+})^{1/4} ( dx_2^2 + dx_5^2 ) + \Delta_{-\ldots-}^{-3/4} (\Delta_{-\ldots} \Delta_{++})^{1/4} ( dy_1^2 + dy_4^2 )
\]

\[
(C_{(2)r})_{01} = -2kp_r \sqrt{\nu_2 \nu_3} \Delta_{-\ldots-}^{-1} - k^{-1} \tilde{p}_r \sqrt{2 \nu_1} \Delta_{-\ldots-}^{-1} 
\]

\[
(C_{(2)r})_{23} = 2kp_r \sqrt{\nu_2 \nu_3} \Delta_{-\ldots-}^{-1} - k^{-1} \tilde{p}_r \sqrt{2 \nu_2} \Delta_{-\ldots-}^{-1} 
\]

\[
(C_{(2)r})_{45} = 2kp_r \sqrt{\nu_1 \nu_3} \Delta_{-\ldots-}^{-1} - k^{-1} \tilde{p}_r \sqrt{2 \nu_3} \Delta_{-\ldots-}^{-1} 
\]

where \( k \) is a real constant related to the asymptotic scalars, \( (p_1, p_2) = (p, q) \) are the 5-brane charges and \( \tilde{p}_r \) is a real doublet, fulfilling \( \epsilon^{rps} \tilde{p}_s = 1 \). We write the 5-form as \( H_5 = d\Delta \wedge G_4 \) plus the hodge dual and this 4-form has the following components

\[
(G_{(4)})_{2345} = -\sqrt{2 \nu_1} (\Delta_{++} \Delta_{+-})^{-1}
\]

\[
(G_{(4)})_{0145} = -\sqrt{2 \nu_2} (\Delta_{--} \Delta_{-+})^{-1}
\]

\[
(G_{(4)})_{0123} = -\sqrt{2 \nu_3} (\Delta_{-\ldots} \Delta_{++})^{-1}
\]

We can then integrate to get the 4-form potential. We have to make a distinction between different ranks of the 2-form. For rank 6 we get (the rank 4 case is just obtained by setting one of the \( \nu \)'s to zero)

\[
(C_{(4)})_{2345} = \frac{\sqrt{2 \nu_1}}{2k^{-1} + \nu_1} \log \frac{\Delta_{++}}{\Delta_{++}}
\]

\[
(C_{(4)})_{0145} = \frac{\sqrt{2 \nu_2}}{2k^{-1} + \nu_2} \log \frac{\Delta_{--}}{\Delta_{--}}
\]

\[
(C_{(4)})_{0123} = \frac{\sqrt{2 \nu_3}}{2k^{-1} + \nu_3} \log \frac{\Delta_{-\ldots}}{\Delta_{-\ldots}}
\] (42)

In the rank 2 case we get by, e.g., putting \( \nu_1 \) and \( \nu_2 \) equal to zero,

\[
(C_{(4)})_{2345} = (C_{(4)})_{0145} = 0
\]

\[
(C_{(4)})_{0123} = \sqrt{2 \nu_3} \Delta_{-}^{-1}
\] (43)
The general expressions for the scalars can be found from the general scalar doublet in \[3\]. We will do the comparison for \((p, q) = (0, 1)\) (and therefore \(\bar{p} = -1\)). In this special case, the scalars take the following simple form

\[
e^\phi = k^{-2} \left( \Delta_{---} \Delta_{++} \right)^{-\frac{1}{2}} \Delta_{++}, \quad \chi = \bar{q} - k^2 \sqrt{8\nu_1 \nu_2 \nu_3 \Delta_{---}^{-1}}
\]

So \(k^{-2}\) is the undeformed asymptotic dilaton.

Now turn to the solution of \[25\], which can be obtained, using T-duality techniques. The solution is given in euclidean space. We can Wick rotate to obtain the lorentzian solution. The solution is furthermore given in the string frame. In the Einstein frame we get

\[
ds^2 = g_s^{-\frac{1}{2}} \left( \frac{\xi_1}{\xi_2} \right)^{\frac{3}{2}} \left( \frac{\xi_1}{\xi_3} \right)^{\frac{1}{2}} d\bar{x}_{0,1}^2 + \left( \frac{\xi_1}{\xi_2} \right)^{\frac{3}{2}} \left( \frac{\xi_1}{\xi_3} \right)^{-\frac{1}{2}} d\bar{x}_{2,3}^2
\]  
\[+ \left( \frac{\xi_1}{\xi_2} \right)^{\frac{1}{2}} \left( \frac{\xi_1}{\xi_3} \right)^{-\frac{3}{2}} d\bar{x}_{4,5}^2 + \left( \frac{\xi_1}{\xi_2} \frac{\xi_1}{\xi_3} \right)^{\frac{1}{2}} d\bar{y}^2
\]

(45)

\[B_{01} = \tanh \theta_1 f^{-1} h_1, \quad B_{23} = \tan \theta_2 f^{-1} h_2\]
\[B_{45} = \tan \theta_3 f^{-1} h_3, \quad e^{2\phi} = g_s^2 f^{-1} h_1 h_2 h_3\]

where for simplicity, a shorthand notation is used for the line elements. Since the solution is lorentzian, a minus sign is understood in front of the time component in the metric. Note that \(g_s\) is the asymptotic value of the deformed dilaton. The harmonic functions are given by

\[f = 1 + \frac{\bar{q}^2}{\cosh \theta_1 \cos \theta_2 \cos \theta_3}, \quad h_1^{-1} = -\sinh^2 \theta_1 f^{-1} + \cosh^2 \theta_1\]
\[h_2^{-1} = \sin^2 \theta_2 f^{-1} \cos^2 \theta_2, \quad h_3^{-1} = \sin^2 \theta_3 f^{-1} + \cos^2 \theta_3\]

(46)

From the form of the metric and the dilaton we get the following relations between the harmonic functions

\[\Delta_{---} = (1 - \nu_1 - \nu_2 - \nu_3) \frac{\xi_1}{\xi_2}, \quad \Delta_{++} = (1 + \nu_1 + \nu_2 - \nu_3) \frac{\xi_1}{\xi_2}\]
\[\Delta_{+-+} = (1 + \nu_1 - \nu_2 + \nu_3) \frac{\xi_1}{\xi_3}, \quad \Delta_{--+} = (1 + \nu_1 - \nu_2 - \nu_3) f\]

(47)

We get exact agreement between the solution if we do the following coordinate transformation

\[
x^{0,1} = g_s^{-1/4} (1 - \nu_1 - \nu_2 - \nu_3)^\frac{1}{\bar{q}} (1 + \nu_1 + \nu_2 - \nu_3)^{-\frac{1}{\bar{q}}} (1 + \nu_1 - \nu_2 + \nu_3)^{-\frac{1}{\bar{q}}} \bar{x}^{0,1}
\]
\[x^{2,3} = g_s^{-1/4} (1 - \nu_1 - \nu_2 - \nu_3)^{-\frac{1}{\bar{q}}} (1 + \nu_1 + \nu_2 - \nu_3)^\frac{1}{\bar{q}} (1 + \nu_1 - \nu_2 + \nu_3)^\frac{1}{\bar{q}} \bar{x}^{2,3}
\]
\[x^{4,5} = g_s^{-1/4} (1 - \nu_1 - \nu_2 - \nu_3)^{-\frac{1}{\bar{q}}} (1 + \nu_1 + \nu_2 - \nu_3)^{-\frac{1}{\bar{q}}} (1 + \nu_1 - \nu_2 + \nu_3)^{-\frac{1}{\bar{q}}} \bar{x}^{4,5}
\]
\[r = g_s^{-1/4} (1 - \nu_1 - \nu_2 - \nu_3)^{-\frac{1}{\bar{q}}} (1 + \nu_1 + \nu_2 - \nu_3)^{-\frac{1}{\bar{q}}} (1 + \nu_1 - \nu_2 + \nu_3)^{-\frac{1}{\bar{q}}} \bar{r}\]

(48)

and we also get the relations between the parameters in the two formulations

\[
cosh^2 \theta_1 = \frac{1 + \nu_1 - \nu_2 - \nu_3}{1 + \nu_1 - \nu_2 + \nu_3}, \quad \cos^2 \theta_2 = \frac{1 + \nu_1 - \nu_2 - \nu_3}{1 + \nu_1 + \nu_2 - \nu_3}
\]
\[\cos^2 \theta_3 = \frac{1 + \nu_1 - \nu_2 - \nu_3}{1 + \nu_1 + \nu_2 + \nu_3}\]

(49)

\^In that paper the solution is expressed in terms of three functions, \(f_{2-}\), \(f_{4-}\) and \(f_1\) which can be written like: \(f_{2-} = \Delta_{+-+}\), \(f_{4-} = \Delta_{---} \Delta_{++} \Delta_{+-+}\), \(f_1 = \Delta_{++} \Delta_{+-+} + 4\nu_2 \nu_3\).
Now turn to the rank two case. In \([7]\), the 4-form is also included in the NS5-brane solution with a rank 2 RR 2-form.

\[
A_{0123} = g_s^{-1} \sin \theta f\nonumber
\]

The metric is the same as before with \(\nu_1=\nu_2=0\) and therefore we have \(\Delta_- = (1 - \nu)f\). The translation of the metric, the NS-NS 2-form and the dilaton from above, of course also holds in this special case. Using the above relations between the coordinates and the parameters, restricted to the rank 2 case, we also get exact agreement for the 4-form.

Thus in the rank two case, it has been shown by comparing all the fields that our solution for the D5-brane is equivalent with the ones obtained by using T-duality techniques. As for the D3-brane, we also get a match in the general \((p,q)\) 5-brane case, since we just need to do an SL(2,\(\mathbb{R}\))-transformation. In the rank six case, the equivalence has also been shown, but not all fields could be compared and not all 5-branes, since our solution is the only complete and SL(2,\(\mathbb{R}\))-covariant one in this case.

4 Conclusion

In this paper we have shown the equivalence of various bound state solutions, obtained in different ways. Using our method, a zero mode analysis is performed, enabling one to write down an Ansatz and then solve the supergravity equations exactly. With the other method, the fields on the brane are obtained via T-duality techniques. The latter method is the easiest one in the simplest cases, since one just has to use the T-duality rules \([30, 31, 32, 33]\). On the other hand, we automatically get the most general solution by finding the allowed zero modes on the brane. In their original form, the solutions \([2, 3]\) look somewhat different from the ones obtained with T-duality techniques, but by rewriting the solutions in a particular basis, the resemblance between the solutions is increased. By performing a rescaling of the coordinates and relating the harmonic functions and the deformation parameters, we have shown that the different methods indeed yield equivalent solutions.

For the solutions obtained through duality techniques, the \(B\)-field is obtained by T-dualising in a direction on the brane, rotating or boosting in the directions where one wants the \(B\)-field and then T-dualising back. Alternatively, one can do a double T-duality, a gauge transformation to get the \(B\)-field and then another double T-duality to obtain these solutions \([\mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}]\). This method of course also yields equivalent solutions and the translation can be done in precisely the same way as in section 3.

We have also clarified some issues regarding the (M2,M5)-brane and its boosted solutions. In particular considering the decoupled theories, we have shown that the V-duality requirement of infinitesimal boosts follows directly from the assumption that the coordinates scale in the same way before and after the boost despite the fact that the boost mixes coordinates that scale differently. Without any extra assumptions, i.e., using coordinates on the brane which scale homogeneously in the limit giving the OM supergravity dual, as done in \([\mathcal{X}]\), we do not get any restrictions
on the boost and V-duality reduces to ordinary Lorentz transformations. We have also shown that there is no problem associated with boosting critical solutions when taking into account the transformation of the constant in the harmonic function.

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A Type IIB supergravity

Type IIB supergravity in ten dimensions has an $\text{SL}(2,\mathbb{R})$ invariance (which is broken to $\text{SL}(2,\mathbb{Z})$ by quantum effects) and contains the following fields: the metric, 2 scalars (the dilaton $\phi$ and the axion $\chi$), the NS-NS 2-form potential $B$, the R-R 2-form potential $C$ and the R-R 4-form potential $C(4)$. There exists a formulation with manifest $\text{SL}(2,\mathbb{R})$ covariance [37, 38]. Here we use the notation of [39, 40]. The two 2-forms can be collected in an $\text{SL}(2,\mathbb{R})$ doublet $C_r$, where $r=1,2$ corresponds to the NS-NS and R-R 2-forms respectively. The scalars can be described by a complex doublet $U^r$, with $\tau = U^1 / U^2 = \chi + i e^{-\phi}$. The scalar doublet fulfills the $\text{SL}(2,\mathbb{Z})$-invariant constraint

$$\frac{i}{2} \epsilon_{rs} U^r U^s = 1 \quad (51)$$

The 2-form doublet has a 3-form doublet of field strengths $H_{(3)r} = dC_r$, which can be combined with the scalar doublet into a complex 3-form

$$H_{(3)} = U^r H_{(3)r}, \quad H_{(3)r} = \epsilon_{rs} \text{Im} \left( U^s \bar{H}_{(3)} \right) \quad (52)$$

From the scalar doublet we can construct the Mauer-Cartan 1-forms $P$ and $Q$

$$Q = \frac{i}{2} \epsilon_{rs} dU^r \bar{U}^s, \quad P = \frac{i}{4} \epsilon_{rs} dU^r U^s \quad (53)$$

The equations of motion can now be written as

$$D*P + \frac{i}{4} \mathcal{H}_3 \wedge \star \mathcal{H}_3 = 0$$
$$D*\mathcal{H}_3 + i P \wedge \star \mathcal{H}_3 - i H_5 \wedge \mathcal{H}_3 = 0$$
$$D\mathcal{H}_3 + i \bar{\mathcal{H}}_3 \wedge P = 0$$
$$dH_5 - \frac{i}{4} \mathcal{H}_3 \wedge \bar{\mathcal{H}}_3 = 0$$
$$R_{MN} = 2 \bar{P}_{(M} P_{N)} + \frac{i}{4} \bar{H}_{(M}^{RS} \mathcal{H}_{N)RS} - \frac{1}{16} g_{MN} \bar{H}_{RST} H_{RST}^{RSU} + \frac{1}{16} \bar{H}_{(M}^{RSU} H_{N)RSU}$$

The first two equations are the equations of motion for $P$ and $\mathcal{H}_3$, respectively. The following two are the Bianchi identities for the 3-forms and the 5-form. The last line is the Einstein equations.
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