On the Hodge Structure of Projective Hypersurfaces in Toric Varieties

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The purpose of this paper is to explain one extension of the ideas of the Griffiths-Dolgachev-Steenbrink method for describing the Hodge theory of smooth (resp. quasi-smooth) hypersurfaces in complex projective spaces (resp. in weighted projective spaces). The main idea of this method is the representation of the Hodge components $H^{d-1-p,p}(X)$ in the middle cohomology group of projective hypersurfaces

$$X = \{ z \in P^d : f(z) = 0 \}$$

in $P^d = \text{Proj} \mathbb{C}[z_1, \ldots, z_{d+1}]$ using homogeneous components of the quotient of the polynomial ring $\mathbb{C}[z_1, \ldots, z_{d+1}]$ by the ideal $J(f) = \langle \partial f / \partial z_1, \ldots, \partial f / \partial z_{d+1} \rangle$. Basic references are [13, 14, 24, 29].

In this paper, we consider hypersurfaces $X$ in compact $d$-dimensional toric varieties $P_\Sigma$ associated with complete rational polyhedral fan $\Sigma$ of simplicial cones $R^d$. According to the theory of toric varieties [12, 22], $P_\Sigma$ is defined by glueing together of affine toric varieties $A_\sigma = \text{Spec} \mathbb{C}[\hat{\sigma} \cap \mathbb{Z}^d]$ ($\sigma \in \Sigma$) where $\hat{\sigma}$ denotes the dual to $\sigma$ cone. Weighted projective spaces are examples of toric varieties.

M. Audin [1] first noticed that there exists another approach to the definition of the toric variety $P_\Sigma$. This definition bases on the representation of $P_\Sigma$ as a quotient of some Zariski open subset $U(\Sigma)$ in an affine space $A^n$ by a linear diagonal action of some algebraic

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subgroup $D(\Sigma) \subset (\mathbb{C}^*)^n$. The group of characters of $D(\Sigma)$ is isomorphic to the group of classes $Cl(\Sigma)$ of divisors on $P_{\Sigma}$ modulo the rational equivalence. The dimension $n$ of the open set $U(\Sigma)$ equals the number of 1-dimensional cones in the fan $\Sigma$, and the dimension of $D(\Sigma)$ equals $n - d$, the rank of the Picard group of $P_{\Sigma}$. In particular, if $P_{\Sigma}$ is smooth, then $U(\Sigma)$ is the universal torsor over $P_{\Sigma}$ (see [21]) and $D(\Sigma)$ is the torus of Neron-Severi.

The codimension of the complement

$$Z(\Sigma) = \mathbb{A}^n \setminus U(\Sigma)$$

is at least 2. So the ring of regular algebraic functions on $U(\Sigma)$ is isomorphic to the polynomial ring

$$S(\Sigma) = \mathbb{C}[z_1, \ldots, z_n].$$

The action $D(\Sigma)$ on $U(\Sigma)$ induces a canonical grading of the ring $S(\Sigma)$ by elements of $Cl(\Sigma)$, i.e., by characters of $D(\Sigma)$. In the paper [3] of the second author, the polynomial ring $S(\Sigma)$ together with the $Cl(\Sigma)$-grading is called the homogeneous coordinate ring of the toric variety $P_{\Sigma}$. One nice feature of this ring is that a hypersurface $X \subset P_{\Sigma}$ has a defining equation $f = 0$ for some $f \in S(\Sigma)_\beta$. Here, $S(\Sigma)_\beta$ is is the graded piece of $S(\Sigma)$ in degree $\beta$, and $\beta$ is the divisor class of $X$ in $Cl(\Sigma).

Let us describe the contents of the paper in more detail:

Sections 1 and 2. We establish notation and review the construction of a simplicial toric variety $P_{\Sigma}$ as a quotient $U(\Sigma)/D(\Sigma)$. We also study the irreducible components and codimension of $Z(\Sigma) = \mathbb{A}^n \setminus U(\Sigma).

Sections 3 and 4. We characterize a quasi-smooth hypersurface $X \subset P_{\Sigma}$ in terms of its defining equation and we examine relations with $V$-submanifolds and toroidal pairs. We also study $T(\Sigma)$-linearized sheaves on $P_{\Sigma}$ (where $T(\Sigma)$ is the torus acting on $P_{\Sigma}$).

Sections 5, 6 and 7. The Bott-Steenbrink-Danilov vanishing theorem for toric varieties was stated without proof in [1, 23]. We give a proof in the case when $P_{\Sigma}$ is simplicial, and more generally we prove a vanishing theorem for the weight filtration on $\Omega^p_{P_{\Sigma}}(\log D) \otimes L$.

Sections 8 and 9. Given an ample hypersurface $X \subset P_{\Sigma}$, we study $H^0(P_{\Sigma}, \Omega^p_{P_{\Sigma}}(X))$. We use $Cl(\Sigma)$-graded $S(\Sigma)$-modules from [3] to describe $\Omega^p_{P_{\Sigma}}$, and we give explicit generators for the global sections when $p = d$ or $d - 1$ (where $d$ is the dimension of $P_{\Sigma}$).

Section 10. Given an ample hypersurface $X \subset P_{\Sigma}$ defined by $f \in S(\Sigma)_\beta$, we show that under the Hodge filtration on $H^d(P_{\Sigma} \setminus X)$, the graded pieces $Gr^p_F H^d(P_{\Sigma} \setminus X)$ are naturally isomorphic to certain graded pieces of $S(\Sigma)/J(f)$, where $J(f)$ is the Jacobian ideal of $f$. We then study the primitive cohomology of $X$ and show how to generalize classical results of Griffiths, Dolgachev and Steenbrink.

Section 11. The first author recently studied the cohomology of the affine hypersurface $Y = X \cap T(\Sigma)$ (see [3]). We show how these results can be stated in terms of the the ideal of $S(\Sigma)$ generated by $z_i \partial f / \partial z_i$ for $i = 1, \ldots, n$. By looking at the weight filtration on $Y$, we also get some results on the Hodge components of $X$.

Sections 12 and 13. The last two sections of the paper give further results on toric varieties. First, we generalize the classic Euler exact sequence and apply it to study $d - 1$ forms and the tangent sheaf on $P_{\Sigma}$. Then we show how the graded piece $(S(\Sigma)/J(f))_\beta$ of the Jacobian ring is related to the moduli of hypersurfaces in $P_{\Sigma}$ defined by $f \in S(\Sigma)_\beta$. We use results on the automorphism group of $P_{\Sigma}$ obtained by the first author in [3].
1 The definition of a simplicial toric variety $P_{\Sigma}$

Let $M$ be a free abelian group of rank $d$, $N = \text{Hom}(M, \mathbb{Z})$ the dual group. We denote by $M_\mathbb{R}$ (resp. by $N_\mathbb{R}$) the $\mathbb{R}$-scalar extension of $M$ (resp. of $N$).

**Definition 1.1** A convex subset $\sigma \subset N_\mathbb{R}$ is called a rational $k$-dimensional simplicial cone $(k \geq 1)$ if there exist $k$ linearly independent elements $e_1, \ldots, e_k \in N$ such that

$$\sigma = \{ \mu_1 e_1 + \cdots + \mu_k e_k \mid l_i \in \mathbb{R}, l_i \geq 0 \}.$$  

We call $e_1, \ldots, e_k \in N$ integral generators of $\sigma$ if for every $e_i$ $(1 \leq i \leq k)$ and any non-negative rational number $\mu$, $\mu \cdot e_i \in N$ only when $\mu \in \mathbb{Z}$. The origin $0 \in N_\mathbb{R}$ is the rational 0-dimensional simplicial cone, and the set of integral generators of this cone is empty.

**Definition 1.2** A rational simplicial cone $\sigma'$ is called a face of a rational simplicial cone $\sigma$ (we write $\sigma' \prec \sigma$) if the set of integral generators of $\sigma'$ is a subset of the set of integral generators of $\sigma$.

**Definition 1.3** A finite set $\Sigma = \{ \sigma_1, \ldots, \sigma_s \}$ of rational simplicial cones in $N_\mathbb{R}$ is called a rational simplicial complete $d$-dimensional fan if the following conditions are satisfied:

(i) if $\sigma \in \Sigma$ and $\sigma' \prec \sigma$, then $\sigma' \in \Sigma$;

(ii) if $\sigma, \sigma'$ are in $\Sigma$, then $\sigma \cap \sigma' \prec \sigma$ and $\sigma \cap \sigma' \prec \sigma'$;

(iii) $N_\mathbb{R} = \sigma_1 \cup \cdots \cup \sigma_s$.

The set of all $k$-dimensional cones in $\Sigma$ will be denoted by $\Sigma^{(k)}$.

**Example 1.4** Let $w = \{ w_1, \ldots, w_{d+1} \}$ be the set of positive integers satisfying the condition $\gcd(w_i) = 1$. Choose $d+1$ vectors $e_1, \ldots, e_{d+1}$ in a $d$-dimensional real space $V$ such $V$ is spanned by $e_1, \ldots, e_{d+1}$ and there exists the linear relation

$$w_1 e_1 + \cdots + w_{d+1} e_{d+1} = 0.$$  

Define $N$ to be the lattice in $V$ consisting of all integral linear combinations of $e_1, \ldots, e_{d+1}$. Obviously, $N_\mathbb{R} = V$. Let $\Sigma(w)$ be the set of all possible simplicial cones in $V$ generated by proper subsets of $\{ e_1, \ldots, e_{d+1} \}$. Then $\Sigma(w)$ is an example of a rational simplicial complete $d$-dimensional fan.

We want to show how every rational simplicial complete $d$-dimensional fan $\Sigma$ defines a compact $d$-dimensional complex algebraic variety $P_{\Sigma}$ having only quotient singularities. For instance, if $\Sigma$ is a fan $\Sigma(w)$ from Example 1.4, then the corresponding variety $P_{\Sigma}$ will be the $d$-dimensional weighted projective space $\mathbb{P}(w_1, \ldots, w_{d+1})$. The standard definition of $\mathbb{P}(w_1, \ldots, w_{d+1})$ describes it as a quotient of $\mathbb{C}^{d+1} \setminus \{ 0 \}$ by the diagonal action of the multiplicative group $\mathbb{C}^*$:

$$(z_1, \ldots, z_{d+1}) \mapsto (t^{w_1} z_1, \ldots, t^{w_{d+1}} z_{d+1}), \quad t \in \mathbb{C}^*.$$  

In Definition 1.11, we will show that the toric variety $P_{\Sigma}$ can be constructed in a similar manner. We first need some definitions.
Definition 1.5 (\[5\]) Let $S(\Sigma) = \mathbb{C}[z_1, \ldots, z_n]$ be the polynomial ring over $\mathbb{C}$ with variables $z_1, \ldots, z_n$, where $\Sigma^{(1)} = \{\rho_1, \ldots, \rho_n\}$ are the 1-dimensional cones of $\Sigma$. Then, for $\sigma \in \Sigma$, let $\tilde{z}_\sigma = \prod_{\rho_i \not\subset \sigma} z_i$, and let $B(\Sigma) = \langle \tilde{z}_\sigma : \sigma \in \Sigma \rangle \subset S(\Sigma)$ be the ideal generated by the $\tilde{z}_\sigma$’s.

Definition 1.6 Let $A^n = \text{Spec } S(\Sigma)$ be the $n$-dimensional affine space over $\mathbb{C}$ with coordinates $z_1, \ldots, z_n$. The ideal $B(\Sigma) \subset S(\Sigma)$ gives the variety $Z(\Sigma) = V(B(\Sigma)) \subset A^n$, and we get the Zariski open set $U(\Sigma) = A^n \setminus Z(\Sigma)$.

Definition 1.7 Consider the injective homomorphism $\alpha : M \to \mathbb{Z}^n$ defined by $\alpha(m) = (\langle m, e_1 \rangle, \ldots, \langle m, e_n \rangle)$. The cokernel of this map is $\mathbb{Z}^n/\alpha(M) \simeq \text{Cl}(\Sigma)$, and we define $D(\Sigma) : = \text{Spec } \mathbb{C}[\text{Cl}(\Sigma)]$ to be $(n - d)$-dimensional commutative affine algebraic $D$-group whose group of characters is isomorphic to $\text{Cl}(\Sigma)$ (see [18]).

Remark 1.8 Notice that the finitely generated abelian group $\text{Cl}(\Sigma)$ defines $D(\Sigma)$ not only as an abstract commutative affine algebraic $D$-group, but also as a canonically embedded into $(\mathbb{C}^*)^n = \text{Spec } \mathbb{C}[\mathbb{Z}^n]$ subgroup associated with the surjective homomorphism $\mathbb{Z}^n \to \mathbb{Z}^n/\alpha(M) \simeq \text{Cl}(\Sigma)$. So we obtain the canonical diagonal action of $D(\Sigma)$ on the affine space $A^n$. Obviously, $U(\Sigma)$ is invariant under this action.

Now we are ready to formulate the key theorem.

Theorem 1.9 Let $\Sigma$ be a rational simplicial complete $d$-dimensional fan. Then the canonical action of $D(\Sigma)$ on $U(\Sigma)$ has the geometric quotient

$$P_\Sigma = U(\Sigma)/D(\Sigma)$$

which is a compact complex $d$-dimensional algebraic variety having only abelian quotient singularities (and hence $P_\Sigma$ is a $V$-manifold).

Proof. The existence of the quotient $U(\Sigma)/D(\Sigma)$ was proved in the analytic case by Audin [1] and in the algebraic case by Cox [3]. It remains to show that $P_\Sigma$ has only abelian quotient singularities. First observe that $Z(\Sigma)$ is defined by the vanishing of $\tilde{z}_\sigma$ for the $d$-dimensional cones $\sigma \in \Sigma^{(d)}$. For such a $\sigma$, set $U_\sigma = \{z \in A^n : \tilde{z}_\sigma \neq 0\}$. Thus $U(\Sigma) = \bigcup_{\sigma \in \Sigma^{(d)}} U_\sigma$, and it suffices to show that $U_\sigma/D(\Sigma)$ has abelian quotient singularities.

If the 1-dimensional cones of $\Sigma$ are $\Sigma^{(1)} = \{\rho_1, \ldots, \rho_n\}$, put $G(\Sigma) = \{e_1, \ldots, e_n\}$ where $e_i$ is the integral generator of $\rho_i$. Now renumber $G(\Sigma)$ so that the generators of $\sigma$ are
$e_1, \ldots, e_d$. Then $U_\sigma = \mathbb{C}^d \times (\mathbb{C}^*)^{n-d}$. To see how $D(\Sigma)$ acts on this set, consider the commutative diagram

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
\mathbb{Z}^{n-d} & = & \mathbb{Z}^{n-d} \\
\downarrow & & \downarrow \\
0 & \to & M \\
\| & & \| \\
0 & \to & \mathbb{Z}^d \\
\downarrow & & \downarrow \\
0 & \to & Cl(\sigma) \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
\]  

(1)

where the map $M \to \mathbb{Z}^d$ is $m \mapsto (\langle m, e_1 \rangle, \ldots, \langle m, e_d \rangle)$. Since $e_1, \ldots, e_d$ are linearly independent, it follows that $Cl(\sigma)$ is a finite group. Then the last column of the diagram gives the exact sequence

\[ 1 \to D(\sigma) \to D(\Sigma) \to (\mathbb{C}^*)^{n-d} \to 1. \]  

(2)

Since $U_\sigma = \mathbb{C}^d \times (\mathbb{C}^*)^{n-d}$ and $D(\sigma)$ acts naturally on $\mathbb{C}^d$, it follows that the map $\mathbb{C}^d \to U_\sigma$ defined by $(t_1, \ldots, t_d) \mapsto (t_1, \ldots, t_d, 1, \ldots, 1)$ is equivariant. It is also easy to see that the induced map $\mathbb{C}^d/D(\sigma) \to U_\sigma/D(\Sigma)$ is an isomorphism. Since $D(\sigma)$ is a finite abelian group, the theorem is proved. \qed

**Remark 1.10** Audin [1] and Cox [5] have shown that $P_\Sigma$ is isomorphic to the complete toric variety associated with the simplicial fan $\Sigma$ in the usual sense of the theory [9, 23] (see [5] for other people who have discovered this result). Also, [5] shows that $A_\sigma = U_\sigma/D(\Sigma) \subset P_\Sigma$ is the affine toric open associated to the cone $\sigma \in \Sigma$. We can thus use Theorem 1.9 as the definition of toric variety.

**Definition 1.11** The complete $d$-dimensional algebraic variety

\[ P_\Sigma = U(\Sigma)/D(\Sigma) \]

is called the toric variety associated with the complete simplicial fan $\Sigma$.

This construction of a toric variety makes it easy to see the torus action.

**Definition 1.12** Denote by $T(\Sigma)$ the quotient of $(\mathbb{C}^*)^n$ by the subgroup $D(\Sigma)$. When $\Sigma$ is fixed, we denote $T(\Sigma)$ simply by $T$.

**Remark 1.13** The group $T(\Sigma)$ is isomorphic to the torus $N \otimes \mathbb{C}^* = (\mathbb{C}^*)^d$. Since the group $(\mathbb{C}^*)^n$ is an open subset of $U(\Sigma)$, and it acts canonically on $U(\Sigma)$, $T(\Sigma)$ is an open subset of $P_\Sigma$ having the induced action by regular automorphisms of $P_\Sigma$. 

5
2 The structure of \( Z(\Sigma) = \mathbb{A}^n \setminus U(\Sigma) \)

We will next discuss the closed subvariety \( Z(\Sigma) = \mathbb{A}^n \setminus U(\Sigma) \). It has an interesting combinatorial interpretation.

**Definition 2.1** ([2]) Let the 1-dimensional cones of \( \Sigma \) be \( \Sigma^{(1)} = \{ \rho_1, \ldots, \rho_n \} \), and put \( G(\Sigma) = \{ e_1, \ldots, e_n \} \) where \( e_i \) is the integral generator of \( \rho_i \). We call a subset \( P = \{ e_{i_1}, \ldots, e_{i_p} \} \subset G(\Sigma) \) a primitive collection if \( \{ e_{i_1}, \ldots, e_{i_p} \} \) is not the set of generators of a \( p \)-dimensional simplicial cone in \( \Sigma \), while any proper subset of \( P \) generates a cone in \( \Sigma \).

**Example 2.2** Let \( \Sigma \) be a fan \( \Sigma(w) \) from Example [4]. Then there exists only one primitive collection in \( G(\Sigma(w)) \) which coincides with \( G(\Sigma(w)) \) itself.

**Definition 2.3** Let \( P = \{ e_{i_1}, \ldots, e_{i_p} \} \) be a primitive collection in \( G(\Sigma) \). Define \( A(P) \) to be the \((n-p)\)-dimensional affine subspace in \( \mathbb{A}^n \) having the equations

\[
z_{i_1} = \cdots = z_{i_p} = 0.
\]

**Remark 2.4** Since every primitive collection \( P \) has at least two elements, the codimension of \( A(P) \) is at least 2.

**Lemma 2.5** Let \( Z(\Sigma) \subset \mathbb{A}^n \) be the variety defined in Definition 1.6. Then the decomposition of \( Z(\Sigma) \) into its irreducible components is given by

\[
Z(\Sigma) = \bigcup_{P} A(P),
\]

where \( P \) runs over all primitive collections in \( G(\Sigma) \).

**Proof.** First, it follows from the definition of \( Z(\Sigma) = V(\hat{z}_\sigma : \sigma \in \Sigma) \) that

\[
Z(\Sigma) = \bigcup_{Q} A(Q),
\]

where \( Q \) runs over all subsets \( Q \subset G(\Sigma) \) which are not the set of generators of any cone in \( \Sigma \). Then note that the set of all such \( Q \)'s are partially ordered by inclusion, and \( Q \subset Q' \) implies \( A(Q') \subset A(Q) \). Hence, in the above union, it suffices to use the minimal \( Q \)'s, which are precisely the primitive collections. It follows that these give the irreducible components of \( Z(\Sigma) \). \( \square \)

**Remark 2.6** In [2], the first author conjectured that for smooth complete toric varieties, the number of primitive collections could be bounded in terms of the Picard number \( \rho \) (= \( n - d \)). Since primitive collections correspond to irreducible components of \( Z(\Sigma) \) by the above lemma, we can reformulate this conjecture as follows.
Conjecture 2.7 For any \( d \)-dimensional smooth complete toric variety with Picard number \( \rho \) defined by a complete regular fan \( \Sigma \), there exists a constant \( N(\rho) \) depending only on \( \rho \) such that \( Z(\Sigma) \subset \mathbb{A}^n \) has at most \( N(\rho) \) irreducible components.

We next study the codimension of \( Z(\Sigma) \). When \( P_\Sigma \) is a weighted projective space, it follows from Example 2.2 that \( Z(\Sigma) = \{0\} \), which is as small as possible. It turns out that in most other cases, \( Z(\Sigma) \) is considerably larger. The precise result is as follows.

Proposition 2.8 Let \( P_\Sigma \) be a complete simplicial toric variety of dimension \( d \), and let \( Z(\Sigma) \subset \mathbb{A}^n \) be as above. Then either

1. \( 2 \leq \text{codim } Z(\Sigma) \leq \lfloor \frac{1}{2}d \rfloor + 1 \), or
2. \( n = d + 1 \) and \( Z(\Sigma) = \{0\} \).

Proof. To illustrate the range of techniques that can be brought to bear on this subject, we will give two proofs of this result. For the first proof, we make the additional assumption that \( P_\Sigma \) is projective. A projective embedding of \( P_\Sigma \) is given by a strictly convex support function on \( \Sigma \), which determines a convex polytope \( \Delta \subset M_\mathbb{R} \) (see [23]). Now consider the dual polytope \( \Delta^* \subset N_\mathbb{R} \). Combinatorially, \( \Delta^* \) is closely related to the fan \( \Sigma \)—in fact, \( \Sigma \) is the cone over \( \Delta^* \).

Let \( k = \lfloor \frac{1}{2}d \rfloor + 1 \) and assume that \( \text{codim } Z(\Sigma) > k \). Now pick any subset \( \mathcal{Q} \subset G(\Sigma) \) consisting of \( k \) elements (this corresponds to picking \( k \) vertices of \( \Delta^* \)). Then \( \text{codim } Z(\Sigma) > k \) implies that \( A(\mathcal{Q}) \not\subset Z(\Sigma) \). From equation (3), it follows that \( \mathcal{Q} \) must be the set of generators for some face \( \sigma \in \Sigma \).

In terms of the polytope \( \Delta^* \), this shows that every set of \( k \) vertices are the vertices of some face of \( \Delta^* \). Since \( k > \lfloor \frac{1}{2}d \rfloor \), standard results about convex polytopes (see Chapter 7 of [16]) imply that \( \Delta^* \) is a simplex, which proves that the number of 1-dimensional cones is \( n = d + 1 \).

Our second proof, which applies to arbitrary complete simplicial toric varieties, uses the Stanley-Reisner ring of a certain monomial ideal associated with \( \Sigma \).

Definition 2.9 Let \( \Sigma \) be a complete simplicial fan of dimension \( d \). The Stanley-Reisner ideal \( I(\Sigma) \) is the ideal in \( S = S(\Sigma) \) generated by all monomials \( z_{e_1} \cdots z_{e_p} \) such that \( \{e_{i_1}, \ldots, e_{i_p}\} \) is a primitive collection in \( G(\Sigma) \). The quotient \( R(\Sigma) = S(\Sigma)/I(\Sigma) \) is called the Stanley-Reisner ring of the fan \( \Sigma \).

As explained in [26, 28], the ring \( R(\Sigma) \) comes from the simplicial complex given by all subsets of \( G(\Sigma) \) which correspond to cones of \( \Sigma \) (this is because \( \Sigma \) is a simplicial fan). But since \( \Sigma \) is also complete, the simplicial complex is a triangulation of the \( d-1 \) sphere. This has some very strong consequences about the ring \( R(\Sigma) \). For instance, by §1 of [26], the irreducible components of the affine variety \( \text{Spec } R(\Sigma) \) correspond to maximal faces of the simplicial complex. It follows that \( \text{Spec } R(\Sigma) \) is a union of \( d \)-dimensional linear subspaces in \( \mathbb{A}^n \), one for each \( d \)-dimensional cone of \( \Sigma \).
Remark 2.10 One should not confuse $Z(\Sigma)$ with $\text{Spec } R(\Sigma)$. For a weighted projective space as in Example 1.4, $Z(\Sigma) = \{0\}$, but $I(\Sigma)$ is a principal ideal generated by the monomial $z_1 \cdots z_{d+1}$, i.e., $\text{Spec } R(\Sigma)$ is the union of $d+1$ linear subspaces of codimension 1 in $A^n = A^{d+1}$.

The ring $R(\Sigma)$ has many nice properties. First, it has the natural grading by elements of $\mathbb{Z}_{\geq 0}$ induced from the $\mathbb{Z}_{\geq 0}$-grading of the polynomials ring $S$. Second, since the simplicial complex is a triangulation of the $d-1$ sphere, it follows from Chapter II, §5 of [28] that $R(\Sigma)$ is a graded Gorenstein ring of dimension $d$. Then, by [4], the minimal free resolution

$$0 \to P_{n-d} \xrightarrow{d_{n-d}} P_{n-d-1} \xrightarrow{d_{n-d-1}} \cdots \xrightarrow{d_3} P_1 \xrightarrow{d_0} P_0 \to R(\Sigma) \to 0$$

of $R(\Sigma)$ as a module over $S$ satisfies the duality property

$$P_i \cong \text{Hom}_S(P_{n-d-i}, P_{n-d})$$

where each $d_i$ is a graded homomorphism of degree 0 between the graded $S$-modules $P_i$ and $P_{i-1}$. Further, the module $P_1$ is isomorphic to $S$, and since we know the generators of $I(\Sigma)$, we get an isomorphism

$$P_1 \cong \bigoplus_{P \subset G(\Sigma)} S(-|P|)$$

where $P$ runs over all primitive collections in $G(\Sigma)$, and $|P|$ is the cardinality of $P$. Finally, the duality (4) shows that $P_{n-d}$ is a free $S$-module of rank 1.

We claim that $P_{n-d} \cong S(-n)$. For each $i$ between 0 and $n-d$, let $h(i)$ be the minimal integer $h$ such that $P_i$ has a nonzero element of degree $h$. The minimality of the free resolution implies that $0 = h(0) < h(1) < \cdots < h(n-d-1) < h(n-d)$. Since $P_{n-d}$ has rank 1, we have $P_{n-d} = S(-h(n-d))$. Hence it suffices to show $h(n-d) = n$. The Hilbert-Poincare series of $S(-j)$ is $H(S(-j), t) = t^j/(1-t)^n$, so that the free resolution of $R(\Sigma)$ implies that

$$H(R(\Sigma), t) = \sum_{i=0}^{n-d} (-1)^i H(P_i, t) = \frac{\text{polynomial of degree } h(n-d)}{(1-t)^n}.$$ 

However, Theorem 1.4 of Chapter II of [28] shows that

$$H(R(\Sigma), t) = \frac{\text{polynomial of degree } d}{(1-t)^d}.$$ 

Comparing these two expressions, we conclude $h(n-d) = n$.

Once we know $P_{n-d} = S(-n)$, the isomorphism (5) and the duality (4) give an isomorphism

$$P_{n-d-1} \cong \bigoplus_{P \subset G(\Sigma)} S(-n + |P|).$$

Assume now that $n > d + 1$. Then $h(n-d-1) > \cdots > h(1)$ implies

$$h(n-d-1) \geq h(1) + n - d - 2.$$ (7)
From the isomorphisms (5) and (6), we see that there exist primitive collections \( P_1 \) and \( P_2 \) such that
\[
h(n - d - 1) = n - |P_1| \quad \text{and} \quad h(1) = |P_2|.
\]
Then the inequality (7) implies that
\[
|P_1| + |P_2| \leq d + 2,
\]
and thus
\[
\min(|P_1|, |P_2|) \leq \left\lfloor \frac{1}{2} d \right\rfloor + 1.
\]
On the other hand, Lemma 2.5 implies
\[
\min_{P \subseteq G(\Sigma)} |P| = \min_{P \subseteq G(\Sigma)} \operatorname{codim} A(P) = \operatorname{codim} Z(\Sigma).
\]
This proves \( \operatorname{codim} Z(\Sigma) \leq \left\lfloor \frac{1}{2} d \right\rfloor + 1 \) when \( n > d + 1 \).

On the other hand, the equality \( n = d + 1 \) is possible only if the minimal projective resolution of \( R(\Sigma) \) consists of \( P_0 \cong S \) and \( P_1 \cong S(-n) \), which means that \( I(\Sigma) \) is the principal ideal generated by the monomial \( z_1 \cdots z_n \) of degree \( n \). This implies that \( \{z_1, \ldots, z_n\} \) is the unique primitive collection, and we obtain \( Z(\Sigma) = \{0\} \). This completes the proof of the proposition.

The condition \( n = d + 1 \) is closely related to \( P_\Sigma \) being a weighted projective space.

**Lemma 2.11** Let \( \Sigma \) be a complete simplicial fan with \( n = d + 1 \) 1-dimensional cones. Then there is a weighted projective space \( P(w_1, \ldots, w_{d+1}) \) and a finite surjective morphism
\[
P(w_1, \ldots, w_{d+1}) \to P_\Sigma.
\]
Furthermore, if \( G(\Sigma) = \{e_1, \ldots, e_{d+1}\} \), then the following are equivalent:
1. \( P_\Sigma \) is a weighted projective space.
2. \( D(\Sigma) \cong \mathbb{C}^* \) (or equivalently, \( Cl(\Sigma) \cong \mathbb{Z} \)).
3. \( e_1, \ldots, e_{d+1} \) generate \( N \) as a \( \mathbb{Z} \)-module.

**Proof.** First note that (1) \( \Rightarrow \) (2) is immediate. Further, if (3) holds, then Example 1.4 shows that \( P_\Sigma \) is a weighted projective space. It remains to show (2) \( \Rightarrow \) (3). But if \( Cl(\Sigma) \cong \mathbb{Z} \), then taking the dual of the exact sequence
\[
0 \to M \xrightarrow{\alpha} \mathbb{Z}^{d+1} \to \mathbb{Z} \to 0
\]
from Definition 1.7 gives an exact sequence
\[
0 \to \mathbb{Z} \to \mathbb{Z}^{d+1} \to N \to 0,
\]
and it follows that \( e_1, \ldots, e_{d+1} \) generate \( N \).

Finally, given an arbitrary fan \( \Sigma \) with \( n = d + 1 \), let \( N' \subset N \) be the sublattice generated by \( e_1, \ldots, e_{d+1} \). Then the fan \( \Sigma \) induces a fan \( \Sigma' \) for the lattice \( N' \), and the natural inclusion \( (N', \Sigma') \to (N, \Sigma) \) induces a finite surjection \( P_{\Sigma'} \to P_\Sigma \) by Corollary 1.16 of [23]. By the above, \( P_{\Sigma'} \) is a weighted projective space, and the proposition is proved.

Lemma 2.11 implies that a smooth complete toric variety with \( n = d + 1 \) is \( P^d \) (this fact is well-known). Thus we get the following corollary of Proposition 2.8.
Corollary 2.12 Let $P_\Sigma$ be a smooth complete toric variety. Then either

1. $2 \leq \text{codim } Z(\Sigma) \leq \left[ \frac{1}{2}d \right] + 1$, or
2. $P_\Sigma = P^d$.

We end this section with some comments about combinatorially equivalent fans.

Definition 2.13 Two rational simplicial complete $d$-dimensional fans $\Sigma$ and $\Sigma'$ are called \textit{combinatorially equivalent} if there exists a bijective mapping $\Sigma \rightarrow \Sigma'$ respecting the face-relation “$\prec$” (see Definition 1.2).

Remark 2.14 It is easy to see that the closed subset $Z(\Sigma) \subset \mathbb{A}^n$ depends only on the combinatorial structure of $\Sigma$, i.e., for two combinatorially equivalent fans $\Sigma$ and $\Sigma'$, we can assume that we are in the same affine space $\mathbb{A}^n$, and then we have $Z(\Sigma) = Z(\Sigma')$ and hence $U(\Sigma) = U(\Sigma')$.

It follows that all toric varieties coming from combinatorially equivalent fans are quotients of the same open set $U \subset \mathbb{A}^n$, though the group actions will differ. For example, all $d$-dimensional weighted projective spaces come from combinatorially equivalent fans and are quotients of $U = \mathbb{A}^{d+1} - \{0\}$.

3 Quasi-smooth hypersurfaces

Throughout this section, let $P = P_\Sigma$ be a fixed $d$-dimensional complete simplicial toric variety. The action of $G = D(\Sigma)$ on $\mathbb{A}^n$ induces an action on $S = S(\Sigma) = \mathbb{C}[z_1, \ldots, z_n]$. The decomposition of this representation gives a grading on $S$ by the character group $Cl(\Sigma)$ of $G$. A polynomial $f$ in the graded piece of $S$ corresponding to $\beta \in Cl(\Sigma)$ is said to be $G$-homogeneous of degree $\beta$.

Such a polynomial $f$ has a zero set $V(f) \subset \mathbb{A}^n$, and $V(f) \cap U(\Sigma)$ is stable under $G$ and hence descends to a hypersurface $X \subset P$ (this is because $P$ is a geometric quotient—see [1] for more details). We call $V(f) \subset \mathbb{A}^n$ the \textit{affine quasi-cone} of $X$.

Definition 3.1 If a hypersurface $X \subset P$ is defined by a $G$-homogeneous polynomial $f$, then we say that $X$ is \textit{quasi-smooth} if the affine quasi-cone $V(f)$ is smooth outside $Z = Z(\Sigma) \subset \mathbb{A}^n$.

To see what this definition says about the hypersurface $X$, we need the concept of a $V$-submanifold of a $V$-manifold. As usual, a $d$-dimensional variety $W$ is a $V$-manifold if for every point $p \in W'$, there is an analytic isomorphism of germs $(\mathbb{C}^d/G, 0) \cong (W, p)$ where $G \subset GL(d, \mathbb{C})$ is a finite small subgroup. (Recall that being \textit{small} means there are no elements with 1 as an eigenvalue of multiplicity $d-1$, i.e., no complex rotations other than the identity). In this case, we say that $(\mathbb{C}^d/G, 0)$ is a \textit{local model} of $W$ at $p$. 
Definition 3.2 If a $d$-dimensional variety $W$ is a $V$-manifold, then a subvariety $W' \subset W$ is a $V$-submanifold if for every point $p \in W'$, there is a local model $(\mathbb{C}^d/G, 0) \cong (W, p)$ such that $G \subset GL(d, \mathbb{C})$ is a finite small subgroup and the inverse image of $W'$ in $\mathbb{C}^d$ is smooth at 0.

Remark 3.3 In [25], Prill proved that $G$ is determined up to conjugacy by the germ $(W, p)$. Hence, if $W' \subset W$ is a $V$-submanifold for one local model at $p$, then it is a $V$-submanifold for all local models at $p$. It is also easy to see that a $V$-submanifold of a $V$-manifold $W$ is again a $V$-manifold. However, the converse is false: a subvariety of $W$ which is a $V$-manifold need not be a $V$-submanifold. This is because the singularities of a $V$-submanifold are intimately related to the singularities of the ambient space.

We will omit the proof of the following easy lemma.

Lemma 3.4 If $G \subset GL(d, \mathbb{C})$ is a small and finite, then a subvariety $W' \subset \mathbb{C}^d/G$ is a $V$-submanifold if and only if the inverse image of $W'$ in $\mathbb{C}^d$ is smooth.

We can now describe when a hypersurface of the toric variety $P$ is a $V$-submanifold.

Proposition 3.5 If a hypersurface $X \subset P$ is defined by a $D$-homogeneous polynomial $f$, then $X$ is quasi-smooth if and only if $X$ is a $V$-submanifold of $P$.

Proof. We will use the notation of the proof of Theorem 1.9. Thus, let $\sigma$ be a $d$-dimensional cone of $\Sigma$ and assume that $\sigma$ is generated by $e_1, \ldots, e_d$. As we saw in the proof of Theorem 1.9, $\sigma$ gives an affine open set $A_\sigma = \mathbb{C}^d/D(\sigma) \cong U_\sigma/D$ of $P$.

We claim that $D(\sigma) \subset (\mathbb{C}^*)^d$ is a small subgroup. Suppose that $g = (\lambda, 1, \ldots, 1) \in D(\sigma)$. We can regard $g$ as a homomorphism $g : Cl(\sigma) \to \mathbb{C}^*$, and diagram (1) shows that $g = 1$ on the image of $((m, e_1), \ldots, (m, e_d))$ whenever $m \in M$. This means that $\lambda^{(m, e_1)} = 1$ for all $m \in M$. Since $e_1$ is not the multiple of any element of $N$ and $M = \text{Hom}(N, \mathbb{Z})$, it follows that $\lambda = 1$ and hence $g$ is the identity.

Since $A_\sigma = \mathbb{C}^d/D(\sigma)$, Lemma 3.4 implies that $X \cap A_\sigma$ is a $V$-submanifold if and only if the inverse image of $X \cap A_\sigma$ in $\mathbb{C}^d$ is smooth. Since the map $\mathbb{C}^d/D(\sigma) \cong U_\sigma/D$ is induced by $(t_1, \ldots, t_d) \mapsto (t_1, \ldots, t_d, 1, \ldots, 1)$, it follows that the inverse image of $X \cap A_\sigma$ is $\mathbb{C}^d$ is defined by $g = 0$, where $g(t_1, \ldots, t_d) = f(t_1, \ldots, t_d, 1, \ldots, 1)$.

Thus $X \cap A_\sigma$ is a $V$-submanifold if and only if the subvariety $g = 0$ is smooth in $\mathbb{C}^d$.

We need to relate this to the smoothness of $V(f) \cap U_\sigma$. Recall the decomposition $U_\sigma = \mathbb{C}^d \times (\mathbb{C}^*)^{n-d}$. Using the surjection $D \to (\mathbb{C}^*)^{n-d}$ from equation (2), we see that $V(f) \cap U_\sigma$ is smooth if and only if it is smooth at all points of the form $(t_1, \ldots, t_d, 1, \ldots, 1)$. Now comes the key observation.

Lemma 3.6 $V(f) \cap U_\sigma$ is smooth at $t = (t_1, \ldots, t_d, 1, \ldots, 1)$ if and only if one of the partials $f_{z_i}(t)$ is nonzero for some $1 \leq i \leq d$. 


Remark 3.7 This lemma and the previous paragraphs show that \( V(f) \cap U_\sigma \) is smooth if and only if \( X \cap A_\sigma \) is a \( V \)-submanifold. Since the quasi-cone of \( X \) is smooth outside \( Z \) if and only if \( V(f) \cap U_\sigma \) is smooth for all \( d \)-dimensional \( \sigma \), Proposition 3.3 follows from Lemma 3.6.

Proof. Suppose \( f_i(t) = 0 \) for \( 1 \leq i \leq d \). Now take any \( j > d \). Since \( e_1, \ldots, e_d \) is a basis of \( N_Q \), we can write \( e_j = \sum_{i=0}^{d} \phi_i e_i \). By Lemma 3.8 below, there is a constant \( \phi(\beta) \) such that

\[
\phi(\beta)f = z_j \frac{\partial f}{\partial z_j} - \sum_{i=1}^{d} \phi_i z_i \frac{\partial f}{\partial z_i}.
\]

Evaluating this at \( t \in V(f) \) and using \( f_i(t) = 0 \) for \( 1 \leq i \leq d \), we see that \( f_j(t) = 0 \) for all \( 1 \leq j \leq n \). Hence, in order to be smooth at \( t \), at least one of the first \( d \) partials must be nonvanishing.

To complete the proof of Proposition 3.3, we need to prove the following lemma.

Lemma 3.8 Suppose that we have complex numbers \( \phi_1, \ldots, \phi_n \) with the property that \( \sum_{i=0}^{n} \phi_i e_i = 0 \) in \( N_C \). Then, for any class \( \beta \in Cl(\Sigma) \), there is a constant \( \phi(\beta) \) with the property that for any \( D \)-homogeneous polynomial \( f \in S \) of degree \( \beta \), we have

\[
\phi(\beta)f = \sum_{i=1}^{n} \phi_i z_i \frac{\partial f}{\partial z_i}.
\]

Proof. From \( \phi_1, \ldots, \phi_n \), we get a map \( \hat{\phi} : \mathbb{Z}^n \to \mathbb{C} \) defined by \( (a_1, \ldots, a_n) \mapsto \sum_{i=1}^{n} \phi_i a_i \). Furthermore, under the map \( \alpha : M \to \mathbb{Z}^n \), note that \( m \in M \) maps to \( \sum_{i=1}^{n} \phi_i \langle m, e_i \rangle = \langle m, \sum_{i=1}^{n} \phi_i e_i \rangle = 0 \). Thus \( \hat{\phi} \) induces a map \( \hat{\phi} : \mathbb{Z}^n / \alpha(M) \cong Cl(\Sigma) \to \mathbb{C} \).

Note that every monomial \( f = \prod_{i=1}^{n} z_i^{a_i} \) of \( S \) is \( D \)-homogeneous. If we let \( \beta = \deg f = \sum_{i=1}^{n} a_i \deg z_i \), then the desired formula for \( \phi(\beta)f \) follows immediately. By linearity, the formula then holds for all \( D \)-homogeneous polynomials of degree \( \beta \).

In light of this lemma, we make the following definition.

Definition 3.9 The identity \( \phi(\beta)f = \sum_{i=1}^{n} \phi_i z_i f_i \) from Lemma 3.8 is called the Euler formula determined by \( \phi_1, \ldots, \phi_n \).

Remark 3.10 It follows easily from the proof of Lemma 3.8 that the set of all Euler formulas form the vector space \( \text{Hom}(Cl(\Sigma), \mathbb{C}) \). For example, if \( P \) is a weighted projective space, then \( \text{Hom}(Cl(\Sigma), \mathbb{C}) \cong \mathbb{C} \), where a basis is given by the usual Euler formula for homogeneous polynomials. Note also that \( \text{Hom}(Cl(\Sigma), \mathbb{C}) = \text{Lie}(D) \), so that an Euler formula can be regarded as a vector field \( \sum_{i=1}^{n} \phi_i z_i \frac{\partial}{\partial z_i} \) which is tangent to the orbits of \( D \).

We will end this section with a discussion of how quasi-smooth hypersurfaces of \( P \) relate to Danilov’s theory of “toroidal pairs” [7].

Definition 3.11 Let \( W' \) be a subvariety of an algebraic variety \( W \). Then the pair \( (W, W') \) is simplicially toroidal if for every point \( p \in W \), there is a simplicial cone \( \sigma \) such that the germ of \( (W, W') \) at \( p \) is analytically isomorphic to the germ of \( (A_\sigma, D) \) at the origin, where \( A_\sigma \) is the toric variety of \( \sigma \) and \( D \) is an irreducible torus-invariant subvariety of \( A_\sigma \).
For hypersurfaces of \( P \), this concept is equivalent to being quasi-smooth.

**Proposition 3.12** A hypersurface \( X \subset P \) is quasi-smooth if and only if the pair \((P, X)\) is simplicially toroidal.

**Proof.** First suppose we have a pair \((A_\sigma, D)\) as in Definition 3.11. We know that \( A_\sigma \cong C^d/D(\sigma) \), and since the inverse image of \( D \) is a coordinate subspace, we see that \( D \) is a \( V \)-submanifold of \( A_\sigma \). Thus \((P, X)\) simplicially toroidal implies that \( X \) is a \( V \)-submanifold and hence quasi-smooth by Proposition 3.3.

Conversely, if \( X \) is quasi-smooth, then it is a \( V \)-submanifold of \( P \). Thus, given \( p \in X \), there is a local model \((C^d/G, 0) \cong (P, p)\) such that \( G \subset (C^*)^d \) is small and finite, and the inverse image \( Y \subset C^d \) of \( X \subset P \) is smooth at the origin. Then \( Y \) is defined by some equation \( h = 0 \), and since \( Y \) is \( G \)-invariant, there is a character \( \lambda : G \to C^* \) such that \( h(g \cdot t) = \lambda(g) h(t) \) for all \( g \in G \). Since \( Y \) is smooth at the origin, we can also assume that the partial derivative \( h_{z_1}(0) \) is nonvanishing.

Then the map \( \phi : C^d \to C^d \) defined by \((t_1, \ldots, t_d) \mapsto (h(t_1, \ldots, t_d), t_2, \ldots, t_d)\) is a local analytic isomorphism which carries \( Y \) to a coordinate hyperplane. Note that \( \phi \) is equivariant provided that \( g = (g_1, \ldots, g_d) \in G \) acts on the target space via

\[
g \cdot (t_1, \ldots, t_d) = (\lambda(g)t_1, g_2t_2, \ldots, g_d t_d).
\]

This gives a map \( \psi : G \to (C^*)^d \).

We thus have a local model \((C^d/\psi(G), 0)\) where the inverse image of \( X \) is a coordinate hyperplane. Since the torus \((C^*)^d/\psi(G)\) acts on the affine variety \( C^d/\psi(G) \), it follows from Theorem 1.5 of [23] that \( C^d/\psi(G) \) is an affine toric variety. Thus \( C^d/\psi(G) \cong A_\sigma \) for some cone \( \sigma \), and note that \( \sigma \) must be simplicial. Finally, the image in \( C^d/\psi(G) \) of a coordinate hyperplane is an irreducible torus-invariant subvariety. This proves that \((P, X)\) has the appropriate simplicial toroidal local model, and the proposition is proved. \( \square \)

**Remark 3.13** Let \( A_\sigma = \text{Spec } C[\hat{\sigma} \cap M] \) be an affine \( d \)-dimensional toric variety corresponding to a \( d \)-dimensional rational simplicial cone \( \sigma \in \Sigma \), where \( \hat{\sigma} \) is the dual cone in \( M_R \).

There are two important cases where one can explicitly describe local toroidal models for a quasi-smooth hypersurface \( X \) in \( A_\sigma \):

**Case I.** \( X \) has transversal intersections with all orbits \( T_\tau \subset A_\sigma \) of the action of the torus \( T \) on \( A_\sigma \). Then at the point of intersection of \( X \) with a \( 1 \)-dimensional stratum \( T_\tau_0 \) corresponding to a \((d - 1)\)-dimensional face \( \tau_0 \prec \sigma \), the local toroidal model of \( X \) is the \((d - 1)\)-dimensional affine toric variety \( A_{\tau_0} \).

**Case II.** \( X \) contains the single closed \( T_\sigma \)-orbit \( p_\sigma \in A_\sigma \) and is “tangent” to a closure of a \((d - 1)\)-dimensional \( T_\sigma \)-orbit corresponding to a \( 1 \)-dimensional cone \( \rho \prec \sigma \). In this case, the local toroidal model of \( X \) at \( p_\sigma \) is the \((d - 1)\)-dimensional affine toric variety \( A_{\sigma/\rho} \), where \( \sigma/\rho \) is the \((d - 1)\)-dimensional projection of the cone \( \sigma \subset N_R \) to the quotient \( N_R/R\rho \).
4 T-linearized sheaves on P

In this section we will assume that $P$ is a complete toric variety. Recall from [22] the notion of a linearization of a sheaf on an algebraic variety having a regular action of an algebraic group. We will apply this to sheaves on $P$ with its action by the torus $T$.

**Definition 4.1** Let $\mu_t : P \to P$ be the automorphism of $P$ defined by $t \in T$. Then a $T$-linearization of a sheaf $E$ is a family of isomorphisms

$$\phi_t : \mu_t^* E \cong E$$

satisfying the co-cycle condition

$$\phi_{t_1 \cdot t_2} = \phi_{t_2} \circ \mu_{t_2}^* \phi_{t_1}, \text{ for all } t_1, t_2 \in T.$$

**Remark 4.2** If $E$ is a $T$-linearized sheaf on $P$, then for any $T$-invariant open subset $U \subset P$, the group $T$ has the natural linear representation in the space of global sections $H^0(U, E)$. Thus $H^0(U, E)$ splits into a direct sum of subspaces $H^0(U, E)_m$ corresponding to characters $m \in M$ of $T$.

**Definition 4.3** ([19]) Let $E$ be a $T$-linearized sheaf on $P$. Then the polytope

$$\Delta(E) = \text{Conv} \{m \in M : H^0(P, E)_m \neq 0\}$$

is called the support polytope for $E$.

**Definition 4.4** Let $L$ be a $T$-linearized invertible sheaf on $P$, $\sigma \in \Sigma^{(d)}$ a $d$-dimensional cone in $\Sigma$. We denote by $m_\sigma(L)$ the unique element of $M$ with the property that

$$\{m \in M : H^0(X_\sigma, L)_m \neq 0\} = m_\sigma(L) + \bar{\sigma} \cap M,$$

where $A_\sigma \subset P$ is the affine toric variety determined by $\sigma$.

**Remark 4.5** The existence of $m_\sigma(L)$ follows from §6.2 of [9], and it is unique because $\sigma$ is $d$-dimensional. Note also that the mapping $h : N_\mathbb{R} \to \mathbb{R}$ defined by $h(n) = \langle m_\sigma(L), n \rangle$ (for $n \in \sigma$) is the support function for the invertible sheaf $L$.

We should also mention that every invertible sheaf $L$ on $P$ has a $T$-linearization and that any two linearizations of $L$ differ by a homomorphism $T \to \mathbb{C}^*$, i.e., by an element of $M$.

**Proposition 4.6** ([9]) Let $L$ be a $T$-linearized invertible sheaf on $P$. Then the corresponding polyhedron $\Delta = \Delta(L)$ is equal to the intersection

$$\bigcap_{\sigma \in \Sigma^{(d)}} (m_\sigma(L) + \bar{\sigma}).$$
**Definition 4.7** Let $\tau$ be a $k$-dimensional cone in $\Sigma$, and let $\text{St}(\tau)$ the set of all cones $\sigma \in \Sigma$ such that $\tau \prec \sigma$. Consider the $(d-k)$-dimensional fan $\Sigma(\tau)$ consisting of projections of cones $\sigma \in \text{St}(\tau)$ into $\mathbb{N}_R/R\tau$. The corresponding $(d-k)$-dimensional toric subvariety in $\mathbb{P}$ will be denoted by $P_\tau$.

**Remark 4.8** The toric subvariety $P_\tau$ is the closure of a $(d-k)$-dimensional orbit $T_\tau$ of $T$. Any $T_\tau$-linearized sheaf $E$ on $P_\tau$ can be considered as a $T$-linearized sheaf on $P$.

**Definition 4.9** Let $L$ be a $T$-linearized ample invertible sheaf on $P$. For any $k$-dimensional cone in $\tau \in \Sigma$, we denote by $\Delta_\tau$ the face of $\Delta$ of codimension $k$ defined as
\[
\bigcap_{\sigma \in \Sigma, \tau \prec \sigma} (m_\sigma(L) + \sigma \cap \tau^\perp).
\]

The next statement follows immediately from the ampleness criterion for invertible sheaves [9].

**Proposition 4.10** One has the one-to-one correspondence between $(d-k)$-dimensional faces $\Delta_\tau$ of the polytope $\Delta = \Delta(L)$ and $k$-dimensional cones $\tau \in \Sigma$ reversing the face relation. Moreover, the $(d-k)$-dimensional polytope $\Delta_\tau$ is the support polytope for the $T$-linearized sheaf $O_P \otimes L$.

We next study the relation between $H^0(P, L)$ and the coordinate ring $S = \mathbb{C}[z_1, \ldots, z_n]$.

**Lemma 4.11** If $L$ is a $T$-linearized invertible sheaf on $P$, and let $\beta \in \text{Cl}(\Sigma)$ be the class of $L$. Then there is a natural isomorphism
\[
H^0(P, L) \cong S_\beta,
\]
where $S_\beta$ is the graded piece of $S$ corresponding to $\beta$. This isomorphism is determined uniquely up to a nonzero constant in $\mathbb{C}$.

**Proof.** Since $P$ is complete, there is a one-to-one correspondence between $T$-linearized invertible sheaves and $T$-invariant Cartier divisors (see §2.2 of [23]). Thus there is a $T$-invariant Cartier divisor such that $L \cong O_P(D)$ as $T$-linearized sheaves. Note that this isomorphism is unique up to a nonzero constant. However, in [4], it is shown that $D$ determines an isomorphism $H^0(P, O_P(D)) \cong S_\beta$, and the lemma follows immediately. $\square$

**Remark 4.12** If in addition $P$ is simplicial, then the polynomial $f \in S_\beta$ corresponding to a global section of $L$ determines a hypersurface $X \subset P$ as in §3, and one can check that $X$ is exactly the zero section of the global section.

**Definition 4.13** Let $f$ be a global section of an ample invertible sheaf $L$ on $P$. Then the hypersurface $X = \{p \in P : f(p) = 0\}$ is called nondegenerate if for any $\tau \in \Sigma$, the affine hypersurface $X \cap T_\tau$ is a smooth subvariety of codimension 1 in $T_\tau$. 

15
Proof. As observed in [8], the first part of the statement follows from Bertini’s theorem, every nondegenerate hypersurface \( X \) need not be nondegenerate.

Remark 4.16 One should remark that a quasi-smooth hypersurface in a toric variety \( X \) is a quasi-coherent sheaf on \( P \). The nondegeneracy of a global section \( L \) as defined in [3] (where \( \Delta = \Delta(\mathcal{L}) \)) was proved in [9] that \( X \) is simplicially toroidal. Thus, it remains to apply Proposition 3.12.

Remark 4.14 The nondegeneracy of a global section \( f \) is equivalent to \( f \) being \( \Delta \)-regular, as defined in [3] (where \( \Delta = \Delta(\mathcal{L}) \)). For a proof of this, see [20].

Proposition 4.15 Let \( f \) be a generic global section of an ample \( T \)-linearized invertible sheaf \( \mathcal{L} \) on \( P \). Then \( X = \{ p \in P : f(p) = 0 \} \) is a nondegenerate hypersurface. Moreover, every nondegenerate hypersurface \( X \subset P \) is quasi-smooth.

Proof. As observed in [8], the first part of the statement follows from Bertini’s theorem, and it was proved in [3] that \( X \) is simplicially toroidal. Thus, it remains to apply Proposition 3.12.

We will conclude this section by studying the relation between \( T \)-linearized sheaves on \( P \) and graded \( S \)-modules. It is known (see [3]) that every \( Cl(\Sigma) \)-graded \( S \)-module \( F \) determines a quasi-coherent sheaf \( \tilde{F} \) on \( P \). What extra structure on \( F \) is needed in order to induce a \( T \)-linearization on \( \tilde{F} \)? To state the answer, note that \( S \) has a natural grading by \( \mathbb{Z}^n \) which is compatible with its grading by \( Cl(\Sigma) \) via the map \( \mathbb{Z}^n \to Cl(\Sigma) \) from Definition 1.7. Then any \( \mathbb{Z}^n \)-graded module can be regarded as a \( Cl(\Sigma) \)-graded module.

Proposition 4.17 If \( F \) is a \( \mathbb{Z}^n \)-graded \( S \)-module, then the sheaf \( \tilde{F} \) on \( P \) has a natural \( T \)-linearization. Furthermore, if \( P \) is simplicial and \( N \) is generated by \( e_1, \ldots, e_n \), then every \( T \)-linearized quasi-coherent sheaf on \( P \) arises in this way.

Proof. Given \( \sigma \in \Sigma \), let \( S_\sigma \) be the localization of \( S \) at \( \tilde{z}_\sigma = \prod_{i \in \mathbb{Z}^\sigma} z_i \), and let \( (S_\sigma)_0 \) be the elements of degree 0 with respect to \( Cl(\Sigma) \). Then the module \( (F \otimes S_\sigma)_0 \) determines a sheaf on the affine piece \( A_\sigma = \text{Spec}((S_\sigma)_0) \) of \( P \), and, as explained in §3 of [3], these sheaves patch to give \( \tilde{F} \).

Since \( S_\sigma \) and \( F \otimes S_\sigma \) have natural \( \mathbb{Z}^n \) gradings, the groups \( (S_\sigma)_0 \) and \( (F \otimes S_\sigma)_0 \) have natural gradings by \( M \) (which is the kernel of \( \mathbb{Z}^n \to Cl(\Sigma) \)). The \( M \) grading on \( (S_\sigma)_0 \) determines the action of \( T \) on \( A_\sigma \), and the \( M \) grading on \( (F \otimes S_\sigma)_0 \) then determines a \( T \)-linearization. These clearly patch to give a \( T \)-linearization of \( \tilde{F} \).

To prove the final part of the proposition, note that \( Cl(\Sigma) \) is torsion free since \( e_1, \ldots, e_n \) generate \( N \). Thus the map \( \mathbb{Z}^n \to Cl(\Sigma) \) has a left inverse \( \phi : Cl(\Sigma) \to \mathbb{Z}^n \). Now, given \( \alpha \in Cl(\Sigma) \), consider the \( S \)-module \( S(\alpha) \), which has the \( Cl(\Sigma) \) grading \( S(\alpha)_{\beta} = S_{\alpha+\beta} \). We can give \( S(\alpha) \) a grading by \( \mathbb{Z}^n \) where \( S(\alpha)_u = S_{\alpha+\phi(\alpha)} \) for \( u \in \mathbb{Z}^n \) (and we are using the usual \( \mathbb{Z}^n \) grading of \( S \)). By the above, this grading on \( S(\alpha) \) gives a \( T \)-linearized sheaf \( \mathcal{O}_P(\alpha) \).

If \( F \) is a quasi-coherent sheaf on \( P \), then §3 of [3] implies that

\[
F = \bigoplus_{\alpha \in Cl(\Sigma)} H^0(P, F \otimes \mathcal{O}_P(\alpha))
\]

is a \( Cl(\Sigma) \)-graded \( S \)-module whose associated sheaf is \( F \). The module structure on \( F \) comes from the natural isomorphism \( H^0(P, \mathcal{O}_P(\alpha)) \cong S_\alpha \).
Now assume that $F$ has a $T$-linearization. Then each $F \otimes \mathcal{O}_P(\alpha)$ has a natural $T$-linearization, so that $H^0(P, F \otimes \mathcal{O}_P(\alpha))$ has a grading by $M$. Then define a $\mathbb{Z}^n$ grading on $F$ by setting

$$F_u = H^0(P, F \otimes \mathcal{O}_P(\alpha))_{u-\phi(\alpha)},$$

where $u$ maps to $\alpha$ under the map $\mathbb{Z}^n \to Cl(\Sigma)$. Since $\phi$ is a homomorphism, it is easy to check that $F$ becomes a $\mathbb{Z}^n$-graded $S$-module and that this grading induces the given $T$-linearization on $F$.

**Remark 4.18** In Definition 8.1, we describe graded $S$ modules which give the sheaves $\Omega^p_P$. The definition shows that these modules have a canonical $\mathbb{Z}^n$ grading, which gives the usual $T$-linearization on $\Omega^p_P$. Note that we use the $\mathbb{Z}^n$ grading in a crucial way in the proof of Proposition 9.12.

## 5 Local properties of differential forms

Let $\Omega^p_P$ denote the sheaf of $p$-differential forms of Zariski on $P$. This means $\Omega^p_P = j_* \Omega^p_W$, where $W$ is the smooth part of $P$, $\Omega^p_W$ is the usual sheaf of $p$-forms on $W$, and $j : W \to P$ is the natural inclusion. The sheaf $\Omega^p_P$ has a canonical $T$-linearization which induces $M$-graded decompositions of sections of $\Omega^p_P$ over the $T$-invariant the affine open subsets $A_\sigma \subset P$.

**Definition 5.1** Let $m$ be an element of $\bar{\sigma} \cap M$, where $\sigma \in \Sigma$. We denote by $\Gamma(m)$ the $C$-subspace in $M_C$ generated by elements of the minimal face of $\bar{\sigma}$ containing $m$.

**Proposition 5.2** ([4], §4) Let $A_\sigma \subset P$ be the affine open corresponding to a $d$-dimensional cone $\sigma \in \Sigma$. Then the sections over $A_\sigma$ of the $T$-linearized sheaf $\Omega^p_P$ decompose into a direct sum of $M$-homogeneous components as follows:

$$\Omega^p_{A_\sigma} := H^0(A_\sigma, \Omega^p_P) = \bigoplus_{m \in \bar{\sigma} \cap M} \Lambda^p \Gamma(m).$$

Besides $\Omega^p_P$, we also have the sheaves $\Omega^p_P(\log D)$ of differential $p$-forms with logarithmic poles along $D = P \setminus T$ (see §15 of [4]). These sheaves have the weight filtration

$$W : 0 \subset W_0 \Omega^p_P(\log D) \subset W_1 \Omega^p_P(\log D) \subset \cdots \subset W_p \Omega^p_P(\log D) = \Omega^p_P(\log D)$$

defined by

$$W_k \Omega^p_P(\log D) = \Omega^{p-k}_P \wedge \Omega^k_P(\log D).$$

Note in particular that $W_0 \Omega^p_P(\log D) \cong \Omega^p_P$ and $W_p \Omega^p_P(\log D) \cong \mathcal{O}_P \otimes \Lambda^p M$.

We have the following local description of the weight filtration.

**Proposition 5.3** ([4], §15.6) If $\sigma$ is a $d$-dimensional cone in $\Sigma$, then the sections over $A_\sigma$ of the $T$-linearized sheaf $W_k \Omega^p_P(\log D)$ decompose into a direct sum of $M$-homogeneous components as follows:

$$W_k \Omega^p_{A_\sigma}(\log D) := H^0(A_\sigma, W_k \Omega^p_P(\log D)) = \bigoplus_{m \in \bar{\sigma} \cap M} \Lambda^{p-k} \Gamma(m) \wedge \Lambda^k M_C.$$
The successive quotients of the weight filtration are described using the Poincaré residue map. Recall from Definition 4.7 that $\mathbf{P}_\tau$ is the Zariski closure of the $\mathbf{T}$-orbit of $\mathbf{P}$ corresponding to $\tau \in \Sigma$.

**Theorem 5.4** ([3], §15.7) For any integer $k$ ($0 \leq k \leq p$), there is a short exact sequence

$$0 \rightarrow W_{k-1}\Omega^p_\mathbf{P}(\log D) \rightarrow W_k\Omega^p_\mathbf{P}(\log D) \xrightarrow{\text{Res}} \bigoplus_{\text{dim } \tau = k} \Omega^p_{\mathbf{P}_\tau} \rightarrow 0$$

where "dim $\tau = k$" means the sum is over all $k$-dimensional cones in $\tau \in \Sigma$, and $\text{Res}$ is the Poincaré residue map.

This short exact sequence has a natural $\mathbf{T}$-action and splits into $M$-homogeneous components. Let’s examine what happens over an affine toric chart $\mathbf{A}_\sigma$, where $\sigma$ is a $d$-dimensional cone in $\Sigma$. Assume that the generators of $\sigma$ are $\{e_1, \ldots, e_d\}$. Then, for any subset $\{i_1, \ldots, i_k\} \subset \{1, \ldots, d\}$, we denote by $\mathbf{A}_{i_1 \ldots i_k}$ the closed affine subvariety in $\mathbf{A}_\sigma$ corresponding to the cone $\tau_{i_1 \ldots i_k} = \mathbb{R}_{\geq 0}e_{i_1} + \cdots + \mathbb{R}_{\geq 0}e_{i_k}$. By §15.7 of [3], we get the following local description of the residue map:

**Proposition 5.5** Given $m \in \hat{\sigma} \cap M$, let $\omega_m$ be an element of $\Lambda^{p-k}\Gamma(m)$, and let $\omega'$ be an element of $\Lambda^k\mathbb{C}$. Then the image of the $m$-homogeneous element $\omega_m \wedge \omega' \in W_k\Omega^p_{\mathbf{A}_\sigma}(\log D)$ under the residue map

$$\text{Res} : W_k\Omega^p_{\mathbf{A}_\sigma}(\log D) \rightarrow \bigoplus_{1 \leq i_1 < \cdots < i_k \leq d} \Omega^p_{\mathbf{A}_{i_1 \ldots i_k}}$$

is given by

$$\text{Res}(\omega_m \wedge \omega')_{i_1 \ldots i_k} = \omega'(e_{i_1}, \ldots, e_{i_k}) \cdot \omega_m \in \Lambda^{p-k}\Gamma(m).$$

**6 Globalization of the Poincaré residue map**

Let $\mathcal{L}$ be an ample $\mathbf{T}$-linearized invertible sheaf on the complete toric variety $\mathbf{P}$, and let $\Delta = \Delta(\mathcal{L})$ be as in Definition 4.3. Tensoring by $\mathcal{L}$ the short exact sequence in Theorem 5.4, we obtain the exact sequence

$$0 \rightarrow W_{k-1}\Omega^p_\mathbf{P}(\log D) \otimes \mathcal{L} \rightarrow W_k\Omega^p_\mathbf{P}(\log D) \otimes \mathcal{L} \xrightarrow{\text{Res}} \bigoplus_{\text{dim } \tau = k} \Omega^p_{\mathbf{P}_\tau} \otimes \mathcal{L} \rightarrow 0.$$

The goal of this section is to give an explicit description of the map of spaces of global sections

$$\gamma : H^0(\mathbf{P}, W_k\Omega^p_\mathbf{P}(\log D) \otimes \mathcal{L}) \rightarrow \bigoplus_{\text{dim } \tau = k} H^0(\mathbf{P}, \Omega^p_{\mathbf{P}_\tau} \otimes \mathcal{L}) \quad (8)$$

induced by the Poincaré residue map.

**Definition 6.1** Let $\mathcal{L}$ be a $\mathbf{T}$-linearized ample invertible sheaf, which determines the convex polytope in $\Delta = \Delta(\mathcal{L}) \subset M_{\mathbb{R}}$. For any $m \in M$, we denote by $\Gamma_\Delta(m)$ the $\mathbb{C}$-subspace in $M_{\mathbb{C}}$ generated by all vectors $s - s'$, where $s, s' \in \Delta_m$, and $\Delta_m$ is the minimal face of $\Delta$ containing $m$. 

18
First we notice the following properties:

**Proposition 6.2** The space of global sections of $T$-linearized sheaf $\Omega^p_P(\log D) \otimes L$ decomposes into a direct sum of $M$-homogeneous components as follows:

$$H^0(P, W_k \Omega^p_P(\log D) \otimes L) = \bigoplus_{m \in \Delta \cap M} \Lambda^{p-k} \Gamma_{\Delta}(m) \wedge \Lambda^k M_C$$

**Proof.** The statement follows from Proposition 5.3 and Proposition 4.6. $$\square$$

**Proposition 6.3** Let $\tau$ be a $k$-dimensional cone in $\Sigma$, $\Delta_\tau$ the corresponding face of $\Delta$ of codimension $k$ (see Proposition 4.10). Then the space of global sections of $T$-linearized sheaf $\Omega^p_{P\tau} \otimes L$ decomposes into a direct sum of $M$-homogeneous components as follows:

$$H^0(P, \Omega^{p-k}_{P\tau} \otimes L) = \bigoplus_{m \in \Delta_\tau \cap M} \Lambda^{p-k} \Gamma_{\Delta_\tau}(m)$$

**Proof.** The statement follows from Proposition 5.2 and Proposition 4.6. $$\square$$

The linear mapping $\gamma$ of (8) is the direct sum $\bigoplus_{\dim \tau = k} \gamma_\tau$, where

$$\gamma_\tau : H^0(P, W_k \Omega^p_P(\log D) \otimes L) \to H^0(P, \Omega^{p-k}_{P\tau} \otimes L).$$

By Proposition 5.5, we can then describe $\gamma_\tau$ as follows.

**Proposition 6.4** Let $\omega_m$ be an element of $\Lambda^{p-k} \Gamma_{\Delta}(m)$ and $\omega'$ be an element of $\Lambda^k M_C$. Thus $\omega_m \wedge \omega'$ is an $m$-homogeneous element in $H^0(P, W_k \Omega^p_P(\log D) \otimes L)$. Choose a $k$-dimensional cone $\tau$ with generators $e_{i_1}, \ldots, e_{i_k}$. Then

1. $\gamma_\tau(\omega_m \wedge \omega') = 0$ if $m \notin \Delta_\tau$;
2. $\gamma_\tau(\omega_m \wedge \omega') = \omega'(e_{i_1}, \ldots, e_{i_k}) \cdot \omega_m$ if $m \in \Delta_\tau$.

### 7 A generalized theorem of Bott-Steenbrink-Danilov

In Theorem 7.5.2 of [9], Danilov formulated without proof the following vanishing theorem generalizing for complete toric varieties the well-known theorem of Bott and Steenbrink:

**Theorem 7.1** Let $\mathcal{L}$ be an ample invertible sheaf on $P$. Then for any $p \geq 0$ and $i > 0$, one has

$$H^i(P, \Omega^p_P \otimes \mathcal{L}) = 0.$$

We prove now for simplicial toric varieties a more general vanishing theorem:

**Theorem 7.2** Let $\mathcal{L}$ be an ample invertible sheaf on a complete simplicial toric variety $P$. Then for any $p \geq 0$, $k \geq 0$ and $i > 0$, one has

$$H^i(P, W_k \Omega^p_P(\log D) \otimes \mathcal{L}) = 0.$$
Proof. We prove this theorem using induction on $p - k$. For $p - k = 0$, the sheaf $W_{p} \Omega_{\mathbb{P}}^{p}(\log D) \otimes \mathcal{L} = \Omega_{\mathbb{P}}^{p}(\log D) \otimes \mathcal{L} = \mathcal{O}_{\mathbb{P}}^{d} \otimes \mathcal{L}$ is the direct sum of $\binom{d}{p}$ copies of $\mathcal{L}$. Thus, the vanishing property for $W_{p} \Omega_{\mathbb{P}}^{p}(\log D) \otimes \mathcal{L}$ is implied by the following general vanishing property for the ample invertible sheaf $\mathcal{L}$.

Proposition 7.3 ([1], §7.3) Let $\mathcal{L}$ be an ample invertible sheaf on a complete toric variety $\mathbb{P}$. Then

$$H^{i}(\mathbb{P}, \mathcal{L}) = 0 \text{ for } i > 0.$$  

On the other hand, for any $k$ ($0 \leq k \leq p$) we can apply the induction assumption to $W_{k} \Omega_{\mathbb{P}}^{p}(\log D) \otimes \mathcal{L}$ and $\Omega_{\mathbb{P}}^{p}$ appearing in the short exact sequence

$$0 \to W_{k-1} \Omega_{\mathbb{P}}^{p}(\log D) \otimes \mathcal{L} \to W_{k} \Omega_{\mathbb{P}}^{p}(\log D) \otimes \mathcal{L} \to \mathcal{L} \otimes \bigoplus_{\dim \tau = k} \Omega_{\mathbb{P}}^{p-k} \otimes \mathcal{L} \to 0.$$  

The required vanishing properties of $W_{k-1} \Omega_{\mathbb{P}}^{p}(\log D) \otimes \mathcal{L}$ follows now from the following lemma.

Lemma 7.4 The mapping

$$\gamma : H^{0}(\mathbb{P}, W_{k} \Omega_{\mathbb{P}}^{p}(\log D) \otimes \mathcal{L}) \to \bigoplus_{\dim \tau = k} H^{0}(\mathbb{P}, \Omega_{\mathbb{P}}^{p-k} \otimes \mathcal{L})$$  

is surjective.

Proof. Since $\mathcal{L}$ is ample, there exists a one-to-one correspondence between $i$-dimensional cones of $\Sigma$ and $(d-i)$-dimensional faces of the convex polytope $\Delta = \Delta(\mathcal{L})$ (see Proposition 4.10). Choose a $k$-dimensional cone $\tau_{0} \in \Sigma$. We know from Proposition 6.3 that the $m$-homogeneous component of $H^{0}(\mathbb{P}, \Omega_{\tau_{0}}^{p-k} \otimes \mathcal{L})$ is non-zero only if $m \in \Delta_{\tau_{0}}$, and when the latter holds, Definition 6.1 shows that the $m$-component is determined by the minimal face $\Delta_{\tau}$ of $\Delta_{\tau_{0}}$ containing $m$. This means that $\tau_{1} \in \Sigma$ is a cone of dimension $c \geq k$ containing $\tau_{0}$.

Fix such a lattice point $m$ and the corresponding $\tau_{0} \subset \tau_{1}$. We can assume that $\tau_{1}$ is a face of a $d$-dimensional cone $\sigma \in \Sigma$ with generators $e_{1}, \ldots, e_{d} \in N$ such that $e_{1}, \ldots, e_{c}$ are generators of $\tau_{1}$ and $e_{1}, \ldots, e_{k}$ are generators of $\tau_{0}$. To describe $\Gamma_{\Delta_{\tau_{0}}}(m)$, suppose that $\Delta$ is defined by inequalities $\langle m', e_{i} \rangle \geq -a_{i}$ for $1 \leq i \leq n$. Then $\Delta_{\tau_{1}} \subset \Delta$ is the face obtained by requiring $\langle m', e_{i} \rangle = -a_{i}$ for $1 \leq i \leq c$, which implies that the subspace $\Gamma_{\Delta_{\tau_{0}}}(m) \subset M_{\mathbb{C}}$ is defined $\langle m', e_{i} \rangle = 0$ for $1 \leq i \leq c$.

Now let $h_{1}, \ldots, h_{d} \in M_{\mathbb{Q}}$ form the dual basis to the basis $e_{1}, \ldots, e_{d}$ of $N_{\mathbb{Q}}$. It follows immediately that $h_{c+1}, \ldots, h_{d}$ form a basis of $\Gamma_{\Delta_{\tau_{0}}}(m)$. Thus, by Proposition 6.3, the $m$-homogeneous component of $H^{0}(\mathbb{P}, \Omega_{\tau_{0}}^{p-k} \otimes \mathcal{L})$ has a basis consisting of $(p - k)$-vectors

$$\omega_{j_{1} \ldots j_{p-k}} = h_{j_{1}} \wedge \cdots \wedge h_{j_{p-k}}$$  

where $\{j_{1}, \ldots, j_{p-k}\}$ is a subset of $\{c + 1, \ldots, d\}$. For any such a $(p - k)$-vector $\omega_{j_{1} \ldots j_{p-k}}$, the $p$-vector

$$h_{1} \wedge \cdots \wedge h_{k} \wedge \omega_{j_{1} \ldots j_{p-k}}$$
defines an element in the $m$-homogeneous component of the space $H^0(P, W_k \Omega^p_P(\log D) \otimes L)$ by Proposition 6.2. Furthermore, Proposition 6.4 shows that

$$\gamma_{\tau_0}(h_1 \wedge \cdots \wedge h_k \wedge \omega_{j_1 \cdots j_{p-k}}) = h_1 \wedge \cdots \wedge h_k (e_1 \wedge \cdots \wedge e_k) \cdot \omega_{j_1 \cdots j_{p-k}} = \omega_{j_1 \cdots j_{p-k}}.$$}

Thus, to prove the lemma, it suffices to show that for $k$-dimensional cones $\tau \in \Sigma$, we have

$$\gamma_{\tau}(h_1 \wedge \cdots \wedge h_k \wedge \omega_{j_1 \cdots j_{p-k}}) = 0 \text{ for } \tau \neq \tau_0.$$

However, by Proposition 6.4, $\gamma_{\tau}(h_1 \wedge \cdots \wedge h_k \wedge \omega_{j_1 \cdots j_{p-k}}) = 0$ for $k$-dimensional cones $\tau \in \Sigma$ such that $m \not\in \Delta_\tau$. Since $\Delta_{\tau_1}$ is the minimal face containing $m$, the condition $m \not\in \Delta_\tau$ holds whenever $\tau$ is not a face of $\tau_1$. It remains to see what happens when $\tau$ is a $k$-dimensional face of $\tau_1$. In this case, the generators of $\tau$ are $e_{i_1}, \ldots, e_{i_k}$, where $\{i_1, \ldots, i_k\} \subset \{1, \ldots, c\}$. Since the $e_i$ are dual to the $h_i$, the value $h_1 \wedge \cdots \wedge h_k (e_{i_1}, \ldots, e_{i_k})$ is nonzero only if $\tau = \tau_0$.

From Proposition 6.4, it follows that $\gamma_{\tau}(h_1 \wedge \cdots \wedge h_k \wedge \omega_{j_1 \cdots j_{p-k}}) = 0$ when $\tau \neq \tau_0$. \hfill \Box

### 8 Differential forms and graded $S$-modules

As usual, $\Omega^p_P$ denotes the sheaf of Zariski differential $p$-forms on a complete simplicial toric variety $P = P_\Sigma$. We will study these sheaves using certain graded modules over the polynomial ring $S = C[z_1, \ldots, z_n]$, which is graded by $Cl(S)$. Given a Cartier divisor $X \subset P$, our goal is to describe $H^0(P, \Omega^p_P(X))$ in terms of $S$.

Recall that the fan $\Sigma$ lies in $N_R \cong R^d$ and that $M = \text{Hom}(N, Z)$. Also recall that $e_1, \ldots, e_n$ are the generators of the 1-dimensional cones of $\Sigma$.

**Definition 8.1** Given $p$ between 0 and $d$, define $\tilde{\Omega}^p_S$ by the exact sequence of graded $S$-modules:

$$0 \to \tilde{\Omega}^p_S \to S \otimes \Lambda^p M \xrightarrow{\gamma} \bigoplus_{i=1}^n (S/z_iS) \otimes \Lambda^{p-1}(M \cap e_i^1),$$

where the $i$th component of $\gamma$ is $\gamma_i(g \otimes \omega) = g \mod z_i \otimes \langle e_i, \omega \rangle$. A careful description of the interior product $\langle e_i, w \rangle \in \Lambda^{p-1}(M \cap e_i^1)$ may be found in §3.2 of [23].

By [2], we know that every finitely generated graded $S$-module $F$ gives a coherent sheaf $\tilde{F}$ on $P$.

**Lemma 8.2** The sheaf $\tilde{\Omega}^p_S$ on $P$ associated to the graded $S$-module $\tilde{\Omega}^p_S$ is naturally isomorphic to $\Omega^p_P$.

**Proof.** From §3 of [2], we get $\tilde{\Sigma} = \mathcal{O}_P$. The primitive element $e_i$ generates the cone $\rho_i \in \Sigma$, which by Definition 4.7 gives the $T$-invariant divisor $D_i = P_{\rho_i} \subset P$. By the example before Corollary 3.8 of [2], the ideal $z_iS$ gives the ideal sheaf of $D_i$. Since $F \mapsto \tilde{F}$ is exact (see Proposition 3.1 of [2]), it follows that $S/z_iS$ gives the sheaf $\mathcal{O}_{D_i}$. Then the exactness of $F \mapsto \tilde{F}$, applied to the sequence defining $\tilde{\Omega}^p_S$, gives the exact sequence of sheaves:

$$0 \to \tilde{\Omega}^p_S \to \mathcal{O}_P \otimes \Lambda^p M \xrightarrow{\gamma} \bigoplus_{i=1}^n \mathcal{O}_{D_i} \otimes \Lambda^{p-1}(M \cap e_i^1).$$
By Theorem 3.6 of [23], we can identify $\hat{\Omega}_S^p$ with $\Omega_P^p$. 

We next discuss the twists of the $\Omega_P^p$. First recall that if $\beta \in Cl(\Sigma)$ and $F$ is a graded $S$-module, then $F(\beta)$ is the graded $S$-module defined by $F(\beta)\gamma = F_{\beta+\gamma}$ for $\gamma \in Cl(\Sigma)$.

**Definition 8.3** Given $\beta \in Cl(\Sigma)$, the sheaf on $P$ associated to the graded $S$-module $\hat{\Omega}_S^p(\beta)$ is denoted $\Omega_P^p(\beta)$.

**Remark 8.4** If $\mathcal{O}_P(\beta)$ is the sheaf associated to $S(\beta)$, then there is a natural isomorphism $\Omega_P^p(\beta) \cong \Omega_P^p \otimes \mathcal{O}_P(\beta)$ whenever $\beta$ is the class of a Cartier divisor. However, when $\beta$ is not Cartier, the sheaves $\Omega_P^p(\beta)$ and $\Omega_P^p \otimes \mathcal{O}_P(\beta)$ may be nonisomorphic.

**Proposition 8.5** For any divisor class $\beta \in Cl(\Sigma)$, there is a natural isomorphism

$$H^0(P, \Omega_P^p(\beta)) \cong (\hat{\Omega}_S^p)_\beta.$$ 

**Proof.** The sequence in Definition 8.1 remains exact after shifting by $\beta$. Then, taking the associated sheaves on $P$, we get an exact sequence

$$0 \to \Omega_P^p(\beta) \to \mathcal{O}_P(\beta) \otimes \Lambda^p M \xrightarrow{\gamma} \bigoplus_{i=1}^n \mathcal{O}_{D_i}(\beta) \otimes \Lambda^{p-1}(M \cap e_i^\perp).$$

Since taking global sections is left exact, we get the exact sequence

$$0 \to H^0(P, \Omega_S^p(\beta)) \to H^0(P, \mathcal{O}_P(\beta)) \otimes \Lambda^p M \xrightarrow{\gamma} \bigoplus_{i=1}^n H^0(P, \mathcal{O}_{D_i}(\beta)) \otimes \Lambda^{p-1}(M \cap e_i^\perp).$$

Using the natural isomorphism $H^0(P, \mathcal{O}_P(\beta)) \cong S_\beta$ from Proposition 1.1 of [3], we get

$$0 \to H^0(P, \Omega_S^p(\beta)) \to S_\beta \otimes \Lambda^p M \xrightarrow{\gamma} \bigoplus_{i=1}^n H^0(P, \mathcal{O}_{D_i}(\beta)) \otimes \Lambda^{p-1}(M \cap e_i^\perp). \quad (9)$$

However, since $z_i$ vanishes on $D_i$, the map $\gamma$ factors

$$S_\beta \otimes \Lambda^p M \to \bigoplus_{i=1}^n (S/z_i S)_\beta \otimes \Lambda^{p-1}(M \cap e_i^\perp) \to \bigoplus_{i=1}^n H^0(P, \mathcal{O}_{D_i}(\beta)) \otimes \Lambda^{p-1}(M \cap e_i^\perp).$$

Assume for the moment that $(S/z_i S)_\beta \to H^0(P, \mathcal{O}_{D_i}(\beta))$ is injective. Then [3] gives the exact sequence

$$0 \to H^0(P, \Omega_S^p(\beta)) \to S_\beta \otimes \Lambda^p M \xrightarrow{\gamma} \bigoplus_{i=1}^n (S/z_i S)_\beta \otimes \Lambda^{p-1}(M \cap e_i^\perp),$$

and the isomorphism $H^0(P, \Omega_S^p(\beta)) \cong (\hat{\Omega}_S^p)_\beta$ follows immediately from Definition 8.1.

It remains to show that the natural map $(S/z_i S)_\beta \to H^0(P, \mathcal{O}_{D_i}(\beta))$ is injective. Let $\sigma$ be a cone of $\Sigma$ containing $e_i$, and let $A_\sigma \subset P$ be the corresponding affine toric variety.
Then it suffices to show that the \((S/z_iS)_\beta \rightarrow H^0(A,\mathcal{O}_{D_i}(\beta))\) is injective. However, as explained in §3 of [1], we have

\[ H^0(A,\mathcal{O}_{D_i}(\beta)) \cong ((S/z_iS)(\beta) \otimes S)_{\beta} = ((S/z_iS) \otimes S)_{\beta}, \]

where \(S_{\sigma}\) is the localization of \(S\) at \(\tilde{z}_{\sigma} = \prod_{\alpha \neq \sigma} z_j\). Thus we need to show that

\[ (S/z_iS)_\beta \rightarrow ((S/z_iS) \otimes S)_{\beta} \]

is injective. This follows easily since \(z_i\) doesn’t divide \(\tilde{z}_{\sigma}\), and the proposition is proved. \(\square\)

We next want to relate \(\hat{\mathcal{O}}_S^p\) to the usual module \(\mathcal{O}_S^p\) of \(p\)-forms in \(dz_1, \ldots, dz_n\). First note that \(\mathcal{O}_S\) can be given the structure of a graded \(S\)-algebra by declaring that \(dz_i\) and \(z_i\) have the same degree in \(\text{Cl}(\Sigma)\). This means that if \(z_i \in S_{\beta_i}\), then we have an isomorphism of graded \(S\)-modules

\[ \mathcal{O}_S^p \cong \bigoplus_{1 \leq i_1 < \cdots < i_p \leq n} S(-\beta_{i_1} - \cdots - \beta_{i_p}). \]

**Lemma 8.6** There is a natural inclusion \(\hat{\mathcal{O}}_S^p \subset \mathcal{O}_S^p\) of graded \(S\)-modules.

**Proof.** Consider the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & \hat{\mathcal{O}}_S^p \\
\downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{O}_S^p
\end{array}
\]

\[
\begin{array}{ccc}
S \otimes \Lambda^p M & \rightarrow & \bigoplus_{i=1}^n (S/z_iS) \otimes \Lambda^{p-1} M \\
\delta & \rightarrow & \bigoplus_{i=1}^n (S/z_iS) \otimes \Lambda^{p-1} \mathbb{Z}^n
\end{array}
\]

The first row is exact by the definition of \(\hat{\mathcal{O}}_S^p\) and the inclusion \(M \cap e_i^+ \subset M\). To understand the second row, let \(h_1, \ldots, h_n\) be the standard basis of \(\mathbb{Z}^n\). Then the map \(\mathcal{O}_S^p \rightarrow S \otimes \Lambda^p \mathbb{Z}^n\) is defined by

\[ dz_{i_1} \wedge \cdots \wedge dz_{i_p} \mapsto z_{i_1} \cdots z_{i_p} \otimes h_{i_1} \wedge \cdots \wedge h_{i_p}, \]

and for \(g \otimes \omega \in S \otimes \Lambda^p \mathbb{Z}^n\), the \(i\)th component of \(\delta(g \otimes \omega)\) is \(\delta_i(g \otimes \omega) = g \mod z_i \otimes \langle h_i^*, \omega \rangle\), where \(h_1^*, \ldots, h_n^*\) is the dual basis to \(h_1, \ldots, h_n\). It is easy to check that the sequence on the bottom is exact.

The vertical maps in (10) are induced by the map \(\alpha : M \rightarrow \mathbb{Z}^n\) from Definition [4], and since the dual \(\alpha^* : \mathbb{Z}^n \rightarrow N\) maps \(h_i^*\) to \(e_i\), it follows that diagram (10) commutes. This gives the desired inclusion. \(\square\)

The final step is to relate \(\hat{\mathcal{O}}_S^p\) to rational \(p\)-forms on \(P\) with poles on a hypersurface \(X \subset P\). We will assume that \(X\) is defined by \(f = 0\) for some \(f \in S_\beta\). Thus \(\beta \in \text{Cl}(\Sigma)\) is the class of \(X\), and we will also assume that \(X\) is a Cartier divisor. Then \(\mathcal{O}_P^p(X) = \mathcal{O}_P^p \otimes \mathcal{O}_P(X)\). Recall that local sections of \(\mathcal{O}_P^p(X)\) are rational \(p\)-forms which become holomorphic when multiplied by the local equation of \(X\).

**Proposition 8.7** If \(X\) is a Cartier divisor on \(P\) defined by \(f = 0\) for \(f \in S_\beta\), then

\[ H^0(P, \mathcal{O}_P^p(X)) = \left\{ \frac{\omega}{f} : \omega \in (\hat{\mathcal{O}}_S^p)_\beta \right\}. \]
Proof. We first observe that multiplication by $1/f$ gives an isomorphism $\mathcal{O}_P(\beta) \cong \mathcal{O}_P(X)$. To see this, we will work locally on an affine open $A_\sigma \subset P$, where $\sigma \in \Sigma$. Recall that $A_\sigma = \text{Spec}(S_\sigma)_0$, where $S_\sigma$ is the localization of $S$ at $\bar{z}_\sigma = \prod_{e_i \in \beta} z_i$. Since $X$ is Cartier, Lemma 3.4 of [5] shows that there is a monomial $z^D \in S_\beta$ which is invertible in $S_\sigma$. It follows that $f/z^D = 0$ is the local equation of $X$ on $A_\sigma$. Hence

$$H^0(A_\sigma, \mathcal{O}_P(X)) = \frac{1}{f/z^D} \cdot H^0(A_\sigma, \mathcal{O}_P) = \frac{1}{f/z^D} \cdot (S_\sigma)_0 = \frac{1}{f} \cdot (S_\sigma)_0,$$

so that $(1/f)S$ is the graded $S$-module that gives $\mathcal{O}_P(X)$.

Thus $1/f : S(\beta) \to (1/f)S$ is a graded isomorphism which induces $\mathcal{O}_P(\beta) \cong \mathcal{O}_P(X)$. This proves that we have an isomorphism $\Omega^P_\beta(\beta) \cong \Omega^P_\beta(X)$ which is given by multiplication by $1/f$ when we represent each sheaf by a graded $S$-module. Then the desired result follows immediately from Proposition 8.3.

Remark 8.8 There is a direct way of seeing that $\omega/f$ descends to $p$-form on $P$. In diagram (11), we can think of $S \otimes \Lambda^p Z^n$ as $\Omega^S_\beta(\log D)$, where $D$ is the union of the coordinate hyperplanes. With this interpretation, the standard basis of $Z^n$ is $dz_i/z_i$, and the map $\Omega^S_\beta \to S \otimes \Lambda^p Z^n$ is given by $dz_i \mapsto z_i \otimes dz_i/z_i$. Now fix a basis $m_1, \ldots, m_d$ of $M$ and let

$$t_j = \prod_{i=1}^n z_i^{(m_j, e_i)},$$

for $1 \leq j \leq d$. The $t_j$ are invariant under the group $D = D(\Sigma)$ and hence descend to rational functions on $P$ (in fact, they are coordinates for the torus $T \subset P$). Note that $dt_j/t_j = \sum_{i=1}^n (m_j, e_i)dz_i/z_i = \alpha(m_j)$, where $\alpha : M \to Z^n$ is the map from Definition 1.7. Then, in $S \otimes \Lambda^p Z^n$, we can write

$$\omega = \sum_{j_1 < \cdots < j_d} g_{j_1 \cdots j_d} \otimes \alpha(m_{j_1}) \wedge \cdots \wedge \alpha(m_{j_d})$$

$$= \sum_{j_1 < \cdots < j_d} g_{j_1 \cdots j_d} \otimes dt_{j_1}/t_{j_1} \wedge \cdots \wedge dt_{j_d}/t_{j_d},$$

where $g_{j_1 \cdots j_d} \in S_\beta$. Thus $\omega/f$ can be written

$$\frac{\omega}{f} = \sum_{j_1 < \cdots < j_d} g_{j_1 \cdots j_d}/f \otimes dt_{j_1}/t_{j_1} \wedge \cdots \wedge dt_{j_d}/t_{j_d}.$$

Since $g_{j_1 \cdots j_d}$ and $f$ have the same degree, $\omega/f$ descends to a rational $p$-form on $P$.

9 Differential forms of degree $d$ and $d - 1$

In this section, we will find module generators for $\Omega^d_\beta$ and $\hat{\Omega}^{d-1}_\beta$, where $d$ is the dimension of the complete simplicial toric variety $P$. This will enable us to give an explicit description of $H^0(P, \Omega^P_\beta(X))$ for $p = d$ and $d - 1$. 

24
**Definition 9.1** Fix an integer basis \( m_1, \ldots, m_d \) for the lattice \( M \). Then, given a subset \( I = \{ i_1, \ldots, i_d \} \subset \{ 1, \ldots, n \} \) consisting of \( d \) elements, define
\[
\det(e_I) = \det(\langle m_j, e_{i_k} \rangle_{1 \leq j, k \leq d}).
\]
We also define \( dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_d} \) and \( \hat{z}_I = \prod_{i \in I} z_i \).

**Remark 9.2** Although \( \det(e_I) \) and \( dz_I \) depend on how the elements of \( I \) are ordered, their product \( \det(e_I)dz_I \) does not.

**Definition 9.3** We define the \( d \)-form \( \Omega_0 \in \Omega^d_S \) by the formula
\[
\Omega_0 = \sum_{|I|=d} \det(e_I)\hat{z}_I dz_I,
\]
where the sum is over all \( d \) element subsets \( I \subset \{ 1, \ldots, n \} \).

**Example 9.4** Suppose that \( P \) is the weighted projective space \( P(w_1, \ldots, w_{d+1}) \). Then one can check that up to a nonzero constant (which depends on the basis of \( M \) chosen in Definition 9.1), we have
\[
\hat{\Omega}_0 = \sum_{i=1}^{d+1} (-1)^i w_i z_1 \wedge \cdots \wedge \hat{d}z_i \wedge \cdots \wedge dz_{d+1}.
\]
This form appears in 2.1.3 of [13], where it is denoted \( \Delta(dT_0 \wedge \cdots \wedge dT_r) \). Also, when \( P = P^d \), we have \( w_i = 1 \) for all \( i \), and we recover the form \( \Omega \) in Corollary 2.11 of [14].

**Proposition 9.5** \( \hat{\Omega}_0^d \subset \Omega^d_S \) is a free \( S \)-module of rank 1 generated by \( \Omega_0 \in \Omega^d_S \).

**Proof.** Using the basis \( m_1, \ldots, m_d \) of \( M \), we see that \( S \otimes \Lambda^d M \) is free of rank 1 with generator \( m_1 \wedge \cdots \wedge m_d \). Furthermore, since \( \langle e_i, m_1 \wedge \cdots \wedge m_d \rangle \) is nonzero in the rank 1 \( \mathbb{Z} \)-module \( \Lambda^{d-1}(M \cap e_i^+) \), the definition of \( \hat{\Omega}_S^d \) shows that for \( A \in S \), \( Am_1 \wedge \cdots \wedge m_d \) lies in \( \hat{\Omega}_S^d \) if and only if \( A \) is divisible by \( z_1 \cdots z_n \). Thus \( \hat{\Omega}_S^d \subset S \otimes \Lambda^d M \) is free of rank 1 with \( z_1 \cdots z_n m_1 \wedge \cdots \wedge m_d \) as generator.

As in Remark 8.8, we will denote the standard basis of \( \mathbb{Z}^n \) by \( dz_1/z_1, \ldots, dz_n/z_n \). Then, using the map \( \alpha : M \to \mathbb{Z}^n \), we have \( \alpha(m) = \sum_{i=1}^n \langle m, e_i \rangle dz_i/z_i \). Thus, inside \( S \otimes \Lambda^d \mathbb{Z}^n \), the generator \( z_1 \cdots z_n m_1 \wedge \cdots \wedge m_d \) of \( \hat{\Omega}_S^d \) is
\[
z_1 \cdots z_n \left( \sum_{i=1}^n \langle m_1, e_i \rangle \frac{dz_i}{z_i} \right) \wedge \cdots \wedge \left( \sum_{i=1}^n \langle m_d, e_i \rangle \frac{dz_i}{z_i} \right) = z_1 \cdots z_n \sum_{|I|=d} \det(e_I)(\prod_{i \in I} z_i^{-1}) dz_I
\]
\[
= \sum_{|I|=d} \det(e_I)\hat{z}_I dz_I = \Omega_0,
\]
and the proposition is proved. \( \square \)
Remark 9.6 If \( z_i \in S_{\beta_i} \) for \( 1 \leq i \leq n \), then we set \( \beta_0 = \sum_{i=1}^{n} \beta_i \). Since \( \Omega_0 \) has degree \( \beta_0 \), the above proposition gives an isomorphism of graded \( S \)-modules

\[
\hat{\Omega}^d_S \cong S(-\beta_0).
\]

Furthermore, since \( \beta_0 \in Cl(\Sigma) \) is the class of \( P \setminus T = \sum_{i=1}^{n} D_i \), we get the well-known isomorphism of sheaves

\[
\Omega^d_P \cong \mathcal{O}_P(-\sum_{i=1}^{n} D_i).
\]

If we combine Propositions 9.5 and 8.7, we get the following way of representing \( d \)-forms on \( P \) with poles on \( X \).

Theorem 9.7 Let \( X \subset P \) be a Cartier divisor defined by \( f = 0 \), where \( f \in S_{\beta} \). Then

\[
H^0(P, \Omega^d_{P}(X)) = \left\{ \frac{A\Omega_0}{f} : A \in S_{\beta - \beta_0} \right\},
\]

where \( \beta_0 = \sum_{i=1}^{n} \beta_i \) and \( z_i \in S_{\beta_i} \).

We next describe generators for \( \hat{\Omega}^{d-1}_S \).

Definition 9.8 Given \( i \) between 1 and \( n \), we define the \((d-1)\)-form \( \Omega_i \in \Omega^{d-1}_S \) by the formula

\[
\Omega_i = \sum_{\substack{|J|=d-1 \\setminus \{i\} \in J \\setminus \{i\}}} \det(e_{J\cup\{i\}}) \hat{z}_{J\cup\{i\}} dz_J,
\]

where \( \det(e_{J\cup\{i\}}) \) is computed by ordering the elements of \( J \cup \{i\} \) so that \( i \) is first (this ensures that \( \det(e_{J\cup\{i\}})dz_J \) is well-defined).

Example 9.9 When \( P = P^1 \times P^1 \), we have \( e_1 = (1,0) \), \( e_2 = -e_1 \), \( e_3 = (1,0) \) and \( e_4 = -e_3 \), and one can check that

\[
\begin{align*}
\Omega_1 &= z_2(z_4dz_3 - z_3dz_4) \\
\Omega_2 &= -z_1(z_4dz_3 - z_3dz_4) \\
\Omega_3 &= z_3(z_1dz_2 - z_2dz_1) \\
\Omega_4 &= -z_4(z_1dz_2 - z_2dz_1)
\end{align*}
\]

In this case, one can show that \( \hat{\Omega}^1_S \) is the submodule of \( \Omega^1_S \) generated by \( z_1dz_2 - z_2dz_1 \) and \( z_4dz_3 - z_3dz_4 \) (this will also follow from Proposition 9.12 below).

As the above example indicates, the \( \Omega_i \) may be multiples of other forms. This happens whenever there is a \( j \) such that \( e_j = -e_i \). In this situation, note that \( \det(e_{J\cup\{i\}}) = 0 \) when \( j \in J \) since the matrix will have one column which is the negative of another. But when \( j \notin J \), then \( z_j \) divides \( \hat{z}_{J\cup\{i\}} \). It follows that \( e_j = -e_i \) implies

\[
\Omega_i = z_j \sum_{|J|=d-1 \setminus \{i,j\} \in J \setminus \{i,j\}} \det(e_{J\cup\{i,j\}}) \hat{z}_{J\cup\{i,j\}} dz_J.
\]

Hence we get the following definition.
Definition 9.10 Let

\[ I_0 = \{ i : -e_i \notin \{ e_1, \ldots, e_n \} \} \]
\[ I_1 = \{ (i, j) : i < j, e_i = -e_j \} \]

Furthermore, for \((i, j) \in I_1\), we define the \((d - 1)\)-form

\[ \Omega_{ij} = \sum_{|J| = d - 1, i,j \notin J} \det(e_{J \cup \{i\}}) \hat{z}_{J \cup \{i,j\}} dz_J. \]

Remark 9.11 For \((i, j) \in I_1\), we have \(\Omega_i = z_j \Omega_{ij}\), and since \(\det(e_{J \cup \{i\}}) = -\det(e_{J \cup \{j\}})\) (remember \(e_j = -e_i\)), we get \(\Omega_j = -z_i \Omega_{ij}\) as in Example 9.9.

Proposition 9.12 With the above notation, \(\hat{\Omega}_S^{d-1}\) is the submodule of \(\Omega_S^{d-1}\) generated by \(\Omega_i\) for \(i \in I_0\) and the \(\Omega_{ij}\) for \((i, j) \in I_1\).

Proof. We begin with the exact sequence

\[ 0 \to \hat{\Omega}_S^{d-1} \to S \otimes \Lambda^{d-1} \xrightarrow{\gamma} \bigoplus_{i=1}^n (S/z_i S) \otimes \Lambda^{d-2}M \]

which follows from the definition of \(\hat{\Omega}_S^{d-1}\) and the inclusion \(M \cap e_i^+ \subset M\). If \(m_1, \ldots, m_d\) is an integer basis of \(M\), then we get the bases

basis of \(\Lambda^{d-1}M\) : \(\omega_j = (-1)^{j+1}m_1 \wedge \cdots \wedge \widehat{m_j} \wedge \cdots \wedge m_d, 0 \leq j \leq d\)

basis of \(\Lambda^{d-2}M\) : \(\omega_{jk} = (-1)^{j+k}m_1 \wedge \cdots \wedge \widehat{m_j} \wedge \cdots \wedge \widehat{m_k} \wedge \cdots \wedge m_d, 0 \leq j < k \leq d\)

We will also set \(\omega_{jk} = -\omega_{kj}\) when \(j > k\). The signs in the definitions of \(\omega_j\) and \(\omega_{jk}\) were chosen to make interior products easy to calculate. In fact, one can check that if \(m_1^*, \ldots, m_d^*\) is the dual basis to \(m_1, \ldots, m_d\), then

\[ \langle m_k^*, \omega_j \rangle = \begin{cases} 0 & k = j \\ \omega_{kj} & k \neq j. \end{cases} \]

Since \(e_i = \sum_{k=1}^d \langle m_k, e_i \rangle m_k^*\), it follows that

\[ \langle e_i, \omega_j \rangle = \sum_{k \neq j} \langle m_k, e_i \rangle \omega_{kj}. \] (12)

For later purposes, we also note that as in the proof of Proposition 9.3, the form \(\omega_j\) can be written inside \(S \otimes \Lambda^{d-1}Z^n\) as

\[ \sum_{|J| = d-1} (-1)^{j+1} \det(e^J_j) (\prod_{l \in J} z_l^{-1}) dz_J \] (13)

where \(\det(e^J_j)\) is the determinant of the \((d - 1) \times (d - 1)\) submatrix of \((\langle m_k, e_l \rangle)_{1 \leq k \leq d, l \in J}\) obtained by deleting the \(j\)th row.
The next observation is that $S$ has a $\mathbb{Z}^n$ grading coming from the action of $(\mathbb{C}^*)^n$ on $\mathbb{A}^n$. The graded pieces are 1-dimensional, each spanned by a single monomial. This induces a grading on $S \otimes \Lambda^{d-1} M$ and $S/\xi S \otimes \Lambda^{d-2} M$, and the map $\gamma$ of (11) is a graded homomorphism. Thus the entire exact sequence has a $\mathbb{Z}^n$ grading. It following that in studying $\hat{\Omega}^{d-1}_S \subset S \otimes \Lambda^{d-1} M$, we can restrict our attention to “homogeneous” elements of $S \otimes \Lambda^{d-1} M$, i.e., elements of the form $\sum_{j=1}^d c_j z^D \otimes \omega_j$, where $\omega_j$ is as above, $c_j \in \mathbb{C}$, and $z^D$ is a monomial.

Take such an element $\sum_{j=1}^d c_j z^D \otimes \omega_j \in S \otimes \Lambda^{d-1} M$. Then, in the exact sequence (11), equation (12) shows that the $i^{th}$ component of $\gamma(\sum_{j=1}^d c_j z^D \otimes \omega_j)$ is given by

$$\sum_{j=1}^d \sum_{k \neq j} (c_j z^D \langle m_k, e_i \rangle \mod z_i) \otimes \omega_{kj} = \sum_{j<k} (\langle c_j m_k - c_k m_j, e_i \rangle z^D) \mod z_i \otimes \omega_{kj}.$$ 

Thus it follows that $\sum_{j=1}^d c_j z^D \otimes \omega_j$ lies in $\hat{\Omega}^{d-1}_S$ if and only if

$$z_i \text{ divides } \langle c_j m_k - c_k m_j, e_i \rangle z^D \text{ for all } i, j \text{ and } k. \quad (14)$$

For $1 \leq i \leq n$, the above criterion shows that $\sum_{j=1}^d \langle m_j, e_i \rangle \hat{z}_j \omega_j$ lies in $\hat{\Omega}^{d-1}_S$, where $\hat{z}_i = \prod_{l \neq i} z_l$. Furthermore, using equation (13), we see that in $S \otimes \Lambda^{d-1} \mathbb{Z}^n$, this form equals

$$\sum_{j=1}^d \langle m_j, e_i \rangle \hat{z}_i (\sum_{|J|=d-1} (-1)^{|J|+1} \det(e_{j \cup i}) (\Pi_{l \not\in J} \hat{z}_l^{-1})) dz_J,$$

which can be written as

$$\sum_{|J|=d-1} (-1)^{|J|+1} \langle m_j, e_i \rangle \det(e_{j \cup i}) \hat{z}_{j \cup i} dz_J = \sum_{|J|=d-1} \det(e_{j \cup i}) \hat{z}_{j \cup i} dz_J = \Omega_i$$

since $\langle m_j, e_i \rangle$ for $1 \leq j \leq d$ gives the first column of the matrix in $\det(e_{j \cup i})$. Similarly, if $(i, j) \in \mathcal{I}_1$, one can show that $\sum_{j=1}^d \langle m_j, e_i \rangle \hat{z}_{ij} \omega_j$ lies in $\hat{\Omega}^{d-1}_S$ and equals $\Omega_{ij}$.

Thus $\Omega_i$, $1 \leq i \leq n$, and $\Omega_{ij}$, $(i, j) \in \mathcal{I}_1$ lie in $\hat{\Omega}^{d-1}_S$. To prove that these forms generate $\hat{\Omega}^{d-1}_S$, suppose $\omega = \sum_{j=1}^d c_j z^D \otimes \omega_j$ satisfies equation (14). We can assume that at least one $c_j \neq 0$. There are three cases to consider:

**Case I.** Suppose that $z_i$ divides $z^D$ for all $i$. It suffices to consider $\omega = z_1 \ldots z_n \omega_j$, and we can assume that $e_1, \ldots, e_d$ are linearly independent. Then for $1 \leq i \leq d$, the $z_i \Omega_i = \sum_{j=1}^n \langle m_j, e_i \rangle z_1 \ldots z_n \omega_j$ have the same span as the $z_1 \ldots z_n \omega_j$ since $\langle \langle m_j, e_i \rangle_{1 \leq k, l \leq d} \rangle$ is invertible. Thus $z_1 \ldots z_n \omega_j$ lies in the submodule generated by the $\Omega_i$.

**Case II.** Suppose that $z_i$ doesn’t divide $z^D$ for some $i \in \mathcal{I}_0$. Then (14) implies that $\langle c_j m_k - c_k m_j, e_i \rangle = 0$ for all $j, k$. Thus $e_i$ is orthogonal to the codimension 1 sublattice spanned by $c_j m_k - c_k m_j$. Since $e_i$ is primitive, we see that $e_i$ is determined uniquely up to $\pm$. In particular, if there were some $j \neq i$ where $z_j$ also didn’t divide $z^D$, then we would have $e_j = -e_i$, which is impossible since $i \in \mathcal{I}_0$. This shows that $z_i$ divides $z^D$. Furthermore, $0 = \langle c_j m_k - c_k m_j, e_i \rangle = c_j \langle m_k, e_i \rangle - c_k \langle m_j, e_i \rangle$ shows that for some constant $\lambda$, we have $c_j = \lambda \langle m_j, e_i \rangle$ for all $j$. Thus $\omega$ is a multiple of $\Omega_i$.

**Case III.** Suppose that $z_i$ doesn’t divide $z^D$ for some $i \notin \mathcal{I}_0$. We can assume $(i, j) \in \mathcal{I}_1$, and the argument of Case II shows that $\hat{z}_{ij}$ divides $z^D$. Then, as in Case II, we see that $\omega$ is a multiple of $\Omega_{ij}$. 

28
Since \( \Omega_i = z_i \Omega_{ij} \) and \( \Omega_j = -z_i \Omega_{ij} \) for \( (i, j) \in \mathcal{I}_1 \), we only need the the \( \Omega_i \) for \( i \in \mathcal{I}_0 \) along with the \( \Omega_{ij} \) to generate \( \hat{\Omega}_S^{d-1} \). This proves the proposition.

We get the following description of \((d - 1)\)-forms on \( P \) with poles on \( X \).

**Theorem 9.13** Let \( X \subset P \) be a Cartier divisor defined by \( f = 0 \), where \( f \in S_\beta \). Then

\[
H^0(P, \Omega_P^{d-1}(X)) = \left\{ \frac{\sum_{i \in \mathcal{I}_0} A_i \Omega_i + \sum_{(i, j) \in \mathcal{I}_0} A_{ij} \Omega_{ij}}{f} : A_i \in S_{\beta - \beta_0 + \beta_i}, \ A_{ij} \in S_{\beta - \beta_0 + \beta_i + \beta_j} \right\},
\]

where \( \beta_0 = \sum_{i=1}^n \beta_i \) and \( z_i \in S_{\beta_i} \). However, if \( S_\beta \subset B(\Sigma) = \langle \hat{\zeta}_\sigma : \sigma \in \Sigma \rangle \) (see Definition 1.3), then

\[
H^0(P, \Omega_P^{d-1}(X)) = \left\{ \frac{\sum_{i=1}^n A_i \Omega_i}{f} : A_i \in S_{\beta - \beta_0 + \beta_i} \right\}.
\]

**Proof.** Since \( \Omega_i \) has degree \( \beta_0 - \beta_i \) and \( \Omega_{ij} \) has degree \( \beta_0 - \beta_i - \beta_j \), the first part of the theorem follows immediately from Propositions 8.7 and 9.12. For the second part, suppose that \( S_\beta \subset B(\Sigma) \), and consider a form \( A \Omega_{ij} \) where \( A \in S_{\beta_0 + \beta_i + \beta_j} \). If \( z^D \) is a monomial that appears in \( A \), then \( z^D \hat{\zeta}_{ij} \in S_{\beta} \subset B(\Sigma) \). This implies that \( z^D \hat{\zeta}_{ij} \) is divisible by \( \hat{\zeta}_\sigma \) for some \( \sigma \in \Sigma \). But \( \sigma \) can’t contain both \( e_i \) and \( e_j \) since \( e_i = -e_j \). Thus \( z_i \) or \( z_j \) divides \( \hat{\zeta}_\sigma \), so that \( z_i \) or \( z_j \) divides \( z^D \hat{\zeta}_{ij} \). It follows that \( z^D \) is divisible by \( z_i \) or \( z_j \), and thus \( z^D \Omega_{ij} \) is a multiple of either \( z_i \Omega_{ij} = -\Omega_j \) or \( z_j \Omega_{ij} = \Omega_j \). Hence \( A \Omega_{ij} \) is in the submodule generated by the \( \Omega_i \), and the theorem is proved.

**Remark 9.14** The reader can check that when \( P = P^d \), the description of \( H^0(P, \Omega_P^{d-1}(X)) \) given above generalizes equation (4.4) from [14].

To exploit the second part of Theorem 9.13, we need to know when \( S_\beta \subset B(\Sigma) \).

**Lemma 9.15** If \( \beta \in \text{Cl}(\Sigma) \) is the class of an ample divisor on \( P \), then \( S_\beta \subset B(\Sigma) \).

**Proof.** Given \( z^D = \prod_{i=1}^n z_i^{a_i} \in S_\beta \), our hypothesis implies that \( D = \sum_{i=1}^n a_i D_i \) is an ample divisor on \( P \). Thus \( \mathcal{L} = \mathcal{O}_P(D) \) is an ample \( T \)-linearized invertible sheaf. Let \( \Delta = \Delta(\mathcal{L}) \subset M_\mathbb{R} \) be its associated convex polytope. Since \( \Delta \) is defined by the inequalities \( \langle m, e_i \rangle \geq -a_i \), we have \( 0 \in \Delta \) since \( a_i \geq 0 \). Then let \( \Delta_0 \) be the minimal face of \( \Delta \) containing \( 0 \).

We know that for some \( \sigma \in \Sigma \), the face \( \Delta_\sigma \) of \( \Delta \) corresponding to \( \sigma \) is \( \Delta_0 \). We claim that \( \hat{\zeta}_\sigma \) divides our monomial \( z^D \). To see why this is true, suppose we have some \( z_i \) which doesn’t divide \( z^D \). This means \( a_i = 0 \). If \( \rho_i \) is the 1-dimensional cone of \( \Sigma \) generated by \( e_i \), then the corresponding facet of \( \Delta \) is \( \Delta_{\rho_i} = \Delta \cap \{ m : \langle m, e_i \rangle \geq 0 \} \) since \( a_i = 0 \). Thus \( 0 \in \Delta_{\rho_i} \), which implies \( \Delta_\sigma \subset \Delta_{\rho_i} \) by the minimality of \( \Delta_\sigma \). It follows that \( \rho_i \subset \sigma \). We have thus proved that \( a_i = 0 \) implies \( e_i \in \sigma \), and it follows immediately that \( \hat{\zeta}_\sigma = \prod_{i_1 \in \sigma} z_i \) divides \( z^D \). Thus \( z^D \in B(\Sigma) \), and the lemma is proved.

If we combine this lemma with Theorem 9.13, we get the following useful corollary.

**Corollary 9.16** Let \( X \subset P \) be an ample Cartier divisor defined by \( f = 0 \), where \( f \in S_\beta \). Then

\[
H^0(P, \Omega_P^{d-1}(X)) = \left\{ \frac{\sum_{i=1}^n A_i \Omega_i}{f} : A_i \in S_{\beta - \beta_0 + \beta_i} \right\},
\]

where \( \beta_0 = \sum_{i=1}^n \beta_i \) and \( z_i \in S_{\beta_i} \).
10 Cohomology of the complement of an ample divisor

In this section, \( P \) will be a \( d \)-dimensional complete simplicial toric variety and \( X \subset P \) will be the zero locus of a global section of a \( T \)-linearized ample invertible sheaf \( L \). If \( \beta \in \text{Cl}(\Sigma) \) is the class of \( L \), then Lemma 4.11 shows that \( X \) is defined by an equation \( f = 0 \) for some \( f \in S_\beta \). We will also assume that \( X \) is quasi-smooth (by Proposition 4.15, this is true for generic \( f \in S_\beta \)). Our goal is to compute the cohomology of \( P \setminus X \) in terms of \( f \in S \). We will also study the cohomology of \( X \). Our results will generalize classical results of Griffiths, Dolgachev and Steenbrink (see [14, 13, 29]).

Since \( X \) is a \( V \)-submanifold of the \( V \)-manifold \( P \), we can compute \( H^i(P \setminus X) \) using the complex \( \Omega_p^{\text{log}}(X) \) (we always use cohomology with coefficients in \( \mathbb{C} \)). Furthermore, the Hodge filtration \( F^p \) on \( H^{p+q}(P \setminus X) \) comes from the spectral sequence \( H^q(P, \Omega_p^{d}(\text{log}X)) \Rightarrow H^{p+q}(P \setminus X) \) which degenerates at \( E_1 \) (see §15 of [3]). Thus we obtain isomorphisms

\[
Gr_F^p H^d(P \setminus X) \cong H^{d-p}(P, \Omega_p^{d}(\text{log}X))
\]

for \( p = 0, \ldots, d \). Moreover, \( \Omega_p^{d}(\text{log}X) \) has the following resolution.

**Proposition 10.1** If \( X \) is a quasi-smooth hypersurface of a complete simplicial toric variety \( P \), then there is a canonical exact sequence

\[
0 \to \Omega_p^{d}(\text{log}X) \to \Omega_p^{d}(X) \xrightarrow{d} \Omega_p^{d}(2X)/\Omega_p^{d+1}(X) \xrightarrow{d} \ldots
\]

\[
\ldots \xrightarrow{d} \Omega_p^{d-1}((d-p)X)/\Omega_p^{d-1}((d-p-1)X) \xrightarrow{d} \Omega_p^{d}((d-p+1)X)/\Omega_p^{d}((d-p)X) \to 0.
\]

**Proof.** The proof of this is similar to the proof of Theorem 6.2 in [3]. \( \square \)

If we combine the above exact sequence with the Bott-Steenbrink-Danilov vanishing theorem (see Theorem 7.2), then we obtain the following corollary.

**Corollary 10.2** There are natural isomorphisms

\[
Gr_F^p H^d(P \setminus X) \cong H^{d-p}(P, \Omega_p^{d}(\text{log}X)) \cong \frac{H^0(P, \Omega_p^{d}((d-p+1)X))}{H^0(P, \Omega_p^{d}((d-p)X) + dH^0(P, \Omega_p^{d-1}((d-p)X))}.
\]

Before we can state our next result, we need some definitions.

**Definition 10.3** Let \( f \in S_\beta \) be a nonzero polynomial. Then the *Jacobian ideal* \( J(f) \subset S \) is the ideal of \( S = \mathbb{C}[z_1, \ldots, z_n] \) generated by the partial derivatives \( \partial f/\partial z_1, \ldots, \partial f/\partial z_n \). Also, the *Jacobian ring* \( R(f) \) is the quotient ring \( S/J(f) \).

**Remark 10.4** Since \( f \in S_\beta \), we have \( \partial f/\partial z_i \in S_{\beta - \beta_i} \), where \( z_i \in S_{\beta_i} \). Thus \( J(f) \) is a graded ideal of \( S \), so that \( R(f) \) has a natural grading by the class group \( \text{Cl}(\Sigma) \).

**Lemma 10.5** If \( f \in S_\beta \), where \( \beta \neq 0 \), then \( f \in J(f) \).
Proof. We will prove this using the Euler formulas from Definition 3.9. Let $\prod_{i=1}^{n} z_i^{b_i}$ be a monomial appearing in $f$. Then, in $Cl(\Sigma)$, we have $\beta = [\sum_{i=1}^{n} b_i D_i]$, where $D_i$ is the divisor in $\mathbf{P}$ corresponding to $e_i$. If we can find $\phi_1, \ldots, \phi_n \in \mathbf{C}$ such that $\sum_{i=1}^{n} \phi_i e_i = 0$, then Lemma 3.8 tells us that

$$
(\sum_{i=1}^{n} \phi_i b_i) f = \sum_{i=1}^{n} \phi_i z_i \frac{\partial f}{\partial z_i}.
$$

Thus we need to find a relation $\sum_{i=1}^{n} \phi_i e_i = 0$ such that $\sum_{i=1}^{n} \phi_i b_i \neq 0$. To do this, pick $j$ such that $b_j > 0$. By completeness, $-e_j$ must lie in some cone $\sigma \in \Sigma$. Since $e_j$ can’t be a generator of $\sigma$, we get $-e_j = \sum_{i \neq j} \phi_i e_i$, where $\phi_i \geq 0$, so that setting $\phi_j = 1$ gives $\sum_{i=1}^{n} \phi_i e_i = 0$. Since $b_j > 0$ and $b_i \geq 0$ for $i \neq j$, we have $\sum_{i=1}^{n} \phi_i b_i > 0$. 

We can now state the first main result of this section.

**Theorem 10.6** Let $\mathbf{P}$ be a d-dimensional complete simplicial toric variety, and let $X \subset \mathbf{P}$ be a quasi-smooth ample hypersurface defined by $f \in S_\beta$. If $R(f)$ is the Jacobian ring of $f$, then there is a canonical isomorphism

$$Gr^p_f H^d(\mathbf{P} \setminus X) \cong R(f)_{(d-p+1)\beta-\beta_0},$$

where $z_i \in S_{\beta_i}$ and $\beta_0 = \sum_{i=1}^{n} \beta_i$.

**Proof.** The arguments are similar to those used in the classical case (see, for example, [24]). By Theorem 9.7, we have

$$H^0(\mathbf{P} , \Omega^d_{\mathbf{P}}((d-p+1)X)) = \left\{ \frac{A\Omega_0}{f^{d-p+1}} : A \in S_{(d-p+1)\beta-\beta_0} \right\},$$

so that the map $\phi(A\Omega_0/f^{d-p+1}) = A$ defines a bijection

$$\phi : H^0(\mathbf{P} , \Omega^d_{\mathbf{P}}((d-p+1)X)) \cong S_{(d-p+1)\beta-\beta_0}.$$

By Corollary 10.2, it suffices to show that the subspace

$$H^0(\mathbf{P} , \Omega^d_{\mathbf{P}}((d-p)X) + dH^0(\mathbf{P} , \Omega^{d-1}_{\mathbf{P}}((d-p)X) \subset H^0(\mathbf{P} , \Omega^d_{\mathbf{P}}((d-p+1)X))$$

maps via $\phi$ to $J(f)_{(d-p+1)\beta-\beta_0} \subset S_{(d-p+1)\beta-\beta_0}$.

If $p = d$, then the desired result follows immediately since $H^0(\mathbf{P} , \Omega^d_{\mathbf{P}})$, $H^0(\mathbf{P} , \Omega^{d-1}_{\mathbf{P}})$ and $J(f)_{\beta-\beta_0}$ all vanish (the last because $\partial f/\partial z_i \in S_{\beta-\beta}$). Now assume $p < d$. Since $A\Omega_0/f^{d-p} = fA\Omega_0/f^{d-p+1}$, we see that $\phi(H^0(\mathbf{P} , \Omega^d_{\mathbf{P}}((d-p)X))) = fS_{(d-p)\beta-\beta_0}$. It remains to see what happens to $dH^0(\mathbf{P} , \Omega^{d-1}_{\mathbf{P}}((d-p)X))$. By Corollary 9.10, we know that

$$H^0(\mathbf{P} , \Omega^{d-1}_{\mathbf{P}}((d-p)X)) = \left\{ \frac{\sum_{i=1}^{n} A_i \Omega_i}{f^{d-p}} : A_i \in S_{(d-p)\beta-\beta_0+\beta_i} \right\}.$$

Since $d-p \neq 0$ and $f \in J(f)$ by Lemma 10.5, the theorem now follows easily from the following lemma.
Lemma 10.7 If $A \in S_{(d-p)\beta - \beta_0 + \beta_1}$, then
\[
d(\frac{A_{O_i}}{f^{d-p}}) = \frac{(f \partial A/\partial z_i - (d-p)A \partial f/\partial z_i)\Omega_0}{f^{d-p+1}}.
\]

Proof. First note that
\[
d(\frac{A_{O_i}}{f^{d-p}}) = \frac{1}{f^{d-p+1}}(f dA \wedge \Omega_i + f Ad\Omega_i - (d-p)df \wedge \Omega_i).
\]
This equals $B\Omega_0/f^{d-p+1}$ for some $B \in S_{(d-p+1)\beta - \beta_0}$, where
\[
\Omega_0 = \sum_{|I|=d} \text{det}(e_I)\hat{z}_I dz_I
\]
(see Definition 9.3). Pick $I_0 \subset \{1, \ldots, n\}$ such that $|I_0| = d$, $i \in I_0$ and $\text{det}(e_{I_0}) \neq 0$. To find $B$, it suffices to determine the coefficient of $dz_{I_0}$ in (15).

From Definition 9.8, we have
\[
\Omega_i = \sum_{|J|=d, i \notin J} \text{det}(e_{J\cup\{i\}})\hat{z}_{J\cup\{i\}} dz_{J\cup\{i\}}.
\]
Since neither $z_i$ nor $dz_i$ appear in $\Omega_i$, $dz_{I_0}$ doesn’t appear in $f Ad\Omega_i$. Furthermore, if we set $J_0 = I_0 \setminus \{i\}$, then the coefficient of $dz_{I_0} = dz_i \land dz_{J_0}$ in $f dA \wedge \Omega_i - (d-p)df \wedge \Omega_i$ is
\[
\frac{\partial A}{\partial z_i} dz_i \land \text{det}(e_{J_0 \cup \{i\}})\hat{z}_{J_0 \cup \{i\}} dz_{J_0} - (d-p)A\frac{\partial f}{\partial z_i} dz_i \land \text{det}(e_{J_0 \cup \{i\}})\hat{z}_{J_0 \cup \{i\}} dz_{J_0}
\]
\[
= (f \partial A/\partial z_i - (d-p)A \partial f/\partial z_i) \text{det}(e_{I_0})\hat{z}_{I_0} dz_{I_0}.
\]
This shows that $B = f \partial A/\partial z_i - (d-p)A \partial f/\partial z_i$ and completes the proof of the lemma. \(\square\)

We next study the cohomology of the hypersurface $X$. Our first result is a Lefschetz theorem.

Proposition 10.8 Let $X$ be a quasi-smooth hypersurface of a $d$-dimensional complete simplicial toric variety $P$, and suppose that $X$ is defined by $f \in S_\beta$. If $f \in B(\Sigma)$ (see Definition 7.3), then the natural map $i^* : H^i(P) \rightarrow H^i(X)$ is an isomorphism for $i < d-1$ and an injection for $i = d-1$. In particular, this holds if $X$ is an ample hypersurface.

Proof. In the affine space $A^n$, $f \in B(\Sigma)$ implies that $Z(\Sigma) = V(B(\Sigma)) \subset V(f)$. Thus $A^n \setminus V(f) \subset A^n - Z(\Sigma) = U(\Sigma)$ is affine. Since $P = U(\Sigma)/D(\Sigma)$, it follows that $P \setminus X = (A^n \setminus V(f))/D(\Sigma)$ is affine. This implies $H^i(P \setminus X) = 0$ for $i > d$ by Corollary 13.6 of [4]. Now consider the Gysin sequence
\[
\ldots \rightarrow H^{i-2}(X) \xrightarrow{i_*} H^i(P) \rightarrow H^i(P \setminus X) \rightarrow H^{i-1}(X) \xrightarrow{i_*} H^{i+1}(P) \rightarrow \ldots
\]
(see the proof of Theorem 3.7 of [1]). Since the Gysin map $i_*$ is dual to $i^*$ under Poincaré duality, we see that $i^*$ has the desired property. Finally, if $X$ is ample, then Lemma 9.15 implies that $f \in B(\Sigma)$. \(\square\)

This result shows that the “interesting” part of the cohomology of $X$ occurs in dimension $d-1$ and consists of those classes which don’t come from $P$. Hence we get the following definition.
**Definition 10.9** The primitive cohomology group $PH^{d-1}(X)$ is defined by the exact sequence

$$0 \to H^{d-1}(P) \to H^{d-1}(X) \to PH^{d-1}(X) \to 0.$$ 

**Remark 10.10** Since $H^{d-1}(P)$ and $H^{d-1}(X)$ have pure Hodge structures, $PH^{d-1}(X)$ is also pure. Its Hodge components are denoted $PH^{p,d-1-p}(X)$.

**Proposition 10.11** When $X \subset P$ is ample, there is an exact sequence

$$0 \to H^{d-2}(P) \cup [X] \to H^d(P) \to H^d(P \setminus X) \to PH^{d-1}(X) \to 0$$

where $[X] \in H^2(P)$ is the cohomology class of $X$.

**Proof.** By (3.3) of [24], we have a commutative diagram

$$
\begin{array}{ccc}
H^i(P) & \xrightarrow{i^*} & H^i(X) \\
\downarrow r & & \downarrow \cup [X] \\
H^i(P) & \xrightarrow{i} & H^{i+2}(P)
\end{array}
$$

(16)

When $i = d - 1$, $\cup [X]$ is an isomorphism by Hard Lefschetz, and it follows that $PH^{d-1}(X)$ is isomorphic to the kernel of $i : H^{d-1}(X) \to H^{d-1}(P)$. Then the Gysin sequence gives us an exact sequence

$$H^{d-2}(X) \xrightarrow{i} H^d(P) \to H^d(P \setminus X) \to PH^{d-1}(X) \to 0.$$ 

Since $\cup [X] : H^{d-2}(P) \to H^d(P)$ is injective by Hard Lefschetz and $i^* : H^{d-2}(P) \to H^{d-2}(X)$ is an isomorphism by Proposition [10.8], the desired exact sequence now follows easily using (16) with $i = d - 2$. □

The maps in Proposition [10.11] are all morphisms of mixed Hodge structures (with appropriate shifts). Since $H^i(P)$ vanishes for $i$ odd and has only $(p,p)$ classes for $i$ even (see Theorem 3.11 of [23]), we get the following corollary of Proposition [10.11].

**Corollary 10.12** When $p \neq d/2$, there is a natural isomorphism

$$Gr_F^p H^d(P \setminus X) \cong PH^{p-1,d-p}(X),$$

and when $p = d/2$, there is an exact sequence

$$0 \to H^{d-2}(P) \cup [X] \to H^d(P) \to Gr_F^{d/2} H^d(P \setminus X) \to PH^{d/2-1,d/2}(X) \to 0.$$ 

If we combine this with Theorem [10.6] and replace $p$ with $p+1$, then we get the following theorem.
Theorem 10.13 Let $P$ be a $d$-dimensional complete simplicial toric variety, and let $X \subset P$ be a quasi-smooth ample hypersurface defined by $f \in S_\beta$. If $R(f)$ is the Jacobian ring of $f$, then for $p \neq d/2 - 1$, there is a canonical isomorphism

$$R(f)_{(d-p)\beta-\beta_0} \cong \text{PH}^{p,d-1-p}(X)$$

where $z_i \in S_{\beta_i}$ and $\beta_0 = \sum_{i=1}^{n} \beta_i$. For $p = d/2 - 1$, we have an exact sequence

$$0 \to H^{d-2}(P) \cup [X] \to H^d(P) \to R(f)_{(d/2+1)\beta-\beta_0} \to \text{PH}^{d/2-1,d/2}(X) \to 0.$$

Remark 10.14 Notice that when $d$ is odd, we always have $R(f)_{(d-p)\beta-\beta_0} \cong \text{PH}^{p,d-1-p}(X)$. The same conclusion holds whenever $P$ is a toric variety with the property that $\cup [X] : H^{d-2}(P) \to H^d(P)$ is an isomorphism. The latter holds when $P$ is a weighted projective space, which explains why the classical case is so nice. Notice also that in all cases, we always have a surjection

$$R(f)_{(d-p)\beta-\beta_0} \to \text{PH}^{p,d-1-p}(X) \to 0.$$ 

Remark 10.15 One consequence of our results is that there is a natural map

$$H^d(P) \to R(f)_{(d/2+1)\beta-\beta_0}.$$ 

It would be interesting to have an explicit description of this map.

11 Cohomology of affine hypersurfaces

In a recent paper [3], the first author obtained some results on the cohomology of the affine hypersurface $Y = X \cap T \subset T$, where $T$ is the torus contained in a projective toric variety $P$ and $X \subset P$ is an ample hypersurface. We will show how the results of [3] can be expressed in terms of various graded ideals of the ring $S = \mathbb{C}[z_1, \ldots, z_n]$. We begin with a definition.

Definition 11.1 If $X \subset P$ is a hypersurface, then we let $Y = X \cap T \subset T$ be its intersection with the torus $T \subset P$. The primitive cohomology group $\text{PH}^{d-1}(Y)$ is then defined to be the cokernel of the map $H^{d-1}(T) \to H^{d-1}(Y)$, where as usual we use cohomology with coefficients in $\mathbb{C}$.

Remark 11.2 In [10], it is shown that $H^{d-1}(T) \to H^{d-1}(Y)$ is injective. Note also that $\text{PH}^{d-1}(Y)$ has a natural mixed Hodge structure. The Hodge filtration on $\text{PH}^{d-1}(Y)$ will be denoted $F^r$.

One of the main results of [3] is a description of the Hodge filtration on $\text{PH}^{d-1}(Y)$. To state this in terms of the ring $S$, we need the following ideal of $S$.

Definition 11.3 Given $f \in S_\beta$, let $J_0(f) \subset S$ denote the ideal generated by $z_i \partial f / \partial z_i$ for $1 \leq i \leq n$. We then let $R_0(f)$ denote the quotient ring $S/J_0(f)$.
Remark 11.4 Since \( f \in S_\beta \), we see that \( J_0(f) \) is a graded ideal of \( S \), and hence \( R_0(f) \) has a natural grading by the class group \( Cl(\Sigma) \).

Theorem 11.5 (3) If \( X \subset \mathbf{P} \) is a nondegenerate (see Definition 4.13) ample divisor defined by \( f \in S_\beta \) and \( Y = X \cap T \) is the corresponding affine hypersurface in the torus \( T \), then there is a natural isomorphism

\[ Gr^p_F PH^{d-1}(Y) \cong R_0(f)(d-p)_\beta. \]

Proof. We first show how certain constructions in \( \mathcal{F} \) can be formulated in terms of the ring \( S \). Since \( X \subset \mathbf{P} \) is ample, we get the \( d \)-dimensional convex polyhedron \( \Delta \subset M_\mathbb{R} \).

Recall that \( \Delta \) is defined by the inequalities \( \langle m, e_i \rangle \geq -b_i \), where \( b_i \geq 0 \) and \( X \) is linearly equivalent to \( D = \sum_{i=1}^n b_i D_i \).

As in \( \mathcal{F} \), we define the ring \( S_{\Delta} \) to be the subring of \( \mathbf{C}[t_0, t_1^{\pm 1}, \ldots, t_d^{\pm 1}] \) spanned over \( \mathbf{C} \) by all Laurent monomials of the form \( t_0^{k_0}t_1^{m_1} \cdots t_d^{m_d} \) where \( k \geq 0 \) and \( m \in k\Delta \). This ring is graded by setting \( \deg(t_0^{k_0}t_1^{m_1} \cdots t_d^{m_d}) = k \). We should think of \( t_1, \ldots, t_d \) as coordinates on the torus \( T \) and \( t_0 \) as an auxiliary variable to keep track of the grading.

A first observation is that there is a natural isomorphism of graded rings

\[ \rho : S_{\Delta} \cong \bigoplus_{k=0}^{\infty} S_{k\beta} \subset S \]

which is defined by

\[ \rho(t_0^{k_0}t_1^{m_1} \cdots t_d^{m_d}) = \prod_{i=1}^{\infty} z_i^{k} t_0^{b_i \langle m, e_i \rangle}. \]

Since the integer points of \( k\Delta \) naturally give a basis of \( H^0(\mathbf{P}, \mathcal{O}_\mathbf{P}(kD)) \), Proposition 1.1 of \( \mathcal{F} \) shows that we get the desired ring isomorphism.

In particular, \( f \in S_\beta \) corresponds to an element \( \sum_{m \in \Delta \cap M} a_m t_0^{m} \in (S_{\Delta})_1 \). Setting \( t_0 = 1 \), we get the Laurent polynomial \( g = \sum_{m \in \Delta \cap M} a_m t_0^{m} \) formed using the lattice points of \( \Delta \). Since \( t_1, \ldots, t_d \) are coordinates on the torus \( T \), one can show that \( Y \) is naturally isomorphic to the subvariety of \( T \) defined by \( g = 0 \).

Following \( \mathcal{F} \), we use the Laurent polynomial \( g \) to define \( F = t_0 g - 1 \), and then we set \( F_i = t_i \partial F / \partial t_i \) for \( i = 0, 1, \ldots, d \). Note that \( F_i \in (S_{\Delta})_1 \) for all \( i \). Finally, we define \( J_{g, \Delta} \) to be the ideal of \( S_{\Delta} \) generated by \( F_i \). Then Corollary 6.10 of \( \mathcal{F} \) gives an isomorphism

\[ Gr^p_F PH^{d-1}(Y) \cong (S_{\Delta}/J_{g, \Delta})_{d-p}. \]

To prove the theorem, it suffices to show that for all \( k \geq 0 \), the isomorphism \( \rho \) of (17) maps the graded piece \(( J_{g, \Delta})_k \subset (S_{\Delta})_k \) to \( J_0(f)_{k\beta} \subset S_{k\beta} \).

First observe that \( F_0 = t_0 g \) maps to \( f \) under the map \( \rho \). To write this explicitly, we introduce the following notation: given \( m \in \Delta \cap M \), let \( z^{D(m)} = \prod_{i=1}^{n} z_i^{b_i \langle m, e_i \rangle} \). Thus \( f = \sum_{m \in \Delta \cap M} a_m z^{D(m)} \). Next note that \( t_1, \ldots, t_d \) are a basis of the character group of \( T \) and hence determine a basis of \( M \). If \( h_1, \ldots, h_d \in N \) is the dual basis, then one computes that for \( i > 0 \), the polynomial \( F_i = t_0 t_i \partial g / \partial t_i \) maps to

\[ \tilde{f}_i = \sum_{m \in \Delta \cap M} a_m \langle m, h_i \rangle z^{D(m)}. \]
In contrast, note that
\[ z_i \partial f / \partial z_i = \sum_{m \in \Delta \cap M} a_m (b_i + \langle m, e_i \rangle) z^{D(m)} = b_i f + \sum_{m \in \Delta \cap M} a_m \langle m, e_i \rangle z^{D(m)}. \]

It suffices to show that each \( z_i \partial f / \partial z_i \) is a \( \mathbb{C} \)-linear combination of \( f, \tilde{f}_1, \ldots, \tilde{f}_d \) and conversely.

To prove this, first note that \( e_i \) can be expressed in terms of the \( h_1, \ldots, h_d \), and hence \( z_i \partial f / \partial z_i \) is a \( \mathbb{C} \)-linear combination of \( f, \tilde{f}_1, \ldots, \tilde{f}_d \). Going the other way, we can label \( e_1, \ldots, e_n \) so that \( e_1, \ldots, e_d \) are linearly independent. Then \( h_i \) can be expressed in terms of \( e_1, \ldots, e_d \), and it follows easily that \( f, \tilde{f}_1, \ldots, \tilde{f}_d \) are \( \mathbb{C} \)-linear combinations of \( f, z_1 \partial f / \partial z_1, \ldots, z_n \partial f / \partial z_n \). To complete the proof, observe that \( f \in J_0(f) \) follows easily from the proof of Lemma \([10, \S]\).

We next study the weight filtration on \( PH^{d-1}(Y) \). Since \( Y \) is quasi-smooth, we have \( W_{d-2}H^{d-1}(Y) = 0 \), which implies that \( W_{d-2}PH^{d-1}(Y) = 0 \). It follows that \( W_{d-1}PH^{d-1}(Y) \) has a pure Hodge structure. We can identify this Hodge structure as follows.

**Proposition 11.6** If \( X \subset P \) is a nondegenerate ample hypersurface, then there is a natural isomorphism of Hodge structures
\[
PH^{d-1}(X) \cong W_{d-1}PH^{d-1}(Y).
\]

**Proof.** Let \( D = P \setminus T \), and recall from Theorem \([5, \S]\) that the complex \( \Omega^*_{P}(\log D) \) has a weight filtration \( W \) with the property that
\[
Gr_k^W \Omega^*_{P}(\log D) \cong \bigoplus_{\dim \tau = k} \Omega^\tau_{P},
\]

The spectral sequence of this filtered complex gives the weight filtration on \( H^* (T) \), and since the spectral sequence degenerates at \( E_2 \), we get an exact sequence
\[
\bigoplus_{i=1}^n H^{d-3}(D_i) \to H^{d-1}(P) \to Gr_{d-1}^W H^{d-1}(T) \to 0,
\]
where \( D = \sum_{i=1}^n D_i \). Notice also that \( Gr_{d-1}^W H^{d-1}(T) = W_{d-1}H^{d-1}(T) \) since \( T \) is smooth. Since \( X \subset P \) is nondegenerate, the same argument applies to \( D \cap X = X \setminus Y \), so that we also have an exact sequence
\[
\bigoplus_{i=1}^n H^{d-3}(D_i \cap X) \to H^{d-1}(X) \to Gr_{d-1}^W H^{d-1}(Y) \to 0,
\]
and as noticed earlier, \( Gr_{d-1}^W H^{d-1}(Y) = W_{d-1}H^{d-1}(Y) \).

To see how this applies to primitive cohomology, consider the commutative diagram:
\[
\begin{array}{c}
0 \\
\uparrow \\
PH^{d-1}(X) \to W_{d-1}PH^{d-1}(Y) \\
\uparrow \\
\bigoplus_{i=1}^n H^{d-3}(D_i \cap X) \\
\uparrow \\
H^{d-1}(P) \to W_{d-1}H^{d-1}(T) \to 0
\end{array}
\]

36
The columns are exact by the definition of primitive cohomology and the strictness of the weight filtration, and we've already seen that the bottom two rows are exact. It follows easily that the map \( \alpha : PH^{d-1}(X) \to W_{d-1}PH^{d-1}(Y) \) exists and is surjective. Notice also that \( \alpha \) is a morphism of Hodge structures.

However, each \( D_i \) is a \((d-1)\)-dimensional complete simplicial toric variety, and \( D_i \cap X \subset D_i \) is quasi-smooth since \( X \) is nondegenerate. Thus Proposition 10.8 implies that

\[
\bigoplus_{i=1}^{n} H^{d-3}(D_i) \to \bigoplus_{i=1}^{n} H^{d-3}(D_i \cap X)
\]

is an isomorphism. An easy diagram chase then shows that \( \alpha \) is injective, and the proposition is proved. \( \square \)

In order to interpret this proposition in terms of the polynomial ring \( S \), we will need the following ideal.

**Definition 11.7** Given the ideal \( J_0(f) = \langle z_1 \partial f \partial z_1, \ldots, z_n \partial f / \partial z_n \rangle \subset S \), we get the ideal quotient \( J_1(f) = J_0(f) : z_1 \cdots z_n \). We put \( R_1(f) = S/J_1(f) \).

**Theorem 11.8** ([3]) If \( X \subset P \) is a nondegenerate ample divisor defined by \( f \in S_{\beta} \), then there is a natural isomorphism

\[
PH^{p,d-1-p}(X) \cong R_1(f)_{(d-p)\beta-\beta_0},
\]

where \( z_i \in S_{\beta_i} \) and \( \beta_0 = \sum_{i=1}^{n} \beta_i \).

**Proof.** As in Definition 2.8 of [3], consider the ideal \( I^{(1)}_{\Delta} \subset S_{\Delta} \) spanned by all monomials \( t^{m} \) such that \( m \) is an interior point of \( k\Delta \). Then let \( H = \oplus H_t \) denote the image of the homogeneous ideal \( I^{(1)}_{\Delta} \) in the graded Artinian ring \( S_{\Delta}/J_{\Delta} \), where \( J_{\Delta} \) is as in the proof of Theorem 11.5. Proposition 9.2 of [3] tells us that under the isomorphism \( (S_{\Delta}/J_{\Delta})_{d-p} \cong Gr_{p}^{d-1}(Y) \), the subspace \( H_{d-p} \) maps to \( W_{d-1}Gr_{p}^{d-1}(Y) \). If we combine this with Proposition 11.6, then we get an isomorphism

\[
H_{d-p} \cong PH^{p,d-1-p}(X).
\]

Under the isomorphism \( \rho \) of (17), a monomial \( t_{0}^{m} \) maps to \( z^{D(m)} = \prod_{i=1}^{n} z_{i}^{b_{i}+(m,e_{i})} \). Since \( m \) is in the interior of \( k\Delta \) if and only if \( \langle m, e_i \rangle > -b_i \) for all \( i \), we see that \( t_{0}^{m} \in (I^{(1)}_{\Delta})_{k} \) exactly when \( z^{D(m)} \) is divisible by \( z_1 \cdots z_n \). Thus \( \rho((I^{(1)}_{\Delta})_{k}) = \langle z_1 \cdots z_n \rangle_{k} \). Since we know by the proof of Theorem 11.3 that \( \rho \) maps \( (J_{\Delta})_{k} \) to \( J_{0}(f)_{k} \), it follows easily that \( H_{k} \subset (S_{\Delta}/J_{\Delta})_{k} \) is isomorphic to the image of \( \langle z_1 \cdots z_n \rangle_{k} \) in \( (S/J_0(f))_{k} \). This last subspace is isomorphic to \( (S/J_1(f))_{k \beta - \beta_0} = R_1(f)_{k \beta - \beta_0} \), and the theorem is proved. \( \square \)

**Remark 11.9** It is interesting to compare this result to Theorem 10.13, which gives a natural surjection

\[
R(f)_{(d-p)\beta-\beta_0} \to PH^{p,d-1-p}(X) \to 0
\]
when $X$ is quasi-smooth. The theorem just proved shows that, under the stronger hypothesis that $X$ is nondegenerate, there is an isomorphism

$$PH^{p,d-1-p}(X) \cong R_1(f)_{(d-p)\beta-\beta_0}.$$  

One can show that the composition of these maps in induced by the obvious inclusion of ideals

$$J(f) = \langle \partial f/\partial z_i \rangle \subset \langle z_i \partial f/\partial z_i \rangle: z_1 \cdots z_n = J_1(f).$$

Since the map of (19) is an isomorphism for $p \neq d/2 - 1$, it follows that the ideals $J(f)$ and $J_1(f)$ agree in degrees $(d-p)\beta-\beta_0$ for $p \neq d/2 - 1$, though for $p = d/2 - 1$, we get an exact sequence

$$0 \to H^{d-2}(\mathcal{P}) \xrightarrow{\cup [X]} H^d(\mathcal{P}) \to (J_1(f)/J(f))_{(d/2+1)\beta-\beta_0} \to 0.$$

When $\mathcal{P}$ is a weighted projective space, the ideals $J(f)$ and $J_1(f)$ are equal in all degrees. For in this case, $f$ being quasi-smooth means that the $\partial f/\partial z_i$ form a regular sequence, while $f$ being nondegenerate means that the $z_i \partial f/\partial z_i$ form a regular sequence. Then standard results from commutative algebra (see part $(\gamma)$ of (1.2) of [27]) imply that $\langle \partial f/\partial z_i \rangle = \langle z_i \partial f/\partial z_i \rangle: z_1 \cdots z_n$, which gives the desired equality. In the general case, the precise relation between $J(f)$ and $J_1(f)$ is not well understood.

12 A generalized Euler short exact sequence and applications

We begin with a generalization of the classical Euler short exact sequence.

**Theorem 12.1** Let $D_1, \ldots, D_n$ the irreducible components of $D = \mathcal{P} \setminus \mathcal{T}$, and let $d$ be the dimension of $\mathcal{P}$. Then there exists the following short exact sequence

$$0 \to \Omega^1_\mathcal{P} \to \bigoplus_{i=1}^n \mathcal{O}_\mathcal{P}(-D_i) \to \mathcal{O}_\mathcal{P}^{n-d} \to 0.$$  

(20)

**Remark 12.2** When $\mathcal{P}$ is projective space, the above short exact sequence coincides with the well-known Euler short exact sequence. So we call (24) the generalized Euler exact sequence.

**Proof.** There are the following two short exact sequences:

$$0 \to \Omega^1_\mathcal{P} \to \Omega^1_\mathcal{P}(\log D) \xrightarrow{r} \bigoplus_{i=1}^n \mathcal{O}_{D_i} \to 0$$

(21)

and

$$0 \to \bigoplus_{i=1}^n \mathcal{O}_\mathcal{P}(-D_i) \to \mathcal{O}^n_\mathcal{P} \xrightarrow{p} \bigoplus_{i=1}^n \mathcal{O}_{D_i} \to 0,$$  

(22)
where $r$ is the Poincaré residue map.

The short exact sequence

$$0 \rightarrow M \rightarrow \mathbb{Z}^n \rightarrow Cl(\Sigma) \rightarrow 0$$

shows that $Cl(\Sigma)$ has rank $n - d$. Since $\Omega^1_{\mathbb{P}}(\log D) \cong \Lambda^1 M \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{P}}$, we can tensor this sequence with $\mathcal{O}_{\mathbb{P}}$ to obtain

$$0 \rightarrow \Omega^1_{\mathbb{P}}(\log D) \rightarrow \mathcal{O}^n_{\mathbb{P}} \rightarrow \mathcal{O}^{n-d}_{\mathbb{P}} \rightarrow 0.$$ 

By global properties of the Poincaré residue map (Section 6), one has $r = p \circ s$. So $s$ induces an injective map $i : \Omega^1_{\mathbb{P}} \hookrightarrow \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}}(-D_i)$, and then the short exact sequences (21) and (22) fit into the following commutative diagram:

$$
\begin{array}{ccccccc}
0 & \rightarrow & \Omega^1_{\mathbb{P}} & \rightarrow & \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}}(-D_i) & \rightarrow & \mathcal{Q} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \Omega^1_{\mathbb{P}}(\log D) & \rightarrow & \mathcal{O}^n_{\mathbb{P}} & \rightarrow & \mathcal{O}^{n-d}_{\mathbb{P}} & \rightarrow & 0 \\
\downarrow r & & \downarrow p & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \bigoplus_{i=1}^n \mathcal{O}_{D_i} & = & \bigoplus_{i=1}^n \mathcal{O}_{D_i} & \rightarrow & 0 & & \\
0 & & 0 & & 0 & & 
\end{array}
$$

Since the first two columns are exact, the snake lemma implies that the sheaf $\mathcal{Q}$ is isomorphic to $\mathcal{O}^{n-d}_{\mathbb{P}}$. \hfill \Box

As a first application of the Euler exact sequence, we will show how to find generators of $H^0(\mathbb{P}, \Omega^d_{\mathbb{P}}(X))$, where $X \subset \mathbb{P}$ is an ample hypersurface defined by $f \in S_\beta$. If we apply the functor $\text{Hom}(\ast, \Omega^d_{\mathbb{P}}(X))$ to (22), then we obtain the exact sequence

$$0 \rightarrow (\Omega^d_{\mathbb{P}}(X))^{n-d} \rightarrow \bigoplus_{i=1}^n \text{Hom}(\mathcal{O}_{\mathbb{P}}(-D_i), \Omega^d_{\mathbb{P}}(X)) \rightarrow \Omega^{d-1}_{\mathbb{P}}(X) \rightarrow 0,$$

since $\text{Hom}(\Omega^k_{\mathbb{P}}, \Omega^d_{\mathbb{P}}) = \Omega^{d-k}_{\mathbb{P}}$ (see 3). Then we get the short exact sequence of global sections

$$0 \rightarrow H^0(\mathbb{P}, (\Omega^d_{\mathbb{P}}(X))^{n-d}) \rightarrow \bigoplus_{i=1}^n H^0(\mathbb{P}, \text{Hom}(\mathcal{O}_{\mathbb{P}}(-D_i), \Omega^d_{\mathbb{P}}(X))) \rightarrow H^0(\mathbb{P}, \Omega^{d-1}_{\mathbb{P}}(X)) \rightarrow 0$$

since $H^1(\mathbb{P}, \Omega^d_{\mathbb{P}}(X)) = 0$ by the Bott-Steenbrink-Danilov vanishing theorem.

However, by Remark 4, we know that $\Omega^d_{\mathbb{P}} \cong \mathcal{O}_{\mathbb{P}}(-D)$, and it follows that we have an isomorphism of $\mathcal{T}$-linearized sheaves

$$\text{Hom}(\mathcal{O}(-D_i), \Omega^d_{\mathbb{P}}(X)) \cong \mathcal{O}(X - D + D_i).$$

Then Lemma 4.11 implies that we have an isomorphism

$$H^0(\mathbb{P}, \text{Hom}(\mathcal{O}(-D_i), \Omega^d_{\mathbb{P}}(X))) \cong S_{\beta - \beta_0 + \beta_i},$$

where as usual $\beta_i$ is the class of $D_i$ and $\beta = \sum_{i=1}^n \beta_i$ is the class of $D$. We have thus proved the following result.

39
**Proposition 12.3** When \( X \) is an ample hypersurface of \( \mathbf{P} \) defined by \( f \in S_\beta \), then there exists a surjective homomorphism

\[
\bigoplus_{i=1}^{n} S_{\beta - \beta_0 + \beta_i} \to H^0(\mathbf{P}, \Omega_{\mathbf{P}}^{d-1}(X)).
\]

**Remark 12.4** The reader should compare this proposition with Theorem 9.13 and Corollary 9.16.

We conclude this section with a discussion of the tangent sheaf of a toric variety.

**Definition 12.5** Let \( \mathcal{T}_\mathbf{P} \) be the sheaf \( \text{Hom}(\Omega^1_{\mathbf{P}}, \mathcal{O}_\mathbf{P}) \). We call \( \mathcal{T}_\mathbf{P} \) the tangent sheaf of \( \mathbf{P} \).

**Remark 12.6** If \( \mathbf{P} \) is smooth, then \( \mathcal{T}_\mathbf{P} \) coincides with the usual tangent sheaf \( \Theta_{\mathbf{P}} \) of \( \mathbf{P} \).

Applying \( \text{Hom}(\ast, \mathcal{O}_\mathbf{P}) \) to (20), we obtain the short exact sequence

\[
0 \to \mathcal{O}^{n-d}_{\mathbf{P}} \to \bigoplus_{i=1}^{n} \mathcal{O}(D_i) \to \mathcal{T}_\mathbf{P} \to 0.
\]

Since \( H^1(\mathbf{P}, \mathcal{O}_\mathbf{P}) = 0 \), we get

\[
\dim H^0(\mathbf{P}, \mathcal{T}_\mathbf{P}) = \sum_{i=1}^{n} (\dim S_{\beta_i} - 1) + d.
\]

It was proved by second author in [5] that

\[
\dim \text{Aut}(\mathbf{P}) = \sum_{i=1}^{n} (\dim S_{\beta_i} - 1) + d.
\]

Thus the global sections of \( \mathcal{T}_\mathbf{P} \) can be identified with the Lie algebra of \( \text{Aut}(\mathbf{P}) \).

### 13 Moduli of ample hypersurfaces

This section will study how the Jacobian ring is related to the moduli of the hypersurfaces \( X \subset \mathbf{P} \) coming from sections of an ample invertible sheaf \( \mathcal{L} \) on \( \mathbf{P} \). As usual, we assume that \( \mathbf{P} \) is a complete simplicial toric variety.

We first study the automorphisms of \( \mathbf{P} \) which preserve a given divisor class. Let \( \text{Aut}(\mathbf{P}) \) denote the automorphism group of \( \mathbf{P} \).

**Definition 13.1** Given \( \beta \in Cl(\Sigma) \), let \( \text{Aut}_\beta(\mathbf{P}) \) denote the subgroup of \( \text{Aut}(\mathbf{P}) \) consisting of those automorphisms which preserve \( \beta \).

**Remark 13.2** If \( \text{Aut}^0(\mathbf{P}) \) is the connected component of the identity of \( \text{Aut}(\mathbf{P}) \), then the results of §3 of [3] imply that \( \text{Aut}^0(\mathbf{P}) \) is a subgroup of finite index in \( \text{Aut}_\beta(\mathbf{P}) \).
When we describe \( P \) as the quotient \( U(\Sigma)/D(\Sigma) \), note that \( \text{Aut}(P) \) doesn’t act on \( U(\Sigma) \). However, in \([5]\), it is shown that there is an exact sequence

\[
1 \to D(\Sigma) \to \tilde{\text{Aut}}(P) \to \text{Aut}(P) \to 1
\]

where \( \tilde{\text{Aut}}(P) \) is the group of automorphisms of \( \mathbb{A}^n \) which preserve \( U(\Sigma) \) and normalize \( D(\Sigma) \). An element \( \phi \in \text{Aut}(P) \) induces an automorphism \( \phi : S \to S \) which for all \( \gamma \in \text{Cl}(\Sigma) \) satisfies \( \phi(S_\gamma) = S_{\phi(\gamma)} \).

**Definition 13.3** Given \( \beta \in \text{Cl}(\Sigma) \), let \( \tilde{\text{Aut}}(P)_\beta \) denote the subgroup of \( \tilde{\text{Aut}}(P) \) consisting of those automorphisms which preserve \( \beta \).

The group \( \tilde{\text{Aut}}(P)_\beta \) has the following obvious properties.

**Lemma 13.4** There is a canonical exact sequence

\[
1 \to D(\Sigma) \to \tilde{\text{Aut}}(P)_\beta \to \text{Aut}(P)_\beta \to 1.
\]

Furthermore, there is a natural action of \( \tilde{\text{Aut}}(P)_\beta \) on \( S_\beta \).

**Remark 13.5** Let \( \tilde{\text{Aut}}^0(P) \) be the connected component of the identity of \( \tilde{\text{Aut}}(P) \). In \([3]\), it is shown that \( \tilde{\text{Aut}}^0(P) \) is naturally isomorphic to the group \( \text{Aut}_g(S) \) of \( \text{Cl}(\Sigma) \)-graded automorphisms of \( S \). Then \( \tilde{\text{Aut}}^0(P) \subset \tilde{\text{Aut}}(P)_\beta \), and the action of \( \tilde{\text{Aut}}(P)_\beta \) on \( S_\beta \) is compatible with the action of \( \text{Aut}_g(S) \).

If \( \beta \in \text{Cl}(\Sigma) \) is an ample class, then we know that \( X = \{ p \in P : f(p) = 0 \} \subset P \) is quasi-smooth for generic \( f \in S_\beta \) (see Proposition 13.15). Then

\[
\{ f \in S_\beta : f \text{ is quasi-smooth} \}/\tilde{\text{Aut}}(P)_\beta
\]

should be the coarse moduli space of quasi-smooth hypersurfaces in \( P \) in the divisor class of \( \beta \). The problem is that \( \tilde{\text{Aut}}(P) \) need not be a reductive group, so that the quotient may not exist. However, it is well-known (see, for example, §2 of \([3]\)) that there is a nonempty invariant open set

\[
U \subset \{ f \in S_\beta : f \text{ is quasi-smooth} \}
\]

such that the geometric quotient

\[
U/\tilde{\text{Aut}}(P)_\beta
\]

exists.

**Definition 13.6** We call the quotient \( U/\tilde{\text{Aut}}(P)_\beta \) a generic coarse moduli space for hypersurfaces of \( P \) with divisor class \( \beta \).

We can relate the Jacobian ring \( R(f) \) to the generic coarse moduli space as follows.
Proposition 13.7 If $\beta$ is ample and $f \in S_\beta$ is generic, then $R(f)_\beta$ is naturally isomorphic to the tangent space of the generic coarse moduli space of quasi-smooth hypersurfaces of $P$ with divisor class $\beta$.

Proof. First note that $\widetilde{\text{Aut}}^0(P) \subset \widetilde{\text{Aut}}_\beta(P)$ has finite index. Thus, by shrinking $U$ if necessary, we may assume that

$$U/\widetilde{\text{Aut}}^0(P) \to U/\widetilde{\text{Aut}}_\beta(P)$$

is étale. Hence it suffices to identify $R(f)_\beta$ with the tangent space to $U/\widetilde{\text{Aut}}^0(P)$. Shrinking $U$ further, we may assume that the map

$$U \to U/\widetilde{\text{Aut}}^0(P)$$

is smooth (see §2 of [6]). Then the tangent space to a point of the generic moduli space is naturally isomorphic to the quotient of $S_\beta$ (= tangent space of $U$) modulo the tangent space to the orbit of $\text{Aut}_g(S) = \widetilde{\text{Aut}}^0(P)$ acting on $f \in S_\beta$.

Hence, to prove the proposition, we need to show that $J(f)_\beta$ is the tangent space to the orbit of $f$. But the tangent space to the orbit is given by the action of the Lie algebra of $\text{Aut}_g(S)$ on $f$. Since the Lie algebra consists of all derivations of $S$ which preserve the grading, its elements can be written in the form $\sum_{i=1}^n A_i \partial/\partial z_i$, where $A_i$ and $z_i$ lie in the same graded piece $S_{\beta_i}$ of $S$ for all $i$. Thus the action of the Lie algebra on $f$ gives the subspace $\{ \sum_{i=1}^n A_i \partial f/\partial z_i : A_i \in S_{\beta_i} \} = J(f)_\beta$, and the proposition is proved. \qed

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