Stabilization of Affine Systems with Polytopic Control Value Sets

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Abstract
The objective of this work is to design continuous feedback controls for global asymptotic stabilization (GAS) of affine systems, with control value set given by a convex polytope. This stabilization problem is solved based on a design of a feedback function restricted to the hyperbox and obtained by means of the CLF theory. By “normalizing” this feedback, the continuous stabilizer restricted to such control value set is obtained.

Keywords Stabilization · Admissible function · Lyapunov function

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1 Introduction
Consider the multiple input continuous-time affine system
\[ \dot{x} = f(x) + G(x)u, \]
where \( x \in \mathbb{R}^n \), \( f, g_i : \mathbb{R}^n \to \mathbb{R}^n \), for \( i = 1, \ldots, m \), are \( C^s(\mathbb{R}^n) \) vector fields (\( s \geq 0 \)), \( g_i(x) \) are the columns of the matrix \( G(x) \), and the control value set (CVS) is a bounded and convex subset of \( \mathbb{R}^m \). Without loss of generality, we shall assume that \( f(0) = 0 \). Such CVS will be required to be a sublevel set
\[ U_\phi = \{ u \in \mathbb{R}^m \mid \phi(u) \leq 1 \}, \]
where \( \phi : \mathbb{R}^m \to \mathbb{R}_+ \) is a convex and positively homogeneous function, that is, \( \phi(ru) = r\phi(u) \) for any real number \( r \geq 0 \); in particular, \( \partial U_\phi \) is given by the level set \( \{ u \in \mathbb{R}^m \mid \phi(u) = 1 \} \). We will assume that the set \( U_\phi \subset \mathbb{R}^m \) is compact and convex with \( 0 \in \text{int } U_\phi \).
The main goal of this work is to provide a continuous feedback function, restricted to any polytope CVS with the origin inside it, that stabilizes a family of nonlinear systems as Eq. 1.

It is well known the usefulness of discontinuous controls in system stabilization, mainly to obtain robustness and stabilization in finite time, see [15]. However, discontinuous controls lead to non-modeled instabilities (such as “chattering”, see [1]), then in this work we return to a continuous control design, with robustness properties, which can be used to stabilize affine systems with different CVS.

In order to obtain smooth stabilization, we consider the set of admissible feedback control functions \( \mathcal{U}_\phi \) defined by

\[
\mathcal{U}_\phi := \{ u : \mathbb{R}^n \to U_\phi \mid u(x) \text{ is continuous} \}.
\]

Given any convex bounded CVS \( U \), we seek to obtain continuous feedback control laws \( u(x) \in U \) that stabilize nonlinear systems of type (1). The relevance of this open problem was stated in [3]: “Find universal formulas for CLF stabilization, for general (convex) control-value sets \( U \)”. To address this important problem, a review of the work carried out to date suggests that it is convenient to separate the bounded and convex sets \( U \) in two classes, the sets \( U \) with smooth boundary and those with a non-smooth boundary. In the second one class, we can find the polytopes, whose boundary \( \partial U \) is piecewise linear.

Since Artstein’s theorem (see [2]) is valid in any bounded and convex CVS \( U \), under the assumption that a “control Lyapunov function” (CLF) is known, we will approach the stabilization problem according to the line of work established in [2, 9, 20]. In the works [13, 16, 17], for a CVS \( U \subset \mathbb{R}^m \) with a smooth boundary \( \partial U \), studies were presented that addressed the stabilization problem using the CLF theory. In the case of CVS a polytope, [19, 20] shows the existence of an optimal feedback control for system (1) that takes values at the vertices of the polytope and an explicit formula for it is obtained. In the case of the CVS represented by an asymmetric hyperbox, i.e., a rectangular parallelepiped whose faces are each one perpendicular to some of the basis vectors, in [8] a continuous and explicit feedback function is presented to globally stabilize the system (1).

In [8, Formula 27] an explicit and decentralized design of admissible feedback controls \( u^x(x) \) restricted to a hyperbox \( H \subset \mathbb{R}^m \) is presented, so that the proposed family of continuous controllers \( u^x(x) \) approaches the control that optimizes the robust stability margin. In this paper we extend the hyperbox constrained continuous stabilizer design to other sets of control values, including the polytope case. With the exception of the hyperbox, in the current literature there are no designs of continuous stabilizers restricted to a polytope.

For the study of the stabilization of nonlinear systems it is advisable to consult the book [14]. The delay stabilization problem is addressed in [4]. In [23] the stabilization problem with uncertainty is addressed.

It is worth mentioning that other approaches such as the use of integral inequalities, Local Approximations of Trajectories, or the use of homogeneous functions have recently been reported for the study of the problem of stabilization of control affine systems (see [10, 22, 25]).

2 Main Results

Given the system (1) with control value set given by the hyperbox \( H \), by means of a Lyapunov function \( V(x) \), a design of admissible feedback functions is explicitly presented in [8], with the property of being continuous, sub-optimal and decentralized.
As was mentioned above, the study of convex sets has been divided in the literature into

two large groups: the convex sets with smooth boundary (strictly convex) and polytopes. In [12] some concepts of convexity handled in this work can be consulted. In general, the problem of stabilizing system (1) is strongly related to the particular characteristics of the CVS $U$, such as the smoothness of its boundary $\partial U$. In the literature of CLF theory there are stabilization studies of affine systems (1), for a CVS $U$ bounded and strictly convex, with a smooth $\partial U$ boundary, articles [11, 16–18] and [21] correspond to this case.

In general, in the stabilization problem of the affine system (1) with a compact and convex set, is necessary a further work in finding explicitly admissible functions $u(x)$, with properties of smoothness and robustness.

2.1 Types of CVS $U_\phi$

Some results about convexity theory, considered implicitly in the development of this work can be found in [12].

Examples of the support function $\phi$ for a non-empty compact convex set $U_\phi$, are the following:

- $\phi_1(u) = L^T |u|$, where $L^T = (l_1, l_2, \ldots, l_m)$, with $l_i$ positive constants.
- $\phi_2(u) = u^T Qu$, where $Q \in \mathbb{R}^{m \times m}$ is a positive definite matrix.
- $\phi_3(u) = \max_{i=1,...,k} \{v_i^T u\}$, for $v_1, v_2, \ldots, v_k \in \mathbb{R}^m$ non-zero vectors.

For each such a support functions $\phi_i$, we assume that $0 \in \text{int} U_{\phi_i}$, so that $\phi_i(u) = 0$ only if $u = 0$. The sets $U_{\phi_i}$ represented by

$$U_{\phi_i} := \{u \in \mathbb{R}^m | \phi_i(u) \leq 1\}, \quad i = 1, 2, 3,$$

are compact and convex subsets of $\mathbb{R}^m$, where $U_{\phi_1}$ is a cross-polytope with $2^m$ faces and $2m$ vertices, and symmetric centered at the origin, $U_{\phi_2}$ is an ellipsoid also centered at the origin.

For every convex polytope $P \subset \mathbb{R}^m$, with $0 \in \text{int} P$, there are vectors \{v_1, v_2, \ldots, v_k\} $\in \mathbb{R}^m$, such that by means of the continuous non-negative function and piecewise linear (see [24, Theorem 1.1], [12, p. 174]),

$$\phi(u) = \max_{i=1,...,k} \{v_i^T u\}$$

so that we can represent the polytope $P$ as

$$P := \{u \in \mathbb{R}^m | \phi(u) \leq 1\},$$

which we can denote as $P_\phi$. In [19] and [20] polytopes are considered as CVS, giving rise to the corresponding set of admissible feedback functions $U_{\phi_i}$. Currently there are no continuous stabilizers restricted to polytopes.

Consider the affine system (1) with $P \subset \mathbb{R}^m$ being the control set, such that $P$ is compact convex given by Eq. 2. Our aim is to find a function $u(x)$ satisfying the following conditions:

1. $u : \mathbb{R}^m \to P$ is a continuous function;
2. $u(0) = 0$;
3. The equilibrium point $x = 0$ of the feedback system

$$\dot{x} = f(x) + G(x)u(x),$$

is global asymptotically stable.
The function \( u(x) \) is called an admissible feedback control function, or admissible stabilizer, equivalently \( u(x) \in \mathcal{U}_\phi \) according to Eq. 3 with
\[
\phi(u) = \max_{i=1, \ldots, k} \{v_i^T u\}. \tag{2}
\]

2.2 Lyapunov Function and Artstein’s Theorem

The admissible stabilizer is obtained based on Artstein’s theorem, see [2]. Suppose that system (1) is regular and \( \mathcal{U}_\phi \subset \mathbb{R}^m \) is a CVS. There is a smooth Lyapunov function \( V(x) \) if there is a continuous control \( u(x) \), except possibly at \( x = 0 \), restricted to taking values in \( \mathcal{U}_\phi \), which generates the stabilization of the system (1).

Given the system (1) and the CVS \( \mathcal{U}_\phi \), to obtain an admissible stabilizer \( u(x) \in \mathcal{U}_\phi \), two conditions must be met: the CLF condition and the SCP property.

The CLF Condition A non-negative function \( V : \mathbb{R}^n \to \mathbb{R}_+ \) is called the control Lyapunov function (CLF), with respect to the system (1) and the constraint \( \mathcal{U}_\phi \), if it happens that
\[
\min_{u \in \mathcal{U}_\phi} \{a(x) + \beta(x) \cdot u\} < 0, \quad \text{for all } x \neq 0, \tag{3}
\]
where
\[
a(x) := L_f V(x) \quad \& \quad \beta(x) := (\beta_1(x), \ldots, \beta_m(x)), \quad \text{with } \beta_i(x) := Lg_i V(x), \quad i = 1, \ldots, m. \tag{4}
\]
This inequality means that there is an optimal stabilizer \( \omega(x) \), which is not admissible because it is discontinuous; if the set \( \mathcal{U}_\phi \) is a polytope, the function \( \omega(x) \) takes values only at the vertices of the polytope (see [7, 8, 19] and [20]), and represents the control that gives the system the “best stabilization rate”, according to the derivative of the Lyapunov function \( V(x) \). A relevant purpose here is to find a continuous function that approaches \( \omega(x) \), without losing the previous inequality.

In [2], Zvi Artstein proved that the existence of a continuous stabilizing feedback control taking values in a convex CVS \( \mathcal{U} \subset \mathbb{R}^m \) is equivalent to the existence of a control Lyapunov function (CLF).

The SCP Property The existence of a continuous stabilizer at the origin is ensured by the small control property (SCP): For each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that we have the inequality
\[
a(x) + \beta(x) \cdot u < 0, \quad \text{for all } x \neq 0,
\]
for \( u \) with \( \|u\|_{\mathcal{U}_\phi} < \varepsilon \), provided that \( 0 < \|x\| < \delta \), \( a(x) \) and \( \beta(x) \) as were defined above for Eq. 3.

We consider that the control value set is given by the hyperbox
\[
H := [-r_1^-, r_1^+] \times \cdots \times [-r_m^-, r_m^+] \subset \mathbb{R}^m, \quad r_i^-, r_i^+ > 0,
\]
which can also be represented as
\[
H := \{u \in \mathbb{R}^m \mid \max_{i=1, \ldots, m} \{|u_i|/r_i \} \leq 1 \}
\]
where \( r_i \) for \( i = 1, 2, \ldots, m \), is defined as
\[
r_i(b) := \begin{cases} 
  r_i^+ & \text{if } b \geq 0, \\
  r_i^- & \text{if } b \leq 0.
\end{cases}
\]
Therefore, for compact sets $H$ and $U_\phi$, with $0 \in \text{int} \ U_\phi \subset H \subset \mathbb{R}^m$, it happens that
\[
\min_{u \in H} dV/dt \leq \min_{u \in U_\phi} dV/dt < 0,
\]
and the CLF condition and SCP property remain when changing the CVS $U_\phi$ to $H$, due mainly that the SCP property of a control system is local, for a neighborhood of $u = 0$ and $x = 0$.

### 2.3 An Explicit Feedback Control Formula with Respect to a Hyperbox.

The $\epsilon$-parameterized design ($\epsilon > 0$) of the family of feedback control functions $u^\epsilon(x)$ presented in [8, Theorem 14] is considered, which was obtained by means of the Artstein’s theorem with the hyperbox $H$ as CVS.

This feedback function $u^\epsilon(x)$ is admissible with the hyperbox $H$, explicitly given, decentralized and sub-optimal, defined as follows:
\[
u^\epsilon(x) := (u^\epsilon_1(x), \ldots, u^\epsilon_m(x))
\]
with
\[
u^\epsilon_i(x) = 
\begin{cases}
\varphi^\epsilon_i(a(x), h_1(\beta, r) \bar{\omega}_i(x)), & \text{if } \beta_i |r_i| > 0, \\
0, & \text{if } \beta_i |r_i| = 0,
\end{cases}
\]
where $\bar{\omega}(x)$ is the best rate control sharing the scheme of $\min_{u \in H} dV/dt$, with the non-negative function $h_1(\beta, r) := |\beta_1| r_1 + \cdots + |\beta_m| r_m$, defined in general for a set of vectors $\{v_1, \ldots, v_n\}$ as the $L_1$-norm of the Hadamard product (element-wise product), i.e.,
\[
h_1(v_1, \ldots, v_n) := \|v_1 \odot \cdots \odot v_n\|_1.
\]

The function $\varphi^\epsilon_i : \mathbb{R} \times [0, \infty] \rightarrow [0, 1]$, which is a regulating function that makes continuous $u^\epsilon_i(x)$ and is defined by
\[
\varphi^\epsilon_i(a, \beta) = \begin{cases} 1 - \left(1 - \frac{|a| + a}{2 h_1(\beta, r)} |\beta_i| |r_i| \right) \exp \left( \tau^\epsilon_i |\beta_i| |r_i| \right) & \text{if } |\beta_i| |r_i| > 0, \\
0 & \text{if } |\beta_i| |r_i| = 0,
\end{cases}
\]
and $\tau^\epsilon_i(x)$ is a non-positive function defined as
\[
\tau^\epsilon_i(x) = \begin{cases} m \frac{\ln(\lambda(x))}{\lambda(x)} - \varepsilon |\beta_i| |r_i| & \text{if } |\beta| |r| > 0, \\
0 & \text{if } |\beta| |r| = 0,
\end{cases}
\]
for $i = 1, \ldots, m$, where $\lambda(x) = 1 - \frac{1}{2}(|a(x)| + a(x))/h_1(\beta, r)$ and $\varepsilon > 0$ is a tuning parameter.

The control (5) - (6) is continuous with respect to $x$, since the regulating function $\varphi^\epsilon(a, \beta)$ cancels the discontinuities of the optimal stabilizer $\bar{\omega}(x)$.

### 3 An Explicit Feedback Control Formula Regarding CVS $U_\phi \subset H$

Given the stabilization problem with control restricted to a polytope, we present a way to design admissible stabilization functions. Consider the continuous feedback controls in a decentralized way $u^\epsilon(x)$, given by Eqs. 5 - 6. The main idea is to extend the feedback...
function \( u^\varepsilon(x) \) restricted to the hyperbox \( H \), so that the feedback function \( u^\varepsilon_{\phi}(x) \), restricted to the new CVS \( U_{\phi} \subset H \).

Once the set \( U_{\phi} \) has been defined, a hyperbox \( H \) such that \( U_{\phi} \subset H \) is chosen (the smallest possible). Let \( M \) be such that

\[
M := \max_H \phi(u),
\]

thus

\[
0 \leq \min_H \phi(u) \leq \phi(u) \leq \max_H \phi(u) = M,
\]

therefore, for the case \( 1 \leq \phi(u) \) and for any non-negative function \( a(x) \), we have

\[
\frac{a(x)}{M} \leq \frac{a(x)}{\phi(u)} \leq a(x).
\]

Proposition 1

If the function \( V(x) \) is CLF and satisfies the SCP with respect to the affine system (1) with CVS the hyperbox \( H \), then the feedback function \( w^\varepsilon_{\phi}(x) \) given by Eq. 8 is admissible and:

1. If \( a(x) > 0 \), then the feedback system (7)-(8) is GAS.
2. If \( a(x) \leq 0 \) for all \( x \neq 0 \), then the feedback system (1)-(8) is GAS.

Proof

Assume that \( a(x) > 0 \). The continuity of \( w^\varepsilon(x) \) is inherited from the continuity of \( u^\varepsilon(x) \), see [8, Prop. 12, Theorem 14]. For the case \( \phi(u^\varepsilon(x)) \leq 1 \) it is immediate, since \( w^\varepsilon(x) = u^\varepsilon(x) \). If \( \phi(u^\varepsilon(x)) \geq 1 \), then

\[
\frac{1}{\phi(u^\varepsilon(x))} u^\varepsilon_i(x),
\]

for \( i = 1, \ldots, m \), is a quotient of continuous functions, in fact, they are the components of the vector function \( \frac{1}{\phi(u^\varepsilon(x))} u^\varepsilon(x) \).

It is satisfied that \( w^\varepsilon(x) \in U_{\phi} \), since for the case \( \phi(u^\varepsilon(x)) \geq 1 \) we have that

\[
\phi \left( \frac{1}{\phi(u^\varepsilon(x))} u^\varepsilon(x) \right) = \frac{1}{\phi(u^\varepsilon(x))} \phi(u^\varepsilon(x)) = 1,
\]

since \( \phi \) is positively homogeneous.

Next, we prove that the feedback system (7)-(8) is globally asymptotically stable. With the design \( u^\varepsilon(x) \in H \) given by Eqs. 5-6, we have

\[
a(x) + b_1 u^\varepsilon_1(x) + \cdots + b_m u^\varepsilon_m(x) < 0 \quad \text{for all } x \neq 0,
\]
such that, for the case $\phi(u) \geq 1$, with $a(x) \geq 0$, we have

$$\frac{1}{M}a(x) \leq \frac{1}{\phi(u)}a(x) \leq a(x).$$

Therefore,

$$\frac{1}{M}a(x) + b_1 \frac{1}{\phi(u)}u_1(x) + \cdots + b_m \frac{1}{\phi(u)}u_m(x) < 0 \quad \text{for all } x \neq 0,$$

we conclude that the feedback system (7) - (8) is globally asymptotically stable.

For the proof of second part of the theorem we address the case $a(x) = 0$ (stable Lyapunov), since for $a(x) < 0$ the system is GAS with $u = 0$. From the CLF condition stated in inequality (3), if $a(x) = 0$, then

$$\min_{u \in U_\phi} \{\beta(x) \cdot u\} = -h_1(\beta(x), r) = -(|\beta_1|r_1 + \cdots + |\beta_m|r_m) < 0$$

for all $x \neq 0$, according to case 1 of Remark 15 of [8, p. 758], for $a(x) \leq 0$, we have

$$\varrho_i^e(\alpha, \beta) = 1 - e^{-\frac{|\beta_i|^2}{h_1(\beta(x), r)}}$$

considering that

$$u_i^e(x) = \begin{cases} -r_i\text{sign}(\beta_i)\varrho_i^e(x) & \text{if } |\beta_i|r_i > 0 \\ 0 & \text{if } |\beta_i|r_i = 0 \end{cases}$$

we obtain

$$\beta_i(x)u_i^e(x) = -\varrho_i^e(x)|\beta_i|r_i,$$

thus

$$\dot{V} = \beta(x) \cdot u^e(x) = -h_1(\rho^e(x), \beta(x), r) = -\left(\varrho_1^e(x)|\beta_1|r_1 + \cdots + \varrho_m^e(x)|\beta_m|r_m\right) < 0,$$

for all $x \neq 0$. Now, with the feedback $w^e(x) \in P$, with $a(x) = 0$, we have that $\dot{V} < 0$ for all $x \neq 0$, since

$$\dot{V} = \beta(x) \cdot w^e(x) = \begin{cases} -h_1(\rho^e(x), \beta(x), r) & \text{if } \phi(u^e(x)) \leq 1 \\ -\frac{1}{\phi(u^e(x))}h_1(\rho^e(x), \beta(x), r) & \text{if } \phi(u^e(x)) > 1. \quad \Box \end{cases}$$

**Remark 2** Rigid spacecraft stabilization problems news with angular velocity are currently being addressed. In [5], objective is to design a controller for the spacecraft under restrictions of the controls to a box, such that the states of the closed-loop system can be stabilized.

In example 21 of [8], a problem of stabilizing the angular velocity of a rigid spacecraft with CVS $H := [-1, 1]^3 \subset \mathbb{R}^3$ and $a(x) = 0$, the term $\dot{V}$ is given by

$$\dot{V} = -h_1(\rho^e(x), \beta(x), r)$$

$$= -\left(1 - e^{-\frac{|x_1|^2}{h_1(\beta(x), r)}}\right)|x_1| - \left(1 - e^{-\frac{|x_2|^2}{h_1(\beta(x), r)}}\right)|x_2| - \left(1 - e^{-\frac{|x_3|^2}{h_1(\beta(x), r)}}\right)|x_3|$$

$$< 0 \quad \text{for all } x \neq 0,$$

where $h_1(\beta(x), r) = |x_1| + |x_2| + |x_3|$. Here, it is also possible to stabilize by any function $\phi(u)$, with CVS defined by $U_\phi := \{u \in \mathbb{R}^3 : \phi(u) \leq 1\} \subset [-1, 1]^3$. 

\[ \text{Springer} \]
By [8, Formula (27)], we have the admissible feedback $u(x) = (u_1(x), \ldots, u_m(x))^T \in H$ with coordinate functions $u_i(x) = \rho_i^\varepsilon(x) \omega_i(x)$ and $\varepsilon > 0$ a tuning parameter, with a rescaling vector $\rho_i^\varepsilon(x) = (\rho_1^\varepsilon(x), \ldots, \rho_m^\varepsilon(x))$, and $\omega(x) = (\omega_1(x), \ldots, \omega_m(x))^T$ being the CLF-optimal solution of Eq. 4, for $u \in H$. So that, the formula (8) has components as follows,

$$
\begin{align*}
w_i^\varepsilon(x) &= \begin{cases} 
\rho_i^\varepsilon(x) \omega_i(x) & \text{if } \phi(u(x)) \leq 1, \\
\frac{1}{\phi(u^\varepsilon(x))} \rho_i^\varepsilon(x) \omega_i(x) & \text{if } \phi(u(x)) \geq 1.
\end{cases}
\end{align*}
$$

From [8, Theorem 14], if $\beta_i(x) \neq 0$, $i = 1, 2, \ldots, m$, then $\lim_{\varepsilon \to \infty} u_i^\varepsilon(x) = \omega(x)$, therefore the control (8) satisfies that,

$$
\lim_{\varepsilon \to \infty} v_i^\varepsilon(x) = \frac{1}{\phi(\omega(x))} \omega(x) \in \partial U_{\phi},
$$

since $\phi \left( \frac{1}{\phi(\omega(x))} \omega(x) \right) = \frac{1}{\phi(\omega(x))} \phi(\omega(x)) = 1$. Then, if $u_i^\varepsilon(x) \in \partial H$, it follows that $w_i^\varepsilon(x) \in \partial U_{\phi}$.

**Remark 3** Given an open-loop unstable system (i.e., $a(x) \geq 0$), from Eq. 3 with control (8) we have that the global instability of the system with feedback can be represented by the inequality

$$
\frac{1}{k} a(x) + \beta(x)w_i^\varepsilon(x) < 0, \text{ for all } x \neq 0 \text{ and for } k \geq M \geq 1,
$$

so that, the admissible formula (8) presents a tradeoff: the magnitude of the constant $M \geq 1$ is directly proportional to the size of the set $H \setminus U_{\phi}$, on such way that decreases the size of the instability $\frac{1}{k} a(x)$ in order to hold the above inequality.

**4 Example**

In order to describe the optimal stabilizer $\omega(x)$ that satisfies (3) with control restricted to a triangle, consider the affine system (1) with $m = 2$ and CVS given by the triangle $T$ defined by $T = \text{conv}\{v_0 = (0, -2), v_1 = (\sqrt{3}, 1), v_2 = (-\sqrt{3}, 1)\}$, depicted in Fig. 1A.

![Figure 1](image-url) **Fig. 1** Figure (A) illustrates a triangular CVS for the affine system (1), while figure (B) depicts the partitioned domain by cones of its optimal stabilizer. Figure (C) displays a typical CVS represented by a hyperbox.
Suppose we know a Lyapunov function \( V : \mathbb{R}^n \to \mathbb{R}_+ \), so that the CLF and SCP properties hold:

\[
\min_{u \in H} \dot{V} = a(x) + b_1 \omega_1 + b_2 \omega_2 < 0 \text{ for all } x \neq 0,
\]

where \( a(x) \), \( b_1(x) \) and \( b_2(x) \) were defined in Eq. 4. Hence, from [19, Formula 21], we obtain

\[
\omega(b_1, b_2) = \begin{cases} 
  v_1 = (\sqrt{3}, 1) & \text{if } (b_1, b_2) \in C_1, \\
  v_2 = (-\sqrt{3}, 1) & \text{if } (b_1, b_2) \in C_2, \\
  v_3 = (0, -2) & \text{if } (b_1, b_2) \in C_3,
\end{cases}
\]

where (see Fig. 1B)

\[
C_1 = \{ (b_1, b_2) \in \mathbb{R}^2 \mid b_1 \geq 0, b_2 \geq -\frac{b_1}{\sqrt{3}} \},
\]

\[
C_2 = \{ (b_1, b_2) \in \mathbb{R}^2 \mid b_1 \leq 0, b_2 \geq \frac{b_1}{\sqrt{3}} \},
\]

and

\[
C_3 = \{ (b_1, b_2) \in \mathbb{R}^2 \mid b_2 \leq 0, \sqrt{3}b_2 \leq b_1 \leq -\sqrt{3}b_2 \},
\]

such that \( \omega(b) = (\omega_1, \omega_2) \) is constant on each open polytopal cone \( \text{int } C_i \), and it is equal to the vertices of the triangle \( T \).

Instead, if the CVS is the hyperbox \( H \), defined by

\[
H := \text{conv}\{v_1 = (\sqrt{3}, 1), v_2 = (-\sqrt{3}, 1), v_3 = (-\sqrt{3}, -2), v_4 = (\sqrt{3}, -2)\},
\]

equivalently, defined as \( H := [-\sqrt{3}, \sqrt{3}] \times [-2, 1] = [-r_1^-, r_1^+] \times [-r_2^-, r_2^+] \subset \mathbb{R}^2 \) (see Fig. 1C), then the optimal stabilizer (with the best rate) is \( (\bar{w}_1, \bar{w}_2) \in H \), defined by the equality

\[
\min_{u \in H} \dot{V} = a(x) + b_1 \bar{w}_1 + b_2 \bar{w}_2 = a(x) - (|b_1| r_1 + |b_2| r_2),
\]

consequently (see example in [8]),

\[
\bar{w}(b_1, b_2) = \begin{cases} 
  v_1 & \text{if } (b_1, b_2) \in \text{cl}(\mathbb{R}^2_{++}), \\
  v_2 & \text{if } (b_1, b_2) \in \text{cl}(\mathbb{R}^2_{+}), \\
  v_3 & \text{if } (b_1, b_2) \in \text{cl}(\mathbb{R}^2_{-}), \\
  v_4 & \text{if } (b_1, b_2) \in \text{cl}(\mathbb{R}^2_{--}).
\end{cases}
\]

The CLF condition for system (1) with CVS \( H \) implies that \( \min_{u \in H} \dot{V} < 0 \) for all \( x \neq 0 \), so that for \( a(x) \geq 0 \), the inequality

\[
0 \leq \frac{a(x)}{\beta(x)} < 1,
\]

is satisfied, or else

\[
0 \leq \frac{a(x) + |a(x)|}{2\beta(x)} < 1.
\]
Hence, we define the following non-negative functions:

\[
    r_i(b_i) := \begin{cases} 
        r_i^+ & \text{if } b_i \geq 0, \\
        r_i^- & \text{if } b_i \leq 0, 
    \end{cases}, \quad i = 1, 2,
\]

\[
    \beta := |b_1|r_1 + |b_2|r_2,
\]

\[
    |a| + a := |L_x V(x)| + L_x V(x)
\]

\[
    \lambda(x) := 1 - \frac{1}{2}(|a(x)| + a(x))/\beta(x)
\]

and a non-positive function defined as

\[
    \tau^\varepsilon_i(x) = \begin{cases} 
        m \ln \left( \frac{\lambda(x)}{\lambda(x)} \right) - \varepsilon |b_i|r_i & \text{if } \beta > 0, \\
        0 & \text{if } \beta = 0,
    \end{cases}
\]

with \( \varepsilon > 0 \) is a tuning parameter. The function \( \varrho_i^\varepsilon : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R} \) is defined by

\[
    \varrho_i^\varepsilon(a, \beta) = \begin{cases} 
        1 - \left( 1 - \frac{|a| + a}{2\beta} \frac{|b_i|r_i}{\beta} \right) \exp \left( \frac{\tau^\varepsilon_i |b_i|r_i}{\beta} \right) & \text{if } |b_i|r_i > 0, \\
        0 & \text{if } |b_i|r_i = 0.
    \end{cases}
\]

This feedback function \( u^\varepsilon(x) \) is admissible with the hyperbox \( H \), explicitly given and sub-optimal, defined as follows:

\[
    u^\varepsilon(x) := (u_1^\varepsilon(x), u_2^\varepsilon(x))
\]

with

\[
    u_i^\varepsilon(x) = \varrho_i^\varepsilon(a(x), \beta(x))\omega_i(x), \quad i = 1, 2.
\]

In particular, for the affine system

\[
    f(x_1, x_2) = \begin{pmatrix} 
        \frac{5\sqrt{3}}{2} & x_2 \\
        5 & \frac{x_1}{1 + x_1^2} + \frac{5}{2} x_2 \\
        \frac{1}{2} + x_1^2 & \frac{1}{2} + x_2^2
    \end{pmatrix}, \quad g_1 = \begin{pmatrix} 
        1 \\
        0
    \end{pmatrix}, \quad g_2 = \begin{pmatrix} 
        0 \\
        1
    \end{pmatrix},
\]

with rectangular CVS \( H = [-\sqrt{3}, \sqrt{3}] \times [-2, 1] \), it is possible to generate (see formulas (29)-(31) in [8]) the continuous control \( u^\varepsilon(x) := (u_1^\varepsilon(x), u_2^\varepsilon(x)) \), given by

\[
    u_i^\varepsilon(x) = \varrho_i^\varepsilon(a(x), \beta(x))\omega_i(x), \quad i = 1, 2.
\]

Suppose we know a Lyapunov function \( V : \mathbb{R}^2 \rightarrow \mathbb{R}_+ \), so that the CLF and SCP properties hold. With

\[
    V(x_1, x_2) = \frac{1}{2} \left( x_1^2 + x_2^2 \right),
\]

then

\[
    \dot{V} = x_1 \left( \sqrt{3} \frac{x_2}{1 + x_2^2} + u_1 \right) + x_2 \left( \frac{x_1}{1 + x_1^2} + \frac{x_2^2}{1 + x_2^2} + u_2 \right)
\]

so that

\[
    a(x) = x_1 \left( \sqrt{3} \frac{x_2}{1 + x_2^2} \right) + x_2 \left( \frac{x_1}{1 + x_1^2} + \frac{x_2^2}{1 + x_2^2} \right)
\]
and \( b_1 = x_1, b_2 = x_2 \), therefore
\[
\begin{align*}
    r_1(x_1) &:= \sqrt{3}, \\
    r_2(x_2) &:= \begin{cases} 
1 & \text{if } x_2 \geq 0, \\
2 & \text{if } x_2 \leq 0,
\end{cases} \\
    \beta &:= \sqrt{3}|x_1| + r_2|x_2|.
\end{align*}
\]

So that the SCP property is satisfied:
\[
\lim_{(x_1, x_2) \to (0,0)} a(x_1, x_2) \beta(x_1, x_2) = \lim_{(x_1, x_2) \to (0,0)} x_1 \left( \sqrt{3} \frac{x_2}{1 + x_2^2} \right) + x_2 \left( \frac{x_1}{1 + x_1^2} + \frac{x_2^2}{1 + x_2^2} \right) = 0,
\]
also the CLF property is satisfied:
\[
a(x_1, x_2) + \min_{u \in H} \{ x_1u_1 + x_2u_2 \} < 0, \text{ for all } (x_1, x_2) \neq (0,0).
\]

The CLF and SCP properties allow the design of the continuous stabilizer \( u(x) := (u_1^x(x), u_2^x(x)) \in H \).

The optimal stabilizer (with the best rate) is \((\bar{\omega}_1, \bar{\omega}_2) \in H\), defined by the equality
\[
\min_{u \in H} \dot{V} = a(x) + b_1\bar{\omega}_1 + b_2\bar{\omega}_2 \\
= a(x) - (|b_1|r_1 + |b_2|r_2),
\]
consequently (see example in [8])
\[
\bar{\omega}(b_1, b_2) = \begin{cases} 
v_1 & \text{if } (b_1, b_2) \in \text{cl}(\mathbb{R}^2_{++}), \\
v_2 & \text{if } (b_1, b_2) \in \text{cl}(\mathbb{R}^2_{+-}), \\
v_3 & \text{if } (b_1, b_2) \in \text{cl}(\mathbb{R}^2_{-+}), \\
v_4 & \text{if } (b_1, b_2) \in \text{cl}(\mathbb{R}^2_{--}).
\end{cases}
\]

so that, by a straightforward calculation we get that,
\[
\min_{u \in H} \dot{V} = a(x) - (|b_1|r_1 + |b_2|r_2) < 0, \text{ if } x \neq 0.
\]

The CLF and SCP properties allow the design of the continuous stabilizer \( u(x) := (u_1^x(x), u_2^x(x)) \in H \).

The phase portrait for this example is shown in Fig. 2, and the script used for its generation is available in [6].

### 5 Results and Discussion

In the present work, we address the problem of the global stabilization of an affine system thought a continuous feedback function restricted to an \( m \)-dimensional CVS \( U_\phi \) convex and bounded, such that \( 0 \in \text{int } U_\phi \) and \( \phi : \mathbb{R}^m \to \mathbb{R} \) is a non-negative positively homogeneous function.

We address specially the case of convex polytopes, defined through the inequality \( \phi(u) := \max_i \{ u_i^T u \} \leq 1 \), where \( \phi(u) \) is a continuous piecewise linear function. For this
case in concrete, we show the solution to the CLF-optimization problem (4), represented by a feedback function $\omega(x)$ taking values at the vertices of the polytope, on such way that is not admissible because it is discontinuous.

In general, for any CVS $U_\phi$ we can design an admissible control for the system (7) using an explicit formula of admissible feedback $u^f(x)$ with CVS given by an $m$-dimensional asymmetric hyperbox $H$, designed to stabilize globally the affine system (1) under the CLF and SCP conditions of the Artstein’s theorem, in such way that restricted to $U_\phi \subset H$ we get an admissible feedback $w^f(x)$ that stabilize globally (7) for some value $M > 1$. Some properties of $u^f(x) \in H$, such as continuity and the extreme values reaching, are inherited to $w^f(x)$; in such way that if $u^f(x) \in \partial H$, then $w^f(x) \in \partial U_\phi$.

**Declarations**

**Conflicts of interest** The authors declare that they have no conflict of interest regarding the publication of this paper.

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