A LEADING TERM FOR THE VELOCITY OF STATIONARY VISCOUS INCOMPRESSIBLE FLOW AROUND A RIGID BODY PERFORMING A ROTATION AND A TRANSLATION

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Abstract. We consider the Navier-Stokes system with Oseen and rotational terms describing the stationary flow of a viscous incompressible fluid around a rigid body moving at a constant velocity and rotating at a constant angular velocity. In a previous paper, we proved a representation formula for Leray solutions of this system. Here the representation formula is used as starting point for splitting the velocity into a leading term and a remainder, and for establishing pointwise decay estimates of the remainder and its gradient.

1. Introduction. Consider a rigid body moving with constant velocity and rotating with constant angular velocity in a viscous incompressible fluid. Suppose the flow around the body is steady (“developed”). Then the usual mathematical model of this flow, with respect to a reference frame in which the body is at rest, is given by the Navier-Stokes system with an Oseen term and rotational terms. Written in normalized form (see [41] for details), this system takes the form

\[ \begin{align*}
-\Delta u(x) + \tau \partial_1 u(x) + \tau (u(x) \cdot \nabla) u(x) - (\varrho e_1 \times x) \cdot \nabla u(x) \\
+ \varrho e_1 \times u(x) + \nabla \pi(x) = f(x), \quad \text{div } u(x) = 0 \quad \text{for } x \in \mathbb{R}^3 \setminus \mathcal{D},
\end{align*} \]

and is supplemented by homogeneous Dirichlet boundary conditions at infinity,

\[ \begin{align*}
|u(x)| \to 0 \quad \text{for } |x| \to \infty.
\end{align*} \]

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In this model, the open bounded set $D \subset \mathbb{R}^3$ corresponds to the rigid body, the unknown functions $u$ and $\pi$ describe respectively the velocity and the pressure field of the flow, whereas the given function $f$ represents a prescribed volume force acting on the fluid. The parameter $\tau \in (0, \infty)$ is the Reynolds number, and $\varrho \in \mathbb{R}\{0\}$ the Taylor number. We assume the Reynolds number and the Taylor number to be fixed throughout the paper. The symbol $e_1$ stands for the unit vector $(1, 0, 0)$. The constant velocity of the body, as seen from an observer at rest, is given by $-\tau e_1$, and its constant angular velocity by $\varrho e_1$. In particular, the direction of translation and the axis of rotation of the rigid body are parallel, as should be expected in the case of a steady-state flow around that body.

We are interested in Leray solutions to (1), (2), that is, solutions $(u, \pi)$ such that $u \in L^3(\mathbb{R}^3)^3$, $\nabla u \in L^2(\mathbb{R}^3)^9$ and $\pi \in L_{loc}^2(\mathbb{R}^3)$. This type of solution is known to exist even for large data, under suitable regularity assumptions on $\partial D$ and $f$ (\cite[Theorem XI.3.1]{36}). Note that the conditions $u \in L^6(\mathbb{R}^3)^3$, $\nabla u \in L^2(\mathbb{R}^3)^9$ mean in particular that (2) holds in a weak sense (\cite[Theorem II.6.1]{36}). Another property of Leray solutions that is well known by now concerns their asymptotics far from $D$. In fact, independently of any boundary condition on $\partial D$, the velocity part $u$ of such a solution satisfies the relation

$$|\partial^\alpha u(x)| = O\left(|x| s_\tau(x)\right)^{-1-|\alpha|/2} \quad \text{for} \quad |x| \to \infty,$$

with $\alpha \in \mathbb{N}_0^3$ with $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3 \leq 1$, if $|f(x)|$ tends to zero sufficiently fast for large values of $|x|$ (\cite[10]{37}). Note that in the case $\alpha = 0$, the relation in (3) describes the decay of $u$, whereas the asymptotic behavior of $\nabla u$ is specified by (3) if $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = 1$. The term $s_\tau(x)$ is defined as

$$s_\tau(x) := 1 + \tau(|x| - x_1) \quad (x \in \mathbb{R}^3).$$

Since the vector $-\tau e_1$ corresponds to the velocity of the body as seen from an observer at rest, the factor $s_\tau(x)^{-1-|\alpha|/2}$ in (3) should be considered as a mathematical manifestation of the wake extending behind a body moving in a viscous incompressible fluid. In view of (3), it is natural to look for a vector-valued function $L$ (“leading term”) with the property that $L(x)$ decays with exactly the rate given by the right-hand side of (3) with $\alpha = 0$, whereas the term $(u-L)(x)$ (“remainder”) decays pointwise at a rate which is higher than the one in (3) with $\alpha = 0$. In addition, an analogous statement should hold for $\partial^\alpha L(x)$ if $|\alpha| = 1$. Identifying such a function $L$ is highly interesting from a physical point of view because it gives a good idea of the structure (“profile”) of the flow as observed at some distance from the rigid body. This interpretation of $L$ explains why a leading term should be expressed, if possible, as an explicit combination of elementary functions. Actually, beyond the physical meaning of such a term, it might even be argued that in any case, the notion of leading term refers to functions that are given in a more or less explicit way.

A leading term is relevant also in some mathematical applications. For example, numerical computation of exterior flows often involve bounded computational domains, with an artificial boundary condition on the outer boundary of such domains. The choice of such boundary conditions, and related error estimates, are determined by the asymptotic structure of the flow field, and thus by a leading term; see [2, 3, 27].

For some types of flows, a leading term with all relevant features could be exhibited. This is true if $\varrho = 0$ and the Oseen term in (1) is dropped (case of a body...
which neither rotates nor translates, or which is rotationally symmetric and rotates but does not translate). In this situation, Korolev and Sverák [51] could show that
$$L$$
is given by the so-called Landau solution of the stationary Navier-Stokes system
\[-\Delta w + \tau (w \cdot \nabla)w + \nabla q = 0, \quad \text{div } w = 0 \quad \text{in } \mathbb{R}^3 \setminus \{0\},
\]
under the assumption that the velocity is small in a suitable sense. In a series of papers due to Farwig, Hishida [20], [21], [22] and Farwig, Galdi, Kyed [17], a satisfactory theory could also be developed for the case that the Oseen term \( \tau \partial_1 u \) is not present in (1) but the parameter \( \varrho \) does not vanish (flow around a body that rotates but does not translate: “purely rotational case”). In this situation, if \( \varrho \) is small, it turned out that a leading term is again given by the Landau solution. On the other hand, if the Oseen term is present but \( \varrho \) vanishes (case of a body that translates but does not rotate; “purely translational case”), it was shown first for solutions that are “strong” in a suitable sense (see [28], [29]), and then for Leray solution (see [3], [36] and the references in [36]) that a leading term may be given by an appropriate linear combination of the columns of the (matrix-valued) velocity part of a fundamental solution to the stationary Oseen system
\[-\Delta v + \tau \partial_1 v + \nabla \gamma = \varrho, \quad \text{div } v = 0.
\]
This matrix-valued function – we denote it by \( \mathcal{O} \) (see Section 2 for its definition) – is completely explicit, in the sense that each entry of \( \mathcal{O}(x) \), for \( x \in \mathbb{R}^3 \setminus \{0\} \), consists of sums, products and quotients of \( x_1, x_2, x_3, |x|, e^{-\tau(|x|-x_1)/\varrho} \) and \( \tau \). No smallness conditions are involved in the purely translational case.

For the flow considered here – around a body performing a rotation and a translation –, only partial results are known. They are due to Kyed [61], who considered a leading term given by the product of a coefficient times a column of \( \mathcal{O} \). He showed that the corresponding remainder \( R(x) \) belongs to \( L^p \)-spaces in the complement \( B_S^c \) of a ball \( B_S := \{ y \in \mathbb{R}^3 : |y| < S \} \) containing \( \mathcal{O} \), where the parameter \( p > 1 \) can be chosen so close to 1 that the function \( (|x| s_+(x))^{-1} \) is excluded from \( L^p(B_S^c) \).

This indicates but does not prove that the remainder might decay faster than \( (|x| s_+(x))^{-1} \) for \( |x| \to \infty \), that is, faster than the rate given by (3) for \( u(x) \). Using a similar reasoning, the author suggests that the gradient of his remainder may exhibit a more rapid pointwise decay than the one given by (3) for \( \nabla u(x) \). Again a similar question remains open whether this is actually the case.

In [62], Kyed indicates that \( |R(x)| \) behaves as \( O(|x|^{-1/(3+\varrho)}) \) if \( |x| \to \infty \), for some arbitrary but fixed \( \epsilon > 0 \).

As a further restriction, the theory in [61] only covers Leray solutions satisfying a no-slip boundary condition on \( \partial \mathcal{O} \), and it requires \( f = 0 \). Concerning the argument adopted in [61] in order to derive these results, it is based on certain estimates of solutions to the time-periodic Oseen system in the whole space \( \mathbb{R}^3 \). These estimates, in turn, are obtained via Fourier expansions of these solutions, an approach which ultimately reduces to a study of solutions to the standard Oseen system (5) in the whole space \( \mathbb{R}^3 \).

It is the purpose of the work at hand and of a companion paper [12] to present an improved theory on a leading term associated with problem (1), (2). Our aim is to bring this theory up to the level attained in the purely translational case (\( \varrho = 0 \)). More precisely, we will study Leray solutions to (1), (2) without requiring any boundary condition on \( \partial \mathcal{O} \).
The leading term we will arrive at is of the same kind as the one in [61], that is, a coefficient times a column of $\mathcal{D}$, which means in particular that the rotation does not contribute anything to that term. We will show that the coefficient in question depends in an explicit way on the restriction of $u$, $\nabla u$ and $\pi$ to $\partial \Omega$. (At the end of this section, we comment on the regularity issues this situation raises.) If the boundary condition imposed in [61] is satisfied, our coefficient coincides with the one in that reference. This was announced in [12] and proved in [12].

Moreover – and this is the focus of the work at hand – we will establish optimal pointwise decay rates for the remainder term and its gradient (Theorem 3.1) which is the main result of this article). Thus we are able to answer the questions left open in [61]. For the leading term we actually look for. As concerns the other two columns, the splitting of $u$, $\nabla u$ and $\partial \Omega$, where the pair $(u, \pi)$ is a given Leray solution of (1), (2). (As we may recall, we will comment further below on the regularity issues this definition raises.) The principal result of the present article (Theorem 3.1) then states that the remainder $R_j(x) := u_j(x) - L_j(x)$ and its gradient decay with an optimal rate for $|x| \to \infty$.

This leaves open the question how to get from the leading term to another one involving $\mathcal{D}(x)$. The answer is given in the companion paper [12], where it is shown that the coefficients in our leading terms and coefficients in [61], coincide if we take into account also the boundary conditions. In [12] we deal with the integral on $(0, \infty)$ appearing in each entry of $\mathcal{D}(x, 0)$. For the entries in the first column of $\mathcal{D}(x, 0)$, we will be able to compute the integral in question, obtaining the first column of $\mathcal{D}(x)$, which, after multiplication by $b_1$, corresponds to the leading term we actually look for. As concerns the other two columns, the integral appearing in their entries turns out to be oscillating, and therefore may be shown to decay more rapidly – as a function of $x$ – than the right-hand side of (3). As a consequence, we may subsume these entries in the remainder term, thus completing our argument.

Let us briefly indicate what is the principal technical difficulty we have to deal with in the work at hand. Recall that we intend to estimate the remainder $R(x)$ and its gradient for large values of $|x|$. To this end we will start with an integral representation of $u(x)$ proved in [77] (weak solutions) and in [11] (Leray solutions), and restated below as Theorem 2.15. One of the integrals appearing in this representation is the volume potential $\int_{\mathcal{D}} \mathcal{D}(x, y) [(u \cdot \nabla)u](y) dy$. When we look for a bound of the gradient of the remainder, this integral gives rise to the problem of estimating the term $\int_{\mathcal{D}} \partial_x \partial_{x_m} \mathcal{D}(x, y) [(u \cdot \nabla)u](y) dy$, for $x$ from outside a ball $B_S$, and for $1 \leq l, m \leq 3$. The factor $[(u \cdot \nabla)u](y)$ in this term can be handled by...
using \( \mathfrak{F} \). As concerns the second derivatives \( \partial_x \partial_x \mathfrak{F}(x, y) \) of \( \mathfrak{F}(x, y) \), however, no suitable bounds are available in literature, so we have to return to the definition of \( \mathfrak{F}(x, y) \), which, as we may recall, involves an integral on \((0, \infty)\). In this way we arrive at an integral on \( \mathfrak{F} \times (0, \infty) \), which we split into several parts. Each of these parts will then be estimated separately (proof of Theorem 3.1).

We mentioned above that the coefficients \( b_j \) \((1 \leq j \leq 3)\) in our leading term are defined via restrictions of \( u \), \( \nabla u \) and \( \pi \) to \( \partial \mathcal{D} \). Of course, this is feasible only if the boundary \( \partial \mathcal{D} \) exhibits some regularity, and if \( \nabla u \) and \( \pi \) are somewhat more regular near \( \partial \mathcal{D} \) than required for Leray solutions. This situation is of interest when the link between boundary conditions and the asymptotic of \( u \) is to be studied. However, our theory does not depend on any additional regularity of \( u \) or \( \pi \) beyond what is required for Leray solutions.

Concerning further articles related to the work at hand, we mention the fundamental paper [6], where the representation formula (Theorem 2.15) is derived. For the properties of the representation formula see [8]–[10], [13]. With respect to the purely rotational linearized case in an \( L^q \) framework, we refer to the work of Farwig et al. see [23]. For an approach with the Kondratiev type weight functions, see [1]. A case where also translation is included see [10]. Concerning nonlinear \( L^q \) setting we refer to the work of Farwig, Hishida [19], Heck et al. [50]. A case with nondecaying initial data was studied by Giga et al. [49]. The spectrum of the “rotating” Stokes operator was studied by Farwig and Neustupa [26]. A weighted analogue can be found in the work of Farwig et al. see [24, 25]. Let us mention the work of Hishida who proved that semigroup generated by the Stokes operator with rotating terms is not analytical. The extention to \( L^p \) case can be found in the work of Hieber et al. [42]. \( L^p - L^q \) estimates were studied by Hishida and Shibata [48] and [49]. An investigation of Leray solution can be found in [38–40]. Anisotropic weights in an \( L^2 \) or \( L^q \) framework were studied in [52–55]. An asymptotic profile of both linearized and nonlinear problems were exhibited in [59–61]. A different subject is rotating fluids where the Coriolis force or centrifugal force play important role. In this point let us mention the work of Feireisl et al. [30, 31], where singular limits are studied. Also the case of rotating fluids was studied for rough boundaries [34].

2. Notation and preliminaries. The open bounded set \( \mathcal{D} \subset \mathbb{R}^3 \) introduced in Section 1 will be kept fixed throughout. We assume its boundary \( \partial \mathcal{D} \) to be of class \( C^2 \), and we denote its outward unit normal by \( n(\mathcal{D}) \). The numbers \( \tau \) and \( \varrho \) and the vector \( \omega \) also introduced in Section 1 will be kept fixed, too. Define the matrix

\[
\Omega := \begin{pmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{pmatrix} = \varrho \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix},
\]
so that \( \omega \times x = \Omega \cdot x \) for \( x \in \mathbb{R}^3 \).

Let us denote \( s(x) := s_1(x) = 1 + (|x| - x_1) \). We recall that the function \( s_r \) was defined in Section 1, as was the notation \( |\alpha| \) for the length of a multi-index \( \alpha \in \mathbb{N}_0^3 \). If \( A \subset \mathbb{R}^3 \), we write \( A^c \) for the complement \( \mathbb{R}^3 \setminus A \) of \( A \). The open ball centered at \( x \in \mathbb{R}^3 \) and with radius \( r > 0 \) is denoted by \( B_r(x) \). If \( x = 0 \), we will write \( B_r \) instead of \( B_r(0) \). Put \( e_1 := (1, 0, 0) \). Let \( x \times y \) denote the usual vector product of \( x, y \in \mathbb{R}^3 \). For \( T \in (0, \infty) \), set \( \mathcal{D}_T := B_T \setminus \overline{B} \) (“truncated exterior domain”). By the symbol \( \mathcal{C} \), we denote constants only depending on \( \mathcal{D} \), \( \tau \) or \( \omega \). We write \( \mathcal{C}(\beta_1, \ldots, \beta_n) \) for positive constants that additionally depend on parameters \( \beta_1, \ldots, \beta_n \in \mathbb{R} \), for some \( n \in \mathbb{N} \). As usual, \( C(\gamma_1, \ldots, \gamma_n) \) means a positive constant only depending on \( \gamma_1, \ldots, \gamma_n \).

We begin by introducing the fundamental solutions used in what follows. We set
\[
\begin{align*}
K(x, t) &= (4\pi t)^{-3/2}e^{-|x|^2/4t}, \quad x \in \mathbb{R}^3, \quad t \in (0, \infty), \\
N_{jk}(x) &= x_j x_k |x|^{-2}, \quad x \in \mathbb{R}^3 \setminus \{0\}, \\
A_{jk}(x, t) &= K(x, t) \left( \delta_jk - N_{jk}(x) - \frac{1}{4t} \left( \frac{|x|^2}{4t} \right) \right) \left( \delta_jk/3 - N_{jk}(x) \right), \\
E_{jk}(x) &= (4\pi)^{-1}x_j |x|^{-3}, \quad 1 \leq j \leq 3, \quad x \in \mathbb{R}^3 \setminus \{0\}, \\
1_{F_1}(1, c, u) &= \sum_{n=0}^{\infty} \left( \frac{\Gamma(n+c)}{\Gamma(n+c)} \right) \cdot u^n \quad \text{for} \quad u \in \mathbb{R}, \quad c \in (0, \infty),
\end{align*}
\]

where \( \Gamma \) denotes the usual Gamma function. In the following, the letter \( \Gamma \) will stand for the matrix-valued function defined by
\[
(\Gamma_{jk}(y, z, t))_{1 \leq j, k \leq 3} := (\Lambda_{jk}(y - \tau t (\varepsilon_1 - e^{-it} \cdot z, t))_{1 \leq j, k \leq 3} \cdot e^{-it}, \quad y, z \in \mathbb{R}^3, \quad t \in (0, \infty) \quad \text{with} \quad y \neq \tau t (\varepsilon_1 - e^{-it} \cdot z) \neq 0.
\]

We will use the ensuing estimate, which was proved in [15].

**Lemma 2.1.** Let \( \beta \in (1, \infty) \). Then \( \int_{\partial B_r} s_r(x)^{-\beta} \, dx \leq \mathcal{C}(\beta) r^{-\beta} \) for \( r \in (0, \infty) \).

Our following lemma restates [6, Corollary 3.1]:

**Lemma 2.2.** The function \( \Gamma \) may be continuously extended to a function from \( C^\infty(\mathbb{R}^3 \times \mathbb{R}^3 \times (0, \infty)) \).

According to [6, Theorem 3.1], we have

**Lemma 2.3.** \( \int_0^\infty |\Gamma_{jk}(y, z, t)| dt < \infty \) for \( y, z \in \mathbb{R}^3 \) with \( y \neq z, \quad 1 \leq j, k \leq 3 \).

Thus we may define
\[
\vartheta_{jk}(y, z) := \int_0^\infty \Gamma_{jk}(y, z, t) dt
\]

for \( y, z \in \mathbb{R}^3 \) with \( y \neq z, \quad 1 \leq j, k \leq 3 \).

We will use the following technical lemmas:

**Lemma 2.4.** Let \( \delta > 0 \). Assuming \( z \in B_\delta(x) \), \( x \in \mathbb{R}^3 \) we have
\[
|z| \geq |x|/2, \quad \text{for} \quad |x| \geq 2\delta, \tag{6}
\]
\[
s_r(z)^{-1} \leq \mathcal{C}(1 + |x - z|) s_r(x)^{-1} \leq \mathcal{C}(\delta) s_r(x)^{-1}. \tag{7}
\]
Proof. For $|x| \geq 2\delta$ we have $|z| \geq |x| - |x - z| \geq |x| - \delta \geq |x|/2$, i.e. the relation (8) is satisfied. For the proof of (8) see [51, Lemma 4.8].

Lemma 2.5 ([8 Corollary 3.1]). Let $j, k \in \{1, 2, 3\}$, $\alpha, \beta \in \mathbb{N}_0^3$ with $|\alpha + \beta| \leq 2$, $y, z \in \mathbb{R}^3$, $t \in (0, \infty)$. Then

$$|\partial_y^\alpha \partial_z^\beta \Gamma_{jk}(y, z)| \leq C (|y - \tau t e_1 - e^{-t \cdot \Omega} \cdot z|^2 + t)^{-3/2 - |\alpha + \beta|/2}.$$  

Lemma 2.6 ([7 Theorem 2.19]). Let $S_1, S \in (0, \infty)$ with $S_1 < S$, $\nu \in (1, \infty)$. Then

$$\int_0^\infty (|y - \tau t e_1 - e^{-t \cdot \Omega} \cdot z|^2 + \tau)^{-\nu} dt \leq C(S_1, S, \nu) (|y| \cdot s_r(y))^{\nu + 1/2}$$  

for $y \in B_S^R$, $z \in \overline{B_{S_1}}$.

Lemma 2.7 ([8 Lemma 3.2]). Let $j, k \in \{1, 2, 3\}$. For $\alpha, \beta \in \mathbb{N}_0^3$ with $|\alpha + \beta| \leq 2$, $y, z \in \mathbb{R}^3$ with $y \neq z$, the function $(0, \infty) \ni t \mapsto \partial_y^\alpha \partial_z^\beta \Gamma_{jk}(y, z, t) \in \mathbb{R}$ is integrable, the derivative $\partial_y^\alpha \partial_z^\beta \Gamma_{jk}(y, z)$ exists, and

$$\partial_y^\alpha \partial_z^\beta \Gamma_{jk}(y, z) = \int_0^\infty \partial_y^\alpha \partial_z^\beta \Gamma_{jk}(y, z, t) dt.$$  

Moreover, for $\alpha, \beta$ as before, the derivative $\partial_y^\alpha \partial_z^\beta \Gamma_{jk}(y, z)$ is a continuous function of $y, z \in \mathbb{R}^3$ with $y \neq z$.

Lemma 2.8. Let $S_1, S \in (0, \infty)$ with $S_1 < S$, $\alpha, \beta \in \mathbb{N}_0^3$ with $|\alpha + \beta| \leq 2$, $1 \leq j, k \leq 3$. Then

$$|\partial_y^\alpha \partial_z^\beta \Gamma_{jk}(y, z)| \leq C(S_1, S) (|y| \cdot s_r(y))^{1 - |\alpha + \beta|/2}$$  

for $y \in B_S^R$, $z \in \overline{B_{S_1}}$,

$$|\partial_y^\alpha \partial_z^\beta \Gamma_{jk}(y, z)| \leq C(S_1, S) (|z| \cdot s_r(z))^{1 - |\alpha + \beta|/2}$$  

for $y \in B_S^R$, $y \in \overline{B_{S_1}}$.

Proof. Lemma 2.6 - 2.7

Lemma 2.9 ([8 Theorem 3.1]). Let $k \in \{0, 1\}$, $R \in (0, \infty)$, $y, z \in B_R$ with $y \neq z$. Then

$$\int_0^\infty (|y - \tau t e_1 - e^{-t \cdot \Omega} \cdot z|^2 + t)^{-3/2 - k/2} dt \leq C(R) |y - z|^{-1 - k}.$$  

Due to Lemma 2.3, this means for $y, z$ as above, and for $j, k \in \{1, 2, 3\}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$ that

$$|\partial_y^\alpha \mathcal{F}(y, z)| + |\partial_z^\alpha \mathcal{F}(y, z)| \leq C(R) |y - z|^{-1 - |\alpha|}.$$  

Lemma 2.10 ([8 Lemma 4.1]). Let $j, k \in \{1, 2, 3\}$, $g \in L^1(\partial \mathcal{D})$, and put

$$F(y) := \int_{\partial \mathcal{D}} \mathcal{F}_{jk}(y, z) g(z) \, dz \quad \text{for } y \in \mathcal{F}.$$  

Then $F \in C^1(\mathcal{F}^c)$ and

$$\partial_m F(y) = \int_{\partial \mathcal{D}} \partial_m \mathcal{F}_{jk}(y, z) g(z) \, dz \quad \text{for } 1 \leq m \leq 3, \ y \in \mathcal{F}^c.$$  

Lemma 2.11. ([8 Lemma 4.2]) Let $j, k, l \in \{1, 2, 3\}$, $R > 0$, $g \in L^1(B_R)$, and put

$$F(y) := \int_{B_R} \partial_{jl} \mathcal{F}_{jk}(y, z) g(z) \, dz \quad \text{for } y \in \overline{B_R}.$$
Then $F \in C^1(B_R^\epsilon)$ and
\[
\partial_m F(y) = \int_{B_R} \partial_{y_m} \partial_z \delta_k(y, z) g(z) \, dz \quad \text{for } y \in \overline{B_R^\epsilon}, \ 1 \leq m \leq 3.
\]

Finally, we need some pointwise estimates of convolutions of the type $\eta_{-b}^\alpha \ast \eta_{-c}^\beta \leq c \eta_{-f}^\gamma$, where $\eta_{-b}^\alpha(x) := (1 + |x|)^\alpha s(x)^\beta$. A logarithmic factor on the right-hand side of these estimates is also admitted. For a systematic study of estimates of the convolutions of the type $\eta_{-b}^\alpha \ast \eta_{-c}^\beta$ of for all real values of parameters $a, b, c, d$, we refer to [15, 56]. This type of estimate will be needed here in two concrete cases:

**Lemma 2.12.** Let $\gamma \in (1/4, \infty)$. Then there is a constant $C(\gamma) > 0$ such that for all $x \in \mathbb{R}^3$:
\[
\int_{\mathbb{R}^\gamma} [(1 + |x - y|) s(x - y)]^{-3/2} [(1 + |y|) s(y)]^{-\gamma} \, dy \leq C(\gamma) (1 + |x|)^{-c} s(x)^{-d} \ln(2 + |x|),
\]
where
\[
c := \begin{cases} 
\gamma - 1/2 & \text{if } \gamma \in (1/4, 2] \\
3/2 & \text{if } \gamma \in (2, +\infty)
\end{cases}
\]
d := \begin{cases} 
\gamma & \text{if } \gamma \in (1/4, 3/2] \\
3/2 & \text{if } \gamma \in (3/2, +\infty)
\end{cases}
\]
k := \begin{cases} 
0 & \text{if } \gamma \neq 2 \\
1 & \text{if } \gamma = 2.
\end{cases}

**Lemma 2.13.** There exist a constant $C > 0$ such that for all $x \in \mathbb{R}^3$:
\[
\int_{\mathbb{R}^\gamma} [(1 + |x - y|) s(x - y)]^{-2} [(1 + |y|) s(y)]^{-2} \, dy \leq C [(1 + |x|) s(x)]^{-2} \ln(2 + |x|)
\]

Starting point of our considerations will be the following theorem about the integrability and pointwise decays of the velocity and its gradient, where by the velocity we mean the velocity part of a solution is to the rotational Navier-Stokes equations:

**Theorem 2.14 ([10] Theorem 1.1).** Let $\tau \in (0, \infty)$, $\omega \in \mathbb{R}^3 \setminus \{0\}$, $\Omega \subset \mathbb{R}^3$ open and bounded. Take $\gamma, \sigma, \gamma_1 \in (0, \infty)$, $p_0 \in (1, \infty)$, $A \in (2, \infty)$, $B \in [0, 0.3/2]$, $f : \mathbb{R}^3 \to \mathbb{R}^3$ measurable with $\Omega \subset B_{S_1}$, $A + \min\{B, 1\} > 3$, $A + B \geq 7/2$, $f|B_{S_1} \in L^{p_0}(B_{S_1})$, $A|f(y)| \leq |y|^{-A} \sigma(y) - B$ for $y \in B_{S_1}$.
\[
(11)
\]

Let $u \in L^6(\Omega)^3 \cap W^{1,1}_{loc}(\Omega)^3$, $\pi \in L^2_{loc}(\Omega)^3$, $\nabla u \in L^2(\Omega)^9$, and
\[
\int_{\Omega} \left( \nabla u \cdot \nabla \varphi + \left( \tau \partial_1 u + \tau (u \cdot \nabla) u \right) \cdot \varphi - \pi \div \varphi - f \cdot \varphi \right) \, dz = 0, \quad \div u = 0 \quad (12)
\]
for $\varphi \in C_{c}^\infty(\Omega)^3$. Let $S \in (S_1, \infty)$. Then
\[
|\partial^\alpha u(y)| \leq D \left( |y| \sigma(y) \right)^{-1-|\alpha|/2} \quad \text{for } x \in B_{S}^\gamma, \ \alpha \in \mathbb{N}_0^3, \ |\alpha| \leq 1, \quad (13)
\]
with the constant $D$ depending on $\tau, \rho, \gamma, S_1, p_0, A, B, \|f\|_{B_{S_1}}, u, \pi, S$, and on an arbitrary but fixed number $S_0 \in (0, S_1)$ with $\Omega \subset B_{S_0}$.

Let $p \in (1, \infty)$, $q \in (1, 2)$, $f \in L^p_{loc}(\mathbb{R}^3)^3$ with $f|B_{S}^\gamma \in L^q(B_{S}^\gamma)^3$ for some $S \in (0, \infty)$. 


For $y \in \mathbb{R}^3$, $j \in \{1, 2, 3\}$, we set
\[
\mathcal{R}_j(f)(y) := \int_{\mathbb{R}^3} \sum_{k=1}^{3} 3_{jk}(y, z) f_k(z) \, dz.
\]

According to [7, Lemma 3.1], the integral appearing in the definition of $\mathcal{R}_j(f)$ is well defined at least for almost every $y \in \mathbb{R}^3$. If $f$ is a function on $\overline{\mathbb{D}}^c$, the function $f$ in the previous definition is to be replaced by the extension of $f$ by zero to $\mathbb{R}^3$.

In order to derive the leading terms of the velocity and its gradient we are going to use a representation formula of a solution of the rotational Navier-Stokes equation:

**Theorem 2.15.** Let $u \in W^{1,1}_\text{loc}(\overline{\mathbb{D}}^c)^3 \cap L^6(\overline{\mathbb{D}}^c)^3$ with $\nabla u \in L^2(\overline{\mathbb{D}}^c)^9$. Let $p \in (1, \infty)$, $q \in (1, 2)$, $f : \overline{\mathbb{D}}^c \to \mathbb{R}^3$ a function with $f|_T \in L^p(\mathcal{D}_T)^3$ for $T \in (0, \infty)$ with $\overline{\mathbb{D}} \subset B_T$, $f|_S \in L^q(B_S)^3$ for some $S \in (0, \infty)$ with $\overline{\mathbb{D}} \subset B_S$. Further assume that $u_t \partial \mathcal{D} \in W^{2-1/p,p}(\partial \mathcal{D})^3$ and $\pi : \overline{\mathbb{D}}^c \to \mathbb{R}$ is a function with $\pi|_T \in L^p(\mathcal{D}_T)$ for $T$ as above.

Suppose that the pair $(u, \pi)$ is a weak solution of the Navier-Stokes system with Oseen and rotational terms, and with right-hand side $f$ in the sense of [12]. Then $u \in W^{2,\min\{p,3/2\}}(\mathcal{D}_T)^3$, $\pi \in W^{1,\min\{p,3/2\}}(\mathcal{D}_T)$ for any $T \in (0, \infty)$ with $\overline{\mathbb{D}} \subset B_T$, $u_j(y) = \mathcal{B}_j(f - \tau \cdot (u \cdot \nabla)u)(y) + \mathcal{B}_j(\mathcal{A}_j)(y)$ for $j \in \{1, 2, 3\}$, a.e. $y \in \overline{\mathbb{D}}^c$, (14) where $\mathcal{B}_j(u, \pi)$ is defined by
\[
\mathcal{B}_j(u, \pi)(y) := \int_{\partial \mathcal{D}} \sum_{k=1}^{3} \sum_{l=1}^{3} \left( 3_{jk}(y, z) \left( -\partial_l u_k(z) + \delta_{kl} \pi(z) + u_k(z) (\tau e_1 - \omega \times z)_l \right) \\ + \partial_2 l 3_{jk}(y, z) u_k(z) \right) \left[ n_l^1(z) + E_{4j}(y - z) u_k(z) n_k^1(z) \right] \, \, dz
\]
for $y \in \overline{\mathbb{D}}^c$.

**Proof.** See [11] Theorem 4.1, and its proof, as well as [7] Theorem 4.4.

In comparison with the linear case we will need some additional lemma:

**Lemma 2.16.** Let $\phi \in W^{1,1}_\text{loc}(U)$ for some open set $U \subset \mathbb{R}^3$ and $A \in \mathbb{R}^{3 \times 3}$ such that $A^{-1} = A^T$. Then:
\[
A \nabla_z (\phi(Az)) = \nabla \phi(Az)
\]

**Proof.** Indeed:
\[
\frac{\partial}{\partial z_l}(\phi(Az)) = \sum_{k=1}^{3} \partial_k \phi(Az) \frac{\partial(Az)_k}{\partial z_l} = \sum_{k=1}^{3} \partial_k \phi(Az)A_{kl} = \sum_{k=1}^{3} A^T_{lk} \partial_k \phi(Az)
\]
for $z \in \overline{B_{S_1}^c}$, which gives the mentioned formula.

**Corollary 1.** In the situation of Theorem 2.14 we get for $z \in \overline{B_{S_1}^c}$ that
\[
\sum_{l=1}^{3} (u_l \partial_l u)(e^\Omega z) = \sum_{l=1}^{3} \partial_l (u_l u)(e^\Omega z)
\]
\[
= \sum_{l=1}^{3} e^{\Omega} \partial_{z_k} \left[ (u_l u)(e^\Omega z) \right] = \sum_{l=1}^{3} (e^{\Omega} \nabla_z) u_l (u_l u)(e^\Omega z).
\]
Lemma 2.17. In the situation of Theorem 2.14, we have
\[
\int_{\mathbb{R}^3} |\partial^\alpha_x \mathfrak{F}(x,y) \cdot \mathbf{u}(y)\| dy < \infty \quad \text{for } x \in \overline{B_{S_1}}, \alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| \leq 1. \tag{16}
\]
Moreover the function \( \mathfrak{M}(x) := \int_{\mathbb{R}^3} \mathfrak{F}(x,y) \cdot \mathbf{u}(y)\| dy \) \( (x \in \overline{B_{S_1}}) \) belongs to \( C^1(\overline{B_{S_1}})^3 \), with
\[
\partial^\alpha \mathfrak{M}(x) = \int_{\mathbb{R}^3} \partial^\alpha_x \mathfrak{F}(x,y) \cdot \mathbf{u}(y)\| dy \quad \text{for } x, \alpha \text{ as in (16).} \tag{17}
\]

Proof. Let \( U \subset \mathbb{R}^3 \) be open and bounded, with \( \mathring{U} \subset \overline{B_{S_1}} \). It is enough to show that (16) holds for \( x \in U \), that \( \mathfrak{M} U \subset C^1(U)^3 \), and (17) is valid for \( x \in U \).

Due to our assumptions on \( U \), we may choose \( R, S \in (S_1, \infty) \) such that \( B_S \cap U = \emptyset \) and \( \mathring{U} \subset B_R \). In particular we have \( \text{dist}(B_S, U) > 0 \) and \( \text{dist}(U, B_R^c) > 0 \). This observation and Lemma 2.8 imply that \( |\partial^\alpha_x \mathfrak{F}(x,y)\| \leq C_0 \) for \( x \in U, y \in B_S \setminus \overline{\mathbf{B}}, \alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| \leq 1 \), where \( C_0 \) is independent of \( x \) and \( y \). We further observe that \( (u \cdot \nabla)u \in L^{3/2}(\overline{\mathbf{B}})^3 \), hence \( (u \cdot \nabla)u|_{B_S \setminus \overline{\mathbf{B}}} \in L^1(B_S \setminus \overline{\mathbf{B}})^3 \). Lemma 2.8 and (13) yield that \( |\partial^\alpha_x \mathfrak{F}(x,y) \cdot \mathbf{u}(y)\| \leq C_1 \| y \|^{-7/2-|\alpha|/2} \) for \( x \in U, y \in B_R^c \), with \( C_1 \) again being independent of \( x \) and \( y \). In view of the last statement of Lemma 2.7 we may thus conclude by Lebesgue’s theorem that the function
\[
y \mapsto \partial^\alpha_x \mathfrak{F}(x,y) \cdot \mathbf{u}(y), \quad y \in A := (B_S \setminus \overline{\mathbf{B}}) \cup B_R^c,
\]
is integrable for \( x \in U, \alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| \leq 1 \), that the function
\[
\mathfrak{M}^{(\alpha)}(x) := \int_{A} \mathfrak{F}(x,y) \cdot \mathbf{u}(y)\| dy, \quad x \in U,
\]
belongs to \( C^1(U)^3 \), and that \( \partial^\alpha \mathfrak{M}^{(\alpha)}(x) = \int_{A} \partial^\alpha_x \mathfrak{F}(x,y) \cdot \mathbf{u}(y)\| dy \) for \( x, \alpha \) as before.

Let \( \varphi \in C_0^\infty(\mathbb{R}^3) \) with \( 0 \leq \varphi \leq 1 \), \( \varphi|_{B_{1/2}} = 0 \), \( \varphi|_{B_1^c} = 1 \), and define \( \varphi_\delta(x) := \varphi(\delta^{-1}x) \) for \( x \in \mathbb{R}^3 \), \( \delta > 0 \). Then \( \varphi_\delta \in C_0^\infty(\mathbb{R}^3) \), \( 0 \leq \varphi_\delta \leq 1 \), \( \varphi_\delta|_{B_{1/2}} = 0 \), \( \varphi_\delta|_{B_3^c} = 1 \) and \( |\nabla \varphi_\delta(x)| \leq C_\delta^{-1} \) for \( x \in \mathbb{R}^3 \), \( \delta > 0 \).

Using Lemma 2.9 and Theorem 2.14 we see there are constants \( C_2, C_3 \) with
\[
|\partial^\alpha_x \left( \mathfrak{F}(x,y) \varphi_\delta(x-y) \right) \cdot \mathbf{u}(y)\| \leq C_2(|x-y|^{-2} + \delta^{-1}|x-y|^{-1}) \chi_{(\delta/2,\infty)}(|x-y|) \leq C_3 \delta^{-2} \tag{18}
\]
for \( x \in U, y \in B_R \setminus B_S, \alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| \leq 1 \). In addition, if \( y \in B_R \setminus B_S \), the function \( x \mapsto \mathfrak{F}(x,y) \varphi_\delta(x-y) \cdot \mathbf{u}(y)\| \) is continuously differentiable, as follows from Lemma 2.6. Now we may conclude from Lebesgue’s theorem that the function \( y \mapsto \partial^\alpha_x \left( \mathfrak{F}(x,y) \varphi_\delta(x-y) \right) \cdot \mathbf{u}(y)\| \) is integrable for any \( \delta > 0, x \in U, \alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| \leq 1 \), the function
\[
\mathfrak{B}_\delta(x) := \int_{B_R \setminus B_S} \mathfrak{F}(x,y) \varphi_\delta(x-y) \cdot \mathbf{u}(y)\| dy, \quad x \in U,
\]
belongs to \( C^1(U)^3 \) for any \( \delta > 0 \), and
\[
\partial^\alpha \mathfrak{B}_\delta(x) = \int_{B_R \setminus B_S} \partial^\alpha_x \left( \mathfrak{F}(x,y) \varphi_\delta(x-y) \right) \cdot \mathbf{u}(y)\| dy.
\]
and functions $f$ is a weak solution of the Navier-Stokes system with Oseen and rotational terms, and

3. Leading term of the velocity and of its gradient. The aim of this part is to find the leading term of the velocity and its gradient for the Navier-Stokes problem with rotational terms. Let us recall that the quantities $\mathbf{u}$ and $\nabla \mathbf{u}$ were fixed in Section 2. We study the case $f$ has a compact support in $\overline{\mathcal{O}}^c$. The result we will prove in the work at hand may be stated as:

**Theorem 3.1.** Let $S_1 \in (0, \infty)$ with $\overline{\mathcal{O}} \subset B_{S_1}$, $p \in (1, \infty)$, $f \in L^p(\overline{\mathcal{O}})^3$ with $\text{supp}(f) \subset B_{S_1}$, $u \in L^6(\overline{\mathcal{O}})^3 \cap W^{1,1}_{\text{loc}}(\overline{\mathcal{O}})^3$ with $\nabla u \in L^2(\overline{\mathcal{O}})^9$ and $u|\partial \mathcal{O} \in W^{2-1/p,p}(\partial \mathcal{O})^3$, $\pi \in L^2_{\text{loc}}(\overline{\mathcal{O}})^3$ with $\pi|\partial \mathcal{O} \in L^p(\partial \mathcal{O})$. Suppose that the pair $(u, \pi)$ is a weak solution of the Navier-Stokes system with Oseen and rotational terms, and with right-hand side $f$ in the sense of \cite{[12]}. Then there are coefficients $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ and functions $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3 \in C^1(B_{S_1})^3$ such that for $j \in \{1, 2, 3\}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, $x \in B_{S_1}$, $\partial^\alpha u_j(x) = \partial^\alpha L_j(x) + \partial^\alpha \mathbf{f}_j(x)$,

where

$$L_j(x) = \left\{ \sum_{k=1}^3 \beta_k \mathbf{f}_{jk}(x, 0) + \left( \int_{\partial \mathcal{O}} u \cdot n(\mathcal{O}) \, d\sigma \right) E_{k\mathcal{O}}(x) \right\},$$

and if $S \in (S_1, \infty)$, $x \in B_{S^c}$,

$$|\partial^\alpha \mathbf{f}(x)| \leq \mathcal{C} \left| \ln \left( \frac{|x| + s_T(x)}{s_T(x)} \right) \right|^{-3/2} - \frac{|\alpha|}{2} \ln (2 + |x|),$$

where $\mathcal{C}$ depends on $\tau, \omega, p, S_1, S$, certain norms of $u, \pi$ and $f$, and on the constant $D$ from \cite{[13]}.

In Theorem 3.1 the estimate presented in \cite{[11]} Theorem 3.14] for the linear case is extended to the nonlinear one. Note that by \cite{[17]} (3.9), the function $\mathbf{f}(x, 0)$ in the leading term on the right-hand side of (19) corresponds to the time integral of a fundamental solution of the evolutionary Oseen system multiplied by a rotation depending on time.

**Remark 1.** The rate of decay stated in Theorem 3.1 is optimal in the following sense:
Firstly, $|R_j(x)|$ will be seen to behave as $O\left(\frac{|x|}{s_x(x)} \cdot \frac{x^n}{|x|^{3/2}} \cdot \ln(2 + |x|)\right)$ for $|x| \to \infty$, and thus in the same way as $|\nabla \mathcal{D}(x)|$ except for the logarithmic factor $\ln(2 + |x|)$; see [56].

Secondly we will show that $|\nabla R_j(x)|$ decays as $O\left(\frac{|x|}{s_x(x)} \cdot \frac{x^n}{|x|^{3/2}} \cdot \ln(2 + |x|)\right)$ for $|x| \to \infty$, that is, with the same rate as $|D^2 \mathcal{D}(x)|$, once more up to the logarithmic factor $\ln(2 + |x|)$; again see [56].

Remark 2. In the case of [61], [62] the term
\[ \left(\int_{\partial \mathcal{D}} u \cdot n(\mathcal{D}) \, dz \right) E_{4j}(x) \]
is zero because of the boundary conditions.

In other words, the decay rates of the remainder and its gradient are optimal in the sense that they correspond to those of an additional derivative of $\mathcal{D}$, apart from a logarithmic factor. A comparison with $\mathcal{O}$ seems to be reasonable in view of the fact that we want to arrive at a leading term of the form coefficient times a column of $\mathcal{O}$.

Proof of Theorem 3.1. The term of (20) contained in braces \{\ldots\} will be called “the leading term”, term $\mathfrak{F}$ we will call “the remainder”. From Theorem 2.15 we have
\[ u_j(x) = \mathfrak{R}_j(f - \tau (u \cdot \nabla)u)(x) + \mathfrak{B}_j(u, \pi)(x), \quad j \in \{1, 2, 3\}, \quad \text{for a.e. } x \in \overline{\mathcal{D}}, \quad (22) \]
where $\mathfrak{B}_j(u, \pi)$ was defined in (15).

We put
\[ \beta_k := \beta_k^{(I)} - \tau \beta_k^{(II)} \]
\[ \beta_k^{(I)} := \int_{B_{1,1}} f_k(y) \, dy \]
\[ + \int_{\partial \mathcal{D}} \sum_{l=1}^{3} \left( -\partial_l u_k(y) + \delta_{kl} \pi(y) + u_k(y) (\tau e_1 - \omega \times y)_l \right) n_l(\mathcal{D})(y) \, dy \]
\[ \beta_k^{(II)} := -\int_{\partial \mathcal{D}} \sum_{m=1}^{3} (n_m(\mathcal{D}) u_m(y) \, dy \]
for $1 \leq k \leq 3$. By the definition of $\beta_k$ the leading term in formula (19) is determined. Because (19) is in fact rearrangement of formula (22), we now define the value $\mathfrak{F}_j$ as the difference of the right-hand side of the representation formula (22) minus the leading term. We will distinguish $\mathfrak{F}^{(I)}_j$ coming from the linear terms and $\mathfrak{F}^{(II)}_j$ arising from the non-linear part, i.e. from $\mathfrak{R}_j(f - \tau (u \cdot \nabla)u)$:
\[ \mathfrak{F}_j(x) := \mathfrak{F}^{(I)}_j(x) - \tau \mathfrak{F}^{(II)}_j(x), \]
\[ \mathfrak{F}^{(I)}_j(x) := \int_{B_{1,1}} \left( \sum_{k=1}^{3} \left( 3_{jk}(x, y) - 3_{jk}(x, 0) \right) f_k(y) \right) \, dy \]
\[ + \int_{\partial \mathcal{D}} \sum_{k=1}^{3} \left( (3_{jk}(x, y) - 3_{jk}(x, 0)) \right) \]
\[ \quad + \sum_{l=1}^{3} \left( -\partial_l u_k(y) + \delta_k \pi(y) + u_k(y) (\tau e_1 - \omega \times y) t \right) n_l(\mathcal{D})(y) \]
\[ + \left( E_{4j}(x - y) - E_{4j}(x) \right) u_k(y) n_k(\mathcal{D})(y) \right) \, dy \]
\[ + \int_{\partial \mathcal{D}} \sum_{k,l=1}^{3} \partial_y l 3_{jk}(x, y) u_k(y) n_l(\mathcal{D})(y) \, dy, \]
\[ \mathbf{g}^{(1)}(x) := \int_{\mathcal{D}} \sum_{k,l=1}^{3} 3_{jk}(x, y) (u_l \partial_l u_k)(y) dy + \]
\[ + \int_{\partial \mathcal{D}} \sum_{k,l=1}^{3} 3_{jk}(x, 0)(n_l(\mathcal{D}) u_l u_k)(y) \, dy \tag{23} \]

for \( x \in B_{S_1}^c \), \( 1 \leq j \leq 3 \). Then by \cite{22}, we get \cite{19}. The assertion of the theorem will be proved in four steps:

1. Estimates and continuity of \( \partial^\alpha \mathbf{g}^{(1)} \), where \( |\alpha| = 0 \) or \( |\alpha| = 1 \).

By exactly the same proof as given in [5, p. 473-474] for [5, Theorem 1.1], we obtain that \( \mathbf{g}^{(1)} \in C^1(B_{S_1}^c)^3 \) and
\[ |\partial^\alpha \mathbf{g}^{(1)}(x)| \leq C \| f \|_1 + \| \nabla u | \partial \mathcal{D} \|_1 + \| \pi | \partial \mathcal{D} \|_1 + \| u | \partial \mathcal{D} \|_1 \| | s_\tau(x) (x) \|^{-3/2 - |\alpha|/2} \]
for \( S \in (S_1, \infty) \), \( x \in B_{S}^c \), \( \alpha \in \mathbb{N}_0^3 \) with \( |\alpha| \leq 1 \), with \( C \) depending on \( \tau, \omega, p, S_1 \) and \( S \).

2. \( C^1 \)-continuity of \( \mathbf{g}^{(1)} \).

By Lemma \ref{2.17}, the function \( \mathbf{g}^{(1)} \in C^1(B_{S_1}^c)^3 \), and first-order derivatives may be moved into the volume integral appearing on the right-hand side of \cite{23}.

3. Estimates of \( \partial^\alpha \mathbf{g}^{(1)} \), first steps.

Let \( x \in B_{S}^c \). Recalling that
\[ \mathbf{g}^{(1)}(x) = \int_{\mathcal{D}} \sum_{l=1}^{3} 3(x, y) (u_l \partial_l u)(y) dy + \int_{\partial \mathcal{D}} \sum_{l=1}^{3} 3(x, 0)(n_l(\mathcal{D}) u_l u)(y) dy \tag{24} \]
we apply firstly the integration by parts and then split the resulting volume integral in an integral \( \mathcal{B}_R \) on the bounded domain \( B_R \setminus \mathcal{D} \) and integral \( \mathcal{E}_R \) on the exterior domain \( (B_R)^c \), where \( R = (S_1 + S)/2 \). Thus \( \mathbf{g}^{(1)}(x) \) becomes
\[ - \int_{\partial \mathcal{D}} \sum_{l=1}^{3} [3(x, y) - 3(x, 0)] (n_l(\mathcal{D}) u_l u)(y) \, dy \]
\[ - \left\{ \int_{B_R \setminus \mathcal{D}} + \int_{(B_R)^c} \right\} \sum_{l=1}^{3} \partial_y 3(x, y) (u_l u)(y) \, dy \]
\[ = S_{\partial \mathcal{D}} - \mathcal{B}_R - \mathcal{E}_R. \tag{25} \]
Of course, here and in similar situations in the following, a partial integration has to be performed first on a bounded domain, where \(B_T \setminus (B_R \cup B_\epsilon(x))\) with \(T > \max\{2R, 2|x|\}\), \(0 < \epsilon\) is a good choice for such a domain. In the next step we let \(\epsilon\) tend to zero. This passage to the limit may be handled by referring to Lemma 2.9 and (13). Finally we let \(T\) tend to infinity. The surface integral on \(\partial B_T\) which came up in the partial integration then vanishes, as follows from Lemma 2.8 and (13). The same references imply that all the volume integrals involved tend to integrals on \(B_R^c\) when \(T \to \infty\).

In volume integral \(\mathcal{E}_R\) over the exterior domain \((B_R)^c\) we use firstly the definition of \(3\), (20) and the Fubini’s theorem, and then the domain invariant transformation \(y = e^{\Omega z}\) for fixed \(t > 0\). The reason why we use the mentioned transformation is that we would like to avoid a periodic term in the right-hand side of (30):

\[
\mathcal{E}_R = \int_{(B_R)^c} \sum_{l=1}^{3} \partial_y \zeta(x, y)[u(t)](y) \, dy = \int_0^\infty \int_{(B_R)^c} \sum_{l=1}^{3} \partial_y \Gamma(x, y, t)[u(t)](y) \, dy \, dt
\]

Finally we split the the exterior domain of integration \((B_R)^c\) on two domains: Let \(\delta\) be a sufficiently small positive number comparing to \(1, R\) and \(S - S_1\), f.e. \(\delta := \min\{1, (S - S_1)/2, R/2\} = \min\{1, S - R, R/2\}\). Note that \(\overline{B_\delta(x)} \subset B_R^c\). We obtain:

\[
\mathcal{E}_R = \left\{ \int_0^\infty \int_{B_R(x)} + \int_0^\infty \int_{(B_R)^c \setminus B_R(x)} \right\} \sum_{l=1}^{3} \partial_y \Gamma(x, y, t)[u(t)](e^{\Omega z}) \, dz \, dt
\]

Substituting the expression of \(\mathcal{E}_R\) into (25) we get:

\[
\delta^{I_{11}} = \mathcal{S}_{\partial \mathcal{D}} - B_R - \mathcal{V}_\delta - \mathcal{V}_{R, \delta}
\]

\[
\partial_x^a B_R, \ \partial_x^a \mathcal{S}_{\partial \mathcal{D}}: \text{ Estimating the first two terms and their derivatives } \partial_x^a \text{ for } |\alpha| = 0, 1, \text{ we get the following estimate:}
\]

\[
|\partial_x^a B_R| + |\partial_x^a \mathcal{S}_{\partial \mathcal{D}}| \leq \mathcal{C}(S_1, S)(|x|s_\tau(x))^{-3/2 - |\alpha|/2}, \ x \in B_S^c
\]

Indeed, from Lemma 2.8 for \(y \in B_R, x \in B_S^c:\)

\[
|\partial_x^a \partial_y \zeta(x, y)| \leq \mathcal{C}(S_1, S)(|x|s_\tau(x))^{-3/2 - |\alpha|/2},
\]

\[
|\partial_x^a (\zeta(x, y) - \zeta(x, 0))| = \left| \sum_{k=1}^{3} \partial_x^a \partial_z \zeta(x, z)|_{z = \theta y_k} y_k \right| \leq \mathcal{C}(S_1, S)(|x|s_\tau(x))^{-3/2 - |\alpha|/2}
\]

for some \(0 \leq \theta \leq 1\). So, with Lemma 2.11 and (28)

\[
|\partial_x^a B_R| \leq \left| \int_{B_R} \sum_{l=1}^{3} \partial_x^a \partial_y \zeta(x, y)(u_j u)(y) \, dy \right| \leq
\]

\[
\mathcal{C}(S_1, S)(|x|s_\tau(x))^{-3/2 - |\alpha|/2} \left[ \int_{B_R} \sum_{l=1}^{3} (u_j u)(y) \, dy \right]
\]

\[
\leq \mathcal{C}(S_1, S)(|x|s_\tau(x))^{-3/2 - |\alpha|/2},
\]
because $|u|^2$ is $L^1$-integrable on bounded domain $B_R \setminus \overline{D}$. Similarly, we have with (10) and (29):

$$ |\partial^\alpha x \mathcal{J} \cap D| \leq C(S_1, R) \left( |x| s_r(x) \right)^{-3/2-|\alpha|/2} \int_{\partial D} \sum_{i=1}^3 |(n_i \mathcal{D} u_i)(y)| dy \leq C(S_1, R) \left( |x| s_r(x) \right)^{-3/2-|\alpha|/2}. $$

4. Estimates of $\partial^\alpha x \mathcal{J}^{(II)}$ for $\alpha = 0$.

$\mathcal{V}_d$: For the estimate of this term we use Lemma 2.5 for the first order derivatives of $\Gamma$: We have (for $x \neq x(t)$)

$$ |\partial y, \Gamma(x, y, t)|_{y=e^{\alpha z}} \leq C \left( |x - \tau t e_1 - z|^2 + t \right)^{-2}. \quad (30) $$

From Theorem 2.14 we have for $y \in B_R^c$

$$ |u(y)|^2 \leq C(R) \left( |y| s_r(y) \right)^{-2}. $$

If $z \in (B_R)^c$ then $e^{\alpha z} \in (B_R)^c$, we get:

$$ |u(e^{\alpha z})|^2 \leq C(R) \left( |e^{\alpha z}| s_r(e^{\alpha z}) \right)^{-2} = C(R) \left( |z| s_r(z) \right)^{-2}. \quad (31) $$

Since $B(x) \subset B_R^c$, we thus get due to (6), (7)

$$ |u(e^{\alpha z})| \leq C(R, \delta) \left( |x| s_r(x) \right)^{-2} \quad \text{for} \quad z \in B(x). \quad (32) $$

So we have

$$ |\mathcal{V}_d| = \left| \int_0^\infty \int_{B_h(x)} \sum_{j=1}^3 \partial y_j \Gamma(x, y, t)|_{y=e^{\alpha z}} \left[ (u_j u) \left( e^{\alpha z} \right) \right] dz dt \right| \leq C(R) \int_{B_h(x)} \int_0^\infty \left( |x - \tau t e_1 - z|^2 + t \right)^{-2} \left( |x| s_r(x) \right)^{-2} dt dz \leq C(R, \delta) \left( |x| s_r(x) \right)^{-2}, $$

where the integral with respect to variable $t$ is estimated using Lemma 2.9 choosing in its application $y := x - z$, $z := 0$.

$\mathcal{V}_{R,\delta}$: Similarly as in the previous case, using (30) and (31) we find

$$ |\mathcal{V}_{R,\delta}| \leq \left| \int_0^\infty \int_{B_h \setminus B_h(x)} \sum_{j=1}^3 \partial y_j \Gamma(x, y, t)|_{y=e^{\alpha z}} \left[ (u_j u) \left( e^{\alpha z} \right) \right] dz dt \right| \leq C(R) \int_{B_h \setminus B_h(x)} \left( |x - \tau t e_1 - z|^2 + t \right)^{-2} \left( |z| s_r(z) \right)^{-2} dt dz $$

Now, the integral with respect to $t$ can be estimated using Lemma 2.6 with $y := x - z$, $z := 0$.

$$ \int_0^\infty \left( |x - \tau t e_1 - z|^2 + t \right)^{-2} dt \leq C(S_1, S) \left( |x - z| s_r(x - z) \right)^{-3/2}, \quad z \in B_R \setminus B_h(x) $$

$$ |\mathcal{V}_{R,\delta}| \leq C(S_1, S) \int_{B_h \setminus B_h(x)} \left( |x - z| s_r(x - z) \right)^{-3/2} \left( |z| s_r(z) \right)^{-2} dz \leq C(S_1, S) \ln(2 + |x|) \left( |x| s_r(x) \right)^{-3/2}. $$

The last inequality follows from lemma 2.12 with $\gamma = 2$.
5. Estimates of $\partial^\alpha \tilde{g}^{(II)}$ for $|\alpha| = 1$.

Let us mention that $S, S_1, R, \delta$ are the same as in the previous section, so $\overline{B_\delta(x)} \subset B_R^c$. The aim of this part is to find the leading term of the gradient of the velocity for the Navier-Stokes problem with rotational terms: Unlike in the previous part is that we cannot apply an integration by parts over the whole domain $\overline{\Omega}$ because we have to exclude the neighbourhood $B_\delta(x)$ due to singularities of the second order derivatives of $3$. On the other hand, to avoid some technical difficulties, we are able to handle the integrals with respect to $t$ only in domains invariant with respect to the transformation $y = e^{\Omega} z$, $t > 0$. These facts cause some additional computations. So we use Lemma 2.17 split the domain of integration into the bounded part $B_R \setminus \overline{\Omega}$ and the exterior domain $(B_R)^c$, and then perform an integration by parts firstly only on the bounded domain:

$$\partial^\alpha_x \tilde{g}^{(II)}(x) = \int_{\Omega} \sum_{k,l=1}^3 \partial^\alpha_x \bar{3}_{jk}(x,y) \left( u_l \partial_l u_k \right)(y) dy$$

$$+ \int_{\partial \Omega} \sum_{k,l=1}^3 \partial^\alpha_x \bar{3}_{jk}(x,0) \left( n_l \bar{D} u_l \right)(y) dy$$

$$= \left\{ \int_{B_R \setminus \overline{\Omega}} + \int_{(B_R)^c} \right\} \sum_{k,l=1}^3 \partial^\alpha_x \bar{3}_{jk}(x,y) \left( u_l \partial_l u_k \right)(y) dy$$

$$+ \int_{\partial B_R} \sum_{k,l=1}^3 \partial^\alpha_x \bar{3}_{jk}(x,0) \left( n_l \bar{D} u_l \right)(y) dy$$

$$= - \int_{\Omega} \sum_{k,l=1}^3 \left[ \partial^\alpha_x \bar{3}_{jk}(x,y) - \partial^\alpha_x \bar{3}_{jk}(x,0) \right] \left( n_l \bar{D} u_l \right)(y) dy$$

$$- \int_{B_R \setminus \overline{\Omega}} \sum_{k,l=1}^3 \partial y_l \partial^\alpha_x \bar{3}_{jk}(x,y) \left( u_l \partial_l u_k \right)(y) dy$$

$$+ \int_{\partial B_R} \sum_{k,l=1}^3 \partial^\alpha_x \bar{3}_{jk}(x,y) \left( u_l \partial_l u_k \right)(y) dy$$

$$+ \int_{(B_R)^c} \sum_{k,l=1}^3 \partial^\alpha_x \bar{3}_{jk}(x,y) \left( u_l \partial_l u_k \right)(y) dy$$

So we get:

$$\partial^\alpha_x \tilde{g}^{(II)}(x) = \partial^\alpha S_{\partial \Omega} - \partial^\alpha B_R + \mathcal{E}'_R + \mathcal{E}_R'$$

Concerning the last term in (34), we find due to (9) that

$$\mathcal{E}'_R(x)_j = \int_{(B_R)^c} \sum_{k,l=1}^3 \partial^\alpha_x \bar{3}_{jk}(x,y) \left( u_l \partial_l u_k \right)(y) dy$$

$$= \int_{(B_R)^c} \int_0^{+\infty} \sum_{k,l=1}^3 \partial^\alpha_x \Gamma_{jk}(x,y,t) \left( u_l \partial_l u_k \right)(y) dy dt$$

The domain of integration of $\mathcal{E}'_R$ is $(B_R)^c$. This exterior domain is invariant with respect to the transformation $y = e^{\Omega} z$, $t > 0$. We may use the same transformation
to avoid periodic terms as in the case $|\alpha| = 0$:

$$
E_R'(x)_j = \int_0^{+\infty} \int_{B_0(x)} \sum_{k,l=1}^3 \partial_x^3 \Gamma_{jk}(x, e^{\tau \Omega} z, t) (u_l \partial_t u_k) (e^{\tau \Omega} z) dz \, dt 
$$

Unlike in the case $|\alpha| = 0$, this transformation is used before the integration by parts. More precisely, we split the domain of integration into the two domains $B_0(x)$ and $(B_R)^c \setminus B_0(x)$. In the integral over the unbounded domain we apply the identity from Corollary 1 and integrate by parts:

$$
E_R'(x)_j = \int_0^{+\infty} \int_{B_0(x)} \sum_{k,l=1}^3 \partial_x^3 \Gamma_{jk}(x, e^{\tau \Omega} z, t) (u_l \partial_t u_k) (e^{\tau \Omega} z) dz \, dt 
$$

$$
+ \int_0^{+\infty} \int_{\partial B_0(x)} \sum_{k,l=1}^3 \partial_x^3 \Gamma_{jk}(x, e^{\tau \Omega} z, t) (u_l \partial_t u_k) (e^{\tau \Omega} z) \left[ (u_l u_k) (e^{\tau \Omega} z) \right] \left( e^{\tau \Omega} (x - z) / \delta \right)_l \, do_z \, dt 
$$

$$
+ \int_0^{+\infty} \int_{\partial B_R} \sum_{k,l=1}^3 \partial_x^3 \Gamma_{jk}(x, e^{\tau \Omega} z, t) (u_l \partial_t u_k) (e^{\tau \Omega} z) \left( e^{\tau \Omega} / (z - R) \right)_l \, do_z \, dt 
$$

$$
- \int_0^{+\infty} \int_{B_0(x) \setminus B_R} \sum_{k,l=1}^3 (e^{\tau \Omega} \nabla z)_l \partial_x^3 \Gamma_{jk}(x, e^{\tau \Omega} z, t) (u_l \partial_t u_k) (e^{\tau \Omega} z) \, dz \, dt 
$$

$$
= (U_0)_j + (S'_0)_j + (- S'_0)_j + (U_R, \delta)_j 
$$

Substituting the expression of $E_R'(x)$ into (34) and using (9), we get finally:

$$
\partial_x^3 \delta^{(1)}(x) = \partial^3 S_{\partial B} - \partial^3 B_R + U_0 + S'_0 + U_R, \delta 
$$

(35)

Now we will estimate all terms of (35) for $|\alpha| = 1$:

$\partial^3 S_{\partial B}, \partial^3 B_R$: From (27) we know that $|\partial_x^3 S_{\partial B}| + |\partial_x^3 B_R| \leq \mathcal{C}_1(S_1, S) (|x| s_r(x))^{-2}$.

$U_0$: The estimate of this term is completely analogous to the evaluation of $V_0$ in the case $|\alpha| = 0$. The only difference is that from Theorem 2.14 we deduce for $y \in (B_R)^c$ $|u(y)||\nabla u(y)| \leq \mathcal{C}(S) (|y| s_r(y))^{-5/2}$. In this way we get

$$
|U_0| \leq \mathcal{C}(S_1, S) (|x| s_r(x))^{-5/2}. 
$$

$S'_0$: Lemma 2.5 applied to the first order derivatives of $\Gamma$, for $x \neq e^{\tau \Omega} z$ that:

$$
|\partial_x^3 \Gamma(x, e^{\tau \Omega} z, t)| \leq \mathcal{C} (|x - \tau te_1 - z|^2 + t)^{-2}. 
$$

By (32)

$$
|u(e^{\tau \Omega} z)|^2 \leq \mathcal{C}(R, \delta) (|x| s_r(x))^{-2} \text{ for } z \in \partial B_0(x). 
$$

It is clear that $|e^{\tau \Omega} (x - z) / \delta| = 1$ for $z \in B_0(x)$.

$S'_0$: Lemma 2.5 applied to the first order derivatives of $\Gamma$, for $x \neq e^{\tau \Omega} z$ that:

$$
|\partial_x^3 \Gamma(x, e^{\tau \Omega} z, t)| \leq \mathcal{C} (|x - \tau te_1 - z|^2 + t)^{-2}. 
$$

By (32)

$$
|u(e^{\tau \Omega} z)|^2 \leq \mathcal{C}(R, \delta) (|x| s_r(x))^{-2} \text{ for } z \in \partial B_0(x). 
$$

It is clear that $|e^{\tau \Omega} (x - z) / \delta| = 1$ for $z \in B_0(x)$.

So we have

$$
|S'_0| \leq \mathcal{C}(R) \int_{\partial B_0(x)} \int_0^{\infty} \left( |x - \tau te_1 - z|^2 + t \right)^{-2} \left( |x| s_r(x) \right)^{-2} dt \, do_z 
$$

$$
\leq \mathcal{C}(R, \delta) (|x| s_r(x))^{-2} \int_{\partial B_0(x)} |x - z|^{-2} \, do_z \leq \mathcal{C}(R, \delta) (|x| s_r(x))^{-2} 
$$

where the integral with respect to variable $t$ is estimated using Lemma 2.5 with $y := x - z$, $z := 0$. So, the integral $S'_0$ may be subsumed into the remainder.
\[ U_{R,\delta} : \text{We shall use Lemma 2.5 for the evaluation of the second order derivatives}\]

\[
\left| (e^{t\Omega} \nabla z)_j \partial_x^2 \Gamma(x, e^{t\Omega} z, t) \right| \leq \mathcal{C}(x - \tau e_1 - z + t)^{-5/2}
\]

The integral with respect to \( t \) of the right-hand side can be estimated by once more applying Lemma 2.6 with \( y = x - z, \ z = 0 \). Hence:

\[
\int_0^\infty \left( |y - \tau e_1 - z|^2 + t \right)^{-5/2} dt \leq \mathcal{C}(S_1, S) (|x - z| s_r(x - z))^{-2}
\]

Using (31), we find

\[
|U_{R,\delta}| \leq \int_0^\infty \int_{B_R \setminus B_1(x)} \sum_{j=1}^3 (e^{t\Omega} \nabla z)_j \partial x_m \Gamma(x, e^{t\Omega} z, t) [(u_j u_j) (e^{t\Omega} z)] \ dz \ dt
\]

\[
\leq \mathcal{C}(S_1, S) \int_{B_R \setminus B_1(x)} (|x - z| s_r(x - z))^{-2} (|z| s_r(z))^{-2} \ dz
\]

\[
\leq \mathcal{C}(S_1, S) (|x| s_r(x))^{-2} \ln(2 + |x|).
\]

The last inequality follows from Lemma 2.13.

\[ \square \]

**Remark 3.** So, we finally get that the leading term of \( \partial^\alpha u_j \) takes the same form as in the linear case, that is

\[
\sum_{k=1}^3 \beta_k \partial_x^3 j_k(x, 0),
\]

but with the definition of \( \beta = \) containing the additional term

\[
\int_{\partial \mathcal{D}} \sum_{j=1}^3 (n(\partial \mathcal{D}))_j u_j(y) u_j(y) \ dy.
\]

**Corollary 2.** Let \( U \subset \mathbb{R}^3 \) be open and bounded, \( S_1 \subset (0, \infty) \) with \( \overline{U} \subset B_{S_1} \), \( p \in (1, \infty) \), \( f \in L^p(U)^3 \) with supp \( (f) \subset B_{S_1}. \) Let \( u \in W^{1,1}_{\text{loc}}(\overline{U})^3 \) with \( u \in L^p(\overline{U})^3 \), \( \nabla u \in L^2(\overline{U})^3 \), \( \pi \in L^2_{\text{loc}}(\overline{U})^3 \), and suppose that \( u, \pi \) satisfy (12) (weak form of rotational Navier-Stokes system, as in Theorem 2.14) with \( U \) in the place of \( \mathcal{D}. \) Let \( S_0 \subset (0, S_1) \) with \( \overline{U} \subset B_{S_0}. \) Then the conclusions of Theorem 3.1 hold, with \( \mathcal{D} \) replaced by \( B_{S_0} \).

**Proof.** Obviously \( (u \cdot \nabla) u \in L^{3/2}(\overline{U})^3 \) so that \( f - (u \cdot \nabla) u \in L^p_{\text{loc}}(\partial \mathcal{D})^3. \) Therefore, by interior regularity of the Stokes system, as stated in [33 Theorem IV.4.1], we have \( u \in W^{2, \min(p,3/2)}(\overline{U})^3 \) and \( \pi \in W^{1, \min(p,3/2)}(\overline{U})^3 \); also see the proof of [7 Theorem 5.5]. Now the corollary follows from Theorem 3.1 with \( p_0 = \min(p,3/2) \) and \( B_{S_0} \) in the place of \( \mathcal{D}. \)

\[ \square \]

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