THE WRONSKI MAP AND SHIFTED TABLEAU THEORY

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ABSTRACT. The Mukhin-Tarasov-Varchenko Theorem, conjectured by B. and M. Shapiro, has a number of interesting consequences. Among them is a well-behaved correspondence between certain points on a Grassmannian — those sent by the Wronski map to polynomials with only real roots — and (dual equivalence classes of) Young tableaux.

In this paper, we restrict this correspondence to the orthogonal Grassmannian \( \text{OG}(n,2n+1) \subset \text{Gr}(n,2n+1) \). We prove that a point lies on \( \text{OG}(n,2n+1) \) if and only if the corresponding tableau has a certain type of symmetry. From this we recover much of the theory of shifted tableaux for Schubert calculus on \( \text{OG}(n,2n+1) \), including a new, geometric proof of the Littlewood-Richardson rule for \( \text{OG}(n,2n+1) \).

1. Introduction

For any non-negative integer \( k \), let \( \mathbb{C}_k[z] \) denote the \((k+1)\)-dimensional complex vector space of polynomials of degree at most \( k \):

\[
\mathbb{C}_k[z] := \{ f(z) \in \mathbb{F}[z] \mid \text{deg } f(z) \leq k \}.
\]

Let \( X = \text{Gr}(n, \mathbb{C}_{2n}[z]) \), the Grassmannian variety of all \( n \)-dimensional subspaces of the \((2n+1)\)-dimensional vector space \( \mathbb{C}_{2n}[z] \). If \( x \in X \) is the span of polynomials \( f_1(z), \ldots, f_n(z) \), the Wronskian

\[
\text{Wr}(x; z) := \begin{vmatrix}
  f_1(z) & \cdots & f_n(z) \\
  f'_1(z) & \cdots & f'_n(z) \\
  \vdots & \ddots & \vdots \\
  f_1^{(n-1)}(z) & \cdots & f_n^{(n-1)}(z)
\end{vmatrix}.
\]

is a non-zero polynomial of degree at most \( n(n+1) \), and up to a scalar multiple, it depends only on \( x \). Hence, \( x \mapsto \text{Wr}(x; z) \) determines a well-defined, morphism schemes \( \text{Wr} : X \to \mathbb{P}(\mathbb{C}_{n(n+1)}[z]) \) called the Wronski map. This morphism is flat and finite [2].

Let \( \text{SYT}(\square) \) denote the set of standard Young tableaux whose shape is an \( n \times (n+1) \) rectangle. The degree of Wronskian map is equal to \( |\text{SYT}(\square)| \); hence one might hope to find a surjective correspondence between \( \text{SYT}(\square) \) and the points of a fibre \( \text{Wr}^{-1}(h(z)) \) of the Wronski map. It turns out that this is possible to do when the roots of \( h(z) \)

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are all real; in this case, we write
\[ h(z) = \prod_{a_i \neq \infty} (z + a_i), \]
a_1, \ldots, a_{n(n+1)} \in \mathbb{R}^P — a polynomial of degree \( n(n+1) - k \), is considered to have a root of multiplicity \( k \) at \( \infty \). Eremenko and Gabrielov first established such a correspondence in an asymptotic setting \([1]\), and the remarkable theorem of Mukhin, Tarasov and Varchenko (see Theorem 3 in Section 2) ensures that it can extended unambiguously to polynomials with only real roots. We refer the reader to the survey article \([11]\) for a discussion of the history, context and other applications of this result.

In this paper, we will use the notation \( X(a) := \text{Wr}^{-1}(\prod_{a_i \neq \infty} (z + a_i)) \), to denote the fibre of the Wronski map associated to the multiset \( a = \{a_1, \ldots, a_{n(n+1)}\} \), and \( x_T(a) \) to denote the specific point in \( X(a) \) that corresponds to the tableau \( T \in \text{SYT}(\square) \). We will review a characterization and other key properties of the correspondence in Section 2. For now it is enough to remark that if \( n > 1 \), \( a \mapsto x_T(a) \) is not a continuous function. As strange as it may seem, this is a feature, not a bug: in \([7]\), we showed that the discontinuities essentially encode Schützenberger’s jeu de taquin, and this fact provides a tight connection between the geometry of \( X \) and the combinatorics of Young tableaux.

Our goal in this paper is to establish similar results for the orthogonal Grassmannian. Let \( \langle \cdot, \cdot \rangle \) be the non-degenerate symmetric bilinear form on \( \mathbb{C}^{2n}[z] \) given by
\[ \langle \sum_{k=0}^{2n} a_k z^k ; \sum_{\ell=0}^{2n} b_\ell z^\ell \rangle = \sum_{m=0}^{2n} (-1)^m a_m b_{2n-m}. \]
The **orthogonal Grassmannian** \( Y = \text{OG}(n, \mathbb{C}^{2n}[z]) \subset X \) is the variety of all \( n \)-dimensional isotropic subspaces of \( \mathbb{C}^{2n}[z] \).

The restriction of the Wronski map to \( Y \) has the interesting property that \( \text{Wr}(y ; z) \) is a perfect square for all \( y \in Y \) \([8]\). This raises the following question. Suppose that \( x \in X \) has the property that \( \text{Wr}(x ; z) \) is a perfect square. Under what conditions can we conclude that \( x \in Y \)?

To give a concrete answer, we will need to assume, moreover, that \( \text{Wr}(x ; z) = \prod_{a_i \neq \infty} (z + a_i) \) has only real roots. This allows us to write \( x = x_T(a) \) for some \( T \in \text{SYT}(\square) \). Suppose that the tableau \( T \) has entry \( k \) in row \( i_k \) and column \( j_k \). We’ll say that \( T \) is **symmetrical** if \( i_{2k} = j_{2k-1} \) and \( j_{2k} = i_{2k-1} + 1 \), for all \( k = 1, \ldots, \frac{n(n+1)}{2} \). See Figure 1 for an example.

![Figure 1](image-url)
We are now ready to state our main result, whose proof will be given in Section 3.

**Theorem 1.** Let \( x \in X \) be such that \( \text{Wr}(x; z) = \prod_{a_i \neq \infty} (z + a_i) \) is a perfect square with only real roots. Then \( x \in Y \) if and only if there exists a symmetrical tableau \( T \in \text{SYT}(\varnothing) \) such that \( x = x_T(a) \).

The combinatorics of symmetrical tableaux are essentially the same as the combinatorics of shifted tableaux; indeed if one deletes the odd entries from a symmetrical tableau, the result is a standard shifted tableau (with entries multiplied by 2). In Section 4 we will use Theorem 1 to show that the results of [7, Section 6] have analogues for \( Y \), where tableaux are replaced by shifted tableaux. This includes a geometric proof of the Littlewood-Richardson rule for \( \text{OG}(n, 2n+1) \).

We had already noted in [8] that it should be possible to prove these analogues by adapting the proofs in [7]. This, however, would be a long and tedious exercise. The approach we take in this paper is considerably more efficient. Rather than reprove everything, we will use Theorem 1, in combination with results from [8], to deduce facts about \( Y \) easily and directly from known facts about \( X \).

### 2. Tableaux and points of \( X \)

Rather than recall exactly how the correspondence \((T, a) \mapsto x_T(a)\) was originally defined, we will state a theorem (Theorem 4) that describes some of its important properties, and prove that these properties characterize the map. Before we do this, we need some additional notation and background.

For each \( a \in \mathbb{C}P^1 \), define a full flag in \( \mathbb{C}_{2n}[z] \)

\[
F_\bullet(a) : \{0\} \subset F_1(a) \subset \cdots \subset F_{2n}(a) \subset \mathbb{C}_{2n}[z].
\]

For \( a \in \mathbb{C} \), \( F_i(a) := (z + a)^{2n+1-i} \mathbb{C}[z] \cap \mathbb{C}_{m-1}[z] \) is the set of polynomials in \( \mathbb{C}_{2n}[z] \) divisible by \((z + a)^{2n+1-i}\). We also set \( F_\infty := \lim_{a \to \infty} F_i(a) \).

Let \( \Lambda \) denote the set of all partitions \( \lambda \) where \( \lambda_1 \leq n+1 \), \( \lambda_n \geq 0 \). The largest partition in \( \Lambda \) is denoted by \( \varnothing \). For each \( \lambda \in \Lambda \) we have a **Schubert variety** in \( X \) relative to the flag \( F_\bullet(a) \):

\[
X_\lambda(a) := \{ x \in X \mid \dim (x \cap F_{n+1-i+\lambda_i}(a)) \geq i, \text{ for } i = 1, \ldots, n \},
\]

which has codimension \( |\lambda| \) in \( X \). We denote its cohomology class by \([X_\lambda] \in H^{2|\lambda|}(X)\).

The relationship between Schubert varieties and the Wronski map is given by the following classical fact (see e.g. [2, 7, 11]).

**Proposition 2.** \( \text{Wr}(x; z) \) is divisible by \((z + a)^k\) if and only if \( x \in X_\lambda(a) \) for some partition \( \lambda \vdash k \).

The Mukhin-Tarasov-Varchenko Theorem asserts, moreover, that intersections of Schubert varieties relative to the flags \( F_\bullet(a) \) are as well behaved as one might possibly hope.
Theorem 3 (Mukhin-Tarasov-Varchenko \[1, 5\]). If \(a_1, \ldots, a_s \in \mathbb{R}P^1\), and \(\lambda_1, \ldots, \lambda_s \in \Lambda\) are partitions with \(|\lambda_1| + \cdots + |\lambda_s| = \dim X\), then the intersection
\[
X_{\lambda_1}(a_1) \cap \cdots \cap X_{\lambda_s}(a_s)
\
\]
is finite, transverse, every point in the intersection is real (i.e. has a basis in \(\mathbb{R}[z]\)).

We will also need some combinatorial notions from tableau theory. If \(\lambda, \mu\) are partitions, \(\lambda \geq \mu\), let \(\text{SYT}(\lambda/\mu)\) denote the set of standard Young tableaux of shape \(\lambda/\mu\). Suppose that \(T \in \text{SYT}(\lambda/\mu)\) and \(U \in \text{SYT}(\mu)\). We can draw \(T\) in red and \(U\) in blue on the same diagram of shape \(\lambda\), with \(U\) “inside” of \(T\). The basic jeu de taquin algorithm can be used to switch \(U\) and \(T\), so that we end up with two new tableaux, \(\hat{T}\) in red on the inside, and \(\hat{U}\) in blue on the outside.

1. Let \(u\) be the largest entry in \(U\).
2. Slide \(u\) through \(T\). (If there are entries of \(T\) to the right of \(u\) and below \(u\), switch the smaller of these entries with \(u\). If only one of these exists, switch it with \(u\). Repeat until \(u\) has reached the “outside” of \(T\).)
3. Let \(u\) be the next largest entry in \(U\), and repeat step (2) until every entry of \(U\) has been moved outside of \(T\).

The resulting \(\hat{T}\) is called the \textit{rectification} of \(T\); its shape is a partition, called the \textit{rectification shape} of \(T\). A theorem of Schützenberger states that \(\hat{T}\) does not depend on \(U\) \[10\]. On the other hand, \(\hat{U}\), may depend on \(T\). We say that \(T\) and \(T'\) are \textit{dual equivalent}, and write \(T \sim^* T'\), if \(T\) and \(T'\) produce the same \(\hat{U}\) for all (equivalently for some) \(U \in \text{SYT}(\mu)\). Both versions of this last definition are due to Haiman \[3\]. It is worth noting that the dual equivalence relation \(T \sim^* T'\) is quite different from \(\hat{T} = \hat{T'}\); in fact, if both are true then \(T = T'\). Dual equivalence classes on \(\text{SYT}(\lambda/\mu)\) are in bijection with Littlewood-Richardson tableaux of shape \(\lambda/\mu\); hence statements involving Littlewood-Richardson numbers may be formulated in terms of counting dual equivalence classes.

If \(T \in \text{SYT}(\square)\), and \(I\) is an interval, let \(T_i\) denote the subtableau of \(T\) consisting of entries in \(I\). We’ll also sometimes write \(T_{<i} := T_{[1,i)}, T_{\geq i} := T_{[i,n(n+1)]}, \) etc. \(T_i\) is essentially a standard Young tableau of some skew shape \(\lambda/\mu\) — the definitions of rectification and dual equivalence make sense despite the fact that the entries are \(\mathbb{Z} \cap I\) instead of \(\{1, \ldots, |\lambda/\mu|\}\).

Finally, let \(A\) denote the set of \(n(n+1)\)-element multisets \(a = \{a_1, \ldots, a_{n(n+1)}\}\), \(a_1, \ldots, a_{n(n+1)} \in \mathbb{R}P^1\). There is a natural map \((\mathbb{R}P^1)^{n(n+1)} \to A, (a_1, \ldots, a_{n(n+1)}) \mapsto \{a_1, \ldots, a_{n(n+1)}\}\); we endow \(A\) with the quotient topology. We will also need to refine the relation \(|a| \leq |b|, a, b \in \mathbb{R}P^1\), to a total order. Any refinement will do, but for the sake of concreteness, define \(a \preceq b\) if either \(a = b\), \(|a| < |b|\) or \(0 < a = -b < \infty\). Define \(a \preceq\)-\textit{zone} to be a subset of \(A\) of the form
\[
\{a_1 \preceq a_2 \preceq \cdots \preceq a_{n(n+1)}\} \in A \mid 0 \leq a_i \epsilon_i \leq \infty \text{ for } i = 1, \ldots, n(n+1)\},
\]
where \(\epsilon_1, \ldots, \epsilon_{n(n+1)} \in \{\pm 1\}\).

**Theorem 4.** There is a unique map \(\text{SYT}(\square) \times A \to X\), denoted \((T, a) \mapsto x_T(a)\), with all of the following properties:
(i) For all \( T \in \text{SYT}(\square) \) and \( a \in A \), \( x_T(a) \in X(a) \).

(ii) For all \( a \in A \), the map \( T \mapsto x_T(a) \) is surjective onto the fibre \( X(a) \). If \( a \) is a set, i.e. \( a_i \neq a_j \) for all \( i \neq j \), then it is also one to one.

(iii) For any \( T \in \text{SYT}(\square) \), the map \( a \mapsto x_T(a) \) is discontinuous at \( a \) only if

\[
\sum_{i,j} a_i a_j \neq \sum_{i,j} a_i a_j
\]

(iv) Assume that \( a_1 \leq a_2 \leq \cdots \leq a_{n(n+1)} \), and that \( a_i = a_{i+1} = \cdots = a_j \). Let

\[
T \in \text{SYT}(\square).
\]

Then \( x_T(a) \in X(\lambda(a)) \) where \( \lambda \) is the rectification shape of \( T_{[i,j]} \).

(v) Under the same hypotheses as (iv), let \( T, T' \in \text{SYT}(\square) \) be two tableaux such that

\[
T_{[i,j]} = T'_{[i,j]}, \quad T'_{[i,j]} = T_{[i,j]}.
\]

Then \( x_T(a) = x_{T'}(a) \) if and only if \( T_{[i,j]} \sim^* T'_{[i,j]} \).

**Proof.** The “existence” part of this theorem is mainly a summary of several of the results in [8]. There, a map satisfying (i) and (ii) is constructed for

\[
\{a \in A | 0 < |a_1| < |a_2| < \cdots < |a_{n(n+1)}| < \infty\}
\]

[8 Corollary 4.10]: it is continuous on that disconnected domain. Since \( W \) is flat and finite, we can extend this to a continuous map on any single \( \leq \)-zone, and (ii) will still hold. If \( 0, \infty \notin a \), then \( a \) is in a unique \( \leq \)-zone, and this defines \( x_T(a) \) unambiguously. Otherwise, \( a \) is in more than one \( \leq \)-zone, and we need [8, Theorem 4.5] to see that \( x_T(a) \) is well-defined. Thus the original correspondence can be extended to all \( a \in A \) in such a way that (i)–(iii) hold.

Statement (iv) and the \( \Rightarrow \) direction of (v) are the content of [8, Theorem 6.4].

As for the \( \Rightarrow \) direction of (v), suppose that

\[
a_1 = \cdots = a_{l_1} < a_{l_1+1} = \cdots = a_{l_2} < \cdots < a_{l_{m+1}} = \cdots = a_{n(n+1)}.
\]

Let \( \sim^*_a \) be the equivalence relation on \( \text{SYT}(\square) \) defined by \( T \sim^*_a T' \) if and only if

\[
T_{[i_0,i_{l+1}]} \sim^* T'_{[i_0,i_{l+1}]} \quad \text{for all } l = 0, 1, \ldots, m.
\]

From (ii) and the \( \Rightarrow \) direction of (v), we know that \( T \mapsto x_T(a) \) is surjective onto \( X(a) \) and constant on the equivalence classes of \( \sim^*_a \). The Littlewood-Richardson rule tells us that the number of \( \sim^*_a \) equivalence classes in \( \text{SYT}(\square) \) is equal to

\[
\int_X \prod_{l=0}^m \left( \sum_{\lambda \vdash (i_{l+1}-i_l)} [X_\lambda] \right).
\]

The transversality statement in Theorem 3 interpreted through Proposition 2 asserts that this is exactly the number of points in \( X(a) \). Thus there cannot be two equivalence classes of \( \sim^*_a \) that map to the same point in \( X(a) \).

It remains to show uniqueness. By the continuity property (iii), it is enough to show that the inverse map \( X(a) \to \text{SYT}(\square) \) is determined by properties (i)–(iv), in the case where \( a \) is a set. Assume that

\[
a_1 < a_2 < \cdots < a_{n(n+1)},
\]

and let \( x \in X(a) \). We will prove that from \( x \), one can uniquely determine the tableau \( T \) such that \( x = x_T(a) \).

Let \( a_{t,k} = \{ta_1, \ldots, ta_k, a_{k+1}, \ldots, a_{n(n+1)}\} \) for \( t \in [0, 1], k \in \{1, \ldots, n(n+1)\} \). By (ii) the map \( T \mapsto x_T(a_{t,k}) \) is one to one for all \( t \in (0,1) \). Thus there is a unique lifting

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of the path \( t \mapsto \mathbf{a}_{t,k} \in A, t \in [0,1] \), to a path \( t \mapsto x_{t,k} \in X(\mathbf{a}_{t,k}) \), with \( x_{1,k} = x \). By (iii), the map \( t \mapsto x_T(\mathbf{a}_{t,k}) \) is also continuous on \([0,1]\), and so we see that if \( T \) is the tableau such that \( x_T(\mathbf{a}) = x \), then \( x_T(\mathbf{a}_{t,k}) = x_{t,k} \), for all \( t \in [0,1] \). In particular, \( x_T(\mathbf{a}_{0,k}) = x_{0,k} \).

Now, since \( \mathbf{a}_{0,k} \) contains 0 with multiplicity \( k \), by (iv) we have \( x_T(\mathbf{a}_{0,k}) \in X_\lambda(0) \) where \( \lambda \) is the shape of \( T_{\leq k} \). Moreover, since 0 does not have multiplicity greater than \( k \) in \( \mathbf{a}_{0,k} \), we cannot have \( x_T(\mathbf{a}_{0,k}) \in X_\mu(0) \) for any \( \mu > \lambda \). It follows that the tableau \( T \) such that \( x = x_T(\mathbf{a}) \) must have the property that the shape of \( T_{\leq k} \) is the largest partition \( \lambda \) such that \( x_{0,k} \in X_\lambda(0) \). Since this is true for all \( k \in \{1, \ldots, n(n+1)\} \), we have determined \( T \), as required.

**Remark 5.** The fact that the discontinuities of the map \( \mathbf{a} \mapsto x_T(\mathbf{a}) \) are at points where \( a_i = -a_j \) for some \( i, j \) has no particular geometric significance: the fibres of the Wronski map are as well behaved at these points as any. However, the uniqueness of the input data for a Schubert variety \( Y \) in \( X \) shows that it is impossible to produce a continuous correspondence satisfying (i), (ii) and (iv). Since these are highly desirable properties, we are forced to have jump discontinuities somewhere, and the points for which \( a_i = -a_j \) are a fairly obvious and convenient choice for where to put them.

### 3. Tableaux and points of \( Y \)

Our goal in this section is to prove Theorem 1. We begin by recalling some of the relevant results from [8].

Let \( \Sigma \) denote the set of all strict partitions \( \sigma : (\sigma^1 > \sigma^2 > \cdots > \sigma^k) \), with \( \sigma^1 \leq n, \sigma^k > 0, k \leq n \). The diagram of \( \sigma \) contains \( \sigma_j \) boxes in the \( j^{th} \) row, with the leftmost box shifted \( j-1 \) boxes to the right. A **standard shifted tableau** of shape \( \sigma \) is a filling of the diagram of \( \sigma \) with entries \( 1, \ldots, |\sigma| \); the entries must increase downwards and to the right; we write \( \text{SST}(\sigma) \) for the set of all standard shifted tableaux of shape \( \sigma \). We will be particularly concerned with \( \text{SST}(\nabla) \), where \( \nabla : (n > n-1 > \cdots > 2 > 1) \) denotes the largest strict partition in \( \Sigma \).

The input data for a Schubert variety \( Y \) are a strict partition \( \sigma \in \Sigma \) and a flag \( F^\bullet \) satisfying \( \langle F_i, F_{2n+1-i} \rangle = \{0\} \) for all \( i = 0, 1, \ldots, 2n+1 \); the flags \( F^\bullet(a) \) satisfy this condition. For our purposes, the most convenient way to define Schubert varieties in \( Y \) is in terms of the Schubert varieties in \( X \). For each strict partition \( \sigma \in \Sigma \), define a partition \( \tilde{\sigma} : (\tilde{\sigma}^1 \geq \tilde{\sigma}^2 \geq \cdots \geq \tilde{\sigma}^n) \),

\[ \tilde{\sigma}^j := \sigma^i + \# \{ j \mid j \leq i < j + \sigma^j \} . \]

The **Schubert variety** in \( Y \) relative to the flag \( F^\bullet(a) \) is

\[ Y_\sigma(a) := Y \cap X_{\tilde{\sigma}}(a) . \]

\( Y_\sigma(a) \) has codimension \( |\sigma| \) in \( Y \); we denote its cohomology class by \( [Y_\sigma] \in H^{2|\sigma|}(Y) \).

Figure 2 shows an example of a strict partition \( \sigma \) and the associated partition \( \tilde{\sigma} \).

The diagram of \( \tilde{\sigma} \) always decomposes as a copy of the diagram of \( \sigma \) and its “transpose”, exhibiting the same type of symmetry as a symmetrical standard young tableau.
If \( a = \{a_1, a_2, \ldots, a_m\} \) is a multiset, let \( a^* := \{a_1, a_1, a_2, a_2, \ldots, a_m, a_m\} \). Let \( A^* \subseteq A \) denote the subspace of multisets of the form \( a^* \), \( a = \{a_1, \ldots, a_{n(n+1)/2}\} \). As mentioned in the introduction, \( \text{Wr}(y; z) \) is always a perfect square for \( y \in \mathcal{Y} \), hence \( \mathcal{Y} \cap \mathcal{X}(a) = \emptyset \) if \( a \not\in A^* \). If \( a = b^* \in A^* \), let
\[
\mathcal{Y}(b) := \mathcal{Y} \cap \mathcal{X}(b^*),
\]
This will be the analogue of the fibre of the Wronski map for \( \mathcal{Y} \).

The next two results from [8] are analogues of Proposition 2 and Theorem 3.

**Proposition 6.** Let \( y \in \mathcal{Y} \). \( \text{Wr}(y; z) \) is divisible by \( (z + a)^{2k} \) if and only if \( y \in \mathcal{X}(\sigma) \) for some strict partition \( \sigma \vdash k \).

**Theorem 7.** If \( b_1, \ldots, b_\ell \in \mathbb{R} \) are distinct real points, and \( \sigma_1, \ldots, \sigma_\ell \in \Sigma \), with \( |\sigma_1| + \cdots + |\sigma_\ell| = \dim \mathcal{Y} \), then the intersection
\[
\mathcal{Y}_{\sigma_1}(b_1) \cap \cdots \cap \mathcal{Y}_{\sigma_\ell}(b_\ell)
\]
is finite, transverse, and every point in the intersection is real.

In the case where \( b_1, \ldots, b_{n(n+1)/2} \) are distinct real numbers and \( \sigma_i = [\ ] \) for \( i = 1, \ldots, \frac{n(n+1)}{2} \), Proposition 6 and Theorem 7 tell us that \( |\mathcal{Y}(b)| \) is equal to the Schubert intersection number \( \int_{\mathcal{Y}} |\mathcal{X}(\bigwedge^n\mathcal{Y})| \). Basic Schubert calculus then gives us
\[
|\mathcal{Y}(b)| = |\text{SST}(\nabla)|.
\]

We need one additional Lemma.

**Lemma 8.** Let \( b = \{0 \prec b_1 \prec \cdots \prec b_{n(n+1)/2}\} \subseteq \mathbb{R} \) be a set, and let \( B \) be the space of multisets \( c = \{c_1 \preceq \cdots \preceq c_{n(n+1)/2}\} \) such that \( b_i c_i \geq 0 \) for all \( i \).

(i) If \( T \in \text{SYT}(\bigwedge^n) \) is a tableau such that \( x_T(b^*) \in \mathcal{Y} \), then for \( x_T(c^*) \in \mathcal{Y} \) for all \( c \in B \).

(ii) If \( x \in \mathcal{Y}(c) \) where \( c \in B \), then there exists a tableau \( T \) such that \( x = x_T(c^*) \) and \( x_T(b^*) \in \mathcal{Y} \).

**Proof.** Let \( B^0 \subseteq B \) be the subspace of sets \( \{c_1 \prec \cdots \prec c_{n(n+1)/2}\} \) such that \( b_i c_i > 0 \) for all \( i \). The fibre \( \mathcal{X}(c^*) \) is reduced if \( c \in B^0 \), and \( \mathcal{Y}(c) \) is a subset of \( \mathcal{X}(c^*) \) varying continuously with \( c \). Since \( B \) is connected, any continuous section \( s : B \to \mathcal{X} \), \( s(c) \in \mathcal{X}(c^*) \) will either have its image entirely in \( \mathcal{Y} \), or \( s(B^0) \cap \mathcal{Y} = \emptyset \). In particular, since \( b \in B^0 \), if \( s(b) \in \mathcal{Y} \) then the former occurs. Statement (i) follows by applying this to the section \( c \mapsto x_T(c^*) \), which is continuous on \( B \) by Theorem 4(iii).

![Figure 2. The strict partition \( \sigma : (4 \geq 3 > 1) \) (left) and the associated partition \( \bar{\sigma} : (5 \geq 5 \geq 3 \geq 2 \geq 2) \) (right).](image)
For statement (ii), choose any path $c_t \in B$, $t \in [0, 1]$, with $c_0 = c$, $c_1 = b$. We can lift this (though not necessarily uniquely) to a path in $y_t \in \text{Y}(c_t)$ such that $y_0 = x$. Then $y_1 = x_T(b^*)$ for some tableau $T$, and since $c \mapsto x_T(c^*)$ is continuous on $B$, we also have $x = y_0 = x_T(c^*)$.

We now turn to our main result.

**Theorem 1** Let $x \in X(a)$, where $a \in A^\ast$. Then $x \in Y$ if and only if there exists a symmetrical tableau $T \in \text{SYT}(\square)$ such that $x = x_T(a)$.

**Proof.** Let $T \in \text{SYT}(\square)$, and let $b = \{0 < b_1 < \cdots < b_{n+1/2}\} \subset \mathbb{R}$ be a set. We’ll first show that if $x_T(b^*) \in Y$, then $T$ is symmetrical.

First, we note that since each element of $b^*$ has multiplicity 2. By Theorem 4(iv), the rectification shape of $T_{[2k-1, 2k]}$ must be equal to the $\lambda$, where $x_T(b^*) \in X_\lambda(b_k)$. Since $x_T(b^*) \in Y$, $\lambda = \square$. For $T$ to have rectification shape $\square$, entry $2k - 1$ in $T$ must be strictly to the left of entry $2k$.

Now, for $k = \{1, \ldots, \frac{n+1}{2}\}$, let $b_k = \{0, \ldots, 0, b_{k+1}, \ldots, b_{(n+1)/2}\}$. By Lemma 8(i), we have $x_T(b_k^*) \in Y$. Since $b_k^*$ contains 0 with multiplicity $2k$, by Theorem 4(iv), shape of $T_{\leq 2k}$ must be equal to the largest partition $\lambda$ such that $x_T(b_k^*) \in X_\lambda(0)$. Since $x_T(b_k^*) \in Y$, by $\lambda = \sigma$ for some strict partition $\sigma$.

Putting these two facts together, we see that $T$ must be symmetrical. This shows that we have an injective map from $Y(b)$ to the symmetrical tableaux in $\text{SYT}(\square)$, or equivalently to $\text{SST}(\bigtriangleup)$. Since the number of points in the fibre $Y(b)$ is equal to $|\text{SST}(\bigtriangleup)|$, this is a bijection.

Finally, let $a$ be as in the statement of the theorem. Then $a = c^*$, for some multiset $c$. We can find a set $b$ as above such that $c \in B$, where $B$ is the set defined in Lemma 8. If $T$ is symmetrical, then since $x_T(b^*) \in Y$, by Lemma 8(i) we have $x_T(c^*) \in Y$. Conversely, if $x \in X(a) \cap Y = Y(c)$, then by Lemma 8(ii) there exists a tableau $T$ such that $x = x_T(a^*)$ and $x_T(b^*) \in Y$. As we’ve just shown, the last statement implies that $T$ is symmetrical.

4. **Shifted tableau theory**

The theory of shifted tableaux, as developed in [3, 4, 9, 12, 13], is parallel to the theory of Young tableaux. In particular, the jeu de taquin theory works in essentially the same way. Given strict partitions $\sigma \geq \tau$ in $\Sigma$, and shifted tableaux $T \in \text{SST}(\sigma/\tau)$ and $U \in \text{SST}(\tau)$, the tableau switching algorithm outlined in Section 2 makes sense exactly as stated. Thus we can define the rectification (and rectification shape) of $T \in \text{SST}(\sigma/\tau)$ as well as the dual equivalence relation $\sim^*$ on $\text{SST}(\sigma/\tau)$.

As already noted in the introduction, for any shifted tableau $T \in \text{SST}(\bigtriangleup)$, there is is a corresponding symmetrical tableau $T^* \in \text{SYT}(\square)$, characterized by the fact that deleting the odd entries of $T^*$ gives $T$, with entries multiplied by 2. This same definition makes sense for any $T \in \text{SST}(\sigma/\tau)$ of skew shape, in which case the corresponding $T^*$ is a skew tableau in $\text{SYT}(\sigma/\tau)$. 
Lemma 9. Let $T \in \text{SST}(\sigma/\tau)$ and $U \in \text{SST}(\tau)$. Let $\hat{T}$ and $\hat{U}$ be the results of applying the tableau switching algorithm to $T$ and $U$. Let $\hat{T}^*$ and $\hat{U}^*$ be the results of applying the switching algorithm to $T^* \in \text{SYT}(\tilde{\sigma}/\tilde{\tau})$ and $U^* \in \text{SYT}(\tilde{\tau})$. Then

$$\hat{T}^* = (\hat{T})^* \quad \text{and} \quad \hat{U}^* = (\hat{U})^*. $$

Proof. One can easily check that each time we slide two entries of $U^*$ through $T^*$, the first never crosses below the diagonal, and the second entry takes a path symmetrical to the first. Thus if $T^*$ and $U^*$ are remain symmetrical throughout the switching algorithm, and if we simply delete the odd entries, we recover the switching algorithm for $T$ and $U$. \hfill \Box

From this observation, we can deduce facts about dual equivalence for shifted tableaux from the corresponding facts about standard Young tableaux. For example:

Proposition 10. Let $\tau \in \Sigma$. Any two tableaux in $\text{SST}(\tau)$ are dual equivalent, as are any two tableaux in $\text{SST}(\tilde{\tau}/\tau)$.

The analogous statement for $\mu \in \Lambda$ is that any two tableaux in $\text{SYT}(\mu)$ are dual equivalent, as are any two tableaux in $\text{SYT}(\tilde{\mu}/\mu)$. This, and Proposition 10 are proved combinatorially in [3]. However, statements about dual equivalence for standard Young tableaux have a geometric interpretation. For example, the assertion above follows from [7, Lemma 6.1] and [7, Theorem 6.4]; the equivalence of the two versions of the definition of $\sim^*$ given in Section 2 is part of the proof of [7, Theorem 6.4]. This motivates us to outline a short proof of Proposition 10 by reduction.

Proof. The statement for $\text{SST}(\tau)$ is immediate from the definition of dual equivalence. For $\text{SST}(\tilde{\tau}/\tau)$, Lemma 9 implies that $T \sim^* T' \in \text{SST}(\tilde{\tau}/\tau)$ if and only if $T^* \sim^* (T')^* \in \text{SYT}(\tilde{\mu}/\mu)$ (for this, we need both definitions of the dual equivalence relation, one for each direction). Since any two tableaux in $\text{SYT}(\tilde{\mu}/\mu)$ are dual equivalent, the result follows. \hfill \Box

For $T \in \text{SST}(\tilde{\tau})$, and $b = \{b_1, \ldots, b_{n(n+1)}\}$, $b_1, \ldots, b_{n(n+1)} \in \mathbb{RP}^1$, let $y_T(b) := x_{T^*}(b^*)$. Theorem 1 tells us that $y_T(b) \in Y(b)$, and every point in $Y(b)$ is of the form $y_T(b)$ for some standard shifted tableau $T$. This key fact will be used implicitly throughout the rest of the paper.

Theorem 11. Suppose that $b_1 \leq b_2 \leq \cdots \leq b_{n(n+1)/2}$, and that $b_i = b_{i+1} = \cdots = b_j$.

(i) Let $T \in \text{SST}(\tilde{\tau})$. Then $y_T(b) \in Y_\sigma(b_i)$ where $\sigma$ is the rectification shape of $T_{[i,j]}$.

(ii) Let $T, T' \in \text{SST}(\tilde{\tau})$ be two tableaux such that $T_{<i} = T'_{<i}$, $T_{>j} = T'_{>j}$. Then $y_T(b) = y_{T'}(b)$ if and only if $T_{[i,j]} \sim^* T'_{[i,j]}$.

Proof. For (i), by Theorem 4(iv), $y_T(b) = x_{T^*}(b^*) \in X_\lambda(b_i)$ where $\lambda$ is the rectification shape of $T_{[2i-1,2j]}$. By Lemma 9, $\lambda = \tilde{\sigma}$ where $\sigma$ is the rectification shape of $T_{[i,j]}$. Since we also know $y_T(b) \in Y$, we deduce $y_T(b) \in X_\lambda(b_i) \cap Y = Y_\sigma(b_i)$. 

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For (ii), we have \( y_T(b) = x_{T'}(b^*) \) and \( y_{T'}(b) = x_{(T')}^*(b^*) \). By Theorem 4(v), \( x_{T'}(b^*) = x_{(T')}^*(b^*) \) if and only if \( T_{[2i-1,2j]} \sim^* (T')_{[2i-1,2j]}^* \). By Lemma 3 this is true if and only if \( T_{[i,j]} \sim^* T'_{[i,j]} \).

**Remark 12.** A related result, [7, Theorem 6.2], explains the geometric significance of the equivalence relation \( \sim \) on standard Young tableaux, defined by \( T \sim T' \) iff the rectifications of \( T \) and \( T' \) are equal. This too has an analogue for \( Y \), which can be proved by similar arguments. Since new notation is required to state it, we will omit further details here.

Similarly, [7, Theorem 3.5], which describes how the correspondence \( (T, a) \rightarrow x_T(a) \) changes at points of discontinuity, has an analogue for \( Y \). The statement is virtually identical, but with tableaux replaced by shifted tableaux. Here, however, a bit more finesse is required in the proof, since a path in \( A^* \) does not satisfy the hypotheses of [7, Theorem 3.5]. This can be resolved by perturbing the path, and we leave the details to the reader.

As an application of Theorem 11 we prove a version of the Littlewood-Richardson rule for the orthogonal Grassmannian (Theorem 14).

**Lemma 13.** For \( \kappa \in \Sigma \), let \( \kappa^\vee \in \Sigma \) be the strict partition such that \( \int_Y [Y_\kappa] \cdot [Y_{\kappa^\vee}] = 1 \) ([\( Y_{\kappa^\vee} \) is dual to \( Y_\kappa \) under the Poincaré pairing]). Every tableau in \( SST(\Sigma/\kappa) \) has rectification shape \( \kappa^\vee \).

**Proof.** Let \( T \in SST(\Sigma) \) be a tableau such that \( T_{\leq |\kappa|} \) has shape \( \kappa \), and suppose \( T_{>|\kappa|} \) has rectification shape \( \sigma \). Let \( b = \{b_1, \ldots, b_{n(n+1)}\} \) where \( b_1 = \cdots = b_{|\kappa|} = 0 \) and \( b_{|\kappa|+1} = \cdots = b_{n(n+1)/2} = \infty \). By Theorem 11(i), \( y_T(b) \in Y_\kappa(0) \cap Y_\sigma(\infty) \), but the fact that this intersection is non-empty implies \( \sigma = \kappa^\vee \).

**Theorem 14** (Littlewood-Richardson rule for \( OG(n, 2n+1) \)). For \( \sigma, \tau, \kappa \in \Sigma \), the Littlewood-Richardson number \( c^\sigma_{\tau\kappa} \) for \( OG(n, 2n+1) \), defined by

\[
[Y_\sigma] \cdot [Y_\tau] = \sum_\kappa c^\sigma_{\tau\kappa} [Y_\kappa]
\]

in \( H^*(Y) \), is equal to the number dual equivalence classes in \( SST(\kappa/\tau) \) with rectification shape \( \sigma \).

**Proof.** With \( \kappa^\vee \) as in Lemma 13 we have

\[
c^\kappa_{\sigma\tau} = \int_Y [Y_\sigma] \cdot [Y_\tau] \cdot [Y_{\kappa^\vee}].
\]

By Theorem 7 this is the number of points in \( Y_\tau(0) \cap Y_\sigma(1) \cap Y_{\kappa^\vee}(\infty) \).

Theorem 11 allows to determine exactly which tableaux correspond to points in this intersection, and when two tableaux correspond to the same point.

Let \( b = \{b_1, \ldots, b_{n(n+1)}\} \) where

\[
b_1 = \cdots = b_{|\tau|} = 0, \quad b_{|\tau|+1} = \cdots = b_{|\kappa|} = 1, \quad \text{and} \quad b_{|\kappa|+1} = \cdots = b_{n(n+1)/2} = \infty,
\]

and let \( T \in SST(\Sigma) \). By Theorem 11(i) we have:
In other words $y_T(b) \in Y_\tau(0) \iff T_{\leq |\tau|}$ has shape $\tau$;
$y_T(b) \in Y_\sigma(1) \iff T_{(|\tau|,|\kappa|)}$ has rectification shape $\sigma$;
$y_T(b) \in Y_\kappa(\infty) \iff T_{>|\kappa|}$ has rectification shape $\kappa^\vee$, or equivalently by Lemma 13, $T_{\leq |\kappa|}$ has shape $\kappa$.

Moreover, by Theorem 11(ii), $T$ and $T'$ correspond to the same point if and only if $T_{\leq |\tau|} \sim^* T'_{\leq |\tau|}$, $T_{(|\tau|,|\kappa|)} \sim^* T'_{(|\tau|,|\kappa|)}$, and $T_{>|\kappa|} \sim^* T'_{>|\kappa|}$. By Proposition 10, the first and last of these are true whenever the subtableaux have the same shape. Thus $T$ and $T'$ correspond to the same point if and only if $T_{(|\tau|,|\kappa|)} \sim^* T'_{(|\tau|,|\kappa|)}$.

These two arguments show that the point $y_T(b)$ depends only on $T_{(|\tau|,|\kappa|)}$. Putting them together, the points in $Y_\tau(0) \cap Y_\sigma(1) \cap Y_\kappa(\infty)$ correspond bijectively to tableaux in $SST(\kappa/\tau)$ with rectification shape $\sigma$, modulo dual equivalence. \hfill $\square$

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