TWO PARTITIONS OF A FLAG MANIFOLD

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INTRODUCTION

0.1. Let $G$ be a connected reductive group over an algebraically closed field $k$. Let $W$ be the Weyl group of $G$ and let $B$ be the variety of Borel subgroups of $G$. In this paper we consider two partitions of $B$ into pieces indexed by the various $w \in W$. One partition (introduced in [L79]) consists of the subvarieties

(a) $Y_{s,w} = \{ B \in B; pos(B, sBs^{-1}) = w \}$

where $s$ is in $G_*$ (the open subset of $G$ consisting of regular semisimple elements) and $pos : B \times B \to W$ is the relative position map. The other partition (introduced in [DL76]) is defined when

(b) $k$ is an algebraic closure of a finite prime field; and

c) $F : G \to G$ is the Frobenius map for an $F_q$-split rational structure on $G$ (with $F_q$ being the subfield of $k$ with $q$ elements); it consists of the subvarieties

d) $X_w = \{ B \in B; pos(B, F(B)) = w \}$.

(The definition of the varieties in (a) was inspired by that of the varieties in (d).) One of the themes of this paper is to point out a remarkable similarity between $Y_{s,w}$ in (a) and the quotient $U^F \backslash X_w$ of $X_w$ in (b), where $U^F$ is the group of rational points of the unipotent radical of an $F$-stable $B \in B$, acting by conjugation.

We show that $Y_{s,w}$ is affine when $w$ has minimal length in its conjugacy class (the analogous result for $U^F \backslash X_w$ was known earlier). We show that the closure of $Y_{s,w}$ has the same type of singularities as the closure of a Bruhat cell (the analogous result for $X_w$ was known earlier). We show that if $k, F, q$ are as in (b), (c), the number of fixed point of $F^t$ on $Y_{s,w}$ and on $U^F \backslash X_w$ is the same (here $t \in \{1, 2, \ldots \}$); this result is implicit in [L78], [L79]. (This number can be expressed as a trace of left multiplication by the standard basis element $T_w$ on the Iwahori-Hecke algebra of $W$.) We show that while the cohomologies of $X_w$ give rise to a virtual $G^F$-module (see [DL76]) (so that the cohomologies of $U^F \backslash X_w$ give rise to a virtual module of the Hecke algebra of $G^F$ with respect to $B^F$), the cohomologies of $Y_{s,w}$ give rise to a virtual $W$-module.

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But there is one key difference between $Y_{s,w}$ and $U^F \setminus X_w$: the former can be defined also in characteristic zero, while the latter cannot. A study of $Y_{s,w}$ over the real numbers can be found in §3.

Another biproduct of our study is a way to associate to any $w$ in $W$ (or more generally any Coxeter group) a subset $E(w)$ of $W$, see 1.10 and §5.

0.2. For any $B \in \mathcal{B}$ let $U_B$ be the unipotent radical of $B$. For $B \in \mathcal{B}, w \in W$ let

\[ \mathcal{B}_{B,w} = \{ B' \in B; \text{pos}(B, B') = w \}, \]

\[ \mathcal{B}_{B, \leq w} = \{ B' \in B; \text{pos}(B, B') \leq w \}. \]

Here $\leq$ is the standard partial order on $W$. For $w$ in $W$ and $B, B'$ in $\mathcal{B}$ let

\[ Z_{B, B', w} = \{ B'' \in B; \text{pos}(B, B'') = z, \text{pos}(B', B'') = w \}, \]

\[ Z_{B, B', \leq w} = \{ B'' \in B; \text{pos}(B, B'') = z, \text{pos}(B', B'') \leq w \} \]

where $z = \text{pos}(B, B')$. For a maximal torus $T$ of $G$ we set $\mathcal{B}^T = \{ B \in \mathcal{B}; T \subset B \}$. For $s \in G_*$ let $T_s$ be the unique maximal torus containing $s$.

**Proposition 0.3.** Let $s \in G_*, w \in W, z \in W$. Let $B \in \mathcal{B}^T, B' \in \mathcal{B}^T$ be such that $\text{pos}(B, B') = z$.

(a) There is a canonical isomorphism $Y_{s,w} \cap \mathcal{B}_{B,z} \sim Z_{B, B', w}$.

(b) Let $Y_{s, \leq w} = \bigcup_{y \in W} Y_{s,y}$ (a closed subset of $\mathcal{B}$). There is a canonical isomorphism $Y_{s, \leq w} \cap \mathcal{B}_{B,z} \sim Z_{B, B', \leq w}$.

Statement (a) is contained in the proof of [L79, 1.2].

**Proposition 0.4.** Assume that $k, F$ are as in 0.1(b),(c). Let $w \in W, z \in W$. Let $B \in \mathcal{B}^F, B' \in \mathcal{B}^F$ be such that $\text{pos}(B, B') = z$.

(a) There is a canonical isomorphism $U_B^F \setminus (X_w \cap \mathcal{B}_{B,z}) \sim Z_{B, B', w}$.

(b) Let $X_{\leq w} = \bigcup_{y \in W} X_y$ (a closed subset of $\mathcal{B}$). There is a canonical isomorphism $U_B^F \setminus (X_{\leq w} \cap \mathcal{B}_{B,z}) \sim Z_{B, B', \leq w}$.

The action of $U_B^F$ is by conjugation.

0.5. Let $q$ be an indeterminate. Let $\mathcal{H}$ be the free $\mathbb{Z}[q]$-module with basis $\{ T_w; w \in W \}$. It is well known that there is a unique structure of associative $\mathbb{Z}[q]$-algebra on $\mathcal{H}$ such that $T_y T_{y'} = T_{yy'}$ if $l(yy') = l(y) + l(y')$ and $(T_y + 1)(T_y - q) = 0$ if $l(y) = 1$. Here $l : W \to \mathbb{N}$ is the length function.

For $w, w'$ in $W$ we have $T_w T_{w'} = \sum_{w'' \in W} N_{w, w', w''} T_{w''}$ where $N_{w, w', w''} \in \mathbb{Z}[q]$.

**Corollary 0.6.** Assume that $k, F, q$ are as in 0.1(b),(c). Let $w \in W, z \in W$.

(a) Assume that $s \in G_* \cap G^F$. Let $B \in \mathcal{B}^T \cap \mathcal{B}^F$. Let $t \in \{1, 2, \ldots \}$. Then $\sharp(Y_{s,w} \cap \mathcal{B}_{B,z})^{F^t} = N_{w, z^{-1}, z^{-1}}(q^t)$.

(b) Let $B \in \mathcal{B}^F$. Then $\sharp(U_B^F \setminus (X_w \cap \mathcal{B}_{B,z}))^{F^t} = N_{w, z^{-1}, z^{-1}}(q^t)$. 
Corollary 0.7. Assume that $k, F, q$ are as in $0.1(b), (c)$ and $t \in \{1, 2, \ldots \}$. Let $w \in W$.

(a) Assume that $s \in G_\ast \cap G^F$ and that $T_s$ is split over $F_q$. Then $\tilde{\tau}(Y_{s,w}^{F^t}) = \sum_{z \in W} N_{w,z^{-1},z^{-1}}(q^t) = \text{tr}(T_w : H \to H)_{q=q^t}$.

(b) Let $B \in B^F$. Then $\tilde{\tau}(U_B^F \setminus X_w)^{F^t} = \sum_{z \in W} N_{w,z^{-1},z^{-1}}(q^t) = \text{tr}(T_w : H \to H)_{q=q^t}$.

Here $T_w : H \to H$ is left multiplication by $T_w$ in $H$. Note that (a) can be deduced from [L79, 1.2]; (b) appears in [L78, 3.10(a)].

Proposition 0.8. Let $w \in W, i \in Z$. Assume that $k, F, q$ are as in $0.1(b), (c)$.

(a) If $s \in G_\ast \cap G^F$ and some/any $B \in B^{T_s}$ satisfies $F(B) = B$, then any eigenvalue of $F$ on $H^i_c(Y_{s,w})$ is in $\{q^j; j \in Z\}$.

(b) If $B \in B^F$, any eigenvalue of $F$ on $H^i_c(U_B^F \setminus X_w)$ is in $\{q^j; j \in Z\}$.

0.9. Let $W$ be the set of conjugacy classes in $W$. For $w \in W$ we denote by $w$ the conjugacy class of $w$. In this subsection $k$ is as in 0.1(b). Assuming that $F$ is as in 0.1(c) we define a map $G_\ast \cap G^F \to W, s \mapsto [s]$ by $\text{pos}(B, F(B)) \in [s]$ for some/any $B \in B^{T_s}$. Let $w \in W$. As shown in [DL76], if $F$ is as in 0.1(c), the finite group group $G^F$ acts naturally on $H^i_c(X_w)$ ($\bar{\mathbb{Q}}$-cohomology with compact support, $i \in Z$), giving rise to a virtual representation $R_w = \sum_i(-1)^i H^i_c(X_w)$ of $G^F$; moreover, $R_w$ depends only on $w$.

For $j \in Z$ we denote by $H^i_c(X_w)_j$ the part of weight $j$ of $H^i_c(X_w)$ and we set $R_{j,w} = \sum_i(-1)^i H^i_c(X_w)_j$ (a virtual representation of $G^F$). Note that $R_w = \sum_j R_{j,w}$.

The following result is proved in §2.

(a) For any $j \in Z$ there is a unique virtual representation $\mathfrak{R}_{j,w}$ of $W$ such that for any $F, q$ as in 0.1(c) and any $s \in G_\ast \cap G^F$ we have $\text{tr}(y, \mathfrak{R}_{j,w}) = \sum_{i \in Z} (-1)^i \text{tr}(F, H^i_c(Y_{s,w})) q^{-j/2}$ for some/any $y \in [s]$. Here $H^i_c(Y_{s,w})_j$ is the part of weight $j$ of $H^i_c(Y_{s,w})$. Moreover, $\mathfrak{R}_w := \sum_j \mathfrak{R}_{j,w}$ depends only on $w$.

Another definition of $\mathfrak{R}_{j,w}$ (valid also in characteristic zero) is given in [L79, 1.2] (where it is denoted by $\rho_{j,w}$). We will not attempt to compare it with the present definition. In [L79] several arguments are based on a statement in [L79, p.327, line -6] which was stated without proof. That statement can be proved using the theory of character sheaves; this will not be done here.

0.10. Let $w \in W$. Let

(a) $\mathcal{E}(w) = \{z \in W; N_{w,z,z} \neq 0\}$.

From 0.6 we see that the following three conditions for $z \in W$ are equivalent.

(i) We have $Y_{s,w} \cap B_{B_0,z^{-1}} \neq \emptyset$ for some, or equivalently, any $s \in G_\ast, B \in B^{T_s}$.

(ii) We have $X_w \cap B_{B_0,z^{-1}} \neq \emptyset$ for some, or equivalently, any $B \in B^F$ (here $k, F$ are as in 0.1(b), (c)).

(iii) We have $z \in \mathcal{E}(w)$.

We use that $N(w, z, z) \neq 0$ implies $N(w, z, z)(q) \neq 0$ for any $q \in \{2, 3, \ldots \}$, since
(b) $N(w, w', w'')$ is a polynomial in $q, q - 1$ with coefficients in $\mathbb{N}$.

I believe that the subsets $E(w)$ of $W$ deserve further study. A beginning of such a study can be found in §5.

0.11. I thank Xuhua He for useful discussions.

1. Proof of Propositions 0.3, 0.4, 0.8

1.1. We prove Proposition 0.3(a). There is a unique $B' - \in \mathcal{B}^{T_{s}}$ such that $B' \cap B' - = T_{s}$. Let $U_{z} = U_{B} \cap U_{B' -}$. We identify $U_{z}$ with $\mathcal{B}_{B,z}$ by $u \mapsto uB'u^{-1}$. Hence we can identify

$$\begin{align*}
Y_{s,w} \cap \mathcal{B}_{B,z} &= \{ u \in U_{z}; \pos(uB'u^{-1}, suB'u^{-1}s^{-1}) = w \} \\
&= \{ u \in U_{z}; \pos(B', u^{-1}suB'u^{-1}s^{-1}u) = w \} \\
&= \{ u \in U_{z}; \pos(B', s^{-1}u^{-1}suB'u^{-1}s^{-1}us) = w \}.
\end{align*}$$

Using the isomorphism $U_{z} \sim U_{z}$ given by $u \mapsto s^{-1}u^{-1}su$ (recall that $s \in G_{s}$) we obtain an identification

\( (a) \ Y_{s,w} \cap \mathcal{B}_{B,z} = \{ u' \in U_{z}; \pos(B', u'B'u'^{-1}) = w \} \).

Now $u' \mapsto u'B'u'^{-1}$ identifies

$$\begin{align*}
\{ u' \in U_{z}; \pos(B', u'B'u'^{-1}) = w \} &= \{ B'' \in \mathcal{B}_{B,z}; \pos(B', B'') = w \} = Z_{B,B',w}.
\end{align*}$$

Combining with (a) we obtain 0.3(a). The same proof (replacing $= w$ by $\leq w$) gives 0.3(b). Note that the isomorphisms in 0.3 are given by $uB'u^{-1} \mapsto s^{-1}u^{-1}suB'u^{-1}s^{-1}us$ with $u \in U_{z}$.

1.2. We prove Proposition 0.4(a). Let $B' - \in \mathcal{B}^{F}$ such that $B' \cap B' -$ is a maximal torus contained in $B$. Let $U_{z} = U_{B} \cap U_{B' -}$. We identify $U_{z}$ with $\mathcal{B}_{B,z}$ by $u \mapsto uB'u^{-1}$. Hence we can identify

$$\begin{align*}
X_{s} \cap \mathcal{B}_{B,z} &= \{ u \in U_{z}; \pos(uB'u^{-1}, F(uB'u^{-1})) = w \} \\
&= \{ u \in U_{z}; \pos(B', u^{-1}F(u)B'F(u^{-1})u) = w \}.
\end{align*}$$

Using the isomorphism $U_{s}^{F} \setminus U_{z} \sim U_{z}$ given by $u \mapsto u^{-1}F(u)$ (coming from Lang’s theorem) we obtain an identification

$$\begin{align*}
U_{s}^{F} \setminus (X_{s} \cap \mathcal{B}_{B,z}) &= \{ u' \in U_{z}; \pos(B', u'B'u'^{-1}) = w \}.
\end{align*}$$

This can be identified with $Z_{B,B',w}$ as in 1.1. This proves 0.4(a). The same proof (replacing $= w$ by $\leq w$) gives 0.4(b). Note that the isomorphisms in 0.4 are given by $uB'u^{-1} \mapsto u^{-1}F(u)B'F(u^{-1})u$ with $u \in U_{z}$.

1.3. Let $T$ be a maximal torus of $G$. Let $w_{0}$ be the longest element of $W$. For any $B \in \mathcal{B}^{F}$ let $B_{x} = \{ B' \in \mathcal{B}; \pos(B', B) = w_{0} \}$, an open set in $\mathcal{B}$. We show:

\( (a) \ B = \bigcup_{B \in \mathcal{B}^{T}} (B_{B}). \)

Let $B' \in \mathcal{B}$. Let $B_{1} \in \mathcal{B}^{T}$. We have $\pos(B_{1}, B') = z$ for some $z \in W$. We can find $B \in \mathcal{B}^{T}$ such that $\pos(B, B_{1}) = w_{0}z^{-1}$. Since $l(w_{0}z^{-1}) + l(z) = l(w_{0})$ we must have $\pos(B, B') = w_{0}$. Thus $B' \in B_{B}$. This proves (a).
1.4. Let \( w \in W, s \in G_*. \) We show:
(a) \( Y_{s,w} \) is smooth of pure dimension \( l(w) \).

From 1.3(a) we have an open covering \( Y_{s,w} = \bigcup_{B \in \mathcal{B}^T_s} (Y_{s,w} \cap B \mathcal{B}) \). It is enough to prove that for any \( B \in \mathcal{B}^T_s \), \( Y_{s,w} \cap B \mathcal{B} \) is smooth of pure dimension \( l(w) \).

Define \( B' \in \mathcal{B}^T_s \) by \( B \cap B' = T_s \). By 0.3(a) we can identify \( Y_{s,w} \cap B \mathcal{B} = Z_{B,B',w} = \{ B'' \in \mathcal{B}; \text{pos}(B', B'') = w_0, \text{pos}(B', B'') = w \} \). (The identification is by \( uB'u^{-1} \mapsto s^{-1}w^{-1}suB'u^{-1}s^{-1}us \) with \( u \in U_B \).) This is the intersection of the smooth irreducible subvariety \( \{ B'' \in \mathcal{B}; \text{pos}(B', B'') = w \} \) of dimension \( l(w) \) of \( \mathcal{B} \) with the open subset \( \{ B'' \in \mathcal{B}; \text{pos}(B, B'') = w_0 \} \) of \( \mathcal{B} \) hence it is smooth of pure dimension \( l(w) \). This proves (a).

Another (longer) proof of (a) is given in [L79, 1.1].

1.5. In this subsection we assume that \( k,F \) are as in 0.1(b),(c). Let \( w \in W \).

The following result appears in [DL76,1.4].
(a) \( X_w \) is smooth of pure dimension \( l(w) \).

We give an alternative proof of (a) similar to that in 1.4. Let \( T \) be a maximal torus of \( G \) such that \( F(T) = T \). From 1.3(a) we have an open covering \( X_w = \bigcup_{B \in \mathcal{B}^T} (X_w \cap B \mathcal{B}) \). It is enough to prove that for any \( B \in \mathcal{B}^T \), \( X_w \cap B \mathcal{B} \) is smooth of pure dimension \( l(w) \).

Define \( B' \in \mathcal{B}^T \) by \( B \cap B' = T \). By 0.4(a) we can identify \( U_{B}^F \setminus (X_w \cap B \mathcal{B}) = Z_{B,B',w} \) where as in the proof in 1.4, \( Z_{B,B',w} \) is smooth of pure dimension \( l(w) \).

The conjugation action of \( U_{B}^F \) on \( X_w \cap B \mathcal{B} \) is free since the conjugation action of \( U_B \) on \( B \mathcal{B} \) is free (and transitive). It follows that \( X_w \cap B \mathcal{B} \) is smooth of pure dimension \( l(w) \). This proves (a).

1.6. Let \( s \in G_*, w \in W \). Let \( T = T_s \). We show:
(a) \( Y_{s,\leq w} \) is equal to the closure of \( Y_{s,w} \) in \( \mathcal{B} \).

Let \( B_1 \in Y_{s,\leq w} \). We have \( B_1 \in Y_{s,y} \) for a unique \( y \in W \). By 1.3(a) we can find \( B \in \mathcal{B}^T \) such that \( B_1 \in B \mathcal{B} \). Define \( B' \in \mathcal{B}^T \) by \( B \cap B' = T \). By 0.3(b) the open set \( Y_{s,\leq w} \cap B \mathcal{B} \) of \( Y_{s,\leq w} \) is identified \( B \mathcal{B} \cap B_{B',\leq w} \), an open set in \( \mathcal{B}_{B',\leq w} \). Under this identification \( B_1 \) becomes an element \( B_2 \in B \mathcal{B} \cap B_{B',y} \). Since \( B \mathcal{B} \) is open in \( \mathcal{B} \) we have

\[ B \mathcal{B} \cap (\text{ closure of } B_{B',w} \text{ in } \mathcal{B}) \subset \text{ closure of } B \mathcal{B} \cap B_{B',w} \text{ in } B \mathcal{B}. \]

In particular we have \( B_2 \subset \text{ closure of } B \mathcal{B} \cap B_{B',w} \text{ in } B \mathcal{B} \). Using again the identification above we deduce that \( B_1 \) is in the closure of \( Y_{s,w} \cap B \mathcal{B} \) in \( B \mathcal{B} \) and in particular \( B_1 \) is in the closure of \( Y_{s,w} \) in \( \mathcal{B} \). This proves (a).

Now let \( B_1, B, B', B_2, y \) be as in the proof of (a). Let \( \mathcal{H}^i_{B_1} \) be the stalk at \( B_1 \) of the \( i \)-th cohomology sheaf of the intersection cohomology complex of \( Y_{s,\leq w} \) with coefficients in \( Q_i|Y_{s,w} \) (this is defined in view of (a) and 1.4(a)). From the proof of (a) we see that
(b) \( \mathcal{H}^i_{B_1} = \mathcal{H}^i_{B_2}, \)

where \( \mathcal{H}^i_{B_2} \) is the stalk at \( B_2 \in B_{B',y} \) of the \( i \)-th cohomology sheaf of the intersection cohomology complex of \( B_{B',\leq w} \) with coefficients in \( Q_i|B_{B',w} \).
Note that if \( w \in W \) and \( k, F \) are as in 0.1(b),(c), then results similar to (a),(b) are known to hold for \( X_{\leq w} \).

1.7. We prove Proposition 0.8. Let \( B \) be as in (a),(b). Using the decompositions

\[
Y_{s,w} = \bigcup_{z \in W} (Y_{s,w} \cap B_{B,z}),
\]

\[
U_B \backslash X_w = \bigcup_{z \in W} (U_B \backslash (X_w \cap B_{B,z})),
\]

we see that it is enough to show that for any \( z \in W \) and any \( i \in \mathbb{Z} \), any eigenvalue of \( F \) on \( H_c(Y_{s,w} \cap B_{B,z}) \) is in \( \{q^j; j \in \mathbb{Z}\} \) (in case (a)) and any eigenvalue of \( F \) on \( H_c(U_B \backslash (X_w \cap B_{B,z})) \) is in \( \{q^j; j \in \mathbb{Z}\} \) (in case (b)). Using 0.3, 0.4, we see that it is enough to show that for any \( z \in W \) and any \( i \in \mathbb{Z} \), any eigenvalue of \( F \) on \( H_c(Z_{B,B'},w) \) is in \( \{q^j; j \in \mathbb{Z}\} \) where \( B' \) is as in 0.3, 0.4. This is a special case of \([L78, 3.7]\).

1.8. We prove Corollary 0.6. Let \( B \) be as in 0.6. Using 0.3, 0.4 we see that it is enough to prove that for some \( B' \in B^F \) with \( \text{pos}(B, B') = z \) we have

(a) \( \sharp(Z_{B,B',w}^F) = N_{w,z^{-1},z^{-1}}(q^t) \).

This is a special case of \([L78, 3.7]\).

2. Construction of \( F_{j,w} \)

2.1. For any \( n \in \mathbb{Z} \) let \( \mathcal{H}_n = \mathbb{Q}_l \otimes_{\mathbb{Z}[q]} \mathcal{H} \) where \( \mathbb{Q}_l \) is viewed as a \( \mathbb{Z}[q] \)-algebra via \( q \mapsto n \). Then \( \mathcal{H}_n \) is a \( \mathbb{Q}_l \)-algebra with \( \mathbb{Q}_l \)-basis \( \{T_w; w \in W\} \). If \( n \neq -1 \), the algebra \( \mathcal{H}_n \) is semisimple and the irreducible \( \mathcal{H}_n \)-modules (up to isomorphism) are in natural bijection \( E_n \leftrightarrow E \) with \( \text{Irr}(W) \), the set of irreducible \( W \)-modules over \( \mathbb{Q}_l \) (up to isomorphism), once \( \sqrt{\frac{1}{n}} \) has been chosen.

In this section we assume that \( k \) is as in 0.1(b). Let \( w \in W, j \in \mathbb{Z} \). Note that the uniqueness of \( F_{j,w} \) in 0.9(a) is clear since for any \( y \in W \) we can find \( F, q \) as in 0.1(c) and \( s \in G_\ast \cap G^F \) such that \( y \in [s] \). We now prove the existence of \( F_{j,w} \) in 0.9(a).

Let \( F, q \) be as in 0.1(c). Let \( F_q \) be the vector space of functions \( B^F \rightarrow \mathbb{Q}_l \). By assigning to the basis element \( T_w \) of \( \mathcal{H}_q \) the linear map \( T_w: F_q \rightarrow F_q \) given by \( f \mapsto f' \) where \( f'(B) = \sum_{B' \in B_{F^*}; \text{pos}(B,B') = w} f(B') \), we identify \( \mathcal{H}_q \) with a subalgebra of \( \text{End}(F_q) \). We have a canonical decomposition \( F_q = \bigoplus_{E \in \text{Irr}(W)} E_q \otimes [E]_q \) (as a \( (\mathcal{H}_q, G^F) \)-module) where \( [E]_q \) is an irreducible representation of \( G^F \).

Let \( s \in G_\ast \cap G^F \). Let \( |W| \) be the order of \( W \). For \( t \in \mathbb{N} \) we set \( F_t = F^{|t||W|}, q_t = q^{1+|t||W|} \). We have

\[
\sum_{i \in \mathbb{Z}} (-1)^i \text{tr}(F_i, H_c(Y_{s,w}))
\]

\[
= \sharp(B \in Y_{s,w}; F_t(B) = B) = \sharp(B \in B^F; \text{pos}(B, sBs^{-1}) = w)
\]

\[
= \text{tr}(sT_w: F_{q_t} \rightarrow F_{q_t}) = \sum_{E \in \text{Irr}(W)} \text{tr}(T_w, E_{q_t}) \text{tr}(s, [E]_{q_t}).
\]
By [DL76, 7.8] we have $tr(s,[E]_{q_t}) = ([E]_{q_t} : R_{y,q_t})$ where $y \in [s]$, $R_{y,q_t}$ is defined as $R_y$ in 0.3(a) with $F,q$ replaced by $F_t,q_t$ and $([E]_{q_t} : R_{y,q_t})$ denotes multiplicity of an irreducible $G^{F_t}$-module in a virtual $G^{F_t}$-module. Note that the conjugacy class $[s]$ in $W$ associated to $s$ and to $F_t$ is independent of $t$ since $F^{|W|}(B) = B$ for any $B \in \mathcal{B}$ that contains $s$. Moreover, from [L84] it is known that $([E]_{q_t} : R_{y,q_t})$ is independent of $t$. Thus we have

$$\sum_{i \in \mathbb{Z}} (-1)^i tr(F_t, H_i^c(Y_{s,w})) = \sum_{E \in \text{Irr}(W)} tr(T_w, E_{q_t}) ([E]_q : R_y).$$

From 0.8 we see that $H^c_i(Y_{s,w})j' = 0$ if $j'$ is odd, while if $j'$ is even $F^{|W|}$ acts on $H^c_i(Y_{s,w})j'$ as $q^{j'/2}|W|/2$ times a unipotent transformation so that $tr(F_t, H^c_i(Y_{s,w})j') = \sum_{E \in \text{Irr}(W)} tr(T_w, E_{q_t}) ([E]_q : R_y)$. We see that

$$\sum_{i,j' \in \mathbb{Z}} (-1)^i tr(F, H^c_i(Y_{s,w})j')q^{-j'/2}q^{j'/2} = \sum_{E \in \text{Irr}(W)} tr(T_w, E_{q_t}) ([E]_q : R_y).$$

It is known that for $E \in \text{Irr}(W)$ we have $tr(T_w, E_{q_t}) = \sum_{j' \in \mathbb{Z}} \mu_{w,E; j'}q^{j'/2}$ for $t \in \mathbb{N}$ where $\mu_{w,E; j'} \in \mathbb{Z}$ are independent of $t$ and are zero for all but finitely many $j'$. Thus we have

(a) $$\sum_{i,j' \in \mathbb{Z}} (-1)^i tr(F, H^c_i(Y_{s,w})j')q^{-j'/2}q^{j'/2} = \sum_{E \in \text{Irr}(W), j' \in \mathbb{Z}} \mu_{w,E; j'}([E]_q : R_y)q^{j'/2}$$

for $t \in \mathbb{N}$. By comparing the coefficient of $q^{j'/2}$ in the two sides of (a), we obtain

(b) $$\sum_{i \in \mathbb{Z}} (-1)^i tr(F, H^c_i(Y_{s,w})j)q^{-j/2} = \sum_{E \in \text{Irr}(W)} \mu_{w,E; j}([E]_q : R_y).$$

Now for $E \in \text{Irr}(W)$ we set $R_E = |W|^{-1} \sum_{z \in W} tr(z,E)R_z$ (a rational linear combination of representations of $G^F$) and we use the equality $([E]_q : R_{E'}) = ([E']_q : R_E)$ for $E,E'$ in $\text{Irr}(W)$ (a known property [L84] of the nonabelian Fourier transform). We obtain

$$\sum_{i \in \mathbb{Z}} (-1)^i tr(F, H^c_i(Y_{s,w})j)q^{-j/2} = \sum_{E,E' \in \text{Irr}(W)} tr(y, E')([E]_q : R_{E'})\mu_{w,E;j}$$

$$= \sum_{E,E' \in \text{Irr}(W)} tr(y, E')([E']_q : R_{E})\mu_{w,E;j}$$

$$= \sum_{E' \in \text{Irr}(W)} tr(y, E')([E']_q : R_E) = \sum_{E' \in \text{Irr}(W)} tr(y, E')([E']_q : R_{j,w})$$

where the last equality can be deduced from [L84, 3.8]. We see that $\mathfrak{R}_{j,w} := \sum_{E' \in \text{Irr}(W)} ([E']_q : R_{j,w})E'$ has the properties stated in 0.9(a). Note that $\mathfrak{R}_w = \sum_{j} \mathfrak{R}_{j,w} = \sum_{E' \in \text{Irr}(W)} ([E']_q : R_w)E'$ depends only on $w$ since $R_w$ has such a property. This completes the proof of 0.9(a).
2.2. From the proof in 2.1 we see that for $E \in \text{Irr}(W), w \in W, j \in 2\mathbb{N}$ we have

(a) $(E : \mathfrak{g}_{j,w}) = ([E]_q : R_{j,w})$

where the left hand side is a multiplicity as a $W$-module and the right hand side is a multiplicity as a $G^F$-module. It follows that

(b) $(E : \mathfrak{g}_w) = ([E]_q : R_w)$.

3. Over real numbers

3.1. In this section we assume that $k = C$ and that $G$ has a given split $R$-structure. Let $G(R), B(R)$ be the set of real points of $G, B$. Let $s \in G_+ \cap G(R)$ be such that for any $B \in \mathcal{B}^{Ts}$ we have $B \in B(R)$. Let $w \in W$ and let $Y_{s,w}(R) = Y_{s,w} \cap B(R)$. This is a smooth manifold of pure (real) dimension equal to $l(w)$. (See 1.4(a)).

For $w$ in $W$ and $B, B'$ in $\mathcal{B}^{Ts}$ let $Z_{B,B',w}$ be as in 0.2. Then $Z_{B,B',w}$ is defined over $R$ and we set $Z_{B,B',w}(R) = Z_{B,B',w} \cap B(R)$. Let $z = \text{pos}(B, B')$. The following result can be deduced from [R98, 6.1]:

(a) $\chi_c(Z_{B,B',w}(R)) = N_{w,z^{-1},z^{-1}}(-1)$

where $\chi_c$ is Euler characteristic in cohomology with compact support. Using (a) and the homeomorphism $Y_{s,w}(R) \cap B_{B,z} \cong Z_{B,B',w}(R)$ deduced from 0.3(a) we see that

$\chi_c(Y_{s,w}(R) \cap B_{B,z}) = N_{w,z^{-1},z^{-1}}(-1)$.

Using this and the partition $Y_{s,w}(R) = \bigcup_{z \in W}(Y_{s,w}(R) \cap B_{B,z})$, we deduce:

**Proposition 3.2.** We have $\chi_c(Y_{s,w}(R)) = \sum_{z \in W} N_{w,z^{-1},z^{-1}}(-1) = \text{tr}(T_w : \mathcal{H}_{-1} \rightarrow \mathcal{H}_{-1})$.

Here $T_w : \mathcal{H}_{-1} \rightarrow \mathcal{H}_{-1}$ is left multiplication by $T_w$ in $\mathcal{H}_{-1}$. 

4. Affineness

4.1. Let $s \in G_+, w \in W$. Assume that $w$ has minimal length in $w$. We show:

(a) $Y_{s,w}$ is affine.

The proof is a modification of the proof of the analogous result for $X_w$ given in [BR08]. As in [loc.cit.] we can assume that $w$ is elliptic and good (as defined in [loc.cit.]). For such $w$, it is shown in [loc.cit.] that for some $n \geq 1$, the variety

$$V = \{(B_0, B_1, B_2, \ldots, B_n) \in \mathcal{B}^{n+1};$$
$$\text{pos}(B_0, B_1) = \text{pos}(B_1, B_2) = \cdots = \text{pos}(B_{n-1}, B_n) = w\}$$

is affine. Define $\phi : \mathcal{B} \rightarrow \mathcal{B}^{n+1}$ by $B \mapsto (B, sBs^{-1}, s^2Bs^{-2}, \ldots, s^nBs^{-n})$. Then $\phi$ is an isomorphism of $\mathcal{B}$ onto a closed subvariety $V'$ of $\mathcal{B}^{n+1}$. Then $V \cap V'$ is closed in $V$ hence is affine. Now $\phi$ restricts to an isomorphism of $Y_{s,w}$ onto $V \cap V'$ hence $Y_{s,w}$ is affine.
5. The subset $\mathcal{E}(w)$ of $W$

5.1. In this section $W$ is allowed to be any Coxeter group. For any $w, w', w''$ in $W$, the polynomials $N_{w,w',w''} \in \mathbb{Z}[q]$ can be defined in terms of the Iwahori-Hecke algebra $H$ with basis $\{T_w; w \in W\}$ attached as in 0.5 to $W$. As in 0.10(a) for any $w \in W$ we define a subset of $W$ by
\[
\mathcal{E}(w) = \{z \in W; N_{w,z,z} \neq 0\}.
\]
For example, if $w = 1$, then $\mathcal{E}(w) = W$; if $w = \sigma$, $l(\sigma) = 1$, then $\mathcal{E}(w) = \{z \in W; l(\sigma z) = l(z) - 1\}$.

(a) If $W$ is finite and $w_0$ is the longest element of $W$, then for any $w$ we have $w_0 \in \mathcal{E}(w)$. In particular, $\mathcal{E}(w) \neq \emptyset$.

We argue by induction on $l(w)$. If $w = 1$ we have $T_w T_{w_0} = T_{w_0}$ and the desired result holds. Assume now that $l(w) > 0$. We have $w = \sigma z$ where $l(\sigma) = 1, l(z) = l(w) - 1$. From the definition we have $N_{w,w_0,w_0} = N_{z,w_0,w_0}(q - 1) + N_{z,w_0,\sigma w_0}$. Since any $N_{a,b,c}$ is a satisfies 0.10(b) and $N_{z,w_0,w_0} \neq 0$ by the induction hypothesis, it follows that $N_{w,w_0,w_0} \neq 0$. This proves (a).

(b) If $W$ is a finite Weyl group and $w$ is a Coxeter element of minimal length in $W$ then $\mathcal{E}(w) = \{w_0\}$.

Assume that $k, F, q$ are as in 0.1(b),(c). Let $B, B'$ be in $B^F$. Using [L76, 2.5] we have $X_w \cap B_{B,z} = \emptyset$ for $z \neq w_0$. Using this and 0.4(a), we see that $Z_{B,B',w} = \emptyset$ if $pos(B, B') \neq w_0$. Using now 0.6, we see that $N_{w,z-1,z-1}(q) = 0$ if $z \neq w_0$. Since $q$ can take infinitely many values we see that $N_{w,z-1,z-1} = 0$ if $z \neq w_0$, so that $\mathcal{E}(w) \subset \{w_0\}$. By (a) we have $\mathcal{E} \neq \emptyset$ and (b) follows.

(c) Assume that $n \in \{2,3,\ldots\}$ and that $W$ has generators $\sigma_1, \sigma_2$ and relations $\sigma_1^2 = \sigma_2^2 = 1$ and $(\sigma_1 \sigma_2)^{2n} = 1$ (a dihedral group of order $4n$). Let $w = (s_1s_2)^k$ where $k \in \{1,2,\ldots,n\}$. We have $\mathcal{E}(w) = \{z \in W; l(z) \geq 2n - k + 1\}$.

The proof is by computation.

(d) Assume that $W$ has generators $\sigma_1, \sigma_2$ and relations $\sigma_1^2 = \sigma_2^2 = 1$ (an infinite dihedral group). Let $w = (s_1s_2)^k$ where $k \in \{1,2,\ldots\}$. We have $\mathcal{E}(w) = \emptyset$. We have $\mathcal{E}(s_1s_2s_1) = \{s_1s_2, s_1s_2s_1, s_1s_2s_1s_2, s_1s_2s_1s_2s_1, \ldots\}$.

The proof is by computation.

5.2. Let $W^* = \{w \in W; \mathcal{E}(w) \neq \emptyset\}$. If $W$ is finite then by 5.1(a) we have $w_0 \in \mathcal{E}(w)$ for any $w \in W$; thus in this case $W^* = W$. If $W$ is infinite then it may happen that $W^* \neq W$. For example in the setup of 5.1(d) we have $s_1s_2 \notin W^*$.

(a) For any $w \in W^*$ and any $z \in \mathcal{E}(w)$ we have $\deg(N_{w,z,z}) \leq l(w)$.

The proof is immediate; see for example [L20, 2(b)].

For any $w \in W^*$ we set
\[
(d(w) = \max_{z \in \mathcal{E}(w)} \deg N_{w,z,z}.
\]
From (a) we have
\[
(c) d(w) \leq l(w) \text{ for all } w \in W^*.
\]
If $W$ is finite we have $d(w) = l(w)$ for all $w \in W = W^*$; indeed, by the proof of 5.1(a), we have $\deg N_{w,w_0,w_0} = l(w)$. If $W$ is infinite then it may happen that
$d(w) < l(w)$ for some $w \in W^\bullet$. For example in the setup of 5.1(d) we have $d(s_1s_2s_1) = 2 < 3 = l(s_1s_2s_1)$.

For $w \in W^\bullet$ we define $\mathcal{E}'(w) = \{ z \in \mathcal{E}(w); \deg N_{w,z,z} = d(w) \}$ (a nonempty set). If $W$ is finite and $w$ is such that any simple reflection of $W$ appears in any reduced decomposition of $w$ then

(d) $\mathcal{E}'(w) = \{ w_0 \}$.

This can be deduced from [L78, p.29, lines 2-4] (which was stated without proof) or it can be deduced from results in [L20, no.2].

5.3. We return to the setup in 0.1 and we assume that $G$ is adjoint, that $s \in G_*$ and that $w \in W$ is elliptic, of minimal length in $w$. Then $T = T_s$ acts on $Y_{s,w}$ by conjugation. The following result can be deduced from [L11, 5.2].

(a) Any isotropy group of the $T$-action on $Y_{s,w}$ is finite.

In the case where $w$ is a Coxeter element of minimal length, we have the following stronger result.

(b) $Y_{s,w}$ is a principal homogeneous space for $T$.

Let $B \in B^T$. Let $B' \in B^T$ be such that $B \cap B' = T$. From 5.1(b) we see (using 0.6) that $Y_{s,w} \cap B_{B,z} = \emptyset$ if $z \neq w_0$. Hence $Y_{s,w} = Y_{s,w} \cap B_{B,w_0}$. From 1.1(a) we can identify $Y_{s,w} = Y_{s,w} \cap B_{B,w_0}$ (with its $T$-action by conjugation) and $\{ u' \in U_B; \text{pos}(B', u'B'^{-1}) = w \}$ (with its $T$-action by conjugation); by [L76, 2.2] this last $T$-space is principal homogeneous. This proves (b).

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