Resummation and the semiclassical theory of spectral statistics

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We address the question as to why, in the semiclassical limit, classically chaotic systems generically exhibit universal quantum spectral statistics coincident with those of Random Matrix Theory. To do so, we use a semiclassical resummation formalism that explicitly preserves the unitarity of the quantum time evolution by incorporating duality relations between short and long classical orbits. This allows us to obtain both the non-oscillatory and the oscillatory contributions to spectral correlation functions within a unified framework, thus overcoming a significant problem in previous approaches. In addition, our results extend beyond the universal regime to describe the system-specific approach to the semiclassical limit.

Keywords: quantum chaos, spectral statistics, semiclassical approximation, periodic orbit theory, random matrix theory

1. Introduction

One of the central ideas in the field of Quantum Chaos is that the quantum energy spectra of classically chaotic systems exhibit universal fluctuation statistics (Bohigas \textit{et al.} 1984, McDonald & Kaufman 1979, Casati \textit{et al.} 1980, Berry 1987): once their mean energy level densities are scaled to be equal, the spectral statistics of different systems coincide, forming a small number of symmetry-dependent universality classes. This idea is supported by overwhelming experimental and numerical evidence for systems as diverse as atomic nuclei, Rydberg atoms and quantum billiards (Stöckmann 1999). Theoretically, it emerges in the semiclassical limit, $\hbar \to 0$, when spectral correlations are measured on the scale of the mean energy level separation.

The fundamental reasons for universality are still not fully understood. Nevertheless, a phenomenological description of the specific universal forms taken by spectral statistics has been developed based on the following insight: accepting universality, one may consider ensemble averages rather than individual chaotic systems. This idea lies at the heart of random-matrix theory (RMT), developed by Wigner and Dyson to describe spectral statistics in atomic nuclei (Wigner 1959, Haake 2001). In RMT, one averages over ensembles of all matrix representations of Hamiltonians belonging to the same symmetry class, with, for convenience, a Gaussian weight. The pertinent ensemble for systems without any symmetries, comprising of all complex Hermitian matrices, is referred to as the Gaussian Unitary Ensemble (GUE); for dynamics whose sole symmetry is time-reversal invariance the...
Gaussian Orthogonal Ensemble (GOE) of real-symmetric matrices is the appropriate one. Ensemble averages then yield predictions for measures of spectral statistics such as the two-point correlation function of the level density of energy levels $E_n$, $ho(E) = \sum_n \delta(E - E_n)$:

$$R(\epsilon) \equiv \frac{1}{\overline{\rho}} \left\langle \rho \left( E + \frac{\epsilon}{2\pi \overline{\rho}} \right) \rho \left( E - \frac{\epsilon}{2\pi \overline{\rho}} \right) \right\rangle - 1.$$  \hspace{1cm} (1.1)

Here $\langle \cdot \rangle$ denotes an ensemble average, replacing an average over the centre energy $E$ for individual systems, and $\overline{\rho}$ is the mean level density. In the large-matrix limit, the random-matrix predictions involve power series of non-oscillatory and oscillatory contributions with coefficients $c_n, d_n$ (Heusler et al. 2007) depending on the symmetry class:

$$R(\epsilon) = \Re \sum_{n=2}^{\infty} (c_n + d_n e^{2i\epsilon}) \left( \frac{1}{\epsilon} \right)^n \hspace{1cm} (1.2)$$

For the GUE, $c_2 = -d_2 = -1/2$ and $c_n = d_n = 0$ for all $n > 2$.

 Establishing precisely when and why random matrix theory describes the spectral statistics of individual classically chaotic systems is one of the central problems in the field. To illustrate its difficulty, we remark that several fully chaotic systems are known, such as the cat maps (Keating 1991) and geodesic motion on compact arithmetic surfaces of constant negative curvature (Bogomolny et al. 1992), for which the spectral statistics do not coincide with any of the random-matrix forms; and there are examples of perturbed cat maps where the spectral statistics do coincide with one of the random-matrix forms, but not the form one would predict based on the symmetries of the classical dynamics (Keating & Mezzadri 2000). In all of these cases the origin of the unexpected behaviour lies in the existence of arithmetical quantum symmetries that have no classical counterpart. There are currently no rigorously established necessary or sufficient conditions for the quantum spectrum of any individual classically chaotic system to exhibit random matrix statistics. For this reason, the relationship is usually conjectured to hold for generic systems, where the term ‘generic’ is sufficient to cover experimental applications, but as yet has not been given a precise mathematical definition.

The fact that we appear to be far from a rigorous understanding of universality in spectral statistics motivates the development of heuristic approaches to justifying the random-matrix conjecture in the expectation that this will lead to further insights into the reasons underlying its success, and its limitations. This is our main purpose here.

A natural starting point for understanding quantum consequences of classical chaos is provided by semiclassical techniques. (Other approaches are discussed in Haake (2001), Muzykantskii & Khmel’nitskii (1995) and Andreev et al. (1996).) By stationary-phase approximation of Feynman’s path integral, Gutzwiller (1990) was able to represent the level density as a sum over classical periodic orbits. The correlation function $R(\epsilon)$ is thus a sum over pairs of periodic orbits $a, b$ (Berry 1985). Importantly, this classical double sum incorporates quantum interference effects through a phase factor $e^{i(S_a - S_b)/\hbar}$ depending on the difference between the two actions $S_a$ and $S_b$. In the semiclassical limit, as $\hbar \to 0$, constructive interference requires small action differences $S_a - S_b$, at most of the order of $\hbar$. Constructively interfering orbit pairs were identified by Hannay & Ozorio de Almeida (1984)
and Berry (1985), Sieber & Richter (2001), Sieber (2002) and Müller et al. (2004, 2005), and shown to yield non-oscillatory contributions to $R(\epsilon)$: Hannay & Ozorio de Almeida (1984) and Berry (1985) showed, using a sum rule that follows from ergodicity, that ‘diagonal’ pairs of identical or mutually time-reversed orbits give a contribution quadratic in $1/\epsilon$ and thus determine $c_2$; Sieber and Richter (2001) derived the cubic term $c_3$ using pairs of orbits that differ from each other only by their connections in an ‘encounter’ of two close orbit stretches; and, building on that insight, it was shown by Müller et al. (2004, 2005) that higher-order contributions to $R(\epsilon)$ are due to an infinity of further families of pairs of orbits differing in arbitrarily many encounters of arbitrarily many close stretches. Summation of all contributions yields the full non-oscillatory power series $\text{Re} \sum_{n=2}^{\infty} c_n (\frac{1}{\epsilon})^n$. In these calculations the essential simplifying assumption is that the orbit pairs identified are the only ones that ultimately contribute. We shall make the same assumption in our approach.

The extension to the oscillatory terms proved to be much more difficult; these terms are related to more subtle correlations between periodic orbit actions (Argmann et al. 1993). However, the leading oscillatory contribution was derived in a modified semiclassical setting by Bogomoly & Keating (1996), and an expansion to all orders realized by Heusler et al. (2007). The approach of Heusler et al. was based on the following detour inspired by RMT (or more precisely a field theoretic implementation of RMT, the so-called nonlinear sigma model): the correlation function $R(\epsilon)$ can be represented through derivatives of a generating function involving a ratio of four spectral determinants $\Delta(E) = \det(E - H)$ with four different energy arguments. (Here $H$ denotes the Hamiltonian.) In fact, such a representation can be realized in two equivalent ways. It turned out that with a Gutzwiller-type approximation for the generating function, one either recovers the non-oscillatory or the oscillatory terms, depending on which of the two representations is chosen. This is somewhat paradoxical, because both representations should in principle give the same result. In each representation a part of the result is missed because in Gutzwiller’s trace formula the energies have to be taken with sufficiently large imaginary parts to guarantee convergence, and these imaginary parts can, for example, cause oscillatory factors to become exponentially small. Nevertheless, it was observed that adding both results one recovers the full correlation function predicted by RMT. This obviously leaves the question as to why the two types of contributions are additive, and why either one by itself does not give the complete answer.

We here justify additivity within the framework of an improved semiclassical approximation developed by Berry & Keating (1990), Keating (1992) and Berry & Keating (1992). Their resummation of the Gutzwiller-type contributions preserves the unitarity of the time evolution (in particular, the fact that energy eigenvalues are real) and in doing so reveals a remarkable relationship between contributions from long and short orbits. It is this relationship that allows non-oscillatory and oscillatory contributions to be described naturally within a unified approach.

We shall first review the resummation technique used and illustrate its application to spectral statistics for an averaged product of two spectral determinants. We will then turn to the generating function and the correlation function $R(\epsilon)$, and finally justify the additivity assumed in Heusler et al. (2007). To keep the presentation simple, we will focus on systems without time-reversal invariance, however
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the ideas generalize straightforwardly. Importantly, the treatment of the leading-order contributions will be carried beyond the universal regime, and so describes the approach to the semiclassical limit.

2. Resummation

To derive a semiclassical approximation for the spectral determinant one may start from Gutzwiller’s formula for $\text{tr}(E - H)^{-1}$. The spectral determinant is then obtained as an exponential involving the smoothed number $N(E)$ of energy levels below $E$ and a sum over classical periodic orbits $a$,

$$
\Delta(E^+) \propto \exp\left(\int_{E^-}^{E^+} dE' \text{tr} \frac{1}{E' - H}\right)
\propto \exp\left(-i\pi N(E^+) - \sum_a F_a e^{iS_a(E^+)/\hbar}\right).
$$

(2.1)

Here, $E^+$ denotes an energy with a small positive imaginary part (needed to ensure convergence), and $S_a$ is the classical action of the orbit $a$. The factor $F_a$ incorporates an amplitude depending on the stability of $a$ and the Maslov phase. It is convenient to expand the exponentiated sum over orbits into a sum over unordered collections of periodic orbits, or *pseudo-orbits* $A$ (Berry & Keating 1990)

$$
\Delta(E^+) \propto e^{-i\pi N(E^+)} \sum_A F_A (-1)^{n_A} e^{iS_A(E^+)/\hbar},
$$

(2.2)

where $n_A$ is the number of orbits inside $A$ and $S_A$ is the sum of their actions. $F_A$ is the product of the factors $F_a$ of the individual orbits (it also includes corrections to the simple sign factor $(-1)^{n_A}$ that appear if $A$ contains several identical copies of a given orbit). For negative imaginary parts, complex conjugation yields

$$
\Delta(E^-) \propto e^{i\pi N(E^-)} \sum_A F_A^* (-1)^{n_A} e^{-iS_A(E^-)/\hbar}.
$$

(2.3)

Importantly (2.2) and (2.3) do not manifestly preserve the unitarity of the quantum dynamics: in the limit of vanishing imaginary parts $\Delta(E^+)$ and $\Delta(E^-)$ should become real (given the reality of the energy levels) and identical to each other. This is not at all obvious from the above formulas. It was shown in Berry & Keating (1990), Keating (1992) and Berry & Keating (1992) that by explicitly incorporating unitarity one arrives at an improved approximation for $\Delta(E)$. Postulating that (2.2) and (2.3) become identical for $E^+, E^- \rightarrow E$ implies duality relations between pseudo-orbits whose duration (i.e., sum of periods of contributing orbits) is larger than half of the Heisenberg time $T_H = 2\pi\hbar/\rho$ and those shorter than $T_H/2$; the overall contribution of long orbits is the complex conjugate of the contribution arising from short orbits. (In the case of billiards the relationship follows directly from the fact that the spectral determinant is an even function of the momentum and the spectral counting function an odd function, see Keating & Sieber (1994).) This leads to the improved semiclassical approximation for $\Delta(E)$ as a sum over pseudo-orbits with durations shorter than $T_H/2$,

$$
\Delta(E) \propto e^{-i\pi N(E)} \sum_{A \ (T_A < T_H/2)} F_A (-1)^{n_A} e^{iS_A/\hbar} + \text{c.c.}
$$

(2.4)

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Equation (2.4) incorporates explicitly the relations between long and short orbits. Taken together with the orbit correlations discussed above this leads to a complete semiclassical picture of spectral fluctuations. It is known as the Riemann-Siegel lookalike formula because of its similarity to a corresponding expression for the Riemann zeta function (Berry & Keating 1990).

We emphasize that resummation solves an important problem inherent in (2.2) and (2.3): whereas (2.2) and (2.3) diverge unless the energy has a sufficiently large imaginary part, the sum in (2.4) is finite and so converges when $E$ is real. Essentially, it is the need to include an imaginary part in the energy in (2.2) and (2.3) that is responsible for the analysis of Heusler et al. (2007) missing a part of the RMT expression for the spectral statistics, depending on the representation used. With (2.4) we are now able to access the complete expression.

3. Product of spectral determinants

To illustrate how resummation can be used in the context of spectral statistics, we first consider the product of spectral determinants

$$\Pi(\alpha) = \langle \Delta(E + \alpha) \Delta(E - \alpha) \rangle \, .$$

(3.1)

We shall see that for individual chaotic systems $\Pi(\alpha)$ conforms in the semiclassical limit to the random-matrix prediction (Ketteman et al. 1997)

$$\Pi(\alpha) = \Pi(0) \sin(2\pi \rho \alpha)/2\pi \rho \alpha \, .$$

(3.2)

To show this, we insert (2.4) into (3.1). $\Pi(\alpha)$ then turns into the following double sum over pseudo-orbits $A, B$

$$\Pi(\alpha) \propto \sum_A \sum_B \left( e^{-i\pi N(E + \alpha)} F_A(-1)^n e^{iS_A(E + \alpha)/\hbar} \times e^{i\pi N(E - \alpha)} F_B(-1)^n e^{-iS_B(E - \alpha)/\hbar} \right) + \text{c.c.} \, .$$

(3.3)

In $\langle \ldots \rangle$ the non-conjugate terms in the approximation (2.4) for $\Delta(E + \alpha)$ are paired with the complex conjugate ones for $\Delta(E - \alpha)$. The term “$+\text{c.c.}$” is due to the opposite combination. All other pairings involve highly oscillatory factors with phases $\propto N(E)$ and thus vanish after averaging over $E$.

Still, the remaining phase factor involving $(S_A(E + \alpha) - S_B(E - \alpha))/\hbar$ oscillates rapidly in the limit $\hbar \to 0$ and thus averages to zero for most pseudo-orbits. Systematic contributions to (3.3) therefore arise only from pairs with action differences at most of the order of $\hbar \rho$. In the spirit of the diagonal approximation, the dominating contributions can be expected to originate from identical pseudo-orbits $A = B$. If we restrict ourselves to such pairs, and expand $N(E \pm \alpha) \approx N(E) \pm \alpha$, $S(E \pm \alpha) \approx S(E) \pm T(E)\alpha$, we obtain

$$\Pi(\alpha) \propto e^{-2i\pi \rho \alpha} \sum_A |F_A|^2 e^{2i\alpha T_A/\hbar} + \text{c.c.} \, .$$

(3.4)

† We neglect contributions where the $S$- and $N$-dependent phase factors both oscillate rapidly but these oscillations mutually compensate, since there is no systematic mechanism giving rise to the pairs of pseudo-orbits needed.
It is now tempting to drop the upper limit $T_H/2$ since it tends to $\infty$ in the semiclassical limit and, given the factor $|F_A|^2$, it is no longer needed for convergence (for $\alpha$ with an arbitrarily small imaginary part). Indeed, dropping the upper bound is justified immediately if semiclassically $\alpha \rho \rightarrow \infty$. The result can then be written as

$$\Pi(\alpha) \propto e^{-2i\pi\rho\alpha}(-2i\alpha/\hbar) + \text{c.c.} \quad (3.5)$$

where the dynamical zeta function $\zeta(s)$ is defined by

$$\zeta(s) = \sum_A |F_A|^2(-1)^n e^{-sT_A} \quad (3.6)$$

or, equivalently,

$$\zeta^{-1}(s) = \sum_A |F_A|^2 e^{-sT_A}. \quad (3.7)$$

Importantly, in chaotic systems $\zeta(s)$ has a simple zero at $s = 0$ (Haake 2001). This is equivalent to the periodic orbit sum rule that follows from classical ergodicity (Hannay & Ozorio de Almeida 1984).

For energy differences of the order of the mean level spacing the same result is obtained after a short calculation: if we incorporate the condition $T_A < T_H/2$ through a step-function $\Theta(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dz e^{xz/\hbar} (\text{with small positive } c)$ $\Pi(\alpha)$ can be written as

$$\Pi(\alpha) \propto e^{-2i\pi\rho\alpha} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{dz}{z} e^{\pi z^2} \sum_A |F_A|^2 e^{2i\alpha z T_A} + \text{c.c.}$$

$$= e^{-2i\pi\rho\alpha} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{dz}{z} e^{\pi z^2} \zeta\left(\frac{z - 2i\alpha}{\hbar}\right)^{-1} + \text{c.c.} \quad (3.8)$$

If we close the contour on the left, the residue at $z = 0$ reproduces (3.5) and the residue due to the zero of $\zeta(s)$ at $s = 0$ is proportional to $(2i\alpha)^{-1} + \text{c.c.} = 0$ and thus vanishes.

We note that equation (3.5) includes non-universal corrections to the random-matrix prediction (3.2), encoded in the analytic structure of $\zeta(s)$. It reduces to (3.2) in the semiclassical limit because $\zeta(s) \propto s$ as $s \rightarrow 0$; that is, as a consequence of classical ergodicity. We have thus seen how oscillatory contributions to spectral statistics arise naturally from a resummed semiclassical approximation, even beyond the universal regime.

### 4. Generating function

Let us now consider the generating function for $R(\epsilon)$,

$$Z(\alpha, \beta, \gamma, \delta) = \left\langle \frac{\Delta(E + \gamma)\Delta(E - \delta)}{\Delta(E + \alpha)\Delta(E - \beta)} \right\rangle. \quad (4.1)$$

Here the two energy differences $\alpha, \beta$ in the denominator are taken with small positive imaginary parts. Then

$$R(\epsilon) = \frac{1}{2\pi^2 \rho^2} \Re \frac{\partial^2 Z}{\partial \alpha \partial \beta} |_{(\alpha, \beta)} - \frac{1}{2} \quad (4.2)$$
where \( (\|) \) indicates the limit \( \alpha, \beta, \gamma, \delta \to \frac{1}{2\pi i} \). Equation (4.2) is easily checked: taking two derivatives of \( Z \) leads to \( \text{tr} \left[ \frac{1}{E+\alpha-H} \right] \text{tr} \left[ \frac{1}{E-\beta-H} \right] \) times the original ratio of four determinants; the two traces ultimately lead to level densities whereas the ratio converges to unity in the limit \( (\|) \).

To obtain a resummed semiclassical approximation for \( Z \) we use the Riemann-Siegel lookalike (2.4) for the two determinants in the numerator. Due to the important role played by the imaginary parts in the denominator, no resummation is possible there, and we rather stick to the unresummed expression

\[
\Delta(E^+)^{-1} \propto e^{i\pi N(E^+)} \sum_A F_A e^{iS_A(E+)/\hbar}
\]

and

\[
\Delta(E^-)^{-1} \propto e^{-i\pi N(E^-)} \sum_A F_A^* e^{-iS_A(E^-)/\hbar}.
\]

Specifically, no resummation of \( \Delta^{-1} \) as a finite sum can reproduce the poles at the energy levels (a finite sum can, of course, reproduce the corresponding zeros of \( \Delta \) itself). The choice of when we use \( \Delta(E^+) \) or \( \Delta(E^-) \) is fixed by the positions of these poles. Collecting the pseudo-orbit sums from all four spectral determinants we then obtain two sums over quadruplets of pseudo-orbits \( A, B, C, D \),

\[
Z = \left\{ e^{i\pi N(E+\alpha)} \sum_A F_A e^{iS_A(E+\alpha)/\hbar} \right. \\
\times e^{-i\pi N(E-\beta)} \sum_B F_B^* e^{-iS_B(E-\beta)/\hbar} \\
\times e^{-i\pi N(E+\gamma)} \sum_C (-1)^n C e^{iS_C(E+\gamma)/\hbar} \\
\times e^{i\pi N(E-\delta)} \sum_D (-1)^n D e^{-iS_D(E-\delta)/\hbar} \\
\left. + \{ \gamma \to -\delta, \ \delta \to -\gamma \} \right\}.
\]

These sums arise from combining the non-conjugate terms in (2.4) for \( \Delta(E+\gamma) \) with the complex conjugate ones for \( \Delta(E-\delta) \) and vice versa.

Interference between pseudo-orbits leads to a phase factor

\[
e^{i(S_A(E+\alpha)-S_B(E-\beta)+S_C(E+\gamma)-S_D(E-\delta))/\hbar}.
\]

The pseudo-orbits will interfere constructively if the cumulative action of \( A \) and \( C \) nearly coincides with the cumulative action of \( B \) and \( D \). The simplest way to realize this is to have the orbits in \( B \) and \( D \) identical to those in \( A \) and \( C \) (neglecting repetitions). The sum over such “diagonal” quadruplets of pseudo-orbits can be split into sums over the intersections \( A \cap B, A \cap D, C \cap B, C \cap D \). If we expand \( S \) and

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\( \mathbb{N} \) as above these sums can be written as
\[
Z = e^{i\pi \rho (\alpha + \beta - \gamma - \delta)} \sum_{A \cap B, A \cap D, C \cap B, C \cap D, \tau_0, \tau_D < T_H/2} |F_{A \cap B}|^2 e^{i \tau_{A \cap B}(\alpha + \beta)/\hbar} \\
\times |F_{A \cap D}|^2 (-1)^n_{A \cap D} e^{i \tau_{A \cap D}(\alpha + \delta)/\hbar} \\
\times |F_{C \cap B}|^2 (-1)^n_{C \cap B} e^{i \tau_{C \cap B}(\gamma + \beta)/\hbar} \\
\times |F_{C \cap D}|^2 e^{i \tau_{C \cap D}(\gamma + \delta)/\hbar} + \{\gamma \to -\delta, \delta \to -\gamma\}. \tag{4.8}
\]

In the same way as for the product of spectral determinants, we now drop the upper limits at \( T_H/2 \). This is clearly justified if semiclassically \( \alpha \tau \to \infty \) etc., and there are no singularities prohibiting analytic continuation to \( \alpha, \beta, \gamma, \delta \propto 1/\tau \). Evaluating the sum in that case gives
\[
Z = e^{i \pi \rho (\alpha + \beta - \gamma - \delta)} \zeta(-i(\alpha + \delta)/\hbar) \zeta(-i(\gamma + \beta)/\hbar) + \{\gamma \to -\delta, \delta \to -\gamma\}. \tag{4.9}
\]

Like (3.5), equation (4.8) is valid even beyond the universal regime; that is, it describes the system-specific approach to the semiclassical limit via the analytic structure of the classical zeta function \( \zeta(s) \). Taking the semiclassical limit we find
\[
Z = e^{i \pi \rho (\alpha + \beta - \gamma - \delta)} \frac{(\alpha + \delta)(\gamma + \beta)}{(\alpha + \beta)(\gamma + \delta)} + \{\gamma \to -\delta, \delta \to -\gamma\} \equiv Z_1 + Z_2 \\
R(\epsilon) = -\frac{1}{2\pi^2} \text{Re} \frac{\partial^2 Z}{\partial \alpha \partial \beta} \bigg|_{(\|)} - \frac{1}{2} = -\left(\frac{\sin \epsilon}{\epsilon}\right)^2. \tag{4.9}
\]

which is exactly the random-matrix prediction for the GUE \((c_2 = -1/2, d_2 = 1/2)\). Here the summands \( Z_1 \) and \( Z_2 \) are responsible for the non-oscillatory and oscillatory contributions to the correlation function, corresponding to two different saddle points in RMT. We stress that \( Z_2 \) becomes accessible to a semiclassical treatment only through resummation. Differentiating (4.8) gives a formula identical to that due to Bogomolny & Keating (1996) for the nonuniversal corrections to the random-matrix expression for \( R(\epsilon) \).

5. Relation to Heusler et al.

In Heusler et al. (2007) no resummation was made and thus only the first summand \( Z_1 \) could be obtained. Hence if one represents \( R(\epsilon) \) as in (3.4) only non-oscillatory contributions are found. To recover oscillatory contributions an alternative representation of \( R(\epsilon) \) through \( Z \) was used, with the limit \((\|)\) substituted by \((\times)\) \( \alpha, \beta \to \frac{\epsilon}{2\pi^2}, \gamma, \delta \to -\frac{\epsilon}{2\pi^2} \). If one replaces \( Z \) by \( Z_1 \) (and drops \(-\frac{1}{2}\)) this time only oscillatory contributions are obtained. It was claimed that summation of both results
\[
-\frac{1}{2\pi^2} \text{Re} \frac{\partial^2 Z_1}{\partial \alpha \partial \beta} \bigg|_{(\|)} - \frac{1}{2\pi^2} \text{Re} \frac{\partial^2 Z_1}{\partial \alpha \partial \beta} \bigg|_{(\times)} - \frac{1}{2} \tag{5.1}
\]

\* Again, convergence is guaranteed when \( \alpha, \beta, \gamma \) and \( \delta \) are given arbitrarily small imaginary parts.
yields the full correlation function. After comparison to (4.2) we see that this claim is equivalent to
\[
\frac{\partial^2 Z_1}{\partial \alpha \partial \beta} \bigg|_\times = \frac{\partial^2 Z_2}{\partial \alpha \partial \beta} \bigg|_\parallel
\]
(5.2)
which is easily verified from Riemann-Siegel resummation: the right-hand side of (5.2) differs from \( \frac{\partial^2 Z_1}{\partial \alpha \partial \beta} \bigg|_\parallel \) because \( \gamma \) and \( \delta \) are interchanged and flipped in sign; on the left the same effect is reached by replacing \( \bigg|_\parallel \) with \( \times \).

6. Conclusions and outlook

By resummation of the generating function we have clarified the semiclassical origin of oscillatory contributions to spectral correlators. This opens the door for a systematic use of generating functions in semiclassical theory, with possible applications in quantum transport or to higher-order correlation functions. Moreover we have extended the semiclassical treatment beyond the universal regime. Within the diagonal approximation, non-universal effects come into play through the dynamical zeta function. It would be interesting to see if this carries over to off-diagonal terms.

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