A Recurrence Relation Related to the Reve’s Puzzle

Abdullah-Al-Kafi Majumdar*

Department of Mathematics, Jahangirnagar University, Savar, Dhaka 1342, Bangladesh

Abstract: In a recent paper, Majumdar [1] studied, to some extent, the generalized recurrence relation, introduced by Matsuura [2]:

\[ MT(n, \beta) = \min_{1 \leq s \leq n} \left\{ \beta MT(n-s, \beta) + 2^{-1} \right\}, \]

where \( n \geq 1 \) and \( \beta \geq 2 \) are integers. It may be mentioned here that, \( \beta = 2 \) corresponds to the Reve’s puzzle, introduced by Dudeney [3]. In this paper, we study more closely the properties of the function \( MT(n, \beta) \), and give a closed-form expression of it when \( \beta = 2^i \) (for any integer \( i \geq 2 \)).

Keywords: Reve’s puzzle, Recurrence relation, Local-value relationships.

1. INTRODUCTION

Matsuura [2] posed the recurrence relation below:

\[ MT(n, \beta) = \min_{0 \leq k \leq n-1} \{\beta MT(k, \beta) + 2^{n-k} - 1\}; \quad n \geq 1, \]

\[ MT(0, \beta) = 0, \]

where \( \beta \geq 2 \) is an integer.

Some of the properties satisfied by \( MT(n, \beta) \) has been studied by Majumdar [1]. In what follows, let, for \( n \geq 1 \) and \( \beta \geq 2 \) fixed:

\[ FT(n, k, \beta) = \beta MT(n, k, \beta) + 2^{n-k} - 1, \quad 0 \leq k \leq n-1. \]

Note that:

\[ MT(n, \beta) = \min_{0 \leq k \leq n-1} \{FT(n, k, \beta)\}. \]

The main results found by Majumdar [1] are reproduced below for reference later.

Lemma 1.1: For any \( \beta \geq 2 \) fixed, let \( FT(n, k, \beta) \) and \( FT(n+1, k, \beta) \) be minimized at the points \( k = k_1 \) and \( k = k_2 \) respectively. Then, \( k_1 \leq k_2 \leq k_1 + 1. \)

The inequality in Lemma 1.1 above needs some explanation. When \( \beta = 2^i \) (for some integer \( i \geq 2 \)), as Lemma 2.4 shows, there are instances when \( MT(n, \beta) \) is attained at a unique value of \( k \), while in other cases, \( MT(n, \beta) \) is attained at exactly two (consecutive) values of \( k \). Thus, in the latter case, \( k_1 \) may be interpreted as the minimum of the two values of \( k \) at which \( MT(n, \beta) \) is attained and then \( k_2 \) is the minimum of the two values (if such a situation arises) at which \( MT(n, \beta) \) is attained. Alternatively, \( k_1 \) may be taken as the maximum of the two values at which \( MT(n, \beta) \) is attained, and then \( k_2 \) is the maximum of the two values where \( MT(n, \beta) \) is attained.

The following lemma shows that, for \( \beta \geq 3 \) fixed,

\( MT(n, \beta) \) is convex with respect to \( n \) (in the sense of the left-hand side inequality).
**Lemma 1.2:** For any $\beta \geq 3$ fixed, for all $n \geq 1$,

$$MT(n + 1, \beta) - MT(n, \beta) \leq MT(n + 2, \beta) - MT(n + 1, \beta) \leq 2[MT(n + 1, \beta) - MT(n, \beta)].$$

**Lemma 1.3:** If $\beta = 2^i$ (for some integer $i \geq 1$), $MT(n, \beta)$ is attained either at a unique $k$, or else at two (consecutive) values.

**Lemma 1.4:** For any $\beta \geq 3$ and $n \geq 1$ fixed, $FT(n, k, \beta)$ is convex in $k$ in the sense that:

$$FT(n, k + 2, \beta) - FT(n, k + 1, \beta) \geq FT(n, k + 1, \beta) - FT(n, k, \beta)$$

for all $0 \leq k \leq n - 3$.

This paper gives an explicit expression of $MT(n, \beta)$ when $\beta = 2^i$ (for some integer $i \geq 3$). This is done in Section 3. Section 2 gives some background materials. We conclude the paper with some remarks in Section 4.

### 2. BACKGROUND MATERIALS

We have the following results, due to Majumdar [1].

**Lemma 2.1:** Let $\beta = 2^i$ (for some integer $i \geq 2$). Then, for all $n \geq 1$, $MT(n + 1, \beta) - MT(n, \beta) = 2^i$ for some integer $s \geq 1$.

**Lemma 2.2:** Let $\beta = 2^i$ (for some integer $i \geq 2$). Let $FT(N, k, \beta)$ be minimized at the two points $k = K, K + 1$ for some integer $N \geq 1$. Let

$$M = 2N - K + i - 1.$$  

Then, $FT(M, k, \beta)$ is minimized at the two values $k = N - 1, N$.

**Lemma 2.3:** Let, for some $N \geq 1$, $\beta \geq 2$, $FT(N, k, \beta)$ be minimized at the two points $k = K, K + 1$. Then:

1. $FT(N - 1, k, \beta)$ is minimized at $k = K$,
2. $FT(n + 1, k, \beta)$ is minimized at $k = K + 1$,
3. $MT(N, \beta) - MT(N - 1, \beta) = 2^{n - K - 1}$

   $\quad = MT(N + 1, \beta) - MT(N, \beta)$.

**Lemma 2.4** below shows that, if $\beta = 2^i$ for some integer $i \geq 2$, there is an integer $N \geq 1$ such that $FT(N, k, \beta)$ is minimized at two values of $k$, and there is an integer $M \geq 1$ such that $FT(M, k, \beta)$ is minimized at a unique value of $k$.

**Lemma 2.4:** Let $\beta = 2^i$ for some integer $i \geq 2$. Then,

1. $FT(i, k, \beta)$ is minimized at the point $k = 0$ only,
2. $FT(i + 1, k, \beta)$ is minimized at the two points $k = 0, 1; FT(i + 2, k, \beta)$ is minimized at the unique point $k = 1$.
3. $FT(i + 3, k, \beta)$ is minimized at the two points $k = 1, 2; FT(i + 4, k, \beta)$ is minimized at the unique point $k = 2$.

**Proof:** To prove part (1), note that:

$$FT(i, 0, \beta) = 2^i - 1 < \beta MT(1, \beta) + 2^i - 1 = FT(i + 1, 1, \beta),$$

so, that by Lemma 1.4, $FT(i, k, \beta)$ is minimized at the unique point $k = 0$.

(2) Since

$$FT(i + 1, 0, \beta) = 2^{i+1} - 1 = \beta MT(1, \beta) + 2^i - 1 = FT(i + 1, 1, \beta),$$

by Lemma 1.4 and Lemma 1.3, $FT(i + 1, k, \beta)$ is minimized at $k = 0, 1$. Again, since

$$FT(i + 2, 0, \beta) = 2^{i+2} - 1 > \beta MT(1, \beta) + 2^{i+1} - 1 = FT(i + 2, 1, \beta),$$

it follows that $FT(i + 2, k, \beta)$ is minimized at the unique point $k = 1$.

(3) Note that:

$$FT(i + 3, 1, \beta) = \beta MT(2, \beta) + 2^{i+1} - 1 = FT(i + 3, 2, \beta).$$
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Therefore, $FT(i + 3, k, \beta)$ is minimized at $k = 1, 2$. Again, since

$$FT(i + 4, 1, \beta) = \beta MT(1, \beta) + 2^{i+3} - 1$$

it follows that $FT(i + 4, k, \beta)$ is minimized at the unique point $k = 2$.

**Lemma 2.5:** Let $\beta = 2^i$ for some integer $i \geq 2$. Let, for $\beta$ fixed, $FT(N, k, \beta)$ be minimized at the unique point $k = K (\neq 0)$ for some integer $N \geq 1$. Then:

1. $FT(N + 1, k, \beta)$ is minimized at $k = K$,
2. $FT(N - 1, k, \beta)$ is minimized at the two points $k = K - 1, K$.

**Proof:** By assumption,

$$MT(N, \beta) = \beta MT(K, \beta) + 2^{N-K} - 1 < \beta MT(K + 1, \beta) + 2^{N-K} - 1,$$

and hence,

$$\beta [MT(K + 1, \beta) - MT(K, \beta)] > 2^{N-K-1}, \quad (1)$$

$$\beta [MT(K, \beta) - MT(K - 1, \beta)] < 2^{N-K}. \quad (2)$$

(1) If $FT(N + 1, k, \beta)$ is not minimized at $k = K$, then by Lemma 1.3, $FT(N + 1, k, \beta)$ is minimized at the unique point $k = K + 1$, so that:

$$MT(N + 1, \beta) = \beta MT(K + 1, \beta) + 2^{N-K} - 1 < \beta MT(K, \beta) + 2^{N-K+1} - 1.$$

Thus,

$$\beta [MT(K + 1, \beta) - MT(K, \beta)] < 2^{N-K},$$

which, together with the inequality (1), contradicts Lemma 2.1. Thus, $FT(N + 1, k, \beta)$ is minimized at $k = K$.

Then,

$$MT(N + 1, \beta) = \beta MT(K, \beta) + 2^{N-K+1} - 1 \leq \beta MT(K + 1, \beta) + 2^{N-K} - 1,$$

giving

$$\beta [MT(K + 1, \beta) - MT(K, \beta)] \geq 2^{N-K}. \quad (3)$$

Observe that,

$$MT(N + 1, \beta) - MT(N, \beta) = 2^{N-K}. \quad (2.1)$$

(2) If $FT(N - 1, k, \beta)$ is not minimized at $k = K$, then by virtue of Lemma 1.3, it is minimized at the unique point $k = K - 1$, so that

$$MT(N - 1, \beta) = \beta MT(K - 1, \beta) + 2^{N-K} - 1$$

$$< \beta MT(K, \beta) + 2^{N-K-1} - 1.$$

Thus,

$$\beta [MT(K, \beta) - MT(K - 1, \beta)] > 2^{N-K-1},$$

which, together with the inequality (2), contradicts Lemma 2.1. Hence, $FT(N - 1, k, \beta)$ is minimized at $k = K$. We now want to show that $FT(N - 1, k, \beta)$ is minimized at $k = K - 1$ as well, for otherwise:

$$MT(N - 1, \beta) = \beta MT(K, \beta) + 2^{N-K-1} - 1 \leq \beta MT(K - 1, \beta) + 2^{N-K} - 1,$$

giving

$$\beta [MT(K, \beta) - MT(K - 1, \beta)] < 2^{N-K-1}.$$ Using the inequality (3), we get:

$$\beta [MT(K + 1, \beta) - MT(K, \beta)] \geq 2^{N-K} \quad > 2 \beta [MT(K, \beta) - MT(K - 1, \beta)],$$

which contradicts Lemma 1.2. Hence, $FT(N - 1, k, \beta)$ is minimized at $k = K - 1$ as well.

Here,

$$MT(N - 1, \beta) = \beta MT(K, \beta) + 2^{N-K-1} - 1$$

$$= \beta MT(K - 1, \beta) + 2^{N-K} - 1,$$

so that

$$MT(N, \beta) - MT(N - 1, \beta) = 2^{N-K-1}. \quad (2.2)$$

Moreover,

$$MT(N, \beta) - MT(N - 1, \beta) = \beta [MT(K, \beta) - MT(K - 1, \beta)]. \quad (2.3)$$
**Corollary 2.1:** Let \( \beta = 2^i \) for some integer \( i \geq 1 \). Let, for \( \beta \) fixed, \( FT(N, k, \beta) \) be minimized at the unique point \( k = K \) for some integer \( N \geq 1 \). Then, \( FT(N + 1, k, \beta) \) is minimized at the two points \( k = K, K + 1 \), with
\[
MT(N + 1, \beta) - MT(N, \beta) = 2^{N - K}.
\]

**Proof:** If \( FT(N + 1, k, \beta) \) is minimized at the unique point \( k = K \), then by part (b) of Lemma 2.5, \( FT(N, k, \beta) \) is minimized at two values of \( k \), contrary to the assumption. Thus,
\[
MT(N + 1, \beta) = \beta MT(K, \beta) + 2^{N-K+1} - 1
\]
\[= \beta MT(K + 1, \beta) + 2^{N-K} - 1,
\]
which gives
\[
\beta [MT(K + 1, \beta) - MT(K, \beta)] = 2^{N-K}
\]
\[= MT(N + 1, \beta) - MT(N, \beta).
\]

**Theorem 2.1:** Let \( \beta = 2^i \) for some integer \( i \geq 2 \). Let, for \( \beta \) fixed, \( FT(N, k, \beta) \) be minimized at the point \( k = K \) for some integer \( N \geq 1 \). Then,
\[
MT(N, \beta) - MT(N - 1, \beta) = 2^{N - K - 1}.
\]

Moreover,
1. If \( FT(N, k, \beta) \) is minimized at the two points \( k = K, K + 1 \), then
\[
MT(N + 1, \beta) - MT(N, \beta) = 2^{N-K-1}
\]
\[= \beta [MT(K + 1, \beta) - MT(K, \beta)],
\]
2. If \( FT(N, k, \beta) \) is minimized at the unique point \( k = K \), then
\[
MT(N + 1, \beta) - MT(N, \beta) = 2^{N-K}
\]
\[= \beta [MT(K + 1, \beta) - MT(K, \beta)].
\]

**Proof:** follows readily from Lemma 2.3, (2.2) and (2.4).

Let \( \beta = 2^i \) for some integer \( i \geq 2 \). Starting with a uniquely attained function \( MT(N, \beta) \) (for \( \beta \) fixed), the lemma below finds another.

**Lemma 2.6:** Let \( \beta = 2^i \) (for any integer \( i \geq 2 \)). Let, for \( \beta \) fixed, \( FT(N, k, \beta) \) be minimized at the unique point \( k = K \). Let
\[
M = 2N - K + i - 1.
\]
Then, \( FT(M + 1, k, \beta) \) is minimized at the unique point \( k = N \).

**Proof:** Since \( FT(N, k, \beta) \) is minimized at the unique point \( k = K \), \( FT(N + 1, k, \beta) \) is minimized at the two points \( k = K, K + 1 \) (by Corollary 2.1). Therefore, using (2.1) and (2.3), we get
\[
\beta [MT(N + 1, \beta) - MT(N, \beta)] = 2^{N-K+i-1} = 2^{M-N},
\]
(2.6)
\[
\beta [MT(N, \beta) - MT(N - 1, \beta)] = 2^{N-K+i-1} = 2^{M-N}.
\]
(2.7)
By part (2) of Lemma 2.5, \( FT(N - 1, k, \beta) \) is minimized at the two points \( k = K - 1, K \), and so, by part (3) of Lemma 2.2,
\[
MT(N - 1, \beta) - MT(N - 2, \beta) = 2^{N-K-1}.
\]
Therefore, by (2.5),
\[
\beta [MT(N - 1, \beta) - MT(N - 2, \beta)] = 2^{N-K+i-1} = 2^{M-N}.
\]
(2.8)
Now, if,
\[
MT(M, \beta) = \beta MT(N - 1, \beta) + 2^{M-N+1}-1
\]
\[= \beta MT(N - 2, \beta) + 2^{M-N+2}-1,
\]
we get,
\[
\beta [MT(N - 1, \beta) - MT(N - 2, \beta)] = 2^{N-K+i-1} = 2^{M-N+1},
\]
contradicting (2.8). Thus, \( FT(M, k, \beta) \) is not minimized at \( k = N - 2 \). Again, if
\[
MT(M, \beta) = \beta MT(N - 1, \beta) + 2^{M-N+1}-1
\]
\[< \beta MT(N, \beta) + 2^{M-N}-1,
\]
we get
\[
\beta [MT(N, \beta) - MT(N - 1, \beta)] > 2^{M-N}.
\]
which contradicts (2.7). Hence, $FT(M, k, \beta)$ is minimized at the two points $k = N - 1, N$. Then, $FT(M + 1, k, \beta)$ is minimized at the unique point $k = N$, because otherwise,

$$MT(M + 1, \beta) = \beta MT(N, \beta) + 2^{M - N + 1} - 1$$

$$= \beta MT(N + 1, \beta) + 2^{M - N} - 1,$$

so that,

$$\beta (MT(N + 1, \beta) - MT(N, \beta)) = 2^{M - N},$$

contradicting (2.6). Thus, $MT(M + 1, \beta)$ is attained at the unique point $k = N$.

**Lemma 2.7:** Let $\beta = 2^i$ for some integer $i \geq 2$. Let, for $\beta$ fixed, $FT(n, k, \beta)$ be minimized at the unique point $k = K$ for some integer $N \geq 1$, so that,

$$MT(N, \beta) - MT(N - 1, \beta) = 2^{N - K - 1},$$

$$MT(N + 1, \beta) - MT(N, \beta) = 2^{N - K}.$$

Let,

$$M = \min \{ n : MT(n, \beta) - MT(n - 1, \beta) = 2^{N - K - 1} \}.$$ 

Then, $FT(n, k, \beta)$ is minimized at two values of $k$, for any $n$ satisfying $M \leq n \leq N - 1$.

**Proof:** By assumption,

$$MT(M, \beta) - MT(M - 1, \beta) = 2^{N - K - 1},$$

but

$$MT(M - 1, \beta) - MT(M - 2, \beta) = 2^{N - K - 2}.$$ 

Let $FT(M, k, \beta)$ be minimized at $k = L$ (if $FT(M, k, \beta)$ is minimized at two values of $k$, $L$ is the minimum of the two values). Then, from Theorem 2.1,

$$MT(M, \beta) - MT(M - 1, \beta) = 2^{M - L - 1}.$$ 

Now, if $FT(M - 1, k, \beta)$ is minimized at the two points $k = L - 1, L$, then by part (3) of Lemma 2.3,

$$MT(M - 1, \beta) - MT(M - 2, \beta) = 2^{M - L - 1},$$

which contradicts the definition of $M$. Consequently, $FT(M - 1, k, \beta)$ is minimized at the unique point $k = L$, and hence, by virtue of Lemma 2.5 and Corollary 2.1, $FT(M, k, \beta)$ is minimized at the two points $k = L, L + 1$, so that, by part (2) of Lemma 2.2, $FT(M + 1, k, \beta)$ is minimized at $k = L + 1$. Now, if $FT(M + 1, k, \beta)$ is minimized at the unique point $k = L + 1$, then by (2.2) and (2.1),

$$MT(M + 1, \beta) - MT(M, \beta) = 2^{M - L - 1},$$

$$MT(M + 2, \beta) - MT(M + 1, \beta) = 2^{M - L},$$

and we must have $M + 1 = N$. Otherwise, $FT(M + 1, k, \beta)$ is minimized at two values of $k$, namely, at $k = L + 1, L + 2$. Continuing in this way, we see that each of the functions $FT(M, k, \beta), FT(M + 1, k, \beta), ..., FT(N - 1, k, \beta)$ is minimized at two values of $k$; more precisely, for $\ell = 0, 1, ..., FT(M + \ell, k, \beta)$ is minimized at the two points $k = L + \ell, L + \ell + 1$.

The next section considers the problem of finding the solution of the recurrence relation (1.1) when $\beta = 2^i$ for some integer $i \geq 2$.

3. **THE SOLUTION OF THE RECURRENC RELATION**

This section derives a closed-form expression of $MT(n, \beta)$ when $\beta = 2^i$ for any integer $i \geq 2$.

For any $\beta \geq 2$ fixed, let:

$$a_n = MT(n, \beta) - MT(n - 1, \beta), \quad n \geq 1. \quad (3.1)$$

Let $\beta = 2^i$ for some integer $i \geq 2$. Let $k_j (j \geq 0)$ be the largest index such that

$$a_{k_j} = 2^i. \quad (3.2)$$
It may be noted here that, when $\beta = 2^i$ for some integer $i \geq 2$, and $k_j$ is the largest index satisfying the condition (3.2), then $MT(k_j, \beta)$ is attained at a unique point, for otherwise, if $MT(k_j, \beta)$ is attained at $k = K, K + 1$, then by part (3) of Lemma 2.3, the definition of $k_j$ is violated.

Clearly, for any $\beta \geq 2$,

$$k_0 = 1, k_1 = 2.$$  

However, the sequence of numbers $\{k_n\}_{n \geq 0}$, $n \geq 3$, depends on $\beta$. For example,

$$k_2(\beta) = \begin{cases} 4, & \text{if } \beta = 3, 4 \\ 3, & \text{if } \beta \geq 5 \end{cases}$$

as can easily be verified.

**Theorem 3.1**: Let $\beta = 2^i$ for some integer $i \geq 2$. Let $k_j$ ($j \geq 0$) be the largest index such that

$$a_{k_j} = 2^j.$$  

Then,

1. $MT(k_j, \beta)$ is attained uniquely at the point $k = k_j - j - 1$,
2. $MT(n, \beta)$ is attained at the two points $k = n - j - 1$, $n - j - 2$ for all $n$ with $k_j + 1 \leq n \leq k_{j+1} - 1$,
3. $MT(k_{j+1}, \beta)$ is attained at the point $k = k_{j+1} - j - 2$.

**Proof**: We prove the theorem by induction on $j$. The results can easily be verified when $j = 0$ and $j = 1$. So, we assume that the results are true for some $j$.

Now, in the notation of Lemma 2.7,

$$N = k_j, N - K - 1 = j, N + M = k_{j+1}.$$  

1. To prove part (1), note that $MT(k_j, \beta)$ is attained at the unique point $k = K = k_j - j - 1$.
2. follows immediately from Lemma 2.7 with $M = k_j + 1, L = M - j - 1 = k_j - j, N = k_{j+1}$, so that $MT(M + \ell, \beta)$ is attained at the two points $k = M + \ell - j - 1, M + \ell - j$.
3. Since (by part (2) above), $MT(k_{j+1} - 1, \beta)$ is attained at the two points;

$$k = k_{j+1} - j - 3, k_{j+1} - j - 2,$$

it follows, by Lemma 2.3, that $MT(k_{j+1}, \beta)$ is attained (uniquely) at the point $k = k_{j+1} - j - 2$. Thus, the results are true for $j + 1$ as well, completing induction.

When $\beta = 2^i$ for some integer $i \geq 1$, an expression of $MT(n, \beta)$ in terms of the numbers $k_j$ can be derived. This is done in the following theorem. We then illustrate the use of Theorem 3.2 by finding $MR\left(\frac{u + 1 + j + 2i}{2}\right)$ corresponding to the Reve’s puzzle in Lemma 3.1.

**Theorem 3.2**: Let $\beta = 2^i$ for some integer $i \geq 1$. Let $k_j$ ($j \geq 0$) be the largest index such that

$$a_{k_j} = 2^j.$$  

Let $k_j \leq n \leq k_{j+1}$ for some $j \geq 0$. Then,

$$MT(n, \beta) = 1 + \sum_{j=0}^{n} (k_j - k_{j-1})2^j + 2^{j+1}(n - k_j).$$

**Proof**: We write $MT(n, \beta)$ as follows:

$$MT(n, \beta) = \sum_{m=1}^{n} [MT(m, \beta) - MT(m-1, \beta)]$$

$$= 1 + \sum_{j=1}^{n} \sum_{m=k_{j-1}+1}^{k_j} [MT(m, \beta) - MT(m-1, \beta)]$$

$$+ \sum_{m=k_{j+1}}^{n} [MT(m, \beta) - MT(m-1, \beta)].$$

Now, noting that
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MT(m, β) − MT(m − 1, β) = 2ℓ if kℓ,1 + 1 ≤ m ≤ kℓ; ℓ = 1, 2, ...

the desired expression follows.

Recall that β = 2 correspond to the Reve’s puzzle.

Let MR(n) is the minimum number of moves required to solve the Reve’s puzzle with n ≥ 1 discs. It is well-known (see, for example, Roth [4], Hinz [5] and Majumdar [6, 7]) that, for j ≥ 0, MR((j + 1)(j + 2)) is attained at the unique point k = j(j + 1)/2. We now give an expression of MT((j + 1)(j + 2)/2, 2) = MR((j + 1)(j + 2)/2), j ≥ 0, in the following lemma, which makes use of Theorem 3.2.

Lemma 3.1: For j ≥ 0,

MT((j + 1)(j + 2)/2, 2) = MR((j + 1)(j + 2)/2) = j2/2 + 1.

Proof: We first note that, when β = 2, kℓ of Theorem 3.2 is given by

kℓ = (ℓ + 1)(ℓ + 2)/2 for all ℓ ≥ 0.

Therefore, by Theorem 3.2,

MR((j + 1)(j + 2)/2) = 1 + \sum_{ℓ=1}^{j} [\frac{(ℓ + 1)(ℓ + 2)}{2} - \frac{ℓ(ℓ + 1)}{2}]2ℓ

= 1 + \sum_{ℓ=1}^{j} (ℓ + 1)2ℓ.

Now, let

S = \sum_{ℓ=1}^{j} (ℓ + 1)2ℓ = \sum_{ℓ=1}^{j} 2ℓ + 2(2ℓ − 1).

Then,

2S = \sum_{ℓ=1}^{j} (ℓ + 1)2ℓ+1 = \sum_{ℓ=1}^{j-1} (ℓ + 1)2ℓ+1 + (j + 1)2ℓ+1

= \sum_{ℓ=1}^{j} k2ℓ + (j + 1)2ℓ+1

= \left(\sum_{ℓ=1}^{j} \frac{ℓ}{2} - 2\right) + (j + 1)2ℓ+1

= S2/2 + (j + 1)2ℓ+1,

so that

S = j2/2 + 1.

Hence, finally we get the desired expression.

The expression of MR((j + 1)(j + 2)/2) is well-known, and can be found in, for example, Roth [4], Hinz [5] and Majumdar [6, 7]). Lemma 3.1 gives an alternative approach to find it. We now state and prove the following theorem.

Theorem 3.3: Let β = 2i for some integer i ≥ 2. Then, for any n ≥ 0, MT((n + 1)(\frac{i}{2} n + ℓ), β) is attained at the unique point k = n[\frac{i}{2}(n − 1) + ℓ], 1 ≤ ℓ ≤ i.

Proof: The proof is by induction on n.

Noting that MT(n, β) is attained at the unique point k = 0 for all 1 ≤ n ≤ i (by virtue of part (1) of Lemma 2.4), the validity of the result for n = 0 follows. So, we assume that the result holds true for some n. Then, by Lemma 2.6, MT(N + 1, β) is attained at the unique point k = (n + 1)(\frac{i}{2} n + ℓ), where;

N + 1 = 2(n + 1)(\frac{i}{2} n + ℓ) − n[\frac{i}{2}(n − 1) + ℓ] + i

= (n + 2)[\frac{i}{2}(n + 1) + ℓ].

This shows that the result is true for n + 1 as well, thereby completing induction.
The following two lemmas deal respectively with the particular cases when $\beta = 4$ and $\beta = 8$.

**Lemma 3.2:** For all $n \geq 1$,
1. $MT(n^2, 4) = \frac{1}{2} \left( (3n - 2)2^{n-1} + 1 \right)$,
2. $MT(n(n + 1), 4) = \frac{1}{4} \left( (3n - 1)2^n + 1 \right)$.

**Proof:** By Theorem 3.3, $MT(n, 4)$ is attained at the (unique) point $k = (n - 1)^2$.

Therefore,

$$MT(n^2, 4) = 4MT((n-1)^2, 4) + 2^{2n-1} - 1 = 4 [4MT((n-2)^2, 4) + 2^{2n-3} - 1 + 2^{2n-1} - 1] = 4^2MT((n-2)^2, 4) + 2^{2n-1} - (1 + 4).$$

Continuing in this way $\ell$ times, we get:

$$MT(n^2, 4) = 4^\ell MT((n-\ell)^2, 4) + \ell \cdot 2^{2n-1} - (1 + 4 + \ldots + 4^{\ell-1}) = 4^\ell MT((n-\ell)^2, 4) + \ell \cdot 2^{2n-1} - \frac{4\ell - 1}{3}.$$

Now, choosing $\ell = n$ and then simplifying, we get the expression desired.

(2) Since $MT(n(n + 1), 4)$ is attained at $k = n(n - 1)$, by repeated application, we get

$$MT(n(n + 1), 4) = 4MT((n-1)^2, 4) + 2^{2n - 1} = 4 [4MT((n-2)^2, 4) + 2^{2n-2} - 1 + 2^{2n-1} - 1] = 4^2MT((n-1)^2, 4) + 2^{2n} - (1 + 4).$$

After $\ell$ iterations, we get

$$MT(n(n + 1), 4) = 4^\ell MT((n-\ell + 1)(n-\ell), 4) + \ell \cdot 2^{2n} - (1 + 4 + \ldots + 4^{\ell-1}) = 4^\ell MT((n-\ell + 1)(n-\ell), 4) + \ell \cdot 2^{2n} - \frac{4\ell - 1}{3}.$$

Finally, putting $\ell = n$, we get the desired result after simplification.

**Corollary 3.1:** For any integer $n \geq 1$,

$$MT(n, 4) = \begin{cases} 
(3n - 2)2^{n-1} + 1 + (N - n)2^{2n-1}, & \text{if } n^2 \leq n \leq (n+1) \\
(3n - 1)2^n + 1 + (N - n(n+1))2^{2n}, & \text{if } n(n+1) \leq N < (n+1)^2 
\end{cases}$$

**Proof:** Since

$$MT(n^2 + m, 4) - MT(n^2 + m - 1, 4) = 2^{2n-1}$$

for all $1 \leq m \leq n$, the result follows from Lemma 3.2.

**Lemma 3.3:** For any integer $n \geq 1$,

1. $MT\left(\frac{3}{2}n(n + 1), 8\right) = \frac{1}{4} \left( (7n - 1)2^{2n} + 1 \right)$,
2. $MT\left(\frac{n + 1}{2}(3n + 2), 8\right) = \frac{1}{4} \left( (7n + 3)2^{2n+1} + 1 \right)$,
3. $MT\left(\frac{n + 1}{2}(3n + 4), 8\right) = \frac{1}{4} \left( (7n + 5)2^{2n+2} + 1 \right)$.

**Proof:** The proofs are given below.

(1) By Theorem 3.3, $MT\left(\frac{3}{2}n(n + 1), 8\right)$ is attained at the point $k = \frac{3}{2}n(n - 1)$, so that

$$MT\left(\frac{3}{2}n(n + 1), 8\right) = 8MT\left(\frac{3}{2}n(n - 1), 8\right) + 2^{2n} - 1 = 8[8MT\left(\frac{3}{2}(n - 1)(n - 2), 8\right) + 2^{2(n-1)} - 1] + 2^{2n} - 1 = 8^2MT\left(\frac{3}{2}(n - 1)(n - 2), 8\right) + 2^{2n} - (1 + 8).$$

Continuing $\ell$ times, we get

$$MT\left(\frac{3}{2}n(n + 1), 8\right) = 8^\ell MT\left(\frac{3}{2}(n - \ell + 1)(n - \ell), 8\right) + \ell \cdot 2^{2n} - (1 + 8 + \ldots + 8^{\ell-1}) = 8^\ell MT\left(\frac{3}{2}(n - \ell + 1)(n - \ell), 8\right) + \ell \cdot 2^{2n} - \frac{1}{4} \left( 8^\ell - 1 \right).$$

In the above expression, putting $\ell = n$, and then simplifying, we get the desired expression.
(2) Since \( MT\left(\frac{(n+1)(3n+2)}{2}, 8\right) \) is attained at the point 
\[ k = \frac{n(3n-1)}{2}, \]
we get 
\[
MT\left(\frac{(n+1)(3n+2)}{2}, 8\right) = 8 MT\left(\frac{n(3n-1)}{2}, 8\right) + 2^{3n+1} - 1 \]
\[ = 8\{8 MT\left(\frac{(n-1)(3n-4)}{2}, 8\right) + 2^{3n-2} - 1\} + 2^{3n+1} - 1 \]
\[ = 8^2 MT\left(\frac{(n-1)(3n-4)}{2}, 8\right) + 2.2^{3n+1} - (1 + 8). \]
And in general, after \( \ell \) iterations, we have:
\[
MT\left(\frac{(n+1)(3n+2)}{2}, 8\right) = 8^\ell MT\left(\frac{1}{2} (n-\ell + 1)(3(n-\ell) + 2), 8\right) + \ell.2^{3n+1} - (1 + 8 + \ldots + 8^{(\ell-1)}) \]
\[ = 8^\ell MT\left(\frac{1}{2} (n-\ell + 1)(3(n-\ell) + 2), 8\right) + \ell.2^{3n+1} \]
\[ - \frac{1}{7} (8^\ell - 1). \]
Now, putting \( \ell = n \), we get
\[
MT\left(\frac{(n+1)(3n+2)}{2}, 8\right) = 8^n + n.2^{3n+1} - \frac{1}{7} (8^n - 1). \]
Simplifying, we get the result desired.

(3) Since \( MT\left(\frac{(n+1)(3n+4)}{2}, 8\right) \) is attained at the point 
\[ k = \frac{n(3n+1)}{2}, \]
we have
\[
MT\left(\frac{(n+1)(3n+4)}{2}, 8\right) = 8 MT\left(\frac{n(3n+1)}{2}, 8\right) + 2^{3n+2} - 1 \]
\[ = 8[8 MT\left(\frac{(n-1)(3n-2)}{2}, 8\right) + 2^{3n-1} - 1] + 2^{3n+2} - 1 \]
\[ = 8^2 MT\left(\frac{(n-1)(3n-2)}{2}, 8\right) + 2.2^{3n+2} - (1 + 8), \]
and after \( \ell \) iterations, we have:
\[
MT\left(\frac{(n+1)(3n+4)}{2}, 8\right) = 8^\ell MT\left(\frac{1}{2} (n-\ell + 1)(3(n-\ell) + 4), 8\right) + \ell.2^{3n+1} \]
\[ - \frac{1}{7} (8^\ell - 1). \]
Finally, putting \( \ell = n \), we get
\[
MT\left(\frac{(n+1)(3n+4)}{2}, 8\right) = 3.8^n + n.2^{3n+2} - \frac{1}{7} (8^n - 1), \]
which gives the desired result after simplification.

4. REMARKS

In this paper, we derive some results in connection with the difference \( MT(n+1, \beta) - MT(n, \beta) \), which plays a vital role in solving the recurrence relation (1.1). These are given in Section 2. In Section 3, an alternative expression of \( MT(n, \beta) \) is given when \( \beta = 2^i \) for some integer \( i \geq 1 \). From Theorem 3.2, we observe that the determination of the numbers \( k_i \), satisfying the condition, \( MT(k_j, \beta) - MT(k_j - 1, \beta) = 2^i \), is required, which is given in Theorem 3.3. This would enable us to find a closed-form expression of \( MT(n, \beta) \), as has been illustrated in Lemma 3.2 and Corollary 3.1 explicitly for the particular case when \( \alpha = 4 \).

It may be mentioned here that, Matsura [2] adopted a different approach to find \( MT(n, \beta) \). More specifically, letting \( \{b_n\}_{n\geq 1} \) be the sequence of numbers defined as follows:

\[ b_n = 2^m \beta^m; m \geq 0, \ell \geq 0, \]
and arranged in non-decreasing order, Matsura [2] showed, by induction on \( n \), using a recurrence relation satisfied by \( b_n \), that \( a_n = b_n \). In this paper, we follow a different approach, which enables us to find an explicit form of \( MT(n, \beta) \). Our analysis reveals many interesting properties and local-value relationships that are inherent in the optimal value function \( MT(n, \beta) \). Moreover, though for small values of \( n, b_n \) may be found out, for
large values of \( n \), finding \( b_n \), even using the recurrence relation satisfied by it, is a challenging problem.

For \( \alpha = 4 \), the first few terms of the sequence \( \{b_n\}_{n \geq 1} \) are:

\[ 1, 2, 4, 8, 8, 16, 16, 32, 32, 32, 32, 64, \ldots \]

We recall that, when \( \beta = 2 \), \( MT(n, \beta) \) satisfies exactly one of the following relationships:

\[
MT(n+2, \beta) - MT(n+1, \beta) = 2[MT(n+1, \beta) - MT(n, \beta)],
\]

(4.1)

\[
MT(n+2, \beta) - MT(n+1, \beta) = MT(n+1, \beta) - MT(n, \beta).
\]

(4.2)

Corollary 4.1: Let \( \beta = 2^i \) (for some integer \( i \geq 1 \)). Then, \( MT(n, \beta) \) satisfies the relationship (4.1) for some integer \( n \geq 1 \) if and only if \( MT(n+1, \beta) \) is attained at a unique value of \( k \).

It has been proved that, when \( \beta \neq 2^i \) for any integer \( i \geq 1 \), \( MT(n, \beta) \) is attained at a unique value of \( k \) (see Corollary 3.5 in Majumdar [1]). It then follows, by Corollary 4.1 above that, in such a case, \( MT(n, \beta) \) does not satisfy the relation (4.2).

Corollary 4.2: For \( \beta = 2^i \) (for some integer \( i \geq 1 \)), the relationship (4.1) cannot hold for all \( n \geq 1 \).

Proof: Let \( MT(n, \beta) \), \( MT(n+1, \beta) \) and \( MT(n+2, \beta) \) satisfy the relationship (4.1). By Lemma 4.1, \( MT(n+1, \beta) \) is attained at a unique point, say, \( k = K \). Then, by Corollary 2.1, \( MT(n+2, \beta) \) is attained at the points \( k = K, K+1 \), so that by Theorem 2.1,

\[
MT(n+3, \beta) - MT(n+2, \beta) = 2^{n-K+1}.
\]

This shows that, \( MT(n+1, \beta) \), \( MT(n+2, \beta) \) and \( MT(n+3, \beta) \) satisfy the relationship (4.2).

Lemma 4.2: Let \( \beta = 2^i \) (for some integer \( i \geq 1 \)). Then, \( MT(n+1, \beta) \) is attained at the unique point \( k = K \) (for some integer \( n \geq 1 \)) if and only if \( MT(K, \beta) \) is attained at a unique value of \( k \).

Proof: Let \( MT(n+1, \beta) \) be attained at the unique point \( k = K \). Then, by Lemma 4.1, \( MT(n+1, \beta) \) satisfies the relationship (4.1), so that by (2.3) and (2.4),

\[
MT(K+1, \beta) - MT(K, \beta) = 2[MT(K, \beta) - MT(K-1, \beta)],
\]

so that, \( MT(K, \beta) \) is attained at a unique value of \( k \). Next, let \( MT(K, \beta) \) be attained at a unique \( k \). Then, by Lemma
2.6, we can find $MT(n + 1, \beta)$ which is attained at the unique point $k = K$.

In Table 4.1 and Table 4.2, we give the values of $MT(n, \beta)$ for $1 \leq n \leq 19$, $\beta = 4, 8, 16$.

**Table 1.** Values of $MT(n, \beta)$ for $1 \leq n \leq 10$ and $\beta = 4, 8, 16$.
In each cell, the number in parenthesis gives the value(s) of $k$ at which $MT(n, \beta)$ is attained.

| $\alpha$ | $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----------|-----|---|---|---|---|---|---|---|---|---|----|
| 4        |     | 1 | 3 | 7 | 11| 19|27 |43 |59 |75 |107 |
|          |     | (0)| (0)| (0,1)| 1 | (1,2)| 2 | (2,3)| 3 |(4,5)|    |
| 8        |     | 1 | 3 | 7 | 15| 23|39 |55 |87 |119|183 |
|          |     | (0)| (0)| (0)| (0,1)| (1)| (1,2)| 2 |(2,3)| 3 |(3,4)|    |
| 16       |     | 1 | 3 | 7 | 15| 31|47 |79 |111|175|239 |
|          |     | (0)| (0)| (0)| (0)| (0,1)| (1)| (1,2)| 2 |(2,3)| 3 |    |

**Table 2.** Values of $MT(n, \beta)$ for $11 \leq n \leq 19$ and $\beta = 4, 8, 16$.
In each cell, the number in parenthesis gives the value(s) of $k$ at which $MT(n, \beta)$ is attained.

| $\alpha$ | $n$ | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|----------|-----|----|----|----|----|----|----|----|----|----|
| 4        |     | 139| 171|235 |299 |363 |427 |555 |683 |811 |
|          |     | (5,6)| (6)| (6,7)| (7,8)| (8,9)| (9)| (9,10)| (10,11)| (11,12)|    |
| 8        |     | 247| 311|439 |567 |695 |951 |1207|1463|1975|
|          |     | (4,5)| (5)| (5,6)| (6,7)| (7)| (7,8)| (8,9)| (9)| (9,10)|    |
| 16       |     | 367| 495|751 |1007|1263|1775|2287|2799|3823|
|          |     | (3,4)| (4)| (4,5)| (5,6)| (6)| (6,7)| (7,8)| (8)| (8,9)|    |

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