Domain Wall Fermions and chiral symmetry restoration rate.

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Domain Wall Fermions utilize an extra space time dimension to provide a method for restoring the regularization induced chiral symmetry breaking in lattice vector gauge theories even at finite lattice spacing. The breaking is restored at an exponential rate as the size of the extra dimension increases. As a precursor to lattice QCD studies the dependence of the restoration rate to the other parameters of the theory and, in particular, the lattice spacing is investigated in the context of the two flavor lattice Schwinger model.

A massive vector theory in $2n + 1$ dimensions with a mass term $m(s)$ that depends on the periodic $2n + 1$ dimension $s$ such that $m(s) = m_0$ for $1 \leq s \leq L_s/2$ and $m(s) = -m_0$ for $L_s/2 < s \leq L_s$, where $L_s$ is the size of the $2n + 1$ dimension, develops a Dirac fermion in $2n$ dimensions with chiral components separated and localized along the $2n + 1$ direction $s$ at $s = 0$ and $s = L_s/2$ [1]. Perturbative calculations indicate exponential localization along $s$ with the mixing of the chiral components vanishing exponentially fast with $L_s$. At $L_s = \infty$ the mixing is zero and the regularization leaves intact the chiral symmetry.

The $L_s = \infty$ case can be handled using the Overlap formalism [2]. Therefore the Overlap is an ideal regulator for vector theories. However, calculations using the Overlap involve diagonalization and determinant evaluation of matrices with size $\sim$ volume $\times$ volume. Because of this large scale dynamical QCD calculations with the Overlap are beyond the capacity of present day supercomputers (however, see [3] for a promising new development).

An alternative is to keep $L_s$ finite and view it as an extra parameter that controls the regularization induced chiral symmetry breaking. This formulation goes under the name of Domain Wall Fermions (DWF). Unlike Wilson or staggered fermions the chiral limit can be approached even at finite lattice spacing by increasing $L_s$. The theory can be simulated using standard Hybrid Monte Carlo techniques. However, before DWF can be used in dynamical simulations of QCD the following questions should be answered:

1) Is the regularization induced chiral symmetry breaking restored exponentially fast as $L_s$ increases in the full dynamical theory?

2) How does the functional behavior of the restoration and the restoration rate depend on the lattice spacing $a$, the fermion mass $m_f$ and the domain wall height $m_0$?

3) For the interesting $a$ and $m_f$ how large should $L_s$ be for the effects of the regularization induced chiral symmetry breaking to be negligible?

This work addresses these questions in the context of the $N_f = 2$ flavor lattice Schwinger model at fixed physical volume. For a detailed analysis see [4]. For some interesting work that also partially addresses these questions see [5]. A variant domain wall model is used [6][7] where $m(s) = m_0$ for $1 \leq s \leq L_s$ but the boundary conditions along the $2n+1$ direction are free. In this model the chiral components are localized at $s = 1$ and $s = L_s$ and as a result the decay region is twice as long for the same $L_s$. Also, an explicit coupling of the two chiral components at $s = 0$ and $s = L_s$ with strength $m_f$ is introduced to allow for linear control over the fermion mass. The action is the same as the one for a $2n + 1$ Wilson type fermion except that the gauge field “lives” only on the $2n$ dimensional lattice, the diagonal term is of the form $2n + 1 - m_0$ and the boundary conditions along the $2n + 1$ dimension are free.

An explicit calculation [8] of the free theory
propagator gives the mass of the lightest mode:

\[ m_{\text{eff}} = m_0 (2 - m_0) [m_f + (1 - m_0) \mu_l] \]

This strongly suggests the pattern for chiral symmetry restoration in the model. Recent calculations of the one loop correction result in a renormalization of \( m_0 \).

Two observables are used to probe the symmetries of the theory: the chiral condensate \( \langle \bar{\psi} \psi \rangle \) and the t’Hooft vertex \( \langle w \rangle \) where

\[
W = \prod_{i=1}^{N_f} \bar{\psi}_R^i \psi_L^i + \prod_{i=1}^{N_f} \bar{\psi}_L^i \psi_R^i.
\]

The chiral condensate is used to probe chiral symmetry. At \( m_f = 0 \) and \( L_s = \infty \), \( \langle \bar{\psi} \psi \rangle = 0 \). When \( L_s \) is finite the regularization breaks chiral symmetry and \( \langle \bar{\psi} \psi \rangle \neq 0 \). From perturbative calculations one expects that \( \langle \bar{\psi} \psi \rangle \) should approach zero exponentially fast with increasing \( L_s \). The t’Hooft vertex is used to probe the anomalously broken \( U(1) \) axial symmetry. At \( m_f = 0 \), \( L_s = \infty \) one should find that \( \langle w \rangle \neq 0 \).

In order to study the approach to the \( L_s \to \infty \) limit it is very useful to calculate the values of these observables at \( L_s = \infty \) using the Overlap formalism. In two dimensions this is not computationally demanding and can be done as in \cite{3}. The results, interesting in their own right, are shown in figures 1 and 2. In figure 1 \( \langle \bar{\psi} \psi \rangle / m_\gamma \) is plotted vs. \( m_f \) for \( L = 6, \ m_0 = 0.9 \) and \( \mu_l \) = 3.0. \( g_0 \) is the coupling constant \( \mu_l = g_0 L/\sqrt{\pi} \), \( m_\gamma = \mu \sqrt{N_f} \) is the photon mass, \( L \) is the lattice size in lattice units and \( l \) in physical units. The fits are to \( \langle \bar{\psi} \psi \rangle = A m_f^{\beta} \). Both fits have a \( \chi^2 \) per degree of freedom of about one. For \( m_f < 0.1 \)

\[
p = 0.996(3), \text{ while for } m_f > 0.1 \ p = 0.32(2).
\]

This is in agreement with the analytical solutions found in \cite{10}. In figure 2 \( < w > / m_\gamma^2 \) is plotted vs. \( m_f \) for the same parameters. The dotted lines are the \( m_f = 0 \) result \( \pm \) the statistical error. As expected at \( m_f = 0 \), \( \langle w \rangle \neq 0 \).

The finite \( L_s \) model is studied using a standard Hybrid Monte Carlo algorithm. In order to reveal the chiral symmetry breaking due to the DWF regularization, \( \langle \bar{\psi} \psi \rangle \) is calculated for various \( L_s \) at \( m_f = 0 \). It is plotted in figure 3 for fixed physical volume \( \mu_l = 3.0, \ m_0 = 0.9 \) and four lattice spacings \( \mu_l/L = \mu a \), with \( L = 6, 8, 10, 12 \) corresponding to the lines from top to bottom. The data is consistent with exponential decay indicated by the fitted lines (\( \chi^2 \) per degree of freedom \( \approx 1 \)). However, at the larger spacing \( L = 6 \) the data also fits well to a power law behavior. This is not so at the smallest spacing \( L = 12 \) (a power law fit has \( \chi^2 \) per degree of freedom \( \approx 31 \)).

![Figure 1. Chiral condensate vs. fermion mass.](image1)

![Figure 2. t’Hooft vertex vs. fermion mass.](image2)

![Figure 3. Chiral condensate vs. \( L_s \) for \( m_f = 0 \).](image3)
tial decay rate for $L_s$ up to some value and with a slower exponential decay rate for $L_s$ above that value. For the range of lattice spacings used, the inflection appeared at $L_s \approx 10$. Using a simple model, it was found in [4] that the first fast decay can be associated with restoration of chiral symmetry in the zero topological sector while the second slower decay can be associated with the regions of gauge field configuration space that connect the $q = 0$ and $q = \pm 1$ topological sectors.

The effects of the size of the lattice spacing $a$ to the two decays are apparent. The fast decay becomes faster as $a$ is decreased. The vanishing of $\langle \bar{\psi}\psi \rangle$ is consistent with a form $e^{-cL_s}$ with $e^{-c}$ being roughly a linear function of $a$. However, more data at smaller $a$ are needed before one can be confident that this is the correct scaling form. The second slower decay also becomes faster as $a$ is decreased and it differs less from the faster decay as $a$ becomes smaller. This behavior can also be understood using the simple model in [4].

For small but non zero $m_f$ the values of $\langle \bar{\psi}\psi \rangle$ and $\langle w \rangle$ are presented in figures 4 and 5 for two different lattice spacings set by $L = 4$ (lower curve) and $L = 10$ (upper curve) at $m_0 = 0.9$. The physical volume and $m_fL$ are fixed at $\mu L = 3.0$ and $m_fL = 2.0$. The fits are to $A + Be^{-cL_s}$. The dotted lines are the $L_s = \infty$ results ± the error. The cross is the coefficient $A$. The $L_s = \infty$ numbers are approached in a way that is consistent with exponential decay with a rate that becomes faster as the lattice spacing decreases. Finally, it was also found in [4] that the larger the fermion mass the sooner the $L_s = \infty$ value was approached (within a few percent at $L_s = 4 \sim 8$).

The next step is to carry out a similar investigation for dynamical QCD. Many of the characteristics of DWF found here are sufficiently generic so that one would expect that they will also be present in QCD. If it turns out that QCD at the presently accessible lattice spacings, volumes and quark masses has similar restoration rates as the ones found here, then DWF will indeed provide a powerful fermion discretization method.

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