A twistor sphere of generalized Kähler potentials on hyperkähler manifolds

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Abstract

We consider the generalized Kähler structures \((g, J_+, J_-)\) that arise on a hyperkähler manifold \((M, g, I, J, K)\) when we choose \(J_+\) and \(J_-\) from the twistor space of \(M\). We find a relation between semichiral and arctic superfields which can be used to determine the generalized Kähler potential for hyperkähler manifolds whose description in projective superspace is fully understood. We use this relation to determine an \(S^2\)-family of generalized Kähler potentials for Euclidean space and for the Eguchi-Hanson geometry. Cotangent bundles of Hermitian symmetric spaces constitute a class of hyperkähler manifolds where our method can be applied immediately since the necessary results from projective superspace are already available. As a non-trivial higher-dimensional example, we determine the generalized potential for \(T^*\mathbb{C}P^n\), which generalizes the Eguchi-Hanson result.

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1 Introduction

Hyperkähler manifolds admit various generalized Kähler structures. The corresponding generalized Kähler potentials can be used to reconstruct the hyperkähler geometry. These generalized potentials are in general quite different from the ordinary Kähler potential and thus provide a new way of studying hyperkähler geometry and finding hyperkähler metrics. To gain insight into this new way of looking at hyperkähler geometry, one first needs to study some examples. In this paper, we make use of the twistor space of hyperkähler manifolds to develop a general method for determining their generalized Kähler potentials and we also explicitly work out some examples.

In the next section, we review the relevant features of generalized Kähler geometry in its bihermitian formulation. This geometry involves two complex structures $J_+, J_-$ on a Riemannian manifold $(M, g)$ and can locally be described by a generalized Kähler potential. In this paper, we consider the case where the kernel of $[J_+, J_-]$ is trivial. Then the potential is defined as the generating function for a symplectomorphism between coordinates $(x_L, y_L)$ and $(x_R, y_R)$ that are holomorphic w.r.t. $J_+$ and $J_-$. Generalized Kähler geometry was initially found as the target space geometry of $2D N = (2, 2)$ supersymmetric sigma models, where the potential is the superspace Lagrangian and the coordinates $x_L, x_R$ describe semichiral superfields.

In section 3, we review aspects of hyperkähler geometry and its twistor space. We parametrize the twistor sphere of complex structures by a complex coordinate $\zeta$ and introduce holomorphic Darboux coordinates $\Upsilon(\zeta), \tilde{\Upsilon}(\zeta)$ for a certain holomorphic symplectic form. This construction is relevant for the projective superspace description of $2D N = (4, 4)$ sigma models, where $\Upsilon, \tilde{\Upsilon}$ are arctic superfields.

Using its twistor space, a hyperkähler manifold can be seen as a generalized Kähler manifold in various ways while keeping the metric fixed. In section 4, we consider a two-sphere of generalized Kähler structures on a hyperkähler manifold and express the coordinates $x_L, x_R, y_L, y_R$ in terms of $\Upsilon, \tilde{\Upsilon}$. This enables us to determine the generalized Kähler potential on a hyperkähler manifold if we can find the decomposition of the arctic superfields $\Upsilon, \tilde{\Upsilon}$ in terms of their $N = (2, 2)$ components, i.e. in terms of coordinates on $M$.

In section 5, we consider four-dimensional hyperkähler manifolds and explicitly determine the partial differential equations that the coordinates describing those arctic superfields have to fulfill. In section 6, we determine the potential for Euclidean space, where the differential equations for $\Upsilon, \tilde{\Upsilon}$ are easy to solve. In section 7, we look at the Eguchi-Hanson metric, where the relevant coordinates $\Upsilon, \tilde{\Upsilon}$ have been found previously in [3]. We give an explicit expression for the $S^2$-family of generalized Kähler potentials for this
geometry, which belongs to the family of gravitational instantons and is thus of interest to physicists.

The Eguchi-Hanson geometry lives on the cotangent bundle of $\mathbb{C}P^1$. For hyperkähler structures on cotangent bundles over arbitrary Kähler manifolds, projective superspace can be used to determine the coordinates $\Upsilon, \bar{\Upsilon}$. This has been done in particular for all Hermitian symmetric spaces. In section 8 we review this procedure and as a non-trivial higher-dimensional example, we use the results for $T^*\mathbb{C}P^n$ to determine its generalized Kähler potential, which generalizes the Eguchi-Hanson result.

In an appendix, we extend our results from section 4 and consider the full $S^2 \times S^2$-family of generalized Kähler structures on a hyperkähler manifold, i.e. we let both $J_+$ and $J_-$ be an arbitrary point on the twistor sphere of complex structures. We also give the explicit potential depending on two complex parameters $\zeta_+, \zeta_-$ for the simplest hyperkähler manifold, namely for Euclidean space.

## 2 The generalized Kähler potential

Generalized Kähler geometry first appeared in the study of 2D $\mathcal{N} = (2, 2)$ nonlinear $\sigma$-models [5] and was later rediscovered by mathematicians as a special case of generalized complex geometry [4]. In its bihermitian formulation, generalized Kähler geometry consists of two (integrable) complex structures $J_+, J_-$ on a Riemannian manifold $(M, g)$, where the metric is hermitian with respect to $J_+$ and $J_-$. Furthermore, the forms $\omega_\pm := g J_\pm$ have to fulfill

$$d c_\pm \omega_\pm + d c_\pm^c \omega_\pm = 0, \quad dd c_\pm \omega_\pm = 0, \quad (2.1)$$

where $d c_\pm = i(\partial_\pm - \bar{\partial}_\pm)$. This allows us to define the closed three-form $H := d c_\pm \omega_\pm = -d c_\pm^c \omega_\pm$, whose local two-form potential we denote by $B$ ($H = dB$). In general, $\omega_\pm$ is not closed and thus $(M, g, J_\pm)$ is not Kähler. In this paper, we will however consider the case where $H = 0$. Then $\partial_\pm \omega_\pm, \bar{\partial}_\pm \omega_\pm$ have to vanish separately, so $d \omega_\pm = 0$, i.e. $(M, g, J_\pm)$ is Kähler.

In [6] it was shown that like ordinary Kähler geometry, generalized Kähler geometry is locally described by a single function, the generalized Kähler potential. On a generalized Kähler manifold, one can define the Poisson structure $\sigma := [J_+, J_-]g^{-1}$ [8]. Here, we consider the case where $[J_+, J_-]$ is invertible and recall how the generalized Kähler potential is defined in this case [13]:

Inverting $\sigma$ gives

$$\Omega_G := \sigma^{-1} = g[J_+, J_-]^{-1}, \quad (2.2)$$
which is a real, closed and non-degenerate two-form that fulfills $J^T \Omega_G J = -\Omega_G$ [8], i.e. it is a real holomorphic symplectic form both w.r.t. $J_+$ and w.r.t. $J_-$. This means that $\Omega_G$ can be split into the sum of a $(2,0)$- and a $(0,2)$-form both w.r.t. $J_+$ and w.r.t. $J_-:
\Omega_G = \Omega_+^{(2,0)} + \Omega_+^{(0,2)} = \Omega_-^{(2,0)} + \Omega_-^{(0,2)},
\tag{2.3}
$ where $\bar{\partial}_\pm \Omega_\pm^{(2,0)} = 0$ and $\Omega_\pm^{(0,2)} = \Omega_\pm^{(2,0)}$ (here the complex conjugate is taken w.r.t. $J_+$ and $J_-$ respectively).

One then introduces Darboux coordinates $x^p_L$ and $y^p_L$, holomorphic w.r.t. $J_+$, for $\Omega_+^{(2,0)}$; and $x^p_R$ and $y^p_R$, holomorphic w.r.t. $J_-$, for $\Omega_-^{(2,0)}$ ($p = 1, ..., n$, where $\dim \mathbb{R} M = 4n$) [13]. Then

\begin{align*}
\Omega_G &= \Omega_+^{(2,0)} + \Omega_+^{(0,2)} = dx^p_L \wedge dy^p_L + dx^p_L \wedge d\bar{y}^p_L, \\
\Omega_G &= \Omega_-^{(2,0)} + \Omega_-^{(0,2)} = dx^p_R \wedge dy^p_R + dx^p_R \wedge d\bar{y}^p_R,
\tag{2.4}
\end{align*}

i.e. the coordinate transformation from $\{x_L, \bar{x}_L, y_L, \bar{y}_L\}$ to $\{x_R, \bar{x}_R, y_R, \bar{y}_R\}$ is a symplectomorphism (canonical transformation) preserving $\Omega_G$. It is thus described by a generating function $P(x_L, x_R, \bar{x}_L, \bar{y}_R)$ such that (omitting indices from now on)

\begin{align*}
\frac{\partial P}{\partial x_L} = y_L, \quad \frac{\partial P}{\partial x_R} = -y_R, \quad \frac{\partial P}{\partial \bar{x}_L} = \bar{y}_L, \quad \frac{\partial P}{\partial \bar{x}_R} = -\bar{y}_R.
\tag{2.5}
\end{align*}

This generating function is the generalized Kähler potential\footnote{We can choose to let the potential depend on other combinations of old and new coordinates as well. The potentials corresponding to the four different choices of variables are then related via Legendre transforms. In previous papers, the roles of $x_R$ and $y_R$ were interchanged. However, formulas in previous papers for reconstructing $g, J_+, J_-$ and $B$ from the potential remain unchanged when using our convention.} and can be used to locally reconstruct all the geometric data of generalized Kähler geometry [13], i.e. the two complex structures $J_+$, $J_-$, the metric $g$ and the $B$-field. It also turns out to be the superspace Lagrangian for the $\mathcal{N} = (2,2)$ $\sigma$-models that led to the discovery of generalized Kähler geometry [6].

3 Hyperkähler manifolds and their twistor spaces

In this paper, we consider generalized Kähler structures $(g, J_+, J_-)$ on a hyperkähler manifold $M$ and investigate their generalized Kähler potentials. For the choice of the two complex structures $J_+$ and $J_-$, we will make use of the twistor space $Z = M \times S^2$ of $M$.

Hyperkähler manifolds appear for instance as the target spaces for hypermultiplet scalars in four-dimensional nonlinear $\sigma$-Models with rigid $\mathcal{N} = 2$ supersymmetry on the
In geometric terms, they are described by the data \((M, g, I, J, K)\), where \(g\) is a Riemannian metric on \(M\) that is Kähler with respect to the three complex structures \(I, J, K\), which fulfill the quaternion algebra (i.e. \(IJ = K = -JI\)). In fact, there exists a whole two-sphere of complex structures on \(M\) with respect to which \(g\) is a Kähler metric, namely \((M, g, J = v_1I + v_2J + v_3K)\) is Kähler for each \((v_1, v_2, v_3) \in S^2\). Using (the inverse of) the stereographic projection, we parametrize this family of complex structures on \(M\) in a chart of \(S^2\) including the north-pole by a complex coordinate \(\zeta\):

\[
J(\zeta) := v_1(\zeta)I + v_2(\zeta)J + v_3(\zeta)K := \frac{1}{1 + \zeta\bar{\zeta}} [(1 - \zeta\bar{\zeta})I + (\zeta + \bar{\zeta})J + i(\bar{\zeta} - \zeta)K].
\] (3.6)

We define the complex two-forms

\[
\omega^{(2,0)} := \omega_2 + i\omega_3, \quad \omega^{(0,2)} := \omega_2 - i\omega_3;
\] (3.7)

where \(\omega_1 = gI, \omega_2 = gJ, \omega_3 = gK\) are the three Kähler forms. Then for each \(\zeta \in \mathbb{C}\)

\[
\Omega_H(\zeta) := \omega^{(2,0)} - 2\zeta \omega_1 - \zeta^2 \omega^{(0,2)}
\] (3.8)

turns out to be a holomorphic symplectic form with respect to the complex structure \(J(\zeta)\) \cite{2}. In particular, \(\omega^{(2,0)} = \Omega_H(\zeta = 0)\) is a \((2,0)\)-form w.r.t. \(I = J(\zeta = 0)\).

Starting from \(\zeta = 0\), we can locally find holomorphic Darboux coordinates \(Y^p(\zeta)\) and \(\tilde{Y}_p(\zeta)\) \((p = 1, ..., n, \text{where } \dim_{\mathbb{R}}M = 4n)\) for \(\Omega_H(\zeta)\) that are analytic in \(\zeta\) such that \cite{3}

\[
\Omega_H(\zeta) = i dY^p(\zeta) \wedge d\tilde{Y}_p(\zeta).
\] (3.9)

These canonical coordinates \(Y, \tilde{Y}(\zeta)\) for \(\Omega_H\) are crucial for the projective superspace formulation of \(\sigma\)-models with eight real supercharges\cite{3}, where they describe ”arctic” superfields. They have been determined for instance in \cite{3} for the Eguchi-Hanson metric and we will use them in this paper to determine the generalized Kähler potential for hyperkähler manifolds.

### 4 Gen. Kähler structures on hyperkähler manifolds

We want to transport the idea of a twistor space from hyperkähler to generalized Kähler geometry, namely we interpret a hyperkähler manifold \((M, g, I, J, K)\) as a generalized Kähler manifold \((M, g, J_+, J_-)\), where we fix the left complex structure \(J_+ = I\) and let

\[\text{If we define } \tilde{Y}(\zeta) := \tilde{Y}(-\frac{1}{\zeta}), \tilde{\tilde{Y}}(\zeta) := \tilde{\tilde{Y}}(-\frac{1}{\zeta}), \text{then } (Y, \tilde{Y}) \text{ and } (\tilde{Y}, \tilde{\tilde{Y}}) \text{ are related by a } \zeta^2\text{-twisted symplectomorphism whose generating function } f(Y, \tilde{Y}; \zeta) \text{ can be interpreted as the projective superspace Lagrangian } \cite{3}.\]
the right complex structure depend on $\zeta$: $J_\pm = J(\zeta)$ (see eq. (3.6)). So for a given hyperkähler manifold, we consider an $S^2$-family of generalized Kähler structures whose generalized Kähler potentials we now try to determine.

First, we need an explicit expression for the symplectic form $\Omega_G$ (eq. (2.2)), which now depends on $\zeta$. The anticommutator of two complex structures on a locally irreducible hyperkähler manifold is equal to a constant times the identity, $\{J_+, J_-\} = c \mathbb{1}$ (see, e.g., [9]). If $J_+ \neq \pm J_-$, then $|c| < 2$ and $\frac{1}{\sqrt{4-c^2}}[J_+, J_-]$ is another complex structure, so in particular it squares to $-\mathbb{1}$. Using this, we have

$$\Omega_G = g[J_+, J_-]^{-1} = -\frac{1}{4-c^2}g[J_+, J_-], \quad (4.10)$$

which in our case, where we have $c = -2v_1 = -2\frac{1-\bar{\zeta}}{1+\zeta}$ and $[J_+, J_-] = 2v_2K - 2v_3J$, gives

$$\Omega_G(\zeta) = -\frac{1}{2 - 2v_1^2} (v_2 \omega_3 - v_3 \omega_2) = -\frac{1 + \bar{\zeta}}{8 \zeta \bar{\zeta}} [(\zeta + \bar{\zeta}) \omega_3 - i(\bar{\zeta} - \zeta) \omega_2]. \quad (4.11)$$

This can be split into the sum of the holomorphic form $\Omega_+^{(2,0)} = i\bar{\zeta} \frac{1 + \zeta}{8 \zeta} \omega_+(0,2)$ and the antiholomorphic form $\Omega_-^{(0,2)} = -i\zeta \frac{1 + \zeta}{8 \zeta} \omega_-(0,2)$ with respect to $J_+$ (see equation (3.7)). Combining equations (3.8) and (3.9), we can choose the following Darboux coordinates for $\Omega_G(\zeta)$:

$$x_L^p = \Upsilon^p(\zeta = 0), \quad y_L^p = -\frac{1}{8 \zeta \bar{\zeta}} \bar{\Upsilon}^p(\zeta = 0);$$

$$\bar{x}_L^p = \bar{\Upsilon}^p(\zeta = 0), \quad \bar{y}_L^p = -\frac{1}{8 \zeta \bar{\zeta}} \bar{\Upsilon}^p(\zeta = 0). \quad (4.12)$$

With respect to $J_- = J(\zeta)$, $\Omega_G$ splits into the sum of $\Omega_-^{(2,0)} = i\bar{\zeta} \frac{1 + \zeta}{8 \zeta} \Omega_H(\zeta)$ and $\Omega_-^{(0,2)} = -i\zeta \frac{1}{8 \zeta} \bar{\Omega}_H(\zeta)$. Consequently, we can choose\footnote{We denote the complex conjugate of $\Upsilon(\zeta)$ by $\bar{\Upsilon} \equiv \bar{\Upsilon}(\bar{\zeta}) \equiv \bar{\Upsilon}(\zeta)$ which is not to be confused with the notation in [3], where $\bar{\Upsilon}$ is shorthand for $\bar{\Upsilon}(\zeta) = \bar{\Upsilon}(-\zeta^{-1})$.}

$$x_R^p = \Upsilon^p(\zeta), \quad y_R^p = -\frac{1}{8 \zeta \bar{\zeta}} \bar{\Upsilon}^p(\zeta);$$

$$\bar{x}_R^p = \bar{\Upsilon}^p(\zeta), \quad \bar{y}_R^p = -\frac{1}{8 \zeta \bar{\zeta}} \bar{\Upsilon}^p(\zeta). \quad (4.13)$$

We are thus able to express the coordinates $x_{L,R}$ and $y_{L,R}$ that describe semichiral superfields in $\mathcal{N} = (2, 2)$ models in terms of the coordinates $\Upsilon(\zeta), \bar{\Upsilon}(\zeta)$ describing arctic superfields in the projective superspace formulation of $\mathcal{N} = (4, 4)$ supersymmetric sigma models. This will enable us to determine the $\zeta$-dependent generalized Kähler potential for hyperkähler manifolds whose projective superspace description is known.
5 The four-dimensional case

In this section, we consider the four-dimensional case and explicitly determine the partial differential equations for \( \Upsilon(\zeta) \) and \( \tilde{\Upsilon}(\zeta) \) in order to be holomorphic w.r.t. \( J(\zeta) \) and to fulfill equation (3.9). A four-dimensional Kähler manifold \((M, g, J)\) is hyperkähler if and only if around each point there are holomorphic coordinates \((z, u)\) on \(M\) such that the Kähler potential \( K(z, u) \) fulfills the following Monge-Ampère equation [10]:

\[
K_{\overline{u}u}K_{z\overline{z}} - K_{u\overline{z}}K_{z\overline{u}} = 1. \tag{5.14}
\]

From a Kähler potential fulfilling this equation, we can construct the three Kähler forms:

\[
\omega_1 = -\frac{i}{2} \partial \overline{\partial} K, \\
\omega_2 = \frac{i}{2} (dz \wedge du - d\overline{z} \wedge d\overline{u}), \\
\omega_3 = \frac{1}{2} (dz \wedge du + d\overline{z} \wedge d\overline{u}). \tag{5.15}
\]

Together with the metric \(g\) whose line element is

\[
ds^2 = K_{u\overline{u}} du d\overline{u} + K_{z\overline{z}} dz d\overline{z} + K_{\overline{z}u} d\overline{z} du + K_{\overline{z}\overline{z}} d\overline{z} d\overline{u}, \tag{5.16}
\]

we get the three complex structures, where equation (5.14) ensures that \(J = g^{-1} \omega_2\) and \(K = g^{-1} \omega_3\) indeed square to \(-1\).

We find the following basis for the \((1, 0)\) forms w.r.t. \(J(\zeta)\):

\[
\theta^1 = dz - \zeta K_{u\overline{u}} du - \zeta K_{z\overline{z}} dz, \quad \theta^2 = du + \zeta K_{z\overline{u}} du + \zeta K_{\overline{z}\overline{z}} d\overline{z}. \tag{5.17}
\]

For \(\Upsilon, \tilde{\Upsilon}\) to be holomorphic w.r.t. \(J(\zeta)\), \(d\Upsilon(\zeta)\) and \(d\tilde{\Upsilon}(\zeta)\) must be linear combinations of \(\theta^1\) and \(\theta^2\) (here the differential does not act on \(\zeta\)):

\[
d\Upsilon = \frac{\partial \Upsilon}{\partial z} \theta^1 + \frac{\partial \Upsilon}{\partial u} \theta^2, \quad d\tilde{\Upsilon} = \frac{\partial \tilde{\Upsilon}}{\partial z} \theta^1 + \frac{\partial \tilde{\Upsilon}}{\partial u} \theta^2. \tag{5.18}
\]

Here the coefficients have been determined by comparing the \(dz\)- and \(du\)-terms on both sides. From equations (5.17) and (5.18), we get the requirement that both \(\Upsilon\) and \(\tilde{\Upsilon}\) have to fulfill the following two PDEs:

\[
\frac{\partial \Psi}{\partial \overline{z}} = \zeta \left( K_{\overline{z}u} \frac{\partial \Psi}{\partial u} - K_{u\overline{u}} \frac{\partial \Psi}{\partial \overline{z}} \right), \quad \frac{\partial \Psi}{\partial \overline{u}} = \zeta \left( K_{\overline{z}\overline{u}} \frac{\partial \Psi}{\partial u} - K_{u\overline{u}} \frac{\partial \Psi}{\partial \overline{z}} \right) \quad (\Psi = \Upsilon, \tilde{\Upsilon}). \tag{5.19}
\]

Furthermore, we find that

\[
\Omega_H(\zeta) = idz \wedge du + i\zeta \partial \overline{\partial} K + i\zeta^2 d\overline{z} \wedge d\overline{u} = i\theta^1 \wedge \theta^2, \tag{5.20}
\]

so using (5.18), we obtain that equation (3.9) corresponds to the requirement

\[
\frac{\partial \Upsilon}{\partial z} \frac{\partial \tilde{\Upsilon}}{\partial u} - \frac{\partial \Upsilon}{\partial u} \frac{\partial \tilde{\Upsilon}}{\partial z} = 1. \tag{5.21}
\]
We now use the relation between $x,y$ and $\Upsilon,\tilde{\Upsilon}$ derived in section 4 to determine the generalized Kähler potential for Euclidean space. Here the Kähler potential is given by

$$K = u\bar{u} + z\bar{z}, \quad (6.22)$$

which clearly fulfills equation (5.14). Assuming that $(z,u)$ are holomorphic coordinates w.r.t. $I$ and setting $\omega^{(2,0)} = idz \wedge du$, we get the complex structures as described in section 5. They are the differentials (pushforwards) of the left action of the imaginary basis quaternions $i,j,k$ on $\mathbb{H} \approx \mathbb{C}^2$, where we make the identification $(z,u) = (x_0 + ix_1, x_2 + ix_3) \mapsto x_0 + ix_1 + jx_2 + kx_3$.

$$\Upsilon(\zeta) = z - \zeta\bar{u}, \quad \tilde{\Upsilon}(\zeta) = u + \zeta\bar{z} \quad (6.23)$$

fulfill equations (5.19) and (5.21), i.e. they are holomorphic w.r.t. $J(\zeta)$ and satisfy equation (3.9). Using equations (4.12) and (4.13), we make the identifications

$$x_L = \Upsilon(\zeta = 0) = z, \quad x_R = \Upsilon(\zeta) \equiv \Upsilon \quad \text{and} \quad y_L = \frac{1 + \zeta\bar{\zeta}}{8\zeta} u, \quad y_R = \frac{1}{8\zeta}\tilde{\Upsilon}. \quad (6.24)$$

Solving for $y_L, y_R$ in terms of $x_L, x_R$, we get

$$y_L = \frac{1 + \zeta\bar{\zeta}}{8\zeta}(\bar{x}_L - \bar{x}_R), \quad y_R = \frac{1}{8\zeta}(1 + \zeta\bar{\zeta})\bar{x}_L - \bar{x}_R, \quad (6.25)$$

which leads (up to an additive constant) to the generating function (see equation (2.5))

$$P = -\frac{1}{8\zeta\bar{\zeta}} \left[ x_R\bar{x}_R + (1 + \zeta\bar{\zeta}) \cdot (x_L\bar{x}_L - x_L\bar{x}_R - \bar{x}_Lx_R) \right]. \quad (6.26)$$

This is the generalized Kähler potential for Euclidean space, where $J_+ = I$ and $J_-$ is an arbitrary point on the twistor-sphere of complex structures, $J_- \neq \pm I$. However, we notice that $P$ only involves the combination $\zeta\bar{\zeta}$, i.e. it only depends on the angle between $J_+$ and $J_-$ in the space spanned by the three complex structures $(I,J,K)$. Also $P$ turns out to be asymmetric between left- and right-coordinates. This can be resolved however, as there are various ambiguities in the generalized Kähler potential. For instance, we could distribute factors differently in (6.24) or even perform a more complicated symplectomorphism, going to new coordinates $x'_{L/R}, y'_{L/R}$. If we make the identifications

$$x'_L = i\sqrt{\frac{1 + \zeta\bar{\zeta}}{8\zeta}} z, \quad x'_R = i\sqrt{\frac{1}{8\zeta}}\Upsilon \quad \text{and} \quad y'_L = i\sqrt{\frac{1 + \zeta\bar{\zeta}}{8\zeta}} u, \quad y'_R = i\sqrt{\frac{1}{8\zeta}}\tilde{\Upsilon}, \quad (6.27)$$

\[5\text{We stick to the convention from previous papers and include the i-factor in the choice of } \omega^{(2,0)}. \]

This interchanges the complex structures $J$ and $K$, s.t. $J$ corresponds to left multiplication by $k$ and $K$ corresponds to left multiplication by $-j$.  

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the potential is left-right-symmetric. Furthermore, we can perform Legendre transforms and express the potential in terms of a different set of variables. If we use for instance \((6.27)\) and in addition exchange the roles of \(x'_R\) and \(y'_R\), we arrive at the potential

\[
P' = \sqrt{1 + \zeta \bar{\zeta} \cdot (x'_L y_R + x'_R y'_L)} + \sqrt{\zeta \bar{\zeta} \cdot (x'_L x'_L + y'_R y'_R)}. \tag{6.28}
\]

7 Example: Eguchi-Hanson geometry

The real function

\[
K = \sqrt{1 + 4u \bar{u}(1 + z \bar{z})^2} + \frac{1}{2} \log \left[ \frac{4u \bar{u}(1 + z \bar{z})^2}{(1 + \sqrt{1 + 4u \bar{u}(1 + z \bar{z})^2})^2} \right] \tag{7.29}
\]

in the two complex variables \(z, u\) fulfills the Monge-Ampère equation \((5.14)\). It thus defines a hyperkähler metric, where the Kähler forms are given by equation \((5.15)\). The first Kähler form takes the form

\[
\omega_1 = -\frac{i}{2} \frac{1 + z \bar{z}}{\sqrt{1 + 4u \bar{u}(1 + z \bar{z})^2}} \left[ (1 + z \bar{z}) du \wedge d \bar{u} + 2u \bar{z} dz \wedge d \bar{u} + 2z \bar{u} \bar{z} dz \wedge d \bar{u} + \frac{1}{(1 + z \bar{z})^3} + 4u \bar{u}) dz \wedge d \bar{z} \right], \tag{7.30}
\]

from which the metric can be read off. This is the well-known Eguchi-Hanson geometry.\(^6\) The holomorphic Darboux coordinates for \(\Omega_H(\zeta)\) (fulfilling equations \((5.19)\) and \((5.21)\)) can be chosen as \(^3\)

\[
\tilde{\Upsilon} = u + \zeta^2 \bar{z}^2 \bar{u} + \frac{\bar{z} \zeta}{1 + z \bar{z}} \sqrt{1 + 4u \bar{u}(1 + z \bar{z})^2},
\]

\[
\Upsilon = z - \frac{2u \zeta (1 + z \bar{z})^2}{1 + \sqrt{1 + 4u \bar{u}(1 + z \bar{z})^2} + 2u \bar{z} \zeta (1 + z \bar{z})}. \tag{7.31}
\]

We solve \(\Upsilon, \tilde{\Upsilon}(z, \bar{z}, u, \bar{u})\) for \(u\) and \(\bar{u}\) to get

\[
u(z, \bar{z}, \Upsilon, \tilde{\Upsilon}) = \frac{\zeta}{1 + z \bar{z}} \cdot \frac{(\bar{z} - \tilde{\Upsilon})(1 + \Upsilon \bar{z})}{\zeta \bar{\zeta}(1 + \Upsilon \bar{z})(1 + \Upsilon \bar{z}) - (z - \Upsilon)(\bar{z} - \tilde{\Upsilon})}, \tag{7.32}
\]

and its complex conjugate. Using this and the identifications derived in section 4 (equation \((6.24)\)), we get \(y_L(x_L, x_R)\). We then integrate \(y_L(x_L, x_R)\) w.r.t. \(x_L\) to get the generalized

\(^6\)Setting \(u = \frac{1}{2}w^2, \quad z = \frac{\bar{z}}{w}\) and \(r := \sqrt{u \bar{u} + z \bar{z}}\) gives the familiar Kähler potential \(K = \sqrt{1 + r^4} + \log_{1 + \sqrt{1 + r^4}}\) for the Eguchi-Hanson metric.\(^3\)
Kähler potential up to a possible additive term that is independent of $x_L$:

$$P = \int y_L(x_L, x_R) dx_L = -\frac{1}{8} \frac{1 + \zeta \bar{\zeta}}{\zeta \bar{\zeta}} \int u(z, \bar{\Upsilon}) dz$$

(7.33)

Plugging $u(z = x_L, \bar{\Upsilon} = x_R)$ into $\bar{\Upsilon}(z = x_L, u)$ (equation (7.31)) gives

$$y_R(x_L, x_R) = -\frac{1}{8} \frac{1 + \zeta \bar{\zeta} \cdot \bar{x}_L - (1 - \zeta \bar{\zeta} \cdot x_L x_R) x_R}{\zeta \bar{\zeta} (1 + x_L x_R)(1 + x_L x_R) - (x_L - x_R)(\bar{x}_L - \bar{x}_R)}$$

(7.34)

which is indeed equal to $-\frac{\partial P}{\partial x_R}$. $P$ is real, so $\frac{\partial P}{\partial x_L} = \bar{y}_L$ and $\frac{\partial P}{\partial x_R} = -\bar{y}_R$ are also fulfilled and thus equation (7.33) gives indeed the $\zeta$-dependent generalized Kähler potential for the Eguchi-Hanson geometry:

$$P(x_L, \bar{x}_L, x_R, \bar{x}_R) = -\frac{1}{8} \frac{1 + |x_L|^2}{\zeta \bar{\zeta} \cdot |1 + x_L x_R|^2 - |x_L - x_R|^2}$$

(7.35)

Again, the generalized Kähler potential turns out to depend only on the combination $\zeta \bar{\zeta}$, i.e. on the angle between $J_+$ and $J_-$. Of course, there are again many ambiguities in the potential, but (7.35) seems to be already in its simplest form.

8 Hyperkähler structures on cotangent bundles of Kähler manifolds and projective superspace

The target space of 4D $\mathcal{N} = 2$ sigma models is constrained to be a hyperkähler manifold [1]. This corresponds to 2D $\mathcal{N} = (4, 4)$ sigma models without $B$-field [1]. Projective superspace provides methods to construct such models and for a large class of examples it can be used to extract the arctic superfields $\Upsilon$ and $\bar{\Upsilon}$ [18,20,21] (see [17] for a review) that we need in order to determine the generalized Kähler potential of the hyperkähler target space using the method derived in chapter 4. This has been done in particular for the hyperkähler structure on cotangent bundles of Hermitian symmetric spaces ([18],[22]). As an example, we use the results from [18],[20],[21] to determine the generalized Kähler potential for $T^*\mathbb{C}P^n = T^*(SU(n+1)/U(n))$. The special case $n = 1$ then corresponds to the Eguchi-Hanson geometry that we considered in the last section.

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7For vanishing $B$-field, all the results from 4D $\mathcal{N} = 2$ projective superspace can be immediately transferred to 2D $\mathcal{N} = (4, 4)$ projective superspace. Actually the results that we are using only depend on the target space geometry, not on the number of space-time dimensions.
It is a well-known fact that a hyperkähler metric exists on (some open subset of) the cotangent bundle of every Kähler manifold \( M \) [15], [16], [19]. In projective superspace, this corresponds to models where the projective superspace Lagrangian \( f(\Upsilon, \bar{\Upsilon}; \zeta) \) (see footnote [3]) does not explicitly depend on \( \zeta \) [18]. To obtain the coordinates \( \Upsilon \) and \( \bar{\Upsilon} \) for \( T^*M \), one takes the \( N = (4, 4) \) projective superspace Lagrangian \( f(\Upsilon, \bar{\Upsilon}) \) to be the Kähler potential \( K(\phi, \bar{\phi}) \) of the base space \( M \), where the arctic and antarctic superfields \( \Upsilon, \bar{\Upsilon} \) replace the chiral and antichiral superfields \( \phi, \bar{\phi} \) of the \( N = (2, 2) \) description, i.e. the holomorphic coordinates on \( M \). In \( N = (2, 2) \) components, \( \Upsilon \) decomposes as

\[
\Upsilon = \phi + \zeta \Sigma + \sum_{j=2}^{\infty} \zeta^j X_j, \tag{8.36}
\]

where \( \phi \) is a chiral superfield describing the coordinates on \( M \), \( \Sigma \) is a complex linear superfield describing coordinates in the fiber of the tangent bundle \( TM \) and all higher order terms are unconstrained auxiliary superfields [19], [22]. Solving the algebraic equations of motion for the auxiliary superfields yields \( \Upsilon \), and thus the Lagrangian in terms of the \( N = (2, 2) \) superfields \( (\phi, \Sigma) \) and the auxiliary complex variable \( \zeta \). Integrating out \( \zeta \) and dualizing\(^\text{8} \) the action, i.e. performing a Legendre transform replacing the complex linear superfields \( \Sigma \) by chiral superfields \( \psi \), gives the transformation \( \Sigma(\psi) \) from which we obtain \( \Upsilon(\phi, \psi) \). Here, \( \psi \) describes coordinates in the fiber of the cotangent bundle \( T^*M \). \( \bar{\Upsilon}(\phi, \psi) \) can then be obtained from \( f \) and \( \Upsilon(\phi, \psi) \) via \( \bar{\Upsilon} = \zeta \frac{\partial f}{\partial \Upsilon} \) [3]. One can also read off the ordinary Kähler potential of \( T^*M \) from the dualized action [19], [22].

### 8.1 Generalized Kähler potential for \( T^*\mathbb{C}P^n \)

The crucial step in the above procedure is to eliminate the infinite tower of unconstrained auxiliary \( N = (2, 2) \) superfields using their algebraic equations of motion. This has been done for instance in [20] for \( T^*\mathbb{C}P^n \).

For the projective sigma model with target space \( T^*\mathbb{C}P^n \), we take the projective superspace Lagrangian \( f \) to be the Kähler potential of the Fubini-Study metric on \( \mathbb{C}P^n \):

\[
f(\Upsilon^i(\zeta), \bar{\Upsilon}^i(\zeta)) = a^2 \log \left( 1 + \frac{\Upsilon^j \bar{\Upsilon}^j}{a^2} \right). \tag{8.37}
\]

Here, \( a \) is a real parameter. The equations of motion for the auxiliary superfields have been

\(^{8}\)The duality between chiral \( \psi \) and complex linear superfields \( \Sigma \) is just an ordinary coordinate transformation that does not change the target space geometry.
solved in \[20\]. This gives \( \Upsilon \) in terms of chiral and complex linear \( \mathcal{N} = (2, 2) \) superfields\(^9\):

\[
\Upsilon^i = z^i + \zeta \frac{\Sigma^j}{1 - \frac{z^k \Sigma^k}{a^2 + z^k z^k}}. \tag{8.38}
\]

Here, we change notation and let \( z \equiv \phi \) parametrize the base space and \( u \equiv \psi \) the fibers of the cotangent bundle.

Dualizing the action of the sigma model to go from complex linear coordinates \( \Sigma \) to chiral coordinates \( u \) gives the equations\(^10\)

\[
u_i = -\frac{g_{ij} \Sigma^j}{1 - \frac{g_{ij} \Sigma^j}{a^2}}, \quad \bar{u}_i = -\frac{g_{ij} \bar{\Sigma}^j}{1 - \frac{g_{ij} \bar{\Sigma}^j}{a^2}}; \tag{8.39}
\]

which have to be solved for the old coordinates \( \Sigma, \bar{\Sigma} \) in terms of \( u, \bar{u} \). Here \( g_{ij} \) is

\[
 g_{ij} = \frac{a^2 \delta_{ij}}{a^2 + z^k \bar{z}^k} - \frac{a^2 \bar{z}^i z^j}{(a^2 + z^l \bar{z}^l)^2}, \tag{8.40}
\]

the Fubini-Study metric on \( \mathbb{C}P^n \), and \( g^{ij} \) is its inverse. We find the following solution:

\[
\Sigma^i = -\frac{2 \bar{u}_j g^{ij}}{1 + \sqrt{1 + 4 \frac{g^{kl} u_k \bar{u}_l}{a^2}}}, \quad \bar{\Sigma}^i = -\frac{2 g_{ij} u_j}{1 + \sqrt{1 + 4 \frac{g^{kl} u_k \bar{u}_l}{a^2}}}. \tag{8.41}
\]

Plugging this into \(8.38\) gives the arctic superfields \( \Upsilon \) in terms of chiral \( \mathcal{N} = (2, 2) \) superfields \( z \) and \( u \):

\[
\Upsilon^i = z^i - \zeta \frac{2 \bar{u}_j g^{ij}}{1 + \sqrt{1 + 4 \frac{g^{kl} u_k \bar{u}_l}{a^2}} + 2 \zeta \frac{g^{i\bar{l}} \bar{z}^l \bar{u}_l}{a^2 + z^m \bar{z}^m}}. \tag{8.42}
\]

Together with

\[
\tilde{\Upsilon}^i = \zeta \frac{\partial f}{\partial \Upsilon^i} = \frac{\zeta \tilde{\Upsilon}^i}{1 + \frac{\Upsilon^i \bar{\Upsilon}^j}{a^2}}, \tag{8.43}
\]

this is all the information we need to determine the generalized Kähler potential for \( T^*\mathbb{C}P^n \) using the identifications found in section \[4\].

Solving \(8.42\) and its complex conjugate for \( u \) and \( \bar{u} \) gives \( u(z, \bar{\Upsilon}) \):

\[
u_i = \zeta \cdot \frac{a^2 (a^2 + z^k \bar{z}^k)(a^2 + z^l \bar{z}^l)}{a^2 \zeta^2 (a^2 + z^k \bar{z}^k)(a^2 + z^l \bar{z}^l) - (z^i - \Upsilon^i) g_{\bar{m} \bar{l}} (z^\bar{m} - \bar{\Upsilon}^\bar{m}) (a^2 + z^k \bar{z}^k)^2}. \tag{8.44}
\]

---

\(^9\)This result was already obtained in the preparation of \[19\] and later independently derived and first published in \[20\].

\(^{10}\)Repeated indices are always summed over \(1, \ldots, n\) and the metric is always written out explicitly, i.e. we never use it to raise or lower indices.
Integrating this with respect to $z^i$ gives:

$$\int u_i dz^i = \frac{\zeta a^2}{1 + \zeta} \log \frac{a^2 + z^T \bar{\Upsilon}}{(a^2 + z^T \Upsilon) - (\bar{z} - \Upsilon)^T g(\bar{z} - \bar{\Upsilon})(a^2 + z^T \bar{z})^2}.$$  \hfill (8.45)

Here, no sum is implied on the left-hand side and on the right-hand side we use vector notation ($z := (z^1, ..., z^n)^T$, etc.) and $g := (g_{ij})_{1 \leq i, j \leq n}$. So, up to an additive term $c(x_R, \bar{x}_R)$, the generalized Kähler potential for $T^* \mathbb{C} P^n$ is

$$P = -\frac{a^2}{8} \log \frac{a^2 + x_L^T \bar{x}_L}{(a^2 + x_L^T x_R) - (x_L - x_R)^T g(x_L - \bar{x}_R)(a^2 + x_L^T \bar{x}_L)^2}.$$  \hfill (8.46)

For $n = 1$ and $a = 1$, we have $g_{zz} = \frac{1}{(1 + z^2)^2}$ and all the results from section 7 are reproduced. Therefore, we assume that $c(x_R, \bar{x}_R)$ can be set to zero.

9 Discussion

There is an increasing number of examples, most notably among Hermitian symmetric spaces, where the decomposition of the $\mathcal{N} = (4,4)$ artctic superfields $\Upsilon, \bar{\Upsilon}$ in terms of their $\mathcal{N} = (2,2)$ components $(z, u)$ has been determined. In these cases, one can apply the methods developed in this paper to determine more examples of generalized Kähler potentials on hyperkähler manifolds. Having whole classes of manifo lds available for our analysis, one could try to find more general statements about the generalized Kähler potential in the case of hyperkähler manifolds.

The Eguchi-Hanson geometry is one of the hyperkähler manifolds that can be obtained from the generalized Legendre transform construction in [2] (generalized T-duality). The manifolds stemming from that construction are $4n$-dimensional hyperkähler manifolds admitting $n$ commuting tri-holomorphic killing vectors. They are called toric hyperkähler manifolds and have been classified in [14]. It should be possible to determine the relevant coordinates $\Upsilon(z, u; \zeta)$ and $\bar{\Upsilon}(z, u; \bar{\zeta})$ for toric hyperkähler manifolds. For four-dimensional toric hyperkähler manifolds, [11] gives a formula for the generalized Kähler potential as a certain threefold Legendre transform in the special case $\zeta \bar{\zeta} = 1$. One could compare this construction with our results at least for the examples given in this paper or try to relate the two methods in general for four-dimensional toric hyperkähler manifolds. As a further explicit example, one could for instance consider the Taub-NUT geometry and determine its generalized Kähler potential.

The generalized Kähler potential for the Eguchi-Hanson geometry can also be obtained from a generalized quotient of Euclidean 8-dimensional space by a $U(1)$-isometry and in this setting turns out to be exactly (7.35) as well [12].
2D $\mathcal{N} = (2, 2)$ sigma models have a target space that is a generalized Kähler manifold and in general, they are described by chiral, twisted chiral and semichiral superfields. The models with hyperkähler target space and $J_+ \neq \pm J_-$ are described purely in terms of semichiral superfields. These models do not in general admit off-shell $\mathcal{N} = (4, 4)$ supersymmetry, since it was shown in \cite{23} and \cite{24} that a sigma model parametrized by semichiral fields can only be extended to off-shell $\mathcal{N} = (4, 4)$ supersymmetry if the target space is $4n$-dimensional with $n > 1$. However, they are always dual to models with $\mathcal{N} = (4, 4)$ supersymmetry that are parametrized by chiral and twisted chiral superfields \cite{11}. The exact relation between the $\mathcal{N} = (4, 4)$ sigma models described by chiral and twisted chiral superfields, and their dual semichiral models will be described in \cite{25}.

The relation between the coordinates $x_{L/R}, y_{L/R}$ and $\Upsilon, \bar{\Upsilon}$ has been obtained in this paper from a purely differential geometric approach. $x_{L/R}, y_{L/R}$ describe left- and right-semichiral superfields in $2D \mathcal{N} = (2, 2)$ sigma models. For a target space that is hyperkähler, these models are dual to models with $\mathcal{N} = (4, 4)$ supersymmetry that are parametrized by chiral and twisted chiral superfields. The coordinates $\Upsilon(\zeta), \bar{\Upsilon}(\zeta)$ however describe arctic superfields in $\mathcal{N} = (4, 4)$ sigma models in projective superspace. The field theoretical interpretation and understanding of this relation between arctic $\mathcal{N} = (4, 4)$ models and the semichiral models that are dual to $\mathcal{N} = (4, 4)$ models remains an open problem. The complex coordinate $\zeta$ is an auxiliary variable that gets integrated out in projective superspace, but for ordinary superspace it is just a constant parameter. Thus in our relation, an arctic model corresponds to a two-sphere (or more precisely to a cylinder) of presumably equivalent semichiral models.

In this paper, we mainly focused on the special case, where the bihermitian structure $(J_+, J_-)$ only depends on one complex parameter $\zeta$ and established a relation to projective superspace. In the appendix, we show that the results from section 4 can be generalized to the full $S^2 \times S^2$-family of generalized complex structures on a hyperkähler manifold parametrized by two complex parameters $\zeta_+$ and $\zeta_-$. Many hints point towards an intimate relation of this formulation to doubly-projective superspace \cite{26,27}.

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A Appendix: \( S^2 \times S^2 \)-family of generalized complex structures

In this appendix, we generalize our results from section \[4\] Instead of fixing one complex structure, we can also let both \( J_+ \) and \( J_- \) depend on an individual complex coordinate and thus consider an \( S^2 \times S^2 \)-family of generalized complex structures \((M, g, J_+, J_-)\) on a given hyperkähler manifold \((M, g, I, J, K)\). We parametrize vectors \( \vec{u}, \vec{v} \in S^2 \setminus \{-1, 0, 0\} \) by complex coordinates \( \zeta_+, \zeta_- \) like in equation \((3.6)\) and define

\[
J_+ := J(\zeta_+) = u_1 I + u_2 J + u_3 K,
J_- := J(\zeta_-) = v_1 I + v_2 J + v_3 K.
\]  
(A.1)

The anticommutator depends only on the angle \( \theta \) between \( \vec{u} \) and \( \vec{v} \):

\[
\{J_+, J_-\} = -2(\vec{u} \cdot \vec{v}) = -2 \cos \theta.
\]  
(A.2)

The commutator turns out to be perpendicular to \( J_+ \) and \( J_- \) in the space spanned by \((I, J, K)\):

\[
[J_+, J_-] = 2(u_2 v_3 - u_3 v_2)I - 2(u_1 v_3 - u_3 v_1)J + 2(u_1 v_2 - u_2 v_1)K = 2(\vec{u} \times \vec{v}) \cdot (I, J, K)^T.
\]  
(A.3)

In order to determine the coordinates \( x_R, y_R \), we need to split \( \Omega_G \) into a \((2, 0)\)- and a \((0, 2)\)-form w.r.t. \( J_- \). Indeed, we find that

\[
g[J_+, J_-] = \frac{i}{(1 + \zeta_+ \zeta_-)^2} \left( (\bar{a}(\zeta_-) \cdot \vec{u}) \Omega_H(\zeta_-) - (\bar{a}(\zeta_-) \cdot \vec{u}) \overline{\Omega_H(\zeta_-)} \right),
\]  
(A.4)

where \( \bar{a}(\zeta) = (-2\zeta, 1 - \zeta^2, i(1 + \zeta^2))^T \), i.e. \( \Omega_H(\zeta) = \bar{a}(\zeta) \cdot \vec{w} \) (see eq. \((3.8)\)). So we find

\[
\Omega_G = -\frac{1}{4 - 4(\vec{u} \cdot \vec{v})^2} g[J_+, J_-] = \Omega_{-,(2,0)}^+ + \Omega_{-,(0,2)}^-,
\]  
(A.5)

where

\[
\Omega_{-,(2,0)}^- = -\frac{i (\bar{a}(\zeta_-) \cdot \vec{u})}{4 \sin^2 \theta (1 + \zeta_- \zeta_-^2)} \Omega_H(\zeta_-) \equiv -i c_- \Omega_H(\zeta_-),
\Omega_{-,(0,2)}^- = \overline{\Omega_{-,(2,0)}^+}.
\]  
(A.6)

Thus knowing that \( \Omega_H(\zeta_-) = id\tilde{Y}^p(\zeta_-) \wedge d\tilde{Y}_p(\zeta_-) \), we can choose (omitting indices)

\[
x_R = \tilde{Y}(\zeta_-), \quad y_R = c_- \tilde{Y}(\zeta_-).
\]  
(A.7)

to get \( \Omega_G = dx_R \wedge dy_R + dx_R \wedge dy_R \).

Exchanging the roles of \( \vec{u}, \vec{v} \) and \( \zeta_+, \zeta_- \) respectively and considering the antisymmetry of \([J_+, J_-]\), we get the following splitting w.r.t. \( J_+ \):

\[
\Omega_{+,(2,0)}^+ = \frac{i (\bar{a}(\zeta_+) \cdot \vec{v})}{4 \sin^2 \theta (1 + \zeta_+ \zeta_+^2)} \Omega_H(\zeta_+) \equiv -i c_+ \Omega_H(\zeta_+), \quad \Omega_{+,(0,2)}^- = \overline{\Omega_{+,(2,0)}^+},
\]  
(A.8)

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which allows us to choose
\[ x_L = \Upsilon(\zeta_+), \quad y_L = c_+ \tilde{\Upsilon}(\zeta_+). \]  
(A.9)

The constants \( c_+, c_-(\zeta_+, \zeta_-) \) can be written as
\[ c_+ = \frac{1 + \zeta_-\tilde{\zeta}_-}{8(1 + \zeta_+\tilde{\zeta}_-)(\zeta_+ - \zeta_-)}, \quad c_- = \frac{1 + \zeta_+\tilde{\zeta}_+}{8(1 + \zeta_+\tilde{\zeta}_-)(\zeta_+ - \zeta_-)}. \]  
(A.10)

We see that by exchanging \( \zeta_+ \) with \( \zeta_- \), we exchange \( x_L \) with \( x_R \) and \( y_L \) with \( -y_R \). In the special case \( \zeta_+ = 0 \) (i.e. \( \vec{u} = (1, 0, 0) \)) and \( \zeta_- = \zeta \), we have \( \sin^2 \theta = \frac{4\zeta \tilde{\zeta}}{(1 + \zeta \tilde{\zeta})^2} \) and (A.7), (A.9) reduce to the results (4.12), (4.13) from section 4.

Using (A.7) and (A.9), we can now also determine an \( S^2 \times S^2 \)-family of generalized Kähler potentials \( P_{\zeta_+\zeta_-} \) for hyperkähler manifolds. For Euclidean space, we find the \( \zeta_+ \)- and \( \zeta_- \)-dependent generalized potential to be
\[ P = -\frac{1}{8(\zeta_+ - \zeta_-)(\zeta_+ - \zeta_-)} \left[ (1 + \zeta_-\tilde{\zeta}_-)x_L\bar{x}_L + (1 + \zeta_+\tilde{\zeta}_+)x_R\bar{x}_R \right. \]
\[ - (1 + \zeta_+\tilde{\zeta}_+)(1 + \zeta_-\tilde{\zeta}_-) \left( \frac{x_L\bar{x}_R}{1 + \zeta_+\tilde{\zeta}_+} + \frac{\bar{x}_L x_R}{1 + \zeta_-\tilde{\zeta}_-} \right) \].  
(A.11)

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