Abstract

We compute the leading behaviour of the quark anti-quark potential from a generalized Nambu-Goto action associated with a curved space-time having an "extra dimension". The extra dimension can be the radial coordinate in the AdS/CFT correspondence, the Liouville field in Polyakov’s approach, or an internal dimension in MQCD. In particular, we derive the condition for confinement, and in the case it occurs we find the string tension and the correction to the linear potential.

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1 Introduction

Recently, the idea of presenting the Wilson loop of non-abelian gauge theories in terms of $< e^{-S}>$, where $S$ is a string worldsheet area, has undergone a Renaissance period. This type of construction of the quark anti–quark potential has emerged mainly in the framework of the novel gravity/gauge field duality [1] but also in the context of the M-theory description of QCD (MQCD) [2, 3, 4] and in Polyakov’s Liouville approach [5].

In the gravity/gauge duality approach, physical quantities of the boundary gauge theory are computed in terms of the bulk gravitational properties. In particular, the string between the quark anti–quark pair is not confined to the four dimensional boundary but rather stretches inside the five dimensional $AdS_5$ part of the ten dimensional space-time. A non-flat five dimensional space is also the picture that Polyakov draws for the non-critical string proposed as a solution of the loop equation originating from non–abelian gauge dynamics. In MQCD the QCD string translates into a membrane M2 ending on the five-brane M5 which describes the super YM (or super QCD) degrees of freedom. One of the M2 coordinates is along a trajectory embedded in the coordinates transverse to the four dimensional space time. Thus, a common concept invoked in these ”modern” calculations is the fact that the Wilson loop is a boundary of the string world sheet which is embedded in a higher (than four) dimensional space-time.

The space-time metric that associates with these three setups is a diagonal one and is a function of only one coordinate. It is the fifth coordinate of the $AdS_5$ (or its analogs in the non-extremal cases), the fifth direction of the M5 brane in MQCD, and the Liouville coordinate in Polyakov’s approach. Denoting this coordinate by $s$, the metric takes the following generic form

$$ds^2 = G_{MN}dx^M dx^N = -G_{00}(s)dt^2 + G_{x||x||}(s)dx_{||}^2 + G_{ss}(s)ds^2 + G_{xTxT}(s)dx_T^2$$

where $x_{||}$ are the $R^3$ ordinary space coordinates (or possible generalization to p-brane space coordinates in the gravity/gauge duality approach) and $x_T$ are the coordinates transverse to the five dimensional space. This case of three dimensional space can be easily generalized to p dimensional space with $x_T$ being the coordinates of the $8 - p$ dimensional transverse space. Upon choosing the world sheet coordinates $\sigma = x$ and $\tau = t$ and assuming translation invariance along $t$, the Nambu–Goto string action takes the form

$$S = \int d\sigma d\tau \sqrt{\text{det}[\partial_{\alpha} X^M \partial_{\beta} X^N G_{MN}]}$$

$$= T \cdot \int dx \sqrt{G_{00}(s(x))G_{x||x||}(s(x)) + G_{00}(s(x))G_{ss}(s(x))(\partial_s x)^2}$$

(1)
With $T$ the total time of the Wilson loop rectangle. In spatial Wilson loop calculations (see for example [10]) the two sides of the Wilson rectangle are taken to be along space directions $(x, x')$. In these cases it is convenient to choose $\tau = x'_\parallel$ and therefore $G_{00}$ has to be replaced with $G_{x_\parallel x'_\parallel}$, and $T$ by $L'$, the total length along $x'_\parallel$ of the Wilson loop rectangle. For convenience we define now

$$f^2(s(x)) \equiv G_{00}(s(x))G_{x_\parallel x'_\parallel}(s(x)) \quad (3)$$

$$g^2(s(x)) \equiv G_{00}(s(x))G_{ss}(s(x)) \quad (4)$$

so that the Nambu–Goto action reads

$$S = T \cdot \int dx \sqrt{f^2(s(x)) + g^2(s(x))(\partial_x s)^2} \quad (5)$$

Naturally, this metric should be positive, and we can assume that the functions $f(s)$ and $g(s)$ are real and non-negative.

To get familiarized with this form of the action we write now the form that $f^2(s(x))$ and $g^2(s(x))$ take in certain examples.

- In the original AdS/CFT case [1], it is customary to use $U$ instead of $s$. The functions are

$$f^2(U(x)) = (2\pi)^{-2}(U/R_{AdS})^4$$

$$g^2(U(x)) = (2\pi)^{-2}$$

where $R_{AdS}^4 = 4\pi g N$.

- In the supergravity setup corresponding to the "pure YM case" [10] one finds

$$f^2(U(x)) = (2\pi)^{-2}(U/R_{AdS})^4$$

$$g^2(U(x)) = (2\pi)^{-2}(1 - (U_T/U)^4)^{-1}$$

with $U_T$ related to the energy density.

- In the near extremal AdS solution corresponding to field theory at finite temperature [8],

$$f^2(U(x)) = (2\pi)^{-2}(U/R_{AdS})^4(1 - (U_T/U)^4)$$

$$g^2(U(x)) = (2\pi)^{-2}$$

when $U_T/(\pi R_{AdS}^2)$ is the Hawking temperature.
In the "pure YM" theory corresponding to the supergravity solution of rotating branes [11], the functions are (in M theory units)

\[ f^2(U(x)) = C \frac{U^6}{U_0^4} \Delta \]
\[ g^2(U(x)) = \frac{CU^2}{1 - a^4/U^4 - U_0^6/U^6} \]

where \( \Delta = 1 - a^4 \cos^2 \theta/U^4 \), \( a \) parameterizes the angular momentum of the brane (whose rotation is limited to a single plane), \( U_0 \) is the location of the horizon, \( \theta \) is a coordinate of the internal space (which is asymptotically \( S^4 \)), and \( C \) is a constant with the correct dimensions.

For the \( SU(2) \) case of MQCD, it was realized [4] that

\[ f^2(s(x)) = 8 \zeta \cosh(s/R) \]
\[ g^2(s(x)) = 8 \zeta \cosh(s/R) \]

with \( R \sim \Lambda^{-1}_{QCD} \) (the radius of the 11-th dimension) and \( \zeta \sim \Lambda^4_{QCD} \) parameterizing the M-theory curve.

In Polyakov’s approach, the fifth coordinate is the Liouville field \( \phi \). In that case [5]

\[ f^2(\phi(x)) = a^4(\phi) \]
\[ g^2(\phi(x)) = a^2(\phi) \]

where \( a(\phi) \) is determined by conformal invariance.

In this note we derive the quark anti–quark potential, namely the Wilson loop, that associates with the Nambu–Goto action (5). Our analysis is based on the classical equations of motion and does not include quantum fluctuations [12]. Our main result, which is stated below in a rigorous way, is that (assuming without loss of generality that \( f(s) \) has a minimum or \( g(s) \) diverges at \( s = 0 \)) confinement occurs if and only if \( f(0) > 0 \) and the corresponding string tension is \( f(0) \). In addition we show that when \( f(0) = 0 \), the potential behaves asymptotically as a (negative) power of the separation of the quark and anti–quark, and we find the exact power and coefficient. When \( f(0) \neq 0 \), apart from the linear potential and a constant term, we find the form of the next correction. For the critical case when the minimum of \( f(s) \) is just deep enough (or the divergence of \( g(s) \) is just strong enough) to allow the separation to diverge as the string approaches the minimum, the correction is exponentially small. At the non critical cases, the correction is power–like. In both cases, we explicitly find the relevant constants.
In section 2 we present the classical analysis of the action, we compute the quark
anti–quark potential and state the main result of the paper. In Section 3 we present a
rigorous proof of our statement. Section 4 is devoted to several examples to which we
apply our general result. In section 5 we deal with a variant of our main analysis. In
section 6 we give summary and conclusions.

2 The quark anti–quark potential

We shall write the Lagrangian density (relative to \(x\)) corresponding to the general Nambu–
Goto action \(\mathcal{L}\). We take the Lagrangian without the factor \(T\), and therefore the action
derived from it represents the quark anti–quark potential.

\[
\mathcal{L}(s, s') = \sqrt{f^2(s) + g^2(s)s'^2} \tag{6}
\]

As \(S = T \cdot \int \mathcal{L} \, dx\) is dimensionless, we see that in our formulation, \(\mathcal{L}\) has dimensions of
mass\(^2\).

The conjugate momentum is

\[
p = \frac{\delta \mathcal{L}}{\delta s'} = \frac{g^2(s)s'}{\sqrt{f^2(s) + g^2(s)s'^2}} \tag{7}
\]

and therefore

\[
\mathcal{H}(s, p) = p \cdot s' - \mathcal{L}(s, s'(s, p)) = \frac{-f^2(s)}{\sqrt{f^2(s) + g^2(s)s'^2}} = -\frac{f^2(s)}{\mathcal{L}} \tag{8}
\]

As the Hamiltonian \(\mathcal{H}\) does not depend explicitly on \(x\), its value is a constant of
motion. We shall deal with the case in which \(s(x)\) is an even function, and therefore there
is a minimal value \(s_0 = s(0)\) for which \(s'(0) = 0\). At that point, we see (7) that
\(p = 0\) also. The constant of motion is, therefore,

\[
\mathcal{H}(s_0, 0) = -f(s_0) \tag{9}
\]

From (8,9) we can express the Lagrangian without taking recourse of \(g(s)\):

\[
\mathcal{L} = \frac{f^2(s)}{f(s_0)} \tag{10}
\]

and we can also extract the differential equation of the geodesic line:

\[
\frac{ds}{dx} = \pm \frac{f(s)}{g(s)} \cdot \frac{\sqrt{f^2(s) - f^2(s_0)}}{f(s_0)} \tag{11}
\]
The distance (in the ordinary space) between two ”quarks” situated at \( s = s_1 \) is, therefore,
\[
l = \int dx = \int \left( \frac{ds}{dx} \right)^{-1} ds = 2 \int_{s_0}^{s_1} \frac{g(s)}{f(s)} f(s_0) \sqrt{f^2(s) - f^2(s_0)} ds \tag{12}
\]

The energy of the configuration is the length of the string according to the metric (6)
\[
E' = \int Ldx = \int \left( \frac{ds}{dx} \right)^{-1} Lds = 2 \int_{s_0}^{s_1} \frac{g(s)}{f(s)} f^2(s) \sqrt{f^2(s) - f^2(s_0)} ds \tag{13}
\]
\[
= f(s_0) \cdot l + 2 \int_{s_0}^{s_1} \frac{g(s)}{f(s)} \sqrt{f^2(s) - f^2(s_0)} ds \tag{14}
\]

In order to get the potential between the ”quarks”, we have to subtract the masses of the two quarks. The masses are independent of \( s_0 \), and their subtraction is needed only to regulate the singularities in (13). Other subtraction schemes (e.g the one used in [10]) are possible, and would result in a constant shift of the potential. As we analyze the behaviour of the potential only for large separations, the constant term has no direct physical meaning. Nevertheless, in our setting it is most natural to take as the ”bare” quark a straight string with a constant value of \( x \), stretching from \( s = 0 \) to \( s = s_1 \). Each quark has, then, a mass of
\[
m_q = \int_{0}^{s_1} g(s) ds \tag{15}
\]
Moreover, we shall find that in this subtraction scheme, the sign of the potential is naturally related to the globality of the minimum. We would like further to comment that this choice sets a limit on the permitted divergence of \( g(s) \) as \( s \to 0 \).

For the potential we get, therefore,
\[
E = f(s_0) \cdot l + 2 \int_{s_0}^{s_1} \frac{g(s)}{f(s)} \left( \sqrt{f^2(s) - f^2(s_0)} - f(s) \right) ds - 2 \int_{0}^{s_0} g(s) ds \tag{16}
\]
\[
\equiv f(s_0) \cdot l - 2K(s_0) \tag{17}
\]

with
\[
K(s_0) = \int_{s_0}^{s_1} \frac{g(s)}{f(s)} \left( f(s) - \sqrt{f^2(s) - f^2(s_0)} \right) ds + \int_{0}^{s_0} g(s) ds \tag{18}
\]
We define also \( \kappa = K(0) \).

The geometric picture we are depicting is quite simple. Under suitable assumptions, The geodesic line can not pass a value of \( s \) for which either \( f(s) \) has a minimum or \( g(s) \) diverges. without loss of generality, we can take this value of \( s \) to be 0. For large \( l \), then,
the geodesic line lies, for most of its part, very close to \( s = 0 \). The first term of the expression (17) tends, then, to \( f(0) \cdot l \), while the second and third ones are \( O(1) \) (bounded by a constant in \( l \)). We therefore get that there is confinement (i.e linear potential) for the metric (8), if and only if \( f(0) \neq 0 \), and then \( f(0) \) is the string tension. Note that \( f(s) \) has the same dimensionality as \( L \), i.e mass\(^2\), which is indeed the dimensionality of string tension. In the following section we prove all of our claims.

3 The proof

Our proof of the statement made above on the functional dependence of \( E(l) \) includes the followings steps. We first prove in theorem 1 that \( s_0 \) is a monotonic decreasing function of \( l \). In theorems 2 and 3 we write down expressions for the asymptotic behavior of \( l(s_0) \) and \( K(s_0) \). Finally the asymptotics of \( E(l) \) are derived in theorem 4.

3.1 Monotonicity of \( l(s_0) \)

The solution (14) of the Euler–Lagrange equations does not have to be the global minimum. It is possible that the global minimum is non differentiable, and therefore (14) is not sensitive to it. However, from the triangle inequality it is clear that a ”corner” can occur in a minimum–action function only if there is a direction in which the metric is zero. Therefore, it is possible in our setting only if \( f(0) = 0 \) and the string reaches \( s = 0 \). In that case it is clear that the best configuration is two bare quarks connected with a string segment on \( s = 0 \) (which does not ”cost” any energy). This configuration, in our conventions, has zero energy (as we subtract exactly the masses of the two quarks). Therefore we conclude that if the energy of the string describing the Euler–Lagrange geodesic has \( E < 0 \), it is the true solution (global minimum).

**Theorem 1** Let \( f(s), g(s) > 0 \) for \( s > 0 \), and let \( f(s) \) be monotone increasing. Assume that \( s(x) \) described by (14) has the global minimum value of \( E \). Then \( s_0 \) is monotone decreasing as a function of \( l \).

Proof: Assume the contrary. Then there are two intersecting geodesic lines \( G^{(1)}, G^{(2)} \) with \( s^{(1)}_0 < s^{(2)}_0, l^{(1)} < l^{(2)} \) (see figure 1a).

We shall now build a new line for \( l^{(2)} \). The new line will consist of the two halves of \( G^{(1)} \), separated so they span the distance \( l^{(2)} \), with a straight segment in the middle, (lying at \( s = s^{(1)}_0 \) and of length \( l^{(2)} - l^{(1)} \)). Obviously, \( s' = 0 \) for that straight segment.
From our assumption $s_0^{(1)} < s_0^{(2)}$, this segment has a smaller value of $s$ than any point of the corresponding curved segment of $G^{(2)}$. As $f(s)$ is monotone increasing, $f(s)$ is smaller for the straight segment than for any point on the curved one. Therefore, $\mathcal{L}$ is also smaller for the straight segment.

The total energy of the curved parts of the new line is smaller than that of the corresponding parts of $G^{(2)}$, for otherwise those latter parts combined would give a line for $l^{(1)}$ with energy smaller than that of $G^{(1)}$ (see figure 1b). (Although that combined line is not differentiable at $s = s_0^{(2)}$, it can be smoothed so that its energy is changed by no more than $\epsilon$ for every $\epsilon > 0$.) Therefore, the new line has a smaller energy then the assumed geodesic global minimum for $G^{(2)}$, which is a contradiction.

![Figure 1](image.png)

Figure 1: (a) the geodesic lines $G^{(1)}, G^{(2)}$. (b) $G^{(2)}$ and the new line having smaller energy.

### 3.2 Asymptotics of $l(s_0)$ and $K(s_0)$

Let us assume that $f(s)$ has a minimum at $s = 0$. We claim that under suitable assumptions, the first term of the expression tends to $f(0) \cdot l$, while the second and third ones are $O(1)$ (bounded by a constant in $l$). We therefore get that there is confinement (i.e linear potential) for the metric if and only if $f(0) \neq 0$ (and then $f(0)$ is the string tension). We shall now make those arguments more exact.
First, we need to state some preliminaries. In what follows, we denote by the symbol $\sim$ that two functions behave alike, up to a non zero multiplicative constant. That is,

**Definition 1** \( h_1(s) \sim h_2(s) \) in a region if there exist constants \( a, A > 0 \) such that for all \( s \) in that region, \( ah_2(s) \leq h_1(s) \leq Ah_2(s) \)

Obviously, the relation $\sim$ is an equivalence relation.

**Lemma 1** Define

\[
C_{n,m}(\tilde{y}) = \int_{1}^{\tilde{y}} \frac{dy}{y^{n}\sqrt{1-y^{-m}}} \quad (19)
\]

\[
C'_{n,m}(\tilde{y}) = \int_{1}^{\tilde{y}} \frac{2(1-\sqrt{1-y^{-m}})dy}{y^{n-m}} \quad (20)
\]

Then, if \( m > 0 \),

1. if \( n > 1 \), then \( 0 < C_{n,m}(\infty) < \infty \), and \( C_{n,m}(\tilde{y}) = C_{n,m}(\infty) - O(\tilde{y}^{-(n-1)}) \).
2. if \( n = 1 \), then \( C_{n,m}(\tilde{y}) = \log \tilde{y} + O(1) \).
3. if \( 0 < n < 1 \), then \( C_{n,m}(\tilde{y}) = \frac{1}{1-n} \tilde{y}^{1-n} + O(1) + O(\tilde{y}^{1-n-m}) \).

Similar relations hold for \( C'_{n,m}(\tilde{y}) \).

Proof: the lower limit of the integrals does not diverge. At the upper limit, the integrands behave as \( y^{-n} \), and the principal behaviours follow. The corrections for assertion 1 follow from the boundaries of the integral of \( y^{-n} \), and from the expansion of \( 1/\sqrt{1-y^{-m}} \). As for assertions 2 and 3, they follow from separating \( y^{-n} \), whose integral gives the divergence, from the integrands.

**Definition 2**

\[
D_{n,m} \equiv \frac{1}{m-n+1} + \frac{1}{2} C'_{n,m}(\infty) - C_{n,m}(\infty) \quad (21)
\]

**Lemma 2** If \( k > 0 \) and \(-1 < j < k-1\) then \( D_{2k-j,2k} > 0 \)

Proof: First we show that for a given value of \( m \), \( D_{n,m} \) is a monotone increasing function of \( n \) (when both \( m \) and \( n \) are in the specified range). In this range, \( D_{n,m} \) may be written as

\[
D_{n,m} = \int_{1}^{\infty} \left( y^{n-m-2} + y^{m-n} \frac{\sqrt{1-y^{-m}} - 1}{\sqrt{1-y^{-m}}} \right) dy \quad (22)
\]
Differentiating with respect to \( n \) gives
\[
\frac{\partial D_{n,m}}{\partial n} = \int_1^\infty dy \frac{y^{n-m} \log y}{\sqrt{1 - y^{-m}}} \left( \sqrt{1 - y^{-m}} - 1 \right) + 1 \tag{23}
\]
The last integrand is obviously positive for \( 1 < y < \infty \) and we find that the derivative is positive.

To complete the proof, we look at \( D_{n,m} \) for the minimal value of \( n \), that is, the maximal value of \( j \), which is \( k - 1 \).

\[
D_{k+1,2k} = \int_1^\infty \left( y^{-(k+1)} + y^{k-1} \frac{\sqrt{1 - y^{-2k}} - 1}{\sqrt{1 - y^{-2k}}} \right) dy
\]
\[
= -\left( \sqrt{1 - y^{-2k}}(y^k - \sqrt{y^{2k} - 1}) \right)_{1}^{\infty} / k
\]
\[
= 0 \tag{24}
\]

We shall investigate the behaviour of \( l \) as \( s_0 \) approaches (without loss of generality) the value 0, assuming it is finite. We shall find that this behaviour is governed by the expansions of the functions \( f(s) \) and \( g(s) \). As the functions used in (6) are \( f^2(s) \) and \( g^2(s) \), it may very well happen that the powers of the leading terms in those expansions will be half integers. Therefore, we do not assume that those powers are integer. Those considerations serve as motivations for the conditions in the following theorem.

**Theorem 2** Let \( f(s) \) be a function for \( s > 0 \), such that for \( s \) close enough to 0,
\[
f(s) = f(0) + a_k s^k + O(s^{k+1}) \tag{25}
\]
with \( k > 0 \), \( a_k > 0 \). Let \( g(s) \) be such that for \( s \) close enough to 0,
\[
g(s) = b_j s^j + O(s^{j+1}) \tag{26}
\]
with \( b_j > 0 \). Assume also that \( f(s), g(s) \geq 0 \) for \( 0 < s < \infty \).

Take any \( 0 < s_1 \leq \infty \) such that \( l = l(s_0) \), as in (12), converges for \( 0 < s_0 < s_1 \). Then, as \( s_0 \) tends to 0 from above,

1. if \( f(0) \neq 0 \),
   
   (a) if \( k < 2(j + 1) \), then \( l \) is bounded.
   
   (b) if \( k = 2(j + 1) \), then \( l = -\frac{2b_j}{\sqrt{2f(0)a_k}} \log s_0 + \lambda \), with
   
   \( i. \ \lambda = O(\log(-\log s_0)) \)
ii. \( \lambda \leq O(\log \log (-\log s_0)) \).

(c) if \( k > 2(j+1) \), then
\[ l = \frac{2b_j}{\sqrt{2}f(0)a_k}C_{k/2-j,k}(\infty) s_0^{-(k/2-j-1)} + O(s_0^{-\frac{(k/2-j-1)k}{k/2-j}}) \]

In particular, \( l \) diverges for \( s_0 \to 0 \) if and only if \( k \geq 2(j+1) \).

2. if \( f(0) = 0 \),

(a) if \( k < (j+1)/2 \), then \( l = O(s_0^k) \).

(b) if \( k = (j+1)/2 \), then \( l \sim -s_0^k \log s_0 \).

(c) if \( k > (j+1)/2 \), then
\[ l = \frac{2b_j}{a_k}C_{2k-j,2k}(\infty)s_0^{j+1-k} + O(s_0^{j+1-k+\frac{2k-j-1}{2k-j}}) \]

In particular, if \( j > -1 \) then \( l \) diverges for \( s_0 \to 0 \) if and only if \( k > j+1 \).

Proof: We choose some \( \tilde{s} \) such that the expansions [23][21] are valid, and \( f(s) \) is increasing, for \( 0 \leq s \leq \tilde{s} \). We shall separate the integration range for \( l \) into two parts - below \( \tilde{s} \) and above it.

The integral in the second range is
\[ \Delta l = 2 \int_{\tilde{s}}^{s_0} \frac{g(s)}{f(s)} \frac{f(s_0)}{\sqrt{f^2(s) - f^2(s_0)}} ds \leq \frac{f(s_0)}{f(\tilde{s})} \cdot 2 \int_{\tilde{s}}^{s_0} \frac{g(s)}{f(s)} \frac{f(\tilde{s})}{\sqrt{f^2(s) - f^2(\tilde{s})}} ds = \frac{f(s_0)}{f(\tilde{s})} \cdot l(\tilde{s}) \]
so \( \Delta l = O(l(\tilde{s})) \) and \( \Delta l \leq l(\tilde{s}) \) for \( f(0) \neq 0 \), and \( \Delta l = O((s_0/\tilde{s})^k l(\tilde{s})) \) for \( f(0) = 0 \).

Now we look at the integral in the first range, which can be the cause of the divergence of \( l(s_0) \). Let us assume first that \( f(0) \neq 0 \). Then \( f^2(s) - f^2(s_0) = (1+O(\tilde{s}))2f(0)a_k(s^k-s_0^k) \) there, so
\[ 2 \int_{s_0}^{\tilde{s}} \frac{g(s)}{f(s)} \frac{f(s_0)}{\sqrt{f^2(s) - f^2(s_0)}} ds = (1+O(\tilde{s})) \frac{2b_j}{2f(0)a_k} \int_{s_0}^{\tilde{s}} \frac{s^j ds}{\sqrt{s^k - s_0^k}} \]
\[ = (1+O(\tilde{s})) \frac{2b_j}{2f(0)a_k} \int_{1}^{\tilde{s}/s_0} \frac{dy}{y^{k/2-j}\sqrt{1-y^{-k}}} \]
\[ = (1+O(\tilde{s})) \frac{2b_j}{2f(0)a_k} \int_{1}^{\tilde{s}/s_0} \frac{dy}{y^{k/2-j}\sqrt{1-y^{-k}}} \cdot C_{k/2-j,k}(\tilde{s}/s_0) \]
with \( y = s/s_0 \). Substituting the behaviour of \( C_{k/2-j,k} \) and taking \( \tilde{s} \) fixed, we get that \( l \) is bounded for \( k < 2(j+1) \), \( l \sim -\log s_0 \) for \( k = 2(j+1) \), and \( l \sim s_0^{j+1-k/2} \) for \( k > 2(j+1) \).

To get sharper results, we now let \( \tilde{s} \) vary with \( s_0 \). By choosing \( \tilde{s} = s_0^{(k/2-j-1)/(k/2-j)} \) when \( k > 2(j+1) \) we prove assertion [1d]. By choosing \( \tilde{s} = -1/\log s_0 \) when \( k = 2(j+1) \) and using \( \Delta l = O(l(\tilde{s})) \) we prove assertion [1(b)]. Using that assertion for \( l(\tilde{s}) \) and \( \Delta l \leq l(\tilde{s}) \), again with \( \tilde{s} = -1/\log s_0 \), we prove assertion [1(b)].
Let us now assume that \( f(0) = 0 \). Now, \( f^2(s) - f^2(s_0) = (1 + O(\tilde{s}))a_k^2(s^{2k} - s_0^{2k}) \) for \( s_0 \leq s \leq \tilde{s} \), and
\[
2 \int_{s_0}^{\tilde{s}} \frac{g(s)}{f(s)} \frac{f(s)}{\sqrt{f^2(s) - f^2(s_0)}} ds = (1 + O(\tilde{s})) \frac{2b_j}{a_k} \int_{s_0}^{\tilde{s}} \frac{s^j}{s_0} \frac{s_0^k ds}{\sqrt{s^{2k} - s_0^{2k}}}
= (1 + O(\tilde{s})) \frac{2b_j}{a_k} \int_{s_0}^{\tilde{s}} \frac{s_0 \cdot dy}{y^{2k-j} \sqrt{1 - y^{-2k}}}
= (1 + O(\tilde{s})) \frac{2b_j}{a_k} \int_{s_0}^{\tilde{s}} s_0^{j+1-k} \cdot C_{2k-j,2k}(\tilde{s}/s_0)
\]
(29)

Substituting the behaviour of \( C_{2k-j,2k} \) and taking \( \tilde{s} \) fixed we get \( l = O(s_0^k) \) for \( k < (j+1)/2 \), \( l \sim -s_0^k \log s_0 \) for \( k = (j+1)/2 \), and \( l \sim s_0^{j+1-k} \) for \( k > (j+1)/2 \). To get a more precise result in the latter case and prove assertion 2c, we can now take \( \tilde{s} = s_0^{(2k-j-1)/(2k-j)} \).

Now we turn to investigate the behaviour of \( K(s_0) \) as \( s_0 \to 0 \). In order for it to be defined in the first place, we need two additional conditions. One of them ensures the convergence of the second integral of (18), and the other ensures the convergence of the first integral even for \( s_1 = \infty \).

**Theorem 3** Let \( f(s), g(s) \) be functions as in theorem 3. Assume also that

1. \( j > -1 \).
2. \( \int_{-\infty}^{\infty} g(s)/f^2(s)ds < \infty \) (i.e the integral converges when its upper limit is taken to infinity).

Take any \( 0 < s_1 \leq \infty \), and define \( K(s_0), \kappa \) as in (18).

1. if \( f(0) \neq 0 \), then we have \( 0 < \kappa < \infty \), and as \( s_0 \) tends to 0 from above,
   (a) if \( k < 2(j+1) \), then \( K(s_0) = \kappa + O(s_0^0) \).
   (b) if \( k = 2(j+1) \), then \( K(s_0) = \kappa - b_j \sqrt{a_k/2f(0)} s_0^k \log s_0 + O(s_0^0 \log(-\log s_0)) \).
   (c) if \( k > 2(j+1) \), then \( K(s_0) = \kappa + b_j \sqrt{a_k/2f(0)} (C_{k/2-j,k}(\infty) + \frac{2}{k/2+j+1}) s_0^{k/2+j+1} + O(s_0^{k/2+j+1 + \frac{j/2-j-1}{k/2-j}}) \).

2. if \( f(0) = 0 \), then we have that as \( s_0 \) tends to 0 from above,
   (a) if \( k < (j+1)/2 \), then \( K(s_0) = O(s_0^{2k}) \).
   (b) if \( k = (j+1)/2 \), then \( K(s_0) = -\frac{1}{2} b_j \frac{2k}{j+1} s_0^2 \log s_0 + O(s_0^{2k} \log(-\log s_0)) \).
   (c) if \( k > (j+1)/2 \), then \( K(s_0) = b_j \left( \frac{1}{2} C_{2k-j,2k}(\infty) + \frac{1}{j+1} \right) s_0^{j+1} + O(s_0^{j+1 + \frac{2k-j-1}{2k-j}}) \).
Proof: We choose some \( \tilde{s} \) such that the expansions (25,26) are valid for \( 0 \leq s \leq \tilde{s} \). We also note that for \( 0 \leq x \leq 1 \), we have \( (1 - \sqrt{1-x}) \sim x \).

The limit \( s \to 0 \) in the integrals for \( K(s_0) \) is \( \sim \int_0 g(s) ds < \infty \) by condition \( 3 \). We shall now show that the first integral also converges in its upper limit even for \( s_1 = \infty \). The ratio of the integrand and the one of condition \( 2 \) is

\[
f^2(s) \left( 1 - \frac{1 - f^2(s_0)}{f^2(s)} \right) \sim f^2(s) \cdot \frac{f^2(s_0)}{f^2(s)} = f^2(s_0).
\]

As the ratio is bounded, \( K(s_0) \) (and \( \kappa \)) also converge.

Now let us look at

\[
K(s_0) - \kappa = K(s_0) - K(0)
\]

\[
= \int_{s_0}^{s_1} \frac{g(s)}{f(s)} \left( \sqrt{f^2(s) - f^2(0)} - \sqrt{f^2(s) - f^2(s_0)} \right) ds
\]

\[
+ \int_0^{s_0} \frac{g(s)}{f(s)} \sqrt{f^2(s) - f^2(0)} ds
\]

\[
\equiv \Delta K_1(s_0) + \Delta K_2(s_0)
\]

Although \( \kappa \equiv K(0) = 0 \) when \( f(0) = 0 \), the above relation will be useful in that case also.

In order to evaluate \( \Delta K_1(s_0) \), we shall divide its integration range into two parts. In the first one,

\[
\int_{\tilde{s}}^{s_1} \frac{g(s)}{f(s)} \left( \sqrt{f^2(s) - f^2(0)} - \sqrt{f^2(s) - f^2(s_0)} \right) ds
\]

\[
= \int_{\tilde{s}}^{s_1} \frac{g(s)}{f(s)} \left( \sqrt{1 - \frac{f^2(0)}{f^2(s)}} - \sqrt{1 - \frac{f^2(s_0)}{f^2(s)}} \right) ds
\]

\[
\sim \int_{\tilde{s}}^{s_1} \frac{g(s) f^2(s_0) - f^2(0)}{2 f^2(s)} ds
\]

\[
= \frac{f^2(s_0) - f^2(0)}{f^2(\tilde{s}) - f^2(0)} \cdot \int_{\tilde{s}}^{s_1} \frac{g(s)}{f(s)} f^2(\tilde{s}) - f^2(0) ds
\]

\[
\sim \frac{f^2(s_0) - f^2(0)}{f^2(\tilde{s}) - f^2(0)} \cdot \int_{\tilde{s}}^{s_1} \frac{g(s)}{f(s)} (\sqrt{f^2(s) - f^2(0)} - \sqrt{f^2(s) - f^2(\tilde{s})}) ds
\]

\[
= \frac{f^2(s_0) - f^2(0)}{f^2(\tilde{s}) - f^2(0)} \cdot \Delta K_1(\tilde{s})
\]

Therefore we see that the contribution of that part is \( \sim (s_0/\tilde{s})^k \Delta K_1(\tilde{s}) \) if \( f(0) \neq 0 \), and \( \sim (s_0/\tilde{s})^{2k} \Delta K_1(\tilde{s}) \) if \( f(0) = 0 \).

In order to evaluate the other terms, we will have to deal separately with the cases \( f(0) \neq 0 \) and \( f(0) = 0 \). First we deal with the former case. The lower part of \( \Delta K_1(s_0) \) is

\[
\int_{s_0}^{\tilde{s}} \frac{g(s)}{f(s)} \left( \sqrt{f^2(s) - f^2(0)} - \sqrt{f^2(s) - f^2(s_0)} \right) ds
\]
Combining all the terms we demonstrate assertion 1a. Letting \( \tilde{s} \) vary with \( s_0 \), and using again lemma \([\text{II}]\) we can sharpen our results. By taking \( \tilde{s} = -1/\log s_0 \) we demonstrate assertion \([\text{II}]\), and by taking \( \tilde{s} = s_0^{(k/2-j-1)/(k/2-j)} \) we demonstrate assertion \([\text{II}3]\).

Next we move to the case \( f(0) = 0 \). The lower part of the integral of \( \Delta K_1(s_0) \) is now

\[
\int_{s_0}^{\tilde{s}} \frac{g(s)}{f(s)} \left( f(s) - \sqrt{f^2(s) - f^2(0)} \right) ds = \quad \quad (39)
\]

\[
(1 + O(\tilde{s}))b_j \int_{s_0}^{\tilde{s}} \sqrt{1 - (\frac{s_0}{s})^{2k}} ds = \quad \quad (40)
\]

\[
(1 + O(\tilde{s}))b_j s_0^{j+1} \int_{1}^{\tilde{s}/s_0} y^{j} \left( 1 - \sqrt{1 - y^{-2k}} \right) dy = \quad \quad (41)
\]

\[
(1 + O(\tilde{s}))b_j s_0^{j+1} \frac{1}{2} C_{2k-j,2k}(\tilde{s}/s_0) \quad \quad (42)
\]

which gives, upon taking \( \tilde{s} \) fixed, that \( \Delta K_1(s_0) = O(s_0^{2k}) \) for \( k < (j + 1)/2 \), \( \Delta K_1(s_0) \sim -s_0^{2k} \log s_0 \) for \( k = (j + 1)/2 \), and \( \Delta K_1(s_0) \sim s_0^{j+1} \) for \( k > (j + 1)/2 \).

\( \Delta K_2(s_0) \) is now very simple

\[
\int_{0}^{s_0} \frac{g(s)}{f(s)} \sqrt{f^2(s) - f^2(0)} ds = \int_{0}^{s_0} g(s) ds = \quad \quad (43)
\]

\[
= (1 + O(s_0))b_j \int_{0}^{s_0} s^j ds = \quad \quad (44)
\]

\[
= b_j \frac{1}{j+1} s_0^{j+1} + O(s_0^{j+2}) \quad \quad (45)
\]

Combining all the terms we demonstrate assertion \([\text{II}4]\). In order to demonstrate assertion \([\text{II}3]\) we use lemma \([\text{II}]\) while taking \( \tilde{s} = -1/\log s_0 \), and for assertion \([\text{II}4]\) we should be taking \( \tilde{s} = s_0^{(2k-j-1)/(2k-j)} \).
3.3 Asymptotics of $E(l)$

In the computation of the behaviour of $E$ as a function of $l$, we shall take the limit $s_1 \to \infty$, so that strictly speaking there are no "quarks" — they are the limits of the geodesic line. The condition ensuring that $K(s_0)$ does not diverge in this limit will ensure also that $l$ does not diverge.

The geodesic line, coming from $s = \infty$, can not "pass the first valley of $f(s)$", or "climb a cliff of $g(s)$". We know, from theorem 3, what to demand in order that $l$ will diverge for $s_0 \to 0$. In order for $l$ not to diverge before that, we demand that $f(s)$ is increasing, (and therefore has no minimum for $s > 0$).

After motivating our demands from $f(s), g(s)$, we are ready to present the main result of the article:

**Theorem 4** Let $\mathcal{L}$ be as in (4), with functions $f(s), g(s)$ such that:

1. $f(s)$ is analytic for $0 < s < \infty$. At $s = 0$, its expansion is:
   \[ f(s) = f(0) + a_k s^k + O(s^{k+1}) \]
   with $k > 0$, $a_k > 0$.

2. $g(s)$ is smooth for $0 < s < \infty$. At $s = 0$, its expansion is:
   \[ g(s) = b_j s^j + O(s^{j+1}) \]
   with $j > -1$, $b_j > 0$.

3. $f(s), g(s) \geq 0$ for $0 \leq s < \infty$.

4. $f'(s) > 0$ for $0 < s < \infty$.

5. \( \int_0^\infty g(s)/f^2(s)ds < \infty \).

Then for (large enough) $l$ there will be an even geodesic line asymptoting from both sides to $s = \infty$, and $x = \pm l/2$. As for the potential (13) related to that configuration,

1. if $f(0) > 0$, then
   \begin{enumerate}
   \item[(a)] if $k = 2(j + 1)$, \( E = f(0) \cdot l - 2\kappa + O((\log l)^\beta e^{-\alpha l}) \)
   \item[(b)] if $k > 2(j + 1)$, \( E = f(0) \cdot l - 2\kappa - d \cdot l^{\frac{k+2(j+1)}{k-2(j+1)}} + O(l^{\frac{k+2(j+1)}{k-2(j+1)} - \frac{1}{2}}}e^{-\alpha l}) \).
   \end{enumerate}
with some $\beta$ and the positive constants

$$\kappa = \int_0^\infty \frac{g(s)}{f(s)} \left( f(s) - \sqrt{f^2(s) - f^2(0)} \right) ds$$

$$\alpha = \frac{\sqrt{2f(0)ak}}{2b_j}$$

$$d = \frac{2b_j}{k/2 + j + 1} \sqrt{\frac{ak}{2f(0)}} \left( \frac{2b_j}{\sqrt{2f(0)ak}} C_{k/2-j,k}(\infty) \right)^{\frac{k+2(j+1)}{k-2(j+1)}}$$

In particular, there is linear confinement.

2. if $f(0) = 0$, then if $k > j + 1$, $E = -d' \cdot l^{-\frac{j+1}{k-j-1}} + O(l^{-\frac{j+1}{k-j-1}} \frac{2k-j-1}{(k-j)(k-j-1)})$ with

$$d' = 2b_j \left( \frac{2b_j}{ak} C_{2k-j,2k}(\infty) \right)^{\frac{j+1}{k-j-1}} D_{2k-j,2k}$$

In particular, there is no confinement.

Proof: Take any $s_0$ and choose arbitrary $\tilde{s} > s_0$. The ratio between the integrand of $l$ (12), and that of condition 5 is $1/\sqrt{1 - (f^2(s_0)/f^2(s))}$. As $f(s)$ is monotone increasing (condition 4), we have

$$1 < 1/\sqrt{1 - (f^2(s_0)/f^2(s))} < 1/\sqrt{1 - (f^2(s_0)/f^2(\tilde{s}))} \quad \text{for } s > \tilde{s}$$

In other words, that ratio is $\sim 1$. As the integral in 5 converges, so does that of (12), and $l$ is finite when $s_1 \to \infty$. Hence, theorem 2 is applicable.

In all the cases we are going to prove, in particular $E < 0$ and so by theorem 1, $s_0$ decreases when $l$ increases, and therefore converges at that limit. We should show that the case $s_0 \to s^* \neq 0$ is impossible. We can apply theorem 2 with any $s = s^*$ instead of $s = 0$. in the case $s^* \neq 0$, we have $k = 1, j \geq 0$ and therefore $k < 2(j + 1)$, so by that theorem, $l \not\to \infty$.

Let us look now at assertion 1. From (16) we get

$$E = f(0) \cdot l - 2\kappa + \Delta E$$

with

$$\Delta E = (f(s_0) - f(0))l - 2(K(s_0) - \kappa)$$

We now evaluate $\Delta E$, first for the case $k = 2(j + 1)$. From theorem 2,

$$l \leq -\frac{2b_j}{\sqrt{2f(0)ak}} \log s_0 + O(\log \log l)$$

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so \( s_0 = O((\log l) e^{-(\alpha/k)l}) \) for some \( \gamma \). From theorem 3 we get

\[
K(s_0) - \kappa = \frac{a_k}{2} l s_0^k + O(\log l s_0^k)
\] (50)

On the other hand,

\[
(f(s_0) - f(0)) l = a_k l s_0^k + O(l s_0^{k+1})
\] (51)

Therefore we have a cancelation, and

\[
\Delta E = O(\log l s_0^k) = O((\log l)^\beta e^{-\alpha l})
\] (52)

as needed, with \( \beta = \gamma k + 1 \).

Now we turn to evaluate \( \Delta E \) in the case \( k > 2(j + 1) \). Again we use the functional dependence \( l(s_0) \) from theorem 2, and get

\[
(f(s_0) - f(0)) l = 2b_j \sqrt{2f(0)a_k} C_{k/2 - j, k}(\infty) s_0^{k/2 + j + 1} + O(s_0^{k/2 + j + 1 + k/2 - j - 1})
\] (53)

We also revert that functional dependence to get

\[
s_0 = \left( \frac{2b_j}{\sqrt{2f(0)a_k}} C_{k/2 - j, k}(\infty) \right)^{k/2 - j - 1} l^{-k/2 - j - 1} + O(l^{-k/2 - j - 1 - k/2 - j})
\] (54)

Using the value of \( K(s_0) \) from theorem 3, we have a partial cancelation, and we get

\[
\Delta E = -2b_j \frac{k/2 + j + 1}{k/2 + j + 1} \sqrt{a_k s_0^{k/2 + j + 1} + O(s_0^{k/2 + j + 1 + k/2 - j - 1})}
\] (55)

Substituting \( s_0 \) we get the desired result.

Now we prove assertion 2. From theorem 2, we get

\[
s_0 = \left( \frac{2b_j}{a_k} C_{2k - j, 2k}(\infty) \right)^{1/k - j - 1} l^{-1/k - j - 1} + O(l^{-1/k - j - 1 - 2k - j - 1})
\] (56)

and

\[
f(s_0) l = 2b_j C_{2k - j, 2k}(\infty) s_0^{j+1} + O(s_0^{j+1 + 2k - j - 1/k - j})
\] (57)

Using theorem 3, we get

\[
E = 2b_j (C_{2k - j, 2k}(\infty) - \frac{1}{2} C'_{2k - j, 2k}(\infty) - \frac{1}{j + 1}) s_0^{j+1} + O(s_0^{j+1 + 2k - j - 1})
\] (58)

which gives the desired result.
4 Applications and verifications of the general result

4.1 invariance under reparameterizations of $s$

The results obtained should be invariant under reparameterizations of $s$. For example, if the "quarks" are supposed, in a certain setup, to reside at $s_1 \neq \infty$, we can reparameterize $s$ so as to "move" them to $s_1 = \infty$. The simplest reparameterization is a dilatation, that is, taking $s \mapsto \lambda s$, with $\lambda \neq 0$ a constant. If, under this dilatation, a magnitude is multiplied by $\lambda^c$, we can say that this magnitude has conformal dimension $c$. It is easy to find the conformal dimensions of various magnitudes we have encountered. For example:

| magnitude $s \times L f(s) a_k g(s) b_j \alpha$ |
|-------|-------|-------|-------|-------|-------|
| dimension 1 0 0 0 $-k$ $-1$ $-j-1$ $j+1-k/2$ |

We see that $\alpha$ has conformal dimension 0, as it should, only when $k = 2(j + 1)$. This is precisely when we have found the exponential correction to occur.

4.2 General considerations

When $f(0) > 0$, we see that a term proportional to $l^{-1}$ (of the order of the quantum correction Lüscher term [12]) can not arise from that correction with $j > -1$, but that it is the limiting case as $k \to \infty$. The classical correction computed in this article is always smaller, for large $l$, than a $l^{-1}$ correction.

When $f(0) = 0$, For the generic case $k = 2, j = 0$ we get a "Coulomb" potential $\sim l^{-1}$. This case agrees with the potential found in [1] for $\mathcal{N} = 4, D = 4$ SYM. Indeed, that theory is conformal and has no natural length scale, so the potential must be $\sim l^{-1}$.

In the case where the correction to the potential is proven to be exponentially small, that is $O((\log l)^\beta e^{-\alpha l})$, we know of no explicit computation exhibiting that behaviour. In the explicit computations, $\lambda = O(1)$ always, and therefore $s_0^k \sim e^{-\alpha l}$. Moreover, the corrections to the behaviour of $K(s_0)$ are then $O(s_0^k)$ and not $O(\log l s_0^k)$, and therefore $\Delta E$ and the corrections to the potential are $\sim e^{-\alpha l}$ or smaller. It may be that the $(\log l)^\beta$ is an artifact of the proof. On the other hand, the cancelation of the $\sim l s_0^k = O((\log l)^\beta l e^{-\alpha l})$ term (which is always $O(l e^{-\alpha l})$ in the explicit calculations), shown in the proof of theorem [4], is generic, and was indeed observed in those calculations, where it was considered accidental.
4.3 The \( AdS_5 \times S^5 \) dual of the \( \mathcal{N} = 4 \) SYM in four dimensions

In this context, it is customary to use \( U \) instead of \( s \). The Lagrangian is \[ \mathcal{L} = \frac{1}{2\pi} \sqrt{U^4/R^4 + (U')^2} \] \( (59) \)

We see that \( R \) is dimensionless and so the theory has no natural length scale. From \( (59) \) we extract \( f(U) = (2\pi)^{-1}U^2/R^2 \) and \( g(U) = (2\pi)^{-1} \). Therefore, \( f(0) = 0, k = 2, a_k = (2\pi)^{-1}R^{-2} \) and \( j = 0, b_j = (2\pi)^{-1} \). Moreover, \( \int_0^\infty g(U)/f^2(U) dU \sim \int_0^\infty U^{-4} dU < \infty \). Therefore, we can apply theorem \( \[ \]

\[ C_{4,4}(\infty) = \frac{\sqrt{2\pi^{3/2}}}{\Gamma(\frac{1}{4})^2} \] \( (60) \)
and so

\[ d' = \frac{2\sqrt{2\pi}R^2}{\Gamma(\frac{1}{4})^2} D_{4,4} \] \( (61) \)

and finally

\[ E = -\frac{2\sqrt{2\pi}R^2}{\Gamma(\frac{1}{4})^2} D_{4,4} \cdot l^{-1} + O(l^{-7/4}) \] \( (62) \)

This result agrees completely with Maldacena’s result \( \[ \]

4.4 Non-conformal cases with sixteen supersymmetries

A generalization of the former \( D_3 \) brane Lagrangian to \( D_p \)-branes \( (p \leq 4) \) with sixteen supersymmetries can be achieved following the steps taken in \( \[ \]

For those cases the Lagrangian is given by

\[ \mathcal{L} = \frac{1}{2\pi} \sqrt{(U/R)^{7-p} + (U')^2} \] \( (63) \)

With \( R^{7-p} \) having the dimension of mass\( ^{3-p} \). When \( p \neq 3 \), the theories are not conformal, so we do not expect a "Coulomb" potential. Indeed, \( f(U) = (2\pi)^{-1}(U/R)^{(7-p)/2} \), so \( f(0) = 0, k = (7-p)/2, a_k = (2\pi)^{-1}R^{-(7-p)/2} \), while \( g(U), j, b_j \) remain as in the former sub-section. Also, \( \int_0^\infty g(U)/f^2(U) dU \sim \int_0^\infty U^{p-7} dU < \infty \). Hence, \( E = -d' \cdot l^{-2/(5-p)} + O(l^{-2/(5-p)} - 2(6-p)/(5-p)(7-p)) \), with \( d' \propto R^{(7-p)/(5-p)} \). This result agrees with the computation for \( p = 2 \), also performed in \( \[ \]

4.5 Dual models of pure YM theory in three and four dimensions

Following a proposal of Witten \( \[ \]

one can write down a gravity solution that corresponds to a pure YM theory. For instance, to get \( YM_3 \) one starts with the near extremal \( D_3 \)
solution in the near horizon limit and compactifies the Euclidean time direction on a circle. Upon taking the Wilson loop along this circle and along a space direction one ends up with a scenario describing the four dimensional theory at finite temperature \[11\]. However, for Wilson loops along two space directions the limit of vanishing radius corresponds to a pure Euclidean YM theory in three dimensions. The string action takes, for that case, the following form

\[
\mathcal{L} = \frac{1}{2\pi} \sqrt{(U/R)^4 + (U')^2(1 - (U_T/U)^4)^{-1}}
\]

where the critical point (where \(g(U)\) diverges) is \(U = U_T\) and not \(U = 0\). There, \(f(U_T) = \frac{1}{2\pi}(U_T/R)^2 \neq 0\), \(k = 1, a_k = \frac{U_T}{\pi R^2}\), while \(g(U) = \frac{1}{2\pi}(1 - (U_T/U)^4)^{-1/2}\) and so \(j = -1/2, b_j = \sqrt{U_T}/4\pi\). Also, \(\int_0^\infty g(U)/f^2(U)dU \sim \int_0^\infty U^{-4} < \infty\). As \(k = 2(j+1)\), we get from theorem \[4\] that \(E = \frac{U_T^2}{2\pi R^2} \cdot l - 2\kappa + O((\log l)^\beta e^{-\alpha l})\) with \(\alpha = 2U_T/R^2\). A detailed computation in \[13\] agrees with the leading term, and does not include the constant one (due to a different subtraction scheme). However, the next correction in \[13\] is claimed to be \(\sim le^{-\alpha l}\). We believe that this is an erroneous result \[14\] and in fact the correction behaves like \(e^{-\alpha l}\).

In the case of QCD\(_4\),

\[
\mathcal{L} = \frac{1}{2\pi} \sqrt{(U/R)^3 + (U')^2(1 - (U_T/U)^3)^{-1}}
\]

so \(f(U_T) = \frac{1}{2\pi}(U_T/R)^{3/2} \neq 0\), \(k = 1, a_k = \frac{3\sqrt{U_T}}{4\pi R^{3/2}}\) and \(j = -1/2, b_j = \sqrt{U_T}/2\sqrt{3}\pi\). Now, \(\int_0^\infty g(U)/f^2(U)dU \sim \int_0^\infty U^{-3} < \infty\). We get \(E = \frac{U_T^{3/2}}{2\pi R^{3/2}} \cdot l - 2\kappa + O((\log l)^\beta e^{-\alpha l})\) when now \(\alpha = U_T^{1/2}/2R^{3/2}\).

### 4.6 Dual models of pure YM theory at finite temperature

When the time coordinate is compactified, and the Wilson loop is along this direction (on top of one space direction) the corresponding theory is a four dimensional theory at finite temperature. It was shown in \[8\] that

\[
\mathcal{L} = \frac{1}{2\pi} \sqrt{(U/R)^4(1 - (U_T/U)^4) + (U')^2}
\]

At the line \(U = U_T\), \(f(s)\) becomes negative. It does not have a minimum there \((k = 1)\), and \(g(s)\) does not diverge (it is constant, \(j = 0\)). As \(k < 2(j + 1)\), \(l\) is bounded as \(U_0\) approaches \(U_T\), and nothing seems to prevent the string from entering the unphysical region \(U < U_T\). However, it is argued in \[8\] that for values of \(l\) above a critical one, in which the string reaches some \(U_c > U_T\), the energy of the geodesic string is positive. In agreement with our general considerations, the physical solution is found to be two ”bare” quarks, and the potential is zero. Evidently, the reasons for the string not to enter the unphysical region are outside the scope of our model.
4.7 Pure YM from rotating branes

Dimensionally reducing the theory of rotating M-branes \[11\] leads to the Lagrangian

\[
L = \sqrt{\mathcal{C}} \frac{U^6}{U_0^6} \Delta + \frac{(U')^2}{1 - a^4/U^4 - U_0^6/U^6} \tag{67}
\]

It is easy to see that there is a singular point \(U_\infty\) for which \(g^2(U)\) has a simple pole, while \(f^2(U)\) is strictly positive for \(U_\infty \leq U\). Hence, we have \(k = 1, j = -1/2\) and by theorem \[4\] we find linear confinement with exponentially small correction.

4.8 MQCD

The QCD string for the M theory version of \(\mathcal{N} = 1, D = 4\) Super Yang–Mills is characterized by \[4\]

\[
L = 2\sqrt{2\zeta} \sqrt{\cosh(s/R)} \sqrt{1 + s'^2} \tag{68}
\]

With \(R\) (the radius of the 11-th dimension), and \(\zeta\), related to \(\Lambda_{QCD}\). We have \(f(s) = g(s) = 2\sqrt{2\zeta} \sqrt{\cosh(s/R)}\), and so we find \(f(0) = 2\sqrt{2\zeta}, k = 2, a_k = \sqrt{2\zeta}/2R^2\) and \(j = 0, b_j = f(0)\). In this case, \(\int_{-\infty}^{\infty} g(s)/f^2(s)ds \sim \int_{-\infty}^{\infty} e^{-s/2R}ds < \infty\). Again \(k = 2(j + 1)\), and therefore \(E = 2\sqrt{2\zeta} \cdot l - 2\kappa + O((\log l)^{j+1}e^{-\alpha l})\) with \(\alpha = 1/\sqrt{2R}\). The exact expression, computed in \[4\], agrees with that result \[3\], and even gives a better estimate for the exponential term, as \(O(\epsilon^2e^{-\alpha l})\). A term \(\sim (\log l)^{j+1}e^{-\alpha l}\) with \(\beta \neq 0\) appears nowhere in the expansion. The cancelation of the \(\sim e^{-\alpha l}\) term, which does appear, is, in a sense, ”accidental”. The cancelation of the \(O(\epsilon^2e^{-\alpha l})\) (or \(O(\epsilon \log \epsilon)\) in the notations of \[4\]), however, which seemed also accidental, is generic, as explained above.

When the supersymmetry is broken, we have \[4\]

\[
L = 2\sqrt{2\zeta} \sqrt{\cosh(s/Rc)} + \mu \sqrt{1 + s'^2} \tag{69}
\]

with \(c \approx 1\) and \(\mu\) a soft supersymmetry breaking parameter. Now \(b_j = f(0) = 2\sqrt{2\zeta} \sqrt{1 + \mu R^2}\), and \(a_k = \sqrt{2\zeta}/2\sqrt{1 + \mu R^2}\), but \(k, j\) do not change for \(\mu > -1\). The changes in the string tension and the constant term of the explicit computation of \(E\) agree with the general result. Now \(\alpha = 1/\sqrt{2\sqrt{1 + \mu R}}\), and the ”accidental” cancelation of the \(e^{-\alpha l}\) term persists, so the exponential term in the explicit computation is shown to be again \(O(\epsilon^2e^{-\alpha l})\). Furthermore, that computation shows that the exponential correction can be either positive (for \(\mu_1 < \mu < \mu_2\), with \(\mu_{1,2} = 27 \mp 16\sqrt{3}\)) or negative. For \(\mu = \mu_{1,2}\) there is a further cancelation, and the correction is \(O(e^{-2\alpha l})\).

\[3\]Note that \(\kappa\) is defined differently in the two contexts.
When $\mu = -1$, we have $f(0) = 0, k = j = 1$. As $k = (j + 1)/2$, $l$ does not grow to infinity as $s_0$ approaches 0, and theorem 4 is not applicable. As mentioned in [4], above a certain value of $l$, the configuration of two bare quarks is energetically favourable to the string obeying the Euler–Lagrange equations. (Very loosely speaking, $D_{1,2} = -\infty$).

4.9 Polyakov’s non–critical string

Polyakov suggests [5] the conformal invariant solution $a(\phi) = e^{\alpha \phi}$ with $\alpha$ constant. The range of $\phi$ is $-\infty < \phi < \infty$ and the ”quarks” are situated at $\phi = \infty$. The function $f(\phi) = a^2(\phi)$ has no minimum at a finite value of $\phi$, and $g(\phi) = a(\phi)$ does not diverge, so our theorem 4 is not directly applicable. We can, however, reparameterize $\phi$ and set $s = a(\phi) = e^{\alpha \phi}$. Now,

$$\mathcal{L} = \sqrt{f^2(\phi) + g^2(\phi)\phi'^2} = \sqrt{e^{4\alpha \phi} + e^{2\alpha \phi} \phi'^2} = \sqrt{s^4 + \alpha^{-2} s'^2}$$ (70)

and we are essentially back in Maldacena’s case which is conformal, as we have indeed seen.

5 The even vs. odd cases

If the functions $f(s), g(s)$ are defined also for $s < 0$, and are even functions of $s$, then it makes sense to look also at the case where the quark and anti–quark are situated at $s = \pm s_1$ and the string describes an odd function $s(x)$, and to compare it to the previous case in which $s(x)$ is even.

The Lagrangian and Hamiltonian are equal to those of the previous case, but, as $s(0) = 0$, the solution is now specified by the value of $r_0 \equiv s'(0)$, and

$$\mathcal{H}(s, s') = \mathcal{H}(0, r_0) = -\frac{f^2(0)}{\sqrt{f^2(0) + g^2(0)r_0^2}}$$ (71)

(from which we see that we must have either $f(0) \neq 0$ or $g(0) \neq 0$). The equations for $\frac{ds}{dx}$, $l$, $E$ can be extracted, and a treatment of this case, following the lines of the previous one, can be achieved. We shall not pursue this course. Instead, we shall give a simple relation between the energies in the two cases.

**Theorem 5** Let $E_{\text{even}}(l)$ be the quark anti–quark potential in the even case as a function of their separation, and let $E_{\text{odd}}(l)$ be the corresponding potential in the odd case. Let $s_0$ be the smallest value of $s$ attained by the even string. Then,

$$0 \leq E_{\text{odd}}(l) - E_{\text{even}}(l) \leq 2 \int_0^{s_0} g(s)ds$$ (72)
proof: If we reflect, for negative $x$, the graph of the odd $s(x)$ string, we get an even one with the same energy (which is not in equilibrium for $x = 0, s(x) = 0$). The solution for the even case has, of course, a smaller energy. Therefore $E_{\text{odd}}(l) \geq E_{\text{even}}(l)$ (see figure (4)). On the other hand, if we break the graph of the even $s(x)$ string into two parts, put them at different sides of $s = 0$, and connect them with a straight string segment at $x = 0, -s_0 \leq s \leq s_0$, as shown in figure (3), we get a string describing an odd function (which is not in equilibrium for $x = 0, s(x) = \pm s_0$). The solution for the odd case has a smaller energy, and hence $E_{\text{odd}}(l) \leq E_{\text{even}}(l) + \int_{-s_0}^{s_0} g(s) ds$.

Figure 2: (a) The minimal energy odd string. (b) The derived even string

This simple theorem is sufficient to give strong estimates on the difference of the potentials in the two cases. Let us examine the MQCD setup discussed in sub–section 4.8. In that setup, $2 \int_0^{s_0} g(s) ds \approx 2\sqrt{2\zeta} \cdot 2s_0 \sim R\sqrt{\zeta} e^{-(\alpha/2)l}$, while the explicit computations give $E_{\text{odd}}(l) - E_{\text{even}}(l) \sim R\sqrt{\zeta} e^{-\alpha l}$. We see that theorem 4 gives an exponentially small estimation of the difference, which is not too bad. It is easy to see that if the correction is exponentially small for the even case, it always remains so also in the odd case.

6 Summary and conclusions

Recently, Wilson loops, or quark anti–quark potentials, were computed from various string actions [1, 3, 4, 8, 9, 10]. In this note we suggest a "unified" framework for discussing all those cases and others. The basic setup includes a "quark" and an "anti–quark"
situated in flat space, connected by a string stretched in that space and in an extra curved dimension. The space–time metric depends only on this extra coordinate. We have identified the cases in which the position of the middle of the string approaches a constant value of the extra dimension as the quark anti–quark separation \( l \) grows, and have computed the potential in that limit from the leading terms of the Taylor expansion of the metric around that constant value. We have shown that the linear coefficient of the potential is equal to the string tension of a string situated at the aforementioned constant value of the extra dimension. Therefore, confinement arises exactly when that string tension is non–vanishing. In more technical terms, we have shown that (assuming without loss of generality that \( f(s) \) has a minimum or \( g(s) \) diverges at \( s = 0 \)) confinement occurs if and only if \( f(0) > 0 \) and the corresponding string tension is \( f(0) \).

In case of confinement, we have shown that the correction to the linear potential (apart from a constant term) is either a negative power of the separation, or exponentially small. In both cases we explicitly find the relevant constants. The exponentially small correction arises in the critical case when the minimum of \( f(s) \) is just deep enough (or the divergence of \( g(s) \) is just strong enough) to allow the separation to diverge as the string approaches the minimum. This case arises when the minimum of \( f(s) \) at \( s = 0 \) is quadratic, and \( g(s) \) neither diverges nor has a zero there, and therefore is the generic one.

We have proven that the exponentially small correction is \( O((\log l)^\alpha e^{-\alpha l}) \), when we
have identified $\alpha$ but not $\beta$. In all explicit calculations $\beta = 0$, and the $(\log l)^\beta$ factor might be an artifact of the proof. We detect a cancelation of the $\sim ls_0^k$ terms (which are $\sim le^{-\alpha l}$ in the explicit calculations). This cancelation was encountered in explicit computations and we argue that it is in fact generic. This result contradicts the results of [13] where it was claimed that the leading correction is of the form $le^{-\alpha l}$. Some of the explicit computations show that the true behaviour of the correction is $e^{-\alpha l}$, so our bound is rather tight.

When there is no confinement, the potential we find is asymptotically a negative power of the separation, and we find the exact power and coefficient. In particular, for the large $N$ CFT of [1], we explicitly re-derive the potential, including the numerical constants.

We demonstrate our general results by applying them to a set of string configurations that were studied recently [1, 4, 4, 6, 7, 8, 9].

The next step in this program of "precision measurements" of the Wilson loops or the quark anti–quark potential is not to compute higher order corrections of the classical computation but rather to determine the quantum fluctuations. For a string that is stretched only along the flat four dimensions with no extra curved dimension, the quantum fluctuations were computed in [12]. The result is a correction of the form $c/L$ where $c$ is a universal coefficient independent of the coupling constant. We are studying [15] a generalization of this construction to the string configurations discussed in this note.

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