Multiparticle correlations from momentum conservation

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Using a generating-function formalism, we compute the contribution of momentum conservation to multiparticle correlations between the emitted particles in high-energy collisions. In particular, we derive a compact expression of the genuine $M$-particle correlation, for arbitrary $M$.

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The main purpose of this paper is to deal with a rather general problem: consider $N$ random variables (in a $D$-dimensional space) $\mathbf{p}_1, \ldots, \mathbf{p}_N$, which are independent except for a constraint $\mathbf{p}_1 + \ldots + \mathbf{p}_N = \mathbf{0}$. What is the multiple correlation between $M$ variables among the $\mathbf{p}_j$ (with $M < N$) induced by this constraint, and in particular, what is the cumulant of the $M$-variable correlation?

One may for example think of the correlation between $M$ monomers of a finite-size-$N$ ring polymer. Another instance of the problem under study, from which we borrow the terminology in this paper, is the constraint arising from global momentum conservation in high-energy collisions: in the center-of-mass frame, the sum of the momenta of the $N$ emitted particles vanishes, and this induces correlations between $M$ arbitrary particles.

An accurate knowledge of these unavoidable multiparticle correlations due to global momentum conservation is important. They represent an ever-present background to multiparticle correlations for $M$ particles, the connected part of the $N$-particle correlation (cumulant) between an arbitrary number of particles to leading order in $1/N$, and show that they follow the same behavior as short-range correlations.

To derive multiparticle correlations of arbitrary order in a systematic way, we introduce a generating function of the distributions, which we define as

$$G(x_1, \ldots, x_N) \equiv 1 + x_1 f(p_1) + x_2 f(p_2) + \cdots + x_1 x_2 f(p_1, p_2) + \cdots, \quad (1)$$

and so on for every order $M$. Thus, the coefficient of the term $x_{j_1} x_{j_2} \cdots x_{j_M}$ is the $M$-particle probability distribution $f(p_{j_1}, p_{j_2}, \ldots, p_{j_M})$.

We shall mostly be interested in the connected part of the $M$-particle distribution, that is, the term which cannot be expressed as a product of correlations between less than $M$ particles. For instance, the two-particle distribution can be decomposed into two parts:

$$f(p_1, p_2) = f(p_1) f(p_2) + f_c(p_1, p_2),$$

where the first term in the right-hand side (rhs) is the mere product of the one-particle probability distributions, while the second one, the “connected part,” reflects correlations between $p_1$ and $p_2$. In particular, $f_c(p_1, p_2)$ vanishes if $p_1$ and $p_2$ are uncorrelated. This genuine correlation, which is also called the cumulant of the distribu-

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tion in probability theory, is obtained by taking the logarithm of the generating function of distributions. The cumulant function, $f_n(p_1, \ldots, p_m)$, is the coefficient of $x_1 \cdots x_m$ in the expansion of $G(x_1, \ldots, x_N)$:

$$G(x_1, \ldots, x_N) \equiv x_1 f_1(p_1) + x_2 f_2(p_2) + \cdots + x_1 x_2 f_2(p_1, p_2) + \cdots (2)$$

If the system splits into independent subsystems or, more generally, if correlations in the system are short-range, the probability distributions of each subsystem add up: $f_n(\{p_j\}) = \sum_A (N_A/N) f_n(\{p_j\})$, where the sum runs over subsystems with sizes $N_A$. It follows that $G$ can be factorized into the product of functions for each subsystem: $G(\{x_j\}) = \prod_A g_A(\{N_A x_j/N\})$. In the limit where each individual particle is a subsystem, $G(\{x_j\}) = [g(\{x_j/N\})]^N$. Therefore, $\ln G$, and the cumulants, are the sums of the corresponding quantities for the subsystems. As a result, inspecting the term in $x_1 \cdots x_m$ shows that, while the $M$-particle probability distribution $f_n(p_1, \ldots, p_m)$ is independent of $N$, the cumulant $f_n(p_1, \ldots, p_m)$ scales like $1/N^{M-1}$. In Sec. III we shall show that the cumulants from the correlations due to the opening momentum conservation follow the same scaling to leading order in $1/N$. More precisely, we shall demonstrate that the generating function of distributions reads

$$G(x_1, \ldots, x_N) \propto e^{N g_f(x_j/N)} \left(1 + \sum_{j=1}^{N} \frac{\{x_j\}^l}{N^l} \right),$$

where the sum runs over terms with $q \geq l$.

II. CUMULANTS IN THE LARGE-$N$ LIMIT

In this Section, we derive a compact expression of the cumulants of multiparticle correlations due to global momentum conservation to leading order in $1/N$. In particular, we show that the $M$-particle cumulants scale like $1/N^{M-1}$.

Let us assume for simplicity that the only source of correlation between the particles is global momentum conservation, $p_1 + \cdots + p_N = 0$. Under this assumption, the $M$-particle distribution of $p_1, \ldots, p_M$, with $M < N$, is defined as

$$f_1(p_1, \ldots, p_M) = \left( \prod_{j=1}^{M} F(p_j) \right) \int \delta^D(p_1 + \cdots + p_N) \prod_{j=M+1}^{N} \frac{[F(p_j) d^D p_j]}{N_D^{N-M}},$$

where $F(p)$ is the one-particle momentum distribution unrenormalized for the momentum conservation constraint, and $N_D = \int F(p) d^D p$ is a normalization constant. In the following, we denote by $\langle \ldots \rangle$ an $F(p)$-weighted average, that is, $\langle g(p) \rangle \equiv \int g(p) F(p) d^D p / N_D$, for any function of momentum $g(p)$.

The denominator in Eq. (3) is a constant, which we denote by $1/C_D$, independent of $M$. The actual value will not influence the forthcoming discussion.

Consider next the numerator of Eq. (3). Introducing a Fourier representation of the Dirac distribution, it reads

$$\int \frac{d^D k}{(2\pi)^D} \left( \prod_{j=1}^{M} F(p_j) e^{i k \cdot p_j} \right) \langle e^{i k \cdot p} \rangle^{N-M} = \int \frac{d^D k}{(2\pi)^D} \langle e^{i k \cdot p} \rangle^N \left( \prod_{j=1}^{M} F(p_j) e^{i k \cdot p_j} \right).$$

Inserting in Eq. (3) the expression of the $M$-particle distribution, Eq. (3), with the numerator replaced by the rhs of Eq. (4), we obtain

$$G(x_1, \ldots, x_N) = C_D \int \frac{d^D k}{(2\pi)^D} \langle e^{i k \cdot p} \rangle^N \prod_{j=1}^{N} \left(1 + x_j F(p_j) e^{i k \cdot p_j} \right)$$

$$\approx C_D \int \frac{d^D k}{(2\pi)^D} \langle e^{i k \cdot p} \rangle^N \exp \left( \sum_{j=1}^{N} x_j F(p_j) e^{i k \cdot p_j} \right).$$

In passing from the first line to the second one, we have used the fact that we shall only consider the coefficient
of $x_{j_1} \ldots x_{j_M}$, where the $M$ indices $j_1$, \ldots, $j_M$ are all different [see Eq. (2)]. One easily checks that the only difference between the two forms of the generating function in Eq. (5) comes from terms in which at least one $x_j$ is raised to some power $m \geq 2$ (corresponding to the “autocorrelation” of particle $j$ with itself). Therefore, as far as we are concerned, Eq. (5) really is an identity, not an approximation.

Please note that in Eq. (5), the variable $x_j$ is always multiplied by a factor $F(p_j)$. We may thus rescale $x_j$ by this factor, and drop it in the following to simplify expressions. This is quite satisfactory, since it means that the measurable multiparticle correlations $f(p_{j_1}, \ldots, p_{j_M})$ or $F(p_{j_1}, \ldots, p_{j_M})$ will not depend on the non-measurable distribution $F(p)$—this justifies our previously calling it “unrenormalized”.

To evaluate the integral in Eq. (5), we rely on the fact that $N$ is large, and use a saddle-point approximation. To simplify the discussion, we introduce the notation

$$
\mathcal{F}(k) \equiv \ln\langle e^{ik\cdot p} \rangle + \sum_{j=1}^N \frac{x_j e^{ik\cdot p_j}}{N} \langle e^{ik\cdot p_j} \rangle \tag{6}
$$

Thus, the integrand in Eq. (5) is $e^{N\mathcal{F}(k)}$. We call $k_0$ the position of the saddle point. Please note that $\mathcal{F}$ and its successive derivatives, which we shall denote by $\mathcal{F}', \mathcal{F}''$, $\mathcal{F}^{(3)}$, \ldots, depend on $x$ only through $x/N$, where $x$ stands for any of the $x_j$. Therefore, $k_0$, which is of course the solution of $\mathcal{F}'(k) = 0$, also depends on $x/N$ only. One should pay attention to the fact that the saddle point $k_0$ is not merely the origin $k = 0$.

We shall now demonstrate that to leading order in $1/N$, all multiparticle cumulants are determined by the saddle-point value $N\mathcal{F}(k_0)$. More precisely, we show that the $M$-particle cumulant is of order $1/N^{M-1}$, and that corrections to the saddle-point calculation only yield subleading terms, suppressed by (positive) powers of $1/N$.

Using a Taylor expansion of $\mathcal{F}(k)$ around the saddle point, the generating function (6) reads

$$
G(x_1, \ldots, x_N) = C_D e^{N\mathcal{F}(k_0)} \int \frac{d^Dk}{(2\pi)^D} e^{N\mathcal{F}''(k_0)(k-k_0)^2/2} \exp \left[ N \sum_{m=1}^{+\infty} \frac{\mathcal{F}^{(m)}(k_0)}{m!} (k-k_0)^m \right] \left( 1 + \sum_{n=1}^{+\infty} \frac{1}{n!} \left[ N \sum_{m=3}^{+\infty} \frac{\mathcal{F}^{(m)}(k_0)}{m!} (k-k_0)^m \right]^n \right). \tag{7}
$$

As recalled in Sec. II, the cumulants are given by the logarithm of the generating function. Now, the logarithm of Eq. (7) will split into three parts. First, there is $\ln C_D$, which does not depend on $x$, and thus does not influence the values of the multiparticle cumulants.

Next, there is $N\mathcal{F}(k_0)$. As noted above, both $\mathcal{F}$ and $k_0$ only involve powers of $x/N$. Thus, the coefficient of $x_{j_1} \ldots x_{j_M}$ in $\mathcal{F}(k_0)$ contains a factor $1/N^M$. Multiplying by the overall factor of $N$, we find that the contribution of $N\mathcal{F}(k_0)$ to the cumulant $f_e(x_{j_1} \ldots x_{j_M})$ scales as $1/N^{M-1}$.

Finally, the cumulants involve the contribution of the logarithm of the integral in Eq. (7). We shall use the fact that the integrand is the sum of a Gaussian function and of combinations of its moments. After integration, this yields a common factor, $1/[2\pi N \mathcal{F}''(k_0)]^{D/2}$, which multiplies a sum $\Sigma$. Let us show that $\Sigma$ only involves terms in $x^l/N^q$ with $q \geq l$. Noting that the values which contribute to the Gaussian integral are of order $(k-k_0)^2 \approx 1/N$, one sees that each term $N\mathcal{F}^{(m)}(k_0)(k-k_0)^m$ is of order $N^{1-m/2}$, multiplied by a function of $x/N$. Since the sum runs over $m \geq 3$, $N^{1-m/2} \leq N^{-1/2}$; therefore, a power $x^l$ goes with at most a factor $1/N^{l+1/2}$. Even after raising the sum to the power $n$, and taking the logarithm, this will always remain of the form $x^l/N^q$ with $q > l$. Finally, the logarithm of the overall factor $1/[2\pi N \mathcal{F}''(k_0)]^{D/2}$ involves $x$ only through the form of powers of $x/N$ in $\ln \mathcal{F}''(k_0)$. All in all, the contribution to the $M$-particle cumulant of the integral is therefore at most of order $1/N^{M-1}$, subleading with respect to the contribution of $N\mathcal{F}(k_0)$.

Therefore, we have shown that the leading contribution to the cumulants of multiparticle correlations due to momentum conservation is the saddle-point value: $\ln G(x_1, \ldots, x_N) = N\mathcal{F}(k_0)$. This means first that the mere knowledge of $k_0$ gives access to all cumulants at once, at least to leading order. One easily checks that $k_0$ is pure imaginary, i.e., $ik_0$ has real-valued components. From Eq. (6), it follows that $\mathcal{F}(k_0)$ is real-valued as well, and so are the cumulants, as should be. We shall illustrate these points by computing explicitly the two- and three-particle cumulants in the following Section. Our result also means that genuine $M$-particle correlations arising from momentum conservation, which is a long-range effect, scale in the same way as correlations from short-range sources, namely as $O(1/N^{M-1})$. This was certainly not obvious a priori.
III. TWO- AND THREE-PARTICLE CUMULANTS

As an example of the order of magnitude derived in the previous Section, let us compute the two- and three-particle cumulants. According to the discussion in Sec. II they are given by the terms in \( x^2 \) and \( x^3 \) in the generating function of cumulants \( G(x_1, \ldots, x_N) \), that is, following Sec. III the corresponding terms in \( N \tilde{F}(k_0) \). We shall for simplicity perform calculations assuming that \( F(p) \) only depends on the modulus \( |p| \), not on the azimuthal angle of \( p \). As a consequence, the average momentum \( \langle p \rangle \) vanishes. Departures from this assumption are discussed at the end of this Section.

Now a straightforward calculation shows that the saddle point \( k_0 \) is given by

\[
\left( \sum_{j=1}^{N} \frac{x_j}{N} e^{i k_0 \cdot p_j} - 1 \right) (p e^{i k_0 \cdot p}) = \sum_{j=1}^{N} \frac{x_j}{N} D_j e^{i k_0 \cdot p_j}. \tag{8}
\]

As mentioned above, \( k_0 \) is a function of \( x/N \). Equation (8) can be solved order by order in \( x/N \), using the fact that one term (in the left-hand side) is of order 0 in \( x/N \) while the other two are linear in \( x/N \). Inspecting Eq. (8), and using the fact that the first term in the rhs is even in \( k \), we see that the calculation of \( F(k_0) \) to order \( x^3 \) requires our knowing \( k_0 \) to order \( x^2 \) (while the calculation to order \( x^2 \) only requires \( k_0 \) to order \( x \)).

We can then compute \( k_0 \), expanding Eq. (8) in powers of \( k_0 \), which gives

\[
i k_0 = - \left[ I_D - \left( X_0 I_D - \frac{D}{\langle p^2 \rangle} X_2 \right) \right]^{-1} \frac{D}{\langle p^2 \rangle} X_1, \tag{9}
\]

where \( I_D \) denotes the unit \( D \times D \) matrix and we have introduced

\[
X_0 \equiv \sum_{j=1}^{N} x_j \frac{1}{N}, \quad X_1 \equiv \sum_{j=1}^{N} x_j p_j, \quad X_2 \equiv \sum_{j=1}^{N} x_j p_j \otimes p_j.
\]

Note that \( i k_0 \) is real-valued, as expected. Reporting its value in Eq. (9), we obtain

\[
\tilde{F}(k_0) = X_0 - \frac{D}{2\langle p^2 \rangle} X_1^2 - \frac{D}{2\langle p^2 \rangle} \left( X_0 I_D - \frac{D}{\langle p^2 \rangle} X_2 \right) \cdot X_1.
\]

Multiplying this result by \( N \), we finally obtain, to leading order in \( 1/N \):

\[
\ln G(x_1, \ldots, x_N) = \sum_{j=1}^{N} x_j - \frac{D}{2N\langle p^2 \rangle} \sum_{j,k} x_j x_k (p_j \cdot p_k)
- \frac{D}{2N^3\langle p^2 \rangle} \sum_{j,k,l} x_j x_k x_l \left[ p_j \cdot p_l - \frac{D}{\langle p^2 \rangle} (p_j \cdot p_k)(p_k \cdot p_l) \right] + \mathcal{O}(x^4). \tag{10}
\]

Hence the coefficients of \( x_1 x_2 \) and \( x_1 x_2 x_3 \):

\[
f_c(p_1, p_2) = -\frac{D}{N\langle p^2 \rangle} p_1 \cdot p_2, \tag{11}
\]

\[
f_c(p_1, p_2, p_3) = -\frac{D}{N^2\langle p^2 \rangle} (p_1 \cdot p_2 + p_1 \cdot p_3 + p_2 \cdot p_3)
+ \frac{D^2}{N^3\langle p^2 \rangle^2} \left[ (p_1 \cdot p_2)(p_1 \cdot p_3) + (p_1 \cdot p_2)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_3) \right]. \tag{12}
\]

We recover, in the case \( D = 2 \), the expression of the two-particle correlation due to transverse momentum conservation already derived in Refs. [2] [3]: the correlation is back-to-back, and stronger between high-momentum particles.

Finally, let us comment on the assumption that \( F(p) \) only depends on the modulus of \( p \), not on its azimuthal angle.

Our expansion of Eq. (5) relies on both \( \langle p \rangle = 0 \) and on the identity \( \langle (k \cdot p)^2 \rangle = k^2\langle p^2 \rangle / D \), which are no longer valid if \( F \) is non-isotropic. In the general case, \( k^2\langle p^2 \rangle / D \) will be replaced by another quadratic form in \( k \), namely \( k \cdot (p' \otimes p') \cdot k \), where \( p' = p - \langle p \rangle \). To perform explicit calculations, it becomes necessary to introduce the principal-axes frame, in which \( p' \otimes p' \) is diagonal. For instance, the two-particle cumulant, Eq. (11), will read

\[
f_c(p_1, p_2) = -\sum_{i=1}^{D} \frac{(p_1)_i(p_2)_i}{N\langle p_i^2 \rangle}, \tag{13}
\]

where the sum runs over the coordinates along the principal axes. The distinction between the various directions...
is for example relevant in a high-energy collision: while the transverse momentum distribution can be isotropic (unless there is some anisotropy, as, e.g., a correlation to the impact parameter direction), the isotropy will not extend to the beam direction. In that case, one may want to use Eq. (11), with $D = 2$, when studying two-particle correlations in the transverse plane, but turn to Eq. (13) if interested in 3-dimensional correlations.

IV. DISCUSSION

We have shown how it is possible to calculate the multiparticle correlations arising from momentum conservation between any number of particles using a generating function formalism:

\[
\ln G(x_1, \ldots, x_N) = NF(k_0),
\]

where $F(k)$ is given by Eq. (6) and $k_0$ by

\[
F'(k_0) = 0.
\]

In particular, we have seen that the $M$-particle cumulant scales as $1/N^{M-1}$, where $N$ is the total number of emitted particles. That means that the correlation scales in the same way as correlations arising from short-range final interactions or from resonance decays, although the underlying reason is less obvious. In the latter cases, the scaling is a simple consequence of combinatorics: the $M$ particles are required either to be altogether in a small phase-space region, or to originate from a single decay, hence the factor $1/N^{M-1}$. However, in the case of global momentum conservation, the same arguments do not apply \textit{a priori}.

The scaling of the genuine $M$-particle correlation due to momentum conservation has several consequences. First, a good feature: in the context of collective flow in heavy-ion collisions, a new method of analysis has been proposed, which relies on the idea that a cumulant expansion allows one to separate genuine collective phenomena from trivial short-range correlations, while they interfere in the data. Since correlations from momentum conservation behave as short-range correlations, they can be removed as efficiently as them, to leave a clean collective flow signal.

Then, some bad news. Since the $M$-particle correlation from momentum conservation is of the same order as that from short-range interactions, as for instance quantum correlations, it may bias measurements of these other correlations. It is thus worth checking that momentum conservation does not contribute significantly to the correlations measured by particle interferometry and attributed to quantum (anti)symmetrization of the wave-function. This is especially true when studying the dependence of HBT parameters on the average momentum of the particles since the correlation from momentum conservation increases with momentum [see Eqs. (11) and (12)].

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