The One-Phase Bifurcation For The $p$-Laplacian

Alaa Akram Haj Ali & Peiyong Wang*

Department of Mathematics
Wayne State University
Detroit, MI 48202

Abstract

A bifurcation about the uniqueness of a solution of a singularly perturbed free boundary problem of phase transition associated with the $p$-Laplacian, subject to given boundary condition is proved in this paper. We show this phenomenon by proving the existence of a third solution through the Mountain Pass Lemma when the boundary data decreases below a threshold. In the second part, we prove the convergence of an evolution to stable solutions, and show the Mountain Pass solution is unstable in this sense.

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1 Introduction

In this paper, one considers the phase transition problem of minimizing the $p$-functional

$$J_{p,\varepsilon}(u) = \int_{\Omega} \frac{1}{p} |\nabla u(x)|^p + Q(x)\Gamma_{\varepsilon}(u(x)) \, dx \quad (1 < p < \infty)$$  \hspace{1cm} (1.1)

which is a singular perturbation of the one-phase problem of minimizing the functional associated with the $p$-Laplacian

$$J_p(u) = \int_{\Omega} \frac{1}{p} |\nabla u(x)|^p + Q(x)\chi_{\{u(x) > 0\}} \, dx,$$  \hspace{1cm} (1.2)

where $\Gamma_{\varepsilon}(s) = \Gamma(s)$ for $\varepsilon > 0$ and for a $C^\infty$ function $\Gamma$ defined by

$$\Gamma(s) = \begin{cases} 
0 & \text{if } s \leq 0 \\
1 & \text{if } s \geq 1, 
\end{cases}$$

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and $0 \leq \Gamma(s) \leq 1$ for $0 < s < 1$, and $Q \in W^{2,2}(\Omega)$ is a positive continuous function on $\Omega$ such that $\inf_{\Omega} Q(x) > 0$. Let $\beta_\varepsilon(s) = \Gamma'_\varepsilon(s) = \frac{1}{\varepsilon} \beta(\frac{s}{\varepsilon})$ with $\beta = \Gamma'$. The domain $\Omega$ is always assumed to be smooth in this paper for convenience. As in the following we will fix the value of $\varepsilon$ unless we specifically examine the influence of the value of $\varepsilon$ on the critical boundary data and will not use the notation $J_p$ for a different purpose, we are going to abuse the notation by using $J_p$ for the functional $J_{p,\varepsilon}$ from now on.

The Euler-Lagrange equation of (1.1) is

$$-\Delta_p u + Q(x) \beta_\varepsilon(u) = 0 \quad x \in \Omega \quad (1.3)$$

One imposes the boundary condition

$$u(x) = \sigma(x), \quad x \in \partial \Omega \quad (1.4)$$

on $u$, for $\sigma \in C(\partial \Omega)$ with $\min_{\partial \Omega} \sigma > 0$, to form a boundary value problem.

In this paper, we take on the task of establishing in the general case when $p \neq 2$ the results proved in [CW] for the Laplacian when $p = 2$. The main difficulty in this generalization lies in the lack of sufficient regularity and the singular-degenerate nature of the $p$-Laplacian when $p \neq 2$. A well-known fact about $p$-harmonic functions is the optimal regularity generally possessed by them is $C^{1,\alpha}$ (e.g. [E] and [Le]). Thus we need to employ more techniques associated with the $p$-Laplacian, and in a case or two we have to make our conclusion slightly weaker. Nevertheless, we follow the overall scheme of approach used in [CW]. In the second section, we prove the bifurcation phenomenon through the Mountain Pass Theorem. In the third section, we establish a parabolic comparison principle. In the last section, we show the convergence of an evolution to a stable steady state in accordance with respective initial data.

### 2 A Third Solution

We first prove if the boundary data is small enough, then the minimizer is nontrivial. More precisely, let $u_0$ be the trivial solution of (1.3) and (1.4), being $p$-harmonic in the weak sense, and $u_2$ be a minimizer of the $p$-functional (1.1), and set

$$\sigma_M = \max_{\partial \Omega} \sigma(x) \quad \text{and} \quad \sigma_m = \min_{\partial \Omega} \sigma(x).$$

If $\sigma_M$ is small enough, then $u_0 \neq u_2$.

In fact, we pick $u \in W^{1,p}(\Omega)$ so that

$$\begin{cases}
  u = 0 & \text{in } \Omega_\delta, \\
  u = \sigma & \text{on } \partial \Omega, \quad \text{and} \\
  -\Delta_p u = 0 & \text{in } \Omega \setminus \Omega_\delta,
\end{cases} \quad (2.1)$$

where $\Omega_\delta = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \delta \}$ and $\delta > 0$ is a small constant independent of $\varepsilon$ and $\sigma$ so that $\int_{\Omega_\delta} Q(x) \, dx$ has a positive lower bound which is also independent of $\varepsilon$ and $\sigma$. Using an approximating domain if necessary, we may assume $\Omega_\delta$ possesses a smooth boundary. Clearly,

$$J_p(u_0) = \int_\Omega \frac{1}{p} |\nabla u_0|^p + Q(x) \, dx \geq \int_\Omega Q(x) \, dx.$$
It is well-known that
\[ \int_{\Omega \setminus \hat{\Omega}_\delta} |\nabla u|^p \leq C \sigma_p \delta^{1-p} \] for \( C = C(n, p, \Omega) \), so that
\[ J_p(u) = \int_{\Omega \setminus \hat{\Omega}_\delta} \frac{1}{p} |\nabla u|^p + \int_{\Omega \setminus \hat{\Omega}_\delta} Q(x) \, dx \]
\[ \leq C \sigma_p \delta^{1-p} + \int_{\Omega \setminus \hat{\Omega}_\delta} Q(x) \, dx. \]
So, for all small \( \varepsilon > 0 \),
\[ J_p(u) - J_p(u_0) \leq C \sigma_p \delta^{1-p} - \int_{\Omega \setminus \hat{\Omega}_\delta} Q(x) \, dx < 0 \]
if \( \sigma_M \leq \sigma_0 \) for some small enough \( \sigma_0 = \sigma_0(\delta, \Omega, Q) \).

Let \( \mathcal{B} \) denote the Banach space \( W_0^1(\Omega) \) we will work with. For every \( v \in \mathcal{B} \), we write \( u = v + u_0 \) and adopt the norm \( ||v||_\mathcal{B} = (\int_\Omega |\nabla v|^p)^{\frac{1}{p}} = (\int_\Omega |\nabla u - \nabla u_0|^p)^{\frac{1}{p}} \). We define the functional
\[ I[v] = J_p(u) - J_p(u_0) = \int_{\Omega} \frac{1}{p} |\nabla u|^p - \int_{\{u < \varepsilon\}} Q(x) (1 - \Gamma_\varepsilon(u)) - \int_{\Omega} \frac{1}{p} |\nabla u_0|^p \] (2.2)
Set \( v_2 = u_2 - u_0 \). Clearly, \( I[0] = 0 \) and \( I[v_2] \leq 0 \) on account of the definition of \( u_2 \) as a minimizer of \( J_p \). If \( I[v_2] < 0 \) which is the case if \( \sigma_M \) is small, we will apply the Mountain Pass Lemma to prove there exists a critical point of the functional \( I \) which is a weak solution of the problem (1.3) and (1.4).

The Fréchet derivative of \( I \) at \( v \in \mathcal{B} \) is given by
\[ I'[v] \varphi = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi + Q(x) \beta_\varepsilon(u) \varphi \] \( \varphi \in \mathcal{B} \) (2.3)
which is obviously in the dual space \( \mathcal{B}^* \) of \( \mathcal{B} \) in light of the Hölder’s inequality. Equivalently
\[ I'[v] = -\Delta_p (v + u_0) + Q(x) \beta_\varepsilon(v + u_0) \in \mathcal{B}^*. \] (2.4)
We see that \( I' \) is Lipschitz continuous on any bounded subset of \( \mathcal{B} \) with Lipschitz constant depending on \( \varepsilon, p, \) and \( \sup Q \). In fact, for any \( v, w, \) and \( \varphi \in \mathcal{B}, \)
\[ |I'[v] \varphi - I'[w] \varphi| = |\int_{\Omega} |\nabla v + \nabla u_0|^{p-2} (\nabla v + \nabla u_0) \cdot \nabla \varphi + Q(x) \beta_\varepsilon(v + u_0) \varphi - (\nabla w + \nabla u_0)^{p-2} (\nabla w + \nabla u_0) \cdot \nabla \varphi - Q(x) \beta_\varepsilon(w + u_0) \varphi| \]
\[ \leq \left| \int_{\Omega} |\nabla v + \nabla u_0|^{p-2} (\nabla v + \nabla u_0) \cdot \nabla \varphi - |\nabla w + \nabla u_0|^{p-2} (\nabla w + \nabla u_0) \cdot \nabla \varphi \right| \]
\[ + \left| \int_{\Omega} Q(x) \beta_\varepsilon(v + u_0) - Q(x) \beta_\varepsilon(w + u_0) \right| \]
Furthermore,
\[
\left| \int_{\Omega} Q(x) \beta_{\varepsilon}(v + u_0) - Q(x) \beta_{\varepsilon}(w + u_0) \right|
\]
\[
= \left| \int_{\Omega} Q(x) \int_0^1 \beta'_{\varepsilon}((1-t)w + tv + u_0) \, dt \, (v(x) - w(x)) \, dx \right|
\]
\[
\leq \sup_{\Omega} |\beta'_{\varepsilon}| \int_{\Omega} |Q(x) (v(x) - w(x))| \, dx
\]
\[
\leq C_{\varepsilon^2} \left( \int_{\Omega} Q_{\beta'}(x) \right)^{\frac{1}{p}} \left( \int_{\Omega} |v(x) - w(x)|^p \, dx \right)^{\frac{1}{p}}
\]

and
\[
\left| \int_{\Omega} |\nabla v + \nabla u_0|^{p-2} (\nabla v + \nabla u_0) \cdot \nabla \varphi - |\nabla w + \nabla u_0|^{p-2} (\nabla w + \nabla u_0) \cdot \nabla \varphi \right|
\]
\[
\leq \left( \int_{\Omega} |\nabla v + \nabla u_0|^p \right)^{\frac{p-2}{p}} \left( \int_{\Omega} |\nabla \varphi|^p \right)^{\frac{1}{p}} \left( \int_{\Omega} |\nabla v - \nabla w|^p \right)^{\frac{1}{p}},
\]

and
\[
\left| \int_{\Omega} (|\nabla v + \nabla u_0|^{p-2} - |\nabla w + \nabla u_0|^{p-2}) (\nabla w + \nabla u_0) \cdot \nabla \varphi \right|
\]
\[
\leq C(p) \int_{\Omega} (||\nabla v||_{L^p} + ||\nabla w||_{L^p} + ||\nabla u_0||_{L^p})^{p-2} \|\nabla v - \nabla w\|_{L^p(\Omega)} \|\nabla \varphi\|_{L^p(\Omega)}.
\]

Therefore $I'$ is Lipschitz continuous on bounded subsets of $\mathcal{B}$.

We note that $f \in \mathcal{B}^*$ if and only if there exist $f^0$, $f^1$, $f^2$, ..., $f^n \in L^{p'}(\Omega)$, where
\[
\frac{1}{p} + \frac{1}{p'} = 1,
\]
such that
\[
< f, u > = \int_{\Omega} f^0 u + \sum_{i=1}^n f^i u_{x_i} \quad \text{holds for all } u \in \mathcal{B}; \quad \text{and} \quad (2.5)
\]
\[
\left\| f \right\|_{\mathcal{B}^*} = \inf \left\{ \left( \int_{\Omega} \sum_{i=0}^n |f^i|^p \, dx \right)^{\frac{1}{p}} : (2.5) \text{ holds.} \right\}
\]

(2.6)
Next we justify the Palais-Smale condition on the functional $I$. Suppose $\{v_k\} \subset B$ is a Palais-Smale sequence in the sense that

$$|I[v_k]| \leq M \quad \text{and} \quad I'[v_k] \to 0 \quad \text{in } B^*$$

for some $M > 0$. Let $u_k = v_k + u_0 \in W^{1,p}(\Omega)$, $k = 1, 2, 3, \ldots$.

That $Q(x)\beta_\varepsilon(v + u_0) \in W^{1,p}_0(\Omega)$ implies that the mapping $v \mapsto Q(x)\beta_\varepsilon(v + u_0)$ from $W^{1,p}_0(\Omega)$ to $B^*$ is compact due to the fact $W^{1,p}_0(\Omega) \subset L^p(\Omega) \subset B^*$ following from the Rellich-Kondrachov Compactness Theorem. Then there exists $f \in L^p(\Omega) \subset B^*$ such that for a subsequence, still denoted by $\{v_k\}$, of $\{v_k\}$, it holds that

$$Q(x)\beta_\varepsilon(u_k) \to -f \quad \text{in } L^p(\Omega).$$

Recall that

$$|I'[v_k]| = \sup_{\|\varphi\|_{B^*} \leq 1} \left| \int_\Omega |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla \varphi + Q(x)\beta_\varepsilon(u_k)\varphi \right| \to 0.\quad (2.7)$$

As a consequence,

$$\sup_{\|\varphi\|_{B^*} \leq M} \left| \int_\Omega |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla \varphi - f \varphi \right| \to 0 \quad \text{for any } M \geq 0.\quad (2.7)$$

Obviously, that $\{I[v_k]\}$ is bounded implies that a subsequence of $\{v_k\}$, still denoted by $\{v_k\}$ by abusing the notation without confusion, converges weakly in $B = W^{1,p}_0(\Omega)$. In particular,

$$\int_\Omega f v_k - f v_m \to 0 \quad \text{as } k, m \to \infty.$$

Then by setting $\varphi = v_k - v_m = u_k - u_m$ in (2.7), one gets

$$\left| \int_\Omega (|\nabla u_k|^{p-2} \nabla u_k - |\nabla u_m|^{p-2} \nabla u_m) \cdot \nabla (u_k - u_m) \right| \to 0 \quad \text{as } k, m \to \infty,\quad (2.8)$$

since

$$\|u_k - u_m\|^p_B = \|v_k - v_m\|^p_B \leq 2pM + 2J_p[u_0].$$

In particular, if $p = 2$, $\{v_k\}$ is a Cauchy sequence in $W^{1,2}_0(\Omega)$ and hence converges. We will apply the following elementary inequalities associated with the $p$-Laplacian, $[L]$, to the general case $p \neq 2$:

$$< |b|^{p-2}b - |a|^{p-2}a, b - a > \geq (p - 1)|b - a|^2(1 + |a|^2 + |b|^2)^{\frac{p-2}{2}}, \quad 1 \leq p \leq 2;\quad (2.9)$$

and

$$< |b|^{p-2}b - |a|^{p-2}a, b - a > \geq 2^{p-2}|b - a|^p, \quad p \geq 2.\quad (2.10)$$

We assume first $1 < p < 2$. Let $K = 2pM + 2J_p[u_0]$. Then the first elementary inequality (2.9) implies

$$(p - 1) \int_\Omega |\nabla u_k - \nabla u_m|^2 \left(1 + |\nabla u_k|^2 + |\nabla u_m|^2\right)^{\frac{p-2}{2}}$$

$$\leq \int_\Omega (|\nabla u_k|^{p-2} \nabla u_k - |\nabla u_m|^{p-2} \nabla u_m) \cdot \nabla (u_k - u_m) \to 0$$
Meanwhile Hölder’s inequality implies
\[
\int_{\Omega} |\nabla v_k - \nabla v_m|^p = \int_{\Omega} |\nabla u_k - \nabla u_m|^p \\
\leq \left( \int_{\Omega} |\nabla u_k - \nabla u_m|^2 \right)^{\frac{p}{2}} \left( \int_{\Omega} \left( |\nabla u_k|^2 + |\nabla u_m|^2 \right)^{\frac{p}{2}} \right)^{\frac{2}{p}} \\
\leq C(p) \left( |\Omega| + K \right)^{\frac{2-p}{2}} \left( \int_{\Omega} |\nabla u_k - \nabla u_m|^2 \right)^{\frac{2-p}{2}} \left( \int_{\Omega} \left( |\nabla u_k|^2 + |\nabla u_m|^2 \right)^{\frac{p}{2}} \right)^{\frac{2}{p}}
\]
Therefore, \( \{v_k\} \) is a Cauchy sequence in \( \mathcal{B} \) and hence converges.

Suppose \( p > 2 \). The second elementary inequality (2.10) implies
\[
\int_{\Omega} |\nabla v_k - \nabla v_m|^p = \int_{\Omega} |\nabla u_k - \nabla u_m|^p \\
\leq 2^{p-2} \int_{\Omega} \left( |\nabla u_k|^{p-2} |\nabla v_k - |\nabla u_m|^{p-2} |\nabla v_m| \right) \cdot (\nabla u_k - \nabla u_m),
\]
which in turn implies \( \{v_k\} \) is a Cauchy sequence in \( \mathcal{B} \) and hence converges, on account of (2.8). The Palais-Smale condition is verified for \( 1 < p < \infty \) for the functional \( I \) on the Banach space \( W^{1,p}_0(\Omega) \).

Before we continue the main proof, let us state an elementary result closely related to the \( p \)-Laplacian, which follows readily from the Fundamental Theorem of Calculus.

**Lemma 2.1** For any \( a \) and \( b \) \( \in \mathbb{R}^n \), it holds
\[
|b|^p \geq |a|^p + p < |a|^{p-2} a, b - a > + C(p)|b - a|^p \quad (p \geq 2)
\]
where \( C(p) > 0 \).

*If \( 1 < p < 2 \), then*
\[
|b|^p \geq |a|^p + p < |a|^{p-2} a, b - a > + C(p)|b - a|^p \int_{0}^{1} \int_{0}^{t} |(1-s)a + sb|^{p-2} \, ds \, dt,
\]
where \( C(p) = p(p-1) \).

We are now in a position to show there is a closed mountain ridge around the origin of the Banach space \( \mathcal{B} \) that separates \( v_2 \) from the origin with the energy \( I \) as the elevation function, which is the content of the following lemma.

**Lemma 2.2** For all small \( \varepsilon > 0 \) such that \( C \varepsilon \leq \frac{1}{2} \sigma_m \) for a large universal constant \( C \), there exist positive constants \( \delta \) and \( \alpha \) independent of \( \varepsilon \), such that, for every \( v \) in \( \mathcal{B} \) with \( \|v\|_{\mathcal{B}} = \delta \), the inequality \( I[v] \geq a \) holds.

**Proof.** It suffices to prove \( I[v] \geq a > 0 \) for every \( v \in C_0^\infty(\Omega) \) with \( \|v\|_{\mathcal{B}} = \delta \) for \( \delta \) small enough, as \( I \) is continuous on \( \mathcal{B} \), and \( C_0^\infty(\Omega) \) is dense in \( \mathcal{B} \).

Let \( \Lambda = \{x \in \Omega : u(x) \leq \varepsilon \} \), where \( u = v + u_0 \). We claim that \( \Lambda = \emptyset \) if \( \delta \) is small enough. If not, one may pick \( z \in \Lambda \). Let \( \mathcal{AC}([a, b], S) \) be the set of absolutely continuous
functions \( \gamma : [a, b] \to S \), where \( S \subseteq \mathbb{R}^n \). For each \( \gamma \in \mathcal{AC}([a, b], S) \), we define its length to be \( L(\gamma) = \int_a^b |\gamma'(t)| \, dt \). For \( x_0 \in \partial \Omega \), we define the distance from \( x_0 \) to \( z \) to be

\[
d(x_0, z) = \inf\{L(\gamma) : \gamma \in \mathcal{AC}([0, 1], \Omega), \text{ s.t. } \gamma(0) = x_0, \text{ and } \gamma(1) = z\}
\]

As shown in [CW], there is a minimizing path \( \gamma_{x_0} \) for the distance \( d(x_0, z) \).

Suppose the domain \( \Omega \) is convex or star-like about \( z \). For any \( x_0 \in \partial \Omega \), let \( \gamma = \gamma_{x_0} \) be a minimizing path of \( d(x_0, z) \). Then it is clear that \( \gamma \) is a straight line segment and \( \gamma(t) \neq z \) for \( t \in [0, 1] \). Furthermore, for any two distinct points \( x_1 \) and \( x_2 \in \partial \Omega \), the corresponding minimizing paths do not intersect in \( \Omega \setminus \{z\} \). For this reason, we can carry out the following computation. Clearly \( v(x_0) = 0 \) and \( v(\gamma(1)) = \varepsilon - u_0(\gamma(1)) \leq \varepsilon - \sigma_m < 0 \). So the Fundamental Theorem of Calculus

\[
v(\gamma(1)) - v(\gamma(0)) = \int_0^1 \nabla v(\gamma(t)) \cdot \gamma'(t) \, dt
\]

implies

\[
\sigma_m - \varepsilon \leq \int_0^1 |\nabla v(\gamma(t))| |\gamma'(t)| \, dt. \tag{2.13}
\]

For each \( x_0 \in \partial \Omega \), let \( e(x_0) \) be the unit vector in the direction of \( x_0 - z \) and \( \nu(x_0) \) the outer normal to \( \partial \Omega \) at \( x_0 \). Then \( \nu(x_0) \cdot e(x_0) > 0 \) everywhere on \( \partial \Omega \). Hence the above inequality (2.13) implies

\[
(\sigma_m - \varepsilon) \int_{\partial \Omega} \nu(x_0) \cdot e(x_0) \, dH^{n-1}(x_0)
\]

\[
\leq \int_{\partial \Omega} \int_0^1 |\nabla v(\gamma(t))| |\gamma'(t)| \, dt \nu(x_0) \cdot e(x_0) \, dH^{n-1}(x_0)
\]

\[
\leq \int_{\partial \Omega} \left( \int_0^1 |\gamma'(t)| \, dt \right)^{\frac{1}{p'}} \left( \int_0^1 |\nabla v(\gamma(t))|^p |\gamma'(t)| \, dt \right)^{\frac{1}{p}} \nu(x_0) \cdot e(x_0) \, dH^{n-1}(x_0),
\]

where \( \frac{1}{p} + \frac{1}{p'} = 1 \),

\[
= \int_{\partial \Omega} L(\gamma_{x_0}) \left( \int_0^1 |\nabla v(\gamma(t))|^p |\gamma'(t)| \, dt \right)^{\frac{1}{p'}} \nu(x_0) \cdot e(x_0) \, dH^{n-1}(x_0)
\]

\[
\leq \left( \int_{\partial \Omega} L(\gamma_{x_0}) \nu(x_0) \cdot e(x_0) \, dH^{n-1} \right)^{\frac{1}{p'}} \left( \int_{\partial \Omega} \int_0^1 |\nabla v(\gamma(t))|^p |\gamma'(t)| \nu \cdot e \, dt \, dH^{n-1} \right)^{\frac{1}{p}}
\]

\[
= C|\Omega| \left( \int_{\Omega} |\nabla v|^p \, dx \right)^{\frac{1}{p'}} \left( \int_{\partial \Omega} \int_0^1 |\nabla v(\gamma(t))|^p |\gamma'(t)| \nu \cdot e \, dt \, dH^{n-1} \right)^{\frac{1}{p}}
\]

\[
\leq C\{|u > \varepsilon\}|^{\frac{1}{p'}} \delta \leq C\{|u > 0\}|^{\frac{1}{p'}} \delta,
\]

where the second and third inequalities are due to the application of the Hölder’s inequality, and the constant \( C \) depends on \( n \) and \( p \). The second equality follows from the two representation formulas

\[
|\Omega| = C(n) \int_{\partial \Omega} L(\gamma_{x_0}) \nu(x_0) \cdot e(x_0) \, dH^{n-1}(x_0)
\]
and
\[ \int_{\Omega} |\nabla v(x)|^p \, dx = C(n) \int_{\partial \Omega} \int_0^1 |\nabla v(\gamma_{x_0}(t))|^p \left| \gamma'_{x_0}(t) \right| \nu(x_0) \cdot e(x_0) \, dt \, dH^{n-1}(x_0). \]

If we take \( \delta \) sufficiently small and independent of \( \varepsilon \) in the preceding inequality
\[ (\sigma_m - \varepsilon) \int_{\partial \Omega} \nu(x_0) \cdot e(x_0) \, dH^{n-1}(x_0) \leq C|\{u > 0\}|^{\frac{p}{p'}} \delta, \]
the measure \(|\{u > 0\}|\) of the positive domain would be greater than that of \( \Omega \), which is impossible, provided that
\[ \int_{\partial \Omega} \nu(x_0) \cdot e(x_0) \, dH^{n-1}(x_0) \geq C, \quad (2.14) \]
for a constant \( C \) which depends on \( n, p \) and \(|\Omega|\), but not on \( z \) or \( v \). Hence \( \Lambda \) must be empty. So we need to justify the inequality (2.14). To fulfil that condition, for \( e = e(x_0) \), we set \( l(e, z) = l(e) = L(\gamma_{x_0}) \). Then
\[ \int_{\partial \Omega} \nu(x_0) \cdot e(x_0) \, dH^{n-1}(x_0) = \int_{e \in \partial B} (l(e))^{n-1} \, d\sigma(e), \]
where \( B \) is the unit ball about \( z \) and \( d\sigma(e) \) is the surface area element on the unit sphere \( \partial B \) which is invariant under rotation and reflection. Clearly,
\[ \left( \int_{\partial B} (l(e))^{n-1} \, d\sigma(e) \right)^{\frac{2}{n-1}} \geq C(n) \int_{\partial B} l^2(e) \, d\sigma(e) \]
Consequently, in order to prove (2.14), one needs only to prove
\[ \int_{\partial B} l^2(e) \, d\sigma(e) \geq C(n, p, |\Omega|). \quad (2.15) \]

Next, we show the integral on the left-hand-side of (2.15) is minimal if \( \Omega \) is a ball while its measure is kept unchanged. In fact, this is almost obvious if one notices the following fact. Let \( \pi \) be any hyperplane passing through \( z \), and \( x_1 \) and \( x_2 \) be the points on \( \partial \Omega \) which lie on a line perpendicular to \( \pi \). Let \( x_1^* \) and \( x_2^* \) be the points on the boundary \( \partial \Omega_{x^*} \), where \( \Omega_{x^*} \) is the symmetrized image of \( \Omega \) about the hyperplane \( \pi \), which lie on the line \( x_1x_2 \). Let \( 2a = |x_1x_2| = |x_1^*x_2^*| \) and \( d \) be the distance from \( z \) to the line \( x_1x_2 \). Then for some \( t \) in \(-a \leq t \leq a\), it holds that
\[ L^2(\gamma_{x_1}) + L^2(\gamma_{x_2}) = (d^2 + (a - t)^2) + (d^2 + (a + t)^2) \geq 2(d^2 + a^2) = 2 \left( L^*(\gamma_{x^*_1}) \right)^2. \]
As a consequence, if \( \Omega^* \) is the symmetrized ball with measure equal to that of \( \Omega \), then
\[ \int_{\partial B} l^2(e) \, d\sigma(e) \geq \int_{\partial B} (l^*(e))^2 \, d\sigma(e) = C(n, |\Omega|), \]
\[ \int_{\partial B} l^2(e) \, d\sigma(e) \geq C(n, |\Omega|). \]
where \( l^* \) is the length from \( z \) to a point on the boundary \( \partial \Omega^* \) which is constant. This finishes the proof of the fact that \( \Lambda = \emptyset \).

In case the domain \( \Omega \) is not convex, the minimizing paths of \( d(x_1, z) \) and \( d(x_2, z) \) for distinct \( x_1, x_2 \in \partial \Omega \) may partially coincide. We form the set \( \mathcal{DA}(\partial \Omega) \) of the points \( x_0 \) on \( \partial \Omega \) so that a minimizing path \( \gamma \) of \( d(x_0, z) \) satisfies \( \gamma(t) \in \Omega \setminus \{z\} \) for \( t \in (0, 1) \). We call a point in \( \mathcal{DA}(\partial \Omega) \) a **directly accessible** boundary point. Let \( \Omega_1 \) be the union of these minimizing paths for the directly accessible boundary points. It is not difficult to see that \( |\Omega_1| > 0 \) and hence \( H^{n-1}(\mathcal{DA}(\partial \Omega)) > 0 \). Then we may apply the above computation to the star-like domain \( \Omega_1 \) with minimal modification. We have

\[
(\sigma_m - C\varepsilon) \int_{\partial \Omega} \nu(x_0) \cdot e(x_0) \, dH^{n-1}(x_0) \leq C|\Omega_1|^{\frac{1}{p}} \delta \leq C|\Omega|^{\frac{1}{p}} \delta. \tag{2.16}
\]

For small enough \( \delta \), this raises a contradiction \( |\Omega| > |\Omega| \). So \( \Lambda = \emptyset \).

Finally we prove that \( ||v||_{\Omega_1} = \delta \) implies

\[
I[v] = \int_{\Omega} \frac{1}{p} |\nabla v + \nabla u_0|^p - \frac{1}{p} |\nabla u_0|^p \geq a \quad \text{for a certain} \quad a > 0. \tag{2.17}
\]

If \( p \geq 2 \), then the elementary inequality (2.11) implies that

\[
I[v] \geq p(p-1) \int_{\Omega} |\nabla v|^2 \int_{0}^{1} \int_{0}^{t} \frac{1}{|\nabla u_0 + s \nabla v|^{2-p}} \, ds \, dt \, dx \geq p(p-1) \int_{\Omega} |\nabla v|^2 \int_{0}^{1} \int_{0}^{t} \frac{1}{(|\nabla u_0| + s |\nabla v|)^{2-p}} \, ds \, dt \, dx.
\]

If \( \int_{\Omega} |\nabla u_0|^p = 0 \), then \( I[v] = \frac{1}{p} \delta^p > 0 \). So in the following, we assume \( \int_{\Omega} |\nabla u_0|^p > 0 \).

Let \( S = S_{\lambda} = \{ x \in \Omega : |\nabla v| > \lambda \delta \} \), where the constant \( \lambda = \lambda(p, |\Omega|) \) is to be taken. Then

\[
\delta^p = \int_{\Omega} |\nabla v|^p = \int_{|\nabla v| \leq \lambda \delta} |\nabla v|^p + \int_{S} |\nabla v|^p \leq (\lambda \delta)^p |\Omega| + \int_{S} |\nabla v|^p
\]

and hence

\[
\int_{S} |\nabla v|^p \geq \delta^p (1 - \lambda^p |\Omega|) \geq \frac{1}{2} \delta^p, \quad \text{if} \quad \lambda \quad \text{satisfies} \quad \frac{1}{4} < \lambda^p |\Omega| \leq \frac{1}{2}.
\]
Meanwhile, for $1 < p < 2$, it holds that
\[
I[v] \geq C(p) \int_S |\nabla v|^2 \int_0^1 \int_0^t \frac{1}{(|\nabla u_0| + s|\nabla v|)^{2-p}} \, ds \, dt \, dx
\]
\[
= C(p) \left( \int_{S \cap \{|\nabla u_0| \leq |\nabla v|\}} |\nabla v|^2 \int_0^1 \int_0^t \frac{1}{(|\nabla u_0| + s|\nabla v|)^{2-p}} \, ds \, dt \, dx \right)
\]
\[
+ \int_{S \cap \{|\nabla u_0| > |\nabla v|\}} |\nabla v|^2 \int_0^1 \int_0^t \frac{1}{(|\nabla u_0| + s|\nabla v|)^{2-p}} \, ds \, dt \, dx .
\]
The first integral on the right satisfies
\[
\int_{S \cap \{|\nabla u_0| \leq |\nabla v|\}} |\nabla v|^2 \int_0^1 \int_0^t \frac{1}{(|\nabla u_0| + s|\nabla v|)^{2-p}} \, ds \, dt \, dx
\]
\[
\geq C(p) \int_{S \cap \{|\nabla u_0| \leq |\nabla v|\}} |\nabla v|^p \int_0^1 \int_0^t \frac{1}{(1 + s)^{2-p}} \, ds \, dt \, dx
\]
\[
= C(p) \int_{S \cap \{|\nabla u_0| \leq |\nabla v|\}} |\nabla v|^p \, dx,
\]
while the second integral on the right satisfies
\[
\int_{S \cap \{|\nabla u_0| > |\nabla v|\}} |\nabla v|^2 \int_0^1 \int_0^t \frac{1}{(|\nabla u_0| + s|\nabla v|)^{2-p}} \, ds \, dt \, dx
\]
\[
\geq C(p) \int_{S \cap \{|\nabla u_0| > |\nabla v|\}} |\nabla v|^2 |\nabla u_0|^{2-p} \int_0^1 \int_0^t \frac{ds \, dt}{(1 + s)^{2-p}} \, dx
\]
\[
= C(p) \int_{S \cap \{|\nabla u_0| > |\nabla v|\}} |\nabla v|^2 \frac{|\nabla u_0|^{2-p}}{dx}.
\]
The H"older's inequality applied with exponents $\frac{2}{p}$ and $\frac{2}{2-p}$ implies that
\[
\int_{S \cap \{|\nabla u_0| > |\nabla v|\}} |\nabla v|^p \leq \left( \int_{S \cap \{|\nabla u_0| > |\nabla v|\}} \frac{|\nabla v|^2}{|\nabla u_0|^{2-p}} \right)^{\frac{p}{2}} \left( \int_{S \cap \{|\nabla u_0| > |\nabla v|\}} |\nabla u_0|^p \right)^{\frac{2-p}{2}} ,
\]
or equivalently
\[
\int_{S \cap \{|\nabla u_0| > |\nabla v|\}} \frac{|\nabla v|^2}{|\nabla u_0|^{2-p}} \geq \left( \int_{S \cap \{|\nabla u_0| > |\nabla v|\}} \frac{|\nabla v|^p}{|\nabla u_0|^{2-p}} \right)^{\frac{2}{p}} \left( \int_{S \cap \{|\nabla u_0| > |\nabla v|\}} |\nabla u_0|^p \right)^{\frac{2-p}{p}}
\]
\[
\geq \left( \int_{S \cap \{|\nabla u_0| > |\nabla v|\}} \frac{|\nabla v|^p}{|\nabla u_0|^{2-p}} \right)^{\frac{2}{p}} \left( \int_{\Omega} |\nabla u_0|^p \right)^{\frac{2-p}{p}} .\]
Consequently,

\[ I[v] \geq C(p) \int_{S \cap \{ |\nabla u_0| \leq |\nabla v| \}} |\nabla v|^p + C(p) \frac{\left( \int_{S \cap \{ |\nabla u_0| > |\nabla v| \}} |\nabla v|^p \right)^{\frac{2}{p}}}{\left( \int_{\Omega} |\nabla u_0|^p \right)^{\frac{2-p}{p}}}, \]

where the last inequality is a consequence of the elementary inequality

\[ a^2 + b^2 \geq C(p) (a + b)^2 \]

for \( a, b \geq 0 \), and the constant

\[ A(u_0) = \min \left\{ 1, \frac{1}{\left( \int_{\Omega} |\nabla u_0|^p \right)^{\frac{2-p}{p}}} \right\}. \]

So we have proved \( I[v] \geq a > 0 \) for some \( a > 0 \) whenever \( v \in C_0^\infty(\Omega) \) satisfies \( \|v\|_B = \delta \), for any \( p \in (1, \infty) \). \( \square \)

Let

\[ \mathcal{G} = \{ \gamma \in C([0,1], H) : \gamma(0) = 0 \text{ and } \gamma(1) = v_2 \} \]

and

\[ c = \inf_{\gamma \in \mathcal{G}} \max_{0 \leq t \leq 1} I[\gamma(t)]. \]

The verified Palais-Smale condition and the preceding lemma allow us to apply the Mountain Pass Theorem as stated, for example, in [J] to conclude that there is a \( v_1 \in \mathfrak{B} \) such that \( I[v_1] = c \), and \( I'[v_1] = 0 \) in \( \mathfrak{B}^* \). That is

\[ \int_{\Omega} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \varphi + Q(x) \beta_\varepsilon(u_1) \varphi dx = 0 \]

for any \( \varphi \in \mathfrak{B} = W_0^{1,p}(\Omega) \), where \( u_1 = v_1 + u_0 \). So \( u_1 \) is a weak solution of the problem (1.3) and (1.4). In essence, the Mountain Pass Theorem is a way to produce a saddle point solution. Therefore, in general, \( u_1 \) tends to be an unstable solution in contrast to the stable solutions \( u_0 \) and \( u_2 \).

In this section, we have proved the following theorem.

**Theorem 2.3** If \( \varepsilon \ll \sigma_m \) and \( J_p(u_2) < J_p(u_0) \), then there exists a third weak solution \( u_1 \) of the problem (1.3) and (1.4). Moreover, \( J_p(u_1) \geq J_p(u_0) + a \), where \( a \) is independent of \( \varepsilon \).
3 A Comparison Principle for Evolution

In this section, we prove a comparison theorem for the following evolution problem.

\[
\begin{cases}
w_t - \Delta_p w + \alpha(x, w) = 0 & \text{in } \Omega \times (0, T) \\
w(x, t) = \sigma(x) & \text{on } \partial \Omega \times (0, T) \\
w(x, 0) = v_0(x) & \text{for } x \in \bar{\Omega},
\end{cases}
\]  

(3.1)

where \( T > 0 \) may be finite or infinite, and \( \alpha \) is a continuous function satisfying \( 0 \leq \alpha(x, w) \leq Kw \) and \( |\alpha(x, r_2) - \alpha(x, r_1)| \leq K|r_2 - r_1| \) for all \( x \in \Omega, r_1 \) and \( r_2 \in \mathbb{R} \), and some \( K \geq 0 \). Let us introduce the notation \( H_p w = w_t - \Delta_p w + \alpha(x, w) \). We recall a weak sub-solution \( w \in L^2(0, T; W^{1,p}(\Omega)) \) satisfies

\[
\int_V w\varphi \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_V -w\varphi_t + |\nabla w|^{p-2}\nabla w \cdot \nabla \varphi + \alpha(x, w)\varphi \leq 0
\]

for any region \( V \subset \subset \Omega \) and any test function \( \varphi \in L^2(0, T; W^{1,p}(\Omega)) \) such that \( \varphi_t \in L^2(\Omega \times \mathbb{R}_T) \) and \( \varphi \geq 0 \) in \( \Omega \times \mathbb{R}_T \), where \( L^2(0, T; W^{1,p}(\Omega)) \) is the subset of \( L^2(0, T; W^{1,p}(\Omega)) \) that contains functions which is equal zero on the boundary of \( \Omega \times \mathbb{R}_T \), where \( \mathbb{R}_T = [0, T] \).

For convenience, we let \( \mathfrak{H} \) denote this set of test functions in the following.

In particular, it holds that

\[
\int_0^T \int_\Omega -w\varphi_t + |\nabla w|^{p-2}\nabla w \cdot \nabla \varphi + \alpha(x, w)\varphi \leq 0
\]

for any test function \( \varphi \in L^2_0(0, T; W^{1,p}(\Omega)) \) such that \( \varphi_t \in L^2(\Omega \times \mathbb{R}_T) \) and \( \varphi \geq 0 \) in \( \Omega \times \mathbb{R}_T \).

The comparison principle for weak sub- and super-solutions is stated as follows.

**Theorem 3.1** Suppose \( w_1 \) and \( w_2 \) are weak sub- and super-solutions of the evolutionary problem (3.1) respectively with \( w_1 \leq w_2 \) on the parabolic boundary \( (\bar{\Omega} \times \{0\}) \cup (\partial \Omega \times (0, +\infty)) \). Then \( w_1 \leq w_2 \) in \( \mathcal{D} \).

Uniqueness of a weak solution of (3.1) follows from the comparison principle, Theorem 3.1, immediately.

**Lemma 3.2** For \( T > 0 \) small enough, if \( H_p w_1 \leq 0 \leq H_p w_2 \) in the weak sense in \( \Omega \times \mathbb{R}_T \) and \( w_1 < w_2 \) on \( \partial_p(\Omega \times \mathbb{R}_T) \), then \( w_1 \leq w_2 \) in \( \Omega \times \mathbb{R}_T \).

**Proof.** For any given small number \( \delta > 0 \), we define a new function \( \tilde{w}_1 \) by

\[
\tilde{w}_1(x, t) = w_1(x, t) - \frac{\delta}{T - t},
\]

where \( x \in \bar{\Omega} \) and \( 0 \leq t < T \). In order to prove \( w_1 \leq w_2 \) in \( \Omega \times \mathbb{R}_T \), it suffices to prove \( \tilde{w}_1 \leq w_2 \) in \( \Omega \times \mathbb{R}_T \) for all small \( \delta > 0 \). Clearly, \( \tilde{w}_1 < w_2 \) on \( \partial_p(\Omega \times \mathbb{R}_T) \), and

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\[ \lim_{t \to T} \tilde{w}_1(x, t) = -\infty \] uniformly on \( \Omega \). Moreover, the following holds for any \( \varphi \in \mathfrak{T}_+ \):

\[
\int_0^T \int_\Omega \tilde{w}_1 \varphi_t + < \nabla \tilde{w}_1 |^{p-2} \nabla \tilde{w}_1, \nabla \varphi > + \alpha(x, \tilde{w}_1) \varphi \\
= \int_0^T \int_\Omega -w_1 \varphi_t + < \nabla w_1 |^{p-2} \nabla w_1, \nabla \varphi > + \frac{\delta}{T-t} \varphi_t + (\alpha(x, \tilde{w}_1) - \alpha(x, w_1)) \varphi \\
\leq \int_0^T \int_\Omega \frac{\delta}{T-t} \varphi_t + K \frac{\delta}{T-t} \varphi, \text{ as } w_1 \text{ is a weak sub-solution}
\]

\[
= \int_0^T \int_\Omega \left( -\frac{\delta}{(T-t)^2} + K \frac{\delta}{T-t} \right) \varphi \\
\leq \int_0^T \int_\Omega \frac{\delta}{2(T-t)^2} \varphi, \text{ for } T \leq \frac{1}{2K} \text{ so that } 2K \leq \frac{1}{T-t} < 0,
\]

i.e.

\[ H_p \tilde{w}_1 \leq -\frac{\delta}{2(T-t)^2} \leq -\frac{\delta}{2T^2} < 0 \] in the weak sense.

That is, if we abuse the notation a little by denoting \( \tilde{w}_1 \) by \( w_1 \) in the following for convenience, it holds for any \( \varphi \in \mathfrak{T}_+ \),

\[ \int_0^T \int_\Omega -w_1 \varphi_t + < \nabla w_1 |^{p-2} \nabla w_1, \nabla \varphi > + \alpha(x, w_1) \varphi \leq \int_0^T \int_\Omega \frac{\delta}{2T^2} \varphi < 0. \]

Meanwhile, for any \( \varphi \in \mathfrak{T}_+ \), \( w_2 \) satisfies

\[ \int_0^T \int_\Omega -w_2 \varphi_t + < \nabla w_2 |^{p-2} \nabla w_2, \nabla \varphi > + \alpha(x, w_2) \varphi \geq 0. \]

Define, for \( j = 1, 2 \), \( v_j(x, t) = e^{-\lambda t} w_j(x, t) \), where the constant \( \lambda > 2K \). Then \( w_j(x, t) = e^\lambda v_j(x, t) \), and it is clear that \( w_1 \leq w_2 \) in \( \Omega \times \mathbb{R}_T \) is equivalent to \( v_1 \leq v_2 \) in \( \Omega \times \mathbb{R}_T \). In addition, for any \( \varphi \in \mathfrak{T}_+ \), the following inequalities hold:

\[
\int_0^T \int_\Omega -e^\lambda v_1 \varphi_t + e^\lambda (p-1)t < |\nabla v_1|^{p-2} \nabla v_1, \nabla \varphi > + \alpha(x, e^\lambda v_1) \varphi \leq -\int_0^T \int_\Omega \frac{\delta}{2T^2} \varphi
\]

and

\[
\int_0^T \int_\Omega -e^\lambda v_2 \varphi_t + e^\lambda (p-1)t < |\nabla v_2|^{p-2} \nabla v_2, \nabla \varphi > + \alpha(x, e^\lambda v_2) \varphi \geq 0.
\]

Consequently, it holds for any \( \varphi \in \mathfrak{T}_+ \)

\[
\int_0^T \int_\Omega -e^\lambda (v_1 - v_2) \varphi_t + e^\lambda (p-1)t < |\nabla v_1|^{p-2} \nabla v_1 - |\nabla v_2|^{p-2} \nabla v_2, \nabla \varphi > + (\alpha(x, e^\lambda v_1) - \alpha(x, e^\lambda v_2)) \varphi \leq -\int_0^T \int_\Omega \frac{\delta}{2T^2} \varphi.
\]
We take \( \varphi = (v_1 - v_2)^+ = \max\{v_1 - v_2, 0\} \) as the test function, since it vanishes on the boundary of \( \Omega \times \mathbb{R}_T \). Then

\[
\int_0^T \int_{\{v_1 > v_2\}} -e^\lambda (v_1 - v_2)(v_1 - v_2)_t + e^\lambda (p-1)t < |\nabla v_1|^{p-2} \nabla v_1 - |\nabla v_2|^{p-2} \nabla v_2, \nabla v_1 - \nabla v_2 > \\
+ \left( \alpha(x, e^\lambda v_1) - \alpha(x, e^\lambda v_2) \right) (v_1 - v_2) \leq -\frac{\delta}{2T^2} \int_0^T \int_{\{v_1 > v_2\}} (v_1 - v_2).
\]

Since \( \{v_1 > v_2\} \subset \Omega \times (0, T) \) due to the facts \( v_1 \leq v_2 \) on \( \partial_p(\Omega \times \mathbb{R}_T) \) and \( v_1 \to -\infty \) as \( t \uparrow T \), the divergence theorem implies

\[
\int_0^T \int_{\{v_1 > v_2\}} -e^\lambda (v_1 - v_2)(v_1 - v_2)_t = \int_0^T \int_{\{v_1 > v_2\}} \lambda e^\lambda \frac{1}{2} (v_1 - v_2)^2.
\]

On the other hand,

\[
\left( \alpha(x, e^\lambda v_1) - \alpha(x, e^\lambda v_2) \right) (v_1 - v_2) \geq -K e^\lambda (v_1 - v_2)^2 \text{ on } \{v_1 > v_2\}.
\]

As a consequence, it holds that

\[
\int_0^T \int_{\{v_1 > v_2\}} \left( \frac{\lambda}{2} - K \right) e^\lambda (v_1 - v_2)^2 + e^\lambda (p-1)t < |\nabla v_1|^{p-2} \nabla v_1 - |\nabla v_2|^{p-2} \nabla v_2, \nabla v_1 - \nabla v_2 > \\
\leq -\frac{\delta}{2T^2} \int_0^T \int_{\{v_1 > v_2\}} (v_1 - v_2).
\]

We call into play two elementary inequalities (\([L]\)) associated with the \( p \)-Laplacian:

\[
< |b|^{p-2}b - |a|^{p-2}a, b - a > \geq (p-1)|b - a|^2 \left( 1 + |b|^2 + |a|^2 \right)^{\frac{p-2}{2}} \quad (1 \leq p \leq 2),
\]

and

\[
< |b|^{p-2}b - |a|^{p-2}a, b - a > \geq 2^{p-2}p|b - a|^p \quad (p \geq 2)
\]

for any \( a, b \in \mathbb{R}^n \).

By applying them with \( b = \nabla v_1 \) and \( a = \nabla v_2 \) in the preceding inequalities, we obtain

\[
\int_0^T \int_{\{v_1 > v_2\}} \left( \frac{\lambda}{2} - K \right) e^\lambda (v_1 - v_2)^2 + (p-1)e^\lambda (p-1)t |\nabla v_1 - \nabla v_2|^2 \left( 1 + |\nabla v_1|^2 + |\nabla v_2|^2 \right)^{\frac{p-2}{2}}
\]

\[
\leq -\frac{\delta}{2T^2} \int_0^T \int_{\{v_1 > v_2\}} (v_1 - v_2) \quad \text{for } 1 < p < 2
\]

and

\[
\int_0^T \int_{\{v_1 > v_2\}} \left( \frac{\lambda}{2} - K \right) e^\lambda (v_1 - v_2)^2 + 2^{2-p}e^\lambda (p-1)t |\nabla v_1 - \nabla v_2|^p
\]

\[
\leq -\frac{\delta}{2T^2} \int_0^T \int_{\{v_1 > v_2\}} (v_1 - v_2) \quad \text{for } p \geq 2.
\]
One can easily see in either case the respective inequality is true only if the measure of the set \( \{ v_1 > v_2 \} \) is zero. The proof is complete.

In the next lemma, we show the strict inequality on the boundary data can be relaxed to a non-strict one.

**Lemma 3.3** For \( T > 0 \) sufficiently small, if \( H_p w_1 \leq 0 \leq H_p w_2 \) in the weak sense in \( \Omega \times \mathbb{R}_T \) and \( w_1 \leq w_2 \) on \( \partial_p (\Omega \times \mathbb{R}_T) \), then \( w_1 \leq w_2 \) on \( \Omega \times \mathbb{R}_T \).

**Proof.** For any \( \delta > 0 \), take \( \tilde{\delta} > 0 \) such that \( \tilde{\delta} \leq \frac{\delta}{4K} \) and define

\[
\tilde{w}_1(x, t) = w_1(x, t) - \delta t - \tilde{\delta}(x, t) \in \bar{\Omega} \times \mathbb{R}^n.
\]

Then \( \tilde{w}_1 < w_1 \leq w_2 \) on \( \partial_p (\Omega \times \mathbb{R}^n) \), and for any \( \varphi \in \mathcal{S}_+ \), the following holds:

\[
\begin{align*}
\int_0^T \int_{\Omega} -\tilde{w}_1 \varphi_t + & < |\nabla \tilde{w}_1|^{p-2} \nabla \tilde{w}_1, \nabla \varphi > + \alpha(x, \tilde{w}) \varphi \\
= & \int_0^T \int_{\Omega} -w_1 \varphi_t + < |\nabla w_1|^{p-2} \nabla w_1, \nabla \varphi > + \alpha(x, w_1) \varphi \\
& - \delta \varphi + \left( \alpha(x, w_1 - \delta t - \tilde{\delta}) - \alpha(x, w_1) \right) \varphi \\
& \leq \int_0^T \int_{\Omega} -\delta \varphi + K \left( \delta t + \tilde{\delta} \right) \varphi \\
& \leq \int_0^T \int_{\Omega} \left( -\delta + \frac{\delta}{2} + \frac{\tilde{\delta}}{4} \right) \varphi \quad \text{for } T \text{ small} \\
= & -\frac{\delta}{4} \int_0^T \int_{\Omega} \varphi.
\end{align*}
\]

The preceding lemma implies \( \tilde{w}_1 \leq w_2 \) in \( \bar{\Omega} \times \mathbb{R}_T \) for small \( T \) and for any small \( \delta > 0 \), and whence the conclusion of this lemma.

Now the parabolic comparison theorem (3.1) follows from the preceding lemma quite easily as shown by the following argument: Let \( T_0 > 0 \) be any small value of \( T \) in the preceding lemma so that the conclusion of the preceding lemma holds. Then \( w_1 \leq w_2 \) on \( \bar{\Omega} \times (0, T_0) \). In particular, \( w_1 \leq w_2 \) on \( \partial_p (\bar{\Omega} \times (T_0, 2T_0)) \). The preceding lemma may be applied again to conclude that \( w_1 \leq w_2 \) on \( \bar{\Omega} \times (T_0, 2T_0) \). And so on. This recursion allows us to conclude that \( w_1 \leq w_2 \) on \( \bar{\Omega} \times \mathbb{R}_T \).

### 4 Convergence of Evolution

Define \( \mathcal{S} \) to be the set of weak solutions of the stationary problem (1.3) and (1.4). The \( p \)-harmonic function \( u_0 \) is the maximum element in \( \mathcal{S} \), while \( u_2 \) denotes the least solution which may be constructed as the infimum of super-solutions. We also use the term *non-minimal solution* with the same definition in [CW]. That is, \( u \) a non-minimal solution of
the problem (1.3) and (1.4) if it is a viscosity solution but not a local minimizer in the sense that for any $\delta > 0$, there exists $v$ in the admissible set of the functional $J_p$ with $v = \sigma$ on $\partial \Omega$ such that $\|v - u\|_{L^\infty} < \delta$, and $J_p(v) < J_p(u)$.

In this section, we consider the evolutionary problem

$$\begin{align*}
\begin{cases}
  w_t - \triangle_p w + Q(x)\beta_\varepsilon(w) = 0 &\text{in } \Omega \times (0, +\infty) \\
  w(x, t) = \sigma(x) &\text{on } \partial \Omega \times (0, +\infty) \\
  w(x, 0) = v_0(x) &\text{for } x \in \Omega,
\end{cases}
\end{align*}$$

(4.1)

and will apply the parabolic comparison principle (3.1) proved in Section 3 to prove the following convergence of evolution theorem. One just notes that the parabolic problem (3.1) includes the above problem (4.1) as a special case so that the comparison principle (3.1) applies in this case.

**Theorem 4.1** If the initial data $v_0$ falls into any of the categories specified below, the corresponding conclusion of convergence holds.

1. If $v_0 \leq u_2$ on $\Omega$, then $\lim_{t \to +\infty} w(x, t) = u_2(x)$ locally uniformly for $x \in \bar{\Omega}$;
2. Define
   $$\bar{u}_2(x) = \inf_{u \in \mathcal{S}, u \geq u_2, u \neq u_2} u(x), \quad x \in \Omega.$$
   If $\bar{u}_2 > u_2$, then for $v_0$ such that $u_2 < v_0 < \bar{u}_2$, $\lim_{t \to +\infty} w(x, t) = u_2(x)$ locally uniformly for $x \in \bar{\Omega}$;
3. Define $\bar{u}_0(x) = \sup_{u \in \mathcal{S}, u \leq u_0, u \neq u_0} u(x)$, $x \in \Omega$. If $\bar{u}_0 < u_0$, then for $v_0$ such that $\bar{u}_0 < v_0 < u_0$, $\lim_{t \to +\infty} w(x, t) = u_0(x)$ locally uniformly for $x \in \Omega$;
4. If $v_0 \geq u_0$ in $\Omega$, then $\lim_{t \to +\infty} w(x, t) = u_0(x)$ locally uniformly for $x \in \bar{\Omega}$;
5. Suppose $u_1$ is a non-minimal solution of (1.3) and (1.4). For any small $\delta > 0$, there exists $v_0$ such that $\|v_0 - u_1\|_{L^\infty(\Omega)} < \delta$ and the solution $w$ of the problem (4.1) does not satisfy

$$\lim_{t \to +\infty} w(x, t) = u_1(x) \quad \text{in } \Omega.$$

**Proof.** We first take care of case 4. We may take new initial data a smooth function $\tilde{v}_0$ so that $D^2\tilde{v}_0 < -KI$ and $|\nabla \tilde{v}_0| \geq \delta > 0$ on $\bar{\Omega}$. According to the parabolic comparison principle (3.1), it suffices to prove the solution $\tilde{w}$ generated by the initial data $\tilde{v}_0$ converges locally uniformly to $u_0$ if we also take $\tilde{v}_0$ large than $v_0$, which can easily be done. So we use $v_0$ and $w$ for the new functions $\tilde{v}_0$ and $\tilde{w}$ without any confusion.

For any $V \subset \subset \Omega$ and any nonnegative function $\varphi$ which is independent of the time variable $t$ and supported in $V$, it holds that

$$\int_V |\nabla v_0|^{p-2} \nabla v_0 \cdot \nabla \varphi = \int_V -\text{div} (|\nabla v_0|^{p-2} \nabla v_0) \varphi \geq \int_V M \varphi \quad \text{for some } M = M(n, p, K, \delta) > 0.$$
The Hölder continuity of $\nabla w$ up to $t = 0$ as stated in [DiB], then implies

$$\int_V |\nabla w|^{p-2} \nabla w \cdot \nabla \varphi \geq \frac{M}{2} \int_V \varphi$$

for any small $t$ in $(0, t_0)$, and any nonnegative function $\varphi$ which is independent of $t$, supported in $V$ and subject to the condition

$$\frac{\int_V |\nabla \varphi|}{\int_V \varphi} \leq A$$

(4.2)

for a fixed constant $A > 0$ and some $t_0 > 0$ dependent on $A$. Then the sub-solution condition on $w$

$$\int_V w_\varphi \bigg|_{t=t_2} - \int_V w_\varphi \bigg|_{t=t_1} + \int_{t_1}^{t_2} \int_V |\nabla w|^{p-2} \nabla w \cdot \nabla \varphi \leq 0$$

implies that

$$\int_V w_\varphi \bigg|_{t=t_2} - \int_V w_\varphi \bigg|_{t=t_1} \leq -\frac{M}{2} (t_2 - t_1) \int_V \varphi$$

for any small $t_2 > t_1$ in $(0, t_0)$, and any nonnegative function $\varphi$ which is independent of $t$, supported in $V$ and subject to (4.2). In particular, $\int_V w_\varphi \bigg|_{t=t_1} \leq 0$ for any nonnegative function $\varphi$ independent of $t$, supported in $V$ and subject to (4.2). So

$$w(x, t_2) \leq w(x, t_1)$$

for any $x \in \Omega$ and $0 \leq t_1 \leq t_2$. Then the parabolic comparison principle readily implies $w$ is decreasing in $t$ for $t$ in $[0, \infty)$. Therefore $w(x, t) \to w^\infty(x)$ locally uniformly as $t \to \infty$ and hence $w^\infty$ is a solution of (1.3) and (1.4). Furthermore, the parabolic comparison principle also implies $w(x, t) \geq u_0(x)$ at any time $t > 0$. Consequently, $u^\infty = u_0$ as $u_0$ is the greatest solution of (1.3) and (1.4).

Next, we briefly explain the proof for case 1. We may take a new smooth initial data $\tilde{v}_0$ such that $\tilde{v}_0$ is very large negative, $D^2 \tilde{v}_0 \geq K I$ and $|\nabla \tilde{v}_0| \geq \delta$ on $\bar{\Omega}$ for large constant $K > 0$ and constant $\delta > 0$. It suffices to prove the solution $\tilde{w}$ generated by the initial data $\tilde{v}_0$ converges to $u_2$ locally uniformly on $\bar{\Omega}$ as $t \to \infty$. Following a computation exactly parallel to that in case 4, we can prove $w$ is increasing in $t$ in $[0, \infty)$. So $w$ converges locally uniformly to a solution $w^\infty$ of (1.3) and (1.4). As $w^\infty \leq u_2$ and $u_2$ is the least solution of (1.3) and (1.4), we conclude $w^\infty = u_2$.

In case 2, we may replace $v_0$ by a strict super-solution of $\triangle_p v - Q_{\beta_\varepsilon}(v) = 0$ in $\bar{\Omega}$ between $u_2$ and $\bar{u}_2$, by employing the fact that $u_2$ is the infimum of super-solutions of (1.3) and (1.4). Using $v_0$ as the initial data, we obtain a solution $w(x, t)$ of (4.1). Then one argues as in case 4 that for any $V \subset \subset \Omega$, there exist constants $A > 0$ and $t_0 > 0$ such that for $t_1 < t_2$ with $t_1, t_2 \in [0, t_0)$, $\int_V w_\varphi \bigg|_{t_1} \leq 0$ for any nonnegative function $\varphi$ independent of $t$, supported in $V$ and subject to the condition $\frac{\int_V |\nabla \varphi|}{\int_V \varphi} \leq A$. As a consequence, $w(x, t_1) \geq w(x, t_2)$ $(x \in \Omega)$. Then the parabolic comparison principle implies $w$ is decreasing in $t$ over $[0, +\infty)$. Therefore $w(x, t)$ converges locally uniformly to
some function $u^\infty$ as $t \to \infty$ which solves (1.3) and (1.4). Clearly $u_2(x) \leq w(x, t) \leq \bar{u}_2(x)$ from which $u_2(x) \leq u^\infty(x) \leq \bar{u}_2(x)$ follows. As $w$ is decreasing in $t$ and $v_0 \neq \bar{u}_2$, $u^\infty \neq \bar{u}_2$. Hence $u^\infty = u_2$.

The proof of case 3 is parallel to that of case 2 with the switch of sub- and supersolutions. Hence we skip it.

In case 5, we pick $v_0$ with $\|v_0 - u_1\|_{L^\infty} < \delta$ and $J_p(v_0) < J_p(u_1)$. Let $w$ be the solution of (4.1) with $v_0$ as the initial data. Clearly, we may change the value of $v_0$ slightly if necessary so that it is not a solution of the equation

$$-\nabla \cdot \left( (\varepsilon + |\nabla u|^2)^{p/2 - 1} \nabla u \right) + Q(x)\beta(u) = 0$$

for any small $\varepsilon > 0$.

Let $w^\varepsilon$ be the smooth solution of the uniformly parabolic boundary-value problem

$$\begin{cases}
  w_t - \nabla \cdot \left( (\varepsilon + |\nabla w|^2)^{p/2 - 1} \nabla w \right) + Q\beta(w) = 0 & \text{in } \Omega \times (0, +\infty) \\
  w(x, t) = \sigma(x) & \text{on } \partial\Omega \times (0, +\infty) \\
  w(x, 0) = v_0(x) & \text{on } \Omega.
\end{cases}$$

$w^\varepsilon$ converges to $w$ in $W^{1,p}(\Omega)$ for every $t \in [0, \infty)$ as $\varepsilon \to 0$.

We define the functional

$$J_{\varepsilon,p}(u) = \frac{1}{p} \int_{\Omega} (\varepsilon + |\nabla u|^2)^{p/2} + Q(x)\Gamma(u) \, dx.$$ 

It is easy to see that

$$\int_0^t \int_{\Omega} (w^\varepsilon_t)^2 - \nabla \cdot \left( (\varepsilon + |\nabla w^\varepsilon|^2)^{p/2 - 1} \nabla w^\varepsilon \right) w^\varepsilon_t + Q\beta(w^\varepsilon)w^\varepsilon_t = 0.$$

As $w^\varepsilon_t = 0$ on $\partial\Omega \times (0, \infty)$, one gets

$$\int_0^t \int_{\Omega} (w^\varepsilon_t)^2 + (\varepsilon + |\nabla w^\varepsilon|^2)^{p/2 - 1} \nabla w^\varepsilon \cdot \nabla w^\varepsilon + Q(x)\Gamma(w^\varepsilon)_t = 0,$$

which implies

$$\int_0^t \int_{\Omega} (w^\varepsilon_t)^2 + \frac{1}{p} \left( (\varepsilon + |\nabla w^\varepsilon|^2)^{p/2} \right)_t + Q(x)\Gamma(w^\varepsilon)_t = 0.$$

Consequently, it holds

$$\int_0^t \int_{\Omega} (w^\varepsilon_t)^2 + \frac{1}{p} \int_{\Omega} \left( (\varepsilon + |\nabla w^\varepsilon(x,t)|^2)^{p/2} + Q\Gamma(w^\varepsilon(x,t)) \right)$$

$$= \frac{1}{p} \int_{\Omega} \left( (\varepsilon + |\nabla w^\varepsilon(x,0)|^2)^{p/2} + Q\Gamma(w^\varepsilon(x,0)) \right)$$

i.e.

$$\int_0^t \int_{\Omega} (w^\varepsilon)^2 + J_{\varepsilon,p}(w^\varepsilon(\cdot, t)) = J_{\varepsilon,p}(w^\varepsilon(\cdot, 0)).$$
Therefore

\[ J_{\varepsilon,p}(w^\varepsilon(\cdot, t)) \leq J_{\varepsilon,p}(v_0), \]

which in turn implies

\[ J_p(w(\cdot, t)) \leq J_p(v_0) < J_p(u_1). \]

In conclusion, \( w \) does not converge to \( u_1 \) as \( t \to \infty \). □

References

[CW] L. A. Caffarelli and P. Wang, “A bifurcation phenomenon in a singularly perturbed one-phase free boundary problem of phase transition”, *Calc. Var. Partial Differential Equations*, 54(2015), no.4, 3517-3529.

[DiB] E. DiBenedetto, “Degenerate Parabolic Equations”, *Springer-Verlag*, 1993.

[E] L. C. Evans, “A new proof of local \( C^{1,\alpha} \) regularity for solutions of certain degenerate elliptic P. D. E.”, *Journal of Differential Equations*, 45(1982), 356-373.

[J] Y. Jabri, “The mountain pass theorem: variants, generalizations, and some applications”, *Encyclopedia of Mathematics and its Applications*, 95. *Cambridge University Press*, Cambridge, 2003.

[Le] J. L. Lewis, “Regularity of the derivatives of solutions to certain degenerate elliptic equations”, *Indiana University Math. J.*, 32(1983), 849-858.

[L] P. Lindqvist, “Notes on the \( p \)-Laplace equation”, *Report. University of Jyväskylä Department of Mathematics and Statistics*, 102. *University of Jyväskylä*, Jyväskylä, 2006. ii+80pp. ISBN 951-39-2586-2.