I. INTRODUCTION

The two-dimensional Edwards-Anderson Ising spin glass with bimodal interactions has been widely used to study properties of spin-glass systems. Despite the fact that the model only orders at zero temperature, its popularity can be ascribed mainly to its ease of implementation and simplicity. There has been a long-standing controversy regarding the behavior of the free energy of the model, in particular the size of the excitation gap and the critical behavior in the thermodynamic limit.

The exponential scaling of the free energy, and thus correspondingly of all other thermodynamic quantities, was first proposed by Wang and Swendsen. They surmised that

\[ C_V \sim \beta^2 e^{-A\beta J}, \]

where \( \beta = 1/T \) represents the inverse temperature, \( J \) is the magnitude of the bonds, and \( A \) is a numerical prefactor. Simple analytical arguments for Ising systems with bimodal interactions on square lattices with coordination 4 suggest that the energy gap should be \( 4J \), i.e., \( A = 4 \). In addition, according to hyperscaling, the singular part of the free energy scales as \( \xi^{-d} \) with \( d = 2 \) and so, if \( C_V \) scales exponentially, we expect that the correlation length \( \xi \) scales as

\[ \xi \sim e^{n\beta J}. \]

with \( A = 2n \), as predicted first by Saul and Kardar. Wang and Swendsen were the first to calculate numerically the specific heat of the model and find \( A = 2 \), thus showing a nontrivial scaling of the free energy; however their results for small system sizes and few disorder realizations implied strong corrections to scaling. These results were later supported by numerical work of Lukic et al. who computed the specific heat of the model for intermediate system sizes (\( L \leq 50 \)) and also showed that \( A = 2 \).

II. MODEL AND OBSERVABLES

The Hamiltonian of the two-dimensional Ising spin glass is given by

\[ H = - \sum_{\langle i,j \rangle} J_{ij} S_i S_j. \]

\( S_i = \pm 1 \) represent Ising spins and the sum is over nearest neighbors on a square lattice with periodic boundary conditions. The interactions \( J_{ij} \in \{ \pm 1 \} \) are bimodally distributed. For the Monte Carlo simulations we use a combination of single-spin flips, parallel tempering updates, and rejection-free cluster moves to speed up equilibration. Equilibration of the method is
tested by performing a logarithmic data binning of all observables, and we require that the last three bins agree with error bars and are independent of the number of Monte Carlo sweeps $N_{\text{sweep}}$. The parameters of the simulation are listed in Table I.

The finite-size correlation length $\xi_L$ is given by

$$\xi_L = \frac{1}{2 \sin(|k_{\text{min}}|/2)} \left[ \frac{\chi_{\text{SG}} (0)}{\chi_{\text{SG}} (k_{\text{min}})} - 1 \right]^{1/2},$$  \hspace{1cm} (4)$$

where $k_{\text{min}} = (2\pi/L, 0)$ is the smallest nonzero wave vector, and $\chi_{\text{SG}} (k)$ is the wave-vector-dependent spin-glass susceptibility:

$$\chi_{\text{SG}} (k) = \frac{1}{N} \sum_{i,j} \langle [S_i S_j]^2 \rangle_{\text{av}} e^{i k \cdot (R_i - R_j)}.$$  \hspace{1cm} (5)$$

In the previous equation $\langle \cdots \rangle_{\text{av}}$ represents a disorder average and $\langle \cdots \rangle$ a thermal average.

In order to compare to the data of Lukic et al., we also compute the specific heat of the system:

$$C_V = \frac{1}{T^2} \left[ \langle H^2 \rangle - \langle H \rangle^2 \right]_{\text{av}}.$$  \hspace{1cm} (6)$$

### III. RESULTS

In Fig. 1 we show data for the natural logarithm of the finite-size correlation length as a function of $1/T$ for different system sizes. We expect data for $\ln (\xi_L)$ vs $1/T$ to approach a slope of $n$. The data, at first sight, show good agreement with $n = 2$ (dashed line). Note that the data peel off the asymptotic behavior thus masking any potential effective variation of $n$ with $L$ and $T$. Data at high temperatures show a slope $n > 2$, thus suggesting that the slope might change for decreasing temperatures and increasing system sizes.

Figure 2 shows data for the derivative of the natural logarithm of the finite-size correlation length with respect to the inverse temperature as a function of temperature for several system sizes. In the bulk regime ($T \gtrsim 0.7$) the data show no system size dependence, the computed values for the slope suggest $2 \lesssim n \lesssim 3$. Interestingly, the data vary linearly with $T$ and then successively peel off the linear behavior at increasingly lower temperatures for increasing system sizes. A linear extrapolation of the data in the bulk regime seems plausible and shows that $n = 1$ at low temperatures in the thermodynamic limit, thus concluding in a nontrivial scaling of the free energy in the two-dimensional Ising spin glass with bimodal interactions.

Our results clearly show that the two-dimensional Ising spin glass with bimodal interactions has strong corrections to scaling. In Ref. 5 a finite-size scaling analysis of the finite-size correlation length suggested compatibility with $n = 2$, whereas a finite-size scaling of the data for $n = 1$ did not seem plausible. Merely the data for the two largest sizes showed a signature of a scaling behavior when $n = 1$. This in turn leads to the conclusion that $n = 2$. While the same observable is being studied here, the results presented contradict these findings and show that to understand the low-temperature properties of this model, the limit of $L \to \infty$ has to be taken before extrapolating to $T = 0$. Finally, in Fig. 3 we show data for the specific heat $C_V$ as a function of temperature plotted as $-T \ln (T^2 C_V)$ vs $T$. The data plotted in this way should tend to $A$ for
FIG. 2: (Color online) Derivative of the natural logarithm of the finite-size correlation length with respect to the inverse temperature as a function of temperature for several system sizes. In agreement with Fig. 1 the slope for the different data sets ranges between 2 and almost 3 for the highest temperatures, suggesting that the slope is temperature dependent. For large temperatures the data are independent of $L$ and well fitted by a straight line (solid line in plot), thus suggesting that corrections to $n$ are quadratic. In addition, the data peel off at low temperatures from the straight-line behavior. An extrapolation of the data to $T = 0$ agrees well with $n = 1$ in the thermodynamic limit. Note that the data show that the results in the thermodynamic limit are incompatible with $n = 2$ for $L \to \infty$. The dotted lines correspond to error estimates to $n$.

$T \to 0$. Our results show that $A \approx 2$ for $T \to 0$ already for intermediate system sizes, in agreement with results by Lukic et al.

IV. CONCLUSIONS

To conclude, we have shown data for the correlation length and the specific heat of the two-dimensional Ising spin glass with bimodal interactions. The data for the correlation length scale exponentially $\sim \exp(n\beta J)$ and seem compatible with $n = 1$ in the thermodynamic limit, in contrast to previous calculations. A detailed study of $n$ as a function of temperature shows that $n$ strongly depends on system size and temperature. This result is supported by data for the specific heat which also scale exponentially with $A \approx 2 = 2n$. This means that the excitation gap for the bimodal spin glass is $\approx 2J$. Our results in the thermodynamic limit agree with work by Lukic et al. as well as Wang and Swendsen.

Our data show that corrections to scaling in the two-dimensional Ising spin glass with bimodal interactions are extremely large. In particular, it is important to take the limit $L \to \infty$ before extrapolating to $T = 0$. Thus system sizes of at least $\sim 100^2$ and larger are required to probe the true thermodynamic behavior. Therefore an analysis with yet larger system sizes at lower temperatures is imperative.

Acknowledgments

We would like to thank F. Hassler and A. P. Young for fruitful discussions. L.W.L. acknowledges support from the National Science Foundation under NSF Grant No. DMR 0337049. The simulations were performed on the Hreidar and Gonzales clusters at ETH Zürich.

1 S. F. Edwards and P. W. Anderson, Theory of spin glasses, J. Phys. F: Met. Phys. 5, 965 (1975).
2 J. Wang and R. H. Swendsen, Low-temperature properties of the $\pm J$ spin glass in two dimensions, Phys. Rev. B 38, 4840 (1988).
3 L. Saul and M. Kardar, Exact integer algorithm for the two-dimensional $\pm J$ Ising spin glass, Phys. Rev. E 48, R3221 (1993).
4 J. Lukic, A. Gallucio, E. Marinari, O. C. Martin, and G. Rinaldi, Critical thermodynamics of the two dimensional $\pm J$ Ising spin glass, Phys. Rev. Lett. 92, 117202 (2004).
5 J. Houdayer, A cluster Monte Carlo algorithm for 2-dimensional spin glasses, Eur. Phys. J. B. 22, 479 (2001).
6 K. Binder, Critical properties from Monte Carlo coarse graining and renormalization, Phys. Rev. Lett. 47, 693 (1981).
7 H. G. Katzgraber and L. W. Lee, Correlation length of the two-dimensional Ising spin glass with bimodal interactions, Phys. Rev. B 71, 134404 (2005).
8 F. Cooper, B. Freedman, and D. Preston, Solving $\phi_{4,2}$ theory with Monte Carlo, Nucl. Phys. B 210, 210 (1982).
9 H. G. Ballesteros, A. Cruz, L. A. Fernandez, V. Martin-Mayor, J. Pech, J. J. Ruiz-Lorenzo, A. Tarancon, P. Tellez, C. L. Ulld, and C. Ungil, Critical behavior of the three-dimensional Ising spin glass, Phys. Rev. B 62, 14237 (2000).
10 K. Hukushima and K. Nemoto, Exchange Monte Carlo method and application to spin glass simulations, J. Phys. Soc. Jpn. 65, 1604 (1996).
11 E. Marinari, Optimized Monte Carlo methods, in Advances in Computer Simulation, edited by J. Kertész and I. Kondor (Springer-Verlag, Berlin, 1998), p. 50, (cond-mat/9612010).
12 J. K. Kim, Asymptotic scaling of the mass gap in the two-dimensional $O(3)$ nonlinear sigma model: A numerical study, Phys. Rev. D 50, 4663 (1994).
13 M. Palassini and S. Caracciolo, Universal Finite-Size Scaling Functions in the 3D Ising Spin Glass, Phys. Rev. Lett. 82, 5128 (1999).
14 L. W. Lee and A. P. Young, Single spin- and chiral-glass transition in vector spin glasses in three dimensions, Phys. Rev. Lett. 90, 227203 (2003).
15 H. G. Katzgraber, L. W. Lee, and A. P. Young, Correlation Length of the Two-Dimensional Ising Spin Glass with Gaussian Interactions, Phys. Rev. B 70, 014417 (2004).