Gauge symmetry and constraints structure in topologically massive AdS gravity: A symplectic viewpoint

Omar Rodríguez-Tzompantzi
Facultad de Ciencias Físico Matemáticas,
Benemérita Universidad Autónoma de Puebla,
Apartado postal 1152, Puebla, Pue., México

Alberto Escalante†
Instituto de Física, Benemérita Universidad Autónoma de Puebla,
Apartado Postal J-48 72570, Puebla Pue., México,
(Dated: September 20, 2018)

By applying the Faddeev-Jackiw symplectic approach we systematically show that both the local
gauge symmetry and the constraint structure of topologically massive gravity with a cosmological
constant $\Lambda$, elegantly encoded in the zero-modes of the symplectic matrix, can be identified.
Thereafter, via an appropriate partial gauge-fixing procedure, the time gauge, we calculate the
quantization bracket structure (generalized Faddeev-Jackiw brackets) for the dynamic variables and
confirm that the number of physical degrees of freedom is one. This approach provides an alternative
to explore the dynamical content of massive gravity models.

PACS numbers: 98.80.-k,98.80.Cq

I. INTRODUCTION

Fundamental issues in modern cosmology, such as inflation, dark matter and dark energy,[1,2],
which attempt to explain the primordial and late time accelerating expansion of our universe, have
long been motivated alternative gravity theories beyond original Einstein’s General Relativity, both
in the ultraviolet (UV) and the infrared (IR) regimes. According to Lovelock’s theorem[3,4]
any modification of General Relativity requires at least one of the following ingredients: i) extra
dimensions, ii) extra degrees of freedom, iii) higher-derivatives terms, and iv) non-locality. Massive
gravity theories are an example of the type-ii ingredients, in which the massless graviton of General
Relativity is given a non-zero mass (see e.g. [5–18]). Along these lines, it has long been known that
the first massive gravity theory was introduced circa 1939 by Pauli and Fierz in Ref. [6], where
they presented a linear action with respect to a spin-2 field on a flat space-time background. The
Fierz-Pauli theory describes five degrees of freedom of positive energy in four dimensions at the
linear level whereas General Relativity has two degrees of freedom. However, Boulware and Deser

*Electronic address: omar.tz2701@gmail.com
†Electronic address: aescalan@ifuap.buap.mx
studied some specific fully non-linear massive gravity theories and pointed out that a general non-linear theory of massive gravity generically contains six propagating degrees of freedom. While the linear theory has five degrees of freedom, the non-linear theories studied by these authors turned out to have an extra degree of freedom, which however is unphysical as it has a negative kinetic energy and renders the whole theory unstable: it was therefore called the Boulware-Deser ghost \[19\].

After a great effort, a non-linear theory free of such a ghost field was at last obtained by de Rham, Gabadadze and Tolley (dRGT) \[7–9\]. The advantage of the dRGT model is that it contains two dynamical constraints that eliminate both the ghost field and its canonically-conjugate momentum. The absence of the Boulware-Deser ghost was shown explicitly by counting the degrees of freedom in the framework of the Hamiltonian formalism \[10–12, 17, 18\]. Unfortunately, the Hamiltonian analysis of these models remains quite complex and, therefore, their symmetry properties have not been studied yet via first-class constraints. On the other hand, in the study of some topics of General Relativity, such as massive gravity, it is always useful to consider toy models that share the conceptual foundations of the four-dimensional theories, but at the same time are free of technical difficulties. This is particularly true in three-dimensional (3D) gravity. In this work, we focus on the simplest 3D version of a massive gravity theory.

To obtain a realistic 3D-Einstein gravity as compared to the higher-dimensional theory, regarding the local propagating modes, one can modify the theory by adding up higher-derivative curvature terms in the Einstein-Hilbert (EH) action, which leads to the simplest 3D-massive gravity theory known as Topologically Massive Gravity (TMG). This theory consists of an EH term, with or without a cosmological constant \(\Lambda\), plus a parity-violating gravitational Chern-Simons (CS) term with coefficient \(\frac{1}{\mu}\) \[20–23\]. At the linear level, this theory describes a single massive state of helicity +2 or -2 (depending on the relative sign between the EH and CS terms) in Minkowski background\[24\] and defines a unitary irreducible representation of the 3D Poincaré group \[25\].

However, while a linearized analysis usually allows a reliable counting of the physical degrees of freedom, it can yield misleading results in some cases. A Lagrangian/Hamiltonian formulation should provide a way to count the number of local physical degrees of freedom without resorting to linearization, that is, taking into account all the physical constraints and gauge invariance (i.e. gauge-independence). In this sense, the identification of the physical degrees of freedom can be addressed by a direct application of Dirac’s method for constrained Hamiltonian systems \[26\], which systematically separates all the constraints into first-and second-class ones \[27, 28\]. As a consequence, the physical degrees of freedom can be separated from the gauge degrees of freedom, and a generator of the gauge symmetry can be constructed out of a combination of first-class constraints \[29\]. Furthermore, the bracket structure (Dirac’s brackets) to quantize a gauge system can be obtained once the second-

---

1 In the presence of a cosmological constant, Minkowski space-time is no longer a vacuum solution and the new maximally symmetric solutions are de Sitter (dS) space-time for positive \(\Lambda\) (dS has isometry group \(SO(3, 1)\)) and anti-de Sitter (AdS) space-time for negative \(\Lambda\) (AdS has isometry group \(SO(2, 2)\)).
class constraints are removed. In the case of the massive gravity theories, however, the separation between first-and second-class constraints is a delicate issue, and the system considered in this paper is not an exception \cite{30-33}. In particular, in Ref. \cite{30} the Hamiltonian structure of TMG was further analyzed via the Dirac formalism. Indeed, these authors obtain the secondary first-class-constraint structure of this model with the help of the theorem: "If $\phi$ is a first-class constraint, then $\{\phi, H_T^T\}$ is also a first-class constraint". Nevertheless, this treatment is quite involved and unsatisfactory. On the other hand, the authors of Refs. \cite{32,33} present a fully Lagrangian analysis, but the right number of physical degrees of freedom in configuration space can only be obtained once an ad hoc extra constraint on the basic variables is invoked. This is the main difficulty and it thus worth exploring whether all the necessary constraints can be systematically obtained via a Lagrangian formulation. Thereby, the analysis of the constraints and the gauge symmetry of massive gravity models, still missing in the literature, is relevant and it is thus mandatory to carry out such an analysis to quantize the theory.

Very interestingly, as an alternative to Dirac’s method, Faddeev and Jackiw \cite{34} proposed a new approach, which is geometrically well motivated and is based on the symplectic structure for constrained systems. This approach, the so-called Faddeev-Jackiw (F-J) symplectic formalism (for a detailed account see \cite{35-42}), is useful to obtain in an elegant way several essential elements of a particular physical theory, such as the physical constraints, the local gauge symmetry, the quantization bracket structure and the number of physical degrees of freedom. It turns out that the F-J approach does not require to classify the constraints into first- and second-class ones. Even more, it does not invoke Dirac’s conjectura. Rather, in this approach, the quantization brackets can be identified as the elements of the inverse matrix of the symplectic one. For a gauge system, the symplectic matrix remains singular unless a gauge-fixing procedure is introduced. In addition, the generators of the gauge symmetries are given in terms of the zero-modes of the symplectic matrix. In this respect, the F-J symplectic method provides an effective tool for dealing with gauge theories.

The purpose of this article is to present a detailed F-J analysis of three-dimensional topologically massive AdS gravity in a completely different context to that presented in Refs. \cite{30-33}. In particular, we study the nature of the physical constraints and obtain the gauge symmetry, as well as its generators, under which all the physical quantities must be invariant. Afterwards, we obtain both the fundamental quantization brackets and the number of physical degrees of freedom by introducing an appropriate gauge-fixing procedure. The remainder of this paper is structured as follows. In section II we briefly review the topologically massive AdS gravity action. Section III is devoted to explore the nature of the constraints within the Faddeev-Jackiw symplectic framework and derive the corresponding symplectic matrix. The full set of physical constraints of the theory are also obtained. In Section IV, the gauge symmetry and its generators are obtained via the zero-modes of the symplectic matrix. We introduce gauge-fixing conditions in order to obtain both the quantization bracket structure and the number of physical degrees of freedom in Section V. We conclude with a brief discussion of our results in Section VI.
II. ACTION AND EQUATIONS OF MOTION OF TOPOLOGICALLY MASSIVE GRAVITY

Our starting point is the action of topologically massive AdS gravity written in the first-order formalism:

\[
S[A, e, \lambda] = \int_{\mathcal{M}} \left[ 2\theta e^i \wedge F[A]^i - \frac{1}{3} \Lambda f_{ijk} e^i \wedge e^j \wedge e^k + \lambda^i \wedge T_i + \frac{\theta}{\mu} A^i \wedge \left( dA_i + \frac{1}{3} f_{ijk} A^j \wedge A^k \right) \right],
\]

(1)

where \( \mu \) is the Chern-Simons parameter, \( \theta = \frac{1}{16\pi G} \) with \( G \) the 3D Newton’s constant, and \( \Lambda \) is a cosmological constant such that \( \Lambda = -\frac{1}{l^2} \), where \( l \) is the AdS radius [24]. Furthermore, the fundamental fields of this action are: the dreibein 1-form \( e^i = e^i_\mu dx^\mu \) that determines a space-time metric via \( g_{\mu\nu} = e^i_\mu e^j_\nu \eta_{ij} \); the auxiliary field 1-form \( \lambda^i \) that ensures that the torsion vanishes \( T_i = 0 \) [43, 44]; and the dualized spin-connection \( A^i = A^i_\mu dx^\mu \) valued on the adjoint representation of the Lie group \( SO(2,2) \), so that, it admits an invariant totally anti-symmetric tensor \( f_{ijk} \). The connection acts on internal indices and defines a derivative operator:

\[
D^\mu V^i \equiv \partial^\mu V^i + f^i_{jk} A^j_\mu V^K,
\]

(2)

where \( \partial \) is a fiducial derivative operator. Finally, \( T_i \) is the local Lorentz covariant torsion 2-form and \( F_i \) is the curvature 2-form of the spin connection \( A^i \), which explicitly read

\[
T_i \equiv de_i + f^i_{jk} A_j^\mu e^K, \quad F_i \equiv dA_i + \frac{1}{2} f^i_{jk} A^j_\mu A^K.
\]

(3)

The convention adopted is the standard one, that is, Greek indices refer to spacetime coordinates and Latin letters correspond to Lorentz indices. The equations of motion that can be extracted by varying the action (1) with respect to \( e^i \), \( A^i \) and \( \lambda^i \), respectively, in addition to some total derivative terms, are given by

\[
(\delta e)^{\alpha i} = \epsilon^{\alpha\nu\rho} \left( 2\theta F_{\nu\rho}^i + D_\nu \lambda^i - \Lambda f^j_{jk} e^i e^K \right) = 0,
\]

(4)

\[
(\delta A)^{\alpha i} = \epsilon^{\alpha\nu\rho} \left( 2\theta T_{\nu\rho}^i + f^j_{jk} \lambda^{ij} e^{iK} + 2\theta \mu^{-1} F_{\nu\rho}^i \right) = 0,
\]

(5)

\[
(\delta \lambda)^{\alpha i} = \epsilon^{\alpha\nu\rho} T_{\nu\rho}^i = 0.
\]

(6)

One can note that Eq. (6) is the condition for the compatibility of \( A^i_\mu \) and \( e^i_\mu \), which implies

\[
A^i_\mu = -e^i_\rho \partial_\mu e^\rho + \Gamma^i_{\alpha\beta} e^\rho e^\alpha e^\beta,
\]

(7)

with \( \Gamma^i_{\alpha\beta} \) the Christoffel symbols of the metric \( g_{\mu\nu} \), and \( A^i_\mu = e^j_\mu A^j_\nu \) the standard connection obtained by dualizing the f-tensor, \( A^i_\mu = -f^{ij}_k A^j_\nu \). Moreover, by inserting Eq. (6) into Eq. (5), one can solve for the Lagrangian multiplier \( \lambda^i_\mu \) in terms of the 3D Schouten tensor of the manifold \( \mathcal{M} \):

\[
\lambda^i_\mu = 2\theta \mu^{-2} S^i_{\mu\nu} \epsilon^{\mu\nu} \quad \text{with} \quad S^i_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R.
\]

(8)

Here we have made use of the fact that the internal and space-time curvature tensors \( F^{ij}_{\mu\nu} \) and \( R^{\alpha\beta}_{\mu\nu} \) are related by

\[
R^{\alpha\beta}_{\mu\nu} = e^{\alpha}_i e^{\beta}_j F^{ij}_{\mu\nu} \quad \text{with} \quad F^{ij}_{\mu\nu} = -f^{ij}_k F^{k\nu}_{\mu}.
\]

(9)
After plugging these results into Eq. (4) and a lengthy calculation, one can find the field equation of TMG in the second-order formalism:

\[ G_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0, \]  

(10)

where \( G_{\mu\nu} \) is the cosmological-constant-modified Einstein tensor defined as

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu}, \]  

(11)

and \( C_{\mu\nu} \) is the symmetric traceless Cotton tensor given by

\[ C_{\mu\nu} = \epsilon_{\mu\alpha\beta} \nabla_\alpha \left( R_{\beta\nu} - \frac{1}{4} g_{\beta\nu} R \right), \]  

(12)

where \( \nabla \) is the covariant derivative defined by \( \Gamma \). Considering small perturbations around an anti-de Sitter background, this theory describes the presence of a single massive graviton mode [20, 24, 25]. However, from a theoretical point of view, it is better to check out the validity of such rough arguments by a careful Hamiltonian or Lagrangian analysis at nonlinear order.

### III. THE NATURE OF THE CONSTRAINTS IN THE FADDEEV-JACKIW SYMPLECTIC FRAMEWORK

In order to apply the Faddeev-Jackiw’s symplectic approach [34], throughout this work we take the spacetime \( M \) to be globally hyperbolic such that it may be foliated as \( M \simeq \Sigma \times \mathbb{R} \), where \( \Sigma \) corresponds to a Cauchy’s surface without boundary (\( \partial \Sigma = 0 \)) and \( \mathbb{R} \) represents an evolution parameter. By performing a 2 + 1 splitting of our fields without breaking the internal symmetry, the TMG action (1) acquires the form,

\[
S[A, e, \lambda] = \int \left[ e^{ab} \theta \left( \frac{1}{\mu} A_{bi} + 2 e_{bi} \right) \dot{A}^i + e^{ab} \lambda_{ab} \dot{e}^i + e^{ab} e^i_0 \left( \theta F_{abi} + D_a \lambda_{bi} - \Lambda f_{ijk} e^j a e^k b \right) + e^{ab} A^i_0 \left( \theta T_{abi} + \frac{1}{\mu} \theta F_{abi} + f_{ijk} \lambda^j a e^k b \right) + \frac{1}{2} e^{ab} \lambda^i_0 T_{abi} \right] d^3 x,
\]  

(13)

up to a boundary term. Here \( F_{abi} = \partial_a A^i b - \partial_b A^i a + f_{ijk} A^j a A^k b \) is the field strength of \( A^i a \), \( T_{abi} = D_a e^i b - D_b e^i a \) and \( D_a \lambda^i b = \partial_a \lambda^i b + f_{ijk} A^j a \lambda^k b \). Besides \( a, b, c, \ldots \) are space coordinates and the dot denotes a derivative with respect to the evolution parameter. We can read off the Lagrangian density from (13) as

\[
\mathcal{L}^{(0)} = e^{ab} \theta \left( \frac{1}{\mu} A_{bi} + 2 e_{bi} \right) \dot{A}^i + e^{ab} \lambda_{ab} \dot{e}^i + e^{ab} e^i_0 \left( \theta F_{abi} + D_a \lambda_{bi} - \Lambda f_{ijk} e^j a e^k b \right) + \frac{1}{2} e^{ab} \lambda^i_0 T_{abi}
\]  

(14)

In particular, this Lagrangian density can be expressed compactly as

\[
\mathcal{L}^{(0)} = a^{(0)} I \dot{\xi}^{(0)} I - V^{(0)},
\]  

(15)

where an initial set of symplectic variable is introduced as follows

\[
\xi^{(0)} I = (A^i a, A^i_0, e^i a, e^i_0, \lambda^i a, \lambda^i_0),
\]  

(16)
which allows us to identify the corresponding symplectic one-form

$$a^{(0)}_I = (\epsilon^{ab}\theta \left( \frac{1}{\mu} A_{bi} + 2 e_{bi} \right), 0, \epsilon^{ab}\lambda_{bi}, 0, 0, 0),$$

whereas the symplectic potential reads as

$$V^{(0)} = -\epsilon^{ab} e^{2}(0) F_{abi} + D_{a} \lambda_{bi} - \Lambda f_{ijk} e_{a}^{j} e_{b}^{k}) - \frac{1}{2} \epsilon^{ab} \lambda_{i}^{0} T_{abi} - \epsilon^{ab} A_{i}^{0} \left( \theta T_{abi} + \frac{1}{\mu} F_{abi} + f_{ijk} \lambda_{a}^{i} e_{b}^{k} \right).$$

On the other hand, the corresponding equations of motion arising from the above Lagrangian can be written as

$$f^{(0)}_{IJ} \dot{\xi}^{(0)J} - \frac{\delta}{\delta \xi^{(0)I}} V(\xi^{(0)}) = 0,$$

with $f^{(0)}_{IJ} \equiv \frac{\delta}{\delta \xi^{(0)J}} a_{I} - \frac{\delta}{\delta \xi^{(0)I}} a_{J}$ the two-form symplectic matrix associated with $\mathcal{L}^{(0)}$, which is clearly antisymmetric. By using the symplectic variables and $\mathcal{L}^{(0)}$, we find that the corresponding symplectic matrix $f^{(0)}_{IJ}(x, y)$ can be written as

$$\delta^{2}(x - y).$$

It is not difficult to see that the matrix $f^{(0)}_{IJ}$ is degenerate in the sense that there are more degrees of freedom in the equations of motion than physical degrees of freedom in the theory. In this case, there are constraints that must remove the unphysical degrees of freedom. In this formalism, the constraints emerge as algebraic relations necessary to maintain the consistency of the equations of motion. Moreover, it is straightforward to determine that the zero-modes of the singular matrix are $(v^{(0)}_{1})^{I} = (0, v^{A}_{a}, 0, 0, 0, 0)$ and $(v^{(0)}_{2})^{I} = (0, 0, v^{e}_{a}, 0, 0)$ and $(v^{(0)}_{3})^{I} = (0, 0, 0, 0, v^{\lambda}_{a})$, with non-vanishing arbitrary components $v^{A}_{a}, v^{e}_{a}$ and $v^{\lambda}_{a}$, respectively.

The zero-modes satisfy the equation $(v^{(0)}_{1,2,3})^{I} f^{(0)}_{IJ} = 0$, therefore from the equation of motion, we have the following constraint relations:

$$\int dx^{2}(v^{(0)}_{1})^{T} \frac{\delta}{\delta \xi^{(0)}} \int dy^{2} V^{(0)} = v^{A}_{a} \left( \theta \epsilon^{ab} T_{abi} + \frac{\theta}{\mu} \epsilon^{ab} F_{abi} + \epsilon^{ab} f_{ijk} \lambda_{a}^{i} e_{b}^{k} \right) = 0,$$

$$\int dx^{2}(v^{(0)}_{2})^{T} \frac{\delta}{\delta \xi^{(0)}} \int dy^{2} V^{(0)} = v^{e}_{a} \left( \theta \epsilon^{ab} F_{abi} + \epsilon^{ab} D_{a} \lambda_{bi} - \Lambda \epsilon^{ab} f_{ijk} e_{a}^{j} e_{b}^{k} \right) = 0,$$

$$\int dx^{2}(v^{(0)}_{3})^{T} \frac{\delta}{\delta \xi^{(0)}} \int dy^{2} V^{(0)} = v^{\lambda}_{a} \left( \frac{1}{2} \epsilon^{ab} T_{abi} \right) = 0,$$

where $v^{A}_{a}, v^{e}_{a}$ and $v^{\lambda}_{a}$ are arbitrary functions. The constraints become

$$\Xi^{(0)}_{i} = \theta \epsilon^{ab} T_{abi} + \frac{\theta}{\mu} \epsilon^{ab} F_{abi} + \epsilon^{ab} f_{ijk} \lambda_{a}^{i} e_{b}^{k} = 0,$$

$$\Theta^{(0)}_{i} = \theta \epsilon^{ab} F_{abi} + \epsilon^{ab} D_{a} \lambda_{bi} - \Lambda \epsilon^{ab} f_{ijk} e_{a}^{j} e_{b}^{k} = 0,$$

$$\Sigma^{(0)}_{i} = \frac{1}{2} \epsilon^{ab} T_{abi} = 0.$$
Now, according to the methodology of the symplectic framework, we will analyze whether there are new constraints. To achieve this, we demand stability (consistency condition) of the constraints (A8), (25) and (26), which guarantees their time-independence. Since \( \Xi_i, \Theta_i \) and \( \Sigma_i \) depend only on the set of symplectic variables \( \xi^{(0)j} \), the consistency condition can be written as

\[
\dot{\Omega}^{(0)} = \frac{\delta \Omega^{(0)}}{\delta \xi^{(0)j}} \dot{x}^{(0)j} = 0 \quad \text{with} \quad \Omega^{(0)} = \Xi_i^{(0)}, \Theta_i^{(0)}, \Sigma_i^{(0)}.
\] (27)

Therefore the consistency of the constraints \( \Omega^{(0)} \), together with the equations of motion (19) can be generally rewritten as

\[
f_{KJ}^{(1)} \dot{\xi}^{(0)j} = Z_K^{(1)}(\xi),
\] (28)

with

\[
f_{KJ}^{(1)} = \left( \frac{f_{IJ}^{(0)}}{\delta \xi^{(0)j} \Omega^{(0)}} \right) \quad \text{and} \quad Z_K^{(1)} = \left( \begin{array}{c} \delta \\ \delta \xi^{(0)j} \end{array} \right).
\] (29)

Furthermore, the new matrix \( f_{KJ}^{(1)} \) can be written as

\[
\left( \begin{array}{cccccc} 2\frac{\partial}{\partial v^i} \eta_{ij} & 0 & -2\theta \eta_{kj} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2\theta \eta_{ij} & 0 & 0 & 0 & -\eta_{ij} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta_{ij} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2\theta (\eta_{ij} \partial_a - f_{ijk} A^k_a - \mu f_{ijk} e^k_a) & 0 & 2\theta (\eta_{ij} \partial_a - f_{ijk} A^k_a - \frac{1}{2\theta} f_{ijk} \lambda^k_a) & 0 & -f_{ijk} e^k_a & 0 \\ 2\theta (\eta_{ij} \partial_a - f_{ijk} A^k_a - \frac{1}{2\theta} f_{ijk} \lambda^k_a) & 0 & 2\Lambda f_{ijk} e^k_a & 0 & 0 & 0 \\ -f_{ijk} e^k_a & 0 & (\eta_{ij} \partial_a - f_{ijk} A^k_a) & 0 & 0 & 0 \end{array} \right) \times e^{ab} \delta^2(x - y).
\] (30)

It is clear that \( f_{KJ}^{(1)} \) is not a square matrix, however, it has linearly independent zero-modes, which turn out to be

\[
(v_1^{(1)})^K = \left( -\partial_a \eta^j_m - f_{ljm} A^l_a, 0, -f_{ljm} e^l_a, 0, -f_{ljm} \lambda^l_a, 0, \eta^j_m, 0, 0 \right),
\] (31)

\[
(v_2^{(1)})^K = \left( -\frac{\mu}{2\theta} f_{ljm} e^l_a, 0, -\partial_a \eta^j_m - f_{ljm} A^l_a, 0, f_{ljm} (\mu \lambda^l_a + 2\Lambda e^l_a), 0, 0, \eta^j_m, 0 \right),
\] (32)

\[
(v_3^{(1)})^K = \left( -\frac{\mu}{2\theta} f_{ljm} e^l_a, 0, 0, 0, -\partial_a \eta^j_m - f_{ljm} A^l_a + \mu f_{ljm} e^l_a, 0, 0, 0, \eta^j_m \right),
\] (33)

such that \( (v_{1,2,3}^{(1)})^K f_{KJ}^{(1)} = 0 \). By using the symplectic potential, we find that the matrix \( Z_K^{(1)} \) is given
by
\[
\begin{pmatrix}
-2\theta \left( D_a e_{0j} + \frac{1}{\mu} D_\alpha A_{0j} \right) + f_{jlm} \left( e_0^l \lambda_a^m + \left( \lambda_0^l + 2\theta A_0^l \right) e_a^m \right) \\
-D_a \lambda_{0j} - 2\theta D_\alpha A_{0j} + f_{jlm} A_0^l \lambda^m_a - 2\Lambda f_{jlm} e_0^l e^m_a \\
-D_a e_{0j} + f_{jlm} A_0^l e^m_a \\
\Xi_i^{(0)} \\
\Theta_i^{(0)} \\
-\Sigma_i^{(0)} \\
0 \\
0 \\
0
\end{pmatrix}
\epsilon^{ab} \delta^2(x - y).
\]

By multiplying both sides of Eq. (28) by the zero-modes of the matrix \( f_K^{(1)} \), and evaluating at \( \Omega^{(0)} = 0 \), we get the following covariant constraint relations (the integration symbols \( \int \) is omitted for clarity):
\[
(v_1^{(1)})_K Z_K^{(1)} |_{\Omega^{(0)} = 0} = 0,
\]
\[
(v_2^{(1)})_K Z_K^{(1)} |_{\Omega^{(0)} = 0} = -\frac{1}{2\theta} \epsilon^{\alpha\beta\gamma} \lambda_{\alpha i} e^j_\gamma,
\]
\[
(v_3^{(1)})_K Z_K^{(1)} |_{\Omega^{(0)} = 0} = \frac{1}{2\theta} \epsilon^{\alpha\beta\gamma} e_{\alpha i} e^j_\gamma.
\]
The substitution \( \Omega^{(0)} = 0 \) guarantees that these constraints will drop from the remainder of the calculation. Then, from (36) and (37), together with the invertibility of \( e_{\alpha i} \) and \( \lambda_{\alpha i} \), we finally obtain
\[
\Phi^\alpha = \epsilon^{\alpha\beta\gamma} e^j_\gamma = 0,
\]
which are known as symmetry conditions \cite{13} and play a crucial role in the relation of the metric and tetrad formulations of massive gravity theories and multi-bigravity ones. Furthermore, one finds that the equation (38) can be split into two equations:
\[
\Phi^a = \epsilon^{ab} \left( e^i_0 \lambda_{ab} - e^i_b \lambda_{0b} \right) = 0,
\]
\[
\Phi^0 = \epsilon^{ab} e^i_a \lambda_{ib} = 0.
\]
We can see that the Eq. (39) has fixed fields \( e^i_0 \) and \( \lambda^i_0 \), whereas Eq. (40) gives us one more constraint. This agrees completely with what was found in \cite{30} by means of the Dirac procedure, however, in that formalism the constraints (39) and (40) arise as tertiary constraints, whereas in \cite{32,33} the constraint (40) was introduced by hand. Now, by imposing the stability condition on the new constraint (40), we have the following equation:
\[
f_{KJ}^{(2)} \dot{\xi}^{(0)J} = Z_K^{(2)}(\xi),
\]
where the matrices \( f_{KJ}^{(2)} \) and \( Z_K^{(1)} \) can be expressed as
\[
f_{KJ}^{(2)} = \left( \begin{array}{c}
\frac{f_{IJ}^{(1)}}{Z_K^{(0)}}, \\
\delta_{KJ}^{(0)}
\end{array} \right) \quad \text{and} \quad Z_K^{(2)} = \left( \begin{array}{c}
Z_K^{(1)} \\
0
\end{array} \right).
\]
Consequently, the new matrix $f_{ij}^{(2)}$ is given by

$$
\begin{pmatrix}
2\theta \eta_{ij} & 0 & -2\theta \eta_{ij} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
2\theta \eta_{ij} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \eta_{ij} & 0 & 0 & 0 \\
0 & 0 & 0 & \eta_{ij} & 0 & 0 \\
2\theta \eta_{ij} \partial_a - f_{ijk} A^k_a - \mu f_{ijk} e^k_a & 0 & 2\theta \eta_{ij} \partial_a - f_{ijk} A^k_a - \frac{1}{2\theta} f_{ijk} \lambda^k_a & 0 & -f_{ijk} e^k_a & 0 \\
2\theta \eta_{ij} \partial_a - f_{ijk} A^k_a - \frac{1}{2\theta} f_{ijk} \lambda^k_a & 0 & 2\Lambda f_{ijk} e^k_a & 0 & (\eta_{ij} \partial_a - f_{ijk} A^k_a) & 0 \\
-f_{ijk} e^k_a & 0 & (\eta_{ij} \partial_a - f_{ijk} A^k_a) & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_{aj} & 0 & e_{aj} \\
\end{pmatrix} \times e^{a_b} \delta^2(x - y).
$$

One can easily verify that $f_{ij}^{(2)}$ is also a singular matrix that has the following linearly independent zero-modes:

$$
(v_1^{(2)})^j = (-\partial_a \eta^j_m - f^{j lm} A^l_a, 0, -f^{j lm} \lambda^l_a, 0, -f^{j lm} \lambda^l_a, 0, 0, 0),
$$

$$
(v_2^{(2)})^j = (-\frac{\mu}{2\theta} f^{j lm} \lambda^l_a, 0, -\partial_a \eta^j_m - f^{j lm} A^l_a, 0, f^{j lm} (\mu \lambda^l_a + 2\Lambda e^l_a), 0, 0, \eta^j_m, 0, 0),
$$

$$
(v_3^{(2)})^j = (-\frac{\mu}{2\theta} f^{j lm} \lambda^l_a, 0, 0, 0, -\partial_a \eta^j_m - f^{j lm} A^l_a + \mu f^{j lm} \lambda^l_a, 0, 0, 0),
$$

$$
(v_4^{(2)})^j = (\mu e^j a, 0, 0, 0, 0, 0, 0, 0),
$$

After performing the contraction of the both of (41) with the new zero-modes, it is not difficult to see that the zero-modes (44), (45) and (46) do not generate any new constraint, whereas from the zero-mode $(v_4^{(2)})^j$ we have the following constraint relation:

$$
(v_4^{(2)})^K Z_K^{(2)} |_{\Omega(0), \Omega(1)} = e^{\alpha \beta \nu} f_{ijk} e^i_{\alpha} e^j_{\beta} (A e^k_{\nu} + \mu \lambda^k_{\nu}) = -2e (3\Lambda + \mu \lambda) = 0,
$$

where we have used $e^{\alpha \beta \nu} e^i_{\alpha} e^j_{\beta} e^k_{\nu} = e f^{ijk}$ with $e = \det | e^i_{\alpha} |$ and $\lambda = e^i_{\alpha} \lambda^i_{\alpha}$. Hence, from Eq. (48) we can identify the following scalar constraint:

$$
\Upsilon = 3\Lambda + \mu \lambda = 0,
$$

which is also in agreement with what was obtained in Ref. [30] via the Dirac procedure, whereas in Ref. [32, 33] such a constraint is missing. Once again, we can introduce the consistency condition on (19) and explore whether there are further constraints in the theory. To this aim, we study the equation

$$
f_{ij}^{(3)} \xi^{(0)j} = Z_K^{(3)} (\xi).
$$

It is easy to verify that even after inserting the above constraint into the matrix $f_{ij}^{(3)}$ and calculating its zero-modes, no new constraint is obtained. Hence, there are no further constraints in the theory and thus our procedure to obtain new constraints via the consistency condition is done. With the above results and the F-J method, we can now introduce the constraints [45, 25, 26, 48] and...
We can then use the symplectic variables (52) and (53) to construct the corresponding square
symplectic matrix \( f \equiv \frac{\delta}{\delta \xi} a_{ij} - \frac{\delta}{\delta \xi} a_{ij} \), which turns out to be
\[
\begin{pmatrix}
2\frac{\delta}{\delta \eta_{ij}} - 2\frac{\partial}{\partial \eta_{ij}} & -2\frac{\partial}{\partial \eta_{ij}} & -2\frac{\partial}{\partial \eta_{ij}} & 0 & E_{aij} & 0 & 0 \\
2\frac{\partial}{\partial a_{ij}} & 0 & 2\frac{\partial}{\partial a_{ij}} & 0 & -E_{aij} & 0 & 0 \\
2\theta_{ij} & 0 & -2\Delta_{aij} & 0 & -2\Lambda_{aij} & -\lambda_{ij} & -\mu e_{ad} \lambda^d_j \\
2\theta_{ij} & 0 & -2\Delta_{aij} & 0 & -2\Lambda_{aij} & -\lambda_{ij} & -\mu e_{ad} \lambda^d_j \\
0 & E_{aij} & \eta_{ij} & -D_{aij}^x & 0 & 0 & -e_{aj} & -\mu e_{ad} \epsilon^d_j \\
0 & 0 & \lambda_{aj} & 0 & e_{aj} & 0 & 0 & 0 \\
0 & 0 & \mu e_{ad} \lambda^d_j & 0 & \mu e_{ad} \epsilon^d_j & 0 & 0 & 0
\end{pmatrix} \epsilon_{ab} \delta^2(x - y).
\]
We thus proceed towards the discussion of the gauge symmetry in the symplectic framework. It is worth noting that, when all the constraints have been considered and the symplectic matrix still has zero-modes but no new constraint can be obtained, one is led to conclude that the theory must have a local gauge symmetry. Therefore the zero-modes act as the generators of the corresponding gauge symmetry $'\delta G'$, that is, the components of the zero-modes give the transformation properties related to the underlying (gauge) symmetry [36–38]. The local infinitesimal transformations of the symplectic variables generated by $(v)^I$ can be expressed as
\[
\delta_G \xi^I = (v_A)^I \epsilon^A,
\]
where $(v_A)$ are the independent zero-modes of the singular symplectic matrix $f_{IJ}$ and $e^A$ are the gauge parameters. For the singular symplectic matrix [51], these zero-modes turn out to be
\[
(v_1)^I = -\partial_a \eta^i_k - f^j_{ik} A^l_{aI}, \eta^i_k, -f^j_{ik} \epsilon^l_{aI}, 0, -f^j_{ik} \lambda^l_{aI}, 0, 0, 0, 0, 0),
\]
\[
(v_2)^I = -\frac{\mu}{2g} f^j_{ik} \lambda^l_{aI}, 0, -\partial_a \eta^i_k - f^j_{ik} \epsilon^l_{aI}, \eta^i_k, f^j_{ik} (\mu \lambda^l_{aI} + 2\Lambda \epsilon^l_{aI}), 0, 0, 0, 0),
\]
\[
(v_3)^I = -\frac{\mu}{2g} f^j_{ik} \epsilon^l_{aI}, 0, 0, 0, -\partial_a \eta^i_k - f^j_{ik} \lambda^l_{aI} + \mu f^j_{ik} \epsilon^l_{aI}, \eta^i_k, 0, 0, 0).
\]
which are orthogonal to the gradient of the symplectic potential and at the same time generate local displacements on the isopotential surface. As one can infer from (55), the infinitesimal gauge transformations that leave the original Lagrangian invariant are given by
\[
\delta_G A^i_a(x) = -D_a \zeta^i - \frac{\mu}{2g} f^j_{ik} (e_{aI} \xi^k + \lambda^l_{aI} \eta^l_k),
\]
\[
\delta_G e^i_a(x) = -D_a \kappa^i - f^j_{ik} \epsilon^l_{aI} \kappa^l,
\]
\[
\delta_G \lambda^i_a(x) = -D_a \zeta^i - f^j_{ik} \lambda^l_{aI} \xi^k + \mu f^j_{ik} (\lambda^l_{aI} \kappa^l + e_{aI} \xi^k) + 2\Lambda f^j_{ik} \epsilon^l_{aI} \kappa^l,
\]
where $\zeta^i$, $\kappa^i$ and $\zeta^i$ are the time-dependent gauge parameters. It is worth remarking that (59), (60) and (61) correspond to the fundamental gauge symmetry of the theory, though the diffeomorphisms have not been found yet. However, it is well-known that an appropriate choice of the gauge parameters does generate the diffeomorphism (on-shell) [46]. Let us redefine the gauge parameters as
\[
\zeta^i = -A^i \mu \epsilon^\mu, \quad \kappa^i = -e^i \mu \epsilon^\mu, \quad \zeta^i = -\lambda^i \mu \epsilon^\mu,
\]
with $\epsilon^\mu$ an arbitrary three-vector. Hence, from the fundamental gauge symmetry (61) and the mapping (62), we obtain
\[
\delta_G A^i_a = \mathcal{L}_\epsilon A^i_a + \mu \epsilon^\mu e_{a\mu\nu} \left[ \frac{1}{2g} (\delta A)^\nu_i + (\delta \lambda)^\nu_i \right],
\]
\[
\delta_G e^i_a = \mathcal{L}_\epsilon e^i_a - \epsilon^\mu e_{a\mu\nu} (\delta \lambda)^\nu_i,
\]
\[
\delta_G \lambda^i_a = \mathcal{L}_\epsilon \lambda^i_a + 2\mu \theta \epsilon^\mu e_{a\mu\nu} \left[ \frac{1}{2\mu \theta} (\delta e)^\nu_i - \frac{1}{2g} (\delta A)^\nu_i + (\delta \lambda)^\nu_i \right].
\]
which are precisely (on-shell) diffeomorphisms. In addition, TMG [11] is also made invariant under Poincaré transformations by construction [45, 46]. Thus, in order to recover the Poincaré symmetry,
we need to map the arbitrary gauge parameters of the fundamental gauge symmetry \( \delta_G \) \(^{(61)}\) into those of the Poincaré symmetry. This is achieved by a mapping of the gauge parameters \([46–48]\), e.g.:

\[
\begin{align*}
\zeta^i &= A^i_{\mu} \varepsilon^\mu + \omega^i, \quad \kappa^i = \delta^i_{\mu} \varepsilon^\mu, \quad \zeta^i = \lambda^i_{\mu} \varepsilon^\mu
\end{align*}
\]

such that \( \varepsilon^\mu \) and \( \omega^i \) are related to local coordinate translations and local Lorentz rotations, respectively, which together constitute the 6 independent gauge parameters of Poincaré symmetries in 3D. By using this map, the gauge symmetries reproduce the Poincaré symmetries modulo terms proportional to the equations of motion

\[
\begin{align*}
\delta_G e^i_{\alpha} &= -\varepsilon^\mu \partial_\alpha e^i_{\mu} - e^j_{\mu} \partial_\alpha \varepsilon^\mu - f^i_{\mu j k} \varepsilon^\mu \omega^k \varepsilon^{\alpha \alpha \gamma \nu} \varepsilon^{\mu \nu}, \\
\delta_G A^i_{\alpha} &= -\partial_\alpha \omega^i - f^i_{\mu j k} A^i_{\alpha} \varepsilon^\mu \partial_\alpha \varepsilon^\mu - A^i_{\mu} \partial_\alpha \varepsilon^\mu - \mu \varepsilon^\gamma \varepsilon^{\alpha \alpha \gamma \nu} \left[ \frac{1}{2 \theta} (\delta A)^{\mu \nu} + (\delta \lambda)^{\mu \nu} \right], \\
\delta_G \lambda^i_{\alpha} &= -\varepsilon^\mu \partial_\alpha \lambda^i_{\mu} - \lambda^j_{\mu} \partial_\alpha \varepsilon^\mu - f^i_{\mu j k} \lambda^j_{\alpha} \omega^k \varepsilon^{\mu \gamma \nu} \left[ \frac{1}{2 \theta} (\delta A)^{\mu \nu} + \frac{1}{2 \theta} (\delta \lambda)^{\mu \nu} \right]
\end{align*}
\]

where the equations of motion \((\delta e)^{\mu \nu}, (\delta A)^{\mu \nu}\) and \((\delta \lambda)^{\mu \nu}\) are defined in \((41)-(53)\). We thus conclude that the Poincaré symmetry \((65)\) as well as the diffeomorphisms \((62)\) are not independent symmetries: they are contained indeed in the fundamental gauge symmetry \((61)\) as on-shell symmetries, that is, only when the equations of motion are imposed. In addition, the generators of such gauge transformations can be represented in terms of the zero-modes, thereby making evident that the zero-modes of the symplectic two-form encode all the information about the gauge structure of this theory.

V. THE FADDEEV-JACKIW BRACKETS AND DEGREE OF FREEDOM COUNT

As was already mentioned in Sec. III, in theories with a gauge symmetry, the symplectic matrix obtained at the end of the procedure is still singular. Nevertheless, in order to obtain a non-singular symplectic matrix and to determine the quantization bracket (F-J brackets) structure between the dynamical fields, we must impose a gauge-fixing procedure, that is, new gauge constraints. In this case, we now partially fix the gauge by imposing the time-gauge, namely, \( A^i_{\mu} = 0, \varepsilon^i_{\mu} = 0, \lambda^i_{\mu} = 0 \) and \( \phi_0 = \text{cte} \) (i.e. \( \phi_0 = 0 \)). In this manner, we also introduce new Lagrange multipliers that enforce these gauge conditions, namely, \( \rho_i, \omega_i, \tau_i \) and \( \sigma^0 \). Thus, the final symplectic Lagrangian after gauge fixing can be written as

\[
\mathcal{L} = \epsilon^{ab} \theta \left( \frac{1}{\mu} A_{bi} + 2 \epsilon_{bi} \right) \dot{A}^i_{\alpha} + \epsilon^{ab} \lambda_{bk} \dot{e}^i_{\alpha} - (\Xi_i - \rho_i) \dot{\beta}^i - (\Theta_i - \omega_i) \dot{\alpha}^i - (\Sigma_i - \tau_i) \dot{\gamma}^i - (\Phi^0 - \sigma^0) \dot{\phi}_0.
\]

From the Lagrangian density \((66)\) one may read off the final set of symplectic variables

\[
\xi^I = \left( A^i_{\alpha}, \beta^i, e^i_{\alpha}, \alpha^i, \lambda^i_{\alpha}, \gamma^i, \phi^0, \rho^i, \omega^i, \tau^i, \sigma_0 \right),
\]

so that, the corresponding symplectic 1-form is given by

\[
a_f = \left( \epsilon^{ab} \theta \left( \frac{1}{\mu} A_{bi} + 2 \epsilon_{bi} \right), -\Xi_i + \rho_i, \epsilon^{ab} \lambda_{bi}, -\Theta_i + \omega_i, 0, -\Sigma_i + \tau_i, -\Phi^0 + \sigma_0, 0, 0, 0 \right).
\]
After some algebra, we obtain the explicit form of the symplectic matrix \( f_{IJ} \)

\[
\left(\begin{array}{ccccccccc}
2\frac{\partial}{\partial \eta_{ij}} & -2\frac{\partial}{\partial x_{ij}} & -2\theta \eta_{ij} & -2\theta \Delta_{aij} & 0 & E_{aij} & 0 & 0 & 0 & 0 & 0 \\
2\frac{\partial}{\partial x_{aij}} & 0 & 2\theta \Delta_{aij} & -2\theta L_{aij} & 0 & 0 & 0 & -\frac{1}{2} e_{ab} \eta_{ij} & 0 & 0 & 0 \\
2\theta \eta_{ij} & -2\theta \Delta_{aij} & 0 & 2\Delta E_{aij} & -\eta_{ij} & 0 & -2\theta \eta_{ij} & 0 & 0 & 0 & 0 \\
2\theta \Delta_{aij} & 0 & 2\Delta E_{aij} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} e_{ab} \eta_{ij} & 0 \\
0 & E_{aij} & \eta_{ij} & D_{aij}^y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-E_{aij} & 0 & D_{aij}^y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} e_{ab} \eta_{ij} \\
0 & 0 & \lambda_{aij} & e_{aij} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} e_{ab} \eta_{ij} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} e_{ab} \eta_{ij} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} e_{ab} & 0 & 0 & 0 \\
\end{array}\right) \times e^{ab} \delta^2(x - y).
\]

(69)

It is clear that such a matrix is not singular. The corresponding inverse matrix \( f_{IJ}^{-1} \) is given by

\[
\left(\begin{array}{ccccccccc}
\frac{\partial}{\partial \eta_{ij}} & 0 & 0 & 0 & -\mu \eta_{ij} & 0 & 0 & D_{aij}^y & -\frac{\partial}{\partial L_{aij}} & -\frac{\partial}{\partial E_{aij}} & \mu e_{ai} \\
0 & 0 & 0 & 0 & 0 & \eta_{ij} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\eta_{ij} & 0 & 0 & 0 & 0 & 0 \\
\mu \eta_{ij} & 0 & -\eta_{ij} & 0 & -2\theta \mu \eta_{ij} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2\theta \mu \eta_{ij} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{\partial}{\partial L_{aij}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\partial}{\partial E_{aij}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{\partial}{\partial \eta_{ij}} & 0 & 0 & 0 & -\frac{\partial}{\partial \eta_{ij}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\mu e_{ai} & 0 & -e_{ai} & 0 & \lambda_{aij} & -1 & 0 & 0 & 0 & -\frac{1}{2} e_{ab} \eta_{ij} & 0 \\
\end{array}\right) \times e^{ab} \delta^2(x - y),
\]

(70)

with \( \xi_{aij} = (\mu L_{aij} + 2\Delta E_{aij}) \). In this way, the quantization bracket, dubbed generalized Faddeev-Jackiw bracket, \( \{\cdot,\cdot\}_{F-J} \) between two elements of the symplectic variable set \( (71) \), is defined as

\[
\{\xi_I(x), \xi_J(y)\}_{F-J} = (f_{IJ})^{-1}.
\]

(71)

The non-vanishing Faddeev-Jackiw brackets for topologically massive AdS gravity can now be easily extracted using \((70)\) and \((71)\). We thus have

\[
\{A^i_a(x), A^j_b(y)\}_{F-J} = \frac{\mu}{2\theta} \eta^{ij} \delta^2(x - y),
\]

(72)

\[
\{A^i_a(x), \lambda^j_b(y)\}_{F-J} = -\mu e_{ab} \eta^{ij} \delta^2(x - y),
\]

(73)

\[
\{\lambda^i_a(x), \lambda^j_b(y)\}_{F-J} = 2\theta \mu e_{ab} \eta^{ij} \delta^2(x - y),
\]

(74)

\[
\{e^i_a(x), \lambda^j_b(y)\}_{F-J} = e_{ab} \eta^{ij} \delta^2(x - y).
\]

(75)

These F-J brackets correspond to the Dirac brackets reported in \(30\). The canonical quantization \( \{\xi_I, \xi_J\}_{F-J} \rightarrow \frac{1}{i\hbar} [\hat{\xi}_I, \hat{\xi}_J] \) can be carried out by using the aforementioned brackets given
by \((72)-(75)\). In addition, we are now ready to perform the counting of physical degrees of freedom: starting with 18 canonical variables \((e^i_a, \lambda^i_a, A_i^a)\), we end up with 17 independent constraints \((\Xi^{(0)}_i, \Theta^{(0)}_i, \Sigma^{(0)}_i, \Phi^0, e^0_i = 0, A^0_i = 0, \varphi_0 = \text{cte})\) after imposing the gauge-fixing term. Therefore, the number of physical degrees of freedom per space point for 3D Topologically Massive AdS Gravity is one, independently of the value of \(\mu\), as it was also found in \([32, 33]\).

VI. CONCLUSIONS AND DISCUSSIONS

In the present paper, the nature of the constraints and gauge structure of the topologically massive AdS gravity theory was studied from the perspective of the Faddeev-Jackiw symplectic approach. The whole set of independent physical constraints was identified through the consistency condition and the zero-modes. It was shown that even when all the physical constraints are found, but the symplectic matrix still has zero-modes, that is, when the zero-modes are orthogonal to the gradient of the symplectic potential on the surface of the constraints, one is led to deduce that the theory has a local gauge symmetry. Therefore, the zero-modes straightforwardly generate the local gauge symmetry under which all physical quantities are invariant. By mapping the gauge parameters appropriately we have also obtained the Poincaré transformations and the diffeomorphism symmetry. Additionally, we have shown that the time-gauge fixing of the density Lagrangian renders the non-degenerate symplectic matrix \(f_{ij}\). We then have identified the quantization bracket (F-J brackets) structure and have proved that there is one physical degree of freedom. It is worth remarking that all the results presented here can be applied to the study of the physical content of models such as massive gravity and bigravity theories in 2+1 dimensions, in which secondary, tertiary, or higher-order constraints are present. Such problems are under study and will be published elsewhere \([51]\). Another line for further research is the application of the procedure used here to explore conceptual and technical issues of gravity models in 3+1 dimensions.

Acknowledgements

This work has been partially supported by CONACyT under grant number CB-2014-01/240781. We would like to thank G. Tavares-Velasco for reading a draft version of this paper and alerting us to various typos.

Appendix A: Faddeev-Jackiw symplectic approach

In this appendix, we summarize the main aspects of the Faddeev-Jackiw symplectic approach \([34]\), which is based on a first-order Lagrangian in time derivative. However, this is not a serious restriction because even if the original Lagrangian is not of first-order, it is always possible to introduce variables of auxiliary fields to obtain a first-order one (usually, the canonical momenta are chosen as auxiliary fields). After introducing variables of the auxiliary fields, we can construct
a first-order Lagrangian for a physical system as follows:

\[ \mathcal{L}(\xi) = a_I(\xi) \dot{\xi}^I - V(\xi), \quad (I = 1, 2, 3, \ldots, N), \]  

(A1)

where \( \xi^I \) is the so-called symplectic variable, which consists of a combination of the original variables along with some auxiliary fields and the canonical momenta. The term \( V(\xi) \), which is called symplectic potential, is assumed to be free of time derivatives of \( \xi^I \), and it is easy to see that it is the negative of the canonical Hamiltonian. Finally, the function \( a_I(\xi) \) is the canonical one-form and is the main focus of interest. The Euler-Lagrange equations of motion for Lagrangian (A1) can be written as

\[ f_{IJ} \dot{\xi}^J - \frac{\partial}{\partial \xi^I} V(\xi) = 0, \]  

(A2)

where \( f_{IJ} \) is the so-called symplectic matrix with the following explicit form:

\[ f_{IJ} = \frac{\partial}{\partial \xi^I} a^J - \frac{\partial}{\partial \xi^J} a^I. \]  

(A3)

When this matrix is non-singular, it can be inverted, and therefore all the symplectic variables can be solved from (A2)

\[ \dot{\xi}^I = (f_{IJ})^{-1} \frac{\partial}{\partial \xi^J} V(\xi). \]  

(A4)

Otherwise, there are some constraints in the theory. In the method of Faddeev-Jackiw, the above equation can be written as

\[ \dot{\xi}^I = \{\xi^I, \xi^J\}_{F_{-J}} \frac{\partial V}{\partial \xi^J}. \]  

(A5)

where the Faddeev-Jackiw bracket \( \{,\}_{F_{-J}} \) is defined by

\[ \{\xi^I, \xi^J\}_{F_{-J}} = (f_{IJ})^{-1}. \]  

(A6)

However, in gauge invariant theories, where in addition to the true dynamical degrees of freedom there are also gauge degrees of freedom, the symplectic matrix turns out to be singular, which implies that the system is endowed with constraints. In this case, the matrix \( f_{IJ} \) necessarily has some zero-modes (\( v_k \)) (with \( k \) all the linearly independent zero-modes that are found for \( f_{IJ} \)), where each \( (v_k) \) is a column vector with \( N \) entries \( (v_k)^I \). By definition, the zero-modes satisfy the following equation

\[ (v_k)^I f_{IJ} = 0, \quad (k = 1, 2, 3, \ldots, \leq N). \]  

(A7)

Consequently, the constraints associated with the symplectic matrix are given by

\[ \phi_k \equiv (v_k)^I \frac{\partial}{\partial \xi^I} V(\xi) = 0, \]  

(A8)

which shows that the zero-modes of \( f_{IJ} \) encode the information of the constraints. Following the prescription of the symplectic formalism, we will analyze whether there are new constraints. To this aim, we impose a consistency condition on the constraints as in the Dirac approach:

\[ \dot{\phi}_k = \frac{\partial \phi_k}{\partial \xi^I} \dot{\xi}^I = 0. \]  

(A9)
The consistency condition on the constraints and equations of motion can be rewritten as

\[ f^{(1)}_{KJ} \dot{\xi}^J = Z^{(1)}_K (\xi), \]  

(A10)

where

\[ f^{(1)}_{KJ} = \left( \begin{array}{c} f_{IJ} \\ \frac{\partial \phi}{\partial \xi^J} \end{array} \right) \quad \text{and} \quad Z^{(1)}_K (\xi) = \left( \begin{array}{c} \frac{\partial V}{\partial \xi^I} \\ 0 \end{array} \right), \]  

(A11)

The new matrix \( f^{(1)}_{IJ} \) is not a square matrix anymore, however, it still contains linearly independent zero-modes \( (v^{(1)}_l) \), which are different from the original ones. Multiplying both sides of Eq. (A10) by these modes, we get the following constraint relations

\[ \left( v^{(1)}_l \right)^K Z^{(1)}_K (\xi) |_{\phi_k = 0} = 0. \]  

(A12)

The substitution \( \phi_k = 0 \) guarantees that these constraints will drop from the remainder of the calculation. If Eqs. (A12) turn out to fulfill the identity \( 0 = 0 \), then there are no further constraints; otherwise, the constraints arising from Eq. (A12) are given by

\[ \phi^{(1)}_l \equiv \left( v^{(1)}_l \right)^K Z^{(1)}_K (\xi) |_{\phi_k = 0}. \]  

(A13)

These new constraints can be treated in the same way as \( \phi_k \). In other words, we can now introduce the consistency condition for \( \phi^{(1)}_l \), as

\[ \dot{\phi}^{(1)}_l = \frac{\partial \phi^{(1)}_l}{\partial \xi^I} \dot{\xi}^I = 0. \]  

(A14)

and combine it with Eq. (A10) in order to construct a set of new linear equations, from which we explore whether there are more constraints. These steps are repeated until there are no further constraints in the system and the identities \( 0 = 0 \) are fulfilled.

Once \( m \) constraints are obtained after \( h \) steps through the consistency conditions of the constraints, we can modify our original Lagrangian by introducing the whole set of constraints multiplied by the corresponding Lagrangian multipliers \( \eta^m \) as follows:

\[ L^{(E)} = a_I (\xi) \dot{\xi}^I + \phi_m (\xi) \eta^m - V(\xi)^{(E)}, \]  

(A15)

where \( V(\xi)^{(E)} = V(\xi) |_{\phi_m = 0} \). We can now also calculate the new symplectic matrix associated with the modified Lagrangian, \( f^{(E)}_{IJ} = \partial a_I^{(E)}/\partial \xi^J - \partial a_J^{(E)}/\partial \xi^I \) with \( \xi^{(E)} I = (\xi^I, \eta^I) \); this new matrix can be either singular or non-singular. In the latter case it has an inverse and therefore all the new symplectic variables can be solved as in (A5). On the other hand, for gauge systems, this symplectic matrix is still singular and has no inverse unless some gauge-fixing terms (gauge conditions) are introduced. In this way, the procedure can be finished and the Faddeev-Jackiw brackets can be identified as in (A6).

[1] Supernova Search Team Collaboration, A. G. Riess et. al., Observational evidence from supernovae for an accelerating universe and a cosmological constant, Astron.J. 116 (1998) 10091038.
[2] R. Durrer and R. Maartens, *Dark Energy and Dark Gravity*, Gen. Rel. Grav. 40 (2008) 301328.
[3] D. Lovelock, *The Einstein tensor and its generalizations*, J. Math. Phys. 12 (1971) 498501.
[4] D. Lovelock, *The four-dimensionality of space and the einstein tensor*, J. Math. Phys. 13 (1972) 874-867.
[5] V.A. Rubakov and P.G. Tinyakov, *Infrared-modified gravities and massive gravitons*, Phys. Usp. 51 (2008) 759.
[6] M. Fierz and W. Pauli, *On relativistic wave equation for particles of arbitrary spin in an electromagnetic field*, Proc. Roy. Soc. Lon. A 173 (1939) 211-232.
[7] C. de Rham and G. Gabadadze, *Generalization of the Fierz-Pauli action*, Phys. Rev. D 82 (2010) 044020.
[8] C. de Rham, G. Gabadadze and A.J. Tolley, *Resummation of massive gravity*, Phys. Rev. Lett. 106 (2011) 231101.
[9] C. de Rham, *Massive Gravity*, Living Rev. Rel 17 (2014) 7.
[10] S. F. Hassan and R.A. Rosen, *Resolving the ghost problem in non-linear massive gravity*, Phys. Rev. Lett. 108 (2012) 041101.
[11] S. F. Hassan and R.A. Rosen, *Confirmation of the secondary constraints and absense of ghost in massive gravity and bimetric gravity*, JHEP 02 (2012) 026.
[12] S. F. Hassan, R.A. Rosen and A. Schimidt-May, *Ghost-free massive gravity with a general reference metric*, JHEP 02 (2012) 126.
[13] K. Hinterbichler and R. A. Rosen, *Interacting Spin-2 Fields*, JHEP 07 (2012) 047.
[14] K. Hinterbichler, *Theoretical Aspects of Massive Gravity*, Rev. Mod. Phys. 84 (2012) 671.
[15] Hamid R. Afshar, Eric A. Bergshoeff and Wout Merbis, *Interacting spin-2 fields in three dimensions*, JHEP 01 (2015) 040.
[16] C. Deffayet, J. Mourad and G. Zahariade, *Covariant Constraints in ghost free massive gravity*, JCAP 01 (2013) 032.
[17] J. Kluson, *Note about Hamiltonian structure of the non-linear massive gravity*, JHEP 01 (2012) 013.
[18] Kurt Hinterbichler and Rachel A. Rosen, *Interacting spin-2 fields* JHEP 07 (2012) 047.
[19] D. Boulware and S. Deser, *Can gravity have a finite range?*, Phys. Rev. D 6 (1972) 3368.
[20] S. Deser, R. Jackiw and S. Templeton, *Topologically Massive Gauge Theories*, Annals Phys. 140 (1982) 372 [Annals Phys. 281 (2000) 409] [Annals Phys. 185 (1988) 406] [INSPIRE].
[21] S. Deser, R. Jackiw, and G. ’t Hooft, *Three-dimensional Einstein gravity: Dynamical of flat space*, Ann. Phys. (N. Y. ) 152, 220 (1984)
[22] S. Deser, R. Jackiw, and G. ’t Hooft, *Three-dimensional cosmological gravity: Dynamical of constant curvature*, Ann. Phys. (N. Y. ) 153, 405 (1984)
[23] E. Witten, *(2+1)-dimensional gravity as an exactly soluble system*, Nucl. Phys. B311, 46 (1988)
[24] W. Li, W. Song, A. Strominger, *Chiral gravity in three dimensions*, JHEP 0804, 082 (2008)
[25] S. S. Deser, R. Jackiw and S. Templeton, *Three-Dimensional Massive Gauge Theories*, Phys. Rev. Lett. 48 (1982) 975 [INSPIRE].
[26] P.A.M. Dirac, *Lectures Notes on Quantum Mechanics*, Yeshiva University, New York, NY (1964).
[27] T. Hanson, A., Egge and C. Teitelboim, *Constraints Hamiltonian Systems*, Roma: Accademia Nazionale dei Lincei 1978.
[28] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, Princeton, New Jersey:Princeton University Press 1991.
[29] L. Castellani, Symmetries in the constrained Hamiltonian system, Ann. Phys. 143 (1982) 357.
[30] M. Blagojevic and B. Cvetkovic, Canonical structure of topologically massive gravity with a cosmological constant, JHEP 05 (2009) 073.
[31] Mu-In Park, Constraint Dynamics and Gravitons in Three Dimensions, JHEP 0809 (2008) 084.
[32] S. Carlip, Constraint Algebra of Topologically Massive AdS Gravity, JHEP 0810 (2008) 078.
[33] D. Grumiller, R. Jackiw and N. Johansson, Canonical analysis of cosmological topologically massive gravity at the chiral point, MIT-CTP 3957, UUITP-12/08, (2008).
[34] L. D. Faddeev and R. Jackiw, Hamiltonian Reduction of Unconstrained and Constrained Systems, Phys. Rev. Lett 60(1988) , 1692.
[35] Barcelos-Neto, J. et al., Symplectic quantization of constrained systems, Mod.Phys.Lett. A 7 (1992) 1737.
[36] J. Barcelos-Neto, C. Wotzasek, Faddeev-Jackiw quantization and constraints, Int.J.Mod.Phys. A 7 (1992) 4981.
[37] H. Montani and R. Montemayor, Lagrangian approach to a symplectic formalism for singular systems, Phys. Rev. D 58 (1998) 125018.
[38] H. Montani and C. Wotzasek, Faddeev-Jackiw quantization of nonabelian systems, Mod.Phys.Lett. A 8 (1993) 3387.
[39] J. Antonio Garcia, Josep M. Pons Equivalence of Faddeev-Jackiw and Dirac approaches for gauge theories, Int. J. Mod. Phys. A 12 (1997) 451.
[40] Leng Liao, Yong Chang Huang, Non-equivalence of Faddeev-Jackiw method and Dirac-Bergmann algorithm and the modification of Faddeev-Jackiw method for keeping the equivalence, Annals of Physics 322 (2007) 2469.
[41] E.M.C. Abreu, A.C.R. Mendes, C. Neves, W. Oliveira, R.C.N. Silva and C. Wotzasek, Obtaining non-Abelian field theories via the Faddeev-Jackiw symplectic formalism, Phys.Lett. A 375 (2010) 3603.
[42] Wotzasek, Clovis, Faddeev-Jackiw approach to hidden symmetries, Annals Phys. 243 (1995) 73.
[43] S. Carlip, Inducing Liouville theory from topologically massive gravity, Nucl. Phys. B 362 (1991) 111-124.
[44] S. Carlip, S. Deser, A. Waldron, and Wise, D. K., Cosmological Topologically Massive Gravitons and Photons, class. Quantum Grav 26 (2009) 075008.
[45] R. Utiyama, Invariant theoretical interpretation of interaction, Phys. Rev. 101 (1956) 1597.
[46] T. W. B. Kibble, Lorentz invariance and the gravitational field, J. Math. Phys. 2, 212 (1961)
[47] R. Banerjee, S. Gangopadhyay, P. Mukherjee and D. Roy, Symmetries of the general topologically massive gravity in the hamiltonian and lagrangian formalisms, JHEP 1002, 075 (2010).
[48] M. Blagojevic, Gravitation and Gauge Symmetries, (IOP, Bristol, United Kingdom, 2002).
[49] D. J. Toms, Faddeev-Jackiw quantization and the path integral, Phys. Rev D 92 105026 (2015).
[50] A. Fuster, Marc Henneaux and Axel. Maas, BRST quantization: A Short review, Int. J. Geom. Meth. Mod. Phys. 2, 939-964 (2005).
[51] Omar Rodríguez-Tzompantzi, in preparation.