Asymptotically exact trial wavefunctions for yrast states of rotating Bose gases

S Viefers and M Taillefumier

Department of Physics, University of Oslo, PO Box 1048 Blindern, N-0316 Oslo, Norway
E-mail: s.f.viefers@fys.uio.no

Received 23 May 2010, in final form 17 June 2010
Published 19 July 2010
Online at stacks.iop.org/JPhysB/43/155302

Abstract
We revisit the composite fermion (CF) construction of the lowest angular momentum yrast states of rotating Bose gases with weak short-range interaction. For angular momenta at and below the single vortex, \( L \leq N \), the overlaps between these trial wavefunctions and the corresponding exact solutions increase with increasing system size and appear to approach unity in the thermodynamic limit. In the special case \( L = N \), this remarkable behaviour was previously observed numerically. Here we present methods to address this point analytically and find strongly suggestive evidence in favour of similar behaviour for all \( L \leq N \). While not constituting a fully conclusive proof of the converging overlaps, our results do demonstrate a striking similarity between the analytic structure of the exact ground state wavefunctions at \( L \leq N \) and that of their CF counterparts. Results are given for two different projection methods commonly used in the CF approach.

1. Introduction

The behaviour of ultracold atomic Bose gases under rotation has been a subject of intense study over the past few years [1]. One of the most studied aspects, involving a large body of theoretical work, is the expected occurrence of strongly correlated states of the fractional quantum Hall (FQH) type in the limit of ultrafast rotation [2]. On the experimental side, progress has been extremely impressive, from the creation of the first vortex in 1999 [3] to arrays of hundreds of vortices [4]. Nevertheless there are practical obstacles to reaching the actual quantum Hall regime this way. Other, more promising, scenarios are now being pursued to try and create the effective ‘magnetic’ fields required for producing bosonic FQH states [5]. Another limit that has attracted considerable interest is that of slow rotation, in the sense that only one or a few vortices are present in the system. In particular, various groups have studied the yrast line [6] of the system, i.e. the states with lowest energy for a given total angular momentum. Theoretical work includes analytical studies [6], exact diagonalization [7], the Gross–Pitaevskii approach [8, 9], as well as trial many-body wavefunctions [10, 11]. Moreover, for the lowest angular momentum regime, \( 2 \leq L \leq N \), exact analytical ground-state wavefunctions were found for bosons with short-range interaction sufficiently weak to assume the particles reside in the lowest Landau level (LLL) [12–16].

The focus of the present paper is this latter case. The composite fermion (CF) approach [17], originally designed to give trial wavefunctions in the FQH regime, was first applied to some low-angular momentum states in [10, 11]. It was noted [11] that overlaps between CF and exact wavefunctions for the ‘single-vortex’ state at \( L = N \) appeared to increase with particle number, at least for small systems. This point was studied more systematically in [18], and overlaps for the single-vortex state computed numerically for up to 45 particles. The results strongly suggested convergence of the overlaps to unity in the thermodynamic limit, with deviation decreasing as \( \sim 1/N \). This kind of behaviour is generically unusual for any kind of non-exact trial wavefunctions. In this case it was even more surprising because \textit{a priori}, the CF construction could not be expected to work well in this low-angular momentum regime—for reasons to be explained below. Unfortunately, the numerical calculations gave little insight into the reasons for this behaviour. Therefore, here we re-examine the issue analytically, including also the states below the single vortex: we study the yrast states in the entire regime \( 2 \leq L \leq N \), comparing the analytic form of the CF wavefunctions to their exact counterparts and present evidence...
that the increase of overlaps with system size occurs for all these states. We address the issue for two different projection methods commonly used in the CF approach. In the case of ‘full projection’ (method I), we demonstrate that the analytic structure of the CF wavefunctions is strikingly similar to that of the exact ones; the latter have the form of fundamental symmetric polynomials in the coordinates \((z_i - Z)\), where \(Z\) is the centre of mass, while the former are merely ‘lacking’ one or several coordinates in the centre of mass term. For the other projection method (method II), it is possible to rewrite the CF wavefunctions, as well as the exact ones, in the form of an \(N \times N\) determinant, divided by a Jastrow factor. We analytically compute overlaps between pairs of corresponding entries in these determinants and show that the overlaps between all pairs of entries converge to unity in the thermodynamic limit. This does not necessarily allow us to conclude that the overlaps between the full determinants (and thus the wavefunctions) go to unity as \(N \to \infty\), since the size of the matrices themselves grows with \(N\); the latter remains a challenging open problem.

Although exact analytic yrast wavefunctions are known for the particular states addressed in this paper, it is nevertheless of interest to examine why the CF trial wavefunctions work so surprisingly well in this regime. In particular, this may help to justify applying the CF construction to nearby states where no exact solutions are known—such as excitations above the present yrast states or other few-vortex yrast states at \(L > N\).

The outline of the paper is as follows: section 2 presents some necessary background on rotating bosons in the LLL, the form of the exact yrast wavefunctions for \(2 \leq L \leq N\), as well as the CF approach. Sections 3 and 4 contain results for the single-vortex state \(L = N\) for the two different projection methods. The methods developed here can be directly transferred to all the lower angular momentum states, \(L < N\); this is the topic of section 5. Section 6 gives some concluding remarks. An appendix is included, providing some details of the overlap calculations in section 4.

2. Background

2.1. Rotating bosons in the lowest Landau level

Consider a system of \(N\) spinless bosons with mass \(m\) in a harmonic trap of strength \(\omega\) rotating with angular frequency \(\Omega\) and interacting via a short-range (delta function) potential \(H_1\). In a rotating frame the full, three-dimensional Hamiltonian can be written as

\[
H = \sum_{i=1}^{N} \left[ \frac{\hat{p}_i^2}{2m} + \frac{1}{2} m \omega^2 \bar{r}_i^2 \right] - \Omega L_z + H_1, \tag{1}
\]

where \(L_z\) denotes the angular momentum around the rotation axis. Completing the square inside the brackets, this may be recast as

\[
H = \sum_{i=1}^{N} \left[ \frac{1}{2m} (\hat{p}_i - \bar{A})^2 + H_{ho}(z_i) \right] + (\omega - \Omega) L_z + H_1, \tag{2}
\]

with \(\bar{A} = m\omega(-y, x)\), \(H_{ho}(z)\) denoting the \(z\)-part of the harmonic oscillator Hamiltonian and \(|\rangle\) denoting the planar \((x, y)\) part of the Hamiltonian. We note that the planar part of \(H\) takes the form of particles moving in an effective ‘magnetic’ field \(B_{\text{eff}} = \nabla \times \vec{A} = 2m\omega \hat{z}\).

Now, the interaction is assumed to be weak in the sense that it does not mix different harmonic oscillator levels. We will be interested, for a given total angular momentum, only in the lowest many-body states (the ‘yrast’ states). In this limit, the model may be rewritten as a LLL problem in the effective ‘magnetic’ field \(B_{\text{eff}} = 2m\omega\) (and of course, \(n_z = 0\), freezing out harmonic oscillator excitations in the \(z\)-direction). In other words, this limit effectively constitutes a dimensional reduction to a system of particles in two dimensions and restricted to the LLL. The Hamiltonian then takes the form

\[
H = (\omega - \Omega) L + g \sum_{i<j} \delta^2 (\vec{r}_i - \vec{r}_j) \tag{3}
\]

(where \(\hbar = 1\)), where we now use \(L\) to denote the total angular momentum, \(L = \sum_i l_i = L_z\).

In the following we shall be concerned with many-body wavefunctions describing this system at some given number of particles and total angular momentum. A convenient basis of single-particle states spanning our Hilbert space (the LLL) is given by

\[
\eta_{0,l} = \frac{1}{\sqrt{2^{l+1} \pi l!}} e^{-z^2/4}, \tag{4}
\]

where \(z = \sqrt{2m\omega}(x + iy)\) are the complex coordinates denoting the particle positions in the plane, and \(l\) is the angular momentum of the state. A general bosonic many-body wavefunction \(\psi(z_1, \ldots, z_N)\) will then be a homogeneous, symmetric polynomial in \(z_i\) times the exponential factor \(\exp \left( - \sum |z_i|^2 / 4 \right)\) (which will be suppressed throughout most of this paper for simplicity). The degree of the polynomial gives the total angular momentum of the state.

In connection with the CF construction, we shall need more general single-particle basis states for the full Landau problem, i.e. all Landau levels. In symmetric gauge, these are given by

\[
\eta_{n,m} = N_{n,m} e^{-|z|^2/4} z^m L_n^m \left( \frac{z^2}{2} \right), \tag{5}
\]

where \(n\) is the Landau level index, \(m\) denotes the angular momentum, \(N_{n,m}\) is a normalization factor and \(L_n^m\) are the associated Laguerre polynomials.

2.2. Exact ground states

Here and in the following we will use the notation \(\psi_{(L,N)}\) for the (ground state) wavefunction of an \(N\)-particle state at total angular momentum \(L\). Exact LLL ground-state wavefunctions for the Hamiltonian discussed above are known for all angular momenta \(2 \leq L \leq N\). As was shown in a series of papers [12–16], the yrast states in this angular momentum interval are simply given by the fundamental symmetric polynomials
\[ S_L(\tilde{z}_i), \text{ where } \tilde{z}_i = z_i - Z, \text{ and } Z = \langle \sum_i \tilde{z}_i \rangle / N \text{ is the centre-of-mass coordinate:} \]

\[ \psi^{\text{ex}}_{[L,N]}(\{z_i\}) = \frac{1}{N!} \prod_{p_1 < p_2 < \cdots < p_L} (z_{p_1} - Z)(z_{p_2} - Z) \cdots (z_{p_L} - Z). \tag{6} \]

The multiple sum over the particle indices \(\{p_i\}\) implements a symmetrization over all particles. So, for example, \(\psi^{[2,N]}_{[1]} = S[(z_1 - Z)(z_2 - Z)]\), with \(S\) denoting symmetrization over all particle coordinates. As a special case, the so-called single-vortex state at \(L = N\) obeys the exact ground-state wavefunction

\[ \psi^{\text{ex}}_{[N,N]} = \prod_{i=1}^{N} (z_i - Z). \tag{7} \]

### 2.3. Composite fermion approach

The phenomenology of CFs [17] was first introduced in the context of the fractional quantum Hall effect (FQHE), where it was used to construct very successful trial LLL many-body wavefunctions for a large number of FQH states. Roughly speaking, the basic idea of this construction is that an even number of vortices is bound to each electron. Each such vortex, in a mean field sense, cancels one flux quantum of magnetic field, cf (6). Literally, LLL projection amounts to the symmetrization over all particles. So, for example, \(\psi^{[2,N]}_{[1]} = S[(z_1 - Z)(z_2 - Z)]\), with \(S\) denoting symmetrization over all particle coordinates. As a special case, the so-called single-vortex state at \(L = N\) obeys the exact ground-state wavefunction

\[ \psi^{\text{ex}}_{[N,N]} = \prod_{i=1}^{N} (z_i - Z). \tag{7} \]

### 2.3. Composite fermion approach

The phenomenology of CFs [17] was first introduced in the context of the fractional quantum Hall effect (FQHE), where it was used to construct very successful trial LLL many-body wavefunctions for a large number of FQH states. Roughly speaking, the basic idea of this construction is that an even number of vortices is bound to each electron. Each such vortex, in a mean field sense, cancels one flux quantum of the external magnetic field, so that the strongly interacting electrons are mapped to weakly interacting CFs, moving in a reduced effective magnetic field. A mathematical way of seeing that these composite objects are weakly interacting is that ‘attaching \(p\) vortices’ amounts to multiplying the many-body wavefunction by \(\prod_{i<j}(z_i - z_j)^p\)—a factor that ‘keeps the particles apart’, or introduces an additional \(p\)-fold correlation hole around each particle, since it can be seen to go to zero as any two particle coordinates approach each other. A modified version of this model was later applied to study quantum Hall-type states in rapidly rotating Bose gases, mapping the LLL bosons to CFs by attaching an odd number of vortices \([2, 10, 11, 19, 20]\). Due to the above qualitative picture of CFs being weakly interacting objects, they are commonly approximated as non-interacting. Trial wavefunctions for LLL bosons at some angular momentum \(L\) are thus constructed as a Slater determinant \(\Phi\) of CFs with angular momentum \(L - pN(N - 1)/2\) times an odd power \(p\) of Jastrow factors,

\[ \psi^{\text{CF}}_{[L,N]}(\{z_i\}) = P \left( \Phi(\{z_i\}, \{\tilde{z}_i\}) \prod_{i<j} (z_i - z_j)^p \right). \tag{8} \]

The operator \(P\) projects the wavefunction to the LLL. This is necessary since, in the intermediate step, the Slater determinant \(\Phi\) typically contains particles occupying states in higher Landau levels in the effective magnetic field, cf (5). Literally, LLL projection amounts to the replacement \(\tilde{z}_i \to \partial / \partial z_i\) in the polynomial part of the wavefunction—the recipe is to replace all \(\tilde{z}\) with derivatives in the final polynomial, after multiplying the Slater determinant and the Jastrow factors and moving all \(\tilde{z}\) to the left. This is what we will refer to as full projection or method I. As this method tends to get computationally heavy in numerical calculations with many particles and a large number of derivatives, slightly different methods of obtaining LLL wavefunctions have been employed in most of the CF literature [17]. These, too, are often referred to as projection, and we will encounter the most common of these as ‘method II’. The use of these ‘approximate’ methods is justified by the fact that numerically the results are known to depend very little on the projection method chosen [17].

We will get back to the details of all this in later sections. However, let us now point to one already important issue: assume the Slater determinant \(\Phi\) is such that a majority of the CFs reside in higher effective Landau levels. This would translate to large powers in \(\tilde{z}\) (see equation (5)) and thus, upon projection, to a large number of derivatives acting on the factor \(\prod_{i<j}(z_i - z_j)^p\). Given the above qualitative arguments, if the number of derivatives is sufficiently large compared to the degree of the polynomial itself, one might expect that these derivatives would destroy the good correlations built in by the ‘vortex attachment’, so that the approximation of non-interacting CFs underlying (8) might be rather poor. In particular, naively applying the recipe (8) to the lowest angular momentum states at \(L \leq N\) involves \(O(N^3)\) derivatives acting on an order \(N^2\) polynomial in \(z_i\), killing off basically the entire vortex attachment factor, to leave a polynomial of degree \(N\) or less. So these states are far outside the quantum Hall regime, and one would not, \(a\ priori\), expect the CF scheme to work very well. Nevertheless one finds the very surprising fact that the overlap of the CF trial wavefunctions with the exact ones (6) increases with increasing particle number (being very large already for small systems) and approaches 1 as \(N \to \infty\)—in other words, the CF wavefunctions appear to be analytically exact in the thermodynamic limit. For \(L = N\), this trend has previously been pointed out in the context of numerical calculations [11, 18]. In the present paper we study the analytic structure of the yrast wavefunctions in the entire interval \(2 \leq L \leq N\) and present analytic evidence of how this highly unusual behaviour comes about.

### 3. Single vortex—projection I

With the preliminaries in place, we study the single-vortex state, \(L = N\). For ‘full projection’ (here referred to as projection method I and explained below), this case was previously studied numerically [18]; in figure 1 we see...
numerically computed overlaps between the CF and exact single-vortex wavefunctions for up to 40 particles. This figure shows how the overlap converges towards unity, which is quite unusual behaviour for trial wavefunctions in general. As argued above, the fact that we are far outside the regime where the CF ansatz is naively expected to work makes this result even more surprising. A full analytic proof for the convergence of these overlaps is still missing. But in this section we show how the CF wavefunction may be recast in a closed form, demonstrating that the analytic structure of the Jain states is extremely similar to that of the exact ground state; the main difference lies in that the centre of mass coordinate in the exact wavefunction is replaced by \( \text{incomplete} \) centre of mass factors, with one or more particle coordinates ‘missing’. The results of the present section can be directly used to find similar, closed expressions for the lower angular momentum Jain states, \( L < N \). These results will be presented in section 5.

According to equation (7), the exact ground-state wavefunction at \( L = N \) is given by \( \prod_i (z_i - Z) \), where the centre of mass coordinate \( Z = \sum_i z_i / N \). The bosonic CF ansatz for the single-vortex state, on the other hand, is given by \([11, 18]\)

\[
\psi^{\text{CF}}_{[N,N]}(\{z_i\}) = \mathcal{P} \left( \Phi_{[N,N]}(\{z_i, \xi_i\}) \prod_{i<j}(z_i - z_j) \right),
\]

where the unprojected CF Slater determinant is

\[
\Phi_{[N,N]}(\{z_i, \xi_i\}) = \begin{vmatrix} z_1 & z_2 & \ldots & z_N \\ 1 & 1 & \ldots & 1 \\ \xi_1 & \xi_2 & \ldots & \xi_N \\ \vdots & \vdots & \ddots & \vdots \\ z_{N-2} & z_{N-2} & \ldots & z_{N-2} \\ \xi_{N-2} & \xi_{N-2} & \ldots & \xi_{N-2} \end{vmatrix}. \tag{10}
\]

Now, what we call projection method I, or full projection, simply amounts to replacing all \( \xi \)'s in the Slater determinant by derivatives, \( \xi^2 \rightarrow \partial^2 \), thus acting on the full Jastrow factor to its right. We now present an iterative method by which we can perform all the derivatives to find closed, analytic expressions for any number of particles. To this end, first note that for \( L = N = 3 \) we know from translation invariance arguments \([18]\) that the CF ansatz has to be identical to the exact wavefunction,

\[
\begin{vmatrix} 1 & 1 & 1 \\ \xi_1 & \xi_2 & \xi_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = \prod_{i<j}(z_i - z_j) = (z_1 - Z)(z_2 - Z)(z_3 - Z), \tag{11}
\]

with \( Z = (z_1 + z_2 + z_3)/3 \), as can of course be checked explicitly. This will help us simplify the corresponding expressions for larger \( N \). To see how this works, consider the case \( L = N = 4 \) and expand the \( 4 \times 4 \) Slater determinant by the row with the largest number of derivatives,

\[
\psi^{\text{CF}}_{[4,4]} = \frac{4}{\prod_{i=1}^{4} (-1)^{\ell_i}} \frac{\partial_k \partial_l \partial_m}{\partial_k \partial_l \partial_m} \prod_{a<b}(z_a - z_b), \tag{12}
\]

where the superscript \((i)\) on the Slater determinant denotes that the coordinate \( i \) is excluded. Next, note that the Jastrow factor in (12) may be split up according to

\[
\prod_{a<b}(z_a - z_b) = \prod_i (z_a - z_b) \cdot (-1)^{\ell_1} \prod_{k \neq i}^3 (z_i - z_k), \tag{13}
\]

where the sign factor comes from moving the coordinate \( i \) to the front. Furthermore, up to irrelevant overall multiplicative factors,

\[
\partial^2 \prod_{k \neq i}^3 (z_i - z_k) \propto (z_i - Z) \propto (z_i - Z^{(i)}), \tag{14}
\]

where \( Z \) is the centre of mass of all four particles, while \( Z^{(i)} = \sum_j z_j / 3 \) is the three-particle centre of mass with coordinate \( i \) omitted. Combining equations (12), (13) and (14), we find

\[
\psi^{\text{CF}}_{[4,4]} = \sum_{i=1}^{4} (z_i - Z) \cdot \psi^{(i)}_{[3,3]} \tag{16}
\]

or, equivalently (again neglecting irrelevant overall multiplicative factors),

\[
\psi^{\text{CF}}_{[4,4]} = \sum_{i=1}^{4} \prod_{k=1}^{3} (z_k - Z^{(j)}). \tag{17}
\]

Note that, except for the one missing coordinate in the centre of mass, this is identical to the exact ground-state wavefunction (7). If this were the case at all \( N \), it would thus not be very surprising that the CF wavefunction approached the exact one in the thermodynamic limit. However, as we shall see now, going to higher particle numbers introduces factors where more and more coordinates are missing from the centre of mass, as compared to the exact state. Thus, the convergence of the CF state to the exact one appears to be quite subtle.

The calculation outlined above for four particles, can be immediately iterated to larger systems. For example, for six particles,

\[
\psi^{\text{CF}}_{[6,6]} = \sum_{i=1}^{6} (z_i - Z) \cdot \psi^{(i)}_{[5,5]} \tag{18}
\]

Another way that this convergence might occur trivially would be that the CF and exact wavefunctions simply shared the same leading term, and all other terms became irrelevant in the large-\( N \) limit. That this is not the case was discussed in \([18]\).
analytically that for each pair of entries, the overlaps typically compare the two determinants entry by entry. We will prove Jastrow factor. This has the advantage that we can, in fact, part obviously approaches

$$\psi_{N,N}^{\text{CP}} = \sum_{i=1}^{N} (z_i - Z) \cdot \psi_{N-1,N-1}^{\text{CP}},$$

(20)

which can be expanded iteratively in analogy with equation (19). Thus, the structure of the CF wavefunctions is remarkably similar to the exact wavefunctions, the only difference being the incomplete centre of mass factors\(^2\). While suggestive, this is not a proof that the CF wavefunctions approach the exact ones in the large N limit, since the number of coordinates ‘missing’ in the centre of mass piece increases with N. Unfortunately, full analytic overlap calculations on these expressions seem quite intractable. Instead, we now revisit the problem within projection method II, where explicit overlap calculations are possible to some extent.

4. Single vortex—projection II

In this section we re-examine the issue using a different LLL projection method, referred to as method II (see below for details). This is the most commonly used projection method in the CF quantum Hall literature, being numerically more manageable (considerably fewer derivatives) than full projection. For quantum Hall states, it is known to make basically no difference numerically which of these projections is used [17]. In our case, this is not entirely obvious; the increase and convergence of the overlaps is a subtle effect, and exactly how the derivatives act in the projection may well make a difference. In order to compare the CF ansatz to the exact wavefunction, we here rewrite each of them in the form of an \(N \times N\) determinant divided by a single Jastrow factor. This has the advantage that we can, in fact, compare the two determinants entry by entry. We will prove analytically that for each pair of entries, the overlaps typically decrease with N for small number of particles, but then start increasing monotonically, and all converge to 1 in the thermodynamic limit. While looking very suggestive, this however does not necessarily prove that the same would be true for the entire determinant, i.e. for the overlaps between the full wavefunctions. This point will be discussed in more detail below.

4.1. A useful identity

We start this section by deriving a mathematical identity that will turn out very useful in the following. Consider the Slater determinant

$$A^{(k)}[z_1, \ldots, z_N] = \frac{1}{\prod_{i<j}(z_i - z_j)},$$

(21)

where \(f_{N-k}^{S}(\{z_i\})\) is some fully symmetric polynomial of degree \(N-k\). In fact, \(f_{N-k}^{S}\) is, by definition, the Schur polynomial [21] \(s_{\lambda}(\{z_i\})\), where the partition \(\lambda = (1, 1, \ldots, 1, 0, 0, \ldots, 0)\) contains \((N-k)\) Is followed by \(k\) zeros. From the Jacobi–Trudy identity [21], which prescribes how to express Schur polynomials in terms of fundamental symmetric polynomials, it furthermore follows that \(f_{N-k}^{S}(\{z_i\})\) simply equals \(S_{N-k}(\{z_i\})\), the fundamental symmetric polynomial of degree \(N-k\) in the \(N\) coordinates \(\{z_i\}\). Thus,

$$S_{N-k}(\{z_i\}) = A^{(k)}[z_1, \ldots, z_N] \prod_{i<j}(z_i - z_j).$$

(23)

Some special cases of this identity have been discussed in the quantum Hall literature, e.g. [22, 23].

4.2. Exact state

Again, the exact ground-state wavefunction at \(L = N\) is given by \(S_N(\{z_i\})\), where \(z_i = z_i - Z\). Using equation (23) with \(k = 0\), and \(z_i\) replaced by \(\tilde{z}_i\) (and noting that \((z_i - z_j) = (\tilde{z}_i - \tilde{z}_j))\), we can write this wavefunction in the form

$$\psi_{L=N}^{\text{ex}} = \frac{A^{(0)}[\tilde{z}_1, \ldots, \tilde{z}_N]}{\prod_{i<j}(\tilde{z}_i - \tilde{z}_j)} = \frac{\det(M^{\text{ex}})}{\prod_{i<j}(\tilde{z}_i - \tilde{z}_j)}.$$  

(24)

with

$$M^{\text{ex}} = \begin{pmatrix} z_1 & z_2 & \ldots & z_N \\ \tilde{z}_1 & z_2 & \ldots & z_N \\ \ldots & \ldots & \ldots & \ldots \\ \tilde{z}_1 & \tilde{z}_2 & \ldots & z_N \end{pmatrix}.$$  

(25)

This form will turn out to be particularly convenient when comparing to the Jain wavefunction.

4.3. Composite fermion ansatz

The unprojected CF wavefunction for the single vortex was given in section 3. To put this into a form comparable to equation (24), we use a standard trick due to Jain [17] to absorb two Jastrow factors into the determinant and perform
where $M$ is to shed light on to what extent these two wavefunctions compute the overlap between here we merely state and analyze the results. The main steps are outlined in the appendix; the corresponding polynomials, one uses the orthonormality relation

\[
\prod_{i \neq j} (z_i - z_j) \prod_{i \neq j} \sum_{k=1}^{\infty} \frac{1}{k!} = \prod_{i \neq j} \frac{1}{(z_i - z_j)^{2\infty}} \prod_{i \neq j} \frac{1}{(z_i - z_j)^{2\infty}}
\]

This is what we refer to as projection method $\Pi^3$.

4.4. Overlaps

To summarize the above manipulations, we have put both the exact and CF wavefunction into the form

\[
\psi = \frac{\det(M)}{\prod_{i < j} (z_i - z_j)},
\]

where $M$ is an $N \times N$ matrix. Recalling that our goal is to shed light on to what extent these two wavefunctions appear to approach each other with increasing $N$, we thus compare the two determinants and do so entry by entry: We analytically compute pairwise overlaps $\langle M^e \rangle^N \langle M^e \rangle^N$ between the corresponding individual entries of the two determinants and show that each one of these overlaps converges towards $1$ in the limit $N \to \infty$. In other words, since in this limit the size of the matrices themselves goes to infinity, this does not strictly speaking prove that the overlap between the full determinants does approach unity as well. We have, so far, not been able to come up with a conclusive argument to settle this mathematically highly nontrivial question.

In order to compute the overlap between LLL polynomials, one uses the orthonormality relation

\[
(z^m | z^n) = \int d^3z z^m \bar{z}^n e^{-|z|^2/2} = 2\pi \delta_{mn} z^n m!
\]

for all coordinates. The actual overlap calculations involve a rather large amount of straightforward but tedious algebra and combinatorics. The main steps are outlined in the appendix; here we merely state and analyze the results.

As a concrete example, let us start with the first row and compute the overlap between $\langle M^e \rangle$ and $\langle M^e \rangle$ as defined through (25) and (27), respectively. The result will only depend on the row index, i.e. be independent of the second index $\alpha$, so we compute the overlap

\[
O_1 = \frac{\langle M_1^e \rangle^N \langle M_1^e \rangle^N}{\sqrt{\langle M_1^e \rangle^N \langle M_1^e \rangle^N}},
\]

where the denominator has to be included since the wavefunctions are not a priori normalized. The result is

\[
O_1(N) = \left[ 1 - \frac{N}{N} \right]^{-N-1} \sum_{k=0}^{N-1} \frac{1}{k!} \left( 1 - \frac{k}{N} \right)^{-1/2}.
\]

Now, using the fact that $\lim_{n \to \infty} (1 + 1/n)^n = e$, and likewise, $\sum_{k=0}^{\infty} (1/k!) = e$, one finds that

\[
\lim_{N \to \infty} O_1(N) = \lim_{N \to \infty} \left[ e^{1/2} \left( e - \frac{e}{N} \right) \right]^{-1/2} = 1.
\]

So the overlap between any two corresponding entries in the first row approaches unity in the thermodynamic limit. Similarly, one finds for the second row

\[
O_2(N) = \left[ 1 - \frac{N}{N} \right]^{-N-1} \sum_{k=0}^{N-1} \frac{1}{k!} \left( 1 - \frac{k}{N} \right)^{-1/2},
\]

which is again seen to converge towards unity in the limit $N \to \infty$. In fact, as shown in the appendix, one can derive an analytic expression for the overlaps in any row $\alpha + 2$, where $\alpha = 1, \ldots, (N - 2)$:

\[
O_{\alpha+2}(N) = \left[ 1 - \frac{N}{N} \right]^{-N-\alpha-1} \sum_{k=0}^{N-\alpha-1} \frac{1}{k!} \left( 1 - \frac{k}{N-\alpha-1} \right)
\]

\[
\times \left( 1 - \frac{k}{N-2} \right) \cdots \left( 1 - \frac{k}{N-\alpha} \right)^{-1/2}.
\]

We already examined some of the lowest rows. Let us also have a look at a couple of examples for the highest row indices. First, note that for the last ($N$th) row, the overlap equals unity for all $N$. This can be shown by evaluating equation (34) for $\alpha = N - 2$, or by noting through direct calculation that $\tilde{\alpha}_0^N \prod_{k=1}^{N-1} (z_k - z_k) = N(z_k - Z) \propto \tilde{z}_k$. For row number $N - 1$, i.e. $\alpha = N - 3$, equation (34) gives, after some algebra,

\[
O_{N-1}(N) = \left[ 1 - \frac{1}{N} \right]^2 \left( 1 + \frac{2}{N-1} + \frac{1}{(N-1)(N-2)} \right)^{-1/2},
\]

which is again seen to approach unity as $N \to \infty$. Similarly,

\[
O_{N-2}(N) = \left[ 1 - \frac{1}{N} \right]^3 \left( 1 + \frac{3}{N-1} + \frac{3}{(N-1)(N-2)} \right)^{-1/2} + \left( \frac{1}{N-1}(N-2)(N-3) \right)^{-1/2},
\]

etc. Figures 2 and 3 show some examples of overlaps versus particle number $N$. They illustrate that $O_{\alpha+1} > O_\alpha$ for given
some row overlaps, given in equations (31)–(34), as functions of particle number. From below, rows 1, 2 (too close to be resolved from each other), 3, 4, 5 and 6.

![Figure 2](image)

**Figure 2.** Some row overlaps, given in equations (31)–(34), as functions of particle number. From below, rows 1, 2 (too close to be resolved from each other), 3, 4, 5 and 6.

![Figure 3](image)

**Figure 3.** Overlaps for first row (lower curve) and row number 600, for up to 4000 particles.

| $N$ | $L$ | Projection I | Projection II |
|-----|-----|--------------|---------------|
| 4   | 4   | 0.980        | 0.944         |
| 5   | 5   | 0.986        | 0.917         |
| 6   | 6   | 0.990        | 0.894         |

Table 1. Numerically computed overlaps between the full exact and CF wavefunctions at $L = N$ for the two projection methods discussed in the text. Note that the overlaps increase with $N$ for full projection (method I), but, for these small particle numbers, appear to decrease with $N$ for projection method II.

$N$, i.e., overlaps increase with increasing row index, and the first row, equation (31), represents a lower bound of how fast the overlaps approach unity for growing $N$. As mentioned above, this does not provide a conclusive answer concerning the overlaps between the full wavefunctions. In fact, from table 1 we note that at small particle numbers, projection method II produces a decreasing overlap between the full wavefunctions as function of $N$, which may look discouraging at first sight. However, as figure 2 illustrates, for system sizes this small, overlaps between the rows themselves actually tend to decrease rather rapidly as well, before passing a minimum and then increasing monotonically towards 1. It is thus fully conceivable that the overlaps between the full wavefunctions start increasing at larger $N$, too—but the present analysis does not provide a conclusive answer to this question. It would be interesting to study this point numerically for rather large systems, say 20–30 particles or more. Incidentally, if it did turn out that the full overlaps do not converge towards unity at large $N$, this would be a very rare example of a system where the choice of projection method makes an important qualitative difference to the behaviour of the CF wavefunctions.

5. Lower angular momenta

The techniques developed for the single vortex in the previous sections can be immediately transferred to the yrast states below the single vortex, $L < N$, and we shall again do this for the two projection methods separately. One apparent complication arising here is that for $L < N$, there will be more than one candidate CF Slater determinant for each state, the number of candidates increasing with decreasing $L$. So na"ively one would expect the ground state to be some linear superposition of these. However, remarkably, it turns out that after projection and antisymmetrization, the different CF candidates for a given $L$ and $N$ typically result in identical wavefunctions, leaving us with only one, representative, candidate state.

5.1. Projection I

We start by discussing the case $L = N - 1$, before generalizing to lower angular momenta. To construct this state using the CF scheme, one finds two candidate Slater determinants for the ground state$^4$:

$$\Phi_{(N-1,N)}(\{z_i, \bar{z}_i\}) = \begin{vmatrix}
 z_1 & z_2 & \ldots & z_N \\
 1 & 1 & \ldots & 1 \\
 \partial_1 & \partial_2 & \ldots & \partial_N \\
 \vdots & \vdots & \ddots & \vdots \\
 \partial_1^{N-1} & \partial_2^{N-1} & \ldots & \partial_N^{N-1}
\end{vmatrix} \quad (37)$$

and

$$\Phi_{(N-1,N)}(\{z_i, \bar{z}_i\}) = \begin{vmatrix}
 1 & 1 & \ldots & 1 \\
 z_1 \partial_1 & z_2 \partial_2 & \ldots & z_N \partial_N \\
 \partial_1 & \partial_2 & \ldots & \partial_N \\
 \vdots & \vdots & \ddots & \vdots \\
 \partial_1^{N-2} & \partial_2^{N-2} & \ldots & \partial_N^{N-2}
\end{vmatrix} \quad (38)$$

Manipulating these, one can show that these two candidates, when acting on a single Jastrow factor, produce identical wavefunctions$^5$. Thus, it is sufficient to consider one of them, and we shall choose the first, equation (37). Proceeding similar to the steps of equations (12)–(16), we expand the Slater determinant (37) by the row with the highest number of

$^4$ CF candidates for ground states have to be compact$^{17}$. This means that the particles in any given CF Landau level occupy consecutive angular momentum states without any ‘holes’, starting from $l_{\text{min}} = -n$.

$^5$ We have checked this explicitly for up to six particles; since the corresponding manipulations will be completely analogous for higher $N$, we believe this to be true in general.
derivatives to cast the CF wavefunction in the form
\[ \psi_{\text{CF}}^{\text{CF},N-1,N} = \sum_{i=1}^{N} \phi_{\text{CF}}^{(i)} \prod_{a \neq b} (z_a - z_b) \cdot \hat{a}_1^{N-1} \prod_{k \neq i}^j (z_i - z_k). \] (39)

Noting that the \( N - 1 \) derivatives simply turn the last part into a constant, and that the first part has the form of a sum over \((N - 1)\)-particle single-vortex states (discussed in section 3) with particle \( i \) omitted, we thus have
\[ \psi_{\text{CF}}^{\text{CF},N-1,N} = \sum_{i=1}^{N} \psi_{\text{CF}}^{(i)} \psi_{\text{CF},N-1,N-1}^{\text{CF}}. \] (40)

So the CF yrast state at \( L = N - 1 \) is simply a symmetrized sum over \( L = N \) states with one particle less; these, in turn, are known explicitly from section 3.

Similar considerations apply for lower \( L \), where there will be even more CF candidates for each state. Although we have not bothered to perform a complete, systematic study for all possible cases, we have checked for a large class of these candidates that they again produce identical wavefunctions (or zero). Since the analytic structures of the wavefunctions at lower \( L \) are very analogous to those encountered above, we believe this to be true in general\(^6\). Thus we, again, concentrate on one representative candidate, most conveniently the one whose Slater determinant has rows \([z_1, 1, \partial, \ldots, \hat{a}_1^{N-k-2}, \hat{a}_1^{N-k}, \ldots, \hat{a}_1^{N-1}]\), the generalization of (37). The above calculation then immediately generalizes to give the result
\[ \psi_{\text{CF}}^{\text{CF},N-k,N} = \sum_{i=1}^{N} \psi_{\text{CF}}^{(i)} \psi_{\text{CF},N-k,N-1}^{\text{CF}}. \] (41)

Thus, using this expression iteratively, the CF yrast state at any \( L = N - k \) can be expressed as a symmetrized (multiple) sum over single-vortex states with \( N-k \) particles, which are known from section 3. For example,
\[ \psi_{\text{CF}}^{\text{CF},N-2,N} = \sum_{i=1}^{N} \psi_{\text{CF}}^{(i)} \psi_{\text{CF},N-2,N-1}^{\text{CF}} = \sum_{i=1}^{N-1} \sum_{j \neq i}^{N} \psi_{\text{CF}}^{(i)} \psi_{\text{CF},N-2,N-2}^{\text{CF}}. \] (42)

and analogously for smaller \( L \). Note again the striking similarity to the structure of the corresponding exact yrast wavefunction (6). Table 2 shows some numerically computed overlaps between the exact and yrast wavefunctions for the \( L = N, N - 1, \) and \( N - 2 \) states. We note that in all cases, the overlaps indeed increase with the number of particles, just as

\[ \text{Table 2.} \text{ Numerically computed overlaps (projection I) between exact and CF wavefunctions at } L = N, N - 1 \text{ and } N - 2 \text{ for different values of } N \text{ (so the values of } N \text{ range from 4–6, 5–7 and 6–8, respectively in the three columns). Note that the overlaps increase with } N \text{ in all three cases. For fixed } L, \text{ they tend to decrease as one moves away from the single vortex towards lower angular momenta.} \]

| \( L \)     | \( N = L \) | \( N = L - 1 \) | \( L = N - 2 \) |
|-----------|------------|----------------|---------------|
| 4         | 0.980      | 0.953          | 0.926         |
| 5         | 0.986      | 0.967          | 0.945         |
| 6         | 0.990      | 0.973          | 0.954         |

for the single-vortex state\(^7\). At the same time, overlaps tend to decrease as one moves from the single-vortex towards lower angular momentum states at fixed \( L \) or \( N \). This is perhaps not very surprising, given that the state at \( L = N - k \) can be expressed through single-vortex states at a smaller number \((N-k)\) of particles. Thus the results of this section strongly suggest that the same kind of behaviour will be found for all yrast states in the interval \( 4 \leq L \leq N \) (not including \( L = 2, 3 \) here since there the CF wavefunction is known to be exact for all \( N \)[11]).

5.2. Projection II

The analysis of the \( L < N \) yrast states with projection II will be very analogous to that of section 4 for the single vortex. The aim of this section is to demonstrate how we can, again, express both the exact and CF states as a determinant divided by the Jastrow factor and compare the determinants entry by entry. Considering first the exact ground state at \( L = N - k \), equation (6), we can again use equation (23), with general \( k \), and \( z_i \) replaced by \( \tilde{z}_i \), to write
\[ \psi_{\text{ex}}^{\text{ex},N-k,N} = S_{N-k}(\{\tilde{z}_i\}) = \frac{A^{(i)}(\tilde{z}_1, \ldots, \tilde{z}_N)}{\prod_{i<j}(\tilde{z}_i - \tilde{z}_j)} \equiv \det(M_{\text{ex}}^{\text{ex},N-k,N}) \] (43)

with
\[ M_{\text{ex}}^{\text{ex},N-k,N} = \begin{pmatrix} \tilde{z}_1^N & \tilde{z}_2^N & \ldots & \tilde{z}_N^N \\ z_1^{k+1} & z_2^{k+1} & \ldots & z_N^{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{z}_1 & \tilde{z}_2 & \ldots & \tilde{z}_N \end{pmatrix}. \] (44)

For the CF yrast state at \( L = N - k \), we again choose to study the representative Slater determinant discussed in connection with equation (41). This immediately leads to the following

---

\(^6\) See also the discussion in [18] on the lowest angular momentum states, \( L = 2 \) and 3. For these cases it is known that the CF scheme produces the exact wavefunctions for all \( N \), and that paper shows how all candidate CF Slater determinants \((N-1)\) candidates for \( L = 2, N - 2 \) for \( L = 3 \) reduce to identical wavefunctions.

\(^7\) This further justifies our choice of one, representative, CF state. Even if there were other, non-equivalent CF candidates, these would not contribute to the CF ground state in a limit where our representative state approaches the exact ground state.
analog of equation (27):
\[
\psi_{[N-k,N]}^{\text{CF}} = \begin{bmatrix}
\prod_{i\leq j}(z_i - z_j) & z_1 \prod_{i\leq j}(z_2 - z_j) & \cdots & z_p \prod_{i\leq j}(z_N - z_j) \\
1 & \prod_{i\leq j}(z_2 - z_j) & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
\delta^{N-k-1} \prod_{i\leq j}(z_1 - z_j) & \delta^{N-k-2} \prod_{i\leq j}(z_2 - z_j) & \cdots & \delta^{N-k-2} \prod_{i\leq j}(z_N - z_j)
\end{bmatrix}
\]
\[
\equiv \frac{1}{\prod_{i\leq j}(z_i - z_j)} \det(M_{[N-k,N]}^{\text{CF}}) 
\]
(45)
(46)

Note that the last row of \(M_{[N-k,N]}^{\text{CF}}\) reduces to a row of identical constants, thus being identical (up to an overall constant) to the last row of \(M_{[N-k,N]}\). All other rows have entries we already encountered in our discussion of the single vortex in section 4. From the results of that section, we can thus immediately conclude that \textit{between each pair of entries of these two determinants, for all} \(4 \leq L \leq N\), the overlaps converge to unity in the thermodynamic limit. In other words, all yrast states below the single vortex behave in a way very analogous to that found for \(L = N\).

6. Conclusions

The aim of this paper was to shed light on the surprising fact that trial CF wavefunctions for yrast states at and below the single vortex improve with system size and appear to converge to the exact ground states in the thermodynamic limit, as suggested by numerics using projection method I. The ultimate goal would of course be a full derivation in terms of analytic expressions for the overlaps between CF and exact wavefunctions for arbitrary \(N\). One might also wish to study analytic expressions for the energies of the CF states as functions of \(L\) and \(N\) and compare to their exact counterparts [16]. Both tasks turn out to be very challenging. Here, we reported progress in this direction by comparing analytically the form of the CF and exact yrast wavefunctions for the two most common projection methods and pointing out their striking similarities. In particular, our analytical overlap calculations (in projection method II) for individual matrix entries should be viewed as suggestive evidence of the converging overlaps of the full wavefunctions but do not provide a conclusive answer—the two main ‘problems’ being that the sizes of the matrices at hand themselves go to infinity in the thermodynamic limit, and that the increase of the individual row overlaps is non-monotonic. Thus it would be interesting to study the full overlaps for projection method II numerically for large systems. If they do not converge to unity, this would be one of very few known cases where the choice of projection method in the CF construction makes an important qualitative difference. One may also hope that some of the identities and manipulations pointed out in this paper may turn out useful in other contexts, where exact analytic wavefunctions are not available. Finally, the calculations in this paper were done entirely in disc geometry; it would be of interest to re-examine the issues discussed here in, e.g. spherical geometry.

Acknowledgments

We thank Stellan Östlund, Steve Simon, Søren Eilers, Hans Hansson and Bernhard Mehlig for enlightening discussions. This work was supported by the Norwegian Research Council.

Appendix. Analytic overlap calculations

In this appendix, we derive the expression for the overlap given by equation (34) for a general number of particles \(N\) and a given row \(\beta\). To prove this result we extensively use the binomial and multinomial expressions:
\[
(a - b)^n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} a^i b^{n-i}
\]
\[
\left( \sum_i a_i \right)^n = \sum_{|\mathbf{k}|=n} \binom{n}{\mathbf{k}} \prod_i a_i^k,
\]
(4.1)
(4.2)
with
\[
\binom{\alpha}{i} = \frac{\alpha!}{(\alpha - i)! i!},
\]
(4.3)
\[
\binom{\alpha}{\mathbf{k}} = \frac{\alpha!}{\prod_i k_i!},
\]
(4.4)

where \(\mathbf{k} = (k_1, \ldots, k_n)\) and \(|\mathbf{k}| = \sum_i k_i = \alpha\). Since we are interested in the overlap between the matrix entries \(M_{\beta i}^{\text{CF}}\) and \(M_{\beta j}^{\text{CF}}\), defined through equations (25) and (27) respectively, we first calculate the norm of each polynomial. The norm of \(M_{\beta i}^{\text{CF}}\) is given by (consistently suppressing factors of \(2\pi\) from here on)
\[
\langle M_{\beta i}^{\text{CF}} | M_{\beta j}^{\text{CF}} \rangle = \left( \left| z_i - \frac{1}{N} \sum_{\alpha=1}^{N} z_{\alpha} \right|^{\beta} \left| z_j - \frac{1}{N} \sum_{\alpha=1}^{N} z_{\alpha} \right|^{\beta} \right)
\]
\[
= \left( \frac{N - 1}{N} \right)^{2\beta} \sum_{\gamma=0}^{\beta} \gamma! \left( \frac{\beta - \gamma}{\gamma} \right) \left( \frac{1}{N - 1} \right)^{2(\beta - \gamma)}
\]
\[
\times \sum_{|\mathbf{k}|=\beta-\gamma} \binom{\beta - \gamma}{\mathbf{k}} \prod_{\alpha} \prod_{\omega} \left( z_{\alpha} - z_{\omega} \right)^{k_{\alpha} - k_{\omega}}
\]
\[
= 2^{\beta} \left( \frac{N - 1}{N} \right)^{2\beta} \sum_{\gamma=0}^{\beta} \gamma! \left( \frac{\beta - \gamma}{\gamma} \right) \left( \frac{1}{N - 1} \right)^{2(\beta - \gamma)}
\]
\[
\times \left( \beta - \gamma \right)! \sum_{|\mathbf{k}|=\beta-\gamma} \binom{\beta - \gamma}{\mathbf{k}}
\]
\[
= 2^{\beta} \beta! \left( \frac{N - 1}{N} \right)^{2\beta} \sum_{\gamma=0}^{\beta} \beta! \left( \frac{\beta - \gamma}{\gamma} \right) \left( \frac{1}{N - 1} \right)^{\beta - \gamma}
\]
\[
= 2^{\beta} \beta! \left( \frac{N - 1}{N} \right)^{\beta}.
\]
In the first step, we applied the above binomial expansion and the orthogonality relation (29) for the variables $z_i$. Then, equation (A.7) can be obtained after expanding the remaining sum with the multinomial expression and the orthogonality relation. Further simplifications give rise to equation (A.9).

To calculate the norm of $M_{\beta i}^{CF}$, we first need to evaluate the following expression:

$$\sum_{\beta \neq 1}^{N} \prod_{\beta \neq 1}^{\beta} (z_1 - z_\beta) = \sum_{\alpha=0}^{N-\beta-1} (-1)^{N-\alpha-1} S_{N-\alpha-1}([z], i \neq 1) z_1^{\alpha} \cdot$$

(A.10)

$$= \sum_{\alpha=0}^{N-\beta-1} (-1)^{\alpha} S_{N-\alpha-1}([z], i \neq 1) z_1^{\alpha} \cdot$$

(A.11)

where $S_\alpha([z])$ are the elementary symmetric polynomials of degree $\alpha$. The norm of equation (A.10) can then easily be deduced by using the orthogonality relation (A.10) for the variables $z_i$. It is given by

$$\left\langle \sum_{\beta \neq 1}^{N} \prod_{\beta \neq 1}^{\beta} (z_1 - z_\beta) \right\rangle \left\langle \sum_{\beta \neq 1}^{N} \prod_{\beta \neq 1}^{\beta} (z_1 - z_\beta) \right\rangle = 2^{N-\beta-1} \sum_{\alpha=0}^{N-\beta-1} \frac{(\alpha + 1)!}{\alpha!(N-\beta-\alpha-1)!}. \quad (A.13)$$

Finally, the overlap is given by

$$\left\langle \sum_{\beta \neq 1}^{N} \prod_{\beta \neq 1}^{\beta} (z_1 - z_\beta) \right\rangle \left\langle \sum_{\beta \neq 1}^{N} \prod_{\beta \neq 1}^{\beta} (z_1 - z_\beta) \right\rangle = \left( \frac{N-1}{N} \right)^{N-\beta-1} \sum_{i=0}^{\lambda \neq 1} \left( \frac{N-\beta-1}{N-\beta-\lambda-1} \right)^{N-\beta-1} \cdot$$

(A.14)

$$\times \sum_{i=2}^{\lambda \neq 1} \left( \sum_{i=2}^{\lambda \neq 1} \right)^{N-\beta-1} \cdot$$

(A.15)

$$= 2^{N-\beta-1} \left( \frac{N-1}{N} \right)^{N-\beta-1} \sum_{i=0}^{\lambda \neq 1} \left( \frac{N-\beta-1}{N-\beta-\lambda-1} \right)^{N-\beta-1} \cdot$$

(A.16)

To obtain equation (A.15), we first applied the orthogonality relation to the variable $z_1$, then expanded the remaining multinomial expression, only keeping terms involving $k_o = 0$ or 1 (as other terms do not give any contributions to the overlap). Combining equation (A.17), (A.13) and (A.9), we find that the correctly normalized overlap for any given row $\beta + 2$ is given by

$$O_{\beta+2} \equiv \left\langle \sum_{\beta \neq 1}^{N} \prod_{\beta \neq 1}^{\beta} (z_1 - z_\beta) \right\rangle \left\langle \sum_{\beta \neq 1}^{N} \prod_{\beta \neq 1}^{\beta} (z_1 - z_\beta) \right\rangle = \frac{1}{\sqrt{\left( \frac{N-1}{N} \right)^{N-\beta-1} \sum_{i=0}^{\lambda \neq 1} \left( \sum_{i=2}^{\lambda \neq 1} \left( \sum_{i=2}^{\lambda \neq 1} \right)^{N-\beta-1} \right)^{N-\beta-1}}. \quad (A.18)$$

By reorganizing the factorials and products in the denominator of equation (A.18), the sum over $\alpha$ can be recast to give the final expression (34).

The overlaps $O_1$ (31) and $O_2$ (33) for the first and the second rows (where the CF determinant does not contain derivatives) are calculated separately, but the calculations follow the same general strategy as those demonstrated here.

References

[1] Fetter A L 2009 Rev. Mod. Phys. 81 647
[2] For recent reviews, see Viefers S 2008 J. Phys.: Condens. Matter 20 123202
[3] Cooper N R 2008 Adv. Phys. 57 539
[4] Matthews M R, Anderson B P, Haljan P C, Hall D S, Wiegman C E and Cornell E A 1999 Phys. Rev. Lett. 83 2498
[5] Madison K W, Chevy F, Wohlleben W and Dalibard J 2000 Phys. Rev. Lett. 84 806
[6] Schweickhard V, Coddington I, Engels P, Morgendorff V P and Cornell E A 2004 Phys. Rev. Lett. 92 040404
[7] Lin Y-J, Compton R L, Jimenez-Garcia K, Porto J V and Spielman I B 2009 Nature 462 628
[8] Mottelson B 1999 Phys. Rev. Lett. 83 2695
[9] Toreblad M, Borgh M, Koskinen M, Manninen M and Reimann S M 2004 Phys. Rev. Lett. 93 090407
[10] Reimann S M, Koskinen M, Yu Y and Manninen M 2006 Phys. Rev. A 74 043603
[11] Kavoulakis G M, Mottelson B and Pethick C J 2000 Phys. Rev. A 62 063605
[12] Lieb E H, Seiringer R and Yngvason J 2009 Phys. Rev. A 79 063626
[13] Cooper N R and Wilkin N K 1999 Phys. Rev. B 60 R16279
[14] Viefers S, Hansson T H and Reimann S M 2000 Phys. Rev. A 62 053604
[15] Wilkin N K, Gunn J M F and Smith R A 1998 Phys. Rev. Lett. 80 2265
[16] Bertsch G F and Papenbrock T 1999 Phys. Rev. Lett. 83 5412
[17] Smith R A and Wilkin N K 2000 Phys. Rev. A 62 061602
[18] Jackson A D and Kavoulakis G M 2000 Phys. Rev. Lett. 85 2854
[19] Papenbrock T and Bertsch G F 2001 Phys. Rev. A 63 023616
[17] Jain J K 2007 Composite Fermions (Cambridge: Cambridge University Press)

[18] Korslund M N and Viefers S 2006 Phys. Rev. A 73 063602

[19] Chang C, Regnault N, Jolicoeur T and Jain J K 2005 Phys. Rev. A 72 013611

[20] Regnault N and Jolicoeur Th 2003 Phys. Rev. Lett. 91 030402
Regnault N and Jolicoeur Th 2004 Phys. Rev. B 69 235309

[21] Macdonald I G 1995 Symmetric Functions and Hall Polynomials 2nd edn (Oxford: Oxford University Press)

[22] MacDonald A H 1995 Introduction to the Physics of the Quantum Hall Regime (Proc. of the 1994 Les Houches Summer School on Mesoscopic Physics) (Amsterdam: Elsevier) (arXiv:condmat/9410047)

[23] Di Francesco P, Aharony O and Yankielowicz S 1994 Int. J. Mod. Phys. A 9 4257 (arXiv:hep-th/9401163)