Analysis of Maximal Topologies and Their DoFs in Topological Interference Management

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ABSTRACT

Topological interference management (TIM) can obtain degrees of freedom (DoF) gains with no channel state information at the transmitters (CSIT) except topological information of network in the interference channel. It was shown that TIM achieves symmetric DoF $1/2$ when internal conflict does not exist among messages [7]. However, it is difficult to assure whether a specific topology can achieve symmetric DoF $1/2$ without scrutinizing internal conflict. It is also hard to derive a specific topology directly from the conventional condition for symmetric DoF $1/2$. Even except for the topology achieving symmetric DoF $1/2$, topology achieving specific DoF less than $1/2$ is not well known. With these problems in mind, we propose a method to derive all maximal topologies directly in TIM, named as alliance construction in $K$-user interference channel. That is, it is proved that a topology is maximal if and only if it is derived from alliance construction. Further we translate a topology design by alliance construction in the alignment-conflict graph into topology matrix and propose conditions for maximal topology matrix (MTM). Moreover, we propose a generalized alliance construction that derives topologies achieving DoF $1/n$ for $n \geq 3$ by generalizing sub-alliances. A topology matrix can also be used to analyze topologies with DoF $1/n$.

INDEX TERMS

Alliance, alliance construction, degrees-of-freedom (DoF), interference channel, alignment graph, conflict graph, internal conflict, maximal topology matrix (MTM), topological interference management (TIM).

I. INTRODUCTION

Recently, there have been many advances in the wireless networks with interference and the most remarkable achievement is the idea of interference alignment (IA) [1]. IA is a scheme to design signals in such a way that interference signals can be overlapped and separated from desired signal at each receiver so that each receiver can recover its desired message with gains of degrees of freedom (DoF). IA greatly enhances research on interference channels and a number of IA-based related studies have been performed. The initial researches on the IA mainly depend on the perfect and instantaneous channel state information at the transmitters (CSIT) [2], [3]. However, perfect CSIT assumption is not practical and challenging, because perfect CSIT is rarely available to transmitter. Also, when the number of users is large or the channel rapidly changes, the burden of CSIT becomes large. Considering the difficulty of perfect CSIT, researchers begin to explore settings with relaxed CSIT assumptions. It is shown that the setting with delayed CSIT can achieve the optimal symmetric DoF using the benefit of reconfigurable antenna, which is the same as perfect CSIT assumption model [4]. Also blind IA could improve DoF with some structured patterns of fading channels of different users beyond the absolutely no CSIT case [5].

Nevertheless, most of the studies are based on the theoretical insights, which remain fragile so far to be applied to practice directly. Also, these traditional interference management schemes based on IA always consider all interference links regardless of their strength, which results in unnecessary waste on resources such as time and antennas. As the strength of interference rapidly decays with distance due to shadowing, blocking, and path loss, interference from some sources is necessarily weaker than others, which is enough to be ignored. There are more opportunities in terms of DoF and resources by utilizing the characteristic of partial connectivity in actual interference channel.

With the more practical assumptions of interference channel and relaxation for heavy CSIT assumptions, interference management with no channel state information except the...
knowledge of the connectivity at the transmitters has been suggested under the name of the “topological interference management (TIM)” [6]. Jafar suggested that index coding problem could be applied to TIM problem only with linear solutions and translated the index coding problem into TIM problem in a way of analyzing DoF gains [6]. It has been shown that under the topology satisfying certain conditions, TIM can obtain gains in terms of DoF and further achieve one half DoF per user, which is optimal for an interference channel with perfect CSIT. And it can be achieved with only topological information.

Inspired by the new framework of topological interference management that has a merit of tremendous reduction of CSIT, there have been a lot of follow-up researches in line with various assumptions such as channel, antenna, cellular network, transmit cooperation, and message passing. Fast fading channel [7] and alternating connectivity [8] were also considered and fundamental limits on multiple antennas in the TIM setting was derived [9]. Furthermore, TIM was studied in the downlink cellular network with hexagonal structure [10] and more gains of DoF is achievable with the help of message passing in uplink cellular network [11]. TIM was also studied in the interference broadcast channels [12], [13] and device-to-device communications [14].

Most of the above existing works on TIM try to establish the conditions of topology for symmetric DoF based on graph theory. In contrast, Shi et al. [15] present algorithmic method to find the achievable DoFs by interpreting DoF problem in TIM as an LRMC (low-rank matrix completion) problem. Riemannian pursuit framework is proposed to detect the rank of matrix to be recovered by iteratively increasing the rank. Shi et al. [16] deliver user admission control that maximizes the number of admitted users for achieving the feasibility of TIM compared to traditional TIM, where all the users are assumed to be admitted and target is to maximize the achievable symmetric DoFs for all the users. In order to handle the problem, sparse and low-rank optimization framework is proposed and the Riemannian trust-region algorithm is developed.

Unlike above follow-up studies of TIM, we further develop the research in [6] in more practical sense rather than changing assumptions or putting some schemes which help to enhance DoF. Theorem 4 in [6] suggests condition for topology achieving symmetric DoF 1/2. However, since the study on TIM in [6] is based on index coding problems which mainly focus on each message, it is hard to design a specific topology achieving symmetric DoF 1/2 directly from Theorem 4 in [6], which is not suitable for dealing with actual network topologies. Furthermore, with the exception of of topologies achieving DoF 1/2, topologies achieving DoF less than 1/2 are not well characterized, where most of them have only the upper bound of their DoFs.

For these problems, we raise a question, “Is it possible to derive and determine all topologies achieving symmetric DoF 1/2 in TIM easily?” This is the motivation of our research. In order to avoid finding unnecessary topologies, which are sub-topologies of other topologies, we focus on finding only maximal topologies, where any interference link cannot be added without degenerating current DoF. In this paper, we reinterpret the previous condition of topology for the DoF 1/2 into more understandable conditions of topology by introducing the subset of messages with constraints, called alliance and propose how to construct a maximal topology.

Maximality conditions for topology are not derived in the previous studies including [6]. Finding maximal topologies is challenging due to the difficulty of identifying alignment sets and internal conflicts in the alignment and conflict graphs. For these reasons, it is meaningful that maximal topology for DoF 1/2 is identified by using alliance in this paper. Even though conditions for maximal topologies for DoF 1/2 are derived, it is still difficult to apply TIM into practical wireless communication network without systematic design method of maximal topologies. In this paper, design of maximal topology matrix is also proposed using conditions for maximal topologies. Through design of maximal topology matrix, TIM can be applied to practical wireless network.

Meanwhile, there is a still unsolved question, “Is it possible to derive a topology achieving symmetric DoF less than 1/2? We derive topologies achieving symmetric DoF 1/n for any n ≥ 3 by generalizing alliance construction. In general, it is not guaranteed that current interference networks contain weak interference links enough to achieve symmetric DoF 1/2 in TIM. Nevertheless, not many studies for topologies that cannot achieve symmetric DoF 1/2 have been done, because these cases are more difficult to be analyzed compared to the case for symmetric DoF 1/2. In this paper, some of topologies and their design methods are proposed.

It seems that our work and [15], and [16] have a common point, that is, matrix completion. However, our work is basically different from the results in [15], and [16]. Compared to dealing with LRMC problem in [15], design of topology matrix in our work is based on index coding problem (i.e., graph theory). For this reason, our topology design covers only some cases, not the whole, of topology achieving symmetric DoF less than 1/2. However, contrast to design of [15], designs of matrix in our paper do not require NP-hard complexity. Also, design objects, matrices in [15] and [16] are not topology matrix, but related to precoding and decoding vectors. On the other hand, topology matrices related to network topology are considered in our paper.

More specifically, our contributions in this paper are summarized as follows:

- We propose a method called alliance construction which derives maximal topology by stipulating several conditions for message relationship in the alignment graph and conflict graph. It is proved that a topology is maximal if and only if it is derived from the alliance construction.
- Properties of alliance construction are derived such as the maximum number of alliances and a method to partition messages into sub-alliances.
• Message relationship based on alliance construction is translated into topology matrix in TIM. Permutation of the topology matrix is used to demonstrate the characteristics of the alliances easily in the topology matrix. The conditions for maximal topology matrix (MTM) are characterized and the discriminant of topology matrix for maximality and transformation of non-MTM into MTM are proposed.

• Alliance construction is generalized by introducing generalized sub-alliances, which extends the range of topologies derived from alliance construction in the achievable DoFs. The analysis of generalized alliance construction in the topology matrix is also proposed.

The rest of this paper is organized as follows: Section II presents the system model and definitions for TIM. The main result of this paper, that is, alliance construction for maximal topology for symmetric DoF 1/2 is proposed in Section III. Properties of alliance construction are suggested in Section IV. The topology analysis in matrix perspective is discussed in Section V. The generalized alliance construction and the maximal topology with symmetric DoF 1/n are proposed in Section VI and Section VII concludes the paper.

II. SYSTEM MODEL AND PRELIMINARIES

Throughout the paper, some notations are defined as follows. \( A, A, \) and \( \mathcal{A} \) represent a variable, a matrix, and a set, respectively. \([A] \) denotes the cardinality of the set \( A \) and \( a_{ij} \) is the \((i, j)\)-th entry of the matrix \( A \). Let \( K = \{1, 2, \ldots, K\} \) be a set of users, \( \mathcal{N} = \{1, 2, \ldots, N\} \), and \( \mathcal{M} = \{1, 2, \ldots, M\} \). A set of all messages is denoted as \( \mathcal{W} \) with \( K = |\mathcal{W}| \). Let \( \mathcal{V} \) and \( \mathcal{E} \) be sets of vertices and edges. A graph \( G = (\mathcal{V}, \mathcal{E}) \) consists of a set of vertices \( \mathcal{V} \) and a set of edges \( \mathcal{E} \) between two vertices. A directed graph \( D = (\mathcal{V}, \mathcal{E}) \) consists of a set of vertices \( \mathcal{V} \) and a set of directed edges \( \mathcal{E} \) between two vertices.

A. CHANNEL MODEL

We consider the TIM setting [6] in a partially connected \( K \)-user interference channel, where \( K \) transmitters want to send \( K \) independent messages to \( K \) receivers equipped with a single antenna. Then, the received signal at the receiver \( j \) through partially connected channel over \( n \)th time slot is represented as

\[
y_j(n) = \sum_{i \in S_j} h_{ij} x_i(n) + z_j(n), \quad j \in K, \quad n \in \mathcal{N},
\]

where \( x_i(n) \) is the transmitted signal with the average power constraint \( \mathbb{E}[x_i^2(n)] \leq P \), \( z_j(n) \) is the Gaussian noise with zero-mean and noise power spectral density \( N_0 \), \( h_{ij} \) is the channel coefficient between transmitter \( i \) and receiver \( j \), and \( S_j \) represents a set of the indices of transmitters that are heard by receiver \( j \).

The network topology is denoted by \( T \), which is directed bipartite graph with transmitters and receivers at each side, and with edges from transmitters to receivers only when they are connected. We also define sub-topology as follows. If a topology \( T_i \) is a sub-graph of \( T \), it is briefly called a sub-topology of \( T \).

Similar to TIM researches in [6] - [11], the following channel state information (CSI) is assumed:

(i) The channel coefficients are assumed to be fixed throughout the duration of communication such that \( h_{ij}(n) = h_{ij} \) and thus the network topology \( T \) is also assumed to be fixed.

(ii) The channel coefficients \( h_{ij} \) for all \( i, j \) are unavailable at the transmitters, but the network topology \( T \) is known to all transmitters and receivers.

(iii) The channel state information at the receiver (CSIR) includes only the information of the desired channel coefficient \( h_{ij} \) at each receiver.

B. PROBLEM STATEMENT

Formally, the achievable rates of the TIM can be defined as follows. Each message \( W \in \mathcal{W} \) is an independent random variable uniformly selected from \( \{1, 2, \ldots, 2^{NR(W)}\} \), where \( N \) is the length of codebook and \( R(W) \) is rate associated with message \( W \). Transmitter \( i \) uses an encoder, which is a mapping from the message \( W_i \) to a sequence of transmitted signals \( x_i(1), x_i(2), \ldots, x_i(N) \) over \( N \) channel uses. Receiver \( j \) decodes, which is a mapping from the sequence of received symbols \( y_j(1), y_j(2), \ldots, y_j(N) \) to a set of decoded values for its desired message. A rate allocation \( R(W) \) which is assigned to all messages \( W \in \mathcal{W} \) is said to be achievable if there exist a series of encoders and decoders with \( N \), such that decoding errors tend to zero when \( N \) goes to infinity. The capacity region \( C \) is defined as the closure of achievable rate allocations. In this paper, we use the strategy for DoF using the linear beamforming scheme in [6] and set the DoF as our main performance criterion.

Definition 1 (Symmetric DoF): The symmetric DoF \( d_{sym} \) of the \( K \)-user interference channel is defined as

\[
d_{sym} = \limsup_{P \to \infty} \sup_{(R(W)_1, \ldots, R(W)_K) \in C} \frac{R(W)}{\log(P)} \quad (2)
\]

A topology matrix from \( T \) is defined as follows.

Definition 2 (Topology Matrix [6]): A topology matrix for \( K \)-user interference channel, \( T_K = [t_{ij}]_{K \times K} \) is defined as \( t_{ij} = 1 \), if there is a link between transmitter \( i \) and receiver \( j \), and \( t_{ij} = 0 \), otherwise.

In fact, the actual \( K \)-user interference channel is not partially connected. However, in TIM framework, each interference link is considered as “weak” or “strong” using one bit CSIT and only strong interference links are considered as “connected”. For the actual \( K \)-user interference channel, the topology matrix defines the weak (disconnected) interference links with following criterion

\[
\sum_{i \neq j, t_{ij} = 0} |h_{ij}|^2 P \leq N_0. \quad (3)
\]

That is, the average received signal power at receiver \( j \) from all weak interferers is less than or equal to the noise floor. For the topology analysis, we use definitions of conflict
Topology and alignment-conflict graph.

**Definition 3 (Conflict Graph [6]):** For a network topology, its conflict graph is a directed graph $D_i = (V_i, E_i)$ such that $i \in V$ represents the message $W_i$ from the transmitter $i$ to the receiver $i$ and conflict edge $e_{ij} \in E_i$ represents the interfering link from the transmitter $i$ to receiver $j$ in the interference network.

We simply say that message $W_i$ conflicts with $W_j$ if there is a conflict edge from message $W_i$ to message $W_j$ in the conflict graph.

**Definition 4 (Alignment Graph [6]):** For a network topology, its alignment graph is a graph $G_a = (V, E_a)$ such that $i \in V$ represents the message $W_i$ from the transmitter $i$ to receiver $i$ and an alignment edge $e_{ij} \in E_a$ exists if the transmitters $i$ and $j$ interfere with receiver $k$ that wants to receive message $W_k$, $k \neq i$ and $k \neq j$.

**Definition 5 (Alignment Set [6]):** Each set of connected vertices in an alignment graph is called an alignment set.

**Definition 6 (Internal Conflict [6]):** If any two messages that belong to the same alignment set have a conflict edge between them in the conflict graph, it is called an internal conflict.

**Example 1:** In Fig. 1 (a), there is a topology for 5-user interference channel. The alignment graph and conflict graph for Fig. 1 (a) are shown in Fig. 1 (b). Since the alignment graph and conflict graph have the same set of vertices, they can be plotted in Fig. 1 (b) together and we briefly call it alignment-conflict graph. There is an alignment edge between message nodes 1 and 3 because both transmitters 1 and 3 interfere with receiver 2 that wants to message $W_2$. Likewise, message vertices 2 and 5 are connected with an alignment edge because they are interference signals for receiver 3 and message vertices 3 and 4 are connected with an alignment edge because they are interference signals for receiver 5. In Fig. 1 (b), there are two alignment sets, $\{W_1, W_3, W_4\}$ and $\{W_2, W_5\}$ and there is an internal conflict in the alignment set $\{W_1, W_3, W_4\}$ due to the conflict edge between message vertices 1 and 4.

Further, we define maximal topology as follows.

**Definition 7 (Maximal Topology):** A topology is maximal if any interference link cannot be added without reducing symmetric DoF that it can currently achieve. A maximal topology for $K$-user interference channel is denoted by $T_M$.

The achievable DoF of interference channel in TIM is determined by the topology of interference channel. In this paper, we mainly focus on the topologies achieving symmetric DoF 1/2 and the condition for symmetric DoF 1/2 will be described in the following subsection.

III. ALLIANCE CONSTRUCTION FOR MAXIMAL TOPOLOGY

In this section, we propose our main results, alliance construction and maximal topology derived from alliance construction in TIM. The condition of topology for symmetric DoF 1/2 per user in TIM is already proposed in Theorem 4 in [6] that a topology can achieve symmetric DoF 1/2 per user in TIM if and only if its alignment sets do not have any internal conflict.

However, since alignment graph and conflict graph take into account each message, not the message set, it is not easy to check whether a given topology achieves symmetric DoF 1/2 or not without drawing alignment-conflict graph for whole messages and investigating the existence of internal conflict, which requires lots of works. In other words, Theorem 4 in [6] does not directly produce a topology achieving symmetric DoF 1/2. This is the beginning of our study and one of our main contributions is to derive all topologies achieving symmetric DoF 1/2 in TIM by combining and reinterpreting the alignment set and the internal conflict into a single concept, referred to as alliance. To this end, we defined a maximal topology in the previous section. Only maximal topology is considered because any non-maximal topology is sub-topology of maximal topology.

But, it is not enough to derive maximal topology for symmetric DoF 1/2 only with the concept of alliance. After defining the alliance, we propose how to associate alliances with each other. We name it alliance construction to define the relationship among alliances in order to derive maximal topology. Using alliance construction, the relationship among messages together with alliances is determined naturally, which makes it possible to derive all maximal topologies achieving symmetric DoF 1/2.

A. ALLIANCE

In the alignment-conflict graph, each message $W_i \in \mathcal{W}$ implicitly represents a pair of transmitter $i$ and receiver $i$ which sends and wants it, where DoF is analyzed based on each message rather than set of messages. However, it is better to consider topology in terms of the alignment set with no internal conflict rather than messages themselves for symmetric DoF 1/2, because internal conflict is not just relationship of two messages but group of messages. Moreover, it is necessary to make each alignment set to have maximality of interference links with maintaining current symmetric DoF that its topology can achieve. Thus, the
alignment set with constraints is needed, that is, no conflict edge between messages within the alignment sets and satisfying maximality of topology. In order to propose how to design a maximal topology in TIM, we introduce alliance of messages as follows.

**Definition 8 (Partition of Set):** A family of sets $\mathcal{P}_W$ is a called partition of $W$ if and only if the following conditions hold:

(i) The union of the sets in $\mathcal{P}_W$ is equal to $W$.

(ii) The intersection of any two distinct sets in $\mathcal{P}_W$ is empty.

Also, each set in $\mathcal{P}_W$ is referred to as a block.

**Definition 9 (Alliance):** Let $\mathcal{P}_W = \{A_1, \ldots, A_M\}$ be a partition of $W$. For each block $A \in \mathcal{P}_W$ satisfying the following two conditions, $A$ is called an alliance.

(i) (No conflict) There is no conflict edge between any two messages in $A$.

(ii) (Set conflict) All messages in $A$ conflict with a subset of messages that are not in $A$.

According to the above definition, there are two conditions for messages in alliance. The first condition is no conflict among messages in an alliance, which prevents internal conflict in each alignment set. The second condition means that if single message $W_i$ in $A$ conflicts with message $W_k \notin A$, then the other messages in $A$ should also conflict with $W_k$, which results that all messages in an alliance are fully connected with alignment edges. The set conflict condition makes the topology to be able to contain as many interference links as possible without changing message relationship, that is, it is the necessary condition for a maximal topology.

**Lemma 1 (Set Conflict):** Set conflict is necessary if a topology of $K$-user interference network is maximal in TIM.

**Proof:** Suppose that messages in an alignment set do not satisfy the set conflict, that is, messages in a subset of the alignment set conflicts with $W_k$. Consider the possible relationship of the remaining messages in the alignment set and $W_k$ without incurring internal conflict. Due to internal conflict, the remaining messages and $W_k$ can be connected not with alignment edges but with conflict edges. Since the messages in the subset and the remaining messages are already in the same alignment set, connecting the remaining messages and $W_k$ with conflict edges does not change the message relationship. Thus, the topology is not maximal and we prove it.

In order to help to understand the above proof more clearly, we will explain the proof by Fig. 2. There is an alignment set, $\{W_1, W_2, W_3, W_4\}$ in Fig. 2 (a) and (b). Fig. 2 (a) does not satisfy the set conflict in the alignment set of the alignment-conflict graph. Then, we can add a conflict edge from $W_4$ to $W_k$ as in Fig. 2 (b).
Further, we also describe differences between an alliance and an alignment set in Fig. 3. The alignment set \( \{W_1, W_2, W_3, W_4\} \) can have internal conflict in the set and does not always follow the set conflict. As a result, it is not a clique in the alignment graph as in Fig. 3 (b). On the contrary, messages in an alliance have no conflict among them, which prevents internal conflict and follows the set conflict as in Fig. 3 (a).

**B. ALLIANCE CONSTRUCTION**

The alliance itself is not enough to derive topology of \( K \)-user interference channel. Because it is just a subset of messages, which has the relationship of messages in the alliance and its conflicting messages as in Definition 9. The definition of alliance does not require relationship among alliances. Now, it is needed to establish inter-alliance relationship as relationship of whole messages in the alignment-conflict graph. Here, we need some definitions.

**Definition 10 (Hostility and Mutual Hostility):** Alliance \( A_m \) is said to be hostile to alliance \( A_l \) if all messages in \( A_m \) conflict with all messages in \( A_l \), denoted by

\[
A_m \to A_l. \tag{4}
\]

Also, alliances \( A_m \) and \( A_l \) are said to be mutually hostile if and only if all messages in \( A_m \) conflict with all messages in \( A_l \) and vice versa, denoted by

\[
A_m \iff A_l. \tag{5}
\]

The possible number of alliances is limited to two if we assume mutual hostility of all alliances as in the following lemma.

**Lemma 2:** If all alliances are mutually hostile, there exist only two alliances.

**Proof:** Suppose that there are three alliances \( A_1, A_2, \) and \( A_3 \) with mutual hostility as in Fig. 4. Due to the mutual hostility of all alliances, both \( A_1 \) and \( A_2 \) are hostile to \( A_3 \). This is contradiction that \( A_1 \) and \( A_2 \) should be combined into a single alliance because all messages in \( A_1 \) and \( A_2 \) have conflict edges with all messages in \( A_3 \), but cannot be combined due to hostility between them called internal conflict. Similarly, more than three alliances cannot exist with the mutual hostility of all alliances. Therefore, there exist only two alliances if all alliances are mutually hostile.

The mutual hostility can be related to maximality of topology as in the following theorem.

**Theorem 1 (2-Alliance With Mutual Hostility):** For \( W \) in \( K \)-user interference channel, there are two alliances \( P_W = \{A_1, A_2\} \). A topology of \( K \)-user interference channel is maximal if and only if \( A_1 \) and \( A_2 \) are mutually hostile

\[
A_1 \iff A_2. \tag{6}
\]

The achievable symmetric DoF of the topology with 2-alliance with mutual hostility is optimal,

\[
d_{sym} = \frac{1}{2}. \tag{7}
\]

**Proof (Necessity):** Assume that \( A_1 \) and \( A_2 \) are not mutually hostile, that is, all messages in \( A_1 \) conflict with some messages in \( A_2 \) or all messages in \( A_2 \) conflict with some messages in \( A_1 \). Then, we can add conflict edges between messages in \( A_1 \) and \( A_2 \) without occurring the internal conflict and thus it is not maximal.

**Sufficiency:** From Lemma 2, there are only two alliances if all alliances are mutually hostile. Then all messages in \( A_1 \) fully conflict with all messages in \( A_2 \) and vice versa. Also, it is not possible to add any conflict edges among messages in the same alliance due to the no conflict of messages in alliance. Thus the topology is maximal.

We omit the proof of DoF optimality because it has already been shown in [6] for the alignment-conflict graph. The achievable scheme is proposed in the following Section.

**Example 2:** In Fig. 5, there are two maximal topologies for 4-user interference channel. They are maximal topologies derived from 2-alliance with mutual hostility. Each solid edge indicates that two messages are connected as alignment edge and belong to the same alliance (alignment set). The dashed edges indicate the conflict edge between them. In Fig. 5 (a), there are two alliances, \( A_1 = \{W_1, W_2\} \) and \( A_2 = \{W_3, W_4\} \) whose alignment-conflict graph is given in Fig. 5(b). The interference channel in Fig. 5(c) is another maximal topology, where \( A_1 = \{W_1, W_2, W_3\} \) and \( A_2 = \{W_4\} \) whose alignment-conflict graph is given in Fig. 5(d).

The two topologies in Example 2 are all possible maximal topologies derived from 2-alliance with mutual hostility for 4-user interference channel if we do not take into account the indices of messages. Note that if there is no hostility between any two alliances, two alliances (alignment sets) are combined by adding conflict edges without internal conflict and thus its topology is not maximal.

However, there are other maximal topologies which are generated from other than design with two alliances. The natural question is, “Is there other way to satisfy maximality with giving up mutual hostility of all alliances?” The alliance construction can be generalized by setting hostility of alliances in a more general way. The key idea is to construct alliances, where each subset of messages in an alliance
is interfered separately from all messages of each alliance. To this end, some definitions are needed as follows.

**Definition 11 (Sub-Alliance):** For alliance $A_m \in \mathcal{P}_W$, $A_{m,k}$ and $A_{m,k'}$ are blocks of partition of $A_m$, $A_{m,k} \cap A_{m,k'} = \emptyset$ for distinct $m$, $k$, and $k'$ and $\bigcup_{k \neq m} A_{m,k} = A_m$. Then $A_{m,k}$ is called a sub-alliance of $A_m$, where all messages in $A_k$ conflict with each message in $A_{m,k}$.

The structure of alliance and its sub-alliances is described in Figs. 6 and 7. There are alliance $A_m$ and its sub-alliances as $A_{m,1}, A_{m,2}, \ldots, A_{m,m-1}, A_{m,m+1}, \ldots, A_{m,M}$. Sub-alliance $A_{m,k}$ is the subset of messages in $A_m$ where there exist conflict edges from all messages in $A_k$ to all messages in $A_{m,k}$ in Fig. 6. That is, the second index $n$ of sub-alliance $A_{m,k}$ is the index of alliance $A_n$, whose messages are connected to messages in sub-alliance $A_{m,n}$ with conflict edges. The directed line in Fig. 7 (a) indicates that all messages in $A_k$ are connected to all messages in $A_{m,k}$ with conflict edges. Fig. 7(a) is equivalent to Fig. 7 (b), which is the alignment-conflict graph for messages in $A_1$ and $A_{m,1}$. In Fig. 7 (b), there exist an alliance $A_1 = \{W_1, W_2\}$ and a sub-alliance $A_{m,1} = \{W_3, W_4, W_5\}$. There exist conflict edges from $\{W_1, W_2\}$ in $A_1$ to $\{W_3, W_4, W_5\}$ in $A_{m,1}$.

**Definition 12 (Partial Hostility):** $A_l$ is said to be partially hostile to $A_m$ if all messages in $A_l$ only conflict with all messages in $A_{m,l}$ of $A_m$, denoted as $A_l \rightarrow A_m$, identically $A_l \rightarrow A_{m,l}$. (8)

**Definition 13 (Mutually Partial Hostility (MPH)):** Alliances $A_m$ and $A_l$ are said to be mutually partial hostile if all messages in $A_m$ only conflict with all message in $A_{l,m}$ of $A_l$ and all messages in $A_l$ only conflict with all messages in $A_{m,l}$ of $A_m$, where at least one of $A_{m,l}$ and $A_{l,m}$ are non empty sets, denoted by $A_m \leftarrow A_l \leftarrow A_{m,l}$ (9) identically $A_m \rightarrow A_{l,m}$ and/or $A_l \rightarrow A_{m,l}$. (10)

Even though one of sub-alliances $A_{m,l}$ and $A_{l,m}$ is an empty set, $A_m$ and $A_l$ are also said to be mutually partial.
hostile in this paper. That is, \( A_m \subseteq A_l \) includes three cases that only \( A_m \) is partially hostile to \( A_l \) or only \( A_l \) is partially hostile to \( A_m \) or both \( A_m \) and \( A_l \) are partially hostile to each other.

**Lemma 3:** If sub-alliances \( A_{m,l} \) and \( A_{l,m} \) are all empty sets, the topology is not maximal.

**Proof:** Since there is no conflict edge between messages in \( A_m \) and \( A_l \) in the conflict graph, two alliances can be merged into an alliance. If we merge them, the merged alliance should follow the set conflict of alliance. However, conflict edges between messages in \( A_l \) and \( A_{k,m} \) and conflict edges between messages in \( A_m \) and \( A_{k,l} \) in the conflict graph can be added for all distinct \( m, l \), and \( k \), which means that the topology is not maximal.

Using the sub-alliances and MPH, Theorem 1 can be modified into the following theorem.

**Theorem 2 (M-Alliances With MPH):** For \( \mathcal{W} \) in \( K \)-user interference channel, there are \( M \)-alliances given as a partition \( \mathcal{P}_{\mathcal{W}} = \{ A_1, A_2, \ldots, A_M \} \). A topology of \( K \)-user interference channel is maximal if and only if any distinct \( A_m \) and \( A_l \) are mutually partially hostile, that is,

\[
A_m \subseteq A_l, \text{ for any } m, l \in M. \tag{11}
\]

Further, its achievable symmetric DoF is optimal,

\[
d_{sym} = \frac{1}{2}. \tag{12}
\]

**Proof (Necessity):** Assume that for some \( m \) and \( l \), \( A_m \) and \( A_l \) are not mutually partially hostile, where there are three cases: i) \( A_{m,l} = A_{l,m} = \emptyset \), ii) \( A_{m,l} \cap A_{m,k} \neq \emptyset \), iii) \( \cup_{i} A_{m,l} \neq A_m \). From Lemma 3, we have already proved the case i). For the second case, since messages in \( A_l \) and \( A_k \) have common messages to conflict with messages in \( A_m \), messages in \( A_l \) and \( A_k \) should be combined into an alignment set. But there exist different edges between messages in \( A_{l,k} \) and \( A_l \) or messages in \( A_{k,l} \) and \( A_l \) in the conflict graph, which means that internal conflict exists and its topology is not maximal. For the third case, there are at least one messages in \( A_m \), which is not interfered with. Then, some interference links can be added and thus the topology is not maximal.

**(Sufficiency):** Assume that \( M \) alliances are MPH. Due to MPH for all pairs of alliances, there exist conflict edges between all messages in \( A_m \) and \( A_{l,m} \) and between all messages in \( A_l \) and \( A_{m,l} \) for any \( m \) and \( l \) in the conflict graph. Let us add a conflict edge from a message \( W_e \) in \( A_{m} \) to \( W_p \) in \( A_{l,k} \) for any distinct \( m, l, \) and \( k \). Then all messages in \( A_{k,m} \) and \( W_e \) are connected with alignment edges, which results that all messages in \( A_m \) and \( A_k \) are tied as an alignment set. Since \( A_m \) and \( A_k \) are hostile to each other, there exists internal conflict in the alignment set \( A_m \cup A_k \). Thus, internal conflict always occurs by adding a conflict edge between any two messages and thus its topology is maximal. Similarly, we omit the proof of DoF optimality.

Then, the following corollary can be stated without proof.

**Corollary 1:** A topology of \( K \)-user interference channel is maximal if and only if all distinct alignment sets are alliances with MPH.

The above corollary tells that it is possible to derive all maximal topologies for symmetric DoF 1/2 by designing \( \mathcal{P}_{\mathcal{W}} \) and its alliances with MPH. A design method of maximal topology is proposed with the definition of sub-alliance graph as follows. For \( \mathcal{W} \), let \( \mathcal{P}_{\mathcal{W}} = \{ A_1, A_2, \ldots, A_M \} \). Here, alliance \( A_m \) is partitioned into sub-alliances \( A_{m,l}, l \in M, l \neq m \). Let \( A_{sub} \) be a set of all sub-alliances given as \( A_{sub} = \{ A_{m,l} | m, l \in M, m \neq l \} \). Then, the sub-alliance graph is defined as follows.

**Definition 4 (Sub-Alliance Graph):** For \( \mathcal{W} \), let \( \mathcal{P}_{\mathcal{W}} = \{ A_1, A_2, \ldots, A_M \} \) be a partition of \( \mathcal{W} \). A directed graph \( D = (A_{sub}, E_{sub}) \) is called a sub-alliance graph, if there exist directed edges from all sub-alliances in \( A_m \) to sub-alliance \( A_{l,m} \) in \( A_l \) for all \( m, l \in M, m \neq l \).

**Proposition 1 (Design of Maximal Topology):** A maximal topology \( T_M \) is derived from sub-alliance graph as follows:

i) There exists a direct link from transmitter \( i \) to receiver \( j \) for each message \( W_i \in \mathcal{W} \).

ii) There exists an interference link from transmitter \( i \), whose message belongs to alliance \( A_m \) to receiver \( j \), whose message belongs to \( A_{l,m} \) for all distinct \( m \) and \( l \).

**Example 3:** A sub-alliance graph is shown in Fig. 8 (a). A maximal topology from Fig. 8 (a) is given in Fig. 8 (b) and its alignment-conflict graph is shown in Fig. 8 (c).

Let \( A_1 = A_{1,2} \cup A_{1,3} \), whose sub-alliances are \( A_{1,2} = \{ W_1 \} \) and \( A_{1,3} = \{ W_2 \} \), \( A_2 = A_{2,1} \cup A_{2,3} \), whose sub-alliances are \( A_{2,1} = \{ W_3 \} \) and \( A_{2,3} = \{ W_4 \} \), and \( A_3 = A_{3,1} \cup A_{3,2} \), whose sub-alliances are \( A_{3,1} = \{ W_5 \} \) and \( A_{3,2} = \{ W_6 \} \). Contrary to 2-alliance with mutual hostility, receivers for messages in each alliance are partially interfered by transmitters of all messages in each alliance, where messages in each alliance are partitioned into sub-alliances indicating interferers. There are many ways to construct alliances by changing the number of alliances and partitioning messages into sub-alliances differently.

**IV. PROPERTIES OF ALLIANCE CONSTRUCTION**

**A. BEAMFORMING VECTOR DESIGN FOR ALLIANCE CONSTRUCTION**

In this subsection, we design a beamforming vector assigned for messages in each alliance and show that the linear beamforming scheme for alliance construction achieves symmetric DoF 1/2 in TIM. Suppose that there are \( M \) alliances with MPH for \( K \)-user interference channel and we use two time extensions for beamforming vectors. The beamforming vectors split each received signal space into two subspaces with desired message and one directional aligned interference signals for all receivers. \( M \) pairwise linearly independent beamforming vectors can be constructed and each of them is allotted to each alliance. Let \( V_m \) be a \( 2 \times 1 \) beamforming vector for messages in alliance \( A_m, m \in M \). There is no conflict edge among messages in an alliance and the
messages in $\mathcal{A}_{m,l}$ are only interfered by all messages in $\mathcal{A}_l$. Consider the receiver $j$ that wants message $W_j$, which belongs to sub-alliance $\mathcal{A}_{m,l}$ after the alliance construction. Then $2 \times 1$ received signal vector at receiver $j$ for two time slots is given as

$$Y_j = h_j V_m W_j + \sum_{W_i \in \mathcal{A}_l} h_{ij} V_l W_k + Z_j. \quad (13)$$

Since $V_m$ and $V_l$ are linearly independent, receiver $j$ can null the aligned interference signals corresponding to messages in $\mathcal{A}_l$ and recover $W_i$. In the same way, every receiver can decode its desired message by only two time extensions, which means that the network achieves symmetric DoF $1/2$ in TIM.

### B. MAXIMUM NUMBER OF ALLIANCES AND PARTITION OF MESSAGES INTO ALLIANCES

In this subsection, we derive the maximum number of alliances for a given number of messages $K$ for the alliance construction. In fact, we derive the minimum number of messages required to construct $M$ alliances with MPH rather than the maximum number of alliances that can be made with $K$ messages. Let $a_M$ be the minimum number of messages which can construct $M$ alliances with MPH. It is trivial that $a_1 = 1$ and $a_2 = 2$. When $M = 3$, the alliance construction requires the minimum number of messages, $K = 3$, where there is only a message in each alliance and hostility between alliances is a tail-bite as in Fig. 9.

It is clear that when all alliances are related with only one-way hostility, where any non-empty sub-alliance has only one message, the alliance construction has the minimum number of messages. That is, for any $m, l \in \mathcal{M}$, one of two sub-alliances $\mathcal{A}_{m,l}$ and $\mathcal{A}_{l,m}$ is empty and the other has only one message.

Every non-empty sub-alliance requires at least one message. And it is enough to make mutually partial hostility for each pair of alliances with only one non-empty sub-alliance of them. Assume that $M$ alliances have already been constructed using the minimum number of messages $a_M$ and we want to add a new alliance $\mathcal{A}_{M+1}$. In this situation, $\mathcal{A}_{M+1}$ should relate hostility with all existing $M$ alliances, which requires non-empty sub-alliance $\mathcal{A}_{m,M+1}$ or $\mathcal{A}_{M+1,m}$ for each $m \in \mathcal{M}$ and this requires at least $M$ additional messages. Thus, the recurrence relation is formulated as

$$a_{M+1} = a_M + M, \quad M \geq 3 \quad (14)$$

and thus $a_M$ is computed as

$$a_M = \binom{M}{2}, \quad M \geq 3. \quad (15)$$

In fact, alliance construction with the minimum number of messages is equivalent to the handshake problem. Using (15), the maximum number of alliances for given $K$ users can be derived as follows. Let $M_{\text{max}}$ be the maximum number of alliances with MPH for a given number of messages $K$. The range of $K$ which can construct alliances up to $M_{\text{max}}$ is given as

$$\binom{M_{\text{max}}}{2} \leq K < \binom{M_{\text{max}} + 1}{2}. \quad (16)$$

It is clear that different alliance constructions are possible for the same numbers of alliances and messages because there are many ways to partition messages into sub-alliances. The partition of messages for a given number of alliances $M$ is summarized as follows.

**Corollary 2 (Partition of Messages):** There exist $M$ alliances with MPH for $K$ user interference channel, if the number of messages in sub-alliance satisfies the following conditions for $K \geq \binom{M}{2}$:

(i) $\sum_{l=1, l \neq m}^{M} |\mathcal{A}_{l,m}| \geq 1, m \in \mathcal{M}$

(ii) $\sum_{l=1, l \neq m}^{M} |\mathcal{A}_{m,l}| \geq 1, m \in \mathcal{M}$
(iii) \(|A_{m,l}| + |A_{l,m}| \geq 1, m, l \in \mathcal{M}\)
(iv) \(\sum_{m=1}^{M} \sum_{\substack{l=1, l \neq m}}^{M} |A_{m,l}| = K\).

The first inequality constraints that every alliance has at least one common message to conflict with. The second one constraints that every alliance has at least one message. The third inequality is necessary and sufficient conditions of MPH between two alliances. The last one means that every message should belong to a sub-alliance. The condition for \(K \geq \binom{M}{2}\) is required to ensure enough messages for constructing \(M\) alliances. We omit the proof of above corollary.

C. DISCRIMINANT AND TRANSFORMATION OF MAXIMAL TOPOLOGY

In this subsection, a method to determine the maximality of topology is proposed using alliance construction with MPH and the transformation of non-maximal topology into maximal one is also proposed.

Proposition 2 (Discriminant of Maximal Topology): The maximality of topology is determined as follows:

(i) Construct all alignment sets (i.e., tentative alliances) for a given topology.
(ii) Investigate all messages in each alignment set whether they follow the conditions of no conflict and set conflict or not. If yes, alignment sets become alliances and otherwise, it is not maximal.
(iii) Investigate whether all alliances are pairwise mutually hostile or not, that is, there is no message which is not interfered and there is no pair of alliances \(A_m\) and \(A_l\) for \(m, l \in \mathcal{M}\), whose sub-alliances \(A_{m,l}\) and \(A_{l,m}\) are both empty sets. If yes, the topology is maximal and otherwise, it is not maximal.

We also propose how to transform non-maximal topology into maximal topology. Only the topology whose alignment sets have no internal conflict can be transformed into maximal topology of symmetric DoF 1/2 by adding interference links properly. The reason why we add disconnected interference links to the topology is that the links treated as disconnected ones (noises) actually degenerate the SINR performance. And thus, it is desirable to consider more interference links without changing the current DoF value in TIM.

Before transformation, alignment sets (i.e., tentative alliances) are constructed such that each alignment set follows the condition of no conflict, where set conflict will be considered later.

Proposition 3 (Transformation of Non-Maximal Topology Into Maximal Topology): Suppose that all alignment sets have no internal conflict. Then non-maximal topology is transformed into maximal one as follows:

(i) Add interference links so that all messages in each alignment set follow the set conflict of alliance, which makes all alignment sets into alliances.
(ii) If there is no hostility between \(A_m\) and \(A_l\), there are two ways to transform the topology into:
   a) Merge two alliances \(A_m\) and \(A_l\) into a single one by combining \(A_{k,l}\) and \(A_{k,m}\) for distinct \(m, l, \) and \(k\), where combining \(A_{k,l}\) and \(A_{k,m}\) requires to add conflict edges from all messages in \(A_{k,l}\) to each message in \(A_{k,m}\) and from all messages in \(A_l\) to each message in \(A_{k,m}\).
   b) If there exist a message \(W \in A_m\) (or \(W \in A_l\)) whose receiver is not given any interference, add corresponding conflict edges from all messages in \(A_l\) to the message \(W\) or from all messages in \(A_m\) to the message \(W\).
(iii) If there still exists a message \(W\) in \(A_m\) whose receiver is not given any interference, add conflict edges from all messages in an alliance except \(A_m\) to message \(W\).

Example 4: A topology for 8-user interference channel is given in Fig. 10. There are four provisional alliances (i.e., alignment sets) with no internal conflict, \(A_1 = \{W_1, W_3, W_7\}\) with sub-alliances \(A_{1,2} = \{W_1\}, A_{1,3} = \{W_7\},\) and \(A_{1,4} = \{W_3\}\), \(A_2 = \{W_2, W_4\}\) with sub-alliances \(A_{2,4} = \{W_2\}, A_3 = \{W_5, W_6\}\) with sub-alliances \(A_{3,1} = \{W_5\}\) and \(A_{3,4} = \{W_6\}\), and \(A_4 = \{W_7\}\). Since \(A_1\) does not follow the set conflict, the interference link (dashed line) from transmitter 7 to receiver 8 should be added. Also, there is no hostility between \(A_2\) and \(A_3,\) and \(W_6\) in \(A_2\) is not interfered. For this situation, there are two ways to transform the topology into the maximal one. The first one is to merge alliances \(A_2\) and \(A_3\) and add interference links from transmitters in an arbitrary alliance to receiver 4. It is also required to add interference links from transmitters in \(A_2\) to receivers in \(A_3\) and from transmitters in \(A_3\) to receivers in \(A_2\). A maximal topology is given in Fig. 10 (b). Another way is to make hostility between \(A_2\) and \(A_3\) by setting \(A_{2,3} = \{W_4\}\). Other maximal topology is given in Fig. 10 (c). Generally, there are many ways to transform non-maximal topology into maximal topology by adding interference links.

V. MAXIMAL TOPOLOGY MATRIX

In the previous section, the conditions for maximal topology were specified as the relationship of messages in alliance and relationship among alliances. However, it is still hard to determine maximality of topology using alliance construction in the sub-alliance graph. In this section, the analysis of topology in the matrix form is proposed to derive an MTM, determine the maximality of topology matrix, and transform non-MTM into MTM.

A. CONDITIONS FOR MAXIMAL TOPOLOGY MATRIX

In this subsection, the sufficient and necessary conditions for MTM are derived based on alliance construction. First, some of definitions related to topology matrix are given as follows.

Definition 15 (Alliance Block and Interference Block): Suppose that the indices of messages in each alliance are ordered consecutively as \(A_m = \{W_i, W_{i+1}, \cdots, W_{i+|A_m|-1}\}\). A principal submatrix of the topology matrix \(T\) is called an alliance block of \(A_m\) if \(T\) satisfies the following conditions:

(i) The principal submatrix corresponding to \(A_m\) is an identity matrix of size \(|A_m| \times |A_m|\).
There exist at least one $j$ in $T$ such that $t_{kj} = 1$ for all $k \in \{i, i+1, \cdots, i+|A_m|-1\}$ and $W_j \notin A_m$.

(iii) There does not exist $j$ in $T$ such that $t_{kj} = 1$ and $t_{lj} = 0$ for some $k, l \in \{i, i+1, \cdots, i+|A_m|-1\}$ and $W_j \notin A_m$.

The above $|A_m| \times 1$ submatrix $[t_{kj}]$ is called an interference block from $A_m$.

The first condition corresponds to no conflict of messages in alliance preventing internal conflict and the second and third ones are the set conflict.

**Example 5** Fig. 11 shows topology and its topology matrix for 6-user interference channel. There are three principal submatrices whose sizes are $3 \times 3, 2 \times 2,$ and $1 \times 1$. The $3 \times 3$ and $1 \times 1$ square matrices are alliance blocks. But $2 \times 2$ one is not due to $t_{5,4} = 1$, which represents internal conflict in alignment set $\{W_4, W_5\}$. There is an interference block from the alliance block $A_1 = \{W_1, W_2, W_3\}$ given as $\{t_{i,j}|i \in \{1, 2, 3\}\}$, which represents the set conflict to $W_5$. Thus the above matrix is not MTM.

It is also required to translate the mutually partial hostility in alliance construction into topology matrix for MTM.

**Theorem 3 (MTM):** Suppose that there are $M$ alliance blocks in a topology matrix and messages in each alliance are ordered consecutively in indices. A topology matrix is MTM if and only if all alliance blocks satisfy the following conditions:

(i) Each column of the alliance blocks has a single interference block.

(ii) There exists at least one interference block between any two alliance blocks.

The first condition ensures that there is no message whose receiver is not interfered and each receiver for every message is interfered from all messages in a single alliance. The second condition ensures that at least one of sub-alliances $A_{m,l}$ and $A_{l,m}$ are not empty-sets for any $m, l \in M$. In fact, the above two conditions correspond to MPH for $M$ alliances. Thus we omit the proof of Theorem.

**Example 6** In Fig. 12, there are two topology matrices for 9-user interference channel. The topology matrix in Fig. 12(a) satisfies all conditions for MTM. On the other hand, the topology matrix in Fig. 12(b) is not MTM because the column of message $W_6$ in $A_2 = \{W_5, W_6, W_7\}$ has two interference blocks from $A_1 = \{W_1, W_2, W_3, W_4\}$ and $A_3 = \{W_8, W_9\}$.

**B. DISCRIMINANT AND TRANSFORMATION OF MTM**

In this subsection, we propose the discriminant of MTM in matrix perspective. The interpretation of the discriminant of maximal topology into the topology matrix is needed because the characteristics of maximal topology are more easily analyzed in topology matrix than sub-alliance graph. It is desirable that the messages that belong to the same alliance are ordered consecutively in indices. However, the indices of messages in a given topology matrix are always not well sorted and the alliance and interference blocks are not easily discerned. Thus, the permutation of messages for consecutive ordering of indices for each alliance should precede the analysis of maximality of topology in matrix perspective.
Proposition 4 (Discriminant of MTM): The maximality of topology matrix is determined as follows:

(i) Construct all tentative alliance blocks by permuting matrix indices in such a way that any $i$th and $j$th rows and $i$th and $j$th columns are simultaneously permuted into consecutive order if $t_{i,k} = t_{j,k} = 1$.

(ii) Investigate whether all tentative alliance blocks are alliance blocks or not. If not, it is not MTM.

(iii) If yes, investigate whether all alliance blocks follow two conditions in Theorem 3 or not.

(iv) If yes, it is an MTM and otherwise, it is not an MTM.

Note that the above (i) corresponds to reordering of message indices in order to make interference blocks.

Example 7 The topology matrix in Fig. 13 represents a 5-user interference channel. However, if we swap message indices 2 and 3 in both rows and columns simultaneously in Fig. 13 (a), $A_1 = \{W_1\}$ and $A_3 = \{W_3\}$ can be combined into a single alliance block because $t_{1,4} = t_{3,4} = 1$ as in Fig. 13 (b). Even though message indices are reordered, this topology matrix is still not an MTM, because every column in the matrix does not have one interference block. Thus it is possible to add more interference links while maintaining the current DoF. It is not trivial to determine which empty spaces should be filled with element 1 (interference link) in the topology matrix. We propose how to transform non-MTM into MTM by filling some empty spaces with element 1 as in the following proposition.

Proposition 5 (Transformation of Non-MTM Into MTM): First, check whether each principal submatrix is an identity matrix or not. If yes, the transformation procedure can be stated as:

(i) Insert element 1 to the topology matrix in such a way that incomplete interference blocks do not exist.

(ii) If two alliance blocks $A_m$ and $A_l$ do not have any corresponding interference block, there are two ways to transform the topology matrix as:

a) Merge them by permuting matrix indices in such a way that all indices in $A_m$ and $A_l$ are rearranged in a consecutive order.

b) If there exists the $i$th column with no interference block for message $W_i \in A_m$ or $W_i \in A_l$, add corresponding interference block to the $i$th column of $A_m$ or $A_l$.

(iii) If there still exists a column with no interference block, add an arbitrary interference block to the column.

Proposition 5 shows that transformation is not unique for a given topology matrix. There are many ways to merge provisional alliance blocks into single alliance block. Also for the column with no interference block, there are many ways to put an arbitrary interference block into the column.
VI. GENERALIZED ALLIANCE CONSTRUCTION

In this section, we propose generalized alliance construction for topologies achieving DoF less than 1/2 by modifying the definition of sub-alliance and derive its topology.

A. GENERALIZED ALLIANCE CONSTRUCTION

Until now, we focus on alliance construction for maximal topologies achieving symmetric DoF 1/2 and analyze characteristics of alliance construction and its topology matrix. In TIM, there exist other topologies whose achievable DoFs are less than 1/2. But it is more difficult to show achievability and optimality of DoF less than 1/2. In this subsection, we propose generalized alliance construction for topologies achieving symmetric DoFs less than 1/2. In order to do so, some definitions in the previous section are modified for the generalized alliance construction as follows.

Definition 16 (Generalized Sub-Alliance): The alliance \( A_m \) is partitioned into \( n_m \) generalized sub-alliances \( A_m, \varepsilon_k^m \), where \( \varepsilon_k^m \) is the set of indices of alliances whose messages give interference to all messages in \( A_m, \varepsilon_k^m \) with \( \cup_{k=1}^{n_m} A_m, \varepsilon_k^m = A_m \) but \( \varepsilon_k^m \) and \( \varepsilon_k^{m'} \) can have a common subset for any distinct \( k_1 \) and \( k_2 \). Then \( A_m, \varepsilon_k^m \) is called a generalized sub-alliance of \( A_m \).

Definition 17 (Mutually Multiple Partial Hostility (MMPH)): For alliances \( A_m \) and \( A_{m'} \), it is called mutually multiple partial hostile if \( l \in \cup_{k=1}^{n_m} \varepsilon_k^m \) and \( m \in \cup_{k=1}^{n_{m'}} \varepsilon_k^{m'} \).

Theorem 4 (Generalized Symmetric DoF): For \( \mathcal{W} \), there is a partition \( \mathcal{P}_\mathcal{W} = \{ A_1, A_2, \ldots , A_M \} \) for \( M \) alliances with generalized sub-alliances and MMPH. Let \( E_M \) be \( \max_{m,k} |\varepsilon_k^m| \) for all \( A_m \in \mathcal{P}_\mathcal{W} \). Then, the achievable symmetric DoF by using the proposed linear beamforming scheme is

\[
d_{\text{sym}} = \frac{1}{E_M + 1}.
\]  

Proof: Suppose that there are \( M \) alliances with generalized sub-alliances and MMPH for \( K \)-user interference channel and we use \( (E_M + 1) \) time extensions for beamforming vectors. It is possible to construct \( M \) beamforming vectors allotted to each alliance, where any \( E_M + 1 \) vectors in \( M \) vectors are linearly independent. Let \( V_m \) be an \( (E_M + 1) \times 1 \) beamforming vector for messages in \( A_m, m \in \mathcal{M} \). There is no conflict edge among messages in each \( A_m \) in the conflict graph. Also each receiver of message in \( A_m, \varepsilon_k^m \) is interfered by all messages in all alliances \( A_l, l \in \varepsilon_k^m \). Consider the receiver \( j \) that wants message \( W_{ij} \), which belongs to \( A_m, \varepsilon_k^m \) after the generalized alliance construction. Then the \( (E_M + 1) \times 1 \) received signal vector at receiver \( j \) for \( (E_M + 1) \) time slots is given as

\[
Y_j = h_{ij} V_m W_j + \sum_{l \in \mathcal{E}_k^m} \sum_{W_i \in A_l} h_{ij} V_l W_i + Z_j. \tag{18}
\]

Since there are at most \( E_M \) alliances with indices in \( \varepsilon_k^m \) and any \( E_M + 1 \) beamforming vectors are linearly independent, receiver \( j \) can null the aligned interference signals and receive \( W_j \). In the same way, every receiver can decode its desired message by only \( E_M + 1 \) time extensions, which means that the interference channel achieves DoFs 1/(\( E_M + 1 \)) in TIM.

The symmetric DoF achieved by linear beamforming scheme is bounded by the number of interfering alliances for all generalized sub-alliances. Note that the interference channel can achieve symmetric DoF 1/2 when each receiver of message in each sub-alliance in the interference channel is interfered by all messages from a single alliance, that is, \( E_M = 1 \), which results in \( d_{\text{sym}} = 1/2 \).

B. TOPOLOGY MATRIX FOR GENERALIZED ALLIANCE CONSTRUCTION

Generalized alliance covers not only maximal topologies for DoF 1/2 but also topologies for DoFs less than 1/2 by generalizing sub-alliances. Theorem 4 tells that the achievable DoF does not change even if \( \varepsilon_k^m = 1 \) for all \( m \) and its \( k, 1 \leq k \leq n_m \). Thus, it is possible to consider maximality of topology matrix with the proposed linear beamforming scheme. The following corollary is the matrix version of Theorem 4 and suggests conditions for MTM with DoF 1/n for \( n \geq 3 \).

Corollary 3 (MTM With DoF 1/(\( E_M + 1 \)) With the Proposed Scheme): Suppose that there are \( M \) alliance blocks in a topology matrix and messages in each alliance are ordered consecutively in indices. A topology matrix is MTM with DoF 1/(1+\( E_M \)) with the proposed scheme, if topology matrix satisfies the following conditions:

(i) Every column has \( E_M \) interference blocks.

(ii) There is at least one interference block between any two alliance blocks.
Example 8: In Fig. 14, there are two topology matrices for 7-user interference channel, which have been already well permuted. Two topology matrices have four tentative alliance blocks, respectively and both matrices can achieve symmetric DoF 1/3 in TIM, because $E_M$ is equal to two. However, the topology matrix in Fig. 14(a) is not MTM with the proposed scheme because there are lots of rooms for additional interference links. The topology matrix in Fig. 14(b) is designed as an example of MTM from the topology matrix in Fig. 14(a). The bold elements are inserted properly to satisfy the maximality of topology in Fig. 14(b). After transformation, it can be seen that the topology matrix in Fig. 14(b) satisfies two conditions in Corollary 3 and thus, it is an MTM with DoF 1/3.

Note that Corollary 3 restricts definition of MTM with the proposed beamforming scheme. This is because it may be possible to achieve higher DoF for the same topology in a way other than the one we propose. The maximality means that topology matrix contains as many as possible with the proposed scheme because all messages in each alliance is connected to conflict edges with messages in exact $E_M$ alliances. For the reason, the discrimination of MTM for DoFs less than 1/2 is not proposed in the paper. It is future work to prove the optimality condition for topology DoFs less than 1/2 and suggest achievable schemes for DoFs.

VII. CONCLUSION

In this paper, we introduced the alliance as a set of messages that follows no internal conflict and set conflict in the alignment-conflict graph. Based on alliance, we proposed the alliance construction with MPH, which results in generating maximal topology. Using alliance construction, some properties of maximal topologies were given and the discriminant and transformation for maximal topology were also proposed. Moreover, we convert alliance construction in the alignment-conflict graph into topology matrix in order to analyze the maximality of topology easily. The sufficient and necessary conditions for MTM were derived and the discriminant of MTM and the transformation of non-MTM into MTM were also proposed. Furthermore, we generalized the alliance construction with generalized sub-alliances dealing with topologies for DoF $1/n$. The generalized alliance construction was represented in matrix form and the conditions of MTM with DoF $1/n$ with the proposed scheme were described.

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