Evolution of homogeneous and isotropic scalar-tensor cosmological models

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Abstract. Homogeneous and isotropic cosmological models are considered in the framework of general scalar-tensor theories of gravity with arbitrary coupling function and scalar potential. Nonlinear approximate equations are derived for the regime close to the so-called limit of general relativity where the local observational constraints are satisfied. Approximate solutions in cosmological time are presented for the potential dominated era and for the era when the energy density of ordinary matter dominates over the energy density of scalar potential.

1. Introduction
Contemporary mathematical cosmology is based on Einstein’s theory of general relativity (GR) or on some of its modifications: scalar-tensor theory of gravity (STG), $f(R)$ theory, low energy limit of string or brane theory, etc. Many of the latter ones can also be presented in the form of scalar-tensor gravity. This motivates us to focus just on STG cosmological models [1].

Present observational data are in good correspondence with general relativistic Friedmann-Lemaître-Robertson-Walker (FLRW) flat ($k=0$) cosmology with homogeneous and isotropic 3-space, a cosmological constant $\Lambda > 0$ and additional cold dark matter ($\Lambda$CDM). However, $\Lambda$CDM model seems to be somewhat phenomenological and fine-tuned: extremely small observational value of $\Lambda$, very special initial and/or boundary conditions, etc. Mathematics invites us to consider a variety of possible cosmological models that may be very different from $\Lambda$CDM. But a realistic model of our Universe must conform to observations and the Solar System experiments.

It has been demonstrated already long ago [2], that wide classes of STG cosmologies dynamically converge to specific late time solutions which are indistinguishable from the corresponding solutions of GR. It follows that for those models the late time local weak field observational constraints are satisfied (attractor mechanism).

In the present paper, we consider flat ($k=0$) FLRW cosmological models of STG with generic coupling function and scalar potential or ordinary dust matter. In Section 2 we present the exact equations and in Section 3, we propose nonlinear approximate ones which hold in the neighbourhood of a point in the phase space (so-called GR point) where the field equations coincide with those of the corresponding model of GR. We review and develop our investigations [3, 4, 5] of behaviour of solutions of approximate equations which can describe different realistic models. In Section 4, we compare our results with analogous ones presented earlier [2]. Section 5 is a summary and outlook.

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2. Scalar-tensor FLRW cosmology

In the flat $\Lambda$CDM model it is assumed that 4-dimensional Universe is a differentiable manifold which is characterized by a line element

$$ds^2 = -dt^2 + a(t)^2 \left( dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right),$$

i.e. it has a homogeneous and isotropic 3-space with a scale factor $a(t)$ which is a solution of the Einstein equations of GR with ordinary and dark matter and cosmological constant $\Lambda$. In STG cosmological models we retain the general form of the metric (1), but the scale factor $a(t)$ is determined by modified equations.

Let us assume that gravity is described by metric tensor $g_{\mu\nu}$ and a scalar field $\Psi$. The general action functional of STG given in the Jordan frame “Brans-Dicke” parametrization reads

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ \Psi R(g_{\mu\nu}) - \frac{\omega(\Psi)}{\Psi} \nabla^\mu \Psi \nabla_\mu \Psi - 2\kappa^2 V(\Psi) \right]$$

In fact it contains a family of theories since each pair of functions $\omega(\Psi)$ and $V(\Psi)$ specifies a theory. It implies a variable gravitational “constant” $8\pi G = \frac{\kappa^2}{\Psi}$ (we assume $0 < \Psi < \infty$). We also assume positive energy density: $2\omega(\Psi) + 3 \geq 0$, $V(\Psi) \geq 0$, and $S_m$ is the usual action for matter fields $\chi$.

In the framework of STG equations of FLRW cosmological models with metric (1) and barotropic matter fluid $p(t) = w\rho(t)$, $w =$ const. read

$$H^2 = -\frac{\dot{\Psi}}{\Psi} + \frac{\dot{\Psi}^2}{6 \Psi^2} \omega(\Psi) + \frac{\kappa^2 \rho}{\Psi^3} + \frac{\kappa^2 V(\Psi)}{\Psi^3},$$

$$2 \dot{H} + 3H^2 = -2\frac{\dot{\Psi}}{\Psi} - \frac{\dot{\Psi}^2}{2 \Psi^2} \omega(\Psi) - \frac{\dot{\Psi}^2}{\Psi} \rho + \frac{\kappa^2}{\Psi} V(\Psi),$$

$$\ddot{\Psi} = -3H \dot{\Psi} + \frac{1}{2\omega(\Psi) + 3} \frac{d\omega(\Psi)}{d\Psi} \dot{\Psi}^2 + \frac{\kappa^2}{2\omega(\Psi) + 3} (1 - 3w) \rho$$

$$+ \frac{2\kappa^2}{2\omega(\Psi) + 3} \left[ 2V(\Psi) - \Psi \frac{dV(\Psi)}{d\Psi} \right],$$

$$\dot{\rho} = -3H (w + 1) \rho,$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. Phase space for system (3)–(6) is 4-dimensional $\{\Psi, \Pi \equiv \dot{\Psi}, H, \rho\}$, but phase trajectories lie on the 3-surface of the Friedmann constraint (3)

$$H = -\frac{\Pi}{2\Psi} \pm \sqrt{\frac{(2\omega(\Psi) + 3) \Pi^2}{12\Psi^2} + \frac{\kappa^2 (\rho + V(\Psi))}{3\Psi}},$$

Tangents of trajectories $(\dot{\Psi}, \Pi, \dot{H}, \dot{\rho})$ are given as

$$\dot{\Psi} = \Pi,$$

$$\dot{\Pi} = -\frac{1}{2\omega(\Psi) + 3} \left[ \frac{d\omega(\Psi)}{d\Psi} \Pi^2 - \kappa^2 (1 - 3w) \rho + 2\kappa^2 \left( \frac{dV(\Psi)}{d\Psi} \Psi - 2V(\Psi) \right) \right] - 3H \Pi,$$

$$\dot{H} = \frac{1}{2\Psi(2\omega(\Psi) + 3)} \left[ \frac{d\omega(\Psi)}{d\Psi} \Pi^2 - \kappa^2 (1 - 3w) \rho + 2\kappa^2 \left( \frac{dV(\Psi)}{d\Psi} \Psi - 2V(\Psi) \right) \right]$$

$$- \frac{1}{2\Psi} \left[ 6H^2 \Psi + 2H \Pi - \kappa^2 (1 - w) \rho - 2\kappa^2 V(\Psi) \right],$$

$$\dot{\rho} = -3H (1 + w) \rho.$$
These equations cannot be integrated without specifying the two arbitrary functions \( \omega(\Psi) \) and \( V(\Psi) \). However, not all possible functions and the corresponding solutions are of immediate physical interest, but only those that satisfy local (Solar System) observational constraints. It is well known that gravity in Solar System is rather precisely described by GR. So we are motivated to consider cosmological models of a modified theory of gravitation which locally differ not much from GR [3, 6, 7].

3. Approximate equations and solutions

In our earlier investigations [6, 7] we have focussed on a point in the phase space where the STG equations tend to coincide with those of GR. We have justified the following definition of this GR point \( (\Psi^*, \Pi^*) \) in the phase space: (a) \( \frac{1}{2\omega(\Psi^*) + 3} = 0 \), (b) \( \Psi^* \equiv \Pi^* = 0 \), (c) \( A^* \equiv \frac{d}{d\Psi} \left( \frac{1}{2\omega(\Psi^*) + 3} \right) \Psi^* \neq 0 \), (d) \( \frac{1}{2\omega(\Psi^*) + 3} \) is differentiable at \( \Psi^* \). Under these conditions the local (Solar System) observational constraints are satisfied within the error bars in the GR point and also in its neighbourhood. Therefore it makes sense to investigate more in detail approximate STG equations for small deviations from GR.

We consider separately two cases: (i) potential of the scalar field dominates over matter density \( (V \neq 0, \rho = 0) \) and the phase trajectories (8)-(10) lie on a two-dimensional surface of the Friedmann constraint (7); (ii) cosmological dust matter density dominates over scalar potential \( (V \equiv 0, \rho \neq 0, w = 0) \) and the phase trajectories lie on a three-dimensional surface of the Friedmann constraint (7).

3.1. Scalar potential dominated models with \( \rho = 0 \)

Let us focus around the point \( (\Psi^*, \Pi^*) \) \( \Psi = \Psi^* + x, \Pi = \Pi^* + y = y \) and expand in series

\[
\frac{1}{2\omega(\Psi)} + \frac{1}{2\omega(\Psi^* + x)} \approx \frac{1}{2\omega(\Psi^*) + 3} + A^* x + ... \approx A^* x ,
\]

\[
(2\omega(\Psi) + 3)\Pi^2 \approx \frac{y^2}{0 + A^* x + ...} = \frac{y^2}{A^* x} (1 + O(x)) \approx \frac{y^2}{A^* x} .
\]

Keeping terms which are of first order in \( x \) and \( y \), the dynamical system (7)-(10) becomes [4]

\[
\dot{x} = y , \\
\dot{y} = \frac{y^2}{2x} - C_1 y + C_2 x ,
\]

where

\[
C_1 \equiv \pm \sqrt{3\kappa^2 V(\Psi^*) \over \Psi^*} , \quad C_2 \equiv 2\kappa^2 A^* \left( 2V(\Psi) - dV(\Psi) \right) \bigg|_{\Psi^*} .
\]

These constants encode the behavior of the functions \( \omega \) and \( V \) near the point \( (\Psi^*, \Pi^*) \) and in fact describe the choice of the underlying theory of STG.

Notice that nonlinear terms prevent us from using standard fixed point analysis, but still allow to investigate phase portraits. In our earlier papers [3, 4] we have provided a complete classification of possible phase portraits of the dynamical system (14)-(15) and argued that the topology of phase trajectories in the approximate case is similar to that of the full theory. This justifies our thorough investigation of the approximation (14)-(15).

We can integrate Eqs. (14)-(15) and find solutions in terms of cosmological time [5]

\[
\pm x(t) = \begin{cases} 
  e^{-C_1 t} \left( M_1 e^{\frac{1}{2} t \sqrt{|C|}} - M_2 e^{-\frac{1}{2} t \sqrt{|C|}} \right)^2 , & C > 0 , \\
  e^{-C_1 t} \left( M_1 t - M_2 \right)^2 , & C = 0 , \\
  e^{-C_1 t} \left( N_1 \sin(\frac{1}{2} t \sqrt{|C|}) - N_2 \cos(\frac{1}{2} t \sqrt{|C|}) \right)^2 , & C < 0 ,
\end{cases}
\]

\[
\pm y(t) = e^{-C_1 t} \left( M_1 e^{\frac{1}{2} t \sqrt{|C|}} - M_2 e^{-\frac{1}{2} t \sqrt{|C|}} \right) , \quad C > 0 ,
\]

\[
\pm y(t) = e^{-C_1 t} \left( M_1 t - M_2 \right) , \quad C = 0 ,
\]

\[
\pm y(t) = e^{-C_1 t} \left( N_1 \sin(\frac{1}{2} t \sqrt{|C|}) - N_2 \cos(\frac{1}{2} t \sqrt{|C|}) \right) , \quad C < 0 .
\]
where $C \equiv C_1^2 + 2C_2$ and $M_1, M_2, \tilde{M}_1, \tilde{M}_2, N_1, N_2$ are constants of integration determined by initial conditions.

Via Friedmann constraint (7) we can get an approximate expression for $H(x(t))$

$$H \approx \frac{C_1}{3} - \frac{1}{2\Psi_*} \dot{x} + \frac{1}{2\Psi_*} \left[ \frac{C_1}{3} - \frac{C_2}{2C_1\Psi_* A_*} \right] x + \frac{1}{8C_1\Psi_*^2 A_*} \frac{\dot{x}^2}{x} + \ldots$$

and can calculate effective barotropic index $w_{\text{eff}}(t)$

$$w_{\text{eff}} \equiv -1 - \frac{2H}{3H^2} = -1 + \frac{1}{C_1^2 \Psi_*} \left[ \frac{3}{2} \left( 1 + \frac{1}{\Psi_* A_*} \right) \frac{\dot{x}^2}{x} - 4C_1 \dot{x} + 3C_2 x \right] + \ldots$$

It is possible to have solutions which have oscillating or non-oscillating $w_{\text{eff}}$, which are crossing the phantom divide ($w_{\text{eff}} = -1$), and not crossing the phantom divide.

Let us consider more in detail solutions (17) of theories with $C > 0$ which have exponential dependence on time $t$. Conditions for them to be of attractor type, i.e. with a factor that guarantees exponential decreasing of $x$ in time, are $C_1 > 0$ and $C_2 < 0$, or equivalently

$$\frac{d}{d\Psi} \left( \frac{1}{2\omega(\Psi) + 3} \right) \left| _{\Psi_*} \left( 2V(\Psi) - \frac{dV(\Psi)}{d\Psi} \Psi \right) \right| _{\Psi_*} < 0.$$  \hspace{1cm} (20)

They converge asymptotically in time to the de Sitter solution with $H_* = \frac{C_1}{3}$. Saddle-type solutions are characterized by a condition $C_2 > 0$ which entails an exponential growth of $x$ in time, so these solutions cannot serve as approximations of exact late time solutions and they can be trusted only near the GR point.

Solutions (17) of theories with $C < 0$ describe models where the Hubble parameter oscillates around its de Sitter value. Solutions (17) of the case $C = 0$ correspond to a STG with rather finetuned constants (16), $C_1^2 = -2C_2$.

3.2. Dust matter dominated models with $V = 0$

The system (3)-(6) is now characterized by three variables $\{\Psi, H, \rho\}$, but one of them is algebraically related to the others via the Friedmann equation. Eliminating $\rho$ yields two equations

$$\ddot{\Psi} = -3H\dot{\Psi} + \frac{1}{2}(2\omega + 3)A(\Psi)\dot{\Psi}^2 + \frac{1}{(2\omega + 3)} \left( 3\Psi H^2 + 3H \dot{\Psi} - \frac{\dot{\Psi}^2}{2\Psi} \omega \right),$$  \hspace{1cm} (21)

$$\dot{H} = -\frac{3}{2}H^2 + H \frac{\dot{\Psi}}{2\Psi} - \frac{\dot{\Psi}^2}{4\Psi^2} \omega - \frac{1}{4}(2\omega + 3)A(\Psi) \frac{\dot{\Psi}^2}{\Psi} - \frac{1}{2(2\omega + 3)} \left( 3H^2 + 3H \frac{\dot{\Psi}}{\Psi} - \frac{\dot{\Psi}^2}{2\Psi^2} \omega \right),$$  \hspace{1cm} (22)

where $A(\Psi) \equiv \frac{d}{d\Psi} \left( \frac{1}{2\omega(\Psi) + 3} \right)$. Notice that in this case the standard fixed point analysis would give a dust dominated universe with $\dot{H} = 0$, which implies a trivial cosmology with $H = 0$ and $\rho = 0$.

As in the case of potential domination, let us (a) define $\Psi_*$ by $\frac{1}{2\omega(\Psi_*) + 3} = 0$ and focus upon the solutions near this point

$$\Psi(t) = \Psi_* + x(t),$$  \hspace{1cm} (23)
where \( x(t) \) is a small deviation. It follows from (23) that \( \dot{\Psi}(t) = \dot{x}(t) \) and (b) we expect \( \dot{x}(t) \) to be also small. Under the two additional mathematical assumptions (c) \( A_* = A(\Psi_*) \neq 0 \) and (d) \( \frac{1}{2\omega+3} \) is differentiable at \( \Psi_* \), we can expand in series

\[
\frac{1}{2\omega(\Psi) + 3} = \frac{1}{2\omega(\Psi_*) + 3} + A_* x + ... \approx A_* x ,
\]

\[
(2\omega(\Psi) + 3) \dot{\Psi} = \frac{\dot{x}^2}{0 + A_* x + ...} = \frac{\dot{x}^2}{A_* x} (1 + O(x)) \approx \frac{\dot{x}^2}{A_* x} .
\]

The latter result actually informs us that in order to avoid a spacetime singularity which can follow from the constraint (7), \( \frac{\dot{x}^2}{x} \) must not diverge, hence we should treat \( x(t) \) and \( \dot{x}(t) \) as the same order (small) quantities.

Let us expand also the Hubble parameter into the main part \( H_*(t) \) and a small deviation \( h(t) \)

\[
H(t) = H_*(t) + h(t)
\]

and define

\[
\dot{H}_* = -\frac{3}{2} H_*^2 , \quad H_*(t) = \frac{2}{3t}
\]

which is a familiar Friedmann solution of GR.

The approximate first order equations now read

\[
\dot{x} = \frac{\dot{x}^2}{2x} - 3H_* \dot{x} + 3A_* \Psi_* H_*^2 x ,
\]

\[
\dot{h} + 3H_* h = -\frac{1}{4\Psi_*} \left( 1 + \frac{1}{2A_* \Psi_*} \right) \frac{\dot{x}^2}{x} + \frac{1}{2\Psi_*} H_* \dot{x} - \frac{3}{2} A_* H_*^2 x
\]

with \( H_* \) given by Eq. (27). Notice that due to \( H_*(t) \) Eqs. (28), (29) depend explicitly on time \( t \). This means that the corresponding system of first order equations is not autonomous and the standard phase space analysis is not applicable.

However, we can find solutions in terms of cosmological time \( t \)

\[
\pm x(t) = \begin{cases} 
\frac{1}{7} \left( M_1 t^{\frac{1}{2}} \sqrt{D} - M_2 t^{\frac{1}{2}} \sqrt{D} \right)^2 , & D > 0 , \\
\frac{1}{7} \left( M_1 \ln t - M_2 \right)^2 , & D = 0 , \\
\frac{1}{7} \left( N_1 \sin \left( \frac{1}{2} \sqrt{|D|} \ln t \right) - N_2 \cos \left( \frac{1}{2} \sqrt{|D|} \ln t \right) \right)^2 , & D < 0 ,
\end{cases}
\]

where \( M_1, M_2, \bar{M}_1, \bar{M}_2, N_1, N_2 \) are constants of integration (determined by initial conditions) and model-dependent constant \( D \) reads \( D = 1 + \frac{3}{2} A_* \Psi_* \).

We can also calculate the approximate values of the Hubble parameter for all three cases

\[
H(t) = \begin{cases} 
\frac{2}{3} \left\{ 1 \pm \frac{1}{7} \left[ M_1 t^{\frac{1}{2}} \left( -a \sqrt{D} + b \right) t^{\sqrt{D}} + M_2 \left( a \sqrt{D} + b \right) t^{-\sqrt{D}} + K \right] \right\} , & D > 0 , \\
\frac{2}{3} \left\{ 1 \pm \frac{1}{7} \left[ \frac{M_1 \left( \ln t \right)}{2} + \left( \bar{M}_1 - \bar{M}_2 \right) \ln t + K \right] \right\} , & D = 0 , \\
\frac{2}{3} \left\{ 1 \pm \frac{1}{7} \left[ \left( \frac{1}{2} \left( N_2^2 - N_1^2 \right) a \sqrt{D} + b \right) \cos \left( \sqrt{|D|} \ln t \right) + \left( N_1 N_2 a \sqrt{D} + \frac{1}{2} \left( N_2^2 - N_1^2 \right) b \right) \sin \left( \sqrt{|D|} \ln t \right) + K \right] \right\} , & D < 0 .
\end{cases}
\]

Here \( K \) is a constant of integration and we have introduced constants \( a, b \) which characterize the underlying STG

\[
a = \frac{3 + 6A_* \Psi_*}{8A_* \Psi_*^2} , \quad b = \frac{3 + 10A_* \Psi_*}{8A_* \Psi_*^2} .
\]
The corresponding effective barotropic index reads

\[ \pm w_{\text{eff}} = \begin{cases} 
- \frac{\sqrt{D}}{3} \left[ M_1^2 \left( -a\sqrt{D} + b \right) t^{\sqrt{D}} - M_2^2 \left( a\sqrt{D} + b \right) t^{-\sqrt{D}} \right], & D > 0, \\
- \frac{M_1}{M_2 \sqrt{D}} \left( \tilde{M}_1 + t \tilde{M}_2 - \tilde{M}_2 \right), & D = 0, \\
- \sqrt{D} \left[ \left( \frac{1}{2} (N_2 - N_2^2) a\sqrt{D} - N_1 N_2 b \right) \cos \left( \sqrt{D} \ln t \right) \\
- \left( N_1 N_2 a\sqrt{D} + \frac{1}{2} (N_2^2 - N_2^2) b \right) \sin \left( \sqrt{D} \ln t \right) \right], & D < 0.
\]  

(33)

Let us consider solutions (30)-(33) with \( D > 0 \) which have polynomial time evolution. Asymptotically at \( t \to \infty \) the solutions exhibit two distinct behaviors. For STGs with \( \sqrt{D} < 1 \) (i.e. \( A_\star \Psi_\star < 0 \)) all cosmological solutions irrespective of their initial conditions monotonically approach the general relativistic dust matter FRW cosmology, \( \Psi(t) \to \Psi_\star = \text{const.}, H(t) \to H_\star(t) = 2/(3t), w_{\text{eff}}(t) \to 0 \), since all first order corrections vanish at this limit. On the other hand STGs with \( \sqrt{D} > 1 \) (i.e. \( A_\star \Psi_\star > 0 \)) allow only solutions that will diverge, \( x(t) \to \infty \), \( h(t) \to \infty \), \( w_{\text{eff}}(t) \to \infty \), meaning that solutions in these theories can linger near general relativity only for a certain period, while as time evolves they will leave and the approximation scheme will break down eventually. The case \( \sqrt{D} = 1 \) would imply \( A_\star = 0 \) or \( \Psi_\star = 0 \), which contradicts the assumptions (c) or \( 0 < \Psi < \infty \) of the present study.

Solutions (30)-(33) with \( D < 0 \) describe oscillatory cosmological models which asymptotically at \( t \to \infty \) approach the general relativistic dust matter FRW cosmology.

Solutions (30)-(33) with \( D = 0 \) are of marginal interest due to the fine-tuning of the STG by the condition \( A_\star \Psi_\star = -\frac{3}{8} \).

4. Discussion

STG dust cosmology equations in the Einstein frame were investigated already long ago by Damour and Nordtvedt [2]. They demonstrated that in the case of coupling function \((2\omega(\Psi) + 3)^{-1/2} \equiv \alpha(\varphi) = k\varphi, k = \text{const.} \) the type of a solution for the Einstein frame scalar field \( \varphi(p) \) with the evolution parameter \( p = (2/3) \ln t \) depends on the numerical value of the model-dependent constant \( k \): the solution is exponential in time parameter \( p \), i.e. polynomial in cosmological time \( t \), if \( 0 < k < 3/8 \), linear-exponential if \( k = 3/8 \) and oscillating if \( k > 3/8 \).

In our earlier papers [6, 7] we investigated the Jordan frame scalar field equation in the linear approximation, found the fixed points and calculated the eigenvalues which determine the type of solutions around these fixed points. Our results were qualitatively similar to those of Damour and Nordtvedt [2], but the critical value of the model-dependent parameter turned out to be \( 3/16 \) instead of \( 3/8 \).

In the present paper we found solutions in the nonlinear approximation for the Jordan frame scalar field \( \Psi(t) \) in the cosmological time \( t \) and obtained the critical value of the model-dependent parameter \( D \) to be given by \( A_\star \Psi_\star = -3/8 \). It is in exact agreement with the results of Damour and Nordtvedt, since the transformation between the Einstein and the Jordan frame quantities

\[ (d\varphi)^2 = \frac{2\omega(\Psi) + 3}{4\Psi^2} (d\Psi)^2 \]

(34)
gives

\[ k = \frac{d\alpha}{d\varphi}_\star = \left[ \frac{2\Psi}{(2\omega + 3)^2} \frac{d\omega}{d\Psi} \right]_\star = -A_\star \Psi_\star. \]

(35)

It follows that the approximation used by Damour and Nordtvedt [2] in the Einstein frame is equivalent to our nonlinear approximation in the Jordan frame and thus can be considered as an additional justification of our expansions (24), (25), (26).
5. Summary
We have derived and solved nonlinear approximate equations for decoupled scalar field in the framework of STG FLRW flat cosmological models in two eras. First we look the era when the energy density of the scalar potential dominates over the energy density of the ordinary matter and the Universe has evolved close to the limit of GR (which acts as an attractor for certain classes of STG). Secondly we consider the era of dust dominated matter close to the limit of GR. In both cases we have presented hints why the nonlinear approximation can be trusted: in the case of potential domination we have demonstrated in our earlier papers [3, 4] that phase portraits determined by exact and approximate equations are qualitatively similar and in the case of dust domination we have indicated analogous solutions obtained by Damour and Nordtvedt [2] using a different approach.

The solutions which approach GR asymptotically in cosmological time $t$ can be classified under two characteristic types: (a) exponential or polynomial convergence, and (b) damped oscillations around general relativity. The classes of STGs which contain these solutions are of particular interest since they naturally satisfy observational constraints and are good candidates to explain possible deviations from the ΛCDM. However, for choosing a distinct realistic model our results need to be supplemented by other types of investigations, e.g. considering evolution of cosmological perturbations.

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