An Integrable Nineteen Vertex Model Lying on a Hypersurface

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Abstract

We have found a family of solvable nineteen vertex model with statistical configurations invariant by the time reversal symmetry within a systematic study of the respective Yang-Baxter relation. The Boltzmann weights sit on a degree seven algebraic threefold which is shown birationally equivalent to the three-dimensional projective space. This permits to write parameterized expressions for both the transition operator and the R-matrix depending on three independent affine spectral parameters. The Hamiltonian limit tells us that the azimuthal magnetic field term is connected with the asymmetry among two types of spectral variables. The absence of magnetic field defines a physical submanifold whose geometrical properties are remarkably shown to be governed by a quartic K3 surface. This expands considerably the class of irrational manifolds that could emerge in the theory of quantum integrable models.

Keywords: Yang-Baxter Equation, Vertex Models, Algebraic Geometry
1 Introduction

At present the method of commuting transfer matrices provides the most important device for constructing exactly solvable lattice systems of statistical mechanics in two spatial dimensions [1]. Let us denote by $T_N(\omega)$ the model transfer matrix defined on a given direction of the lattice with length $N$. In order to make notation simpler we have represented the lattice Boltzmann weights $\omega_1, \ldots, \omega_m$ by the single vector $\omega \in \mathbb{C}^m$. The commutativity of two transfer matrices with distinct weights implies the condition,

$$\left[ T_N(\omega'), T_N(\omega'') \right] = 0, \quad \forall \, \omega' \text{ and } \omega'', \quad (1)$$

for arbitrary length $N$.

The fact that such commutation relation depends on the size $N$ seems that one needs to verify an infinite number of relations among the Boltzmann weights to conclude that two different transfer matrices indeed commute. This is fortunately not the case since Baxter [1] argued that it is sufficient to solve only a finite set of algebraic relations to built up a family of commuting transfer matrices for any size $N$. This local condition is often referred as Yang-Baxter equation and its specific structure depends much on the class of lattice system under consideration. In this paper we are interested to investigate novel solutions to this relation in the case of lattice vertex models. We recall that the fluctuation variables of vertex models lie on the bonds between neighboring lattice points and the interaction energies depend on the allowed vertices configurations. The main feature of these models are their inherent tensor structure which allows us to construct the corresponding transfer matrices out of a single local transition operator. In the simplest case of rectangular lattices this operator acts on the direct product of the auxiliary and quantum spaces associated respectively to the horizontal and vertical edges statistical configurations. Assuming that each edge of the rectangular lattice can take values on $q$ possible states one can represent the transition operator
L(\omega) on the auxiliary space as the following $q \times q$ matrix,

$$
L(\omega) = \\
\begin{pmatrix}
W_{1,1} & W_{1,2} & \cdots & W_{1,q} \\
W_{2,1} & W_{2,2} & \cdots & W_{2,q} \\
\vdots & \vdots & \ddots & \vdots \\
W_{q,1} & W_{q,2} & \cdots & W_{q,q}
\end{pmatrix}.
$$

(2)

The entries $W_{a,b}$ are also $q \times q$ operators but now acting on the space of quantum vertical degrees of freedom. Their matrix elements $W_{a,b}(c,d)$ represent the Boltzmann weights for the edge horizontal states $a,b$ and the edge vertical configurations $c,d$. The maximum number $m$ of distinct Boltzmann weights the vertex model can have is therefore $m = q^4$.

The transition operators can be combined to construct for instance the row-to-row transfer matrix represented as operators in the quantum space variables with an arbitrary number $N$ of columns. Considering periodic boundary conditions on the horizontal direction the transfer matrix takes the form,

$$
T_N(\omega) = \text{Tr}_q [L_1(\omega) L_2(\omega) \cdots L_N(\omega)],
$$

(3)

where the matrix multiplication and the trace operations are performed on the auxiliary space. The subscript index for the transition operator $L_j(\omega)$ means that its matrix elements act non-trivially only at the $j$-th vertical quantum space of states.

A sufficient condition for the commutativity of $T_N(\omega')$ and $T_N(\omega'')$ assumes the existence of a non-singular $q^2 \times q^2$ numerical matrix $R(w)$ which together with the transition operator fulfill the renowned Yang-Baxter relation [1],

$$
R(w) \left[ L(\omega') \otimes I_q \right] \left[ I_q \otimes L(\omega'') \right] = \left[ I_q \otimes L(\omega'') \right] \left[ L(\omega') \otimes I_q \right] R(w),
$$

(4)

where $I_q$ denotes the $q \times q$ unity matrix and the tensor product is considered within the auxiliary space. We have used the bold symbol $w$ to emphasize that the entries of the $R$-matrix should not be confused with the set of Boltzmann weights $\omega$ defining the transition operator.

Nowadays we are aware of several series of integrable vertex models for every value of $q$ and different types of statistical configurations. Notable examples are the lattice models with trigonometric weights associated to the representation theory of deformed Lie algebras [2,3] and their
elliptic generalizations \[4, 5\]. However, we still do not know any criterion to find the most general statistical configurations of the Boltzmann weights for which the existence of non-trivial solutions of the Yang-Baxter equation could be assured. Even for a given lattice statistical configuration it is an open problem to describe by what means the explicit form of all possible corresponding matrices \( L(\omega) \) and \( R(\omega) \) would be obtained. This latter question is equivalent to find the irreducible zeroes sets of a large number of homogeneous polynomials with many distinct monomials arising from the Yang-Baxter equation \([3]\). This is the typical problem one faces in Algebraic Geometry which in our case is formulated on the product of three projective spaces denoted here by \( \mathbb{CP}^{m-1} \times \mathbb{CP}^{m-1} \times \mathbb{CP}^{m-1} \). To the best of our knowledge it is not clear how the wonderful results in this field of mathematics could be used to shed some light into the theory of classification of solutions of the Yang-Baxter equation. In fact, it appears that so far some basic statements using Algebraic Geometry methods has been restricted to vertex models with two states per edge yet under additional assumption on the transition operator properties \([6]\). In spite of that we can still sketch some practical guidelines for searching solutions of the Yang-Baxter relation in the realm of Algebraic Geometry. We hope that this approach could be useful at least for vertex models with a specific given statistical configuration and for moderate number of edge states. These basic points to be described below are certainly influenced by the celebrated analysis of the Yang-Baxter relation for the eight-vertex model \([4]\). The manner we elaborate upon this method has been however inspired in our previous experience in dealing with several types of functional relations associated to a three-state vertex model \([7]\).

We start by recalling that the number of functional relations coming from Eq.(4) is generically much larger than the corresponding number of Boltzmann weights we need to determine. This leads us to solve a very overdetermined system of homogenous polynomial equations which are however linear on the entries of the R-matrix. We can use this feature to fix the basic structure of the matrix \( R(\omega) \) by means of standard linear elimination of its elements out of a suitable subset of independent functional relations. The R-matrix will be ultimately dependent on the set of weights \( \omega' \) and \( \omega'' \) and this means that we can formally rewrite the Yang-Baxter equation as,

\[
R(\omega', \omega'') \left[ L(\omega') \otimes I_q \right] \left[ I_q \otimes L(\omega'') \right] = \left[ I_q \otimes L(\omega'') \right] \left[ L(\omega') \otimes I_q \right] R(\omega', \omega'').
\]

(5)
The fact that $T_N(\omega)$ always commutes with itself should be encoded as particular solution of the Yang-Baxter equation for general transition operators. Direct inspection of Eq. (5) at the point $\omega' = \omega''$ tells us that the R-matrix simply switches the order of the tensor product of two transition operators with equal weights. We then conclude that such trivial solution for arbitrary transition operators is attained imposing the initial condition,

$$R(\omega, \omega) = \xi(\omega)P_q,$$

where $\xi(\omega)$ is a normalization and the operator $P_q$ denotes the $q^2 \times q^2$ permutator.

We next note that the Yang-Baxter relation (5) provides us certain consistent condition on the R-matrix upon exchange of the weights $\omega'$ and $\omega''$. In order to see that we first interchange single primed and double primed weights labels in Eq. (5) and afterwards we use the help of the permutator $P_q$ to reorder the tensor product of two different transition operators. As a final result we obtain,

$$P_q R(\omega'', \omega') P_q \left[ I_q \otimes L(\omega') \right] \left[ L(\omega') \otimes I_q \right] = \left[ L(\omega') \otimes I_q \right] \left[ I_q \otimes L(\omega'') \right] P_q R(\omega'', \omega') P_q,$$

(7)

The above expression can be further simplified once we apply on its left hand side the R-matrix $R(\omega', \omega'')$ and after that we use the Yang-Baxter equation (5) to rearrange the order of the right hand side transition operators. Considering these manipulations we find that Eq. (7) can be rewritten as the following commutator,

$$\left[ R(\omega', \omega'') P_q R(\omega'', \omega') P_q, \left[ I_q \otimes L(\omega'') \right] [L(\omega') \otimes I_q] \right] = 0,$$

(8)

The left term of the commutator (8) is a scalar on the tensor product of two auxiliary spaces whereas the right term for an arbitrary $L(\omega)$ turns out to be a complicated operator in this same space. This means that a generic solution to Eq. (8) occurs when its left term becomes proportional to the identity matrix in the product of auxiliary spaces. This leads us to what is usually called unitarity condition or inversion relation (8) for the R-matrix, namely

$$R(\omega', \omega'') P_q R(\omega'', \omega') P_q = \rho(\omega', \omega'') I_q \otimes I_q,$$

(9)
where $\rho(\omega', \omega'')$ is a scalar normalization. We observe that unitarity of the R-matrix is fully compatible with the initial condition (6).

From now on we shall assume that we are dealing with integrable vertex models whose R-matrices satisfy the unitarity property (9) which assures us that they are invertible. At this point it is natural to ask whether or not the unitarity relation together with the Yang-Baxter equation are capable to impose any relevant restriction on the functional equations when their single and double primed weights labels are exchanged. This idea has already been explored in our recent work [7] and we shall here present only the main conclusion. Let $F_j(\omega', \omega'')$ be the polynomials derived from the Yang-Baxter equation after we have performed the elimination of the R-matrix elements. It has been shown that such polynomials have to satisfy the following anti-symmetrical property,

$$F_j(\omega', \omega'') + F_j(\omega'', \omega') = 0.$$ (10)

For a general vertex model it is not expected that all the functional equations to be satisfied are automatically anti-symmetrical upon the exchange of the weights $\omega'$ and $\omega''$. In fact, the requirement that $F_j(\omega', \omega'')$ should satisfy the property (10) has been decisive to simplify cumbersome high degree polynomials expressions emerging in the analysis of a three-state vertex model [7].

In order to make further progress it is crucial that we are able to recast at least part of the anti-symmetrical polynomials $F_j(\omega', \omega'')$ in the following particular factorized form,

$$F_j(\omega', \omega'') = H_j(\omega')G_j(\omega'') - H_j(\omega'')G_j(\omega'), \quad j = 1, \cdots, n,$$ (11)

for some integer $n$. The homogeneous polynomials $H_j(\omega)$ and $G_j(\omega)$ are assumed to be irreducible having the same degree on the weights.

The above step provides us the basic ingredient to start the construction of two commuting transfer matrices whose weights will be sited on the same algebraic variety. This can be achieved imposing that each factorized functional relation (11) vanishes upon the choice of the same polynomial restriction for both set of variables $\omega'$ and $\omega''$. Such special solution to Eq.(11) in which the weights with distinct labels are separated is clearly given by,

$$\frac{H_j(\omega)}{G_j(\omega)} = \Lambda_j, \quad j = 1, \cdots, n,$$ (12)
where $\Lambda_1, \cdots, \Lambda_n$ are free parameters.

In the language of Algebraic Geometry the particular solution (12) can be seen as a prime divisor over the algebraic set made out of the zeroes of the factorized polynomial we have started with. For the technical details concerning this interpretation see [7] and in what follows we shall refer to such special solutions as divisors.

We next have to deal with the remaining functional relations which could not be brought into the suitable factorized form (11). In these cases we hope that such polynomials can also be set equal to zero at the expense of imposing additional constraints on the available free parameters $\Lambda_j$. Being successful on such last step we then have solutions of the Yang-Baxter equation whose properties are formally governed by the algebraic variety,

$$Y = \{ \omega \in \mathbb{C} \mathbb{P}^{m-1} | H_1(\omega) - \Lambda_1 G_1(\omega) = 0, H_2(\omega) - \Lambda_2 G_2(\omega) = 0, \cdots, H_n(\omega) - \Lambda_n G_n(\omega) = 0 \}, \quad (13)$$

where now a subset of the parameters $\Lambda_1, \cdots, \Lambda_n$ may be fixed.

The complete characterization of the geometrical features of the projective variety $Y$ certainly depends much on the polynomial form of the generators of its ideal. For example, the intersection of many polynomials can in principle give rise to a number of irreducible varieties none of which being superfluous. However, there exists an important invariant of the variety $Y$ for which we can make a concrete statement without the knowledge of the specific structure of the polynomials (12). This turns out to be the maximal of the dimensions of the irreducible components of $Y$ denoted here by the symbol $\dim(Y)$. The Algebraic Geometry theory predicts a lower bound for such invariant which is [9],

$$\dim(Y) \geq (m - 1) - n, \quad (14)$$

and when the equality holds such component of $Y$ is named a complete intersection.

The dimension of the variety underlying a solution of the Yang-Baxter equation dictates the number of free weights or spectral parameters expected to be present in the uniformization of the respective transition operator. We believe that it is of great interest to search for integrable systems whose weights lie on high dimensional algebraic varieties. For instance even a rational three-dimensional variety can contain several non-rational surfaces and some of them could still
represent a submanifold of physical interest. This study provides us a clear route to discover examples of solvable models lying on irrational varieties more involved than those uniformized by high genus curves such as the chiral Potts model \[11,12\]. In fact, we are not aware of examples of integrable models with weights lying on non-rational surfaces which are not ruled by algebraic curves.

The main purpose of this paper is to investigate the above possibility in the case of a rather general three-state vertex model with ice-rule statistical configurations. The corresponding transition operator commutes with the azimuthal component of the spin-1 generators and this degree of freedom could somewhat be interpreted as the presence of an extra spectral parameter. In general, we expect that generic solvable U(1) invariant vertex models will contain the minimum number of two free spectral variables and consequently their weights should at least be sited on two-dimensional algebraic manifolds. Another motivation comes from the existence of a solvable spin-1 quantum chains with three main free coupling constants discovered by Crampré, Frappat and Ragoucy \[13\] within the coordinate Bethe ansatz. The respective spin-1 Hamiltonian is a generalization of the one built out of colored transition operators based on representations of the algebra U[SU(2)]$_q$ when q is at roots of unity \[14,15\]. It is therefore conceivable that some of the Hamiltonian couplings could originate through the presence of additional spectral variables rather than from the usual constants associated to divisors. We found that this is indeed the situation of the spin chain denominated SpR in the reference \[13\]. We shall show that the integrability properties are governed by a transition operator sitting on a degree seven algebraic threefold with polynomial coefficients depending on two arbitrary constants. As a result we then have three independent spectral parameters at our disposal and one of them reflects the solvability of the Hamiltonian in the presence of any azimuthal magnetic field. Remarkably enough, the submanifold giving rise to the spin-1 Hamiltonian in absence of the magnetic field is governed by the geometric properties of an algebraic surface on the K3 class. Recall here that K3 surfaces have zero Kodaira dimension\[1\].

\[1\] There exists a rough relationship between positive(negative) Kodaira dimension and the negative(positive) curvature of the surface. A zero value for the Kodaira dimension corresponds to flatness and for details of definitions and properties we refer to the book \[16\].
being two-dimensional Calabi-Yau manifolds which do not have a group structure. This appears to be the first example of a solution of the Yang-Baxter equation lying on such famous family of compact complex surfaces.

We have organized this paper as follows. In next Section we present the structure of the transition operator of the nineteen vertex model. We assume that the model is invariant by time reversal symmetry leading us to the parameter subspace of fourteen non-null weights. The analysis of the functional relations is performed in Section 3 and we find an integrable vertex model sitting on an algebraic threefold with two free couplings. In Section 4 we show that such threefold is birationally equivalent to the projective space $\mathbb{CP}^3$. This mapping is used to present the parameterized form of the respective transition operator. We discuss the Hamiltonian limit of the vertex model and some of its couplings originate from combinations of three independent spectral weights. In particular the presence of an arbitrary magnetic field is related to the asymmetry of two types of weights. In Section 5 we discuss the the submanifold associated with the absence of any magnetic field and we show that its geometrical properties are governed by K3 surfaces. The expression of the respective transition operator lying on a quartic K3 surface with only canonical singularities is provided. We have summarized our concluding remarks in Section 6. In three Appendices we have presented certain technical details omitted in the main text, the expressions of the R-matrices on three different embeddings as well as the computations of the Hamiltonian limit.

\section{The Nineteen Vertex Model}

The nineteen vertex model has three states per bond and its statistical configurations are restricted by the ice rule. This means that the weights $W_{a,b}(c,d)$ are non null only when the state variables at the vertex satisfy the condition $a + c = b + d$. Here we shall consider a subclass of such models whose Boltzmann weights are invariant when we rotate the lattice of 180 degrees. In analogy with relativistic 1+1 dimensional scattering theory\cite{10} this invariance is often denominated time reversal symmetry,

$$W_{a,b}(c,d) = W_{b,a}(d,c).$$

(15)
We would like to remark that the request of time reversal invariance forces us from the very beginning to be far away of the recent found integrable genus five manifold \[7\]. We also note that this symmetry is not that stringent since the vertex model space of parameters is reduced to still fourteen distinct weights. These facts favor the possibility of uncovering new solvable nineteen vertex models which hopefully will be sited on high dimensional varieties. The explicit distinct matrix elements of the transition operator are,

\[
W_{1,1} = \begin{pmatrix} a_+ & 0 & 0 \\ 0 & b_+ & 0 \\ 0 & 0 & f_+ \end{pmatrix}, \quad W_{2,2} = \begin{pmatrix} \bar{b}_+ & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & \bar{b}_- \end{pmatrix}, \quad W_{3,3} = \begin{pmatrix} f_- & 0 & 0 \\ 0 & b_- & 0 \\ 0 & 0 & a_- \end{pmatrix},
\]
\[
W_{1,2} = \begin{pmatrix} 0 & 0 & 0 \\ c_+ & 0 & 0 \\ 0 & d_+ & 0 \end{pmatrix}, \quad W_{1,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ h & 0 & 0 \end{pmatrix}, \quad W_{2,3} = \begin{pmatrix} d_- & 0 & 0 \\ 0 & c_- & 0 \end{pmatrix},
\]

where the other matrix elements are determined by the time reversal symmetry \((15)\).

The above notation for the transition operator bears the charge conjugation operation which exchange the subscripts \(+ \leftrightarrow -\) of the weights. We shall see for instance that in the Hamiltonian limit the asymmetry among the weights \(c_+\) and \(c_-\) implies the presence of a non-null azimuthal magnetic field. This choice will be also convenient to write compact expressions for the functional equations coming from the Yang-Baxter equation. We now substitute the matrix expression of the transition operator in the Yang-Baxter equation using as an ansatz for \(R(w)\) the most general \(9 \times 9\) matrix. The analysis of the corresponding functional relations derived from Eq.\((14)\) reveals us that for a generic vector \(\omega \in \mathbb{CP}^{13}\) the non null entries of the R-matrix are also constrained by the ice rule. This leads us to conclude that the basic form for the \(R(w)\)-matrix is similar to that of the
transition operator, namely

\[ R(w) = \begin{bmatrix}
    a_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & b_+ & 0 & c_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & f_+ & 0 & d_+ & 0 & h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & c_+ & 0 & \mathbf{E}_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & d_+ & 0 & g & 0 & d_- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & \mathbf{E}_- & 0 & c_- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & h & 0 & d_- & 0 & f_- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & c_- & 0 & b_- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad (17) \]

where we have distinguished the R elements by means of bold letters.

In next section we shall investigate the solutions of the functional equations derived by substituting the expressions of the transition operator (16) and the R-matrix (17) into the Yang-Baxter equation. This leads to fifty-seven distinct polynomial relations but fortunately only a subset of them are enough to decide on the existence of a hypersurface solution.

3 The Functional Relations

The polynomial equations can be classified in terms of their number of monomials involving the R-matrix entries and two different set of Boltzmann weights. The minimum number of such monomials is three while the maximum one turns out to be five and this information has been summarized in Table (1).

The first step to solve the Yang-Baxter relation is to perform the elimination of the R-matrix elements. Because we are dealing with homogeneous system of equations this is equivalent of the vanishing of a number of determinants whose coefficients depend on the Boltzmann weights. The main point is to choose suitable system of relations whose determinants could be factorized in the convenient form (11). We shall start this analysis considering the simplest family of functional equations which are those involving three different monomials.
Table 1: The number of functional relations with a given number monomials.

| Number of Relations | Number of Monomials |
|---------------------|---------------------|
| 18                  | Three               |
| 27                  | Four                |
| 12                  | Five                |

3.1 Three Monomials

The expressions of the eighteen functional relations involving three different monomials are,

\[
\begin{align*}
\mathbf{a}_\pm c'_\pm a''_\pm - \mathbf{b}_\pm c'_\pm b''_\pm - c'_\pm a''_\pm c''_\pm &= 0, \\
\mathbf{c}_\pm c'_\pm b''_\pm + b'_\pm a'_\pm c''_\pm - a'_\pm b'_\pm c''_\pm &= 0, \\
\mathbf{a}_\pm d'_\pm b''_\pm - c'_\pm b'_\pm d''_\pm - \mathbf{b}_\pm d'_\pm f''_\pm &= 0, \\
\mathbf{b}_\pm b'_\pm d''_\pm - a'_\pm f'_\pm d''_\pm + c'_\pm d'_\pm f''_\pm &= 0, \\
\mathbf{c}_\pm b'_\pm a''_\pm - \mathbf{b}_\pm c'_\pm c''_\pm - c'_\pm a''_\pm \overline{b''}_\pm &= 0, \\
\mathbf{b}_\pm d'_\pm d''_\pm + c'_\pm \overline{b''}_\pm f''_\pm - c'_\pm f'_\pm \overline{b''}_\pm &= 0, \\
\mathbf{f}_\pm b'_\pm c''_\pm - b'_\pm f'_\pm c''_\pm + d'_\pm d''_\pm \overline{b''}_\pm &= 0, \\
\mathbf{d}_\pm f'_\pm a''_\pm - f'_\pm d'_\pm c''_\pm - d'_\pm \overline{b''}_\pm c''_\pm &= 0, \\
\mathbf{b}_\pm d'_\pm a''_\pm - f'_\pm d''_\pm b''_\pm - d'_\pm \overline{b''}_\pm c''_\pm &= 0.
\end{align*}
\]

We emphasize that each of the equations (18-26) splits into two different functional relations associated to the two possible subscripts values ± for the weights. We start the solution of these equations by first noticing that out of Eqs. (18-23) we can construct two independent homogeneous linear systems for the R-matrix entries \(\mathbf{a}_\pm, \mathbf{b}_\pm, \overline{\mathbf{b}}_\pm\) and \(\mathbf{c}_\pm\). We find that the determinants of coefficients associated to Eqs. (18-21) have the nice property that they can be written in the following factorized form,

\[
\begin{align*}
\left[ (b'_\pm^2 - a'_\pm f'_\pm) c''_\pm d''_\pm - (b''_\pm^2 - a''_\pm f''_\pm) c'_\pm d'_\pm \right] [a'_\pm d'_\pm c''_\pm f''_\pm - b''_\pm d''_\pm b''_\pm c''_\pm].
\end{align*}
\]
In order to have a non-trivial solution for the R-matrix entries \( a_{\pm}, b_{\pm}, \overline{b}_{\pm} \) and \( c_{\pm} \) the above determinants must vanish. This can be achieved if either of the two factors of Eq. (27) vanishes which gives us two branches to be analyzed. We stress that the purpose of this paper is not to pursue a classification of possible integrable nineteen vertex models even within the subclass of systems invariant by the time reversal symmetry. Here we are mainly interested to point out an example of solvable nineteen vertex model whose weights would be lying on high-dimensional algebraic varieties. In this sense we choose the branch associated to the first factor of Eq. (27) since its polynomial expression is clearly less restrictive than that of the second factor. We also note that the former factor has the suitable polynomial form (11) and the condition that it vanishes leads us to our first divisor,

\[
\frac{b^2_{\pm} - a_{\pm} f_{\pm}}{c_{\pm} d_{\pm}} = \Lambda^\pm_1. \tag{28}
\]

We now can solve Eqs. (18, 21) for the R-matrix entries by means of linear elimination of such variables. After using Eq. (28) and taking \( c_{\pm} \) as a common normalization we find that their expressions are,

\[
\frac{a_{\pm}}{c_{\pm}} = \frac{b'_{\pm} c'_{\pm} b''_{\pm} - a''_{\pm} d''_{\pm} c''_{\pm}}{\Lambda^+_1 c'_+ d'_+ c''_+ d''_+}, \tag{29}
\]

\[
\frac{\overline{b}_{\pm}}{c_{\pm}} = \frac{b'_{\pm} c'_{\pm} b''_{\pm} - a''_{\pm} d''_{\pm} b''_{\pm} c''_{\pm}}{\Lambda^+_1 c'_+ d'_+ c''_+ d''_+}, \tag{30}
\]

\[
\frac{b_{\pm}}{c_{\pm}} = \frac{f'_{\pm} c'_{\pm} b''_{\pm} - b'_{\pm} d''_{\pm} f''_{\pm}}{\Lambda^+_1 c'_+ d'_+ c''_+ d''_+}, \tag{31}
\]

where we are tacitly assuming that the free parameters \( \Lambda^\pm_1 \) are non null. Along the lines of our work [7] it is possible to show that the particular point \( \Lambda^+_1 = 0 \) corresponds to the branch in which the second factor of the determinant (27) is set to zero. As remarked before we shall not consider such rather special branch in what follows.

In order to complete the solution of the first twelve functional relations we have to substitute these results for R-matrix entries into the remaining equations (22, 23). After few simplifications with the help of the divisor (28) we find that the Eqs. (22) can be rewritten as,

\[
(b'_{\pm} c'_{\pm} - \Lambda^+_1 b''_{\pm} d''_{\pm}) a''_{\pm} d''_{\pm} - (b''_{\pm} c''_{\pm} - \Lambda^+_1 b''_{\pm} d''_{\pm}) a'_{\pm} d'_{\pm} = 0, \tag{32}
\]

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while Eqs. (23) becomes proportional to the polynomial,

\[
(b'_\pm d'_\pm - \Lambda^\pm_1 c'_\pm \bar{b}'_\pm) c''_\pm f''_\pm - (b''_\pm d''_\pm - \Lambda^\pm_1 c''_\pm \bar{b}''_\pm) c'_\pm f'_\pm = 0.
\]  

The above polynomials are both in the convenient factorized form \((\text{II})\) and they are solved by the following divisors,

\[
\frac{b_\pm c_\pm - \Lambda^\pm_1 \bar{b}_\pm d_\pm}{a_\pm d_\pm} = \Lambda^\pm_2,
\]  

and

\[
\frac{b_\pm d_\pm - \Lambda^\pm_1 \bar{b}_\pm c_\pm}{c_\pm f_\pm} = \Lambda^\pm_3.
\]

From the above analysis we are able to conclude that the number of independent weights can be reduced by five variables. In fact, we first note that the divisors \((28, 34)\) can be easily resolved by means of linear elimination of the weights \(d_\pm\) and \(f_\pm\) and as result we obtain,

\[
d_\pm = \frac{b_\pm c_\pm}{\Lambda^\pm_2 a_\pm + \Lambda^\pm_1 \bar{b}_\pm},
\]

and

\[
f_\pm = \frac{b_\pm}{a_\pm} \left[ \frac{\Lambda^\pm_2 a_\pm + \Lambda^\pm_1 \bar{b}_\pm}{\Lambda^\pm_2 a_\pm + \Lambda^\pm_1 \bar{b}_\pm} \right].
\]

We next substitute the above variables into the last divisor \((35)\) and notice that the weights \(\bar{b}_\pm\) are still linearly encoded in the resulting expressions. This means that we can for instance extract the weight \(\bar{b}_-\) from Eq. \((35)\) by using the channel with subscript plus. Substituting this result back to Eq. \((35)\) but now with subscript minus produces however a non-linear constraint among the variables \(a_\pm, b_\pm, c_\pm\) and \(\bar{b}_+\). Considering these steps we find that the expression for the weight \(\bar{b}_-\) is,

\[
\bar{b}_- = \frac{b_+}{\Lambda^+_1 a_+} \left[ \frac{(1 - \Lambda^+_2 \Lambda^+_3) a_+ b_+ - \Lambda^+_1 \Lambda^+_3 (b_+ \bar{b}_+ - c^2_+)}{\Lambda^+_2 a_+ + \Lambda^+_1 \bar{b}_+} \right],
\]

while the constraint turns out to be a degree five hypersurface leaving on \(\mathbb{C}P^6\) given by,

\[
S[a_\pm, b_\pm, \bar{b}_+, c_\pm] = [\Lambda^+_1]^2 \left( [1 - \Lambda^+_2 \Lambda^+_3] a_- b_-^2 + \Lambda^+_1 \Lambda^+_3 b_- c_-^2 - \Lambda^+_1 \Lambda^+_2 a_-^2 \bar{b}_+ \right) a_+ \bar{b}_+ + \Lambda^+_1 \Lambda^+_2 \left( [1 - \Lambda^+_2 \Lambda^+_3] a_- b_-^2 + \Lambda^+_1 \Lambda^+_3 b_- c_-^2 - \Lambda^+_1 \Lambda^+_2 a_-^2 \bar{b}_+ \right) a_+^2 + \Lambda^+_1 \Lambda^+_1 \Lambda^+_3 \left( \Lambda^+_3 b^2_+ + \Lambda^+_1 a_- b_+ \right) (b_+ \bar{b}_+ - c^2_+) b_+ + \Lambda^+_1 [\Lambda^+_2 \Lambda^+_3 - 1] \left( \Lambda^+_3 b^2_+ + \Lambda^+_1 a_- \bar{b}_+ \right) a_+ b_+^2 = 0.
\]
Let us come back to discuss the solution of the remaining functional relations \((24-26)\). Direct inspection of such equations tells us that we can use two of them to eliminate the R-matrix entries \(d_±\) and \(f_±\). Choosing Eqs.\((24,25)\) to be solved we find that such matrix elements are given by,

\[
\begin{align*}
\frac{d_±}{c_±} &= \frac{b_±}{c_±} \left[ \frac{d'_± f''_± c''_±}{b'_± f''_± a''_± + [d'_±]^2 b''_± - b'_± b''_±} \right], \\
\frac{f_±}{c_±} &= \frac{d_±}{c_±} \left[ \frac{f'_± a''_± - b'_± b''_±}{d'_± c''_±} \right],
\end{align*}
\]

where the ratio \(b_± / c_±\) is obtained from Eq.\((31)\).

We have now reached a point in which only Eqs.\((26)\) remain to be analyzed. They lead to polynomials having as unknowns the R-matrix entries \(c_±\) whose coefficients consist of very complicated expressions depending on the weights \(a_±, b_±, c_±\) and \(\bar{b}_±\). We shall postpone their analysis until next section since many of the free parameters \(\Lambda_1^±, \Lambda_2^±, \Lambda_3^±\) are going to be fixed later on. This fact will be responsible for the cancellation a large number of monomials resulting in much simpler polynomial expressions. In spite of that it is possible to make a prediction on the lowest dimension value of the variety \(Y\) associated to a potential solution of the Yang-Baxter equation. First we should note that Eqs.\((26)\) are only capable to produce at most two more additional divisors. This fact together with the results obtained so far imply that the maximum number of divisors solving the full set of functional equations \((18-26)\) should be therefore eight. We next recall that these divisors are embedded in a \(\mathbb{CP}^{11}\) projective space whose coordinates are the twelve weights \(a_±, b_±, \bar{b}_±, c_±, d_±\) and \(f_±\). Considering these information on formula \((14)\) we conclude that the dimension of the underlying variety must satisfy,

\[
\dim(Y) \geq 12 - 1 - 8 = 3,
\]

and therefore we have a concrete possibility of the Boltzmann weights being sited at least on an algebraic threefold.

The above conclusion assumes that the many other functional relations coming from the Yang-Baxter equation can be solved without additional divisors other than that necessary to resolve Eqs.\((26)\) as well as those that determine the last two weights \(h\) and \(g\). It is exactly at this point
that the free parameters $\Lambda_1^\pm, \Lambda_2^\pm$ and $\Lambda_3^\pm$ we have at our disposal are going to play a very important role. In fact, this freedom will be used to cancel out several relevant polynomial relations avoiding the need of the proposal of any extra divisors.

### 3.2 Four Monomials

The weights $h$ and $g$ start to emerge on the functional relations containing four different monomials. The key point to solve these relations is to notice that few of them still have as unknowns some R-matrix entries that have already been sorted out in subsection 3.1. The condition of consistency of our elimination procedure compels us to search for linear combinations among such relations and four special functional equations involving three monomials. The vanishing of the determinants of these linear combinations lead us to determine the remaining weights $h$ and $g$ with the help of only two independent divisors. This makes it possible to maintain the lower bound (42) for the dimension of the underlying algebraic variety. Let us first describe this reasoning for the weight $h$.

#### 3.2.1 The variables $h$ and $h$

Among the several four monomials functional relations solely four of them have as unknowns the R-matrix elements we have previously eliminated whose coefficients also contain the weights $h'$ and $h''$. These relations involve the entries $b_{\pm, \overline{b}_{\pm}}$ and $c_{\pm}$ and their expressions are,

\begin{align*}
  c_+ d_+ d_- - c_- d_- d_+ + \overline{B}_+ \bar{b}_- h'' - \overline{B}_+ \bar{b}_+ h'' = 0, \quad (43)
  b_- c_+ c_- - b_+ d_+ d_- - c_+ \bar{b}_- h'' + c_- h' \bar{b}_- = 0, \quad (44)
  \overline{B}_+ c_+ c_+ - b_- d_- d_+ - c_- \bar{b}_+ h'', c_+ h' \bar{b}_+ = 0, \quad (45)
  c_- c_+ c_+ - c_- c_+ c_- + b_- h' \bar{b}_- - b_+ h' \bar{b}_+ = 0. \quad (46)
\end{align*}

In order to build out a consistent homogeneous linear system we have to search for two more relations involving the six matrix elements $b_{\pm, \overline{b}_{\pm}}$ and $c_{\pm}$. Direct inspection of the functional equations with three monomials reveals us that such relations are in fact given by the four possible
channels of Eqs. (22,23). The composition of the eight functional relations (22,23,43-46) gives rise to twenty-eight different linear systems but we find that only two of them are able to produce determinants that factorize into smaller pieces. This turns out to be the combination of the above relations (43,46) with either the plus or the minus subscript component of the three monomials equations (22,23). The vanishing of such determinants can be written as,

\[
\left[ c'_+ d'_- c''_+ d''_+ - h' \tilde{b}'_+ h'' \tilde{b}''_+ \right] F_\pm (\omega', \omega'') = 0,
\]  

where the polynomial \( F_\pm (\omega', \omega'') \) has the following more complicated expression,

\[
F_\pm (\omega', \omega'') = c'_+ d'_- \tilde{b}'_+ h' a''_+ d''_- \tilde{b}''_+ - a'_+ c'_+ d'_- \tilde{b}'_+ c''_+ d''_+ h'' + c'_+ d'_+ \tilde{b}'_+ h' c''_+ d''_+ f''_+ \tilde{b}''_+ \\
- c'_+ d'_- f'_+ \tilde{b}'_+ c''_+ d''_+ h'' + [c'_+ d'_+]^2 c''_+ d''_+ d''_+ - c'_+ c''_+ d''_+ [c''_+ d''_+]^2 \\
+ c'_+ d'_- \tilde{b}'_+ b''_+ \left( a''_+ c''_+ d''_+ h'' + c''_+ d''_+ f''_+ h'' - c''_+ d''_+ [h'']^2 \right) \\
- c''_+ d''_+ \tilde{b}'_+ b''_+ \left( a''_+ c''_+ d''_+ h'' + c''_+ d''_+ f''_+ h'' - c''_+ d''_+ [h'']^2 \right).
\]  

(48)

We encounter once again the situation in which we have in principle two branches that have to be analyzed depending on the factor in Eq. (47) we set to zero. As already emphasized this work is not concerned with the complete classification of nineteen vertex models and in what follows we will choose the branch in which the weights \( h' \) and \( h'' \) can be determined in a linear manner. This leads us to impose the vanishing of the first term in Eq. (47) since the polynomials \( F_\pm (\omega', \omega'') \) contain clearly quadratic terms on the weights \( h' \) and \( h'' \). The corresponding divisors in which the single and double primed variables are separated from each other are,

\[
\frac{c'_+ d'_+}{h' \tilde{b}'_+} = \Lambda^+_4 \quad \text{and} \quad \frac{c''_+ d''_+}{h'' \tilde{b}''_+} = \Lambda^-_4.
\]  

(49)

We now have the necessary condition to start the solution of the functional relations (43,46). The first step is to extract the ratio among the R-matrix elements \( c_+ \) and \( c_- \) from one of these equations. We find that Eq. (43) gives us the simplest possible expression for such ratio which is,

\[
\frac{c_-}{c_+} = \frac{\Lambda^- b'_+ [c'_-]^2 b''_+ + b'_+ (\Lambda^+_4 a'_+ + \Lambda^+_4 b'_+ (b'_+ a'_+ - a'_+ b'_+))}{\Lambda^-_4 b'_+ [c'_-]^2 b''_+ + b'_+ (\Lambda^-_4 a'_+ + \Lambda^-_4 b'_+ (b'_- a'_- - a'_- b'_-))}.
\]  

(50)
We next substitute this result in Eq. (44) which leads us to a degree three bihomogenous polynomial on the weights \( a_+, b_+, \tilde{b}_+ \) and \( c_+ \). We find that such polynomial satisfies the anti-symmetrical property (10) only when the free parameters \( \Lambda^\pm_4 \) are related by,

\[
\Lambda^-_4 = \Lambda^+_4, \tag{51}
\]

bringing the divisors (49) to have the same form on both single and double primed indices as expected. These divisors can now be used in order to determine the weights \( h \) and \( b_- \) in terms of the so far free amplitudes \( a_+, b_+, \bar{b}_+ \) and \( c_+ \). By substituting Eqs. (36,38) in Eqs. (49) and after few simplifications their expressions become,

\[
h = \frac{\Lambda^+_1 \Lambda^\pm_4 a_+ c_+ c_-}{[1 - \Lambda^+_2 \Lambda^+_3]a_+ b_+ + \Lambda^+_1 \Lambda^+_3 [c^2_- - b_+ \bar{b}_+]}, \tag{52}
\]

\[
b_- = \frac{\Lambda^-_1 [1 - \Lambda^-_2 \Lambda^-_3]a_+ b^-_+ + \Lambda^-_1 \Lambda^-_2 [\Lambda^-_2 a_+ + \Lambda^-_1 \bar{b}_+] a_- a_+ + \Lambda^-_1 \Lambda^-_3 \Lambda^-_3 [c^2_- - b_+ \bar{b}_+] b_+}{[1 - \Lambda^-_2 \Lambda^-_3]a_+ b_+ + \Lambda^-_1 \Lambda^-_3 [c^2_- - b_+ \bar{b}_+]}
\times \frac{[\Lambda^+_4]^2 \bar{b}_+}{\Lambda^+_2 a_+ + \Lambda^+_3 b_+}. \tag{53}
\]

Let us now return to the analysis of Eq. (44). We find that using the constraint (51) the vanishing of this functional equation becomes equivalent to the following relation,

\[
\tilde{b}_+ a_+^{\prime\prime} H(a_+^{\prime\prime}, b_+^{\prime\prime}, \bar{b}_+^{\prime\prime}) G(a_+^{\prime\prime}, b_+^{\prime\prime}, \bar{b}_+^{\prime\prime}) - b_+ a_+^{\prime\prime} H(a_+^{\prime\prime}, b_+^{\prime\prime}, \bar{b}_+^{\prime\prime}) G(a_+^{\prime\prime}, b_+^{\prime\prime}, \bar{b}_+^{\prime\prime}) = 0, \tag{54}
\]

where the expressions for the polynomials \( H(a_+, b_+, \bar{b}_+) \) and \( G(a_+, b_+, \bar{b}_+, c_+) \) are,

\[
H(a_+, b_+, \bar{b}_+) = (1 - \Lambda^+_2 \Lambda^+_3) a_+ b_+ + \Lambda^+_1 \Lambda^+_3 (c^2_+ - b_+ \bar{b}_+), \tag{55}
\]

\[
G(a_+, b_+, \bar{b}_+, c_+) = \Lambda^+_4 (1 - \Lambda^+_2 \Lambda^+_3) a_+ b_+ + \Lambda^+_1 (\Lambda^+_3 - \Lambda^+_4) \left[ \Lambda^+_3 c^2_+ - (\Lambda^+_3 + \Lambda^+_4) b_+ \bar{b}_+ \right]. \tag{56}
\]

We see that Eq. (54) has the desirable factorized form (11) and in principle could be solved by means of an extra divisor between the weights \( a_+, b_+, \tilde{b}_+ \) and \( c_+ \). This certainly is going to lower the bound of the underlying algebraic variety and therefore such solution should be discarded whenever possible. It is however fortunate that Eq. (54) can be trivially satisfied provide we impose the following relations among the free parameters,

\[
\Lambda^+_4 = \Lambda^+_3 = \frac{1}{\Lambda^+_2}, \tag{57}
\]

\[
\Lambda^+_2 \Lambda^+_3 + \Lambda^+_1 \Lambda^+_4 = 0.
\]
which sets the polynomial \( G(a_+, b_+, \bar{b}_+, c_+) \) identically to zero.

The above reasoning can also be implemented to Eqs.\((45, 46)\) which at this point are proportional to each other. The basic difference is that now we have the presence of the weight \( c_- \) in their expressions however solely through even powers. This variable can then be eliminated in a systematic way with the help of the hypersurface expression \((39)\) since its dependence on the weight \( c_- \) is quadratic. It is not difficult to perform this algebraic operation within the Mathematica computer system as exemplified in Appendix A. After this operation we have an involved degree eight bihomogenous polynomial which fortunately vanishes by imposing the additional constraints,

\[
\Lambda_3^- = \frac{1}{\Lambda_2} = \Lambda_2^+.
\]

(58)

At this point we have now gathered the basic ingredients to finally come back to the solution of the functional relations \((26)\). Considering the value for the weight \( b_- \) given by Eq.\((53)\) as well as the constraints \((57, 58)\) we find that the resulting polynomials coming from Eqs.\((26)\) are magically canceled out provided that the weight \( a_- \) is fixed by the following expression,

\[
a_- = \frac{(b_+ \bar{b}_+ - c_+^2) (\Lambda_1^- a_+ b_+ - \Lambda_2^+ [c_+^2 - b_+ \bar{b}_+])}{a_+^2 (\Lambda_2^+ a_+ + \Lambda_1^+ b_+)}.
\]

(59)

We start the solution of the above relations solving one of them for the ratio \( h/c_+ \). Here we
choose the component of Eqs. (60) with subscript plus and we find that this ratio is given by,

$$\frac{h}{c_+} = \left[\frac{[\Lambda^+_2]^2(\Lambda^+_2 a'_+ + \Lambda^+_2 b'_+) [\Lambda^-_2 a'_+ b'_+ + \Lambda^+_2 (b'_+ b'_+) - [c'_+]^2]}{[\Lambda^+_2]^2(\Lambda^+_2 a'_+ + \Lambda^+_2 b'_+) [\Lambda^-_2 a'_+ b'_+ + \Lambda^+_2 (b'_+ b'_+) - [c'_+]^2]} - a'_+ b'_+ b'_+ \right]$$

$$\times \left[\frac{[a'']^2 c'_- (b'_+ b'_+ - [c'_+]^2)(\Lambda^+_2 a''_+ + \Lambda^+_2 b''_+)}{[a'']^2 c'_- (b'_+ b'_+ - [c'_+]^2)(\Lambda^+_2 a''_+ + \Lambda^+_2 b''_+)} \right]. \quad (66)$$

We now substitute the results we have obtained so far in the remaining relations (60-65) resulting in polynomials depending on the weights $a_+, b_+, \bar{b}_+, c_+$ and $c_-$. As before the dependence on the variable $c_-$ occurs only via even powers and this weight can once again be eliminated by using the hypersurface constraint (39). After performing this step we find that all the above functional relations are satisfied at the expense of fixing the parameter $\Lambda^-_1$ as,

$$\Lambda^-_1 = \frac{1 - [\Lambda^+_2]^2 + [\Lambda^+_2]^4}{\Lambda^+_2 [\Lambda^+_2]^2}. \quad (67)$$

We conclude observing that so far we have not introduced any additional divisor besides the three expected ones. In fact, two of them are used to resolve of Eqs. (60) and are directly associated with the determination of the weights $a_-$ and $b_-$ by means of the free variables $a_+, b_+, \bar{b}_+$ and $c_+$. The third one is responsible for the elimination of the weight $h$ again in terms of the same variables. This means that the general character of our solution concerning the variety dimension bound (42) remains unchanged.

3.2.2 The variables $g'$ and $g''$

We have six functional relations encoding the weights $g', g''$ and the matrix element $g$ together with the previous determined variables. Their expressions are given by,

$$c_\pm g'_+ b'_+ - b_\pm c'_+ c'_+ - \bar{b}_\pm d'_+ d'_+ - c_\pm b'_+ g'' = 0, \quad (68)$$

$$g d'_+ b'_+ + d_\pm b'_+ c'_+ - c_\pm b'_+ d'_+ - b_\pm d'_+ g'' = 0, \quad (69)$$

$$g \bar{b}_+ c'_+ = d_\pm d'_+ b'_+ - \bar{b}_\pm d'_+ \bar{b}'_+ - c_\pm c'_+ b''_+ = 0. \quad (70)$$

We note that the unknowns of Eqs. (68) are again the R-matrix elements $b_\pm, \bar{b}_\pm, c_\pm$ and thus we have to make linear combinations with the same three monomials functional relations. This leads us
to set to zero the determinants built out of either the plus or the minus channels of Eqs.\,(22,23,68).

After using the help of divisors (34,35) these vanishing conditions become,

\[
b_{\pm} c_{\pm} d_{\pm} \left( \Lambda_{1}^{\pm} c_{\pm} d_{\pm} g'' + a_{\pm} [d_{\pm}']^2 + [c_{\pm}']^2 f'' \right) - b_{\pm} c_{\pm} d_{\pm} \left( \Lambda_{1}^{\pm} c_{\pm} d_{\pm} g' + a_{\pm} [d_{\pm}']^2 + [c_{\pm}']^2 f' \right) = 0. \quad (71)
\]

Clearly, the above determinants give rise to polynomials of the factorized form (11) and the corresponding divisors are given by,

\[
\frac{\Lambda_{1}^{\pm} c_{\pm} d_{\pm} g + a_{\pm} d_{\pm}^2 + c_{\pm}^2 f_{\pm}}{b_{\pm} c_{\pm} d_{\pm}} = \Lambda_{5}^{\pm}. \quad (72)
\]

Because we have two divisors to eliminate a single weight \( g \) they must be compatible otherwise they will give origen to an extra constrain. This would be an undesirable situation since it will expoil the earlier bound (32) for the variety dimension. Fortunately, this compatibility can be achieved once the parameters \( \Lambda_{5}^{\pm} \) are fixed by the relation,

\[
\Lambda_{5}^{+} = \Lambda_{5}^{-} = \frac{1 + [\Lambda_{2}^2]^2}{\Lambda_{2}^+}, \quad (73)
\]

and by using Eqs.(36,37) the weight \( g \) is uniquely determined by the expression,

\[
g = \frac{a_{+} b_{+} \bar{b}_{+} + \Lambda_{2}^{+} (\Lambda_{2}^{+} a_{+} + \Lambda_{1}^{+} \bar{b}_{+})(c_{+}^2 - b_{+} \bar{b}_{+})}{\Lambda_{2}^{+} a_{+}(\Lambda_{2}^{+} + \Lambda_{1}^{+} \bar{b}_{+})}. \quad (74)
\]

In order to complete the solution of the above functional relations we just need to solve one of them for the matrix element \( g \). We can for instance extract this R-matrix entry from the plus component of Eq.(69) to obtain,

\[
\frac{g}{c_{+}} = \left[ \frac{b_{+} d_{+} g'' - d_{+} b_{+}'}{c_{+}'} - \bar{b}_{+} d_{+}' \right] / (d_{+}' \bar{b}_{-}''), \quad (75)
\]

where the ratios \( b_{+}/c_{+} \) and \( d_{+}/c_{+} \) are given by Eqs.(31,40).

By substituting the result (75) back into Eqs.(68,70) we find that they are either zero or produce polynomials depending on the variables \( a_{+}, b_{+}, \bar{b}_{+}, c_{+} \) and \( c_{-} \). In the latter case we use the hypersurface (39) to eliminate the even powers of the weight \( c_{-} \) and after this procedure they are immediately satisfied. At this point we mention that we still have to verify some additional functional relations involving four and five monomials which so far have not been mentioned in the
text. These extra relations are presented in Appendix A in which we have discussed the algebraic procedure used to check that they are indeed satisfied.

As a result the solution of the Yang-Baxter equation is determined by the intersection of the hypersurface \((39)\) with the expressions of the weights \(a_-\) and \(b_-\) given by Eqs.(53-59). This is clearly a complete intersection leading us to a three-dimensional variety whose polynomial can be written as,

\[
T(a_+, b_+, \bar{b}_+, c_+, c_-) = \left( c_+^2 - b_+ \bar{b}_+ \right)^2 \left[ \Lambda_2^+ (c_+^2 - b_+ \bar{b}_+) (\Lambda_2^+ a_+ + \Lambda_1^+ \bar{b}_+) - \Lambda_1^- \Lambda_2^+ a_+^2 b_+ 
- (\Lambda_2^+)^2 - 1 \right] a_+ b_+ \bar{b}_+ \right] - a_+^3 c_-^2 \left[ \Lambda_2^+ a_+ + \Lambda_1^+ \bar{b}_+ \right]^2,
\]

where \(\Lambda_1^+\) and \(\Lambda_2^+\) are free couplings while \(\Lambda_1^-\) is determined by these parameters using Eq.(67).

The transition operator weights other than those entering as variables in the above threefold can be determined with the help of the divisors \((38, 37, 36, 53, 59, 74)\). For sake of clarify we list below their explicit expressions \(^2\)

\[
d_+ = \frac{b_+ c_+}{\Lambda_2^+ a_+ + \Lambda_1^+ b_+}, \quad f_+ = \frac{b_+ \left( \Lambda_2^+ a_+ b_+ - \Lambda_1^+ \left( c_+^2 - b_+ \bar{b}_+ \right) \right)}{a_+ \left[ \Lambda_2^+ a_+ + \Lambda_1^+ b_+ \right]},
\]

\[
b_- = \frac{\bar{b}_+ \left[ c_+^2 - b_+ \bar{b}_+ \right]}{\Lambda_2^+ a_+ \left[ \Lambda_2^+ a_+ + \Lambda_1^+ b_+ \right]}, \quad \bar{b}_- = \frac{b_+ \left[ c_+^2 - b_+ \bar{b}_+ \right]}{\Lambda_2^+ a_+ \left[ \Lambda_2^+ a_+ + \Lambda_1^+ b_+ \right]},
\]

\[
g = \frac{a_+ b_+ \bar{b}_+ + \Lambda_2^+ \left[ c_+^2 - b_+ \bar{b}_+ \right] \left[ \Lambda_2^+ a_+ + \Lambda_1^+ b_+ \right]}{\Lambda_2^+ a_+ \left[ \Lambda_2^+ a_+ + \Lambda_1^+ b_+ \right]}, \quad h = \frac{a_+ c_+ c_-}{c_+^2 - b_+ \bar{b}_+},
\]

\[
a_- = \frac{\left[ c_+^2 - b_+ \bar{b}_+ \right] \left( \Lambda_2^+ \left[ c_+^2 - b_+ \bar{b}_+ \right] - \Lambda_1^- a_+ b_+ \right)}{a_+^2 \left[ \Lambda_2^+ a_+ + \Lambda_1^+ b_+ \right]}, \quad d_- = \frac{a_+ \bar{b}_+ c_-}{\Lambda_2^+ \left[ c_+^2 - b_+ \bar{b}_+ \right]},
\]

\[
f_- = \frac{\bar{b}_+ \left( \Lambda_1^- \Lambda_2^+ a_+^3 c_+^2 \left[ \Lambda_2^+ a_+ + \Lambda_1^+ \bar{b}_+ \right]^2 - \bar{b}_+ \left[ c_+^2 - b_+ \bar{b}_+ \right]^3 \right)}{\left[ \Lambda_2^+ \right]^2 \left[ \Lambda_2^+ a_+ + \Lambda_1^+ b_+ \right] \left[ c_+^2 - b_+ \bar{b}_+ \right]^2 \left( \Lambda_1^- a_+ b_+ - \Lambda_2^+ \left[ c_+^2 - b_+ \bar{b}_+ \right] \right)},
\]

The explicit form of the R-matrix as function of the weights with distinct labels are in general very cumbersome. In order to avoid overcrowding this section with extra heavier formulae we have

\[^2\text{We observe the existence of the simple relation } b_+ b_- = \bar{b}_+ \bar{b}_-.\]
presented the R-matrix elements in Appendix B. In what follows we shall however show that the threefold (76) is birationally equivalent to a rational variety. As a consequence we will be able to express both the transition operator and the R-matrix without the need of any algebraic constraint.

4 The Threefold Geometry

The understanding of the geometrical properties of a specific algebraic variety require in general its desingularization and this problem can be very difficult for high dimensional manifolds [17]. Despite of the fact that the threefold (76) is not normal because of the presence of two-dimensional singularities we have been able to resolve them with the help of birational morphisms. The concept of birational equivalence is unique to Algebraic Geometry and through these mappings basic invariants of the variety are preserved such as the generalization of the concept of geometric genus [9].

We recall that the singular locus of this threefold is determined by the zeroes set of first order partial derivatives on the variables \( a_+, b_+, \bar{b}_+, c_+ \) and \( c_- \). We find that they are constituted of three quadric surfaces,

\[
\text{Sing}(T) = \{[a_+:b_+:+c_+:c_-] \in \mathbb{CP}^4 | \phi_1(b_+, \bar{b}_+, c_+) = 0 \land \phi_2(a_+, \bar{b}_+) = 0 \lor a_+ = 0 \lor c_- = 0\},
\]

(82)

where the polynomials \( \phi_1(b_+, \bar{b}_+, c_+) = 0 \) and \( \phi_2(a_+, \bar{b}_+) \) are,

\[
\phi_1(b_+, \bar{b}_+, c_+) = c_+^2 - b_+\bar{b}_+, \quad \phi_2(a_+, \bar{b}_+) = \Lambda_2^+ a_+ + \Lambda_1^+ \bar{b}_+.
\]

(83)

The partial desingularization can be implemented by observing that the threefold (76) has two factorized terms both containing quadratic forms. By exploring this fact we are able to decrease the dimensionality of the singular locus and the degree of the image polynomial. This is done with the help of the following rational map,

\[
T(a_+, b_+, \bar{b}_+, c_+, c_-) \subset \mathbb{CP}^4 \xrightarrow{\phi} T_1(a_+, b_+, \bar{b}_+, c_+, c_-) \subset \mathbb{CP}^4
\]

(84)

\[
[a_+:b_+:+c_+:c_-] \longmapsto [a_+:b_+:+c_:c_+ : \frac{a_+c_- c_+ \phi_2(a_+, \bar{b}_+)}{\phi_1(b_+, \bar{b}_+, c_+)}].
\]
where the map image is the cubic threefold,

\[
T_1(a_+, b_+, \bar{b}_+, c_+, c_-) = \Lambda_2^+ \left[ c_+^2 - b_+ \bar{b}_+ \right] - a_+ c_+^2 - \Lambda_1^- a_+^2 b_+ - (\Lambda_2^+)^2 (a_+ b_+ \bar{b}_+) .
\]

(85)

We note that the map \( \phi \) is everywhere defined except at the closed subsets consisted of the singular locus (82) which is allowed by the notion of birational equivalence. Informally speaking, two varieties are considered birational if they are isomorphic up to lower dimensional subsets. This is exactly the case of the map \( \phi \) since it is bijective over the non-singular locus of the threefold (76) with the inverse,

\[
T_1(a_+, b_+, \bar{b}_+, c_+, c_-) \subset \mathbb{CP}^4 \quad \phi^{-1} \quad T(a_+, b_+, \bar{b}_+, c_+, c_-) \subset \mathbb{CP}^4
\]

(86)

\[
[a_+ : b_+ : \bar{b}_+ : c_+ : c_-] \quad \mapsto \quad \left[ a_+ : b_+ : \bar{b}_+ : \frac{c_+ \phi_1(b_+, \bar{b}_+, c_+, c_-)}{a_+ \phi_2(a_+, \bar{b}_+)} \right].
\]

Although the cubic threefold \( T_1 \) is still singular we have drastically reduced the dimension of its singular locus. In fact, the singularities are now made of three isolated points with the following projective coordinates,

\[
P_1 = [0 : 1 : 0 : 0 : 0] \quad \text{and} \quad P_\pm = [1 : 0 : \frac{1 - 2[\Lambda_2^+]^2 \pm i\sqrt{3}}{2\Lambda_1^- \Lambda_2^+} : 0 : 0].
\]

(87)

It is not difficult to show that any cubic threefold with isolated singularities is birationally equivalent to projective space \( \mathbb{CP}^3 \). This fact is a consequence of the property that a generic set of lines passing through a given singular point will have a further intersection point with the cubic threefold. Choosing \( P_1 \) as the singular point we can build the rational map,

\[
\mathbb{CP}^3 \rightarrow \psi \quad T_1(a_+, b_+, \bar{b}_+, c_+, c_-) \subset \mathbb{CP}^4
\]

(88)

\[
[a_+ : b_+ : c_+ : c_-] \quad \mapsto \quad \left[ a_+ : \frac{\psi_1(a_+, b_+, \bar{b}_+, c_+, c_-)}{\psi_2(a_+, b_+)} : \bar{b}_+ : c_+ : c_- \right],
\]

where the expressions of the polynomials \( \psi_1(a_+, b_+, \bar{b}_+, c_+, c_-) \) and \( \psi_2(a_+, \bar{b}_+) \) are,

\[
\psi_1(a_+, b_+, \bar{b}_+, c_+, c_-) = -a_+ c_-^2 + \Lambda_2^+ \left[ \Lambda_2^+ a_+ + \Lambda_1^+ \bar{b}_+ \right] c_+^2 ,
\]

\[
\psi_2(a_+, \bar{b}_+) = \Lambda_1^- \Lambda_2^+ a_+^2 + (2[\Lambda_2^+]^2 - 1) a_+ b_+ + \Lambda_1^+ \Lambda_2^+ \bar{b}_+^2 .
\]

(89)
The map $\psi$ is regular when restricted to dense open subsets of $\mathbb{CP}^3$ being once again birational. Its inverse is the projection,

$$T_1(a_+, b_+, \bar{b}_+, c_+, c_-) \subset \mathbb{CP}^4 \overset{\psi^{-1}}{\longrightarrow} \mathbb{CP}^3$$

$$[a_+ : b_+ : c_+] \longmapsto [a_+ : \bar{b}_+ : c_+] .$$

(90)

The composition of birational mappings keeps the birationality property and consequently we have been able to establish the following equivalence,

$$T(a_+, b_+, \bar{b}_+, c_+, c_-) \setminus \text{Sing}(T) \cong \mathbb{CP}^3 .$$

(91)

We shall next use the above correspondence to exhibit a rational parameterization of the transition operator.

### 4.1 Parameterized Transition Operator

We have now the basic ingredients to present an explicit expression for the transition operator without any algebraic constraint among the weights. The map $\phi$ compels us to introduce a new variable $c_-$ as follows,

$$c_- = \frac{a_+ c_- \phi_2(a_+, \bar{b}_+)}{\phi_1(b_+, \bar{b}_+, c_-)} ,$$

(92)

which now replaces $c_-$ as an independent variable.

The second map $\psi$ makes it possible the elimination of the weight $b_+$. Taking into account these steps together we are able to extract in a linear way the weights $b_+$ and $c_-$,

$$b_+ = \frac{\psi_1(a_+, \bar{b}_+, c_-, c_-)}{\psi_2(a_+, b_+)} , \quad c_- = c_- \frac{\psi_2(a_+, \bar{b}_+) - b_+ \psi_1(a_+, b_+, c_+, c_-)}{a_+ \phi_2(a_+, b_+) \psi_2(a_+, b_+)} .$$

(93)

After these transformations the transition operator elements become parameterized by four projective weights that is $a_+, \bar{b}_+, c_+$ and $c_-$. Since one of them can be used as normalization this solution has three affine independent spectral parameters. Considering the matrix structure
we can represent the transition operator as,

\[
L(\omega) = \begin{bmatrix}
a_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b_+ & c_+ & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & f_+ & 0 & d_+ & 0 & h & 0 \\
0 & c_+ & 0 & \bar{b}_+ & 0 & 0 & 0 & 0 \\
0 & 0 & d_+ & 0 & g & 0 & d_- & 0 \\
0 & 0 & 0 & 0 & \bar{b}_- & 0 & c_- & 0 \\
0 & 0 & h & 0 & d_- & 0 & f_- & 0 \\
0 & 0 & 0 & 0 & c_- & 0 & b_- & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_-
\end{bmatrix},
\]

where the components of the set \( \omega \) are the free variables \( a_+, \bar{b}_+, c_+ \) and \( c_- \).

The other entries of the transition operator are fixed by the components of the set \( \omega \) with the help of Eqs.\((77\text{--}81)\). By substituting the weights \( b_+ \) and \( c_- \) \((93)\) in the relations \((77\text{--}81)\) and after
performing few simplifications we obtain,

\[
d_+ = \frac{c_+ \psi_1(a_+, \bar{b}_+, c_+, c_-)}{\psi_2(a_+, b_+) \phi_2(a_+, b_+)}; \quad d_- = \frac{c_- \bar{b}_+}{\Lambda_2^- \phi_2(a_+, b_+)}; \quad h = \frac{c_- c_+}{\phi_2(a_+, b_+)}.,
\]

\[
f_+ = \frac{\psi_1(a_+, \bar{b}_+, c_+, c_-) [\phi_2(a_+, \bar{b}_+) \psi_1(a_+, \bar{b}_+, c_+, c_-) - \Lambda_1^+ c_+^2 \psi_2(a_+, \bar{b}_+)\]}{a_+ \phi_2(a_+, b_+) \psi_2(a_+, b_+)^2},
\]

\[
b_- = \frac{\bar{b}_+ [\Lambda_1^- \Lambda_2^+ a_+ c_+^2 + [c_+^2 + ([\Lambda_2^+]^2 - 1)c_+^2] \bar{b}_+]}{\Lambda_2^+ \phi_2(a_+, b_+) \psi_2(a_+, b_+)^2},
\]

\[
a_- = \frac{[c_+^2 \psi_2(a_+, \bar{b}_+) - \bar{b}_+ \psi_1(a_+, \bar{b}_+, c_+, c_-)] [\Lambda_1^- \Lambda_2^+ a_+ c_+^2 + ([\Lambda_2^+]^2 - c_+^2) \bar{b}_+]}{\Lambda_2^+ a_+ \phi_2(a_+, b_+) \psi_2(a_+, b_+)^2},
\]

\[
g = \frac{c_+^2}{a_+} + \frac{\bar{b}_+ \psi_1(a_+, \bar{b}_+, c_+, c_-) [a_+ - \Lambda_2^+ \phi_2(a_+, \bar{b}_+)\]}{\Lambda_2^+ a_+ \phi_2(a_+, b_+) \psi_2(a_+, b_+).}
\]

The parameterized expressions for the corresponding R-matrix elements are somehow involved since contains six free affine spectral parameters. For sake of completeness they have been also explicitely exhibited in Appendix B.

We conclude by noticing that the above transition operator is regular since there exists a special value \(\omega_0\) of the spectral variables such that \(L(\omega_0)\) becomes the permutator. Direct inspection of the transition operator entries reveals us that the projective coordinates of this point are,

\[
\omega_0 = [a_+ : \bar{b}_+ : c_+ : c_-] = [1 : 0 : 1 : \Lambda_2^+].
\]

making it possible the construction of local conserved charges being the Hamiltonian of the respective spin chain one of them.
4.2 The Spin-1 Hamiltonian

The Hamiltonian limit of the vertex model is obtained by expanding the logarithm of the transfer matrix around the regular point of the transition operator. Here we shall discuss the form of this operator in a general situation which is able to cover potential interesting submanifolds. To this end it is wise to consider the perturbation around the regular point directly on the weights of the degree seven threefold. The expansion giving the permutator at zero order is as follows,

\[ a_+ = 1 + \epsilon \dot{a}_+, \quad b_+ = 0 + \epsilon \dot{b}_+, \quad \bar{b}_+ = 0 + \epsilon \dot{\bar{b}}_+, \quad c_+ = 1 + \epsilon \dot{c}_+, \quad c_- = 1 + \epsilon \dot{c}_-, \quad (103) \]

where \( \epsilon \) denotes the expansion parameter.

The coefficients \( \dot{a}_+, \dot{b}_+, \dot{\bar{b}}_+, \dot{c}_+ \) and \( \dot{c}_- \) need to satisfy the threefold polynomial (76) up to the first order in the parameter \( \epsilon \). This requires that they are constrained by the relation,

\[ \Lambda_1^- \dot{b}_+ + \Lambda_1^+ \dot{\bar{b}}_+ + 4\Lambda_2^+ \dot{a}_+ + 2\Lambda_2^+ \dot{\bar{b}}_- - 6\Lambda_2^+ \dot{c}_+ = 0. \quad (104) \]

We shall see that combinations of these coefficients will play the role of two additional coupling constants for the Hamiltonian. This is a manifestation of the fact that the vertex model we are considering sits on a three-dimensional variety. To obtain the spin chain expression we have to expand the transition operator up to the first order on the parameter \( \epsilon \). The technical details of this computation have been summarized in Appendix C and in what follows we present only the final result. We find that the Hamiltonian is given by,

\[ H = \sum_{j=1}^{N} \left\{ (S_j^- S_{j+1}^+ + \frac{\dot{b}_-}{b_+} S_j^+ S_{j+1}^-) (1 + \frac{[\Lambda_1^+ - 1]^2}{\Lambda_2^+} S_j^z S_{j+1}^z) - \frac{\Lambda_1^+}{2} [S_j^+ S_{j+1}^+]^2 - \frac{\Lambda_1^-}{2} \frac{\dot{b}_+}{b_+} [S_j^+ S_{j+1}^-]^2 \right. \]

\[ + [\Lambda_2^+ - 1] \left( S_j^- S_j^+ S_{j+1}^- + \frac{\dot{b}_+}{b_+} S_j^+ S_j^+ S_{j+1}^- + \frac{[\Lambda_2^+ - 1]}{\Lambda_2^+} (S_j^- S_{j+1}^+ S_j^z S_{j+1}^- + \frac{\dot{b}_+}{b_+} S_j^+ S_j^z S_{j+1}^- S_{j+1}^-) \right) \]

\[ - (\Lambda_1^+ + \frac{\Lambda_1^- \dot{b}_+}{b_+}) |S_j^z|^2 + 2\frac{\Lambda_2^+}{b_+} (\dot{c}_+ - \dot{c}_-) S_j^z \}, \quad (105) \]

where \( S_j^+, S_j^- \) and \( S_j^z \) are the spin-1 matrices of the SU(2) algebra.

In order to compare the Hamiltonian (105) with the one found previously by Crampé, Frappat and Ragoucy [13] we just have to apply a gauge transformation on our solution of the Yang-Baxter
equation. This correspondence together with the explicit matching of the Hamiltonian couplings can also be found in Appendix C.

An interesting feature of deriving the Hamiltonian from the integrable vertex model is that it tells us the presence of an arbitrary azimuthal magnetic field is directly connected with the asymmetry among the weights \( c_+ \) and \( c_- \). This means that the simplest way to obtain a Hamiltonian in absence of any linear magnetic field is to consider the threefold subvariety,

\[
c_- - c_+ = 0. \tag{106}
\]

The physical properties of the spin chain is expected to be dependent whether or not we have the presence of the azimuthal magnetic field. It is conceivable that this fact should be reflected as a drastic change of the geometrical properties of the variety underlying the vertex model. We shall next confirm this feeling by investigating the geometry of the hyperplane section \((106)\).

5 The Integrable Submanifold

The weights of the vertex model whose Hamiltonian limit does not contain any azimuthal magnetic field sits on the intersection of \( c_+ - c_- = 0 \) with the original threefold \((76)\). This leads us to a surface defined by the polynomial,

\[
S(a_+, b_+, \bar{b}_+, c_+) = (c_+^2 - b_+ \bar{b}_+)^2 \left[ \Lambda_+^2 (c_+^2 - b_+ \bar{b}_+) (\Lambda_2^+ a_+ + \Lambda_1^+ \bar{b}_+) - \Lambda_1^2 \Lambda_2^+ a_+^2 b_+ \right]
\]

\[
- \left( [\Lambda_2^+]^2 - 1 \right) a_+ b_+ \bar{b}_+ - a_+^2 \Lambda_2^+ [\Lambda_2^+ a_+ + \Lambda_1^+ \bar{b}_+]^2, \tag{107}
\]

which has two non coplanar singular lines given by,

\[
\text{Sing}(S) = \{(a_+ : 0 : \bar{b}_+ : 0) \cup [a_+ : b_+ : 0 : 0]\}. \tag{108}
\]

The classification of surfaces has been mostly done by Enriques who divided them into four basic families according to what nowadays is called Kodaira dimension. In order to find out which family the above surface belongs we have to perform its desingularization again with the help of
birational maps. As before the basic idea is lower the dimensionality of the singular locus as well as the degree of the image polynomial. Inspired by what has been done for the threefold we are able to propose the following rational map,

\[
S(a_+, b_+, \bar{b}_+, c_+) \subset \mathbb{CP}^3 \xrightarrow{\sigma} S_1(a_+, b_+, \bar{b}_+, c_+) \subset \mathbb{CP}^3
\]

where the respective target variety is a surface of degree four,

\[
S_1(a_+, b_+, \bar{b}_+, c_+)= \Lambda_1^2 - \Lambda_2^2 (\Lambda_2^2 a_+^2 - b_+ c_+) a_+ c_+ + \Lambda_2^2 (\Lambda_1^2 - 2[\Lambda_2^2] - 1) a_+ \bar{b}_+ c_+ + (1 - [\Lambda_2^2]) b_+ \bar{b}_+ c_+ + \Lambda_1^4 3[\Lambda_2^2] - 1) a_+ \bar{b}_+ c_+ + [\Lambda_1^2] \Lambda_2^2 \bar{b}_+ c_+ - b_+^2 \bar{b}_+.
\]

(110)

We see that the above map satisfies the expected properties of a birational equivalence onto its image. In fact, the map \(\sigma\) only fails on lower dimensional subsets of the surface (107) and it is easily invertible,

\[
S_1(a_+, b_+, \bar{b}_+, c_+) \subset \mathbb{CP}^3 \xrightarrow{\sigma^{-1}} S(a_+, b_+, \bar{b}_+, c_+) \subset \mathbb{CP}^3
\]

where the respective target variety is a surface of degree four,

\[
S_1(a_+, b_+, \bar{b}_+, c_+)\]

We now are left to analyze the geometrical properties of the birationally equivalent surface (110). This surface is fortunately normal because its singularities consist of four isolated points with the following projective coordinates,

\[
P_1 = [0 : 0 : 0 : 1], \quad P_2 = [1 : 0 : -\Lambda_2^2 / \Lambda_1^2 : 0], \quad \text{and} \quad P_\pm = [1 : 0 : 1 - 2[\Lambda_2^2] \pm i\sqrt{3} / 2\Lambda_1^4 \Lambda_2^2 : 0].
\]

(112)

It is known that the minimal model of a normal quartic surface can be either the celebrated K3 surface, a ruled surface over an elliptic curve or still a rational surface [18]. In order to decide the actual class of the surface (110) we have to investigate the local topological behavior of the singular points according to the classification by Arnold et al. [19]. This problem can be sorted out for example within the computer algebra system Singular [20] which enables us to compute
the singularities modality. We found out that the singular points \((\mathbf{112})\) are canonical rational double points whose local forms are that of the so-called \(A_2\) singularities \([19]\). As long as the singularities are rational the minimal resolutions of singular quartics are known to be K3 surfaces see for instance \([21]\). Technically speaking, such resolutions do not affect adjunction and the geometric properties are equivalent to that of a smooth quartic which is the simplest type of K3 surface. This discussion together with the map \(\sigma\) are sufficient to characterize the birational class of the degree seven surface \((\mathbf{107})\) that is,

\[
S(a_+, b_+, b^\prime_+, c_+) \setminus \text{Sing}(S) \cong \text{K3 surface}.
\]

For practical purposes it is of clear interest to have the explicit form of the transition operator with weights constrained by the K3 surface with the lowest possible degree. This can be achieved with the help of the birational map \(\sigma\) which makes it possible to replace the weight \(b_+\) by a new variable \(b^\prime_+\) as follows,

\[
b^\prime_+ = \frac{c_+ [b_+ c_+ - a_+ \phi_2(a_+, b^\prime_+)]}{b_+ b^\prime_+}.
\]

After this transformation the transition operator elements will become dependent on the projective coordinates \(a_+, b_+, b^\prime_+\) and \(c_+\). Its matrix representation is as before except that now \(c_- = c_+\), namely

\[
L(\omega) = \begin{bmatrix}
a_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b_+ & 0 & c_+ & 0 & 0 & 0 & 0 \\
0 & 0 & f_+ & 0 & d_+ & 0 & h & 0 \\
0 & c_+ & 0 & \bar{b}_+ & 0 & 0 & 0 & 0 \\
0 & 0 & d_+ & 0 & g & 0 & d_- & 0 \\
0 & 0 & 0 & 0 & 0 & \bar{b}_- & 0 & c_+ \\
0 & 0 & h & 0 & d_- & 0 & f_- & 0 \\
0 & 0 & 0 & 0 & 0 & c_+ & 0 & b_- \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_-
\end{bmatrix},
\]

where now \(\omega\) stands for the variables \(a_+, b_+, b^\prime_+\) and \(c_+\).
The respective matrix elements can be computed by substituting the transformation (114) in the general relations (77-81) and as a result we obtain,

\[
d_+ = c^2_+ \frac{[b_+ c_+ - a_+ \phi_2(a_+, \bar{b}_+)]}{b_+ \bar{b}_+ \phi_2(a_+, b_+)}, \quad b_- = \frac{\bar{b}_+ c_+}{\Lambda^+_2 b_+}, \quad d_- = \frac{b_+ \bar{b}_+}{\Lambda^+_2 \phi_2(a_+, b_+)},
\]

(116)

\[
f_+ = \frac{c^2_+}{b_+} \frac{[b_+ c_+ - a_+ \phi_2(a_+, \bar{b}_+)][\Lambda^+_2 b_+ c_+ - \phi_2(a_+, \bar{b}_+)^2]}{[b_+ b_+]^2 \phi_2(a_+, b_+)}, \quad b_- = \frac{\bar{b}_+ c_+}{\phi_2(a_+, b_+)}, \quad h = \frac{b_+ c_+}{\phi_2(a_+, b_+)},
\]

(117)

\[
g = \frac{c^2_+ [b_+ c_+ - a_+ \phi_2(a_+, \bar{b}_+) + \Lambda^+_2 \phi_2(a_+, \bar{b}_+)^2]}{\Lambda^+_2 b_+ \phi_2(a_+, b_+)}, \quad \bar{b}_- = \frac{c^2_+ [b_+ c_+ - a_+ \phi_2(a_+, \bar{b}_+)]}{\Lambda^+_2 b_+ \phi_2(a_+, b_+)},
\]

(118)

\[
a_- = \frac{c^2_+ [\Lambda^+_2 \bar{b}_+ + \Lambda^-_1 a_+ \phi_2(a_+, \bar{b}_+) - \Lambda^-_1 b_+ c_+]}{b_+ \bar{b}_+}, \quad \bar{b}_- = \frac{c^2_+ [b_+ c_+ - a_+ \phi_2(a_+, \bar{b}_+)]}{\Lambda^+_2 b_+ \phi_2(a_+, b_+)},
\]

(119)

\[
f_- = \frac{\bar{b}_+^2 [\bar{b}_+ c_+ \phi_2(a_+, \bar{b}_+) - \Lambda^-_1 b_+^3]}{[\Lambda^+_2]^2 c_+ \phi_2(a_+, b_+) (\Lambda^+_2 \bar{b}_+ + \Lambda^-_1 a_+ \phi_2(a_+, \bar{b}_+) - \Lambda^-_1 b_+ \phi_2(a_+, b_+) - \Lambda^-_1 b_+ c_+)}. \quad (120)
\]

Here we emphasize that the variables \(a_+, b_+, \bar{b}_+\) and \(c_+\) are constrained by the quartic K3 surface \(S_1(a_+, b_+, \bar{b}_+, c_+)\). In other words the polynomial given by Eq.(110) in which the variable \(b_+\) is replaced by the new auxiliary weight \(b_+\). We finally remark that the respective R-matrix entries as function of such variables have been presented in Appendix B. Although we have the symmetry \(c_+ = c_-\) for the transition operator the same does not occur for the equivalent entries of the R-matrix. This probably is related to the fact that K3 surfaces do not have an underlying group structure.

6 Conclusions

In this work we have discussed some guidelines to search for solutions of the Yang-Baxter equation from the point of view of Algebraic Geometry. We have applied this approach in the case of nineteen vertex models with the time-reversal symmetry which restrict the parameters space to fourteen distinct Boltzmann weights. Even in this subspace we have to deal with a large number of independent functional relations being therefore a satisfactory test of the practical utility of our
framework. We have been able to uncover a family of such integrable models lying on a degree seven algebraic threefold whose polynomial coefficients depends on two free couplings. By means of birational mappings we have shown that this variety is equivalent to the projective space $\mathbb{CP}^3$. These transformations are used to express both the transition operator and the R-matrix in a parameterized form with three independent affine spectral variables. We have verified explicitly that the R-matrix satisfies the Yang-Baxter equation for three distinct sets of non-additive spectral parameters.

We have discussed the Hamiltonian limit of this family of nineteen vertex model and show how it can be related to the spin-1 chain discovered in the previous work [13]. We have argued that the presence of an arbitrary azimuthal magnetic field in the Hamiltonian comes from the fact that it is proportional to the asymmetry between two kinds of spectral weights. This prompted us to define the physical interesting submanifold in which the magnetic field is absent. It turns out that the geometric properties of such submanifold are on the class of the famous K3 surfaces. This finding has enlarged in substantial way the type of non-rational varieties that can emerge as solution of the Yang-Baxter equation. A natural question to be asked is whether this is an isolated result or actually a tip of an iceberg? At least from the perspective of Algebraic Geometry K3 surfaces can exist in vast quantities as certain sections of a family of threefolds called Fano [22]. This suggests that the answer of the above question is somehow related to the existence or not of an abundance of integrable models sitting on three-dimensional Fano varieties.

Finally, we believe that the algebraic approach discussed in this work could be extended to provide the complete classification of three-states vertex models invariant by the U(1) symmetry together with the proper identification of the respective algebraic varieties. We hope to report on this problem in a forthcoming work.

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Appendix A: Extra Functional Relations

In the main text we have solved thirty-nine functional equations out of the fifty-seven relations coming from the Yang-Baxter equation (4). The remaining eighteen functional equations mixed several weights and R-matrix elements previously determined. Their explicit expression are given by,

\[ \begin{align*}
    c_\pm d_\pm^\prime a_\pm^\prime' - ha_\pm^\prime d_\pm^\prime' - f_\pm h_\prime d_\pm^\prime' - d_\pm a_\pm^\prime' g_\prime = 0, \\
    a_\pm d_\pm^\prime c_\pm^\prime' - gc_\pm^\prime d_\pm^\prime' - d_\mp h_\prime f_\mp^\prime' - d_\mp a_\mp^\prime' h_\prime' = 0, \\
    c_\mp f_\mp^\prime c_\mp^\prime' - d_\mp g_\prime d_\mp^\prime' - h c_\mp^\prime f_\mp^\prime' - f_\mp c_\mp^\prime h_\prime = 0, \\
    c_\mp g_\prime^\prime c_\mp^\prime' - d_\mp a_\mp d_\mp^\prime' - d_\mp h_\prime f_\mp^\prime' - g c_\mp^\prime g_\prime + b_\mp c_\mp^\prime \tilde{b}_\mp = 0, \\
    h d_\mp^\prime c_\mp^\prime' - g c_\mp^\prime d_\mp^\prime' - d_\mp a_\mp^\prime f_\mp^\prime' - d_\mp h_\prime f_\mp^\prime' + d_\pm \tilde{b}_\mp' \tilde{b}_\mp = 0, \\
    f_\mp a_\mp^\prime d_\mp^\prime' - b_\mp b_\mp^\prime d_\mp^\prime' + h h_\prime d_\mp^\prime' + d_\pm c_\pm^\prime g_\prime - c_\mp d_\mp^\prime' h_\prime' = 0, \\
    \tilde{b}_\mp c_\mp^\prime \tilde{b}_\mp + c_\mp h_\prime c_\mp^\prime' - d_\mp g_\prime d_\mp^\prime' - f_\mp c_\mp^\prime f_\mp^\prime' - h c_\mp^\prime h_\prime = 0, \\
    g g_\prime d_\mp^\prime' - \tilde{b}_\mp \tilde{b}_\mp^\prime d_\mp^\prime' + d_\mp c_\mp^\prime f_\mp^\prime' - c_\mp c_\mp^\prime d_\mp^\prime' + d_\pm c_\pm^\prime h_\prime' = 0, \\
    d_\mp b_\mp^\prime b_\mp^\prime' + g d_\mp^\prime c_\mp^\prime' - h c_\mp^\prime d_\mp^\prime' - f_\mp c_\mp^\prime d_\mp^\prime' - d_\mp g_\prime g_\prime' = 0.
\end{align*} \]

(A.1) \hspace{1cm} (A.2) \hspace{1cm} (A.3) \hspace{1cm} (A.4) \hspace{1cm} (A.5) \hspace{1cm} (A.6) \hspace{1cm} (A.7) \hspace{1cm} (A.8) \hspace{1cm} (A.9)

By substituting the data of the solution described in section (3) we are going to end with eighteen polynomials depending solely on the weights \(a_+, b_+, \tilde{b}_+, c_+\) and \(c_-\). The main point is that the dependence of such polynomials on the variable \(c_-\) appears only by means of even powers. It turns out that the lowest possible power \(c_-^2\) can be extracted from the threefold (76). Denoting this power by \(\text{aux}\) it is given by,

\[ \text{aux} = \left[ (-1 + \left[ \Lambda^+_2 \right]^2 - \left[ \Lambda^+_2 \right]^4) a_+^2 b_+ + \Lambda^+_1 \left[ \Lambda^+_2 \right]^2 (c_-^2 - b_+ \tilde{b}_+) (\Lambda^+_2 a_+ + \Lambda^+_1 \tilde{b}_+) + \Lambda^+_1 \Lambda^+_2 (1 - \left[ \Lambda^+_2 \right]^2) a_+ b_+ \tilde{b}_+ + \right] \\
\times \left[ \frac{(c_-^2 - b_+ \tilde{b}_+)^2}{\Lambda^+_1 \Lambda^+_2 a_+^2 (\Lambda^+_2 a_+ + \Lambda^+_1 \tilde{b}_+)^2} \right]. \]  

(A.10)

The main idea is to replace the even powers on the weight \(c_-\) by the above auxiliary variable in a nested way starting from the highest power. Direct inspection of the polynomials associated to Eqs. (A.1)-(A.9) reveals us that the highest power is \(c_-^6\). Denoting a given functional equation by
eq[*] we can replace the powers on the weight $c_-$ by using the following Mathematica code,

\[
\begin{align*}
eq1 & = \text{Factor}[eq[*]], \\
eq2 & = \text{Factor}[eq1 /. \{[c']^6 \to \text{aux'}, [c'']^4 \to \text{aux''} [c'']^4\}], \\
eq3 & = \text{Factor}[eq2 /. \{[c']^4 \to \text{aux'}, [c'']^4 \to \text{aux''} [c'']^2\}], \\
eq\text{end} & = \text{Factor}[eq3 /. \{[c']^2 \to \text{aux'}, [c'']^2 \to \text{aux''}\}],
\end{align*}
\]

where $\text{aux'}$ and $\text{aux''}$ are given by Eq.(A.10) with weights labeled by ’ and ’” respectively.

It is not difficult to check that the simplified relation $\text{eq}\text{end}$ is always zero. In this way we are able to verify that the whole Yang-Baxter equation is algebraically verified.

**Appendix B: The $R$-matrix**

The $R$-matrix depends on both sets of spectral variables $\omega$ and $\omega'$ and without loss of generality it can be normalized by the element $c_+$. Its matrix representation becomes,

\[
\begin{pmatrix}
a_+(\omega, \omega') & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b_+(\omega, \omega') & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & f_+(\omega, \omega') & 0 & d_+(\omega, \omega') & 0 & h(\omega, \omega') & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_+(\omega, \omega') & 0 & g(\omega, \omega') & 0 & d_-(\omega, \omega') & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & f_-(\omega, \omega') & 0 & c_-(\omega, \omega') \\
0 & 0 & h(\omega, \omega') & 0 & d_-(\omega, \omega') & 0 & f_-(\omega, \omega') & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_-(\omega, \omega') & 0 & b_-(\omega, \omega') \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_-(\omega, \omega')
\end{pmatrix}.
\]

(B.1)

Let us first discuss the form of the $R$-matrix in the generic situation where the components of the set $\omega$ are the variables $a_+, b_+, \bar{b}_+, c_+, c_-$ which satisfy the threefold polynomial (76). We first eliminate the even powers on the variables $c_-$ and $c'_-$ of the $R$-matrix elements as explained in the previous Appendix. After this step we still end up with some complicated polynomials but we find that they can be expressed in closed forms once we define the auxiliary bi-homogeneous
polynomial,

\[
Q(a_+, a'_+, b_+, b'_+, \bar{b}_+, \bar{b}'_+, c_+, c'_+) = \Lambda_1 \Lambda_2^+ a_+ b_+ a'_+ b'_+ + (1 - [\Lambda_2^+]^2) \left( c^2_+ a'_+ b'_+ - b_+ \bar{b}_+ a'_+ b'_+ \right) - [\Lambda_2^+]^2 \left( a_+ b_+ [c'_+]^2 - a_+ b'_+ \bar{b}'_+ \right) + \Lambda_1^+ \Lambda_2^+ \left( c^2_+ - b_+ \bar{b}_+ \right) \left( [c'_+]^2 - b'_+ \bar{b}'_+ \right). \tag{B.2}
\]
Considering the above polynomial the expressions of the R-matrix are given by,

\[ a_+(\omega, \omega') = \frac{b_+ a'_+ b_+ + a_+ \phi_1(b'_+, b'_+, c'_+)}{c_+ a'_+ c'_+}, \quad b_+(\omega, \omega') = \frac{b_+ a'_+ - a_+ b_+}{c_+ c'_+}, \]  
\[ (B.3) \]

\[ b_+(\omega, \omega') = \frac{a_+ b_+ \phi_1(b'_+, b'_+, c'_+) - \phi_1(b_+, b_+, c_+) a'_+ b_+}{c_+ a'_+ c'_+}, \]  
\[ (B.4) \]

\[ d_+(\omega, \omega') = -\frac{b_+(\omega, \omega') c_+ [a'_+]^2 c'_+ \phi_2(a'_+, b_+)}{\phi_1(b'_+, b'_+, c'_+) Q(a_+, a'_+, 1, b'_+, b_+, b_+', 0, c'_+)}, \]  
\[ (B.5) \]

\[ f_+(\omega, \omega') = \frac{b_+(\omega, \omega') Q(a_+, a'_+, b_+, b_+', b_+, b_+', b_+, c_+)}{a_+ Q(a_+, a'_+, 1, b'_+, b_+, b_+', 0, c'_+)}, \]  
\[ (B.6) \]

\[ h(\omega, \omega') = \frac{[a'_+]^2 [c'_+]^2 \phi_1(b_+, b_+, c_+) \phi_2(a'_+, b_+)}{a_+^2 c_+ \phi_1(b'_+, b'_+, c'_+) \phi_2(a'_+, b_+) Q(a_+, a'_+, 1, b'_+, b_+, b_+', 0, c'_+)} \]  
\[ (B.7) \]

\[ b_-(\omega, \omega') = -\frac{b_-(\omega, \omega') a'_+ [a_+ b_+ b'_+ + \phi_1(b_+, b_+, c_+)]}{a_+ Q(a_+, a'_+, 1, b'_+, b_+, b_+', 0, c'_+)}, \]  
\[ (B.8) \]

\[ c_-(\omega, \omega') = -\frac{c_- [a'_+]^2 c'_+ \phi_2(a'_+, b_+)}{c_+ c'_+ \phi_1(b_+, b_+, c_+) \phi_1(b'_+, b'_+, c'_+) Q(a_+, a'_+, 1, b'_+, b_+, b_+', 0, c'_+)}, \]  
\[ (B.9) \]

\[ d_-(\omega, \omega') = -\frac{b_+(\omega, \omega') a_+ c_- a'_+ c'_+ \phi_2(a_+, b_+)}{\phi_1(b_+, b_+, c_+) Q(a_+, a'_+, 1, b'_+, b_+, b_+', 0, c'_+)}, \]  
\[ (B.10) \]

\[ f_-(\omega, \omega') = \frac{b_+(\omega, \omega') a'_+ Q(a_+, a'_+, 1, 1, b'_+, b_+', 0, 0)}{Q(a_+, a'_+, 1, b'_+, b_+, b_+', 0, c'_+)}, \]  
\[ (B.11) \]

\[ a_-(\omega, \omega') = \frac{a'_+ [Q(a_+, a'_+, b_+, b_+', b_+', c_+, 0) - a_+ b_+ b'_+ - \phi_1(b_+, b_+, c_+) a'_+]}{a_+^2 c_+ c'_+ Q(a_+, a'_+, 1, b'_+, b_+, b_+', 0, c'_+)} \times \left[ a_+ b_+ b'_+ + \phi_1(b_+, b_+, c_+) a'_+ \right], \]  
\[ (B.12) \]

\[ g(\omega, \omega') = -\frac{b_+(\omega, \omega') Q(a_+, a_+, b_+, b_+', b_+', b_+', b_+', c_+, c_+)}{a_+ Q(a_+, a'_+, 1, b'_+, b_+, b_+', 0, c'_+)} + \frac{c_+ Q(a_+, a'_+, 1, b'_+, b_+, b_+', 0, c'_+)}{c'_+ Q(a_+, a'_+, 1, b'_+, b_+, b_+', 0, c'_+)}, \]  
\[ (B.13) \]
where the polynomials $\phi_1(b_+, \bar{b}_+, c_+) \text{ and } \phi_2(a_+, \bar{b}_+)$ have been defined in Eq.(83).

It is not difficult to check that this R-matrix indeed satisfy the regularity condition (6). By a systematic use of the threefold (76) as explained in the previous Appendix we are able to verified it also satisfies the Yang-Baxter equation for non-additive operators, namely

$$R_{12}(\omega, \omega')R_{13}(\omega, \omega'')R_{23}(\omega', \omega'') = R_{23}(\omega', \omega'')R_{13}(\omega, \omega'')R_{12}(\omega, \omega')$$ (B.14)

defined on three distinct sets of weights lying on the threefold (76). As usual the subscript labels of the R-matrix indicates its non-trivial action on the product of the three distinct auxiliary spaces. Recall that this relation is a sufficient condition for the associativity of the quadratic algebra we have started with (4).

- **Parameterized R-matrix**

The threefold (76) was shown to be birationally equivalent to the projective space $\mathbb{CP}^3$ and therefore it is possible to present the R-matrix entries as function of four free parameters. In this situation we recall that the set $\omega$ is constituted of the independent variables $a_+, \bar{b}_+, c_+, c_-$. Once again we find that the R-matrix elements can be better expressed with the help of an auxiliary polynomial, namely

$$Q_1(a_+, a'_+, \bar{b}_+, \bar{b}'_+, x_1, x_2, y_1, y_2) = \left[\Lambda_1^{-1} \Lambda_2^+ a_+ a'_+ + (-1 + [\Lambda_2^+]^2) \bar{b}_+ a'_+ + \Lambda_2^+ (\Lambda_2^+ a_+ + \Lambda_1^+ \bar{b}_+) \bar{b}_+ \right] (x_1 y_2)^2 + \left[-\Lambda_1^{-1} \Lambda_2^+ a_+ a'_+ + a_+ \bar{b}_+ - \Lambda_2^+ (\Lambda_2^+ \bar{b}_+ a'_+ + \Lambda_2^+ a_+ \bar{b}_+ + \Lambda_1^+ \bar{b}_+ \bar{b}_+) \right] (x_2 y_1)^2 + \left[\bar{b}_+ a'_+ - a_+ \bar{b}_+ \right] [(x_1 x_2)^2 + (y_1 y_2)^2],$$ (B.15)

where $x_1, x_2, y_1, y_2$ are parameters taken values on the specific set $0, 1, c_-, c'_-.$

Considering the above polynomial and after some cumbersome simplifications we find that the
parameterized expressions for the matrix elements are,

\[
a_+(\omega, \omega') = -\frac{Q_1(a_+, a'_+, \bar{b}_+, \bar{b}'_+, 0, c'_+, 1, c')}{c_+ c'_+ \psi_2(a'_+, b'_+)}, \quad b_+(\omega, \omega') = \frac{Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, c_+, c'_+, c_-, c_-)}{c_+ c'_+ \psi_2(a_+, b_+) \psi_2(a'_+, b'_+)}, \tag{B.16}
\]

\[
\bar{b}_+(\omega, \omega') = \frac{\bar{b}_+ a'_- - a_+ \bar{b}_+^\prime}{c_+ c'_+}, \quad d_+(\omega, \omega') = \frac{c'_- Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, c_+, c'_+, c_-, c_-)}{c_+ c'_+ Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, 1, c_+, 0, c'_-) \psi_2(a_+, b_+)} \tag{B.17}
\]

\[
f_+(\omega, \omega') = \left[ \frac{Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, c_-, c'_+, c_+, c'_-)}{c_+ c'_+ Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, 1, c_+, 0, c'_-) \psi_2(a'_+, b'_+) [\psi_2(a_+, b_+)^2]} \right] \times \frac{Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, c_+, c'_-, c_-, c'_-)}{c_+ c'_+ Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, 1, c_+, 0, c'_-) \psi_2(a_+, b_+)^2}, \tag{B.18}
\]

\[
a_-(\omega, \omega') = -\frac{Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, c_+, c_-, 1, 0, c_-) Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, c_+, 0, c_-, 1 \psi_2(a'_+, b'_+) \psi_2(a_+, b_+)^2}{c_+ c'_+ Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, 1, c'_-, 0, c'_-)}, \tag{B.19}
\]

\[
b_- (\omega, \omega') = \frac{[\bar{b}_+ a'_- - a_+ \bar{b}_+^\prime]}{c_+ c'_+ Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, c_+, c_-, 1, 0, c_-) Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, c_+, 0, c_-, 1 \psi_2(a'_+, b'_+) \psi_2(a_+, b_+)^2}{c_+ c'_+ Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, 1, c'_-, 0, c'_-)} \tag{B.20}
\]

\[
\bar{b}_- (\omega, \omega') = \frac{Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, c_+, c'_-, c_-, c'_-) Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, c_+, 0, c_-, 1, \psi_2(a'_+, b'_+) \psi_2(a_+, b_+)^2}{c_+ c'_+ Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, 1, c'_-, 0, c'_-)}, \tag{B.21}
\]

\[
c_- (\omega, \omega') = \frac{c_- c'_- Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, c_+, 0, c_-, 1) \psi_2(a'_+, b'_+) \psi_2(a_+, b_+)^2}{c_+ c'_+ Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, 1, c'_-, 0, c'_-)} \tag{B.22}
\]

\[
d_- (\omega, \omega') = \frac{c_- [\bar{b}_+ a'_- - a_+ \bar{b}_+^\prime] \psi_2(a'_+, \bar{b}_+^\prime)}{c_+ Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, 1, c'_-, 0, c'_-)} \times \frac{c_- \psi_2(a'_+, b'_+)}{Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, 1, c'_-, 0, c'_-)}, \tag{B.23}
\]

\[
f_- (\omega, \omega') = \frac{[\bar{b}_+ a'_- - a_+ \bar{b}_+^\prime] Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, 0, 1, 1, 1) \psi_2(a'_+, \bar{b}_+^\prime)}{c_+ c'_+ Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, 1, c'_-, 0, c'_-)} \tag{B.24}
\]

\[
g(\omega, \omega') = -\frac{Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, c_+, c'_-, c_-, c'_-) Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, 0, 1, 1, 1) + [c_- c'_-] \psi_2(a_+, \bar{b}_+) \psi_2(a'_+, \bar{b}_+^\prime)}{c_+ c'_+ Q_1(a_+, a'_+, \bar{b}_+^\prime, \bar{b}_+^\prime, 1, c'_-, 0, c'_-) \psi_2(a_+, b_+)} \tag{B.25}
\]

- **Submanifold R-matrix**

In the submanifold $c_- = c_+$ we have shown that the underlying variety is birational equivalent.
to the quartic K3 surface $S_1(a_+, b_+, \bar{b}_+, c_+)$ defined by the polynomial (110). In this situation the components of the set $\omega$ are given by the weights $a_+, b_+, \bar{b}_+, c_+$ and the auxiliary bi-homogenous polynomial turns out to be,

\[
Q_2(a_+, a'_+, b_+, b'_+, \bar{b}_+, c_+, c'_+) = [\Lambda_1^+]^2 (2[\Lambda_2^+][\Lambda_2^-] - 1) \bar{b}_+^2 a'_+ b'_+ + [\Lambda_1^+]^3 \Lambda_2^+ \bar{b}_+ [b'_+]^2 \\
+ [\Lambda_2^+]^2 a_+^2 \left( \Lambda_1^- [a'_+]^2 + \Lambda_1^+ [b'_+]^2 \right) + \Lambda_2^+ ([\Lambda_2^+]^2 + \Lambda_1^- \Lambda_1^+ - 1) a_+ b_+ \left( \Lambda_2^+ [a'_+]^2 - b'_+ c'_+ \right) \\
- \Lambda_1^- [\Lambda_2^+]^2 \left( a_+^2 b'_+ c'_+ + b_+ c_+ [a'_+]^2 \right) + \Lambda_1^+ \Lambda_2^+ \left( \Lambda_1^- \Lambda_1^+ + 3[\Lambda_2^+]^2 - 1 \right) a_+ b_+ a'_+ b'_+ \\
+ \Lambda_2^+ \left( \Lambda_1^- \Lambda_1^+ + [\Lambda_2^+]^2 \right) \left( \Lambda_2^+ a_+^2 - b_+ c_+ \right) a'_+ b'_+ + 2[\Lambda_1^+ \Lambda_2^+]^2 a_+ b_+ [b'_+]^2 \\
+ \Lambda_1^+ \left( [\Lambda_2^+]^2 - 1 \right) \bar{b}_+^2 \left( \Lambda_2^+ [a'_+]^2 - b'_+ c'_+ \right) + \Lambda_2^+ b_+ c_+ \left( \Lambda_1^- b'_+ c'_+ - \Lambda_1^+ \Lambda_2^+ [b'_+]^2 \right).
\]

(B.26)

The expressions of the entries of the R-matrix are simplified by using systematically the quartic
K3 surface $S_1(a_+, b_+, \bar{b}_+, c_+)$. The final results are,

$$a_+(\omega, \omega') = \frac{\bar{b}_+ b'_+ c'_+ - \bar{b}_+ a'_+ - a_+ b'_+}{c_+ b'_+ b'_+} \phi_2(a'_+, \bar{b}'_+), \quad b_+(\omega, \omega') = \frac{\bar{b}_+ a'_+ - a_+ \bar{b}'_+}{c_+ c'_+}, \quad (B.27)$$

$$b_+(\omega, \omega') = \frac{\bar{b}_+ \phi_2(a_+, \bar{b}_+)[\Lambda_1^+ [a'_+^2 - b'_+ c'_+] - [\Lambda_2^+ a_+^2 - b_+ c_+] \phi_2(a'_+, \bar{b}'_+) \bar{b}'_+}{b_+ b'_+ b'_+}, \quad (B.28)$$

$$b_-(\omega, \omega') = \frac{\bar{b}_+ \phi_2(a_+, \bar{b}_+)[a'_+ - a_+ \bar{b}'_+] \phi_2(a_+, \bar{b}_+ + b_+ c_+ \bar{b}'_+)}{b_+ b'_+ c_+ Q_2(a_+, a'_+, 1, b'_+, b_+, b'_+, 0, c'_+)}, \quad (B.29)$$

$$c_-(\omega, \omega') = \frac{b_-(\omega, \omega') b_+ b'_+}{b_+(\omega, \omega') c_+ c'_+}, \quad d_+(\omega, \omega') = \frac{b_+(\omega, \omega') c_+ [b'_+ ^2 \phi_2(a_+, \bar{b}_+)]}{c_+ Q_2(a_+, a'_+, 1, b'_+, b_+, b'_+, 0, c'_+)}, \quad (B.30)$$

$$f_+(\omega, \omega') = -\frac{b_+(\omega, \omega') c_+ \phi_2(a_+, \bar{b}_+ Q_2(a_+, a'_+, b_+, \bar{b}'_+, b_+, c_+, c'_+))}{b_+ b'_+ Q_2(a_+, a'_+, 1, b'_+, b_+, b'_+, 0, c'_+)}, \quad (B.32)$$

$$h(\omega, \omega') = \frac{c_+ [b'_+ ^2 \phi_2 Q_2(a_+, a'_+, b_+, \bar{b}_+, c_+, 0) + b_+ \bar{b}_+ c_+ \phi_2(a_+, \bar{b}_+)]}{b_+ ^2 b'_+ c_+ Q_2(a_+, a'_+, 1, b'_+, b_+, b'_+, 0, c'_+)}, \quad (B.33)$$

$$f_-(\omega, \omega') = -\frac{\bar{b}_+ \phi_2(a_+, \bar{b}_+ Q_2(a_+, a'_+, 1, b'_+, b_+, b'_+, 0, c'_+))}{c_+ c'_+ \phi_2(a_+, \bar{b}_+ Q_2(a_+, a'_+, 1, b'_+, b_+, b'_+, 0, c'_+)}, \quad (B.34)$$

$$a_-(\omega, \omega') = \frac{\phi_2(a_+, a'_+, b_+, b'_+, c_+, 0) + \phi_2(a_+, b'_+) [b_+ c_+ \bar{b}'_+ + (b_+ a'_+ - a_+ b'_+) \phi_2(a_+, \bar{b}_+) \bar{b}'_+)}{b_+ b'_+ c'_+ \phi_2(a_+, a'_+, b_+, b'_+, b'_+, 0, c'_+)}, \quad (B.35)$$

$$g(\omega, \omega') = \frac{\phi_2(a_+, \bar{b}_+ Q_2(a_+, a'_+, b_+, b'_+, b_+, \bar{b}_+, c_+, c'_+))}{b_+ b'_+ Q_2(a_+, a'_+, 1, b'_+, b_+, b'_+, 0, c'_+) c_+ \phi_2(a_+, \bar{b}_+ Q_2(a_+, a'_+, 1, b'_+, b_+, b'_+, 0, c'_+) + b_+ \bar{b}_+ c'_+ \phi_2(a_+, \bar{b}_+) \bar{b}'_+)} c'_+ \phi_2(a_+, \bar{b}_+ Q_2(a_+, a'_+, 1, b'_+, b_+, b'_+, 0, c'_+) \quad (B.36)$$
Appendix C: The Hamiltonian Limit

It is well known that out of an integrable vertex model we are able to construct multiparametric solutions of the Yang-Baxter equation with the help of the so-called gauge transformations. The simplest one is a diagonal twist with constant coefficients preserving the U(1) symmetry of the nineteen vertex model. This gauge transformation leads us to the following family of transition operators,

\[
L(\omega, \Delta) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & b_+ & 0 & c_+ & 0 & 0 \\
0 & 0 & f_+ & 0 & \Delta d_+ & 0 \\
0 & c_+ & 0 & \bar{b}_+ & 0 & 0 \\
0 & 0 & \frac{d_+}{\Delta} & 0 & g & 0 \\
0 & 0 & 0 & 0 & \bar{b}_- & 0 \\
0 & 0 & h & 0 & \Delta d_- & 0 \\
0 & 0 & 0 & 0 & c_- & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{d_-}{\Delta} \\
\end{bmatrix}, \tag{C.1}
\]

where \( \Delta \) is the twist parameter and the weights \( a_+, b_+, \bar{b}_+, c_+ \) and \( c_- \) are constrained by the threefold polynomial (76). The other entries of the transition operator (C.1) are expressed in terms of the threefold variables by means of Eqs. (77-81).

The Hamiltonian limit of the vertex model is obtained considering the expansion of the transition operator (C.1) as defined in the subsection 4.2. Considering the expansion of this operator according to Eq. (103) we obtain,

\[
L(\omega, \Delta) \sim P_3(1 + \epsilon H_{j,j+1}), \tag{C.2}
\]

where \( P_3 \) denotes the three-dimensional permutator and \( H_{j,j+1} \) represents the two-body Hamiltonian.
whose matrix expression is:

\[
H_{j,j+1} = \begin{bmatrix}
\dot{a}_+ & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \dot{c}_+ & 0 & \dot{b}_+ & 0 & 0 & 0 \\
0 & 0 & \dot{a}_+ + \dot{c}_- - \dot{c}_+ & 0 & \Delta \dot{b}_+ & 0 & -\Lambda^{-1}_1 \dot{b}_+ \\
0 & \dot{b}_+ & 0 & \dot{c}_+ & 0 & 0 & 0 \\
0 & 0 & \Delta \dot{b}_+ & 0 & \Delta \dot{b}_+ & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \dot{c}_- & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dot{a}_- \\
\end{bmatrix}
\]  

From threefold expansion constraint (104) we can for instance extract the weight \( \dot{a}_+ \) and as a result the Hamiltonian matrix elements will become dependent solely on the coefficients \( \dot{b}_+ \), \( \dot{b}_+ \), \( \dot{c}_+ \) and \( \dot{c}_- \). We find that the expressions of the remaining entries \( \dot{a}_+ \), \( \dot{g} \) and \( \dot{a}_- \) are,

\[
\dot{a}_+ = -\frac{\Lambda_1 \dot{b}_+}{4 \Lambda_2^2} - \frac{\Lambda_1^+ \dot{b}_+}{4 \Lambda_2^2} - \frac{\dot{c}_-}{2} + \frac{3 \dot{c}_+}{2},
\]

\[
\dot{a}_- = -\frac{\Lambda_1^{-1} \dot{b}_+}{4 \Lambda_2^2} - \frac{\Lambda_1^+ \dot{b}_+}{4 \Lambda_2^2} + \frac{3 \dot{c}_-}{2} - \frac{\dot{c}_+}{2},
\]

\[
\dot{g} = \frac{\Lambda_1 \dot{b}_+}{4 \Lambda_2^2} + \frac{\Lambda_1^+ \dot{b}_+}{4 \Lambda_2^2} + \frac{\dot{c}_-}{2} + \frac{\dot{c}_+}{2}.
\]

The two-body Hamiltonian \([C.3]\) can also be represented by means of the spin-1 generators of the SU(2) algebra,

\[
S^+_j = \sqrt{2} \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}_j, \quad S^-_j = \sqrt{2} \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{bmatrix}_j, \quad S^z_j = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1 \\
\end{bmatrix}_j
\]  

\(^3\)For real values of the couplings this operator is Hermitian when \( \Delta = \pm, \Lambda_1 = \Lambda_1^+ \) and \( \dot{b}_+ = \dot{b}_+ \).
and after few algebraic manipulations \(H_{j,j+1}\) is rewritten as,

\[
H_{j,j+1} = \frac{\Delta \dot{b}_+}{2\Lambda_2^+} \left(S_j^+ S_{j+1}^+ + \frac{\dot{b}_+}{b_+} S_j^+ S_{j+1}^-\right) \left(1 + \frac{(\Lambda_2^+ - \Delta)(\Lambda_2^+ \Delta - 1)}{\Delta^2 \Lambda_2^+} S_j^z S_{j+1}^z\right)
- \frac{\dot{b}_+}{4\Lambda_2^+} \left(\Lambda_1^+ [S_j^+ S_{j+1}^+]^2 + \frac{\Lambda_1^+ \dot{b}_+}{b_+} [S_j^+ S_{j+1}^-]^2\right) + \frac{[\Lambda_2^+ - \Delta] \dot{b}_+}{2\Lambda_2^+} \left(S_j^- S_j^z S_{j+1}^+ + \frac{\dot{b}_+}{b_+} S_j^+ S_j^- S_{j+1}^z\right)
+ \frac{[\Delta \Lambda_2^+ - 1] \dot{b}_+}{2[\Lambda_2^+]^2} \left(S_j^- S_{j+1}^z S_j^+ S_{j+1}^- + \frac{\dot{b}_+}{b_+} S_j^+ S_j^z S_{j+1}^- S_{j+1}^z\right) - \frac{[\Lambda_2^+ \dot{b}_+ + \Lambda_2^+ \dot{\bar{b}}_+]}{4\Lambda_2^+} ([S_j^z]^2 + [S_j^z_{j+1}]^2)
+ \frac{[\dot{c}_+ - \dot{c}_-]}{2} (S_j^z + S_{j+1}^z) + \left(\frac{\Lambda_1^- \dot{b}_+}{4\Lambda_2^+} + \frac{\Lambda_1^+ \dot{\bar{b}}_+}{4\Lambda_2^+} + \frac{\dot{c}_+}{2} + \frac{\dot{c}_-}{2}\right) I_3 \otimes I_3. 
\]

(C.6)

Assuming periodic boundary conditions the two-body operator (C.6) with \(\Delta = 1\) leads us to the bulk Hamiltonian (105) up to an overall normalization factor \(\frac{b_+}{2\Lambda_2^+}\) and a trivial additive term.

We finally can compare our two-body Hamiltonian (C.6) with the one previously presented in reference [13], see Eq.(5.19). Denoting such Hamiltonian by \(H_{\text{red}}(\tau_p, \tau_3, \theta)\) we find that the relationship is,

\[
H_{\text{red}}(\tau_p, \tau_3, \theta) = H_{j,j+1} + \frac{(-1 + \tau_3 - \tau_3^2 - \theta \tau_p^2 + 2\dot{c}_- - 2\dot{c}_+)}{4} [S_j + S_{j+1}]
+ \frac{(1 - \tau_3 + \tau_3^2 + \theta \tau_p^2 - 2\dot{c}_- - 2\dot{c}_+)}{4} I_3 \otimes I_3,
\]

(C.7)

where the free parameters \(\tau_p, \tau_3\) and \(\theta\) of the work [13] are related to the coupling constants used here by,

\[
\dot{b}_+ = \tau_p \theta, \quad \dot{\bar{b}}_+ = \tau_p, \quad \Delta = \Lambda_2^+ = \frac{1}{\sqrt{\tau_3}}, \quad \Lambda_1^+ = \frac{\tau_3 - \tau_3^2 - 1}{\tau_p \sqrt{\tau_3}}. \quad \text{(C.8)}
\]

References

[1] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, New York, 1982.

[2] M. Jimbo, *Comm.Math.Phys. 102* (1986) 537.

[3] V.V. Bazhanov, *Phys.Lett.B 159* (1985) 321.

[4] R.J. Baxter, *Ann.Phys. 70* (1972) 193.
[5] A.A. Belavin, *Nucl.Phys. B* 180 (1981) 189.

[6] I.M. Krichever, *Funct.Anal.Appl.* 15 (1981) 92.

[7] M.J. Martins, *Nucl.Phys.B* 874 (2013) 243.

[8] Y. Stroganov *Phys.Lett.A* 74 (1979) 116.

[9] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, 52, Spring-Verlag, New York, 1977.

[10] A.B. Zamolodchikov and Al.B. Zamolodchikov, *Nucl.Phys.B* 133 (1978) 525, *Ann.Phys.(N.Y)* 120 (1979) 253.

[11] H. Au-Yang, B.M. McCoy, J.H.H. Perk, S. Tang and M.L. Yan, *Phys.Lett.A* 123 (1987) 219; B.M. McCoy, J.H.H. Perk, S. Tang and C.H. Sah, *Phys.Lett.A* 125 (1987) 9.

[12] R.J. Baxter, J.H.H. Perk and H. Au-Yang, *Phys.Lett.A* 128 (1988) 138.

[13] C. Crampé, L. Frappat and E. Ragoucy, *J.Phys.A* 46 (2013) 405001.

[14] T. Deguchi and Y. Akutsu, *J.Phys.Soc.Jpn* 60 (1991) 4051; T. Deguchi and Y. Akutsu, *Phys.Rev.Lett.* 67 (1991) 777.

[15] C. Gomez and G. Sierra, *Nucl.Phys.B* 373 (1985) 63; C. Gomez, M. Ruiz-Altaba and G. Sierra, *Phys.Lett.B* 265 (1991) 95.

[16] A. Beauville, *Complex Algebraic Surfaces*, London Mathematical Society Student Texts, 34, Cambridge University Press, New York, 1996.

[17] M. Reid, *Proceedings Symposia Pure Mathematics, American Mathematical Society, Part 1* 46 (1987) 345.

[18] Y. Umezu, *Tokyo J.Math.* 4 (1981) 343.
[19] V.I. Arnold, S.M. Gusein-Zade, A.N. Varchenko, *Singularities of Differential Maps*, Vol. I, Birkhäuser, 1995.

[20] W. Decker, G.-W. Greuel, G. Pfister and H. Schönemann, *Singular 3.1.6, A computer algebra system for polynomial computations*, 2012, available at [http://www.singular.uni-kl.de](http://www.singular.uni-kl.de).

[21] B. Saint-Donat, *Amer.J.Math. 96 (1974) 602*; T. Urabe, *Invent.Math. 87 (1987) 549*.

[22] S. Altmok, G. Brown and M. Reid, *Fano 3-folds, K3 surfaces and graded rings*, preprint (2002) [math.AG/0202092](http://arxiv.org/abs/math.AG/0202092).