Understanding chiral symmetry breaking with the overlap action

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A chiral fermion action allows one to do very clean studies of chiral symmetry breaking in QCD. I will briefly describe how to compute with the overlap action (relatively) cheaply, and then turn to physics: Low modes of the Dirac operator show a “lumped” chiral density which peaks at the locations of instantons and anti-instantons. These modes dominate correlation functions at small quark mass in many channels. The picture qualitatively (and in some cases quantitatively) resembles an instanton liquid model.

Instantons are a plausible candidate for the source of chiral symmetry breaking. Would-be fermion zero modes sitting on individual instantons mix into a band, and the density of low modes is connected to the chiral condensate via the Banks-Casher relation. An elaborate phenomenology built on the interactions of fermions with instantons can account for many of the low energy properties of QCD [1]. But is this picture true?

In principle, lattice simulations can address this issue. However, the usual lattice fermion actions themselves distort chiral symmetry. This can cloud the lattice results, since what is observed might just be due to the bad behavior of the action. Lattice actions which implement an exact chiral symmetry without doubling (namely, the overlap action [2]) allow one to study these questions in a theoretically clean context. This note describes recent work by Anna Hasenfratz and me, studying chiral symmetry breaking using an overlap action.

A generic overlap operator is

\[ D(0) = x_0(1 + \frac{z}{\sqrt{2}\lambda}) \]  

(1)

where \( z = d(-x_0)/x_0 \) and \( d(m) = d + m \) is some massive nonchiral Dirac operator for mass \( m \). My choice of “kernel” \( d \) is an action with nearest and next-nearest neighbors and fat links as a gauge connection, designed to “look like” an overlap action. I find eigenmodes of the squared Dirac operator \( D^\dagger D \) using a conjugate gradient routine [3]. This algorithm takes a set of trial vectors and iteratively improves them (with many multiplications of trial vectors by \( D^\dagger D \)). Any way it is implemented, an overlap action is much more expensive to use than a non-overlap action. The crucial trick I use to speed up the calculation is to begin the computation of eigenmodes of \( D^\dagger D \) with eigenmodes of \( d(0)^\dagger d(0) \). If these modes are close to eigenmodes of \( D^\dagger D \), fewer iterations are needed. This can give a gain of up to a factor of 20 in time needed to find eigenvectors, compared to the cost of finding eigenmodes of the overlap with \( d \) given by the Wilson action. In principle this would work for any action, but in practice I could not discover a good approximation to begin the calculation of eigenmodes of the overlap with Wilson or clover kernels.

All of the studies reported here were done in quenched approximation at \( \beta = 5.9 \) (lattice spacing 0.11-0.13 fm), on \( 12^4 \) and \( 12^3 \times 24 \) lattices.

We collected a set of eigenmodes, extracted the local chiral density \( \omega(x) = \langle \psi(x) | \gamma_5 | \psi(x) \rangle \) for each mode, and measured autocorrelation functions \( C_{\omega\omega}(r) = \langle \omega(r) \omega(0) \rangle \) and correlators of the chiral density with the topological charge density \( Q(r) \) (as measured from a pure gauge observable), \( C_{\omega Q}(r) = \langle \omega(r) Q(0) \rangle \). All modes showed a strong peaking in both correlators at small \( r \). Chiral zero modes “sit” on one sign of bumps of topological charge, while nonchiral fermion modes are localized on both signs of topological charge. As the fermion eigenvalue rises, these correlations slowly die away, but they persisted out to eigenvalues of 500 MeV or so. An example of a chirality auto-
correlator is shown in Fig. 1.

![Figure 1](image)

Figure 1. Chirality autocorrelator, normalized at the origin, and showing chiral zero modes (the upper lines) and nonchiral modes.

We could extract a density profile from the shape of the bumps. Interpreting this profile a being due to a mixture of fermion zero modes sitting on instantons, we infer a typical instanton radius of \( \langle \rho \rangle \approx 0.3 \) fm, a familiar number from instanton phenomenology.

But are these few modes (typically 10-20 out of 125) important? That is a qualitative question, but it is relevant to phenomenology. To test this, we constructed quark propagators “exactly,” and propagators truncated to include only a few low eigenmodes. We then looked at ordinary \( \vec{k} = 0 \) propagators (as would be used in spectroscopy).

I also computed point-to-point correlators,

\[
\Pi_i(x) = \text{Tr}(J_i^a(x)J_i^a(0)).
\]

(2)

In the latter case it is customary to divide out by the free field correlator \( \Pi_i^0(x) \) and measure

\[
R_i(x) = \Pi_i(x)/\Pi_i^0(x).
\]

(3)

When the quark mass was small enough (pseudoscalar/vector mass ratio 0.5 or less), a few modes were sufficient to saturate the pseudoscalar and scalar correlators. This was not the case for the vector or axial channels. Baryon channels are noisy, but low modes seemed to make a substantial contribution there, too. What we saw was basically consistent with instanton liquid phenomenology: channels with spin-0 \( q\bar{q} \) pairs or diquarks are supposed to couple strongly to instantons, as the pair propagates by hopping from instanton to instanton. Presumably this means that quark eigenmodes which are most sensitive to topology dominate the correlator. In channels with spin-1 diquarks or \( q\bar{q} \) pairs, the fermions cannot couple to the same instanton, suggesting that low modes do not saturate their correlators.

As one example among many, consider the the correlator ratios \( R_{V+A} \) and \( R_{V-A} \). In the sum rule/operator product expansion approach \( R_{V+A} \) is dominated by perturbative physics and is expected to take a value very close to unity, while \( R_{V-A} \) is zero at small \( x \) and receives only nonperturbative contributions which are relevant to chiral symmetry breaking. A comparison of full and low mode truncated correlators (at \( m_{PS}/m_V \approx 0.5 \)) is shown in Figs. 2 and 3. Where phenomenology expects instanton-sensitive modes, there they dominate. An extrapolation of this lattice data to zero quark mass is in reasonably good agreement with an instanton model and with tau-decay data.

Finally, in Fig. 4 I show a comparison of several pionic matrix elements computed with complete quark propagators and from quark propagators truncated to the lowest 20 eigenmodes of \( D \). Shown are the PCAC quark mass (from the divergence of the axial vector current), \( f_{\pi} \), and \( f_5 = \langle 0|\bar{\psi}\gamma_5\psi|\pi \rangle \). Low modes make a large contribution to these observables.

To conclude: Low modes of fermions show structure which is strongly reminiscent of the expectations of instanton liquid models. Low modes show a peaked chirality density correlated with gauge field topology. These low modes contribute strongly in the pseudoscalar channel and other places where chiral symmetry is important, when the quark mass is small. It seems that instants are connected, after all, with chiral symmetry breaking.
Figure 2. Comparison of $R_{V+A}$ from “exact” propagators (octagons) and from propagators truncated to the lowest 10 modes of $D^\dagger D$ (squares) at $m_{PS}/m_V = 0.5$, on $12^4$ lattices at $\beta = 5.9$.

Figure 3. Comparison of $R_{V-A}$ as in Fig. 2.

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