Holographic CBK Relation

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The Crewther-Broadhurst-Kataev (CBK) relation connects the Bjorken function for deep-inelastic sum rules (or the Gross - Llewellyn Smith function) with the Adler function for electron-positron annihilation in QCD; it has been checked to hold up to four loops in perturbation theory. Here we study non-perturbative terms in the CBK relation using a holographic dual theory that is believed to capture properties of QCD. We show that for the large invariant momenta the perturbative CBK relation is exactly satisfied. For the small momenta non-perturbative corrections enter the relation and we calculate their significant effects. We also give an exact holographic expression for the Bjorken function, as well as for the entire three-point axial-vector-vector correlation function, and check their consistency in the conformal limit.
1 Introduction and summary

Our goal is to derive from holography a non-perturbative analogue of the Crewther-Broadhurst-Kataev (CBK) relation [1, 2]. In perturbative QCD (pQCD) with an arbitrary number of colors and flavors the CBK relation connects the Bjorken function for deep-inelastic sum rules (or the Gross-Llewellyn Smith (GLS) function) with the Adler function for the electron-positron annihilation. While the two functions have perturbative corrections of their own, most of those corrections cancel out in their product; the remaining ones assemble themselves into the QCD $\beta$-function $\beta(\alpha_s)$, as was first discovered by Broadhurst and Kataev in [2] in the three-loop approximation. Hence, the CBK relation reads:

$$C_{Bj}(\alpha_s) \cdot D_5(\alpha_s) = 1 + \beta(\alpha_s)r(\alpha_s).$$

(1.1)

Here $r$ is a polynomial of its argument, that also depends on the number of colors and flavors, and $C_{Bj}$ and $D_5$ are the Bjorken and Adler functions defined in (2.1) and (2.2), respectively.\footnote{Using the GLS function instead of $C_{Bj}$ would lead to additional terms on the right hand side (RHS) of (1.1) due to the light-by-light type diagrams; for simplicity we will focus on the Bjorken function in what follows.} It was subsequently argued that the CBK relation should be satisfied to all orders in pQCD [3–6], while the explicit calculations were done up to four loops [7], confirming the validity of the relation to that order (see also [8–10] for further work on various aspects of the CBK relation in pQCD). However, the fate of non-perturbative corrections to the CBK relation – of potential importance from the particle physics point of view [8] – have not been studied.

In this paper we study the status of the CBK relation at the non-perturbative level. While such a quest is possible within QCD itself, in an order-by-order Operator Product Expansion (OPE) of the correlators relevant for the Bjorken and Adler functions, we will adopt a different path that enables exact calculations via the holographic duality [11–13]. The latter is a powerful framework to study strongly coupled conformal quantum field theories with the large number of colors $N_c$. The real world QCD is neither strongly coupled (at high energies) nor is conformal, and has $N_c = 3$. Nevertheless, it is useful to think of a conformal cousin of QCD, with an arbitrary number of colors and flavors; in this case, the CBK relations should simplify to, $C_{Bj}(\alpha_s)D_5(\alpha_s) = 1$. As to the strong coupling imposed by holographic computations, one could only rely on an assumption that no singular behavior is encountered as one interpolates from a strongly coupled regime to a weakly coupled one.

There exist two approaches to holographic QCD in the literature: the top-down approach where a string theory model generates symmetries and degrees of freedom relevant to QCD via intersecting D-branes (see, e.g., [13, 14] and works referencing them), or a bottom-up approach in which the minimal field content of the gravity dual is postulated just to capture the relevant physics of QCD (see, [15–17]). Neither of the above two approaches is ideal: the former typically generates other particles at the scale of QCD, and hence describes a strongly coupled theory that is significantly different from QCD, while the latter approach is less justifiable a priori.
Nevertheless, both of the above approaches have been used, and often produced reasonable agreement with the known QCD results. In particular, the bottom-up models were successful in describing the spectrum of the light mesons and the correlation functions of quark currents [15–17]. Since our quest in this work has relevance to low energy hadron physics, we adopt the bottom-up approach (with all the above-mentioned caveats in mind); in particular, we will use the model proposed in Ref. [17] amended by the 5D Chen-Simons term required to study the 4D correlation functions exhibiting the axial anomaly [18–20].

We derive the CBK relation using the above holographic model. The relation explicitly exhibits the hadronic resonances in the vector and (non-singlet) axial channels, and thus, captures non-perturbative dynamics of QCD. In the limit of the large invariant momenta we recover the perturbative CBK relation of a conformal field theory. For the small invariant momenta non-perturbative corrections enter the relation and give rise to the resonance behavior which is straightforwardly obtained. *En route*, we obtain an exact holographic expression for the Bjorken function, as well as the entire three-point axial-vector-vector correlation function. The latter, we show, agrees in the conformal limit with the one-loop expression obtained in a conformal field theory.

2 A brief summary of the pQCD results

Crewther [1] found that two seemingly independent quantities, the Bjorken coefficient, $C_{Bj}$, entering the deep-inelastic sum rules, and the Adler function of the $e^+e^-$ annihilation, both appearing in the OPE of quark currents, were in fact related to each other. To make this explicit we recall the definitions of these two quantities via respective OPE’s:

$$i \int d^4x e^{ipx} T \{ J_{\mu}^{em}(x) J_{\nu}^{em}(0) \} = \frac{2i}{3} C_{Bj} \left( \ln \left( \frac{\mu^2}{-p^2} \right), \alpha_s(\mu^2) \right) \varepsilon_{\mu\nu\rho\sigma} \frac{p^\rho}{p^2} J_5^{(3)}(0) + \cdots$$

$$i \int d^4x e^{ipx} T \{ J_{5\mu}^{(3)}(x) J_{5\nu}^{(3)}(0) \} = (p^2 \eta_{\mu\nu} - p_\mu p_\nu) \left( \frac{-N_c}{12\pi^2} \frac{1}{2} \Pi_5 \left( \ln \left( \frac{\mu^2}{-p^2} \right), \alpha_s(\mu^2) \right) \right) + \cdots,$$

where $J_{\mu}^{em}$ is a quark electromagnetic current, $\alpha_s(\mu)$ is a scale dependent strong coupling constant, and $N_c$ denotes the number of colors. Furthermore, we ignore the electromagnetic loop corrections, and also the quark masses; in this approximation the flavor non-singlet axial current, $J_{5\mu}^{(3)}$, is conserved and thus has zero anomalous dimension. Moreover, our $D_5$ then coincides, in the perturbative approximation, with the Adler’s $D$ function defined via the symmetric part of the vector-vector correlation function. In the approximation of massless quarks and switched off electromagnetic loop corrections, both $C_{Bj}$ and a log derivative of $\Pi_5$ (the analog of the Adler function, $D_5$) are renormalisation group invariants and satisfy

$$C_{Bj} \left( \ln \left( \frac{\mu^2}{-p^2} \right), \alpha_s(\mu^2) \right) = C_{Bj} \left( 1, \alpha_s(P^2) \right),$$
\[
\frac{\partial}{\partial \ln(-p^2)} \Pi_5 \left( \ln \left( \frac{\mu^2}{-p^2} \right), \alpha_s(\mu^2) \right) = \frac{\partial}{\partial \ln(P^2)} \Pi_5 \left( 1, \alpha_s(P^2) \right) \equiv D_5 ,
\]

where \( P^2 = -p^2 \), and \( \alpha_s(P^2) \) denotes the QCD running coupling. Now we sketch how the CBK relation emerges in QCD (see [3], and references therein). One starts with the anomalous three-point function and does the OPE of the two electromagnetic currents in the antisymmetric channel with respect to their indices (See Fig. 1):

\[
\Delta_{\mu\nu\rho}^{em,em,(3)} = i \int d^4x d^4y e^{ik_1 x + ik_2 y} \langle 0 | T \left\{ J^em_\mu(x), J^em_\nu(y), J^{\bar{5},(3)}_\rho(0) \right\} | 0 \rangle \bigg|_{q^2 \to +\infty} = (2.4)
\]

here \( Q^2 = -q^2 \) and we introduced new variables \( k_1 = p + q \) and \( k_2 = p - q \), with the kinematical constraint \( p \cdot q = 0 \) [21]. In the one loop approximation the three-point function was calculated by Rosenberg [22], and its anomalous behavior was discovered by Adler, and by Bell and Jackiw [23, 24].

The Adler-Bardeen theorem on the one-loop nature of the axial anomaly [26], can be formulated as the existence of a definition of the composite operator of the axial current for which the anomaly in the operator form is exactly given by the one loop diagram. Schreier [27] argued that if the theory is conformally invariant then one loop result is exact and no corrections from higher loops are allowed in the entire three-point function (and not only in its anomalous part; conformal symmetry constrains the form of the three-point function up to a constant factor, which is fixed through the one-loop anomaly). In QCD this was studied at two loops [28] and at three loops [29]. The two loop result is null, while Mondejar and Melnikov showed that at three loops there are nonzero terms proportional to the QCD beta function [29], consistent with Schrödinger's earlier work [25] derives an expression for the axial anomaly via the background field method. However, the expression is obtained as a matrix element of the pseudoscalar operator, \( 2m \bar{\psi} \gamma_5 \psi \), acting on two photon states in the limit when all the momenta are negligible as compared to the physical fermion mass \( m \). This coincides with the axial anomaly (with an opposite sign) but this is not the axial anomaly itself; the latter should appear in addition to \( 2m \bar{\psi} \gamma_5 \psi \), and is missing in Schwinger's work [25]. We are grateful to Arkady Vainshtein for remembering and insisting that Schwinger had it wrong, contrary to the statement in our version 1.

This definition however is not unique and might or might not be enforced by underlying symmetries in a full theory; a well-know example where it’s not enforced is in a supersymmetric theory.
an expectation from the CBK relation. Hence, gathering these results we can write:

$$\frac{N_c}{9\pi^2 q^2} C_{Bj} \left( \alpha_s(Q^2) \right) \Pi_5 \left( \alpha_s(4P^2) \right) =$$

$$= F_2^{PT}(p^2, q^2) \left|_{q^2 \to +\infty} \right. + \frac{N_c}{9\pi^2 q^2} \beta_{QCD}(\alpha_s(Q^2)) R \left( \alpha_s(4P^2), \alpha_s(Q^2) \right),$$

where $F_2^{PT}$ is a form factor within the anomalous three point function defined explicitly in Appendix A, while the function $R$ is some polynomial of its arguments.\(^4\) Taking now the logarithmic derivative with respect to $P^2$ we obtain:

$$C_{Bj} \left( \alpha_s(Q^2) \right) D_5 \left( \alpha_s(4P^2) \right) = 1 + \beta_{QCD}(\alpha_s) \tilde{R} \left( \alpha_s(4P^2), \alpha_s(Q^2) \right), \quad (2.5)$$

with $\tilde{R}$ related to $R$. Furthermore, using the definition

$$\alpha_s(4P^2) = \alpha_s(Q^2) - \int_{\ln Q^2}^{\ln 4P^2} \beta_{QCD}(\alpha_s(e^u)) du, \quad (2.6)$$

we can schematically write:

$$C_{Bj} \left( \alpha_s(Q^2) \right) D_5 \left( \alpha_s(4P^2) \right) = 1 + \sum_{n=0}^{\infty} \sum_{j=1}^{r_{n,j}} \beta_n^j r_{n,j} \left( \alpha_s(4P^2), \alpha_s(Q^2), \ln \left( \frac{4P^2}{Q^2} \right) \right), \quad (2.7)$$

where $\beta^j$s are the coefficients of the QCD beta function, and $r_{n,j}$’s some functions of their arguments. Thus, in a conformal theory the RHS equals to 1; this is the result we’d like to obtain from holographic QCD.

In what follows, we will also need the antisymmetric part of the two-point correlation function of the electromagnetic currents. If we just sandwich the LHS of equation (2.1) between vacuum states, the RHS would be zero due to Lorentz invariance. For this reason we introduce a small constant background axial field with the quantum numbers carried by $J_5^{(3)}$. In the presence of this constant background field, and in the leading order in its value, we have for the antisymmetric part of the correlator:

$$i \int d^4 x e^{iqx} \langle 0 \mid T \left\{ J_{\mu}^{em}(x) J_{\nu}^{em}(0) \right\} \mid 0 \rangle_{A^{(3)}} = i \frac{-N_c}{12\pi^2} \left( 1 + C_{Bj} \frac{2\pi^2}{3} \frac{f_0^2}{q^2} + \ldots \right) \varepsilon_{\mu\nu\rho\sigma} q^\rho A^{(3)}_{\sigma}. \quad (2.8)$$

Note that the first term on the RHS is the one loop axial anomaly. The next term is the leading power correction, and higher power corrections are denoted by the dots (note that even when the operator expression for the axial anomaly has an exact one-loop form, the anomalous matrix element in the three-point function can receive higher loop perturbative, as well as non-perturbative corrections\(^3\)). The numerical coefficient of the $1/q^2$ term is fixed from the axial-axial correlation function in the

\(^4\) Strictly speaking one could in general get some function of the beta-function on the RHS, as long as that function vanishes when the beta-function does. For simplicity, and also due to the three loop result\(^2\), we keep just the beta-function.
\chi PT framework (our coefficient differs from that of Ref. [33] by an extra factor of 1/3):

\[ i \int d^4x e^{ipx} \langle 0 | T \left\{ J_{5\mu}^{(3)}(x) J_{5\nu}^{(3)}(0) \right\} | 0 \rangle = -\frac{N_c}{9} \left( p^2 \eta_{\mu\nu} - p_\mu p_\nu \right) \frac{f_\pi^2}{p^2} \ldots \] (2.9)

We are now ready to proceed to the derivation of these results from holographic QCD.

3 Holographic Crewther-Broadhurst-Kataev Relation

3.1 The Gravity Dual

For simplicity we will only consider two light quarks and set their masses to zero. Classically this model exhibits global \( U_L(2) \otimes U_R(2) \) symmetry and corresponding eight currents are conserved. To describe the correlation functions of these currents in the holographic framework we introduce 4 + 4 vector fields, \( L_M \) and \( R_M \), living in the 5D AdS bulk and promote the global \( U_L(2) \otimes U_R(2) \) symmetry to local gauge invariance for these vector fields. It is more convenient to work with their linear combinations \( L = V + A \) and \( R = V - A \), referred as Vector (V) and Axial (A) fields. These 5D fields will be the actual duals to the 4D vector and axial currents. The 5D AdS space action consists of three pieces, the Yang-Mills, Chern-Simons, and a boundary term:

\[ S = S_{YM} + S_{CS} + S_{Boundary}. \] (3.1)

The boundary piece is introduced to recover the gauge invariance for the vector fields entering the 5D Chern-Simons term. The relevant part of the action reads:

\[ S_{YM}^{(2)} \supset -\frac{1}{4g_5} \int d^4 x \int_0^{z_m} dz \frac{1}{z} \left[ V^a_{MN} V^a_{MN} + A^a_{MN} A^a_{MN} \right], \] (3.2)

\[ S_{CS} + S_{Boundary} \supset \frac{3\kappa}{2g_5} \varepsilon^{\mu\nu\rho\sigma} \int d^4 x \left[ \int_0^{z_m} dz \left( \partial_\mu V_\nu \partial_\rho V_\sigma^{(3)} - \partial_\rho V_\nu^{(3)} \partial_\sigma V_\mu \right) A_\sigma^{(3)} \right]. \] (3.3)

Here the capital Latin indices run through \((0, 1, 2, 3, z)\), with \( z \) being the bulk coordinate, while the Greek indices take the values, \((0, 1, 2, 3)\). The indices in \( S_{YM}^{(2)} \) are contracted via the 5D flat metric (in our conventions the Minkowski metric is mostly negative, \( \eta_{\mu\nu} = (1, -1, -1, -1) \)). The parameter \( z_m \) is the position of IR brane, which is introduced in order to model the confining scale. Matching the results obtained from the 5D gravity to the 4D logarithm in the polarization function, and to the 4D axial anomaly require: \( g_5 = 12\pi^2/N_c \) and \( \kappa = 1 \). Furthermore, we impose the axial gauge:

\[ L_z = R_z = 0. \quad \rightarrow \quad V_z = A_z = 0. \] (3.4)

The vector currents are conserved and we only need to consider the transverse part of \( V^a_\mu \). It’s easier to work in the momentum space:

\[ V^a_\mu(x, z) = \frac{1}{(2\pi)^4} \int d^4q \ e^{-iqx} \mathcal{T}_\mu^a(q) V^a_\mu(q) v(q, z), \] (3.5)

\(^5\)Upper lower case Latin indices run from 0 to 3, where 0 corresponds to an iso-scalar part and 1, 2, 3 to iso-vector ones; 0 is suppressed from iso-scalars.
\[ A^b_\mu(x,z) = \frac{1}{(2\pi)^4} \int d^4 p \, e^{-ipx} \, A^b_\alpha(p) \left( T^\alpha_\mu(p) \, a(p,z) + \mathcal{L}^\alpha_\mu(p) \, \pi(p,z) \right). \]  

(3.6)

We also split off the sources, \( V^\alpha_a(q) \), \( A^\alpha_a(q) \), from their bulk profiles \( v(q,z), a(q,z), \pi(q,z) \). Here, \( T \) and \( L \) are transverse and longitudinal projectors (see [34] for a similar parametrization) and \( a = 0, 1, 2, 3; \ b = 1, 2, 3 \).

\[ T^\alpha_\mu(p) = \frac{\delta^\alpha_\mu - p^\alpha p_\mu}{p^2}; \quad \mathcal{L}^\alpha_\mu(p) = \frac{p^\alpha p_\mu}{p^2}. \]  

(3.7)

Then, the linearized equations of motion and boundary conditions for the vector and axial bulk profiles take the form (as before, \( Q^2 = -q^2 > 0 \)):

\[ \partial_z \left( \frac{1}{z} \partial_z v \right) - \frac{1}{z} Q^2 v = 0, \quad v(Q,0) = 1, \quad \partial_z v(Q,z_m) = 0; \]  

(3.8)

\[ \partial_z \left( \frac{1}{z} \partial_z a \right) - \frac{1}{z} Q^2 a = 0, \quad a(Q,0) = 1, \quad a(Q,z_m) = 0; \]  

(3.9)

\[ \partial_z \left( \frac{1}{z} \partial_z \pi \right) = 0, \quad \pi(0) = 1, \quad \partial_z \pi(z_m) = 0. \]  

(3.10)

Solutions to these equations can be written in terms of the Bessel functions:

\[ v(Q,z) = Qz \left( K_1(Qz) + I_1(Qz) \right) \frac{K_0(Qz_m)}{I_0(Qz_m)}, \]  

(3.11)

\[ a(Q,z) = Qz \left( K_1(Qz) - I_1(Qz) \right) \frac{K_1(Qz_m)}{I_1(Qz_m)}, \quad \pi(z) = 1. \]  

(3.12)

Note that the boundary condition on the IR brane for the longitudinal component of the axial profile coincides with the one for a vector. This is a consequence of the \( z \)-component of the equation of motion for the bulk axial field, \( \partial_z \partial_z A^\mu(x,z) = 0 \), that requires the longitudinal part to be independent of \( z \). In that case, chiral symmetry is broken only spontaneously, and not explicitly, consistent with massless QCD.

### 3.2 The Meson Spectra

One of the successes of holographic QCD is an impressive approximate description of the meson mass spectra [15–17, 20]. To get it in the axial channel, we calculate the transverse part of the two point functions using the standard holographic dictionary

\[ i \int d^4 x e^{ipx} \langle 0 | T \left\{ J^\alpha_{5\mu}(x) J^b_{5\nu}(0) \right\} | 0 \rangle^H = \frac{1}{g_5^2} \frac{1}{z} a'(\sqrt{-p^2},z) \delta^{ab} T^\mu_\nu(p) \bigg|_{z=\epsilon_{UV}\to 0} = \]  

\[ = \left( p^\mu \eta_{\mu\nu} - p_\mu p_\nu \right) \frac{-\delta^{ab}}{2g_5} \Pi_5(p^2), \]  

(3.13)

\[ \Pi_5(p^2) = 2\gamma + \log \left( \frac{e_{UV}^2}{4} \right) - 2 \frac{K_1 \left( z_m \sqrt{-p^2} \right)}{I_1 \left( z_m \sqrt{-p^2} \right)} + \log (-p^2), \]  

(3.14)
Figure 2. The residues as a function of the poles. The red circles are for the axial sector, while the blue squares for the vector sector.

where $\epsilon_{\text{UV}}$ is a UV cut-off scale. This expression captures well the correct leading perturbative log behavior at high energies, but does not contain power corrections – all the non-perturbative terms are exponentials encoded in the Bessel functions.

We should keep in mind that the momentum square $p^2 < 0$, and to get the particle spectrum analytic continuation to the $p^2 > 0$ region is needed. Physical particles correspond to the poles of $\Pi_5$ for $p^2 \geq 0$. These are the zeros of the Bessel $J_1$. The masses of axial mesons and residues corresponding to them are ($c_i^n$ is the $n^{\text{th}}$ zero of the Bessel $J_i$):

$$m_5^n = \frac{c_1^n}{z_m}, \quad R_5^n = -2\pi c_1^n \frac{Y_1(c_1^n)}{J_0(c_1^n)},$$

(3.15)

The $J_1(x)$ function has its first zero at $x = 0$, corresponding to a massless pion with the residue, $R_1^n = 4$. The next state should be the $a_1(1260)$ meson. The residues are all positive (see Fig. 2). By comparing the residues of the poles in (3.13) and (2.9) we get the following relation:

$$\frac{1}{z_m} = \sqrt{\frac{2}{3}} \pi f_\pi.$$

(3.16)

The numerical value $f_\pi = 92.3\text{MeV}$ leads to $z_m^{-1} = 237\text{MeV}$, which is a good approximation to $\Lambda_{\text{QCD}}$. Moreover, the present setup enables us to interpret $z_m^{-1}$ as the scale at the chiral symmetry breaking (which is close to the confinement scale in QCD), so that $z_m^{-1}$ should be dual to the quark condensate. The numerical value of quark condensate is $\langle \bar{\psi} \psi \rangle \approx (-240\text{MeV})^3$ so the predicted value looks good.

Last but not least, we can perform a similar analysis to calculate the vector-vector correlator, with the result for the transverse formfactor:

$$\Pi(p^2) = 2\gamma + \log \left( \frac{\epsilon_{\text{UV}}^2}{4} \right) + 2 \frac{K_0 \left( z_m \sqrt{-p^2} \right)}{I_0 \left( z_m \sqrt{-p^2} \right)} + \log \left( -p^2 \right),$$

(3.17)

and find the masses and residues of the vector particles:

$$m_n = \frac{c_0^n}{z_m}, \quad R_n = 2\pi \frac{c_0^n Y_0(c_0^n)}{J_1(c_0^n)}.$$

(3.18)
Since $J_0$ does not have a zero at the origin, there is no massless vector state. This is consistent with QCD. On the other hand, like in the axial-axial two-point function the above expression captures the leading perturbative log behavior at high energies, but has no power corrections; the non-perturbative terms are exponentials encoded in the Bessel functions. Thus, holographic gauge theory differs from QCD where the power corrections determine the properties of hadrons [35].

From the above expressions it is possible to predict the ratio of the masses of the vector $\rho$ and the axial $a_1$ meson: $\frac{m_\rho}{m_{a_1}} = \frac{c_1}{c_2} \approx 0.63$, which is in agreement with the observed ratio, $\frac{m_\rho(770)}{m_{a_1}(1260)} = 0.63$. Even though this model correctly predicts the ratio of the $\rho$ and $a_1$ meson masses, the predicted absolute values of these quantities, $m_\rho = 570\,\text{MeV}$ and $m_{a_1} = 910\,\text{MeV}$, are off by 25% or so. This is perhaps expected since the holographic model is minimalistic, has zero quark masses, and does not account for the gluon condensate; the latter makes a contribution to the vector meson masses in the QCD sum rules [35]. Accounting for these effects might potentially bring the masses of the mesons closer to the observed ones, as suggested by the relations derived in [36].

### 3.3 Non-perturbative Physics in CBK

In order to calculate the Adler and Bjorken functions we need the symmetric and antisymmetric parts of the axial and electromagnetic current two point functions, respectively. We already got the symmetric part for the axial current correlator in (3.13). Comparing this formula with (2.2), and using $D_5 = d\Pi_5/d\ln(P^2)$, we get:

$$D_5(p^2) = 1 + \frac{1}{I_1(z_m\sqrt{-p^2})^2}.$$  (3.19)

Finding the exact Bjorken function is more tricky, one needs to calculate the electromagnetic current two point function in the presence of a constant background axial field. The relevant part of the 5D action is the Chern-Simons term. Although the background field is constant on the UV boundary, one still has to consider its non-constant bulk profile. A bit of a calculus yields:

$$i \int d^4x e^{iqx} \langle 0|T \left\{ J^e_{\mu}(x) J^e_{\nu}(0) \right\} |0 \rangle_{A(3)}^H =$$

$$= \frac{i\kappa}{g_5} \int_0^{z_m} dz \left( \frac{3}{4} a(0, z) + \frac{1}{4} \pi(z) \right) \partial_z v^2(q, z) \varepsilon^{\mu\nu\rho\sigma} q_\rho A_\sigma =$$

$$= -\frac{i\kappa}{g_5} \left[ 1 + \frac{1}{q^2 z_m^2} - \left( \frac{1}{2} + \frac{1}{q^2 z_m^2} \right) I_0 \left( z_m \sqrt{-q^2} \right)^{-2} \right] \varepsilon^{\mu\nu\rho\sigma} q_\rho A_\sigma.$$  (3.20)

\[6\]By $\rho$ meson we mean the first pole that appears in the vector-vector two point function corresponding to $\rho^0, \rho^\pm$ and $\omega$, all degenerate here since we ignored the electromagnetism, the up and down quark masses, and the strange quark.
One has to compare this formula with its field theory counterpart (2.8), which together with (3.16) gives the exact Bjorken function:

$$C_{Bj}(q^2) = 1 - \left(1 + \frac{q^2 z_m^2}{2}\right) I_0 \left(z_m \sqrt{-q^2}\right)^{-2}. \quad (3.21)$$

Multiplying (3.19) by (3.21) gives the holographic CBK relation:

$$C_{Bj}(q^2) D_5(q^2) = 1 - \left(1 + \frac{q^2 z_m^2}{2}\right) I_0 \left(z_m \sqrt{-q^2}\right)^{-2} + I_1 \left(z_m \sqrt{-q^2}\right)^{-2} - \left(1 + \frac{q^2 z_m^2}{2}\right) I_0 \left(z_m \sqrt{-q^2}\right)^{-2} I_1 \left(z_m \sqrt{-q^2}\right)^{-2}. \quad (3.22)$$

This expression is exact! The two asymptotic values of this relation are (See Fig. 3):

$$C_{Bj}(q^2) D_5(q^2) \bigg|_{q^2 \to \infty} = 1; \quad C_{Bj}(q^2) D_5(q^2) \bigg|_{q^2 \to 0} = 4. \quad (3.23)$$

The high momentum limit corresponds to perturbative QCD; hence, the obtained value of the product is consistent with the pQCD calculations in the conformal limit \([1-3, 6, 8]\). The low momentum-square formula is one of our main results; we emphasize that the low energy numerical value is determined by the residue of a massless pion. At intermediate momenta, \(q^2 \sim z_m^{-2}\), the product \(C_{Bj}(q^2) D_5(q^2)\) deviates from 1 by terms related to various resonances. The latter suggest that in QCD the CBK relation should be modified by non-perturbative corrections, in similarity with eq. (3.22)

![Figure 3](image-url)

**Figure 3.** The product of the Bjorken and Adler functions, as a function of the dimensionless momentum square, \((x = z_m \sqrt{-q^2})\). The numerical value at \(x = 0\) comes from the residue of a massless pion.

### 3.4 Consistency with Three-point Function

The 5D Chern-Simons term can be used to calculate the anomalous three point function as well. A lengthy but straightforward calculation gives (see Appendix A for
\( i \int d^4x \, d^4y \, e^{i k_1 x + i k_2 y} \langle 0 | T \{ J^\text{em}_\mu(x) J^\text{em}_\nu(y) J^\text{em}_\rho(0) \} | 0 \rangle^H = \)
\[ = F_3^H \, \varepsilon_{\mu \nu \rho \sigma} q^\sigma \]
\[ + F_2^H \, (p_\mu \varepsilon_{\nu \rho \sigma} - p_\nu \varepsilon_{\mu \rho \sigma}) p^\sigma q^\tau + F_1^H \, (q_\mu \varepsilon_{\nu \rho \sigma} + q_\nu \varepsilon_{\mu \rho \sigma}) p^\rho q^\sigma, \]

where:
\[ F_1^H = \frac{\kappa}{g_5} R_T(p^2, q^2), \]
\[ F_2^H = -\frac{\kappa}{g_5} \left( \frac{q^2 R_T(p^2, q^2)}{p^2 + q^2} - \frac{R_L(p^2, q^2)}{p^2} \right), \]
\[ F_3^H = -\frac{\kappa}{g_5} R_L(p^2, q^2); \]
\[ R_T(p^2, q^2) \equiv \int_0^{z_m} dz \, a(2 \sqrt{-p^2}, z) \, \partial_z v^2(\sqrt{-p^2 - q^2}, z), \]
\[ R_L(p^2, q^2) \equiv \int_0^{z_m} dz \, \pi(z) \, \partial_z v^2(\sqrt{-p^2 - q^2}, z). \]

The integrand in (3.28) is a product of three Bessel functions with different arguments, for which it’s hard find a general analytic expression. Luckily, it’s sufficient for us to have the integral evaluated at \( q^2 \gg p^2 \). For the details of the calculation see Appendix B. Having this done, we can then follow the procedure used in Section 2, and get:
\[ C_{Bj}(Q^2) \, D_5(4P^2) \bigg|_{q^2 \to +\infty} = 1 + \frac{1}{I_1 \left( 2z_m \sqrt{-p^2} \right)^2}. \]

Dividing this formula by (3.19), we get the asymptotic value of the Bjorken function in agreement with the expression (3.21).

### 3.5 A Low Energy Relation

Let us look at the antisymmetric part of the two-point function of the electromagnetic fields. As before, we assume the presence of a constant background axial field with the quantum numbers of the pion current. In QCD so defined, and in the leading order in the external field, we can split this correlator as a sum of the axial anomaly plus the rest:
\[ i \int d^4x \, e^{i q x} \langle 0 | T \{ J^\text{em}_\mu(x) J^\text{em}_\nu(0) \} | 0 \rangle_{A(3)} = -i \frac{N_c}{12\pi^2} \left[ 1 + f(q^2) \right] \varepsilon_{\mu \nu \rho \sigma} q^\rho A^{(3)\sigma}, \]

where the first term is due to the axial anomaly, while the form-factor \( f \) parametrizes the rest, including all the non-perturbative parts. This two-point function can be calculated in homographic QCD and is given by the first equation in (3.20). Comparison of (3.31) with (3.20) gives:
\[ 1 + f(q^2) = 1 + \frac{1}{q^2 z_m^2} \left( \frac{1}{2} + \frac{1}{q^2 z_m^2} \right) I_0 \left( z_m \sqrt{-q^2} \right)^{-2}. \]
In the limit $q^2 \to 0$, the RHS of (3.32) vanishes and we are left with:

$$1 = -f(q^2) \bigg|_{q^2=0}. \quad (3.33)$$

The latter is a low energy relation that connects the axial anomaly, on the LHS, with the rest of the contributions to the two-point function in holographic QCD, represented on the RHS. While the high-$q^2$ behavior of $f(q^2)$ can be calculated in QCD, the $q^2 \to 0$ limit is perturbatively unaccessible.

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**A Appendix: Conformal Three Point Functions**

Adopting the kinematical condition that the square momenta of the two photons equal to each other, we can write the most general expression for the anomalous three point function as follows [21]:

$$\Delta^{em, em, (3)}_{\mu\nu\rho} = i \int d^4 x d^4 y e^{ik_1 x + ik_2 y} \langle 0 \big| T \{ J_{\mu}^{em}(x) J_{\nu}^{em}(y) J_\rho^{5(3)}(0) \} \big| 0 \rangle = F_3 \varepsilon_{\mu\nu\rho\tau} q^\tau + F_2 (p_\mu \varepsilon_{\nu\rho\sigma} - p_\nu \varepsilon_{\mu\rho\sigma}) p^\sigma q^\tau + F_1 (q_\mu \varepsilon_{\nu\rho\sigma} + q_\nu \varepsilon_{\mu\rho\sigma}) p^\sigma q^\tau,$$

were $k_1 = p + q$ and $k_2 = p - q$ and the kinematic constraint specified above translates into, $p \cdot q = 0$. As before, the quarks are massless. One loop expressions for the form factors $F_i$ can be found in [22]. $F_1$ and $F_2$ are finite functions, while $F_3$ is defined through the vector Ward identity and is just a number

$$F_1^{PT} = \frac{N_c}{6\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{(x+y) - (x+y)^2}{-(x+y) (p^2 + q^2) + p^2 (x-y)^2 - q^2 (x+y)^2},$$

$$F_2^{PT} = \frac{N_c}{6\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{x+y - (x-y)^2}{-(x+y) (p^2 + q^2) + p^2 (x-y)^2 + q^2 (x+y)^2},$$

$$F_3^{PT} = - \left( p^2 F_2^{PT} + q^2 F_1^{PT} \right) = \frac{N_c}{12\pi^2}. \quad (A.4)$$

Within the holographic framework, conformal limit corresponds to $z_m \to +\infty$, in this case we recover the exact AdS metric, and the results of the holographic calculation for the three-point function should agree with the one-loop perturbative field theory calculation, since the latter is all-loop exact in a conformal theory [27]. Integrating the profile functions (3.28) and (3.29) we calculate:

$$R^{AdS}_{T}(p^2, q^2) = -1 + \frac{\sqrt{\pi}}{2} G_{3,3}^{3,2} \left( \begin{array}{c} 0, 0, \frac{3}{2} \\ 0, 1, 2 \end{array} \right) \left( 1 + \frac{q^2}{p^2} \right), \quad (A.5)$$
\[ R_{L}^{AdS}(p^2, q^2) = -1, \] (A.6)

where \( G \) stands for the Meijer G-function. After plugging these expressions into (3.25)-(3.27), we obtain:

\[ F_{3}^{AdS} = \frac{N_c}{12\pi^2}, \] (A.7)

\[ F_{2}^{AdS} = -\frac{N_c}{12\pi^2 \frac{1}{p^2 + q^2}} \left[ 1 + \frac{\sqrt{\pi}}{2} \frac{q^2}{p^2} G_{3,3}^{3,2} \left( \begin{array} { c c c } { 0, 0, \frac{3}{2} } \\ { 0, 1, 2 } \end{array} \right| 1 + \frac{q^2}{p^2} \right] \]. (A.8)

Furthermore, by doing a numerical analysis of these expressions as well as perturbative expansions, we convinced ourselves that the relation

\[ F_{2}^{PT} = F_{2}^{AdS} \equiv F_{2}, \] (A.9)

holds for arbitrary \( p \) and \( q \); hence, the AdS/CFT conjecture works well in this case! It is possible to express \( F_{2} \) in terms of the DiLogarithm as follows:

\[ F_{2} = -\frac{N_c}{12\pi^2 q^2} \left\{ \frac{1}{2x} - \frac{1+3x}{4x} \ln \left( \frac{4x}{1+x} \right) - \frac{(1+x)(1-3x)}{4 x^{3/2}} \text{Im} \left[ \text{Li}_2 \left( e^{2i \tan^{-1} \sqrt{x}} \right) \right] \right\}; \] (A.10)

where \( x \equiv p^2/q^2 \) and “Im” stands for the imaginary part.

### B Appendix: Beyond Conformal Limit

Calculation of the integral in (3.28) is not an easy task, since it contains a product of three Bessel functions with different arguments. To tackle this problem, let us first perform partial integration and write:

\[ R_{T}(p^2, q^2) = -1 - \int_{0}^{z_m} dz \, v^2(\sqrt{-p^2 - q^2}, z) \partial z a(2\sqrt{-p^2}, z). \] (B.1)

In the approximation when \(-q^2 \gg z_m^{-2}\), or \(-p^2 \gg z_m^{-2}\), the integrand gets localized near the UV boundary and the integral becomes essentially independent of the IR cut-off \( z_m \) (See Fig. 4). Using this fact, we extend the range of integration from \([0, z_m]\) to \([0, +\infty]\). Note that in the range \([z_m, +\infty]\) the integrand has bad behavior and the integral is divergent. \textit{Mathematica} can explicitly solve this integral. After removing the divergent part, we find non-conformal corrections to (A.5) and (A.6):

\[ R_{T}(p^2, q^2) = -1 + \frac{\sqrt{\pi}}{2} G_{3,3}^{3,2} \left( \begin{array} { c c c } { 0, 0, \frac{3}{2} } \\ { 0, 1, 2 } \end{array} \right| 1 + \frac{1}{x} \right) + \]

\[ + (1 + x) \left( 1 + 3x + \frac{(1+x)(1-3x)}{4 x^{3/2}} \tan^{-1}(\sqrt{x}) \right) K_{1} \left( 2z_{m} \sqrt{-p^2} \right) I_{1} \left( 2z_{m} \sqrt{-p^2} \right) + \]

\[ + \mathcal{O} \left( e^{-2z_{m} \sqrt{-p^2 - q^2}} \right), \] (B.2)

\[ R_{L}(p^2, q^2) = -1 + \frac{1}{I_{0} \left( z_{m} \sqrt{-p^2 - q^2} \right)^{2}} = -1 + \mathcal{O} \left( e^{-2z_{m} \sqrt{-p^2 - q^2}} \right). \] (B.3)
Here again $x \equiv p^2/q^2$. The small terms come from the ratios of the Bessel functions; since we chose $-q^2 \gg z_m^{-2}$, or $-p^2 \gg z_m^{-2}$, the arguments of the Bessel functions are large, and the functions themselves are exponentially suppressed. After plugging these into (3.25-3.27), and assuming that $p^2/q^2 \ll 1$, we get:

$$F_2^H \approx \frac{N_c}{9\pi^2} q^2 \left[ -\frac{11}{12} + \ln \left( \frac{4p^2}{q^2} \right) - 2 \frac{K_1 \left( 2z_m \sqrt{-p^2} \right)}{I_1 \left( 2z_m \sqrt{-p^2} \right)} + \frac{2p^2}{5q^2} \left( -\frac{503}{120} + \ln \left( \frac{4p^2}{q^2} \right) - 2 \frac{K_1 \left( 2z_m \sqrt{-p^2} \right)}{I_1 \left( 2z_m \sqrt{-p^2} \right)} \right) \right] + O \left( \frac{p^4}{q^6} \right),$$

$$F_3^H \approx \frac{N_c}{12\pi^2} + O \left( e^{-2z_m \sqrt{-p^2} - q^2} \right).$$

The first two terms in each line in the expansion of $F_2^H$ come from the conformal part, the rest are non-perturbative corrections. Note that the non-perturbative effects also renormalize the anomalous part. Taking logarithmic derivative of (B.4) gives (3.30).

**Figure 4.** The plots of the integrand in (B.1). In the left solid blue plot (a) corresponds to $q^2/p^2 = 100$ and the right dashed purple (b) to $q^2/p^2 = 10000$. In both cases $-p^2 z_m = 1$. Increasing the value of $-p^2$, while keeping the ratio $q^2/p^2$ fixed, makes our approximation even better.

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