ASYMPTOTIC PROPERTIES OF STEADY SOLUTIONS TO THE 3D AXISYMMETRIC NAVIER-STOKES EQUATIONS WITH NO SWIRL

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Abstract. We study the asymptotic behavior of axisymmetric solutions with no swirl to the steady Navier-Stokes equations in the outside of the cylinder. We prove an a priori decay estimate of the vorticity under the assumption that the velocity has generalized finite Dirichlet integral. As an application, we obtain a Liouville-type theorem.

1. Introduction

We study the asymptotic behavior of axisymmetric solutions with no swirl to the steady Navier-Stokes equations
\[
\begin{align*}
(v \cdot \nabla)v + \nabla p &= \Delta v, \\
\nabla \cdot v &= 0,
\end{align*}
\]
where \(D\) is the outside of cylinders in \(\mathbb{R}^3\) specified later, and \(v = v(x) = (v_1(x), v_2(x), v_3(x))\) and \(p = p(x)\) denote the velocity vector field and the scalar pressure at the point \(x = (x_1, x_2, x_3) \in D\), respectively.

To state previous results, we temporary append the condition at infinity
\[
\lim_{|x| \to \infty} v(x) = v_{\infty}
\]
with a given constant vector \(v_{\infty}\). For the stationary Navier-Stokes equations (1.1)–(1.2) in general exterior domains \(D\) with the condition \(v = 0\) at the compact boundary \(\partial D\) and a given smooth external force \(f\), Leray [24] proved that there exists at least one smooth solution \((v, p)\). The solution constructed in [24] has the finite Dirichlet integral. In general, the solution \(v\) of (1.1) having the bounded Dirichlet integral, i.e., \(\int_D |\nabla v(x)|^2 dx < \infty\) is called a D-solution. Although the convergence (1.2) had been shown in such a weak sense as \(\int_D |v(x) - v_{\infty}|^p dx < \infty\), later on, Finn [10] proved that any D-solution converges to the prescribed constant vector uniformly at infinity. After that, Finn [11] introduced the notion of the physically reasonable (PR) solution \(v\) of (1.1) which satisfies \(v(x) = O(|x|^{-1})\) if \(v_{\infty} = 0\) and \(v(x) - v_{\infty} = O(|x|^{-\frac{1}{2} + \varepsilon})\) if \(v_{\infty} \neq 0\) with some \(\varepsilon > 0\). Then, Finn [12] constructed a PR-solution, provided that the data are sufficiently small. Furthermore, in the case where \(v_{\infty} = 0\) and the external force is sufficiently small, Galdi–Simader [15], Novotny–Padula [27], and Borchers–Miyakawa [4] constructed a solution satisfying \(v(x) = O(|x|^{-1})\) and \(\nabla v(x) = O(|x|^{-\frac{3}{2}})\). We note that, in

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particular, this solution satisfies $\nabla v \in L^q$ for all $q > 3/2$ (we refer [22] to the first author and Yamazaki for construction of the solution in the class $\nabla v \in L^{3/2, \infty}$).

It has been an important problem to study the relation between the D-solution and the PR-solution. It is easily proved that every PR-solution is necessarily a D-solution, however, the converse implication, namely, the precise asymptotic behavior of D-solutions, had been an open question. For that question, Babenko [2] proved that if $v_\infty \neq 0$ and the external force $f$ is compactly supported, then any D-solution is a PR-solution. On the other hand, in the case of $v_\infty = 0$, much less is known. Galdi [13] showed the same result as [2] when $v_\infty = 0$, provided that the data are sufficiently small.

In order to study further the asymptotic behavior of solutions when $v_\infty = 0$, recently, axisymmetric solutions are fully investigated. For the axisymmetric solutions, we may expect that the situation becomes similar to that of the 2-dimensional case in which the asymptotic behavior of the solution is well-studied. For the literature of 2-dimensional problems, we refer the reader to [10] [11] [20] [21].

In what follows, we use the cylindrical coordinates $r = \sqrt{x_1^2 + x_2^2}$, $\theta = \tan^{-1}(x_2/x_1)$, $z = x_3$, and let $e_r = (x_1/r, x_2/r, 0)$, $e_\theta = (-x_2/r, x_1/r, 0)$, and $e_z = (0, 0, 1)$. Using such $\{e_r, e_\theta, e_z\}$ as an orthogonal basis in $\mathbb{R}^3$, we express the vector field $v = v(x)$ as

$$v(x) = v^r(r, \theta, z)e_r + v^\theta(r, \theta, z)e_\theta + v^z(r, \theta, z)e_z.$$  

By axisymmetry we mean that $v_r$, $v_\theta$ and $v_z$ are independent of $\theta$. Choe–Jin [9], Weng [20], and Carrillo–Pan–Zhang [3] showed that an axisymmetric solution of (1.1)–(1.2) in $D = \mathbb{R}^3$ with $v_\infty = 0$ in the class the finite Dirichlet integral satisfies

$$v(x) = \mathcal{O}\left(\frac{(\log r)^{3/2}}{r^{3/2}}\right),$$

(1.3)  

$$|\omega^\theta| = \mathcal{O}\left(\frac{(\log r)^{3/2}}{r^{2}}\right), \quad |\omega^r| + |\omega^z| = \mathcal{O}\left(\frac{(\log r)^{3/2}}{r^{2}}\right)$$

uniformly in $z$ as $r \to \infty$, where $\omega^r, \omega^\theta, \omega^z$ are the components of the vorticity $\omega$ defined by

$$\omega^r = -\partial_z v^\theta, \quad \omega^\theta = \partial_z v^r - \partial_r v^z, \quad \omega^z = \frac{1}{r} \partial_r (rv^\theta).$$

Recently, Li–Pan [26] studied a similar asymptotic behavior of solutions in the class of the finite generalized Dirichlet integral

$$\int_{\mathbb{R}^3} |\nabla v(x)|^q \, dx < +\infty$$

(1.5)

for some $q \in (2, \infty)$. They first showed

$$\exists v^*_\infty \in \mathbb{R}, \quad |v(r, z) - (0, 0, v^*_\infty)| = \mathcal{O}(r^{3/2})$$

(1.6)

for any $r_0 > 0$ and $r > r_0$, \( |v(r, z) - v(r_0, z)| = \mathcal{O}\left(\frac{1}{r^{1/2}}\right) \), \( q = 3 \),

as $r \to \infty$ uniformly in $z \in \mathbb{R}$. Then, for the behavior of $\omega$, they obtained

$$|\omega^\theta(r, z)| = \mathcal{O}(r^{-(\frac{1}{4} + \frac{1}{2q} + \varepsilon)}), \quad |\omega^r(r, z)| + |\omega^z(r, z)| = \mathcal{O}(r^{-(\frac{1}{4} + \frac{1}{2q} + \frac{1}{2} \varepsilon)}),$$

(1.7)
for an arbitrary $\varepsilon > 0$ provided that $q \in [3, \infty)$ and that $\sup_{z \in \mathbb{R}} |u(r_0, z)| \leq C$ for some $r_0 > 0$ hold, or, $q \in (2, 3)$ and $v^z_\infty = 0$ hold. Besides them, they also showed

$$
|\omega^\theta(r, z)| = O(r^{-\frac{2}{3} + \varepsilon}), \quad |\omega^r(r, z)| + |\omega^z(r, z)| = O(r^{-\frac{1}{3} + \frac{2}{3} + \varepsilon}),
$$

for an arbitrary $\varepsilon > 0$ provided that $q \in (2, 3)$ and that $v^z_\infty \neq 0$ hold.

In this paper, we treat general axisymmetric exterior domains, and we prove that if the velocity has no swirl, that is, $v^\theta \equiv 0$, then the decay rates of $\omega^\theta$ obtained in [5] and [26] are further improved. We note that, as described before, solutions satisfying (1.5) are constructed in [15], [27], [4], [22], and concerning the decay property at infinity, the condition (1.5) with $q > 2$ is weaker than that in the class of the finite Dirichlet integral, namely with $q = 2$. Furthermore, as a byproduct of our result, we show a Liouville-type theorem. Although our result needs an additional assumption that $\lim_{|z| \to \infty} \omega^\theta(r, z) = 0$, we may treat the case when the velocity grows at infinity. For other Liouville-type theorems, we refer the reader to [18], [19], [25], [31], and [29] for axisymmetric solutions, and [14], [6], [7], [8], [28], [23] for general cases, respectively.

To state our main results, we introduce some notations. We consider the cylindrical domain $D = \{(r, \theta, z) \in \mathbb{R}_+ \times (0, 2\pi) \times \mathbb{R}; r > r_0\}$ with some constant $r_0 > 0$, and let $D_0 = \{(r, z) \in \mathbb{R}_+ \times \mathbb{R}; r > r_0\}$. By the axisymmetric velocity with no swirl, we mean that $v^r$ and $v^z$ are independent of $\theta$ and $v^\theta \equiv 0$. From this, we rewrite the equation (1.1) as

$$
\begin{cases}
(v^r \partial_r + v^z \partial_z)v^r + \partial_r p = \left(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2}\right)v^r, \\
(v^r \partial_r + v^z \partial_z)v^z + \partial_z p = \left(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2\right)v^z, \\
\partial_r v^r + \frac{v^r}{r} + \partial_z v^z = 0,
\end{cases} \quad (r, z) \in D_0.
$$

(1.9)

Since $v^\theta = 0$, it follows from (1.4) that $\omega^r = \omega^z = 0$, and we see that $\omega^\theta = \partial_z v^r - \partial_r v^z$ satisfies the vorticity equation

$$
(v^r \partial_r + v^z \partial_z)\omega^\theta - \frac{v^r}{r}\omega^\theta = \left(\partial_r^2 + \frac{1}{r} + \partial_z^2 - \frac{1}{r^2}\right)\omega^\theta.
$$

(1.10)

Moreover, $\Omega = \frac{\omega^\theta}{r}$ is subject to the identity

$$
- \left(\partial_r^2 + \frac{3}{r} \partial_r\right)\Omega + (v^r \partial_r + v^z \partial_z)\Omega = 0.
$$

(1.11)

The equation (1.11) has a similar structure to the vorticity equation of the 2-dimensional Navier-Stokes equations. In particular, every solution $\Omega$ to (1.11) satisfies the maximum principle, which means that for each bounded subdomain $D_1 \subset D_0$, $\Omega$ attains its maximum and minimum on the boundary of $D_1$. In addition to the assumption (1.5), we suppose that there exist $k \in \mathbb{R}$ and $C > 0$ such that

$$
|v(r, z)| \leq C(1 + |r|)^k
$$

(1.12)

holds for all $(r, z) \in D_0$.

Under the assumptions (1.5) and (1.12), we have the following asymptotic behavior of the vorticity $\Omega$ and $\omega^\theta$. 

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Theorem 1.1. Let \((v, p)\) be a smooth axisymmetric solution of \((1.1)\) with no swirl satisfying \(\int_D |\nabla v(x)|^q \, dx < \infty\) with some \(q \in [2, \infty)\). Assume that \(v\) fulfills \((1.12)\). Then, we have

\[
\lim_{r \to \infty} r^{1 + \frac{3}{q} - \frac{k}{2}} \sup_{z \in \mathbb{R}} |\Omega(r, z)| = 0,
\]

\[
\lim_{r \to \infty} r^{\frac{3}{q} - \frac{k}{2}} \sup_{z \in \mathbb{R}} |\omega^\theta(r, z)| = 0
\]
as \(r \to \infty\).

Remark 1.1. We note that the asymptotic behavior \((1.6)\) obtained by \([26]\) is also true in the outside \(D\) of cylinder. Thus, we can choose \(k\) in the assumption \((1.12)\) from \((1.6)\). For example, when \(q \in (2, 3)\) and \(v_\infty^z \neq 0\), the assumption \((1.12)\) is valid with \(k = 0\), and when \(q \in (2, 3)\) and \(v_\infty^z = 0\), the assumption \((1.12)\) is valid with \(k = 1 - \frac{3}{q}\).

Remark 1.2. (i) Compared with the estimates of \(\omega^\theta\) in \((1.3)\), we note that the above theorem in the case where \(D = \mathbb{R}^3\) and \(q = 2\) gives a better decay. Indeed, we already know \(|v(x)| = O(r^{-\frac{3}{2}} \sqrt{\log r})\) from the first estimate of \((1.3)\). Thus, \(v\) satisfies the assumption \((1.12)\) with \(k = -\frac{3}{2} + \varepsilon\) for arbitrary small \(\varepsilon > 0\). Then, the estimate \((1.14)\) implies \(|\omega^\theta(r, z)| = o(r^{-\frac{3}{2} + \varepsilon})\), which is slightly better than that of \((1.3)\). As we pointed out, in the case of no swirl, both \(\omega^r\) and \(\omega^z\) vanish, and we emphasize that our result is valid not only in \(\mathbb{R}^3\) but also in exterior domains of cylinders.

(ii) Let us compare our result with that of \([26]\). We first note that the estimates of \(\omega^\theta\) in \((1.7)\) and \((1.8)\) are also true in the outside \(D\) of the cylinder, while the estimates of \(\omega_r\) and \(\omega_z\) are proved only in \(\mathbb{R}^3\). In comparison with the estimates \((1.7)\) and \((1.8)\) for \(q \in (2, \infty)\), we note that Theorem \(1.1\) gives slightly better estimates. Indeed, for example, when \(q \in (2, 3)\) and \(v_\infty^z \neq 0\) in \((1.6)\), the assumption \((1.12)\) is valid for \(k = 0\). Then, Theorem \(1.1\) implies \(|\omega^\theta(r, z)| = o(r^{-\frac{3}{4} + \varepsilon})\), while \((1.8)\) shows \(|\omega^\theta(r, z)| = O(r^{-\frac{3}{4} + \varepsilon})\). The other cases are also discussed similarly.

As a byproduct of Theorem \(1.1\) by the maximum principle to \(\Omega\), we immediately have the following Liouville-type theorem.

Corollary 1.2. Let \(D = \mathbb{R}^3\) and let \((v, p)\) be a smooth axisymmetric solution of \((1.1)\) with no swirl having the finite generalized Dirichlet integral as in \((1.9)\) for some \(q \in [2, \infty)\) with the condition \((1.12)\) for some \(k \leq 2q + 5\). Moreover, for every \(r > 0\) we assume that \(\lim_{|z| \to \infty} |\omega^\theta(r, z)| = 0\). Then, we have \(\omega^\theta \equiv 0\), and hence \(v\) is harmonic on \(\mathbb{R}^3\).

2. Proof of Theorem \(1.1\)

In what follows, we denote by \(C\) generic constants. In particular, \(C = C(\ast, \ldots, \ast)\) denotes constants depending only on the quantities appearing in the parenthesis. We sometimes use the symbols \(\text{div}_{r,z}, \nabla_{r,z}, \text{and } \Delta_{r,z}\), which mean the differential operators defined by \(\text{div}_{r,z}(f_1, f_2)(r, z) = \partial_r f_1(r, z) + \partial_z f_2(r, z), \nabla_{r,z} f(r, z) = (\partial_r f, \partial_z f)(r, z), \text{and } \Delta_{r,z} f(r, z) = \partial^2_r f(r, z) + \partial^2_z f(r, z), \text{respectively.} \)
2.1. $L^q$-energy estimates. Since
\[ |\nabla_x v(x)|^2 = |\nabla_{r,z} v|^2 + \frac{1}{r^2} |v^r(r, z)|^2 \]
for the axisymmetric vector field $v$ without swirl, we first note that the condition (2.1) implies
\[ \infty > \int_D |\nabla v|^q \ dx \geq C \int_{D_0} [|\nabla_{r,z} v(r, z)|^q + r^{-q}|v^r(r, z)|^q] \ r \ dr \ dz, \]
and hence, $\omega^\theta$ and $\Omega$ satisfy
\[ \int_{D_0} |\omega^\theta(r, z)|^q r \ dr \ dz < \infty, \quad \int_{D_0} r^{q+1} |\Omega(r, z)|^q r \ dr \ dz < \infty, \]
respectively.

From the above bound of $\Omega$ and the equation (1.11), we prove the following estimate.

**Lemma 2.1.** Suppose the assumptions of Theorem 1.1. Let $r_1 > r_0$ and $D_1 = \{(r, z) \in \mathbb{R}_+ \times \mathbb{R}; \ r \geq r_1\}$. Let $\alpha \leq \min\{q + 3, q + 2 - k\}$. Then, we have
\[ \int_{D_1} r^\alpha |\Omega(r, z)|^{q-2} |\nabla \Omega(r, z)|^2 \ dr \ dz \leq C \int_{D_0} r^{q+1} |\Omega(r, z)|^q r \ dr \ dz, \]
where $C = C(q, \alpha, k, r_0, r_1)$.

**Proof.** Let $r_1, r_2$ be $r_2 > r_1 > r_0$, and let $\xi_1(r), \xi_2(r) \in C^\infty((0, \infty))$ be nonnegative functions satisfying
\[ \xi_1(r) = \begin{cases} 1 & (r \geq r_2), \\ 0 & (r_0 < r \leq r_1), \end{cases} \quad \xi_2(r) = \begin{cases} 1 & (0 \leq r \leq 1/2), \\ 0 & (r \geq 1). \end{cases} \]
For $R > 0$, we define a test function
\[ \eta_R(r, z) = \xi_1(r) \xi_2 \left( \frac{r}{R} \right) \xi_2 \left( \frac{|z|}{R} \right). \]
Then, we see that
\[ |\nabla_{r,z} \eta_R(r, z)| \leq C(|\xi_1'(r)| + R^{-1}), \quad |\Delta_{r,z} \eta_R(r, z)| \leq C(|\xi_2''(r)| + R^{-1}|\xi_1'(r)| + R^{-2}). \]

Let $h = h(\Omega)$ be a $C^2$ function determined later. We start with the following identity (see also [10], p.385):
\[ \text{div}_{r,z} [r^\alpha \eta_R \nabla_{r,z} h(\Omega) - h(\Omega) \nabla_{r,z} (r^\alpha \eta_R) - (r^\alpha \eta_R) h(\Omega) v] = r^\alpha \eta_R h'(\Omega) |\nabla_{r,z} \Omega|^2 - h(\Omega) |\Delta_{r,z} (r^\alpha \eta_R) + v \cdot \nabla_{r,z} (r^\alpha \eta_R)| + r^\alpha \eta_R h'(\Omega) [\Delta_{r,z} \Omega - v \cdot \nabla_{r,z} \Omega] - r^\alpha \eta_R h(\Omega) \text{div}_{r,z} v. \]
By the equation (1.11), we have
\[ r^\alpha \eta_R h'(\Omega) [\Delta_{r,z} \Omega - v \cdot \nabla_{r,z} \Omega] = r^\alpha \eta_R h'(\Omega) \left( -\frac{3}{r} \partial_r \Omega \right) = -3 \partial_r \left( r^{\alpha-1} \eta_R h(\Omega) \right) + 3 \partial_r \left( r^{\alpha-1} \eta_R \right) h(\Omega). \]
Also, from the third line of (1.9), we obtain \(-r^\alpha \eta_R h(\Omega) \operatorname{div}_{r, z} v = r^{\alpha - 1} \eta_R h(\Omega) v\). These observations with a straightforward computation lead to

\[
\begin{align*}
\operatorname{div}_{r, z} [(r^\alpha \eta_R) \nabla_{r, z} h(\Omega) - h(\Omega) \nabla_{r, z} (r^\alpha \eta_R) - (r^\alpha \eta_R) h(\Omega) v] + 3 \partial_r (r^{\alpha - 1} \eta_R h(\Omega)) \\
= r^\alpha \eta_R \eta''(\Omega) |\nabla_{r, z} \Omega|^2 - h(\Omega) [\Delta_{r, z} (r^\alpha \eta_R) + v \cdot \nabla_{r, z} (r^\alpha \eta_R)] \\
+ 3 \partial_r (r^{\alpha - 1} \eta_R) h(\Omega) + r^{\alpha - 1} \eta_R h(\Omega) v \\
= r^\alpha \eta_R \eta''(\Omega) |\nabla_{r, z} \Omega|^2 - h(\Omega) [r^\alpha \Delta_{r, z} \eta_R + (2\alpha - 3)r^{\alpha - 1} \partial_r \eta_R] \\
- h(\Omega) [r^\alpha v \nabla_{r, z} \eta_R + (\alpha - 1)r^{\alpha - 1} v \eta_R] - (\alpha - 3)(\alpha - 1) h(\Omega) r^{\alpha - 2} \eta_R.
\end{align*}
\]

Taking \(h(\Omega) = |\Omega|^q\) and integrating the above identity over \(D_0\), we deduce

\[
\begin{align*}
q(q - 1) & \int_{D_0} r^\alpha \eta_R |\Omega(r, z)| |\nabla_{r, z} \Omega(r, z)|^2 drdz \\
= & \int_{D_0} |\Omega(r, z)|^q [r^\alpha \Delta_{r, z} \eta_R + (2\alpha + 3)r^{\alpha - 1} \partial_r \eta_R] drdz \\
& + \int_{D_0} |\Omega(r, z)|^q [r^\alpha v \nabla_{r, z} \eta_R + (\alpha - 1)r^{\alpha - 1} v \eta_R] drdz \\
& + (\alpha + 3)(\alpha - 1) \int_{D_0} |\Omega(r, z)|^q r^{\alpha - 2} \eta_R drdz.
\end{align*}
\]

Applying (2.2) and the assumption (1.12), we have by the property of the support of \(\eta_R\) and its derivatives that

\[
\begin{align*}
\int_{D_0} r^\alpha \eta_R |\Omega(r, z)| |\nabla_{r, z} \Omega(r, z)|^2 drdz \\
& \leq C \sum_{l=0}^{2} R^{-l} \int_{r_0 < r < R} \int_{-\infty}^{\infty} r^{|-2+l}|\Omega(r, z)|^q drdz \\
& \quad + CR^{-1} \int_{r_0 < r < R} \int_{-\infty}^{\infty} r^{k+\alpha}|\Omega(r, z)|^q drdz + C \int_{D_0} r^{k+\alpha-1}|\Omega(r, z)|^q drdz \\
& \quad + C \int_{r_1 < r < r_2} \int_{-\infty}^{\infty} |\Omega(r, z)|^q drdz \\
& \leq C \int_{D_0} r^\alpha \eta_R |\Omega(r, z)|^q drdz + C \int_{D_0} r^{k+\alpha-1}|\Omega(r, z)|^q drdz.
\end{align*}
\]

From the assumption \(\alpha \leq \min\{q + 3, q + 2 - k\}\), we obtain

\[
\int_{D_0} r^\alpha \eta_R |\Omega(r, z)| |\nabla_{r, z} \Omega(r, z)|^2 drdz \leq C \int_{D_0} r^{q+1}|\Omega(r, z)|^q drdz.
\]

Finally, since the left-hand side is bounded from below by

\[
\int_{D_1} r^\alpha |\Omega(r, z)| |\nabla_{r, z} \Omega(r, z)|^2 drdz,
\]

we conclude that

\[
\int_{D_1} r^\alpha |\Omega(r, z)| |\nabla_{r, z} \Omega(r, z)|^2 drdz \leq C \int_{D_0} r^{q+1}|\Omega(r, z)|^q drdz.
\]

This completes the proof of Lemma 2.1. \(\square\)
2.2. Pointwise behavior via maximum principle. The condition (2.1) and Lemma 2.1 give the boundedness
\[ \int_{D_0} r^{q+1}|\Omega(r,z)|^q \, drdz < \infty, \quad \int_{D_1} r^{\alpha}|\Omega(r,z)|^{q-2}\left|\nabla\Omega(r,z)\right|^2 \, drdz < \infty \]
with \( \alpha \leq \min\{q + 3, q + 2 - k\} \). The following proposition shows that the above bounds with the maximum principle yield a pointwise behavior of \( \Omega \) as \( r \to \infty \).

**Proposition 2.2.** Let \( r_1 > 0 \), \( D_1 = \{(r,z) \in \mathbb{R}^+ \times \mathbb{R}; \ r > r_1\} \), and let \( f = f(r,z) \in C^\infty(D_1) \) satisfy
\[ \int_{D_1} r^\alpha|f(r,z)|^{q-2}\left|\nabla_{r,z} f(r,z)\right|^2 \, drdz + \int_{D_1} r^{q+1}|f(r,z)|^q \, drdz < \infty \]
with some \( q \in [2, \infty) \) and \( \alpha \leq q + 3 \). Moreover, we assume that \( f \) satisfies the maximum principle, that is, for each bounded domain \( D \subset D_1 \), the function \( f|_D \) does not attain its maximum or minimum value in the interior of \( D \). Then, we have
\[ \lim_{r \to \infty} r^{-q-1}\sup_{z \in \mathbb{R}}|f(r,z)| = 0. \]

In order to prove this proposition, we first show that the bounds (2.3) lead to a pointwise behavior along with a certain sequence \( \{r_n\}_{n=1}^\infty \) satisfying \( \lim_{n \to \infty} r_n = \infty \).

**Lemma 2.3.** Let \( r_1 > 0 \), \( D_1 = \{(r,z) \in \mathbb{R}^+ \times \mathbb{R}; \ r > r_1\} \), and let \( f = f(r,z) \in C^\infty(D_1) \) satisfy the condition (2.3) with some \( q \in [2, \infty) \) and \( \alpha \leq q + 3 \). Then, there exists a sequence \( \{r_n\}_{n=1}^\infty \) satisfying \( \lim_{n \to \infty} r_n = \infty \) such that
\[ \lim_{n \to \infty} r_n^{-q-1}\sup_{z \in \mathbb{R}}|f(r_n,z)| = 0. \]

**Proof.** Let \( n \in \mathbb{N} \) satisfy \( 2^n > r_1 \). By the assumption and the Schwarz inequality, we have
\[ \int_{r>2^n} \int_{-\infty}^\infty |f(r,z)|^{q-2}\left(r^{q+1}|f(r,z)|^2 + r^{\frac{\alpha+q+3}{2}}|\nabla_{r,z} f(r,z)|\right) \, drdz \]
\[ \leq C \int_{r>2^n} \int_{-\infty}^\infty |f(r,z)|^{q-2}\left(r^{q+1}|f(r,z)|^2 + r^{\alpha}\left|\nabla_{r,z} f(r,z)\right|^2\right) \, drdz \]
\[ < \infty. \]

Here, we note that the Lebesgue dominated convergence theorem shows
\[ \lim_{n \to \infty} \int_{r>2^n} \int_{-\infty}^\infty |f(r,z)|^{q-2}\left(r^{q+1}|f(r,z)|^2 + r^{\frac{\alpha+q+3}{2}}|\nabla_{r,z} f(r,z)|\right) \, drdz = 0. \]

The mean value theorem for integration implies that there exists \( r_n \in [2^n, 2^{n+1}] \) such that
\[ \int_{-\infty}^\infty |f(r_n,z)|^{q-2}\left(r^{q+2}_n|f(r_n,z)|^2 + r^{\frac{\alpha+q+3}{2}}|\nabla_{r,z} f(r_n,z)|\right) \, dz \]
\[ = \frac{1}{\log 2} \int_{2^n}^{2^{n+1}} dr \int_{-\infty}^\infty |f(r,z)|^{q-2}\left(r^{q+2}|f(r,z)|^2 + r^{\frac{\alpha+q+3}{2}}|\nabla_{r,z} f(r,z)|\right) \, dz \]
\[ \leq C \int_{r>2^n} \int_{-\infty}^\infty |f(r,z)|^{q-2}\left(r^{q+1}|f(r,z)|^2 + r^{\frac{\alpha+q+3}{2}}|\nabla_{r,z} f(r,z)|\right) \, drdz. \]
We denote the left-hand side by $J_n$:
\[
J_n := \int_{-\infty}^{\infty} |f(r_n, z)|^{q-2} \left( r_n^{q+2} |f(r_n, z)|^2 + r_n^{\frac{\alpha+q+3}{2}} |f(r_n, z)||\nabla_{r,z}f(r_n, z)| \right) dz.
\]
From the estimate above and (2.4), we obtain
\[
\lim_{n \to \infty} J_n = 0.
\]
On the other hand, by the fundamental theorem of calculus, we see that for any $z_1, z_2 \in \mathbb{R}$,
\[
r_n^{\frac{\alpha+q+1}{2}} |f(r_n, z_1)|^q - r_n^{\frac{\alpha+q+1}{2}} |f(r_n, z_2)|^q = r_n^{\frac{\alpha+q+1}{2}} \int_{z_2}^{z_1} \partial_z (|f(r_n, z)|^q) \, dz \\
\leq C r_n^{\frac{\alpha+q+3}{2}} \int_{-\infty}^{\infty} |f(r_n, z)|^{q-1} |\nabla_{r,z}f(r_n, z)| \, dz.
\]
Integrating it over $[-r_n, r_n]$ with respect to $z_2$, we have
\[
r_n^{\frac{\alpha+q+1}{2}} |f(r_n, z_1)|^q \leq C r_n^{\frac{\alpha+q+1}{2}} \int_{-r_n}^{r_n} |f(r_n, z_2)|^q \, dz_2 \\
+ C r_n^{\frac{\alpha+q+3}{2}} \int_{-\infty}^{\infty} |f(r_n, z)|^{q-1} |\nabla_{r,z}f(r_n, z)| \, dz \\
\leq C J_n,
\]
where we have used that $\frac{\alpha+q+1}{2} \leq q+2$, which follows from the assumption $\alpha \leq q+3$. Since $z_1 \in \mathbb{R}$ is arbitrary, we conclude from (2.6) that
\[
\lim_{n \to \infty} r_n^{\frac{\alpha+q+3}{2}} \sup_{z \in \mathbb{R}} |f(r_n, z)|^q = 0,
\]
and the proof of Lemma 2.3 is now complete. \(\square\)

**Proof of Proposition 2.2.** Let $\{r_n\}_{n=1}^{\infty}$ be the sequence given by Lemma 2.3. Then, we have
\[
\lim_{n \to \infty} r_n^{\frac{\alpha+q+3}{2}} \sup_{z \in \mathbb{R}} |f(r_n, z)| = 0.
\]
To prove Proposition 2.2, we apply the maximum principle to obtain the pointwise estimate for general $r$ in the interval $[r_n, r_{n+1}]$. However, before doing it, we also need to control the pointwise behavior for $z$-direction. Therefore, we claim that there exists a sequence $\{z_m\}_{m \in \mathbb{Z}}$ satisfying $\lim_{m \to \pm \infty} z_m = \pm \infty$ such that
\[
\lim_{m \to \pm \infty} |z_m| \sup_{r_n \leq r \leq r_{n+2}} r^{\alpha+q+3} |f(r, z_m)|^q = 0.
\]
The reason why we take the interval $[r_n, r_{n+2}]$ instead of $[r_n, r_{n+1}]$ is that the length of the former interval $[r_n, r_{n+2}]$ has the bound from both above and below such as $r/4 \leq r_{n+2} - r_n \leq 8r$ for all $r \in [r_n, r_{n+2}]$ (note that $r_n \in [2^n, 2^{n+1}]$).

Let us prove (2.6). We fix $n \in \mathbb{N}$, namely, fix the interval $[r_n, r_{n+2}]$. In the same way as in the proof of Lemma 2.3, we see that there exists a sequence $\{z_m\}_{m \in \mathbb{Z}}$
satisfying \( z_{\pm l} \in [\pm 2^l, \pm 2^{l+1}] \) for each \( l \in \mathbb{N} \)

\[
\int_{r_n}^{r_{n+2}} |z_{\pm l}||f(r, z_{\pm l})|^q - 2 \left( r^{q+1}|f(r, z_{\pm l})|^2 + r^{\frac{q+1}{q-1}}|f(r, z_{\pm l})||\nabla_r z f(r, z_{\pm l})| \right) dr
\]

\[
\leq C \int_{r_n}^{r_{n+2}} |f(r, z)|^{q-2} \left( r^{q+1}|f(r, z)|^2 + r^{\frac{q+1}{q-1}}|f(r, z)||\nabla_r z f(r, z)| \right) dz dr.
\]

We denote the left-hand side by \( K_{\pm l} \):

\[
K_{\pm l} := \int_{r_n}^{r_{n+2}} |z_{\pm l}||f(r, z_{\pm l})|^q - 2 \left( r^{q+1}|f(r, z_{\pm l})|^2 + r^{\frac{q+1}{q-1}}|f(r, z_{\pm l})||\nabla_r z f(r, z_{\pm l})| \right) dr.
\]

We note that the above estimate and the Lebesgue dominated convergence theorem yield

\[
(2.7) \quad \lim_{l \to \infty} K_{\pm l} = 0.
\]

In what follows, for simplicity we abbreviate in such a way that \( m = \pm l \) and \( K_m = K_{\pm l} \). For every \( r, \tilde{r} \in [r_n, r_{n+2}] \), by the fundamental theorem of calculus, we deduce

\[
|z_m| r^{\frac{q+1}{2}} |f(r, z_m)|^q - |z_m| \tilde{r}^{\frac{q+1}{2}} |f(\tilde{r}, z_m)|^q
\]

\[
= |z_m| \int_\tilde{r}^r \partial_\rho \left( \rho^{\frac{q+1}{2}} |f(\rho, z_m)|^q \right) d\rho
\]

\[
\leq C |z_m| \int_{r_n}^{r_{n+2}} \left[ \rho^{\frac{q+1}{2}} |f(\rho, z_m)|^q + \rho^{\frac{q+1}{q-1}} |f(\rho, z_m)|^{q-1} |\nabla_r z f(\rho, z_m)| \right] d\rho.
\]

Since \( \alpha \leq q + 3 \), we have \( \frac{q+1}{2} \leq q + 1 \), and hence integration over \([r_n, r_{n+2}]\) with respect to \( \tilde{r} \) of both sides of the above estimate yields

\[
(r_{n+2} - r_n) |z_m| r^{\frac{q+1}{2}} |f(r, z_m)|^q
\]

\[
\leq |z_m| \int_{r_n}^{r_{n+2}} \tilde{r}^{\frac{q+1}{2}} |f(\tilde{r}, z_m)|^q d\tilde{r}
\]

\[
+ (r_{n+2} - r_n) |z_m| \int_{r_n}^{r_{n+2}} \left[ \rho^{\frac{q+1}{2}} |f(\rho, z_m)|^q + \rho^{\frac{q+1}{q-1}} |f(\rho, z_m)|^{q-1} |\nabla_r z f(\rho, z)| \right] d\rho
\]

\[
\leq K_m.
\]

Combining it with \( r_{n+2} - r_n \sim r \) and (2.4), we obtain the claim (2.6).

Finally, we apply the maximum principle to complete the proof of Proposition 2.2. Let \( r > r_1 \) be sufficiently large and take \( n \in \mathbb{N} \) so that \( r \in [r_n, r_{n+2}] \) with \( \{r_n\}_{n=1}^\infty \) satisfying (2.5). We also take \( \{z_m\}_{m \in \mathbb{Z}} \) so that (2.6) holds. Let \( D_{n,m} = \{(r, z) \in \mathbb{R}^+ \times \mathbb{R}; r \in [r_n, r_{n+2}] \times [z_m, z_{m}]\} \). Then, by the maximum principle, for
every \((r, z) \in D_{n, m}\), we have
\[
\lim_{r \to \infty} r^{\alpha q + 3} |f(r, z)| = 0,
\]
which yields the desired estimate. This completes the proof of Proposition 2.2. \(\square\)

2.3. **Proof of Theorem 1.1.** By the assumptions on Theorem 1.1, we apply Lemma 2.1 and Proposition 2.2 with \(\alpha = \min\{q + 3, q + k\}\) to obtain
\[
\lim_{r \to \infty} r^\frac{3}{q} \max_{z \in \mathbb{R}} |\Omega(r, z)| = 0.
\]
Concerning the estimate of \(\omega^\theta\), we have by the relation \(\omega^\theta = r \Omega\) that
\[
\lim_{r \to \infty} r^{\frac{3}{q} - \frac{3}{2q} \max\{0, 1 + k\}} \max_{z \in \mathbb{R}} |\omega^\theta(r, z)| = 0.
\]
This completes the proof of Theorem 1.1.

2.4. **Proof of Corollary 1.2.** Since \(k \leq 2q + 5\), we see that \(1 + \frac{3}{q} - \frac{3}{2q} \max\{0, 1 + k\} \geq 0\), and hence it follows from 1.13 that
\[
\lim_{r \to \infty} \max_{z \in \mathbb{R}} |\Omega(r, z)| = 0.
\]
Since \(\lim_{|z| \to \infty} |\Omega(r, z)| = 0\) for each fixed \(r > 0\), we obtain from the maximum principle that \(\Omega \equiv 0\) on \(\mathbb{R}^3\). Since \(\omega^\theta \in C(\mathbb{R}^3)\), we conclude that \(\omega^\theta \equiv 0\) on \(\mathbb{R}^3\). This proves Corollary 1.2.

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