The Two-dimensional String as a Topological Field Theory

Sunil Mukhi

Tata Institute of Fundamental Research
Homi Bhabha Rd, Bombay 400 005, India

ABSTRACT

A certain topological field theory is shown to be equivalent to the compactified $c = 1$ string. This theory is described in both Kazama-Suzuki coset and Landau-Ginzburg formulations. The genus-$g$ partition function and genus-0 multi-tachyon correlators of the $c = 1$ string are shown to be calculable in this approach. The KPZ formulation of non-critical string theory has a natural relation to this topological model.

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1 Introduction

For many years it used to be said that string theory is consistent only in 26 spacetime dimensions. With the understanding that any CFT with total central charge \( c = 26 \) is a consistent background for bosonic string propagation, this statement had to be modified. It remained true that in some sense the simplest known background for the bosonic string was the one with 26 flat, noncompact spacetime dimensions.

The relevance of the concept of “simplest known background” stems from the fact that we do not yet fully understand background-independent string field theory. In this situation, the theory in the simplest background may reasonably be expected to furnish clues about the nature of the full theory. For example, arbitrary backgrounds in a field theory tend to spontaneously break some or all of the global symmetries. In such backgrounds we would be hard pressed to discover the full symmetry structure of the theory. Thus, we must look for backgrounds which preserve as many symmetries as possible – indeed, this is probably the most reasonable definition of “simplest background”.

With this in mind, it becomes clear that the noncompact 26-dimensional background is far from being the simplest one. Indeed, just compactifying the spacetime on a torus of appropriate shape can produce large numbers of Kac-Moody currents, whose integrals are unbroken symmetry generators. But even better is the background with just 2 spacetime dimensions, traditionally described by scalar fields \( X(z, \bar{z}) \) and \( \phi(z, \bar{z}) \) on the world-sheet. The unbroken symmetries of this theory correspond to infinitely many holomorphic currents satisfying the wedge subalgebra of \( W_\infty \). This is the most symmetric known phase of the bosonic string. More precisely, the maximum number of unbroken symmetries arise when the field \( X(z, \bar{z}) \) is compactified on a circle of radius \( \frac{1}{\sqrt{2}} \), the so-called self-dual radius.

The two-dimensional string background (also known as “\( c = 1 \) string theory”) has the following energy-momentum tensor for its matter sector:

\[
T(z) = -\frac{1}{2} \partial X \partial X - \frac{1}{2} \partial \phi \partial \phi + \sqrt{2} \partial^2 \phi
\]  

(1)

where \( X(z, \bar{z}) \) and \( \phi(z, \bar{z}) \) are conventionally known as the “matter” and “Liouville” fields respectively. The presence of a total derivative term in \( T(z) \) corresponds to an extra term in the worldsheet action proportional to \( \int \sqrt{g} R^{(2)} \). This term renders the functional integral ill-defined, and to stabilise the theory we must add a cosmological term \( \mu \int \sqrt{g} \exp(\alpha \phi) \). This means that Liouville momentum is not conserved. Every correlation function carries a power of \( \mu \) equal to minus the amount by which it violates Liouville momentum conservation, in multiples of \( \sqrt{2} \).
In particular, it is easy to see that the partition function of the \( c = 1 \) string in genus \( g \) has the behaviour
\[
Z_g(\mu) \sim \mu^{2-2g}
\]
Indeed, for the two-dimensional string compactified on the self-dual radius, the partition function is known explicitly from matrix models\[^1\] and is given by
\[
Z_{\text{self-dual}}^g(\mu) = \mu^{2-2g} \chi_g, \quad \chi_g = \frac{B_{2g}}{2g(2g-2)}
\]
where \( \chi_g \) is the virtual Euler characteristic of the moduli space of genus-\( g \) Riemann surfaces, which is proportional to the Bernoulli number \( B_{2g} \).

It has long been a challenge to reproduce this result from a continuum formulation of two-dimensional string theory. Other matrix-model results for this theory, particularly the correlation functions of “tachyons” in genus 0, have been re-derived in the continuum only after considerable difficulty and after resorting to somewhat arbitrary continuations of the parameters of the theory\[^8, 9, 10, 11, 12\]. One of the crucial difficulties in continuum calculations is that the theory is not perturbative in the cosmological constant \( \mu \) (as is evident from Eq.\(^2\)). Another problem is that standard CFT techniques produce the amplitudes for local vertex operators inserted at fixed points on the worldsheet, which must then be integrated over the worldsheet to give the physical amplitudes. In a theory coupled to gravity, physical answers for correlations of integrated operators (which are typically quite simple) should not have to depend on first obtaining the local answer and explicitly performing a complicated integral.

In what follows I will show that there is a topological field theory which is equivalent to \( c = 1 \) string theory at the self-dual radius, and that the manifestly topological formulation permits the computation of the partition function and amplitudes in any genus, and at nonzero cosmological constant, without resorting to analytic continuations. This means that we have a continuum formulation of this string theory which is apparently as powerful as the very successful matrix-model description. This discovery may lend itself to generalization and teach us something about string theory itself rather than about some specific background, unlike the matrix models which have so far resisted attempts to go beyond the specific backgrounds which they represent.

2 Topological Symmetry of String Backgrounds

Topological symmetry was thought to be a special property of some string backgrounds. More recently, it has become clear that it is present in \textit{any} string background in which there is at least one (possibly anomalous) \( U(1) \) current\[^13\]. The basic idea is that in
conformal gauge, even the bosonic string possesses four chiral fields of the right spins and statistics to form a twisted (hence topological) \( N = 2 \) superconformal algebra. In general, this algebra is generated by bosonic fields \( T(z) \) and \( J(z) \) of spins 2 and 1 respectively, and fermionic fields \( G^+(z) \) and \( G^-(z) \) which also have spins 2 and 1 respectively. They obey the following singular OPE relations:

\[
\begin{align*}
T(z)T(w) &\sim \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \\
T(z)G^\pm(w) &\sim \frac{1}{2} \frac{(3 \mp 1)G^\pm(w)}{(z-w)^2} + \frac{\partial G^\pm(w)}{(z-w)} \\
T(z)J(w) &\sim \frac{c^T/3}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{(z-w)} \\
J(z)G^\pm(w) &\sim \pm \frac{G^\pm(w)}{(z-w)} \\
J(z)J(w) &\sim \frac{c^T/3}{(z-w)^2} \\
G^\pm(z)G^\pm(w) &\sim 0 \\
G^+(z)G^-(w) &\sim \frac{c^T/3}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{T(w)}{(z-w)}
\end{align*}
\]

(4)

For the \( c = 1 \) string, the bosonic fields \( T(z), J(z) \) in the above algebra arise from the full stress-energy tensor (including ghost contribution) and the ghost-number current, while the fermionic fields \( G^\pm(z) \) are the BRS current and the antighost. It turns out that the OPEs among these fields are not actually closed, but on modifying some of the fields by total derivative terms, one finds a closed algebra which is precisely the twisted \( N = 2 \) superconformal algebra above, with some fixed value of the topological central charge \( c^T \).

Explicitly, the generators of the twisted \( N = 2 \) algebra are as follows:

\[
\begin{align*}
T(z) &= T(z)^{\text{matter}} + T(z)^{\text{ghost}} \\
G^+(z) &= c(z)T(z)^{\text{matter}} + \frac{1}{2} : c(z)T(z)^{\text{ghost}} : + x\partial^2 c(z) + y\partial(c(z)\partial\eta(z)) \\
G^-(z) &= b(z) \\
J(z) &= : c(z)b(z) : - y\partial\eta(z)
\end{align*}
\]

(5)

where \( \partial\eta(z) \) is a \( U(1) \) current formed out of the fields in the matter sector of the background. (Here we will work with the case where the current arises as the derivative of a free scalar field, although the more general case is obvious.) If the \( \eta \) field has a
background charge $Q_\eta$, then it is easy to check that the constants $x$ and $y$ must take the following values in order to get a closed algebra:

\[
\begin{align*}
  x &= \frac{1}{2} (3 + Q_\eta y) \\
  y &= \frac{1}{2} (-Q_\eta + \sqrt{Q_\eta^2 - 8})
\end{align*}
\]  

(6)

The topological central charge of this algebra is found to be $c^T = 6x$.

Clearly, the structure described above depends on the choice of both a background and a particular $U(1)$ current of that background. For example, if we choose the $c = 1$ string, the current can be associated to either the Liouville scalar $\phi(z, \bar{z})$ or the matter scalar $X(z, \bar{z})$. Thus, we have:

\[
\begin{align*}
  \eta = \phi, & \quad Q_\phi = 2\sqrt{2} \quad \rightarrow \quad x = -\frac{1}{2}, \quad y = -\sqrt{2}, \quad c^T = -3 \\
  \eta = X, & \quad Q_X = 0 \quad \rightarrow \quad x = \frac{3}{2}, \quad y = i\sqrt{2}, \quad c^T = 9
\end{align*}
\]  

(7)

Another interesting case is the critical bosonic string, in which one can choose the $U(1)$ current to come from any of 26 free scalars, all non-anomalous, and one obtains $c^T = 9$ in each case.

The case of interest here will be the non-critical $c = 1$ background. Thus, in principle both the choices of $U(1)$ current above are available to us. The first choice, in which it is associated to the Liouville field, may seem more natural since this field is generally present in noncritical backgrounds\[14\]. However, that choice has the following disadvantage. If the action is perturbed by the cosmological operator, as it must be to produce a well-defined functional integral, the Liouville field satisfies

\[
\bar{\partial} \partial \phi + \mu e^\phi = 0
\]  

(8)

Thus, the current $\partial \phi$ is no longer holomorphic, and we conclude that the $\eta$-current in the $N = 2$ algebra cannot be identified with the Liouville field except at zero cosmological constant. No such restriction applies if we choose instead the $c = 1$ matter field (the “time” coordinate of the string). Hence we make that choice, and conclude that $c = 1$ string theory is described by a topological $N = 2$ algebra with central charge $c^T = 9$.

This observation has one more or less immediate consequence. The various states in the BRS cohomology of the $c = 1$ string must be classified by an $N = 2$ topological algebra. It is well-known\[14, 16, 17, 18\] that there is a plethora of physical states in this theory, including tachyons, discrete states of ghost number 1 (the “remnants” of transverse gravitons and other tensor excitations of the string propagating in two dimensions), and discrete states of ghost number 0 (the so-called “ground ring” elements). It can be argued that the only states among these which are actually primaries of the
topological algebra are the tachyons. This applies at any compactification radius for the
$X$ coordinate, and both the types of tachyons that generically exist (the special discrete
tachyons as well as the intermediate ones) are always primary. The discrete “remnant”
states of ghost number 1 are secondaries of the $N = 2$ algebra. On the other hand, the
discrete ground ring states, of ghost number 0, are both primary and secondary. These
states would normally be termed null, except that (as is familiar\cite{18,15}) the projection
from the chiral algebra to the Fock representation fails to be an isomorphism, due to
the presence of background charges. Hence null vectors in the module of the chiral
algebra can be non-vanishing Fock-space states.

In our notation, tachyon operators are represented by

$$T_k^\pm = c \exp \frac{1}{\sqrt{2}} (2 \mp k) \phi \exp \frac{i}{\sqrt{2}} kX$$

(9)

where the label $k$ takes all real values in the noncompact case, and integer values at the
self-dual radius. The total conformal dimension of these operators is of course 0, which
is a necessary condition for them to be in the BRS cohomology. However, one can also
consider more general operators (“off-shell tachyons”) described by

$$T_{k_1,k_2}^\pm = c \exp \frac{1}{\sqrt{2}} (2 \mp k_1) \phi \exp \frac{i}{\sqrt{2}} k_2X$$

(10)

Here, we fix the Liouville momentum $k_1$ to be a positive integer, while the matter mo-
mentum $k_2$ takes values in $k_1, k_1 - 2, \ldots, -k_1$. These fields have negative conformal
dimension $k_2^2 - k_1^2$, hence by well-known theorems they cannot be in the BRS coho-
mology. In fact, they are BRS exact. But some secondaries of the topological algebra
above these off-shell tachyons are in the BRS cohomology, and in fact are just the
various types of discrete states.

Thus we see that the presence of a twisted $N = 2$ topological algebra in the $c = 1$
string actually explains the variety of physical states which were discovered over
the last few years\cite{19,20,21}. The tachyons, whose existence was expected and is
rather well-understood, are really the only basic states of the theory. The two types
of discrete states, both of which seemed a little exotic at first, are in fact nothing but
particular secondaries with respect to the topological algebra. Some details concerning
this phenomenon may be found in Ref.\cite{1}, although the full picture probably contains
more that has not yet been worked out.

3 A Coset Model of the $c = 1$ String

Besides organising the various physical states of the $c = 1$ string elegantly, the topolog-
ical symmetry described above does something more fundamental. As it stands, this
symmetry looks accidental, in that there is nothing known about the basics of string theory which would predict it. But since it exists, one could imagine looking for alternative formulations of various string backgrounds in which the topological symmetry is manifest. These formulations might then be exactly solvable in the way that topological theories often are. For the special case of the \( c = 1 \) background (at the self-dual radius) we will argue that there is a manifestly topological model which fulfils these requirements: it is equivalent to the \( c = 1 \) string, but is “more solvable”, in the sense that it permits the computation of amplitudes which are not calculable (as far as we know) within the conventional KPZ\(^{[22]}\) or DDK\(^{[23]}\) formulations.

The model for which we are searching cannot be predicted from some general principles, but we do know from the discussion of the previous section that it must possess a topological symmetry with central charge \( c^T = 9 \). We will look for this model among a general class of \( N = 2 \) supersymmetric CFT’s, the Landau-Ginzburg and Kazama-Suzuki models. First we concentrate on the Kazama-Suzuki (KS) description.

The construction of KS models is based on the following facts. Let us start with an \( N = 1 \) supersymmetric WZW model, based on a group \( G \), and gauge the adjoint action of a subgroup \( H \). Then, if and only if the coset \( G/H \) is a Kähler manifold, the model so obtained has \( N = 2 \) supersymmetry\(^{[24]}\). Applying this idea to the case where \( G = SL(2, R) \) and \( H = U(1) \), we find a series of models with central charge \( c = 3k/(k - 2) \), where \( k \) is the level of the \( SL(2, R) \) current algebra. After twisting the model so obtained to make it topological, one finds as usual that the central charge of the theory becomes zero, but there is a topological central charge \( c^T \) which has the same value as the central charge before twisting. Accordingly, if we want \( c^T = 9 \) in this class of models, the unique choice is to take level \( k = 3 \). Curiously, \( SL(2, R) \) at level 3 also occurs in the KPZ description of \( c = 1 \) string theory, and we will see below that this is not a coincidence – in fact, the construction described here will illuminate a few long-standing mysteries of the KPZ approach.

Let us now look at the KS model at level 3 in some detail. The stress-energy tensor of the supersymmetric \( G \) model is, as usual,

\[
T^{(G)}(z) =: J^+(z)J^-(z) + (J^3(z))^2 - \frac{1}{2}(b(z)\partial c(z) + c(z)\partial b(z))
\]  

(11)

where \( J^+, J^- \), and \( J^3 \) are the spin-1 \( SL(2, R) \) currents, \( b, c \) are spin-\( \frac{1}{2} \) fermions which serve to supersymmetrise the theory, and the coefficient of the first term is unity precisely at \( k = 3 \).

One can write the stress-energy tensor of the coset model as the difference of the (supersymmetric) \( G \) and \( H \) stress-energy tensors. However, for our purposes it is more convenient to use an alternative formulation in which to the \( G \) stress-energy we add a
gauge contribution and then pass to a BRS cohomology on this larger space \[25\]. Thus we have

\[ T(z) = T^{(G)}(z) + T^{(\text{gauge})}(z) \]  

(12)

We will write this out explicitly below, but first let us see what the other generators of the \( N = 2 \) algebra look like. Following Kazama-Suzuki \[24\], we find (at \( k = 3 \))

\[
\begin{align*}
G^+(z) &= c(z)J^+(z) \\
G^-(z) &= c(z)J^-(z) \\
J_{N=2}(z) &= 3 : c(z)b(z) : - 2J^3(z)
\end{align*}
\]

(13)

Here we have labelled the \( U(1) \) current of the \( N = 2 \) algebra explicitly to distinguish it from other \( U(1) \) currents in the problem.

So far, this is an untwisted \( N = 2 \) superconformal algebra, with central charge \( 3k/(k-2) \) as mentioned earlier. Now we render it topological through the twist

\[ T(z) \rightarrow T(z) + \frac{1}{2} \partial J_{N=2}(z) \]  

(14)

As a result, the final stress-energy tensor (to be compared with Eqs.(11),(12)) is

\[ T(z) = : J^+(z)J^-(z) : + (J^3(z))^2 - \partial J^3(z) - 2b(z)\partial c(z) + c(z)\partial b(z) + T^{(\text{gauge})}(z) \]  

(15)

This twist has a rather miraculous consequence. The spins of all the fields in \( T^{(G)} \) have changed, and we now have an \( SL(2,R) \) current multiplet \( (J^+, J^3, J^-) \) of spins \( (2,1,0) \) respectively. Moreover, the free fermions \( (b,c) \) have changed their spins from \( (\frac{1}{2}, \frac{1}{2}) \) to \( (2, -1) \). The currents are now reminiscent of those in the KPZ description of \( c = 1 \) string theory, where they described the gravitational or Liouville sector, while the fermions have become identical to the usual ghost system of bosonic string theory! The central charges are found to be \( c = 27 \) for the twisted currents and of course \( c = -26 \) for the fermions.

Now we add in the \( U(1) \) gauge sector of the theory by making the gauge choice

\[ A_z(z) = \partial X(z), \quad \bar{A}_{\bar{z}}(\bar{z}) = -\bar{\partial} \bar{X}(\bar{z}) \]  

(16)

and defining the free scalar field \( X(z, \bar{z}) = X(z) - \bar{X}(\bar{z}) \). Fixing the gauge in this way requires the introduction of a pair of fermionic ghosts \( (B,C) \) of spins \( (1,0) \), hence we get

\[ T^{(\text{gauge})}(z) = -\frac{1}{2} \partial X(z)\partial X(z) - B(z)\partial C(z) \]  

(17)
along with the $U(1)$ BRS charge

$$Q_{U(1)} = \int c(z)(J^3(z) - :c(z)b(z) - \frac{i}{\sqrt{2}}\partial X(z)) + \text{c.c} \tag{18}$$

So far, everything except the current-algebra sector has been reduced to free fields. We now choose to represent the currents also in terms of free fields, using the Wakimoto representation:

$$J^+(z) = :\beta(z)\gamma(z)^2 - \sqrt{2}\gamma(z)\partial\phi(z) + 3\partial\gamma(z)$$

$$J^3(z) = :\beta(z)\gamma(z) - \frac{1}{\sqrt{2}}\partial\phi(z)$$

$$J^-(z) = \beta(z) \tag{19}$$

where $(\beta, \gamma)$ are commuting ghosts and $\phi$ is a free scalar field. Since we know the spins of the currents (after twisting of course), it is easy to deduce that the ghosts have spins $(0,1)$ respectively. This means they contribute a total central charge of $+2$. But the total central charge of the twisted current algebra is $27$, so we conclude that the field $\phi$ has a central charge $c_\phi = 25$, which means it must have a background charge $Q_\phi = 2\sqrt{2}$. Thus, $\phi$ is clearly identical to the Liouville field in $c = 1$ string theory.

To summarise, the full Hilbert space of our topological theory is

$$\mathcal{H} : \mathcal{H}_\phi \oplus \mathcal{H}_X \oplus \mathcal{H}_{b,c} \oplus \mathcal{H}_{B,C} \oplus \mathcal{H}_{\beta,\gamma}$$

| spins   | (0) | (0) | (2, -1) | (1, 0) | (0, 1) |
|---------|-----|-----|---------|--------|--------|
| central charge | 25  | 1   | -26     | -2     | 2      |

while the physical Hilbert space is obtained from this by taking the quotient with the two BRS charges

$$G^+ = \int cJ^+ = \int c(\beta\gamma^2 - \sqrt{2}\gamma\partial\phi + 3\partial\gamma)$$

$$Q_{U(1)} = \int C(\beta\gamma - cb - \partial X^-) \tag{21}$$

(We have defined $X^\pm = \frac{1}{\sqrt{2}}(\phi \mp iX)$.)

Note the remarkable fact that the first three Hilbert spaces above are isomorphic to the Hilbert space of the conventional $c = 1$ string quantised in the DDK formalism. The Liouville field arises from the Wakimoto representation for the $SL(2,\mathbb{R})$ current algebra, the $c = 1$ matter scalar field comes from the $U(1)$ gauge field, and the ghosts are just the fermions of the original supersymmetric WZW model, after twisting. The remaining two Hilbert spaces consist of first-order pairs of the same spins and opposite
statistics, so it is quite reasonable to expect that they cancel out in some sense. Thus, we have found a very likely candidate for a manifestly topological description of $c = 1$ string theory.

There are two ways to make this connection more convincing. One is to directly compute the double cohomology of the BRS charges above on the full Hilbert space. This has been described in detail in Ref.[1] and will not be repeated here. The result turns out to be isomorphic to the complete cohomology in the DDK formalism, described in detail in Ref.[5]. Another way is to explore the relation of this model to the KPZ formalism of $c = 1$ string theory.

4 Relation to KPZ

Let us recall the KPZ[22] formulation of non-critical string theory. One starts with an $SL(2,R)$ current algebra at level $k$, and twists the Sugawara stress-energy tensor by

$$T_{SL(2,R)} \rightarrow T_{SL(2,R)} - \partial J^3(z)$$  \hspace{1cm} (22)

The total central charge of this twisted system is

$$c = \frac{3k}{k-2} + 6k$$  \hspace{1cm} (23)

Next, one gauges the parabolic subgroup generated by $J^-(z)$[26]. This introduces a pair of anticommuting ghosts $(B, C)$ of spins $(1, 0)$, along with a BRS charge

$$Q_{KPZ} = \int B(z)(J^-(z) - 1)$$  \hspace{1cm} (24)

Note that in this procedure, the gauge field disappears completely, unlike in the case where we gauge the subgroup generated by $J^3$. This is due to the fact that the constraint is first-class[27] in the present case.

Now, to the above system (which is collectively supposed to represent the Liouville, or gravitational, degree of freedom on the worldsheet) we couple by hand a $c = 1$ matter field $X(z, \bar{z})$ and a pair of anticommuting ghosts $(b, c)$ of spins $(2, -1)$, and impose the usual string BRS cohomology via

$$Q_{BRS} = \int c(T_{SL(2,R)} + T_X + \frac{1}{2}T_{\text{ghost}})$$  \hspace{1cm} (25)

Summarising, the total Hilbert space is precisely the same as in Eq.(20) above, but instead of the two BRS charges in Eq.(21), we have to impose the ones in Eqs.(24) and (25).
Now to argue the equivalence of our coset model to the KPZ theory, it only remains to prove that the two double cohomologies on the same Hilbert space are isomorphic. The proof follows most simply from the following result\[28\]

\[ \beta^{-1}Q_{BRS} = G^+ + \{Q_{U(1)}, *\} \]  

(26)

where "*" represents some combination of the fields in the theory, whose detailed form is unimportant. Now on the left hand side, if we pass to the cohomology of \(Q_{KPZ}\) then we can set \(\beta = 1\) (recalling that in the Wakimoto representation, \(J^- = \beta\)) and we are left with \(Q_{BRS}\). On the right hand side, if we pass to the cohomology of \(Q_{U(1)}\) then the second term vanishes. Now the equation just tells us that the operators with respect to which we want to take the next cohomology are equal, hence the double cohomologies are equivalent.

This completes the proof that the topological coset model is equivalent to \(c = 1\) string theory. In the next sections we will see why it is a “more solvable” formulation of the theory.

5 Correlation Functions

In order to show that the \(k = 3\) coset model is exactly solvable, we need to investigate the available techniques to solve models of this kind. Let us first consider a different class of models: the \(N = 2\) supersymmetric minimal models labelled by a positive integer \(k\). After twisting and coupling to gravity, these models are believed to represent special points in the moduli space of the \(k + 1\) matrix models\[29\]. In continuum language, these represent the \((k,1)\) minimal models of central charge \(1 - 6(k - 1)^2/k\) coupled to ordinary gravity, from which the \((k,k')\) models can be obtained by adding marginal perturbations.

For this class of models, the first powerful technique to be discovered was the Landau-Ginzburg description\[30\]. In this approach, one identifies the above theories with the infrared fixed points of the \(N = 2\) Landau-Ginzburg (LG) theory with a single superfield, and superpotential \(X^{k+2}\). In this description, it is known how to explicitly compute correlation functions (after coupling to gravity) at least in genus 0\[31, 32\].

The gravitational primaries in the LG theory coupled to gravity are described by \(1, X, X^2, \ldots, X^k\). Let us denote the primary \(X^r\) by \(U_r\). Gravitational secondaries are obtained by multiplying these with the usual fields \(\sigma_n\) of pure topological gravity. The selection rules for the correlator \(\langle \sigma_{n_1}(U_{r_1}) \ldots \sigma_{n_N}(U_{r_N}) \rangle\) are

\[ \sum_{i=1}^{N} \left( \frac{r_i}{k+2} - 1 \right) + \sum_{i=1}^{N} n_i = (2g - 2) \frac{k + 3}{k + 2} \]  

(27)
This picture can be suitably modified for our purposes. For positive $k$, Landau-Ginzburg theory flows in the infrared to the Kazama-Suzuki coset $SU(2)_k/U(1)$. Since for many purposes the $SL(2, R)_k/U(1)$ coset is like the coset of $SU(2)$ at level minus $k$, the model we have described above should be the infrared fixed point of the LG theory with $k = -3$, hence with superpotential $X^{-1}$. This requires some extra work to define carefully, but we will see below that sphere correlators for $c = 1$ string theory can be extracted from this formalism by making only some very broad assumptions.

To start with, let us restrict to gravitational primaries and set $k = -3$ in the selection rule above. The resulting equation is genus-independent, and remarkably simple:

$$ \sum_{i=1}^{N} (r_i + 1) = 0 \quad (28) $$

This gives us the first clue about the identification of these fields with the physical fields in the $c = 1$ string (at the self-dual radius). The tachyons in this string theory have discrete momenta, labelled by integers $k_i$, satisfying momentum conservation in every genus:

$$ \sum_{i=1}^{N} k_i = 0 \quad (29) $$

Thus, we are tempted to make the identification $k_i = r_i + 1$, as a result of which we would claim that the tachyons in the LG formulation are gravitational primaries:

$$ T_k = X^{k-1} \quad (LG) \quad (30) $$

Note that tachyons with all positive and negative integer momenta are required, so we must allow fields $U_r = X^r$ for all positive and negative integer values of $r$. It is less clear that all of these are gravitational primaries, and for us it will be enough to treat only $X^r$ for positive $r$ as primaries.

In this framework, it was first noticed by Cecotti and Vafa\[33\] that the correct 4-point function of tachyons in $c = 1$ string theory can be obtained, using techniques that were developed to deal with polynomial superpotentials. Indeed, it is now clear that one can obtain the tachyon $N$-point function for all $N$, using Landau-Ginzburg theory with superpotential $X^{-1}$. This has been worked out in Ref.\[2\] using a generalization (to $k = -3$) of a formalism due to Losev\[32\] in which contact terms are dealt with explicitly, and one finds a recursion formula (on the sphere) determining correlators of $N$ tachyons in terms of those of $N - 1$ tachyons.

Let us illustrate this first for the 4-point function. Losev’s formula for this is

$$ \langle X^{r_1} X^{r_2} X^{r_3} X^{r_4} \rangle_V = \frac{d}{dt_4} \langle X^{r_1} X^{r_2} X^{r_3} \rangle_V |_{t_4 = 0} 
+ \langle C_V (X^{r_1}, X^{r_4}) X^{r_2} X^{r_3} \rangle_V 
+ \langle X^{r_1} C_V (X^{r_2}, X^{r_4}) X^{r_3} \rangle_V 
+ \langle X^{r_1} X^{r_2} C_V (X^{r_3}, X^{r_4}) \rangle_V \quad (31) $$

11
where \( C_V(X^i, X^j) \) is a contact term between two operators, for which an expression is known in the case of polynomial LG theory, where it is described by the so-called Saito pairing. For \( k = -3 \), it turns out[2] that the correct contact term is much simpler, and is given by

\[
C_V(X^i, X^j) = (i + j)X^{i+j}, \quad (i + j) < 0
\]

\[
= 0, \quad (i + j) > 0
\]

The only remaining information required is that the three-point function is given by[31]

\[
\langle X^{r_1}X^{r_2}X^{r_3} \rangle_V = \text{res}(\frac{X^{r_1}X^{r_2}X^{r_3}}{V'})
\]

(33)

With all this, the four-point function is easily calculated and one gets

\[
\langle T_{k_1}T_{k_2}T_{k_3}T_{k_4} \rangle_{g=0} = -\frac{1}{2}|k_1 + k_2| - \frac{1}{2}|k_1 + k_3| - \frac{1}{2}|k_2 + k_3| + 1
\]

(34)

which is precisely the tachyon 4-point correlator coming from matrix models of \( c = 1 \) string theory[34] and from perturbative techniques in the DDK formalism[8].

The same technique can now be applied to the recursive formula for \( N \)-point functions, due to Losev and Polyubin in the context of polynomial LG theories:

\[
\langle X^{r_1}X^{r_2} \ldots X^{r_N} \rangle_V = \frac{d}{dt_N}\langle X^{r_1}X^{r_2} \ldots X^{r_{N-1}} \rangle_{V+t_NX^{r_N}}|_{t_N=0}
\]

\[
+ \langle C_V(X^{r_1}, X^{r_N})X^{r_2} \ldots X^{r_{N-1}} \rangle_V + \ldots
\]

\[
+ \langle X^{r_1}X^{r_2} \ldots C_V(X^{r_{N-1}}, X^{r_N}) \rangle_V
\]

(35)

Choosing the kinematic region \( k_1 < 0 \) and \( k_2, k_3, \ldots, k_N > 0 \), we find[2]

\[
\langle T_{k_1}T_{k_2} \ldots T_{k_N} \rangle_{g=0} = (k_1 + 1)(k_1 + 2)\cdots(k_1 + N - 3)
\]

(36)

which is precisely the matrix-model result!

6 Partition Function

An alternative method which is in principle even more powerful, is to identify these theories with the Kazama-Suzuki (KS) coset models based on \( SU(2)_k/U(1) \). It has been shown[35] that algebraic-geometry techniques can be brought to bear on this problem, resulting in expressions for arbitrary correlators in arbitrary genus. These are described as integrals of products of Chern classes of certain bundles over moduli space.

In the KS formulation, the gravitational primaries are \( 1, g_{11}, g_{11}^2, \ldots, g_{11}^k \) (where \( g_{ab} \) is the \( SL(2, R) \)-valued matrix field of the WZW model). The primary \( g_{11}^r \) is equivalent to
the Landau-Ginzburg primary $X^r$ so we again assign it the label $U_r$. Gravitational secondaries, and the selection rules for generic correlators, are the same as in the previous section.

Starting again with genus 0, the 4-point function of gravitational primaries was explicitly computed for $k > 0$ using algebraic-geometry techniques, in Ref. [35]. This computation is performed by choosing a section of the relevant bundle and finding its divisor. The final result is

$$
\langle U_{r_1} U_{r_2} U_{r_3} U_{r_4} \rangle_{g=0} = \frac{1}{2} \left( \min(q_1 + q_2, q_3 + q_4) + \min(q_1 + q_3, q_2 + q_4) \right)
+ \min(q_1 + q_4, q_2 + q_3)) - \frac{k+1}{k+2}
$$

(37)

where $q_i = r_i/(k+2)$ are the $U(1)$ charges of the fields.

This result can now explicitly be continued to the case of interest to us. Inserting $k = -3$ and using the identification in Eq.(30), we easily find the expression in Eq.(34). Thus the KS formulation also permits the calculation of the tachyon four-point function, although in practice the LG formulation was the simpler one for this case.

Let us finally consider the case of higher genus. Here, two basic results are known. From Ref. [35], we know that the genus-$g$ partition function of the $SU(2)/U(1)$ KS model, when continued to $k = -3$, is precisely the (virtual) Euler characteristic $\hat{\chi}_g$ of the moduli space of genus-$g$ Riemann surfaces with no punctures. Now, from the matrix model compactified at the self-dual radius, we have an explicit result for the genus-$g$ partition function (subject to the assumption that the nonsinglet sector can be ignored), and the answer is

$$
Z_g = \frac{B_{2g}}{2g(2g-2)}
$$

(38)

where $B_{2g}$ are the Bernoulli numbers. Remarkably, it has been shown [36, 37, 38] that this is just the virtual Euler characteristic of the moduli space of genus-$g$ Riemann surfaces. So, our topological formulation of $c = 1$ string theory is powerful enough to reproduce the genus-$g$ partition function, directly in a continuum approach. It is easy to check that genus-$g$ correlation functions of the cosmological operator ($T_0 = X^{-1} = g_{11}^{-1}$) are also obtained correctly in our approach. Correlators of arbitrary (discrete) tachyons in higher genus have yet to be computed explicitly.

7 Conclusions

We have collected enough evidence to show that not only does our manifestly topological model correctly describe $c = 1$ string theory, but it also allows the explicit computation
of correlators which are either impossible, or very difficult to calculate in any other continuum formulation (including the conventional conformal-gauge DDK description). It should be stressed that the correlators described above are at nonzero cosmological constant, and powers of the cosmological constant can easily be inserted in the right places in the above formulae, although we have chosen to omit them for simplicity of presentation. In contrast, the DDK continuum formulation requires a perturbative treatment of the cosmological operator, in addition to certain prescriptions to allow negative numbers of insertions of this operator.

The work described here puts \( c = 1 \) string theory on the same footing as the \( c < 1 \) theory, for which a manifestly topological description has long been known. However, it leaves some interesting open questions which need to be addressed in the Lagrangian approaches (LG or KS). First of all, the role of gravitational descendants in the topological theory needs to be elucidated. Since such fields are labelled by two integers, it is tempting to conjecture that they are related to the discrete states of the \( c = 1 \) string. However, this identification has so far not been clarified sufficiently. Well-known properties of \( c = 1 \) string theory like the presence of a ground ring, and the \( W_\infty \) symmetry algebra, appear explicitly in the coset CFT$[1]$, but are less obvious in the Lagrangian formulations. Indeed, the important question is not how the KS coset (described in terms of conformal fields) is related to the DDK \( c = 1 \) string, but rather, how exactly the Lagrangian formulations of this theory (which are exactly solvable) are related to the CFT formulation.

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