KOSZUL’S SPLITTING THEOREM AND THE SUPER ATIYAH CLASS

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ABSTRACT. In this article we present a self-contained account of two important results in complex supergeometry: (1) Koszul’s Splitting theorem and (2) Donagi and Witten’s decomposition of the super Atiyah class. These results are related in the same sense that global holomorphic connections on a holomorphic vector bundle are ‘related’ to the Atiyah class of that vector bundle—the latter being the obstruction to the existence of the former. In complex supergeometry: Koszul’s theorem pertains to the existence of supermanifold splittings whereas the super Atiyah class accordingly pertains to obstructions to the existence of splittings.

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A foundational problem in the area of complex supergeometry concerns the classification of supermanifolds. Berezin gave the first steps toward such a classification, detailed posthumously in [Ber87, p. 164]. Subsequently this classification was, in a sense, resolved by Batchelor in [Bat79] in the smooth setting; and, in the holomorphic setting, extended by Green in [Gre82], Manin in [Man88] and Onishchik in [Oni99]. In this article we focus on the investigation into this classification by Koszul in [Kos94]; and aim to see it as consistent with another investigation offered more recently by Donagi and Witten in [DW14], en route to their proof of the non-splitness of the supermoduli space of marked curves.

Koszul’s Theorem, being the focus of Part II of this article, asserts simply: any global, even, holomorphic connection a supermanifold furnishes it with a unique splitting. Donagi and Witten’s decomposition then, the focus of Part III, asserts contrapositively: the (affine) Atiyah class of a supermanifold projects onto the primary obstruction to splitting that supermanifold. More classically on holomorphic vector bundles over complex manifolds, Atiyah in [Ati57] constructed a cohomology class which measures precisely the failure for the vector bundle in question to admit a global, holomorphic connection. This result has been generalised to the supermanifold setting by Bruzzo et. al. in [BBHR91, p. 161]. Specialising to the tangent bundle, we have: the affine Atiyah class\(^1\) studied by Donagi and Witten in [DW14] measures the obstruction to the existence of a global holomorphic connection, as studied by Koszul in [Kos94]. Therefore, we might expect a relation between the (affine) Atiyah class and the obstruction classes that goes beyond the primary level. Such a relation was proposed in the author’s doctoral thesis [Bet16].

In this article we will be constrained to a largely pedagogical, self-contained account of Koszul’s theorem and its proof in Part II; and Donagi and Witten’s decomposition and proof in Part III. While these main results are known, there are two results we obtain en route to our proof of these main results which we might claim is new. The

\(^1\)By ‘affine’ Atiyah class, we mean the Atiyah class of the tangent bundle.
first result concerns the reduction of our proof of Koszul’s theorem to that of lifting the Euler vector field in Theorem II.2.5. The second result concerns the relation between the affine Atiyah class of split models and supermanifolds in §III.3, Theorem III.3.3. More generally, we hope the treatment given in this article of Koszul’s theorem and Donagi and Witten’s decomposition respectively will complement existing treatments in the literature; in addition to providing insight into the splitting problem for complex supermanifolds. The question of relating the (affine) Atiyah class to the higher obstructions to splitting and applications to the characterisation of algebraic connections on supermanifolds will be deferred to a planned sequel.

In what follows we present a summary of the essential points in this article.

**Article Summary.** This article is divided into three distinguished parts, starting with preliminaries in Part I.

**Summary of Part I.** Here one can find key definitions such as that of a model in Definition I.1.1, supermanifold framings in Definition I.1.2 and morphisms in Definition I.1.3 leading thereby to categories of ‘supermanifolds’ and ‘framed supermanifolds’. Proposition I.1.5 is a general result relating these categories, clarifying that any supermanifold of a prescribed dimension will admit a framing, albeit non-canonically. With the notion of a splitting then established in Definition I.1.6, the following Lemma I.1.7 gives equivalent characterisations of splittings, serving as the foundation for deciding when splittings exist. The tangent sheaf of a supermanifold and its strata are then introduced in (I.2.1.1). Unlike the general case, the tangent sheaf of split models (i.e., split supermanifolds) can be decomposed into a \( \mathbb{Z} \)-graded module. Each graded component is related to the modelling data in Lemma I.2.2 while Corollary I.2.3 concerns the tangent sheaf itself. We conclude with Lemma I.2.5 relating both the structure sheaf and tangent sheaf of a supermanifold with those of its split model, adapting similar constructions in algebra relating filtered objects to their associated graded objects. The relations in Lemma I.2.5 will be essential ingredients in the conceptual and technical analyses of supermanifolds to be presented in the sections to follow.

**Summary of Part II.** As the title suggests, Part II concerns Koszul’s Splitting Theorem. It begins with Definition II.1.1, establishing what is meant by a connection
on a supermanifold and in Definition II.1.3 specialised to what is meant by Koszul in [Kos94] (see the following Remark II.1.4). Theorem II.1.2 is a straightforward adaptation to the supermanifold setting of the affine space of connections. Its proof is deferred to Appendix A. With this preamble established, we state Koszul’s Splitting Theorem in Theorem II.1.5. Its proof concerns the subsequent sections in Part II. Our method of proof loosely follows the original, in that we aim to establish a supermanifold splitting based on an inductive argument. Rather than focussing on a global connection directly, we follow the intuitive remarks made by Koszul in the introduction in [Kos94]. That is, we consider the problem of lifting a certain vector field—the Euler vector field defined in Definition II.2.1, which exists on any split model \( \hat{X} \), to a general supermanifold \( X \) with \( \hat{X} \) as its split model. With the relations between the structure sheaves and tangent sheaves of \( X \) and \( \hat{X} \) from Lemma I.2.2, the problem of lifting the Euler vector field on \( \hat{X} \) to \( X \), defined in Definition II.2.2, becomes a cohomological problem. This leads to the notion of Euler differential of \( X \) in Definition II.2.3. Theorem II.2.5, whose proof is deferred to §II.4.1, then clarifies the motivation for studying the Euler differential, being: the Euler differential of \( X \) vanishes, if and only if \( X \) is split. In §II.3 we show explicitly how a splitting of \( X \) can be obtained if we can lift the Euler vector field from \( \hat{X} \) to \( X \). The afore-mentioned Theorem II.2.5 will follow as a consequence of the Euler differential being the obstruction to lifting the Euler vector field.

Our proof of Koszul’s Theorem is now reduced to the following: show that any global (even), affine connection on \( X \) can be used to solve \( \delta \epsilon_{\hat{X}} = 0 \), where \( \delta \epsilon_{\hat{X}} \) denotes the Euler differential of \( \hat{X} \). Equivalently, that it will lift the Euler vector field. This is the subject of the concluding §II.4. The base case of our argument-by-induction is established via a generic, cohomological vanishing property in Proposition II.4.4. It implies the existence of a ‘1-lift’ of the Euler differential (Corollary II.4.6); and hence a ‘mod-1 lift’ of the Euler vector field (see Proposition II.4). The rest of §II.4 is then devoted to showing: if there exists a ‘mod-\( \ell \) lift’ of the Euler vector field, then a global (even), affine connection \( \nabla \) can be used to define a mod-(\( \ell + 1 \)) lift. This is achieved by the important, shear-like property that \( \nabla \) will fix any 1-lift \( H \) of Euler vector vector field and shift the other components of \( H \) to a higher stratum.
in the tangent sheaf. The assumption that $\nabla$ be even is essential and emphasised in the proof of Proposition II.4.13. By induction then, we can lift the Euler vector field and thereby split $\mathfrak{X}$, whence Koszul’s Theorem.

In Appendix B we review earlier work by Onischik in [Oni98] on liftings of vector fields from the split model to supermanifolds more generally. While this work is not directly related to ours, there is an important subtlety to resolve, being Onischik’s Theorem B.4(i), the mapping in (B.4) and Theorem II.2.5 concerning the Euler vector field lift. In the comments succeeding Theorem II.2.5, we argue that Onischik’s results are consistent with Theorem II.2.5; and subsequently deduce a new characterisation of splitting cotangent supermanifolds by reference to the Euler vector field in Theorem B.5.

Summary of Part III. Our focus now turns to the obstructions to the existence of supermanifold splittings, rather than the splittings directly. As such, it is largely independent of Part II. Before looking at the Atiyah class of supermanifolds directly, we begin with a brief review of one of the main results on Atiyah classes in Theorem III.1.1, obtained originally by Atiyah in [Ati57]. Where its generalisation to the supermanifold setting is concerned, we defer to [BBHR91, p. 161]. In contrast to the case of manifolds, the affine Atiyah space of a supermanifold\(^2\) is naturally $\mathbb{Z}_2$-graded. In Lemma III.1.2 we see that it suffices to only consider the even component of this space, as it is here where the affine Atiyah class will be valued. Theorem III.1.3 concerns its relation to global, even, holomorphic connections. Now before we can state Donagi and Witten’s decomposition theorem, whose proof forms the focal point of this Part III, it is necessary to digress on obstruction theory more generally for supermanifolds in §III.2. The material presented herein can mostly be found in the literature. Where our treatment of obstruction theory differs however lies in our attempt to preserve ‘functoriality in supermanifolds’. After reviewing the relevant material on obstruction theory as it can be found in the literature, we look give a more natural treatment of this material in §III.2.2.\(^3\) Proposition III.2.10 confirms

\(^2\) see (III.1.1.1)
\(^3\) c.f., Theorem III.2.7 in contrast to Theorem III.2.1
that our treatment of obstruction theory in §III.2.2 is consistent with that in the literature, given in §III.2.1.

We include, in Appendix C, a digression on the classification of supermanifolds necessary for understanding the material on obstruction theory given in §III.2.2. As with our treatment of obstruction theory, our treatment of the classification in Appendix C is tailored toward a viewpoint ‘functorial in supermanifolds’. See the concluding section of Appendix §C.

Our study of Atiyah classes on supermanifolds is divided into two sections. In §III.3 we study the Atiyah class of split models (i.e., split supermanifolds); with the more general case, leading to Donagi and Witten’s decomposition, deferred to §III.4. Concerning the split case in §III.3 we begin a preliminary decomposition, preempting the forthcoming, general decomposition in Proposition III.3.1. That is, we obtain a decomposition of the Atiyah space of the split model in III.3.1.1; and subsequently a decomposition of a projection of the Atiyah class. In the spirit of Lemma I.2.5 on the relation between supermanifolds and their split models, we give a relation between the Atiyah class of split models with supermanifolds more generally in Theorem III.3.3. The body of our article concludes with §III.4, which is devoted to the statement and proof of Donagi and Witten’s decomposition of the affine Atiyah class in Theorem III.4.1. The statement of Theorem III.4.1 is almost identical to Proposition III.3.1 concerning split models. A crucial difference here being the appearance of the primary obstruction class in (III.4.0.2). The proof of Theorem III.4.1, which mainly concerns the appearance of this obstruction class, occupies what remains of §III.4. Our method for proving many of the statements in §III.3 and §III.4 reduce to forming appropriate, commutative diagrams of sheaves, inducing commutative diagrams on cohomology whence our proposed statements follow.

In the concluding remarks we speculate on some future directions in which Koszul’s Theorem and Donagi and Witten’s decomposition might lead. Among these include the characterisation of connections on supermanifolds; generalising Donagi and Witten’s decomposition to the ‘full, affine Atiyah class’, rather than its restriction to the reduced space; and a question on the nature of Atiyah classes of sheaves on
supermanifolds more generally. From a broader perspective, the central constructs in this article on which the main results essentially depend are: the Euler differential of supermanifolds in Part II; and the primary obstruction to splitting in Part III. In Appendix D we include, for completeness, a commentary on these constructs including a relation in Theorem D.2.
Part I. Preliminary Theory

I.1. Definitions and Notation

I.1.1. Supermanifolds. A supermanifold $\mathfrak{X}$ is a locally ringed space $(X, \mathcal{O}_X)$ together with a sheaf of local, supercommutative rings $\mathcal{O}$ augmented over $\mathcal{O}_X$, i.e., equipped with an epimorphism $\alpha : \mathcal{O} \to \mathcal{O}_X \to 0$. We can then write $\mathfrak{X} = ((X, \mathcal{O}_X), \mathcal{O}, \alpha)$. The sheaf $\mathcal{O}$ is referred to as the structure sheaf of the super-space $\mathfrak{X}$ and so, accordingly, we denote: $\mathcal{O} \cong \mathcal{O}_X$ and $\mathfrak{X} \cong (X, \mathcal{O}_X)$. We say $\mathfrak{X}$ is a supermanifold if there exists a locally free (l.f.) $\mathcal{O}_X$-module $E$ such that $\mathcal{O} \otimes_{\mathcal{O}_X} E$ and $\wedge^\bullet \mathcal{O}_X E$ are locally isomorphic.\(^4\) Fixing some l.f., $\mathcal{O}_X$-module $E$, a supermanifold $\mathfrak{X} = (X, \mathcal{O}_X)$ such that $\mathcal{O} \otimes_{\mathcal{O}_X} E$ and $\wedge^\bullet \mathcal{O}_X E$ are locally isomorphic is said to be modelled on $((X, \mathcal{O}_X), E)$ or simply $(X, E)$. The data $(X, E)$ is referred to as modelling data.

**Definition I.1.1.** Let $(X, E)$ be modelling data. We refer to the space $X$ as the reduced space; and the l.f., sheaf $E$ as the odd contangent bundle. This motivates the notation $T^*_{X, -} \cong E$.

More intrinsically, if we are given a supermanifold $\mathfrak{X}$, its modelling data will be referred to by: $((\mathfrak{X}), T^*_{\mathfrak{X}, -})$. The dimension of any supermanifold $\mathfrak{X}$ modelled on $(X, T^*_{X, -})$ is $(\dim X | \text{rank } T^*_{X, -})$, with:

$$\dim_+ \mathfrak{X} = \dim X \quad \text{ and } \quad \dim_- \mathfrak{X} = \text{rank } T^*_{X, -}. \quad (I.1.1.1)$$

With $\mathfrak{X} = (X, \mathcal{O}_X)$ a supermanifold modelled on $(X, T^*_{X, -})$, the augmentation $\mathcal{O}_X \to \mathcal{O}_X$ corresponds, geometrically, to an embedding of spaces $|\mathfrak{X}| \subset \mathfrak{X}$. The supermanifold $(X, \wedge^\bullet \mathcal{O}_X T^*_{X, -})$, where the augmentation is the projection of $\wedge^\bullet \mathcal{O}_X T^*_{X, -}$ onto its degree zero component, is referred to as the split model. We denote this supermanifold by $\hat{\mathfrak{X}}$. By definition, any supermanifold $\mathfrak{X}$ with $\dim \mathfrak{X} = \dim \hat{\mathfrak{X}}$ will be locally isomorphic to $\hat{\mathfrak{X}}$. Hence we can also refer to the split model $\hat{\mathfrak{X}}$ intrinsically, by referring to it as the split model associated to a given supermanifold $\mathfrak{X}$.

I.1.2. Framings, Models and Splittings. The structure sheaf $\mathcal{O}_X$, by virtue of being supercommutative is globally $\mathbb{Z}_2$-graded, so: $\mathcal{O}_X \cong \mathcal{O}_{X, +} \oplus \mathcal{O}_{X, -}$, where $\mathcal{O}_{X, +} \subset \mathcal{O}_X$ is the sheaf of even subalgebras; and $\mathcal{O}_{X, -}$ is an $\mathcal{O}_{X, +}$-module, referred to as the

\(^4\)Note, the topology of $\mathfrak{X}$ comes from the topological space $X$.
fermionic module. The kernel of the augmentation \( \mathcal{O}_X \to \mathcal{O}_X \), denoted \( \mathcal{J}_X \), is referred to as the fermionic ideal. We have the following relations between the fermionic ideal, the reduced space \( X \) and the fermionic module:

\[
\mathcal{O}_X \cong \frac{\mathcal{O}_X}{\mathcal{J}_X} \cong \frac{\mathcal{O}_X}{\mathcal{J}_X^+} \quad \text{and} \quad \frac{\mathcal{J}_X}{\mathcal{J}_X^2} \cong \frac{\mathcal{O}_X}{\mathcal{J}_X^2} \quad \text{(I.1.2.1)}
\]

where the above isomorphisms are as \( \mathcal{O}_X \)-modules. Now more generally we have, for any \( m > 0 \), a short exact sequence

\[
0 \to \frac{\mathcal{J}_X^m}{\mathcal{J}_X^{m+1}} \to \frac{\mathcal{O}_X}{\mathcal{J}_X^m} \to \frac{\mathcal{O}_X}{\mathcal{J}_X^{m+1}} \to 0. \quad \text{(I.1.2.2)}
\]

The isomorphisms in (I.1.2.1) reveal that the sequence in (I.1.2.2) splits when \( m = 1 \).

**Definition I.1.2.** A choice of splitting of the sequence in (I.1.2.2) for \( m = 1 \) is referred to as a framing for \( \mathfrak{X} \).

In order to give a more categorical understanding of framings we need to establish what is meant by morphisms between supermanifolds. Starting from their definition as locally ringed spaces we have:

**Definition I.1.3.** Let \( \mathfrak{X} \) and \( \mathfrak{X}' \) be supermanifolds with respective structure sheaves \( \mathcal{O}_X \) and \( \mathcal{O}_{X'} \) and reduced spaces \( X \) and \( X' \). A morphism \( f : \mathfrak{X} \to \mathfrak{X}' \) consists of:

(i) a morphism \( |f| : X \to X' \) of reduced spaces;

(ii) a morphism of algebras \( f^\sharp : \mathcal{O}_{X'} \to \mathcal{O}_X \) which commutes with the respective augmentations, i.e., a commutative diagram:

\[
\begin{array}{ccc}
|f|^* \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X \\
\downarrow & & \downarrow \\
|f|^* \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X \\
\end{array}
\]

If \( f \) is invertible, it is an isomorphism.

**Definition I.1.4.** A morphism of supermanifolds \( f : \mathfrak{X} \to \mathfrak{X}' \) is said to be even resp. odd if, with respect to the extant \( \mathbb{Z}_2 \)-gradings \( \mathcal{O}_X \cong \mathcal{O}_X^+ \oplus \mathcal{O}_X^- \) and \( \mathcal{O}_{X'} \cong \mathcal{O}_{X'}^+ \oplus \mathcal{O}_{X'}^- \) that

\[
\text{im } f^* \mathcal{O}_{X',\pm} \subset \mathcal{O}_{X,\pm} \quad \text{resp.} \quad \text{im } f^* \mathcal{O}_{X,\pm} \subset \mathcal{O}_{X',\pm}
\]
where the image is taken under the algebra morphism $f^\sharp$ from Definition I.1.3(ii).

We can now form two categories of supermanifolds. The first is of framed supermanifolds and the second of modelled supermanifolds. As suggested by the name, this first category comprises framed supermanifolds of a given dimension $(p|q)$, denoted $\text{SM}_{(p|q)}^{\text{fr}}$. Morphisms in this category are morphisms of supermanifolds which fix the framing. The second category is that of supermanifolds modelled on $(X, T^*_X, -)$. Here again we have a category with supermanifolds modelled on $(X, T^*_X, -)$ as its objects; and morphisms being those which do not change the modelling data $(X, T^*_X, -)$. This category is denoted $\text{SM}_{(X,T^*_X,-)}$.

**Proposition I.1.5.** There exists an equivalence of categories:

$$\text{SM}_{(p|q)}^{\text{fr}} \cong \text{SM}_{(X,T^*_X,-)}$$

for some $(X, T^*_X, -)$.

**Proof.** Recall that the modelling data determines the dimension of any supermanifold it models by (I.1.1.1). Hence the proposed equivalence makes sense dimensionally if $\dim X = p$ and $\text{rank } T^*_X = q$. Now by definition we know that a framing of any supermanifold $\mathfrak{X}$ consists of an isomorphism $O_X \cong O_X \oplus (J_X / J^2_X)$, where $J_X / J^2_X$ is an l.f., sheaf on $X$. Setting $T^*_X, - \cong J_X / J^2_X$ shows that any supermanifold $\mathfrak{X}$ with a fixed framing will be modelled on $(X, T^*_X, -)$ and so establishes the desired, categorical equivalence. □

The notion of supermanifold morphisms from Definition I.1.3 leads to splittings as follows:

**Definition I.1.6.** A supermanifold $\mathfrak{X}$ is split if it is isomorphic to its associated split model $\hat{\mathfrak{X}}$. Any such isomorphism is referred to as a splitting.

**Lemma I.1.7.** The following statements are equivalent:

(i) $\mathfrak{X}$ is split;
(ii) the sequence in (I.1.2.2) splits for all $\ell$;
(iii) taking W.L.O.G., $\mathfrak{X}$ modelled on $(X, T^*_X, -)$, that $O_X \cong \wedge_{O_X} T^*_X, -$.

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<sup>5</sup>c.f., Proposition I.1.5.
(iv) the following two short exact sequences split:

\[ 0 \to \mathcal{J}_X \to \mathcal{O}_X \to \mathcal{O}_X \to 0 \quad \text{and} \quad 0 \to \mathcal{J}_X^2 \to \mathcal{J}_X \to T^*_{X,-} \to 0 \]

where we have identified \( T^*_{X,-} = \mathcal{J}_X / \mathcal{J}_X^2 \).

\[ \square \]

Lemma I.1.7(ii) and (iii) follow straightforwardly from the definition; while Lemma I.1.7(iv) is a classical result in supermanifold theory. A proof was provided by the author in [Bet18a, Appendix A]. Note that by Lemma I.1.7(ii), any splitting of \( X \) modulo \( \mathcal{J}_X^2 \) will define a framing.

I.2. THE TANGENT SHEAF

Most of the material in this section can be found in [Oni98, Oni99]. We present it here largely to establish notation and key, foundational results to be called upon in later parts of this article.

I.2.1. The Adic Filtration. The tangent sheaf of a supermanifold \( X \) is defined as the sheaf of \( \mathcal{O}_X \)-derivations \( T_X \cong \text{Der}_{\mathcal{O}_X} \mathcal{O}_X \). With \( \mathcal{J}_X \subset \mathcal{O}_X \) the fermionic ideal, powers of this ideal define a filtration of the tangent sheaf \( T_X \). With respect to this filtration the \( m \)-th strata, denoted \( T_X^{(m)} \), is defined as:

\[ T_X^{(m)} \cong \{ \nu \in T_X \mid \nu(\mathcal{J}_X^\ell) \subset \mathcal{J}_X^{\ell+m} \} \quad (I.2.1.1) \]

for any \( \ell \geq 0 \) and where \( \mathcal{J}_X^0 = \mathcal{O}_X \). While we have \( T_X^{(m+1)} \subset T_X^{(m)} \), note that \( T_X \neq T_X^{(0)} \). Rather, we have the following characterisation:

Lemma I.2.1.

\[ T_X^{(0)} \cong \left\{ \nu \in T_X \mid (\mathcal{J}_X \xrightarrow{\nu} \mathcal{O}_X \to \mathcal{O}_X) = 0 \right\} . \]

Proof. Using firstly that \( \mathcal{O}_X = \mathcal{O}_X / \mathcal{J}_X \); and secondly that \( \nu(\mathcal{J}_X) \subset \mathcal{J}_X \) if and only if \( \nu \in T_X^{(0)} \) it follows that the composition \( \mathcal{J}_X \xrightarrow{\nu} \mathcal{O}_X \to \mathcal{O}_X \) must vanish. \[ \square \]

We set \( T_X^{(-1)} = T_X \). By Lemma I.2.1, non-trivial elements of the quotient \( T_X^{(-1)} / T_X^{(0)} \) comprise derivations \( \nu \) such that the composition \( \mathcal{J}_X \xrightarrow{\nu} \mathcal{O}_X \to \mathcal{O}_X \) is non-trivial.
As the fermionic ideal $\mathcal{J}_X$ has nilpotence degree\(^6\) $\dim_- \mathfrak{X}$ it follows that the adic filtration on $T_{\mathfrak{X}}$ in (I.2.1.1) will define a filtration of length $\dim_- \mathfrak{X} + 1$.

### I.2.2. Split Tangents.

In the case where $\mathfrak{X} = \hat{\mathfrak{X}}$ is the split model its structure sheaf is $\mathbb{Z}$-graded as an $\mathcal{O}_X$-module.\(^7\) Hence this filtration will reduce to a $\mathbb{Z}$-grading and we can write $T_{\mathfrak{X}} \cong \bigoplus_{m \geq -1} T^m_{\mathfrak{X}}$ as $\mathcal{O}_X$-modules.\(^8\) Each summand can subsequently be understood as follows in relation to a model.

**Lemma I.2.2.** Let $\hat{\mathfrak{X}}$ be modelled on $(X, T^*_X, -)$. Then for each $m$ we have a short exact sequence of $\mathcal{O}_X$-modules:

$$0 \rightarrow \wedge^{m+1} T^*_{X, -} \otimes_{\mathcal{O}_X} T_{X, -} \rightarrow T^m_{\hat{\mathfrak{X}}} \rightarrow \wedge^m T^*_{X, -} \otimes_{\mathcal{O}_X} T_X \rightarrow 0$$

For $m = -1$ we have: $T^{-1}_{\hat{\mathfrak{X}}} \cong T_{X, -}$.

**Proof.** See Onishchik in [Oni98, p. 3]. □

**Corollary I.2.3.** Let $\hat{\mathfrak{X}}$ be the split model, modelled on the data $(X, T^*_X, -)$. Then

$$T_{\hat{\mathfrak{X}}} \otimes_{\mathcal{O}_{\hat{\mathfrak{X}}}} \mathcal{O}_X \cong T_X \oplus T_{X, -}$$

**Proof.** By Lemma I.2.2 we have:

$$0 \rightarrow T^*_X \otimes T_{X, -} \rightarrow T^{(0)} \rightarrow T_{\hat{\mathfrak{X}}} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow T_{X, -} \rightarrow T^{-1}_{\hat{\mathfrak{X}}} \rightarrow 0 \quad \text{(I.2.2.1)}$$

The fermionic ideal $\mathcal{J}_{\hat{\mathfrak{X}}} \subset \mathcal{O}_{\hat{\mathfrak{X}}}$ can be identified with $\bigoplus_{m > 0} \wedge^m T^*_{X, -}$. Set $T_{\hat{\mathfrak{X}}, +} = \bigoplus m T^{(2m)}_{\hat{\mathfrak{X}}}$ and $T_{\hat{\mathfrak{X}}, -} = \bigoplus m T^{(2m+1)}_{\hat{\mathfrak{X}}}$. Note that $T_{\hat{\mathfrak{X}}, \pm}$ are $\mathcal{O}_{\hat{\mathfrak{X}}, \pm}$-modules. Now with $\mathcal{O}_X = \mathcal{O}_{\hat{\mathfrak{X}}}/\mathcal{J}_{\hat{\mathfrak{X}}}$ and the general rule: for any l.f., $\mathcal{O}_{\hat{\mathfrak{X}}}$-module $\mathcal{M}$ that $\mathcal{M}/\mathcal{J}_{\hat{\mathfrak{X}}} \mathcal{M} \cong \mathcal{M} \otimes_{\mathcal{O}_{\hat{\mathfrak{X}}}} (\mathcal{O}_{\hat{\mathfrak{X}}}/\mathcal{J}_{\hat{\mathfrak{X}}}) = \mathcal{M} \otimes_{\mathcal{O}_{\hat{\mathfrak{X}}}} \mathcal{O}_X$ we can deduce:

$$T_{\hat{\mathfrak{X}}} \otimes_{\mathcal{O}_{\hat{\mathfrak{X}}}} \mathcal{O}_X \cong \big( T_{\hat{\mathfrak{X}}, +} \otimes_{\mathcal{O}_{\hat{\mathfrak{X}}, +}} \mathcal{O}_X \big) \oplus \big( T_{\hat{\mathfrak{X}}, -} \otimes_{\mathcal{O}_{\hat{\mathfrak{X}}, -}} \mathcal{O}_X \big)$$

$$\cong \left( \frac{T^{(0)}_{\hat{\mathfrak{X}}}}{\mathcal{J}_{\hat{\mathfrak{X}}} T_{\hat{\mathfrak{X}}} \cap T^{(0)}_{\hat{\mathfrak{X}}} \oplus \cdots} \right) \oplus \left( \frac{T^{-1}_{\hat{\mathfrak{X}}}}{\mathcal{J}_{\hat{\mathfrak{X}}} T_{\hat{\mathfrak{X}}} \cap T^{-1}_{\hat{\mathfrak{X}}} \oplus \cdots} \right)$$

$$\cong T_X \oplus T_{X, -} \quad \text{(by (I.2.2.1))}$$

\(^6\)i.e., that $\mathcal{J}_X^m = (0)$ for $m > \dim_- \mathfrak{X}$
\(^7\)c.f., Lemma I.1.7(iii).
\(^8\)We use a different bracket style $\{-\}$ in the superscript, in contrast to $(-)$, to distinguish from the generic case. Their relation on the split model is: $T^{(m)}_{\hat{\mathfrak{X}}} \cong \bigoplus_{l \geq m} T^{(l)}_{\hat{\mathfrak{X}}}$. 
I.2.3. **Initial Forms.** An important observation for the purposes of this article concerns the relation between the structure sheaves $\mathcal{O}_\mathcal{X}$ and $\hat{\mathcal{O}}_\mathcal{X}$; and between their corresponding derivations. The latter, as has been observed earlier, is a $\mathbb{Z}$-graded sheaf. On $\mathcal{X}$, with $\mathcal{J}_\mathcal{X} \subset \mathcal{O}_\mathcal{X}$ the fermionic ideal, its powers define an adic filtration on $\mathcal{O}_\mathcal{X}$. With respect to this filtration we can realise $\hat{\mathcal{O}}_\mathcal{X}$ as its associated graded sheaf, i.e., that

$$\mathcal{O}_\mathcal{X} \cong \bigoplus_{m \geq 0} \mathcal{J}_\mathcal{X}^m / \mathcal{J}_\mathcal{X}^{m+1} \tag{I.2.3.1}$$

Appropriating a classical construction from commutative algebra now\(^9\) we have a mapping of sheaves of sets: $\text{in} : \mathcal{O}_\mathcal{X} \to \hat{\mathcal{O}}_\mathcal{X}$ sending any $F \in \mathcal{O}_\mathcal{X}$ to its initial form defined as follows: for $\ell$ the least integer such that $F \in \mathcal{O}_\mathcal{X}^{(\ell)}$ we set $\text{in}(F) \triangleq F \mod \mathcal{J}_\mathcal{X}^{\ell+1}$, which is an element of $\mathcal{O}_\mathcal{X}$ by (I.2.3.1).

**Remark I.2.4.** The initial form is a mapping $\text{in} : \mathcal{O}_\mathcal{X} \to \hat{\mathcal{O}}_\mathcal{X}$ as sheaves of sets. Both of these sheaves are over $\mathbb{C}$, i.e., that $\mathbb{C} \subset H^0(X, \mathcal{O}_\mathcal{X})$ and $\mathbb{C} \subset H^0(X, \hat{\mathcal{O}}_\mathcal{X})$, meaning we can multiply germs of sections by complex numbers. Multiplication by complex scalars will not change the $\mathcal{J}_\mathcal{X}$-adic filtration on $\mathcal{O}_\mathcal{X}$ however, so it follows that the initial form will be a $\mathbb{C}$-linear mapping.

Where the tangent sheaf is concerned we have a parallel description. As has been observed earlier, the tangent sheaf of the split model $T_\mathcal{X}$ is $\mathbb{Z}$-graded. Onishchik in [Oni98, p. 3] illustrates how it can be realised as the associated graded sheaf to the filtration on $T_\mathcal{X}$ from (I.2.1.1). That is:

$$T_\mathcal{X}^{(m)} \cong \frac{T_\mathcal{X}^{(m)}}{T_\mathcal{X}^{(m+1)}} \quad \text{and so} \quad T_\hat{\mathcal{X}} \cong \bigoplus_{m \geq -1} \frac{T_\mathcal{X}^{(m)}}{T_\mathcal{X}^{(m+1)}}. \tag{I.2.3.2}$$

On $\mathcal{X}$ now, set $\mathcal{O}_\mathcal{X}^{(m)} = \mathcal{J}_\mathcal{X}^m$; and denote by $\mathcal{O}_\mathcal{X}[n]$ resp. $T_\mathcal{X}[n]$ the stratum shifted by $n$, e.g., $(\mathcal{O}_\mathcal{X}[n])^{(m)} = \mathcal{O}_\mathcal{X}^{(m+n)}$ and $(T_\mathcal{X}[n])^{(m)} = T_\mathcal{X}^{(m+n)}$. We summarise the observations in (I.2.3.1) and (I.2.3.2) in the following:

---

\(^9\)see e.g., [Eis95, §5.1, p. 147].
Lemma I.2.5. There exists a short exact sequences of sheaves of sets on $X$:

\[ 0 \rightarrow \mathcal{O}_X[1] \rightarrow \mathcal{O}_\hat{x} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathcal{T}_X[1] \rightarrow \mathcal{T}_\hat{x} \rightarrow 0. \]

The sequences above will be referred to as initial form sequences.

Remark I.2.6. The comments in Remark I.2.4 pertaining to the structure sheaf apply also to the tangent sheaf. That is, the initial form mappings on germs of tangent vectors will be $\mathbb{C}$-linear.
Part II. Koszul’s Theorem

II.1. Introduction and Statement

Following classical conventions, an affine connection on a supermanifold $\mathfrak{X}$ is a connection $\nabla$ on its tangent bundle. It is global if it can be defined as a mapping of sheaves $^{10} T_\mathfrak{X} \otimes_C T_\mathfrak{X} \to T_\mathfrak{X}$ subject to $\mathcal{O}_\mathfrak{X}$-linearity in the first argument and the $\mathcal{O}_\mathfrak{X}$-Leibnitz rule in its second. More precisely:

**Definition II.1.1.** A global, affine connection on a supermanifold $\mathfrak{X}$ is a mapping of sheaves $\nabla : T_\mathfrak{X} \otimes_C T_\mathfrak{X} \to T_\mathfrak{X}$ such that:

(i) for any $u, v \in T_\mathfrak{X}$ and $f, g \in \mathcal{O}_\mathfrak{X}$ that:

$$\nabla(fu \otimes gv) = f \nabla(u \otimes gv)$$

$$= f(u(g)v + (-1)^{|u||g|} \nabla(u \otimes v))$$

where $|u|$ and $|g|$ denote the respective $\mathbb{Z}_2$-parities, and;

(ii) for any $^{11} m', m'' \geq 0$ that $\nabla : T_\mathfrak{X}^{(m')} \otimes T_\mathfrak{X}^{(m'')} \to T_\mathfrak{X}^{(m'+m'')}$. We write $\nabla(u \otimes v) \triangleq \nabla_u v$.

A consequence of the algebra of differential forms on a supermanifold is:

**Theorem II.1.2.** The space of affine connections on a supermanifold $\mathfrak{X}$ is modelled on the sections $H^0(X, \circ^2 T_\mathfrak{X}^* \otimes T_\mathfrak{X})$, where ‘$\circ$’ refers to the symmetric tensor product. The space of even affine connections is likewise modelled on $H^0(X, (\circ^2 T_\mathfrak{X}^* \otimes T_\mathfrak{X})_+)$ where $(\circ^2 T_\mathfrak{X}^* \otimes T_\mathfrak{X})_+$ denotes the even component. $^{12}$

**Proof.** A more detailed account of the terminology introduced and proof is deferred to Appendix A. \(\square\)

---

$^{10}$Note, the tensor product is over $\mathbb{C}$ rather than $\mathcal{O}_\mathfrak{X}$. This is because we cannot identify $u \otimes (gv)$ with $(gu) \otimes v$ for all $g \in \mathcal{O}_\mathfrak{X}$ as their respective images under $\nabla$ will not coincide, as illustrated by the Leibnitz rule. They will coincide if $g$ is constant however.

$^{11}$for $m = m' = -1$ recall that $T_\mathfrak{X}^{(-1)} = T_\mathfrak{X}$ in which case we recover the connection $T_\mathfrak{X} \otimes T_\mathfrak{X} \nabla \to T_\mathfrak{X}$

$^{12}$The sheaves $T_\mathfrak{X}$ and $T_\mathfrak{X}^*$ are both $\mathbb{Z}_2$-graded sheaves of $\mathcal{O}_\mathfrak{X,+}$-modules and so are their tensor and symmetric products. As such it makes sense to form even and odd components of the resultant products.
In addition to the above definition, we will also need recourse to a notion of parity. For any vector field $u$, the mapping $u \mapsto \nabla_u$ sends $T_X \to \text{End}_\mathbb{C}T_X$. The extant $\mathbb{Z}_2$-grading of the structure sheaf $\mathcal{O}_X$ induces a $\mathbb{Z}_2$-grading of the tangent sheaf $T_X \cong T_{X,+} \oplus T_{X,-}$ and so also a grading of the endomorphisms $\text{End}_\mathbb{C}T_X \cong (\text{End}_\mathbb{C}T_X)_+ \oplus (\text{End}_\mathbb{C}T_X)_-$. This leads to the following notion of parity for affine connections.

**Definition II.1.3.** A connection $\nabla$ on $T_X$ is said to be *even* if the linear mapping $u \mapsto \nabla_u$ is valued in $(\text{End}_\mathbb{C}T_X)_+$ for all $u \in T_X$. As a tensor product over $\mathbb{C}$ an even connection induces mappings

$$T_{X,+} \otimes_\mathbb{C} T_{X,+} \to T_{X,+} \quad T_{X,+} \otimes_\mathbb{C} T_{X,-} \to T_{X,-} \quad \text{and} \quad T_{X,-} \otimes_\mathbb{C} T_{X,-} \to T_{X,+}.$$ 

**Remark II.1.4.** Koszul in [Kos94, p. 155] considers all connections to be ‘even’ in the sense of Definition II.1.3. Hence, in restricting our attention to these even connections, we will not lose the generality inherent in Koszul’s theorem.

In addition to the equivalent conditions for deducing splittings of supermanifolds in Lemma I.1.7, Koszul’s theorem from [Kos94] relates another, sufficient condition for the existence of splittings. We can now state this theorem below, whose proof will occupy Section II.4.

**Theorem II.1.5.** *(Koszul’s Theorem)* Let $\hat{X}$ be a supermanifold and suppose it admits a global, even, affine connection $\nabla$. Then $\nabla$ will define a unique splitting $X \cong \hat{X}$.

### II.2. THE EULER VECTOR FIELD AND DIFFERENTIAL

A central construct in our proof of Koszul’s Theorem concerns the Euler vector field. It is a natural, global object which exists on any split model and the problem of lifting it to a global object on given a supermanifold is equivalent to splitting that supermanifold. This is, in essence, our main observation here.

**II.2.1. Construction.** Let $\hat{X}$ be the split model associated to $(X, T_{X,-}^*)$. Its tangent sheaf $T_{\hat{X}}$ is $\mathbb{Z}$-graded and is related to exterior powers of the fermionic module $T_{X,-}^*$ as in Lemma I.2.2. We are interested in this sequence in degree zero which we present
below for convenience:
\[ 0 \longrightarrow T^{*}_{X,-} \otimes_{\mathcal{O}_X} T_{X,-} \overset{i}{\longrightarrow} T^{(0)}_{\hat{X}} \longrightarrow T_X \longrightarrow 0 \quad (\text{II.2.1.1}) \]

Identifying \( T^{*}_{X,-} \otimes_{\mathcal{O}_X} T_{X,-} \cong \mathcal{E}nd_{\mathcal{O}_X} T_{X,-} \), we have on global sections the embedding \( H^0(X, \mathcal{E}nd_{\mathcal{O}_X} T^{*}_{X,-}) \overset{i_*}{\subset} H^0(X, T^{(0)}_{\hat{X}}) \). Hence any global endomorphism of the odd cotangent bundle \( T^{*}_{X,-} \) will define a global, degree zero tangent vector on the split model.

**Definition II.2.1.** Let \( \hat{X} \) be the split model associated to \((X, T^{*}_{X,-})\). The Euler vector field on \( \hat{X} \), denoted \( \epsilon_{\hat{X}} \), is the global, degree zero vector field in the image of the identity \( 1_{T^{*}_{X,-}} \in H^0(X, \mathcal{E}nd_{\mathcal{O}_X} T^{*}_{X,-}) \) under the embedding \( H^0(X, \mathcal{E}nd_{\mathcal{O}_X} T^{*}_{X,-}) \overset{i_*}{\subset} H^0(X, T^{(0)}_{\hat{X}}) \). That is,

\[ \epsilon_{\hat{X}} \overset{\Delta}{=} i_* 1_{T^{*}_{X,-}} . \]

Hence, any split model will admit at least one global, non-zero vector field—the Euler vector field.

**II.2.2. The Euler Differential.** In contrast to the split model \( \hat{X} \), supermanifolds \( \mathfrak{X} \) more generally need not admit global, non-zero tangent vectors. To see why, recall the sequence in Lemma I.2.5 relating the tangent sheaf \( T_X \) to that of \( T_{\hat{X}} \). Conjoin this sequence in degree zero with (II.2.1.1), giving:

\[ 0 \longrightarrow T^{(1)}_X \longrightarrow T^{(0)}_X \overset{p}{\longrightarrow} T^{(0)}_{\hat{X}} \longrightarrow 0 \quad (\text{II.2.2.1}) \]
and so, on cohomology:

\[
\begin{array}{cccccccc}
0 & \downarrow &  & \downarrow &  & \downarrow &  & \downarrow &  \\
& H^0(\mathcal{E}nd_{O_X} T^*_X,-) & & \cdots & & H^0(T^0(\hat{X})) & p^* & H^0(T^0(\hat{X})) & \delta & H^1(T^1(\hat{X})) & \cdots
\end{array}
\]  

(II.2.2.2)

where \( H^j(-) = H^j(X, -) \). We arrive now at important definitions for the purposes of this article.

**Definition II.2.2.** Let \( \mathfrak{X} \) be a supermanifold with split model \( \hat{\mathfrak{X}} \). The Euler vector field \( \epsilon_{\hat{\mathfrak{X}}} \) is said to **lift to** \( \mathfrak{X} \) if there exists some global vector field \( H \in H^0(X, T^0(\hat{X})) \) such that \( p_* H = \epsilon_{\hat{\mathfrak{X}}} \).

**Definition II.2.3.** The **Euler differential** of \( \mathfrak{X} \) is the image of the Euler vector field \( \epsilon_{\hat{\mathfrak{X}}} \) under the cohomological boundary mapping \( \delta \) in (II.2.2.2).

**Remark II.2.4.** Onishchik in [Oni98] also considers the problem of lifting vector fields from the split model to supermanifolds. Accordingly, a more general definition of vector field lifting is given in [Oni98], subsuming our Definition II.2.2. While Onishchik’s paper is not directly related to the present work, it is perhaps tangentially related. For completeness therefore we present a short review of this paper in Appendix B in addition to some comments relating it to the present work.

Exactness of the bottom row in (II.2.2.2) ensures that the Euler differential of \( \mathfrak{X} \) measures the obstruction to the existence of a lift of the Euler vector field to \( \hat{\mathfrak{X}} \).\(^{13}\) Now if \( \mathfrak{X} = \hat{\mathfrak{X}} \) then obviously \( \delta \epsilon_{\hat{\mathfrak{X}}} = 0 \). This vanishing will hold in the case where \( (\mathfrak{X} = \hat{\mathfrak{X}}) \) is replaced by \( (\mathfrak{X} \cong \hat{\mathfrak{X}}) \). Our observation is that that the vanishing of the Euler differential will imply this splitting also. That is, we have:

**Theorem II.2.5.** Any supermanifold \( \mathfrak{X} \) is split if and only if its Euler differential vanishes.

\(^{13}\)c.f., [Oni98, Proposition 3.1, p. 59]
A proof of Theorem II.2.5 will become apparent following §II.3 concerning supermanifold diffeomorphisms.

II.2.3. Useful Properties. In a local coordinate system \((x|\theta)\) on \(\mathcal{X}\) the Euler vector field \(\epsilon_{\mathcal{X}}\) can be represented as the derivation:

\[
\epsilon_{\mathcal{X}}(x|\theta) = \sum_{i=1}^{\dim \mathcal{X}} \theta_i \frac{\partial}{\partial \theta_i}.
\]  

Since \(\epsilon_{\mathcal{X}}\) is global we can equate \(\epsilon_{\mathcal{X}}(x|\theta) = \epsilon_{\mathcal{X}}(y|\eta)\) on intersections of coordinate neighbourhoods \((x|\theta) \cap (y|\eta)\). Therefore we can justify deriving certain global statements from the local expression for \(\epsilon_{\mathcal{X}}\) in (II.2.3.1). Now with \(\mathcal{O}_{\mathcal{X}}\) denoting the structure sheaf of the split model \(\mathcal{X}\) we know from Lemma I.1.7(iii) that it isomorphic to an exterior algebra. Denote by \(\mathcal{O}_{\mathcal{X}}^{(m)}\) each graded piece so that, as an \(\mathcal{O}_{\mathcal{X}}\)-module, we have \(\mathcal{O}_{\mathcal{X}} \cong \bigoplus_{m \geq 0} \mathcal{O}_{\mathcal{X}}^{(m)}\). If \(\mathcal{X}\) is associated to \((X, T^*_X, -)\) then \(\mathcal{O}_{\mathcal{X}}^{(m)} \cong \wedge^m T^*_X, -\).

Direct calculation with the local expression for \(\epsilon_{\mathcal{X}}\) in (II.2.3.1) reveals the following properties which will be essential in forthcoming calculations:

**Lemma II.2.6.** The Euler vector field \(\epsilon_{\mathcal{X}}\) satisfies:

1. \(\epsilon_{\mathcal{X}}(f) = mf\) iff \(f \in \mathcal{O}_{\mathcal{X}}^{(m)}\) and;
2. \(\text{ad}_{\epsilon_{\mathcal{X}}}(v) = mv\) iff \(v \in T_{\mathcal{X}}^{(m)}\).

where \(\text{ad}_u(v) = [u, v]\).

II.3. Diffeomorphisms and Splittings

Before embarking on a proof of Koszul’s Theorem (Theorem II.1.5) we will motivate our method by explicating the relation between diffeomorphisms and splittings.

II.3.1. The Inverse Function Theorem. En route to our forthcoming definition of a supermanifold diffeomorphism, we recall the classical Inverse Function Theorem on manifolds for motivation. Let \(M\) and \(N\) be smooth manifolds and \(f : M \to N\) a differentiable function. It defines a linear map \(Tf : TM \to T^*FN\) of vector bundles over \(M\), given by \((Tf)_p : T_pM \to T_{f(p)}N\). If \((Tf)_p\) is invertible, then \(f\) will diffeomorphically map an open neighbourhood of \(p\) onto its image. If \(Tf\) is an isomorphism of vector bundles over \(M\), \(f\) is said to be a local diffeomorphism of
manifolds. Hence $f$ is locally $C^\infty$ and invertible. It is globally $C^\infty$ and invertible, i.e., a diffeomorphism, if it is, in addition, bijective as a mapping of point-sets. Where supermanifolds are concerned, a generalisation of this theorem to this case can be found in [Lei80, §3]. We will take it here as the basis for a definition however.

**Definition II.3.1.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be supermanifolds and suppose $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ is an even morphism.\(^{14}\) We say $f$ is a **diffeomorphism** if and only if:

(i) $|f|$ is a bijection and;

(ii) $Tf$ is an isomorphism.

**Lemma II.3.2.** Let $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a diffeomorphism. Then $|f|: X \rightarrow Y$ is a diffeomorphism and the odd cotangents $T^*_X, -$ and $T^*_Y, -$ are isomorphic.

**Proof.** We know that $|f|$ is a bijection. It remains to argue that $Tf$ will give an isomorphism between the respective tangent bundles $T_X$ and $T_Y$, whence we can apply the classical Inverse Function Theorem. To deduce this now we will use:

(i) let $R$ be a supercommutative ring with fermionic ideal $J \subset R$; and let $M$ and $N$ be $R$-modules. Then if $f: M \rightarrow N$ is an isomorphism, it will define an isomorphism between $M$ and $N$ modulo $J$ (see e.g., [Lei80, Proposition 1.7.2]);

(ii) With $\mathcal{O}_X$ the structure sheaf of $\mathfrak{X}$ and any flat, $\mathcal{O}_X$-module $\mathcal{M}$ we have the identification $\mathcal{M}/\mathcal{J}_X \mathcal{M} \cong \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X$.

Now with $T_{\mathfrak{X}}$ the tangent sheaf of $\mathfrak{X}$ we have the decomposition $T_{\mathfrak{X}} = T_{\mathfrak{X}},+ \oplus T_{\mathfrak{X}},-$ induced from the global, $\mathbb{Z}_2$-grading on $\mathcal{O}_X$. The tangent sheaf is locally free (c.f., [Lei80, §2]) and hence flat so we can use (ii) above. Now from Lemma I.2.5 we can identify $T_{\mathfrak{X}}$ and $\hat{T}_{\mathfrak{X}}$ modulo the fermionic ideal, i.e., that there exists an isomorphism $T_{\mathfrak{X}} \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \hat{T}_{\mathfrak{X}} \otimes_{\mathcal{O}_X} \mathcal{O}_X$. Using this, (ii) and Corollary I.2.3 gives:

$$\frac{T_{\mathfrak{X}}}{J_{\mathfrak{X}} T_{\mathfrak{X}}} \cong T_{\mathfrak{X}} \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong T_{\hat{\mathfrak{X}}} \otimes_{\mathcal{O}_X} \mathcal{O}_X = T_X \oplus T_{X, -}. \quad \text{(II.3.1.1)}$$

Now recall that we are assuming $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ is a diffeomorphism; and therefore that $Tf: T_{\mathfrak{X}} \xrightarrow{\cong} f^* T_{\mathfrak{Y}}$. By (i) and (II.3.1.1) it follows that

$$T_X \oplus T_{X, -} \cong |f|^*(T_Y \oplus T_{Y, -}). \quad \text{(II.3.1.2)}$$

\(^{14}\text{c.f., Definition I.1.4.}\)
Since $f$ must even by Definition II.3.1, the isomorphism in (II.3.1.2) will be an isomorphism of summands, i.e., it necessitates $T_X \cong |f|^* T_Y$ and $T_{X,-} \cong |f|^* T_{Y,-}$ respectively. Lemma II.3.2 now follows.

Definition I.1.3 concerned morphisms and isomorphisms of supermanifolds. How do they compare with Definition II.3.1 concerning diffeomorphisms? It may well be routine to show isomorphisms and diffeomorphisms coincide. For our purposes in this article however, we will only need to establish a particular case, being the following:

**Proposition II.3.3.** Any diffeomorphism between $X$ and its split model $\hat{X}$ will define a splitting of $X$.

Our proof of Proposition II.3.3 will resort to the following general construction.

**II.3.2. Splitting Operators.** In Lemma I.2.5 we documented that, for any smooth supermanifold $X$, its tangent sheaf fits naturally into a graded, short exact sequence of sheaves, referred to as the initial form sequence. That is, for each $m$, we have a short exact sequence,

$$0 \rightarrow T_{X}^{(m+1)} \rightarrow T_{X}^{(m)} \rightarrow T_{\hat{X}}^{(m)} \rightarrow 0.$$

More generally now, suppose $\mathcal{F}$ is a sheaf over $\mathbb{C}$ equipped with a filtration $\mathcal{J} = (\cdots \mathcal{F}^{(m+1)} \subset \mathcal{F}^{(m)} \cdots)$. Set $\hat{\mathcal{F}}^{(m)} = \mathcal{F}^{(m)}/\mathcal{F}^{(m+1)}$ as the $m$-th component of the associated graded sheaf so that, in analogy with the tangent sheaf in Lemma I.2.5, we have an initial form sequence:

$$\mathcal{S}(\mathcal{F}) : 0 \rightarrow \mathcal{F}[1] \rightarrow \mathcal{F} \rightarrow \hat{\mathcal{F}} \rightarrow 0.$$

For each $m$ we denote $\mathcal{S}^{(m)} \mathcal{F} : 0 \rightarrow \mathcal{F}^{(m+1)} \rightarrow \mathcal{F}^{(m)} \rightarrow \hat{\mathcal{F}}^{(m)} \rightarrow 0$. A splitting operator on $(\mathcal{F}, \mathcal{J})$, denoted $O^{(m)}(\mathcal{F})$ or more simply $O^{(m)}$ if confusion is unlikely, is then a certain, $\mathbb{C}$-linear operator $O^{(m)} : \mathcal{F}^{(m)} \rightarrow \mathcal{F}^{(m)}$ contrived precisely to split the sequence $\mathcal{S}^{(m)} \mathcal{F}$.

**Lemma II.3.4.** Let $O^{(m)} : \mathcal{F}^{(m)} \rightarrow \mathcal{F}^{(m)}$ be a $\mathbb{C}$-linear operator such that:

(i) $\ker O^{(m)} = \mathcal{F}^{(m+1)}$ and;

(ii) $O^{(m)}(v) \equiv v \mod \mathcal{F}^{(m+1)}$.

Then $O^{(m)}$ will define a splitting of $\mathcal{S}^{(m)} \mathcal{F}$. 
Proof. We begin by noting the existence of a commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{F}^{(m+1)} & \longrightarrow & \mathcal{F}^{(m)} & \longrightarrow & \hat{\mathcal{F}}^{\{m\}} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \text{im } \mathcal{O}^{(m)} & \longrightarrow & \hat{\mathcal{F}}^{(m)} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathcal{F}^{(m)} & \longrightarrow & \hat{\mathcal{F}}^{\{m\}} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \text{im } \mathcal{O}^{(m)} & \longrightarrow & \hat{\mathcal{F}}^{\{m\}} & \longrightarrow & 0
\end{array}
\]

The second, horizontal row of arrows show \( \text{im } \mathcal{O}^{(m)} \cong \hat{\mathcal{F}}^{\{m\}} \). The third column of arrows shows we have a commutative diagram:

\[
\begin{array}{ccc}
\mathcal{F}^{(m)} & \leftarrow & \text{im } \mathcal{O}^{(m)} \\
& \leftarrow & \text{im } \mathcal{O}^{(m)} \\
\end{array}
\]

Hence the short exact sequence \( 0 \to \ker \mathcal{O}^{(m)} \to \mathcal{F}^{(m)} \to \text{im } \mathcal{O}^{(m)} \to 0 \) is split, giving \( \mathcal{F}^{(m)} \cong \ker \mathcal{O}^{(m)} \oplus \text{im } \mathcal{O}^{(m)} \cong \mathcal{F}^{(m+1)} \oplus \hat{\mathcal{F}}^{\{m\}} \), as required. \( \square \)

The following is now a direct consequence of Lemma II.3.4.

**Corollary II.3.5.** If there exist splitting operators \( \mathcal{O}^{(m)} \) on \((\mathcal{F}, \mathcal{J})\) for each \( m \), then \( \mathcal{F} \) is split i.e., there exists an isomorphism \( \mathcal{F} \cong \bigoplus_m \hat{\mathcal{F}}^{\{m\}} \).\(^{15}\) \( \square \)

II.3.3. **Proof of Proposition II.3.3.** Let \( \mathfrak{X} \) be our supermanifold modelled on \((X, T^*_X)\). Assuming \( \mathfrak{X} \) is diffeomorphic to its split model, Definition II.3.1(ii) asserts \( T_\mathfrak{X} \) and \( T^*_\mathfrak{X} \) will be isomorphic as sheaves on \( X \). Hence the sequence relating \( T_\mathfrak{X} \) with \( T^*_\mathfrak{X} \) in Lemma I.2.5 will be split. If \( spl. \) denotes this splitting in degree zero, then it induces the mapping \( spl. : H^0(X, T^*_X) \to H^0(X, T^*_\mathfrak{X}) \) on cohomology and evidently will lift the Euler vector field to \( \mathfrak{X} \). We set:

\[
\epsilon_\mathfrak{X} \overset{\Delta}{=} spl.* \epsilon_\mathfrak{X}.
\]  

\(^{15}\)Note, we are not assuming any restrictions on \( \mathcal{J} \) such as finiteness or nilpotency. This result is therefore very general and formal.
Our strategy for proving Proposition II.3.3 now will be to construct splitting operators on the structure sheaf \((\mathcal{O}_X, \mathcal{J}_X)\), where \(\mathcal{J}_X\) is the fermionic ideal whose powers define a filtration on \(\mathcal{O}_X\). We can then deduce splitness by Lemma I.1.7(iv). This filtration on \(\mathcal{O}_X\) has length \(\dim X\) and so Corollary II.3.5 requires the existence of \(\dim X\)-many splitting operators. However, note by Lemma I.1.7(iv) that it will be sufficient to only have two such operators. Returning now to the lifted Euler vector field \(\epsilon_X\) in (II.3.3.1), since \(spl\). splits the initial form sequence for the tangent sheaf in Lemma I.2.5 in degree zero, it splits the bottom row in (II.2.2.1) and hence

\[ p_*\epsilon_X = p_* spl_*, \epsilon_X = \epsilon_X, \]

where \(p_*\) is the map induced on cohomology in (II.2.2.2). By Lemma II.2.6(i) we therefore have for any \(F \in \mathcal{O}_X\),

\[ p_*\epsilon_X(F) = \epsilon_X(\text{in}(F)) = |\text{in}(F)| \text{ in}(F) \quad (\text{II.3.3.2}) \]

where \(|\text{in}(F)|\) is the homogeneous degree of the initial form \(\text{in}(F) \in \mathcal{O}_\hat{X}\). Now recall the sequence for \(\mathcal{O}_X\) in (I.1.2.2). Using (I.2.3.1) we obtain the following for each \(m\):

\[ 0 \rightarrow \mathcal{O}_\hat{X}^{(m)} \rightarrow \mathcal{O}_X/\mathcal{J}_X^{m+1} \rightarrow \mathcal{O}_X/\mathcal{J}_X^m \rightarrow 0 \quad (\text{II.3.3.3}) \]

where \(\mathcal{O}_\hat{X}^{(m)}\) corresponds to the \(m\)-th summand in (I.2.3.1). On \(\mathcal{O}_X\) now we can form the operator,

\[ \tilde{O}^{(m)} \triangleq m - \text{ad}_{\epsilon_X} \]

where \(m = m1_{\mathcal{O}_X}\) and \(1_{\mathcal{O}_X} : \mathcal{O}_X = \mathcal{O}_\hat{X}\) is the identity map. We have:

**Lemma II.3.6.** For each \(m\) and \(\ell\), \(\tilde{O}^{(m)} : \mathcal{J}^{\ell} \rightarrow \mathcal{J}^{\ell}\) is \(\mathbb{C}\)-linear with \(\ker \tilde{O}^{(m)} = \mathcal{O}_\hat{X}^{(m)}\).

**Proof.** That \(\tilde{O}^{(m)}\) is \(\mathbb{C}\)-linear follows from \(\mathbb{C}\)-linearity of its constituent components \(m\) and \(\text{ad}_{\epsilon_X}\). It remains to verify \(\text{im} \tilde{O}^{(m)}|\mathcal{J}^{\ell}_X \subset \mathcal{J}^{\ell}_X\) for all \(\ell\). But this is also clear upon inspection of these components. To make sense of the statement \(\ker \tilde{O}^{(m)} = \mathcal{O}_\hat{X}^{(m)}\), observe: since \(\mathcal{J}^{\ell+1}_X \subset \mathcal{J}^{\ell}_X\); and \(\tilde{O}^{(m)}\) sends \(\mathcal{J}^{\ell+1}_X \rightarrow \mathcal{J}^{\ell+1}_X\), it will induce a mapping of the sequence in (II.3.3.3) and hence of the quotient \(\mathcal{J}^{\ell}_X/\mathcal{J}^{\ell+1}_X = \mathcal{O}_\hat{X}^{(\ell)} \rightarrow \mathcal{O}_\hat{X}^{(\ell)}\). Now
by (II.3.3.2) note that for any \( j^\ell \in \mathcal{J}^\ell \),
\[
\text{in}(\tilde{O}^{(m)}(j^\ell)) = \text{in}(m(j^\ell)) - \text{in}(\text{ad}_{\epsilon_X}(j^\ell)) \quad \text{(c.f., Remark I.2.4)}
\]
\[
= (m - \ell) \text{in}(j^\ell).
\]  
(II.3.3.4)

Thus \( \tilde{O}^{(m)} \) sends \( O^{(\ell)}_X \rightarrow 0 \) iff \( \ell = n \). □

By Lemma II.3.6 and its proof we have the following diagram for each \( m \):
\[
\begin{array}{cccccc}
0 & \rightarrow & J^{m+1}_X & \rightarrow & J^m_X & \rightarrow & O^{(m)}_X & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & \text{im} \tilde{O}^{(m)} & \rightleftharpoons & \text{im} \tilde{O}^{(m)} & \rightarrow & 0
\end{array}
\]  
(II.3.3.5)

The isomorphism in the bottom, horizontal row of (II.3.3.5) asserts: to any \( j^m \in J^m_X \) there will exist some unique \( j^{m+1} \in J^{m+1}_X \) such that \( \tilde{O}^{(m)}(j^m) = j^{m+1} \). Hence for each \( m \) we get the \( \mathbb{C} \)-linear mapping \( \tilde{O}^{(m)}: J^m_X \rightarrow J^{m+1}_X \). With \( q = \dim_X \mathfrak{x} \), the composition \( O^{(m+1,q+1)} \Delta \tilde{O}^{(q+1)} \circ \tilde{O}^{(q-1)} \circ \ldots \tilde{O}^{(m+1)} \) maps \( J^{m+1}_X \rightarrow 0 \). Hence for any \( \ell < m + 1 \) we obtain the diagram:
\[
\begin{array}{cccccc}
0 & \rightarrow & J^{m+1}_X & \rightarrow & J^m_X & \rightarrow & J^\ell_X / J^{m+1}_X & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & \text{im} O^{(m+1,q+1)} & \rightarrow & \text{im} O^{(m+1,q+1)} & \rightarrow & 0
\end{array}
\]  
(II.3.3.6)

Specialising to the case \( \ell = m \) above, we want to identify \( \text{im} O^{(m+1,q+1)} \) with \( O^{(m)}_X \). This will not be the case however since, by (II.3.3.4), for any \( j^m \in J^m_X \) we have \( \text{in} \tilde{O}^{(m')}(j^m) = (m' - m) \text{in}(j^m) \). That is, each operator in the composite \( O^{(n+1,q+1)} \) will add an integral factor modulo \( J^{m+1}_X \). We illustrate this explicitly below.

**Illustration II.3.7.** We will look at the composite \( \tilde{O}^{(m+2)} \circ \tilde{O}^{(m+1)} \). By \( \mathbb{C} \)-linearity of the operators involved we have:
\[
\tilde{O}^{(m+2)} \circ \tilde{O}^{(m+1)} = \tilde{O}^{(m+2)} \circ (m + 1 - \text{ad}_{\epsilon_X})
\]
\[
= \tilde{O}^{(m+2)} \circ (m + 1) - \tilde{O}^{(m+2)} \circ \text{ad}_{\epsilon_X}
\]
\[
= (m + 2)(m + 1) - (m + 2) \circ \text{ad}_{\epsilon_X} + \text{ad}_{\epsilon_X} \circ \text{ad}_{\epsilon_X}.
\]
Now from (II.3.3.2) and using that $\text{ad}_{\epsilon_X}(F) = \epsilon_X(F)$ for any $F \in \mathcal{O}_X$, we can deduce:

$$\text{in}(\text{ad}_{\epsilon_X} \circ \text{ad}_{\epsilon_X}(F)) = |\text{in}(F)|^2 \text{in}(F)$$

For $j^m \in \mathcal{J}_X^m$ then we find:

$$\text{in}((\tilde{\mathcal{O}}^{(m+2)} \circ \tilde{\mathcal{O}}^{(m+1)}(j^m)) = \left( (m + 2)(m + 1) - (m + 2)m + m^2 \right) \text{in}(j^m) = (m^2 + 2m + 2) \text{in}(j^m)$$

(II.3.3.7)

Now define $Q^{(m+1;m+2)} = \frac{1}{m^2 + 2m + 2} \tilde{\mathcal{O}}^{(m+2)} \circ \tilde{\mathcal{O}}^{(m+1)}$. By our calculation in (II.3.3.7) we know that $\text{in}(Q^{(m+1;m+2)}(j^m)) = \text{in}(j^m)$. Hence we have the diagram:

$$\begin{array}{ccc}
0 & \longrightarrow & \mathcal{J}_X^{m+1} \longrightarrow \mathcal{J}_X^m \longrightarrow \mathcal{O}_X^{(m)} \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{J}_X^{m+3} & \longrightarrow & \text{im } Q^{(m+1;m+2)} & \longrightarrow & \mathcal{O}_X^{(m)} \longrightarrow 0
\end{array}$$

(II.3.3.8)

In this way we can view $Q^{(m+1;m+2)}$ as a splitting operator on $\mathcal{J}_X^m$ ‘modulo 2’.

Following Example II.3.7, we aim to construct ‘modulo-(q + 1)’ and ‘modulo q’ splitting operators on $\mathcal{O}_X$ and $\mathcal{O}_X[1] = \mathcal{J}_X$ respectively. As remarked earlier, Lemma I.1.7(iv) will guarantee that such operators will give our desired splitting of $\mathcal{O}_X$. We begin by noting the following algebraic expansions:

$$\prod_{j=1}^{q}(j - x) = q! + (-1)(q - 1)! x + \cdots + (-1)^q x^q$$

(II.3.3.9)

And

$$\prod_{j=2}^{q}(j - x) = q! + (-1)(q - 1)! x + \cdots + (-1)^{q-1} x^{q-1}.$$  

(II.3.3.10)

The formal intermediate $x$ represents the operator $\text{ad}_{\epsilon_X}$ and, by (II.3.3.2), will satisfy:

$$\text{in}(x^k F) = |\text{in}(F)|^k \text{in}(F)$$

(II.3.3.11)

for all $F \in \mathcal{O}_X$. Promoting the left-hand sides in (II.3.3.9) and (II.3.3.10) to operators $\tilde{\mathcal{O}}^{(1;q)}$ and $\tilde{\mathcal{O}}^{(2;q)}$ on $\mathcal{O}_X$ respectively, see that they will send $\mathcal{J}_X$ resp. $\mathcal{J}_X^2$ to zero. In writing $j^m$ now we will mean an element of $\mathcal{J}_X^m$ so that $\text{in}(j^m) \in \mathcal{O}_X^{(m)}$. From the
expansion on right hand sides of \((\text{II.3.3.9})\) resp. \((\text{II.3.3.10})\) in addition to \((\text{II.3.3.11})\), we have:

\[
in(\tilde{O}^{(1; q)}(j^0)) = q! \; in(j^0) \quad \text{and} \quad in(\tilde{O}^{(2; q)}(j^1)) = (q! - (q - 1)! + (q - 2)! + \cdots + (-1)^{q-1}) \; in(j^1)
\]

\[
= \sum_{k=0}^{q-1} (-1)^k (q - k)! \; in(j^1).
\]

Accordingly, set:

\[
O^{(1)} \triangleq \frac{1}{q!} \tilde{O}^{(1; q)} \quad \text{and} \quad O^{(2)} \triangleq \frac{1}{\sum_{k=0}^{q-1} (-1)^k (q - k)!} \tilde{O}^{(2; q)}.
\]

Then \(O^{(1)}\) and \(O^{(2)}\) above will define splitting operators for the sequences in Lemma \(\text{I.1.7} (iv)\) respectively and hence define a splitting of \(\mathfrak{X}\). This completes the proof of Proposition \(\text{II.3.3}\). \(\square\)

### II.3.4. Further Commentary on Splittings

While we have not equated diffeomorphisms, defined in Definition \(\text{II.3.1}\), with isomorphisms, defined independently in Definition \(\text{I.1.3}\); we have nevertheless established in Proposition \(\text{II.3.3}\) that for any supermanifold \(\mathfrak{X}\) with split model \(\hat{\mathfrak{X}}\),

\[
\text{Diffeo.}(\mathfrak{X}, \hat{\mathfrak{X}}) \cong \text{Splittings}(\mathfrak{X}) \triangleq \text{Isom.}(\mathfrak{X}, \hat{\mathfrak{X}}).
\]

Hence, at least for \(\mathfrak{X} = \hat{\mathfrak{X}}\) we have established an equivalence of definitions, i.e., that \(\text{Diffeo.}(\mathfrak{X}, \hat{\mathfrak{X}}) \cong \text{Isom.}(\mathfrak{X}, \hat{\mathfrak{X}})\). Concerning the proof of Proposition \(\text{II.3.3}\), note that we only required the existence of a splitting of the tangent sequence \(T_{\mathfrak{X}}\) in Lemma \(\text{I.2.5}\) in degree zero. That is, we only made use of a splitting \(\text{spl.}\) of the sequence \(0 \to T^{(1)}_{\hat{\mathfrak{X}}} \to T^{(0)}_{\hat{\mathfrak{X}}} \to T^{(0)}_{\hat{\mathfrak{X}}} \to 0\). Furthermore, the splitting \(\text{spl.}\) was only used to lift the Euler vector field \(\epsilon_{\hat{\mathfrak{X}}}\) as in \((\text{II.3.3.1})\) to some global vector field \(\epsilon_{\mathfrak{X}}\) on \(\mathfrak{X}\). And so we come now to our main observation here being: in the proof of Proposition \(\text{II.3.3}\), all that was important was the initial form formula in \((\text{II.3.3.2})\). Hence, the same proof given for Proposition \(\text{II.3.3}\) will also yield the following:

**Theorem II.3.8.** Let \(H\) be a global vector field on \(\mathfrak{X}\) with initial form the Euler vector field \(\epsilon_{\hat{\mathfrak{X}}}\). Then \(H\) will define a splitting of \(\mathfrak{X}\). \(\square\)
A classical result in supergeometry is Batchelor’s theorem, originally appearing in [Bat79], which asserts that any smooth supermanifold splits. We can recover this result from Theorem II.3.8 as follows.

**Corollary II.3.9. (Batchelor’s Theorem) Any smooth supermanifold is split.**

*Proof.* If $\mathfrak{X}$ is smooth, then its tangent sheaf is fine. This means $H^j(X, T^{(\ell)}_\mathfrak{X}) = (0)$ for all $\ell$ and all $j > 0$. Now from the initial form relation between $T_\mathfrak{X}$ and $T_\mathfrak{X}^\wedge$ in Lemma I.2.5, note that in degree zero\(^{16}\) we have the following sequence on cohomology:

$$
\cdots \longrightarrow H^0(X, T^{(0)}_\mathfrak{X}) \longrightarrow H^0(X, T^{(0)}_\mathfrak{X}^\wedge) \longrightarrow H^1(X, T^{(1)}_\mathfrak{X}) = (0). \quad (\text{II.3.4.1})
$$

Hence $H^0(X, T^{(0)}_\mathfrak{X})$ surjects onto $H^0(X, T^{(0)}_\mathfrak{X}^\wedge)$ and so there will necessarily exist at least one global vector field on $\mathfrak{X}$ with the Euler vector field as its initial form. This vector field will then split $\mathfrak{X}$ by Theorem II.3.8. \qed

Splittings of a smooth supermanifold $\mathfrak{X}$ are, in general, ‘non-canonical’. That is, they need not exist uniquely, and so there may be many different splitting maps. We can interpret this statement to mean, by surjectivity in (II.3.4.1), that there may exist many non-identical, global vector fields $H$ on $\mathfrak{X}$ with $\epsilon_\mathfrak{X}$ as their initial form (and thereby which split $\mathfrak{X}$). In the smooth setting then, Koszul’s splitting theorem (Theorem II.1.5) can be taken to guarantee the existence of a unique, smooth splitting. In the complex analytic setting, splittings need not exist however since globally defined, holomorphic vector fields on arbitrary supermanifolds need not exist. As such, Koszul’s theorem amounts to a statement about both existence and uniqueness of splitting maps.

II.4. **Proof of Koszul’s Theorem**

We begin by tying up a loose end leftover at the end of §II.2, being Theorem II.2.5 concerning the relation between splittings and the Euler differential.

II.4.1. **Proof of Theorem II.2.5.** Let $\mathfrak{X}$ be a superanifold with Euler differential $\delta \epsilon_\mathfrak{X}$. As observed in the paragraph preceeding the statement of Theorem II.2.5, the Euler differential measures precisely the failure for the Euler vector field to lift to

\(^{16}\text{c.f., (II.2.2.2)}\)
Now, if $\delta\epsilon_X \neq 0$, then $\mathfrak{X}$ cannot be split for the reason that if it were split, then there would exist a lift of $\epsilon_{\mathfrak{X}}$ to $\mathfrak{X}$, therefore implying $\delta\epsilon_X = 0$ and contradicting our assumption. If we assume $\delta\epsilon_X = 0$ now, then there will exist some lift of $\epsilon_{\mathfrak{X}}$ to $\mathfrak{X}$, i.e., that there exists some global vector field $H$ on $\mathfrak{X}$ with initial form $\epsilon_{\mathfrak{X}}$. We can now use Theorem II.3.8 to conclude $\mathfrak{X}$ will split. We have therefore established the truth of the following statements:

$$\delta\epsilon_{\mathfrak{X}} = 0 \implies \mathfrak{X} \text{ is split} \quad \text{and} \quad \delta\epsilon_{\mathfrak{X}} \neq 0 \implies \mathfrak{X} \text{ is non-split}$$

whence Theorem II.2.5 follows.\(^{17}\)

II.4.2. **Proof Sketch.** In the same vein as Koszul’s proof in [Kos94], ours will also proceed by induction. To describe the inductive step it will be useful to introduce an intermediate notion of the Euler vector field lift.

II.4.2.1. **The Euler Differential Lift.** Recall that the tangent sheaf $T_{\mathfrak{X}}$ is filtered with strata $(T_{\mathfrak{X}}^m)_{m \in \mathbb{Z}}$ defined as in (I.2.1.1). This filtration is finite with length $\dim_{\mathfrak{X}} + 1$, i.e., that $T_{\mathfrak{X}}^m = (0)$ for all $m < -1$ and $m > \dim_{\mathfrak{X}}$. With $T_{\mathfrak{X}}^{(-1)} = T_{\mathfrak{X}}$ the filtration is descending so that $T_{\mathfrak{X}} = T_{\mathfrak{X}}^{(-1)} \supset T_{\mathfrak{X}}^{(0)} \supset T_{\mathfrak{X}}^{(1)} \supset \cdots \supset T_{\mathfrak{X}}^{(n)} \supset (0)$. Now by definition of the Euler differential in Definition II.2.3, it is valued in $H^1(X, T_{\mathfrak{X}}^{(1)})$. Its lifts are defined as follows.

**Definition II.4.1.** Let $\mathfrak{X}$ be a supermanifold with Euler differential $\delta\epsilon_{\mathfrak{X}}$. We say that this differential admits an $\ell$-lift, for $\ell > 0$, if there exists some $\omega \in H^1(X, T_{\mathfrak{X}}^{( \ell + 1)})$ such that $\omega^* \mapsto \delta\epsilon_{\mathfrak{X}}$ under the induced map\(^{18}\) $H^1(X, T_{\mathfrak{X}}^{(\ell + 1)}) \to H^1(X, T_{\mathfrak{X}}^{(1)})$.

In relation to the Euler vector field lift we have:

**Proposition II.4.2.** Let $\mathfrak{X}$ be a supermanifold and suppose its Euler differential admits a $\dim_{\mathfrak{X}}$-lift. Then the Euler vector field will also lift to $\mathfrak{X}$.

**Proof.** We will use that $T_{\mathfrak{X}}^{(m)} = (0)$ for all $m > \dim_{\mathfrak{X}}$ and hence that $T_{\mathfrak{X}}^{(\dim_{\mathfrak{X}} + 1)} = (0)$. Now if $\delta\epsilon_{\mathfrak{X}}$ admits a $\dim_{\mathfrak{X}}$-lift, there exists $\omega \in H^1(X, T_{\mathfrak{X}}^{(\dim_{\mathfrak{X}} + 1)})$ mapping

\(^{17}\)To more clearly see why Theorem II.2.5 follows, note from first-order logic:

$$((P \implies Q) \text{ and } (\neg P \implies \neg Q)) \iff (P \iff Q).$$

Apply this to: $P = (\delta\epsilon_{\mathfrak{X}} = 0)$ and $Q = (\mathfrak{X} \text{ is split})$.

\(^{18}\)this mapping is induced by the inclusion $T_{\mathfrak{X}}^{(\ell)} \subset T_{\mathfrak{X}}^{(1)}$ for any $\ell \geq 1$. 

onto $\delta \epsilon_{\hat{X}}$; but $H^1(X, T^{\dim_\mathfrak{X} + 1}_\hat{X}) = (0)$ and so $\omega = 0$, giving $\delta \epsilon_{\hat{X}} = 0$ which is precisely the condition for $\epsilon_{\hat{X}}$ to lift to $\mathfrak{X}$.

\[ \square \]

A consequence of Theorem II.2.5 and Proposition II.4.2 is the following.

**Corollary II.4.3.** A supermanifold $\mathfrak{X}$ is split if and only if its Euler differential admits a $\dim_\mathfrak{X}$-lift.

\[ \square \]

### II.4.2.2. Koszul’s Theorem Proof Sketch.

We turn now to a proof sketch of Koszul’s theorem (Theorem II.1.5). Our strategy is as follows: under the assumption of a global, affine, even connection $\nabla$ on $\mathfrak{X}$ we will show: if its Euler differential admits a 1-lift, then it will admit a 2-lift. Hence, upon establishing the existence of a 1-lift \textit{à priori}, we can use $\nabla$ to lift $\delta \epsilon_{\hat{X}}$ indefinitely. Koszul’s theorem will then follow from Corollary II.4.3.

### II.4.3. The 1-Lift.

The initial form sequence from Lemma I.2.5 is exact in each degree and so gives long exact sequences on cohomology. In order to deduce, then, the existence of a 1-lift of the Euler differential $\delta \epsilon_{\hat{X}}$ of a supermanifold $X$, it will suffice to show its image in $H^1(X, T^{(1)}_{\hat{X}})$ vanishes. For clarity we present the following composition, extending the diagram in (II.2.2.2),

\[ (II.4.3.1) \]

where $p_*$ here is induced from $0 \to T^{(2)}_{\hat{X}} \to T^{(1)}_{\hat{X}} \xrightarrow{p} T^{(1)}_{\hat{X}} \to 0$. The dashed arrow above interpolates between cohomologies of sheaves of differing weights (i.e., from even to odd) and so ought to vanish. This is the subject of what follows.

\[ ^{19}\text{c.f., the paragraph preceding Theorem II.2.5.} \]
**Proposition II.4.4.** For any $m$ the composition of maps on cohomology induced from the initial form sequence in Lemma I.2.5, represented below by the dashed arrow:

$$
\begin{array}{ccc}
H^0(T^{(m)}_{\hat{X}}) & \longrightarrow & H^1(T^{(m+1)}_{\hat{X}}) \\
\downarrow & & \downarrow \\
& H^1(T^{(m+1)}_{\hat{X}}) & \\
\end{array}
$$

(II.4.3.2)

generalising (II.4.3.1), vanishes.

**Proof.** Recall the initial form sequence from Lemma I.2.5. This is a sequence of even morphisms so, with the decomposition $T_X \cong T_{X,+} \oplus T_{X,-}$, we have for each $m$ an exact sequence $T^{(m+2)}_{X,p(m)} \to T^{(m)}_{X,p(m)} \to T^{(m)}_{\hat{X}}$, where $p(m) \in \{+, -\}$ denotes the parity of $m$. This reveals: for any global section $\nu \in H^0(X, T^{(m)}_{\hat{X}})$ and with respect to any covering $(U_\alpha)_\alpha$, we can always represent $\nu$ on intersections $U_\alpha \cap U_\beta$ by the difference $\nu|_{U_\alpha \cap U_\beta} = \nu_\beta - \nu_\alpha \in T^{(m+2)}_{\hat{X}}(U_\alpha \cap U_\beta)$. In particular, this difference vanishes upon projection onto $T^{(m+1)}_{\hat{X}}(U_\alpha \cap U_\beta)$. Hence, on cohomology, it follows that the arrow $H^0(X, T^{(m)}_{\hat{X}}) \to H^1(X, T^{(m+1)}_{\hat{X}})$ necessarily vanishes. \(\square\)

**Remark II.4.5.** Note that Proposition II.4.4 does not necessarily imply there do not exist odd morphisms $T^{(m)}_{\hat{X}} \to T^{(m+1)}_{\hat{X}}$; only that the composition in (II.4.3.2) vanishes. Indeed, in Appendix B, it will be clear that non-trivial, odd morphisms indeed exist.

Specialising Proposition II.4.4 to $m = 0$ now reveals what we originally wanted:

**Corollary II.4.6.** For any supermanifold $\mathfrak{X}$ there exists a 1-lift of its Euler differential, i.e., that $p_* \delta \epsilon_{\mathfrak{X}} = 0$. \(\square\)

II.4.4. **Partial Liftings of the Euler Vector Field.** Recall from Theorem II.3.8 that a global vector field on $\mathfrak{X}$ with $\epsilon_{\mathfrak{X}}$ as its initial form will define a splitting of $\mathfrak{X}$. The Euler differential measures the failure for the Euler vector field to lift and thereby also relates information on splitting, albeit more indirectly, in Theorem II.2.5. In Proposition II.4.2 this relation to the Euler vector field lift was clarified. Presently, we look to study a notion of partial liftings of the Euler vector field.
Proposition II.4.7. On any supermanifold $\mathfrak{X}$ there exists a smooth, global vector field $H_{\infty}$ such that, modulo $T^{(2)}_{\mathfrak{X}}$, it is holomorphic and maps onto the Euler vector field $\hat{\varepsilon}_{\mathfrak{X}}$.

Proof. Note that we have the following nine-term lattice:

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & \downarrow & \downarrow & \downarrow & 0 \\
0 & T^{(2)}_{\mathfrak{X}} & T^{(2)}_{\mathfrak{X}} & T^{(1)}_{\mathfrak{X}} & \downarrow & \downarrow & 0 \\
0 & T^{(1)}_{\mathfrak{X}} & T^{(0)}_{\mathfrak{X}} & T^{(0)}_{\mathfrak{X}} & \downarrow & \downarrow & 0 \\
0 & T^{(1)}_{\mathfrak{X}} & T^{(0)}_{\mathfrak{X}} / T^{(2)}_{\mathfrak{X}} & T^{(0)}_{\mathfrak{X}} & \downarrow & \downarrow & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

(II.4.4.1)

The right-most column and the bottom row are exact and induce the following maps between cohomologies:

\[
\begin{array}{ccc}
H^0(T^{(0)}_{\mathfrak{X}}) & \overset{\delta}{\longrightarrow} & H^1(T^{(1)}_{\mathfrak{X}}) \\
\downarrow & & \downarrow p_* \\
H^0(T^{(0)}_{\mathfrak{X}} / T^{(2)}_{\mathfrak{X}}) & \longrightarrow & H^0(T^{(0)}_{\mathfrak{X}}) \longrightarrow H^1(T^{(1)}_{\mathfrak{X}})
\end{array}
\]

(II.4.4.2)

where $\delta$ and $p_*$ are the maps from the diagram in (II.4.3.1). By Corollary II.4.6, the image of $\hat{\varepsilon}_{\mathfrak{X}}$ in $H^1(X, T^{(1)}_{\mathfrak{X}})$ vanishes. Hence by exactness of the bottom row in (II.4.4.2), there exists a global vector field $\mathfrak{H} \in H^0(X, T^{(0)}_{\mathfrak{X}} / T^{(2)}_{\mathfrak{X}})$ mapping onto the Euler vector field $\hat{\varepsilon}_{\mathfrak{X}}$. To complete the proof, observe that the middle column in (II.4.4.1) is exact and gives on cohomology:

\[
\cdots \longrightarrow H^0(T^{(0)}_{\mathfrak{X}}) \longrightarrow H^0(T^{(0)}_{\mathfrak{X}} / T^{(2)}_{\mathfrak{X}}) \longrightarrow H^1(T^{(2)}_{\mathfrak{X}}) \longrightarrow \cdots
\]
We label by an ‘∞’ superscript the sheaves of smooth objects—e.g., $T^\infty_X$ denotes the sheaf of smooth vector fields as opposed to $T^\infty_X$ which denotes holomorphic vector fields if $X$ is assumed to be complex.\(^{20}\) As holomorphy is a stronger condition than smoothness, we have $T^\infty_X \subset T^\infty_X$. Any sheaf of smooth vector fields is fine and so $H^1(X, T^{(2);\infty}_X) = (0)$. Therefore, there will always exist some smooth, global vector field $H^\infty \in H^0(X, T^{(0);\infty}_X)$ mapping onto any global section in $H^0(X, T^{(0)}_X / T^{(2)}_X) \subset H^0(X, T^{(0);\infty}_X / T^{(2);\infty}_X)$. We have therefore argued the existence of a smooth, global vector field $H^\infty$ on $X$ such that, modulo $T^{(2);\infty}_X$, it is holomorphic and maps onto $\epsilon^\infty_X$. This completes the proof. \(\square\)

As with the $\ell$-lifts of the Euler differential from Definition II.4.1, Proposition II.4.7 is suggestive of a similar notion for the Euler vector field itself. To give this notion recall that we can identify $T^{(0)}_X \cong T^{(1)}_X$ by Lemma I.2.5. Hence, from the filtration on $T_X$ we have more generally a surjection $T^{(0)}_X / T^{(\ell)}_X \to T^{(0)}_{\hat{X}} \to 0$ for any $\ell \geq 1$. Upon recalling that $\epsilon^\infty_X \in H^0(X, T^{(0)}_{\hat{X}})$ by Definition II.2.1 we arrive at the following.

**Definition II.4.8.** We say the Euler vector field on $\hat{X}$ will admit a mod $\ell$ lift to $X$ if and only if there exists a global section $\overline{H} \in H^0(X, T^{(0)}_{\hat{X}} / T^{(\ell)}_{\hat{X}})$ which maps onto $\epsilon^\infty_X$ via $H^0(X, T^{(0)}_{\hat{X}} / T^{(\ell)}_{\hat{X}}) \to H^0(X, T^{(0)}_{\hat{X}})$.

**Definition II.4.9.** Any smooth, global vector field $H^\infty \in H^0(X, T^{(0);\infty}_X)$ which defines a mod $\ell$ lift $\overline{H}$ will be referred to as a *smooth extension* of $\overline{H}$.

In the language introduced in Definition II.4.8 and II.4.9 then, Proposition II.4.7 asserts firstly that the Euler vector field of a split model $\hat{X}$ will admit a mod 2 lift to any supermanifold $X$ to which $\hat{X}$ is associated; and secondly that this lift furthermore admits a smooth extension. Similarly to Corollary II.4.3 we now have:

**Corollary II.4.10.** Any supermanifold $X$ which admits a mod $\ell$ lift of the Euler vector field with $\ell > \dim \_ X$ is split.

*Proof.* This follows from Theorem II.3.8 upon observing that $T^{(m)}_X = (0)$ for $m > \dim \_ X$. \(\square\)

\(^{20}\)As for complex manifolds, there exists a Dolbeault-like operator $\tilde{\partial}$ on complex supermanifolds. A smooth function is holomorphic iff $\tilde{\partial}f = 0$. 
II.4.5. Shear-Like Transformations. Central to our proof of Koszul’s theorem will be the construction of certain transformations on local sections of the tangent sheaf reminiscent of shearing.

II.4.5.1. Local Coordinates. Despite our efforts, we are unable to give a completely coordinate-free proof of Koszul’s theorem. For the reader interested in learning about local coordinate formulations on supermanifolds, we refer them to the classical texts [Lei80, Man88]. See also [Bet16, §5] where Koszul’s theorem is studied largely from a local coordinate perspective. We digress here briefly in order to establish notation. Let \( \mathfrak{X} \) be a supermanifold with tangent sheaf \( T_\mathfrak{X} \). Recall that it is filtered according to powers of the fermionic ideal \( J_m^{\mathfrak{X}} \) as in (I.2.1.1). Let \( U \subset \mathfrak{X} \) be a local coordinate neighbourhood. Note that \( U \) is itself a supermanifold and is split so we can regard it as an open set in \( \widehat{\mathfrak{X}} \) also. We denote by \(|U| \subset X\) its reduced space. A splitting of \( U \) is furnished by choice of local coordinates \( (x|\theta) \). Hence we can regard \( \mathcal{O}_X(U) \) as a \( \mathcal{O}_X(|U|) \)-module. The variables \( x \) and \( \theta \) are even and odd respectively; \( (\theta) \) generates the fermionic ideal \( J_X(U) \) over \( \mathcal{O}_X(|U|) \) and so the \( m \)-th powers \( (\theta^m) \) generate \( J_X^m(U) \) over \( \mathcal{O}_X(|U|) \). Where the tangent sheaf is concerned, we have locally an isomorphism\(^{21}\)

\[
T_X^{(m)}(U) \cong \bigoplus_{\ell \geq m} T_{\widehat{\mathfrak{X}}}^{(\ell)}(|U|). \tag{II.4.5.1}
\]

Now the sequence for the sheaf \( T_{\widehat{\mathfrak{X}}}^{(\ell)} \) in Lemma I.2.2 will split over \( U \) so therefore, in coordinates \( (x|\theta) \), \( T_{\widehat{\mathfrak{X}}}^{(\ell)}(U) \) is generated as an \( \mathcal{O}_X(|U|) \)-module by the sections \( \theta^\ell \partial/\partial x \) and \( \theta^{\ell+1} \partial/\partial \theta \) respectively.\(^{22}\) Accordingly, from the isomorphism in (II.4.5.1) sections of \( T_X^{(m)}(U) \) are generated by tuples,

\[
T_X^{(m)}(U) = \left( \theta^m \frac{\partial}{\partial x}, \theta^{m+1} \frac{\partial}{\partial \theta}, \theta^{m+1} \frac{\partial}{\partial x}, \theta^{m+2} \frac{\partial}{\partial \theta}, \ldots \right) \tag{II.4.5.2} \]

over \( \mathcal{O}_X(|U|) \). The global \( \mathbb{Z}_2 \)-grading of the functions on \( \mathfrak{X} \) induce a \( \mathbb{Z}_2 \)-grading \( T_\mathfrak{X} \cong T_{\mathfrak{X},+} \oplus T_{\mathfrak{X},-} \), decomposing sections into their even and odd constituents. Over

\(^{21}\)c.f., footnote 8.

\(^{22}\)In this way, with \( T_{\mathfrak{X},-} = T_{\mathfrak{X}}^{(-1)} \) we can see the motivation behind referring to \( T_{\mathfrak{X},-} \) as the bundle of odd cotangents as in Definition I.1.1.
Then and in the notation (II.4.5.2) we have over $\mathcal{O}_X(|U|)$ the even strata:

$$T^{(2m)}_{\mathfrak{x},+}(U) = \left( \frac{\partial}{\partial x}, \theta \frac{\partial}{\partial \theta}, \theta^2 \frac{\partial}{\partial x}, \ldots \right)$$

and:

$$T^{(2m)}_{\mathfrak{x},-}(U) = \left( \frac{\partial}{\partial x}, \theta \frac{\partial}{\partial \theta}, \theta^2 \frac{\partial}{\partial x}, \ldots \right).$$

(II.4.5.3)

Regarding the odd strata we have $T^{(2m+1)}_{\mathfrak{x},+} = T^{(2m+2)}_{\mathfrak{x},+}$ and similarly, $T^{(2m+1)}_{\mathfrak{x},-} = T^{(2m)}_{\mathfrak{x},-}$.

II.4.5.2. Algebraic Sections. Let $\nabla$ be a global, even, affine connection on $X$. With $\nabla$ we will show that the boundary of any mod 2 lift, $\partial \mathcal{H}$, will vanish. Before presenting the relevant calculations we digress to present the following useful definition concerning tangent vectors.

Definition II.4.11. Let $v \in T_X$ be a section with initial form $\text{in}(v) \in T^{(m)}_{\mathfrak{x}}$. We will refer to this section as algebraic if its initial form vanishes modulo $J^{m+1}_{\mathfrak{x}}$.

Note, in terms of the sequence in Lemma I.2.2, the condition for a section $v \in T^{(m)}_{\mathfrak{x}}$ to be algebraic is for the projection of its initial form under $T^{(m)}_{\mathfrak{x}} \to \wedge^m_{\mathcal{O}_X} T^*_{\mathfrak{x},-} \otimes_{\mathcal{O}_X} T_X$ to vanish, in which case it will be in the image of $\wedge^{m+1}_{\mathcal{O}_X} T^*_{\mathfrak{x},-} \otimes_{\mathcal{O}_X} T_{\mathfrak{x},-}$. As a result we have the following.

Lemma II.4.12. Any mod \ell lift of the Euler vector field is algebraic.

Proof. Recall from Definition II.4.8 that any mod \ell lift $\overline{H}$ will have the Euler vector field $\epsilon_{\mathfrak{x}}$ as its initial form and so gives section in $T^{(0)}_{\mathfrak{x}}$. From Definition II.2.1 the Euler vector field will be a section of $T^{(0)}_{\mathfrak{x}}$. It is in the image of $\text{End}_{\mathcal{O}_X} T^*_{\mathfrak{x},-}$ and so, upon inspection of the sequence in Lemma I.2.2 specialised to $m = 0$, its projection onto $T_X$ vanishes. Hence it vanishes modulo $J_{\mathfrak{x}}$ and so, by Definition II.4.11, the mod \ell lift $\overline{H}$ will be algebraic.

II.4.5.3. Shearing. Let $H$ be a global vector field on $\mathfrak{x}$ with initial form $\epsilon_{\mathfrak{x}}$. Note that it need not be holomorphic. With a global, affine connection $\nabla$ we can define an $\mathcal{O}_X$-morphism $\nabla H : T_X \to T_X$ given by $v \mapsto (\nabla H)(v) \overset{\Delta}{=} \nabla_v H$.

\textsuperscript{23}The terms introduced in Definition II.4.11 is adapted from a similar usage in \cite[p.88]{KMS93}.
Proposition II.4.13. Let $\nabla$ be even. Then for any algebraic section $v \in T^{(m)}_X$, 

$$(\nabla H)(v) \equiv v \mod T^{(m+1)}_X$$

where $H$ is a global vector field with initial form $\hat{\epsilon}_\xi$.

Proof. With respect to a covering $(\mathcal{U}_\alpha)$ of $X$ and the isomorphism between tangent sheaves in (II.4.5.1), we can write $H$ locally as follows:

$$H|_{\mathcal{U}_\alpha} = \epsilon_\alpha + Q_{\alpha}^{(1)} + Q_{\alpha}^{(2)} + \cdots$$

(II.4.5.5)

where $Q_{\alpha}^{(j)} \in T^{(j)}_\xi([U_\alpha])$ and $\epsilon_\alpha = \hat{\epsilon}_{\xi|_{\mathcal{U}_\alpha}}$. 24 Let $(x|\theta)$ denote coordinates on $\mathcal{U}_\alpha$. Then for any section $v \in T_X$ note that locally we can always project onto the algebraic component by:

$$v \mapsto v(\theta) \frac{\partial}{\partial \theta}.$$  

(II.4.5.6)

This merely emphases that the sequence in Lemma I.2.2 splits over any coordinate neighbourhood. Now with the local expression for the Euler vector field from (II.2.3.1) observe that for any $v \in T^{(m)}_X$,

$$(\nabla H)(v)|_{\mathcal{U}_\alpha} = \nabla_v H|_{\mathcal{U}_\alpha}$$

$$= \nabla_v (\epsilon_\alpha + Q_{\alpha}^{(1)} + Q_{\alpha}^{(2)} + \cdots)$$

(from (II.4.5.5))

$$\equiv \nabla_v \epsilon_\alpha \mod T^{(m+1)}_X$$

(c.f., Definition II.1.1(ii))

$$= v(\theta) \frac{\partial}{\partial \theta} + \theta \nabla_v \frac{\partial}{\partial \theta}$$

(from (II.2.3.1)). (II.4.5.7)

It remains to argue $\theta \nabla_v \frac{\partial}{\partial \theta} \equiv 0$ modulo $T^{(m+1)}_X(\mathcal{U}_\alpha)$. To see this it will be essential to use the assumption that $\nabla$ is even and $v$ is algebraic. Indeed, from the notation in (II.4.5.2), if $v \in T^{(m)}_X$ is algebraic then its image in $\wedge^m_{\mathcal{O}_X} T^*_X \otimes \mathcal{O}_X T_X$ will vanish. Hence

$$v|_{\mathcal{U}_\alpha} \in \left(\theta^{m+1} \frac{\partial}{\partial \theta}, \theta^{m+1} \frac{\partial}{\partial x}, \ldots\right).$$

(II.4.5.8)

24The sections $Q_{\alpha}^{(j)}$ are holomorphic. Patching the expressions (II.4.5.5) together over $X$ with a partition of unity recovers $H$. And while $H$ can locally be expressed as a holomorphic section, the resultant need not longer be holomorphic since the partition of unity are not holomorphic functions.
Now recall from Definition II.1.3 that if $\nabla$ is even, then $\nabla_v \in \text{End}_\mathbb{C} T_X$ will be an even endomorphism, hence $\nabla_v : T_{X,\pm} \to T_{X,\pm}$. As a mapping of tensor products over $\mathbb{C}$ we recall:

$$T_{X,+} \otimes_\mathbb{C} T_{X,+} \xrightarrow{\nabla} T_{X,+} \quad T_{X,+} \otimes_\mathbb{C} T_{X,-} \xrightarrow{\nabla} T_{X,-} \quad \text{and} \quad T_{X,-} \otimes_\mathbb{C} T_{X,-} \xrightarrow{\nabla} T_{X,+}. \quad (\text{II.4.5.9})$$

From the latter most mapping above and the local characterisation of $T_{X,\pm}$ in (II.4.5.3) and (II.4.5.4) we must then have $\nabla_{\partial/\partial \theta}(\partial/\partial \theta) \in (\partial/\partial x, \theta \partial/\partial \theta, \theta^2 \partial/\partial x, \ldots)$. As a result of this in addition to (II.4.5.8) we are therefore led to,

$$\theta \nabla_v \frac{\partial}{\partial \theta} \in \left( \theta^{m+2} \frac{\partial}{\partial \theta}, \theta^{m+2} \frac{\partial}{\partial x}, \ldots \right) \in T^{(m+1)}_X(U_\alpha).$$

Hence $\theta \nabla_v \frac{\partial}{\partial \theta} \equiv 0$ modulo $T^{(m+1)}_X$ and so with the identification of the algebraic component of $v$ in (II.4.5.6) this lemma now follows from (II.4.5.7). \hfill \square

Recall that shear transformations of a vector space are linear transformations which fix a subspace and translate its complement parallel-wise. Analogously, the morphism $\nabla H : T_X \to T_X$ from Proposition II.4.13 will fix the subsheaf of algebraic sections modulo $T_X[1]$ and so we might view it as a kind of shearing. From Lemma II.4.12 and Proposition II.4.13 now we have immediately:

**Corollary II.4.14.** For any global, even, affine connection $\nabla$,

$$\nabla H H \equiv H \mod T^{(1)}_X$$

where $H$ is any global vector field with initial form $\epsilon_{\hat{X}}$. \hfill \square

In the case where $\bar{X} = \hat{X}$ and $H = \epsilon_{\hat{X}}$ we have:

**Lemma II.4.15.** For any global, even, affine connection $\nabla$ on $\hat{X}$,

$$\nabla_{\epsilon_{\hat{X}}} \epsilon_{\hat{X}} = \epsilon_{\hat{X}}.$$
Proof. We will use the coordinate representation $\epsilon_{\tilde{X}}|_{U} = \theta_i \partial/\partial \theta_i$ from (II.2.3.1), where the index $i$ is implicitly summed. This leads to:

\[
\nabla_{\epsilon_{\tilde{X}}} \epsilon_{\tilde{X}}|_{U} = \nabla_{\theta_i \frac{\partial}{\partial \theta_i}} \left( \theta_j \frac{\partial}{\partial \theta_j} \right)
\]

\[
= \theta_i \left( \frac{\partial}{\partial \theta_i} \theta_j \right) \frac{\partial}{\partial \theta_j} - \theta_i \theta_j \nabla_{\theta_j} \frac{\partial}{\partial \theta_j} \theta_i \text{ (by Definition II.1.1)(i)}
\]

\[
= \theta_j \frac{\partial}{\partial \theta_j} - \sum_{i<j} \theta_i \theta_j \nabla_{\theta_i} \frac{\partial}{\partial \theta_j} + \sum_{i>j} \theta_i \theta_j \nabla_{\theta_j} \frac{\partial}{\partial \theta_i}
\]

\[
= \epsilon_{\tilde{X}}|_{U} - \theta_i \theta_j \left( \nabla_{\theta_i} \frac{\partial}{\partial \theta_j} + \nabla_{\theta_j} \frac{\partial}{\partial \theta_i} \right) \text{ (II.4.5.10)}
\]

\[
= \epsilon_{\tilde{X}}|_{U}.
\]

where we used that the latter, bracketed term in (II.4.5.10) vanishes identically by Theorem II.1.2 (c.f., (A.6)).

We return now to an important relation between global, even, affine connections and algebraic sections.

**Lemma II.4.16.** Let $\nabla$ be a global, even, affine connection on $\tilde{X}$. Then for any global vector field $H$ on $\tilde{X}$ with initial form $\epsilon_{\tilde{X}}$ and any section $v \in T_{\tilde{X}}$, the section $\nabla_{v}H$ will be algebraic.

Proof. If $v$ is itself algebraic then $\nabla_{v}H$ will be algebraic by Proposition II.4.13. Assume now that $v$ is not algebraic and W.L.O.G., suppose $v \in T_{\tilde{X}}^{(m)}$. Then locally on a coordinate neighbourhood $U$ with coordinates $(x|\theta)$,

\[
v|_{U} \in \left( \theta^{m} \frac{\partial}{\partial x}, \theta^{m+1} \frac{\partial}{\partial \theta}, \ldots \right). \text{ (II.4.5.11)}
\]

Now using the local expansion for $H$ as in (II.4.5.5) gives

\[
(\nabla_{v}H)|_{U} \equiv v(\theta) \frac{\partial}{\partial \theta} + \theta \nabla_{v} \frac{\partial}{\partial \theta} \text{ mod } T_{\tilde{X}}^{(m+1)}. \text{ (II.4.5.12)}
\]

Since $\nabla$ is even we know from (II.4.5.3), (II.4.5.4) and (II.4.5.9) that $\nabla_{\theta/\partial \theta}(\partial/\partial x) \in (\partial/\partial \theta, \theta \partial/\partial x, \ldots)$. Then from (II.4.5.11), $\theta \nabla_{v}(\partial/\partial \theta) \in (\theta^{m+1} \partial/\partial \theta, \theta^{m+2} \partial/\partial x, \ldots)$ and therefore is algebraic. Furthermore, we know that the term $v(\theta) \frac{\partial}{\partial \theta}$ is algebraic\(^{25}\)

\(^{25}\text{c.f., (II.4.5.6)}\)
and so $\nabla_v H$ in (II.4.5.12) is a sum of algebraic vector fields and so is, in particular, itself algebraic.

II.4.5.4. A Shear-Like Transformation. Any choice of global, affine connection allows for the formulation of the vector field Lie bracket. And so, on a supermanifold $\mathfrak{X}$ with global, affine connection $\nabla$ we can form, for any two sections $u, v \in T_{\mathfrak{X}}$, their bracket:

$$[u, v] \overset{\Delta}{=} \nabla_u v - \nabla_v u + \text{tor.}\nabla(u, v) \quad (\text{II.4.5.13})$$

where $\text{tor}^\nabla : T_{\mathfrak{X}} \otimes_{O_{\mathfrak{X}}} T_{\mathfrak{X}} \to T_{\mathfrak{X}}$ is a homomorphism referred to as the torsion of $\nabla$.

Lemma II.4.17. Let $\nabla$ be even, $H$ a lift of $\tilde{\epsilon}_X$ and $v$ any algebraic section of $T^{(m)}_{\mathfrak{X}}$. Then $\text{tor}^\nabla (H, v) \equiv 0 \mod T^{(m+1)}_{\mathfrak{X}}$.

Proof. Since $\nabla$ is even we have the data of mappings from (II.4.5.9). Its torsion must be a morphism in the same way and so will give a morphism $T_{\mathfrak{X},-} \otimes_{O_{\mathfrak{X}}} T_{\mathfrak{X},-} \to T_{\mathfrak{X},+}$ whereby,

$$\text{tor}^\nabla \left( \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \right) \in \left( \frac{\partial}{\partial x}, \theta \frac{\partial}{\partial \theta}, \ldots \right). \quad (\text{II.4.5.14})$$

With a lift $H$ of $\epsilon_{\mathfrak{X}}$ and with $v \in T_{\mathfrak{X}}^{(m)}$ algebraic we have the local expression

$$\text{tor}^\nabla (H, v) \equiv \text{tor}^\nabla \left( \theta \frac{\partial}{\partial \theta}, \theta^m \frac{\partial}{\partial \theta} \right) \mod T^{(m+1)}_{\mathfrak{X}}$$

$$= \theta^{m+1} \text{tor}^\nabla \left( \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \right)$$

$$\in \left( \theta^{m+1} \frac{\partial}{\partial x}, \theta^{m+2} \frac{\partial}{\partial \theta}, \ldots \right) = T^{(m+1)}_{\mathfrak{X}} \quad \text{from (II.4.5.14)}.$$ 

Hence $\text{tor}^\nabla (H, v) \equiv 0$ modulo $T^{(m+1)}_{\mathfrak{X}}$, as required. \qed

The results presented so far culminate in the following.

Proposition II.4.18. Let $\nabla$ be a global, even, affine connection on $\mathfrak{X}$. For any algebraic section $v \in T_{\mathfrak{X}}^{(m)}$ and lift $H$ of $\epsilon_{\mathfrak{X}}$,

$$\nabla_H v \equiv (m+1)v \mod T^{(m+1)}_{\mathfrak{X}}.$$
Proof. This is immediate from the relation to the Lie bracket in (II.4.5.13), the scaling property of $\hat{\epsilon}$ from Lemma II.2.6(ii), Proposition II.4.13 and Lemma II.4.17. □

Unlike the shearing described in Proposition II.4.13, the mapping in Proposition II.4.18 above also dilates the algebraic sections. Hence, relative to the mapping in Proposition II.4.13, we might refer to $\nabla_H$ in Proposition II.4.18 as *shear-like*. More generally, if $v \in T^{(m)}_\mathfrak{X}$ from Proposition II.4.18 is not algebraic, we have:

**Proposition II.4.19.** Let $\nabla$ be a global, even, affine connection on $\mathfrak{X}$. For any non-algebraic section $v \in T^{(m)}_\mathfrak{X}$ and lift $H$ of $\hat{\epsilon}$, $\nabla_H v \equiv mv \mod T^{(m)}_\mathfrak{X} \cap \{\text{algebraic sections}\}$.

Proof. The proof of this proposition can be given by direct calculation. With $H$ a lift of $\hat{\epsilon}$ and $(\mathcal{U}_\alpha)$ a covering of $\mathfrak{X}$, write $H$ as in (II.4.5.5). Then we have

$$
(\nabla_H v)|_{\mathcal{U}_\alpha} = \nabla_{\epsilon + Q_\alpha^{(1)}} \ldots v
\equiv \nabla_{\epsilon \alpha} v|_{\mathcal{U}_\alpha} \mod T^{(m+1)}_\mathfrak{X}(\mathcal{U}_\alpha) \quad \text{(by Definition II.1.1(ii)).}
$$

In coordinates $(x|\theta)$ on $\mathcal{U}_\alpha$ consider the following local expression:

$$
v(x|\theta) \equiv f_m \theta^m \frac{\partial}{\partial x} + g_{m+1} \theta^{m+1} \frac{\partial}{\partial \theta} \mod T^{(m+1)}_\mathfrak{X}(\mathcal{U}_\alpha).
$$

Then we have

$$
\nabla_{\epsilon \alpha} v|_{\mathcal{U}_\alpha} = \epsilon_{\alpha} (f_m \theta^m) \frac{\partial}{\partial x} + \epsilon_{\alpha} (g_{m+1} \theta^{m+1}) \frac{\partial}{\partial \theta}
+ f_m \theta^m \nabla_{\epsilon \alpha} \frac{\partial}{\partial x} + g_{m+1} \theta^{m+1} \nabla_{\epsilon \alpha} \frac{\partial}{\partial \theta}
= mv + g_{m+1} \theta^{m+1} \frac{\partial}{\partial \theta} \quad \text{(by Lemma II.2.6(i))} \quad (\text{II.4.5.15})
$$

$$
+ f_m \theta^m \nabla_{\epsilon \alpha} \frac{\partial}{\partial x} + g_{m+1} \theta^{m+1} \nabla_{\epsilon \alpha} \frac{\partial}{\partial \theta}. \quad (\text{II.4.5.16})
$$

With $\epsilon_{\alpha} = \theta \partial / \partial \theta$ and the property that $\nabla$ is even, the lattermost term in (II.4.5.16), being $g_{m+1} \theta^{m+1} \nabla_{\epsilon \alpha} (\partial / \partial \theta)$, will be an element of $T^{(m+1)}_\mathfrak{X}(\mathcal{U}_\alpha)$; while the former term $f_m \theta^m \nabla_{\epsilon \alpha} (\partial / \partial x)$ will be an algebraic section of $T^{(m)}_\mathfrak{X}(\mathcal{U}_\alpha)$. Note moreover that the latter term in (II.4.5.15), being $g_{m+1} \theta^{m+1} \partial / \partial \theta$, is also an algebraic section of $T^{(m)}_\mathfrak{X}(\mathcal{U}_\alpha)$. 
Hence, up to algebraic sections of $T_x^{(m)}(\mathcal{U}_\alpha)$ we can equate $\nabla_H v$ with $m v$. This proposition now follows. \hfill \Box

We will now turn to our proof of Koszul’s theorem.

II.4.6. **Proof of Theorem II.1.5.** Let $X$ be our supermanifold and suppose it is equipped with a mod $\ell$ lift of the Euler vector field $\overline{H}$. Denote by $H^\infty$ a smooth extension of $\overline{H}$ to global vector field on $\mathfrak{X}$. The argument for why such an extension will always exist follows along very similar lines to that in Proposition II.4.7 for the mod 2 lift. In brief: firstly note the existence of a short exact sequence $T_x^{(f)} \rightarrow T_x^{(0)} \rightarrow T_x^{(0)}/T_x^{(f)}$ for any $\ell$ giving the boundary mapping $\partial : H^0(X, T_x^{(0)}/T_x^{(f)}) \rightarrow H^1(X, T_x^{(f)})$. Recall for the mod $\ell$ lift $\overline{H}$ that $\overline{H} \in H^0(X, T_x^{(0)}/T_x^{(f)})$. Where smooth sections are concerned we know that $\forall j > 0, H^j(X, T_x^{(f):\infty}) = (0)$; and so we have $H^0(X, T_x^{(0):\infty}) \rightarrow H^0(X, T_x^{(0):\infty}/T_x^{(f):\infty}) \overline{\partial} \rightarrow H^1(X, T_x^{(f)}) = (0)$. From the inclusion $H^0(X, T_x^{(0)/T_x^{(f)})}) \subset H^0(X, T_x^{(0):\infty}/T_x^{(f):\infty})$ we can therefore conclude the existence of some smooth extension $H^\infty$ of our given, mod $\ell$ lift. Now let $(\mathcal{U}_\alpha)$ be an open covering of $\mathfrak{X}$. Over $\mathcal{U}_\alpha$ we can generically write

$$H^\infty|_{\mathcal{U}_\alpha} = \epsilon_\alpha + Q_\alpha$$

$$= \epsilon_\alpha + Q^{(1)}_\alpha + Q^{(2)}_\alpha + \cdots$$

where $\epsilon_\alpha = \epsilon_x|_{\mathcal{U}_\alpha}$ is a local expression for the Euler vector field; $Q_\alpha \in T_x^{(1)}(\mathcal{U}_\alpha)$ and $Q^{(j)}_\alpha \in T_x^{(j)}(\mathcal{U}_\alpha)$.

The local isomorphism in (II.4.5.1) justifies the expression in (II.4.6.2). Now recall the boundary mapping $\partial : H^0(X, T_x^{(0)}/T_x^{(f)}) \rightarrow H^1(X, T_x^{(f)})$. Since $H^\infty$ is a smooth extension of a mod $\ell$ lift of $\epsilon_x$ we have on intersections $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$,\footnote{Note, while $H^\infty$ given by (II.4.6.1) is holomorphic over $\mathcal{U}_\alpha$,this does not imply $H^\infty$ will be holomorphic as a global vector field on $\mathfrak{X}$. See footnote 24.}

$$\partial(H^\infty \mod T_x^{(f)}(\mathcal{U}_\alpha)) = \partial(H^\infty \mod T_x^{(f)}(\mathcal{U}_\beta)) - (H^\infty_\beta \mod T_x^{(f)}(\mathcal{U}_\beta)) \in T_x^{(f)}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta).$$

Comparing this with the expression in (II.4.6.1) and using that $\epsilon_\alpha = \epsilon_\beta$ on $\mathcal{U}_{\alpha\beta}$, we see also that $Q_\alpha \equiv Q_\beta \mod T_x^{(f)}(\mathcal{U}_{\alpha\beta})$. As such there exists some smooth, global

$$\partial(H^\infty_0 \mod T_x^{(f)}(\mathcal{U}_0)) = \partial(H^\infty \mod T_x^{(f)}(\mathcal{U}_0)) - (H^\infty \mod T_x^{(f)}(\mathcal{U}_0)) \in T_x^{(f)}(\mathcal{U}_0).$$

where $H^\infty_0$ is a smooth extension of $H^\infty$ on $\mathcal{U}_0$.
vector field $Q^\infty \in H^0(X, T_X^{(1):\infty})$ such that $Q^\infty|_{U_\alpha} \equiv Q_\alpha \mod T_X^{(\ell)}(U_\alpha)$ and for all $\alpha$. Forming the difference $H^\infty - Q^\infty$ of smooth vector fields on $\mathfrak{X}$ now reveals,

$$(H^\infty - Q^\infty)|_{U_\alpha} \equiv \epsilon_\alpha \mod T_X^{(\ell)}(U_\alpha)$$

Hence W.L.O.G., and referencing the expression (II.4.6.2), any smooth extension of a mod $\ell$ lift of $\epsilon_\mathfrak{X}$ can be taken to be locally of the form

$$(H^\infty|_{U_\alpha} = \epsilon_\alpha + Q_\alpha^{(\ell)} + Q_\alpha^{(\ell+1)} + \cdots). \tag{II.4.6.4}$$

We will suppose now that $H^\infty$ is a smooth extension of a mod $\ell$ lift of $\epsilon_\mathfrak{X}$ with $\ell > 1$ and is given as in (II.4.6.4). In the statement of Koszul’s theorem (Theorem II.1.5) recall that we assume the existence of a global, even, affine connection on $\mathfrak{X}$. Consider the construct $\nabla H^\infty$. Over $U_\alpha$ we have:

$$(\nabla H^\infty)|_{U_\alpha} = \nabla_{H^\infty|_{U_\alpha}} (H^\infty|_{U_\alpha})$$

$$= \nabla \epsilon_\alpha + \nabla Q_\alpha^{(\ell)} \epsilon_\alpha + \nabla \epsilon_\alpha Q_\alpha^{(\ell)} + \cdots$$

$$= \epsilon_\alpha + \nabla Q_\alpha^{(\ell)} \epsilon_\alpha + \ell Q_\alpha^{(\ell)} + w_\alpha^{alg.} + \cdots. \tag{II.4.6.5}$$

where (II.4.6.5) follows from Lemma II.4.15 and Proposition II.4.19 for $w_\alpha^{alg.} \in T_X^{(\ell+1)}(U_\alpha)$ some algebraic section. Importantly, the ellipses ‘…” above denote terms in $T_X^{(\ell+1)}(U_\alpha)$. By Lemma II.4.16 the section $\nabla Q_\alpha^{(\ell)} \epsilon_\alpha$ will be algebraic. Hence the only not-necessarily-algebraic section in (II.4.6.5) is $\ell Q_\alpha^{(\ell)}$. In order to eliminate it, we form $H^\infty_{alg.} \overset{\Delta}{=} \frac{1}{\ell - 1} (\nabla H^\infty - \ell H^\infty).$\footnote{recall that we are assuming $\ell > 1$.} We then have

$$H^\infty_{alg.|_{U_\alpha}} \equiv \epsilon_\alpha + W_\alpha^{alg.} \mod T_X^{(\ell+1)}(U_\alpha).$$

for $W_\alpha^{alg.} \in T_X^{(\ell)}(U_\alpha)$ some algebraic section. Form again now:

$$H^\infty[1] \overset{\Delta}{=} \frac{1}{\ell - 1} \left( \nabla H^\infty_{alg.} - \ell H^\infty_{alg.} \right). \tag{II.4.6.6}$$

From the shear-like transformation in Proposition II.4.18 then,

$$(H^\infty[1]|_{U_\alpha}) \equiv \epsilon_\alpha \mod T_X^{(\ell+1)}(U_\alpha).$$

We have therefore shown if $H^\infty$ is as in (II.4.6.4) for any $\ell$ and defines thereby a smooth extension of a mod $\ell$ lift of $\epsilon_\mathfrak{X}$, then with the global, even connection $\nabla$
we can modify $H^\infty$ to $H^\infty[1]$ so that now $H^\infty[1]$ will define a smooth extension of a mod $(\ell+1)$ lift of $\epsilon_\hat{X}$. Since $\nabla$ is global, the modification $H^\infty[1]$ is well defined and so we have established an inductive step. To see how Koszul’s theorem can now follow we need to establish the base case. But this is precisely the content of Proposition II.4.7 which asserts there will always exist a mod 2 lift of the Euler vector field to any supermanifold. We can now apply induction to derive a mod $(\dim_\mathcal{X}+1)$ lift of $\epsilon_\hat{X}$. Koszul’s theorem then follows from Corollary II.4.10. □

Our proof of Koszul’s theorem above invoked Corollary II.4.10 concerning lifts of the Euler vector field. For an alternative perspective, observe from (II.4.6.3) that we are using the global, even, affine connection to solve the equation $\delta\epsilon_\hat{X} = 0$ in $H^1(X, T^{(1)}_\hat{X})$. Hence, such a connection represents a solution to this equation and, by Theorem II.2.5, any solution to this equation results in a splitting.

II.4.7. Further Commentary: A Unique Splitting. The Euler vector field exists on any split model $\hat{X}$ and is unique. As we can infer from Theorem II.3.8 however, we do not necessarily need the existence of a unique vector field to split a given supermanifold $\mathcal{X}$. Once we find some global vector field $H$ on $\mathcal{X}$ with initial form $\epsilon_\hat{X}$, we can use $H$ to split $\mathcal{X}$. This $H$ need not be unique and any other $H'$ with initial form $\epsilon_\hat{X}$ will suffice. In the presence of a global, even, affine connection $\nabla$ on $\mathcal{X}$ however we can in fact solve for a unique such $H$ on $\mathcal{X}$ with initial form $\epsilon_\hat{X}$. This was observed by Koszul in [Kos94]. We present a statement and proof below.

**Theorem II.4.20.** Let $\nabla$ be a global, even, affine connection on $\mathcal{X}$. Then there exists a unique, degree zero vector field $H^\nabla \in H^0(X, T^{(0)}_\mathcal{X})$ such that

$$\nabla_{H^\nabla} H^\nabla = H^\nabla.$$  

(II.4.7.1)

**Proof.** We know from our proof of Koszul’s theorem that there exists some $H \in H^0(X, T^{(0)}_\mathcal{X})$ mapping onto $\epsilon_\hat{X}$. To see that there will exist some unique such $H$ associated to $\nabla$ and satisfying (II.4.7.1), consider starting with some mod $\ell$ lift $\overline{H} \in H^0(X, T^{(0)}_\mathcal{X}/T^{(\ell)}_\mathcal{X})$ for some $\ell > 1$, mapping onto $\epsilon_\hat{X}$ and let $H^\infty$ be a smooth extension of $\overline{H}$. The process $^{28} H^\infty \xrightarrow{\nabla} H^\infty[1] \xrightarrow{\nabla} (H^\infty[1])[1] \xrightarrow{\nabla} \cdots$ will eventually,

$^{28}$c.f., (II.4.6.6)
and after finitely many steps, \(^{29}\) stabilise to some \(H^\nabla\) which, in a given atlas \(\mathcal{U} = (\mathcal{U}_\alpha)\) satisfies, \(H^\nabla|_{\mathcal{U}_\alpha} = \epsilon_\alpha = \epsilon_\widehat{\mathcal{X}}|_{\mathcal{U}_\alpha}, \forall \alpha\). In particular, \(H^\nabla\) is independent of choice of \(\ell\). Furthermore, by Lemma II.4.15 we have:

\[
(\nabla_{H^\nabla} H^\nabla)|_{\mathcal{U}_\alpha} = \nabla_{\epsilon_\alpha} \epsilon_\alpha = \epsilon_\alpha = H^\nabla|_{\mathcal{U}_\alpha}.
\]

(II.4.7.2)

Since \(\nabla\) and \(H^\nabla\) are global, (II.4.7.2) implies \(\nabla_{H^\nabla} H^\nabla = H^\nabla\). Now suppose \(H \in H^0(\mathcal{X}, T_{\mathcal{X}}^{(0)})\) maps onto \(\epsilon_\widehat{\mathcal{X}}\) and satisfies \(\nabla H = H\). Locally we can write \(H|_{\mathcal{U}_\alpha} = \epsilon_\alpha + Q_\alpha\) for some \(Q_\alpha \in T^{(1)}(\mathcal{U}_\alpha)\).\(^{30}\) As such, by Lemma II.4.15 and Proposition II.4.19:

\[
(\nabla_{H^\nabla} H)|_{\mathcal{U}_\alpha} = \nabla_{\epsilon_\alpha+Q_\alpha}(\epsilon_\alpha + Q_\alpha) \\
= \epsilon_\alpha + Q^{(1)} + w^{alg.}_{\alpha} + 2Q^{(2)} + w'^{alg.} + \cdots + \nabla Q_\alpha Q_\alpha
\]

where \(Q^{(j)}_\alpha \in T^{(j)}(|\mathcal{U}_\alpha|)\) denotes the projection of \(Q_\alpha\) under the local isomorphism (II.4.5.1). Evidently, we can equate \(\nabla_{H^\nabla} H = H\) if and only if \(Q_\alpha = 0\) (which obviously implies \(\nabla Q_\alpha Q_\alpha = 0\)); and in this case we find \(H|_{\mathcal{U}_\alpha} = H^\nabla|_{\mathcal{U}_\alpha}\) for all \(\alpha\), so thereby we must identify \(H\) and \(H^\nabla\).

From the proof of Theorem II.4.20 note that the unique vector field \(H^\nabla\) associated to \((\mathcal{X}, \nabla)\), where \(\mathcal{X} = \widehat{\mathcal{X}}\) is the split model, is precisely the Euler vector field \(\epsilon_{\widehat{\mathcal{X}}}\). As a consequence now of Theorem II.4.20, we have:

**Corollary II.4.21.** Associated to any global, even, affine connection on a supermanifold \(\mathcal{X}\) is a unique splitting \(\mathcal{X} \xrightarrow{\xi} \widehat{\mathcal{X}}\).

---

\(^{29}\)More precisely, after \((\dim_\mathcal{X} - \ell + 1)\)-many steps.

\(^{30}\)C.f., (II.4.6.1).
Part III. The Super Atiyah Class

III.1. Atiyah Classes: Preliminaries

III.1.1. Preliminaries: On Manifolds. Koszul’s theorem relates the existence of a global, even, affine connection on a supermanifold $\mathfrak{X}$ with a splitting of $\mathfrak{X}$. Hence, the existence of an obstruction class to splitting $\mathfrak{X}$ will also obstruct the existence of such a connection. Classically, the Atiyah class of a complex manifold measures precisely the obstruction to the existence of a global, holomorphic connection. This is the central construction in Atiyah’s paper [Ati57] which we paraphrase as follows.

**Theorem III.1.1.** Let $E$ be a holomorphic vector bundle on a complex manifold $X$ and with sheaf of holomorphic sections $\mathcal{E}$. Associated to $E$ is the Atiyah sequence

\[ 0 \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \Omega^1_X \otimes \mathcal{E}) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{J}^1\mathcal{E}) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \longrightarrow 0 \]

where $\mathcal{J}^1\mathcal{E}$ is the sheaf of holomorphic 1-jets of sections of $E$; and $\mathcal{O}_X$ is the structure sheaf of holomorphic functions on $X$. The Atiyah sequence induces on cohomology the mapping $H^0(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) \overset{\delta}{\longrightarrow} H^1(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \Omega^1_X \otimes \mathcal{E}))$. The following two statements are now equivalent:

(i) $E$ admits a global, holomorphic connection;
(ii) $\delta(1_{\mathcal{E}}) = 0$.

where $1_{\mathcal{E}}$ is the identity mapping $\mathcal{E} \overset{\sim}{\to} \mathcal{E}$.

The equivalence of statements Theorem III.1.1(i) and Theorem III.1.1(ii) justifies referring to $\delta(1_{\mathcal{E}})$ as a ‘complete obstruction’ to the existence of a global, holomorphic connection on $E$. We introduce now the following notation, to any vector bundle $E$ on $X$ with sheaf of holomorphic sections $\mathcal{E}$:

\[ \mathcal{A}t \mathcal{E} \overset{\Delta}{=} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \Omega^1_X \otimes \mathcal{E}); \quad (\text{III.1.1.1}) \]

\[ \text{AT } \mathcal{E} \overset{\Delta}{=} H^1(X, \mathcal{A}t \mathcal{E}); \quad (\text{III.1.1.2}) \]

\[ \text{at } \mathcal{E} \overset{\Delta}{=} \delta(1_{\mathcal{E}}). \quad (\text{III.1.1.3}) \]

We refer to $\mathcal{A}t \mathcal{E}$ as the Atiyah sheaf of $\mathcal{E}$; AT $\mathcal{E}$ as the Atiyah space of $\mathcal{E}$; and at $\mathcal{E}$ as the Atiyah class of $\mathcal{E}$. 

III.1.2. The Affine Atiyah Class. If $E$ is the tangent sheaf on $X$, then the term ‘Atiyah’ is prefixed by the term ‘affine’. In this special case, the calculus of differential forms will ensure that the Atiyah class at $T_X$ will be valued in symmetric $2$-forms.\(^{31}\)

And so, in the case $E = T_X$ we modify the Atiyah sheaf in (III.1.1) to:

$$\text{At } T_X \overset{\Delta}{=} \circ^2 T^*_X \otimes T_X.$$  \hspace{1cm} (III.1.2.1)

The sheaf in (III.1.1.1) will be referred to as the affine Atiyah sheaf of $X$; and similarly for the corresponding space $\text{AT } T_X$ and class at $T_X$. As for vector bundles generally in Theorem III.1.1, the affine Atiyah class of $X$ measures the failure for there to exist a global, affine, holomorphic connection on $X$. The generalisation to supermanifolds is now immediate. We have already encountered the notion of tangent sheaves for supermanifolds in §I.2. More generally, Bruzzo et. al. in [BBHR91, p. 161] present a generalisation of Theorem III.1.1 to supermanifolds. And so we have straightforwardly the notion of an affine Atiyah sheaf, space and class for any supermanifold $\mathfrak{X}$.

III.1.3. The Even Part. A new feature in supergeometry relative to (commutative) geometry is the presence of a global dichotomy between even and odd. On a supermanifold $\mathfrak{X}$, the global $\mathbb{Z}_2$-grading on its functions $\mathcal{O}_X \cong \mathcal{O}_{X,+} \oplus \mathcal{O}_{X,-}$ induces a grading on its tangents $T_X \cong T_{X,+} \oplus T_{X,-}$ and so also on its Atiyah sheaf $\text{At } T_X \cong (\text{At } T_X)_+ \oplus (\text{At } T_X)_-$, realising it as a graded, $\mathcal{O}_{X,+}$-module. Accordingly, we have a decomposition $\text{AT } T_X \cong (\text{AT } T_X)_+ \oplus (\text{AT } T_X)_-$ with respect to which:

**Lemma III.1.2.**

$$\text{at } T_X \in (\text{AT } T_X)_+.$$ 

**Proof.** Recall that $\text{End}_{\mathcal{O}_X} T_X$ is an $\mathcal{O}_X$-algebra and the section $1_{T_X} \in H^0(X, \text{End}_{\mathcal{O}_X} T_X)$ is invertible, i.e., is in the image of $H^0(X, \text{Aut}_{\mathcal{O}_X} T_X)$. It must therefore be in the even component of $\text{End}_{\mathcal{O}_X} T_X$, where the decomposition into even and odd components is as $\mathcal{O}_{X,+}$-modules. The Atiyah sequence from Theorem III.1.1 adapted to supermanifolds will be a sequence of even, $\mathcal{O}_X$-morphisms so therefore,

$$\text{at } T_X \overset{\Delta}{=} \delta(1_{T_X}) \in H^1(X, (\text{At } T_X)_+) \overset{\Delta}{=} (\text{AT } T_X)_+,$$

\(^{31}\)C.f., the proof of Theorem II.1.2 in Appendix A.
We will hereafter be concerned specifically with the even Atiyah class of supermanifolds. As might be expected from Theorem III.1.1, the even Atiyah class is related to global, even, affine connections thusly:

**Theorem III.1.3.** The even, affine Atiyah class of any supermanifold \( X \) measures the failure for there to exist a global, even, affine connection on \( X \).

**Proof.** By Theorem II.1.2 the space of global, even, affine connections forms a torsor over the sections \( H^0(X, (At T_X)_+) \). Arguing in analogy with [Ati57] will reveal that the obstruction to the existence of a global connection will lie in \( (AT T_X)_+ \). See also the more general argument in [BBHR91, p. 161].

**Remark III.1.4.** In dropping the prefix ‘even’, the affine Atiyah class will more generally measure the failure for there to exist a global, affine connection on \( X \) from Definition II.1.1.

### III.2. Preliminaries on Obstruction Theory

**III.2.1. Preliminaries.** The term ‘obstruction theory’ has many connotations in a number of mathematical disciplines and is typically associated with the study of the obstructions to the lifting of some structure. In topology, obstruction theory might pertain the obstructions to lifting topological to smooth structures; smooth structures to orientable structures; and these latter to spin structures, as in [MS74, §6–9]. In complex and algebraic geometry, obstruction theory might pertain to the obstructions to the existence of, and liftings of, versal deformations as in [Kod86, Ch. 5] and [Har10, Ch. 2]. In complex supergeometry, following the work of Berezin in [Ber87, Ch. 4], Green in [Gre82] and Manin in [Man88, Ch. 4], obstruction theory concerns the obstructions to the existence of supermanifold splittings.

In this section we present some of the rudiments of obstruction theory for supermanifolds essential for subsequent investigations into the super Atiyah class. Material on the classification of supermanifolds are further prerequisites for the material presented here and so we have included a review of this classification in Appendix C.
In contrast to existing treatments in the literature, our review is tailored toward a viewpoint which is functorial in supermanifolds (see the concluding section of Appendix §C). To begin now, fix a model \((X, T^*_X, -)\) and recall the automorphism sheaves \(G^{(m)}_{(X, T^*_X, -)}\) from (C.1), presented below for convenience:

\[
G^{(m)}_{(X, T^*_X, -)} \triangleq \left\{ \alpha \in (\text{Aut} \wedge \bullet T^*_X, -) \mid \alpha(u) - u \in \bigoplus_{\ell \geq m} \wedge^\ell T^*_X \right\}.
\]

Green in [Gre82] established the following.

**Theorem III.2.1.** To any model \((X, T^*_X, -)\) the automorphism sheaves \((G^{(m)}_{(X, T^*_X, -)})_{m \in \mathbb{Z}}\) satisfy the following:

(i) \(\forall m > \text{rank } T^*_X, G^{(m)}_{(X, T^*_X, -)} = \{1\}\);

(ii) \(\forall m < 0, G^{(m)}_{(X, T^*_X, -)} = \emptyset\);

(iii) \(\forall m \geq 0, G^{(m+1)}_{(X, T^*_X, -)} \subset G^{(m)}_{(X, T^*_X, -)}\) is normal;

(iv) \(\forall m \geq 2, G^{(m)}_{(X, T^*_X, -)}/G^{(m+1)}_{(X, T^*_X, -)}\) is abelian.

\[\square\]

If \(G\) is a non-abelian sheaf of groups then its cohomology is only defined in degrees zero and one. In degree zero it is a group and in degree one it is a pointed set. Grothendieck in [Gro55] nevertheless observed that to any short exact sequence of sheaves of not-necessarily-abelian groups there will be induced a long exact sequence on cohomology as in the case of abelian sheaves, albeit truncated to degrees zero and one and viewed as pointed sets. For any \(m \geq 2\) set

\[
\mathcal{O}^{(m)}_{b_{(X, T^*_X, -)}} \triangleq \frac{G^{(m)}_{(X, T^*_X, -)}}{G^{(m+1)}_{(X, T^*_X, -)}} \quad \text{and} \quad \text{OB}^{(m)}_{(X, T^*_X, -)} \triangleq H^1(X, \mathcal{O}^{(m)}_{b_{(X, T^*_X, -)}}). \tag{III.2.1.1}
\]

Note that \(\mathcal{O}^{(m)}_{b_{(X, T^*_X, -)}}\) makes sense by Theorem III.2.1(iii); and by Theorem III.2.1(iv) it will be abelian.

**Definition III.2.2.** The sheaf \(\mathcal{O}^{(m)}_{b_{(X, T^*_X, -)}}\) in (III.2.1.1) will be referred to as an *obstruction sheaf* and the space \(\text{OB}^{(m)}_{(X, T^*_X, -)}\) as an *obstruction space*. In the special case \(m = 2\) we refer to this sheaf as the *primary obstruction sheaf* and write \(\mathcal{O}^{\text{primary}}_{b_{(X, T^*_X, -)}}\).
Accordingly, its 1-cohomology will be referred to as the obstruction space and primary obstruction space respectively with the latter written $\text{OB}^{\text{primary}}_{(X,T\hat{X}_-)}$.

As a consequence of Theorem III.2.1(iii) we have a long exact sequence of pointed sets which, on cohomology, ends on the piece:

$$
\cdots \longrightarrow \hat{H}^1(G^{(m+1)}_{(X,T\hat{X}_-)}) \longrightarrow \hat{H}^1(G^{(m)}_{(X,T\hat{X}_-)}) \xrightarrow{\omega_*} \text{OB}^{(m)}_{(X,T\hat{X}_-)}.
$$

(III.2.1.2)

Exactness ensures that $\text{im}\{\hat{H}^1(G^{(m+1)}_{(X,T\hat{X}_-)}) \rightarrow \hat{H}^1(G^{(m)}_{(X,T\hat{X}_-)})\} \cong \ker \omega_*$.

32 From Green’s classification in Theorem C.1 any framed supermanifold $(\mathfrak{X}, \phi)$ will define an element $[\mathfrak{X}, \phi]$ in $\hat{H}^1(G^{(2)}_{(X,T\hat{X}_-)})$ and so, to framed supermanifolds $(\mathfrak{X}, \phi)$ we have naturally associated an element $\omega_*[\mathfrak{X}, \phi] \in \text{OB}^{\text{primary}}_{(X,T\hat{X}_-)}$.

**Definition III.2.3.** To any framed supermanifold $(\mathfrak{X}, \phi)$ the element $\omega_*[\mathfrak{X}, \phi]$ will be referred to as its primary obstruction to splitting.

A vanishing primary obstruction class is an almost a meaningless contribution to the question of whether a supermanifold splits. Its non-vanishing is highly instructive however and formed an integral fact in Donagi and Witten’s study of the supermoduli space of curves. More precisely, we have the following:

**Theorem III.2.4.** Let $\mathfrak{X}$ be a supermanifold and suppose, with respect to some framing $\phi$, that its primary obstruction $\omega_*([\mathfrak{X}, \phi]) \in \text{OB}^{\text{primary}}_{(X,T\hat{X}_-)}$ is non-vanishing.

Then $\mathfrak{X}$ is non-split.

*Proof.* See e.g., [Bet18b, Appx. A].

**Remark III.2.5.** A consequence of Theorem III.2.4 is that the class $\omega_*([\mathfrak{X}, \phi])$ depends essentially on the isomorphism class of $\mathfrak{X}$ and not the particular choice of framing $\phi$. Henceforth, by abuse of notation, we will simply refer to the primary obstruction class of $(\mathfrak{X}, \phi)$ by $\omega_*[\mathfrak{X}]$ and omit explicit reference to a framing $\phi$.

\[32\] The kernel of a mapping between pointed sets comprise all those elements which, under the mapping in question, map to the base-point. This includes, in particular, the base-point in the former set and so the kernel will not be empty. Now by Theorem III.2.1(iv) the obstruction space $\text{OB}^{(m)}_{(X,T\hat{X}_-)}$ will be a complex vector space. Its base-point corresponds then to the zero vector $0$.

\[33\] C.f., footnote 32.
Remark III.2.6. The sequence in (III.2.1.2) suggests the contrivance of higher obstruction classes to framed supermanifolds. These would be elements in $\text{OB}^{(m)}_{(X,T^{\ast}_X,-)}$ somehow associated to a given, framed supermanifold $(X, \phi)$ and general $m$. Such an association cannot be naturally associated to $X$ however due to the existence of ‘exotic structures’. These were identified by Donagi and Witten in [DW15] as obfuscating the resolution of the splitting problem for supermanifolds by reference to obstruction theory alone. The author in [Bet18b] presented a further study in higher obstruction theory concerning cases where such a theory can be naturally associated to supermanifolds. As the setting involving the primary obstruction class will suffice for our purposes in this article, we will not consider ‘higher’ obstruction theory here.

III.2.2. Naturality. In Remark III.2.6 it was noted that ‘higher’ obstruction classes need not be naturally associated to supermanifolds. This is in contrast to the primary obstruction class which, as mentioned in the comment succeeding Theorem III.2.4, depends essentially on the isomorphism class of supermanifolds. In order to arrive then at a more intrinsic description of the primary obstruction class, we need to appeal to the intrinsic description of the automorphism groups $G_{(O_X)}^{(m)}$ in (C.3) in contrast to the extrinsic description $G_{(X,T^{\ast}_X,-)}^{(m)}$ in (C.1). And so with $G_{(O_X)}^{(m)}$ playing the role of $G_{(X,T^{\ast}_X,-)}^{(m)}$ in Theorem III.2.1 we have:

**Theorem III.2.7.** For any supermanifold $X$, the automorphism sheaves $(G_{(O_X)}^{(m)})_{m \in \mathbb{Z}}$ satisfy:

(i) \( \forall m > \dim_X, G_{(O_X)}^{(m)} = \{1\} \);
(ii) \( \forall m < 0, G_{(O_X)}^{(m)} = \emptyset \);
(iii) \( \forall m \geq 0, G_{(O_X)}^{(m)} \subset G_{(O_X)}^{(m+1)} \) is normal;
(iv) \( \forall m \geq 2, G_{(O_X)}^{(m)}/G_{(O_X)}^{(m+1)} \) is abelian.

**Proof.** Regarding (i), for $\mathcal{J}_X \subset O_X$ the fermionic ideal, it is nilpotent with degree \( \dim_X + 1 \), i.e., that $\mathcal{J}_X^{\dim_X + 1} = (0)$. Hence $G_{(O_X)}^{(m)}$ is the trivial group for $m > \dim_X$; (ii) is vacuously true since $G_{(O_X)}^{(m)}$ is not defined for $m < 0$;\(^{34}\) (iii) the proof given by Green in [Gre82] applies straightforwardly in this general case; and lastly

\(^{34}\) alternatively, see the proof of Lemma C.3.
(iv), note that if we can express $G^{(m+1)}$ as the kernel of a homomorphism $G^{(m)} \xrightarrow{f} H$ for some abelian sheaf $H$, then the quotient, which exists by (iii), will necessarily be abelian. This is because it will be isomorphic to the image $\text{im}\{ f : G^{(m)}_\mathcal{O}_X \to H \}$ which, being a subsheaf of an abelian sheaf $H$, must itself be abelian. And so, onto the particulars of the morphism $f$ and sheaf $H$, recall that $G^{(m)}_\mathcal{O}_X$ is a sheaf of automorphisms of $\mathcal{O}_X$ of a certain kind. Hence it embeds in the endomorphisms $G^{(m)}_\mathcal{O}_X \subset \mathcal{E}nd_{\mathcal{O}_X} \mathcal{O}_X$ which is itself a sheaf of $\mathcal{O}_X$-algebras. In $\mathcal{E}nd_{\mathcal{O}_X} \mathcal{O}_X$ it makes sense to form binary operations such as the sum or difference of sections of $G^{(m)}_\mathcal{O}_X$ and so, similarly to Onishchik in [Oni99, p. 56] in the split case, to any $g \in G^{(m)}_\mathcal{O}_X$ we can use the formal logarithm to map $G^{(m)}_\mathcal{O}_X \xrightarrow{\ln} \mathcal{E}nd_{\mathcal{O}_X} \mathcal{O}_X$ by

$$
g \ln \mapsto (g - 1) + \frac{(g - 1)^2}{2} - \cdots$$

(III.2.2.1)

where $1 \in G^{(m)}_\mathcal{O}_X$ is the identity. Observe that, by definition, $g - 1 : \mathcal{O}_X \to \mathcal{J}^m_{\mathcal{O}_X}$ so therefore the formal sum in (III.2.2.1) will only contain finitely many terms. Furthermore it will be additive, i.e., $\ln(g \circ h) = \ln g + \ln h$ for all $g, h \in G^{(m)}_\mathcal{O}_X$. Now observe for any $\ell > 0$ that $(g - 1)^\ell \in \mathcal{J}^{m+\ell}_{\mathcal{O}_X}$ and so $\ln g \equiv g - 1 \mod \mathcal{J}^{m+1}_{\mathcal{O}_X}$. If $g \in G^{(m+1)}_\mathcal{O}_X$ then $g - 1 : \mathcal{O}_X \to \mathcal{J}^{m+1}_{\mathcal{O}_X}$ giving therefore $\ln g \equiv 0 \mod \mathcal{J}^{m+1}_{\mathcal{O}_X}$. We have therefore a left exact sequence of sheaves on $X$:

$$1 \longrightarrow G^{(m+1)}_\mathcal{O}_X \longrightarrow G^{(m)}_\mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathcal{O}_X} \left( \mathcal{O}_X, \frac{\mathcal{J}^m_{\mathcal{O}_X}}{\mathcal{J}^{m+1}_{\mathcal{O}_X}} \right).$$

Hence $G^{(m)}_\mathcal{O}_X / G^{(m+1)}_\mathcal{O}_X \subset \mathcal{H}om_{\mathcal{O}_X} \left( \mathcal{O}_X, \frac{\mathcal{J}^m_{\mathcal{O}_X}}{\mathcal{J}^{m+1}_{\mathcal{O}_X}} \right)$ will be abelian. □

As in (III.2.1.1) we set

$$\mathcal{O}b^{(m)}_{\mathcal{O}_X} \triangleq \frac{G^{(m)}_\mathcal{O}_X}{G^{(m+1)}_\mathcal{O}_X} \quad \text{and} \quad \mathcal{O}b^{(m)}_{\mathcal{O}_X} \triangleq H^1(X, \mathcal{O}b^{(m)}_{\mathcal{O}_X}).$$

By Theorem III.2.7(iv), $\mathcal{O}b^{(m)}_{\mathcal{O}_X}$ will be a finite dimensional, complex vector space. Definition III.2.2 now adapts straightforwardly.

\footnote{In contrast to (iii), Green’s proof in [Gre82] cannot be readily adapted for, in [Gre82], Green makes explicit use of the $\mathbb{Z}$-grading on the exterior algebra.}
Definition III.2.8. To any supermanifold $\mathfrak{X}$, the sheaf $\mathcal{O}_{\mathfrak{X}}(m)$ will be referred to as the $m$-th obstruction sheaf of $\mathfrak{X}$ and the space $\text{OB}_X(m)$ as the $m$-th obstruction space of $\mathfrak{X}$. In the special case $m = 2$ we refer to these objects as the primary obstruction sheaf and space and accordingly denote $\mathcal{O}\text{b}_{\mathfrak{X}}^{\text{primary}} \triangleq \mathcal{O}_{\mathfrak{X}}(2)$ and $\text{OB}_X^{\text{primary}} \triangleq \text{OB}_X(2)$.

Since $G(2)\mathcal{O}_{\mathfrak{X}} = G(2)(\mathfrak{X}, T\mathfrak{X}^*)$ for all $m$, we readily recover the constructions from §III.2.1 upon specialising to the split model, i.e., in taking $\mathfrak{X} = \hat{\mathfrak{X}}$.

III.2.3. The Primary Obstructions. From our classification in Theorem C.5, the isomorphism class of any supermanifold $\mathfrak{X}$ will define the base-point in the pointed set $\mathfrak{M}_X$.

If $\mathfrak{X}$ is, in addition, equipped with a framing $\phi$, then by the characterisation in (C.4), we can identify $G(2)\mathcal{O}_{\mathfrak{X}} \cong (\text{Aut}_{\mathfrak{X}} \mathcal{O}_{\mathfrak{X}})^\phi$, i.e., that $G(2)\mathcal{O}_{\mathfrak{X}}$ will comprise those automorphisms which preserve the framing.

Hence, under this identification, the isomorphism class of any supermanifold $\mathfrak{X}$ equipped with a framing $\phi$ will define the base-point $[\mathfrak{X}, \phi] \in \tilde{H}^1(\mathfrak{X}, (\text{Aut}_{\mathfrak{X}} \mathcal{O}_{\mathfrak{X}})^\phi)$. Since $G(2)\mathcal{O}_{\mathfrak{X}} \cong (\text{Aut}_{\mathfrak{X}} \mathcal{O}_{\mathfrak{X}})^\phi$ we have $\tilde{H}^1(\mathfrak{X}, (\text{Aut}_{\mathfrak{X}} \mathcal{O}_{\mathfrak{X}})^\phi) \cong \tilde{H}^1(\mathfrak{X}, G(2)\mathcal{O}_{\mathfrak{X}})$ and so the isomorphism class of any $(\mathfrak{X}, \phi)$ will define the base-point in $\tilde{H}^1(\mathfrak{X}, G(2)\mathcal{O}_{\mathfrak{X}})$, which we will also denote by $[\mathfrak{X}, \phi]$. Now from Theorem III.2.7(iii) we have induced a sequence on cohomology,

$$
\cdots \to \tilde{H}^1(\mathfrak{X}, G(3)\mathcal{O}_{\mathfrak{X}}) \to \tilde{H}^1(\mathfrak{X}, G(2)\mathcal{O}_{\mathfrak{X}}) \xrightarrow{\eta_*} \text{OB}_X^{\text{primary}}.
$$

Hence, to any framed supermanifold $(\mathfrak{X}, \phi)$ we have associated a class $\eta_*[\mathfrak{X}, \phi]$.

Definition III.2.9. To any framed supermanifold $(\mathfrak{X}, \phi)$ the class $\eta_*[\mathfrak{X}, \phi]$ will be referred to as the primary obstruction to splitting $\mathfrak{X}$.

Now in Definition III.2.3 we presented another class associated to framed supermanifolds $(\mathfrak{X}, \phi)$ in the obstruction space $\text{OB}_X^{\text{primary}}$. The two classes are related as follows.

Proposition III.2.10. For any supermanifold $\mathfrak{X}$ with associated split model $\hat{\mathfrak{X}}$ there is an isomorphism of primary obstruction spaces $\text{OB}^{\text{primary}}_\mathfrak{X} \cong \text{OB}^{\text{primary}}_{\hat{\mathfrak{X}}}$ under which, for any framing $\phi$, that $\eta_*[\mathfrak{X}, \phi] \mapsto \omega_*[\hat{\mathfrak{X}}]$.

$^{36}$Recall (C.6).

$^{37}$Recall from Definition I.1.2 that a framing is an isomorphism $\mathcal{O}_X/\mathcal{J}_X^2 \cong \mathcal{O}_X \oplus (\mathcal{J}_X/\mathcal{J}_X^2)$.

$^{38}$c.f., Remark III.2.5.
Proof. Recall that a choice of framing $\phi$ on $\mathcal{X}$ is an isomorphism $\mathcal{O}_X/\mathcal{J}_X^2 \cong \mathcal{O}_X \oplus (\mathcal{J}_X/\mathcal{J}_X^2)$. In particular, $\mathcal{O}_X/\mathcal{J}_X^2 \cong \mathcal{O}_\hat{X}/\mathcal{J}_{\hat{X}}^2$. Now recall the left exact sequence in (III.2.2.2) for any $\mathcal{X}$. That this sequence is in fact a short exact sequence of sheaves of groups follows from the characterisations of these group elements in (C.4) and (C.5). Hence,

$$\mathcal{O}_{primary}^{\mathcal{O}_X} \cong \mathcal{O}_{primary}^{\mathcal{O}_\hat{X}} \cong \mathcal{O}_X/\mathcal{J}_X^2 \cong \mathcal{O}_\hat{X}/\mathcal{J}_{\hat{X}}^2.$$

Using that any framing gives an isomorphism $\mathcal{O}_X/\mathcal{J}_X^2 \cong \mathcal{O}_\hat{X}/\mathcal{J}_{\hat{X}}^2$, and furthermore that $\mathcal{J}_X^m/\mathcal{J}_{\hat{X}}^m \cong \mathcal{J}_{\hat{X}}^m/\mathcal{J}_{\hat{X}}^{m+1}$ for any $m$ allows us to link the isomorphisms in (III.2.3.1) with:

$$\mathcal{H}om_{\mathcal{O}_X} \left( \frac{\mathcal{O}_X}{\mathcal{J}_X^2}, \frac{\mathcal{J}_X^2}{\mathcal{J}_X^3} \right) \cong \mathcal{H}om_{\mathcal{O}_\hat{X}} \left( \frac{\mathcal{O}_\hat{X}}{\mathcal{J}_{\hat{X}}^2}, \frac{\mathcal{J}_{\hat{X}}^2}{\mathcal{J}_{\hat{X}}^3} \right) \cong \mathcal{G}_{\mathcal{O}_X}^{(2)} \cong \mathcal{G}_{\mathcal{O}_\hat{X}}^{(2)} \cong \mathcal{O}_{primary}^{\mathcal{O}_X}.$$

Hence from any framing $\phi$ we obtain an isomorphism $\mathcal{O}_{primary}^{\mathcal{O}_X} \cong \mathcal{O}_{primary}^{\mathcal{O}_\hat{X}}$; and therefore a isomorphism of obstruction spaces $\text{OB}_{primary}^{\mathcal{O}_X} \cong \text{OB}_{primary}^{\mathcal{O}_\hat{X}}$. With regards to the primary obstruction class itself, of which we presently have two incarnations, they can readily be identified by appealing to classifications of supermanifolds. Rather than insisting on an isomorphism of sheaves of groups, observe that we have at least an isomorphism of 1-cohomologies: for any supermanifold $\mathcal{X}$ with associated split model $\hat{\mathcal{X}}$ that,

$$\check{H}^1(X, \mathcal{G}_{\mathcal{O}_X}^{(2)}) \cong \check{H}^1(X, \mathcal{G}_{\mathcal{O}_\hat{X}}^{(2)}).$$

The justification behind (III.2.3.2) proceeds as follows: each cohomology is a pointed set and (1) by Green’s classification in Theorem C.1; and (2) its variant in Theorem C.5, each 1-cohomology in (III.2.3.2) will classify the same objects for the reason that $\mathcal{X}$ and $\hat{\mathcal{X}}$ are locally isomorphic. As a result they will in turn lie in bijective correspondence and so are isomorphic as sets. Note, they certainly need not be isomorphic as pointed sets, for the base-point in $\check{H}^1(X, \mathcal{G}_{\mathcal{O}_X}^{(2)})$ is $[\mathcal{X}, \phi]$ and, if $\mathcal{X}$ is non-split, cannot be mapped to the base-point in $\check{H}^1(X, \mathcal{G}_{\mathcal{O}_\hat{X}}^{(2)})$, which corresponds to the split model, since this point will not correspond to $[\mathcal{X}, \phi]$ by Theorem C.1 (Green’s classification). Now with a choice of framing $\phi$ we can generate:
(i) the base-point \([\mathfrak{X}, \phi] \in \tilde{H}^1(X, G_{\mathcal{O}_X}^{(2)})\);
(ii) a corresponding point \([\mathfrak{X}, \phi'] \in \tilde{H}^1(X, G_{\mathcal{O}_X}^{(2)})\);
(iii) an isomorphism \(O_{\mathfrak{X}}^{primary} \cong O_{\hat{\mathfrak{X}}}^{primary}\).

And so with (III.2.3.2) any framing \(\phi\) will generate a commutative diagram:

\[
\begin{array}{ccc}
\tilde{H}^1(X, G_{\mathcal{O}_X}^{(2)}) & \xrightarrow{\eta_*} & O_{\mathfrak{X}}^{primary} \\
\cong & & \cong \\
\tilde{H}^1(X, G_{\mathcal{O}_{\hat{\mathfrak{X}}}^{(2)}}) & \xrightarrow{\omega_*} & O_{\hat{\mathfrak{X}}}^{primary}
\end{array}
\]

with \([\mathfrak{X}, \phi] \xrightarrow{\eta_*} [\mathfrak{X}, \phi]\) and \([\mathfrak{X}, \phi'] \xrightarrow{\omega_*} [\hat{\mathfrak{X}}, \phi']\).

Hence \(\eta_*[\mathfrak{X}, \phi]\) will correspond to \(\omega_*[\mathfrak{X}, \phi'] = \omega_*[\mathfrak{X}]\) as required. □

With Proposition III.2.10 we can now immediately adapt Theorem III.2.4.

**Theorem III.2.11.** Let \((\mathfrak{X}, \phi)\) be a framed supermanifold with non-vanishing, primary obstruction \(\eta_*[\mathfrak{X}, \phi] \in O_{\mathfrak{X}}^{primary}\). Then \(\mathfrak{X}\) is non-split. □

**Remark III.2.12.** As in Remark III.2.5 then, a consequence of Theorem III.2.11 is that the primary obstruction of any supermanifold—in the sense of Definition III.2.9, depends essentially on the isomorphism class of \(\mathfrak{X}\) and not the particular choice of framing. As such, by abuse of notation, we will refer to its primary obstruction without referring to a choice of framing.

### III.3. The Atiyah Class of Split Models

With the affine Atiyah sheaf, space and class established for supermanifolds in §III.1.2, we can now look to inspect this class in more detail. We begin here with a characterisation of the affine Atiyah class of split models. Subsequently, we establish a relation between the affine Atiyah class of supermanifolds \(\mathfrak{X}\) more generally and between that of its split model \(\hat{\mathfrak{X}}\).

**III.3.1. On the Split Model.** Our objective here will be to prove the following.

**Proposition III.3.1.** Let \(\hat{\mathfrak{X}}\) be the split model associated to \((X, T_{X,-}^*)\) and denote by \(i : X \subset \hat{\mathfrak{X}}\) the inclusion of the reduced space. Then on X we have a decomposition
of the even, affine Atiyah space,

\[(i^\sharp \text{AT}_{\hat{X}})_+ \cong \text{AT}_{X} \oplus \text{AT}_{X,-} \oplus \text{OB}_{\hat{X}}^{\text{primary}}.\]  

(III.3.1.1)

Let \( pr : (i^\sharp \text{AT}_{\hat{X}})_+ \to \text{AT}_{X} \oplus \text{AT}_{X,-} \) be the map projecting out the primary obstruction space. Then under this projection the affine Atiyah class of \( \hat{X} \) satisfies, \( ^39 \)

\[ pr \circ i^\sharp \text{at} \text{at}_{\hat{X}} = \text{at}_{X} \oplus \text{at}_{X,-} \cdot \]

Proof. With \( i : X \subset \hat{X} \) the inclusion given in the statement of this theorem, it corresponds to the quotient on function algebras \( i^\sharp : \mathcal{O}_{\hat{X}} \to \mathcal{O}_{X} = \mathcal{O}_{\hat{X}} / \mathcal{J}_{\hat{X}} \) where \( \mathcal{J}_{\hat{X}} \subset \mathcal{O}_{\hat{X}} \) is the fermionic ideal. Recall from Corollary I.2.3 that on tangents we have:

\[ i^\sharp T_{\hat{X}} = T_{\hat{X}} \otimes_{\mathcal{O}_{X}} \left( \frac{\mathcal{O}_{\hat{X}}}{\mathcal{J}_{\hat{X}}} \right) \cong T_{X,-} \oplus T_{X}. \]  

(III.3.1.2)

Now pullbacks under inclusions will commute with tensor products. Hence,

\[ i^\sharp \text{AT}_{\hat{X}} = i^\sharp (\circ^2 \Omega^1_\hat{X} \otimes T_{\hat{X}}) \]

(recall (III.1.2.1))

\[ = \circ^2 (T^*_{X,-} \oplus \Omega^1_{\hat{X}}) \oplus (T_{X,-} \oplus T_{X}) \]  

(by (III.3.1.2)).

To identify the even components now we will use that dualising is an ‘odd’ operation’, i.e., for any \( \mathcal{O}_{X} \)-module\(^{40} \) \( \mathcal{M} \) with \( \mathbb{Z}_2 \)-decomposition \( \mathcal{M} \cong \mathcal{M}_+ \oplus \mathcal{M}_- \) that \( (\mathcal{M}^*)_+ = \mathcal{M}^{*_+} \). In particular \( \Omega^1_{\hat{X},+} = T^*_{\hat{X},+} \). This leads to,

\[ \circ^2 \Omega^1_{\hat{X}} = \circ^2 (\Omega^1_{\hat{X},+} \oplus \Omega^1_{\hat{X},-}) \]

\[ = \circ^2 \Omega^1_{\hat{X},+} \oplus \wedge^2 \Omega^1_{\hat{X},-} \oplus (\Omega^1_{\hat{X},+} \otimes \Omega^1_{\hat{X},-}) \]

\[ = \wedge^2 T^*_{\hat{X},-} \oplus \circ^2 T^*_{\hat{X},+} \oplus (T^*_{\hat{X},-} \otimes T^*_{\hat{X},+}) \cdot \]

\(^{39} \text{c.f., Lemma III.1.2.} \)

\(^{40} \text{note, here we are considering a general supermanifold } \hat{X}. \)
Using that the parity distributes additively over tensor products, the even component of the Atiyah sheaf can be identified thusly:

\[(\mathcal{A} t \tilde{T} \hat{X})^+ = (\odot^2 \Omega^1_{\hat{X}} \otimes \tilde{T} \hat{X})^+ = (\odot^2 \Omega^1_{\hat{X}^+} \otimes \tilde{T} \hat{X}^+ \otimes (\wedge^2 \Omega^1_{\hat{X}^-} \otimes \tilde{T} \hat{X}^-) + (\Omega^1_{\hat{X}^+} \otimes \Omega^1_{\hat{X}^-} \otimes \tilde{T} \hat{X}^+ \otimes \tilde{T} \hat{X}^-) \cong (\odot^2 \Omega^1_{\hat{X}^+} \otimes \tilde{T} \hat{X}^+ \otimes (\wedge^2 \Omega^1_{\hat{X}^-} \otimes \tilde{T} \hat{X}^+) \oplus (\Omega^1_{\hat{X}^+} \otimes \operatorname{End} \tilde{T} \hat{X}^-) \cong (\odot^2 \Omega^1_{\hat{X}^+} \otimes \tilde{T} \hat{X}^+) \otimes \tilde{T} \hat{X}^+ \oplus (\Omega^1_{\hat{X}^+} \otimes \operatorname{End} \tilde{T} \hat{X}^-) \oplus (\wedge^2 \Omega^1_{\hat{X}^-} \otimes \tilde{T} \hat{X}^-) \oplus \tilde{T} \hat{X}^-).\]

Pullbacks will commute with tensor, symmetric and anti-symmetric products and so on

We have arrived now at the desired decomposition of the Atiyah space in (III.3.1.1).

Where the decomposition of the Atiyah class is concerned, firstly recall the Atiyah sequences for \(T_X\) and \(T_{X,-}\) respectively. From Theorem III.1.1 we set, for notational convenience:

\[\tilde{\mathcal{J}}^1 \mathcal{E} \triangleq \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{J}^1 \mathcal{E}).\]

The Atiyah sequences for \(T_X\) and \(T_{X,-}\) are now:

\[0 \rightarrow \mathcal{A} t \tilde{T}_X \rightarrow \mathcal{J}^1 \tilde{T}_X \rightarrow \operatorname{End}_{\mathcal{O}_X} T_X \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathcal{A} t \tilde{T}_{X,-} \rightarrow \mathcal{J}^1 \tilde{T}_{X,-} \rightarrow \operatorname{End}_{\mathcal{O}_X} T_{X,-} \rightarrow 0\]

The sequences above can be combined to form the sum:

\[0 \rightarrow \mathcal{A} t \tilde{T}_X \oplus \mathcal{A} t \tilde{T}_{X,-} \rightarrow \mathcal{J}^1 \tilde{T}_X \oplus \mathcal{J}^1 \tilde{T}_{X,-} \rightarrow \operatorname{End} T_X \oplus \operatorname{End} T_{X,-} \rightarrow 0\]

Now note the following:
(i) $\tilde{i}^*(\mathcal{E}nd T\hat{X})_+ \cong \mathcal{E}nd T_X \oplus \mathcal{E}nd T_{X,-}$ and;

(ii) as $\mathcal{O}_X$-modules, $\tilde{J}^1T\hat{X} \cong \mathcal{A}ti(T\hat{X}) \oplus \mathcal{E}nd\mathcal{O}_X T_X$ and $\tilde{J}^1T\hat{X}_{-} \cong \mathcal{A}ti(T\hat{X}_{-}) \oplus \mathcal{E}nd\mathcal{O}_X T_{X,-}$.

Observation (ii) above generalises to $\tilde{J}^1T\hat{X} \cong \mathcal{A}ti(T\hat{X}) \oplus \mathcal{O}_{\hat{X}}^{primary} \oplus \mathcal{E}nd T_X \oplus \mathcal{E}nd T_{X,-}$.

Therefore, pulling back to $X$ and projecting out $\mathcal{O}_{\hat{X}}^{primary}$ yields a morphism of Atiyah sequences:

$$
\begin{array}{cccccc}
0 & \longrightarrow & (At T\hat{X})_+ & \longrightarrow & (\tilde{J}^1T\hat{X})_+ & \longrightarrow & (\mathcal{E}nd T\hat{X})_+ & \longrightarrow & 0 \\
0 & \longrightarrow & At T_X \oplus At T_{X,-} & \longrightarrow & \tilde{J}^1T_X \oplus \tilde{J}^1T_{X,-} & \longrightarrow & E_n T_X \oplus E_n T_{X,-} & \longrightarrow & 0
\end{array}
$$

Importantly, we have: $\tilde{i}^*(1_{T\hat{X}}) = 1_{T_X} \oplus 1_{T_{X,-}}$. By Theorem III.1.1 the Atiyah class of $T_X$ resp. $T_{X,-}$ is $\delta(1_{T_X})$ resp. $\delta(1_{T_{X,-}})$ where $\delta$ is the boundary map on cohomology induced the respective Atiyah sequences for $T_X$ and $T_{X,-}$. Similarly, the (even) Atiyah class of $T\hat{X}$ is $\delta(1_{T\hat{X}})$ where $\delta$ here is the boundary map on cohomology induced by the (even) Atiyah sequence for $T\hat{X}$. Now from our morphism of Atiyah sequences above we have the following commutative diagram on cohomology:

$$
\begin{array}{ccc}
H^0(X, (\mathcal{E}nd T\hat{X})_+) & \longrightarrow & H^1(X, (At T\hat{X})_+) \\
\mathcal{i} & \downarrow & \mathcal{i} \\
H^0(X, \mathcal{E}nd T_X) \oplus H^0(X, \mathcal{E}nd T_{X,-}) & \mathcal{\oplus} & H^1(X, At T_X) \oplus H^1(X, At T_{X,-})
\end{array}
$$

Using $\mathcal{i}^*(1_{T\hat{X}}) = 1_{T_X} \oplus 1_{T_{X,-}}$ gives:

$\mathcal{i}^*\delta(1_{T\hat{X}}) = (\delta \oplus \delta)(\mathcal{i}^*1_{T\hat{X}}) = \delta(1_{T_X}) \oplus \delta(1_{T_{X,-}}) = at T_X \oplus at T_{X,-}$.

The proposition now follows. \hfill \Box

---

41see e.g., [Ati57, §4, p. 193]. Indeed, in the notation from Theorem III.1.1, $J^1\mathcal{E} \cong (\Omega^1_X \otimes \mathcal{O}_X \mathcal{E}) \oplus \mathcal{E}$. A holomorphic connection is a splitting of the sequence $0 \rightarrow \Omega^1_X \otimes \mathcal{E} \rightarrow J^1\mathcal{E} \rightarrow \mathcal{E} \rightarrow 0$ with respect to a ‘twisted’ $\mathcal{O}_X$-module structure on $J^1\mathcal{E}$ given by $f \cdot (\omega \otimes s + t) \equiv f \omega \otimes s + df \otimes t + ft$. 
III.3.2. **On Supermanifolds More Generally.** Recall that the affine Atiyah class is defined for any supermanifold $X$, split or otherwise. We intend here to obtain a relation between the Atiyah class of supermanifolds with that of its split model $\hat{X}$. Central to obtaining such a relation is the initial form sequence from Lemma I.2.5. We recall it below for convenience:

\[
0 \to T_X[1] \to T_X \to T_{\hat{X}} \to 0. \tag{III.3.2.1}
\]

Preempting a relation between the classes now, the following lemma details a relation between Atiyah sheaves.

**Lemma III.3.2.** There exists a commutative diagram of sheaves on $X$:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \odot^2 T_{\hat{X}}^* \otimes T_X & \longrightarrow & \text{At } T_X & \\
& & \downarrow & \swarrow \pi & & \\
0 & \longrightarrow & \text{At } T_{\hat{X}} & \longrightarrow & \odot^2 T_{\hat{X}}^* \otimes T_{\hat{X}} & \\
\end{array} \tag{III.3.2.2}
\]

*Proof.* We begin with a general fact on powers of modules (see e.g., [Har77, p. 121]). To any short exact sequence of sheaves of modules $0 \to F' \to F \to F'' \to 0$ the $r$-th symmetric or anti-symmetric power of $F$, denoted $AS^r F$, admits a length-$(r + 1)$ filtration:

\[
AS^r F = F^0 \supset F^1 \supset \cdots F^r \supset 0
\]

with successive quotients:

\[
F^p / F^{p+1} \simeq AS^p F' \otimes AS^{r-p} F''.
\]

Returning now to the initial form sequence in (III.3.2.1) for the tangent sheaf of a supermanifold $T_X$, dualising gives

\[
0 \longrightarrow T_{\hat{X}}^* \longrightarrow T_X^* \longrightarrow T_X[1]^* \longrightarrow 0
\]

and so by our observation on the symmetric and anti-symmetric powers above we know that $\odot^2 T_X^*$ will be filtered as follows:

\[
\odot^2 T_X^* = F^0 \supset F^1 \supset F^2 \supset 0
\]
with quotients:

\[ F^0 / F^1 = \mathcal{O}^2 \mathcal{T}_X^* / F^1 \cong \mathcal{O}^2 \mathcal{T}_{\hat{X}}^* [1]; \]
\[ F^1 / F^2 \cong \mathcal{T}_X^* \otimes \mathcal{T}_X [1]^* \] and;
\[ F^2 / F^3 = F^2 = \mathcal{O}^2 \mathcal{T}_{\hat{X}}^*. \]

Hence

\[ \mathcal{O}^2 \mathcal{T}_X^* \subset \mathcal{O}^2 \mathcal{T}_{\hat{X}}^* \to \mathcal{O}^2 \mathcal{T}_{\hat{X}}^*[1]^* \quad (\text{III.3.2.3}) \]

Now recall that:

\[ \mathcal{A} \at \mathcal{T}_X = \mathcal{O}^2 \mathcal{T}_X^* \otimes \mathcal{T}_X \quad \text{and} \quad \mathcal{A} \at \mathcal{T}_{\hat{X}} = \mathcal{O}^2 \mathcal{T}_{\hat{X}}^* \otimes \mathcal{T}_{\hat{X}}. \quad (\text{III.3.2.4}) \]

Since tensor products with locally free sheaves will preserve injections, we obtain the top horizontal arrow \( 0 \to \mathcal{O}^2 \mathcal{T}_X^* \otimes \mathcal{T}_X \to \mathcal{A} \at \mathcal{T}_X \) in (III.3.2.2). The vertical arrow \( \mathcal{O}^2 \mathcal{T}_X^* \otimes \mathcal{T}_X \to \mathcal{A} \at \mathcal{T}_{\hat{X}} \) follows from tensoring the surjection of tangents \( \mathcal{T}_X \to \mathcal{T}_{\hat{X}} \to 0 \) from (III.3.2.1) with \( \mathcal{O}^2 \mathcal{T}_{\hat{X}}^* \). The injection \( 0 \to \mathcal{A} \at \mathcal{T}_{\hat{X}} \to \mathcal{O}^2 \mathcal{T}_{\hat{X}}^* \otimes \mathcal{T}_{\hat{X}} \) on the bottom row of (III.3.2.2) follows from tensoring the inclusion in (III.3.2.3) with \( \mathcal{T}_{\hat{X}} \). Finally, the vertical map \( \mathcal{A} \at \mathcal{T}_{\hat{X}} \to \mathcal{O}^2 \mathcal{T}_{\hat{X}}^* \otimes \mathcal{T}_{\hat{X}} \) follows from tensoring the surjection \( \mathcal{T}_X \to \mathcal{T}_{\hat{X}} \to 0 \) with \( \mathcal{O}^2 \mathcal{T}_{\hat{X}}^* \). \( \square \)

The purpose of Lemma III.3.2 is to make sense of the statement of the following theorem which concerns our relation between Atiyah classes.

**Theorem III.3.3.** With \( \iota \) and \( \pi \) the morphisms in the diagram from Lemma III.3.2, let \( \iota_* \) and \( \pi_* \) denote the respective induced morphisms on 1-cohomology. Then we have the following relation between Atiyah classes:

\[ \pi_* \at \mathcal{T}_X = \iota_* \at \mathcal{T}_{\hat{X}}. \]

**Proof.** Recall from (III.3.2.1) that \( \mathcal{T}_X \), the tangent sheaf of \( \mathfrak{X} \), surjects onto \( \mathcal{T}_{\mathfrak{X}} \), the tangent sheaf of \( \mathfrak{X} \). Not every endomorphism of \( \mathcal{T}_X \) will induce an endomorphism of \( \mathcal{T}_{\mathfrak{X}} \) however. To that end, consider the subsheaf of endomorphisms \( \mathcal{E}nd^* \mathcal{T}_X \subset \mathcal{E}nd \mathcal{T}_X \) which induce endomorphisms of \( \mathcal{T}_{\mathfrak{X}} \). Elements of \( H^0 \mathfrak{X}, \mathcal{E}nd^* \mathcal{T}_X \) define

---

\( ^{42} \)Note, this sequence is not exact. However \( \mathcal{O}^2 \mathcal{T}_{\mathfrak{X}}^* \) is in the kernel of \( \mathcal{O}^2 \mathcal{T}_{\mathfrak{X}}^* \to \mathcal{O}^2 \mathcal{T}_X [1]^* \to 0 \).

\( ^{43} \)Note that the vertical maps in (III.3.2.2) are not necessarily surjective.
endomorphisms of the following short exact sequence, represented by the dashed arrows:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & T_X[1] & \longrightarrow & T_X & \longrightarrow & T_\hat{X} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_X[1] & \longrightarrow & T_X & \longrightarrow & T_\hat{X} & \longrightarrow & 0 \\
\end{array}
\] (III.3.2.5)

Since the identity \(1_{T_X} : T_X \rightarrow T_X\) will induce a morphism as in (III.3.2.5), it follows that \(1_{T_X} \in \text{End}^* T_X\), i.e., that \(\text{End}^* T_X\) is non-empty. Furthermore, this induced endomorphism will be the identity \(T_\hat{X} \rightarrow T_\hat{X}\). Hence we can conclude:

under the maps \(\text{End}^* T_X \rightarrow \text{End} T_X\) that \(1_{T_X} \mapsto 1_{T_X}\) (III.3.2.6)

Now form the sheaf

\[J^* \overset{\Delta}{=} (\otimes^2 T_\hat{X}^* \otimes T_X) \oplus \text{End}^* T_X\]

Then by Lemma III.3.2 and the definition of \(\text{End}^* T_X\) we have,

\[
\begin{array}{ccccccc}
J^* & \longrightarrow & \widetilde{J^1 T_X} & \longrightarrow & \widetilde{J^1 T_\hat{X}} \\
\downarrow & & \downarrow & & \downarrow & & \\
& & \widetilde{J^1 T_\hat{X}} & & \\
\end{array}
\]

Note that for any sheaf \(\mathcal{F}\) of \(\mathcal{O}\)-modules we have a natural map \(H^0(\mathcal{F}) \otimes \mathcal{O} \rightarrow \mathcal{F}\) given by \((\phi \otimes f)_x = \phi(x) \otimes [f]_x\), where \([f]_x\) denotes the germ of \(f\) at the point \(x\). If \(\mathcal{O}\) is unital, then we have the mapping \(H^0(\mathcal{F}) \rightarrow \mathcal{F}\) given by \((\phi)_x \mapsto \phi(x) \otimes [1]_x\). Then with \(\mathcal{O}_X\) the function algebra for a supermanifold \(X\) with unit the constant function \((x|\theta) \mapsto 1 \in \mathbb{C}\); and with \(\text{End} T_X\) an \(\mathcal{O}_X\)-module, we can consider \(1_{T_X}\) as an element of \(\text{End} T_X\) or \(\text{End}^* T_X\).
The twisted $\mathcal{O}_X$-module structure on $\tilde{J^1 T_X}$ can then be induced on $J^*$ yielding therefore a morphism of exact sequences:

\[
\begin{array}{cccccc}
\mathcal{A}t \ T \hat{X} & \longrightarrow & \tilde{J^1 T_X} & \longrightarrow & \mathcal{E}nd \ T \hat{X} \\
\odot^2 T^* \hat{X} \otimes T_X & \longrightarrow & J^* & \longrightarrow & \mathcal{E}nd T \hat{X} \\
\mathcal{A}t \ T \hat{X} & \longrightarrow & \tilde{J^1 T_X} & \longrightarrow & \mathcal{E}nd T \hat{X}
\end{array}
\]

(III.3.2.7)

Importantly, the bottom and top-right sequences in (III.3.2.7) are precisely the Atiyah sequences of $T_X$ and $T_{\hat{X}}$ respectively. On cohomology, the boundary maps give:

\[
\begin{array}{cccccc}
H^0(X, \mathcal{E}nd T_X) & \longrightarrow & \mathcal{A}T \ T_X \\
H^0(X, \mathcal{E}nd^* T_X) & \longrightarrow & H^1(X, \odot^2 T^* \hat{X} \otimes T_X) \\
H^0(X, \mathcal{E}nd T_{\hat{X}}) & \longrightarrow & \mathcal{A}T \ T_{\hat{X}}
\end{array}
\]

(III.3.2.8)

Since the above diagram commutes and the sequences in (III.3.2.7) inducing this diagram are Atiyah sequences, looking at image of $1^*_T \in H^0(X, \mathcal{E}nd^* T_X)$ from (III.3.2.6)
implies:

\[
\begin{array}{c}
\delta 1_T^* \xrightarrow{i_*} \delta 1_T^\Delta \xrightarrow{\Delta} \text{at } T_X
\\
\downarrow p_* \downarrow \downarrow
\\
\delta 1_T^\Delta \xrightarrow{\Delta} \text{at } T_{\tilde{X}}
\end{array}
\]  

(III.3.2.9)

where \(i_*\) and \(p_*\) are the induced maps in (III.3.2.8). To deduce Theorem III.3.3 now we can ‘complete the prism’ on the right in (III.3.2.8) by Lemma III.3.2, resulting in the commuting square:

\[
\begin{array}{c}
H^1(\odot^2 T_{\tilde{X}}^* \otimes T_{\tilde{X}}) \xrightarrow{i_*} \text{AT } T_X
\\
\downarrow p_* \downarrow \downarrow \pi_*
\\
\text{AT } T_{\tilde{X}} \xrightarrow{i_*} H^1(\odot^2 T_{\tilde{X}}^* \otimes T_{\tilde{X}})
\end{array}
\]  

(III.3.2.10)

Theorem III.3.3 now follows as a consequence of the mappings in (III.3.2.9) and commutativity of (III.3.2.10). □

III.4. DONAGI AND WITTEN’S DECOMPOSITION

A key component to Donagi and Witten’s second proof of the non-projectedness of the supermoduli space of punctured curves in [DW14] is their decomposition of the affine Atiyah class of supermanifolds. In [DW14, Theorem 2.5, p. 14 (arXiv version)], Donagi and Witten show, on any supermanifold \(\mathfrak{X}\), that at \(T_{\tilde{X}}\) restricts to the Atiyah classes of the underlying spaces and the primary obstruction class to splitting. We have partly derived this decomposition into the underlying Atiyah classes in Proposition III.3.1. The component valued in the obstruction space is given by the following.

**Theorem III.4.1.** Let \(\mathfrak{X}\) be a supermanifold modelled on \((X, T_{X,-}^*)\). Then with \(\iota: X \subset \mathfrak{X}\) the given inclusion of the reduced space, we have on \(X\) a decomposition of the even Atiyah space

\[
(i^\# \text{AT } T_{\mathfrak{X}})_+ \cong \text{AT } T_X \oplus T_{X,-}^* \oplus \text{OB}_X^\text{primary}.
\]

(III.4.0.1)
Correspondingly, the affine Atiyah class of $\mathfrak{X}$ decomposes as follows:

$$\iota^* \mathrm{at}_{T_X} = \mathrm{at}_{T_X} \oplus \eta_\ast [\mathfrak{X}].$$  \hfill (III.4.0.2)

where $\eta_\ast [\mathfrak{X}]$ is the primary obstruction to splitting $\mathfrak{X}$.\textsuperscript{45}

The contents of Theorem III.4.1 is contained in [DW14, Theorem 2.5, p. 14 (arXiv)].

In what remains of this section, we will give a proof of Theorem III.4.1 distinct from the argument by Donagi and Witten in the afore-cited article. We proceed as follows, starting with the decomposition of the Atiyah space.

**Lemma III.4.2.** By abuse of notation we will denote by $\iota$ the embeddings $X \subset \mathfrak{X}$ and $\tilde{X} \subset \mathfrak{X}$. On $X$ then, there exists an isomorphism of Atiyah spaces $\iota^* \mathrm{AT}_{T_X} \cong \iota^* \mathrm{AT}_{T_{\tilde{X}}}$.

\textbf{Proof.} Note that $\iota^* T_X[m] = \iota^* T_{\tilde{X}}[m] = (0)$ for all $m > 0$ and any supermanifold $\mathfrak{X}$.\textsuperscript{46} Hence under $\iota^*$ the diagram in Lemma III.3.2 will comprise isomorphisms and so, as sheaves, $\iota^* \mathrm{AT}_{T_X} \cong \iota^* \mathrm{AT}_{T_{\tilde{X}}}$. The lemma now follows. \hfill \Box

Since $\mathrm{OB}_X^{primary} \cong \mathrm{OB}_{\tilde{X}}^{primary}$ by Proposition III.2.10, we recover (III.4.0.1). Furthermore, the argument in Proposition III.3.1 now applies, identifying the component of $\iota^* \mathrm{at}_{T_X}$ in the Atiyah spaces $\mathrm{AT}_{T_X} \oplus \mathrm{AT}_{T_{X,-}}$ with the Atiyah classes $\mathrm{at}_{T_X} \oplus \mathrm{at}_{T_{X,-}}$. It remains to deduce that the component in the primary obstruction space will be the primary obstruction class.

**III.4.1. Proof Outline.** We return now to the affine Atiyah sequence for $\mathfrak{X}$ which, following Theorem III.1.1, we recall:

$$
0 \longrightarrow \mathrm{AT}_{T_X} \longrightarrow \mathcal{J}^1 T_X \longrightarrow \mathcal{E} \mathrm{nd}_{\mathcal{O}_X} T_X \longrightarrow 0
$$

where $\mathcal{J}^1 T_X = \mathcal{H} \mathrm{om}_{\mathcal{O}_X}(T_X, \mathcal{J}^1 T_X)$. Our method for completing the proof of Theorem III.4.1 will be to firstly obtain a ‘correspondent sequence’ between the affine Atiyah sequence above and that for $\mathcal{G}_X^{(2)}$ from Proposition C.2. That is, we will form a

\textsuperscript{45}c.f., Remark III.2.12.

\textsuperscript{46}This is because, on functions, $\iota^* : \mathcal{O}_X \rightarrow \mathcal{O}_X/\mathcal{J}_X$ is reduction modulo the fermionic ideal $\mathcal{J}_X$. Then for any $\mathcal{O}_X$-module $\mathcal{M}$ see that $\mathcal{M}[m] = \mathcal{J}_X^m \mathcal{M}$ so therefore $\forall m > 0, \iota^* \mathcal{M}[m] = (0)$.\hfill
morphism of sequences:

\[
\begin{array}{ccccccc}
\mathcal{G}^{(2)}_{\mathcal{O}_x} & \to & \mathcal{G}^{(1)}_{\mathcal{O}_x} & \to & \text{Aut}_{\mathcal{O}_x} T^*_x, & \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(At T^*_x)_+ & \to & (J^1 T^*_x)_+ & \to & \text{End} T^*_x, & \to \\
\downarrow \downarrow & & \downarrow \downarrow & & \downarrow \downarrow & & \downarrow \downarrow \\
\text{Ob}_{\mathcal{O}_x}^{\text{primary}} & \to & \mathcal{H} & \to & \text{End}_{\mathcal{O}_x} T^*_x, & \to \\
\end{array}
\]

(III.4.1.1)

where we have identified $T^*_X, = J^2_X / J^1_X$. The idea now is that on cohomology we obtain a commutative diagram:

\[
\begin{array}{ccccccc}
H^0(\text{Aut}_{\mathcal{O}_x} T^*_X) & \to & \tilde{H}^1(\mathcal{G}^{(2)}_{\mathcal{O}_x}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^0(\text{End}_{\mathcal{O}_x} T^*_X) & \to & (AT T^*_x) & \to \\
\downarrow & & \downarrow \downarrow & & \downarrow \downarrow & & \downarrow \downarrow \\
H^0(\text{End}_{\mathcal{O}_x} T^*_X) & \to & \text{Ob}_{\mathcal{O}_x}^{\text{primary}} & \to \\
\end{array}
\]

(III.4.1.2)

Commutativity of each square above will guarantee that the projection of the Atiyah class onto the component in the primary obstruction space will coincide precisely with the primary obstruction class. It remains therefore to construct a morphism of sequences as in (III.4.1.1).

III.4.2. The Automorphism Sheaves. Recall from Theorem III.2.7(iii) and (iv) that for each $m$, $\mathcal{G}^{(m+1)}_{\mathcal{O}_x} \subset \mathcal{G}^{(m)}_{\mathcal{O}_x}$ is normal has abelian quotient. In analogy with Green in [Gre82] in the split case, we have:

**Lemma III.4.3.** For any $m > 1$ there is an isomorphism of abelian sheaves,

\[
\begin{align*}
\mathcal{G}^{(m)}_{\mathcal{O}_x} & \cong \mathcal{G}^{(m+1)}_{\mathcal{O}_x} \quad m \text{ is even;} \\
\text{Hom}_{\mathcal{O}_x}(\mathcal{O}_x / J^m_X, J^m_X / J^{m+1}_X) & \quad m \text{ is odd.}
\end{align*}
\]
\textbf{Proof.} Note that the automorphisms in \( \mathcal{G}_{\mathcal{O}_x}^{(m)} \) are even. Hence, the inclusion of the quotient \( \mathcal{G}_{\mathcal{O}_x}^{(m)}/\mathcal{G}_{\mathcal{O}_x}^{(m+1)} \subset \text{Hom}_{\mathcal{O}_x}(\mathcal{O}_x, \mathcal{J}_x^m/\mathcal{J}_x^{m+1}) \) from (III.2.2.2) will lie in the even component. With the global \( \mathbb{Z}/2 \)-grading \( \mathcal{O}_x \cong \mathcal{O}_{x,+} \oplus \mathcal{O}_{x,-} \) we will use, for any even morphism \( \omega : \mathcal{O}_x \to \mathcal{J}_x^m \), that:

\[
\begin{align*}
\omega|_{\mathcal{O}_{x,-}} &\equiv 0 \pmod{\mathcal{J}_x^{m+1}} & \text{if } m \text{ is even}; \\
\omega|_{\mathcal{O}_{x,+}} &\equiv 0 \pmod{\mathcal{J}_x^{m+1}} & \text{if } m \text{ is odd}. 
\end{align*}
\]  

(III.4.2.1)

(III.4.2.2)

Now from the sequence \( 0 \to \mathcal{J}_x \to \mathcal{O}_x \to \mathcal{O}_x/\mathcal{J}_x \to 0 \) see that we have on hom-sheaves and by contravariance,

\[
0 \to \text{Hom}_{\mathcal{O}_x} \left( \mathcal{O}_x/\mathcal{J}_x, \mathcal{J}_x^{m}/\mathcal{J}_x^{m+1} \right) \to 0.
\]  

(III.4.2.3)

Recall \( \mathcal{G}_{\mathcal{O}_x}^{(m)}/\mathcal{G}_{\mathcal{O}_x}^{(m+1)} \subset \text{Hom}_{\mathcal{O}_x}(\mathcal{O}_x, \mathcal{J}_x^m/\mathcal{J}_x^{m+1}) \) from (III.2.2.2). Using \( \mathcal{O}_x \cong \mathcal{O}_{x,+} \oplus \mathcal{O}_{x,-} \) in addition to \( \mathcal{O}_x/\mathcal{J}_x = \mathcal{O}_{x,+}/(\mathcal{J}_x \cap \mathcal{O}_{x,+}) \) see that, if \( m \) is even, then by (III.4.2.1) the image of \( \mathcal{G}_{\mathcal{O}_x}^{(m)}/\mathcal{G}_{\mathcal{O}_x}^{(m+1)} \) under the mapping in (III.4.2.3) will vanish. Thus when \( m \) is even we have \( \mathcal{G}_{\mathcal{O}_x}^{(m)}/\mathcal{G}_{\mathcal{O}_x}^{(m+1)} \cong \text{Hom}_{\mathcal{O}_x} \left( \mathcal{O}_x/\mathcal{J}_x, \mathcal{J}_x^{m}/\mathcal{J}_x^{m+1} \right) \) by exactness.

When \( m \) is odd, (III.4.2.2) will guarantee \( \mathcal{G}_{\mathcal{O}_x}^{(m)}/\mathcal{G}_{\mathcal{O}_x}^{(m+1)} \cap \text{im} \text{Hom}_{\mathcal{O}_x} \left( \mathcal{O}_x/\mathcal{J}_x, \mathcal{J}_x^{m}/\mathcal{J}_x^{m+1} \right) = \emptyset \), so we can conclude \( \mathcal{G}_{\mathcal{O}_x}^{(m)}/\mathcal{G}_{\mathcal{O}_x}^{(m+1)} \subset \text{Hom}_{\mathcal{O}_x} \left( \mathcal{J}_x^2 \mathcal{J}_x^m/\mathcal{J}_x^{m+1} \right) \). Now consider the sequence \( \mathcal{J}_x^2 \to \mathcal{J}_x \to \mathcal{J}_x^2/\mathcal{J}_x^3 \) giving, contravariantly, the sequence of hom-sheaves,

\[
0 \to \text{Hom}_{\mathcal{O}_x} \left( \mathcal{J}_x^2 \mathcal{J}_x^m/\mathcal{J}_x^{m+1} \right) \to 0.
\]  

(III.4.2.4)

The image of \( \mathcal{G}_{\mathcal{O}_x}^{(m)}/\mathcal{G}_{\mathcal{O}_x}^{(m+1)} \) under the mapping (III.4.2.4) will vanish so therefore by exactness again we have, when \( m \) is odd, \( \mathcal{G}_{\mathcal{O}_x}^{(m)}/\mathcal{G}_{\mathcal{O}_x}^{(m+1)} \cong \text{Hom}_{\mathcal{O}_x} \left( \mathcal{J}_x^2 \mathcal{J}_x^m/\mathcal{J}_x^{m+1} \right) \).

\hfill \Box

Theorem III.2.7(iii) and (iv) can be generalised to the following.

**Lemma III.4.4.** For any \( m > 0 \), the embedding \( \mathcal{G}_{\mathcal{O}_x}^{(m+2)} \subset \mathcal{G}_{\mathcal{O}_x}^{(m)} \) is normal and the quotient \( \mathcal{G}_{\mathcal{O}_x}^{(m)}/\mathcal{G}_{\mathcal{O}_x}^{(m+2)} \) is abelian.
Proof. A subgroup $H < G$ is normal with abelian quotient if $H$ contains the commutator $[G, G]$. We refer to [Bet18a] for an argument justifying $\mathcal{G}^{(m)}_{\mathcal{O}_x}, \mathcal{G}^{(m)}_{\mathcal{O}_x} \subset \mathcal{G}^{(m+2)}_{\mathcal{O}_x}$, from which the lemma follows. □

Accepting normality of $\mathcal{G}^{(m)}_{\mathcal{O}_x} \subset \mathcal{G}^{(m+2)}_{\mathcal{O}_x}$ another way to see that the quotient will be abelian is by inspecting the sequence (III.2.2.2) in the proof of Theorem III.2.7(iv). With $\mathcal{G}^{(m)}_{\mathcal{O}_x} \subset \mathcal{E}nd_{\mathcal{O}_x} \mathcal{O}_x$, we can form the formal logarithm as in (III.2.2.1). For any $g \in \mathcal{G}^{(m)}_{\mathcal{O}_x}$ and $u \in \mathcal{O}_x$ we have $(g - 1)^2(u) \in \mathcal{J}^m_\mathcal{X} \subset \mathcal{J}^{m+2}_\mathcal{X}$ for any $m > 0$. Hence $\mathcal{G}^{(m+2)}_{\mathcal{O}_x} \subset \ker \{ \mathcal{G}^{(m)}_{\mathcal{O}_x} \to \mathcal{H}om_{\mathcal{O}_x}(\mathcal{O}_x/\mathcal{J}^m_\mathcal{X}, \mathcal{J}^m_\mathcal{X}/\mathcal{J}^{m+2}_\mathcal{X}) \}$ and so, as in the proof of Theorem III.2.7(iv), we find:

$$\frac{\mathcal{G}^{(m)}_{\mathcal{O}_x}}{\mathcal{G}^{(m+2)}_{\mathcal{O}_x}} \subset \mathcal{H}om_{\mathcal{O}_x}\left(\mathcal{O}_x, \frac{\mathcal{J}^m_\mathcal{X}}{\mathcal{J}^{m+2}_\mathcal{X}}\right). \quad (III.4.2.5)$$

This shows the quotient will be abelian. Arguing as in the proof of Lemma III.4.3, the inclusion in (III.4.2.5) can be refined to the following:

Lemma III.4.5.

$$\frac{\mathcal{G}^{(m)}_{\mathcal{O}_x}}{\mathcal{G}^{(m+2)}_{\mathcal{O}_x}} \subset \mathcal{H}om_{\mathcal{O}_x}\left(\mathcal{O}_x, \frac{\mathcal{J}^m_\mathcal{X}}{\mathcal{J}^{m+2}_\mathcal{X}}\right).$$

Proof. Since any element of $\mathcal{G}^{(m)}_{\mathcal{O}_x}$ will map $\mathcal{J}^2_\mathcal{X} \to \mathcal{J}^{m+2}_\mathcal{X}$, the composition $\mathcal{G}^{(m)}_{\mathcal{O}_x} \to \mathcal{H}om_{\mathcal{O}_x}(\mathcal{O}_x, \mathcal{J}^m_\mathcal{X}) \to \mathcal{H}om_{\mathcal{O}_x}(\mathcal{J}^2_\mathcal{X}, \mathcal{J}^m_\mathcal{X})$ vanishes and so vanishes modulo $\mathcal{J}^{m+2}_\mathcal{X}$. Hence we have a commuting diagram of inclusions,

$$\begin{tikzcd}
\mathcal{H}om_{\mathcal{O}_x}\left(\mathcal{O}_x, \frac{\mathcal{J}^m_\mathcal{X}}{\mathcal{J}^{m+2}_\mathcal{X}}\right) \\
\frac{\mathcal{G}^{(m)}_{\mathcal{O}_x}}{\mathcal{G}^{(m+1)}_{\mathcal{O}_x}} \arrow[r] \arrow[ur, dashed] & \mathcal{H}om_{\mathcal{O}_x}\left(\mathcal{O}_x, \frac{\mathcal{J}^m_\mathcal{X}}{\mathcal{J}^{m+2}_\mathcal{X}}\right).
\end{tikzcd}$$

This lemma now follows. □
Using $\mathcal{G}^{(1)}_{\mathcal{O}_X}/\mathcal{G}^{(2)}_{\mathcal{O}_X} \cong \text{Aut}_{\mathcal{O}_X} T_{X,-}^* \subset \text{End}_{\mathcal{O}_X} T_{X,-}^*$ by Proposition C.2; Lemma III.4.3 with $m = 2$; and Lemma III.4.5 above, we obtain a morphism of abelian sheaves:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{G}^{(2)}_{\mathcal{O}_X}/\mathcal{G}^{(3)}_{\mathcal{O}_X} & \longrightarrow & \mathcal{G}^{(1)}_{\mathcal{O}_X}/\mathcal{G}^{(3)}_{\mathcal{O}_X} & \longrightarrow & \mathcal{G}^{(1)}_{\mathcal{O}_X}/\mathcal{G}^{(2)}_{\mathcal{O}_X} & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow & & \downarrow & & \downarrow & \\
\mathcal{H}om_{\mathcal{O}_X} \left( \mathcal{O}_X, \frac{\mathcal{J}^2_{\mathcal{O}_X}}{\mathcal{J}_{\mathcal{O}_X}} \right) & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X} \left( \mathcal{O}_X, \frac{\mathcal{J}^3_{\mathcal{O}_X}}{\mathcal{J}_{\mathcal{O}_X}} \right) & \longrightarrow & \mathcal{E}nd_{\mathcal{O}_X} T_{X,-}^* \\
\end{array}
\]

(III.4.2.6)

III.4.3. **An Exact Lattice.** Unlike the top row in (III.4.2.6), the bottom row will not be exact. Indeed, for any $m > 0$, using the short exact sequences:

\[
0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O} \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \mathcal{J}^m \longrightarrow \mathcal{J}^{m+1} \longrightarrow \mathcal{J}^{m+2} \longrightarrow \mathcal{J}^{m+1} \longrightarrow 0
\]

we can form the 9-term, exact lattice:\[47\]

\[
\begin{array}{cccccc}
\mathcal{H}om_{\mathcal{O}_X} \left( \mathcal{O}_X, \frac{\mathcal{J}^{m+1}}{\mathcal{J}^{m+2}} \right) & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X} \left( \mathcal{O}_X, \frac{\mathcal{J}^{m+1}}{\mathcal{J}^{m+2}} \right) & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X} \left( \mathcal{O}_X, \frac{\mathcal{J}^{m+1}}{\mathcal{J}^{m+2}} \right) \\
\mathcal{H}om_{\mathcal{O}_X} \left( \mathcal{O}_X, \frac{\mathcal{J}^{m+1}}{\mathcal{J}^{m+2}} \right) & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X} \left( \mathcal{O}_X, \frac{\mathcal{J}^{m+1}}{\mathcal{J}^{m+2}} \right) & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X} \left( \mathcal{O}_X, \frac{\mathcal{J}^{m+1}}{\mathcal{J}^{m+2}} \right) \\
\mathcal{H}om_{\mathcal{O}_X} \left( \mathcal{O}_X, \frac{\mathcal{J}^{m+1}}{\mathcal{J}^{m+2}} \right) & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X} \left( \mathcal{O}_X, \frac{\mathcal{J}^{m+1}}{\mathcal{J}^{m+2}} \right) & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X} \left( \mathcal{O}_X, \frac{\mathcal{J}^{m+1}}{\mathcal{J}^{m+2}} \right) \\
\end{array}
\]

(III.4.3.1)

For $m = 1$ and with the identification $T_{X,-}^* = \mathcal{J}/\mathcal{J}^2_{\mathcal{O}_X}$, the diagonal morphisms above, indicated by the dotted arrows, are precisely the morphisms in the bottom row of (III.4.2.6). Note that by (III.2.3.1), the sheaf in the top-left corner of (III.4.3.1) is the primary obstruction sheaf $\mathcal{O}_{\mathcal{O}_X}^{primary}$. Regarding the middle term in (III.4.3.1) we have the following.

\[\text{an exact lattice is a two-dimensional array of morphisms whose rows and columns are exact}\]
Lemma III.4.6. For any $m$, 
\[ \text{Hom}_{O_X} \left( \frac{O_X}{J^2_X}, \frac{I^m_X}{I^{m+2}_X} \right) \cong \text{Hom}_{O_X} \left( \frac{O_X}{J^2_X}, \frac{I^{m+1}_X}{I^{m+2}_X} \right) \oplus \text{Hom}_{O_X} \left( \frac{J^2_X}{J^m_X}, \frac{J^m_X}{J^{m+1}_X} \right) \oplus \text{Hom}_{O_X} \left( \frac{O_X}{J^2_X}, \frac{I^m_X}{I^{m+1}_X} \right) \oplus \text{Hom}_{O_X} \left( \frac{J^2_X}{J^m_X}, \frac{J^m_X}{J^{m+1}_X} \right) \]

Proof. Recall that \( O_X/J^2_X \cong (O_X/J_X) \oplus (J_X/J^2_X) \). This gives
\[ \text{Hom}_{O_X} \left( \frac{O_X}{J^2_X}, \frac{I^m_X}{I^{m+2}_X} \right) \cong \text{Hom}_{O_X} \left( \frac{O_X}{J^2_X}, \frac{I^m_X}{I^{m+2}_X} \right) \oplus \text{Hom}_{O_X} \left( \frac{J^2_X}{J^m_X}, \frac{J^m_X}{J^{m+1}_X} \right) \]

Regarding the ideal quotient \( I^m_X/I^{m+2}_X \), assuming \( J_X \) is flat over \( O_X \) allows for the following deductions, which makes use again of the decomposition \( O_X/J^2_X \cong (O_X/J_X) \oplus (J_X/J^2_X) \):
\[ \frac{I^m_X}{I^{m+2}_X} = \frac{J^m_X}{J^{m+2}_X} \cong J^m_X \otimes_{O_X} \frac{O_X}{J_X} \]
\[ \cong J^m_X \otimes_{O_X} \left( \frac{O_X}{J_X} \oplus \frac{J_X}{J^2_X} \right) \]
\[ \cong \frac{J^m_X}{J^{m+1}_X} \oplus \frac{J^m_X}{J^{m+2}_X} \]

The lemma follows upon substituting (III.4.3.3) into (III.4.3.2). \( \square \)

III.4.4. The Affine Atiyah Sequence. An important consequence of the decomposition in Lemma III.4.6 is the following.

Lemma III.4.7. With \( \iota : X \subset \bar{X} \) the embedding of the reduced space, we have a morphism,
\[ 0 \to \iota^* (\text{At}_{T_X})_+ \to \iota^* (\text{At}^1_{T_X})_+ \to \iota^* (\text{End}_{O_X} T_X)_+ \to 0 \]
\[ \text{Hom}_{O_X} \left( \frac{O_X}{J^2_X}, \frac{J^m_X}{I_X} \right) \to \text{Hom}_{O_X} \left( \frac{O_X}{J^2_X}, \frac{J^m_X}{I_X} \right) \to \text{End}_{O_X} T^*_X \]
Proof. By Lemma III.4.2 we know that $i^\ast \mathcal{A}t\ T_X \cong i^\ast \mathcal{A}t\ \hat{T}_X$; from (III.3.1.3) that $\text{Ob}_{\mathcal{O}_X}^{\text{primary}} \subset i^\ast \mathcal{A}t\ T_X$; from the proof of Proposition III.2.10 that $\mathcal{O}_{\mathcal{O}_X}^{\text{primary}} \cong \text{Ob}_{\mathcal{O}_X}^{\text{primary}}$; and finally by (III.2.3.1) that $\text{Ob}_{\mathcal{O}_X}^{\text{primary}} \subset i^\ast \mathcal{A}t\ \hat{T}_X$; from the proof of Proposition III.2.10 that $\text{Ob}_{\mathcal{O}_X}^{\text{primary}} \cong \text{Ob}_{\mathcal{O}_X}^{\text{primary}}$; and finally by (III.2.3.1) that $\text{Ob}_{\mathcal{O}_X}^{\text{primary}} \cong \text{Ob}_{\mathcal{O}_X}^{\text{primary}}$. Hence $\text{Hom}_{\mathcal{O}_X} \left( \mathcal{O}_{\mathcal{O}_X}^{\text{primary}}, \mathcal{O}_{\mathcal{O}_X}^{\text{primary}} \right)$ will be a direct summand of $i^\ast \mathcal{A}t\ T_X$ and, accordingly, we can define a projection $i^\ast \mathcal{A}t\ T_X$ leading thereby to the projection $i^\ast \mathcal{A}t\ T_X$ leading thereby to the projection $i^\ast \mathcal{A}t\ T_X \left( \mathcal{O}_{\mathcal{O}_X}^{\text{primary}}, \mathcal{O}_{\mathcal{O}_X}^{\text{primary}} \right)$.

III.4.5. Proof Completion of Theorem III.4.1. With (III.4.2.6) and Lemma III.4.7 see that we obtain a correspondent sequence between the affine Atiyah sequence and that of the automorphism sheaves as in (III.4.1.1)—this correspondent sequence is the bottom row in (III.4.2.6), or equivalently in Lemma III.4.7. As mentioned in the remarks following (III.4.2.6), this sequence need not be exact and hence we need-not-necessarily obtain the diagram on cohomology as in (III.4.1.2). That we will in fact obtain such a diagram is a consequence of the 9-term, exact lattice in (III.4.3.1) specialised to $m = 1$. In this case, the boundary map $H^0(\mathcal{E}nd_{\mathcal{O}_X} T_X) \rightarrow \text{OB}_{\mathcal{O}_X}^{\text{primary}}$ can be defined by composing the top-horizontal boundary with the right-vertical boundary; or equivalently the bottom-horizontal boundary with the left-vertical boundary. Either way, with this composition we arrive at the diagram on cohomology in (III.4.1.2), commutativity of which justifies the decomposition of the Atiyah class in (III.4.0.2). Theorem III.4.1 now follows.

III.4.6. On the Split Model. Recall in Proposition III.3.1 that we obtained an expression for the Atiyah class of split models with the obstruction space projected out. With the general decomposition in Theorem III.4.1 along with Theorem III.2.11 which implies: for any split model $\hat{X}$ that its primary obstruction satisfies $\eta_{\mathcal{O}_X} \hat{X} = 0$; we can thus conclude:
Corollary III.4.8. Let $\hat{\mathcal{X}}$ be a split model with reduced space $X$, embedding $\iota : X \subset \hat{\mathcal{X}}$; and odd cotangent bundle $T^*_X$. Then on $X$,

$$\iota^* \text{at } T_{\hat{\mathcal{X}}} = \text{at } T_X \oplus \text{at } T^*_X.$$


**Concluding Remarks**

**The Full Atiyah Class.** An immediate corollary of Donagi and Witten’s decomposition in Theorem III.4.1 is the following:

**Theorem.** Let $\mathfrak{X}$ be a complex supermanifold whose affine Atiyah class vanishes upon restriction to the reduced space. Then its reduced space and odd cotangent bundle admit global, holomorphic connections.$^{48}$

For $(X, T^*_X, -)$ the model for $\mathfrak{X}$, note in particular that the existence of global holomorphic connections on $T_X$ and $T^*_X, -$ will be insufficient to define a global, affine, even connection on $\mathfrak{X}$. This is for the reason that, even if $\iota^*$ at $T_X$ vanishes, $\mathfrak{X}$ need not be split.$^{49}$ Regarding the full Atiyah class however, we know from Theorem III.1.3 that: *if at $T_X = 0$, then there will exist a global, even, affine connection on $\mathfrak{X}$*. Hence by Koszul’s theorem (Theorem II.1.5) and the theorem stated above it follows:

**Theorem.** Let $\mathfrak{X}$ be a complex supermanifold with vanishing affine Atiyah class. Then:

(i) the reduced space and odd cotangent bundle of $\mathfrak{X}$ admit global, holomorphic connections and;

(ii) $\mathfrak{X}$ is split.$^{50}$

Part (ii) of the above theorem strongly suggests the existence of a relation between the affine Atiyah class and the higher obstructions to splitting or, equivalently, the Euler differential. In the author’s doctoral thesis [Bet16] a relation along the lines of Theorem III.4.1, utilising thickenings$^{50}$ was proposed, being: *suppose $\mathfrak{X}$ is endowed with a covering in which an $\ell$-th obstruction to splitting, $\eta^{(\ell)}$ is defined. Let $\iota^{(\ell)} : \hat{\mathfrak{X}}^{(\ell)} \subset \mathfrak{X}$ be the associated thickening. Then $\iota^{(\ell)*}$ at $T_X = at T_X \oplus T^*_X, - \oplus \eta^{(\ell)}$. In a planned sequel to this article, tentatively containing a ‘Part IV’, we will investigate the relation between the full affine Atiyah class, higher obstructions; and the Euler differential*.

---

$^{48}$By ‘global holomorphic connection on the reduced space’ it is meant a global, affine connection. If $\mathfrak{X}$ is modelled on $(X, T^*_X, -)$, then $X$ is the reduced space and $T^*_X, -$ is its odd cotangent bundle.

$^{49}$c.f., the remarks following Definition III.2.3.

$^{50}$For more on thickenings in supergeometry, see [Bet19].
For completeness and for the interested reader we clarify, in Appendix D, the relation between the Euler differential and primary obstructions of complex supermanifolds, as ought to be expected from Theorem II.2.5.

**Beyond Affine.** Classically, as established by Atiyah in [Ati57], the Atiyah class of a vector bundle contains important information about the base manifold, being the Chern classes of that bundle. In this article we have seen how the affine Atiyah class, generalised to complex supermanifolds, will contain important information pertaining to the classification of its complex structure via splitting. The Atiyah class itself however can be generalised to any sheaf of modules on a space as explained in [Sta20, Tag 09DF]; and to any supermanifold as in [BBHR91, p. 161]. Given the results in affine case then, it is natural to speculate as to what kind of information will be captured by the Atiyah class of a sheaf of modules more generally. Where the complex structure is concerned we conclude with the following open question:

> let $\mathfrak{X}$ be a complex supermanifold and $\mathcal{F}$ a sheaf of $\mathcal{O}_\mathfrak{X}$-modules.
> Then what information, if any, pertaining to the classification of $\mathfrak{X}$ is contained in the Atiyah class of $\mathcal{F}$?

---

Note in particular that with such a relation, Koszul’s splitting theorem will be immediate.
Appendix A. Proof of Theorem II.1.2

Recall that an affine connection $\nabla$ on $\mathfrak{X}$ is the $\mathbb{C}$-linear mapping of sheaves $\nabla : T_\mathfrak{X} \otimes T_\mathfrak{X} \to T_\mathfrak{X}$. It is $\mathcal{O}_\mathfrak{X}$-linear in its first argument; and is an $\mathcal{O}_\mathfrak{X}$-derivation in its second argument. It can be represented locally by $\nabla \sim d + A$ where $d$ is the universal de Rham differential and $A \in \text{Hom}_{\mathcal{O}_\mathfrak{X}}(T_\mathfrak{X} \otimes T_\mathfrak{X}, T_\mathfrak{X})$. Now for tensor squares we have the decomposition into symmetric ‘$\circ$’ and anti-symmetric ‘$\wedge$’ tensors, giving:

$$\text{Hom}_{\mathcal{O}_\mathfrak{X}}(T_\mathfrak{X} \otimes T_\mathfrak{X}, T_\mathfrak{X}) \cong (\circ^2 T^*_\mathfrak{X} \oplus \wedge^2 T^*_\mathfrak{X}) \otimes T_\mathfrak{X}. \quad (A.1)$$

We need to argue the connection form $A$ will necessarily be valued in $\circ^2 T^*_\mathfrak{X} \otimes T_\mathfrak{X}$. This is easiest to see in local coordinates $(x|\theta)$. Since the coordinates $x$ and $\theta$ differ in parity, so do their differentials. Hence for vector fields $H$ and $Z$ we have:

$$\left(dx \circ \wedge dy\right)(H, Z) = H(x)Z(y) \pm H(y)Z(x) \text{ and;} \quad (A.2)$$

$$\left(d\theta \circ \wedge d\eta\right)(H, Z) = H(\theta)Z(\eta) \mp H(\eta)Z(\theta) \quad (A.3)$$

where by $\circ$ it is meant $\circ$ and $\wedge$ respectively; and $(y|\eta)$ are other even-odd coordinates. Now let $A$ denote the connection form for $\nabla$. In coordinates we can write $A$ in tensor components,

$$A(x|\theta) = \alpha_{\mu\nu}^\sigma dx^\mu \otimes dx^\nu \otimes \frac{\partial}{\partial x^\sigma} + \alpha_{\mu k} dx^\mu \otimes dx^\nu \otimes \frac{\partial}{\partial \theta^k} + \alpha_{\mu}^{j\sigma} dx^\mu \otimes d\theta^j \otimes \frac{\partial}{\partial x^\sigma}$$

$$+ \alpha_{\mu k}^j dx^\mu \otimes d\theta^j \otimes \frac{\partial}{\partial \theta^k} + \alpha^{ij\sigma}_k d\theta^i \otimes d\theta^j \otimes \frac{\partial}{\partial x^\sigma} + \alpha^{ij}_k d\theta^i \otimes d\theta^j \otimes \frac{\partial}{\partial \theta^k}.$$

In order to see $A$ will be symmetric in its differentials we will need to recall the following: for any two vector fields $H$ and $Z$, their bracket can be defined over the whole space by setting:

$$[H, Z] \triangleq \nabla_H Z - \nabla_Z H + \text{Tor.}(\nabla H, Z).$$

The above expression generalises to supermanifolds by replacing the bracket by the graded-commutative bracket giving (c.f., Definition II.1.1(i)):

$$[H, Z] \triangleq \nabla_H Z - (-1)^{|Z||H|} \nabla_Z H + \text{Tor.}(\nabla H, Z) \quad (A.4)$$
where \(|Z|\) and \(|H|\) refers to the \(\mathbb{Z}_2\)-parity of \(Z\) and \(H\) as vector fields on \(X\). Now in a coordinate system \((x|\theta)\), the frame for the tangent bundle \((\partial/\partial x|\partial/\partial \theta)\) will be torsion free. Hence by (A.4) we find:

\[
\nabla \frac{\partial}{\partial x^\rho} \frac{\partial}{\partial x^\gamma} - \nabla \frac{\partial}{\partial x^\gamma} \frac{\partial}{\partial x^\rho} = \left[ \frac{\partial}{\partial x^\rho}, \frac{\partial}{\partial x^\gamma} \right] = 0 \quad \text{and;}
\]

\[
\nabla \frac{\partial}{\partial \theta_l} + \nabla \frac{\partial}{\partial \theta_l} \frac{\partial}{\partial \theta_k} = \left[ \frac{\partial}{\partial \theta_k}, \frac{\partial}{\partial \theta_l} \right] = 0. \tag{A.6}
\]

And so where the connection form is concerned, we have:

\[
A(x|\theta) \left( \frac{\partial}{\partial x^\rho}, \frac{\partial}{\partial x^\gamma} \right) = \bigotimes^2 A \left( \frac{\partial}{\partial x^\rho}, \frac{\partial}{\partial x^\gamma} \right) + \bigwedge^2 A \left( \frac{\partial}{\partial x^\rho}, \frac{\partial}{\partial x^\gamma} \right)
\]

\[
= \left( (\alpha^\rho_{\gamma} + \alpha^\gamma_{\rho}) + (\alpha^\rho_{\gamma} - \alpha^\gamma_{\rho}) \right) \otimes \frac{\partial}{\partial x^\sigma}
\]

\[
+ \left( (\alpha^\gamma_{\rho k} + \alpha^\gamma_{\rho k}) + (\alpha^\gamma_{\rho k} - \alpha^\gamma_{\rho k}) \right) \otimes \frac{\partial}{\partial \theta_k}
\]

\[
\text{set } A(x|\theta) \left( \frac{\partial}{\partial x^\rho}, \frac{\partial}{\partial x^\gamma} \right) \quad \text{by (A.5)}
\]

\[
= \left( (\alpha^\rho_{\gamma} + \alpha^\gamma_{\rho}) + (\alpha^\rho_{\gamma} - \alpha^\gamma_{\rho}) \right) \otimes \frac{\partial}{\partial x^\sigma}
\]

\[
+ \left( (\alpha^\gamma_{\rho k} + \alpha^\gamma_{\rho k}) + (\alpha^\gamma_{\rho k} - \alpha^\gamma_{\rho k}) \right) \otimes \frac{\partial}{\partial \theta_k}
\]

\[
\quad \iff \quad \alpha^\rho_{\gamma} - \alpha^\gamma_{\rho} = 0 \quad \text{and} \quad \alpha^\gamma_{\rho k} - \alpha^\gamma_{\rho k} = 0;
\]

\[
\iff \quad \bigwedge^2 A \left( \frac{\partial}{\partial x^\rho}, \frac{\partial}{\partial x^\gamma} \right) = 0 \quad \text{(A.7)}
\]

A similar calculation for the odd derivations, but making use (A.3) and (A.6) now, will reveal:

\[
\quad \bigotimes^2 A \left( \frac{\partial}{\partial \theta_k}, \frac{\partial}{\partial \theta_l} \right) = 0. \tag{A.8}
\]

Viewed as a supermatrix\(^{52}\) then, the constraints in (A.7) and (A.8) can be summarised by \(\bigwedge^2 A = 0\). Hence, by (A.1), it follows that \(A\) is valued in \(\bigotimes^2 T^*_X \otimes T_X\). □

\(^{52}\)Following convention in [Man88], for a module \(M = M_+ \oplus M_-\) over a supercommutative ring \(R\) we have: \(\bigwedge^\bullet M \cong \bigwedge_{R_+}^\bullet M \otimes \bigotimes_{R_-}^\bullet M_-\) and \(\bigotimes^\bullet M \cong \bigotimes_{R_+}^\bullet M_+ \otimes \bigwedge_{R_-}^\bullet M_-\).
Appendix B. Cotangent Supermanifolds and Liftings

Onishchik’s Characterisation. Green’s classification in Theorem C.1 applies to supermanifolds generally. Onishchik in [Oni98] specialises the class of supermanifolds considered to the following kind, which we refer to as ‘cotangent supermanifolds’.

**Definition B.1.** Any supermanifold modelled on a complex manifold $X$ and its cotangent bundle $T^*_X$ will be referred to as a *cotangent supermanifold*.

Let $(X, T^*_X)$ be a model for cotangent supermanifolds and denote by $\hat{X}$ the split model. By Lemma I.2.2 we have an exact sequence for the tangent sheaf for each $m$, $\forall m > 0$,

$$0 \to \Omega^{m+1}_X \otimes T_X \to i \circ T^{(m)}_{\hat{X}} \to \Omega^m_X \otimes T_X \to 0.$$  \hspace{1cm} (B.1)

Now let $d_X : \mathcal{O}_X \to \Omega^1_X$ be the universal, Kähler derivation. Note that it defines a global vector field on the split model of degree one, i.e., $d^X \in H^0(X, T^{(1)}_{\hat{X}})$. Accordingly, we will refer to $d^X$ as the *de Rham vector field*. Note furthermore that the inclusion $i$ in (B.1) is defined for any $m$, i.e., that $i : \Omega^m_X \otimes T_X \to T^{(m-1)}_{\hat{X}}$. With these observations and the (super) Lie bracket on $T_{\hat{X}}$, we can form a mapping: $\ell : \Omega^m_X \otimes T_X \to T^{(m)}_{\hat{X}}$ given by $\psi \mapsto [i(\psi), d^X]$. In contrast with the general case now, the mapping $\ell$ defines a holomorphic splitting of the sequence (B.1). Hence, for any $m$, we have:

$$T^{(m)}_{\hat{X}} \cong i(\Omega^{m+1}_X \otimes T_X) \oplus \ell(\Omega^m_X \otimes T_X).$$  \hspace{1cm} (B.2)

With this splitting we have a mapping $\Omega^m_X \otimes T^{(m)}_{\hat{X}} \to i(\Omega^{m+1}_X \otimes T_X) \oplus \ell(\Omega^m_X \otimes \Omega^{m+1}_X \otimes T_X) \to i(\Omega^{m+1}_X \otimes T_X) \oplus \ell(\Omega^m_X \otimes \Omega^{m+1}_X \otimes T_X) \cong T^{(m+n)}_{\hat{X}}$, where we used the wedge product on differential forms, $\otimes \mapsto \wedge$. Onishchik in [Oni98] then studies the mapping $\mu : \Omega^1_X \to \Omega^1_X \otimes T^{(1)}_{\hat{X}} \to T^{(2)}_{\hat{X}}$ given by $\psi \mapsto \psi \otimes d^X \mapsto \psi d^X$. Now for any $m > 0$, the formal exponential of vector fields in $T^{(m)}_{\hat{X}}$, for any $\hat{X}$, will be a finite sum and defines a mapping of sheaves of sets $\exp : T^{(m)}_{\hat{X}} \to G^{(m)}_{\hat{X}}$. Upon specialising now to the case where $\hat{X}$ is a split, cotangent supermanifold with model $(X, T^*_X)$, we can compose with $\mu$ to get a mapping of sheaves of sets:

$$\exp \mu : \Omega^1_X \to G^{(2)}_{\hat{X}}$$ given by $\psi \mapsto \exp \psi d^X$.  \hspace{1cm} (B.3)

\footnote{Products here are wedge products, so $\psi d^X = \psi \wedge d^X$.}
Hence we have induced on cohomology $(\exp \mu)_* : H^1(X, \Omega^1_X) \to \check{H}^1(X, G^{(2)}_{\hat{X}})$. Applying Green’s classification in Theorem C.1 reveals, when $\hat{X}$ is a split, cotangent supermanifold, that any element of $H^1(X, \Omega^1_X)$ will define a cotangent supermanifold with $\hat{X}$ as its split model. Onishchik in [Oni98, §2] derived the following characterisation.

**Theorem B.2.** Let $\omega \in H^1(X, \Omega^1_X)$ and denote by $\check{X}_\omega \overset{\Delta}{=} (\exp \mu)_* (\omega)$ the associated, cotangent supermanifold. If $\omega \neq 0$, $\check{X}_\omega$ is non-split. □

From the explicit construction in the proof of Theorem B.2 in [Oni98], the primary obstruction $\eta_* [\check{X}_\omega]$ of the cotangent supermanifold $\check{X}_\omega$ can be identified with $pr \mu_* (\omega)$, where $\mu_* : H^1(X, \Omega^1_X) \to H^1(X, T^{(2)}_{\hat{X}})$ is the induced mapping on cohomology; and $pr$ is the projection $H^1(X, T^{(2)}_{\hat{X}}) \to \text{OB}_{\hat{X}}^{\text{primary}} \cong H^1(X, \Omega^2_X \otimes T_X)$. Note that, having established just this, Theorem B.2 will follow from the more general Theorem III.2.4.

With respect to a covering $(U_\alpha)_\alpha$, we can see from (B.3) that a cocycle representative of the primary obstruction $\eta_* [\check{X}_\omega]$ is given by $(\omega_{\alpha\beta} d^X)_{\alpha, \beta}$, where $(\omega_{\alpha\beta})_{\alpha, \beta}$ is a cocycle representation of the given class $\omega$.

**Relation to the Euler Vector Field.** Observe that the sequence in (B.1) can continued further, giving:

\[
0 \longrightarrow \Omega^{m+1}_X \otimes T_X \longrightarrow T^{(m)}_{\hat{X}} \longrightarrow \Omega^m_X \otimes T_X \longrightarrow 0
\]

\[
\check{T}^{(m-1)}_{\hat{X}} \downarrow
\]

\[
\Omega^{m-1}_X \otimes T_X \downarrow
\]

\[
0
\]

54That is, up to (framed) isomorphism.
The composition $T^{(m)}_{\hat{X}} \to T^{(m-1)}_{\hat{X}}$ indicated above is, by construction, non-zero. There is a natural isomorphism of sheaves of $\mathcal{O}_X$-modules, $T_X \cong T^{(-1)}_{\hat{X}}$ given by $\partial/\partial x \mapsto \partial/\partial(dx)$. Evidently, the morphism $T^{(m)}_{\hat{X}} \to T^{(m-1)}_{\hat{X}}$ is defined by reference to this isomorphism, i.e., is the composition $T^{(m)}_{\hat{X}} \to \Omega^m_X \otimes T_X \cong \Omega^m_X \otimes T^{(-1)}_{\hat{X}} \subset T^{(m-1)}_{\hat{X}}$. For $m = 1$ we have the mapping $T^{(1)}_{\hat{X}} \to T^{(0)}_{\hat{X}}$. Comparing local coordinate descriptions of the de Rham vector field and the Euler vector field in (II.2.3.1), and using that they are global sections, it is evident that:

$$d^X \mapsto \epsilon_{\hat{X}},$$

under the induced mapping $H^0(X, T^{(1)}_{\hat{X}}) \to H^0(X, T^{(0)}_{\hat{X}})$.

**Vector Field Liftings.** In Definition II.2.2 the notion of a vector field lift was adapted to the Euler vector field. Such a notion can be made sense of more generally however with the initial form sequence in Lemma I.2.5. That is:

**Definition B.3.** Let $\mathfrak{X}$ be a supermanifold with split model $\hat{\mathfrak{X}}$. For a given $m$, any global vector field on $\hat{\mathfrak{X}}$ of degree $m$, i.e., any element of $H^0(X, T^{(m)}_{\hat{X}})$, is said to *lift* to a vector field on $\mathfrak{X}$ if it is in the image of the mapping on global sections $H^0(X, T^{(m)}_{\hat{X}}) \to H^0(X, T^{(m)}_{\hat{X}})$.

Our proof of Koszul’s theorem in this article relied essentially on Theorem II.2.5, which identified liftings of the Euler vector field with splittings.\(^{56}\) Onishchik in [Oni98] also studied the splitting problem, specialised to cotangent supermanifolds; and from the viewpoint of vector field liftings. We summarise these results in the following.

**Theorem B.4.** For $\omega \in H^1(X, \Omega^1_X)$ let $\mathfrak{X}_\omega$ be the cotangent supermanifold from Theorem B.2. Then:

(i) the de Rham vector field $d^X \in H^0(X, T^{(1)}_{\hat{X}})$ will lift to $\mathfrak{X}_\omega$;

---

\(^{55}\)Recall that in writing $T_{\hat{X}} = \bigoplus_{m \geq -1} T^{(m)}_{\hat{X}}$, we have decomposed $T_{\hat{X}}$ into a $\mathbb{Z}$-graded sheaf of $\mathcal{O}_X$-modules with $\Omega^j_X \otimes T^{(j)}_{\hat{X}} \subset T^{(j+j')}_{\hat{X}}$.

\(^{56}\)c.f., §II.4.7.
Recall the splitting of $T^{(m)}_{\mathcal{X}}$ in (B.2) for any $m$. Specialising to $m = 0$, for any $u \in H^0(X, T_X)$, if
\[ \ell(u) \sim \mu_* \omega = 0, \]  
then $\ell(u)$ lifts to $\mathcal{X}_\omega$.  

\[ \square \]

We wish to remark here that Theorem B.4 is indeed consistent with, and independent of, Theorem II.2.5, which concerns the Euler vector field. Indeed, on a first reading, Theorem B.4(i), the mapping in (B.4) and Theorem II.2.5 might jointly imply every cotangent supermanifold $\mathcal{X}_\omega$ splits, contradicting Theorem B.2. But such a conclusion would be based on the error of forgetting the parity, i.e., how the vector fields act on functions. With $\mathcal{X}$ the split, cotangent supermanifold modelled on $(X, T_X^*)$, its structure sheaf is $\mathcal{O}_X = \Omega_X^\bullet$. The de Rham vector field acts by $d^X : \Omega_X^\bullet \to \Omega_X^\bullet[1]$. In contrast, the Euler vector field acts as $\epsilon_{\mathcal{X}} : \Omega_X^\bullet \to \Omega_X^\bullet[0]$. Hence we cannot identify $d^X$ and $\epsilon_{\mathcal{X}}$ as global vector fields on $\mathcal{X}$, despite the mapping in (B.4). In particular, for any cotangent supermanifold $\mathcal{X}$ with split model $\mathcal{X}$, the image of $d^X$ in $H^1(X, T^{(2)}_{\mathcal{X}})$ will not correspond to the Euler differential. Now with Theorem B.4(i) taken to mean: the image of $d^X$ in $H^1(X, T^{(2)}_{\mathcal{X}})$ vanishes for any $\mathcal{X}$; in stating that Theorem B.4 is consistent with Theorem II.2.5, we observe that Theorem B.4(i) need not imply vanishing of the Euler differential.

Splitting Cotangent Supermanifolds. The result on lifting in Theorem B.4(i) is suggestive and we might wonder under what condition it will imply the existence of a supermanifold splitting. This is the subject of the following and might be viewed as an analogue of Theorem B.4(ii) for the Euler vector field.

---

57Recall that on cohomology the cup product is the mapping $H^j(X, \mathcal{F}) \otimes H^j(X, \mathcal{G}) \to H^{j+j'}(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})$. The equation in (B.5) is evidently in $H^1(X, T^{(0)}_{\mathcal{X}} \otimes T^{(2)}_{\mathcal{X}})$.  
58Note moreover, for any $f \in \mathcal{O}_X$ and $\omega \in \Omega_X^\bullet$, that $d^X(f \omega) = (d^X f) \omega + f d^X \omega$; while by Lemma II.2.6(ii), $\epsilon_{\mathcal{X}}(f \omega) = f \epsilon_{\mathcal{X}}(\omega)$. That is, $\epsilon_{\mathcal{X}}$ is an $\mathcal{O}_X$-linear derivation of $\Omega_X^\bullet$ while $d^X$ is certainly not $\mathcal{O}_X$-linear.  
59Recall by Proposition II.4.4 and Corollary II.4.6 that the Euler differential will also be valued in $H^1(X, T^{(2)}_{\mathcal{X}})$.  

**Theorem B.5.** Let $\mathfrak{X}_\omega$ be a cotangent supermanifold as in Theorem B.2 and suppose there exists a global, degree-$(-1)$ vector field $\gamma \in H^0(X, T_X^{(-1)})$ such that,

(i) $[\gamma, d^X] = \epsilon_X$ and;
(ii) $\gamma$ lifts to $\mathfrak{X}_\omega$.

Then $\mathfrak{X}_\omega$ is split.

**Proof.** Our proof makes use of the following claim: for vector fields $u, v \in H^0(X, T_X)$, if $u$ and $v$ lift to some supermanifold $\mathfrak{X}$ with split model $\hat{\mathfrak{X}}$, then so does their Lie bracket. To see why this claim holds, recall that if $u$ and $v$ are homogeneous with parities $p(u)$ and $p(v)$ respectively, then $[u, v] \in H^0(X, T_X^{(p(u) + p(v))})$. Note that also makes sense to form Lie brackets of vector fields on $\mathfrak{X}$. Now let $U$ and $V$ denote the respective liftings of $u$ and $v$ to $\mathfrak{X}$. Since $U$ and $V$ are globally defined, it suffices to consider them locally wherein we can write $U = u + u^{(p(u)+1)}$ and $V = v + v^{(p(v)+1)}$, for sections $u^{(p(u)+1)} \in T_X^{(p(u)+1)}$ and $v^{(p(v)+1)} \in T_X^{(p(v)+1)}$. Evaluating their bracket gives,

$$[U, V] = [u, v] + [u, v^{(p(v)+1)}] + [u^{(p(u)+1)}, v] + [u^{(p(u)+1)}, v^{(p(v)+1)}]$$

(B.6)

See that (B.6) is a section of $T_X^{(p(u)+p(v)+1)}$. Hence $[U, V] \mod T_X^{(p(u)+p(v)+1)} = [u, v]$, which proves the claim. Applying this to our case, recall from Theorem B.4(ii) that $d^X$ will always lift to $\mathfrak{X}_\omega$ and so, with our assumption in Theorem B.5(ii) we know that $[\gamma, d^X]$ will be a degree-zero vector field on $\hat{\mathfrak{X}}$ which lifts to $\mathfrak{X}_\omega$. Assuming Theorem B.5(i), we see that the Euler vector field lifts to $\mathfrak{X}_\omega$ and hence its differential vanishes. This theorem now follows from Theorem II.2.5. \[\square\]

**Remark B.6.** Note that if we know Theorem B.5(i) and (ii), then by Theorem B.2 it follows that $\omega = 0$. Hence Theorem B.5(i) and (ii) might be viewed as ‘supergeometric conditions’ guaranteeing the vanishing of cohomology classes in $H^1(X, \Omega^1_X)$, or equivalently, the $(1, 1)$-Dolbeault classes in $H^{1,1}(X, \mathbb{C})$.\[60\]
APPENDIX C. ON THE CLASSIFICATION OF SUPERMANIFOLDS

The title of this section coincides largely with that of a paper by Onishchik [Oni99] and not coincidentally so. In [Oni99], Hodge-theoretic methods are adapted to establish the moduli problem for complex supermanifolds, following the set-theoretic underpinnings provided by Green in [Gre82]. Similarly, we will also begin from the foundational work by Green and look to present a moduli-theoretic extension, with a view toward functoriality.

We begin by reviewing this work by Green below.

**Green’s Classification.** Let \((X, T^*_X, -)\) be a model on which we can form a category of supermanifolds.\(^{61}\) Associated to this model is following sheaf of groups on \(X\):

\[
G^{(m)}_{(X,T^*_X, -)} \triangleq \left\{ \alpha \in (\text{Aut} \wedge ^\bullet T^*_X, -) \mid \alpha(u) - u \in \bigoplus_{\ell \geq m} \wedge ^\ell T^*_X \right\}. \quad (C.1)
\]

The sheaf of groups \(G^{(m)}_{X,T^*_X, -} |_{m=2}\) is of particular importance and plays the central role in Greens’s classification. In this case we have a short exact sequence of sheaves of groups on \(X\),\(^{62}\)

\[
\{1\} \longrightarrow G^{(2)}_{(X,T^*_X, -)} \longrightarrow G^{(1)}_{(X,T^*_X, -)} \longrightarrow \text{Aut}_{\mathcal{O}_X} T^*_X \longrightarrow \{1\} \quad (C.2)
\]

from whence we obtain a group action \(H^0(X, \text{Aut}_{\mathcal{O}_X} T^*_X)\) on \(\tilde{H}^1(X, G^{(2)}_{(X,T^*_X, -)})\), an observation first documented by Grothendieck in [Gro55] in a much more general setting. Green in [Gre82] establishes:

**Theorem C.1.** (Green’s Theorem) The set of supermanifolds modelled on \((X, T^*_X, -)\) up to isomorphism lies in bijective correspondence with the set of cosets:

\[
\frac{\tilde{H}^1(X, G^{(2)}_{(X,T^*_X, -)})}{H^0(X, \text{Aut}_{\mathcal{O}_X} T^*_X)}.
\]

We will refer to the above orbit set as Green’s orbit set. \(\square\)

\(^{61}\)c.f., Proposition I.1.5.

\(^{62}\)Green in [Gre82] refers to automorphisms of connected sheaves \(\mathcal{A}\), which are \(\mathbb{Z}\)-graded sheaves with degree zero component \(\mathcal{O}_X\). As automorphisms of such sheaves preserve \(\mathcal{O}_X\) it follows, in the case where \(\mathcal{A} = \wedge ^\bullet T^*_X\), that \(G^{(1)}_{(X,T^*_X, -)}\) will be its sheaf of automorphisms.
The set $H^1(X, G^{(2)}_{X, T_{X,-}^*})$ is itself a pointed set and the $H^0(X, Aut_{O_X} T_{X,-}^*)$-orbit of this point defines the base-point in Green’s orbit set from Theorem C.1 above. Under the correspondence with isomorphism classes of supermanifolds, this base-point corresponds to the isomorphism class of the split model associated to $(X, T_{X,-}^*)$.

**A Generalised Classification.** Green’s classification in Theorem C.1 does not involve supermanifolds intrinsically, i.e., it is predicated on the exterior algebra. Recall however from Lemma I.1.7(iii) that the exterior algebra $\bigwedge^*_{O_X} T_{X,-}^*$ forms the structure sheaf of the split model $\widehat{\mathfrak{X}}$. Indeed, the definition of the automorphism sheaves $G^{(m)}_{X, T_{X,-}^*}$ in (C.1), central to Green’s classification, only requires the data of the structure sheaf and fermionic ideal. More generally therefore we can form, for any supermanifold $\mathfrak{X}$ with structure sheaf $O_{\mathfrak{X}}$ and fermionic ideal $J_{\mathfrak{X}}$, the sheaf of groups:

$$G^{(m)}_{O_{\mathfrak{X}}} \triangleq \{ \alpha \in (Aut_{O_{\mathfrak{X}}} O_{\mathfrak{X}})_+ \mid \alpha(u) - u \in J_{\mathfrak{X}}^m \}. \quad \text{(C.3)}$$

As already observed, in the case where $\mathfrak{X} = \widehat{\mathfrak{X}}$ we recover (C.1) by Lemma I.1.7(iii).

In analogy with (C.2) now we have:

**Proposition C.2.** For any supermanifold $\mathfrak{X}$ there exists a short exact sequence of sheaves of groups,

$$\{1\} \longrightarrow G^{(2)}_{O_{\mathfrak{X}}} \longrightarrow G^{(1)}_{O_{\mathfrak{X}}} \longrightarrow Aut_{O_{\mathfrak{X}}} T_{X,-}^* \longrightarrow \{1\}$$

where we have identified $J_{\mathfrak{X}}/J_{\mathfrak{X}}^2$ with $T_{X,-}^*$.

Our proof of Proposition C.2 will involve the following preliminary results.

**Lemma C.3.** Any automorphism $\alpha : O_{\mathfrak{X}} \rightarrow O_{\mathfrak{X}}$ sends $J_{\mathfrak{X}}^m \rightarrow J_{\mathfrak{X}}^{m - m_\alpha}$, for any $m$.

**Proof.** Suppose there exists an automorphism $\alpha$ which reduces the power of the fermionic ideal, say $J_{\mathfrak{X}}^m \rightarrow J_{\mathfrak{X}}^{m - m_\alpha}$ for some integer $m_\alpha$. Then the $n$-fold composition $\alpha^n$ sends $J_{\mathfrak{X}}^m \rightarrow J_{\mathfrak{X}}^{m - nm_\alpha}$. Since $J_{\mathfrak{X}}^{m - nm_\alpha} = (0)$ for $m - nm_\alpha < 0$, there will exist some $n$ such that $\alpha^n = 0$, i.e., $\alpha$ will be nilpotent. But then $\alpha$ cannot be invertible and so cannot be an automorphism. Thus any automorphism preserves the power of the fermionic ideal. $\square$
Corollary C.4. Any automorphism of $\mathcal{O}_X$ will induce an automorphism of $\mathcal{O}_X/\mathcal{J}_X^m$ for any $m$. □

Recall the sequence of sheaves modulo powers of the fermionic ideal from (I.1.2.2). By Corollary C.4, note that any $\alpha \in \mathcal{G}_{\mathcal{O}_X}^{(m)}$ will induce the following diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{J}_X^m/\mathcal{J}_X^{m+1} & \longrightarrow & \mathcal{O}_X/\mathcal{J}_X^m & \longrightarrow & 0 \\
\alpha \downarrow & & \pi & & \downarrow & & \\
0 & \longrightarrow & \mathcal{J}_X^m/\mathcal{J}_X^{m+1} & \longrightarrow & \mathcal{O}_X/\mathcal{J}_X^m & \longrightarrow & 0
\end{array}
$$

(C.4)

Hence we obtain a map $\mathcal{G}_{\mathcal{O}_X}^{(m)} \to \text{Aut}_{\mathcal{O}_X} \mathcal{J}_X^m/\mathcal{J}_X^{m+1}$ given by: $\alpha \mapsto \pi|_{\mathcal{J}_X^m/\mathcal{J}_X^{m+1}}$. The kernel of this restriction comprise those automorphisms $\beta \in \mathcal{G}_{\mathcal{O}_X}^{(m)}$ inducing:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{J}_X^m/\mathcal{J}_X^{m+1} & \longrightarrow & \mathcal{O}_X/\mathcal{J}_X^m & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{J}_X^m/\mathcal{J}_X^{m+1} & \longrightarrow & \mathcal{O}_X/\mathcal{J}_X^m & \longrightarrow & 0
\end{array}
$$

(C.5)

We now arrive at the proof of the proposition motivating these subsequent considerations.

Proof of Proposition C.2. We will argue that:

$$
\ker \left\{ \mathcal{G}_{\mathcal{O}_X}^{(1)} \to \text{Aut}_{\mathcal{O}_X} T^*_X, \right\} \cong \mathcal{G}_{\mathcal{O}_X}^{(2)}.
$$

Since $\mathcal{O}_X$ is globally $\mathbb{Z}_2$-graded we know that $\mathcal{O}_X/\mathcal{J}_X^2$ splits. That is, $\mathcal{O}_X/\mathcal{J}_X^2 \cong (\mathcal{O}_X/\mathcal{J}_X) \oplus (\mathcal{J}_X/\mathcal{J}_X^2)$. Recall furthermore that any automorphism in $\mathcal{G}_{\mathcal{O}_X}^{(1)}$ will induce the identity modulo $\mathcal{J}_X$. Now on inspection of the diagram in (C.5), specialised to $m = 1$, splitness of the middle term necessitates $\beta = 1$, the identity. From our characterisation of automorphisms in (C.4), automorphisms in $\mathcal{G}_{\mathcal{O}_X}^{(1)}$ inducing the identity on $\mathcal{O}_X/\mathcal{J}_X^2$ are precisely elements in $\mathcal{G}_{\mathcal{O}_X}^{(2)}$, whence Proposition C.2 follows. □

Applying Grothendieck’s observation\(^{63}\) now to the sequence in Proposition C.2, there will exist an action $H^0(X, \text{Aut}_{\mathcal{O}_X} T^*_X, \left\{-\right\})$ on the pointed set $\tilde{H}^1(X, \mathcal{G}_{\mathcal{O}_X}^{(2)})$. For the orbit

\(^{63}\)c.f., remarks succeeding (C.2)
set under this action we write:
\[ \mathcal{M}_X \triangleq \frac{\check{H}^1(X, \mathcal{G}_{\mathcal{O}_x}^{(2)})}{H^0(X, Aut_{\mathcal{O}_x} T_{X,-}^*)}. \] (C.6)

In the case \( X = \hat{X} \) we recover Green’s orbit set. In analogy then with Green’s classification in Theorem C.1, we have the following generalisation.

**Theorem C.5.** For any supermanifold \( \mathfrak{X} \) the set \( \mathcal{M}_X \) is a pointed set which classifies isomorphism classes of supermanifolds \( \mathfrak{X}' \) which are locally isomorphic to \( \mathfrak{X} \). The base-point in \( \mathcal{M}_X \) corresponds to the isomorphism class of \( \mathfrak{X} \).

**Proof.** More generally, to see that \( \check{H}^1(X, \mathcal{G}_{\mathcal{O}_x}^{(m)}) \) is a pointed set for any \( m \), its base-point can be described as follows: since \( \mathcal{G}_{\mathcal{O}_x}^{(m)} \) is a sheaf of groups there exists a natural homomorphism from the sheaf of trivial groups \( \{1\} \rightarrow \mathcal{G}_{\mathcal{O}_x}^{(m)} \), given by \( 1 \mapsto 1_{\mathcal{O}_x} \). The Čech cohomology \( \check{H}^1(X, \{1\}) \) is a one-point set and the base-point in \( \check{H}^1(X, \mathcal{G}_{\mathcal{O}_x}^{(m)}) \) is the image of this point under the inclusion \( \{pt\} = \check{H}^1(X, \{1\}) \rightarrow \check{H}^1(X, \mathcal{G}_{\mathcal{O}_x}^{(m)}) \), which is induced by the morphism \( \{1\} \rightarrow \mathcal{G}_{\mathcal{O}_x}^{(m)} \). Upon identifying \( \mathcal{G}_{\mathcal{O}_x}^{(2)} \) with the sheaf of automorphisms of \( \mathfrak{X} \) with preserve a given framing \( \phi \), the set \( \check{H}^1(X, \mathcal{G}_{\mathcal{O}_x}^{(2)}) \) can be identified with ‘twisted forms’. These are precisely supermanifolds \( \mathfrak{X}' \) with framing \( \phi \) and which are locally isomorphic to \( \mathfrak{X} \). The base-point corresponds to the trivially twisted form \( \mathfrak{X} \) itself. For further details on twisted forms more generally we refer to [Mil80, p. 134]. The group \( H^0(X, Aut_{\mathcal{O}_x} T_{X,-}^*) \) acts by permuting the choice of framing \( \phi \) and so the orbit set will classify isomorphism classes of supermanifolds, locally isomorphic to \( \mathfrak{X} \). \( \square \)

**Toward a Functor of Supermanifolds.** In analogy with work in algebraic geometry pertaining to classification and moduli,\(^{65}\) it is desirable to form a functor from supermanifolds to sets, \( SM \Rightarrow Set \). By Proposition I.1.5 we know that any supermanifold \( \mathfrak{X} \) will have some model \( (X, T_{X,-}^*) \); and Green’s theorem (Theorem C.1) affirms that its isomorphism class will define an element in a set, Green’s orbit set, which itself only uses information about the model \( (X, T_{X,-}^*) \). The determination of a model

\(^{64}\) i.e., is in bijective correspondence with the set of isomorphism classes of supermanifolds \( \mathfrak{X}' \) which are locally isomorphic to \( \mathfrak{X} \)

\(^{65}\) see e.g., [Har10, §4]
$(X, T^r_{\hat{X}})$ from the data of a supermanifold $\hat{X}$ is not unique however since it requires the choice of a framing which, while extant, is by Definition I.1.2 non-canonical. Hence we cannot functorially assign points in Green’s orbit set from the data of a supermanifold. The purpose behind presenting the abstraction $G_{\mathcal{O}_{\hat{X}}}(m)$ in (C.3) is to emphasise a construction which is intrinsic to the data of a supermanifold and does not require a choice of framing. The notation in (C.6) is therefore suggestive of our desired, $\text{Set}$-valued functor of supermanifolds. At the level of objects we have, to each $\mathcal{X} \in \text{Ob}(\text{SM})$ the assignment $\mathcal{X} \mapsto \mathfrak{M}_X$. Modelling $\text{SM}$ as a discrete category then, with any morphism the identity mapping we obtain a functor $F : \text{SM} \Rightarrow \text{Set}$. We will not further develop this line of thought now however and so will conclude on these open remarks. Theorem C.5 will suffice for our purposes in this article.

**Appendix D. Primary Obstructions and the Euler Differential**

Recall from Theorem III.2.4 that the primary obstruction to splitting a supermanifold $\mathcal{X}$ measures the failure for $\mathcal{X}$ to split. As mentioned in the succeeding comments however, its vanishing does not necessarily contribute any useful information on this splitting question. In contrast, by Theorem II.2.5, the Euler differential $\delta \varepsilon_{\hat{X}}$ is a complete measure of splitting. Therefore, we might expect a natural relation between the Euler differential and the primary (and higher) obstructions and indeed such a relation exists. It is this relation which we will establish presently. To formulate the statement, recall Proposition II.4.4 on the vanishing of the composition $H^0(X, T^r_{\hat{X}}) \to H^1(X, T^{r+1}_{\hat{X}})$. It follows then, for any $m$, that we obtain a mapping on cohomology $\delta' : H^0(X, T^m_{\hat{X}}) \to H^1(X, T^{m+2}_{\hat{X}})$ represented in the following
diagram by the dotted arrow:\footnote{Note, the existence of the dashed arrow $\tilde{\delta} : H^0(X, T^{(m)}_{\hat{x}}) \rightarrow H^1(X, T^{(m+2)}_{\hat{x}})$ follows from exactness of the column.}

\[
\begin{array}{c}
H^1(T^{(m+2)}_{\hat{x}}) \xrightarrow{\delta'} H^1(T^{(m+2)}_{\hat{x}}) \\
\cdots \xrightarrow{\delta} H^0(T^{(m)}_{\hat{x}}) \xrightarrow{\delta} H^1(T^{(m+1)}_{\hat{x}}) \xrightarrow{\delta} \cdots \\
\end{array}
\]

(D.1)

Where the obstruction sheaves are concerned, we have the general characterisation from Green in [Gre82]:

**Lemma D.1.** Let $\hat{x}$ be the split model with reduced space $X$ and odd cotangent bundle $T^*_{\hat{x},-}$. Then for any $m$, the obstruction sheaf of the split model satisfies,

\[
\mathcal{O}b_{\mathcal{O}_{\hat{x}}}^{(m)} \cong \begin{cases} 
\wedge^m T^*_{X,-} \otimes \mathcal{O}_X T_X & \text{if } m = 2\ell \text{ is even} \\
\wedge^m T^*_{X,-} \otimes \mathcal{O}_X T_{X,-} & \text{if } m = 2\ell + 1 \text{ is odd}.
\end{cases}
\]

\[\square\]

Hence Lemma I.2.2 and Lemma D.1 above give: $T^{(2)}_{\hat{x}} / \mathcal{O}b_{\mathcal{O}_{\hat{x}}}^{(3)} \cong \mathcal{O}b_{\mathcal{O}_{\hat{x}}}^{\text{primary}}$ and so the exact piece on cohomology:

\[
\cdots \xrightarrow{\mathcal{O}b_{\mathcal{O}_{\hat{x}}}^{(3)}} H^1(X, T^{(2)}_{\hat{x}}) \xrightarrow{q_*} \mathcal{O}b_{\mathcal{O}_{\hat{x}}}^{\text{primary}} \xrightarrow{\cdots} \]

(D.2)

Upon specialising (D.1) to $m = 0$ and utilising $q_*$ from (D.2), we have:

**Theorem D.2.** For any supermanifold $\mathfrak{x}$,

\[q_* \tilde{\delta} \epsilon_{\hat{x}} = \eta_* [\mathfrak{x}]\]

where $\epsilon_{\hat{x}}$ is the Euler vector field and $\eta_* [\mathfrak{x}]$ is the primary obstruction to splitting $\mathfrak{x}$.

**Remark D.3.** Recall from Definition II.2.3 that the Euler differential is the image of $\epsilon_{\hat{x}}$ in $H^1(X, T^{(1)}_{\hat{x}})$. From the diagram in (D.1) we can, W.L.O.G., identify the Euler
differential with $\delta' \epsilon_{\hat{X}}$. Hence Theorem D.2 can be considered a relation between the Euler differential and the primary obstruction.

**Proof.** As with many of the other arguments provided in this article, our method for proving Theorem D.2 will be to deduce it from commutativity of an appropriate diagram of cohomologies. To that extent we begin by observing, for any $m, \ell, \ell'$ with $m \leq \ell \leq \ell'$, that there will exist a short exact sequence:

$$0 \rightarrow \frac{T^{(\ell)}_X}{T^{(\ell')}_{\hat{X}}} \rightarrow \frac{T^{(m)}_X}{T^{(m)}_{\hat{X}}} \rightarrow \frac{T^{(m)}_X}{T^{(m)+1}_{\hat{X}}} \rightarrow 0$$

Appropriately specialising the above sequence gives the following diagram:

$$0 \rightarrow T^{(2)}_{\hat{X}} \rightarrow T^{(0)}_X/T^{(3)}_{\hat{X}} \rightarrow T^{(0)}_X/T^{(2)}_{\hat{X}} \rightarrow 0 \quad (D.3)$$

$$0 \rightarrow T^{(1)}_{\hat{X}}/T^{(3)}_{\hat{X}} \rightarrow T^{(0)}_X/T^{(3)}_{\hat{X}} \rightarrow T^{(0)}_X/T^{(1)}_{\hat{X}} \rightarrow 0$$

where we have used that $T^{(m)}_{\hat{X}} \cong T^{(m)}_X/T^{(m+1)}_{\hat{X}}$. Regarding the middle row in (D.3) observe that it comes from reducing the initial form sequence in Lemma I.2.5 by $T^{(3)}_{\hat{X}}$. That is, we have:

$$T^{(1)}_X \rightarrow T^{(0)}_X \rightarrow T^{(0)}_X \rightarrow H^0(T^{(0)}_{\hat{X}}) \rightarrow \delta \rightarrow H^1(T^{(1)}_X)$$

$$T^{(1)}_X/T^{(3)}_X \rightarrow T^{(0)}_X/T^{(3)}_X \rightarrow T^{(0)}_X/T^{(1)}_{\hat{X}} \rightarrow H^0(T^{(0)}_{\hat{X}}) \rightarrow \partial \rightarrow H^1(T^{(1)}_X/T^{(3)}_{\hat{X}}) \quad (D.4)$$

where the latter diagram above on cohomology commutes. On considering a morphism of tangent sheaves analogous to the former in (D.4) shifted in degree we can form another commuting diagram, essentially reducing that in cohomology in (D.4)
by \( T^{(2)}_\mathfrak{X} \). Combining this with (D.4) results in:

\[
\begin{array}{c}
H^0(T^{(0)}_\mathfrak{X}) \\
\downarrow \delta \\
H^1(T^{(1)}_\mathfrak{X}) \\
\downarrow p_* \\
H^1(T^{(1)}_\mathfrak{X} / T^{(3)}_\mathfrak{X})
\end{array} 
\xrightarrow{\partial}
\begin{array}{c}
H^1(T^{(1)}_\mathfrak{X}) \\
\downarrow p_* \\
H^1(T^{(1)}_\mathfrak{X} / T^{(3)}_\mathfrak{X})
\end{array}
\]

Hence,

\[
\delta = \partial \mod T^{(3)}_\mathfrak{X} \quad \text{and} \quad 0 = p_*\delta = p_*\partial. \quad (D.5)
\]

We return now to the diagram in (D.3). It induces the following on cohomology:

\[
\begin{array}{c}
H^0(T^{(0)}_\mathfrak{X} / T^{(2)}_\mathfrak{X}) \\
\downarrow \partial' \\
H^0(T^{(0)}_\mathfrak{X}) \\
\downarrow \partial \\
H^1(T^{(1)}_\mathfrak{X} / T^{(3)}_\mathfrak{X})
\end{array} 
\xrightarrow{\partial'}
\begin{array}{c}
H^1(T^{(1)}_\mathfrak{X}) \\
\downarrow p_* \\
H^1(T^{(1)}_\mathfrak{X})
\end{array}
\]

Since \( p_*\partial = 0 \) from (D.5), the existence of \( \partial' \), represented above by the dotted arrow, is guaranteed. With the former relation in (D.5) then, we therefore have:

\[
\delta' \epsilon_{\mathfrak{X}} = \partial' \epsilon_{\mathfrak{X}}. \quad (D.7)
\]

To complete the proof of this theorem, we will need to recall from Lemma III.4.4 that, for any \( m \), \( \mathcal{G}^{(m+2)}_{\mathcal{O}_{\mathfrak{X}}} \subset \mathcal{G}^{(m)}_{\mathcal{O}_{\mathfrak{X}}} \) is normal. Onishchik in [Oni99] shows:

\[
\frac{\mathcal{G}^{(2m)}_{\mathcal{O}_{\mathfrak{X}}}}{\mathcal{G}^{(2m+2)}_{\mathcal{O}_{\mathfrak{X}}}} \cong T^{(2m)}_{\mathfrak{X}}. \quad (D.8)
\]

Now for any supermanifold \( \mathfrak{X} \) the formal exponential of derivations of positive degree will be finite. This leads to an isomorphism of sheaves of sets, \( T^{(m)}_{\mathfrak{X}} \cong \mathcal{G}^{(m)}_{\mathcal{O}_{\mathfrak{X}}} \), given by
\[ \nu \mapsto \exp \nu = 1 + \nu + \frac{1}{2!} \nu^2 + \cdots \] We obtain therefore a morphism of sheaves, where it should be remembered that only the middle, vertical morphism is as sheaves of sets:

\[
\begin{array}{cccc}
G_{O_X}^{(2)} & \rightarrow & G_{O_X}^{(1)} & \rightarrow & \text{Aut}_{O_X} T_{X,-}^* \\
\downarrow & & \downarrow & & \downarrow \\
T_{\hat{x}}^{(2)} & \rightarrow & T_{\hat{x}}^{(1)} / T_{\hat{x}}^{(3)} & \rightarrow & \text{End}_{O_X} T_{X,-}^* \\
\downarrow & & \downarrow & & \downarrow \\
T_{\hat{x}}^{(1)} / T_{\hat{x}}^{(3)} & \rightarrow & T_{\hat{x}}^{(0)} / T_{\hat{x}}^{(3)} & \rightarrow & T_{\hat{x}}^{(0)}
\end{array}
\]

Note that even though the middle-vertical arrow is as sheaves of sets, the left- and right-most, vertical morphisms are as sheaves of groups and so we have induced a commuting diagram on cohomology:

\[
\begin{array}{cccc}
H^0(\text{Aut}_{O_X} T_{X,-}) & \rightarrow & \check{H}^1(G_{O_X}^{(2)}) & \rightarrow & H^1(T_{\hat{x}}^{(2)}) \\
\downarrow & & \downarrow \eta' & & \downarrow q_* \\
H^0(T_{\hat{x}}^{(1)}) & \rightarrow & \check{H}^1(T_{\hat{x}}^{(1)} / T_{\hat{x}}^{(3)}) & \rightarrow & \check{H}^1(T_{\hat{x}}^{(1)} / T_{\hat{x}}^{(3)})
\end{array}
\]

where, by construction, \( \partial \) is the boundary mapping from (D.4) with lift \( \partial' \) from (D.6). Note that we have also appended the triangle of mappings \( \eta', \eta, \) and \( q_* \). By Definition II.2.1, the Euler vector field \( \epsilon_{\hat{x}} \) is in the image of \( 1_{T_{X,-}^*} \in H^0(X, \text{Aut}_{O_X} T_{X,-}^*) \). Furthermore the class of \( X, [X] \in \check{H}^1(X, G_{O_X}^{(2)}) \), also lies in the image of \( 1_{T_{\hat{x}}^{(1)}} \) since it corresponds to the base-point (see e.g., Theorem C.5 and the proof of Proposition III.2.10). Now since \( \partial' \) lifts \( \partial \), it follows that the sub-diagram in (D.9) formed by \( \partial' \) and \( \eta' \) commute. Hence, and by (D.7), we have

\[
\delta' \epsilon_{\hat{x}} = \partial' \epsilon_{\hat{x}} = \eta'[X].
\]
Finally, the appended triangle of mappings \((\eta', \eta_*, q_*)\) in (D.9) will commute since it is obtained from the following commuting diagram

\[
\begin{array}{ccc}
\mathcal{G}_{\mathcal{O}_X}^{(2)} & \rightarrow & \mathcal{G}_{\mathcal{O}_X}^{(2)}/\mathcal{G}_{\mathcal{O}_X}^{(4)} \\
\downarrow & & \downarrow \\
\mathcal{G}_{\mathcal{O}_X}^{(2)}/\mathcal{G}_{\mathcal{O}_X}^{(4)} & \rightarrow & \mathcal{G}_{\mathcal{O}_X}^{(2)}/\mathcal{G}_{\mathcal{O}_X}^{(3)}
\end{array}
\]

where we used that \(\mathcal{G}_{\mathcal{O}_X}^{(2)}/\mathcal{G}_{\mathcal{O}_X}^{(4)} \cong T_{\bar{X}}^{(2)}\) by (D.8); and \(\mathcal{G}_{\mathcal{O}_X}^{(2)}/\mathcal{G}_{\mathcal{O}_X}^{(3)} \cong \mathcal{O}h_{\mathcal{O}_X}^{primary}\) by (III.2.1.1). Theorem D.2 now follows from the definition of the primary obstruction in Definition III.2.9, the identities in (D.10); and commutativity of the triangle of mappings \((\eta', \eta_*, q_*)\) in (D.9).
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