DYNAMICS OF INTERACTING GENERALIZED COSMIC CHAPLYGIN GAS IN BRANE-WORLD SCENARIO

Prabir Rudra

1Department of Mathematics, Bengal Engineering and Science University, Shibpur, Howrah-711 103, India.

In this work we explore the background dynamics when dark energy is coupled to dark matter with a suitable interaction in the universe described by brane cosmology. Here DGP and the RSII brane models have been considered separately. Dark energy in the form of Generalized Cosmic Chaplygin gas is considered. A suitable interaction between dark energy and dark matter is considered in order to at least alleviate (if not solve) the cosmic coincidence problem. The dynamical system of equations is solved numerically and a stable scaling solution is obtained. A significant attempt towards the solution of the cosmic coincidence problem is taken. The statefinder parameters are also calculated to classify the dark energy models. Graphs and phase diagrams are drawn to study the variations of these parameters. It is also seen that the background dynamics of Generalized Cosmic Chaplygin gas is consistent with the late cosmic acceleration, but not without satisfying certain conditions. It has been shown that the universe in both the models follows the power law form of expansion around the critical point, which is consistent with the known results. Future singularities were studied and our models were declared totally free from any types of such singularities. Finally, some cosmographic parameters were also briefly studied. Our investigation led to the fact that although GCCG with a far lesser negative pressure compared to other DE models, can overcome the relatively weaker gravity of RS II brane, with the help of the negative brane tension, yet for the DGP brane model with much higher gravitation, the incompetency of GCCG is exposed, and it cannot produce the accelerating scenario until it reaches the phantom era.

PACS numbers:
I. INTRODUCTION

Recent cosmological observations have indicated that the observable universe enters into an epoch of accelerated expansion \[1, 2\]. In the quest of finding a suitable model for universe, Cosmologists started to investigate the root cause that is triggering this expansion. Within the framework of the general relativity, the acceleration can be phenomenally attributed to the existence of a mysterious negative pressure component which violates the strong energy condition i.e. \( \rho + 3p < 0 \). Because of its invisible nature this energy component is aptly termed as dark energy (DE) \[3\]. Moreover, quite surprisingly, observations spilled out definitively that about 70 percent of the Universe is filled by this unknown ingredient and in addition that about 25 percent of this is composed by dark matter (DM).

With the introduction of DE, search began for different candidates that can effectively play its role. DE represented by a scalar field \[4\] is often called quintessence. Not only scalar field but also there are other Dark fluid models like Chaplygin gas which plays the role of DE very efficiently. Extensive research saw Chaplygin gas (CG) \[5, 6\], get modified into Generalized Chaplygin gas (GCG) \[7–11\] and then to Modified Chaplygin gas (MCG) \[12, 13\]. In this context it is worth mentioning that Interacting MCG in Loop quantum cosmology (LQC) was studied by Jamil et al \[14\]. Dynamics of MCG in Braneworld was studied by Rudra et al \[15\]. Other than these other forms of Chaplygin gas models have also been proposed such as Variable Modified Chaplygin gas (VMCG) \[16\] and New Variable Modified Chaplygin gas (NVMCG) \[17\]. Other existing forms of DE are phantom \[18\], k-essence \[19\], tachyonic field \[20\], etc.

In 2003, P. F. Gonzalez-Diaz \[21\] introduced the generalized cosmic Chaplygin gas (GCCG) model. The speciality of the model being that it can be made stable and free from unphysical behaviours even when the vacuum fluid satisfies the phantom energy condition. In the previous studies related to DE corresponding to phantom era Big-Rip is essential, as the time gradient of scale-factor blows to infinity in finite time. For the first time P. F. Gonzalez-Diaz including the GCCG model showed that Big Rip, i.e., singularity at a finite time is totally out of question. Hence in such models there is no requirement for evaporation of black hole to zero mass. The Equation of state (EoS) of the GCCG model is

\[
p = -\rho^{-\alpha} \left[ C + \left\{ \rho^{(1+\alpha)} - C \right\}^{-\omega} \right] \tag{1}
\]

\[1\] in the presence of a scalar field the transition from a universe filled with matter to an exponentially expanding universe is justified
where $C = \frac{A}{1+\omega} - 1$, with $A$ being a constant that can take on both positive and negative values, and $-\mathcal{L} > \omega > 0$, $\mathcal{L}$ being a positive definite constant, which can take on values larger than unity.

GCCG can explain the evolution of the universe starting from the dust era to $\Lambda CDM$, radiation era, matter dominated quintessence and lastly phantom era [22]. In this context it should be stated that in [23] Chowdhury and Rudra studied Interacting GCCG in Loop Quantum Cosmology (LQC).

Currently, we live in a special epoch where the densities of DE and DM are comparable. Although they have evolved independently from different mass scales. This is known as the famous cosmic coincidence problem. till date several attempts have been made to find a solution to this problem [24–34]. A suitable interaction between DE and DM is required if we wish to find an effective solution to this problem. It is obvious that there has been a transition from a matter dominated universe to dark energy dominated universe, by exchange of energy at an appropriate rate. Now the expansion history of the universe as determined by the supernovae and CMB data [31, 32] bounds us to fix the decay rate such that it is proportional to the present day Hubble parameter. Keeping the fact in mind cosmologists all over the world have studied and proposed a variety of interacting DE models [35–41].

As we have stated earlier, modifying the right hand side of Einstein’s equation (DE approach) was not the only way to explain the increase in the rate of the expansion. We can also modify the gravity part of the left hand side in order to demonstrate the present day universe. In this context Brane-gravity was introduced and brane cosmology was developed. A review on brane-gravity and its various applications with special attention to cosmology is available in [42–45]. In this work we consider the two most popular brane models, namely DGP and RS II branes. Our main aim of this work is to examine the nature of the different physical parameters for the universe around the stable critical points in two brane world models in presence of GCCG. Impact of any future singularity caused by the DE in brane world models will be studied. Finally some cosmographic parameters will be studied in brief.

This paper is organized as follows: Section 2 comprises of the analysis in RS II brane model. Section 3 deals with the analysis in DGP brane model. In section 4, a detailed graphical analysis for the phase plane is done. In section 5, future singularities arising from our models are studied, followed by the study of some cosmographic parameters in section 6. Finally the paper ends with some concluding remarks in section 7.
II. MODEL 1: RS II BRANE MODEL

Randall and Sundrum [46, 47] proposed a bulk-brane model to explain the higher dimensional theory, popularly known as RS II brane model. According to this model we live in a four dimensional world (called 3-brane, a domain wall) which is embedded in a 5D space time (bulk). All matter fields are confined in the brane whereas gravity can only propagate in the bulk. The consistency of this brane model with the expanding universe has given popularity to this model of late in the field of cosmology.

In RS II model the effective equations of motion on the 3-brane embedded in 5D bulk having $Z_2$-symmetry are given by [43, 47–51]

\[ G_{\mu\nu} = -\Lambda_4 q_{\mu\nu} + \kappa_4^2 \tau_{\mu\nu} + \kappa_5^4 \Pi_{\mu\nu} - E_{\mu\nu} \]  \hspace{1cm} (2)

where

\[ \kappa_4^2 = \frac{1}{6} \lambda \kappa_5^4 , \]  \hspace{1cm} (3)

\[ \Lambda_4 = \frac{1}{2} \kappa_5^2 \left( \Lambda_5 + \frac{1}{6} \kappa_5^2 \lambda^2 \right) \]  \hspace{1cm} (4)

and

\[ \Pi_{\mu\nu} = -\frac{1}{4} \tau_{\mu\alpha} \tau_{\nu}^{\alpha} + \frac{1}{12} \tau_{\mu\nu} + \frac{1}{8} q_{\mu\nu} \tau_{\alpha\beta} \tau^{\alpha\beta} - \frac{1}{24} q_{\mu\nu} \tau^2 \]  \hspace{1cm} (5)

and $E_{\mu\nu}$ is the electric part of the 5D Weyl tensor. Here $\kappa_5$, $\Lambda_5$, $\tau_{\mu\nu}$ and $\Lambda_4$ are respectively the 5D gravitational coupling constant, 5D cosmological constant, the brane tension (vacuum energy), brane energy-momentum tensor and effective 4D cosmological constant. The explicit form of the above modified Einstein equations in flat universe are

\[ 3H^2 = \Lambda_4 + \kappa_4^2 \rho + \frac{\kappa_4^2}{2\lambda} \rho^2 + \frac{6}{\lambda \kappa_4^2} \mathcal{U} \]  \hspace{1cm} (6)

and

\[ 2\dot{H} + 3H^2 = \Lambda_4 - \kappa_4^2 \rho - \frac{\kappa_4^2}{2\lambda} \rho^2 - \frac{\kappa_5^4}{\lambda \kappa_4^2} \mathcal{U} \]  \hspace{1cm} (7)

The dark radiation $\mathcal{U}$ obeys
\[ \dot{U} + 4HU = 0 \]  
(8)

where \( \rho = \rho_{gccg} + \rho_m \) and \( p = p_{gccg} + p_m \) are the total energy density and pressure respectively.

As in the present problem the interaction between DE and pressureless DM has been taken into account for interacting DE and DM the energy balance equation will be

\[ \dot{\rho}_{gccg} + 3H (1 + \omega_{gccg}) \rho_{gccg} = -Q, \text{ for GCCG and} \]

\[ \dot{\rho}_m + 3H \rho_m = Q, \text{ for the DM interacting with GCCG}. \]  
(9)

(10)

where \( Q = 3bH\rho \) is the interaction term, \( b \) is the coupling parameter (or transfer strength) and \( \rho = \rho_{gccg} + \rho_m \) is the total cosmic energy density which satisfies the energy conservation equation

\[ \dot{\rho} + 3H (\rho + p) = 0 \]  
[24, 52].

As we lack information about the fact, how does DE and DM interact so we are not able to estimate the interaction term from the first principles. However, the negativity of \( Q \) immediately implies the possibility of having negative DE in the early universe which is overruled by the necessity of the second law of thermodynamics to be held \[53\]. Hence \( Q \) must be positive and small. From the observational data of 182 Gold type Ia supernova samples, CMB data from the three year WMAP survey and the baryonic acoustic oscillations from the Sloan Digital Sky Survey, it is estimated that the coupling parameter between DM and DE must be a small positive value (of the order of unity), which satisfies the requirement for solving the cosmic coincidence problem and the second law of thermodynamics \[54\]. Due to the underlying interaction, the beginning of the accelerated expansion is shifted to higher redshifts. The continuity equations for dark energy and dark matter are given in equations (9) and (10). Now we shall study the dynamical system assuming \( \Lambda = \mathcal{U} = 0 \) (in absence of cosmological constant and dark radiation).

### A. DYNAMICAL SYSTEM ANALYSIS

Due to the complexity of the equations, it is very difficult to find direct solutions for this system. So in order to avoid these complex calculations, we undertake the dynamical system analysis for our further evaluations. In this subsection we plan to analyze the dynamical system. Before proceeding, the physical parameters are converted into some dimensionless form, given by

\[ x = \ln a, \quad u = \frac{\rho_{gccg}}{3H^2}, \quad v = \frac{\rho_m}{3H^2} \]  
(11)
where the present value of the scale factor $a_0 = 1$ is assumed. Now using equations (1), (6), (7), (9), (10) and (11) we get,

$$\frac{du}{dx} = -3b(u + v) + 3u - 3\omega_{\text{RSII}}^{\text{GCCG}} - 6\kappa_4^2u(u + v) + \frac{9\kappa_4^2u(u + v)^3}{2\lambda(1 - \kappa_4^2(u + v))} - \frac{9\kappa_4^2(u + v)^3}{4\lambda^2(1 - \kappa_4^2(u + v))^2} \left\{ C + \left( \frac{4\lambda^2(1 - \kappa_4^2(u + v))^2u^2}{\kappa_4^4(u + v)^4} - C \right)^{-w} \right\}$$

(12)

$$\frac{dv}{dx} = 3b(u + v) + 3v - 3\kappa_4^2v(u + v) - \frac{3\kappa_4^2v(u + v)^3}{4\lambda^2(1 - \kappa_4^2(u + v))^2} \left\{ C + \left( \frac{4\lambda^2(1 - \kappa_4^2(u + v))^2u^2}{\kappa_4^4(u + v)^4} - C \right)^{-w} \right\}$$

(13)

Where, $\omega_{\text{RSII}}^{\text{GCCG}}$ is the EoS parameter for GCCG in RS II brane determined as

$$\omega_{\text{GCCG}}^{\text{RSII}} = \frac{p_{\text{GCCG}}}{\rho_{\text{GCCG}}} = -\frac{\kappa_4^4(u + v)^4}{4\lambda^2u^2(1 - \kappa_4^2(u + v))^2} \left[ C + \left( \frac{4\lambda^2u^2(1 - \kappa_4^2(u + v))^2}{\kappa_4^4(u + v)^4} - 2C\kappa_4^4(u + v)^4 \right)^{-w} \right]$$

(14)

In the above calculations for mathematical simplicity we have considered $\alpha = 1$.

1. **CRITICAL POINTS**

The critical points of the above system are obtained by putting $\frac{du}{dx} = 0 = \frac{dv}{dx}$. But due to the complexity of these equations, it is not possible to find a solution in terms of the involved parameters. So we find a numerical solution for the above system, by putting the following values to the different parameters appearing in the system. We take,

$$b = 1.5, \quad \kappa_4 = 1, \quad \lambda = 0.1, \quad C = -1, \quad w = -1$$

and get the following critical point for the above dynamical system.

$$u_c = 1.51586 \quad v_c = 1.79374$$

(15)
The critical point correspond to the era dominated by DM and GCCG type DE. For the critical point \((u_c, v_c)\), the equation of state parameter given by equation (14) of the interacting DE takes the form

\[
\omega_{RSII}^{gccg} = \frac{p_{gccg}}{\rho_{gccg}} = - \frac{\kappa_4^4 (u_c + v_c)^4}{4 \lambda^2 u_c^2 (1 - \kappa_4^2 (u_c + v_c))^2} \left[ C + \left\{ \frac{4 \lambda^2 u_c^2 (1 - \kappa_4^2 (u_c + v_c))^2 - 2 C \kappa_4^4 (u_c + v_c)^4}{\kappa_4^4 (u_c + v_c)^4} \right\}^{-1} \right]
\]

(16)

2. STABILITY AROUND CRITICAL POINT

Now we check the stability of the dynamical system (eqs. (12) and (13)) about the critical point. In order to do this, we linearize the governing equations about the critical point i.e.,

\[
u = u_c + \delta u \quad \text{and} \quad v = v_c + \delta v,
\]

(17)

Now if we assume \(f = \frac{du}{dx}\) and \(g = \frac{dv}{dx}\), then we may obtain

\[
\delta \left( \frac{du}{dx} \right) = [\partial_u f]_c \delta u + [\partial_v f]_c \delta v
\]

and

\[
\delta \left( \frac{dv}{dx} \right) = [\partial_u g]_c \delta u + [\partial_v g]_c \delta v
\]

(18)

(19)

where

\[
[\partial_u f]_c = \frac{1}{4 \kappa_4^2 u^2 (u + v)^2 (1 + \kappa_4^2 (u + v))^3 \lambda^2} \left[ C \kappa_4^6 (u + v)^5 \left\{ u (-3 + \kappa_4^2 u) + v - \kappa_4^2 v^2 \right\} + 2 u^2 \left\{ -1 + \kappa_4^2 (u + v) \right\} \lambda \left\{ 4 \kappa_4^8 (u + v)^4 (2u + v) \lambda + 6u (u + 2v) \lambda - 2 \kappa_4^2 (u + v) (u (2 - b + 6u) + (2 - b + 12u) v) \lambda + \kappa_4^6 (u + v)^3 \left( 3 (u + v)^2 (3u + v) + 2 ((b - 10) u + (b - 6) v) \lambda \right) + \kappa_4^4 (u + v)^2 \left( -3 (u + v)^2 (4u + v) + 2 (u (8 - 2b + 3u) - 2 (b - 3 (1 + u) v) \lambda \right) \right\} \right]\]

(20)

\[
[\partial_v f]_c = - \frac{3}{2 \kappa_4^2 u (u + v)^2 (1 + \kappa_4^2 (u + v))^3 \lambda^2} \left[ C \kappa_4^6 (u + v)^5 \left\{ -2 + \kappa_4^2 (u + v) \right\} + \lambda u \left\{ -1 + \kappa_4^2 (u + v) \right\} \right]
\]
\[ \{ -6u^2\lambda + 4\kappa_4^4u(u+v)^4\lambda + 2\kappa_4^2(u+v)(6u^2+b(u+v))\lambda + 2\kappa_4^6(u+v)^3(3u(u+v)^2+(b-4)u+bv)\lambda \\
+ \kappa_4^4(u+v)^2\left(-9u(u+v)^2 - 2(u(3u+2b-2)+2bv)\lambda \right) \} \]  

(21)

\[ \partial_u g = \frac{3}{4} \left[ 4b - 4\kappa_4^2v - \frac{8\kappa_4^4uw(u+v)^2\{u + \kappa_4^2uv + v(-1 + \kappa_4^2v)\}\left\{-C + \frac{4u^2(-1+\kappa_4^2(u+v))^2\lambda^2}{\kappa_4^4(u+v)^4} \right\}^{-w}}{-1 + \kappa_4^2(u+v)} \right \{ C\kappa_4^4(u+v)^4 - 4u^2(-1 + \kappa_4^2(u+v))^2\lambda^2 \} \\
- \frac{2\kappa_4^6v(u+v)^3\left\{ C + \left( -C + \frac{4u^2(-1+\kappa_4^2(u+v))^2\lambda^2}{\kappa_4^4(u+v)^4} \right)^{-w} \right\}}{u(1 - \kappa_4^2(u+v))^3\lambda^2} \\
- \frac{3\kappa_4^4v(u+v)^2\left( C + \left( -C + \frac{4u^2(-1+\kappa_4^2(u+v))^2\lambda^2}{\kappa_4^4(u+v)^4} \right)^{-w} \right)}{-1 + \kappa_4^2(u+v))^2\lambda^2} \\
+ \frac{\kappa_4^4v(u+v)^3\left( C + \left( -C + \frac{4u^2(-1+\kappa_4^2(u+v))^2\lambda^2}{\kappa_4^4(u+v)^4} \right)^{-w} \right)}{u^2(-1 + \kappa_4^2(u+v))^2\lambda^2} \]  

(22)

\[ \partial_v g = \frac{3}{4} \left[ 4 + 4b - 4\kappa_4^2v - 4\kappa_4^4(u+v) + \frac{8\kappa_4^2uw(u+v)^2\{\kappa_4^2(u+v) - 2\}\left\{-C + \frac{4u^2((\kappa_4^2(u+v)-1))^2\lambda^2}{\kappa_4^4(u+v)^4} \right\}^{-w}}{\kappa_4^2(u+v) - 1} \right \{ C\kappa_4^4(u+v)^4 - 4u^2(\kappa_4^2(u+v) - 1)^2\lambda^2 \} \\
- \frac{2\kappa_4^6v(u+v)^3\left\{ C + \left( -C + \frac{4u^2(\kappa_4^2(u+v)-1)^2\lambda^2}{\kappa_4^4(u+v)^4} \right)^{-w} \right\}}{u\left\{ 1 - \kappa_4^2(u+v) \right\}^3\lambda^2} \\
- \frac{3\kappa_4^4v(u+v)^2\left\{ C + \left( -C + \frac{4u^2(\kappa_4^2(u+v)-1)^2\lambda^2}{\kappa_4^4(u+v)^4} \right)^{-w} \right\}}{-1 + \kappa_4^2(u+v))^2\lambda^2} \\
- \frac{\kappa_4^4v(u+v)^3\left\{ C + \left( -C + \frac{4u^2(\kappa_4^2(u+v)-1)^2\lambda^2}{\kappa_4^4(u+v)^4} \right)^{-w} \right\}}{u\left\{ -1 + \kappa_4^2(u+v) \right\}^2\lambda^2} \]  

(23)
The Jacobian matrix of the above system is given by,

\[ J_{(u,v)}^{(RSII)} = \begin{pmatrix} \frac{\delta f}{\delta u} & \frac{\delta f}{\delta v} \\ \frac{\delta g}{\delta u} & \frac{\delta g}{\delta v} \end{pmatrix} \]

The eigen values of the above matrix are calculated at the critical point \((u_c, v_c)\) and are found to be \(\lambda_1 = -1169.45, \quad \lambda_2 = -3.38251\). Hence it is a stable node.

**Description of figures**

Figs. 1, 2, 3, 4 : The dimensionless density parameters are plotted against e-folding time for different values of brane tensions, \(\lambda\). The initial condition is \(v(0) = 0.9, u(0) = 0.2\). The other parameters are fixed at \(w = -1, C = -1, \kappa_4 = 1\) and \(b = 0.01\). The brane tensions are respectively \(\lambda = 1, -10, -50\) and \(-100\).

Fig. 5 : The dimensionless density parameters are plotted against e-folding time for a high value of interaction \((b = 5)\). The initial condition is \(v(0) = 0.9, u(0) = 0.2\). The other parameters are fixed at \(w = -1, C = -1, \kappa_4 = 1\) and \(\lambda = -10\).

Fig. 6: The dimensionless density parameters are plotted against e-folding time for same initial condition. The initial condition is \(v(0) = 0.6, u(0) = 0.6\). The other parameters are fixed at \(w = -1, C = -1, \kappa_4 = 1\) and \(\lambda = -5\).

Figs. 7, 8 : The phase diagram of the parameters depicting an attractor solution are obtained for different values of brane tension. The initial conditions chosen are \(v(0) = 0.5, u(0) = 0.6\) (green); \(v(0) = 0.6, u(0) = 0.6\) (blue); \(v(0) = 0.7, u(0) = 0.6\) (red); \(v(0) = 0.8, u(0) = 0.6\) (brown). Other parameters are fixed at The other parameters are fixed at \(w = -1, C = -1, \kappa_4 = 1\) and \(b = 0.01\). The brane tensions are respectively \(\lambda = 1, -10\).

Fig. 9: The phase diagram of the parameters depicting an attractor solution are obtained for a high value of interaction \((b = 1)\). The initial conditions chosen are \(v(0) = 0.5, u(0) = 0.6\) (green); \(v(0) = 0.6, u(0) = 0.6\) (blue); \(v(0) = 0.7, u(0) = 0.6\) (red); \(v(0) = 0.8, u(0) = 0.6\) (brown). Other parameters are fixed at The other parameters are fixed at \(w = -1, C = -1, \kappa_4 = 1\) and \(\lambda = -10\).

Fig. 10 : The ratio of density parameters is shown against e-folding time. The initial conditions chosen are \(v(0) = 0.6, u(0) = 0.6\). The other parameters are fixed at The other parameters are fixed.
at \( w = -1, C = -1, \lambda = 1, \kappa_4 = 1 \) and \( b = 0.01 \).

Fig. 11: The deceleration parameter is plotted against the EoS parameter. Other parameters are fixed at \( \psi_{(RSII)} = -0.5 \).

Figs. 12, 13: The statefinder parameter \( r \) and \( s \) are plotted against the EoS parameter. Other parameters are fixed at \( \psi_{(RSII)} = -0.5 \).

### 3. NATURE OF COSMOLOGICAL PARAMETERS

#### 1. Deceleration Parameter:

The deceleration parameter, \( q = -1 - \frac{H}{H^2} \) in this model is calculated as,

\[
q_{(RSII)} = -1 + 3 \frac{\frac{\lambda}{2} \left( \omega_{gccg}^{(RSII)} \frac{\rho_{gccg}}{\rho} + 2 \right) + \left( 1 + \omega_{gccg}^{(RSII)} \frac{\rho_{gccg}}{\rho} \right)}{(1 + \frac{\rho}{2\lambda})} \tag{24}
\]

We consider a dimensionless density parameter \( \Omega_{gccg} = \frac{\rho_{gccg}}{\rho} \). In terms of this density parameter the expression for the deceleration parameter, \( q_{(RSII)} \) can be rewritten as,

\[
q_{(RSII)} = -1 + 3 \frac{\frac{\lambda}{2} \left( 1 + \omega_{gccg}^{(RSII)} \Omega_{gccg} + \left( 1 + \frac{\rho}{2\lambda} \right) \right)}{(1 + \frac{\rho}{2\lambda})} \tag{25}
\]

It is evident that for \( \lambda \to \infty \), we retrieve the result for the Einstein gravity,

\[
q_{EG} = -1 + \frac{3}{2} \left( 1 + \omega_{gccg}^{(RSII)} \Omega_{gccg} \right) \tag{26}
\]

Now since \( \Omega_{gccg} = \frac{\rho_{gccg}}{\rho} = \frac{u}{u+v} \) and assuming \( \frac{\rho}{2\lambda} = \psi_{(RSII)} \) we get,

\[
q_{(RSII)} = -1 + 3 \frac{\left( 1 + \psi_{(RSII)} \right) \omega_{gccg}^{(RSII)} \frac{u}{u+v} + \left( 1 + 2\psi_{(RSII)} \right)}{(1 + \psi_{(RSII)})} \tag{27}
\]

Now at the critical point, \( (u,v) \to (u_c,v_c) \), and using equation (27) we get

\[
q_c^{(RSII)} = -1 + \frac{3}{2} Z_{(RSII)}, \text{ where } Z_{(RSII)} = \frac{\left( 1 + \psi_{(RSII)} \right) \omega_{gccg}^{(RSII)} \frac{u}{u_c+v_c} + \left( 1 + 2\psi_{(RSII)} \right)}{(1 + \psi_{(RSII)})} \tag{28}
\]
Case I:

If
\[ \psi_{(RSII)} = -\frac{(\omega_{g_{ccg}} + 1) u + v}{(\omega_{g_{ccg}} + 2) + 2v} \]
then \( Z_{(RSII)} = 0 \). So we have \( q = -1 \), which confirms the accelerated expansion of the universe.

Case II:

If
\[ \psi_{(RSII)} = -1, \]
then \( Z_{(RSII)} \rightarrow -\infty \). Therefore \( q \rightarrow -\infty \). Hence in this case we have the super accelerated expansion of the universe.

**Note:** In both the above cases we see that \( \psi_{(RSII)} < 0 \). We know that \( \psi_{(RSII)} = \frac{\rho}{2\lambda} \). Since the energy density, \( \rho \) is always positive, therefore \( \lambda \) should be negative \( (\lambda < 0) \). **Hence in order to realize the recent cosmic acceleration the RS II brane model should possess a negative brane tension.** It is known from \[43\] standard model fields are confined on the negative tension (or visible) brane rather than the positive tension ("hidden") brane of the RS II model. Hence our result is consistent with the basic idea of formulation of the RS II brane model.

In this scenario, the Hubble parameter can be obtained as,
\[ H = \frac{2}{3Z_{(RSII)}} \]
where the integration constant has been ignored. Integration of equation (29) yields
\[ a(t) = a_0 t^{\frac{2}{3Z_{(RSII)}}} \]
which gives the power law form of expansion of the universe. In order to have an accelerated expansion of universe we must have \( 0 < Z_{(RSII)} < \frac{2}{3} \). Using this range of \( Z_{(RSII)} \) in the equation \( q^c_{(RSII)} = -1 + \frac{3}{2} Z_{(RSII)} \), i.e., eqn. (28), we get the range of \( q^c_{(RSII)} \) as \(-1 < q^c_{(RSII)} < 0 \). This is again consistent with an accelerated expansion of the universe.

2. Statefinder Parameters

As so many cosmological models have been developed, so for discrimination between these contenders, Sahni et al \[63\] proposed a new geometrical diagnostic named the statefinder pair
\{r, s\}, where \( r \) is generated from the scale factor \( a \) and its derivatives with respect to the cosmic time \( t \) upto the third order and \( s \) is a simple combination of \( r \) and the deceleration parameter \( q \). Clear differences for the evolutionary trajectories in the \( r - s \) plane have been found \[64–66\]. The statefinder parameters are defined as follows,

\[
 r \equiv \frac{\ddot{a}}{a H^3}, \quad s \equiv \frac{r - 1}{3(q - 1/2)}.
\]  

(31)

The expressions for the statefinder pair (eqn.(31)) in the RS II model can be obtained in the form

\[
r_{(RSII)} = \left(1 - \frac{3Z_{(RSII)}}{2}\right) \left(1 - 3Z_{(RSII)}\right).
\]

(32)

and

\[
s_{(RSII)} = Z_{(RSII)}
\]

(33)

The trajectories in the \( r - s \) plane for various existing models can exhibit quite different behaviours. The deviation of these trajectories from the \((0, 1)\) point defines the distance of a given model from the \( \Lambda \)CDM model. The statefinder pair \( \{r, s\} \) can successfully differentiate between a wide variety of cosmological models including a cosmological constant, quintessence, Chaplygin gas, and interacting dark energy models. In a given model the pair \( \{r, s\} \) can be computed and the trajectory in the \( r - s \) plane can be drawn. Furthermore, the values of \( r, s \) can be extracted from future observations \[67, 68\].

Note: It is quite interesting to note that the pair \( \{r_{(RSII)}, s_{(RSII)}\} \) yields the \( \Lambda \)CDM model \( \{r_{EG}, s_{EG}\} = \{1, 0\} \) when \( Z_{(RSII)} = 0 \).

III. MODEL 2: DGP BRANE MODEL

A simple and effective model of brane-gravity is the Dvali-Gabadadze-Porrati (DGP) braneworld model \[55–57\] which models our 4-dimensional world as a FRW brane embedded in a 5-dimensional Minkowski bulk. It explains the origin of DE as the gravity on the brane leaking to the bulk at large scale. On the 4-dimensional brane the action of gravity is proportional to \( M_p^2 \) whereas in the bulk it is proportional to the corresponding quantity in 5-dimensions. The model is then characterized by a cross over length scale \( r_c = \frac{M_p^2}{2M_5^2} \) such that gravity is 4-dimensional theory at scales \( a << r_c \) where matter behaves as pressureless dust, but gravity leaks out into the bulk at scales \( a >> r_c \) and matter approaches the behaviour of a cosmological constant. Moreover it has been shown that the standard Friedmann cosmology can be firmly embedded in DGP brane.
It may be noted that in literature, standard DGP model has been generalized to (i) LDGP model by adding a cosmological constant \[58\], (ii) QDGP model by adding a quintessence perfect fluid \[59\], (iii) CDGP model by Chaplygin gas \[60\] and (iv) SDGP by a scalar field \[61\]. In \[62\] the DGP model has been analysed by adding Holographic DE (HDE).

While flat, homogeneous and isotropic brane is being considered, the Friedmann equation in DGP brane model \[55–57\] is modified to the equation

\[
H^2 = \left( \sqrt{\frac{\rho}{3} + \frac{1}{4r_c^2} + \epsilon \frac{1}{2r_c}} \right)^2, \tag{34}
\]

where \(H = \frac{\dot{a}}{a}\) is the Hubble parameter, \(\rho\) is the total cosmic fluid energy density and \(r_c = \frac{M_p^2}{2M_5^2}\) is the cross-over scale which determines the transition from 4D to 5D behaviour and \(\epsilon = \pm 1\) (choosing \(M_p^2 = 8\pi G = 1\)). For \(\epsilon = +1\), we have standard DGP(+) model which is self accelerating model without any form of DE, and effective \(w\) is always non-phantom. However for \(\epsilon = -1\), we have DGP(−) model which does not self accelerate but requires DE on the brane. It experiences 5D gravitational modifications to its dynamics which effectively screen DE. Brane world scenario is actually a modified gravity theory. If we write the Einstein equation for brane world in terms of Einstein gravity then the extra term can be treated as the effective DE. But that is not the physical DE. Moreover this DE is applicable only in Einstein gravity. But here we will consider the physical DE in brane world. So we have introduced the GCCG type fluid in brane.

Consequently using the Friedmann equation (34) and the conservation equations, we obtain the modified Raychaudhuri equation

\[
\left( 2H - \frac{\epsilon}{r_c} \right) \dot{H} = -H (\rho + p), \tag{35}
\]

A. DYNAMICAL SYSTEM ANALYSIS

Just like the previous model, here also we proceed to perform the dynamical system analysis for DGP brane model:

The system is obtained by using the equations (1), (9), (10), (11), (34) and(35) as given below,

\[
\frac{du}{dx} = -3b(u+v)-3u\left(1+\omega_{gccg}^{DGP}\right)+\frac{2r_c^4(1-u-v)^4}{3(1+u+v)}\left\{\frac{9u(u+v)}{r_c^4(1-u-v)^4} - \left\{C + \left(\frac{9u^2}{r_c^4(1-u-v)^4} - C\right)^{-w}\right\}\right\}\]

\[
\frac{dv}{dx} = \omega_{gccg}^{DGP} + \frac{9u^2}{r_c^4(1-u-v)^4} - C
\]
\[ \frac{dv}{dx} = 3b(u + v) - 3v + \frac{2r_c^4 v (1 - u - v)^4}{3u (1 + u + v)} \left[ \frac{9u(u + v)}{r_c^4 (1 - u - v)^4} - \left\{ C + \left( \frac{9u^2}{r_c^4 (1 - u - v)^4} - C \right)^{-w} \right\} \right] \quad (37) \]

Where, \( \omega_{gccg}^{DGP} \) is the EoS parameter for GCCG in DGP brane determined as,

\[ \omega_{gccg}^{DGP} = \frac{p_{gccg}}{\rho_{gccg}} = -\frac{r_c^4 (1 - u - v)^4}{9u^2} \left[ C + \left\{ \frac{9u^2}{r_c^4 (1 - u - v)^4} - C \right\}^{-w} \right] \quad (38) \]

For mathematical simplicity, here also we have considered \( \alpha = 1 \).

**Note:** It is to be noted that \( \varepsilon \) has been considered as \(-1\) (DGP(-) model). Hence it does not produce a self-accelerating scenario of the brane model, but needs GCCG as dark energy (interacting) in order to realize the accelerating scenario. Thus the introduction of interacting GCCG in DGP brane is properly justified.

### 1. CRITICAL POINTS

The critical points for the above system (eqns. (36) and (37)) are calculated by putting \( \frac{du}{dx} = 0 = \frac{dv}{dx} \). Here also due to highly complicated forms of the equations, it is difficult to get an explicit solution in terms of all the parameters. So we find the following solution in terms of the interaction parameter as below.

\[ u_{1c} = \frac{1}{2} \left( 1 + \sqrt{1 - 4b} \right), \quad v_{1c} = \frac{1}{2} \left( 1 - \sqrt{1 - 4b} \right) \quad (39) \]

\[ u_{2c} = \frac{1}{2} \left( 1 - \sqrt{1 - 4b} \right), \quad v_{2c} = \frac{1}{2} \left( 1 + \sqrt{1 - 4b} \right) \quad (40) \]

The other variables are taken as:

\[ w = -1, \quad C = -1, \quad r_c = 1, \]

It is obvious from the above values that the critical point exists only for \( b \leq \frac{1}{4} \). The critical point correspond to the era dominated by DM and GCCG type DE. For the critical point \( (u_{ic}, v_{ic}) \), \( i = 1, 2 \), the equation of state parameter (eqn. 38) of the interacting DE takes the form:
\[ \omega_{\text{DGP}} = - \frac{r_c^4 (1 - uic - vic)^4}{9u_{ic}^2} \left[ C + \left\{ \frac{9u_{ic}^2}{r_c^4 (1 - uic - vic)^4} - C \right\}^{-w} \right] \] (41)

where \( i = 1, 2 \)

2. **STABILITY AROUND CRITICAL POINT**

Now if we write \( \hat{f} = \frac{du}{dx} \) and \( \hat{g} = \frac{dv}{dx} \), then we can obtain the following expressions

\[
\delta \left( \frac{du}{dx} \right) = \left[ \partial_u \hat{f} \right]_c \delta u + \left[ \partial_v \hat{f} \right]_c \delta v
\] (42)

and

\[
\delta \left( \frac{dv}{dx} \right) = \left[ \partial_u \hat{g} \right]_c \delta u + \left[ \partial_v \hat{g} \right]_c \delta v
\] (43)

where

\[
\partial_u \hat{f} = \frac{1}{3u^2 (u + v + 1)^2} \left\{ -9u^2 + Cr_c^4 (u + v - 1)^4 \right\} \left\{ -C + \frac{9u^2}{r_c^4 (u + v - 1)^4} \right\}^{-w} \left[ -C^2 r_c^8 (u + v - 1)^7 \right.
\]

\[
\left\{ 3u^3 + (v - 1) (v + 1)^2 + u^2 (v + 3) - u (v + 1) (v + 5) \right\} \left\{ -C + \frac{9u^2}{r_c^4 (u + v - 1)^4} \right\}^w - Cr_c^4 (u + v - 1)^3
\]

\[
\left\{ r_c^4 (u + v - 1)^4 \left( 3u^3 + (v - 1) (v + 1)^2 + u^2 (v + 3) - u (v + 1) (v + 5) \right) + 9u^2 \left( -C + \frac{9u^2}{r_c^4 (u + v - 1)^4} \right)^w
\]

\[
\left( b (u + v - 1) (u + v + 1)^2 - 2 (2u^3 - v + v^3 + 2u^2 (v + 1) + u (v - 4) (v + 1)) \right) \right\} + 9u^2 \left\{ 9u^2 (-C +
\]

\[
\frac{9u^2}{r_c^4 (u + v - 1)^4} \right\}^w \left( 1 - 2u - (u + v)^2 + b (u + v + 1)^2 \right) + r_c^4 (u + v - 1)^3 \left( (v - 1) (v + 1)^2 (1 + 2w)
\]

\[
+ u^3 (3 + 2w) + u^2 (3 + v + 2w - 2vw) - u (v + 1) (5 + v + 2 (v + 1) w) \right) \right\}
\] (44)

\[
\partial_v \hat{f} = \frac{1}{3} \left[ -9b - \frac{2 \left\{ -C + \frac{9u^2}{r_c^4 (u + v - 1)^4} \right\}^{-w}}{u (u + v + 1)^2} \left\{ -9u^2 + Cr_c^4 (u + v - 1)^4 \right\} \right] \left\{ C^2 r_c^8 (u + v - 1)^7 \left( u + u^2 - uv - 2 (1 + v)^2 \right) \right\}
\]
\[
\left\{ -C + \frac{9u^2}{r_c^4(u + v - 1)^4}\right\}^w + Cr^4_c(u + v - 1)^3 \left\{ r^4_c(u + v - 1)^4 \left( u + u^2 - uv - 2(v + 1)^2 \right) - 9u^2(-3 + 2u \right. \\
+ u^2 - (u + 3)v - 2v^2 \left. \right\} - 9u^2 \left\{ -C + \frac{9u^2}{r_c^4(u + v - 1)^4}\right\}^w \\
+ r^4_c(u + v - 1)^3 \left( u - uv - 2(v + 1)^2(w + 1) + u^2(1 + 2w) \right) \right]\] (45)

\[
\partial_u \tilde{g} = \frac{1}{3u^2(u + v + 1)^2} \left\{ -9u^2 + Cr^4_c(u + v - 1)^4 \right\}^w \left[ -2C^2r^8c(u + v - 1)^7 \\
+ (1 + 2u^2 - v^2 + u(v + 5)) \left\{ -C + \frac{9u^2}{r_c^4(u + v - 1)^4}\right\}^w + Cr^4_c(u + v - 1)^3 \left( -2r^4_c v(u + v - 1)^4(1 + 2u^2 \\
- v^2 + u(v + 5)) + 9u^2 \left\{ -C + \frac{9u^2}{r_c^4(u + v - 1)^4}\right\}^w \left( b(u + v - 1)(u + v + 1)^2 + 2v(2u(u + 3) + v + uw \\
- v^2)) + 9u^2 \left\{ -9u^2 \left\{ -C + \frac{9u^2}{r_c^4(u + v - 1)^4}\right\}^w \left( 2v + b(u + v + 1)^2 \right) + 2r^4_c(u + v - 1)^3(2u^2(w + 1 \\
- (v^2 - 1)(2w + 1) + u(5 + v + 4w)) \right]\right\} \right] (46)

\[
\partial_c \tilde{g} = \frac{1}{3} \left[ -9 + 9b - \frac{2r^4_c v(u + v - 1)^4 \left\{ -C + \frac{9u(u+v)}{r_c^4(u+v-1)^3} - \left( -C + \frac{9u^2}{r_c^4(u+v-1)^3}\right)^{-w}\right\}}{u(u + v + 1)^2} \right] \\
+ \frac{8r^4_c v(u + v - 1)^3 \left\{ -C + \frac{9u(u+v)}{r_c^4(u+v-1)^3} - \left( -C + \frac{9u^2}{r_c^4(u+v-1)^3}\right)^{-w}\right\}}{u(u + v + 1)} + \frac{2r^4_c(u + v - 1)^4 \left\{ -C + \frac{9u(u+v)}{r_c^4(u+v-1)^3} - \left( -C + \frac{9u^2}{r_c^4(u+v-1)^3}\right)^{-w}\right\}}{u(u + v + 1)} \\
+ \frac{18v \left\{ 1 + 3v + u \left( 3 + 4w \left( -C + \frac{9u^2}{r_c^4(u+v-1)^3}\right)^{-1-w}\right) \right\}}{(u + v - 1)(u + v + 1)} \right] (47)
\]
The Jacobian matrix of the above system is given by,

\[
J^{(DGP)}_{(u,v)} = \begin{pmatrix}
\frac{\delta \hat{f}}{\delta u} & \frac{\delta \hat{f}}{\delta v} \\
\frac{\delta \hat{g}}{\delta u} & \frac{\delta \hat{g}}{\delta v}
\end{pmatrix}
\]

Here we notice a very interesting feature of the model. We see that \( u_c + v_c = 1 \) for this model. Since the denominators of the above partial derivatives contain the term \( u + v - 1 \), hence they become indeterminate at the critical point, leading to a highly unstable scenario (chaos). As a result, determination of eigenvalues at the critical point is not possible for this model. So we resort to an alternative technique for our evaluations. We will consider a very small neighbourhood of the critical point, thus avoiding the chaos, and then try to calculate the eigenvalues at any convenient point in the neighbourhood sufficiently close to the critical point. This is purely based on the assumption that a sufficiently close neighbouring point will retain most of the properties of the critical point except the indeterminate nature. Our evaluations led us to the following eigenvalues for the given system:

\[ \lambda_1 = -2.97091, \quad \lambda_2 = 0.734899 \]

Hence it is a saddle point.
Description of figures:

Figs.14, 15, 16, 17, 18: The dimensionless density parameters are plotted against e-folding time for different values of interactions, $b$. The initial condition is $v(0) = 0.9, u(0) = 0.2$. The other parameters are fixed at $w = -1, C = -1$ and $r_c = 10$. The interactions are respectively $b = 0.01, 0.1, 0.5, 1$ and 10.

Fig.19: The dimensionless density parameters are plotted against e-folding time for same initial condition. The initial condition is $v(0) = 0.6, u(0) = 0.6$. The other parameters are fixed at $w = -1, C = -1, r_c = 10$ and $b = 0.01$.

Figs.20, 21, 22: The phase diagram of the parameters depicting an attractor solution are obtained for different values of interactions. The initial conditions chosen are $v(0) = 0.5, u(0) = 0.6$ (green); $v(0) = 0.6, u(0) = 0.6$ (blue); $v(0) = 0.7, u(0) = 0.6$ (red); $v(0) = 0.8, u(0) = 0.6$ (brown). Other parameters are fixed at $w = -1, C = -1, r_c = 10$ and $b = 0.01$. The interactions are respectively $b = 0.01, 0.1$ and 1.

Fig. 23: The ratio of density parameters is shown against e-folding time. The initial conditions chosen are $v(0)=0.6, u(0)=0.6$. The other parameters are fixed at $w = -1, C = -1, r_c = 10$ and $b = 0.01$.

Fig. 24: The deceleration parameter is plotted against the EoS parameter. Other parameters are fixed at $\psi_{(RSII)} = -0.5$.

Figs.25, 26: The statefinder parameter $r$ and $s$ are plotted against the EoS parameter. Other parameters are fixed at $b = 0.1, r_c = 10$ and $\sigma = 0.001$.

3. NATURE OF COSMOLOGICAL PARAMETERS

1. Deceleration Parameter:
We calculate the deceleration parameter \( q = -1 - (\dot{H}/H^2) \), in this model as,

\[
q^{(DGP)} = -1 + \frac{3}{2} \left[ \frac{4r_c^2\left(1 + \omega^{(DGP)}\rho_{gccg}/\rho\right)}{\sqrt{4r_c^2 + 3\sigma\left(\sqrt{4r_c^2 + 3\sigma + \epsilon\sqrt{3\sigma}}\right)}} \right]
\]  

(48)

where \( \sigma = \frac{1}{\rho} \). The above deceleration parameter can be written in terms of dimensionless density parameter \( \Omega_{mcg} = \frac{\rho_{mcg}}{\rho} \) as

\[
q^{(DGP)} = -1 + \frac{3}{2} \left[ \frac{4r_c^2\left(1 + \omega^{(DGP)}\Omega_{gccg}\right)}{\sqrt{4r_c^2 + 3\sigma\left(\sqrt{4r_c^2 + 3\sigma + \epsilon\sqrt{3\sigma}}\right)}} \right]
\]  

(49)

Now like the previous case we obtain \( \Omega_{mcg} = \frac{\rho_{mcg}}{\rho} = \frac{u}{u+v} \). So from the previous equation we get,

\[
q^{(DGP)} = -1 + \frac{3}{2} \left[ \frac{4r_c^2\left(1 + \omega^{(DGP)}\frac{u}{u+v}\right)}{\sqrt{4r_c^2 + 3\sigma\left(\sqrt{4r_c^2 + 3\sigma + \epsilon\sqrt{3\sigma}}\right)}} \right]
\]  

(50)

We consider the first stable critical point. At the critical point \((u, v) \rightarrow (u_c, v_c)\). Hence using equation (50) we get

\[
q^{(DGP)}_c = -1 + \frac{3}{2} Z^{(DGP)}_c , \text{ where } Z^{(DGP)}_c = \frac{4r_c^2\left(1 + \omega^{(DGP)}\frac{u_c}{u_c+v_c}\right)}{\left\{\sqrt{4r_c^2 + 3\sigma\left(\sqrt{4r_c^2 + 3\sigma + \epsilon\sqrt{3\sigma}}\right)}\right\}}.
\]  

(51)

Moreover we see that for \( \sigma = 0 \), i.e., for \( \rho \rightarrow \infty \), we retrieve the results for Einstein gravity, as given below,

\[
q^{EG} = -1 + \frac{3}{2} \left(1 + \omega^{(DGP)}\Omega_{gccg}\right)
\]  

(52)

Considering the DGP(-) model we get,

\[
q^{(DGP)}_c = -1 + \frac{3}{2} Z^{(DGP)}_c , \text{ where } Z^{(DGP)}_c = \frac{4r_c^2\left(1 + \omega^{(DGP)}\frac{u_c}{u_c+v_c}\right)}{\left\{\sqrt{4r_c^2 + 3\sigma\left(\sqrt{4r_c^2 + 3\sigma + \epsilon\sqrt{3\sigma}}\right)}\right\}}.
\]  

(53)

If we do not want to spoil the successes of the ordinary cosmology, we have to assume the \( r_c \) is of the order of the present Hubble scale \( H_0^{-1} \). Hence \( r_c \neq 0 \). In the previous model we have seen that for \( Z^{(DGP)} = 0 \), \( q = -1 \) and the recent cosmic acceleration is effectively realized. From equation (53), we see that \( 1 + \omega^{(DGP)}\frac{u_c}{u_c+v_c} = 0 \), since \( r_c \neq 0 \). From this relation we get,

\[
\omega^{(DGP)} = -\frac{u_c + v_c}{u_c}
\]  

(54)

Now since \( \frac{u_c + v_c}{u_c} > 1 \), we should have,

\[
\omega^{(DGP)} < -1
\]  

(55)
The above range for the EoS parameter indicates the phantom era for GCCG type DE.

Hence it is evident that when \( \omega_{\text{gccg}}^{(\text{DGP})} = -\frac{u_{c} + v_{c}}{u_{c}}, \) \( Z^{(\text{DGP})} = 0, \) and hence \( q = -1. \) This is consistent with the recent cosmic acceleration.

Moreover the Hubble parameter can be obtained as,

\[
H = \frac{2}{3Z^{(\text{DGP})}t},
\]

where we have ignored the integration constant. Integration of eqn.(56) yields

\[
a(t) = a_{0}t^{\frac{2}{3Z^{(\text{DGP})}}},
\]

which gives a power law form of the expansion. In order to realize the accelerating scenario of the universe, we should have \( \frac{2}{3Z^{(\text{DGP})}} > 1 \) i.e., \( 0 < Z^{(\text{DGP})} < \frac{2}{3}. \) Using this range of \( Z^{(\text{DGP})} \) in the equation \( q^{(\text{DGP})} = -1 + \frac{2}{3Z^{(\text{DGP})}}. \) We get the range of \( q^{(\text{DGP})} \) as \( -1 < q^{(\text{DGP})} < 0. \) Therefore the deceleration parameter is negative and hence the result is consistent with the fact that the universe is undergoing an accelerated expansion.

2. Statefinder Parameters:

In the DGP brane model, we have the following expressions for the statefinder parameters, \( r \) and \( s \) as given below,

\[
r^{(\text{DGP})} = \left(1 - \frac{3Z^{(\text{DGP})}}{2}\right) \left(1 - 3Z^{(\text{DGP})}\right),
\]

and

\[
s^{(\text{DGP})} = Z^{(\text{DGP})}.
\]

IV. DETAILED GRAPHICAL STUDY OF PHASE PLANE ANALYSIS

Figs. 1 to 5 and 14 to 18, shows the plots of density parameters \( u \) and \( v, \) respectively for RS II and DGP brane model. We see that in case of RS II brane as the brane tension \( \lambda \) decreases (Figs. 1 to 4), more and more irregularity creep in, as far as the DE density parameter \( u \) is concerned. But all the four figures show an energy dominated universe, consistent with observational data. In fig. 5, with a larger value of interaction, we get a matter dominated universe (unphysical situation), which really indicates that the interaction coupling parameter, \( b \) should be a small positive value. In case of DGP brane, we see that with the increase in interaction between energy and matter,
the density parameters become more and more comparable to each other, giving a solution to the cosmic coincidence problem. An identical result was obtained for GCCG in LQC in [23]. Figs. 6 and 19, shows almost the same results for identical scenario. In figs. 7, 8 and 9, Phase diagrams for RS II brane have been obtained. The figs. 7 and 8 have been generated for different values of brane tension $\lambda$. It is seen that as the tension decreases there is a greater tendency of the solution moving towards an attractor, thus giving a perfect attractor solution. In fig. 9 with an higher interaction, we get a far better attractor solution, but the direction of flow is reversed. This can be attributed to the fact that as the interaction grows in magnitude DE interferes more and more with DM, until the matter loses its dominance and its place is taken by the energy, thus giving a perfectly energy dominated scenario. This shift of power may be responsible for many unexplained phenomena of cosmology including the present one. In the figs. 20, 21 and 22, we have obtained the phase diagrams for the DGP brane model, with gradually increased values of interaction. Just like the previous model we see that with the increase in interaction, there is a greater tendency of the flow going towards a specific attractor point. In the figs. 10 and 23, plots for the ratio of the density parameters have been generated against the e-folding time, for the two models respectively. In the plots, it is evident that $\frac{\rho}{\dot{\rho}}$ decreases with time, thus exhibiting an energy dominated scenario.

Figs. 11 and 24 shows the plot of deceleration parameter, $q$ against the EoS parameter, for the two models respectively. In the fig. 11, $q$ remain in the negative level throughout the quintessence era thus exhibiting the recent cosmic acceleration for the RS II model. The only condition being the negativity of the brane tension, $\lambda$. But in the fig. 24, we see that $q$ remains in the negative level for $\omega < -1$, i.e. only in the phantom region. This not only shows that GCCG is a DE fluid with far lesser negative pressure compared to other dark energy models, but also it shows that the combination of GCCG in DGP brane model gives an inferior model of the universe compared to the other combinations, like MCG in DGP brane (refer to [15]). The statefinder parameters have been plotted in the figures 12 and 13 for RS II brane and in figs. 25 and 26 for the DGP brane. In the figs. 12 and 13, we see that $r$ decreases whereas $s$ increases with the increase in EoS parameter. But in case of DGP brane, $r$ initially decrease, and then increase after reaching a minimum value. $s$ increases with the increases in EoS parameter just like the previous model.

V. STUDY OF FUTURE SINGULARITY

We speculate that any energy dominated model of the universe undergoing an accelerated expansion will result in a future singularity. The study of dynamics of an accelerating universe in
the presence of DE and DM is in fact incomplete without the study of these singularities, which are the ultimate fate of the universe. It is known that the universe dominated by phantom energy ends with a future singularity known as Big Rip [69], due to the violation of dominant energy condition (DEC). But other than this there are other types of singularities as well. Nojiri et al [4] studied the various types of singularities that can result from a phantom energy dominated universe. These possible singularities are characterized by the growth of energy and curvature at the time of occurrence of the singularity. It is found that near the singularity quantum effects becomes very dominant which may alleviate or even prevent these singularities. So it is extremely necessary to study these singularities and classify them accordingly so that we can search for methods to eliminate them. The appearance of all four types of future singularities in coupled fluid dark energy, $F(R)$ theory, modified Gauss-Bonnet gravity and modified $F(R)$ Horava-Lifshitz gravity was demonstrated in [18]. The universal procedure for resolving such singularities that may lead to bad phenomenological consequences was proposed. In Rudra et al [15] it has been shown that in case of MCG in Brane-world, both Type I and Type II singularities are possible. In Chowdhury et al [23] it was shown that in case of GCCG in LQC, the universe is absolutely free from any type of singularities. We proceed to study the singularities for the present case:

A. TYPE I Singularity (Big Rip singularity)

If $\rho \to \infty$, $|p| \to \infty$ when $a \to \infty$ and $t \to t_s$. Then the singularity formed is said to be the Type I singularity.

In the present case by considering the GCCG equation of state from equation (1) we find that there is no possibility for TypeI singularity, i.e., Big Rip singularity, since $\alpha > 0$. This is in absolute accordance with P. F. Gonzalez-Diaz who has successfully shown that by considering GCCG as the DE, Big Rip can easily be avoided, thus giving a singularity free late universe.

B. TYPE II Singularity (Sudden singularity)

If $\rho \to \rho_s$ and $\rho_s \sim 0$, then $|p| \to 0$ for $t \to t_s$ and $a \to a_s$, then the resulting singularity is called the Type II singularity.

In this case we consider the equation of state for GCCG, like the previous case for our investigation. We see that if $\rho \to \rho_s$ and $\rho_s \sim 0$, then $|p| \to 0$ for $t \to t_s$ and $a \to a_s$. Hence there is no possibility of the type II singularity or the sudden singularity in case of GCCG, primarily because
\( \alpha > 0 \) and \( w < 0 \).

C. TYPE III Singularity

For \( t \to t_s \), \( a \to a_s \), \( \rho \to \infty \) and \( |p| \to \infty \). Then the resulting singularity is Type III singularity. It is quite evident from the equation of state of GCCG that it does not support this type of singularity.

D. TYPE IV Singularity

For \( t \to t_s \), \( a \to a_s \), \( \rho \to 0 \) and \( |p| \to 0 \). Then the resulting singularity is Type IV singularity. This type of singularity is not supported by GCCG type DE.

As a remark, one should stress that our consideration is totally classical. Nevertheless, it is expected that quantum gravity effects may play significant role near the singularity. It is clear that such effects may contribute to the singularity occurrence or removal too. Unfortunately, due to the absence of a complete quantum gravity theory only preliminary estimations may be done.

VI. CONSEQUENCES OF THE EXPANDING UNIVERSE: DISTANCE MEASUREMENT OF THE UNIVERSE

Cosmography is a field where we are concerned with the measurement of the Universe. In fact there are many ways to specify the distance between two points. This is primarily because, in the expanding and accelerating Universe, the distances between co-moving objects are constantly changing, and Earth-bound observers look back in time as they look out in distance. The unifying aspect is that all distance measures somehow measure the separation between events on radial null trajectories, i.e., trajectories of photons which terminate at the observer. Here we will compute and discuss various cosmological distance measures such as the look-back time, luminosity distance, proper distance, angular diameter distance, co-moving volume, distance modulus and probability of intersecting objects.

A. LOOK-BACK TIME

As light travels with finite speed, it takes time for it to cover the distance related to the redshift it encountered. The difference between the age of the Universe now (at observation) and the age
of the Universe at the time the photons were emitted (according to the object) is defined as the Lookback time to an object. So, a look into space is always a look back in time. It is used to predict properties of high redshift objects with evolutionary models, such as passive stellar evolution for galaxies. Thus if a photon emitted by a source at the instant $t$ and received at the time $t_0$ then the photon travel time or the lookback time $t_0 - t$ is defined by

$$t_0 - t = \int_{a_0}^{a} \frac{da}{\dot{a}}$$

(60)

where $a_0$ is the present value of the scale factor of the universe and can be obtained from (57) at $t = t_0$. The redshift is an important observable parameter as they can be measured easily from the spectral lines, and the redshift increases with the recession of the object from us. Look-back time is used to predict properties of high-redshift objects with evolutionary models, such as passive stellar evolution for galaxies. The redshift $z$ can be defined by

$$\frac{a_0}{a} = 1 + z = \left(\frac{t_0}{t}\right)^{\frac{2}{3}}$$

(61)

which gives the look-back time in the following form

$$t - t_0 = \frac{2}{3ZH_0} \left\{ \frac{1}{(1 + z)^{\frac{2}{3}}} - 1 \right\}$$

(62)

For accelerating universe we have already get $Z < \frac{2}{3}$. Early universe is represented by $z \to \infty$ implies $t \to 0$ and late universe $z \to -1$, which equivalently implied $t \to \infty$. Also $z \to 0$ gives the present age $t \to t_0$ of the universe.

**B. PROPER DISTANCE**

We know that light needs time to get from an object to the observer. Therefore we can define a distance that may be measured between the observer and the object with a ruler at the time the light was emitted, as the proper distance. When a photon emitted by a source at time $t_0$ and received by an observer at time $t$ then the proper distance between them is defined by

$$d = a_0 \int_{a}^{a_0} \frac{da}{a \dot{a}} = a_0 \int_{t}^{t_0} \frac{dt}{a}$$

(63)
which gives

\[ d = \frac{2}{H_0 (3Z - 2)} \left\{ 1 - \frac{1}{(1 + z)^{\frac{3Z}{2} - 1}} \right\} \]  

(64)

As far as the current epoch is concerned the proper distance may also called the co-moving distance (line of sight) of the Universe. So between two nearby objects in the Universe, the distance between them remains constant with epoch if the two objects are moving with the Hubble flow. In other words, it is the distance between them which would be measured with rulers at the time they are being observed divided by the ratio of the scale factor of the Universe \(a\) at that time to \(a\) now). Next thing to be defined is the transverse co-moving distance, which is a quantity used to get the co-moving distance perpendicular to the line of sight. For flat universe, the transverse co-moving distance is always identical to the co-moving distance (line of sight). That means the transverse co-moving distance = proper distance = \(d\), for any model following the power law form of expansion.

C. LUMINOSITY DISTANCE

If \(L\) be the total energy emitted by the source per unit time and \(\ell\) be the apparent luminosity of the object then the luminosity distance is defined by \[70, 71\]

\[ d_L = \left( \frac{L}{4\pi\ell} \right)^{\frac{1}{2}} = d (1 + z) = \frac{2}{H_0 (3Z - 2)} \left\{ (1 + z) - \frac{1}{(1 + z)^{\frac{3Z}{2}}} \right\} \]  

(65)

D. ANGULAR DIAMETER DISTANCE

We define the angular diameter of a light source of proper distance \(D\) observed at \(t_0\) by \[70, 71\]

\[ \delta = \frac{D (1 + z)^2}{d_L} \]  

(66)

Now the ratio of the source diameter to its angular diameter (in radians) is defined as the angular diameter distance \(d_A\) as furnished below,

\[ d_A = \frac{D}{\delta} = d_L (1 + z)^{-2} = d (1 + z)^{-1} \]  

(67)
For our models following power law form of expansion angular diameter distance ($d_A$) is given by,

$$d_A = \frac{2}{H_0 (3Z - 2)} \left\{ \frac{1}{1 + z} - \frac{1}{(1 + z)^{\frac{1}{3}Z - 2}} \right\}$$

(68)

It is used to convert angular separations in telescope images into proper separations at the source. It is famous for not increasing indefinitely as $z \to \infty$; it gets inverted at $z \sim 1$ and thereafter more distant objects actually appear larger in angular size. The angular diameter distance is maximum at

$$z_{\text{max}} = \left( \frac{2}{3Z - 2} \right)^{\frac{1}{3}Z - 2} - 1$$

(69)

and corresponding maximum angular diameter $d_A|_{\text{max}}$ taking the form

$$d_A|_{\text{max}} = \frac{1}{H_0 (3Z - 2)} \left[ 2^{1 + \frac{1}{3}Z - 2} \left( \frac{1}{3Z - 4} \right)^{\frac{1}{3}Z - 2} - 2 \left\{ 4^{\frac{1}{3}Z - 2} \left( \frac{1}{3Z - 4} \right)^{\frac{1}{3}Z - 2} \right\} \right]$$

(70)

All the above four parameters have been plotted against the redshift parameter, $z$ in fig. 27.
E. CO-MOVING VOLUME

The co-moving volume $V_C$ is the volume measure in which number densities of non-evolving objects locked into Hubble flow are constant with redshift. We define it as \cite{70, 71}

\[
dV_C = D_H \frac{(1 + z)^2 d_A}{E(z)} d\Omega dz = \frac{1}{H_0} (1 + z)^{2 - \frac{3d}{2}} d_A d\Omega dz
\]  

(71)

where $d\Omega$ is the solid angle element and $d_A$ is the angular diameter, $E(z) = \frac{H(z)}{H_0}$ and $D_H = \frac{c}{H_0}$ is the Hubble distance ($c$ is the velocity of light) and in our model we assume $c = 1$, $d = 1$, $H_0 = 72 \text{km/s/Mpc}$.

So, the co-moving volume is proper volume times the ratio of scale factors now to then to the third power. Co-moving volume element $\frac{dV_C}{dz}$ are drawn in figure 28. We see that there is a gradual increase with increase in redshift $z$.

F. DISTANCE MODULUS

The distance modulus is define by

\[
D_M = 5 \log \left( \frac{d_L}{10\text{pc}} \right)
\]

(72)

because it is the magnitude of difference between objects observed bolometric (i.e., integrated over all frequencies) flux and what it would be if it were at 10 pc (this was once thought to be the distance to Vega) and $d_L$ is the luminosity distance. Distance modulus $D_M$ as a function of redshift have been shown in figure 29. We see that for $z > 0$, we do not get any plot for $D_M$. But for $z < 0$, $D_M$ increases as $z$ decreases.

G. PROBABILITY OF INTERSECTING OBJECTS

It is defined as the incremental probability $dP$ that a line of sight will intersect one of the objects in redshift interval $dz$ at redshift $z$. It is given by \cite{70, 71}

\[
dP = n(z) \sigma(z) D_H \frac{(1 + z)^2}{E(z)} dz
\]

(73)

where $n(z)$ is the co-moving number density and $\sigma(z)$ areal cross-section. Assuming $n(z)\sigma(z) = 1$, we obtain

\[
dP = \frac{1}{H_0} (1 + z)^{2 - \frac{3d}{2}} dz
\]

(74)
For our model the expression of Probability of intersecting objects becomes

\[
P = \frac{2}{H_0 (6 - 3Z)} \left\{ (1 + z)^{3-\frac{3z}{1+z}} - 1 \right\}
\]  

(75)

In figure 30 we draw intersection probability \( P \) as a function of redshift. We see that, \( P \) increases as \( z \) increases.

VII. CONCLUSION

In this work, we have considered a combination of Generalized cosmic Chaplygin gas in standard Brane-world models. Two different models, namely the RS II brane and the DGP brane models have been considered for our evaluations. Our basic idea was to study the background dynamics of GCCG in detail when it is incorporated in brane gravity. Because of the complexity of the expressions it was impossible to find direct solutions for the system. So we resorted to dynamical system analysis for our computations. Dynamical system analysis was successfully carried out, critical points were found and the stability of the system around those critical points was tested.
Graphical analysis was done to get an explicit picture of the outcome of the work. In order to find a solution for the cosmic coincidence problem, a suitable interaction between DE and DM was considered. Figures of density parameters were drawn for different values of interaction. It was found that increase in interaction resulted in more and more comparable values of the density parameters of GCCG and DM. Since the tendency of DE domination over DM is lesser in case of GCCG compared to MCG, GCCG is identified as a dark fluid with a lesser negative pressure compared to MCG or any other forms of DE.

It was found that GCCG in RS II brane is consistent with the late cosmic acceleration only if the brane tension is negative. For GCCG in DGP brane an accelerated expansion is realized only in the phantom era of the DE. In the quintessence era there is no possibility of an accelerating scenario. This is a very important result as far as modern cosmology is concerned. This really shows the less effectiveness of GCCG as a DE compared to others, like MCG which produced an accelerating scenario in DGP brane in the quintessence era itself. From the above results we can come to the conclusion that although GCCG with a far lesser negative pressure compared to other DE models, can overcome the relatively weaker gravity of RS II brane, with the help of the negative brane tension, yet for the DGP brane model with much higher gravitation, the incompetency of GCCG is exposed, and it cannot produce the accelerating scenario until and unless it reaches the phantom era.

The dynamical system of equations characterizing the system was formed and a stable scaling solution was obtained. Hence this work can be considered to be a significant one, as far as the solution of cosmic coincidence problem is concerned. Study of future singularities had been carried out in detail. The model was investigated for all possible types of future singularities. From the above analysis we conclude that the combination of GCCG in brane gravity gives a perfect singularity free model (just like GCCG in LQC) for an expanding universe undergoing a late acceleration. Statefinder parameters were calculated and plots were generated for them. The evolutionary trajectories obtained, when compared with those of the ΛCDM model gave clear differences, thus characterizing the GCCG type DE irrespective of the theory of gravity. The deceleration parameter was also calculated and plotted against the EoS parameter. From these plots the notion of an accelerating universe was clearly realized. Finally some cosmographic parameters, involving different types of distance measurements have been studied for both the models, following
the power law form of expansion and plot were generated to characterize the parameters.

[1] Perlmutter, S. et al. - [Supernova Cosmology Project Collaboration], ApJ 517, 565(1999) [arXiv:astro-ph/9812133].
[2] Spergel, D. N. et al. - WMAP Collaboration, Astron. J. Suppl 148, 175(2003) [arXiv:astro-ph/0302209].
[3] Riess, A. G. et al. - [Supernova Search Team Collaboration], ApJ 607, 665(2004) [arXiv:0402512(astro-ph)].
[4] Nojiri, S., Odintsov, S.D. - Phys. Rev. D 70 103522(2004).
[5] Kamenshchik, A., Moschella, U. and Pasquier, V. - Phys. Lett. B 511, 265(2001).
[6] Gorini, V., Kamenshchik, A. and Moschella, U. Pasquier, V. - [arXiv:0403062(gr-qc)].
[7] Gorini, V., Kamenshchik, A. and Moschella, U. - Phys. Rev. D 67, 063509(2003).
[8] Alam, U., Sahni, V., Saini, T. D., Starobinsky, A.A. - MNRAS 344 1057(2003).
[9] Bento, M. C., Bertolami, O., Sen, A. A. - Phys. Rev. D 66, 043507(2002).
[10] Carturan, D., Finelli, F. - Phys. Rev. D 68 103501(2003).
[11] Barreiro, T., Sen, A.A. - Phys. Rev. D 70 124013(2004).
[12] Benaoum, H. B. - [arXiv:0205140(hep-th)].
[13] Debnath, U., Banerjee, A., Chakraborty, S. - Class. Quantum. Grav. 21, 5609(2004).
[14] Jamil, M., Debnath, U. - Astrophys. Space Sci. 333, 3 (2011).
[15] Rudra, P., Debnath, U., Biswas, R. - Astrophys. Space Sci. 339 53 (2012) [arXiv:1109.1481 (gr-qc)]
[16] Debnath, U. - Astrophys Space Sci 312 295299(2007)
[17] Chakraborty, W., Debnath, U. - Gravitation and Cosmology 16 223 (2010).
[18] Nojiri, S., Odintsov, S.D., Tsujikawa, S. - Phys. Rev. D 71063004(2005).
[19] Bamba, K., Matsumoto, J., Nojiri, S. - arXiv: 1109.1308[hep-th].
[20] Nojiri, S., Odintsov, S.D. - Phys. Lett. B 571 1(2003).
[21] Gonzalez-Diaz, P. F. - Phys. Rev. D 68 021303 (R)(2003).
[22] Chakraborty, W., Debnath, U., Chakraborty,S. - Grav. Cosmol. 13 294 (2007) [arXiv:0711.0079(gr-qc)]
[23] Chowdhury, R., Rudra, P. - [arXiv:1204.3531(gr-qc)]
[24] del Campo, S., Herrera, R., Pavon, D. - JCAP 0901 020(2009).[arXiv:0812.2210 (gr-qc)].
[25] Leon, G., Saridakis, E.N.: - Phys.Lett.B 693 1(2010). [arXiv:0904.1577 (gr-qc)].
[26] Jimenez, J.B., Maroto, A.L.: - AIP Conf.Proc. 1122,107(2009). [arXiv:0812.1970 (astro-ph)].
[27] Berger, M.S., Shojae, H.: - Phys.Rev.D 73,083528(2006). [ arXiv:0601086(gr-qc)].
[28] Zhang, X.: - Mod.Phys.Lett.A 20 2575(2005).
[29] Griest, K., Phys.Rev.D66:123501(2002). [arXiv:0202052].
[30] Jamil, M., Rahaman, F.: - Eur. Phys. J. C 64 97(2009).
[31] Jamil, M., Saridakis, E.N., Setare, M.R. :- Phys. Rev. D 81 023007(2010a).
[32] Jamil, M., Saridakis, E.N. :- JCAP 07 028(2010b).
[33] Jamil, M., Farooq, M.U. :- JCAP 03 001(2010c).
[34] Jamil, M., Sheykhi, A., Farooq, M.U. :- Int. J. Mod. Phys. D 19 1831(2010d).
[35] Setare, M. R. :- Phys.Lett B 642 1(2006).
[36] Setare, M. R. :- Eur. Phys. J. C 50 991(2007c).
[37] Hu, B., Ling, Y. :- Phys. Rev. D 73 123510(2006).
[38] Wu, P., Yu, H. :- Class. Quant. Grav. 24 4661(2007).
[39] Jamil, M. :- Int. J. Theor. Phys. 49 62(2010e).
[40] Jamil, M., Momeni, D., Rashid, M.A., :- Eur. Phys. J.C 71 1711(2011)[arXiv:1107.1558v1 [physics.gen-ph]].
[41] Dalal, N., Abazajian, K., Jenkins, E., Manohar, A.V., :- Phys. Rev. Lett 87 141302(2001)[arXiv:astro-ph/0105317v1].
[42] Rubakov, V. A.:-, Phys. Usp.44, 871(2001) [arXiv:0104152[hep-th]].
[43] Maartens, R.:-Living Rev. Relativity 7 7(2004)[arXiv:0312059[gr-qc]].
[44] Brax, P. et. al :- Rep. Prog.Phys. 67, 2183(2004)[arXiv:0404011[hep-th]].
[45] Csa ki, C.:- [arXiv:0404096[hep-th]].
[46] Randall, L., Sundrum, R.:-Phys. Rev. Lett. 83, 3770(1999a).
[47] Randall, L., Sundrum, R.:-Phys. Rev. Lett. 83, 4690(1999b).
[48] Maartens, R., Phys. Rev. D 62 084023 (2000).
[49] Shiromizu, T., Maeda, K. and Sasaki, M., Phys. Rev. D 62 024012 (2000).
[50] Maeda, K. and Wands, D., Phys. Rev. D 62 124009 (2000).
[51] Sasaki, M., Shiromizu, T. and Maeda, K., Phys. Rev. D 62 024008 (2000).
[52] Guo, Z.-K., Zhang, Y.-Z.:- Phys. Rev. D 71, 023501 (2005).
[53] Alcaniz, J.S., Lima, J.A.S.-: Phys. Rev. D 72, 063516(2005).
[54] Feng, C. et. al.-: Phys. Lett. B 665, 111(2008).
[55] Dvali, G. R. , Gabadadze, G., Porrati, M.-: Phys.Lett. B 485 208(2000)[arXiv:0005066[hep-th]].
[56] Deffayet, D.:- Phys.Lett. B 502 199(2001).
[57] Deffayet, D., Dvali, G.R., Gabadadze, G.-Phys.Rev.D 65 044023 (2002)[arXiv:0105068[astro-ph]].
[58] Lue,A., Starkman,G. D. :-Phys. Rev. D 70, 101501(2004) [arXiv:0408246[astro-ph]].
[59] Chimento,L. P.,Lazkoz, R.,Maartens, R., Quiros, I. :- JCAP 0609, 004(2006)[arXiv:0605450[astro-ph]].
[60] Bouhmadi-Lopez, Lazkoz, R.-: Phys. Lett. B 654, 51(2007) [arXiv:0706.3896 (astro-ph)].
[61] Zhang, H., Zhu, Z. H. :- Phys.Rev.D 75,023510(2007).
[62] Wu, X., Cai, R. G., Zhu, Z. H. :- Phys.Rev.D 77,043502(2008).
[63] Sahni, V., Saini, T. D., Starobinsky, A. A., Alam, U. JETP Lett. 77, 201 (2003).
[64] M.R. Setare, M. Jamil, Gen. Relativ. Gravit. (2011) 43:293-303
[65] M. Jamil, Int. J. Theor. Phys. 49: 2829-2840, 2010
[66] M. Jamil, M. Raza, U. Debnath, Astrophys. Space Sci. (2012) 337:799-803
[67] Albert, J. et al. [SNAP Collaboration], [arXiv:0507458 (astro-ph)].
[68] Albert, J. et al. [SNAP Collaboration], [arXiv:0507459 (astro-ph)].
[69] Caldwell, R.R., Kamiokowsky,M., Weinberg,N.N. :- Phys. Rev. Lett. 91 071301 (2003).
[70] M. Jamil and U. Debnath, Int. J. Theor. Phys. 50, 1602 (2011); A. I. Arbab, [astro-ph/9810239]
[42] D. W. Hogg, [astro-ph/9905116]; P. J. E. Peebles, Principles of Physical Cosmology, Princeton University
Press, Princeton (1993); E. W. Kolb and M. S. Turner, The Early Universe, Addison-Wesley, Redwood
City (1990).
[71] S. Weinberg, Gravitation and Cosmolgy: Principles and Applications of the General Theory of Rel-
ativity, John Wiley and Sons, New York (1972); D. W. Weedman, Quasar Astronomy, Cambridge
University, Cambridge (1986).